# On scalar growth systems governed by delayed nonlinear negative feedback

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**Marcus R.W. Martin**
aus Fritzlar-Obermöllrich (geb. 11.12.1972)

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# Introduction

Growth processes play a fundamental role in economics, ecology, engineering, chemistry, and many other important fields of science and technology. Therefore, these processes have been the subject of numerous investigations and this thesis is devoted to yield a first insight into the global dynamics of a special class of this kind of processes, namely, scalar growth processes governed by delayed nonlinear negative feedback.

### Prologue

Mathematically valuable growth models date back at least to Th. R. MALTHUS' "*Essay on the Principles of Population*" in 1798 where an instantaneous constant growth rate $r > 0$ (for a single population) was assumed. This naive approach provides the simplest model for scalar growth processes, and the dynamics of the population density $x$ is described by the ordinary differential equation

$$\dot{x} = rx \ . \tag{1}$$

Though mathematically uninteresting and biologically unrealistic, the naive model (1) can be regarded to be at the "core" of some more realistic growth models proposed later on if we take the following control-theoretic point of view:

Except for the stationary solution $x = 0$ every other solution of (1) is unbounded and exponentially growing ("escaping"). So, in some sense, one may ask for a procedure to "stabilize" the (exponentially unstable) system (1), i.e. to modify the growth law in such a way that one obtains more than the trivially bounded zero solution. (Biologically, this idea corresponds to the fact that resources of food and environment are usually limited which is neglected in the model (1)).

In other words, we aim to introduce an additional control term $u$ in order to diminish the growth of $x$. There seem to be two basic possibilities for doing so. First, one may think of a *multiplicative control* which leads to growth models of the form

$$\dot{x} = rx \cdot u \ ,$$

or, alternatively, a second possible modification might be to add an *additive control* term which yields

$$\dot{x} = rx + u \ .$$

Surprisingly enough, only the first approach usually occurs in literature (cf., e.g., Cushing [15], Frauenthal [19], Amann [3] or Wu [71]). The most prominent examples are the logistic equation

$$\dot{x} = rx \cdot (a - bx)$$

due to P.L. Verhulst (1838) and its "delayed version"

$$\dot{x}(t) = rx(t) \cdot (a - bx(t - \tau)) \tag{2}$$

introduced by G.E. Hutchinson (1948) where $a$, $b$ and $\tau$ denote positive constants. For a discussion of the underlying biological assumptions we refer to Cushing [15, p. 13f.] and the references therein.

It is noteworthy to mention the properties of the control term $u$ in the above examples. In both cases $u$ depends on the values of $x$, either instantaneously as in the classical equation or it depends on the past values of $x$ ($\tau$ time units ago) in the delayed logistic equation (2): such delays occur as simple realization of maturation times in population dynamics.

Equations in which the rate of change of $x$ involves the current as well as the past values of $x$ are called *differential delay equations* and serve as the simplest models which take the behaviour of a system in the past into account.

Furthermore, our control $u$ displays a so called *negative feedback* property with respect to the non-trivial stationary solution $x = \frac{a}{b}$: whenever the solution $x$ exceeds this "critical value", $u$ becomes negative such that the rate of change of $x$ is slowed down in order to prevent the solution from escaping. On the other hand, if the solution has values below $\frac{a}{b}$, the sign of $u$ is positive in order to accelerate solutions towards this value (or, to draw them back).

Hence, the Hutchinson equation (2) provides a model of a self-adjusting (and, therefore, autonomous) scalar growth system which is governed by a multiplicative negative feedback mechanism.

In view of the lack of knowledge about additive control mechanisms it is convenient to consider an additive control approach in more detail.

As in the multiplicative case we want to use control functions $u$ which incorporate the influence of the past and a negative feedback property (with respect to the trivial solution $x = 0$) since we still intend to stabilize (1). Therefore, $u$ should have the form

$$u := g(x(\cdot - \tau))$$

with some fixed time delay $\tau > 0$ and a continuous function $g : \mathbb{R} \to \mathbb{R}$ which has always the reverse sign than its argument $x(\cdot - \tau)$. Consequently, the rate of change of $x$ at time $t$ is decreased by $u$ whenever $x$ had a value above zero at time $t - \tau$ , or it is increased whenever $x$ was below zero at time $t - \tau$. Scaling the time variable with respect to the delay $\tau$ yields the model

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{$\mu, f$}$$

wherein $-\mu := r\tau$, $f := \tau g$, and $x(\tau\cdot)$ is replaced by $x$ again.

Hence, equation $(\mu, f)$ serves as a model for a scalar growth process with instantaneous growth rate $-\mu > 0$ which is additively controlled by delayed nonlinear negative feedback (as proposed in our title).

The choice of $\mu$ to be a negative real number may seem strange at first sight but there are two reasons for doing so: first, the wealth of results for the well-known equation $(\mu, f)$ with positive $\mu$ which apply for the general case $\mu \neq 0$ is directly accessible for our investigations such that elementary results can be taken over without further comments. Second, our results are directly comparable with the well-studied case $\mu \in \mathbb{R}^+$ (cf., e.g., [12, 33, 39, 63, 65] and the references therein).

Completely analogous models occur in economics where $x$ usually denotes the price of a single good or an interest rate (cf. MACKEY [38], UNDERWOOD and DAVIS [14], CHIARELLLA [13], or BÉLAIR and MACKEY [7] which contains an extensive bibliography), in electrical engineering and neural networks where $x$ is the current or voltage in a relay or neuron (cf. MINORSKY [46] or MILTON [45] and WU [72]), or even in the theory of automatic control of motors and robots (cf. UTKIN [59] and the references in the work of SHUSTIN and his prominent collaborators [20, 21, 22, 50, 54]).

Among all properties of dynamical systems oscillating behaviour seems to be the most desirable and interesting one. The reason for this is that most of the applications mentioned above display periodic changes of the scalar variable $x$: these refer to biological fluctuations of the size of a single species, self-sustained oscillations in electricity and mechanics, to business and growth cycles in stock or commodity markets or even to so called sliding modes in control theory.

Furthermore, we are to mention at this point that so called *slowly oscillating solutions* of $(\mu, f)$ play an important or – more precisely – dominating role in the global dynamics of decay delay equations $(\mu, f)$ where $\mu \geq 0$. Here, a solution of $(\mu, f)$ is called eventually slowly oscillating, if there exists a $t_0 \in \mathbb{R}_0^+$ such that any two of its zeros in $[t_0; +\infty)$ are distanced larger than the delay, i.e. whenever

$$\zeta - \zeta' > 1$$

holds for any two zeros $\zeta \neq \zeta'$, $\{\zeta, \zeta'\} \subset [t_0; +\infty)$, of a solution $x : [-1; +\infty) \to \mathbb{R}$ of $(\mu, f)$. As conjectured by KAPLAN and YORKE [30] and finally proved by MALLET-PARET and WALTHER [43], the set $\mathcal{S}$ of initial data for eventually slowly oscillating solutions $x : [-1; +\infty) \to \mathbb{R}$ is open and dense in the phase space $C := C([-1; 0], \mathbb{R})$.

Analogously, numerical experiments indicate that the set of initial data for bounded eventually slowly oscillating solutions of scalar growth systems $(\mu, f)$ is open and dense in the set of initial values that yield bounded solutions.

Therefore, in order to gain a first insight into the global dynamics of $(\mu, f)$ and because of their practical importance, we will focus on the existence and uniqueness of slowly oscillating periodic solutions of $(\mu, f)$ in this treatise. These problems will be the content of the central chapters, Chapter 3 and 4, of this work.

In answering these questions for scalar growth systems governed by nonlinear delayed negative feedback we generalize several results well-known from scalar decay processes.

Furthermore, we study a (discontinuous) model equation which reflects the basic components (autocatalytic growth and negative feedback) of the growth processes under consideration: for this equation we are able to describe the global dynamics in full detail.

All in all we hope to prepare the ground for a further investigation of this simplest model for a growth process governed by nonlinear delayed negative feedback.

Before turning to a short description of the contents and organization of the thesis we should add some comments about related mathematical problems.

### Mathematical context

Summarizing the considerations of the preceding section we are interested in scalar differential delay equations of type

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \qquad (\mu, f)$$

which serve as a model for ***scalar autocatalytic growth governed by delayed nonlinear negative feedback*** if we assume

$$-\mu > 0$$

and $f$ to possess the negative feedback property, i.e.

$$f(\xi) \cdot \xi < 0 \quad \text{for all } \xi \in \mathbb{R} \setminus \{0\} . \qquad (NF)$$

Evidently, these equations can be regarded as the simplest examples of an interesting but not yet intensively investigated class of dynamical systems which are governed by two competitive mechanisms: a growth mechanism on one hand (induced by $-\mu > 0$) and, contrary to this, a delayed negative feedback mechanism (given by ($NF$) and the retarded argument).

One should mention that this situation differs essentially from the corresponding equations of decay type ($-\mu < 0$) with delayed positive feedback (i.e., $-f$ satisfies ($NF$)): such equations arise in neural networks and were studied, e.g., by Krisztin, Walther and Wu in [33] and by Krisztin and Walther in [32] (for a mathematical introduction into the dynamics of neural networks, we refer to the forthcoming book of Wu [72]).

Although these equations have a structure very similar to our problem, they are to some extent a little easier to handle, since the positive feedback property of the nonlinearity guarantees that the generated semiflow is *strongly order preserving* (cf. Smith and Thieme [56, 57]) such that one can apply the whole wealth of results about order preserving dynamical systems (see, e.g., Smith [55]).

On the other hand, the delayed logistic equation (2) is also extensively studied in literature: we refer to the original paper [28] by Hutchinson and to the monographs [15, 19, 26, 16] (as well as the references therein).

In contrast to our situation Hutchinson's equation (2) permits the explicit determination of the set of initial values that yield bounded solutions. This turns out to be the basis for almost all studies of oscillating behaviour of solutions of equation (2) and, in particular, of existence and uniqueness of periodic orbits. By this reason, most of the arguments developed there are not available in our situation and had to be replaced by modified and alternative approaches.

Since we focus on slowly oscillating periodic solutions of $(\mu, f)$ in the third and fourth chapter, the corresponding results for decay equations certainly play an important role: our approach in Chapter 3 is based on the same ideas as the work of Walther [67, 68], while Chapter 4 is deeply influenced by Cao's uniqueness result [12].

Throughout the whole work we will refer to the monographs of Diekmann, van Gils, Verduyn Lunel and Walther [16] and of Hale and Verduyn Lunel [26] as the standard sources for basic and well-known results about delay differential equations.

### Synopsis

This thesis is organized as follows:

The first chapter contains basic material such as the hypotheses (H1)–(H3) we want to impose throughout the whole treatise, prototype nonlinearities $f_{\alpha,M}$ satisfying these assumptions, elementary results on bounded and unbounded as well as slowly oscillating solutions of $(\mu, f)$, and the definition of a discrete Lyapunov functional introduced by Mallet-Paret [39], Cao [11] and Arino [6]. In the last section of Chapter 1 we consider the limiting case of equations $(\mu, f_{\alpha,M})$ for $\alpha \to \infty$ and obtain a growth equation with discontinuous delayed nonlinear feedback,

$$\dot{x}(t) = -\mu x(t) - M \operatorname{sign}(x(t-1)) \ , \tag{$s$}$$

which reflects only the "essential" mechanisms: autocatalytic growth and delayed negative feedback of the scalar variable $x$.

Therefore, we extensively investigate the global dynamics of equations $(s)$ in the second chapter for three reasons: First, as already mentioned above, these equations reflect the essential mechanisms whose mutual interaction we want to understand. Second, such equations arise in several models of automatic control (see the references in [20, 21, 22, 50, 54]) and, thus, are also of some interest in applications. Third, we want to use the results on the solutions of $(s)$ to derive results for equations $(\mu, f)$ for nonlinearities $f$ which are close to the sign-nonlinearity in some sense made precise in the third chapter.

After solving the problem of the lack of continuity for the semiflow induced by $(s)$ by introducing the phase space

$$X := \left\{\varphi \in C \ : \ |\varphi^{-1}(0)| < \infty\right\}$$

in Section 2.1, we compute the periodic solutions of $(s)$ explicitly in the second section of Chapter 2. Section 2.3 contains detailed information about the dynamics in the set of bounded solutions which do not converge to one of the two steady states $u_j$, $j \in \{-,+\}$. These results were completed by the fourth section which contains a geometric description of the stable sets of these steady states.

Following and generalizing the ideas of WALTHER [67] in Chapter 3, we prove the existence of slowly oscillating periodic solutions of delay equations $(\mu, f)$ for nonlinearities $f$ which belong to the class $N(\beta,\varepsilon)$: these continuous functions $f$ are in some sense close to the discontinuous nonlinearity $g := -a\,\text{sign}$, $a \in \mathbb{R}^+$, and allow the definition of a POINCARÉ map $R_f$ on the set

$$A(\beta) := \left\{\psi \in C \ : \ \|\psi\| \leq -\frac{M_f}{\mu}, \psi(t) \geq \beta \ \forall t \in [-1;0], \psi(0) = \beta\right\} .$$

Since $R_f$ turns out to be completely continuous and LIPSCHITZ continuous for a LIPSCHITZ continuous nonlinearity $f$, the fixed points of $R_f$ define periodic solutions of $(\mu, f)$ with segments in $A(\beta)$ (cf. THEOREM 3.2.2).

In case that $R_f$ is a contraction, we prove in Section 3.3 that the orbit of the slowly oscillating periodic solution corresponding to the unique fixed point of $R_f$ in $A(\beta)$ is hyperbolic, stable and exponentially attractive with asymptotic phase.

Chapter 4 is devoted to prove the uniqueness of the orbit of a slowly oscillating periodic solution of $(\mu, f)$ under assumptions (H1), (H2), and an additional convexity assumption (H4). It provides a generalization of an approach of CAO [12] for decay delay equations.

The central aspect of the method is a geometric criterion which describes the mutual position of the $(x, \dot{x})$-projections of orbits of slowly oscillating periodic solutions into the plane. These $\mathbb{R}^2$-orbits turn out to be JORDAN curves which have to be nested in a certain way (as shown in PROPOSITION 4.3.1). The method of proof we used is based on a contradiction argument and differs essentially from CAO [12].

Combining the results of Chapter 3 with those of Chapter 4 we obtain existence and uniqueness of the slowly oscillating periodic solution of $(\mu, f)$ for a subclass of nonlinearities in $N(\beta,\varepsilon)$. In particular, the prototype nonlinearities (introduced in Chapter 1) are included within the range of PROPOSITION 4.4.1.

The last chapter addresses several questions motivated by the investigation of the model equation $(s)$ in Chapter 2. It should be seen as a prospect for further research on the global dynamics of scalar growth systems governed by nonlinear delayed negative feedback.

Furthermore, each chapter (except for the introductionary and the final chapter) is supplemented with a section that contains open problems and further references to related work. The aim of these sections is to set the results of the chapter in perspective and to point out directions for future research.

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# 1

# Hypotheses and elementary results

This preliminary chapter serves as a source for elementary and rather general remarks concerning different topics revisited in later chapters. After recalling the hypotheses we want to impose on the range of the real parameter $\mu$ and some smoothness and boundedness assumptions on the nonlinearity $f$, we start with the existence and uniqueness of solutions of

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \ . \tag{1.1}$$

The solutions of this equation constitute a continuous semiflow $F_{\mu,f}$ on the phase space $C := C([-1;0],\mathbb{R})$ of continuous real-valued functions defined on the interval $[-1;0]$, and we note some elementary properties of this semiflow in the second section. Then we will consider the linearization along the stationary solutions in the case where $f$ is assumed to be smooth and strictly monotone. Further elementary properties of the solutions which are needed in subsequent chapters, such as oscillatory behaviour and boundedness, as well as some basic facts about non-autonomous equations can also be found in this chapter.

## 1.1 Hypotheses

For convenience, we state the basic hypotheses which we are going to use throughout the whole thesis (except for the discussion of the discontinuous nonlinearity $f := -a\,\mathrm{sign}$).

(H1) The real parameter $\mu$ is a negative real number, i.e. $\mu \in \mathbb{R}^- := (-\infty; 0)$.

(H2) The nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a smooth, strictly monotonically decreasing, and bounded function with $f(0) = 0$. More precisely:

(H2.1) $f$ is continuously differentiable on $\mathbb{R}$ and $f(0) = 0$, (H2.2) $f$ is strictly monotonically decreasing on $\mathbb{R}$,

(H2.3) $f$ is bounded, i.e. there exists $M_f > 0$ such that

$$|f(\xi)| \leq M_f \quad \text{for all } \xi \in \mathbb{R} \ .$$

(H3) The shape of the graph of $f$ and the real parameter $\mu$ are related in the following way:

(H3.1) If $\mu \in (-1; 0)$, set $\alpha_* := -\frac{\mu}{\cos \vartheta_\mu}$ where $\vartheta_\mu \in (0; \frac{\pi}{2})$ solves $\vartheta_\mu = -\mu \tan \vartheta_\mu$, and set $\alpha_* := -\mu$ otherwise. Then we assume

$$\alpha_0 := -f'(0) \in (\alpha_*; +\infty) \setminus \left\{ -\frac{\mu}{\cos \vartheta_{\mu,n}} \ : \ n \in \mathbb{N} \right\}$$

where $\vartheta_{\mu,n} \in (n\pi; n\pi + \frac{\pi}{2})$ solves $\vartheta_{\mu,n} = -\mu \tan \vartheta_{\mu,n}$ for $n \in \mathbb{N}$, and

(H3.2) suppose the existence of a *unique* negative solution $u := \xi^- \in \mathbb{R}^-$ and a *unique* positive solution $u = \xi^+ \in \mathbb{R}^+$ of the equilibrium equation

$$-\mu u + f(u) = 0 \ .$$

Furthermore, we assume that

$$0 \leq -f'(u) < -\mu$$

holds for $u \in \{\xi^-, \xi^+\}$.

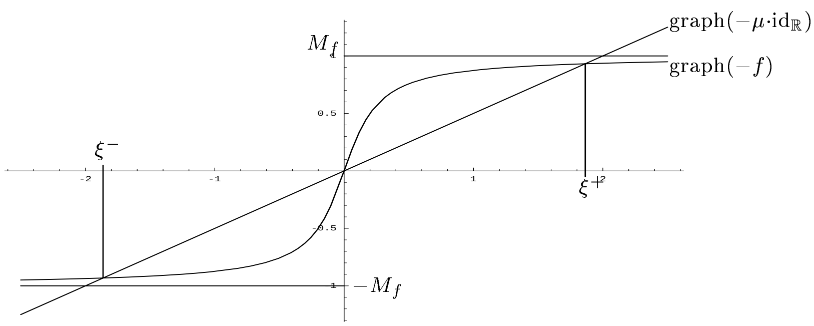
Typical examples of nonlinearities $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfying these assumptions are the two-parameter families of smooth functions

**Example 1.1.1** $f_{\alpha,M} : \mathbb{R} \ni \xi \mapsto -\frac{2M}{\pi} \arctan(\alpha\xi) \in \mathbb{R}$

and

**Example 1.1.2** $f_{\alpha,M} : \mathbb{R} \ni \xi \mapsto -M \tanh(\alpha\xi) \in \mathbb{R}$

for appropriately chosen parameters $M \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}^+$, as depicted below:



As a consequence of (H2.1) and (H2.2), $f$ satisfies a negative feedback property with respect to the trivial equilibrium $\xi^0 = 0$, i.e. we have

$$\xi \cdot f(\xi) < 0 \quad \text{for all } \xi \in \mathbb{R} \setminus \{0\} \ . \tag{1.2}$$

The choice of $\mu$ to be a negative real number may seem strange at first sight but there are two advantages for doing so: first, the wealth of results for the well-known equation (1.1) with positive $\mu$ which apply for the general case $\mu \neq 0$ is directly accessible for our investigations such that elementary results can be taken over without further comments. Second, our results are directly comparable with the well-studied case $\mu \in \mathbb{R}^+$. For many considerations we will need only a subset of the hypotheses stated above. This was the reason for itemizing the properties of $f$ in the second hypothesis (H2). On the other hand, we will add further assumptions where necessary.

In particular, in some sections we consider only *odd* nonlinearities but in most cases this is done only to simplify the formulation and to clarify the investigations. Thus, most of the results hold for the general case, too. For example, the nonlinearities $f := f_{\alpha,M}$ defined in EXAMPLE 1.1.1 and 1.1.2 above satisfy the additional hypothesis

(H2.4) $f$ is odd, i.e. we have $f(-\xi) = -f(\xi)$ for all $\xi \in \mathbb{R}$.

A ***solution of* (1.1)** is either a continuous function $x : [t_0 - 1; +\infty) \to \mathbb{R}$ for $t_0 \in \mathbb{R}$ which satisfies the differential delay equation on $(t_0; +\infty)$ for some $t_0 \in \mathbb{R}$, or a differentiable function $x : \mathbb{R} \to \mathbb{R}$ that satisfies (1.1) on $\mathbb{R}$. In the latter case we call $x : \mathbb{R} \to \mathbb{R}$ a ***global solution*** of (1.1).

Note that the hypotheses guarantee the existence of exactly two non-trivial ***stationary solutions***

$$\mathbb{R} \ni t \mapsto \xi^- \in \mathbb{R}^- \qquad \text{and} \qquad \mathbb{R} \ni t \mapsto \xi^+ \in \mathbb{R}^+$$

beside the trivial zero solution

$$\mathbb{R} \ni t \mapsto \xi^0 \in \mathbb{R}$$

with $\xi^0 := 0$. The corresponding restrictions

$$u_j : [-1; 0] \ni t \mapsto \xi^j \in \mathbb{R}, \quad j \in \{-, 0, +\} \ ,$$

are elements of our phase space $C$ and will be called ***steady states*** of (1.1).

The third hypothesis contains information about the linearizations of (1.1) along these stationary solutions and, thus, on the local behaviour near the steady states. In essence, it implies that the steady states are hyperbolic as we will recall in Section 1.3. For the moment it is sufficient to notice that

$$-f'(0) > -\mu$$

holds (since the consequence of this is the existence of the non-trivial stationary solutions).

## 1.2 The semiflow and some of its properties

The initial value problem

$$\begin{cases} \dot{x}(t) = -\mu x(t) + f(x(t-1)) & , t \in \mathbb{R}^+ \\ x(t) = \varphi(t) & , t \in [-1;0] \end{cases} \tag{1.3}$$

for given $\varphi \in C := C([-1;0],\mathbb{R})$ has a unique solution $x : [-1;+\infty) \to \mathbb{R}$ in the sense of Section 1.1: that is a continuous function on $[-1;+\infty)$ which satisfies the differential equation on $\mathbb{R}^+$ and coincides with $\varphi$ on $[-1;0]$.

This is most easily seen applying the ***variation-of-constants formula***

$$x(t) = e^{-\mu(t-(n-1))}x(n-1) + \int\limits_{n-1}^{t} e^{-\mu(t-s)} f(x(s-1))ds \tag{1.4}$$

for $t \in [n-1;n]$ and $n \in \mathbb{N}$. This method of constructing a solution $x : [-1;\infty) \to \mathbb{R}$ successively on the intervals $[n-1;n]$, $n \in \mathbb{N}$, is usually called *method of steps* (see, e.g., Driver [17]).

The solutions obtained are denoted by $\boldsymbol{x^{\varphi}}$ or sometimes by $\boldsymbol{x^{\varphi,f}}$ as we will do it in Chapter 3 when we have to compare solutions of different delay equations. Furthermore, the variation-of-constants formula (1.4) yields the continuous dependence on the initial value $\varphi \in C$ in the following sense.

**Remark 1.2.1** *For any $\varepsilon > 0$, $t_0 \in \mathbb{R}_0^+$, and $\varphi \in C$ there exists a $\delta > 0$ such that for all $\psi \in U_\delta(\varphi) := \{\chi \in C \, : \, \|\chi - \varphi\| < \delta\}$ we have*

$$|x^{\varphi}(t) - x^{\psi}(t)| < \varepsilon \quad \textit{for all } t \in [0;t_0] \ ,$$

*where $C$ is endowed with the maximum norm on $[-1;0]$ defined by*

$$\|\cdot\| : C \ni \varphi \mapsto \max_{\tau\in[-1;0]} |\varphi(\tau)| \in \mathbb{R}_0^+ \ .$$

As usual we introduce the notion of a ***segment*** $\boldsymbol{x_t^{\varphi} \in C}$ of a solution $x^{\varphi}$ at time $t \in \mathbb{R}_0^+$ by

$$x_t^{\varphi} : [-1;0] \ni s \mapsto x^{\varphi}(t+s) \in \mathbb{R} \ .$$

These phase curves define a continuous semiflow

$$F_f : \mathbb{R}_0^+ \times C \ni (t,\varphi) \mapsto x_t^{\varphi} \in C$$

generated by the delay differential equation (1.1). In the sequel we note some properties of $F_f$. The strict monotonicity of $f$ combined with the variation-of-constants formula (1.4) yields the following remark which is taken from Walther [63, Remark 3.3].

**REMARK 1.2.2** *Let $f$ satisfy* (H2.2). *Then each map $F_f(t,\cdot)$, $t \in \mathbb{R}_0^+$, is injective.*

The restriction of $F_f$ to $(1;+\infty) \times C$ is of class $C^1$, and for $t \in (1;+\infty)$ and $\varphi \in C$ we have

$$D_1 F_f(t,\varphi)1 = (\dot{x}^{\varphi})_t =: \dot{x}_t^{\varphi} \ .$$

The partial derivatives with respect to the state variable exist on all of $\mathbb{R}^+ \times C$, and the maps $D_2 F_f(t,\varphi)$, $(t,\varphi) \in \mathbb{R}^+ \times C$, are injective (cf. WALTHER [63, REMARK 3.3]), too. They are given by

$$D_2 F_f(t,\varphi)\psi = y_t \ ,$$

where $y : [-1;+\infty) \to \mathbb{R}$ is a solution of the initial value problem

$$\begin{cases} \dot{y}(t) & = -\mu y(t) + f'(x^{\varphi}(t-1))y(t-1) \quad , t \in \mathbb{R}^+ \\ y_0 & = \psi \in C \end{cases} .$$

It follows that

$$D_2 F_f(t,x_0)x_0 = \dot{x}_t \quad \text{for all } t \in \mathbb{R}_0^+$$

for every global solution $x : \mathbb{R} \to \mathbb{R}$ of (1.1). In particular, we obtain in case of the stationary solutions $x : \mathbb{R} \ni t \mapsto \xi^j \in \mathbb{R}$, $j \in \{-,0,+\}$, the linear autonomous equation

$$\dot{y}(t) = -\mu y(t) - \alpha_j y(t-1) \tag{1.5}$$

with $\alpha_j := -f'(\xi^j) > 0$.

## 1.3 Linearization along the stationary solutions

For the study of the local behaviour of solutions near the stationary solutions $u_j$ we need information about the linearization along these particular solutions. For further details and proofs of most of the results stated in this paragraph the interested reader may consult the monographs [16, Chapter XI] or [26, Chapter 7] or the articles of MALLET-PARET [39] and WALTHER [63]. Fix $j \in \{-,0,+\}$ and set

$$\alpha_j := -f'(\xi^j) \ .$$

The linear variational equation along the equilibrium solution $u_j$, $j \in \{-,0,+\}$, takes the form (1.5),

$$\dot{y}(t) = -\mu y(t) - \alpha_j y(t-1) \ ,$$

and the operators

$$T_j(t) := D_2 F_f(t,u_j), \quad t \in \mathbb{R}_0^+ \ ,$$

form a strongly continuous semigroup with $T_j(t)\varphi = y_t^{\varphi}$ where $y^{\varphi} : [-1;+\infty) \to \mathbb{R}$ is the solution of the linear delay equation (1.5) with initial value $y_0 = \varphi$. Let $T_j^{\mathbb{C}}(t)$, $t \in \mathbb{R}_0^+$, denote the operators of the strongly continuous semigroup on $C_{\mathbb{C}} := C([-1;0],\mathbb{C})$ which is defined by the complex-valued solutions of (1.5) on $[-1;+\infty)$. The spectrum $\sigma_j := \sigma(A_j)$ of the generator $A_j$ of the $C^0$-semigroup $\big(T_j^{\mathbb{C}}(t)\big)_{t\in\mathbb{R}_0^+}$ consists of isolated eigenvalues with finite multiplicities, given by the roots of the corresponding characteristic equation

$$z + \mu + \alpha_j e^{-z} = 0 \ . \tag{1.6}$$

For every eigenvalue $\lambda \in \sigma_j$, the function $e^{\lambda\cdot} \in C_{\mathbb{C}}$ is an eigenvector, and the associated generalized eigenprojection

$$\mathrm{pr}_j(\lambda) : C_{\mathbb{C}} \to C_{\mathbb{C}}, \quad \lambda \in \sigma_j \ ,$$

onto the generalized one-dimensional eigenspace $G_j(\lambda) = \mathbb{C}\cdot e^{\lambda\cdot}$ satisfies

$$\overline{\mathrm{pr}_j(\lambda)\varphi} = \mathrm{pr}_j(\overline{\lambda})\varphi \ . \tag{1.7}$$

**A. General remarks on the position of eigenvalues**

The results of this subsection hold for the linearization at each of the steady states such that we state them *en bloc* before specializing according to our hypotheses (or, equivalently, according to where we linearize). In either case there exist countably many isolated roots $\lambda_k^{(j)}$, $k \in \mathbb{Z}$, of the characteristic equation (1.6) as we already mentioned above. For $k \in \mathbb{Z}\setminus\{0\}$, these roots are complex conjugates,

$$\overline{\lambda_k^{(j)}} = \lambda_{-k}^{(j)} \ ,$$

each of these is simple, and $\lambda_k^{(j)}$ is the only root contained in the strip

$$\Sigma_k := \{z \in \mathbb{C} \ : \ 2k\pi < \mathrm{Im}(z) < (2k+1)\pi\}$$

while $\lambda_{-k}^{(j)}$ is the unique root in the strip

$$\Sigma_{-k} := \{z \in \mathbb{C} \ : \ 2k\pi < -\mathrm{Im}(z) < (2k+1)\pi\} \ .$$

For $k = 0$ we have two roots counting multiplicity in the strip

$$\Sigma_0 := \{z \in \mathbb{C} \ : \ |\mathrm{Im}(z)| < \pi\}$$

which could either be real numbers $\lambda_{00}^{(j)}$ and $\lambda_0^{(j)}$ (with $\lambda_{00}^{(j)} \le \lambda_0^{(j)}$) or conjugated complex numbers $\lambda_\pm^{(j)}$. The real parts of the roots are ordered and tend to $-\infty$ for $k \to \infty$, i.e. it is

$$\mathrm{Re}(\lambda_k^{(j)}) \ge \mathrm{Re}(\lambda_{k+1}^{(j)})$$

for all $k \in \mathbb{Z}\setminus\{0\}$ and $\lim\limits_{k\to\infty} \mathrm{Re}(\lambda_k^{(j)}) = -\infty$. According to our hypothesis (H3) we have to distinguish between the linearization along the trivial and the non-trivial stationary solutions, i.e., between $\mu + \alpha_0 > 0$ and $\mu + \alpha_j < 0$, $j \in \{-,+\}$, respectively.

## B. Linearization along the zero solution

Hypothesis (H3.1) yields

$$\alpha_0 + \mu > 0 \ ,$$

and in this case both roots of (1.6) contained in $\Sigma_0$ lie on the same side of the imaginary axis (cf. MALLET-PARET [39, THEOREM 6.1]). More precisely, we have the following situations:

(i) For $\mu \in (-1; 0)$ and

$$\alpha_0 > \alpha_* := -\frac{\mu}{\cos\vartheta_\mu} \ ,$$

where $\vartheta_\mu \in (0; \frac{\pi}{2})$ solves $\vartheta_\mu = -\mu \tan \vartheta_\mu$, we have

$$(\sigma_0 \cap \Sigma_0) \cap \{z \in \mathbb{C} \,:\, \mathrm{Re}(z) \leq 0\} = \emptyset \ ,$$

i.e. both zeros of (1.6) contained in $\Sigma_0$ lie in the right half plane.

(ii) Let $\mu \in (-1; 0)$. If $\alpha \in (0; \alpha_*)$, then

$$\sigma_0 \cap \Sigma_0 \subset \{z \in \mathbb{C} \,:\, \mathrm{Re}(z) < 0\} \ ,$$

and if $\alpha_0 = \alpha_*$ we have

$$\sigma_0 \cap \Sigma_0 \subset i\mathbb{R} \ .$$

(iii) If $\mu \in (-\infty; -1]$, then $\sigma_0 \cap \mathbb{R} = \emptyset$ and, furthermore,

$$(\sigma_0 \cap \Sigma_0) \cap \{z \in \mathbb{C} \,:\, \mathrm{Re}(z) \leq 0\} = \emptyset \ ,$$

too.

**REMARK 1.3.1** *Under the assumption* (H3) *the trivial steady state* $u_0 = 0$ *is hyperbolic.*

The earliest form of the following result goes back to WRIGHT [70] (compare also WALTHER [63]) and concerns the behaviour of the eigenvalues as the parameter $\alpha_0$ increases (while $\mu$ is fixed).

**REMARK 1.3.2** *Let us denote by* $\sigma_0(\alpha)$ *the set of solutions of the characteristic equation* (1.6) *for* $\alpha = \alpha_0$*. Then the function*

$$(-\mu; +\infty) \ni \alpha \mapsto \Big| \sigma_0(\alpha) \cap \{z \in \mathbb{C} \,:\, \mathrm{Re}(z) > 0\} \Big| \in 2\mathbb{N}_0$$

*is monotonically increasing.*

### C. Linearization at the non-trivial steady states

According to assumption (H3.2) we have $\alpha_j + \mu < 0$ such that we obtain that the elements of $\sigma_j \cap \Sigma_0$, $j \in \{-,+\}$, are real and ordered:

$$\lambda_{00}^{(j)} < 0 < \lambda_0^{(j)} < -\mu \ . \tag{1.8}$$

In view of **A.** this readily implies $\sigma_j \cap i\mathbb{R} = \emptyset$ for $j \in \{-,+\}$ such that we obtain

**Remark 1.3.3** *The non-trivial steady states $u_j$, $j \in \{-,+\}$, are hyperbolic.*

The fact that $\lambda_0^{(j)} \in \mathbb{R}$ together with the relation (1.7) imply that

$$C \ni \psi \mapsto \operatorname{Re}\left(\operatorname{pr}_j(\lambda_0^{(j)})\psi\right) = \operatorname{pr}_j(\lambda_0^{(j)})\psi \in C$$

defines a projection $\operatorname{Pr}_{P_j} := \operatorname{pr}_j(\lambda_0^{(j)})\Big|_C$ from $C$ onto its one-dimensional real subspace

$$P_j := \operatorname{Re}\left(G_j(\lambda_0^{(j)})\right) = \mathbb{R} \cdot e^{\lambda_0^{(j)} \cdot} \subset C \ .$$

Furthermore, setting

$$Q_j := \left(\mathrm{id}_C - \operatorname{Pr}_{P_j}\right)(C)$$

we obtain the following ***spectral decomposition of the phase space***

$$C = P_j \oplus Q_j$$

where $Q_j$ has codimension 1 and is the generalized real eigenspace corresponding to the eigenvalues in $\sigma_j \setminus \left\{\lambda_0^{(j)}\right\}$.

The spectral projection onto $P_j$ along $Q_j$ is explicitly given by

$$\operatorname{Pr}_{P_j} : C \ni \varphi \mapsto \frac{1}{1+\mu+\lambda_0^{(j)}} \left(\varphi(0) + (\mu + \lambda_0^{(j)}) \int_{-1}^{0} e^{-\lambda_0^{(j)} s}\varphi(s)ds\right) \cdot e^{\lambda_0^{(j)} \cdot} \in P_j \ , \tag{1.9}$$

whereas the spectral projection onto $Q_j$ along $P_j$ can be obtained from this and the well-known identity $\operatorname{Pr}_{Q_j} := \mathrm{id}_C - \operatorname{Pr}_{P_j}$. Furthermore, recall

$$\ker \operatorname{Pr}_{Q_j} = P_j$$

for the kernel of the spectral projection onto $Q_j$.

## 1.4 Bounded and unbounded solutions

In contrast to the case $\mu \in \mathbb{R}^+$ where all solutions of (1.1) remain bounded on $\mathbb{R}_0^+$ (cf. Walther [63, 65, 66]), the existence of unbounded solutions of (1.1) is evident from the hypotheses (H1) and (H2.3) as will be shown in Lemma 1.4.1 below. In conclusion, there cannot exist a global attractor of solutions on the whole phase space $C$. Therefore, one key problem in the study of the delay equation (1.1) under the given hypotheses is to separate bounded and unbounded solutions. For convenience, we introduce some new notation.

**Definition 1.4.1** We denote by $\boldsymbol{\mathcal{B}}$ the set of all $\varphi \in C$ for which $x^\varphi$ is bounded on $\mathbb{R}_0^+$.

Clearly, $\mathcal{B}$ is not empty since the segments of the stationary solutions $u_j$, $j \in \{-, 0, +\}$, are trivially contained in $\mathcal{B}$. Furthermore, $C \setminus \mathcal{B}$ is the set of all initial values that yield unbounded solutions. As we will prove in the sequel these unbounded solutions have a special monotonicity property.

**Definition 1.4.2** A solution $x : [-1; +\infty) \to \mathbb{R}$ of (1.1) is called ***ultimately strictly monotonic*** if there is a $t_x \in \mathbb{R}_0^+$ such that $\dot{x}(t) \neq 0$ for all $t \in [t_x; +\infty)$.

Furthermore, every unbounded solution converges after some finite time monotonically to infinity, as we state in

**Lemma 1.4.1** *Set*

$$E := \left\{ \varphi \in C \ : \ |\varphi(0)| > -\frac{M_f}{\mu} \right\}.$$

(1) *Let $\varphi \in E$. Then $x^\varphi$ is ultimately strictly monotonic.*

(2) *For any $\varphi \in C \setminus \mathcal{B}$ there exists a $t_\varphi \in \mathbb{R}_0^+$ such that $x_{t_\varphi}^\varphi \in E$.*

Proof: Since part (2) is a trivial consequence of Definition 1.4.2 (choose $t_0 \in \mathbb{R}^+$ such that $|x^\varphi(t_0)| > -\frac{M_f}{\mu}$ which is possible for an unbounded solution) it remains to prove only the first assertion.
Without loss of generality we consider the case $\varphi(0) > -\frac{M_f}{\mu}$ and set $\eta := \varphi(0) + \frac{M_f}{\mu} > 0$. Because of the boundedness of the nonlinearity (cf. (H2.3)) we have $f(\varphi(t-1)) \geq -M_f$ for all $t \in [0; 1]$ and all $\varphi \in C$. Consequently,

$$\dot{x}(t) = -\mu x(t) + f(\varphi(t-1)) \geq -\mu x(t) - M_f$$

for $t \in [0;1]$ and $x^\varphi(0) = \varphi(0)$ yield

$$x(t) \geq \left(\varphi(0) + \frac{M_f}{\mu}\right) e^{-\mu t} - \frac{M_f}{\mu} > \eta e^{-\mu t} - \frac{M_f}{\mu}$$

for all $t \in [0;1]$ which gives $\dot{x}^\varphi(t) > -\mu\eta > 0$ on $[0;1]$ and thus, by the method-of-steps, on $\mathbb{R}_0^+$.

□

**DEFINITION 1.4.3** We denote by $\boldsymbol{\mathcal{E}}^+$ the set of all $\varphi \in C$ for which $x^\varphi$ tends monotonically to $+\infty$,

$$x^\varphi(t) \nearrow +\infty \quad \text{as } t \to \infty ,$$

and by $\boldsymbol{\mathcal{E}}^-$ the set of all $\varphi \in C$ for which $x^\varphi(t) \searrow -\infty$ as $t \to \infty$.

For obvious reasons we call the sets $\mathcal{E}^\pm$ *escape sets*. Clearly, the above lemma states that the escape sets are disjoint and contain all unbounded solutions:

$$C \setminus \mathcal{B} = \mathcal{E}^+ \dot{\cup} \mathcal{E}^- .$$

Since the dynamics of the unbounded solutions is rather boring (every unbounded solution becomes strictly monotonic after some finite time) we now turn to the more interesting set of initial values of bounded solutions. One of the central questions is, therefore, to find an appropriate description of the set $\mathcal{B}$ since the interesting dynamics of (1.1) will take place in this part of the phase space. We will return to this problem in Chapter 5 and – in the special case of a discontinuous nonlinearity – in Chapter 2. A trivial but rather important observation is that all bounded solutions are uniformly bounded as we conclude from LEMMA 1.4.1.

**LEMMA 1.4.2** *For every $\varphi \in \mathcal{B}$ it is*

$$|x^\varphi(t)| = |x_t^\varphi(0)| \leq -\frac{M_f}{\mu}$$

*for all $t \in \mathbb{R}_0^+$.*

PROOF: *Assume to the contrary* that there exists $t_0 \in \mathbb{R}_0^+$ with $x^\varphi(t_0) > -\frac{M_f}{\mu}$ (the case that $x^\varphi(t_0) < \frac{M_f}{\mu}$ can be treated similarly). Then $x_{t_0}^\varphi \in \mathcal{E}^+$ and LEMMA 1.4.1 implies $x^\varphi(t) \nearrow +\infty$ *in contradiction* to $\varphi \in \mathcal{B}$.

□

In particular, LEMMA 1.4.2 implies that every bounded solution is contained in a ball of radius $-\frac{M_f}{\mu}$ around the zero solution $u_0$ after at least one time step and stays there forever. This is of importance for numerical simulations and also in the context of periodic solutions.

**COROLLARY 1.4.1** *We have*

$$F_f\left([1;+\infty)\times\mathcal{B}\right)\subset\left\{\chi\in C\ :\|\chi\|\leq-\frac{M_f}{\mu}\right\}\ .$$

Another simple, yet important implication of LEMMA 1.4.2 concerns global bounded solutions of (1.1) and will be needed in Chapter 4.

**COROLLARY 1.4.2** *If $x:\mathbb{R}\to\mathbb{R}$ is a global bounded solution of* (1.1) *then*

$$x(\mathbb{R})\subset\left[\tfrac{M_f}{\mu};-\tfrac{M_f}{\mu}\right]=:I_\infty\ .$$

Typical global bounded solutions, beside the stationary solutions, are periodic ones. The existence and uniqueness of (especially slowly oscillating) periodic solutions is one of the most challenging questions in a first attempt to understand the dynamics of a delay equation and we will give partial answers to this questions in Chapters 3 and 4. Before we can do this we have to prepare the ground for a detailed investigation of oscillating (not necessarily periodic) solutions. This will be done in the next sections.

## 1.5 Oscillating solutions

A first simple observation concerning oscillatory behaviour of solutions of equation (1.1) is that all oscillating solutions are necessarily uniformly bounded. This will turn out to be extremely helpful in the context of proving the uniqueness of slowly oscillating solutions around the trivial solution.
Once more we have to specify some notation in our situation where we have three equilibria (instead of a single equilibrium as for $\mu\in\mathbb{R}^+$) such that there exist oscillating solutions around each of these.

**DEFINITION 1.5.1** Let $j\in\{-,0,+\}$. A solution $x:I_x\to\mathbb{R}$ of (1.1) with $I_x=\mathbb{R}$ or $I_x=[t_0-1;+\infty)$ for some $t_0\in\mathbb{R}$ is called ***oscillating around (the equilibrium) $\boldsymbol{\xi^j}$***, iff

$$\left|x^{-1}(\xi^j)\cap[t;+\infty)\right|=\infty\qquad\text{for all } t\in I_x\ .$$

Clearly, due to LEMMA 1.4.1 we know that every unbounded solution leaves the ball $\left\{\chi\in C\ :\|\chi\|\leq-\frac{M_f}{\mu}\right\}$ in finite time and is ultimately strictly monotonic. Hence, there cannot exist any unbounded oscillating solutions around either of the three equilibria $\xi^j$, $j\in\{-,0,+\}$.

**Remark 1.5.1** *For every $j \in \{-, 0, +\}$ we have*

$$\left\{\varphi \in C \;:\; |(x^\varphi)^{-1}(\xi^j) \cap [t; +\infty)| = \infty \;\forall t \in \mathbb{R}_0^+\right\} \subset \mathcal{B} \;,$$

*i.e. all solutions that oscillate around the equilibrium $\xi^j$ remain bounded.*

This indicates that oscillating solutions may play an as important role for the dynamics in the set $\mathcal{B}$ as they do for the dynamics in the whole phase space when $\mu \in \mathbb{R}^+$. In particular, so called slowly oscillating solutions are fundamental for the dynamics of differential delay equations (1.1) with $\mu \in \mathbb{R}^+$ and we will adopt the notion of slowly oscillating solutions to our problem now.

**Definition 1.5.2** A solution $x$ of (1.1) is called ***(eventually) slowly oscillating around $\boldsymbol{\xi^0 = 0}$*** if there is a $t_0 \in \mathbb{R}_0^+$ such that

$$|\zeta - \zeta'| > 1 \tag{1.10}$$

holds for all zeros $\zeta \neq \zeta'$ of $x$ in $[t_0; +\infty)$.

In a completely similar manner one can define the notion of slowly oscillating solutions around $\xi^j$ for $j \in \{+, -\}$ by reducing this to the previous definition.

**Definition 1.5.3** A solution $x$ of (1.1) is called ***(eventually) slowly oscillating around $\boldsymbol{\xi^j}$***, $j \in \{-, +\}$, if the corresponding solution $z := x - u_j$ is slowly oscillating (in sense of Definition 1.5.2) for the differential delay equation

$$\dot{z}(t) = -\mu z(t) + g(z(t-1)) \;, \tag{1.11}$$

where

$$g : \mathbb{R} \ni \zeta \mapsto f(\zeta + \xi^j) - f(\xi^j) \in \mathbb{R} \;. \tag{1.12}$$

**Remark 1.5.2** *Let $j \in \{-, +\}$. By definition, $g \in C^1$ and*

$$g' : \mathbb{R} \ni \zeta \mapsto f'(\zeta + \xi^j) \in \mathbb{R} \;,$$

*such that $g$ is strictly decreasing if* (H2.1) *and* (H2.2) *hold for $f$. Furthermore,* (1.11) *has three stationary solutions, namely*

$$\mathbb{R} \ni t \mapsto 0 \in \mathbb{R} \;, \quad \mathbb{R} \ni t \mapsto -\xi^j \in \mathbb{R} \;, \quad \textit{and} \quad \mathbb{R} \ni t \mapsto \xi^k - \xi^j \in \mathbb{R} \;,$$

*where $k \in \{-, +\} \setminus \{j\}$.*

We already mention here for completeness that slowly oscillating solutions around the non-trivial equilibria $\xi^j$, $j \in \{-,+\}$, do not exist. This will be proved in Chapter 5 when we investigate the stable sets of the non-trivial stationary solutions in more detail. Now, we turn our interest to the slowly oscillating solutions around 0. Therefore, we note some elementary properties of solutions of (1.1) that evolve essentially from the negative feedback property (1.2).

**Lemma 1.5.1** *Let $x$ be a solution of (1.1) and $t_0 \in \mathbb{R}^+$ be given.*

(1) *If $x(t_0-1) < 0$ and $x(t_0) > 0$ then $\dot{x}(t_0) > 0$.*

(2) *If $x(t_0-1) > 0$ and $x(t_0) < 0$ then $\dot{x}(t_0) < 0$.*

A first step in our quest for slowly oscillating solutions around $\xi^0 = 0$ is to exclude bounded monotone solutions converging to the zero solution $u_0$.

**Lemma 1.5.2** *Let $\mu \in (-1;0)$ and $-f'(0) = \alpha_0 > 1 + \frac{1}{e-1}$. Then there does not exist a non-trivial eventually monotonically decreasing solution $x : [-1;+\infty) \to [0;\xi^+)$.*

Proof: *Assume to the contrary* that we can find a $t_0 \in \mathbb{R}_0^+$ with $x([t_0;+\infty)) \subset [0;\xi^+)$ and $\dot{x} \le 0$ on $[t_0;+\infty)$.

1. By assumption, we have
$$\lim_{t\to+\infty} x(t) = \xi \in [0;\xi^+) \ .$$
Necessarily, $\xi = 0$ since otherwise
$$\lim_{t\to+\infty} \dot{x}(t) = -\mu\xi + f(\xi) < 0$$
would imply $x(t) \to -\infty$ as $t \to \infty$ in contradiction to $x([-1;+\infty)) \subset [0;\xi^+)$.

2. Since $\dot{x} \le 0$ by assumption and $\lim\limits_{t\to\infty} x(t) = 0$, either $x(t) > 0$ for all $t \in [t_0;+\infty)$ or there exists a $t_1 \in [t_0;+\infty)$ such that $x_t = 0$ for all $t \in [t_1;+\infty)$.
We *claim* that $x(t) > 0$ for all $t \ge t_0$ and argue once more by *contradiction*: If this was not the case there would exist
$$t_1 := \inf\{t \in [t_0;+\infty) \ : \ x_t = 0\} \ ,$$
such that $x_t = 0$ for all $t > t_1$ and $x_{t_1} \ne 0$ (because $x$ is not the trivial solution). Consequently, due to Remark 1.2.2 the injectivity of $F_f(t,\cdot)$, $t \in \mathbb{R}_0^+$, yields
$$F_f(t, x_{t_1}) \ne 0 = F_f(t, 0)$$
for all $t \in [0;+\infty)$ *contradicting* the definition of $t_1$.

3. We fix
$$\varepsilon \in \left(0; \alpha_0 - 1 - \tfrac{1}{e-1}\right) \ ,$$
and choose $\delta = \delta_\varepsilon > 0$ such that
$$f(\xi) \leq (\varepsilon - \alpha_0) \cdot \xi \quad \text{ for all } \xi \in [0; \delta] \ .$$
Here, $\alpha_0 := -f'(0) > 1 > -\mu$ due to our assumptions such that (H3.1) is satisfied.

4. Since $x(t) \searrow 0$ as $t \to \infty$ there exists a $s \geq t_0 + 1$ such that $x(t) \in [0; \delta]$ for all $t \in [s; +\infty)$. Clearly,
$$x(s+2) - x(s+1) = \int_{s+1}^{s+2} \dot{x} = \int_{s+1}^{s+2} [-\mu x] + \int_{s+1}^{s+2} f(x(\cdot - 1)) \ , \tag{1.13}$$
and we are going to estimate this expression now.

5. Using the monotonicity of $x$ as well as the monotonicity of $f$ we get
$$f(x(t)) \leq f(x(s+1)) \quad \text{ for all } t \in [s; s+1] \ .$$
Hence,
$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \leq -\mu x(t) + f(x(s+1))$$
for all $t \in [s+1; s+2]$ implies
$$x(t) \leq \left( x(s+1) - \frac{f(x(s+1))}{\mu} \right) e^{-\mu(t-(s+1))} + \frac{f(x(s+1))}{\mu}$$
for all $t \in [s+1; s+2]$. Note further that
$$\frac{1 - e^{-\mu}}{\mu} > 1 \quad \text{ for all } \mu \in \mathbb{R}^- \supset (-1; 0) \ .$$
As a consequence,
$$\begin{aligned} \int_{s+1}^{s+2} [-\mu x] \ &\leq \ \int_{s+1}^{s+2} [(-\mu x(s+1) + f(x(s+1)))e^{-\mu(t-(s+1))} - f(x(s+1))] dt = \\ &= \ [-\mu x(s+1) + f(x(s+1))] \int_{s+1}^{s+2} e^{-\mu(t-(s+1))} dt - f(x(s+1)) = \\ &= \ [-\mu x(s+1) + f(x(s+1))] \frac{1 - e^{-\mu}}{\mu} - f(x(s+1)) = \\ &= \ -(1 - e^{-\mu}) x(s+1) + \left( \frac{1 - e^{-\mu}}{\mu} - 1 \right) f(x(s+1)) \leq \end{aligned}$$

$$\begin{aligned}
&\le -(1-e^{-\mu})x(s+1) + \left(\frac{1-e^{-\mu}}{\mu} - 1\right)(\varepsilon-\alpha_0)x(s+1) = \\
&= \left(-(1-e^{-\mu}) + \frac{1-e^{-\mu}}{\mu}(\varepsilon-\alpha_0)\right)x(s+1) - (\varepsilon-\alpha_0)x(s+1) \ ,
\end{aligned}$$

which gives an estimate of the first integral in equation (1.13).

6. The second integral in (1.13) can be estimated roughly as follows:

$$\int_{s+1}^{s+2} f(x(t-1))dt \ \le \ (\varepsilon-\alpha_0)\int_{s}^{s+1} x(t)dt \le (\varepsilon-\alpha_0)x(s+1) \ ,$$

taking advantage of the monotonicity of $x$ which yields $\min_{t\in[s;s+1]} x(t) = x(s+1)$.

7. Summarizing steps 4., 5., and 6., we obtain

$$\begin{aligned}
x(s+2)-x(s+1) &\le \left(-(1-e^{-\mu}) + \frac{1-e^{-\mu}}{\mu}(\varepsilon-\alpha_0)\right)x(s+1) = \\
&= (1-e^{-\mu})\left(-1+\frac{\varepsilon-\alpha_0}{\mu}\right)x(s+1) \ ,
\end{aligned}$$

and, finally,

$$x(s+2) \le \left[(1-e^{-\mu})\left(-1+\frac{\varepsilon-\alpha_0}{\mu}\right)+1\right]x(s+1) \ . \tag{1.14}$$

8. It is rather elementary to check that for every $\nu \in (-\infty; -1-\frac{1}{e-1})$ each function

$$h_\nu : (-1;0) \ni \mu \mapsto (1-e^{-\mu})\left(-1+\frac{\nu}{\mu}\right)+1 \in \mathbb{R}$$

has range $h_\nu((-1;0)) \subset \mathbb{R}^-$: To see this, verify that the auxiliary function

$$g : [-1;0) \ni \mu \mapsto \mu\left(1-\frac{1}{1-e^{-\mu}}\right) \in \mathbb{R}$$

is strictly increasing on $[-1;0)$ because the well-known inequality $e^x < \frac{1}{1-x}$ for all $x \in \mathbb{R}^-$ implies $g'(\mu) = \frac{1+e^\mu(\mu-1)}{(e^\mu-1)^2} > 0$ for all $\mu \in [-1;0)$. Now, use

$$\nu < -1-\frac{1}{e-1} = g(0) < g(\mu) = \mu\left(1-\frac{1}{1-e^{-\mu}}\right) \qquad \text{for all } \mu \in (-1;0)$$

to derive via

$$\begin{aligned}
&\nu < \mu\left(1-\frac{1}{1-e^{-\mu}}\right) \iff \frac{\nu}{\mu} > 1-\frac{1}{1-e^{-\mu}} \iff -1+\frac{\nu}{\mu} > -\frac{1}{1-e^{-\mu}} \iff \\
&\iff (1-e^{-\mu})\left(-1+\frac{\nu}{\mu}\right) < -1
\end{aligned}$$

that $h_\nu(\mu) < 0$ for all $\mu \in (-1;0)$ and $\nu \in (-\infty; -1-\frac{1}{e-1})$.

9. Since the choice of $\varepsilon$ in 3. implies $\nu := \varepsilon - \alpha_0 \in (-\alpha_0; -1 - \frac{1}{e-1})$, we conclude from step 8. and (1.14) that
$$x(s+2) < 0$$
which in fact *contradicts* $x([t_0; +\infty)) \subset (0; \xi_+)$.

$\square$

## 1.6 A discrete LYAPUNOV functional

A basic tool in dealing with oscillatory behaviour is a discrete LYAPUNOV functional which was introduced for solutions in the global attractor $\mathcal{A}$ of certain differential delay equations

$$\dot{x}(t) = g(x(t), x(t-1))$$

by MALLET-PARET [39] in order to give a MORSE decomposition of $\mathcal{A}$. This concept was generalized by CAO [12] for all solutions of the non-autonomous differential delay equations

$$\dot{x}(t) = g(x(t), x(t-1), t) \tag{1.15}$$

which we will follow here and which will be applied in Chapter 4. A slightly modified and simplified approach is due to ARINO [6] and we will also use some of his results later.

Further examples for the application of discrete LYAPUNOV functionals can be found in the treatises of MALLET-PARET and SELL (cf. [41],[42]) or in the monograph of KRISZTIN, WALTHER and WU [33] as well as in the article [32] of KRISZTIN and WALTHER.

The material in this section will be stated without proofs for two reasons: first, most of it is taken from CAO [12] or ARINO [6] with only slight modification in the notation, such that we refer to this well-written articles. Second, we will extend many of the mentioned results to a discontinuous limiting case in Chapter 2 where one can also get a good impression of how to prove these results here.

**DEFINITION 1.6.1** We define a ***count function*** $\# : C \to \mathbb{N}_0 \cup \{\infty\}$ as follows: For $\varphi \in C$ we denote by $\#(\varphi) \in \mathbb{N}_0 \cup \{\infty\}$ the number of zeros of $\varphi$ in $[-1; 0]$ (not counting multiplicities) where we count a subinterval $[\alpha; \beta] \subsetneqq [-1; 0]$ as a single zero if $\varphi\Big|_{[\alpha;\beta]} = 0$ and if there is no interval $[\gamma; \delta] \subsetneqq [-1; 0]$ such that $[\alpha; \beta] \subsetneqq [\gamma; \delta]$ and $\varphi\Big|_{[\gamma;\delta]} = 0$. Finally, we set $\#(\varphi) := \infty$ if $\varphi = 0$.

**DEFINITION 1.6.2** Let

$$J : C \times \mathbb{R}_0^+ \ni (\varphi, t) \mapsto (x^\varphi)^{-1}(0) \cap [t; +\infty) \in \mathfrak{P}(\mathbb{R})$$

and set

$$\sigma : C \times \mathbb{R}_0^+ \ni (\varphi, t) \mapsto \left\{ \begin{array}{cl} \infty & , \text{ if } J(\varphi,t) = \emptyset \\ \inf J(\varphi,t) & , \text{ otherwise} \end{array} \right\} \in \mathbb{R}_0^+ \cup \{\infty\} \ .$$

Then we call

$$V : C \times \mathbb{R}_0^+ \ni (\varphi, t) \mapsto \left\{ \begin{array}{cl} 0 & , \text{ if } \sigma(\varphi,t) = \infty \\ \#(x^\varphi_{\sigma(\varphi,t)}) & , \text{ otherwise} \end{array} \right\} \in \mathbb{N}_0 \cup \{\infty\}$$

a ***discrete* LYAPUNOV *functional*** for (1.15).

Let us fix $j \in \{-, 0, +\}$ throughout the remainder of this section. A first consequence of the definitions above is the following remark (which is obvious in view of REMARK 1.2.2).

