# A scalar curvature flow in low dimensions

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**Abstract**

Let $(M^n, g_0)$ be a $n = 3, 4, 5$ dimensional, closed Riemannian manifold of positive Yamabe invariant. For a smooth function $K > 0$ on $M$ we consider a scalar curvature flow, that tends to prescribe $K$ as the scalar curvature of a metric $g$ conformal to $g_0$. We show global existence and in case $M$ is not conformally equivalent to the standard sphere smooth flow convergence and solubility of the prescribed scalar curvature problem under suitable conditions on $K$.

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# 1 Introduction

## 1.1 Overview and related works

We study the problem of prescribing the scalar curvature of a closed Riemannian manifold within its conformal class, called the prescribed scalar curvature problem. Many work has been devoted to this topic in the last decades and we refer to [2], [22] and the references therein for an overview. More precisely we consider the problem of conformally prescribing a smooth function $K > 0$ as the scalar curvature in case the underlying manifold already admits a conformal metric of positive scalar curvature.

The problem has variational structure and solutions of the prescribed scalar curvature problem then correspond to critical points of a non negative energy functional $J$, which does not satisfy a compactness criterion known as the Palais-Smale condition. So direct variational methods can not be applied. Indeed considering a minimizing or more general a Palais-Smale sequence the possible obstacle of finding a minimizer or a critical point of the associated energy functional is, what we call a critical point at infinity - a blow up phenomenon, whose profile however is well understood [26].

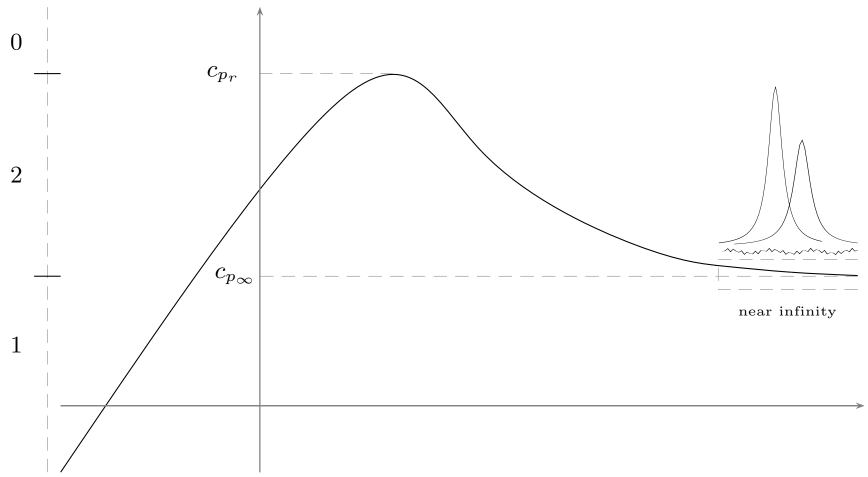


Figure 1: Blow up at infinity and topological contribution

The problem of prescribing a constant scalar curvature is known as the Yamabe problem. In this case the critical energy levels, at which a blow up may occur, are quantized. Thus to prove existence of a minimizer, it is sufficient to find a test function, whose energy is below the least critical energy level [3], [25]. Even, if this is not possible, one can show existence of critical points by analysing the critical points at infinity and their topological contribution to the underlying space as indicated in the above figure, cf. [7], [8], [9] and [11] for some genuine algebraic topological argument.

In addition to these two approaches one may recover solutions by perturbation arguments [1], [16].

Besides pure existence results it is a natural idea to find critical points as the limit of the gradient flow or more general of a pseudo gradient flow related to the energy functional. In this context one has to show long time existence and flow convergence with the crucial task being to ensure, that a flow line does not escape from the variational space towards a critical point at infinity. In the Yamabe case the question of flow convergence reduces to proving, that along a flow line, which becomes highly concentrated, the associated will eventually be below the critical energy levels, at which blow up may occur, and thus can not blow up at all [13], [18], [27], [29].

When prescribing the scalar curvature however the critical energy levels are not necessarily quantized. Nonetheless to show existence of a minimizer one may construct a test function with energy strictly below the least critical energy like for the Yamabe problem [5], [19] and one may use as well topological arguments to show existence of solutions as critical points [4], [10], [12], [23], [24].

The strategy of finding solutions by starting a flow is more complicated. The first task is to show long time existence. Secondly one has to prove, that the flow or at least one flow line does not converge to a critical point at infinity instead of a critical point - the ingredient of quantized energy levels being not available. To overcome this deficit one may impose assumptions on the function to be prescribed and therefore on the energy functional to be considered, which ensure a quantization of the critical energy levels [17].

One may object, that, when using deformations in the context of topological arguments, some pseudo gradient flow is always used, so there is nothing new. But the freedom of possibly choosing another more suitable pseudo gradient flow, in case some lines of a given flow do blow up, as sketched in figure 2, is lost, once we limit ourselves to considering one fixed pseudo gradient flow. And a priori there is no equivalence in using different flows.

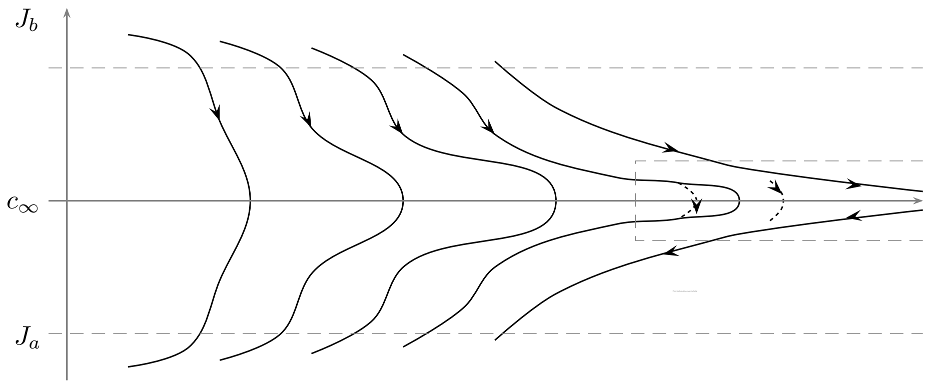


Figure 2: Suitable deformation to avoid infinity

However, if we do not limit ourselves to use pseudo gradient flows with just the purpose of finding solutions of the prescribed scalar curvature problem, it is of its own interest to describe the asymptotic behaviour of flow lines qualitatively - those converging to critical points and those diverging to critical points at infinity. And this is the aim of this work within its restrictive setting.

We would like to point out, that blowing up flow lines are not an unusual feature of the prescribed scalar curvature problem. On the contrary only under very restrictive assumptions blowing up flow lines can be excluded.

## 1.2 Exposition

We wish to give a quick overview on our main arguments.

In subsection 1.3 we provide the setting of this work, introduce the pseudo gradient flow to be considered, its basic properties and state two theorems, that provide full flow convergence and solubility of the prescribed scalar curvature problem under sufficient conditions on the function $K$ to be prescribed.

Section 2 is devoted to prove long time existence and weak convergence of the first variation $\partial J$ along a flow line $u$ in a sense to be made precise. The arguments, we use, are straight forward adaptations from the Yamabe setting [13], [27]; cf. [17] for a similar reasoning.

Section 3 describes the flow near infinity. Since a flow line $u$ restricted to any time sequence tending to infinity is a Palais-Smale sequence, well known blow up and concentration compactness arguments [26] provide a suitable parametrization. Namely $u$ can up to a small error term $v$ be written as a linear combination of a solution $\omega$ and finitely many bubbles

$$u = \alpha\omega + \alpha^i \delta_{a_i,\lambda_i} + v,\ i = 1, \dots, p,$$

where locally around $a_i$ the bubble $\delta_{a_i,\lambda_i}$ has the form

$$\delta_{a_i,\lambda_i}(x) = (\frac{\lambda_i}{1 + \lambda_i^2 d(a_i, x)^2})^{\frac{n-2}{2}}.$$

Thus a blow up corresponds to $\lambda_i \longrightarrow \infty$.

We then refine the representation by choosing more suitable bubbles $\varphi_{a_i,\lambda_i}$ instead of $\delta_{a_i,\lambda_i}$ and take care of a possible degeneracy of the representation in the spirit of [13]. Degeneracy in this context refers to the degeneracy of $\partial^2 J(\omega)$. Subsequently the representation is made unique by means of a Lyapunow-Schmidt reduction, that implies some orthogonality properties of the error term $v$ with respect to the solution $\omega$ and the bubbles $\varphi_{a_i,\lambda_i}$. In particular we obtain smallness of linear interactions of $v$ with $\omega$ and $\varphi_{a_i,\lambda_i}$ - a crucial aspect, that will enable us to identify the principal forces, that move $\lambda_i$ for instance or $a_i$.

Finally we show by Lojasiewicz inequality type arguments [15], [21], that, if a flow line is precompact, it is fully compact, thus convergent and this generically with exponential speed.

In section 4 we then consider the case, that a flow line $u$ near infinity can up to a small error term $v$ be thought of as a linear combination of bubbles

$$u = \alpha^i \varphi_{a_i,\lambda_i} + v,$$

so no solution $\omega$ is there. By suitable testing of the pseudo gradient flow equation in the spirit of [6] we analyse the movement of the bubbles by establishing explicit evolution equations of those three parameters, that constitute the bubbles, namely the scaling parameter $\alpha_i$, height $\lambda_i$ and position $a_i$. At this point the special choice of the Lyapunow-Schmidt reduction implies, that the evolution equations of the aforementioned parameters are independent of the time derivative of the error term $v$, which is difficult to control.

Using the fact, that the second variation $\partial^2 J(u)$ is positive definite in this case, when applied to the error term $v$, we are able to give a suitable a priori estimate on $v$ - indeed $\partial J(u)$ is square integrable in time, since we are dealing with a pseudo gradient flow and $\partial J(\alpha^i \varphi_{a_i})$ is small.

In conclusion we obtain a precise description of the behaviour of the flow line in terms of $\lambda_i$ as the only non compact variable and $a_i$.

Section 5 deals analogously to section 4 with the case, that a flow line $u$ near infinity can be written as a linear combination of a non trivial solution $\omega > 0$ and finitely many bubbles - up to a small error term. We then follow the same scheme as in the previous section. The main difference is, that there are more parameters to be considered beyond the scaling factor, height and position of the bubbles. Namely we have to deal with a scaling factor $\alpha$ for the solution $\omega$ plus finitely many parameters $\beta_i$ to describe the degenerate space of the solution $\omega$ and the implicit function theorem yields a suitable parametrization $u_{\alpha,\beta} = \alpha u_{1,\beta}$ for this purpose. So

$$u = u_{\alpha,\beta} + \alpha^i \varphi_{a_i,\lambda_i} + v.$$

We would like to point out, that generically a solution $\omega$ is non degenerate, in which case $u_{\alpha,\beta}$ reduces to $\alpha\omega$. Moreover the second variation $\partial^2 J(u)$ is not necessarily positive definite. But, since we have taken care of the degenerate space, the second variation is sort of non degenerate, when applied to the space, that the error term $v$ lives on. Thence we still get a sufficient estimate on $v$.

In section 6, subsection 6.1 we proceed considering the flow near infinity and, under a suitable assumption on the energy functional, that the flow behaves as one would expect, e.g. that a flow line does not only converge to a solution, once this is true for a time sequence as seen at the end of section 3, but that the same holds true for a critical point at infinity. This means, that, if for some time sequence the flow line blows up, this is true for the full flow line as well. Moreover we show, that the critical set $[\nabla K = 0]$ attracts the concentration points $a_i$ of a flow line near infinity.

The following subsection 6.2 contains the very essence of the proof of the theorem. Under suitable conditions on $K$, which already imply, that the flow behaves in the sense of the foregoing subsection, we explicitly construct some functions adapted to the dimension and the case, whether $\omega$ is trivial or not, with the basic property of becoming arbitrarily negative in case the flow line blows up, while on the other hand their time derivative is basically non negative. So they can be thought of as a way to check the compactness of a flow line near infinity. This idea originates from [6], where it was used in case $M = \mathbb{S}^3$ to exclude a multi bubble blow up, and our constructions are somewhat technical, but natural generalisations to the non spherical situation in dimensions $n = 3, 4, 5$.

For the construction the explicit evolution equations of the parameters $\lambda_i$ and $a_i$ of the bubbles $\varphi_{a_i,\lambda_i}$ obtained in sections 4 and 5 are used. Besides the necessity of controlling the error term $v$ there are two basic features to be considered.

The first one concerns self-interaction phenomen, whereby we mean quantities, which are attributed solely to a one bubble situation. In this case, the question of what moves a bubble is simply answered by saying, a bubble is moved, by what prevents a bubble from being a solution. E.g. on the standard sphere a bubble is a solution of the Yamabe problem, but not of the prescribed scalar curvature problem for $K$ non constant. Thus we expect a bubble to be moved by the non vanishing derivatives of $K$, for instance the gradient of $K$ moves $a_i$ as $\lambda_i$ is moved by the laplacian

If in addition we are dealing with an arbitrary manifold we expect other geometric quantities to move the bubbles as well - thereby the positive mass theorem comes into play.

The second feature is due to interaction quantities arising from the presence of several bubbles or from bubbles and a solution $\omega$. On the standard sphere for example, while each bubble is a solution of the Yamabe problem, their linear combination is not. Thus the movement of the bubbles is caused solely by the interaction phenomena and in the context of proving flow convergence, one has to ensure, that the interaction terms rather decrease the possibly non compact variables $\lambda_i$ instead of increasing them.

In subsection 6.3 we put all the previous informations together and show flow convergence by contradiction based on the functions constructed in foregoing subsection 6.2. Thus proving theorem 1. In order to prove theorem 2 we basically prove the existence of a converging flow line - using the same arguments as for proving theorem 1.

The final subsection 6.4 exposes a non trivial scenario of a blowing up flow line. In this example the function $K$ to be prescribed as the scalar curvature satisfies at one of its maximum points a flatness condition, that due to [19] guarantees the existence of a minimizer of $J$ in case $M$ is not conformally equivalent to the standard sphere. On the other hand the flow line constructed blows up at the same maximum point.

## 1.3 Preliminaries and statement of the theorems

We consider a smooth, closed Riemannian manifold

$$M = (M^n, g_0),\ n = 3, 4, 5$$

with volume measure $\mu_{g_0}$ and scalar curvature $R_{g_0}$. The Yamabe invariant

$$Y(M,g_0)=\inf_{\mathcal{A}}\frac{\int c_n|\nabla u|^2_{g_0}+R_{g_0}u^2d\mu_{g_0}}{(\int u^{\frac{2n}{n-2}}d\mu_{g_0})^{\frac{n-2}{n}}},$$

where $c_n=4\frac{n-1}{n-2}$ and

$$\mathcal{A}=\{u\in W^{1,2}_{g_0}(M)\mid u\geq 0, u\not\equiv 0\},$$

is assumed to be positive, $Y(M,g_0)>0$. The conformal laplacian

$$L_{g_0}=-c_n\Delta_{g_0}+R_{g_0}$$

then forms a positive, self-adjoint operator with Green's function

$$G_{g_0}:M\times M\longrightarrow\mathbb{R}_+$$

and we may assume for the background metric

$$R_{g_0}>0\ \text{ and }\ \int Kd\mu_{g_0}=1.$$

Considering a conformal metric $g=g_u=u^{\frac{4}{n-2}}g_0$ there holds

$$d\mu=d\mu_{g_u}=u^{\frac{2n}{n-2}}d\mu_{g_0}$$

for the volume element and for the scalar curvature

$$R=R_{g_u}=u^{-\frac{n+2}{n-2}}(-c_n\Delta_{g_0}u+R_{g_0}u)=u^{-\frac{n+2}{n-2}}L_{g_0}u.$$

Let $0<K\in C^\infty(M)$ and

$$r=r_u=\int Rd\mu,\ k=k_u=\int Kd\mu,\ \bar{K}=\bar{K}_u=\frac{K}{k}.$$

Note, that

$$c\|u\|_{W^{1,2}}\leq r_u=\int L_{g_0}uud\mu_{g_0}=\int c_n|\nabla u|^2_{g_0}+R_{g_0}u^2d\mu_{g_0}\leq C\|u\|_{W^{1,2}}$$

and

$$c\|u\|^{\frac{2n}{n-2}}_{L^{\frac{2n}{n-2}}}\leq k_u=\int Ku^{\frac{2n}{n-2}}d\mu_{g_0}\leq C\|u\|^{\frac{2n}{n-2}}_{L^{\frac{2n}{n-2}}}.$$

In particular we may define

$$\|u\|=\int L_{g_0}uud\mu_{g_0}$$

and use $\|\cdot\|$ as an equivalent norm on $W^{1,2}$. The aim of this paper is a study of

$$\partial_t u = -\frac{1}{K}(R - r\bar{K})u,\ u(\cdot,0) = u_0 > 0$$

as an evolution equation for the conformal factor. Obviously

$$\partial_t k = \partial_t \int K u^{\frac{2n}{n-2}} d\mu_{g_0} = 0.$$

Thus, if we choose as an initial value

$$u(\cdot,0) = u_0 > 0 \ \text{ satisfying } \ k_{u_0} = \int K u_0^{\frac{2n}{n-2}} = 1,$$

then the unit volume $k \equiv 1$ is preserved and in case

$$u \longrightarrow u_\infty > 0 \ \text{ in } \ W^{1,2}_{g_0}(M),$$

where $u_\infty$ is a stationary point, there necessarily holds

$$\int K u_\infty^{\frac{2n}{n-2}} d\mu_{g_0} = 1 \ \text{ and } \ R_{u_\infty} = r_{u_\infty} K.$$

In what follows we will simply call any maximal solution

$$u : M \times [0,T) \longrightarrow \mathbb{R},\ T \in (0,\infty]$$

of

$$\partial_t u = -\frac{1}{K}(R - r\bar{K}),\ u(\cdot,0) = u_0 > 0 \ \text{ with } \ \int K u_0^{\frac{2n}{n-2}} = 1$$

a flow line with initial value $u_0$. Let us consider the energy

$$J(u) = \frac{\int c_n |\nabla u|^2_{g_0} + R_{g_0} u^2 d\mu_{g_0}}{(\int K u^{\frac{2n}{n-2}} d\mu_{g_0})^{\frac{n-2}{n}}} \ \text{ for } \ u \in \mathcal{A}.$$

**Proposition 1.1** (Derivatives of $J$).
*We have*

*(i)*

$$J(u) = \frac{r_u}{k_u^{\frac{n-2}{n}}}$$

*(ii)*

$$\begin{aligned}\frac{1}{2}\partial J(u)v =& \frac{1}{k_u^{\frac{n-2}{n}}}[\int L_{g_0} uv - \frac{r_u}{k_u} \int K u^{\frac{n+2}{n-2}} v] \\ =& \frac{1}{k_u^{\frac{n-2}{n}}} \int (R_u - \frac{r_u}{k_u} K) u^{\frac{n+2}{n-2}} v\end{aligned}$$

*(iii)*

$$\begin{aligned}\frac{1}{2}\partial^2 J(u)vw =& \frac{1}{k_u^{\frac{n-2}{n}}}[\int L_{g_0}vw - \frac{n+2}{n-2}\frac{r_u}{k_u}\int Ku^{\frac{4}{n-2}}vw] \\ &- \frac{2}{k_u^{\frac{n-2}{n}+1}}[\int L_{g_0}uv\int Ku^{\frac{n+2}{n-2}}w + \int L_{g_0}uw\int Ku^{\frac{n+2}{n-2}}v] \\ &+ 4\frac{n-1}{n-2}\frac{r_u}{k_u^{\frac{n-2}{n}+2}}\int Ku^{\frac{n+2}{n-2}}v\int Ku^{\frac{n+2}{n-2}}w.\end{aligned}$$

*Moreover $J$ is $C^{2,\alpha}_{loc}$ and uniformly Hölder continuous on each*

$$U_\epsilon = \{u \in \mathcal{A} \mid \epsilon < \|u\|,\ J(u) \leq \epsilon^{-1}\} \subset \mathcal{A}.$$

The derivatives stated above are obtained by straight forward calculation. Moreover note, that $u \in U_\epsilon$ implies

$$\epsilon^2 \leq r_u \leq \epsilon^{-2} \ \text{ and } \ c\epsilon^3 \leq k_u^{\frac{n-2}{n}} = J(u)^{-1}r_u \leq C\epsilon^{-3}.$$

Thus uniform Hölder continuity on $U_\epsilon$ follows from the pointwise estimates

$$||a|^p - |b|^p| \leq C_p|a-b|^p \ \text{ in case } 0 < p < 1$$

and

$$||a|^p - |b|^p| \leq C_p \max\{|a|^{p-1}, |b|^{p-1}\}|a-b| \ \text{ in case } \ p \geq 1.$$

So the problem of prescribing the scalar curvature has a variational structure, since a critical point $\omega > 0$ of $J$ satisfies

$$R_\omega = \frac{r_\omega}{k_\omega}K, \ \text{ where } \ r_\omega = \int L_{g_0}\omega\omega, k_\omega = \int K\omega^{\frac{2n}{2n}},$$

whence the scalar curvature $R_\omega$ of $g_\omega = \omega^{\frac{4}{n-2}}g_0$ equals $K$ up to a coefficient. Note, that the standard norm of $\partial J(u)$

$$\|\partial J(u)\| = \|\partial J(u)\|_{W^{-1,2}_{g_0}(M)}$$

may be estimated by

$$\frac{1}{2}\|\partial J(u)\| \leq \frac{1}{k^{\frac{n-2}{n}}}\|R - r\bar{K}\|_{L_\mu^{\frac{2n}{n+2}}} \leq \frac{1}{k^{\frac{n-2}{n}}}\|R - r\bar{K}\|_{L^2_\mu}.$$

We therefore define by a slight abuse of notation

$$|\delta J(u)| = \frac{2}{k^{\frac{n-2}{n}}}\|R - r\bar{K}\|_{L^2_\mu}$$

as a natural majorant of $\|\partial J(u)\|$. Since $k \equiv 1$ along a flow line, we get

$$\partial_t J(u) = \partial J(u)\partial_t u = -2\int \frac{1}{K}|R - r\bar{K}|^2 u^{\frac{2n}{n-2}} \leq -\frac{1}{2\max_M K}|\delta J(u)|^2.$$

This justifies the notion of $\partial_t u = -\frac{1}{K}(R - r\bar{K})u$ as a pseudo gradient flow related to $J$ and, since $J$ is bounded from below, we have a priori integrability

$$\int_0^T |\delta J(u)|^2 dt < C(K)J(u_0).$$

On the other hand the positivity of the Yamabe invariant implies

$$J(u) > \frac{Y(M, g_0)}{\max_M K^{\frac{n-2}{n}}} > c.$$

Thus we may assume, that along a flow line $c < J(u) = r_u < C$ due to $k \equiv 1$. Recalling proposition 1.1 this shows $u \in U_\epsilon$ for some $\epsilon > 0$ small and fix, whence $J$ is uniformly Hölder continuous along and close by every flow line.

Consider the following conditions in cases $n = 3, 4, 5$, which are obviously satisfied, if $M$ is not conformally equivalent to the standard $\mathbb{S}^n$ and $K \equiv 1$. They are scaling invariant with respect to $K$ as one should expect due to the scaling invariance of $J$.

**Hypothesis 1.2** (Dimensional conditions)**.**

*$Cond_3$ : M is not conformally equivalent to the standard sphere* $\mathbb{S}^3$

*$Cond_4$ : M is not conformally equivalent to the standard sphere* $\mathbb{S}^4$ *and*

$$[\nabla K = 0] \subseteq [\frac{\Delta K}{K} > -c] \ \textit{ for some } \ c = c(M) > 0$$

*$Cond_5$ : M is not conformally equivalent to the standard sphere* $\mathbb{S}^5$ *and*

$$\langle \nabla \Delta K, \nabla K \rangle > \frac{1}{3}|\Delta K|^2$$

*holds on* $[\Delta K < 0] \cap U$ *for an open neighbourhood* $U$ *of* $[\nabla K = 0]$.

*Moreover let* $Cond'_n$ *denote* $Cond_n$ *with* $[\nabla K = 0]$ *replaced by* $[K = \max K]$.

Theorem 1 below generalizes the convergence of the Yamabe flow in these dimensions proven in [13], however by a different strategy.

**Theorem 1.**
*Let* $M = (M^n, g_0)$, $n = 3, 4, 5$ *be a smooth, closed Riemannian manifold of positive Yamabe-invariant. Then for* $0 < K \in C^\infty(M)$ *every flow line*

$$\partial_t u = -\frac{1}{K}(R - r\bar{K})u, \ u(\cdot, 0) = u_0 > 0 \ \textit{ with } \int K u_0^{\frac{2n}{n-2}} = 1$$

*exists for all times and remains positive.*

*Moreover we have convergence in the sense, that*

$$u \longrightarrow u_\infty > 0 \text{ in } C^\infty \text{ solving } R_{u_\infty} = r_{u_\infty} K,$$

*provided the dimensional condition $Cond_n$ is satisfied.*

So $Cond_n$ implies compactness of the flow, whereas $Cond'_n$ is at least sufficient to solve the prescribed scalar curvature problem.

**Theorem 2.**
*Let $M = (M^n, g_0)$, $n = 3, 4, 5$ be a smooth, closed Riemannian manifold of positive Yamabe-invariant. Then for $0 < K \in C^\infty(M)$ there exists*

$$u_\infty > 0 \text{ in } C^\infty \text{ solving } R_{u_\infty} = r_{u_\infty} K,$$

*provided the dimensional condition $Cond'_n$ is satisfied.*

# 2 Long time existence and weak convergence

In this section adapted from [13] and [27] we derive global existence and weak convergence in the sense, that $\|R - r\bar{K}\|_{L^p_\mu} \longrightarrow 0$ as $t \longrightarrow \infty$.

## 2.1 Long time existence

**Lemma 2.1** (Lower bounding the scalar curvature)**.**
*Along a flow line the scalar curvature $R$ is uniformly lower bounded.*

***Proof of lemma 2.1.***
Letting

$$\tilde{R} = e^{\frac{4}{n-2}\int_0^t \frac{r}{k}(\tau)d\tau} R \tag{2.1}$$

we have in view of lemma 7.1

$$\begin{aligned} \partial_t \tilde{R} = & e^{\frac{4}{n-2}\int_0^t \frac{r}{k}(\tau)d\tau} [c_n \Delta_g \frac{R}{K} + \frac{4}{n-2}(R - r\bar{K})\frac{R}{K}] + \frac{4}{n-2}\frac{r}{k}\tilde{R} \\ = & c_n \Delta_g \frac{\tilde{R}}{K} + \frac{4}{n-2} R \frac{\tilde{R}}{K} \geq c_n \Delta_g \frac{\tilde{R}}{K}. \end{aligned} \tag{2.2}$$

The parabolic maximum principle then shows

$$\min_{\{t\}\times M} \frac{\tilde{R}}{K} \geq \min_{\{0\}\times M} \frac{\tilde{R}}{K}, \tag{2.3}$$

whence

$$\min_{\{t\}\times M} R \geq C(K) e^{-\frac{4}{n-2}\int_0^t \frac{r}{k}(\tau)d\tau} \min_{\{0\}\times M} R. \tag{2.4}$$

Since $\frac{r}{k} = r \geq r_\infty > 0$ along a flow line, the assertion follows. □

Due to Gronwall's lemma this lower bound implies an upper bound on $u$.

**Lemma 2.2** (Upper bound)**.**
*Along a flow line $u$ there exists $C > 0$ such, that for $0 \leq t < T$ we have*

$$\sup_M u(t, \cdot) \leq e^{Ct}.$$

***Proof of lemma 2.2.***
From lemma 2.1 we infer

$$\partial_t u = -\frac{1}{K}(R - r\bar{K})u \leq cu. \tag{2.5}$$

The claim follows from Gronwall's inequality. □

The Harnack inequality now implies a lower bound on $u$.

**Lemma 2.3** (Lower bound)**.**
*Along a flow line $u$ there exists for $\Theta > 0$ some $C = C(\Theta) > 0$ such, that*

$$\sup_{M\times[0,T)} u \leq \Theta \Longrightarrow \inf_{M\times[0,T)} u \geq C.$$

***Proof of lemma 2.3.***
Let us choose $c > 0$, such that $R + c > 0$ according to lemma 2.1. Then for

$$P = R_{g_0} + cu^{\frac{4}{n-2}} \tag{2.6}$$

we have

$$-c_n\Delta_{g_0}u + Pu = L_{g_0}u - R_{g_0}u + Pu = Ru^{\frac{n+2}{n-2}} + cu^{\frac{n+2}{n-2}}. \tag{2.7}$$

Thus the weak Harnack inequality gives

$$k = \int Ku^{\frac{2n}{n-2}} \leq \sup_M(Ku^{\frac{n+2}{n-2}}) \int u \leq C \sup_M(Ku^{\frac{n+2}{n-2}}) \inf_M u, \tag{2.8}$$

where $C = C(\|P\|_{L^\infty})$. The claim follows. □

As a consequence of the positivity of the Yamabe invariant we obtain a logarithmic type estimate on the first variation of $J$.

**Lemma 2.4** (Logarithmic-type estimate on the first variation)**.**
*For $p > \frac{n}{2}$ there exist constants*

$$c = c(p) > 0 \ \text{ and } \ C = C(p) > 0$$

*such, that along a flow line we have*

$$\begin{aligned}\partial_t \int |R - r\bar{K}|^p d\mu + c\Big(\int |R - r\bar{K}|^{\frac{pn}{n-2}} d\mu\Big)^{\frac{n-2}{n}} \\ \leq C\Big(\int |R - r\bar{K}|^p d\mu\Big)^{\frac{2p+2-n}{2p-n}} + C \int |R - r\bar{K}|^p d\mu.\end{aligned}$$

***Proof of lemma 2.4.***
In view of lemma 7.1 we have

$$\begin{aligned}\partial_t \int &|R - r\bar{K}|^p d\mu \\ &= p \int \partial_t (R - r\bar{K})(R - r\bar{K})|R - r\bar{K}|^{p-2} d\mu + \int |R - r\bar{K}|^p \partial_t d\mu \\ &= pc_n \int \Delta_g \frac{R - r\bar{K}}{K}(R - r\bar{K})|R - r\bar{K}|^{p-2} d\mu \\ &\quad + \frac{4p}{n-2}\int \frac{R}{K}|R - r\bar{K}|^p d\mu - \frac{2n}{n-2}\int |R - r\bar{K}|^p \frac{R - r\bar{K}}{K} d\mu.\end{aligned} \tag{2.9}$$

Integrating by parts we obtain

$$\begin{aligned}\partial_t \int |R - r\bar{K}|^p d\mu \leq &- c(p)\int \frac{1}{K}|\nabla(R - r\bar{K})|_g^2 |R - r\bar{K}|^{p-2} d\mu \\ &+ C(p)(\int |R - r\bar{K}|^{p+1} d\mu + \int |R - r\bar{K}|^p d\mu).\end{aligned} \tag{2.10}$$

Using $|\nabla(R - r\bar{K})|_g \overset{a.e.}{=} |\nabla|R - r\bar{K}||_g$ this gives

$$\begin{aligned}\partial_t \int |R - r\bar{K}|^p d\mu \leq &- c(p)\int c_n |\nabla|R - r\bar{K}|^{\frac{p}{2}}|_g^2 d\mu \\ &+ C(p)(\int |R - r\bar{K}|^{p+1} d\mu + \int |R - r\bar{K}|^p d\mu)\end{aligned} \tag{2.11}$$

Then $Y(M, g_0) > 0$ implies

$$\begin{aligned}\partial_t \int |R - r\bar{K}|^p d\mu \leq &- c(p)(\int |R - r\bar{K}|^{\frac{pn}{n-2}} d\mu)^{\frac{n-2}{n}} \\ &+ C(p)(\int |R - r\bar{K}|^{p+1} d\mu + \int |R - r\bar{K}|^p d\mu).\end{aligned} \tag{2.12}$$

Since $p > \frac{n}{2}$, we may apply Hölder's inequality to $f = |R - r\bar{K}|^p$ via

$$\begin{aligned}\|f^{\frac{p+1}{p}}\|_{L^1_{g_0}} = &\|f\|^{\frac{p+1}{p}}_{L^{\frac{p+1}{p}}_{g_0}} \leq \|f\|^{\lambda\frac{p+1}{p}}_{L^{\Lambda}_{g_0}} \|f\|^{(1-\lambda)\frac{p+1}{p}}_{L^{\Theta}_{g_0}} \leq \|f\|^{\frac{n}{2p}}_{L^{\frac{n}{n-2}}_{g_0}} \|f\|^{\frac{2p+2-n}{2p}}_{L^1_{g_0}} \\ \leq &\varepsilon \|f\|_{L^{\frac{n}{n-2}}_{g_0}} + c(p, \varepsilon)\|f\|^{\frac{2p+2-n}{2p-n}}_{L^1_{g_0}},\end{aligned} \tag{2.13}$$

where $\Lambda = \frac{n}{n-2},\ \Theta = 1,\ \lambda = \frac{n}{2(p+1)}$ to conclude by absorption

$$\begin{aligned}\partial_t \int &|R - r\bar{K}|^p d\mu \\ &\leq - c(p)(\int |R - r\bar{K}|^{\frac{pn}{n-2}} d\mu)^{\frac{n-2}{n}} \\ &\quad + C(p)[(\int |R - r\bar{K}|^p d\mu)^{\frac{2p+2-n}{2p-n}} + \int |R - r\bar{K}|^p d\mu].\end{aligned} \tag{2.14}$$

This is the desired result. □

The next proposition is a typical parabolic type estimate.

**Proposition 2.5** (Main observation for long time existence)**.**
*Along a flow line there holds for* $1 \leq p \leq \frac{n}{2}$

$$\partial_t \int \frac{R_+^p}{K^{p-1}} d\mu \leq -4\frac{p-1}{p} c_n \int |\nabla(\frac{R_+}{K})^{\frac{p}{2}}|_g^2 d\mu - \frac{2n-4p}{n-2} \int \frac{1}{K^p} |R_+ - r\bar{K}|^{p+1} d\mu.$$

Here $R_+ = \min\{R, 0\}$.

***Proof of proposition 2.5***.
In view of lemma 7.1 we have

$$\begin{aligned}\partial_t \int \frac{R_+^p}{K^{p-1}} d\mu =& p \int \partial_t R R_+^{p-1} d\mu + \int R_+^p \partial_t d\mu \\ =& pc_n \int \Delta_g \frac{R}{K} (\frac{R_+}{K})^{p-1} d\mu + \frac{4p-2n}{n-2} \int (R - r\bar{K})(\frac{R_+}{K})^p d\mu \\ =& -4\frac{p-1}{p} c_n \int |\nabla(\frac{R_+}{K})^{\frac{p}{2}}|_g^2 d\mu \\ &+ \frac{4p-2n}{n-2} \int (R_+ - r\bar{K})[(\frac{R_+}{K})^p - (\frac{r}{k})^p] d\mu \\ &+ \frac{4p-2n}{n-2} (\frac{r}{k})^p \int (R_+ - r\bar{K}) d\mu.\end{aligned} \tag{2.15}$$

Due to $(a^p - b^p)(a-b) \geq |a-b|^{p+1}$ and $\int (R - r\bar{K}) d\mu = 0$ one obtains

$$\partial_t \int \frac{R_+^p}{K^{p-1}} d\mu \leq -4\frac{p-1}{p} c_n \int |\nabla(\frac{R_+}{K})^{\frac{p}{2}}|_g^2 d\mu + \frac{4p-2n}{n-2} \int \frac{1}{K^p} |R_+ - r\bar{K}|^{p+1} d\mu. \tag{2.16}$$

This is the desired result. □

The following is by now an easy consequence.

**Corollary 2.6.**
*Along a flow line there holds*

$$\begin{aligned}\sup_{0 \leq t < T} \int \frac{R_+^p}{K^{p-1}} d\mu + 4\frac{p-1}{p} c_n \int_0^T \int |\nabla(\frac{R_+}{K})^{\frac{p}{2}}|_g^2 d\mu dt \\ + \frac{2n-4p}{n-2} \int_0^T \int \frac{1}{K^p} |R_+ - r\bar{K}|^{p+1} d\mu dt \\ \leq \int \frac{R_+^p}{K^{p-1}} d\mu\lfloor_{t=0}\end{aligned}$$

This implies via Sobolev embedding higher integrability, which applied to lemma 2.4 proves the following time dependent bound.

**Corollary 2.7** ($L^p$-bound on the first variation)**.**
*For* $1 \leq p \leq \frac{n^2}{2(n-2)}$ *and* $T > 0$ *there exists* $C = C(p,T)$ *such, that*

$$\sup_{0 \leq t < T} \int |R - r\bar{K}|^p d\mu \leq C \quad \text{along a flow line.}$$

***Proof of corollary 2.7.***
From corollary 2.6 for $p = \frac{n}{2}$ we infer

$$\sup_{0 \leq t < T} \int R_+^{\frac{n}{2}} d\mu + \int_0^T \int |\nabla (\frac{R_+}{K})^{\frac{n}{4}}|_g^2 d\mu dt \leq C. \tag{2.17}$$

Sobolev's embedding then implies

$$\int_0^T (\int (\frac{R_+}{K})^{\frac{n^2}{2(n-2)}} d\mu)^{\frac{n-2}{n}} dt \leq C. \tag{2.18}$$

Since $R$ is uniformly bounded from below according to lemma 2.1 we get

$$\int_0^T (\int |R|^{\frac{n^2}{2(n-2)}} d\mu)^{\frac{n-2}{n}} dt \leq C, \tag{2.19}$$

whence

$$\int_0^T (\int |R - r\bar{K}|^{\frac{n^2}{2(n-2)}} d\mu)^{\frac{n-2}{n}} dt \leq C. \tag{2.20}$$

But from lemma 2.4 with $p = \frac{n^2}{2(n-2)} > \frac{n}{2}$ we infer

$$\partial_t \ln \int |R - r\bar{K}|^{\frac{n^2}{2(n-2)}} d\mu \leq C (\int |R - r\bar{K}|^{\frac{n^2}{2(n-2)}} d\mu)^{\frac{n-2}{n}} + C. \tag{2.21}$$

This proves the claim. □

With the above bounds at hand one uses Morrey's inequality to prove Hölder regularity.

**Proposition 2.8** (Time-dependent Hölder regularity)**.**
*Along a flow line there exists for* $0 < \alpha < \min\{\frac{4}{n}, 1\}$ *and* $T > 0$ *a constant*

$$C = C(\alpha, T)$$

*such, that we have*

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C(|t_1 - t_2|^{\frac{\alpha}{2}} + d(x_1, x_2)^\alpha)$$

*for all* $x_1, x_2 \in M$ *and* $0 \leq t_1, t_2 < T$ *with* $|t_1 - t_2| \leq 1$

***Proof of proposition 2.8.***
Let $\alpha = 2 - \frac{n}{p}$ and $\frac{n}{2} < p < \min\{\frac{n^2}{2(n-2)}, n\}$. Lemma 2.1 and 2.6 show

$$\int |R|^p d\mu \leq C \tag{2.22}$$

with $C = C(T)$, whence by conformal invariance and lemmata 2.2, 2.3

$$\int |\Delta_{g_0} u|^p \leq C. \tag{2.23}$$

On the other hand corollary 2.7 shows

$$\int |\frac{\partial_t u}{u}|^p d\mu \leq C, \quad \text{in particular} \quad \int |\partial_t u|^p \leq C \tag{2.24}$$

From this it follows via Morrey

$$|u(x,t) - u(y,t)| \leq C d(x,y)^\alpha \quad \text{for all} \quad x, y \in M, \tag{2.25}$$

where $0 < \alpha < \min\{\frac{4}{n}, 1\}$, and

$$\begin{aligned}
&|u(x,t_1) - u(x,t_2)| \\
&\quad = |t_1 - t_2|^{-\frac{n}{2}} \int_{B_{\sqrt{|t_1-t_2|}}(x)} |u(x,t_1) - u(x,t_2)| d\mu_{g_0}(y) \\
&\quad \leq |t_1 - t_2|^{-\frac{n}{2}} \int_{B_{\sqrt{|t_1-t_2|}}(x)} |u(y,t_1) - u(y,t_2)| d\mu_{g_0}(y) + C|t_1 - t_2|^{\frac{\alpha}{2}} \\
&\quad \leq |t_1 - t_2|^{-\frac{n}{2}+1} \sup_{0 \leq t < T} \int_{B_{\sqrt{|t_1-t_2|}}(x)} |\partial_t u(t,y)| d\mu_{g_0}(y) + C|t_1 - t_2|^{\frac{\alpha}{2}} \\
&\quad \leq |t_1 - t_2|^{-\frac{n-2}{2}} |t_1 - t_2|^{\frac{n}{2}\frac{p-1}{p}} \sup_{0 \leq t < T} (\int |\partial_t u|^p d\mu)^{\frac{1}{p}} + C|t_1 - t_2|^{\frac{\alpha}{2}}
\end{aligned} \tag{2.26}$$

for all $|t_1 - t_2| \leq 1$. The claim follows from $-\frac{n-2}{2} + \frac{n}{2}\frac{p-1}{p} = \frac{\alpha}{2}$. $\square$

With Hölder regularity at hand standard regularity arguments show

**Corollary 2.9** (Long-time existence)**.**
*Each flow line exists for all times.*

***Proof of corollary 2.9.***
This follows from short time existence and proposition 2.8. $\square$

## 2.2 Integrability and weak convergence

Now, that long time existence has been established, we examine in which sense the first variation of $J$ vanishes as $t \longrightarrow \infty$.

**Lemma 2.10** (Integrability and weak convergence)**.**
*For $1 \leq p < \frac{n}{2}$ we have along a flow line*

$$\int_0^\infty \int |R - r\bar{K}|^{p+1} d\mu dt \leq C \quad \text{and} \quad \liminf_{t \nearrow \infty} \int |R - r\bar{K}|^{p+1} d\mu = 0.$$

***Proof of lemma 2.10****.*
Clearly the first inequality above implies the second one. Note, that

$$\int_0^\infty \int |R_+ - r\bar{K}|^{p+1} d\mu dt \leq C \tag{2.27}$$

with time independent $C$ according to corollary 2.6. Moreover we have

$$\min_{\{t\}\times M} R \geq C(K) e^{-\frac{4}{n-2}\int_0^t \frac{r}{k}(\tau)d\tau} \min_{\{0\}\times M} R, \tag{2.28}$$

cf. (2.4). Since along a flow line $k = 1$ and $r \searrow r_\infty > 0$ this gives

$$R_- \leq Ce^{-ct}, R_- = -\min\{R, 0\} \tag{2.29}$$

for suitable constants $c, C > 0$. From this the assertion follows. $\square$

Interpolating via lemma 2.4 we obtain weak convergence.

**Proposition 2.11** (Weak convergence of the first variation)**.**
*Along a flow line we have for any $1 \leq p < \infty$*

$$\lim_{t \nearrow \infty} \int |R - r\bar{K}|^p d\mu = 0.$$

*In particular we have $|\delta J(u)| \longrightarrow 0$ as $t \longrightarrow \infty$.*

***Proof of proposition 2.11*** *(cf. [27], Lemma 3.3 and equation (43)).*
Due to lemma 2.10 for any $\max\{2, \frac{n}{2}\} < p_0 < \frac{n+2}{2}$ there holds

$$\int_0^\infty \int |R - r\bar{K}|^{p_0} d\mu dt \leq C \text{ and } \liminf_{t \nearrow \infty} \int |R - r\bar{K}|^{p_0} d\mu = 0. \tag{2.30}$$

Thus we may choose a sequence $\tau_k^0 \nearrow \infty$ satisfying

$$\int |R - r\bar{K}|^{p_0} d\mu \lfloor_{\tau_k^0} \leq \frac{1}{2k} \text{ and } \int_{\tau_k^0}^\infty \int |R - r\bar{K}|^{p_0} d\mu dt < \frac{1}{4Ck}, \tag{2.31}$$

where $C = C(p)$ is the constant appearing in lemma 2.4. Define

$$\theta_k^0 = \sup\{\tau > \tau_k^0 \mid \forall \tau_k^0 < t < \tau \;:\; \int |R - r\bar{K}|^{p_0} d\mu < \frac{2}{k}\} > \tau_k^0. \tag{2.32}$$

Then we infer from lemma 2.4 for $\tau_k^0 < t < \theta_k^0$

$$\begin{aligned}\int |R - r\bar{K}|^{p_0} d\mu\lfloor_t + c\int_{\tau_k^0}^t (\int |R - r\bar{K}|^{p_0 \frac{n}{n-2}} d\mu)^{\frac{n-2}{n}} dt \\ \leq \int |R - r\bar{K}|^{p_0} d\mu\lfloor_{\tau_k^0} \\ + C\int_{\tau_k^0}^t (\int |R - r\bar{K}|^{p_0} d\mu)^{1+\frac{2}{2p-n}} dt \\ + C\int_{\tau_k^0}^t \int |R - r\bar{K}|^{p_0} d\mu dt \\ \leq \frac{1}{2k} + 2C\int_{\tau_k^0}^\infty \int |R - r\bar{K}|^{p_0} d\mu dt \leq \frac{1}{k}.\end{aligned} \tag{2.33}$$

If $\theta_k^0 < \infty$, then $\frac{2}{k} = \int |R - r\bar{K}|^{p_0} d\mu\lfloor_{\theta_k^0} \leq \frac{1}{k}$, whence $\theta_k^0 = \infty$ and

$$\int |R - r\bar{K}|^{p_0} d\mu \leq \frac{2}{k} \quad \text{on} \quad [\tau_k^0, \infty). \tag{2.34}$$

We conclude $\lim_{t \nearrow \infty} \int |R - r\bar{K}|^{p_0} d\mu = 0$ and in particular, cf. (2.33),

$$\int_0^\infty (\int |R - r\bar{K}|^{p_1} d\mu)^{\frac{n-2}{n}} dt < \infty \quad \text{and} \quad \liminf_{t \nearrow \infty} \int |R - r\bar{K}|^{p_1} d\mu = 0 \tag{2.35}$$

letting

$$p_1 = \frac{n}{n-2} p_0. \tag{2.36}$$

As before we may choose a sequence $\tau_k^1 \nearrow \infty$ satisfying

$$(\int |R - r\bar{K}|^{p_1} d\mu)^{\frac{n-2}{n}} \lfloor_{\tau_k^1} \leq \frac{1}{2k} \tag{2.37}$$

and

$$\int_{\tau_k^1}^\infty (\int |R - r\bar{K}|^{p_1} d\mu)^{\frac{n-2}{n}} dt < \frac{n}{4Ck(n-2)}, \tag{2.38}$$

where $C = C(p)$ is the constant appearing in lemma 2.4. Define

$$\theta_k^1 = \sup\{\tau > \tau_k^1 \mid \forall \tau_k^1 < t < \tau \ : \ (\int |R - r\bar{K}|^{p_1} d\mu)^{\frac{n-2}{n}} < \frac{2}{k}\} > \tau_k^1. \tag{2.39}$$

Then we infer from lemma 2.4 for $\tau_k^0 < t < \theta_k^0$

$$
\begin{aligned}
(\int |R-r\bar{K}|^{p_1} d\mu)^{\frac{n-2}{n}}\lfloor_t + c\frac{n-2}{n}\int_{\tau_k^1}^t \frac{(\int |R-r\bar{K}|^{p_1\frac{n}{n-2}} d\mu)^{\frac{n-2}{n}}}{(\int |R-r\bar{K}|^{p_1})^{\frac{2}{n}}} dt \\
\leq (\int |R-r\bar{K}|^{p_0} d\mu)^{\frac{n-2}{n}}\lfloor_{\tau_k^1} \\
+ C\frac{n-2}{n}\int_{\tau_k^1}^t (\int |R-r\bar{K}|^{p_1} d\mu)^{\frac{n-2}{n}+\frac{2}{2p-n}} dt \\
+ C\frac{n-2}{n}\int_{\tau_k^1}^t (\int |R-r\bar{K}|^{p_1} d\mu)^{\frac{n-2}{n}} dt \\
\leq \frac{1}{2k} + 2C\frac{n-2}{n}\int_{\tau_k^1}^\infty \int |R-r\bar{K}|^{p_0} d\mu dt \leq \frac{1}{k}.
\end{aligned}
\tag{2.40}
$$

If $\theta_k^1 < \infty$, then $\frac{2}{k} = \int |R-r\bar{K}|^{p_1} d\mu\lfloor_{\theta_k^1} \leq \frac{1}{k}$, whence $\theta_k^1 = \infty$ and

$$
\int |R-r\bar{K}|^{p_1} d\mu \leq \frac{2}{k} \quad \text{on} \quad [\tau_k^1, \infty). \tag{2.41}
$$

We conclude $\lim_{t\nearrow\infty} \int |R-r\bar{K}|^{p_1} d\mu = 0$ and in particular, cf. (2.40),

$$
\int_0^\infty (\int |R-r\bar{K}|^{p_2} d\mu)^{\frac{n-2}{n}} dt < \infty \quad \text{and} \quad \liminf_{t\nearrow\infty} \int |R-r\bar{K}|^{p_2} d\mu = 0 \tag{2.42}
$$

letting $p_2 = p_1(\frac{n}{n-2})$. Note, that from this we may start an induction yielding

$$
\lim_{t\nearrow\infty} \int |R-r\bar{K}|^{p_k} d\mu = 0 \tag{2.43}
$$

and

$$
\int_0^\infty (\int |R-r\bar{K}|^{p_{k+1}} d\mu)^{\frac{n-2}{n}} dt < \infty \quad \text{and} \quad \liminf_{t\nearrow\infty} \int |R-r\bar{K}|^{p_{k+1}} d\mu = 0 \tag{2.44}
$$

letting $p_{k+1} = \frac{n}{n-2} p_k$ for $k \geq 1$. Thereby the claim is evidently proven. □

# 3 The flow near infinity

## 3.1 Blow-up analysis

For a Palais-Smale sequence of decreasing energy, say $u_k = u(t_k)$ for a flow line $u$ and $t_k \longrightarrow \infty$, the lack of compactness is described as follows.

**Proposition 3.1** (Concentration-Compactness)**.**
*Let* $(u_m) \subset W^{1,2}_{g_0}(M, \mathbb{R}_{>0})$ *satisfy* $k_{u_m} = \int K u_m^{\frac{2n}{n-2}} d\mu_{g_0} = 1$ *and*

$$
\sup_{m\in\mathbb{N}} J(u_m) < \infty \quad \textit{and} \quad \|\partial J(u_m)\| \longrightarrow 0.
$$

*Passing to a subsequence we then have*

$$J(u_m) = r_{u_m} \longrightarrow J_\infty = r_\infty$$

*and there exist* $0 \leq u_\infty \in W^{1,2}_{g_0}(M)$ *with either* $u_\infty \equiv 0$ *or* $u_\infty > 0$ *solving*

$$L_{g_0} u_\infty = r_\infty K u_\infty^{\frac{n+2}{n-2}}$$

*and for some* $p \in \mathbb{N}_0$ *sequences* $(a_{i_m}) \subset M,\ (\lambda_{i_m}) \subset \mathbb{R}_{>0},\ i = 1, \ldots, p$ *with*

$$a_{i_m} \longrightarrow a_{i_\infty} \ \ \text{and} \ \ \lambda_{i_m} \longrightarrow \infty \ \ \text{as} \ \ m \longrightarrow \infty$$

*such, that*

$$\|u_m - u_\infty - \sum_{i=1}^{p} \hat{\delta}_{a_{i_m}, \lambda_{i_m}}\| \longrightarrow 0,$$

*where*

$$\hat{\delta}_{a_{i_m}, \lambda_{i_m}} = (\frac{4n(n-1)}{r_\infty K(a_i)})^{\frac{n-2}{4}} \eta_{a_{i_m}} (\frac{\lambda_{i_m}}{1 + \lambda_{i_m}^2 |\exp^{-1}_{a_{i_m}}(\cdot)|^2_{g_0}})^{\frac{n-2}{2}}$$

*with a cut-off function* $\eta_{a_{i_m}} = \eta(|\exp^{-1}_{a_{i_m}}(\cdot)|^2_{g_0})$, *where*

$$\eta \in C^\infty(B_2(0), \mathbb{R}_{\geq 0}),\ \eta \equiv 1 \ \ \text{on} \ \ B_1(0).$$

*More precisely there holds for each* $i \neq j = 1, \ldots, p$

$$\frac{\lambda_{i_m}}{\lambda_{j_m}} + \frac{\lambda_{j_m}}{\lambda_{i_m}} + \lambda_{i_m} \lambda_{j_m} d^2_{g_0}(a_{i_m}, a_{j_m}) \longrightarrow \infty \ \ \text{as} \ \ m \longrightarrow \infty.$$

This characterization is classical and we refer to [26]. The proposition is proven by straight forward adaptation. For the last statement cf. [14].

