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DISSERTATION

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Virtual Reeb Flows

and

Odd-Symplectic Surgery

submitted by

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Abstract

We consider odd-symplectic manifolds admitting a cover by a contact manifold of bounded geometry. The characteristic foliation of these manifolds defines a dynamic that is the projection of the Reeb dynamic of the covering manifold. We prove the existence of closed contractible characteristics in several cases. Furthermore we define a surgery construction along isotropic spheres in an odd-symplectic manifold that leads to a symplectic cobordism.

Abstract

Wir betrachten ungerad-dimensionale symplektische Mannigfaltigkeiten, die eine Überlagerung durch eine Kontaktmannigfaltigkeit mit beschränkter Geometrie besitzen. Die charakteristische Blätterung dieser Mannigfaltigkeiten definiert eine Dynamik, welche mit der projizierten Reebdynamik der überlagernden Mannigfaltigkeit übereinstimmt. Wir zeigen die Existenz einer geschlossenen, zusammenziehbaren Charakteristik in verschiedenen Fällen. Des weiteren definieren wir eine Chirurgiekonstruktion entlang isotroper Sphären in ungerad-dimensionalen symplektischen Mannigfaltigkeiten, welche symplektischen Kobordismen liefert.

Contents

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CHAPTER 1

Introduction

A Hamiltonian function on a symplectic manifold gives rise to a dynamical system. Each characteristic of this system stays on an energy hypersurface of the Hamiltonian function. It is interesting to study these characteristics. For a general Hamiltonian function there is little hope to fully understand the characteristics on an arbitrary hypersurface as the Horocycle flow shows, see [3]. Therefore, we have to make some assumptions about the hypersurface. Energy surfaces of contact type are a particular understandable class. In 1978, Alan Weinstein conjectured that every closed hypersurface of contact type carries a closed characteristic. This conjecture has been proven in several cases: starshaped hypersurfaces in $\mathbb{R}^{2n}$ [46], overtwisted contact manifolds [29] and three-dimensional contact manifolds [45]. Recently, Fish–Hofer [15] proved the existence of a non-dense characteristic on any closed, non-empty, regular energy hypersurface in $\mathbb{R}^4$.

These hypersurfaces are odd-symplectic in the sense of Chapter 2. We will prove the existence of a closed characteristic for odd-symplectic manifolds that admit a covering by a certain contact manifold.

We begin our discussion with an outline of the concepts we will work with. In particular we will give the definition of a virtually contact structure on a covering $\pi : M' \to M$ of an odd-symplectic manifold (M, ω), this definition was introduced in [12]. Afterwards, in Chapter 3 and 4, we discuss the geometry of the covering space M' and consider holomorphic discs in the symplectisation of M', in particular those subject to a certain boundary condition that we will specify later. To be more precise, we will carry out a bubbling off analysis as in [29,30], but adapt the technique to a non-compact base manifold M', see Chapter 6. In Chapter 7 we will use the technical preliminaries to conclude the existence of closed characteristics in several odd-symplectic manifolds supporting a virtually contact structure. In particular, this yields a some huge classes of non-compact contact manifolds admitting a closed Reeb orbit. Note that all these results were already published as [5] by Bae–Zehmisch and the author, but we will give more detailed explanations on the bubbling off analysis and on the methods that yield closed characteristics.

In Chapter 8 we will describe a surgery construction for odd-symplectic manifolds. For some special cases of this surgery we will explain how this construction is compatible with a possible virtually contact structure. Moreover, our surgery construction gives a symplectic cobordism between odd-symplectic manifolds. We continue with a more abstract consideration of cobordisms between odd-symplectic manifolds and end our discussion with the definition of a cobordism between virtually contact structures.

CHAPTER 2

Odd-Symplectic Geometry

We give the basic definitions of odd-symplectic manifolds and virtually contact structures. Further we formulate some elementary properties of these manifolds and present an alternative perspective on some of these subjects.

Definition 2.1. A pair (M^{2n-1}, ω) consisting of a smooth, oriented, $(2n-1)$-dimensional manifold M without boundary and a closed 2-form ω is called **odd-symplectic manifold** if ω is maximally non-degenerate, i.e., the kernel $\ker \omega$ of ω is a line field distribution. This distribution is called **characteristic foliation** of (M, ω).

An **odd-symplectic diffeomorphism** or **odd-symplectomorphism** is an orientation preserving diffeomorphism $\varphi : (M_0, \omega_0) \to (M_1, \omega_1)$ between odd-symplectic manifolds with $\varphi^*\omega_1 = \omega_0$.

Examples 2.2. The easiest example of an odd-symplectic manifold is an oriented hypersurface M of codimension 1 in a symplectic manifold (W, Ω) together with the 2-form $\omega := \Omega|_{TM}$. It is obvious that ω is closed and as we will see in Remark 2.3 it is easy to see that it is also non-degenerate.

A second large class of examples is provided by contact manifolds. Indeed, if (M, α) is a contact manifold, then $(M, \mathrm{d}\alpha)$ is an odd-symplectic manifold. The characteristic foliation is spanned by the Reeb vector field.

In a given example it is often easier to check the following alternative condition for non-degeneracy.

Remark 2.3. The 2-form ω is non-degenerate if and only if ω^{n-1} is nowhere vanishing, i.e., at each point $p \in M$ there exist tangent vectors $v_1, \dots, v_{2n-2} \in T_pM$ such that $\omega_p^{n-1}(v_1, \dots, v_{2n-2}) \neq 0$.

Proof. Let us first assume that ω is maximally non-degenerate. Let p be a point in M. By a parametric version of the standard form for skew-symmetric bilinear maps (see [8, Theorem 1.1] for the unparametric version) we find local vector fields $U_1, \dots, U_k, E_1, \dots, E_m, F_1, \dots, F_m$ on a neighbourhood U of p which form a basis of

TU and satisfy

$$\begin{aligned}
&\omega(U_i, V) = 0, && \text{for all } 1 \leq i \leq k \text{ and } V \in \Gamma(TU),\\
&\omega(E_i, E_j) = 0 = \omega(F_i, F_j), && \text{for all } 1 \leq i, j \leq m,\\
&\omega(E_i, F_j) = \delta_{ij}, && \text{for all } 1 \leq i, j \leq m.
\end{aligned}$$

Since we know that $\dim(\ker \omega) = 1$, we get that $k = 1$ and therefore

$$\begin{aligned}
\omega^{n-1}(E_1, F_1, \ldots, E_{n-1}, F_{n-1}) &= (n-1)! \cdot \omega(E_1, F_1) \cdot \ \ldots \ \cdot \omega(E_{n-1}, F_{n-1})\\
&= (n-1)!\,.
\end{aligned}$$

Thus for each point $p \in M$ we find vector fields defined near p such that ω^{n-1} does not vanish on these vector fields.

On the other hand, if we assume that ω^{n-1} is nowhere vanishing then by the same standard form argument we have $\dim(\ker \omega) \geq 1$ at each point. Near a point $p \in M$ we choose a local basis as in said standard form and for convenience of notation we write $E_{m+i} := F_i$ for $1 \leq i \leq m$. Now take vector fields $V_1, \ldots, V_{2n-2}$ such that ω^{n-1} does not vanish on this tuple near p. By expressing these vector fields in the chosen basis and by linearity of ω^{n-1} we get

$$0 \neq \omega^{n-1}(V_1, \ldots, V_{2n-2}) = \sum_{\varphi} a(\varphi) \cdot \omega^{n-1}(E_{\varphi(1)}, \ldots, E_{\varphi(2n-2)}),$$

where the sum is taken over all maps $\varphi : \{1, \ldots, 2n-2\} \to \{1, \ldots 2m\}$ and $a(\varphi)$ denotes some real coefficient. Since each vector field U_j lies in the kernel of ω, all summands containing at least one argument equal to some U_j vanish. We assert that the only way for the $E_{\varphi(i)}$ to be linear independent is if φ is injective and therefore m satisfies $m \geq n-1$. For dimensional reasons we must have $m = n-1$ and thus $k = 1$. Hence the kernel is a line bundle distribution. □

In many situations it is useful to complete an odd-symplectic form to a volume form. This completion is achieved by a 1-form.

Definition 2.4. Let (M, ω) be an odd-symplectic manifold oriented by some volume form vol_M. A 1-form γ satisfying $\gamma \wedge \omega^{n-1} = \mathrm{vol}_M$ is called **framing** of (M, ω).

Lemma 2.5. *Let (M, ω) be an odd-symplectic manifold oriented by some volume form vol_M. Then there exists a framing γ. Furthermore there exists a global vector field X with $\iota_X \mathrm{vol}_M = \omega^{n-1}$. In particular, X spans the kernel of ω and is nowhere vanishing.*

Proof. If there exists a vector field X satisfying $\iota_X \mathrm{vol}_M = \omega^{n-1}$, the fact that ω^{n-1} is nowhere vanishing tells us that the same is true for X. Moreover, we get

$$0 = \iota_X\left(\iota_X \mathrm{vol}_M\right) = \iota_X \omega^{n-1} = (n-1) \cdot \iota_X \omega \wedge \omega^{n-2}.$$

Since ω^{n-2} is also nowhere vanishing, we see that $\iota_X\omega = 0$, so X lies in the kernel of ω, is non-vanishing and therefore spans the kernel of ω.

To obtain X, notice that we have a bundle map

$$\begin{aligned}\Phi : \Gamma(M) &\longrightarrow \Omega^{2n-2}(M)\\ X &\longmapsto \iota_X \mathrm{vol}_M.\end{aligned}$$

By dimensional reasons this map is fibrewise an isomorphism if and only if it is fibrewise injective. If $\iota_{X_p}(\mathrm{vol}_M)_p$ vanishes, X_p has to be zero. Otherwise we could complete X_p to a basis and inserting the remaining basis vectors into $\iota_{X_p}(\mathrm{vol}_M)_p$ would give a non-zero value.

Hence, we obtain the desired vector field X as $\Phi^{-1}(\omega^{n-1})$. Since ω^{n-1} and vol_M are smooth, we get a smooth vector field X.

Choosing an arbitrary Riemannian metric g on M, we can define a 1-form $\widetilde{\gamma}$ by $\widetilde{\gamma} := \iota_X g$ where X is the vector field determined by $\iota_X \mathrm{vol}_M = \omega^{n-1}$. By inserting X into $\widetilde{\gamma} \wedge \omega^{n-1}$ we obtain $\|X\|_g^2 \cdot \omega^{n-1}$, which is a nowhere vanishing $(2n-2)$-form. So $\widetilde{\gamma} \wedge \omega^{n-1}$ is nowhere vanishing and hence a multiple of the volume form, i.e., $\widetilde{\gamma} \wedge \omega^{n-1} = f \cdot \mathrm{vol}_M$ for some nowhere vanishing function f. We conclude that $\gamma := \frac{1}{f}\widetilde{\gamma}$ is a framing. □

Remark 2.6. For our later studies we are interested in the dynamics of an odd-symplectic manifold (M, ω) defined by the kernel of ω. Thus it is helpful to have a global vector field that spans this kernel as constructed in the previous lemma. The construction of this vector field used the existence of a volume form.

Note that the orientation requirement in the definition of odd-symplectic manifolds is necessary and does not follow from the other conditions: Let M denote the Möbius strip and consider the product $M \times \mathbb{R}$. Thinking of the Möbius strip as a bundle over the circle S^1, we denote the S^1-coordinate by θ and the $\mathbb{R}$-coordinate by t. In this case $\mathrm{d}\theta \wedge \mathrm{d}t$ is a maximally non-degenerate 2-form.

A closed example is obtained by replacing the Möbius strip by the Klein bottle and the line $\mathbb{R}$ by the circle S^1.

Both of these manifolds are non-orientable and hence not odd-symplectic.

The next lemma expands the first part of Example 2.2.

Lemma 2.7. *Let (W^{2n}, Ω) be a symplectic manifold, $M \subset W$ a hypersurface with respect to the orientation induced on M by Ω and Y, where Y is a vector field defined in a neighbourhood and transverse to M. Then $(M, \Omega|_{TM})$ is an odd-symplectic manifold and $\iota_Y \Omega|_{TM}$ is a framing for $\Omega|_{TM}$.*

Proof. By Remark 2.3 it suffices to show that $\Omega_p^{n-1} \neq 0$ for all $p \in M$. At $p \in M$ take $X_p \in T_pM$ with $\Omega_p(X_p, Y_p) \neq 0$. Complete (Y_p, X_p) to a symplectic basis

$(Y_p, X_p, V_1, \ldots, V_{2n-2})$ of T_pW with $V_1, \ldots, V_{2n-2} \in T_pM$. We obtain

$$\begin{aligned} 0 &\neq \Omega^n(Y_p, X_p, V_1, \ldots, V_{2n-2}) \\ &= n \cdot \Omega(Y_p, X_p) \cdot \Omega^{n-1}(V_1, \ldots, V_{2n-2}), \end{aligned}$$

so $\Omega^{n-1}|_{TM}$ does not vanish. The same argument shows that

$$\begin{aligned} 0 &\neq \Omega_p^n(Y_p, X_p, V_1, \ldots, V_{2n-2}) \\ &= (\iota_{Y_p}\Omega_p^n)(X_p, V_1, \ldots, V_{2n-2}) \\ &= n \cdot ((\iota_{Y_p}\Omega_p) \wedge \Omega_p^{n-1})(X_p, V_1, \ldots, V_{2n-2}). \end{aligned}$$

Therefore, $\iota_Y \Omega \wedge \Omega^{n-1}|_{TM}$ is a volume form on M. □

The definition of a framing behaves well under structure preserving maps. The following lemma states that framings are pulled back to framings under odd-symplectic diffeomorphism up to scaling.

Lemma 2.8. *Let $(M_i, \omega_i), i = 0, 1$, be orientable odd-symplectic manifolds and $\varphi : M_0 \to M_1$ an odd-symplectomorphism. Then $\varphi^*\gamma_1 \wedge \omega_0^{n-1}$ is a positive volume form for any framing γ_1 of ω_1.*

Proof. Let vol_{M_i} be a volume form on M_i. Since $\varphi^*\omega_1 = \omega_0$, we compute

$$\varphi^*\gamma_1 \wedge \omega_0^{n-1} = \varphi^*\gamma_1 \wedge (\varphi^*\omega_1)^{n-1} = \varphi^*(\gamma_1 \wedge \omega_1^{n-1}) = \varphi^*\mathrm{vol}_{M_1} = f \cdot \mathrm{vol}_{M_0}$$

for some smooth non-vanishing function $f : M_0 \to \mathbb{R}^+$. □

Definition 2.9. Let M be a smooth manifold, g a Riemannian metric and η a k-form on M. The **pointwise $\mathbf{C^0}$-norm** of η with respect to g at a point $p \in M$ is given by

$$\|\eta_p\|_{C^0} = \sup_{v_1, \ldots, v_k} |\eta_p(v_1, \ldots, v_k)|,$$

where the supremum is taken over all g-unit vectors in T_pM. The **$\mathbf{C^0}$-norm** of η is

$$\|\eta\|_{C^0} = \sup_{p \in M} \|\eta_p\|_{C^0}.$$

Definition 2.10 ([12])**.** An odd symplectic manifold (M, ω) supports a **virtually contact structure** $(\pi : M' \to M, \omega, \alpha, g)$ if there exist a covering $\pi : M' \to M$ of M by a contact manifold (M', α) with $\mathrm{d}\alpha = \pi^*\omega$ and a constant $K > 0$ such that

i) the form α is bounded with respect to the lifted Riemannian metric $g' := \pi^*g$, i.e., $\|\alpha\|_{C^0} \leq K$,

ii) for all $v \in \ker \mathrm{d}\alpha$ it holds that $|\alpha(v)| \geq \frac{1}{K}\|v\|_{g'}$.

The manifold M is oriented by the volume form vol_M. With respect to the orientation of M' given by $\pi^*\mathrm{vol}_M$ we additionally require that $\alpha \wedge \mathrm{d}\alpha^{n-1} > 0$.

The listed properties will be referred to as **boundedness** conditions on the contact manifold (M, α).

An odd-symplectic manifold (M, ω) supporting a virtually contact structure is called **virtually contact manifold**.

A virtually contact structure is **non-trivial** if ω is not the exterior derivative of a contact form on M.

These objects are studied in [4, 5, 12, 48].

Remark 2.11. If (M, ω) is compact and supports a virtually contact structure with respect to the metric g then any other choice of a Riemannian metric $\widehat{g}$ also leads to a virtually contact structure $(\pi : M' \to M, \omega, \alpha, \widehat{g})$. This is due to the fact that any two Riemannian metrics on a compact manifold are equivalent.

Examples 2.12. The first examples of virtually contact structures were given in [12, Chapter 5]. These examples are energy hypersurfaces in the twisted cotangent bundle of energy above the Mañé critical value.

Further examples were constructed in [48]. We repeat these constructions in Section 7.2 and 8.1 and expand the class of possible surgery operations.

There is an alternative description of virtually contact structures that is more focused on the geometric phenomena.

Definition 2.13. An odd symplectic manifold (M, ω) supports a **virtually contact structure** $(\pi : M' \to M, \omega, \alpha, g)$ if there exist a covering $\pi : M' \to M$ of M by a contact manifold (M', α) with $\mathrm{d}\alpha = \pi^*\omega$ and a constant $K > 0$ such that

i) the length of the Reeb vector field R of α with respect to the lifted Riemannian metric $g' := \pi^*g$ is bounded, i.e., $\|R\|_{g'} \leq K$,

ii) with respect to the g'-orthogonal splitting $TM' = \xi^\perp \oplus \xi$, where $\xi = \ker \alpha$, the $\xi^\perp$-component $R^\perp$ satisfies $\|R^\perp\|_{g'} > \frac{1}{K}$.

Lemma 2.14. *The Definitions 2.10 and 2.13 are equivalent.*

Proof. Let $(\pi : M' \to M, \omega, \alpha, g)$ denote the data as in the definitions and set $\xi := \ker \alpha$ and $g' := \pi^*g$. In both cases it suffices to show the estimates in the respective definition.

Starting with 2.10 we have to show that

$$\|R\|_{g'} \leq K \quad \text{and} \quad \tfrac{1}{K} \leq \|R^\perp\|_{g'},$$

where R denotes the Reeb vector field and $R^\perp$ is the $\xi^\perp$-component of R in the g'-orthogonal splitting $\xi^\perp \oplus \xi$. By Definition 2.10 we have $|\alpha(v)| \geq \frac{1}{C} \cdot \|v\|_{g'}$ for all $v \in \ker \pi^*\omega$. We can apply this to the Reeb vector field and obtain

$$\|R\|_{g'} \leq K \cdot |\alpha(R)| = K.$$

For the second estimate we write $R = R^{\perp} + Y \in \xi^{\perp} \oplus \xi$ as in Definition 2.13. We know that

$$1 = |\alpha(R)| = |\alpha(R^{\perp}) + \alpha(Y)|.$$

Since Y is contained in the kernel of α this equation becomes $1 = |\alpha(R^{\perp})|$. By the first estimate in Definition 2.10 we have $|\alpha(R^{\perp})| \leq C \cdot \|R^{\perp}\|_{g'}$. So we can conclude

$$\frac{1}{K} \leq \|R^{\perp}\|_{g'}.$$

Therefore every virtually contact structure in the sense of Definition 2.10 is a virtually contact structure in the sense of Definition 2.13.

Let us assume we begin with a virtually contact structure in the sense of Definition 2.13. We have to show that

$$\tfrac{1}{K} \cdot \|v\|_{g'} \leq |\alpha(v)| \quad \text{and} \quad |\alpha(w)| \leq K \cdot \|w\|_{g'}$$

for all $v \in \ker \pi^*\omega$ and $w \in TM'$. Take $v \in \ker \pi^*\omega$ and write $v = \mu \cdot R$ for some $\mu \in \mathbb{R}$. Using the length estimate for the Reeb vector field we get

$$\|v\|_{g'} = |\mu| \cdot \|R\|_{g'} \leq |\mu| \cdot K.$$

It follows that

$$|\alpha(v)| = |\alpha(\mu \cdot R)| = |\mu| \geq \frac{\|v\|_{g'}}{K}.$$

It remains to show that $|\alpha(w)| \leq K \cdot \|w\|_{g'}$ for all $w \in TM'$. We write $w = \mu \cdot R + w_\xi \in \mathbb{R} \cdot R \oplus \xi$ and replace R by $R^{\perp} + Y \in \xi^{\perp} \oplus \xi$ and hence

$$w = \mu \cdot R^{\perp} + (\mu \cdot Y + w_\xi).$$

By the Pythagorean theorem we have

$$\|w\|_{g'}^2 = \|\mu \cdot R^{\perp}\|_{g'}^2 + \|\mu \cdot Y + w_\xi\|_{g'}^2 \geq |\mu|^2 \cdot \|R^{\perp}\|_{g'}^2,$$

and therefore

$$\|w\|_{g'} \geq |\mu| \cdot \|R^{\perp}\|_{g'}.$$

Note that $w = \mu \cdot R + w_\xi \in \mathbb{R} \cdot R \oplus \xi$ implies $\alpha(w) = \mu$. Combining this with the previous estimates and $\frac{1}{K} \leq \|R^{\perp}\|_{g'}$ we get

$$|\alpha(w)| = |\mu| \leq \frac{\|w\|_{g'}}{\|R^{\perp}\|_{g'}} \leq K \cdot \|w\|_{g'}.$$

This shows that every virtually contact structure in the sense of Definition 2.13 is a virtually contact structure in the sense of Definition 2.10. □

Definition 2.15. An odd-symplectic manifold (M, ω) supporting a virtually contact structure $(\pi : M' \to M, \omega, \alpha, g)$ is **somewhere contact** if there exists a non-empty, open set $U \subset M$ such that $\omega|_{TU}$ admits a primitive α_U on U with $\alpha_U \wedge (\mathrm{d}\alpha_U)^{n-1} \neq 0$ and $\pi^*\alpha_U = \alpha|_{\pi^{-1}(U)}$.

Lemma 2.16 (Darboux theorem for odd-symplectic manifolds)**.** *Let (M, ω) be an odd-symplectic manifold and $p \in M$. Then there exists a neighbourhood U of p and a 1-form α_U such that $\mathrm{d}\alpha_U = \omega|_U$. Moreover, α_U can be chosen to be a contact form on U and there are local coordinates $(z, \mathbf{x}, \mathbf{y})$ such that $\omega = \mathrm{d}\mathbf{x} \wedge \mathrm{d}\mathbf{y}$.*

Proof. Let U be a contractible neighbourhood of p. Using the Poincaré lemma we obtain a 1-form α' on U with $\omega|_U = \mathrm{d}\alpha'$. We may assume that U is the flow-box of some local vector field spanning the kernel of ω, for example the restriction of X as in Lemma 2.5 to U. Without loss of generality we consider $M = \mathbb{R}^{2n-1}$ with coordinates $(z, x_1, y_1, \ldots, x_{n-1}, y_{n-1})$ and $\ker \omega|_U = \mathbb{R} \cdot \partial_z$. Adding $\mathrm{d}z$ to α', if necessary, we can ensure that $\alpha(\partial_z) \neq 0$ and still $\mathrm{d}\alpha = \omega$. We claim that α is a contact form and check this by applying the definition. First observe that $\alpha \wedge (\mathrm{d}\alpha)^{n-1} = \alpha \wedge \omega^{n-1}$ and second that $\iota_{\partial_z}(\alpha \wedge \omega^{n-1}) = \alpha(\partial_z) \cdot \omega^{n-1}$ is non-vanishing. Therefore $\alpha \wedge (\mathrm{d}\alpha)^{n-1}$ is also non-vanishing and α is a contact form. The rest follows from the Darboux theorem for contact manifolds [19, Theorem 2.5.1]. □

Remark 2.17. Note that Lemma 2.16 does not imply that every virtually contact structure is somewhere contact: The contact form on the covering is in general not invariant with respect to deck transformations.

In the following we consider ourselves with non-trivial virtual contact structures.

Remark 2.18 ([4])**.** If (M, ω) is supports a virtually contact structure $(\pi : M' \to M, \omega, \alpha, g)$ with a finite covering π, then $(M, \omega = \mathrm{d}\alpha_M)$ is already a contact manifold. To see this, assume that $\pi : M' \to M$ is a finite covering, i.e., each point has finitely many preimages under π. Let G be the (finite) group of deck transformations. Recall that $\alpha \wedge (\mathrm{d}\alpha)^{n-1}$ is positive with respect to the orientation given by $\pi^* \mathrm{vol}_M$.

We define an alternative contact form $\widetilde{\alpha}$ on M' by

$$\widetilde{\alpha} = \frac{1}{|G|} \sum_{\varphi \in G} \varphi^* \alpha,$$

where α is the contact form on the cover M' with $\mathrm{d}\alpha = \pi^*\omega$. Then $(\pi : M' \to M, \omega, \widetilde{\alpha}, g)$ is a trivial virtually contact structure. First note that $\widetilde{\alpha}$ is well-defined and invariant under the group action. Further it satisfies

$$
\begin{aligned}
\mathrm{d}\widetilde{\alpha} &= \frac{1}{|G|}\sum_{\varphi\in G}\varphi^*\mathrm{d}\alpha = \frac{1}{|G|}\sum_{\varphi\in G}\varphi^*\pi^*\omega \\
&= \frac{1}{|G|}\sum_{\varphi\in G}(\pi\circ\varphi)^*\omega = \frac{1}{|G|}\sum_{\varphi\in G}\pi^*\omega \\
&= \frac{1}{|G|}|G|\pi^*\omega = \pi^*\omega,
\end{aligned}
$$

so $\widetilde{\alpha}$ is indeed a primitive of $\pi^*\omega$. Here $\pi\circ\varphi = \pi$ since all $\varphi \in G$ are deck transformations.

The next step is to show that $\widetilde{\alpha}$ is indeed a contact structure. To this end, we check the contact condition.

$$
\begin{aligned}
\widetilde{\alpha}\wedge(\mathrm{d}\widetilde{\alpha})^{n-1} &= \widetilde{\alpha}\wedge(\pi^*\omega)^{n-1} = \widetilde{\alpha}\wedge\pi^*(\omega^{n-1}) \\
&= \left(\frac{1}{|G|}\sum_{\varphi\in G}\varphi^*\alpha\right)\wedge\pi^*(\omega^{n-1}) = \frac{1}{|G|}\sum_{\varphi\in G}\varphi^*\alpha\wedge\pi^*\omega^{n-1}.
\end{aligned}
$$

By the G-invariance of $\pi^*\omega$ this becomes

$$
\frac{1}{|G|}\sum_{\varphi\in G}\varphi^*(\alpha\wedge\pi^*\omega^{n-1}) = \frac{1}{|G|}\sum_{\varphi\in G}\varphi^*(f\cdot\mathrm{vol}_{M'})
$$

for some positive function f. Since f is positive, we obtain

$$
\begin{aligned}
\widetilde{\alpha}\wedge(\mathrm{d}\widetilde{\alpha})^{n-1} &= \frac{1}{|G|}\sum_{\varphi\in G}\varphi^*(f\cdot\mathrm{vol}_{M'}) = \frac{1}{|G|}\sum_{\varphi\in G}(f\circ\varphi)\cdot\varphi^*\mathrm{vol}_{M'} \\
&\geq \frac{1}{|G|}\left(\min_{M'} f\right)\cdot\sum_{\varphi\in G}\mathrm{vol}_{M'} = \left(\min_{M'} f\right)\cdot\mathrm{vol}_{M'} \\
&> 0,
\end{aligned}
$$

where we used that any deck transformation preserves the orientation of M'.

Since $\widetilde{\alpha}$ is invariant under the action of the deck transformation group it descents to a 1-form on M. As $\widetilde{\alpha}$ is a contact form on M' with $\pi^*\alpha_M = \widetilde{\alpha}$ we see that α_M is also a contact form which satisfies $\mathrm{d}\alpha_M = \omega$. Indeed,

$$
\begin{aligned}
0 < \widetilde{\alpha}\wedge(\mathrm{d}\widetilde{\alpha})^{n-1} &= \pi^*\alpha_M\wedge(\mathrm{d}\pi^*\alpha_M)^{n-1} \\
&= \pi^*\left(\alpha_M\wedge(\mathrm{d}\alpha_M)^{n-1}\right),
\end{aligned}
$$

and thus we have

$$
0 < \alpha_M\wedge(\mathrm{d}\alpha_M)^{n-1},
$$

since π is orientation preserving.

CHAPTER 3

A Tame Geometry

The covering space M' that appears in a virtually contact structure is of a bounded geometry. We explore how the lifted metric g' is related to a metric induced by α and an almost complex structure j. We use these structures to prove an isoperimetric inequality for certain curves in M'.

3.1. The Induced Structure

Let $(\pi : M' \to M, \omega, \alpha, g)$ be a contact manifold with compact base manifold (M, ω). Further denote by ξ the kernel of α and by g' the lift of g with respect to the covering map π. On the contact structure $\xi = \ker \alpha$ the 2-form $\mathrm{d}\alpha$ is non-degenerate and we can solve the equation

$$\mathrm{d}\alpha = g'(\Phi(\cdot), \cdot)$$

to obtain a skew adjoint vector bundle isomorphism $\Phi : \xi \to \xi$. Then $-\Phi^2$ is self adjoint and positive definite, hence its square root is well defined. Using [19, Proposition 2.4.5] and the construction in [19, Proposition 1.3.10] the complex structure on ξ obtained by

$$j := \Phi \circ \sqrt{-\Phi^2}^{-1}$$

is compatible with $\mathrm{d}\alpha$, i.e.,

$$g_j := \mathrm{d}\alpha(\cdot, j\cdot)$$

is a bundle metric on ξ. In this situation we say that the 2-form $\mathrm{d}\alpha$ tames the complex structure j. In the following we compare the geometries of ξ induced by the restriction of g' and by g_j with each other.

Lemma 3.1. *The norm $\|\cdot\|$ induced by g_j and the restriction of the norm $\|\cdot\|_{g'}$ to ξ are uniformly equivalent, i.e., there exist constants $c_1, c_2 > 0$ such that on ξ*

$$\tfrac{1}{c_1}\|\cdot\|_{g'} \leq \|\cdot\|_j \leq c_2\|\cdot\|_{g'}.$$

Remark 3.2. Let us take a closer look at the second condition in Definition 2.13. By computing the sine of the angle ϱ between the Reeb vector field R and the contact structure $\xi = \ker \alpha$ we see that ϱ is bounded away from 0 and π. To make this precise, write $R = R^\perp + Y \in \xi^\perp \oplus \xi \cong TM'$. The sine of ϱ is given by

$$\sin(\varrho) = \frac{\|R^\perp\|_{g'}}{\|R\|_{g'}}.$$

Applying the estimates given in Definition 2.13 we obtain

$$\sin(\varrho) > \frac{1}{K^2} > 0.$$

Hence there exists an $\varepsilon > 0$ such that the angle ϱ is contained in $[\varepsilon, \pi - \varepsilon]$.

Remember that $T\pi(R)$ spans the kernel of ω, so the Reeb vector field is invariant under deck transformations up to rescaling. In general this will not be true for the contact structure ξ. That said the previous discussions show that the image of the contact structure under the projection π is contained in a cone like subset of TM that stays away from the kernel of ω.

Sketch of proof of Lemma 3.1. The proof is based on the observation that the eigenvalues of $\sqrt{-\Phi^2}$ are uniformly bounded away from 0 and from above. This observation is achieved by considering the map $\Omega : TM \to T^*M, v \mapsto \iota_v\omega$. The kernel of Ω is the characteristic line bundle of (M, ω). We regard the cone like subset C of TM consisting of all tangent vectors whose angle to the characteristic distribution is greater or equal than some uniform constant $c_0 > 0$. As we just saw, we have

$$T\pi(\xi) \subset C$$

if c_0 is chosen small enough. The map Ω has uniform upper and lower bounds on the compact subset $C \cap STM$, where STM is the unit cotangent bundle with respect to g. The set C lifts to a cone like subbundle C' of TM' and the bounds for Ω imply the existence of the same bounds for the map $\Omega' : TM' \to T^*M', v \mapsto \iota_v \mathrm{d}\alpha$ and $C' \cap STM'$. The next point we have to address is that

$$\|\Omega'(v)\|_{(g')^\flat} = \|\Phi(v)\|_{g'}$$

for all $v \in \xi$. Here $(g')^\flat$ denotes the dual metric of g' on M' defined by

$$\begin{aligned}(g'_p)^\flat : T^*_pM' \times T^*_pM' &\longrightarrow \mathbb{R}\\ (\alpha_1, \alpha_2) &\longmapsto (g'_p)^\flat(\alpha_1, \alpha_2) = g'_p(v_1, v_2),\end{aligned}$$

where v_i is the dual vector of α_i uniquely determined by $\alpha_i = g'_p(v_i, \cdot)$. That is we have the same bounds for Φ. To conclude the lemma we observe that

$$g_j = g'\left(\sqrt{-\Phi^2}\,\cdot\,,\,\cdot\,\right).$$

Indeed,

$$\begin{aligned} g_j &= \mathrm{d}\alpha\left(\,\cdot\, , \Phi \circ \sqrt{-\Phi^2}^{-1} \right) \\ &= g'\left(\Phi \,\cdot\, , \Phi \circ \sqrt{-\Phi^2}^{-1} \,\cdot\, \right). \end{aligned}$$

Since Φ is skew adjoint, this becomes

$$g'\left(-\Phi^2 \,\cdot\, , \sqrt{-\Phi^2}^{-1} \,\cdot\, \right) = g'\left(\sqrt{-\Phi^2} \circ \sqrt{-\Phi^2} \,\cdot\, , \sqrt{-\Phi^2}^{-1} \,\cdot\, \right).$$

The square root of a self adjoint linear map is also self adjoint, thus we obtain

$$g_j = g'\left(\sqrt{-\Phi^2} \,\cdot\, , \,\cdot\, \right).$$

Therefore,

$$\lambda_1 \|\cdot\|_{g'}^2 \leq \|\cdot\|_j^2 \leq \lambda_2 \|\cdot\|_{g'}^2,$$

where λ_1 and λ_2 are the smallest and the largest eigenvalue of $\sqrt{-\Phi^2}$, respectively. Observe that the smallest eigenvalue λ_1 is given by the operator norm $\|\Phi^{-1}\|$ and λ_2 by $\|\Phi\|$, both of which are uniformly bounded. $\square$

We extend the metric g_j on $\xi = \ker \alpha$ to a Riemannian metric on the covering space M' via

$$g_\alpha := \alpha \otimes \alpha + g_j$$

with respect to the splitting $TM' = \mathbb{R} \cdot R \oplus \xi$. As we have done for $g'|_\xi$ and g_j, we compare the metrics g' and g_α.

Lemma 3.3 ([5, Lemma 2.4.1])**.** *The norm $\|\cdot\|_\alpha$ induced by g_α and the norm $\|\cdot\|_{g'}$ are uniformly equivalent on M', i.e., there exist constants $c_1, c_2 > 0$ such that*

$$\tfrac{1}{c_1} \|\cdot\|_{g'} \leq \|\cdot\|_\alpha \leq c_2 \|\cdot\|_{g'}.$$

3.2. Length and Area

As we just saw, the norms induced by g_α and g' are uniformly equivalent, hence we can use both to formulate isoperimetric inequalities. By that we mean if we can prove an isoperimetric inequality with respect to one of the metrics, we also obtain it for the other one after adjusting the constant. For the further discussion we emphasise that the base manifold M is closed.

Since g' is defined as the lift of g with respect to the covering map $\pi : M' \to M$, the metrics are locally isometric and therefore g' is of bounded geometry in the sense of [44, Definition 2.4]. Additionally, g' is geodesically complete since this holds for g, see [10, Theorem I.7.2]. The bounded geometry implies that the absolute value of the sectional curvature is uniformly bounded and the injectivity radius of g' is bounded away from zero by, say $2i_0 > 0$, see [9, 18]. Further, the bounded injectivity radius implies that for all $p \in M'$ the exponential map that is defined on the whole tangent space T_pM' becomes a diffeomorphism when restricted to the tangent vectors $v \in T_pM'$ of length $\|v\|_{g'} < i_0$, that is, we have a diffeomorphism

$$T_pM' \supset B_{i_0}(0) \longrightarrow B_{i_0}(p) \subset M'$$

between the open ball of radius i_0 in T_pM' and the open g'-geodesic ball $B_{i_0}(p)$ in M'. We denote this restriction of $\exp_p$ to $B_{i_0}(0) \subset T_pM'$ by $\mathcal{E}_p$. As explained in [42, p. 318], the linearisation of $\mathcal{E}_p$ and $\mathcal{E}_p^{-1}$ are uniformly bounded in the operator norm with respect to g', i.e., there exists a constant $C > 0$ such that for all $p \in M'$

$$\|T\mathcal{E}_p\|_{g'}, \|T\mathcal{E}_p^{-1}\|_{g'} < C.$$

We formulate an isoperimetric inequality for smooth loops that are contained in a geodesic ball of radius i_0. Consider a 2π-periodic map $c : \mathbb{R} \to M'$ with image in $B_{i_0}(c(0))$. We associate to c a loop of tangent vectors in $T_{c(0)}M'$ as the unique solution of $\exp_{c(0)} X(\theta) = c(\theta)$ or, in other words, $X(\theta) = \mathcal{E}_{c(0)}^{-1}(c(\theta))$. This loop extends to a map $f_c : \mathbb{D} \to M'$ on the closed unit disc $\mathbb{D} \subset \mathbb{C}$ via

$$f_c(r\mathrm{e}^{\mathrm{i}\theta}) = \exp_{c(0)}(rX(\theta)),$$

with polar coordinates $z = r\mathrm{e}^{\mathrm{i}\theta}$ on $\mathbb{D}$.

Lemma 3.4. *With the notation as above and C as a bound on both the linearisation of $\mathcal{E}_{c(0)}$ and $\mathcal{E}_{c(0)}^{-1}$ we have*

$$\|\partial_r f_c(r\mathrm{e}^{\mathrm{i}\theta})\|_{g'} \leq \tfrac{C}{2}\,\mathrm{length}_{g'}(c)$$

and

$$\|\partial_\theta f_c(r\mathrm{e}^{\mathrm{i}\theta})\|_{g'} \leq C^2 \|\dot{c}(\theta)\|_{g'},$$

where the length of a curve $c : [0, 2\pi] \to M'$ with respect to the metric g' is given as

$$\mathrm{length}_{g'}(c) := \int_0^{2\pi} \|\dot{c}(\theta)\|_{g'} \mathrm{d}\theta.$$

Proof. We can estimate the g'-norm of X using the g'-length of the curve c as follows. The map $\mathcal{E}_{c(0)}^{-1}$ is a radial isometry and therefore the norm $\|X(\theta)\|_{g'}$ equals the distance between $c(0)$ and $c(\theta)$. Since the distance is the infimum over the length of all possible paths connecting $c(0)$ and $c(\theta)$ it is always a lower bound for length of a specific path connecting these points. In our case we use this to say

$$\mathrm{dist}_{g'}(c(0), c(\theta)) \leq \int_0^\theta \|\dot{c}(\theta')\|_{g'} \mathrm{d}\theta'$$

and

$$\mathrm{dist}_{g'}(c(0), c(\theta)) \leq \int_\theta^{2\pi} \|\dot{c}(\theta')\|_{g'} \mathrm{d}\theta'.$$

Adding these equations and dividing by 2 we obtain

$$\|X(\theta)\|_{g'} \leq \tfrac{1}{2}\,\mathrm{length}_{g'}(c),$$

which is independent of θ. Given this estimate, the first inequality in the lemma follows by

$$\begin{aligned}\|\partial_r f_c(r\mathrm{e}^{\mathrm{i}\theta})\|_{g'} &= \|\partial_r \left((\mathcal{E}_{c(0)}(rX(\theta)) \right) \|_{g'} \\ &\leq \|T\mathcal{E}_{c(0)}\|_{g'} \cdot \|X(\theta)\|_{g'} \\ &\leq \tfrac{C}{2}\,\mathrm{length}_{g'}(c).\end{aligned}$$

The second estimate is more elementary to obtain. We just calculate

$$\begin{aligned}\|\partial_\theta f_c(r\mathrm{e}^{\mathrm{i}\theta})\|_{g'} &= \|\partial_\theta(\exp_{c(0)}(r \cdot X(\theta))\|_{g'} \\ &= \|\partial_\theta \mathcal{E}_{c(0)}(r \cdot \mathcal{E}_{c(0)}^{-1}(c(\theta)))\|_{g'} \\ &= \|T_{rX}\mathcal{E}_{c(0)} \cdot r \cdot T_{\mathcal{E}(X)}\mathcal{E}_{c(0)}^{-1}\dot{c}(\theta)\|_{g'} \\ &\leq C^2\|\dot{c}(\theta)\|_{g'},\end{aligned}$$

where we used the uniform bound on $\mathcal{E}_{c(0)}$ and $\mathcal{E}_{c(0)}^{-1}$ as well as the fact that $r \leq 1$. □

Corollary 3.5 (Isoperimetric Inequality)**.** *Keeping our notation, we estimate the area of the disc $f_c(\mathbb{D})$ by*

$$\mathrm{Area}_{g'}(f_c(\mathbb{D})) \leq \tfrac{C^3}{2} \left(\mathrm{length}_{g'}(c)\right)^2.$$

Proof. The area of the disc $f(\mathbb{D})$ is

$$\mathrm{Area}_{g'}(f_c(\mathbb{D})) = \int_{(0,1)\times(0,2\pi)} \sqrt{\det(f_c^* g')_{ij}}\ \mathrm{d}r \wedge \mathrm{d}\theta.$$

We can estimate the determinant of the pulled back metric by

$$\begin{aligned}\det(f_c^* g')_{ij} &= \|f_r\|_{g'}^2 \|f_\theta\|_{g'}^2 - g(f_r, f_\theta)^2 \\ &\leq \|f_r\|_{g'}^2 \|f_\theta\|_{g'}^2.\end{aligned}$$

Combining this with the results of the previous lemma we obtain

$$\begin{aligned}
\mathrm{Area}_{g'}(f_c(\mathbb{D})) &\leq \int_{(0,1)\times(0,2\pi)} \|f_r\|_{g'}\|f_\theta\|_{g'} \mathrm{d}r \wedge \mathrm{d}\theta \\
&\leq \int_{(0,1)\times(0,2\pi)} \tfrac{C^3}{2} \,\mathrm{length}_{g'}(c)|\dot{c}(\theta)| \mathrm{d}r \wedge \mathrm{d}\theta \\
&\leq \tfrac{C^3}{2} \,\mathrm{length}_{g'}(c) \int_{(0,2\pi)} |\dot{c}(\theta)| \mathrm{d}\theta \\
&\leq \tfrac{C^3}{2} \left(\mathrm{length}_{g'}(c)\right)^2 . \qquad \square
\end{aligned}$$

This isoperimetric inequality will be extended to more general maps in Section 4.4 and will also be used to prove the monotonicity lemma in Section 4.5.

