Homoclinic and stable periodic solutions for differential delay equations from physiology

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Chapter 1

Introduction

1.1 Overview

Some physiological processes can be described by differential delay equations. For example, the Mackey-Glass model describes the regulation of blood cell production [9]. The primary symptom of chronic granulocytic leukemia is periodicity of the fluctuations in white blood cell counts. The Mackey-Glass model reproduces the qualitative features of normal and pathological function. There is a significant delay between the initiation of cellular production in the bone marrow and the release of mature cells into the blood. The original Mackey-Glass equation can be written as [9]:

$$\dot{x}(t) = -ax(t) + \frac{\beta_0 \theta^m x(t-r)}{\theta^m + x^m(t-r)}, \tag{1.1}$$

where $x(t)$ is the density of mature circulating cells in blood at time moment t, a is the mortality rate, $\dfrac{\beta_0 \theta^m x(t-r)}{\theta^m + x^m(t-r)}$ is the blood cell reproduction rate, β_0, θ, m, a are positive constants, r is the time required to produce a blood cell. As r is increased an initially stable equilibrium becomes unstable and stable periodic solutions appear [9].

Another example is Lasota-Wazewska-Czyzewska equation [8] which models the survival of red blood cells in an animal:

$$\dot{x}(t) = -ax(t) + pe^{-\gamma x(t-r)}, \tag{1.2}$$

where $x(t)$ denotes the number of red blood cells at time t, a is the probability of death of a red blood cell, p and γ are positive constants related to the production of red blood cells per unit time, r is the time required to produce a red blood cell.

This thesis was inspired by Mackey-Glass and Lasota-Wazewska-Czyzewska equations, by the properties of their nonlinear functions and the significance of delay. There are a lot of works on existence of periodic solutions for Mackey-Glass type delay differential equations and for other classes of delay differential equations [5, 3, 11]. The aim of our work is to prove the existence of a homoclinic solution for a class of delay differential equations. The homoclinic trajectory that we construct is a trajectory which is defined on $\mathbb{R}$, not constant and tends to an equilibrium point as $t \to +\infty$, $t \to -\infty$.

In our work we study equations of the following form

$$\dot{x}(t) = -ax(t) + f_\alpha(x(t-1)) \tag{1.3}$$

with $a > 0$ and a nonlinear function $f_\alpha \in C^2$, $f_\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $f_\alpha(x) \geq 0$ for all $x \geq 0$, $f_\alpha(0) = 0$, $\alpha \geq 0$, $\alpha \in A \subset \mathbb{R}$ is a parameter. We assume that there exists a point of maximal value p_m^α: $f'_\alpha(x) > 0$ if $0 < x < p_m^\alpha$, $f'_\alpha(x) < 0$ if $x > p_m^\alpha$. We suppose that there exist stationary points p_0^α and p_1^α of equation (1.3) such that $0 < p_0^\alpha < p_m^\alpha < p_1^\alpha$, and that we have one-dimensional unstable manifold at p_0^α for all $\alpha \in A$.

So, we consider one-parameter family of equations (1.3) with f_α satisfying a list of conditions. Our goal is to study and prove analytically the existence of a homoclinic solution which joins p_0^α to itself for some critical value of parameter α. In order to achieve this, we consider solutions which start on the "upper branch" of the local unstable manifold at p_0^α, that is, with initial functions $\phi_\alpha(t) > p_0^\alpha$, $t \in [-1, 0]$.

After proving the existence of a homoclinic solution we also study the existence of stable periodic solutions for equations (1.3) for values of parameter close to the critical one.

As far as we know there are no papers on proof of the existence of a homoclinic solution for Mackey-Glass type delay differential equations. Although there are works on numerical computation of homoclinic solutions for different types of delay differential equations. For example, in article [10] the numerical computation of homoclinic orbits in delay differential equations is discussed. The authors develop a method to compute connecting orbits in delay differential equations based on projection boundary conditions and demonstrate this method using a model for neural activity.

The structure of this thesis is following. In section 2.4 we formulate a theorem with sufficient conditions for the existence of a homoclinic orbit of equation (1.3). In chapter 3 we study the existence of stable periodic solutions on the basis of the fact that we have a homoclinic orbit and using bifurcation theorem from paper [11]. Note that in paper [11] the possibility of homoclinic solution of equation (1.3) is touched upon. In section 4.2 we show that the sufficient conditions of the theorems can be satisfied for concrete parameters and concrete nonlinear functions f_α of equation (1.3). Also in section 4.2 we provide numerical results which correspond to our theoretical results.

1.2 Exposition

We give an outline of the most important ideas and methods of this work. We assume that the functions f_α, $\alpha \in A$, satisfy the list of conditions (conditions (i)-(xi) from chapter 2). We prove that the only positive eigenvalue of the linearised equation of equation (1.3) at p_0^α has the following property: $\lambda_0^\alpha < a$ and $\lambda_0^\alpha < |\mathrm{Re}(\lambda_1^\alpha)|$, where λ_1^α is the eigenvalue with the smallest absolute real part among those with negative real part, i.e. the dominant stable eigenvalue. This is the condition under which one expects stable periodic solutions to bifurcate from a homoclinic solution.

We consider solutions starting on the "upper branch" of the local unstable manifold of p_0^α and use results on monotone feedback from [6] as long as they stay in monotone region. Particularly, such solutions grow up to p_1^α and remain above p_1^α on a time interval of length 1.

We have one-parameter family of nonlinear functions of equation (1.3), and we prove that for some parameters solutions converge to zero (which is stable), for other parameters solutions "grow up" again. Wazewski-type argument gives a solution oscillating about p_0^α for some value of parameter, which then has to converge to p_0^α. More precisely, the set of parameters under consideration

$A := [\alpha_0, \alpha_1]$ is closed. We consider two subsets of A: $A^1 = \{\alpha \in A \mid \exists t_*^\alpha > \tau_m^\alpha + 1 : x^\alpha(t_*^\alpha) \geq p_m^\alpha\}$ and $A^2 = \{\alpha \in A \mid x^\alpha(t) < p_m^\alpha$ for all $t > \tau_m^\alpha$, $\exists T^\alpha > \tau_m^\alpha + 1 : x_{T^\alpha}^\alpha < p_0^\alpha\}$, where $\tau_m > 0 : x^\alpha(\tau_m) = p_m^\alpha$, $\dot{x}^\alpha(\tau_m) < 0$. We prove that $\alpha_1 \in A^1$ and $\alpha_0 \in A^2$. We show that A^1 and A^2 are open subsets of A. Then we prove that for a critical value of parameter $\alpha_* : \alpha_* \notin A^1$, $\alpha_* \notin A^2$, the solution oscillates rapidly about $p_0^{\alpha_*}$ for $t \geq \tau_m^{\alpha_*}$.

For bifurcation the condition that $x_{t_+}^\alpha$ lies on the same side of the local stable manifold as $x_0^{\alpha_*}$, where x^{α_*} is the homoclinic solution, $t_+ \in \mathbb{R} : x^{\alpha_*}(t) \in B(p_0^{\alpha_*}, \delta)$ for all $t \geq t_+$ for some $\delta > 0$, $B(p_0^{\alpha_*}, \delta)$ is a ball with center in $p_0^{\alpha_*}$ and radius δ, has to be verified for $\alpha_* < \alpha < \alpha_* + \epsilon$ with some small $\epsilon > 0$. This is related to to the behaviour of the solutions for parameters, where solutions "grow up" again.

As an example we consider the class of equations (1.3) with $f_\alpha(x) = \dfrac{x^p}{1+x^q}$ for $x \le p_1 + \delta_1$ with some small $\delta_1 > 0$, $f_\alpha(x) = \alpha p_1$ for $x \ge p_1 + \delta_2$ with some small $\delta_2 > \delta_1$. Verification of sufficient conditions of our theorems for this class of equations is computer assisted.

Chapter 2

The existence of a homoclinic solution for a class of delay differential equations

2.1 Local dynamics

Let us consider equations of the form

$$\dot{x}(t) = -ax(t) + g(x(t-1)), \tag{2.1}$$

where $g : \mathbb{R} \to \mathbb{R}$, $g \in C^2$, $a > 0$ and there exists $p_0 \in \mathbb{R} : g(p_0) = ap_0$, so that the constant function $[-1, 0] \ni \theta \mapsto p_0$ is an equilibrium point of the semiflow generated by equation (2.1). Assume that $a < g'(p_0) < \frac{3\pi}{2} e^{-a}$ is satisfied.

Let us consider the linearised equation at p_0 of equation (2.1)

$$\dot{y}(t) = -ay(t) + g'(p_0)y(t-1). \tag{2.2}$$

Lemma 1. *There exists only one positive real eigenvalue λ_0 at p_0 for the linearised equation* (2.2) *and all the other eigenvalues at p_0 have negative real parts with $Re(\lambda_j) < -a$ for $j \geq 1$. It follows that there is one-dimensional local unstable manifold at the stationary point p_0.*

Proof. Consider characteristic equation for equation (2.2)

$$\lambda = -a + g'(p_0)e^{-\lambda}. \tag{2.3}$$

Introduce $\tilde{\lambda} = \lambda + a$, $k = g'(p_0)e^a > 0$ and rewrite the characteristic equation as $\tilde{\lambda} = ke^{-\tilde{\lambda}}$. There is one real eigenvalue $\tilde{\lambda}_0$, and the others form a sequence of complex conjugate pairs $(\tilde{\lambda}_j, \bar{\tilde{\lambda}}_j)$ with

$\mathrm{Re}(\tilde{\lambda}_{j+1}) < \mathrm{Re}(\tilde{\lambda}_j) < \tilde{\lambda}_0$ and $(2j-1)\pi < \mathrm{Im}(\tilde{\lambda}_j) < 2\pi j$ for all integers $j \geq 1$, and $\mathrm{Re}(\tilde{\lambda}_j) \to -\infty$ as $j \to +\infty$ [6].

Note that $\lambda_0 = 0 \Leftrightarrow \tilde{\lambda}_0 = a \Leftrightarrow a = ke^{-a} = g'(p_0)e^a e^{-a} = g'(p_0)$. Notice that $\tilde{\lambda}_0$ increases if k increases. It is clear that if $g'(p_0) > a$ then there is one positive real eigenvalue $\lambda_0 = \tilde{\lambda}_0 - a$. It is known that if $k \in (0, \frac{3\pi}{2})$ then $\mathrm{Re}(\tilde{\lambda}_j) < 0$ for all $j \geq 1$ [7]. And we obtain:

$$0 < k < \frac{3\pi}{2} \Leftrightarrow 0 < g'(p_0)e^a < \frac{3\pi}{2} \Leftrightarrow 0 < g'(p_0) < \frac{3\pi}{2}e^{-a}.$$

So, we have that $\mathrm{Re}(\lambda_j) = \mathrm{Re}(\tilde{\lambda}_j) - a < -a$ for $j \geq 1$. $\square$

Lemma 2. *If* $g'(p_0) < 2ae^a$ *then the only real positive eigenvalue of equation* (2.2) λ_0 *satisfies:* $\lambda_0 < a$. *And it follows that* $\lambda_0 < |Re(\lambda_1)|$.

Proof. According to lemma 1 we have one real positive eigenvalue of equation (2.2) $\lambda_0 > 0$ and $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\tilde{\lambda}_1) - a < -a$. Note that $\lambda_0 = a \Leftrightarrow \tilde{\lambda}_0 = 2a \Leftrightarrow 2a = ke^{-2a} = g'(p_0)e^a e^{-2a} = g'(p_0)e^{-a}$. So, if $g'(p_0) < 2ae^a$ then $\tilde{\lambda}_0 < 2a$. It follows that $\lambda_0 < a$, $\mathrm{Re}(\lambda_1) < -a$ and, hence, $\lambda_0 < |\mathrm{Re}(\lambda_1)|$. $\square$

The eigenspace U of the generator associated with λ_0 is 1-dimensional and spanned by the function $\chi_0 : [-1, 0] \ni t \mapsto e^{\lambda_0 t} \in \mathbb{R}$. Denote by S the realified generalized eigenspace of the generator which is given by the spectral set of all λ_j, $\bar{\lambda}_j$, $j \geq 1$ (here S corresponds to $Q + L$ in [6]). The relations $\Phi_g(t, \phi) = x_t$, $x = x^\phi$, $x_t(s) = x(t + s)$, $s \in [-1, 0]$, define a continuous semiflow $\Phi_g : \mathbb{R}^+ \times C \to C$. Consider the time-one-map $F_g = \Phi_g(1, \cdot) : C \mapsto C$. We have $F_g(p_0) = p_0$, $F_g \in C^1$, $DF_g(p_0)\psi = v_1^\psi$, where v satisfies

$$\dot{v}^\psi(t) = -av^\psi(t) + g'(p_0)v^\psi(t-1), v_0^\psi = \psi. \tag{2.4}$$

The system (2.4) generates a semigroup [4] $\{T(t)\}_{t\geq 0}$, $T(t) : C \mapsto C$ is a linear operator and $T(1) = DF_g(p_0)\psi$. Define $T := T(1)$. Let us denote the original norm in the space of continuous functions by $||y||_\infty = \sup_{[-1,0]} |y(\theta)|$. There exists a norm $||\cdot||$ equivalent to $||\cdot||_\infty$ so that $||\cdot|| \leq 2\tilde{K}||\cdot||_\infty$ with some $\tilde{K} > 0$ and $||\cdot||_\infty \leq 2||\cdot||$, and there exist $\beta > 1$, $\gamma < 1$ such that for all $u \in U$: $||Tu|| \geq \beta||u||$, for all $s \in S$: $||Ts|| \leq \gamma||s||$, and $||u + s|| = \max\{||u||, ||s||\}$ for $u \in U$, $s \in S$. The proof of this fact you can find in lemma 26, where we introduce new adapted norm by $||u|| = \sup_{n\leq 0} e^{-bn}||T^n u||_\infty$ for $u \in U$ with some $b > 0$ and $||s|| = \sup_{n\geq 0} e^{-an}||T^n s||_\infty$ for $s \in S$ with some $a < 0$.

Denote by N_S, N_U the open balls with center 0 and radius δ for some $\delta > 0$ in S, U, respectively, by means of new adapted norm. Introduce $N_{\mathrm{loc}} = N_S + N_U$. Denote by pr_U the projection to the space U and by pr_S the projection to the space S. For ϕ, $\psi \in C$ define $\phi << \psi$ if and only if $\phi(s) < \psi(s)$ for all $s \in [-1, 0]$. We will denote by $W^u_{\text{loc},g}$ the local unstable manifold at p_0 of the semiflow Φ_g, which is tangent to the eigenspace of the eigenvalue λ_0 and denote by $W^u_{\text{loc},g}(F_g)$ the local unstable manifold at p_0 of the time-one map F_g.

Choose $1 < \beta < e^{\lambda_0}$. Theorem I.4 [6] implies that there exist ϵ_0 and a C^1-map $\omega : (-\epsilon_0, \epsilon_0)\chi_0 \to S$, $\omega(0) = 0$, $D\omega(0) = 0$ so that for every $\epsilon \in (-\epsilon_0, \epsilon_0)$ there exists a unique trajectory (uniqueness follows from Theorem I.3 (iii) in [6]) $(\phi_n)^0_{-\infty}$ of $\Phi_g(1, \cdot)$ with $\phi_0 = p_0 + \omega(\epsilon\chi_0) + \epsilon\chi_0$, $(\phi_n - p_0)\beta^{-n} \in N_{\text{loc}}$ for all integers $n \le 0$ and $(\phi_n - p_0)\beta^{-n} \to 0$ as $n \to -\infty$. Notice that if $(-\epsilon_0, \epsilon_0)\chi_0 = N_U$ then $p_0 + \text{graph }\omega = W^u_{\text{loc},g}(F_g)$.

Lemma 3. *For ψ of the form $\psi = p_0 + \omega(\epsilon\chi_0) + \epsilon\chi_0$ with $\epsilon \in (-\epsilon_0, \epsilon_0)$ there exists a unique solution $x^\psi : \mathbb{R} \to \mathbb{R}$ of equation* (2.1) *such that $x^\psi(t) \to p_0$ as $t \to -\infty$.*

Proof. Namely, one can consider $x^\psi_n = \phi_n$ for $n \le 0$ and $x^\psi_{n+t} = \Phi_g(t, x^\psi_n)$ for $0 \le t \le 1$, it means that we have a backward solution of equation (2.1). And $x^\psi_n \to p_0$ as $n \to -\infty$ implies $x^\psi(t) \to p_0$ as $t \to -\infty$. Uniqueness of $(\phi_n)^0_{-\infty}$ implies uniqueness of x^ψ. □

Note that the local stable manifold $W^s_{\text{loc},g}$ is a graph: $W^s_{\text{loc},g} = p_0 + \text{graph } s \cap N_{loc}$, $s : S \cap N_{\text{loc}} \to U$. We denote by $W^u_g(F_g, p_0, \bar{\delta})$ and by $W^s_g(F_g, p_0, \bar{\delta})$ the local unstable manifold and the local stable manifold for F_g in $\bar{\delta}-$neighbourhood of p_0, which we denote by $B(p_0, \bar{\delta})$. Analogously we denote by $W^u_g(\Phi_g, p_0, \tilde{\delta})$, $W^s_g(\Phi_g, p_0, \tilde{\delta})$ the local unstable manifold and the local stable manifold for the semiflow Φ_g. Note that we consider balls and neighbourhoods with respect to adapted norm.

Lemma 4. *For $\tilde{\delta} > 0$ there exists $\bar{\delta} < \tilde{\delta}$: $W^u_g(F_g, p_0, \bar{\delta}) \subset W^u_g(\Phi_g, p_0, \tilde{\delta})$ and $W^s_g(F_g, p_0, \bar{\delta}) \subset W^s_g(\Phi_g, p_0, \tilde{\delta})$, $s_{F_g}|_{B(0,\bar{\delta})\cap S} = s_{\Phi_g}|_{B(0,\bar{\delta})\cap S}$ and $\omega_{F_g}|_{B(0,\bar{\delta})\cap U} = \omega_{\Phi_g}|_{B(0,\bar{\delta})\cap U}$.*

Proof. There exists $\bar{\delta} \in (0, \tilde{\delta}] : \Phi_g\big([0, 1], B(p_0, \bar{\delta})\big) \subset B(p_0, \tilde{\delta})$ due to uniform continuity of Φ_g on the compact set $[0, 1] \times \{p_0\}$. It follows that if $x^\psi_n \in B(p_0, \bar{\delta})$ and $x^\psi_n \in W^u_g(F_g, p_0, \bar{\delta})$ then $x^\psi_{n+t} = \Phi_g(t, x^\psi_n) \in B(p_0, \tilde{\delta})$ and $x^\psi_{n+t} \in W^u_g(\Phi_g, p_0, \tilde{\delta})$ for $0 \le t \le 1$. So, we have $W^u_g(F_g, p_0, \bar{\delta}) \subset W^u_g(\Phi_g, p_0, \tilde{\delta})$. This implies that ω_{F_g} and ω_{Φ_g} coincide on $B(0, \bar{\delta}) \cap U$. The result for $W^s_g(F_g, p_0, \bar{\delta})$, $W^s_g(\Phi_g, p_0, \tilde{\delta})$ and $s_{F_g}|_{B(0,\bar{\delta})\cap S}$, $s_{\Phi_g}|_{B(0,\bar{\delta})\cap S}$ can be proved analogously. □

We will denote by W^u_ϵ the unstable manifold $W^u_g(\Phi_g, p_0, \epsilon)$ for the semiflow Φ_g, the Lipschitz constant of ω_{Φ_g} by $L_\omega := L_{\omega_{\Phi_g}}$, $\omega := \omega_{\Phi_g}$.

Choose $\epsilon_2 \in (0, \frac{\epsilon_0}{4K})$ so small that the Lipschitz constant L_ω of ω with respect to the original norm in the space of continuous functions satisfies: $L_\omega \leq \frac{1}{2}e^{-\lambda_0}$ on $(-\epsilon_2, \epsilon_2)\chi_0$. Fix $\epsilon \in (0, \epsilon_2)$ and set $\eta = p_0 + \omega(\epsilon\chi_0) + \epsilon\chi_0$.

