A local Hopf Bifurcation Theorem for differential equations with state - dependent delays

Doctoral Dissertation

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Contents

Zusammenfassung

Das Ziel dieser Dissertation ist der Beweis eines Hopfverzweigungssatzes für Differentialgleichungen mit zustandsabhängiger Verzögerung, wie es ihn für gewöhnliche und partielle Differentialgleichungen sowie für Differentialgleichungen mit konstanter Verzögerung bereits gibt.
Dieser Satz sollte auf Gleichungen der folgenden Form angewandt werden können:

$$x'(t) = f\Big(\alpha, x\big(t - r(x_t)\big)\Big), \quad t \in \mathbb{R}, \quad \alpha \in J \subset \mathbb{R}.$$

Hierbei ist

$$f : J \times \mathbb{R}^n \to \mathbb{R}^n,$$

$J \subset \mathbb{R}$ ein Intervall, eine 2 - fach stetig differenzierbare Abbildung. Das Segment x_t, $t \in \mathbb{R}$, ist ein Element des Raumes $C^1([-h, 0]|\mathbb{R}^n)$ stetig differenzierbarer Funktionen $\phi : [-h, 0] \to \mathbb{R}^n$, $h > 0$. In vielen Beispielen wird r implizit durch eine Gleichung wie

$$r = x(-r) + x(0) + D,$$

D eine nichtnegative Konstante, gegeben.
Wenn man von Hopfverzweigung spricht, erwartet man die folgende Situation:
Es sei $C([-h, 0]|\mathbb{R}^n)$ *der Raum stetiger reellwertiger Funktionen* $\phi : [-h, 0] \to \mathbb{R}^n$.
Für $\alpha \in J$ *sei* $L(\alpha) : C([-h, 0]|\mathbb{R}^n) \to \mathbb{R}^n$ *ein beschränkter linearer Ope-rator. Es sei* $F(\alpha, \phi) := f\Big(\alpha, \phi\big(-r(\phi)\big)\Big) - L(\alpha)\phi$*, für* $\phi \in C^1([-h, 0]|\mathbb{R}^n)$ *und* $\alpha \in J$.
Damit wird die Ausgangsgleichung in die Form

$$x'(t) = L(\alpha)x_t + F(\alpha, x_t), \quad t \in \mathbb{R}, \quad \alpha \in J$$

umgestellt.
Es gebe ein Gleichgewicht, das heisst eine konstante Funktion $\phi^* \in C([-h, 0]|\mathbb{R}^n)$*, so dass* $F\big(\alpha, \phi^*\big) = 0$ *für alle* $\alpha \in J$ *gilt.*
Darüberhinaus gebe es einen Zweig einfacher Eigenwerte $\lambda(\alpha)$, $\alpha \in I \subset J$, $I \subset J$ *ein Intervall, die zur linearen parametrisierten Gleichung*

$$y'(t) = L(\alpha)y_t, \quad t \in \mathbb{R}, \quad \alpha \in J,$$

gehören und die imaginäre Achse bei einem kritischen Parameter $\alpha_0 \in I$ *kreuzen.*
Dann erwartet man, eine Parametrisierung

$$\mathbb{R} \supset Q \ni a \mapsto \big(\alpha(a), \phi(a), T(a)\big) \in J \times C^1([-h,0]|\mathbb{R}^n) \times \mathbb{R},$$

$Q \subset \mathbb{R}$ *ein Intervall, zu erhalten, für die das Folgende gilt:*
$0 \in Q$. *Für jedes* $a \in Q$ *existiert eine* $T(a)$ *- periodische Lösung* $x^*(a)$ *der obigen Gleichung zum Parameter* $\alpha(a)$ *mit* $x^*(a)_{t=0} = \phi(a)$ *für* $a \in Q$*, sowie* $\alpha(0) = \alpha_0$*,* $\lambda(0) = \lambda_0$ *und* $\phi(0) = \phi^*$.
Eine der ersten Arbeiten zum Thema wurde von Hal Smith (siehe [9]) veröffentlicht:
Bei der Betrachtung einer Differentialgleichung mit zustandsabhängiger Verzögerung setzte er $L(\alpha)\chi := D_2 f\big(\alpha, \phi^*(0)\big)\big(\chi(-r(\phi^*))\big)$ für $\chi \in C([-h,0]|\mathbb{R}^n)$, und erhielt einen Zweig von Eigenwerten $\lambda(\alpha)$, $\alpha \in I$, in Verbindung mit $L(\alpha)$, $\alpha \in I$, der die imaginäre Achse bei einem kritischen Parameter α_0 kreuzte. Der Ansatz für $L(\alpha)$, $\alpha \in I$, stammte dabei von Cooke und Huang (siehe [1]). Smith ging davon aus, dass bei dem Beispiel eine Hopfverzweigung vorlag. Aber er konnte seine Vermutung nicht beweisen.
Als die Arbeit an dieser Dissertation begann, war es naheliegend zu versuchen, ähnlich wie in [7] die Hopfverzweigung mit Hilfe von Dimensionsreduktion über eine invariante Zentrumsmannigfaltigkeit zu beweisen. Dann hätte man auf das reduzierte System einen Verzweigungssatz für gewöhnliche Differentialgleichungen, wie er in [7] zu finden ist, anwenden können. Die Existenz einer Lipschitzstetigen Zentrumsmannigfaltigkeit in der Nähe eines Gleichgewichtes bei Differentialgleichungen mit zustandsabhängiger Verzö - gerung wurde in [4] bewiesen. Darüberhinaus gibt es derzeit noch keine weiteren Erkenntnisse zur Glattheit von Zentrumsmannigfaltigkeiten für Differentialgleichungen mit zustandsabhängiger Verzögerung. Bekannt sind bislang lediglich Ergebnisse über die Existenz glatter instabiler Mannigfaltigkeiten in der Nähe eines hyperbolischen Gleichgewichtes, veröffentlicht in [6] und [5], sowie über die Existenz eines Halbflusses auf einer Lösungs - mannigfaltigkeit, veröffentlicht in [11].
Wir werden deshalb die Hopfverzweigung mittels eines funktionalanalytischen Ansatzes beweisen, der ohne die Existenz von Zentrumsmannig-

faltigkeiten und eines Halbflusses auskommt. Dieser Ansatz verwendet den Satz über die Fredholm - Alternative, zu finden als Satz 1.1.4.1 im ersten Abschnitt unseres ersten Kapitels. Mit Hilfe dieses Satzes konnte man Hopfverzweigungen für Differentialgleichungen mit konstanter Verzögerung in [3], Kapitel 11.1, *Hopf bifurcation*, beweisen.

Im ersten Kapitel dieser Dissertation, *Hopf bifurcation*, werden wir die Beweisschritte des Hopfverzweigungssatzes aus [3] unter Berücksichtigung der besonderen Differenzierbarkeitseigenschaften der Abbildung F, zusammengefasst als H 1) bis H 6) am Anfang des ersten Kapitels, modifizieren. Auf diese Weise werden wir einen Hopfverzweigungssatz für Differentialgleichungen mit zustandsabhängiger Verzögerung mit allen dazu notwendigen Voraussetzungen präsentieren und beweisen können. Die exakte Herangehensweise an diesen Beweis wird im ersten Kapitel, Abschnitt *General approach of the proof of local Hopf- bifurcation*, beschrieben werden.

Im zweiten Kapitel, *The robot arm*, werden wir eine Anwendung beschreiben, um zu zeigen, dass das Phänomen Hopfverzweigung im Falle von Differentialgleichungen mit zustandsabhängiger Verzögerung tatsächlich auftritt:

Das Differentialgleichungssystem, welches wir betrachten werden, beschreibt die Bewegung eines Roboterarmes über einem darunterliegenden Objekt. Der Roboterarm berechnet seine Position aus der Laufzeit eines Signales, das zum Zeitpunkt $t - r$ vom Arm ausgesandt und vom Objekt reflektiert wird, um zum Zeitpunkt t wieder vom Arm empfangen zu werden.

Dieses System wurde in [11] als ein Beispiel für den Halbfluss auf der Lösungsmannigfaltigkeit präsentiert.

Das letzte Kapitel *Appendix* enthält wichtige Werkzeuge, die für den Beweis der Hopverzweigung nützlich sein werden.

Wir werden mit einem Abschnitt *General settings* beginnen, in dem wir Definitionen und Schreibweisen präsentieren werden, die für die gesamte Arbeit gültig sein werden.

Introduction

The goal of this doctoral dissertation is the proof of a local Hopf bifurcation Theorem for delay differential equations with state - dependent delays such as it is known for ordinary differential equations, partial differential equations or delay differential equations with constant delays.
This theorem should be applicable to parametrized delay differential equations of the form

$$x'(t) = f\Big(\alpha, x\big(t - r(x_t)\big)\Big), \quad t \in \mathbb{R}, \quad \alpha \in J \subset \mathbb{R}.$$

Here

$$f : J \times \mathbb{R}^n \to \mathbb{R}^n,$$

$J \subset \mathbb{R}$ an interval, is 2 times continuously differentiable. The segment x_t, $t \in \mathbb{R}$, is an element of the space $C^1([-h,0]|\mathbb{R}^n)$ of continuously differentiable real - valued functions $\phi : [-h,0] \to \mathbb{R}^n$, $h > 0$. In many examples r is implicitly given by an equation like

$$r = x(-r) + x(0) + D,$$

D a nonnegative constant.
When dealing with Hopf bifurcation one considers the following situation:
Let $C([-h,0]|\mathbb{R}^n)$ be the space of continuous real - valued functions $\phi : [-h,0] \to \mathbb{R}^n$.
For $\alpha \in J$ let $L(\alpha) : C([-h,0]|\mathbb{R}^n) \to \mathbb{R}^n$ be a bounded linear operator. Let $F(\alpha, \phi) := f\Big(\alpha, \phi\big(-r(\phi)\big)\Big) - L(\alpha)\phi$, for $\phi \in C^1([-h,0]|\mathbb{R}^n)$ and $\alpha \in J$.
Hence, our equation becomes

$$x'(t) = L(\alpha)x_t + F(\alpha, x_t), \quad t \in \mathbb{R}, \quad \alpha \in J.$$

Suppose there exists an equilibrium, i.e., a constant function $\phi^* \in C^1([-h,0]|\mathbb{R}^n)$, such that $F\big(\alpha, \phi^*\big) = 0$ for all $\alpha \in J$.
Furthermore, suppose there exists a branch of simple eigenvalues $\lambda(\alpha)$, $\alpha \in I \subset J$, $I \subset J$ an interval, belonging to the linear parametrized functional differential equation

$$y'(t) = L(\alpha)y_t, \quad t \in \mathbb{R}, \quad \alpha \in J,$$

which crosses the imaginary axis at a critical parameter $\alpha_0 \in I$*. (See detailed explanation in 1.1.2 and 1.1.3)*
Then one expects to get a parametrization

$$Q \subset \mathbb{R} \ni a \mapsto \big(\alpha, (a), \phi(a), T(a)\big) \in J \times C^1(\mathbb{R}|\mathbb{R}^n) \times \mathbb{R},$$

$Q \subset \mathbb{R}$ *an interval, such that the following holds:*
$0 \in Q$*. For every* $a \in Q$*, there exists a periodic solution* $x^*(a) : \mathbb{R} \to \mathbb{R}^n$ *of the equation above with parameter* $\alpha(a)$*, period* $T(a)$ *and* $x(a)_{t=0} = \phi(a)$*. Furthermore,* $\alpha(0) = \alpha_0$*,* $\phi(0) = \phi^*$ *and* $\lambda(0) = \lambda_0$*.*
One of the first papers on Hopf bifurcation for state dependent delay equations was published by Hal Smith, [9]: When investigating an example of delay differential equations he set $L(\alpha)\chi := D_2 f\big(\alpha, \phi^*(0)\big)\big(\chi(-r(\phi^*))\big)$, for $\chi \in C([-h, 0]|\mathbb{R}^n)$, and observed a branch of simple eigenvalues $\lambda(\alpha)$, $\alpha \in I$ associated with $L(\alpha)$, $\alpha \in I$, crossing the imaginary axis at a critical parameter α_0. The Ansatz for $L(\alpha)$, $\alpha \in I$, was motivated by Cooke and Huang, [1]. Consequently, he supposed there might be a Hopf bifurcation. But he could not give a proof of his hypothesis.
When starting the work on this thesis the first idea of proving Hopf bifurcation was to follow the standard approach of reducing the dimension via invariant finite - dimensional center manifolds as described in [7] and to apply a Hopf bifurcation Theorem for ordinary differential equations also to be found in [7]. But yet, we have no center manifolds of class C^2 associated with our problem of delay differential equations. There are only results on the following issues:
Existence of centermanifolds close to a nonhyperbolic equilibrium as stated in [4], existence of unstable manifolds close to a hyperbolic equilibrium as stated in [6] and [5], and existence of a C^1 - semiflow on a solution manifold as stated in [11].
Therefore, we will proof Hopf bifurcation by applying a functionalanalytic approach which avoids the existence of a semiflow and a centermanifold. This approach uses the *Fredholm alternative Theorem* 1.1.4.1 as stated in the first section of our first chapter. Given that theorem one was able to prove Hopf bifurcation in the case of differential equations with constant delays in [3], chapter 11.1, *Hopf bifurcation.* In the first chapter of this thesis,

Hopf bifurcation, we will modify the steps of the proof of Hopf bifurcation in [3]. The differentiability properties of the mapping F (assumed as H 1) to H 6) at the beginning of the first chapter) make this approach more difficult than the original one in [3]. A detailed description of the proof is contained in the first section of chapter 1, *General approach of the proof of local Hopf-bifurcation*.

In the second chapter, *The robot arm*, we will give an example in order to show that Hopf bifurcation really occurs in the case of delay differential equations with state dependent delays:

The system of delay differential equations we will concentrate on describes the movement of a robot arm over an object below. The robot arm computes its position from the running time r of a signal of speed c, emitted at time $t - r$, reflected by the object and absorbed at time t.

This system was introduced in [11] as an example for the *semiflow on the solution manifold*.

The last chapter, *Appendix*, contains some important tools which will be useful for the proof of Hopf bifurcation.

We will start by introducing some *general settings* which will be used throughout all chapters.

0.1 General Settings

Let $T \in \mathbb{R}$ be a positive real number, let $n \in \mathbb{N}$, $i \in \{0, 1, 2\}$.
C^i denotes the Banach space of i - times continuously differentiable functions $u : [0, T] \to \mathbb{R}^n$ equipped with the norm

$$\|u\|_{C^i} := \max_{0 \le s \le i} \sup_{t \in [0,T]} \|u^{(s)}(t)\|_{\mathbb{R}^n}$$

for $u \in C^i$ where $u^{(i)}(t)$ denotes the ith derivative of u in $t \in [0, T]$.
For any (2 times) continuously differentiable function $x : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval, we will also denote the first (and second derivative) of x in $s \in I$ by $x'(s)$ (and $x''(s)$).
C^i_T denotes the space of i - times continuously differentiable T - periodic functions $u : \mathbb{R} \to \mathbb{R}^n$, equipped with the norm $\|u\|_{C^i_T} := \|u|_{[0,T]}\|_{C^i}$ for $u \in C^i_T$.
Let $h > 0$ be a real number. C^i_h denotes the space of i - times continuously differentiable functions $\phi : [-h, 0] \to \mathbb{R}^n$, equipped with the norm

$$\|\phi\|_{C^i_h} := \max_{0 \le s \le i} \sup_{\theta \in [-h,0]} \|\phi^{(s)}(\theta)\|_{\mathbb{R}^n}$$

for $\phi \in C^i_h$. We set $C^0_h = C_h$. For any i - times continuously differentiable mapping $x : \mathbb{R} \to \mathbb{R}^n$ and $t \in \mathbb{R}$ we define the segment $x_t \in C^i_h$ by $x_t(\theta) := x(t + \theta)$, for $\theta \in [-h, 0]$.
Let B and D be Banach spaces. Let $L(B|D)$ denote the space of bounded linear mappings $\chi : B \to D$, equipped with the norm

$$\|\chi\|_{L(B|D)} := \sup_{\substack{b \in B \\ \|b\|_B = 1}} \|\chi(b)\|_D$$

for $\chi \in L(B|D)$.

Let B_1 and B_2 be Banach spaces, let $O_1 \subset B_1$ and $O_2 \subset B_2$ open subsets. Then $L(B_1, B_2|D)$ denotes the space of bounded bilinear mappings $\chi : B_1 \times B_2 \to D$, equipped with the norm

$$\|\chi\|_{L(B_1,B_2|D)} := \sup_{\substack{(b_1, b_2) \in B_1 \times B_2 \\ \|b_1\|_{B_1} = 1, \|b_2\|_{B_2} = 1}} \|\chi(b_1)(b_2)\|_D$$

for $\chi \in L(B_1, B_2|D)$. Let $L^2(B|D)$ denote the space of bounded symmetric bilinear mappings $\chi : B \times B \to D$. We can equip that space with the norm

$$\|\chi\| := \sup_{\substack{(b_1, b_2) \in B \times B \\ \|b_1\| = 1, \|b_2\|_B = 1}} \|\chi(b_1)(b_2)\|_D$$

for $\chi \in L^2(B|D)$.
One can easily show that this norm is equivalent to

$$\|\chi\| := \sup_{\substack{b \in B \\ \|b\| = 1}} \|\chi(b)(b)\|_D$$

for $\chi \in L^2(B|D)$. So we set

$$\|\chi\|_{L^2(B|D)} := \sup_{\substack{b \in B \\ \|b\| = 1}} \|\chi(b)(b)\|_D$$

for $\chi \in L^2(B|D)$.
For any continuously differentiable mapping

$$\gamma : (O_1 \times O_2) \subset (B_1 \times B_2) \to D,$$

$j \in \{1, 2\}$, let

$$D_j\gamma(x_1, x_2) \in L(B_j|D)$$

denote the partial derivative of γ with respect to x_j in $(x_1, x_2) \in O_1 \times O_2$.
For any 2 times continuously differentiable mapping

$$\gamma : (O_1 \times O_2) \subset (B_1 \times B_2) \to D,$$

$j \in \{1, 2\}$, $i \in \{1, 2\}$,

$$D_j(D_i\gamma)(x_1, x_2) \in L\big(B_j|L(B_i|D)\big)$$

denotes the partial derivative of $D_i\gamma(x_1, x_2)$ with respect to x_j in $(x_1, x_2) \in O_1 \times O_2$. γ being 2 times continuously differentiable the identity

$$D_i(D_j)\gamma(x_1, x_2)(b_j)(b_i) = D_j(D_i)\gamma(x_1, x_2)(b_i)(b_j)$$

holds for $i, j = 1, 2$, $b_i \in B_i$ and $b_j \in B_j$.
Therefore, we can define $D_iD_j\gamma(x_1, x_2) = D_iD_j\gamma(x_1, x_2)$ as an element of $L^2(B_i, B_j|D)$ by setting $D_iD_j\gamma(x_1, x_2)(b_i)(b_j) := D_i(D_j^2)\gamma(x_1, x_2)(b_i)(b_j)$ for $(b, b') \in B_j \times B_j$. In the case $i = j = 2$ we say that $D_iD_i\gamma(x_1, x_2) = D_i^2\gamma(x_1, x_2) \in L^2(B_i|D)$ is the second partial derivative of γ with respect to x_i in $(x_1, x_2) \in O_1 \times O_2$.
If

$$\gamma : I \subset \mathbb{R} \to D,$$

$I \subset \mathbb{R}$ an open subset, is a continuously differentiable mapping then we set $D\gamma(x)1 := D\gamma(x)(1)$
We will often write $\gamma'(x)$ instead of $D\gamma(x)1$, for $x \in I$.
If

$$\gamma : I \subset \mathbb{R} \to D$$

is a 2 times continuously differentiable mapping then we will often write $\gamma''(x)$ instead of $D^2\gamma(x)(1)(1)$, for $x \in I$.
We finish by recalling the definition of some standard symbols - and notations which will be valid throughout the whole dissertation:

a) For any matrix $B \in \mathbb{R}^{n\times m}$ the transposed matrix is denoted by $B^t \in \mathbb{R}^{m\times n}$.

b) $\mathbb{N}$ denotes the set of natural numbers $\{1, 2...\}$.

c) $\mathbb{Z}$ denotes the set of integers $\{... - 2, -1, 0, 1, 2...\}$.

d) $\mathbb{Q}$ denotes the set of rational numbers $\{\frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N} \setminus \{0\}\}$.

e) $\mathbb{R}$ denotes the set of real numbers.

f) $\mathbb{C}$ denotes the set of complex numbers.

Chapter 1

Hopf bifurcation

1.1 General approach of the proof of local Hopf-bifurcation

1.1.1 General Assumptions

Let $J \subset \mathbb{R}$ be an interval, $\Omega \subset C_h^1$ and $\Omega^* \subset C_h^2$ be open subsets such that $\Omega^* = \Omega \cap C_h^2$. Let $g : J \times \Omega \to \mathbb{R}^n$ be a mapping satisfying the following assumptions:

H 1): The mapping $g : J \times \Omega \to \mathbb{R}^n$ is continuously differentiable. There exists a constant function $\phi^* \in \Omega^*$ such that $g(\alpha, \phi^*) = 0$ for all $\alpha \in J$.

H 2): For any $(\alpha, \phi) \in J \times \Omega$ the partial derivative $D_2 g(\alpha, \phi) \in L(C_h^1 | \mathbb{R}^n)$ of g with respect to ϕ extends to a bounded linear mapping $D_{2,ext} g(\alpha, \phi) : C_h \to \mathbb{R}^n$.

H 3): The mapping

$$J \times \Omega \times C_h \ni (\alpha, \phi, \chi) \mapsto D_{2,ext} g(\alpha, \phi)(\chi) \in \mathbb{R}^n$$

is continuous.

H 4): The mapping $g^* := g|_{J \times \Omega^*}$ is 2 times continuously differentiable.

H 5): The second partial derivative $D_2^2 g^*(\alpha,\phi) \in L^2(C_h^2|\mathbb{R}^n)$ of g^* with respect to ϕ in $(\alpha,\phi) \in J \times \Omega^*$ extends to a continuous bilinear mapping $D^2_{2,ext}g^*(\alpha,\phi) : C_h^1 \times C_h^1 \to \mathbb{R}^n$.

H 6): Let $J_{C_h^2,C_h^1}$ denote the continuous embedding from C_h^2 to C_h^1.
The mappings

$$J \times \Omega^* \times C_h^1 \times C_h^1 \ni (\alpha,\phi,\chi_1,\chi_2) \mapsto D^2_{2,ext}g^*(\alpha,\phi)(\chi_1)(\chi_2) \in \mathbb{R}^n$$

and

$$D^2_{2,ext,1}\mathbf{g}^* : J \times \Omega^* \times C_h^1 \to L(C_h^2|\mathbb{R}^n),$$

defined by

$$D^2_{2,ext}g^*(\alpha,\phi)(\chi)\big(J_{C_h^2,C_h^1}(\psi)\big),$$

for $(\alpha,\phi,\chi) \in J \times \Omega^* \times C_h^1$ and $\psi \in C_h^2$, are continuous.

Note that H 3) does not include the continuity of

$$J \times \Omega \ni (\alpha,\phi) \mapsto D_{2,ext}g(\alpha,\phi) \in L(C_h|\mathbb{R}^n).$$

H 6) does not include the continuity of

$$J \times \Omega^* \ni (\alpha,\phi) \mapsto D^2_{2,ext}g^*(\alpha,\phi) \in L^2(C_h^1|\mathbb{R}^n).$$

In this work we will consider the problem of Hopf bifurcation for a differential equation like

$$x'(t) = g(\alpha, x_t), \quad t \in \mathbb{R}, \quad \alpha \in J \tag{1.1}$$

where the function g satisfies all assumptions $H1$) to $H6$).
We assume the linearization $L(\alpha) := D_{2,e}g(\alpha,\phi^*)$, $\alpha \in J$, to have a branch of simple eigenvalues $\lambda(\alpha) \in \mathbb{C}$, $\alpha \in I \subset J$ which crosses the imaginary axis at a critical parameter $\alpha_0 \in I$ (an exact definition follows in section 1.1.3). As we described in the introduction we want to get an open intervall $0 \in Q \subset \mathbb{R}$ such that for every $a \in Q$ there exists a periodic solution $x^*(a)$ of equation (1.1) with parameter $\alpha(a)$, period $T(a)$ and $x^*(a)_{t=0} = \phi(a)$.

Furthermore, the identities $\alpha(0) = \alpha_0$, $\phi(0) = \phi^*$ and $\lambda(0) = \lambda_0$ should hold.
We will prove the Hopf Bifurcation Theorem by applying the *Fredholm Alternative Theorem* as stated by 1.1.4.1. This theorem yields necessary and sufficient conditions for the existence of periodic solutions of the following nonautonomous equation

$$x'(t) = Lx_t + f(t), \tag{1.2}$$

where $L \in L(C_h|\mathbb{R}^n)$ and $f : \mathbb{R} \to \mathbb{R}^n$ is a continuous T - periodic function. Therefore, we will consider equation 1.1 as a perturbation of the linear autonomous equation

$$y'(t) = L(\alpha_0)y_t.$$

First, we recall some basic knowledge about linear problems which we will need throughout the whole thesis.

1.1.2 Linear problems

Let $L \in L(C_h|\mathbb{R}^n)$. The linear functional differential equation

$$y'(t) = Ly_t, \quad t \in \mathbb{R} \tag{1.3}$$

leads to a semigroup of continuous operators $T(t)_{t\geq 0}$:
For $t \geq 0$ the operator $T(t)$ maps $\phi \in C_h$ onto the segment $y_t^\phi \in C_h$ which is the solution of (1.3) at time $t \geq 0$, satisfying $y_0^\phi = \phi$.
The infinitesimal generator $A : D(A) \subset C_h \to C_h$ is defined in the following way:

$$D(A) := \{\phi \in C_h| \lim_{\epsilon \to 0} \frac{T(\epsilon)(\phi) - \phi}{\epsilon} \quad \text{exists in} \quad C_h\}$$

and

$$A(\phi) := \lim_{\epsilon \to 0} \frac{T(\epsilon)(\phi) - \phi}{\epsilon}$$

for $\phi \in D\big(A\big)$.
One can show that $D(A) = \{\phi \in C_h^1|\phi'(0) = L\phi\}$ and $A(\phi) = \phi'$ for $\phi \in D(A)$.
The spectrum of A consists of the roots of the characteristic function

$$char(z) = \det\Big(\big(L\exp(z)_j\big)_{1\leq j\leq n} - Id_{\mathbb{R}^n} \cdot z\Big), \quad z \in \mathbb{C}.$$

Here, $\big(\exp(z)_j\big)_{1\le j\le n} \in C_h$ is given by $\exp(z)_j(\theta) := e^{z\cdot\theta} \cdot e_j$, where e_j denotes the jth unit - vector in $\mathbb{R}^n$, for $\theta \in [-h, 0]$ and $1 \le j \le n$.
The spectrum is discrete, and every eigenvalue λ of A generates a finite - dimensional generalized eigenspace $E_\lambda \subset C_h$, which is invariant under $T(t)_{t\ge 0}$.
$T(t)$, $t \ge 0$ extends to a group $T(t)$, $t \in \mathbb{R}$, on E_λ. Hence, $T(t)\phi = y_t^\phi$ solves (1.3) for all $t \in \mathbb{R}$ and $\phi \in E_\lambda$.
Furthermore, all eigenfunctions $\phi \in E_\lambda$ are C^∞. If $\phi \in E_\lambda$, all ϕ', ϕ'' etc. are elements of E_λ.
Thus, we can define continuous embeddings J_{E_λ,C^0}, J_{E_λ,C^1} and J_{E_λ,C^2} from E_λ to C^0, C^1 and C^2 respectively:
Let $y^\phi(t) := y_t^\phi(0)$, for $\phi \in E_\lambda$ and $t \in [0, T]$. Then $J_{E_\lambda,C^i}(\phi)(t) := y^\phi(t)$, for $i \in \{0, 1, 2\}$. The continuity of J_{E_λ,C^0} is a consequence of the fact that all mappings $C_h \ni \phi \mapsto y_t^\phi$, $t \ge 0$, are continuous.
The continuity of J_{E_λ,C^i}, $i \in \{1, 2\}$, follows from the fact that all $J_{E_\lambda,C^i}(E_\lambda)$, $i \in \{0, 1, 2\}$, are equal and finite - dimensional, and that norms on finite spaces are equivalent.
If for any eigenvalue $\lambda_0 \in \mathbb{C}$ of A, E_{λ_0} is such that y_t^ϕ, $t \in \mathbb{R}$, is T - periodic for all $\phi \in E_{\lambda_0}$, we can analogously define continuous embeddings $J_{E_{\lambda_0},C_T^0}$, $J_{E_{\lambda_0},C_T^1}$ and $J_{E_{\lambda_0},C_T^2}$ from E_{λ_0} to C_T^0, C_T^1 and C_T^2 respectively.
Now we can exactly explain what we mean by a *branch of eigenvalues* $\lambda(\alpha) \in \mathbb{C}$, $\alpha \in I \subset J$, *associated with* $L(\alpha)$, *which crosses the imaginary axis at a critical parameter* α_0.

1.1.3 First hypothesis for Hopf bifurcation

When dealing with Hopf bifurcation we assume that the linearization $L(\alpha)$, $\alpha \in J$, satisfies the following conditions:
There exists an interval $I \subset J$ and $\alpha_0 \in I$ and a parametrization $I \ni \alpha \mapsto \lambda(\alpha) \in \mathbb{C}$ onto eigenvalues of the infinitesimal generator $A(\alpha)$ belonging to the continuous semigroup $T(\alpha)(t)_{t\ge 0}$ associated with $L(\alpha)$.
The parametrization $I \ni \alpha \mapsto \lambda(\alpha) \in \mathbb{C}$ must satisfy the following properties:

L 1): $\lambda(\alpha_0) = \lambda_0 = \omega \cdot i$, $\omega = \frac{2\pi}{T}$ a real number, is a purely imaginary simple eigenvalue of the infinitesimal generator $A(\alpha_0)$ of the semigroup $T(\alpha_0)(t)_{t\geq 0}$ associated with $L(\alpha_0)$. There exists no further eigenvalue of $A(\alpha_0)$ but $\bar{\lambda}_0 = -\omega \cdot i$.

L 2): The mapping $I \ni \alpha \to \lambda(\alpha) \in \mathbb{C}$ is continuously differentiable with $\Re[(\lambda'(\alpha_0))] \neq 0$.

L 3): $\lambda(\alpha)$ for $\alpha \in I$ is a simple eigenvalue of the infinitesimal generator $A(\alpha)$ belonging to the semigroup $T(\alpha)(t)_{t\geq 0}$ associated with $L(\alpha)$.

1.1.4 The Fredholm alternative Theorem, necessary and sufficient conditions for the existence of periodic solutions of 1.1

Let $k \in \mathbb{N}$ and $\mathbf{P}$ be the k - dimensional space of continuous T - periodic solutions of

$$y'(t) = Ly_t. \tag{1.4}$$

There exists a basis $\phi_1, ..., \phi_k$ of $\mathbf{P}$ such that $\phi_j^t(s) \cdot \phi_l(s) = \delta_{jl}$ for $j, l \in \{1, ..., k\}$ and $s \in \mathbb{R}$. We set $\Phi := \big(\phi_1, ..., \phi_k\big)$.
The Fredholm alternative Theorem can be stated as follows:

Theorem 1.1.4.1. *(Fredholm alternative, see Corollary 4.1 in Chapter 6 of [3]) The necessary and sufficient condition for the existence of T - periodic solutions of (1.2) is*

$$\int_0^T \Phi^t(s) f(s) ds = 0.$$

Furthermore, there exists a continuous projection $\mathbf{J} : C_T^0 \to \mathbf{P}$ and a bounded linear operator $\mathbf{K} : (Id - \mathbf{J})(C_T^0) \to C_T^0$ such that $\mathbf{K}(f)$ is the unique T - periodic solution of (1.2) for $f \in (Id - \mathbf{J})(C_T^0)$.

This leads us to the approach of finding periodic solutions of (1.1), which was used in [3], chapter 11.1, *Hopf bifurcation*:
Let $\Phi^* : \mathbb{R} \to \mathbb{R}^n$ be a constant function such that $\Phi^*|_{C_h} = \phi^*$. Let $x : \mathbb{R} \to \mathbb{R}^n$ be a continuously differentiable function. Let $\beta \in (-1/2, 1/2)$, $t \in \mathbb{R}$. Let $u : \mathbb{R} \to \mathbb{R}^n$ be defined by $u(t) := x\big(t \cdot (1 + \beta)\big) + \Phi^*\big(t \cdot (1 + \beta)\big)$. Then

the identity $u\big(\tau+\theta/(1+\beta)\big) := x(t+\theta) - \Phi^*(t+\theta)$ holds for $\tau := t/(1+\beta)$ and $\theta \in [-h,0]$.
Let $u_\tau \in C_h^1$ be defined by $u_\tau(\theta) := u(\tau+\theta)$ for $\tau \in \mathbb{R}$ and $\theta \in [-h,0]$.
Let $u_{\tau,\beta} \in C_h^1$ be defined by $u_{\tau,\beta}(\theta) := u\big(\tau+\theta/(1+\beta)\big)$ for $\tau \in \mathbb{R}$ and $\theta \in [-h,0]$.
Then $x_t = u_{\tau,\beta}$ for $t = (1+\beta)\tau$ and x is a periodic solution of (1.1) with period $(1+\beta)\cdot T$ if and only if u is a periodic solution of

$$u'(\tau) = (1+\beta)\cdot g\big(\alpha, u_{\tau,\beta}+\phi^*\big), \quad \tau \in \mathbb{R}, \quad \alpha \in J \tag{1.5}$$

with period T.
Let $L : J \to L(C_h|\mathbb{R}^n)$ be given by $L(\alpha) = D_{2,ext}g\big(\alpha,\phi^*\big)$ for $\alpha \in J$.
We define

$$\mathbf{g} : J \times \Omega \times \mathbb{R} \to \mathbb{R}^n$$

by

$$\mathbf{g}(\alpha,\phi,\beta) := (1+\beta)g(\alpha,\phi+\phi^*),$$

for $(\alpha,\phi,\beta) \in J \times \Omega \times \mathbb{R}$, and

$$\mathbf{G} : J \times \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}^n$$

by

$$\mathbf{G}(\alpha,\phi,\chi,\beta) := \mathbf{g}(\alpha,\phi,\beta) - L(\alpha_0)\chi,$$

for $(\alpha,\phi,\chi,\beta) \in J \times \Omega \times \Omega \times \mathbb{R}$. Then we may rewrite (1.5) as

$$u'(\tau) = L(\alpha)u_\tau + \mathbf{G}(\alpha, u_{\tau,\beta}, u_\tau, \beta), \quad \tau \in \mathbb{R}, \quad \alpha \in J, \quad \beta \in (-1/2, 1/2)$$

which is a perturbation of the linear autonomous equation

$$y'(t) = L(\alpha_0)y_t. \tag{1.6}$$

From here on we assume the space $\mathbf{P}$ of T - periodic solutions of (1.6) to have dimension 2. (Note that in L 1) we requested $\lambda_0 = \omega \cdot i$ to be simple)
Let $\Phi_1(\alpha_0), \Phi_2(\alpha_0)$ denote a basis of $\mathbf{P}$ and let $\Phi(\alpha_0) = \big(\Phi_1(\alpha_0), \Phi_2(\alpha_0)\big)$.
We recall a fact which is well known from the case of linear ordinary differential equations in dimension 2:

For any $p \in \mathbf{P}$ there exist $(a, b) \in \mathbb{R} \times \mathbb{R}$ such that $p(\tau - b) = \Phi(\alpha_0)(\tau)(a, 0)^t$ for all $\tau \in \mathbb{R}$.