**REMARK 1.6.1** *Let $x$ be a solution of* (1.1). *If $x^\varphi_{t_*} \neq u_j$ for some $t_* \in \mathbb{R}^+$, then $x^\varphi_t \neq u_j$ for all $t \in \mathbb{R}_0^+$.*

That $V$ behaves indeed like a LYAPUNOV functional is the main result of the first section of CAO's paper [12, THEOREM 1.5] (see also ARINO [6, PROPOSITION 4]) which we restate as

**THEOREM 1.6.1** *Let $g : \mathbb{R}^2 \times \mathbb{R}_0^+ \to \mathbb{R}$ be in $C^1$ with $g(0,0,t) = 0$ for all $t \in \mathbb{R}_0^+$ and suppose the existence of a $m > 0$ such that $g$ satisfies $g(0,y,t) \neq 0$ for all $t \in \mathbb{R}^+$ and all $y \in [-m; m] \setminus \{0\}$.*
*Let $\varphi \in C$ be given such that $x^\varphi$ is the solution of* (1.15) *with initial value $x^\varphi_0 = \varphi$ and $\|x^\varphi_t\| \leq m$ for all $t \in \mathbb{R}^+$. Then the discrete LYAPUNOV functional $V$ is non-increasing in $t$, i.e.*

$$V(\varphi, t) \leq V(\varphi, t_0) \quad \text{for all } t \in [t_0; +\infty) \ .$$

By assumptions (H1)–(H3), we have

$$-\mu \cdot 0 + f(y + \xi^j) - f(\xi^j) = f(y + \xi^j) - f(\xi^j) = 0$$

if and only if $y = 0$, such that the following remark will prove the applicability of CAO's results for the delay equations (1.1).

**REMARK 1.6.2** *Each function*

$$g_j : \mathbb{R}^2 \times \mathbb{R}^+ \ni (x, y, t) \mapsto -\mu x + f(y + \xi^j) - f(\xi^j) \in \mathbb{R}$$

*fulfills the assumptions in of* THEOREM 1.6.1. *Thus,* (1.15) *with $g := g_j$ defines a discrete* LYAPUNOV *functional $V_j$ and a count function $\#_j$ according to* DEFINITION 1.6.2 *and* DEFINITION 1.6.1, *respectively.*

Notice that $\#_j(\varphi)$ counts the number of zeros of $z := \varphi - u_j$ in $[-1;0]$, or, equivalently, the number of pre-images of the value $\xi^j$ of $\varphi$ in $[-1;0]$ (counting again a subinterval $[\alpha;\beta] \subsetneqq [-1;0]$ where $\varphi\Big|_{[\alpha;\beta]} = u_j\Big|_{[\alpha;\beta]}$ as a single pre-image of $\xi^j$ according to DEFINITION 1.6.1).

Therefore, we can apply the results of [12] to gain a deeper insight into the oscillatory behaviour of solutions of (1.1) because every oscillating solution around $\xi^j$ is necessarily bounded (cf. REMARK 1.5.1). Clearly, this remark enables us to call solutions of (1.1) eventually slowly oscillating around $\xi^j$, if there exists a $t_\varphi \in \mathbb{R}^+$ such that $V_j(\varphi, t) = 1$ holds for all $t \in [t_\varphi; +\infty)$.

**COROLLARY 1.6.1** *If $\varphi \in C$ is the initial value of an oscillating solution of* (1.1) *around $\xi^j$ with $V_j(\varphi, 0) = n \in \mathbb{N}_0$, then $V_j(\varphi, t) \leq n$ for all $t \in \mathbb{R}_0^+$.*

This means for the particular case of oscillating solutions around $\xi^0 = 0$ that the number of zeros of each segment $x_t^\varphi$, $t \in \mathbb{R}_0^+$, does not increase in time, or, in other words: the (asymptotic or final) "frequency" of a solution $x^\varphi$ of (1.1) oscillating around $\xi^0 = 0$ is bounded from above by the "initial frequency" $V_0(\varphi, 0) = n$.

If we have a zero of multiplicity two (or higher) at $t_0 \in \mathbb{R}^+$, then the LYAPUNOV functional decreases strictly at this time $t_0$. This is the assertion of [12, LEMMA 1.6] which we state here as

**COROLLARY 1.6.2** *Let $\varphi \in \mathcal{B}$ and $x^\varphi$ be a solution of* (1.1). *If, for some $t_* \in \mathbb{R}_0^+$, $x^\varphi(t_*) = \dot{x}^\varphi(t_*) = \xi^j$, then $V_j(\varphi, t) \leq V_j(\varphi, t_*) - 1$ for all $t \in (t_*; +\infty)$.*

A trivial consequence of this corollary is the fact that oscillating solutions of eventually finite frequency must have simple zeros from some time $t_\varphi \in \mathbb{R}^+$ on (as is proved in [12, COROLLARY 1.7]):

**COROLLARY 1.6.3** *Let $\varphi \in \mathcal{B}$ and $x^\varphi$ be a solution of* (1.1). *If*

$$\lim_{t \to \infty} V_j(\varphi, t) \in \mathbb{N}_0 \ ,$$

*then there exists a $t_\varphi \in \mathbb{R}^+$ such that all zeros of $x^\varphi$ are simple in $[t_\varphi; +\infty)$.*

Especially every slowly oscillating periodic solution (around either of the equilibria) has simple zeros. Even more can be said about the oscillatory behaviour of solutions $x^\varphi$ if the initial value $\varphi$ is in the stable or unstable set of the (by assumption hyperbolic) steady states $u_j$ of (1.1).

**DEFINITION 1.6.3** The set

$$W^s(u_j) := \{\varphi \in C \ : \ x_t^\varphi \to u_j \ \ (t \to +\infty)\}$$

is called the ***(global) stable set of the steady state $u_j$***, whereas we denote by

$$W^u(u_j) := \{\varphi \in C \ : \ [x^\varphi : \mathbb{R} \to \mathbb{R}],\ x_t^\varphi \to u_j \ \ (t \to -\infty)\}$$

the ***(global) unstable set of the steady state $u_j$.***

By definition $W^u(u_j)$ contains only global solutions in the sense of Section 1.1. Evidently, both sets are non-empty since each contains the stationary solution $u_j$, but even more could be said:

**REMARK 1.6.3** *The sets $W^s(u_j)$ and $W^u(u_j)$ are immersed submanifolds of $C$.*

PROOF: We know from Section 1.2 that $F_f(t,\cdot)$ and $D_2F_f(t,\varphi)(\cdot)$ are injective such that the assertion follows from HALE & VERDUYN LUNEL [26, p. 311] or HALE [24, p. 49].

□

As we know from REMARK 1.3.1 and REMARK 1.3.3, each stationary solution $u_j$ is hyperbolic (as a consequence of the hypothesis (H3)) such that we can apply [6, PROPOSITION 5] to obtain

**THEOREM 1.6.2** *Let $\mu \in (-1;0)$.*

(1) *Let $\varphi \in W^s(u_j)$ with $V_j(\varphi,0) \in \mathbb{N}_0$. Then there exists $T \in \mathbb{R}_0^+$ such that*

$$V_j(\varphi,t) =: N^+ \in \mathbb{N}_0 \qquad \textit{for all } t \in [T;+\infty)\ .$$

*Furthermore, it is $\limsup_{t\to\infty} \#_j(x_t^\varphi) = N^+$.*

(2) *Suppose now that $z$ is a global solution of (1.1) with $z_0 =: \varphi \in W^u(u_j)$. Then there exists a $T \in \mathbb{R}_0^-$ such that $V_j(\varphi,t) =: N^- \in \mathbb{N}_0$ for all $t \in (-\infty;T]$. We also have $\limsup_{t\to-\infty} \#_j(z_t) = N^-$.*

(3) *Suppose finally that $z$ is a global solution with $z_0 =: \varphi \in W^s(u_j)$. Then*

$$N^+ \le N^- := \lim_{t\to-\infty} V(\varphi,t)\ .$$

Obviously, Lemma 1.5.2 and the preceding theorem give some insight into the behaviour of solutions in the stable set of the trivial stationary solution $u_0$.

**Corollary 1.6.4** *If $\mu \in (-1;0)$, $\varphi \in W^s(u_0)$, then $N^+ \in \mathbb{N} \cup \{\infty\}$. In particular, every solution in the stable set of the trivial stationary solution has to oscillate around $\xi^0 = 0$. Furthermore, if there exists a global solution in the stable set of $u_0$, then it has to oscillate around $\xi^0 = 0$ on $\mathbb{R}^-$, too.*

Further results can be found in Cao's treatise [12] which can be applied to (1.1) (assuming (H1)–(H3)), but we won't need them here such that we conclude this section stating a last corollary which excludes the existence of a homoclinic orbit through $u_0$.

**Corollary 1.6.5** *A homoclinic orbit through $u_0$ does not exists for* (1.1).

Proof: This is a trivial consequence of Cao's Theorem 4.1, our assumption (H3.1) (on the linearization at the trivial steady state), and the choice of $\mu$ to be negative.

□

## 1.7 A limiting case

Up to this point we have collected basic results and developed some tools that we will need in the sequel but we didn't get deeper into the qualitative structure of the set $\mathcal{B}$. A first step in this direction could be to look for an appropriate model nonlinearity which

- reflects the "essential" properties of our rather general class of nonlinearities defined by (H2) and (H3), and which
- is easy enough to handle but general enough to infer at least some information for the delay equations defined for a subclass of all nonlinearities and parameter values determined by (H1)–(H3).

Against this background it is tempting to try smooth nonlinearities that are monotonic, odd, and bounded, such as those in Example 1.1.1 or Example 1.1.2. The disadvantage of this idea is that the corresponding equations are still too difficult to handle (analytically) and it would be preferable to find nonlinearities for which one can compute the solutions explicitly. Therefore, for fixed $M > 0$ we take an even crude approach by considering the *discontinuous* differential delay equations

$$\dot{x}(t) = -\mu x(t) - M \operatorname{sign}(x(t-1)) \qquad (s)$$

which may be regarded (purely mathematically) as the simplest examples of delay equations reflecting the negative feedback property and the boundedness of the original nonlinearities from which they evolve as pointwise limits of the sequences of delay equations

$$\dot{x}(t) = -\mu x(t) + f_\alpha(x(t-1)) \tag*{$(1.1)_\alpha$}$$

for the smooth, monotone, odd, and bounded $f_\alpha := f_{\alpha,M}$ from Example 1.1.1 or Example 1.1.2 as $\alpha$ tends to infinity. Observe that there exist three equilibria $\xi^- := \frac{M}{\mu}$, $\xi^0 := 0$, and $\xi^+ := -\frac{M}{\mu}$ yielding steady states $u_j$, $j \in \{-, 0, +\}$, in complete analogy the smooth case.

In this sense we can say that the discontinuous delay equation $(s)$ "caricatures" the smooth delay equation $(1.1)_\alpha$ or is a rough simplification or approximation of it which still displays a negative feedback property of the bounded nonlinear part and has three stationary solutions. Clearly, our hope is that the dynamics of the limit delay equation $(s)$ somehow reflects the rudimentary structure of the global dynamics of each delay equation $(1.1)_\alpha$. For a special class of nonlinearities which are "close enough" to the sign nonlinearity we will reconsider this question in Chapter 3.

This motivation is borrowed from Section XVI.2 of the monograph [16] where the simpler case $\mu = 0$ is treated (and it also appears in the articles [51] and [52] of Peters). Further references and an alternative approach to these model equations will be given at the beginning of the following chapter which is devoted to the study of the global dynamics of this discontinuous nonlinear differential delay equations.

# 2

# A discontinuous model nonlinearity

In this chapter we shall investigate the qualitative behaviour of solutions of the *discontinuous* differential delay equation

$$\dot{x}(t) = -\mu x(t) - a\,\mathrm{sign}(x(t-1)) \tag{2.1}$$

where

$$\mathrm{sign} : \mathbb{R} \ni \xi \mapsto \left\{ \begin{array}{cc} \frac{\xi}{|\xi|} & , \xi \neq 0 \\ 0 & , \xi = 0 \end{array} \right\} \in \{-1, 0, +1\}$$

denotes the ***sign function*** and $a \in \mathbb{R}^+$ is a positive constant.

Such equations arise in simple control systems where only the minimal information about the (shape of the) phase state in the past is known, namely, whether these states had positive or negative sign. In this sense this discontinuous delay equation is the minimal knowledge negative feedback system which can be considered. The first consideration of equations of this type dates back at least to the fourties of the last century; see, e.g., the surveys of André and Seibert [4, 5]. Further progress in this subject has been made especially by Shustin and his prominent collaborators, and we refer the reader to [20], [21], [22], [54], [50] as well as to the work of Akian *et al.* [2] and the references therein.

Following the lines of the monograph [16, pp. 430–439] we start with the definition of an appropriate phase state for which the semiflow generated by equation (2.1) becomes continuous. Thereafter, we will explicitly compute the oscillating solutions in Section 2 and prepare the description of the action of the semiflow on the phase space in the third section. These parts of the chapter generalize the results from Section XVI.2 of Diekmann *et al.* [16] and render more precisely some of the results of Fridman *et al.* [20, pp. 1165-1166]. Section 2.4 is an attempt to understand the structure of the stable sets of the non-trivial steady states and should be seen in connection with Chapter 5 where we will study the same question for the smooth case (1.1).

## 2.1 Existence and semiflow of solutions

In accordance to Browder's terminology a ***solution*** of equation (2.1) is a continuous function $x : I \to \mathbb{R}$, $I = \mathbb{R}$ or $I = [t_0 - 1; +\infty)$ for some $t_0 \in \mathbb{R}$, which satisfies the integral equation

$$x(t) = e^{-\mu(t-\alpha)}x(\alpha) - a \cdot \int\limits_{\alpha-1}^{t-1} e^{-\mu(t-s-1)}\mathrm{sign}(x(s))ds \tag{2.2}$$

for all $t \geq \alpha$ with $\alpha - 1 \in I$. I.e., a solution of (2.1) can be obtained from the ***variation-of-constants formula*** (2.2) at least for initial values with finitely many zeros (for the general case see Akian and Bliman [2] or Shustin *et al.* [22]).

Evidently, the constant functions

$$\mathbb{R} \ni t \mapsto \frac{a}{\mu} \in \mathbb{R}, \quad \mathbb{R} \ni t \mapsto 0 \in \mathbb{R}, \quad \text{and} \quad \mathbb{R} \ni t \mapsto -\frac{a}{\mu} \in \mathbb{R}$$

are stationary solutions of (2.1) defined by the equilibria $\xi^- := \frac{a}{\mu}$, $\xi^0 := 0$ and $\xi^+ := -\frac{a}{\mu}$, respectively. Furthermore, we denote by

$$u_j : [-1; 0] \ni t \mapsto \xi^j \in \mathbb{R} \qquad , j \in \{-, 0, +\} ,$$

the initial segments of the stationary solutions and call them ***steady states*** of (2.1) again.

As in Section 1.2 we define the segment $x_t$ of a solution $x$ of (2.1) at time $t$ (for all $t$ with $t - 1 \in I$), and wish to define a semiflow generated by the segments $x_t^\varphi$, $t \in \mathbb{R}_0^+$, of solutions $x^\varphi$ of the initial value problem

$$\begin{cases} \dot{x}(t) = -\mu x(t) - a\,\mathrm{sign}(x(t-1)) & , t \in \mathbb{R}^+ \\ x_0 = \varphi \end{cases} \tag{2.3}$$

for $\varphi \in C$. As a first step in this direction one may ask for the continuous dependence of the solutions $x^\varphi$ of (2.3) on the initial value $\varphi$. Before we handle this question it is convenient to write down explicitly how solutions of (2.3) look provided that we know (at least the sign distribution of) the initial value.

For every solution $x : [-1; +\infty) \to \mathbb{R}$ the restriction to $\mathbb{R}_0^+$ is composed of straight lines of slope zero or branches of exponentials. More precisely: let $\alpha \in \mathbb{R}_0^+$ and $\beta \in (\alpha; +\infty)$ be given such that

$$\mathrm{sign}(x(t)) = s_\alpha := (\mathrm{sign} \circ x)((\alpha - 1)+) = \lim_{\substack{t \to \alpha-1 \\ t > \alpha-1}} \mathrm{sign}(x(t))$$

for all $t \in (\alpha - 1; \beta - 1) \subset [-1; +\infty)$, then we evidently obtain

$$x\Big|_{(\alpha;\beta)} : (\alpha; \beta) \ni t \mapsto \left(x(\alpha) + s_\alpha \frac{a}{\mu}\right) e^{-\mu(t-\alpha)} - s_\alpha \frac{a}{\mu} \in \mathbb{R} \ . \tag{2.4}$$

Now, if we consider $\varphi = 0$ and $\psi := \frac{\delta}{2} \cdot \mathbb{I} \in U_\delta(0)$ for any $\delta \in (0; \min\{\xi^+, 2(1-e^\mu)\xi^+\})$, formula (2.4) yields

$$|x^\varphi(1) - x^\psi(1)| = |x^\psi(1)| = \left| \frac{\delta}{2} e^{-\mu} + (e^{-\mu} - 1)\xi^- \right| \geq (e^{-\mu} - 1)\xi^+ - \frac{\delta}{2} e^{-\mu} > 0$$

such that we won't be able to make $|x^\varphi(1) - x^\psi(1)|$ arbitrarily small for sufficiently small $\psi \in U_\delta(\varphi)$. Thus, our solutions for initial values in $C$ won't depend *continuously* on the initial value (and, thus, the semiflow generated by the segments of this solutions won't be continuous, too).

In order to circumvent this problem which is a consequence of the discontinuity of our nonlinearity, we have to choose an appropriate phase space in which solutions depend continuously on the initial value. As it turns out, an adequate choice is

$$X := \left\{ \varphi \in C \ : \ |\varphi^{-1}(0)| < \infty \right\}$$

endowed with the topology induced by the maximum norm of $C$. It is a dense subset of $C$ because all polynomials (except for the zero polynomial) are contained in $X$ such that this assertion follows from the classical theorem of WEIERSTRAß.

**REMARK 2.1.1** *For every $\varphi \in X$ there exists a unique solution $x^\varphi : [-1; +\infty) \to \mathbb{R}$ of* (2.3) *which can be computed by repeated application of formula* (2.4).

**LEMMA 2.1.1** *The solutions $x^\varphi$ of* (2.3) *depend continuously on the initial value $\varphi \in X$, i.e.: For any $\varepsilon > 0$, $t_0 \in \mathbb{R}_0^+$ and $\varphi \in X$ there exists a $\delta > 0$ such that for all*

$$\psi \in U_\delta(\varphi) := \{\chi \in X \ : \ \|\chi - \varphi\| < \delta\}$$

*we have*

$$|x^\varphi(t) - x^\psi(t)| < \varepsilon \quad \textit{for all } t \in [0; t_0] \ .$$

PROOF: Let $\varepsilon > 0$ and $\varphi \in X$ be given. We show the continuous dependence on $\varphi$ for $t \in [0; 1]$ since the general case $t \in \mathbb{R}_0^+$ can easily be derived from this situation using the method of steps. Set $\varepsilon_0 := \frac{1}{2a}\varepsilon \cdot e^\mu$ and denote by $\lambda$ the LEBESGUE measure on $\mathbb{R}$.

1. Set $N := N(\varphi) = |\varphi^{-1}(0) \cap (-1; 0)|$, and denote the zeros of $\varphi$ in $(-1; 0)$ by $z_n$, $n \in \{1, ..., N\}$. Now, we define

   $$U_j := \left( -\tfrac{\varepsilon_0}{2N} + z_j; z_j + \tfrac{\varepsilon_0}{2N} \right) \quad \text{for } j \in \{1, ..., N\}$$

   and choose

   $$\delta' := \tfrac{1}{2} \cdot \min_{\tau \in [0;1] \setminus \bigcup_{j=1}^N U_j} |\varphi(\tau)| > 0 \ .$$

For any $\psi \in U_{\delta'}(\varphi)$ this yields

$$\operatorname{sign}(\varphi(\tau)) = \operatorname{sign}(\psi(\tau)) \quad \text{for all } \tau \in [-1;0] \setminus \bigcup_{j=1}^{N} U_j$$

and

$$\lambda\left(\{\tau \in [-1;0] \ : \ \operatorname{sign}(\varphi(\tau)) \neq \operatorname{sign}(\psi(\tau))\}\right) \leq \lambda\left(\bigcup_{j=1}^{N} U_j\right) \leq \varepsilon_0 \ .$$

2. Therefore, we obtain from (2.2) for $t \in [0;1]$ with $\delta := \min\left\{\delta', \frac{1}{2}e^{\mu}\varepsilon\right\}$.

$$\begin{aligned} |x^{\varphi}(t) - x^{\psi}(t)| \ &\leq \ |\varphi(0) - \psi(0)|e^{-\mu} + \left| a \cdot \int_{-1}^{t-1} e^{-\mu(t-1-\cdot)}[\operatorname{sign} \circ \varphi - \operatorname{sign} \circ \psi] \right| \leq \\ &\leq \ \|\varphi - \psi\| e^{-\mu} + ae^{-\mu}\lambda\left(\{\tau \in [-1;0] \ : \ \operatorname{sign}(\varphi(\tau)) \neq \operatorname{sign}(\psi(\tau))\}\right) \leq \\ &\leq \ \delta e^{-\mu} + ae^{-\mu}\varepsilon_0 \\ &< \ \varepsilon \ . \end{aligned}$$

□

Before we can define the semiflow on $X$ in the more or less "usual" way we need a last preparation: in order to obtain a semiflow we have to guarantee that all segments of solutions starting in $X$ remain in $X$, i.e., have finitely many isolated zeros in each time interval $[t-1;t]$, $t \in [1;+\infty)$.

**Lemma 2.1.2** *Zeros of solutions $x$ with $x_0 \in X$ are isolated, or, in other words: If $\varphi \in X$, then $x_t^{\varphi} \in X$ for all $t \in \mathbb{R}_0^+$.*

Proof: Once more, it is sufficient to prove the assertion for $t \in [0;1]$ since the method of steps will then evidently give the full generality.
Since $\varphi \in X$, the graph of $x_1^{\varphi}$ consists of finitely many branches of exponentials or straight lines due to (2.4) such that it is enough to consider the case of the existence of a small interval $[\alpha;\beta] \subset [0;1]$, $\alpha < \beta$, where $x^{\varphi}$ vanishes identically, i.e., where $x^{\varphi}\Big|_{[\alpha;\beta]} = 0$. Let us *assume* the existence of such an interval. Since $x^{\varphi}$ is a solution of (2.1) this would imply

$$\dot{x}^{\varphi}(t) = -\mu x^{\varphi}(t) - a\operatorname{sign}(x^{\varphi}(t)) = 0 \quad \text{for all } t \in (\alpha;\beta) \ ,$$

hence

$$0 = x^{\varphi}(t) = -\frac{a}{\mu}\operatorname{sign}(\varphi(t-1)) \quad \text{for all } t \in (\alpha;\beta) \ .$$

Thus, we would end up with $\varphi\Big|_{(\alpha-1;\beta-1)} = 0$ *contradicting* the choice $\varphi \in X$.

□

This enables us to conclude that

$$F_{-a\,\mathrm{sign}} : \mathbb{R}_0^+ \times X \ni (t,\varphi) \mapsto x_t^\varphi \in X$$

defines a continuous semiflow on $X$, and in analogy to Section 1.2 we will note some of its properties for later use.

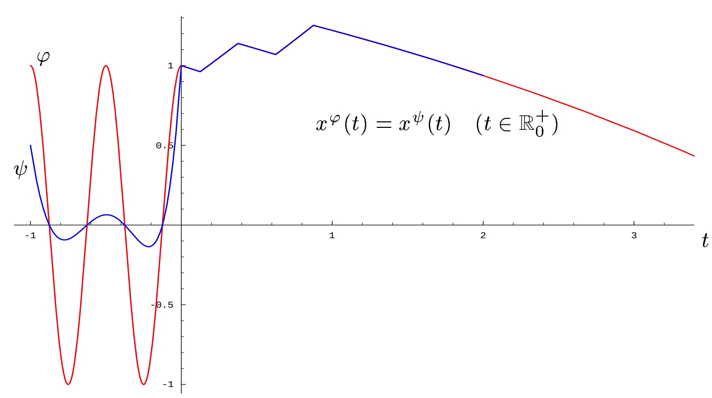
Observe here that the limiting process that led from the smooth delay equations $(1.1)_\alpha$ to the discontinuous delay equation (2.1) had some negative consequences: first, we had to modify our phase space (and to leave $C$ loosing the trivial steady state $u_0 = 0$ on this way) in order to keep the limiting semiflow continuous. Second, we lost the injectivity of the maps $F_{-a\,\mathrm{sign}}(t,\cdot)$, $t \in \mathbb{R}^+$, as is most easily seen by considering two initial values $\varphi$ and $\psi$ in $X$ with

$$\varphi \neq \psi\,, \quad \mathrm{sign}\circ\varphi = \mathrm{sign}\circ\psi \ \text{ on } [-1;0]\,, \quad \text{and} \quad \varphi(0)=\psi(0) \ :$$

Then (2.4) immediately implies $x^\varphi(t) = x^\psi(t)$ for all $t \in \mathbb{R}_0^+$ and, hence,

$$F_{-a\,\mathrm{sign}}(t,\varphi) = F_{-a\,\mathrm{sign}}(t,\psi) \quad \text{for all } t \in [1;+\infty)\ .$$

This explains what was meant above with the sequence that the shape of the initial value is not as important for determining a solution as its sign distribution, and that (2.1) can be understood as a minimal knowledge feedback system.




Clearly, this is the price we have to pay for our rather strong simplification but on the other hand the advantages should also be mentioned now: Due to the choice of our phase space and the existence of the solution formula (2.4) we are able to compute the solutions of (2.1) *explicitly*, and this feature will prove extremely helpful in the sequel. Moreover, $X$ is dense in $C$ and still contains the non-trivial steady states $u_j$, $j \in \{-,+\}$, at which the semiflow $F_{-a\,\mathrm{sign}}$ is differentiable with respect to the state variables, and the partial derivatives with respect to the state variable at $u_j$, $j \in \{-,+\}$, are given by

$$D_2 F_{-a\,\mathrm{sign}}(t, u_j)\psi = y_t \ ,$$

where $y : [-1; +\infty) \to \mathbb{R}$ is a solution of the initial value problem

$$\left\{ \begin{array}{rll} \dot{y}(t) & = -\mu y(t) & , \, t \in \mathbb{R}^+ \\ y_0 & = \psi \in X & \end{array} \right. ,$$

such that we are able to linearize (2.1) at the non-trivial steady states. This is the basis for a detailed investigation of the stable sets of the non-trivial steady states that will be tackled in Section 4 of this chapter. To prepare this we introduce now some subsets of $X$ that will help us to characterize the stable sets as well as the escape sets (cf. Section 1.4).

For all $\varphi \in W_1^+$, where

$$W_1^+ := \left\{ \psi \in X \ : \ \psi \geq 0, \, \psi(0) = -\frac{a}{\mu} \right\} \ ,$$

we obtain $x_1^\varphi = u_+$ from (2.4) and, thus, $x_t^\varphi = u_+$ for all $t \in [1; +\infty)$. Analogously, $\varphi \in W_1^-$, where

$$W_1^- := \left\{ \psi \in X \ : \ \psi \leq 0, \, \psi(0) = \frac{a}{\mu} \right\} \ ,$$

yields $x_t^\varphi = u_-$ for all $t \in [1; +\infty)$. Consequently, we have

$$W_1^j \subset W^s(u_j) \quad \text{ for } j \in \{-,+\} \tag{2.5}$$

which shows that the stable sets $W^s(u_j)$ are not trivial.

Furthermore, since we have $M_{-a\,\mathrm{sign}} := \sup_{\mathbb{R}} |-a\,\mathrm{sign}| = a$ we infer with the same notation and similar arguments as in Section 1.4 that

$$W_2^+ := \left\{ \psi \in X \ : \ \psi(0) > -\frac{a}{\mu} \right\} \subset \mathcal{E}^+ \tag{2.6}$$

since $\varphi \in W_2^+$ yields $\dot{x}^\varphi(t) > 0$ for all $t \in \mathbb{R}^+$, such that $x^\varphi$ is ultimately strictly increasing and "escaping" to infinity; cf. Definition 1.4.2 and 1.4.3. Also, every initial state in

$$W_2^- := \left\{ \psi \in X \ : \ \psi(0) < \frac{a}{\mu} \right\}$$

continues to an ultimately strictly decreasing solution of (2.1) escaping to $-\infty$, and this proves

$$W_2^- \subset \mathcal{E}^- \ . \tag{2.7}$$

Clearly, the sets $W_k^j$, $(k,j) \in \{1,2\} \times \{-,+\}$, are positively invariant under the semiflow $F_{-a\,\mathrm{sign}}$, i.e.

$$F_{-a\,\mathrm{sign}}(\mathbb{R}^+ \times W_k^j) \subset W_k^j \quad \text{ for } (k,j) \in \{1,2\} \times \{-,+\} \ ,$$

and we are now able to give the announced characterization of $W^s(u_j)$, $j \in \{-,+\}$.

**PROPOSITION 2.1.1** *Let $j \in \{-,+\}$. Then*

$$W^s(u_j) = \left\{\varphi \in X \;:\; (\exists\, t \in \mathbb{R}_0^+ \;:\; x_t^\varphi \in W_1^j)\right\} \;.$$

PROOF: We prove the assertion for $j = +$, since the proof for $j = -$ is completely analogous. Furthermore, since we have already proved $W_1^+ \subset W^s(u_+)$ above (cf. inclusion (2.5)), it remains only to show the reverse inclusion

$$W^s(u_+) \subset \left\{\varphi \in X \;:\; (\exists\, t \in \mathbb{R}_0^+ \;:\; x_t^\varphi \in W_1^+)\right\} \;.$$

*Suppose* that this inclusion does *not* hold, and take $\varphi \in W^s(u_+)$: then there exists $t_1 \in \mathbb{R}_0^+$, such that $x_t^\varphi \geq \frac{1}{2}u_+ > 0$ for all $t \in [t_1;\infty)$ (since $x_t^\varphi \to u_+$ as $t \to \infty$). By assumption, $x_t^\varphi \notin W_1^+$, such that

$$x_t^\varphi(0) < \xi^+ \quad \text{for all } t \in [t_1;+\infty)$$

because otherwise either $x_t^\varphi \in W_1^+$ (which is not allowed by assumption) or $x_t^\varphi \in W_2^+$ (which would imply $x^\varphi(t) \nearrow +\infty$ in contradiction to $x_t^\varphi \to u_+$ for $t \to \infty$) would yield contradictions.
Hence, (2.1) would give $\dot{x}^\varphi(t) < 0$ for all $t \in [t_1;+\infty)$ meaning

$$\lim_{t\to\infty} x^\varphi(t) \leq x^\varphi(t_1) = x_{t_1}^\varphi(0) < \xi^+ \;,$$

in *contradiction* to $\varphi \in W^s(u_+)$.

$\square$

In view of the results on $W_1^j$ and $W_2^j$, $j \in \{-,+\}$, one may ask now what we can say about solutions evolving from initial values $\varphi \in X$ satisfying $|\varphi(0)| < -\frac{a}{\mu}$. These yield (eventually) oscillating solutions and we remind the reader of the definitions in Section 1.5 that will be used without further mentioning.

## 2.2 Explicit computation of periodic solutions

We now turn to the investigation of periodic or, more generally, slowly or rapidly oscillating solutions. It is by no means surprising that for small absolute values of $\mu \in \mathbb{R}^-$ we expect a behaviour which is similar to the case $\mu = 0$ described, e.g., in Chapter XVI of the monograph [16].

**LEMMA 2.2.1** *For $\mu \in (-\infty; -\log 2]$ there cannot exist any (eventually) slowly oscillating solutions around $\xi^0 = 0$.*

PROOF: *Assume* that one can find an (eventually) slowly oscillating solution $x$ around $\xi^0 = 0$, and denote by $(t_n)_{n\in\mathbb{N}_0}$ the ordered sequence of its zeros in $\mathbb{R}^+$.

1. If there exists a $n \in \mathbb{N}$ such that $\operatorname{sign}(x(t_n-)) = -1$, then consider $\psi := x_{t_n}$:
   For $x$ being slowly oscillating, i.e. $t_n - t_{n-1} > 1$, we have $\operatorname{sign}(x(t)) = -1$ for all $t \in [t_n - 1, t_n]$ such that
   $$x(t) = -\frac{a}{\mu}e^{-\mu(t-t_n)} + \frac{a}{\mu} \quad \text{for all } t \in [t_n; t_n + 1] \ .$$
   Since $\mu \le -\log 2$, we obtain from
   $$-\frac{a}{\mu}e^{-\mu} + \frac{a}{\mu} \ge -\frac{a}{\mu}$$
   that
   $$x(t_n + 1) = \frac{a}{\log 2} \quad \text{if } \mu = -\log 2$$
   such that $x_{t_n+1} \in W_1^+$ in this case, and
   $$x(t_n + 1) > -\frac{a}{\mu} \quad \text{if } \mu < -\log 2$$
   such that $x_{t_n+1} \in W_2^+$ in the other case.
   Therefore, in either case this yields a *contradiction* to our assumption: because in the first case we infer from PROPOSITION 2.1.1 that $x_0 \in W^s(u_+)$ such that $x_t = u_+ > 0$ for all $t \in [t_{n+1}; +\infty)$. In case $\mu \in (-\infty; -\log 2)$ we obtain from the inclusion (2.6) that $x(t) \to \infty$ as $t \to +\infty$ such that $x$ does not oscillate around zero in either case.

2. If there does not exists a $n \in \mathbb{N}$ such that $\operatorname{sign}(x(t_n-)) = -1$, we have
   $$\operatorname{sign}(x(t_n-)) = +1 \quad \text{for all } n \in \mathbb{N} \ .$$
   As in the first step we obtain
   $$x_{t_1} \in W_1^- \quad \text{if } \mu = -\log 2 \ ,$$
   and
   $$x_{t_1} \in W_2^- \quad \text{if } \mu < -\log 2 \ ,$$
   which gives again a *contradiction* to our assumption.

$\square$

Loosely speaking, the proof of the preceding lemma shows that we cannot obtain slowly oscillating solutions around zero for $\mu$ being too large in absolute value since then the steady states $u_j$, $j \in \{-,+\}$, will be too close to $u_0 = 0$ to allow the existence of amplitudes of slowly oscillating solutions. The reason for this lies in the inclusions (2.6) and (2.7) that prevent bounded solutions from crossing the values $\xi^+ := -\frac{M_{-a\,\mathrm{sign}}}{\mu}$ and $\xi^- := \frac{M_{-a\,\mathrm{sign}}}{\mu}$ (as well as in the fact that every unbounded solution escapes strictly monotonically either to $+\infty$ or $-\infty$). Summarizing this arguments, we have proved the following result.

**Lemma 2.2.2** *There cannot exist oscillating solutions of* (2.1) *around the steady states* $u_j$, $j \in \{-,+\}$.

As expected, for the remaining parameter values we can prove the existence of slowly oscillating periodic solutions around $\xi^0 = 0$.

The special feature that the nonlinearity in (2.1) is an odd function simplifies the treatment of periodic solutions since periodic solutions then also display a certain symmetry which can be used to find or to construct them.

**Lemma 2.2.3**

(1) *For all* $\varphi \in X$ *and all* $t \in \mathbb{R}_0^+$ *we have* $x^{(-\varphi)}(t) = -x^{\varphi}(t)$.

(2) *Let* $\varphi \in X$ *with* $\varphi(0) = 0$ *be given such that the smallest positive zero*

$$z := \min((y\big|_{\mathbb{R}^+})^{-1}(0))$$

*of the solution* $y := x^{\varphi}$ *of* (2.3) *satisfies the relation* $y_z = -y_0$. *Then* $y$ *is periodic with minimal period* $p = 2z$ *and*

$$y(t) = -y(t+z) \quad \text{for all } t \in \mathbb{R}_0^+ \ . \tag{2.8}$$

Proof:

1. It is sufficient to prove the first assertion only for $t \in [0;1]$ and then to apply the method of steps. For $t \in [0;1]$ the variation-of-constants formula (2.2) yields

$$\begin{aligned} x^{(-\varphi)}(t) &= e^{-\mu t}(-\varphi)(0) - a \cdot \int_{-1}^{t-1} e^{-\mu(t-1-s)} \operatorname{sign}((-\varphi)(s))ds = \\ &= -\left( e^{-\mu t}\varphi(0) - a \cdot \int_{-1}^{t-1} e^{-\mu(t-1-s)} \operatorname{sign}(\varphi(s))ds \right) = -x^{\varphi}(t) \ . \end{aligned}$$

2. If we choose $\psi := y_z$, then initial value problem (2.3) has $x^{\psi} = y(z+\cdot)$ as its unique solution, and we have, by the first assertion and because of $-\psi = -x_z^{\psi} = y_0$,

$$y(t) = x^{y_0}(t) = x^{-\psi}(t) = -x^{\psi}(t) = -y(z+t) \quad \text{for all } t \in \mathbb{R}_0^+ \ ,$$

hence

$$y(t+2z) = y(z+(t+z)) = -y(t+z) = -(-y(t)) = y(t)$$

for all $t \in \mathbb{R}_0^+$ such that $y$ is periodic with period $p = 2z$ on $\mathbb{R}_0^+$.
Let us *assume* the existence of a $p' \in (0; p)$ with $y(t + p') = y(t)$ for all $t \in \mathbb{R}$. From $0 = y(0) = y(0 + p') = y(p')$ we see that $p'$ would have to be a zero of $y$ in $\mathbb{R}^+$. Therefore, $p' \in [z; p) = [z; 2z)$ because, by assumption, $z$ is the smallest positive zero of $y$ in $\mathbb{R}^+$. Since $p' = z$ would contradict (2.8) we must have $p' \in (z; 2z)$. But then (2.8) implies that $z' := p' - z \in (0; z)$ is a zero of $y$ since

$$y(z') = -y(z' + z) = -y(p') = 0$$

which yields a *contradiction* to the choice of $z$.

□

Observe that we didn't use the special form of the nonlinearity such that the statements of the last lemma will also hold in the context of any odd continuous function $f$ (if we use (1.4) instead of (2.2) in part 1. of the proof). Now we are in a position to prove one of the main results of this section.

**PROPOSITION 2.2.1** *For $\mu \in (-\log 2; 0)$ there exists a slowly oscillating periodic solution $x^{(0)}$ around $\xi^0 = 0$ with $x^{(0)}(0) = 0$, $\operatorname{sign}(x^{(0)}(0-)) = -1$ and minimal period*

$$p_\mu = p_\mu^{(0)} = 2\left(1 + \log \sqrt[\mu]{2 - e^{-\mu}}\right) .$$

PROOF: We choose

$$\varphi : [-1; 0] \ni t \mapsto -\frac{a}{\mu} e^{-\mu t} + \frac{a}{\mu} \in \mathbb{R}$$

as initial value and show that this leads to the wanted slowly oscillating periodic solution. Since $\operatorname{sign}(\varphi(\tau)) = -1$ for all $\tau \in [-1; 0)$, we obtain

$$x^\varphi\Big|_{[0;1]} : [0; 1] \ni t \mapsto -\frac{a}{\mu} e^{-\mu t} + \frac{a}{\mu} \in \left[0; -\tfrac{a}{\mu}\right) ;$$

note that $x^\varphi(1) = -\frac{a}{\mu} e^{-\mu} + \frac{a}{\mu} < -\frac{a}{\mu}$ because of the choice of $\mu \in (-\log 2; 0)$. Now, $\operatorname{sign}(x^\varphi(\tau)) = +1$ for all $\tau \in (0; 1)$ implies

$$x^\varphi\Big|_{[1;t_1+1]} : [1; t_1 + 1] \ni t \mapsto \left(-\frac{a}{\mu} e^{-\mu} + \frac{2a}{\mu}\right) e^{-\mu(t-1)} - \frac{a}{\mu} \in \left(\tfrac{a}{\mu}; -\tfrac{a}{\mu}\right)$$

where

$$t_1 := 1 + \log \sqrt[\mu]{2 - e^{-\mu}} = 1 - \frac{1}{\mu} \log \frac{1}{2 - e^{-\mu}}$$

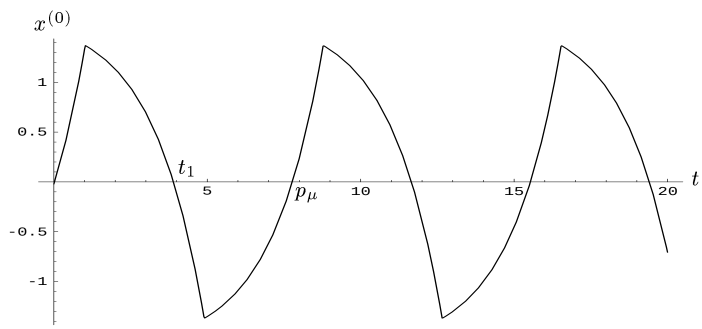
is the first zero of $x^\varphi$ in $(0; +\infty)$. Furthermore,

$$x_{t_1}^\varphi(s) = x^\varphi(t_1 + s) = \left(-\frac{a}{\mu} e^{-\mu} + \frac{2a}{\mu}\right) e^{-\mu s} e^{\log \frac{1}{2 - e^{-\mu}}} - \frac{a}{\mu} = \frac{a}{\mu} e^{-\mu s} - \frac{a}{\mu} = -x_0^\varphi(s)$$

for all $s \in [-1;0]$ such that we obtain $x^{\varphi}_{t_1} = -x^{\varphi}_0$. Now the assertion follows from LEMMA 2.2.3 (2) with $y := x^{\varphi}$ and $z := t_1$.

□

We included this rather extensive proof here since we will need the special shape of the graph of the slowly oscillating solution in Chapter 3 when we try to establish the existence of slowly oscillating solutions for a certain class of continuous nonlinearities close to $-a\,\mathrm{sign}$. A sketch of the graph of $x^{(0)}$ (for $\mu = -\frac{1}{2}$ and $a = 1$) is depicted below.




The next remark may be interpreted as a trivial consequence of the frequently mentioned fact that our nonlinearity $-\alpha\,\mathrm{sign}$ uses only minimal information about the initial phase states.

**REMARK 2.2.1** (1) *Every $\varphi \in X$ with $\varphi \neq 0$ on $[-1;0)$ and $\varphi(0) = 0$ initiates a slowly oscillating solution $x^{\varphi}$ of (2.1) which coincides on $\mathbb{R}^+_0$ either with $x^{(0)}\big|_{\mathbb{R}^+_0}$, if $\varphi(t) < 0$ for all $t \in [-1;0)$, or with $y := -x^{(0)}\big|_{\mathbb{R}^+_0}$, otherwise.*

(2) *With $K^+_s := \left\{\varphi \in X \ : \ 0 < \varphi,\, \varphi(0) < -\frac{a}{\mu}\right\}$ let us define the truncated cone*

$$K_s := K^+_s \cup (-K^+_s) \ .$$

*Setting $t_* : K_s \ni \varphi \mapsto \frac{1}{\mu} \log\left(1 + \frac{\mu|\varphi(0)|}{a}\right) \in \mathbb{R}^+$ we obtain*

$$x^{\varphi}_{t+t_*(\varphi)} = x^{(0)}_t \qquad \text{or} \qquad x^{\varphi}_{t+t_*(\varphi)} = y_t$$

*for all $t \in [1;+\infty)$ and all $\varphi \in K_s$.*

(3) *For every (eventually) slowly oscillating solution $x : [-1;+\infty) \to \mathbb{R}$ of (2.1) exists a time $T_x \in \mathbb{R}^+$ such that*

$$x_t \in \mathcal{O}_0 \qquad \text{for all } t \in [T_x;+\infty) \ ,$$

*where*

$$\mathcal{O}_0 := \{x^{(0)}_t \ : \ t \in \mathbb{R}\}$$

*denotes the* **orbit** *of $x^{(0)}$ in $C$.*

PROOF:

1. Obviously, every initial value $\varphi \in X$ with $\varphi(0) = 0$ and $\mathrm{sign}(\varphi(t)) = -1$ for all $t \in [-1; 0)$ yields $x_1^\varphi = x_1^{(0)}$ by virtue of (2.4). Hence, $x^\varphi\Big|_{\mathbb{R}_0^+} = x^{(0)}\Big|_{\mathbb{R}_0^+}$, and the full assertion follows from LEMMA 2.2.3.

2. Without loss of generality assume $\varphi \in K_s^+$. By definition of $t_*$ we obtain $x^\varphi(t_*) = 0$ from (2.4). Now, $\psi := x_{t_*}^\varphi$ fulfills the assumptions of (1) which implies

$$x_{t+t_*}^\varphi = x_t^{x_{t_*}^\varphi} = x_t^\psi = x_t^{(0)}$$

for all $t \in [1; +\infty)$.

3. For every eventually slowly oscillating solution $x : [-1; +\infty) \to \mathbb{R}$ of (2.1) there exists a $T_1 \in \mathbb{R}_0^+$ such that any two zeros $z > z'$ of $x$ in $[T_1; +\infty)$ have a distance $z - z' > 1$. Let $z$ be any zero of $x$ in $[T_1 + 1; +\infty)$. Thus, the assumptions of (1) are met by $\varphi := x_z$ which yields $x_t \in \mathcal{O}_0$ for all $t \in [z + 1; +\infty)$.

$\square$

A particular aspect of the preceding lemma is that $x^{(0)} : \mathbb{R} \to \mathbb{R}$ is the *unique* global slowly oscillating solution of (2.1) up to translations of time. In Chapter 3 we will need a special time translation of $x^{(0)}$, namely $y := x^{(0)}\left(\cdot + \frac{p_\mu}{2}\right) = -x^{(0)}$, and we draw some of its properties up from the proof of PROPOSITION 2.2.1.

**REMARK 2.2.2** *Consider the slowly oscillating periodic solution $y := -x^{(0)}$ of (2.1).*

(1) *The unique first zero of $y\Big|_{\mathbb{R}+}$ is given by*

$$z(\mu) := 1 + \frac{1}{\mu} \log\left(2 - e^{-\mu}\right) \ ,$$

*and $y$ has minimal period $p_\mu = 2z(\mu)$.*
*Note that $\cosh(-\mu) = \frac{1}{2}(e^{-\mu} + e^{\mu}) > 1$ for all $\mu \in (-\log 2; 0)$ implies $\mu > \log(2 - e^{-\mu})$ and, hence, $z(\mu) \in (2; +\infty)$ for $\mu \in (-\log 2; 0)$.*

(2) *More explicitly, we have*

$$y(t) = \begin{cases} \frac{a}{\mu}(e^{-\mu t} - 1) & , \ \ \textit{if} \ \ t \in [0; 1]\,, \\ \frac{a}{\mu}\left((1 - 2e^{\mu})e^{-\mu t} + 1\right) & , \ \ \textit{if} \ \ t \in [1; z(\mu) + 1]\,, \end{cases}$$

*such that $y$ is strictly negative on $(0; z)$ and strictly positive on $(z; 2z)$, and we have*

$$\dot{y}(t) < 0 \ \ \textit{for } t \in (0; 1) \qquad \textit{and} \qquad \dot{y}(t) > 0 \ \ \textit{for} \ \ t \in (1; z(\mu) + 1)\,.$$

*Evidently, we have*

$$\max_{\mathbb{R}} |y| = |y(1)| = -y(1) = -\frac{a}{\mu}(e^{-\mu} - 1) \ .$$

(3) *The periodic solution $y$ is continuous on $\mathbb{R}$, satisfies* (2.1) *on $\mathbb{R} \setminus (\mathbb{Z}z(\mu) + 1)$, and has the symmetry property*

$$y(t) = -y(t + z(\mu)) \quad \textit{for all} \quad t \in \mathbb{R} \ .$$

Finally, we add some remarks to clarify the transition from $\mu \in (-\log 2; 0)$ to the well-known case $\mu = 0$ (cf. Fridman *et al.* [20, 21, 22] and Diekmann *et al.* [16]).

**Remark 2.2.3** *Let us denote the zeros of the slowly oscillating solution $x^{(0)} =: x^{(0,\mu)}$ of* (2.1) *for $\mu \in \mathbb{R}^-$ by $t_n^{(\mu)}$ , $n \in \mathbb{N}$, and its minimal period by $p_\mu$.*

(1) *It is evident from our construction that $t_n^{(\mu)} = n t_1^{(\mu)} = n\left(1 + \log \sqrt[\mu]{2 - e^{-\mu}}\right)$, $n \in \mathbb{N}_0$.*

(2) *Since $t_1^{(\mu)} = 1 + \log \sqrt[\mu]{2 - e^{-\mu}} \to 2$ as $\mu \to 0$, we easily obtain $t_n^{(\mu)} \to t_n^{(0)} = 2n$, $n \in \mathbb{N}_0$, and $p_\mu \to p_0 = 4$ as $\mu \to 0$.*

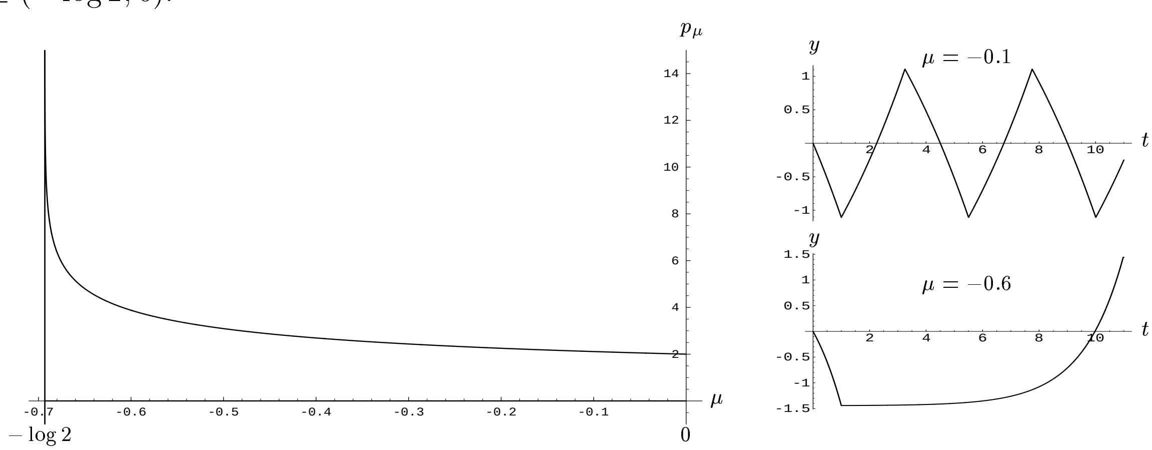
(3) *Furthermore, the segments of the slowly oscillating solution $x^{(0,\mu)}$ constructed above converge pointwise to the segments of the periodic slowly oscillating solution $x^{(0,0)}$ of* (2.1) *with $\mu = 0$. More precisely, we have locally compact convergence of $x^{(0,\mu)}$ to $x^{(0,0)}$ on $\mathbb{R} \setminus (\mathbb{Z}t_1^{(0)} + 1)$ as $\mu \to 0$.*

(4) *It should also be mentioned that*

$$\lim_{\mu \searrow -\log 2} t_1^{(\mu)} = \infty \ ,$$

*which reflects the fact that, for $\mu = -\log 2$, every element $\varphi \in X$ as in* Remark 2.2.1 (1) *gives rise to $x_1^\varphi \in W_1^j$, $j \in \{-, +\}$, such that $\varphi \in W^s(u_j)$.*

We illustrate the results of this remark which will be revisited several times (in this as well as in the following chapter) by a sketch of the period $p_\mu$ of $x^{(0)}$ as a function of $\mu \in (-\log 2; 0)$.



Beside $x^{(0)}$ and $y := -x^{(0)}$ and translates thereof there also exist a countable number of rapidly oscillating solutions $x^{(N)}$, $N \in 2\mathbb{N}$, as we intend to show now.

**PROPOSITION 2.2.2** *For $\mu \in (-\log 2; 0)$ and for each $m \in \mathbb{N}$, let $\tau_*(\mu, m) \in \left(0; \frac{1}{2\cdot(2m)+1}\right)$ be the unique solution of*

$$(2m+1)\tau + \frac{2m}{\mu}\log(2 - e^{-\mu\tau}) = 1 \ . \tag{2.9}$$

*Then there exists a rapidly oscillating solution $x^{(2m)}$ around zero with $x^{(2m)}(0) = 0$, $\operatorname{sign}(x^{(2m)}(0-)) = -1$ and minimal period*

$$p_\mu^{(2m)} := 2 \cdot \left(\tau_*(\mu, m) + \frac{1}{\mu}\log(2 - e^{-\mu\tau_*(\mu,m)})\right) = \frac{1 - \tau_*(\mu, m)}{m} \ .$$

PROOF: Let $\mu \in (-\log 2; 0)$ be given and fix an integer $m \in \mathbb{N}$.

1. For any $\tau \in \mathbb{R}^+$ we have $2 - e^{-\mu\tau} < 1$. Note further that for $\mu \in (-\log 2; 0)$ we have the estimate $\frac{1}{2m+1} \leq 1 < \frac{\log 2}{-\mu}$ (for all $m \in \mathbb{N}$) such that

   $$-\frac{a}{\mu}e^{-\mu\tau} + \frac{a}{\mu} < -\frac{a}{\mu} \qquad \text{for all } \tau \in \left(0; \tfrac{1}{2m+1}\right)$$

   and

   $$2 - e^{-\mu\tau} > 0 \qquad \text{for all } \tau \in \left(0; \tfrac{1}{2m+1}\right) \ .$$

   From $\cosh\left(\frac{\mu}{4m+1}\right) = \frac{1}{2}\left(e^{\frac{\mu}{4m+1}} + e^{-\frac{\mu}{4m+1}}\right) > 1$ we obtain $e^{\frac{\mu}{4m+1}} > 2 - e^{-\frac{\mu}{4m+1}}$, hence

   $$\frac{1}{4m+1} < \frac{1}{\mu}\log(2 - e^{-\frac{\mu}{4m+1}})$$

   implies

   $$-\frac{2m}{4m+1} + \frac{2m}{\mu}\log(2 - e^{-\frac{\mu}{4m+1}}) > 0 \ .$$

2. The function

   $$h : \left(0; \tfrac{1}{4m+1}\right) \ni \tau \mapsto (2m+1)\tau + \frac{2m}{\mu}\log(2 - e^{-\mu\tau}) - 1 \in \mathbb{R}$$

   is strictly increasing on $\left(0; \frac{1}{4m+1}\right)$ since recalling $2 - e^{-\mu\tau} \in (0; 1)$ from step 1. yields

   $$h'(\tau) = 2m + 1 + 2m\frac{e^{-\mu\tau}}{2 - e^{-\mu\tau}} > 2m + 1 + 2me^{-\mu\tau} > 4m + 1 > 0$$

for all $\tau \in \left(0; \frac{1}{4m+1}\right)$. Furthermore, $h(0) = -1$ and

$$h(\tfrac{1}{4m+1}) = -\frac{2m}{4m+1} + \frac{2m}{\mu}\log(2 - e^{\frac{-\mu}{4m+1}}) \overset{1.}{>} 0$$

such that the existence of $\tau_* \in \left(0; \frac{1}{2\cdot(2m)+1}\right)$ is a consequence of the Intermediate Value Theorem (while the uniqueness is evident from the strict monotonicity of $h$).

3. Now, we can define the initial value $\varphi := x_0^{(2m)}$ that continues to the announced rapidly oscillating periodic solution. Therefore, set

$$\tau_{**} := \tau_* + \frac{1}{\mu}\log(2 - e^{-\mu\tau_*}) \overset{(2.9)}{=} \frac{1-\tau_*}{2m} \ ,$$

let

$$\overline{x} : [0; \tau_{**}] \ni t \mapsto \left\{ \begin{array}{ll} -\frac{a}{\mu}e^{-\mu t} + \frac{a}{\mu} & , \ t \in [0; \tau_*] \\ \left(-\frac{a}{\mu}e^{-\mu\tau_*} + \frac{2a}{\mu}\right) e^{-\mu(t-\tau_*)} - \frac{a}{\mu} & , \ t \in [\tau_*; \tau_{**}] \end{array} \right\} \in \mathbb{R}_0^+$$

and define $\varphi := x_0^{(2m)}$ as follows:

(1) For $l \in \{0, ..., 2m-1\}$ let

$$\varphi\Big|_{[-(l+1)\tau_{**}; -l\tau_{**}]} : [-(l+1)\tau_{**}; -l\tau_{**}] \ni t \mapsto (-1)^{l+1}\overline{x}(t + (l+1)\tau_{**}) \in \mathbb{R},$$

such that we have defined $\varphi$ in this step on the interval $[-2m\tau_{**}; 0] = [-1+\tau_*; 0]$.

(2) Finally, let

$$\varphi\Big|_{[-1; -1+\tau_*]} : [-1; -1+\tau_*] \ni t \mapsto -\overline{x}(t + \tau_{**} + 1 - \tau_*) \in \mathbb{R} \ .$$

4. It remains to show that $\varphi$ continues indeed to a (rapidly oscillating) periodic solution $x^{(2m)}$ which is now an easy calculation: from equation (2.4) we readily obtain $x^\varphi\Big|_{[0;\tau_*]} = \overline{x}\Big|_{[0;\tau_*]}$ since $\text{sign}(\varphi(\tau)) = -1$ for all $\tau \in [-1; -1+\tau_*)$ and $\varphi(0) = 0$. In the next step we have $\text{sign}\varphi(\tau) = +1$ for $\tau \in [-2m\tau_{**}; -(2m-1)\tau_{**})$ which implies in particular $x^\varphi\Big|_{[0;\tau_{**}]} = \overline{x}$ and, hence,

$$x^\varphi_{\tau_{**}} = -\varphi \ ,$$

such that $x^\varphi = x^{(2m)}$ is a rapidly oscillating periodic solution of equation (2.1) with minimal period $p_\mu^{(2m)} = 2\tau_{**}$ (as a consequence of Lemma 2.2.3(2)).

□

The existence of these rapidly oscillating periodic solutions should – as in the case $\mu = 0$ – be seen in connection with the fact that, as $\alpha = \alpha_0$ increases to $+\infty$, more and more complex conjugate pairs of characteristic values of the linearization

$$\dot{x}(t) = -\mu x(t) + f'_\alpha(0)x(t-1)$$

of equation $(1.1)_\alpha$ (at the trivial steady state) move into the right half plane (cf. REMARK 1.3.2), giving rise to HOPF bifurcations (see, e.g., [16, Chapter X]).

