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DISSERTATION

The prescribed Mean Curvature Problem on four dimensional manifolds with boundary

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Abstract

In this thesis we study the prescribed Mean Curvature Problem for Riemannian manifolds with boundary. Given a compact four-dimensional Riemannian manifold with boundary (M, g), the prescribed Mean Curvature Problem asks for conditions on $K : \partial M \to \mathbb{R}$, such that K can be realized as the mean curvature $h_{\tilde{g}}$ of a conformal metric $\tilde{g} \in [g]$ with vanishing scalar curvature $R_{\tilde{g}}$ in M. The prescribed Mean Curvature Problem is equivalent to the existence of a solution to the following non-linear boundary value problem:

$$\begin{cases} -\Delta_g u + \frac{1}{6} R_g u = 0 & \text{in } M \\ \partial_\nu u + h_g u = K(x) u^2 & \text{on } \partial M \\ u > 0. \end{cases}$$

These solutions are in one-to-one correspondence to critical points of a functional, defined on a Sobolev-space. Since this functional does not satisfy the Palais-Smale condition, standard variational methods can not be applied.
We use the method of **critical points at infinity**, developed by Abbas Bahri, to study non-converging flow lines of a suitable pseudo gradient vector field. We understand "limit sets" of these flow lines and understand the difference of topology in the variational space, induced by the non-converging flow lines. Comparing this difference of topology to the topology of the variational space yields existence results for critical points of the given functional. And therefore conditions on K such that K can be realized as the mean curvature of a conformal metric.

Contents

1. Introduction

1.1. The prescribed Mean Curvature Problem

A very famous problem, which has been solved during the last decades, is the **Yamabe Problem**. To introduce the Yamabe Problem let (M, g) be a compact Riemannian manifold of dimension greater than or equal to 3. The Yamabe Problem asks for the existence of a metric $\bar{g}$ in the conformal class of g with constant scalar curvature. Combining [35, 34, 6, 31] the authors H. Yamabe, N. Trudinger, T. Aubin and R. Schoen were able to prove the existence of a conformal metric with constant scalar curvature for any compact Riemannian manifold of dimension greater than or equal to 3. A very comprehensive survey about the Yamabe Problem was written by J. Lee and T. Parker (see [24]).

A variation of the Yamabe Problem to manifolds with boundary is given as follows: Let (M, g) be a compact Riemannian manifold with boundary ∂M of dimension greater than or equal to 3. Find a metric $\bar{g}$ conformal to g with zero scalar curvature in M and constant mean curvature on ∂M. This problem was first introduced by Escobar in [19].

As a generalization we will now introduce the **prescribed Mean Curvature Problem**: Let (M, g) be a Riemannian manifold with boundary of dimension n greater than or equal to 3 and $K : \partial M \to \mathbb{R}$ a smooth function. Does there exist a metric $\bar{g}$ conformal to g with zero scalar curvature in M and mean curvature precisely given by K on ∂M? This problem was first introduced by Cherrier in [16].
The prescribed Mean Curvature Problem is equivalent to a non-linear boundary value problem on M. To be more precise let $\bar{g} = u^{\frac{4}{n-2}} g$ be a conformal metric to g and u a positive, smooth function on M. Then (see [16]) the metric $\bar{g}$ has zero scalar curvature in M and mean curvature given by K iff u solves the boundary value problem

$$(PMCP) \quad \begin{cases} -\Delta_g u + \frac{n-2}{4(n-1)} R_g u = 0 & \text{in } M \\ \partial_\nu u + \frac{n-2}{2} h_g u = \frac{n-2}{2} K(x) u^{\frac{n}{n-2}} & \text{on } \partial M \\ u > 0. \end{cases} \tag{1.1}$$

Here ν is the outward normal vector field on ∂M, R_g the scalar curvature in M and h_g the mean curvature on ∂M.
Let us first look into the case where K is constant. The boundary value problem then

has variational structure. To be more precise, let $J : \Sigma^+ \cap U \to \mathbb{R}$, where

$$J(u) = \frac{\int_M \left(|\nabla u|_g^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) dV_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 d\sigma_g}{\left(\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_g \right)^{\frac{n-2}{n-1}}},$$

$\Sigma^+ := \{u \in H^1(M) \ : \ ||u||_{H^1(M)} = 1, \quad u \geq 0 \text{ a.e.}\}$ and

$$U := \left\{ u \in H^1(M) \ : \ \int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_g > 0 \right\}.$$

Here $H^1(M)$ is the Sobolev-space of functions on M such that one weak derivative exists.