Hence, as a consequence of Theorem 1.1.4.1 $u \in C_T^1$ will be a T - periodic solution of (1.5) if and only if there exist $(a, b) \in \mathbb{R} \times \mathbb{R}$ such that v, defined by $v(\tau) := u(\tau - b)$, for $\tau \in \mathbb{R}$, is a solution of both equations

$$v(\cdot) = \Phi(\alpha_0)(\cdot)(a, 0)^t + (\mathbf{K} \circ [Id - \mathbf{J}])\Big(\mathbf{G}(\alpha, v_{\cdot,\beta}, v_{\cdot}, \beta)\Big) \tag{1.7}$$

and

$$0 = \int_0^T \Phi(\alpha_0)^t(s)\mathbf{G}(\alpha, v_{s,\beta}, v_s, \beta)ds \tag{1.8}$$

for $\tau \in \mathbb{R}$, $\alpha \in J$ and $\beta \in (-1/2, 1/2)$. Considering that the map g is now restricted by the properties H1) to H6) we need to show that the equations (1.7) and (1.8) yield a continuously differentiable mapping $\hat{O} \ni a \mapsto \hat{u} \in C_T^1$ onto solutions of both equations.

1.1.5 Steps of the proof of Hopf bifurcation

We will begin in Section 1.2 dealing with the general case of higher derivatives of mappings with restricted differentiability properties. In Section 1.3 we will apply these results to the mapping

$$J \times \hat{\Omega} \times (-1/2, 1/2) \ni (\alpha, u, \beta) \mapsto \mathbf{G}(\alpha, u_{\cdot,\beta}, u_{\cdot}, \beta) \in C_T^0. \tag{1.9}$$

Here, $\hat{\Omega} \subset C_T^1$ is an open subset such that $(u_{\tau,\beta}, u_\tau) \in \Omega \times \Omega$ for $u \in \hat{\Omega}$ and $(\tau, \beta) \in \mathbb{R} \times (-1/2, 1/2)$.

Having established the differentiability properties of the map 1.9 we will concentrate on the proof of Hopf bifurcation in Section 1.4. This proof will be divided into two parts:

First part:

We will use the results on the mapping (1.9) in order to solve equation (1.7). This is more difficult than proofs of Hopf bifurcation results which apply to partial differential equations or delay equations with constant delay.

Due to the particular differentiability properties of g from assumption H1) - H6), the Implicit Function Theorem will yield mappings

$$\tilde{O} \ni (\alpha, a, \beta) \to \tilde{u}(\alpha, a, \beta) \in C_T^1 \tag{1.10}$$

and

$$(1.11) \qquad O^* \ni (\alpha, a, \beta) \to u^*(\alpha, a, \beta) \in C_T^2$$

onto solutions of (1.7) which are continuously differentiable. Here $\tilde{O}$ and O^* are suitable subsets of $J \times \mathbb{R} \times (-1/2, 1/2)$. But we need one of these mappings to be 2 times continuously differentiable. This will be achieved by an application of Theorem 3.3.2 from Appendix III, to both mappings (1.10) and (1.11). Theorem 3.3.2 will yield a subset $\hat{O} \subset J \times \mathbb{R} \times (-1/2, 1/2)$ and a 2 times continuously differentiable mapping

$$(1.12) \qquad \hat{O} \ni (\alpha, a, \beta) \to \hat{u}(\alpha, a, \beta) \in C_T^1$$

which satisfies $\tilde{u}|_{\hat{O}} = \hat{u}$.

Second part:

We will then follow a standard approach of proving Hopf bifurcation:

We will insert (1.10), (1.11) and (1.12) into equation (1.8) and solve the resulting equation for α and β as a function of a.

1.2 Higher derivatives for mappings with restricted differentiability properties

In this section we suppose the following:
Let A, B, C, D, E be Banach spaces such that both B and C are dense in C and E respectively. Let $\Omega_1 \subset C$, $\Omega_1^* \subset B$ and $\Omega_2 \subset A$ be open subsets such that $\Omega_1^* = \Omega_1 \cap B$. Let $k \in \mathbb{N}$ be an integer and $\Delta \subset \mathbb{R}^k$ be an open bounded subset.
Let $\mathbf{h}$, j and $\mathbf{j}$ be mappings for which we make the following assumptions:

h 1): $\mathbf{h} : \Omega_1 \subset C \to D$ is continuously differentiable.

h 2): For every $c \in \Omega_1 \subset C$ the first derivative $D\mathbf{h}(c) \in L(C|D)$ of $\mathbf{h}$ with respect to c extends to a linear continuous mapping

$$D_{ext}\mathbf{h}(c) : E \to D.$$

h 3): The mapping

$$\Omega_1 \times E \ni (c, e) \mapsto D_{ext}\mathbf{h}(c)(e) \in D$$

is continuous.

h 4): $\mathbf{h}^* := \mathbf{h}|_{\Omega_1^*} : \Omega_1^* \subset B \to D$ is 2 times continuously differentiable.

h 5): For every $b \in \Omega_1^* \subset B$ the second derivative $D^2\mathbf{h}^*(b) \in L^2(B|D)$ of $\mathbf{h}^*$ with respect to b extends to a bilinear continuous mapping

$$D^2_{ext}\mathbf{h}^*(b) : C \times C \to D.$$

h 6): Let $J_{B,C}$ denote the continuous embedding from B to C.
Both mappings

$$\Omega_1^* \times C \times C \ni (b, c, c') \mapsto D^2_{ext}\mathbf{h}^*(b)(c)(c') \in D$$

and

$$D^2_{ext,1}\mathbf{h}^* : \Omega_1^* \times C \to L(B|D),$$

defined by

$$D^2_{ext}\mathbf{h}(b)(c)\big(J_{B,C}(b')\big),$$

for $(b,c) \in \Omega_1^* \times C$ and $b' \in B$, are continuous.

j 1): $j : \Omega_2 \times \Delta \to \Omega_1^* \subset B$ is continuous. For every $s \in \Delta$ the mapping $j(\cdot, s) : A \to B$ is linear and bounded. Furthermore, $\sup_{s\in\Delta} \|j(\cdot,s)\|_{L(A|B)} < \infty$.

j 2): The mapping

$$j^* := J_{B,C} \circ j : \Omega_2 \times \Delta \to \Omega_1 \subset C$$

is continuously differentiable. For every $s \in \Delta$ the mapping $D_2 j^*(\cdot, s) : A \to L(\mathbb{R}^k|C)$ is linear and bounded.
Furthermore, $\sup_{s\in\Delta} \|D_2 j^*(\cdot,s)\|_{L(A,\mathbb{R}^k|C)} < \infty$.

j 3): Let $J_{C,E}$ denote the continuous embedding from C to E. The mapping $j^{**} := J_{C,E} \circ j^* : \Omega_2 \times \Delta \to E$ is 2 times continuously differentiable.

j 4): $\mathbf{j} : \Omega_2 \times \Delta \mapsto \Omega_1 \subset C$ is continuous. For every $s \in \Delta$ the mapping $\mathbf{j}(\cdot, s) : A \to C$ is linear and bounded. Furthermore, $\sup_{s\in\Delta} \|\mathbf{j}(\cdot,s)\|_{L(A|C)} < \infty$.

j 5): Let $J_{C,E}$ denote the continuous embedding from C to E. The mapping $\mathbf{j}^* := J_{C,E} \circ \mathbf{j} : \Omega_2 \times \Delta \to E$ is continuously differentiable.

Note that do not assume the following properties:

- The continuity of

$$\Omega_1 \subset C \ni c \mapsto D_{ext}\mathbf{h}(c) \in L(E|D)$$

  in **h** 3)

- The continuity of

$$\Omega_1^* \subset B \ni b \mapsto D^2_{ext}\mathbf{h}^*(b) \in L^2(C|D)$$

  in **h** 6)

- The continuity of
$$\Delta \ni s \mapsto j(\cdot, s) \in L(A|B)$$
in j 1)

- The continuity of
$$\Delta \ni s \mapsto D_2 j^*(\cdot, s) \in L(A, \mathbb{R}^k|C)$$
in j 2)

- The continuity of
$$\Delta \ni s \mapsto \mathbf{j}(\cdot, s) \in L(A|C)$$
in j 4)

Lemma 1.2.0.1. *Let* $\mathbf{h}$ *be a mapping such that* $\mathbf{h}$ *1),* $\mathbf{h}$ *2) and* $\mathbf{h}$ *3) are satisfied.*
Let $\mathbf{j}$ *be a mapping such that* j *4) and* j *5) are satisfied.*
Then the mapping

$$\Omega_2 \times \Delta \ni (a, s) \mapsto D\mathbf{h}\big(\mathbf{j}(a, s)\big)\big(\mathbf{j}(\cdot, s)\big) \in L(A|D)$$

is continuous.

Proof. For simplicity we only consider the case $\Omega_1 = C$ and $\Omega_2 = A$.
We show that the mapping

$$C \times \Delta \ni (c, s) \mapsto D\mathbf{h}(c)\big(\mathbf{j}(\cdot, s)\big) \in L(A|D)$$

is continuous. The claim of the lemma then follows from the continuity of the mapping
$$A \times \Delta \ni (a, s) \mapsto \mathbf{j}(a, s) \in C.$$

We have to show that for given $(c, s) \in C \times \Delta$ and given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{\substack{v \in A \\ \|v\|_A = 1}} \|D\mathbf{h}(c)\big(\mathbf{j}(v, s)\big) - D\mathbf{h}(c')\big(\mathbf{j}(v, s')\big)\|_D < \epsilon$$

for $(c', s') \in C \times \Delta$ with $\|c - c'\|_C \leq \delta$, $|s - s'|_{\mathbb{R}^k} \leq \delta$.
We know that the mapping

$$C \ni c' \mapsto D\mathbf{h}(c') \in L(C|D)$$

is continuous and that for given $c' \in C$ the mapping

$$D\mathbf{h}_{ext}(c') : E \to D$$

is linear and bounded. Also we know that $\sup\limits_{s' \in \Delta} \|\mathbf{j}(\cdot, s')\|_{L(A|C)} < \infty$.
Hence, for given $c \in C$ there exists a $\tilde{\delta} > 0$ such that

$$\|D_{ext}\mathbf{h}(c)(e - e')\|_D \leq 1/2 \cdot \epsilon$$

and

$$\|D\mathbf{h}(c) - D\mathbf{h}(c')\|_{L(C|D)} \cdot \sup_{s' \in \Delta} \|\mathbf{j}(\cdot, s')\|_{L(A|C)} \leq 1/2 \cdot \epsilon$$

for $(e, e') \in E \times E$, with $\|e - e'\|_E \leq \tilde{\delta}$, and $c' \in C$, with $\|c - c'\|_C \leq \tilde{\delta}$.
Hypothesis j 5) implies the continuity of

$$\Delta \ni s \mapsto \mathbf{j}^*(\cdot, s) \in L(A|E).$$

Hence, one gets the existence of $\hat{\delta} > 0$ such that for given $s \in \Delta$

$$\sup_{\substack{v \in A \\ \|v\|_A = 1}} \|\mathbf{j}^*(v, s) - \mathbf{j}^*(v, s')\|_E \leq \tilde{\delta}$$

for $s' \in \Delta$, with $\|s - s'\|_{\mathbb{R}^k} \leq \hat{\delta}$.
We set $\delta := \min\{\tilde{\delta}, \hat{\delta}\}$.
Hence, by observing the identity $D\mathbf{h}(c')\big(\mathbf{j}(v, s')\big) = D_{ext}\mathbf{h}(c')\big(\mathbf{j}^*(v, s')\big)$ for all $(c', s') \in C \times \Delta$, the inequality

$$\sup_{\substack{v \in A \\ \|v\|_A = 1}} \|D\mathbf{h}(c)\big(\mathbf{j}(v, s)\big) - D\mathbf{h}(c')\big(\mathbf{j}(v, s')\big)\|_D$$

$$\leq \sup_{\substack{v \in A \\ \|v\|_A = 1}} \|D\mathbf{h}(c)\big(\mathbf{j}(v, s')\big) - D\mathbf{h}(c')\big(\mathbf{j}(v, s')\big)\|_D$$

$$+ \sup_{\substack{v \in A \\ \|v\|_A = 1}} \|D_{ext}\mathbf{h}(c)\big(\mathbf{j}^*(v,s)\big) - D_{ext}\mathbf{h}(c)\big(\mathbf{j}^*(v,s')\big)\|_D$$

$$\leq \|D\mathbf{h}(c) - D\mathbf{h}(c')\|_{L(C|D)} \cdot \sup_{s' \in \Delta} \|\mathbf{j}(\cdot, s')\|_{L(A|C)}$$

$$+ \sup_{\substack{v \in A \\ \|v\|_A = 1}} \|D_{ext}\mathbf{h}(c)\big(\mathbf{j}^*(v,s) - \mathbf{j}^*(v,s')\big)\|_D$$

$$\leq 1/2 \cdot \epsilon + 1/2 \cdot \epsilon = \epsilon$$

holds for $(c', s') \in C \times \Delta$ with $\|c - c'\|_C \leq \delta$, $|s - s'|_{\mathbb{R}^k} \leq \delta$.
Thus,

$$C \times \Delta \ni (c,s) \mapsto D\mathbf{h}(c)\big(\mathbf{j}(\cdot, s)\big) \in L(A|D)$$

is continuous. □

Lemma 1.2.0.2. *Let* **h** *be a mapping such that* **h** *1),***h** *2),* **h** *3),* **h** *4),* **h** *5),* **h** *6)are satisfied.*
Let j be a mapping such that j 1), j 2) and j 3) are satisfied.
Then the mapping

$$\Omega_2 \times \Delta \ni (a,s) \mapsto D^2\mathbf{h}^*\big(j(a,s)\big)\big(j(\cdot,s)\big)\big(j(\cdot,s)\big) \in L^2(A|D)$$

is continuous.

Proof. For simplicity we only consider the case $\Omega_1 = C$, $\Omega_1^* = B$ and $\Omega_2 = A$. We show that the mapping

$$B \times \Delta \ni (b,s) \mapsto D^2\mathbf{h}^*(b)\big(j(\cdot,s)\big)\big(j(\cdot,s)\big) \in L^2(A|D)$$

is continuous. The claim of the lemma then follows from the continuity of the mapping

$$A \times \Delta \ni (a,s) \mapsto \big(j(a,s), s\big) \in B \times \Delta.$$

We have to show that for given $(b, s) \in B \times \Delta$ and given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{\substack{v \in A \\ \|v\|_A = 1}} \|D^2\mathbf{h}^*(b)\big(j(v,s)\big)\big(j(v,s)\big) - D^2\mathbf{h}^*(b)\big(j(v,s')\big)\big(j(v,s')\big)\|_D < \epsilon$$

for $(b', s') \in B \times \Delta$ with $\|b - b'\|_B \leq \delta$, $|s - s'|_{\mathbb{R}^k} \leq \delta$.
We know that the mapping

$$B \ni b' \mapsto D^2\mathbf{h}^*(b') \in L^2(B|D)$$

is continuous and that for given $b' \in B$ the mapping

$$D^2_{ext}\mathbf{h}^*(b') : C \times C \to D$$

is bilinear and bounded.
Also we know that $\sup_{s' \in \Delta} \|j(\cdot, s')\|_{L(A|B)} < \infty$.
Hence, for given $b \in B$ there exists a $\tilde{\delta} > 0$ such that

$$\|D^2_{ext}\mathbf{h}^*(b)(c)(c) - D^2_{ext}\mathbf{h}^*(b)(c')(c')\|_D \leq 1/2 \cdot \epsilon$$

and

$$\|D^2\mathbf{h}^*(b) - D^2\mathbf{h}^*(b')\|_{L^2(B|D)} \cdot \sup_{s' \in \Delta} \|j(\cdot, s')\|_{L(A|B)} \leq 1/2 \cdot \epsilon$$

for $(c, c') \in C \times C$, with $\|c - c'\|_C \leq \tilde{\delta}$, and $b' \in B$, with $\|b - b'\|_B \leq \tilde{\delta}$.
Hypothesis j 3) yields the continuity of

$$\Delta \ni s \mapsto j^*(\cdot, s) \in L(A|C).$$

Hence, one gets the existence of $\hat{\delta} > 0$ such that for given $s \in \Delta$

$$\sup_{\substack{v \in A \\ \|v\|_A = 1}} \|j^*(v,s) - j^*(v,s')\|_C \leq \tilde{\delta}$$

for $s' \in \Delta$, with $\|s - s'\|_{\mathbb{R}^k} \leq \hat{\delta}$.
We set $\delta := \min\{\tilde{\delta}, \hat{\delta}\}$.

Hence, by observing the identity $D^2\mathbf{h}^*(b')\big(j(\cdot,s')\big) = D^2_{ext}\mathbf{h}^*(b')\big(j^*(\cdot,s')\big)$ for all $(b',s') \in B \times \Delta$, the inequality

$$\sup_{\substack{v \in A \\ \|v\|_A = 1}} \|D^2\mathbf{h}^*(b)\big(j(v,s)\big)\big(j(v,s)\big) - D^2\mathbf{h}^*(b')\big(j(v,s')\big)\big(j(v,s')\big)\|_D$$

$$\leq \sup_{\substack{v \in A \\ \|v\|_A = 1}} \|D^2\mathbf{h}^*(b)\big(j(v,s')\big)\big(j(v,s')\big) - D^2\mathbf{h}^*(b')\big(j(v,s')\big)\big(j(v,s')\big)\|_D$$

$$+ \sup_{\substack{v \in A \\ \|v\|_A = 1}} \|D^2_{ext}\mathbf{h}^*(b)\big(j^*(v,s)\big)\big(j^*(v,s)\big) - D^2_{ext}\mathbf{h}^*(b)\big(j^*(v,s')\big)\big(j^*(v,s')\big)\|_D$$

$$\leq \|D^2\mathbf{h}^*(b) - D^2\mathbf{h}^*(b')\|_{L^2(B|D)} \cdot \sup_{s' \in \Delta} \|j(\cdot,s')\|_{L(A|B)}$$

$$+ \sup_{\substack{v \in A \\ \|v\|_A = 1}} \|D^2_{ext}\mathbf{h}^*(b)\big(j^*(v,s)\big)\big(j^*(v,s)\big) - D^2_{ext}\mathbf{h}^*(b)\big(j^*(v,s')\big)\big(j^*(v,s')\big)\|_D$$

$$\leq 1/2 \cdot \epsilon + 1/2 \cdot \epsilon = \epsilon$$

holds for $(b',s') \in B \times \Delta$ with $\|b - b'\|_B \leq \delta$, $|s - s'|_{\mathbb{R}^k} \leq \delta$.
Thus,

$$B \times \Delta \ni (b,s) \mapsto D^2\mathbf{h}^*(b)\big(j(\cdot,s)\big)\big(j(\cdot,s)\big) \in L^2(A|D)$$

is continuous. □

Lemma 1.2.0.3. *Let* $\mathbf{h}$ *be a mapping such that* $\mathbf{h}$ *1),* $\mathbf{h}$ *2) and* $\mathbf{h}$ *3) are satisfied.*
Let $\mathbf{j}$ *be a mapping such that* j *4) and* j *5) are satisfied.*
Then the mapping $H : \Omega_2 \times \Delta \ni (a,s) \mapsto \mathbf{h}\big(\mathbf{j}(a,s)\big) \in D$ *has a partial derivative* $D_2H(a,s) \in L(\mathbb{R}^k|D)$ *with respect to* s *in every* $(a,s) \in \Omega_2 \times \Delta$. *Furthermore, the mapping* $\Omega_2 \times \Delta \ni (a,s) \mapsto D_2H(a,s) \in L(\mathbb{R}^k|D)$ *is continuous.*

Proof. We study the case where $k=1$. The case $k>1$ would follow the steps of the proof of $k=1$ by examining the existence and continuity of all directional derivatives $D_i(\cdot,\cdot)H1 : \Omega_2 \times \Delta \to D$, $i \in \{2, ..., k+1\}$.
We claim that $D_2H(a,s)1 = D_{ext}\mathbf{h}\big(\mathbf{j}(a,s)\big)\big(D_2\mathbf{j}^*(a,s)1\big) \in D$ is the derivative of H with respect to s in $(a,s) \in \Omega_2 \times \Delta$ and that the mapping

$$D_2H(\cdot,\cdot)1 : \Omega_2 \times \Delta \ni (a,s) \mapsto D_{ext}\mathbf{h}\big(\mathbf{j}(a,s)\big)\big(D_2\mathbf{j}^*(a,s)1\big) \in D$$

is continuous:
The identity

$$\mathbf{h}(c+c') - \mathbf{h}(c) = \int_0^1 D\mathbf{h}(c + q\cdot c')(c')dq$$

holds for $(c,c') \in C \times C$ with $c + c' \in \Omega_1$ due to the fact that $\mathbf{h}|_C$ is continuously differentiable. By replacing $D\mathbf{h}$ with its extension $D_{ext}\mathbf{h}$ we rewrite this identity as

$$\mathbf{h}(c+c') - \mathbf{h}(c) = \int_0^1 D_{ext}\mathbf{h}(c + q\cdot c')\big(J_{C,E}(c')\big)dq.$$

Thus, we get that the identity

$$\tfrac{1}{\epsilon}\cdot H(a,s+\epsilon) - H(a,s) =$$

$$\tfrac{1}{\epsilon}\cdot \mathbf{h}\big(\mathbf{j}(a,s+\epsilon)\big) - \mathbf{h}\big(\mathbf{j}(a,s)\big) =$$

$$\int_0^1 D_{ext}\mathbf{h}\big(\mathbf{j}(a,s) + q[\mathbf{j}(a,s+\epsilon) - \mathbf{j}(a,s)]\big)\big(\tfrac{1}{\epsilon}\cdot\big(\mathbf{j}^*(a,s+\epsilon) - \mathbf{j}^*(a,s)\big)dq$$

holds for $(a,s,\epsilon) \in \Omega_2 \times \Delta \times \Delta$, $\epsilon \neq 0$ sufficiently small. The last expression then tends to $D_{ext}\mathbf{h}\big(\mathbf{j}(a,s)1\big)\big(D_2\mathbf{j}^*(a,s)\big)$, as $\epsilon \to 0$, due to the continuity of the mappings

$$C \times E \ni (c,e) \mapsto D_{ext}\mathbf{h}(c)(e) \in D,$$

$$\mathbf{j} : \Omega_2 \times \Delta \to C,$$

and due to $A \times \Delta \ni (a,s) \mapsto \mathbf{j}^*(a,s) \in E$ being continuously differentiable. The continuity of $D_2H : A \times \Delta \to D$ again is a consequence of the continuity of the mappings

$$C \times E \ni (c,e) \mapsto D_{ext}\mathbf{h}(c)(e) \in D,$$

$$\mathbf{j} : \Omega_2 \times \Delta \to C,$$

and

$$D_2\mathbf{j}^* : \Omega_2 \times \Delta \to E.$$

□

Lemma 1.2.0.4. *Let* $\mathbf{h}$ *be a mapping such that* $\mathbf{h}$ *1),*$\mathbf{h}$ *2),* $\mathbf{h}$ *3),* $\mathbf{h}$ *4),* $\mathbf{h}$ *5),* $\mathbf{h}$ *6)are satisfied.*
Let j *be a mapping such that* j *1),* j *2) and* j *3) are satisfied.*
Then the mapping

$$H^* : \Omega_2 \times \Delta \times \Delta \ni (a, s, s') \mapsto D\mathbf{h}^*\big(j(a,s)\big)\big(j(\cdot, s')\big) \in L(A|D)$$

has a partial derivative $D_2H^*(a, s, s') \in L\big(\mathbb{R}^k, L(A|D)\big)$ *with respect to* s *in every* $(a, s, s') \in \Omega_2 \times \Delta \times \Delta$*. Furthermore, the mapping*

$$\Omega_2 \times \Delta \times \Delta \ni (a, s, s') \mapsto D_2H^*(a, s, s') \in L\big(\mathbb{R}^k, L(A|D)\big)$$

is continuous.

Proof. We study the case where $k = 1$. The case $k > 1$ would follow the steps of the proof of $k = 1$ by examining the existence and continuity of all directional derivatives $D_iH^*(\cdot, \cdot, \cdot)1 : \Omega_2 \times \Delta \times \Delta \to L(A|D)$, $i \in \{1, ..., k\}$.
We claim that the mapping

$$l : \Omega_2 \times \Delta \ni (a, s) \mapsto D\mathbf{h}^*\big(j(a, s)\big) \in L(B|D)$$

has a partial derivative with respect to s in every $(a, s) \in \Omega_2 \times \Delta$.
Let

$$\frac{\boldsymbol{\Delta} j(a, s)}{\boldsymbol{\Delta} t} := \frac{1}{t}[j(a, s + t) - j(a, s)]$$

and

$$\frac{\boldsymbol{\Delta} j^*(a, s)}{\boldsymbol{\Delta} t} := \frac{1}{t}[j^*(a, s + t) - j^*(a, s)]$$

for $(a, s, t) \in \Omega_2 \times \Delta \times \Delta$, $t \neq 0$ sufficiently small.
The identity

$$D\mathbf{h}^*(b_1 + b_2)(b_3) - D\mathbf{h}^*(b_1)(b_3) = \int_0^1 D^2\mathbf{h}^*(b_1 + q \cdot b_2)(b_2)(b_3)dq$$

holds for $(b_1, b_2, b_3) \in B \times B \times B$ such that $b_1 + b_2 \in \Omega_1^*$ due to the fact that $\mathbf{h}^*$ is 2 times continuously differentiable. By replacing $D^2\mathbf{h}^*$ with $D^2_{ext,1}\mathbf{h}^*$ we rewrite this identity as

$$D\mathbf{h}^*(b_1 + b_2)(b_3) - D\mathbf{h}^*(b_1)(b_3) = \int_0^1 D^2_{ext,1}\mathbf{h}^*(b_1 + q \cdot b_2)\big(J_{B,C}(b_2)\big)(b_3)dq.$$

Therefore, the inequality

$$\|\tfrac{1}{t}[D\mathbf{h}^*\big(j(a, s + t)\big)(b) - D\mathbf{h}^*\big(j(a, s)\big)(b)] - D^2_{ext,1}\mathbf{h}^*\big(j(a, s)\big)\big(D_2 j^*(a, s)1\big)(b)\|_D =$$

$$\|\int_0^1 D^2_{ext,1}\mathbf{h}^*\big(j(a, s + q \cdot t \cdot \tfrac{\mathbf{\Delta} j(a,s)}{\mathbf{\Delta} t})\big)\big(\tfrac{\mathbf{\Delta} j^*(a,s)}{\mathbf{\Delta} t}\big)(b) - D^2_{ext,1}\mathbf{h}^*\big(j(a, s)\big)\big((D_2 j^*(a, s)1\big)(b)dq\|_D \leq$$

$$\int_0^1 \|D^2_{ext,1}\mathbf{h}^*\big(j(a, s + q \cdot t \cdot \tfrac{\mathbf{\Delta} j(a,s)}{\mathbf{\Delta} t})\big)\big(\tfrac{\mathbf{\Delta} j^*(a,s)}{\mathbf{\Delta} t}\big) - D^2_{ext,1}\mathbf{h}^*\big(j(a, s)\big)\big((D_2 j^*(a, s)1\big)\|_{L(B|D)}\|b\|_B dq$$

holds for all $(a, s, t, b) \in \Omega_2 \times \Delta \times \Delta \times B$, $t \neq 0$ sufficiently small. Hence, due to the continuity of

$$D^2_{ext,1}\mathbf{h}^* : \Omega_1^* \times C \to L(B|D)$$

and due to the fact that for $a \in A$

$$\frac{\mathbf{\Delta} j^*(a, s)}{\mathbf{\Delta} t} \to D_2 j^*(a, s)1,$$

as $t \to 0$, we get that

$$\lim_{t \to 0} \|\tfrac{1}{t}[l(a, s + t) - l(a, s)] - D^2_{ext,1}\mathbf{h}^*\big(j(a, s)\big)\big((D_2 j^*(a, s)1\big)\|_{L(B|D)} =$$

$$\lim_{t \to 0} \sup_{\substack{v \in B \\ \|v\|_B = 1}} \|\tfrac{1}{t}[D\mathbf{h}^*\big(j(a, s + t)\big)(v) - D\mathbf{h}^*\big(j(a, s)\big)(v)] - D^2_{ext,1}\mathbf{h}^*\big(j(a, s)\big)\big((D_2 j^*(a, s)1\big)(v)\|_D \leq$$

$$\lim_{t \to 0} \int_0^1 \|D^2_{ext,1}\mathbf{h}^*\big(j(a, s + q \cdot t \cdot \tfrac{\mathbf{\Delta} j(a,s)}{\mathbf{\Delta} t})\big)\big(\tfrac{\mathbf{\Delta} j^*(a,s)}{\mathbf{\Delta} t}\big) - D^2_{ext,1}\mathbf{h}^*\big(j(a, s)\big)\big((D_2 j^*(a, s)1\big)\|_{L(B|D)} dq = 0.$$

Therefore, the mapping

$$l : \Omega_2 \times \Delta \ni (a, s) \mapsto D\mathbf{h}^*\big(j(a, s)\big) \in L(B|D)$$

has a partial derivative with respect to s in every $(a, s) \in \Omega_2^* \times \Delta$ which is given by $D_2 l(a, s) := D^2_{ext,1}\mathbf{h}^*\big(j(a, s)\big)\big((D_2 j^*(a, s)1\big) \in L(B|D)$. The decomposition $H^*(\cdot, \cdot, s') = \mathbb{B} \circ \mathbb{A}_{s'}$ with

$$\mathbb{A}_{s'} : A \times \Delta \ni (a, s) \mapsto \big(l(a, s), j(\cdot, s')\big) \in L(B|D) \times L(A|B)$$

and

$$\mathbb{B} : L(B|D) \times L(A|B) \ni (T, S) \mapsto T \circ S \in L(A|D)$$

yields that the mapping H^* has a partial derivative with respect to s, given by

$$D_2 H^*(a, s, s')1 = D^2_{ext,1}\mathbf{h}^*\big(j(a, s)\big)\big(D_2 j^*(a, s)1\big)\big(j(\cdot, s'\big) \in L(A|D)$$

in $(a, s, s') \in A \times \Delta \times \Delta$. The proof of the continuity of

$$\Omega_2 \times \Delta \times \Delta \ni (a, s, s') \mapsto D^2_{ext,1}\mathbf{h}^*\big(j(a, s)\big)\big(D_2 j^*(a, s)1\big)\big(j(\cdot, s'\big) \in L(A|D)$$

would follow the same steps as the proof of the continuity of

$$\Omega_2 \times \Delta \ni (a, s) \mapsto D\mathbf{h}\big(\mathbf{j}(a, s)\big)\big(\mathbf{j}(\cdot, s)\big) \in L(A|D)$$

in Lemma 1.2.0.1:
In the situation of that lemma we showed that the mapping

$$\Omega_1 \times \Delta \ni (c, s) \mapsto D\mathbf{h}(c)\big(\mathbf{j}(\cdot, s)\big) \in L(A|D)$$

is continuous. The continuity of

$$\Omega_2 \times \Delta \ni (a, s) \mapsto D\mathbf{h}\big(\mathbf{j}(a, s)\big)\big(\mathbf{j}(\cdot, s)\big) \in L(A|D)$$

then followed from the continuity of

$$\Omega_2 \times \Delta \ni (a, s) \mapsto \mathbf{j}(a, s) \in C.$$

The following conditions were satisfied by assumption:
$D\mathbf{h}\big(c\big) \in L(C|D)$ extends to $D_{ext}\mathbf{h}\big(c\big) \in L(E|D)$ for $c \in \Omega_1$.
Both mappings

$$\Omega_1 \times E \ni (c, e) \mapsto D_{ext}\mathbf{h}\big(c\big)(e) \in D$$

and

$$\Delta \ni s \mapsto \mathbf{j}^*(\cdot, s) \in L(A|E)$$

are continuous.

The inequality

$$\sup_{s\in\Delta} \|\mathbf{j}(\cdot,s)\|_{L(A|C)} < \infty$$

holds.