**REMARK 2.2.4** *For $\mu \in (-\log 2; 0)$ and $m \in \mathbb{N}$ the rapidly oscillating solutions $x^{(2m)}$ defined in* PROPOSITION 2.2.2 *have the following properties:*

(1) *The distance of two consecutive zeros is*

$$\tau_{**}(\mu, m) := \tau_*(\mu, m) + \frac{1}{\mu}\log(2 - e^{-\mu\tau_*(\mu,m)}) = \frac{1 - \tau_*(\mu, m)}{2m} \ .$$

(2) *The number of zeros of the initial values $\varphi := x_0^{(2m)}$ in the interval $(-1; 0)$ is*

$$N(\varphi) = 2m \in 2\mathbb{N} \ .$$

(3) *For $\mu \to 0$ we obtain $p_\mu^{(2m)} \searrow p_0^{(2m)} = \frac{4}{4m+1}$ for all $m \in \mathbb{N}$, and the segments of $x^{(2m)}$ converge pointwise to the segments of solutions of* (2.1) *for $\mu = 0$, where the convergence is locally compact on $\mathbb{R} \setminus (\mathbb{Z}\tau_{**} + 1)$.*

IDEA OF THE PROOF: While (1) and (2) are obvious from PROPOSITION 2.2.2 we only sketch the proof of (3). Fix $m \in \mathbb{N}$.

1. By $p_\mu^{(2m)} = \frac{1-\tau_*(\mu,m)}{m}$ it suffices to show the existence of $\lim_{\mu\to 0} \tau_*(\mu, m) =: \tau_*$. In this case (2.9) implies $\tau_* = \frac{1}{4m+1}$ such that $\lim_{\mu\to 0} p_\mu^{(2m)} = \frac{1-\tau_*}{m} = \frac{4}{4m+1}$ will prove the assertion. Since $\tau_*(\mu, m) \in \left(0; \frac{1}{4m+1}\right)$ it is enough to show that $\tau_*(\cdot, m)$ is monotonic.

2. For this purpose consider

$$h : (-\log 2; 0) \times \left(0; \tfrac{1}{4m+1}\right) \ni (\mu, \tau) \mapsto (2m + 1)\tau + \frac{2m}{\mu}\log(2 - e^{-\mu\tau}) - 1 \in \mathbb{R}$$

and observe that $h_\tau(\mu, \tau) = 2m + 1 + 2m \cdot \frac{e^{-\mu\tau}}{2-e^{-\mu\tau}} > 0$ for all $\mu \in (-\log 2; 0)$ (as we already know from step 2. of the previous proof). Therefore, the Implicit Function Theorem yields $\tau_*(\cdot, m) \in C^1(-\log 2; 0)$ with $h(\mu, \tau_*(\mu, m)) = 0$ for all $\mu \in (-\log 2; 0)$ and

$$(\tau_*)_\mu(\mu, m) = -\frac{h_\mu(\mu, \tau)}{h_\tau(\mu, \tau)}\Big|_{(\mu,\tau_*(\mu,m))} = \frac{-\frac{2m\tau}{\mu}\frac{e^{-\mu\tau}}{2-e^{-\mu\tau}} + \frac{2m}{\mu^2}\log(2 - e^{-\mu\tau})}{1 + 2m \cdot \left[1 + \frac{e^{-\mu\tau}}{2-e^{-\mu\tau}}\right]}\Big|_{(\mu,\tau_*(\mu,m))} \ .$$

Now, $h(\mu,\tau)=0$ for $(\mu,\tau)=(\mu,\tau_*(\mu,m))$ implies $\frac{2m}{\mu}\log(2-e^{-\mu\tau})=1-(2m+1)\tau$. Inserting this into the expression for $(\tau_*)_\mu$ above yields

$$\begin{aligned}(\tau_*)_\mu(\mu,m) &= -\frac{h_\mu(\mu,\tau)}{h_\tau(\mu,\tau)}\Big|_{(\mu,\tau_*(\mu,m))} = \frac{-\frac{1}{\mu}\left[2m\tau\left(\frac{e^{-\mu\tau}}{2-e^{-\mu\tau}}+1\right)-1+\tau\right]}{1+2m\cdot\left[1+\frac{e^{-\mu\tau}}{2-e^{-\mu\tau}}\right]}\Big|_{(\mu,\tau_*(\mu,m))} = \\ &= -\frac{1}{\mu}\left(\tau-\frac{1}{1+\frac{4m}{2-e^{-\mu\tau}}}\right)\Big|_{(\mu,\tau_*(\mu,m))}.\end{aligned}$$

3. Finally, proving $\tau-\frac{1}{1+\frac{4m}{2-e^{-\mu\tau}}}>0$ for $\tau=\tau_*(\mu,m)$ will complete the proof. Because $\tau\in(0;\frac{1}{4m+1})$,

$$(2-e^{-\mu\tau})^{-1}=\exp\left(-\frac{\mu}{2m}(1-(2m+1)\tau)\right)>\exp\left(-\frac{\mu}{4m+1}\right),$$

and

$$\tau-\frac{1}{1+\frac{4m}{2-e^{-\mu\tau}}}>\tau-\frac{1}{1+4m\exp(-\frac{\mu}{4m+1})}$$

it suffices to show $h(\mu,\frac{1}{1+4m\exp(-\frac{\mu}{4m+1})})<0$ since the monotonicity of $h(\mu,\cdot)$ yields $\tau_*(\mu,m)=\tau>\frac{1}{1+4m\exp(-\frac{\mu}{4m+1})}$. The map

$$H:(-\log 2;0)\ni\mu\mapsto\frac{2m+1}{1+4m\exp(-\frac{\mu}{4m+1})}+\frac{2m}{\mu}\log\left(2-e^{-\frac{\mu}{1+4m\exp(-\frac{\mu}{4m+1})}}\right)-1\in\mathbb{R}$$

is monotonically increasing because of $H_\mu(\mu)>0$ for all $\mu\in(-\log 2;0)$ and

$$H(0)=\lim_{\mu\to 0-}H(\mu)=\frac{2m+1}{1+4m}+\frac{2m}{1+4m}-1=0$$

which proves $H(\mu)=h(\mu,\frac{1}{1+4m\exp(-\frac{\mu}{4m+1})})<0$ for all $\mu\in(-\log 2;0)$. Therefore, $(\tau_*)_\mu(\mu,m)>0$ for all $\mu\in(-\log 2;0)$ and the assertion is proved.

$\square$

In this context it is reasonable to refer to the work of NUSSBAUM and SHUSTIN [50] as well as to work of AKIAN and BLIMAN [2] which contains a discrete LYAPUNOV functional for a class of equations containing equation (2.1) completely analogous to that for smooth delay equations treated in Section 1.6. In the next section we will develop a similar approach.

## 2.3 Description of the semiflow on a subset of $\mathcal{B}$

The last paragraph provided us with a first insight into the dynamics in $X$: we were able to compute explicitly (infinitely many) periodic solutions of (2.1). The next step is to determine sets into which each solution $x^\varphi$ enters after some finite time and stays in.

### A. Criteria for the boundedness of solutions

The boundedness or unboundedness of a solution $x^\varphi$ for given $\varphi \in X$ can be determined completely within one time unit. This is the assertion of

**Lemma 2.3.1** *Let $\mu \in (-\log 2; 0)$. For every $\varphi \in X$ we have*

(1) either *there exists a $t_\varphi \in [0;1]$ with $|x^\varphi(t_\varphi)| > -\frac{a}{\mu}$ (and, thus, $\varphi \in \mathcal{E}^+ \cup \mathcal{E}^-$),*

(2) or *it is $x^\varphi(\mathbb{R}_0^+) \subset [\frac{a}{\mu}; -\frac{a}{\mu}]$.*

Proof: Let $x^\varphi$ be the solution of (2.1) with $x_0 = \varphi$ and *assume* that there exists a $t_0 \in (1; +\infty)$ such that $x^\varphi(t_0) \notin [\frac{a}{\mu}; -\frac{a}{\mu}]$ but $x^\varphi([0;1]) \subset [\frac{a}{\mu}; -\frac{a}{\mu}]$; without loss of generality let $x^\varphi(t_0) > -\frac{a}{\mu}$ which means $\varphi \in \mathcal{E}^+$ because of $x^\varphi_{t_0} \in W_2^+$ by (2.6).

1. Let
$$t_* := \sup\left\{t \in [1; t_0] \,:\, x^\varphi(t) \le -\frac{a}{\mu}\right\} .$$
By definition it is $t_* \in [1; t_0)$ and $x^\varphi(t_*) = -\frac{a}{\mu}$.

2. We claim that $x^\varphi_{t_*} > 0$.
`Assume` the existence of a $\tau \in [t_* - 1; t_*]$ with $x^\varphi(\tau) \le 0$. Since all zeros of $x^\varphi_{t_*}$ are isolated (cf. Lemma 2.1.2), set
$$z := t_* + \max\left(x^\varphi_{t_*}\right)^{-1}(0) .$$
Clearly, $z \in [t_* - 1; t_*)$ since $x^\varphi(t_*) = -\frac{a}{\mu} > 0$. Because of $\dot{x}^\varphi(t) \le -\mu x(t) + a$ for all $t \in [z; t_*]$, formula (2.4) implies
$$x^\varphi(t) \le -\frac{a}{\mu}e^{-\mu(t-z)} + \frac{a}{\mu} \qquad \text{for all } t \in [z; t_*] ,$$
and together with $x^\varphi(t_*) = -\frac{a}{\mu}$ we must necessarily have
$$t_* - z \ge -\frac{1}{\mu}\log 2 > 1 ,$$
such that $z < t_* - 1$ in `contradiction` to $z \in [t_* - 1; t_*)$.

3. Now we conclude from steps 1. and 2. that $x^\varphi_{t_1} \in W_1^+ \subset W^s(u_+)$ *in contradiction* to $\varphi \in \mathcal{E}^+$.

□

Evidently, the preceding lemma contains the assertion that there cannot exist any (neither slowly nor rapidly) oscillating solutions around the nontrivial stationary points $u_j$, $j \in \{-,+\}$, and, thus, yields an alternative proof of LEMMA 2.2.2.

For the remainder of this chapter let us assume

$$\text{(H1}'\text{)} \qquad \mu \in (-\log 2; 0) \ ,$$

instead of (H1). Note that this assumption gives bounds on the slope of our solutions by (2.4) such that we can draw a number of extremely helpful conclusions from LEMMA 2.3.1 now.

**COROLLARY 2.3.1** *For $\mu \in (-\log 2; 0)$ we have*

$$\mathcal{B} = \left\{ \varphi \in X \ : \ x^\varphi(\mathbb{R}_0^+) \subset [\tfrac{a}{\mu}; -\tfrac{a}{\mu}] \right\} = \left\{ \varphi \in X \ : \ \|x_1^\varphi\| \leq -\tfrac{a}{\mu} \right\} \ .$$

Moreover, due to LEMMA 2.3.1 we are in a position to improve PROPOSITION 2.1.1. In order to check whether $\varphi$ belongs to $W^s(u_j)$ (for some $j \in \{-,+\}$) or not, it is sufficient to compute $x_1^\varphi$.

**COROLLARY 2.3.2** *Let $j \in \{-,+\}$. Then*

$$W^s(u_j) = \left\{ \varphi \in X \ : \ (\exists t_0 \in [0;1] \ : \ x^\varphi_{t_0} \in W_1^j) \right\} \ .$$

The following corollary is essentially THEOREM 3.6 of FRIDMAN *et al.* [20] and gives a sufficient (and easy to verify) criterion on $\varphi$ to be the initial value of a bounded solution of (2.1). In some sense, it reflects the influence of the "affinity" of (2.1) to ordinary differential equations which evolves from the fact that the nonlinearity in (2.1) is piecewise constant such that the single value $\varphi(0)$ is enough to yield information about the solution $x^\varphi$.

**COROLLARY 2.3.3** *For $\mu \in (-\log 2; 0)$ we have*

$$\left\{ \varphi \in X \ : \ |\varphi(0)| < -\frac{a}{\mu}(2e^\mu - 1) \right\} \subset \mathcal{B} \ .$$

PROOF: By LEMMA 2.3.1 we only have to check $x^{\varphi}([0;1]) \subset [\frac{a}{\mu}; -\frac{a}{\mu}]$. We consider the case $\varphi(0) \geq 0$ without loss of generality. The trivial estimate $\operatorname{sign}(x^{\varphi}(t-1)) \geq -1$ yields

$$x^{\varphi}(t) \leq \left(\varphi(0) - \frac{a}{\mu}\right) e^{-\mu t} + \frac{a}{\mu} \leq \left(\varphi(0) - \frac{a}{\mu}\right) e^{-\mu} + \frac{a}{\mu} < -\frac{a}{\mu} \quad \text{for all } t \in [0;1]$$

by choice of $\varphi(0) < -\frac{a}{\mu}(2e^{\mu} - 1)$.

□

Furthermore, since $\mathcal{B}$ contains the stable sets of the nontrivial stationary solutions, let

$$\mathcal{Z} := \mathcal{B} \setminus (W^s(u_+) \cup W^s(u_-)) \ .$$

In view of LEMMA 2.3.1 we singled out again the initial values that yield bounded solutions and, therein, those which stay bounded but do not "converge" to one of the steady states.

We introduce a last subset of $X$ which will turn out to be a subset of $\mathcal{Z}$ and will be needed in the next section again. Let

$$W_3 := \left\{ \varphi \in X \ : \ |\varphi(0)| < -\frac{a}{\mu}(2e^{\mu} - 1) \right\} \ .$$

As a consequence of COROLLARY 2.3.3 we readily obtain

**COROLLARY 2.3.4** *We have $W_3 \subset \mathcal{Z}$. Observe that the inverse implication can not be true (as can be seen by choosing $\varphi := r \cdot \mathbb{I}$ for $r \in [-\frac{a}{\mu}(2e^{\mu} - 1); -\frac{a}{\mu})$ as initial value which yields a slowly oscillating solution by* REMARK 2.2.1*).*

We prepare the introduction of a discrete LYAPUNOV functional on $\mathcal{B}$ by a series of lemmata. Our first aim is to show the existence of an unbounded sequence of zeros of a solution $x$ of (2.1) in $\mathbb{R}^+$ where $x$ changes sign.

**DEFINITION 2.3.1** Let $\varphi \in X$ be given. A zero $z$ of $\varphi$ in $(-1;0)$ or of $x^{\varphi}$ in $(-1;+\infty)$ is called ***simple***, iff $\operatorname{sign}(x^{\varphi}(z-)) = -\operatorname{sign}(x^{\varphi}(z+))$. Otherwise we call $z$ a ***multiple*** zero.

Please notice that we deviate from the usual denotation of multiple zeros because by our definition a function mustn't have a sign change at the multiple zero.

**LEMMA 2.3.2** *For every $\varphi \in \mathcal{Z}$ the solution $x^{\varphi}$ has infinitely many zeros in $\mathbb{R}^+$, and all these zeros of $x^{\varphi}$ in $\mathbb{R}^+$ form an unbounded strictly increasing sequence $(\tau_n(\varphi))_{n \in \mathbb{N}}$.*

In particular, this lemma shows that every solution starting in $\mathcal{Z}$ is necessarily oscillating around $\xi^0 := 0$ while all other solutions (the escaping as well as all solutions with initial value in the stable set of a steady state) stay away from zero within one time step.

PROOF: Let $\varphi \in \mathcal{Z}$. If $\varphi^{-1}(0) \neq \emptyset$, set $z_1 := \inf \varphi^{-1}(0)$. If $z_1 = -1$, then let $z_2 := \inf(\varphi\big|_{(-1;0]})^{-1}(0)$ or $z_2 = 0$ in case that $(\varphi\big|_{(-1;0]})^{-1}(0) = \emptyset$. If $\varphi^{-1}(0) = \emptyset$, set $z_1 := -1$ and $z_2 := 0$. Furthermore, define

$$s_\alpha := \left\{ \begin{array}{ll} \operatorname{sign}(\varphi\big|_{[-1;z_1)}) & , z_1 \neq -1 \\ \operatorname{sign}(\varphi\big|_{(-1;z_2)}) & , z_1 = -1 \end{array} \right\}.$$

In case that $\varphi(0) \neq 0$, the continuity of $x^\varphi$ yields $|x^\varphi(t)| > 0$ for some $\varepsilon > 0$. In case $\varphi(0) = 0$ we obtain from (2.4)

$$x^\varphi(t) = s_a \frac{a}{\mu} e^{-\mu t} - s_\alpha \frac{a}{\mu} \neq 0$$

for all $t \in (0;\varepsilon)$, $\varepsilon \in (0;\varepsilon_0)$ where

$$\varepsilon_0 := \left\{ \begin{array}{ll} z_1 + 1 & , z_1 \neq -1 \\ z_2 + 1 & , z_1 = -1 \end{array} \right\}.$$

Hence, we have $|x^\varphi| > 0$ on some interval $(0;\varepsilon)$, $\varepsilon > 0$; we assume $x^\varphi(t) > 0$ for all $t \in (0;\varepsilon)$ without loss of generality.

1. The *assumption* $x^\varphi(t) \in \left(0; -\frac{a}{\mu}\right)$ for $t \in \mathbb{R}_0^+$ leads to

$$x^\varphi(t) = \left(x^\varphi(1) + \tfrac{a}{\mu}\right) e^{-\mu(t-1)} - \tfrac{a}{\mu} \qquad \text{for all } t \in [1;+\infty)\,.$$

But since $x^\varphi(1) \in \left(\frac{a}{\mu}; -\frac{a}{\mu}\right)$, this would imply $\lim\limits_{t\to+\infty} x(t) = -\infty$ in *contradiction* to $x^\varphi > 0$ on $[0;+\infty)$.

2. Thus, there must exist a $\tau_1(\varphi) > 0$ with $x^\varphi(\tau_1(\varphi)) = 0$ given by

$$\tau_1(\varphi) := \inf\{t \in \mathbb{R}^+ \ : \ x^\varphi(t) \le 0\}\ .$$

Furthermore, there must exist infinitely many zeros of $x^\varphi$ in $\mathbb{R}^+$, recursively defined as

$$\tau_{n+1}(\varphi) := \inf((x^\varphi)^{-1}(0) \cap (\tau_n(\varphi);+\infty))$$

forming the sequence $(\tau_n(\varphi))_{n\in\mathbb{N}}$ since *otherwise* one could apply the same reasoning as above to $x^\varphi_{\tau_N(\varphi)}$ instead of $\varphi$ (where $\tau_N(\varphi)$ is the largest of the finitely many zeros) and derive a *contradiction* since all segments of $x^\varphi$ have finitely many isolated zeros by LEMMA 2.1.2.

$\square$

Analyzing the proof of the foregoing lemma, it is clear that there must necessarily exist simple zeros of $x^\varphi$ in $\mathbb{R}^+$, at least from time to time. Moreover, it is not difficult to prove the existence of infinitely many simple zeros of $x^\varphi$ in $\mathbb{R}^+$ refining the proof above. Our aim is to prove more than this rather crude information, namely, to derive an analogue of Corollary 1.6.3 which guarantees that all zeros of $x^\varphi$ are simple from some finite time $t_\varphi \in \mathbb{R}^+$ on.

## B. A discrete Lyapunov functional for (2.1) on $\mathcal{Z}$

Guided by Lemma 2.3.2, we want to determine more explicitly the behaviour of the solutions in $\mathcal{Z}$. Thus, our next step is to define a discrete Lyapunov functional (completely analogous to the approach of Mallet-Paret–Cao–Arino in Section 1.6) and to prove that for every $\varphi \in \mathcal{Z}$ there is a time $t_0(\varphi) \in \mathbb{R}^+$ such that all zeros are simple from that time on.

**Definition 2.3.2** We set

$$\sigma : \mathbb{R}_0^+ \times \mathcal{Z} \ni (t, \varphi) \mapsto \inf \left( (x^\varphi)^{-1}(0) \cap [t; +\infty) \right) \in \mathbb{R}_0^+ ,$$

and call

$$U : \mathbb{R}_0^+ \times \mathcal{Z} \ni (t, \varphi) \mapsto \left| (x^\varphi_{\sigma(\varphi,t)})^{-1}(0) \cap [-1; 0] \right| \in \mathbb{N}_0$$

a ***discrete LYAPUNOV functional*** for (2.1) on $\mathcal{Z}$.

Before we can proceed to give a detailed description of the dynamics in $\mathcal{Z}$, we collect the basic properties of $U$ which are completely analogous to those in [12, pp. 368-371]; cf. also Section 1.6 for the "continuous counterpart". This shows the good applicability of the discrete Lyapunov functional (due to Mallet-Paret, Cao and Arino) even to this discontinuous situation. Furthermore, the discrete Lyapunov functional $U$ is similar to the frequency function of Fridman *et al.* [20, 21, 22] from which it differs slightly in its definition.

An elementary observation concerns the "computation" of $U(t_0, \varphi)$ for given $\varphi \in \mathcal{Z}$ and $t_0 \in \mathbb{R}_0^+$: the definition of $\sigma$ and $U$ yield

$$U(t, \varphi) = U(t_0, \varphi) \quad \text{for all } t \in [t_0; \sigma(t_0, \varphi)] , \tag{2.10}$$

or, setting $n := \inf\{k \in \mathbb{N} : t_0 \in (\tau_{k-1}(\varphi); \tau_k(\varphi)]\}$ we see that

$$\sigma(t_0, \varphi) = \tau_n(\varphi)$$

such that

$$U(t, \varphi) = U(\tau_n(\varphi), \varphi) \quad \text{for all } t \in (\tau_{n-1}(\varphi); \tau_n(\varphi)]$$

where $(\tau_n(\varphi))_{n \in \mathbb{N}}$ is the unbounded strictly increasing sequence of all zeros of $x^\varphi$ in $\mathbb{R}^+$ from Lemma 2.3.2(1). Trivially, note that $U(t, \varphi) = U(\sigma(t, \varphi), \varphi)$ holds for all $t \in \mathbb{R}_0^+$.

**Lemma 2.3.3** *Let $\varphi \in \mathcal{Z}$ be given.*

(1) *For any two zeros $t_j \in \mathbb{R}_0^+$, $j \in \{1,2\}$, of $x^\varphi$ with $t_1 < t_2$ there exists a simple zero $s$ of $x^\varphi$ in $(t_1 - 1; t_2 - 1)$.*

(2) *If there exist zeros $t_j \in \mathbb{R}_0^+$, $j \in \{1,2\}$, of $x^\varphi$ with $t_2 \in (t_1; t_1 + 1]$, then we have*

$$U(t_2, \varphi) \le U(t_1, \varphi) \ .$$

*Moreover, if $t_1 \in \mathbb{R}^+$ is a multiple zero of $x^\varphi$, then $t_1 - 1$ is also a zero of $x^\varphi$ and*

$$U(t_2, \varphi) \le U(t_1, \varphi) - 1 \ .$$

(3) *The mapping*

$$\mathbb{R}^+ \ni t \mapsto U(t, \varphi) \in \mathbb{N}_0$$

*is non-increasing.*

(4) *If $\tau_n(\varphi) \in \mathbb{R}^+$ is a multiple zero for some $n \in \mathbb{N}$, then*

$$U(t, \varphi) \le U(\tau_n(\varphi), \varphi) - 1 \quad \textit{for all } t \in [\tau_n(\varphi) + 1; +\infty) \ .$$

Proof: We fix $\varphi \in \mathcal{Z}$ throughout the whole proof.

1. Under the *assumption* that there is no simple zero of $x^\varphi$ in $(t_1 - 1; t_2 - 1)$, we may assume that $x^\varphi \ge 0$ on $(t_1 - 1; t_2 - 1)$ without loss of generality. Since zeros of $x^\varphi$ have to be isolated by definition of $X$ and Lemma 2.1.2, the variation-of-constants formula (2.2) would give

$$x^\varphi(t) = -\frac{a}{\mu} e^{-\mu(t - t_1)} + \frac{a}{\mu} \quad \text{for all } t \in [t_1; t_2) \ ,$$

hence $x^\varphi(t_2) = -\frac{a}{\mu} e^{-\mu(t_2 - t_1)} + \frac{a}{\mu} > 0$ *contradicting* $x^\varphi(t_2) = 0$.

2. We denote the zeros of $x^\varphi$ in $[t_1; t_2]$ by $z_j$, $j \in \{1, ..., N_1\}$, such that $z_1 := t_1$, $z_{N_1} := t_2$, and $z_j < z_{j+1}$ for all $j \in \{1, ..., N_1 - 1\}$. Part (1) of this lemma yields corresponding simple zeros $s_j \in (z_j - 1; z_{j+1} - 1)$ for all $j \in \{1, ..., N_1 - 1\}$.

2.1 We denote the zeros of $x^\varphi$ in $[t_2 - 1; t_1] \subset [t_1 - 1; t_1]$ by $z'_j$, $j \in \{1, ..., N_2\}$, such that $z'_1 \ge t_2 - 1$, $z'_{N_2} = t_1$ and $z'_j < z'_{j+1}$ for all $j \in \{1, ..., N_2 - 1\}$. The total number of zeros in $[t_2 - 1; t_2]$ is therefore

$$\begin{aligned} U(t_2, \varphi) &= |(x^\varphi)^{-1}(0) \cap [t_2 - 1; t_1)| + |(x^\varphi)^{-1}(0) \cap [t_1; t_2]| = \\ &= |\{z'_1, ..., z'_{N_2 - 1}\}| + |\{z_1, ..., z_{N_1}\}| = N_2 - 1 + N_1 \ . \end{aligned}$$

Since $s_{N_1-1} \in (z_{N_1-1}-1; z_{N_1}-1) = (z_{N_1-1}-1; t_2-1)$ is a simple zero of $x^\varphi$, we have

$$\{s_1, ..., s_{N_1-1}, z'_1, ..., z'_{N_2}\} \subset (x^\varphi)^{-1}(0) \cap [t_1-1; t_1]$$

such that

$$U(t_1, \varphi) \geq \left|\{s_1, ..., s_{N_1-1}, z'_1, ..., z'_{N_2}\}\right| = N_1 - 1 + N_2 \ ,$$

which gives $U(t_1, \varphi) \geq U(t_2, \varphi)$.

2.2 If $t_1 \in \mathbb{R}^+$ is a multiple zero of $x^\varphi$, then $x^\varphi(t_1) = 0$ but $x^\varphi$ does not change sign at $t_1$ which means that there exists an $\varepsilon > 0$ such that either

$$\mathrm{sign}(x^\varphi(t)) = +1 \quad \text{ for all } t \in (t_1-\varepsilon; t_1+\varepsilon) \setminus \{t_1\}$$

or

$$\mathrm{sign}(x^\varphi(t)) = -1 \quad \text{ for all } t \in (t_1-\varepsilon; t_1+\varepsilon) \setminus \{t_1\} \ .$$

Without loss of generality let us consider only the first case (since the second can be treated in a completely similar fashion). We claim that

$$x^\varphi(t_1-1) = 0 \ .$$

To prove this, we *assume to the contrary* that $x^\varphi(t_1-1) \neq 0$.

2.2.1 If $x^\varphi(t_1-1) > 0$, then there exists an $\eta \in (0;\varepsilon)$ such that $x^\varphi(t) > 0$ for all $t \in [t_1-1; t_1-1+\eta)$ (by continuity of $x^\varphi$). Hence, formula (2.4) gives

$$x^\varphi(t) = \frac{a}{\mu} e^{-\mu(t-t_1)} - \frac{a}{\mu} \quad \text{ for all } t \in [t_1; t_1+\eta)$$

such that $x^\varphi(t) < 0$ for all $t \in (t_1; t_1+\eta)$. In fact, this *contradicts* our assumption $\mathrm{sign}(x^\varphi(t)) = 1$ for all $t \in (t_1; t_1+\varepsilon) \supset (t_1; t_1+\eta)$.

2.2.2 In case of $x^\varphi(t_1-1) < 0$ we we can find an $\eta \in (0; \min\{\varepsilon, 1-t_1\})$ such that $x^\varphi(t) < 0$ for all $t \in (t_1-1-\eta; t_1-1]$. Consequently, formula (2.4) implies that $x^\varphi(t) < 0$ for all $t \in (t_1-\eta; t_1)$ in *contradiction* to our assumption.

2.3 By definition we have

$$z'_k > s_j > z_1 - 1 = t_1 - 1$$

for all $k \in \{1, ..., N_2\}$ and all $j \in \{1, ..., N_1-1\}$. Since $t_1$ is a multiple zero, $t_1 - 1$ is also a zero of $x^\varphi$ (by step 2.2) and we get

$$\{t_1-1, s_1, ..., s_{N_1-1}, z'_1, ..., z'_{N_2}\} \subset (x^\varphi)^{-1}(0) \cap [t_1-1; t_1] \ .$$

Hence, we obtain

$$U(t_1, \varphi) - 1 \geq |\{t_1-1, s_1, ..., s_{N_1-1}, z'_1, ...z'_{N_2}\}| - 1 = N_1 + N_2 - 1 = U(t_2, \varphi) \ .$$

3. Choose any $t_0 \in \mathbb{R}_0^+$; we want to show $U(t,\varphi) \leq U(t_0,\varphi)$ for all $t \in [t_0;+\infty)$. By virtue of (2.10), we have $U(t,\varphi) = U(t_0,\varphi)$ for all $t \in [t_0;\sigma(t_0,\varphi)]$. Thus, instead of considering all $t \in [t_0;+\infty)$, we discuss $t \in (\sigma(t_0,\varphi);\sigma(t_0,\varphi)+1]$ since the assertion follows then by a method of steps (renaming $\sigma(t_0,\varphi)+1$ by $t_0$ and repeating the arguments).

   3.1 For $t \in (\sigma(t_0,\varphi);\sigma(t_0,\varphi)+1]$ we define

$$t_1 := \max\{\tau \in [\sigma(t_0,\varphi),t] \ : \ x^\varphi(\tau) = 0\}$$

   and claim

$$U(t_1,\varphi) \leq U(\sigma(t_0,\varphi),\varphi) \ .$$

   To see this, let us first consider the case that $t_1 = \sigma(t_0,\varphi)$. Then we have

$$U(t_1,\varphi) = U(\sigma(t_0,\varphi),\varphi) \leq U(\sigma(t_0,\varphi),\varphi) \ .$$

   Otherwise we can apply part (2) because of $t_1 \in (\sigma(t_0,\varphi);\sigma(t_0,\varphi)+1]$ and obtain

$$U(t_1,\varphi) \leq U(\sigma(t_0,\varphi),\varphi) \ ,$$

   too.

   3.2 Let $t \in (\sigma(t_0,\varphi);\sigma(t_0,\varphi)+1]$. We distinguish between the following cases.

   3.2.1 In case of $\sigma(t,\varphi) = t_1$ we have

$$U(\sigma(t,\varphi),\varphi) = U(t_1,\varphi) \overset{3.1}{\leq} U(\sigma(t_0,\varphi),\varphi) \ .$$

   3.2.2 If $\sigma(t,\varphi) \in (t_1;t_1+1]$, we can apply (2) again which leads to

$$U(\sigma(t,\varphi),\varphi) \leq U(t_1,\varphi) \overset{3.1}{\leq} U(\sigma(t_0,\varphi),\varphi) \ .$$

   3.2.3 In case of $\sigma(t,\varphi) > t_1+1$ we have $x^\varphi(\tau) \neq 0$ for all $\tau \in (t_1;\sigma(t,\varphi))$ by Definition 2.3.2 and

$$U(\sigma(t,\varphi),\varphi) = 1 \leq U(t_1,\varphi) \overset{3.1}{\leq} U(\sigma(t_0,\varphi),\varphi) \ .$$

   In either case, we have

$$U(t,\varphi) = U(\sigma(t,\varphi),\varphi) \leq U(\sigma(t_0,\varphi),\varphi) = U(t_0,\varphi)$$

   for all $t \in [t_0;\sigma(t_0,\varphi)+1)$ which proves the assertion.

4. To shorten notation, set $t_0 := \tau_n(\varphi)$ for the given $n \in \mathbb{N}$.

   4.1 First, the second part of assertion (2) yields that $t_0 - 1$ is also a zero of $x^\varphi$ since $t_0 \in \mathbb{R}^+$ is assumed to be a multiple zero. This guarantees $U(t_0,\varphi) \geq 2$.

4.2 Now, set $t_1^* := \tau_{n+1}(\varphi)$ such that $t_1^* > t_0$ by definition of the sequence $(\tau_n(\varphi))_{n\in\mathbb{N}}$. Furthermore,

$$t_1^* \leq \sigma(t_0+1,\varphi)$$

because it is $\sigma(t_0+1,\varphi) = \tau_m(\varphi)$ for some $m \in \mathbb{N}$ with $m > n$ such that the monotonicity of $(\tau_k(\varphi))_{k\in\mathbb{N}}$ yields

$$\sigma(t_0+1,\varphi) \geq \tau_{n+1}(\varphi) = t_1^* \ .$$

4.3 If $t_1^* \leq t_0 + 1$, then assertion (2) yields

$$U(t_1^*,\varphi) \leq U(t_0,\varphi) - 1 \ ,$$

and in case $t_1^* > t_0 + 1$ we have $U(t_1^*,\varphi) = 1$ such that part 4.1 of this proof yields

$$U(t_1^*,\varphi) = 2 - 1 \leq U(t_0,\varphi) - 1 \ .$$

Thus, in any case, $U(t_1^*,\varphi) \leq U(t_0,\varphi) - 1$ holds.

4.4 Since $t_1^* \leq \sigma(t_0+1,\varphi)$ holds by 4.2, part (2) of this lemma yields

$$U(t_1^*,\varphi) \geq U(\sigma(t_0+1,\varphi),\varphi) = U(t_0+1,\varphi)$$

such that we have

$$U(t_1^*,\varphi) \geq U(t_0+1,\varphi) \geq U(t,\varphi) \quad \text{ for all } t \in [t_0+1;+\infty)$$

by assertion (3).
Now, together with step 4.3 we obtain

$$U(t,\varphi) \leq U(t_0,\varphi) - 1 \quad \text{ for all } t \in [t_0+1;+\infty)$$

which completes the proof of this assertion.

□

This technical lemma enables us to describe the behaviour of solutions of (2.1) in terms of oscillation numbers of their segments since we can infer from assertions (3) and (4) that the zeros of $x^\varphi$ are eventually simple and decompose $\mathcal{Z}$ identifying those solutions that have the same limit frequency (oscillation number).

**PROPOSITION 2.3.1** *Let $\varphi \in \mathcal{Z}$.*

(1) *There exists a smallest number $n_1 := n_1(\varphi) \in \mathbb{N}$ such that all zeros $\tau_n(\varphi) \in \mathbb{R}^+$, $n \in \mathbb{N} \cap [n_1(\varphi); +\infty)$, are simple.*

(2) *The map $n_0 : \mathcal{Z} \to \mathbb{N}$ given by $n_0(\varphi) := \min M(\varphi)$ for $\varphi \in \mathcal{Z}$ where*

$$M(\varphi) := \Big\{ n \in \mathbb{N} : x^{\varphi}_{\tau_k(\varphi)} \text{ has only simple zeros in } (-1;0) \text{ and } x^{\varphi}_{\tau_k(\varphi)}(-1) \neq 0 \text{ for all } k \in \mathbb{N} \cap [n;\infty) \Big\}$$

*is well-defined. Furthermore, we have $n_0(\varphi) \geq n_1(\varphi)$.*

(3) *Finally, setting*

$$t_k(\varphi) := \tau_{k+n_0(\varphi)-1}(\varphi) \quad \text{for } k \in \mathbb{N} ,$$

*we conclude that $x^{\varphi}_{t_n(\varphi)}(-1) \neq 0$ and $U(t_n(\varphi), \varphi) \in 2\mathbb{N}_0 + 1$ for every $n \in \mathbb{N}$.*

PROOF:

1. Since the mapping $\mathbb{R}^+ \ni t \mapsto U(t, \varphi) \in \mathbb{N}_0$ is monotonic (by LEMMA 2.3.3(3)) and bounded, there exists a $t_0 \in \mathbb{R}^+$ and a $N \in \mathbb{N}_0 \cap [0; U(0, \varphi)]$, such that $U(t, \varphi) = N$ for all $t \in [t_0; +\infty)$. Now, LEMMA 2.3.3(4) shows that there cannot exist multiple zeros in $[t_0 + 1; +\infty)$. Consequently, the set

$$A := \{k \in \mathbb{N} \ : \ \tau_k(\varphi) \geq t_0 + 1\}$$

is non-empty and

$$A \subset \{n \in \mathbb{N} \ : \ \tau_k(\varphi) \text{ is a simple zero for all } k \in \mathbb{N} \cap [n; +\infty)\} \ .$$

Thus,

$$n_1(\varphi) := \min \{n \in \mathbb{N} \ : \ \tau_k(\varphi) \text{ is a simple zero for all } k \in \mathbb{N} \cap [n; +\infty)\}$$

is well-defined and this definition yields that all zeros $\tau_n(\varphi)$ are simple for $n \geq n_1(\varphi)$.

2. By step 1., all zeros $\tau_n(\varphi)$, $n \in \mathbb{N} \cap [n_1(\varphi); +\infty)$ are simple and $U(\tau_n(\varphi), \varphi)$ is constant for all $n \in A$. This together with LEMMA 2.3.3(2) implies that $M(\varphi)$ is a non-empty subset of $\mathbb{N} \cap [n_1(\varphi); +\infty)$. Hence, $n_0 : \mathcal{Z} \to \mathbb{N}$ is well-defined and $n_0(\varphi) \geq n_1(\varphi)$ for all $\varphi \in \mathcal{Z}$.

3. Let $n \in \mathbb{N}$ be given and set $\psi := x^{\varphi}_{t_n(\varphi)}$.

3.1 The definitions of $n_0(\varphi)$ and $(t_n(\varphi))_{n\in\mathbb{N}}$ yield

$$\psi(-1) = x^{\varphi}_{t_n(\varphi)}(-1) \neq 0 = x^{\varphi}_{t_n(\varphi)}(0) = \psi(0) \ .$$

3.2 By 3.1 it suffices to prove that $\psi := x^{\varphi}_{t_n(\varphi)}$ has an even number of zeros in $(-1;0)$.
Denote the finitely many simple zeros of $\psi$ in $(-1;0)$ by $z_j$, $j \in \{1,...,N\}$, chosen such that $z_j > z_{j+1}$ for all $j \in \{1,...,N-1\}$. Furthermore, set $z_0 := 0$, $z_{N+1} = -1$, and $s_j := (\mathrm{sign} \circ \psi)(z_j+)$ for $j \in \{1,...,N+1\}$.
Clearly, we have $s_{j+1} = -s_j$ for all $j \in \{1,...,N\}$ and $s_0 := (\mathrm{sign} \circ \psi)(0-) = s_1$, since $z_1$ is the largest simple zero of $\psi$ in $(-1;0)$ such that $\psi$ does not change sign between $z_1$ and $z_0 = 0$. Therefore, we get

$$s_{N+1} = (-1)^N s_0 \ .$$

If we *assume to the contrary* that $N$ is odd, i.e. $N \in 2\mathbb{N}_0 + 1$, the last equation would give $(\mathrm{sign} \circ \psi)(z_{N+1}+) = -(\mathrm{sign} \circ \psi)(z_0-)$. Assuming without loss of generality $s_0 = (\mathrm{sign} \circ \psi)(z_0-) = +1$ we could find an $\varepsilon > 0$ such that $\psi(t) < 0$ for all $t \in (-1; -1+\varepsilon)$ (by continuity of $\psi$). But this would give $x^{\psi}(t) = -\frac{a}{\mu}e^{-\mu t} + \frac{a}{\mu} > 0$ for $t \in (0;\varepsilon)$ by (2.4) *contradicting* the simplicity of $t_n(\varphi)$ since we would have

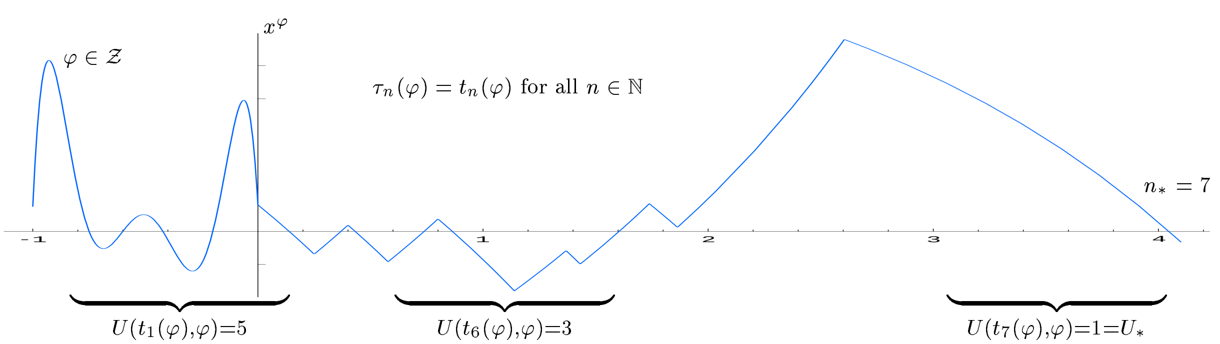
$$\begin{aligned}(\mathrm{sign} \circ x^{\varphi})(t_n(\varphi)+) &= (\mathrm{sign} \circ x^{\psi})(0+) = \\ &= 1 \neq -1 = -s_0 = \\ &= -(\mathrm{sign} \circ x^{\psi})(0-) = -(\mathrm{sign} \circ x^{\varphi})(t_n(\varphi)-) \ .\end{aligned}$$

Consequently, $N \in 2\mathbb{N}_0$ for every $\varphi \in \mathcal{Z}$.

Thus, $U(t_n(\varphi), \varphi) = N + 1 \in 2\mathbb{N}_0 + 1$.

□

Notice that $U(t_n(\varphi), \varphi)$, $\varphi \in \mathcal{Z}$ fixed, is *not* necessarily constant for *all* $n \in \mathbb{N}$. Clearly, there is a $n_* := n_*(\varphi) \in \mathbb{N}$ such that $U(t_n(\varphi), \varphi) = U_* \in 2\mathbb{N}_0 + 1$ for $n \in \mathbb{N} \cap [n_*; +\infty)$, but we will have $n_* > 1$ in general (as illustrated in the figure below).



We can interpret the last result as follows: every trajectory

$$\gamma_\varphi : \mathbb{R}_0^+ \ni t \mapsto F_{-a\,\mathrm{sign}}(t,\varphi) \in X$$

which started at a point $\varphi = F_{-a\,\mathrm{sign}}(0,\varphi)$ in $\mathcal{Z}$ enters the set

$$\mathcal{Z}_1 := \{\varphi \in \mathcal{Z} \,:\, \varphi(0) = 0 \neq \varphi(-1),\, U(0,\varphi) \in 2\mathbb{N}_0 + 1,\, \varphi \text{ has only simple zeros in } (-1;0)\}$$

after some finite time $t_1(\varphi) = \tau_{n_0}(\varphi)$ and from this time on all segments $x^\varphi_{t_k(\varphi)}$, $k \in \mathbb{N}$, belong to this set.

Observe that the initial values $x_0^{(2m)}$, $m \in \mathbb{N}_0$, of the periodic solutions of (2.1) defined in the previous section are contained in the set $\mathcal{Z}_1$. Also, all segments $x^\varphi_{t_n(\varphi)}$, $n \in \mathbb{N}$, from the example in the figure on the previous page evidently belong to $\mathcal{Z}_1$.

All in all, the asymptotic behaviour of a solution $x^\varphi$ evolving from $\varphi \in \mathcal{Z}$ will be described by the behaviour of a solution $x^\chi$ starting in $\chi := x^\varphi_{t_1(\varphi)} \in \mathcal{Z}_1$ whose segments $x^\chi_{\tau_n(\chi)}$ belong to $\mathcal{Z}_1$ for all $n \in \mathbb{N}$. This motivates a deeper study of these solutions.

## C. Behaviour of solutions starting in $\mathcal{Z}_0$

In view of the results just mentioned, it is sufficient to consider only those initial values $\chi$ in $\mathcal{Z}_1$ whose segments $x^\chi_{\tau_n(\chi)}$ belong to $\mathcal{Z}_1$ for all $n \in \mathbb{N}$. Therefore, we assume that

$$\chi := x^\varphi_{t_1(\varphi)} \in \mathcal{Z}_1 \quad \text{and} \quad x^\chi_{\tau_n(\chi)} \in \mathcal{Z}_1 \text{ for all } n \in \mathbb{N}$$

throughout the whole subsection, i.e. we consider only initial values from

$$\mathcal{Z}_0 := \left\{\varphi \in \mathcal{Z}_1 \;:\; x^\varphi_{\tau_n(\varphi)} \in \mathcal{Z}_1 \text{ for all } n \in \mathbb{N}\right\} .$$

In particular, for every $\chi \in \mathcal{Z}_0$ we have $\tau_n(\chi) = t_n(\chi)$ for all $n \in \mathbb{N}$. We will use this fact several times (without further mentioning).

In order to apply formula (2.4) to compute the solution $x^\chi$, $\chi \in \mathcal{Z}_0$, we need some more notation which follows [16, pp. 433ff.].

**DEFINITION 2.3.3** For $\varphi \in \mathcal{Z}_0$ set

$$N(\varphi) := U(0,\varphi) - 1 \in 2\mathbb{N}_0 \quad \text{and} \quad z_0(\varphi) := 0 .$$

If $N(\varphi) \in 2\mathbb{N}$, denote the simple zeros of $\varphi$ in $(-1;0)$ by $z_j(\varphi)$, $j \in \{1,...,N(\varphi)\}$, such that $z_j > z_{j+1}$ holds for all $j \in \{1,...,N(\varphi)-1\}$. Furthermore, we define

$$s_n(\varphi) := (\mathrm{sign} \circ \varphi)(z_n -) \quad \text{for all } n \in \{0,...,N(\varphi)\} .$$

Since all zeros of $\varphi \in \mathcal{Z}_0$ in $(-1;0)$ are simple, it is clear that

$$s_n(\varphi) = (-1)^n s_0(\varphi) \quad \text{for all } n \in \{0, ..., N(\varphi)\} \ , \tag{2.11}$$

and, thus, we notice $\operatorname{sign}(\varphi(-1)) = s_{N(\varphi)}(\varphi) = s_0(\varphi)$.

**DEFINITION 2.3.4** Let $\varphi \in \mathcal{Z}_0$ be given. In case $N(\varphi) = 0$, set $v_1(\varphi) := 1$, otherwise, if $N(\varphi) \in 2\mathbb{N}$, set

$$\begin{aligned} v_n(\varphi) &:= z_{n-1}(\varphi) - z_n(\varphi) \quad \text{for all } n \in \{1, ..., N(\varphi)\} \ , \\ v_{N(\varphi)+1}(\varphi) &:= 1 - \sum_{n=1}^{N(\varphi)} v_n(\varphi) \ . \end{aligned}$$

To illustrate the definitions above we consider the initial states of the periodic solutions $x^{(2m)}$, $m \in \mathbb{N}_0$, as examples. Clearly, we have $N(x_0^{(2m)}) = 2m$ for all $m \in \mathbb{N}_0$, and the zeros of $x_0^{(2m)}$ in $(-1;0)$ for $m \in \mathbb{N}$ are given by

$$z_l(x_0^{(2m)}) := -l \cdot \left( \tau_* + \frac{1}{\mu} \log(2 - e^{-\mu \tau_*}) \right), \quad l \in \{1, ..., 2m\} \, ,$$

where $\tau_* := \tau_*(\mu, m)$ denotes the unique solution of (2.9). Hence, the distances are given by

$$v_l(x_0^{(2m)}) = \tau_* + \frac{1}{\mu} \log(2 - e^{-\mu \tau_*}) = \frac{1 - \tau_*}{2m}, \quad l \in \{1, ..., 2m\} \ ;$$

see also REMARK 2.2.4.

With this preliminaries, the next result is an immediate consequence of (2.4) (successively applied to the intervals $(\alpha; \beta) := (z_{k+1}(\varphi) + 1; z_k(\varphi) + 1)$ for $k \in \{0, ..., N(\varphi)\}$).

**PROPOSITION 2.3.2** *For $\varphi \in \mathcal{Z}_0$, set $w_{-1}(\varphi) := \varphi(0) = 0$ and, recursively,*

$$w_\ell(\varphi) := \left( w_{\ell-1}(\varphi) + s_{N(\varphi)-\ell}(\varphi) \tfrac{a}{\mu} \right) e^{-\mu v_{N(\varphi)+1-\ell}(\varphi)} - s_{N(\varphi)-\ell}(\varphi) \tfrac{a}{\mu} \quad \textit{for } \ell \in \{0, ..., N(\varphi)\} \ .$$

*Then the local extrema of the corresponding solution $x^\varphi$ in the interval $[0;1]$ are given by*

$$x^\varphi \left( \sum_{k=0}^{\ell} v_{N(\varphi)+1-k}(\varphi) \right) = x^\varphi (z_{N(\varphi)-\ell}(\varphi) + 1) = w_\ell(\varphi)$$

*for $\ell \in \{0, ..., N(\varphi)\}$. More precisely, for $t \in J_\ell := \left[ z_{N(\varphi)-\ell+1}(\varphi) + 1; z_{N(\varphi)-\ell}(\varphi) + 1 \right]$, $\ell \in \{0, ..., N(\varphi)\}$, we have*

$$x^\varphi(t) = \left( w_{\ell-1}(\varphi) + s_{N(\varphi)-\ell}(\varphi) \tfrac{a}{\mu} \right) e^{-\mu (t - z_{N(\varphi)-\ell+1} - 1)} - s_{N(\varphi)-\ell}(\varphi) \tfrac{a}{\mu} \ .$$

PROOF: Let $\varphi \in \mathcal{Z}_0$ be given.

1. Recall from the definitions above that

$$\begin{aligned}\sum_{k=0}^{\ell} v_{N(\varphi)+1-k}(\varphi) &= v_{N(\varphi)+1}(\varphi) + \sum_{k=1}^{\ell} v_{N(\varphi)+1-k}(\varphi) = 1 - \sum_{k=1}^{N(\varphi)-\ell} v_k(\varphi) = \\ &= 1 - \sum_{k=1}^{N(\varphi)-\ell} (z_{k-1}(\varphi) - z_k(\varphi)) = 1 - (z_0(\varphi) - z_{N(\varphi)-\ell}(\varphi))\end{aligned}$$

   for $\ell \in \{0, ..., N(\varphi)\}$ such that

$$z_{N(\varphi)-\ell}(\varphi) + 1 = \sum_{k=0}^{\ell} v_{N(\varphi)+1-k}(\varphi) \qquad \text{for } \ell \in \{0, ..., N(\varphi)\} \ .$$

2. Now, we can apply (2.4) to the intervals $(\alpha; \beta) := (z_{N(\varphi)-\ell+1}(\varphi) + 1; z_{N(\varphi)-\ell}(\varphi) + 1)$ for $\ell \in \{0, ..., N(\varphi)\}$ and obtain the desired results because of

$$\operatorname{sign}(\varphi(t)) = s_{N(\varphi)-\ell} =: s_\alpha \qquad \text{for all } t \in (z_{N(\varphi)-\ell+1}(\varphi); z_{N(\varphi)-\ell}(\varphi)) \ .$$

$\square$

It should be useful to find an explicit expression for the values $w_\ell(\varphi)$, $\ell \in \{0, ..., N(\varphi)\}$, instead of the above recursion formula. The elementary computation is rather lengthy and yields

$$w_\ell(\varphi) = \frac{s_0(\varphi)a}{\mu} \left( \sum_{k=0}^{\ell} \Big[ (-1)^{N(\varphi)-\ell+k} \exp\Big( (-\mu) \cdot \sum_{m=0}^{k} v_{N(\varphi)+1-\ell+m}(\varphi) \Big) \Big] - 1 \right) \qquad (2.12)$$

for $\ell \in \{0, ..., N(\varphi)\}$ as we mention here without proof.

Recall that $\varphi \in \mathcal{Z}_0$ does not only exclude any multiple zeros on $\mathbb{R}^+$. For these initial values we obtain $t_n(\varphi) = \tau_n(\varphi)$ for all $n \in \mathbb{N}$. In particular, $t_1(\varphi)$ is the first zero of $x^\varphi$ in $\mathbb{R}^+$ and is necessarily simple.

**COROLLARY 2.3.5** *For every* $\varphi \in \mathcal{Z}_0$ *we have* $\operatorname{sign}(w_0(\varphi)) = -s_{N(\varphi)}(\varphi) = -s_0(\varphi) \neq 0$. *In case that*

$$\operatorname{sign}(w_n(\varphi)) = \operatorname{sign}(w_0(\varphi)) \qquad \textit{for all } n \in \{0, ..., N(\varphi)\}, \qquad (2.13)$$

$x^\varphi$ *does not change sign on* $(0; 1]$, *which implies* $t_1(\varphi) \in [1; +\infty)$ *and*

$$\operatorname{sign}(\dot{x}^\varphi(t)) = s_{N(\varphi)}(\varphi) = s_0(\varphi) \qquad \textit{for all } t \in (1; 2) \ .$$

*Hence* $\operatorname{sign}(x^\varphi(t)) = -s_{N(\varphi)}(\varphi)$ *for all* $t \in (0; 1]$.

PROOF: By definition of $w_0(\varphi)$ in PROPOSITION 2.3.2 (or by (2.12)), the first assertion follows at once from

$$\begin{aligned} \operatorname{sign}(w_0(\varphi)) &= \operatorname{sign}\left(\frac{s_{N(\varphi)}(\varphi)a}{\mu}\left[e^{-\mu v_{N(\varphi)+1}(\varphi)} - 1\right]\right) = \\ &= s_{N(\varphi)}(\varphi) \cdot \operatorname{sign}(a) \cdot \operatorname{sign}\left(\frac{e^{-\mu v_{N(\varphi)+1}(\varphi)} - 1}{\mu}\right) = -s_{N(\varphi)}(\varphi) \neq 0 \end{aligned}$$

since $\mu \in \mathbb{R}^-$, $a \in \mathbb{R}^+$, and $v_{N(\varphi)+1}(\varphi) \in \mathbb{R}^+$. All other assertions follow easily from (2.4) or (2.1).

$\square$

**DEFINITION 2.3.5** We define a map $j : \mathcal{Z}_0 \to \mathbb{N}$ as follows: for $\varphi \in \mathcal{Z}_0$ set

$$j(\varphi) := \begin{cases} N(\varphi) + 1 & \text{, if (2.13) holds,} \\ \min\{k \in \{1, ..., N(\varphi)\} \ : \ \operatorname{sign}(w_k(\varphi)) = -\operatorname{sign}(w_0(\varphi))\} & \text{, otherwise.} \end{cases}$$

**REMARK 2.3.1** *The range of $j : \mathcal{Z}_0 \to \mathbb{N}$ is the set of odd non-negative integers, i.e. $j(\mathcal{Z}_0) = 2\mathbb{N}_0 + 1$.*

PROOF: Let $\varphi \in \mathcal{Z}_0$ and set $N := N(\varphi)$, $w_k := w_k(\varphi)$, $z_k := z_k(\varphi)$, and $s_k := s_k(\varphi)$ for $k \in \{0, ..., N\}$.

1. Obviously, the assertion is true if (2.13) is satisfied (since $N(\varphi) \in 2\mathbb{N}_0$). So, we only have to deal with the case that (2.13) does not hold.

2. We *assume* $j := j(\varphi) \in 2\mathbb{N}$ for $\varphi \in \mathcal{Z}_0$. Recall that $x^\varphi$ has only simple zeros in $\mathbb{R}^+$.

   2.1 If $\operatorname{sign}(w_j) = -1$, then $\operatorname{sign}(w_0) = \operatorname{sign}(w_k) = +1$ for all $k \in \{0, ...., j-1\}$ by definition of $j$. Note that the first part of COROLLARY 2.3.5 yields

   $$s_N = -\operatorname{sign}(w_0) = -1 \ .$$

   On the other hand we infer from PROPOSITION 2.3.2

   $$\begin{aligned} w_j &= \left(w_{j-1} + s_{N-j}\frac{a}{\mu}\right) e^{-\mu v_{N+1-j}} - s_{N-j}\frac{a}{\mu} = \\ &= w_{j-1}e^{-\mu v_{N+1-j}} + s_{N-j}\frac{a}{\mu}\left(e^{-\mu v_{N+1-j}} - 1\right) \end{aligned}$$

   such that $w_j < 0$, $w_{j-1} > 0$, and $\frac{a}{\mu}\left(e^{-\mu v_{N+1-j}} - 1\right) < 0$ imply $s_{N-j} = +1$. Since $j \in 2\mathbb{N}$ (by assumption), $N \in 2\mathbb{N}_0$, and $s_{N-j} = (-1)^{N-j}s_0$ this gives $s_0 = +1$ and, thus, the *contradiction*

   $$-1 = s_N = (-1)^N s_0 = +1$$

   by means of (2.11)

2.2 If $\operatorname{sign}(w_j) = +1$, one can argue in a completely analogous fashion as in 2.1 to derive a *contradiction*, too.

Therefore, $j = j(\varphi) \in 2\mathbb{N}_0 + 1$.

$\square$

**COROLLARY 2.3.6** *If $\varphi \in \mathcal{Z}_0$ is such that* (2.13) *does not hold, then the first simple zero of $x^\varphi$ in $(0;1]$ is given by*

$$\begin{aligned} t_1(\varphi) &= \sum_{k=0}^{j(\varphi)-1} v_{N(\varphi)+1-k}(\varphi) + \frac{1}{\mu}\log\left(1 - |\frac{\mu}{a} w_{j(\varphi)-1}(\varphi)|\right) = \\ &= z_{N(\varphi)-j(\varphi)+1} + 1 + \frac{1}{\mu}\log\left(1 - |\frac{\mu}{a} w_{j(\varphi)-1}(\varphi)|\right) \end{aligned}$$

*and satisfies*

$$t_1(\varphi) \in J_{j(\varphi)} := \left(\sum_{k=0}^{j(\varphi)-1} v_{N(\varphi)+1-k}(\varphi); \sum_{k=0}^{j(\varphi)} v_{N(\varphi)+1-k}(\varphi)\right) = \left(z_{N(\varphi)-j(\varphi)+1}+1; z_{N(\varphi)-j(\varphi)}+1\right).$$

PROOF: As in the previous proof, we use the abbreviations $j := j(\varphi)$, $N := N(\varphi)$, $s_k := s_k(\varphi)$, and $w_k := w_k(\varphi)$ for given $\varphi \in \mathcal{Z}_0$ to simplify the notation.

1. The first zero of $x^\varphi$ in $\mathbb{R}^+$ is necessarily simple with $x^\varphi_{t_1(\varphi)} \in \mathcal{Z}_1$ by choice of $\mathcal{Z}_0$. In particular, this proves $t_1(\varphi) \neq 1$.