Due to a regularity result by Cherrier (see [16]) critical points u of J correspond to smooth, positive solutions of (1.1) with $K = \frac{2}{n-2} l(u)$, where

$$l(u) = \frac{\int_M \left(|\nabla u|_g^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) dV_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 d\sigma_g}{\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_g}.$$

Unfortunately the functional J does not satisfy the Palais-Smale condition. The Sobolev trace embedding $H^1(M) \hookrightarrow L^{\frac{2(n-1)}{n-2}}(\partial M)$ is critical and hence not compact. This lack of compactness makes the variational theory complicated and standard methods can not be applied. Nonetheless in [19, 20] Escobar was able to show that

$$Q(M, \partial M, [g]) := \inf \left\{ J(u) \ : \ u \in \Sigma^+ \cap U \right\}. \tag{1.2}$$

is achieved provided

$$-\infty < Q(M, \partial M, [g]) < Q(B^n, \partial B^n, [g_{eucl}]). \tag{1.3}$$

Thus the Yamabe Problem on manifolds with boundary is solved if (1.3) holds. Here B^n is the unit ball in $\mathbb{R}^n$ and g_{eucl} the Euclidean metric. Furthermore Escobar (see [18]) proved that

$$Q(B^n, \partial B^n, [g_{eucl}]) = \inf \left\{ \frac{\int_{\mathbb{R}^n_+} |\nabla u|^2}{\left(\int_{\partial \mathbb{R}^n_+} |u|^{\frac{2(n-1)}{n-2}} \right)^{\frac{n-2}{n-1}}} \ \middle| \ u \in C_0^\infty \left(\overline{\mathbb{R}^n_+} \right); \quad u_{|\partial \mathbb{R}^n_+} \neq 0 \right\}$$

is the sharp constant in the Sobolev trace embedding. Furthermore the minimum is achieved by a function u iff u belongs to the following family of functions:

$$\alpha \left(\frac{\lambda}{(1 + \lambda t)^2 + \lambda^2 |x - a|^2} \right)^{\frac{n-2}{2}} \quad ; \quad \lambda > 0, \ a \in \partial \mathbb{R}^n_+, \ \alpha \neq 0, \quad (x, t) \in \mathbb{R}^{n-1} \times \mathbb{R}_+ = \mathbb{R}^n_+. \tag{1.4}$$

Using these functions Escobar (see [19]) showed that the inequality

$$Q(M, \partial M, [g]) \leq Q(B^n, \partial B^n, [g_{eucl}])$$

holds true for every compact manifold with boundary. Therefore the solution of the Yamabe Problem on manifolds with boundary was reduced to proving the strict inequality (1.3). Through the works of Escobar [19, 21], Marques [28, 27], Almaraz [2], Chen [15] as well as Mayer and Ndiaye [30], the Yamabe Problem on manifolds with boundary is completely solved and we have the following Theorem due to the previous authors.

Theorem. *Let (M, g) be a compact Riemannian manifold with boundary of dimension greater than or equal to 3. If $Q(M, \partial M, [g]) > -\infty$, then there exists a conformal metric with zero scalar curvature in M and constant mean curvature on ∂M. Moreover the constant can be chosen to be $sign\,(Q(M, \partial M, [g]))$, where $sign(0) := 0$.*

It is worth to mention that the authors in [19, 21, 28, 27, 2, 15] use some appropriate test functions "close" to (1.4) to prove (1.3), whereas the authors in [30] use an algebraic topological argument, developed by Bahri and Coron [9].