Lemma 5. *Under the definition of η above, the following is satisfied: (i) $p_0 << \eta \in W^u_{\epsilon_0}$, (ii) $||pr_U(\eta - p_0)||_\infty \geq \frac{1}{2}||\eta - p_0||_\infty$.*

Proof. It is clear that $\eta \in W^u_{\epsilon_0}$. We have that

$$\epsilon\chi_0 + \omega(\epsilon\chi_0) \geq \epsilon\chi_0(v) - \frac{1}{2}e^{-\lambda_0}\epsilon||\chi_0||_\infty \geq \epsilon(e^{\lambda_0 v} - \frac{1}{2}e^{-\lambda_0}) > 0$$

for all $v \in [-1, 0]$. Notice that $||pr_U(\eta - p_0)||_\infty = \epsilon$ and $||\eta - p_0||_\infty \leq \epsilon(||\chi_0||_\infty + \frac{1}{2}e^{-\lambda_0}||\chi_0||_\infty) \leq$ $\leq \epsilon(1 + \frac{1}{2}e^{-\lambda_0}) \leq 2\epsilon$. □

2.2 Monotone nonlinearities

Let us consider equations of the form

$$\dot{x}(t) = -ax(t) + g(x(t-1)), \tag{2.5}$$

where $g : \mathbb{R} \to \mathbb{R}$ is monotone increasing, $g \in C^2$, $g(x) > 0$ for $x > 0$, $g(0) = 0$, $a > 0$ and there exist p_* and p_0 such that $\frac{g(x)}{x} < a$ for $0 < x < p_0$, $\frac{g(x)}{x} > a$ for $p_0 < x < p_*$, $\frac{g(x)}{x} < a$ for $x > p_*$, $a < g'(p_0) < \frac{3\pi}{2}e^{-a}$. Equation (2.5) has 3 stationary points: 0, p_0 and p_*. So, we have $0 < p_0 < p_*$. We apply lemmas 1, 3, 5 to the equation (2.5) and consider η from the previous section. Proposition 2.2 from [6] yields $0 << \phi << p_*$ for every $\phi \in \Phi_g(\mathbb{R} \times W^u_{\mathrm{loc},g})$.

Lemma 6. *Let us consider the unique solution $x : \mathbb{R} \to \mathbb{R}$ of equation (2.5) with $x_0 = \eta$. For such solution the following is satisfied: $x_t - x_s$ has no zero for $s \neq t$.*

Proof. We have that $x_{s_1} \neq x_{s_2}$ for $s_1 \neq s_2$, $s_1 < 0$, $s_2 < 0$, where $x_{s_1} \in W^u_{\mathrm{loc},g}$, $x_{s_2} \in W^u_{\mathrm{loc},g}$ since otherwise x is periodic with $x_0 \neq p_0$ which leads to a contradiction to $x_s \to p_0$ as $s \to -\infty$. Let real numbers $s \neq t$ be given. We note that there exists $j \in \mathbb{N}$: $x_{t-2k} \in W^u_{\epsilon_2}$, $x_{s-2k} \in W^u_{\epsilon_2}$ for all integers $k \geq j$. We have $pr_U(x_v - p_0) \to 0$ as $v \to -\infty$ since $x_0 \in W^u_{\epsilon_0}$. It follows that there is an integer $k \geq j$ such that for each integer $m \geq k$:

$$x_{t-2m} - x_{s-2m} = p_0 + pr_U(x_{t-2m} - p_0) - p_0 - pr_U(x_{s-2m} - p_0) + \omega(pr_U(x_{t-2m} - p_0)) - \omega(pr_U(x_{s-2m} - p_0))$$

and $||\omega(pr_U(x_{t-2m}-p_0))-\omega(pr_U(x_{s-2m}-p_0))||_\infty \leq \frac{1}{2}e^{-\lambda_0}||pr_U(x_{t-2m}-p_0)-pr_U(x_{s-2m}-p_0)||_\infty$ since $L_\omega < \frac{1}{2}e^{-\lambda_0}$. The injectivity of pr_U on $W^u_{\epsilon_2}$ gives $pr_U(x_{t-2m}-p_0)-pr_U(x_{s-2m}-p_0)=c_m\chi_0$ with $c_m \neq 0$ for all integers $m \geq k$. Now it follows that

$$|x_{t-2m}(v)-x_{s-2m}(v)| \geq |c_m|\chi_0(v) - \frac{e^{-\lambda_0}}{2}|c_m|\ ||\chi_0||_\infty = |c_m|(e^{\lambda_0 v} - \frac{e^{-\lambda_0}}{2}) > 0$$

for all integers $m \geq k$ and for all $v \in [-1,0]$. Hence Proposition 2.2 [6] yields that $x_t - x_s$ has no zero. □

Lemma 7. *The solution $x:\mathbb{R}\to\mathbb{R}$ of equation* (2.5) *with $x_0=\eta$ is strictly increasing.*

Proof. We have that x is injective since otherwise there exist $s \neq t$ with $x(s)=x(t)$, and $x_t - x_s$ has a zero at 0, a contradiction to lemma 6. It follows that x is strictly monotone. If x is strictly decreasing, then $p_0 < \eta(0) \leq x(t)$ for all $t \leq 0$, contradicting $x_t \to p_0$ as $t \to -\infty$. So, x is strictly increasing. □

Lemma 8. *The solution $x:\mathbb{R}\to\mathbb{R}$ of equation* (2.5) *with $x_0=\eta$ has the following property: $x(t)\to p_*$ as $t\to+\infty$.*

Proof. We have $x(t) \leq p_*$ for all $t \in \mathbb{R}$ due to monotonicity property of the semiflow. It follows that $x(t)$ converges to some $\xi \in (0,p_*)$ as $t \to +\infty$. By equation (2.5), $\dot{x}(t) \to -a\xi + g(\xi)$ as $t \to +\infty$. In case $-a\xi + g(\xi) \neq 0$, $\dot{x}$ is bounded away from 0 on some unbounded interval in $\mathbb{R}^+$, and we obtain a contradiction to $\lim_{t\to+\infty} x(t) = \xi$. Thus, $a\xi = g(\xi)$ and, hence, $\xi = p_*$. □

2.3 Assumptions and solution behaviour

Let us consider the following class of equations

$$\dot{x}(t) = -ax(t) + f(x(t-1)) \tag{2.6}$$

with $a > 0$ and a nonlinear function $f \in \Gamma$. Γ is the class of functions f for which the following conditions are satisfied:

(i) $f \in C^2$, $f:\mathbb{R}\to\mathbb{R}$;

(ii) $f(x) \geq 0$ for all $x \geq 0$, $f(0)=0$, $f'(0)=0$;

(iii) there exists a unique point of maximal value p_m: $f'(x) > 0$ if $0 < x < p_m$, $f'(x) < 0$ if $x > p_m$,

$f''(p_m) \neq 0$;

(iv) there exist a unique stationary point p_0 and a unique stationary point p_1 of equation (2.6), such that $0 < p_0 < p_m < p_1$;

(v) $f(x) < ax$ on $(0, p_0) \cup (p_1, +\infty)$, $f(x) > ax$ on (p_0, p_1);

(vi) $a < f'(p_0) < \dfrac{3\pi}{2} e^{-a}$;

(vii) $f'(p_1) < -1$;

(viii) $e^{-a} p_{**} + f(p_m) \dfrac{1 - e^{-a}}{a} < p_m$, where $p_{**} : f(p_{**}) = a p_m$, $p_{**} < p_m$;

(ix) $\dfrac{\max_{x \in [0, p_m]} f'(x)}{a} (1 - e^{-a}) < 1$;

(x) $f'(p_0) < 2ae^a$.

Figure 2.1 shows the shape of functions from the class Γ.

Notice that p_0, p_m, p_1 and the eigenvalues depend on a concrete function $f \in \Gamma$.

For auxiliary purposes we also consider equations of the form

$$\dot{x}(t) = -ax(t) + g(x(t-1)), \tag{2.7}$$

where $a > 0$, g is monotone increasing and $f(x) = g(x)$ on the interval $[0, p_m)$, and there exists $p_* > p_m$ such that $\dfrac{g(x)}{x} < a$ for $x > p_*$ and $\dfrac{g(x)}{x} > a$ for $p_0 < x < p_*$. Equation (2.7) has 3 stationary points: 0, p_0 and p_*, $p_* > p_m$, p_* depends on g.

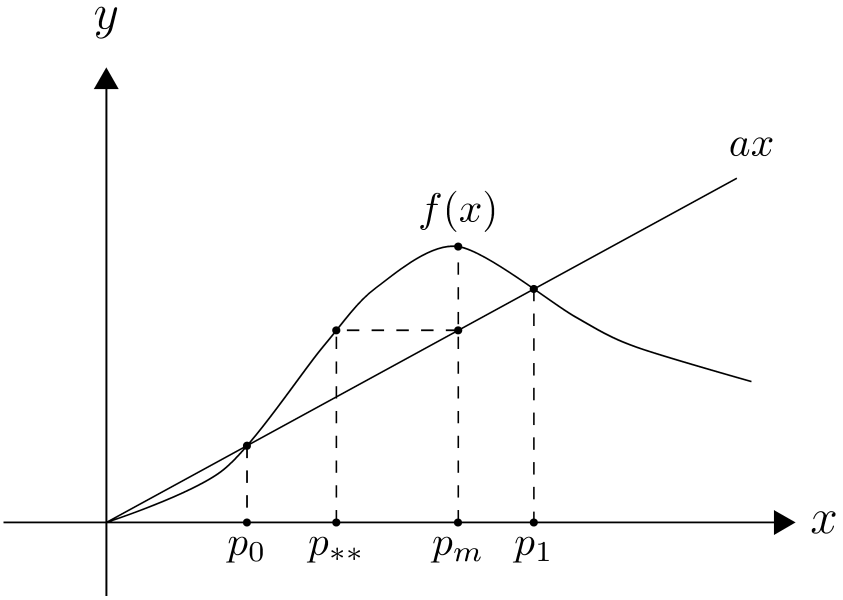

Figure 2.1: Functions from class Γ

Remark Notice that lemma 1 is true for equation (2.6) since the sufficient condition is satisfied for f according to condition (vi).

Remark Note that lemma 2 is true for equation (2.6) since the sufficient condition is satisfied according to condition (x).

Recall that $W^u_{\mathrm{loc},f}$ is the local unstable manifold at p_0 for the semiflow Φ_f.

Lemma 9. *For all* $f \in \Gamma$ *and for all* $\eta \in W^u_{loc,f}$, $\eta >> p_0$, *the solution* $x : \mathbb{R} \to \mathbb{R}$ *of equation* (2.6) *with* $x_0 = \eta$ *has the following property :* x *grows up to the level* p_m.

Proof. Notice that $W^u_{\mathrm{loc},f}$ is one-dimensional. If there are 2 functions $\eta, \tilde{\eta} \in W^u_{\mathrm{loc},f}$, $\eta >> p_0$, $\tilde{\eta} >> p_0$, then $x^{\tilde{\eta}}$ is just a phase shifted version of x^{η}.
Note that the functions g and f coincide on the interval $[0, p_m)$, and the behaviour of the solutions of equations (2.6) and (2.7) coincide while $x(t) < p_m$. For function g we can use the results of lemmas 3-8. More precisely, the solution of equation (2.6) with initial value $x_0 = \eta$ coincides with the solution of equation (2.7) with the same initial value up to the moment $t_m + 1$, where t_m is the first time moment with $x(t_m) = p_m$. So, we showed that solution x with $x_0 = \eta$ of equation (2.6) grows up to time moment $t_m + 1$ and intersects p_m at some finite time moment t_m. ☐

Lemma 10. *For all* $f \in \Gamma$ *and for all* $\eta \in W^u_{loc,f}$, $\eta >> p_0$, *the following is satisfied: the solution* x *with* $x_0 = \eta$ *has the properties:*

> x *intersects* p_m, x *grows up to* p_1 *and* x *remains strictly above* p_1 *at least on a time interval of length* 1.
>
> (2.8)

Proof. In lemma 9 it was proved that the solution of equation (2.6) with $x_0 = \eta$ intersects p_m and increases on $[t_m, t_m + 1)$, where $t_m : x(t_m) = p_m$. So, $x(t) > p_m$ if $t \in (t_m, t_m + 1]$. Consider the behaviour of the solution on $[t_m + 1, t_m + 2]$: if $x < p_1$ on $(t_m, t_m + 2]$, then $\dot{x}(t) = -ax(t) + f(x(t-1))$ and $f(x(t-1)) \in [f(p_1), f(p_m)]$ for $t \in [t_m + 1, t_m + 2]$. It follows that
$\dot{x}(t) > -ax(t) + f(p_1) > -ap_1 + f(p_1) = 0$ for $t \in [t_m + 1, t_m + 2]$. We can use the same argumentation for further time intervals and show that the derivative of the solution $\dot{x}(t) > 0$ while $x < p_1$ on the previous time interval of length one and for the current time moment t. So, there are 2 possibilities:

1) $\dot{x}(t) > 0$ forever and $x(t) \to p_1$ as $t \to +\infty$ (a limit less then p_1 is impossible), or 2) $\dot{x}(t) > 0$ up to the moment t_1: $x(t_1) = p_1$, i.e. there exists such moment t_1 that the solution intersects p_1.

Let us change variables in equation (2.6): $\tilde{x}(t) = x(t) - p_1$, $y(t) = e^{at}\tilde{x}(t)$. Then we have $\dot{y}(t) = e^{at}\dot{\tilde{x}}(t) + \tilde{x}(t)ae^{at} = e^{at}\big(-ax(t) + f(x(t-1)) + a(x(t) - p_1)\big) = e^{at}\big(-ap_1 + f(\tilde{x}(t-1) + p_1)\big)$. Now we can consider the following equation instead of equation (2.6):

$$\dot{y}(t) = e^{at}\big(-ap_1 + f(y(t-1)e^{-a(t-1)} + p_1)\big). \tag{2.9}$$

Notice that if $x(t-1) > p_m$, i.e. we are in the monotone region $[p_m, \infty)$ of f, the following is satisfied:

1. if $y(t-1)e^{-a(t-1)} < 0$, then $\dot{y}(t) > 0$,

2. if $y(t-1)e^{-a(t-1)} > 0$, then $\dot{y}(t) < 0$. It means that the solution is under influence of negative feedback.

Using lemma 11 below, one can prove that the solution $y(t)$ of equation (2.9) intersects 0. Then $y(t)$ remains above 0 on a time interval of length 1 since the derivative of the solution remains positive on a time interval of length 1 after the intersection of 0. It follows that the solution $x(t)$ of equation (2.6) intersects p_1 and remains strictly above p_1 on a time interval of length 1. So, we have shown that there exists such moment t_1 that the solution intersects p_1, and if $t_1 > t_m + 1$ then $x(t) > p_1$ on a time interval of length 1 due to condition 1. Notice that if $t_1 \in (t_m, t_m + 1]$ then we have that $\dot{x}(t) > 0$ for $t \in (t_1, t_m + 1)$, and for $t \in (t_m, t_1)$ we have $y(t) < 0$, hence, $\dot{y}(t) > 0$ for $t \in (t_m + 1, t_1 + 1)$ and $x(t) > p_1$ for $t \in (t_m + 1, t_1 + 1]$. It follows that $x(t) > p_1$ for $t \in (t_1, t_1 + 1]$. $\square$

Corollary 1. *The properties in* (2.8) *of lemma 10 are true for all solutions of equation* (2.6) *such that there exists* $T : p_0 < x_T(s) < p_m$ *for all* $s \in [-1, 0]$*. In particular, there exist* $t_m > T$*,* $t_2 > t_m + 1$*:* $x(t_m) = p_m$*,* $x_{t_2} > p_1$*.*

Lemma 11. *Consider equations* (2.9)*,* (2.6)*,* $f \in \Gamma$ *and assume that solution* $x(\theta) > p_m$ *for* $\theta \in [t-1, \infty)$*. Then there exists* $s > t$ *with sign* $y(s) = -$*sign* $y(t)$*.*

Proof. The main idea of the proof is analogous to the proof of Proposition 2.2 of section XV.1 in [2]. Let $y(t) < 0$. Assume $y < 0$ on $[t, +\infty)$. Then $\dot{y}(t) \geq 0$ on $[t + 1, +\infty)$, $y(s) \to 0$ as $s \to +\infty$ (a negative limit is impossible).

Setting $\tilde{g}(s, y) = e^{as}\big(-ap_1 + f(e^{-as}y + p_1)\big)$ for $s, y \in \mathbb{R}$, equation (2.9) transforms to

$$\dot{y}(t) = e^{a}\tilde{g}(t-1, y). \tag{2.10}$$

Note that $\tilde{g}(t,0) = 0$ for all $t \in \mathbb{R}$, $\partial_2\tilde{g}(t,0) = f'(p_1) < -1$ for all $t \in \mathbb{R}$ according to condition (vii). Choose $\epsilon > 0$ with $e^a(1-\epsilon) \geq 1$. We have $\tilde{g}(t,y) - \tilde{g}(t,0) = \int_0^y \partial_2\tilde{g}(t,\tilde{y})d\tilde{y}$. There exists $s > t+1$ with $-\delta < y(\theta) < 0$ for $\theta \in [s,+\infty)$, i.e. $|y(\theta)| < \delta$ for $\theta \in [s,+\infty)$. The map $(t,y) \to \partial_2\tilde{g}(t,y)$ is continuous and uniformly continuous on $I \times \{0\}$, where I is a compact time interval, for example, $I = [s,s+1]$. In particular, for any $0 < \epsilon < 1$ there exists δ sufficiently small so that $|\partial_2\tilde{g}(t,y) - \partial_2\tilde{g}(t,0)| < \epsilon$ for all $t \in I$, $|y| < \delta$ and $|\partial_2\tilde{g}(y)| \geq 1-\epsilon$.
So, $|\tilde{g}(t,y) - \tilde{g}(t,0)| = |\int_0^y \partial_2\tilde{g}(t,\tilde{y})d\tilde{y}| \geq (1-\epsilon)|y|$ for $|y| < \delta$. It follows that $|\tilde{g}(t,y)| \geq (1-\epsilon)|y|$ for $t \in I$, $|y| < \delta$.
Consider $y(s+2) - y(s+1) = e^a \int_s^{s+1} \tilde{g}(t,y(t))dt \geq e^a(1-\epsilon)\int_s^{s+1}|y|dt \geq -y(s+1)$. Hence, $y(s+2) \geq 0$ and we obtain a contradiction to $y(s+2) < 0$. ☐

Remark In lemma 11 we use the assumption (vii) that $f'(p_1) < -1$ for simplification, this condition is not necessary but is sufficient for the result. It is possible to use the condition $f'(p_1)e^a < -e^{-1}$ instead, which excludes real negative eigenvalues at p_1. Otherwise it is possible that $y(t) \to 0$ as $t \to +\infty$ and remains negative (or positive), i.e., no sign change happens.

Lemma 12. *For all $f \in \Gamma$ we have the following property: if any solution of equation (2.6) satisfies: $x(t) < p_0$ on a time interval of length 1 for $t \in [t^*, t^*+1]$, then $x(t) \to 0$ as $t \to +\infty$ and $x(t) < p_0$ for $t > t^*$.*

Proof. First of all one can prove that there exists $\eta' \in W^u_{\mathrm{loc},f}$, $\eta' << p_0$, so that the solution x^* of equation (2.6) with initial value $x^*_0 = \eta'$ has the following properties: x^* is less than p_0 for all $t \in \mathbb{R}$, strictly decreasing and $x^*(t) \to 0$ as $t \to +\infty$. The proof of this fact is analogous to the proofs of lemmas 3-8, using that $f \in \Gamma$ is monotone on $[0,p_0]$.
For ϕ, $\psi \in C$ we define $\phi \leq \psi$ if and only if $\phi(s) \leq \psi(s)$ for all $s \in [-1,0]$. It is clear that $x^*(t) \to p_0$ as $t \to -\infty$. So, if a solution $x(t)$ of equation (2.6) remains below p_0 on a time interval $[t^*, t^*+1]$ of length 1, then there exists $\tilde{t} : x_{t^*+1} \leq x^*_{\tilde{t}} < p_0$. Using monotonicity of f and Proposition 2.2 from [6] one can show that $0 \leq x(t) \leq x^*(t + \tilde{t} - (t^*+1))$ for all $t \geq t^*+1$, hence x is as asserted. ☐

Lemma 13. *For all $f \in \Gamma$ if a solution of equation (2.6) satisfies: $0 < x < p_m$ on a time interval of length 1 $[\tilde{t}, \tilde{t}+1]$ then for $t > \tilde{t}+1$ the solution x can cross the level p_m for the first time only transversally (with positive derivative).*

Proof. Consider p_{**} from condition (viii). Assume that there exists the first $t'' > \tilde{t}+1 : x(t'') = p_m$. According to assumption of the lemma we know that $x(t''-1) < p_m$. We have:

$$x(t'') = e^{-a}x(t''-1) + \int_{t''-1}^{t''} e^{-a(t''-s)} f(x(s-1))ds \le e^{-a}x(t''-1) + \frac{1-e^{-a}}{a} f(p_m).$$

It follows that $x(t''-1) \ge \big(p_m - \frac{1-e^{-a}}{a} f(p_m)\big)e^{a} > p_{**}$ according to condition (viii). So, we have that $\dot{x}(t'') = -ap_m + f(x(t''-1)) > -ap_m + f(p_{**}) \ge 0$. $\square$

Assume that functions f_α from a subset of Γ are parametrised by some parameter $\alpha \in A \subset \mathbb{R}$, so we have $\Gamma \supset \{f_\alpha\}_{\alpha\in A}$. Also we assume that for all $R > 0$, $\alpha \mapsto f_\alpha$ is continuous with respect to $||\cdot||_{C^1([0,R],\mathbb{R})}$. Note that p_0^α, p_1^α, p_m^α depend continuously on α as follows from conditions (vi), (vii) and (iii) for Γ. We denote $W^u_{\mathrm{loc},f_\alpha}(\Phi_\alpha(\cdot,\cdot),p_0^\alpha)$ by $W^u_{\mathrm{loc},\alpha}$.