Now we are in the situation that for $(b,c)\in\Omega_1^*\times C$

$$D^2_{ext,1}\mathbf{h}^*\big(b\big)\big(c\big)\in L(B|D)$$

extends to

$$D^2_{ext}\mathbf{h}^*\big(b\big)\big(c\big)\in L(C|D)$$

where the mapping

$$\Omega_1^*\times C\times C\ni(b,c,c')\mapsto D^2_{ext}\mathbf{h}^*\big(b\big)\big(c\big)\big(c')\big)\in D$$

is assumed to be continuous. Furthermore, the mapping

$$\Delta\ni s\mapsto j^*(\cdot,s)\in L(A|C)$$

is continuous and j satisfies the inequality

$$\sup_{s\in\Delta} \|j(\cdot,s)\|_{L(A|B)} < \infty$$

Thus, the steps of the proof of the continuity of

$$\Omega_1\times\Delta\ni(c,s)\mapsto D\mathbf{h}(c)\big(\mathbf{j}(\cdot,s)\big)\in L(A|D)$$

in Lemma 1.2.0.1 may be analogously applied to the mapping

$$\Omega_1^*\times C\times\Delta\ni(b,c,s)\mapsto D^2_{ext,1}\mathbf{h}^*(b)(c)(\big(j(\cdot,s)\big)\in L(A|D)$$

which therefore is continuous.

The continuity of

$$\Omega_2\times\Delta\times\Delta\ni(a,s,s')\mapsto D^2_{ext,1}\mathbf{h}^*\big(j(a,s)\big)\big(D_2j^*(a,s)\big)\big(j(\cdot,s')\in L(A|D)$$

then follows from the continuity of the mappings

$$\Omega_2\times\Delta\ni(a,s)\mapsto j(a,s)\in B$$

and

$$\Omega_2 \times \Delta \ni (a,s) \mapsto D_2 j^*(a,s) \in C.$$

Hence, the partial derivative $D_2 H^*(a,s,s')$ of H^* with respect to s exists in every $(a,s,s') \in \Omega_2 \times \Delta \times \Delta$ and the mapping $\Omega_2 \times \Delta \times \Delta \ni (a,s,s') \mapsto D_2 H^*(a,s,s') \in L(A|D)$ is continuous. $\square$

Lemma 1.2.0.5. *Let* $\mathbf{h}$ *be a mapping such that* $\mathbf{h}$ *1),*$\mathbf{h}$ *2),* $\mathbf{h}$ *3),* $\mathbf{h}$ *4),* $\mathbf{h}$ *5),* $\mathbf{h}$ *6)are satisfied.*
Then the mapping

$$\mathbf{H}^{**} : \Omega_1^* \times C \ni (b,c) \to D\mathbf{h}\big(J_{B,C}(b)\big)(c) \in D$$

has a partial derivative $D_1\mathbf{H}^{**}(b,c) = D^2_{ext,1}\mathbf{h}^*(b)(c) \in L(B|D)$ *with respect to* b *in every* $(b,c) \in \Omega_1^* \times C$*. Furthermore, the mapping* $\Omega_1^* \times C \ni (b,c) \mapsto D_1\mathbf{H}^{**}(b,c) \in L(B|D)$ *is continuous.*

Proof. We have to show that

$$\tfrac{1}{\|h\|_B} \cdot \|D\mathbf{h}\big(J_{B,C}(b+h)\big)(c) - D\mathbf{h}\big(J_{B,C}(b)\big)(c) - D^2_{ext,1}\mathbf{h}^*(b)(c)(h)\|_D \to 0$$

as $h \to 0$. Suppose b' is an element of B. Due to the fact that $\mathbf{h}^* : \Omega_1^* \to D$ is 2 times continuously differentiable the identity

$$D\mathbf{h}^*(b+h)(b') - D\mathbf{h}^*(b)(b') = \int_0^1 D^2\mathbf{h}^*(b+s\cdot h)(h)(b')ds$$

holds for $(b,b',h) \in \Omega_1^* \times \Omega_1^* \times B$, h sufficiently small.
As $D^2\mathbf{h}^*(b)(b')(b'') = D^2_{ext,1}\mathbf{h}^*(b)\big(J_{B,C}(b')\big)(b'')$ and $D\mathbf{h}^*(b)(b') = D_{ext}\mathbf{h}^*(b)\big(J_{B,C}(b')\big)$ for $(b,b',b'') \in \Omega_1^* \times B \times B$ we rewrite this identity as

$$D_{ext}\mathbf{h}^*(b+h)\big(J_{B,C}(b')\big) - D_{ext}\mathbf{h}^*(b)\big(J_{B,C}(b')\big) = \int_0^1 D^2_{ext,1}\mathbf{h}^*(b+s\cdot h)\big(J_{B,C}(b')\big)(h)ds.$$

Now suppose c is an element of C. $B \subset C$ being dense one gets a sequence $(b')_{n\in\mathbb{N}} \in$ such that $b'_n \to c$ in C, as $n \to \infty$. Then $D\mathbf{h}\big(J_{B,C}(b)\big)(c) = \lim_{n\to\infty} D_{ext}\mathbf{h}^*(b)\big(J_{B,C}(b'_n)\big)$ for any $b \in \Omega^*$. Therefore and due to the continuity of

$$\Omega_1^* \times C \ni (b,c) \to D^2_{ext,1}\mathbf{h}^*\big(b)(c) \in L(B|D),$$

one gets that the following identity

$$D\mathbf{h}\big(J_{B,C}(b+h)\big)(c) - D\mathbf{h}\big(J_{B,C}(b)\big)(c) =$$

$$\lim_{n\to\infty} D_{ext}\mathbf{h}^*(b+h)\big(J_{B,C}(b'_n)\big) - D_{ext}\mathbf{h}^*(b)\big(J_{B,C}(b'_n)\big) =$$

$$\lim_{n\to\infty} \int_0^1 D^2_{ext,1}\mathbf{h}^*(b+s\cdot h)(b'_n)(h)ds =$$

$$\int_0^1 D^2_{ext,1}\mathbf{h}^*(b+s\cdot h)(c)(h)ds$$

holds for $c \in C$, $b \in \Omega_1^*$ and $h \in B$ sufficiently small. With this identity holding we get the estimation

$$\frac{1}{\|h\|_B} \cdot \|D\mathbf{h}\big(J_{B,C}(b+h)\big)(c) - D\mathbf{h}\big(J_{B,C}(b)\big)(c) - D^2_{ext,1}\mathbf{h}^*(b)(h)(c)\|_D =$$

$$\frac{1}{\|h\|_B} \cdot \|\int_0^1 D^2_{ext,1}\mathbf{h}^*(b+s\cdot h)(c)(h) - D^2_{ext,1}\mathbf{h}^*(b)(c)(h)ds\|_D \leq$$

$$\frac{1}{\|h\|_B} \cdot \int_0^1 \|D^2_{ext,1}\mathbf{h}^*(b+s\cdot h)(c) - D^2_{ext,1}\mathbf{h}^*(b)(c)\|_{L(B|D)}ds\|h\|_B =$$

$$\int_0^1 \|D^2_{ext,1}\mathbf{h}^*(b+s\cdot h)(c) - D^2_{ext,1}\mathbf{h}^*(b)(c)\|_{L(B|D)}ds$$

for $(b,h) \in \Omega_1^* \times \Omega_1^*$ and $c \in C$.
Due to the continuity of

$$\Omega_1^* \times C \ni (b,c) \to D^2_{ext,1}\mathbf{h}^*\big(b)(c) \in L(B|D).$$

the expression

$$\int_0^1 \|D^2_{ext,1}\mathbf{h}^*(b+s\cdot h)(c) - D^2_{ext,1}\mathbf{h}^*(b)(c)\|_{L(B|D)}ds$$

tends to 0, as $h \to 0$. Thus, $\mathbf{H}^{**}$ is partially differentiable with respect to b in every $(b,c) \in \Omega_1^* \times C$.
The continuity of $\Omega_1^* \times C \ni (b,c) \mapsto D\mathbf{H}^{**}(b,c) \in L(B|D)$ again is a consequence of h 6). □

Lemma 1.2.0.6. *Let* $\mathbf{h}$ *be a mapping such that* $\mathbf{h}$ *1),*$\mathbf{h}$ *2),* $\mathbf{h}$ *3),* $\mathbf{h}$ *4),* $\mathbf{h}$ *5),* $\mathbf{h}$ *6)are satisfied.*
Let j *be a mapping such that* j *1),* j *2),*j *3) are satisfied.*
Then the mapping

$$\tilde{H} : \Omega_2 \times \Delta \times \Delta \ni (a, s, s') \mapsto D\mathbf{h}\big(j^*(a, s')\big)\big(j^*(\cdot, s)\big) \in L(A|D)$$

has a partial derivative $D_2\tilde{H}(a, s, s') \in L\big(\mathbb{R}^k, L(A|D)\big)$ *with respect to* s *in every* $(a, s, s') \in \Omega_2 \times \Delta \times \Delta$*. Furthermore, the mapping*

$$\Omega_2 \times \Delta \times \Delta \ni (a, s, s') \mapsto D_2\tilde{H}(a, s, s') \in L\big(\mathbb{R}^k, L(A|D)\big)$$

is continuous.

Proof. We study the case where $k = 1$. The case $k > 1$ would follow the steps of the proof of $k = 1$ by examining the existence and continuity of all directional derivatives $D_i\tilde{H}(\cdot, \cdot, \cdot)1 : \Omega_2 \times \Delta \times \Delta \to L(A|D)$, $i \in \{2, ..., k+1\}$.
$j^{**} : A \times \Omega_2 \to E$ is 2 times continuously differentiable by assumption. Therefore, and by the fact that $j^{**} : A \times \Delta \to E$ is 2 times continuously differentiable with $D_1j^{**}(a, s)(\hat{a}) = j^{**}(\hat{a}, s)$ and $D_2(D_1j^{**})(a, s)(\hat{a})1 = D_2j^{**}(\hat{a}, s)1$ for $(a, \hat{a}, s) \in A \times A \times \Delta$ one gets that

$$\lim_{\epsilon \to 0} \|\frac{1}{\epsilon} \cdot \big(j^{**}(\cdot, s+\epsilon) - j^{**}(\cdot, s)\big) - D_2j^{**}(\cdot, s)1\|_{L(A|E)} = 0.$$

As $D\mathbf{h}\big(j^*(a, s')\big) \in L(C|D)$ extends to $D_{ext}\mathbf{h}\big(j^*(a, s')\big) \in L(E|D)$ one gets the following result:

$$\lim_{\epsilon \to 0} \|D\mathbf{h}\big(j^*(a, s')\big)[\tfrac{1}{\epsilon} \cdot \big(j^*(\cdot, s+\epsilon) - j^*(\cdot, s)\big)] - D_{ext}\mathbf{h}\big(j^*(a, s')\big)[D_2j^{**}(\cdot, s)1]\|_{(L(A|D)} =$$

$$\lim_{\epsilon \to 0} \|D_{ext}\mathbf{h}\big(j^*(a, s')\big)[\tfrac{1}{\epsilon} \cdot \big(j^{**}(\cdot, s+\epsilon) - j^{**}(\cdot, s)\big) - D_2j^{**}(\cdot, s)1]\|_{L(A|D)} = 0$$

Therefore,

$$\tilde{H} : \Omega_2 \times \Delta \times \Delta \ni (a, s, s') \mapsto D\mathbf{h}\big(j^*(a, s')\big)\big(j^*(\cdot, s)\big) \in L(A|D)$$

has a partial derivative $D_2\tilde{H}(a, s, s') = D_{ext}\mathbf{h}\big(j^*(a, s')\big)\big(D_2j^{**}(\cdot, s)1\big) \in L(A|D)$ with respect to s in every $(a, s, s') \in \Omega_2 \times \Delta \times \Delta$.
The continuity of

$$D_2H^* : \Omega_2 \times \Delta \times \Delta \ni (a, s, s') \mapsto D_{ext}\mathbf{h}\big(j^*(a, s')\big)\big(D_2j^{**}(\cdot, s)1\big) \in L(A|D)$$

can be shown in the following way:
First we observe that the identity

$$D_{ext}\mathbf{h}\big(j^*(a,s')\big)\big(D_2 j^{**}(v,s)1\big) = D\mathbf{h}\big(j^*(a,s')\big)\big(D_2 j^*(v,s)1\big)$$

holds for all $(a,s,s') \in \Omega_2 \times \Delta \times \Delta$ and $v \in A$. The proof of the continuity of

$$\Omega_2 \times \Delta \times \Delta \ni (a,s,s') \mapsto D\mathbf{h}\big(j^*(a,s')\big)\big(D_2 j^*(\cdot,s)1\big) \in L(A|D)$$

is similar to the proof of the continuity of

$$\Omega_2 \times \Delta \ni (a,s) \mapsto D\mathbf{h}\big(\mathbf{j}(a,s)\big)\big(\mathbf{j}(\cdot,s)\big) \in L(A|D)$$

in Lemma 1.2.0.1:
In the situation of that lemma we showed that the mapping

$$\Omega_1 \times \Delta \ni (c,s) \mapsto D\mathbf{h}(c)\big(\mathbf{j}(\cdot,s)\big) \in L(A|D)$$

is continuous. The continuity of

$$\Omega_2 \times \Delta \ni (a,s) \mapsto D\mathbf{h}\big(\mathbf{j}(a,s)\big)\big(\mathbf{j}(\cdot,s)\big) \in L(A|D)$$

then followed from the continuity of

$$\Omega_2 \times \Delta \ni (a,s) \mapsto \mathbf{j}(a,s) \in C.$$

The following conditions were satisfied by assumption:
$D\mathbf{h}\big(c\big) \in L(C|D)$ extends to $D_{ext}\mathbf{h}\big(c\big) \in L(E|D)$ for $c \in \Omega_1$.
Both mappings

$$\Omega_1 \times E \ni (c,e) \mapsto D_{ext}\mathbf{h}\big(c\big)(e) \in D$$

and

$$\Delta \ni s \mapsto \mathbf{j}^*(\cdot,s) \in L(A|E)$$

are continuous.
The inequality

$$\sup_{s\in\Delta} \|\mathbf{j}(\cdot,s)\|_{L(A|C)} < \infty$$

holds.
Now like in Lemma 1.2.0.1 we are in the situation that for $c \in \Omega_1$

$$D\mathbf{h}\big(c\big) \in L(C|D)$$

extends to

$$D_{ext}\mathbf{h}\big(c\big) \in L(E|D)$$

where the mapping

$$\Omega_1 \times E \ni (c, e) \mapsto D_{ext}\mathbf{h}\big(c\big)\big(e\big) \in D$$

is assumed to be continuous.
Again, by the fact that $j^{**} : A \times \Delta \mapsto E$ is 2 times continuously differentiable with $D_2(D_1 j^{**})(a, s)(\hat{a})1 = D_2 j^{**}(\hat{a}, s)1$ for $(a, \hat{a}, s) \in A \times A \times \Delta$ one gets that

$$\Delta \ni s \mapsto D_2 j^{**}(\cdot, s)1 \quad \in L(A|E)$$

is continuous.
Furthermore, in j 2) we assumed that $D_2 j^*$ satisfies the inequality

$$\sup_{s \in \Delta} \|D_2 j^*(\cdot, s)1\|_{L(A|C)} < \infty.$$

Thus, the steps of the proof of the continuity of the mapping

$$\Omega_1 \times \Delta \ni (c, s) \mapsto D\mathbf{h}(c)\big(\mathbf{j}(\cdot, s)\big)$$

in Lemma 1.2.0.1 may be analogously applied to the mapping

$$\Omega_1 \times \Delta \ni (c, s) \mapsto D\mathbf{h}(c)\big(D_2 j^*(\cdot, s)1\big) \in L(A|D)$$

which therefore is continuous.
The continuity of

$$\Omega_2 \times \Delta \times \Delta \ni (a, s, s') \mapsto D\mathbf{h}\big(j^*(a, s')\big)\big(D_2 j^*(\cdot, s)1\big) \in L(A|D)$$

then follows from the continuity of

$$\Omega_2 \times \Delta \ni (a, s) \mapsto j^*(a, s) \in C.$$

□

Lemma 1.2.0.7. *Let* $\mathbf{h}$ *be a mapping such that* $\mathbf{h}$ *1),*$\mathbf{h}$ *2),* $\mathbf{h}$ *3),* $\mathbf{h}$ *4),* $\mathbf{h}$ *5),* $\mathbf{h}$ *6)are satisfied.*

Let j be a mapping such that j 1), j 2),j 3) are satisfied.
Then the mapping

$$\hat{H} : \Omega_2 \times \Delta \ni (a, s) \mapsto D\mathbf{h}\big(j^*(a, s)\big)\big(D_2 j^*(a, s)\big) \in L(\mathbb{R}^k|D)$$

has a partial derivative $D_1\hat{H}(a, s) \in L(A, \mathbb{R}^k|D)$ with respect to a in every $(a, s) \in \Omega_2 \times \Delta$. Furthermore, the mapping

$$\Omega_2 \times \Delta \ni (a, s) \mapsto D_1\hat{H}(a, s) \in L(A, \mathbb{R}^k|D)$$

is continuous.

Proof. We study the case where $k = 1$.
We prove the claim in two steps:
First step:
We show that

$$\hat{H}_1 : \Omega_2 \times \Omega_2 \times \Delta \ni (a, a', s) \mapsto D\mathbf{h}\big(j^*(a', s)\big)\big(D_2 j^*(a, s)1\big) \in D$$

has a partial derivative $D_1\hat{H}_1(a, a', s) \in L(A|D)$ with respect to a in every $(a, a', s) \in \Omega_2 \times \Omega_2 \times \Delta$ and that the mapping

$$\Omega_2 \times \Omega_2 \times \Delta \ni (a, a', s) \mapsto D_1\hat{H}_1(a, a', s) \in L(A|D)$$

is continuous.
For fixed $s \in \Delta$ the mapping $D_2 j^{**}(\cdot, s)1 : A \to E$ is linear and bounded. On the other hand $D\mathbf{h}\big(j^*(a', s)\big) \in L(C|D)$ extends to $D_{ext}\mathbf{h}\big(j^*(a', s)\big) \in L(E|D)$. Therefore, the identity

$$\hat{H}_1(a, a', s) = D_{ext}\mathbf{h}\big(j^*(a', s)\big)\big(D_2 j^{**}(a, s)1\big)$$

holds for all $(a, a', s) \in \Omega_2\times\Omega_2\times\Delta$. Hence, the partial derivative $D_1\hat{H}(a, a', s) \in L(A|D)$ of $\hat{H}_1$ with respect to a in $(a, a', s) \in \Omega_2 \times \Omega_2 \times \Delta$ is given by $D_{ext}\mathbf{h}\big(j^*(a', s)\big)\big(D_2 j^{**}(\cdot, s)1\big) \in L(A|D)$. The proof of the continuity of

$$D_1\hat{H} : \Omega_2 \times \Omega_2 \times \Delta \ni (a, a', s) \mapsto D_{ext}\mathbf{h}\big(j^*(a', s)\big)\big(D_2 j^{**}(\cdot, s)1\big) \in L(A|D)$$

is similar to the proof of the continuity of

$$D_2\tilde{H} : \Omega_2 \times \Delta \times \Delta \ni (a, s, s') \mapsto D_{ext}\mathbf{h}\big(j^*(a, s')\big)\big(D_2 j^{**}(\cdot, s)1\big) \in L(A|D)$$

in the previous lemma.
Second step:
We show that

$$\hat{H}_2 : \Omega_2 \times \Omega_2 \times \Delta \ni (a, a', s) \mapsto D\mathbf{h}\big(j^*(a,s)\big)\big(D_2 j^*(a',s)1\big) \in D$$

has a partial derivative $D_1\hat{H}_2(a, a', s) \in L(A|D)$ with respect to a in every $(a, a', s) \in \Omega_2 \times \Omega_2 \times \Delta$ and that the mapping

$$\Omega_2 \times \Omega_2 \times \Delta \ni (a, a', s) \mapsto D_1\hat{H}_2(a, a', s) \in L(A|D)$$

is continuous
In Lemma 1.2.0.5 we proved that the mapping

$$\mathbf{H}^{**} : \Omega_1^* \times C \ni (b, c) \mapsto D\mathbf{h}\big(J_{B,C}(b)\big)(c) \in D$$

has a partial derivative $D_1\mathbf{H}^{**}(b, c) \in L(B|D)$ with respect to b in every $(b, c) \in \Omega_1^* \times C$ which is given by $D_1\mathbf{H}^{**}(b, c) = D^2_{ext,1}\mathbf{h}^*(b)(c) \in L(B|D)$. On one hand $D_2 j^*(a', s)$ is an element of C for every $(a', s) \in \Omega_2 \times \Delta$. On the other hand, as for fixed $s \in \Delta$ the mapping $j(\cdot, s) : A \to B$ is linear and bounded we have that $D_1 j(a, s)$ exists with $D_1 j(a, s)(\hat{a}) = j(\hat{a}, s)$ for $(a, \hat{a}) \in A \times A$.
Therefore, the decomposition $\hat{H}_2(\cdot, a', s) = \mathbf{H}^{**}\big(\cdot, D_2 j^*(a', s)\big) \circ j(\cdot, s)$ for $(a', S) \in A \times \Delta$ yields that

$$\Omega_2 \times \Omega_2 \times \Delta \ni (a, a', s) \mapsto D\mathbf{h}\big(j^*(a,s)\big)\big(D_2 j^*(a',s)\big) \in D$$

has a partial derivative with respect to a in every $(a, a', s) \in \Omega_2 \times \Omega_2 \times \Delta$ which is given by

$$D^2_{ext,1}\mathbf{h}^*\big(j(a,s)\big)\big(D_2 j^*(a',s)\big)\big(\big(j(\cdot,s)\big) \in L(A|D).$$

The proof of the continuity of

$$\Omega_2 \times \Omega_2 \times \Delta \ni (a, a', s) \mapsto D^2_{ext,1}\mathbf{h}^*\big(j(a,s)\big)\big(D_2 j^*(a',s)\big)\big(\big(j(\cdot,s)\big) \in L(A|D)$$

is similar to the proof of the continuity of

$$D_2 H^* : \Omega_2 \times \Delta \times \Delta \ni (a, s, s') \mapsto D^2_{ext,1}\mathbf{h}^*\big(j(a,s)\big)\big(D_2 j^*(a,s)\big)\big(\big(j(\cdot,s')\big) \in L(A|D)$$

in Lemma 1.2.0.4.
Both of these steps combined yield that the mapping

$$\hat{H} : \Omega_2 \times \Delta \ni (a, s) \mapsto D\mathbf{h}\big(j^*(a, s)\big)\big(D_2 j^*(a, s)\big) \in D$$

has a partial derivative $D_1\hat{H}(a, s) \in L(A|D)$ with respect to a in every $(a, s) \in \Omega_2 \times \Delta$ and that the mapping

$$\Omega \times \Delta \ni (a, s) \mapsto D_1\hat{H}(a, s) \in L(A|D)$$

is continuous. □

Lemma 1.2.0.8. *Let* $\mathbf{h}$ *be a mapping such that* $\mathbf{h}$ *1),*$\mathbf{h}$ *2),* $\mathbf{h}$ *3),* $\mathbf{h}$ *4),* $\mathbf{h}$ *5),* $\mathbf{h}$ *6)are satisfied.*
Let j *be a mapping such that* j *1),* j *2),*j *3) are satisfied.*
Then the mapping

$$\hat{H} : \Omega_2 \times \Delta \ni (a, s) \mapsto D\mathbf{h}\big(j^*(a, s)\big)\big(D_2 j^*(a, s)1\big) \in L(\mathbb{R}^k|D)$$

has a partial derivative $D_2\hat{H}(a, s) \in L^2(\mathbb{R}^k|D)$ *with respect to* s *in every* $(a, s) \in \Omega_2 \times \Delta$*. Furthermore, the mapping*

$$\Omega_2 \times \Delta \ni (a, s) \mapsto D_2\hat{H}(a, s) \in L^2(\mathbb{R}^k|D)$$

is continuous.

Proof. We study the case where $k = 1$. The case $k > 1$ would follow the steps of the proof of $k = 1$ by examining the existence and continuity of all directional derivatives $D_i\hat{H}(\cdot, \cdot, \cdot)1 : \Omega_2 \times \Delta \times \Delta \to L(A|D)$, $i \in \{2, ..., k+1\}$.
Again we prove the claim in two steps:
First step:
We show that the mapping

$$\hat{H}^1 : \Omega_2 \times \Delta \times \Delta \ni (a, s, s') \mapsto D\mathbf{h}\big(j^*(a, s')\big)\big(D_2 j^*(a, s)\big)1 \in D$$

has a partial derivative $D_2\hat{H}^1(a, s, s') \in D$ with respect to s in every $(a, s, s') \in \Omega_2 \times \Delta \times \Delta$ and that the mapping

$$\Omega_2 \times \Delta \times \Delta \ni (a, s, s') \mapsto D_2\hat{H}^1(a, s, s')1 \in D$$

is continuous.
$D\mathbf{h}\big(j^*(a,s')\big) \in L(C|D)$ extending to $D_{ext}\mathbf{h}\big(j^*(a,s')\big) \in L(E|D)$ and j^{**} : $\Omega_2 \times \Delta \to E$ being 2 times continuously differentiable it is clear that the partial derivative $D_2\hat{H}^1(a,s,s')1$ of $\hat{H}$ with respect to s in $(a,s,s') \in \Omega_2 \times \Delta \times \Delta$ is given by $D_2\hat{H}^1(a,s,s')1 = D_{ext}\mathbf{h}\big(j^*(a,s')\big)\big(D_2^2 j^{**}(a,s)(1)(1)\big) \in D$:

$$\lim_{\epsilon\to 0} \tfrac{1}{\epsilon}\Big(D\mathbf{h}\big(j^*(a,s')\big)\big(D_2 j^*(a,s+\epsilon)1\big) - D\mathbf{h}\big(j^*(a,s')\big)\big(D_2 j^*(a,s)1\big)\Big) =$$

$$\lim_{\epsilon\to 0}\Big(D_{ext}\mathbf{h}\big(j^*(a,s')\big)\big(\tfrac{1}{\epsilon}[D_2 j^{**}(a,s+\epsilon)1 - D_2 j^{**}(a,s)1]\big)\Big) =$$

$$D_{ext}\mathbf{h}\big(j^*(a,s')\big)\big(D_2^2 j^{**}(a,s)(1)(1)\big)$$

The continuity of

$$\Omega_2 \times \Delta \times \Delta \ni (a,s,s') \mapsto D_{ext}\mathbf{h}\big(j^*(a,s')\big)\big(D_2^2 j^{**}(a,s)(1)(1)\big) \in D$$

is a consequence of the fact that all three mappings

$$\Omega_1 \times E \ni (c,e) \mapsto D_{ext}\mathbf{h}(c)(e) \in D,$$

$$j^* : \Omega_2 \times \Delta \to C$$

and

$$D_2^2 j^{**} : \Omega_2 \times \Delta \to E.$$

are assumed to be continuous.
Second step:
We show that the mapping

$$\hat{H}^2 : \Omega_2 \times \Delta \times \Delta \ni (a,s,s') \mapsto D\mathbf{h}\big(j^*(a,s)\big)\big(D_2 j^*(a,s')1\big) \in D$$

has a partial derivative $D_2\hat{H}^2(a,s,s')1 \in D$ with respect to s in every $(a,s,s') \in \Omega_2 \times \Delta \times \Delta$ and that the mapping

$$\Omega_2 \times \Delta \times \Delta \ni (a,s,s') \mapsto D_2\hat{H}^2(a,s,s')1 \in D$$

is continuous.
Lemma 1.2.0.5 tells us that the mapping

$$\mathbf{H}^{**} : \Omega_1^* \times C \ni (b,c) \mapsto D\mathbf{h}\big(J_{B,C}(b)\big)(c) \in D$$

has a partial derivative $D_1\mathbf{H}^{**}(b,c) \in L(B|D)$ with respect to b in $(b,c) \in \Omega_1^* \times C$ which is given by

$$D^2_{ext,1}\mathbf{h}^*(b)(c) \in L(B|D).$$

As $D_2 j^*(a,s')1 \in C$ for $a \in \Omega_2$ and $s' \in \Delta$ the identity

$$D\mathbf{h}\big(j^*(a,s+\epsilon)\big)\big(D_2 j^*(a,s')1\big) - D\mathbf{h}\big(j^*(a,s)\big)\big(D_2 j^*(a,s')1\big) =$$

$$\mathbf{H}^{**}\big(j(a,s+\epsilon), D_2 j^*(a,s')1\big) - \mathbf{H}^{**}\big(j(a,s), D_2 j^*(a,s')1\big) =$$

$$\int_0^1 D^2_{ext,1}\mathbf{h}^*\big(j(a,s) + q\cdot[j(a,s+\epsilon) - j(a,s)]\ \big)(D_2 j^*(a,s')1)\big(j(a,s+\epsilon) - j(a,s)\big)dq$$

holds for $a \in \Omega_2$, $(s,s') \in \Delta \times \Delta$ and $\epsilon \in \mathbb{R}$ sufficiently small.
On the other hand $j^* : \Omega_2 \times \Delta \to C$ is continuously differentiable. By replacing $D^2_{ext,1}\mathbf{h}^*$ with $D^2_{ext}\mathbf{h}^*$ and due to the continuity of

$$\Omega_1^* \times C \times C \ni (b,c,c') \mapsto D^2_{ext}\mathbf{h}^*(b)(c)(c') \in D$$

one gets that

$$\lim_{\epsilon\to 0} \tfrac{1}{\epsilon}\Big(D\mathbf{h}\big(j^*(a,s+\epsilon)\big)\big(D_2 j^*(a,s')1\big) - D\mathbf{h}\big(j^*(a,s)\big)\big(D_2 j^*(a,s')1\big)\Big) =$$

$$\lim_{\epsilon\to 0} \int_0^1 D^2_{ext}\mathbf{h}^*\big(j(a,s) + q\cdot[j(a,s+\epsilon) - j(a,s)\ ]\big)(D_2 j^*(a,s')1)\big(\tfrac{1}{\epsilon}[j^*(a,s+\epsilon) - j^*(a,s)]\big)dq =$$

$$D^2_{ext}\mathbf{h}^*\big(j(a,s)\big)\big(D_2 j^*(a,s')1\big)\big(D_2 j^*(a,s)1\big).$$

Thus, the partial derivative of

$$\Omega_2 \times \Delta \times \Delta \ni (a,s,s') \mapsto D\mathbf{h}\big(j^*(a,s)\big)\big(D_2 j^*(a,s')1\big) \in D$$

with respect to s exists in every $(a,s,s') \in \Omega_2 \times \Delta \times \Delta$.
The continuity of

$$\Omega_2 \times \Delta \times \Delta \ni (a,s,s') \mapsto D^2_{ext}\mathbf{h}^*\big(j(a,s)\big)\big(D_2 j^*(a,s')1\big)\big(D_2 j^*(a,s)1\big) \in D$$

is again a consequence of the continuity of

$$\Omega_1^* \times C \times C \ni (b,c,c') \mapsto D^2_{ext}\mathbf{h}^*(b)(c)(c') \in D$$

and the continuity of both

$$j : \Omega_2 \times \Delta \to B$$

and

$$D_2 j^* : \Omega_2 \times \Delta \to C.$$

Both steps combined yield that the mapping

$$\hat{H} : \Omega_2 \times \Delta \ni (a,s) \mapsto D\mathbf{h}\big(j^*(a,s)\big)\big(D_2 j^*(a,s)\big) \in D$$

has a partial derivative $D_2\hat{H}(a,s)1 \in D$ with respect to s in every $(a,s) \in \Omega_2 \times \Delta$ and that the mapping

$$\Omega_2 \times \Delta \ni (a,s) \mapsto D_2\hat{H}(a,s)1 \in D$$

is continuous. □

Theorem 1.2.1. 1. *Let* $\mathbf{h}$ *be a mapping such that* $\mathbf{h}$ *1),* $\mathbf{h}$ *2) and* $\mathbf{h}$ *3) are satisfied.*
Let $\mathbf{j}$ *be a mapping such that* j *4) and* j *5) are satisfied.*
Then the mapping $\mathbf{h} \circ \mathbf{j} : \Omega_2 \times \Delta \to D$ *is continuously differentiable.*

2. *Let* $\mathbf{h}$ *be a mapping such that* $\mathbf{h}$ *1),*$\mathbf{h}$ *2),* $\mathbf{h}$ *3),* $\mathbf{h}$ *4),* $\mathbf{h}$ *5),* $\mathbf{h}$ *6)are satisfied.*
Let j *be a mapping such that* j *1),* j *2),*j *3) are satisfied.*
Then $\mathbf{h}^* \circ j : \Omega_2 \times \Delta \to D$ *is 2 times continuously differentiable.*