## 3.2 Bubbles and interaction estimates

We refine the definition of blow up functions $\hat{\delta}_{a,\lambda}$ given in proposition 3.1, referred to as bubbles, since they form a spherical geometry around $a$.

**Definition 3.2** (Bubbles)**.**
*For* $a \in M$ *let* $u_a$ *introduce normal conformal coordinates around* $a \in M$ *via*

$$g_a = u_a^{\frac{4}{n-2}} g_0.$$

*Let* $G_{g_a}$ *be the Green's function of the conformal laplacian*

$$L_{g_a} = -c_n \Delta_{g_a} + R_{g_a},\ c_n = 4\frac{n-1}{n-2}.$$

*For $\lambda > 0$ let*

$$\varphi_{a,\lambda} = u_a\left(\frac{\lambda}{1+\lambda^2\gamma_n G_a^{\frac{2}{2-n}}}\right)^{\frac{n-2}{2}},\ G_a = G_{g_a}(a,\cdot),\ \gamma_n = (4n(n-1)\omega_n)^{\frac{2}{n-2}}.$$

*One may expand*

$$G_a = \frac{1}{4n(n-1)\omega_n}(r_a^{2-n} + H_a),\ r_a = d_{g_a}(a,\cdot),\ H_a = H_{r,a} + H_{s,a}.$$

*There holds $H_{r,a} \in C^{2,\alpha}_{loc}$ and in conformal normal coordinates*

$$H_{s,a} = O\begin{pmatrix} 0 & \text{for } n=3 \\ r_a^2 \ln r_a & \text{for } n=4 \\ r_a & \text{for } n=5 \end{pmatrix}$$

*In addition it follows from the positive mass theorem, that*

$$H_a(a) = 0 \ \text{ for } \ M \simeq \mathbb{S}^n \ \text{ and } \ H_a(a) > 0 \ \text{ for } \ M \not\simeq \mathbb{S}^n,$$

*so $H_a(a)$ is always non negative with strict positivity unless $M$ is conformally equivalent to the standard sphere $\mathbb{S}^n$ .*

For the expansion of the Green's function stated cf. [22], Theorem 6.5. Ibidem conformal normal coordinates are introduced in section 5, see also the improvement due to [20]. Note, that we may and will replace $\hat{\delta}_{a,\lambda}$ by $\varphi_{a,\lambda}$ in proposition 3.1, since

$$\|\varphi_{a,\lambda} - \hat{\delta}_{a,\lambda}\| \longrightarrow 0 \ \text{ as } \ \lambda \longrightarrow \infty.$$

The reason for the above redefinition of bubbles is the simple way to calculate their conformal laplacian in terms of its Green's function, see the lemma below, whose proof we delay to the appendix.

**Lemma 3.3** (Emergence of the regular part)**.**
*One has $L_{g_0}\varphi_{a,\lambda} = O(\varphi_{a,\lambda}^{\frac{n+2}{n-2}})$ and on a geodesic ball $B_\alpha(a)$ for $\alpha > 0$ small*

$$\begin{aligned} L_{g_0}\varphi_{a,\lambda} =& 4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}} - 2nc_n r_a^{n-2}((n-1)H_a + r_a\partial_{r_a}H_a)\varphi_{a,\lambda}^{\frac{n+2}{n-2}} \\ & + \frac{u_a^{\frac{2}{n-2}} R_{g_a}}{\lambda}\varphi_{a,\lambda}^{\frac{n}{n-2}} + o(r_a^{n-2})\varphi_{a,\lambda}^{\frac{n+2}{n-2}}, \end{aligned}$$

*where $r_a = d_{g_a}(a,\cdot)$. Note, that $R_{g_a} = O(r_a^2)$ in geodesic normal coordinates.*

We would like to point out, that the term $\frac{R_{g_a}}{\lambda}\varphi_{a,\lambda}^{\frac{n}{n-2}}$ is negligible for our discussion, whereas it plays a crucial role in higher dimensions.

To abbreviate the notation we make the following definitions.

**Definition 3.4** (Relevant quantities)**.**
*For* $k,l=1,2,3$ *and* $\lambda_i>0$, $a_i\in M$, $i=1,\dots,p$ *define*

(i) $\varphi_i=\varphi_{a_i,\lambda_i}$ *and* $(d_{1,i},d_{2,i},d_{3,i})=(1,-\lambda_i\partial_{\lambda_i},\frac{1}{\lambda_i}\nabla_{a_i})$

(ii) $\phi_{1,i}=\varphi_i,\ \phi_{2,i}=-\lambda_i\partial_{\lambda_i}\varphi_i,\ \phi_{3,i}=\frac{1}{\lambda_i}\nabla_{a_i}\varphi_i,$ *so* $\phi_{k,i}=d_{k,i}\varphi_i$

We collect some useful estimates, which are well known, so we delay their proof to the appendix. They are essential for the rest of our discussion and will be heavily used.

**Lemma 3.5** (Interactions)**.**
*Let* $k,l=1,2,3$ *and* $i,j=1,\dots,p$. *We have*

(i) $|\phi_{k,i}|,|\lambda_i\partial_{\lambda_i}\phi_{k,i}|,|\frac{1}{\lambda_i}\nabla_{a_i}\phi_{k,i}|\le C\varphi_i$

(ii) $\int\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\phi_{k,i}=c_k\cdot id+O(\frac{1}{\lambda_i^{n-2}}+\frac{1}{\lambda_i^2}),\ c_k>0$

(iii) $\int\varphi_i^{\frac{n+2}{n-2}}\phi_{k,j}=b_kd_{k,i}\varepsilon_{i,j}+o_{\varepsilon}(\varepsilon_{i,j})=\frac{n+2}{n-2}\int\phi_{k,i}\varphi_i^{\frac{4}{n-2}}\varphi_j,\ b_k>0,\ i\neq j$

(iv) $\int\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\phi_{l,i}=O(\frac{1}{\lambda_i^{n-2}}+\frac{1}{\lambda_i^2})$ *for* $k\neq l$, $\int\varphi_i^{\frac{2n}{n-2}}=c_1+O(\frac{1}{\lambda_i^{n-2}})$ *and*

$$\int\varphi_i^{\frac{n+2}{n-2}}\phi_{k,i}=O(\frac{1}{\lambda_i^{n-2}})\ \text{ for }\ k=2,3$$

(v) $\int\varphi_i^{\alpha}\varphi_j^{\beta}=O(\varepsilon_{i,j}^{\beta})$ *for* $i\neq j$ *and* $\alpha+\beta=\frac{2n}{n-2},\ \frac{n}{n-2}>\alpha>\beta\ge1$

(vi) $\int\varphi_i^{\frac{n}{n-2}}\varphi_j^{\frac{n}{n-2}}=O(\varepsilon_{i,j}^{\frac{n}{n-2}}\ln\varepsilon_{i,j}),\ i\neq j$

(vii) $(1,\lambda_i\partial_{\lambda_i},\frac{1}{\lambda_i}\nabla_{a_i})\varepsilon_{i,j}=O(\varepsilon_{i,j}),\ i\neq j,$

*where* $\epsilon=\min\{\frac{1}{\lambda_i},\frac{1}{\lambda_j},\varepsilon_{i,j}\}$ *and*

$$\varepsilon_{i,j}=(\frac{\lambda_j}{\lambda_i}+\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_nG_{g_0}^{\frac{2}{2-n}}(a_i,a_j))^{\frac{2-n}{2}}.$$

Here we used and will use later on $a=o_{\epsilon}(b)$ as short hand for

$$|a|\le\omega(\varepsilon)|b|\ \text{ with }\ \omega(\epsilon)\longrightarrow0\text{ as }\ \epsilon\longrightarrow0.$$

## 3.3 Degeneracy and pseudo critical points

In order to obtain a precise description of the dynamical behaviour of a flow line we have to take care of a possible degeneracy of $J$ at a critical point.

**Lemma 3.6** (Spectral theorem and degeneracy).
*Let* $\omega > 0$ *solve* $L_{g_0}\omega = K\omega^{\frac{n+2}{n-2}}$.

*Then there exists a set of solutions*

$$L_{g_0}w_i = \mu_{w_i}K\omega^{\frac{4}{n-2}}w_i,\ \mu_{w_i} \longrightarrow \infty$$

*such, that*

$$\langle w_i, w_j\rangle_{L_{g_0}} = \delta_{ij}, \langle w_i \mid i \in \mathbb{N}\rangle = W^{1,2}_{g_0}(M)$$

*and for any eigenspace* $E_\mu(\omega) = \langle w_i \mid \mu_{w_i} = \mu\rangle$ *we have* $\dim E_\mu < \infty$. *Moreover we have* $\partial J(\omega) = 0$ *and isomorphy*

$$\partial^2 J(\omega)\lfloor_{H_0(\omega)^{\perp_{L_{g_0}}}} : H_0(\omega)^{\perp_{L_{g_0}}} \xrightarrow{\simeq} (H_0(\omega)^{\perp_{L_{g_0}}})^*,$$

*where*

$$H_0(\omega) = \langle \omega\rangle \oplus \langle \mathrm{e}_i \mid i = 1, \ldots, m\rangle$$

*with*

$$\langle \mathrm{e}_i \mid i = 1, \ldots, m\rangle = E_{\frac{n+2}{n-2}}(\omega),\ \langle \mathrm{e}_i, \mathrm{e}_j\rangle_{L_{g_0}} = \delta_{ij}$$

*denotes the kernel of* $\partial^2 J$ *at* $\omega$ *and* $H_0(\omega)^{\perp_{L_{g_0}}}$ *is the orthogonal of* $H_0(\omega)$ *with respect* $\langle\cdot,\cdot\rangle_{L_{g_0}}$. *The case* $E_{\frac{n+2}{n-2}}(\omega) = \emptyset$ *is generic.*

Please note, that due to scaling invariance of the functional the kernel always contains $\omega$ itself. We may thus call $\omega$ (essentially) non degenerate, if simply $H_0(\omega) = \langle\omega\rangle$, or equivalently, if $E_{\frac{n+2}{n-2}}(\omega) = \emptyset$. The foregoing lemma asserts, that non degeneracy is generic.

***Proof of lemma 3.6.***
The statement on the basis $\{w_i \mid i \in \mathbb{N}\}$ of eigenfunctions is a direct application of the spectral theorem for compact operators. Moreover

$$r_\omega = \int L_{g_0}\omega\omega = \int K\omega^{\frac{2n}{n-2}} = k_\omega \tag{3.1}$$

for a solution $L_{g_0}\omega = K\omega^{\frac{n+2}{n-2}}$. Thus proposition 1.1 shows

$$\partial J(\omega), \partial^2 J(\omega)\omega, \partial^2 J(\omega)\mathrm{e}_j = 0, \tag{3.2}$$

which is easy to check. Likewise for $v \perp_{L_{g_0}} \omega$ one obtains

$$\frac{1}{2}\partial^2 J(\omega)vf = k_\omega^{\frac{2-n}{n}} \int (L_{g_0}v - \frac{n+2}{n-2}\omega^{\frac{4}{n-2}}v)f. \tag{3.3}$$

This proves the claim with isomorphy of

$$\partial^2 J(\omega)\lfloor_{H_0(\omega)^{\perp_{L_{g_0}}}} : H_0(\omega)^{\perp_{L_{g_0}}} \xrightarrow{\simeq} (H_0(\omega)^{\perp_{L_{g_0}}})^* \tag{3.4}$$

given by

$$w_i \longrightarrow 2k_\omega^{\frac{2-n}{n}}(1-\mu_{w_i}^{-1}\frac{n+2}{n-2})\langle w_i, \cdot\rangle_{L_{g_0}}. \tag{3.5}$$

We are left with proving genericity of $E_{\frac{n+2}{n-2}}(\omega) = \emptyset$.
To that end consider the scalar curvature mapping

$$R : C^{2,\alpha}(M, A_\epsilon) \longrightarrow C^{0,\alpha}(M) : \omega \longrightarrow R_\omega = \omega^{-\frac{n+2}{n-2}} L\omega, \tag{3.6}$$

where $A_\epsilon = (\epsilon, \epsilon^{-1})$ for some $\epsilon > 0$, with derivative

$$\partial R_\omega \cdot v = \omega^{-\frac{n+2}{n-2}}(L_{g_0} v - \frac{n+2}{n-2} R_\omega \omega^{\frac{4}{n-2}} v). \tag{3.7}$$

Note, that for $\omega \in C^{2,\alpha}(M, A_\epsilon)$ fixed we have isomorphy of

$$C^{2,\alpha}(M) \longrightarrow C^{0,\alpha}(M) : v \longrightarrow \omega^{-\frac{n+2}{n-2}} L_{g_0} v \tag{3.8}$$

and compactness of

$$C^{2,\alpha}(M) \longrightarrow C^{0,\alpha}(M) : v \longrightarrow R_\omega \omega^{\frac{4}{n-2}} v. \tag{3.9}$$

Thus $\partial R$ is a Fredholm operator and the Smale-Sard lemma gives

$$R[\partial R \neq 0] = \cap_{k=1}^\infty O_k \tag{3.10}$$

with countably many open and dense subsets $O_k \subset Im(R)$. Covering

$$\mathbb{R}_{>0} = \cup_{k=1}^\infty A_{\frac{1}{k}} \tag{3.11}$$

we obtain the same result for $R : C^{2,\alpha}(M, \mathbb{R}_{>0}) \longrightarrow C^{0,\alpha}(M)$.

Thus, if $K \in C^{0,\alpha}(M)$ is the scalar curvature of a conformal metric

$$K = R_\omega = \omega^{-\frac{n+2}{n-2}} L\omega,\ \omega \in C^{2,\alpha}(M, \mathbb{R}_{>0}), \tag{3.12}$$

then obviously $K \in Im(R)$ and generically $K \in R[\partial R \neq 0]$, so

$$L_{g_0} v - \frac{n+2}{n-2} K\omega^{\frac{4}{n-2}} v \neq 0 \text{ for all } 0 \neq v \in C^{2,\alpha}(M), \tag{3.13}$$

whenever $K = R_\omega$. Consequently for a solution $L\omega = K\omega^{\frac{n+2}{n-2}}$

$$\partial^2 J(\omega) = \frac{2}{k^{\frac{n-2}{n}}}(L_{g_0} u - \frac{n+2}{n-2} K\omega^{\frac{n+2}{n-2}}) \tag{3.14}$$

is for a generic $K$ invertible, which is equivalent to $E_{\frac{n+2}{n-2}}(\omega) = \emptyset$.
Please note, that we may replace $C^{2,\alpha}, C^{0,\alpha}$ by any $C^{k+2,\alpha}, C^{k,\alpha}$. □

In light of the foregoing lemma the following parametrization is a natural application of the implicit function theorem.

**Lemma 3.7** (Degeneracy and pseudo critical points)**.**
*For* $\omega > 0$ *solving* $L_{g_0}\omega = K\omega^{\frac{n+2}{n-2}}$ *let*

$$\Pi = \Pi_{H_0(\omega)^{\perp_{L_{g_0}}}}$$

*be the projection on* $H_0(\omega)^{\perp_{L_{g_0}}}$*.*

*Then there exist* $\epsilon > 0$*, an open neighbourhood* $U$ *of* $\omega$

$$\omega \in U \subset W^{1,2}_{g_0}(M)$$

*and a smooth function* $h : B^{\mathbb{R}^{m+1}}_{\epsilon}(0) \longrightarrow H_0(\omega)^{\perp_{L_{g_0}}}$ *such, that*

$$\begin{aligned}&\{w \in U \mid \Pi\nabla J(w) = 0\}\\ &\qquad=\{u_{\alpha,\beta} = (1+\alpha)\omega + \beta^i \mathrm{e}_i + h(\alpha,\beta) \mid (\alpha,\beta) \in B^{m+1}_{\epsilon}(0)\}\end{aligned}$$

*with*

$$\|h(\alpha,\beta)\| = O(|\alpha|^2 + \|\beta\|^2),$$

*where* $\nabla J$ *is gradient of* $\partial J$ *with respect to the scalar product* $\langle\cdot,\cdot\rangle_{L_{g_0}}$*.*
*We call* $w \in U$ *a pseudo critical point related to* $\omega$*, if* $\Pi_{H_0(\omega)^{\perp_{L_{g_0}}}}\nabla J(w) = 0$*.*

Thus the construction above parametrizes in a neighbourhood of $\omega$ the set of pseudo critical points related to $\omega$; and clearly every critical point of $J$ is a pseudo critical point related to $\omega$ as well.

For the sake of clarity consider $u_{\alpha,\beta} > 0$ close to $\omega$ solving

$$\Pi_{H_0(\omega)^{\perp_{L_{g_0}}}}\nabla J(u_{\alpha,\beta}) = 0.$$

Then

$$\partial J(u_{\alpha,\beta})f = 2k_{u_{\alpha,\beta}}^{\frac{2-n}{n}}\int(L_{g_0}u_{\alpha,\beta} - \frac{r_{u_{\alpha,\beta}}}{k_{u_{\alpha,\beta}}}Ku_{\alpha,\beta}^{\frac{n+2}{n-2}})f,$$

so $\nabla J(u_{\alpha,\beta}) = \overline{u}_{\alpha,\beta}$ solves

$$L_{g_0}\overline{u}_{\alpha,\beta} = 2k_{u_{\alpha,\beta}}^{\frac{2-n}{n}}(L_{g_0}u_{\alpha,\beta} - \frac{r_{u_{\alpha,\beta}}}{k_{u_{\alpha,\beta}}}u_{\alpha,\beta}^{\frac{n+2}{n-2}}).$$

Thus $\Pi\overline{u}_{\alpha,\beta} = 0$ implies

$$\begin{aligned}L_{g_0}u_{\alpha,\beta} - \frac{r_{u_{\alpha,\beta}}}{k_{u_{\alpha,\beta}}}Ku_{\alpha,\beta}^{\frac{n+2}{n-2}} &= \frac{k_{u_{\alpha,\beta}}^{\frac{n-2}{n}}}{2}L_{g_0}\overline{u}_{\alpha,\beta}\\ &=\frac{k_{u_{\alpha,\beta}}^{\frac{n-2}{n}}}{2}\langle\overline{u}_{\alpha,\beta},\frac{\omega}{\|\omega\|}\rangle_{L_{g_0}}L_{g_0}\frac{\omega}{\|\omega\|} + \frac{k_{u_{\alpha,\beta}}^{\frac{n-2}{n}}}{2}\sum_{j=1}^{m}\langle\overline{u}_{\alpha,\beta},\mathrm{e}_j\rangle_{L_{g_0}}L_{g_0}\mathrm{e}_j\\ &=[\int(L_{g_0}u_{\alpha,\beta} - \frac{r_{u_{\alpha,\beta}}}{k_{u_{\alpha,\beta}}}Ku_{\alpha,\beta}^{\frac{n+2}{n-2}})\frac{w}{\|w\|}]L_{g_0}\frac{w}{\|w\|}\\ &\quad+\sum_{j=1}^{m}[\int(L_{g_0}u_{\alpha,\beta} - \frac{r_{u_{\alpha,\beta}}}{k_{u_{\alpha,\beta}}}Ku_{\alpha,\beta}^{\frac{n+2}{n-2}})\mathrm{e}_j]L_{g_0}\mathrm{e}_j\end{aligned}$$

***Proof of lemma 3.7.***
The statement is a mere application of the implicit function theorem to

$$W^{1,2}(M) = H_0(\omega) \oplus_{L_{g_0}} H_0(\omega)^{\perp_{L_{g_0}}} \longrightarrow H_0(\omega)^{\perp_{L_{g_0}}} : u \longrightarrow \Pi\nabla J(u). \quad (3.15)$$

Indeed $\Pi\nabla J(\omega) = 0$, since $\nabla J(\omega) = 0$. Moreover

$$\nabla(\Pi\nabla J)(\omega) = \Pi\nabla^2 J(\omega). \quad (3.16)$$

and from (3.4) and (3.5) we have isomorphy

$$\nabla^2 J(\omega)\lfloor_{H_0(\omega)^{\perp_{L_{g_0}}}} : H_0(\omega)^{\perp_{L_{g_0}}} \xrightarrow{\simeq} H_0(\omega)^{\perp_{L_{g_0}}}. \quad (3.17)$$

As $\Pi$ is the identity operator on $H_0(\omega)^{\perp_{L_{g_0}}}$, we obtain

$$\nabla_{H_0(\omega)^{\perp_{L_{g_0}}}}(\Pi\nabla J)(\omega) = \nabla^2 J(\omega)\lfloor_{H_0(\omega)^{\perp_{L_{g_0}}}} \quad (3.18)$$

and therefore isomorphy of $\nabla_{H_0(\omega)^{\perp_{L_{g_0}}}}(\Pi\nabla J)(\omega)$ as well.

Finally the estimate on $h$ follows from (3.2). □

Using Moser iteration one may improve this result to a smooth setting.

**Proposition 3.8** (Smoothness of $u_{\alpha,\beta}$)**.**
*For any $k \in \mathbb{N}$ we have $w, \mathrm{e}_i, u_{\alpha,\beta}, h_{\alpha,\beta} \in C^k$ and*

$$\|h(\alpha,\beta)\|_{C^k} \longrightarrow 0 \quad as \quad |\alpha| + \|\beta\| \longrightarrow 0.$$

***Proof of proposition 3.8.***
In view of lemma 3.6 let us write

$$u_{\alpha,\beta} = (1+\alpha)\omega + \beta^i \mathrm{e}_i + h(\alpha,\beta). \quad (3.19)$$

The equation solved by $u_{\alpha,\beta}$ is $\Pi\nabla J = 0$, which is equivalent to

$$\begin{aligned} L_{g_0}u_{\alpha,\beta} - (r\bar{K})_{u_{\alpha,\beta}} u_{\alpha,\beta}^{\frac{n+2}{n-2}} =& [\int (L_{g_0}u_{\alpha,\beta} - (r\bar{K})_{u_{\alpha,\beta}} u_{\alpha,\beta}^{\frac{n+2}{n-2}})\frac{\omega}{\|\omega\|}]L_{g_0}\frac{\omega}{\|\omega\|} \\ &+ \sum_{i=1}^m [\int (L_{g_0}u_{\alpha,\beta} - (r\bar{K})_{u_{\alpha,\beta}} u_{\alpha,\beta}^{\frac{n+2}{n-2}})\mathrm{e}_i]L_{g_0}\mathrm{e}_i \end{aligned} \quad (3.20)$$

In particular $L_{g_0}u_{\alpha,\beta} = Pu_{\alpha,\beta} + v_{\alpha,\beta}$ with $\|v_{\alpha,\beta}\|_{W_{g_0}^{-1,2}(M)} = O(|\alpha| + \|\beta\|)$ and

$$\|P\|_{L^{\frac{n}{2}}(B_r(x_0))} \overset{r\to 0}{\longrightarrow} 0 \text{ for all } x_0 \in M. \quad (3.21)$$

Let $p \geq 1$ and consider a suitable cut-off function $\eta \in C_0^1(B_{2r}(x_0))$. For

$$w_{\alpha,\beta} = u_{\alpha,\beta}^{2p-1}\eta^2 \text{ and } \mathrm{w}_{\alpha,\beta} = u_{\alpha,\beta}^p \eta \quad (3.22)$$

one obtains using Young's inequality and absorption

$$|\nabla \mathrm{w}_{\alpha,\beta}|^2_{g_0} \leq c_p(\langle \nabla u_{\alpha,\beta}, \nabla w_{\alpha,\beta}\rangle_{g_0} + u^{2p}_{\alpha,\beta}|\nabla \eta|^2_{g_0}) \tag{3.23}$$

and thus

$$\begin{aligned}\int L_{g_0}\mathrm{w}_{\alpha,\beta}\mathrm{w}_{\alpha,\beta} =& \int c_n|\nabla \mathrm{w}_{\alpha,\beta}|^2_{g_0} + R_{g_0}\mathrm{w}^2_{\alpha,\beta}\\ \leq& c_{n,p}\int L_{g_0}u_{\alpha,\beta}w_{\alpha,\beta} + u^{2p}_{\alpha,\beta}|\nabla\eta|^2_{g_0}\\ =& c_{n,p}\int Pu_{\alpha,\beta}w_{\alpha,\beta} + v_{\alpha,\beta}w_{\alpha,\beta} + u^{2p}_{\alpha,\beta}|\nabla\eta|^2_{g_0}.\end{aligned} \tag{3.24}$$

As $\mathrm{w}^2_{\alpha,\beta} = u_{\alpha,\beta}w_{\alpha,\beta}$ and $w_{\alpha,\beta} = \mathrm{w}_{\alpha,\beta}u^{p-1}_{\alpha,\beta}\eta$ one may absorb via (3.21) to get

$$\int L_{g_0}\mathrm{w}_{\alpha,\beta}\mathrm{w}_{\alpha,\beta} \leq C_{n,p}(\|v_{\alpha,\beta}u^{p-1}_{\alpha,\beta}\|^2_{L^{\frac{2n}{n+2}}} + \|u^{2p}_{\alpha,\beta}\|_{L^1_{g_0}}). \tag{3.25}$$

Suppose $u_{\alpha,\beta} \in L^r$, $r \geq \frac{2n}{n-2}$. We then get for $p = \frac{r}{2}$ using Hölder's inequality

$$\begin{aligned}\int L_{g_0}\mathrm{w}_{\alpha,\beta}\mathrm{w}_{\alpha,\beta} \leq& C_{n,p}(\|v_{\alpha,\beta}\|^2_{L^{\frac{nr}{n+r}}_{g_0}}\|u_{\alpha,\beta}\|^{r-2}_{L^r_{g_0}} + \|u_{\alpha,\beta}\|^r_{L^r_{g_0}})\\ \leq& C_{n,p}(\|v_{\alpha,\beta}\|^r_{L^{\frac{nr}{n+r}}_{g_0}} + \|u_{\alpha,\beta}\|^r_{L^r_{g_0}}).\end{aligned} \tag{3.26}$$

whence using a suitable covering $M = \sum_{i=1}^m B_{r_i}(x_i)$ we get

$$\|u_{\alpha,\beta}\|^2_{L^{\frac{n}{n-2}r}_{g_0}} \leq C_{n,p}(\|v_{\alpha,\beta}\|^r_{L^{\frac{nr}{n+r}}_{g_0}} + \|u_{\alpha,\beta}\|^r_{L^r_{g_0}}). \tag{3.27}$$

Note, that in case $|\alpha| + \|\beta\| = 0$ we have $u_{\alpha,\beta} = \omega$ and $v_{\alpha,\beta} = 0$, whence by iteration of (3.27) one obtains $w \in L^p_{g_0}$ for all $1 \leq p < \infty$. Due to

$$L_{g_0}\omega = K\omega^{\frac{n+2}{n-2}} \ \text{ and } \ L_{g_0}\mathrm{e}_j = \frac{n+2}{n-2}K\omega^{\frac{4}{n-2}}\mathrm{e}_j$$

this gives $\omega, \mathrm{e}_j \in C^\infty$ by standard regularity arguments.
Recalling (3.20) this implies $v_{\alpha,\beta} \in C^k$ and

$$\|v_{\alpha,\beta}\|_{C^k} = O(|\alpha| + \|\beta\|) \tag{3.28}$$

Thus we obtain by iteration of (3.27)

$$\forall\, 1 \leq q < \infty \ : \ \sup_{|\alpha|+\|\beta\|<\epsilon} \|u_{\alpha,\beta}\|_{L^q_{g_0}} < \infty. \tag{3.29}$$

and therefore $\sup_{|\alpha|+\|\beta\|<\epsilon}\|u_{\alpha,\beta}\|_{C^k} < \infty$. Since by the very definition of $u_{\alpha,\beta}$

$$\|h(\alpha,\beta)\| \longrightarrow 0 \ \text{ for } \ |\alpha| + \|\beta\| \longrightarrow 0, \tag{3.30}$$

this convergence generalizes to all $C^k$ by compact embedding. □

Note, that due to scaling invariance

$$\Pi\nabla J(\omega)=0 \Longleftrightarrow \forall\ \alpha>0\ :\ \Pi\nabla J(\alpha\omega)=0.$$

Thus we may reparametrise the pseudo critical points related to $\omega$ as

$$u_{\alpha,\beta}=\alpha(\omega+\beta^i \mathrm{e}_i+h(\beta)), h(\beta)\perp_{L_{g_0}} H_0(\omega),$$

where $\|h(\beta)\|=O(\|\beta\|^2)$ and $\|h(\beta)\|_{C^k}\longrightarrow 0$ as $\|\beta\|\longrightarrow 0$.

## 3.4 Critical points at infinity

**Definition 3.9** (A neighbourhood of critical points at infinity)**.**
*Let $\omega\geq 0$ solve $L_{g_0}\omega=K\omega^{\frac{n+2}{n-2}}$, $p\in\mathbb{N}$ and $\varepsilon>0$ sufficiently small. For $u\in W^{1,2}_{g_0}(M)$ we define*

$$\begin{aligned}A_u(\omega,p,\varepsilon)=\{&(\alpha,\beta_k,\alpha_i,\lambda_i,a_i)\in(\mathbb{R}_+,\mathbb{R}^m,\mathbb{R}^p_+,\mathbb{R}^p_+,M^p)\mid\\ &\forall_{i\neq j}\ \lambda_i^{-1},\lambda_j^{-1},\varepsilon_{i,j},|1-\frac{r\alpha_i^{\frac{4}{n-2}}K(a_i)}{4n(n-1)k}|,\\ &|1-\frac{r\alpha^{\frac{4}{n-2}}}{k}|,\|\beta\|,\|u-u_{\alpha,\beta}-\alpha^i\varphi_{a_i,\lambda_i}\|<\varepsilon\ \},\end{aligned}$$

*where*

$$\varepsilon_{i,j}=(\frac{\lambda_j}{\lambda_i}+\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j))^{\frac{2-n}{2}}.$$

*In case $p>0$ we call*

$$V(\omega,p,\varepsilon)=\{u\in W^{1,2}_{g_0}(M)\mid A_u(\omega,p,\varepsilon)\neq\emptyset\}$$

*a neighbourhood of a critical point at infinity.*

Keep in mind, that $k\equiv 1$ and $r\searrow r_\infty$ along a flow line. We would like to make a remark on two special cases.

(i) If $\omega=0$, then $u_{\alpha,\beta}=0$. So the conditions on $\alpha$ and $\beta_k$ are trivial. Thus the sets $A_u(0,p,\varepsilon)$ and $V(0,p,\varepsilon)$ naturally reduce to

$$\begin{aligned}A_u(p,\varepsilon)=\{&(\alpha_i,\lambda_i,a_i)\in(\mathbb{R}^p_+,\mathbb{R}^p_+,M^p)\mid\\ &\forall_{i\neq j}\ \lambda_i^{-1},\lambda_j^{-1},\varepsilon_{i,j},|1-\frac{r\alpha_i^{\frac{4}{n-2}}K(a_i)}{4n(n-1)k}|,\|u-\alpha^i\varphi_{a_i,\lambda_i}\|<\varepsilon\ \}\end{aligned}$$

and $V(p,\varepsilon)=\{u\in W^{1,2}_{g_0}(M)\mid A_u(p,\varepsilon)\neq\emptyset\}$.

(ii) $V(\omega,0,\varepsilon)$ corresponds to a neighbourhood the critical point line

$$\{\alpha\omega\mid\alpha>0\}.$$

So proposition 3.1 states, that every sequence $u(t_k)$ is precompact with respect to $V(\omega, p, \varepsilon)$ in the sense, that up to a subsequence for any $\varepsilon > 0$ we find an index $k_0$, for which $u_{t_k} \in V(w, p, \varepsilon)$ for some $p \geq 0$ and all $k \geq k_0$.

The subsequent reduction by minimization, whose prove we postpone to the appendix, makes the representation in $V(\omega, p, \varepsilon)$ unique.

**Proposition 3.10** (Optimal choice)**.**
*For every $\varepsilon_0 > 0$ there exists $\varepsilon_1 > 0$ such, that for $u \in V(\omega, p, \varepsilon)$ with $\varepsilon < \varepsilon_1$*

$$\inf_{(\tilde\alpha, \tilde\beta_k, \tilde\alpha_i, \tilde a_i, \tilde\lambda_i) \in A_u(\omega, p, 2\varepsilon_0)} \int K u^{\frac{4}{n-2}} |u - u_{\tilde\alpha, \tilde\beta} - \tilde\alpha^i \varphi_{\tilde a_i, \tilde\lambda_i}|^2$$

*admits an unique minimizer $(\alpha, \beta_k, \alpha_i, a_i, \lambda_i) \in A_u(\omega, p, \varepsilon_0)$ and we define*

$$\varphi_i = \varphi_{a_i, \lambda_i}, v = u - u_{\alpha, \beta} - \alpha^i \varphi_i, \;\; \varepsilon_{i,j} = \Big(\frac{\lambda_j}{\lambda_i} + \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i, a_j)\Big)^{\frac{2-n}{2}}.$$

*Moreover $(\alpha, \beta_k, \alpha_i, a_i, \lambda_i)$ depends smoothly on $u$.*

Thus for a sequence $u_l \in V(\omega, p, \varepsilon_l)$, $\varepsilon_l \longrightarrow 0$ we may assume, that for each $u_l$ there exists an unique representation in $A_{u_l}(\omega, p, \varepsilon_0)$, say

$$u_l = u_{\alpha_l, \beta_l} + \alpha^{i,l} \varphi_{a_{i,l}, \lambda_{i,l}} + v_l, \;\; (\alpha_l, \beta_{k,l}, \alpha_{i,l}, a_{i,l}, \lambda_{i,l}) \in A_{u_l}(\omega, p, \varepsilon_0)$$

and we have $(\alpha_l, \beta_{k,l}, \alpha_{i,l}, a_{i,l}, \lambda_{i,l}) \in A_{u_l}(\omega, p, \epsilon_l)$ for suitable $\epsilon_l \longrightarrow 0$.

The error term $v = u - u_{\alpha, \beta} - \alpha^i \varphi_i$ is with respect to the scalar product

$$\langle \cdot, \cdot \rangle_{K u^{\frac{4}{n-2}}} = \int \cdot K u^{\frac{4}{n-2}} \cdot$$

orthogonal to

$$\langle u_{\alpha, \beta}, \partial_{\beta_i} u_{\alpha, \beta}, \varphi_i, -\lambda_i \partial_{\lambda_i} \varphi_i, \frac{1}{\lambda_i} \nabla_{a_i} \varphi_i \rangle$$

and due to $|\delta J(u)| \longrightarrow 0$ almost orthogonal with respect to

$$\langle \cdot, \cdot \rangle_{L_{g_0}} = \int \cdot L_{g_0} \cdot .$$

**Definition 3.11** (The orthogonal bundle $H(\omega, p, \varepsilon)$)**.**
*For $u \in V(\omega, p, \varepsilon)$ let*

$$H_u(\omega, p, \varepsilon) = \langle u_{\alpha, \beta}, \partial_{\beta_i} u_{\alpha, \beta}, \varphi_i, -\lambda_i \partial_{\lambda_i} \varphi_i, \frac{1}{\lambda_i} \nabla_{a_i} \varphi_i \rangle^{\perp_{K u^{\frac{4}{n-2}}}}$$

*in case $\omega > 0$ and in case $\omega = 0$*

$$H_u(p, \varepsilon) = \langle \varphi_i, -\lambda_i \partial_{\lambda_i} \varphi_i, \frac{1}{\lambda_i} \nabla_{a_i} \varphi_i \rangle^{\perp_{K u^{\frac{4}{n-2}}}}$$

Orthogonality of the error term $v$ implies smallness of linear interactions. Subsequently we will even show, that essentially $v$ is negligible.

**Lemma 3.12** (Linear $v$-type interactions)**.**
*On $V(\omega, p, \varepsilon)$ for $\varepsilon > 0$ small we have*

*(i)* $\int L_{g_0}\phi_{k,i}v = o(\frac{1}{\lambda_i^{\frac{n-2}{2}}} + \sum_{i\neq j=1}^{p}\varepsilon_{i,j}) + O(\|v\|^2)$

*(ii)* $\int L_{g_0}u_{\alpha,\beta}v = o(\sum_r \frac{1}{\lambda_r^{\frac{n-2}{2}}}) + O(\|v\|^2 + |\delta J(u)|^2)$

*(iii)* $\int K u^{\frac{n+2}{n-2}}\phi_{k,i} = \int K(u_{\alpha,\beta} + \alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} + O(\|v\|^2)$

*(iv)* $\int K u^{\frac{n+2}{n-2}}u_{\alpha,\beta} = \int K(u_{\alpha,\beta} + \alpha^j\varphi_j)^{\frac{n+2}{n-2}}u_{\alpha,\beta} + O(\|v\|^2)$

*and more precisely for $u \in V(p, \varepsilon)$*

$$\int L_{g_0}\phi_{k,i}v = o(\frac{1}{\lambda_i^{n-2}} + \sum_{i\neq j=1}^{p}\varepsilon_{i,j}) + O(\frac{|\nabla K_i|^2}{\lambda_i^2} + \|v\|^2).$$

We use $K_i$ as a short hand notation for $K(a_i)$, $\nabla K_i$ for $\nabla K(a_i)$ etc.

***Proof of lemma 3.12***.
We first calculate the bubble type interactions. Recall

$$\phi_{k,i} = d_{k,i}\varphi_i, \text{ where } (d_{k,i})_{k=1,2,3} = (1, -\lambda_i\partial_{\lambda_i}, \frac{1}{\lambda_i}\nabla_{a_i}). \tag{3.31}$$

By lemma 3.3 one obtains

$$\begin{aligned}\int L_{g_0}\phi_{k,i}v = & \int d_{k,i}L_{g_0}\varphi_i v \\ = & 4n(n-1)\int_{B_\alpha(a_i)} d_{k,i}\varphi_i^{\frac{n+2}{n-2}}v + o(\frac{1}{\lambda_i^{n-2}}) + O(\|v\|^2),\end{aligned} \tag{3.32}$$

whence with $c_k > 0$

$$\int L_{g_0}\phi_{k,i}v = c_k\int \varphi_i^{\frac{4}{n-2}}\phi_{k,i}v + o(\frac{1}{\lambda_i^{n-2}}) + O(\|v\|^2). \tag{3.33}$$

Moreover we have

$$\int (K-K_i)\varphi_i^{\frac{4}{n-2}}\phi_{k,i}v = o(\frac{1}{\lambda_i^{n-2}}) + O(\frac{|\nabla K_i|^2}{\lambda_i^2} + \|v\|^2) \tag{3.34}$$

and thus

$$\int L_{g_0}\phi_{k,i}v = c_k\int \frac{K}{K_i}\varphi_i^{\frac{4}{n-2}}\phi_{k,i}v + o(\frac{1}{\lambda_i^{n-2}}) + O(\frac{|\nabla K_i|^2}{\lambda_i^2} + \|v\|^2). \tag{3.35}$$

Expanding $u^{\frac{4}{n-2}} = (\alpha^j \varphi_j + v)^{\frac{4}{n-2}}$ in case $u \in V(p, \varepsilon)$ we have

$$0 = \int K u^{\frac{4}{n-2}} \phi_{k,i} v = \int_{[\alpha^j \varphi_j \geq v]} K(\alpha^j \varphi_j)^{\frac{4}{n-2}} \phi_{k,i} v + O(\|v\|^2), \tag{3.36}$$

whence

$$\int K(\alpha^j \varphi_j)^{\frac{4}{n-2}} \phi_{k,i} v = O(\|v\|^2). \tag{3.37}$$

Thus we obtain, since $|\phi_{k,i}| \leq C\varphi_i$,

$$\begin{aligned} O(\|v\|^2) = & \int K(\alpha^j \varphi_j)^{\frac{4}{n-2}} \phi_{k,i} v \\ = & \int_{[\alpha_i \varphi_i \geq \sum_{i \neq j=1}^p \alpha_j \varphi_j]} K(\alpha_i \varphi_i + \sum_{i \neq j=1}^p \alpha_j \varphi_j)^{\frac{4}{n-2}} \phi_{k,i} v \\ & + \int_{[\alpha_i \varphi_i < \sum_{i \neq j=1}^p \alpha_j \varphi_j]} K(\alpha_i \varphi_i + \sum_{i \neq j=1}^p \alpha_j \varphi_j)^{\frac{4}{n-2}} \phi_{k,i} v \\ = & \int_{[\alpha_i \varphi_i \geq \sum_{i \neq j=1}^p \alpha_j \varphi_j]} K(\alpha_i \varphi_i)^{\frac{4}{n-2}} \phi_{k,i} v + O(\sum_{i \neq j=1}^p \int \varphi_j^{\frac{4}{n-2}} \varphi_i |v|) \\ = & \int K(\alpha_i \varphi_i)^{\frac{4}{n-2}} \phi_{k,i} v + O(\sum_{i \neq j=1}^p \int \varphi_j^{\frac{4}{n-2}} \varphi_i |v|). \end{aligned} \tag{3.38}$$

Using lemma 3.5 we have $\|\varphi_j^{\frac{4}{n-2}} \varphi_i\|_{L^{\frac{2n}{n+2}}} = O(\varepsilon_{i,j})$ for $i \neq j$. This gives

$$\int K(\alpha_i \varphi_i)^{\frac{4}{n-2}} \phi_{k,i} v = o(\varepsilon_{i,j}) + O(\|v\|^2) \tag{3.39}$$

Plugging this into (3.35) we conclude

$$\int L_{g_0} \phi_{k,i} v = o(\frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j=1}^p \varepsilon_{i,j}) + O(\frac{|\nabla K_i|^2}{\lambda_i^2} + \|v\|^2). \tag{3.40}$$

Expanding $u^{\frac{4}{n-2}} = (u_{\alpha,\beta} + \alpha^i \varphi_i + v)^{\frac{4}{n-2}}$ in case $u \in V(\omega, p, \varepsilon)$ we have

$$\begin{aligned} 0 = & \int K u^{\frac{4}{n-2}} \phi_{k,i} v = \int K(u_{\alpha,\beta} + \alpha^j \varphi_j + v)^{\frac{4}{n-2}} \phi_{k,i} v \\ = & \int_{[u_{\alpha,\beta} + \alpha^j \varphi_j \geq v]} K(u_{\alpha,\beta} + \alpha^j \varphi_j)^{\frac{4}{n-2}} \phi_{k,i} v + O(\|v\|^2) \end{aligned} \tag{3.41}$$

and thus

$$
\begin{aligned}
O(\|v\|^2) = & \int K(u_{\alpha,\beta} + \alpha^j \varphi_j)^{\frac{4}{n-2}} \phi_{k,i} v \\
= & \int_{[\varphi_i \geq u_{\alpha,\beta} + \sum_{i\neq j=1}^p \varphi_j]} K(u_{\alpha,\beta} + \alpha^j \varphi_j)^{\frac{4}{n-2}} \phi_{k,i} v \\
& + \int_{[\varphi_i < u_{\alpha,\beta} + \sum_{i\neq j=1}^p \varphi_j]} K(u_{\alpha,\beta} + \alpha^j \varphi_j)^{\frac{4}{n-2}} \phi_{k,i} v \\
= & \int_{[\varphi_i \geq u_{\alpha,\beta} + \sum_{i\neq j=1}^p \varphi_j]} K(\alpha_i \varphi_i)^{\frac{4}{n-2}} \phi_{k,i} v \\
& + O(\int_{[\varphi_i \geq u_{\alpha,\beta} + \sum_{i\neq j=1}^p \varphi_j]} \varphi_i^{\frac{4}{n-2}} (u_{\alpha,\beta} + \sum_{i\neq j=1}^p \varphi_j)|v| \\
& \quad + \int_{[\varphi_i < u_{\alpha,\beta} + \sum_{i\neq j=1}^p \varphi_j]} (u_{\alpha,\beta} + \sum_{i\neq j=1}^p \varphi_j)^{\frac{4}{n-2}} \varphi_i |v|).
\end{aligned}
\tag{3.42}
$$

This gives

$$
\begin{aligned}
& \int K(\alpha_i \varphi_i)^{\frac{4}{n-2}} \phi_{k,i} v \\
= & O(\int_{[\varphi_i \geq u_{\alpha,\beta}]} \varphi_i^{\frac{4}{n-2}} u_{\alpha,\beta}|v| + \int_{[\varphi_i \geq \sum_{i\neq j=1}^p \varphi_j]} \varphi_i^{\frac{4}{n-2}} \sum_{i\neq j=1}^p \varphi_j |v| \\
& + \int_{[\varphi_i < u_{\alpha,\beta}]} (u_{\alpha,\beta})^{\frac{4}{n-2}} \varphi_i |v| + \int_{[\varphi_i < \sum_{i\neq j=1}^p \varphi_j]} (\sum_{i\neq j=1}^p \varphi_j)^{\frac{4}{n-2}} \varphi_i |v|),
\end{aligned}
\tag{3.43}
$$

whence by Hölder's inequality, direct integration and lemma 3.5

$$
\int K \varphi_i^{\frac{4}{n-2}} \phi_{k,i} v = o(\frac{1}{\lambda_i^{\frac{n-2}{2}}} + \sum_{i\neq j=1}^p \varepsilon_{i,j}) + O(\|v\|^2). \tag{3.44}
$$

Plugging this into (3.35) we conclude

$$
\int L_{g_0} \phi_{k,i} v = o(\frac{1}{\lambda_i^{\frac{n-2}{2}}} + \sum_{i\neq j=1}^p \varepsilon_{i,j}) + O(\|v\|^2). \tag{3.45}
$$

Next we calculate for $u \in V(\omega, p, \varepsilon)$ as before

$$
\begin{aligned}
0 = & \int K u^{\frac{4}{n-2}} u_{\alpha,\beta} v = \int K(u_{\alpha,\beta} + \alpha^i \varphi_i)^{\frac{4}{n-2}} u_{\alpha,\beta} v + O(\|v\|^2) \\
= & \int K u_{\alpha,\beta}^{\frac{n+2}{n-2}} v \\
& + O(\int_{u_{\alpha,\beta} \geq \alpha^i \varphi_i} u_{\alpha,\beta}^{\frac{4}{n-2}} \varphi_i |v| + \int_{u_{\alpha,\beta} < \alpha^i \varphi_i} (\alpha^i \varphi_i)^{\frac{4}{n-2}} u_{\alpha,\beta}|v| + \|v\|^2) \\
= & \int K u_{\alpha,\beta}^{\frac{n+2}{n-2}} v + o(\sum_r \frac{1}{\lambda_r^{\frac{n-2}{2}}}) + O(\|v\|^2),
\end{aligned}
\tag{3.46}
$$

whence due to (3.46) and $\Pi\nabla J(u_{\alpha,\beta})=0$, cf. the remark on lemma 3.7

$$\begin{aligned}\int L_{g_0}u_{\alpha,\beta}v=&\int(L_{g_0}u_{\alpha,\beta}-(r\bar{K})_{u_{\alpha,\beta}}u_{\alpha,\beta}^{\frac{n+2}{n-2}})v\\&+o(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}})+O(\|v\|^2)\\=&\int(L_{g_0}u_{\alpha,\beta}-(r\bar{K})_{u_{\alpha,\beta}}u_{\alpha,\beta}^{\frac{n+2}{n-2}})\frac{\omega}{\|\omega\|}\int L_{g_0}\frac{\omega}{\|\omega\|}v\\&+\sum_{i=1}^m\int(L_{g_0}u_{\alpha,\beta}-(r\bar{K})_{u_{\alpha,\beta}}u_{\alpha,\beta}^{\frac{n+2}{n-2}})\mathrm{e}_i\int L_{g_0}\mathrm{e}_iv\\&+o(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}})+O(\|v\|^2).\end{aligned}\tag{3.47}$$

This gives

$$\begin{aligned}\int L_{g_0}u_{\alpha,\beta}v=&\int(L_{g_0}u-(r\bar{K})_{u_{\alpha,\beta}}u^{\frac{n+2}{n-2}})\omega\int L_{g_0}\omega v\\&+\sum_{i=1}^m\int(L_{g_0}u-(r\bar{K})_{u_{\alpha,\beta}}u^{\frac{n+2}{n-2}})\mathrm{e}_i\int L_{g_0}\mathrm{e}_iv\\&+o(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}})+O(\|v\|^2)\\=&O(|(\frac{r}{k})_{u_{\alpha,\beta}}-(\frac{r}{k})_u|^2)\\&+o(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}})+O(\|v\|^2+|\delta J(u)|^2).\end{aligned}\tag{3.48}$$

Note, that

$$\begin{aligned}\int(L_{g_0}u-(r\bar{K})_uu^{\frac{n+2}{n-2}})u_{\alpha,\beta}=&\int L_{g_0}u_{\alpha,\beta}u_{\alpha,\beta}-(r\bar{K})_uu_{\alpha,\beta}^{\frac{2n}{n-2}}\\&+O(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}}+\|v\|),\end{aligned}\tag{3.49}$$

whence as a rough estimate

$$(\frac{r}{k})_{u_{\alpha,\beta}}-(\frac{r}{k})_u=O(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}}+\|v\|+|\delta J(u)|).\tag{3.50}$$

This proves

$$\int L_{g_0}u_{\alpha,\beta}v=o(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}})+O(\|v\|^2+|\delta J(u)|^2).\tag{3.51}$$

Moreover for $u \in V(\omega, p, \varepsilon)$

$$\begin{aligned}\int K u^{\frac{n+2}{n-2}} \phi_{k,i} = & \int K(u_{\alpha,\beta} + \alpha^j \varphi_j)^{\frac{n+2}{n-2}} \phi_{k,i} \\ & + \frac{n+2}{n-2} \int K(u_{\alpha,\beta} + \alpha^j \varphi_j)^{\frac{4}{n-2}} \phi_{k,i} v + O(\|v\|^2)\end{aligned} \tag{3.52}$$

and we simply estimate

$$0 = \int K u^{\frac{4}{n-2}} \phi_{k,i} v = \int K(u_{\alpha,\beta} + \alpha^j \varphi_j)^{\frac{4}{n-2}} \phi_{k,i} v + O(\|v\|^2). \tag{3.53}$$

□

## 3.5 Convergence versus critical points at infinity

Due to the Lojasiewicz inequality one has along a flow line either convergence or a time sequence blowing up.

**Proposition 3.13** (Unicity of a limiting critical point)**.**
*If a sequence $u(t_k)$ converges in $L^{\frac{2n}{n-2}}$ to a critical point $u_\infty$ of $J$, then*

$$u \longrightarrow u_\infty \ \ in \ \ C^\infty \ \ as \ \ t \longrightarrow \infty$$

*with at least polynomial, but generically exponential convergence rate in $C^{k,\alpha}$.*

More precisely genericity arises from the fact, that generically the second variation is non degenerate, cf. lemma 3.6, and exponential speed of convergence holds true, whenever the limiting critical point is non degenerate.

In particular the proposition implies, that in order to show flow convergence we have to exclude the case of blow up, so we may assume the latter case arguing by contradiction.