2. Without loss of generality we assume $w_j = x^\varphi(z_{N-j}+1) < 0$, i.e. $\operatorname{sign}(w_j) = -1$. By definition of $j$ we have $\operatorname{sign}(w_n) = \operatorname{sign}(w_0) = +1$ for all $n \in \{1, ..., j-1\}$. Consequently, PROPOSITION 2.3.2 implies that $x^\varphi(t) > 0$ for all $t \in (0; z_{N+1-j}+1)$. Therefore, a change of sign and, thus, the first simple zero of $x^\varphi$ in $(0;1]$ occurs in the interval $J_j$ by continuity of $x^\varphi$. Due to

$$z_{N+1-j} + 1 = \sum_{k=0}^{j-1} v_{N+1-k}$$

PROPOSITION 2.3.2 implies

$$x^\varphi(t) = \left(w_{j-1} + s_{N-j}\tfrac{a}{\mu}\right)\exp\left(-\mu(t - \sum_{k=0}^{j-1} v_{N+1-k})\right) - s_{N-j}\tfrac{a}{\mu}$$

for all $t \in J_j$ such that solving the equation $x^\varphi(t_1) = 0$ gives the unique simple zero $t_1 = t_1(\varphi) \in J_j$. First, we obtain

$$\exp\left(-\mu(t_1 - \sum_{k=0}^{j-1} v_{N+1-k})\right) = \frac{s_{N-j}\frac{a}{\mu}}{w_{j-1} + s_{N-j}\frac{a}{\mu}} = \left(1 + \frac{\mu w_{j+1}}{s_{N-j}a}\right)^{-1}.$$

Since $F_{-a\,\mathrm{sign}}(\mathbb{R}^+\times\mathcal{Z})\subset\mathcal{Z}$, we have $w_k\in\left(\frac{a}{\mu};-\frac{a}{\mu}\right)$ and, consequently, $\frac{w_k\mu}{s_l a}\in(-1;+1)$ for all $k\in\{0,...,N\}$ and all $l\in\{0,...,N\}$. Furthermore, we have $\mathrm{sign}(w_{j-1})=-\mathrm{sign}(w_j)=\mathrm{sign}(w_0)=-s_N$ by definition of $j=j(\varphi)$, such that

$$\mathrm{sign}\left(\frac{w_{j-1}\mu}{s_{N-j}a}\right)=\frac{\mathrm{sign}(w_{j-1})\cdot\mathrm{sign}(\mu)}{\mathrm{sign}(s_{N-j})\cdot\mathrm{sign}(a)}=\frac{(-s_N)\cdot(-1)}{(-1)^{N-j}s_0\cdot 1}=(-1)^j=-1\ ,$$

since $j\in 2\mathbb{N}_0+1$ by Remark 2.3.1. Thus, $\frac{\mu w_{j-1}}{s_{N-j}a}=-|\frac{\mu}{a}w_{j-1}|$ and

$$-\mu(t_1-\sum_{k=0}^{j-1}v_{N+1-k})=\log\left(1+\frac{\mu w_{j+1}}{s_{N-j}a}\right)^{-1}=-\log\left(1-|\frac{\mu}{a}w_{j-1}|\right)\ .$$

yields the desired expression for $t_1:=t_1(\varphi)\in J_j$.

$\square$

Although there are infinitely many simple zeros of $x^\varphi$ in $\mathbb{R}^+$ (cf. Proposition 2.3.1), and in view of the "standard approach" to the investigation of oscillating solutions via return maps, it is tempting to ask *whether $t_1(\varphi)$ depends continuously on $\varphi\in\mathcal{Z}_0$ ?*

In this context our approach via a Lyapunov functional that led to the "asymptotic phase space" $\mathcal{Z}_0$ in $\mathcal{Z}$ gets a further justification since the next remark will guarantee the continuity of the return map on $\mathcal{Z}_0$.

**Remark 2.3.2** *The map $\mathcal{Z}_0\ni\varphi\mapsto t_1(\varphi)\in\mathbb{R}^+$ is continuous.*

Proof:

1. We prove an auxiliary (and additional) statement first, namely, that $t_1$ is continuous on $\mathcal{Z}_*:=\{\psi\in\mathcal{Z}\,:\,\psi(0)\neq 0,\ x^\psi_{\tau_n(\psi)}\in\mathcal{Z}_1 \text{ for all } n\in\mathbb{N}\}$.
   To see this, let $\varphi\in\mathcal{Z}_*$ and $\varepsilon\in(0;\min\{t_1(\varphi),t_2(\varphi)-t_1(\varphi)\})$ be given and set $t_k:=t_k(\varphi)$ for $k\in\mathbb{N}$ as in Proposition 2.3.1.

   1.1 For all $\delta>0$ and for every $\chi\in U_\delta(\varphi)\cap\mathcal{Z}_*$ there must exist a first zero $t_1(\chi)\in\mathbb{R}^+$ of $x^\chi$ which satisfies

   $$x^\chi(t_1(\chi))=0\neq(\mathrm{sign}\circ x^\chi)(t_1(\chi)-)=-(\mathrm{sign}\circ x^\chi)(t_1(\chi)+)$$

   due to Proposition 2.3.1. Without loss of generality we assume

   $$(\mathrm{sign}\circ x^\varphi)(t_1-)=+1$$

   and choose $\tau_0\in(\varepsilon;t_2-t_1)$ such that $\mathrm{sign}(x^\varphi(t))=-1$ for all $t\in(t_1;t_1+\tau_0)$.

1.2 LEMMA 2.1.1 yields for

$$\bar{t} := t_1(\varphi) + \tau_0$$

and

$$\eta = \min\left\{\frac{1}{2}\min\{x^\varphi(\tau) \,:\, \tau \in [0; t_1(\varphi) - \varepsilon]\}, -\frac{1}{2}x^\varphi(t_1(\varphi) + \varepsilon)\right\} > 0$$

the existence of a $\delta(\varphi, \varepsilon) > 0$ such that

$$|x^\psi(t) - x^\varphi(t)| < \eta \quad \text{for all } t \in [0; \bar{t}\,]$$

and all $\psi \in U_\delta(\varphi) \supset U_\delta(\varphi) \cap \mathcal{Z}_*$.

1.2.1 Since $x^\varphi(t) \geq 2\eta$ for all $t \in [0; t_1 - \varepsilon]$ (by choice of $\eta$), we obtain

$$x^\psi(t) \geq \eta > 0 \quad \text{for all } t \in [0; t_1 - \varepsilon] \ .$$

Hence, it is

$$t_1(\psi) > t_1 - \varepsilon \quad \text{for all } \psi \in U_\delta(\varphi) \cap \mathcal{Z}_* \ .$$

1.2.2 By assumption $x^\varphi(t_1 + \varepsilon) < 0$, such that we obtain

$$x^\psi(t_1 + \varepsilon) < \frac{1}{2}x^\varphi(t_1 + \varepsilon) < 0 \ .$$

Therefore, we conclude

$$t_1(\psi) < t_1 + \varepsilon \quad \text{for all } \psi \in U_\delta(\varphi) \cap \mathcal{Z}_*$$

because of the continuity of $x^\psi$ on $[-1; +\infty)$.

This proves the assertion.

2. Now, let $\varphi \in \mathcal{Z}_0$.

2.1 By definition of $\mathcal{Z}_0$ we have $\operatorname{sign}(\varphi(-1)) = s_0 \neq 0$, so we may assume $\varphi(-1) < 0$ without loss of generality. Furthermore, let any $\varepsilon \in (0; -\varphi(-1))$ be given. We denote the first zero of $\varphi + \varepsilon\mathbb{I}$ by

$$z_\varepsilon := \inf\left((\varphi + \varepsilon\mathbb{I})^{-1}(0)\right) \in (-1; 0] \ .$$

Clearly, we have $\psi(t) < 0$ for $t \in [-1; z_\varepsilon)$ for all $\psi \in U_\varepsilon(\varphi) \supset U_\varepsilon(\varphi) \cap \mathcal{Z}_0$. Finally, let $z'_\varepsilon$ denote the first zero of $x^\varphi - \varepsilon\mathbb{I}_{[-1;+\infty)}$ in $\mathbb{R}^+$,

$$z'_\varepsilon := \inf\left(((x^\varphi - \varepsilon\mathbb{I}_{[-1;+\infty)})\Big|_{\mathbb{R}^+})^{-1}(0)\right) \ ,$$

where $\mathbb{I}_{[-1;+\infty)} : [-1; +\infty) \ni t \mapsto 1 \in \mathbb{R}$, and set $t_\varepsilon := \max\{z_\varepsilon + 1, z'_\varepsilon\} \in \mathbb{R}^+$.

2.2 As a consequence of the continuous dependence on the initial value $\varphi$ (LEMMA 2.1.1) we can find a $\widehat{\delta} \in (0;\varepsilon)$ such that

$$|x^{\varphi}(t) - x^{\psi}(t)| < \varepsilon \qquad \text{for } t \in [0;t_\varepsilon]$$

holds for all $\psi \in U_{\widehat{\delta}}(\varphi) \subset U_\varepsilon(\varphi)$. Step 2.1 guarantees

$$\dot{x}^{\psi}(t) > 0 \quad \text{and} \quad x^{\psi}(t) > 0 \qquad \text{for } t \in (0;t_\varepsilon]$$

for all $\psi \in U_{\widehat{\delta}}(\varphi) \cap \mathcal{Z}_0$ which implies $x^{\psi}_{t_\varepsilon} \in \mathcal{Z}_*$ for all $\psi \in U_{\widehat{\delta}}(\varphi) \cap \mathcal{Z}_0$.
Now, fix $t_\varepsilon > 0$ and observe that $\widetilde{\varphi} := x^{\varphi}_{t_\varepsilon}$ is in $\mathcal{Z}_*$. As we know from step 1., $t_1$ is continuous on $\mathcal{Z}_*$, i.e. we can find a $\widetilde{\delta} \in (0;\widehat{\delta})$ such that

$$|t_1(\widetilde{\psi}) - t_1(\widetilde{\varphi})| < \varepsilon$$

for all $\widetilde{\psi} \in U_{\widetilde{\delta}}(\widetilde{\varphi}) \cap \mathcal{Z}_*$.
Furthermore, by continuous dependence on the initial value $\varphi \in \mathcal{Z}_0$ there exists a $\delta \in (0;\widetilde{\delta})$ for given $t_\varepsilon > 0$, $\widetilde{\delta} \in (0;\widehat{\delta})$ and $\varphi \in \mathcal{Z}_0$ with

$$x^{\psi}_{t_\varepsilon} \in U_{\widetilde{\delta}}(\widetilde{\varphi}) = U_{\widetilde{\delta}}(x^{\varphi}_{t_\varepsilon}) \qquad \text{for all } \psi \in U_\delta(\varphi) \cap \mathcal{Z}_0 \ .$$

Using

$$t_1(\chi) = t_\varepsilon + t_1(x^{\chi}_{t_\varepsilon}) \qquad \text{for all } \chi \in U_\delta(\varphi) \cap \mathcal{Z}_0$$

(which follows from $|x^{\chi}| > 0$ on $(0;t_\varepsilon]$ and by definition of $t_1(x^{\chi}_{t_\varepsilon})$) we have found a $\delta > 0$ such that

$$\begin{aligned} |t_1(\psi) - t_1(\varphi)| &= |(t_\varepsilon + t_1(x^{\psi}_{t_\varepsilon})) - (t_\varepsilon + t_1(x^{\varphi}_{t_\varepsilon}))| = \\ &= |t_1(x^{\psi}_{t_\varepsilon}) - t_1(x^{\psi}_{t_\varepsilon})| < \varepsilon \end{aligned}$$

for all $\psi \in U_\delta(\varphi) \cap \mathcal{Z}_0$, such that $t_1$ is continuous on $\mathcal{Z}_0$.

$\square$

Finally, these preparations give us the possibility to describe $\psi := x^{\varphi}_{t_1(\varphi)}$, $\varphi \in \mathcal{Z}_0$, in detail.

**COROLLARY 2.3.7** *Let $\varphi \in \mathcal{Z}_0$ be given. Then the segment $\psi := x^{\varphi}_{t_1(\varphi)}$ of the solution $x^{\varphi}$ has the following properties:*

(1) *If $j(\varphi) = N(\varphi) + 1$, then $N(\psi) = 0$.*

(2) *If $j(\varphi) \in \{1, ..., N(\varphi) - 1\} \cap (2\mathbb{N}_0 + 1)$, then $N(\psi) = N(\varphi) - j(\varphi) + 1$, such that $N(\psi) \in 2\mathbb{N}_0$, and it is*

$$v_1(\psi) = t_1(\varphi), \quad \text{and} \quad v_k(\psi) = v_{k-1}(\varphi) \ \text{ for } k \in \{2, ..., N(\varphi) - j(\varphi) + 1\} \ .$$

PROOF: To prove the assertions, set $N := N(\varphi)$, $j := j(\varphi)$, $s_k := s_k(\varphi)$, $v_k := v_k(\varphi)$, $w_k := w_k(\varphi)$, and $t_1 := t_1(\varphi)$.

1. Assertion (1) is trivial by virtue of COROLLARY 2.3.5.

2. First, recall that $\psi := x_{t_1}^{\varphi} \in \mathcal{Z}_1$ such that all zeros of $\psi$ are simple. Now, COROLLARY 2.3.6 yields $t_1 - 1 \in (z_{N-j+1}; z_{N-j})$. Hence, the zeros of $x^{\varphi}$ in $(t_1 - 1; t_1)$ are given by $\{z_{N-j}, z_{N-j-1}, ..., z_1, z_0\}$ since $|x^{\varphi}(t)| > 0$ for all $t \in (0; t_1)$. Thus, we have

$$N(\psi) = |\{z_{N-j}, z_{N-j-1}, ..., z_1, z_0\}| = N(\varphi) - j(\varphi) + 1$$

as well as $z_0(\psi) = 0$ and

$$z_k(\psi) = z_{k-1}(\varphi) - t_1(\varphi) \quad \text{for } k \in \{1, ..., N(\psi)\} = \{1, ..., N(\varphi) - j(\varphi) + 1\}$$

which yields the third assertion by virtue of $v_k(\psi) := z_{k-1}(\psi) - z_k(\psi)$, $k \in \{1, ..., N(\psi)\}$. Recalling $j(\varphi) \in 2\mathbb{N}_0 + 1$ from REMARK 2.3.1 shows $N(\psi) \in 2\mathbb{N}_0$.

□

The central aspect of COROLLARY 2.3.7 that we want to focus on is that this corollary enables us to define a return map

$$R : \mathcal{Z}_0 \ni \varphi \mapsto F_{-a\,\mathrm{sign}}(t_1(\varphi), \varphi) \in \mathcal{Z}_0 \ .$$

The goal is therefore to study the dynamics of (2.1) on $\mathcal{Z}_0$ in terms of this POINCARÉ map which is continuous by REMARK 2.3.2 and the continuity of the semiflow. For convenience, we note the most obvious elementary property of the return map $R$.

**REMARK 2.3.3** *We have*

$$R(x_0^{(2m)}) = -x_0^{(2m)} \quad \textit{for all } m \in \mathbb{N}_0 \ ,$$

*such that the initial values $x_0^{(2m)} \in \mathcal{Z}_0$, $m \in \mathbb{N}_0$, define fixed points of $R^2 = R \circ R$.*

The investigation of the return map $R$ can be done in the following way: we associate with $R$ a transformation on the vectors

$$v(\varphi) = (v_1(\varphi), ..., v_{N(\varphi)}(\varphi))$$

which are uniquely determined by $\varphi \in \mathcal{Z}_0$ since these vectors together with $s_0(\varphi)$ contain all the information needed about $\varphi$ to determine $x_{t_1(\varphi)}^{\varphi} = R(\varphi)$ completely.

## D. A conjugated discrete dynamical system

Using the abbreviation $\mathbb{R}_+^N := (\mathbb{R}^+)^N$ let us define

$$\Omega_0 := \{1\} \times \{-1, +1\} ,$$

and

$$\Omega_N := \left\{ v \in \mathbb{R}_+^N \ : \ \sum_{n=1}^{N} v_n < 1 \right\} \times \{-1, +1\} \quad \text{for } N \in 2\mathbb{N} .$$

Moreover, set

$$v_{N+1}(v, \sigma) := 1 - \sum_{n=1}^{N} v_n =: 1 - \mathbf{1}_N \cdot v$$

with $\mathbf{1}_N := (1, ..., 1) \in \mathbb{R}^N$, as wells as (in view of (2.12))

$$w_\ell(v, \sigma) := \frac{\sigma a}{\mu} \left( \sum_{k=0}^{\ell} \left[ (-1)^{N-\ell+k} \exp\Big( (-\mu) \cdot \sum_{m=0}^{k} v_{N+1-\ell+m} \Big) \right] - 1 \right), \ \ \ell \in \{0, ..., N\},$$

for $(v, \sigma) \in \Omega_N$, $N \in 2\mathbb{N}$. Observe, that $\operatorname{sign}(w_0(v, \sigma)) = -\sigma$ for all $(v, \sigma) \in \Omega_N$.

Finally, let $\Omega$ be the disjoint union of the topological spaces $\Omega_N$, $N \in 2\mathbb{N}_0$,

$$\Omega := \biguplus_{N \in 2\mathbb{N}_0} \Omega_N .$$

The mapping which "extracts" the necessary information from any given $\varphi \in \mathcal{Z}_0$ (namely, the distribution of simple zeros in $(-1; 0)$ and $s_0(\varphi)$), and transforms it into an element of $\Omega$ can now easily be defined in view of Definition 2.3.3, Definition 2.3.4 and Corollary 2.3.5.

**Definition 2.3.6** For $N \in 2\mathbb{N}$ let

$$V : \mathcal{Z}_0 \ni \varphi \mapsto \left\{ \begin{array}{ll} \Big( (v_1(\varphi), ..., v_N(\varphi)), s_0(\varphi) \Big) & , \text{ if } N(\varphi) \in 2\mathbb{N} \\ (1, s_0(\varphi)) & , \text{ if } N(\varphi) = 0 \end{array} \right\} \in \Omega .$$

Clearly, the "coordinate map" $V$ is surjective, since for any given $\omega \in \Omega$ we can choose at once a $\varphi \in \mathcal{Z}_0$ with the given sign distribution on $[-1; 0]$, e.g., an appropriately chosen interpolation polynomial.

While the evolution of $\varphi \in \mathcal{Z}_0$ under $R$ is described in Corollary 2.3.7 in the original setting, we have to find a corresponding self map of $\Omega$ which reflects the same dynamics, i.e. we want to find a map $f : \Omega \to \Omega$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{Z}_0 & \xrightarrow{R} & \mathcal{Z}_0 \\ V \downarrow & & \downarrow V \\ \Omega & \xrightarrow{f} & \Omega \end{array} \tag{2.14}$$

**Lemma 2.3.4** *Define $f : \Omega \to \Omega$ as follows*

(1) *for $\omega = (1, \sigma) \in \Omega_0$ set $f(\omega) := (1, -\sigma)$,*

(2) *if $\omega = (v, \sigma) \in \Omega_N$, $N \in 2\mathbb{N}$, with $\operatorname{sign}(w_n(v, \sigma)) = -\sigma$ for all $n \in \{0, ..., N\}$, let*

$$f(\omega) := (1, -\sigma) \in \Omega_0 \ ,$$

(3) *and in case $(v, \sigma) \in \Omega_N$, $N \in 2\mathbb{N}$, but $\operatorname{sign}(w_n(v, \sigma)) = \sigma$ for some $n \in \{1, ..., N\}$, set*

$$f(\omega) := \left( \Big( \sum_{k=0}^{j-1} v_{N+1-k} + \frac{1}{\mu} \log \Big( 1 - |\frac{\mu}{a} w_{j-1}| \Big) \, , v_1, ..., v_{N-j} \Big), -\sigma \right) \in \Omega_{N-j+1} \ ,$$

*where*

$$j := j(v, \sigma) = \min \{ n \in \{1, ..., N\} \ : \ \operatorname{sign}(w_n(v, \sigma)) = \sigma \} \in 2\mathbb{N}_0 + 1 \ ,$$

*and $w_{j-1} := w_{j(v,\sigma)-1}(v, \sigma)$.*

*Then $V \circ R = f \circ V$, i.e. the diagram* (2.14) *commutes. Furthermore, we have*

$$V \circ R^n = f^n \circ V$$

*for all $n \in \mathbb{N}$.*

Proof:

1. Let $\varphi \in \mathcal{Z}_0$. Without loss of generality let us assume $s_0(\varphi) = +1$ (since the case $s_0(\varphi) = -1$ can be treated in a completely similar way).

   1.1 If $N(\varphi) = 0$, then we have $V(\varphi) = (1, 1) \in \Omega_0$ and $f(V(\varphi)) = (f \circ V)(\varphi) = (1, -1)$ by definition of $f$ (cf. (1)). On the other hand, we obtain $x_t^{\varphi} = y_t$ for all $t \in [1; +\infty)$ by (2.4) (cf. also Remark 2.2.1). Hence, $t_1(\varphi) = z(\mu)$ and $R(\varphi) = y_{z(\mu)}$. Because of $V(y_{z(\mu)}) = (1, -1)$ we have

   $$(V \circ R)(\varphi) = V(R(\varphi)) = V(y_{z(\mu)}) = (1, -1) = f(V(\varphi)) = (f \circ V)(\varphi)$$

   which proves the assertion in this case.

   1.2 If $N(\varphi) \in 2\mathbb{N}$ set $N := N(\varphi)$ and $v := v(\varphi)$. Observe $V(\varphi) = (v, +1) \in \Omega_N$.

1.2.1 In case that (2.13) holds, we have $\operatorname{sign}(w_n(v,1)) = -s_0(\varphi) = -1$ for all $n \in \{0,...,n\}$ and the definition of $f$ in (2) yields $(f \circ V)(\varphi) = f(V(\varphi)) = f((v,1)) = (1,-1)$. Furthermore, COROLLARY 2.3.5 yields $t_1(\varphi) > 1$ and we obtain $R(\varphi) = x^{\varphi}_{t_1(\varphi)} = y_{z(\mu)}$ again. Hence, this proves

$$(V \circ R)(\varphi) = V(R(\varphi)) = V(y_{z(\mu)}) = (1,-1) = f((v,1)) = (f \circ V)(\varphi) \ .$$

1.2.2 If (2.13) does not hold, we are to use (3) and obtain $(f \circ V)(\varphi) = f(V(\varphi)) =$

$$f((v,1)) = \left( \Big( \sum_{k=0}^{j-1} v_{N+1-k} + \frac{1}{\mu} \log \Big( 1 - |\frac{\mu}{a} w_{j-1}| \Big) \, , v_1, ..., v_{N-j} \Big), -1 \right) \ .$$

On the other hand, COROLLARY 2.3.6 and COROLLARY 2.3.7 yield

$$v(x^{\varphi}_{t_1}) = (t_1(\varphi), v_1(\varphi), ..., v_{N-j}(\varphi)) \in \Omega_{N-j+1} \ .$$

Hence, for the first component of $V(R(\varphi))$ we obtain

$$(V(R(\varphi)))_1 = v(x^{\varphi}_{t_1(\varphi)}) = (t_1(\varphi), v_1(\varphi), ..., v_{N-j}(\varphi))$$

and $s_0(\psi) = s_0(x^{\varphi}_{t_1(\varphi)}) = -s_N(\varphi) = -s_0(\varphi) = -1$ gives $(V(R(\varphi)))_2 = -1$ which proves $(f \circ V)(\varphi) = (V \circ R)(\varphi)$ for those $\varphi \in \mathcal{Z}_0$ for which (2.13) does not hold.

Therefore, we have $(V \circ R)(\varphi) = (f \circ V)(\varphi)$ for all $\varphi \in \mathcal{Z}_0$ which proves that the diagram (2.14) commutes.

2. The second assertion follows by induction on $n \in \mathbb{N}$ using

$$V \circ R^n = (V \circ R^{n-1}) \circ R = (f^{n-1} \circ V) \circ R = f^n \circ V \qquad \text{for } 2 \leq n \in \mathbb{N} \ .$$

□

For every element $(v, \sigma)$ in the subset

$$\Omega_{N0} := \{(v,\sigma) \in \Omega_N \ : \ \operatorname{sign}(w_1(v,\sigma)) = \sigma\} \quad , \ N \in 2\mathbb{N},$$

of $\Omega_N$ we have $j(v,\sigma) = 1$, and therefore $f$ takes the value

$$f(v,\sigma) = \left( \Big( v_{N+1} + \frac{1}{\mu} \log\left(2 - \exp\left(-\mu v_{N+1}\right)\right), v_1, ..., v_{N-1} \Big), -\sigma \right) \in \Omega_N$$

by LEMMA 2.3.4. Its first component, $(f(v,\sigma))_1 \in \mathbb{R}^N_+$, is given by the restriction $g_N$ of the nonlinear map

$$\widehat{g}_N : \Sigma^o_N \ni v \mapsto \left( 1 - \mathbf{1}_N \cdot v + \frac{1}{\mu} \log\left(2 - \exp\left((-\mu)(1 - \mathbf{1}_N \cdot v)\right)\right), v_1, ..., v_{N-1} \right) \in \mathbb{R}^N_+$$

to the subset

$$\Omega_{N01} := \left\{ v \in \mathbb{R}^N_+ \ : \ v_N > 1 - \mathbf{1}_N \cdot v > 0 \right\}$$

of the open standard simplex $\Sigma^o_N := \{v \in \mathbb{R}^N_+ \ : \ \mathbf{1}_N \cdot v < 1\}$ in $\mathbb{R}^N$.

**Remark 2.3.4** *On the complementary subsets of* $\Omega_{N0}$*, denoted as*

$$\Omega_{N1} := \Omega_N \setminus \Omega_{N0} \quad , N \in 2\mathbb{N} \ ,$$

*we have* either $f(v,\sigma) \in \Omega_0$ or

$$j(v,\sigma) \in [3; N(v,\sigma)-1] \cap (2\mathbb{N}_0+1) \quad \textit{and} \quad f(v,\sigma) \in \Omega_{N-j(v,\sigma)+1}$$

*by definition of* $f$ *in* Lemma 2.3.4.

In some sense, the subset $\Omega_{N1}$ determines the "exit set" of $\Omega_N$ because every point in $\Omega_{N1}$ is transported to a set $\Omega_{N-l}$, $l \in \{2,...,N\} \cap (2\mathbb{N})$, under the action of $f$. So, it remains to discover the behaviour of trajectories $(f^n(\omega))_{n\in\mathbb{N}}$ starting in $\Omega_{N0} \ni \omega$.

**Lemma 2.3.5** *Let* $N \in 2\mathbb{N}$*. Each iterate* $(g_N)^k$*,* $k \in \mathbb{N}$*, of* $g_N := \widehat{g}_N\Big|_{\Omega_{N01}}$ *has exactly one fixed point, namely,*

$$v^{(N)} := \left(\tfrac{1-\tau_*}{N}, ..., \tfrac{1-\tau_*}{N}\right) \in \Omega_{N01} \ ,$$

*where* $\tau_* := \tau_*(\mu, \frac{N}{2})$ *denotes the unique solution of* (2.9)*.*

Proof: We fix $N \in 2\mathbb{N}$ throughout the whole proof and set $g := g_N$.

1. First, we claim that $v^{(N)}$ is a fixed point of $g$ in $\Omega_{N01}$. A vector $v = (v_1, v_2, ..., v_N)$ in $\Omega_{N01}$ is by definition a fixed point of $g$, iff $v = g(v)$, such that

   $$(v_1, v_2, ..., v_N) = \left(1 - \sum_{k=1}^{N} v_k + \frac{1}{\mu}\log\left(2 - \exp\left((-\mu)(1 - \sum_{k=1}^{N} v_k)\right)\right), v_1, ..., v_{N-1}\right)$$

   yields

   $$v_j = v_1 \quad \text{for all } j \in \{2,...,N\}$$

   and, thus,

   $$v_1 = 1 - Nv_1 + \frac{1}{\mu}\log\left(2 - \exp\left((-\mu)(1-Nv_1)\right)\right) \ .$$

   Making a change of variable, replacing $v_1$ by

   $$v_1 := \frac{1-\tau}{N}$$

   with some $\tau \in (0;1)$ which is to be calculated, we end up with

   $$\frac{1-\tau}{N} = \tau + \frac{1}{\mu}\log\left(2 - \exp(-\mu\tau)\right)$$

which is equation (2.9) with $m = \frac{1}{2}N$, and has the unique solution $\tau = \tau_* \in \left(0; \frac{1}{2N+1}\right)$. Observe that $\mathbf{1}_N \cdot v^{(N)} = 1 - \tau_* < 1$ and

$$v_N^{(N)} = \frac{1-\tau_*}{N} > \tau_* = 1 - \mathbf{1}_N \cdot v^{(N)}$$

because of $\tau_* \in \left(0; \frac{1}{2N+1}\right)$ such that $v^{(N)}$ is indeed in $\Omega_{N01}$.

2. We claim that $g^2 = g \circ g^1$ has no other fixed point than $v^{(N)}$ in $\Omega_{N01}$.
Let $u = (u_1, u_2, ..., u_n) \in \Omega_{N01}$ be a fixed point of $g^2$. We will prove $u = v^{(N)}$. Setting

$$v := g(u) = \left(1 - \sum_{k=1}^{N} u_k + \frac{1}{\mu} \log\left(2 - \exp\left((-\mu)(1 - \sum_{k=1}^{N} u_k)\right)\right), u_1, ..., u_{N-1}\right)$$

we calculate

$$\begin{aligned} g^2(u) \; &= g(g(u)) = g(v) = ((g(v))_1, (g(v))_2, ..., (g(v))_N) = \\ &= \left(1 - \sum_{k=1}^{N} v_k + \frac{1}{\mu} \log\left(2 - \exp\left((-\mu)(1 - \sum_{k=1}^{N} v_k)\right)\right), v_1, ..., v_{N-1}\right) = \\ &= \left((g(v))_1, 1 - \sum_{k=1}^{N} u_k + \frac{1}{\mu} \log\left(2 - \exp\left((-\mu)(1 - \sum_{k=1}^{N} u_k)\right)\right), u_1, ..., u_{N-2}\right). \end{aligned}$$

From the fixed point equation $g^2(u) = u$ we get the following system of equations in $u_k$, $k \in \{1, ..., N\}$:

$$\left\{ \begin{array}{lcl} u_1 &=& 1 - (v_1 + \sum_{k=1}^{N-1} u_k) + \frac{1}{\mu} \log\left(2 - \exp\left((-\mu)(1 - (v_1 + \sum_{k=1}^{N-1} u_k))\right)\right) \\ u_2 &=& 1 - \sum_{k=1}^{N} u_k + \frac{1}{\mu} \log\left(2 - \exp\left((-\mu)(1 - \sum_{k=1}^{N} u_k)\right)\right) \\ u_3 &=& u_1 \\ u_4 &=& u_2 \\ \vdots & & \vdots \\ u_N &=& u_{N-2} \end{array} \right.$$

where

$$v_1 = 1 - \sum_{k=1}^{N} u_k + \frac{1}{\mu} \log\left(2 - \exp\left((-\mu)(1 - \sum_{k=1}^{N} u_k)\right)\right) .$$

Consequently, the last $(N-2)$ equations in the above system yield

$$u_j = \left\{ \begin{array}{ll} u_1 & , \text{ if } j \in [3; N] \cap (2\mathbb{N}_0 + 1) \\ u_2 & , \text{ if } j \in [3; N] \cap 2\mathbb{N}_0 \end{array} \right.$$

for $j \in \{3, ..., N\}$ (in case $4 \leq N \in 2\mathbb{N}$) such that the above system in $N$ unknowns reduces to a system in two unknowns $u_1$ and $u_2$: using $N \in 2\mathbb{N}$,

$$\sum_{k=1}^{N} u_k = \frac{N}{2}(u_1 + u_2)\,, \quad \text{and} \quad \sum_{k=1}^{N-1} u_k = \frac{N}{2}u_1 + \frac{N-2}{2}u_2$$

to simplify the expressions, we end up with

$$\left\{ \begin{array}{rcl} u_1 & = & u_2 - \frac{1}{\mu}\log\left(2 - e^{-\mu(1-\frac{N}{2}(u_1+u_2))}\right) + \frac{1}{\mu}\log\left(2 - [2 - e^{-\mu(1-\frac{N}{2}(u_1+u_2))}]e^{-\mu u_2}\right) \\ u_2 & = & 1 - \frac{N}{2}(u_1+u_2) + \frac{1}{\mu}\log\left(2 - \exp\left((-\mu)(1-\frac{N}{2}(u_1+u_2))\right)\right) \end{array} \right. . \tag{2.15}$$

We aim to derive detailed information about $u_1 + u_2$ from this system in the sequel. First, inserting the second into the first term of the first equation of (2.15) yields

$$u_1 = 1 - \frac{N}{2}(u_1+u_2) + \frac{1}{\mu}\log\left(2 - [2 - e^{-\mu(1-\frac{N}{2}(u_1+u_2))}]e^{-\mu u_2}\right)\ ,$$

hence

$$\begin{aligned} u_1 + u_2 &= 1 - \frac{N}{2}(u_1+u_2) + \frac{1}{\mu}\log(e^{\mu u_2}) + \frac{1}{\mu}\log\left(2 - [2 - e^{-\mu(1-\frac{N}{2}(u_1+u_2))}]e^{-\mu u_2}\right) = \\ &= 1 - \frac{N}{2}(u_1+u_2) + \frac{1}{\mu}\log\left(2e^{\mu u_2} - [2 - e^{-\mu(1-\frac{N}{2}(u_1+u_2))}]\right)\ . \end{aligned}$$

Because of the second equation of (2.15) we have

$$2e^{\mu u_2} = 2e^{\mu\left(1-\frac{N}{2}(u_1+u_2)\right)} \cdot [2 - e^{-\mu(1-\frac{N}{2}(u_1+u_2))}]$$

such that we infer

$$\begin{aligned} u_1 + u_2 &= 1 - \frac{N}{2}(u_1+u_2) + \frac{1}{\mu}\log\left(2e^{\mu\left(1-\frac{N}{2}(u_1+u_2)\right)} - 1\right) + \\ &\qquad + \frac{1}{\mu}\log\left(2 - e^{-\mu(1-\frac{N}{2}(u_1+u_2))}\right)\ . \end{aligned}$$

But since

$$\begin{aligned} \frac{1}{\mu}\log\left(2e^{\mu\left(1-\frac{N}{2}(u_1+u_2)\right)} - 1\right) &= \frac{1}{\mu}\log\left(e^{\mu(1-\frac{N}{2}(u_1+u_2))}\left(2 - e^{-\mu(1-\frac{N}{2}(u_1+u_2))}\right)\right) = \\ &= 1 - \frac{N}{2}(u_1+u_2) + \frac{1}{\mu}\log\left(2 - e^{-\mu(1-\frac{N}{2}(u_1+u_2))}\right) \end{aligned}$$

we arrive at

$$u_1 + u_2 = 2\left(1 - \frac{N}{2}(u_1+u_2)\right) + \frac{2}{\mu}\log\left(2 - e^{-\mu(1-\frac{N}{2}(u_1+u_2))}\right)$$

or, equivalently, at

$$\xi = 1 - N\xi + \frac{1}{\mu}\log(2 - e^{-\mu(1-N\xi)}) \qquad \text{with } \xi := \frac{u_1 + u_2}{2} \ . \tag{2.16}$$

Now, using the change of variables

$$\xi := \frac{1-\tau}{N}$$

for some $\tau \in (0;1)$ we get equation (2.9) with $m = \frac{1}{2}N$ again (exactly as in step 1.). Therefore, we infer from PROPOSITION 2.2.2 that $\xi = \frac{u_1+u_2}{2} = \frac{1-\tau_*}{N}$ is the unique solution of (2.16). Hence, *every* solution $(u_1, u_2)$ of (2.15) has to satisfy

$$\frac{u_1 + u_2}{2} = \frac{1-\tau_*}{N} \ ,$$

and inserting this into the second equation of (2.15) gives

$$\begin{aligned} u_2 &= 1 - N \cdot \frac{1-\tau_*}{N} + \frac{1}{\mu}\log\left(2 - \exp\left((-\mu)(1 - N\cdot\frac{1-\tau_*}{N}\right)\right) = \\ &= \tau_* + \frac{1}{\mu}\log(2 - e^{-\mu\tau_*}) \overset{(2.9)}{=} \\ &= \frac{1-\tau_*}{N} \ , \end{aligned}$$

which implies $u_1 = u_2 = \frac{1-\tau_*}{N}$. Recall that this yields $u = v^{(N)} = (\frac{1-\tau_*}{N}, ..., \frac{1-\tau_*}{N})$ which proves that there cannot exist other fixed points of $g^2$.

3. It is now easy to conclude that $v^{(N)}$ is the unique fixed point of $g^k$ in $\Omega_{N01}$ for $k \in \mathbb{N}$ since step 2. contains the essential ideas required for induction on $k \in \mathbb{N}$: whenever $g^k$ has $v^{(N)}$ as unique fixed point in $\Omega_{N01}$ for some $k \in \mathbb{N}$, similar arguments as in 2. provide that $g^{k+1} = g \circ g^k$ cannot have an additional fixed point in $\Omega_{N01} \setminus \{v^{(N)}\}$.

□

In order to determine the stability properties of these fixed points under the action of the map $g_N$, $N \in 2\mathbb{N}$, we have to linearize $g_N$ at $v^{(N)}$, which is the content of following lemma.

**LEMMA 2.3.6** *For $N \in 2\mathbb{N}$ let $\tau_* := \tau_*(\mu, \frac{N}{2})$ denote the unique solution of (2.9) in $\left(0; \frac{1}{2N+1}\right)$. The Jacobian of $g_N$ at $v^{(N)}$ is given by*

$$J_{g_N}(v^{(N)}) = \begin{pmatrix} \alpha^* & \alpha^* & \cdots & \alpha^* & \alpha^* \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

*where $\alpha^* := \alpha^*(\mu, N) = -1 - \exp\left(-\frac{\mu}{N}(1 - \tau^*(\mu, \frac{N}{2}))\right) \in (-\infty; -2)$.*

PROOF: Since $J_{g_N}(v^{(N)}) = \left(\frac{\partial (g_N)_i}{\partial v_j}(v^{(N)})\right)_{(i,j)\in\{1,...,N\}^2}$, we only have to indicate how to obtain the first row of the Jacobian. For $j \in \{1, ..., N\}$ an easy computation shows

$$\begin{aligned}\frac{\partial (g_N)_1}{\partial v_j}(v^{(N)}) &= -1 - \frac{\exp\left((-\mu)(1 - \sum_{k=1}^N v_k^{(N)})\right)}{2 - \exp\left((-\mu)(1 - \sum_{k=1}^N v_k^{(N)})\right)} = \\ &= -1 - \frac{\exp(-\mu\tau_*)}{2 - \exp(-\mu\tau_*)} ,\end{aligned}$$

and because of

$$\frac{1-\tau_*}{N} = \tau_* + \frac{1}{\mu}\log\left(2 - e^{-\mu\tau_*}\right)$$

we get

$$\frac{\partial (g_N)_1}{\partial v_j}(v^{(N)}) = -1 - \exp\left(-\frac{\mu}{N}(1-\tau_*)\right) =: \alpha^*(\mu, N) .$$

$\square$

For completeness, we mention without proof (which can easily be done combining the continuity of the map $\alpha^*(\cdot, N)$, $N \in 2\mathbb{N}$, with the proof of REMARK 2.2.4(3)).

**REMARK 2.3.5** *For every $N \in 2\mathbb{N}$ we have $\lim_{\mu\to 0} \alpha^*(\mu, N) = -2$.*

Certainly, we are now interested in the local behaviour of $g_N$ near the unique fixed point $v^{(N)}$. Therefore, we have to determine the position of the eigenvalues of $J_{g_N}(v^{(N)})$.

**LEMMA 2.3.7** *Let $N \in 2\mathbb{N}$. Then $\sigma\left(J_{g_N}(v^{(N)})\right) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$.*

PROOF: For any $\lambda \in \sigma\left(J_{g_N}(v^{(N)})\right)$ we have to prove $|\lambda| > 1$. Choose a corresponding eigenvector $v = (v_1, ..., v_N) \in \mathbb{C}^N \setminus \{0\}$ to the given eigenvalue $\lambda$. Then

$$\left(\alpha^* \cdot \sum_{k=1}^N v_k, v_1, ..., v_{N-1}\right) = (\lambda v_1, \lambda v_2, ..., \lambda v_N)$$

yields the system of equations

$$\left\{\begin{array}{rcll} \alpha^* \cdot \sum_{k=1}^N v_k &=& \lambda v_1 & \\ v_{k-1} &=& \lambda v_k & \quad \text{for } j \in \{2, ..., N\} \end{array}\right.$$

which is the subject of our investigations in the sequel.

1. Notice first, that $1 \notin \sigma(J_{g_N}(v^{(N)}))$ since *otherwise* $\lambda = 1$ yields $v_{k-1} = v_k$ for all $k \in \{2, ..., N\}$ such that $v_1 = v_k$ for all $k \in \{1, ..., N\}$. Thus,

$$\alpha^* \cdot \sum_{k=1}^{N} v_k = \alpha^* N \cdot v_1 = \lambda v_1 = v_1$$

such that we obtain *contradiction*

$$1 = \alpha^* N < 0 \ .$$

Therefore, we assume $\lambda \neq 1$ for the remainder of this proof without further mentioning.

2. From the last $(N-1)$ equations of the above system we obtain by induction

$$v_k = \lambda^{N-k} v_N \quad \text{ for all } k \in \{1, ..., N\} \ ,$$

which implies, inserted into the first equation,

$$\alpha^* \sum_{k=1}^{N} \lambda^{N-k} = \alpha^* \sum_{k=0}^{N-1} \lambda^k = \alpha^* \frac{1 - \lambda^N}{1 - \lambda} = \lambda \cdot \lambda^{N-1} = \lambda^N \ .$$

Hence, we must have for $\lambda \neq 1$,

$$\lambda^{N+1} - (1 + \alpha^*)\lambda^N + \alpha^* = 0 \ . \tag{2.17}$$

Now, we want to apply ROUCHÉ's Theorem to prove that there is no solution of the "characteristic" equation (2.17) in $\overline{\mathbb{D}} \setminus \{1\}$. For this purpose, set

$$h_1 : \mathbb{C} \ni \lambda \mapsto \lambda^N \in \mathbb{C} \quad \text{ and } \quad h_2 : \mathbb{C} \ni \lambda \mapsto -(1 + \alpha^*)\lambda^N + \alpha^* \in \mathbb{C}$$

such that $h_1(\lambda) + h_2(\lambda) = \lambda^{N+1} - (1 + \alpha^*)\lambda^N + \alpha^*$ holds for all $\lambda \in \mathbb{C}$.

2.1 For $r \in (0; 1)$ set

$$\gamma_r : [0; 1) \ni \vartheta \mapsto r e^{2\pi i \vartheta} \in \mathbb{C} \ ,$$

and denote the trace of the JORDAN curve $\gamma_r$ by $U_r := |\gamma_r| = \{z \in \mathbb{C} \ : \ |z| = r\}$.

2.2 Evidently, $h_1(\lambda) \neq 0$ for all $\lambda \in U_r$, while we can only have $h_2(\lambda) = 0$ for those $\lambda \in \mathbb{C}$ which satisfy

$$\lambda^N = 1 - \frac{1}{\alpha^* + 1} \ .$$

By virtue of $\alpha^* \in (-\infty; -2)$ we have $1 - \frac{1}{\alpha^*+1} > 1$, such that the last identity implies that $h_2$ does not have a zero in $\overline{\mathbb{D}} \supset U_r$.

2.3 Clearly, it is

$$|h_2(\lambda)| \geq |\alpha^*| - |1+\alpha^*|r^N > (|\alpha^*| - |1+\alpha^*|)r^N = r^N > r^{N+1} = |h_1(\lambda)|$$

for all $\lambda \in U_r$ (where we used $\alpha^* \in (-\infty; -2)$ to conclude $|\alpha^*| - |1+\alpha^*| = -\alpha^* - (-(1+\alpha^*)) = 1$).

2.4 Now, ROUCHÉ's Theorem [53, p. 218] yields that $h_2$ and $h_1 + h_2$ must have the same number of zeros in $\operatorname{int}|\gamma_r|$, the closed interior of the trace of the JORDAN curve $\gamma_r$, which proves that all solutions $\lambda$ of (2.17) satisfy $|\lambda| > r$.

Since $r$ was chosen arbitrarily in step 2.1, we conclude $\sigma(J_{g_N}(v^{(N)})) \subset \mathbb{C} \setminus \mathbb{D}$.

3. It remains to prove that there are no eigenvalues on the unit circle.

3.1 By LEMMA 2.3.6 we know that $\alpha^* \in (-\infty; -2)$, such that $\alpha^* + 1 \in (-\infty; -1)$. Thus, the circle

$$c_{\alpha^*} := \{z \in \mathbb{C} \ : \ |z - (1+\alpha^*)| = |\alpha^*|\}$$

has center $1 + \alpha^* \in (-\infty; -1)$ and radius $|\alpha^*| > 2$, such that it intersects the unit circle tangentially at $z = +1$, i.e.

$$c_{\alpha^*} \cap \partial\mathbb{D} = \{+1\} \ .$$

3.2 Now, let us *assume* that there exist eigenvalues $\lambda \in \partial\mathbb{D} \setminus \{1\}$. These eigenvalues have to satisfy

$$|\lambda|^N \cdot |\lambda - (1+\alpha^*)| = |-\alpha^*|$$

as we conclude from equation (2.17), such that

$$\lambda \in ((\partial\mathbb{D}) \setminus \{1\}) \cap c_{\alpha^*} \overset{3.1}{=} \emptyset$$

which gives the *contradiction*.

$\square$

Using the Principle of Linearized Stability for maps we obtain that the fixed point $v^{(N)}$, $N \in 2\mathbb{N}$, is unstable (or, more precisely, is a source). But we need more.

**LEMMA 2.3.8** *Fix $N \in 2\mathbb{N}$.*

(1) *The set*

$$A_N := \left\{ v \in \Omega_{N01} \ : \ g_N^k(v) \in \Omega_{N01} \ \text{ for all } k \in \mathbb{N}_0 \right\}$$

*has* LEBESGUE*-measure zero in $\mathbb{R}^N$.*

(2) *Moreover, $A_N$ is a closed subset of $\Omega_{N01}$ and, therefore, nowhere dense in $\Omega_{N01}$.*

(3) *There is an open neighborhood $U \subset \Omega_{N01}$ of $v^{(N)}$ with the following property: for every $v \in (U \cap A_N) \setminus \{v^{(N)}\}$ there is a $k \in \mathbb{N}$ such that $g_N^k(v) \cap U = \emptyset$.*

(4) *Let $v \in A_N \setminus \{v^{(N)}\}$. Then there does not exist a $p \in \mathbb{N}$ and a $k_0 \in \mathbb{N}_0$ such that*

$$g_N^{k+p}(v) = g_N^k(v) \quad \textit{for all } k \in [k_0; +\infty) \cap \mathbb{N}_0 \ .$$

*In other words: $A_N$ does not contain any cycles.*

Proof: We write $A := A_N$ for short throughout the whole proof.

1. Let us *assume* that $\lambda_N(A) \neq 0$.

   1.1 By definition of $A$ we have

   $$g_N^k(A) \subset \Omega_{N01} \quad \text{for all } k \in \mathbb{N} \ .$$

   In particular, we get

   $$\lambda_N\left(g_N^k(A)\right) \leq \lambda_N\left(\Omega_{N01}\right) \quad \text{for all } k \in \mathbb{N} \ .$$

   1.2 For every $v \in \Omega_{N01}$ we obtain (exactly as in Lemma 2.3.6)

   $$J_{g_N}(v) = \begin{pmatrix} \alpha(v) & \alpha(v) & \cdots & \alpha(v) & \alpha(v) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

   with

   $$\alpha : \Omega_{N01} \ni v \mapsto -1 - \frac{\exp\left((-\mu)(1 - \mathbf{1}_N \cdot v)\right)}{2 - \exp\left((-\mu)(1 - \mathbf{1}_N \cdot v)\right)} \in \mathbb{R} \ .$$

   Consequently, we can compute the determinant of the Jacobian expanding it by its first row minors: this yields

   $$\det\left(J_{g_N}(v)\right) = (-1)^{N+1}\alpha(v) = -\alpha(v) = 1 + \frac{\exp\left((-\mu)(1 - \mathbf{1}_N \cdot v)\right)}{2 - \exp\left((-\mu)(1 - \mathbf{1}_N \cdot v)\right)}$$

   for all $v \in \Omega_{N01}$.

1.3 Because of $\mu \in (-\log 2; 0)$ and $v \in \Omega_{N01}$ we have $\exp((-\mu)(1-\mathbf{1}_N v)) > 1$ and, thus,

$$-\alpha(v) = 1 + \frac{\exp\left((-\mu)(1-\mathbf{1}_N \cdot v)\right)}{2 - \exp\left((-\mu)(1-\mathbf{1}_N \cdot v)\right)} > 2$$

for all $v \in \Omega_{N01}$. Hence,

$$\det\left(J_{g_N}(v)\right) > 2 \quad \text{ for all } v \in \Omega_{N01} \ .$$

1.4 Let $k \in \mathbb{N}$. We notice that $g_N$ is a $C^1$-diffeomorphism with $C^1$-inverse given by the restriction of the map

$$h_N : \mathbb{R}_+^N \ni w \mapsto \left( w_2, w_3, ..., w_N, 1 - \sum_{k=2}^N w_k - \frac{1}{\mu} \log\left(\frac{1}{2}(1+e^{\mu w_1})\right)\right) \in \mathbb{R}^N$$

to $g(\Omega_{N01}) \subset \mathbb{R}_+^N$.

To check this, note that $h_N$ is a $C^1$ map, let $w \in \Delta := g(\Omega_{N01}) \subset \mathbb{R}_+^N$, and set $x := \sum_{k=1}^N w_k = 1 - \frac{1}{\mu}\log\left(\frac{1}{2}(1+e^{\mu w_1})\right)$. Now, we have

$$\begin{aligned}
(g_N \circ h_N\big|_\Delta)(w) &= g_N\left(w_2, w_3, ..., w_N, 1 - \sum_{k=2}^N w_k - \tfrac{1}{\mu}\log\left(\tfrac{1}{2}(1+e^{\mu w_1})\right)\right) = \\
&= \left(1 - x + \frac{1}{\mu}\log\left(2 - e^{-\mu(1-x)}\right), w_2, w_3, ..., w_N\right) = \\
&= \left(\frac{1}{\mu}\log\left(\frac{1}{2}(1+e^{\mu w_1})\right) + \frac{1}{\mu}\log\left(2 - \exp\left(-\log\left(\frac{1}{2}(1+e^{\mu w_1})\right)\right)\right), w_2, w_3, ..., w_N\right) = \\
&= \left(\frac{1}{\mu}\log\left(\frac{1}{2}(1+e^{\mu w_1})\right) + \frac{1}{\mu}\log\left(2 - \frac{2}{1+e^{\mu w_1}}\right), w_2, w_3, ..., w_N\right) = \\
&= \left(\frac{1}{\mu}\log\left(\frac{1}{2}(1+e^{\mu w_1})\right) + \frac{1}{\mu}\log\left(2 \cdot \frac{1+e^{\mu w_1} - 1}{1+e^{\mu w_1}}\right), w_2, w_3, ..., w_N\right) = \\
&= \left(\frac{1}{\mu}\log\left(\frac{1}{2}(1+e^{\mu w_1})\right) + \frac{1}{\mu}\log\left(\frac{2}{1+e^{\mu w_1}} \cdot e^{\mu w_1}\right), w_2, w_3, ..., w_N\right) = \\
&= \left(\frac{1}{\mu}\log\left(\frac{1}{2}(1+e^{\mu w_1})\right) + w_1 - \frac{1}{\mu}\log\left(\frac{1+e^{\mu w_1}}{2}\right), w_2, w_3, ..., w_N\right) = \\
&= (w_1, w_2, ..., w_N) = w
\end{aligned}$$

such that $g_N \circ h_N\big|_\Delta = \mathrm{id}_\Delta$. On the other hand, we also have $h_N\big|_\Delta \circ g_N = \mathrm{id}_{\Omega_{N01}}$ since we get for $v \in \Omega_{N01}$

$$\begin{aligned}
(h_N\big|_\Delta \circ g_N)(v) &= h_N\big|_\Delta\left(1 - \sum_{k=1}^N v_k + \tfrac{1}{\mu}\log\left(2 - \exp(-\mu(1-\sum_{k=1}^N v_k))\right), v_1, ..., v_{N-1}\right) = \\
&= (v_1, ..., v_{N-1}, \\
&\quad 1 - \sum_{k=1}^{N-1} v_k - \tfrac{1}{\mu}\log\left(\tfrac{1}{2}(1+\exp\left(\mu(1-\sum_{k=1}^N v_k + \tfrac{1}{\mu}\log\left(2-\exp(-\mu(1-\sum_{k=1}^N v_k)))\right))\right)\right)\right) \\
&= (v_1, ..., v_{N-1},
\end{aligned}$$

$$1-\sum_{k=1}^{N-1} v_k - \frac{1}{\mu}\log\left(\frac{1}{2}(1+\exp\left(\mu(1-\sum_{k=1}^{N} v_k)\right)\left(2-\exp(-\mu(1-\sum_{k=1}^{N} v_k))\right))\right) =$$

$$\begin{aligned}
&= \left(v_1, ..., v_{N-1}, 1-\sum_{k=1}^{N-1} v_k - \tfrac{1}{\mu}\log\left(\tfrac{1}{2}(1+2\exp\left(\mu(1-\sum_{k=1}^{N} v_k)\right)-1)\right)\right) = \\
&= \left(v_1, ..., v_{N-1}, 1-\sum_{k=1}^{N-1} v_k - \tfrac{1}{\mu}\log\left(\exp\left(\mu(1-\sum_{k=1}^{N} v_k)\right)\right)\right) = \\
&= \left(v_1, ..., v_{N-1}, 1-\sum_{k=1}^{N-1} v_k - (1-\sum_{k=1}^{N} v_k)\right) = \\
&= (v_1, ..., v_{N-1}, v_N) = v\ .
\end{aligned}$$

Therefore, $h_N\Big|_{\Delta} = g_N^{-1}$ and $g_N : \Omega_{N01} \to \Delta$ is a $C^1$-invertible map as needed for applying the transformation rule.

Hence, using the transformation rule yields

$$\lambda_N\left(g_N^k(A)\right) = \int\limits_{g_N^k(A)} d\lambda_N = \int\limits_{g_N^{k-1}(A)} |\det\left(J_{g_N}(v)\right)| d\lambda_N(v) > 2 \int\limits_{g_N^{k-1}(A)} d\lambda_N\ ,$$

such that induction on $k \in \mathbb{N}$ implies

$$\lambda_N\left(g_N^k(A)\right) > 2^k \lambda_N(A) \quad \text{for all } k \in \mathbb{N}\ .$$

1.5 Finally, choosing $k := \left\lceil \frac{\log\lambda_N(\Omega_{N01}) - \log\lambda_N(A)}{\log 2} \right\rceil$ we obtain

$$\lambda_N(g_N^k(A)) > 2^k\lambda_N(A) \geq \lambda_N(\Omega_{N01})$$

from 1.4 in *contradiction* to step 1.1.

2. Let $w \in \overline{\Omega_{N01}}$ be a limit point of $A$. We claim that $w \in A$. *Otherwise*, $w \in \overline{\Omega_{N01}} \setminus A$ and there is a sequence $(w^j)_{j\in\mathbb{N}}$ in $A \subset \Omega_{N01}$ with $\lim\limits_{j\to\infty} w^j = w$.

2.1 First, we show the existence of a $\varkappa \in \mathbb{N}$ such that

$$g_N^{\varkappa}(w) \notin \Omega_{N01}\ .$$

Since $w \in \overline{\Omega_{N01}} \setminus A$, either $w \in \Omega_{N01} \setminus A$ or $w \in \partial\Omega_{N01}$ and we investigate these cases separately below. Before we do this, we need a last preparation. Observe that $\mu \in (-\log 2; 0)$ yields that $g_N$ can be extended continuously to the open set

$$\left\{v \in \mathbb{R}^N\ :\ 1 + \frac{\log 2}{\mu} < \sum_{k=1}^{N} v_k\right\} \supset [0; +\infty)^N \supset \overline{\Omega_{N01}}\ .$$

For simplicity, we denote this extension by $g_N$ again.

2.1.1 If $w \in \Omega_{N01} \setminus A$, then there exists a (minimal) $\varkappa \in \mathbb{N}$ such that

$$g_N^k(w) \in \Omega_{N01} \quad \text{for all } k \in \{0, ..., \varkappa - 1\} \qquad \text{and} \qquad g_N^{\varkappa}(w) \notin \Omega_{N01}$$

by definition of $A$.