Proof. 1. We prove the claim in two steps.
First step:
We show that there exists a partial derivative $D_1(\mathbf{h} \circ \mathbf{j})(a,s) \in L(A|D)$ of $\mathbf{h} \circ \mathbf{j}$ with respect to a in every $(a,s) \in \Omega_2 \times \Delta$ and that the mapping $\Omega_2 \times \Delta \ni (a,s) \mapsto D_1(\mathbf{h} \circ \mathbf{j})(a,s) \in L(A|D)$ is continuous.
$\mathbf{h} : \Omega_1 \to D$ is continuously differentiable and for fixed $s \in \Delta$ the mapping $\mathbf{j}(\cdot, s) : A \to C$ is linear and bounded. Therefore, the derivative of $\mathbf{h} \circ \mathbf{j}$ with respect to a is given by $D\mathbf{h}\big(\mathbf{j}(a,s)\big)\big(\mathbf{j}(v,s)\big)$ for every $(a,s) \in \Omega_2 \times \Delta$ and $v \in A$. The continuity of

$$\Omega_2 \times \Delta \ni (a,s) \to D\mathbf{h}\big(\mathbf{j}(a,s)\big)\big(\mathbf{j}(\cdot,s)\big) \in L(A|D)$$

is shown in Lemma 1.2.0.1.
Second step:
We have to show that there exists a partial derivative $D_2(\mathbf{h} \circ \mathbf{j})(a,s) \in L(\mathbb{R}^k|D)$ of $\mathbf{h} \circ \mathbf{j}$ with respect to s in every $(a,s) \in \Omega_2 \times \Delta$ and that the mapping $\Omega_2 \times \Delta \ni (a,s) \mapsto D_2(\mathbf{h} \circ \mathbf{j})(a,s) \in L(\mathbb{R}^k|D)$ is continuous.
But this is a consequence of Lemma 1.2.0.3.
Both steps combined yield that the mapping $\mathbf{h} \circ \mathbf{j} : \Omega_2 \times \Delta \to D$ is continuously differentiable.
2.
As the identity $(\mathbf{h}^* \circ j)(a,s) = (\mathbf{h} \circ j^*)(a,s)$ holds for all $(a,s) \in \Omega_2 \times \Delta$ the first part of this theorem yields that the mapping $\mathbf{h}^* \circ j : \Omega_2 \times \Delta \to D$ is continuously differentiable with its derivative being given by

$$\begin{aligned} D(\mathbf{h}^* \circ j)(a,s)(v,s') &= D\mathbf{h}\big(j^*(a,s)\big)\big(j^*(v,s)\big) \\ &+ D\mathbf{h}\big(j^*(a,s)\big)\big(D_2 j^*(a,s)(s')\big) \end{aligned} \tag{1.13}$$

for $(a,s) \in \Omega_2 \times \Delta$ and $(v,s') \in A \times \mathbb{R}^k$. Observing the identity

$$D\mathbf{h}\big(j^*(a,s)\big)\big(j^*(\cdot,s)\big) = D\mathbf{h}^*\big(j(a,s)\big)\big(j(\cdot,s)\big), \tag{1.14}$$

for all $(a,s) \in \Omega_2 \times \Delta$ we may rewrite (1.13) as

$$\begin{aligned} D(\mathbf{h}^* \circ j)(a,s)(v,s') &= D\mathbf{h}^*\big(j(a,s)\big)\big(j(v,s)\big) \\ &+ D\mathbf{h}\big(j^*(a,s)\big)\big(D_2 j^*(a,s)(s')\big) \end{aligned} \tag{1.15}$$

for $(a,s) \in \Omega_2 \times \Delta$ and $(v,s') \in A \times \mathbb{R}^k$. First we look at the first term of (1.15), which is

$$D\mathbf{h}^*\big(j(a,s)\big)\big(j(v,s)\big)$$

for $(a,s) \in \Omega_2 \times \Delta$ and $v \in A$, and show that the mapping

$$\mathbf{H} : \Omega_2 \times \Delta \ni (a,s) \mapsto D\mathbf{h}^*\big(j(a,s)\big)\big(j(\cdot,s)\big) \in L(A|D)$$

is continuously differentiable.
$\mathbf{h}^* : \Omega_1^* \to D$ is 2 times continuously differentiable and for fixed $s \in \Delta$ the

mapping $j(\cdot, s) : A \to B$ is linear and bounded. Therefore, the decomposition $\mathbf{H} = \mathbb{F} \circ \mathbb{H}$ with

$$\mathbb{H} : A \times \Delta \ni (a,s) \mapsto \big(j(a,s), j(\cdot,s)\big) \in B \times L(A|B)$$

and

$$\mathbb{F} : B \times L(A|B) \ni (b,T) \mapsto D\mathbf{h}^*(b) \circ T \in L(A|D)$$

and an application of the chainrule yields that the mapping

$$\mathbf{H} : \Omega_2 \times \Delta \ni (a,s) \mapsto D\mathbf{h}^*\big(j(a,s)\big)\big(j(\cdot,s)\big) \in L(A|D)$$

has a partial derivative $D_1\mathbf{H}(a,s) = D^2\mathbf{h}^*\big(j(a,s)\big)\big(j(\cdot,s)\big)\big(j(\cdot,s)\big) \in L^2(A|D)$ with respect to a in every $(a,s) \in \Omega_2 \times \Delta$. The continuity of

$$\Omega_2 \times \Delta \ni (a,s) \mapsto D^2\mathbf{h}^*\big(j(a,s)\big)\big(j(\cdot,s)\big)\big(j(\cdot,s)\big) \in L^2(A|D)$$

is shown in Lemma 1.2.0.2.
Furthermore, we have to show that the mapping

$$\mathbf{H} : \Omega_2 \times \Delta \ni (a,s) \mapsto D\mathbf{h}^*\big(j(a,s)\big)\big(j(\cdot,s)\big) \in L(A|D)$$

has a partial derivative $D_2\mathbf{H}(a,s) \in L(\mathbb{R}^k, A|D)$ with respect to s in every $(a,s) \in \Omega_2 \times \Delta$ and that the mapping $\Omega_2 \times \Delta \ni (a,s) \mapsto D_2\mathbf{H}(a,s) \in L(\mathbb{R}^k, A|D$ is continuous:
We prove this in two steps:
First step:
We observe the identity (1.14), and apply Lemma 1.2.0.6 which shows that the mapping

$$\tilde{H} : \Omega_2 \times \Delta \times \Delta \ni (a,s,s') \mapsto D\mathbf{h}\big(j^*(a,s')\big)\big(j^*(\cdot,s)\big) \in L(A|D)$$

has a partial derivative $D_2\tilde{H}(a,s,s') \in L(\mathbb{R}^k, A|D)$ with respect to s in every $(a,s,s') \in \Omega_2 \times \Delta \times \Delta$ and that the mapping $\Omega_2 \times \Delta \times \Delta \ni (a,s,s') \mapsto D_2\tilde{H}(a,s,s') \in L(\mathbb{R}^k, A|D)$ is continuous.
Thus,

$$\mathbf{H}_1 : \Omega_2 \times \Delta \times \Delta \ni (a,s,s') \mapsto D\mathbf{h}^*\big(j(a,s')\big)\big(j(\cdot,s)\big) \in L(A|D)$$

has a partial derivative $D_2\mathbf{H}_1(a,s,s') \in L(\mathbb{R}^k, A|D)$ with respect to s in every $(a,s,s') \in \Omega_2 \times \Delta \times \Delta$ and the mapping $\Omega_2 \times \Delta \times \Delta \ni (a,s,s') \mapsto D_2\mathbf{H}(a,s,s') \in L(\mathbb{R}^k, A|D)$ is continuous.

Second step:

Lemma 1.2.0.4 shows that the mapping

$$H^* : \Omega_2 \times \Delta \times \Delta \ni (a,s,s') \mapsto D\mathbf{h}^*\big(j(a,s)\big)\big(j(\cdot,s')\big) \in L(A|D)$$

has a partial derivative $D_2H^*(a,s,s') \in L(\mathbb{R}^k, A|D)$ with respect to s in every $(a,s,s') \in \Omega_2 \times \Delta \times \Delta$ and that the mapping $\Omega_2 \times \Delta \times \Delta \ni (a,s,s') \mapsto D_2H^*(a,s,s') \in L(\mathbb{R}^k, A|D)$ is continuous.

Both steps combined yield that the mapping

$$\mathbf{H} : \Omega_2 \times \Delta \ni (a,s) \mapsto D\mathbf{h}^*\big(j(a,s)\big)\big(j(\cdot,s)\big) \in L(A|D)$$

has a partial derivative $D_2\mathbf{H}(a,s) \in L(\mathbb{R}^k, A|D)$ with respect to s in every $(a,s) \in \Omega_2 \times \Delta$ and that the mapping $\Omega_2 \times \Delta \ni (a,s) \mapsto D_2\mathbf{H}(a,s) \in L(\mathbb{R}^k, A|D)$ is continuous.

Therefore, the mapping

$$\mathbf{H} : \Omega_2 \times \Delta \ni (a,s) \mapsto D\mathbf{h}^*\big(j(a,s)\big)\big(j(\cdot,s)\big) \in L(A|D)$$

is continuously differentiable.

Now we look at the second term of (1.15), which is

$$D\mathbf{h}\big(j^*(a,s)\big)\big(D_2j^*(a,s)(s')\big)$$

for $(a,s) \in \Omega_2 \times \Delta$ and $s' \in \mathbb{R}$, and show that the mapping

$$\hat{H} : \Omega_2 \times \Delta \ni (a,s) \mapsto D\mathbf{h}\big(j^*(a,s)\big)\big(D_2j^*(a,s)\big) \in L(\mathbb{R}^k|D)$$

is continuously differentiable. But this is a consequence of Lemma 1.2.0.7 and Lemma 1.2.0.8.

Both results on both terms of (1.15) yield that the mapping

$$\Omega_2 \times \Delta \ni (a,s) \mapsto D(\mathbf{h}^* \circ j)(a,s) \in L(\mathbb{R}^k, A|D)$$

is continuously differentiable.

Hence, $\mathbf{h}^* \circ j : \Omega_2 \times \Delta \to D$ is 2 times continuously differentiable. $\square$

1.3 Differentiability properties of the mapping (1.9)

In this section we will state a theorem (Theorem 1.3.2.1) on the differentiability properties of the mapping (1.9), we defined in section 1.1.5:

$$J \times \hat{\Omega} \times (-1/2, 1/2) \ni (\alpha, u, \beta) \mapsto \mathbf{G}(\alpha, u_{\cdot,\beta}, u_{\cdot}, \beta) \in C_T^0$$

Recall the definition of $\mathbf{G}$ which was

$$\mathbf{G}(\alpha, \phi, \chi, \beta) = \mathbf{g}(\alpha, \phi, \beta) - L(\alpha_0)\chi$$

for $(\alpha, \phi, \chi, \beta) \in J \times \Omega \times \Omega \times \mathbb{R}$. The mapping

$$\mathbf{g} : J \times \Omega \times \mathbb{R} \to \mathbb{R}^n$$

was defined by $\mathbf{g}(\alpha, \phi, \beta) := (1+\beta) g(\alpha, \phi + \phi^*)$ for $(\alpha, \phi, \beta) \in J \times \Omega \times \mathbb{R}$. As g is satisfying H 1) to H6) $\mathbf{g}$ satisfies the following assumptions:

- $\tilde{H}$ 1): The mapping $\mathbf{g} : J \times \Omega \times \mathbb{R} \to \mathbb{R}^n$ is continuously differentiable and the identity $\mathbf{g}(\alpha, 0, \beta) = 0$ holds for all $\beta \in (-1/2, 1/2)$ and $\alpha \in J$.
- $\tilde{H}$ 2): The partial derivative $D_2\mathbf{g}(\alpha, \phi, \beta) \in L(C_h^1|\mathbb{R}^n)$ of g with respect to ϕ in $(\alpha, \phi, \beta) \in J \times \Omega \times \mathbb{R}$ extends to a linear bounded mapping $D_{2,ext}\mathbf{g}(\alpha, \phi, \beta) : C_h \to \mathbb{R}^n$.
- $\tilde{H}$ 3): The mapping

  $$J \times \Omega \times \mathbb{R} \times C_h \ni (\alpha, \phi, \beta, \chi) \mapsto D_{2,ext}\mathbf{g}(\alpha, \phi, \beta)(\chi) \in \mathbb{R}^n$$

  is continuous.
- $\tilde{H}$ 4): The mapping $\mathbf{g}^* := \mathbf{g}|_{J \times \Omega^* \times \mathbb{R}}$ is 2 times continuously differentiable.
- $\tilde{H}$ 5): The second partial derivative $D_2^2\mathbf{g}^*(\alpha, \phi, \beta) \in L^2(C_h^2|\mathbb{R}^n)$ of g with respect to ϕ in $(\alpha, \phi, \beta) \in J \times \Omega^* \times \mathbb{R}$ extends to a continuous bilinear mapping $D_{2,ext}^2\mathbf{g}^*(\alpha, \phi, \beta) : C_h^1 \times C_h^1 \to \mathbb{R}^n$.
- $\tilde{H}$ 6): Let $J_{C_h^2, C_h^1}$ denote the continuous embedding from C_h^2 to C_h^1. The mappings

  $$J \times \Omega^* \times \mathbb{R} \times C_h^1 \times C_h^1$$

  $$\ni (\alpha, \phi, \beta, \chi_1, \chi_2) \mapsto D_{2,ext}^2\mathbf{g}^*(\alpha, \phi, \beta)(\chi_1)(\chi_2) \in \mathbb{R}^n$$

and

$$D^2_{2,ext,1}\mathbf{g}^* : J \times \Omega^* \times \mathbb{R} \times C^1_h \to L(C^2_h|\mathbb{R}^n),$$

defined by

$$D^2_{2,ext}\mathbf{g}^*(\alpha,\phi,\beta)(\chi)\big(J_{C^2_h,C^1_h}(\psi)\big),$$

for $(\alpha,\phi,\beta,\chi) \in J \times \Omega^* \times \mathbb{R} \times C^1_h$ and $\psi \in C^2_h$, are continuous.

Note that in $\tilde{H}$ 3) does not include the continuity of

$$J \times \Omega \times \mathbb{R} \ni (\alpha,\phi,\beta) \mapsto D_{2,ext}\mathbf{g}(\alpha,\phi,\beta) \in L(C_h|\mathbb{R}^n).$$

$\tilde{H}$ 6) does not include the continuity of

$$J \times \Omega^* \times \mathbb{R} \ni (\alpha,\phi,\beta) \mapsto D^2_{2,ext}\mathbf{g}^*(\alpha,\phi,\beta) \in L^2(C^1_h|\mathbb{R}^n).$$

We divide this section into two subsections:
We start with a subsection of preparations for the proof of Theorem 1.3.2.1. Then we state the theorem itself.

1.3.1 Preparations

Lemma 1.3.1.0.1. *Let $I \subset \mathbb{R}$ be an interval.*

1. *The mapping*

$$H^1 : C^1_T \times I \ni (u,s) \mapsto u(s) \in \mathbb{R}^n$$

is continuously differentiable.

2. *The mapping*

$$H^2 : C^2_T \times I \ni (u,s) \mapsto u(s) \in \mathbb{R}^n$$

is 2 times continuously differentiable.

Proof. We only show 2. The proof of 1. is similar and simple.
As, for fixed $s \in I$, the mapping $C^2_T \ni u \mapsto H^2(u,s) = u(s) \in \mathbb{R}^n$ is linear and bounded the partial derivative of H^2 in $(u,s) \in C^2_T \times I$ with respect to u is given by $D_1H^2(u,s)(v) = v(s)$ for $v \in C^2_T$. The continuity of $C^2_T \times I \ni (u,s) \mapsto D_1H^2(u,s) \in L(C^2_T|\mathbb{R}^n)$ is obtained in the following way:

We take a sequence $(u_n, s_n)_{n\in\mathbb{N}} \in C_T^2 \times I$ such that $(u_n, s_n) \to (u, s)$ as $n \to \infty$. Then the inequality

$$\sup_{\substack{v \in C_T^2 \\ \|v\|_{C_T^2} = 1}} \|v(s) - v(s_n)\|_{\mathbb{R}^n} \leq$$

$$\sup_{\substack{v \in C_T^2 \\ \|v\|_{C_T^2} = 1}} \Big(\sup_{t\in[0,T]} \|v'(t)\|_{\mathbb{R}^n} \cdot |s - s_n| \Big) \leq$$

$$\sup_{\substack{v \in C_T^2 \\ \|v\|_{C_T^2} = 1}} \|v\|_{C_T^2} \cdot |s - s_n| = |s - s_n|$$

holds for all $n \in \mathbb{N}$. The last expression $|s - s_n|$ tends to 0, as $n \to \infty$.
Thus, $C_T^2 \times I \ni (u, s) \mapsto D_1H^2(u, s) \in L(C_T^2|\mathbb{R}^n)$ is continuous.
The partial derivative of H^2 with respect to s in $(u, s) \in C_T^2 \times I$ is given by $DH^2(u, s)1 = u'(s)$. The continuity of $C_T^2 \times I \ni (u, s) \mapsto D_2H^2(u, s)1 \in \mathbb{R}^n$ is obtained in the following way:
Being given any sequence $(u_n, s_n)_{n\in\mathbb{N}} \in C_T^2 \times I$ such that $(u_n, s_n) \to (u, s)$, as $n \to \infty$, the inequality

$$\|u'(s) - u_n'(s_n)\|_{\mathbb{R}^n} \leq \|u'(s) - u'(s_n)\|_{\mathbb{R}^n} + \|u'(s_n) - u_n'(s_n)\|_{\mathbb{R}^n}$$

holds for $n \in \mathbb{N}$. The first term of the right hand side of this inequality tends to 0, as $n \to \infty$, by the fact that $s_n \to s$, as $n \to \infty$ and due to the continuity of $u' : \mathbb{R} \to \mathbb{R}^n$. We may estimate the second term from above by

$$\sup_{t\in[0,T]} \|u'(t) - u_n'(t)\|_{\mathbb{R}^n} \leq \|u - u_n\|_{C_T^2}$$

for $n \in \mathbb{N}$. The right hand side of this inequality tends to 0, as $n \to \infty$, by assumption. Therefore, $C_T^2 \times I \ni (u, s) \mapsto D_2H^2(u, s)1 \in \mathbb{R}^n$ is continuous.
Note that for the proof of existence and continuity of D_1H^2 and D_2H^2 we did not need u to be in C_T^2. It would have been sufficient to have $u \in C_T^1$. Thus, we even proved that $H^1 : C_T^1 \times I \ni (u, s) \mapsto u(s) \in \mathbb{R}^n$ is continuously differentiable.

The partial derivative of $D_1H^2(u,s)$ with respect to u in (u,s) is obviously zero. The partial derivative of $D_1H^2(u,s)$ with respect to s in (u,s) is given by $D_2D_1H^2(u,s)(v)1 = v'(s)$ for $v \in C_T^2$. For the continuity of

$$C_T^2 \times I \ni (u,s) \mapsto D_2D_1H^2(u,s)1 \in L(C_T^2|\mathbb{R}^n)$$

we take a a sequence $(u_n,s_n)_{n\in\mathbb{N}} \in C_T^2 \times I$ such that $(u_n,s_n) \to (u,s)$, as $n \to \infty$. Then we get by estimation that

$$\sup_{\substack{v \in C_T^2 \\ \|v\|_{C_T^2} = 1}} \|v'(s) - v'(s_n)\|_{\mathbb{R}^n} \leq$$

$$\sup_{\substack{v \in C_T^2 \\ \|v\|_{C_T^2} = 1}} \Big(\sup_{t\in[0,T]} \|v''(t)\|_{\mathbb{R}^n} \cdot |s - s_n|\Big) \leq$$

$$\sup_{\substack{v \in C_T^2 \\ \|v\|_{C_T^2} = 1}} \|v\|_{C_T^2} \cdot |s - s_n| = |s - s_n|$$

holds for all $n \in \mathbb{N}$. The last expression $|s - s_n|$ tends to 0, as $n \to \infty$. Thus, $C_T^2 \times I \ni (u,s) \mapsto D_2D_1H^2(u,s)1 \in L(C_T^2|\mathbb{R}^n)$ is continuous.
Considering that for fixed $s \in I$ the mapping $C_T^2 \ni u \mapsto D_2H^2(u,s) = u'(s) \in \mathbb{R}^n$ is linear and bounded the derivative of D_2H^2 with respect to u in $(u,s) \in C_T^2 \times I$ is given by $D_1D_2H^2(u,s)(v)1 = v'(s) = D_2D_1H^2(u,s)(v)1$, for $v \in C_T^2$. The continuity of $C_T^2 \times I \ni (u,s) \mapsto D_2D_1H^2(u,s)1 \in L(C_T^2|\mathbb{R}^n)$ was already shown. The derivative of D_2H^2 in $(u,s) \in C_T^2 \times I$ with respect to s is given by $D_2^2H^2(u,s)(1)(1) = u''(s)$. The continuity of $C_T^2 \times I \ni (u,s) \mapsto D_2^2H^2(u,s)(1)(1) \in \mathbb{R}^n$ is obtained in the following way:
Being given any sequence $(u_n,s_n)_{n\in\mathbb{N}} \in C_T^2 \times I$ such that $(u_n,s_n) \to (u,s)$, as $n \to \infty$ one gets that the inequality

$$\|u''(s) - u_n''(s_n)\|_{\mathbb{R}^n} \leq \|u''(s) - u''(s_n)\|_{\mathbb{R}^n} + \|u''(s_n) - u_n''(s_n)\|_{\mathbb{R}^n}$$

holds for $n \in \mathbb{N}$. The first term of the right hand side of this inequality tends to 0, as $n \to \infty$, by the fact that $s_n \to s$, as $n \to \infty$, and due to the continuity of $u'' : \mathbb{R} \to \mathbb{R}^n$. We estimate the second term from above by

$$\sup_{t\in[0,T]} \|u''(t) - u_n''(t)\|_{\mathbb{R}^n} \leq \|u - u_n\|_{C_T^2}$$

for $n \in \mathbb{N}$. The right hand side of this estimation tends to 0, as $n \to \infty$, by assumption. Therefore, $C_T^2 \times I \ni (u,s) \mapsto D_2^2 H^2(u,s)(1)(1) \in \mathbb{R}^n$ is continuous.

Hence, the mapping

$$C_T^1 \times I \ni (u,s) \mapsto H^1(u,s) \in \mathbb{R}^n$$

is continuously differentiable and the mapping

$$C_T^2 \times I \ni (u,s) \mapsto H^2(u,s) \in \mathbb{R}^n$$

is 2 times continuously differentiable. □

Lemma 1.3.1.0.2. *Let $S^* \subset \mathbb{R}$ be an open bounded interval such that $[0,T] \subset S^*$. Then the following properties hold:*

1. *Let $\beta \in (-1/2, 1/2)$ and $\tau \in S^*$ be real numbers.*
   If $u \in C_T^1$, then $u_{\tau,\beta}$, defined by $u_{\tau,\beta}(\theta) := u\big(\tau + \theta/(1+\beta)\big)$, for $-h \leq \theta \leq 0$ is an element of C_h^1.
   The mapping $\Xi_1^1 : C_T^1 \times (-1/2,1/2) \times S^* \ni (u,\beta,\tau) \mapsto u_{\tau,\beta} \in C_h^1$ is continuous. For every $(\beta,\tau) \in (-1/2,1/2) \times S^*$ the mapping

   $$C_T^1 \ni v \mapsto \Xi_1^1(v,\beta,\tau) \in C_h^1$$

   is linear and bounded. Furthermore, the inequality

   $$\sup_{(\beta,\tau)\in(-1/2,1/2)\times S^*} \left\{ \|\Xi_1^1(\cdot,\beta,\tau)\|_{L(C_T^1|C_h^1)} \right\} < \infty$$

   holds.

2. *Let $\beta \in (-1/2, 1/2)$ and $\tau \in S^*$ be real numbers.*
   If $u \in C_T^2$, then $u_{\tau,\beta}$, defined by $u_{\tau,\beta}(\theta) := u\big(\tau + \theta/(1+\beta)\big)$, for $-h \leq \theta \leq 0$ is an element of C_h^2. The mapping $\Xi_2^2 : C_T^2 \times (-1/2,1/2) \times S^* \ni (u,\beta,\tau) \mapsto u_{\tau,\beta} \in C_h^2$ is continuous. For every $(\beta,\tau) \in (-1/2,1/2) \times S^*$ the mapping

   $$C_T^2 \ni v \mapsto \Xi_2^2(v,\beta,\tau) \in C_h^2$$

is linear and bounded. Furthermore, the inequality

$$\sup_{(\beta,\tau)\in(-1/2,1/2)\times S^*} \left\{ \|\Xi_2^2(\cdot,\beta,\tau)\|_{L(C_T^2|C_h^2)} \right\} < \infty$$

holds.

3. *The mapping* $\Xi_0^1 : C_T^1 \times (-1/2, 1/2) \times S^* \ni (u, \beta, \tau) \mapsto u_{\tau,\beta} \in C_h$ *is continuously differentiable.*

4. *The mapping* $\Xi_1^2 : C_T^2 \times (-1/2, 1/2) \times S^* \ni (u, \beta, \tau) \mapsto u_{\tau,\beta} \in C_h^1$ *is continuously differentiable.*
   For $(u, \beta, \tau) \in C_T^2 \times (-1/2, 1/2) \times S^*$ *let* $D_{(\beta,\tau)}\Xi_1^2(u, \beta, \tau) \in L(\mathbb{R}^2|C_h^1)$ *denote the derivative of* Ξ_1^2 *with respect to* (β, τ) *in* $(u, \beta, \tau) \in C_T^2 \times (-1/2, 1/2) \times S^*$.
   Then for every $(\beta, \tau) \in (-1/2, 1/2) \times S^*$ *the mapping*

   $$C_T^2 \ni v \mapsto D_{(\beta,\tau)}\Xi_1^2(v, \beta, \tau) \in L(\mathbb{R}^2|C_h^1)$$

   is linear and bounded. Furthermore, the inequality

   $$\sup_{(\beta,\tau)\in(-1/2,1/2)\times S^*} \left\{ \|D_{(\beta,\tau)}\Xi_1^2(\cdot,\beta,\tau)\|_{L(\mathbb{R}^2, C_T^2|C_h^1)} \right\} < \infty$$

   holds.

5. *The mapping* $\Xi_0^2 : C_T^2 \times (-1/2, 1/2) \times S^* \ni (u, \beta, \tau) \mapsto u_{\tau,\beta} \in C_h$ *is 2 times continuously differentiable.*

Proof. We only show 2., 5. and 4. The proofs of 1. and 3. is similar to 2. and 5. respectively and simple.
2. In the previous lemma we have seen that for any interval $I \subset \mathbb{R}$ the mapping $H^2 : C_T^2 \times I \ni (u, s) \mapsto u(s) \in \mathbb{R}^n$ is 2 times continuously differentiable. Thus, by applying the chain rule the mapping

$$C_T^2 \times (-1/2, 1/2) \times S^* \times [-h, 0] \ni (u, \beta, \tau, \theta) \mapsto u_{\tau,\beta}(\theta) = H^2\big(u, \tau + \theta/(1+\beta)\big) \in \mathbb{R}^n$$

is 2 times continuously differentiable.
Therefore, an application of part six of Theorem 3.1.1 in Appendix I yields the continuity of the mapping

$$\Xi_2^2 : C_T^2 \times (-1/2, 1/2) \times S^* \ni (u, \beta, \tau) \mapsto u_{\tau,\beta} \in C_h^2.$$

For $v \in C_T^2$ let $v^{(0)} := v$, $v^{(1)} := v'$ and $v^{(2)} := v''$. Then it is obvious that for every $s \in \mathbb{R}$ all mappings

$$C_T^2 \ni v \mapsto v^{(i)}(s) \in \mathbb{R}^n$$

$i \in \{0, 1, 2\}$ are linear and bounded. It is easy to see that for any $(\beta, \tau) \in (-1/2, 1/2) \times S^*$ the mapping

$$C_T^2 \ni v \mapsto v_{\tau,\beta} \in C_h^2$$

is linear and bounded. If we define $b : (-1/2, 1/2) \times S^* \times [-h, 0] \to \mathbb{R}$ by

$$b(\beta, \tau, \theta) := \tau + \theta/(1 + \beta)$$

for $\theta \in [-h, 0]$ and $(\beta, \tau) \in (-1/2, 1/2) \times S^*$ then the mapping b is continuously differentiable. $D_3 b(\beta, \tau, \theta) = 1/(1 + \beta) < 2$ for all $(\beta, \tau) \in (-1/2, 1, 2) \times S^*$. Thus, the inequality

$$\sup_{(\beta,\tau,\theta)\in(-1/2,1/2)\times S^*\times[-h,0]} \sup_{\substack{v \in C_T^2 \\ \|v\|_{C_T^2} = 1}} \left\{ \max_{i\in\{0,1,2\}} \|v^{(i)}\big(b(\beta,\tau,\theta)\big)\|_{\mathbb{R}^n} \big(D_3 b(\beta,\tau,\theta)\big)^i \right\} < \infty$$

holds.
Hence, we get that even the inequality

$$\sup_{(\beta,\tau)\in(-1/2,1/2)\times S^*} \left\{ \|\Xi_2^2(\cdot, \beta, \tau)\|_{L(C_T^2|C_h^2)} \right\} < \infty$$

holds.
5. In the proof of 2. we have seen that the mapping

$$C_T^2 \times (-1/2, 1/2) \times S^* \times [-h, 0] \ni (u, \beta, \tau, \theta) \mapsto u_{\tau,\beta}(\theta) = H^2\big(u, \tau + \theta/(1+\beta)\big) \in \mathbb{R}^n$$

is 2 times continuously differentiable.
By applying the fourth part of Theorem 3.1.1 we get that the mapping

$$\Xi_0^2 : C_T^2 \times (-1/2, 1/2) \times S^* \ni (u, \beta, \tau) \mapsto u_{\tau,\beta} \in C_h$$

is 2 times continuously differentiable.
4. An application of the fifth part of Theorem 3.1.1 in Appendix I shows

that the mapping $\Xi_1^2 : C_T^2 \times (-1/2, 1/2) \times S^* \ni (u, \beta, \tau) \mapsto u_{\tau,\beta} \in C_h^1$ is continuously differentiable.
As the identity

$$\Xi_1^2(u, \beta, \tau)(\theta) = u\big(\tau + \theta/(1+\beta)\big) = b(\beta, \tau, \theta)$$

holds for $(u, \beta, \tau, \theta) \in C_T^2 \times (-1/2, 1/2) \times S^* \times [-h, 0]$ and by the fact that u and b are continuously differentiable an application of part 2 of Theorem 3.1.1 in Appendix I yields that the derivative $D_{(\beta,\tau)}\Xi_1^2(u, \beta, \tau) \in L(\mathbb{R}^2 | C_h^1)$ of Ξ_1^2 with respect to (β, τ) in $(u, \beta, \tau) \in C_T^2 \times (-1/2, 1/2) \times S^*$ is given by

$$\big(D_{(\beta,\tau)}\Xi_1^2(u, \beta, \tau)(\tilde{\beta}, \tilde{\tau})\big)(\theta) = u'_{\tau,\beta}(\theta) D_{(\beta,\tau)} b(\beta, \tau, \theta)(\tilde{\beta}, \tilde{\tau})$$

for $\theta \in [-h, 0]$ and $(\tilde{\beta}, \tilde{\tau}) \in \mathbb{R}^2$.
Like in the proof of 2. we can show that for any $(\beta, \tau) \in (-1/2, 1/2) \times S^*$ the mapping

$$C_T^2 \ni v \mapsto v'_{\tau,\beta} D_{(\beta,\tau)} b(\beta, \tau, \cdot) \in L(\mathbb{R}^2 | C_h^1)$$

is linear and bounded and that the inequality

$$\sup_{(\beta,\tau)\in(-1/2,1/2)\times S^*} \Big\{ \| D_{(\beta,\tau)}\Xi_1^2(\cdot, \beta, \tau) \|_{L(,\mathbb{R}^2, C_T^2 | C_h^1)} \Big\} < \infty$$

holds. □

Lemma 1.3.1.0.3. *Let* $\mathbf{g} : J \times \Omega \times \mathbb{R} \to \mathbb{R}^n$ *be a mapping such that all assumptions* $\tilde{H}$ *1) to* $\tilde{H}$ *6) are satisfied. And let the mappings* Ξ_1^1, Ξ_2^2, Ξ_0^1, Ξ_1^2, Ξ_0^2 *be defined like in the previous lemma.*
Let $\hat{\Omega} \subset C_T^1$ *be an open subset such that* $u_{\tau,\beta} \in \Omega$ *for* $u \in \hat{\Omega}$ *and* $(\beta, \tau) \in (-1/2, 1/2) \times S^*$.
Let $\tilde{\Omega} := \hat{\Omega} \cap C_T^2$.
Then the following properties hold:

1. *The mapping*

   $$\xi_1 : J \times \hat{\Omega} \times (-1/2, 1/2) \times S^* \ni (\alpha, u, \beta, \tau) \mapsto \mathbf{g}\big(\alpha, \Xi_1^1(u, \beta, \tau), \beta\big) \in \mathbb{R}^n$$

   is continuously differentiable.

2. The mapping

$$\xi_2 : J \times \tilde{\Omega} \times (-1/2, 1/2) \times S^* \ni (\alpha, u, \beta, \tau) \mapsto \mathbf{g}^*\big(\alpha, \Xi_2^2(u, \beta, \tau), \beta\big) \in \mathbb{R}^n$$

is 2 times continuously differentiable.