***Proof of proposition 3.13***. ([13], proposition 2.6)
Suppose $\|u(\tau_l) - \omega\|_{L^{\frac{2n}{n-2}}} \longrightarrow 0$ as $\tau_l \nearrow \infty$, but $\|u - \omega\|_{L^{\frac{2n}{n-2}}} \not\longrightarrow 0$ as $t \longrightarrow \infty$.

For $\varepsilon_0 > 0$ small we then find a decomposition

$$a_1 < b_1 < a_2 < b_2 < \ldots < b_{m-1} < a_m < b_m < a_{m+1} < \ldots \tag{3.54}$$

such, that

$$\sum_m (a_m, b_m) = \{t > 0 \mid \|u - \omega\|_{L^{\frac{2n}{n-2}}} < \varepsilon_0\} \tag{3.55}$$

and for a subsequence $\tau_l \in (a_{m_l}, b_{m_l})$.

$$\begin{aligned}\|u(b_{m_l}) - u(\tau_l)\|^{\frac{n}{n-2}}_{L^{\frac{2n}{n-2}}} = & (\int |u(b_{m_l}) - u(\tau_l)|^{\frac{2n}{n-2}})^{\frac{1}{2}} \\ \leq & c(\int |u^{\frac{n}{n-2}}(b_{m_l}) - u^{\frac{n}{n-2}}(\tau_l)|^2)^{\frac{1}{2}} = c\|u^{\frac{n}{n-2}}(b_{m_l}) - u^{\frac{n}{n-2}}(\tau_l)\|_{L^2} \\ \leq & c\int_{\tau_l}^{b_{m_l}} \|\partial_t u^{\frac{n}{n-2}}\|_{L^2} \leq c\int_{a_{m_l}}^{b_{m_l}} |\delta J(u)|,\end{aligned} \tag{3.56}$$

whence according to proposition 2.11 we may assume

$$b_{m_l} - a_{m_l} \longrightarrow \infty. \tag{3.57}$$

Passing to a subsequence we thus may inductively decompose

$$[a_{m_{l_1}}, b_{m_{l_1}}) = \sum_{k=1}^{m_1} [s_k, t_k),\ 2^k \leq t_k - s_k < c2^{k+1},\ c \in [1,3) \tag{3.58}$$

and

$$[a_{m_{l_2}}, b_{m_{l_2}}) = \sum_{k=m_1+1}^{m_2} [s_k, t_k),\ 2^k \leq t_k - s_k < c2^{k+1},\ c \in [1,3) \tag{3.59}$$

and so on.
By analyticity of $J$ we may use the Lojasiewicz inequality

$$\exists\, C > 0, \gamma \in (0,1]\ \forall\, u \in B_{\varepsilon_0}(\omega)\ :\ |J(u) - J(\omega)| \leq C \|\partial J(u)\|^{1+\gamma}, \tag{3.60}$$

cf. [21], Theorem 4.1. Clearly $J(\omega) = J_\infty = r_\infty$ and along a flow line we have

$$\|\partial J(u)\| \leq C |\delta J(u)|. \tag{3.61}$$

Thus for $t \in (s_k, t_k)$

$$\partial_t J(u) \leq -\, c|\delta J(u)|^2 \leq -C(J(u) - J_\infty)^{\frac{2}{\gamma+1}}. \tag{3.62}$$

Without loss of generality $\gamma < 1$, whence $\partial_t (J(u) - J_\infty)^{\frac{\gamma-1}{\gamma+1}} \geq c$ and

$$(J(u(t_k)) - J_\infty)^{\frac{\gamma-1}{\gamma+1}} \geq (J(u(s_k)) - J_\infty)^{\frac{\gamma-1}{\gamma+1}} + c(t_k - s_k) \tag{3.63}$$

and in particular $J(u(t_k)) - J_\infty \leq c(t_k - s_k)^{\frac{\gamma+1}{\gamma-1}}$. We conclude

$$\begin{aligned}
&(\int_{s_k}^{t_k} |\delta J(u)|)^2 \\
&\leq (t_k - s_k) \int_{s_k}^{t_k} |\delta J(u)|^2 \leq c(t_k - s_k)(J(u(s_k)) - J(u(t_k))) \\
&\leq c(t_k - s_k)(J(u(s_k)) - J_\infty) \leq c(t_k - s_k)(J(u(t_{k-1})) - J_\infty) \\
&\leq c(t_k - s_k)(t_{k-1} - s_{k-1})^{\frac{\gamma+1}{\gamma-1}} \leq c_2 2^{k+1} (2^{k-1})^{\frac{\gamma+1}{\gamma-1}} \leq c(2^{\frac{2\gamma}{\gamma-1}})^{k-1}
\end{aligned} \tag{3.64}$$

having used Jensen's inequality. Consequently

$$\sum_{m_l} \int_{a_{m_l}}^{b_{m_l}} |\delta J(u)| = \sum_k \int_{s_k}^{t_k} |\delta J(u)| < \infty, \tag{3.65}$$

whence $\lim_{l\to\infty}\int_{a_{m_l}}^{b_{m_l}}|\delta J(u)|=0$. This contradicts (3.56) and we conclude

$$u\longrightarrow\omega \text{ in } L^{\frac{2n}{n-2}} \text{ as } t\longrightarrow\infty. \tag{3.66}$$

Now let $x_0\in M$. Then $\|R\|_{L_\mu^{\frac{n}{2}}(B_r(x_0))}=o(r)$ by proposition 2.11, whence

$$L_{g_0}u=Ru^{\frac{n+2}{n-2}}=Pu \text{ with } \|P\|_{L_{g_0}^{\frac{n}{2}}(B_r(x_0))}=o(r). \tag{3.67}$$

Lemma 7.2 then shows $\sup_{t\geq 0}\|u\|_{L_{g_0}^p}<\infty$ for all $p\geq 1$ and due to

$$-c_n\Delta_{g_0}u=(R-r\bar{K})u^{\frac{n+2}{n-2}}+r\bar{K}u^{\frac{n+2}{n-2}}-R_{g_0}u \tag{3.68}$$

and proposition 2.11 it follows, that $(-\Delta u)\subset L^p$ and applying Calderon-Zygmund estimates, that $(u)\subset W^{2,p}\hookrightarrow L^\infty$ is uniformly bounded.
Then lemma 2.3 shows $0<c<u<C<\infty$. Due to proposition 2.11 we have $\int|R-r\bar{K}|^p d\mu\longrightarrow 0$ for all $p\geq 1$. With this at hand one may repeat the arguments proving proposition 2.8 to show

$$|u(x_1,t_1)-u(x_2,t_2)|\leq C(\alpha)(|t_1-t_2|^{\frac{\alpha}{2}}+d(x_1,x_2)^\alpha), \tag{3.69}$$

for all $x_1,x_2\in M$ and $0\leq t_1,t_2<\infty$, $|t_1-t_2|\leq 1$, where

$$0<\alpha<\min\{\frac{4}{n},1\}. \tag{3.70}$$

By standard regularity arguments then $(u)\subset C^{k,\alpha}$ is uniformly bounded. As for the speed of convergence note, that as before we have

$$\partial_t(J(u)-J_\infty)^{\frac{\gamma-1}{\gamma+1}}\geq c. \tag{3.71}$$

From this we obtain polynomial convergence of $J(u)$, namely

$$0<J(u)-J_\infty<\frac{C}{(1+t)^{\frac{1+\gamma}{1-\gamma}}}. \tag{3.72}$$

Moreover

$$\partial_t\|u^{\frac{n}{n-2}}-\omega^{\frac{n}{n-2}}\|_{L^2}\leq c|\delta J(u)| \tag{3.73}$$

and applying once more the Lojasiewicz inequality (3.60)

$$\begin{aligned}\partial_t(J(u)-J_\infty)^{\frac{\gamma}{1+\gamma}}\leq&-c(J(u)-J_\infty)^{\frac{\gamma}{1+\gamma}-1}|\delta J(u)|^2\\ \leq&-c(J(u)-J_\infty)^{-\frac{1}{1+\gamma}}\|\partial J(u)\||\delta J(u)|\leq -c|\delta J(u)|,\end{aligned} \tag{3.74}$$

whence

$$\partial_t\|u^{\frac{n}{n-2}}-\omega^{\frac{n}{n-2}}\|_{L^2}\leq -C\partial_t(J(u)-J(_\infty))^{\frac{\gamma}{1+\gamma}}. \tag{3.75}$$

We conclude polynomial convergence $u \longrightarrow \omega$ in $L^{\frac{2n}{n-2}}$ via

$$\begin{aligned}\|u-\omega\|_{L^{\frac{2n}{n-2}}}^{\frac{n}{n-2}} \leq & C\|u^{\frac{n}{n-2}}-\omega^{\frac{n}{n-2}}\|_{L^2} \leq C(J(u)-J(\infty))^{\frac{\gamma}{1+\gamma}} \\ \leq & \frac{C}{(1+t)^{\frac{\gamma}{1-\gamma}}}.\end{aligned} \tag{3.76}$$

With uniform boundedness at hand we may use Sobolev space interpolation

$$\|v\|_{W^{k,p}} \leq C(k,p)\|v\|_{W^{k-1,p}}^{\frac{1}{2}}\|v\|_{W^{k+1,p}}^{\frac{1}{2}} \tag{3.77}$$

to conclude polynomial convergence at least in each Sobolev or Hölder space.

Note, that in case $\gamma = 1$ we have

$$\partial_t(J(u)-J_\infty) \leq -c|\delta J(u)|^2 \leq -C|J(u)-J_\infty|, \tag{3.78}$$

whence $J(u) \searrow J_\infty$ with convergence at exponential rate. Moreover

$$\partial_t\|u^{\frac{n}{n-2}}-\omega^{\frac{n}{n-2}}\|_{L^2} \leq c|\delta J(u)| \tag{3.79}$$

and

$$\partial_t(J(u)-J_\infty))^{\frac{1}{2}} \leq -c(J(u)-J_\infty)^{-\frac{1}{2}}|\delta J(u)|^2 \leq -C|\delta J(u)|. \tag{3.80}$$

By the same arguments as before we conclude $u \longrightarrow \omega$ at exponential rate in every Sobolev or Hölder space in case $\gamma = 1$.

In the generic case $E_{\frac{n+2}{n-2}}(\omega) = \emptyset$, cf. lemma 3.6, however the Lojasiewicz inequality (3.60) holds with optimal exponent $\gamma = 1$.

Indeed $J(u) = J(\omega)$ for $u \in \langle\omega\rangle = H_0(\omega)$ by scaling invariance and

$$|J(u)-J(\omega)| \leq |u-\omega|^2 \text{ and } |\delta J(u)| \geq c|u-\omega| \tag{3.81}$$

for $u \in \langle\omega\rangle^{\perp_{L_{g_0}}} = H_0(\omega)^{\perp_{L_{g_0}}} = kern(\partial^2 J(\omega))$. □

# 4 Case ω=0

The starting point in this section is a flow line $u \in V(p,\varepsilon)$, that we study by analysing the evolution of the parameters $\alpha_i, \lambda_i, a_i$ in the representation

$$u = \alpha^i\varphi_i + v = \alpha^i\varphi_{a_i,\lambda_i} + v$$

given by proposition 3.10. To that end we test the flow equation

$$\partial_t u = -\frac{1}{K}(R - r\bar{K})$$

with $\varphi_i, \lambda_i\partial_{\lambda_i}\varphi_i$ and $\frac{1}{\lambda_i}\nabla_{a_i}\varphi_i$, cf. definition 3.4.

**Lemma 4.1** (The shadow flow)**.**
*For* $u\in V(p,\varepsilon)$ *with* $\varepsilon>0$ *and*

$$\sigma_{k,i}=-\int(L_{g_0}u-r\bar{K}u^{\frac{n+2}{n-2}})\phi_{k,i},\ i=1,\ldots,p,\ k=1,2,3$$

*we have by testing* $K\partial_t u=-(R-r\bar{K})u$ *with* $u^{\frac{4}{n-2}}\phi_{k,i}$

*(i)*
$$\frac{\dot{\alpha}_i}{\alpha_i}=\frac{\alpha_i^{\frac{n+2}{2-n}}}{c_1K_i}\sigma_{1,i}(1+o_{\frac{1}{\lambda_i}}(1))+R_{1,i}$$

*(ii)*
$$-\frac{\dot{\lambda}_i}{\lambda_i}=\frac{\alpha_i^{\frac{n+2}{2-n}}}{c_2K_i}\sigma_{2,i}(1+o_{\frac{1}{\lambda_i}}(1))+R_{2,i}$$

*(iii)*
$$\lambda_i\dot{a}_i=\frac{\alpha_i^{\frac{n+2}{2-n}}}{c_3K_i}\sigma_{3,i}(1+o_{\frac{1}{\lambda_i}}(1))+R_{3,i}$$

*with constants* $c_k>0$ *given in lemma 3.5 and*

$$R_{k,i}=O(\sum_{r\neq s}\varepsilon_{r,s}^2+\|v\|^2+|\delta J(u)|^2)_{k,i}.$$

***Proof of lemma 4.1.***
For each $i,j=1,\ldots,p,\ k=1,2,3$ let

$$(\dot{\xi}_{1,j},\dot{\xi}_{2,j},\dot{\xi}_{3,j})=(\dot{\alpha}_j,-\alpha_j\frac{\dot{\lambda}_j}{\lambda_j},\alpha_j\lambda_j\dot{a}_j)\tag{4.1}$$

and recall

$$\phi_{k,i}=d_{k,i}\varphi_i=(\varphi_i,-\lambda_i\partial_{\lambda_i}\varphi_i,\frac{1}{\lambda_i}\nabla_{a_i}\varphi_i).\tag{4.2}$$

Testing $K\partial_t u=-(R-r\bar{K})u$ with $u^{\frac{4}{n-2}}\phi_{k,i}$ we obtain using $\int Ku^{\frac{4}{n-2}}\phi_{k,i}v=0$

$$\begin{aligned}\sigma_{k,i}=&\int\partial_t uKu^{\frac{4}{n-2}}\phi_{k,i}=\int\partial_t(\alpha^j\varphi_j+v)Ku^{\frac{4}{n-2}}\phi_{k,i}\\=&\dot{\xi}^{l,j}\int Ku^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i}-\int Kv[\partial_t u^{\frac{4}{n-2}}\phi_{k,i}+u^{\frac{4}{n-2}}\partial_t\phi_{k,i}].\end{aligned}\tag{4.3}$$

Note, that

$$\begin{aligned}\int Ku^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i}=&\int K(\alpha^m\varphi_m)^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i}+O(\|v\|)_{k,i,l,j}\\=&c_k\alpha_i^{\frac{4}{n-2}}K_i\delta_{kl}\delta_{ij}+O(\frac{|\nabla K_i|}{\lambda_i}+\frac{1}{\lambda_i^2}+\frac{1}{\lambda_i^{n-2}})_{k,l}\delta_{ij}\\&+O(\sum_{i\neq m=1}^{p}\varepsilon_{i,m}+\|v\|)_{k,i,l,j}.\end{aligned}\tag{4.4}$$

Indeed

$$
\begin{aligned}
&\int K(\alpha^m\varphi_m)^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i} \\
&\quad= \int_{[\varphi_i\geq\sum_{i\neq m=1}^p\varphi_m]} K(\alpha_i\varphi_i)^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i} \\
&\qquad+\sum_{i\neq m=1}^p O(\int_{[\varphi_i\geq\sum_{i\neq m=1}^p\varphi_m]}\varphi_i^{\frac{4}{n-2}}\varphi_j\varphi_m+\int_{[\varphi_i<\sum_{i\neq m=1}^p\varphi_m]}\varphi_m^{\frac{4}{n-2}}\varphi_j\varphi_i),
\end{aligned}
\tag{4.5}
$$

whence by means of lemma 3.5 we have

$$
\begin{aligned}
&\int K(\alpha^m\varphi_m)^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i} \\
&\qquad= \int_{[\varphi_i\geq\sum_{i\neq m=1}^p\varphi_m]} K(\alpha_i\varphi_i)^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i}+O(\sum_{i\neq m=1}^p\varepsilon_{i,m}) \\
&\qquad=\int K(\alpha_i\varphi_i)^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i} \\
&\qquad\quad+O(\int_{[\varphi_i<\sum_{i\neq m=1}^p\varphi_m]}\varphi_i^{\frac{n+2}{n-2}}\varphi_j+\sum_{i\neq m=1}^p\varepsilon_{i,m}) \\
&\qquad=\alpha_i^{\frac{4}{n-2}}\int K\varphi_i^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i}+O(\sum_{i\neq m=1}^p\varepsilon_{i,m}) \\
&\qquad=\alpha_i^{\frac{4}{n-2}}\delta_{ij}\int K\varphi_i^{\frac{4}{n-2}}\phi_{l,i}\phi_{k,i}+O(\sum_{i\neq m=1}^p\varepsilon_{i,m}) \\
&\qquad=\alpha_i^{\frac{4}{n-2}}\delta_{ij}\delta_{kl}\int K\varphi_i^{\frac{4}{n-2}}\phi_{k,i}^2+O(\frac{1}{\lambda_i^2}+\frac{1}{\lambda_i^{n-2}})\delta_{ij}+O(\sum_{i\neq m=1}^p\varepsilon_{i,m}).
\end{aligned}
\tag{4.6}
$$

From this (4.4) follows. Moreover we may write

$$
\int Ku^{\frac{4}{n-2}}\partial_t\phi_{k,i}v=O(\|v\|)_{i,k,l,j}\dot{\xi}^{l,j}
\tag{4.7}
$$

using $|\partial_\alpha\phi_{k,i}|,|\lambda_i\partial_{\lambda_i}\phi_{k,i}|,|\frac{1}{\lambda_i}\nabla_{a_i}\phi_{k,i}|\leq C\varphi_i$ and estimate

$$
\begin{aligned}
|\int Kv\partial_t u^{\frac{4}{n-2}}\Phi_{k,i}|=&\frac{4}{n-2}|\int v(R-r\bar{K})u^{\frac{4}{n-2}}\phi_{k,i}| \\
\leq& C\int|R-r\bar{K}|u^{\frac{4}{n-2}}\varphi_i|v|=C\int|R-r\bar{K}|u^{\frac{4}{n-2}}|u-v||v| \\
\leq& C\int|R-r\bar{K}|u^{\frac{n+2}{n-2}}|v|+C\int|R-r\bar{K}|u^{\frac{4}{n-2}}|v|^2 \\
\leq& C(\|R-r\bar{K}\|_{L_\mu^{\frac{2n}{n+2}}}\|v\|+\|R-r\bar{K}\|_{L_\mu^{\frac{n}{2}}}\|v\|^2)
\end{aligned}
\tag{4.8}
$$

using $|\phi_{k,i}| \leq C\varphi_i$, whence according to proposition 2.11 we obtain

$$\int K v \partial_t u^{\frac{4}{n-2}} \Phi_{k,i} = O(|\delta J(u)|^2 + \|v\|^2). \tag{4.9}$$

Thus plugging (4.4), (4.7) and (4.9) into (4.3) we obtain for

$$\begin{aligned}\Xi_{k,i,l,j} =& c_k \alpha_i^{\frac{4}{n-2}} K_i \delta_{kl}\delta_{ij} \\ & + O(\frac{|\nabla K_i|}{\lambda_i} + \frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^{n-2}})_{k,l}\delta_{ij} + O(\sum_{i\neq m=1}^{p} \varepsilon_{i,m} + \|v\|)_{k,i,l,j}\end{aligned} \tag{4.10}$$

the identity

$$\Xi_{k,i,l,j}\dot{\xi}^{l,j} = \sigma_{k,i} + O(\|v\|^2 + |\delta J(u)|^2)_{k,i}. \tag{4.11}$$

For the inverse $\Xi^{-1}$ of $\Xi$ we then have

$$\begin{aligned}\Xi^{-1}_{k,i,l,j} =& \frac{\alpha^{\frac{4}{2-n}}}{c_k K_i} \delta_{kl}\delta_{ij} \\ & + O(\frac{|\nabla K_i|}{\lambda_i} + \frac{1}{\lambda_i^2} + \frac{1}{\lambda_i^{n-2}})_{k,l}\delta_{ij} + O(\sum_{i\neq m=1}^{p} \varepsilon_{i,m} + \|v\|)_{k,i,l,j}\end{aligned} \tag{4.12}$$

and the claim follows, since by definition $\sigma_{k,i} = O(|\delta J(u)|)$. □

Consequently our task is two folded. We have to carefully evaluate $\sigma_{k,i}$ by expansion and find suitable estimates on the error term $v$.

**Proposition 4.2** (Analysing $\sigma_{k,i}$).
*On $V(p,\varepsilon)$ for $\varepsilon > 0$ small we have with constants $b_1, \ldots, e_4 > 0$*

*(i)*

$$\begin{aligned}\sigma_{1,i} =& 4n(n-1)\alpha_i[\frac{r\alpha_i^{\frac{4}{n-2}} K_i}{4n(n-1)k} - 1] \int \varphi_i^{\frac{2n}{n-2}} \\ & + 4n(n-1) \sum_{i\neq j=1}^{p} \alpha_j [\frac{r\alpha_j^{\frac{4}{n-2}} K_j}{4n(n-1)k} - 1] b_1 \varepsilon_{i,j} \\ & + d_1 \alpha_i \frac{H_i}{\lambda_i^{n-2}} + e_1 \frac{r\alpha_i^{\frac{n+2}{n-2}}}{k} \frac{\Delta K_i}{\lambda_i^2} + b_1 \frac{r\alpha_i^{\frac{4}{n-2}} K_i}{k} \sum_{i\neq j=1}^{p} \alpha_j \varepsilon_{i,j} + R_{1,i}\end{aligned}$$

*(ii)*

$$\begin{aligned}\sigma_{2,i} =& -4n(n-1)\alpha_i[\frac{r\alpha_i^{\frac{4}{n-2}} K_i}{4n(n-1)k} - 1] \int \varphi_i^{\frac{n+2}{n-2}} \lambda_i \partial_{\lambda_i} \varphi_i \\ & - 4n(n-1) b_2 \sum_{i\neq j=1}^{p} \alpha_j [\frac{r\alpha_j^{\frac{4}{n-2}} K_j}{4n(n-1)k} - 1] \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} + d_2 \alpha_i \frac{H_i}{\lambda_i^{n-2}} \\ & + e_2 \frac{r\alpha_i^{\frac{n+2}{n-2}}}{k} \frac{\Delta K_i}{\lambda_i^2} - b_2 \frac{r\alpha_i^{\frac{4}{n-2}} K_i}{k} \sum_{i\neq j=1}^{p} \alpha_j \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} + R_{2,i}\end{aligned}$$

*(iii)*

$$\begin{aligned}\sigma_{3,i} =&4n(n-1)\alpha_i[\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{4n(n-1)k}-1]\int\varphi_i^{\frac{n+2}{n-2}}\frac{1}{\lambda_i}\nabla_{a_i}\varphi_i\\ &+4n(n-1)b_3\sum_{i\neq j=1}^{p}\alpha_j[\frac{r\alpha_j^{\frac{4}{n-2}}K_j}{4n(n-1)k}-1]\frac{1}{\lambda_i}\nabla_{a_i}\varepsilon_{i,j}\\ &+\frac{r\alpha_i^{\frac{n+2}{n-2}}}{k}[e_3\frac{\nabla K_i}{\lambda_i}+e_4\frac{\nabla\Delta K_i}{\lambda_i^3}]\\ &+b_3\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{k}\sum_{i\neq j=1}^{p}\frac{\alpha_j}{\lambda_i}\nabla_{a_i}\varepsilon_{i,j}+R_{3,i},\end{aligned}$$

*where* $R_{k,i}=o_\varepsilon(\frac{1}{\lambda_i^{n-2}}+\sum_{i\neq j=1}^{p}\varepsilon_{i,j})_{k,i}+O(\sum_{r\neq s}\varepsilon_{r,s}^2+\|v\|^2)_{k,i}$.

***Proof of proposition 4.2.***
By definition and conformal invariance

$$\sigma_{k,i}=-\int(L_{g_0}u-r\bar{K}u^{\frac{n+2}{n-2}})\phi_{k,i}=-\int(R-r\bar{K})u^{\frac{n+2}{n-2}}\phi_{k,i}. \qquad (4.13)$$

We start evaluating

$$\int L_{g_0}u\phi_{k,i}=\int L_{g_0}(\alpha^j\varphi_j+v)\phi_{k,i}=\alpha^j\int L_{g_0}\varphi_j\phi_{k,i}+\int L_{g_0}\phi_{k,i}v. \qquad (4.14)$$

Using lemmata 3.3 and 3.5 we obtain for $\alpha>0$ small

$$\begin{aligned}\alpha^j\int L_{g_0}\varphi_j\phi_{k,i}=&\alpha_i\int L_{g_0}\varphi_i\phi_{k,i}+\sum_{i\neq j=1}^{p}\alpha_j\int L_{g_0}\varphi_j\phi_{k,i}\\ =&4n(n-1)\alpha_i\int_{B_\alpha(a_i)}\varphi_i^{\frac{n+2}{n-2}}\phi_{k,i}\\ &-2nc_n\alpha_i\int_{B_\alpha(a_i)}(((n-1)H_i+r_i\partial_{r_i}H_i)r_i^{n-2}\varphi_i^{\frac{n+2}{n-2}})\phi_{k,i}\\ &+4n(n-1)\sum_{i\neq j=1}^{p}\alpha_j\int_{B_\alpha(a_j)}\varphi_j^{\frac{n+2}{n-2}}\phi_{k,i}+o_\varepsilon(\frac{1}{\lambda_i^{n-2}}+\sum_{i\neq j=1}^{p}\varepsilon_{i,j}) \qquad (4.15)\\ =&4n(n-1)\alpha_i\int_{B_\alpha(a_i)}\varphi_i^{\frac{n+2}{n-2}}\phi_{k,i}+4n(n-1)b_k\sum_{i\neq j=1}^{p}\alpha_jd_{k,i}\varepsilon_{i,j}\\ &-2nc_n\alpha_i\int_{B_\alpha(a_i)}((n-1)H_i+r_i\partial_{r_i}H_i)r_i^{n-2}\varphi_i^{\frac{n+2}{n-2}})\phi_{k,i}\\ &+o_\varepsilon(\frac{1}{\lambda_i^{n-2}}+\sum_{i\neq j=1}^{p}\varepsilon_{i,j}).\end{aligned}$$

Indeed the curvature related term arising from lemma 3.3 is of order

$$\int_{B_\alpha(0)} \frac{r^2}{\lambda_i}\left(\frac{\lambda_i}{1+\lambda_i^2 r^2}\right)^{\frac{n-2}{2}(\frac{n}{n-2}+1)} = \lambda_i^{-4} O(\lambda_i, \ln \lambda_i, 1) = o\left(\frac{1}{\lambda_i^{n-2}}\right). \tag{4.16}$$

Thus

$$\begin{aligned}
&\alpha^j \int L_{g_0}\varphi_j \phi_{k,i} \\
=&4n(n-1)[\alpha_i \int \varphi_i^{\frac{n+2}{n-2}}\phi_{k,i} + b_k \sum_{i\neq j=1}^{p} \alpha_j d_{k,i}\varepsilon_{i,j}] \\
&-(n-1)(n-2)c_n\alpha_i H_i \int_{B_\alpha(0)} r^{n-2}(1, -\lambda_i\partial_{\lambda_i}, \frac{1}{\lambda_i}\nabla)\left(\frac{\lambda_i}{1+\lambda_i^2 r^2}\right)^n \\
&-(n-2)c_n\alpha_i \nabla H_i \int_{B_\alpha(0)} \nabla r r^{n-1}(1, -\lambda_i\partial_{\lambda_i}, \frac{1}{\lambda_i}\nabla)\left(\frac{\lambda_i}{1+\lambda_i^2 r^2}\right)^n \\
&+o_\varepsilon\left(\frac{1}{\lambda_i^{n-2}} + \sum_{i\neq j=1}^{p}\varepsilon_{i,j}\right)
\end{aligned} \tag{4.17}$$

using $\gamma_n \nabla_{a_i} G_{a_i}^{\frac{2}{2-n}} = 2x + O(r^{n-1})$. By radial symmetry we then get

$$\begin{aligned}
\alpha^j \int L_{g_0}\varphi_j \phi_{k,i} =&4n(n-1)[\alpha_i \int \varphi_i^{\frac{n+2}{n-2}}\phi_{k,i} + b_k \sum_{i\neq j=1}^{p} \alpha_j d_{k,i}\varepsilon_{i,j}] \\
&-\alpha_i\left(d_1\frac{H_i}{\lambda_i^{n-2}}, d_2\frac{H_i}{\lambda_i^{n-2}}, d_3\frac{\nabla H_i}{\lambda_i^{n-1}}\right) \\
&+o_\varepsilon\left(\frac{1}{\lambda_i^{n-2}} + \sum_{i\neq j=1}^{p}\varepsilon_{i,j}\right)
\end{aligned} \tag{4.18}$$

with $d_k > 0$. Inserting this into (4.14) and applying lemma 3.12 gives

$$\begin{aligned}
&\int L_{g_0} u \phi_{k,i} = \int L_{g_0}(\alpha^j\varphi_j + v)\phi_{k,i} \\
=&4n(n-1)[\alpha_i \int \varphi_i^{\frac{n+2}{n-2}}\phi_{k,i} + b_k \sum_{i\neq j=1}^{p} \alpha_j d_{k,i}\varepsilon_{i,j}] \\
&-\alpha_i\left(d_1\frac{H_i}{\lambda_i^{n-2}}, d_2\frac{H_i}{\lambda_i^{n-2}}, d_3\frac{\nabla H_i}{\lambda_i^{n-1}}\right) + o_\varepsilon\left(\frac{1}{\lambda_i^{n-2}} + \sum_{i\neq j=1}^{p}\varepsilon_{i,j}\right) + O(\|v\|^2).
\end{aligned} \tag{4.19}$$

Next from lemma 3.12 we infer

$$\int K u^{\frac{n+2}{n-2}}\phi_{k,i} = \int K(\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} + O(\|v\|^2). \tag{4.20}$$

Clearly

$$
\begin{aligned}
&\int K(\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} \\
&= \int\limits_{[\alpha_i\varphi_i \geq \sum_{i\neq j=1}^{p}\alpha_j\varphi_j]} K(\alpha_i\varphi_i)^{\frac{n+2}{n-2}}\phi_{k,i} + \frac{n+2}{n-2}(\alpha_i\varphi_i)^{\frac{4}{n-2}}\sum_{i\neq j=1}^{p}\alpha_j\varphi_j\phi_{k,i} \\
&\quad + \int\limits_{[\alpha_i\varphi_i < \sum_{i\neq j=1}^{p}\alpha_j\varphi_j]} K(\sum_{i\neq j=1}^{p}\alpha_j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} \\
&\quad + O(\int\limits_{[\varphi_i\geq\epsilon\sum_{i\neq j=1}^{p}\varphi_j]}\varphi_i^{\frac{4}{n-2}}\sum_{i\neq j=1}^{p}\varphi_j^2 + \int\limits_{[\epsilon\varphi_i<\sum_{i\neq j=1}^{p}\varphi_j]}\sum_{i\neq j=1}^{p}\varphi_j^{\frac{4}{n-2}}\varphi_i^2),
\end{aligned}
\tag{4.21}
$$

whence

$$
\begin{aligned}
&\int K(\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} \\
&= \int K(\alpha_i\varphi_i)^{\frac{n+2}{n-2}}\phi_{k,i} + \frac{n+2}{n-2}(\alpha_i\varphi_i)^{\frac{4}{n-2}}\sum_{i\neq j=1}^{p}\alpha_j\varphi_j\phi_{k,i} \\
&\quad + \int K(\sum_{i\neq j=1}^{p}\alpha_j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} \\
&\quad + O(\int\limits_{[\varphi_i\geq\epsilon\sum_{i\neq j=1}^{p}\varphi_j]}\varphi_i^{\frac{4}{n-2}}\sum_{i\neq j=1}^{p}\varphi_j^2 + \int\limits_{[\epsilon\varphi_i<\sum_{i\neq j=1}^{p}\varphi_j]}\sum_{i\neq j=1}^{p}\varphi_j^{\frac{4}{n-2}}\varphi_i^2).
\end{aligned}
\tag{4.22}
$$

Therefore we obtain applying lemma 3.5

$$
\begin{aligned}
\int K(\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i}
&= \int K(\alpha_i\varphi_i)^{\frac{n+2}{n-2}}\phi_{k,i} + \frac{n+2}{n-2}(\alpha_i\varphi_i)^{\frac{4}{n-2}}\sum_{i\neq j=1}^{p}\alpha_j\varphi_j\phi_{k,i} \\
&\quad + \int K(\sum_{i\neq j=1}^{p}\alpha_j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} + o_{\varepsilon}(\sum_{i\neq j=1}^{p}\varepsilon_{i,j}).
\end{aligned}
\tag{4.23}
$$

Moreover note, that for $\epsilon > 0$ sufficiently small

$$
M = \cup_{i=1}^{p}[\varphi_i > \epsilon\sum_{i\neq j=1}^{p}\varphi_j] = \cup_{i=1}^{p}A_i,
\tag{4.24}
$$

whence for $B_i = A_i \setminus \cup_{i\neq j=1}^p A_j$ we have $M = \sum_{i=1}^p B_i$. This gives

$$\int K(\sum_{i\neq j=1}^{p} \alpha_j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} = \sum_{i\neq j=1}^{p} \int_{B_j} K(\sum_{i\neq j=1}^{p} \alpha_j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} + o(\sum_{i\neq j=1}^{p} \varepsilon_{i,j})$$
$$= \sum_{i\neq j=1}^{p} \int K(\alpha_j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} + O(\sum_{\substack{s\neq i, r\neq i\\ r\neq s}} \int \varphi_r^{\frac{4}{n-2}}\varphi_s\varphi_i) + o(\sum_{i\neq j=1}^{p} \varepsilon_{i,j}) \tag{4.25}$$

and we obtain using Hölder's inequality and lemma 3.5

$$\int K(\sum_{i\neq j=1}^{p} \alpha_j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} = \sum_{i\neq j=1}^{p} \int K(\alpha_j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} + o(\sum_{i\neq j=1}^{p} \varepsilon_{i,j}) + O(\sum_{r\neq s} \varepsilon_{r,s}^2). \tag{4.26}$$

Therefore

$$\begin{aligned}\int K(\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} =& \alpha_i^{\frac{n+2}{n-2}} \int K\varphi_i^{\frac{n+2}{n-2}}\phi_{k,i} + \sum_{i\neq j=1}^{p} \alpha_j^{\frac{n+2}{n-2}} \int K\varphi_j^{\frac{n+2}{n-2}}\phi_{k,i}\\ &+ \frac{n+2}{n-2}\alpha_i^{\frac{4}{n-2}} \sum_{i\neq j=1}^{p} \alpha_j \int K\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\varphi_j\\ &+ o_\varepsilon(\sum_{i\neq j=1}^{p} \varepsilon_{i,j}) + O(\sum_{r\neq s} \varepsilon_{r,s}^2).\end{aligned} \tag{4.27}$$

By a simple expansion we then get

$$\begin{aligned}\int K(\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} =& \alpha_i^{\frac{n+2}{n-2}} K_i \int \varphi_i^{\frac{n+2}{n-2}}\phi_{k,i} + \sum_{i\neq j=1}^{p} \alpha_j^{\frac{n+2}{n-2}} K_j \int \varphi_j^{\frac{n+2}{n-2}}\phi_{k,i}\\ &+ \frac{n+2}{n-2}\alpha_i^{\frac{4}{n-2}} K_i \sum_{i\neq j=1}^{p} \alpha_j \int \varphi_i^{\frac{4}{n-2}}\phi_{k,i}\varphi_j\\ &+ \alpha_i^{\frac{n+2}{n-2}}(e_1\frac{\Delta K_i}{\lambda_i^2}, e_2\frac{\Delta K_i}{\lambda_i^2}, e_3\frac{\nabla K_i}{\lambda_i} + e_4\frac{\nabla\Delta K_i}{\lambda_i^3})\\ &+ o_\varepsilon(\frac{1}{\lambda_i^{n-2}} + \sum_{i\neq j=1}^{p} \varepsilon_{i,j}) + O(\sum_{r\neq s} \varepsilon_{r,s}^2).\end{aligned} \tag{4.28}$$

Indeed using (7.11), (7.12), (7.13) we have in case $k = 1$,

$$
\begin{aligned}
&\int (K - K_i)\varphi_i^{\frac{n+2}{n-2}}\phi_{k,i} \\
&= \int_{B_{\lambda_i\alpha}(0)} \frac{K(\frac{1}{\lambda_i}\cdot) - K(0)}{(1 + r^2(1 + \frac{1}{\lambda_i^{n-2}}r^{n-2}H_{a_i}(\frac{\cdot}{\lambda_i}))^{\frac{2}{2-n}})^n} + O(\frac{1}{\lambda_i^n}) \\
&= \int_{B_{\lambda_i\alpha}(0)} \frac{K(\frac{1}{\lambda_i}\cdot) - K(0)}{(1 + r^2)^n} + O(\frac{1}{\lambda_i^{n-1}}) = e_1\frac{\Delta K_i}{\lambda_i^2} + o(\frac{1}{\lambda_i^{n-2}}),
\end{aligned}
\tag{4.29}
$$

where $e_1 = \frac{1}{2n}\int_{\mathbb{R}^n}\frac{r^2}{(1+r^2)^n}$. In case $k = 2$ we get

$$
\begin{aligned}
&\int (K - K_i)\varphi_i^{\frac{n+2}{n-2}}\phi_{k,i} \\
=&\frac{n-2}{2}\frac{1}{\lambda_i}\int_{B_{\lambda_i\alpha}(0)} \frac{(K(\frac{1}{\lambda_i}\cdot) - K_i)(r^2 - 1)}{(1 + r^2)^{n+1}} + O(\frac{1}{\lambda_i^{n-1}}) \\
=&e_2\frac{\Delta K_i}{\lambda_i^2} + o(\frac{1}{\lambda_i^{n-2}}),
\end{aligned}
\tag{4.30}
$$

where $e_2 = \frac{(n-2)}{4n}\int_{\mathbb{R}^n}\frac{r^2(r^2-1)}{(1+r^2)^{n+1}}$ and in case $k = 3$

$$
\begin{aligned}
\int (K - K_i)\varphi_i^{\frac{n+2}{n-2}}\phi_{k,i} = &\frac{n-2}{2n}\int (K - K_i)\frac{1}{\lambda_i}\nabla_{a_i}\varphi_i^{\frac{2n}{n-2}} \\
=&\frac{n-2}{2n}\frac{\nabla K_i}{\lambda_i}\int \varphi_i^{\frac{2n}{n-2}} + \frac{n-2}{2n}\frac{\nabla_{a_i}}{\lambda_i}\int (K - K_i)\varphi_i^{\frac{2n}{n-2}} \\
=&e_3\frac{\nabla K_i}{\lambda_i} + e_4\frac{\nabla\Delta K_i}{\lambda_i^3} + o(\frac{1}{\lambda_i^{n-2}})
\end{aligned}
\tag{4.31}
$$

with $e_3 = \frac{n-2}{2n}\int_{\mathbb{R}^n}\frac{1}{(1+r^2)^n}, e_4 = \frac{n-2}{4n^2}\int_{\mathbb{R}^n}\frac{r^2}{(1+r^2)^n}$.

Plugging (4.28) into (4.20) gives

$$
\begin{aligned}
&\int Ku^{\frac{n+2}{n-2}}\phi_{k,i} \\
&=\alpha_i^{\frac{n+2}{n-2}}K_i\int \varphi_i^{\frac{n+2}{n-2}}\phi_{k,i} + \sum_{i\neq j=1}^{p}\alpha_j^{\frac{n+2}{n-2}}K_j\int \varphi_j^{\frac{n+2}{n-2}}\phi_{k,i} \\
&\quad + \frac{n+2}{n-2}\alpha_i^{\frac{4}{n-2}}K_i\sum_{i\neq j=1}^{p}\alpha_j\int \varphi_i^{\frac{4}{n-2}}\phi_{k,i}\varphi_j \\
&\quad + \alpha_i^{\frac{n+2}{n-2}}(e_1\frac{\Delta K_i}{\lambda_i^2}, e_2\frac{\Delta K_i}{\lambda_i^2}, e_3\frac{\nabla K_i}{\lambda_i} + e_4\frac{\nabla\Delta K_i}{\lambda_i^3}) \\
&\quad + o_\varepsilon(\frac{1}{\lambda_i^{n-2}} + \sum_{i\neq j=1}^{p}\varepsilon_{i,j}) + O(\sum_{r\neq s}\varepsilon_{r,s}^2 + \|v\|^2)
\end{aligned}
\tag{4.32}
$$

and inserting finally (4.19) and (4.32) into (4.13) we conclude

$$\begin{aligned}\sigma_{k,i} =& -4n(n-1)[\alpha_i\int\varphi_i^{\frac{n+2}{n-2}}\phi_{k,i}+b_k\sum_{i\neq j=1}^p\alpha_j d_{k,i}\varepsilon_{i,j}]\\ &+\alpha_i(d_1\frac{H_i}{\lambda_i^{n-2}},d_2\frac{H_i}{\lambda_i^{n-2}},d_3\frac{\nabla H_i}{\lambda_i^{n-1}})\\ &+\alpha_i^{\frac{n+2}{n-2}}\frac{r}{k}K_i\int\varphi_i^{\frac{n+2}{n-2}}\phi_{k,i}+\sum_{i\neq j=1}^p\alpha_j^{\frac{n+2}{n-2}}\frac{r}{k}K_j\int\varphi_j^{\frac{n+2}{n-2}}\phi_{k,i}\\ &+\frac{n+2}{n-2}\alpha_i^{\frac{4}{n-2}}\frac{r}{k}K_i\sum_{i\neq j=1}^p\alpha_j\int\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\varphi_j\\ &+\alpha_i^{\frac{n+2}{n-2}}\frac{r}{k}(e_1\frac{\Delta K_i}{\lambda_i^2},e_2\frac{\Delta K_i}{\lambda_i^2},e_3\frac{\nabla K_i}{\lambda_i}+e_4\frac{\nabla\Delta K_i}{\lambda_i^3})\\ &+o_\varepsilon(\frac{1}{\lambda_i^{n-2}}+\sum_{i\neq j=1}^p\varepsilon_{i,j})+O(\sum_{r\neq s}\varepsilon_{r,s}^2+\|v\|^2).\end{aligned}\tag{4.33}$$

The claim follows. □

As $\sigma_{1,i}=O(|\delta J(u)|)$ the equations for $\sigma_{2,i},\sigma_{3,i}$ simplify significantly.

**Corollary 4.3** (Simplifying $\sigma_{k,i}$)**.**
*On $V(p,\varepsilon)$ for $\varepsilon>0$ small we have with constants $b_2,\ldots,e_4>0$*

*(i)*
$$\sigma_{2,i}=d_2\alpha_i\frac{H_i}{\lambda_i^{n-2}}+e_2\frac{r\alpha_i^{\frac{n+2}{n-2}}}{k}\frac{\Delta K_i}{\lambda_i^2}-b_2\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{k}\sum_{i\neq j=1}^p\alpha_j\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}+R_{2,i}$$

*(ii)*
$$\sigma_{3,i}=\frac{r\alpha_i^{\frac{n+2}{n-2}}}{k}[e_3\frac{\nabla K_i}{\lambda_i}+e_4\frac{\nabla\Delta K_i}{\lambda_i^3}]+b_3\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{k}\sum_{i\neq j=1}^p\frac{\alpha_j}{\lambda_i}\nabla_{a_i}\varepsilon_{i,j}+R_{3,i},$$

*where*

$$R_{k,i}=o_\varepsilon(\frac{1}{\lambda_i^{n-2}}+\sum_{i\neq j=1}^p\varepsilon_{i,j})_{k,i}+O(\sum_{r\neq s}\varepsilon_{r,s}^2+\|v\|^2+|\delta J(u)|^2)_{k,i}.$$

***Proof of corollary 4.3.***
We have

$$C|\delta J(u)|\geq|\int(R-r\bar{K})u^{\frac{n+2}{n-2}}\varphi_i|=|\sigma_{1,i}|,\tag{4.34}$$

whence due to proposition 4.2 for $k=1$

$$\begin{aligned}\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{4n(n-1)k}=&1+O(\frac{1}{\lambda_i^{n-2}}+\frac{|\Delta K_i|}{\lambda_i^2}+\sum_{i\neq j=1}^p\varepsilon_{i,j}\\ &+\sum_{r\neq s}\varepsilon_{r,s}^2+\|v\|^2+|\delta J(u)|).\end{aligned}\tag{4.35}$$

Inserting (4.35) into proposition 4.2 for $k=2,3$ proves the claim, since

$$\frac{\nabla_{a_i}}{\lambda_i}\int \varphi_i^{\frac{2n}{n-2}}, \lambda_i\partial_{\lambda_i}\int \varphi_i^{\frac{2n}{n-2}} = O(\frac{1}{\lambda_i^{n-2}}). \tag{4.36}$$

□

We turn to estimate the error term $v$. To do so we characterize the first two derivatives of $J$ at $\alpha^i\varphi_i = u - v$.

**Proposition 4.4** (Derivatives on $H(p,\varepsilon)$)**.**
*For $\varepsilon > 0$ small let $u = \alpha^i\varphi_i + v \in V(p,\varepsilon)$ and $h_1, h_2 \in H = H_u(p,\varepsilon)$.*
*We then have*

*(i)*
$$\|\partial J(\alpha^i\varphi_i)\lfloor_H\| = O(\sum_r \frac{|\nabla K_r|}{\lambda_r} + \frac{|\Delta K_r|}{\lambda_r^2} + \frac{1}{\lambda_r^{n-2}} + \sum_{r\neq s}\varepsilon_{r,s} + \|v\|^2 + |\delta J(u)|)$$

*(ii)*
$$\frac{1}{2}\partial^2 J(\alpha^i\varphi_i)h_1h_2 = k_{\alpha^i\varphi_i}^{\frac{2-n}{n}}[\int L_{g_0}h_1h_2 - c_n n(n+2)\sum_i\int \varphi_i^{\frac{4}{n-2}}h_1h_2] + o_\varepsilon(\|h_1\|\,\|h_2\|)$$

***Proof of proposition 4.4.***
Let in addition $h \in H_u(p,\varepsilon)$ with $\|h\| = 1$. From proposition 1.1 we then infer

$$\frac{1}{2}\partial J(\alpha^i\varphi_i)h = k_{\alpha^i\varphi_i}^{\frac{2-n}{n}}[\int L_{g_0}(\alpha^i\varphi_i)h - \int (r\bar{K})_{\alpha^i\varphi_i}(\alpha^i\varphi_i)^{\frac{n+2}{n-2}}h] \tag{4.37}$$

and

$$\begin{aligned}\frac{1}{2}\partial^2 J(\alpha^i\varphi_i)h_1h_2 =& k_{\alpha^i\varphi_i}^{\frac{2-n}{n}}[\int L_{g_0}h_1h_2 - \frac{n+2}{n-2}\int (r\bar{K})_{\alpha^i\varphi_i}(\alpha^i\varphi_i)^{\frac{4}{n-2}}h_1h_2] \\ &+ o_\varepsilon(\|h_1\|\|h_2\|),\end{aligned} \tag{4.38}$$

since, when considering the formula for the second variation, we have

$$\begin{aligned}\int L_{g_0}uh_i =& \frac{r}{k}\int Ku^{\frac{n+2}{n-2}}h_i + O(|\delta J(u)|\|h_i\|) \\ =& \frac{r}{k}\int Ku^{\frac{4}{n-2}}vh_i + O(|\delta J(u)|\|h_i\|) \\ =& O(\|v\| + |\delta J(u)|)\|h_i\|.\end{aligned} \tag{4.39}$$

Using $\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{k} = 4n(n-1) + o_\varepsilon(1)$ and $c_n = 4\frac{n-1}{n-2}$ we obtain

$$\begin{aligned}\frac{1}{2}\partial^2 J(\alpha^i\varphi_i)h_1h_2 =& k_{\alpha^i\varphi_i}^{\frac{2-n}{n}}[\int L_{g_0}h_1h_2 - c_n n(n+2)\int \sum_i \tilde{\varphi}_i^{\frac{4}{n-2}}h_1h_2] \\ &+ o_\varepsilon(\|h_1\|\|h_2\|),\end{aligned} \tag{4.40}$$

This shows the statement on the second derivative. Moreover by lemma 3.12

$$\frac{r_{\alpha^i\varphi_i}}{k_{\alpha^i\varphi_i}} =\frac{r}{k}+o(\sum_r\frac{1}{\lambda_r^{n-2}}+\sum_{r\neq s}\varepsilon_{r,s})+O(\sum_r\frac{|\nabla K_r|^2}{\lambda_r^2}+\|v\|^2). \tag{4.41}$$

We obtain with $r\bar{K}=\frac{r}{k}K=(\frac{r}{k})_u K$

$$\begin{aligned}\frac{1}{2}\partial J(\alpha^i\varphi_i)h =& k_{\alpha^i\varphi_i}^{\frac{2-n}{n}}[\int L_{g_0}(\alpha^i\varphi_i)h-\int r\bar{K}(\alpha^i\varphi_i)^{\frac{n+2}{n-2}}h] \\ & +o(\sum_r\frac{1}{\lambda_r^{n-2}}+\sum_{r\neq s}\varepsilon_{r,s})+O(\sum_r\frac{|\nabla K_r|^2}{\lambda_r^2}+\|v\|^2),\end{aligned} \tag{4.42}$$

where due to lemmata 3.3 and 3.5

$$\begin{aligned}\int K(\alpha^i\varphi_i)^{\frac{n+2}{n-2}}h =& \sum_i\frac{\alpha_i^{\frac{n+2}{n-2}}K_i}{4n(n-1)}\int L_{g_0}\varphi_i h \\ & +O(\sum_r\frac{|\nabla K_r|}{\lambda_r}+\frac{|\Delta K_r|}{\lambda_r^2}+\frac{1}{\lambda_r^{n-2}}+\sum_{r\neq s}\varepsilon_{r,s}).\end{aligned} \tag{4.43}$$

This gives

$$\begin{aligned}\frac{1}{2}\partial J(\alpha^i\varphi_i)h =& k_{\alpha^i\varphi_i}^{\frac{2-n}{n}}\alpha^i(1-\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{4n(n-1)k})\int L_{g_0}\varphi_i h \\ & +O(\sum_r\frac{|\nabla K_r|}{\lambda_r}+\frac{|\Delta K_r|}{\lambda_r^2}+\frac{1}{\lambda_r^{n-2}}+\sum_{r\neq s}\varepsilon_{r,s})+\|v\|^2.\end{aligned} \tag{4.44}$$

From this the assertion on the first derivative follows from (4.35). □

The second variation at $\alpha^i\varphi_i$ turns out to be positive definite.