Assume that for all $f_\alpha \in \Gamma$, in addition to conditions (i)-(x), the following conditions are satisfied:

(xi) there exist $\alpha_1, \alpha_0 \in A : \alpha_0 < \alpha_1$; constants $c_2^\alpha \ge c_1^\alpha \ge 0$ for all $\alpha \in [\alpha_0, \alpha_1]$; $\delta_2^\alpha \ge 0$ for all $\alpha \in [\alpha_0,\alpha_1]$ with the following properties:

1) for solutions with $x_0^\alpha = \eta^\alpha$, $\eta^\alpha \in W^u_{\mathrm{loc},\alpha}$, $\eta^\alpha >> p_0^\alpha$ we have: $x^\alpha(t) \ge p_1^\alpha + \delta_2^\alpha$ on an interval $[\tau_1^\alpha, \tau_2^\alpha]$ with $\tau_2^\alpha \ge \tau_1^\alpha + 1$, $x^\alpha(\tau_2^\alpha) = p_1^\alpha + \delta_2^\alpha$, $\alpha \in [\alpha_0,\alpha_1]$;

2) $f_\alpha(x) \in [c_1^\alpha, c_2^\alpha]$ for $x \ge p_1^\alpha + \delta_2^\alpha$ for all $\alpha \in [\alpha_0,\alpha_1]$;

3) $\dfrac{c_2^{\alpha_0}}{a} + \left(p_1^{\alpha_0} + \delta_2^{\alpha_0} - \dfrac{c_2^{\alpha_0}}{a}\right)e^{-a} + f_{\alpha_0}(p_m^{\alpha_0})\dfrac{\ln(p_1^{\alpha_0} + \delta_2^{\alpha_0} - \frac{c_2^{\alpha_0}}{a}) - \ln(p_0^{\alpha_0} - \frac{c_2^{\alpha_0}}{a})}{a} < p_0^{\alpha_0}$;

4) $p_m^\alpha - \dfrac{c_2^\alpha}{a} > 0$ for all $\alpha \in [\alpha_0,\alpha_1]$, $p_0^{\alpha_0} - \dfrac{c_2^{\alpha_0}}{a} > 0$;

5) $c_1^{\alpha_1} > \dfrac{a\big(p_0^{\alpha_1} - (p_1^{\alpha_1} + \delta_2^{\alpha_1})e^{-a}\big)}{1-e^{-a}}$;

6) $\left((p_1^{\alpha_1} + \delta_2^{\alpha_1} - \dfrac{c_1^{\alpha_1}}{a})e^{-a} + \dfrac{c_1^{\alpha_1}}{a}\right)\dfrac{p_1^{\alpha_1} - \dfrac{c_2^{\alpha_1}}{a}}{p_1^{\alpha_1} + \delta_2^{\alpha_1} - \dfrac{c_2^{\alpha_1}}{a}} > p_0^{\alpha_1}$;

7) $\dfrac{c_2^\alpha}{a} + (p_1^\alpha + \delta_2^\alpha - \dfrac{c_2^\alpha}{a})e^{-a} + f_\alpha(\dfrac{p_1^\alpha + p_m^\alpha}{2})\dfrac{\ln(p_1^\alpha + \delta_2^\alpha - \frac{c_2^\alpha}{a}) - \ln(\frac{p_1^\alpha + p_m^\alpha}{2} - \frac{c_2^\alpha}{a})}{a} < p_m^\alpha$ for all $\alpha \in [\alpha_0,\alpha_1]$;

8) $\dfrac{\dfrac{p_1^\alpha + p_m^\alpha}{2} - \dfrac{c_1^\alpha}{a}}{p_1^\alpha + \delta_2^\alpha - \dfrac{c_1^\alpha}{a}}\left(\dfrac{c_2^\alpha}{a} + (p_1^\alpha + \delta_2^\alpha - \dfrac{c_2^\alpha}{a})e^{-a}\right) + f_\alpha(p_m^\alpha)\dfrac{p_1^\alpha + \delta_2^\alpha - p_m^\alpha}{a(p_1^\alpha + \delta_2^\alpha - \frac{c_2^\alpha}{a})} < p_m^\alpha$ for all $\alpha \in [\alpha_0,\alpha_1]$;

9) $c_2^\alpha < c_1^\alpha + a\left(p_1^\alpha + \delta_2^\alpha - \dfrac{c_1^\alpha}{a}\right)e^{-a}$ for all $\alpha \in [\alpha_0, \alpha_1]$.
An example of functions satisfying conditions (i)-(xi) will be given in section 4.2.

Remark Let us fix $\hat{\delta}$ such that there exist $W_\alpha^s(\Phi_\alpha, p_0^\alpha, \hat{\delta})$ and $W_\alpha^u(\Phi_\alpha, p_0^\alpha, \hat{\delta})$ for $\alpha \in [\alpha_0, \alpha_1]$. Denote by $\eta^\alpha(\theta) = p_0^\alpha + \dfrac{\hat{\delta}}{2}e^{\lambda_0^\alpha \theta} + \omega_{\Phi^\alpha}^\alpha(\dfrac{\hat{\delta}}{2}e^{\lambda_0^\alpha \theta})$ the initial value of solutions under consideration. Notice that, as in [11], $\hat{\delta}$ can be chosen uniformly for α in the compact interval $[\alpha_0, \alpha_1]$.

Lemma 14. *For all $\alpha \in [\alpha_0, \alpha_1]$, $f_\alpha \in \Gamma$, the solution with $x_0^\alpha = \eta^\alpha$, $\eta^\alpha \in W_{loc,\alpha}^u$, $\eta^\alpha >> p_0^\alpha$, has the following properties: $\dot{x}^\alpha(t) < 0$ for $t \in [\tau_2^\alpha, \tau_2^\alpha + 1]$ and $x^\alpha(\tau_2^\alpha) > p_m^\alpha > x^\alpha(\tau_2^\alpha + 1)$.*

Proof. For $t \in [\tau_2^\alpha, \tau_2^\alpha + 1]$ we obtain using (xi,2): $-ax^\alpha(t) + c_1^\alpha \le \dot{x}^\alpha(t) \le -ax^\alpha(t) + c_2^\alpha$. It follows that

$$(p_1^\alpha + \delta_2^\alpha - \frac{c_1^\alpha}{a})e^{-a(t-\tau_2^\alpha)} + \frac{c_1^\alpha}{a} \le x^\alpha(t) \le \frac{c_2^\alpha}{a} + (p_1^\alpha + \delta_2^\alpha - \frac{c_2^\alpha}{a})e^{-a(t-\tau_2^\alpha)} \tag{2.11}$$

for $t \in [\tau_2^\alpha, \tau_2^\alpha + 1]$. Notice that $\min_{t\in[\tau_2^\alpha, \tau_2^\alpha+1]} x(t) \ge (p_1 + \delta_2 - \dfrac{c_1^\alpha}{a})e^{-a} + \dfrac{c_1^\alpha}{a}$. So, we have that for $t \in [\tau_2^\alpha, \tau_2^\alpha + 1]$: $\dot{x}^\alpha(t) \le c_2^\alpha - c_1^\alpha - a\left(p_1^\alpha + \delta_2^\alpha - \dfrac{c_1^\alpha}{a}\right)e^{-a} < 0$ according to condition (xi,9). Using (2.11), we obtain that

$$(p_1^\alpha + \delta_2^\alpha - \frac{c_1^\alpha}{a})e^{-a} + \frac{c_1^\alpha}{a} \le x^\alpha(\tau_2^\alpha + 1) \le \frac{c_2^\alpha}{a} + (p_1^\alpha + \delta_2^\alpha - \frac{c_2^\alpha}{a})e^{-a} < p_m^\alpha \tag{2.12}$$

according to condition (xi,7). It follows that for $t \in [\tau_2^\alpha, \tau_2^\alpha + 1]$: $\dot{x}^\alpha(t) < 0$ and
$x^\alpha(\tau_2^\alpha) > p_m^\alpha > x^\alpha(\tau_2^\alpha + 1)$. □

Corollary 2. *For all $\alpha \in [\alpha_0, \alpha_1]$, $f_\alpha \in \Gamma$, the solution with $x_0^\alpha = \eta^\alpha$, $\eta^\alpha \in W_{loc,\alpha}^u$, $\eta^\alpha >> p_0^\alpha$, has the property: there exist unique time moments $\tau_m^\alpha \in (\tau_2^\alpha, \tau_2^\alpha + 1), t'^\alpha \in (\tau_2^\alpha, \tau_2^\alpha + 1), t_2^\alpha \in (\tau_2^\alpha, \tau_2^\alpha + 1)$ such that $x^\alpha(\tau_m^\alpha) = p_m^\alpha$, $x^\alpha(t'^\alpha) = \dfrac{p_1^\alpha + p_m^\alpha}{2}$, $x^\alpha(t_2^\alpha) = p_1^\alpha$. According to (2.11) we have:*
$\dfrac{c_1^\alpha}{a} + (p_1^\alpha + \delta_2^\alpha - \dfrac{c_1^\alpha}{a})e^{-a(\tau_m^\alpha - \tau_2^\alpha)} \le x^\alpha(\tau_m^\alpha) \le \dfrac{c_2^\alpha}{a} + (p_1^\alpha + \delta_2^\alpha - \dfrac{c_2^\alpha}{a})e^{-a(\tau_m^\alpha - \tau_2^\alpha)}$ *and*

$$\frac{\ln(p_1^\alpha + \delta_2^\alpha - \frac{c_1^\alpha}{a}) - \ln(p_m^\alpha - \frac{c_1^\alpha}{a})}{a} \le \tau_m^\alpha - \tau_2^\alpha \le \frac{\ln(p_1^\alpha + \delta_2^\alpha - \frac{c_2^\alpha}{a}) - \ln(p_m^\alpha - \frac{c_2^\alpha}{a})}{a}. \tag{2.13}$$

Analogously one can obtain using (2.11)

$$\frac{\ln(p_1^\alpha + \delta_2^\alpha - \frac{c_1^\alpha}{a}) - \ln(\frac{p_1^\alpha + p_m^\alpha}{2} - \frac{c_1^\alpha}{a})}{a} \le t'^\alpha - \tau_2^\alpha \le \frac{\ln(p_1^\alpha + \delta_2^\alpha - \frac{c_2^\alpha}{a}) - \ln(\frac{p_1^\alpha + p_m^\alpha}{2} - \frac{c_2^\alpha}{a})}{a} \tag{2.14}$$

and

$$\frac{\ln(p_1^\alpha + \delta_2^\alpha - \frac{c_1^\alpha}{a}) - \ln(p_1^\alpha - \frac{c_1^\alpha}{a})}{a} \leq t_2^\alpha - \tau_2^\alpha \leq \frac{\ln(p_1^\alpha + \delta_2^\alpha - \frac{c_2^\alpha}{a}) - \ln(p_1^\alpha - \frac{c_2^\alpha}{a})}{a}. \tag{2.15}$$

Notice that $\ln(p_1^\alpha + \delta_2^\alpha - \frac{c_1^\alpha}{a})$*,* $\ln(p_m^\alpha - \frac{c_1^\alpha}{a})$*,* $\ln(p_1^\alpha - \frac{c_2^\alpha}{a})$*,* $\ln(p_1^\alpha + \delta_2^\alpha - \frac{c_2^\alpha}{a})$*,* $\ln(p_m^\alpha - \frac{c_2^\alpha}{a})$ *are well defined according to condition (xi,4).*

Lemma 15. *For all* $\alpha \in [\alpha_0, \alpha_1]$*,* $f_\alpha \in \Gamma$*, the solution with* $x_0^\alpha = \eta^\alpha$*,* $\eta^\alpha \in W^u_{loc,\alpha}$*,* $\eta^\alpha >> p_0^\alpha$*, has the following property:* $x^\alpha(t) < p_m^\alpha$ *for* $t \in (\tau_m^\alpha, \tau_m^\alpha + 1]$*, where* $\tau_m^\alpha : x^\alpha(\tau_m^\alpha) = p_m^\alpha$*,* $\tau_2^\alpha < \tau_m^\alpha < \tau_2^\alpha + 1$*.*

Proof. Let us omit the index α. According to condition (xi,1) we have $x(t) \geq p_1 + \delta_2$ for $t \in [\tau_1, \tau_2]$. According to corollary 2 we know that there exist unique $\tau_m \in (\tau_2, \tau_2 + 1)$, $t' \in (\tau_2, \tau_2 + 1)$: $x(\tau_m) = p_m$, $x(t') = \frac{p_1 + p_m}{2}$. According to lemma 14 we have that $x(t) < p_m$ for $t \in (\tau_m, \tau_2 + 1]$ since $\dot{x}(t) < 0$ for $t \in [\tau_2, \tau_2 + 1]$. Let us estimate the solution for $t \in (\tau_2 + 1, \tau_m + 1]$. Let us consider the following decomposition $[\tau_2 + 1, \tau_m + 1] = [\tau_2 + 1, t' + 1] \cup [t' + 1, \tau_m + 1]$.

For $t \in [\tau_2 + 1, t' + 1]$ we obtain, using (2.12), (2.14):

$$\begin{aligned}
x(t) &= e^{-a(t-\tau_2-1)}x(\tau_2 + 1) + \int_{\tau_2+1}^{t} e^{-a(t-s)} f(x(s-1))ds \leq \\
&\leq x(\tau_2 + 1) + f(\frac{p_1 + p_m}{2})(t - \tau_2 - 1) \leq \\
&\leq x(\tau_2 + 1) + f(\frac{p_1 + p_m}{2})(t' - \tau_2) \leq \\
&\leq \frac{c_2}{a} + \left(p_1 + \delta_2 - \frac{c_2}{a}\right) e^{-a} + f(\frac{p_1 + p_m}{2}) \frac{\ln\left(p_1 + \delta_2 - \frac{c_2}{a}\right) - \ln\left(\frac{p_1 + p_m}{2} - \frac{c_2}{a}\right)}{a} < \\
&< p_m
\end{aligned}$$

according to condition (xi,7).

For $t \in (t' + 1, \tau_m + 1]$ we have, using (2.12), (2.13), (2.14):

$$\begin{aligned}
x(t) &= e^{-a(t-\tau_2-1)}x(\tau_2 + 1) + \int_{\tau_2+1}^{t} e^{-a(t-s)} f(x(s-1))ds \leq \\
&\leq e^{-a(t'-\tau_2)}x(\tau_2 + 1) + f(p_m)\frac{1 - e^{-a(\tau_m - \tau_2)}}{a} \leq \\
&\leq \frac{\frac{p_1 + p_m}{2} - \frac{c_1}{a}}{p_1 + \delta_2 - \frac{c_1}{a}} \left(\frac{c_2}{a} + (p_1 + \delta_2 - \frac{c_2}{a})e^{-a}\right) + f(p_m)\frac{p_1 + \delta_2 - p_m}{a(p_1 + \delta_2 - \frac{c_2}{a})} < \\
&< p_m
\end{aligned}$$

according to condition (xi,8).

So, we know that $x^{\alpha}(t) < p_m^{\alpha}$ for $t \in (\tau_m^{\alpha}, \tau_m^{\alpha}+1]$, $\alpha_0 \le \alpha \le \alpha_1$. $\square$

Lemma 16. *For $f_{\alpha_0} \in \Gamma$ the solution with $x_0^{\alpha_0} = \eta^{\alpha_0}$, $\eta^{\alpha_0} \in W^u_{loc,\alpha_0}$, $\eta^{\alpha_0} >> p_0^{\alpha_0}$, has the property: there exists some $t_0^{\alpha_0} \in (\tau_2^{\alpha_0}, \tau_2^{\alpha_0}+1) : x^{\alpha_0}(t_0^{\alpha_0}) = p_0^{\alpha_0}$ and $x^{\alpha_0}(t) < p_0^{\alpha_0}$ for $t \in (t_0^{\alpha_0}, t_0^{\alpha_0}+1]$.*

Proof. Let us omit the index α_0. Figure 2.2 shows the approximate shape of the solution for $\alpha = \alpha_0$.

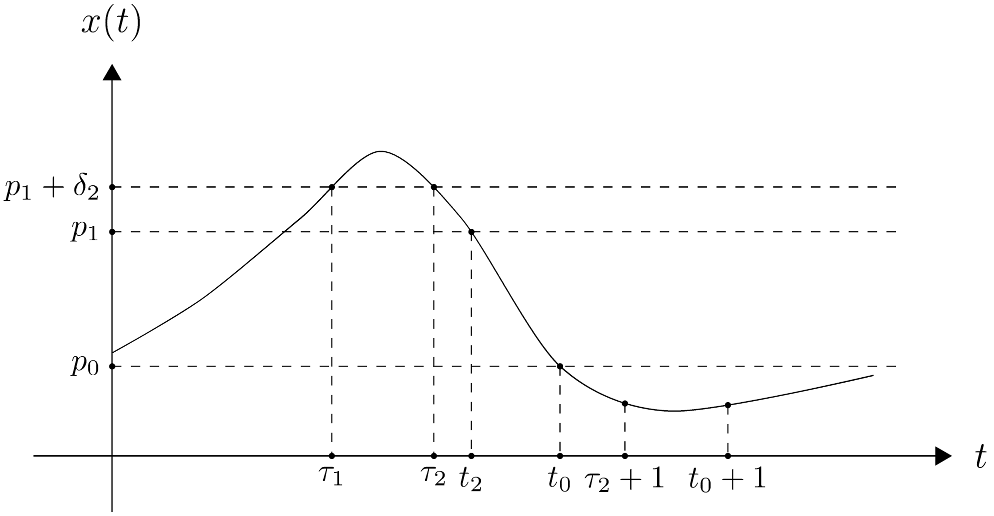

Figure 2.2: Approximate shape of the solution for f_{α_0}

According to condition (xi,1) we have $x(t) \ge p_1 + \delta_2$ for $t \in [\tau_1, \tau_2]$. For $t \in [\tau_2, \tau_2+1]$ we obtain using (xi,2): $-ax(t) \le \dot{x}(t) \le -ax(t) + c_2$. We know that $\dot{x}(t) < 0$ for $t \in [\tau_2, \tau_2+1]$ according to lemma 14. Using (2.12) we obtain $x(\tau_2+1) \le \frac{c_2}{a} + (p_1 + \delta_2 - \frac{c_2}{a})e^{-a} < p_0$ according to condition (xi,3). It follows that there exists a unique $t_0 \in (\tau_2, \tau_2+1) : x(t_0) = p_0$.

From (2.11) we conclude: $t_0 \le \frac{\ln(p_1 + \delta_2 - \frac{c_2}{a}) - \ln(p_0 - \frac{c_2}{a}) + a\tau_2}{a}$. Notice that $\ln(p_0 - \frac{c_2}{a})$ is well defined according to condition (xi,4). According to lemma 14 we have that $x(t) < p_0$ for $t \in (t_0, \tau_2+1]$ since $\dot{x}(t) < 0$ for $t \in [\tau_2, \tau_2+1]$. For $t \in (\tau_2+1, t_0+1]$ we obtain, using (2.12):

$$\begin{aligned} x(t) &= e^{-a(t-\tau_2-1)}x(\tau_2+1) + \int_{\tau_2+1}^{t} e^{-a(t-s)} f(x(s-1))ds \le \\ &\le \frac{c_2}{a} + (p_1+\delta_2-\frac{c_2}{a})e^{-a} + f(p_m)(t_0+1-\tau_2-1) \le \\ &\le \frac{c_2}{a} + (p_1+\delta_2-\frac{c_2}{a})e^{-a} + f(p_m)\frac{\ln(p_1+\delta_2-\frac{c_2}{a}) - \ln(p_0-\frac{c_2}{a})}{a} < \\ &< p_0 \end{aligned}$$

according to condition (xi,3). So, $x(t) < p_0$ for $t \in (t_0, t_0+1]$. $\square$

Lemma 17. *For $f_{\alpha_1} \in \Gamma$ the solution with $x_0^{\alpha_1} = \eta^{\alpha_1}$, $\eta^{\alpha_1} \in W^u_{loc,\alpha_1}$, $\eta^{\alpha_1} >> p_0^{\alpha_1}$, has the property: $x^{\alpha_1}(t) \in (p_0^{\alpha_1}, p_m^{\alpha_1})$ for $t \in (\tau_m^{\alpha_1}, \tau_m^{\alpha_1}+1]$, where $\tau_m^{\alpha_1} : x^{\alpha_1}(\tau_m^{\alpha_1}) = p_m^{\alpha_1}$, $\tau_2^{\alpha_1} < \tau_m^{\alpha_1} < \tau_2^{\alpha_1}+1$.*

Proof. Let us omit the index α_1. Figure 2.3 shows the approximate shape of the solution.