Proof. We want to apply Theorem 1.2.1:
$\mathbf{g}$ satisfying $\tilde{H}$ 1) - $\tilde{H}$ 6) we set $\mathbf{h} := \mathbf{g}$, $\mathbf{h}^* := \mathbf{g}^*$, $C := \mathbb{R} \times C_h^1 \times \mathbb{R}$, $B := \mathbb{R} \times C_h^2 \times \mathbb{R}$, $E := \mathbb{R} \times C_h \times \mathbb{R}$, $\Omega_1 := J \times \Omega \times \mathbb{R}$, $\Omega_1^* := J \times \Omega^* \times \mathbb{R}$ and $D := \mathbb{R}^n$.
Thus, $\mathbf{h}$ satisfies all assumptions $\mathbf{h}$ 1) - $\mathbf{h}$ 6).
For the proof of 1. we set $A := \mathbb{R} \times \mathbb{R} \times C_T^1$, $\Omega_2 := J \times (-1/2, 1/2) \times \hat{\Omega}$, $k := 3$. Let $\Delta = (-1/2, 1/2) \times S^*$ which is an open bounded subset of $\mathbb{R}^3$. We define the mapping

$$\mathbf{j} : \Omega_2 \times \Delta \to \Omega_1 \subset C$$

by

$$\mathbf{j}(a, s) := (\alpha, u_{\tau,\beta'}, \beta) = \big(\alpha, \Xi_1^1(u, \beta', \tau), \beta\big)$$

for $a := (\alpha, \beta, u)$ and $s := (\beta', \tau)$. Then, due to the first part of the previous lemma, $\mathbf{j}$ is continuous. The linearity and boundedness of the mapping

$$A \ni a \mapsto \mathbf{j}(a, s) \in C,$$

for fixed $s \in \Delta$, is a consequence of the fact that for any
$(\beta', \tau) \in (-1/2, 1/2) \times S^*$ the mapping

$$C_T^1 \ni v \mapsto \Xi_1^1(v, \beta', \tau) \in C_h^1$$

is linear and bounded (see again the first part of the previous lemma). Furthermore, the inequality

$$\sup_{s \in \Delta} \|\mathbf{j}(\cdot, s)\|_{L(A|C)} < \infty$$

holds because of the fact that as a result of the first part of the previous lemma the inequality

$$\sup_{(\beta', \tau) \in (-1/2, 1/2) \times S^*} \Big\{ \|\Xi_1^1(\cdot, \beta', \tau)\|_{L(C_T^1|C_h^1)} \Big\} < \infty$$

holds.
We define

$$\mathbf{j}^* : \Omega_2 \times \Delta \to E$$

by $\mathbf{j}^* = J_{C,E} \circ \mathbf{j}$ if $J_{C,E}$ denotes the embedding from C to E. Then the identity $\mathbf{j}^*(a,s) = \big(\alpha, \Xi_0^1\big(u, \beta', \tau\big), \beta\big)$ holds for $a = (\alpha, \beta, u) \in \mathbb{R} \times \mathbb{R} \times C_T^1$ and $s = (\beta', \tau) \in (-1/2, 1/2) \times S^*$. Hence, due to the third part of the previous lemma, $\mathbf{j}^*$ is continuously differentiable. Therefore, $\mathbf{j}$ satisfies j 4) and j 5).
Thus, by applying the first part of Theorem 1.2.1 one gets that the mapping

$$\Omega_2 \times \Delta \ni (a,s) \mapsto \mathbf{h}\big(\mathbf{j}(a,s)\big) \in D$$

which here is

$$\big(J \times (-1/2,1/2) \times \hat{\Omega}\big) \times \big((-1/2,1/2) \times S^*\big) \ni \big((\alpha,\beta,u),(\beta',\tau)\big) \mapsto \mathbf{g}\big(\alpha, \Xi_1^1(u,\beta',\tau),\beta\big) \in \mathbb{R}^n$$

is continuously differentiable. An application of the chainrule yields that the mapping

$$\xi^1 : J \times \hat{\Omega} \times (-1/2,1/2) \times S^* \ni (\alpha,u,\beta,\tau) \mapsto \mathbf{g}\big(\alpha, \Xi_1^1(u,\beta,\tau),\beta\big) \in \mathbb{R}^n$$

is continuously differentiable.
For the proof of 2. we set $A := \mathbb{R} \times \mathbb{R} \times C_T^2$, $\Omega_2 := J \times (-1/2,1/2) \times \tilde{\Omega}$, $k := 3$ and $\Delta := (-1/2,1/2) \times S^*$ which is a bounded open subset of $\mathbb{R}^3$.
We define the mapping

$$j : \Omega_2 \times \Delta \to \Omega_1^* \subset B$$

by

$$j(a,s) := (\alpha, u_{\tau,\beta'}, \beta) = \big(\alpha, \Xi_2^2\big(u,\beta',\tau\big),\beta\big)$$

for $a := (\alpha,\beta,u)$ and $s := (\beta',\tau)$. Then, due to the second part of the previous lemma, j is continuous. The linearity and boundedness of the mapping

$$A \ni a \mapsto j(a,s) \in B,$$

for fixed $s \in \Delta$, is a consequence of the fact that for any $(\beta',\tau) \in (-1/2,1/2) \times S^*$ the mapping

$$C_T^2 \ni v \mapsto \Xi_2^2(v,\beta',\tau) \in C_h^2$$

is linear and bounded (see again the second part of the previous lemma). Furthermore, the inequality

$$\sup_{s\in\Delta} \|j(\cdot, s)\|_{L(A|B)} < \infty$$

holds because of the fact that as a result of the second part of the previous lemma the inequality

$$\sup_{(\beta',\tau)\in(-1/2,1/2)\times S^*} \left\{\|\Xi_2^2(\cdot, \beta', \tau)\|_{L(C_T^2|C_h^2)}\right\} < \infty$$

holds.

We define

$$j^* : A \times \Delta \to C$$

by $j^* = J_{B,C} \circ j$ if $J_{B,C}$ denotes the embedding from B to C. Then the identity $j^*(a, s) = (\alpha, \Xi_1^2(u, \beta', \tau), \beta)$ holds for $a = (\alpha, \beta, u) \in \mathbb{R}\times\mathbb{R}\times C_T^2$ and $s = (\beta', \tau) \in (-1/2, 1/2) \times S^*$. Hence, due to the fourth part of the previous lemma, j^* is continuously differentiable. The linearity and boundedness of the mapping

$$A \ni a \mapsto D_2 j^*(a, s) \in L(\mathbb{R}^3|C),$$

for fixed $s \in \Delta$, is a consequence of the fact that for any $(\beta', \tau) \in (-1/2, 1/2) \times S^*$ the mapping

$$C_T^2 \ni v \mapsto D_{(\beta',\tau)}\Xi_1^2(v, \beta', \tau) \in L(\mathbb{R}^2|C_h^1)$$

is linear and bounded (see again the fourth part of the previous lemma). Furthermore, the inequality

$$\sup_{s\in\Delta} \|D_2 j^*(\cdot, s)\|_{L(\mathbb{R}^3,A|C)} < \infty$$

holds because of the fact that as a result of the fourth part of the previous lemma the inequality

$$\sup_{(\beta',\tau)\in(-1/2,1/2)\times S^*} \left\{\|D_{(\beta',\tau)}\Xi_1^2(\cdot, \beta', \tau)\|_{L(\mathbb{R}^2,C_T^2|C_h^1)}\right\} < \infty$$

holds.

We define

$$j^{**} : A \times \Delta \to E$$

by $j^{**} = J_{C,E} \circ j^*$, if $J_{C,E}$ denotes the embedding from C to E. Then the identity $j^{**}(a,s) = \big(\alpha, \Xi_0^2\big(u,\beta',\tau\big),\beta\big)$ holds for $a = (\alpha,\beta,u) \in \mathbb{R}\times\mathbb{R}\times C_T^2$ and $s = (\beta',\tau) \in (-1/2,1/2)\times S^*$. Hence, due to the last part of the previous lemma, j^{**} is 2 times continuously differentiable. Therefore, j satisfies j 1), j 2) and j 3).
Thus, by applying the second part of Theorem 1.2.1 one gets that the mapping

$$\Omega_2 \times \Delta \ni (a,s) \mapsto \mathbf{h}^*\big(j(a,s)\big) \in D$$

which here is

$$\big(J\times(-1/2,1/2)\times\tilde{\Omega}\big)\times\big((-1/2,1/2)\times S^*\big) \ni \big((\alpha,u),(\beta,\beta',\tau)\big) \mapsto \mathbf{g}^*\big(\alpha,\Xi_2^2(u,\beta',\tau),\beta\big) \in \mathbb{R}^n$$

is 2 times continuously differentiable. An application of the chain rule yields that the mapping

$$\xi^2 : J\times\tilde{\Omega}\times(-1/2,1/2)\times S^* \ni (\alpha,u,\beta,\tau) \mapsto \mathbf{g}^*\big(\alpha,\Xi_2^2(u,\beta,\tau),\beta\big) \in \mathbb{R}^n$$

is 2 times continuously differentiable. □

1.3.2 Theorem on the differentiability properties of the mapping (1.9)

Now we are able to state a theorem on the differentiability - properties of the mapping (1.9). Recall the definition of the mappings Ξ_i^j, $i,j \in \{0,1,2\}$ and ξ_j, $j \in \{1,2\}$ which we introduced in the last section. Therefore, the mapping (1.9) will also be presented in new notation.

Theorem 1.3.2.1. *Let $\mathbf{g} : J\times\Omega\times\mathbb{R} \to \mathbb{R}^n$ be given with all assumptions $\tilde{H}$ 1) to $\tilde{H}$ 6) being satisfied. Let $\alpha_0 \in I$ be the critical parameter as stated in L 1). Let $\hat{\Omega} \subset C_T^1$, $\tilde{\Omega} \subset C_T^2$ and the mappings ξ^1, ξ^2 be defined like in Lemma 1.3.1.0.3.*
Let the mappings Ξ_0^1, Ξ_0^2 be defined like in Lemma 1.3.1.0.2.
Furthermore, suppose that $u_\tau \in \Omega$ for $u \in \hat{\Omega}$ and $\tau \in \mathbb{R}$.
Then the following properties hold:

1. *The mapping*

$$\mathbf{G}^0 : J\times\hat{\Omega}\times(-1/2,1/2) \to C_T^0,$$

defined by

$$\mathbf{G}^0(\alpha,u,\beta)(\tau) := \xi^1(\alpha,u,\beta,\tau) - L(\alpha_0)\Xi_0^1(u,0,\tau),$$

for $(\alpha,u,\beta) \in J \times \hat{\Omega} \times (-1/2,1/2)$ *and* $\tau \in \mathbb{R}$*, is continuously differentiable.*
The identities $\mathbf{G}^0(\alpha_0,0,0) = 0$ *and* $D_2\mathbf{G}^0(\alpha_0,0,0) = 0$ *hold.*

2. *The mapping*

$$\mathbf{G}^1 : J \times \tilde{\Omega} \times (-1/2,1/2) \to C_T^1,$$

defined by

$$\mathbf{G}^1(\alpha,u,\beta)(\tau) := \xi^2(\alpha,u,\beta,\tau) - L(\alpha_0)\Xi_0^2(u,0,\tau)$$

and

$$\big(\mathbf{G}^1(\alpha,u,\beta)\big)'(\tau) := D_4\xi^2(\alpha,u,\beta,\tau) - L(\alpha_0)D_3\Xi_0^2(u,0,\tau),$$

for $(\alpha,u,\beta) \in J \times \tilde{\Omega} \times (-1/2,1/2)$ *and* $\tau \in \mathbb{R}$*, is continuously differentiable.*
The identities $\mathbf{G}^1(\alpha_0,0,0) = 0$ *and* $D_2\mathbf{G}^1(\alpha_0,0,0) = 0$ *hold.*

3. *The mapping*

$$\mathbf{G}^2 : J \times \tilde{\Omega} \times (-1/2,1/2) \to C_T^0,$$

defined by

$$\mathbf{G}^2(\alpha,u,\beta)(\tau) := \xi^2(\alpha,u,\beta,\tau) - L(\alpha_0)\Xi_0^2(u,0,\tau),$$

for $(\alpha,u,\beta) \in J \times \tilde{\Omega} \times (-1/2,1/2)$ *and* $\tau \in \mathbb{R}$*, is 2 times continuously differentiable.*
The identities $\mathbf{G}^2(\alpha_0,0,0) = 0$ *and* $D_2\mathbf{G}^2(\alpha_0,0,0) = 0$ *hold.*

Proof. With no loss of generality let $\phi^* = 0$.
1. In Lemma 1.3.1.0.3 we have seen that the mapping

$$\xi^1 : J \times \hat{\Omega} \times (-1/2,1/2) \times S^* \ni (\alpha,u,\beta,\tau) \mapsto \mathbf{g}(\alpha,u_{\tau,\beta},\beta) \in \mathbb{R}^n$$

is continuously differentiable.
Lemma 1.3.1.0.2 yields that the mapping

$$\Xi_0^1 : C_T^1 \times (-1/2,1/2) \times S^* \ni (u,\beta,\tau) \mapsto u_{\tau,\beta} \in C_h$$

is continuously differentiable.
Hence, both restrictions $\xi^1|_{J\times\hat{\Omega}\times(-1/2,1/2)\times[0,T]}$ and $\Xi_0^1|_{C_T^1\times(-1/2,1/2)\times[0,T]}$ are continuously differentiable.
An application of the second part of Theorem 3.1.1 in Appendix I yields that

$$\mathbf{g}^0 : J \times \hat{\Omega} \times (-1/2, 1/2) \to C^0$$

defined by

$$\mathbf{g}^0(\alpha, u, \beta)(\tau) := \xi^1(\alpha, u, \beta, \tau) - L(\alpha_0)\Xi_0^1(u, 0, \tau),$$

for $(\alpha, u, \beta) \in J \times \hat{\Omega} \times (-1/2, 1/2)$ and $\tau \in [0, T]$, is continuously differentiable. As u was T - periodic we can extend $\mathbf{g}^0(\alpha, u, \beta)$ to a T - periodic continuous function $\mathbf{G}^0(\alpha, u, \beta)$ on $\mathbb{R}$ for $(\alpha, u, \beta) \in J \times \hat{\Omega} \times (-1/2, 1/2)$:
If $\tau \in [nT, (n+1)T]$, for $n \in \mathbb{Z}$, we set $\mathbf{G}^0(\alpha, u, \beta)(\tau) := \mathbf{g}^0(\alpha, u, \beta)(\tau)$.
By the fact that $\|v\|_{C_T^0} = \|v\|_{C^0}$ for $v \in C_T^1$ it easily follows that $\mathbf{g}^0$ extends to a continuously differentiable mapping

$$\mathbf{G}^0 : J \times \hat{\Omega} \times (-1/2, 1/2) \to C_T^0.$$

The identities $\mathbf{G}^0(\alpha_0, 0, 0) = 0$ and $D_2\mathbf{G}^0(\alpha_0, 0, 0) = 0$ are a consequence of the definition of ξ^1 and Ξ_0^1.
2. In Lemma 1.3.1.0.3 we have seen that the mapping

$$\xi^2 : J \times \tilde{\Omega} \times (-1/2, 1/2) \times S^* \ni (\alpha, u, \beta, \tau) \mapsto \mathbf{g}^*(\alpha, u_{\tau,\beta}, \beta) \in \mathbb{R}^n$$

is 2 times continuously differentiable.
Lemma 1.3.1.0.2 yields that the mapping

$$\Xi_0^2 : C_T^2 \times (-1/2, 1/2) \times S^* \ni (u, \beta, \tau) \mapsto u_{\tau,\beta} \in C_h$$

is 2 times continuously differentiable.
Hence, both restrictions $\xi^2|_{J\times\tilde{\Omega}\times(-1/2,1/2)\times[0,T]}$ and $\Xi_0^2|_{C_T^2\times(-1/2,1/2)\times[0,T]}$ are 2 times continuously differentiable.
An application of the fifth part of Theorem 3.1.1 in Appendix I yields that

$$\mathbf{g}^1 : J \times \tilde{\Omega} \times (-1/2, 1/2) \to C^1$$

defined by

$$\mathbf{g}^1(\alpha, u, \beta)(\tau) := \xi^2(\alpha, u, \beta, \tau) - L(\alpha_0)\Xi_0^2(u, 0, \tau)$$

and

$$\big(\mathbf{g}^1(\alpha,u,\beta)\big)'(\tau) := D_4\xi^2(\alpha,u,\beta,\tau) - L(\alpha_0)D_3\Xi_0^2(u,0,\tau),$$

for $(\alpha,u,\beta) \in J \times \tilde{\Omega} \times (-1/2,1/2)$ and $\tau \in [0,T]$, is continuously differentiable. As u was T - periodic we can extend $\mathbf{g}^1(\alpha,u,\beta)$ to a T - periodic continuously differentiable function $\mathbf{G}^1(\alpha,u,\beta)$ on $\mathbb{R}$ for $(\alpha,u,\beta) \in J \times \tilde{\Omega} \times (-1/2,1/2)$:
If $\tau \in [nT,(n+1)T]$, for $n \in \mathbb{Z}$, we set $\mathbf{G}^1(\alpha,u,\beta)(\tau) := \mathbf{g}^1(\alpha,u,\beta)(\tau)$ and $\big(\mathbf{G}^1(\alpha,u,\beta)\big)'(\tau) := \big(\mathbf{g}^1(\alpha,u,\beta)\big)'(\tau)$.
By the fact that $\|v\|_{C^1_T} = \|v\|_{C^1}$ for $v \in C^1_T$ it easily follows that $\mathbf{g}^1$ extends to a continuously differentiable mapping

$$\mathbf{G}^1 : J \times \tilde{\Omega} \times (-1/2,1/2) \to C^1_T.$$

The identities $\mathbf{G}^1(\alpha_0,0,0) = 0$ and $D_2\mathbf{G}^1(\alpha_0,0,0) = 0$ are a consequence of the definition of ξ^2 and Ξ_0^2.
3. In Lemma 1.3.1.0.3 we have seen that the mapping

$$\xi^2 : J \times \tilde{\Omega} \times (-1/2,1/2) \times S^* \ni (\alpha,u,\beta,\tau) \mapsto \mathbf{g}^*(\alpha,u_{\tau,\beta},\beta) \in \mathbb{R}^n$$

is 2 times continuously differentiable.
Lemma 1.3.1.0.2 yields that the mapping

$$\Xi_0^2 : C^2_T \times (-1/2,1/2) \times S^* \ni (u,\beta,\tau) \mapsto u_{\tau,\beta} \in C_h$$

is 2 times continuously differentiable.
Hence, both restrictions $\xi^2|_{J\times\tilde{\Omega}\times(-1/2,1/2)\times[0,T]}$ and $\Xi_0^2|_{C^2_T\times(-1/2,1/2)\times[0,T]}$ are 2 times continuously differentiable.
An application of the fourth part of Theorem 3.1.1 in Appendix I yields that

$$\mathbf{g}^2 : J \times \tilde{\Omega} \times (-1/2,1/2) \to C^0$$

defined by

$$\mathbf{g}^2(\alpha,u,\beta)(\tau) := \xi^2(\alpha,u,\beta,\tau) - L(\alpha_0)\Xi_0^2(u,0,\tau),$$

for $(\alpha,u,\beta) \in J \times \tilde{\Omega} \times (-1/2,1/2)$ and $\tau \in [0,T]$, is 2 times continuously differentiable. As u was T - periodic we can extend $\mathbf{g}^2(\alpha,u,\beta)$ to a T - periodic continuous function $\mathbf{G}^2(\alpha,u,\beta)$ on $\mathbb{R}$ for $(\alpha,u,\beta) \in J \times \tilde{\Omega} \times$

$(-1/2, 1/2)$:
If $\tau \in [nT, (n+1)T]$, for $n \in \mathbb{Z}$, we set $\mathbf{G}^2(\alpha, u, \beta)(\tau) := \mathbf{g}^2(\alpha, u, \beta)(\tau)$.
By the fact that $\|v\|_{C^0_T} = \|v\|_{C^0}$ for $v \in C^0_T$ it easily follows that $\mathbf{g}^2$ extends to a 2 times continuously differentiable mapping

$$\mathbf{G}^2 : J \times \tilde{\Omega} \times (-1/2, 1/2) \to C^0_T.$$

The identities $\mathbf{G}^2(\alpha_0, 0, 0) = 0$ and $D_2\mathbf{G}^2(\alpha_0, 0, 0) = 0$ are a consequence of the definition of ξ^2 and Ξ^2_0. $\square$

1.4 Hopf Bifurcation Theorem

We repeat the conditions which are necessary for Hopf bifurcation as stated in section 1.1.3:
There exists an interval $I \subset J$ and a parametrization $I \ni \alpha \mapsto \lambda(\alpha) \in \mathbb{C}$ onto eigenvalues of the infinitesimal generator of $L(\alpha)$.
This parametrization has the following properties:

L 1): $\lambda(\alpha_0) = \lambda_0 = \omega \cdot i$, $\omega = \frac{2\pi}{T}$ a real number, is a purely imaginary simple eigenvalue of the infinitesimal generator $A(\alpha_0)$ of the semigroup $T(\alpha_0)(t)_{t\geq 0}$ associated with $L(\alpha_0)$. There exists no further eigenvalue of $A(\alpha_0)$ but $\bar{\lambda}_0 = -\omega \cdot i$.

L 2): The mapping $I \ni \alpha \to \lambda(\alpha) \in \mathbb{C}$ is continuously differentiable with $\Re[(\lambda'(\alpha_0))] \neq 0$.

L 3): $\lambda(\alpha)$ for $\alpha \in I$ is a simple eigenvalue of the infinitesimal generator $A(\alpha)$ belonging to the semigroup $T(\alpha)(t)_{t\geq 0}$ associated with $L(\alpha)$.

We divide this section into two subsections.
In the first one we will prove that there exists a 2 times continuously differentiable mapping

$$\hat{O} \ni (\alpha, a, \beta) \to \hat{u}(\alpha, a, \beta) \in C_T^1$$

onto solutions of (1.7) , where $\hat{O}$ is a suitable open subsets of $J \times \mathbb{R} \times (-1/2, 1/2)$.
In the second subsection we will follow a standard approach to Hopf bifurcation which is plugging $\hat{u}$ into equation (1.8) and solving it for γ and β, given the assumptions L 1) to L 3). This will be done in the proof of the Hopf bifurcation Theorem (Theorem 1.4.2.1).

1.4.1 Solutions of (1.7)

Lemma 1.4.1.0.1. *Let $L \in L(C_h|\mathbb{R}^n)$ and f be a T - periodic function. Let $x : \mathbb{R} \to \mathbb{R}^n$ be a continuous T - periodic solution of*

$$x'(t) = Lx_t + f(t), \quad t \in \mathbb{R} \tag{1.16}$$

Then the following results hold:

1. *If $f \in C^0_T$, then $x \in C^1_T$ and the inequality*

$$\|x\|_{C^1_T} \leq \Big(\|L\|_{L\big(C_h|\mathbb{R}^n\big)} + 1\Big) \cdot \|x\|_{C^0_T} + \|f\|_{C^0_T} \tag{1.17}$$

holds.

2. *If $f \in C^1_T$, then $x \in C^2_T$ and the inequality*

$$\begin{aligned} \|x\|_{C^2_T} \leq & \Big(\|L\|^2_{L\big(C_h|\mathbb{R}^n\big)} + \|L\|_{L\big(C_h|\mathbb{R}^n\big)} + 1\Big) \cdot \|x\|_{C^0_T} \\ & + \Big(\|L\|_{L\big(C_h|\mathbb{R}^n\big)} + 1\Big) \cdot \|f\|_{C^1_T} \end{aligned} \tag{1.18}$$

holds.

Proof. 1. x is a solution of equation (1.16). Then, due to the continuity of the right hand side of (1.16) the mapping $x : \mathbb{R} \to \mathbb{R}^n$ needs to be continuously differentiable. Hence, one gets by estimation that the inequality

$$\sup_{t\in[0,T]} \|x'(t)\|_{\mathbb{R}^n} \leq \sup_{t\in[0,T]} (\|Lx_t\|_{\mathbb{R}^n} + \|f(t)\|_{\mathbb{R}^n}) \leq$$

$$\|L\|_{L\big(C_h|\mathbb{R}^n\big)} \|x\|_{C^0_T} + \|f\|_{C^0_T}$$

holds. On the other hand

$$\sup_{t\in[0,T]} \|x(t)\|_{\mathbb{R}^n} = \|x\|_{C^0_T}.$$

Thus, the inequality

$$\|x\|_{C^1_T} = \max\{\sup_{t\in[0,T]} \|x(t)\|_{\mathbb{R}^n}, \sup_{t\in[0,T]} \|x'(t)\|_{\mathbb{R}^n}\} \leq$$

$$\Big(\|L\|_{L\big(C_h|\mathbb{R}^n\big)} + 1\Big) \cdot \|x\|_{C^0_T} + \|f\|_{C^0_T}$$

holds.

2. x satisfying (1.16), where

$$\mathbb{R} \ni t \mapsto f(t) \in \mathbb{R}^n$$

and

$$\mathbb{R} \ni t \mapsto x_t \in C_h$$

are continuously differentiable one may again differentiate the whole equation (1.16) which then becomes

$$x''(t) = Lx'_t + f'(t), \quad t \in \mathbb{R} \tag{1.19}$$

Thus, $x : \mathbb{R} \to \mathbb{R}^n$ is 2 times continuously differentiable and, analogously to 1., one gets by estimation that the inequality

$$\sup_{t\in[0,T]} \|x''(t)\|_{\mathbb{R}^n} \le \sup_{t\in[0,T]} (\|Lx'_t\|_{\mathbb{R}^n} + \|f'(t)\|_{\mathbb{R}^n}) \le$$

$$\|L\|_{L\left(C_h|\mathbb{R}^n\right)} \|x'\|_{C_T^0} + \|f'\|_{C_T^0}$$

holds. In 1.we have seen that x' satisfies

$$\|x'\|_{C_T^0} = \sup_{t\in[0,T]} \|x'(t)\|_{\mathbb{R}^n} \le \sup_{t\in[0,T]} (\|Lx_t\|_{\mathbb{R}^n} + \|f(t)\|_{\mathbb{R}^n}) \le$$

$$\|L\|_{L\left(C_h|\mathbb{R}^n\right)} \|x\|_{C_T^0} + \|f\|_{C_T^0}.$$

Thus, the inequality

$$\sup_{t\in[0,T]} \|x''(t)\|_{\mathbb{R}^n} \le$$

$$\|L\|_{L\left(C_h|\mathbb{R}^n\right)} \left(\|L\|_{L\left(C_h|\mathbb{R}^n\right)} \cdot \|x\|_{C_T^0} + \|f\|_{C_T^0}\right)$$

$$+\|f'\|_{C_T^0} \le$$

$$\|L\|^2_{L\left(C_h|\mathbb{R}^n\right)} \cdot \|x\|_{C_T^0} + \left(\|L\|_{L\left(C_h|\mathbb{R}^n\right)} + 1\right) \cdot \|f\|_{C_T^1}$$

holds. Hence, the inequality

$$\|x\|_{C_T^2} = \max\{\sup_{t\in[0,T]} \|x(t)\|_{\mathbb{R}^n}, \sup_{t\in[0,T]} \|x'(t)\|_{\mathbb{R}^n}, \sup_{t\in[0,T]} \|x''(t)\|_{\mathbb{R}^n}\} \le$$

$$\left(\|L\|^2_{L\left(C_h|\mathbb{R}^n\right)} + \|L\|_{L\left(C_h|\mathbb{R}^n\right)} + 1\right) \cdot \|x\|_{C_T^0}$$

$$+\left(\|L\|_{L\left(C_h|\mathbb{R}^n\right)} + 1\right) \cdot \|f\|_{C_T^1}$$

holds. □

Lemma 1.4.1.0.2. *Let the results of the previous lemma be given and let the operators* $\mathbf{K}$ *and* $\mathbf{J}$ *be defined like in Theorem 1.1.4.1.*

1. *If* $f \in C_T^0$ *then the function* $(\mathbf{K} \circ [Id - \mathbf{J}])(f)$ *is an element of* C_T^1 *and the operator* $(\mathbf{K} \circ [Id - \mathbf{J}])^1 : C_T^0 \to C_T^1$ *defined by* $(\mathbf{K} \circ [Id - \mathbf{J}])(f)$ *for* $f \in C_T^0$ *is linear and bounded.*

2. *If* $f \in C_T^1$ *then the function* $(\mathbf{K} \circ [Id - \mathbf{J}])(f)$ *is an element of* C_T^2 *and the operator* $(\mathbf{K} \circ [Id - \mathbf{J}])^2 : C_T^1 \to C_T^2$ *defined by* $(\mathbf{K} \circ [Id - \mathbf{J}])(f)$ *for* $f \in C_T^1$ *is linear and bounded.*

Proof. 1. As $(\mathbf{K} \circ [Id - \mathbf{J}])(f)$ is the unique continuous T - periodic solution of

$$x'(t) = Lx_t + [Id - \mathbf{J}](f)(t)$$

the first part of Lemma 1.4.1.0.1 tells us that $(\mathbf{K} \circ [Id - \mathbf{J}])(f) \in C_T^1$, Furthermore, $(\mathbf{K} \circ [Id - \mathbf{J}])(f)$ has to satisfy inequality (1.17). Thus, we get by estimation that the inequality

$$\|(\mathbf{K} \circ [Id - \mathbf{J}])^1(f)\|_{C_T^1} \leq$$

$$\left(\|L\|_{L\left(C_h|\mathbb{R}^n\right)} + 1\right) \cdot \|(\mathbf{K} \circ [Id - \mathbf{J}])(f)\|_{C_T^0} + \|[Id - \mathbf{J}](f)\|_{C_T^0} \leq$$

$$\left[\left(\|L\|_{L\left(C_h|\mathbb{R}^n\right)} + 1\right) \cdot \|(\mathbf{K} \circ [Id - \mathbf{J}])\|_{L(C_T^0|C_T^0)} + \|[Id - \mathbf{J}]\|_{L(C_T^0|C_T^0)}\right)\Big] \|f\|_{C_T^0}$$

holds.

2. As $(\mathbf{K} \circ [Id - \mathbf{J}])(f)$ is the unique continuously differentiable T - periodic solution of

$$x'(t) = Lx_t + [Id - \mathbf{J}](f)(t)$$

the second part of Lemma 1.4.1.0.1 tells us that $(\mathbf{K} \circ [Id - \mathbf{J}])(f) \in C_T^2$. Furthermore, $(\mathbf{K} \circ [Id - \mathbf{J}])(f)$ has to satisfy inequality (1.18). Thus, we get the by estimation that the inequality

$$\|(\mathbf{K} \circ [Id - \mathbf{J}])^2(f)\|_{C_T^2} \leq$$

$$\left(\|L\|^2_{L\left(C_h|\mathbb{R}^n\right)} + \|L\|_{L\left(C_h|\mathbb{R}^n\right)} + 1\right) \cdot \|(\mathbf{K} \circ [Id - \mathbf{J}])(f)\|_{C_T^0}$$

$$+\Big(\|L\|_{L\big(C_h|\mathbb{R}^n\big)}+1\Big)\|[Id-\mathbf{J}](f)\|_{C_T^1}\leq$$

$$\Big[\Big(\|L\|^2_{L\big(C_h|\mathbb{R}^n\big)}+\|L\|_{L\big(C_h|\mathbb{R}^n\big)}+1\Big)$$

$$\cdot\|(\mathbf{K}\circ[Id-\mathbf{J}])\|_{L(C_T^0|C_T^0)}+\Big(\|L\|_{L\big(C_h|\mathbb{R}^n\big)}+1\Big)\|[Id-\mathbf{J}]|_{C_T^1}\|_{L(C_T^1|C_T^1)}\Big]\|f\|_{C_T^1}$$

holds. In the last step we used the continuity of $\mathbf{J}|_{C_T^1}$ which is a consequence of the two following facts:

1. The restriction of $\mathbf{J}$ to C_T^1 is the projection along the kernel of the continuous linear functional

$$C_T^1\ni f\mapsto\int_0^T\Phi^t(s)f(s)ds\in\mathbb{R}^2,$$

as defined in Theorem 1.1.4.1.

2. This kernel has finite codimension.

□

Lemma 1.4.1.0.3. *Let the results of the previous lemma be given. Let the mappings* $\mathbf{G}^0$, $\mathbf{G}^1$ *and* $\mathbf{G}^2$ *be defined like in Theorem 1.3.2.1: Then the following results hold:*

1. *The mapping*

$$(\mathbf{K}\circ[Id-\mathbf{J}])^1\circ\mathbf{G}^0:J\times\hat{\Omega}\times(-1/2,1/2)\to C_T^1$$

is continuously differentiable.

2. *The mapping*

$$(\mathbf{K}\circ[Id-\mathbf{J}])^2\circ\mathbf{G}^1:J\times\tilde{\Omega}\times(-1/2,1/2)\to C_T^2$$

is continuously differentiable.

3. *The mapping*

$$(\mathbf{K}\circ[Id-\mathbf{J}])^1\circ\mathbf{G}^2:J\times\tilde{\Omega}\times(-1/2,1/2)\to C_T^1$$

is 2 times continuously differentiable.

Proof. 1. The mapping

$$\mathbf{G}^0 : J \times \hat{\Omega} \times (-1/2, 1/2) \to C_T^0$$

being continuously differentiable and

$$(\mathbf{K} \circ [Id - \mathbf{J}])^1 : C_T^0 \to C_T^1$$

being linear and bounded it is clear that the mapping

$$(\mathbf{K} \circ [Id - \mathbf{J}])^1 \circ \mathbf{G}^0 : J \times \hat{\Omega} \times (-1/2, 1/2) \to C_T^1$$

is continuously differentiable.
2. The mapping

$$\mathbf{G}^1 : J \times \tilde{\Omega} \times (-1/2, 1/2) \to C_T^1$$

being continuously differentiable and

$$(\mathbf{K} \circ [Id - \mathbf{J}])^2 : C_T^1 \to C_T^2$$

being linear and bounded it is clear that the mapping

$$(\mathbf{K} \circ [Id - \mathbf{J}])^2 \circ \mathbf{G}^1 : J \times \tilde{\Omega} \times (-1/2, 1/2) \to C_T^2$$

is continuously differentiable.
3. The mapping

$$\mathbf{G}^2 : J \times \tilde{\Omega} \times (-1/2, 1/2) \to C_T^0$$

being 2 times continuously differentiable and

$$(\mathbf{K} \circ [Id - \mathbf{J}])^1 : C_T^0 \to C_T^1$$

being linear and bounded it is clear that the mapping

$$(\mathbf{K} \circ [Id - \mathbf{J}])^1 \circ \mathbf{G}^2 : J \times \tilde{\Omega} \times (-1/2, 1/2) \to C_T^1$$

is 2 times continuously differentiable.

□

Theorem 1.4.1.1. *Let all assumptions $\tilde{H}$ 1) to $\tilde{H}$ 6) be satisfied.*
Let all assumptions L 1) to L 3) be satisfied.
let the mappings $\mathbf{G}^0$, $\mathbf{G}^1$ and $\mathbf{G}^2$ be defined like in Theorem 1.3.2.1.

Let $\phi_1(\alpha_0)$, $\phi_2(\alpha_0)$ denote a basis of $E_{\lambda(0)}$. Let $J_{E_{\lambda(\alpha_0)},C_T^1}$ and $J_{E_{\lambda(\alpha_0)},C_T^2}$ denote the continuous embeddings of the eigenspace $E_{\lambda(\alpha_0)}$ into C_T^1 and C_T^2, respectively, as defined in Section 1.1.2
Let $b : \mathbb{R} \to E_{\lambda(\alpha_0)}$ be defined by $b(a) := a \cdot \phi_1(\alpha_0)$, the scalar multiplication of a scalar a with the vector $\phi_1(\alpha_0)$.
Then there exist neighborhoods $\mathbf{S} \subset J \times \mathbb{R} \times (-1/2, 1/2)$ of $(\alpha_0, 0, 0) \in \mathbb{R}^3$ and $\mathbb{A} \subset \hat{\Omega}$ of $0 \in C_T^1$ such that for every $(\alpha, a, \beta) \in \mathbf{S}$ there is a solution $u^*(\alpha, a, \beta) \in \mathbb{A} \cap \tilde{\Omega}$ of the equation

$$u = J_{E_{\lambda(\alpha_0)},C_T^2}\big(b(a)\big) + (\mathbf{K} \circ [Id - \mathbf{J}])^2 \circ \big(\mathbf{G}^1(\alpha, u, \beta)\big)$$

which in fact is equation (1.7) rewritten in different notation.
Any solution u^* of (1.7) in $\mathbb{A}$ at parameter $(\alpha, a, \beta) \in \mathbf{S}$ must have the form $u^* = u^*(\alpha, a, \beta)$.
There are subsets $\hat{O} \subset O^* \subset \mathbf{S}$ such that the following properties hold:
The identity $u^*(\alpha, 0, \beta) = 0$ holds for all (α, β) such that $(\alpha, 0, \beta) \in O^*$.
The mapping

$$u^* : O^* \to \mathbb{A} \cap \tilde{\Omega} \subset C_T^2$$

is continuously differentiable.
If $\hat{u} := J_{C_T^2,C_T^1} \circ u^*|_{\hat{O}}$ then the mapping

$$\hat{u} : \hat{O} \to \mathbb{A} \subset \hat{\Omega} \subset C_T^1$$

is 2 times continuously differentiable.