CHAPTER 4

Holomorphic Curves

We want to discuss holomorphic curves in in the symplectisation $(\mathbb{R} \times M', \mathrm{d}(\mathrm{e}^t\alpha))$. This discussion prepares the ground for Chapter 6 and 7. Where we will discuss compactness for families of holomorphic discs and conclude the existence of closed characteristics from it. In addition to the usual non-compactness in $\mathbb{R}$-direction we have to deal with the non-compactness of M': Our holomorphic curves can not only escape to $-\infty$ in $\mathbb{R}$-direction but their projection to M' can also get arbitrary far away from the image of their boundary. To handle these problems we refer to Hofer's analysis for holomorphic curves for the $\mathbb{R}$-direction and use the structure of M' as a covering space as well as monotonicity phenomena of holomorphic curves.

4.1. An Almost Complex Structure

For the discussion of holomorphic discs which are maps $u : \mathbb{D} \to \mathbb{R} \times M'$ that satisfy the Cauchy–Riemann equation, we need to equip $(\mathbb{R} \times M', \alpha, g')$ with an almost complex structure J. We require that J is invariant under the $\mathbb{R}$-action, maps ∂_t to the Reeb vector field R of α and coincides with j on $\xi = \ker \alpha$, where j is as constructed in Section 3.1. Remember that j defines a bundle metric g_j on ξ by

$$g_j := \mathrm{d}\alpha(\cdot, j\cdot).$$

As we have seen in Lemma 3.1, the norm $\|\cdot\|_j$ induced by this metric is equivalent to the restriction of the norm $\|\cdot\|_{g'}$ defined by the metric g' to ξ.

Definition 4.1. A smooth map $u : \mathbb{D} \to \mathbb{R} \times M'$ defined on the closed unit disc $\mathbb{D}$ is **holomorphic** if it satisfies the Cauchy–Riemann equation

$$Tu \circ \mathrm{i} = J_u \circ Tu$$

in the interior of the disc, i.e., for all $z \in \mathrm{Int}(\mathbb{D})$ and $v \in T_z\mathbb{D}$ holds

$$T_z u(\mathrm{i} \cdot v) = J_{u(z)} \cdot T_z u(v).$$

Usually we want to require a boundary condition for holomorphic discs. The most common condition in our situation will be $u(\partial\mathbb{D}) \subset \{0\} \times M'$. If we want to emphasise this condition we write $u : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{R} \times M', \{0\} \times M')$. Writing $u = (a, f)$ with respect to the splitting $\mathbb{R} \times M'$, the Cauchy–Riemann equation takes the form

$$\begin{cases} -\mathrm{d}a \circ \mathrm{i} &= f^*\alpha, \\ \pi_\xi Tf \circ \mathrm{i} &= j_f \circ \pi_\xi Tf, \end{cases}$$

where π_ξ denotes the projection onto ξ along R.

Proof. After writing u as (a, f) the Cauchy–Riemann equation attains the form

$$\begin{pmatrix} \mathrm{d}a \\ Tf \end{pmatrix} \circ \mathrm{i} = Tu \circ \mathrm{i} = J_u \circ Tu = J_{(a,f)} \circ \begin{pmatrix} \mathrm{d}a \\ Tf \end{pmatrix}.$$

Splitting TM' into $\mathbb{R} \cdot R$ and ξ and using our conditions on J we get

$$\begin{aligned} (\mathrm{d}a\ \partial_t + \pi_R Tf + \pi_\xi Tf) \circ \mathrm{i}(\cdot) &= J_{(a,f)}(\mathrm{d}a(\cdot)\partial_t + \pi_R Tf(\cdot) + \pi_\xi Tf(\cdot)) \\ &= J_{(a,f)}(\mathrm{d}a(\cdot)\partial_t + \alpha(Tf(\cdot))\, R + \pi_\xi Tf(\cdot)) \\ &= \mathrm{d}a(\cdot)R - \alpha(Tf(\cdot))\, \partial_t + j_f \pi_\xi Tf(\cdot). \end{aligned}$$

Comparing the coefficients gives $-\mathrm{d}a \circ \mathrm{i} = \alpha(Tf\cdot) = f^*\alpha$ and $\pi_\xi Tf \circ \mathrm{i} = j_f \pi_\xi Tf$. $\square$

From this formulation of the Cauchy–Riemann equation we see that a is subharmonic, i.e., $\Delta a \geq 0$. To check this we have to use that (by construction of j) the symmetric form $\mathrm{d}\alpha(\cdot, j\cdot)$ is a bundle metric on ξ. Therefore a satisfies a maximum principle [14, Section 6.4] and the image $u(\mathbb{D})$ of u lies in $(-\infty, 0] \times M'$. Indeed,

$$\begin{aligned} a_{xx} := \mathrm{d}\left(\mathrm{d}a(\partial_x)\right)(\partial_x) &= \mathrm{d}\left(-\mathrm{d}a(\mathrm{i}^2 \cdot \partial_x)\right)(\partial_x) \\ &= -\mathrm{d}(f^*\alpha\,(\mathrm{i} \cdot \partial_x))\,(\partial_x) = -f^*\mathrm{d}\alpha(\mathrm{i} \cdot \partial_x,\ \partial_x) \\ &= -\mathrm{d}\alpha(Tf\ \mathrm{i} \cdot \partial_x,\ Tf\partial_x) = -\mathrm{d}\alpha(\pi_\xi Tf\ \mathrm{i} \cdot \partial_x, \pi_\xi Tf\ \partial_x) \\ &= -\mathrm{d}\alpha(j_f \cdot \pi_\xi Tf\ \partial_x, \pi_\xi Tf\ \partial_x) \geq 0. \end{aligned}$$

Note that the calculation for a_{yy} is analogous.

Lemma 4.2. *Let $u = (a, f) : \mathbb{D} \to \mathbb{R} \times M'$ be a holomorphic disc. Then*

$$u^*(\mathrm{d}t \wedge \alpha) = (a_x^2 + a_y^2)\mathrm{d}x \wedge \mathrm{d}y$$

and

$$f^*\mathrm{d}\alpha = \tfrac{1}{2}\left(\|f_x\|_{g_j}^2 + \|f_y\|_{g_j}^2\right)\mathrm{d}x \wedge \mathrm{d}y,$$

where $a_x := \mathrm{d}a(\partial_x)$ *with respect to the canonical basis* (∂_x, ∂_y) *of* $T\mathbb{D}$ *and similarly for* a_y, f_x *and* f_y.

Proof. We make the computation

$$\begin{aligned} u^*(\mathrm{d}t \wedge \alpha) &= \mathrm{d}a \wedge f^*\alpha \\ &= \mathrm{d}a \wedge (-\mathrm{d}a \circ \mathrm{i}) \\ &= -(a_x \mathrm{d}x + a_y \mathrm{d}y) \wedge (-a_x \mathrm{d}y + a_y \mathrm{d}x) \\ &= a_x^2 \mathrm{d}x \wedge \mathrm{d}y - a_y^2 \mathrm{d}y \wedge \mathrm{d}x \\ &= (a_x^2 + a_y^2) \mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

For the second equation we begin with

$$\begin{aligned} f^*\mathrm{d}\alpha &= \mathrm{d}\alpha(Tf\,\cdot\,, Tf\,\cdot\,) \\ &= -\mathrm{d}\alpha(\pi_\xi Tf\,\cdot\,, j \circ j \cdot \pi_\xi Tf\,\cdot\,) \\ &= -g_j(\pi_\xi Tf\,\cdot\,, j\pi_\xi \cdot Tf\,\cdot\,) \\ &= -g_j(\pi_\xi Tf\,\cdot\,, \pi_\xi Tf \circ \mathrm{i}\,\cdot\,). \end{aligned}$$

Evaluating this on the basis (∂_x, ∂_y) we get

$$f^*\mathrm{d}\alpha(\partial_x, \partial_y) = -g_j(\pi_\xi Tf\partial_x, -\pi_\xi Tf\partial_x) = \|\pi_\xi f_x\|_{g_j},$$

or, after switching the order of ∂_x and ∂_y,

$$f^*\mathrm{d}\alpha(\partial_x, \partial_y) = -f^*\mathrm{d}\alpha(\partial_y, \partial_x) = -\big(- g_j(\pi_\xi Tf\partial_y, \pi_\xi Tf\partial_y)\big) = \|\pi_\xi f_y\|_{g_j}.$$

We extend g_j to TM to a pseudometric on TM' by precomposing with the projection on π_ξ and omit π_ξ in the following. Adding these equations and dividing by 2 we obtain

$$f^*\mathrm{d}\alpha(\partial_x, \partial_y) = \tfrac{1}{2}\left(\|f_x\|_{g_j}^2 + \|f_y\|_{g_j}^2\right),$$

which leads to the conclusion

$$f^*\mathrm{d}\alpha = \tfrac{1}{2}\left(\|f_x\|_{g_j}^2 + \|f_y\|_{g_j}^2\right) \mathrm{d}x \wedge \mathrm{d}y. \qquad \square$$

4.2. Area Growth

We will investigate how much area of a holomorphic disc is contained in an open cylinder of radius r centred at a point $p \in M'$. The radius is measured with respect to the metric g'.

To be more precise, let

$$u = (a, f) : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{R} \times M', \{0\} \times M')$$

be a holomorphic disc, $p \in M'$ and $0 < t \leq i_0$, where $2i_0$ is a lower bound for the injectivity radius of the exponential map with respect to g'. We consider the open solid cylinder over the open t-ball $B_t^{g'}(p)$, that is $\mathbb{R} \times B_t^{g'}(p)$, and, depending on t, we define

$$G_t = u^{-1}(\mathbb{R} \times B_t(p)) = f^{-1}(B_t(p)).$$

We assume that the image of $\partial\mathbb{D}$ under f lies outside the geodesic ball $B_{i_0}(p)$, i.e., $f(\partial\mathbb{D}) \subset M' \setminus B_{i_0}(p)$ and therefore that G_t is disjoint from the boundary of the disc, $G_t \cap \partial\mathbb{D} = \emptyset$.

We denote the radial distance function of g' at p by

$$\begin{aligned} r : B_{i_0}(p) &\longrightarrow [0, i_0) \\ x &\longmapsto \mathrm{dist}_{g'}(p, x), \end{aligned}$$

where, as usual, $\mathrm{dist}_{g'}(p, x)$ denotes the distance between p and x. It is the infimum of the length over all paths connecting p and x. By [42, Lemma 12] we can use Gauss's lemma to show that the pointwise operator norm of Tr with respect to g' equals 1. The restriction of r to the image of the holomorphic disc can be understood as the map

$$\begin{aligned} F : G_{i_0} &\longrightarrow [0, i_0) \\ z &\longmapsto r(f(z)). \end{aligned}$$

With this function we can characterise the boundary of G_t as $\partial G_t = F^{-1}(t)$. We denote the set of regular values of F which are not contained in the image $r(\{\pi_\xi Tf = 0\})$ by $\mathrm{Reg} \subset [0, i_0)$. Note that by [23, Proof of Lemma 7] the image $r(\{\pi_\xi Tf = 0\})$ is a finite set. Indeed, the set $\{\pi_\xi Tf = 0\}$ is finite. We remark that f has no critical points on $F^{-1}(\mathrm{Reg})$.

Assume that h is a metric on $F^{-1}(\mathrm{Reg}) \subset \mathbb{C}$ such that the gradient of F is bounded from above with respect to the metric h, i.e., there exists a constant $c_0 > 0$ with

$$\| \operatorname{grad}_h F \|_h \leq \tfrac{1}{c_0}.$$

Using the techniques that are presented in [31, pp. 27] we obtain a lower bound for the derivative of the area of a holomorphic curve contained in a slim cylinder.

Lemma 4.3. *For all $t \in \mathrm{Reg}$ the t-derivative of the area of G_t exists and satisfies*

$$(\mathrm{Area}_h(G_t))' \geq c_0 \cdot \mathrm{length}_h(\partial G_t),$$

where

$$\mathrm{Area}_h(G_t) = \int_{G_t} \sqrt{\det(h)}\, \mathrm{d}x \wedge \mathrm{d}y.$$

Note that we did not use the structure of our situation in the previous discussion. We will use it now to find a metric h on $F^{-1}(\mathrm{Reg})$ that admits a bounded gradient of F and compute the constant c_0 for this case. Let $\mathcal{T}$ denote the set of all smooth strictly increasing functions $\tau : (-\infty, 0] \to [0, 1]$ with $\tau(0) = 1$.

For $\tau \in \mathcal{T}$ we equip $(-\infty, 0] \times M'$ with the symplectic form $\mathrm{d}(\tau\alpha) = \tau' \mathrm{d}t \wedge \alpha + \tau \mathrm{d}\alpha$. This symplectic form extends to a symplectic form on $\mathbb{R} \times M'$ by extending τ to a strictly increasing function in $\mathbb{R}$. Note that the precise form of the extension is not important since all our holomorphic discs will have image in $(\infty, 0] \times M'$.

The most common choice for $\tau \in \mathcal{T}$ and the one that we will use later is $\tau = \mathrm{e}^t$. We observe that J is compatible with $\mathrm{d}(\tau\alpha)$ for all $\tau \in \mathcal{T}$. That is we obtain a metric through

$$g_\tau := \mathrm{d}(\tau\alpha)(\cdot, J\cdot) = \tau'(\mathrm{d}t \otimes \mathrm{d}t + \alpha \otimes \alpha) + \tau g_j.$$

We choose h as the conformal metric $u^* g_\tau|_{F^{-1}(\mathrm{Reg})}$ and show the existence of a c_0 with $\| \operatorname{grad}_h F \|_h \leq \frac{1}{c_0}$. Moreover, we give a specific formula for c_0 only depending on the $\mathbb{R}$-coordinate of the holomorphic disc. To this end consider the h-unit vector

$$v = \frac{\operatorname{grad}_h F}{\| \operatorname{grad}_h F \|_h}$$

and begin by using the definition of the gradient as the unique vector field with $\mathrm{d}F(\cdot) = h(\operatorname{grad}_h(F), \cdot)$

$$\| \operatorname{grad}_h F \|_h = \frac{\| \operatorname{grad}_h F \|_h^2}{\| \operatorname{grad}_h F \|_h} = \left| \mathrm{d}F \left(\frac{\operatorname{grad}_h F}{\| \operatorname{grad}_h F \|_h} \right) \right|.$$

Since $F = r \circ f$ we can use the chain rule and the equality $\|Tr\|_{g'} = 1$ for the operator norm of r to obtain

$$|\mathrm{d}F(v)| \leq \|Tf(v)\|_{g'}.$$

The norm induced by g' is equivalent to the norm $\| \cdot \|_\alpha$ induced by $g_\alpha = \alpha \otimes \alpha + g_j$ so

$$\|Tf(v)\|_{g'} \leq c_1 \|Tf(v)\|_\alpha,$$

with c_1 as in Lemma 3.3. Comparing g_α with g_τ along the image of the holomorphic curve $u = (a, f)$ we observe

$$\begin{aligned} g_\alpha &\leq \mathrm{d}t \otimes \mathrm{d}t + \alpha \otimes \alpha + g_j \\ &= \tfrac{1}{\tau'}\tau' \cdot (\mathrm{d}t \otimes \mathrm{d}t + \alpha \otimes \alpha) + \tfrac{1}{\tau}\tau \cdot g_j \\ &\leq \max_{a(\mathbb{D})} \left(\tfrac{1}{\tau'}, \tfrac{1}{\tau}\right) [\tau'(\mathrm{d}t \otimes \mathrm{d}t + \alpha \otimes \alpha) + \tau g_j] \\ &= \max_{a(\mathbb{D})} \left(\tfrac{1}{\tau'}, \tfrac{1}{\tau}\right) g_\tau. \end{aligned}$$

Applying this estimate to the induced norms we get $\| \cdot \|_\alpha \leq \sqrt{\max_{a(\mathbb{D})} \left(\frac{1}{\tau'}, \frac{1}{\tau}\right)}\, \| \cdot \|_\tau$. Using this we can continue our estimate for $\| \operatorname{grad}_h F \|_h$.

$$\| \operatorname{grad}_h F \|_h \leq c_1 \| Tf(v) \|_\alpha \leq c_1 \sqrt{\max_{a(\mathbb{D})} \left(\tfrac{1}{\tau'}, \tfrac{1}{\tau}\right)}\, \| Tf(v) \|_\tau.$$

Using $Tu(v) = (\mathrm{d}a(v), Tf(v))$ and

$$\| Tf(v) \|_\tau^2 \leq \tau' \cdot |\mathrm{d}a(v)|^2 + \| Tf(v) \|_\tau^2 = \| Tu(v) \|_\tau^2$$

we can estimate $\| Tf(v) \|_\tau$ by $\| Tu(v) \|_\tau^2$. Which is nothing else but the norm of v with respect to the metric $u^* g_\tau = h$. Since v is an h-unit vector we obtain

$$\| \operatorname{grad}_h F \|_h \leq c_1 \| Tf(v) \|_\alpha \leq c_1 \sqrt{\max_{a(\mathbb{D})} \left(\tfrac{1}{\tau'}, \tfrac{1}{\tau}\right)}.$$

Thus the constant required in Lemma 4.3 is given by

$$c_0 = \frac{1}{c_1 \sqrt{\max_{a(\mathbb{D})} \left(\frac{1}{\tau'}, \frac{1}{\tau}\right)}}.$$

Observe that this constant depends on the $\mathbb{R}$-coordinate of the holomorphic curve.

Remark 4.4. For each $\tau \in \mathcal{T}$, τ is less or equal to 1 on the $\mathbb{R}$-coordinate of the holomorphic disc. Therefore $\frac{1}{\tau}$ is always greater or equal to 1. The same is true for the maximum that appears in the estimate of $\| \operatorname{grad}_h F \|_h$, this implies that the square root of said value is less or equal to the value itself. All in all, $c_1 \max_{a(\mathbb{D})} \left(\frac{1}{\tau'}, \frac{1}{\tau}\right)$ is also an upper bound for $\| \operatorname{grad}_h F \|_h$ as presented in [5].

4.4. An Isoperimetric Inequality

As in the previous discussion, we choose $\tau \in \mathcal{T}$ and estimate the area of the holomorphic curve with respect to the symplectic form $\mathrm{d}(\tau\alpha)$ inside the solid open cylinder $B_t(p) \times \mathbb{R}$. The **symplectic area** A is defined by the formula

$$A(t) := \int_{G_t} u^* \mathrm{d}(\tau\alpha)$$

for $t \in [0, i_0]$. The isoperimetric inequality in Corollary 3.5 compares the area of a single disc f_c to the length of its boundary. We can use this to estimate $A(t)$ in terms of the length of the boundary curves $u(\partial G_t)$ measured with respect to the metric

$$g_0 = \mathrm{d}t \otimes \mathrm{d}t + g_\alpha.$$

The precise statement is

Lemma 4.5. *There exists a positive constant $c_3 > 0$, which only depends on the geometry of (M', g') such that*

$$A(t) \leq c_3 \Big(1 + \max_{G_t} \left(\tau'(a)\right)\Big) \Big(\mathrm{length}_{g_0}\left(u(\partial G_t)\right)\Big)^2$$

where $\mathrm{length}_{g_0}(u(\partial G_t))$ *is the sum of the lengths of all boundary components.*

Proof. The idea of the proof is to use Stokes' theorem to reduce the question for the area to a question about length, then fill each boundary component with a disc as in Section 3.2 and work out the estimate for these special discs.

Let N be the number of connected components of $\partial u(G_t)$. We parametrise the l-th boundary component $\partial u(G_t)_l$ of $u(G_t)$ as $(\gamma_{a,l}, \gamma_{f,l})$ with respect to the splitting $\mathbb{R} \times M'$, where $\gamma_{a,l} : S^1 \to \mathbb{R}$ and $\gamma_{f,l} : S^1 \to M'$. The choice of parametrisation is not important, since the estimates will ultimately be independent of the chosen parametrisation. A filling disc f_l of $\gamma_{f,l}$ is given as in Section 3.2. Using convex interpolation the map $\gamma_{a,l}$ extends to a disc map a_l by

$$a_l\Big(r\mathrm{e}^{\mathrm{i}\theta}\Big) = r \cdot \gamma_{a,l}(\theta) + (1-r) \cdot \gamma_{a,l}(0),$$

Then (a_l, f_l) is a disc map with boundary $(\gamma_{a,l}, \gamma_{f,l})$. We choose orientations for the boundary components according to the requirements in Stokes' theorem. Applying Stokes' theorem twice yields

$$A(t) = \int_{G_t} u^* \mathrm{d}(\tau\alpha) = \int_{\partial G_t} u^*(\tau\alpha) = \sum_{l=1}^{N} \int_{\mathbb{D}} (a_l, f_l)^* \mathrm{d}(\tau\alpha).$$

We do not have to worry about the choice of orientations, since we will estimate the integrals with their absolut value. We will treat each boundary component separately and take the sum later. The integrand equals

$$(a_l, f_l)^* \mathrm{d}(\tau\alpha) = \tau'(a_l)\, \mathrm{d}a_l \wedge f_l^* \alpha + \tau(a_l) f_l^* \mathrm{d}\alpha. \tag{4.1}$$

For the second summand we have

$$|\tau(a_l)\cdot f_l^*\mathrm{d}\alpha| \le |f_l^*\mathrm{d}\alpha| = |\mathrm{d}\alpha(Tf_l\partial_r, Tf_l\partial_\theta)\mathrm{d}r\wedge\mathrm{d}\theta|,$$

where we used $\tau(a_l)\le 1$. Using that $\mathrm{d}\alpha(\cdot,\, j\cdot) = g_j$ is a metric, the Cauchy–Schwarz inequality, the observation that j is a g_j-isometry and the equivalence of metrics in Lemma 3.1, we obtain

$$\begin{aligned}
\mathrm{d}\alpha(Tf_l\partial_r, Tf_l\partial_\theta) &= \mathrm{d}\alpha(\pi_\xi Tf\partial_r,\, \pi_\xi Tf\partial_\theta)\\
&= g_j(\pi_\xi Tf\partial_r,\, -j\pi_\xi Tf\partial_\theta)\\
&\le \|Tf\partial_r\|_{g_j}\|j\|_{g_j}\|Tf\partial_\theta\|_{g_j}\\
&\le c_2^2\|Tf\partial_r\|_{g'}\|Tf\partial_\theta\|_{g'}.
\end{aligned}$$

With the estimate from Lemma 3.4 we conclude

$$\begin{aligned}
\int_{\mathbb{D}}\tau(a_l)\cdot f_l^*\mathrm{d}\alpha &\le \frac{c_2^2C^3}{2}\int_{(0,1]\times[0,2\pi)}\mathrm{length}_{g'}(\gamma_{f,l})\cdot\|\dot\gamma_{f,l}(\theta)\|_{g'}\mathrm{d}r\wedge\mathrm{d}\theta\\
&\le \frac{c_2^2C^3}{2}\Big(\mathrm{length}_{g'}(\partial G_{t,l})\Big)^2. \qquad (4.2)
\end{aligned}$$

Since the length of the boundary curves is non-negative and $a^2+b^2\le(a+b)^2$ for non-negative numbers, the sum of these terms over all boundary components can be estimated by

$$\frac{c_2^2C^3}{2}\Big(\mathrm{length}_{g'}(\partial G_t)\Big)^2,$$

where $\mathrm{length}_{g'}(\partial G_t) := \sum_l \mathrm{length}(\partial G_{t,l})$. The other summand $\tau'(a_l)\mathrm{d}a_l\wedge f_l^*\alpha$ of (4.1) equals

$$\tau'(a_l)\cdot\big(a_{l,r}\cdot\alpha(f_{l,\theta}) - a_{l,\theta}\cdot\alpha(f_{l,r})\big).$$

For the derivatives of a_l we have

$$\begin{aligned}
a_{l,r} &= \gamma_{a,l}(\theta)-\gamma_{a,l}(0)\le \mathrm{osc}(\gamma_{a,l})\le \mathrm{length}_{g_0}(\partial G_{t,l})\\
a_{l,\theta} &= r\cdot\dot\gamma_{a,l}(\theta)
\end{aligned}$$

where osc denotes the oscillation, i.e., the difference between the maximum and the minimum of the function. For the terms including α we use that it is bounded by the definition of a virtually contact structure (Definition 2.10) and that by Lemma 3.4 we then get

$$\begin{aligned}
\alpha(f_{l,r}) &\le K\cdot\|f_{l,r}\|_{g'}\le \tfrac{KC}{2}\cdot\mathrm{length}_{g'}(\gamma_{f,l})\\
\alpha(f_{l,\theta}) &\le K\cdot\|f_{l,\theta}\|_{g'}\le KC^2\cdot\ \|\dot\gamma_{f,l}\|_{g'},
\end{aligned}$$

where K is a constant as in the definition of a virtually contact structure. In the following we will write const. for some constant that only depends on the geometry that will not specify. Inserting all the above estimates we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} \tau'(a_l)\mathrm{d}a_l \wedge f_l^*\alpha \\
\leq &\int_{(0,1]\times[0,2\pi)} \tau'(a_l)\Big(|a_{l,r}|\cdot|\alpha(f_{l,\theta})| + |a_{l,\theta}|\cdot|\alpha(f_{l,r})|\Big)\,\mathrm{d}r\wedge\mathrm{d}\theta \\
\leq &\text{ const.}\int_{(0,1]\times[0,2\pi)} \tau'(a_l)\cdot \mathrm{length}_{g_0}(\partial G_{t,l})\cdot\|\dot\gamma_{f,l}\|_{g'}\,\mathrm{d}r\wedge\mathrm{d}\theta \\
&\quad + \text{const.}\int_{(0,1]\times[0,2\pi)} \tau'(a_l)\cdot r\cdot|\dot\gamma_{a,l}(\theta)|\cdot \mathrm{length}_{g'}(\gamma_{f,l})\,\mathrm{d}r\wedge\mathrm{d}\theta.
\end{aligned}$$

We can use $r\leq 1$, $\mathrm{length}_{g'}(\gamma_{f,l})\leq \mathrm{length}_{g_0}(\partial G_{t,l})$ and $\tau'(a_l)\leq \max_{G_t}\tau(a)$, where a denotes the $\mathbb{R}$-coordinate of the holomorphic disc u, to make the further estimate

$$\text{const.}\cdot\max_{G_t}\tau(a)\cdot \mathrm{length}_{g_0}(\partial G_{t,l})\cdot\left(\int_{[0,2\pi)}|\dot\gamma_{a,l}(\theta)|\mathrm{d}\theta + \int_{[0,2\pi)}\|\dot\gamma_{f,l}(\theta)\|_{g'}\mathrm{d}\theta\right).$$

In the end this leads to the estimate

$$\begin{aligned}
&\int_{\mathbb{D}} \tau'(a_l)\mathrm{d}a_l \wedge f_l^*\alpha \\
&\leq \text{const.}\cdot\max_{G_t}\tau(a)\cdot\Big(\mathrm{length}_{g_0}(\partial G_{t,l})\Big)^2.
\end{aligned}\tag{4.3}$$

Taking the sum over all boundary components we can argue as above and bound the sum over all squared lengths with the square of their sum, i.e.,

$$\sum_l \Big(\mathrm{length}_{g_0}(\partial G_{t,l})\Big)^2 \leq \Big(\mathrm{length}_{g_0}(\partial G_t)\Big)^2.$$

Combing the two estimates (4.2) and (4.3) we get

$$\begin{aligned}
A(t) &= \sum_l\int_{\mathbb{D}}(a_l,f_l)^*\mathrm{d}(\tau\alpha) \\
&\leq \sum_l\int_{\mathbb{D}}|(a_l,f_l)^*\mathrm{d}(\tau\alpha)| \\
&\leq \sum_l\frac{c_2^2C^3}{2}\Big(\mathrm{length}_{g'}(\partial G_{t,l})\Big)^2 + \text{const.}\cdot\max_{G_t}\tau(a)\cdot\Big(\mathrm{length}_{g_0}(\partial G_{t,l})\Big)^2 \\
&\leq c_3\cdot(1+\max_{G_t}\tau(a))\cdot\Big(\mathrm{length}_{g_0}(\partial G_t)\Big)^2,
\end{aligned}$$

for an appropriate choice of the constant c_3. □

4.5. Monotonicity

We go back to the discussion that started in Section 4.2. Recall that h denotes the metric on G_t induced by u^*g_τ.

Proposition 4.6 (Monotonicity Lemma)**.** *Let*

$$u = (a, f) : (\mathbb{D}, \partial\mathbb{D}) \longrightarrow (\mathbb{R} \times M', \{0\} \times M')$$

be a holomorphic disc. Consider $p \in f(\mathbb{D}) \subset M'$ and assume that the g'-geodesic ball $B_{i_0}(p)$ of radius i_0 and the image of the boundary circle $f(\partial\mathbb{D})$ have empty intersection. Then

$$A(t) \geq \mathrm{m}^2 t^2,$$

for all $t \in [0, i_0]$, where

$$A(t) = \int_{f^{-1}(B_t(p))} u^* \mathrm{d}(\tau\alpha)$$

is the symplectic area functional and $\mathrm{m} = \mathrm{m}(\tau(a))$ is a positive constant depending on the $\mathbb{R}$-coordinate of the holomorphic disc $u = (a, f)$.

Proof. At first we want to see that $A(t) = \mathrm{Area}_h(G_t)$. For this equation we remind ourselves that

$$\mathrm{Area}_h(G_t) = \int_{G_t} \sqrt{\det(h)}\ \mathrm{d}x \wedge \mathrm{d}y = \int_{G_t} \sqrt{\det(u^* g_\tau)}\ \mathrm{d}x \wedge \mathrm{d}y$$

and by Lemma 4.2 we conclude

$$A(t) = \int_{G_t} u^* \mathrm{d}(\tau\alpha) = \int_{G_t} \left[\tau'(a)(a_x^2 + a_y^2) + \tfrac{1}{2}\tau(a)\left(\|f_x\|_{g_j}^2 + \|f_y\|_{g_j}^2\right)\right]\ \mathrm{d}x \wedge \mathrm{d}y.$$

To proof the equality we have to calculate u^*g_τ. Observe that u^*g_τ is given by

$$\begin{pmatrix} \tau'(a)(a_x^2 + a_y^2) + \tau(a)\|f_x\|_{g_j}^2 & 0 \\ 0 & \tau'(a)(a_x^2 + a_y^2) + \tau(a)\|f_y\|_{g_j}^2 \end{pmatrix}.$$

The desired equation follows once we have noticed that due to holomorphicity of u we have $\|f_x\|_{g_j} = \|f_y\|_{g_j}$.

With Lemma 4.3 we conclude

$$\begin{aligned} A'(t) &= \left(\mathrm{Area}_h(G_t)\right)' \\ &\geq c_0 \cdot \mathrm{length}_h(\partial G_t). \end{aligned}$$

By the definition of h this equals

$$c_0 \cdot \mathrm{length}_{g_\tau}(u(\partial G_t)).$$

For the further estimate we have to compare the length induced by g_τ with the one induced by g_0. We have already seen that $g_0 \leq \max_{a(\mathbb{D})}\left(\frac{1}{\tau'}, \frac{1}{\tau}\right) g_\tau$. Therefore the according lengths satisfy

$$\mathrm{length}_{g_\tau}(\cdot) \geq \frac{1}{\sqrt{\max_{a(\mathbb{D})}\left(\frac{1}{\tau'}, \frac{1}{\tau}\right)}} \mathrm{length}_{g_0}(\cdot).$$

Hence, we can estimate $A'(t)$ from below by

$$\frac{c_0}{\sqrt{\max_{a(\mathbb{D})}\left(\frac{1}{\tau'}, \frac{1}{\tau}\right)}} \mathrm{length}_{g_0}(u(\partial G_t)).$$

With Lemma 4.5 we get

$$\mathrm{length}_{g_0}(u(\partial G_t)) \geq \sqrt{\frac{1}{c_3\left(1+\max_{a(\mathbb{D})}(\tau')\right)} A(t)}$$

and therefore

$$A'(t) \geq 2\mathrm{m}\sqrt{A(t)}$$

for all $t \in \mathrm{Reg}$, where

$$\mathrm{m} = \mathrm{m}(\tau(a)) := \frac{1}{2c_1\sqrt{c_3}} \frac{1}{\max_{a(\mathbb{D})}\left(\frac{1}{\tau'}, \frac{1}{\tau}\right)} \sqrt{\frac{1}{1+\max_{a(\mathbb{D})}(\tau')}}.$$

Arguing as in [31, p. 28] the estimate $A' \geq 2\mathrm{m}\sqrt{A}$ implies the monotonicity lemma in symplectisation. □

4.6. A Distance Estimate

For the canonical choice $\tau(t) = \mathrm{e}^t$ we can compute the monotonicity constant m as

$$\mathrm{m}(\mathrm{e}^a) = c_4 \, \mathrm{e}^{-\max_{\mathbb{D}} |a|}$$

for a positive constant c_4 only depending on the geometry of (M', g'). For this choice of τ we will estimate the maximal distance between the M'-coordinate f of the holomorphic curve $u = (a, f)$ and a maximally J-totally real submanifold

$L \subset M'$ with compact closure in M', where we assume the boundary condition $f(\partial\mathbb{D}) \subset L$. The maximal distance is expressed by

$$\mathrm{dist}_{g'}(L, f(\mathbb{D})) := \sup_{f(\mathbb{D})} \mathrm{dist}_{g'}(L, \cdot),$$

where $\mathrm{dist}_{g'}(L, f(z))$ measures the minimal length, with respect to g', of paths connecting a point in L with $f(z)$. To be precise

$$\mathrm{dist}_{g'}(L, f(z)) = \inf_{p \in L} \inf_{\gamma} \mathrm{length}_{g'}(\gamma),$$

where the second infimum is taken over all path γ starting in p and ending in $f(z)$.

Proposition 4.7. *Let $L \subset (M', g')$ be a relative compact, J-maximally totally real submanifold and denote the symplectic energy*

$$\int_{\mathbb{D}} u^* \mathrm{d}(\mathrm{e}^t \alpha)$$

of a holomorphic disc u by $E(u)$. Then there exist constants K_1, K_2 depending only on the geometry of (M', g') such that for all holomorphic discs

$$u = (a, f) : \ (\mathbb{D}, \partial\mathbb{D}) \longrightarrow (\mathbb{R} \times M', \{0\} \times L)$$

the estimate

$$\mathrm{dist}_{g'}\left(L, f(\mathbb{D})\right) \leq \max\left\{K_1 \mathrm{e}^{\max_{\mathbb{D}} |a|} \sqrt{E(u)},\ K_2 \mathrm{e}^{2\max_{\mathbb{D}} |a|} E(u)\right\}$$

holds.

Proof. We write the distance $\mathrm{dist}(L, f(\mathbb{D}))$ as $2Ni_0 + d_0$ for a unique $N \in \mathbb{N}_0$ and $d_0 \in [0, 2i_0)$. The case $N = 0$ is covered by Proposition 4.6. Indeed if $\mathrm{dist}(L, f(\mathbb{D})) = d_0 < 2i_0$ we can choose a point $p_0 \in f(\mathbb{D}) \setminus L$ whose ball neighbourhood of radius $\frac{d_0}{2}$ has empty intersection with L. This yields

$$E(u) = \int_{\mathbb{D}} u^* \mathrm{d}(\mathrm{e}^t \alpha) \ \geq \ \int_{f^{-1}\left(B_{d_0/2}(p_0)\right)} u^* \mathrm{d}(\mathrm{e}^t \alpha).$$

Since $\frac{d_0}{2} < i_0$ we can apply Proposition 4.6 and see

$$\begin{aligned} E(u) &\geq \int_{f^{-1}\left(B_{d_0/2}(p_0)\right)} u^* \mathrm{d}(\mathrm{e}^t \alpha) \\ &\geq \mathrm{m}^2 \tfrac{d_0^2}{4} \\ &= \tfrac{\mathrm{m}^2}{4} \mathrm{dist}_{g'}\left(L, f(\mathbb{D})\right)^2. \end{aligned}$$

We have $\mathrm{m} = c_4 \mathrm{e}^{-\max_{\mathbb{D}} |a|}$ and obtain the estimate in this case.

For $N \geq 1$ we choose points $p_1, \dots, p_N \in f(\mathbb{D})$ with

$$\operatorname{dist}_{g'}(L, p_l) = \operatorname{dist}\big(L, f(\mathbb{D})\big) - 2i_0(N-l) + i_0.$$

The triangle inequality shows that

$$\operatorname{dist}_{g'}(L, p_l) - i_0 \leq \operatorname{dist}_{g'}(L, q) \leq \operatorname{dist}_{g'}(L, p_l) + i_0$$

for all $q \in B_{i_0}(p_l)$, so that the distance function $\operatorname{dist}_{g'}(L, \cdot)$ maps B_{i_0} into the shifted interval

$$\Big(\operatorname{dist}_{g'}(L, f(\mathbb{D})) - 2i_0(N-l)\ ,\ \operatorname{dist}_{g'}(L, f(\mathbb{D})) - 2i_0(N-l-1)\Big).$$

We estimate the symplectic energy of the curve from below

$$\begin{aligned} E(u) &= \int_{\mathbb{D}} u^* \mathrm{d}(\mathrm{e}^t \alpha) \\ &\geq \int_{f^{-1}(\cup_l B_{i_0}(p_l))} u^* \mathrm{d}(\mathrm{e}^t \alpha). \end{aligned}$$

Since the i_0-balls around p_l and $p_{l'}$ are disjoint for $l \neq l'$ we can take the sum over the individual integrals

$$E(u) \geq \sum_l \int_{f^{-1}\left(B_{i_0}(p_l)\right)} u^* \mathrm{d}(\mathrm{e}^t \alpha).$$

By our choice of points p_l we have that the $B_{i_0}(p_l)$ and the image of the boundary $f(\partial\mathbb{D}) \subset L$ are disjoint, so we can apply Proposition 4.6 and get

$$E(u) \geq \sum_l \mathrm{m}^2 i_0^2 = N \mathrm{m}^2 i_0^2.$$

Since $d \leq 2i_0$ and N is at least 1 we have

$$2Ni_0 + d \leq (2N+1)i_0 \leq 4Ni_0.$$

Inserting this in the estimate above we get

$$E(u) \geq \tfrac{\mathrm{m}^2 i_0}{4}(2Ni_0 + d) = \tfrac{\mathrm{m}^2 i_0}{4} \operatorname{dist}_{g'}\big(L, u(\mathbb{D})\big).$$

Inserting $\mathrm{m} = c_4 \cdot \mathrm{e}^{-\max_{\mathbb{D}}|a|}$ and rearranging the estimate, we obtain the claimed statement. □

Now consider a family $u_\nu = (a_\nu, f_\nu)$ of holomorphic disc maps which satisfy a common boundary condition $u(\partial\mathbb{D}) \subset \{0\} \times L$, have uniformly bounded energy and uniformly bounded $\mathbb{R}$-component a_ν, i.e., the family stays above a certain slice $\{-R\} \times M'$ for some $R > 0$. In this case the preceding Proposition 4.7 tells us that the maximal distance $\mathrm{dist}_{g'}(L, f(\mathbb{D}))$ with respect to g' is bounded from above. Accordingly, we can apply Gromov's compactness theorem presented in [16] and [17, Theorem 1.1] and obtain:

Corollary 4.8. *Let L be a relative compact submanifold in M' that is maximally totally real with respect to J and*

$$u_\nu = (a_\nu, f_\nu) : \ (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{R} \times M', \{0\} \times L)$$

a sequence of J-holomorphic discs. Assume that their projections to the $\mathbb{R}$-coordinate, denoted by a_ν, are uniformly bounded and that the family admits a uniform energy bound, i.e.,

$$\sup_{\nu \in \mathbb{N}} \max_{\mathbb{D}} |a_\nu| < \infty \quad \textit{and} \quad \sup_{\nu \in \mathbb{N}} E(u_\nu) < \infty.$$

Assume further that all boundaries $u(\partial\mathbb{D})$ are contained in a compact subset of L. Then there exists a Gromov convergent subsequence of $(u_\nu)_\nu$ and the limit is a stable holomorphic disc.

For the notation of stable holomorphic discs and Gromov convergence consider the detailed work of Urs Frauenfelder [16].

Remark 4.9. Note that no sphere bubbling can occur in Corollary 4.8. That is because each holomorphic sphere in a symplectic manifold with exact symplectic form has vanishing energy and is therefore constant. By the definition of the convergence there cannot be any constant holomorphic spheres in the limit.

CHAPTER 5

Higher Order Bounds on Primitives

We discuss a covariant derivative for 1-forms and define the induced C^k-norms. According to these norms we show C^∞_{loc}-convergence on a covering $\pi : (M', \mathrm{d}\alpha) \to (M, \omega)$ for a sequence of contact forms α_ν obtained from α by pull back with a sequence of deck transformations. That is we show compactness for the action of the deck transformation group of a virtually contact structure.

5.1. Higher Order Covariant Derivatives

Let (M', g') be a connected Riemannian manifold and denote the Levi–Civita connection by ∇, see for example [9], [10] or [42] for the basic notion of Riemannian geometry. We extend the concept of covariant derivatives to $(0, k)$-tensors τ for $k \in \mathbb{N}$ in the same way as explained in [18, p. 73] and [32, p. 52]. Let X be a vector field on M'. The **covariant derivative of τ in the direction of X** evaluated on test vector fields $Y_1, \dots, Y_k$ is

$$(\nabla_X \tau)(Y_1, \dots, Y_k) = X\big(\tau(Y_1, \dots, Y_k)\big) - \sum_{j=1}^{k} \tau(Y_1, \dots, Y_{j-1}, \nabla_X Y_j, Y_{j+1}, \dots, Y_k).$$

This defines a $(0, k+1)$ tensor $\nabla\tau$ by

$$\nabla\tau(X, Y_1, \dots, Y_k) := (\nabla_X \tau)(Y_1, \dots, Y_k).$$

A 1-form α on M' can be thought of as a $(0, 1)$-tensor and we define its **k-th covariant derivative** inductively through

$$\nabla^k \alpha = \nabla(\nabla^{k-1}\alpha),$$

where we start with $\nabla^0\alpha = \alpha$. It make sense to define the k-th derivative this way since, as we said, the covariant derivative of a tensor is again a tensor. Let us illustrate this definition by computing the covariant derivatives of a 1-form α up to the second grade:

$$\begin{aligned}(\nabla^0\alpha)(Y) &= \alpha(Y)\\ (\nabla\alpha)(X,Y) &= \nabla(\nabla^0\alpha)(X,Y) = X\big(\alpha(Y)\big) - \alpha(\nabla_X Y)\\ (\nabla^2\alpha)(X_1,X_2,Y) &= \nabla(\nabla\alpha)(X_1,X_2,Y) = (\nabla_{X_1}\nabla\alpha)(X_2,Y)\\ &= X_1\big(\nabla\alpha(X_2,Y)\big) - \nabla\alpha(\nabla_{X_1}X_2,Y) - \nabla\alpha(X_2,\nabla_{X_1}Y)\\ &= X_1\Big(X_2\big(\alpha(Y)\big) - \alpha(\nabla_{X_2}Y)\Big)\\ &\quad - (\nabla_{X_1}X_2)\left(\alpha(Y)\right) + \alpha\left(\nabla_{\nabla_{X_1}X_2}Y\right)\\ &\quad - X_2\big(\alpha(\nabla_{X_1}Y)\big) + \alpha\big(\nabla_{X_2}(\nabla_{X_1}Y)\big).\end{aligned}$$

Using the definition of the k-th covariant derivative we can define a C^k-norm for 1-forms. We begin with the **pointwise norm**

$$\|\nabla^k\alpha\|_p = \sup\left|\left(\nabla^k\alpha\right)_p(v,w_1,\dots,w_k)\right|$$

where the supremum is taken over all g'-unit vectors $v, w_1\dots,w_k \in T_pM'$. Further taking the supremum of all pointwise norms over M' yields to the $\mathbf{C^0}$**-norm** of $\nabla^k\alpha$, i.e.,

$$\|\nabla^k\alpha\|_{C^0} := \sup_{p\in M'}\|\nabla^k\alpha\|_p.$$

The $\mathbf{C^k}$**-norm** of α is given by

$$\|\alpha\|_{C^k} := \sup_{0\le l\le k}\|\nabla^l\alpha\|_{C^0}.$$

Remark 5.1. The higher order covariant derivatives we just discussed can naturally be extended to functions on M'. For a smooth function $f\in C^\infty(M')$ we set $\nabla^0 f = f$ and $\nabla^k f = \nabla^{k-1}\mathrm{d}f$. The C^k-norm extends in the same fashion

$$\|f\|_{C^k} = \sup_{0\le l\le k}\|\nabla^l f\|_{C^0} = \sup\{\|f\|_{C^0}, \|\mathrm{d}f\|_{C^{k-1}}\}.$$

Each C^k-norm establishes a topology on the space of 1-forms on M'. Considering the subspace of all 1-forms that are bounded in all C^k-norms we equip it with the C^∞-topology induced by intersecting all C^k-topologies. If we restrict ourselves to open relatively compact subsets of M' this topology leads to the same convergence as the compact open topology that is discussed in [28, Section 2.1]. The convergence on these subsets is called C^∞_{loc}-convergence. In the same way we endow the space of smooth functions that are bounded in each C^k-norm with the C^∞-norm inherited from the C^k-norms. Then C^∞_{loc}-convergence is understood that as for 1-forms.