2.1.2 For $w \in \partial\Omega_{N01}$ we have $g_N(w) \notin \Omega_{N01}$.
To see this, set

$$B_j := \left\{ v \in [0;1]^N \ : \ v_j = 0 \text{ and } 1 - \sum_{l=1}^{N} v_l \in [0; v_N] \right\}$$

for $j \in \{1, ..., N\}$,

$$B_0 := \left\{ v \in [0;1]^N \ : \ 1 - \sum_{j=1}^{N} v_j = 0 \right\} \ ,$$

and

$$B_{N+1} := \left\{ v \in \Sigma_N^o \ : \ 1 - \sum_{j=1}^{N} v_j = v_N \right\} \ ,$$

and note that

$$\partial\Omega_{N01} = \bigcup_{j=0}^{N+1} B_j$$

because of $\Omega_{N01} = \Sigma_N^o \cap \left\{ v \in \mathbb{R}^N \ : \ 1 - \sum_{j=1}^{N} v_j < v_N \right\} =: \Sigma_N^o \cap H_N$,

$$\partial\Sigma_N^o = \left\{ v \in [0;1]^n \ : \sum_{j=1}^{N} v_j = 1 \right\} \cup \bigcup_{j=1}^{N} \partial\Sigma_{N,j} = B_0 \cup \bigcup_{j=1}^{N} \partial\Sigma_{N,j}$$

with

$$\partial\Sigma_{N,j} := \left\{ v \in [0;1]^N \ : \ v_j = 0 \text{ and } (v_1, ..., v_{j-1}, v_{j+1}, ..., v_N) \in \overline{\Sigma_{N-1}^o} \right\}$$

for $j \in \{1, ..., N\}$, and

$$\begin{aligned}
\partial\Omega_{N01} &= (\partial\Sigma_N^o \cap \overline{H_N}) \cup (\Sigma_N^o \cap \overline{H_N}) = \\
&= (B_0 \cap \overline{H_N}) \cup \left( (\bigcup_{j=1}^{N} \partial\Sigma_{N,j}) \cap \overline{H_N} \right) \cup B_{N+1} = \\
&= B_0 \cup \left( \bigcup_{j=1}^{N} (\partial\Sigma_{N,j} \cap \overline{H_N}) \right) \cup B_{N+1} \ .
\end{aligned}$$

Now, we compute $g_N(B_j)$ for every $j \in \{0, ..., N+1\}$. First,

$$g_N(B_{N+1}) = \left\{ \left(v_N + \frac{1}{\mu} \log\left(2 - e^{-\mu v_N}\right), v_1, ..., v_{N-1}\right) \ : \ v \in B_{N+1} \right\}$$

together with $v_N + \frac{1}{\mu} \log\left(2 - e^{-\mu v_N}\right) > 2v_N$ shows that we have

$$1 - \sum_{j=1}^{N} u_j = 1 - u_1 - \sum_{j=2}^{N} u_j < 1 - v_N - \sum_{j=1}^{N} v_j = -v_n + v_N = 0$$

for

$$u := \left(v_N + \frac{1}{\mu} \log\left(2 - e^{-\mu v_N}\right), v_1, ..., v_{N-1}\right) \qquad \text{with } v \in B_{N+1}$$

such that $g_N(B_{N+1}) \cap \overline{\Omega_{N01}} = \emptyset$.
Similarly, we see that

$$g_N(B_0) = \{(0, v_1, ..., v_{N-1}) \ : \ v \in B_0\}$$

which yields $g_N(B_0) \cap \Omega_{N01} = \emptyset$.
Now, consider

$$g_N(B_1) = \left\{ g_N(0, v_2, ..., v_N) \ : \ 1 - \sum_{k=2}^{N} v_k \in [0; v_N] \right\} .$$

For every $v \in B_1$ we have

$$u = g_N(v) = \left(1 - \sum_{k=2}^{N} v_k + \frac{1}{\mu} \log(2 - e^{-\mu(1 - \sum_{k=2}^{N} v_k)}), 0, v_2, ..., v_{N-1}\right)$$

such that $u_2 = 0$ implies $u = g(v) \notin \Omega_{N01}$. Hence,

$$g_N(B_1) \cap \Omega_{N01} = \emptyset$$

and completely analogous arguments yield

$$g_N(B_j) \cap \Omega_{N01} = \emptyset$$

for all $j \in \{1, ..., N-1\}$. For $j = N$ and any $v \in B_N$ we obtain

$$g_N(v) = g_N(v_1, ..., v_{N-1}, 0) = (0, v_1, ..., v_{N-1})$$

because of $1 - \sum_{k=1}^{N} v_K = 0 = v_N$ such that

$$g_N(B_N) \cap \Omega_{N01} = \emptyset$$

holds, too.

Therefore, we proved the assertion 2.1.

2.2 Notice that a repeated application of the arguments in step 2.1.2 will show that for any $\omega \in \partial\Omega_{N01}$ there is a $\widetilde{\varkappa} := \widetilde{\varkappa}(\omega) \in \mathbb{N}_0$ with

$$g_N^{\widetilde{\varkappa}}(\omega) \notin \overline{\Omega_{N01}} \ .$$

To clarify this, we only need to consult step 2.1.2 again: If $\omega \in \Omega_{N01}$, then there is a $j \in \{0, ..., N+1\}$ with $\omega \in B_j$: Since we have already shown

$$g_N(B_{N+1}) \cap \overline{\Omega_{N01}} = \emptyset$$

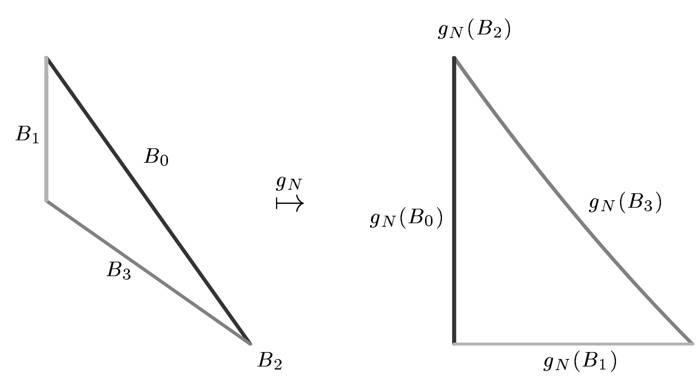
there it remains to deal with the cases $j = 0$ and $j \in \{1, ..., N\}$.
If $\omega \in B_0$, then either $g_N(\omega) \notin \overline{\Omega_{N01}}$ or $g_N(\omega) \in B_1$. Now, whenever there is a $\omega' \in B_j$, $j \in \{1, ..., N-1\}$, we conclude that $g_N(\omega')$ either belongs to the complement of $\overline{\Omega_{N01}}$ or to $B_{j+1}$ as follows easily from the definition of $g_N$ (because of the "shift part" of $g_N$).
Therefore, for any $\omega \in B_j$, $j \in \{0, ..., N-1\}$ we either have $g_N(\omega) \notin \overline{\Omega_{N01}}$ (and we are done) or there is a $k \in \{1, ..., N\}$ with $g_N^k(\omega) \in B_N$.
Finally, for all $\omega'' \in B_N$ we have

$$g_N(\omega'') = (0, \omega_1'', ..., \omega_{N-1}'') \qquad \text{with } 1 - \sum_{j=1}^{N-1} \omega_j'' = 0$$

such that

$$g_N^2(\omega'') = (0, 0, \omega_1'', ..., \omega_{N-2}'') \notin \overline{\Omega_{N01}}$$

which proves the existence of a $\widetilde{\varkappa} \in \{1, ..., N+1\}$ with $g_N^{\widetilde{\varkappa}}(\omega) \notin \overline{\Omega_{N01}}$ for every $\omega \in \partial\Omega_{N01}$.
We shall illustrate the results of steps 2.1 and 2.2 for $N = 2$ in the following figure wherein $\partial\Omega_{N01}$ and $g_N(\partial\Omega_{N01})$ are plotted.




Different gray levels refer to the different sets $B_j$, $j \in \{0, 1, 3\}$, and $B_2$ is the singleton $\{(1, 0)\}$ in this special case. Thus, $g_N$ expands and rotates $\partial\Omega_{N01}$ in

such a way that the first or second iterate of every $w \in \partial\Omega_{N01}$ can not longer belong to $\overline{\Omega_{N01}}$.
Furthermore, the figure above demonstrates that $g_N$ expands the volume by a factor larger than 2 as we have already seen in step 1. of this proof.

2.3 Since $g_N^{\varkappa}$ is continuous on $\overline{\Omega_{N01}}$ we have

$$\lim_{j\to\infty} g_N^{\varkappa}(w^j) = g_N^{\varkappa}(w) \notin \Omega_{N01} \ .$$

Therefore, for any given $\varepsilon > 0$ there exists a $j_\varepsilon \in \mathbb{N}_0$ such that

$$\|g_N^{\varkappa}(w^j) - g_N^{\varkappa}(w)\|_1 < \varepsilon$$

for all $j \in [j_\varepsilon; +\infty) \cap \mathbb{N}_0$.

2.4 If $g_N^{\varkappa}(w) \notin \overline{\Omega_{N01}}$, then we choose $\varepsilon \in \left(0; \frac{1}{2}\mathrm{dist}_{\mathbb{R}^N}(g_N^{\varkappa}(w), \overline{\Omega_{N01}})\right)$. Hence, the previous step implies

$$g_N^{\varkappa}(w^j) \notin \Omega_{N01} \qquad \text{for all } j \in [j_\varepsilon; +\infty) \cap \mathbb{N}_0$$

such that

$$w^j \notin A \qquad \text{for all } j \in [j_\varepsilon; +\infty) \cap \mathbb{N}_0$$

in *contradiction* to $(w^j)_{j\in\mathbb{N}} \subset A$.

2.5 In case $g_N^{\varkappa}(w) =: \omega \in \partial\Omega_{N01}$ step 2.2 enables us to find a $\widetilde{\varkappa} \in \mathbb{N}_0$ with

$$g_N^{\widetilde{\varkappa}}(\omega) = g_N^{\varkappa+\widetilde{\varkappa}}(w) \notin \overline{\Omega_{N01}} \ .$$

Consequently, the arguments from step 2.4 apply for $\varkappa$ replaced by $\varkappa + \widetilde{\varkappa}$ and yield the desired *contradiction*, too.

2.6 Hence $A$ is a closed subset of $\Omega_{N01}$. Since $A$ must not contain inner points by assertion (1), we conclude that

$$\left(\overline{A}\right)^o = A^o = \emptyset \ ,$$

i.e. $A$ is nowhere dense.

3. The third assertion is a trivial consequence of the Principle of Linearized Stability for maps and LEMMA 2.3.7.

4. Assertion (4) is only another formulation of LEMMA 2.3.5 (for $A \subset \Omega_{N01}$).

□

The preceding lemma states that $g_N$ is "ejective" almost everywhere on $\Omega_{N01}$, i.e. every orbit $(g_N^k(v))_{k\in\mathbb{N}}$ starting in $\Omega_{N01} \setminus A_N$ has to leave this set in a finite time. The exceptional set $A_N$ of initial values of orbits that do not leave $\Omega_{N01}$ contains the fixed point $v^{(N)}$ and is a nowhere dense zero set in $\Omega_{N01}$. It is tempting to conjecture $A_N = \{v^{(N)}\}$ but this is still an open question which should be addressed in further investigations.

Apart from this we obtain some elementary consequences stated as

**COROLLARY 2.3.8** *Every periodic solution of* (2.1) *is a translate of some* $x^{(N)}$, $N \in 2\mathbb{N}_0$*. Moreover, all periodic solutions* $x^{(N)}$, $N \in 2\mathbb{N}$, *are unstable.*

IDEA OF THE PROOF: For $N = 0$ use REMARK 2.2.1(3) to show that all slowly oscillating periodic solutions are translates of $x^{(0)}$.
Since the LYAPUNOV functional has a constant value $U_* = 2N + 1$ on the orbit of every periodic solution for some $N \in 2\mathbb{N}$, a periodic solution defines necessarily a fixed point of $g_N$. Consequently, we can use LEMMA 2.3.5 to prove that all periodic solutions are translates of $x^{(N)}$, $N \in 2\mathbb{N}$. Finally, the instability of these rapidly oscillating solutions is an immediate consequence of LEMMA 2.3.8(4).

□

Finally, we can draw the similar conclusions as in [16, p. 439] concerning the global dynamics of (2.1) on $\mathcal{Z}$.

## E. Geometric description of the action of the semiflow on $\mathcal{Z}$

Let $\varphi \in \mathcal{Z}$ be given. PROPOSITION 2.3.1 asserts the existence of a $t_1(\varphi) \in \mathbb{R}^+$ such that $x^{\varphi}_{t_1(\varphi)} \in \mathcal{Z}_0$ and

$$V(x^{\varphi}_{t_1(\varphi)}) \in \Omega_N \quad \text{for some } N \in 2\mathbb{N}_0 \ .$$

If $N = 0$, then the trajectory

$$\gamma_\varphi : \mathbb{R}_0^+ \ni t \mapsto x_t^{\varphi} \in \mathcal{Z}$$

of $\varphi$ in $\mathcal{Z}$ merges into the orbit $\mathcal{O}_0$ of the slowly oscillating solution $x^{(0)}$ as we know from LEMMA 2.2.1(3).

For $N \in 2\mathbb{N}$ we have to distinguish the following cases.

I. If $(V(x^{\varphi}_{t_1(\varphi)}))_1 = v^{(N)}$, then the segment $x^{\varphi}_{t_1(\varphi)}$ has the same sign distribution as $(V(x^{\varphi}_{t_1(\varphi))})_2 \cdot x_0^{(N)}$ in $[-1;0]$ such that (2.1) yields that the trajectory $\gamma_\varphi$ of $\varphi$ in $\mathcal{Z}$ merges into the orbit of the corresponding periodic solution $x^{(N)}$ in $\mathcal{Z}$,

$$\mathcal{O}_N := \{x_t^{(N)} \ : \ t \in \mathbb{R}\} \ .$$

II. If $(V(x^{\varphi}_{t_1(\varphi)}))_1 \neq v^{(N)}$, either $V(x^{\varphi}_{t_1(\varphi)}) \in \Omega_{N1}$ or $V(x^{\varphi}_{t_1(\varphi)}) \in \Omega_{N0}$.

II.1 If we have $V(x^{\varphi}_{t_1(\varphi)}) \in \Omega_{N0}$, then we have to distinguish two possibilities.

II.1.1 If $(V(x^{\varphi}_{t_1(\varphi)}))_1 \in A_N$ where $A_N$ is the nowhere dense zero set defined in LEMMA 2.3.8, then the trajectory $\gamma_\varphi$ remains in $\Omega_N$ from that moment on.

II.1.2 In case $(V(x^{\varphi}_{t_1(\varphi)}))_1 \in \Omega_{N01} \setminus A_N$ the trajectory $\gamma_\varphi$ reaches the set $\Omega_{N1}$ at some time $t' \in (t_1(\varphi); +\infty)$, due to LEMMA 2.3.8.

II.2 If $V(x^{\varphi}_{t_1(\varphi)}) \in \Omega_{N1}$, then set $t' := t_1(\varphi)$.

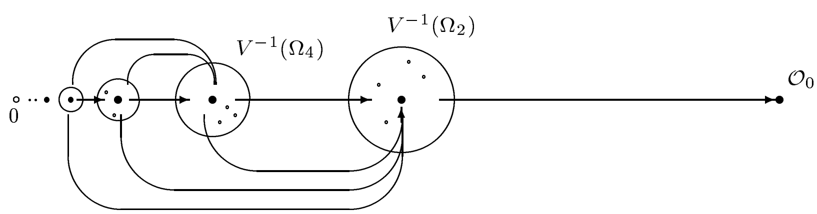
Consequently, in cases II.1.2 and II.2 there is a $t'' \in [t'; +\infty)$ such that

$$x^{\varphi}_{t''} \in \mathcal{Z}_0 \quad \text{ and } V(x^{\varphi}_{t''}) \in \Omega_l \ ,$$

where $l \in [0; N-2] \cap (2\mathbb{N}_0)$, as follows from REMARK 2.3.4.

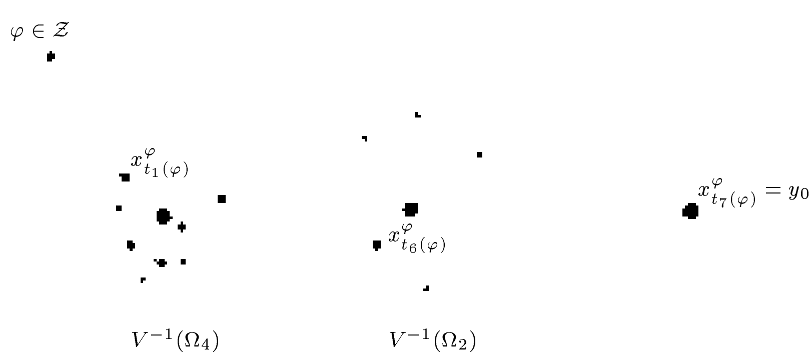
Observe that almost all trajectories $\gamma_\varphi$, $\varphi \in \mathcal{Z}$, eventually merge into one of the periodic orbits $\mathcal{O}_N$ for some $N \in 2\mathbb{N}_0$. Furthermore, there is only a nowhere dense zero set of initial values in each level set $V^{-1}(\Omega_N)$, $N \in 2\mathbb{N}$, which stays on that level.

We may visualize the dynamics in the set $\mathcal{Z}_0$ (and, thus, in $\mathcal{Z}$) roughly as follows:




Herein the disks correspond to the sets $V^{-1}(\Omega_N)$, $N \in 2\mathbb{N}$, where the centers of these disks represent the orbits $\mathcal{O}_N$ of the periodic solutions $x^{(N)}$. The other points in each disk represent those trajectories that remain in $V^{-1}(\Omega_N)$. Finally, the arrows indicate the action of $R$ in $\mathcal{Z}_0$.

For the example given in the figure on page 49, one can draw the following picture to demonstrate the action of the semiflow on $\mathcal{Z}$ (or $\mathcal{Z}_0$, respectively):




Note that the orbits $\mathcal{O}_N$ (and, therefore, the disks) approach a "hole" in $\mathcal{Z}_0$, namely $0 \notin \mathcal{Z}_0$.

If the conjecture $A_N = \{v^{(N)}\}$ is true, then we can remove the additional points in each disk in the above figures. In this case one can also hope to prove that the domain of attraction of $\mathcal{O}_0$ is open and dense in $\mathcal{Z}_0$.

## 2.4 The stable sets of the non-trivial steady states

Now, we address to the question of a description of the remaining part of $\mathcal{B}$ which is formed by the stable sets of the non-trivial stationary points $u_j$, $j \in \{-,+\}$. Our aim is to gain a deeper understanding of the geometry of these sets that form in some sense the "boundary" of the set $\mathcal{B}$.

For simplicity, we will only consider the stable set $W^s(u_+)$ throughout the whole section since the treatment is the same for $W^s(u_-)$. Furthermore, remind that due to the oddness of $-a\,\mathrm{sign}$ and LEMMA 2.2.3(1) we have $W^s(u_-) = -W^s(u_+)$ since $u_+ = -u_-$.

As in the smooth case one would like to start with the linearization at the steady state $u_+$ in order to understand the local behaviour of the semiflow near this equilibrium. Unfortunately, the fact that $X$ is not a linear space causes some difficulties which will be overcome by a formal affine phase space decomposition.

### A. A formal affine phase space decomposition

We have already seen in Section 2.1 that $F_{-a\,\mathrm{sign}}$ is a continuous semiflow on $X$ which is differentiable at the steady state $u_+$. In this subsection we want to describe the linearization at $u_+$ in more detail in order to derive some kind of an affine phase space decomposition similar to the results in Section 1.3.

Using the translation

$$z := x - u_+$$

and setting

$$g := -a\,\mathrm{sign}(\cdot - \frac{a}{\mu}) + a\,\mathrm{sign}(-\frac{a}{\mu})$$

we obtain – since $g$ is continuously differentiable in a sufficiently small neighborhood of $z = 0$ and $g'(0) = 0$ (cf. also DEFINITION 1.5.3 as well as REMARK 1.5.2) – as linearized equation

$$\begin{cases} \dot{z}(t) &= -\mu z(t) \quad , \quad t \in \mathbb{R}^+, \\ z_0 &= \psi \in X_+ \quad , \end{cases} \tag{2.18}$$

where

$$X_+ := \{\varphi - u_+ \,:\, \varphi \in X\} = \left\{\varphi \in C \,:\, |\varphi^{-1}(-\xi^+)| < \infty\right\} \;.$$

Clearly, (2.18) does not only generate a continuous semiflow on $X_+$ but also on the vector space $C \supset X_+$. Therefore, we may interpret (2.18) as an initial value problem for a linear differential delay equation in $C$ restricting the range of initial values to $X_+$.

According to Section 1.3 we find the well-known decomposition

$$C = Q \oplus P$$

of the linear space $C \supset X_+$, where

$$P := \left\{\psi(0)e^{-\mu\cdot} \ : \ \psi \in C\right\} = \mathbb{R} \cdot e^{-\mu\cdot} \quad \text{and} \quad Q := \left\{\psi - \psi(0)e^{-\mu\cdot} \ : \ \psi \in C\right\} \ .$$

Notice that $Q$ has codimension 1, and that the linear projection onto $Q$ along $P$ is explicitly given by

$$\mathrm{Pr}_Q : C \ni \varphi \mapsto \varphi - \varphi(0)e^{-\mu\cdot} \in Q \ .$$

We want to stretch the fact that it was necessary to leave $X_+$ for the above considerations since $X_+$ is obviously not a vector space. In order to return to the phase space $X_+$ we introduce a *formal* decomposition with respect to the above vector space spectral decomposition in the following way.

Let any $\psi \in X_+ \subset C$ be given. Due to the spectral decomposition of $C$ there exist uniquely defined $\varphi \in P$ and $\chi \in Q$ such that $\psi = \varphi + \chi = \mathrm{Pr}_P(\psi) + \mathrm{Pr}_Q(\psi)$, namely,

$$\varphi = \psi(0)e^{-\mu\cdot} \quad \text{and} \quad \chi = \psi - \psi(0)e^{-\mu\cdot} \quad \text{with } \psi \in X_+ \ .$$

Thus, we may write suggestively

$$X_+ = P_+ \oplus Q_+ \ , \tag{2.19}$$

defining

$$P_+ := \mathrm{Pr}_P(X_+) = P \quad \text{and} \quad Q_+ := \mathrm{Pr}_Q(X_+) = \left\{\varphi - \varphi(0)e^{-\mu\cdot} \ : \ \varphi \in X_+\right\} \ ,$$

where we call (2.19) the ***formal decomposition*** of $X_+$ with respect to (2.18).

Since we prefer to work in $X$ instead of $X_+$ we have to decompose $X = u_+ + X_+$ as a sum of two formal "affine" subspaces,

$$X = (u_+ + P_+) \oplus (u_+ + Q_+) = u_+ + (P_+ \oplus Q_+) \ , \tag{2.20}$$

which means that for every $\varphi \in X$ there exists a uniquely defined $\psi \in X_+$ such that

$$\varphi = u_+ + \psi = u_+ + \psi(0)e^{-\mu\cdot} + (\psi - \psi(0)e^{-\mu\cdot}) \in u_+ + (P_+ \oplus Q_+) \ .$$

For this reason we call (2.20) the ***formal affine decomposition*** of our phase space $X$ (with respect to the linearization at $u_+$). This is the setting in which we will work for the remainder of this section.

A standard approach to obtain at least a local description of a stable (or unstable) set of a hyperbolic equilibrium is to try to find a graph representation over the underlying linear stable (or unstable) manifold of the linearization at the equilibrium (see, e.g., DIEKMANN *et al.* [16, Chapter VIII] or HALE *et al.* [26, Chapter 10]). For this purpose let us define the formal projection

$$\mathrm{Pr}_{u_++Q_+} : X \ni \varphi \mapsto u_+ + \mathrm{Pr}_Q(\varphi - u_+) \in u_+ + Q_+$$

onto the formal affine space $u_+ + Q_+$ of $X$ which corresponds to the mentioned stable set of the linearization at $u_+$.

Inspired by Section 3 of KRISZTIN, WALTHER and WU [33, pp. 17–21] it is tempting to ask for the properties of this formal affine projection which will be the subject of the following subsections.

## B. Non-injectivity of $\mathrm{Pr}_{u_++Q_+}$ on $W^s(u_+)$

Recall from COROLLARY 2.3.2 that for every $\varphi \in W^s(u_+)$ the trajectory

$$\mathbb{R}_0^+ \ni t \mapsto x_t^\varphi \in X$$

enters $W_1^+$ (for the definition of $W_1^+$ see page 27) within one time unit. In other words, there exists a map which associates with each $\varphi \in W^s(u_+)$ its first entry time to $W_1^+$,

$$t_0 : W^s(u_+) \ni \varphi \mapsto \inf\left\{t \in [0;1) \ : \ x_t^\varphi \in W_1^+\right\} \in [0;1) \ .$$

Clearly, this map is surjective, since for every $\tau \in t_0(W^s(u_+)) = [0;1)$ *every* initial value $\varphi \in X$ satisfying

$$\begin{cases} \varphi(t) < 0 & , \quad t \in [-1;-1+\tau) \ , \\ \varphi(t) = 0 & , \quad t = -1+\tau \ , \\ \varphi(t) > 0 & , \quad t \in (-1+\tau;0) \ , \\ \varphi(t) = -\frac{a}{\mu}(2e^{\mu\tau}-1) & , \quad t = 0 \ , \end{cases} \tag{2.21}$$

gives rise to $x_\tau^\varphi \in W_1^+$ by (2.4), such that $\varphi \in W^s(u_+)$ due to COROLLARY 2.3.2, and $t_0(\varphi) = \tau$.

It is of importance to recognize once again that a solution $x^\varphi$ of (2.1) is uniquely defined by the sign distribution of $\varphi$ in $(-1;0)$ and the value $\varphi(0)$, i.e. that the particular shape of the graph of $\varphi \in X$ in the intervals $[-1;-1+\tau)$ and $(-1+\tau;0)$ satisfying (2.21) has no influence on the fact that $\varphi \in W^s(u_+)$. This "high degree of non-injectivity" of the semiflow $F_{-a\,\mathrm{sign}}$ enables us to prove

**Remark 2.4.1** *The restricted projection* $\mathrm{Pr}_{u_+ + Q_+}\Big|_{W^s(u_+)}$ *of* $W^s(u_+)$ *onto* $u_+ + Q_+$ *is* not *injective.*

Proof: We have to show the existence of $\varphi$ and $\psi$ in $W^s(u_+)$ with

$$\mathrm{Pr}_{u_+ + Q_+}(\varphi) = \mathrm{Pr}_{u_+ + Q_+}(\psi) \quad \text{but} \quad \varphi \neq \psi \ .$$

This means $\varphi - \psi \in \ker \mathrm{Pr}_{Q_+} \setminus \{0\} = P_+ \setminus \{0\} = (\mathbb{R} \setminus \{0\}) e^{-\mu \cdot}$. Hence, we are to construct $\varphi$ and $\psi$ in $W^s(u_+)$ such that

$$\varphi - \psi = r e^{-\mu \cdot}$$

holds for some $r \in \mathbb{R} \setminus \{0\}$.
Fix any $\tau \in (0; 1)$ and choose for this $\tau$ a $\varphi \in X$ satisfying (2.21) and, additionally,

$$\frac{a}{\mu}(1 - e^{\mu\tau}) < \varphi(t) < 0 \quad \text{for all} \quad t \in [-1; -1 + \tau) \ , \tag{$*$}$$

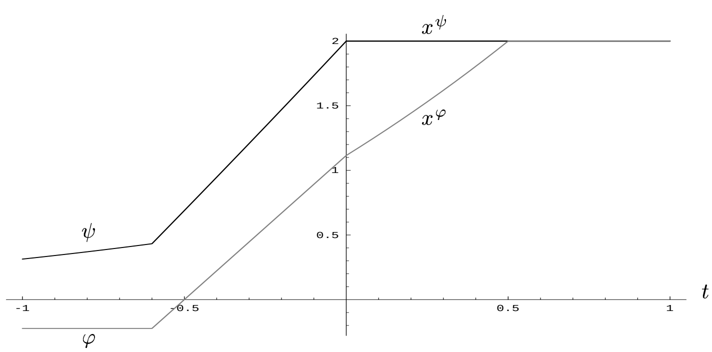
such that $\varphi \in W^s(u_+)$. Now, let us choose

$$r := -\frac{a}{\mu} - \varphi(0) = -2\frac{a}{\mu}(1 - e^{\mu\tau}) > 0 \ ,$$

and set

$$\psi := \varphi + r e^{-\mu \cdot} \ .$$

Thus, $\psi$ satisfies $\psi > 0$ on $[-1; 0]$ due to $(*)$ and the choice of $r$, and $\psi(0) = \varphi(0) + r = -\frac{a}{\mu}$ such that $\psi \in W_1^+ \subset W^s(u_+)$.
Explicit examples of such $\varphi$ and $\psi$ can easily be constructed, as the next figure indicates.




$\square$

Therefore, the set $W^s(u_+)$ is "not flat enough" with respect to the projection onto the formal affine subspace $u_+ + Q_+$ to enable a representation as a graph over $\mathrm{Pr}_{u_+ + Q_+}(W^s(u_+))$. The reason for this lies in the non-injectivity of the nonlinear semiflow.

Analyzing the proof we see that $\varphi$ and $\psi$ given there are the initial values of trajectories which enter $W_1^+$ at *different* times. It might be interesting to ask whether there may also exist $\varphi$ and $\psi$ in $W^s(u_+)$ such that $t_0(\varphi) = t_0(\psi)$ and $\varphi - \psi \in P_+ \setminus \{0\}$.

For $\tau \in [0;1)$ let us define

$$M_\tau^+ := t_0^{-1}(\tau) = \{\varphi \in W^s(u_+) \ : t_0(\varphi) = \tau\} \ ,$$

such that

$$W^s(u_+) = \bigcup_{\tau \in [0;1)} M_\tau^+ = \bigcup_{\tau \in [0;1)} t_0^{-1}(\tau) \ ,$$

i.e. $W^s(u_+)$ is the disjoint union of its ***fibers*** with respect to $t_0$. Note that this could also be interpreted as a partition of $W^s(u_+)$ with respect to the equivalence relation "$\approx$" on $W^s(u_+)$ given by

$$\varphi \approx \psi : \Longleftrightarrow \ t_0(\varphi) = t_0(\psi) \ .$$

**REMARK 2.4.2** *For every $\tau \in [0;1)$, $\mathrm{Pr}_{u_+ + Q_+}\Big|_{M_\tau^+}$ is injective on $M_\tau^+$.*

PROOF: Let $\varphi$ and $\psi$ in $W^s(u_+)$ be given with $t_0(\varphi) = t_0(\psi) = \tau \in [0;1)$ such that

$$\mathrm{Pr}_{u_+ + Q_+}(\varphi) = \mathrm{Pr}_{u_+ + Q_+}(\psi) \ .$$

Then, $\varphi - \psi = re^{-\mu \cdot}$ for some $r \in \mathbb{R}$, and we have to show $r = 0$ in order to prove the injectivity of $\mathrm{Pr}_{u_+ + Q_+}\Big|_{M_\tau^+}$ on $M_\tau^+$.

1. If $\tau = 0$, then $\varphi(0) = \psi(0) = \xi^+$ by definition of $W_1^+$ such that $r = \varphi(0) - \psi(0) = 0$ implies the injectivity of $\mathrm{Pr}_{u_+ + Q_+}\Big|_{M_\tau^+}$ on $M_0^+ = W_1^+$.

2. For $\tau \in (0;1)$ we argue as follows:

   2.1 Let $\chi \in \{\varphi, \psi\}$. By definition of $\tau := t_0(\chi)$ and continuity of $x^\chi$ there is an $\varepsilon \in (0; \min\{\tau, 1-\tau\})$ such that

   $$x^\chi(t) \in (0;\xi^+) \quad \text{ for all } t \in (\tau - \varepsilon; \tau) \subset (0;\tau)$$

   and $x^\chi(t) = \xi^+$ for all $t \in (\tau; \tau + \varepsilon) \subset (\tau; 1]$. Therefore, $\chi \in X$,

   $$\dot{x}^\chi(t) > 0, \quad \text{ and } \quad x^\chi(t) \in (0;\xi^+)$$

   yields $x^\chi(t-1) = \chi(\tau - 1) < 0$ for all $t \in (\tau - \varepsilon'; \tau)$, $\varepsilon' \in (0;\varepsilon)$, by virtue of (2.1) while

   $$\dot{x}^\chi(t) = 0 \quad \text{ and } \quad x^\chi(t) = \xi^+ = -\frac{a}{\mu}$$

   implies $x^\chi(t-1) = \chi(t-1) > 0$ for all $t \in (\tau; \tau + \varepsilon')$ by (2.1). Consequently, the continuity of $\chi$ gives $x^\chi(\tau - 1) = \chi(\tau - 1) = 0$.

2.2 It is evident from step 2.1 that $\varphi(\tau-1) = 0 = \psi(\tau-1)$ because of $t_0(\varphi) = \tau = t_0(\psi)$. Hence,

$$\varphi(\tau-1) - \psi(\tau-1) = re^{-\mu(\tau-1)} = 0$$

gives $r = 0$.

$\square$

Although $\mathrm{Pr}_{u_+ + Q_+}$ is not injective on $W^s(u_+)$, it is injective on each of the $t_0$-fibers of $W^s(u_+)$. On the other hand, we can still ask whether the projection of $W^s(u_+)$ onto $u_+ + Q_+$ is already $u_+ + Q_+$.

Up to this point we did not have to bother about

$$u_+ + Q_+ = u_+ + \mathrm{Pr}_Q(X_+) \subset C \ .$$

But in order to use the results of the previous sections we are to modify this "crude approach" in order to guarantee

$$u_+ + \chi + se^{-\mu\cdot} \in X$$

for all $s \in \mathbb{R}$ for any given $\chi$ which "stems" from $(u_+ + Q_+) \cap X$ in a sense to be described in detail now.

First, we introduce the equivalence relation "$\sim$" on $X$ by

$$\psi \sim \varphi \ :\Longleftrightarrow \ [\mathrm{sign} \circ \psi = \mathrm{sign} \circ \varphi \quad \text{and} \quad \psi(0) = \varphi(0)]$$

for $\{\varphi, \psi\} \subset X$, and denote the equivalence class of any $\varphi \in X$ by

$$[\varphi] := \{\psi \in X \ : \ \varphi \sim \psi\} \ .$$

Clearly, given $\varphi \in X$ we recall from Section 2.1 that

$$x^\psi(t) = x^\varphi(t) \quad \text{ for all } t \in \mathbb{R}_0^+ \text{ and all } \psi \in [\varphi] \ .$$

From this point of view every $[\varphi] \in \{[\psi] \ : \ \psi \in X\} = [X]$ gives rise to a single "solution" $x^{[\varphi]}\Big|_{\mathbb{R}_0^+} := x^\psi\Big|_{\mathbb{R}_0^+}$ of (2.1) given by the trace of the solution $x^\psi\Big|_{\mathbb{R}_0^+}$ of *any* of its representatives $\psi \in [\varphi]$. Keeping this in mind we return to our original problem.

Since we are only interested in a description of the dynamics of (2.1) on $X$, we prefer to consider only that part of the set $u_+ + Q_+$ that lies in $X$, i.e. $(u_+ + Q_+) \cap X$. Defining

$$u_+ + \widetilde{Q_+} := [(u_+ + Q_+) \cap X] = \{[\psi] \ : \ \psi \in \ (u_+ + Q_+) \cap X\} \, ,$$

it is easy to see that for any $u_+ + \varphi \in (u_+ + Q_+) \cap X$ there exists a representative $u_+ + \chi \in X$ of $[u_+ + \varphi]$ such that

$$u_+ + \chi + se^{-\mu\cdot} \in X \quad \text{ for all } s \in \mathbb{R}$$

(choose, e.g., an appropriate piecewise linear function $g =: u_+ + \chi \in X$ with "saw-tooth" graph such that $g(0) = -\frac{a}{\mu} + \psi(0)$ and $\mathrm{sign} \circ g = \mathrm{sign} \circ (u_+ + \psi)$ on $[-1; 0]$).

## C. The image of $W^s(u_+)$ under $\mathrm{Pr}_{u_+ + Q_+}$

Using the elementary machinery developed in Sections 2.1–2.3 we can show

**PROPOSITION 2.4.1** $[\mathrm{Pr}_{u_+ + Q_+}(W^s(u_+)) \cap X] \supset u_+ + \widetilde{Q_+}$.

PROOF: For simplicity we denote $F_{-a\,\mathrm{sign}}$ by $F$ throughout this proof. Let $u_+ + \psi \in u_+ + \widetilde{Q_+}$ be given and choose a representative $u_+ + \chi \in [u_+ + \psi]$ such that

$$\varphi_\sigma := u_+ + \chi + \sigma\, e^{-\mu\cdot} \in X \quad \text{ for all } \sigma \in \mathbb{R}.$$

In order to prove the assertion we have to show the existence of a real number $\alpha = \alpha(\chi)$ such that

$$\varphi_\alpha = u_+ + \chi + \alpha\, e^{-\mu\cdot} \in W^s(u_+)$$

because of $[\mathrm{Pr}_{u_+ + Q_+}(\varphi_\alpha)] = [u_+ + \chi] = [u_+ + \psi]$.

1. There exists a $\sigma \in \mathbb{R}$ such that $F(0, \varphi_\sigma) \in W_3$ as follows from the definition of $W_3$ (on page 41) choosing $\sigma \in \mathbb{R}$ such that

$$-\frac{a}{\mu} + \chi(0) + \sigma \cdot 1 \in \left( \frac{a}{\mu}(2e^\mu - 1); -\frac{a}{\mu}(2e^\mu - 1) \right) .$$

2. There exist $\sigma_j \in \mathbb{R}$, $j \in \{-, +\}$, such that $F(t, \varphi_{\sigma_j}) \in \mathcal{E}^j$ for some $t \in [0; 1]$. To see this, take any $\varepsilon > 0$ and set

$$\sigma_0 := \max \left\{ \Big| \min_{t \in [-1;0]} (u_+ + \chi)(t) \Big|, \Big| \max_{t \in [-1;0]} (u_+ + \chi)(t) \Big| \right\} + \varepsilon \, .$$

   Consequently, $F(0, \varphi_{\sigma_0}) \in W_2^+$ and $F(0, \varphi_{-\sigma_0}) \in W_2^-$ which gives the assertion with $\sigma_- := -\sigma_0$ and $\sigma_+ := \sigma_0$.

3. Now we can introduce the following sets.

$$\begin{aligned} A_+ &:= \left\{ \sigma \in \mathbb{R} \ : \ \exists\, t \in [0;1] \text{ such that } F(t, \varphi_\sigma) \in W_2^+ \right\} \ , \\ A_- &:= \left\{ \sigma \in \mathbb{R} \ : \ \exists\, t \in [0;1] \text{ such that } F(t, \varphi_\sigma) \in W_2^- \right\} \ , \\ A_o &:= \left\{ \sigma \in \mathbb{R} \ : \ \exists\, t \in \mathbb{R}_0^+ \text{ such that } F(t, \varphi_\sigma) \in W_3 \right\} \ , \end{aligned}$$

   and let $\mathcal{A} := \{A_+, A_-, A_o\}$. We need some elementary facts about the elements of $\mathcal{A}$. Let $A \in \mathcal{A}$ be given.

   3.1 The set $A$ is open, since $W_2^j$, $j \in \{-, +\}$, and $W_3$ are open subsets of $X$:

3.1.1 Let $\varphi \in W_2^+$ be given and fix any $t_\varphi \in [0;1]$ with $x^\varphi(t_\varphi) > -\frac{a}{\mu}$. Now, the continuous dependence on the initial value $\varphi \in X$ yields that we can find a $\delta \in \mathbb{R}^+$ for given $\varepsilon \in (0; x^\varphi(t_\varphi) + \frac{a}{\mu})$ with

$$|x^\varphi(t) - x^\psi(t)| < \varepsilon \quad \text{for all } t \in [0; t_\varphi] \text{ and all } \psi \in U_\delta(\varphi) \ .$$

Therefore, $x^\psi(t_\varphi) > x^\varphi(t_\varphi) - \varepsilon > x^\varphi(t_\varphi) - x^\varphi(t_\varphi) - \frac{a}{\mu} = -\frac{a}{\mu}$ for all $\psi \in U_\delta(\varphi)$ which yields $U_\delta(\varphi) \subset W_2^+$ and, hence, the openness of $W_2^+$.

3.1.2 Analogous arguments as in 3.1.1 yield that $W_2^-$ is open, too.

3.1.3 Let $\varphi \in W_3$ be given. Choosing $\delta \in (0; |-\frac{a}{\mu}(2e^\mu - 1) - |\varphi(0)||)$ yields $U_\delta(\varphi) \subset W_3$ which shows the openness of $W_3$.

3.2 Step 1. and step 2. show that $A \neq \emptyset$.

3.3 If $B \in \mathcal{A} \setminus \{A\}$, then $B \cap A = \emptyset$.
To see this we prove two auxiliary statements first.

3.3.1 Let $j \in \{-, +\}$. Then we have $\sigma \in A_j$ if and only if $\varphi_\sigma \in \mathcal{E}^j$.
If $\sigma \in A_j$, then $\varphi_\sigma \in \mathcal{E}^j$ by Lemma 2.3.1(1). On the other hand, let $\varphi_\sigma \in \mathcal{E}^j$. Then the analogue of Lemma 1.4.1 implies the existence of a $t \in \mathbb{R}$ such that $|x_t^{\varphi_\sigma}(0)| > -\frac{a}{\mu}$. Now, we infer $t \in [0;1]$ and $x_t^{\varphi_\sigma} \in W_2^j$ as in the proof of Lemma 2.3.1 . Thus, $\sigma \in A_j$.

3.3.2 We have $\sigma \in A_o$ if and only if $\varphi_\sigma \in \mathcal{Z} \subset \mathcal{B}$.
If $\sigma \in A_o$, then there exists a $t \in \mathbb{R}_0^+$ with $x_t^{\varphi_\sigma} \in W_3$. Recalling $W_3 \subset \mathcal{Z}$ from Corollary 2.3.4) implies $\varphi_\sigma \in \mathcal{Z}$. Now, let $\varphi_\sigma \in \mathcal{Z}$. We infer from Lemma 2.3.2 that there is an unbounded sequence $(\tau_n(\varphi_\sigma))_{n\in\mathbb{N}}$ of zeros of $x^{\varphi_\sigma}$ in $\mathbb{R}^+$. Therefore, we obtain $0 = |x^{\varphi_\sigma}(\tau_1(\varphi_\sigma))| = |x^{\varphi_\sigma}_{\tau_1(\varphi_\sigma)}(0)| < -\frac{a}{\mu}(2e^\mu - 1)$ such that $x^{\varphi_\sigma}_{\tau_1(\varphi_\sigma)} \in W_3$ for $t = \tau_1(\varphi_\sigma)$ and, thus, $\sigma \in A_o$.

Hence, 3.3.1 and 3.3.2 together with

$$X = \mathcal{E}^- \,\dot\cup\, \mathcal{B} \,\dot\cup\, \mathcal{E}^+$$

yield $B \cap A = \emptyset$ for any $A \in \mathcal{A}$, $B \in \mathcal{A} \setminus \{A\}$.

Thus, the connectedness of $\mathbb{R}$ shows that

$$\mathbb{R} \setminus (A_+ \cup A_- \cup A_o) =: M \neq \emptyset \ .$$

Now,

$$X = \mathcal{E}^- \,\dot\cup\, W^s(u_-) \,\dot\cup\, \mathcal{Z} \,\dot\cup\, W^s(u_+) \,\dot\cup\, \mathcal{E}^+$$

and the fact that $\varphi_\sigma \in \mathcal{E}^- \,\dot\cup\, \mathcal{Z} \,\dot\cup\, \mathcal{E}^+$ if and only if $\sigma \in A_- \dot\cup A_o \dot\cup A_+$ (cf. step 3.3) yield that the elements of $M$ have the following property.

3.4 If $\sigma \in M$, then $\varphi_\sigma = u_+ + \chi + \sigma \cdot e^{-\mu \cdot} \in W^s(u_-) \,\dot\cup\, W^s(u_+)$.

4. We claim that $A_+$ is bounded from below.
*Otherwise*, there should exist $\alpha \in A_+$ with $\alpha < \alpha_\# := 2\frac{a}{\mu} - \|\chi\|$; for these $\alpha$

$$\varphi_\alpha < \varphi_{\alpha_\#} = u_+ + \chi + (2\frac{a}{\mu} - \|\chi\|) \cdot e^{-\mu \cdot} < \frac{a}{\mu}\mathbb{I} = u_-$$

would imply $\varphi_\alpha \in \mathcal{E}^-$ *contradicting* $\mathcal{E}^+ \cap \mathcal{E}^- = \emptyset$. Hence, $A_+ \subset [\alpha_\#; +\infty)$.
Consequently, we can define

$$\alpha := \inf A_+ \; .$$

The next step is devoted to prove that $\alpha \in M$.

5. We have $\alpha \in \mathbb{R} \setminus (A_+ \cup A_- \cup A_o)$.

5.1 Since $A_+$ is open (cf. step 3.1) we have

$$\alpha = \inf A_+ \notin A_+ \; .$$

5.2 As a consequence of step 5.1 and the definition of $\alpha$, there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $A_+$ with $\alpha_n > \alpha$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \alpha_n = \alpha$.
Observe that

$$\|\varphi_\alpha - \varphi_{\alpha_n}\| = |\alpha - \alpha_n| = \alpha_n - \alpha \; ,$$

such that $\varphi_{\alpha_n} \in U_\delta(\varphi_\alpha)$ whenever $\alpha_n - \alpha < \delta$.

5.3 *If* $\alpha$ belongs to $A_-$, then the openness of $A_-$ (cf. step 3.1) would imply the existence of a $\delta \in \mathbb{R}^+$ such that $U_\delta(\alpha) \subset A_-$. For this $\delta \in \mathbb{R}^+$ we could find a $n_\delta \in \mathbb{N}$ with $\alpha_n \in A_+$ for all $n \in \mathbb{N} \cap [n_\delta; +\infty)$ and $|\alpha - \alpha_n| = \alpha_n - \alpha < \delta$ by step 5.2. Now, this would give $\alpha_n \in A_- \cap A_+$ for all $n \in \mathbb{N} \cap [n_\delta; +\infty)$ which *contradicts* the fact that $A_+$ and $A_-$ are disjoint as we know from step 3.3. Therefore, $\alpha \notin A_-$.

5.4 Finally, $\alpha \notin A_o$ since *otherwise* the openness of $A_o$ will imply a *contradiction* to the disjointness of $A_o$ and $A_+$ again (similar to the previous step).

6. Now, we have $\alpha \in M$ by step 5., such that 3.4 implies that $\varphi_\alpha \in W^s(u_-) \,\dot{\cup}\, W^s(u_+)$. Clearly, we must have $\varphi_\alpha \in W^s(u_+)$ since the other case will obviously lead to a contradiction to the continuity of the semiflow because of Lemma 2.3.1 as we will show in the sequel to complete the proof.
If we *assume*

$$\varphi_\alpha = u_+ + \chi + \alpha\, e^{-\mu \cdot} \in W^s(u_-) \; ,$$

there would exist a $t_\alpha \in [0; 1]$ such that $x^{\varphi_\alpha}_{t_\alpha} \in W_1^-$ by Corollary 2.3.2. Due to the continuous dependence on the initial value $\varphi_\alpha$ there exists for every $\varepsilon \in (0; -2\frac{a}{\mu})$ a $\delta = \delta(t_\alpha + 1, \varepsilon) > 0$ such that the estimate

$$x^\psi(t) \leq x^{\varphi_\alpha}(t) + \varepsilon$$

holds on $[0; t_\alpha + 1]$ for all $\psi \in U_\delta(\varphi_\alpha)$.
By step 5.2 there exists a $n_\delta \in \mathbb{N}$ such that $\alpha_n - \alpha < \delta$ for all $n \in \mathbb{N} \cap [n_\delta; +\infty)$. Note that $\alpha_n \in A_+$ for all $n \in \mathbb{N} \cap [n_\delta; +\infty)$. Consequently, there exists a $t'_n \in [0;1]$ with $x^{\varphi_{\alpha_n}}(t) > -\frac{a}{\mu}$ for all $t \in [t'_n; +\infty) \supset [1; +\infty)$ for every $n \in \mathbb{N} \cap [n_\delta; +\infty)$ as a consequence of LEMMA 2.3.1. Hence, $x^{\varphi_{\alpha_n}}(t) > -\frac{a}{\mu}$ for all $t \in [1; +\infty)$ and all $n \in \mathbb{N} \cap [n_\delta; +\infty)$.
In fact, we arrive at the *contradiction*

$$-\frac{a}{\mu} < x^{\varphi_{\alpha_n}}(t) \leq x^{\varphi_\alpha}(t) + \varepsilon < \frac{a}{\mu} - 2\frac{a}{\mu} = -\frac{a}{\mu} \quad \text{for all } t \in [\max\{t_\alpha, 1\}; t_\alpha + 1] \ .$$

□

The proof above seems to be long-winded and unnecessarily lengthy. In our opinion this can be justified by the fact that only slight modification will enable us to generalize most parts of the proof to the case of continuous nonlinearities.

Obviously, LEMMA 2.3.1 (as well as its corollaries) and the continuous dependence on the initial value (or, equivalently, the continuity of the semiflow) are the crucial tools that provided the proof.

## D. Geometric description of $\boldsymbol{W^s(u_+)}$

We are now in a position to draw a geometric picture of the stable set of $u_+$. PROPOSITION 2.4.1 yields that for every $u_+ + \psi \in (u_+ + Q_+) \cap X$ there is a $u_+ + \chi \in [u_+ + \psi]$ such that we can find a $s \in \mathbb{R}$ with

$$u_+ + \chi + se^{-\mu \cdot} \in W^s(u_+) \ .$$

Therefore, for every $[u_+ + \psi] \in u_+ + \widetilde{Q_+}$ the set

$$I([u_+ + \psi]) := \left\{ r \in \mathbb{R} \ : \ \exists\, u_+ + \chi \in [u_+ + \psi] \text{ with } u_+ + \chi + re^{-\mu \cdot} \in W^s(u_+) \right\}$$

is non-empty and contains in general more than one element according to REMARK 2.4.1. This enables us to write

$$[W^s(u_+)] = \bigcup_{[u_+ + \psi] \in u_+ + \widetilde{Q_+}} \left( [u_+ + \psi] + I([u_+ + \psi]) e^{-\mu \cdot} \right) \ ,$$

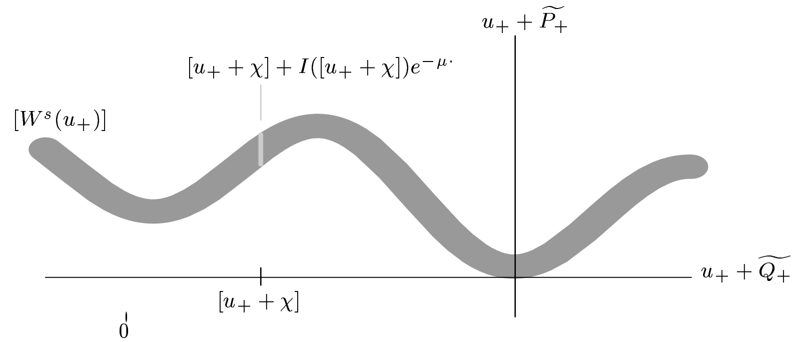
which gives a representation of $[W^s(u_+)]$ over the formal affine space $u_+ + \widetilde{Q_+}$ in terms of a graph of the set-valued function

$$\mathfrak{Sep} : u_+ + \widetilde{Q_+} \ni [u_+ + \psi] \mapsto [u_+ + I([u_+ + \psi]) \cdot e^{-\mu \cdot}] \in \mathfrak{P}(u_+ + \widetilde{P_+}) \ ,$$

such that we may write

$$[W^s(u_+)] = \mathrm{graph}(\mathfrak{Sep}) \ ,$$

and illustrate this in the following figure.




We want to refine this picture in view of Remark 2.4.2: therefore, let us rewrite $I([u_+ + \psi])$ as the disjoint union of all those $t_0$-fibers of $W^s(u_+)$ that intersect with the set $[u_+ + \psi] + I([u_+ + \psi])e^{-\mu\cdot}$. Let $[u_+ + \psi] \in u_+ + \widetilde{Q_+}$ be given. For $\tau \in [0;1)$ we introduce

$$I_\tau([u_+ + \psi]) := \left\{ r \in \mathbb{R} \ : \ \exists\, u_+ + \chi \in [u_+ + \psi] \text{ with } u_+ + \chi + re^{-\mu\cdot} \in M_\tau^+ \right\} \ ,$$

such that

$$I([u_+ + \psi]) = \bigcup_{\tau \in [0;1)} I_\tau([u_+ + \psi]) \ .$$

For $\tau \in [0;1)$, we infer from Remark 2.4.2 that $I_\tau([u_+ + \psi])$ contains at most a single real number, but it could also be the empty set, depending on $[u_+ + \psi]$. Therefore, we define

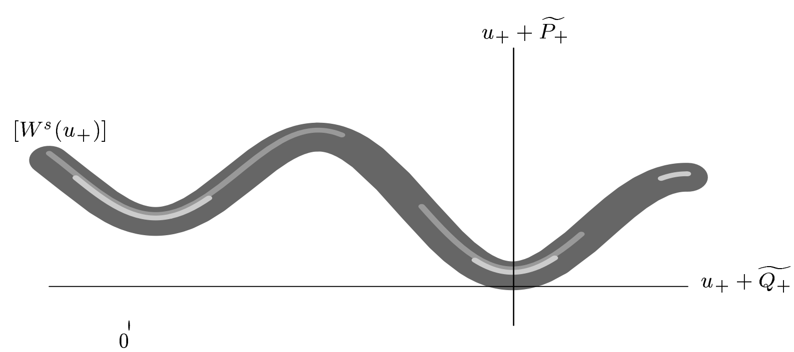
$$T : u_+ + \widetilde{Q_+} \ni [u_+ + \psi] \mapsto \{\tau \in [0;1) \ : \ I_\tau([u_+ + \psi]) \neq \emptyset\} \in \mathfrak{P}(\mathbb{R})$$

which measures the "thickness" of $W^s(u_+)$ over the point $[u_+ + \psi] \in u_+ + \widetilde{Q_+}$. Thus, we have

$$[W^s(u_+)] \ = \bigcup_{[u_+ + \psi] \in u_+ + \widetilde{Q_+}} \ \bigcup_{\tau \in T([u_+ + \psi])} \left( [u_+ + \psi] + I_\tau([u_+ + \psi])e^{-\mu\cdot} \right)$$

where $I_\tau(\cdot)$ is a real number for every $\tau \in T(\cdot)$ such that we can describe $[W^s(u_+)]$ in form of this ***sections of the set-bundle*** over $u_+ + \widetilde{Q_+}$.

This representation is illustrated in the following figure where the different gray levels on $[W^s(u_+)]$ correspond to different sections of the set-bundle.



We conclude this section with a short list of open problems that will initiate further research on this topic.

To begin with, we could ask for any kind of "smoothness properties" of the mapping $\mathfrak{Sep}$. Typically, one introduces the notation of *upper semicontinuity* for set-valued functions (cf., e.g., EISENACK and FENSKE [18, p. 209] or ZEIDLER [75, Section 9.2] and the references therein), and it will be the subject of further research to clarify whether $\mathfrak{Sep}$ is upper semincontinous or, more generally, whether this is the *right* concept of "smoothness" in our setting: In particular, is $[W^s(u_+)]$ "tangential" to $u_+ + \widetilde{Q_+}$ in some sense ?

Furthermore, one may ask for a more detailed description of the sets $I([u_+ + \psi])$ for $[u_+ + \psi] \in u_+ + \widetilde{Q_+}$ which would give a deeper insight into the structure of $[W^s(u_+)]$. Obvious questions could be: Is $I([u_+ + \psi])$ an interval ? If not, does it contain inner points, or can we say anything about its LEBESGUE measure ? Do there exist $[u_+ + \psi] \in u_+ + \widetilde{Q_+}$ such that $I([u_+ + \psi])$ is a CANTOR set ?

Following ZEIDLER [75, p. 463] we call a single-valued map $f : W \to M$ satisfying

$$f(w) \in \mathfrak{F}(w) \quad \text{for all } w \in W$$

a ***selection*** of the set-valued mapping $\mathfrak{F} : W \to \mathfrak{P}(M)$. Naturally, this raises the question: Are the sections of the set-bundle over $u_+ + \widetilde{Q_+}$ continuous (or even smooth) selections of $\mathfrak{Sep}$ or of restrictions of $\mathfrak{Sep}$ ?

## 2.5 Supplementary remarks

This final section is devoted to set the results of this chapter into perspective, and to give, by the way, further references to related work and questions.

As already mentioned, the main part of our presentation in Section 2.1–2.3 follows the lines of DIEKMANN *et. al.* [16, Section XVI.2] and serves as generalization of this work on one hand, as well as a rudimentary model from which we hope to derive information about the dynamics generated by (1.1) for continuous nonlinearities $f$ on the other hand: this will be the content of Chapter 3.

In order to draw a complete picture of the action of the semiflow generated by (2.1) on $\mathcal{Z}$ we had to introduce a discrete LYAPUNOV functional which is adapted from CAO [11]. There are similar (and slightly different) definitions of so called "frequency functions" due to SHUSTIN *et al.* [20, 21, 22, 54, 50]. In fact, the main issue of most of this articles is not to give a detailed description of the behaviour of the semiflow: These are concerned with the question of non-existence of so called *super-high-frequency solutions* (SHFS), and we refer to the work of NUSSBAUM and SHUSTIN [50] or AKIAN and BLIMAN [2] for a state-of-the-art overview.

Summarizing this, we can say that Sections 2.1–2.3 combine the merits of the other approaches: a detailed study of the long-term behaviour of the solutions of (2.1) is provided, and CAO's approach to discrete LYAPUNOV functionals is applied for the first time in a discontinuous setting yielding some results which were not covered by the papers of SHUSTIN *et al.*.

Section 2.4 is completely independent of all problems and questions treated in the literature, and, to the author's knowledge, the first investigation of a stable set for discontinuous differential delay equations. It provides a geometric visualization of the stable set in terms of sections of a set-bundle over the affine subspace $u_+ + \widetilde{Q}_+$ that correspond to the entry times into the set $W_1^+$ which gives rise to further research.

There were two principal goals we wanted to reach with this chapter:

First, we were able to give an almost complete description of the dynamics of a model equation which is of the special feedback type (an instantaneous growth process governed by delayed negative feedback) in which we are interested.

Second, we hope that equation (2.1), as the limiting case of the prototype equations $(1.1)_\alpha$, will display the rudimentary dynamical structures that may occur for (1.1) such that this chapter provides in some way a "program" for the further investigations: In fact, our Chapters 3 and 4 as well as Chapter 5 resemble problems in context of the continuous equation (1.1) which were also investigated in Sections 2.2–2.3 as well as 2.4, respectively, for the discontinuous model equation (2.1).

The first step from the discontinuous limiting equation (2.1) back to the delay equation (1.1) is done in the subsequent chapter, where we will prove the existence of slowly oscillating periodic solutions of equation (1.1) for continuous nonlinearities $f$ which are close to the sign nonlinearity in some special sense.

# 3

# Contracting return maps for a class of differential delay equations

In this chapter we intend to return to the consideration of equations of type

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1.1}$$

for continuous nonlinearities $f : \mathbb{R} \to \mathbb{R}$. We recall briefly that the segment $x_{n+1}^{\varphi,f}$ of a solution $x^{\varphi,f}$ of (1.1) with initial value $x_0^{\varphi,f} = \varphi \in C$ is given by

$$x_{n+1}^{\varphi,f}(t) = x^{\varphi,f}(n+1+t) = x^{\varphi,f}(n)e^{-\mu(t+1)} + \int_n^{n+1+t} e^{-\mu(n+1+t-s)} f(x^{\varphi,f}(s-1))ds \tag{3.1}$$

for every integer $n \in \mathbb{N}_0$ and all $t \in [-1;0]$, as we infer from the variation-of-constants formula (1.4).