From now on we turn back to the prescribed Mean Curvature Problem. First we consider the case $-\infty < Q(M, \partial M, [g]) \leq 0$, which is much simpler to handle than the positive case. Since we have not found any reference for the case $Q(M, \partial M, [g]) < 0$, let us briefly explain how to solve (1.1) in this case, if $K(x) < 0$ on ∂M. We essentially use the method of sub- and supersolutions like in [21] or [23]. To write (1.1) in a shorter form we introduce the conformal Laplacian $L_g u := -\Delta_g u + \frac{n-2}{4(n-1)} R_g u$ and the conformal boundary operator $B_g u = \partial_\nu u + \frac{n-2}{2} h_g u$, which are conformally covariant. Thus if $\bar{g} = v^{\frac{4}{n-2}} g$ for some positive, smooth function, then

$$L_g(uv) = v^{\frac{n+2}{n-2}} L_{\bar{g}}(u), \quad B_g(uv) = v^{\frac{n}{n-2}} B_{\bar{g}}(u). \tag{1.5}$$

Due to the previous Theorem we can find a metric $\bar{g}$, conformal to g, with zero scalar curvature and mean curvature constant -1. With this metric and (1.5), (1.1) is equivalent to :

$$\begin{cases} L_{\bar{g}} u &= 0 \qquad \text{in } M \\ B_{\bar{g}} u &= \frac{n-2}{2} K(x) u^{\frac{n}{n-2}} \quad \text{on } \partial M \\ & u > 0. \end{cases} \tag{1.6}$$

We call a positive function $u \in C^2(M)$ **subsolution (supersolution)** to (1.6) if

$$L_{\bar{g}} u \leq (\geq)\ 0 \quad \text{and} \quad B_{\bar{g}} u \leq (\geq)\ \frac{n-2}{2} K(x) u^{\frac{n}{n-2}}.$$

Since $-\infty < Q(M, \partial M, [g]) < 0$ the first eigenvalue λ_1 of the problem

$$\begin{cases} L_{\bar{g}} u = 0 & \text{in } M \\ B_{\bar{g}} u = \lambda u & \text{on } \partial M \end{cases} \tag{1.7}$$

is negative (see [19]). Using that the first eigenfunction φ_1 can be chosen to be positive, we observe that $\alpha\varphi_1$ is a subsolution if α is positive but small. Furthermore a large constant function C can be chosen to be a supersolution, because $K < 0$.
Let $(u_k)_{k\in\mathbb{N}}$ be the sequence of smooth functions, defined by: $u_0 = \alpha\varphi_1$,

$$\begin{cases} L_{\bar{g}} u_k & = 0 & \text{in } M \\ B_{\bar{g}} u_k + M u_k & = \frac{n-2}{2} K(x) u_{k-1}^{\frac{n}{n-2}} + M u_{k-1} & \text{on } \partial M \end{cases}$$

for $k \geq 1$ and a large positive constant M. Due to the maximum principle

$$\alpha\varphi_1 \leq u_{k-1} \leq u_k \leq u_{k+1} \leq C$$

for all $k \in \mathbb{N}$. Hence, using methods, similar to those used in [23], it is possible to prove the existence of a smooth positive solution u of (1.6) by showing that $(u_k)_k$ converges in an appropriate space (weak convergence in $H^1(M)$ is sufficient). This proves the existence of a conformal metric with zero scalar curvature and mean curvature given by K.

In case $Q(M, \partial M, [g]) = 0$ Escobar was able to give a complete answer to the prescribed Mean Curvature Problem:

Theorem ([21]). *Let (M, g) be a compact Riemannian manifold of dimension greater than or equal to 3 such that $Q(M, \partial M, [g]) = 0$ and $K : \partial M \to \mathbb{R}$ smooth. Then K is the mean curvature of a conformal metric with zero scalar curvature if and only if*

$$K \quad \textit{changes sign and} \quad \int_{\partial M} K d\sigma_g < 0.$$

This proof also uses the method of sub- and supersolutions.

Lastly, we turn to the case $Q(M, \partial M, [g]) > 0$. This case is much more complicated and the techniques are quite different to the previous ones. Like in the constant mean curvature case, the problem has variational structure and smooth, positive solutions of (1.1) are critical points of the functional $J : U \to \mathbb{R}$, where

$$J(u) = \frac{\int_M \left(|\nabla u|_g^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) dV_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 d\sigma_g}{\left(\int_{\partial M} K(x) |u|^{\frac{2(n-1)}{n-2}} d\sigma_g \right)^{\frac{n-2}{n-1}}} \tag{1.8}$$

and

$$U := \left\{ u \in H^1(M) \ : \ \int_{\partial M} K(x)|u|^{\frac{2(n-1)}{n-2}} d\sigma_g \neq 0, \quad u \geq 0 \text{ a.e.} \right\}.$$