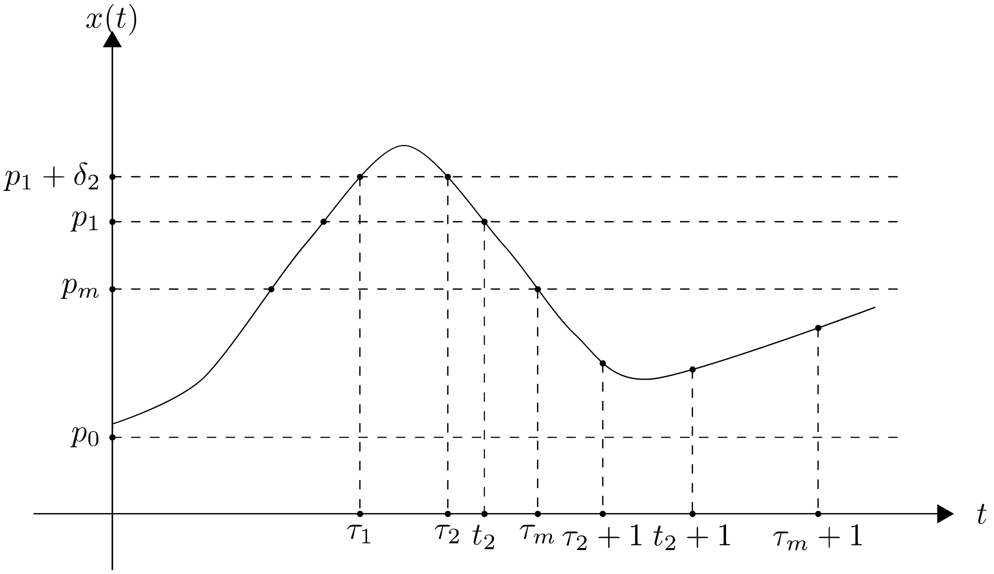

Figure 2.3: Approximate shape of the solution for f_{α_1}

According to lemma 15 we know that $x(t) < p_m$ for $t \in (\tau_m, \tau_m+1]$ with $\tau_m \in (\tau_2, \tau_2+1)$, and $\dot{x}(t) < 0$ for $t \in [\tau_2, \tau_2+1]$ according to lemma 14. It follows that $ax(t) < ap_m$ for $t \in (\tau_m, \tau_m+1]$. For $t \in [\tau_2+1, t_2+1]$ with $t_2 \in (\tau_2, \tau_m)$, $x(t_2) = p_1$, we obtain using (2.12) and (2.15):

$$x(t) \geq x(\tau_2+1)e^{-a(t-\tau_2-1)} \geq x(\tau_2+1)e^{-a(t_2-\tau_2)} \geq$$

$$\geq \left((p_1+\delta_2-\frac{c_1}{a})e^{-a}+\frac{c_1}{a}\right)\frac{p_1-\dfrac{c_2}{a}}{p_1+\delta_2-\dfrac{c_2}{a}} >$$

$$> p_0$$

according to condition (xi,6).

For $t \in [t_2+1, \tau_m+1]$ we have: $\dot{x}(t) = -ax(t)+f(x(t-1)) > -ap_m+ap_1 > 0$ since $x(t-1) \in [p_m, p_1]$ and $f(x(t-1)) \geq f(p_1)$. It follows that $p_0 < x(t) < p_m$ for $t \in (\tau_m, \tau_m+1]$. □

2.4 Critical value of parameter

Consider equation (2.6) with $f_\alpha \in \Gamma$ and solutions $x^\alpha(t)$ with $x_0^\alpha = \eta^\alpha$, $\eta^\alpha \in W^u_{\text{loc},\alpha}$, $\eta^\alpha >> p_0^\alpha$. Let us consider the set of parameters $A := [\alpha_0, \alpha_1]$. It was proved in lemma 15 that for all $\alpha_0 \leq \alpha \leq \alpha_1$

the solution $x^\alpha(t) < p_m^\alpha$ for $t \in (\tau_m^\alpha, \tau_m^\alpha + 1]$. We will introduce the following sets:

$$A^1 := \{\alpha \in A \mid \exists t_*^\alpha > \tau_m^\alpha + 1 : x^\alpha(t_*^\alpha) \geq p_m^\alpha\},$$
$$A^2 := \{\alpha \in A \mid x^\alpha(t) < p_m^\alpha \text{ for all } t > \tau_m^\alpha,\ \exists T^\alpha > \tau_m^\alpha + 1 : x_{T^\alpha}^\alpha < p_0^\alpha\}.$$

The set A^1 is an open set in $[\alpha_0, \alpha_1]$ since solutions $x^\alpha(t)$ can cross the level p_m^α only transversally for the first time $t_*^\alpha > \tau_m^\alpha + 1$ due to lemma 13 and since solutions depend continuously on α.

Lemma 18. $\alpha_1 \in A^1$.

Proof. It was proved in lemma 17 that $x^{\alpha_1}(t) \in (p_0^{\alpha_1}, p_m^{\alpha_1})$ for $t \in (\tau_m^{\alpha_1}, \tau_m^{\alpha_1} + 1]$. According to corollary 1 $x^{\alpha_1}(t)$ intersects the level $p_m^{\alpha_1}$ for some time moment $t_*^\alpha > \tau_m^{\alpha_1} + 1$. It follows that $\alpha_1 \in A^1$ by the definition of A^1. $\square$

Lemma 19. $\alpha_0 \in A^2$.

Proof. According to lemma 16, $x^{\alpha_0}(t) < p_0^{\alpha_0}$ for $t \in (t_0^{\alpha_0}, t_0^{\alpha_0} + 1]$ with $t_0^{\alpha_0} \in (\tau_2^{\alpha_0}, \tau_2^{\alpha_0} + 1)$, and, hence, $x^{\alpha_0}(t) < p_0^{\alpha_0}$ for $t > t_0^{\alpha_0}$ according to lemma 12. We know that $\dot{x}^{\alpha_0}(t) < 0$ for $t \in [\tau_2^{\alpha_0}, \tau_2^{\alpha_0} + 1]$ according to lemma 14. It follows that $x^{\alpha_0}(t) < p_m^{\alpha_0}$ for $t > \tau_m^{\alpha_0}$. So, we have that $\alpha_0 \in A^2$ by the definition of A^2. $\square$

Lemma 20. *The set A^2 is open.*

Proof. Fix some $\alpha_2 \in A^2$. We know that $\dot{x}^{\alpha_2}(\tau_m^{\alpha_2}) < 0$. Choose $\delta > 0$ such that $\dot{x}^{\alpha_2}(t) < 0$ for $t \in [\tau_m^{\alpha_2} - \delta, \tau_m^{\alpha_2} + \delta]$. Fix $\delta' = \dfrac{\min_{t \in [\tau_m^{\alpha_2} - \delta, \tau_m^{\alpha_2} + \delta]} |\dot{x}^{\alpha_2}(t)|}{2}$. We obtain that $x^{\alpha_2}(\tau_m^{\alpha_2} + \delta) < p_m^{\alpha_2} - \delta'\delta$ and $x^{\alpha_2}(\tau_m^{\alpha_2} - \delta) > p_m^{\alpha_2} + \delta'\delta$. Fix $\delta'' = \dfrac{\delta'\delta}{2}$. We know that for $\alpha = \alpha_2$ there exists $T^{\alpha_2} > \tau_m^{\alpha_2} + 1 : x_{T^{\alpha_2}}^{\alpha_2} < p_0^{\alpha_2}$. For $t \in [\tau_m^{\alpha_2} + \delta, T^{\alpha_2}]$ we have: $x^{\alpha_2}(t) < p_m^{\alpha_2}$ since $\alpha_2 \in A^2$. It follows that $\max_{t \in [\tau_m^{\alpha_2} + \delta, T^{\alpha_2}]} x^{\alpha_2}(t) < p_m^{\alpha_2}$. For $t \in [T^{\alpha_2} - 1, \infty)$ we have $x^{\alpha_2}(t) < p_0^{\alpha_2} < p_m^{\alpha_2}$. We have continuous dependence of solutions on parameter α, and neighbouring solutions are close in C^1 on $[0, r]$ for every $r > 0$. It follows that there exists $\epsilon > 0$ such that if $|\alpha - \alpha_2| < \epsilon$ then:

1) $||x^\alpha - x^{\alpha_2}||_{C^1([\tau_m^{\alpha_2} - \delta, \tau_m^{\alpha_2} + \delta])} < \delta'$,

2) $||x^\alpha - x^{\alpha_2}||_{C^1([\tau_m^{\alpha_2} - \delta, \tau_m^{\alpha_2} + \delta])} < \delta''$,

3) $\max_{t \in [\tau_m^{\alpha_2} + \delta, T^{\alpha_2}]} x^\alpha(t) < p_m^\alpha$,

4) there exists $T^\alpha > \tau_m^\alpha + 1 : x_{T^\alpha}^\alpha < p_0^\alpha$ with $|T^\alpha - T^{\alpha_2}| < \dfrac{1}{2}$,

5) $x^\alpha(t) < p_m^\alpha$ for $t \in [T^{\alpha_2} - 1, T^\alpha]$,

6) $||p_m^{\alpha_2} - p_m^{\alpha}|| < \delta''$.

Notice that from conditions 4) and 5) we obtain: 7) $x^{\alpha}(t) < p_m^{\alpha}$ for $t \in [T^{\alpha_2} - 1, \infty)$ according to lemma 12.

According to property 1) it follows that $\dot{x}^{\alpha}(t) < \delta' + \dot{x}^{\alpha_2}(t) \leq -\delta' < 0$ for $t \in [\tau_m^{\alpha_2} - \delta, \tau_m^{\alpha_2} + \delta]$. Due to property 2) and 6) we have that $x^{\alpha}(\tau_m^{\alpha_2} + \delta) < x^{\alpha_2}(\tau_m^{\alpha_2} + \delta) + \delta'' < p_m^{\alpha_2} - \delta'' < p_m^{\alpha}$ and $x^{\alpha}(\tau_m^{\alpha_2} - \delta) > x^{\alpha_2}(\tau_m^{\alpha_2} - \delta) - \delta'' > p_m^{\alpha_2} + \delta'' > p_m^{\alpha}$. So, there exists a unique τ_m^{α} as it was shown in corollary 2 such that $\tau_m^{\alpha} \in (\tau_m^{\alpha_2} - \delta, \tau_m^{\alpha_2} + \delta) : x^{\alpha}(\tau_m^{\alpha}) = p_m^{\alpha}$ and $x^{\alpha}(t) < p_m^{\alpha}$ for $t \in (\tau_m^{\alpha}, \tau_m^{\alpha_2} + \delta]$. Due to property 3) we have that $x^{\alpha}(t) < p_m^{\alpha}$ for $t \in [\tau_m^{\alpha_2} + \delta, T^{\alpha_2}]$. Due to property 4) there exists $T^{\alpha} : x_{T^{\alpha}}^{\alpha} < p_0^{\alpha}$ and according to property 7) it follows that $x^{\alpha}(t) < p_m^{\alpha}$ for $t \in (\tau_m^{\alpha}, \infty)$. So, we have that if $|\alpha - \alpha_2| < \epsilon$ then $\alpha \in A^2$. $\square$

It is clear that the sets A^1 and A^2 are disjoint. So, from connectedness of A we obtain the following decomposition $A = A^1 \cup A^2 \cup A^*$, where $A^* := \{\alpha \in A \mid x^{\alpha}(t) < p_m^{\alpha}$ for all $t > \tau_m^{\alpha}$, $\nexists T > \tau_m^{\alpha} + 1 : x_T^{\alpha} < p_0^{\alpha}\}$ is not empty.

Definition 1. *We say that solution $x^{\alpha} : \mathbb{R} \to \mathbb{R}$ of equation* (2.6) *with $f_{\alpha} \in \Gamma$ oscillates rapidly about p_0^{α} for $t \geq t_3$, where t_3 is some time moment $\Leftrightarrow \nexists T \geq t_3 + 1 : x_T^{\alpha} < p_0^{\alpha}$ and $\nexists T' \geq t_3 + 1 : x_{T'}^{\alpha} > p_0^{\alpha}$.*

Lemma 21. *If a solution of equation* (2.6) *oscillates rapidly about p_0^{α} for $t \geq t_3$, $t_3 \in \mathbb{R}$, and $x^{\alpha}(t) < p_m^{\alpha}$ for $t \in [t_3, t_3 + 1]$, then $x^{\alpha}(t) < p_m$ for all $t \geq t_3$.*

Proof. Let us omit the index α. Consider $z_0 > t_3$: $|z_0 - t_3| \leq 1$ and $x(z_0) = p_0$. Such z_0 exists according to definition 1. We have $\dot{x}(t) \leq -ax(t) + f(p_m)$ for $t \in [z_0, z_0 + 1]$. It follows that $x(t) \leq \frac{f(p_m)}{a} + (p_0 - \frac{f(p_m)}{a})e^{-a(t-z_0)}$ for $t \in [z_0, z_0 + 1]$. Notice that $p_0 - \frac{f(p_m)}{a} < 0$ and $\max_{t \in [z_0, z_0+1]} x(t) \leq \frac{f(p_m)}{a} + (p_0 - \frac{f(p_m)}{a})e^{-a} = e^{-a}p_0 + f(p_m)\frac{1 - e^{-a}}{a} < e^{-a}p_{**} + f(p_m)\frac{1 - e^{-a}}{a} < p_m$ according to condition (viii). One can conclude that $x(t) < p_m$ for $t \geq t_3$. $\square$

Lemma 22. *For all $\alpha \in A^*$ the solution of equation* (2.6) *with $x_0^{\alpha} = \eta^{\alpha}$, $\eta^{\alpha} \in W_{loc,\alpha}^u$, $\eta^{\alpha} >> p_0$, oscillates rapidly about p_0^{α} for $t \geq \tau_m^{\alpha}$, in particular $\nexists T \geq \tau_m^{\alpha} + 1 : x_T^{\alpha} < p_0^{\alpha}$ and $\nexists T' \geq \tau_m^{\alpha} + 1 : x_{T'}^{\alpha} > p_0^{\alpha}$, and $x^{\alpha}(t) < p_m^{\alpha}$ for $t > \tau_m^{\alpha}$.*

Proof. Let us consider some $\alpha_* \in A^*$. We have that $\alpha_* \notin A^1$, it follows that $\nexists t_*^{\alpha_*} > \tau_m^{\alpha_*} + 1 : x^{\alpha_*}(t_*^{\alpha_*}) \geq p_m^{\alpha_*}$. Also we know that $x^{\alpha_*}(t) < p_m^{\alpha_*}$ for $t \in (\tau_m^{\alpha_*}, \tau_m^{\alpha_*} + 1]$ according to lemma 15. It follows that $x^{\alpha_*}(t) < p_m^{\alpha_*}$ for all $t > \tau_m^{\alpha_*}$.

It is clear that $\nexists T' : x^{\alpha_*}_{T'} > p^{\alpha_*}_0$ with $T' \geq \tau^{\alpha_*}_m + 1$ otherwise there exists $t^{\alpha_*}_* > \tau^{\alpha_*}_m + 1 : x^{\alpha_*}(t^{\alpha_*}_*) \geq p^{\alpha_*}_m$ according to corollary 1. Also, it is clear that $\nexists T \geq \tau^{\alpha}_m + 1 : x^{\alpha_*}_T < p^{\alpha_*}_0$ since $\alpha_* \notin A^2$. Hence, x^{α_*} has to oscillate rapidly about $p^{\alpha_*}_0$. $\square$

Let us consider the following auxiliary equation

$$\dot{x}(t) = -ax(t) + b(t)x(t-1) \tag{2.16}$$

with $a > 0$, $\dfrac{\sup_{t\geq 0} b(t)}{a}(1 - e^{-a}) < 1$, $b(t) \geq 0$ for $t \geq 0$, b is continuous.

Lemma 23. *Assume that equation* (2.16) *has rapidly oscillating solution* $x : \mathbb{R} \to \mathbb{R}$ *about* 0 *for* $t \geq 0$*. Then* $x(t) \to 0$ *as* $t \to +\infty$*.*

Proof. Assume that $x(0) = 0$ without loss of generality. Then we can estimate for $t \in [0, 1]$:

$$\begin{aligned}|x(t)| &= |\int_0^t e^{-a(t-v)} b(v)x(v-1)dv| \leq \\ &\leq \big(\sup_{t\geq 0} b(t)\big)||x|_{[-1,0]}||_\infty \int_0^1 e^{-a(t-v)}dv = \\ &= \big(\sup_{t\geq 0} b(t)\big)||x|_{[-1,0]}||_\infty \frac{1-e^{-at}}{a} \leq \\ &\leq ||x|_{[-1,0]}||_\infty q\end{aligned}$$

with $q = \dfrac{\sup_{t\geq 0} b(t)}{a}(1 - e^{-a})$. Let us consider $Z := \{z \in \mathbb{R}, z \geq 0 : x(z) = 0\}$. We know that for all $z \in Z$ there exists $z' > z$, $z' \in Z : |z - z'| \leq 1$ according to the definition of rapidly oscillating solution. Let us fix $z_0 := \max\{z \in Z | z \in (0, 1]\}$ and consider the following sequence $\{z_k\}$, $k \in \mathbb{N}$, such that $z_{k+1} := \max\{z \in Z | z \in (z_k, z_k + 1]\}$. It is clear that $|z_{k+1} - z_k| \leq 1$, $|z_{k+2} - z_k| > 1$, $z_k \to \infty$ as $k \to \infty$. Notice that $\bigcup_{k=0}^{\infty} [z_k, z_{k+1}] = [z_0, \infty)$ and $[z_0 - 1, z_0] \subset [-1, 1]$.
Note that $||x|_{[z_0-1,0]}||_\infty \leq ||x|_{[-1,0]}||_\infty$ and $||x|_{[0,z_0]}||_\infty \leq ||x|_{[0,1]}||_\infty \leq ||x|_{[-1,0]}||_\infty q < ||x|_{[-1,0]}||_\infty$ since $q < 1$. So, we can estimate solution for $t \in [z_0, z_0 + 1]$:

$$|x(t)| = |\int_{z_0}^t e^{-a(t-v)} b(v)x(v-1)dv| \leq q||x|_{[z_0-1,z_0]}||_\infty \leq ||x|_{[-1,0]}||_\infty q.$$

For $t \in [z_1, z_1 + 1]$ we have

$$|x(t)| = |\int_{z_1}^t e^{-a(t-v)} b(v)x(v-1)dv| \leq q||x|_{[z_1-1,z_1]}||_\infty \leq ||x|_{[-1,0]}||_\infty q^2,$$

since $||x|_{[z_1-1,z_1]}||_\infty \leq ||x|_{[0,z_0+1]}||_\infty \leq ||x|_{[-1,0]}||_\infty q$.

Let us introduce $\nu := ||x|_{[-1,0]}||_\infty$, $m_k := ||x|_{[z_k,z_k+1]}||_\infty$. So, we have $m_0 \leq \nu q$, $m_1 \leq \nu q^2$. Let us estimate for $k \geq 1$:

$$\begin{aligned} m_{k+1} &\leq q||x|_{[z_{k+1}-1,z_{k+1}]}||_\infty \leq \\ &\leq q \max\{||x|_{[z_{k+1}-1,z_k]}||_\infty, ||x|_{[z_k,z_k+1]}||_\infty\} \leq \\ &\leq q \max\{||x|_{[z_{k-1},z_k]}||_\infty, m_k\} \leq \\ &\leq q \max\{||x|_{[z_{k-1},z_{k-1}+1]}||_\infty, m_k\} = \\ &= q \max\{m_{k-1}, m_k\}. \end{aligned}$$

It follows that $m_2 \leq \nu q^2$, $m_3 \leq \nu q^3$, $m_4 \leq \nu q^3$... So, we obtain that $|x(t)| \to 0$ as $t \to +\infty$.

□

Lemma 24. *Assume that a solution of equation* (2.6) *with* $f_\alpha \in \Gamma$, $x_0^\alpha = \eta^\alpha$, $\eta^\alpha \in W^u_{loc,\alpha}$, $\eta^\alpha >> p_0$, *satisfies:* x^α *oscillates rapidly about* p_0^α *for* $t \geq t_3$, $t_3 \in \mathbb{R}$, *and* $x^\alpha(t) < p_m$ *for* $t \in [t_3, t_3+1]$. *Then* $x^\alpha(t) \to p_0^\alpha$ *as* $t \to +\infty$.

Proof. Notice that if a solution of equation (2.6) starts above 0, then it always remains positive since $f^\alpha(x) \geq 0$ for $x \in [0,\infty)$. Note that $x^\alpha(t) < p_m$ for $t \geq t_3$ according to lemma 21. Consider the following equivalent equation for $t > t_3 + 1$ instead of equation (2.6)

$$\dot{x}(t) = -ax(t) + f_\alpha(p_0^\alpha) + \int_0^1 f'_\alpha(p_0^\alpha + s(x(t-1) - p_0^\alpha))(x(t-1) - p_0^\alpha)ds. \tag{2.17}$$

We introduce $\tilde{x}^\alpha(t) = x^\alpha(t) - p_0^\alpha$, $b_\alpha(t) = \int_0^1 f'_\alpha(p_0^\alpha + s(x^\alpha(t-1) - p_0^\alpha))ds \leq \max_{x\in[0,p_m^\alpha]} f'_\alpha(x)$ for $t > t_3 + 1$ and rewrite equation (2.17) as

$$\dot{\tilde{x}}(t) = -a\tilde{x}(t) + b_\alpha(t)\tilde{x}(t-1).$$

We shift time and assume that $\tilde{x}^\alpha$ oscillates rapidly about 0 for $t \geq 0$. Notice that $\frac{\max_{x\in[0,p_m^\alpha]} f'_\alpha(x)}{a}(1 - e^{-a}) < 1$ according to condition (ix). Then one can use lemma 23 for $\tilde{x}^\alpha$ and obtain that $|\tilde{x}^\alpha(t)| \to 0$ as $t \to +\infty$. It follows that $x^\alpha(t) \to p_0^\alpha$ as $t \to +\infty$.