**Proposition 4.5** (Positivity of the second variation)**.**
*There exist $\gamma,\varepsilon_0>0$ such, that for any*

$$u=\alpha^i\varphi_i+v\in V(p,\varepsilon)$$

*with $0<\varepsilon<\varepsilon_0$ we have*

$$\partial^2 J(\alpha^i\varphi_i)\lfloor_H>\gamma,\quad H=H_u(p,\varepsilon).$$

***Proof of proposition 4.5***. (Cf. [13], proposition 5.4)
In view of proposition 4.4 there would otherwise exist

$$\epsilon_k\searrow 0 \text{ and } (w_k)\subset H_{u_k}(p,\epsilon_k) \tag{4.45}$$

such, that

$$1=\int c_n|\nabla w_k|_{g_0}^2+R_{g_0}w_k^2\leq c_n n(n+2)\lim_{k\nearrow\infty}\int\sum_i\varphi_{i,k}^{\frac{4}{n-2}}w_k^2. \tag{4.46}$$

We order $\frac{1}{\lambda_{1_k}} \leq \ldots \leq \frac{1}{\lambda_{p_k}}$ and choose $\gamma_k \nearrow \infty$ tending to infinity slower than

$$\frac{1}{\lambda_{i_k}}, \varepsilon_{i_k,j_k} \longrightarrow 0 \tag{4.47}$$

does tend to zero in the sense, that for all $i < j$

$$\frac{\lambda_{i_k}}{\gamma_k}, \; \frac{\frac{\lambda_{i_k}}{\lambda_{j_k}} + \lambda_{i_k} G^{\frac{1}{2-n}}(a_{i_k}, a_{j_k})}{\gamma_k} \nearrow \infty \tag{4.48}$$

as $k \longrightarrow \infty$. Define inductively

$$\Omega_{j,k} = B_{\frac{\gamma_k}{\lambda_{j_k}}}(a_{j_k}) \setminus \cup_{i<j} B_{\frac{\gamma_k}{\lambda_{i_k}}}(a_{i_k}). \tag{4.49}$$

Then there exists $j = 1, \ldots, p$ such, that

$$\lim_{k \longrightarrow \infty} \int \varphi_{j,k}^{\frac{4}{n-2}} w_k^2 > 0 \tag{4.50}$$

and

$$\lim_{k \longrightarrow \infty} \int_{\Omega_{j,k}} c_n |\nabla w_k|^2_{g_0} + R_{g_0} w_k^2 \leq c_n n(n+2) \lim_{k \longrightarrow \infty} \int \varphi_{j,k}^{\frac{4}{n-2}} w_k^2. \tag{4.51}$$

Blowing up on $\Omega_{j,k}$ one obtains $\tilde{w}_k \rightharpoonup: \tilde{w}$ locally with $\tilde{w} \in W^{1,2}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} |\nabla \tilde{w}|^2 \leq n(n+2) \int_{\mathbb{R}^n} (\frac{1}{1+r^2})^2 \tilde{w}^2, \; \int_{\mathbb{R}^n} (\frac{1}{1+r^2})^2 \tilde{w}^2 > 0. \tag{4.52}$$

In particular $\tilde{w} \neq 0$. But due to orthogonality $w_k \in H_{u_k}(p, \varepsilon)$ one finds

$$\int_{\mathbb{R}^n} (\frac{1}{1+r^2})^{\frac{n+2}{2}} \tilde{w}, \int_{\mathbb{R}^n} (\frac{1}{1+r^2})^{\frac{n+2}{2}} \frac{1-r^2}{1+r^2} \tilde{w} = 0 \tag{4.53}$$

and

$$\int_{\mathbb{R}^n} (\frac{1}{1+r^2})^{\frac{n+2}{2}} \frac{x}{1+r^2} \tilde{w}(x) = 0. \tag{4.54}$$

This is a contradiction, cf. [28] Appendix D, pp.49-51. □

Smallness of the first and positivity of the second derivative give a suitable estimate on the error term $v$.

**Corollary 4.6** (A-priori estimate on $v$)**.**
*On $V(p, \varepsilon)$ for $\varepsilon > 0$ small we have*

$$\|v\| = O(\sum_r \frac{|\nabla K_r|}{\lambda_r} + \frac{|\Delta K_r|}{\lambda_r^2} + \frac{1}{\lambda_r^{n-2}} + \sum_{r \neq s} \varepsilon_{r,s} + |\delta J(u)|).$$

***Proof of corollary 4.6.***
Note, that $\partial^2 J$ is uniformly Hölder continuous on $V(p,\varepsilon)$ by proposition 1.1 and the remarks following, whence in view of proposition 4.5 we have

$$\begin{aligned}\partial J(u)v =&\partial J(\alpha^i\varphi_i+v)v = \partial J(\alpha^i\varphi_i)v+\partial^2 J(\alpha^i\varphi_i)v^2+o(\|v\|^2)\\ \geq&\partial J(\alpha^i\varphi_i)v+\gamma\|v\|^2+o(\|v\|^2).\end{aligned} \tag{4.55}$$

Since $v\in H_u(p,\varepsilon)$ the claim follows from proposition 4.4 by absorption. □

Thus having analysed $\sigma_{k,i}$ and the error term $v$ the shadow flow reads as

**Corollary 4.7** (Simplifying the shadow flow)**.**
*For $u\in V(p,\varepsilon)$ with $\varepsilon>0$ small we have*

*(i)*
$$\begin{aligned}-\frac{\dot\lambda_i}{\lambda_i} =&\frac{r}{k}[\frac{d_2}{c_2}\frac{H_i}{\lambda_i^{n-2}}+\frac{e_2}{c_2}\frac{\Delta K_i}{K_i\lambda_i^2}-\frac{b_2}{c_2}\sum_{i\neq j=1}^{p}\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}](1+o_{\frac{1}{\lambda_i}}(1))\\ &+R_{2,i}\end{aligned}$$

*(ii)*
$$\begin{aligned}\lambda_i\dot a_i =&\frac{r}{k}[\frac{e_3}{c_3}\frac{\nabla K_i}{K_i\lambda_i}+\frac{e_4}{c_3}\frac{\nabla\Delta K_i}{K_i\lambda_i^3}+\frac{b_3}{c_3}\sum_{i\neq j=1}^{p}\frac{\alpha_j}{\alpha_i}\frac{1}{\lambda_i}\nabla_{a_i}\varepsilon_{i,j}](1+o_{\frac{1}{\lambda_i}}(1))\\ &+R_{3,i},\end{aligned}$$

*where*

$$\begin{aligned}R_{2,i},R_{3,i} =&o_\varepsilon(\frac{1}{\lambda_i^{n-2}}+\sum_{i\neq j=1}^{p}\varepsilon_{i,j})\\ &+O(\sum_r\frac{|\nabla K_r|^2}{\lambda_r^2}+\frac{|\Delta K_r|^2}{\lambda_r^4}+\frac{1}{\lambda_r^{2(n-2)}}+\sum_{r\neq s}\varepsilon_{r,s}^2+|\delta J(u)|^2).\end{aligned}$$

Thus the movement of $a_i$ and $\lambda_i$ is primarily ruled by quantities arising from self-interaction of $\varphi_i$ and direct interaction of $\varphi_i$ with other bubbles $\varphi_j$.

***Proof of corollary 4.7.***
This follows immediately from corollaries 4.3, 4.6 applied to lemma 4.1 and using (4.35) for the $H_i$ term; we have replaced $\frac{d_2}{4n(n-1)}$ by $d_2$ □

# 5 Case ω>0

Analogously to the case $\omega=0$ we establish the shadow flow.

**Lemma 5.1** (The shadow flow)**.**
*For $u\in V(\omega,p,\varepsilon)$ with $\varepsilon>0$ small and*

$$\sigma_{k,i}=-\int(L_{g_0}u-r\bar K u^{\frac{n+2}{n-2}})\phi_{k,i},\ i=1,\dots,p,\ k=1,2,3$$

*we have suitable testing of $K\partial_t u=-(R-r\bar K)u$*

*(i)*
$$\frac{\dot\alpha_i}{\alpha_i} = \frac{\alpha_i^{\frac{n+2}{2-n}}}{c_1 K_i}\sigma_{1,i}(1+o_{\frac{1}{\lambda_i}}(1)) + R_{1,i}.$$

*(ii)*
$$-\frac{\dot\lambda_i}{\lambda_i} = \frac{\alpha_i^{\frac{n+2}{2-n}}}{c_2 K_i}\sigma_{2,i}(1+o_{\frac{1}{\lambda_i}}(1)) + R_{2,i}$$

*(iii)*
$$\lambda_i \dot a_i = \frac{\alpha_i^{\frac{n+2}{2-n}}}{c_3 K_i}\sigma_{3,i}(1+o_{\frac{1}{\lambda_i}}(1)) + R_{3,i}$$

*with constants $c_k > 0$ given in lemma 3.5 and*

$$R_{k,i} = O(\sum_r \frac{1}{\lambda_r^{n-2}} + \sum_{r\neq s}\varepsilon_{r,s}^2 + \|v\|^2 + |\delta J(u)|^2)_{k,i}.$$

One should not be surprised, that in contrast to lemma 4.1 there appear $\frac{1}{\lambda_r^{n-2}}$ terms in $R_{k,i}$. Indeed, just like $\varepsilon_{i,j}$ measures the interaction of the bubbles $\varphi_i$ and $\varphi_j$, the interaction of $u_{\alpha,\beta}$ and $\varphi_i$ is measured by $\frac{1}{\lambda_i^{\frac{n-2}{2}}}$.

***Proof of lemma 5.1***.
Let
$$(\dot\xi_{1,j}, \dot\xi_{2,j}, \dot\xi_{3,j}) = (\dot\alpha_j, -\alpha_j\frac{\dot\lambda_j}{\lambda_j}, \alpha_j\lambda_j\dot a_j). \tag{5.1}$$

Testing as indicated in the statement we get

$$\begin{aligned}\sigma_{k,i} =& \int K u^{\frac{4}{n-2}}\partial_t u \phi_{k,i} = \int K u^{\frac{4}{n-2}}\partial_t(u_{\alpha,\beta} + \alpha^j\varphi_j + v)\phi_{k,i} \\ =&\dot\alpha\int K u^{\frac{4}{n-2}}\partial_\alpha u_{\alpha,\beta}\phi_{k,i} + \dot\beta^m\int K u^{\frac{4}{n-2}}\partial_{\beta_m}u_{\alpha,\beta}\phi_{k,i} \\ &+ \dot\xi^{l,j}\int K u^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i} - \int K v[\partial_t u^{\frac{4}{n-2}}\phi_{k,i} + u^{\frac{4}{n-2}}\partial_t\phi_{k,i}].\end{aligned} \tag{5.2}$$

The first two integrals on the right hand side above may be estimated via

$$\begin{aligned}\int u^{\frac{4}{n-2}}\varphi_i =& \int (u_{\alpha,\beta} + \alpha^q\varphi_q)^{\frac{4}{n-2}}\varphi_i + O(\|v\|) \\ \leq& C\int \varphi_i + \varphi_i^{\frac{n+2}{n-2}} + C\sum_{i\neq q=1}^p \int \varphi_q^{\frac{4}{n-2}}\varphi_i + O(\|v\|) \\ \leq& C\sum_{i\neq q=1}^p \|\varphi_q^{\frac{4}{n-2}}\varphi_i\|_{L^{\frac{2n}{n+2}}} + O(\frac{1}{\lambda_i^{\frac{n-2}{2}}} + \|v\|) \\ =& O(\frac{1}{\lambda_i^{\frac{n-2}{2}}} + \sum_{i\neq q=1}^p \varepsilon_{i,q} + \|v\|)\end{aligned} \tag{5.3}$$

where we made use of lemma 3.5, yielding

$$\dot{\alpha}\int Ku^{\frac{4}{n-2}}\partial_{\alpha}u_{\alpha,\beta}\phi_{k,i}+\dot{\beta}^{m}\int Ku^{\frac{4}{n-2}}\partial_{\beta_m}u_{\alpha,\beta}\phi_{k,i}$$
$$=\begin{pmatrix} O(\frac{1}{\lambda_i^{\frac{n-2}{2}}}+\sum_{i\neq q=1}^{p}\varepsilon_{i,q}+\|v\|)_{k,i} \\ O(\frac{1}{\lambda_i^{\frac{n-2}{2}}}+\sum_{i\neq q=1}^{p}\varepsilon_{i,q}+\|v\|)_{k,i,m}\end{pmatrix}\begin{pmatrix}\frac{\dot{\alpha}}{\alpha}\\ \dot{\beta}^{m}\end{pmatrix} \tag{5.4}$$

Turning to the third summand on the right hand side of (5.2) note, that

$$\int Ku^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i}=\int K(u_{\alpha,\beta}+\alpha^{m}\varphi_{m})^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i}+O(\|v\|) \tag{5.5}$$

and

$$\begin{aligned}\int K(u_{\alpha,\beta}+\alpha^{m}\varphi_{m})^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i} \\ =\int_{[\alpha^{m}\varphi_{m}\geq u_{\alpha,\beta}]}K(\alpha^{m}\varphi_{m})^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i} \\ +O(\int\limits_{[\alpha^{m}\varphi_{m}\geq u_{\alpha,\beta}]}(\alpha^{m}\varphi_{m})^{\frac{6-n}{n-2}}u_{\alpha,\beta}\varphi_{j}\varphi_{i}+\int\limits_{[\alpha^{m}\varphi_{m}<u_{\alpha,\beta}]}u_{\alpha,\beta}^{\frac{4}{n-2}}\varphi_{j}\varphi_{i}) \\ =\int K(\alpha^{m}\varphi_{m})^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i} \\ +O(\int\limits_{[\alpha^{m}\varphi_{m}\geq u_{\alpha,\beta}]}(\alpha^{m}\varphi_{m})^{\frac{6-n}{n-2}}u_{\alpha,\beta}\varphi_{j}\varphi_{i}+\int\limits_{[\alpha^{m}\varphi_{m}<u_{\alpha,\beta}]}u_{\alpha,\beta}^{\frac{4}{n-2}}\varphi_{j}\varphi_{i}).\end{aligned} \tag{5.6}$$

Using

$$\int\varphi_{j}\varphi_{i}\leq C(\int\varphi_{i}+\int\varphi_{j}^{\frac{n+2}{n-2}}\varphi_{i})=O(\lambda_{i}^{\frac{n-2}{2}}+\varepsilon_{i,j}) \tag{5.7}$$

and

$$\begin{aligned}&\int\limits_{[\alpha^{m}\varphi_{m}\geq u_{\alpha,\beta}]}(\alpha^{m}\varphi_{m})^{\frac{6-n}{n-2}}u_{\alpha,\beta}\varphi_{j}\varphi_{i} \\ &\leq C\int\limits_{[\alpha^{m}\varphi_{m}\geq u_{\alpha,\beta}]\cap[\varphi_{i}\geq\sum_{i\neq q=1}^{p}\varphi_{q}]}(\alpha^{m}\varphi_{m})^{\frac{4}{n-2}}u_{\alpha,\beta}\varphi_{i} \\ &+C\int\limits_{[\alpha^{m}\varphi_{m}\geq u_{\alpha,\beta}]\cap[\varphi_{i}<\sum_{i\neq q=1}^{p}\varphi_{q}]}(\alpha^{m}\varphi_{m})^{\frac{4}{n-2}}u_{\alpha,\beta}\varphi_{i} \\ &\leq C(\int\varphi_{i}^{\frac{n+2}{n-2}}+\int(\sum_{i\neq q=1}^{p}\varphi_{q})^{\frac{n+2}{n-2}}\varphi_{i})=O(\lambda_{i}^{\frac{n-2}{2}}+\sum_{i\neq q=1}^{p}\varepsilon_{i,q})\end{aligned} \tag{5.8}$$

we obtain

$$\begin{aligned}\int K(u_{\alpha,\beta}+\alpha^m\varphi_m)^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i}\\ &=\int K(\alpha^m\varphi_m)^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i}+O(\lambda_i^{\frac{n-2}{2}}+\sum_{i\neq q=1}^{p}\varepsilon_{i,j})\\ &=\alpha_i^{\frac{4}{n-2}}\int K\varphi_i^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i}+O(\lambda_i^{\frac{n-2}{2}}+\sum_{i\neq q=1}^{p}\varepsilon_{i,j}),\end{aligned} \tag{5.9}$$

where we made use of (4.6). Plugging this into (5.5) we obtain

$$\begin{aligned}&\int Ku^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i}\\ &=\alpha_i^{\frac{4}{n-2}}\int K\varphi_i^{\frac{4}{n-2}}\phi_{l,j}\phi_{k,i}+O(\lambda_i^{\frac{n-2}{2}}+\sum_{i\neq q=1}^{p}\varepsilon_{i,j}+\|v\|)\\ &=c_k\alpha_i^{\frac{4}{n-2}}K_i\delta_{kl}\delta_{ij}+O(\frac{|\nabla K_i|}{\lambda_i})\delta_{ij}+O(\lambda_i^{\frac{n-2}{2}}+\sum_{i\neq q=1}^{p}\varepsilon_{i,j}+\|v\|).\end{aligned} \tag{5.10}$$

Moreover arguing as for (4.7) and (4.9) we have

$$\int Ku^{\frac{4}{n-2}}\partial_t\phi_{k,i}v=O(\|v\|)_{i,k,l,j}\dot{\xi}^{l,j}, \tag{5.11}$$

and

$$\int Kv\partial_t u\phi_{k,i}=O(\|v\|^2+|\delta J(u)|^2). \tag{5.12}$$

Thus plugging (5.4), (5.10), (5.11) and (5.12) into (5.2) we conclude

$$\begin{aligned}\sigma_{k,i}=&\begin{pmatrix}O(\frac{1}{\lambda_i^{\frac{n-2}{2}}}+\sum_{i\neq q=1}^{p}\varepsilon_{i,q}+\|v\|)_{k,i}\\ O(\frac{1}{\lambda_i^{\frac{n-2}{2}}}+\sum_{i\neq q=1}^{p}\varepsilon_{i,q}+\|v\|)_{k,i,m}\\ \Xi_{k,i,l,j}\end{pmatrix}^T\begin{pmatrix}\frac{\dot{\alpha}}{\alpha}\\ \dot{\beta}^m\\ \dot{\xi}^{l,j}\end{pmatrix}\\ &+O(\|v\|^2+|\delta J(u)|^2)_{k,i}.\end{aligned} \tag{5.13}$$

where

$$\begin{aligned}\Xi_{k,i,l,j}=c_k\alpha_i^{\frac{4}{n-2}}K_i\delta_{kl}\delta_{ij}+O(\frac{|\nabla K_i|}{\lambda_i})_{k,l}\delta_{ij}\\ +O(\frac{1}{\lambda_i^{\frac{n-2}{2}}}+\sum_{i\neq q=1}^{p}\varepsilon_{i,q}+\|v\|)_{k,i,l,j}.\end{aligned} \tag{5.14}$$

Next let

$$\sigma=-\int(L_{g_0}u-r\bar{K}u^{\frac{n+2}{n-2}})u_{\alpha,\beta}. \tag{5.15}$$

We then have

$$\begin{aligned}\sigma =&\int Ku^{\frac{4}{n-2}}\partial_t u u_{\alpha,\beta} = \int Ku^{\frac{4}{n-2}}\partial_t(u_{\alpha,\beta}+\alpha^i\varphi_i+v)u_{\alpha,\beta}\\ =&\frac{\dot{\alpha}}{\alpha}\int Ku^{\frac{4}{n-2}}u^2_{\alpha,\beta}+\dot{\beta}^m\int Ku^{\frac{4}{n-2}}\partial_{\beta_m}u_{\alpha,\beta}u_{\alpha,\beta}\\ &+\dot{\xi}^{l,j}\int Ku^{\frac{4}{n-2}}\phi_{l,j}u_{\alpha,\beta}-\int Kv\partial_t u^{\frac{4}{n-2}}u_{\alpha,\beta}\end{aligned}\tag{5.16}$$

and therefore recalling $\alpha\partial_\alpha u_{\alpha,\beta}=u_{\alpha,\beta}$

$$\begin{aligned}\sigma =&\begin{pmatrix}\int Ku^{\frac{4}{n-2}}u^2_{\alpha,\beta}\\ \int Ku^{\frac{4}{n-2}}\partial_{\beta_m}u_{\alpha,\beta}u_{\alpha,\beta}\\ O(\frac{1}{\lambda_j^{\frac{n-2}{2}}}+\sum_{j\neq q=1}^p\varepsilon_{i,q}+\|v\|)_{l,j}\end{pmatrix}^T\begin{pmatrix}\frac{\dot{\alpha}}{\alpha}\\ \dot{\beta}^m\\ \dot{\xi}^{l,j}\end{pmatrix}\\ &+O(\|v\|^2+|\delta J(u)|^2).\end{aligned}\tag{5.17}$$

Likewise we obtain for $\sigma_n=-\int(L_{g_0}u-r\bar{K}u^{\frac{n+2}{n-2}})\partial_{\beta_n}u_{\alpha,\beta}$

$$\begin{aligned}\sigma_n =&\int Ku^{\frac{4}{n-2}}\partial_t u\partial_{\beta_n}u_{\alpha,\beta} = \int Ku^{\frac{4}{n-2}}\partial_t(u_{\alpha,\beta}+\alpha^i\varphi_i+v)\partial_{\beta_n}u_{\alpha,\beta}\\ =&\dot{\alpha}\int Ku^{\frac{4}{n-2}}\partial_\alpha u_{\alpha,\beta}\partial_{\beta_n}u_{\alpha,\beta}+\dot{\beta}^m\int Ku^{\frac{4}{n-2}}\partial_{\beta_m}u_{\alpha,\beta}\partial_{\beta_n}u_{\alpha,\beta}\\ &+\dot{\xi}^{l,j}\int Ku^{\frac{4}{n-2}}\phi_{l,j}\partial_{\beta_n}u_{\alpha,\beta}\\ &-\int Kv[\partial_t u^{\frac{4}{n-2}}\partial_{\beta_n}u_{\alpha,\beta}+u^{\frac{4}{n-2}}\partial_t\partial_{\beta_n}u_{\alpha,\beta}]\\ =&\begin{pmatrix}\int Ku^{\frac{4}{n-2}}u_{\alpha,\beta}\partial_{\beta_n}u_{\alpha,\beta}+O(\|v\|)\\ \int Ku^{\frac{4}{n-2}}\partial_{\beta_m}u_{\alpha,\beta}\partial_{\beta_n}u_{\alpha,\beta}+O(\|v\|)\\ O(\frac{1}{\lambda_j^{\frac{n-2}{2}}}+\sum_{j\neq l=q}^p\varepsilon_{i,q}+\|v\|)_{n,l,j}\end{pmatrix}^T\begin{pmatrix}\frac{\dot{\alpha}}{\alpha}\\ \dot{\beta}^m\\ \dot{\xi}^{l,j}\end{pmatrix}\\ &+O(\|v\|^2+|\delta J(u)|^2)_n.\end{aligned}\tag{5.18}$$

Summing up we conclude

$$(A+R)^T_{i,k,j,l,n,m}\begin{pmatrix}\frac{\dot{\alpha}}{\alpha}\\ \dot{\beta}^m\\ \dot{\xi}^{l,j}\end{pmatrix}=\begin{pmatrix}\sigma\\ \sigma_{k,i}\\ \sigma_n\end{pmatrix}+O(\|v\|^2+|\delta J(u)|^2)_{k,i,n},\tag{5.19}$$

where

$$A_{i,\dots,m}=\begin{pmatrix}\langle u_{\alpha,\beta},u_{\alpha,\beta}\rangle & \langle u_{\alpha,\beta},\partial_{\beta_m}u_{\alpha,\beta}\rangle & 0\\ \langle u_{\alpha,\beta},\partial_{\beta_n}u_{\alpha,\beta}\rangle & \langle\partial_{\beta_n}u_{\alpha,\beta},\partial_{\beta_m}u_{\alpha,\beta}\rangle & 0\\ 0 & 0 & \tilde{\Xi}\end{pmatrix}\tag{5.20}$$

with

$$\tilde{\Xi}=a_kK_i\alpha_i^{\frac{4}{n-2}}\delta_{kl}\delta_{ij}+O(\frac{|\nabla K_i|}{\lambda_i})_{k,l}\delta_{ij}\tag{5.21}$$

and

$$R_{i,\dots,m} = O(\sum_r \frac{1}{\lambda_r^{\frac{n-2}{2}}} + \sum_{r\neq s}\varepsilon_{r,s} + \|v\|)_{i,\dots,m}. \tag{5.22}$$

Using $\sigma, \sigma_{k,i}, \sigma_n = O(|\delta J(u)|)$ we obtain

$$A_{i,k,j,l,m,n}\begin{pmatrix}\dot\alpha\\ \dot\xi^{l,j}\\ \dot\beta^m\end{pmatrix} = \begin{pmatrix}\sigma\\ \sigma_{k,i}\\ \sigma_n\end{pmatrix} + R_{k,i,n} \tag{5.23}$$

with

$$R_{k,i,n} = O(\sum_r \frac{1}{\lambda_r^{n-2}} + \sum_{r\neq s}\varepsilon_{r,s}^2 + \|v\|^2 + |\delta J(u)|^2)_{k,i,n}.$$

Note, that we may write $A = A_{i,k,j,l,n,m}$ as

$$A = \begin{pmatrix}B & C & 0\\ C & D & 0\\ 0 & 0 & E\end{pmatrix} = \begin{pmatrix}I & CD^{-1} & 0\\ CB^{-1} & I & 0\\ 0 & 0 & I\end{pmatrix}\begin{pmatrix}B & 0 & 0\\ 0 & D & 0\\ 0 & 0 & E\end{pmatrix}, \tag{5.24}$$

whence we obtain via Neumann series

$$A^{-1} = \begin{pmatrix}B^{-1} & 0 & 0\\ 0 & D^{-1} & 0\\ 0 & 0 & E^{-1}\end{pmatrix}\sum_{k=0}^{\infty}(-1)^k\begin{pmatrix}0 & CD^{-1} & 0\\ CB^{-1} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}^k. \tag{5.25}$$

Last note, that the third row of $A^{-1}$ is just $E^{-1}$, where $E = \tilde{\Xi}$. □

As before our task is two folded, namely to analyse $\sigma_{k,i}$ and to provide a suitable estimate on $v$.

**Proposition 5.2** (Analysing $\sigma_{k,i}$).
*On $V(\omega, p, \varepsilon)$ for $\varepsilon > 0$ small we have with constants $b_1, \dots, d_3 > 0$*

*(i)*

$$\begin{aligned}\sigma_{1,i} =& 4n(n-1)\alpha_i[\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{4n(n-1)k} - 1]\int\varphi_i^{\frac{2n}{n-2}}\\ &+ 4n(n-1)b_1\sum_{i\neq j=1}^{p}\alpha_j[\frac{r\alpha_j^{\frac{4}{n-2}}K_j}{4n(n-1)k} - 1]\varepsilon_{i,j}\\ &- \int(L_{g_0}u_{\alpha,\beta} - r\bar{K}u_{\alpha,\beta}^{\frac{n+2}{n-2}})\varphi_i\\ &+ b_1\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{k}\sum_{i\neq j=1}^{p}\alpha_j\varepsilon_{i,j} + d_1\frac{r\alpha_i^{\frac{4}{n-2}}}{k}\frac{\alpha K_i\omega_i}{\lambda_i^{\frac{n-2}{2}}} + R_{1,i}\end{aligned}$$

*(ii)*

$$\begin{aligned}\sigma_{2,i} =& -4n(n-1)\alpha_i[\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{4n(n-1)k}-1]\int \varphi_i^{\frac{n+2}{n-2}}\lambda_i\partial_{\lambda_i}\varphi_i\\ &-4n(n-1)b_2\sum_{i\neq j=1}^{p}\alpha_j[\frac{r\alpha_j^{\frac{4}{n-2}}K_j}{4n(n-1)k}-1]\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}\\ &-\int(L_{g_0}u_{\alpha,\beta}-r\bar{K}u_{\alpha,\beta}^{\frac{n+2}{n-2}})\lambda_i\partial_{\lambda_i}\varphi_i\\ &-b_2\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{k}\sum_{i\neq j=1}^{p}\alpha_j\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}+d_2\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{k}\frac{\alpha\omega_i}{\lambda_i^{\frac{n-2}{2}}}+R_{2,i}\end{aligned}$$

*(iii)*

$$\begin{aligned}\sigma_{3,i} =& 4n(n-1)\alpha_i[\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{4n(n-1)k}-1]\int \varphi_i^{\frac{n+2}{n-2}}\frac{1}{\lambda_i}\nabla_{a_i}\varphi_i\\ &+4n(n-1)b_3\sum_{i\neq j=1}^{p}\alpha_j[\frac{r\alpha_j^{\frac{4}{n-2}}K_j}{4n(n-1)k}-1]\frac{1}{\lambda_i}\nabla_{a_i}\varepsilon_{i,j}\\ &-\int(L_{g_0}u_{\alpha,\beta}-r\bar{K}u_{\alpha,\beta}^{\frac{n+2}{n-2}})\frac{1}{\lambda_i}\nabla_{a_i}\varphi_i\\ &+b_3\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{k}\sum_{i\neq j=1}^{p}\alpha_j\frac{1}{\lambda_i}\nabla_{a_i}\varepsilon_{i,j}+d_3\frac{r\alpha_i^{\frac{n+2}{n-2}}}{k}\frac{\nabla K_i}{\lambda_i}+R_{3,i},\end{aligned}$$

*where* $R_{k,i}=o_\varepsilon(\frac{1}{\lambda_i^{\frac{n-2}{2}}}+\sum_{i\neq j=1}^{p}\varepsilon_{i,j})+O(\sum_{r\neq s}\varepsilon_{r,s}^2+\|v\|^2+|\delta J(u)|^2)$.

Here and in what follows $\omega_i$ is short hand for $\omega(a_i)$ analogously to $K_i=K(a_i)$.

***Proof of proposition 5.2.***
We evaluate by means of lemma 3.12

$$\begin{aligned}&\int(L_{g_0}u-r\bar{K}u^{\frac{n+2}{n-2}})\phi_{k,i}\\ &=\int L_{g_0}u_{\alpha,\beta}\phi_{k,i}+\alpha^j\int L_{g_0}\varphi_j\phi_{k,i}-\int r\bar{K}(u_{\alpha,\beta}+\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i}\\ &\quad+o(\frac{1}{\lambda_i^{\frac{n-2}{2}}}+\sum_{i\neq j=1}^{p}\varepsilon_{i,j})+O(\|v\|^2+|\delta J(u)|^2).\end{aligned} \tag{5.26}$$

From (4.18) we infer

$$\begin{aligned}&\int L_{g_0}u_{\alpha,\beta}\phi_{k,i}+\alpha^j\int L_{g_0}\varphi_j\phi_{k,i}\\ &=4n(n-1)[\alpha_i\int\varphi_i^{\frac{n+2}{n-2}}\phi_{k,i}+b_k\sum_{i\neq j=1}^{p}\alpha_j d_{k,i}\varepsilon_{i,j}]\\ &\quad+\int L_{g_0}u_{\alpha,\beta}\phi_{k,i}+o_\varepsilon(\frac{1}{\lambda_i^{\frac{n-2}{2}}}+\sum_{i\neq j=1}^{p}\varepsilon_{i,j}),\end{aligned} \tag{5.27}$$

where $(d_{1,i}, d_{2,i}, d_{3,i}) = (1, -\lambda_i\partial_{\lambda_i}, \frac{1}{\lambda_i}\nabla_{a_i})$. On the other hand we may expand

$$\int K(u_{\alpha,\beta}+\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} = \int\limits_{[u_{\alpha,\beta}\geq\alpha^j\varphi_j]} K(u_{\alpha,\beta}+\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} + \int\limits_{[u_{\alpha,\beta}<\alpha^j\varphi_j]} K(u_{\alpha,\beta}+\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i}$$

$$\begin{aligned} &= \int\limits_{[u_{\alpha,\beta}\geq\alpha^j\varphi_j]} K(u_{\alpha,\beta})^{\frac{n+2}{n-2}}\phi_{k,i} + \int\limits_{[u_{\alpha,\beta}<\alpha^j\varphi_j]} K(\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} \\ &+ \frac{n+2}{n-2}\int_{[u_{\alpha,\beta}<\alpha^j\varphi_j]} K(\alpha^j\varphi_j)^{\frac{4}{n-2}}u_{\alpha,\beta}\phi_{k,i} \\ &+ O(\int\limits_{[u_{\alpha,\beta}\geq\alpha^j\varphi_j]} u_{\alpha,\beta}^{\frac{4}{n-2}}\alpha^j\varphi_j\varphi_i) + \int\limits_{[u_{\alpha,\beta}<\alpha^j\varphi_j]} (\alpha^j\varphi_j)^{\frac{6-n}{n-2}}u^2_{\alpha,\beta}\varphi_i). \end{aligned} \tag{5.28}$$

This gives

$$\begin{aligned} &\int K(u_{\alpha,\beta}+\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} \\ &\quad= \int K(u_{\alpha,\beta})^{\frac{n+2}{n-2}}\phi_{k,i} + \int K(\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} \\ &\quad+ \frac{n+2}{n-2}\int K(\alpha^j\varphi_j)^{\frac{4}{n-2}}u_{\alpha,\beta}\phi_{k,i} \\ &\quad+ O(\int\limits_{[u_{\alpha,\beta}\geq\alpha^j\varphi_j]} u_{\alpha,\beta}^{\frac{4}{n-2}}\alpha^j\varphi_j\varphi_i) + \int\limits_{[u_{\alpha,\beta}<\alpha^j\varphi_j]} (\alpha^j\varphi_j)^{\frac{6-n}{n-2}}u^2_{\alpha,\beta}\varphi_i). \end{aligned} \tag{5.29}$$

Note, that $\int\limits_{[u_{\alpha,\beta}\geq c_0\varphi_i]}\varphi_i^2 = o(\frac{1}{\lambda_i^{\frac{n-2}{2}}})$ and for suitable $\epsilon > 0$ we have

$$\begin{aligned} &\int\limits_{[u_{\alpha,\beta}<\alpha^j\varphi_j]} (\alpha^j\varphi_j)^{\frac{6-n}{n-2}}(u_{\alpha,\beta})^2\varphi_i \\ &\qquad= \int\limits_{[u_{\alpha,\beta}<\alpha^j\varphi_j]\cap[\varphi_i\geq\sum_{i\neq j=1}^p\varphi_j]} (\alpha^j\varphi_j^{\frac{6-n}{n-2}}(u_{\alpha,\beta})^2\varphi_i) \\ &\qquad+ \int\limits_{[u_{\alpha,\beta}<\alpha^j\varphi_j]\cap[\varphi_i<\sum_{i\neq j=1}^p\varphi_j]} (\alpha^j\varphi_j)^{\frac{6-n}{n-2}}(u_{\alpha,\beta})^2\varphi_i, \end{aligned} \tag{5.30}$$

whence

$$\begin{aligned}\int\limits_{[u_{\alpha,\beta}<\alpha^j\varphi_j]} & (\alpha^j\varphi_j)^{\frac{6-n}{n-2}}(u_{\alpha,\beta})^2\varphi_i \\ & \leq C\int_{B_{\frac{\epsilon}{\sqrt{\lambda_i}}}a_i}\varphi_i^{\frac{4}{n-2}}+C\int_{\cup_{i\neq j=1}^{p}B_{\frac{\epsilon}{\sqrt{\lambda_j}}}(a_j)}(\sum_{i\neq j=1}^{p}\varphi_j)^{\frac{n+2}{n-2}-\epsilon}\varphi_i \\ & \leq o(\frac{1}{\lambda_i^{\frac{n-2}{2}}})+|[\cup_{i\neq j=1}^{p}B_{\epsilon\sqrt{\lambda_j}}(0)]|^{\frac{\epsilon(n-2)}{2n}}\sum_{i\neq j=1}^{p}\varepsilon_{i,j}.\end{aligned} \tag{5.31}$$

Plugging thus (5.31) into (5.29) we get

$$\begin{aligned}\int & K(u_{\alpha,\beta}+\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} \\ & =\int K(u_{\alpha,\beta})^{\frac{n+2}{n-2}}\phi_{k,i}+\int K(\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} \\ & \quad +\frac{n+2}{n-2}\int K(\alpha_i\varphi_i)^{\frac{4}{n-2}}u_{\alpha,\beta}\phi_{k,i}+o(\frac{1}{\lambda_i^{\frac{n-2}{2}}}+\sum_{i\neq j=1}^{p}\varepsilon_{i,j}).\end{aligned} \tag{5.32}$$

Then (4.28) shows

$$\begin{aligned}\int & K(u_{\alpha,\beta}+\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i} \\ & =\alpha_i^{\frac{n+2}{n-2}}K_i\int\varphi_i^{\frac{n+2}{n-2}}\phi_{k,i}+\int Ku_{\alpha,\beta}^{\frac{n+2}{n-2}}\phi_{k,i}+\sum_{i\neq j=1}^{p}\alpha_j^{\frac{n+2}{n-2}}K_j\int\varphi_j^{\frac{n+2}{n-2}}\phi_{k,i} \\ & \quad +\frac{n+2}{n-2}\alpha_i^{\frac{4}{n-2}}K_i\sum_{i\neq j=1}^{p}\alpha_j\int\varphi_i^{\frac{4}{n-2}}\phi_{k,i}\varphi_j \\ & \quad +\alpha_i^{\frac{n+2}{n-2}}(e_1\frac{\Delta K_i}{\lambda_i^2},e_2\frac{\Delta K_i}{\lambda_i^2},e_3\frac{\nabla K_i}{\lambda_i}+e_4\frac{\nabla\Delta K_i}{\lambda_i^3}) \\ & \quad +\frac{n+2}{n-2}\alpha_i^{\frac{4}{n-2}}\int K\varphi_i^{\frac{4}{n-2}}u_{\alpha,\beta}\phi_{k,i} \\ & \quad +o_\varepsilon(\frac{1}{\lambda_i^{\frac{n-2}{2}}}+\sum_{i\neq j=1}^{p}\varepsilon_{i,j})+O(\sum_{r\neq s}\varepsilon_{r,s}^2)\end{aligned} \tag{5.33}$$

and we obtain letting

$$
\begin{aligned}
&\int K(u_{\alpha,\beta}+\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i}\\
&=\alpha_i^{\frac{n+2}{n-2}}K_i\int\varphi_i^{\frac{n+2}{n-2}}\phi_{k,i}+\int Ku_{\alpha,\beta}^{\frac{n+2}{n-2}}\phi_{k,i}+\sum_{i\neq j=1}^{p}\alpha_j^{\frac{n+2}{n-2}}K_jb_kd_{k,i}\varepsilon_{i,j}\\
&\quad+\sum_{i\neq j=1}^{p}\alpha_i^{\frac{4}{n-2}}\alpha_jK_ib_kd_{k,i}\varepsilon_{i,j}+\alpha_i^{\frac{n+2}{n-2}}(0,0,e_3\frac{\nabla K_i}{\lambda_i})\\
&\quad+\alpha_i^{\frac{4}{n-2}}\int Kd_{k,i}\varphi_i^{\frac{n+2}{n-2}}u_{\alpha,\beta}+o_\varepsilon(\frac{1}{\lambda_i^{\frac{n-2}{2}}}+\sum_{i\neq j=1}^{p}\varepsilon_{i,j})+O(\sum_{r\neq s}\varepsilon_{r,s}^2).
\end{aligned}
\tag{5.34}
$$

Since $u_{\alpha,\beta}(a_i)=\alpha\omega(a_i)+o_\varepsilon(1)$, we get in cases $k=1,2$ with $d_k>0$

$$
\int Kd_{k,i}\varphi_i^{\frac{n+2}{n-2}}u_{\alpha,\beta}=d_k\frac{\alpha K_i\omega_i}{\lambda_i^{\frac{n-2}{2}}}+o_\varepsilon(\frac{1}{\lambda_i^{\frac{n-2}{2}}}),
\tag{5.35}
$$

and in case $k=3$ by radial symmetry

$$
\int K\omega\frac{1}{\lambda_i}\nabla_{a_i}\varphi_i^{\frac{n+2}{n-2}}=o(\frac{1}{\lambda_i^{\frac{n-2}{2}}}).
\tag{5.36}
$$

We get

$$
\begin{aligned}
&\int K(u_{\alpha,\beta}+\alpha^j\varphi_j)^{\frac{n+2}{n-2}}\phi_{k,i}\\
&=\alpha_i^{\frac{n+2}{n-2}}K_i\int\varphi_i^{\frac{n+2}{n-2}}\phi_{k,i}+\int Ku_{\alpha,\beta}^{\frac{n+2}{n-2}}\phi_{k,i}+\sum_{i\neq j=1}^{p}\alpha_j^{\frac{n+2}{n-2}}K_jb_kd_{k,i}\varepsilon_{i,j}\\
&\quad+\alpha_i^{\frac{4}{n-2}}(d_1\frac{\alpha K_i\omega_i}{\lambda_i^{\frac{n-2}{2}}},d_2\frac{\alpha K_i\omega_i}{\lambda_i^{\frac{n-2}{2}}},d_3\frac{\alpha_i\nabla K_i}{\lambda_i})\\
&\quad+\sum_{i\neq j=1}^{p}\alpha_i^{\frac{4}{n-2}}\alpha_jK_ib_kd_{k,i}\varepsilon_{i,j}+o_\varepsilon(\frac{1}{\lambda_i^{\frac{n-2}{2}}}+\sum_{i\neq j=1}^{p}\varepsilon_{i,j})+O(\sum_{r\neq s}\varepsilon_{r,s}^2).
\end{aligned}
\tag{5.37}
$$

Plugging (5.27) and (5.37) into (5.26) yields

$$
\begin{aligned}
&\int (L_{g_0}u - r\bar{K}u^{\frac{n+2}{n-2}})\phi_{k,i} \\
=&4n(n-1)[\alpha_i \int \varphi_i^{\frac{n+2}{n-2}}\phi_{k,i} + b_k \sum_{i\neq j=1}^{p} \alpha_j d_{k,i}\varepsilon_{i,j}] + \int L_{g_0}u_{\alpha,\beta}\phi_{k,i} \\
&- \frac{r\alpha_i^{\frac{n+2}{n-2}}K_i}{k}\int \varphi_i^{\frac{n+2}{n-2}}\phi_{k,i} - \frac{r}{k}\int K u_{\alpha,\beta}^{\frac{n+2}{n-2}}\phi_{k,i} \\
&- b_k \sum_{i\neq j=1}^{p} \frac{r\alpha_j^{\frac{n+2}{n-2}}K_j}{k} d_{k,i}\varepsilon_{i,j} - b_k \sum_{i\neq j=1}^{p} \frac{r\alpha_i^{\frac{4}{n-2}}\alpha_j K_i}{k} d_{k,i}\varepsilon_{i,j} \\
&- \frac{r\alpha_i^{\frac{4}{n-2}}}{k}(d_1 \frac{\alpha K_i\omega_i}{\lambda_i^{\frac{n-2}{2}}}, d_2 \frac{\alpha K_i \omega_i}{\lambda_i^{\frac{n-2}{2}}}, d_3 \frac{\alpha_i \nabla K_i}{\lambda_i}) \\
&+ o_{\varepsilon}(\frac{1}{\lambda_i^{\frac{n-2}{2}}} + \sum_{i\neq j=1}^{p} \varepsilon_{i,j}) + O(\sum_{r\neq s} \varepsilon_{r,s}^2 + \|v\|^2 + |\delta J(u)|^2).
\end{aligned}
\tag{5.38}
$$

From this the assertion follows. □

The equation on $\sigma_{1,i} = O(|\delta J(u)|)$ and the fact, that $u_{\alpha,\beta}$ is almost a solution, simplify the equations on $\sigma_{2,i}$ and $\sigma_{3,i}$ significantly.

**Corollary 5.3** (Simplifying $\sigma_{k,i}$)**.**
*On $V(\omega, p, \varepsilon)$ for $\varepsilon > 0$ small we have*

*(i)*

$$
\sigma_{2,i} = d_2 \frac{r\alpha_i^{\frac{4}{n-2}}}{k}\frac{\alpha\omega_i}{\lambda_i^{\frac{n-2}{2}}} - b_2 \frac{r\alpha_i^{\frac{4}{n-2}}K_i}{k}\sum_{i\neq j=1}^{p} \alpha_j \lambda_i \partial_{\lambda_i}\varepsilon_{i,j} + R_{2,i},
$$

*(ii)*

$$
\sigma_{3,i} = d_3 \frac{r\alpha_i^{\frac{n+2}{n-2}}}{k}\frac{\nabla K_i}{\lambda_i} + b_3 \frac{r\alpha_i^{\frac{4}{n-2}}K_i}{k}\sum_{i\neq j=1}^{p} \alpha_j \frac{1}{\lambda_i}\nabla_{a_i}\varepsilon_{i,j} + R_{2,i},
$$

*where* $R_{k,i} = o_{\varepsilon}(\frac{1}{\lambda_i^{\frac{n-2}{2}}} + \sum_{i\neq j=1}^{p} \varepsilon_{i,j}) + O(\sum_{r\neq s} \varepsilon_{r,s}^2 + \|v\|^2 + |\delta J(u)|^2)$.

***Proof of corollary 5.3.***
Note, that

$$
\begin{aligned}
\int (L_{g_0}u_{\alpha,\beta} - r\bar{K}u_{\alpha,\beta}^{\frac{n+2}{n-2}})\phi_{k,i} = &\int (L_{g_0}u_{\alpha,\beta} - \frac{r_{u_{\alpha,\beta}}}{k_{u_{\alpha,\beta}}} K u_{\alpha,\beta}^{\frac{n+2}{n-2}})\phi_{k,i} \\
&+ ((\frac{r}{k})_{u_{\alpha,\beta}} - (\frac{r}{k})_u) \int K u_{\alpha,\beta}^{\frac{n+2}{n-2}}\phi_{k,i}
\end{aligned}
\tag{5.39}
$$

Due to $\Pi\nabla J(u_{\alpha,\beta})=0$, cf. lemma 3.7 and the remarks following, we have

$$\begin{aligned}L_{g_0}u_{\alpha,\beta}-(r\bar{K})_{u_{\alpha,\beta}}u_{\alpha,\beta}^{\frac{n+2}{n-2}}=&[\int(L_{g_0}u_{\alpha,\beta}-(r\bar{K})_{u_{\alpha,\beta}}u_{\alpha,\beta}^{\frac{n+2}{n-2}})\frac{\omega}{\|\omega\|}]L_{g_0}\frac{\omega}{\|\omega\|}\\&+\sum_{i=1}^{m}[\int(L_{g_0}u_{\alpha,\beta}-(r\bar{K})_{u_{\alpha,\beta}}u_{\alpha,\beta}^{\frac{n+2}{n-2}})\mathrm{e}_i]L_{g_0}\mathrm{e}_i\end{aligned}\tag{5.40}$$

and there holds

$$\begin{aligned}&\int(L_{g_0}u_{\alpha,\beta}-(r\bar{K})_{u_{\alpha,\beta}}u_{\alpha,\beta}^{\frac{n+2}{n-2}})\omega\\&\quad=\int(L_{g_0}u_{\alpha,\beta}-(r\bar{K})_{u}u_{\alpha,\beta}^{\frac{n+2}{n-2}})\omega+O(|(\frac{r}{k})_{u_{\alpha,\beta}}-(\frac{r}{k})_u|)\\&\quad=\int(L_{g_0}(u-\alpha^i\varphi_i)-(r\bar{K})_u(u-\alpha^i\varphi_i)^{\frac{n+2}{n-2}})\omega\\&\qquad+O(|(\frac{r}{k})_{u_{\alpha,\beta}}-(\frac{r}{k})_u|+\|v\|).\end{aligned}\tag{5.41}$$

Clearly

$$\int L_{g_0}\varphi_i\omega=O(\frac{1}{\lambda_i^{\frac{n-2}{2}}})\tag{5.42}$$

and we have

$$\begin{aligned}&\int K(u-\alpha^i\delta_i)^{\frac{n+2}{n-2}}\omega\\&\quad=\int_{[u>\alpha^i\delta_i]}K(u-\alpha^i\delta_i)^{\frac{n+2}{n-2}}\omega+\int_{[u<\alpha^i\delta_i]}K(u-\alpha^i\delta_i)^{\frac{n+2}{n-2}}\omega\\&\quad=\int_{[u>\alpha^i\delta_i]}Ku^{\frac{n+2}{n-2}}\omega+O(\sum_i\frac{\ln^{\frac{n-2}{n}}\lambda_i}{\lambda_i^{\frac{n-2}{2}}})\\&\quad=\int Ku^{\frac{n+2}{n-2}}\omega+O(\sum_i\frac{\ln^{\frac{n-2}{n}}\lambda_i}{\lambda_i^{\frac{n-2}{2}}}).\end{aligned}\tag{5.43}$$

We obtain

$$\begin{aligned}&\int(L_{g_0}u_{\alpha,\beta}-(r\bar{K})_{u_{\alpha,\beta}}u_{\alpha,\beta}^{\frac{n+2}{n-2}})\omega\\&\quad=\int(L_{g_0}u-(r\bar{K})_u u^{\frac{n+2}{n-2}})\omega\\&\qquad+O(|(\frac{r}{k})_{u_{\alpha,\beta}}-(\frac{r}{k})_u|+\sum_i\frac{\ln^{\frac{n-2}{n}}\lambda_i}{\lambda_i^{\frac{n-2}{2}}}+\|v\|)\\&\quad=O(|(\frac{r}{k})_{u_{\alpha,\beta}}-(\frac{r}{k})_u|+\sum_i\frac{\ln^{\frac{n-2}{n}}\lambda_i}{\lambda_i^{\frac{n-2}{2}}}+\|v\|+|\delta J(u)|).\end{aligned}\tag{5.44}$$

and the same estimate holds for $\omega$ replaced by $\mathrm{e_i}$. Plugging this into (5.40) we obtain for (5.39) the estimate

$$\begin{aligned}\int&(L_{g_0}u_{\alpha,\beta}-(r\bar{K})u_{\alpha,\beta}^{\frac{n+2}{n-2}})\phi_{k,i}\\&=O((|(\frac{r}{k})_{u_{\alpha,\beta}}-(\frac{r}{k})_u|+\sum_r\frac{\ln^{\frac{n-2}{n}}\lambda_r}{\lambda_r^{\frac{n-2}{2}}}+\|v\|+|\delta J(u)|)\frac{1}{\lambda_i^{\frac{n-2}{2}}}),\end{aligned}\tag{5.45}$$

whence using (3.50) we conclude

$$\begin{aligned}\int&(L_{g_0}u_{\alpha,\beta}-(r\bar{K})u_{\alpha,\beta}^{\frac{n+2}{n-2}})\phi_{k,i}\\&=o(\frac{1}{\lambda_i^{\frac{n-2}{2}}})+O(\sum_r\frac{1}{\lambda_r^{n-2}}+\|v\|^2+|\delta J(u)|^2).\end{aligned}\tag{5.46}$$

Consequently equation (i) of proposition 5.2 shows

$$\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{4n(n-1)k}=1+O(\frac{1}{\lambda_i^{\frac{n-2}{2}}}+\sum_{i\neq j=1}^{p}\varepsilon_{i,j}+\sum_{r\neq s}\varepsilon_{r,s}^2+\|v\|^2+|\delta J(u)|).\tag{5.47}$$

Thus the claim follows from proposition 5.2. □

We turn to estimate the error term term $v$. To do so we first characterize the first two derivatives of $J$ at $u_{\alpha,\beta}+\alpha^i\varphi_i=u-v$.