Motivated by analogous results of Walther [67, 68] for decay delay equations and numerical experiments, we will take the following approach: in Section 3.1 we define a certain class of bounded continuous odd nonlinearities which are close to

$$g := -a \operatorname{sign}$$

outside a small neighborhood of 0. These nonlinearities $f$ lead to solutions $x^{\varphi,f}$ which are close to the slowly oscillating solution $y$ of (2.1) on some interval containing $[0; z(\mu)]$ for some closed convex set $A$ of initial values $\varphi \in C$. This permits us to prove that the solutions $x^{\varphi,f}$ with initial values $\varphi \in C$ enter $-A$ at some time $q := q(\varphi, f)$ which leads to the definition of a return map $R_f$ in Section 3.2. The fixed points of $R_f$ define slowly oscillating solutions for (1.1). Applying the Schauder Fixed Point Theorem, we can guarantee the existence of slowly oscillating solutions and show in the fourth section some stability properties of the unique periodic solution for nonlinearities for which the return map is a strict contraction.

We will follow the notation of WALTHER [67] throughout this chapter rather closely as far as possible, whereas it turns out that we have to make essential changes concerning some techniques used in [67].

With respect to the results of Chapter 2 let us fix

$$\mu \in (-\log 2; 0)$$

for the remainder of this chapter, i.e. we assume (H1') to be valid.

## 3.1 A class of nonlinearities for (1.1)

The aim of this section is to answer (at least partially) the question

(∗) *If the nonlinearity $f$ in* (1.1) *is – in some sense – close to* $g := -a$ sign, *what do solutions of* (1.1) *have to do with those of* (2.1) *?*

In a way, the results of this section help to understand why we have considered the *discontinuous* limiting equation (2.1) in order to infer more information about the dynamics of (1.1) for *continuous* nonlinearities $f$.

We fix $a \in \mathbb{R}^+$ and remind the reader of the slowly oscillating solution $y = -x^{(0)}$ for the discontinuous model equation (2.1). The properties of $y$ that will be needed throughout this section could be found in REMARK 2.2.2.

**DEFINITION 3.1.1** For $a \in \mathbb{R}^+$ fix $b \in (a; +\infty)$. For every $\beta \in \left(0; -\frac{a}{\mu}\right)$ and every $\varepsilon \in (0; a)$ denote by $\boldsymbol{N(\beta, \varepsilon)}$ the set of all functions $f \in C(\mathbb{R}, \mathbb{R})$ satisfying the following four conditions:

($N_1$) $f(\xi) = -f(-\xi)$ for all $\xi \in \mathbb{R}$,

($N_2$) $|f(\xi)| \leq b$ for all $\xi \in (0; \beta)$,

($N_3$) $|f(\xi) - (-a)| < \varepsilon$ for all $\xi \in [\beta; \infty)$,

($N_4$) The equation $-\mu\xi + f(\xi) = 0$ has exactly two solutions in $\mathbb{R} \setminus [-\beta; \beta]$, denoted as $\xi_f^+ \in (\beta; +\infty)$ and $\xi_f^- = -\xi_f^+ \in (-\infty; -\beta)$, for which we have

$$\frac{a}{\mu} < \xi_f^- < \frac{a}{\mu}(e^{-\mu} - 1) < 0 < -\frac{a}{\mu}(e^{-\mu} - 1) < \xi_f^+ < -\frac{a}{\mu} \; .$$

Hypothesis $(N_1)$ is only included to clarify the investigations and to permit shorter proofs here.

**REMARK 3.1.1** *Assumption $(N_1)$ is not essential for the results of this chapter (e.g., all results hold for $f_1 \in C(\mathbb{R}, \mathbb{R})$ as depicted below, too).*

The last condition, $(N_4)$, assures that the non-trivial stationary solutions of (1.1) with $f \in N(\beta, \varepsilon)$ have values which are larger than the values of the corresponding slowly oscillating periodic solution $y$ of the "nearby" equation (2.1) because of

$$\max_{\mathbb{R}} |y| = -y(1) = -\frac{a}{\mu}(e^{-\mu} - 1) \ .$$

**REMARK 3.1.2** *Condition $(N_4)$ can certainly be replaced by $(N_4')$: the equation*

$$-\mu\xi + f(\xi) = 0$$

*has more than two solutions in $\mathbb{R} \setminus [-\beta; \beta]$, and $\xi_f^{\pm}$ should be understood as*

$$\xi_f^- := \max\{\xi \in (-\infty; -\beta) \ : \ -\mu\xi + f(\xi) = 0\}$$

*and*

$$\xi_f^+ := \min\{\xi \in (\beta; +\infty) \ : \ -\mu\xi + f(\xi) = 0\}$$

*in the above estimate.*

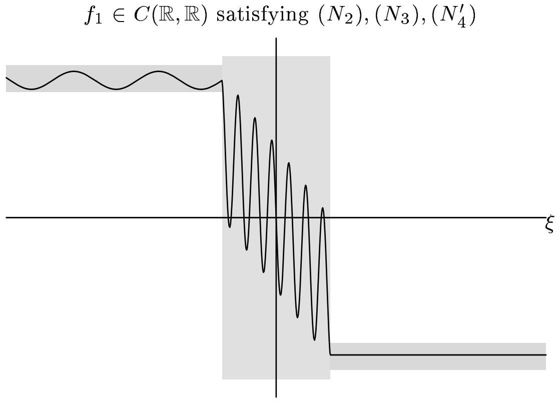
Observe that $M_f = \sup_{\mathbb{R}} |f| \leq \max\{b, a + \varepsilon\}$ for $f \in N(\beta, \varepsilon)$ such that $N(\beta, \varepsilon)$ forms a subset of all continuous nonlinearities which satisfy the hypotheses (H2.3) and (H2.4); for the moment we do not need further smoothness assumptions.

Furthermore, it is noteworthy to mention that the set $N(\beta, \varepsilon)$ contains nonlinearities which

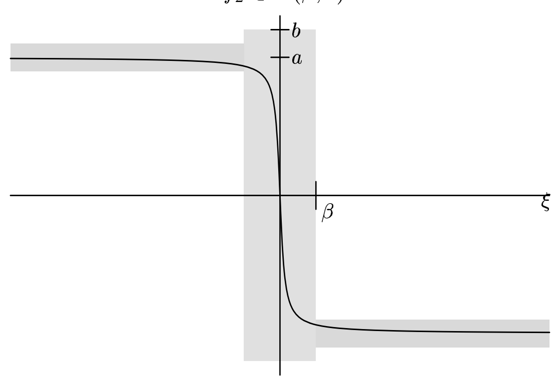
- do *not* satisfy a negative feedback condition (1.2), or
- are *not* monotonic on $\mathbb{R}$, or
- have finitely many stationary states with values in $(-\beta; \beta)$,

as well as the prototype nonlinearities from EXAMPLE 1.1.1 and EXAMPLE 1.1.2 for appropriately chosen parameters.

$f_1 \in C(\mathbb{R}, \mathbb{R})$ satisfying $(N_2), (N_3), (N_4')$




$f_2 \in N(\beta, \varepsilon)$



**DEFINITION 3.1.2** Let $(\beta,\varepsilon) \in (0; -\frac{a}{\mu}) \times (0; a)$ be chosen as in DEFINITION 3.1.1. For $f \in N(\beta,\varepsilon)$ let us define

$$A(\beta) := \left\{ \varphi \in C \ : \ \|\varphi\| \leq -\frac{M_f}{\mu}, \varphi(t) \geq \beta \ \forall t \in [-1; 0], \varphi(0) = \beta \right\} .$$

**REMARK 3.1.3** *It is evident from* DEFINITION 3.1.2 *that $A(\beta)$ is a non-empty, closed, bounded, and convex subset of $C \setminus \{0\}$.*

In contrast to the situation considered by WALTHER in [67] and [68], where $\mu \in \mathbb{R}^+$, a slowly oscillating periodic solution for $\mu \in (-\log 2; 0)$ has an arbitrarily large minimal period depending on $\mu$ since $p_\mu = 2z(\mu) \to +\infty$ as $\mu \searrow -\log 2$ (see LEMMA 2.2.3). Therefore, we introduce the following number since we want to take advantage of the variations-of-constants formula (3.1) using a method of steps.

For $\mu \in (-\log 2; 0)$ let

$$n(\mu) := \begin{cases} \lceil z(\mu) \rceil & , \quad z(\mu) \notin \mathbb{N} \\ \lceil z(\mu) \rceil + \frac{1}{2} & , \quad z(\mu) \in \mathbb{N} \end{cases} , \tag{3.2}$$

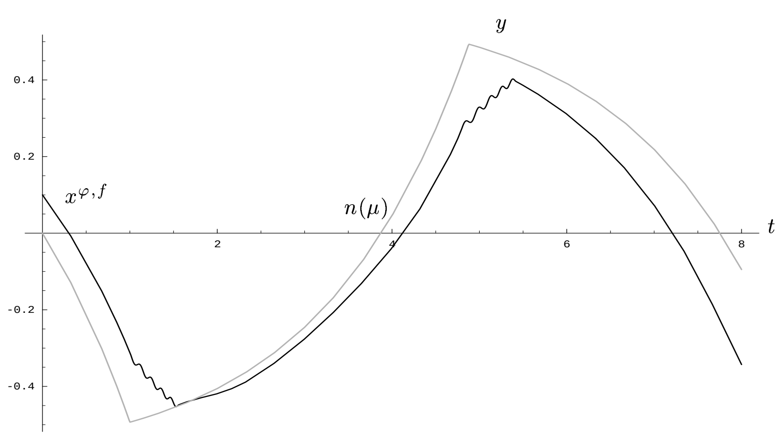
and

$$r(\mu) := \lfloor n(\mu) \rfloor ,$$

where $\lceil \cdot \rceil : \mathbb{R} \ni \xi \mapsto \inf \{\zeta \in \mathbb{Z} \ : \ \xi \leq \zeta\} \in \mathbb{Z}$ and $\lfloor \cdot \rfloor : \mathbb{R} \ni \xi \mapsto \sup \{\zeta \in \mathbb{Z} \ : \ \zeta \leq \xi\} \in \mathbb{Z}$.

By this definition and REMARK 2.2.2 we have $n(\mu) \in [3; +\infty)$, $r(\mu) \in \mathbb{N} \cap [3; +\infty)$, $y_j < 0$ for $j \in \{2, ..., \lfloor n(\mu) - 1 \rfloor\} \cup \{n(\mu) - 1\}$, $y_1 \leq 0$, and $y_{n(\mu)}(0) = y(n(\mu)) > 0$ (cf. the figure below for illustration).

For parameter values $(\beta,\varepsilon)$ close to $(0,0)$, the nonlinearities $f \in N(\beta,\varepsilon)$ are close to $g := -a\,\mathrm{sign}$: furthermore, the solutions $x^{\varphi,f}$ of (1.1) with $f \in N(\beta,\varepsilon)$ and initial values in $A(\beta)$ are close to the slowly oscillating periodic solution $y$ of (2.1) on $[0; n(\mu)]$, as numerical simulations indicate (e.g., we obtain the subsequent picture of $x^{\varphi,f}$ for the nonlinearity $f = f_1$ (depicted on the left hand side of the figure on page 93) and initial datum $\varphi = \beta\mathbb{I} \in A(\beta)$):



The following proposition will explain this observation and gives a precise answer to question $(*)$.

**Proposition 3.1.1** *We have*

$$\lim_{(\beta,\varepsilon)\to(0,0)} \left( \sup_{f\in N(\beta,\varepsilon),\varphi\in A(\beta),t\in[0;n(\mu)]} \left|x^{\varphi,f}(t) - y(t)\right| \right) = 0 \ .$$

Proof:

1. For all $(\beta,\varepsilon) \in (0; -\frac{a}{\mu}) \times (0; a)$, $f \in N(\beta,\varepsilon)$, $\varphi \in A(\beta)$, $t \in [-1; 0]$ we have the estimate

$$\begin{aligned}
|F_f(1,\varphi)(t) \quad - & \quad y_1(t)| = \\
= & \quad \left|\beta e^{-\mu(1+t)} + \int_0^{1+t} e^{-\mu(1+t-s)} f(\varphi(s-1))ds - (0 - \int_0^{1+t} e^{-\mu(1+t-s)} ads)\right| \leq \\
\leq & \quad \beta e^{-\mu} + \left| \int_0^{1+t} e^{\mu s}(f(\varphi(s-1)) - (-a))ds \right| \cdot e^{-\mu} \leq (\beta+\varepsilon)e^{-\mu}
\end{aligned}$$

as a consequence of $(N_3)$. Thus, we obtain

$$\lim_{(\beta,\varepsilon)\to(0,0)} \left( \sup_{f\in N(\beta,\varepsilon),\varphi\in A(\beta)} \|F_f(1,\varphi) - y_1\| \right) = 0 \ .$$

2. Now we claim

$$\lim_{(\beta,\varepsilon,\psi)\to(0,0,y_j)} \left( \sup_{f\in N(\beta,\varepsilon)} \|F_f(1,\psi) - y_{j+1}\| \right) = 0$$

for all $j \in \{1, 2, ..., \lfloor n(\mu) - 1\rfloor\} \cup \{n(\mu) - 1\}$.
Fix $j \in \{1, 2, ..., \lfloor n(\mu) - 1\rfloor\} \cup \{n(\mu) - 1\}$ throughout this part of the proof and recall from Remark 2.2.2 that we have $y_k < 0$ for all $k \in \{2, ..., \lfloor n(\mu) - 1\rfloor\} \cup \{n(\mu) - 1\}$ and $y_1(\tau) < 0$ for all $\tau \in (-1; 0]$ which implies

$$g(y(s-1)) = -a \cdot \text{sign}(y(s-1)) = +a$$

for all $s \in (j; j+1+t]$, $t \in [-1; 0]$.

2.1 Because of

$$
\begin{aligned}
|F_f(1,\psi)(t) - y_{j+1}(t)| \; &= \\
= \Big|\psi(0)e^{-\mu(1+t)} \; &+ \; \int\limits_0^{1+t} e^{-\mu(1+t-s)} f(\psi(s-1))ds \\
&- \; \Big(y(j)e^{-\mu(1+t)} + \int\limits_j^{j+1+t} e^{-\mu(j+1+t-s)}(+a)ds\Big)\Big| \leq \\
\leq \Big(\|\psi - y_j\| \; &+ \; \int\limits_{-1}^{0} |f(\psi(s)) - a|ds\Big) \cdot e^{-\mu}
\end{aligned}
$$

for all $(\beta, \varepsilon) \in (0; -\frac{a}{\mu}) \times (0; a)$, $f \in N(\beta, \varepsilon)$, $\psi \in C$, and $t \in [-1; 0]$, it is sufficient to prove

$$
\sup_{f \in N(\beta,\varepsilon)} \int\limits_{-1}^{0} |f(\psi(t)) - a|dt \to 0 \quad \text{ as } (\beta, \varepsilon, \psi) \to (0, 0, y_j) \; .
$$

2.2 Let $\eta > 0$ be given.
For $\delta = \delta(\eta) \in \Big(0; \frac{\eta}{2(M_f + a)}\Big)$, we obtain due to the boundedness of $f$ for all $(\beta, \varepsilon) \in (0; -\frac{a}{\mu}) \times (0; a)$, $f \in N(\beta, \varepsilon)$, and $\psi \in C$,

$$
\int\limits_{-1}^{-1+\delta} \Big|f(\psi(s)) - a\Big|ds \leq (M_f + a)\delta < \frac{\eta}{2} \; .
$$

Since $y_j < 0$ and $y_j \geq y(1) = \frac{a}{\mu}(e^{-\mu} - 1) > \xi_f^-$ on $[-1 + \delta; 0]$, there exists $\beta(\delta) \in \Big(0; \min\{-\frac{y(\delta)}{2}, y(1) - \xi_f^-\}\Big)$ such that

$$
\xi_f^- + \beta(\delta) < y_j < -2\beta(\delta) \quad \text{ on } [-1 + \delta; 0] \; ,
$$

and for all $\psi \in C$ with $\|\psi - y_j\| < \beta(\delta)$, this yields

$$
\xi_f^- < \psi < -\beta(\delta) \quad \text{ on } [-1 + \delta; 0] \; .
$$

For all $(\beta, \varepsilon) \in (0; \beta(\delta)) \times (0; a)$, $f \in N(\beta, \varepsilon)$, and $\psi \in C$ with $\|\psi - y_j\| < \beta(\delta)$, we get $\xi_f^- < \psi < -\beta$ on $[-1 + \delta; 0]$, and, as a consequence of $|f(\xi) - a| < \varepsilon$ for all $\xi \in (-\infty; -\beta)$,

$$
\int\limits_{-1+\delta}^{0} \Big|f(\psi(s)) - a\Big|ds \leq (1 - \delta)\varepsilon < \varepsilon \; .
$$

Hence we obtain for $(\beta,\varepsilon) \in (0;\beta(\delta)) \times (0;\frac{\eta}{2})$, $f \in N(\beta,\varepsilon)$, $\psi \in U_{\beta(\delta)}(y_j)$, and $j \in \{1, ..., \lfloor n(\mu)-1\rfloor\} \cup \{n(\mu)-1\}$,

$$\int_{-1}^{0} \Big|f(\psi(s)) - a\Big| ds \le \int_{-1}^{-1+\delta} \Big|f(\psi(s)) - a\Big| ds + \int_{-1+\delta}^{0} \Big|f(\psi(s)) - a\Big| ds < \eta \ .$$

3. Finally, let $\eta > 0$ be given.

3.1 We define recursively some variables and notations which we will need in step 3.2. Set $\delta_{n(\mu)-1} := \eta$. In case that $z(\mu) \in \mathbb{N}$, we can apply part 2. to find a $\delta_{\lfloor n(\mu)-1\rfloor} \in (0;\delta_{n(\mu)-1})$ such that

$$\Big\|F_f(1,\psi) - y_{n(\mu)}\Big\| < \delta_{n(\mu)-1}$$

holds for all $\psi \in U_{\delta_{\lfloor n(\mu)-1\rfloor}}(y_{n(\mu)-1})$, $(\beta,\varepsilon) \in (0;\delta_{\lfloor n(\mu)-1\rfloor})^2$, and all $f \in N(\beta,\varepsilon)$. Now, $\delta_{\lfloor n(\mu)-1\rfloor}$ is defined in both cases ($z(\mu) \in \mathbb{N}$ and $z(\mu) \notin \mathbb{N}$) such that we can return to a more general consideration.
Applying part 2. enables us to find a $\delta_{\lfloor n(\mu)-2\rfloor} \in (0;\delta_{\lfloor n(\mu)-1\rfloor})$ such that

$$\Big\|F_f(1,\psi) - y_{\lfloor n(\mu)\rfloor}\Big\| < \delta_{\lfloor n(\mu)-1\rfloor}$$

holds for all $\psi \in U_{\delta_{\lfloor n(\mu)-2\rfloor}}(y_{\lfloor n(\mu)-1\rfloor})$, $(\beta,\varepsilon) \in (0;\delta_{\lfloor n(\mu)-2\rfloor})^2$, and all $f \in N(\beta,\varepsilon)$. Clearly, we can repeat this procedure of finding $\delta_{j-1} \in (0;\delta_j)$ recursively: for $j \in \{1, ..., \lfloor n(\mu)-1\rfloor\}$ we obtain $\delta_{j-1} \in (0;\delta_j)$ (applying part 2.) such that

$$\|F_f(1,\psi) - y_{j+1}\| < \delta_j$$

holds for all $\psi \in U_{\delta_{j-1}}(y_j)$, $(\beta,\varepsilon) \in (0;\delta_{j-1})^2$, and all $f \in N(\beta,\varepsilon)$. Thus, the obtained finite sequence $(\delta_k)_{k\in\{0,...,\lfloor n(\mu)-1\rfloor\}\cup\{n(\mu)-1\}}$ satisfies

$$0 < \delta_k < \delta_{k+1} \le \delta_{n(\mu)-1} = \eta$$

for all $k \in \{0, ..., \lfloor n(\mu)-2\rfloor\}$.

3.2 Part 1. guarantees now the existence of a $\delta \in (0;\delta_0)$ such that for all $(\beta,\varepsilon) \in (0;\delta)^2$, $f \in N(\beta,\varepsilon)$, and $\varphi \in A(\beta)$, we have

$$\|F_f(1,\varphi) - y_1\| < \delta_0 \ ,$$

which gives in particular

$$|x^{\varphi,f}(t) - y(t)| < \delta_0 < \eta \quad \text{ for all } t \in [0;1] \ .$$

Choosing $\psi^1 := F_f(1,\varphi) \in U_{\delta_0}(y_1)$ we obtain, applying 3.1,

$$\|F_f(1,\psi^1) - y_2\| = \|F_f(2,\varphi) - y_2\| < \delta_1 \ ,$$

such that

$$|x^{\varphi,f}(t) - y(t)| < \delta_1 < \eta \quad \text{for all } t \in [1;2] \; .$$

Setting $\psi^j := F_f(j, \varphi) \in U_{\delta_{j-1}}(y_j)$ we can repeat these arguments for all $j \in \{2, ..., \lfloor n(\mu) - 2 \rfloor\}$ and obtain

$$|x^{\varphi,f}(t) - y(t)| \leq \eta \quad \text{for all } t \in [0; \lfloor n(\mu) - 1 \rfloor]$$

and all $\beta$, $\varepsilon$, $f$, and $\varphi$ as above.

3.2.1 If we have $z(\mu) \notin \mathbb{N}$, then $\lfloor n(\mu) - 1 \rfloor = n(\mu) - 1 \in \mathbb{N}$ and we can repeat the previous argument once more with $j = \lfloor n(\mu) - 2 \rfloor + 1 = n(\mu) - 1$. This yields

$$|x^{\varphi,f}(t) - y(t)| < \delta_{n(\mu)-1} = \eta \quad \text{for all } t \in [0; n(\mu)]$$

and all $\beta$, $\varepsilon$, $f$, and $\varphi$ as above. Thus, the proposition is proved in this case.

3.2.2 In case $z(\mu) \in \mathbb{N}$ we have to make an additional step first: As above we obtain

$$\begin{aligned} \|F_f(1, \psi^{\lfloor n(\mu)-1 \rfloor}) - y_{\lfloor n(\mu) \rfloor}\| &= \|F_f(\lfloor n(\mu) \rfloor, \varphi) - y_{\lfloor n(\mu) \rfloor}\| < \\ &< \delta_{\lfloor n(\mu)-1 \rfloor} \end{aligned}$$

for $j = \lfloor n(\mu) - 2 \rfloor + 1 = \lfloor n(\mu) - 1 \rfloor$. Therefore, we conclude from this estimate together with $\|F_f(\lfloor n(\mu) - 1 \rfloor, \varphi) - y_{\lfloor n(\mu)-1 \rfloor}\| < \delta_{\lfloor n(\mu)-2 \rfloor}$ that

$$\|F_f(\lfloor n(\mu) \rfloor - \frac{1}{2}, \varphi) - y_{\lfloor n(\mu) \rfloor - \frac{1}{2}}\| < \delta_{\lfloor n(\mu)-1 \rfloor} \; ,$$

and a further application of part 2. (for $j = \lfloor n(\mu) \rfloor - \frac{1}{2} = n(\mu) - 1$) yields

$$|x^{\varphi,f}(t) - y_{n(\mu)}(t)| < \delta_{n(\mu)-1} = \eta \quad \text{for all } t \in [0; n(\mu)]$$

and all $\beta$, $\varepsilon$, $f$, and $\varphi$ as above (since $\lfloor n(\mu) \rfloor - \frac{1}{2} + 1 = \lfloor n(\mu) \rfloor + \frac{1}{2} = n(\mu)$ by (3.2)). Thus, the proposition is proved in this case, too.

□

**COROLLARY 3.1.1** *There exist $\beta_0 \in (0; -\frac{a}{\mu})$, $\varepsilon_0 \in (0; a)$ such that for all $\beta \in (0; \beta_0)$, $\varepsilon \in (0; \varepsilon_0)$, $f \in N(\beta, \varepsilon)$, and $\varphi \in A(\beta)$ we have*

$$|x^{\varphi,f}(t)| \leq \xi_f^+ < -\frac{M_f}{\mu} \quad \textit{for all } t \in [0; n(\mu)] \; .$$

PROOF: By $(N_4)$ we have $\xi_f^+ > -y(1) = \max_{\mathbb{R}} |y|$. Thus, for any given $\eta \in (0; \xi_f^+ + y(1))$ we can find $\beta_0 \in (0; -\frac{a}{\mu})$, $\varepsilon_0 \in (0; a)$ such that we have

$$|x^{\varphi,f}(t) - y(t)| < \eta$$

for all $t \in [0; n(\mu)]$ by PROPOSITION 3.1.1, which yields the desired estimate.

$\square$

It should be mentioned that up to this point it is not clear whether the intersection $W^s(u_+) \cap A(\beta)$ is empty or not. In order to prove that solutions which start in $A(\beta)$ return to this set (or, more precisely, enter the set $-A(\beta)$) and do *not* converge to the stationary solution associated with $u_+$, we have to introduce some further denotations.

Define, for $c \in \left(0; \frac{n(\mu)-2}{z(\mu)-1}\right]$,

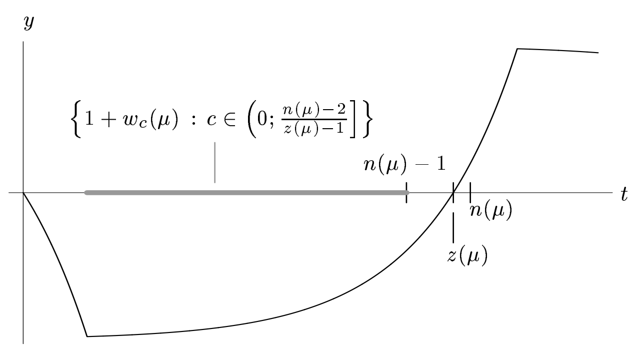
$$w_c(\mu) := (z(\mu) - 1) \cdot c = \frac{c}{\mu} \log\left(2 - e^{-\mu}\right) \ . \tag{3.3}$$

**REMARK 3.1.4** *For every $c \in \left(0; \frac{n(\mu)-2}{z(\mu)-1}\right]$ we have*

$$1 + c < 1 + w_c(\mu) \leq n(\mu) - 1 < z(\mu) < n(\mu) \ ,$$

*while $y < 0$ on $[w_\gamma(\mu); n(\mu) - 1]$ holds for all $\gamma \in (0; 1)$, i.e. in particular for $\gamma = c$.*

To illustrate the preceding remark we include the following figure that displays the mutual positions of $n(\mu)$, $z(\mu)$, and $1 + w_c(\mu)$ for $c \in \left(0; \frac{n(\mu)-2}{z(\mu)-1}\right]$.




**PROPOSITION 3.1.2** *Let $c \in \left(0; \frac{n(\mu)-2}{z(\mu)-1}\right]$. There exist $\beta_c \in (0; \beta_0)$, $\varepsilon_c \in (0; \varepsilon_0)$ such that for each $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$, $f \in N(\beta, \varepsilon)$, and $\varphi \in A(\beta)$ we have the following properties.*

(1) *The function* $x = x^{\varphi,f}$ *satisfies* $-\beta < x(n(\mu))$ *and*

$$\begin{array}{rll} x < -\beta & \text{on} & [w_c(\mu); n(\mu)-1]\,, \\ \dot{x} < 0 & \text{on} & (0;1)\,, \\ 0 < \dot{x} & \text{on} & (1+w_c(\mu); n(\mu))\,. \end{array}$$

(2) *For the unique solution* $q = q(\varphi, f)$ *of the equation*

$$x^{\varphi,f}(t) = -\beta$$

*in* $(n(\mu)-1; n(\mu))$ *we have*

$$x^{\varphi,f}_{q(\varphi,f)} \in -A(\beta)\ .$$

(3) *Furthermore, if* $\psi \in A(\beta)$ *with* $F_f(1+w_c(\mu),\varphi) = F_f(1+w_c(\mu),\psi)$ *then*

$$q(\varphi, f) = q(\psi, f)\ .$$

PROOF: Set for short $w := w_c(\mu)$, $z := z(\mu)$, and $n := n(\mu)$.

1. Since $y < 0$ on $[w; n-1]$ we can choose a $\delta_* \in (0;\beta_0) \cap (0;\varepsilon_0)$ such that $y < -\delta_*$ on $[w; n-1] \subset (0;z)$. Hence, PROPOSITION 3.1.1 implies the existence of a $\delta \in (0;\delta_*)$ such that for all $(\beta,\varepsilon) \in (0;\delta) \times (0;\delta)$, $f \in N(\beta,\varepsilon)$, and $\varphi \in A(\beta)$ we have

$$x^{\varphi,f} < -\delta_* < -\delta < -\beta < 0 \quad \text{on } [w; n-1]$$

and

$$x^{\varphi,f}(n) > \frac{y(n)}{2} > 0 > -\beta\ .$$

Thus, we proved the first two assertions in (1).

The monotonicity properties of $y$ yield

$$\begin{array}{ll} -\mu y(t) - a \le -\mu y(0) - a = -a < 0 & \text{for } t \in (0;1) \text{ and} \\ -\mu y(t) + a \ge -\mu y(1) + a \ge a(2 - e^{-\mu}) > 0 & \text{for } t \in (1;n)\ . \end{array}$$

Choose $\eta > 0$ such that

$$-a + \eta(1-\mu) < 0 < a(2-e^{-\mu}) - (1-\mu)\eta\,,$$

and then $\beta_c = \varepsilon_c \in (0;\delta) \cap (0;\eta)$ so that PROPOSITION 3.1.1 once again yields for all $(\beta,\varepsilon) \in (0;\beta_c) \times (0;\varepsilon_c)$, $f \in N(\beta,\varepsilon)$, and $\varphi \in A(\beta)$,

$$\left|x^{\varphi,f}(t) - y(t)\right| < \eta \quad \text{for all } t \in [0;n]\ .$$

For such $\beta$, $\varepsilon$, $f$, and $\varphi$ as before the solution $x := x^{\varphi,f}$ satisfies

$$\dot{x}(t) \;=\; -\mu x(t) + f(x(t-1)) \le -\mu(y(t)+\eta) - a + \varepsilon \le$$

$$
\begin{aligned}
&\leq \ -\mu(y(0)+\eta) - a + \varepsilon < -a + \eta(1-\mu) < \\
&< \ 0
\end{aligned}
$$

for $t \in (0;1)$.
Now let $t \in (1+w;n) \subset (1;n)$ and recall from the first paragraph of this part of the proof that we have $x(t-1) < -\beta$ since $t-1 \in (w;n-1)$. Consequently,

$$
\begin{aligned}
\dot{x}(t) &= -\mu x(t) + f(x(t-1)) \geq -\mu(y(t)-\eta) + a - \varepsilon > \\
&> -\mu(y(1)-\eta) + a - \eta \geq a(2-e^{-\mu}) - (1-\mu)\eta > \\
&> 0
\end{aligned}
$$

for $t \in (w+1;n)$.

2. We give a rather comprehensive proof of assertion (2) in this part in order to comment on the special choice of $c$ and its consequences later.

   2.1 Since $x := x^{\varphi,f}$ is continuous on $[-1;+\infty)$ and because of $x < -\beta$ on $(w;n-1)$ and $x(n) > -\beta$ by assertion (1), the Intermediate Value Theorem implies the existence of $q = q(\varphi,f) \in (n-1;n)$ with $x(q) = -\beta$.

   2.2 We choose $q_0 := \inf\{t \in (n-1;n) \ : \ x(t) = -\beta\} \in (n-1;n)$. Because $x < -\beta$ on $(w;n-1)$ and this particular choice of $q_0$ we obtain $x < -\beta$ on $(w;q)$ and, thereby, $x_q \in -A(\beta)$ (since otherwise we would obtain a contradiction to the definition of $q_0$).

   2.3 Due to $c \in \left(0; \frac{n(\mu)-2}{z(\mu)-1}\right]$, the solution $x = x^{\varphi,f}$ is strictly increasing on

   $$[n-1;n) = (w;n-1] \cap (1+w;n) \ ,$$

   which implies the uniqueness of $q = q(\varphi,f) \in (n-1;n)$.

3. The last assertion is shown using $x^{\varphi,f} = x^{\psi,f}$ on $[1+w;+\infty)$, $1+w < q(\varphi,f)$ and $1+w < q(\psi,f)$.

□

In contrast to Walther [67] where $c \in (0;1)$ could be chosen independent of $\mu$, we had to choose

$$c = c(\mu) \in \left(0; \frac{n(\mu)-2}{z(\mu)-1}\right]$$

here. As part 2.3 of the last proof indicates, it was necessary to introduce this restricted parameter range for $c$ in order to ensure that $1+w_c(\mu) \leq n(\mu)-1$, which guarantees the *uniqueness* of $q = q(\varphi,f)$ whereas the *existence* of $q \in (n(\mu)-1;n(\mu))$ with $x_q^{\varphi,f} \in -A(\beta)$ remains valid for all $c \in (0;1)$ as one can infer from parts 2.1 and 2.2 of the proof. In other words, we have

**Remark 3.1.5** *Let $c = c(\mu) \in \left(0; \frac{n(\mu)-2}{z(\mu)-1}\right]$ be given and choose $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$. Then*

$$A(\beta) \ni \varphi \mapsto q(\varphi, f) \in (n(\mu) - 1; n(\mu))$$

*is a map for every $f \in N(\beta, \varepsilon)$.*

Furthermore, it is evident from this remark and part (3) of Proposition 3.1.2 that the relations

$$s_f(\psi) = q(\varphi, f) - n(\mu) + 1 \in (0; 1) \ , \quad \psi = F_f(n(\mu) - 1, \varphi) \ , \quad \varphi \in A(\beta)$$

define a map $s_f : F_f(n(\mu) - 1, A(\beta)) \to \mathbb{R}$ for $f \in N(\beta, \varepsilon)$ and $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$.

## 3.2 A Lipschitz continuous return map

The crucial conclusion of Proposition 3.1.2 is that solutions evolving from $A(\beta)$ return to this set (by virtue of the oddness of $f$, cf. also Lemma 2.2.3(1)). Therefore, we can consider the return map

$$R_f : A(\beta) \ni \varphi \mapsto -F_f(q(\varphi, f), \varphi) \in A(\beta)$$

for $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$ and $f \in N(\beta, \varepsilon)$.

This self-map of $A(\beta)$ is extremely helpful for proving the existence of slowly oscillating periodic solutions of (1.1) with $f \in N(\beta, \varepsilon)$ as the following lemma shows.

**Lemma 3.2.1** *For every fixed point $\varphi \in A(\beta)$ of $R_f$ the corresponding solution $x = x^{\varphi,f}$ defines a periodic solution $P$ of equation (1.1) with minimal period $2q(\varphi, f)$ and symmetry*

$$P(t) = -P(t + q(\varphi, f)) \quad \text{for all} \quad t \in \mathbb{R} \ .$$

Proof: To see this observe that the function

$$w : [-1; +\infty) \ni t \mapsto -x^{\varphi,f}(t + q(\varphi, f)) \in \mathbb{R}$$

is a solution of (1.1) since $f$ is odd by $(N_1)$, and use $w_0 = R_f(\varphi) = \varphi = x_0$, which yields $x(t) = -x(t + q(\varphi, f))$ in $[-1; +\infty)$, thus $x(t) = x(t + 2q(\varphi, f))$ on $[-1; +\infty)$.

□

In view of Lemma 3.2.1, we wish to apply fixed point theorems to $R_f$ in order to prove the existence of periodic solutions. As a first step towards this direction, we show that

$R_f$ is LIPSCHITZ continuous if $f \in N(\beta, \varepsilon)$ is assumed to be LIPSCHITZ continuous. For this purpose we rewrite $R_f$ as a composition of the following three maps: the restricted time-1-map

$$F_1 := F_f(1, \cdot)\Big|_{A(\beta)} ,$$

followed by

$$F_{n-2} := F_f(n(\mu) - 2, \cdot)\Big|_{F_f(1, A(\beta))} ,$$

and then by the map

$$S_f : F_f(n(\mu) - 1, A(\beta)) \ni \varphi \mapsto -F_f(s_f(\varphi), \varphi) \in C ,$$

where $s_f : F_f(n(\mu) - 1, A(\beta)) \to (0; 1)$ is the map defined at the end of the last section. Thus, we obtain

$$R_f = S_f \circ F_{n-2} \circ F_1 = -F_f(q(\cdot, f), \cdot) .$$

Recall that LIPSCHITZ constants of maps $T : X \supset D_T \to Y$ are given by

$$L(T) := \sup_{\substack{x \in D_T \\ y \in D_T \setminus \{x\}}} \frac{\|T(x) - T(y)\|_Y}{\|x - y\|_X} ,$$

when $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ were normed linear spaces.

Combining the subsequent four lemmata which yield LIPSCHITZ constants for the mappings $F_1$, $F_{n-2}$, $s_f$ and $S_f$, respectively, we can prove the LIPSCHITZ continuity of the return map $R_f$.

**LEMMA 3.2.2** *Let $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$. Suppose $f \in N(\beta, \varepsilon)$ is* LIPSCHITZ *continuous and let $L_\beta := L(f|_{[\beta;\infty)})$ be the* LIPSCHITZ *constant for the restriction $f\Big|_{[\beta, +\infty)}$. Then $L(F_1) := e^{-\mu} L_\beta$ is a* LIPSCHITZ *constant for $F_1$.*

PROOF: For $\varphi, \psi$ in $A(\beta)$ and $t \in [-1; 0]$ we get, using $\varphi(0) = \beta = \psi(0)$, $\beta \le \varphi$ and $\beta \le \psi$,

$$\begin{aligned} |F_f(1, \varphi)(t) - F_f(1, \psi)(t)| &\le 0 + \int_0^{1+t} e^{-\mu(1+t-s)} |f(\varphi(s-1)) - f(\psi(s-1))| ds \le \\ &\le e^{-\mu} L_\beta \cdot \|\varphi - \psi\| , \end{aligned}$$

which yields the LIPSCHITZ constant for $F_1$.

$\square$

It is important to recognize that functions in $N(\beta,\varepsilon)$ have in general large global LIPSCHITZ constants

$$L(f) \geq \frac{a-\varepsilon}{\beta}$$

as follows from $f(0) = 0$ (due to $(N_1)$ and the continuity of $f$) and $|f(\xi)| \geq a - \varepsilon$ for all $\xi \in \mathbb{R} \setminus (-\beta;\beta)$ by $(N_3)$. In fact, the LIPSCHITZ constants $L(f)$ become large as $(\beta,\varepsilon) \to (0,0)$.

On the other hand, $N(\beta,\varepsilon)$ always contains functions with small LIPSCHITZ constants $L_\beta := L(f|_{[\beta;\infty)})$ of the restrictions of $f \in N(\beta,\varepsilon)$ to $[\beta;\infty)$. The smallness of these LIPSCHITZ constants $L_\beta := L(f|_{[\beta;\infty)})$ is of crucial importance because it ensures that the LIPSCHITZ constant $L(R_f)$ of the return map $R_f$ becomes small.

**LEMMA 3.2.3** *Let $(\beta,\varepsilon) \in (0;\beta_c) \times (0;\varepsilon_c)$. Suppose $f \in N(\beta,\varepsilon)$ is* LIPSCHITZ *continuous with* LIPSCHITZ *constant $L := L(f)$. Then $L(F_{n-2}) := ((1+L)e^{-\mu})^{\lceil n(\mu)-2 \rceil}$ is a* LIPSCHITZ *constant for $F_{n-2}$.*

PROOF: Set $n := n(\mu)$. Using the same arguments as in the proof of LEMMA 3.2.2 (replacing $L_\beta$ with $L$ as necessary) we readily obtain for $\omega$ and $\chi$ in $C \supset F_f(1, A(\beta))$

$$\|F_f(1,\omega) - F_f(1,\chi)\| \leq e^{-\mu}(1+L) \cdot \|\omega - \chi\| \ .$$

If $z(\mu) \notin \mathbb{N}$, then $n-2 = \lceil n-2 \rceil \in \mathbb{N}$ and repeated application of this formula $(n-2)$-times using the semiflow property $F_f(t,\cdot) = F_f(1, F_f(t-1,\cdot))$ for $t > 1$ yields

$$\|F_f(n-2,\omega) - F_f(n-2,\chi)\| \leq \left(e^{-\mu}(1+L)\right)^{\lceil n-2 \rceil} \cdot \|\omega - \chi\|$$

which proves the assertion in this case. For the case $z(\mu) \in \mathbb{N}$ we have $\lceil n \rceil \in \mathbb{N} \cap [4;+\infty)$ such that we obtain in a first step

$$\|F_f(\lceil n-3 \rceil,\omega) - F_f(\lceil n-3 \rceil,\chi)\| \leq \left(e^{-\mu}(1+L)\right)^{\lceil n-3 \rceil} \cdot \|\omega - \chi\| \ .$$

As in the proof of PROPOSITION 3.1.1 we have to make a further (half-) step because of $n - 2 = \lceil n-3 \rceil + \frac{1}{2}$ by (3.2). Hence,

$$\begin{aligned} \|F_f(n-2,\omega) - F_f(n-2,\chi)\| &= \|F_f(\lceil n-3 \rceil, F_f(\frac{1}{2},\omega)) - F_f(\lceil n-3 \rceil, F_f(\frac{1}{2},\chi))\| \leq \\ &\leq \left(e^{-\mu}(1+L)\right)^{\lceil n-3 \rceil} \cdot \|F_f(\frac{1}{2},\omega) - F_f(\frac{1}{2},\chi)\| \end{aligned}$$

together with the rather rough estimate

$$\|F_f(\frac{1}{2},\omega) - F_f(\frac{1}{2},\chi)\| \leq e^{-\mu}(1+L) \cdot \|\omega - \chi\|$$

(which can be obtained exactly as in the proof of LEMMA 3.2.2) implies

$$\|F_f(n-2,\omega) - F_f(n-2,\chi)\| \le \left(e^{-\mu}(1+L)\right)^{\lceil n-2 \rceil} \cdot \|\omega - \chi\| \ .$$

$\square$

**REMARK 3.2.1** *One can certainly get a sharper estimate for the* LIPSCHITZ *constant of* $F_{n-2}$ *taking into consideration that* PROPOSITION 3.1.2(1) *implies*

$$x^{\varphi,f} < -\beta \qquad \text{on } [w_c(\mu); n(\mu)-1]$$

*for any* $\varphi \in A(\beta)$ *such that we can use the smaller* LIPSCHITZ *constant* $L_\beta = L(f|_{[\beta;\infty)})$ *on this time interval: again it will be of the form*

$$\left((1+L)e^{-\mu}\right)^l \cdot \left((1+L_\beta)e^{-\mu}\right)^k$$

*with appropriate* $l$ *and* $k$ *such that* $l + k = \lceil n(\mu) - 2 \rceil$*. This will not affect our subsequent investigations too much such that we will use the rougher estimate from* LEMMA 3.2.3 *instead.*

Since $S_f$ is composed of $s_f$ and $F_f$ we begin studying the LIPSCHITZ continuity of the map $s_f : C \to \mathbb{R}$ before calculating a LIPSCHITZ constant for $S_f$.

**LEMMA 3.2.4** *Let* $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$ *such that*

$$-\beta\mu < a - \varepsilon \ .$$

*If the restriction of* $f \in N(\beta, \varepsilon)$ *to* $[\beta; +\infty)$ *is* LIPSCHITZ *continuous with* LIPSCHITZ *constant* $L_\beta := L(f|_{[\beta;\infty)})$*, then* $s_f$ *is* LIPSCHITZ *continuous with* LIPSCHITZ *constant*

$$L(s_f) = \frac{1 + L_\beta}{a - \varepsilon + \beta\mu} \ .$$

PROOF: Once again we use the abbreviations $n := n(\mu)$, $w := w_c(\mu)$, and $z := z(\mu)$. Let $\varphi$ and $\overline{\varphi}$ belong to $F_f(n-1, A(\beta))$ and set $s := s_f(\varphi)$ and $\overline{s} := s_f(\overline{\varphi})$. Without loss of generality let us assume $\overline{s} \le s$. Since $\sigma \in (0;1)$ and $x^{\varphi,f}(\sigma) = -\beta$ for $\sigma \in \{s, \overline{s}\}$, we get

$$-\beta = \psi(0)e^{-\mu\sigma} + \int_0^\sigma e^{-\mu(\sigma - t)} f(\psi(t-1)) dt$$

with $\psi = \varphi$ or $\psi = \overline{\varphi}$ according to $\sigma = s$ or $\sigma = \overline{s}$, respectively. Evidently, we obtain from this

$$\begin{aligned}\beta|e^{\mu\overline{s}} - e^{\mu s}| &= \Big|\int_{\overline{s}}^{s} e^{\mu t} f(\varphi(t-1))dt + [\varphi(0) - \overline{\varphi}(0)] - \int_{0}^{\overline{s}} e^{\mu t}[f(\overline{\varphi}(t-1)) - f(\varphi(t-1))]dt\Big| \\ &\geq \Big|\int_{\overline{s}}^{s} e^{\mu t} f(\varphi(t-1))dt\Big| - \|\varphi - \overline{\varphi}\| - \Big|\int_{0}^{\overline{s}} e^{\mu t}[f(\overline{\varphi}(t-1)) - f(\varphi(t-1))]dt\Big| \ .\end{aligned}$$

By assertions (1) and (2) of PROPOSITION 3.1.2 we have $\varphi \leq -\beta$ and $\overline{\varphi} \leq -\beta$ on $[-1;0]$, such that we get

$$\min_{t\in[\overline{s};s]} |f(\varphi(t-1))| \geq a - \varepsilon$$

by $(N_3)$ since $[\overline{s}-1; s-1] \subset [-1;0]$, and because of $[-1;\overline{s}-1] \subset [-1;0]$ we have furthermore

$$|f(\overline{\varphi}(t-1)) - f(\varphi(t-1))| \leq L_\beta \cdot |\overline{\varphi}(t-1) - \varphi(t-1)| \leq L_\beta \cdot \|\overline{\varphi} - \varphi\| \quad \text{for all } t \in [0;\overline{s}] \ .$$

These estimates lead to

$$\beta|e^{\mu\overline{s}} - e^{\mu s}| \geq |\overline{s} - s| \cdot (a - \varepsilon) - \|\varphi - \overline{\varphi}\| - L_\beta\|\overline{\varphi} - \varphi\| \ ,$$

and together with

$$\beta|e^{\mu\overline{s}} - e^{\mu s}| \leq \beta|\mu| \cdot |\overline{s} - s| = -\beta\mu \cdot |\overline{s} - s|$$

we obtain

$$(a - \varepsilon + \beta\mu) \cdot |\overline{s} - s| \leq (1 + L_\beta) \cdot \|\overline{\varphi} - \varphi\|$$

which proves the assertion.

□

Finally, the following last lemma of this series of lemmata guarantees the boundedness of the LIPSCHITZ constants of $S_f$ in case when $(\beta, \varepsilon)$ is close to $(0,0)$.

**LEMMA 3.2.5** *Let* $(\beta, \varepsilon) \in (0;\beta_c) \times (0;\varepsilon_c)$ *such that*

$$-\beta\mu < a - \varepsilon \ .$$

*If the restriction of* $f \in N(\beta, \varepsilon)$ *to* $[\beta; +\infty)$ *is* LIPSCHITZ *continuous with* LIPSCHITZ *constant* $L_\beta = L(f|_{[\beta;\infty)})$*, then* $S_f$ *is* LIPSCHITZ *continuous with* LIPSCHITZ *constant*

$$L(S_f) = L(s_f) \cdot \big((-\mu) \cdot \xi_f^+ + M_f\big) + (1 + L_\beta)e^{-\mu} \ .$$

PROOF: We adopt the notation and abbreviations from the proof of LEMMA 3.2.4. Thus, let $\varphi$ and $\overline{\varphi}$ be in $F_f(n-1, A(\beta))$, i.e. there is $\psi \in A(\beta)$ such that $\varphi = F_f(n-1, \psi)$, set again $s := s_f(\varphi)$ and $\overline{s} := s_f(\overline{\varphi})$, and assume $\overline{s} \leq s$ without loss of generality. Because of

$$\begin{aligned} S_f(\varphi) - S_f(\overline{\varphi}) &= -F_f(s,\varphi) + F_f(\overline{s},\varphi) - F_f(\overline{s},\varphi) + F_f(\overline{s},\overline{\varphi}) = \\ &= -(x^{\psi}_{n-1+s} - x^{\psi}_{n-1+\overline{s}}) - (F_f(\overline{s},\varphi) - F_f(\overline{s},\overline{\varphi})) = \\ &= -(x^{\psi}(n-1+s+\cdot) - x^{\psi}(n-1+\overline{s}+\cdot)) - (F_f(\overline{s},\varphi) - F_f(\overline{s},\overline{\varphi})) = \\ &= \underbrace{\int\limits_{n-1+\overline{s}+\cdot}^{n-1+s+\cdot} \dot{x}^{\psi}(u)du}_{=:S_1=S_1((s,\varphi),(\overline{s},\overline{\varphi}))} - \underbrace{(F_f(\overline{s},\varphi) - F_f(\overline{s},\overline{\varphi}))}_{=:S_2=S_2((s,\varphi),(\overline{s},\overline{\varphi}))} . \end{aligned}$$

we estimate the two parts separately.
For $t \in [-1;0]$, denoting by $I_t$ the interval $I_t := [n-1+\overline{s}+t; n-1+s+t] \subset [0;n]$, we have

$$\begin{aligned} |S_1(t)| &= \Big| \int\limits_{I_t} \dot{x}^{\psi}(u)du \Big| = \Big| \int\limits_{I_t} [-\mu x^{\psi}(u) + f(x^{\psi}(u-1))]du \Big| \leq \\ &\leq |s-\overline{s}| \cdot \Big( (-\mu) \cdot \max_{u\in I_t} |x^{\psi}(u)| + \max_{u \in I_t} |f(x^{\psi}(u-1))| \Big) . \end{aligned}$$

Using COROLLARY 3.1.1, the boundedness of $f$, and LEMMA 3.2.4, this implies

$$\begin{aligned} \|S_1\| &\leq |s-\overline{s}| \cdot \big((-\mu)\cdot \xi_f^+ + M_f\big) \leq \\ &\leq L(s_f) \cdot \|\varphi - \overline{\varphi}\| \cdot \big((-\mu)\cdot \xi_f^+ + M_f\big) . \end{aligned}$$

Now, set $J_1 := [-1; -\overline{s}]$ and $J_2 := [-\overline{s}; 0]$. For $t \in [-1;0] = J_1 \cup J_2$ it is

$$S_2(t) = \begin{cases} \varphi(\overline{s}+t) - \overline{\varphi}(\overline{s}+t) & , t \in J_1 \\ e^{-\mu(\overline{s}+t)}[\varphi(0) - \overline{\varphi}(0)] + \int\limits_0^{\overline{s}+t} e^{-\mu(\overline{s}+t-u)}[f(\varphi(u-1)) - f(\overline{\varphi}(u-1))]du & , t \in J_2 \end{cases} .$$

So, it is obvious from PROPOSITION 3.1.2 (2) that

$$\|S_2\| \leq (1+L_\beta) \cdot e^{-\mu} \cdot \|\varphi - \overline{\varphi}\| ,$$

which yields

$$L(S_f) = L(s_f) \cdot \big((-\mu)\cdot \xi_f^+ + M_f\big) + (1+L_\beta)\cdot e^{-\mu} .$$

□

Summarizing the auxiliary lemmata we have proved the LIPSCHITZ continuity of $R_f$.

**THEOREM 3.2.1** *Let $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$ with $-\beta\mu < a - \varepsilon$. Suppose $f \in N(\beta, \varepsilon)$ is* LIPSCHITZ *continuous with* LIPSCHITZ *constant $L(f)$, and that $L_\beta := L(f|_{[\beta;\infty)})$ is the* LIPSCHITZ *constant for $f\Big|_{[\beta;+\infty)}$. Then $R_f$ is* LIPSCHITZ *continuous with*

$$L(R_f) = e^{-\mu} L_\beta \cdot \left((1 + L(f))e^{-\mu}\right)^{\lceil n(\mu)-2 \rceil} \cdot \left[\frac{1 + L_\beta}{a - \varepsilon + \beta\mu} \cdot \left((-\mu) \cdot \xi_f^+ + M_f\right) + (1 + L_\beta)e^{-\mu}\right] \,.$$

Since $A(\beta) \subset C \backslash \{0\}$ is a bounded, closed and convex subset of $C$, and $R_f$ a (LIPSCHITZ) continuous self-map of $A(\beta)$, it is tempting to try to apply

**SCHAUDER'S FIXED POINT THEOREM** (cf. ZEIDLER [75, p. 57]) *If $U$ is a nonempty closed bounded convex subset of a* BANACH *space and $T : U \to U$ is completely continuous, then $T$ has a fixed point.*

Herein a map $T : X \supset U \to X$ defined on a subset $U$ of a BANACH space $X$ is called ***completely continuous*** (or *compact*) if $T$ is continuous and for any bounded set $B \subset U$ the closure of $TB$ is compact (i.e. $TB$ is relatively compact).

**LEMMA 3.2.6** *The return map $R_f$ is completely continuous.*

PROOF: Let $B \subset A(\beta)$ be given; we are to show that $R_f B$ is relatively compact. COROLLARY 3.1.1 implies $|(R_f\varphi)(t)| \leq \xi_f^+$ for all $t \in [-1; 0]$ and all $\varphi \in B$ such that $R_f B$ is pointwise bounded. Furthermore, $R_f B$ is equicontinuous since for all $t \in [-1; 0]$ we have

$$\begin{aligned}
\left|\frac{d}{dt}(R_f\varphi)(t)\right| &= |\dot{x}^{\varphi,f}_{q(\varphi,f)}(t)| = |\dot{x}^{\varphi,f}(q(\varphi, f) + t)| = \\
&= |-\mu x^{\varphi,f}(q(\varphi, f) + t) + f(x^{\varphi,f}(q(\varphi, f) + t - 1))| \leq \\
&\leq -\mu\xi_f^+ + M_f \;,
\end{aligned}$$

such that the relative compactness of $R_f B$ follows from the ARZELA-ASCOLI Theorem (cf., e.g., ZEIDLER [75, p. 772]).

□

Combining THEOREM 3.2.1, REMARK 3.1.3 and LEMMA 3.2.6, the SCHAUDER Fixed Point Theorem implies the existence of fixed points of $R_f$ in $A(\beta)$ that yield periodic solutions according to LEMMA 3.2.1.

**THEOREM 3.2.2** *Let $\mu \in (-\log 2; 0)$, $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$ with $-\beta\mu < a - \varepsilon$, and let $f \in N(\beta, \varepsilon)$ be* LIPSCHITZ *continuous. Then there exist fixed points of $R_f$ in $A(\beta)$, and each fixed point of $R_f$ is the initial state of a (nontrivial) periodic solution to* (1.1).

In case that $R_f$ is a strict contradiction, i.e.

$$L(R_f) < 1 \ ,$$

we get a unique fixed point $\varphi$ in $A(\beta)$ from the classical BANACH Fixed Point Theorem (cf. ZEIDLER [75, p. 17]). Nevertheless there may still exist slowly oscillating periodic solutions whose orbits in $C$ do not intersect $A(\beta)$, i.e. slowly oscillating periodic solutions with (initial) segments which do **not** belong to $A(\beta)$!

Can we hope to find any nonlinearities $f \in N(\beta, \varepsilon)$ for which $R_f$ is a strict contraction on $A(\beta)$ ? Indeed, we will outline a short sketch of how to "construct" some nonlinearities of this type now. For this purpose, let

$$\mu \in (-\log 2; 0), \ a \in \mathbb{R}^+, \ b \in (a; +\infty) \quad \text{ and } c \in \left(0; \tfrac{n(\mu)-2}{z(\mu)-1}\right]$$

be given. Furthermore, according to THEOREM 3.2.1 choose $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$ with

$$-\mu\beta < a - \varepsilon \ .$$

Now, choose

- a LIPSCHITZ continuous odd function $g : [-\beta; +\beta] \to [-b; +b]$ satisfying
$$g(\beta) \in [-a - \varepsilon; -a + \varepsilon]$$
with LIPSCHITZ constant $L(g)$, and
- a LIPSCHITZ continuous function $h : [\beta; +\infty) \to [-a - \varepsilon; -a + \varepsilon]$ with $h(\beta) = g(\beta)$ whose LIPSCHITZ constant $L(h)$ satisfies $L(h) \leq L(g)$ and
$$e^{-\mu} L(h) \cdot \left((1 + L(g)) e^{-\mu}\right)^{\lceil n(\mu) - 2 \rceil} \cdot \left[\frac{1 + L(h)}{a - \varepsilon + \beta\mu} \left(2 \cdot \max\{b, a + \varepsilon\}\right) + (1 + L(h)) e^{-\mu}\right] < 1.$$
Finally, $h$ should be chosen such that there exists a unique solution $\xi_f^+$ of the equation $-\mu\xi + f(\xi) = 0$ in $(\beta; +\infty)$, which satisfies $\xi_f^+ \in (-y(1); -\frac{a}{\mu})$.

Thus,

$$f : \mathbb{R} \ni \xi \mapsto \left\{ \begin{array}{ll} -h(-\xi) & , \ \xi \in (-\infty; -\beta) \\ g(\xi) & , \ \xi \in [-\beta; \beta] \\ h(\xi) & , \ \xi \in (\beta; +\infty) \end{array} \right\} \in \mathbb{R}$$

defines a LIPSCHITZ continuous odd function that belongs to $N(\beta, \varepsilon)$ with (global) LIPSCHITZ constant $L = L(f) = \max\{L(g), L(h)\} = L(g)$ whose restriction to $[\beta; +\infty)$ has the LIPSCHITZ constant $L_\beta = L(h)$ such that THEOREM 3.2.1 implies

$$L(R_f) < 1 \ .$$

Consequently, $R_f$ is a strict contraction and has, by BANACH's Fixed Point Theorem, a unique fixed point that defines a periodic solution $P$ of equation (1.1).

## 3.3 Attraction and hyperbolicity for nonlinearities that yield a strict contraction $R_f$

Certainly, we can also obtain continuously differentiable functions $f \in N(\beta, \varepsilon)$ for which $R_f$ is a strict contraction with unique fixed point $P_0 := \varphi \in A(\beta)$ initiating a slowly oscillating periodic solution $P : \mathbb{R} \to \mathbb{R}$. For the remainder of this section we are now interested in such smooth nonlinearities which yield a contracting $R_f$ only: for these we add the proof of the following theorem which is almost the same as in [67] but we include it here for the sake of completeness.