Proof. With no loss of generality let $\phi^* = 0$ and $\alpha_0 = 0$.
We want to apply Theorem 3.3.2 in Appendix III.
As a result of the previous lemma the mappings $(\mathbf{K} \circ [Id - \mathbf{J}])^1 \circ \mathbf{G}^0$ and $(\mathbf{K} \circ [Id - \mathbf{J}])^2 \circ \mathbf{G}^1$ are continuously differentiable.
The mapping $(\mathbf{K} \circ [Id - \mathbf{J}])^1 \circ \mathbf{G}^2$ is 2 times continuously differentiable. Furthermore, the mappings

$$\mathbb{R} \ni a \mapsto J_{E_{\lambda(0)},C_T^1}\big(b(a)\big) \in C_T^1$$

and

$$\mathbb{R} \ni a \mapsto J_{E_{\lambda(0)},C_T^2}\big(b(a)\big) \in C_T^2$$

are 2 times continuously differentiable.

Thus, the following holds:

1. Let $Y^1 := C^1_T$ and $X := \mathbb{R}^3$. Let $B_1 \subset X$ be an open subset such that $(0,0,0) \in B_1 \subset J \times \mathbb{R} \times (-1/2, 1/2)$ and let $B_2 := \hat{\Omega}$.

The mapping

$$\tilde{K} : B_1 \times B_2 \to Y^1 (= C^1_T),$$

defined by

$$\tilde{K}\big((\alpha, a, \beta), u\big) := J_{E_{\lambda(0)}, C^1_T}\big(b(a)\big) + (\mathbf{K} \circ [Id - \mathbf{J}])^1 \circ \mathbf{G}^0(\alpha, u, \beta),$$

for

$$\big((\alpha, a, \beta), u\big) \in B_1 \times B_2,$$

is continuously differentiable.

2. Let $B^*_2 := B_2 \cap C^2_T = \tilde{\Omega}$ and $Y^2 := C^2_T$. The mapping

$$K^* : B_1 \times B^*_2 \to Y^2 (= C^2_T),$$

defined by

$$K^*\big((\alpha, a, \beta), u\big) := J_{E_{\lambda(0)}, C^2_T}\big(b(a)\big) + (\mathbf{K} \circ [Id - \mathbf{J}])^2 \circ \mathbf{G}^1(\alpha, u, \beta),$$

for

$$\big((\alpha, a, \beta), u\big) \in B_1 \times B^*_2,$$

is continuously differentiable.

3. The mapping

$$\hat{K} : B_1 \times B^*_2 \to Y^1 (= C^1_T),$$

defined by

$$\hat{K}\big((\alpha, a, \beta), u\big) := J_{E_{\lambda(0)}, C^1_T}\big(b(a)\big) + (\mathbf{K} \circ [Id - \mathbf{J}])^1 \circ \mathbf{G}^2(\alpha, u, \beta),$$

for

$$\big((\alpha, a, \beta), u\big) \in B_1 \times B^*_2,$$

is 2 times continuously differentiable.

Clearly, $\tilde{K}(B_1 \times B^*_2) \subset Y^2$.

We know from Theorem 1.3.2.1 that the identities $\mathbf{G}^0(0,0,0) = 0$, $\mathbf{G}^1(0,0,0) = 0$, $D_2\mathbf{G}^0(0,0,0) = 0$ and $D_2\mathbf{G}^1(0,0,0) = 0$ hold. Thus, the identities

$$K^*\big((0,0,0),0\big) = \tilde{K}\big((0,0,0),0\big) = 0,$$

$$Id_{Y^1} - D_2\tilde{K}(\big(0,0,0),0\big) = Id_{Y^1}$$

and

$$Id_{Y^2} - D_2K^*\big((0,0,0),0\big) = Id_{Y^2}$$

hold.
Here, $D_2\tilde{K}(\big(0,0,0),0\big) \in L(Y^1|Y^1)$ and $D_2K^*(\big(0,0,0),0\big) \in L(Y^2|Y^2)$ denote the partial derivatives of $\tilde{K}$ and K^* with respect to u in $\big((0,0,0),0\big) \in Y^1$ and $\big((0,0,0),0\big) \in Y^2$ respectively.
Therefore, the Implicit Function Theorem yields the existence of open neighborhoods $\tilde{O} \subset B_1$ and $O^* \subset B_1$ of $(0,0,0)$ and $\mathbb{A} \subset \hat{\Omega}$ of $0 \in Y^2 \subset Y^1$ such that the follwing holds:
There are continuously differentiable mappings

$$\tilde{u} : \tilde{O} \to B_2 \subset Y^1$$

and

$$u^* : O^* \to B_2^* \subset Y^2,$$

respectively, satisfying $\tilde{K}\big(\tilde{O},\mathbb{A}\big) \subset \mathbb{A}$ and

$$\tilde{K}\big((\alpha,a,\beta),\tilde{u}(\alpha,a,\beta)\big) = \tilde{u}(\alpha,a,\beta), \quad (\alpha,a,\beta) \in \tilde{O},$$

and $K^*\big(O^*,\mathbb{A}\cap\tilde{\Omega}\big) \subset \mathbb{A}\cap\tilde{\Omega}$ and

$$K^*\big((\alpha,a,\beta),u^*(\alpha,a,\beta)\big) = u^*(\alpha,a,\beta), \quad (\alpha,a,\beta) \in O^*,$$

respectively. By the fact that $K^*\big((\alpha,0,\beta),0\big) = 0$ it follows that $u^*(\alpha,0,\beta) = 0$ for all (α,β) such that $(\alpha,0,\beta) \in \tilde{O}$. With no loss of generality we suppose that $O^* \subset \tilde{O}$. It is clear that for reasons of uniqueness $\tilde{u}$ must satisfy $\tilde{u}|_{O^*} = (J_{C_T^2,C_T^1} \circ u^*)$.
Therefore, the assumptions of Theorem 3.3.2 are satisfied and thus there is a neighborhood $\hat{O} \subset B_1$ of $(0,0,0)$ such that the mapping $\hat{u} := \tilde{u}|_{\hat{O}}$ is 2 times continuously differentiable. Furthermore, the identity

$$\hat{K}\big((\alpha,a,\beta),\hat{u}(\alpha,a,\beta)\big) = \hat{u}(\alpha,a,\beta)$$

holds for $(\alpha, a, \beta) \in \hat{O}$. With no loss of generality we suppose that $\hat{O} \subset O^* \subset \tilde{O}$ and set $\mathbf{S} := \tilde{O}$. This completes the proof. $\square$

Corollary 1.4.1.1.1. *Let all assumptions of the previous theorem be satisfied. Let u^* be the solution of*

$$u = J_{E_{\lambda(\alpha_0)}, C_T^2}\big(b(a)\big) + (\mathbf{K} \circ [Id - \mathbf{J}])^2 \circ \mathbf{G}^1(\alpha, u, \beta)$$

which we found in the previous Theorem.
Then the mapping

$$\hat{O} \ni (\alpha, a, \beta) \mapsto \mathbf{G}^2\big(\alpha, u^*(\alpha, a, \beta), \beta\big) \in C_T^0$$

is 2 times continuously differentiable.

Proof. With no loss of generality let $\alpha_0 = 0$.
We want to apply Theorem 3.2.1 in Appendix II:
We know the following:

- If $A_1 \subset J \times \hat{\Omega} \times (-1/2, 1/2)$ is an open subset such that $\hat{u}(\hat{O}) \subset A_1$ then the mapping $\mathbf{G}^0|_{A_1}$ is continuously differentiable.
- The set
$$A_2 := A_1 \cap (J \times \tilde{\Omega} \times (-1/2, 1/2))$$
is an open subset of $J \times \tilde{\Omega} \times (-1/2, 1/2)$ such that $u^*(\hat{O}) \subset A_2$. Furthermore, the mapping $\mathbf{G}^2|_{A_2}$ is 2 times continuously differentiable.

On the other hand u^* has the following properties:

- The mapping
$$u^* : \hat{O} \to \tilde{\Omega} \subset C_T^2$$
is continuously differentiable. (Recall that as a result of the previous theorem $\hat{O} \subset O^*$. Thus, we may restrict u^* to $\hat{O}$.)
- The mapping
$$\hat{u} = J_{C_T^2, C_T^1} \circ u^* : \hat{O} \to \hat{\Omega} \subset C_T^1$$
is 2 times continuously differentiable.

Hence, by considering the map

$$\hat{O} \ni (\alpha, a, \beta) \mapsto \big(\alpha, u^*(\alpha, a, \beta), \beta\big) \in J \times \tilde{\Omega} \times (-1/2, 1/2)$$

we realize that we are exactly in the situation of Theorem 3.2.1. Thus, the mapping

$$\hat{O} \ni (\alpha, a, \beta) \mapsto \mathbf{G}^2\big(\alpha, u^*(\alpha, a, \beta), \beta\big) \in C_T^0$$

is 2 times continuously differentiable. □

1.4.2 Standard approach of proving Hopf bifurcation

In this subsection we continue the proof of Hopf bifurcation as stated in [3] by plugging $\hat{u}$ into equation (1.8) and solving it for α and β, given the assumptions L 1) to L 3) of the linearization $L(\alpha)$, for $\alpha \in J$.
The presentation of the steps on the proof will be more explicit than in [3].
We suppose that the following properties associated with the linearization $L(\alpha)$, for $\alpha \in J$, are satisfied:
(Compare [3], section 7)

- For $\alpha \in I$, $i \in \{0, 1, 2\}$, let $J_{E_{\lambda(\alpha)}, C^i}$ denote the continuous embedding from $E_{\lambda(\alpha)}$ to C^i, such as defined in section 1.1.2.
  For $\alpha = \alpha_0$, $i \in \{0, 1, 2\}$, let $J_{E_{\lambda(\alpha_0)}, C_T^i}$ denote the continuous embedding from $E_{\lambda(\alpha_0)}$ to C_T^i, such as defined in section 1.1.2.
  For $\alpha \in I$ let $\{\phi_1(\alpha), \phi_2(\alpha)\}$ denote a basis of the eigenspace $E_{\lambda(\alpha)}$ of the infinitesimal generator $A(\alpha)$ of $L(\alpha)$ corresponding to the eigenvalue $\lambda(\alpha)$. This basis satisfies $\phi_i^t(s) \cdot \phi_j(s) = \delta_{ij}$, $i, j = 1, 2$.
  Let $\Phi_1(\alpha) = J_{E_{\lambda(\alpha)}, C^0}\big(\phi_1(\alpha)\big)$ and $\Phi_2(\alpha) = J_{E_{\lambda(\alpha)}, C^0}\big(\phi_2(\alpha)\big)$ and $\Phi(\alpha) = \big(\Phi_1(\alpha), \Phi_2(\alpha)\big)$.
  Hence, the identity

  $$\Phi(\alpha)^t(s)\Phi(\alpha)(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  holds for $\alpha \in I$ and $s \in [0, T]$.
  Analogously, for $\alpha = \alpha_0$, $\Phi(\alpha_0) = \big(\Phi_1(\alpha_0), \Phi_2(\alpha_0)\big)$ where $\Phi_1(\alpha_0) = J_{E_{\lambda(\alpha_0)}, C_T^0}\big(\phi_1(\alpha_0)\big)$ and $\Phi_2(\alpha_0) = J_{E_{\lambda(\alpha_0)}, C_T^0}\big(\phi_2(\alpha_0)\big)$.
  Recall the definition of the mapping $b : \mathbb{R} \to E_{\lambda_0}$ in Theorem 1.4.1.1

which is $b(a) := \phi_1(\alpha_0) \cdot a$, for $a \in \mathbb{R}$.

Then,

$$J_{E_{\lambda_0}, C_T^i}\big(b(a)\big)(s) = \Phi(0)(s)(a,0)^t$$

for $s \in [0,T]$ and $i \in \{0,1,2\}$.

- The mapping

$$I \ni \alpha \mapsto \Phi(\alpha) \in C^0 \times C^0$$

is continuously differentiable.

- There exists a continuously differentiable mapping

$$I \ni \alpha \mapsto B(\alpha) \in \mathbb{R}^{2\times 2}$$

such that the identities

$$\Phi(\alpha)(s) = \Phi(\alpha)(0) \exp\big(B(\alpha)s\big)$$

and

$$\Phi^t(\alpha)(s) = \exp\big(-B(\alpha)s\big)\Phi^t(\alpha)(0)$$

hold for $s \in [0,T]$ and $\alpha \in I$

- The identity

$$B(\alpha) = \begin{pmatrix} \Re\big(\lambda(\alpha)\big) & -\Im\big(\lambda(\alpha)\big) \\ \Im\big(\lambda(\alpha)\big) & \Re\big(\lambda(\alpha)\big) \end{pmatrix}$$

holds for $\alpha \in I$.

Lemma 1.4.2.0.1. *Let the assumptions $\tilde{H}$ 1) - $\tilde{H}$ 6) and L 1) to L 3)be satisfied and the results of Section 1.4.1 be given. If we define*

$$\eta : C_T^0 \to \mathbb{R}^2$$

by

$$\eta(\phi) = \int_0^T \Phi(0)^t(s)\phi(s)ds$$

for $\phi \in C_T^0$ then the mapping

(1.20)

$$\mathbf{\Gamma} : \hat{O} \ni (\alpha, a, \beta) \mapsto \left\{ \begin{array}{ll} \frac{1}{a}\eta\big[\mathbf{G}^2\big(\alpha, u^*(\alpha,a,\beta),\beta\big)\big], & a \neq 0 \\ \\ \eta\big[D_2\mathbf{G}^2\big(\alpha, u^*(\alpha,a,\beta),\beta\big)\big(\big(D_2u^*(\alpha,a,\beta)1\big)\big], & a = 0 \end{array} \right\}$$

is continuously differentiable.

Proof. With no loss of generality let $\phi^* = 0$ and $\alpha_0 = 0$.
In Corollary 1.4.1.1.1 we have seen that the mapping

$$\hat{O} \ni (\alpha, a, \beta) \mapsto \mathbf{G}^2\big(\alpha, u^*(\alpha, a, \beta), \beta\big) \in C^0_T$$

is 2 times continuously differentiable. On the other hand

$$\eta : C^0_T \to \mathbb{R}^2$$

is a bounded linear mapping. Hence, the claim follows from a well known fact:
If $H : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^2$, $\Omega \subset \mathbb{R}^3$ an open subset, is a 2 times continuously differentiable mapping, which satisfies $H(0, y, z) = 0$, for all (y, z) such that $(0, y, z) \in \Omega$, then the modified mapping

$$\Omega \ni (x, y, z) \mapsto \left\{ \begin{array}{ll} \frac{1}{x} H(x, y, z) \in \mathbb{R}^2 & x \neq 0 \\ \\ D_1 H(x, y, z) 1 \in \mathbb{R}^2 & x = 0 \end{array} \right\}$$

is continuously differentiable. □

Lemma 1.4.2.0.2. *Let the assumptions $\tilde{H}$ 1) - $\tilde{H}$ 6) and L 1) to L 3) be satisfied and let the results of Section 1.4.1 be given. Let*

$$\mathbf{\Gamma} : \hat{O} \to \mathbb{R}^2$$

be defined like in the previous lemma. Then $D_3\mathbf{\Gamma}(\alpha_0, 0, 0)1 = \begin{pmatrix} 0 \\ 2\pi \end{pmatrix}$.

Proof. With no loss of generality let $\phi^* = 0$ and $\alpha_0 = 0$.
Let $S \subset (-1/2, 1/2)$ be an open set such that $(0, 0, \beta) \in \hat{O}$ for all $\beta \in S$.
$\hat{u}(\alpha, a, \beta) \in C^1_T$ satisfying the fixed point equation

$$u = J_{E_{\lambda(0)}, C^1_T}\big(b(a)\big) + (\mathbf{K} \circ [Id - \mathbf{J}])^1 \circ \mathbf{G}^0(\alpha, u, \beta)$$

for $(\alpha, a, \beta) \in \hat{O}$ it follows that its derivative $D_2\hat{u}(\alpha, a, \beta)$ with respect to a in every $(\alpha, a, \beta) \in \hat{O}$ is given by

$$D_2\hat{u}(\alpha, a, \beta)1 =$$
$$J_{E_{\lambda(0)}, C^1_T}\big(b'(a)\big) + (\mathbf{K} \circ [Id - \mathbf{J}])^1 \circ D_2\mathbf{G}^0\big(\alpha, \hat{u}(\alpha, a, \beta), \beta\big)\big(D_2\hat{u}(\alpha, a, \beta)1\big).$$

Recall that in Lemma 1.3.1.0.2 we defined

$$\Xi_1^1 : C_T^1 \times (-1/2, 1/2) \times [0,T] \to C_h^1$$

by

$$\Xi_1^1(u, \beta, \tau) := u_{\tau,\beta}$$

for $(u, \beta, \tau) \in C_T^1 \times (-1/2, 1/2) \times [0,T]$. Furthermore, we set $\Xi_0^1 := J_{C_T^1, C_0^T} \circ \Xi_1^1$.

We proved in the same lemma that, for fixed $(\beta, \tau) \in (-1/2, 1/2) \times [0,T]$, Ξ_1^1 and Ξ_0^1 are linear with respect to u and that Ξ_0^1 is continuously differentiable. We observe the identities $u^*(0,0,\beta) = 0$ and $D_{2,ext}\mathbf{g}(0,0,\beta) = (1+\beta)L(0)$, for all $\beta \in S$, as well as the definition of $\mathbf{G}^0$ which is

$$\mathbf{G}^0(\alpha, u, \beta)(s) = \mathbf{g}\big(\alpha, \Xi_1^1(u, \beta, s), \beta\big) - L(0)\Xi_0^1(u, 0, s) = \mathbf{g}(\alpha, u_{s,\beta}, \beta) - L(0)u_s,$$

for $(\alpha, u, \beta) \in I \times \hat{\Omega} \times (-1/2, 1/2)$ and $s \in [0,T]$. Then, by an application of the chain rule which involves the evaluation map in $s \in [0,T]$ it follows that

$$\big(D_2\mathbf{G}^0\big(0, \hat{u}(0,0,\beta), \beta\big)(\phi)\big)(s) = \big(D_2\mathbf{G}^0(0,0,\beta)(\phi)\big)(s) =$$

$$D_2\mathbf{g}(0,0,\beta)\big(\Xi_1^1(\phi,\beta,s)\big) - L(0)\Xi_0^1(\phi,0,s) = D_{2,ext}\mathbf{g}(0,0,\beta)\big(\Xi_0^1(\phi,\beta,s)\big) - L(0)\Xi_0^1(\phi,0,s) =$$

$$(1+\beta)L(0)\big(\Xi_0^1(\phi,\beta,s)\big) - L(0)\phi_s = (1+\beta)L(0)\phi_{s,\beta} - L(0)\phi_s$$

for all $\beta \in S$, $\phi \in C_T^1$ and $s \in [0,T]$. Furthermore, $\big(D_2\hat{u}(0,0,0)1\big)(s) = J_{E_{\lambda(0)}, C_T^1}\big(b'(0)\big)(s) = \Phi(0)(s)(1,0)^t$, for $s \in [0,T]$.

Therefore,

$$\mathbf{\Gamma}(0,0,\beta) = \int_0^T \Phi(0)^t(s)\Big[(1+\beta)L(0)\Xi_0^1\big(D_2\hat{u}(0,0,\beta)1, \beta, s\big) - L(0)\Xi_0^1\big(D_2\hat{u}(0,0,\beta)1, 0, s\big)\Big]ds$$

for all $\beta \in S$. Hence,

$$D_3\mathbf{\Gamma}(0,0,\beta)1 =$$

$$\int_0^T \Phi(0)^t(s)\Big[L(0)\Xi_0^1\big(D_2\hat{u}(0,0,\beta)1, \beta, s\big)\Big]ds +$$

$$\int_0^T \Phi(0)^t(s)\Big[(1+\beta)L(0)\Xi_0^1\big(D_3D_2\hat{u}(0,0,\beta)1,\beta,s\big)+(1+\beta)L(0)D_2\Xi_0^1\big(D_2\hat{u}(0,0,\beta)1,\beta,s\big)\Big]ds$$

$$-\int_0^T \Phi(0)^t(s)L(0)\Xi_0^1\big(D_3D_2\hat{u}(0,0,\beta)(1)(1),0,s\big)\Big]ds$$

for all $\beta \in S$. Thus,

$$D_3\mathbf{\Gamma}(0,0,0)1 = \int_0^T \Phi(0)^t(s)\Big[L(0)\Xi_0^1\big(D_2\hat{u}(0,0,0)1,0,s\big)+L(0)D_2\Xi_0^1\big(D_2\hat{u}(0,0,0)1,0,s\big)]ds.$$

By observing the identities $\big(D_2\hat{u}(0,0,0)1\big)(s) = \Phi(0)(s)(1,0)^t$ and

$$\Xi_0^1\big(D_2\hat{u}(0,0,\beta)1,\beta,s\big) = J_{C_T^1,C_T^0}\big(D_2\hat{u}(0,0,\beta)_{s,\beta}\big) = \Phi(0)_{s,\beta},$$

for $\beta \in S$ and $s \in [0,T]$, it follows that the derivative $\gamma'(0)$ of the mapping

$$\gamma : S \ni \beta \mapsto \int_0^T \Phi(0)^t(s)\Big[(1+\beta)L(0)\Phi_{s,\beta}(0) - L(0)\Phi_s(0)\Big](1,0)^t ds$$

in $\beta = 0$ is equal to $D_3\mathbf{\Gamma}(0,0,0)1$.
(Recall the definition of $\Phi(0)_{s,\beta} \in C_h \times C_h$ which is
$\Phi(0)_{s,\beta}(\theta) = \Phi(0)\big(s+\theta/(1+\beta)\big)$, for $\theta \in [-h,0]$)
Separating the integral in two parts yields

$$\gamma(\beta) = \int_0^T \Phi(0)^t(s)\Big[(1+\beta)L(0)\Phi_{s,\beta}(0)\Big](1,0)^t ds$$

$$-\int_0^T \Big[\Phi(0)^t(s)L(0)\Phi_s(0)\Big](1,0)^t ds$$

for $\beta \in S$. Changing s into $z := s/(1+\beta)$ in the first integral yields

$$\int_0^{T/(1+\beta)} \Phi(0)^t(z)\Big[(1+\beta)^2L(0)\Phi_{z,\beta}(0)\Big](1,0)^t dz \tag{1.21}$$

for $\beta \in S$. By the fact that $\Phi(0) = \big(\Phi_1(0),\Phi_2(0)\big)$ where $\Phi(0)_1, \Phi(0)_2$ denote a basis of the subspace $\mathbf{P} \subset C_T^2$ of T - periodic solutions of

$$\frac{du(s)}{ds} = L(0)u_s$$

it follows that

$$\frac{d\Phi(0)(s)}{ds}(1,0)^t = L(0)\Phi_s(1,0)^t, \quad s \in \mathbb{R}.$$

($\frac{du(s)}{ds}$ means that we derive u with respect to s)
Recall that the segment $\Phi_{s,\beta}(1,0)^t \in C_h^1$ was defined such that

$$\frac{d\Phi(0)(s)}{ds}(1,0)^t = (1+\beta)L(0)\Phi_{s,\beta}(1,0)^t, \quad s \in \mathbb{R}.$$

holds for $\beta \in (-1/2, 1/2)$.
Hence,

$$\frac{d\Phi(0)(z)}{dz}(1,0)^t = L(0)\Phi_{z,\beta}(1,0)^t$$

holds for $z = s/(1+\beta)$.
Thus, we can rewrite (1.21) as

$$\int_0^{T/(1+\beta)} \Phi(0)^t(z)\Big[(1+\beta)^2\frac{d\Phi(0)(z)}{dz}\Big](1,0)^t dz$$

for $\beta \in S$. Changing z back into $s = z(1+\beta)$ yields

$$\int_0^T \Phi(0)^t(s)\Big[(1+\beta)\frac{d\Phi(0)(s)}{ds}\Big](1,0)^t ds.$$

Observing the identities

$$\frac{d\Phi(0)(s)}{ds} = \Phi(0)(0)e^{B(0)s}B(0) = \Phi(0)(s)B(0),$$

and

$$\Phi^t(0)(s)\Phi(0)(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for $s \in [0,T]$, one gets that

$$\gamma(\beta) = \int_0^T \beta \cdot B(0)(1,0)^t ds$$

and

$$\gamma'(\beta) = \int_0^T B(0)(1,0)^t ds$$

for $\beta \in S$.
Therefore, and by observing

$$B(0) = \begin{pmatrix} 0 & -\Im\lambda(0) \\ \\ \Im\lambda(0) & 0 \end{pmatrix},$$

it follows that

$$D_3\mathbf{\Gamma}(0,0,0)1 = \gamma'(0) = \begin{pmatrix} 0 \\ T\Im\lambda(0) \end{pmatrix}.$$

Thus, by observing $\Im\lambda(0) = \omega$ one gets that

$$D_3\mathbf{\Gamma}(0,0,0)1 = \begin{pmatrix} 0 \\ \omega T \end{pmatrix} = \begin{pmatrix} 0 \\ 2\pi \end{pmatrix}.$$

□

Lemma 1.4.2.0.3. *Let the assumptions L1) to L3) be satisfied. Let* $L_2(\alpha)(\chi) := D_2 g^*(\alpha, \phi^*)$ *for* $\alpha \in I$ *and* $\chi \in C_h^2$*. Then* L_2 *defines a continuously differentiable mapping from* I *to* $L(C_h^2|\mathbb{R}^n)$ *and the identity*

$$\Phi(\alpha_0)^t(s) L_2'(\alpha_0)\Phi_s(\alpha_0) = B'(\alpha_0)$$

holds for all $s \in [0,T]$.

Proof. With no loss of generality let $\alpha_0 = 0$.
The fact that L_2 is continuously differentiable is a consequence of H 4).
We recall that for $\alpha \in I$, $\{\phi_1(\alpha), \phi_2(\alpha)\}$ denotes a base of $E_{\lambda(\alpha)}$. Furthermore, $\Phi(\alpha) \in C^0 \times C^0$ is defined by

$$\big(\Phi_1(\alpha), \Phi_2(\alpha)\big) := \Big(J_{E_{\lambda(\alpha)}, C^0}\big(\phi_1(\alpha)\big), J_{E_{\lambda(\alpha)}, C^0}\big(\phi_2(\alpha)\big), \Big)$$

for $\alpha \in I$. By the fact that for $\alpha \in I$, $T(\alpha)(s)$, $s \geq 0$, extends to a group $T(\alpha)(s)$, $s \in \mathbb{R}$, on $E_{\lambda(\alpha)}$, we can extend $\Phi(\alpha) : [0,T] \to \mathbb{R}^n \times \mathbb{R}^n$ to a mapping $\Phi(\alpha)_{ext} : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n$ in the following way:
We set $\Phi(\alpha)_{ext}(s) := \Big(T(\alpha)(s)\big(\phi_1(\alpha)\big), T(\alpha)(s)\big(\phi_2(\alpha)\big)\Big)$, for $s \in \mathbb{R}$, $s \notin [0,T]$, and $\Phi(\alpha)_{ext}(s) := \big(\Phi_1(\alpha), \Phi_2(\alpha)\big)(s)$, for $s \in [0,T]$.

Analogously, for $\alpha \in I$, let $\Phi(\alpha)^t_{ext}(s)$, $s \in \mathbb{R}$ be defined such that the identities $\Phi(\alpha)^t_{ext}(s) = \big(\Phi_1(\alpha), \Phi_2(\alpha)\big)^t(s)$, $s \in [0,T]$, and

$$\Phi(\alpha)^t_{ext}(s)\Phi(\alpha)_{ext}(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$s \in \mathbb{R}$, hold.

Lemma 3.9 in section 7.3 of [3] tells us that the derivative of the mapping $B : I \to \mathbb{R}^{2\times 2}$ satisfies

$$B'(\alpha) = \Phi(\alpha)^t_{ext}(0)L_2'(\alpha)\big(\Phi(\alpha)_{ext}\big)_0$$

in all $\alpha \in I$. This identity is independent from the choice of the basis of $E_{\lambda(\alpha)}$. $E_{\lambda(\alpha)}$ being invariant under $T(\alpha)(s)$, $s \in [0,T]$, $\{\big(\Phi_1(\alpha)_{ext}\big)_s, \big(\Phi_2(\alpha)_{ext}\big)_s\}$, for $s \in [0,T]$, denotes another base of $E_{\lambda(\alpha)}$. Thus, the identity

$$\Phi(\alpha)^t_{ext}(s)L'(\alpha)\big(\Phi(\alpha)_{ext}\big)_s = B'(\alpha)$$

holds for this new base.

As for $\alpha = 0$, $\Phi_1(0) \in C^2_T$ and $\Phi_2(0) \in C^2_T$, the identities $\Phi(0)(s) = \Phi(0)_{ext}(s)$ and $\Phi(0)^t(s) = \Phi(0)^t_{ext}(s)$ hold for $s \in \mathbb{R}$.

Hence,

$$\Phi(0)^t(s)L_2'(0)\Phi(0)_s = B'(0)$$

for $s \in [0,T]$. □

Lemma 1.4.2.0.4. *Let the assumptions $\tilde{H}$ 1) - $\tilde{H}$ 6) and L 1) to L 3)be satisfied and the results of Section 1.4.1 be given. Let*

$$\mathbf{\Gamma} : \hat{O} \to \mathbb{R}^2$$

be defined like in Lemma 1.4.2.0.1. Then the following holds: The partial derivative $D_1\mathbf{\Gamma}(\alpha_0, 0, 0)1$ of $\mathbf{\Gamma}$ in $(\alpha_0, 0, 0)$ is given by

$$D_1\mathbf{\Gamma}(\alpha_0, 0, 0)1 = T \cdot \begin{pmatrix} \Re\big(\lambda'(0)\big) \\ \Im\big(\lambda'(0)\big) \end{pmatrix}.$$

Proof. With no loss of generality let $\alpha_0 = 0$.

Let $\mathbf{B} \subset I$ be an open set such that $(\alpha, 0, 0) \in \hat{O}$ for all $\alpha \in \mathbf{B}$. First we realize that

$$\mathbf{\Gamma}(\alpha, 0, 0) = \int_0^T \Big(\Phi(0)^t(s)L(\alpha)\Xi_0^1\big(D_2\hat{u}(\alpha, 0, 0)1, 0, s\big) - \Phi(0)^t(s)L(0)\Xi_0^1\big(D_2\hat{u}(\alpha, 0, 0)1, 0, s\big)\Big)ds$$

holds for all $\alpha \in \mathbf{B}$. $L_2 : I \to L\big(C_h^2|\mathbb{R}^n\big)$ and $\mathbf{B} \ni \alpha \mapsto D_2\hat{u}(\alpha,0,0)1 \in C_T^1$ are continuously differentiable with $L_2(\alpha) = L(\alpha)|_{C_h^2}$, $\alpha \in I$. Furthermore

$$\Xi_0^1\big(D_2\hat{u}(\alpha,0,0)1,0,s\big) = \Xi_2^2\big(D_2u^*(\alpha,0,0)1,0,s\big)$$

for α in a sufficiently small neighborhood of α_0 and $s \in [0,T]$.
Therefore, and by applying Theorem 3.2.1 in Appendix II one can easily show that the derivative of the mapping $\mathbf{B} \ni \alpha \mapsto \mathbf{\Gamma}(\alpha,0,0) \in \mathbb{R}^2$ in $\alpha \in \mathbf{B}$ is given by

$$D_1\mathbf{\Gamma}(\alpha,0,0)1 = \int_0^T \Phi(0)^t(s)L_2'(\alpha)\Xi_2^2\big(D_2u^*(\alpha,0,0)1,0,s\big)ds+$$

$$\int_0^T\Big(\Phi(0)^t(s)L(\alpha)\Xi_0^1\big(D_1D_2\hat{u}(\alpha,0,0)(1)(1),0,s\big) - \Phi(0)^t(s)L(0)\Xi_0^1\big(D_1D_2\hat{u}(\alpha,0,0)(1)(1),0,s\big)\Big)ds.$$

Hence, by observing $\big(D_2\hat{u}(0,0,0)1\big)(s) = \Phi(0)(s)(1,0)^t$ for $s \in [0,T]$, the identity

$$D_1\mathbf{\Gamma}(0,0,0)1 = \int_0^T \Phi^t(0)(s)L_2'(0)\Phi(0)_s(1,0)^t ds$$

holds.
By applying the previous lemma one gets that $D_1\mathbf{\Gamma}(0,0,0)1 = B'(0)(T,0)^t$.
Thus,

$$D_1\mathbf{\Gamma}(0,0,0) = T\cdot\begin{pmatrix}\Re\big(\lambda'(0)\big)\\ \Im\big(\lambda'(0)\big)\end{pmatrix}.$$

□

Theorem 1.4.2.1. *(Hopf bifurcation)*
Let all assumptions H 1) to H 6) on g be satisfied.
Let all assumptions L 1) to L 3) be satisfied.
Then there exists a continuously differentiable mapping

$$Q \ni a \mapsto \big(\phi(a),\alpha(a),T(a)\big) \in \Omega \times I \times [0,\infty)$$

$Q \subset \mathbb{R}$ an interval, such that the following properties are satisfied:
$0 \in Q$. For every $a \in Q$, there exists a nontrivial periodic solution $x^*(a) : \mathbb{R} \to \mathbb{R}^n$ of the equation above with parameter $\alpha(a)$, period $T(a)$ and $x^*(a)_{t=0} = \phi(a)$. Furthermore, $\alpha(0) = \alpha_0$, $\phi(0) = \phi^*$, $T(0) = T$ and $\lambda(0) = \lambda_0$.