□

We have proved the result below.

Theorem 1. *Consider equation* (2.6) *with* $f_\alpha \in \Gamma$. *Under assumptions (i) - (xi), for* $\alpha \in A^*$, *the solution* $x^\alpha : \mathbb{R} \to \mathbb{R}$ *of equation* (2.6) *with* $x_0^\alpha = \eta^\alpha$, $\eta^\alpha \in W^u_{loc,\alpha}$, $\eta^\alpha >> p_0^\alpha$, *satisfies:* $x^\alpha(t) \to p_0^\alpha$ *as* $t \to \pm\infty$. *So, the solution* x^α *of equation* (2.6) *with* $\alpha \in A^*$ *is homoclinic. The "leading" stable eigenvalue* λ_1^α *and the unstable eigenvalue* λ_0^α *satisfy* $\lambda_0^\alpha < |Re(\lambda_1^\alpha)|$.

Chapter 3

Stable periodic solutions for a class of delay differential equations

3.1 Appropriate definition of the critical parameter

From the previous section we know that the set A^* is closed. Define α_* as the maximum of A^*.

Lemma 25. *If* $\alpha > \alpha_*$ *then* $\alpha \in A^1$.

Proof. Assume that there exists $\tilde{\alpha} > \alpha_*$ such that $\tilde{\alpha} \in A^2$. Let us consider the interval $[\tilde{\alpha}, \alpha_1]$. The intersections $A^1 \cap [\tilde{\alpha}, \alpha_1]$ and $A^2 \cap [\tilde{\alpha}, \alpha_1]$ are open with respect to $[\tilde{\alpha}, \alpha_1]$ and disjoint. So, the following decomposition is valid $[\tilde{\alpha}, \alpha_1] = (A^1 \cap [\tilde{\alpha}, \alpha_1]) \cup (A^2 \cap [\tilde{\alpha}, \alpha_1]) \cup \tilde{A}^*$ with $\tilde{A}^* \subset A^*$. It follows that there exists $\alpha' \in \tilde{A}^* \subset A^*$ such that $\alpha' > \alpha_*$, and we obtain a contradiction to the definition of α_*. $\square$

3.2 Invariant cones for maps in Banach space

Now we are going to discuss some properties of maps in Banach spaces. Let F be a C^1 map in a Banach space $(X, ||\cdot||_X)$, $F(0) = 0$, $DF(0)$ is hyperbolic so that $X = U \oplus S$ with the unstable space U and the stable space S of $DF(0)$. Denote $T := DF(0)$. The ideas of the following lemmas were taken from the paper [1].

Lemma 26. *There exists a norm* $||\cdot||$ *equivalent to* $||\cdot||_X$ *and* $\beta > 1$, $\gamma < 1$ *such that for all* $u \in U$: $||Tu|| \geq \beta ||u||$, *for all* $s \in S$: $||Ts|| \leq \gamma ||s||$, *and* $||u + s|| = \max\{||u||, ||s||\}$ *for* $u \in U$, $s \in S$.

Proof. According to [4] there exist constants $K > 1, a < 0, b > 0$ such that the following conditions are satisfied:

1) for all $u \in U$: $||T^n u||_X \leq K e^{bn} ||u||_X$, $n \leq 0$;

2) for all $s \in S$: $||T^n s||_X \leq K e^{an} ||s||_X$, $n \geq 0$.

For $s \in S$ we define a new norm $||s|| = \sup_{n \geq 0} e^{-an} ||T^n s||_X$. It is clear that $||s||_X \leq ||s||$ and $||s|| \leq K ||s||_X$, so the norms $|| \cdot ||$ and $|| \cdot ||_X$ are equivalent on S. Consider $||T^m s|| = \sup_{n \geq 0} e^{-an} ||T^n T^m s||_X = \sup_{n \geq m} e^{-an} ||T^n s||_X e^{am} \leq ||s|| e^{am}$. It follows that for $\gamma = e^a < 1$ we have $||Ts|| \leq \gamma ||s||$.

Let us consider $||T^n u||_X \leq K e^{bn} ||u||_X$ for $n \leq 0$. Denote by $v = (T|_U)^{-n} u$. We have $||T^n v||_X \leq K e^{bn} ||v||_X$, it follows that $||u||_X \leq K e^{bn} ||T^{-n} u||_X$, and hence $||T^{-n} u||_X \geq \dfrac{1}{K} e^{-bn} ||u||_X$. For $m = -n > 0$ we have $||T^m u||_X \geq \dfrac{1}{K} e^{bm} ||u||_X$.

For $u \in U$ we define a new norm $||u|| = \sup_{n \leq 0} e^{-bn} ||T^n u||_X$. It is clear that $||u|| \leq K ||u||_X$ and $||u||_X \leq ||u||$, so the norms $|| \cdot ||$ and $|| \cdot ||_X$ are equivalent on U.

Let us consider for $m \geq 0$

$||T^m u|| = \sup_{n \leq 0} e^{-bn} ||T^n T^m u||_X = \sup_{n \leq 0} e^{-b(n+m)} ||T^{n+m} u||_X e^{bm} = \sup_{n \leq m} e^{-bn} ||T^n u||_X e^{bm} \geq$ $\geq ||u|| e^{bm}$. For $\beta = e^b > 1$ we have $||Tu|| \geq \beta ||u||$. Define $||u + s|| = \max\{||u||, ||s||\}$ for $u \in U$, $s \in S$. Notice that

$$||u+s|| \leq \max\{||u||, ||s||\} \leq K \max\{||u||_X, ||s||_X\} \leq K \max\{||pr_U||_X, ||pr_S||_X\} ||u+s||_X = \\ = \tilde{K} ||u+s||_X \tag{3.1}$$

and

$$||u+s||_X \leq ||u||_X + ||s||_X \leq ||u|| + ||s|| \leq 2||u+s|| \tag{3.2}$$

for $u \in U$, $s \in S$. □

Definition 2. *For $c > 0$ define the cone $K_c = \{x = u + s \in X : ||u|| \geq c||s||\}$.*

Denote by $pr_S : X \mapsto S$ the projection operator to the space S and by $pr_U : X \mapsto U$ the projection operator to the space U. Notice that $||pr_S|| = 1$, $||pr_U|| = 1$. We can represent $F(u+s) = T(u+s) + r(u+s)$ with a nonlinear part $r(u+s) = F(u+s) - T(u+s)$. We have $F(x) = F(u+s) = u_1 + s_1$, where $u_1 \in U$, $s_1 \in S$. Let us consider a ball B_δ with center in 0 and radius δ by means of new norm for some $\delta > 0$ and denote by L the Lipschitz constant of r in B_δ.

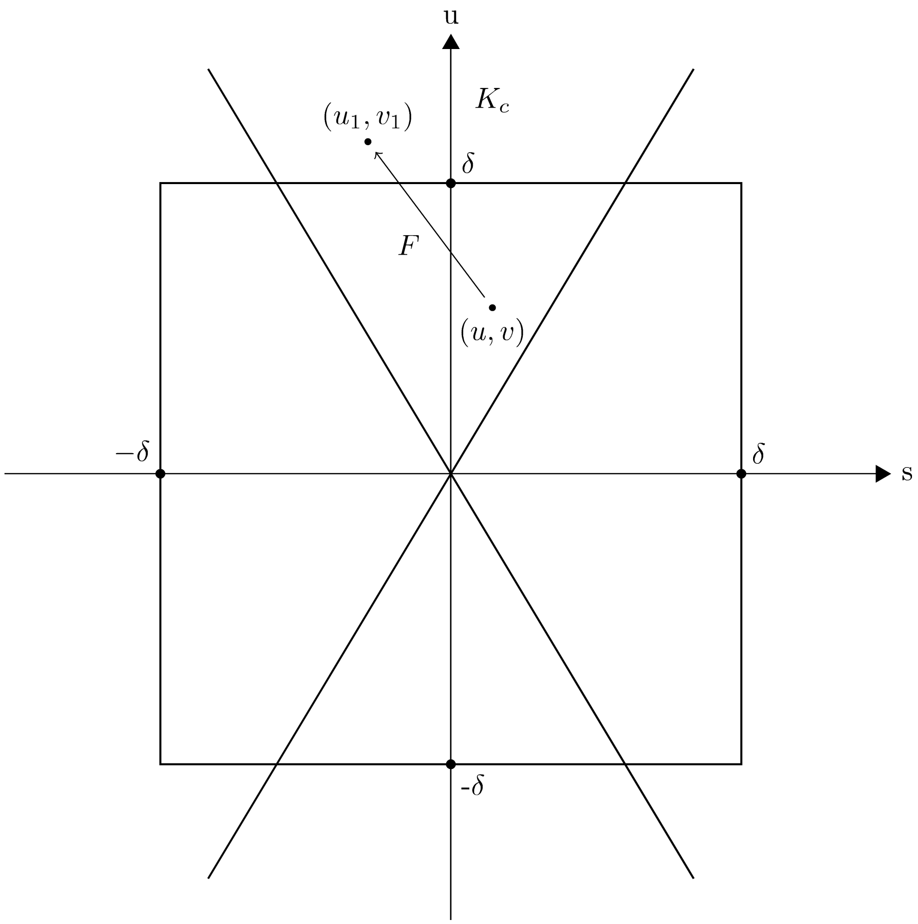

Figure 3.1: Invariant cone

Lemma 27. *For* $c > 0$ *there exists* $\delta' > 0$ *such that* $L \leq \dfrac{(1-\gamma)c^{-1}}{\max\{1, c^{-1}\}}$, $L < \dfrac{\beta - 1}{\max\{1, c^{-1}\}}$ *in* $B_{\delta'}$. *Then for all* $\delta \in (0, \delta')$ *one has with* $B_\delta = \{x \in X : ||x|| < \delta\}$ *that* $F(B_\delta \cap K_c) \subset K_c$.

Proof. Figure 3.1 shows the shape of the invariant cone. We have $F(x) = F(u+s) = u_1 + s_1$. Let us estimate $||s_1|| \leq ||Ts|| + ||pr_S r(u+s)|| \leq \gamma ||s|| + ||pr_S|| L \max\{||u||, ||s||\}$ since $||r(u+s)|| \leq L \max\{||u||, ||s||\}$ with Lipschitz constant L in $B_{\delta'}$. Recall that $||pr_U|| = ||pr_S|| = 1$. We have : $||s|| \leq c^{-1}||u||$. It follows that

$$\begin{aligned} ||s_1|| &\leq ||u||(\gamma c^{-1} + ||pr_S|| L \max\{1, c^{-1}\}) \leq \\ &\leq ||u||\big(\gamma c^{-1} + (1-\gamma)c^{-1}\big) \leq \\ &\leq c^{-1}||u||. \end{aligned}$$

Let us estimate

$$\begin{aligned}
||u_1|| &\geq ||Tu|| - ||pr_U r(u+s)|| \geq \\
&\geq \beta||u|| - ||pr_U||L\max\{||u||,||s||\} \geq \\
&\geq ||u||\cdot(\beta - L||pr_U||\max\{1,c^{-1}\}) \geq \\
&\geq ||u||(\beta-(\beta-1)) = \\
&= ||u||.
\end{aligned}$$

So, we obtain that $||s_1|| \leq c^{-1}||u_1||$. It follows that $u_1 + s_1 \in K_c$. $\square$

Lemma 28. *For $c > 1$ there exists $\delta' > 0$ such that the following conditions are satisfied in $B_{\delta'}$: $L \leq \dfrac{(1-\gamma)c^{-1}}{\max\{1,c^{-1}\}}$, $L < \dfrac{\beta-1}{\max\{1,c^{-1}\}}$, $L < \dfrac{1-\gamma}{\max\{1,c\}}$. There exists $\bar{\delta} \in (0,\delta')$ such that $c\bar{\delta} < \delta'$. Then if $x \in B_{\bar{\delta}}$, $F^j(x) \in B_{\delta'}$, $j = 0,...,n-1$ and $F^n(x) \notin B_{\delta'}$, it follows that $F^n(x) \in K_c$.*

Proof. Let us consider the first case when $x \in K_c \cap B_{\bar{\delta}}$. It follows that $F(x) \in K_c$ due to lemma 27. Using induction principle one can show that $F^n(x) \in K_c$.

Let us consider the second case when $x \in B_{\bar{\delta}} \setminus K_c$. It follows that $x = u_0 + s_0$ with $||u_0|| < c||s_0||$, $u_0 \in U$, $s_0 \in S$. Since $F^n(x) \notin B_{\delta'}$, we have $F^n(x) = u_n + s_n$ with $||u_n + s_n|| \geq \delta'$, $u_n \in U$, $s_n \in S$ and $||u_n + s_n|| = \max\{||u_n||,||s_n||\}$. We have 2 possibilities: 1) there exists $j \leq n$: $F^j(x) \in K_c$; 2) $F^j(x) \notin K_c$ for all $j = 1,...,n$. In the first situation it is clear that $F^n(x) \in K_c$ since K_c is invariant under F as long as iterates remain in $B_{\delta'}$ for $j = 1,...,n-1$. Let us discuss the second situation. We can estimate

$$||s_j|| \leq ||Ts_{j-1}|| + ||pr_S r(u_{j-1},s_{j-1})|| \leq \gamma||s_{j-1}|| + ||pr_S||L\max\{||u_{j-1}||,||s_{j-1}||\}.$$

Since $L < \dfrac{1-\gamma}{\max\{1,c\}}$, we have that $\gamma + ||pr_S||L\max\{1,c\} < 1$. It follows that $||s_j|| < ||s_{j-1}||$ for all $j = 1,..,n$ and, hence, $||s_n|| < ||s_0|| < \delta'$ and $||u_n|| < c||s_n|| < c||s_{n-1}|| < c||s_0|| < c\bar{\delta} < \delta'$. So, $u_n + s_n \in B_{\delta'}$, contradicting $F^n(x) \notin B_{\delta'}$. Case 2 is, hence, impossible. $\square$

3.3 Theorem about periodic solutions

Let us quote some results from paper [11] keeping our notations. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a C^2-function with $f(0) = 0$ and $f'(0) = 1$. Consider equation $\dot{x}(t) = \alpha f(x(t-1))$ with parameter $\alpha \in \mathbb{R}$. For all $\alpha \in (0, \dfrac{3\pi}{2})$ there is a decomposition $C = U_\alpha + S_\alpha$ into invariant closed subspaces $U_\alpha = \mathbb{R}\chi_{0,\alpha}$ and S_α, where $\chi_{0,\alpha}(t) = e^{\lambda_0^\alpha t}$ for $t \in [-1,0]$, λ_0^α is a unique positive zero of characteristic equation at

0. Let $pr_{U,\alpha}$ and $pr_{S,\alpha}$ denote the projections onto U_α and S_α, defined by the decomposition above. Note that for $\phi \in U_\alpha$, $\phi = \phi(0)\chi_{0,\alpha}$, $||\phi||_\infty = |\phi(0)|$ so that $p_{\mathbb{R}} : \phi \mapsto \phi(0)$ maps U_α onto $\mathbb{R}$, with $p_{\mathbb{R}}\chi_{0,\alpha} = 1$ and $|p_{\mathbb{R}}\phi| = ||\phi||_\infty$ ($\phi \in U_\alpha$). Let us denote by B_j open balls in C with center 0 for $j = 0, .., 3$ with respect to the original norm in C. The following version of the saddle point property [11] for the parametrized semiflow $\Phi : \mathbb{R}_0^+ \times C \times \mathbb{R} \to \mathbb{R}$, $\Phi(t, \phi, \alpha) = x_t^\alpha$, $x^\alpha = x^\alpha(\phi)$, is used :

"Let an open set $A \subset (0, \frac{3\pi}{2})$ and some B_0 be given. There exist B_1, B_2 with $B_2 \subset B_1$, constants $c > 0$, $\gamma > 0$ and maps $\omega_\alpha : U_\alpha \cap B_1 \mapsto S_\alpha$, $s_\alpha : S_\alpha \cap B_1 \mapsto U_\alpha$, $\alpha \in A$, with the following properties.
I. For every $\alpha \in A$ we have that
(i) $\omega_\alpha(0) = 0$, $s_\alpha(0) = 0$, ω_α and s_α satisfy a Lipschitz condition with constant $\frac{1}{2}$. The graphs $W_\alpha^u := \{\phi + \omega_\alpha(\phi) : \phi \in U_\alpha \cap B_1\}$ and $W_\alpha^s := \{s_\alpha(\phi) + \phi : \phi \in S_\alpha \cap B_1\}$ are tangent to U_α and S_α respectively, at $\phi = 0$.
(ii) For every $\phi \in U_\alpha \cap B_1$ there is a unique trajectory $x^* = x^*(\phi + \omega_\alpha(\phi)) : \mathbb{R} \mapsto C$ of $\Phi(\cdot, \cdot, \alpha)$ with $x_0^* = \phi + \omega_\alpha(\phi)$ and $||x_t^*||_\infty \leq ce^{\gamma t}||\phi||_\infty$ for all $t \leq 0$. If $x : \mathbb{R} \mapsto C$ is a trajectory of $\Phi(\cdot, \cdot, \alpha)$ with $x_t \in B_2$ for all $t \leq 0$ then $x = x^*(\phi + \omega_\alpha(\phi))$ for some $\phi \in U_\alpha \cap B_1$.
(iii) $\phi \in S_\alpha \cap B_1$ implies $||\Phi(t, s_\alpha(\phi) + \phi, \alpha)||_\infty \leq ce^{-\gamma t}||\phi||_\infty$ for all $t \geq 0$. If $\phi \in C$ and $\Phi(t, \phi, \alpha) \in B_2$ for all $t \geq 0$ then $\phi = s_\alpha(\psi) + \psi$ for some $\psi \in S_\alpha \cap B_1$.
II. For every $\alpha \in A$, $pr_{U,\alpha}B_2 \subset B_1$ and $pr_{S,\alpha}B_2 \subset B_1$, the maps $B_2 \times A \ni (\phi, \alpha) \mapsto \omega_\alpha(pr_{U,\alpha}\phi) \in C$, $B_2 \times A \ni (\phi, \alpha) \mapsto s_\alpha(pr_{S,\alpha}\phi) \in C$ are of class C^2."

$f(0) = 0$ implies $\Phi(t, 0, \alpha) = 0$ on $\mathbb{R}_0^+ \times \mathbb{R}$, and we have $D_2\Phi(t, 0, \alpha)\phi = y_t$, where $y : [-1, \infty) \to \mathbb{R}$ is the solution of the linear equation $\dot{y}(t) = \alpha y(t-1)$ with $y_0 = \phi$. The semigroups $T(\cdot, \cdot, \alpha) :$ $(t, \phi) \to T(t, \phi, \alpha) = y_t = D_2\Phi(t, 0, \alpha)\phi$, $\alpha \in \mathbb{R}$, are strongly continuous. The spectra $\sigma(\alpha)$ of their generators are given be the characteristic equation, i.e. by the zeros of the analytic function $z \to z - \alpha e^{-z}$.

Let us consider the results of the paper [11], we are interested in the following bifurcation theorem.