**Theorem 3.3.1** *Let $\mu \in (-\log 2; 0)$, $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$ with $-\beta\mu < a - \varepsilon$, and let $f \in N(\beta, \varepsilon) \cap C^1$ such that $L(R_f) < 1$. Furthermore, let $\varphi \in A(\beta)$ denote the unique fixed point of the contraction $R_f$. Then the orbit*

$$\mathcal{O}_P := \{P_t \ : \ t \in \mathbb{R}\} \subset C$$

*of the periodic solution $P$ of equation (1.1) given by the initial value $P_0 = \varphi \in A(\beta)$ is hyperbolic, stable, and exponentially attractive with asymptotic phase.*

Before we proceed to prove this theorem, we should recall some basic definitions from Chapter XIV of the monograph [16] which serves as a source for the theoretical background in all what follows.

**Definition 3.3.1** The orbit $\mathcal{O}_P$ of a periodic solution $P : \mathbb{R} \to \mathbb{R}$ of (1.1) is called ***asymptotically stable***, if for any given open neighborhood $V$ of $\mathcal{O}_P$ we can find an open neighborhood $W$ of $\mathcal{O}_P$ in $C$ such that

$$F_f(t, \varphi) \in V \quad \text{for all } t \in \mathbb{R}^+ \quad \text{and} \quad \mathrm{dist}_C(x_t^\varphi, \mathcal{O}_P) \to 0 \quad \text{as } t \to \infty \ .$$

**Definition 3.3.2** The orbit $\mathcal{O}_P$ of a periodic solution $P : \mathbb{R} \to \mathbb{R}$ of (1.1) is called ***stable and exponentially attractive with asymptotic phase*** if $\mathcal{O}_P$ is asymptotically stable, and, furthermore, there exist positive constants $\gamma > 0$, $K > 0$, a neighborhood $W$ of $\mathcal{O}_P$, and for every $\varphi \in W$ a $t_\varphi \in \mathbb{R}$ such that

$$|x^\varphi(t) - P(t + t_\varphi)| \le K e^{-\gamma t} \|\varphi - P_{t_\varphi}\|$$

holds for all $t \in \mathbb{R}_0^+$.

The basic method to prove stability properties of a periodic solution $P$ is to consider a POINCARÉ map $\Pi$ associated with this periodic solution, and we refer the reader to [16, Section XIV.3] for a detailed presentation and further information about POINCARÉ maps. Notice that all results of [16, Chapter XIV] on periodic orbits and POINCARÉ maps apply in our situation, since the restriction of $F_f$ to the set $(1;+\infty) \times C$ is continuously differentiable (cf. Section 1.2).

Thus, we intend to construct a POINCARÉ map $\Pi$ associated with the periodic solution $P$, and the subsequent two propositions are devoted to prepare the definition of $\Pi$.

Let $H := H(\beta)$ denote the closed hyperplane

$$H := H(\beta) = \{\psi \in C \ : \ \psi(0) = \beta\} \ ,$$

such that $P_0 = \varphi \in H$.

**PROPOSITION 3.3.1** *There exist a bounded open neighborhood $U$ of $\varphi$ in $C$ and a $C^1$-map $\zeta : U \to (n(\mu)-1; n(\mu))$ so that for all $\psi \in U$ we have*

$$F_f(\zeta(\psi), \psi) \in -A(\beta) \ ,$$

*and for all $\psi \in U \cap A(\beta)$,*

$$\zeta(\psi) = q(\psi, f) \ .$$

PROOF: The curve

$$\mathbb{R} \ni t \mapsto P_t \in C$$

is differentiable at $t = q(\varphi, f)$, and $\dot{P}_{q(\varphi,f)}(0) = \dot{P}(q(\varphi, f)) > 0$ implies

$$D(s \mapsto P_s)(q(\varphi, f))1 = \dot{P}_{q(\varphi,f)} \notin \{\psi \in C \ : \ \psi(0) = 0\}$$

where $\{\psi \in C \ : \ \psi(0) = 0\} = T_{F_f(q(\varphi,f),\varphi)}(-H)$.
Using the Implicit Function Theorem we find an open neighborhood $V$ of $\varphi = P_0$ in $C$ and a $C^1$-map $\tau : V \to \mathbb{R}$ with

$$\tau(\varphi) = q(\varphi, f) \quad \text{ and } \quad F_f(\tau(\psi), \psi) \in -H \ \text{ for all } \psi \in V \ .$$

Choose a neighborhood $U$ of $\varphi$ in $V$ so small that

$$x^{\psi,f}_{n(\mu)-1} < -\beta \ \text{ on } [-1;0] \quad \text{ and } \quad \tau(\psi) \in (n(\mu)-1; n(\mu))$$

for all $\psi \in U$. Note that one can choose $U$ (by continuous dependence on the initial value $\varphi \in A(\beta)$) so small that we obtain furthermore $|x^{\psi,f}(t)| \leq \xi_f^+ < -\frac{M_f}{\mu}$ for all $\psi \in U$ and $t \in [0; n(\mu)]$ (cf. COROLLARY 3.1.1).

For all $\psi \in U \cap A(\beta)$, PROPOSITION 3.1.2 yields that we have $\dot{x}^{\psi,f}(t) > 0$ on $(n(\mu)-1; n(\mu))$ and $x^{\psi,f}(q(\psi,f)) = -\beta$, i.e.

$$x^{\psi,f}_{q(\psi,f)} \in -H \quad \text{and} \quad x^{\psi,f}_t \notin -H \quad \text{for } t \in (n(\mu)-1; n(\mu)) \setminus \{q(\psi,f)\} \ .$$

It follows that on $U \cap A(\beta)$, $\tau(\psi) = q(\psi,f)$. Therefore, the second assertion is already proved and setting $\zeta := \tau\Big|_U$ will conclude the proof of the theorem: First, recall

$$F_f(\zeta(\psi),\psi) = F_f(\tau(\psi),\psi) \in -H \quad \text{for all } \psi \in U \subset V$$

which implies $x^{\psi,f}_{\zeta(\psi)}(0) = -\beta$. Furthermore, the choice of $U$ above gives

$$x^{\psi,f}_{\zeta(\psi)}\Big|_{[-1;n(\mu)-1-\zeta(\psi)]} < -\beta \quad \text{for all } \psi \in U \ .$$

Finally, the continuity of $x^{\psi,f}$ and the fact that $\zeta := \tau\Big|_U$ is a (first return time) map yield for each $\psi \in U$

$$x^{\psi,f}_{\zeta(\psi)}\Big|_{[n(\mu)-1-\zeta(\psi);0)} < -\beta \ ,$$

too, such that $x^{\psi,f}_{\zeta(\psi)} \leq -\beta$ for all $\psi \in U$. Hence, $F_f(\zeta(\psi),\psi) = x^{\psi,f}_{\zeta(\psi)} \in -A(\beta)$ for all $\psi \in U$.

$\square$

The $C^1$-map

$$Q : U \cap H \ni \psi \mapsto -F_f(\zeta(\psi),\psi) \in H$$

has $\varphi = P_0$ as fixed point and has its values in $A(\beta)$, with $Q = R_f$ on $U \cap A(\beta)$. For $j \in \mathbb{N}$ the iterates $Q^j : D_j \to H$ of $Q$ are defined by

$$\begin{aligned} D_1 &:= U \cap H, & Q^1 &:= Q, \\ D_{j+1} &:= \{\psi \in D_j \ : \ Q^j(\psi) \in U \cap H\}, & Q^{j+1}(\psi) &:= Q(Q^j(\psi)) \ . \end{aligned}$$

Although $(D_j)_{j\in\mathbb{N}}$ is a decreasing sequence of open subsets of $U \cap H$, the intersection of all $D_j$, $j \in \mathbb{N}$, contains an open neighborhood of $\varphi$.

**PROPOSITION 3.3.2** *Let $L := L(R_f) \in [0;1)$ be a* LIPSCHITZ *constant for $R_f$. There exists an open neighborhood $V$ of $\varphi$ in $U$ such that*

$$V \cap H \subset \bigcap_{j\in\mathbb{N}} D_j$$

*and*

$$\|Q^j(\psi) - Q^j(\chi)\| \leq L^{j-1}\|Q(\psi) - Q(\chi)\|$$

*for all $\psi$ and $\chi$ in $V \cap H$ and all $j \in \mathbb{N}$.*

PROOF: For $k \in \mathbb{N}$ let $\mathbb{N}_{\geq k} := \{n \in \mathbb{N} \ : \ n \geq k\}$ and $\mathbb{N}_{\leq k} := \{1, ..., k\}$.

1. Since $U$ is a bounded neighborhood of $\varphi$, there exists an $r_1 := r_1(U) \in \mathbb{R}^+$ with $\|\varphi - \psi\| < r_1$ for all $\psi \in U$. Furthermore, the openness of $U$ guarantees the existence of an $r_2 := r_2(U) \in (0; r_1(U))$ with $U_{r_2}(\varphi) := \{\chi \in C \ : \ \|\chi - \varphi\| < r_2\} \subset U$. Now, the contraction property of $R_f$ and the fixed point equation $R_f(\varphi) = \varphi$ combined yield
$$\|R_f^j(\psi) - \varphi\| = \|R_f^j(\psi) - R_f^j(\varphi)\| \leq L^j \cdot \|\psi - \varphi\| < L^j \cdot r_1 \quad \text{for all } \psi \in U \cap A(\beta) \ ,$$
which, in particular, shows
$$R_f^j(\psi) \in U_{r_2}(\varphi) \subset U$$
for all $\psi \in U \cap A(\beta)$ and all $j \geq \left\lceil \frac{\log \frac{r_2(U)}{r_1(U)}}{\log L(R_f)} \right\rceil$. Thus, there exists a natural number $k \in \mathbb{N}_{\geq 2}$ such that
$$R_f^j(U \cap A(\beta)) \subset U \quad \text{for all } j \in \mathbb{N}_{\geq k-1} \ .$$
Using $Q(\varphi) = \varphi$, $Q(U \cap H) \subset A(\beta)$, and $Q|_{U \cap A(\beta)} = R_f|_{U \cap A(\beta)}$ we obtain from this an open neighborhood $V$ of $\varphi$ in $U$ so that $V \cap H \subset D_k$, and the monotonicity of $(D_j)_{j \in \mathbb{N}}$ implies
$$V \cap H \subset D_j \quad \text{for all } j \in \mathbb{N}_{\leq k} \ .$$
Note, that we have (by definition of $D_j$, $j \in \mathbb{N}$)
$$Q^j(\psi) \in U \cap H \quad \text{for all } \psi \in D_j \text{ and all } j \in \mathbb{N}_{\leq k-1} \ .$$

2. We prove $Q^j(\psi) = R_f^{j-1}(Q(\psi))$ for all $j \in \mathbb{N}$ and all $\psi \in D_j$ by induction on $j$:
   (i) For $j = 1$ the assertion is obvious.
   (ii) Suppose that $Q^j(\psi) = R_f^{j-1}(Q(\psi))$ holds for some $j \in \mathbb{N}$ and all $\psi \in D_j$.
   (iii) Now, let $\psi \in D_{j+1}$. Then $Q^j(\psi) \in U \cap H$ and
   $$Q^{j+1}(\psi) = Q(Q^j(\psi)) \overset{\text{(ii)}}{=} Q(R_f^{j-1}(Q(\psi)) \ .$$
   Using $Q = R_f$ on $U \cap A(\beta)$ and $Q(U \cap H) \subset A(\beta)$ we get $Q^{j+1}(\psi) = R_f^j(Q(\psi))$.

3. Proof of $V \cap H \subset D_j$ for all $j \in \mathbb{N}_{\geq k}$ by induction on $j$:
   (i) For $j = k$, see the definition of $V$ in part 1.
   (ii) Suppose that $V \cap H \subset D_j$ holds for some $j \in \mathbb{N}_{\geq k}$.

(iii) Part 2 gives $Q^j(\psi) = R_f^{j-1}(Q(\psi))$ on $D_j \supset V \cap H$. By the choice of $V$, it is $Q(V \cap H) \subset U$. Using $Q(U \cap H) \subset A(\beta)$ and $j-1 \geq k-1$ we infer

$$R_f^{j-1}(Q(V \cap H)) \subset U \ .$$

Thereby, $Q^j(V \cap H) \subset U$, which implies $Q^j(V \cap H) \subset U \cap H$, or

$$V \cap H \subset D_{j+1} \ .$$

4. Combining steps 1. and 3., we obtain

$$V \cap H \subset \bigcap_{j \in \mathbb{N}} D_j \ .$$

5. Furthermore, steps 2. and 4. together with the contraction property of $R_f$ yield

$$\|Q^j(\psi) - Q^j(\chi)\| = \|R_f^{j-1}(Q(\psi)) - R_f^{j-1}(Q(\chi))\| \leq L^{j-1}\|Q(\psi) - Q(\chi)\|$$

for all $j \in \mathbb{N}$ and all $\psi$ and $\chi$ in $V \cap H$.

$\square$

The oddness of $f$ implies

$$-F_f(t,\psi) = F_f(t,-\psi) \quad \text{on } \mathbb{R}^+ \times C$$

as follows easily by virtue of the variations-of-constants formula (3.1) (cf. also the proof of LEMMA 2.2.3(1)). From this and the semiflow property it follows that the $C^1$-map $\Pi := Q^2$ satisfies

$$\begin{aligned} \Pi(\psi) &= Q(Q(\psi)) = Q(-F_f(\zeta(\psi),\psi)) = -F_f(\zeta(-F_f(\zeta(\psi),\psi)), -F_f(\zeta(\psi),\psi)) = \\ &= F_f(\zeta(-F_f(\zeta(\psi),\psi)), F_f(\zeta(\psi),\psi)) = F_f(\zeta(-F_f(\zeta(\psi),\psi)) + \zeta(\psi),\psi) \end{aligned}$$

for all $\psi \in D_2 \subset H$. Hence,

$$\Pi : V \cap H \ni \varphi \mapsto F_f(\sigma(\varphi),\varphi) \in H$$

is a POINCARÉ map in the sense of [16, p. 370] for the periodic solution $P$, where the return time map is given by

$$\sigma : V \cap H \ni \psi \mapsto q(-F_f(\zeta(\psi),\psi), f) + \zeta(\psi) \in \mathbb{R}^+$$

since

$$-F_f(\zeta(\psi),\psi) = Q(\psi) \in A(\beta) \cap U$$

and

$$\zeta(\chi) = q(\chi, f) \quad \text{for all } \chi \in U \cap A(\beta)$$

due to PROPOSITION 3.3.1 and the choice of the domain of $\Pi$. Clearly, $\varphi := P_0$ is a fixed point of $\Pi$, since in this case we have $\sigma(\varphi) = q(-F_f(\zeta(\varphi),\varphi), f) + \zeta(\varphi) = 2q(\varphi, f)$.

To point out the connection between POINCARÉ maps and the stability properties of periodic orbits, we restate the central result of [16, Section XIV.3], Theorem XIV.3.3, here in the following proposition.

**PROPOSITION 3.3.3** *If $P : \mathbb{R} \to \mathbb{R}$ is a periodic solution of (1.1) with associated* POINCARÉ *map $\Pi$ which satisfies*

$$\sigma(D\Pi(P_0)) \subset \mathbb{D} \ ,$$

*then $\mathcal{O}_P$ is hyperbolic, stable, and exponentially attractive with asymptotic phase.*

In order to prove THEOREM 3.3.1 it is therefore sufficient to show that the spectrum of the derivative

$$D\Pi(\varphi) : T_\varphi H \to T_\varphi H := \{\psi \in C \ : \ \psi(0) = 0\}$$

at the fixed point $\varphi = P_0$ is contained in the open unit disk $\mathbb{D} := \{\lambda \in \mathbb{C} \, : \, |\lambda| < 1\}$ in the complex plane. Recall that spectra of continuous linear operators in BANACH spaces over $\mathbb{R}$ are defined as the spectra of their complexifications, and that $D\Pi(\varphi)$ is a compact map (cf. [16, Proposition XIV.3.5(ii)]).

**PROPOSITION 3.3.4** *It is $\sigma(D\Pi(\varphi)) \subset \mathbb{D}$.*

PROOF: In view of the Spectral Radius Formula

$$\sup_{\lambda \in \sigma(D\Pi(\varphi))} |\lambda| = \lim_{j\to\infty} \|D\Pi(\varphi)^j\|^{\frac{1}{j}}$$

(see, e.g., Theorem V.3.5 of TAYLOR & LAY [58, p. 280]), the definition of complexifications, and $L(R_f) < 1$ it is sufficient to show

$$\limsup_{j\to\infty} \left\|D\Pi(\varphi)^j\right\|^{\frac{1}{j}} \le L(R_f) \ .$$

There is a convex open neighborhood $W$ of $\varphi$ in $C$ with $W \cap H \subset V \cap H$ such that derivatives of the $C^1$-extension of $Q$,

$$\overline{Q} : W \ni \psi \mapsto -F_f(\zeta(\psi), \psi) \in C \ ,$$

are bounded by some $c \ge 0$. For all $\psi$ and $\chi$ in $W \cap H$ and all $j \in \mathbb{N}$, PROPOSITION 3.3.2 yields

$$\left\|\Pi^j(\psi) - \Pi^j(\chi)\right\| \ = \ \left\|Q^{2j}(\psi) - Q^{2j}(\chi)\right\| \le$$

$$\begin{aligned} &\leq L(R_f)^{2j-1} \|Q(\psi) - Q(\chi)\| \leq \\ &\leq L(R_f)^j c \|\psi - \chi\| \ , \end{aligned}$$

which in turn gives

$$\left\|D\Pi(\phi)^j\right\| = \left\|D\Pi^j(\phi)\right\| \leq L(R_f)^j c \ ,$$

or

$$\left\|D\Pi(\phi)^j\right\|^{\frac{1}{j}} \leq c^{\frac{1}{j}} L(R_f) \quad \text{for all } j \in \mathbb{N} \ .$$

$\square$

An application of PROPOSITION 3.3.3 using PROPOSITION 3.3.4 proves THEOREM 3.3.1.

## 3.4 Possible improvements and comments

As in the originating paper of WALTHER [67] we only made weak assumptions on the shape of $f \in N(\beta, \varepsilon)$ up to this moment. In particular, $f$ could be chosen almost arbitrary on $(-\beta; \beta)$ (except for LIPSCHITZ continuity, boundedness, and oddness, of course). Furthermore, it is not obvious whether we can get strict contractions $R_f$ for the usual nonlinearities from EXAMPLE 1.1.1 or EXAMPLE 1.1.2.
It seems to be possible to improve the results of this chapter and to derive sharper estimates for the LIPSCHITZ constant of $R_f$ by taking into account

- that
$$f(\xi) \in [-a - \varepsilon; -a + \varepsilon] \quad \text{for } \xi \in [\beta; +\infty) \ ,$$
giving
$$y_{\pm}^{\psi,\varepsilon} : \mathbb{R} \ni t \mapsto \left(\psi(0) + \frac{a \pm \varepsilon}{\mu}\right) e^{-\mu t} - \frac{a \pm \varepsilon}{\mu} \in \mathbb{R}$$
as comparison functions for the segments $x_t^{\varphi,f}$, $t \in [1; +\infty)$, (instead of $y$) if the segment $\psi := x_{t-1}^{\varphi,f}$ satisfies $\psi \geq \beta$, and

- that we can derive better a-priori information about solutions $x^{\varphi,f}$ starting in $A(\beta)$ if, additionally,
$$f' \text{ is strictly negative on } (-\beta; \beta) \text{ and satisfies a growth condition there.}$$
The monotonicity of $f$ will enter this refined approach in a similar way as we used it in step 5. of the proof of LEMMA 1.5.2 yielding a better comparison function for measuring the diverging behaviour one time unit after the solution traversed the $\beta$-neighborhood of 0 where the nonlinearity is monotone and steep.

This will generalize the results of WALTHER [68] to the case of growth systems governed by monotone negative feedback, and is work in progress that will be contained in a forthcoming paper [44].

Nevertheless, it is clear that the smooth nonlinearities from EXAMPLE 1.1.1 and EXAMPLE 1.1.2 belong to $N(\beta, \varepsilon)$ for certain parameters such that we can infer from THEOREM 3.2.2 the existence of slowly oscillating periodic solutions of (1.1) for all these nonlinearities (which could be found via the return map $R_f$).

**EXAMPLE 3.4.1** *Let* $\mu \in (-\log 2; 0)$, $a \in \mathbb{R}^+$, *and* $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$ *with* $-\beta\mu < a - \varepsilon$ *be given. If we choose* $M := a$ *and* $\alpha \in \left(\frac{1}{\beta} \tan \frac{\pi(a-\varepsilon)}{2a}; +\infty\right)$, *then each function* $f := f_{\alpha,M}$ *from* EXAMPLE 1.1.1 *belongs to* $N(\beta, \varepsilon)$, *and for every equation* (1.1) *with* $f := f_{\alpha,M}$ *exist slowly oscillating periodic solutions.*

**EXAMPLE 3.4.2** *Let* $\mu \in (-\log 2; 0)$, $a \in \mathbb{R}^+$, *and* $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$ *with* $-\beta\mu < a - \varepsilon$ *be given. If we choose* $M := a$ *and* $\alpha \in \left(\frac{1}{\beta}\mathrm{Artanh}\frac{a-\varepsilon}{a}; +\infty\right)$, *then each function* $f := f_{\alpha,M}$ *from* EXAMPLE 1.1.2 *belongs to* $N(\beta, \varepsilon)$, *and for every equation* (1.1) *with* $f := f_{\alpha,M}$ *exist slowly oscillating periodic solutions.*

Consequently, we still raise the question whether there may exist more than one slowly oscillating solution with initial value in $A(\beta)$ for these smooth odd nonlinearities.

To answer this we will generalize an approach of CAO [12] in the next chapter which will give even more information than we would expect from our above investigations: for a class of smooth nonlinearities whose derivatives satisfy a certain convexity condition we will prove the uniqueness of slowly oscillating periodic solutions. Since this class covers the nonlinearities from EXAMPLE 3.4.1 and EXAMPLE 3.4.2 we conclude that the slowly oscillating periodic solution with initial value in $A(\beta)$ is not only unique in $A(\beta)$ but in the whole phase space $C$ !

Before we turn to this investigation we should include some further comments and remarks about the results of this chapter.

One of the main differences of our approach to that of WALTHER [67] concerns the problem of boundedness of solutions which is also the reason for the difficulties that arise if one tries to use the approach towards the existence of slowly oscillating periodic solutions via the Ejective Fixed Point Principle (cf. NUSSBAUM [48] as well as the monographs [26, 16] and the references therein): the boundedness of the solutions starting in $A(\beta)$ is guaranteed by COROLLARY 3.1.1 and is a consequence of hypothesis $(N_4)$.

Furthermore, the fact that the period of the comparison solution $y$ of (2.1) may be extremely long causes some problems as we already mentioned in the text.

We should also mention that there is also another possibility to obtain results analogous to THEOREM 3.3.1 (as also outlined in WALTHER [67]). One can find nonlinearities which coincide with $g := -a\,\text{sign}$ outside a small neighborhood of 0, such that (1.1) has a periodic solution $P$ similar to the slowly oscillating solution $y$ of (2.1) with an orbit $\mathcal{O}_P$ into which many solution curves $F_f(\cdot, \varphi)$ will merge. An associated POINCARÉ map $\Pi$ is then constant, and, thus, $D\Pi(P_0) = 0$. A perturbation theorem as in LANI-WAYDA [35] would then lead to a set of $C^1$–nonlinearities so that equation (1.1) defines a periodic orbit near $\mathcal{O}_P$ which is hyperbolic and stable as above.

Another result (for a "complementary" situation to our situation) which is based on this approach can be found in IVANOV, LANI-WAYDA and WALTHER [29, Corollary 4.2].

All in all, this chapter extends the method of WALTHER [67, 68] to scalar growth systems governed by negative feedback and yields a first existence result for slowly oscillating periodic solutions around $\xi^0 = 0$ in this setting. Furthermore, there are also forthcoming extensions of this method to systems of delay equations due to WU [73] and to state-dependent delay equations due to WALTHER [69].

# 4

# Uniqueness of slowly oscillating periodic solutions

In the preceding chapter we proved the existence of slowly oscillating periodic solutions around $\xi^0 := 0$ for the differential delay equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1.1}$$

with $f$ belonging to a rather general class of nonlinearities and $\mu \in (-\log 2; 0)$.

Following the approach of CAO [12] here we will prove the uniqueness of slowly oscillating periodic solutions for equation (1.1) with $\mu$ and $f$ satisfying hypothesis (H1)–(H2) from Section 1.1 as well as an additional hypothesis (H4) which will be introduced and discussed in Section 2. Here, by "uniqueness of a slowly oscillating periodic solution $x : \mathbb{R} \to \mathbb{R}$" we mean the uniqueness of its orbit $\mathcal{O}_x$ in $C$: if $x$ is such a unique slowly oscillating periodic solution and $y : \mathbb{R} \to \mathbb{R}$ is any (other) slowly oscillating periodic solution of (1.1), then $\mathcal{O}_y = \mathcal{O}_x$, i.e., there is a $t_y \in \mathbb{R}$ such that $x_t = y_{t+t_y}$ for all $t \in \mathbb{R}$.

Surprisingly at first sight, the method of CAO [12] does *not* rely on arguments in the phase space $C$ concerning the orbits $\mathcal{O}$ of slowly oscillating periodic solutions. It is based on earlier work of KAPLAN and YORKE [30, 31], WALTHER [61] and NUSSBAUM [49], and considers the "projection" of the orbit $\mathcal{O}_x \subset C$ into the real $(x, \dot{x})$-plane (somehow analogous to ordinary differential equations). These "projections" into $\mathbb{R}^2$ are JORDAN curves for slowly oscillating periodic solutions and one can obtain the uniqueness from a condition prescribing the mutual positions of these JORDAN curves corresponding to different slowly oscillating solutions: this key result (PROPOSITION 4.3.1) will be proved in full detail whereas we will only sketch the preliminary results of the first section. The general reference for this part is, of course, CAO's article [12] which contains the basic material which can be adapted to our situation. Furthermore, we should mention the Diploma Thesis [23] of GOMBERT who worked out the article of CAO [12] in great detail.

Observe that the hypothesis (H2) implies

$$\xi \cdot f(\xi) < 0 \quad \text{for all } \xi \in \mathbb{R} \setminus \{0\} \tag{4.1}$$

and, in particular,

$$f'(\xi) < 0 \quad \text{for all } \xi \in \left[\tfrac{M_f}{\mu}; -\tfrac{M_f}{\mu}\right] \ . \tag{4.2}$$

## 4.1 SOP-solutions and their orbits in $\mathbb{R}^2$

Throughout the whole chapter we assume (H1) and (H2) without further mentioning. Since we want to show uniqueness properties of slowly oscillating periodic solutions we introduce the following normalization of a slowly oscillating periodic solution around $\xi^0 = 0$.

**DEFINITION 4.1.1** A periodic solution $x : \mathbb{R} \to \mathbb{R}$ of (1.1) with minimal period $q$ oscillating around $\xi^0 = 0$ is called a ***SOP-solution*** (slowly oscillating periodic solution around $\xi^0 = 0$) if there exists $p \in (1; +\infty)$ such that $q - p > 1$,

$$x(t) > 0 \quad \text{for all } t \in (0; p)$$

and

$$x(t) < 0 \quad \text{for all } t \in (p; q) \ .$$

We should note some facts about SOP-solutions $x : \mathbb{R} \to \mathbb{R}$ for later use. Obviously, it is

$$0 = x(0) = x(p) = x(q) \quad \text{and} \quad \dot{x}(p) < 0 < \dot{x}(q) = \dot{x}(0)$$

for every SOP-solution $x : \mathbb{R} \to \mathbb{R}$. Furthermore, notice that the $q$-periodicity of $x$ yields that $\dot{x}$ is also periodic since

$$\dot{x}(t+q) = -\mu x(t+q) + f(x(t+q-1)) = -\mu x(t) + f(x(t-1)) = \dot{x}(t)$$

holds for all $t \in \mathbb{R}$, and recall from COROLLARY 1.4.2 and REMARK 1.5.1 that a SOP-solution $x$ of (1.1) is necessarily bounded and satisfies

$$x(\mathbb{R}) \subset I_\infty := \left(\tfrac{M_f}{\mu}; -\tfrac{M_f}{\mu}\right) \ . \tag{4.3}$$

The smoothness assumptions on $f$ force SOP-solutions $x$ to have very simple graphs without multiple relative extrema (as already proved by MALLET-PARET & NUSSBAUM [40, Corollary 3.1]) and allow us to obtain a-priori information about $x$ which is collected in the lemma below.

**LEMMA 4.1.1** *If $x$ is a SOP-solution of (1.1), then $\dot{x}$ is also slowly oscillating and the zeros $t_1 \in (0;p)$ and $t_2 \in (p;q)$ of $\dot{x}$ satisfy $t_2 - t_1 > 1$, $t_1 + q - t_2 > 1$, such that $\dot{x}|_{[0;t_1)\cup(t_2;q]} > 0$ and $\dot{x}|_{(t_1;t_2)} < 0$.*

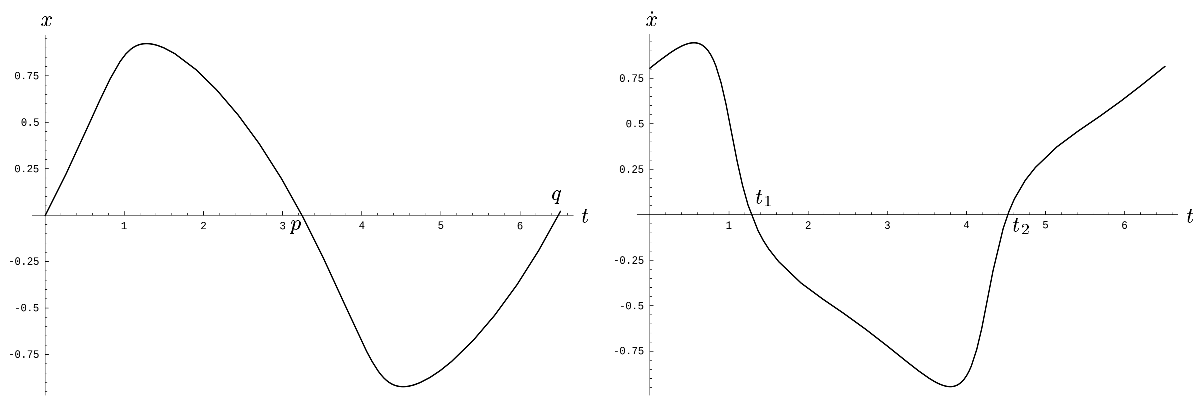
PROOF: From $\dot{x}(p) < 0 < \dot{x}(0) = \dot{x}(q)$ the existence of zeros $t_1 \in (0;p)$ and $t_2 \in (p;q)$ of $\dot{x}$ is obvious. Then, the boundedness property (4.3) together with a slight adaptation of the approach of MALLET-PARET & NUSSBAUM [40, pp. 66–76] yield the assertions about the sign of $\dot{x}$ in $[0;q]$. This can be done in exactly the same way as in CAO [12, pp. 49–50]. Now, notice that $y := \dot{x}$ solves the non-autonomous delay equation

$$\dot{y}(t) = -\mu y(t) + f'(x(t-1))y(t-1)$$

on $\mathbb{R}$. This equation is of type (1.15) such that we can define the discrete LYAPUNOV functional $V$ as in DEFINITION 1.6.2, and the arguments of the proof of [12, LEMMA 3] yield that $y = \dot{x}$ is slowly oscillating with $t_2 - t_1 > 1$ and $t_1 + q - t_2 > 1$.
Alternatively, the assertions can also be derived by elementary but rather lengthy and technical arguments, cf. GOMBERT [23, pp. 12–42] (notice that the methods in [23] apply with only negligible changes to the general case $\mu \neq 0$).

$\square$

We are now able to draw a precise picture of the shape of a SOP-solution $x$ in $[0;q]$ by virtue of the last lemma:




As it turns out, we will also need a scaled version of (1.1) for our investigations,

$$\dot{z}(t) = -\mu z(t) + \lambda \cdot f\left(\lambda^{-1} \cdot z(t-1)\right) \tag{4.4}$$

for $\lambda \in [1;+\infty)$, and $\mu$ and $f$ as above. The connection between slowly oscillating solutions of (1.1) and those of (4.4) is given in

**REMARK 4.1.1** *Let $\lambda \in \mathbb{R}^+$. A function $x$ is a SOP-solution of (1.1) if and only if $z := \lambda \cdot x$ is a SOP-solution of (4.4).*

PROOF: Let $z$ be a SOP-solution of (4.4). Since

$$\begin{aligned}\lambda\dot{x}(t) &= \dot{z}(t) = -\mu z(t) + \lambda f(\lambda^{-1}z(t-1)) = -\mu\lambda x(t) + \lambda f(x(t-1)) = \\ &= \lambda(-\mu x(t) + f(x(t-1)))\ ,\end{aligned}$$

we see that $x$ satisfies (1.1) and therefore it is a slowly oscillating solution of (1.1). On the other hand, if $x$ is a slowly oscillating solution of (1.1), we obtain from multiplying (1.1) by $\lambda$ that $z$ solves (4.4) and is therefore a slowly oscillating solution of (4.4).

$\square$

We now turn to the principal object of our interest in this chapter, the projection of the orbits of SOP-solutions into the real plane $\mathbb{R}^2$.

**DEFINITION 4.1.2** The $\mathbb{R}^2$–***orbit*** of a SOP-solution $x : \mathbb{R} \to \mathbb{R}$ is the trace of the curve

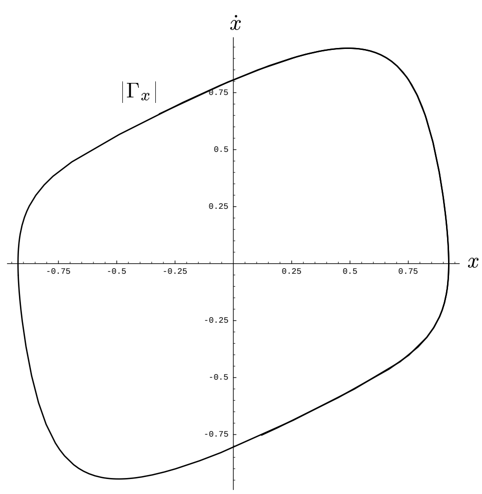
$$\Gamma_x : \mathbb{R} \ni t \mapsto \Big(x(t), \dot{x}(t)\Big) \in \mathbb{R}^2\ ,$$

denoted as

$$|\Gamma_x| := \Gamma_x(\mathbb{R}) = \Big\{\Big(x(t), \dot{x}(t)\Big) \in \mathbb{R}^2\ :\ t \in \mathbb{R}\Big\}\ .$$

The $q$-periodicity of a SOP-solution $x$ and the regular shape of $x$ described by LEMMA 4.1.1 permit a more detailed description of the $\mathbb{R}^2$-orbit of $x$ which is the content of the following remark that restates COROLLARY 3 of CAO [12, p. 49].

**REMARK 4.1.2** *Let $|\Gamma_x|$ be the $\mathbb{R}^2$–orbit of a SOP-solution $x : \mathbb{R} \to \mathbb{R}$ of* (1.1). *Then* $|\Gamma_x| = \{(x(t), \dot{x}(t))\ :\ t \in [0; q)\}$, *$\Gamma_x$ is a* JORDAN *curve and* $(0,0) \in \operatorname{int}|\Gamma_x|$.




Even more could be said about the regularity of $\mathbb{R}^2$–orbits of SOP-solutions: the trace of the corresponding JORDAN curve is the union of two graphs of functions defined on the interval $x(\mathbb{R})$.

**LEMMA 4.1.2** *Let $x$ be a SOP-solution of* (1.1) *and set $J := x(\mathbb{R})$. Then there exist continuously differentiable functions $\varphi^\kappa : J \to \mathbb{R}_0^\kappa$, $\kappa \in \{-,+\}$, with $\varphi^\kappa(J^o) \subset \mathbb{R}^\kappa$ for $\kappa \in \{-,+\}$ and*

$$|\Gamma_x| = \mathrm{graph}(\varphi^+) \cup \mathrm{graph}(\varphi^-) \ .$$

IDEA OF THE PROOF: From LEMMA 4.1.1 we obtain $J = [x(t_2); x(t_1)]$. Furthermore, the sign conditions on $\dot{x}$ imply that $x$ is strictly monotonically increasing on $[0; t_1) \cup (t_2; q]$. Therefore, the inverses of $x$ on these intervals, $\tau_1 := x|_{[0;t_1]}^{-1}$ and $\tau_2 := x|_{[t_2;q]}^{-1}$, give rise to

$$\varphi^+ : [x(t_2); x(t_1)] \ni \xi \mapsto \left\{ \begin{array}{ccl} \dot{x}(\tau_1(\xi)) & , & \text{if } \xi \in [0; x(t_1)] \\ \dot{x}(\tau_2(\xi)) & , & \text{if } \xi \in [x(t_2); 0] \end{array} \right\} \in \mathbb{R}_0^+ \ .$$

Clearly, $\varphi^+|_{(0;x(t_1))}$ and $\varphi^+|_{(x(t_2);0)}$ are continuously differentiable, and with some additional work one can establish $\varphi^+ \in C^1(J^o) = C^1((x(t_2); x(t_1)))$: this is similar to the ideas we will use in step 3.3 of the proof of PROPOSITION 4.3.1 such that we omit the details here.

□

Since $\mathbb{R}^2$–orbits of SOP-solutions are JORDAN curves around the origin, we note some elementary geometric properties of such constellations that will be of importance in the sequel. For simplicity, we endow $\mathbb{R}^2$ with the euclidean norm,

$$\|\cdot\|_2 : \mathbb{R}^2 \ni (\xi, \eta) \mapsto \sqrt{\xi^2 + \eta^2} \in \mathbb{R}_0^+ \ .$$

**LEMMA 4.1.3** *Let $\Gamma_j$, $j \in \{1,2\}$, be* JORDAN *curves in $\mathbb{R}^2$ with $\Gamma_1 \neq \Gamma_2$ and $(0,0) \in \mathrm{int}\,|\Gamma_j|$ for $j \in \{1,2\}$. Then there exists a $\varrho \in (1; +\infty)$ such that*

$$\textit{either} \quad \varrho|\Gamma_1| \not\subset \mathrm{ext}\,|\Gamma_2| \qquad \textit{or} \quad \varrho|\Gamma_2| \not\subset \mathrm{ext}\,|\Gamma_1| \ .$$

IDEA OF THE PROOF: Since $(0,0) \in \mathrm{int}\,|\Gamma_1|$ set

$$r_1 := \sup\left\{r \in \mathbb{R}^+ \ : \ \{(\xi,\eta) \in \mathbb{R}^2 \ : \ \|(\xi,\eta)\|_2 < r\} \subset \mathrm{int}\,|\Gamma_1|\right\}$$

and

$$r_2 := \max\{\|(\xi,\eta)\|_2 \ : \ (\xi,\eta) \in |\Gamma_2|\} \ .$$

Now, we have $r|\Gamma_1| \subset \overline{\mathrm{ext}\,|\Gamma_2|}$ for all $r \in \left[\frac{r_2}{r_1}; +\infty\right)$, such that we can set

$$\varrho := \sup\{r \in \mathbb{R}^+ \ : \ r|\Gamma_1| \not\subset \overline{\mathrm{ext}\,|\Gamma_2|}\} \ .$$

In case $\varrho \notin (1; +\infty)$ exchange the roles of $\Gamma_1$ and $\Gamma_2$ above.

□

**Definition 4.1.3** For $\vartheta \in [-\frac{3\pi}{2}; \frac{\pi}{2})$ we define the ray

$$\ell(\vartheta) := \left\{ r \cdot (\cos\vartheta, \sin\vartheta) \in \mathbb{R}^2 \; : \; r \in \mathbb{R}^+ \right\} .$$

Clearly, each ray $\ell(\vartheta)$, $\vartheta \in [-\frac{3\pi}{2}; \frac{\pi}{2})$, intersects a Jordan curve around the origin at least once. Generally, it is not clear whether these intersections are singletons, or in geometric terms, whether the orbit of a SOP-solution is star-shaped (as in the figure above).

Anyway, we can define the following functions that give the "earliest" and the "latest" intersection point of a $\mathbb{R}^2$–orbit with the ray $\ell(\vartheta)$ for every angle $\vartheta \in [-\frac{3\pi}{2}; \frac{\pi}{2})$, respectively.

**Definition 4.1.4** Let $x$ be a SOP-solution of (1.1) of period $q$ and, for $n \in \mathbb{Z}$, let $D_n := [(n-1)q; nq)$. Then we can define

$$\begin{aligned} \psi_{n,x} \; &: \; \left(-\tfrac{3\pi}{2}; \tfrac{\pi}{2}\right) \ni \vartheta \mapsto \inf\{t \in D_n \; : \; (x(t), \dot{x}(t)) \in \ell(\vartheta)\} \in D_n, \\ \Psi_{n,x} \; &: \; \left(-\tfrac{3\pi}{2}; \tfrac{\pi}{2}\right) \ni \vartheta \mapsto \sup\{t \in D_n \; : \; (x(t), \dot{x}(t)) \in \ell(\vartheta)\} \in D_n. \end{aligned}$$

As it turns out, the functions $\psi_{n,x}$ and $\Psi_{n,x}$ are strictly monotone for every fixed $n \in \mathbb{Z}$. This is the assertion of Proposition 3.3 of Krisztin & Walther [32, p. 16] which we restate here as

**Lemma 4.1.4** *Let $n \in \mathbb{Z}$. Then the maps $\psi_{n,x}$ and $\Psi_{n,x}$ are strictly decreasing.*

## 4.2 An additional assumption

The approach of Cao necessitates the introduction of the following assumption in addition to (H1) and (H2). We will discuss the role of this at the end of Section 4.3.

(H4) Let the nonlinearity $f$ be given such that the auxiliary function

$$h : \mathbb{R}\backslash\{0\} \ni \xi \mapsto \frac{\xi f'(\xi)}{f(\xi)} \in \mathbb{R}$$

- (H4.1) has range $h(\mathbb{R}\backslash\{0\}) \subset (0;1)$, and
- (H4.2) is monotonically decreasing on $\mathbb{R}^+$, and monotonically increasing on $\mathbb{R}^-$.

$\diamond$

Notice that assumption (H4) is valid, e.g., for our prototype nonlinearities from Section 1.1 as an easy calculation shows.

**EXAMPLE 4.2.1** *The hypothesis* (H4) *is valid for all* $f := f_{\alpha,M}$ *from* EXAMPLE 1.1.1.

**EXAMPLE 4.2.2** *The hypothesis* (H4) *is valid for all* $f := f_{\alpha,M}$ *from* EXAMPLE 1.1.2.

So, there is a large class of interesting nonlinearities which satisfy (H1), (H2) and (H4). We will need a last preparatory result which is an easy consequence of (H4.1).

**LEMMA 4.2.1** *Let* $f$ *be such that* (H4.1) *is satisfied, and fix* $x \in \mathbb{R} \setminus \{0\}$. *Then*

$$H(\cdot, x) : \mathbb{R}^+ \ni \lambda \mapsto \lambda f(\lambda^{-1}x) \in \mathbb{R}$$

*is (strictly) monotonically decreasing if* $x \in \mathbb{R}^+$, *and (strictly) monotonically increasing in case* $x \in \mathbb{R}^-$.

PROOF: Let $\lambda \in \mathbb{R}^+$. From

$$\frac{dH(\lambda, x)}{d\lambda} = f(x\lambda^{-1}) + \lambda f'(x\lambda^{-1})(-x\lambda^{-2}) = f(x\lambda^{-1}) \cdot (1 - h(x\lambda^{-1}))$$

we obtain, due to (4.1) and $h(\mathbb{R} \setminus \{0\}) \subset (0; 1)$,

$$\frac{dH(\lambda, x)}{d\lambda} \begin{cases} < 0 & , \ x > 0, \\ > 0 & , \ x < 0, \end{cases}$$

which proves the lemma.

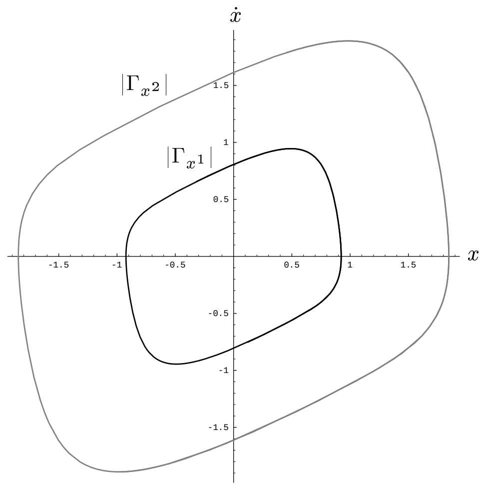
□

## 4.3 Influence of parameters on the shape of the orbits

The key result of this chapter gives a geometric description of the general position of the $\mathbb{R}^2$–orbits of SOP-solution $x^j$ of (4.4) for $\lambda := \lambda_j$, $j \in \{1, 2\}$, in case that the parameters satisfy $\lambda_2 > \lambda_1$.

**PROPOSITION 4.3.1** *Let* $x^j$ *be SOP-solutions of* (4.4) *for* $\lambda = \lambda_j$, $j \in \{1, 2\}$. *If* $\lambda_2 > \lambda_1$, *then*

$$|\Gamma_{x^2}| \subset \text{ext}\, |\Gamma_{x^1}| \ .$$




PROOF: Set $\alpha := 1$. For $j \in \{1, 2\}$ let $x^j$ be a SOP-solution of (4.4), i.e., $x^j$ satisfies

$$\dot{x}^j(t) = -\mu x^j(t) + \lambda_j f(\lambda_j^{-1} x^j(t - \alpha)) \ . \tag{$\lambda_j$}$$

We set $|\Gamma_j| := |\Gamma_{x^j}|$ for $j \in \{1, 2\}$ and *assume to the contrary* that

$$|\Gamma_2| \not\subset \text{ext}\, |\Gamma_1| \ .$$

Thus, $|\Gamma_2|$ is not totally in the exterior of $|\Gamma_1|$, and a completely analogous reasoning as in the (sketch of the) proof of LEMMA 4.1.3 shows the existence of

$$\varrho := \max \left\{ r \in \mathbb{R}^+ \ : \ r|\Gamma_2| \not\subset \text{ext}\, |\Gamma_1| \right\} \in [1; +\infty) \ .$$

With this $\varrho \in [1; +\infty)$ we define

$$\lambda_0 := \varrho \lambda_2, \quad \text{ and } \quad x^0 := \varrho x^2 \ ,$$

such that $x^0$ solves $(\lambda_0)$ and we have $|\Gamma_0| := |\Gamma_{x^0}| = \varrho |\Gamma_2|$.

**1.** Clearly, $|\Gamma_0| \cap |\Gamma_1| \neq \emptyset$ implies the existence of (at least) one intersection point, i.e., the existence of $t^j \in \mathbb{R}$, $j \in \{0, 1\}$, such that

$$(x^0(t^0), \dot{x}^0(t^0)) = (x^1(t^1), \dot{x}^1(t^1)) \ . \tag{4.5}$$

**2.** We claim that

$$b := \dot{x}^0(t^0) = \dot{x}^1(t^1) \neq 0 \ . \tag{4.6}$$

2.1 To prove this assertion we derive a contradiction from the `assumption`

$$\dot{x}^0(t^0) = \dot{x}^1(t^1) = 0 \ .$$

Because of (4.5) and $(0,0) \notin |\Gamma_0| \cup |\Gamma_1|$ it is

$$x^0(t^0) = x^1(t^1) =: c \neq 0$$

such that we may assume without loss of generality $c > 0$ since the treatment for $c < 0$ is similar. From LEMMA 4.1.1 we know that $\dot{x}^j$, $j \in \{0,1\}$, is slowly oscillating such that we obtain from $x^j(t^j) = c > 0$ and DEFINITION 4.1.1,

$$\dot{x}^j(t) > 0 \quad \text{for all } t \in [t^j - \alpha; t^j),\ j \in \{0,1\}.$$

2.2 We have

$$0 < x^0(t^0 - \alpha) < x^1(t^1 - \alpha) \ . \tag{4.7}$$

To see this, let $j \in \{0,1\}$. Since $t^j$ is a local extremum of $x^j$, we obtain from (H1)

$$0 = \dot{x}^j(t^j) = -\mu x^j(t^j) + f(x^j(t^j - \alpha)) > +f(x^j(t^j - \alpha)) \ .$$

Consequently, the negative feedback property of $f$, (4.1) yields

$$x^j(t^j - \alpha) > 0 \quad \text{for } j \in \{0,1\} \ .$$

Subtracting equation $(\lambda_0)$ at time $t^0$ from equation $(\lambda_1)$ at time $t^1$ and using $\dot{x}^0(t^0) = \dot{x}^1(t^1) = 0$ as well as $x^0(t^0) = x^1(t^1) = c$ we get

$$\lambda_0 f\left(\frac{x^0(t^0 - \alpha)}{\lambda_0}\right) = \lambda_1 f\left(\frac{x^1(t^1 - \alpha)}{\lambda_1}\right) \ . \tag{4.8}$$

Now, an application of LEMMA 4.2.1, recalling $\lambda_1 < \lambda_0$ and $x^j(t^j - \alpha) > 0$ for $j \in \{0,1\}$, yields

$$\begin{aligned} \lambda_0 f\left(\frac{x^0(t^0 - \alpha)}{\lambda_0}\right) &= \lambda_1 f\left(\frac{x^1(t^1 - \alpha)}{\lambda_1}\right) = H\left(\lambda_1, x^1(t^1 - \alpha)\right) > \\ &> H\left(\lambda_0, x^1(t^1 - \alpha)\right) = \lambda_0 f\left(\frac{x^1(t^1 - \alpha)}{\lambda_0}\right), \end{aligned}$$

such that we have

$$f\left(\frac{x^0(t^0 - \alpha)}{\lambda_0}\right) > f\left(\frac{x^1(t^1 - \alpha)}{\lambda_0}\right) \ .$$

Thus, the strict monotonicity of $f$ on $\left[\frac{M_f}{\mu}; -\frac{M_f}{\mu}\right]$, (4.2), gives

$$\frac{x^0(t^0 - \alpha)}{\lambda_0} < \frac{x^1(t^1 - \alpha)}{\lambda_0} \ ,$$

which proves $x^0(t^0 - \alpha) < x^1(t^1 - \alpha)$.

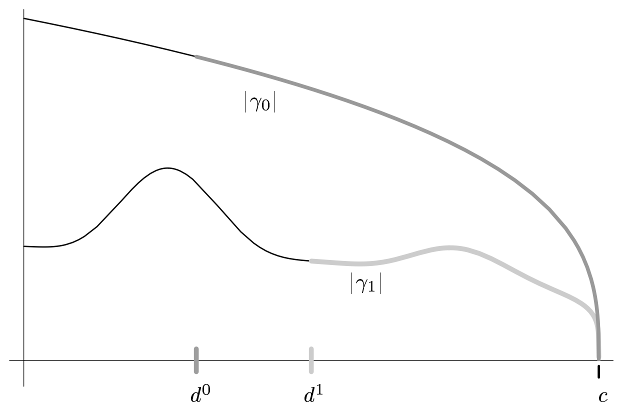
2.3 For $j \in \{0,1\}$ set $d_j := x^j(t^j - \alpha)$ and consider the JORDAN arcs

$$\gamma_j : [t^j - \alpha; t^j] \ni t \mapsto \big(x^j(t), \dot{x}^j(t)\big) \in \mathbb{R}^2$$

whose traces lie in the first quadrant $\mathbb{R}^2_+$ except for the right endpoint $(c, 0) \in |\gamma_j|$. As a consequence of $|\gamma_j| \subset |\Gamma_j| \cap (\mathbb{R}^+_0)^2$, $\dot{x}^j > 0$ on $[t^j - \alpha, t^j)$, and LEMMA 4.1.2, we obtain

$$|\gamma_j| = \operatorname{graph}(\varphi_j) \quad \text{ with } \varphi_j := \varphi_j^+\Big|_{[d_j;c]} ,$$

and $\varphi_j(\xi) > 0$ for all $\xi \in [d_j; c)$, $j \in \{0,1\}$.




2.4 By construction, $|\Gamma_0|$ is in the closure of the exterior of $|\Gamma_1|$. Thus, we infer using the JORDAN Curve Theorem after some rather lengthy but elementary plane-topological considerations (cf. also GOMBERT [23])

$$\varphi_0(\xi) \geq \varphi_1(\xi) > 0 \quad \text{for all } \xi \in [d_1; c) \ . \tag{4.9}$$

Observe that $d_0 = x^0(t^0 - \alpha) < x^1(t^1 - \alpha) = d_1$ by step 2.2.
Obviously, we obtain from (4.9) the estimate

$$\int_{d_1}^{c} \frac{1}{\varphi_0} - \int_{d_1}^{c} \frac{1}{\varphi_1} = \int_{d_1}^{c} \frac{\varphi_1 - \varphi_0}{\varphi_0 \varphi_1} \leq 0 \ . \tag{4.10}$$

2.5 Fix $j \in \{0,1\}$. For every $t \in [t^j - \alpha; t^j]$ we have $(x^j(t), \dot{x}^j(t)) \in |\gamma_j| = \operatorname{graph}(\varphi_j)$, such that there exists a $\xi \in [d_j; c]$ with $x^j(t) = \xi$ and $\dot{x}^j(t) = \varphi^j(\xi) = \varphi(x^j(t))$. This means, that $x^j$ satisfies the ordinary differential equation

$$\dot{x}^j(t) = \varphi_j(x^j(t)) \quad \text{ on } [t^j - \alpha; t^j)$$

Furthermore, step 2.3 guarantees the positivity of $\varphi_j \circ x^j$ on $[t^j - \alpha; t^j)$ for $j \in \{0,1\}$, and we obtain for $t \in [t^j - \alpha; t^j)$

$$t - t^j + \alpha \ = \int_{t^j - \alpha}^{t} d\tau = \int_{t^j - \alpha}^{t} \frac{\dot{x}^j(\tau)}{\varphi_j(x^j(\tau))} d\tau = \int_{x^j(t^j - \alpha)}^{x^j(t)} \frac{d\sigma}{\varphi_j(\sigma)} \ .$$

Now, the continuity of $x^j$ at $t^j$ yields

$$\alpha = \lim_{t \nearrow t^j}(t - t^j + \alpha) = \lim_{t \nearrow t^j} \int\limits_{x^j(t^j-\alpha)}^{x^j(t)} \frac{d\sigma}{\varphi_j(\sigma)} = \int\limits_{d_j}^{c} \frac{d\sigma}{\varphi_j(\sigma)}$$

using the abbreviations $d_j = x^j(t^j - \alpha)$, and our assumption $c = x^j(t^j)$, $j \in \{0, 1\}$.

2.6 The Mean Value Theorem guarantees the existence of $\tau_j \in (t^j - \alpha; t^j)$ such that

$$\alpha = \frac{x^j(t^j) - x^j(t^j - \alpha)}{\dot{x}^j(\tau_j)} = \frac{c - d_j}{\dot{x}^j(\tau_j)}$$

for $j \in \{0, 1\}$. Hence, this identity combined with (4.7) yields

$$\dot{x}^0(\tau_0) = \frac{c - d_0}{c - d_1}\dot{x}^1(\tau_1) < \dot{x}^1(\tau_1) \ . \tag{4.11}$$

Furthermore,

$$\begin{aligned} d_1 - d_0 &= x^1(t^1 - \alpha) - c + c - x^0(t^0 - \alpha) = \\ &= x^1(t^1 - \alpha) - x^1(t^1) + x^0(t^0) - x^0(t^0 - \alpha) = \\ &= -\alpha\dot{x}^1(\tau_1) + \alpha\dot{x}^0(\tau_0) \end{aligned}$$

yields

$$d_1 - d_0 = \alpha(-\dot{x}^1(\tau_1) + \dot{x}^0(\tau_0)) \ .$$

2.7 Finally, setting

$$M_{0,1} := \max_{\xi \in [d_0; d_1]} \frac{1}{\varphi_0(\xi)} > 0$$

we obtain from

$$\begin{aligned} 0 &= \alpha - \alpha = \int\limits_{d_0}^{c} \frac{1}{\varphi_0} - \int\limits_{d_1}^{c} \frac{1}{\varphi_1} \overset{(4.10)}{\leq} \int\limits_{d_0}^{d_1} \frac{1}{\varphi_0} \leq (d_1 - d_0) \cdot M_{0,1} = \\ &= (-\dot{x}^1(\tau_1) + \dot{x}^0(\tau_0)) \cdot \alpha \cdot M_{0,1} \end{aligned}$$

the `contradiction`

$$\dot{x}^1(\tau_1) \leq \dot{x}^0(\tau_0)$$

to (4.11).

Thus, we established (4.6).

**3.** Now, we prove that (4.6) implies a contradiction such that the assumption

$$|\Gamma_2| \not\subset \operatorname{ext}|\Gamma_1|$$

was false.

This step is for the most part a reproduction of the proof of "Claim (B)" of Cao [12, pp. 52–55] (except for steps 3.5 and 3.6), and is included only for completeness. Our presentation of this part follows the lines of Gombert [23] rather closely.
For simplicity, we continue in our notation from above and recall the definitions

$$c := x^1(t^0) = x^1(t^1) \quad \text{ and } \quad b := \dot{x}^0(t^0) = \dot{x}^1(t^1)$$

as well as

$$d_j := x^j(t^j - \alpha) \quad \text{ for } j \in \{0,1\}.$$

Recall that we `assume` $\dot{x}^0(t^0) = \dot{x}^1(t^1) \neq 0$.