If $\Phi^* : \mathbb{R} \to \mathbb{R}^n$ is a constant function such that $\Phi^*_{t=0} = \phi^*$ then there exists a neighborhood $\mathbf{M}$ of $(\alpha_0, T, \Phi^*) \in I \times \mathbb{R} \times C^1_T$ such that the following holds: Let $\tilde{T} > 0$. If there exists a nontrivial periodic solution $\tilde{x} : \mathbb{R} \to \mathbb{R}^n$ of (1.1) with parameter $\tilde{\alpha}$ and period $\tilde{T}$ such that $(\tilde{\alpha}, \tilde{T}, \tilde{x}) \in \mathbf{M}$, then there must be $a \in Q$ and $b \in \mathbb{R}$ such that $\tilde{x}_{t=b} = \phi(a)$, $\tilde{T} = T(a)$ and $\tilde{\alpha} = \alpha(a)$.

Proof. With no loss of generality let $\phi^* = 0$ and $\alpha_0 = 0$.
g satisfying H 1) to H 6) we know that $\mathbf{g}$ satisfies $\tilde{H}$ 1) to $\tilde{H}$ 6). Therefore, Theorem 1.4.1.1 yields the existence of a 2 times continuously differentiable mapping $\hat{u} : \hat{O} \mapsto C^1_T$ satisfying equation (1.7). On the other hand Lemma 1.4.2.0.2 and Lemma 1.4.2.0.4 show that the partial derivative $D_{(\alpha,\beta)}\mathbf{\Gamma}(0,0,0)$ of $\mathbf{\Gamma} : \hat{O} \mapsto \mathbb{R}^2$ with respect to (α, β) in $(0,0,0)$ is given by

$$D_{(\alpha,\beta)}\mathbf{\Gamma}(0,0,0) = T \cdot \begin{pmatrix} D\Re\big(\lambda(0)\big) & 0 \\ D\Im\big(\lambda(0)\big) & 2\pi/T \end{pmatrix}.$$

As in L 3) we assumed $D\Re(\lambda(0)) \neq 0$ it follows that $D_{(\alpha,\beta)}\mathbf{\Gamma}(0,0,0)$ is invertible. Furthermore, we observe that $\mathbf{\Gamma}(0,0,0) = 0$.
Thus, the assumptions of the Implicit Function Theorem are satisfied. Hence there exists an interval $0 \in Q \subset \mathbb{R}$ and a continuously differentiable mapping $Q \ni a \to \big(\alpha(a), \beta(a)\big) \in I \times (-1/2, 1/2)$ such that the identity $\mathbf{\Gamma}\big(\alpha(a), a, \beta(a)\big) = 0$ holds for all $a \in Q$. Thus,

$$0 = a \cdot \mathbf{\Gamma}\big(\alpha(a), a, \beta(a)\big) =$$

$$\int_0^T \Phi^t(s)\mathbf{g}\big(\alpha(a), \hat{u}\big(\alpha(a), a, \beta(a)\big)_{s,\beta}, \beta(a)\big)ds$$

holds for all $a \in Q$. The last expression belongs to equation (1.8) which now is satisfied. Therefore, $\hat{u}\big(\alpha(a), a, \beta(a)\big)$ is solving both equations (1.7) and (1.8) for all $a \in Q$ and thus solving

$$u'(\tau) = (1 + \beta)g\big(\alpha(a), u_{\tau,\beta}\big), \quad \tau \in \mathbb{R}$$

with period T.
If $x^*(a) : \mathbb{R} \to \mathbb{R}^n$ is defined by

$$x^*(a)(t + \theta) = \hat{u}\big(\alpha(a), a, \beta(a)\big)\big((t + \theta)/[1 + \beta(a)]\big),$$

for $t \in \mathbb{R}$ and $\theta \in [-h, 0]$, then $x^*(a)$ is a solution of equation (1.1) with parameter $\alpha(a)$ and period $T(a) := \big(1 + \beta(a)\big) \cdot T$.

Clearly, $\alpha(0) = 0$, $T(0) = T$ and $x^*(0) = 0$.

Furthermore, $x^*_t(a) = u^*\big(\alpha(a), a, \beta(a)\big)_{\tau,\beta(a)}$ holds for all $t \in \mathbb{R}$ with $t = (1 + \beta(a))\tau$, in particular $x^*_{t=0}(a) = u^*\big(\alpha(a), a, \beta(a)\big)_{\tau=0,\beta(a)}$

We claim that the mapping

$$Q \ni a \mapsto \hat{u}\big(\alpha(a), a, \beta(a)\big)_{0,\beta(a)} \in C^1_h$$

is continuously differentiable:

By observing the properties of the mapping u^* in Theorem 1.4.1.1 one sees that the identity

$$\hat{u}\big(\alpha(a), a, \beta(a)\big)_{0,\beta(a)} = \Xi^2_1\Big(u^*\big(\alpha(a), a, \beta(a)\big), \beta(a), 0\Big)$$

holds for $a \in Q$.

The claim then follows from the fact that the mappings

$$Q \ni a \mapsto u^*\big(\alpha(a), a, \beta(a)\big) \in C^2_T$$

and

$$C^2_T \times (-1/2, 1/2) \times \mathbb{R} \ni (u, \beta, \tau) \mapsto \Xi^2_1(u, \beta, \tau) \in C^1_h$$

are continuously differentiable.

We set $\phi(a) := \hat{u}\big(\alpha(a), a, \beta(a))_{0,\beta(a)}$ for $a \in Q$.

Clearly, the Implicit Function Theorem guaranties the existence of a neighborhood $\tilde{\mathbf{M}}$ of $(0, 0, 0) \in I \times \mathbb{R} \times C^1_h$ such that the following holds:

If there exist $\tilde{\beta} \in (-1/2, 1/2)$, $\tilde{\alpha} \in I$, $b \in \mathbb{R}$ and x^* a periodic solution of (1.1) with parameter $\tilde{\alpha}$, period $(1+\tilde{\beta})T$ and $(\tilde{\alpha}, (1+\tilde{\beta})T, \tilde{x}_{t=b}) \in \tilde{\mathbf{M}}$ and $x^*_{t=b}$ a solution to (1.8), then there must be an $a \in Q$ with $\tilde{\alpha} = \alpha(a)$, $\tilde{\beta} = \beta(a)$ and $\hat{u}(a)(\tau + \theta/(1 + \tilde{\beta})\big) = x^*(t + b + \theta)$, for $t = (1 + \tilde{\beta}) \cdot \tau$, $\theta \in [-h, 0]$.

Hence, there exists a neighborhood $\mathbf{M}$ of $(0, T, 0) \in I \times \mathbb{R} \times C^1_T$ such that the following uniqueness property is satisfied:

If there exists a nontrivial periodic solution $\tilde{x} : \mathbb{R} \to \mathbb{R}^n$ of (1.1) with parameter $\tilde{\alpha}$ and period $\tilde{T}$ such that $(\tilde{\alpha}, \tilde{T}, \tilde{x}) \in \mathbf{M}$, then there must be $a \in Q$ and $b \in \mathbb{R}$ such that $\tilde{x}_{t=b} = \phi(a)$, $\tilde{T} = T(a)$ and $\tilde{\alpha} = \alpha(a)$.

This completes the proof. $\square$

Chapter 2

The robot arm

In this chapter we want to apply our results on Hopf bifurcation to a problem of differential equations with state dependent delay from robotics:
Let D, c and α be nonnegative reals and let $a : \mathbb{R} \to \mathbb{R}$ be a function having continuous first and second derivatives, and satisfying $a(0) = 0$. Consider the following system:

$$
\begin{aligned}
x''(t) &= -\alpha x'(t) + a(\tfrac{c}{2} r(x_t) - D) \\
cr &= x(t-r) + x(t) + 2D.
\end{aligned}
\tag{2.1}
$$

This system was studied in [10] and [11] and models an object (a robot arm) which moves on a ray given by $-D < x$, and regulates its distance x from the position $x = 0$ in the following way:
Signals travel at constant speed c from the object to the reference point at $-D$, are reflected, and then received by the object. At time t the object measures the running time $r(t)$ of the signal emitted at time $t - r(t)$, computes from $r = r(t)$ a position $\hat{x}$ according to

$$\hat{x} = \frac{c}{2} r - D.$$

This gives the true position at least if $x(t) = 0 = x\big(t - r(x_t)\big)$, and then uses Newton's law with force $a(\hat{x})$ and damping to react.

Like in [11] we rewrite the system as

$$x'(t) = y(t)$$

$$y'(t) = -\alpha y(t) + a(\hat{x}_t)$$

$$cr(x_t) = x(t-r) + x(t) + 2D,$$

where $\hat{x}_t = \frac{c}{2} r(x_t) + D$. We set $c = 1$, and $h > 2D$.
For technical reasons we rewrite the spaces $C^0([-h,0]|\mathbf{R^2})$, $C^1([-h,0]|\mathbf{R^2})$ and $C^2([-h,0]|\mathbf{R^2})$ as $C^0([-h,0]|\mathbb{R}) \times C^0([-h,0]|\mathbb{R})$, $C^1([-h,0]|\mathbb{R}) \times C^1([-h,0]|\mathbb{R})$ and $C^2([-h,0]|\mathbb{R}) \times C^2([-h,0]|\mathbb{R})$ respectively. We set

$$C_h^1 := C^1([-h,0]|\mathbb{R}),$$

$$C_h^2 := C^2([-h,0]|\mathbb{R})$$

and

$$C_h := C^0([-h,0]|\mathbb{R}).$$

Lemma 2.0.0.1. *There are open subsets $\Omega \subset C_h^1 \times C_h^1$, $\Omega^* \subset C_h^2 \times C_h^2$ and an open interval $J \subset \mathbb{R}$ such that the following holds:*

$$\Omega^* = \Omega \cap \left(C_h^2 \times C_h^2\right)$$

and $a(\hat{\phi}_x)$ is well defined for all $(\phi_x, \phi_y) \in \Omega$.
The mapping $g : J \times \Omega \to \mathbb{R}^2$ defined by

$$g(\alpha, \phi_x, \phi_y) := \begin{pmatrix} \phi_y(0) \\ -\alpha\phi_y(0) + a(\hat{\phi}_x) \end{pmatrix} \tag{2.2}$$

for $(\alpha, \phi_x, \phi_y) \in \mathbb{R} \times \Omega$ satisfy all properties H1) to H6).

Proof. H1):
Proposition 8 in [11] shows the existence of an open subset $U \subset C_h^1$ such that for every $\phi \in U$ there is an $r(\phi) \in [-h, 0]$ satisfying $r(\phi) = \phi(-r) + \phi(0) + 2D$.

The mapping $r : U \to [0,h]$ is continuously differentiable. The derivative of r with respect to ϕ is given by

$$Dr(\phi)(\chi) = \frac{\chi\big(-r(\phi)\big) + \chi(0)}{1 + \phi'\big(-r(\phi)\big)},$$

for $(\phi, \chi) \in U \times C_h^1$.

Thus, the mapping $U \ni \phi \to a(\hat{\phi}) \in \mathbb{R}$, defined by $a(\hat{\phi}) = \frac{1}{2}r(\phi) + D$, for $\phi \in U$, is continuously differentiable. Hence, by setting $\Omega := U \times C_h^1$ and choosing an open interval $J \subset \mathbb{R}$ the mapping $g : J \times \Omega \to \mathbb{R}^2$ is continuously differentiable. The partial derivative $D_{(x,y)}g(\alpha, \phi_x, \phi_y) \in L(C_h^1 \times C_h^1|\mathbb{R}^2)$ of g with respect to (ϕ_x, ϕ_y) in $(\alpha, \phi_x, \phi_y) \in J \times \Omega$ is given by

$$D_{(x,y)}g(\alpha, \phi_x, \phi_y)(\chi_x, \chi_y) = \begin{pmatrix} \chi_y(0) \\ \\ 1/2 \cdot a'(\hat{\phi}_x) Dr(\phi_x)(\chi_x) - \alpha \chi_y(0), \end{pmatrix}$$

for $(\chi_x, \chi_y) \in C_h^1 \times C_h^1$. Obviously the mapping

$$J \times \Omega \ni (\alpha, \phi_x, \phi_y) \mapsto D_{(x,y)}g(\alpha, \phi_x, \phi_y) \in L(C_h^1 \times C_h^1|\mathbb{R}^2)$$

has a partial derivative

$$D_{\alpha(x,y)}g(\alpha, \phi_x, \phi_y) \in L(C_h^1 \times C_h^1|\mathbb{R}^2)$$

with respect to α in every $\alpha \in J$ and $(\phi_x, \phi_y) \in \Omega$ and the mapping $D_{\alpha(x,y)}g : J \times \Omega \to L(C_h^1 \times C_h^1|\mathbb{R}^2)$ is obviously continuous.

Clearly the right hand side of (2.2) has first and second partial derivatives $D_1 g(\alpha, \phi_x, \phi_y) \in \mathbb{R}^2$ and $D_1^2 g(\alpha, \phi_x, \phi_y) \in \mathbb{R}^2$ with respect to the parameter α in every $(\alpha, \phi_x, \phi_y) \in J \times \Omega$.

The mappings $D_1 g : J \times \Omega \to \mathbb{R}^2$ and $D_1^2 g : J \times \Omega \to \mathbb{R}^2$ are obviously continuous.

H2): It is also remarked in [11] that $D_{(x,y)}g\big(\alpha, \phi_x, \phi_y)\big) \in L(C_h^1 \times C_h^1|\mathbb{R}^2)$ extends to a bounded linear mapping

$$D_{(x,y),ext}g\big(\alpha, \phi_x, \phi_y)\big) : C_h \times C_h \to \mathbb{R}^2$$

for $\alpha \in J$ and $(\phi_x, \phi_y) \in \Omega$:

Dr satisfying

$$Dr(\phi)(\chi) = \frac{\chi\big(-r(\phi)\big) + \chi(0)}{1 + \phi'\big(-r(\phi)\big)},$$

for $(\phi, \chi) \in U \times C_h^1$, $Dr(\phi) \in L(C_h^1|\mathbb{R})$ obviously extends to a linear bounded mapping

$$D_{ext}r(\phi) : C_h \ni \chi \mapsto \frac{\chi\big(-r(\phi)\big) + \chi(0)}{1 + \phi'\big(-r(\phi)\big)} \in \mathbb{R}.$$

Hence, $D_{(x,y)}g(\alpha, \phi_x, \phi_y) \in L(C_h^1 \times C_h^1|\mathbb{R}^2)$ extends to a bounded linear mapping

$$D_{(x,y),ext}g(\alpha, \phi_x, \phi_y) : C_h \times C_h \to \mathbb{R}^2.$$

H3) We claim that the mapping

$$D_{(x,y),ext,1}g^* : J \times \Omega \times C_h \times C_h \ni \big(\alpha, \phi_x, \phi_y, \chi_x, \chi_y\big)$$

$$\mapsto D_{(x,y),ext}g\big(\alpha, \phi_x, \phi_y)\big)(\chi_x, \chi_y) \in \mathbb{R}^2$$

is continuous:
Obviously the mapping

$$U \times C_h \ni (\phi, \chi) \mapsto Dr(\phi)(\chi) = \frac{\chi\big(-r(\phi)\big) + \chi(0)}{1 + \phi'\big(-r(\phi)\big)}$$

is continuous. Hence, by the continuity of $U \ni \phi \mapsto a(\hat{\phi}) \in \mathbb{R}$ and by the fact that $C_h \ni \chi \to \chi(0) \in \mathbb{R}$ is continuous the mapping

$$J \times \Omega \times C_h \times C_h \ni \big(\alpha, \phi_x, \phi_y, \chi_x, \chi_y\big) \mapsto \begin{pmatrix} \chi_y(0) \\ \\ 1/2 \cdot a'(\hat{\phi}_x) Dr(\phi_x)(\chi_x) - \alpha\chi_y(0) \end{pmatrix} \in \mathbb{R}^2$$

is continuous.
H4) We set $\Omega^* := \Omega \cap (C_h^2 \times C_h^2)$. We have to prove the existence of a second partial derivative $D^2_{(x,y)}g^*(\alpha, \phi_x, \phi_y) \in L^2(C_h^2 \times C_h^2|\mathbb{R}^2)$ of $g^* = g|_{J \times \Omega^*}$ with respect to (ϕ_x, ϕ_y) in every $(\alpha, \phi_x, \phi_y) \in J \times \Omega^*$.
Furthermore, we have to show that the mapping

$$J \times \Omega^* \ni (\alpha, \phi_x, \phi_y) \mapsto D^2_{(x,y)}g^*(\alpha, \phi_x, \phi_y) \in L^2(C_h^2 \times C_h^2|\mathbb{R}^2)$$

is continuous.
It is obvious that there exists a first partial derivative $D_{(x,y)}g^*(\alpha, \phi_x, \phi_y)$

of g^* with respect to (ϕ_x, ϕ_y) in every $(\alpha, \phi_x, \phi_y) \in J \times \Omega^*$ and that it is given the same way as the first partial derivative $D_{(x,y)}g(\alpha, \phi_x, \phi_y)$ of g with respect to (ϕ_x, ϕ_y) in $(\alpha, \phi_x, \phi_y) \in J \times \Omega$. Furthermore, g^* has first and second partial derivatives $D_1 g^*(\alpha, \phi_x, \phi_y) \in \mathbb{R}^2$ and $D_1^2 g^*(\alpha, \phi_x, \phi_y) \in \mathbb{R}^2$ with respect to the parameter α in every $(\alpha, \phi_x, \phi_y) \in J \times \Omega^*$.
The mappings $D_1 g^* : J \times \Omega^* \to \mathbb{R}^2$ and $D_1^2 g^* : J \times \Omega^* \to \mathbb{R}^2$ are obviously continuous.
Therefore, we have to concentrate on the existence of a second partial derivative $D^2_{(x,y)} g^*(\alpha, \phi_x, \phi_y) \in L^2(C_h^2 \times C_h^2 | \mathbb{R}^2)$ and the continuity of

$$D^2_{(x,y)} g^* : J \times \Omega^* \to L^2(C_h^2 \times C_h^2 | \mathbb{R}^2).$$

Let $Q^* := U \cap C_h^2$.
We first show that $r^* := r|_{Q^*}$ is 2 times continuously differentiable. Clearly r^* is continuously differentiable with its derivative being given by

$$Dr^*(\phi)(\chi) = \frac{\chi\big(-r(\phi)\big) + \chi(0)}{1 + \phi'\big(-r(\phi)\big)},$$

for $\phi \in Q^*$ and $\chi \in C_h^2$
Hence, we have to prove the existence of a derivative $D^2 r^*(\phi) \in L^2(C_h^2 | \mathbb{R})$ of

$$Dr^* : Q^* \to L\big(C_h^2 | \mathbb{R}\big). \tag{2.3}$$

in every $\phi \in Q^*$ and, furthermore, we have to show the continuity of

$$Q^* \ni \phi \mapsto D^2 r^*(\phi) \in L^2(C_h^2 | \mathbb{R}).$$

In Lemma 1.3.1.0.1 we have seen that for any interval $I \subset \mathbb{R}$ the mapping

$$H^2 : C_T^2 \times I \ni (u, s) \mapsto u(s) \in \mathbb{R}^n$$

is 2 times continuously differentiable with $D_1 H^2(u,s)(v) = v(s)$ and $D_2 D_1 H^2(u,s)(v)(1) = v'(s)$ for $(u, s) \in C_T^2 \times I$ and $v \in C_T^2$. Analogously one can show that the mapping

$$\mathbb{H} : C_h^2 \times [-h, 0] \ni (\psi, s) \mapsto \psi(s) \in \mathbb{R}^n$$

is continuously differentiable with $D_1\mathbb{H}(\psi,s)(\chi) = \chi(s)$ and $D_2D_1\mathbb{H}(\psi,s)(\chi)(1) = \chi'(s)$ for $(\chi,s) \in C_h^2 \times [-h,0]$ and $v \in C_T^2$.
Hence, the mapping

$$Ev : [-h,0] \to L\big(C_h^2 \big| \mathbb{R}\big)$$

defined by $Ev(s)(\chi) := D_1\mathbb{H}(0,s)\chi(-s) = \chi(s)$ for $s \in [-h,0]$ and $\chi \in C_h^2$ is continuously differentiable with

$$Ev'(s)(\chi) = -\chi'(-s).$$

Therefore, an application of the product rule yields that the derivative of (2.3) is given by

$$D^2 r^*(\phi)(\chi,\psi) = \frac{Z^*(\phi)(\chi,\psi)}{N(\phi)} \tag{2.4}$$

for $\phi \in Q^*$, $\chi \in C_h^2$ and $\psi \in C_h^2$, where

(2.5)
$$Z^*(\phi)(\chi,\psi) = -\chi'\big(-r^*(\phi)\big)Dr^*(\phi)(\psi)\cdot\big(1+\phi'\big(-r^*(\phi)\big)\big)$$
$$-\chi\big(-r^*(\phi)\big)\cdot\big[\phi''\big(-r^*(\phi)\big)Dr^*(\phi)(\psi) + \psi'\big(-r^*(\phi)\big)\big]$$

and

$$N(\phi) = \big(1+\phi'\big(-r(\phi)\big)\big)^2.$$

Again, by applying chain - and product rule, the mapping $D_{(x,y)}g^* : J \times \Omega^* \to L(C_h^2 \times C_h^2 | \mathbb{R}^2)$ is continuously differentiable.
The derivative $D^2_{(x,y)}g^*(\alpha,\phi_x,\phi_y)$ of $D_{(x,y)}g^*$ with respect to (ϕ_x,ϕ_y) in $(\alpha,\phi_x,\phi_y) \in J \times \Omega^*$ is given by

$$D^2_{(x,y)}g^*(\alpha,\phi_x,\phi_y)(\chi_x,\chi_y)(\psi_x,\psi_y) = \begin{pmatrix} 0 \\ \\ M^*(\phi_x)(\chi_x)(\psi_x) \end{pmatrix}, \tag{2.6}$$

for $(\chi_x,\chi_y) \in C_h^2 \times C_h^2$ and $(\psi_x,\psi_y) \in C_h^2 \times C_h^2$, where

$$M^*(\phi_x)(\chi_x)(\psi_x) := 1/2\cdot a'(\hat{\phi}_x)D^2r^*(\phi_x)(\chi_x,\psi_x) + 1/4\cdot a''(\hat{\phi}_x)Dr^*(\phi_x)(\chi_x)Dr^*(\phi_x)(\psi_x).$$

H5) Let $J_{2,1}$ denote the continuous embedding from C_h^2 to C_h^1.
Then for every $\phi \in Q^*$ the mapping

$$Z(\phi) : C_h^1 \times C_h^1 \to \mathbb{R}$$

defined by

$$\begin{aligned} Z(\phi)(\chi, \psi) = \quad & -\chi'\big(-r^*(\phi)\big) Dr\big(J_{2,1}(\phi)\big)(\psi) \cdot \big(1 + \phi'\big(-r^*(\phi)\big)\big) \\ & -\chi\big(-r^*(\phi)\big) \cdot \big[\phi''\big(-r^*(\phi)\big) Dr\big(J_{2,1}(\phi)\big)(\psi) + \psi'\big(-r^*(\phi)\big)\big] \end{aligned}$$

for $(\chi, \psi) \in C_h^1 \times C_h^1$, is continuous.
Obviously, $Z(\phi)\big(J_{2,1}(\chi)\big)\big(J_{2,1}(\psi)\big) = Z^*(\phi)(\chi)(\psi)$ for all $\phi \in Q^*$ and $(\psi, \chi) \in C_h^2 \times C_h^2$.
Hence, for every $\phi \in Q^*$, $D^2 r^*(\phi) \in L^2(C_h^2|\mathbb{R})$ extends to a continuous bilinear mapping

$$D_{ext}^2 r^*(\phi) : C_h^1 \times C_h^1 \to \mathbb{R}$$

which is given by

$$D_{ext}^2 r^*(\phi)(\chi, \psi) = \frac{Z(\phi)(\chi)(\psi)}{N(\phi)}$$

for $\phi \in Q^*$, $\chi \in C_h^1$ and $\psi \in C_h^1$.
Next, for $\phi \in Q^*$, we define the continuous bilinear mapping

$$M(\phi) : C_h^1 \times C_h^1 \to \mathbb{R}$$

by

$$M(\phi)(\chi)(\psi) := 1/2 \cdot a'(\hat{\phi}) D_{ext}^2 r^*(\phi)(\chi, \psi) + 1/4 \cdot a''(\hat{\phi}) Dr\big(J_{2,1}(\phi)\big)(\chi) Dr\big(J_{2,1}(\phi)\big)(\psi),$$

for $\chi \in C_h^1$ and $\psi \in C_h^1$.
Clearly, $M(\phi)\big(J_{2,1}(\chi)\big)\big(J_{2,1}(\psi)\big) = M^*(\phi)(\chi)(\psi)$ for all $\phi \in Q^*$ and $(\psi, \chi) \in C_h^2 \times C_h^2$.
Therefore, $D_{(x,y)}^2 g^*(\alpha, \phi_x, \phi_y) \in L(C_h^2 \times C_h^2|\mathbb{R}^2)$ extends to a continuous bilinear mapping

$$D_{(x,y),ext}^2 g^*(\alpha, \phi_x, \phi_y) : (C_h^1 \times C_h^1) \times (C_h^1 \times C_h^1) \to \mathbb{R}^2,$$

for $(\alpha, \phi_x, \phi_y) \in J \times \Omega^*$, which is given by

$$D^2_{(x,y),ext} g^*(\alpha, \phi_x, \phi_y)(\chi_x, \chi_y)(\psi_x, \psi_y) = \begin{pmatrix} 0 \\ \\ M(\phi_x)(\chi_x)(\psi_x) \end{pmatrix},$$

for $(\chi_x, \chi_y) \in C^1_h \times C^1_h$ and $(\psi_x, \psi_y) \in C^1_h \times C^1_h$.

H6) All mappings

$$Q^* \times C^1_h \ni (\phi, \chi) \mapsto -\chi'\big(-r^*(\phi)\big) \in \mathbb{R},$$

$$Q^* \times C^1_h \ni (\phi, \chi) \mapsto -\chi\big(-r^*(\phi_x)\big) \in \mathbb{R},$$

$$Q^* \ni \phi \mapsto \phi'\big(-r^*(\phi)\big) \in \mathbb{R},$$

$$Q^* \ni \phi \mapsto \phi''\big(-r^*(\phi)\big) \in \mathbb{R}$$

and

$$Q^* \times C^1_h \ni (\phi, \chi) \mapsto Dr\big(J_{2,1}(\phi)\big)(\chi) \in \mathbb{R}$$

are continuous. Therefore, the mapping

$$Q^* \times C^1_h \times C^1_h \ni (\phi, \chi, \psi) \mapsto Z(\phi)(\chi, \psi) \in \mathbb{R}$$

is continuous.

Observing the continuity of

$$Q^* \ni \phi \mapsto N(\phi) \in \mathbb{R}$$

the mapping

$$Q^* \times C^1_h \times C^1_h \ni (\phi, \chi, \psi) \mapsto D^2_{ext} r^*(\phi)(\chi)(\psi) \in \mathbb{R}$$

is continuous and, consequently, the mapping

$$Q^* \times C^1_h \times C^1_h \ni (\chi, \psi) \mapsto M(\phi)(\chi)(\psi) \in \mathbb{R}$$

is continuous. Hence,

$$J \times \Omega^* \times \left(C^1_h\right)^4 \ni (\alpha, \phi_x, \phi_y, \chi_x, \chi_y, \psi_x, \psi_y) \mapsto$$

$$D^2_{(x,y),ext} g^*(\alpha, \phi_x, \phi_y)(\chi_x, \chi_y)(\psi_x, \psi_y) \in \mathbb{R}^2$$

is continuous.

We know that

$$D_{ext}^2 r^*(\phi)(\chi,\psi) = \frac{Z(\phi)(\chi)(\psi)}{N(\phi)}$$

holds for all $(\phi,\chi,\psi) \in Q^* \times C_h^1 \times C_h^1$. On the other hand

$$\begin{aligned} Z(\phi)(\chi)(\psi) = \; & -\chi'\big(-r^*(\phi)\big)Dr\big(J_{2,1}(\phi)\big)(\psi)\cdot\big(c+\phi'\big(-r^*(\phi)\big)\big) \\ & -\chi\big(-r^*(\phi)\big)\cdot\big[\phi''\big(-r^*(\phi)\big)Dr\big(J_{2,1}(\phi)\big)(\psi)+\psi'\big(-r^*(\phi)\big)\big] \end{aligned}$$

holds for $(\phi,\chi,\psi) \in Q^* \times C_h^1 \times C_h^1$.

Then,

$$Z^{**} : Q^* \times C_h^1 \to L\big(C_h^2|\mathbb{R}\big),$$

defined, by

$$\begin{aligned} Z^{**}(\phi)(\chi)(\psi) := \; & -\chi'\big(-r^*(\phi)\big)Dr^*(\phi)(\psi)\cdot\big(c+\phi'\big(-r^*(\phi)\big)\big) \\ & -\chi\big(-r^*(\phi)\big)\cdot\big[\phi''\big(-r^*(\phi)\big)Dr^*(\phi)(\psi) - Ev'(\psi)\big] \end{aligned}$$

for $(\phi,\chi) \in Q^* \times C_h^1$ and $\psi \in C_h^2$, is continuous.

Clearly, $Z^{**}(\phi)(\chi)(\psi) = Z(\phi)(\chi)\big(J_{2,1}(\psi)\big)$ for $(\phi,\chi) \in Q^* \times C_h^1$ and $\psi \in C_h^2$.

Observing the continuity of

$$Q^* \ni \phi \mapsto N(\phi) \in \mathbb{R}$$

the mapping

$$D_{ext,1}^* r^* : Q^* \times C_h^1 \ni (\phi,\chi) \mapsto$$

$$\tfrac{Z^{**}(\phi)(\chi)}{N(\phi)} \in L\big(C_h^2|\mathbb{R}\big)$$

is continuous.

Therefore, the mapping

$$M^{**} : Q^* \times C_h^1 \to L\big(C_h^2|\mathbb{R}\big)$$

defined by

$$M^{**}(\phi)(\chi) := 1/2 \cdot a'(\hat{\phi})D_{ext,1}^2 r^*(\phi)(\chi) + 1/4 \cdot a''(\hat{\phi})Dr\big(J_{2,1}(\phi)\big)(\chi)Dr^*(\phi),$$

for $(\phi,\chi) \in Q^* \times C_h^1$, is continuous. The identity $M^{**}(\phi)(\chi)(\psi) = M(\phi)(\chi)\big(J_{2,1}(\psi)\big)$ holds for all $(\phi,\chi) \in Q^* \times C_h^1$ and $\psi \in C_h^2$.
Hence, the mapping

$$D^2_{(x,y),ext,1}g^* : J \times \Omega^* \times C_h^1 \times C_h^1 \to L\Big(\big(C_h^2\big)^2 \Big| \mathbb{R}^2\Big),$$

defined by

$$\begin{pmatrix} 0 \\ \\ M^{**}(\phi_x)(\chi_x)(\psi_x) \end{pmatrix},$$

for $(\alpha,\phi_x,\phi_y,\chi_x,\chi_y) \in J \times \Omega^* \times C_h^1 \times C_h^1$ and $(\psi_x,\psi_y) \in C_h^2 \times C_h^2$, is continuous.
It is clear that

$$D^2_{(x,y),ext,1}g^*(\alpha,\phi_x,\phi_y)(\chi_x,\chi_y)(\psi_x,\psi_y) =$$

$$D^2_{(x,y),ext}g^*(\alpha,\phi_x,\phi_y)(\chi_x,\chi_y)\big(J_{2,1}(\psi_x), J_{2,1}(\psi_y)\big)$$

for $(\alpha,\phi_x,\phi_y,\chi_x,\chi_y) \in J \times \Omega^* \times C_h^1 \times C_h^1$ and $(\psi_x,\psi_y) \in C_h^2 \times C_h^2$. □

Lemma 2.0.0.2. *If a satisfies $a'(0) = -\pi^2/8D^2$ and $J \subset \mathbb{R}$ is an interval such that $\alpha_0 := \pi/4D \in J$, then the linearization*

$$L(\alpha) := D_{(x,y),ext}g(\alpha,0,0)$$

of the right hand side of (2.2) in the equilibrium $(\phi_x^*,\phi_y^*) = (0,0)$ satisfies all assumptions L1) to L3) for $\alpha \in J$.