Remark 5.2. For an isometry φ of (M', g') the generalized *theorema egregium* as discussed in [43, Theorem 5.3.1 (ii)] states that

$$\varphi_*(\nabla_X Y) = \nabla_{\varphi_* X} \varphi_* Y$$

for all smooth vector fields X, Y on M'.

Lemma 5.3. *We can expand the theorema egregium to the covariant derivative for tensors, i.e.,*

$$\varphi^*(\nabla\tau) = \nabla(\varphi^*\tau)$$

for all $(0, k)$*-tensors* τ*.*

Remark 5.4. By an inductive argument we can extend this rule to the k-th covariant derivative of a 1-form α and get

$$\varphi^*(\nabla^k\alpha) = \nabla^k(\varphi^*\alpha).$$

As an isometry φ induces a bundle isomorphism on the $(k+1)$-fold unit cotangent bundle $\oplus^{k+1} STM'$, i.e.,

$$\|\varphi^*\alpha\|_{C^k} = \|\alpha\|_{C^k}.$$

Proof of Lemma 5.3. Let us begin with the left hand side $\varphi^*(\nabla\tau)$ and evaluate it on test vector fields $X, Y_1, \ldots, Y_k$. By definition

$$\varphi^*(\nabla\tau)(X, Y_1, \ldots, Y_k) = (\nabla\tau)_{\varphi(\cdot)}\big(T\varphi(X), T\varphi(Y_1), \ldots, T\varphi(Y_k)\big).$$

Recall that $\varphi_*(X) = T_{\varphi^{-1}(\cdot)}\varphi(X \circ \varphi^{-1})$ and therefore $T\varphi(X) = (\varphi_* X) \circ \varphi$. This implies

$$\begin{aligned}(\nabla\tau)_{\varphi(\cdot)}\big(T\varphi(X), T\varphi(Y_1), \ldots, T\varphi(Y_k)\big) &= \big((\nabla\tau)(\varphi_* X, \varphi_* Y_1, \ldots, \varphi_* Y_k)\big) \circ \varphi \\ &= \varphi^*\big((\nabla\tau)(\varphi_* X, \varphi_* Y_1, \ldots, \varphi_* Y_k)\big).\end{aligned}$$

Taking into account the pulled back function only and using the definition of the covariant derivative, we have

$$\begin{aligned}&(\nabla\tau)(\varphi_* X, \varphi_* Y_1, \ldots, \varphi_* Y_k) \\ &= (\varphi_* X)\big(\tau(\varphi_* Y_1, \ldots, \varphi_* Y_k)\big) \\ &\quad - \sum_j \tau(\varphi_* Y_1, \ldots, \varphi_* Y_{j-1}, \nabla_{\varphi_* X}\varphi_* Y_j, Y_{j+1}, \ldots, \varphi_* Y_k).\end{aligned}$$

By the aforementioned theorema egregium we can replace $\nabla_{\varphi_* X}\varphi_* Y_j$ with $\varphi_*\nabla_X Y_j$.

Putting this back into the equation above gives

$$\begin{aligned}
&\varphi^*((\nabla\tau)(X, Y_1, \dots, Y_k)) \\
&= \varphi^*\Big((\varphi_* X)\big(\tau(\varphi_* Y_1, \dots, \varphi_* Y_k)\big)\Big) \\
&\quad - \sum_j \big(\tau(\varphi_* Y_1, \dots, \varphi_* Y_{j-1}, \varphi_* \nabla_X Y_j, Y_{j+1}, \dots, \varphi_* Y_k)\big) \circ \varphi \\
&= \varphi^*\Big((\varphi_* X)\big(\tau(\varphi_* Y_1, \dots, \varphi_* Y_k)\big)\Big) \\
&\quad - \sum_j (\varphi^*\tau)(Y_1, \dots Y_{j-1}, \nabla_X Y_j, Y_{j+1}, \dots, Y_k). \qquad (5.1)
\end{aligned}$$

This finishes the discussion of $\varphi^*(\nabla\tau)$ for the moment. By the definition of the covariant derivative $\nabla(\varphi^*\tau)$ satisfies

$$\begin{aligned}
\nabla(\varphi^*\tau)(X, Y_1, \dots, Y_k) &= \big(\nabla_X(\varphi^*\tau)\big)(Y_1, \dots, Y_k) \\
&= X\big(\varphi^*\tau(Y_1, \dots, Y_k)\big) \\
&\quad - \sum_j (\varphi^*\tau)(Y_1, \dots, \nabla_X Y_j, \dots Y_k). \qquad (5.2)
\end{aligned}$$

To show the equality of the expressions (5.1) and (5.2) we take their difference

$$\begin{aligned}
&\big(\varphi^*(\nabla\tau) - \nabla(\varphi^*\tau)\big)(X, Y_1, \dots, Y_k) \\
&= \varphi^*\Big((\varphi_* X)\big(\tau(\varphi_* Y_1, \dots, \varphi_* Y_k)\big)\Big) - X\big(\varphi^*\tau(Y_1, \dots, Y_k)\big).
\end{aligned}$$

By the formula for the Lie derivative using the Lie bracket for tensors this equals

$$\begin{aligned}
&\varphi^*\big((\mathcal{L}_{\varphi_* X}\tau)(\varphi_* Y_1, \dots, \varphi_* Y_k)\big) + \left(\sum_j \tau(\varphi_* Y_1, \dots, [\varphi_* X, \varphi_* Y_j], \dots, \varphi_* Y_k)\right) \circ \varphi \\
&- \big(\mathcal{L}_X(\varphi^*\tau)\big)(Y_1, \dots, Y_k) - \sum_j \varphi^*\tau(Y_1, \dots, [X, Y_j], \dots, Y_k).
\end{aligned}$$

The second and forth term cancel out since $[\varphi_* X, \varphi_* Y_j] = \varphi_*[X, Y_j]$, so we are left with

$$\big(\varphi^*\left(\mathcal{L}_{\varphi_* X}\tau\right) - \mathcal{L}_X(\varphi^*\tau)\big)(Y_1, \dots, Y_k).$$

We claim that this term vanishes. Let ψ_t denote the flow of X, then the local flow of $\varphi_* X$ is given by $\varphi \circ \psi_t \circ \varphi^{-1}$. Using the definition of the Lie derivative via flows we have

$$\begin{aligned}\varphi^*(\mathcal{L}_{\varphi_* X}\tau) &= \varphi^*\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\varphi \circ \psi_t \circ \varphi^{-1})^*\tau\right) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi_t^*\varphi^*\tau = \mathcal{L}_X(\varphi^*\tau).\end{aligned}$$

We conclude that $\varphi^*(\nabla\tau) = \nabla(\varphi^*\tau)$ for all tensors τ and deck transformations φ. □

5.2. Local Computations

Let $x^1, \dots, x^{2n-1}$ be local coordinates on M'. Expressing the metric in these coordinates we obtain $g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ and the 1-form α looks like $\alpha = \alpha_j \mathrm{d}x^j$. In the following we denote by Γ_{ij}^k the Christoffel symbols of the Levi-Civita connection ∇ with respect to g and use the Einstein summation convention. We can compute the coefficients $(\nabla\alpha)_{ij}$ of $\nabla\alpha$ as

$$\begin{aligned}(\nabla\alpha)_{ij} &= (\nabla\alpha)(e_i, e_j) \\ &= e_i(\alpha(e_j)) - \alpha(\nabla_{e_i} e_j) \\ &= e_i(\alpha(e_j)) - \alpha(\Gamma_{ij}^l e_l) \\ &= \alpha_{j,i} - \Gamma_{ij}^l \alpha_l,\end{aligned}$$

where $\alpha_{j,i}$ denotes the derivative of the function $\alpha_j := \alpha(e_j)$ in the direction e_i. With the same reasoning we can express the coefficients of the higher order covariant derivatives in these local coordinates as

$$\begin{aligned}(\nabla^{k+1}\alpha)_{ij_1\dots j_{k+1}} &= e_i(\nabla^k\alpha(e_{j_1}, \dots, e_{j_k+1})) - \sum_{l=1}^{k+1} \nabla^k\alpha(e_{j_1}, \dots, \nabla_{e_i} e_{j_l}, \dots, e_{j_{k+1}}) \\ &= (\nabla^k\alpha)_{j_1\dots j_{k+1},i} - \sum_{l=1}^{k+1} \Gamma_{ij_l}^m (\nabla^k\alpha)_{j_1\dots m\dots j_{k+1}}\end{aligned}$$

where the m is at the l-th position.

For later discussions it is interesting to have estimates for the C^k-norm of α. We know that for each tangent space there is an orthogonal matrix A such that $(g_{ij})_{ij} = A^T D A$ with a diagonal matrix $D = \mathrm{diag}(\lambda_1, \dots, \lambda_{2n-1})$ where $\lambda_1 \leq \ldots \leq \lambda_{2n-1}$ are the eigenvalues of $(g_{ij})_{ij}$. Writing $v = v^i\partial_i$ with respect to the local coordinates of M' we obtain $\|v\|_{g'}^2 \geq \lambda_1 |v^i|^2$ for all $1 \leq i \leq 2n-1$.

Evaluating $\nabla^k\alpha$ on the g'-unit vectors $v, w_1, \ldots, w_k$ we obtain

$$
\begin{aligned}
|\nabla^k\alpha(v, w_1, \ldots, w_k)| &= \left| \sum_{i,j_1,\ldots,j_k=1}^{2n-1} v^i w^{j_1} \ldots w^{j_k} (\nabla^k\alpha)_{ij_1\ldots j_k} \right| \\
&\leq \sum_{i,j_1,\ldots,j_k=1}^{2n-1} |v^i w^{j_1} \ldots w^{j_k}| \cdot \left|(\nabla^k\alpha)_{ij_1\ldots j_k}\right| && (5.3) \\
&\leq \sum_{i,j_1,\ldots,j_k=1}^{2n-1} \frac{\|v\|_{g'}}{\sqrt{\lambda_1}} \cdot \frac{\|w_1\|_{g'}}{\sqrt{\lambda_1}} \cdot \ldots \cdot \frac{\|w_k\|_{g'}}{\sqrt{\lambda_1}} \left|(\nabla^k\alpha)_{ij_1\ldots j_k}\right| && (5.4) \\
&\leq \left(\frac{2n-1}{\sqrt{\lambda_1}}\right)^{k+1} \max_{i,j_1,\ldots,j_k} \{|(\nabla^k\alpha)_{ij_1\ldots j_k}|\}.
\end{aligned}
$$

In the last step we used that all vector are of length 1 and that the sum contains $(2n-1)^{k+1}$ summands, each of which is bounded by

$$
\left(\frac{1}{\sqrt{\lambda_1}}\right)^{k+1} \max_{i,j_1,\ldots,j_k} \{|(\nabla^k\alpha)_{ij_1\ldots j_k}|\}.
$$

5.3. Uniform C^∞-Bounds – An Example

Let $\mathbb{H}^+ := \{y > 0\}$ be the upper half plane in $\mathbb{R}^2$ and consider $M' = \mathbb{R} \times \mathbb{H}^+$ provided with the metric $g' = \mathrm{d}t \otimes \mathrm{d}t + \frac{1}{y^2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)$. To simplify the notation we denote the canonical coordinates $(t, x, y) \in \mathbb{R} \times \mathbb{H}^+$ by (x^1, x^2, x^3) in the differential forms, but keep y as the third component of the base point. The Christoffel symbols of g' in these coordinates are

$$
\Gamma^k_{ij} = \tfrac{1}{y}\big(\delta_{k3}(\delta_{i2}\delta_{j2} - \delta_{i3}\delta_{j3}) - \delta_{k2}(\delta_{i2}\delta_{j3} + \delta_{i3}\delta_{j2})\big) = \tfrac{1}{y}\gamma^k_{ij},
$$

where δ_{ij} denotes the Kronecker delta and γ^k_{ij} is a constant. Observe that γ^k_{ij} vanishes if at least one index equals 1. Using the summation convention, the coordinates (x^1, x^2, x^3) and the Kronecker delta δ_{ij} we write the 1-form $\alpha = \mathrm{d}t + \frac{1}{y}\mathrm{d}x$ as

$$
\left(\delta_{i1} + \tfrac{1}{y}\delta_{i2}\right)\mathrm{d}x^i.
$$

Lemma 5.5. *The 1-form α is bounded in all C^k-norms with respect to g'.*

Proof. The first covariant derivative of α is

$$
(\nabla\alpha)_{ij} = \tfrac{1}{y^2}\delta_{i2}\delta_{j3} = \tfrac{1}{y^2}\Delta_{ij},
$$

for constants Δ_{ij} that vanish if at least one index equals 1. Inductively we obtain the coefficients of the higher order covariant derivatives as

$$(\nabla^k \alpha)_{j_1 \dots j_{k+1}} = \tfrac{1}{y^{k+1}} \Delta_{j_1 \dots j_{k+1}},$$

where $\Delta_{j_1 \dots j_k}$ are constants that vanish if at least one index equals 1. Therefore for all $k \in \mathbb{N}$ there exists a constant $c_k > 0$ such that for all $(k+1)$-tuples $(j_1, \dots, j_{k+1})$

$$|(\nabla^k \alpha)_{j_1 \dots j_{k+1}}| \leq \tfrac{c_k}{y^{k+1}}.$$

Note that the coefficient $(\nabla^k \alpha)_{j_1 \dots j_{k+1}}$ vanishes if at least one index equals 1. This observation allows us to improve the estimate

$$\|\nabla^k \alpha\|_{C^0} \leq \left(\frac{2n-1}{\sqrt{\lambda_1}} \right)^{k+1} \max_{i, j_1, \dots, j_k} \{ |(\nabla^k \alpha)_{i j_1 \dots j_k}| \},$$

given in the previous section, to

$$\|\nabla^k \alpha\|_{C^0} \leq \left(\frac{2n-2}{\sqrt{\lambda_2}} \right)^{k+1} \max_{i, j_1, \dots, j_k} \{ |(\nabla^k \alpha)_{i j_1 \dots j_k}| \}.$$

The reasoning is as follows: In the sum (5.4) each summand that contains an index equal to 1 vanishes, so only $(2n-2)^{k+1}$ not $(2n-1)^{k+1}$ terms contribute to the sum. The same argument shows that only the second and third coefficient of each vector appear in (5.3). These components satisfy $|v^i| \leq \frac{1}{\sqrt{\lambda_2}} \|v\|_{g'}$, for $i = 2, 3$, where λ_2 is the eigenvalue associated to the eigenvalue ∂_{x^2}. Since we consider the explicit metric $g' = \mathrm{d}t \otimes \mathrm{d}t + \frac{1}{y}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)$ we have $\lambda_2 = \frac{1}{y^2}$. Moreover, we consider a 3-dimensional manifold and have $2n - 1 = \dim M' = 3$. Combining all these statements yields

$$\begin{aligned} \|\nabla^k \alpha\|_{C^0} &\leq \left(\frac{2n-2}{\sqrt{\lambda_2}} \right)^{k+1} \max_{i, j_1, \dots, j_k} \{ |(\nabla^k \alpha)_{i j_1 \dots j_k}| \} \\ &= \left(\frac{2}{\sqrt{1/y^2}} \right)^{k+1} \max_{i j_1 \dots j_k} \{ |(\nabla^k \alpha)_{i j_1 \dots j_k}| \} \\ &\leq 2^{k+1} \, y^{k+1} \, \frac{c_k}{y^{k+1}} \\ &\leq 2^{k+1} \, c_k =: C_k. \end{aligned}$$

Since the C^k-norm of α is defined via the C^0-norms of the covariant derivatives, we see that $\|\alpha\|_{C^k}$ is globally bounded for each $k \in \mathbb{N}_0$. □

Remark 5.6. This example is the universal covering of $M = S^1 \times \Sigma$ for a surface Σ with genus greater than 1. Endowing M with the pull-back of the area form on Σ leads to an odd-symplectic manifold (M, ω) that supports the virtually contact structure $(\pi : M' \to M, \omega, \alpha, g)$. Therefore we have found a 3-dimensional virtually contact structure whose contact form is C^k-bounded for all k. Taking further products with hyperbolic surfaces yields higher dimensional examples.

5.4. An Arzelà–Ascoli Argument

Let $\pi : M' \to M$ be a covering, g a Riemannian metric and ω an odd-symplectic form both on M. We lift g and ω to g' and ω' via π, respectively and obtain the same type of structures on M'. Denote the group of deck transformations of π by G. As we have seen in Remark 2.18 deck transformations act by odd-symplectomorphisms on M', i.e., $\varphi^*\omega' = \omega'$ for all $\varphi \in G$. The same argument shows that G acts by g'-isometries, i.e., $\varphi^* g' = g'$ for all $\varphi \in G$. Let us assume that the lifted odd-symplectic form ω' admits a primitive 1-form α, $\omega' = \mathrm{d}\alpha$.

Now let $(\varphi_\nu)_\nu \subset G$ be a sequence of deck transformations and denote the pull-back of α under φ_ν by $\alpha_\nu := \varphi_\nu^*\alpha$. Since the φ_ν are isometries we have $\|\alpha_\nu\|_{C^k} = \|\alpha\|_{C^k}$ by Remark 5.4. By the observation that $\mathrm{d}\alpha_\nu = \omega'$ we see that $\alpha_\nu - \alpha$ is closed.

Proposition 5.7. *We assume that the base manifold M is closed and that for all $k \in \mathbb{N}$ there exist $C_k > 0$ such that*

$$\|\alpha\|_{C^k} < C_k.$$

Then α_ν has a convergent subsequence in $C^\infty_{\mathrm{loc}}(M')$.

Proof. We have already observed that the deck transformation group acts by odd-symplectomorphisms. Therefore the form $\alpha_\nu - \alpha$ is closed for all $\nu \in \mathbb{N}$. Let us for the moment assume that it is also exact for all $\nu \in \mathbb{N}$, i.e., there exists a sequence of primitive functions f_ν with $\mathrm{d}f_\nu = \alpha_\nu - \alpha$ and $f(o) = 0$ where o is any chosen base point of M'. We claim that C^∞_{loc}-convergence of f_ν up to subsequence implies the same convergence for α_ν. Indeed if f_ν converges to f with respect to the C^∞_{loc}-topology it follows that $\mathrm{d}f_\nu$ converges to $\mathrm{d}f$ and therefore

$$\alpha_\nu - \alpha = \mathrm{d}f_\nu \xrightarrow{C^\infty_{\mathrm{loc}}} \mathrm{d}f,$$

so α_ν converges to $\alpha + \mathrm{d}f$. To finish the proof in the case that $\alpha_\nu - \alpha$ admits a global primitive we will show convergence of f_ν using the Arzelà–Ascoli theorem. Afterwards we will discuss how to conclude convergence on α_ν in the general case.

To apply the Arzelà–Ascoli theorem we have to show that the sequence f_ν is uniformly bounded and equicontinuous. If the base manifold M is compact, the cover (M', g') is geodesically complete [10, Theorem I.7.2]. An application of the Hopf–Rinow theorem [10, Theorem I.7.1] shows that for any point $p \in M'$ we can find a g'-unit speed geodesic $c : [0, T] \to M'$ with $c(0) = o$ and $c(T) = p$ realising the distance between o and p. The mean value theorem applied to f_ν along the curve c

yields a $t_0^\nu \in (0,T)$ with

$$\frac{f_\nu(p) - f_\nu(o)}{T} = T_{c(t_0^\nu)} f_\nu(\dot{c}(t_0^\nu)).$$

By the choice of f_ν holds $f_\nu(o) = 0$. This implies that $f_\nu(p)$ is at most $\mathrm{dist}(o,p) \cdot \|T_{c(t_0^\nu)} f_\nu\|_{g'}$. So we have

$$\|f_\nu\|_{C^0(B_r(o))} \leq r \cdot \|\mathrm{d}f_\nu\|_{C^0(B_r(o))} \leq 2rC_0.$$

Since the right hand side does no longer depend on ν, we have found a bound of $f_\nu|_{B_r(o)}$ that is uniform in ν. An analogous argument show equicontinuity for $f_\nu|_{B_r(o)}$ by replacing o with any $q \in B_r(o)$. By the Hopf–Rinow theorem the ball $\overline{B_r(o)}$ is compact so we can apply the Arzelà–Ascoli theorem and obtain a subsequence of f_ν that converges on $B_r(o)$ with respect to the C^0-topology. The balls $B_r(o)$ are a compact exhaustion of M' so we deduce

$$f_\nu \xrightarrow{C^0_{\mathrm{loc}}} f$$

up to choosing a subsequence.

For the higher order convergence we observe that

$$\|\mathrm{d}f_\nu\|_{C^k} = \|\alpha_\nu - \alpha\|_{C^k} \leq \|\alpha_\nu\|_{C^k} + \|\alpha\|_{C^k} = 2\|\alpha\|_{C^k} < 2C_k$$

for all k, where we used the condition that the C^k-norm of α is bounded and that the deck transformations act by isometries, see Remark 5.4. By our definition of the C^k-norm in Section 5.1 we have

$$\|f_\nu\|_{C^k(B_r(o))} \leq \max\{2rC_0, 2C_{k-1}\}.$$

If the closure $\overline{B_r(o)}$ of $B_r(o)$ is contained in a coordinate chart domain we can think of $B_r(o)$ as a bounded, open subset of $\mathbb{R}^{2n-1}$. In this case, [2, Theorem 8.6] states that the embedding operator

$$C^k(\overline{B_r(o)}) \hookrightarrow C^{k-1}(\overline{B_r(o)})$$

is compact. By the definition of a compact operator this shows that the C^k-bounds of f_ν imply convergence of a subsequence on $B_r(o)$ in the C^k-topology. Taking successive subsequences and invoking a diagonal sequence argument shows C^∞_{loc}-convergence on $B_r(o)$ up to subsequence and it turns out that the limit is indeed smooth.

If $\overline{B_r(o)}$ is not contained in a chart domain, we cover it by finitely many chart domains say $B^1, \ldots, B^l$. Applying the discussion above to these chart domains successively, we obtain a subsequence of f_ν that converges on all B^i and hence on $\overline{B_r(o)}$. A diagonal sequence argument applied to an exhausting sequence $\{B_r(o)\}_{r \in \mathbb{N}}$ of M'

shows $C^\infty_{\mathrm{loc}}(M')$-convergence up to subsequences. This ends the discussion for the case where $\alpha_\nu - \alpha$ admits a global primitive.

If the 1-form $\alpha_\nu - \alpha$ does not admit a global primitive, we work with an exhaustion by compact sets covered by chart domains as we did in the end of the argument above. To be precise on this, let $B \subset M'$ be a bounded open ball such that $\overline{B}$ is contained in a chart domain. By the Poincaré lemma we can select primitive functions f_ν of $(\alpha_\nu - \alpha)|_B$ on B that vanish at the center of B. Arguing as in the case above we can select a subsequence of α_ν that converges on B to a 1-form α_B in C^∞. We consider the sequence of open subsets $B_r(o)$ for $r \in \mathbb{N}$ that exhausts M' and the closure of each of these subsets is compact by the Hopf–Rinow theorem. Cover the set $\overline{B_r(o)}$ by finitely many chart domains $B_r^1, \dots, B_r^{k_r}$. As before we can find a subsequence of α_ν that converges on B_r^1 in C^∞. Taking a further subsequence we can achieve convergence on $B_r^1 \cup B_r^2$. Repeating the finitely many times we obtain a subsequence of α_ν that converges on B_r in C^∞. This can be done for all $r \in \mathbb{N}$ so we end up with a sequence of subsequences each of which converges on a set $B_r(o)$. Taking a diagonal sequence we obtain C^∞_{loc}- convergence on M' as we wanted. □

Corollary 5.8. *Assume that the base manifold M is closed and the C^k-norm of α is finite for some $k \geq 1$. Then α_ν admits a subsequence α_μ converging in $C^{k-1}_{\mathrm{loc}}(M')$ and its limit α_0 is a 1-form of class C^{k-1}.*

5.5. Induced Convergence on Complex Structures

We further specify our situation and assume that $(\pi : M' \to M, \omega, \alpha, g)$ is a virtually contact structure. Let φ_ν be a sequence of deck transformations and use the notation $\alpha_\nu = \varphi_\nu^* \alpha$ and $\xi_\nu = \ker \alpha_\nu$. Note that $\omega' = \pi^* \omega$ equals the exterior derivative $\mathrm{d}\alpha_\nu$ of α_ν and therefore defines a symplectic form on ξ_ν by restriction. As discussed before we find endomorphisms $\Phi_\nu : \xi_\nu \to \xi_\nu$ with

$$\omega' = g'(\Phi_\nu(\cdot), \cdot) \quad \text{on } \xi_\nu$$

for all $\nu \in \mathbb{N}$. These define complex structures j_ν on (ξ_ν, ω') by $j_\nu := \Phi_\nu \circ \left(\sqrt{-\Phi_\nu^2}\right)^{-1}$ which induce a bundle metric g_{j_ν} on ξ_ν through $g_{j_\nu} := \omega'(\cdot, j_\nu \cdot)$, see Section 3.1 for details. We extend these structures to the symplectisation $\mathbb{R} \times M'$ as hinted at in Section 4.1. Let t denote the $\mathbb{R}$-coordinate of the symplectisation and define a sequence of non-degenerate 2-forms η_ν by

$$\eta_\nu := \mathrm{d}t \wedge \alpha_\nu + \omega'$$

whose exterior derivatives

$$\mathrm{d}\eta_\nu = -\mathrm{d}t \wedge \mathrm{d}\alpha_\nu = -\mathrm{d}t \wedge \omega'$$

coincide for all $\nu \in \mathbb{N}$. We extend Φ_ν to an endomorphism field Ψ_ν on $\mathbb{R} \times M'$ that is uniquely defined by demanding its invariance under the $\mathbb{R}$-translation, that it restricts to Φ_ν on ξ_ν and that it maps ∂_t to the Reeb vector field R_ν of α_ν as well as R_ν to $-\partial_t$. By splitting the tangent space of $\mathbb{R} \times M'$ into the $\mathbb{R}$-component, the Reeb direction and the contact plane, i.e., $T(\mathbb{R} \times M') = \mathbb{R}\partial_t \oplus \mathbb{R}R_\nu \oplus \xi_\nu$, we can split Ψ_ν into the map $\mathrm{i} \oplus \Phi_\nu$. Using this splitting we see that the almost complex structure J_ν induced by Ψ_ν through $J_\nu := \Psi_\nu \circ \left(\sqrt{-\Psi_\nu^2}\right)^{-1}$ splits as $\mathrm{i} \oplus j_\nu$. This J_ν is the unique $\mathbb{R}$-invariant almost complex structure that maps ∂_t to R_ν and restricts to j_ν on ξ_ν.

Note that $\alpha_\nu = -\mathrm{d}t \circ J_\nu$ and therefore

$$\eta_\nu(\cdot, J_\nu \cdot) = \mathrm{d}t \otimes \mathrm{d}t + \alpha_\nu \otimes \alpha_\nu + g_{j_\nu}.$$

Using the splitting of the tangent space one checks that $\eta_\nu(\cdot, J_\nu \cdot)$ is positive definite and hence a metric on $\mathbb{R} \times M'$. In a similar fashion we define

$$g_\nu := \mathrm{d}t \otimes \mathrm{d}t + \alpha_\nu \otimes \alpha_\nu + g'|_{\xi_\nu},$$

these two metrics only differ on the ξ_ν-component where their behaviour on ξ_ν is described. The restriction of g' to ξ_ν is given by

$$g'|_{\xi_\nu}(v, w) = g'(v - \alpha_\nu(v)R_\nu \ , \ w - \alpha_\nu(w)R_\nu)$$

for all $v, w \in TM'$ since $v - \alpha_\nu(v)R_\nu$ gives the projection of v to ξ_ν along R_ν and therefore leads to our chosen splitting. The endomorphism field Ψ_ν defined above is the unique solution to the equation $\eta_\nu = g_\nu(\Psi_\nu \cdot, \cdot)$.

Lemma 5.9. *Assume that α satisfies the lower bound condition $|\alpha(v)| \geq K\|v\|_{g'}$ for all $v \in \ker \mathrm{d}\alpha$ and a positive constant K. Let $\alpha_\nu = \varphi_\nu^* \alpha$ be a sequence of contact forms converging to α_0 in $C^\infty_{\mathrm{loc}}(M')$. Then the limit α_0 is a contact form and the sequence of associated almost complex structures J_ν converges to an almost complex structure J_0 in $C^\infty_{\mathrm{loc}}(\mathbb{R} \times M')$. Moreover, J_0 is the almost complex structure that is associated to α_0 by the construction above.*

Sketch of Proof. We divide the proof into several parts. First we show that α_0 is indeed a contact form. This part uses that $\mathrm{d}\alpha_\nu$ equals ω' for all ν and by C^∞_{loc}-convergence the same is true for $\mathrm{d}\alpha_0$. Since all φ_ν are isometries the lower bound $|\alpha_\nu(v)| \geq c\|v\|_{g'}$ holds for all ν and all $v \in \ker \omega'$ and hence for the limit 1-form α_0. So α_0 does not vanish on the kernel of ω' and therefore $\alpha_0 \wedge (\mathrm{d}\alpha_0)^{n-1} = \alpha_0 \wedge (\omega')^{n-1}$ is a volume form.

In a second step we show that the Reeb vector fields R_ν converge to the Reeb vector field R_0 of α_0. To do so we note that the sequence η_ν converges to $\eta_0 := \mathrm{d}t \wedge \alpha_0 + \omega'$ in $C^\infty_{\mathrm{loc}}(\mathbb{R} \times M')$ and that R_ν is the unique $\mathbb{R}$-invariant vector field satisfying $\iota_{R_\nu}\eta_\nu = -\mathrm{d}t$. Using local coordinates and expressing the coefficients of R_ν and R_0 in terms of the coefficients of η_ν and η_0, respectively, we obtain the convergence of R_ν.

This implies that the sequence $g'|_{\xi_\nu} \to g'|_{\xi_0}$ converges in $C^\infty_{\mathrm{loc}}(\mathbb{R}\times M')$. The reasoning is as follows. Let π_{ξ_ν} denote the projection of TM' to ξ_ν along R_ν. This projection extends as $P_\nu := 0 \oplus \pi_{\xi_\nu}$ to $T(\mathbb{R}\times M')$. Ignoring the subscript ν we express the coefficients of $g'|_{\xi_\nu}$ as

$$(g'|_{\xi_\nu})_{ij} = P_i^k P_j^l (g')_{kl}.$$

Since $\pi_{\xi_\nu}(v) = v - \alpha_\nu(v)R_\nu$ we obtain have $P_i := P(\partial_i) = \partial_i - \alpha_i R$ and the k-th coordinate of this vector is $P_i^k = \delta_{ik} - \alpha_i R^k$. The convergence of α_ν and R_ν imply the convergence of the projection $P_\nu : T(\mathbb{R}\times M') \to \xi_\nu$ to the projection $P_0 : T(\mathbb{R}\times M') \to \xi_0$. This implies the convergence of $g'|_{\xi_\nu}$ to $g'|_{\xi_0}$.

The last ingredient in the proof is that the bundle isomorphisms Ψ_ν converge to Ψ_0 in C^∞_{loc} where Ψ_0 is induced by α_0 as described above. The convergence we just discussed implies

$$g_\nu \xrightarrow{C^\infty_{\mathrm{loc}}} \mathrm{d}t\otimes\mathrm{d}t + \alpha_0\otimes\alpha_0 + g'|_{\xi_0}.$$

As remarked, Ψ_ν is uniquely defined by $\eta_\nu = g_\nu(\Psi_\nu\cdot,\cdot)$ and can therefore be expressed by η and g_ν. In local coordinates this is given by $\Psi_i^j = \eta_{il} g_\nu^{lj}$. Hence, the convergence of g_ν and η_ν gives the convergence $\Psi_\nu \to \Psi_0$ in C^∞_{loc}.

The convergence of the almost complex structures follows since

$$J_\nu = \Psi_\nu \circ \left(\sqrt{-\Psi_\nu^2}\right)^{-1}$$

and all operations on the right hand side are continuous. □

Remark 5.10. In a situation as above we write $(\alpha_\nu, J_\nu) \to (\alpha_0, J_0)$ where J_ν and J_0 are the initially constructed almost complex structures associated to α_ν and α_0, respectively. In the proof of Lemma 5.9 we explicitly used C^1_{loc}-convergence to conclude that the differentials of the 1-forms α_ν converge to the differential of the limit 1-form α_0, which then still equals ω'. However, note that we did not use any higher regularity assumption on the convergence. Consequently we can require C^k_{loc}-convergence for some $k \geq 1$ in the lemma and end up with C^k_{loc}-convergence for J_ν.

Combining this with Corollary 5.8 we get the following statement: Assume that the C^k-norm of α is finite for some $k \geq 2$, then the sequences α_ν and J_ν converge in C^{k-1}_{loc} to α_0 and J_0, respectively, and the C^{k-1}-norm of α_0 is finite. Additionally, if α satisfies the lower bound assumption the same is true for α_0. That is if we start with C^k-bounds on α for $k \geq 3$ we obtain at least C^2-bounds for the limit 1-form α_0 and a further application of Corollary 5.8 to $\alpha_0^\nu := \varphi_\nu^*\alpha_0$ for a new sequence of deck transformations φ_ν yields a subsequence of α_0^ν converging to a 1-form α_∞ in C^{k-2}_{loc}. Since we started with $k \geq 3$ we also obtain convergence for the associated almost complex structures $J_\nu \to J_0$ and $J_0^\nu \to J_\infty$ in C^{k-1} and C^{k-2}, respectively. The significance of this Remark lies in the discussion at the end of Section 6.5, where we pass from C^∞-bounds to C^3-bounds.

Remark 5.11. In Section 4.1 we defined the almost complex structure J on $\mathbb{R}\times M'$ as $\mathrm{i}\oplus j$ with respect to the splitting $T(\mathbb{R}\times M') = (\mathbb{R}\oplus\mathbb{R}\cdot R)\oplus\xi$. Note that J equals

$$J = \Psi\circ\left(\sqrt{-\Psi^2}\right)^{-1},$$

where Ψ is the bundle endomorphism defined by

$$\eta = \mathrm{d}t\wedge\alpha + \omega' = \big(\mathrm{d}t\otimes\mathrm{d}t + \alpha\otimes\alpha + g'|_\xi\big)\big(\Psi(\cdot),\cdot\big).$$

The claimed equality for J follows from the observation $\Psi = \mathrm{i}\oplus\Phi$ where Φ was constructed in Section 4.1.

Let F_ν be the diffeomorphism $a_\nu\times\varphi_\nu$ on $\mathbb{R}\times M'$, where a_ν is the translation by a real number a_ν and φ_ν is a sequence of deck transformations. In Section 6 we will encounter diffeomorphisms like this again. We claim that

$$J_\nu = F_\nu^* J$$

where J_ν is the almost complex structure associated to $\alpha_\nu = \varphi_\nu^*\alpha$. A quick computation shows $F_\nu^*\eta = \mathrm{d}t\wedge\alpha_\nu + \omega' = \eta_\nu$. Recall that the projection to ξ along R is given by $\pi_\xi(v) = v - \alpha(v)R$ and that $g'|_\xi$ equals $(\pi_\xi)^*g'$. Using the equation $\pi_\xi\circ T\varphi_\nu = T\varphi_\nu\circ\pi_{\xi_\nu}$ we obtain

$$\varphi_\nu^*(g'|_\xi) = \varphi_\nu^*\pi_\xi^* g' = \pi_{\xi_\nu}^*\varphi_\nu^* g' = \pi_{\xi_\nu}^* g' = g'|_{\xi_\nu}.$$

Therefore we have

$$F_\nu^*(\mathrm{d}t\otimes\mathrm{d}t + \alpha\otimes\alpha + g'|_\xi) = g_\nu.$$

Adding these result we get

$$g_\nu\big(\Psi_\nu(\cdot),\cdot\big) = \eta_\nu = g_\nu\big(F_\nu^*\Psi(\cdot),\cdot\big),$$

where $F_\nu^*\Psi = (TF_\nu)^{-1}\Psi\, TF_\nu$. But since Ψ_ν is uniquely characterized by this equation we come to the conclusion that $\Psi_\nu = F_\nu^*\Psi$. Indeed,

$$\begin{aligned}(\eta_\nu)_p &= (F^*\eta)_p = \eta_{F(p)}(TF\,\cdot, TF\,\cdot)\\ &= (\mathrm{d}t\otimes\mathrm{d}t + \alpha\otimes\alpha + g'|_\xi)_{F(p)}(\Psi\, TF\,\cdot, TF\,\cdot)\end{aligned}$$

and by the definition of the pull back

$$\begin{aligned}(\eta_\nu)_p &= \big(F^*(\mathrm{d}t\otimes\mathrm{d}t + \alpha\otimes\alpha + g'|_\xi)\big)_p\big((TF)^{-1}\Psi\, TF\,\cdot\,,\,\cdot\big)\\ &= g_\nu(F^*\Psi\,\cdot\,,\,\cdot).\end{aligned}$$

For the second claim we can use that the square root of the conjugate of a matrix A is the conjugate of the square root of this matrix, i.e., $\sqrt{CAC^{-1}} = C\sqrt{A}C^{-1}$. We use this to observe

$$
\begin{aligned}
J_\nu &= \Psi_\nu \circ \left(\sqrt{-\Psi_\nu^2}\right)^{-1} \\
&= (TF_\nu)^{-1}\Psi\, TF_\nu \left(\sqrt{-(TF_\nu)^{-1}\Psi^2\, TF_\nu}\right)^{-1} \\
&= (TF_\nu)^{-1}\Psi \left(\sqrt{-\Psi^2}\right)^{-1} TF_\nu \\
&= F_\nu^* J.
\end{aligned}
$$

CHAPTER 6

Compactness

Let $(\pi : M' \to M, \alpha, \omega, g)$ be a virtually contact structure and J the almost complex structure on $\mathbb{R} \times M'$ constructed in Section 3.1 and 4.1. Let $o \in M'$ be a base point of the covering. Denote the group of deck transformations by G and assume that it acts transitively on the fibres of π, that is for every $p \in M$ and all $x, y \in \pi^{-1}(p)$ there exists a $\varphi \in G$ with $\varphi x = y$. A covering π with transitive deck transformation group is called **regular**. We assume that for any sequence $(\varphi_\nu)_{\nu \in \mathbb{N}}$ of deck transformations the induced sequence of contact forms $\alpha_\nu = \varphi_\nu^* \alpha$ has a subsequence converging in C^∞_{loc}. As mentioned before, the corresponding subsequence of J_ν also converges in C^∞_{loc}, see Lemma 5.9. Let α_0 be an accumulation point of α_ν. As pointed out in Remark 5.10, the sequence $\psi_\nu^* \alpha_0$ and the associated sequence of almost complex structures have converging subsequences for all sequences of deck transformations ψ_ν. We denote their limits by $(\alpha_\infty, J_\infty)$. We consider the metric $g_0' := \mathrm{d}t \otimes \mathrm{d}t + g'$ on $\mathbb{R} \times M'$.

Let $u_\nu = (a_\nu, f_\nu) : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{R} \times M', \{0\} \times L)$ be a sequence of J-holomorphic discs with boundary in an open relatively compact subset K_L of a maximally J-totally real submanifold $L \subset M'$. We assume that the Hofer energy

$$E_{\mathrm{Hofer}}(u) = \sup_\tau \int_{\mathbb{D}} u^* \mathrm{d}(\tau\alpha)$$

is uniformly bounded by some positive constant $E > 0$ for all $u = u_\nu$. The supremum is taken over all smooth strictly increasing functions $\tau : \mathbb{R} \to [0, 1]$ with $\tau(0) = 1$.

The aim of this chapter is to prove the following relation between contractible closed characteristics and the compactness of moduli spaces of holomorphic discs.

Proposition. *In the situation described above we assume that the C^3-norm of α defined in Section 5.1 is finite. If (M, ω) has no contractible closed characteristic, then u_ν has a Gromov converging subsequence that converges to a stable J-holomorphic disc with boundary on $\overline{K_L} \subset L$ whose underlying bubble tree consists of discs only.*

As mentioned before, the maximum principle applied to the subharmonic function a tells us that the image of the holomorphic curve lies in $(-\infty, 0] \times M'$. Moreover E. Hopfs boundary value lemma, see [14, Section 6.4.2], implies that 0 is a regular value of a. By Stokes theorem we have

$$\int_{\mathbb{D}} u^* \mathrm{d}(\tau\alpha) = \int_{\partial\mathbb{D}} f^* \alpha$$

for all functions τ with $\tau(0) = 1$. The same equation holds for the symplectic energy $E(u) = \int_{\mathbb{D}} u^* \mathrm{d}(\tau\alpha)$, which therefore is also bounded by E for all $u = u_\nu$.

We will carry out a bubbling off analysis for this situation similar to the one described in [20, 21]. The issue in our setup is the non-compactness of M'.

The proposition will be used in Section 7.1, where we will examine several conditions on M and M', respectively, that lead to non-compactness and in the end to the existence of closed contractible characteristics on M.

6.1. Strategy of Argument

The idea is to argue by contradiction. We assume that we have a sequence of holomorphic discs $u_\nu = (a_\nu, f_\nu)$, subject to the assumptions above, that does not admit a converging subsequence and want to show that this yields a closed contractible characteristic of (M, ω).