**Proposition 5.4** (Derivatives on $H(\omega,p,\varepsilon)$)**.**
*For $\varepsilon>0$ small let $u=u_{\alpha,\beta}+\alpha^i\varphi_i+v\in V(p,\varepsilon)$ and $h_1,h_2\in H=H_u(\omega,p,\varepsilon)$. We then have*

*(i)*
$$\begin{aligned}&\|\partial J(u_{\alpha,\beta}+\alpha^i\varphi_i)\lfloor_H\|\\&=o_\varepsilon(\|v\|)+O(\sum_r\frac{|\nabla K_r|}{\lambda_r}+\frac{1}{\lambda_r^{\frac{n-2}{2}}}+\sum_{r\neq s}\varepsilon_{r,s}+|\delta J(u)|)\end{aligned}$$

*(ii)*
$$\begin{aligned}&\frac{1}{2}\partial^2J(u_{\alpha,\beta}+\alpha^i\varphi_i)h_1h_2\\&=k_{u_{\alpha,\beta}+\alpha^i\varphi}^{\frac{2-n}{n}}[\int L_{g_0}h_1h_2-c_nn(n+2)\int(\frac{K\omega^{\frac{4}{n-2}}}{4n(n-1)}+\sum_i\varphi_i^{\frac{4}{n-2}})h_1h_2]\\&\quad+o_\varepsilon(\|h_1\|\|h_2\|).\end{aligned}$$

***Proof of proposition 5.4.***
Let in addition $h\in H_u(\omega,p,\varepsilon)$ with $\|h\|=1$. From proposition 1.1 we infer

$$\begin{aligned}&\frac{1}{2}\partial J(u_{\alpha,\beta}+\alpha^i\varphi_i)h\\&=k_{u_{\alpha,\beta}+\alpha^i\varphi_i}^{\frac{2-n}{n}}[\int L_{g_0}(u_{\alpha,\beta}+\alpha^i\varphi_i)h\\&\qquad-\int(r\bar{K})_{u_{\alpha,\beta}+\alpha^i\varphi_i}(u_{\alpha,\beta}+\alpha^i\varphi_i)^{\frac{n+2}{n-2}}h]\end{aligned}\tag{5.48}$$

and

$$\begin{aligned}\frac{1}{2}&\partial^2 J(u_{\alpha,\beta}+\alpha^i\varphi_i)h_1h_2\\ &=k_{u_{\alpha,\beta}+\alpha^i\varphi_i}^{\frac{2-n}{n}}[\int L_{g_0}h_1h_2\\ &\qquad -\frac{n+2}{n-2}\int(r\bar{K})_{u_{\alpha,\beta}+\alpha^i\varphi_i}(u_{\alpha,\beta}+\alpha^i\varphi_i)^{\frac{4}{n-2}}h_1h_2]\\ &\quad+o_\varepsilon(\|h_1\|\|h_2\|),\end{aligned}\tag{5.49}$$

since, when considering the formula for the second variation, we have

$$\begin{aligned}\int L_{g_0}uh_i=&\frac{r}{k}\int Ku^{\frac{n+2}{n-2}}h_i+O(|\delta J(u)|\|h_i\|)\\ =&\frac{r}{k}\int Ku^{\frac{4}{n-2}}vh_i+O(|\delta J(u)|\|h_i\|)\\ =&O(\|v\|+|\delta J(u)|)\|h_i\|.\end{aligned}\tag{5.50}$$

By (3.50) there holds

$$(\frac{r}{k})_u=(\frac{r}{k})_{u_{\alpha,\beta}}+o_\varepsilon(1)\tag{5.51}$$

and $\frac{r\alpha_i^{\frac{4}{n-2}}K_i}{k}=4n(n-1)+o_\varepsilon(1)$ by (5.47). Consequently

$$\begin{aligned}\frac{1}{2}&\partial^2 J(u_{\alpha,\beta}+\alpha^i\varphi_i)h_1h_2\\ &=k_{u_{\alpha,\beta}+\alpha^i\varphi_i}^{\frac{2-n}{n}}[\int L_{g_0}h_1h_2\\ &\qquad -c_nn(n+2)(\int\frac{K\omega^{\frac{4}{n-2}}}{4n(n-1)}h_1h_2-\sum_i\int\varphi_i^{\frac{4}{n-2}})h_1h_2)]\\ &\quad+o_\varepsilon(\|h_1\|\|h_2\|).\end{aligned}\tag{5.52}$$

This shows the statement on the second derivative. Moreover by lemma 3.12

$$(\frac{r}{k})_{u_{\alpha,\beta}+\alpha^i\varphi_i}=\frac{r}{k}+o(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}}+\sum_{r\neq s}\varepsilon_{r,s})+O(\|v\|^2+|\delta J(u)|^2).\tag{5.53}$$

We obtain

$$\begin{aligned}\frac{1}{2}&\partial J(u_{\alpha,\beta}+\alpha^i\varphi_i)h\\ &=k_{u_{\alpha,\beta}+\alpha^i\varphi_i}^{\frac{2-n}{n}}[\int L_{g_0}(u_{\alpha,\beta}+\alpha^i\varphi_i)h-\int r\bar{K}(u_{\alpha,\beta}+\alpha^i\varphi_i)^{\frac{n+2}{n-2}}h]\\ &\quad+o_\varepsilon(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}}+\sum_{r\neq s}\varepsilon_{r,s})+O(\|v\|^2+|\delta J(u)|^2),\end{aligned}\tag{5.54}$$

whence by estimates familiar by now

$$
\begin{aligned}
\frac{1}{2}&\partial J(u_{\alpha,\beta}+\alpha^i\varphi_i)h\\
&=k_{u_{\alpha,\beta}+\alpha^i\varphi_i}^{\frac{2-n}{n}}[\int(L_{g_0}u_{\alpha,\beta}-r\bar{K}u_{\alpha,\beta}^{\frac{n+2}{n-2}})h\\
&\quad+\sum_i\alpha_i\int(L_{g_0}\varphi_i-r\bar{K}\alpha_i^{\frac{4}{n-2}}\varphi_i^{\frac{n+2}{n-2}})h]\\
&\quad+O(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}}+\sum_{r\neq s}\varepsilon_{r,s}+\|v\|^2+|\delta J(u)|^2).
\end{aligned}
\tag{5.55}
$$

Using (5.47) we get

$$
\begin{aligned}
\frac{1}{2}&\partial J(u_{\alpha,\beta}+\alpha^i\varphi_i)h\\
&=k_{u_{\alpha,\beta}+\alpha^i\varphi_i}^{\frac{2-n}{n}}[\int(L_{g_0}u_{\alpha,\beta}-r\bar{K}u_{\alpha,\beta}^{\frac{n+2}{n-2}})h\\
&\quad+\sum_i\alpha_i\int(L_{g_0}\varphi_i-4n(n-1)\varphi_i^{\frac{n+2}{n-2}})h]\\
&\quad+O(\sum_r\frac{|\nabla K_r|}{\lambda_r}+\frac{1}{\lambda_r^{\frac{n-2}{2}}}+\sum_{r\neq s}\varepsilon_{r,s}+\|v\|^2+|\delta J(u)|)
\end{aligned}
\tag{5.56}
$$

and we deduce using lemma 3.3

$$
\begin{aligned}
\frac{1}{2}&\partial J(u_{\alpha,\beta}+\alpha^i\varphi_i)h\\
&=k_{u_{\alpha,\beta}+\alpha^i\varphi_i}^{\frac{2-n}{n}}\int(L_{g_0}u_{\alpha,\beta}-r\bar{K}u_{\alpha,\beta}^{\frac{n+2}{n-2}})h\\
&\quad+O(\sum_r\frac{|\nabla K_r|}{\lambda_r}+\frac{1}{\lambda_r^{\frac{n-2}{2}}}+\sum_{r\neq s}\varepsilon_{r,s}+\|v\|^2+|\delta J(u)|).
\end{aligned}
\tag{5.57}
$$

We proceed estimating

$$
\int(L_{g_0}u_{\alpha,\beta}-r\bar{K}u_{\alpha,\beta}^{\frac{n+2}{n-2}})h=\frac{k_{u_{\alpha,\beta}}^{\frac{n}{n-2}}}{2}\langle\partial J(u_{\alpha,\beta}),h\rangle+O(|(\frac{r}{k})_{u_{\alpha,\beta}}-\frac{r}{k}|),
\tag{5.58}
$$

to whose end we will improve (3.50). Due to lemma 3.12 we have

$$
\begin{aligned}
\int&(L_{g_0}u-(r\bar{K})_u u^{\frac{n+2}{n-2}})u_{\alpha,\beta}\\
&=\int(L_{g_0}(u_{\alpha,\beta}+\alpha^i\varphi_i)-(r\bar{K})_u(u_{\alpha,\beta}+\alpha^i\varphi_i)^{\frac{n+2}{n-2}})u_{\alpha,\beta}\\
&\quad+o(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}}+\sum_{r\neq s}\varepsilon_{r,s})+O(\|v\|^2+|\delta J(u)|^2),
\end{aligned}
\tag{5.59}
$$

whence in particular

$$\begin{aligned}\int(L_{g_0}u-(r\bar{K})_u u^{\frac{n+2}{n-2}})u_{\alpha,\beta} \\ =\int(L_{g_0}u_{\alpha,\beta}-(r\bar{K})_u u_{\alpha,\beta}^{\frac{n+2}{n-2}})u_{\alpha,\beta} \\ +O(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}}+\sum_{r\neq s}\varepsilon_{r,s}+\|v\|^2+|\delta J(u)|^2)\end{aligned}\tag{5.60}$$

and therefore

$$(\frac{r}{k})_{u_{\alpha,\beta}}-(\frac{r}{k})_u=O(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}}+\sum_{r\neq s}\varepsilon_{r,s}+\|v\|^2+|\delta J(u)|).\tag{5.61}$$

Plugging (5.61) with $\frac{r}{k}=(\frac{r}{k})_u$ into (5.58) gives recalling lemma 3.7

$$\begin{aligned}&\int(L_{g_0}u_{\alpha,\beta}-r\bar{K}u_{\alpha,\beta}^{\frac{n+2}{n-2}})h \\ =&\frac{k_{u_{\alpha,\beta}}}{2}\langle\partial J(u_{\alpha,\beta}),h\rangle+O(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}}+\sum_{r\neq s}\varepsilon_{r,s}+\|v\|^2+|\delta J(u)|) \\ =&\int(L_{g_0}u_{\alpha,\beta}-(r\bar{K})_{u_{\alpha,\beta}}u_{\alpha,\beta}^{\frac{n+2}{n-2}})\omega\int L_{g_0}\omega h \\ &+\sum_{i=1}^m\int(L_{g_0}u_{\alpha,\beta}-(r\bar{K})_{u_{\alpha,\beta}}u_{\alpha,\beta}^{\frac{n+2}{n-2}})\mathrm{e}_i\int L_{g_0}\mathrm{e}_i h \\ &+O(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}}+\sum_{r\neq s}\varepsilon_{r,s}+\|v\|^2+|\delta J(u)|).\end{aligned}\tag{5.62}$$

Applying (5.61) we then get

$$\begin{aligned}\int(L_{g_0}u_{\alpha,\beta}-r\bar{K}u_{\alpha,\beta}^{\frac{n+2}{n-2}})h \\ =\int(L_{g_0}u_{\alpha,\beta}-(r\bar{K})_u u_{\alpha,\beta}^{\frac{n+2}{n-2}})\omega\int L_{g_0}\omega h \\ +\sum_{i=1}^m\int(L_{g_0}u_{\alpha,\beta}-(r\bar{K})_u u_{\alpha,\beta}^{\frac{n+2}{n-2}})\mathrm{e}_i\int L_{g_0}\mathrm{e}_i h \\ +O(\sum_r\frac{1}{\lambda_r^{\frac{n-2}{2}}}+\sum_{r\neq s}\varepsilon_{r,s}+\|v\|^2+|\delta J(u)|),\end{aligned}\tag{5.63}$$

whence

$$\begin{aligned}
&\int (L_{g_0}u_{\alpha,\beta} - r\bar{K}u_{\alpha,\beta}^{\frac{n+2}{n-2}})h \\
&= \int (L_{g_0}(u_{\alpha,\beta}+\alpha^i\varphi_i) - (r\bar{K})_u(u_{\alpha,\beta}+\alpha^i\varphi_i)^{\frac{n+2}{n-2}})\omega \int L_{g_0}\omega h \\
&\quad + \sum_{i=1}^m \int (L_{g_0}(u_{\alpha,\beta}+\alpha^i\varphi_i) - (r\bar{K})_u(u_{\alpha,\beta}+\alpha^i\varphi_i)^{\frac{n+2}{n-2}})\mathrm{e}_i \int L_{g_0}\mathrm{e}_i h \\
&\quad + O(\sum_r \frac{1}{\lambda_r^{\frac{n-2}{2}}} + \sum_{r\neq s}\varepsilon_{r,s} + \|v\|^2 + |\delta J(u)|).
\end{aligned} \tag{5.64}$$

Since $\int L_{g_0}\omega h, \int L_{g_0}\mathrm{e}_i h = o_\varepsilon(1)$ as $h \in H_u(\omega,p,\varepsilon)$ and $|h|=1$, we conclude

$$\begin{aligned}
&\int (L_{g_0}u_{\alpha,\beta} - r\bar{K}u_{\alpha,\beta}^{\frac{n+2}{n-2}})h \\
&\quad = o_\varepsilon(\|v\|) + O(\sum_r \frac{1}{\lambda_r^{\frac{n-2}{2}}} + \sum_{r\neq s}\varepsilon_{r,s} + |\delta J(u)|).
\end{aligned} \tag{5.65}$$

Plugging this into (5.57) proves the statement on the first derivative. □

In contrast to the case $\omega = 0$ the second variation at $u_{\alpha,\beta} + \alpha^i\varphi_i$ is not necessarily positive definite. It is however sufficient to have non degeneracy.

**Proposition 5.5** (Decomposition of the second variation on $H_u(\omega,p,\varepsilon)$)**.**
*There exist $\gamma, \varepsilon_0 > 0$ such, that for any*

$$u = u_{\alpha,\beta} + \alpha^i\varphi_i + v \in V(\omega,p,\varepsilon) \tag{5.66}$$

*with $0 < \varepsilon < \varepsilon_0$ we may decompose*

$$H_u(\omega,p,\varepsilon) = H = H_+ \oplus_{L_{g_0}} H_- \quad \textit{with} \quad \dim H_- < \infty$$

*and for any $h_+ \in H_+, h_- \in H_-$ there holds*

*(i)* $\partial^2 J(u_{\alpha,\beta} + \alpha^i\varphi_i)\lfloor_{H_+} > \gamma$

*(ii)* $\partial^2 J(u_{\alpha,\beta} + \alpha^i\varphi_i)\lfloor_{H_-} < -\gamma$

*(iii)* $\partial^2 J(u_{\alpha,\beta} + \alpha^i\varphi_i)h_+h_- = o_\varepsilon(\|h_+\|\|h_-\|)$.

***Proof of proposition 5.5***.
Let $H = H_u(\omega,p,\varepsilon)$ and note, that $H$ is a closed subspace of $W$, since

$$H = \langle v, v_j, v_{k,i}\rangle^{\perp_{L_{g_0}}} \tag{5.67}$$

according to definition 3.11 for $v, v_{k,i}, v_j \in W^{1,2}_{g_0}(M)$ solving

$$L_{g_0}v = Ku^{\frac{4}{n-2}}u_{\alpha,\beta},\ L_{g_0}v_j = \frac{n+2}{n-2}Ku^{\frac{4}{n-2}}\partial_{\beta_j}u_{\alpha,\beta} \tag{5.68}$$

and

$$L_{g_0}v_{k,i} = Ku^{\frac{4}{n-2}}\phi_{k,i} \tag{5.69}$$

cf. definitions 3.4 and 3.11. In view of proposition 5.4 we consider

$$T : H \times H \longrightarrow \mathbb{R} : (a,b) \longrightarrow T(a,b) \tag{5.70}$$

with

$$\begin{aligned} T(h_1,h_2) = & \int L_{g_0}h_1h_2 \\ & - c_n n(n+2)\int \left[\frac{K\omega^{\frac{4}{n-2}}}{4n(n-1)} + \sum_i \varphi_i^{\frac{4}{n-2}}ab\right]h_1h_2. \end{aligned} \tag{5.71}$$

Due to the spectral theorem for compact operators there exist

$$(h_i)_{i\in\mathbb{N}} \subset H \;\text{ and }\; (\mu_{h_i}) \subset \mathbb{R} \;\text{ with }\; \mu_{h_i} \longrightarrow 0 \;\text{ as }\; i \longrightarrow \infty \tag{5.72}$$

such, that $(h_i)_{i\in\mathbb{N}}$ forms an orthonormal basis of $H$

$$H = \langle h_i \mid i \in \mathbb{N}\rangle \;\text{ and }\; \langle h_i, h_j\rangle_{L_{g_0}} = \int L_{g_0}h_ih_j = \delta_{ij}, \tag{5.73}$$

and we have $K\omega^{\frac{4}{n-2}}h_i = \mu_{h_i}L_{g_0}h_i$ weakly, so

$$\int K\omega^{\frac{4}{n-2}}h_ih = \mu_i\int L_{g_0}h_ih \;\text{ for all }\; h \in H. \tag{5.74}$$

Likewise there exists an orthonormal basis of $W = W^{1,2}(M)$

$$W = \langle w_q \mid q \in \mathbb{N}\rangle \;\text{ and }\; \langle w_p, w_q\rangle_{L_{g_0}} = \int L_{g_0}w_pw_q = \delta_{pq} \tag{5.75}$$

satisfying for a sequence $(\mu_{w_q}) \subset \mathbb{R}$ with $\mu_{w_q} \longrightarrow 0$ as $q \longrightarrow \infty$

$$K\omega^{\frac{4}{n-2}}w_q = \mu_{w_q}L_{g_0}w_q. \tag{5.76}$$

Below we will prove, that for any $q, l \in \mathbb{N}$ there holds

$$(\mu_{w_q} - \mu_{h_l})\langle w_q, h_l\rangle_{L_{g_0}} \longrightarrow 0 \;\text{ as }\; \varepsilon \longrightarrow 0. \tag{5.77}$$

Moreover recall, that according to proposition 4.5 we have

$$\int L_{g_0}hh - c_n n(n+2)\sum_i \int \varphi_i^{\frac{4}{n-2}}h^2 \geq c\int L_{g_0}hh \tag{5.78}$$

for some positive constant $c > 0$. Thus for any

$$\bar{h} \in H_1 = \langle h_i \mid \frac{n+2}{n-2}\mu_{h_i} \leq \frac{c}{2}\rangle \tag{5.79}$$

we have $T(\bar{h},\bar{h}) \geq \frac{c}{2}\|\bar{h}\|^2$. Let $\epsilon > 0$ such, that

$$\{w_q \mid 1-2\epsilon \leq \frac{n+2}{n-2}\mu_{w_q} \leq 1+2\epsilon\} = \{\mathrm{e}_j \mid j=1,\ldots,m\}, \tag{5.80}$$

where $E_{\frac{n+2}{n-2}}(\omega) = \langle \mathrm{e}_j \mid j=1,\ldots,m\rangle$, cf. lemma 3.6, and define

$$H_2 = \langle h_i \mid \frac{c}{2} < \frac{n+2}{n-2}\mu_{h_i} < 1-\epsilon\rangle \tag{5.81}$$

and

$$W_2 = \langle w_q \mid \frac{c}{2} < \frac{n+2}{n-2}\mu_{w_q} < 1-\epsilon\rangle. \tag{5.82}$$

Then for $0 \neq \tilde{h} \in H_2$ we have due (5.77)

$$\|\tilde{h}\|^2 = \|\Pi_{W_2}\tilde{h}\|^2 + \|\Pi_{W_2^\perp}\tilde{h}\|^2,\ \|\Pi_{W_2^\perp}\tilde{h}\| = o_\varepsilon(\|\tilde{h}\|), \tag{5.83}$$

whence for $\bar{h}+\tilde{h} \in H_1 \oplus H_2$ we obtain

$$\begin{aligned} T(\bar{h}+\tilde{h},\bar{h}+\tilde{h}) =& T(\bar{h},\bar{h}) + 2T(\bar{h},\tilde{h}) + T(\tilde{h},\tilde{h}) \\ \geq& \frac{c}{2}\|\bar{h}\|^2 - 2\frac{n+2}{n-2}\int\sum_i \tilde{\varphi}_i^{\frac{4}{n-2}}\bar{h}(\Pi_{W_2}\tilde{h}) \\ &+ T((\Pi_{W_2}\tilde{h}),(\Pi_{W_2}\tilde{h})) + o_\varepsilon(\|\bar{h}\|^2 + \|\tilde{h}\|^2). \end{aligned} \tag{5.84}$$

Since $W_2$ is fix and finite dimensional, we get

$$\int \tilde{\varphi}_i^{\frac{4}{n-2}}\bar{h}(\Pi_{W_2}\tilde{h}) = o_\varepsilon(\|\bar{h}\|^2 + \|\tilde{h}\|^2) \tag{5.85}$$

and

$$\begin{aligned} &T((\Pi_{W_2}\tilde{h}),(\Pi_{W_2}\tilde{h})) \\ &= \int L_{g_0}(\Pi_{W_2}\tilde{h})(\Pi_{W_2}\tilde{h}) - \frac{n+2}{n-2}\int K\omega^{\frac{4}{n-2}}(\Pi_{W_2}\tilde{h})^2 + o_\varepsilon(\|\overline{h}_2\|^2) \\ &\geq \epsilon\|(\Pi_{W_2}\tilde{h})\|^2 = \epsilon(\|\tilde{h}\|^2 - \|\Pi_{W_2^\perp}\tilde{h}\|^2) \end{aligned} \tag{5.86}$$

Thus $T$ is positive on $H_1 \oplus H_2$. Let

$$H_3 = \langle h_i \mid 1-\varepsilon \leq \frac{n+2}{n-2}\mu_{h_i} \leq 1+\varepsilon\rangle \tag{5.87}$$

and

$$W_3 = \langle w_q \mid 1-\varepsilon \leq \frac{n+2}{n-2}\mu_{w_q} \leq 1+\varepsilon\rangle = \langle \mathrm{e}_j \mid j=1,\ldots,m\rangle. \tag{5.88}$$

Then for $0 \neq \hat{h} \in H_3$ we have due to (5.77) and (5.80)

$$\|\hat{h}\|^2 = \|\Pi_{W_3}\hat{h}\|^2 + \|\Pi_{W_3^\perp}\hat{h}\|^2,\ \|\Pi_{W_3^\perp}\hat{h}\| = o_\varepsilon(\|\hat{h}\|). \tag{5.89}$$

Since $\Pi_{W_3}\hat{h} = \sum_{j=1}^m \langle \mathrm{e}_j, \hat{h}\rangle_{L_{g_0}} \mathrm{e}_j$ and

$$\langle \upsilon_j, \hat{h}\rangle_{L_{g_0}} = 0 \tag{5.90}$$

we obtain

$$\|\Pi_{W_3}\hat{h}\| = o_\varepsilon(\|\hat{h}\|), \tag{5.91}$$

once we know $\|\upsilon_j - \mathrm{e}_j\| = o_\varepsilon(1)$ and we will show this below, cf (5.103).

Thus $H_3 = \{0\}$ is trivial for $\varepsilon > 0$ sufficiently small.
Finally let

$$H_4 = \langle h_i \mid \frac{n+2}{n-2}\mu_{h_i} \geq 1 + \epsilon\rangle = (H_1 \oplus H_2)^{\perp_{L_{g_0}}} \tag{5.92}$$

and

$$W_4 = \langle w_q \mid \frac{n+2}{n-2}\mu_{w_q} \geq 1 + \epsilon\rangle. \tag{5.93}$$

$W_4$ is fixed and finite dimensional. Arguing as for $H_2$ one obtains, that $T$ is strictly negative on $H_4$. We conclude for $H = \tilde{H}_1 \oplus \tilde{H}_2$, where

$$\tilde{H}_1 = H_1 \oplus H_2 \text{ and } \tilde{H}_2 = H_4, \; \dim \tilde{H}_2 < \infty, \tag{5.94}$$

that $T\lfloor_{\tilde{H}_1} > \gamma$ and $T\lfloor_{\tilde{H}_2} < -\gamma$ for some $\gamma > 0$ small, whence

$$\partial^2 J(u_{\alpha,\beta} + \alpha^i\varphi_i)\lfloor_{\tilde{H}_1} > \tilde{\gamma} \text{ and } \partial^2 J(u_{\alpha,\beta} + \alpha^i\varphi_i)\lfloor_{\tilde{H}_2} < -\tilde{\gamma} \tag{5.95}$$

for some $\tilde{\gamma} > 0$ by proposition 5.4. Moreover for $\tilde{h}_1 \in \tilde{H}_1, \; \tilde{h}_2 \in \tilde{H}_2$

$$\int L_{g_0}\tilde{h}_1\tilde{h}_2 = \int K\omega^{\frac{4}{n-2}}\tilde{h}_1\tilde{h}_2 = 0, \tag{5.96}$$

whence

$$T(\tilde{h}_1, \tilde{h}_2) = -\, c_n n(n+2)\sum_i \int \varphi_i^{\frac{4}{n-2}}\tilde{h}_1\tilde{h}_2. \tag{5.97}$$

Thus arguing as for (5.85) we get

$$\partial^2 J(u_{\alpha,\beta} + \alpha^i\varphi_i)\tilde{h}_1\tilde{h}_2 = o_\varepsilon(\|\tilde{h}_1\|\|\tilde{h}_2\|). \tag{5.98}$$

We are left with proving (5.77) and (5.103). First observe, that by definition

$$L_{g_0}\omega = K\omega^{\frac{n+2}{n-2}}, \; L_{g_0}\mathrm{e}_j = \frac{n+2}{n-2}K\omega^{\frac{n+2}{n-2}}\mathrm{e}_j \tag{5.99}$$

and

$$u_{\alpha,\beta} = \alpha(\omega + \beta^j\mathrm{e}_j) + O(\|\beta\|^2). \tag{5.100}$$

Consequently (5.68) implies

$$\|L_{g_0}(\upsilon-\alpha^{\frac{n+2}{n-2}}\omega)\|_{L^{\frac{2n}{n+2}}}, \|L_{g_0}(\upsilon_j-\alpha^{\frac{n+2}{n-2}}\mathrm{e}_j)\|_{L^{\frac{2n}{n+2}}}=o_{\varepsilon}(1). \tag{5.101}$$

Likewise one obtains recalling definition 3.4 and lemma 3.5

$$\|L_{g_0}(\upsilon_{k,i}-c_k\alpha_i^{\frac{4}{n-2}}K_i\phi_{k,i})\|_{L^{\frac{2n}{n+2}}}=o_{\varepsilon}(1). \tag{5.102}$$

Therefore we obtain with $o_{\varepsilon}(1)\longrightarrow 0$ in $W^{1,2}$ as $\varepsilon\longrightarrow 0$

$$\upsilon=\alpha\omega+o_{\varepsilon}(1), \upsilon_j=\alpha\mathrm{e}_j+o_{\varepsilon}(1) \text{ and } \upsilon_{k,i}=c_k\alpha_i^{\frac{4}{n-2}}K_i\phi_{k,i}+o_{\varepsilon}(1). \tag{5.103}$$

Let us write now

$$w_q=\langle w_q,h^i\rangle_{L_{g_0}}h_i+\alpha_q\upsilon+\alpha_q^{k,i}\upsilon_{k,i}+\alpha_q^j\upsilon_j. \tag{5.104}$$

Then on the one hand

$$\int K\omega^{\frac{4}{n-2}}w_qh_l=\mu_{w_q}\langle w_q,h_l\rangle_{L_{g_0}}, \tag{5.105}$$

while on the other one

$$\begin{aligned}\int K\omega^{\frac{4}{n-2}}w_qh_l=&\langle w_q,h^i\rangle_{L_{g_0}}\int K\omega^{\frac{4}{n-2}}h_ih_l+\alpha_q\int K\omega^{\frac{4}{n-2}}\upsilon h_l\\ &+\alpha_q^j\int K\omega^{\frac{4}{n-2}}\upsilon_jh_l+\alpha_q^{k,i}\int K\omega^{\frac{4}{n-2}}\upsilon_{k,i}h_l\\ =&\mu_{h_l}\langle w_q,h_l\rangle_{L_{g_0}}+o_{\varepsilon}(|\alpha_q|+\sum_j|\alpha_q^j|+\sum_{k,i}|\alpha_q^{k,i}|)_l.\end{aligned} \tag{5.106}$$

The last equality above follows easily from (5.103) and the orthogonal properties of $H_u(\omega,p,\varepsilon)$. Combining (5.105) and (5.106) we get

$$(\mu_{w_q}-\mu_{h_l})\langle w_q,h_l\rangle_{L_{g_0}}=o_{\varepsilon}(|\alpha_q|+\sum_j|\alpha_q^j|+\sum_{k,i}|\alpha_q^{k,i}|)_l. \tag{5.107}$$

Moreover

$$\begin{aligned}\langle w_q,\upsilon\rangle_{L_{g_0}}=&\alpha_q\langle\upsilon,\upsilon\rangle_{L_{g_0}}+\alpha_q^j\langle\upsilon_j,\upsilon\rangle+\alpha_q^{l,p}\langle\upsilon_{l,p},\upsilon_{l,p}\rangle_{L_{g_0}}\\ \simeq&\alpha_q+o_{\varepsilon}(\sum_j|\alpha_q^j|+\sum_{l,p}|\alpha_q^{l,p}|),\end{aligned} \tag{5.108}$$

likewise

$$\begin{aligned}\langle w_q,\upsilon_j\rangle_{L_{g_0}}=&\alpha_q\langle\upsilon,\upsilon_j\rangle_{L_{g_0}}+\alpha_q^p\langle\upsilon_p,\upsilon_j\rangle+\alpha_q^{l,p}\langle\upsilon_{l,p},\upsilon_j\rangle_{L_{g_0}}\\ \simeq&\alpha_q^p\delta_{p,j}+o_{\varepsilon}(|\alpha_q|+\sum_j|\alpha_q^j|+\sum_{l,p}|\alpha_q^{l,p}|)\end{aligned} \tag{5.109}$$

and

$$\begin{aligned}\langle w_q, \upsilon_{k,i}\rangle_{L_{g_0}} =&\alpha_q\langle \upsilon, \upsilon_{k,i}\rangle_{L_{g_0}} + \alpha_q^j\langle \upsilon_j, \upsilon_{k,i}\rangle + \alpha_q^{l,p}\langle \upsilon_{l,p}, \upsilon_{k,i}\rangle_{L_{g_0}} \\ \simeq&\alpha_j^{l,p}\delta_{l,k}\delta_{p,i} + o_\varepsilon(|\alpha_q| + \sum_j |\alpha_q^j| + \sum_{l,p} |\alpha_q^{l,p}|)_{k,i}.\end{aligned} \tag{5.110}$$

Summing up we obtain by Parseval's identity

$$\begin{aligned}&\|v\|^2 + \sum_{k,i} \|\upsilon_{k,i}\|^2 + \sum_j \|\upsilon_j\|^2 \\ &\qquad =(1 + o_\varepsilon(1))[\sum_q |\alpha_q|^2 + \sum_{q,k,i} |\alpha_q^{k,i}|^2 + \sum_{q,j} |\alpha_q^j|^2]\end{aligned} \tag{5.111}$$

and the left hand side is uniformly bounded. Thus (5.107) gives

$$(\mu_{w_q} - \mu_{h_l})\langle w_q, h_l\rangle_{L_{g_0}} =o_\varepsilon(1). \tag{5.112}$$

The proof is thereby complete. □

As before smallness of the first and definiteness of the second variation provide an appropriate estimate on the error term $v$.

**Corollary 5.6** (A-priori estimate on $v$)**.**
*On $V(\omega, p, \varepsilon)$ for $\varepsilon > 0$ small we have*

$$\|v\| =O(\sum_r \frac{|\nabla K_r|}{\lambda_r} + \frac{1}{\lambda_r^{\frac{n-2}{2}}} + \sum_{r\neq s} \varepsilon_{r,s} + |\delta J(u)|).$$

***Proof of corollary 5.6****.*
Note, that $\partial^2 J$ is uniformly Hölder continuous on $V(\omega, p, \varepsilon)$ according to proposition 1.1 and the remarks following. Decomposing $v = v_+ + v_- \in H_+ \oplus H_-$ according to proposition 5.5 we readily have

(i) $$\partial J(u)v_+ \geq\partial J(u_{\alpha,\beta} + \alpha^i\varphi_i)v_+ + \gamma\|v_+\|^2 + o_\varepsilon(\|v_+\|\|v_-\|) \tag{5.113}$$

(ii) $$\partial J(u)v_- \leq\partial J(u_{\alpha,\beta} + \alpha^i\varphi_i)v_- - \gamma\|v_-\|^2 + o_\varepsilon(\|v_+\|\|v_-\|). \tag{5.114}$$

This gives $\|v\|^2 = O(|\delta J(u)|^2 + |\delta J(u_{\alpha,\beta} + \alpha^i\varphi_i)\lfloor_H|^2)$ and the claim follows from proposition 5.4 □

Next we combine lemma 5.1 and corollaries 5.3, 4.6.

**Corollary 5.7** (The simplified shadow flow)**.**
*For $u \in V(\omega, p, \varepsilon)$ with $\varepsilon > 0$ we have*

*(i)* $$-\frac{\dot\lambda_i}{\lambda_i} =\frac{r}{k}[\frac{d_2}{c_2}\frac{\alpha\omega_i}{\alpha_i K_i \lambda_i^{\frac{n-2}{2}}} - \frac{b_2}{c_2}\sum_{i\neq j=1}^{p} \frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}](1 + o_{\frac{1}{\lambda_i}}(1)) + R_{2,i}$$

*(ii)*
$$\lambda_i \dot{a}_i = \frac{r}{k}[\frac{d_3}{c_3}\frac{\nabla K_i}{K_i\lambda_i} + \frac{b_3}{c_3}\sum_{i\neq j=1}^{p}\frac{\alpha_j}{\alpha_i}\frac{1}{\lambda_i}\nabla_{a_i}\varepsilon_{i,j}](1+o_{\frac{1}{\lambda_i}}(1)) + R_{3,i},$$

*where*

$$R_{k,i} = o_\varepsilon(\frac{1}{\lambda_i^{\frac{n-2}{2}}} + \sum_{i\neq j=1}^{p}\varepsilon_{i,j}) + O(\sum_r \frac{|\nabla K_r|^2}{\lambda^2} + \frac{1}{\lambda_r^{n-2}} + \sum_{r\neq s}\varepsilon_{r,s}^2 + |\delta J(u)|^2).$$

***Proof of proposition 5.7.***
This follows from lemma 5.1 and corollaries 5.3, 5.6. □

# 6 The flow on V(ω, p, ε)

## 6.1 Principal behaviour

For $u \in V(\omega, p, \varepsilon)$ corollaries 4.3 and 5.3 give a hint on the principal terms of $\partial J(u)$. The following definition assumes these quantities to give a lower bound on the first variation of $J$.

**Definition 6.1** (Principal lower bound of the first variation)**.**
*We call $\partial J$ principally lower bounded,*
*if for every $p \geq 1$ there exist $c, \varepsilon > 0$ such, that*

$$|\delta J(u)| \geq c(\sum_r \frac{|\nabla K_r|}{K_r\lambda_r} + \frac{|\Delta K_r|}{K_r\lambda_r^2} + \frac{1}{\lambda_r^{n-2}} + \sum_{r\neq s}\varepsilon_{r,s}) \text{ for all } u \in V(p,\varepsilon).$$

*and*

$$|\delta J(u)| \geq c(\sum_r \frac{|\nabla K_r|}{K_r\lambda_r} + \frac{1}{\lambda_r^{\frac{n-2}{2}}} + \sum_{r\neq s}\varepsilon_{r,s}) \text{ for all } u \in V(\omega, p,\varepsilon).$$

Under this mild assumption we have uniformity in $V(\omega, p, \varepsilon)$ as follows.

**Proposition 6.2** (Uniformity in $V(\omega, p, \varepsilon)$)**.**
*Assume $\partial J$ to be principally lower bounded.*
*For $u = u_{\alpha,\beta} + \alpha^i\varphi_i + v \in V(\omega, p, \varepsilon)$ with $k_u = \int K u^{\frac{2n}{n-2}} \equiv 1$ we then have*

*(i)*
$$\lambda_i^{-1}, \varepsilon_{i,j}, |1 - \frac{r_\infty \alpha_i^{\frac{4}{n-2}} K_i}{4n(n-1)}|, \|v\| \longrightarrow 0$$

*(ii)*
$$|(\frac{r}{k})_{u_{1,\beta}} - r_\infty \alpha^{\frac{4}{n-2}}|, |\delta J(u_{1,\beta})| \longrightarrow 0$$

*uniformly as $|\delta J(u)| \longrightarrow 0$ and $J(u) = r \longrightarrow J_\infty = r_\infty$.*

In view of (i) above and definition 3.9 we would expect to have as well

$$|1 - r_\infty \alpha^{\frac{4}{n-2}}|, \|\beta\| \longrightarrow 0 \tag{6.1}$$

as $|\delta J(u)| \longrightarrow 0$ and $J(u) = r \longrightarrow J_\infty = r_\infty$.
But, since critical points of $J$ are not necessarily isolated, some $u_{\alpha,\beta}$ with $0 \neq \|\beta\| < \varepsilon$ could be a critical point of $J$ itself.

***Proof of proposition 6.2*** .
Of course $\frac{1}{\lambda_i}, \varepsilon_{i,j} \longrightarrow 0$ as $|\delta J(u)| \longrightarrow 0$ by assumption and the same holds true for $\|v\|$ due to corollaries 4.6, 5.6. Then due to (4.35) and (5.47)

$$1 - \frac{r\alpha_i^{\frac{4}{n-2}} K_i}{4n(n-1)} \longrightarrow 0 \text{ as } |\delta J(u)| \longrightarrow 0 \tag{6.2}$$

as well and $(\frac{r}{k})_{u_{\alpha,\beta}} - (\frac{r}{k})_u \longrightarrow 0$ as $|\delta J(u)| \longrightarrow 0$ due to (5.61). From (5.40) and (5.44) we infer $|\delta J(u_{\alpha,\beta})| \longrightarrow 0$ as $|\delta J(u)| \longrightarrow 0$ and we have $\partial J(u_{\alpha,\beta}) = \alpha J(u_{1,\beta})$, since $u_{\alpha,\beta} = \alpha u_{1,\beta}$ and scaling invariance of $J$. Thereby

$$(\frac{r}{k})_{u_{\alpha,\beta}} = (\frac{r}{k})_{u_{1,\beta}} \alpha^{-\frac{4}{n-2}}, \tag{6.3}$$

whence due to $(\frac{r}{k})_u = r_u \longrightarrow r_\infty$ we have $(\frac{r}{k})_{u_{1,\beta}} - r_\infty \alpha^{\frac{4}{n-2}} \longrightarrow 0$. □

As indicated above $\|\beta\| \longrightarrow 0$ is not necessary. On the other hand we may assume due to proposition 3.1, that along a flow line

$$u = u_{\alpha,\beta} + \alpha^i \varphi_i + v \in V(\omega, p, \varepsilon)$$

we have $\|\beta_{t_k}\| \longrightarrow 0$ for a time sequence $t_k \longrightarrow \infty$.

We then have to show $|1 - r_\infty \alpha^{\frac{4}{n-2}}|$, $\|\beta\| \longrightarrow 0$ along the full flow line.
For $p = 0$ this is true due to the unicity of a limiting critical point, cf. proposition 3.13. The following proposition yields the same result for $p \geq 1$.

**Proposition 6.3** (Unicity of a limiting critical point at infinity)**.**
*Assume $\partial J$ to be principally lower bounded.*
*If a sequence $u(t_k)$ converges to a critical point at infinity of $J$*
*in the sense, that*

$$\exists\, p > 1, \varepsilon_k \searrow 0 \;:\; u(t_k) \in V(\omega, p, \varepsilon_k),$$

*then $u$ converges as well*
*in the sense, that*

$$\exists\, p > 1\, \forall\, \varepsilon > 0\, \exists\, T > 0\, \forall\, t > T \;:\; u(t) \in V(\omega, p, \varepsilon).$$

***Proof of proposition 6.3***.
Since

$$k \equiv 1, J(u) = r \searrow r_\infty \text{ and } \partial J(u) \longrightarrow 0 \tag{6.4}$$

along a flow line we have on $V(\omega, p, \varepsilon)$ according to proposition 6.2

$$\begin{aligned} J(u) = \int L_{g_0} uu = & \int L_{g_0} u_{\alpha,\beta} u_{\alpha,\beta} + \sum_i \alpha_i^2 \int L_{g_0} \varphi_i \varphi_i + o(1) \\ = & \alpha^2 (c_\omega + \|\beta\|^2 + o(\|\beta\|^2)) + c_0 r_\infty^{\frac{2-n}{2}} \sum_i K_i^{\frac{2-n}{2}} + o(1), \end{aligned} \tag{6.5}$$

where $c_\omega = \int L_{g_0} \omega\omega$. On the other hand

$$\begin{aligned} (\frac{r}{k})_{u_{1,\beta}} = & \frac{\int L_{g_0} u_{1,\beta} u_{1,\beta}}{\int K u_{1,\beta}^{\frac{2n}{n-2}}} \\ = & \frac{\int L_{g_0} \omega\omega + L_{g_0} \beta^i \mathrm{e}_i \beta^j \mathrm{e}_j + o(\|\beta\|^2)}{\int K \omega^{\frac{2n}{n-2}} + \frac{2n}{n-2} \frac{n+2}{n-2} K \omega^{\frac{4}{n-2}} \beta^i \mathrm{e}_i \beta^j \mathrm{e}_j + o(\|\beta\|^2)} \\ = & \frac{c_\omega + \|\beta\|^2}{c_\omega + \frac{2n}{n-2} \|\beta\|^2} + o(\|\beta\|^2) = 1 - \frac{n+2}{n-2} \frac{\|\beta\|^2}{c_\omega} + o(\|\beta\|^2) \end{aligned} \tag{6.6}$$

whence still according to proposition 6.2

$$\alpha^{-\frac{4}{n-2}} (1 - \frac{n+2}{n-2} \frac{\|\beta\|^2}{c_\omega} + o(\|\beta\|^2)) = r_\infty + o(1). \tag{6.7}$$

In particular $\alpha$ is fixed in terms of $\|\beta\|^2$ by

$$\alpha^2 = (\frac{c_\omega - \frac{n+2}{n-2} \|\beta\|^2 + o(\|\beta\|^2)}{c_\omega r_\infty})^{\frac{n-2}{2}}. \tag{6.8}$$

Plugging this into (6.5) we obtain, since $J(u) = r_\infty + o(1)$

$$\begin{aligned} c_\omega^{\frac{n-2}{2}} r_\infty^{\frac{n}{2}} = & (c_\omega - \frac{n+2}{n-2} \|\beta\|^2)^{\frac{n-2}{2}} (c_\omega + \|\beta\|^2) \\ & + c_0 c_\omega^{\frac{n-2}{2}} \sum_i K_i^{\frac{2-n}{2}} + o(1) + o(\|\beta\|^2) \\ = & c_\omega^{\frac{n}{2}} - \frac{n}{2} c_\omega^{\frac{n-2}{2}} (1 + o(1)) \|\beta\|^2 + c_0 c_\omega^{\frac{n-2}{2}} \sum_i K_i^{\frac{2-n}{2}} + o(1). \end{aligned} \tag{6.9}$$

Thus, if $\|\beta\|^2$ increases significantly, then $\sum_i K_i^{\frac{2-n}{2}}$ has to increase significantly as well. But

$$\begin{aligned} \partial_t K_i^{\frac{2-n}{2}} = & \frac{2-n}{2} K_i^{-\frac{n}{2}} \frac{\nabla K_i}{\lambda_i} \lambda_i \dot{a}_i \\ \leq & - c \frac{|\nabla K_i|^2}{\lambda_i^2} + O(\sum_i \frac{1}{\lambda_i^{2(2-n)}} + \sum_{r \neq s} \varepsilon_{r,s}^2 + |\delta J(u)|^2) \end{aligned} \tag{6.10}$$

due to corollaries 4.7, 5.7, whence

$$\partial_t K_i^{\frac{2-n}{2}} \leq O(|\delta J(u)|^2) \tag{6.11}$$

due to definition 6.1. If the proposition were false, there would exist

$$s_0 < s_0' < s_1 < s_1' < \ldots < s_n < s_n' < \ldots$$

such, that $u\lfloor_{[s_k,s_k']} \in V(\omega, p, \varepsilon_0)$ and

$$u(s_k) \in V(\omega, p, \varepsilon_k), \varepsilon_k \longrightarrow 0,\, u(s_k') \in \partial V(\omega, p, \varepsilon_0). \tag{6.12}$$

However due to proposition 6.2 we may assume

$$\frac{1}{\lambda_i}, \varepsilon_{i,j}, 1 - \frac{r_\infty \alpha_i^{\frac{4}{n-2}} K_i}{4n(n-1)}, \|v\| \leq \varepsilon_k \text{ during } (s_k, s_k'). \tag{6.13}$$

Thus by the very definition 3.9 of $V(\omega, p, \varepsilon)$ the only possibility for $u$ to escape from $V(\omega, p, \varepsilon_0)$ during $(s_k, s_k')$ is, that $|1 - r_\infty \alpha^{\frac{4}{n-2}}|$ or $\|\beta\|$ has to increase during $(s_k, s_k')$ for at least a quantity $\varepsilon_0 - \varepsilon_k$. This possibility has already been ruled out for $\|\beta\|$ and is thus as well for $|1 - r_\infty \alpha^{\frac{4}{n-2}}|$ by (6.8). □

The only lack in the discussion so far is a missing compactness result on the blow up points. A straight forward use of the evolution equations given by corollaries 4.7 and 5.7 provides at least a weak form of convergence.

**Lemma 6.4** (Critical points of $K$ as attractors)**.**
*Suppose $\partial J$ to be principally lower bounded.*
*We then have*

$$K(a_i) \longrightarrow K_{i_\infty} \text{ and } |\nabla K(a_i)| \longrightarrow 0 \text{ for all } i = 1, \ldots, p$$

*for every flow line $u \in V(\omega, p, \varepsilon)$ converging to a critical point at infinity.*

***Proof of lemma 6.4***.
In case $\partial J$ is principally lower bounded lemmata 4.7 and 5.7 show

$$\partial_t \sum_i K_i = \sum_i \frac{\nabla K_i}{\lambda_i} \lambda \dot{a}_i = O(|\delta J(u)|^2) \tag{6.14}$$

As a consequence

$$K_i = K(a_i) \longrightarrow K_{i_\infty} \text{ for all } i = 1, \ldots, p. \tag{6.15}$$

Then still according to lemmata 4.7 and 5.7 we observe

$$\partial_t |\nabla K_i|^2 = 2 \frac{\nabla^2 K_i(\nabla K_i, \lambda_i \dot{a}_i)}{\lambda_i} = O(|\delta J(u)|^2),$$

whence $|\nabla K_i| \longrightarrow c_{i_\infty}$. Letting

$$P=\{1,\ldots,p\}, Q=\{i\in P \mid c_{i_\infty}\neq 0\} \text{ and } q=\sharp\{Q\} \tag{6.16}$$

we may assume without loss of generality, that

$$Q=\{1,\ldots,q\} \text{ and } \min_{i\in Q, j\in P\setminus q} d(a_i,a_j)>\epsilon_0>0 \tag{6.17}$$

We then reorder, if necessary, the elements of $q$ by

$$\frac{1}{K_1}\ln\frac{1}{\lambda_1}\geq\ldots\geq\frac{1}{K_q}\ln\frac{1}{\lambda_q}. \tag{6.18}$$

In case $u\in V(p,\varepsilon)$ we consider $\psi=\sum_{i=1}^q\frac{C^i}{K_i}\ln\frac{1}{\lambda_i}$. Then corollary 4.7 gives

$$\begin{aligned}\psi'=&\sum_{i=1}^q\frac{C^i}{K_i}[\frac{\ln\lambda_i}{\lambda_i}\frac{\nabla K_i}{K_i}\lambda_i\dot{a}_i-\frac{\dot{\lambda}_i}{\lambda_i}]\\ \geq& c\sum_{i=1}^q\frac{C^i}{K_i}[\gamma_1\frac{|\nabla K_i|^2}{K_i^2}\frac{\ln\lambda_i}{\lambda_i^2}+\gamma_2\frac{H_i}{\lambda_i^{n-2}}+\gamma_3\frac{\Delta K_i}{\lambda_i^2}\\ &\qquad -\gamma_4\sum_{i\neq j=1}^p\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}](1+o_{\frac{1}{\lambda_i}}(1))\\ &+o_\varepsilon(\sum_{i=1}^q\frac{1}{\lambda_i^{n-2}}+\sum_{i=1}^q\sum_{i\neq j=1}^p\varepsilon_{i,j})+O(|\delta J(u)|)^2,\end{aligned} \tag{6.19}$$

where we made use of the principal lower boundedness of $\partial J$. We obtain

$$\psi'\geq -c(1+o_{\frac{1}{\lambda_i}}(1))\sum_{\substack{i\neq j\\ i\in Q}}\frac{C^i}{K_i}\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}+o(\sum_{\substack{i\neq j\\ i\in Q}}\varepsilon_{i,j})+O(|\delta J(u)|^2) \tag{6.20}$$

by definition of $q$. Note, that for $i\in Q$ and $j\in P\setminus Q$ we may assume

$$-\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}=\frac{n-2}{2}\frac{\frac{\lambda_i}{\lambda_j}-\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)}{(\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j))^{\frac{n}{2}}}\geq\frac{n-2}{4}\varepsilon_{i,j}, \tag{6.21}$$

since in that case $d(a_i,a_j)\geq\varepsilon_0>0$, and we obtain

$$\psi'\geq -c(1+o_{\frac{1}{\lambda_i}}(1))\sum_{\substack{i\neq j\\ i\in Q}}\frac{C^i}{K_i}\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}+o(\sum_{\substack{i\neq j\\ i\in Q}}\varepsilon_{i,j})+O(|\delta J(u)|^2). \tag{6.22}$$

Moreover for sufficiently small $\varepsilon>0$ and $C>1$ large we have

$$-\sum_{\substack{i\neq j\\ i,j\in Q}}\frac{C^i}{K_i}\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}\geq c\sum_{\substack{i>j\\ i,j\in Q}}\varepsilon_{i,j}. \tag{6.23}$$

To prove (6.23) note, that by definition we have

$$(C^i - C^j)\frac{\ln\frac{1}{\lambda_i}}{K_i} \leq (C^i - C^j)\frac{\ln\frac{1}{\lambda_j}}{K_j} \tag{6.24}$$

for any $i > j$ with $i, j \in Q$ or equivalently

$$\frac{C^i - C^j}{K_i}\ln\frac{1}{\lambda_i} + \frac{C^j - C^i}{K_j}\ln\frac{1}{\lambda_j} \leq 0. \tag{6.25}$$

We then have

$$\frac{\lambda_j}{\lambda_i} = o(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G^{\frac{2}{2-n}}(a_i, a_j)). \tag{6.26}$$

Otherwise we may assume for some $c > 0$

$$\frac{\lambda_j}{\lambda_i} \geq c(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G^{\frac{2}{2-n}}(a_i, a_j)). \tag{6.27}$$

This implies $\frac{\lambda_j}{\lambda_i} \gg 1 \gg \frac{\lambda_i}{\lambda_j}$ and $d(a_i, a_j) = O(\frac{1}{\lambda_i})$. Consequently

$$\frac{C^i - C^j}{K_j}\ln\frac{\lambda_j}{\lambda_i} \leq O(\frac{\ln\lambda_i}{\lambda_i}), \tag{6.28}$$

yielding a contradiction. Thus (6.26) for $i > j$ is established. Write

$$\begin{aligned} -\sum_{\substack{i\neq j\\ i,j\in Q}} C^i\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} = & -\sum_{\substack{i>j\\ i,j\in Q}} C^i\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} + \sum_{\substack{i<j\\ i,j\in Q}} C^i\frac{\alpha_j}{\alpha_i}\lambda_j\partial_{\lambda_j}\varepsilon_{i,j} \\ & -\sum_{\substack{i<j\\ i,j\in Q}} C^i\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} - \sum_{\substack{i<j\\ i,j\in Q}} C^i\frac{\alpha_j}{\alpha_i}\lambda_j\partial_{\lambda_j}\varepsilon_{i,j} \\ = & -\sum_{\substack{i>j\\ i,j\in Q}}[C^i\frac{\alpha_j}{\alpha_i} - C^j\frac{\alpha_i}{\alpha_j}]\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} - \sum_{\substack{i<j\\ i,j\in Q}} C^i\frac{\alpha_j}{\alpha_i}[\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} + \lambda_j\partial_{\lambda_j}\varepsilon_{i,j}]. \end{aligned} \tag{6.29}$$

We have

$$-\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} - \lambda_j\partial_{\lambda_j}\varepsilon_{i,j} = (n-2)\varepsilon_{i,j}^{\frac{n}{n-2}}\lambda_i\lambda_j\gamma_n G^{\frac{2}{2-n}}(a_i, a_j) > 0 \tag{6.30}$$

and for $i > j$ due to (6.26)

$$-\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} = \frac{n-2}{2}\varepsilon_{i,j}^{\frac{n}{n-2}}(\frac{\lambda_i}{\lambda_j} - \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j\gamma_n G^{\frac{2}{2-n}}(a_i, a_j)) \geq \frac{n-2}{4}\varepsilon_{i,j}. \tag{6.31}$$

This shows (6.23).
Thus plugging (6.23) into (6.22) shows $\psi' \geq O(|\delta J(u)|^2)$ for $C > 1$ sufficiently large, whereas $\psi \longrightarrow -\infty$ by definition as a continuous, piecewise differentiable function in time; a contradiction.