As already mentioned, this functional does not satisfy the Palais-Smale condition which makes it a priori impossible to apply standard variational techniques. Nevertheless, using similar methods as in the constant mean curvature case, it is possible to find conditions on K such that minimizing sequences of J, under the constraint

$$\int_{\partial M} K(x)|u|^{\frac{2(n-1)}{n-2}} d\sigma_g = 1,$$

are still relative compact. First, Escobar [21] used this method and obtained first existence results for general manifolds. To state the Theorem, we need further notations. Therefore let ∇ be the Levi-Civita connection of (M, g). We denote by $h(X, Y) := g(\nabla_X \nu, Y)$ the second fundamental form on ∂M. A point $a \in \partial M$ is called **umbilic** if the umbilicity tensor

$$\Pi := h - h_g g \tag{1.9}$$

vanishes at a. Let us remark that the norm $|\Pi(a)|^2$ is conformal invariant if $\tilde{g}$ is a metric, conformal to g, such that $\tilde{g}(a) = g(a)$.

Theorem. *([21]) Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ and $Q(M, \partial M, [g]) > 0$. If $K : \partial M \to \mathbb{R}$ is smooth and positive somewhere, then g is conformal to a metric with zero scalar curvature and mean curvature given by K if*

1. *$n = 3$ and M is not conformally equivalent to the ball B^3.*
2. *$n = 4$, M is not conformally equivalent to B^4, ∂M is umbilic and $\nabla^2 K(x) = 0$ for a global maximum point x.*
3. *$n \geq 5$, M is locally conformally flat, not conformally equivalent to B^n with umbilic boundary such that $\nabla^l K(x) = 0$ for some global maximum x and $1 \leq l \leq n - 2$.*
4. *$n \geq 6$ and K has a global maximum point x, which is not umbilic, such that $\Delta_g K(x) \leq c(n)|\Pi(x)|^2$, where $c(n)$ is some dimensional constant.*

Here $|\Pi(x)|$ is the norm of the umbilicity tensor at x.

Since the ball is *umbilic* the previous Theorem excludes the ball. Existence results of (1.1) for the ball were obtained by Escobar and Garcia [22] , Chang et al. [14] as well as Ahmedou et al. [1].
As far as we know there are no more works, which prove results for the prescribed Mean Curvature Problem.
In the next section we will state our results and give some analytical background.

1.2. Preliminaries and statement of results

From now on let (M, g) be an n-dimensional compact Riemannian manifold with boundary ∂M and positive Sobolev quotient $Q(M, \partial M, [g])$. Furthermore let $K : \partial M \to \mathbb{R}$ be a smooth, positive function. Since $Q(M, \partial M, [g]) > 0$, there exists a metric $\bar{g}$ conformal to g with positive scalar curvature and zero mean curvature at the boundary (see [19]). Henceforth we assume the metric g to have the previous properties. On $H^1(M)$ we define the scalar product

$$< u, v >:= \int_M \left(\nabla u \cdot \nabla v + \frac{n-2}{4(n-1)} R_g uv \right) dV_g, \tag{1.10}$$

which induces a norm $|| \cdot ||$, equivalent to the standard norm on $H^1(M)$. Our aim is to prove the existence of critical points of

$$J(u) = \frac{||u||^2}{\left(\int_{\partial M} K(x) |u|^{\frac{2(n-1)}{n-2}} d\sigma_g \right)^{\frac{n-2}{n-1}}}$$

on $\Sigma^+ \cap U$, where $\Sigma^+ := \left\{ u \in H^1(M) : \ ||u|| = 1, \ u \geq 0 \text{ a.e.} \right\}$, which leads to smooth, positive solutions of (1.1). Because of technical reasons, which will become clear in chapter 3, we can not restrict to $\Sigma^+ \cap U$. For $\varepsilon_0 > 0$ small we define

$$V_{\varepsilon_0}\left(\Sigma^+\right) := \left\{ u \in U \ : \ ||u|| = 1, \ J(u)^{\frac{n-1}{2}} ||u^-||_{L^{\frac{2(n-1)}{n-2}}(\partial M)} < \varepsilon_0 \right\},$$