Recall that Corollary 4.8 said that uniform bounded energy and uniform bounded $\mathbb{R}$-component of a sequence of holomorphic discs $u_\nu : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{R} \times M', \{0\} \times K_L)$ result in a converging subsequence. Our sequence has uniformly bounded energy by assumption, so it cannot admit a uniformly bounded $\mathbb{R}$-component, i.e.,

$$\sup_\nu \max_{z \in \mathbb{D}} |a(z)| = \infty.$$

In Section 6.2 we will show that in this situation the sequence of gradients ∇u_ν is unbounded in the g_0'-norm. We will consider a sequence of points z_ν with $\|\nabla u_\nu(z_\nu)\|_{g_0'} = \max_{z\in\mathbb{D}} \|\nabla u_\nu(z)\|_{g_0'}$ and rescale the holomorphic disc restricted to a small neighbourhood of z_ν. These rescaled holomorphic disc will either converge to a finite energy plane $v : \mathbb{C} \to \mathbb{R} \times M'$ or a finite energy half-plane $v : (\mathbb{H}, \partial\mathbb{H}) \to (\mathbb{R} \times M', \{0\} \times \overline{K}_L)$.

If the limit is a finite energy plane we will reparametrise it to a cylinder and argue that it yields a closed α_∞-Reeb orbit that projects to a closed contractible characteristic with respect to the covering map $\pi : M' \to M$. See Section 6.3 for details.

If the limit is a finite energy half-plane we will show that it extends to a holomorphic disc, see Section 6.4.

In Section 6.5 we will combine these results to conclude the proposition. By the assumption of aperiodicity we can rule out that the rescaled maps in Section 6.2 converge to a finite energy plane. The last step in the proof is to argue that we find a converging subsequence of u_ν if all rescaled maps in Section 6.2 converge to finite energy half-planes. Thus, aperiodicity implies the existence of a converging subsequence.

6.2. Bubbling Off Analysis

We want to study non-compactness phenomena for families of holomorphic discs. Let

$$u_\nu = (a_\nu, f_\nu) : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{R} \times M', \{0\} \times K_L)$$

be a sequence of holomorphic discs with uniformly bounded energy and boundary on the open relatively compact set $K_L \subset L$ inside the maximally J-totally real submaifold L, as described in the beginning of this chapter. We assume that this sequence does not admit a Gromov converging subsequence.

In case the sequence of real parts $|a_\nu|$ is uniformly bounded we can apply Corollary 4.8 and obtain a Gromov convergent subsequence of u_ν whose limit is a stable holomorphic disc whose underlying bubble tree consists of discs only. Hence, it suffices to discuss the case that $|a_\nu|$ is not bounded. This is the case if and only if the sequence of maxima $\max |a_\nu|$ is not bounded. In this case we find a sequence of points $\zeta_\nu \in \mathbb{D}$ such that $a_\nu(\zeta_\nu)$ tends to $-\infty$. By the mean value theorem there exists a sequence of points $z'_\nu \in \mathbb{D}$ on the line segment connecting ζ_ν and 1 in $\mathbb{D}$ with

$$a_\nu(\zeta_\nu) = T_{z'_\nu} a_\nu \cdot (\zeta_\nu - 1).$$

Actually, the mean value theorem yields a point z'_ν on said line segment with $a_\nu(\zeta_\nu) - a_\nu(1) = T_{z'_\nu} a_\nu(\zeta_\nu - 1)$, but $a_\nu(1)$ equals 0 by our boundary condition. Since the diameter of the disc $\mathbb{D}$ equals 2 we obtain

$$|a_\nu(\zeta_\nu)| \leq |T_{z'_\nu} a_\nu| \cdot |\zeta_\nu - 1| \leq \|T_{z'_\nu} u_\nu\|_{g'_0} \cdot 2$$

with respect to the metric $g'_0 = \mathrm{d}t \otimes \mathrm{d}t + g'$. Hence $\|Tu_\nu\|_{g'_0}$ is not uniformly bounded. By passing to a subsequence, still denoted by u_ν, and writing

$$\|\nabla u_\nu\|_{g'_0} = \sqrt{\|\partial_x u_\nu\|^2_{g'_0} + \|\partial_y u_\nu\|^2_{g'_0}}$$

for $\|Tu_\nu\|_{g'_0}$, we can find a sequence $z_\nu \in \mathbb{D}$ with

$$R_\nu := \max_{\mathbb{D}} \|\nabla u_\nu\|_{g'_0} = \|\nabla u_\nu(z_\nu)\|_{g'_0} \to \infty.$$

By the compactness of the disc we can pass to a converging subsequence $z_\nu \to z_0$ again without changing the notation. The points of this sequence are called **bubble points**. For the further discussion we distinguish the cases where z_0 lies in the interior $B_1(0)$ of the disc $\mathbb{D}$ and on the boundary $\partial\mathbb{D}$ of the disc.

Before we begin the discussions we need two more terms.

Definition 6.1. A non-constant holomorphic map v defined on $\mathbb{C}$ with finite energy is called **finite energy plane**.

A non-constant holomorphic map v defined on $\mathbb{H}$ with finite energy is called **finite energy half-plane**.

Case 1

We begin with the discussion of $z_0 \in B_1(0)$.

Lemma. *Let u_ν be a sequence of holomorphic discs and z_ν a sequence of bubble points with $\|\nabla u_\nu(z_\nu)\|_{g_0'} = \max_{\mathbb{D}} \|\nabla u_\nu\|_{g_0'} =: R_\nu$. Then there exists an $\varepsilon > 0$ and a sequence of holomorphic discs*

$$v_\nu : B_{R_\nu \varepsilon}(0) \to \mathbb{R} \times M',$$

obtained from $u_\nu|_{B_\varepsilon(z_\nu)}$ by rescaling and composing with an g_0'-isometry, such that v_ν converges to a holomorphic map $v : \mathbb{C} \to \mathbb{R} \times M'$. Moreover, v is non-constant and has finite energy, i.e., v is a finite energy plane.

Proof. We assume that all z_ν lie in the interior $B_1(0)$, possibly after passing to yet another subsequence. Therefore, we can choose an $\varepsilon > 0$ such that $B_\varepsilon(z_\nu)$ is contained in $B_1(0)$ for all $\nu \in \mathbb{N}$. Let $\mathcal{D} \subset M'$ be a fundamental domain of the covering π that contains the base point $o \in M'$, see [10, §IV.3] for the definition and properties of a fundamental domain. We choose a sequence $\varphi_\nu \in G$ of deck transformations whose inverses map $f_\nu(z_\nu)$ into the closure of the fundamental domain, i.e., $\varphi_\nu^{-1}(f_\nu(z_\nu)) \in \overline{\mathcal{D}}$. To obtain uniform gradient bounds for the holomorphic discs we consider the rescaled sequence $v_\nu = (b_\nu, h_\nu) : B_{R_\nu \varepsilon}(0) \to \mathbb{R} \times M'$ defined via

$$b_\nu(z) := a_\nu\left(z_\nu + \tfrac{z}{R_\nu}\right) - a_\nu(z_\nu)$$

and

$$h_\nu(z) := \varphi_\nu^{-1}\left(f_\nu\left(z_\nu + \tfrac{z}{R_\nu}\right)\right)$$

for all $z \in B_{R_\nu \varepsilon}(0)$. Observe that the image of v_ν coincides with the image of $u_\nu|_{B_\varepsilon(z_\nu)}$ shifted by the map $-a_\nu(z_\nu) \times \varphi^{-1}$. Each rescaled function satisfies $v_\nu(0) \in \{0\} \times \overline{\mathcal{D}}$ and $B_{R_\nu \varepsilon}(0)$ is the maximal domain where we can ensure existence. Introducing a new function $F_\nu = a_\nu(z_\nu) \times \varphi_\nu$, i.e., $F_\nu(t,p) = (t + a_\nu(z_\nu), \varphi_\nu(p))$, we can express the rescaled function by $v_\nu = F_\nu^{-1} \circ u_\nu \circ \psi$ where ψ is the Möbius transformation

$$z \mapsto z_\nu + \tfrac{z}{R_\nu}.$$

Observe that the function F_ν is an isometry with respect to $g_0' = \mathrm{d}t \otimes \mathrm{d}t + g'$, since any deck transformation φ is a g'-isometry, see Section 5.4. Writing v_ν in this way shows that v_ν is indeed obtained form $u_\nu|_{B_\varepsilon(z_\nu)}$ by rescaling and composing with an isometry. Moreover, since F_ν is an isometry with respect to g_0' we obtain $\|\nabla v_\nu(0)\|_{g_0'} = 1$ and $\|\nabla v_\nu\|_{g_0'} \leq 1$ on $B_{R_\nu \varepsilon}(0)$ for all $\nu \in \mathbb{N}$ using the chain rule and $R_\nu = \max_{\mathbb{D}} \|\nabla u_\nu\|_{g_0'}$. Further, v_ν is J_ν holomorphic with respect to the almost complex structure $J_\nu = F_\nu^* J = TF_\nu^{-1} \circ J \circ TF_\nu$, since

$$\begin{aligned} Tv \circ \mathrm{i} &= TF_\nu^{-1} \circ Tu_\nu \circ T\psi \circ \mathrm{i} = TF^{-1} \circ Tu_\nu \circ \mathrm{i} \circ T\psi \\ &= TF_\nu^{-1} \circ J \circ Tu_\nu \circ T\psi = TF_\nu^{-1} \circ J \circ Tu_\nu \circ T\psi \\ &= J_\nu Tv_\nu. \end{aligned}$$

Associated to the sequence φ_ν of deck transformations we obtain a sequence of contact structures $\alpha_\nu = \varphi_\nu^* \alpha$. Note that by Remark 5.11 the almost complex structure associated to α_ν in Section 4.1 equals J_ν. Remember that in the beginning of the chapter, we made the assumption that α_ν admits a converging subsequence and therefore the associated sequence of almost complex structures J_ν converges, see Lemma 5.9. We denote the C^∞_{loc}-limit of these subsequences by α_0 and J_ν, respectively. Finally notice that, by the transformation formula and non-negativity of $u_\nu^* \mathrm{d}(\tau\alpha)$, the Hofer energy of the rescaled function

$$\sup_\tau \int_{B_{R_\nu \varepsilon}(0)} v_\nu^* \mathrm{d}(\tau \alpha_\nu)$$

is uniformly bounded by E thanks to the energy bound on u_ν Again the supremum is taken over all smooth strictly increasing functions $\tau : \mathbb{R} \to [0,1]$ with $\tau(0) = 1$.

For $k \in \mathbb{N}$ choose $N_0 = N_0(k)$ such that $\overline{B_k(0)} \subset B_{R_\nu \varepsilon}(0)$ for all $\nu \geq N_0$, this is possible since R_ν tends to infinity. Using the uniform gradient bound $\|\nabla v_\nu\|_{g_0'} \leq 1$ we obtain that the g_0'-distance between $v_\nu(0)$ and $v_\nu(z)$ is at most k for all $z \in B_k(0)$ and $\nu \geq N_0$, i.e., $\mathrm{dist}_{g_0'}(v_\nu(0), v_\nu(z)) < k$. In other words, for all $z \in B_k(0)$ the image point $v_\nu(z)$ is contained in the compact set $\overline{B_k(v_\nu(0))} \subset \mathbb{R} \times M'$.

In [10, p. 117] the notion of cut locus of a covering $\pi : M' \to M$ are introduced and it is explained how a to choose a fundamental domain $\mathcal{D}$ of π such that $\pi(\mathcal{D})$ equals the complement of the cut locus of $\pi(o)$ where $o \in M'$ denotes the base point of M'. Naïvely speaking we think of a tangential cut locus $C(p)$ of p as all the points in T_pM where the exponential map at p first fails to be injective and the cut locus as the image of this set under the exponential map. As explained in [10, Theorem III.2.2], the domain $D(p)$ bounded by the tangential cut locus has the property $\exp_p(D(p)) = M \setminus \exp_p(C(p))$. The mentioned fundamental domain is obtained by applying the exponential map of M' to the preimage of $D(p)$ under $T\pi$. We set

$$\mathcal{D} := \exp_o \Big(T\pi^{-1} \Big(D\big(\pi(o)\big) \Big) \Big).$$

Indeed we can construct a fundamental domain in this fashion since the exponential map restricts to a diffeomorphism from $D(o)$ to $\mathcal{D}$ and $C(o)$ is contained in the closure of $D(o)$.

In our situation the base manifold (M, g) is compact and hence its diameter d_0 is finite. This implies that the g'-distance of o and $h_\nu(0)$ is bounded by d_0 as

$h_\nu(0)$ lies in the closure of the fundamental domain whose diameter is bounded by the diameter of M, since $\mathcal{D}$ is isometric to $M \setminus C(\pi(o))$. Combining this with the previous observation that $v_\nu(B_k(0)) \subset B_k(v_\nu(0)) = B_k((0, h_\nu(0)))$, we achieve

$$v_\nu\left(\overline{B_k(0)}\right) \subset [-k, 0] \times \overline{B_{d_0+k}(o)}.$$

Note that the last factor is compact by the Hopf–Rinow theorem. By the definition of C^∞_{loc}-convergence we have that the restriction of (α_ν, J_ν) to $[-k, 0] \times \overline{B_{d_0+k}(o)}$ converges with all derivatives to the restriction of (α_0, J_0).

We want to apply the elliptic regularity explained in [37, Theorem B.4.2] to the sequence $v_\nu : B_{R_\nu\varepsilon}(0) \to \mathbb{R} \times M'$: To do this we quote the theorem and discuss the setting therein. In the following statement Σ is an oriented 2-dimensional manifold, N is a not necessarily compact symplectic manifold and $\mathcal{J}^l(N, N')$ denotes the set of all almost complex structures of class l on N for which the closed submanifold N' is totally real. Otherwise the theorem uses the notation of this chapter.

Theorem 6.2 ([37, Theorem B.4.2]). *Fix an $l \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ and a number $p > 2$. Let $J_\nu \in \mathcal{J}^l(N, N')$ be a sequence of almost complex structures on N converging to $J \in \mathcal{J}^l(N, N')$ in the C^l-topology and j_ν be a sequence of complex structures on Σ converging to j in the C^∞-topology. Let $U_\nu \subset \Sigma$ be an increasing sequence of open subsets whose union is Σ and $u_\nu : U_\nu \to N$ be a sequence of (j_ν, J_ν)-holomorphic curves of class $W^{1,p}$ such that*

$$u_\nu(U_\nu \cap \partial\Sigma) \subset N'.$$

Assume that for every compact set $Q \subset \Sigma$ (with smooth boundary) there exists a compact set $K \subset N$ and a constant $c > 0$ such that

$$\|\mathrm{d}u_\nu\|_{L^p(Q)} \leq c, \qquad u_\nu(Q) \subset K$$

for ν sufficiently large. Then there exists a subsequence of u_ν which converges in C^{l-1}-topology on every compact subset of Σ.

Ideally we would apply this theorem for $\Sigma = \mathbb{C}$, $U_\nu = B_{R_\nu\varepsilon}(0)$, $N = \mathbb{R} \times M'$ and the functions $v_\nu : B_{R_\nu\varepsilon}(0) \to \mathbb{R} \times M'$. Unfortunately we have only C^∞_{loc}-convergence of the almost complex structures on $\mathbb{R} \times M'$. To avoid this problem we perform a little trick. We will apply the theorem to $\Sigma = B_k(0)$ and $U_\nu = B_{R_\nu\varepsilon}(0) \cap B_k(0)$ for each $k \in \mathbb{N}$ and the functions v_ν restricted to U_ν. This will yield subsequences $v^k_\nu : B_k(0) \to (-k, 0) \times B_{d_0+k}(o)$ converging to a J_0-holomorphic map $v^k : B_k(0) \to (-k, k) \times B_{d_0+k}(o)$ in C^∞_{loc}-topology. Taking a diagonal sequence leads to a J_0-holomorphic map $v : \mathbb{C} \to \mathbb{R} \times M'$ that is the C^∞_{loc}-limit of the v_ν.

We give more details on the application of Theorem 6.2. As hinted at above we consider the 2-manifold $\Sigma = B_k(0)$ and the exhausting sets $U_\nu = B_{R_\nu\varepsilon}(0)$ for a fixed k. The sequence j_ν of complex structures on $B_k(0)$ constantly equals the standard complex structure i. The J_ν holomorphic maps $v_\nu|_{U_\nu}$ have image in the relative compact open set $(-k,0)\times B_{d_0+k}(o)$, where the almost complex structures J_ν converge to J_0 in C^∞-topology. Thus we choose $N=(-k,0)\times B_{d_0+k}(0)$. Moreover, the maps v_ν extend to smooth functions on compact sets and hence they are of class $W^{1,p}$. It remains to show the condition that for every compact subset $Q\subset B_k(0)$ there exists a compact subset $K\subset(-k,0)\times B_{d_0+k}(0)$ and a constant $c>0$ such that

$$\|\mathrm{d}v_\nu\|_{L^p(Q)}\leq c \quad \text{and} \quad u_\nu(Q)\subset K$$

for ν sufficiently large. Since we have the uniform gradient bound $\|\nabla v_\nu\|_{g_0'}\leq 1$ we can take the constant c in the first estimate to be the area of Q. For the second statement we can use that Q is contained in some ball $B_R(0)$ for an $R<k$ and therefore its image is contained in $[-R,0]\times\overline{B_{d_0+R}(0)}=:K$ which is compact. Now the theorem gives us a subsequence of v_ν^k that converges on every compact subset to the restriction of a holomorphic function $v^k:B_k(0)\to(-k,0)\times B_{d_0+k}(0)$ with respect to the C^∞-norm, i.e., in C^∞_{loc}-topology.

By a variant of Fatou's lemma [33, VI, Corollary 5.7] for convergent sequences we obtain an energy estimate for the limit function v. Namely we have for all $k\in\mathbb{N}$ and all smooth strictly increasing functions $\tau:\mathbb{R}\to[0,1]$ with $\tau(0)=1$ that

$$\begin{aligned}\int_{B_k(0)} v^*\mathrm{d}(\tau\alpha_0) &= \int_{B_k(0)}\lim_{\nu\to\infty} v_\nu^*\mathrm{d}(\tau\alpha_\nu)\\ &\leq \liminf_{\nu\to\infty}\int_{B_k(0)} v_\nu^*\mathrm{d}(\tau\alpha_\nu)\leq E.\end{aligned}$$

where the last step used the uniform energy bound on v_ν that carries over from u, by the transformation theorem. Taking the limit over k and the supremum we conclude that v has finite Hofer energy

$$\sup_\tau\int_{\mathbb{C}} v^*\mathrm{d}(\tau\alpha)\leq E.$$

Recall that we observed $\|\nabla v_\nu(0)\|_{g_0'}=1$, by C^∞_{loc}-convergence the same is true for the limit function v, so it is non-constant and therefore it is a finite energy plane. □

Case 2

We now discuss the case $z_0\in\partial\mathbb{D}$. In this situation it is convenient to pass to the upper half-plane to obtain more useful coordinates. Therefore, we identify the punctured disc $\mathbb{D}\setminus\{-z_0\}$ with the closed upper half-plane $\mathbb{H}=\{y\geq 0\}\subset\mathbb{C}$ where the points $(z_0,0,-z_0)$ correspond to $(0,\mathrm{i},\infty)$. Precomposing the function

$$u_\nu:(\mathbb{D},\partial\mathbb{D})\longrightarrow(\mathbb{R}\times M',\{0\}\times L)$$

with the inverse of the identification gives a J-holomorphic map

$$\widehat{u}_\nu : (\mathbb{H}, \mathbb{R}) \longrightarrow (\mathbb{R} \times M', \{0\} \times L).$$

Ignoring finitely many elements we can assume that the images of the bubble points, which converge to 0, are contained in $\mathbb{D}^+ := \mathbb{D} \cap \mathbb{H}$. In the following we will denote the images of the bubble points with respect to the coordinate change by z_ν. Since the gradient of the identification map is bounded on $\mathbb{D}^+ \subset \mathbb{H}$ with respect to the standard norms we find a constant $c > 0$ such that for all $\nu \in \mathbb{N}$

$$\tfrac{1}{c} R_\nu \leq \|\nabla \widehat{u}_\nu(z_\nu)\|_{g_0'} \quad \text{and} \quad \|\nabla \widehat{u}_\nu(z)\|_{g_0'} \leq c R_\nu \qquad \text{for all } z \in \mathbb{D}^+.$$

By conformal equivalence, i.e., the transformation theorem, the Hofer energy

$$\sup_\tau \int_{\mathbb{D}^+} \widehat{u}_\nu^* \mathrm{d}(\tau\alpha)$$

stays unchanged and is still uniformly bounded by E. As before the supremum is taken over all smooth strictly increasing functions $\tau : \mathbb{R} \to [0,1]$ with $\tau(0) = 1$. After passing to a subsequence there exists an $\varepsilon > 0$ such that for all $\nu \in \mathbb{N}$

$$B_\varepsilon^+(z_\nu) := B_\varepsilon(z_\nu) \cap \mathbb{H} \subset \mathbb{D}^+$$

and

$$R_\nu y_\nu \to \varrho \in [0, \infty],$$

writing $z_\nu = x_\nu + \mathrm{i} y_\nu$. Using the convergence of z_ν to 0 it is pretty obvious that the first condition can be satisfied for some subsequence and some $\varepsilon > 0$. For the second condition either the sequence $R_\nu y_\nu$ is unbounded in which case we choose a subsequence of $R_\nu y_\nu$ that tends to infinity as ν tends to infinity. On the other hand if the sequence is bounded we can choose a converging subsequence $R_\nu y_\nu \to \varrho \in \mathbb{R}^+$ by the Bolzano–Weierstrass theorem.

Lemma. *Let $\widehat{u}_\nu$ be a sequence of holomorphic half-planes and $z_\nu = x_\nu + \mathrm{i} y_\nu$ a sequence of points with $\|\nabla \widehat{u}_\nu(z_\nu)\|_{g_0'} \geq \frac{1}{c} R_\nu$. Further let $\varrho \in [0, \infty]$ denote the limit of $R_\nu y_\nu$. Then there exists an $\varepsilon > 0$ and*

Case 2(a), $\varrho = \infty$***:*** *a sequence of holomorphic maps*

$$v_\nu : B_{R_\nu \varepsilon}(0) \cap \{y \geq -y_\nu R_\nu\} \longrightarrow \mathbb{R} \times M$$

obtained from $\widehat{u}_\nu|_{B_\varepsilon^+(z_\nu)}$ by rescaling and composing with an g_0'-isometry, such that v_ν converges to a holomorphic map $v : \mathbb{C} \to \mathbb{R} \times M'$. Moreover, v is non-constant and has finite energy, i.e., v is a finite energy plane.

Case 2(b), $\varrho < \infty$***:*** *a sequence of holomorphic maps*

$$v_\nu : B^+_{R_\nu\varepsilon}(\mathrm{i}R_\nu y_\nu) \longrightarrow \mathbb{R} \times M$$

obtained from $\widehat{u}_\nu|_{B^+_\varepsilon(z_\nu)}$ *by rescaling, such that* v_ν *converges to a holomorphic map* $v : \mathbb{H} \to \mathbb{R} \times M'$. *Moreover,* v *is non-constant and has finite energy, i.e.,* v *is a finite energy half-plane.*

Proof. **Case 2(a):** Let us begin with the case $\varrho = \infty$ that is quite similar to Case 1. We rescale $\widehat{u}_\nu = (\widehat{a}_\nu, \widehat{f}_\nu)$ to a map $v_\nu = (b_\nu, h_\nu)$ on $B_{R_\nu\varepsilon}(0) \cap \{y \geq -y_\nu R_\nu\}$ with

$$b_\nu(z) := \widehat{a}_\nu\left(z_\nu + \tfrac{z}{R_\nu}\right) - \widehat{a}_\nu(z_\nu)$$

and

$$h_\nu(z) := \varphi_\nu^{-1}\left(\widehat{f}_\nu\left(z_\nu + \tfrac{z}{R_\nu}\right)\right)$$

where φ_ν is a deck transformation with $\varphi^{-1}(\widehat{f}_\nu(z_\nu)) \in \overline{\mathcal{D}}$ for the fundamental domain $\mathcal{D}$. Observe that the image of v_ν equals the image $\widehat{u}_\nu|_{B^+_\varepsilon(z_\nu)}$ up to a shift by the isometry $-\widehat{a}_\nu(z_\nu) \times \varphi^{-1}$. We define v_ν on this set, since we can control the derivative of $\widehat{u}_\nu$ on $B^+_\varepsilon(z_\nu)$. Note that $v_\nu(0) \in \{0\} \times \overline{\mathcal{D}}$ and by our previous estimates on $\widehat{u}_\nu$

$$\tfrac{1}{c} \leq \|\nabla v_\nu(0)\|_{g_0'} \quad \text{and} \quad \|\nabla v_\nu(z)\|_{g_0'} \leq c$$

for all $z \in B_{R_\nu\varepsilon}(0) \cap \{y \geq -y_\nu R_\nu\}$. Furthermore, v_ν is J_ν-holomorphic with respect to $J_\nu = (\widehat{a}_\nu(z_\nu), \varphi_\nu)^* J$ and its Hofer energy with respect to $\alpha_\nu = \varphi_\nu^*\alpha$ is less or equal than E. As before we can choose a subsequence such that (α_ν, J_ν) converges in C^∞_{loc} with limit (α_0, J_0). Since $B_{R_\nu\varepsilon}(0) \cap \{y \geq -R_\nu y_\nu\}$ is a sequence of exhausting subsets of $\mathbb{C}$ we can apply elliptic regularity, see Theorem 6.2, as in Case 1 and obtain a C^∞_{loc}-converging subsequence whose limit is a non-constant J_0-holomorphic finite energy plane with Hofer energy less or equal than E with respect to α_0.

Case 2(b): If $\varrho < \infty$ is finite we rescale $\widehat{u}_\nu$ to

$$v_\nu(z) := \widehat{u}_\nu\left(x_\nu + \tfrac{z}{R_\nu}\right)$$

for all $z \in B^+_{R_\nu\varepsilon}(\mathrm{i}R_\nu y_\nu)$, where y_ν denotes the imaginary part of $z_\nu = x_\nu + \mathrm{i}y_\nu$. Note that v_ν is (i, J)-holomorphic so we do not have to adjust our almost complex structure on $\mathbb{R} \times M'$ and α stays unchanged, too. Since $y_\nu R_\nu$ converges, the sets $B^+_{R_\nu\varepsilon}(\mathrm{i}R_\nu y_\nu)$ are an exhausting sequence of open subsets of $\mathbb{H}$. Using our estimates on u_ν we obtain

$$\tfrac{1}{c} \leq \|\nabla v_\nu(\mathrm{i}R_\nu y_\nu)\|_{g_0'}$$

independent of ν, and

$$\|\nabla v_\nu\|_{g_0'} \leq c$$

uniformly on $B^+_{R_\nu\varepsilon}(\mathrm{i}R_\nu y_\nu)$. Similar to Case 1 we find for each $k \in \mathbb{N}$ a $\nu \in \mathbb{N}$ such that $B^+_k(0) \subset B^+_{R_\nu\varepsilon}(\mathrm{i}R_\nu y_\nu)$ and the uniform gradient bound yields the distance estimate $\mathrm{dist}_{g_0'}(v_\nu(z), \{0\} \times K_L) \leq ck$ for all $z \in B_k(0)$. Hence, if ν is sufficiently large the image $v_\nu\big(B^+_k(0)\big)$ of $B^+_k(0)$ is contained in a ck-neighbourhood of $\{0\} \times K_L$ with compact closure. Furthermore we have $v_\nu(B^+_k(0) \cap \partial\mathbb{H}) \subset \{0\} \times K_L$ by our boundary condition on $\widehat{u}_\nu$. Again we are in the situation of Theorem 6.2 and obtain a C^∞_{loc}-converging subsequence $v_\nu \to v$ where

$$v : (\mathbb{H}, \mathbb{R}) \longrightarrow (\mathbb{R} \times M', \{0\} \times \overline{K_L})$$

is a J-holomorphic half-plane. Note that v satisfies the same boundary condition as v_ν. Further observe that the Hofer energy of v_ν with respect to α is still bounded by E. Thus, the same is true for the Hofer energy $E_{\mathrm{Hofer}}(v) \leq E$ with respect to α. The lower estimate $\frac{1}{c} \leq \|\nabla v_\nu(\mathrm{i}R_\nu y_\nu)\|_{g_0'}$ for the gradients of v_ν imply the same type of estimate for v. All in all we see that v is a finite energy half-plane. □

Summary. We started with a sequence u_ν of holomorphic disc with uniformly bounded Hofer energy and assumed that it does not admit a converging subsequence. Using Corollary 4.8 we concluded that the sequence of gradients ∇u_ν is not bounded with respect to the g_0'-norm. Distinguishing three cases we found sequences of holomorphic maps v_ν that either converge to a finite energy plane or a finite energy half-plane. These sequences were obtained from u_ν by restriction to a neighbourhood of bubble points, rescaling and shift by a g_0'-isometry.

The limit objects, the finite energy planes and the finite energy half-planes, are studied in Sections 6.3 and 6.4, respectively.

6.3. A Finite Energy Cylinder

Let $\widehat{v} = (\widehat{b}, \widehat{h})$ be a non-constant J_0-holomorphic finite energy plane with Hofer energy $E_{\mathrm{Hofer}}(\widehat{v}) \leq E$ with respect to α_0, for example the finite energy plane obtained in Cases 1 or 2(a).

The aim of this section is to find a periodic α_∞-Reeb orbit. Here α_∞ is obtained as the C^∞_{loc}-limit of $\alpha_0^\nu := \varphi_\nu^*\alpha_0$. To obtain the Reeb orbit we reparametrise a part of the finite energy plane $\widehat{v}$ to a holomorphic cylinder v.

Let $T^1 = \mathbb{R}/2\pi\mathbb{Z}$ be the 1-torus. Precomposing $\widehat{v}$ with the conformal map

$$\begin{aligned} \mathbb{R} \times T^1 &\longrightarrow \mathbb{C} \\ (s,t) &\longmapsto \mathrm{e}^{s+\mathrm{i}t} \end{aligned}$$

we obtain a finite energy cylinder, denoted by $v = (b, h) : \mathbb{R} \times T^1 \to \mathbb{R} \times M'$. Since the gradient of the coordinate change is bounded neither from below nor from above we have to consider the gradient of v.

Lemma. *Let $v = (b, h) : \mathbb{R} \times T^1 \to \mathbb{R} \times M'$ be a J_0-holomorphic energy cylinder, obtained from a finite energy plane $\widehat{v}$ by rescaling with the map $(s, t) \mapsto \mathrm{e}^{s+\mathrm{i}t}$. Then the gradient ∇v is globally bounded on $\mathbb{R} \times T^1$.*

We will argue by contradiction and assume that the gradient is unbounded. As in the previous Section we will trace the points where the gradient blows up and perform a rescaling as in Case 1.

Proof. We pass to the universal cover $\mathbb{R}^2$ and argue by contradiction analogous to the proofs of Proposition 27 and 30 in [29]. Note that this passing to the universal cover is merely for convenience of notation, all arguments can be worked out for $\mathbb{R} \times T^1$. Assuming that $\|\nabla v\|_{g_0'}$ is unbounded, we choose sequences $p_\nu \in \mathbb{R}^2, \varepsilon_\nu' \in (0, \infty)$ with $|p_\nu| \to \infty, \varepsilon_\nu' \to 0$ and $\varepsilon_\nu' \cdot \|\nabla v(p_\nu)\|_{g_0'} \to \infty$. Note that the condition $|p_\nu| \to \infty$ is superfluous since it follows from $\|\nabla v(p_\nu)\|_{g_0'} \to \infty$, we just list it for convenience. Applying the Hofer lemma [29, Lemma 26] to each element $(p_\nu, \varepsilon_\nu')$ in this sequence, we obtain a new sequence (z_ν, ε_ν) with

(a) $\varepsilon_\nu \leq \varepsilon_\nu'$ and $\varepsilon_\nu \cdot \|\nabla v(z_\nu)\|_{g_0'} \geq \varepsilon_\nu' \cdot \|\nabla v(p_\nu)\|_{g_0'}$,

(b) $d(p_\nu, z_\nu) \leq 2\varepsilon_\nu'$,

(c) $2\|\nabla v(z_\nu)\|_{g_0'} \geq \|\nabla v(y)\|_{g_0'}$ for all $y \in B_{\varepsilon_\nu}(z_\nu)$.

Observe that Property (a) implies $\varepsilon_\nu \to 0$ and $R_\nu \varepsilon_\nu \to \infty$ using the abbreviation $R_\nu := \|\nabla v(z_\nu)\|_{g_0'}$. Further, Property (b) implies that $|z_\nu|$ still tends to infinity.

We rescale the function $v = (b, h)$ by a sequence of deck transformations $\varphi_\nu \in G$ whose inverses map $h(z_\nu)$ into the closure of the fundamental domain $\overline{\mathcal{D}}$, i.e., we define a new sequence of functions $u_\nu = (a_\nu, f_\nu)$ by

$$a_\nu(z) := b\left(z_\nu + \tfrac{z}{R_\nu}\right) - b(z_\nu)$$

and

$$f_\nu(z) := \varphi_\nu^{-1}\left(h\left(z_\nu + \tfrac{z}{R_\nu}\right)\right).$$

Similar to Case 1 in Section 6.2 we can write $u_\nu = F_\nu^{-1} \circ v_\nu \circ \psi_\nu$, for a Möbius transformation ψ_ν and a g_0'-isometry $F_\nu = b(z_\nu) \times \varphi_\nu$. Moreover, we observe the following properties of the new sequence. It maps 0 into the closure of the fundamental domain $u_\nu(0) \in \{0\} \times \overline{\mathcal{D}}$. The g_0'-norm of its gradient is uniformly bounded by 2 on $B_{R_\nu \varepsilon_\nu}(0)$ and equals 1 in 0. In addition, u_ν is J_0^ν-holomorphic with respect to $J_0^\nu = F_\nu^* J_0$. At last, the Hofer energy

$$\sup_\tau \int_{B_{R_\nu \varepsilon_\nu}(0)} u_\nu^* \mathrm{d}(\tau \alpha_0^\nu)$$

with respect to $\alpha_0^\nu = \varphi_\nu^* \alpha_0$ is uniformly bounded by E.

We also obtain

$$\int_{B_{R_\nu \varepsilon_\nu}(0)} f_\nu^* \mathrm{d}\alpha_0^\nu = \int_{B_{R_\nu \varepsilon_\nu}(0)} (\varphi_\nu^{-1} \circ h \circ \psi_\nu)^* \mathrm{d}(\varphi_\nu^* \alpha_0) = \int_{B_{\varepsilon_\nu}(z_\nu)} h^* \mathrm{d}\alpha_0,$$

where the last term tends to 0 as ν tends to infinity, since ε_ν tends to 0 and the total area is bounded by E. By the assumption made in the introduction of this chapter, we find a subsequence of (α_0^ν, J_0^ν) that converges in C^∞_{loc} to some $(\alpha_\infty, J_\infty)$.

Arguing as in Case 1 and using Theorem 6.2 we find a $C^\infty_{\mathrm{loc}}(\mathbb{C})$-converging subsequence u_ν whose limit is a non-constant J_∞-holomorphic finite energy plane with Hofer energy

$$\sup_\tau \int_{\mathbb{C}} u^* \mathrm{d}(\tau \alpha_\infty) \le E.$$

On the other hand we claim that the contact area $\int_{\mathbb{C}} f^* \mathrm{d}\alpha_\infty$ of $u = (a, f)$ vanishes. That is because

$$\begin{aligned}\int_{B_k(0)} f^* \mathrm{d}\alpha_\infty &= \int_{B_k(0)} \lim_{\nu \to \infty} f_\nu^* \mathrm{d}\alpha_0^\nu \\ &\le \liminf_{\nu \to \infty} \int_{B_{R_\nu \varepsilon_\nu}(0)} f_\nu^* \mathrm{d}\alpha_0^\nu = 0.\end{aligned}$$

Where we used Fatou's lemma to obtain the estimate and that the contact area tends to 0 for the last term. Concerning [29, Lemma 28] in the case of non-compact manifolds a non-constant finite energy plane cannot have vanishing contact area. Therefore the g_0'-norm of ∇v is globally bounded on $\mathbb{R} \times T^1$. □

The next step is to shift the cylinder $v = (b, h)$ and to obtain a limit cylinder $u = (a, f)$. Finally we will show that the map $f : \mathbb{R} \times T^1$ is independent of the $\mathbb{R}$-variable and a reparametrisation of the α_∞-Reeb flow. Naïvely speaking this means that the maps $\varphi_\nu^{-1}(h(\nu, \cdot)) : T^1 \to M'$ converge to a closed α_∞-Reeb orbit, where φ_ν^{-1} is a deck transformation that maps $h(\nu, 0)$ into the closure of the fundamental domain and α_∞ is the limit 1-form of $\varphi_\nu^* \alpha_0$. This argument is a modification of [29, Theorem 31].

We choose a sequence $\varphi_\nu \in G$ of deck transformations such that $\varphi_\nu^{-1}(h(\nu, 0)) \in \overline{\mathcal{D}}$ for the fundamental domain $\mathcal{D}$ and define a shifted sequence of holomorphic cylinders $u_\nu = (a_\nu, f_\nu)$ via

$$a_\nu(s, t) := b(s + \nu, t) - b(\nu, 0)$$

and

$$f_\nu(s, t) := \varphi_\nu^{-1}\big(h(s + \nu, t)\big).$$

In other words, we define u_ν to be the map $F_\nu^{-1} \circ v \circ (\nu, 0)$, where $F_\nu = b(\nu, 0) \times \varphi_\nu$ is a g_0'-isometry and $(\nu, 0)$ denotes the shift on the cylinder $\mathbb{R} \times T^1$. Let us list some properties of the new sequence u_ν. First of all $u_\nu(0,0) \in \{0\} \times \overline{\mathcal{D}}$ and u_ν is J_0^ν holomorphic with respect to $F_\nu^* J_0^\nu$. Further the gradient ∇u_ν is globally bounded with respect to the g_0'-norm because F_ν is a g_0'-isometry and the gradient of v is globally bounded. Since J_0^ν is associated to α_0^ν the α_0^ν-Hofer energy of u_ν satisfies

$$\sup_\tau \int_{\mathbb{R}\times T^1} u_\nu^* \mathrm{d}(\tau\alpha_0^\nu) \leq E$$

for all $\nu \in \mathbb{N}$. For any given $k \in \mathbb{N}$, the transformation formula shows

$$\int_{[-k,k]\times T^1} f_\nu^* \mathrm{d}\alpha_0^\nu = \int_{[-k+\nu,k+\nu]\times T^1} h^* \mathrm{d}\alpha_0 \overset{\nu\to\infty}{\longrightarrow} 0, \tag{6.1}$$

the convergence follows since the contact area is non-negative and the total integral $\int_{\mathbb{C}} h^* \mathrm{d}\alpha_0$ is bounded by E. Similarly we obtain

$$\int_{\{0\}\times T^1} f_\nu^* \alpha_0^\nu = \int_{\{\nu\}\times T^1} h^* \alpha_0 = \int_{(-\infty,\nu]\times T^1} h^* \mathrm{d}\alpha_0 \to \int_{\mathbb{C}} h^* \mathrm{d}\alpha_0$$

where the last expression is the contact area. Assuming that (α_0^ν, J_0^ν) converges in C^∞_{loc} to $(\alpha_\infty, J_\infty)$ as explained in the introduction to this chapter we can argue analogous to Case 1 and find a $C^\infty_{\mathrm{loc}}(\mathbb{C})$-converging subsequence $u_\nu \to u$ whose limit is a non-constant J_∞-holomorphic finite energy cylinder $u = (a, f) : \mathbb{R}\times T^1 \to \mathbb{R}\times M'$ of Hofer energy

$$\sup_\tau \int_{\mathbb{R}\times T^1} u^* \mathrm{d}(\tau\alpha_\infty) \leq E.$$

Further we can apply Fatou's lemma to show that u has vanishing contact area. Indeed,

$$\begin{aligned}
\int_{\mathbb{R}\times T^1} u^* \mathrm{d}\alpha_\infty &= \lim_{k\to\infty} \int_{[-k,k]\times T^1} u^* \mathrm{d}\alpha_\infty \\
&= \lim_{k\to\infty} \int_{[-k,k]\times T^1} \lim_{\nu\to\infty} u_\nu^* \mathrm{d}\alpha_0^\nu \\
&\leq \lim_{k\to\infty} \liminf_{\nu\to\infty} \int_{[-k,k]\times T^1} u_\nu^* \mathrm{d}\alpha_0^\nu \\
&= 0,
\end{aligned} \tag{6.2}$$

in the last step we used the computation in (6.1). Note that this does not contradict [29, Lemma 28] since the lemma does not apply to the cylinder. The symplectic action

$$\int_{\{0\}\times T^1} f^* \alpha_\infty = \int_{\mathbb{C}} h^* \mathrm{d}\alpha_0 \neq 0 \tag{6.3}$$

is non vanishing by [29, Lemma 28] since otherwise v would be constant. For the computation of the symplectic energy we have used that h extends to a map on $\mathbb{C}$, because it was defined as a rescaling of a map defined on $\mathbb{C}$. Furthermore the gradient ∇u is globally bounded on $\mathbb{R} \times T^1$ with respect to g_0' since the same is true for the gradient of v.

Lemma. *Let $u : \mathbb{R} \times T^1 \to \mathbb{R} \times M'$ be a finite energy holomorphic cylinder obtained from the finite energy cylinder v with uniformly bounded gradient as above. Then $f(s,t)$ is independent of the $\mathbb{R}$-variable and $f(s,\cdot) : T^1 \to M'$ is a reparametrisation of a contractible periodic α_∞-Reeb orbit. Moreover, the α_∞-action of this orbit equals the contact area $\int_{\mathbb{C}} \widehat{h}^* \mathrm{d}\alpha_0$ of the initial finite energy plane $\widehat{v}$ up to multiplication by 2π.*

The argument follows the last part of [29, Theorem 31].