3.1 By continuity of $\dot{x}^j$ and $b \neq 0$ there exists a neighborhood $U_j$ of $t^j$ such that $0 \notin x^j(U_j)$ for $j \in \{0,1\}$. Thus, each $x^j|_{U_j}$ is an invertible $C^1$-map with $C^1$-inverse

$$y^j := \left(x^j|_{U_j}\right)^{-1} \quad ,$$

and $V_j := x^j(U_j)$ is an open neighborhood of $c = x^j(t^j)$ for $j \in \{0,1\}$.
Furthermore, the maps

$$\varphi_j := \dot{x}^j \circ y^j : V_j \to \mathbb{R}, \; j \in \{0,1\},$$

are well-defined and in $C^1$ because $\dot{x}^j$ is continuously differentiable by $(\lambda_j)$.
For $j \in \{0,1\}$ we define the Jordan arcs

$$\omega_j : U_j \ni t \mapsto \left(x^j(t), \dot{x}^j(t)\right) \in \mathbb{R}^2$$

and claim

$$|\omega_j| = \operatorname{graph}(\varphi_j) := \{(\xi, \varphi_j(\xi)) \; : \; \xi \in V_j\} \; .$$

To see this let $z \in \operatorname{graph}(\varphi_j)$. Then there is a $\xi \in V_j$ with $z = (\xi, \varphi_j(\xi))$. By definition of $V_j$ there is a $s \in U_j$ such that $\xi = x^j(s)$ and $\varphi_j(\xi) = (\dot{x}^j \circ y^j \circ x^j)(s) = \dot{x}^j(s)$. Hence, we have $z = (x^j(s), \dot{x}^j(s)) \in |\omega_j|$. For the proof of the reverse inclusion let $z \in |\omega_j|$, i.e. there is a $s \in U_j$ with $z = (x^j(s), \dot{x}^j(s)$. Consequently, we obtain $\xi := x^j(s) \in V_j$ and $s = y^j(\xi)$ such that $\dot{x}^j(s) = (\dot{x}^j \circ y^j)(\xi) = \varphi_j(\xi)$ which proves $z \in \operatorname{graph}(\varphi_j)$.

3.2 In this step of the proof we show that the traces $|\omega_0|$ and $|\omega_1|$ intersect tangentially at the point $(x^j(t^j), \dot{x}^j(t^j)) = (c, b)$ (as one would expect by definition of $\Gamma_0$).

3.2.1 We show that

$$\varphi_0(c) = \varphi_1(c) \ . \tag{4.12}$$

This assertion follows immediately from 3.1 and

$$\begin{aligned}\varphi_0(c) &= (\dot{x}^0 \circ y^0)(x^0(t^0)) = \dot{x}^0(t^0) = b = \dot{x}^1(t^1) = \\ &= (\dot{x}^1 \circ y^1)(x^1(t^1)) = \varphi_1(c).\end{aligned}$$

3.2.2 Now, we can claim

$$\varphi_0'(c) = \varphi_1'(c) \ . \tag{4.13}$$

In order to establish this identity we assume (once more) to the contrary that $\varphi_0'(c) \neq \varphi_1'(c)$. By (4.12), we have

$$s := \varphi_0'(c) - \varphi_1'(c) = \lim_{h\to 0} \frac{\varphi_0(c+h) - \varphi_1(c+h)}{h} \neq 0, \tag{4.14}$$

and we may assume $s > 0$ without loss of generality (because similar arguments apply in the case $s < 0$).

According to the sign of $b$ one has to distinguish the cases $b < 0$ and $b > 0$.

3.2.2.1 In case $b > 0$ we infer from $|\Gamma_0| \subset \overline{\operatorname{ext}|\Gamma_1|}$ (using the Jordan Curve Theorem; cf. Gombert [23] for details)

$$\varphi_0(\xi) \geq \varphi_1(\xi) \quad \text{for all } \xi \in V_0 \cap V_1 \ . \tag{4.15}$$

Clearly, $V_0 \cap V_1$ is again an open neighborhood of $c$ such that we can find a natural number $n_0$ with $c - \frac{1}{n_0} \in V_0 \cap V_1$ and

$$\frac{\varphi_0(c - \frac{1}{n_0}) - \varphi_1(c - \frac{1}{n_0})}{-\frac{1}{n_0}} > 0$$

according to (4.14). But this would imply $\varphi_0(c - \frac{1}{n_0}) < \varphi_1(c - \frac{1}{n_0})$ in contradiction to (4.15).

3.2.2.2 In the case $b < 0$ one uses $\varphi_0(\xi) \leq \varphi_1(\xi)$ for all $\xi \in V_0 \cap V_1$ to derive the contradiction in a completely similar fashion.

This completes the proof of (4.13)

3.3 We prove

$$\ddot{x}^0(t^0) = \ddot{x}^1(t^1) \ .$$

For $j \in \{0,1\}$ and $\xi \in V_0 \cap V_1$ the identity

$$\varphi_j'(\xi) = (\dot{x}^j \circ y^j)'(\xi) = \ddot{x}^j(y^j(\xi)) \cdot \frac{1}{\dot{x}^j(y^j(\xi))} = \frac{\ddot{x}^j(y^j(\xi))}{\varphi_j(\xi)}$$

together with $y^j(c) = t^j$ and

$$\frac{\ddot{x}^0(t^0)}{\varphi_0(c)} = \varphi_0'(c) \overset{(4.13)}{=} \varphi_1'(c) = \frac{\ddot{x}^1(t^1)}{\varphi_1(c)} \overset{(4.12)}{=} \frac{\ddot{x}^1(t^1)}{\varphi_0(c)}$$

shows the validity of our assertion.

3.4 As in part 2.2 we obtain (4.8) here, too. Furthermore, equation (4.8) guarantees

$$\operatorname{sign}(d_0) = \operatorname{sign}(d_1) \ .$$

Differentiating $(\lambda_j)$ and using step 3.3 yields

$$f'\left(\frac{x^0(t^0-\alpha)}{\lambda_0}\right) \cdot \dot{x}^0(t^0-\alpha) = f'\left(\frac{x^1(t^1-\alpha)}{\lambda_1}\right) \cdot \dot{x}^1(t^1-\alpha) \ . \tag{4.16}$$

We may assume $d_j > 0$, $j \in \{0,1\}$, without loss of generality, possibly after some transformation of time as we shall explain now for short: If $d_1 = 0$ we infer $d_0 = 0$ from (4.8), and (4.16) implies

$$\dot{x}^0(t^0-\alpha) = \dot{x}^1(t^1-\alpha)$$

because of $f'(0) < 0$. Setting $\widetilde{t^j} = t^j - \alpha$ and $d_0 = 0 = d_1$, the point

$$\left(x^0(\widetilde{t^0}), \dot{x}^0(\widetilde{t^0})\right) = \left(x^1(\widetilde{t^1}), \dot{x}^1(\widetilde{t^1})\right)$$

is yet another intersection point of the Jordan arcs $\omega_0$ and $\omega_1$ and we can repeat all arguments of this step if we assume $\dot{x}^0(\widetilde{t^0}) = x^1(\widetilde{t^1}) = \widetilde{b} \neq 0$. Now, the fact that the solutions $x^j$, $j \in \{0,1\}$ are assumed to be slowly oscillating yields $\widetilde{d_j} := x^j(\widetilde{t^j} - \alpha) \neq 0$ such that we would proceed with these instead of the original $d_j$.

3.5 Exactly as in the second part of step 2.2 we obtain again from $d_j > 0$, $j \in \{0,1\}$,

$$0 < d_0 < d_1 \ .$$

At this point we use the assumption that $h$ is monotonically decreasing on $\mathbb{R}^+$ and $\lambda_1 < \lambda_0$, to infer from

$$0 < \frac{d_0}{\lambda_0} < \frac{d_1}{\lambda_0} < \frac{d_1}{\lambda_1}$$

the estimate

$$h\left(\frac{d_0}{\lambda_0}\right) \geq h\left(\frac{d_1}{\lambda_1}\right) > 0 \tag{4.17}$$

3.6 From (4.8) and (4.16) we conclude that

$$h\left(\frac{x^0(t^0-\alpha)}{\lambda_0}\right) \cdot \frac{\dot{x}^0(t^0-\alpha)}{x^0(t^0-\alpha)} = h\left(\frac{x^1(t^1-\alpha)}{\lambda_1}\right) \cdot \frac{\dot{x}^1(t^1-\alpha)}{x^1(t^1-\alpha)} \tag{4.18}$$

as one can easily establish using the definition of $h$. Consequently, (4.18) and (4.17) yield three possibilities:

3.6.1 Either

$$\dot{x}^0(t^0-\alpha) = \dot{x}^1(t^1-\alpha) = 0 \ ,$$

3.6.2 or
$$0 < \frac{\dot{x}^0(t^0-\alpha)}{x^0(t^0-\alpha)} \leq \frac{\dot{x}^1(t^1-\alpha)}{x^1(t^1-\alpha)} \ ,$$
3.6.3 or
$$\frac{\dot{x}^1(t^1-\alpha)}{x^1(t^1-\alpha)} \leq \frac{\dot{x}^0(t^0-\alpha)}{x^0(t^0-\alpha)} < 0 \ .$$

PROOF of this claim: Because of
$$h\left(\frac{d_0}{\lambda_0}\right)\cdot\frac{\dot{x}^0(t^0-\alpha)}{x^0(t^0-\alpha)} \overset{(4.18)}{=} h\left(\frac{d_1}{\lambda_1}\right)\cdot\frac{\dot{x}^1(t^1-\alpha)}{x^1(t^1-\alpha)} \overset{(4.17)}{\leq} h\left(\frac{d_0}{\lambda_0}\right)\cdot\frac{\dot{x}^1(t^1-\alpha)}{x^1(t^1-\alpha)}$$
and $h\left(\frac{d_0}{\lambda_0}\right) > 0$, it suffices to consider the cases
$$(i)\ \frac{\dot{x}^0(t^0-\alpha)}{x^0(t^0-\alpha)} = 0, \quad (ii)\ \frac{\dot{x}^0(t^0-\alpha)}{x^0(t^0-\alpha)} > 0, \quad (iii)\ \frac{\dot{x}^0(t^0-\alpha)}{x^0(t^0-\alpha)} < 0,$$
since
$$\operatorname{sign}\left(\frac{\dot{x}^0(t^0-\alpha)}{x^0(t^0-\alpha)}\right) = \operatorname{sign}\left(\frac{\dot{x}^1(t^1-\alpha)}{x^1(t^1-\alpha)}\right) \tag{4.19}$$
which is an immediate consequence of (4.18) and (4.17).
In case $(i)$, (4.19) yields $\frac{\dot{x}^1(t^1-\alpha)}{x^1(t^1-\alpha)} = 0$ such that we obtain 3.6.1 in this case.
If we are in case $(ii)$, we obtain $\frac{\dot{x}^1(t^1-\alpha)}{x^1(t^1-\alpha)} > 0$ from (4.19). Hence, (4.18) and (4.17) imply
$$0 < \frac{\dot{x}^0(t^0-\alpha)}{x^0(t^0-\alpha)} = \frac{h\left(\frac{d_1}{\lambda_1}\right)}{h\left(\frac{d_0}{\lambda_0}\right)}\cdot\frac{\dot{x}^1(t^1-\alpha)}{x^1(t^1-\alpha)} \leq \frac{\dot{x}^1(t^1-\alpha)}{x^1(t^1-\alpha)}$$
which gives 3.6.2.
Similarly, case $(iii)$ leads to 3.6.3.

3.7 Let $\vartheta \in [-\frac{\pi}{2};\frac{3\pi}{2})$ be given. The choice of $\varrho$ above guarantees that the intersection $|\Gamma_0|\cap\ell(\vartheta)$ is not closer to the origin than the intersection $|\Gamma_0|\cap\ell(\vartheta)$. More precisely, we have
$$\|u^1\| \leq \|u^0\| \quad \text{ for all } u^j \in |\Gamma_j|\cap\ell(\vartheta),\ j\in\{0,1\} \ .$$

3.8 Let $I := \left(-\frac{3\pi}{2};\frac{\pi}{2}\right)$, $n\in\mathbb{Z}$, and let $q$ denote the period of $x^0$. Defining $D_n := [(n-1)q;nq)$ for $n\in\mathbb{Z}$ (as in DEFINITION 4.1.4), we obtain from
$$\mathbb{R} = \bigcup_{n\in\mathbb{Z}} D_n$$
the existence of $n_0$ and $n_1$ in $\mathbb{Z}$ such that $t^0\in D_{n_0}$ and $t^1\in D_{n_1}$. For $j\in\{0,1\}$ let us define
$$z_j := \left(x^j(t^j-\alpha), \dot{x}^j(t^j-\alpha)\right)$$

and

$$\vartheta_j := \arctan\left(\frac{x^j(t^j-\alpha)}{\dot{x}^j(t^j-\alpha)}\right) \in \left(-\tfrac{\pi}{2};\tfrac{\pi}{2}\right) \ .$$

3.8.1 Obviously, this settings imply

$$z_j \in \ell(\vartheta_j), \quad j \in \{0,1\} \ .$$

3.8.2 If 3.6.2 or 3.6.3 hold, then

$$0 < |\vartheta_0| \le |\vartheta_1| < \frac{\pi}{2} \ , \tag{4.20}$$

provided that $\vartheta_0 \cdot \vartheta_1 > 0$.

3.9 According to Definition 4.1.4 let $t^* \in \{\psi_{n_0,x^0}(\vartheta_1), \Psi_{n_0,x^0}(\vartheta_1)\}$. Consequently, it is

$$u^0 := \left(x^0(t^*), \dot{x}^0(t^*)\right) \in \ell(\vartheta_1) \cap |\Gamma_0| \ .$$

Since

$$u^1 := z_1 = \left(x^1(t^1-\alpha), \dot{x}^1(t^1-\alpha)\right) \in \ell(\vartheta_1) \cap |\Gamma_1|$$

we can apply step 3.7 to obtain $\|u^0\|\cos(\vartheta_1) \ge \|u^1\|\cos(\vartheta_1)$ and, thus,

$$x^0(t^*) = \|u^0\|\cos(\vartheta_1) \ge \|u^1\|\cos(\vartheta_1) = \|z_1\|\cos(\vartheta_1) = d_1 \ .$$

Hence, we have proved

$$x^0(t^*) \ge d_1 > 0 \ . \tag{4.21}$$

3.10 If either 3.6.2 or 3.6.3 is valid, then

$$x^0(t^0-\alpha) \ge x^0(t^*) \ . \tag{4.22}$$

We will derive this assertion using the monotonicity of $x^0$ and the following conclusions from Definition 4.1.4, namely

$$t^0 - \alpha \ge \psi_{n_0,x^0}(\vartheta_0) \ge \psi_{n_0,x^0}(\vartheta_1) \quad \text{ if } \vartheta_1 \ge \vartheta_0 > 0 \ ,$$

and

$$t^0 - \alpha \le \Psi_{n_0,x^0}(\vartheta_0) \le \Psi_{n_0,x^0}(\vartheta_1) \quad \text{ if } \vartheta_1 \le \vartheta_0 < 0 \ .$$

3.10.1 In case that 3.6.2 holds, we have $\vartheta_1 \ge \vartheta_0 > 0$ by definition of $\vartheta_j$, $j \in \{0,1\}$. Now, choosing $t^* = \psi_{n_0,x^0}(\vartheta_1)$ yields (4.21).

3.10.1 If 3.6.3 is valid, we observe that $\vartheta_1 \le \vartheta_0 < 0$, such that we choose $t^* = \Psi_{n_0,x^0}(\vartheta_1)$ to obtain (4.21) again.

3.11 Finally, we can conclude that
$$d_0 \geq d_1 > 0$$
must hold in either of the cases 3.6.1, 3.6.2 or 3.6.3 which `contradicts` 3.5 and accomplishes the proof of this part. To see this we distinguish between the mentioned cases.

3.11.1 Let us assume that 3.6.1 holds, i.e., it is $\dot{x}^0(t^0 - \alpha) = \dot{x}^1(t^1 - \alpha) = 0$. Thus,
$$z_j = (x^j(t^j - \alpha), \dot{x}^j(t^j - \alpha)) \in \ell(0) \cap |\Gamma_j|$$
for $j \in \{0, 1\}$ such that step 3.7 yields
$$0 < d_1 = x^1(t^1 - \alpha) = \|z_1\| \leq \|z_0\| = x^0(t^0 - \alpha) = d_0 \ .$$

3.11.2 If 3.6.2 holds, we set $t^* = \psi_{n_0, x^0}(\vartheta_1)$. Hence, (4.22) and (4.21) show
$$d_0 = x^0(t^0 - \alpha) \geq x^0(t^*) \geq d_1 > 0 \ .$$

3.11.3 In case 3.6.3 is valid, we choose $t^* = \Psi_{n_0, x^0}(\vartheta_1)$ (instead of $t^* = \psi_{n_0, x^0}(\vartheta_1)$) and use (4.22) and (4.21) to obtain the desired assertion.

Finally, we arrived at a *contradiction* in either case due to steps **2.** and **3.**. Thus, $|\Gamma_2| \not\subset \text{ext}\,|\Gamma_1|$ must not hold.

□

The interested reader has already noted that we have proved indeed

**REMARK 4.3.1** *Let $x^j$, $j \in \{1, 2\}$, be SOP-solutions of*
$$\dot{z}(t) = -\mu z(t) + \lambda \cdot f\left(\lambda^{-1} \cdot z(t - \alpha)\right) \tag{4.4$_\alpha$}$$
*with $\alpha \in \mathbb{R}^+$ and $\lambda_2 > \lambda_1$. Then $|\Gamma_{x^2}| \subset \text{ext}\,|\Gamma_{x^1}|$.*

Please recall that the general framework for the proof is based upon CAO's method [12] but differs essentially from this in step **2.** as a consequence of (H1) (whereas only minor changes concern the steps 3.5 and 3.6).

Part **2.** is more involved in our situation and does not yield the full generality of CAO's corresponding result [12, THEOREM 2] as one easily infers by comparison of step 2.7 and [12, p. 52]:

**REMARK 4.3.2** *The above method of proof does **not** work for SOP-solutions $x^j$ of $(4.4)_{\alpha_j}$, $j \in \{1, 2\}$, with $\alpha_2 > \alpha_1 > 0$ and $\lambda_2 > \lambda_1$.*

A thorough inspection of the proof shows that assumption (H4) cannot be weakened if we want to follow the above scheme of proof: In particular, (H4.1) enters the proof via LEMMA 4.2.1 in step 2.2 and step 3.4, while the monotonicity property (H4.2) is essential for step 3.5.

## 4.4 Uniqueness of SOP-solutions

We are now in a position to derive

**Theorem 4.4.1** *If* (H1), (H2) *and* (H4) *are valid, then there is at least one SOP-solution of* (1.1).

Proof: We argue by contradiction: Let $x$ and $y$ be two *different* SOP-solutions of (1.1) and denote by $\Gamma_x$ and $\Gamma_y$ their orbits in $\mathbb{R}^2$.

1. By assumption, $\Gamma_x \neq \Gamma_y$. Now, Remark 4.1.2 and Lemma 4.1.3 imply the existence of a $\varrho > 1$ such that either $\varrho|\Gamma_x| \not\subset \operatorname{ext}|\Gamma_y|$ or $\varrho|\Gamma_y| \not\subset \operatorname{ext}|\Gamma_x|$.
2. In either case we will derive a contradiction.
   - 2.1 If $\varrho|\Gamma_x| \not\subset \operatorname{ext}|\Gamma_y|$, set $z := \varrho x$. By Remark 4.1.1, $z$ is a solution of (4.4) where $\lambda := \varrho > 1$. Since $y$ is a solution of (4.4) with $\lambda = 1$ we can apply Proposition 4.3.1 to obtain $\varrho|\Gamma_x| = |\Gamma_z| \subset \operatorname{ext}|\Gamma_y|$ in *contradiction* to our assumption.
   - 2.2 If $\varrho|\Gamma_y| \not\subset \operatorname{ext}|\Gamma_x|$, set $z := \varrho y$. By Remark 4.1.1, $z$ is a solution of (4.4) where $\lambda := \varrho > 1$. Since $x$ is a solution of (4.4) with $\lambda = 1$ we can apply Proposition 4.3.1 to obtain $\varrho|\Gamma_y| = |\Gamma_z| \subset \operatorname{ext}|\Gamma_x|$ in *contradiction* to our assumption.

Thus, the assertion is proved.

□

Combining this uniqueness result with the existence result from Chapter 3 we obtain

**Proposition 4.4.1** *Let* $\mu \in (-\log 2; 0)$, $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$ *with* $-\beta\mu < a - \varepsilon$, *and let* $f \in N(\beta, \varepsilon)$ *satisfy* (H2) *and* (H4). *Then there exists exactly one slowly oscillating periodic solution of* (1.1) *around* $\xi^0 = 0$.

Proof: This is the conclusion of Theorem 3.2.2 and Theorem 4.4.1.

□

Clearly, well-known examples in the class of nonlinearities described in the assumptions of Proposition 4.4.1 are provided by our prototype nonlinearities. This follows easily from Example 4.2.1 and 4.2.2 combined with Example 3.4.1 and 3.4.2, respectively.

**EXAMPLE 4.4.1** *Let* $\mu \in (-\log 2; 0)$, $a \in \mathbb{R}^+$, *and* $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$ *with* $-\beta\mu < a - \varepsilon$ *be given. Then there exists a unique SOP-solution of*

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1.1}$$

*for every nonlinearity*

$$f : \mathbb{R} \ni \xi \mapsto -\frac{2a}{\pi} \arctan(\alpha\xi) \in \mathbb{R} \qquad \textit{with } \alpha \in \left(\tfrac{1}{\beta} \tan \tfrac{\pi(a-\varepsilon)}{2a}; +\infty\right) .$$

**EXAMPLE 4.4.2** *Let* $\mu \in (-\log 2; 0)$, $a \in \mathbb{R}^+$, *and* $(\beta, \varepsilon) \in (0; \beta_c) \times (0; \varepsilon_c)$ *with* $-\beta\mu < a - \varepsilon$ *be given. Then there exists a unique SOP-solution of*

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1.1}$$

*for every nonlinearity*

$$f : \mathbb{R} \ni \xi \mapsto -a \tanh(\alpha\xi) \in \mathbb{R} \qquad \textit{with } \alpha \in \left(\tfrac{1}{\beta} \mathrm{Artanh} \tfrac{a-\varepsilon}{a}; +\infty\right) .$$

Although all these examples are odd functions, remember that condition $(N_1)$ was only included to clarify the investigations and to permit shorter proofs in Chapter 3: It is not essential and the assertions of PROPOSITION 4.4.1 are also valid without the oddness hypothesis on the nonlinearity $f$.

Finally, we should emphasize the fact that the slowly oscillating solutions of EXAMPLE 4.4.1 and EXAMPLE 4.4.2 could be found as the fixed points of the return map $R_f$ in the set $A(\beta)$ (see Chapter 3): This is of particular interest for numerical simulations, e.g., to obtain estimates for the (minimal) period of the SOP-solution.

## 4.5 Comments and open problems

Clearly, this approach does *not* give any stability properties of the unique slowly oscillating periodic orbit $\mathcal{O}_x$. We *conjecture*, that one may prove that $\mathcal{O}_x$ is stable and locally attractive. However, the question of hyperbolicity of $\mathcal{O}_x$ seems not to be accessible on this way. Thus, an alternative approach that proves hyperbolicity of $\mathcal{O}_x$ at least for the prototype equations from EXAMPLE 1.1.1 and 1.1.2 is still desirable (and will be addressed in [44]).

Related to these questions is the work of XIE [74] who obtained stability and hyperbolicity of (given) periodic solutions to decay equations (1.1) with bounded nonlinearity.

In a way, the method of CAO itself should be the subject of further investigations. As already mentioned in the introduction to this chapter, it seems to be somehow "unnatural" to leave the phase space $C$ (in which the orbit $\mathcal{O}_x$ lies) in order to consider the projection of this orbit into the $(x, \dot{x})$-plane. This initiates the following questions:

What are the geometric consequences of assumption (H4) for the set of solutions *in the phase space* ? What are geometric conditions *in the phase space* $C$ that guarantee the uniqueness of the orbit of a slowly oscillating periodic solution of (1.1) ?

In particular, these questions are of interest in view of desirable extensions of uniqueness results to systems of delay equations or state-dependent delay equations.

KRISZTIN and WALTHER [32] recently applied CAO's method to the "mathematical counterpart" of our feedback situation, a decay equation governed by delayed positive feedback. Beside the first application of the method to a positive feedback situation, they also adapt the approach even to periodic solutions with higher oscillation frequencies. Using tools from MALLET-PARET and SELL [41, 42] prevents an easy modification to growth systems governed by negative feedback.

The results are then applied to prove that the global attractor of this equation, which occurs in models of neural networks (cf. WU [72]), has the shape of a spindle (as conjectured by KRISZTIN, WALTHER and WU [33]). This indicates the importance of uniqueness results on periodic orbits for the global dynamics.

The importance of eventually slowly oscillating (not necessarily periodic) solutions for the global dynamics of (1.1) is well-known for decay equations. For $\mu \in \mathbb{R}_0^+$, MALLET-PARET and WALTHER [43] proved that the phase curves of all rapidly oscillating solutions form a graph in $C$, given by a map with domain in a subspace of codimension 2 and range in a complementary subspace. Consequently, the set of initial data for eventually slowly oscillating solutions is open and dense in $C$ which proved a long-standing conjecture by KAPLAN and YORKE [30].

This suggests the question whether the set of initial data for eventually slowly oscillating solutions is also dense in $\mathcal{B}$ in case of $\mu \in \mathbb{R}^-$.

Another proof of the KAPLAN-YORKE conjecture can be obtained via an approach which is based on the following observation: Rapidly oscillating solutions for decay equations with strictly monotone nonlinearity are necessarily unstable (cf. [37, 41]). – Is this also true for growth systems governed by delayed monotone negative feedback ?

In fact, note that the situations of EXAMPLE 4.4.1 and EXAMPLE 4.4.2 display similar properties as our limiting discontinuous equation (2.1): in both cases we obtain a unique SOP-solution of (1.1) in analogy to the slowly oscillating solution $y$ of (2.1).

# 5

# Bounded solutions: an outlook

We conclude this treatise with a short description of two further research problems mainly initiated by the results of the previous chapters. These could be the next steps to take in the investigation of scalar growth systems governed by nonlinear delayed negative feedback.

## 5.1 The stable sets of the non-trivial steady states

A central question in dealing with growth systems evokes from the problem of guaranteeing the boundedness of solutions. In our particular case of a growth system governed by delayed negative feedback,

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \ , \tag{1.1}$$

we still have to specify conditions on initial values $\varphi \in C$ that guarantee the boundedness of the solution $x^{\varphi}$ of (1.1) with $x_0^{\varphi} = \varphi$. In other words, we are interested in a characterization of the set $\mathcal{B}$ of initial values whose solutions remain bounded.

In the previous two chapters we dealt with special bounded solutions, namely, slowly oscillating periodic solutions of (1.1). A careful choice of (the set of) nonlinearities $f \in N(\beta, \varepsilon)$ and initial values $\varphi \in A(\beta)$ in Chapter 3 prevented solutions $x^{\varphi,f}$ from escaping to infinity. In Chapter 4 we did not have to care for boundedness since oscillating solutions are necessarily bounded, as we already knew from Remark 1.5.1.

The general question arises if one wants to study the whole set $\mathcal{B}$ instead of the oscillating solutions. Then we are faced with the problem of characterizing conditions for boundedness.

Following our "general strategy" it could be helpful to recall the approach for the discontinuous model (2.1). In Chapter 2 the border or boundary of the set $\mathcal{B}$ was formed by the stable sets of the non-trivial steady states and there is some (numerical) evidence to hope that this property persists in the smooth case (1.1).

By virtue of the linearization at $u_j$, $j \in \{-,+\}$, (cf. Section 1.3 for details and notation) we obtain an affine phase space decomposition of $C$ at $u_j$,

$$C = (u_j + P_j) \oplus (u_j + Q_j) = u_j + (P_j \oplus Q_j) \ ,$$

where $P_j$ denotes the linear unstable subspace

$$P_j = \mathbb{R}e^{\lambda_j^{(0)}} \cdot$$

and $Q_j$ its complementary subspace in $C$. Furthermore, we denote by

$$\mathrm{Pr}_{u_j+Q_j} : C \to u_j + Q_j$$

the affine linear projection onto the affine linear subspace $u_j + Q_j$ (along $u_j + P_j$).

Now, it is comparatively easy to prove the existence of a graph representation of the local stable manifold of $u_j$, $j \in \{-,+\}$, exploiting the assumptions (H2.2) and (H3.2).

**LEMMA 5.1.1** *Fix $j \in \{-,+\}$. Then there exists an open neighborhood $U_j$ of $u_j$ in $C$ and a map*

$$\mathrm{sep}_{j,\mathrm{loc}} : \mathrm{Pr}_{u_j+Q_j}(W^s(u_j) \cap U_j) \to u_j + P_j$$

*such that*

$$W^s(u_j) \cap U_j = \left\{\chi + \mathrm{sep}_{j,\mathrm{loc}}(\chi) \ : \chi \in \mathrm{Pr}_{u_j+Q_j}(W^s(u_j) \cap U_j)\right\} = \mathrm{graph}(\mathrm{sep}_{j,\mathrm{loc}}) \ .$$

Certainly, this result initiates the question for a "global version": does there exist a graph representation of the global stable set of the non-trivial steady state $u_j$, $j \in \{-,+\}$? We conjecture the existence of such a graph representation.

**CONJECTURE 5.1.1** *Fix $j \in \{-,+\}$. Under hypotheses* (H1)–(H3) *there exists a map*

$$\mathrm{sep}_j : \mathrm{Pr}_{u_j+Q_j}(W^s(u_j)) \to u_j + P_j$$

*such that*

$$W^s(u_j) = \left\{\chi + \mathrm{sep_j}(\chi) \ : \ \chi \in \mathrm{Pr}_{u_j+Q_j}(W^s(u_j)\right\} = \mathrm{graph}(\mathrm{sep}_j) \ .$$

The reason for calling the maps above "sep" is motivated by the analogous denotation in Section 2.4 and will be explained in more detail in the following paragraph.

Clearly, $W^s(u_j)$, $j \in \{-,+\}$, is locally a $C^1$-graph over the affine subspace $u_j + Q_j$, i.e. $W^s_{\mathrm{loc}}(u_j)$ is the graph of a $C^1$-map in a neighborhood of $u_j$ in $u_j + Q_j$ (cf. the monographs [16, Section VIII.6] or [26, Section 10.1]).

But, as the well-known example due to HALE and LIN [25, EXAMPLE 2.2] demonstrates, this is not enough to conclude that $W^s(u_j)$ is a $C^1$-manifold (see also [26, pp. 310–311]). So, in case that CONJECTURE 5.1.1 holds, we can then ask for the smoothness properties of $\mathrm{sep}_j$.

Basic material about invariant manifolds is contained in HALE and LIN [25] and in the Diploma Thesis of NEUGEBAUER [47] which is based upon [25]. Related are also WALTHER [66, 64] where the stable and unstable manifolds of periodic solutions are considered in the context of decay delay equations.

## 5.2 Description of the semiflow on a subset of $\mathcal{B}$

An obvious question was left open in the preceding section: Is there another description of $\mathrm{Pr}_{u_j+Q_j}(W^s(u_j))$ or, more specifically, does possibly the inclusion

$$u_j + Q_j \subset \mathrm{Pr}_{u_j+Q_j}(W^s(u_j))$$

hold ? The validity of this inclusion would enable us to follow the lines of Section 2.4 here to find at least a set-valued graph representation for $W^s(u_j)$.

Moreover, if additionally CONJECTURE 5.1.1 is true, then $W^s(u_-)$ is a hypersurface and one could follow the ideas of KRISZTIN, WALTHER and WU [33, Section 3] to show that $W^s(u_j)$ serves as a separatrix in $C$.

In the discontinuous model case (2.1) the above inclusion holds true, so there is some evidence to conjecture the validity of it in our situation, too. Reconsidering the proof of PROPOSITION 2.4.1 (pp. 84ff.) shows that we will most probably need a deeper knowledge about the behaviour of solutions of (1.1) starting in

$$\mathcal{Z} := \mathcal{B} \setminus (W^s(u_-) \cup W^s(u_+)) \ .$$

Certainly, beside from this motivation the dynamics in $\mathcal{Z}$ is of independent interest. It is convenient to ask the following questions.

In how far does the dynamics of the discontinuous model (2.1) reflect the behaviour in the continuous situation in the set of bounded solutions which do not converge to one of the non-trivial steady states ? Can we transfer results on the dynamics of (2.1) in $\mathcal{Z}$ from Section 2.3 to delay equations (1.1) for (a subset of) nonlinearities in $N(\beta, \varepsilon)$ ? For instance, is the set of slowly oscillating solutions around zero dense in $\mathcal{Z}$ if $f$ is a strictly decreasing steep nonlinearity ?

Does there exist a global attractor in $\mathcal{B}$ (cf. HALE [24]) as one may expect in view of decay delay equations (1.1) (cf. WALTHER [65]) ?

# Zusammenfassung
## (Abstract)

Gemäß §7 Absatz 2 der Promotionsordnung der naturwissenschaftlichen Fachbereiche der Justus-Liebig-Universität Gießen wird in diesem Anhang eine ausführliche Zusammenfassung der in der vorliegenden Arbeit enthaltenen Resultate und Ergebnisse in deutscher Sprache gegeben. Die Verweise beziehen sich auch in diesem Abschnitt stets auf die zur Arbeit gehörende Literaturliste (siehe Seite 152 ff.).

### 0. Einleitung

In der vorliegenden Dissertation *On scalar growth systems governed by delayed nonlinear negative feedback* wird die Klasse

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)) \tag{1.1}$$

von Differentialgleichungen mit Verzögerung unter den im folgenden näher aufgeführten Annahmen über die Nichtlinearität $f$ und den reellen Parameter $\mu$ untersucht:

(H1) Für den reellen Parameter $\mu$ sollen ausschließlich negative Werte zugelassen werden, d.h. es sei $\mu \in \mathbb{R}^- := (-\infty; 0)$.

(H2) Die Nichtlinearität $f : \mathbb{R} \to \mathbb{R}$ genüge den folgenden Voraussetzungen:

- (H2.1) $f$ sei stetig differenzierbar auf $\mathbb{R}$,
- (H2.2) $f$ sei streng monoton fallend auf $\mathbb{R}$ und
- (H2.3) $f$ sei beschränkt, d.h. es gebe ein $M_f > 0$ mit $|f(\xi)| \leq M_f$ für alle $\xi \in \mathbb{R}$.

(H3) Die Nichtlinearität $f$ und der reelle Parameter $\mu$ seien wie folgt gekoppelt:

- (H3.1) Es gelte

$$-f'(0) > -\mu \ ,$$

und

(H3.2) es gebe genau eine negative Lösung $u = \xi^- \in \mathbb{R}^-$ und genau eine positive Lösung $u = \xi^+ \in \mathbb{R}^+$ der Gleichgewichtsgleichung

$$-\mu u + f(u) = 0 \ .$$

Für diese nicht-trivialen Lösungen der Gleichgewichtsgleichung gelte ferner

$$0 \leq -f'(u) < -\mu$$

für $u \in \{\xi^-, \xi^+\}$.

Diese Klasse von Differentialgleichungen beschreibt somit die zeitliche Entwicklung einer skalaren Größe $x$, die einerseits autokatalytisch wächst, was durch die Wahl des Parameters (der Wachstumsrate) $-\mu > 0$ beschrieben wird, andererseits aber zeitverzögert (mit einer Reaktionszeit $r = 1$) diese momentan steuernden Prozesse zu regeln versucht. Letzteres geschieht durch verzögerte negative Rückkopplung bezüglich des Gleichgewichtes $0 \in C := C([-1;0], \mathbb{R})$, wie durch

$$\xi \cdot f(\xi) < 0 \quad \text{für alle } \xi \in \mathbb{R} \setminus \{0\} \ . \tag{$NF$}$$

analytisch verbalisiert wird: Diese Eigenschaft ist eine direkte Folge der Annahmen (H2) und (H3) und begründet den zweiten Teil des Titels der vorliegenden Dissertation, dessen erster Teil durch die Voraussetzung (H1) motiviert wird.

Prototypen von Nichtlinearitäten, die diesen Voraussetzungen unter geeigneter Wahl der Parameter genügen, stellen die beiden Zweiparameterfamilien

$$f_{\alpha,M} : \mathbb{R} \ni \xi \mapsto -\frac{2M}{\pi} \arctan(\alpha\xi) \in \mathbb{R}$$

sowie

$$f_{\alpha,M} : \mathbb{R} \ni \xi \mapsto M \tanh(\alpha\xi) \in \mathbb{R}$$

dar. Auf diese wird im Fortschreiten der Arbeit immer wieder zurückgegriffen, um die erzielten Ergebnisse daran exemplarisch zu demonstrieren.

In den folgenden Abschnitten werden zusammengefaßt die Ergebnisse der Untersuchungen der Gleichung (1.1) unter oben aufgeführten Generalvoraussetzungen dargelegt, wobei stets Bezug auf Formeln, Sätze und Definitionen aus der vorstehenden Arbeit genommen wird. Der Aufbau ist dabei an die Reihenfolge und Struktur der Kapitel angelehnt.

## 1. Elementare Ergebnisse

Mit Hilfe der Variation-der-Konstanten-Formel und der sogenannten *method of steps* konstruiert man sukzessive für jedes gegebene $\varphi \in C := C([-1;0], \mathbb{R})$ eine *Lösung* des Anfangswertproblems

$$\begin{cases} \dot{x}(t) &= -\mu x(t) + f(x(t-1)) \ , \qquad t \in \mathbb{R}^+ \ , \\ x_0 &= \varphi \ . \end{cases}$$

Darunter versteht man eine stetige Funktion $x^\varphi : [-1;\infty) \to \mathbb{R}$, die (1.1) auf $\mathbb{R}^+$ und $x_0^\varphi = \varphi$ erfüllt, wobei wir mit

$$x_t^\varphi : [-1;0] \ni s \mapsto x^\varphi(t+s) \in \mathbb{R}$$

das Segment der Lösung $x^\varphi$ (zum Anfangswert $\varphi \in C$) zum Zeitpunkt $t \in \mathbb{R}_0^+$ bezeichnen. Dadurch wird ein stetiger Halbfluß

$$F_f : \mathbb{R}_0^+ \times C \ni (t,\varphi) \mapsto x_t^\varphi \in C$$

erklärt, dessen Linearisierung an den drei Gleichgewichtspunkten

$$u_j : [-1;0] \ni t \mapsto \xi^j \in \mathbb{R} \ , \quad j \in \{-,0,+\} \ ,$$

welches die Anfangswerte der drei stationären Lösungen sind, in Abschnitt 1.3 ausführlich behandelt wird.

Im Gegensatz zum Fall $\mu \in \mathbb{R}^+$ treten in der von uns untersuchten Situation im allgemeinen auch unbeschränkte Lösungen auf, wie man sich anhand der Beschränktheitsvoraussetzung (H2.3) verdeutlichen kann, welche für hinreichend große Werte der Größe $x$ den Wachstumsterm nicht mehr in hinreichendem Maße „zu bremsen" in der Lage ist.

Alle unbeschränkten Lösungen weisen infolgedessen die spezielle Eigenschaft auf, von einem Zeitpunkt an streng monoton zu werden, so daß die Menge der Anfangswerte mit unbeschränkten Lösungen in die beiden disjunkten Teilmengen $\mathcal{E}^+$ und $\mathcal{E}^-$ zerfällt, je nach bestimmter Divergenz der Lösung $x^\varphi$ gegen $+\infty$ oder $-\infty$.

Daher genügt es, sich hinsichtlich der Betrachtung der Dynamik auf die Menge $\mathcal{B}$ der Anfangswerte zu konzentrieren, welche beschränkte Lösungen besitzen. Für Anfangswerte $\varphi \in \mathcal{B}$ erhalten wir (in Lemma 1.4.2)

$$|x^\varphi(t)| \leq -\frac{M_f}{\mu} \qquad \text{für alle } t \in \mathbb{R}_0^+ \ , \tag{A.1}$$

was von zentraler Bedeutung für die Untersuchung periodischer Lösungen von (1.1) in den Kapiteln 3 und 4 ist.

Die Vorbereitung der Behandlung oszillierender Lösungen von (1.1) steht dann auch im Mittelpunkt der Paragraphen 1.5 und 1.6, in denen neben der verwendeten Terminologie auch ein auf Mallet-Paret, Cao und Arino zurückgehendes Lyapunov-Funktional erklärt wird. Dieses Hilfsmittel erlaubt es, bereits in diesem Vorstadium der Untersuchungen zu folgern, daß es weder monoton gegen Null konvergierende Lösungen noch einen homoklinen Orbit durch den Gleichgewichtspunkt $u_0$ geben kann.

## 2. Eine unstetige Modellgleichung

Um zu einem besseren Verständnis der rudimentären dynamischen Strukturen, wie sie skalare Wachstumsprozesse mit zeitlich verzögerter negativer Rückkopplung zeigen, durchzudringen, empfiehlt es sich, zu einer einfacheren, leicht handhabbaren Modellgleichung überzugehen, die eben jene Grundstrukturen aber immer noch aufweist.

Dies ist gerade für die unstetige Delay-Differentialgleichung

$$\dot{x} = -\mu x(t) - a\,\mathrm{sign}(x(t-1)) \tag{2.1}$$

für $a > 0$ der Fall, deren Untersuchung das gesamte Kapitel 2 gewidmet ist.

Zwar zwingt uns die Unstetigkeit der Nichtlinearität $f := -a\,\mathrm{sign}$ erstens dazu, von $C$ zum Phasenraum

$$X := \left\{\varphi \in C \;:\; |\varphi^{-1}(0)| < \infty\right\}$$

überzugehen, um einen stetigen Halbfluß der Segmente der Lösungen von (2.1) garantieren zu können, und verlieren wir zweitens aufgrund der speziellen Struktur zudem noch die Injektivität der Zeit-$t$-Abbildungen (gegenüber dem Fall der bis dahin betrachteten monotonen Nichtlinearitäten $f$), so werden diese Nachteile jedoch weitestgehend aufgewogen durch die Möglichkeit der expliziten Berechenbarkeit der Lösungen von (2.1).

In erster Konsequenz können wir damit bereits in Paragraph 2.2 alle – sowohl die langsam als auch die schnell schwingenden – periodischen Lösungen von (2.1) konstruieren und Aussagen über deren Eigenschaften machen, was einen ersten Einblick in die Struktur der Menge $\mathcal{B}$ der Anfangswerte der beschränkten Lösungen von Gleichung (2.1) ermöglicht. Auf diese wird extensiv im Rahmen des dritten Kapitels zurückgegriffen, in dem wir die Existenz langsam schwingender periodischer Lösungen für eine Klasse stetiger Nichtlinearitäten zeigen.

Ferner ist zu beachten, daß in unserem Speziallfall $\xi^+ = -\frac{a}{\mu} = -\frac{M_f}{\mu}$ (und $\xi^- = -\xi^+$) gilt, so daß nur für den Parameterbereich $(-\log 2; 0)$ überhaupt langsam um Null schwingende periodische Lösungen von (2.1) existieren können, da bekanntlich alle beschränkten Lösungen unterhalb der Schranke $-\frac{M_f}{\mu}$ bleiben müssen (vgl. auch $(A.1)$). Aus diesem Grunde existieren auch keine langsam schwingenden Lösungen von (2.1) um die nichttrivialen Gleichgewichtspunkte $u_j$, $j \in \{-,+\}$.

Darüberhinaus wird in Paragraph 2.3 ein tieferer Einblick in die geometrische Struktur der Teilmenge

$$\mathcal{Z} = \mathcal{B} \setminus (W^s(u_-) \cup W^s(u_+))$$

aller beschränkten, nicht gegen eine der stationären Lösungen konvergierenden Lösungen gegeben.

Dazu wird zunächst eine alternative Charakterisierung von $\mathcal{B}$ anhand einfach zu überprüfender Kriterien in Abschnitt 2.3.A gegeben, bevor in Abschnitt 2.3.B das diskrete Lyapunov-Funktional nach Mallet-Paret [39], Cao [11] und Arino [6] für die Anwendung auf Gleichung (2.1) in der Menge $\mathcal{Z}$ verallgemeinert wird. Dieses Hilfsmittel erlaubt es, uns für die Betrachtung der Dynamik auf die Menge

$$\mathcal{Z}_0 := \{\varphi \in \mathcal{Z} \;:\; \varphi(0) = 0,\ \varphi \text{ hat geradzahlig viele und nur einfache Nullstellen in } (-1;0)\}$$

zu konzentrieren, da alle in $\mathcal{Z}$ startenden Lösungen nach endlicher Zeit in dieser Menge „landen", was in dem Sinne zu verstehen ist, daß für jedes $\varphi \in \mathcal{Z}$ ein $t_0(\varphi) \in \mathbb{R}^+$ existiert, so daß

$$x_t^\varphi \in \mathcal{Z}_0 \quad \text{für alle } t \in (x^\varphi)^{-1}(0) \cap [t_0(\varphi); +\infty)$$

gilt. Daher studieren wir in Abschnitt 2.3.C das Verhalten von Lösungen mit Anfangswerten in $\mathcal{Z}_0$ genauer, was in der Einführung einer POINCARÉ-Abbildung $R : \mathcal{Z}_0 \to \mathcal{Z}_0$ kulminiert, die im Abschnitt 2.3.D zu einem diskreten dynamischen System konjugiert wird, welches eine detaillierte Beschreibung des Lösungsverhaltens erlaubt:

Diese wird in Abschnitt 2.3.E gegeben und kann kurz wie folgt umrissen werden. Fast jede in $\mathcal{Z}$ startende Lösung „fließt" nach endlicher Zeit in den Orbit einer periodischen Lösung von (2.1) hinein, was dort durch eine zur MORSE-Zerlegung ähnlichen Struktur beschrieben wird.

Schließlich wenden wir uns im vierten Paragraphen des zweiten Kapitels einer Untersuchung der stabilen Mengen der nicht-trivialen Gleichgewichte $u_j$, $j \in \{-, +\}$ zu, was die Gesamtbetrachtung der Dynamik von (2.1) komplettiert.

Hierbei stellt sich heraus, daß wir zwar einerseits die Surjektivität der restringierten Projektion

$$\mathrm{Pr}_{u_+ + \widetilde{Q_+}}\Big|_{[W_s(u_+)]} : [W^s(u_+)] \to u_+ + \widetilde{Q_+}$$

auf die zum (formalen) affinen Unterraum $u_+ + Q_+$ gehörende Nebenklasse

$$u_+ + \widetilde{Q_+} := \big\{[u_+ + \psi - \psi(0)e^{-\mu\cdot}] \;:\; \psi \in X\big\} = [(u_+ + Q_+) \cap X]$$

erhalten, andererseits aber auch nachweisen können, daß die Abbildung $\mathrm{Pr}_{u_+ + Q_+}$ nicht injektiv ist, was die „traditionelle" globale Graphdarstellung von $[W^s(u_+)]$ über $u_+ + \widetilde{Q_+}$ unmöglich macht.

Stattdessen nutzen wir nur die Surjektivität der Abbildung $\mathrm{Pr}_{u_+ + \widetilde{Q_+}}|_{[W^s(u_+)]}$, indem wir $[W^s(u_+)]$ als Graph der mengenwertigen Abbildung

$$\mathfrak{Sep} : u_+ + \widetilde{Q_+} \ni [u_+ + \psi] \mapsto [u_+ + I([u_+ + \psi])e^{-\mu\cdot}] \in \mathfrak{P}(u_+ + \widetilde{P_+})$$

mit $I([u_+ + \psi]) := \{r \in \mathbb{R} \,:\, \exists\, u_+ + \chi \in [u_+ + \psi] \text{ mit } u_+ + \chi + re^{-\mu\cdot} \in W^s(u_+)\}$ beschreiben und diese geometrische Darstellung durch Betrachtung der Schnitte der Mengenbündel über dem affinen Raum noch weiter verfeinern.

Diese Methode wirft einige Fragen, beispielsweise nach Glattheit(sbegriffen) und einer noch genaueren Beschreibung der Funktionswerte von $\mathfrak{Sep}$, auf, die als offene Probleme formuliert und die Grundlage für weitere Untersuchungen sein werden.

## 3. Existenz langsam schwingender periodischer Lösungen

Zentrales Anliegen des Kapitels 3 ist es, ausgehend von den für die unstetige Modellgleichung (2.1) gewonnenen Erkenntnissen, Rückschlüsse über die Dynamik der Delay-Gleichung (1.1) für stetige Nichtlinearitäten $f$ zu gewinnen. Dabei liegt der Focus auf der Frage nach der Existenz langsam schwingender periodischer Lösungen von (1.1) für Nichtlinearitäten $f$, die in einem gewissen Sinne hinreichend „nahe" an der Signum-Nichtlinearität sind.

Als geeignet für unsere Zwecke erweist sich bei geeigneter Parameterwahl $(\beta, \varepsilon) \in \mathbb{R}_+^2$ die Klasse $N(\beta, \varepsilon)$ aller stetigen reellen Abbildungen, die ungerade sind, außerhalb einer $\beta$-Umgebung der Null nur Werte im Intervall $(-\varepsilon+a; a+\varepsilon)$ annehmen und genau zwei nichttriviale Äquilibria mit Absolutbetrag größer als $\beta$ besitzen. Dabei kann auf die Forderung, daß $f$ ungerade sein soll, sogar verzichtet werden, sie ist einzig aus technischen Gründen zur Vereinfachung der Argumentation aufgenommen.

Für Differentialgleichungen (1.1) mit $f \in N(\beta, \varepsilon)$ zeigt man nun für hinreichend kleine $\beta > 0$ und $\varepsilon > 0$, daß Segmente $x_t^\psi$ von Lösungen von (1.1), die in der abgeschlossenen, beschränkten und konvexen Menge

$$A(\beta) := \left\{ \psi \in C \ : \ \|\psi\| \leq -\frac{M_f}{\mu}, \psi(t) \geq \beta \ \forall t \in [-1; 0], \psi(0) = \beta \right\}$$

starten, wieder in diese Menge zurückkehren, so daß man unter Verwendung der eindeutig bestimmten Wiederkehrzeit $q_f(\psi) \in \mathbb{R}^+$ die Wiederkehr- oder POINCARÉ-Abbildung

$$R_f : A(\beta) \ni \psi \mapsto -F_f(q_f(\psi), \psi) \in A(\beta)$$

erklären kann.

Ist $f$ zusätzlich noch LIPSCHITZ-stetig, so ist auch die Wiederkehrabbildung $R_f$ LIPSCHITZ- und vollstetig, womit der Weg für die Anwendung des SCHAUDERschen Fixpunktsatzes geebnet ist, welcher die Existenz periodischer Lösungen von (1.1) liefert, da jeder Fixpunkt von $R_f$ der Anfangswert einer langsam um Null schwingenden periodischen Lösung von (1.1) ist (vgl. THEOREM 3.2.2).

Für stetig differenzierbare Nichtlinearitäten $f \in N(\beta, \varepsilon)$, die eine kontrahierende Rückkehrabbildung $R_f$ definieren, zeigen wir dann in THEOREM 3.3.1, daß der durch den eindeutig bestimmten Fixpunkt $\varphi \in A(\beta)$ von $R_f$ gegebene Orbit der zugehörigen periodischen Lösung hyperbolisch, stabil und exponentiell attraktiv mit asymptotischer Phase ist. Daß stetig differenzierbare $f \in N(\beta, \varepsilon)$, für welche $R_f$ kontrahierend ist, tatsächlich existieren, wird durch die explizite Konstruktion einer solchen Nichtlinearität gezeigt.

Die Resultate und Methoden des dritten Kapitels folgen und verallgemeinern zugleich den erstmals von WALTHER in [67] beschrittenen Weg zum Nachweis periodischer Lösungen von Delay-Gleichungen des Typs (1.1). Es steht zu vermuten, daß für glatte monotone Nichtlinearitäten in $N(\beta, \varepsilon)$ analog zu WALTHER [68] einige technische Abschätzungen weiter verfeinert werden können, so daß die Aussagen von THEOREM 3.3.1 insbesondere auf oben aufgeführte Prototypen $f_{\alpha,M}$ (mit geeignet gewählten Parametern $\alpha$ und $M$) angewendet werden können. Dies wird aber Gegenstand einer weiteren Arbeit sein wird.

# 4. Eindeutigkeit langsam schwingender periodischer Lösungen

Nachdem wir in Kapitel 3 bereits die Existenz langsam schwingender periodischer Lösungen von (1.1) für Nichtlinearitäten der Klasse $N(\beta, \varepsilon)$ nachgewiesen haben, wenden wir uns im vierten Kapitel der Frage nach der Eindeutigkeit der Orbits langsam schwingender periodischer Lösungen von (1.1) zu.

Hierzu folgen wir einem Ansatz von Cao [12], den wir auf die Klasse der skalaren Wachstumsgleichungen mit negativer Rückkopplung verallgemeinern. Dabei betrachten wir Nichtlinearitäten $f$, die neben (H1) und (H2) noch der folgenden Konvexitätsvoraussetzung genügen:

(H4) Die Abbildung

$$h : \mathbb{R}\backslash\{0\} \ni \xi \mapsto \frac{\xi f'(\xi)}{f(\xi)} \in \mathbb{R}$$

habe den Wertebereich

$$h(\mathbb{R}\backslash\{0\}) \subset (0;1)$$

und sei monoton fallend auf $\mathbb{R}^+$ sowie monoton wachsend auf $\mathbb{R}^-$.

Da die $(x, \dot{x})$-Projektionen von Orbits langsam schwingender periodischer Lösungen stets Jordan-Kurven im $\mathbb{R}^2$ darstellen, die nach Proposition 4.3.1 nur eine ganz bestimmte gegenseitige geometrische Lage im $\mathbb{R}^2$ einnehmen können, stellt der Beweis dieser Proposition den zentralen Schritt beim Nachweis der Eindeutigkeit des Orbits der langsam schwingenden periodischen Lösung von (1.1) dar. Wie schon in der Arbeit von Cao [12] geschieht der Beweis von Proposition 4.3.1 in zwei Schritten durch einen Widerspruchsbeweis, wobei in unserem Falle der erste fundamental von dem in [12] abweicht, während der zweite Schritt jedoch weitgehend analog mit nur kleineren Modifikationen geführt wird.

Wie bereits erwähnt, dient dann Proposition 4.3.1 dazu, die Eindeutigkeit des Orbits langsam schwingender periodischer Lösungen von (1.1) in Theorem 4.4.1 unter den Annahmen (H1), (H2) und (H4) nachzuweisen.

Kombinieren wir nun die Resultate der Kapitel 3 und 4, so muß festgestellt werden, daß für alle Nichtlinearitäten $f$, welche sowohl den Voraussetzungen aus Theorem 3.2.2 als auch denen aus Theorem 4.4.1 genügen, die verzögerten Differentialgleichungen (1.1) einen eindeutig bestimmten Orbit einer langsam (um Null) schwingenden periodischen Lösungen besitzen.

Insbesondere ist dies wieder für die Prototypnichtlinearitäten $f_{\alpha,M}$ (bei geeigneter Parameterwahl) der Fall, wie man leicht durch konkretes Nachrechnen der Bedingung (H4) verifizieren kann.

# 5. Beschränkte Lösungen: ein Ausblick

Alleinige Aufgabe des abschließenden Kapitels 5 ist es, einen Ausblick auf eine weitere mögliche Forschungsrichtung zu geben, welche wiederum durch die für die unstetige Modellgleichung (2.1) in Paragraph 2.4 erzielten Ergebnisse motiviert ist: Die Frage nach einer genauere Beschreibung der Menge $\mathcal{B}$ der Anfangswerte, die eine beschränkte Lösung von (1.1) initiieren, führt daher quasi zwangsläufig auf die Notwendigkeit der Beschreibung der stabilen Mengen der nicht-trivialen Gleichgewichte $u_j$, $j \in \{-,+\}$. Hierzu werden einige Vermutungen geäußert.

# Notations and symbols

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ — the set of positive integers, non-negative integers, integers, real and complex numbers, respectively.

$\mathbb{M}u + v := \{mu + v \ : \ m \in \mathbb{M}\}$ for $\mathbb{M} \in \{\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}\}$, $u \in \mathbb{R}$ and $v \in \mathbb{R}$.

$\mathbf{1}_N := (1, ..., 1) \in \mathbb{R}^N$, $N \in 2\mathbb{N}$.

$C := C([-1;0], \mathbb{R})$

$C_{\mathbb{C}} := C([-1;0], \mathbb{C})$

$\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| < 1\}$.

$\mathrm{dist}_C(\varphi, U) := \inf\limits_{\psi \in U} \|\varphi - \psi\|$ for $\varphi \in C$ and $U \subset C$.

$\mathrm{dist}_{\mathbb{R}^N}(x, U) := \inf\limits_{y \in U} \|x - y\|_1$ for $x \in \mathbb{R}^N$ and $U \subset \mathbb{R}^N$, $N \in \mathbb{N}$.

$\mathrm{ev}_a : C \ni \varphi \mapsto \varphi(a) \in \mathbb{R}$, $a \in [-1;0]$.

$\mathbb{I} : [-1;0] \ni t \mapsto 1 \in \mathbb{R}$.

$\mathrm{id}_X : X \ni x \mapsto x \in X$, $X$ any Banach space.

$\ker v := v^{-1}(0)$ for $v \in \mathcal{L}(X, X)$, $X$ any Banach space.

$\mathfrak{P}(M) := \{N \ : \ N \subset M\}$

$\|\cdot\| : C \ni \varphi \mapsto \max\limits_{t \in [-1;0]} |\varphi(t)| \in \mathbb{R}_0^+$.

$\|\cdot\|_1 : \mathbb{R}^N \ni (\xi_1, ..., \xi_N) \mapsto \sum_{j=1}^N |\xi_j| \in \mathbb{R}_0^+$, $N \in \mathbb{N}$.

$\|\cdot\|_2 : \mathbb{R}^N \ni (\xi_1, ..., \xi_N) \mapsto \sqrt{\sum_{j=1}^N \xi_j^2} \in \mathbb{R}_0^+$, $N \in \mathbb{N}$.

$\mathbb{R}_+^N := \left\{v = (v_1, ..., v_N) \in \mathbb{R}^N \ : \ v_j > 0 \, \forall j \in \{1, ..., N\}\right\}$, $N \in \mathbb{N}$.

$$\mathrm{sign} : \mathbb{R} \ni \xi \mapsto \left\{ \begin{array}{cc} \frac{\xi}{|\xi|} & , \ \xi \neq 0 \\ 0 & , \ \xi = 0 \end{array} \right\} \in \{-1, 0, +1\} \ .$$

$\Sigma_N^o := \{v \in \mathbb{R}_+^N \ : \ \mathbf{1}_N \cdot v < 1\}$

$\lceil \cdot \rceil : \mathbb{R} \ni \xi \mapsto \inf \{\zeta \in \mathbb{Z} \ : \ \xi \leq \zeta\} \in \mathbb{Z}$.

$\lfloor \cdot \rfloor : \mathbb{R} \ni \xi \mapsto \sup \{\zeta \in \mathbb{Z} \ : \ \zeta \leq \xi\} \in \mathbb{Z}$.

$C = P \oplus Q$ – direct sum of the subspaces $P$ and $Q$.

$A \dot{\cup} B$ – disjoint union of the sets $A$ and $B$.

$\Omega_1 \uplus \Omega_2$ – disjoint union of the topological spaces $\Omega_1$ and $\Omega_2$.

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