where $u^- = \max(0, -u)$ is the negative part. If $u \in V_{\varepsilon_0}\left(\Sigma^+\right)$ is a critical point of J then

$$\begin{aligned} -||u^-||^2 =< u, u^- > &= J(u)^{\frac{n-1}{n-2}} \int_{\partial M} K(x) u |u|^{\frac{2}{n-2}} u^- d\sigma_g \\ &= -J(u)^{\frac{n-1}{n-2}} \int_{\partial M} K(x) |u^-|^{\frac{2(n-1)}{n-2}} d\sigma_g \end{aligned}$$

and hence

$$||u^-||^2_{L^{\frac{2(n-1)}{n-2}}(\partial M)} \leq C \max K J(u)^{\frac{n-1}{n-2}} ||u^-||^{\frac{2(n-1)}{n-2}}_{L^{\frac{2(n-1)}{n-2}}(\partial M)},$$

which implies $u^- = 0$ if ε_0 is small. Thus u will be a positive solution of (1.1). From now on we assume ε_0 to be small enough. As already mentioned J does not satisfy the Palais-Smale condition. Nevertheless the non-compactness is well understood. Therefore let $\lambda > 0$ and define

$$\delta_\lambda(x, t) := \left(\frac{\lambda}{(1 + \lambda t)^2 + \lambda^2 |x|^2} \right)^{\frac{n-2}{2}} \tag{1.11}$$

as in (1.4). This family of functions solves the boundary value problem:

$$\begin{cases} \Delta\delta_\lambda = 0 & \text{in } \mathbb{R}^n_+, \\ \partial_t\delta_\lambda = (2-n)\delta_\lambda^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{1.12}$$

Furthermore, for $a \in \partial M$ let U_a be an open neighbourhood of a in M and $\psi_a : U_a \to B^+_{2\rho_0}$ be Fermi-coordinates around a. A very detailed description of Fermi-coordinates is given in appendix A. Since ∂M is compact we can choose ρ_0 such that $\psi_a : U_a \to B^+_{2\rho_0}$ is a diffeomorphism for all $a \in \partial M$. Finally let $\chi_\rho : \mathbb{R} \to [0,1]$ be a smooth function such that

$$\begin{cases} \chi_\rho(t) = 1 & \text{if } t \le \rho, \\ \chi_\rho(t) = 0 & \text{if } t \ge 2\rho. \end{cases}$$

For $a \in \partial M$ define the smooth function

$$\delta_{a,\lambda} : M \to \mathbb{R},\ \delta_{a,\lambda}(x) := \chi_\rho\left(|\psi_a(x)|\right)\delta_\lambda\left(\psi_a(x)\right).$$

For $p \in \mathbb{N}$ and $\varepsilon > 0$ define

$$W(p,\varepsilon) := \Bigg\{ u \in V_{\varepsilon_0}(\Sigma^+) \;\Bigg|\; \exists a_1,\cdots,a_p \in \partial M;\ \lambda_1,\cdots,\lambda_p \in \left(\frac{1}{\varepsilon},\infty\right)\ s.t. \\ \left|\left| u - \frac{1}{J(u)^{\frac{n-1}{2}}}\sum_{i=1}^p \left(\frac{n-2}{K(a_i)}\right)^{\frac{n-2}{2}} \delta_{a_i,\lambda_i}\right|\right| < \varepsilon;\ \varepsilon_{ij} < \varepsilon\ \forall\, i \neq j \Bigg\}, \tag{1.13}$$

where

$$\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j d_g(a_i,a_j)^2\right)^{\frac{2-n}{2}}. \tag{1.14}$$

Here $d_g(\cdot,\cdot)$ is the distance (on ∂M) with respect to the metric g. Now we are prepared to understand the non-compactness of J.