Proof. We consider the M'-component f of the limit cylinder $u = (a, f)$. Since

$$0 = \int_{\mathbb{R} \times T^1} u^* \mathrm{d}\alpha_\infty = \int_{\mathbb{R} \times T^1} f^* \mathrm{d}\alpha_\infty \tag{6.4}$$

by our previous computation, see (6.2), we obtain $\pi \circ Tf = 0$ where π denotes the projection to $\xi_\infty = \ker \alpha_\infty$ along the Reeb vector field R_∞ associated to α_∞. This is because the integrand of contact area $f^* \mathrm{d}\alpha_\infty$ is non-negative and therefore has to vanish but also measures the length of $(\pi \circ T_p f)(\partial_s)$ and $(\pi \circ T_p f)(\partial_t)$ with respect to the metric $\mathrm{d}\alpha_\infty(\cdot, j_\infty \cdot)$ constructed analogous to Section 3.1. Let $\tau \mapsto x(\tau)$ be the Reeb flow of R_∞ starting at any point in the image of f, say $x(0) = f(0,0)$. Our result about Tf tells us that we find a smooth function $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with

$$f(s,t) = x(\gamma(s,t))$$

and

$$\tfrac{\partial}{\partial s} \gamma = -\tfrac{\partial}{\partial t} a \quad , \quad \tfrac{\partial}{\partial t} \gamma = \tfrac{\partial}{\partial s} a, \tag{6.5}$$

where we lift $a : \mathbb{R} \times T^1$ to a map on $\mathbb{R} \times \mathbb{R}$, still denoted by a. The map γ does not descent to a map on the cylinder $\mathbb{R} \times T^1$ that is $\gamma(s,0) \neq \gamma(s,2\pi)$, since this would imply vanishing symplectic action for u. Accurately, if $\gamma(s,0) = \gamma(s,2\pi)$ we have

$$\int_{\{0\} \times T^1} f^* \alpha_\infty = \int_{\{s\} \times S^1} f^* \alpha_\infty - \int_{[0,s] \times S^1} u^* \mathrm{d}\alpha_\infty,$$

the second term vanishes since the integrand is non-negative and the total integral vanishes, see (6.4). Thus,

$$\begin{aligned}\int_{\{0\}\times T^1} f^*\alpha_\infty &= \int_{\{s\}\times T^1} f^*\alpha_\infty\\ &= \int_{\{s\}\times[0,2\pi]} x\big(\gamma(s,t)\big)^*\alpha_\infty\\ &= \int_{\{s\}\times[0,2\pi]} \partial_t\gamma(s,t)\mathrm{d}t\\ &= 0,\end{aligned}$$

contradicting our previous observation (6.3). Since f is defined on the cylinder, we obtain

$$x\big(\gamma(s,0)\big) = f(s,0) = f(s,2\pi) = x\big(\gamma(s,2\pi)\big)$$

and hence x is periodic because $\gamma(s,0) \neq \gamma(s,2\pi)$. Let T be the minimal period of x. Then there exists a number n such that $\gamma(s,0)+nT = \gamma(s,2\pi)$, since x does not have any double points. Observe that $\gamma(s,t)+nT$ and $\gamma(s,t+2\pi)$ solve the same differential equation

$$\begin{gathered}\partial_t\big(\gamma(s,t)+nT\big) = \partial_t a(s,t) = \partial_t a(s,t+2\pi) = \partial_t\big(\gamma(s,t+2\pi)\big)\\ \gamma(s,0)+nT = \gamma(s,0+2\pi)\end{gathered}$$

and therefore agree for all t. Using this rule iteratively we obtain

$$\gamma(s,2\pi k) = \gamma(s,2\pi l) + (k-l)\cdot nT. \tag{6.6}$$

for all integers k,l. We consider the function $\gamma - \mathrm{i}a : \mathbb{C}\to\mathbb{C}$, where we identified $\mathbb{R}^2$ with $\mathbb{C}$ in the canonical way. By our remarks about the partial derivatives of γ in (6.5) this map is holomorphic. Furthermore its growth, i.e., its gradient, is globally bounded and therefore it is polynomial of degree 1, i.e., it takes the form

$$(\gamma-\mathrm{i}a)(s+\mathrm{i}t) = \kappa\cdot(s+\mathrm{i}t)+\mu$$

for some complex numbers $\kappa,\mu\in\mathbb{C}$. By the definition of $\gamma-\mathrm{i}a$ we have

$$\begin{aligned}\mathrm{i}\cdot\kappa\cdot 2\pi(k-l) =&(\gamma-\mathrm{i}a)(s+\mathrm{i}2\pi k) - (\gamma-\mathrm{i}a)(s+\mathrm{i}2\pi l)\\ \overset{(6.6)}{=}&(k-l)\cdot nT - \mathrm{i}\big(a(s,2\pi k)-a(s,2\pi l)\big)\\ =&(k-l)\cdot nT,\end{aligned}$$

since a is the lift of a map defined on $\mathbb{R}\times T^1$. As we see $\kappa = -\mathrm{i}\frac{nT}{2\pi} = -\mathrm{i}c$ is purely imaginary and γ has the form

$$\begin{aligned}\gamma(s,t) &= \mathfrak{Re}\big((\gamma - \mathrm{i}a)(s+\mathrm{i}t)\big)\\ &= \mathfrak{Re}\big(-\mathrm{i}c\cdot(s+\mathrm{i}t)+\mu\big)\\ &= ct + \mathfrak{Re}(\mu) = ct + d\end{aligned}$$

for a suitable real coefficients c, d and $c \neq 0$. The same reasoning leads to $a = cs + e$. Combining the equations we get

$$u(s,t) = (a,f)(s,t) = \big(cs+e, x\circ\gamma(s,t)\big) = \big(cs+e, x(ct+d)\big).$$

We end this discussion by computing the α_∞-action of the closed orbit. We have

$$\begin{aligned}\int_{T^1} x(c\cdot+d)^*\alpha_\infty &= \int_0^{2\pi} \alpha_\infty\left((T_t x)(ct+d)\right)\mathrm{d}t\\ &= \int_0^{2\pi} \alpha_\infty\big(c\,(R_\infty)_{x(ct+d)}\big)\mathrm{d}t = 2\pi\, c\end{aligned}$$

On the other hand we already computed, see (6.3), that

$$\int_{T^1} x(c\cdot+d)^*\alpha_\infty = \int_{\mathbb{C}} h^*\mathrm{d}\alpha_0.$$

In particular we observe that the action and the constant c are positive. □

6.4. Removal of Boundary Singularity

Let $\{0\}\times L \subset \mathbb{R}\times M'$ be a maximally totally real submanifold with respect to J. We consider the non-constant J-holomorphic half-plane

$$\widehat{v} : (\mathbb{H},\mathbb{R}) \longrightarrow (\mathbb{R}\times M', \{0\}\times\overline{K}_L)$$

with finite Hofer energy $E_{\mathrm{Hofer}}(\widehat{v}) \leq E$ with respect to α, where K_L is an open relatively compact subset of L. Recall that the limit maps obtained in Case 2(b) fall in these class.

The aim of this section is to show that this finite energy half-plane extends to a smooth map on the disc $\mathbb{D}$ with similar properties.

We reparametrise a restriction of $\widehat{v}$ as a map on the strip $S := \mathbb{R}\times[0,\pi]$ by precomposing with the conformal map

$$\begin{aligned}\mathbb{R}\times[0,\pi] &\longrightarrow \mathbb{H}\setminus\{0\}\\ (s,t) &\longmapsto \mathrm{e}^{s+\mathrm{i}t}\end{aligned}$$

and obtain a finite energy strip

$$v : (S, \partial S) \longrightarrow (\mathbb{R} \times M', \{0\} \times \overline{K}_L).$$

Observe that the map v extends to a smooth map on $\{-\infty\} \cup S$. Note further that the gradient of the coordinate change is not bounded from below or from above. Nonetheless we obtain the following statement.

Lemma. *Let $v : (S, \partial S) \to (\mathbb{R} \times M', \{0\} \times \overline{K}_L)$ be a non-constant finite energy strip obtained from a finite energy half-plane $\widehat{v}$ by precomposing with the coordinate change $(s,t) \mapsto \mathrm{e}^{s+\mathrm{i}t}$. Then the gradient of v is globally bounded on S.*

Note that this does not follow form the gradient bounds for the initial function defined on $\mathbb{H}$, since the gradient of the coordinate change is unbounded.

Proof. We argue by contradiction. Assuming that the the gradient is unbounded we can argue as in [29, Theorem 32]. As in the previous discussion, see Section 6.3, we apply the Hofer lemma [29, Lemma 26] to obtain sequences $\varepsilon_\nu \in (0, \infty)$ and $z_\nu = x_\nu + \mathrm{i}y_\nu \in S$ with $\varepsilon_\nu \to 0, |x_\nu| \to \infty$ and $R_\nu := \|\nabla v(z_\nu)\|_{g_0'} \to \infty$ such that $R_\nu \varepsilon_\nu \to \infty$ and $\|\nabla v\|_{g_0'} \leq 2R_\nu$ on $B_{\varepsilon_\nu}(z_\nu)$ at least for a subsequence. After passing to a further subsequence we assume that $R_\nu y_\nu$ and $R_\nu(\pi - y_\nu)$ converge in $[0, \infty]$ to ϱ_0 and ϱ_π, respectively. We distinguish the case where $\varrho_0 = \varrho_\pi = \infty$ and the cases where one of the limits is finite. Observe that not both limits can be finite, since if $\lim_{\nu\to\infty} R_\nu y_\nu = \varrho_0$ is finite, then $\lim_{\nu\to\infty} R_\nu(\pi - y_\nu)$ cannot be finite, because R_ν tends to infinity. The other case can be excluded with a similar argument.

If $\varrho_0 = \varrho_\pi = \infty$ we argue as in the first half of Section 6.3. That is we rescale $v = (b, h)$ to a sequence of maps $u_\nu = (a_\nu, f_\nu)$ given by

$$a_\nu(z) := b\left(z_\nu + \tfrac{z}{R_\nu}\right) - b(z_\nu)$$

and

$$f_\nu(z) := \varphi_\nu^{-1}\left(h\left(z_\nu + \tfrac{z}{R_\nu}\right)\right)$$

with domain $B_{R_\nu\varepsilon_\nu}(0) \cap (\mathbb{R} \times [-R_\nu y_\nu, R_\nu(\pi - y_\nu)])$, where φ_ν is a deck transformation whose inverse maps $h(z_\nu)$ into the closure of the fundamental domain $\mathcal{D}$. Invoking Theorem 6.2 we repeat said argument and obtain a non-constant J_0-holomorphic finite energy plane with respect to α_0 with vanishing contact area, contradicting [29, Lemma 28].

If at least one of the limits is finite we can assume it is ϱ_0 after precomposing with $z \mapsto -(z - \mathrm{i}\pi)$ if necessary. In this case we have $y_\nu \to 0$ and therefore we find an $\varepsilon > 0$ such that $B_\varepsilon^+(z_\nu) := B_\varepsilon(z_\nu) \cap \mathbb{H}^+$ is contained in S. Let φ_ν be a sequence of deck transformations whose inverses map $h(z_\nu)$ into the closure of the fundamental domain $\mathcal{D}$. We define a rescaled sequence $u_\nu = (a_\nu, f_\nu)$ by

$$a_\nu(z) := b\left(x_\nu + \tfrac{z}{R_\nu}\right) - b(z_\nu)$$

and

$$f_\nu(z) := \varphi_\nu^{-1}\left(h\left(x_\nu + \tfrac{z}{R_\nu}\right)\right)$$

with domain $B^+_{R_\nu\varepsilon}(R_\nu y_\nu)$. Arguing as in Section 6.2, Case 2(b), we obtain a subsequence whose limit is a non-constant J_0-holomorphic finite energy half-plane with respect to α_0. Computing the contact area as in the first half of Section 6.3 we see $\int_{\mathbb{H}} u^* \mathrm{d}\alpha_0 = 0$. We imitate the argument in the first half of [29, Theorem 32] to show that there are no non-constant finite energy half-planes with vanishing contact area. Note that this case is not covered by [29, Theorem 28].

Combining these cases we conclude that $\|\nabla v\|_{g_0'}$ is globally bounded on S. □

As we have seen in similar situations before the bounded gradient implies that the point $v(s,t)$ can not get arbitrary far away from the point $v(s,0)$. In fact,

$$\mathrm{dist}_{g_0'}(v(s,0), v(s,t)) \leq C \cdot \pi,$$

where C is an upper bound for the norm $\|\nabla v\|_{g_0'}$ of the gradient of v.The situation, where the image of the boundary ∂S is contained in the compact set $\{0\} \times \overline{K}_L$. Hence, the distance estimate combined with the triangle inequality shows that the image $v(S)$ of the strip is contained in a ball of finite radius around the point $v(0,0)$. The closure of this ball is compact by the Hopf–Rinow theorem. On this set we have bounds on the curvature of g_0' and can compare the geometry with the euclidean one. This allows us to apply [37, Theorem 4.1.2(ii)], which says that the finite energy strip extends to a smooth map from $S \cup \{+\infty\}$ to $\mathbb{R} \times M'$. The quoted theorem is only formulated for compact manifolds and for maps from $\mathbb{D}^+ \setminus \{0\}$ to M. We will explain why our preparation allows use to use the theorem nonetheless. A conformal change of coordinates that maps $[0 \times \infty) \times [0, \pi]$ onto $\mathbb{D}^+ \setminus \{0\}$, similar to the one in the beginning of this section, yields a holomorphic map defined on $\mathbb{D}^+ \setminus \{0\}$ of finite energy whose image is still contained in a compact set. Actually, the proof of the theorem only uses local arguments which go through in our situation. Therefore, the considered holomorphic map on $S \cup \{\infty\} \cong \mathbb{D}^+ \setminus \{0\}$ extends to a smooth map on $\mathbb{D}^+$. Inverting the coordinate changes we see that the finite energy half-plane $\widehat{v}$ we started with extends to a smooth disc

$$\overline{v} : (\mathbb{D}, \partial\mathbb{D}) \cong (\mathbb{H} \cup \{\infty\}, \mathbb{R} \cup \{\infty\}) \longrightarrow (\mathbb{R} \times M', \{0\} \times \overline{K}_L).$$

Observe that $\overline{v}$ is holomorphic.

Remark 6.3. The result [37, Theorem 4.1.2] is formulated for closed symplectic manifolds. It can be applied in the present situation because the discussed phenomena are contained in a compact subset so that we can ignore the non-compactness. In fact the proof takes place in $\mathbb{R}^{2n}$. Therefore the bounded geometry of virtually contact structures discussed in Chapter 3 allows us to adapt the result.

Remark. We recapitulate what we have achieved in Case 2(b) and this section. We started with a sequence of holomorphic discs $u_\nu : \mathbb{D} \to \mathbb{R} \times M'$ and a sequence of points $z_\nu \in \text{interior}(\mathbb{D})$ for which the sequence of gradients $\nabla_\nu(z_\nu)$ blows up and that converged to a point on the boundary $\partial\mathbb{D}$. In Case 2(b) we rescaled the maps u_ν on a neighbourhood of the limit point. We argued that this sequence converges to a finite energy half-plane v up to subsequence. In this section we saw that v extends to a disc map with the same boundary condition $v(\partial\mathbb{D}) \subset \{0\} \times \overline{K_L}$ as u_ν. This is called the bubbling of a disc. Observe that the $\mathbb{R}$-coordinate in this case is not unbounded even if the gradients blow up.

6.5. Aperiodicity and Gromov Convergence

Let us summarise the discussion in the preceding sections. Let $(\pi : M' \to M, \omega, \alpha, g)$ be a virtually contact structure with closed base manifold M. Denote by J the associated almost complex structure on $\mathbb{R} \times M'$ constructed in Section 4.1. Consider a submanifold $L \subset M$ such that $\{0\} \times L$ is maximally J-totally real in $\mathbb{R} \times M'$. Further, consider a sequence

$$u_\nu = (a_\nu, f_\nu) : (\mathbb{D}, \partial\mathbb{D}) \longrightarrow (\mathbb{R} \times M', \{0\} \times L)$$

of J-holomorphic discs whose boundaries lie in an open relatively compact subset K_L of L and whose Hofer energies $E_{\text{Hofer}}(u_\nu)$ are uniformly bounded by a positive constant E.

In analogy to the Weinstein conjecture in contact geometry we are interested in finding **closed characteristics** on M. A closed characteristic is a compact leaf of the 1-dimensional foliation $\ker\omega$. To be more precise, a closed characteristic is an embedded copy of S^1 whose tangent space is contained in $\ker\omega$. After choosing any orientation on a closed characteristic it represents a homotopy class. We say a closed characteristic is **contractible** if there is a non-trivial multiple of this characteristic that is nullhomotopic.

Proposition 6.4. *In the situation described above we assume that the covering π is regular and that the C^3-norm of α defined in Section 5.1 is finite. If (M, ω) has no contractible closed characteristic, then u_ν has a Gromov converging subsequence that converges to a stable J-holomorphic disc with boundary on $\overline{K_L} \subset L$ whose underlying bubble tree consists of discs only.*

Proof. Let us first assume that not only the C^3-norm but all C^k-norms of α are finite. In this situation Proposition 5.7 tells us that for all sequences $(\varphi_\nu)_{\nu\in\mathbb{N}} \subset G$ of deck transformations there exists a C^∞_{loc}-converging subsequence of $\alpha_\nu := \varphi_\nu^*\alpha$. As observed in Lemma 5.9, the associated sequence of almost complex structures J_ν

on $\mathbb{R} \times M'$ converges as well. By Remark 5.10 we can replace (α, J) with the limit (α_0, J_0) and obtain an analogous convergence statement. We call the resulting limit $(\alpha_\infty, J_\infty)$.

Since the kernel of $\omega' = \mathrm{d}\alpha_\infty$ is spanned by the Reeb vector field R_∞, any periodic α_∞ Reeb orbit, possibly obtained as in Section 6.3, is a closed characteristic of ω'. Further, it projects down to a closed characteristic of ω via π, of which there are none by assumption. This means that only Case 2(b) can occur, because Cases 1 and 2(a) imply the existence of a closed Reeb orbit. As observed in Section 6.4, any finite energy half-plane resulting form Case 2(b) extends to a J-holomorphic disc with Hofer energy less or equal to E. In this case the Hofer energy equals the contact area, since we are considering holomorphic discs. Adapting the bubbling off analysis in [29, Lemma 35] as in the Sections 6.2, 6.3 and 6.4 we conclude that the contact area of the finite energy disc obtained by bubbling, i.e., by a convergence as in Section 6.4, is uniformly bounded from below.

After these preliminary notes we come to the actual proof of the proposition. As argued in the beginning of Section 6.2 it suffices to show that $|a_\nu|$ is bounded. That is, we have to rule out the existence of a sequence $\zeta_\nu \in \mathbb{D}$ with $a_\nu(\zeta_\nu) = \max |a_\nu|$ for all ν and $a_\nu(\zeta_\nu) \to -\infty$ for a subsequence as ν tends to infinity. By the maximum principle we have that ζ_ν lies in the interior $B_1(0)$ of $\mathbb{D}$. After precomposing u_ν with a Möbius transformation ψ_ν we may assume that ζ_ν equals 0 for all $\nu \in \mathbb{N}$.

As discussed before, only Case 2(b) can occur and therefore all limits of bubbling points lie on the boundary $\partial\mathbb{D}$. Each resulting bubbled disc comes with a contact area that is bounded from below, see [29, Lemma 35]. The contact area is additive and the sum of all these areas is bounded from above by the Hofer energy and hence by E. Therefore there are only finitely many limit points of bubbling points.

Using [17, Section 2.5], [29, pp. 542] and [37, Theorem 4.6.1] yields finitely many boundary points $\{z_0^1, \ldots, z_0^N\}$ of $\mathbb{D}$ such that for all $\varepsilon > 0$ the gradient $\|\nabla u_\nu\|_{g_0'}$ is uniformly bounded when restricted to

$$\mathbb{D} \setminus \bigcup_{j=1}^{N} B_\varepsilon(z_0^j).$$

Using a mean value argument as in Section 6.2 leads to bounds for the distance between the image of u_ν and the boundary condition $\{0\} \times \overline{K}_L$. Combining this with Theorem 6.2 we find a C^∞_{loc}-converging subsequence of u_ν on $\mathbb{D} \setminus \{z_0^1, \ldots, z_0^N\}$. But since all limit points of bubbling points are contained in the boundary, this convergence contradicts the divergence $a_\nu(0) \to -\infty$. This proves the proposition with the assumption of C^∞-bounds instead of C^3-bounds.

In the next step we argue why the argument above works with the weaker assumption of C^3-bounds. As in the C^∞-case we will rule out the Cases 1 and 2(a) and extend the limit obtained in Case 2(b) to a holomorphic disc. That is we have to repeat the bubbling off argument with weaker assumptions. To do this we use

Corollary 5.8 and Remark 5.10. By these statements we have

$$\alpha_\nu \longrightarrow \alpha_0, \quad J_\nu \longrightarrow J_0$$

in C^2_{loc} and that α_0 admits C^2-bounds. Applying the results to the limit form α_0 we have that

$$\alpha_0^\nu \longrightarrow \alpha_\infty, \quad J_0^\nu \longrightarrow J_\infty$$

in C^1_{loc}.

In Section 6.2 we make the following changes. In Case 1 and 2(a) we refer to [37, Remark B.4.3] instead of Theorem 6.2 to see that the rescaled sequence v_ν converges to v in C^2_{loc}. In Case 2(b) we do not change the almost complex structure and can still obtain C^∞_{loc}-convergence of v_ν.

In Section 6.3 we repeat the rescaling procedure with the C^2-bounded form α_0. Referring to [37, Remark B.4.3] we obtain C^1_{loc}-converging subsequences of u_ν. The arguments in [29, Theorem 31] hold for C^1-regular finite energy planes and we end up with a periodic C^1-Reeb orbit in the Cases 1 and 2(a). This yields a closed characteristic of (M, ω) which is excluded by assumption. In Section 6.4 the only change occures in the argument for the gradient bounds. Instead of C^∞_{loc}-convergence for a subsequence of u_ν, we have C^2_{loc}-convergence. The rest of the argument remains unchanged. □

CHAPTER 7

Contractible Closed Characteristics

In view of Chapter 6 especially Proposition 6.4 we look for conditions that contradict compactness for some moduli space of holomorphic curves on M'. Finding these conditions implies the existence of contractible closed characteristics in the discussed cases.

7.1. Germs of Holomorphic Discs

Since [29] holomorphic curves have been used to prove the existence of periodic Reeb orbits in compact contact manifolds, see [20,22,26]. We extend the techniques therein to our situation. That is we want to conclude the existence of closed Reeb orbits in the covering spaces of virtually contact structures. The existence of such orbits implies that the base manifold carries a closed characteristic. For the most of this section we will only explain the changes that are necessary to adjust the proofs.

Let $(\pi : M' \to M, \omega, \alpha, g)$ be a virtually contact structure whose base manifold M is closed and connected. We endow $\mathbb{R} \times M'$ with the almost complex structure associated to α and g' constructed in Section 3.1 and 4.1. As usual we denote the contact structure $\ker \alpha$ on M' by ξ.

Theorem 7.1. *If the C^3-norm of α is finite and if (M', ξ) is a 3-dimensional overtwisted contact manifold, then (M, ω) admits a contractible closed characteristic.*

See [19, Section 4.5] for the notation of overtwisted discs.

This statement extends a result in [29] from compact manifolds to virtually contact structures. The argument adapts the methods and arguments from [20].

Proof. After passing to the universal covering if necessary we can assume that $\pi : M' \to M$ is regular.

Let $D \subset (M', \xi)$ be an overtwisted disc such that the characteristic foliation

$$D_\xi := (TD \cap \xi|_D)^{\perp_{\omega'}} = TD \cap \xi|_D$$

has a unique singularity e in the interior of D and ∂D is the unique closed leaf of the foliation. By [29, Section 5.1 and Theorem 45] we can perturb a given overtwisted disc to obtain one that has the requested properties. We use $\perp_{\omega'}$ to indicate the symplectic orthogonal complement with respect to ω'. In our situation of a surface in a 3-dimensional manifold, taking the symplectic complement has no effect in regular points. Let $U \subset M'$ be a open relatively compact ball neighbourhood of e. We replace the almost complex structure J on the symplectisation $\mathbb{R} \times M'$ with an almost complex structure J_U that allows a local J_U-holomorphic Bishop disc family emerging from e, see [29, Section 4.2]. Further let J_U be translation invariant under the $\mathbb{R}$-action given by translation, send the vector field ∂_t spanning the $\mathbb{R}$-coordinate of $T(\mathbb{R} \times M')$ to the Reeb vector field R of α and restrict to a complex structure on (ξ, ω'). Note that $J_U|_\xi$ does not necessarily coincide with the complex structure $j = J|_\xi$ constructed in Chapter 3 and 4. Finally, we assume that J_U agrees with J on a neighbourhood of $\mathbb{R} \times (M' \setminus U)$. The existence of such an almost complex structure J_U is discussed in the begining of [29, Section 4.2]. This choice of J_U turns the punctured disc $D \setminus \{e\}$ into a totally real submanifold. Remember that a Bishop disc family is a continuous map $\Phi : [0, \varepsilon) \times \mathbb{D} \to \mathbb{R} \times M'$ with $\Phi(0, z) = e$ for $z \in \mathbb{D}$ and

$$\Phi(\tau, z) \in D \setminus (\{e\} \cup \partial D) \quad \text{for } \tau \in (0, \varepsilon), z \in \partial\mathbb{D}$$

such that $\Phi : (0, \varepsilon) \times \mathbb{D} \to \mathbb{R} \times M'$ is an embedding and

$$T\Phi(\tau, \cdot) \circ \mathrm{i} - (J_U)_{\Phi(\tau,\cdot)} \circ T\Phi(\tau, \cdot) = 0 \quad \text{for } \tau \in (0, \varepsilon).$$

See [29, Section 4.2] and [30, Section 3.1] for more informations on Bishop disc families and the inspiration for this choice.

We consider the moduli space $\mathcal{M}$ of all J_U-holomorphic discs with three marked points p_i geometrically fixed by three mutually distinct leaves l_i of D_ξ other than ∂D for $i = 0, 1, 2$, i.e., the marked point $p_i \in \partial\mathbb{D}$ is mapped to l_i for each map $u \in \mathcal{M}$. Moreover each $u \in \mathcal{M}$ is homologous relative $D \setminus \{e\}$ to one of the Bishop discs, see [27, p. 115] for an introduction to relative homology. Automatic transversality, positivity of intersection and the relative adjunction inequality as discussed in [20, Sections 7–9] imply that the evaluation map to one of the distinguished leaves is a local diffeomorphism

$$\begin{aligned} \mathrm{ev}_i : \mathcal{M} &\longrightarrow l_i \subset \mathbb{R} \times M' \\ u &\longmapsto u(p_i), \end{aligned}$$

where p_i are the marked points and the Bishop discs have unique preimages, see [20, Proposition 5.1]. The maximum principle by E. Hopf in its formulation for boundary values, see [14, Section 6.4], implies that all holomorphic discs with boundary on the totally real punctured disc $D \setminus \{e\}$ are transverse to D_ξ, i.e., $TD = D_\xi \oplus Tu(\partial\mathbb{D})$. Further we conclude that no holomrphic disc can touch the boundary ∂D. If we assume the moduli space to be compact then the evaluation maps ev_i are surjective to l_i, see [20, Proposition 5.1]. This is a contradiction since either a disc touches the boundary circle ∂D or the image is contained in the interior of the overtwisted disc.

But in the second case we would have that $l_i = \mathrm{im}(\mathrm{ev}_i)$ is compact which is not the case, since the leaves of the characteristic distribution are not compact.

In the next step we want to show that the moduli space above is compact if there are no closed characteristics. Using [29, Lemma 33] we achieve that all curves $\widetilde{u}$ in $\mathcal{M}$ satisfy

$$\int_{\mathbb{D}} \widetilde{u}^*(\mathrm{d}(\varphi^*\alpha)) \leq \frac{1}{2}\int_D \|\mathrm{d}\lambda\|_{g'}.$$

To achieve compactness we use Proposition 6.4 modified as follows. The modification is necessary since the results in Section 4.6, 5.5 and Chapter 6 are only valid for the almost complex structure J constructed in Section 4.1. In Proposition 4.7 we compare the distance to $D \cup U$ instead of $L = D$, this is necessary to assure that we can still apply Proposition 4.6 which only holds for the almost complex structure J. After these adjustment Corollary 4.8 still holds. We modify the bubbling off that we discussed in Section 6.2. If the distance $\mathrm{dist}\big(f_\nu(z_\nu), U\big)$ in the Cases 1 and 2(a) is boundedand thus we may have to work with J_U instead of J, we can pretend that M' is compact and argue as in [29, 30]. If $\mathrm{dist}\big(f_\nu(z_\nu), U\big)$ is unbounded we replace $R_\nu\varepsilon$ by R'_ν where

$$R'_\nu = \max\left\{R \leq R_\nu\varepsilon \mid f_\nu(B_R(0)) \subset M' \setminus U\right\}.$$

Then the images of the rescaled holomorphic discs do not intersect the region where we changed the almost complex structure and we can precede with the argument as before. In Case 2(b) we can also argue as in [29, 30]. In the bubbling off analysis of finite energy cylinders as in Section 6.3 we make similar changes. That is, if the distance between $h(z_\nu)$ and U is bounded we argue as in [29, 30]. If the distance is unbounded we change $R_\nu\varepsilon_\nu$ to

$$R'_\nu := \max\{R \leq R_\nu\varepsilon_\nu \mid f_\nu(B_R(0)) \subset M' \setminus U\}.$$

Again these changes are necessary because the previous discussions are only valid for J. After these modifications the proof of Proposition 6.4 goes through, that is we can argue with J_U instead of J. □

Theorem 7.2. *If the C^3-norm of α is finite and if M is a 3-dimensional manifold with non-trivial $\pi_2(M)$, then (M, ω) admits a contractible closed characteristic.*

In the proof we use results from [20].

Proof. First note that $\pi_2(M')$ is also non-trivial since $\pi_2(M') \cong \pi_2(M)$, see [27, Proposition 4.1]. In this situation the 3-dimensional sphere theorem [41] says that there exists a non-zero element in $\pi_2(M')$ represented by an embedding $\iota : S^2 \to M'$. We choose such a non-zero element and denote the image of the embedded sphere by $S := \iota(S^2)$. Note that S is not contractible in M'. The case that the contact structure is overtwisted is covered by 7.1, so we can assume that (M', ξ) is tight. We can further assume that the characteristic foliation $(TS\cap\xi)^{\perp_{\mathrm{d}\alpha}}$ of S has precisely two singular points e^+ and e^-, and both are elliptic, see [19, Section 4.6]. Additionally, there is no limit cycle of the characteristic foliation. We argue by a filling with holomorphic discs. To do so, we consider the Bishop disc families emerging from e^+ respectively e^-. Arguing as in [20, Section 5], compactness of the corresponding moduli spaces would imply that there exists a 3-ball in M' filling S. This is a contradiction to our assumption that S represents a non-trivial homotopy class. Therefore said moduli spaces can not be compact. By Proposition 6.4 this show the existence of a contractible closed characteristic. □

Definition 7.3. An embedded $(2n-2)$-sphere S in (M',ξ) is called **standard**, provided that the restriction of the contact form α to TS equals the restriction of $\frac{1}{2}(\mathbf{x}\mathrm{d}\mathbf{y}-\mathbf{y}\mathrm{d}\mathbf{x})$ to TS, where we identify S with the unit-sphere S^{2n-2} in $\mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$ equipped with coordinates $(w,\mathbf{x},\mathbf{y})$.

Theorem 7.4. *Let $n\geq 3$. If the C^3-norm of α is finite and if (M',ξ) is a $(2n-1)$-dimensional contact manifold that contains a standard sphere S whose class $[S]$ in $\pi_{2n-2}(M')$ is non-trivial, then (M,ω) admits a contractible closed characteristic.*

This theorem is based on [22].

Proof. We equip $\mathbb{R}\times S^{2n-2}$ with the standard contact structure ξ_{st}, that is the contact structure that one uses on $\mathbb{R}\times S^{2n-2}$ as the upper boundary of the index 1-handle, see [22, p. 329]. That is the contact structure induced by the contact form $\alpha_{\mathrm{st}}=w\mathrm{d}v+\frac{1}{2}(\mathbf{x}\mathrm{d}\mathbf{y}-\mathbf{y}\mathrm{d}\mathbf{x})$, where we use the coordinates $(v,w,\mathbf{x},\mathbf{y})$ on $\mathbb{R}\times S^{2n-2}\subset\mathbb{R}\times(\mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$. Consulting [13, Proposition 6.4] we find a contact embedding of $(-2,2)\times S^{2n-2}$ into (M',ξ) mapping $\{0\}\times S^{2n-2}$ onto S, i.e., there is an embedding $\iota:((-2,2)\times S^{2n-2},\xi_{\mathrm{st}})\to(M',\xi=\ker\alpha)$ with $T\iota(\xi_{\mathrm{st}})=\xi$ and $\iota(\{0\}\times S^{2n-2})=S$. We use this embedding to identify $(-2,2)\times S^{2n-2}$ with its image U and introduce a new contact form α_1 by requiring that it agrees with the given contact form α on a neighbourhood of $M'\setminus U$ and with the push forward of α_{st} on a neighbourhood of $[-1,1]\times S^{2n-2}$. On the remaining subset of $(-2,2)\times S^{2n-2}$ we consider a convex interpolation of the two contact forms. The interpolation is possible since both forms are positive contact forms defining the same contact structure. A reversed contact surgery shows that M' is the result of an index 1 surgery on some contact manifold (N,η). By reverse contact surgery we mean a surgery that removes the handle $(-2,2)\times S^{2n-2}$ and undoes an index 1 surgery [19, Section 6]. The complement of the surgery equals $M'\setminus U$ and η admits a defining contact form that coincides with α on $M'\setminus U$.

Now we are in the situation of [22, Chapter 2]. To perform the index 1 surgery on N we choose disjoint Darboux charts in the region where N differs from M'. In local coordinates on this charts the contact form has the form $\mathrm{d}w+\frac{1}{2}(\mathbf{x}\mathrm{d}\mathbf{y}-\mathbf{y}\mathrm{d}\mathbf{x})$. Since the contact structure induced by this form is invariant under the map $(w,\mathbf{x},\mathbf{y})\mapsto(\lambda^2w,\lambda\mathbf{x},\lambda\mathbf{y})$ we can assume that a ball of radius 1 is contained in each of the Darboux charts. We consider the map $(w,\mathbf{x},\mathbf{y})\mapsto(w,\mathbf{x},\mathbf{y})/R$ defined on the ball of radius R in $\mathbb{R}^{2n-1}$, the pull back of standard contact form with this map is $(R\mathrm{d}w + \frac{1}{2}(\mathbf{x}\mathrm{d}\mathbf{y} - \mathbf{y}\mathrm{d}\mathbf{x}))/R^2$. So we may think of the Darboux charts as two balls of large radius R with a contact structure induced by $\pm R\mathrm{d}w + \frac{1}{2}(\mathbf{x}\mathrm{d}\mathbf{y} - \mathbf{y}\mathrm{d}\mathbf{x})$. We choose R large enough such that [22, Lemma 6] holds. We perform the contact surgery on the Darboux charts using the thin handle described in [22, Section 2.1] and assume that the contact from obtained from surgery equals α_1. Here a thin handle is a cylinder with a small radius. This maybe requires a contactomorphism on U along which α is pulled back. Since we used balls of radius R as Darboux charts we can also perform a contact surgery with a thick handle, i.e., a cylinder with a large radius close to R. The resulting contact form is called α_R and coincides with α on a neighbourhood of $M' \setminus U$. Note that the rescaling of the Darboux chart and the requirement that α on these sets pulls back to $\pm R\mathrm{d}w + \frac{1}{2}(\mathbf{x}\mathrm{d}\mathbf{y} - \mathbf{y}\mathrm{d}\mathbf{x})$ leads to a multiplication of α with a large constant. This global change does not change the following argument, so we will ignore it for the rest of the proof.

As in [22, Section 2.2] we define W as the $2n$-dimensional manifold obtained by gluing the region between the thin and the thick handle to $(-\infty, 0] \times M'$ along $\{0\} \times U$. This gluing uses that both, the thin and the thick handle are contained in $\mathbb{R}^{2n}$. Similar to [22, Section 2.2] we consider the symplectic form $\mathrm{d}\lambda$ on W, where λ on the model region, i.e., the region between the thin and the thick handle in $\mathbb{R}^{2n}$, is the dual of the Liouville vector field Y defined on [22, p. 330] with respect to the standard symplectic form $\omega_0 = \mathrm{d}v \wedge \mathrm{d}w + \mathrm{d}\mathbf{x} \wedge \mathrm{d}\mathbf{y}$. The contact forms α and α_1 define the same contact structure on M' and they agree outside of U. Hence we can find a function h on M' with $\alpha = \mathrm{e}^h \alpha_1$ and h has compact support in U. At $(t, p) \in (-\infty, 0] \times M'$ we define λ as $\lambda = \mathrm{e}^{t+b(t)\cdot h}\alpha_1$ for a smooth function b on $\mathbb{R}$. We choose b to vanish on $[0, \infty)$, to equal 1 on $(-\infty, t_0]$ for a suitable $t_0 < 0$ and such that $b'(t) \cdot h > -1$ on M' for all t. This constructions interpolates between α and α_1 on the half-symplectisation $(-\infty, 0] \times M'$, i.e., the upper boundary looks like (M', α_1) and the negative end looks like the negative end of $((-\infty, 0] \times M', \mathrm{d}(\mathrm{e}^t\alpha))$. Moreover, the construction glues a cobordism that thickens the handle to the upper boundary (M', α_1).

We remove the neighbourhood $\mathrm{Int}\big((t_0, 0) \times U\big)$ of the support of $b \cdot h$ from W and obtain two connected components. On the unbounded component, λ equals $\mathrm{e}^t\alpha$. Indeed for $(t, p) \in [t_0, 0] \times M \setminus U$ we have

$$\begin{aligned}\lambda &= \mathrm{e}^{t+b(t)h(p)}\alpha_1 \\ &= \mathrm{e}^{t+b(t)\cdot 0}\alpha_1 \\ &= \mathrm{e}^t\alpha,\end{aligned}$$

since α and α_1 coincide outside of U For $(t, p) \in (-\infty, t_0] \times M'$ the form λ also amounts to

$$\begin{aligned}\lambda &= \mathrm{e}^{t+b(t)h(p)}\alpha_1\\ &= \mathrm{e}^{t+h(p)}\alpha_1\\ &= \mathrm{e}^{t}\alpha.\end{aligned}$$

On the unbounded component of $W\setminus\operatorname{Int}\big((t_0,0)\times U\big)$ we consider the almost complex structure J defined in Section 3.1 and 4.1. We extend J to a $\mathrm{d}\lambda$-compatible almost complex structure J_U on $(W,\mathrm{d}\lambda)$ that

- equals the standard complex structure on the model region, see [22, Section 2.3],
- is generic on $(t_0,0)\times U$ in the sense of [22, Section 3.5],
- is such that the boundary $\partial W \cong M'$ of W with $\lambda|_{T\partial W}=\alpha_R$, is J_U-convex in the sense of [37, Definition 9.2.5]

This leaves us in the situation of [22, Chapter 3] except that the manifold (M',ξ) is non-compact.

We will consider holomorphic discs with boundary on a family of Lagrangian submanifolds. These submanifolds are Lagrangian cylinders $L^{\mathbf{t}}\subset\mathbb{C}^n$ intersected with the region between the thin and the thick handle as described in [22, Section 3.1]. On the complex space $\mathbb{C}^n$ we use the coordinates $(v+\mathrm{i}w;\mathbf{h};z_n)$ with $\mathbf{h}\in\mathbb{C}^{n-2}$ and $z_n\in\mathbb{C}$. For any $\mathbf{t}\in\mathbb{R}^{n-2}$ we consider the cylinder

$$L^{\mathbf{t}} := (\{0\}\times\mathbb{R}_w)\times(\mathbb{R}^{n-2}_{\mathrm{Re}(\mathbf{h})}\times\{\mathrm{i}\mathbf{t}\})\times\partial D^2_{z_n}\subset\mathbb{C}\times\mathbb{C}^{n-2}\times\mathbb{C},$$

here $\mathbb{R}^{n-2}_{\mathrm{Re}(\mathbf{h})}\subset\mathbb{C}^{n-2}$ denote the real part of the complex coordinate $\mathbf{h}$. Note the these Lagrangians are filled by standard discs $u^{\mathbf{t}}_{w,\mathbf{s}}$ given by

$$\begin{aligned}\mathbb{D} &\longrightarrow \mathbb{C}^n\\ z &\longmapsto u^{\mathbf{t}}_{w,\mathbf{s}}(z)=(0,w;\mathbf{s}+\mathrm{i}\mathbf{t};z),\end{aligned}$$

with $w\in\mathbb{R}$ and $\mathbf{s}\in\mathbb{R}^{n-1}$.

We consider the moduli space $\mathcal{W}$ of holomorphic discs as in [22, Section 3.2], that is $\mathcal{W}$ consists of all holomorphic discs $u:\mathbb{D}\to W$ that are smooth up to the boundary and map the boundary circle to one of the Lagrangian cylinders, $u(\partial\mathbb{D})\subset L^{\mathbf{t}}\cap W$ for some $\mathbf{t}\in\mathbb{R}^{n-2}$. The intersection makes sense for the model region between the thin and the thick handle contained in W. Further we require that

- the relative homology class $[u]\in H_2(W,L^{\mathbf{t}}\cap W)$ equals that of some standard disc $u^{\mathbf{t}}_{\mathbf{s}}$ in W.
- the parametrisation of u is fixed by the requirement $u(\mathrm{i}^k)\in L^{\mathbf{t}}\cap\{z_{n-1}=\mathrm{i}^k\}$ for $k=0,1,2$.

Note the by our choice of the radius R each Lagrangian cylinder will contain the boundary of a standard disc.

The properties stated in [22, Chapter 3] transfer to the present situation with the following modification in the proof of compactness. We assume that the Reeb flow on the universal cover M' does not admit a closed orbit. We prove compactness for all J_U holomorphic discs belonging to $\mathcal{W}$ that have uniform bounded projections to M' as in [21, Chapter 6] and [22, Section 3.4]. We exclude the existence of a sequence of holomorphic discs with unbounded projection to M' as in the end of the proof to Theorem 7.1, i.e., by a modifications of Proposition 4.7 and the bubbling off analysis. After these modification we can use energy bounds obtained as in [22, Proposition 5] and the bounded $\mathbb{R}$-component obtained by a bubbling off analysis to conclude that all discs in $\mathcal{W}$ have bounded projection to M' and therefore $\mathcal{W}$ is compact. We see as in [22] that $\mathcal{W}$ is a $(2n-3)$-dimensional compact manifold with boundary.