Theorem 2 (Walther). *Let* $\alpha_* \in (0, \frac{3\pi}{2})$ *be given with* $Re(z) < -\lambda_0^{\alpha_*}$ *for all* $z \in \sigma(\alpha_*) \setminus \{\lambda_0^{\alpha_*}\}$. *Let* $A \subset (0, \frac{3\pi}{2})$ *be an open neighbourhood of* α_*. *Let balls* $B_2 \subset B_1$, *constants* $c > 0$ *and* $\gamma > 0$ *and two families of maps* $\omega_\alpha : U_\alpha \cap B_1 \mapsto S_\alpha$, $s_\alpha : S_\alpha \cap B_1 \mapsto U_\alpha$, $\alpha \in A$, *with properties I and II be given. Suppose there exist* $\epsilon \in \mathbb{R}$ *with* $\epsilon\chi_{0,\alpha} \in B_1$ *for all* $\alpha \in A$, *a ball* $B_3 \subset B_2$, $t_+ \in \mathbb{R}$, $t_- < t_+$, $\epsilon' > 0$

such that the trajectory $x^{\alpha_*} = x^*(\epsilon\chi_{0,\alpha_*} + \omega_{\alpha_*}(\epsilon\chi_{0,\alpha_*}))$ *of* $\Phi(\cdot,\cdot,\alpha_*)$ *is homoclinic to zero with*

$$p_{\mathbb{R}} pr_{U,\alpha_*} x_t^{\alpha_*} < p_{\mathbb{R}} s_{\alpha_*}(pr_{S,\alpha_*} x_t^{\alpha_*}) \text{ for all } t \leq t_- \text{ and } x_t^{\alpha_*} \in B_3 \text{ for all } t \geq t_+ \text{ (and } \lim_{t\to\infty} x_t^{\alpha_*} = 0), \tag{3.3}$$

while for every $\alpha \in A$ *with* $\alpha_* - \epsilon' < \alpha < \alpha_*$ *the trajectories* $x^{\alpha} = x^*(\epsilon\chi_{0,\alpha} + \omega_{\alpha}(\epsilon\chi_{0,\alpha}))$ *of* $\Phi(\cdot,\cdot,\alpha)$ *satisfy*

$$p_{\mathbb{R}} pr_{U,\alpha} x_{t_+}^{\alpha} < p_{\mathbb{R}} s_{\alpha}(pr_{S,\alpha} x_{t_+}^{\alpha}). \tag{3.4}$$

Then there exist a neighbourhood V *of* $\{x_t^{\alpha_*} : t \in \mathbb{R}\} \cup \{0\}$, *an open neighbourhood* $A' \subset A$ *of* α_* *and a differentiable curve* $A' \ni \alpha \mapsto \pi_{0,\alpha} \in C$ *such that*
(i) $\pi_{0,\alpha_*} \in \{x_t^{\alpha_*} : t \in \mathbb{R}\}$,
(ii) for every $\alpha \in A'$ *with* $\alpha < \alpha_*$ *there is a periodic trajectory* $y^{\alpha} : \mathbb{R} \mapsto C$ *of* $\Phi(\cdot,\cdot,\alpha)$ *with* $y_0^{\alpha} = \pi_{0,\alpha}$ *and* $\{y_t^{\alpha} : t \in \mathbb{R}\} \subset V$, y^{α} *is orbitally asymptotically stable with asymptotic phase,*
(iii) there is no periodic trajectory $y : \mathbb{R} \mapsto C$ *of* $\Phi(\cdot,\cdot,\alpha_*)$ *with orbit* $\{y_t : t \in \mathbb{R}\}$ *in* V,
(iv) for every $\alpha \in A'$ *with* $\alpha < \alpha_*$ *and for every periodic trajectory* $y : \mathbb{R} \mapsto C$ *of* $\Phi(\cdot,\cdot,\alpha)$ *with* $\{y_t : t \in \mathbb{R}\} \subset V$ *there exists* $t \in \mathbb{R}$ *with* $y_s = y_{t+s}^{\alpha}$ *for all* $s \in \mathbb{R}$.

Remark Notice that condition (3.4) means that $x_{t_+}^{\alpha}$ lies on the same side (here called "below") of the local stable manifold W_{α}^s as $x_{t_-}^{\alpha_*}$.

We will need a modification of theorem 2. Instead of condition (3.3) we will consider

$$p_{\mathbb{R}} pr_{U,\alpha_*} x_t^{\alpha_*} > p_{\mathbb{R}} s_{\alpha_*}(pr_{S,\alpha_*} x_t^{\alpha_*}) \text{ for all } t \leq t_- \text{ and } x_t^{\alpha_*} \in B_3 \text{ for all } t \geq t_+ \text{ (and } \lim_{t\to\infty} x_t^{\alpha_*} = 0), \tag{3.5}$$

and instead of condition (3.4) we will consider the following condition: for every $\alpha \in A$ with $\alpha_* < \alpha < \alpha_* + \epsilon'$ the trajectories $x^{\alpha} = x^*(\epsilon\chi_{0,\alpha} + \omega_{\alpha}(\epsilon\chi_{0,\alpha}))$ of $\Phi(\cdot,\cdot,\alpha)$ satisfy

$$p_{\mathbb{R}} pr_{U,\alpha} x_{t_+}^{\alpha} > p_{\mathbb{R}} s_{\alpha}(pr_{S,\alpha} x_{t_+}^{\alpha}). \tag{3.6}$$

Remark An analogue of theorem 2 can be proved for equation of type (2.6) with $f_{\alpha} \in \Gamma$ if we consider p_0^{α} instead of 0 and conditions (vi) and (x) instead of "$\alpha \in (0, \frac{3\pi}{2})$" which guaranty the required spectrum at p_0^{α}. For equation (2.6) we consider $A = [\alpha_0, \alpha_1]$. The proof of theorem 2 only requires smoothness properties of semiflow Φ that we have also for equation (2.6) with decay term and does not depend on the structure of equation whether there is a decay term or not.

Now we consider our equation (2.6) with $f_\alpha \in \Gamma$. According to theorem 1 for $\alpha = \alpha_*$ we have a homoclinic solution which tends to $p_0^{\alpha_*}$ as t goes to ∞. From lemma 5 (ii) and from $Ds_{\alpha_*}(0) = 0$, we conclude that for the homoclinic solution there exists t_- such that for all $t \leq t_-$:
$p_{\mathbb{R}} pr_{U,\alpha_*}(x_t^{\alpha_*} - p_0^{\alpha_*}) \geq p_{\mathbb{R}} \frac{1}{2} ||x_t^{\alpha_*} - p_0^{\alpha_*}||_\infty > p_{\mathbb{R}} s_{\alpha_*}(pr_{S,\alpha_*}(x_t^{\alpha_*} - p_0^{\alpha_*}))$. So, condition (3.5) is satisfied for $p_0^{\alpha_*}$ instead of 0.
Let us fix some small, uniform for $\alpha \in (\alpha_*, \alpha_* + \delta_*]$, δ_7, δ_*, $\tilde{\delta} < \delta_7$, $\delta_3 < \frac{p_m^\alpha - p_0^\alpha}{4}$ and $\delta_3 < \delta_7$, and fix $c : c > 1, c > Ke^{\lambda_0^\alpha}$ (K is from the proof of lemma 26) for $\alpha \in (\alpha_*, \alpha_* + \delta_*]$ so that there exist the local stable manifolds $W_\alpha^s(\Phi^\alpha, p_0^\alpha, \tilde{\delta})$, the local unstable manifolds $W_\alpha^u(\Phi^\alpha, p_0^\alpha, \tilde{\delta})$ for the semiflow in $\tilde{\delta}$-neighbourhood of p_0^α for $\alpha \in (\alpha_*, \alpha_* + \delta_*]$, the local stable manifolds $W_\alpha^s(F^\alpha, p_0^\alpha, \delta_7)$ and the local unstable manifolds $W_\alpha^u(F^\alpha, p_0^\alpha, \delta_7)$ for $\alpha \in (\alpha_*, \alpha_* + \delta_*]$, there exist $\delta_4 > \delta_3$, $\delta_5 \geq \delta_4$, $\delta_6 > \delta_5$, $\delta_6 < \delta_7$ such that $p_0^\alpha + \text{graph } s_{F^\alpha}^\alpha \cap B(p_0^\alpha, \delta_3) \subset W_\alpha^s(F^\alpha, p_0^\alpha, \delta_4)$, $W_\alpha^s(F^\alpha, p_0^\alpha, \delta_5) \subset p_0^\alpha + \text{graph } s_{F^\alpha}^\alpha \cap B(p_0^\alpha, \delta_6)$, the Lipschitz constant L_2^α of $s_{F^\alpha}^\alpha$ satisfies $L_2^\alpha < 1$ in $B(0, \delta_3)$ with respect to adapted norm $|| \cdot ||$. Notice that balls and neighbourhoods we take with respect to adapted norm $|| \cdot ||$ on $C^0([-1, 0], \mathbb{R})$.
Then there exist $\delta' < \delta_3$ and $\bar{\delta}$ such that the properties from lemmas 28 and 4 hold for all
$\alpha \in (\alpha_*, \alpha_* + \delta_*]$, $\delta' < \frac{p_{**}^\alpha - p_0^\alpha}{4}$, $\delta' < \frac{p_0^\alpha}{4}$ for $\alpha \in (\alpha_*, \alpha_* + \delta_*]$, there exist the local stable manifolds $W_\alpha^s(F^\alpha, p_0^\alpha, \delta')$, $W_\alpha^s(F^\alpha, p_0^\alpha, \bar{\delta})$ and the local unstable manifolds $W_\alpha^u(F^\alpha, p_0^\alpha, \delta')$, $W_\alpha^u(F^\alpha, p_0^\alpha, \bar{\delta})$ at p_0^α, $\Phi([0, 1] \times B(p_0^\alpha, \bar{\delta})) \subset B(p_0^\alpha, \tilde{\delta})$, $\Phi([0, 1] \times B(p_0^\alpha, \delta')) \subset B(p_0^\alpha, \delta_3)$. Notice that δ' and $\bar{\delta}$ can be chosen uniformly for $\alpha \in [\alpha_*, \alpha_* + \delta_*]$.
Let us notice that $s_{F^\alpha}^\alpha|_{B(0,\bar{\delta}) \cap S} = s_{\Phi^\alpha}^\alpha|_{B(0,\bar{\delta}) \cap S}$ according to lemma 4, where $s_{F^\alpha}^\alpha$ corresponds to function s_α from the property I.(i), but for the time-one-map, and $s_{\Phi^\alpha}^\alpha$ corresponds to function s_α from the property I.(i); $\omega_{F^\alpha}^\alpha$ corresponds to function ω_α from the property I.(i) of theorem 2, but for the time-one-map, and $\omega_{\Phi^\alpha}^\alpha$ corresponds to function ω_α from the property I.(i).
We know that $e^{\lambda_0^\alpha \cdot}$ is eigenvector of $DF_\alpha(p_0^\alpha)$ with the only positive λ_0^α. Notice that
$p_{\mathbb{R}} pr_{U,\alpha} \phi = c_1^\alpha(\phi) \in \mathbb{R}$ for $\phi \in C$, $c_1^\alpha := p_{\mathbb{R}} \circ pr_{U,\alpha} : C \mapsto \mathbb{R}$ is a linear functional.

Lemma 29. *For any $\phi \in K_c$, $\theta \in [-1, 0]$, the following is satisfied:*

$$\phi > 0 \Leftrightarrow c_1^\alpha(\phi) > 0,$$

$$\phi < 0 \Leftrightarrow c_1^\alpha(\phi) < 0.$$

Proof. We have $||pr_{U,\alpha}\phi|| \geq c||pr_{S,\alpha}\phi||$ since $\phi \in K_c$. It follows that

$$||pr_{S,\alpha}\phi||_\infty \leq ||pr_{S,\alpha}\phi|| \leq \frac{1}{c}||pr_{U,\alpha}\phi|| \leq \frac{1}{c}K||pr_{U,\alpha}\phi||_\infty \leq \frac{K|c_1^\alpha(\phi)|}{c}.$$

For all $\theta \in [-1, 0]$ we have:

$$|\phi(\theta)| = |(pr_{U,\alpha}\phi + pr_{S,\alpha}\phi)(\theta)| = |c_1^\alpha(\phi)e^{\lambda_0^\alpha \theta} + (pr_{S,\alpha}\phi)(\theta)| \geq |c_1^\alpha(\phi)e^{-\lambda_0^\alpha}| - \frac{K|c_1^\alpha(\phi)|}{c} = |c_1^\alpha(\phi)|(e^{-\lambda_0^\alpha} - \frac{K}{c})$$

with $e^{-\lambda_0^\alpha} - \frac{K}{c} > 0$ since $c > Ke^{\lambda_0^\alpha}$. So, if $c_1^\alpha(\phi) < 0$ then $\phi(\theta) \leq c_1^\alpha(\phi)(e^{-\lambda_0^\alpha} - \frac{K}{c}) < 0$. If $c_1^\alpha(\phi) > 0$ then $\phi(\theta) \geq c_1^\alpha(\phi)(e^{-\lambda_0^\alpha} - \frac{K}{c}) > 0$. □

For the homoclinic solution with α_* we know that there exists $\bar{t}^{\alpha_*}$ such that $x_t^{\alpha_*} \in B(p_0^{\alpha_*}, \bar{\delta})$ for all $t \geq \bar{t}^{\alpha_*} + 1$. Recall that $x^{\alpha_*}(t) < p_m^{\alpha_*}$ for all $t > \tau_m^{\alpha_*}$ since $\alpha_* \notin A^1$. Continuity with respect to α implies that there exists $\delta_1 \leq \delta_*$:

$$\text{for } \alpha \in (\alpha_*, \alpha_* + \delta_1) \text{ we have: } x^\alpha_{\bar{t}^{\alpha_*}+1} \in B(p_0^\alpha, \bar{\delta}) \subset B(p_0^\alpha, \delta') \text{ and } x^\alpha(t) < p_m^\alpha \text{ for } t \in [\tau_m^\alpha + 1, \bar{t}^{\alpha_*} + 1]. \tag{3.7}$$

Lemma 30. *For $\alpha \in (\alpha_*, \alpha_* + \delta_1)$ there exists $n^\alpha \in \mathbb{N} : x^\alpha_{\bar{t}^{\alpha_*}+1+n^\alpha} \notin B(p_0^\alpha, \delta')$.*

Proof. Let $\alpha \in (\alpha_*, \alpha_* + \delta_1)$. There are 3 possible situations :

1) exists $t_3^\alpha > \bar{t}^{\alpha_*} + 1$ with $x^\alpha_{t_3^\alpha} > p_0^\alpha$;

2) exists $t_4^\alpha > \bar{t}^{\alpha_*} + 1$ with $x^\alpha_{t_4^\alpha} < p_0^\alpha$;

3) exist neither $t_4^\alpha > \bar{t}^{\alpha_*} + 1$ with $x^\alpha_{t_4^\alpha} < p_0^\alpha$ nor $t_3^\alpha > \bar{t}^{\alpha_*} + 1$ with $x^\alpha_{t_3^\alpha} > p_0^\alpha$.

Let us consider situation 3). It follows that x^α oscillates rapidly about p_0^α for $t \geq \bar{t}^{\alpha_*}$. We have that $x^\alpha(t) < p_m^\alpha$ for $t \in [\bar{t}^{\alpha_*}, \bar{t}^{\alpha_*} + 1]$ since $x^\alpha_{\bar{t}^{\alpha_*}+1} \in B(p_0^\alpha, \bar{\delta})$. According to lemma 24 $x^\alpha(t) \to p_0^\alpha$ as $t \to \infty$. So, we have $\alpha \in A^*$ and $\alpha > \alpha_*$, it contradicts to the definition of α_* as the maximum of A^*.

Let us consider situation 1). According to corollary 1 there exists some $t_2^\alpha > \bar{t}^{\alpha_*} + 1 : x^\alpha_{t_2^\alpha} > p_1^\alpha$. Then it is clear that there exists some $t_5^\alpha \in (\bar{t}^{\alpha_*} + 1, t_2^\alpha] : x^\alpha(t) > p_0^\alpha + 2\delta'$ for $t \in [t_5^\alpha, t_5^\alpha + 1]$ since $\delta' < \frac{p_m^\alpha - p_0^\alpha}{4}$. So, there exists $n^\alpha \in \mathbb{N} : \bar{t}^{\alpha_*} + 1 + n^\alpha \in [t_5^\alpha, t_5^\alpha + 1]$ and $x^\alpha(\bar{t}^{\alpha_*} + 1 + n^\alpha) > p_0^\alpha + 2\delta'$. So, we know that $x^\alpha_{\bar{t}^{\alpha_*}+1+n^\alpha} \notin B(p_0^\alpha, \delta')$ since $\delta' \leq \frac{||x^\alpha_{\bar{t}^{\alpha_*}+1+n^\alpha} - p_0^\alpha||_\infty}{2} \leq ||x^\alpha_{\bar{t}^{\alpha_*}+1+n^\alpha} - p_0^\alpha||$ due to (3.2).

Let us consider situation 2). It follows that $x^\alpha(t) \to 0$ as $t \to \infty$ according to lemma 12. So, there exists $n^\alpha \in \mathbb{N} : x^\alpha_{\bar{t}^{\alpha_*}+1+n^\alpha} \notin B(p_0^\alpha, \delta')$ since $\delta' < \frac{p_0^\alpha}{4}$.

□

Definition 3.

$$B(p_0^\alpha,\bar\delta)^+ := \{\phi\in B(p_0^\alpha,\bar\delta) : c_1^\alpha(\phi-p_0^\alpha) > c_1^\alpha(s_{F^\alpha}^\alpha(pr_{S,\alpha}(\phi-p_0^\alpha)))\},$$

$$B(p_0^\alpha,\bar\delta)^- := \{\phi\in B(p_0^\alpha,\bar\delta) : c_1^\alpha(\phi-p_0^\alpha) < c_1^\alpha(s_{F^\alpha}^\alpha(pr_{S,\alpha}(\phi-p_0^\alpha)))\}.$$

Analogously one can define $B(p_0^\alpha,\delta_3)^+$ *and* $B(p_0^\alpha,\delta_3)^-$.

Let us consider the following decomposition:
$B(p_0^\alpha,\bar\delta) = B(p_0^\alpha,\bar\delta)^+ \cup \big(B(p_0^\alpha,\bar\delta)\cap W_\alpha^s(F^\alpha,p_0^\alpha,\bar\delta)\big)\cup B(p_0^\alpha,\bar\delta)^-$.
Since there exists $n^\alpha\in\mathbb{N}$, $x^\alpha_{\bar t^{\alpha_*}+1+n^\alpha}\notin B(p_0^\alpha,\delta')$ according to lemma 30, it follows that there exists the first $n_1^\alpha\in\mathbb{N} : F_\alpha^{n_1^\alpha}(x^\alpha_{\bar t^{\alpha_*}+1})\notin B(p_0^\alpha,\delta')$ and $F_\alpha^{n_1^\alpha}(x^\alpha_{\bar t^{\alpha_*}+1})\in B(p_0^\alpha,\delta_3)$ since
$\Phi([0,1]\times B(p_0^\alpha,\delta'))\subset B(p_0^\alpha,\delta_3)$ for $\alpha\in(\alpha_*,\alpha_*+\delta_1)$.
According to lemma 28 we have that $F_\alpha^{n_1^\alpha}(x^\alpha_{\bar t^{\alpha_*}+1})\in p_0^\alpha+K_c$. Define $y^\alpha := F_\alpha^{n_1^\alpha}(x^\alpha_{\bar t^{\alpha_*}+1})$,
$\alpha\in(\alpha_*,\alpha_*+\delta_1)$. We have that $y^\alpha-p_0^\alpha\in K_c\setminus\{0\}$ and one can use lemma 29.

Lemma 31. *If* $y^\alpha\in B(p_0^\alpha,\delta_3)^+\cap(p_0^\alpha+K_c)$ *then* $y^\alpha(\theta)-p_0^\alpha>0$ *for all* $\theta\in[-1,0]$.
If $y^\alpha\in B(p_0^\alpha,\delta_3)^-\cap(p_0^\alpha+K_c)$ *then* $y^\alpha(\theta)-p_0^\alpha<0$ *for all* $\theta\in[-1,0]$.

Proof. Let us omit the index α. Assume $y\in B(p_0,\delta_3)^+\cap(p_0+K_c)$ and set $z:=y-p_0\in K_c$. We have $||pr_U z||\geq c||pr_S z||\geq c||s_F(pr_S z)||$ since s_F has Lipschitz constant $L_2<1$. Linearity of c_1 and the fact that U is one-dimensional imply $|c_1(z)|\geq c|c_1(s_F(pr_S z))|$ and hence $c>1$ shows $c_1(z)>0\Leftrightarrow c_1(z)>c_1(s_F(pr_S z))$ and $c_1(z)<0\Leftrightarrow c_1(z)<c_1(s_F(pr_S z))$. According to definition 3 and lemma 29 one can see that $z>0\Leftrightarrow y\in B(p_0,\delta_3)^+$; $z<0\Leftrightarrow y\in B(p_0,\delta_3)^-$. □

Lemma 32. *For* $\alpha\in(\alpha_*,\alpha_*+\delta_1)$ *the following is satisfied:* $x^\alpha_{\bar t^{\alpha_*}+1}\in B(p_0^\alpha,\bar\delta)^+$.