Proposition 1. *Assume that J does not have any critical point in $V_{\varepsilon_0}(\Sigma^+)$. Let $(u_n)_{n\in\mathbb{N}} \subset V_{\varepsilon_0}(\Sigma^+)$ be a Palais-Smale sequence of J, then there exists $p \in \mathbb{N}$ and a sequence $\varepsilon_n \searrow 0$ such that $u_n \in W(p,\varepsilon_n)$ along a subsequence.*

The proof of Proposition 1 is, up to minor modifications, the same as the proof by Almaraz [3] in the case where K is constant. Similar results in domains have been obtained by Struwe [32]. Se also Bahri-Coron [9], Brezis-Coron [13], Bahri [7] and Mayer [29].
For our purpose the functions (bubbles) $\delta_{a,\lambda}$ will not be good enough. In chapter 3 we will define the sets $W(p,\varepsilon)$ with new bubbles, which we will call $\varphi_{a,\lambda}$. This is possible, because $||\delta_{a,\lambda} - \varphi_{a,\lambda}|| \to 0$ for $\lambda \to \infty$.

From now on let (M,g) be a four-dimensional Riemannian manifold with boundary and $K:\partial M\to(0,\infty)$ a Morse-function. Since $Q(M,\partial M,[g])$ is positive there exists a unique positive Green's function $G(\cdot,\cdot)$ of the operator (L_g,B_g) such that

$$u(x)=\int_M G(x,y)L_g u(y)dV_g+\int_{\partial M}G(x,y)B_g u(y)d\sigma_g \quad \forall\ u\in C^2(M)$$

For $a\in\partial M$ let $g_a=u_a^2 g$ be a family of metrics such that

$$\sqrt{g_a}(x,t)=1+O(|(x,t)|^{10})$$

in Fermi-coordinates w.r.t. g_a at a. The existence was proved in [27] by Marques. The normalized Green's function $G_a(a,\cdot)$ at a with respect to the operator (L_{g_a},B_{g_a}) is expanded in appendix E (see Proposition 32). Here normalized means:

$$\lim_{x\to a}d_{g_a}(a,x)^2G_a(x)=1.$$

It can be written as

$$G_a(a,x)=\Gamma_a(x)+H_a(x),$$

where Γ_a is singular at a and H_a is regular (in $C^{2,\alpha}$ for some α). Let $crit(K):=\{x_1,\cdots,x_m\}$ be the set of critical points of K. Henceforth we assume

$$2|S^3_+|H_x(x)+\frac{2I_4}{9}\frac{\Delta K(x)}{K(x)}\neq 0 \quad \forall\ x\in\ crit(K). \tag{1.15}$$

Here S^3_+ is the upper half sphere in $\mathbb{R}^4$ and

$$I_4=\int_{\mathbb{R}^3}\frac{|x|^2}{(1+|x|^2)^3}dx.$$

For $1\le p\le m$ define

$$\mathcal{F}_p:=\left\{(x_1,\cdots,x_p)\in crit(K)^p\ \middle|\ 2|S^3_+|H_{x_i}(x_i)+\frac{2I_4}{9}\frac{\Delta K(x_i)}{K(x_i)}<0\ \forall i;\ x_i\neq x_j\ \forall\ i\neq j\right\}.$$

For $y=(y_1,\cdots,y_p)\in\mathcal{F}_p$ we define the matrix $M(y)=M_{ij}\in\mathbb{R}^{p\times p}$ by

$$M_{ii}:=-2|S^3_+|\frac{H_{y_i}(y_i)}{K(y_i)^2}-\frac{2I_4}{9}\frac{\Delta K(y_i)}{K(y_i)^3},\quad M_{ij}:=-2I_1\frac{G(y_i,y_j)}{K(y_i)K(y_j)}\ \text{for}\ i\neq j.$$

Since the Green's function is symmetric, also $M(y)$ is symmetric. Let $\rho_1(y)$ be the least eigenvalue of $M(y)$. From now on we assume

$$\rho_1(x)\neq 0\ \text{for all}\ x\in\mathcal{F}_p. \tag{1.16}$$

Finally define

$$\mathcal{F}_p^\infty=\{x\in\mathcal{F}_p\mid\rho_1(x)>0\}.$$

We assume $crit(K)$ to be an ordered subset. Thus $x_1 < x_2 < \cdots < x_m$. Lastly we set

$$\mathcal{F}^\infty := \left\{ y = (y_1, \cdots, y_p) \in \bigcup_{1 \le q \le m} \mathcal{F}_q^\infty \;\middle|\; y_1 < y_2 < \cdots < y_p \right\}.$$

We are now able to state our Theorems:

Theorem 1. *Let (M, g) be a four-dimensional compact Riemannian manifold with boundary such that $Q(M, \partial M, [g]) > 0$. Let $K : \partial M \to (0, \infty)$ be a Morse-function such that (1.15) and (1.16) hold. Furthermore assume that all critical points of K are also umbilic points, then (PMCP) has a solution if*

$$2|S_+^3|H_x(x) + \frac{2I_4}{9}\frac{\Delta K(x)}{K(x)} > 0$$

at a point $x \in \partial M$ where $K(x) = \sup K$.