If $\mathcal{W}$ is not connected, we consider the connected component of $\mathcal{W}$ with a boundary that consists of standard holomorphic discs that is still denoted $\mathcal{W}$. A deformation of the evaluation map as in [22, Chapter 4 and Section 5.1] yields a continuous map

$$f : (\mathcal{W} \times \mathbb{D}, \partial(\mathcal{W} \times \mathbb{D})) \longrightarrow (M', S)$$

that is transverse to the sphere S, restricts to a map of degree 1 on the boundary of $\mathcal{W}\times\mathbb{D}$ and maps $\mathcal{W}\times\{1\}$ to a $(2n-3)$-dimensional cell in S. Therefore the homology class defined by S in M' vanishes. Indeed, a triangulation of $\mathcal{W} \times \mathbb{D}$ yields a cell structure whose boundary is contained in S and since the map f has degree 1 it is surjective on the boundary. Therefore we have found a cell complex with boundary S. Since S is homologically trivial it separates M'. Indeed, if it would not separate M' then $M'\backslash S$ would be connected and therefore path connected. For a base point in S we consider the path γ in the closure of $M'\backslash S$ that connects this base point in one boundary component with its copy in the other boundary component of $M' \setminus S$. We consider the homological intersection number of S and the path γ. On the one hand the intersection number of S and the path corresponding to γ in M' is 1, because they have exactly one intersection point. On the other hand this intersection number is a bilinear form and S is homologically trivial, so the intersection number vanishes. This contradiction shows that S has to be separating. We denote by M_1 and M_2 the closures of the connected components of $M'\backslash S$. Since the virtually contact structure is assumed to be non-trivial we know that M' is not compact, so the same must be true for at least one of the components M_i. We denote by $V_i \subset \mathcal{W} \times \mathbb{D}$ the preimage of M_i with respect to the deformed evaluation f. As argued in [22, Section 5.2] the mapping degree of f is well-defined, when restricted to V_i. Counting the number of preimages of generic points near S of different components of $M \setminus S$ with signs yields to

$$\deg f_1 - \deg f_2 = \pm 1$$

as in [22, Lemma 8], where $f_i = f|_{V_i}$. We can conclude that one component of $M \setminus S$, say M_1, is compact and that $\deg f_1 = \pm 1$. Using [22, Proposition 11] and the fact that $\mathcal{W} \times \{1\}$ is mapped to a cell by construction, we conclude that M_1 is simply connected. In fact, we choose a base point of M_1 on its boundary S and consider a loop in M_1 that starts and ends in this base point. Since the map f_1 has degree ± 1, this loop lifts to a loop in $\mathcal{W} \times \mathbb{D}$. Contracting the disc $\mathbb{D}$ to the point $\{1\}$ gives a homotopy of the lift to a loop in $\mathcal{W} \times \{1\}$. This deformation yields a deformation of the loop in M_1 whose result is a loop that is contained in the image of $\mathcal{W} \times \mathbb{D}$ with respect to f. As noted before, the image of $\mathcal{W} \times \{1\}$ with respect to f is a cell and hence it is contractible. Therefore, the considered loop is homotopic to a point. Thus M_1 is simply connected. An application of [22, Proposition 12] shows that M_1 has the homology of a ball. Moreover, M_1 is bounded by a $(2n-2)$-sphere for $n \geq 3$. For $n \geq 4$ we can use the h-cobordism theorem and conclude that M_1 is a ball, see [38, Proposition A on p. 108]. The case $n = 3$ is covered by [38, Proposition C on p. 110]. Therefore the sphere S is contractible in M' which contradicts the assumption that the class $[S]$ in $\pi_{2n-2}(M')$ defined by S is non-trivial. Hence the α-Reeb flow must admit a closed orbit. □

Remark 7.5. Let $(\pi : M' \to M, \omega, \alpha, g)$ be a virtually contact structure whose base manifold (M, ω) is closed and connected. Assume that (M, ω) is the connected sum of two odd-symplectic manifolds admitting virtually contact structures and that $(\pi : M' \to M, \omega, \alpha, g)$ is obtained by a covering connected sum as defined in [48, Section 2.2] and recalled in Construction 8.1, up to the rescaling of α with a positive function that equals 1 outside the attached handles. This rescaling gets rid of periodic orbits that where contained in the handles. We assume that the C^3-norm of α is bounded. For example this is given if the virtually contact forms of both summands are C^3-finite, see Remark 7.12. If none of the summands is a homotopy sphere, the base manifold (M, ω) admits a contractible closed characteristic. The connected sum of manifolds different from homotopy spheres is called **non-trivial**.

If M is 3-dimensional we can apply Theorem 7.2, since the belt sphere represents a non-trivial element in the homotopy group $\pi_2(M)$ because otherwise one of the summands would be a 3-dimensional sphere, see [22, Theorem 1]. For the higher dimensional case we are in the situation described in the beginning of the proof of Theorem 7.4. We assume that the α-Reeb flow does not admit a closed orbit. We argue as in that proof and see that the lift of the belt sphere bounds a ball B of the same dimension as M. This ball may contain several copies of the belt sphere. If this is the case we argue as in [38, Proposition 3.10] and consider an innermost belt sphere, i.e., a copy S of the belt sphere in M' such that the interior connected component of $B \backslash S$ does not contain another copies of the belt sphere. The existence of an innermost sphere is guaranteed by the compactness of B. Repeating the topological argument in the proof of Theorem 7.4 shows that S bounds a ball. We restrict the covering map to this ball and obtain a covering map again. This covering is one-to-one on the boundary and therefore the same is true in the interior. Therefore, the belt sphere in M also bounds a ball and the sum is trivial.

Definition 7.6. The **standard contact handle of index** k is the upper boundary $D^k \times S^{2n-1-k}$ of a symplectic handle $D^k \times D^{2n-k}$.

The sphere $\{0\} \times S^{2n-1-k}$ and its image are referred to as the **belt sphere** of the handle.

The index k is **subcritical** if $k \leq n-1$.

The following theorem and the remarks thereafter are inspired by [26].

Theorem 7.7. *Let $n \geq 3$. If the C^3-norm of α is finite and if (M', ξ) is a $(2n-1)$-dimensional contact manifold that admits an contact embedding of the standard contact handle of subcritical index k with belt sphere S whose class $[S]$ in the oriented bordism group $\Omega^{SO}_{2n-1-k}(M')$ is non-trivial, then (M, ω) admits a contractible closed characteristic.*

Proof. In the given situation we can begin to argue as in Theorem 7.4 and get the description of (M', ξ) as the result of a surgery. We add a thickened handle and consider the modified symplectisation. This allows us to modify the handle as in [26, Section 3]. Note that this modification has compact support. We assume that the α-Reeb flow on M' does not admit a closed Reeb orbit and argue as in [26, Section 7]. To be a little bit more precise, we consider the moduli space of holomorphic discs with boundary on a family of Legendrian open books. A deformation of the evaluation map on the surgered moduli space of holomorphic discs as in [26, Section 6] leads to a null-bordism of S which contradicts the assumptions of this theorem. Note that we have to adapt the analysis to a non-compact manifold M', we do this as in Chapter 6 and Theorem 7.1. □

Remark 7.8. Assume the situation described in Theorem 7.7 except that the class $[S]$ of the belts sphere S^{2n-1-k} is non-trivial in $\pi_3(M')$ if $n = 3$ and $k = 2$ or in $\pi_4(M')$ if $n = 4$ and $k = 3$. Then (M, ω) carries a contractible closed orbit. For this statement we can argue as in [26, Section 5] and modify it as indicated in the proof of Theorem 7.7.

Remark 7.9. Theorem 7.7 uses moduli spaces of holomorphic discs with boundary on a family of Legendrian open books in (M', ξ). Replacing the boundary condition by Legendrian open books with boundary, the analysis for holomorphic curves in the symplectisation of M' is modifed as in Chapter 6. Similar to [1,26,35,39] modified as in the proof of Theorem 7.1 we obtain: If $(M'\xi)$ admits a Legendrian open book with boundary and the C^3-norm of α is finite, then (M, ω) admits a contractible closed characteristic. See [48, Proposition 2.6.1 and Proposition 2.6.2] for constructions that yield examples.

7.2. Magnetic Energy Surfaces

The following discussion is motivated by the examples in [12] that were expanded and studied in [48], by Zehmisch in collaboration with the author.

Let (Q, h) be a closed oriented n-dimensional Riemannian manifold and $\tau : T^*Q \to Q$ its cotangent bundle. Denote by D^Q the Levi-Civita connection of h and by $h^\flat$ the dual metric of h. As explained in Appendix A we split the tangent space $TT^*Q = \mathcal{H} \oplus \mathcal{V}$ of T^*Q into a horizontal distribution $\mathcal{H}$ and a vertical distribution $\mathcal{V}$. This splitting yields a metric m on T^*Q that is defined as

$$m\big((v, a), (w, b)\big) = h\big(T\tau(v), T\tau(w)\big) + h^\flat(a, b)$$

for $(v, a), (w, b) \in \mathcal{H} \oplus \mathcal{V}$. Denote the Levi–Civita connection of m by D. This metric turns τ into a Riemannian submersion as defined in [6, Chapter 9.B]. Moreover the fibres of τ are totally geodesic with respect to m.

The **twisted symplectic form** ω_σ on T^*Q is $\omega_\sigma = \mathrm{d}\lambda + \tau^*\sigma$, where λ denotes the Liouville 1-form of τ and σ is a magnetic, i.e., closed, 2-form on Q. Consider the Hamiltonian function

$$H = \tfrac{1}{2}\|\cdot\|^2_{h^\flat} + V \circ \tau \;:\; T^*Q \longrightarrow \mathbb{R}$$

where $V : Q \to \mathbb{R}$ is a so-called **potential function** on Q. Pick an energy $e > \max_Q(V)$ and regard the energy surface $M := \{H = e\} \subset T^*Q$. It turns out to be a regular level set. We restrict the geometry of T^*Q to M and obtain an odd-symplectic form $\omega := \omega_\sigma|_{TM}$ and a metric $g := m|_{TM}$. Moreover, let ∇ be the Levi-Civita connection on (M, g).

Continuing our discussion from [48] we have the following situation that we will explain after the diagram.

$$\begin{array}{ccccc}
(M', \omega', \widetilde{\lambda} + \widetilde{\tau}^*\vartheta|_{TM'}, \widetilde{g}) & \xhookrightarrow{\{\widetilde{H}=e\}} & (T^*\widetilde{Q}, \widetilde{\omega}_\sigma, \widetilde{\lambda} + \widetilde{\tau}^*\vartheta, \widetilde{m}, \widetilde{H}) & \xrightarrow{\widetilde{\tau}} & (\widetilde{Q}, \widetilde{\sigma}, \vartheta, \widetilde{h}) \\
\downarrow \pi & & \downarrow T^*\mu & & \downarrow \mu \\
(M, \omega, g) & \xhookrightarrow{\{H=e\}} & (T^*Q, \omega_\sigma, m, H) & \xrightarrow{\tau} & (Q, \sigma, h)
\end{array}$$

We assume that there exists a 1-form ϑ on the universal cover $\mu : \widetilde{Q} \to Q$ such that $\mu^*\sigma = \mathrm{d}\vartheta$. Choosing the metric $\widetilde{h} = \mu^*h$ on $\widetilde{Q}$ turns μ into a local isometry. Denote the induced Levi-Civita connection of $\widetilde{h}$ on $\widetilde{Q}$ by $\widetilde{D}^Q$. The universal cover $\mu : \widetilde{Q} \to Q$ induces the universal cover $T^*\mu : T^*\widetilde{Q} \to T^*Q$. We lift the Hamiltonian function H along $T^*\mu$ to a Hamiltonian function $\widetilde{H}$ on $T^*\widetilde{Q}$. Restricting $T^*\mu$ to the hypersurface $M' := \{\widetilde{H} = e\}$ gives a cover $\pi := T^*\mu|_{M'}$. Lifting the geometry (T^*Q, m, D) to $(T^*\widetilde{Q}, \widetilde{m}, \widetilde{D})$ turns $T^*\mu$ into a local isometry. We denote the restriction of $\widetilde{m}$ to M' by $g' := \widetilde{m}|_{TM'}$. It turns out that g' coincides with the lift of the metric g on M and hence π is a local isometry, too.

The 1-form $\alpha := (\widetilde{\lambda} + \widetilde{\tau}^*\vartheta)|_{TM'}$ is a primitive for the lifted twisted odd-symplectic form $\omega' = \pi^*(\omega|_{TM})$, where $\widetilde{\lambda}$ is the Liouville 1-form of $\widetilde{\tau} : T^*\widetilde{Q} \to \widetilde{Q}$. By [48, Proposition 2.4.1] this defines a virtually contact structure $(\pi : M' \to M, \alpha, \omega, g)$ for all $e > \sup_{\widetilde{Q}} \widetilde{H}(\vartheta)$, provided $\|\vartheta\|_{C^0}$ is finite.

Proposition 7.10. *Let $k \in \mathbb{N}$ be a natural number. Assume that the C^k-norm of ϑ with respect to $(\widetilde{Q}, \widetilde{h}, \widetilde{D}^Q)$ is finite. Then the C^k-norm of α with respect to (M', g', ∇') is also finite.*

The proof is technical and was published in [5, Proposition 6.2.1].

7.3. Truncating the Magnetic Field

We continue the discussion from Section 7.2 and keep the notation. Let us recap the truncation we explained in [48, Proposition 2.5.1] before we look at its effect on C^k-bounds. Consider a closed disc U embedded into Q and denote the image of the origin by q. We assume that the preimage of U under μ decomposes into closed sets $U^p, p \in \mu^{-1}(q)$ diffeomorphic to U where the diffeomorphism is $\mu|_{U^p}$. This can be achieved by choosing U sufficiently small. In fact, the diffeomorphisms are isometries by our choice of metrics. We further consider a cut off function χ on Q that is identically 1 on $Q \setminus U$ and vanishes on an open disc neighbourhood W of q whose closure is contained in the interior of U. We define a perturbed magnetic form $\widehat{\sigma}$ on Q by demanding that it agrees with σ on $Q \setminus U$ and equals $\mathrm{d}(\chi\vartheta_U)$ on U, where ϑ_U is a primitive of $\sigma|_U$ obtained by the Poincaré lemma [34, Corollary 17.15]. Note that $\widehat{\sigma}$ vanishes on W and that $\widehat{\sigma}$ and σ are cohomologous. Indeed $\widehat{\sigma} - \sigma = \mathrm{d}\big((\chi - 1)\vartheta_U\big)$. We define a primitive ϑ' of $\mu^*\widehat{\sigma}$ on $\widetilde{Q}$ by adding a lift of $(\chi - 1)\vartheta_U$ to the primitive ϑ of $\mu^*\sigma$. This results in the 1-form

$$\vartheta' := \vartheta + (\widetilde{\chi} - 1)\mu^*\vartheta_U.$$

Note that $\vartheta'|_{W^p}$ is closed for all $p \in \mu^{-1}(q)$ and hence exact, where W^p denotes the connected component of $\mu^{-1}(W)$ containing $p \in \mu^{-1}(q)$. As in the proof of [48, Proposition 2.5.1] we select a primitive f^p of $\vartheta'|_{W^p} = \mathrm{d}f^p$ via the Poincaré lemma. Let χ_W be a cut off function that is equals 1 on $Q \setminus W$ and vanishes near q, i.e., on a disc neighbourhood of q whose closure is contained in W. We define a primitive $\widehat{\vartheta}$ of $\mu^*\widehat{\sigma}$ that also vanishes near $\mu^{-1}(q)$ by setting $\widehat{\vartheta} = \vartheta'$ on $\widetilde{Q} \setminus \mu^{-1}(W)$ and $\widehat{\vartheta} = \mathrm{d}(\widetilde{\chi}_W f^p)$ on W^p, here $\widetilde{\chi}_W$ is the lift of χ_W to $\widetilde{Q}$ with respect to μ. The pair $(\widehat{\vartheta}, \widehat{\sigma})$ is called a **truncation** of (ϑ, σ).

Lemma 7.11. *Let $k \in \mathbb{N}$ be a natural number. If the C^k-norm $\|\vartheta\|_{C^k}$ of ϑ with respect to $(\widetilde{Q}, \widetilde{h}, \widetilde{D}^Q)$ is finite, then the same is true for the C^k-norm $\|\widehat{\vartheta}\|_{C^k}$ of $\widehat{\vartheta}$ with respect to $(\widetilde{Q}, \widetilde{h}, \widetilde{D}^Q)$.*

Proof. First note that ϑ' is C^k-finite, since $\vartheta' = \vartheta + \mu^*\big((\chi - 1)\vartheta_U\big)$ where ϑ is C^k-finite by assumption and $\mu^*\big((\chi - 1)\vartheta_U\big)$ is the lift of a 1-form with compact support by a local isometry. The C^0-norm of $\widehat{\vartheta}$ is finite as we showed in [48, Proposition 2.5.1]. Therefore, for the l-th covariant derivative of $\widehat{\vartheta}$ it suffices to consider

$\mathrm{d}(\widetilde{\chi}_W f^p) = \vartheta|_{W^p}$ for $p \in \mu^{-1}(q)$. By Remark 5.1 the l-th covariant derivative of $\widehat{\vartheta}$ is given by the $(l+1)$-th covariant derivative of $\widetilde{\chi}_W f^p$. Applying the Leibniz rule we obtain

$$(\widetilde{D}^Q)^{l+1}(\widetilde{\chi}_W f^p) = \sum_{j=0}^{l+1} (\widetilde{D}^Q)^j \widetilde{\chi}_W \otimes (\widetilde{D}^Q)^{l+1-j} f^p.$$

Here $\otimes$ denotes the product of two tensors and for a k-tensor A, an l-tensor B and test vector fields $X_1, \dots, X_{k+l}$ it is defined as

$$A \otimes B\,(X_1, \dots, X_{k+l}) := \sum_\tau A\left(X_{\tau(1)}, \dots, X_{\tau(k)}\right) B\left(X_{\tau(k+1)}, \dots, X_{\tau(k+l)}\right)$$

where the sum is taken over all permutations $\tau : \{1, \dots, k+l\} \to \{1, \dots, k+l\}$ with

$$\tau(1) < \tau(2) < \dots < \tau(k)$$

and

$$\tau(k+1) < \tau(k+2) < \dots < \tau(k+l).$$

According to Remark 5.4 all covariant derivatives of $\widetilde{\chi}_W$ are finite since this is the case for χ_W. Further it we showed in [48, Proposition 2.5.1] that the C^0-norm of f^p is finite. For all higher derivatives we refer to Remark 5.1 again to conclude

$$(\widetilde{D}^Q)^{l+1-j} f^p = (\widetilde{D}^Q)^{l-j} \vartheta'|_{W^p}.$$

As we noted at the beginning of the proof the C^k-norm of ϑ' is finite. □

Remark 7.12. If $\|\vartheta\|_{C^k}$ is finite we combine [48, Proposition 2.5.1] with Lemma 7.11 to obtain a virtually contact structure $(\pi : M' \to M, \widehat{\omega}, \widehat{\alpha}, g)$ with a C^k-finite contact form $\widehat{\alpha}$ that is somewhere contact in the sense of Definition 2.15. Given two C^k-finite somewhere contact virtually contact structures, performing a covering connected sum in the sense of [48, Section 2.5] and Construction 8.1 results in a virtually contact structure supported by the connected sum that is also C^k-finite.

7.4. Classical Hamiltonians and Magnetic Fields

We consider the setup introduced in Section 7.2 and assume additionally that Q is the product of closed hyperbolic surfaces. We consider an $\mathbb{R}$-linear combination of the area forms corresponding to the factors. The lift of this 2-form to the universal cover $\mu : \widetilde{Q} \to Q$ has a primitive ϑ that is the corresponding linear combination of $\frac{1}{y}\mathrm{d}x$ for $(x, y) \in \mathbb{H}^+$, this primitive is C^k-finite for all $k \in \mathbb{N}$ as we discussed in Section 5.3. We choose σ to be a 2-form whose cohomology class lies in the span of the area forms of the factors. Indeed we can lift the primitive of the difference between the sum of the area forms and σ. Adding it to the sum of the primitives

$\frac{1}{y}\mathrm{d}x$ we obtain a primitive for the lift of σ. Applying the results of Section 7.2 we obtain a rich class of examples of virtually contact type energy surfaces in classical mechanics with magnetic fields.

We describe a particular class of Hamiltonian systems (Q, h), where σ and ϑ are chosen as above and the Hamiltonian function equals $\frac{1}{2}\|\cdot\|^2_{h^\flat} + V \circ \tau$. Further we assume that the potential V is a Morse function on Q with the following properties:

1. V has a unique local maximum, which we assume to be positive.
2. All critical values of V corresponding to critical points of index $n-1$ are strictly smaller than $-\frac{1}{2}t_0^2$, where $t_0 := \max_{\widetilde{Q}} \|\vartheta\|_{(\widetilde{h})^\flat}$.
3. Let c_{n-1} be the largest critical value of V distinguished from the maximum of V. We require there exists a regular value $-v_0 < 0$ with $-v_0 \in \left(c_{n-1}, -\frac{1}{2}t_0^2\right)$ such that σ and ϑ vanish on the disc $\{V \geq -v_0\} \subset Q$ and $\{\widetilde{V} \geq -v_0\} \subset \widetilde{Q}$, respectively, where $\widetilde{V}$ denotes the lift of V to $\widetilde{Q}$ along μ.

Note that the truncation construction in Section 7.3 allows us to achieve the third condition for a given Morse function V. The second condition can be obtained by composing V with a strictly increasing function.

Theorem 7.13. *Let Q be a closed hyperbolic surface. Let V be a Morse function on Q that has a unique local maximum which is required to be positive. Let $v_0 > 0$ be a positive real number such that $\{V \geq -v_0\}$ contains no critical value of V other than the maximum. Let σ be a 2-form on Q that vanishes on the disc $\{V \geq -v_0\}$. If*

$$v_0 > \inf_{\vartheta} \sup_{\widetilde{Q}} \tfrac{1}{2}\|\vartheta\|^2,$$

where the infimum is taken over all C^3-bounded primitive 1-forms ϑ of the lift of σ to the universal cover $\widetilde{Q}$ that vanish on $\{\widetilde{V} \geq -v_0\}$ for the lifted potential $\widetilde{V}$, then the equations of motion of a charged particle on Q under the influence of the magnetic field σ and the presence of the potential V have a non-constant periodic solution of energy $H = 0$ that is contractible in $\{V \leq 0\}$.

Proof. We assume the situation described in the proceeding section for $n = 2$. By [48, Section 3.2] the second homotopy class $\pi_2(M)$ of $M = \{H = 0\}$ is non-trivial. Therefore we are in the situation of Theorem 7.2 and obtain a closed contractible characteristic of X_H on M whose projection to Q via τ is contractible in $\{V \leq 0\}$ since $\tau(M) = \{V \leq 0\}$ and the characteristic is already contractible in M. We have to make sure that the solution is not constant after application of the projection. We take a look at the equation of motion

$$\dot{\gamma} = X_H$$

where X_H is the Haimitonian vector field defined by

$$\iota_{X_H}(\mathrm{d}\lambda + \tau^*\sigma) = \iota_{X_H}\omega_\sigma = -\mathrm{d}H = -\mathrm{d}\left(\tfrac{1}{2}\|\cdot\|^2_{h^\flat} + V\circ\tau\right).$$

The Hamiltonian vector field spans the kernel of $\omega_\sigma|_{T\{H=0\}}$, so any closed characteristic satisfies the equation of motion. A computation in local coordinates shows

$$X_H = (X_H)_{\mathbf{q}}\partial_{\mathbf{q}} + (X_H)_{\mathbf{p}}\partial_{\mathbf{p}} = \mathbf{p}\partial_{\mathbf{q}} + (X_H)_{\mathbf{p}}\partial_{\mathbf{p}}.$$

Therefore the projection of a closed characteristic can only be constant if the fibre coordinate of the initial solution vanishes. By the choice of our hypersurface, this is only the case if $0 = H(\gamma) = V\circ\tau(\gamma)$. By construction of $M = \{H = 0\}$ the projection τ is trivial at $\{V = 0\}$, so the initial characteristic has to be constant, but this contradicts the fact that X_H is nowhere vanishing. □

Example 7.14. We remark that the solution we obtained for the magnetic system does in general not stay in the zero set of the magnetic form. In the following we will discuss the phenomenon in an example. Let Q be a surface and V a Morse function on Q satisfying the requirements above. By [Hirsch, p. 157] we can integrate the gradient flow lines of V to obtain a diffeomorphism $[-v_0, 0]\times S^1 \cong \{-v_0 \le V \le 0\}$ such that the metric tensor on $[-v_0, 0]\times S^1$ is diagonal with respect to the natural (r,θ)-coordinates and the level sets of V are mapped to the circles of constant r-coordinate, i.e., $\{V(r,\theta) = r\}$. In addition we assume that $h_{11} = 1$, perhaps after applying a conformal change, that is we consider the metric $f\cdot h$ with $f|_{-v_0\le V\le 0} = \|\mathrm{grad}V\|^2_h$. Now we choose a magnetic field σ on Q according to the requirements of Theorem 7.13.

Let $\widehat{\gamma}$ be a non-constant periodic solution of the Hamiltonian system with zero energy that is contractible in $\{V \le 0\}$. We claim that the trace of $\widehat{\gamma}$ has to leave the set $\{-v_0 \le V \le 0\}$. We argue by contradiction. If the trace of $\widehat{\gamma}$ is contained in said set we can use the gradient flow to homotope it into the upper boundary $\{V = 0\}$ which is diffeomorphic to a circle. Therefore we can consider the mapping degree of $\widehat{\gamma}$. But since Q is not the 2-sphere no multiple of the boundary circle $\{V = 0\}$ is contractible in $V \le 0$, so the mapping degree of $\widehat{\gamma}$ has to vanish. Therefore $\widehat{\gamma}$ is contractible in $\{-v_0 \le V \le 0\}$ and it lifts to a closed curve $\gamma(t) = (r(t), \theta(t))$ on the universal cover $[-v_0, 0]\times\mathbb{R}$ with (r,θ)-coordinates, where the lifted metric h is diagonal and $h_{11} = 1$. Further the gradient $\mathrm{grad}(V)$ of the lifted potential equals ∂_r. The θ-coordinate of the curve attains an extremum, say at t_0. For this critical point we have $(\dot r(t_0), \dot\theta(t_0)) = \left(\pm\sqrt{-2r(t_0)}, 0\right)$, since

$$(\dot r(t), \dot\theta(t)) = \dot\gamma(t) = T\tau X_H = \mathbf{p}\partial_{\mathbf{q}},$$

where $\mathbf{p}$ denotes the fibre coordinates of the closed characteristic $\widetilde{\gamma}$ that projects to γ. The closed characteristic satisfies

$$\begin{aligned} 0 = H(\widetilde{\gamma}(t)) &= \tfrac{1}{2}\|\mathbf{p}\|^2 + V \circ \tau(\widetilde{\gamma}) \\ &= \tfrac{1}{2}\big(p_1(t)^2 + p_2(t)^2\big) + r(t). \end{aligned}$$

At the critical point t_0 of θ we have $0 = \dot{\theta}(t_0) = p_2(t_0)$ and therefore

$$\dot{r}(t_0) = p_1(t_0) = \sqrt{-2r(t_0)}.$$

Note that the Christoffel symbols $\Gamma^1_{11}, \Gamma^2_{11}$ vanish since the metric is diagonal. We consider the curve $\beta(t) = (b(t), \theta(t_0))$ with a quadratic polynomial $b(t)$ of the form $-\frac{1}{2}t^2 + a_1 t + a_0$ with $b(t_0) = r(t_0)$ and $\dot{b}(t_0) = \sqrt{-2r(t_0)}$, this curve β satisfies the equation $D^Q_{\dot{\beta}}\dot{\beta} = (-1, 0)$ and has energy $H(\beta, \dot{\beta}) = 0$. The considered solution γ satisfies the same differential equation with the same initial values and therefore the curves agree where they are defined. Therefore γ either connects the lower boundary $\{V = -v_0\}$ with the upper boundary $\{V = 0\}$ along a gradient flow line of V or the r-coordinate attempts a minimum. In the first case the velocity of γ has to vanish at the lower boundary since the trace of γ is contained in the cylinder $\{-v_0 \leq V \leq 0\}$ by our assumption. In either case we obtain a point t_1 with $\dot{\gamma}(t_1) = 0$ and $V(\gamma(t_1)) \neq 0$. This is a contradiction to the equation of motion, see the proof of Theorem 7.13.

Theorem 7.15. *Let Q be a product of closed hyperbolic surfaces. Assume that Q admits a potential function V together with a choice of regular value $-v_0 < 0$ and a closed magnetic 2-form σ satisfying the conditions described in Theorem 7.13. In addition, assume that σ is cohomologous to a $\mathbb{R}$-linear combination of the area forms corresponding to the factors of Q. Then the magnetic flow of the Hamiltonian H carries a non-constant periodic solution of zero energy that is contractible in $\{V \leq 0\}$.*

Proof. The statement for $n = \dim Q = 2$ is covered by Theorem 7.13 so we can assume $n \geq 3$. The idea is to find a contact embedding of the standard $(n-1)$-handle $D^{n-1} \times D^{n+1}$ into $(M', \ker\alpha)$ such that the belt sphere represents a non-trivial homology class of degree n in M', where $M = \{H = 0\}$ is supports a virtually contact structure. In the following we will verify the contact type property for M', explain how to find such an embedding and how to derive the existence of a closed characteristic.

We begin by checking the contact type property for M'. Let X be the gradient vector field of τ^*V with respect to the metric m in T^*Q and denote by F the function

$$\begin{aligned} F : T^*Q &\longrightarrow \mathbb{R} \\ u &\longmapsto \lambda_u(X). \end{aligned}$$

Observe that $F(u) = u \circ T\tau X = h^\flat(u, \mathrm{d}V)$ for all covectors u on Q, since $T\tau X =$

$\operatorname{grad}_h V$. We lift F to a Function $\widetilde{F}$ on $T^*\widetilde{Q}$ and for $\varepsilon > 0$ and $\widetilde{u} \in T^*\widetilde{Q}$ we consider

$$(\widetilde{\lambda} + \widetilde{\tau}^*\vartheta - \varepsilon \mathrm{d}\widetilde{F})_{\tilde{u}}(X_{\widetilde{H}})\tilde{u}$$

where $X_{\widetilde{H}}$ is the Hamiltonian vector field of the system $(\widetilde{\omega}_{\mathrm{d}\vartheta}, \widetilde{H})$ defined by $\mathrm{d}\widetilde{H} = -\widetilde{\omega}_{\mathrm{d}\vartheta}(X_{\widetilde{H}}, \cdot)$. By the considerations in Appendix B this equals the sum of

$$\|\widetilde{u}\|^2_{(\tilde{h})^\flat} + (\widetilde{h})^\flat(\widetilde{u}, \vartheta)$$

and

$$\varepsilon \cdot \left((-\mathrm{Hess}_{\tilde{h}}\widetilde{V})(\widetilde{u}^\#, \widetilde{u}^\#) + \|\operatorname{grad}_{\tilde{h}} \widetilde{V}\|^2_{\tilde{h}} + (\tau^*\mu^*\sigma)(\widetilde{u}^\#, \operatorname{grad}_{\tilde{h}} \widetilde{V})\right)$$

where $\widetilde{u}^\# \in T\widetilde{Q}$ is dual vector of $\tilde{u}$ with respect to $\widetilde{h}$ and $\mathrm{Hess} = \widetilde{D}^Q \mathrm{d}\widetilde{V}$ denotes the Hessian of $\widetilde{V}$.

By our assumptions in the theorem we can assume that we are in the situation discussed in the beginning of Section 7.4. Distinguishing the cases $\frac{1}{2}\|\widetilde{u}\|^2_{(\tilde{h})^\flat} \geq v_0$ and $\frac{1}{2}\|\widetilde{u}\|^2_{(\tilde{h})^\flat} < v_0$ we conclude that the sum above is uniformly positive along

$$\left\{\tfrac{1}{2}\|u\|^2_{(\tilde{h})^\flat} = -\widetilde{V}(\widetilde{\tau}(\widetilde{u}))\right\} = \{\widetilde{H} = 0\}$$

for some $\varepsilon > 0$, as explained in Appendix B. Therefore M' is of contact type. Indeed, we can define a Liouville vector field Y by $\iota_Y \widetilde{\omega}_{\mathrm{d}\vartheta} = \widetilde{\lambda} + \tau^*\vartheta - \varepsilon \mathrm{d}\widetilde{F}$ and obtain

$$\mathrm{d}\widetilde{H}(Y) = \widetilde{\omega}_{\mathrm{d}\vartheta}(Y, X_{\tilde{H}}) = (\widetilde{\lambda} + \tau^*\vartheta - \varepsilon \mathrm{d}\widetilde{F})(X_{\tilde{H}})$$

which is uniformly positive along $\{\widetilde{H} = 0\}$, so Y is transversal to M' and by [19, Lemma/Definition 1.4.5] it is of contact type.

Moreover, considering the family $t\vartheta$ of 1-forms for $t \in [0, 1]$ that corresponds to the family $t\sigma$ of magnetic forms, we obtain a family of contact forms

$$\alpha_t = (\widetilde{\lambda} + t\widetilde{\tau}^*\vartheta - \varepsilon \mathrm{d}\widetilde{F})|_{TM'}$$

on M' that connects α_0 with $\alpha = \alpha_1$. See Appendix B for the reasoning of this statement. Note that α_0 descents to the contact form $\alpha_W := (\lambda - \varepsilon \mathrm{d}F)|_{TM}$ on M. We would like to apply the Gray stability argument as in [19, Theorem 2.2.2] to show that the corresponding contact structures $\xi_t = \ker \alpha_t$ are contactomorphic, but as M' is non-compact we have to ensure that the flow obtained by the Moser trick exists for a sufficiently long time. Indeed, the concerned time-dependent vector field on M' is bounded with respect to the complete metric g' because α_t is C^1-bounded, see Sections 7.2 and 7.3, and we can use local computations as in Section 5.5. Hence the contact structure $\ker \alpha_0$ and $\ker \alpha$ are contactomorphic.

As argued in [11, Example 11.12(2)], the hypersurface $(M, \ker \alpha_W)$ is the contact type boundary of the Weinstein domain $\{H \leq 0\} \subset (T^*Q, \mathrm{d}\lambda)$ with Morse function H and Liouville vector field $\mathbf{p}\partial_\mathbf{p} + \varepsilon X_F$. Observe that

$$\begin{aligned}\mathrm{d}H(\mathbf{p}\partial_\mathbf{p} + \varepsilon X_F) =& -\omega(X_H, \mathbf{p}\partial_\mathbf{p} + \varepsilon X_F) \\ =&(\lambda - \varepsilon \mathrm{d}F)(X_H),\end{aligned}$$

where X_F denotes the gradient vector field of F, is positive along M, so the Liouville field is transverse to the hypersurface. By the exhaustion argument in the end of [11, Section 11.4], the set $\{H \leq 0\}$ is even a Weinstein handle body. As we discussed in [48, Section 3.1], the Morse property of V implies the Morse property for H. In addition u is a critical point of H if and only if u is contained in $Q \subset T^*Q$ and a critical point of V. Moreover, the Morse indices with respect to these functions coincide. By our assumptions on V we have that the Weinstein handle body is subcritical, that is, all critical points of H on this sublevel set have index $\leq n-1$.

According to Morse–Smale theory for Weinstein structures we can assume that there exists a critical point of index $n-1$, see the Creation Theorem [11, Theorem 10.11]. By a further manipulation of the Morse function we can achieve that the largest critical value is attained at exactly one critical point p_0, see [11, Proposition 10.10]. These deformations of the Morse function correspond to Weinstein handle moves. These handle moves yield a smooth family of deformations of the contact structure on the boundary. By Gray stability the change of the contact structure can be obtained from a smooth contact isotopy. By Morse theory there exists a neighbourhood of the critical point p_0 diffeomorphic to $D^{n-1} \times D^{n+1}$. The standard handle in contact surgery as described in [19, Section 6.2] carries a Weinstein structure and we obtain a local Weinstein structure in a neighbourhood of p_0. By [11, Proposition 12.12] we can perturb the given Weinstein structure by a Weinstein isotopy supported near p_0 such that resulting Weinstein structure equals the one obtained from the standard model handle near p_0. Flowing along the Liouville flow we can deform ∂W to a regular level set with regular value slightly bigger than the critical value of p_0. This results in a further contact isotopy. If the level set is close enough to the critical point p_0 we can assume that it contains the upper boundary $D^{n-1} \times S^n$ of the standard handle. Reversing all the contact isotopies that we performed we obtain a contact embedding of the upper boundary $D^{n-1} \times S^n$ of the standard handle into $(M, \ker \alpha_W)$. By [48, Section 3.2] the belt sphere of the embedded handle is non-trivial in the homology of M, i.e., it represents a non-zero element. It also lifts to a contact handle in $(M', \ker \pi^*\alpha_W)$ and hence to a contact handle in $(M', \ker \alpha)$. The image of the belt-sphere S^n is non-trivial in the homology of M'. Combining this with Theorem 7.7 we obtain the theorem. □

7.5. Summary of Results

We quote the statements about the existence of virtually contact structures, that we published in [48] in collaboration with Zehmisch and the results about the existences of contractible closed characteristics as discussed in [5], by Bae–Zehmisch and the author. Some of these were discussed before, in these cases we give the reference to the corresponding sections.

The following theorem summarises the existence results for closed contractible characteristics in odd-symplectic manifolds that support a virtually contact structure. The theorem was published as Theorem 1.1 in [5], in the same formulation.

Theorem. *Let $(M', \xi = \ker \alpha)$ be the total space of a virtually contact structure on a closed odd-dimensional symplectic manifold (M, ω). Assume that the contact form α is C^3-bounded. Then the Reeb vector field of α on M' admits a contractible periodic orbit provided that one of the following conditions for the $(2n-1)$-dimensional contact manifold (M', ξ) is satisfied:*

1. *$n = 2$ and ξ is overtwisted,*
2. *$n = 2$ and $\pi_2(M') \neq 0$,*
3. *$n \geq 3$ and (M', ξ) contains a Legendrian open book with boundary,*
4. *$n \geq 3$ and (M', ξ) contains the upper boundary of the standard symplectic handle of index $1 \leq k \leq n-1$ whose belt sphere $S^{2n-1-k} \subset M'$ represents a non-trivial element in*
   - a) *in $\pi_{2n-2}(M')$ if $k = 1$,*
   - b) *in $\pi_3(M')$ if $n = 3$ and $k = 2$,*
   - c) *in $\pi_4(M')$ if $n = 4$ and $k = 3$,*
   - d) *in the oriented bordism group $\Omega^{SO}_{2n-1-k}M'$ if $k \geq 2$,*
5. *$n \geq 3$ and (M', ξ) is obtained by covering contact connected sum as introduced in [48] such that the underlying connected sum decomposition of M is non-trivial and ω is not the exterior differential of a contact form on M.*

References. The first statement about virtually contact structures with overtwisted covering (M, ξ) is explained in Theorem 7.1. The second case in which the second homotopy group $\pi_2(M')$ does not vanish is covered by Theorem 7.2. The arguments follow [29, 30] and [20]. The third point where (M', ξ) contains a Legendrian open book with boundary is discussed in Remark 7.9. The reasoning is as in [1,26,35,39]. The situation in 4.a) is described in Theorem 7.4. The discussion is similar to [22]. The conditions b) to d) in the fourth instance are presented in Remark 7.8 and Theorem 7.7. The proofs are inspired by [26]. The last point is explored in Theorem 7.5. □

In [48], we describe classes of examples that fall in various categories of the previous theorem. These examples are results of the following theorems. Moreover, they are interesting in their own right since they show that there are virtually contact manifolds with rich topology. We pulished them as Theorem 1.1 and Theorem 1.2 in [48].

Theorem. *For all $n \geq 2$ there exist non-trivial closed virtually contact manifolds M of dimension $2n-1$ that are topologically connected sums such that the corresponding belt spheres represents a non-trivial homotopy class in $\pi_{2n-2}(M)$. The involved covering spaces M' are obtained by covering contact connected sums.*

Theorem. *For any $n \geq 2$ and given $b \in \mathbb{N}$ there exists a closed virtually contact manifold M of dimension $2n-1$ such that $\pi_n(M)$ and the image in $H_n(M)$ under the Hurewicz homomorphism, respectively, contain a subgroup isomorphic to $\mathbb{Z}^b$. The virtually contact manifold M appears as the energy surface of a classical Hamiltonian function in a twisted cotangent bundle T^*Q. The rank b of the subgroup $\mathbb{Z}^b$ is the first Betti number of the configuration space Q. If $n \geq 3$ the virtually contact structure on M is non-trivial.*

The initial motivation to study virtually contact structures are hypersurfaces in magnetic cotangent bundles as in [12]. The examples in second theorem are of this form, some of their properties are recalled in Section 7.2. With some additional assumptions we were able to prove the existence of contractible closed characteristics, see Theorem 7.13 and Theorem 7.15. These statements were already published in [5] as Theorem 1.2 and Theorem 1.3.

Theorem. *Let Q be a closed hyperbolic surface. Let V be a Morse function on Q that has a unique local maximum which is required to be positive. Let $v_0 > 0$ be a positive real number such that $\{V \geq -v_0\}$ contains no critical point of V other than the maximum. Let σ be a 2-form on Q that vanishes on the disc $\{V \geq -v_0\}$. If*

$$v_0 > \inf_{\theta} \sup_{\widetilde{Q}} \tfrac{1}{2}\|\vartheta\|^2 \,,$$

where the infimum is taken over all C^3-bounded primitive 1-forms ϑ of the lift of σ to the universal cover $\widetilde{Q}$ that vanish on $\{\widetilde{V} \geq -v_0\}$ for the lifted potential $\widetilde{V}$, then the equations of motion of a charged particle on Q under the influence of the magnetic field σ and the presence of the potential V have a non-constant periodic solution of energy $H = 0$ that is contractible in $\{V \leq 0\}$.

Theorem. *Let Q be a product of closed hyperbolic surfaces. Assume that Q admits a potential function V together with a choice of regular value $-v_0 < 0$ and a closed magnetic 2-form σ satisfying the conditions described in Theorem 7.5. In addition, assume that σ is cohomologous to a $\mathbb{R}$-linear combination of the area forms corresponding to the factors of Q. Then the magnetic flow of the Hamiltonian H carries a non-constant periodic solution of zero energy that is contractible in $\{V \leq 0\}$.*

CHAPTER 8

Surgery

Surgery operations, handle attachments and the corresponding symplectic cobordimsms have been useful in the obstruction theory for holomorphic curves and the study of periodic orbits in contact manifolds, see [22, 26, 29, 35, 40]. In this chapter we want to study these concepts for odd-symplectic manifolds.

8.1. Two Surgery Constructions

We discuss two ways to perform surgery for virtually contact structures. First we take a look at the connected sum construction as presented and used in [5, 48]. Then we present a higher index handle attachments that takes place in a Darboux neighbourhood of a point.

Construction 8.1. We introduce the **contact connected sum**. This is a useful tool to create examples of virtually contact structures with non-trivial homotopy group π_{2n-2}, see [48, Theorem 1.1]. As stated in Remark 7.5 we can find periodic orbits in the covering contact manifolds obtained by this construction if the connected sum in non-trivial. Later we will give a generalization to higher index virtually contact surgery.

Let $(\pi_i : M_i' \to M_i, \omega_i, \alpha_i, g_i), i = 1, 2$, be two somewhere contact virtually contact structures and b a bijection between the fibres of the coverings π_i. After shrinking the open sets U_i where (M_i, ω_i) is somewhere contact in the sense of Definition 2.15, we may assume that the sets carry Darboux coordinates for the contact forms α_{U_i}. Inside the open sets U_i we choose closed embedded discs D^{2n-1} which now also carry Darboux coordinates. We identify the boundaries ∂D_i^{2n-1} with the upper boundaries $\{i\} \times S^{2n-2}$ of the 1-handle $[1, 2] \times D^{2n-1}$. As explained in [19, Section 6] we can perform an index-1 surgery that yields the connected sum $M_1 \# M_2$ and is such that $U_1 \# U_2$ carries a contact form $\alpha_{U_1} \# \alpha_{U_2}$. Given this form we can equip $M_1 \# M_2$ with the odd-symplectic form

$$\omega = \begin{cases} \mathrm{d}(\alpha_{U_1} \# \alpha_{U_2}) & \text{on } U_1 \# U_2, \\ \omega_i & \text{on } M_i \setminus U_i. \end{cases}$$

In a similar fashion we extend the metric over the handle such that it agrees with the old metrics on $M_i \setminus U_i$.