The case $u \in V(\omega, p, \varepsilon)$ is proven analogously. □

The following lemma assures $\partial J$ to be principally lower bounded in the case the dimensional conditions $Cond_n$, on which theorem 1 relies, hold true.

**Proposition 6.5** (Principal lower bound of the first variation under $Cond_n$). *$\partial J$ is principally lower bounded, if $Cond_n$ as in definition 1.2 is satisfied.*

***Proof of proposition 6.5***.
In case $\omega = 0$ corollaries 4.3, 4.6 and (4.35) show, that

(i)
$$\sigma_{2,i} = \tilde{\gamma}_1 \alpha_i \frac{H_i}{\lambda_i^{n-2}} + \gamma_2 \alpha_i \frac{\Delta K_i}{K_i \lambda_i^2} - \tilde{\gamma}_5 b_2 \sum_{i \neq j=1}^{p} \alpha_j \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} + R_{2,i} \tag{6.32}$$

(ii)
$$\sigma_{3,i} = \tilde{\gamma}_3 \alpha_i \frac{\nabla K_i}{K_i \lambda_i} + \gamma_4 \alpha_i \frac{\nabla \Delta K_i}{K_i \lambda_i^3} + \gamma_6 \sum_{i \neq j=1}^{p} \frac{\alpha_j}{\lambda_i} \nabla_{a_i} \varepsilon_{i,j} + R_{3,i}, \tag{6.33}$$

where

$$\begin{aligned} R_{k,i} = & o_{\varepsilon}(\frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j=1}^{q} \varepsilon_{i,j}) \\ & + O(\sum_r \frac{|\nabla K_r|^2}{\lambda_r^2} + \frac{|\Delta K_r|^2}{\lambda_r^4} + \frac{1}{\lambda_r^{2(n-2)}} + \sum_{r \neq s} \varepsilon_{r,s}^2 + |\delta J(u)|^2). \end{aligned} \tag{6.34}$$

Letting $0 < \underline{\kappa} \leq \kappa_i \leq \overline{\kappa} < \infty$ for $|\nabla K_i| \neq 0$ and $\kappa_i = 0$ for $|\nabla K_i| = 0$ we get

$$\begin{aligned} & \sum_i C^i (\sigma_{2,i} + \kappa_i \langle \sigma_{3,i}, \frac{\nabla K_i}{|\nabla K_i|} \rangle) \\ & \geq \sum_i \alpha_i C^i [\gamma_1 \frac{H_i}{\lambda_i^{n-2}} + \gamma_2 \frac{\Delta K_i}{K_i \lambda_i^2} + \gamma_3 \kappa_i \frac{|\nabla K_i|}{K_i \lambda_i} + \gamma_4 \kappa_i \frac{\langle \nabla \Delta K_i, \nabla K_i \rangle}{K_i |\nabla K_i| \lambda_i^3}] \\ & - \tilde{\gamma}_5 \sum_{i \neq j} C^i \alpha_j \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} + o_{\varepsilon}(\sum_{r \neq s} \varepsilon_{r,s}) + O(\sum_{i \neq j} \frac{C^i}{\lambda_i} |\nabla_{a_i} \varepsilon_{i,j}|) \\ & + O(\frac{|\Delta K_r|^2}{\lambda_r^4} + |\delta J(u)|^2). \end{aligned} \tag{6.35}$$

Note, that we do not try to construct a continuous pseudo gradient, so there is no need to choose $\kappa_i$ continuously. As before we order

$$\frac{1}{\lambda_1} \geq \ldots \geq \frac{1}{\lambda_p}. \tag{6.36}$$

We then have for sufficiently small $\varepsilon > 0$ and $C > 1$ large

$$\sum_{i \neq j} C^i \alpha_j \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \geq c \sum_{i > j} C^i \varepsilon_{i,j} \tag{6.37}$$

and

$$\sum_{i\neq j}\frac{C^i}{\lambda_i}|\nabla_{a_i}\varepsilon_{i,j}| = O(\sum_{i>j}C^j\varepsilon_{i,j}) \tag{6.38}$$

To prove (6.37) and (6.38) note, that

$$\begin{aligned}-\sum_{i\neq j}C^i\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} = & -\sum_{i>j}C^i\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} + \sum_{i<j}C^i\frac{\alpha_j}{\alpha_i}\lambda_j\partial_{\lambda_j}\varepsilon_{i,j} \\ & -\sum_{i<j}C^i\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} - \sum_{i<j}C^i\frac{\alpha_j}{\alpha_i}\lambda_j\partial_{\lambda_j}\varepsilon_{i,j} \\ = -\sum_{i>j}[C^i\frac{\alpha_j}{\alpha_i} - C^j\frac{\alpha_i}{\alpha_j}]\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} & - \sum_{i<j}C^i\frac{\alpha_j}{\alpha_i}[\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} + \lambda_j\partial_{\lambda_j}\varepsilon_{i,j}].\end{aligned} \tag{6.39}$$

One has

$$-\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} - \lambda_j\partial_{\lambda_j}\varepsilon_{i,j} = (n-2)\varepsilon_{i,j}^{\frac{n}{n-2}}\lambda_i\lambda_j\gamma_n G^{\frac{2}{2-n}}(a_i,a_j) > 0 \tag{6.40}$$

and for $i>j$

$$-\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} = \frac{n-2}{2}\varepsilon_{i,j}^{\frac{n}{n-2}}(\frac{\lambda_i}{\lambda_j} - \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j\gamma_n G^{\frac{2}{2-n}}(a_i,a_j)) \geq \frac{n-2}{4}\varepsilon_{i,j}. \tag{6.41}$$

Thus (6.37) is proven. We are left with estimating

$$\begin{aligned}\sum_{i\neq j}\frac{C^i}{\lambda_i}|\nabla_{a_i}\varepsilon_{i,j}| = & \frac{n-2}{2}\sum_{i<j}C^i\varepsilon_{i,j}|\frac{(\frac{\lambda_j}{\lambda_i})^{\frac12}(\lambda_i\lambda_j)^{\frac12}\gamma_n\nabla_{a_i}G^{\frac{2}{2-n}}(a_i,a_j)}{\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G^{\frac{2}{2-n}}(a_i,a_j)}| \\ & + o(\sum_{i\neq j}\varepsilon_{i,j}),\end{aligned} \tag{6.42}$$

whence we immediately obtain (6.38).
Plugging (6.38) and (6.38) into (6.35) we obtain for $C>1$ sufficiently large

$$\begin{aligned}& \sum_i C^i(\sigma_{2,i} + \kappa_i\langle\sigma_{3,i}, \frac{\nabla K_i}{|\nabla K_i|}\rangle) \\ & \geq \sum_i \alpha_i C^i[\gamma_1\frac{H_i}{\lambda_i^{n-2}} + \gamma_2\frac{\Delta K_i}{K_i\lambda_i^2} + \gamma_3\kappa_i\frac{|\nabla K_i|}{K_i\lambda_i} + \gamma_4\kappa_i\frac{\langle\nabla\Delta K_i, \nabla K_i\rangle}{K_i|\nabla K_i|\lambda_i^3\lambda_i^3}] \\ & \quad + \gamma_5\sum_{i>j}C^i\varepsilon_{i,j} + O(\frac{|\Delta K_r|^2}{\lambda_r^4} + |\delta J(u)|^2).\end{aligned} \tag{6.43}$$

In case $\Delta K_i \geq 0$ or $|\nabla K_i| > \epsilon$ for $\epsilon > 0$ small we immediately obtain

$$\begin{aligned}\gamma_i\frac{H_i}{\lambda_i^{n-2}} + \gamma_2\frac{\Delta K_i}{K_i\lambda_i^2} + \gamma_3\kappa_i\frac{|\nabla K_i|}{K_i\lambda_i} + \gamma_4\kappa_i\frac{\langle\nabla\Delta K_i, \nabla K_i\rangle}{K_i|\nabla K_i|\lambda_i^3} & \\ \geq c[\frac{H_i}{\lambda_i^{n-2}} + \frac{|\Delta K_i|}{K_i\lambda_i^2} + \frac{|\nabla K_i|}{K_i\lambda_i}] &\end{aligned} \tag{6.44}$$

for some $c>0$ and all $\lambda_i>0$ sufficiently large choosing $\kappa_i$ such, that

$$\gamma_i \frac{H_i}{\lambda_i^{n-2}} + \gamma_4\kappa_i \frac{\langle \nabla\Delta K_i, \nabla K_i\rangle}{K_i|\nabla K_i|\lambda_i^3} \geq c\frac{H_i}{\lambda_i^{n-2}} \tag{6.45}$$

Moreover (6.44) holds true as well for $n=3$ and by $Cond_4$ for $n=4$. For

$$n=5,\ \Delta K_i<0 \ \text{ and } \ |\nabla K_i|<\varepsilon \tag{6.46}$$

we may according to $Cond_5$ assume, that $\langle \nabla\Delta K_i, \nabla K_i\rangle > \frac{1}{3}|\Delta K_i|^2$. Thus

$$\frac{\Delta K_i}{K_i\lambda_i^2} > -\frac{3}{2}\frac{|\nabla K_i|}{K_i\lambda_i} - \frac{3}{2}\frac{\langle \nabla\Delta K_i, \nabla K_i\rangle}{K_i|\nabla K_i|\lambda_i^3}. \tag{6.47}$$

Choosing therefore $\kappa_i$ such, that $\frac{3}{2}\gamma_2<\gamma_3\kappa_i, \frac{3}{2}\gamma_2<\gamma_4\kappa_i$, then (6.44) holds true as well and thus in any case. We conclude

$$\begin{aligned}&\sum_i C^i(\sigma_{2,i}+\kappa_i\langle\sigma_{3,i}, \frac{\nabla K_i}{|\nabla K_i|}\rangle)\\ \geq & c\sum_i[\frac{H_i}{\lambda_i^{n-2}} + \frac{|\Delta K_i|}{K_i\lambda_i^2} + \frac{|\nabla K_i|}{K_i\lambda_i}] + c\sum_{i>j}\varepsilon_{i,j} + O(|\delta J(u)|^2).\end{aligned} \tag{6.48}$$

Since $\sigma_{k,i}=O(|\delta J(u)|)$ by definition, the claim follows.

In case $\omega>0$ we have due to corollaries 5.3, 5.6 and (5.47)

(i)
$$\sigma_{2,i} = \tilde{\gamma}_1\alpha\frac{\omega_i}{K_i\lambda_i^{\frac{n-2}{2}}} - \tilde{\gamma}_3\sum_{i\neq j=1}^{p}\alpha_j\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} + R_{2,i} \tag{6.49}$$

(ii)
$$\sigma_{3,i} = \tilde{\gamma}_2\alpha_i\frac{\nabla K_i}{K_i\lambda_i} + \gamma_4\sum_{i\neq j=1}^{p}\alpha_j\frac{1}{\lambda_i}\nabla_{a_i}\varepsilon_{i,j} + R_{3,i} \tag{6.50}$$

where

$$R_{k,i} = o_\varepsilon(\sum_r \frac{1}{\lambda_r^{\frac{n-2}{2}}} + \sum_{r\neq s}\varepsilon_{r,s}) + O(|\delta J(u)|^2) \tag{6.51}$$

and the same arguments apply in a simpler way. □

## 6.2 Leaving V(ω, p, ε)

In this subsection we consider a flow line

$$u = u_{\alpha,\beta} + \alpha^i\varphi_i + v \in V(\omega, p, \varepsilon)$$

and we wish to define piecewise differentiable continuous function in time

$$\psi : (a_i, \lambda_i)_{i=1,\dots,p} \longrightarrow \psi((a_i,\lambda_i)_{i=1,\dots,p})$$

with the fundamental properties

(i) $\psi \longrightarrow -\infty$ as $\lambda_i \longrightarrow \infty$ for some $i = 1, \ldots, p$

(ii) $\psi' \in L^1(\mathbb{R}_+)$ is integrable in time.

The existence of such a function implies, that a flow line cannot at once remain in $V(\omega, p, \varepsilon)$ for all times and concentrate in the sense, that $\lambda_i \longrightarrow \infty$.

The subsequent propositions are devoted to prove their existence under the dimensional conditions $Cond_n$, cf. definition 1.2.

**Proposition 6.6** (Case $n = 3$, $\omega = 0$).
*Let $n = 3$ and $Cond_3$ hold true. Ordering*

$$\frac{1}{\lambda_1} \geq \ldots \geq \frac{1}{\lambda_p}$$

*the piecewise differentiable continuous function $\psi = \sum_i C^i \ln \frac{1}{\lambda_i}$ satisfies*

$$\psi' \geq \sum_i \frac{H_i}{\lambda_i} + \sum_{i>j} \varepsilon_{i,j} + O(|\delta J(u)|^2),$$

*provided $C > 1$ is sufficiently large*

In view of corollary 4.7 the positive sign of the mass related terms $\frac{H_i}{\lambda_i}$ is rather obvious and the ordering $\frac{1}{\lambda_1} \geq \ldots \geq \frac{1}{\lambda_p}$ and choice of $C \gg 1$ ensure, that the interaction related terms are of positive sign as well.

***Proof of proposition 6.6.***
As $M$ is not conformally equivalent to the standard sphere $\mathbb{S}^3$, the positive mass theorem holds. Thus $H_i > 0$ in the statement of corollary 4.7

$$\begin{aligned} -\frac{\dot{\lambda}_i}{\lambda_i} = & \frac{r}{k}[\gamma_0 \frac{H_i}{\lambda_i} - \gamma_1 \sum_{i \neq j=1}^{p} \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j}](1 + o_{\frac{1}{\lambda_i}}(1)) \\ & + o_\varepsilon(\sum_r \frac{1}{\lambda_r} + \sum_{r \neq s} \varepsilon_{r,s}) + O(|\delta J(u)|^2) \end{aligned} \tag{6.52}$$

for suitable $\gamma_0, \gamma_1 > 0$. Then for $\psi = \sum_i C^i \ln \frac{1}{\lambda_i}$, $C > 1$ there holds

$$\begin{aligned} \psi' = & \frac{r}{k}[\tilde{\gamma}_0 \sum_i C^i \frac{H_i}{\lambda_i} - \gamma_1 \sum_{i \neq j} C^i \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j}](1 + o_{\frac{1}{\lambda_i}}(1)) \\ & + o_\varepsilon(\sum_{r \neq s} \varepsilon_{r,s}) + O(|\delta J(u)|^2). \end{aligned} \tag{6.53}$$

We complete the definition of $\psi$ by ordering

$$\ln \frac{1}{\lambda_1} \geq \ldots \geq \ln \frac{1}{\lambda_p} \tag{6.54}$$

and claim, that there exists $c > 0$ such, that for any $C > 1$ sufficiently large

$$-\gamma_1 \sum_{i \neq j} C^i \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} \geq c \sum_{i>j} C^i \varepsilon_{i,j}. \tag{6.55}$$

Readily the statement of the proposition follows from this fact.

To prove (6.55) note, that

$$\begin{aligned}
-\sum_{i \neq j} C^i \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} = & -\sum_{i>j} C^i \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} + \sum_{i<j} C^i \frac{\alpha_j}{\alpha_i} \lambda_j \partial_{\lambda_j} \varepsilon_{i,j} \\
& -\sum_{i<j} C^i \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} - \sum_{i<j} C^i \frac{\alpha_j}{\alpha_i} \lambda_j \partial_{\lambda_j} \varepsilon_{i,j} \qquad (6.56) \\
= -\sum_{i>j} [C^i \frac{\alpha_j}{\alpha_i} - C^j \frac{\alpha_i}{\alpha_j}] \lambda_i \partial_{\lambda_i} \varepsilon_{i,j} & - \sum_{i<j} C^i \frac{\alpha_j}{\alpha_i} [\lambda_i \partial_{\lambda_i} \varepsilon_{i,j} + \lambda_j \partial_{\lambda_j} \varepsilon_{i,j}].
\end{aligned}$$

One has

$$-\lambda_i \partial_{\lambda_i} \varepsilon_{i,j} - \lambda_j \partial_{\lambda_j} \varepsilon_{i,j} = (n-2) \varepsilon_{i,j}^{\frac{n}{n-2}} \lambda_i \lambda_j \gamma_n G^{\frac{2}{2-n}}(a_i, a_j) > 0 \tag{6.57}$$

and for $\frac{\lambda_j}{\lambda_i} \leq 1$, so for $i > j$, and $\varepsilon > 0$ sufficiently small

$$-\lambda_i \partial_{\lambda_i} \varepsilon_{i,j} = \frac{n-2}{2} \varepsilon_{i,j}^{\frac{n}{n-2}} (\frac{\lambda_i}{\lambda_j} - \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j \gamma_n G^{\frac{2}{2-n}}(a_i, a_j)) \geq \frac{n-2}{4} \varepsilon_{i,j}. \tag{6.58}$$

Thus (6.55) follows. □

**Proposition 6.7** (Case $n = 4$, $\omega = 0$)**.**
*Let $n = 4$ and $Cond_4$ hold true. Ordering*

$$\frac{1}{K_1} \ln \frac{1}{\lambda_1} \geq \ldots \geq \frac{1}{K_p} \ln \frac{1}{\lambda_p}$$

*the piecewise differentiable continuous function $\psi = \sum_i \frac{C^i}{K_i} \ln \frac{1}{\lambda_i}$ satisfies*

$$\psi' \geq \sum_i \frac{H_i}{\lambda_i^2} + \frac{|\nabla K_i|^2 \ln \lambda_i}{K_i^2 \lambda_i^2} + \sum_{i>j} \varepsilon_{i,j} + O(|\delta J(u)|^2),$$

*provided $C > 1$ is sufficiently large.*

The interaction terms are of correct sign again. Differentiating $\frac{1}{K_i}$ in time leads to the quantity $\frac{|\nabla K_i|^2 \ln \lambda_i}{K_i^2 \lambda_i^2}$, which enforces a blow up point $a_i$ to come close to $[\nabla K = 0]$. $Cond_4$ then ensures the $\frac{\Delta K_i}{\lambda_i^2}$ terms to be controlled by the positive mass related terms $\frac{H_i}{\lambda_i^2}$.

***Proof of proposition 6.7.***
As $M$ is not conformally equivalent to the standard sphere $\mathbb{S}^4$, the positive mass theorem holds. Thus $H_i > 0$ in the statement of corollary 4.7

(i)
$$-\frac{\dot\lambda_i}{\lambda_i} = \frac{r}{k}[\gamma_0\frac{H_i}{\lambda_i^2} + \gamma_1\frac{\Delta K_i}{K_i\lambda_i^2} - \gamma_3\sum_{i\neq j=1}^{p}\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}](1+o_{\frac{1}{\lambda_i}}(1))$$
$$+ o_\varepsilon(\sum_r\frac{1}{\lambda_r^2} + \sum_{r\neq s}\varepsilon_{r,s}) + O(\sum_r\frac{|\nabla K_r|^2}{\lambda_r^2} + |\delta J(u)|^2) \tag{6.59}$$

(ii)
$$\dot K_i = \frac{r}{k}\gamma_2\frac{|\nabla K_i|^2}{K_i\lambda_i^2}(1+o_{\frac{1}{\lambda_i}}(1))$$
$$+ \frac{\nabla K_i}{\lambda_i}O(\sum_r\frac{1}{\lambda_r^2} + \sum_{r\neq s}\varepsilon_{r,s} + |\delta J(u)|^2) \tag{6.60}$$

for suitable $\gamma_0,\gamma_1,\gamma_2,\gamma_3>0$. Then for $\psi=\sum_i\frac{C^i}{K_i}\ln\frac{1}{\lambda_i},\ C>1$ there holds

$$\psi' \geq \frac{r}{k}\sum_i\frac{C^i}{K_i\lambda_i^2}(\tilde\gamma_0 H_i + \gamma_1\frac{\Delta K_i}{K_i} + \tilde\gamma_2\frac{|\nabla K_i|^2}{K_i^2}\ln\lambda_i)(1+o_{\frac{1}{\lambda_i}}(1))$$
$$-\gamma_3\frac{r}{k}\sum_{i\neq j}\frac{C^i}{K_i}\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}(1+o_{\frac{1}{\lambda_i}}(1)) \tag{6.61}$$
$$+ o_\varepsilon(\sum_{r\neq s}\varepsilon_{r,s}) + O(|\delta J(u)|^2).$$

We complete the definition of $\psi$ by ordering

$$\frac{1}{K_1}\ln\frac{1}{\lambda_1} \geq \ldots \geq \frac{1}{K_p}\ln\frac{1}{\lambda_p}$$

and claim, that there exists $c>0$ such, that for any $C>1$ sufficiently large

$$-\sum_{i\neq j}\frac{C^i}{K_i}\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} \geq c\sum_{i>j}C^i\varepsilon_{i,j}. \tag{6.62}$$

To prove (6.62) note, that by definition for any pair $i>j$ we have

$$(C^i-C^j)\frac{\ln\frac{1}{\lambda_i}}{K_i} \leq (C^i-C^j)\frac{\ln\frac{1}{\lambda_j}}{K_j} \tag{6.63}$$

or equivalently

$$\frac{C^i-C^j}{K_i}\ln\frac{1}{\lambda_i} + \frac{C^j-C^i}{K_j}\ln\frac{1}{\lambda_j} \leq 0 \tag{6.64}$$

We then have

$$\frac{\lambda_j}{\lambda_i} = o(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G^{\frac{2}{2-n}}(a_i,a_j)), \tag{6.65}$$

from which the claim follows as when proving (6.55). Otherwise we have

$$\frac{\lambda_j}{\lambda_i} \geq c(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G^{\frac{2}{2-n}}(a_i, a_j)) \tag{6.66}$$

for some $c > 0$. This implies $\frac{\lambda_j}{\lambda_i} \gg 1 \gg \frac{\lambda_i}{\lambda_j}$ and $d((a_i, a_j)) = O(\frac{1}{\lambda_i})$. Thus

$$\frac{C^i - C^j}{K_j} \ln \frac{\lambda_j}{\lambda_i} \leq O(\frac{\ln \lambda_i}{\lambda_i}), \tag{6.67}$$

yielding a contradiction.
We conclude

$$\begin{aligned} \psi' \geq & \frac{r}{k} \sum_i \frac{C^i}{K_i \lambda_i^2}(\tilde{\gamma}_0 H_i + \gamma_1 \frac{\Delta K_i}{K_i} + \tilde{\gamma}_2 \frac{|\nabla K_i|^2}{K_i^2} \ln \lambda_i)(1 + o_{\frac{1}{\lambda_i}}(1)) \\ & + \tilde{\gamma}_3 \sum_{i>j} C^i \varepsilon_{i,j} + O(|\delta J(u)|^2). \end{aligned} \tag{6.68}$$

Thereby the assertion follows immediately due to $Cond_4$. □

**Proposition 6.8** (Case $n = 5$, $\omega = 0$)**.**
*Let $n = 5$ and $Cond_5$ hold true. For $\underline{\epsilon} > 0$ small let $\eta_{\underline{\epsilon}} \in C_0^\infty(\mathbb{R}, [0, 1])$ with*

$$\eta_{\underline{\epsilon}}(r) \equiv 0 \text{ for } r \leq \epsilon,\ \eta_{\underline{\epsilon}}(r) \equiv 1 \text{ for } r \geq 2 \text{ and } 0 \leq \eta'_{\underline{\epsilon}} \leq \frac{2}{\underline{\epsilon}}$$

*and*

$$\theta_i = \eta_{\underline{\epsilon}}(-\lambda_i \Delta K_i) \ln \frac{-\lambda_i \Delta K_i}{\underline{\epsilon}} \geq 0.$$

*Ordering for some $\kappa > 0$*

$$\frac{\ln \frac{1}{\lambda_1}}{K_1} - \kappa\theta_1 \geq \ldots \geq \frac{\ln \frac{1}{\lambda_p}}{K_p} - \kappa\theta_p$$

*the piecewise differentiable continuous function*

$$\psi = \sum_i \left( \frac{C^i}{K_i} \ln \frac{1}{\lambda_i} - \kappa C^i \theta_i \right)$$

*satisfies for $C \gg 1$ and a suitable choice of $\kappa$*

$$\psi' \geq \sum_i \frac{H_i}{\lambda_i^3} + \frac{|\nabla K_i|^2}{K_i^2 \lambda_i^2} \ln \lambda_i + \sum_{i \neq j} \varepsilon_{i,j} + O(|\delta J(u)|^2),$$

*provided $d(a_i, [\nabla K = 0]) \ll 1$ is sufficiently small for all $i = 1, \ldots, p$.*

Note, that closeness of the blow up points to the critical set $[\nabla K = 0]$ is not a serious restriction, cf. lemma 6.4 and proposition 6.5.

The interaction terms however are of correct sign again and one is left with comparing $\frac{H_i}{\lambda_i^3}$ to $\frac{\Delta K_i}{\lambda_i^2}$. $Cond_5$ then ensures by differentiating in time, that $\lambda_i \Delta K_i$ can be absorbed.

***Proof of proposition 6.8.***
As $M$ is not conformally equivalent to the standard sphere $\mathbb{S}^5$, the positive mass theorem holds. Thus $H_i > 0$ in the statement of corollary 4.7

(i)
$$\begin{aligned}-\frac{\dot{\lambda}_i}{\lambda_i} =& \frac{r}{k}[\gamma_1 \frac{H_i}{\lambda_i^3} + \gamma_2 \frac{\Delta K_i}{K_i \lambda_i^2} - \gamma_4 \sum_{i \neq j=1}^{p} \frac{\alpha_j}{\alpha_i} \lambda_i \partial_{\lambda_i} \varepsilon_{i,j}](1 + o_{\frac{1}{\lambda_i}}(1)) \\ &+ o_\varepsilon(\sum_r \frac{1}{\lambda_r^3} + \sum_{r \neq s} \varepsilon_{r,s}) + O(|\delta J(u)|^2)\end{aligned} \tag{6.69}$$

(ii)
$$\begin{aligned}\dot{K}_i =& \gamma_3 \frac{r}{k} \frac{|\nabla K_i|^2}{K_i \lambda_i^2}(1 + o_{\frac{1}{\lambda_i}}(1)) \\ &+ O(\sum_r \frac{1}{\lambda_r^3} + \sum_{r \neq s} \varepsilon_{r,s} + |\delta J(u)|^2) \frac{|\nabla K_i|}{\lambda_i}\end{aligned} \tag{6.70}$$

(iii)
$$\begin{aligned}(\Delta K_i)' =& \frac{r}{k}[\gamma_3 \frac{\langle \nabla \Delta K_i, \nabla K_i \rangle}{K_i \lambda_i^2} + \gamma_5 \frac{|\nabla \Delta K_i|^2}{K_i \lambda_i^4} \\ &+ \gamma_6 \sum_{i \neq j=1}^{p} \frac{\alpha_j}{\alpha_i} \frac{\nabla \Delta K_i \nabla_{a_i} \varepsilon_{i,j}}{\lambda_i^2}](1 + o_{\frac{1}{\lambda_i}}(1)) \\ &+ o_\varepsilon(\frac{1}{\lambda_i}(\sum_r \frac{1}{\lambda_r^3} + \sum_{r \neq s} \varepsilon_{r,s})) + O(\frac{1}{\lambda_i}|\delta J(u)|^2)\end{aligned} \tag{6.71}$$

with suitable constants $\gamma_1, \ldots, \gamma_6 > 0$. Here we have used

$$\sum_r \frac{|\nabla K_r|}{K_r \lambda_r} + \frac{|\Delta K_r|}{K_r \lambda_r^2} + \frac{1}{\lambda_r^{n-2}} + \sum_{r \neq s} \varepsilon_{r,s} \leq C|\delta J(u)|$$

according to lemma 6.5. In view of (6.69) we wish to compare $\Delta K_i$ to $\frac{H_i}{\lambda_i}$ in a neighbourhood of a critical point with non positive laplacian and this is done as follows. For $\eta_{\underline{\epsilon}}$ as in statement of the proposition consider

$$\theta_i = \eta_{\underline{\epsilon}}(-\lambda_i \Delta K_i) \ln \frac{-\lambda_i \Delta K_i}{\underline{\epsilon}} \geq 0. \tag{6.72}$$

Letting $s_i = -\lambda_i \Delta K_i$ we calculate

$$\begin{aligned}\theta_i' =& \eta_{\underline{\epsilon}}'(s_i) s_i' \ln \frac{s_i}{\underline{\epsilon}} + \eta_{\underline{\epsilon}}(s_i)(\ln \frac{s_i}{\underline{\epsilon}})' \\ =& [\underline{\epsilon} \eta_{\underline{\epsilon}}'(s_i) \frac{s_i}{\underline{\epsilon}} \ln \frac{s_i}{\underline{\epsilon}} + \eta_{\underline{\epsilon}}(s_i)](\ln \frac{s_i}{\underline{\epsilon}})' = \vartheta_{\underline{\epsilon}, i} \cdot (\ln \frac{s_i}{\underline{\epsilon}})',\end{aligned} \tag{6.73}$$

where readily

(i) $\vartheta_{\underline{\epsilon},i} = 0$ for $\frac{s_i}{\underline{\epsilon}} \leq 1$

(ii) $0 \leq \vartheta_{\underline{\epsilon},i} \leq 4\ln 2 + 1$ for $1 \leq \frac{s_i}{\underline{\epsilon}} \leq 2$

(iii) $\vartheta_{\underline{\epsilon},i} = 1$ for $\frac{s_i}{\underline{\epsilon}} \geq 2$.

From (6.69) and (6.71) we infer

$$\begin{aligned}\theta_i' =& \frac{r}{k}\vartheta_{\underline{\epsilon},i}[-\gamma_1\frac{H_i}{\lambda_i^3} + (-\gamma_2 + \frac{\gamma_3\langle\nabla\Delta K_i, \nabla K_i\rangle}{|\Delta K_i|^2})\frac{\Delta K_i}{K_i\lambda_i^2} \\ & + \gamma_4\sum_{i\neq j=1}^{p}\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} \\ & + \gamma_5\frac{|\nabla\Delta K_i|^2}{K_i\Delta K_i\lambda_i^4} + \gamma_6\sum_{i\neq j=1}^{p}\frac{\alpha_j}{\alpha_i}\frac{\nabla\Delta K_i}{\Delta K_i\lambda_i^2}\nabla_{a_i}\varepsilon_{i,j}](1 + o_{\frac{1}{\lambda_i}}(1)) \\ & + o_\varepsilon(\sum_r\frac{1}{\lambda_r^3} + \sum_{r\neq s}\varepsilon_{r,s}) + O(|\delta J(u)|^2).\end{aligned} \tag{6.74}$$

Note, that we have $-\lambda_i\Delta K_i \geq \underline{\epsilon}$ for $\vartheta_{\underline{\epsilon},i} \neq 0$, whence

$$\vartheta_{\underline{\epsilon},i}\frac{|\nabla\Delta K_i|^2}{K_i\Delta K_i\lambda_i^4} \leq 0 \quad\text{and}\quad \vartheta_{\underline{\epsilon},i}\sum_{i\neq j=1}^{p}\frac{\nabla\Delta K_i}{\Delta K_i\lambda_i^2}\nabla_{a_i}\varepsilon_{i,j} = O(\frac{1}{\lambda_i}\nabla_{a_i}\varepsilon_{i,j}). \tag{6.75}$$

This gives

$$\begin{aligned}\theta_i' \leq& \frac{r}{k}\vartheta_{\underline{\epsilon},i}[-\gamma_1\frac{H_i}{\lambda_i^3} + (-\gamma_2 + \frac{\gamma_3\langle\nabla\Delta K_i, \nabla K_i\rangle}{|\Delta K_i|^2})\frac{\Delta K_i}{K_i\lambda_i^2} \\ & + \gamma_4\sum_{i\neq j=1}^{p}\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} + O(\sum_{i\neq j=1}^{p}\frac{1}{\lambda_i}\nabla_{a_i}\varepsilon_{i,j})](1 + o_{\frac{1}{\lambda_i}}(1)) \\ & + o_\varepsilon(\sum_r\frac{1}{\lambda_r^3} + \sum_{r\neq s}\varepsilon_{r,s}) + O(|\delta J(u)|^2).\end{aligned} \tag{6.76}$$

Consider for some $\kappa > 0$ to be defined later on

$$\psi = \sum_i\frac{C^i}{K_i}\ln\frac{1}{\lambda_i} - \sum_i\kappa C^i\theta_i. \tag{6.77}$$

By (6.69) and (6.70) we have

$$\begin{aligned}(\frac{\ln\frac{1}{\lambda_i}}{K_i})' =& \frac{r}{kK_i}[\gamma_1\frac{H_i}{\lambda_i^3} + \gamma_2\frac{\Delta K_i}{K_i\lambda_i^2} + \gamma_3\frac{|\nabla K_i|^2\ln\lambda_i}{K_i^2\lambda_i^2} \\ & - \gamma_4\sum_{i\neq j=1}^{p}\frac{\alpha_j}{\alpha_i}\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}](1 + o_{\frac{1}{\lambda_i}}(1)) \\ & + o_\varepsilon(\sum_r\frac{1}{\lambda_r^3} + \sum_{r\neq s}\varepsilon_{r,s}) + O(|\delta J(u)|^2),\end{aligned} \tag{6.78}$$

whence in conjunction with (6.76) there holds

$$\begin{aligned}\psi' \geq & \frac{r}{k}\sum_i \frac{C^i}{K_i\lambda_i^2}(\tilde{\gamma}_1\frac{H_i}{\lambda_i}+\gamma_2\frac{\Delta K_i}{K_i}+\tilde{\gamma}_3\frac{|\nabla K_i|^2}{K_i^2}\ln\lambda_i)(1+o_{\frac{1}{\lambda_i}}(1)) \\ & -\gamma_4\frac{r}{k}\sum_{i\neq j}C^i\frac{\alpha_j}{\alpha_i}[\frac{1}{K_i}+\kappa\vartheta_{\underline{\epsilon},i}]\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}(1+o_{\frac{1}{\lambda_i}}(1)) \\ & -\kappa\frac{r}{k}\sum_i C^i\vartheta_{\underline{\epsilon},i}(-\gamma_2+\frac{\gamma_3\langle\nabla\Delta K_i,\nabla K_i\rangle}{|\Delta K_i|^2})\frac{\Delta K_i}{K_i\lambda_i^2}(1+o_{\frac{1}{\lambda_i}}(1)) \\ & +O(\sum_{i\neq j}\frac{C^i}{\lambda_i}|\nabla_{a_i}\varepsilon_{i,j}|)+o_{\varepsilon}(\sum_{r\neq s}\varepsilon_{r,s})+O(|\delta J(u)|^2).\end{aligned} \tag{6.79}$$

We complete the definition of $\psi$ by ordering

$$\frac{\ln\frac{1}{\lambda_1}}{K_1}-\kappa\theta_1\geq\ldots\geq\frac{\ln\frac{1}{\lambda_p}}{K_p}-\kappa\theta_p \tag{6.80}$$

and claim, that there exists $c>0$ such, that for any $C>1$ sufficiently large

$$-\sum_{i\neq j}C^i\frac{\alpha_j}{\alpha_i}[\frac{1}{K_i}+\kappa\vartheta_{\underline{\epsilon},i}]\lambda_i\partial_{\lambda_i}\varepsilon_{i,j}\geq\epsilon\sum_{i>j}C^i\varepsilon_{i,j} \tag{6.81}$$

and

$$\sum_{i\neq j}\frac{C^i}{\lambda_i}|\nabla_{a_i}\varepsilon_{i,j}| =O(\sum_{i>j}C^j\varepsilon_{i,j}). \tag{6.82}$$

To prove (6.81), (6.82) note, that by definition for any $i>j$ we have

$$(C^i-C^j)(\frac{\ln\frac{1}{\lambda_i}}{K_i}-\kappa\theta_i)\leq(C^i-C^j)(\frac{\ln\frac{1}{\lambda_j}}{K_j}-\kappa\theta_j) \tag{6.83}$$

or equivalently

$$\begin{aligned}\frac{C^i-C^j}{K_i}\ln\frac{1}{\lambda_i}+\frac{C^j-C^i}{K_j}\ln\frac{1}{\lambda_j} & \\ +\kappa(C^j-C^i)\theta_i+\kappa(C^i-C^j)\theta_j\leq 0.\end{aligned} \tag{6.84}$$

We then have

$$\frac{\lambda_j}{\lambda_i}=o(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_nG^{\frac{2}{2-n}}(a_i,a_j)). \tag{6.85}$$

Otherwise we may assume for some $c>0$

$$\frac{\lambda_j}{\lambda_i}\geq c(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_nG^{\frac{2}{2-n}}(a_i,a_j)). \tag{6.86}$$

This implies $\frac{\lambda_j}{\lambda_i} \gg 1 \gg \frac{\lambda_i}{\lambda_j}$ and $d(a_i, a_j) = O(\frac{1}{\lambda_i})$. Consequently

$$\frac{C^i - C^j}{K_j} \ln \frac{\lambda_j}{\lambda_i} + \kappa(C^j - C^i)\theta_i + \kappa(C^i - C^j)\theta_j \leq O(\frac{\ln \lambda_i}{\lambda_i}), \tag{6.87}$$

whence due to the definition of $\theta_i$, see (6.72), there necessarily holds

$$\ln \frac{-\lambda_i \Delta K_i}{\underline{\epsilon}} \gg 1, \quad \text{so} \quad -\lambda_i \Delta K_i \gg 1 \tag{6.88}$$

and we get

$$\begin{aligned} &\frac{C^i - C^j}{K_j} \ln \frac{\lambda_j}{\lambda_i} + \kappa(C^j - C^i) \ln \frac{-\lambda_i \Delta K_i}{\underline{\epsilon}} \\ &+ \kappa(C^i - C^j)\eta_{\underline{\epsilon}}(-\lambda_j \Delta K_j) \ln \frac{-\lambda_j \Delta K_j}{\underline{\epsilon}} \leq O(\frac{\ln \lambda_i}{\lambda_i}). \end{aligned} \tag{6.89}$$

On the other hand $d(a_i, a_j) = O(\frac{1}{\lambda_i})$ and therefore

$$1 \ll -\lambda_i \Delta K_i = -\lambda_i \Delta K_j + O(1) = -\frac{\lambda_i}{\lambda_j} \lambda_j \Delta K_j + O(1). \tag{6.90}$$

This shows at once $1 \ll -\lambda_i \Delta K_i \ll -\lambda_j \Delta K_j$ and we conclude

$$\frac{C^i - C^j}{K_j} \ln \frac{\lambda_j}{\lambda_i} + \kappa(C^i - C^j) \ln \frac{-\lambda_j \Delta K_j}{-\lambda_i \Delta K_i} \leq O(\frac{\ln \lambda_i}{\lambda_i}) \tag{6.91}$$

yielding a contradiction. Thus (6.85) is established, whence (6.81) follows as when proving (6.55). We are left with estimating

$$\begin{aligned} \sum_{i \neq j} \frac{C^i}{\lambda_i} |\nabla_{a_i} \varepsilon_{i,j}| =& \frac{n-2}{2} \sum_{i<j} C^i \varepsilon_{i,j} \Big| \frac{(\frac{\lambda_j}{\lambda_i})^{\frac{1}{2}} (\lambda_i \lambda_j)^{\frac{1}{2}} \gamma_n \nabla_{a_i} G^{\frac{2}{2-n}}(a_i, a_j)}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j \gamma_n G^{\frac{2}{2-n}}(a_i, a_j)} \Big| \\ &+ o(\sum_{i \neq j} \varepsilon_{i,j}), \end{aligned} \tag{6.92}$$

whence we immediately obtain (6.82).

We conclude for $C > 1$ sufficiently large

$$\begin{aligned} \psi' \geq& \frac{r}{k} \sum_i \frac{C^i}{K_i \lambda_i^2} (\tilde{\gamma}_1 \frac{H_i}{\lambda_i} + \gamma_2 \frac{\Delta K_i}{K_i} + \tilde{\gamma}_3 \frac{|\nabla K_i|^2}{K_i^2} \ln \lambda_i)(1 + o_{\frac{1}{\lambda_i}}(1)) \\ &- \kappa \frac{r}{k} \sum_i C^i \vartheta_{\underline{\epsilon}, i} (-\gamma_2 + \frac{\gamma_3 \langle \nabla \Delta K_i, \nabla K_i \rangle}{|\Delta K_i|^2}) \frac{\Delta K_i}{K_i \lambda_i^2} (1 + o_{\frac{1}{\lambda_i}}(1)) \\ &+ \tilde{\gamma}_4 \sum_{i>j} C^i \varepsilon_{i,j} + O(|\delta J(u)|^2). \end{aligned} \tag{6.93}$$

This gives

$$\psi' \geq \frac{r}{k}\sum_i C^i[\frac{\gamma_2}{K_i} - \kappa\vartheta_{\underline{\epsilon},i}(-\gamma_2 + \frac{\gamma_3\langle\nabla\Delta K_i, \nabla K_i\rangle}{|\Delta K_i|^2})]\frac{\Delta K_i}{K_i\lambda_i^2}(1+o_{\frac{1}{\lambda_i}}(1))$$
$$+ c(\sum_i \frac{H_i}{\lambda_i^3} + \frac{|\nabla K_i|^2}{K_i^2\lambda_i^2}\ln\lambda_i + \sum_{i>j}\varepsilon_{i,j}) + O(|\delta J(u)|^2). \tag{6.94}$$

We now decompose $P = \{1, \dots, p\} = P_1 + P_2 + P_3$ with

(i) $P_1 = \{i \in \{1, \dots, p\} \mid -\Delta K_i < \frac{\underline{\epsilon}}{\lambda_i}\}$

(ii) $P_2 = \{i \in \{1, \dots, p\} \mid \frac{\underline{\epsilon}}{\lambda_i} \leq -\Delta K_i \leq 2\frac{\underline{\epsilon}}{\lambda_i}\}$

(iii) $P_3 = \{i \in \{1, \dots, p\} \mid -\Delta K_i > 2\frac{\underline{\epsilon}}{\lambda_i}\}$.

Note, that for $i \in P_2 \cup P_3$ we have $\Delta K_i < 0$, whence according to $Cond_5$

$$\langle\nabla\Delta K_i, \nabla K_i\rangle > \frac{1}{3}|\Delta K_i|^2, \tag{6.95}$$

in particular $\nabla K_i \neq 0$ for $i \in P_2 \cup P_3$.

For $i \in P_1$ there holds $\vartheta_{\underline{\epsilon},i} = 0$, thus

$$(\frac{\gamma_2}{K_i} - \kappa\vartheta_{\underline{\epsilon},i}(-\gamma_2 + \frac{\gamma_3\langle\nabla\Delta K_i, \nabla K_i\rangle}{|\Delta K_i|^2}))\frac{\Delta K_i}{K_i\lambda_i^2} \geq -\frac{\gamma_2}{K_i^2}\frac{\underline{\epsilon}}{\lambda_i^3}. \tag{6.96}$$

For $i \in P_2$

$$(\frac{\gamma_2}{K_i} - \kappa\vartheta_{\underline{\epsilon},i}(-\gamma_2 + \frac{\gamma_3\langle\nabla\Delta K_i, \nabla K_i\rangle}{|\Delta K_i|^2}))\frac{\Delta K_i}{K_i\lambda_i^2} \geq -2\frac{\gamma_2}{K_i^2}\frac{\underline{\epsilon}}{\lambda_i^3}, \tag{6.97}$$

since indeed $Cond_5$ imposed on $K$ can be rewritten as

$$-\gamma_2 + \frac{\gamma_3\langle\nabla\Delta K_i, \nabla K_i\rangle}{|\Delta K_i|^2} \geq c_0 > 0 \text{ for } a_i \in U(\mathcal{N}) \cap [\Delta K_i < 0], \tag{6.98}$$

as $\frac{\gamma_3}{\gamma_2} = 3$ by precise calculation, see below.

Choosing therefore

$$\underline{\epsilon} \leq c\min_{a\in M} H(a) \text{ with } c = c(K) \tag{6.99}$$

we get

$$\psi' \geq \frac{r}{k}\sum_{i\in P_3} C^i[\frac{\gamma_2}{K_i} - \kappa c_0]\frac{\Delta K_i}{K_i\lambda_i^2}(1+o_{\frac{1}{\lambda_i}}(1))$$
$$+ c(\sum_i \frac{H_i}{\lambda_i^3} + \frac{|\nabla K_i|^2}{K_i^2\lambda_i^2}\ln\lambda_i + \sum_{i>j}\varepsilon_{i,j}) + O(|\delta J(u)|^2). \tag{6.100}$$

since $\vartheta_{\underline{\epsilon},i} = 1$ on $i \in P_3$. Letting $\kappa = \frac{\gamma_2}{c_0\cdot\min_M K}$ we get as $\Delta K_i < 0$ for $i \in P_3$

$$\psi' \geq c(\sum_i \frac{H_i}{\lambda_i^3} + \frac{|\nabla K_i|^2}{K_i^2\lambda_i^2}\ln\lambda_i + \sum_{i>j}\varepsilon_{i,j}) + O(|\delta J(u)|^2). \tag{6.101}$$

We are left with checking $\frac{\gamma_3}{\gamma_2} = 3$. $\gamma_2$ and $\gamma_3$ arise from differentiating

$$-\frac{\dot{\lambda}_i}{\lambda_i} = \gamma_2 \frac{r}{k} \frac{\Delta K_i}{K_i \lambda_i^2} + \dots \quad \text{and} \quad \lambda_i \dot{a}_i = \gamma_3 \frac{r}{k} \frac{\nabla K_i}{K_i \lambda_i} + \dots, \tag{6.102}$$

where $\gamma_2 = \frac{e_2}{c_2}$, $\gamma_3 = \frac{e_3}{c_3}$, cf. corollary 4.7. According to (7.18) and (7.20)

$$c_2 = \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{|r^2-1|^2}{(1+r^2)^{n+2}}, \; c_3 = \frac{(n-2)^2}{n} \int_{\mathbb{R}^n} \frac{r^2}{(1+r^2)^{n+2}}, \tag{6.103}$$

whereas according to (4.30) and (4.31)

$$e_2 = \frac{(n-2)}{4n} \int_{\mathbb{R}^n} \frac{r^2(r^2-1)}{(1+r^2)^{n+1}} \quad \text{and} \quad e_3 = \frac{n-2}{n} \int_{\mathbb{R}^n} \frac{r^2}{(1+r^2)^{n+1}} \tag{6.104}$$

One obtains

$$\frac{\gamma_3}{\gamma_2} = \frac{c_2 e_3}{c_3 e_2} = 3. \tag{6.105}$$

The proof is thereby complete. □

The strategy in case $\omega > 0$ is independent of the dimension the same as when proving proposition 6.6. Note, that in comparison to propositions 6.6, 6.7 and 6.8 the contribution of the positive mass related term $\frac{H_i}{\lambda^{n-2}}$ is replaced by the positive terms $\frac{\omega_i}{\lambda_i^{\frac{n-2}{2}}}$.

**Proposition 6.9** (Case $\omega > 0$)**.**
*Let $n = 3, 4, 5$. Ordering*

$$\frac{1}{\lambda_1} \geq \dots \geq \frac{1}{\lambda_p}$$

*the function $\psi = \sum_i C^i \ln \frac{1}{\lambda_i}$ satisfies*

$$\psi' \geq \sum_i \frac{\omega_i}{\lambda_i^{\frac{n-2}{2}}} + \sum_{i>j} \varepsilon_{i,j} + O(|\delta J(u)|^2),$$

*provided $C > 1$ is sufficiently large.*

***Proof of proposition 6.9.***
This follows analogously to the proof of proposition 6.6. □

## 6.3 Proving the theorems

### 6.3.1 Proof of theorem 1

Let us consider a flow line, which is a solution of the evolution equation

$$\partial_t u = -\frac{1}{K}(R - r\bar{K})u, \; u_0 = u(\cdot, 0) > 0 \;\text{ with } \int K u_0^{\frac{2n}{n-2}} = 1. \tag{6.106}$$

The flow line exists for all times according to corollary 2.9 and we know

$$J(u) = r \searrow J_\infty = r_\infty \text{ and } |\delta J(u)| \longrightarrow 0 \text{ as } t \longrightarrow \infty. \tag{6.107}$$

due to proposition 2.11.

Thus a flow line is of Palais-Smale type and due to the concentration-compactness principle, cf. proposition 3.1, the flow line is precompact in some $V(\omega, p, \varepsilon)$, cf. definition 3.9 and the remarks following.

Taking the unicity result on a limiting critical point into account, cf. proposition 3.13, we obtain convergence of the flow line to a critical point of $J$, once the flow line is precompact in $V(\omega, 0, \varepsilon)$. In other words the flow line converges strongly, if and only if it converges along a sequence in time, and in this case we are done.

Thus we wish to lead to a contradiction the scenario, that for some $p \geq 1$ the flow line is precompact in some $V(\omega, p, \varepsilon)$.

By assumption of theorem 1 the dimensional condition $Cond_n$ hold true, so $\partial J$ is principally lower bounded, cf. proposition 6.5. Taking the unicity result on a limiting critical point at infinity into account, cf. proposition 6.3, we may assume, that the flow line remains for all times in $V(\omega, p, \varepsilon)$ and goes deeper and deeper in the sense, that

$$\forall\, 0 < \epsilon < \varepsilon\, \exists\, T > 0\, \forall\, t > T\ :\ u(t) \in V(\omega, p, \epsilon). \tag{6.108}$$

In particular the unique representation $u = u_{\alpha,\beta} + \alpha^i \varphi_{a_i,\lambda_i} + v$ given by proposition 3.10 is well defined for all times and we have $\lambda_i \longrightarrow \infty$ as $t \longrightarrow \infty$. Moreover the blow up points $a_i$ converge to $[\nabla K = 0]$, cf. lemma 6.4. Recalling the explanatory introduction of the previous subsection the functions given by propositions 6.6,6.7,6.8 and 6.9 then yield the desired contradiction.

### 6.3.2 Proving theorem 2

First of all note, that on $V(p, \varepsilon)$ we have according to definition 3.9

$$J(u) = \frac{\sum \alpha_i^2 \int L_{g_0} \varphi_i \varphi_i}{(\sum_i \alpha_i^{\frac{2n}{n-2}} \int K \varphi_i^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} + o_\varepsilon(1) = c_0 \frac{\sum_i \alpha_i^2}{(\sum_i \alpha_i^{\frac{2n}{n-2}} K_i)^{\frac{n-2}{n}}} + o_\varepsilon(1) \tag{6.109}$$

with $\alpha_i^{\frac{4}{n-2}} = \frac{4n(n-1)k}{rK_i} + o_\varepsilon(1)$. Therefore

$$J(u) = c_0 (\sum_i \frac{1}{K_i^{\frac{n-2}{2}}})^{\frac{2}{n}} + o_\varepsilon(1). \tag{6.110}$$

From this it is clear, that the least critical energy level at infinity is

$$J_{\infty,\min} = \frac{c_0}{(\max K)^{\frac{n-2}{n}}} \tag{6.111}$$

Thus, if we start a flow line $u$ with $u(0, \cdot) = u_0$, where

$$u_0 = \alpha_0 \varphi_{a_0,\lambda_0} \in V(1, \varepsilon),\ d(a_0, [K = \max K]) < \varepsilon$$

and $\varepsilon > 0$ is sufficiently small, we may assume, that $u$ remains in $V(1,\varepsilon)$ for all times and $d(a,[K = \max K]) = o_{\varepsilon}(1)$.