Proof. We have to find the solutions of the characteristic equation of the robot - problem. The linearization of the right hand side of (2.2) in $(0,0)$ is given by

$$L(\alpha)(\phi_x,\phi_y) = \begin{pmatrix} \phi_y(0) \\ \\ 1/2 \cdot a'(0)\big(\phi_x(-2D) + \phi_x(0)\big) - \alpha\phi_y(0) \end{pmatrix}$$

for $\alpha \in J$ and $(\phi_x,\phi_y) \in C_h^1 \times C_h^1$.
The characteristic function is given by

$$char(z,\alpha) = z^2 + \alpha z - 1/2 \cdot a'(0)\big(\exp(-2Dz) + 1\big)$$

for $\alpha \in J$ and $z \in \mathbb{C}$.
Plugging $z = k \cdot \pi/4D \cdot i$, $a'(0) = -\pi^2/8D^2$ and $\alpha_0 = \pi/4D$ into the characteristic functions yields

$$char(z, \alpha) = \frac{\pi^2}{16D^2}(-k^2 + ki - i + 1), \quad k \in \mathbb{N}.$$

The right hand side of the last expression is not equal to zero unless $k = 1$. Hence, $z_0 = \pi/4D \cdot i$ is a purely imaginary root of the characteristic equation $char(z, \alpha) = 0$ given the parameter $\alpha_0 = \pi/4D$ and $a'(0) = -\pi^2/8D^2$ and there is no $k \in \mathbb{N}$, $k > 1$, such that $k \cdot \pi/4D \cdot i$ is another solution of the characteristic equation:
The characteristic function $char : \mathbb{C} \times J \to \mathbb{C}$ is continuously differentiable. By deriving $char$ with respect to z one gets

$$D_1 char(z, \alpha) = 2z + \alpha + 2D \cdot 1/2 \cdot a'(0) \exp(-2Dz)$$

for $z \in \mathbb{C}$ and $\alpha \in J$. Substituting α_0, $a'(0)$ and z_0 as above yields

$$D_1 char(\pi/4D \cdot i, \pi/4D) = \pi/4D + (\pi/2D + \pi^2/8D)i \neq 0.$$

Therefore, the Implicit Function Theorem yields the existence of an interval $I \subset J$ and a parametrization $I \ni \alpha \to z(\alpha) \in \mathbb{C}$ such that $\alpha_0 \in I$, $z(\alpha_0) = z_0$ and $z(\alpha)$ is a simple root of the characteristic equation for all $\alpha \in I$.
It remains to investigate whether the derivative z' in $\alpha_0 = \pi/4D$ has a non-vanishing real part. Replacing z with $z(\alpha)$ for all $\alpha \in I$ in the characteristic equation and differentiating it with respect to alpha one gets

$$D_1 char\big(z(\alpha), \alpha\big) \cdot z'(\alpha) + D_2 char\big(z(\alpha), \alpha\big) =$$

$$z'(\alpha) \cdot \big(2z(\alpha) + \alpha + 2D \cdot 1/2 \cdot a'(0) \exp\big(-2Dz(\alpha)\big)\big) + z(\alpha) = 0.$$

Plugging in $\alpha_0 = \pi/4D$, $a'(0) = -\pi^2/8D^2$ and $z_0 = \pi/4D \cdot i$ and solving this equation for z' yields

$$z'(\pi/4D) = \frac{-\pi^2/8D^2 - \pi^3/32D^2 - \pi^2/16D^2 \cdot i}{|D_1 char(\pi/4D, \pi/4D \cdot i)|^2}$$

whose real part does not vanish. Therefore, the robot problem satisfies all assumptions L1) to L3). □

Theorem 2.0.1. *Suppose that the results of the previous lemma are given. Then the robot - problem (2.1) has a Hopf bifurcation from the equilibrium* $(0,0)$ *at parameter* $\alpha = \pi/4D$.

Proof. We have seen that the right hand side of (2.2) satisfies all assumptions H1) to H6). On the other hand by setting $\alpha_0 = \pi/4D$, $a'(0) = -\pi^2/8D^2$ and $z_0 = \pi/4D \cdot i$ the previous lemma shows that the linearization $L(\alpha)$ in the equilibrium $(0,0) \in C_h^1 \times C_h^1$ satisfies all assumptions L1) to L3). Hence, the Hopf Bifurcation Theorem (Theorem 1.4.2.1) can be applied which yields an open interval $Q \subset \mathbb{R}$ such that $0 \in Q$ and a continuously differentiable mapping

$$Q \ni b \mapsto \big(\phi_x(b), \phi_y(b), \alpha(b), T(b)\big) \in C_h^1 \times C_h^1 \times I \times \mathbb{R}$$

such that $\big(\phi_x(0), \phi_y(0)\big) = (0,0)$ and $T(0) = 8D$ hold. Hence, for all $b \in Q$, there exists a periodic solution $\big(x^*(b), y^*(b)\big) : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ of the robot problem with parameter $\alpha(b)$ and period $T(b)$. Furthermore, $\big(x^*(b), y^*(b)\big)_{t=0} = \big(\phi_x(b), \phi_y(b)\big)$ for all $b \in Q$ and $\big(x^*(0), y^*(0)\big)_{t=0} = (0,0)$. $\square$

Chapter 3

Appendix

3.1 Appendix I: Smoothness of the substitution operator

Theorem 3.1.1. *Let X, Y be Banach spaces, $[a,b] \subset \mathbb{R}$ an interval, $U \subset X$ an open subset and $f : X \times [a,b] \to Y$ a mapping:*
Let $C([a,b],Y)$ be the space of continuous functions $\phi : [a,b] \to Y$ equipped with the norm $\|\phi\|_{C([a,b]|Y)} := \sup\limits_{s\in[a,b]} \|\phi(s)\|_Y$, for $\phi \in C([a,b]|Y)$ and suppose f is continuous. If we set $F(x)(t) := f(x,t)$ for $x \in X$ and $t \in [a,b]$ then F defines a mapping from X to $C([a,b],Y)$.
Let $C^1([a,b],Y)$ be the space of continuously differentiable functions $\phi : [a,b] \to Y$ with the norm

$$\|\phi\|_{C^1([a,b],Y)} := \max\{\|\phi\|_{C([a,b],Y)}, \|\phi'\|_{C([a,b],Y)}\}$$

for $\phi \in C^1([a,b],Y)$ and suppose f is continuously differentiable:
If we set $\tilde{F}(x)(t) := f(x,t)$ for $x \in X$ and $t \in [a,b]$ then $\tilde{F}$ defines a mapping from X to $C^1([a,b],Y)$ with $\big(\tilde{F}(x)\big)'(t) := D_2 f(x,t)1$ for $x \in X$ and $t \in [a,b]$.
Let $C^2([a,b],Y)$ be the space of continuously differentiable functions $\phi : [a,b] \to Y$ with the norm

$$\|\phi\|_{C^1([a,b],Y)} := \max\{\|\phi\|_{C([a,b],Y)}, \|\phi'\|_{C([a,b],Y)}, \|\phi''\|_{C([a,b],Y)}\}$$

for $\phi \in C^2([a,b],Y)$ and suppose f is 2 times continuously differentiable.
If we set $\hat{F}(x)(t) := f(x,t)$ for $x \in X$ and $t \in [a,b]$ then $\hat{F}$ defines a mapping

from X to $C^1([a,b],Y)$ with $\big(\hat{F}(x)\big)'(t) := D_2 f(x,t)1$ and $\big(\hat{F}(x)\big)''(t) := D_2^2 f(x,t)(1)(1)$, for $x \in X$ and $t \in [a,b]$.

Furthermore, the following results hold:

1. *If f is continuous then F is continuous.*
2. *If f is continuously differentiable then F is continuously differentiable*
3. *If f is continuously differentiable then $\tilde{F}$ is continuous.*
4. *If f is 2 times continuously differentiable then F is 2 times continuously differentiable.*
5. *If f is 2 times continuously differentiable then $\tilde{F}$ is continuously differentiable.*
6. *If f is 2 times continuously differentiable then $\hat{F}$ is continuous.*

Proof. With no loss of generality let $U := X$.
For the proof of 1., 2. and 4. we apply Lemma 1.5 in [2], Appendix IV, which states the following:
Let E and E' be Banach spaces, $I \subset \mathbb{R}$ a compact interval. Let $C(I|E)$ and $C(I|E')$ be the spaces of continuous mappings $\phi : I \to E$ and $\phi : I \to E'$, equipped with the supremum -norm. Let $g : E \to E'$ be a k times continuously differentiable mapping.
Then $g \circ \phi$, for $\phi \in C(I|E)$ defines a continuous mapping from I to E'. The substitution operator $G : C(I|E) \to C(I|E')$, defined by $G(\phi) := g \circ \phi$, for $\phi \in C(I|E)$, is k times continuously differentiable.
We set $E' := Y$, $E := X \times \mathbb{R}$, $I := [a,b]$ and $g := f$. The mapping $\phi : X \to C(I|E)$, defined by $\phi(x)(t) := (x,t)$, for $x \in X$ and $t \in I$, is linear and bounded.
Hence, Lemma 1.5 in [2], Appendix IV, yields that $g \circ \phi(x)$ is an element of $C([a,b]|Y)$ for $x \in X$ and that the mapping $F : X \to C([a,b]|Y)$, defined by

$$F(x)(t) := \big(g \circ \phi(x)\big)(t), \quad t \in [a,b],$$

for $x \in X$, is continuous, continuously differentiable, 2 times continuously differentiable, if f is continuous, continuously differentiable, 2 times continuously differentiable, respectively, with

$$\big(DF(x)(h)\big)(t) := \Big(D\big(g \circ \phi(x)(\cdot)\big)(h, \cdot)\Big)(t) = D_1 f(x,t)(h), \quad t \in [a,b],$$

and

$$\big(D^2 F(x)(h)\big)(t) := \Big(D^2\big(g \circ \phi(x)(\cdot)\big)(h_1, \cdot)(h_2, \cdot)\Big)(t) = D_1^2 f(x,t)(h_1)(h_2), \quad t \in [a,b],$$

for $(x, h, h_1, h_2) \in X^4$.

5. First we claim that $\tilde{F}(x)$, $x \in X$ is an element of $C^1([a,b]|Y)$. Again, we set $E' := Y$, $E := X \times \mathbb{R}$, $I := [a,b]$, $g := f$ and $\phi : X \to C(I|E)$, by $\phi(x)(t) := (x,t)$, for $x \in X$ and $t \in I$.

Then Lemma 1.5. in [2], Appendix IV, yields that $g \circ \phi(x)$ is an element of $C([a,b]|Y)$. Deriving $\tilde{F}(x) = g \circ \phi(x)$ with respect to t yields

$$\big(\tilde{F}(x)\big)'(t) := Dg\big(\phi(x)(t)\big)\Big(\big(\phi(x)\big)'(t)\Big) = D_2 f(x,t)1, \quad t \in [a,b].$$

Analogously to 1., 2. and 4. one may now apply Lemma 1.5. in [2], Appendix IV, to get that

$$D\big(g \circ \phi(x)(\cdot)\big)\Big(\big(\phi(x)\big)'(\cdot)\Big)$$

is an element of $C([a,b]|Y)$.

Hence $\tilde{F}(x)$ is an element of $C^1([a,b]|Y)$ with $\big(\tilde{F}(x)\big)'(t) := D_2 f(x,t)1$ for $x \in X$ and $t \in [a,b]$.

Again, analogously to 1., 2. and 4. we can show that

$$X \ni x \mapsto D\big(g \circ \phi(x)(\cdot)\big)\Big(\big(\phi(x)\big)'(\cdot)\Big) = D_2 f(x,\cdot)1 \in C([a,b]|Y)$$

is continuously differentiable with

$$D\Big[D\big(g \circ \phi(x)(\cdot)\big)\Big(\big(\phi(x)\big)'(\cdot)\Big)\Big](h) = D_1 D_2 f(x,\cdot)(h)1$$

for $(x,h) \in X^2$.

Therefore, $\tilde{F} : X \to C^1([a,b]|Y)$ is continuously differentiable with

$$D\tilde{F}(x)(v)(t) = D_1 f(x,t)(v)$$

and

$$\big(D\tilde{F}(x)(v)\big)'(t) = D_2 D_1 f(x,t)(v)(1)$$

for $v \in X$ and $t \in [a,b]$.

The proof of 3. and 6. is similar to the proof of 5. □

3.2 Appendix II: Chain rule for mappings with restricted differentiability conditions

Theorem 3.2.1. *Let X, Y, Y', Z be Banach spaces, let Y be continuously embedded in Y' and $A_0 \subset X$ and $A_2 \subset Y$ and $A_1 \subset Y'$ be open subsets such that $A_2 = A_1 \cap Y$.*
Let the mapping $\tilde{K} : A_1 \subset Y' \to Z$ be continuously differentiable and let the restriction $K := \tilde{K}|_{A_2} : A_2 \subset Y \to Z$ be 2 times continuously differentiable.
Let the mapping $u : A_0 \subset X \to A_2 \subset Y$ be continuously differentiable.
Let $J_{Y,Y'}$ denote the continuous embedding from Y to Y' and let mapping $\tilde{u} := J_{Y,Y'} \circ u$ be 2 times continuously differentiable.
Then the mapping $K \circ u : A_0 \to Z$ is 2 times continuously differentiable.

Proof. Clearly by applying the chain rule the first derivative of $K \circ u$ in $x \in A_0$ is given by $DK\big(u(x)\big)\big(Du(x)\big) \in L(X|Z)$ and is continuous with respect to x.
Now we have to show that the mapping

$$A_0 \subset X \ni x \to DK\big(u(x)\big)\big(Du(x)\big) \in L(X|Z)$$

is continuously differentiable.
We prove this in two steps:
First step:
We define $\mathbb{A} : L(Y|Z) \times L(X|Y) \to L(X|Z) \mapsto$ by $\mathbb{A}(T, S) := T \circ S$ for $(T, S) \in L(Y|Z) \times L(X|Y)$ and $\mathbb{B} : (A_0 \times A_0) \subset (X \times X) \to L(Y|Z) \times L(X|Y)$ by $\Big(DK\big(u(x)\big), Du(x')\Big)$ for $(x, x') \in A_0 \times A_0$.
We show that the mapping

$$\mathbb{A} \circ \mathbb{B} : (A_0 \times A_0) \subset (X \times X) \to L(X|Z)$$

has a partial derivative with respect to x in every $(x, x') \in A_0 \times A_0$ which is given by $D_1(\mathbb{A} \circ \mathbb{B})(x, x') = D^2K\big(u(x)\big)\big(Du(x)\big)\big(Du(x')\big) \in L\big(X\big|L(X|Z)\big)$.
Furthermore, we show that the mapping

$$D_1(\mathbb{A} \circ \mathbb{B}) : (A_0 \times A_0) \subset (X \times X) \to L\big(X\big|L(X|Z)\big)$$

is continuous.
On one hand $K : A_2 \subset Y \to Z$ is 2 times continuously differentiable. Thus,

the mapping $A_2 \subset Y \ni y \mapsto DK(y) \in L(Y|Z)$ has a derivative with respect to y in every $y \in A_2$ which is given by $D^2K(y) \in L^2(Y|Z)$ for $y \in A_2$.
$u : A_0 \subset X \to Y$ being continuously differentiable it is clear that the mapping

$$A_0 \subset X \ni x \mapsto DK\big(u(x)\big) \in L(Y|Z)$$

has a derivative with respect to x in every $x \in A_0$ given by $D^2K\big(u(x)\big)\big(Du(x)\big) \in L\big(X\big|L(Y|Z)\big)$. On the other hand $Du(x') \in L(X|Y)$ for all $x' \in A_0$.
Thus, an application of the chain rule yields that the mapping

$$\mathbb{A} \circ \mathbb{B} : (A_0 \times A_0) \subset (X \times X) \to L(X|Z)$$

has a partial derivative with respect to x in every $(x, x') \in A_0 \times A_0$ which is given by

$$D_1(\mathbb{A} \circ \mathbb{B})(x, x') = D^2K\big(u(x)\big)\big(Du(x)\big)\big((Du(x')\big) \in L\big(X\big|L(X|Z)\big).$$

for $(x, x') \in A_0 \times A_0$. The continuity of

$$D_1(\mathbb{A} \circ \mathbb{B}) : (A_0 \times A_0) \subset (X \times X) \to L\big(X\big|L(X|Z)\big)$$

is a consequence of the continuity of

$$A_0 \subset X \ni x' \mapsto Du(x') \in L(X|Y)$$

and

$$A_0 \subset X \ni x \mapsto D^2K\big(u(x)\big) \in L^2(Y|Z)$$

Second step:
Let $\mathbb{A}$ and $\mathbb{B}$ be defined like in the first step of the proof.
We show that the mapping

$$\mathbb{A} \circ \mathbb{B} : (A_0 \times A_0) \subset (X \times X) \ni (x', x) \to L(X|Z)$$

has a partial derivative with respect to x in every $(x', x) \in A_0 \times A_0$ which is given by $D_2(\mathbb{A} \circ \mathbb{B})(x', x) = D\tilde{K}\big(u(x')\big)\big(D^2u(x)\big) \in L^2(X|Z)$.
Then we show that

$$D_2(\mathbb{A} \circ \mathbb{B}) : (A_0 \times A_0) \subset (X \times X) \to L^2(X|Z)$$

is continuous.

In order to show this we replace DK with $D\tilde{K}$ and u with $\tilde{u}$ and define $\tilde{\mathbb{B}} : (A_0 \times A_0) \subset (X \times X) \to L(Y'|Z) \times L(X|Y')$ by $\tilde{\mathbb{B}}(x', x) := \Big(D\tilde{K}\big(u(x')\big), D\tilde{u}(x)\Big)$ and $\tilde{\mathbb{A}} : L(Y'|Z) \times L(X|Y') \to L(X|Z)$ by $\tilde{\mathbb{A}}(T, S) := T \circ S$ for $(T, S) \in L(Y'|Z) \times L(X|Y')$.

Then the identity $\mathbb{A} \circ \mathbb{B} = \tilde{\mathbb{A}} \circ \tilde{\mathbb{B}}$ holds.

Now we show that

$$\tilde{\mathbb{A}} \circ \tilde{\mathbb{B}} : (A_0 \times A_0) \subset (X \times X) \to L(X|Z)$$

has a partial derivative with respect to x in every $(x, x') \in A_0 \times A_0$. We know that $\tilde{u} : A_0 \subset X \to Y'$ is 2 times continuously differentiable. Therefore, the mapping

$$A_0 \subset X \ni x \mapsto D\tilde{u}(x) \in L(X|Y')$$

has a derivative in $x \in A_0$ which is given by $D^2\tilde{u}(x) \in L^2(X|Y')$.

On the other hand $D\tilde{K}\big(\tilde{u}(x')\big) \in L(Y'|Z)$ for every $x' \in A_0$.

Thus, by an application of the chain rule one gets that the mapping

$$\tilde{\mathbb{A}} \circ \tilde{\mathbb{B}} : (A_0 \times A_0) \subset (X \times X) \to L(X|Z)$$

has a partial derivative with respect to x in every $(x, x') \in A_0 \times A_0$ which is given by

$$D_2(\tilde{\mathbb{A}} \circ \tilde{\mathbb{B}})(x', x) = D\tilde{K}\big(\tilde{u}(x')\big)\big(D^2\tilde{u}(x)\big) \in L^2(X|Z)$$

for $(x, x') \in A_0 \times A_0$. The continuity of

$$D(\tilde{\mathbb{A}} \circ \tilde{\mathbb{B}}) : (A_0 \times A_0) \subset (X \times X) \to L^2(X|Z)$$

is a consequence of the continuity of

$$A_0 \subset X \ni x' \mapsto D\tilde{K}\big(\tilde{u}(x')\big) \in L(Y'|Z)$$

and

$$A_0 \subset X \ni x \mapsto D^2\tilde{u}(x) \in L^2(X|Y').$$

Both steps combined yield that the mapping

$$A_0 \subset X \ni x \to DK\big(u(x)\big)\big(Du(x)\big) \in L(X|Z)$$

is continuously differentiable. □

3.3 Appendix III: A theorem about the existence of second derivatives

In this section we want to state a theorem about existence of second derivatives. For the proof of that theorem we will mainly apply a version of a theorem of van Gils and Vanderbauwhede (to be found as Lemma 2.8 in chapter 5 of [12]) which we present here:

Theorem 3.3.1. *Suppose that Y_0, Y, Y_1, and Λ are Banach spaces and that Y_0 is continuously embedded in Y and Y is continuously embedded in Y_1 with embedding operators $J_0 : Y_0 \to Y$ and $J : Y \to Y_1$, respectively. Assume that a mapping $f : Y \times \Lambda \to Y$ satisfies the following properties:*

1. *$Jf : Y \times \Lambda \to Y_1$ has a partial derivative $D_1(Jf)(y,x) \in L(Y|Y_1)$ with respect to y in every $(y,x) \in Y \times \Lambda$.*
   Furthermore,
   $$D_1(Jf) : Y \times \Lambda \to L(Y|Y_1)$$
   is continuous.
   There exist mappings $f^1 : Y \times \Lambda \to L(Y|Y)$ and $f_1^1 : Y \times \Lambda \to L(Y_1|Y_1)$ such that the identity
   $$D_1(fJ)(y,x) = Jf^1(y,x) = f_1^1(y,x) \circ J, \quad (y,x) \in Y \times \Lambda,$$
   holds.

2. *The mapping $f_0 : Y_0 \times \Lambda \to Y$, defined by*
   $$f_0(y,x) := f\big(J_0(y),x\big), \quad (y,x) \in Y_0 \times \Lambda,$$
   has a partial derivative $D_2 f_0(y,x) \in L(\Lambda|Y)$ with respect to x in every $(y,x) \in Y_0 \times \Lambda$. Furthermore, the mapping
   $$D_2 f_0 : Y_0 \times \Lambda \to L(\Lambda|Y)$$
   is continuous.

3. *There exists a constant $0 \le q < 1$ such that the inequalities*
   $$\|f(y,x) - f(y',x)\|_Y \le q\|y - y'\|_Y,$$

$$\|f^1(y,x)\|_{L(Y|Y)} \leq q$$

and

$$\|f_1^1(y,x)\|_{L(Y_1|Y_1)} \leq q$$

hold for $(y,y') \in Y \times Y$ *and* $x \in \Lambda$.

4. *There exists a continuous mapping* $y_0^* : \Lambda \to Y_0$ *such that for any* $x \in \Lambda$, $y^*(x) := (J_0 \circ y_0)(x)$ *is the unique solution of the fixed point equation* $y = f(y,x)$.

Then the mapping $y^* : \Lambda \to Y$ *is Lipschitz continuous and* $y_1^* := J \circ y^* : \Lambda \to Y_1$ *is continuously differentiable with* $Dy_1^*(x) = J\big(A^*(x)\big)$, $x \in \Lambda$, *where, for each* $x \in \Lambda$, $A^*(x) \in L(\Lambda|Y)$ *is the unique solution of the equation*

$$A = f^1\big(y^*(x),x\big)\big(A\big) + D_2 f_0\big(y_0^*(x),x\big)$$

in $L(\Lambda|Y)$.

Theorem 3.3.2. *Let* Y^1, Y^2, X *be Banach spaces such that* Y^2 *is continuously embedded in* Y^1 *and let* $B_1 \subset X$, $B^2 \subset Y^1$, $B_2^* \subset Y^2$ *be open subsets such that* $B_2^* = Y^2 \cap B_2$. *Let the mapping* $\tilde{K} : (B_1 \times B_2) \subset (X \times Y^1) \to Y^1$ *be continuously differentiable. Suppose that* $\tilde{K}(B_1 \times B_2^*) \subset Y^2$ *and let the induced mapping* $K^* : (B_1 \times B_2^*) \subset (X \times Y^2) \to Y^2$, *defined by* $K^*(x,y) := \tilde{K}(x,y)$ *for* $(x,y) \in B_1 \times B_2^*$, *be continuously differentiable. Let* J_{Y^2,Y^1} *denote the continuous embedding from* Y^2 *to* Y^1 *and let the mapping* $\hat{K} := J_{Y^2,Y^1} \circ K^*$ *be 2 times continuously differentiable.*
Suppose that for both equations $y = K^*(x,y)$ *in* Y^2 *and* $y = \tilde{K}(x,y)$ *in* Y^1 *the following conditions are satisfied in a point* $(x_0,y_0) \in B_1 \times B_2^*$: $\tilde{K}(x_0,y_0) = y_0 = K^*(x_0,y_0)$ *and*
$\|D_2K^*(x_0,y_0)\|_{L(Y^2|Y^2)} < 1$ *and* $\|D_2K(x_0,y_0)\|_{L(Y^1|Y^1)} < 1$ *such that*

$$Id_{Y^2} - D_2K^*(x_0,y_0) : Y^2 \to Y^2$$

and

$$Id_{Y^1} - D_2\tilde{K}(x_0,y_0) : Y^1 \to Y^1$$

are isomorphisms.
Let $O^* \subset B_1$ *and* $\tilde{O} \subset B_1$ *be open subsets and* u^* *and* $\tilde{u}$ *be continuously*

differentiable mappings from O^* to B_2^* and from $\tilde{O}$ to B_2, respectively, with $u^*(x_0) = y_0 = \tilde{u}(x_0)$ and solving $\tilde{K}(x, \tilde{u}(x)) = \tilde{u}(x)$ for all $x \in \tilde{O}$ and $K^*(x, u^*(x)) = u^*(x)$ for all $x \in O^*$ respectively.
Then there exists an open subset $\hat{O} \subset \tilde{O}$ such that the mapping $\hat{u} := \tilde{u}|_{\hat{O}}$ is 2 times continuously differentiable.

Proof. We want to apply the Theorem of van Gils and Vanderbauwede (Theorem 3.3.1) to the fixed point equations for the first derivatives of $\tilde{u}$ and u^*. As the conditions of the Implicit Function Theorem are satisfied we get that the following identities for the derivatives of u^* and $\tilde{u}$,

$$Du^*(x) = D_2K^*\big(x, u^*(x)\big)\big(Du^*(x)\big) + D_1K^*\big(x, u^*(x)\big)$$

in $L(X|Y^2)$ and

$$D\tilde{u}(x) = D_2\tilde{K}\big(x, \tilde{u}(x)\big)\big(D\tilde{u}(x)\big) + D_1\tilde{K}\big(x, \tilde{u}(x)\big)$$

in $L(X|Y^1)$, hold. This leads to the fixed point equations

$$A^* = D_2K^*\big(x, u^*(x)\big)\big(A^*\big) + D_1K^*\big(x, u^*(x)\big)$$

in $L(X|Y^2)$ and

$$\tilde{A} = D_2\tilde{K}\big(x, \tilde{u}(x)\big)\big(\tilde{A}\big) + D_1\tilde{K}\big(x, \tilde{u}(x)\big)$$

in $L(X|Y^1)$.
We will now check all four conditions that are needed in the Theorem of van Gils and Vanderbauwhede. Following the notation of van Gils and Vanderbauwhede we set $Y_0 := L(X|Y^2)$, $Y := L(X|Y^1)$ and $Y_1 := Y$. Let $J_0 : Y_0 \to Y$ be the given continuous embedding from Y_0 to Y. We may suppose that the Theorem of van Gils and Vanderbauwhede is valid if Λ is an open subset of a Banach space and set $\Lambda := O^* \cap \tilde{O}$. Thus, $\tilde{u}(x) = J_0\big(u^*(x)\big)$ for all $x \in \Lambda$.

1. The mapping $f : Y \times \Lambda \to Y$ defined by

$$f\big(\tilde{A}, x\big) = D_2\tilde{K}\Big(x, J_0\big(u^*(x)\big)\Big)\big(\tilde{A}\big) + D_1\tilde{K}\Big(x, J_0\big(u^*(x)\big)\Big),$$

for $(\tilde{A}, x) \in Y \times \Lambda$ has a partial derivative with respect to $\tilde{A}$ given by

$$D_1 f\big(\tilde{A}, x\big) = D_2 \tilde{K}\Big(x, J_0\big(u^*(x)\big)\Big) \in L(Y|Y)$$

in every $\big(\tilde{A}, x\big) \in Y \times \Lambda$.
Furthermore, the mapping

$$Y \times \Lambda \ni \big(\tilde{A}, x\big) \mapsto D_1 f\big(\tilde{A}, x\big) \in L(Y|Y)$$

is continuous.
As in this case $Y_1 = Y$, the following points are clearly satisfied:

- The mapping

  $$J \circ f : Y \times \Lambda \to Y_1,$$

  if J denotes the identity in $Y = Y_1$, has a partial derivative $D_1(J \circ f)(\tilde{A}, x) \in L(Y|Y_1)$ with respect to $\tilde{A}$ in in every $\big(\tilde{A}, x\big) \in Y \times \Lambda$ and the mapping

  $$D_1(J \circ f) : Y \times \Lambda \to L(Y|Y_1)$$

  is continuous.

- By defining

  $$f^{(1)} : Y \times \Lambda \to L(Y|Y)$$

  and

  $$f_1^{(1)} : Y \times \Lambda \to L(Y_1|Y_1)$$

  by $f^{(1)}\big(\tilde{A}, x\big) = D_1 f\big(\tilde{A}, x\big)$ and $f_1^{(1)}\big(\tilde{A}, x\big) = D_1 f\big(\tilde{A}, x\big)$ for $(\tilde{A}, x) \in Y \times \Lambda$ it follows that the identities

  $$D_1(J \circ f)\big(\tilde{A}, x\big) = (J \circ f^{(1)})\big(\tilde{A}, x\big) = \Big(f_1^{(1)}\big(\tilde{A}, x\big) \circ J\Big)$$

  are satisfied for $\big(\tilde{A}, x\big) \in Y \times \Lambda$.

2. We claim that the mapping

$$f_0 : Y_0 \times \Lambda \to Y$$

defined by

$$f_0\big(A^*, x\big) := f\big(J_0(A^*), x\big)$$

for $\big(A^*, x\big) \in Y_0 \times \Lambda$ has a partial derivative $D_2 f_0\big(A^*, x\big) \in L(\Lambda|Y)$ with respect to x in every $\big(A^*, x\big) \in Y_0 \times \Lambda$.
Furthermore, we claim that the mapping

$$Y_0 \times \Lambda \ni \big(A^*, x\big) \mapsto D_2 f_0\big(A^*, x\big) \in L(\Lambda|Y)$$

is continuous.
Observing the identity

$$f_0(A^*, x) = D_2 \hat{K}\big(x, u^*(x)\big)\big(A^*\big) + D_1 \hat{K}\big(x, u^*(x)\big),$$

for $\big(A^*, x\big) \in Y_0 \times \Lambda$, it is clear that this is an ultimate consequence of the assumption on $\hat{K}$ which is:

$$(B_1 \times B_2^*) \subset (X \times Y^2) \ni (x, y) \to \hat{K}(x, y) \in Y^1$$

is 2 times continuously differentiable.
Therefore and due to the fact that $u^* : \Lambda \to B_2^* \subset Y^2$ is continuously differentiable, both mappings

$$Y_0 \times \Lambda \ni (A^*, x) \to D_2 \hat{K}\big(x, u^*(x)\big)\big(A^*\big) \in Y$$

and

$$\Lambda \ni x \to D_1 \hat{K}\big(x, u^*(x)\big) \in Y$$

are continuously differentiable.
Therefore, the sum f_0 of both mappings must be continuously differentiable.

3. We claim that there exists a constant $q < 1$ such that both inequalities

$$\|f\big(A, x\big) - f\big(B, x\big)\|_Y \leq q\|A - B\|_Y,$$

and

$$\|D_1 f\big(A, x\big)\|_{L(Y|Y)} \leq q$$

hold for $\big(A, B, x\big) \in Y \times Y \times \Lambda$.

Due to the continuity of

$$\Lambda \ni x \mapsto D_2 \tilde{K}\Big(x, J_0\big(u^*(x)\big)\Big) \in L(Y^1|Y^1),$$

and by the fact that

$$\|D_2\tilde{K}\Big(x_0, J_0\big(u^*(x_0)\big)\Big)\|_{L(Y^1|Y^1)} = \|D_2\tilde{K}(x_0, y_0)\|_{L(Y^1|Y^1)} \le q' < 1$$

by assumption, there must be a neighborhood $O' \subset B_1$ of x_0 and a constant $q < 1$ such that the inequality

$$\|D_2\tilde{K}\Big(x, J_0\big(u^*(x)\big)\Big)\|_{L(Y^1|Y^1)} < q$$

holds for all $x \in O'$. One can easily show that $D_2\tilde{K}\Big(x, J_0\big(u^*(x)\big)\Big)$ defines a bounded linear mapping from $L(X|Y^1)$ to $L(X|Y^1)$ with

$$D_2\tilde{K}\Big(x, J_0\big(u^*(x)\big)\Big)(A) := D_2\tilde{K}\Big(x, J_0\big(u^*(x)\big)\Big) \circ A$$

for $A \in L(X|Y^1)$ and

$$\|D_2\tilde{K}\Big(x, J_0\big(u^*(x)\big)\Big)\|_{L(Y^1|Y^1)} = \|D_2\tilde{K}\Big(x, J_0\big(u^*(x)\big)\Big)\|_{L\Big(L(X|Y^1)\Big|L(X|Y^1)\Big)}.$$

Hence, the inquality

$$\|D_2\tilde{K}\Big(x, J_0\big(u^*(x)\big)\Big)\|_{L\Big(L(X|Y^1)\Big|L(X|Y^1)\Big)} < q$$

holds for all $x \in O'$. With no loss of generality we suppose that $O' = \Lambda$. Therefore, both inequalities

$$\|D_1 f\big(A, x\big)\|_{L(Y|Y)} =$$

$$\|D_2\tilde{K}\Big(x, J_0\big(u^*(x)\big)\Big)\|_{L\Big(L(X|Y^1)\Big|L(X|Y^1)\Big)} \le q$$

and

$$\|f\big(A, x\big) - f\big(B, x\big)\|_Y =$$

$$\|D_2\tilde{K}\Big(x, J_0\big(u^*(x)\big)\Big)\big(A - B\big)\|_Y$$

$$\le q\|A - B\|_Y = q\|A - B\|_Y$$

hold for all $(A, B, x) \in Y \times Y \times \Lambda$ as requested. As in this case the identity $D_1 f\big(A, x\big) = f^{(1)}\big(A, x\big) = f_1^{(1)}\big(A, x\big)$ holds for $(A, x) \in Y \times \Lambda$ both inequalities

$$\|f^{(1)}\big(A, x\big)\|_{L(Y|Y)} \le q, \quad \|f_1^{(1)}\big(A, x\big)\|_{L(Y_1|Y_1)} \le q,$$

have to hold, too.

4. The last condition is satisfied by assumption:
   There exists a continuously differentiable mapping $u^* : \Lambda \to B_2^* \subset Y^2$ such that $u^*(x) = K^*\big(x, u^*(x)\big)$ for all $x \in \Lambda$. Therefore, the derivative of u^* with respect to x satisfies

   $$Du^*(x) = D_2K^*\big(x, u^*(x)\big)\big(Du^*(x)\big) + D_1K^*\big(x, u^*(x)\big)$$

   in all $x \in \Lambda$. Hence, $J_0\big(Du^*(x)\big) \in Y$ is the unique solution of

   $$\tilde{A} = f\big(\tilde{A}, x\big)$$

   for all $x \in \Lambda$.

These four conditions being satisfied Theorem 3.3.1 yields that $J_0 \circ Du^* : \Lambda \to Y$ is continuously differentiable. By setting $\hat{O} := \Lambda$ we get that the mapping $\hat{u} := J_0 \circ u^*|_{\hat{O}} = \tilde{u}|_{\hat{O}}$ is 2 times continuously differentiable. □

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