Proof. Recall that for $\alpha\in(\alpha_*,\alpha_*+\delta_1)$ according to lemma 25

$$\text{there exists } t_*^\alpha>\tau_m^\alpha+1 : x^\alpha(t_*^\alpha)\geq p_m^\alpha. \tag{3.8}$$

According to lemma 30 and lemma 28 we know that for $\alpha\in(\alpha_*,\alpha_*+\delta_1)$ there exists the first $n^\alpha\in\mathbb{N} : x^\alpha_{\bar t^{\alpha_*}+1+n^\alpha}\notin B(p_0^\alpha,\delta')$ and $x^\alpha_{\bar t^{\alpha_*}+1+n^\alpha}\in p_0^\alpha+K_c$, $x^\alpha_{\bar t^{\alpha_*}+1+n^\alpha}\in B(p_0^\alpha,\delta_3)$.
If $x^\alpha_{\bar t^{\alpha_*}+1}\in\big(B(p_0^\alpha,\bar\delta)\cap W_\alpha^s(F^\alpha,p_0^\alpha,\bar\delta)\big)$, then x_t^α never leaves $B(p_0^\alpha,\delta_3)$ for $t>\bar t^{\alpha_*}+1$. So, we have a contradiction to (3.8) in this case.
If $x^\alpha_{\bar t^{\alpha_*}+1}\in B(p_0^\alpha,\bar\delta)^-$ then we have for $t\geq\bar t^{\alpha_*}+1$ with $x_t^\alpha\in B(p_0^\alpha,\delta_3)$:

$$p_{\mathbb{R}}pr_{U,\alpha}(x_t^\alpha-p_0^\alpha) < p_{\mathbb{R}}s_{F^\alpha}^\alpha(pr_{S,\alpha}^\alpha(x_t^\alpha-p_0^\alpha)), \tag{3.9}$$

since if there exists some $t_5 : p_{\mathbb{R}} pr^{\alpha}_{U,\alpha}(x^{\alpha}_{t_5} - p^{\alpha}_0) = p_{\mathbb{R}} s^{\alpha}_{F^{\alpha}}(pr^{\alpha}_{S,\alpha}(x^{\alpha}_{t_5} - p^{\alpha}_0))$ and $x^{\alpha}_{t_5} \in B(p^{\alpha}_0, \delta_3)$ then $x^{\alpha}_{t_5} \in p^{\alpha}_0 + \text{graph } s^{\alpha}_{F^{\alpha}} \cap B(p^{\alpha}_0, \delta_3) \subset W^s_{\alpha}(F^{\alpha}, p^{\alpha}_0, \delta_4)$ with $\delta_4 > \delta_3$. It follows then that
$x^{\alpha}_{\bar{t}^{\alpha_*}+1} \in W^s_{\alpha}(F^{\alpha}, p^{\alpha}_0, \delta_5) \subset p^{\alpha}_0 + \text{graph } s^{\alpha}_{F^{\alpha}} \cap B_{\delta_6}$ with $\delta_5 \geq \delta_4$ and $\delta_6 > \delta_5$. It contradicts to the fact that $x^{\alpha}_{\bar{t}^{\alpha_*}+1} \in B(p^{\alpha}_0, \bar{\delta})^-$ since $x^{\alpha}_{\bar{t}^{\alpha_*}+1} \notin p^{\alpha}_0 + \text{graph } s^{\alpha}_{F^{\alpha}}$. So, we have that $x^{\alpha}_{\bar{t}^{\alpha_*}+1+n^{\alpha}} \in B(p^{\alpha}_0, \delta_3)^-$ due to (3.9). It follows according to lemma 31 and lemma 28 that $x^{\alpha}_{\bar{t}^{\alpha_*}+1+n^{\alpha}} - p^{\alpha}_0 < 0$, and according to lemma 12 $x^{\alpha}(t) < p^{\alpha}_0$ for all $t \geq \bar{t}^{\alpha_*} + n^{\alpha}$. For $t \in [\tau^{\alpha}_m, \bar{t}^{\alpha_*} + 1 + n^{\alpha}]$ we have $x^{\alpha}(t) < p^{\alpha}_m$ due to (3.7), lemma 15 and since $x^{\alpha}_t \in B(p^{\alpha}_0, \delta_3)$ for $t \in [\bar{t}^{\alpha_*} + 1, \bar{t}^{\alpha_*} + 1 + n^{\alpha}]$ with $\delta_3 < \dfrac{p^{\alpha}_m - p^{\alpha}_0}{4}$.
So, if $x^{\alpha}_{\bar{t}^{\alpha_*}+1} \in B(p^{\alpha}_0, \bar{\delta})^-$ then there does not exist $t > \bar{t}^{\alpha_*} + 1$ with $x^{\alpha}_t > p^{\alpha}_m$ contradicting (3.8). It follows that the only remaining possibility is that $x^{\alpha}_{\bar{t}^{\alpha_*}+1} \in B(p^{\alpha}_0, \bar{\delta})^+$.
So, we have that $x^{\alpha}_{\bar{t}^{\alpha_*}+1}$ lies "above" $W^s(F^{\alpha}, p^{\alpha}_0, \bar{\delta})$ and "above" $W^s(\Phi^{\alpha}, p^{\alpha}_0, \tilde{\delta})$.

□

So, we know that condition (3.6) is satisfied. Now we are ready to use an analogue of theorem 2 for equation (2.6) considering $\tilde{\Phi}^{\alpha}(t, \phi) := \Phi^{\alpha}(t, \phi + p^{\alpha}_0) - p^{\alpha}_0$.
Recall that a periodic trajectory y of a semiflow Φ is orbitally asymptotically stable with asymptotic phase if there exists a neighbourhood V of $\{y_t : t \in \mathbb{R}\}$ such that for all $\phi \in V$ there exists $\theta(\phi) \in \mathbb{R}$: $||\Phi(t, \phi) - y(t + \theta(\phi))||_{\infty} \to 0$ as $t \to \infty$.

Theorem 3. *Consider equation* (2.6) *with* $f_{\alpha} \in \Gamma$ *and assume that conditions (i)-(xi) are satisfied. Then there exist open neighbourhood* A' *of* α_* *and a differentiable curve* $\alpha \in A' \mapsto \pi_{0,\alpha} \in C$ *such that for every* $\alpha \in A'$ *with* $\alpha > \alpha_*$ *there is a periodic trajectory* $y^{\alpha} : \mathbb{R} \mapsto C$ *of* $\Phi(\cdot, \cdot)$ *with* $y^{\alpha}_0 = \pi_{0,\alpha}$, y^{α} *is orbitally asymptotically stable with asymptotic phase.*

Chapter 4

Example of the nonlinear function

4.1 Modification of the nonlinear function

In order to simplify the verification of conditions (i)-(xi) we will consider the following modifications of the nonlinear functions $f_\alpha \in \Gamma$. Let us consider

$$\dot{x}(t) = -ax(t) + f_\alpha(x(t-1)), \tag{4.1}$$

where $f_\alpha \in \Gamma$:

$$f_\alpha(x) = \begin{cases} f(x), & x < p_1 + \delta_1; \\ f_\alpha^*(x), & x \geq p_1 + \delta_1; \end{cases} \tag{4.2}$$

$0 \leq \delta_1 \leq \delta_2$ and f_α^* satisfies: $f_\alpha^*(p_1+\delta_1) = f(p_1+\delta_1)$, $\frac{d}{dx} f_\alpha^*(p_1+\delta_1) = \frac{d}{dx} f(p_1+\delta_1)$, $\frac{d^2}{dx^2} f_\alpha^*(p_1+\delta_1) = \frac{d^2}{dx^2} f(p_1+\delta_1)$, $f_\alpha^*(x) = \alpha p_1$ for $x \geq p_1 + \delta_2$, $\frac{d}{dx} f_\alpha^*(p_1+\delta_2) = 0$, $\frac{d^2}{dx^2} f_\alpha^*(p_1+\delta_2) = 0$, $f_\alpha^*(x)$ decreases on $[p_1+\delta_1, p_1+\delta_2)$ and parameter $\alpha \in [0, a-\epsilon]$, $(a-\epsilon)p_1 = f(p_1+\delta_1)$. Notice that p_0, p_m, p_1 are independent of α. We assume that f_α^* is monotone for $x > p_1 + \delta_1$ with respect to α, $f_\alpha^* \in C^2$ and for all $R > 0$, $\alpha \mapsto f_\alpha$ is continuous with respect to $|| \cdot ||_{C^1([0,R],\mathbb{R})}$. Figure 4.1 shows the shape of such $f_\alpha(x)$.

So, we need such f and a that conditions (i)-(x) are satisfied. It is clear that the function f_α also satisfies conditions (i)-(x).

Notice that if a solution of equation (4.1) satisfies: $x^\alpha(t) > p_1$ for $t \in (t_1, t_1+1]$ then there exists $t_2 > t_1 + 1$: $x(t_2) = p_1$ according to lemma 11. According to lemma 10 and notice above we can take δ_1 and δ_2 small enough, so that for the solutions $x^\alpha(t)$ of equation (4.1) with $x_0^\alpha = \eta$,

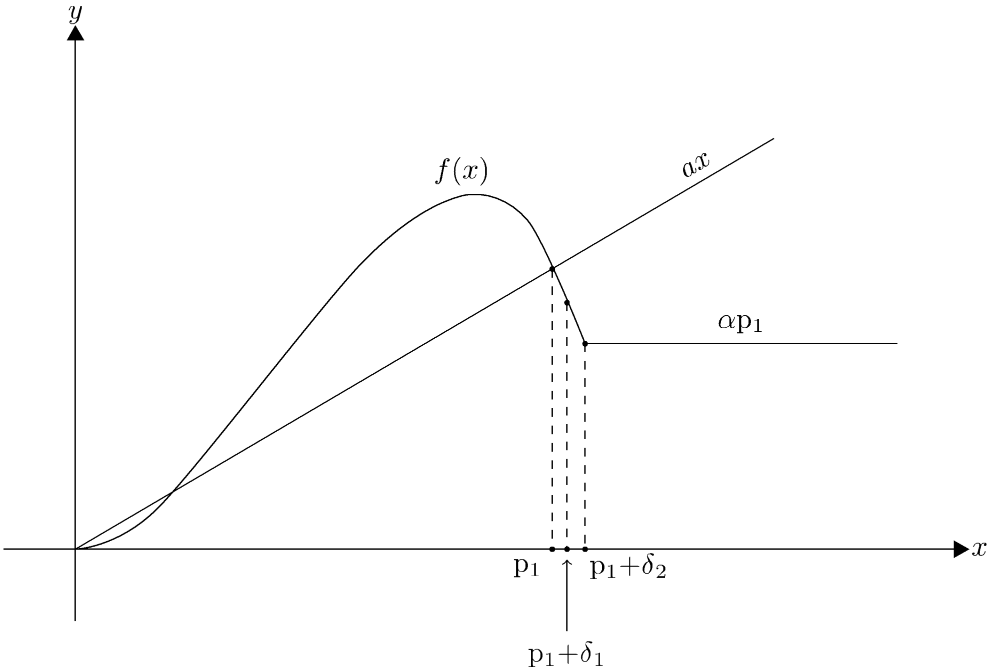

Figure 4.1: Nonlinear function $f_\alpha(x)$

$\eta \in W^u_{\mathrm{loc},f}(\Phi(\cdot,\cdot))$, $\eta >> p_0$, the following is satisfied: there exist τ_1 and $\tau_2^\alpha \geq \tau_1 + 1$ such that $x^\alpha(t) \geq p_1 + \delta_2$ for $t \in [\tau_1, \tau_2^\alpha]$, τ_1 is the first time moment with $x(\tau_1) = p_1 + \delta_2$, $x(\tau_2^\alpha) = p_1 + \delta_2$. Notice that behaviour of solutions $x^\alpha(t)$ of equation (4.1) is the same for all α up to the time moment $\tau_1' + 1$, where τ_1' is the first time moment with $x(\tau_1') = p_1 + \delta_1$. We assume that f_α satisfies condition (xi) with $c_1^\alpha = c_2^\alpha = \alpha p_1$ for $\alpha \in [\alpha_0, \alpha_1]$, where $\alpha_0 = 0$, $\alpha_1 \leq (a - \epsilon) < a$. Note that condition (xi) will be satisfied by the example later.

Let us introduce a new condition for f_α:

(xi,8′) $\dfrac{p_1 + p_m}{2(p_1 + \delta_2)}\left(\dfrac{\alpha_1 p_1}{a} + (p_1 + \delta_2 - \dfrac{\alpha_1 p_1}{a})e^{-a}\right) + f(p_m)\dfrac{p_1 + \delta_2 - p_m}{a(p_1 + \delta_2 - \dfrac{\alpha_1}{a}p_1)} < p_m.$

Remark Notice that if condition (xi,7) is satisfied only for $\alpha = \alpha_1$ for equation (4.1) then it follows that condition (xi,7) is satisfied for all $\alpha \in [0, \alpha_1]$ for equation (4.1) since

$$\frac{\alpha p_1}{a} + (p_1 + \delta_2 - \frac{\alpha p_1}{a})e^{-a} < \frac{\alpha_1 p_1}{a} + (p_1 + \delta_2 - \frac{\alpha_1 p_1}{a})e^{-a} \tag{4.3}$$

for $0 \leq \alpha < \alpha_1$ and

$$\ln\left(\frac{p_1 + \delta_2 - \dfrac{\alpha p_1}{a}}{\dfrac{p_1 + p_m}{2} - \dfrac{\alpha p_1}{a}}\right) < \ln\left(\frac{p_1 + \delta_2 - \dfrac{\alpha_1 p_1}{a}}{\dfrac{p_1 + p_m}{2} - \dfrac{\alpha_1 p_1}{a}}\right)$$

for $0 \leq \alpha < \alpha_1$ since

$$\frac{d}{dv}\left(\frac{p_1+\delta_2-v}{\frac{p_1+p_m}{2}-v}\right)=\frac{-(\frac{p_1+p_m}{2}-v)+(p_1+\delta_2-v)}{(\frac{p_1+p_m}{2}-v)^2}=\frac{p_1+\delta_2-\frac{p_1+p_m}{2}}{(\frac{p_1+p_m}{2}-v)^2}>0.$$

Note that if condition (xi,8′) is satisfied for equation (4.1) then condition (xi,8) is satisfied for all $\alpha \in [0,\alpha_1]$ for equation (4.1) since $\frac{p_1+\delta_2-p_m}{a(p_1+\delta_2-\frac{\alpha p_1}{a})} < \frac{p_1+\delta_2-p_m}{a(p_1+\delta_2-\frac{\alpha_1 p_1}{a})}$ for $0 \leq \alpha < \alpha_1$, $\frac{\frac{p_1+p_m}{2}-\frac{\alpha p_1}{a}}{p_1+\delta_2-\frac{\alpha p_1}{a}} < \frac{p_1+p_m}{2(p_1+\delta_2)}$ for $\alpha \in (0,\alpha_1]$ since

$$\frac{d}{dv}\left(\frac{\frac{p_1+p_m}{2}-v}{p_1+\delta_2-v}\right)<0$$

and according to (4.3).

So, it suffices to verify condition (xi,7) for $\alpha = \alpha_1$ and (xi,8′) for equation (4.1) instead of (xi,7) and (xi,8) for all $\alpha \in [0,\alpha_1]$.

Then one can use theorem 1 and theorem 3 for equation (4.1).

4.2 Verification of conditions

In this section we will give an example, which satisfies conditions (i)-(xi) of the previous sections, based on numerical calculation. While some expressions are evaluated by computer, part of the calculations can be followed 'by hand'. Let us consider the following equation

$$\dot{x}(t) = -ax(t) + b\frac{x^p}{1+x^q} \tag{4.4}$$

with constant parameters $a = 0.455$, $b = 0.5$, $p = 2$ and $q = 136$. Graphic of the nonlinear function $f(x) := b\frac{x^p}{1+x^q}$ and the function $x \mapsto ax$ you can see below (Figure 4.2).

We will show that conditions (i)-(x) are satisfied for these parameters. First of all we will estimate stationary points p_0 and p_1 for equation (4.4). One can show that $0.91 < p_0 < 0.9101$ since $\frac{b\cdot(0.91)^2}{1+0.91^{136}} < 0.414049 < a\cdot 0.91 = 0.41405$ and $a\cdot 0.9101 = 0.4140955 < 0.414139 < \frac{b\cdot(0.9101)^2}{1+0.9101^{136}}$.

Also one can prove that $0.9814 < p_1 < 0.9815$ since $a\cdot 0.9814 = 0.446537 < 0.4468 < \frac{b\cdot(0.9814)^2}{1+0.9814^{136}}$ and $\frac{b\cdot(0.9815)^2}{1+0.9815^{136}} < 0.44645 < a\cdot 0.9815 = 0.4465825$.

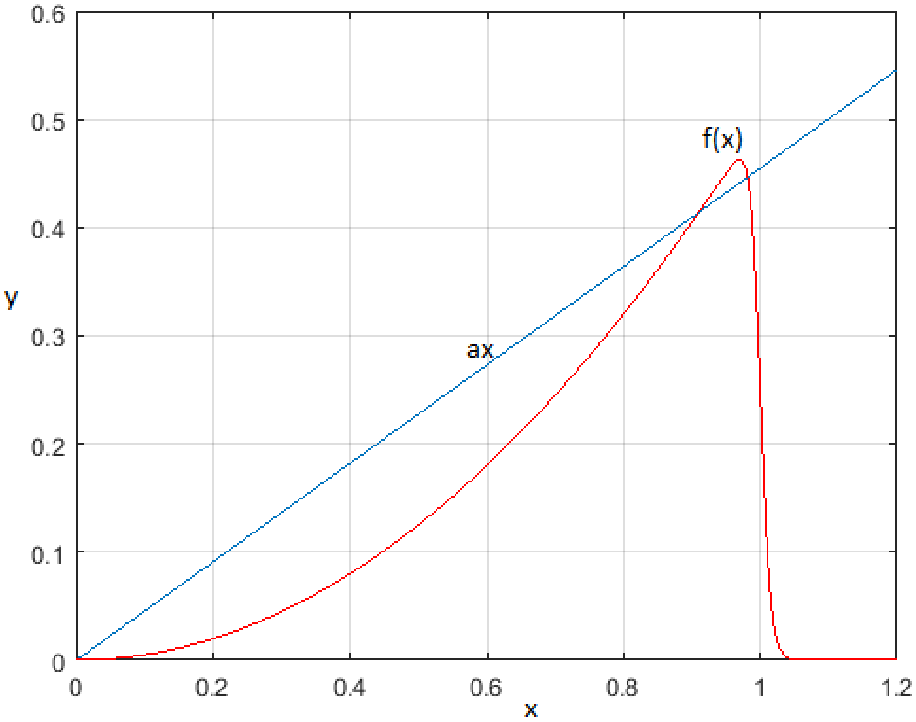

Figure 4.2: Nonlinear function

Let us consider the derivative $f'(x) = \dfrac{bx^{p-1}\big(p + x^q(p-q)\big)}{(1+x^q)^2}$ and the second derivative $f''(x) = \dfrac{b(\gamma_1(x^q)^2 + \gamma_2 x^q + \gamma_3)}{(1+x^q)^3}$, where $\gamma_1 = (p-1)(p-q) + q(p-q) - 2q(p-q) = 18090$, $\gamma_2 = (p-1)(p-q+p) + q(p-q) - 2pq = -18900$ and $\gamma_3 = (p-1)p = 2$. We will study solutions of the following equation

$$\gamma_1(x^q)^2 + \gamma_2 x^q + \gamma_3 = 0 \tag{4.5}$$

in order to find zeros of the second derivative. So, we have $D = \gamma_2^2 - 4\gamma_1\gamma_3 = 357065280$ and $18896 < D^{1/2} < 18897$. It follows that $1.0446 < x_1^q = \dfrac{18900 + D^{1/2}}{2 \cdot 18090} < 1.0447$ and $0.00008 < x_2^q = \dfrac{18900 - D^{1/2}}{2 \cdot 18090} < 0.00012$ and, hence, $1.0003 < x_1 < 1.00033$, $0.932 < x_2 < 0.936$. So, on the interval $[0, x_2)$ the second derivative is positive, on the interval (x_2, x_1) the second derivative is negative and on $(x_1, +\infty)$ the second derivative is positive again.

Let us estimate the point of maximal value of f. It is clear that $p_m = \left(\dfrac{-p}{p-q}\right)^{1/q}$ and we have: $0.9695 < \left(\dfrac{-p}{p-q}\right)^{1/q} < 0.9696$, so $0.9695 < p_m < 0.9696$. It follows that $0.4625 < f(p_m) < 0.464$. It is clear that conditions (i)-(v) are satisfied. Let us discuss condition (vi). We know that

$p_0 \in (0, x_2)$, where f' increases monotonically. It follows that $f'(0.91) < f'(p_0) < f'(0.9101)$ and by numerical evaluation $f'(0.91) > 0.908$, $f'(0.9101) < 0.91$. So, we have that
$a = 0.455 < 0.908 < f'(0.91) < f'(p_0) < f'(0.9101) < 0.91 < 2.826 < \frac{3\pi}{2}e^{-a}$ and condition (vi) is satisfied.

Let us check condition (vii). We know that $p_1 \in (x_2, x_1)$, where f' decreases monotonically. It follows that $f'(0.9815) < f'(p_1) < f'(0.9814) < -3.51 < -1$ and condition (vii) is satisfied.

One can show that $0.939 < p_{**} < 0.9397$ since $0.4411 < ap_m < 0.4412$ and
$f(0.939) < 0.44082 < ap_m$, $f(0.9397) > 0.4414 > ap_m$. Also one can prove that
$0.6344 < e^{-a} < 0.6345$. So, we have that $e^{-a}p_{**} + f(p_m)\frac{1-e^{-a}}{a} < 0.9691 < p_m$, and it means that condition (viii) is satisfied.

Let us discuss condition (ix). It is clear that $\max_{x\in[0,p_m]} f'(x) = f'(x_2)$ since $[0, p_m] \subset [0, x_1]$ and $\max_{x\in[0,x_1]} f'(x) = f'(x_2)$. By numerical evaluation we have that $f'(x_2) < 0.932$ and
$\frac{\max_{x\in[0,p_m]} f'(x)}{a}(1 - e^{-a}) < 0.76 < 1$, so condition (ix) is satisfied.

Let us check conditions (x). We know that $0.908 < f'(p_0) < 0.91$. We can estimate:
$1.5761 < e^a < 1.5762$ and $1.4342 < 2ae^a < 1.4344$, it follows that $f'(p_0) < 0.91 < 2ae^a$, so condition (x) is satisfied.