Indeed according to definition 3.11 and the remarks following $u$ is precompact with respect to $V(\omega,p,\varepsilon)$. Since we want to prove the existence of a non trivial solution $\omega > 0$, we may argue by contradiction and assume, that no non trivial solution exists, that is $\omega = 0$. So $u$ is precompact with respect to $V(p,\varepsilon)$. Moreover, if for some time sequence $t_k \longrightarrow \infty$ we had $u_{t_k} \in V(p,\varepsilon_k)$ with $\varepsilon_k \searrow 0$ and $p \geq 2$, then (6.110) would imply

$$J(u_{t_k}) = c_0(\sum_i \frac{1}{K_i^{\frac{n-2}{2}}})^{\frac{2}{n}} + o_{\varepsilon_k}(1) \geq c_0 \frac{p^{\frac{2}{n}}}{(\max K)^{\frac{n-2}{n}}} + o_{\varepsilon_k}(1), \tag{6.112}$$

whence without loss of generality $J(u_{t_k}) > J(u_0)$; contradicting $\partial_t J(u) \leq 0$.

Therefore $u$ is precompact with respect to $V(1,\varepsilon)$. Likewise we obtain $d(a,[K = \max K]) = o_{\varepsilon}(1)$, since otherwise $J(u_{t_k}) > J(u_0)$.

Repeating now the arguments for proposition 6.5 it is obvious, that $\partial J$ is principally lower bounded along the flow line $u$, since due to $Cond'_n$ the dimensional conditions $Cond_n$, cf. definition 1.2 are satisfied at the critical level $[K = \max K]$, to which $a$ is close. Therefore the results on the principal behaviour proven in subsection 6.1 hold true for the flow line $u$, in particular $d(a,[\nabla K = 0]) \longrightarrow 0$. On the other hand we have

$$\max K - K\lfloor_{[\nabla K=0]\setminus[K=\max K]} > \delta \tag{6.113}$$

for some $\delta > 0$ and $d(a,[K = \max K]) = o_{\varepsilon}(1)$. Thus we may assume

$$a \longrightarrow [K = \max K]. \tag{6.114}$$

Finally note, that the statement of propositions 6.6,6.7 and 6.8 remain valid for the functions constructed there, since as before $Cond'_n$ implies, that $Cond_n$ is satisfied at the critical level $[K = \max K]$, to which $a$ is close. Thus we arrive at the same contradiction as before, whence $u$ has to be precompact in some $V(\omega,p,\varepsilon)$ with $w > 0$ being a non trivial solution. The proof is thereby complete.

## 6.4 A diverging scenario

We give a non trivial example of a non compact flow line.

**Lemma 6.10** (Non-compact flow line with flatness).
*Let $n = 5$ and $u_0 = \alpha_0\varphi_{a_0,\lambda_0}$ with $a_0$ close to $0 \in M$, where*

$$K(x) = 1 - \sum_{i=1,\dots,5} |x_i|^4$$

*in local normal conformal coordinates.*

*Then for $\varepsilon > 0$ small there exists $0 < \varepsilon_0 < \varepsilon$ such, that the flow line $u$ with initial data $u_0$ remains in $V(1,\varepsilon)$ for all times, provided* *(i)* $\alpha_0\varphi_{a_0,\lambda_0} \in V(1,\varepsilon_0)$ *and* $k_{u_0} = \int K u_0^{\frac{2n}{n-2}} = 1$

*(ii)* $\|a_0\| < \varepsilon_0$ *and* $\lambda_0\|a_0\|^2 > \varepsilon_0^{-1}$

*(iii)* $(a_i)_0 = (a_j)_0 > 0$ *for all* $i, j = 1, \ldots, 5$.

*Moreover* $u$ *converges to a critical point at infinity in the sense, that*

$$\lambda \longrightarrow \infty \ \ \textit{and} \ \ \|a\| \longrightarrow 0 \ \ \textit{as} \ \ t \longrightarrow \infty.$$

Note, that $K$ does not satisfy condition $Cond_5$, cf. definition 1.2, since

$$\langle \nabla\Delta K, \nabla K\rangle = \frac{7}{9}|\Delta K|^2 \ \text{ on } \ B_\varepsilon(0),$$

but $K$ satisfies the flatness condition of Theorem 0.1 in [23], cf. [24], [19].

*Proof.* In order to prove, that $u$ remains in $V(1,\varepsilon)$ for all times let us define

$$\begin{aligned} T = \sup\{\tau > 0 \mid \ \forall\, 0 \leq t < \tau \ : u \in V(1,\varepsilon),\ \|a\| < \varepsilon,\ \lambda\|a\|^2 > \varepsilon^{-1} \\ \frac{a_i}{a_j} < \sqrt[4]{\frac{5}{2}} \ \text{ for all } \ i, j = 1, \ldots, n\}. \end{aligned} \tag{6.115}$$

We then have to show $T = \infty$.

Note, that we may assume $J(u_0) \leq C$ independent of $0 < \varepsilon_0 \ll 1$, whence

$$\int_0^\infty |\delta J(u)|^2 \leq c(K) \tag{6.116}$$

independent of the smallness of $0 < \varepsilon \ll 1$.

According to corollary 4.7 the relevant evolution equations are

$$\begin{aligned} -\frac{\dot\lambda}{\lambda} = \frac{r}{k}(\gamma_1\frac{H(a)}{\lambda^3} + \gamma_2\frac{\Delta K(a)}{K(a)\lambda^2})(1 + o_{\frac{1}{\lambda}}(1)) \\ + o(\frac{1}{\lambda^3}) + O(\frac{|\nabla K(a)|^2}{\lambda^2} + |\delta J(u)|^2) \end{aligned} \tag{6.117}$$

and

$$\begin{aligned} \lambda\dot a = \frac{r}{k}(\gamma_3\frac{\nabla K(a)}{K(a)\lambda} + \gamma_4\frac{\nabla\Delta K(a)}{K(a)\lambda^3})(1 + o_{\frac{1}{\lambda}}(1)) \\ + o(\frac{1}{\lambda^3}) + O(\frac{|\nabla K(a)|^2}{\lambda^2} + |\delta J(u)|^2), \end{aligned} \tag{6.118}$$

where $\frac{r}{k} = 4n(n-1) + o(1) = 80(1 + o_\varepsilon(1))$ according to (4.35). Moreover

$$\nabla K(a) = -4\|a\|^2 a,\ \Delta K(a) = -12\|a\|^2 \ \text{ and } \ \nabla\Delta K(a) = -24a. \tag{6.119}$$

We obtain during $(0,T)$ the simplified evolution equations

$$-\frac{\dot\lambda}{\lambda} = 80\gamma_2\frac{\Delta K(a)}{\lambda^2}(1 + o_\varepsilon(1)) + O(|\delta J(u)|^2) \tag{6.120}$$

and

$$\lambda \dot{a} = 80\gamma_3 \frac{\nabla K(a)}{\lambda}(1+o_\varepsilon(1)) + O(|\delta J(u)|^2). \tag{6.121}$$

First note, that during $(0,T)$

$$\begin{aligned}\partial_t \|a\|^2 =& \frac{2}{\lambda}\langle a, \lambda \dot{a}\rangle \\ =& c\frac{\langle \nabla K(a), a\rangle}{\lambda^2}(1+o_\varepsilon(1)) + O(\frac{\|a\||\delta J(u)|^2}{\lambda}) \\ \leq& O(\frac{\|a\||\delta J(u)|^2}{\lambda}),\end{aligned} \tag{6.122}$$

whence

$$\partial_t \ln \|a\|^2 \leq O(\frac{|\delta J(u)|^2}{\lambda \|a\|}).$$

But $\lambda\|a\| = \lambda^{\frac{1}{2}}(\lambda\|a\|^2)^{\frac{1}{2}} > c\varepsilon^{-1}$ during $(0,T)$ by definition. Therefore $\|a\|$ remains uniformly small, e.g.$\|a\| \leq C\varepsilon_0$. Let us calculate

$$\begin{aligned}(\lambda \Delta K(a))' =& \frac{\dot{\lambda}}{\lambda}\lambda \Delta K(a) + \langle \nabla \Delta K(a), \lambda \dot{a}\rangle \\ =& -80\gamma_2 \frac{|\Delta K(a)|^2}{\lambda}(1+o_\varepsilon(1)) \\ &+ 80\gamma_3 \frac{\langle \nabla \Delta K(a), \nabla K(a)\rangle}{\lambda}(1+o_\varepsilon(1)) \\ &+ O((|\lambda \Delta K(a)| + |\nabla \Delta K(a)|)|\delta J(u)|^2).\end{aligned} \tag{6.123}$$

Since $|\lambda \Delta K(a)| = 12\lambda\|a\|^2 \geq c\varepsilon^{-1}$ during $(0,T)$, we obtain

$$\begin{aligned}\frac{(\lambda \Delta K(a))'}{80} =& (-12^2\gamma_2(\sum_{i=1}^{5}|a_i|^2)^2 + 4\cdot 24\gamma_3 \sum_{i=1}^{5}|a_i|^4)\frac{1+o_\varepsilon(1)}{\lambda} \\ &+ O(|\lambda \Delta K(a)||\delta J(u)|^2) \\ \leq& (-12^2\cdot 5^2\,\gamma_2\,a_{\min}^4 + 4\cdot 5\cdot 24\,\gamma_3\,a_{\max}^4)\frac{1+o_\varepsilon(1)}{\lambda} \\ &+ O(|\lambda \Delta K(a)||\delta J(u)|^2),\end{aligned} \tag{6.124}$$

where we used $a_i > 0$ during $(0,T)$ and let

$$a_{\min} = \min\{a_i \mid i=1,\ldots,n\} \text{ and } a_{\max} = \max\{a_i \mid i=1,\ldots,n\}. \tag{6.125}$$

Due to $\frac{\gamma_3}{\gamma_2} = 3$, cf. (6.105), and $a_{\max}^4 < \frac{5}{2}a_{\min}^4$ during $(0,T)$ we get

$$(\lambda \Delta K(a))' \leq O(|\lambda \Delta K(a)||\delta J(u)|^2). \tag{6.126}$$

Therefore

$$\partial_t \ln(-\lambda \Delta K(a)) \geq O(|\delta J(u)|^2) \tag{6.127}$$

and we conclude using (6.116), that

$$12\lambda \|a\|^2 = -\lambda \Delta K(a) \geq -\lambda_0 \Delta K(a_0) e^{-C\int_0^\infty |\delta J(u)|^2} = 12\lambda_0 \|a_0\|^2 \tag{6.128}$$

remains during $(0, T)$ uniformly large, say $\lambda \|a\|^2 \geq c\varepsilon_0^{-1}$. Moreover

$$\begin{aligned} -\frac{\dot{\lambda}}{\lambda} =& 80\gamma_2 \frac{\Delta K(a)}{\lambda^2}(1 + o_\varepsilon(1)) + O(|\delta J(u)|^2) \\ =& -c\frac{\|a\|^2}{\lambda^2} + O(|\delta J(u)|^2) \\ \leq& -c\frac{\varepsilon^{-1}}{\lambda^3} + O(|\delta J(u)|^2), \end{aligned} \tag{6.129}$$

whence

$$\partial_t \lambda^3 + \lambda^3 O(|\delta J(u)|^2) \geq C\varepsilon^{-1}. \tag{6.130}$$

Letting $\vartheta = \lambda^3$ this becomes

$$\dot{\vartheta} + \vartheta O(|\delta J(u)|^2) \geq C\varepsilon^{-1}. \tag{6.131}$$

Thus for $\tau(t) = \vartheta(t) e^{\int_0^t O(|\delta J(u)|^2)}$ there holds

$$\dot{\tau}(t) = (\dot{\vartheta} + \vartheta O(|\delta J(u)|^2))(t) e^{\int_0^t O(|\delta J(u)|^2)} \geq C\varepsilon^{-1} e^{\int_0^t O(|\delta J(u)|^2)} \tag{6.132}$$

and therefore

$$\dot{\tau}(t) \geq c\varepsilon^{-1}, \tag{6.133}$$

whence

$$\vartheta(0) = \tau(0) \leq \tau(t) = \vartheta(t) e^{\int_0^t O(|\delta J(u)|^2)} \leq C\vartheta(t), \tag{6.134}$$

so $\vartheta$ and thereby $\lambda$ remain uniformly large, say $\lambda \geq c\varepsilon_0^{-1}$. Finally note, that

$$\begin{aligned} (\frac{a_i}{a_j})' =& \frac{\lambda \dot{a}_i}{\lambda a_j} - \frac{a_i}{a_j}\frac{\lambda \dot{a}_j}{\lambda a_j} \\ =& -c(\frac{|a_i|^2 a_i}{\lambda^2 a_j} - \frac{a_i}{a_j}\frac{|a_j|^2 a_j}{\lambda^2 a_j})(1 + o_\varepsilon(1)) + O(\frac{|\delta J(u)|^2}{\lambda a_j}) \\ =& -c\frac{a_i}{a_j}\frac{1}{\lambda^2}(|a_i|^2 - |a_j|^2)(1 + o_\varepsilon(1)) + O(\frac{|\delta J(u)|^2}{\lambda a_j}), \end{aligned} \tag{6.135}$$

whence without loss of generality we may assume

$$(\frac{a_{\max}}{a_{\min}})' \leq C\frac{|\delta J(u)|^2}{\lambda a_{\min}} \text{ in case } \frac{a_{\max}}{a_{\min}} \geq \sqrt[4]{\frac{5}{4}}.$$

But during $(0,T)$ we have

$$\lambda a_{\min} \geq \sqrt[4]{\frac{2}{5}}\lambda a_{\max} \geq c\lambda\|a\| \tag{6.136}$$

and $\lambda\|a\| = \lambda^{\frac{1}{2}}(\lambda\|a\|^2)^{\frac{1}{2}} > \varepsilon^{-1}$, whence

$$\partial_t \ln(\frac{a_{\max}}{a_{\min}}) \leq C\varepsilon|\delta J(u)|^2 \text{ in case } \frac{a_{\max}}{a_{\min}} \geq \sqrt[4]{\frac{5}{4}}. \tag{6.137}$$

Consequently we may assume

$$\frac{a_i}{a_j} < \sqrt[4]{\frac{5}{3}} \text{ during } (0,T). \tag{6.138}$$

So far we have seen, that during $(0,T)$ we may assume

$$\|a\| < C\varepsilon_0, \lambda\|a\|^2 > c\varepsilon_0^{-1}, \lambda > c\varepsilon_0^{-1} \text{ and } \frac{a_i}{a_j} < \sqrt[4]{\frac{5}{3}}. \tag{6.139}$$

In order to show $T=\infty$ it remains to prove

$$u \in V(1,\frac{\varepsilon}{2}) \text{ during } (0,T). \tag{6.140}$$

By definition 3.9 and the remarks thereafter this is equivalent to showing

$$|1-\frac{r\alpha^{\frac{4}{n-2}}K(a)}{4n(n-1)k}|, \|u-\alpha\varphi_{a,\lambda}\| = \|v\| < \frac{\varepsilon}{2}. \tag{6.141}$$

To that end let us expand using $k \equiv 1$

$$\begin{aligned} J(u) =& r = \int L_{g_0}uu = \int L_{g_0}(\alpha\varphi_{a,\lambda}+v)(\alpha\varphi_{a,\lambda}+v) \\ =&\alpha^2\int L_{g_0}\varphi_{a,\lambda}\varphi_{a,\lambda} + 2\alpha\int L_{g_0}\varphi_{a,\lambda}v + \int L_{g_0}vv. \end{aligned} \tag{6.142}$$

Due to lemmata 3.3 and 3.5 we have with $n=5$

$$\int L_{g_0}\varphi_{a,\lambda}\varphi_{a,\lambda} = 4n(n-1)c_0 + o_{\frac{1}{\lambda}}(1). \tag{6.143}$$

Moreover from lemma 3.3 we get

$$\begin{aligned} \frac{\int L_{g_0}\varphi_{a,\lambda}v}{4n(n-1)} =& \int \varphi_{a,\lambda}^{\frac{n+2}{n-2}}v + o_{\frac{1}{\lambda}}(1) = \int K\varphi_{a,\lambda}^{\frac{n+2}{n-2}}v + o_{\frac{1}{\lambda}}(1) \\ =&\alpha^{-\frac{4}{n-2}}\int K(u-v)^{\frac{4}{n-2}}\varphi_{a,\lambda}v + o_{\frac{1}{\lambda}}(1) \\ =& -\frac{4}{n-2}\alpha^{-\frac{4}{n-2}}\int Ku^{\frac{6-n}{n-2}}\varphi_{a,\lambda}v^2 + o(\|v\|^2) + o_{\frac{1}{\lambda}}(1) \\ =& -\frac{4}{n-2}\alpha^{-1}\int K\varphi_{a,\lambda}^{\frac{4}{n-2}}v^2 + o(\|v\|^2) + o_{\frac{1}{\lambda}}(1). \end{aligned} \tag{6.144}$$

We conclude

$$
\begin{aligned}
J(u) =& 4n(n-1)c_0\alpha^2 + \int L_{g_0}vv - \frac{32n(n-1)}{n-2}\int \varphi_{a,\lambda}^{\frac{4}{n-2}}v^2 \\
&+ o_{\frac{1}{\lambda}}(1) + o(\|v\|^2)
\end{aligned}
\tag{6.145}
$$

On the other hand we have

$$
\begin{aligned}
1 \equiv & k = \int K u^{\frac{2n}{n-2}} = \int K(\alpha\varphi_{a,\lambda}+v)^{\frac{2n}{n-2}} \\
=& \alpha^{\frac{2n}{n-2}}\int K\varphi_{a,\lambda}^{\frac{2n}{n-2}} + \frac{2n}{n-2}\alpha^{\frac{n+2}{n-2}}\int K\varphi_{a,\lambda}^{\frac{n+2}{n-2}}v \\
&+ \frac{n}{n-2}\frac{n+2}{n-2}\alpha^{\frac{4}{n-2}}\int K\varphi_{a,\lambda}^{\frac{4}{n-2}}v^2 + o(\|v\|^2).
\end{aligned}
\tag{6.146}
$$

Considering the second summand above we obtain using (6.144)

$$
1 = \alpha^{\frac{2n}{n-2}}c_0 + \frac{n(n-6)}{(n-2)^2}\alpha^{\frac{4}{n-2}}\int \varphi_{a,\lambda}^{\frac{4}{n-2}}v^2 + o_{\frac{1}{\lambda}}(1) + o(\|v\|^2),
\tag{6.147}
$$

whence

$$
\alpha = c_0^{-\frac{n-2}{2n}} + \frac{6-n}{2(n-2)}c_0^{-\frac{n+2}{2n}}\int \varphi_{a,\lambda}^{\frac{4}{n-2}}v^2 + o_{\frac{1}{\lambda}}(1) + o(\|v\|^2)
\tag{6.148}
$$

and therefore

$$
c_0\alpha^2 = c_0^{-\frac{2}{n}} + \frac{6-n}{n-2}\int \varphi_{a,\lambda}^{\frac{4}{n-2}}v^2 + o_{\frac{1}{\lambda}}(1) + o(\|v\|^2).
\tag{6.149}
$$

We conclude

$$
\begin{aligned}
J(u) =& 4n(n-1)c_0^{\frac{2}{n}} \\
&+ \int L_{g_0}vv - 4n(n-1)\frac{n+2}{n-2}\int \varphi_{a,\lambda}^{\frac{4}{n-2}}v^2 + o_{\frac{1}{\lambda}}(1) + o(\|v\|^2) \\
\geq & 4n(n-1)c_0^{\frac{2}{n}} \\
&+ c_n\int\left(|\nabla v|_{g_0}^2 - n(n+2)\int \varphi_{a,\lambda}^{\frac{4}{n-2}}v^2\right) + o_{\frac{1}{\lambda}}(1) + o(\|v\|^2)
\end{aligned}
\tag{6.150}
$$

and thus by means of proposition 4.5,

$$
J(u) \geq 4n(n-1)c_0^{\frac{2}{n}} + o_{\frac{1}{\lambda}}(1) + c\|v\|^2.
\tag{6.151}
$$

But $J(u) \leq J(u_0) = 4n(n-1)c_0^{\frac{2}{n}} + O(\frac{1}{\lambda_0})$ and therefore

$$
\|v\|^2 = o_{\frac{1}{\lambda}+\frac{1}{\lambda_0}}(1)
\tag{6.152}
$$

remains uniformly small during $(0,T)$. Finally we infer from (6.148), that $\alpha$ remains uniformly close to $c_0^{-\frac{n-2}{2n}}$, in particular

$$\begin{aligned}\frac{r\alpha^{\frac{4}{n-2}}K(a)}{4n(n-1)k} =&\frac{\alpha^{\frac{4}{n-2}}J(u)}{4n(n-1)}+O(\|a\|)\\ =&\alpha^{\frac{4}{n-2}}c_0^{\frac{2}{n}}+O(\|a\|+\|v\|)+o_{\frac{1}{\lambda}}(1)\\ =&1+O(\|a\|+\|v\|)+o_{\frac{1}{\lambda}}(1),\end{aligned} \tag{6.153}$$

whence

$$|1-\frac{r\alpha^{\frac{4}{n-2}}K(a)}{4n(n-1)k}| \tag{6.154}$$

remains uniformly small. This completes the proof of $T=\infty$, which is to say, that $u$ remains in $V(1,\varepsilon)$. Turning back to (6.133) we then get $\tau\geq t$ as $t\longrightarrow\infty$, whence according to (6.134)

$$\vartheta=\lambda^3\geq ct. \tag{6.155}$$

Finally (6.119) and (6.122) show

$$\partial_t\|a\|^2\leq -c\frac{\|a\|^4}{\lambda^2}+O(\frac{\|a\||\delta J(u)|^2}{\lambda})=\|a|^2(-\frac{\|a\|^2}{\lambda^2}+O(\frac{|\delta J(u)|^2}{\|a\|\lambda})). \tag{6.156}$$

Since $\lambda\|a\|^2$ and therefore $\lambda\|a\|$ as well remain large we obtain

$$\partial_t\ln\|a\|^2\leq -c\frac{\|a\|^2}{\lambda^2}+O(|\delta J(u)|^2), \tag{6.157}$$

whence due to (6.119) and (6.120)

$$\partial_t\ln\|a\|^2\leq -c\frac{\dot{\lambda}}{\lambda}+O(|\delta J(u)|^2)=-c\partial_t\ln\lambda+O(|\delta J(u)|^2). \tag{6.158}$$

Therefore $\lambda\longrightarrow\infty$ implies $\|a\|\longrightarrow 0$. □

# 7 Appendix

**Lemma 7.1.**
*Let $(M^n,g_0)$ be a Riemannian manifold, $g(t)=u^{\frac{4}{n-2}}(t)g_0$, $u>0$. There holds*

*(i)*
$$d\mu_g=u^{\frac{2n}{n-2}}d\mu_{g_0}$$

*(ii)*
$${}_g\Gamma^k_{i,j}={}_{g_0}\Gamma^k_{i,j}+\frac{2}{n-2}u^{-1}(\partial_i u\delta^k_j+\partial_j u\delta^k_i-\partial_l u g^{k,l}g_{i,j})$$

*(iii)*

$$\begin{aligned}\tilde{R}^l_{i,j,k} =& R^l_{i,j,k} + \frac{2}{n-2}u^{-1}[\nabla^2_{i,k}u\delta^l_j - \nabla^2_{i,p}ug^{l,p}g_{j,k} \\ &\qquad - \nabla^2_{j,k}u\delta^l_i + \nabla^2_{j,p}ug^{l.p}g_{i,k}] \\ &- \frac{2n}{(n-2)^2}u^{-2}[\nabla_i u\nabla_k u\delta^l_j - \nabla_i u\nabla_p ug^{l,p}g_{j,k} \\ &\qquad - \nabla_j u\nabla_k u\delta^l_i + \nabla_j u\nabla_p ug^{l,p}g_{i,k} \\ &\qquad + \frac{2}{n}|\nabla u|^2 g_{j,k}\delta^l_i - \frac{2}{n}|\nabla u|^2 g_{i,k}\delta^l_j]\end{aligned}$$

*(iv)*

$$\begin{aligned}\tilde{R}_{i,k} = R_{i,k} &+ \frac{2}{n-2}u^{-1}[(n-2)\nabla^2_{i,k}u - \Delta u g_{i,k}] \\ &- \frac{2}{n-2}u^{-2}[n\nabla_i u\nabla_k u - |\nabla u|^2 g_{i,k}]\end{aligned}$$

*(v)* $R = R_g = u^{-\frac{n+2}{n-2}}[-c_n\Delta u + R_{g_0}u] = u^{-\frac{n+2}{n-2}}L_{g_0}u$, *i.e.*

$$u^{-\frac{n+2}{n-2}}L_{g_0}(uv) = L_g(v)$$

*(vi) and for* $\partial_t u = -\frac{1}{K}(R - r\bar{K})u$ *we have*

$$\partial_t R = c_n\Delta_g\frac{R}{K} + \frac{4}{n-2}(R - r\bar{K})\frac{R}{K}.$$

**Lemma 7.2.** *[Local bound and higher integrability, cf. [27], Theorem A.1.] Let* $P \in C^\infty(M)$, $p > \frac{2n}{n-2}$ *and* $r > 0$ *small.*

*There exists* $C = C(p, r)$ *such, that for* $u > 0$ *solving* $L_{g_0}u = Pu$ *with*

$$\|P\|_{L^{\frac{n}{2}}_{g_0}(B_{2r}(x_0))} < \frac{2n}{n-2}\frac{Y(M, g_0)}{p}$$

*we have*

$$\|u\|_{L^p_{g_0}(B_r(x_0))} \leq C\|u\|_{L^{\frac{2n}{n-2}}_{g_0}(B_{2r}(x_0))}.$$

***Proof of lemma 3.3***.
A straight forward calculation shows

$$\begin{aligned}\Delta_{g_a}(\frac{\lambda}{1+\lambda^2\gamma_n G_a^{\frac{2}{2-n}}})^{\frac{n-2}{2}} =& \frac{n}{2-n}\gamma_n(\frac{\varphi_{a,\lambda}}{u_a})^{\frac{n+2}{n-2}}|\nabla G_a|^2_{g_a}G_a^{2\frac{n-1}{2-n}} \\ &+ \gamma_n\lambda(\frac{\varphi_{a,\lambda}}{u_a})^{\frac{n}{n-2}}G_a^{\frac{n}{2-n}}\Delta_{g_a}G_a,\end{aligned} \tag{7.1}$$

which is due to

$$|\nabla G_a|^2_{g_a} G_a^{2\frac{n-1}{2-n}} = (n-2)^2 |\nabla G_a^{\frac{1}{2-n}}|^2_{g_a} \text{ and } \Delta_{g_a} G_a = \frac{-\delta_a + R_{g_a} G_a}{c_n}, \tag{7.2}$$

where $\delta_a$ denotes the Dirac measure at $a$, equivalent to

$$\begin{aligned}\Delta_{g_a}(\frac{\lambda}{1+\lambda^2\gamma_n G_a^{\frac{2}{2-n}}})^{\frac{n-2}{2}} =& n(2-n)\gamma_n(\frac{\varphi_{a,\lambda}}{u_a})^{\frac{n+2}{n-2}}|\nabla G_a^{\frac{1}{2-n}}|^2_{g_a} \\ &+ \frac{R_{g_a}\gamma_n}{c_n}\lambda(\frac{\varphi_{a,\lambda}}{u_a})^{\frac{n}{n-2}} G_a^{\frac{2}{2-n}}.\end{aligned} \tag{7.3}$$

Since $L_{g_a} = -c_n \Delta_{g_a} + R_{g_a}$ with $c_n = 4\frac{n-1}{n-2}$ we obtain

$$\begin{aligned}L_{g_a}\frac{\varphi_{a,\lambda}}{u_a} =& 4n(n-1)(\frac{\varphi_{a,\lambda}}{u_a})^{\frac{n+2}{n-2}}\gamma_n|\nabla G_a^{\frac{1}{2-n}}|^2_{g_a} \\ &+ R_{g_a}\frac{\varphi_{a,\lambda}}{u_a}(1-\lambda\gamma_n G_a^{\frac{2}{2-n}}(\frac{\varphi_{a,\lambda}}{u_a})^{\frac{2}{2-n}}) \\ =& 4n(n-1)(\frac{\varphi_{a,\lambda}}{u_a})^{\frac{n+2}{n-2}}\gamma_n|\nabla G_a^{\frac{1}{2-n}}|^2_{g_a} + \frac{R_{g_a}}{\lambda}(\frac{\varphi_{a,\lambda}}{u_a})^{\frac{n}{n-2}}.\end{aligned} \tag{7.4}$$

By conformal invariance, cf. lemma 7.1, we conclude

$$\begin{aligned}L_{g_0}\varphi_{a,\lambda} =& u_a^{\frac{n+2}{n-2}} L_{g_a}\frac{\varphi_{a,\lambda}}{u_a} \\ =& 4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}}\gamma_n|\nabla G_a^{\frac{1}{2-n}}|^2_{g_a} + \frac{u_a^{\frac{2}{n-2}} R_{g_a}}{\lambda}\varphi_{a,\lambda}^{\frac{n}{n-2}},\end{aligned} \tag{7.5}$$

in particular $L_{g_0}\varphi_{a,\lambda} = O(\varphi_{a,\lambda}^{\frac{n+2}{n-2}})$. Expanding

$$G_a = \frac{1}{4n(n-1)\omega_n}(r_a^{2-n} + H_a),\ r_a = d_{g_a}(a,\cdot) \tag{7.6}$$

we derive

$$\begin{aligned}\gamma_n|\nabla G_a^{\frac{1}{2-n}}|^2_{g_a} =& |\nabla(r_a(1+r_a^{n-2}H_a)^{\frac{1}{2-n}})|^2_{g_a} \\ =& |\nabla r_a(1+\frac{1}{2-n}r_a^{n-2}H_a + O(|r_a^{n-2}H_a|^2)) \\ &+ r_a(-r_a^{n-3}\nabla r_a H_a + \frac{1}{2-n}r_a^{n-2}\nabla H_a \\ &+ O(|r_a^{n-2}H_a||\nabla(r_a^{n-2}H_a)|))|^2_{g_a},\end{aligned} \tag{7.7}$$

whence

$$\gamma_n|\nabla G_a^{\frac{1}{2-n}}|^2_{g_a} = 1 - \frac{2}{n-2}((n-1)H_a + r_a\partial_{r_a}H_a)r_a^{n-2} + o(r_a^{n-2}). \tag{7.8}$$

Thus we conclude

$$\begin{aligned}L_{g_0}\varphi_{a,\lambda} =&4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}}-2nc_n((n-1)H_a+r_a\partial_{r_a}H_a)r_a^{n-2}\varphi_{a,\lambda}^{\frac{n+2}{n-2}}\\&+o(r_a^{n-2}\varphi_{a,\lambda}^{\frac{n+2}{n-2}})+\frac{u_a^{\frac{2}{n-2}}R_{g_a}}{\lambda}\varphi_{a,\lambda}^{\frac{n}{n-2}}.\end{aligned} \tag{7.9}$$

For $R_{g_a}=O(r_a^2)$ and $\Delta R_{g_a}=-\frac{1}{6}|W(a)|^2$ cf. [22]. □

***Proof of lemma 3.5.***
These kind of expansions are well known, cf. [6]. Using but just slightly modified bubbles we nonetheless repeat their proves.

(i) We have

$$(\phi_{k,i})_{k=1,2,3}=(\varphi_i,-\lambda_i\partial_{\lambda_i}\varphi_i,\frac{1}{\lambda_i}\nabla_{a_i}\varphi_i), \tag{7.10}$$

so

$$\phi_{1,i}=u_{a_i}(\frac{\lambda_i}{1+\lambda_i^2\gamma_n G_{a_i}^{\frac{2}{2-n}}})^{\frac{n-2}{2}} \tag{7.11}$$

and

$$\phi_{2,i}=\frac{n-2}{2}\frac{\lambda_i^2\gamma_n G_{a_i}^{\frac{2}{2-n}}-1}{\lambda_i^2\gamma_n G_{a_i}^{\frac{2}{2-n}}+1}\varphi_i \tag{7.12}$$

and

$$\phi_{3,i}=-\frac{n-2}{2}u_{a_i}\frac{\lambda_i\gamma_n\nabla_{a_i}G_{a_i}^{\frac{2}{2-n}}}{1+\lambda_i^2\gamma_n G_{a_i}^{\frac{2}{2-n}}}\varphi_i+\frac{\nabla_{a_i}u_{a_i}}{u_{a_i}\lambda_i}\varphi_i. \tag{7.13}$$

Note, that in $x\simeq \exp_{g_{a_i}} x$ coordinates

$$\gamma_n G_{a_i}^{\frac{2}{2-n}}(x)=r^2+O(r^n), \tag{7.14}$$

cf. definition 3.2 and

$$\gamma_n(\nabla_{a_i}G_{a_i}^{\frac{2}{2-n}})(x)=-2x+O(r^{n-1}). \tag{7.15}$$

Moreover $u_{a_i}=1+O(r^2)$. The assertion readily follows.

(ii) ($\alpha$) Case $k=1$
We have $\phi_{k,i}=\varphi_i$ for $k=1$ and thus for $c>0$ small

$$\int\varphi_i^{\frac{2n}{n-2}}=\int_{B_c(a_i)}(\frac{\lambda_i}{1+\lambda_i^2\gamma_n G_{a_i}^{\frac{2}{2-n}}})^n d\mu_{g_{a_i}}+O(\frac{1}{\lambda_i^n}). \tag{7.16}$$

By definition 3.2 one has passing to $x \simeq \exp_{g_{a_i}} x$ coordinates

$$\begin{aligned}\int \varphi_i^{\frac{2n}{n-2}} &= \int_{B_c(0)} (\frac{\lambda_i}{1+\lambda_i^2 r^2})^n (1 + O(\frac{\lambda_i^2 O(r^n H_a)}{1+\lambda_i^2 r^2})) + O(\frac{1}{\lambda_i^n}) \\ &= \int_{B_{c\lambda_i}(0)} \frac{1}{(1+r^2)^n} + O(\frac{1}{\lambda_i^{n-2}}) = c_1 + O(\frac{1}{\lambda_i^{n-2}}).\end{aligned} \tag{7.17}$$

($\beta$) Case $k = 2$

The proof runs analogously to the one of case $k = 1$ above yielding

$$c_2 = \frac{(n-2)^2}{4} \int \frac{|r^2-1|^2}{(1+r^2)^{n+2}} \tag{7.18}$$

($\gamma$) Case $k = 3$

We have

$$\phi_{k,i} = \frac{2-n}{2} \frac{\lambda_i \gamma_n \nabla_{a_i} G_{a_i}^{\frac{2}{2-n}}}{1+\lambda_i^2 \gamma_n G_{a_i}^{\frac{2}{2-n}}} \varphi_i + \frac{\nabla_{a_i} u_{a_i}}{\lambda_i u_{a_i}} \varphi_i, \tag{7.19}$$

whence using $\gamma_n(\nabla_{a_i} G_{a_i}^{\frac{2}{2-n}})(x) = -2x + O(r^{n-1})$ and $u_{a_i} = O(r^2)$

$$\begin{aligned}\int |\phi_{k,i}|^2 \varphi_i^{\frac{4}{n-2}} &= \frac{(n-2)^2}{n} \int \frac{r^2}{(1+r^2)^{n+2}} + O(\frac{1}{\lambda_i^{n-2}} + \frac{1}{\lambda_i^2}) \\ &= c_3 + O(\frac{1}{\lambda_i^{n-2}} + \frac{1}{\lambda_i^2})\end{aligned} \tag{7.20}$$

(iii) ($\alpha$) Case $k = 1$

Due to lemma 3.3 and case $(v)$ we have for $c > 0$ small

$$\begin{aligned}\int \varphi_i^{\frac{n+2}{n-2}} \varphi_j &= \int_{B_c(a_i)} \frac{L_{g_0} \varphi_i \varphi_j}{4n(n-1)} + o(\varepsilon_{i,j}) \\ &= \int \frac{L_{g_0} \varphi_i \varphi_j}{4n(n-1)} + o(\varepsilon_{i,j}),\end{aligned} \tag{7.21}$$

whence by $\int L_{g_0} \varphi_i \varphi_j = \int \varphi_i L_{g_0} \varphi_j$ and backward calculation

$$\int \varphi_i^{\frac{n+2}{n-2}} \varphi_j = \int \varphi_i \varphi_j^{\frac{n+2}{n-2}} + o(\varepsilon_{i,j}). \tag{7.22}$$

Thus we may assume $\frac{1}{\lambda_i} \leq \frac{1}{\lambda_j}$. We get

$$\begin{aligned}\int \varphi_i^{\frac{n+2}{n-2}} \varphi_j = \int_{B_c(a_i)} & (\frac{\lambda_i}{1+\lambda_i^2 \gamma_n G_{a_i}^{\frac{2}{2-n}}})^{\frac{n+2}{2}} \\ & \frac{u_{a_j}}{u_{a_i}} (\frac{\lambda_j}{1+\lambda_j^2 \gamma_n G_{a_j}^{\frac{2}{2-n}}})^{\frac{n-2}{2}} d\mu_{g_{a_i}} \\ & + O(\frac{1}{\lambda_i^{\frac{n+2}{2}}} \frac{1}{\lambda_j^{\frac{n-2}{2}}}).\end{aligned} \tag{7.23}$$

Clearly $\lambda_i^{-\frac{n+2}{2}} \lambda_j^{-\frac{n-2}{2}} = o(\varepsilon_{i,j})$ and in $x \simeq \exp_{g_{a_i}}(x)$ coordinates

$$u_{a_i}(x) = 1 + O(r^2) \text{ and } \gamma_n G_{a_i}^{\frac{2}{2-n}}(x) = r^2 + O(r^n), \tag{7.24}$$

whence using case (v)

$$\begin{aligned}\int \varphi_i^{\frac{n+2}{n-2}} \varphi_j = \int_{B_{c\lambda_i}(0)} & \frac{u_{a_j}(a_i)}{(1+r^2)^{\frac{n+2}{2}}} \\ & (\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}} \frac{x}{\lambda_i})})^{\frac{n-2}{2}} + o(\varepsilon_{i,j}).\end{aligned} \tag{7.25}$$

Due to $\frac{1}{\lambda_i} \leq \frac{1}{\lambda_j}$ we have

$$\varepsilon_{i,j}^{\frac{2}{2-n}} \sim \lambda_i \lambda_j \gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i, a_j) \text{ or } \varepsilon_{i,j}^{\frac{2}{2-n}} \sim \frac{\lambda_i}{\lambda_j} \tag{7.26}$$

and may expand on

$$\mathcal{A} = \left[ |\frac{x}{\lambda_i}| \leq \epsilon \sqrt{\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)} \right] \cup \left[ |\frac{x}{\lambda_i}| \leq \epsilon \frac{1}{\lambda_j} \right] \subset B_{c\lambda_i}(0) \tag{7.27}$$

for $\epsilon > 0$ sufficiently small

$$\begin{aligned}& (\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}} \frac{x}{\lambda_i}))^{\frac{2-n}{2}} \\ & = (\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{2-n}{2}} \\ & \quad + \frac{2-n}{2} \frac{\gamma_n \nabla G_{a_j}^{\frac{2}{2-n}}(a_i) \lambda_j x + O(\frac{\lambda_j}{\lambda_i}|x|^2)}{(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n}{2}}}.\end{aligned} \tag{7.28}$$

Thus by (7.25)

$$\int \varphi_i^{\frac{n+2}{n-2}} \varphi_j = \sum_{k=1}^{4} I_k + o(\varepsilon_{i,j}) \tag{7.29}$$

with

$$I_1 = \frac{u_{a_j}(a_i)}{(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n-2}{2}}} \int_{\mathcal{A}} \frac{1}{(1+r^2)^{\frac{n+2}{2}}} \tag{7.30}$$

and

$$I_2 = - \frac{\frac{n-2}{2} u_{a_j}(a_i)\gamma_n}{(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n}{2}}} \int_{\mathcal{A}} \frac{\nabla G_{a_j}^{\frac{2}{2-n}}(a_i)\lambda_j x}{(1+r^2)^{\frac{n+2}{2}}} \tag{7.31}$$

and

$$I_3 = \frac{u_{a_j}(a_i)}{(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n}{2}}} \int_{\mathcal{A}} \frac{O(\frac{\lambda_j}{\lambda_i}|x|^2)}{(1+r^2)^{\frac{n+2}{2}}} \tag{7.32}$$

and

$$I_4 = \int_{\mathcal{A}^c} \frac{u_{a_j}(a_i)}{(1+r^2)^{\frac{n+2}{2}}} \left(\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}} \frac{x}{\lambda_i})}\right)^{\frac{n-2}{2}}. \tag{7.33}$$

Note, that since $\lambda_i \geq \lambda_j$, $\mathcal{A}$ tends to cover $\mathbb{R}^n$ as $\varepsilon_{i,j} \longrightarrow 0$. Thus

$$I_1 = b_1 \frac{u_{a_j}(a_i)}{(\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n-2}{2}}} + o(\varepsilon_{i,j}), \tag{7.34}$$

whereas $I_2 = 0$ by radial symmetry and $I_3 = o(\varepsilon_{i,j})$. Moreover

$$I_4 = o(\varepsilon_{i,j}) \tag{7.35}$$

in case $\varepsilon_{i,j}^{\frac{2}{2-n}} \sim \frac{\lambda_i}{\lambda_j}$. Otherwise we decompose

$$\mathcal{A}^c \subseteq \mathcal{B}_1 \cup \mathcal{B}_2, \tag{7.36}$$

where for a sufficiently large constant $E > 0$

$$\mathcal{B}_1 = [\varepsilon \sqrt{\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)} \leq |\frac{x}{\lambda_i}| \leq E \sqrt{\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)}] \tag{7.37}$$

and

$$\mathcal{B}_2 = [E \sqrt{\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)} \leq |\frac{x}{\lambda_i}| \leq c]. \tag{7.38}$$

We then may estimate

$$
\begin{aligned}
I_4^1 =&\int_{\mathcal{B}_1} \frac{u_{a_j}(a_i)}{(1+r^2)^{\frac{n+2}{2}}}\Big(\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})}\Big)^{\frac{n-2}{2}} \\
\leq & \frac{C(\frac{\lambda_j}{\lambda_i})^{\frac{n+2}{2}}}{(1+\lambda_i^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n+2}{2}}} \\
& \int_{[|\frac{x}{\lambda_j}|\leq E\sqrt{\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)}]} \Big(\frac{1}{1+\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_j})}\Big)^{\frac{n-2}{2}}.
\end{aligned}
\tag{7.39}
$$

Changing coordinates via $d_{i,j}=\exp_{g_{a_i}}^{-1}\exp_{g_{a_j}}$ we get

$$
I_4^1 \leq \frac{C}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n+2}{2}}} \int_{[|\frac{x}{\lambda_j}|\leq \tilde{E}d(a_i,a_j)]} \Big(\frac{1}{1+r^2}\Big)^{\frac{n-2}{2}}
\tag{7.40}
$$

and thus $I_4^1=o(\varepsilon_{i,j})$, since we may assume $\lambda_j \ll \lambda_i$. Moreover

$$
\begin{aligned}
I_{4,2} =&\int_{\mathcal{B}_2} \frac{u_{a_j}(a_i)}{(1+r^2)^{\frac{n+2}{2}}}\Big(\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})}\Big)^{\frac{n-2}{2}} \\
\leq & \frac{C}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n-2}{2}}} \\
& \int_{[|x|\geq\sqrt{\lambda_i^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)}]} \frac{1}{(1+r^2)^{\frac{n+2}{2}}} \\
=& o(\varepsilon_{i,j}),
\end{aligned}
\tag{7.41}
$$

since $\lambda_i^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)\gg 1$ in this case. Therefore

$$
I_4 \leq I_4^1 + I_4^2 = o(\varepsilon_{i,j}).
\tag{7.42}
$$

Collecting terms we get

$$
\int \varphi_i^{\frac{n+2}{n-2}}\varphi_j = I_1 + o(\varepsilon_{i,j}).
\tag{7.43}
$$

Due to conformal invariance there holds

$$
G_{a_j}(a_j,a_i) = u_{a_j}^{-1}(a_i)u_{a_j}^{-1}(a_j)G_{g_0}(a_i,a_j)
\tag{7.44}
$$

and we conclude

$$
\int \varphi_i^{\frac{n+2}{n-2}}\varphi_j = \frac{b_1}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j))^{\frac{n-2}{2}}} + o(\varepsilon_{i,j}).
\tag{7.45}
$$

The claim follows.

$(\beta)$ Case $k=2$

First we deal with the case $\frac{1}{\lambda_i}\leq\frac{1}{\lambda_j}$. For $c>0$ small we get

$$
\begin{aligned}
-\lambda_j & \int \varphi_i^{\frac{n+2}{n-2}}\partial_{\lambda_j}\varphi_j \\
= & \frac{n-2}{2}\int_{B_c(a_i)} \Big(\frac{\lambda_i}{1+\lambda_i^2\gamma_n G_{a_i}^{\frac{2}{2-n}}}\Big)^{\frac{n+2}{2}}\frac{u_{a_j}}{u_{a_i}} \\
& \Big(\frac{\lambda_j}{1+\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}}\Big)^{\frac{n-2}{2}}\frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}-1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}+1}d\mu_{g_{a_i}} \\
& + O\Big(\frac{1}{\lambda_i^{\frac{n+2}{2}}}\frac{1}{\lambda_j^{\frac{n-2}{2}}}\Big).
\end{aligned}
\tag{7.46}
$$

Clearly $\lambda_i^{-\frac{n+2}{2}}\lambda_j^{-\frac{n-2}{2}}=o(\varepsilon_{i,j})$, whence as before

$$
\begin{aligned}
-\lambda_j & \int \varphi_i^{\frac{n+2}{n-2}}\partial_{\lambda_j}\varphi_j \\
= & \frac{n-2}{2}\int_{B_{c\lambda_i}(0)} \frac{u_{a_j}(a_i)}{(1+r^2)^{\frac{n+2}{2}}}\frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})-1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})+1} \\
& \Big(\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})}\Big)^{\frac{n-2}{2}} \\
& + o(\varepsilon_{i,j}).
\end{aligned}
\tag{7.47}
$$

Due to $\frac{1}{\lambda_i}\leq\frac{1}{\lambda_j}$ we have

$$\varepsilon_{i,j}^{\frac{2}{2-n}}\sim\lambda_i\lambda_j\gamma_n G^{\frac{2}{2-n}}(a_i,a_j)\ \text{ or }\ \varepsilon_{i,j}^{\frac{2}{2-n}}\sim\frac{\lambda_i}{\lambda_j} \tag{7.48}$$

and may expand on

$$\mathcal{A}=[|\frac{x}{\lambda_i}|\leq\epsilon\sqrt{\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)}]\cup[|\frac{x}{\lambda_i}|\leq\epsilon\frac{1}{\lambda_j}] \tag{7.49}$$

for $\epsilon>0$ sufficiently small

$$
\begin{aligned}
&\frac{1}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i}))^{\frac{n-2}{2}}} \\
&\qquad\qquad\qquad\qquad \frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})-1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}}\frac{x}{\lambda_i})+1} \\
&=(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{2-n}{2}}\frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)-1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)+1} \\
&\quad+\frac{2-n}{2}\frac{\gamma_n\nabla G_{a_j}^{\frac{2}{2-n}}(a_i)\lambda_j x}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n}{2}}}\frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)-1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)+1} \\
&\quad+\frac{2}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n}{2}}}\frac{\gamma_n\nabla G_{a_j}^{\frac{2}{2-n}}(a_i)\lambda_j x}{1+\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)} \\
&\quad+\frac{O(\frac{\lambda_j}{\lambda_i}|x|^2)}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n}{2}}}.
\end{aligned}
\tag{7.50}
$$

By radial symmetry we then get with $b_2=\frac{n-2}{2}\int\frac{1}{(1+r^2)^{\frac{n+2}{2}}}$

$$
\begin{aligned}
&-\lambda_j\int\varphi_i^{\frac{n+2}{n-2}}\varphi_j \\
&=\frac{b_2 u_{a_j}(a_i)}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n-2}{2}}}\frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)-1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)+1}+o(\varepsilon_{i,j})
\end{aligned}
\tag{7.51}
$$

and thus by conformal invariance

$$
\begin{aligned}
-\lambda_j\int\varphi_i^{\frac{n+2}{n-2}}\partial_{\lambda_j}\varphi_j=&\frac{b_2(\lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)-\frac{\lambda_i}{\lambda_j})}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j))^{\frac{n}{2}}} \\
&+o(\varepsilon_{i,j}).
\end{aligned}
\tag{7.52}
$$

We turn to the case $\frac{1}{\lambda_i}\geq\frac{1}{\lambda_j}$. By the same reasoning as for (7.23)

$$
-\lambda_j\int\varphi_i^{\frac{n+2}{n-2}}\partial_{\lambda_j}\varphi_j=-\lambda_j\int\varphi_i\partial_{\lambda_j}\varphi_j^{\frac{n+2}{n-2}}+o(\varepsilon_{i,j}).\tag{7.53}
$$

For $c>0$ small we get

$$\begin{aligned}
-\lambda_j \int \varphi_i \partial_{\lambda_j} \varphi_j^{\frac{n+2}{n-2}}
=& \frac{n+2}{2} \int\limits_{B_c(a_i)} (\frac{\lambda_i}{1+\lambda_i^2\gamma_n G_{a_i}^{\frac{2}{2-n}}})^{\frac{n-2}{2}} \\
& \frac{u_{a_i}}{u_{a_j}} (\frac{\lambda_j}{1+\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}})^{\frac{n+2}{2}} \frac{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}-1}{\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}+1} d\mu_{g_{a_j}} \\
& + O(\frac{1}{\lambda_j^{\frac{n+2}{2}}}\frac{1}{\lambda_i^{\frac{n-2}{2}}}),
\end{aligned} \tag{7.54}$$

whence

$$\begin{aligned}
-\lambda_j \int \varphi_i \partial_{\lambda_j} \varphi_j^{\frac{n+2}{n-2}}
=& \frac{n+2}{2} \int\limits_{B_{c\lambda_i}(0)} \frac{r^2-1}{r^2+1} (\frac{1}{1+r^2})^{\frac{n+2}{2}} \\
& \frac{u_{a_i}(a_j)}{(\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{a_i}^{\frac{2}{2-n}}(\exp_{g_{a_j}}\frac{x}{\lambda_j}))^{\frac{n+2}{2}}} + o(\varepsilon_{i,j}).
\end{aligned} \tag{7.55}$$

We may expand on

$$\mathcal{A} = [|\frac{x}{\lambda_j} \leq \varepsilon\sqrt{\gamma_n G_{a_i}^{\frac{2}{2-n}}(a_j)}] \cup [|\frac{x}{\lambda_j}| \leq \epsilon\frac{1}{\lambda_i}]$$

for $\epsilon>0$ sufficiently small

$$\begin{aligned}
& (\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{a_i}^{\frac{2}{2-n}}(\exp_{g_{a_j}}\frac{x}{\lambda_i}))^{\frac{2-n}{2}} \\
& = (\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{a_i}^{\frac{2}{2-n}}(a_j))^{\frac{2-n}{2}} \\
& \quad + \frac{2-n}{2}\frac{\gamma_n \nabla G_{a_i}^{\frac{2}{2-n}}(a_j)\lambda_i x + O(\frac{\lambda_i}{\lambda_j}|x|^2)}{(\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{a_i}^{\frac{2}{2-n}}(a_j))^{\frac{n}{2}}}.
\end{aligned} \tag{7.56}$$

This gives with indeed $b_2 = \frac{n+2}{2}\int \frac{r^2-1}{r^2+1}(\frac{1}{1+r^2})^{\frac{n+2}{2}} = \frac{n-2}{2}\int(\frac{1}{1+r^2})^{\frac{n+2}{2}}$

$$\begin{aligned}
-\lambda_j \int \varphi_i^{\frac{n+2}{n-2}} \partial_{\lambda_j}\varphi_j = & b_2 \frac{u_{a_i}(a_j)}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_i}^{\frac{2}{2-n}}(a_j))^{\frac{n-2}{2}}} \\
& + o(\varepsilon_{i,j})
\end{aligned} \tag{7.57}$$

and we conclude by conformal invariance

$$-\lambda_j \int \varphi_i^{\frac{n+2}{n-2}} \partial_{\lambda_j} \varphi_j = \frac{b_2}{(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i, a_j))^{\frac{n-2}{2}}} + o(\varepsilon_{i,j}). \tag{7.58}$$

(7.52) and (7.58) then prove the claim.