Theorem 2. *Let (M, g) be a four-dimensional compact Riemannian manifold with boundary such that $Q(M, \partial M, [g]) > 0$. Let $K : \partial M \to (0, \infty)$ be a Morse-function such that (1.15) and (1.16) hold. Furthermore assume that all critical points of K are also umbilic points, then (PMCP) has a solution if*

$$1 \neq \sum_{x \in \mathcal{F}^\infty} (-1)^{\sum_{i=1}^p ind(x_i, K) + 1}, \tag{1.17}$$

where $x = (x_1, \cdots, x_p)$ and $ind(x, K)$ is the Morse-index of K at x.

From Theorem 1 we can deduce a Corollary. Therefore we need to introduce the *ADM*-mass of an asymptotically flat manifold with boundary.

Definition 1. A Riemannian manifold with boundary (N, g) is called asymptotically flat of order $\tau > 0$ if there exists a compact set $K \subset M$ and a diffeomorphism $\phi : M \setminus K \to \mathbb{R}_+^n \setminus B_1(0)$ such that

$$|g_{ij}(x) - \delta_{ij}| + |x| \cdot |\nabla g_{ij}(x)| + |x|^2 \cdot |\nabla^2 g_{ij}(x)| = O(|x|^{-\tau}) \quad (|x| \to \infty),$$

where g_{ij} are the coefficients of the metric in the chart ϕ.

If $\tau > \frac{n-2}{2}$, R_g is integrable on N and h_g is integrable on ∂N, then the ADM-mass

$$m(g, N) := \lim_{r \to \infty} \left(\int_{S_{r,+}^{n-1}} (\partial_\mu g_{\mu\nu} - \partial_\nu g_{\mu\mu}) \frac{x_\nu}{r} dS + \int_{S_r^{n-2}} g_{in} \frac{x_i}{r} dS \right)$$

is well defined (see [4, 5]). Here $S^{n-1}_{r,+}$ is the upper half-sphere in $\mathbb{R}^n$ with radius r and S^{n-2}_r is the sphere of radius r in $\mathbb{R}^{n-1}$. Furthermore the positive mass theorem for manifolds with boundary (see [5]) asserts that $m(g,N)>0$ if $R_g\geq 0, h_g\geq 0$, $dim(N)\leq 7$ and (N,g) is not isometric to $(\mathbb{R}^n_+, g_{st})$.

Let $G_a(x,\cdot)$ be the Green's function of the metric g_a and (M,g) a four-dimensional Riemannian manifold with boundary. Define $(\hat{M},\hat{g})=(M\setminus\{a\}, G_a(a,\cdot)^2 g_a)$ then $(\hat{M},\hat{g})$ is a four-dimensional asymptotically flat manifold with boundary of order $\tau=2$, if a is an umbilic point. Furthermore $R_{\hat{g}}=0$ and $h_{\hat{g}}=0$. If (M,g) is not conformally diffeomorphic to (B^4, g_{st}) then $(\hat{M},\hat{g})$ is not isometric to $(\mathbb{R}^4_+, g_{st})$. The positive mass theorem implies $m(\hat{g},\hat{M})>0$ in this case. If a is umbilic it is not difficult to show the equality

$$H_a(a)=\frac{1}{12|S^3_+|}m(\hat{g},\hat{M})>0 \tag{1.18}$$

by using the expansion of the Green's function. Hence, the following Corollary is an immediate consequence from (1.18) and Theorem 1.

Corollary. *Let (M,g) be a four dimensional compact Riemannian manifold with boundary such that $Q(M,\partial M,[g])>0$. Moreover assume that (M,g) is not conformally diffeomorphic to (B^4,g_{st}). Let $K:\partial M\to(0,\infty)$ be a Morse-function such that* (1.15) *and* (1.16) *holds. Furthermore assume that all critical points of K are also umbilic points. Then there exists $\varepsilon>0$ such that (PMCP) has a solution if*

$$\frac{\Delta K(x)}{K(x)}>-\varepsilon$$

at a point $x\in\partial M$ where $K(x)=\sup K$.