Now we will consider equation (4.1) with modified nonlinear functions, where $f_\alpha(x) = b\frac{x^p}{1+x^q}$ for $x \le p_1 + \delta_1$. One can choose δ_1 and δ_2 so small that condition (xi,1) is satisfied. We will discuss condition (xi) with $c_1^\alpha = c_2^\alpha = \alpha p_1$. Note that $\alpha_0 = 0$ implies $c_1^{\alpha_0} = c_2^{\alpha_0} = 0$. Condition (xi,2) is clear. Let us check condition (xi,3) taking into account that we can take small δ_2. So, we will check if $e^{-a}p_1 + f(p_m)\frac{\ln(p_1) - \ln(p_0)}{a} < p_0$ is satisfied. One can show that

$$e^{-a}p_1 + f(p_m)\frac{\ln(p_1) - \ln(p_0)}{a} < 0.62277 + \frac{0.464 \cdot 0.0764}{0.455} < 0.70077 < p_0.$$

Let us consider condition (xi,4). We need only $c_2^{\alpha_1} < ap_m$ since $p_m - \frac{c_2^\alpha}{a} > p_m - \frac{c_2^{\alpha_1}}{a}$ for $0 \le \alpha < \alpha_1$. So, we need $\alpha_1 p_1 < ap_m$. By numerical evaluation we have that $\frac{ap_m}{p_1} > 0.449$. It follows that for $0 \le \alpha \le \alpha_1 < 0.449$ condition (xi,4) is satisfied.

Let us consider condition (xi,5). We need $\frac{p_0 - p_1e^{-a}}{\frac{p_1}{a} - \frac{p_1}{a}e^{-a}} < \frac{0.2875}{0.7879} < 0.364895 < \alpha_1 < 0.499$. Let us take $\alpha_1 = 0.3649$. Then $c_1^{\alpha_1} > 0.3581 > \frac{0.4550.2875}{0.3655} > \frac{a(p_0 - p_1e^{-a})}{1 - e^{-a}}$. So, condition (xi,5) is satisfied.

Let us discuss condition (xi,6). We have:

$$(p_1 - \frac{\alpha_1 p_1}{a})e^{-a} + \frac{\alpha_1 p_1}{a} = p_1 e^{-a} + \frac{\alpha_1 p_1}{a}(1 - e^{-a}) > p_0$$

according to condition (xi,5). It is clear that one can take δ_2 so small that condition (xi,6) is satisfied.

Let us check conditions (xi,7) for $\alpha = \alpha_1$ and (xi,8′). We have:

$$\frac{\alpha_1}{a}p_1 + (p_1 - \frac{\alpha_1}{a}p_1)e^{-a} + f(\frac{p_1 + p_m}{2})\frac{\ln(p_1 - \frac{\alpha_1}{a}p_1) - \ln(\frac{p_1 + p_m}{2} - \frac{\alpha_1}{a}p_1)}{a} <$$
$$< 0.7872 + 0.12342 + 0.461 \cdot \frac{0.036}{0.455} < 0.94714 < p_m$$

is satisfied.

And
$$\frac{p_1 + p_m}{2p_1}\Big(\frac{\alpha_1}{a}p_1 + (p_1 - \frac{\alpha_1}{a}p_1)e^{-a}\Big) + f(p_m)\frac{p_1 - p_m}{a\Big(p_1 - \frac{\alpha_1}{a}p_1\Big)} <$$
$$< 0.99425 \cdot (0.7872 + 0.12342) + 0.464 \cdot 0.13591 < 0.9688 < p_m$$

is satisfied.

Obviously condition (xi,9) is satisfied since $c_1^\alpha = c_2^\alpha$.

So, we can use theorem 1 for equation (4.1) with $f(x) = b\frac{x^p}{1 + x^q}$ for $x \le p_1 + \delta_1$ in order to prove the existence of a homoclinic solution for a critical parameter α_* and theorem 3 in order to prove the existence of periodic solutions for parameters $\alpha > \alpha_*$, $\alpha \in A' \subset A$.

Let us demonstrate a numerical approximation of solutions of equation (4.1) with $f(x) = b\frac{x^p}{1 + x^q}$ for $x \le p_1 + \delta_1$, $\delta_1 = 0.00001$, $\delta_2 = 0.000015$, made by means of MATLAB. We take $x|_{[-1,0]} = 0.9101$. In order to find the approximation of solutions we use a standard function dde23 of the package MATLAB. Text of the program can be found in appendix A. Figure 4.3 shows approximation of the solution with parameter $\alpha = 0$, which tends to zero.

Figure 4.4 shows approximation of the solution with parameter $\alpha_1 = 0.3649$, where the solution lies between p_0 and p_m even on a time interval of length larger than 1 after excursion above p_1, approximately for $t = [29.1, 34.4]$.

Figure 4.5 shows approximation of the homoclinic solution with parameter $\alpha = 0.340435$.

Figure 4.6 shows approximation of the periodic solution for parameter $\alpha = 0.34182$.

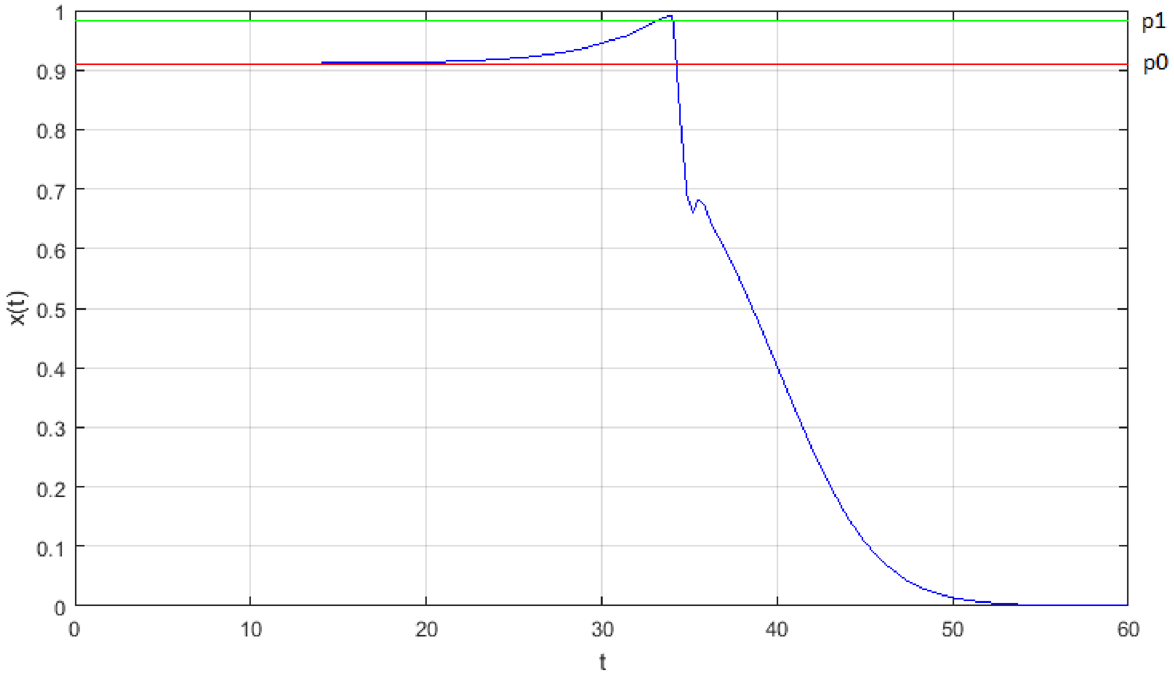

Figure 4.3: Solution with $\alpha = 0$

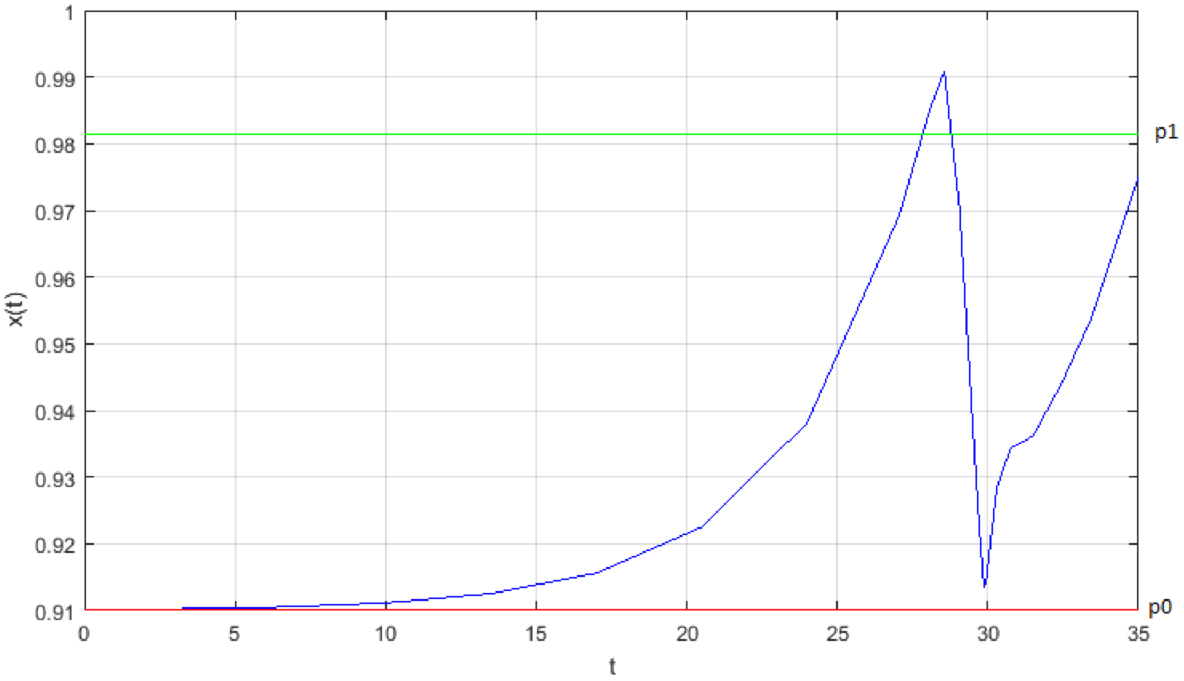

Figure 4.4: Solution with $\alpha = 0.3649$

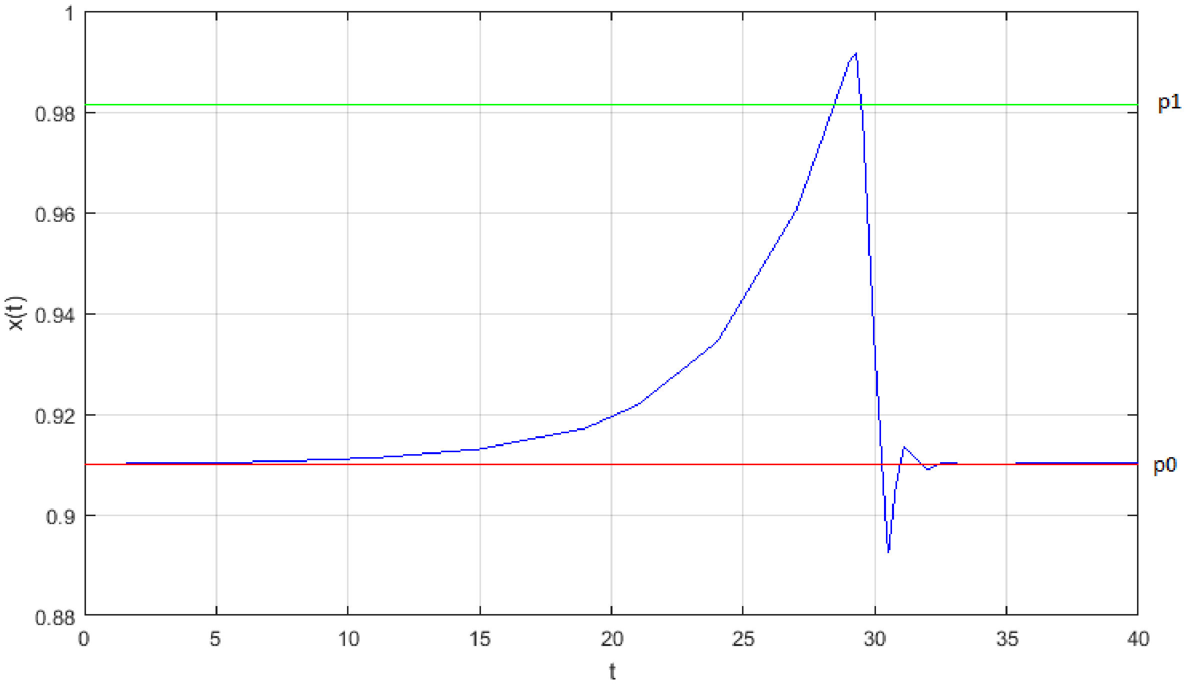

Figure 4.5: Solution with $\alpha = 0.340435$

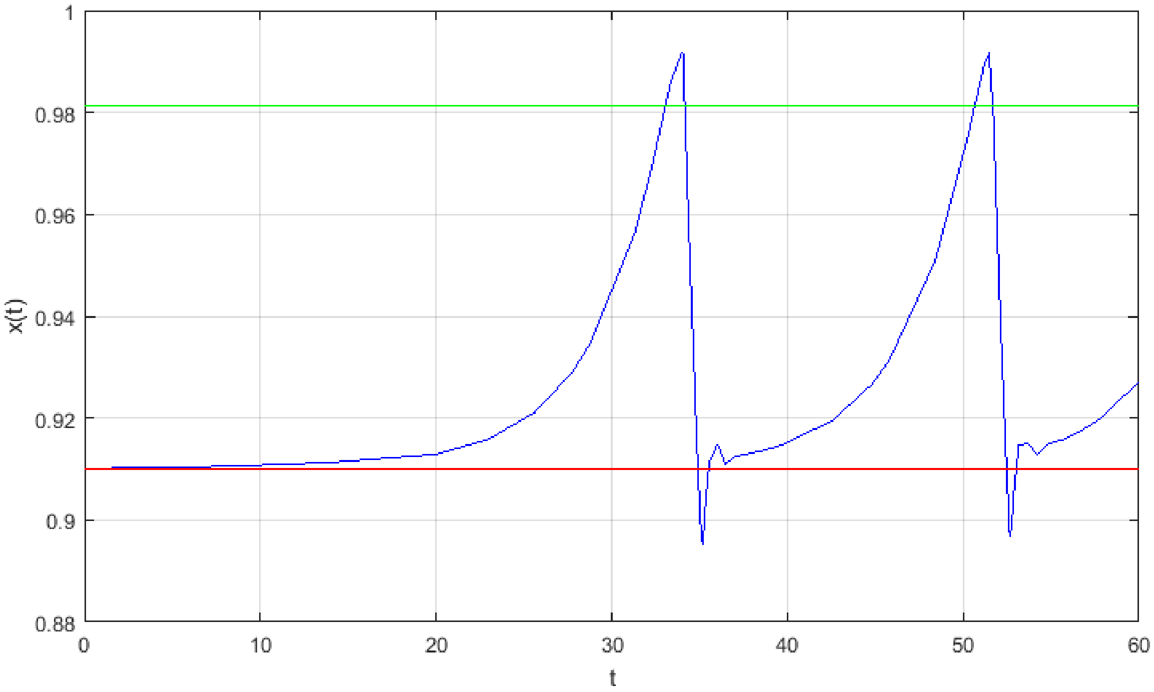

Figure 4.6: Periodic solution for $\alpha = 0.34182$

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Appendix A

Program in MATLAB for numerical results

Below you can see text of the program in MATLAB, which constructs an approximation of solutions of equation (4.1) with $f_\alpha(x) = b\dfrac{x^p}{1+x^q}$ for $x \leq p_1 + \delta_1$ with constant parameters $a = 0.455$, $b = 0.5$, $p = 2$ and $q = 136$, where we take $\delta_1 = 0.00001$ and $\delta_2 = 0.000015$. Also this program checks trueness of conditions (i)-(xi) for this equation.

```
function [ dxdt ] = ddex1de( t,x,Z )
global a b p q;
xlag1 = Z;
dxdt = -a*x+b*((xlag1^p)/(1+(xlag1^q)));
end
```

```
function [ S ] = ddex1hist( t )
global a b p q;
S = 0.9101;
end
```

```
function [ dxdt ] = ddex2de( t,x,Z )
global a b p q s x1 c1 c2 delta1 delta2;
xlag1 = Z;
if (xlag1 < (x1+delta1))
     dxdt = -a*x+b*((xlag1^p)/(1+(xlag1^q)));
   elseif (xlag1 < (x1+delta2))
```

```
          dxdt = -a*x+c1*xlag1+c2;
        else
            dxdt = -a*x+s*x1;
end;



end

function [ y] = derivativeofnonlinearfunction( x )
global  b p q;
y = (b*(x^(p-1))*(p+p*(x^q)-q*(x^q)))/((1+x^q)^2);

end

function [ y ] = eq1( x )
global a b p q;
y = -a*x+b*((x^p)/(1+(x^q)));

end

function [ y ] = eq2( x )
global a b p q x1;
y = b*((x^p)/(1+(x^q))) - a*x1;

end

function [ y ] = eq3( x )
global a b p q xm;
y = b*((x^p)/(1+(x^q))) - a*xm;

end

global a b p q;
```

```
a = 0.455;
b = 0.5;
p = 2;
q = 136;

figure
        fplot('linearfunction', [0 1.2])
        hold on
        fplot('nonlinearfunction', [0 1.2], 'r')
        grid on

function [ y ] = linearfunction( x )
global a;
y = a*x;

end

function [ y] = nonlinearfunction( x )
global a b p q;
y = b*((x^p)/(1+(x^q)));
end

global a b p q x0 x1 xm s c1 c2 delta1 delta2;
a = 0.455;
b = 0.5;
p = 2;
q = 136;
z1 = fzero('eq1', [0.8,0.95]);%[0.8,0.95], [0.8,0.9]
x0 = double(z1);
z2 = fzero('eq1', [0.95,2]);%[0.95,2], [0.9,2]
x1 = double(z2);
xm = (p/(q-p))^(1/q);
```

```
xstar = fzero('eq2', [0.8,0.96]); %[0.8,0.96], [0.8,0.93]
xdoublestar = fzero('eq3', [0.5, 0.96]);
derivative = @(x) (b*(x^(p-1))*(p+p*x^q-q*x^q))/((1+x^q)^2);
pointofmaxderivative = fminbnd(@(x) -derivative(x),0,1);
maxderiv = derivativeofnonlinearfunction(pointofmaxderivative);
derivinx0 = derivativeofnonlinearfunction(x0);
condforderiveinx0 = a < derivinx0 < (3*pi/2)*exp(-a);
derivinx1 = derivativeofnonlinearfunction(x1);
condforderivinx1 = derivinx1 < -1;
condition9 = (maxderiv/a)*(1-exp(-a)) < 1;
condition8 = exp(-a)*xdoublestar + nonlinearfunction(xm)* (1-exp(-a))/a < xm;
check7 = a*xm/x1;
check8 = (x0-x1*exp(-a))/(x1/a-x1*exp(-a)/a);
%s = check8 + 0.00001;
s = 0.3649;
check9 = s*x1/a+(x1-s*x1/a)*exp(-a);
condition103 = exp(-a)*x1+nonlinearfunction(xm)*(log(x1)-log(x0))/a < x0;
condition107 = nonlinearfunction(x1) > a*xm;
condition108 = s*x1/a+(x1-s*x1/a)*exp(-a)+nonlinearfunction(xm+(x1-xm)/2)*(log(

condition109 = (s*x1/a+(x1-s*x1/a)*exp(-a))*((xm+(x1-xm)/2)/x1)+nonlinearfunctio

condition11 = derivinx0 < 2*a*exp(a);


n1 = nonlinearfunction(x1+delta1);
d1 = s * x1;
c1 = (d1-n1)/((x1+delta2)-(x1+delta1));
c2 = n1-c1*(x1+delta1);

clear all;
global a b p q x0 x1  xm delta1 delta2 s c1 c2;
```

```
a = 0.455;
b = 0.5;
p = 2;
q = 136;
z1 = fzero('eq1', [0.8,0.95]);%[0.8,0.95], [0.8,0.9]
x0 = double(z1);
z2 = fzero('eq1', [0.95,2]);%[0.95,2], [0.9,2]
x1 = double(z2);
xm = (p/(q-p))^(1/q);
check8 = (x0-x1*exp(-a))/(x1/a-x1*exp(-a)/a);
%s = 0;
s = 0.3649;
%s = check8 + 0.00001;
%s = 0.340435; %critical value of parameter
%s = 0.34182;
delta1 = 0.00001;
delta2 = 0.000015;
n1 = nonlinearfunction(x1+delta1);
d1 = s * x1;
c1 = (d1-n1)/((x1+delta2)-(x1+delta1));
c2 = n1-c1*(x1+delta1);
lags = 1;
sol = dde23(@ddex2de, lags, @ddex1hist, [0,35]); %[0,40] [0,60]
T = linspace(0, 35); %40
P0 = x0;
P1 = x1;
Pm = xm;
plot(sol.x, sol.y,'b');
grid on;
hold on;
```

```
plot(T, ones(size(T))*P0,'r');
hold on;
plot(T, ones(size(T))*P1,'g');
hold on;
plot(T, ones(size(T))*Pm, 'y');
hold off;
```