($\gamma$) Case $k = 3$

First we consider the case $\frac{1}{\lambda_i} \leq \frac{1}{\lambda_j}$. For $c > 0$ small we get

$$\begin{aligned} \frac{1}{\lambda_j} \int \varphi_i^{\frac{n+2}{n-2}} \nabla_{a_j} \varphi_j =& \frac{2-n}{2} \int_{B_c(a_i)} (\frac{\lambda_i}{1 + \lambda_i^2 \gamma_n G_{a_i}^{\frac{2}{2-n}}})^{\frac{n+2}{2}} \frac{u_{a_j}}{u_{a_i}} \\ & (\frac{\lambda_j}{1 + \lambda_j^2 \gamma_n G_{a_j}^{\frac{2}{2-n}}})^{\frac{n-2}{2}} \frac{\lambda_j \gamma_n \nabla_{a_j} G_{a_j}^{\frac{2}{2-n}}}{1 + \lambda_j^2 \gamma_n G_{a_j}^{\frac{2}{2-n}}} d\mu_{g_{a_i}} \\ & + O(\int_{B_c(a_i)} \varphi_i^{\frac{n+2}{n-2}} |\frac{\nabla_{a_j} u_{a_j}}{u_{a_i} \lambda_j}| \varphi_j + \frac{1}{\lambda_i^{\frac{n+2}{2}}} \frac{1}{\lambda_j^{\frac{n-2}{2}}}). \end{aligned} \tag{7.59}$$

Due to case ($v$) we obtain passing to $x \simeq \exp_{g_{a_i}}(x)$ coordinates

$$\begin{aligned} \frac{1}{\lambda_j} \int \varphi_i^{\frac{n+2}{n-2}} \nabla_{a_j} \varphi_j =& \frac{2-n}{2} \int_{B_{c\lambda_i}(0)} \frac{u_{a_j}(a_i)}{(1+r^2)^{\frac{n+2}{2}}} \frac{\lambda_j \gamma_n \nabla_{a_j} G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}} \frac{x}{\lambda_i})}{1 + \lambda_j^2 \gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}} \frac{x}{\lambda_i})} \\ & (\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j \gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_i}} \frac{x}{\lambda_i})})^{\frac{n-2}{2}} + o(\varepsilon_{i,j}). \end{aligned} \tag{7.60}$$

Since

$$\varepsilon_{i,j}^{\frac{2}{2-n}} \sim \lambda_i \lambda_j \gamma_n G^{\frac{2}{2-n}}(a_i, a_j) \text{ or } \varepsilon_{i,j}^{\frac{2}{2-n}} \sim \frac{\lambda_i}{\lambda_j} \tag{7.61}$$

we may expand on

$$\mathcal{A} = [|\frac{x}{\lambda_i}| \leq \epsilon \sqrt{\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)}] \cup [|\frac{x}{\lambda_i}| \leq \epsilon \frac{1}{\lambda_j}] \tag{7.62}$$

for $\epsilon > 0$ sufficiently small as before to obtain

$$\begin{aligned}
&\frac{1}{\lambda_j}\int \varphi_i^{\frac{n+2}{n-2}}\nabla_{a_j}\varphi_j \\
&\quad = - b_3 \frac{u_{a_j}(a_i)}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n-2}{2}}}\frac{\lambda_j\gamma_n\nabla_{a_j}G_{a_j}^{\frac{2}{2-n}}(a_i)}{1+\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i)} \\
&\quad\quad + o(\varepsilon_{i,j})
\end{aligned} \tag{7.63}$$

with $b_3 = \frac{n-2}{2}\int\frac{1}{(1+r^2)^{\frac{n+2}{2}}}$. This gives

$$\begin{aligned}
\frac{1}{\lambda_j}\int \varphi_i^{\frac{n+2}{n-2}}\nabla_{a_j}\varphi_j = & -b_3\frac{u_{a_j}(a_i)\lambda_i\gamma_n\nabla_{a_j}G_{a_j}^{\frac{2}{2-n}}(a_i)}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n}{2}}} \\
& + o(\varepsilon_{i,j}),
\end{aligned} \tag{7.64}$$

whence by conformal invariance

$$\frac{1}{\lambda_j}\int \varphi_i^{\frac{n+2}{n-2}}\nabla_{a_j}\varphi_j = \frac{-b_3\lambda_i\gamma_n\nabla_{a_j}G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)}{(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j))^{\frac{n}{2}}} + o(\varepsilon_{i,j}). \tag{7.65}$$

We turn to the case $\frac{1}{\lambda_i}\geq\frac{1}{\lambda_j}$. As before

$$\frac{1}{\lambda_j}\int \varphi_i^{\frac{n+2}{n-2}}\nabla_{a_j}\varphi_j = \frac{1}{\lambda_j}\int \varphi_i\nabla_{a_j}\varphi_j^{\frac{n+2}{n-2}} + o(\varepsilon_{i,j}) \tag{7.66}$$

and for $c > 0$ small we obtain by arguments familiar by now

$$\begin{aligned}
&\frac{1}{\lambda_j}\int \varphi_i\nabla_{a_j}\varphi_j^{\frac{n+2}{n-2}} \\
&\quad = -\frac{n+2}{2}\int_{B_{c\lambda_j}(a_j)}\Big(\frac{1}{\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{a_i}^{\frac{2}{2-n}}(\exp_{g_{a_j}}\frac{x}{\lambda_j})}\Big)^{\frac{n-2}{2}} \\
&\quad\quad\quad \frac{u_{a_i}(a_j)\lambda_j\gamma_n\nabla_{a_j}G_{a_j}^{\frac{2}{2-n}}(\exp_{g_{a_j}}\frac{x}{\lambda_j})}{(1+r^2)^{\frac{n+4}{2}}} \\
&\quad\quad + o(\varepsilon_{i,j}),
\end{aligned} \tag{7.67}$$

whence

$$\begin{aligned}
&\frac{1}{\lambda_j}\int \varphi_i\nabla_{a_j}\varphi_j^{\frac{n+2}{n-2}} \\
&\quad =(n+2)\int_{B_{c\lambda_j}(a_j)}\frac{u_{a_i}(a_j)x}{(1+r^2)^{\frac{n+4}{2}}} \\
&\quad\quad\quad \Big(\frac{1}{\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{a_i}^{\frac{2}{2-n}}(\exp_{g_{a_j}}\frac{x}{\lambda_j})}\Big)^{\frac{n-2}{2}} \\
&\quad\quad + o(\varepsilon_{i,j}).
\end{aligned} \tag{7.68}$$

Expanding on

$$\mathcal{A} =[|\frac{x}{\lambda_j}| \leq \epsilon\sqrt{\gamma_n G_{a_i}^{\frac{2}{2-n}}(a_j)}] \cup [|\frac{x}{\lambda_j}| \leq \epsilon\frac{1}{\lambda_i}] \tag{7.69}$$

for $\epsilon > 0$ sufficiently small we derive

$$\frac{1}{\lambda_j}\int \varphi_i \nabla_{a_j}\varphi_j^{\frac{n+2}{n-2}} =b_3\frac{u_{a_i}(a_j)\lambda_i\gamma_n\nabla_{a_j}G_{a_i}^{\frac{2}{2-n}}(a_j)}{(\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(a_i))^{\frac{n}{2}}} + o(\varepsilon_{i,j}) \tag{7.70}$$

with indeed $b_3 = \frac{(n+2)(n-2)}{2n}\int \frac{r^2}{(1+r^2)^{\frac{n+4}{2}}} = \frac{n-2}{2}\int (\frac{1}{1+r^2})^{\frac{n+2}{2}}$. Thus

$$\begin{aligned}\frac{1}{\lambda_j}\int \varphi_i \nabla_{a_j}\varphi_j^{\frac{n+2}{n-2}} =&b_3\frac{\lambda_i\gamma_n\nabla_{a_j}G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)}{(\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j))^{\frac{n}{2}}} \\ &+ o(\varepsilon_{i,j})\end{aligned} \tag{7.71}$$

by conformal invariance. From (7.65), (7.71) the claim follows.

(iv) Due to (7.11), (7.12), (7.13) and

$$\gamma_n G_{a_i}^{\frac{2}{2-n}} = r^2 + O(r^n),\ \gamma_n \nabla_{a_i} G_{a_i}^{\frac{2}{2-n}} = -2x + O(r^{n-1}), \tag{7.72}$$

cf. definition 3.2 we have on $B_c(a_i)$ for $c > 0$ small

($\alpha$)

$$\phi_{1,i} =u_{a_i}(\frac{\lambda_i}{1+\lambda_i^2r^2})^{\frac{n-2}{2}} + O(r^{n-2}(\frac{\lambda_i}{1+\lambda_i^2r^2})^{\frac{n-2}{2}}) \tag{7.73}$$

($\beta$)

$$\begin{aligned}\phi_{2,i} =&\frac{n-2}{2}u_{a_i}\frac{\lambda_i^2r^2-1}{\lambda_i^2r^2+1}(\frac{\lambda_i}{1+\lambda_i^2r^2})^{\frac{n-2}{2}} \\ &+ O(r^{n-2}(\frac{\lambda_i}{1+\lambda_i^2r^2})^{\frac{n-2}{2}})\end{aligned} \tag{7.74}$$

($\gamma$)

$$\begin{aligned}\phi_{3,i} =&-\frac{n-2}{2}u_{a_i}\frac{\lambda_i x}{1+\lambda_i^2r^2}(\frac{\lambda_i}{1+\lambda_i^2r^2})^{\frac{n-2}{2}} \\ &+ O(r^{n-2}(\frac{\lambda_i}{1-\lambda_i^2r^2})^{\frac{n-2}{2}}) + O(\frac{r}{\lambda_i}(\frac{\lambda_i}{1+\lambda_i^2r^2})^{\frac{n-2}{2}})\end{aligned} \tag{7.75}$$

($\varphi$)

$$\varphi_i^{\frac{4}{n-2}} =u_{a_i}^{\frac{4}{n-2}}(\frac{\lambda_i}{1+\lambda_i^2r^2})^2 + O(r^{n-2}(\frac{\lambda_i}{1+\lambda_i^2r^2})^2). \tag{7.76}$$

Consequently

$$\begin{aligned}\int \phi_{1,i}\varphi_i^{\frac{4}{n-2}}\phi_{2,i} =& \int_{B_c(a_i)} \phi_{1,i}\varphi_i^{\frac{4}{n-2}}\phi_{2,i} + O(\frac{1}{\lambda_i^n})\\ =& \int_{B_{\frac{c}{\lambda_i}}(0)} \frac{r^2-1}{r^2+1}(\frac{1}{1+r^2})^n + O(\frac{1}{\lambda_i^{n-2}})\\ =& \int \frac{r^2-1}{r^2+1}(\frac{1}{1+r^2})^n + O(\frac{1}{\lambda_i^{n-2}}) = O(\frac{1}{\lambda_i^{n-2}}),\end{aligned} \tag{7.77}$$

since the integral above vanishes. Alike using radial symmetry

$$\int \phi_{1,i}\varphi_i^{\frac{4}{n-2}}\phi_{3,i}, \int \phi_{2,i}\varphi_i^{\frac{4}{n-2}}\phi_{3,i} = O(\frac{1}{\lambda_i^{n-2}} + \frac{1}{\lambda_i^2}). \tag{7.78}$$

Moreover we have readily have

$$\int \varphi_i^{\frac{n+2}{n-2}}\phi_{k,i} = (1, -\lambda_i\partial_i, \frac{1}{\lambda_i}\nabla_{a_i})\int \varphi_i^{\frac{2n}{n-2}} = \delta_{1k} + O(\frac{1}{\lambda_i}^{n-2}). \tag{7.79}$$

(v) Let $\alpha' = \frac{n-2}{2}\alpha$, $\beta' = \frac{n-2}{2}\beta$, so $\alpha' + \beta' = n$. We distinguish

($\alpha$) $\varepsilon_{i,j}^{\frac{2}{2-n}} \sim \frac{\lambda_i}{\lambda_j} \vee \varepsilon_{i,j}^{\frac{2}{2-n}} \sim \lambda_i\lambda_j\gamma_n G^{\frac{2}{2-n}}(a_i, a_j)$

We estimate for $c > 0$ small

$$\begin{aligned}&\int \varphi_i^\alpha\varphi_j^\beta\\ \leq& C\int_{B_c(0)} (\frac{\lambda_i}{1+\lambda_i^2r^2})^{\alpha'}(\frac{\lambda_j}{1+\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{a_i} x)})^{\beta'}\\ &+ C\frac{1}{\lambda_i^{\alpha'}}\int_{B_c(0)}(\frac{\lambda_j}{1+\lambda_j^2r^2})^{\beta'} + O(\frac{1}{\lambda_i^{\alpha'}}\frac{1}{\lambda_j^{\beta'}})\\ =& C\int_{B_{c\lambda_i}(0)}(\frac{1}{1+r^2})^{\alpha'}(\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{a_i}\frac{x}{\lambda_i})})^{\beta'}\\ &+ C\frac{1}{\lambda_i^{\alpha'}}\frac{1}{\lambda_j^{n-\beta'}}\int_{B_{c\lambda_j}(0)}(\frac{1}{1+r^2})^{\beta'} + o(\varepsilon_{i,j}^\beta).\end{aligned} \tag{7.80}$$

Thus by $\int_{B_{c\lambda_j}(0)}(\frac{1}{1+r^2})^{\beta'} \leq C\frac{1}{\lambda_j^{2\beta'-n}}$ we get

$$\begin{aligned}&\int \varphi_i^\alpha\varphi_j^\beta\\ \leq& C\int_{B_{c\lambda_i}(0)}(\frac{1}{1+r^2})^{\alpha'}(\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{a_i}\frac{x}{\lambda_i})})^{\beta'}\\ &+ o(\varepsilon_{i,j}^\beta).\end{aligned} \tag{7.81}$$

This shows the claim in cases

$$\frac{\lambda_j}{\lambda_i}+\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)\sim\frac{\lambda_i}{\lambda_j}\quad\text{or}\quad d(a_i,a_j)>3c. \tag{7.82}$$

Else we may assume $d(a_i,a_j)<3c$ and

$$\frac{\lambda_j}{\lambda_i}+\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)\sim\lambda_i\lambda_j d^2(a_i,a_j). \tag{7.83}$$

We then get with $\mathcal{B}=[\frac{1}{2}d(a_i,a_j)\leq|\frac{x}{\lambda_i}|\leq 2d(a_i,a_j)]$

$$\begin{aligned}
&\int\varphi_i^\alpha\varphi_j^\beta\\
\leq&C\int_{\mathcal{B}}(\frac{1}{1+r^2})^{\alpha'}(\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j d^2(a_j,\exp_{a_i}(\frac{x}{\lambda_i}))})^{\beta'}+O(\varepsilon_{i,j}^\beta)\\
\leq&C(\frac{1}{1+|\lambda_i d(a_i,a_j)|^2})^{\alpha'}\int_{[|\frac{x}{\lambda_i}|\leq 4d(a_i,a_j)]}(\frac{1}{\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}r^2})^{\beta'}\\
&+O(\varepsilon_{i,j}^\beta)\\
\leq&C\frac{(\frac{\lambda_j}{\lambda_i})^{\beta'-n}}{(1+|\lambda_i d(a_i,a_j)|^2)^{\alpha'}}\int_{[r\leq 4\lambda_j d(a_i,a_j)]}(\frac{1}{1+r^2})^{\beta'}+O(\varepsilon_{i,j}^\beta).
\end{aligned} \tag{7.84}$$

Note, that in case $\lambda_j d(a_i,a_j)$ remains bounded, we are done. Else

$$\begin{aligned}
\int\varphi_i^\alpha\varphi_j^\beta\leq&C\frac{(\frac{\lambda_j}{\lambda_i})^{\beta'-n}(\lambda_j d(a_i,a_j))^{n-2\beta'}}{(1+|\lambda_i d(a_i,a_j)|^2)^{\alpha'}}+O(\varepsilon_{i,j}^\beta)\\
\leq&C(\frac{1}{1+|\lambda_i d(a_i,a_j)|^2})^{\alpha'-\frac{n}{2}+\beta'}(\frac{\lambda_i}{\lambda_j})^{\beta'}+O(\varepsilon_{i,j}^\beta),
\end{aligned} \tag{7.85}$$

whence due to $\alpha'>\frac{n}{2}$ the claim follows.

($\beta$) $\varepsilon_{i,j}^{\frac{2}{2-n}}\sim\frac{\lambda_j}{\lambda_i}$.

We estimate for $c>0$ small

$$\begin{aligned}
&\int\varphi_i^\alpha\varphi_j^\beta\\
\leq&C\int_{B_{c\lambda_j}(0)}(\frac{1}{\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{a_i}^{\frac{2}{2-n}}(\exp_{a_j}(\frac{x}{\lambda_j}))})^{\alpha'}(\frac{1}{1+r^2})^{\beta'}\\
&+C\frac{1}{\lambda_j^{\beta'}}\int_{B_c(0)}(\frac{\lambda_i}{1+\lambda_i^2r^2})^{\alpha'}+O(\frac{1}{\lambda_i^{\alpha'}}\frac{1}{\lambda_j^{\beta'}}),
\end{aligned} \tag{7.86}$$

which by $\int_{B_c(0)}(\frac{\lambda_i}{1+\lambda_i^2r^2})^{\alpha'}\leq C\frac{1}{\lambda_i^{n-\alpha'}}=C\frac{1}{\lambda_i^{\beta'}}$ gives

$$
\begin{aligned}
&\int \varphi_i^\alpha \varphi_j^\beta \\
\le & C\int_{B_{\lambda_j c}(0)} \Big(\frac{1}{\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j\gamma_n G_{a_i}^{\frac{2}{2-n}}(\exp_{a_j}(\frac{x}{\lambda_j}))}\Big)^{\alpha'}\Big(\frac{1}{1+r^2}\Big)^{\beta'} \qquad (7.87)\\
& + o(\varepsilon_{i,j}^\beta).
\end{aligned}
$$

By assumption $d(a_i,a_j)\le \frac{2}{\lambda_i}$, whence we may replace as before

$$
\gamma_n G_{a_i}^{\frac{2}{2-n}}(\exp_{a_j}(\frac{x}{\lambda_j})) \sim d^2(a_i,\exp_{a_j}(\frac{x}{\lambda_j})) \text{ on } B_{\lambda_j c}(0). \qquad (7.88)
$$

Thus for $\gamma > 3$

$$
\begin{aligned}
&\int \varphi_i^\alpha \varphi_j^\beta \\
\le & C\int_{[\gamma\frac{\lambda_j}{\lambda_i}\le |x|\le \lambda_j c]} \Big(\frac{1}{\frac{\lambda_j}{\lambda_i}+\lambda_i\lambda_j d^2(a_i,\exp_{a_j}(\frac{x}{\lambda_j}))}\Big)^{\alpha'}\Big(\frac{1}{1+r^2}\Big)^{\beta'} \\
& + C\Big(\frac{\lambda_i}{\lambda_j}\Big)^{\alpha'}\int_{[|x|<\gamma\frac{\lambda_j}{\lambda_i}]}\Big(\frac{1}{1+r^2}\Big)^{\beta'} + o(\varepsilon_{i,j}^\beta) \qquad (7.89)\\
\le & C\int_{[\gamma\frac{\lambda_j}{\lambda_i}\le |x|\le \lambda_j c]} \Big(\frac{1}{\frac{\lambda_j}{\lambda_i}+\frac{\lambda_i}{\lambda_j}r^2}\Big)^{\alpha'}\Big(\frac{1}{1+r^2}\Big)^{\beta'} \\
& + C\Big(\frac{\lambda_i}{\lambda_j}\Big)^{\alpha'}\Big(\frac{\lambda_j}{\lambda_i}\Big)^{n-2\beta} + o(\varepsilon_{i,j}^\beta),
\end{aligned}
$$

since for $|x|\ge \gamma\frac{\lambda_j}{\lambda_i}$ we may assume using $d(a_i,a_j)\le \frac{1}{\lambda_i}$

$$
d(a_i,\exp_{a_j}(\frac{x}{\lambda_j})) \ge \frac{r}{\lambda_j}. \qquad (7.90)
$$

Therefore

$$
\int \varphi_i^\alpha \varphi_j^\beta \le C\Big(\frac{\lambda_j}{\lambda_i}\Big)^{\alpha'}\int_{[|x|\ge\gamma\frac{\lambda_j}{\lambda_i}]} r^{-2n} + O(\varepsilon_{i,j}^\beta) = O(\varepsilon_{i,j}^\beta). \qquad (7.91)
$$

(vi) By symmetry we may assume $\frac{1}{\lambda_i}\le\frac{1}{\lambda_j}$ and thus

$$
\varepsilon_{i,j}^{\frac{2}{2-n}} \sim \frac{\lambda_i}{\lambda_j} \;\vee\; \varepsilon_{i,j}^{\frac{2}{2-n}} \sim \lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j) \qquad (7.92)
$$

We estimate for $c>0$ small

$$\begin{aligned}\int \varphi_i^{\frac{n}{n-2}}\varphi_j^{\frac{n}{n-2}} &\\ \leq & C\int_{B_c(0)}(\frac{\lambda_i}{1+\lambda_i^2r^2})^{\frac{n}{2}}(\frac{\lambda_j}{1+\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{a_i}x)})^{\frac{n}{2}} \\ & + C\frac{1}{\lambda_i^{\frac{n}{2}}}\int_{B_c(0)}(\frac{\lambda_j}{1+\lambda_j^2r^2})^{\frac{n}{2}}+O(\frac{1}{\lambda_i^{\frac{n}{2}}}\frac{1}{\lambda_j^{\frac{n}{2}}}) \\ = & C\int_{B_{c\lambda_i}(0)}(\frac{1}{1+r^2})^{\frac{n}{2}}(\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{a_j}^{\frac{2}{2-n}}(\exp_{a_i}(\frac{x}{\lambda_i}))})^{\frac{n}{2}} \\ & + O(\ln \lambda_j \varepsilon_{i,j}^{\frac{n}{n-2}}).\end{aligned} \tag{7.93}$$

Thus in cases

$$\frac{\lambda_j}{\lambda_i}+\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)\sim\frac{\lambda_i}{\lambda_j} \text{ or } d(a_i,a_j)>3c \tag{7.94}$$

we obtain

$$\int \varphi_i^{\frac{n}{n-2}}\varphi_j^{\frac{n}{n-2}} \leq C\ln\lambda_i\varepsilon_{i,j}^{\frac{n}{n-2}}+C\ln\lambda_j\varepsilon_{i,j}^{\frac{n}{n-2}}\leq C\ln(\lambda_i\lambda_j)\varepsilon_{i,j}^{\frac{n}{n-2}}, \tag{7.95}$$

thus $\int\varphi_i\varphi_j=O(\varepsilon_{i,j}^{\frac{n}{n-2}}\ln\varepsilon_{i,j})$. Else we may assume $d(a_i,a_j)<3c$ and

$$\frac{\lambda_j}{\lambda_i}+\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j\gamma_n G^{\frac{2}{2-n}}(a_i,a_j)\sim\lambda_i\lambda_j d^2(a_i,a_j). \tag{7.96}$$

We then get with $\mathcal{B}=[\frac{1}{2}d(a_i,a_j)\leq|\frac{x}{\lambda_i}|\leq 2d(a_i,a_j)]$

$$\begin{aligned}\int \varphi_i^{\frac{n}{n-2}}\varphi_j^{\frac{n}{n-2}} &\\ \leq & C\int_{\mathcal{B}}(\frac{1}{1+r^2})^{\frac{n}{2}}(\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j d^2(a_j,\exp_{a_i}(\frac{x}{\lambda_i}))})^{\frac{n}{2}} \\ & + O(\varepsilon_{i,j}^{\frac{n}{n-2}}\ln\varepsilon_{i,j}) \\ \leq & C(\frac{1}{1+|\lambda_i d(a_i,a_j)|^2})^{\frac{n}{2}}\int_{[|\frac{x}{\lambda_i}|\leq 4d(a_i,a_j)]}(\frac{1}{\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}r^2})^{\frac{n}{2}} \\ & + O(\varepsilon_{i,j}^{\frac{n}{n-2}}\ln\varepsilon_{i,j}) \\ \leq & C(\frac{1}{1+|\lambda_i d(a_i,a_j)|^2})^{\frac{n}{2}}(\frac{\lambda_i}{\lambda_j})^{\frac{n}{2}}\int_{[r\leq 4\lambda_j d(a_i,a_j)]}(\frac{1}{1+r^2})^{\frac{n}{2}} \\ & + O(\varepsilon_{i,j}^{\frac{n}{n-2}}\ln\varepsilon_{i,j}) \\ \leq & C\varepsilon_{i,j}^{\frac{n}{n-2}}\ln(\lambda_j d(a_i,a_j))+O(\varepsilon_{i,j}^{\frac{n}{n-2}}\ln\varepsilon_{i,j}).\end{aligned} \tag{7.97}$$

The claim follows, as $\lambda_j\leq\lambda_i$ by assumption.

(vii) $\varepsilon_{i,j} = O(\varepsilon_{i,j})$ is trivial and $\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} = O(\varepsilon_{i,j})$ follows readily due to

$$\lambda_i\partial_{\lambda_i}\varepsilon_{i,j} = \frac{2-n}{2}\varepsilon_{i,j}\frac{\frac{\lambda_i}{\lambda_j} - \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)}. \tag{7.98}$$

Last $\frac{1}{\lambda_i}\nabla_{a_i}\varepsilon_{i,j} = O(\varepsilon_{i,j})$ follows from

$$\frac{1}{\lambda_i}\nabla_{a_i}\varepsilon_{i,j} = \frac{2-n}{2}\varepsilon_{i,j}\frac{\lambda_j\gamma_n\nabla_{a_i}G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)}. \tag{7.99}$$

immediately in case $d(a_i,a_j) > c > 0$. In the contrary case we estimate

$$\frac{\lambda_j\gamma_n|\nabla_{a_i}G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)|}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)} \leq \frac{\frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j\gamma_n|\nabla_{a_i}G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)|^2}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j)} \tag{7.100}$$

with the right hand side being bounded for $d(a_i,a_j)$ small.

□

***Proof of proposition 3.10****(Cf. [9], Appendix A).*
Let us denote by $w(\varepsilon)$ any quantity, for which $|w(\varepsilon)| \overset{\varepsilon\to 0}{\longrightarrow} 0$ and consider for

$$u \in V(\omega,p,\varepsilon) \text{ with } \varepsilon \longrightarrow 0 \tag{7.101}$$

a representation

$$u = u_{\hat{\alpha},\hat{\beta}} + \hat{\alpha}^i\varphi_{\hat{a}_i,\hat{\lambda}_i} + \hat{v},\ (\hat{\alpha},\hat{\beta}_k,\hat{\alpha}_i,\hat{a}_i,\hat{\lambda}_i) \in A_u(\omega,p,\varepsilon), \|\hat{v}\| \leq \varepsilon. \tag{7.102}$$

Since $A_u(\omega,p,\varepsilon) \subset A_u(\omega,p,2\varepsilon_0)$ we have

$$\inf_{(\tilde{\alpha},\tilde{\beta}_k,\tilde{\alpha}_i,\tilde{a}_i,\tilde{\lambda}_i)\in A_u(\omega,p,2\varepsilon_0)} \int Ku^{\frac{4}{n-2}}|u - u_{\tilde{\alpha},\tilde{\beta}} - \tilde{\alpha}^i\varphi_{\tilde{a}_i,\tilde{\lambda}_i}|^2 = w(\varepsilon), \tag{7.103}$$

whence we may consider $(\tilde{\alpha},\tilde{\beta}_k,\tilde{\alpha}_i,\tilde{a}_i,\tilde{\lambda}_i) \in A_u(\omega,p,2\varepsilon_0)$ such, that

$$\int Ku^{\frac{4}{n-2}}|u - u_{\tilde{\alpha},\tilde{\beta}} - \tilde{\alpha}^i\varphi_{\tilde{a}_i,\tilde{\lambda}_i}|^2 = w(\varepsilon). \tag{7.104}$$

Expanding this gives in a first step

$$\int K(\hat{\alpha}\omega + \hat{\alpha}^i\varphi_{\hat{a}_i,\hat{\lambda}_i})^{\frac{4}{n-2}}|u_{\hat{\alpha},\hat{\beta}} - u_{\tilde{\alpha},\tilde{\beta}} + \hat{\alpha}^i\varphi_{\hat{a}_i,\hat{\lambda}_i} - \tilde{\alpha}^i\varphi_{\tilde{a}_i,\lambda_i}|^2 = w(\varepsilon) \tag{7.105}$$

and using lemma 3.5 and proposition 3.8 we derive

$$\begin{aligned} w(\varepsilon) = & \int K\omega^{\frac{4}{n-2}}|u_{\hat{\alpha},\hat{\beta}} - u_{\tilde{\alpha},\tilde{\beta}} - \tilde{\alpha}^i\varphi_{\tilde{a}_i,\tilde{\lambda}_i}|^2 \\ & + \sum_i \int K\varphi_{\hat{a}_i,\hat{\lambda}_i}^{\frac{4}{n-2}}|\hat{\alpha}_i\varphi_{\hat{a}_i,\hat{\lambda}_i} - \tilde{\alpha}^j\varphi_{\tilde{a}_j,\tilde{\lambda}_j}|^2 \end{aligned} \tag{7.106}$$

Consequently for at least one $j = j_i$ the quantity

$$\frac{\tilde{\lambda}_{j_i}}{\hat{\lambda}_i} + \frac{\hat{\lambda}_i}{\tilde{\lambda}_{j_i}} + \tilde{\lambda}_{j_i}\hat{\lambda}_i\gamma_n G_{g_0}^{\frac{2}{2-n}}(\tilde{a}_{j_i}, \hat{a}_i) \tag{7.107}$$

has to stay bounded, whereas on the other hand

$$\hat{\varepsilon}_{i,j} = (\frac{\hat{\lambda}_i}{\hat{\lambda}_j} + \frac{\hat{\lambda}_j}{\hat{\lambda}_i} + \hat{\lambda}_i\hat{\lambda}_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(\hat{a}_i, \hat{a}_j))^{\frac{2-n}{2}} < \varepsilon. \tag{7.108}$$

Thus for any $j = i, \ldots, p$ there exists exactly one $j_i \in \{1, \ldots, p\}$ such, that

$$\frac{\tilde{\lambda}_{j_i}}{\hat{\lambda}_i} + \frac{\hat{\lambda}_i}{\tilde{\lambda}_{j_i}} + \tilde{\lambda}_{j_i}\hat{\lambda}_i\gamma_n G_{g_0}^{\frac{2}{2-n}}(\tilde{a}_{j_i}, \hat{a}_i) \tag{7.109}$$

remains bounded and we may assume $j_i = i$. From this we deduce

$$w(\varepsilon) = \int K\omega^{\frac{4}{n-2}}|u_{\hat{\alpha},\hat{\beta}} - u_{\tilde{\alpha},\tilde{\beta}}|^2 + \sum_i \int K\varphi_{\hat{a}_i,\hat{\lambda}_i}^{\frac{4}{n-2}}|\hat{\alpha}_i\varphi_{\hat{a}_i,\hat{\lambda}_i} - \tilde{\alpha}_i\varphi_{\tilde{a}_i,\tilde{\lambda}_i}|^2. \tag{7.110}$$

Note, that

$$\begin{aligned}\int K\omega^{\frac{4}{n-2}}|u_{\hat{\alpha},\hat{\beta}} - u_{\tilde{\alpha},\tilde{\beta}}|^2 =& |\hat{\alpha} - \tilde{\alpha}|^2 \int K\omega^{\frac{2n}{n-2}} + \sum_i |\hat{\alpha}\hat{\beta}i - \tilde{\alpha}\tilde{\beta}_i|^2 \int K\omega^{\frac{4}{n-2}}\mathrm{e}_i^2 \\ &+ \int K\omega^{\frac{4}{n-2}}|\hat{\alpha}h(\hat{\beta}) - \tilde{\alpha}h(\tilde{\beta})|^2,\end{aligned} \tag{7.111}$$

whence due to $\|h\hat{\beta}\| = O(\|\hat{\beta}\|^2)$ we obtain

$$\int K\omega^{\frac{4}{n-2}}|u_{\hat{\alpha},\hat{\beta}} - u_{\tilde{\alpha},\tilde{\beta}}|^2 \geq C(|\hat{\alpha} - \tilde{\alpha}|^2 + \|\hat{\beta} - \tilde{\beta}\|^2). \tag{7.112}$$

Moreover in $g_{\hat{a}_i}$ normal coordinates with

$$\gamma(\tau) = \tau(\tilde{\alpha}_i, \tilde{a}_i, \tilde{\lambda}_i) + (1-\tau)(\hat{\alpha}_i, \hat{a}_i, \hat{\lambda}_i) \tag{7.113}$$

we have for some $\tau \in (0,1)$

$$\begin{aligned}&\tilde{\alpha}_i\varphi_{\tilde{a}_i,\tilde{\lambda}_i} - \hat{\alpha}_i\varphi_{\hat{a}_i,\hat{\lambda}_i} \\ &\quad= \begin{pmatrix}\partial_\alpha \\ \nabla_a \\ \partial_\lambda\end{pmatrix}(\alpha\varphi_{a,\lambda})\lfloor_{(\alpha,a,\lambda=\gamma(0))}\begin{pmatrix}\tilde{\alpha}-\hat{\alpha} \\ \tilde{a}-\hat{a} \\ \tilde{\lambda}-\hat{\lambda}\end{pmatrix} \\ &\qquad+ \begin{pmatrix}\partial_\alpha^2 & \partial_\alpha\nabla_a & \partial_\alpha\partial_\lambda \\ \nabla_a\partial_\lambda & \nabla_a^2 & \nabla_a\partial_\lambda \\ \partial_\lambda & \partial_\lambda\nabla_a & \partial_\lambda^2\end{pmatrix}(\alpha\varphi_{a,\lambda})\lfloor_{(\alpha,a,\lambda)=\gamma(\tau)}\begin{pmatrix}\tilde{\alpha}-\hat{\alpha} \\ \tilde{a}-\hat{a} \\ \tilde{\lambda}-\hat{\lambda}\end{pmatrix}^2,\end{aligned} \tag{7.114}$$

whence due to lemma 3.5 (i) and $c < \frac{\tilde{\lambda}_i}{\lambda_i} < C$ we obtain

$$\begin{aligned}\|\tilde{\alpha}_i\varphi_{\tilde{a}_i,\tilde{\lambda}_i} - \hat{\alpha}_i\varphi_{\hat{a}_i,\hat{\lambda}_i} - \begin{pmatrix}\varphi_{\hat{a}_i,\hat{\lambda}_i}\\ \frac{1}{\hat{\lambda}_i}\nabla_{\hat{a}_i}\varphi_{\hat{a}_i,\hat{\lambda}_i}\\ \hat{\lambda}_i\partial_{\hat{\lambda}_i}\varphi_{\hat{a}_i,\hat{\lambda}_i}\end{pmatrix}\begin{pmatrix}\tilde{\alpha}_i-\hat{\alpha}_i\\ \hat{\lambda}_i(\tilde{a}_i-\hat{a}_i)\\ \frac{\tilde{\lambda}_i-\hat{\lambda}_i}{\hat{\lambda}_i}\end{pmatrix}\| \\ =O(|\tilde{\alpha}_i-\hat{\alpha}_i|^2+\hat{\lambda}_i^2|\tilde{a}_i-\hat{a}_i|^2+|\frac{\tilde{\lambda}_i-\hat{\lambda}_i}{\hat{\lambda}_i}|^2).\end{aligned} \tag{7.115}$$

So lemma 3.5 (ii) and (iv) yield

$$\begin{aligned}\int K\varphi_{\hat{a}_i,\hat{\lambda}_i}^{\frac{4}{n-2}}|\hat{\alpha}_i\varphi_{\hat{a}_i,\hat{\lambda}_i} - \tilde{\alpha}_i\varphi_{\tilde{a}_i,\tilde{\lambda}_i}|^2 \\ \geq C(|\hat{\alpha}_i-\tilde{\alpha}_i|^2+\hat{\lambda}_i^2|\hat{a}_i-\tilde{a}_i|^2+|\frac{\tilde{\lambda}_i}{\hat{\lambda}_i}-1|^2)+w(\varepsilon).\end{aligned} \tag{7.116}$$

Collecting terms we arrive at

$$\begin{aligned}|\hat{\alpha}-\tilde{\alpha}|^2+\|\hat{\beta}-\tilde{\beta}\|^2 \\ +\sum_i(|\hat{\alpha}_i-\tilde{\alpha}_i|^2+\hat{\lambda}_i^2|\hat{a}_i-\tilde{a}_i|^2+|\frac{\tilde{\lambda}_i}{\hat{\lambda}_i}-1|^2)=w(\varepsilon).\end{aligned} \tag{7.117}$$

Consequently, if we consider a minimizing sequence

$$(\tilde{\alpha}_l,\tilde{\beta}_{k,l},\tilde{\alpha}_{i,l},\tilde{a}_{i,l},\tilde{\lambda}_{i,l})_l \subseteq A_u(\omega,p,2\varepsilon_0) \tag{7.118}$$

for the functional

$$\int Ku^{\frac{4}{n-2}}|u-u_{\tilde{\alpha},\tilde{\beta}}-\tilde{\alpha}^i\varphi_{\tilde{a}_i,\tilde{\lambda}_i}|^2 \tag{7.119}$$

with $u \in V(\omega,p,\varepsilon)$ fixed, e.g.

$$u = u_{\hat{\alpha},\hat{\beta}}+\hat{\alpha}^i\varphi_{\hat{a}_i,\hat{\lambda}_i}+\hat{v},\ (\hat{\alpha},\hat{\beta}_k,\hat{\alpha}_i,\hat{a}_i,\hat{\lambda}_i)\in A_u(\omega,p,\varepsilon), \|\hat{v}\|\leq\varepsilon, \tag{7.120}$$

then there necessarily holds

$$(\tilde{\alpha}_l,\tilde{\beta}_{k,l},\tilde{\alpha}_{i,l},\tilde{a}_{i,l},\tilde{\lambda}_{i,l})_l \subseteq A_u(\omega,p,\varepsilon+w(\varepsilon)). \tag{7.121}$$

for all $l$ sufficiently large. Moreover, since $\tilde{\lambda}_{i,l} \overset{l\to\infty}{\longrightarrow} \infty$ is not possible due to

$$|\frac{\tilde{\lambda}_{i,l}}{\hat{\lambda}_i}-1|^2 = w(\varepsilon) \tag{7.122}$$

the infimum of the functional is attained for some

$$(\alpha,\beta_k,\alpha_i,a_i,\lambda_i)\in\overline{A}_u(\omega,p,\varepsilon+w(\varepsilon))\subset A_u(\omega,p,\varepsilon_0), \tag{7.123}$$

provided $\varepsilon \ll \varepsilon_0$ is sufficiently small.
To show uniqueness we argue by contradiction and assume, that for some

$$u = u_{\hat{\alpha},\hat{\beta}} + \hat{\alpha}^i\varphi_{\hat{a}_i,\hat{\lambda}_i} + \hat{v},\ (\hat{\alpha},\hat{\beta}_k,\hat{\alpha}_i,\hat{a}_i,\hat{\lambda}_i) \in A_u(\omega,p,\varepsilon), \|v\| < \varepsilon, \tag{7.124}$$

in other words for some $u \in V(\omega,p,\varepsilon)$ with suitable representation there exist

$$(\alpha,\beta_k,\alpha_i,a_i,\lambda_i),(\tilde{\alpha},\tilde{\beta}_k,\tilde{\alpha}_i,\tilde{a}_i,\tilde{\lambda}_i) \in A_u(\omega,p,\varepsilon_0) \tag{7.125}$$

such, that

$$\begin{aligned}\inf_{\bar{\alpha},\bar{\beta},\bar{\alpha}_i,\bar{a}_i,\bar{\lambda}_i} & \int Ku^{\frac{4}{n-2}}|u - u_{\bar{\alpha},\bar{\beta}} - \bar{\alpha}^i\varphi_{\bar{a}_i,\bar{\lambda}_i}|^2 \\ & = \int Ku^{\frac{4}{n-2}}|u - u_{\alpha,\beta} - \alpha^i\varphi_{a_i,\lambda_i}|^2 Ku^{\frac{4}{n-2}} \\ & = \int Ku^{\frac{4}{n-2}}|u - u_{\tilde{\alpha},\tilde{\beta}} - \tilde{\alpha}^i\varphi_{\tilde{a}_i,\tilde{\lambda}_i}|^2 Ku^{\frac{4}{n-2}}.\end{aligned} \tag{7.126}$$

By what was shown before the quantities

$$\begin{aligned}& A = |\tilde{\alpha} - \alpha|,\ B_k = |\tilde{\beta}_k - \beta_k|, \\ & A_i = |\tilde{\alpha}_i - \alpha_i|,\ L_i = |\frac{\tilde{\lambda}_i}{\lambda_i} - 1|,\ D_i^2 = \tilde{\lambda}_i\lambda_i d^2(\tilde{a}_i,a_i)\end{aligned} \tag{7.127}$$

are well defined and we will prove the proposition by showing

$$A, B_k, A_i, D_i, L_i = w(\varepsilon) \tag{7.128}$$

and

$$A + \sum_{k=1}^m B_k + \sum_{i=1}^p A_i + D_i + L_i = o(A + \sum_{k=1}^m B_k + \sum_{i=1}^p A_i + D_i + L_i). \tag{7.129}$$

The first statement if rather obvious. Indeed (7.117) shows

$$|\alpha - \hat{\alpha}|, |\tilde{\alpha} - \hat{\alpha}| = w(\varepsilon), \tag{7.130}$$

so $A = w(\varepsilon)$ and the same argument applies to $B_k, A_i, D_i, L_i$ as well.
We are left with proving (7.129). Note, that

$$\begin{aligned}\varphi_{a_j,\lambda_j} - \varphi_{\tilde{a}_j,\tilde{\lambda}_j} = & u_{a_j}(\frac{\lambda_j}{1+\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}})^{\frac{n-2}{2}} - u_{\tilde{a}_j}(\frac{\tilde{\lambda}_j}{1+\tilde{\lambda}_j^2\gamma_n G_{\tilde{a}_j}^{\frac{2}{2-n}}})^{\frac{n-2}{2}} \\ = & \varphi_{a_j,\lambda_j}(1 - \frac{u_{\tilde{a}_j}}{u_{a_j}}(\frac{\tilde{\lambda}_j}{\lambda_j}\frac{1+\lambda_j^2\gamma_n G_{a_j}^{\frac{2}{2-n}}}{1+\tilde{\lambda}_j^2\gamma_n G_{\tilde{a}_j}^{\frac{2}{2-n}}})^{\frac{n-2}{2}})\end{aligned} \tag{7.131}$$

and therefore

$$|\varphi_{a_j,\lambda_j} - \varphi_{\tilde{a}_j,\tilde{\lambda}_j}| \leq c(D_j + L_j)\varphi_{a_j,\lambda_j}. \tag{7.132}$$

First we make use of

$$\partial_\alpha \int K u^{\frac{4}{n-2}} |u - u_{\alpha,\beta} - \alpha^i \varphi_{a_i,\lambda_i}|^2 = 0. \tag{7.133}$$

Differentiating we obtain

$$\begin{aligned}
0 =& \int K u^{\frac{4}{n-2}} (u - u_{\alpha,\beta} - \alpha^i \varphi_{a_i,\lambda_i}) \partial_\alpha u_{\alpha,\beta} \\
=& \int K u^{\frac{4}{n-2}} (u_{\tilde{\alpha},\tilde{\beta}} - u_{\alpha,\beta}) \partial_\alpha u_{\alpha,\beta} + (\tilde{\alpha}^i - \alpha^i) \int K u^{\frac{4}{n-2}} \varphi_{\tilde{a}_i,\tilde{\lambda}_i} \partial_\alpha u_{\alpha,\beta} \\
& + \alpha^i \int K u^{\frac{4}{n-2}} (\varphi_{\tilde{a}_i,\tilde{\lambda}_i} - \varphi_{a_i,\lambda_i}) \partial_\alpha u_{\alpha,\beta} \\
& + \int K u^{\frac{4}{n-2}} \tilde{v} (\partial_\alpha u_{\alpha,\beta} - \partial_{\tilde{\alpha}} u_{\tilde{\alpha},\tilde{\beta}}),
\end{aligned} \tag{7.134}$$

whence $A = o(A + \sum_{k=1}^m B_k + \sum_{i=1}^p A_i + D_i + L_i)$.

Similarly we make use of

$$\partial_{\beta_k} \int K u^{\frac{4}{n-2}} |u - u_{\alpha,\beta} - \alpha^i \varphi_{a_i,\lambda_i}|^2 = 0 \tag{7.135}$$

yielding $B_k = o(A + \sum_{k=1}^m B_k + \sum_{i=1}^p A_i + D_i + L_i)$.

We proceed using

$$\partial_{\alpha_j} \int K u^{\frac{4}{n-2}} |u - u_{\alpha,\beta} - \alpha^i \varphi_{a_i,\lambda_i}|^2 = 0. \tag{7.136}$$

This gives

$$\begin{aligned}
0 =& \int K u^{\frac{4}{n-2}} (u - u_{\alpha,\beta} - \alpha^i \varphi_{a_i,\lambda_i}) \varphi_{a_j,\lambda_j} \\
=& \int K u^{\frac{4}{n-2}} \varphi_{a_j,\lambda_j} (u_{\tilde{\alpha},\tilde{\beta}} - u_{\alpha,\beta}) + (\tilde{\alpha}^i - \alpha^i) \int K u^{\frac{4}{n-2}} \varphi_{a_j,\lambda_j} \varphi_{a_i,\lambda_i} \\
& + \tilde{\alpha}^i \int K u^{\frac{4}{n-2}} \varphi_{a_j,\lambda_j} (\varphi_{\tilde{a}_i,\tilde{\lambda}_i} - \varphi_{a_i,\lambda_i}) \\
& + \int K u^{\frac{4}{n-2}} v (\varphi_{a_j,\lambda_j} - \varphi_{\tilde{a}_j,\tilde{\lambda}_j}),
\end{aligned} \tag{7.137}$$

whence due to (7.132) and lemma 3.5

$$\begin{aligned}
0 =& (\tilde{\alpha}_j - \alpha_j) \int K u^{\frac{4}{n-2}} \varphi_{a_j,\lambda_j}^2 + \tilde{\alpha}_j K_j \alpha_j^{\frac{4}{n-2}} \int \varphi_{a_j,\lambda_j}^{\frac{n+2}{n-2}} (\varphi_{\tilde{a}_j,\tilde{\lambda}_j} - \varphi_{a_j,\lambda_j}) \\
& + o(A + \sum_{k=1}^m B_k + \sum_{i=1}^p A_i + D_i + L_i).
\end{aligned} \tag{7.138}$$

Arguing as for (7.115) we obtain passing to $g_{a_i}$ normal coordinates

$$\begin{aligned}\int \varphi_{a_j,\lambda_j}^{\frac{n+2}{n-2}}(\varphi_{\tilde{a}_j,\tilde{\lambda}_j}-\varphi_{a_j\lambda_j}) = & \int \varphi_{a_j,\lambda_j}^{\frac{n+2}{n-2}}\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}\,\lambda_j(\tilde{a}_j-a_j) \\ & + \int \varphi_{a_j,\lambda_j}^{\frac{n+2}{n-2}}\lambda_j\partial_{\lambda_j}\varphi_{a_j,\lambda_j}\,(\frac{\tilde{\lambda}_j}{\lambda_j}-1) \\ & + o(D_j+L_j),\end{aligned} \tag{7.139}$$

whence according to lemma 3.5 (iv) we obtain

$$\int \varphi_{a_j,\lambda_j}^{\frac{n+2}{n-2}}(\varphi_{\tilde{a}_j,\tilde{\lambda}_j}-\varphi_{a_j\lambda_j}) = o(D_j+L_j). \tag{7.140}$$

We conclude

$$A_j = o(A+\sum_{k=1}^m B_k + \sum_{i=1}^p A_i + D_i + L_i). \tag{7.141}$$

Analogously one obtains

$$L_j, D_j = o(A+\sum_{k=1}^m B_k + \sum_{i=1}^p A_i + D_i + L_i) \tag{7.142}$$

by exploiting

$$\partial_{\lambda_j}\int K u^{\frac{4}{n-2}}|u-u_{\alpha,\beta}-\alpha^i\varphi_i|^2 = 0 \tag{7.143}$$

and

$$\nabla_{a_j}\int K u^{\frac{4}{n-2}}|u-u_{\alpha,\beta}-\alpha^i\varphi_i|^2 = 0 \tag{7.144}$$

using

$$\begin{aligned}&\int \varphi_{a_j,\lambda_j}^{\frac{4}{n-2}}\lambda_j\partial_{\lambda_j}\varphi_{a_j,\lambda_j}(\varphi_{a_j,\lambda_j}-\varphi_{\tilde{a}_j,\tilde{\lambda}_j}) \\ &\qquad = -\int \varphi_{a_j,\lambda_j}^{\frac{4}{n-2}}|\lambda_j\partial_{\lambda_j}\varphi_{a_j,\lambda_j}|^2(\frac{\tilde{\lambda}_j}{\lambda_j}-1)+o(D_j+L_j)\end{aligned} \tag{7.145}$$

and

$$\begin{aligned}&\int \varphi_{a_j,\lambda_j}^{\frac{4}{n-2}}\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}(\varphi_{a_j,\lambda_j}-\varphi_{\tilde{a}_j,\tilde{\lambda}_j}) \\ &\qquad = -\int \varphi_{a_j,\lambda_j}^{\frac{4}{n-2}}(\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j})^2\,\lambda_j(\tilde{a}_j-a_j)+o(D_j+L_j)\end{aligned} \tag{7.146}$$

Finally we show smooth dependence. To that end consider

$$F(u,(\bar{\alpha},\bar{\beta}_k,\bar{\alpha}_i,\bar{a}_i,\bar{\lambda}_i)) = \int K u^{\frac{4}{n-2}}|u-u_{\bar{\alpha},\bar{\beta}}-\bar{\alpha}^i\varphi_{\bar{a}_i,\bar{\lambda}_i}|^2. \tag{7.147}$$

If $(\alpha, \beta_k, \alpha_i, a_i, \lambda_i)$ denotes the minimizer constructed for $u \in V(\omega, p, \varepsilon)$, then

$$D_{(\alpha,\beta_k,\alpha_i,a_i,\lambda_i)}F(u,(\alpha,\beta_k,\alpha_i,a_i,\lambda_i)) = 0. \tag{7.148}$$

Moreover in view of lemma 3.5 we easily find, that

$$D^2_{(\alpha,\beta_k,\alpha_i,a_i,\lambda_i)}F(u,(\alpha,\beta_k,\alpha_i,a_i,\lambda_i)) > 0 \tag{7.149}$$

is positive, provided $\varepsilon > 0$ is sufficiently small. Thus the implicit function theorem provides a smooth parametrization of

$$[D_{(\alpha,\beta_k,\alpha_i,a_i,\lambda_i)}F(u,(\alpha,\beta_k,\alpha_i,a_i,\lambda_i)) = 0]. \tag{7.150}$$

This proves the statement. □

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