For $(M_1 \# M_2, \omega)$ to be virtually contact we need to specify a covering and make sure that the boundedness conditions are satisfied. To do so we assume, after shrinking U_i again, that the preimage of U_i under π_i decomposes into disjoint open sets U_i^y, where y runs through the preimages of the basepoint $x_i \in U_i$ of M_i. Using the bijection b between the fibres we can define a family of connected sums $U_1^y \# U_2^{b(y)}$. Since α_i coincides with the lift of α_{U_i} on $\pi_i^{-1}(U_i)$ we can choose the contact form $\alpha_{U_1} \# \alpha_{U_2}$ on each $U_1^y \# U_2^{b(y)}$ equivariantly. Gluing $M_1' \setminus \pi_1^{-1}(U_1) \cup M_2' \setminus \pi_2^{-1}(U_2)$ with $U_1^y \# U_2^{b(y)}$ along their boundaries leads to a manifold denoted by $M_1' \#_b M_2'$. We obtain a cover

$$\pi : M_1' \#_b M_2' \longrightarrow M_1 \# M_2$$

that restricts to π_i on $M_i' \setminus \pi_i^{-1}(U_i)$ and is the obvious cover on

$$U_1 \#_b U_2 := \bigcup_{y \in \pi^{-1}(x)} U_1^y \# U_2^{b(y)}.$$

The manifold $M_1' \# M_2'$ carries a contact form α given by

$$\alpha := \begin{cases} \pi^*(\alpha_{U_1} \# \alpha_{U_2}) & \text{on } U_1 \#_b U_2, \\ \alpha_i & \text{on } M_i' \setminus \pi_i^{-1}(U_i). \end{cases}$$

As mentioned above it does not suffice to give a covering, we also need to check the boundedness conditions. Remember that $(\pi_i : M_i' \to M_i, \omega_i, \alpha_i, g_i)$ are assumed to be virtually contact, so the bounds descent to $(M_1' \#_b M_2') \setminus (U_1 \#_b U_2)$. On the other hand the part $U_1 \#_b U_2$ is a trivial cover of a compact set, so all bounds concerning the contact form $\alpha|_{U_1 \#_b U_2} = \pi^*(\alpha_1 \# \alpha_2)$ are satisfied as long as they are satisfied on $U_1 \# U_2$. Therefore we indeed obtain a virtually contact structure $(\pi : M' \to M, \omega, \alpha, g)$ on $M := M_1 \# M_2$ covered by $M' := M_1' \#_b M_2'$ via π.

Construction 8.2. Now we come to the general case of a virtually contact surgery. In many concerns it is similar to the construction of connected sum above. Let $(\pi : M' \to M, \omega, \alpha, g)$ be a virtually contact structure and assume that (M, ω) is somewhere contact. To be precise say that we have base point $p \in M$ and a neighbourhood U of p such that $\omega|_U = d\alpha_U$ for some 1-form α_U on U with $\pi^* \alpha_U = \alpha$ on $\pi^{-1}(U)$. Assume additionally that $\pi^{-1}(U)$ decomposes in disjoint open subsets of M' diffeomorphic to U. Further assume that there exists an embedding of S^{k-1} into U such that the image of S^{k-1} has trivial symplectic normal bundle. This is always possible for subcritical indices $k \leq n-1$, see [19, Examples 2.5.6]. We perform contact surgery as usual. To be really precise one performs contact surgery for (U, α_U) and glues the result of the surgery into $M \setminus U$. The surgered manifold has the form

$$N := M \setminus (S^{k-1} \times \mathrm{Int}(D^{2n-k})) \cup_{S^{k-1} \times S^{2n-k-1}} (D^{k-1} \times S^{2n-k-1}).$$

A neighbourhood of the surgery is given by

$$V := U \setminus (S^{k-1} \times \mathrm{Int}(D^{2n-k})) \cup_{S^{k-1} \times S^{2n-k-1}} (D^{k-1} \times S^{2n-k-1}).$$

We define an odd-symplectic form ω_N on N by

$$\omega_N := \begin{cases} \omega & \text{on } N \setminus V \cong M \setminus U, \\ \mathrm{d}\alpha_V & \text{on } V, \end{cases}$$

where α_V denotes the contact form obtained from α_U through surgery. Let ψ be the diffeomorphism used to identify the boundaries of $M \setminus S^{k-1} \times \mathrm{Int}(D^{2n-k})$ and $D^k \times S^{2n-k-1}$ and define a covering of N by

$$N' := M' \setminus \left(\pi^{-1}(S^{k-1} \times \mathrm{Int}(D^{2n-k}))\right) \cup^{\psi}_{\pi^{-1}(S^{k-1} \times S^{2n-k-1})} \bigsqcup_{y \in \pi^{-1}(p)} (D^k \times S^{2n-k-1}),$$

where the ψ indicates that we always use the same identification for the boundaries. The covering map π_N is given by π on $M' \setminus \pi^{-1}(V)$ and the trivial covering on the neighbourhood V of the surgery. We specify a contact form on N' by

$$\alpha_{N'} := \begin{cases} \alpha & \text{on } N' \setminus \pi^{-1}(V) \cong M' \setminus \pi^{-1}(U), \\ \pi_N^* \alpha_V & \text{on } \pi_N^{-1}(V). \end{cases}$$

Choosing the metric g_N similar to the metric in the case of the connected sum we can repeat the argument given in that case to verify that the constructed tuple $(\pi_N : N' \to N, \omega_N, \alpha_{N'}, g_N)$ is indeed virtually contact, i.e., that it satisfies the boundedness conditions.

Questions. These construction can be used as a starting point for further research. The constructions above take place in a Darboux chart, but as explained in the following discussion this assumption is not necessary. The question whether the connected sum is non-trivial was discussed in [48]. Finally, one asks if the belt spheres can serve as germs of holomorphic curves. This is of interest in view of the discussion in Chapter 7 and the references therein.

8.2. An Odd-Symplectic Neighbourhood Theorem

We begin our study of a more general surgery theory with a neighbourhood theorem for odd-isotropic submanifolds which states that a neighbourhood of an isotropic submanifold is characterized by the isomorphism type of the symplectic normal bundle. The analogue result in the contact case is essential for performing contact surgery. See [19, Chapter 2.5 and 6] for a detailed discussion of the contact case.

Let (M^{2n-1}, ω) be an oriented odd-symplectic manifold. We begin with the necessary notations.

Definition 8.3. A submanifold $L \subset (M, \omega)$ is called **isotropic** if $TL \cap \ker \omega = \{0\}$ and $i^*\omega = 0$, where i denotes the embedding of L into M.

A **Legendrian submanifold** is an isotropic submanifold of dimension $n-1$ in a $(2n-1)$-dimensional odd-symplectic manifold.

There is an alternative description of isotropic manifolds that uses the existence of a specific framing.

Corollary 8.4. *A submanifold L of an orientable odd-symplectic manifold (M, ω) is isotropic if and only if there exists a framing γ of (M, ω) such that*

1. $TL \subset \ker \gamma$,
2. $T_pL \subset (\ker \gamma)_p$ *is isotropic with respect to* $\omega|_{\ker \gamma_p}$ *for all* $p \in L$.

Proof. Let us assume that L is isotropic in the sense of Definition 8.3. We choose a Riemannian metric g on M with the property that TL is perpendicular to the kernel of ω, $TL \perp \ker \omega$, which is possible since $TL \cap \ker \omega = \{0\}$. Let X be a vector field generating the kernel of ω, see Lemma 2.5, and define the 1-form

$$\gamma := \iota_X g.$$

It satisfies $\gamma|_{TL} = 0$, i.e., $TL \subset \ker \gamma$. As said in the proof of Lemma 2.5 γ is a framing and therefore $\omega|_{\ker \gamma}$ is symplectic. The condition $i^*\omega = 0$ implies that $T_pL \subset (\ker \gamma)_p$ is isotropic with respect to $\omega|_{\ker \gamma_p}$ for all $p \in L$.

On the other hand if the conditions in the corollary are satisfied we know that $\omega|_{\ker \gamma}$ is a symplectic form by the definition of a framing. So it makes sense to say that $T_pL \subset (\ker \gamma)_p$ is isotropic with respect to ω. This implies that $\omega|_{TL} = 0$ which is the same as saying $i^*\omega = 0$ for the embedding i of L into M. The property $TL \cap \ker \omega = \{0\}$ follows from

$$TL \cap \ker \omega \subset \ker \gamma \cap \ker \omega = \{0\}$$

where the last equation is a consequence of γ being a framing. $\square$

The symplectic normal bundle is defined coherently with the definiton in the contact case [19, Definition 6.2.1].

Definition 8.5. Let (M, ω) be a odd-symplectic manifold with framing γ and L an isotropic submanifold. The quotient bundle

$$SN_M(L) = (TL)^{\perp}/TL$$

with the symplectic structure induced by ω is called **symplectic normal bundle** of L in M with respect to γ. Here $(TL)^\perp$ denotes the symplectic orthogonal complement of TL in $\ker\gamma$ with respect to the symplectic form $\omega|_{\ker\gamma}$.

For the following discussion let L be an isotropic submanifold and γ a framing with $TL \subset \ker\gamma$.

Our first aim is a neighbourhood theorem that describes a neighbourhood of an isotropic submanifold up to odd-symplectomorphism depending on the isomorphism type of $SN_M(L)$ as a symplectic bundle. We consider the normal bundle $NL := TM/TL$ of L in M. It splits as

$$NL \cong \ker\omega|_L \oplus (\ker\gamma|_L)/(TL)^\perp \oplus SN_M(L). \tag{8.1}$$

For $\dim M = 2n-1$ and $\dim L = k \leq n-1$ the dimension of the vector spaces on the right hand side are $1, k$, and $2(n-k-1)$, respectively.

By Lemma 2.5 the kernel $\ker\omega$ is a trivial line bundle spanned by the global vector field X defined by $\iota_X \mathrm{vol}_M = \omega^{n-1}$. Therefore we identify $\ker\omega$ with the line bundle spanned by X.

The following statements are the odd-symplectic counterparts of the results in the contact setting found in [19, pp. 69–71]. The following lemma tells us that the isomorphism type of the second summand only depends on the topology of L. Thus the isomorphism type of NL is completely determined by the isomorphism type of $SN_M(L)$.

Lemma 8.6. *The map*

$$\begin{aligned}\Psi : (\ker\gamma|_L)/(TL)^\perp &\longrightarrow T^*L\\ [Y] &\longmapsto \iota_Y\omega|_{TL}.\end{aligned}$$

is a well defined bundle isomorphism between $(\ker\gamma|_L)/(TL)^\perp$ *and* T^*L.

Proof. The proof is analogue to the proof of [19, Lemma 2.5.4] replacing $\mathrm{d}\alpha$ with ω. □

Lemma 8.7 ([19, Lemma 2.5.7])**.** *Combining the isomorphism* Ψ *with the identity on* TL *we obtain a symplectic isomorphism*

$$\mathrm{id}_{TL} \oplus \Psi : (TL \oplus J(TL), \omega) \to (TL \oplus T^*L, \Omega_L),$$

where $\Omega_L\big((v,\alpha),(w,\beta)\big) = \alpha(w) - \beta(v)$.

Sometimes it is convenient to think of all summands in (8.1) as subbundles of TM. For the first one we take the subbundle $\langle X\rangle$ spanned by X. For the other two we use the following proposition.

Proposition 8.8 ([19, Proposition 2.5.5])**.** *Denote the complex bundle structure on* $\ker\gamma$ *constructed in [19, Proposition 2.4.5] by* $J : \ker\gamma \to \ker\gamma$*. By construction* J

is compatible with the symplectic bundle structure $\omega|_{\ker\gamma}$. *Then there exist (abstract) isomorphisms between* $(\ker\gamma|_L)/(TL)^\perp$ *and* $J(TL)$ *and between the symplectic normal bundle* $SN_M(L)$ *and* $(TL \oplus J(TL))^\perp$. *Moreover the second isomorphism respects the symplectic bundle structure.*

With these result we can describe the normal bundle

$$NL \cong \langle X\rangle \oplus J(TL) \oplus (TL \oplus J(TL))^\perp$$

as a subbundle of TM as desired. To avoid confusion we will stay with the notation $SN_M(L)$ but keep in mind that we can think of it as a subbundle of TM if convenient.

Theorem 8.9 (Odd-symplectic neighbourhood theorem)**.** *Let* L_i *be closed isotropic submanifolds of the odd-symplectic manifolds* (M_i, ω_i), $i = 0, 1$. *Assume there is an isomorphism of the symplectic normal bundles* $\Phi : SN_{M_0}(L_0) \to SN_{M_1}(L_1)$ *that respects the symplectic structures and covers a diffeomorphism* $\phi : L_0 \to L_1$. *Then there exists an odd-symplectomorphism* $\psi : \mathcal{N}(L_0) \to \mathcal{N}(L_1)$ *defined on suitable neighbourhoods* $\mathcal{N}(L_i)$ *that restricts to* ϕ *on* L_0 *and satisfies* $T\psi|_{SN_{M_0}(L_0)} = \Phi$.

The proof is strongly inspired by the proofs of [19, Theorems 2.5.8, 6.2.2].

Proof. We identify $\ker\omega_i$ with the trivial line bundle spanned by X_i. In total, this identifies

$$NL_i = \langle X_i\rangle \oplus J(TL_i) \oplus SN_{M_i}(L_i)$$

as a subbundle of $TM_i|_{L_i}$.

Mapping $X_0(p)$ to $X_1(\phi(p))$ yields the obvious line bundle isomorphism $\Phi_X : \langle X_0\rangle|_{L_0} \to \langle X_1\rangle|_{L_1}$.

We denote the isomorphism defined by the interior product with ω_i by $\Psi_i : J_i(TL_i) \to T^*L_i$, see Lemma 8.6 and Proposition 8.8. We have the symplectic bundle isomorphism

$$T\phi \oplus (\phi^*)^{-1} : (TL_0 \oplus T^*L_0, \Omega_{L_0}) \to (TL_1 \oplus T^*L_1, \Omega_{L_1}).$$

Combining this with Lemma 8.7 gives us the symlectic vector bundle isomorphism

$$T\phi \oplus (\Psi_1^{-1} \circ (\phi^*)^{-1} \circ \Psi_0) \ : \ (TL_0 \oplus J_0(TL_0), \omega_0) \to (TL_1 \oplus J_1(TL_1), \omega_1).$$

We combine these maps to a bundle isomorphism covering ϕ

$$\widetilde{\Phi} : NL_0 \to NL_1$$

defined by

$$\widetilde{\Phi} := \Phi_X \oplus \left(\Psi_1^{-1} \circ (\phi^*)^{-1} \circ \Psi_0\right) \oplus \Phi.$$

Take tubular neighbourhoods $\tau_i : NL_i \to M_i$. Recall that the τ_i are embeddings with the following properties:

- with respect to the identification of L_i with the zero section of NL_i, τ_i restricts to the inclusion.
- the differential $T\tau_i$ induces the identity on NL_i along L_i, where we us the splitting $T(NL_i)|_L = TL_i \oplus NL_i$.

The concatenation $\tau_1 \circ \widetilde{\Phi} \circ \tau_0^{-1}$ of the tubular maps with $\widetilde{\Phi}$ is a diffeomorphism of suitable neighbourhoods $\mathcal{N}(L_i)$ of L_i that induces the bundle map

$$T\phi \oplus \widetilde{\Phi} : TM_0|_{L_0} \longrightarrow TM_1|_{L_1}.$$

This bundle map pulls ω_1 back to ω_0 by construction.

Summarising what we have achieved and abbreviating the notation we may assume the following situation:

- $M_0 = M_1 =: M$, $L_0 = L_1 =: L$, $\phi = \mathrm{id}_L$
- $\omega_0 = \omega_1$ on $TM|_L$ and $\Phi = \mathrm{id}|_{SN_M(L)}$.

Since the odd-symplectic forms coincide along L we can assume that the vector fields X_0 and X_1 that span the kernels of ω_0 and ω_1, respectively, also coincide along L. Therefore we can choose framings γ_i that agree along L and satisfy $TL \subset \ker \gamma_i$. Now consider an open neighbourhood $\mathcal{N}(L)$ and a hypersurface Σ in $\mathcal{N}(L)$ with

- $L \subset \Sigma$,
- Σ is transverse to X_0 and X_1, and each integral curve of X_0 or X_1 in $\mathcal{N}(L)$ intersects Σ exactly once,
- $\ker \gamma_0|_L = \ker \gamma_1|_L = T\Sigma|_L$.

With these conditions we obtain that the restrictions of ω_0 and ω_1 induce symplectic forms on Σ, since Σ is transverse to the kernels of the odd-symplectic forms. Note that these forms satisfy $\omega_0 = \omega_1$ on $T\Sigma|_L$. Denote the inclusion of Σ into M by $j : \Sigma \hookrightarrow M$ and the convex interpolation between the forms by $\widetilde{\omega}_t := (1-t)\omega_0 + t\omega_1$. Further, define the restriction to Σ as $\widehat{\omega}_t := j^*\widetilde{\omega}_t$ and note that $\widehat{\omega}_t$ agrees with ω_0 on $T\Sigma|_L$ for all t. Therefore, $\widehat{\omega}_t$ is a symplectic form on Σ for all $t \in [0,1]$, after shrinking $\mathcal{N}(L)$ and Σ if necessary, since the symplectic condition is open.

We are looking for an isotopy ψ_t of a neighbourhood of $L \subset \Sigma$ with $\psi_t^*\widehat{\omega}_t = \widehat{\omega}_0$, and furthermore $\psi_t|_L = \mathrm{id}_L$ and $T\psi_t = \mathrm{id}$ on $T\Sigma|_L$. We do so by applying the symplectic Moser trick to the family $\widehat{\omega}_t$ of symplectic forms.

We apply the generalised Poincaré lemma [19, Corollary A.4] to the 2-form

$$\eta := \widehat{\omega}_1 - \widehat{\omega}_0 = \dot{\widehat{\omega}}_t,$$

which yields a 1-form ζ in a neighbourhood of $L \subset \Sigma$, vanishing to second order on L, such that $\eta = \mathrm{d}\zeta$.

We assume the ψ_t is the flow of a time-dependent vector field Y_t and differentiate the desired equation $\psi_t^* \widehat{\omega}_t = \widehat{\omega}_0$, which yields the equation

$$\psi_t^*(\mathrm{d}\zeta + \mathrm{d}(\iota_{Y_t}\widehat{\omega}_t)) = 0.$$

To solve this equation it suffices to solve

$$\zeta + \iota_{Y_t}\widehat{\omega}_t = 0,$$

which is uniquely possible because $\widehat{\omega}_t$ is a symplectic form in a neighbourhood of $L \subset \Sigma$ for all $t \in [0,1]$. The vector field Y_t vanishes to second order on L since the same is true for ζ. Hence the local flow of Y_t fixes L, and therefore exists up to time 1 in a neighbourhood of L. The vanishing of Y_t along L implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^* Z) = \psi_t^* L_{Y_t} Z = \psi_t^*([Y_t, Z]) = 0$$

on L for all vector fields Z on L, where ψ_t^* is defined as $T(\psi_t^{-1})$ on vector fields. Together with the initial condition $\psi_0 = \mathrm{id}_\Sigma$ we have $T\psi_t = \mathrm{id}$ on $T\Sigma|_L$ for all $t \in [0,1]$. We extend the diffeomorphism $\psi_1 : \Sigma \to \Sigma$ to a diffeomorphism $\psi : \mathcal{N}(L) \to \mathcal{N}_1(L)$ of the neighbourhood $\mathcal{N}(L)$ of L in M, again after shrinking $\mathcal{N}(L)$ and Σ, if necessary, to another neighbourhood $\mathcal{N}_1(L)$ by requiring that ψ sends flow lines of the vector field X_0 to these of X_1. Since $X_0 = X_1$ along L, this gives $T\psi = \mathrm{id}$ on $TM|_L$. This implies $\psi^*\widehat{\omega}_1 = \widehat{\omega}_0$ on $\mathcal{N}_1(L)$, since $\iota_X \omega = 0$ and $\mathcal{L}_X\omega = 0$ for a vector field spanning the kernel of any ω. □

Question. In view of the aforementioned surgery, see Construction 8.2, it is interesting to classify neighbourhoods an isotropic sphere up to odd-symplectomorphism. In view of the neighbourhood theorem this is the same as understanding the symplectic bundle isomorphisms of the symplectic normal bundle.

Corollary 8.10. *Let $L_0 \subset (M_0, \omega)$ be a closed Legendrian submanifold and $L_1 \subset (M_1, \alpha)$ be a closed Legendrian submanifold in a strict contact manifold (M_1, α) diffeomorphic to L_0. Then there exist a neighbourhood $\mathcal{N}(L_0)$ of L_0 and a contact form α_0 on $\mathcal{N}(L_0)$ with $\mathrm{d}\alpha_0 = \omega$.*

Proof. The symplectic normal bundle of a Legendrian submanifold is the zero-bundle so the only non-empty condition in the odd-symplectic neighbourhood theorem is the existence of a diffeomorphism between the isotropic submanifolds. Therefore, we have a diffeomorphism $\psi : (\mathcal{N}(L_0), \omega) \to (\mathcal{N}(L_1), \mathrm{d}\alpha)$. The pullback of α with ψ gives a contact form on $\mathcal{N}(L_0)$ whose exterior derivative equals ω, indeed

$$\mathrm{d}\psi^*\alpha = \psi^*\mathrm{d}\alpha = \omega.$$ □

With a similar argument we obtain

Corollary 8.11. *Let $L_i \subset (M_i, \omega_i)$ be diffeomorphic (closed) Legendrian submanifolds. Then they admit odd-symplectomorphic neighbourhoods.*

Corollary 8.12. *Let $L_0 \subset (M_0, \omega)$ be a closed isotropic submanifold and $L_1 \subset (M_1, \alpha)$ be a closed isotropic submanifold in a strict contact manifold (M_1, α) diffeomorphic to L_0. Assume further that the symplectic normal bundles $SN_{M_0}(L_0)$ and $SN_{M_1}(L_1)$ are isomorphic as symplectic bundles. Then there exist a neighbourhood $\mathcal{N}(L_0)$ of L_0 and a contact form α_0 on $\mathcal{N}(L_0)$ with $\mathrm{d}\alpha_0 = \omega$.*

Questions. Corollary 8.12 leads to odd-symplectic surgery as explained in the next section. For a better understanding of this surgery it can be interesting to have a better understanding of the contact neighbourhoods given by the corollary. For example one asks how rigid the assumptions in Corollary 8.10 and 8.12 are. That is, given a Legendrian submanifold L in (M_0, ω), are there conditions that enable us to find a contact manifold (M_1, α) that contains L as a Legendrian submanifold. The same question is valid for isotropic submanifolds. That is, given a symplectic normal bundle over an isotropic submanifold L, can we find a contact manifold (M, α) that contains L as an isotropic submanifold such that $SN_M(L)$ is isomorphic to the given bundle?

8.3. Odd-Symplectic Surgery and Symplectic Cobordisms

Based on the neighbourhood theorem (Theorem 8.9) for odd-symplectic manifolds in the previous section we will construct a symplectic cobordism whose lower boundary is a given odd-symplectic manifold and whose upper boundary is diffeomorphic to the result of a surgery: Let (M_0, ω_0) be an oriented, odd-symplectic manifold and $S \subset M_0$ an isotropic $(k-1)$-sphere with trivial symplectic normal bundle. Additionally, we assume that there exists a contact manifold (N, α) that contains a $(k-1)$-sphere S' such that the symplectic normal bundles $SN_M(S)$ of $S \subset M_0$ and $SN_N(S')$ of $S' \subset N$ are isomorphic as symplectic bundles. Surgery along this sphere yields a manifold M_1. Note that this is possible since triviality of the symplectic normal bundle implies that the normal bundle is also trivial by the discussion in Section 8.2. The aim of the following argument is to find a symplectic manifold (W, Ω) with $\partial W = M_1 \sqcup -M_0$ and $\Omega|_{TM_0} = \omega$. This procedure gives an odd-symplectic structure on M_1 by restriction of Ω. In contrast to the situation in contact geometry, the dynamics on the upper boundary of the cobordism differ from those on the lower boundary even away from the surgery region.

Question. For a full understanding of the odd-symplectic surgery it is interesting how restrictive it is to assume the existence of S', i.e., how many k-spheres with trivial symplectic normal bundle are there up to an isomorphism covering a diffeomorphism. For a answer one should understand the situation in contact geometry.

Let us give a precise definition of the subjects mentioned above.

Definition 8.13. Let M_0, M_1 be oriented manifolds. A **symplectic cobordism** from M_0 to M_1 is a compact, connected, symplectic manifold (W, Ω), oriented by

Ω, with oriented boundary

$$\partial W = M_1 \sqcup -M_0,$$

where $-M_0$ denotes M_0 with the opposite orientation.

Definition 8.14. An odd-symplectic manifold (M_0, ω_0) is **symplectically cobordant** to (M_1, ω_1) provided there exists a connected symplectic cobordism (W, Ω) from M_0 to M_1 such that $\Omega|_{TM_0} = \omega_0$ and $\Omega|_{TM_1} = \omega_1$.

Remark 8.15. Note that the neighbourhood theorem (Theorem 8.9) allows us to glue arbitrary odd-symplectic manifolds of the same dimension with diffeomorphic isotropic submanifolds that satisfy conditions along neighbourhoods of these submanifolds. But it turns out that it is helpful for further applications to have some control or understanding about one of the summands. In practice it is useful to work with handles, i.e., disc products, that admit a contact structure on the lower boundary, see [47]. In the following we will discuss the attachment of such a handle.

The construction uses a symplectisation and a symplectic model handle. The symplectic model handle is a modification of the one used in the contact case. We shall only discuss our changes and refer to [19, Section 6.2] for more details. The difference to the contact case is that the symplectisation does not always admit a global Liouville vector field.

For the symplectisation of an odd-symplectic manifold (M, ω) we consider $\mathbb{R} \times M$ together with the 2-form

$$\begin{aligned}\Omega_\gamma &= \mathrm{d}(t\gamma) + \omega \\ &= \mathrm{d}t \wedge \gamma + t\mathrm{d}\gamma + \omega,\end{aligned}$$

where t denotes the $\mathbb{R}$-coordinate and γ is a framing. Note that Ω_γ is a closed 2-form with $\Omega_\gamma|_{T(\{0\}\times M)} = \omega$. We compute

$$\Omega_\gamma^n = n \cdot \mathrm{d}t \wedge \gamma \wedge \omega^{n-1} + \mathcal{O}(t).$$

Therefore we find an $\varepsilon > 0$, depending on ω and γ, such that Ω_γ is non-degenerated on $[-\varepsilon, \varepsilon] \times M$, i.e., $([-\varepsilon, \varepsilon] \times M, \Omega_\gamma)$ is a symplectic manifold with boundary $(\{-\varepsilon\} \times M) \cup (\{\varepsilon\} \times M)$. The orientation on the lower boundary differs from the orientation of M and the orientation on the upper boundary coincides with the one of M. For the handle attachment we restrict ourselves to $[0, \varepsilon] \times M$.

For a handle attachment as described in [19, Section 6.2] we need a Liouville vector field of Ω_γ, at least in a neighbourhood of $[0, \varepsilon] \times S^{k-1}$, where S^{k-1} is the odd-isotropic sphere along which we perform surgery. By Corollary 8.12 we find a neighbourhood U of $S^{k-1} \subset M$ and a contact form α on U with $\mathrm{d}\alpha = \omega$. Using the next lemma we extend α to a framing γ on M. Note that this extension may fail to be contact and even if it is contact we can not ensure that $\mathrm{d}\gamma = \omega$.

Lemma 8.16. *Let (M, ω) be an oriented, odd-symplectic manifold, $U \subset M$ open and assume there is a contact form α on U with $\mathrm{d}\alpha = \omega$. Then for any open subset V with $\overline{V} \subset U$ there exists a framing γ on M with $\gamma|_V = \alpha$.*

Proof. After shrinking U we can assume that there exists a volume form vol_M on M with $\mathrm{vol}_M|_U = \alpha \wedge (\mathrm{d}\alpha)^{n-1} = \alpha \wedge \omega^{n-1}$. Let $V \subset \overline{V} \subset U$ be an open subset of M and choose a cut-off function χ with

$$\chi = \begin{cases} 0 & \text{on } M \setminus U, \\ 1 & \text{on } V. \end{cases}$$

By Lemma 2.5 we find a framing γ with $\gamma \wedge \omega^{n-1} = \mathrm{vol}_M$. We claim that $\widetilde{\gamma} := (1-\chi)\gamma + \chi\alpha$ is a framing as desired. First note that $\widetilde{\gamma}|_V = (1-1)\gamma + 1 \cdot \alpha = \alpha$ by definition of χ and it is indeed a framing since

$$((1-\chi)\gamma + \chi \cdot \alpha) \wedge \omega^{n-1} = (1-\chi)\mathrm{vol}_M + \chi \mathrm{vol}_M = \mathrm{vol}_M.$$

Note that $\widetilde{\gamma}$ is well defined since χ vanishes where α is not defined. □

We replace the framing γ with one that is contact in a neighbourhood $\mathcal{N}(S^{k-1})$ of S^{k-1}, still denoted by γ. In our situation this means that Ω_γ takes the form

$$\Omega_\gamma = \mathrm{d}(t\alpha) + \mathrm{d}\alpha = \mathrm{d}\big((1+t)\alpha\big)$$

on $[0, \varepsilon] \times \mathcal{N}(S^{k-1})$

$$\Phi^{s_0}(0, p) = (\varepsilon, p)$$

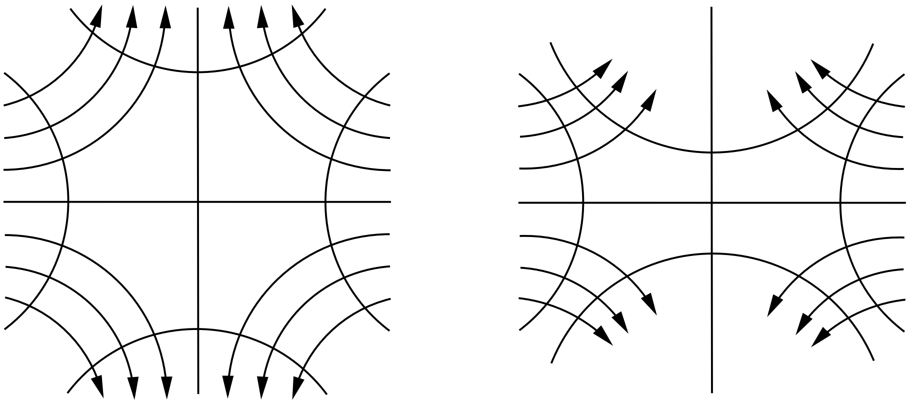
for $s_0 = -1 + \sqrt{1 + 2\varepsilon}$, i.e., the flow of boundary points $(0, p)$ exists for at most time s_0. Note that $\Omega_\gamma|_{T(\{\varepsilon\} \times M)} = \varepsilon \mathrm{d}\gamma + \omega$ will differ from ω in general.

The symplectic model handle is described in [19, pp. 296] as a subset of $(\mathbb{R}^{2n}, \mathrm{d}\mathbf{x} \wedge \mathrm{d}\mathbf{y})$. We would like to use [19, Lemma 5.2.4] to glue the model handle to the symplectisation. This lemma translates to

Lemma 8.17 ([19, Lemma 5.2.4]). *For $i = 0, 1$ let $M_i \subset (W_i, \omega_i)$ be a hypersurface in a symplectic manifold and Y_i a Liouville vector field defined in a neighbourhood of and transversal to M_i. Denote the induced contact form $\iota_{Y_i}\omega_i$ by α_i. Assume that $\phi : (M_0, \alpha_0) \to (M_1, \alpha_1)$ is a contactomorphism, extended to a diffeomorphism $\widetilde{\phi}$ on suitable neighbourhoods via the Liouville flow. Then $\widetilde{\phi}$ is a symplectomorphism.*

The reason we cannot glue the standard model handle to our symplectistion is that the Liouville flow on the boundary component $\{0\} \times M$ of the symplectisation only exist for a short time s_0. Hence the image of $[0, \varepsilon] \times \mathcal{N}(S^{k-1})$ under the symplectomorphism given in the preceding lemma may has empty intersection with the upper boundary. We replace the standard model handle with a **pancake handle**, see Figure 8.1. Given a symplectomorphism as in the lemma defined on

Figure 8.1.: Two types of handles



On the left: The standard handle with Liouville flow
On the right: The pancake handle with shortened Liouville flow

$[0,\varepsilon] \times \mathcal{N}(S^{k-1})$ with image in $\mathbb{R}^{2n}$ such that $\{0\} \times \mathcal{N}(S^{k-1})$ is mapped into the lower boundary of the standard model handle. We replace the upper boundary of the standard handle with a hypersurface diffeomorphic to $D^k \times S^{2n-k-1}$ and transversal to the Liouville vector field such that $(D \setminus D_{1-\delta}) \times S^{2n-k-1}$ is contained in the image of $[0,\varepsilon] \times \mathcal{N}(S^{k-1})$ for some $\delta > 0$.

8.4. An Abstract Point of View

For a closed $(2n-1)$-dimensional odd-symplectic manifold (M,ω) with framing γ the symplectisation is given by

$$\big((-\varepsilon,\varepsilon) \times M, \mathrm{d}(t\gamma) + \omega\big)$$

for an appropriate $\varepsilon > 0$.

We take a closer look at the definition of a symplectic cobordism. Let (W,Ω) be a compact connected symplectic manifold with boundary

$$\partial W = M_1 \cup -M_0,$$

and oriented by Ω^n, where n is half the dimension of W. The boundary of W is oriented with respect to the outward pointing normal vector field ν. Suppose that each boundary component carries its own orientation. We denote by M_1 the union of all boundary components whose orientation coincides with the given one and by M_0 the union of all components where the orientations differ.

By the symplectic neighbourhood theorem for hypersurfaces, see [36, Example 3.36], there exist collar neighbourhoods of the boundaries M_0 and M_1 of the form

$$\big([0,\varepsilon) \times M_0, \mathrm{d}(t\gamma_0) + \omega_0\big)$$

and

$$\big((-\varepsilon, 0] \times M_1, \mathrm{d}(t\gamma_1) + \omega_1\big),$$

where $\omega_i = \Omega|_{TM_i}$ is an odd-symplectic form on M_i and γ_i is a framing of ω_i inducing the given orientation of M_i, for $i = 0, 1$.

In general the relation of being symplectically cobordant described in Definition 8.14 is neither reflexive nor symmetric. We redefine the term symplectically cobordant to achieve reflexivity. The disadvantage of this new definition is that transitivity is no longer obvious. We discuss the properties of the relation after the definition.

Definition 8.18. Two odd-symplectic manifolds (M_0, ω_0) and (M_1, ω_1) are **symplectically cobordant** directed from (M_0, ω_0) to (M_1, ω_1) if there exists an $s_0 > 0$ and a framing γ_1 of (M_1, ω_1) such that for all $s \in (0, s_0)$ there exists a symplectic cobordism (W_s, Ω_s) with

$$\partial(W_s, \Omega_s) = (M_1, \omega_1 + s\mathrm{d}\gamma_1) - (M_0, \omega_0).$$

The **odd-symplectic cobordism relation** is defined as

$$\begin{aligned}(M_0, \omega_0) \preccurlyeq (M_1, \omega_1) \quad :\Leftrightarrow \quad & (M_0, \omega_0) \text{ is symplectically cobordant to } (M_1, \omega_1)\\ & \text{directed from } (M_0, \omega_0) \text{ to } (M_1, \omega_1).\end{aligned}$$

We will now discuss properties of this relation. It turns out that it is reflexive and transitive on odd-symplectic manifolds, but results by Geiges–Zehmisch [25] show that it is not symmetric in general.

Lemma 8.19. *The relation $\preccurlyeq$ is reflexive.*

Proof. Let (M, ω) be an odd-symplectic manifold and γ a corresponding framing. There exists a $\varepsilon_0 > 0$ such that $\big([0, s] \times M, \mathrm{d}(t\gamma) + \omega\big)$ is a symplectic cobordism from (M, ω) to $(M, s\mathrm{d}t + \omega)$ for all $s \in (0, \varepsilon_0)$. See also the discussion about the symplectisation of an odd-symplectic manifold in Section 8.3. □

Lemma 8.20. *The relation $\preccurlyeq$ is transitive.*

Proof. Assume the situation

$$(M_0, \omega_0) \preccurlyeq (M, \omega) \preccurlyeq (M_1, \omega_1)$$

with all necessary requirements on the odd-symplectic manifolds. We need to show $(M_0, \omega_0) \preccurlyeq (M_1, \omega_1)$, i.e., we want to find a symplectic cobordism from (M_0, ω_0) to $(M_1, \omega_1 + r\mathrm{d}\gamma_1)$ for all $r \in (0, r_0)$ and an appropriate $r_0 > 0$. The idea is to glue a cobordism (W_0, Ω_0) from (M_0, ω_0) to $(M, \omega + s\mathrm{d}\gamma)$ and a cobordism (W_1, Ω_1) from (M, ω) to $(M_1, \omega_1 + r\varepsilon_1\mathrm{d}\gamma_1)$ along (M, ω).

Let γ be a framing on (M, ω). Pick s_0 as in Definition 8.18 and $s \in (0, s_0)$. Consider the collar neighbourhood of the upper boundary $(M, \omega + s\mathrm{d}\gamma)$ in the cobordism (W_0, Ω_0). By the symplectic neighbourhood theorem [36, Exercise 3.36] we have a collar neighbourhood

$$\mathrm{Collar}(M, \omega + s\mathrm{d}\gamma) = \big((-\varepsilon, 0] \times M, \mathrm{d}(t\gamma) + \omega + s\mathrm{d}\gamma\big),$$

where t is the coordinate on the interval. The collar of (M, ω) in (W_1, Ω_1) looks like

$$\mathrm{Collar}(M, \omega) = \big([0, \varepsilon_1) \times M, \mathrm{d}(t\gamma) + \omega\big).$$

We change the symplectic form on W_1 to

$$\Omega_1' = \Omega_1 + s\mathrm{d}(\chi\gamma)$$

where χ is a cut-off function supported in $\mathrm{Collar}(M, \omega)$ that only depends on the interval coordinate t with

$$\chi \equiv 1 \text{ near } \{0\} \times M \quad \text{and} \quad \chi \equiv 0 \text{ near } \{\varepsilon_1\} \times M.$$

For s sufficiently small Ω_1' is a symplectic form on W_1 that coincides with Ω_1 near the upper boundary. Therefore there exists an $s > 0$ such that (W_0, Ω_0) and (W_1, Ω_1') can be glued along $(M, \omega + s\mathrm{d}\gamma)$ resulting in a cobordism (W, Ω) with

$$\partial(W, \Omega) = (M_1, \omega_1 + r\mathrm{d}\gamma_1) \sqcup (-M_0, \omega_0)$$

for all $r \in (0, r_0)$ and r_0 as in Definition 8.18. □

Remark 8.21. The easiest way to achieve symmetry for the odd-symplectic cobordism relation would be an operation that "flips" the cobordism, interchanging the lower and upper boundary. But due to our orientation requirement and the condition $\Omega|_{TM_\pm} = \omega_\pm$ such an operation cannot exist. Nonetheless, symmetry holds for the subclass of odd-symplectic manifolds that admit an orientation reversing odd-symplectic diffeomorphism.

One class of examples admitting such diffeomorphisms are symplectic surface bundles over S^1. These examples are classified in [24]. It turns out that such a manifold is cobordant to exactly one of the following (symplecticaly fillable) odd-symplectic manifolds:

i) $(S^1 \times S^2, a\mathrm{d}\varphi \wedge \mathrm{d}z)$ for some $a \in \mathbb{R}^+$,
ii) $(S^1 \times T^2, \mathrm{d}y \wedge \mathrm{d}x)$.

We emphasize that these cobordisms can be turned around to cobordisms from $S^1 \times \Sigma$ to the given surface bundle. Hence all closed symplectic surface bundles over S^1 can be represented by one of these examples.

On the other hand there exists an exotic odd-symplectic form on S^3 constructed in [25] such that (S^3, ω) is symplectically fillable and not cobordant to (S^3, ξ_{st}). However, (S^3, ξ_{st}) is symplectic cobordant to (S^3, ω). In fact, let (W, Ω) be a symplectic filling of (S, ω). The symplectic Darboux theorem [36, Theorem 3.15] applied to an interior point of W yields an embedding of (S^3, ξ_{st}). Removing the interior of this standard sphere results in a cobordism from (S^3, ξ_{st}) to (S, ω). Therefore, the relation can not be symmetric in general.

Remark 8.22. This relation allows us to consider a category where the objects are odd-symplectic manifolds and the morphisms are symplectic cobordisms. This is the largest category with symplectic cobordisms as relation since the boundaries of symplectic cobordisms are always be odd-symplectic. There are two notable subcategories, the category of odd-symplectic manifolds admitting a virtually contect structure, discussed in the next section, and the category of contact manifolds. The second is contained in the first.

8.5. Cobordisms for Virtually Contact Manifolds

Let (M, ω) be a connected odd-symplectic manifold that supports a virtually contact structure $(\pi : M' \to M, \omega, \alpha, g)$. We take a look at the previously defined cobordisms in this situation. For the beginning we talk about fillings, i.e., cobordisms to the empty set.

Definition 8.23. A pair $(\Pi : W' \to W, \Omega)$ consisting of a compact connected symplectic manifold (W, Ω) and a covering $\Pi : W' \to W$ is called **strong (symplectic) filling** of the virtually contact structure $(\pi : M' \to M, \omega, \alpha, g)$ if $\partial W = M$, $\partial W' = M'$, $\Pi|_{\partial W'} = \pi$ and there exists an outward pointing Liouville vector field Y defined on a neighbourhood of $\partial W'$ with $\ker \alpha = \ker \iota_Y \Omega'$, where $\Omega' = \Pi^* \Omega$.

Remark 8.24. In general the Liouville vector field Y will not be invariant under deck transformations, because Ω' is invariant, but the contact structure $\xi = \ker \alpha$ is not. Indeed, if Y is invariant, the same is true for $\ker \iota_Y \Omega' = \ker \alpha$. In this case we can conclude that α is invariant under the action of the deck transformation group. Thus α descends to a contact form on M and the virtually contact structure is trivial. Summarised we have seen that a non-trivial virtually contact structure will not admit a Liouville vector field that is invariant under the deck transformation group.

Note that we do not require that Ω restricted to TM equals ω. But if this is the case we obtain that $\iota_Y \Omega'$ restricted to TM' equals α.

Example 8.25. The hypersurfaces introduced in [12] and studied in Section 7.2 are fillable in the sense of our definition. We have the commutative diagram

$$\begin{array}{ccccccc}
\{H'=c\} & \hookrightarrow & \{H'\le c\} & \hookrightarrow & T^*Q' & \xrightarrow{\tau'} & Q' \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle T^*\rho} & & \downarrow{\scriptstyle \rho} \\
\{H=c\} & \hookrightarrow & \{H\le c\} & \hookrightarrow & T^*Q & \xrightarrow{\tau} & Q.
\end{array}$$

It remains to show the existence of a Liouville vector field with the required properties. We can define a Liouville vector field Y by

$$\iota_Y \mathrm{d}\big(\lambda + (\tau')^*\vartheta\big) = \lambda + (\tau')^*\vartheta,$$

where λ is the canonical 1-form on the cotangent bundle and ϑ is a primitive of the lifted magnetic form $\rho^*\sigma$. As in [12, Lemma 5.1] we can show that the vector field Y is outward pointing by showing that $\mathrm{d}H(Y) > 0$. Also consider Section 7.3 and 7.4 for a discussion of this situation but note that in this example c is larger than the Mañé critical value, especially $c > \max V$.

This example is part of a more general situation. Let (W, Ω) be a compact symplectic manifold with boundary $(M, \omega = \Omega|_{TM})$. Assume there exists a covering $\Pi : W' \to W$ of W by the symplectic manifold $(W', \Omega' = \Pi^*\Omega)$ such that the covering restricted to the boundary $\partial W' = M'$ defines a virtually contact structure $(\pi = \Pi|_{M'} : M' \to M, \omega, \alpha, g)$. We summarise the situation in a commutative diagram

$$\begin{array}{ccc}
(M', \mathrm{d}\alpha = \Omega'|_{TM'}) & \hookrightarrow & (W', \Omega' = \Pi^*\Omega) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \Pi} \\
(M, \omega = \Omega|_{TM}) & \hookrightarrow & (W, \Omega).
\end{array}$$

Finally assume that W and W' are oriented by Ω and Ω', respectively, and that the orientation of M' as boundary of W' and as a contact manifold coincide.

Lemma 8.26. *In the situation above,* $(\Pi : W' \to W, \Omega)$ *is a filling of* $(\pi : M' \to M, \omega, \alpha, g)$.

Proof. It only remains to find a Liouville vector field Y. As observed in Remark 8.24 the vector field Y will satisfy $\iota_Y \Omega'|_{TM'} = \alpha$. We can extend α to a 1-form on a neighbourhood of the boundary and define Y as the unique vector field solving the equation $\iota_Y \Omega = \alpha$. Then we get $\ker(\iota_Y \Omega')|_{TM'} = \ker \alpha$ for free. We have to show that Y is pointing outwards. Note that

$$\begin{aligned}
\big(\iota_Y(\Omega^n)\big)|_{TM'} &= n\big((\iota_Y \Omega) \wedge \Omega^{n-1}\big)|_{TM'} \\
&= n(\iota_Y \Omega)|_{TM'} \wedge (\Omega|_{TM'})^{n-1} \\
&= n\alpha \wedge (\mathrm{d}\alpha)^{n-1}.
\end{aligned}$$

This shows that Y is transversal to M'. Since the orientation induced on M' by $\iota_Y \Omega$ coincides with the one given by the contact form, we conclude that Y is pointing outwards. □

We end the discussion of virtual fillings with an observation about a type of filling that is not allowed for a simple reason.

Lemma 8.27. *A non-trivial virtual contact structure* $(\pi : M' \to M, \alpha, \omega, g)$ *with connected base manifold* M *cannot be filled by a simply connected manifold.*

Proof. By [27] there is a one-to-one correspondence between coverings of a manifold and subgroups of the fundamental group of that manifold. Let $(\Pi : W' \to W, \Omega)$ be a filling of the virtually contact structure $(\pi : M' \to M, \omega, \alpha, g)$ and denote by $H < \pi_1(W)$ and $G < \pi_1(M)$ the subgroups of the fundamental groups of W and M corresponding to Π and π, respectively. Since M and W are connected the number of leaves above a point is constant and coincides for both coverings, because the covering Π equals π when restricted to $M' = \partial W'$. With [27, Proposition 1.32] we obtain

$$\begin{aligned} [\pi_1(W) : H] &= \text{ number of leaves of } \Pi \\ &= \text{ number of leaves of } \pi = [\pi_1(M) : G]. \end{aligned}$$

If W is simply connected we have $[\pi_1(W) : H] = 1$, but $\pi : M' \to M$ has to be an infinite cover for the virtually contact structure to be non-trivial, see Remark 2.18. Therefore the filling of a non-trivial virtually contact structure cannot be simply connected. □

We generalise the definition of a symplectic filling of a virtually contact manifold to a symplectic cobordism between virtually contact manifolds. This definition is compatible with the usual convention that a filling is a cobordims whose lower boundary equals the empty set.

Definition 8.28. A pair $(\Pi : W' \to W, \Omega)$ consisting of a compact connected symplectic manifold (W, Ω) and a covering $\Pi : W' \to W$ is a **cobordism** between the virtually contact structures $(\pi_i : M_i' \to M_i, \omega_i, \alpha_i, g_i), i = 0, 1$, if $\partial W = M_1 \sqcup -M_0, \partial W' = M_1' \sqcup M_0'$, $\Pi|_{M_i'} = \pi_i$ and there exist Liouville vector fields Y_i defined near M_i, satisfying $\ker \iota_{Y_i} \Omega'|_{TM_i} = \ker \alpha_i$ and pointing inwards along M_0 and outwards along M_1.

CHAPTER 9

Further Studies

Starting from this thesis there are several aspects for further studies that result in a better understanding of the field. As hinted at in Chapter 8 there are some points that can be improved. The most general way to phrase this is to ask for consequences of the surgery construction. This requires one to take a closer look at the surgery construction in contact geometry and to understand the structure of statements that are proved with this technique. One question that is of particular interest in this situation is the structure of odd-symplectic cobordisms. To be even more precise one may ask for all odd-symplectic manifolds that admit a symplectic cobordism whose upper boundary is either overtwisted or has a connected component contactomorphic to the standard tight contact 3-sphere.

In view of the recent results by Fish–Hofer [15] concerning feral curves and non-dense orbits, it is interesting to study how these methods can be used for odd-symplectic manifolds. Inspired by results in contact geometry one asks for conditions on the upper boundary of a symplectic cobordism between odd-symplectic manifolds that yield conclusions about non-dense orbits in the lower boundary.

For virtually contact manifolds a deeper understanding of Bae's Lutz twist construction in [4] could yield further understanding of these examples. Moreover, one can ask if a similar construction is possible for odd-symplectic manifolds in general.

Appendices

APPENDIX A

Splitting the Tangent Bundle TT^*Q

Throughout this chapter we will use the sum convention as in the rest of this dissertation.

Let Q be an n-dimensional manifold and ∇ a torsion-free connection of $TQ \to Q$, see [9, 10, 18, 42] for the properties of a connection. As in Section 5.1 we define the induced connection ∇^* of $\tau : T^*Q \to Q$ as

$$(\nabla^*\beta)(X,Y) := (\nabla^*_X\beta)(Y) := X\big(\beta(Y)\big) - \beta\big(\nabla_X Y\big)$$

for a 1-form β and vector fields X, Y on Q. Observe that ∇^* is indeed a connection with image in the 1-forms on Q.

The Christoffel symbols of ∇^* with respect to local coordinates $q^1, \dots, q^n$ are denoted Γ^* and given by

$$\nabla^*_{\partial_{q^i}} \mathrm{d}q^j = \sum_k (\Gamma^*)^k_{ij} \mathrm{d}q^k.$$

They can be expressed in terms of the Christoffel symbols Γ of ∇ as follows:

$$\begin{aligned}(\Gamma^*)^k_{ij} &= \left(\nabla^*_{\partial_{q^i}} \mathrm{d}q^j\right)(\partial_{q^k}) \\ &= \partial_{q^i}(\mathrm{d}q^j(\partial_{q^k})) - \mathrm{d}q^j(\nabla_{\partial_{q^i}}\partial_{q^k}) \\ &= -\mathrm{d}q^j(\Gamma^l_{ik}\partial_{q^l}) \\ &= -\Gamma^j_{ik}.\end{aligned}$$

That is we have $\nabla^*_{\partial_{q^i}} \mathrm{d}q^j = -\Gamma^j_{ik}\mathrm{d}q^k$. The symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ implies $(\Gamma^*)^k_{ij} = (\Gamma^*)^i_{kj}$ and therefore

$$\left(\nabla^*_{\partial_{q^i}} \mathrm{d}q^j\right)(\partial_{q^k}) = \left(\nabla^*_{\partial_{q^k}} \mathrm{d}q^j\right)(\partial_{q^i}).$$

Let $p_1, \dots, p_n$ be the local coordinates on T^*Q dual to $q^1, \dots, q^n$, i.e., $(\mathbf{q}, \mathbf{p}) = p_j \mathrm{d}q^j|_q$. In these local coordinates the tangent vectors on T^*Q have the form

$$\dot{\mathbf{q}}\partial_{\mathbf{q}} + \dot{\mathbf{p}}\partial_{\mathbf{p}} = (\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}, \dot{\mathbf{p}}).$$

For $\mathbf{q} \in Q$ we consider the bilinear map $\Gamma_\mathbf{q}$ on $T_\mathbf{q}Q \times T^*_\mathbf{q}Q$ defined by

$$\begin{aligned}\Gamma_\mathbf{q}(\dot{\mathbf{q}}, \mathbf{p}) &= \Gamma_\mathbf{q}(\dot{q}^i \partial_{q^i}, p_j \mathrm{d}q^j) \\ &:= \dot{q}^i p_j (\Gamma^*)^k_{ij}(\mathbf{q}) \mathrm{d}q^k\end{aligned}$$

with image in $T^*_\mathbf{q}Q$ and set

$$K(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}, \dot{\mathbf{p}}) = \big(\mathbf{q}, \dot{\mathbf{p}} + \Gamma_\mathbf{q}(\dot{\mathbf{q}}, \mathbf{p})\big).$$

Observe that the derivative of the 1-form $p_j \mathrm{d}q^j$ (more precise of $\mathbf{q} \mapsto (\mathbf{q}, p_j(q)\mathrm{d}q^j)$) in the direction $\dot{q}^i \partial_{q^i}$ in $\mathbf{q}$ is

$$\begin{aligned}(T_\mathbf{q} p_j \mathrm{d}q^j)(\dot{q}^i \partial_{q^i}) &= \mathrm{d}p_j(\dot{q}^i \partial_{q^i}) \mathrm{d}q^j|_\mathbf{q} \\ &= \big(\mathbf{q}, p_j \mathrm{d}q^j, \dot{q}^i \partial_{q^i}, \mathrm{d}p_j(\dot{q}^i \partial_{q^i}) \partial_{p_j}\big).\end{aligned}$$

Applying K to the last term yields

$$\begin{aligned}K\big((\mathbf{q}, p_j \mathrm{d}q^j, \dot{q}^i \partial_{q^i}, \mathrm{d}p_j(\dot{q}^i \partial_{q^i}) \partial_{p_j})\big) &= (\mathbf{q}, \mathrm{d}p_j(\dot{q}^i \partial_{q^i}) \partial_{p^j} + \Gamma_\mathbf{q}(\dot{q}^i \partial_{q^i}, p_j \mathrm{d}q^j)) \\ &= \dot{q}^i (\partial_{q^i} p_j) \mathrm{d}q^j + p_j \nabla^*_{\dot{q}^i \partial_{q^i}} \mathrm{d}q^j \\ &= \nabla^*_{\dot{q}^i \partial_{q^i}} (p_j \mathrm{d}q^j)\end{aligned}$$

where we used the Leibniz rule for connections. Hence, in local coordinates

$$K \circ T = \nabla^*.$$

Definition A.1. A smooth map $K : TT^*Q \to T^*Q$ is called a **connection map** on $\tau : T^*Q \to Q$, provided that K is given by

$$(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}, \dot{\mathbf{p}}) \mapsto \big(\mathbf{q}, \dot{\mathbf{p}} + \Gamma_\mathbf{q}(\dot{\mathbf{q}}, \mathbf{p})\big)$$

in local coordinates, where $\Gamma_\mathbf{q}$ is a bilinear map.

Given a connection map K, a covariant derivative is defined via $D := K \circ T$, i.e., $D_X \beta = K\big((T\beta)(X)\big)$ is a connection.

Lemma A.2. *The Christoffel symbols Γ^* of ∇^* define a connection map $K : TT^*Q \to T^*Q$ such that $\nabla^* = K \circ T$.*

Idea of Proof. Take the local description of K given above and show that it is independent of the chosen local coordinates. This yields a smooth map K that satisfies the requested property by the above computation. □

We use the connection map K associated to ∇^* to construct the splitting used in Section 7.2. We define distributions on T^*Q by

$$\begin{aligned}\mathcal{H} &:= \ker(K : TT^*Q \to T^*Q),\\ \mathcal{V} &:= \ker(T\tau : TT^*Q \to TQ).\end{aligned}$$

Locally we have

$$\begin{aligned}K(\mathbf{q},\mathbf{p},\dot{\mathbf{q}},\dot{\mathbf{p}}) &= (\mathbf{q},\dot{\mathbf{p}} + \Gamma_{\mathbf{q}}(\dot{\mathbf{q}},\mathbf{p})),\\ (T\tau)(\mathbf{q},\mathbf{p},\dot{\mathbf{q}},\dot{\mathbf{p}}) &= (\mathbf{q},\dot{\mathbf{q}})\end{aligned}$$

so that in local coordinates

$$\begin{aligned}\mathcal{H}_{(\mathbf{q},\mathbf{p})} &= \{(\mathbf{q},\mathbf{p},\dot{\mathbf{q}},-\Gamma_{\mathbf{q}}(\dot{\mathbf{q}},\mathbf{p})) \mid \dot{\mathbf{q}} \in \mathbb{R}^n\},\\ \mathcal{V}_{(\mathbf{q},\mathbf{p})} &= \{(\mathbf{q},\mathbf{p},0,\dot{\mathbf{p}}) \mid \dot{\mathbf{p}} \in \mathbb{R}^n\}.\end{aligned}$$

We obtain a splitting

$$TT^*Q = \mathcal{H} \oplus \mathcal{V}$$

into the **horizontal** and the **vertical distribution**. Observe that for $u \in T^*Q$

$$\mathcal{V}_u = \ker T_u\tau = T_u T^*_{\tau(u)}Q = T^*_{\tau(u)}Q$$

can be identified with the fibre $\tau^{-1}(\tau(u))$ canonically.

The natural symplectic form $\omega = \mathrm{d}\lambda$ on T^*Q is defined as the exterior derivative of the Liouville form λ that is given as

$$\lambda_u = u \circ T_u\tau.$$

This turns $\mathcal{V} \leqslant TT^*Q$ into a Lagrangian distribution. Indeed, let $v, w \in \mathcal{V}_u$ be tangent vectors and X, Y vector fields on T^*Q that are constant along the fibres of τ, everywhere tangent to $\mathcal{V}$ and with $X_u = v, Y_u = w$. Then

$$\begin{aligned}\omega_u(v,w) = \mathrm{d}\lambda(X,Y)_u &= X\lambda(Y)_u - Y\lambda(X)_u - \lambda([X,Y])_u\\ &= 0,\end{aligned}$$

because $\lambda(X) = 0 = \lambda(Y)$ and by integrability of $\mathcal{V}$ (obtained using Frobenius' theorem), $[X,Y]$ is tangent to $\mathcal{V}$, so that $\lambda([X,Y]) = 0$. Note that we used [34, Proposition 14.29] in the first line. This can also be obtained using local coordinates where $\lambda_{(\mathbf{q},\mathbf{p})} = \mathbf{p}\mathrm{d}\mathbf{q}$ and $\omega_{(\mathbf{q},\mathbf{p})} = \mathrm{d}\mathbf{p} \wedge \mathrm{d}\mathbf{q}$.

By choosing a torsion free connection ∇ the horizontal distribution $\mathcal{H} \leqslant TT^*Q$ becomes Lagrangian, too. In local coordinates we write

$$\mathcal{H}_{(\mathbf{q},p_j\mathrm{d}q^j)} = \operatorname{span}\{\partial_{q^i} - \gamma_i^k\partial_{p^k} \mid i = 1,\ldots,n\},$$

where $\gamma_i^k := p_j(\Gamma^*)_{ij}^k$. Applying the natural symplectic form yields

$$\begin{aligned} \mathrm{d}\mathbf{p} \wedge \mathrm{d}\mathbf{q}(\partial_{q^i} - \gamma_i^k \partial_{p^k}, \partial_{q^s} - \gamma_s^t \partial_{p^t}) &= -\gamma_i^k \delta_{ks} + \gamma_s^t \delta_{it} \\ &= \gamma_s^i - \gamma_i^s \\ &= p_j(\Gamma^*)_{sj}^i - p_j(\Gamma^*)_{ij}^s \\ &= p_j((\Gamma^*)_{sj}^i - (\Gamma^*)_{ij}^s) \\ &= 0, \end{aligned}$$

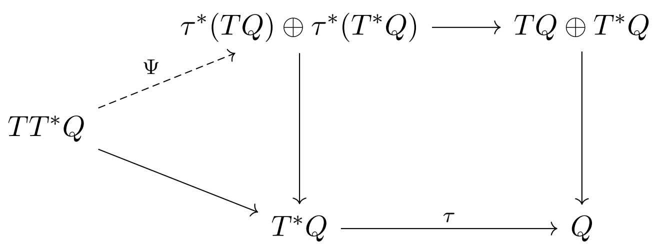
since $(\Gamma^*)_{sj}^k = (\Gamma^*)_{kj}^s$ by the symmetry we observed at the beginning of this chapter. Therefore $\mathcal{H}$ is Lagrangian in the torsion free case. Similarly we evaluate the natural symplectic form with respect to the splitting $\mathcal{H} \oplus \mathcal{V}$ in local coordinates. Both distributions are Lagrangian, hence

$$\begin{aligned} &\mathrm{d}\mathbf{p} \wedge \mathrm{d}\mathbf{q}(v^j(\partial_{q^j} - \gamma_j^k \partial_{p^k}) \oplus a^i \partial_{p^i}, w^m(\partial_{q^m} - \gamma_m^s \partial_{p^s}) \oplus b^l \partial_{p^l}) \\ &= \mathrm{d}\mathbf{p} \wedge \mathrm{d}\mathbf{q}(v^j(\partial_{q^j} - \gamma_j^k \partial_{p^k}), b^l \partial_{p^l}) + \mathrm{d}\mathbf{p} \wedge \mathrm{d}\mathbf{q}(a^i \partial_{p^i}, w^m(\partial_{q^m} - \gamma_m^s \partial_{p^s})) \\ &= -b^l v^j \delta_{jl} + a^i w^m \delta_{im} \\ &= a^i w^i - b^l v^l \\ &= a^i \mathrm{d}q^i(w^m \partial_{q^m}) - b^l \mathrm{d}q^l(v^j \partial_{q^j}). \end{aligned}$$

We use local coordinates to express $v \oplus a \in \mathcal{H} \oplus \mathcal{V}$ as $v = v^j(\partial_{q^j} - \gamma_j^k \partial_{p^k})$, $a = a^i \partial_{p^i}$ and define α by $\alpha := \iota_a \omega = a^i \mathrm{d}q^i$ and similar for $w \oplus b$ and β. Then the formula for ω reads

$$\omega(v \oplus a, w \oplus b) = \alpha(T\tau(w)) - \beta(T\tau(v)).$$

We end this discussion by relating the splitting to the Whitney sum of TQ and T^*Q pulled back along τ, see [7, Definitions 3.9 and 4.1] for basic information about induced bundles and the Whitney sum.



Where the bundle (iso-)morphism Ψ is defined as

$$\begin{aligned} \Psi : TT^*Q = \mathcal{H} \oplus \mathcal{V} &\longrightarrow \tau^*(TQ) \oplus \tau^*(T^*Q) \\ (v, a) &\longmapsto (T\tau(v), \alpha), \end{aligned}$$

using the splitting into the horizontal and vertical distribution. Here $v \in \mathcal{H}_u$ is mapped to $T_u\tau(v) \in T_{\tau(u)}Q = (\tau^*(TQ))_u$ and $\alpha = \iota_a\omega$ under the identification

$$a \in \mathcal{V}_u \cong T_u T^*_{\tau(u)}Q \cong T^*_{\tau(u)}Q = \big(\tau^*(T^*Q)\big)_u \ni \alpha.$$

APPENDIX B

Classical Hamiltonians and Magnetic Fields

In this chapter we explain the calculations in the proof of Theorem 7.15. Let us recall the setting in Section 7.4. We consider an n-dimensional Riemannian manifold (Q, h) and its cotangent bundle $\tau : T^*Q \to Q$. The cotangent bundle carries the canonical 1-form λ. We consider the metric m on T^*Q and the splitting $TT^*Q = \mathcal{H} \oplus \mathcal{V}$, both introduced in Section 7.2 and explained in Appendix A. Recall that

$$m(v \oplus a, w \oplus b) = h\big(T\tau(v), T\tau(w)\big) + h^\flat(a, b)$$

with respect to the splitting $\mathcal{H} \oplus \mathcal{V}$, where $h^\flat$ denotes the dual metric of h. The covering $\mu : \widetilde{Q} \to Q$ of Q induces a covering of T^*Q which we denote by

$$T^*\mu : T^*\widetilde{Q} \longrightarrow T^*Q,$$

where $\widetilde{\tau} : T^*\widetilde{Q} \to \widetilde{Q}$ denotes the cotangent bundle of $\widetilde{Q}$. The canonical 1-form on $T^*\widetilde{Q}$ is denoted by $\widetilde{\lambda}$ and satisfies $\widetilde{\lambda} = (T^*\mu)^*\lambda$. Further, consider the Hamiltonian function

$$\begin{aligned} H : T^*Q &\longrightarrow \mathbb{R} \\ u &\longmapsto \tfrac{1}{2}\|u\|^2_{h^\flat} + V\big(\tau(u)\big), \end{aligned}$$

where V is a smooth function on Q. The lift of H to $T^*\widetilde{Q}$ is denoted by $\widetilde{H}$ and its Hamiltonian vector field $X_{\widetilde{H}}$ is given by

$$\iota_{X_{\widetilde{H}}}(\mathrm{d}\lambda + \widetilde{\tau}^*\widetilde{\sigma}) = -\mathrm{d}\widetilde{H},$$

where $\widetilde{\sigma} = \tau^*\sigma$ is the lift of a closed 2-form σ on Q. Denote the gradient $\mathrm{grad}_m(\tau^*V)$ of τ^*V with respect to m by X, then X is given by

$$m(X, \cdot) = \mathrm{d}(\tau^*V).$$

Note that X is a horizontal vector field, because for all $U \in \mathcal{V} = \ker(T\tau)$ we have

$$0 = \mathrm{d}(\tau^*V)(U) = m(X, U).$$

So X is orthogonal to $\mathcal{V}$ and we have

$$\mathrm{d}(\tau^* V) = m(X, \cdot) = (\tau^* h)(X, \cdot),$$

which yields to

$$T_u \tau(X_u) = (\mathrm{grad}_h V)_{\tau(u)}.$$

We define a function F by

$$\begin{aligned} F : T^*Q &\longrightarrow \mathbb{R} \\ u &\longmapsto \lambda_u(X_u). \end{aligned}$$

With the coordinate free description $\lambda_u = u \circ T\tau$ we obtain

$$\begin{aligned} F(u) &= u \circ T_u \tau(X_u) \\ &= u\big((\mathrm{grad}_h V)_{\tau(u)}\big) \\ &= h^\flat(u, \mathrm{d}V). \end{aligned}$$

Finally, let ϑ be a 1-form on $\widetilde{Q}$ with $\mathrm{d}\vartheta = \widetilde{\sigma}$.

Lemma B.1. *It holds that*

$$(\widetilde{\lambda} + \widetilde{\tau}^* \vartheta - \varepsilon \mathrm{d}\widetilde{F})(X_{\widetilde{H}})(\widetilde{u})$$

equals the sum of

$$\|\widetilde{u}\|^2_{(\widetilde{h})^\flat} + (\widetilde{h})^\flat(\widetilde{u}, \vartheta)$$

and

$$\varepsilon \cdot \big((-\mathrm{Hess}_{\tilde{h}} \widetilde{V})(\widetilde{u}^\#, \widetilde{u}^\#) + \|\,\mathrm{grad}_{\tilde{h}}\, \widetilde{V}\|^2_{\tilde{h}} + (\tau^* \mu^* \sigma)(\widetilde{u}^\#, \mathrm{grad}_{\tilde{h}}\, \widetilde{V})\big)$$

for all $\widetilde{u} \in T^*\widetilde{Q}$.

Proof. In local coordinates $(\mathbf{q}, \mathbf{p})$ the Hamiltonian vector field $X_{\widetilde{H}}$ has the form

$$\begin{aligned} (X_{\widetilde{H}})_{\mathbf{q},\mathbf{p}} &= \frac{\partial \widetilde{H}}{\partial p_i} \partial_{q_i} - \frac{\partial \widetilde{H}}{\partial q_i} \partial_{p_i} - \frac{\partial \widetilde{H}}{\partial p_j} \widetilde{\sigma}_{ji} \partial_{p_i} \\ &= (h^{ij}(\mathbf{q}) p_j) \partial_{q_i} - \frac{\partial \widetilde{H}}{\partial q_i} \partial_{p_i} - \frac{\partial \widetilde{H}}{\partial p_j} \widetilde{\sigma}_{ji} \partial_{p_i}. \end{aligned}$$

Hence,

$$T_{(\mathbf{q},\mathbf{p})} \tau(X_{\widetilde{H}}) = h^{ij}(\mathbf{q}) p_j \partial_{q_i}$$

and therefore

$$\widetilde{\lambda}(X_{\widetilde{X}}) = p_i h^{ij} p_j = \|\mathbf{p}\|_h^2 = \|\widetilde{u}\|^2_{(\widetilde{h})^\flat}.$$

We write $\vartheta = \vartheta_i \mathrm{d}q_i$ and obtain

$$\begin{aligned}\tau^*\vartheta(X_{\widetilde{H}}(\mathbf{q},\mathbf{p})) &= \vartheta_{\mathbf{q}}(T\tau(X_{\widetilde{H}}))\\ &= (\widetilde{h})^\flat(\widetilde{u},\vartheta).\end{aligned}$$

Adding these we obtain the first summand. Note that using the Cauchy-Schwarz inequality as in [48, Proposition 2.4.1] we obtain

$$(\widetilde{\lambda} + \widetilde{\tau}^*\vartheta)(X_{\widetilde{H}})(\widetilde{u}) \geq \|\widetilde{u}\|_{(\widetilde{h})^\flat}\left(\|\widetilde{u}\|_{(\widetilde{h})^\flat} - \|\vartheta\|_{(\widetilde{h})^\flat}\right). \tag{B.1}$$

For the last term we consider $\mathrm{d}\widetilde{F}(X_{\widetilde{H}})$. Observe that

$$T(T^*\mu)(X_{\widetilde{H}}) = X_H$$

and since $\widetilde{F} = F \circ T^*\mu = (T^*\mu)^*F$ it holds that

$$\mathrm{d}\widetilde{F} = (T\mu)^*\mathrm{d}F.$$

This implies

$$\mathrm{d}\widetilde{F}(X_{\widetilde{H}}) \circ T^*\mu = \mathrm{d}F(X_H),$$

so it suffices to compute $\mathrm{d}F(X_H) = X_H(F)$. As above we have $H = \frac{1}{2}h^{ij}p_ip_j + V$ and therefore

$$X_H = h^{ij}p_i\partial_{q^j} - \tfrac{1}{2}(h^{kl})_j p_k p_l \partial_{p_j} - V_{q^j}\partial_{p_j} - h^{il}p_l\sigma_{ij}\partial_{p_j}.$$

Combining this with $F(u) = h^\flat(u, \mathrm{d}V) = h^{st}p_sV_{q^t}$ we can make the computation in local coordinates.

$$\begin{aligned}X_HF = {} & h^{ij}p_i\partial_{q^j}(h^{st}p_sV_{q^t})\\ & - \tfrac{1}{2}(h^{kl})_j p_k p_l \partial_{p_j}(h^{st}p_sV_{q^t})\\ & - V_{q^j}\partial_{p_j}(h^{st}p_sV_{q^t})\\ & - h^{il}p_l\sigma_{ij}\partial_{p_j}(h^{st}p_sV_{q^t})\\ = {} & h^{ij}p_i\big((h^{st})_j p_s V_{q^t} + h^{st}p_sV_{q^jq^t}\big)\\ & - \tfrac{1}{2}(h^{kl})_j p_k p_l h^{jt}V_{q^t}\\ & - V_{q^j}h^{jt}V_{q^t}\\ & - h^{il}p_l\sigma_{ij}h^{jt}V_{q^t}.\end{aligned}$$

For a better overview note that

$$V_{q^j} h^{jt} V_{q^t} = h^\flat(\mathrm{d}V, \mathrm{d}V) = \|\mathrm{grad}_h V\|_h^2. \tag{B.2}$$

The dual vector field $u^\#$ of u is defined by

$$u = h(u^\#, \cdot).$$

Using local coordinates $u = p_j \mathrm{d}q^j$ and $u^\# = v^i \partial_{q^i}$ we see that

$$p_j \mathrm{d}q^j = h_{ij} v^i \mathrm{d}q^j$$

and hence $p_j = h_{ij} v^i$. Using this we get

$$h^{kl} p_l = h^{kl} h_{il} v^i = v^k.$$

This allows us to rewrite the last line in our calculation as

$$h^{il} p_l \sigma_{ij} h^{jt} V_{q^t} = v^i \sigma_{ij} (\mathrm{grad}_h(V))^j = \sigma(u^\#, \mathrm{grad}_h V). \tag{B.3}$$

It remains to take care of the term

$$\begin{aligned} &h^{ij} p_i \big((h^{st})_j p_s V_{q^t} + h^{st} p_s V_{q^j q^t}\big) - \tfrac{1}{2} (h^{kl})_j p_k p_l h^{jt} V_{q^t} \\ &= v^j v^t V_{q^j q^t} + \big(v^j (h^{st})_j h_{ks} v^k - \tfrac{1}{2} (h^{kl})_j h_{mk} v^m h_{sl} v^s h^{jt}\big) V_{q^t}, \end{aligned}$$

where we used $p_j = h_{ij} v^i$ and $v^j = h^{ij} p_i$. By the Leibniz rule, it holds that

$$(h^{st})_j h_{ks} = \partial_{q^j}(h^{st} h_{ks}) - h^{st} (h_{ks})_j = \partial_{q^j}(\delta^t_k) - h^{st}(h_{ks})_j = -h^{st}(h_{ks})_j.$$

Using this in the computation we have

$$\begin{aligned} &v^j v^t V_{q^j q^t} + \big(v^j (h^{st})_j h_{ks} v^k - \tfrac{1}{2} (h^{kl})_j h_{mk} v^m h_{sl} v^s h^{jt}\big) V_{q^t} \\ &= v^j v^t V_{q^j q^t} + \big(- v^j v^k h^{st} (h_{ks})_j + \tfrac{1}{2} v^m v^s h^{kl} (h_{mk})_j h_{sl} h^{jt}\big) V_{q^t}. \end{aligned}$$

Splitting $-v^j v^k h^{st}(h_{ks})_j = -\frac{1}{2} v^j v^k h^{st} (h_{ks})_j - \frac{1}{2} v^j v^k h^{st} (h_{ks})_j$ and renaming some indices we get

$$\begin{aligned} &v^j v^t V_{q^j q^t} + \big(- v^j v^k h^{st} (h_{ks})_j + \tfrac{1}{2} v^m v^s h^{kl} (h_{mk})_j h_{sl} h^{jt}\big) V_{q^t} \\ &= v^j v^t V_{q^j q^t} + \big(- \tfrac{1}{2} v^j v^k h^{st} (h_{ks})_j - \tfrac{1}{2} v^k v^j h^{st} (h_{js})_k + \tfrac{1}{2} v^j v^k (h_{jk})_s h^{st}\big) V_{q^t} \\ &= v^j v^t V_{q^j q^t} - v^j v^k \cdot \Big(\tfrac{1}{2} h^{st} \left((h_{ks})_j + (h_{js})_k - (h_{jk})_s \right) \Big) V_{q^t} \\ &= v^j v^t V_{q^j q^t} - v^j v^k \Gamma^t_{jk} V_{q^t} \\ &= v^i v^j V_{q^i q^j} - v^i v^j \Gamma^t_{ij} V_{q^t}. \end{aligned}$$

Using the formula for the covariant derivative of 1-forms in Section 5.2 we get that

$$\begin{aligned} v^i v^j V_{q^i q^j} - v^i v^j \Gamma^t_{ij} V_{q^t} &= v^i v^j (\nabla \mathrm{d}V)_{ij} \\ &= \nabla \mathrm{d}V(u^\#, u^\#). \end{aligned} \tag{B.4}$$

Adding the partial results (B.1) to (B.4) yields the claimed equality. □

In the following we will assume that the primitive ϑ of $\mu^*\sigma$ is bounded and that V is a Morse function. Further we require the following properties for ϑ, σ and V.

1. The function V has a unique maximum and the maximum value is positive.
2. For all critical points other than the maximum the critical values are less than $-\frac{1}{2}t_0^2$, where

$$t_0 := \sup_{\widetilde{Q}} \|\vartheta\|_{(\widetilde{h})^\flat}.$$

3. Let c_{n-1} be the largest critical value other than the maximum and choose a regular value $v_0 > \frac{1}{2}t_0^2$ of V such that $-v_0 \in (c_{n-1}, -\frac{1}{2}t_0^2)$. We require that σ vanishes on the disc

$$D^n_{-v_0} = \{V \geq -v_0\} \subset Q$$

and ϑ vanishes on

$$\{\widetilde{V} \geq -v_0\} \subset \widetilde{Q}.$$

Lemma B.2. *The hypersurface*

$$M' = \{\widetilde{H} = 0\} = \left\{\tfrac{1}{2}\|u\|^2_{(\widetilde{h})^\flat} = -\widetilde{V}\right\}$$

is of contact type.

Proof. As explained in the proof of Theorem 7.15 it suffices to show that the function

$$(\widetilde{\lambda} + \widetilde{\tau}^*\vartheta - \varepsilon \mathrm{d}\widetilde{F})(X_{\widetilde{H}})$$

is uniformly positive on M' for a sufficiently small $\varepsilon > 0$. More precise we will show that there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a $\delta > 0$ with

$$(\widetilde{\lambda} + \widetilde{\tau}^*\vartheta - \varepsilon \mathrm{d}\widetilde{F})(X_{\widetilde{H}})(\widetilde{u}) \geq \delta$$

for all $\widetilde{u} \in M'$. In view of Lemma B.1 we distinguish the cases $\widetilde{u} \in \{\widetilde{V} \leq -v_0\}$ and $\{\widetilde{u} \in \widetilde{V} > -v_0\}$.

Case 1: Since we consider the set $M' = \{\widetilde{H} = 0\}$ it holds that $\widetilde{u}$ is contained in $\{\widetilde{V} \le -v_0\}$ if and only if $\frac{1}{2}\|\widetilde{u}\|^2 \ge v_0$. We introduce the constants

$$\begin{aligned} V_0 &:= \max_{\tilde{Q}} \|\mathrm{Hess}_{\tilde{h}}\widetilde{V}\|_{\tilde{h}} = \max_{Q} \|\mathrm{Hess}_h V\|_h \\ g_0 &:= \max_{\tilde{Q}} \|\mathrm{grad}_{\tilde{h}}\widetilde{V}\|_{\tilde{h}} = \max_{Q} \|\mathrm{grad}_h V\|_h \\ s_0 &:= \max_{\tilde{Q}} \|\tilde{\sigma}\|_{\tilde{h}} = \max_{Q} \|\sigma\|_h. \end{aligned}$$

Using the formula in Lemma B.1 we obtain

$$\begin{aligned} &(\widetilde{\lambda} + \widetilde{\tau}^*\vartheta - \varepsilon \mathrm{d}\widetilde{F})(X_{\widetilde{H}})(\widetilde{u}) \\ \ge\ & \|\widetilde{u}\|^2_{(\tilde{h})^\flat} - t_0\|\widetilde{u}\|_{(\tilde{h})^\flat} - \varepsilon H_0\|\widetilde{u}\| - \varepsilon s_0 g_0\|\widetilde{u}\| \\ =\ & \|\widetilde{u}\|_{(\tilde{h})^\flat}\left(\|\widetilde{u}\|_{(\tilde{h})^\flat}(1-\varepsilon H_0) - t_0 - \varepsilon s_0 g_0\right). \end{aligned}$$

Choosing ε_0' sufficiently small we can achieve that $1 - \varepsilon H_0 > 0$ for all $\varepsilon \in (0, \varepsilon_0')$ and estimating $\|\widetilde{u}\|_{(\tilde{h})^\flat}$ by $\sqrt{2v_0}$ from below we get

$$\begin{aligned} &\|\widetilde{u}\|_{(\tilde{h})^\flat}\left(\|\widetilde{u}\|_{(\tilde{h})^\flat}(1-\varepsilon H_0) - t_0 - \varepsilon s_0 g_0\right) \\ \ge\ & \sqrt{2v_0}\left(\sqrt{2v_0}(1-\varepsilon H_0) - t_0 - \varepsilon s_0 g_0\right) \end{aligned}$$

Since $\sqrt{2v_0} > t_0$ we can ensure $\sqrt{2v_0}(1-\varepsilon H_0) - t_0 > 0$, possibly after choosing a smaller constant ε_0', e.g., $\varepsilon H_0 < 1 - \frac{t_0}{\sqrt{2v_0}}$. Taking an $\varepsilon_1 > 0$ such that

$$\sqrt{2v_0}(1-\varepsilon_1 H_0) - t_0 > 0$$

we find an $\hat{\varepsilon}_0$ such that

$$\varepsilon s_0 g_0 < \tfrac{1}{2}(\sqrt{2v_0}(1-\varepsilon_1 H_0) - t_0)$$

for all $\varepsilon \in (0, \hat{\varepsilon}_0)$. For $\varepsilon_0 = \min\{\varepsilon_0', \hat{\varepsilon}\}$ we end up with

$$(\widetilde{\lambda} + \widetilde{\tau}^*\vartheta - \varepsilon \mathrm{d}\widetilde{F})(X_{\widetilde{H}})(\widetilde{u}) \ge \tfrac{\sqrt{2v_0}}{2}\left(\sqrt{2v_0}(1-\varepsilon_1 H_0) - t_0\right) =: \delta.$$

Case 2: For $\widetilde{u} \in \{\widetilde{V} > -v_0\}$ we have $\|\widetilde{u}\|^2 < v_0$ and ϑ and σ vanish by assumption. Using the formula in Lemma B.1 we obtain

$$\begin{aligned} &(\widetilde{\lambda} + \widetilde{\tau}^*\vartheta - \varepsilon \mathrm{d}\widetilde{F})(X_{\widetilde{H}})(\widetilde{u}) \\ =\ & \|\widetilde{u}\|^2_{(\tilde{h})^\flat} - \varepsilon(\mathrm{Hess}\widetilde{V})(\widetilde{u}^\#, \widetilde{u}^\#) + \varepsilon\|\mathrm{grad}\widetilde{V}\|^2 \\ \ge\ & \|\widetilde{u}\|^2_{(\tilde{h})^\flat}(1-\varepsilon H_0) + \varepsilon\|\mathrm{grad}\widetilde{V}\|^2 \end{aligned}$$

As in Case 1 we find an ε_0 such that $1 - \varepsilon H_0$ is positive. Therefore the first term is uniformly positive on $\{|\widetilde{u}| > u_0\}$ for all $u_0 \in (0, v_0)$. Since $\{0 \geq \widetilde{V} > -v_0\}$ does not contain any critical points of $\widetilde{V}$, the gradient does not vanish and

$$\delta_0 = \min_{\{-v_0 < \widetilde{V} \leq 0\}} |\operatorname{grad} \widetilde{V}|^2 = \min_{\{-v_0 < V \leq 0\}} |\operatorname{grad} V|^2 > 0$$

and hence,

$$(\widetilde{\lambda} + \widetilde{\tau}^* \vartheta - \varepsilon \mathrm{d}\widetilde{F})(X_{\widetilde{H}})(\widetilde{u}) \geq \varepsilon \delta_0 =: \delta.$$

Always choosing the minimum between Case 1 and 2 for ε_0 and δ ends the proof of the lemma. □

Remark B.3. For $t \in [0, 1]$ the family

$$\alpha_t = (\widetilde{\lambda} + t\widetilde{\tau}^* \vartheta - \varepsilon \mathrm{d}\widetilde{F})|_{TM'}$$

of 1-forms on M' is a family of contact forms. Indeed for fixed t the proofs of Lemma B.1 and Lemma B.2 go through with the obvious changes for the new 1-form and we can use the same constants.

APPENDIX C

List of Terms

M	(closed) manifold
M'	manifold, usually non-compact
$\pi : M' \to M$	covering of M by M' via the covering map π
φ	deck transformation of π
g	Riemannian metric
ω	closed (maximally) non-degenerate 2-form
g', ω'	lift of the corresponding structures to a cover via π
α	contact form on M'
ξ	contact structure, kernel of α
$(\pi : M' \to M, \omega, \alpha, g)$	virtually contact structure, see Definition 2.10
α_0	contact form obtained as the limit of $\varphi_\nu^*\alpha$, for a sequence of deck transformations φ_ν, up to subsequence
α_∞	contact form obtained as the limit of $\varphi_\nu^*\alpha_0$, for a sequence of deck transformations φ_ν, up to subsequence
R, R_0, R_∞	Reeb vector field of α, α_0 and α_∞, respectively
Φ	skew adjoint bundle isomorphism $\xi \to \xi$ with $\mathrm{d}\alpha = g'(\Phi\cdot, \cdot)$, see Section 3.1
j	complex structure on ξ, $j = \Phi \circ (\sqrt{-\Phi^2})^{-1}$, see Section 3.1
J	almost complex structure on $\mathbb{R} \times M'$, translation invariant, restricts to j on ξ, maps ∂_t to R, Section 4.1
$\mathcal{T}$	the set of all smooth, strictly increasing functions $\tau : (-\infty, 0] \to [0, 1]$ with $\tau(0) = 1$
$\mathrm{d}(\tau\alpha)$	symplectic form on $(-\infty, 0] \times M'$
g_j	metric on ξ, $g_j = \mathrm{d}\alpha(\cdot, j\cdot)$
g_α	metric on M', $g_\alpha = \alpha \otimes \alpha + g_j$
g_τ	metric on $(-\infty, 0] \times M'$, $g_\tau = \mathrm{d}(\tau\alpha)(\cdot, J\cdot)$, Section 4.3
g_0	metric on $\mathbb{R} \times M'$, $g_0 = \mathrm{d}t \otimes \mathrm{d}t + g_\alpha$, Section 4.4
g_0'	metric on $\mathbb{R} \times M'$, $g_0' = \mathrm{d}t \otimes \mathrm{d}t + g'$, Chapter 6
$u = (a, f), v = (b, h)$	holomorphic maps with image in $\mathbb{R} \times M'$

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Selbständigkeitserklärung

Hiermit versichere ich, die vorgelegte Thesis selbständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt zu haben, die ich in der Thesis angegeben habe. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben, die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht. Bei den von mir durchgeführten und in der Thesis erwähnten Untersuchungen habe ich die Grundsätze guter wissenschaftlicher Praxis, wie sie in der "Satzung der Justus-Liebig-Universität zur Sicherung guter wissenschaftlicher Praxis" niedergelegt sind, eingehalten. Gemäß § 25 Abs. 6 der Allgemeinen Bestimmungen für modularisierte Studiengänge dulde ich eine Überprüfung der Thesis mittels Anti-Plagiatssoftware.

Gießen, den 11. April 2019

Kevin Emanuel Wiegand