1.3. Outline of the thesis

In chapter 2 we define appropriate test functions (bubbles) $\varphi_{a,\lambda}$ for $\lambda>0$ and $a\in\partial M$. These are "close" to the test function, defined in the previous section. We prove some first estimates that are needed in subsequent chapters. Furthermore we justify the definition of $W(p,\varepsilon)$, where $\delta_{a,\lambda}$ is replaced by $\varphi_{a,\lambda}$.
Since it will be important for our theory we introduce new variables $\alpha_1,\cdots,\alpha_p>0$ such that every $u\in W(p,\varepsilon)$ can be written as follows:

$$u=\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v,$$

where the reminder

$$v\in E_{(\alpha,a,\lambda)}:=\left\langle \varphi_{a_i,\lambda_i}, \frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i}, \frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i} \;\middle|\; 1\leq i\leq p,\ 1\leq m\leq 3\right\rangle^{\perp}\subset H^1(M).$$

This will be proved in chapter 3 by using a minimization argument. Moreover we define new neighbourhoods $V(p,\varepsilon)$ of non-compact Palais-Smale sequences, which are equivalent to $W(p,\varepsilon)$ and will be used in the rest of this thesis. Since the functional J is not as smooth as needed for the theory, we introduce a slightly different functional such that critical points of the new functional lead to critical points of J. In chapter 4 we expand the functional for

$$u=\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v\in V(p,\varepsilon),$$

which helps to understand the behaviour of J with respect to the variables α_i, a_i, λ_i for $1\leq i\leq p$ and v.
We use this expansion in chapter 5 to show that

$$E_{(\alpha,a,\lambda)}\ni v\mapsto J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v\right)$$

has a local minimizer $\bar{v}$, close to zero. This will become very important for the theory.
In chapter 6 we expand the gradient of J in $V(p,\varepsilon)$ to prove that the expansion in chapter 4 is valid also in a C^1-sense.
Based on the gradient expansions in chapter 5 and 6, we construct a pseudo-gradient vector field X, which allows us to understand flow lines of $\dot{u}=-X(u)$. Under the assumption that J does not have any critical point, all flow lines do not converge. We call these flow lines **critical flow lines at infinity**.
Using this pseudo-gradient in chapter 7 we prove that **critical flow lines at infinity** have to remain in some $V(p,\varepsilon)$ for t large. More precisely we understand the behaviour of those flow lines with respect to the variables (α,a,λ,v).
In chapter 8 we show that **critical flow lines at infinity** in $V(p,\varepsilon)$ have to accumulate around $(a,\lambda,v)=(x,\infty,0)$, where $x=(x_1,\cdots,x_p)\in\mathcal{F}_p^\infty$ is called a **critical point at infinity**. Using this knowledge we prove a deformation lemma and a Morse lemma at infinity to compute the change of topology, induced by this **critical points at infinity**.
Finally, under the assumption that J does not have any critical point, the topology of the variational space can be compared to the difference of topology, induced by non-compact flow lines, which proves Theorem 1 and 2.

2. Definition of the test functions and preliminary expansions

As already mentioned in the introduction the standard bubbles $\delta_{a,\lambda}$, which appear in the definition of $W(p,\varepsilon)$, are not good enough. Since they are local they do not carry any information about the global geometry of the manifold. Therefore we glue the standard test functions $\delta_{a,\lambda}$ with the Green's function of the conformal operator (L_g, B_g). This is motivated by Schoen [31].

For $(x,t) \in \mathbb{R}^3 \times \mathbb{R}_+ = \mathbb{R}^4_+$ and $r > 0$ let us introduce the sets

$$B_r^+ = B_r^+(0) = \{(x,t) \in \mathbb{R}^4_+ \mid |(x,t)| < r\}, \; B_r = B_r(0) = \{x \in \mathbb{R}^3 \mid |x| < r\}.$$

From now we assume the reader to be familiar with Fermi-coordinates at points $a \in \partial M$ (see appendix A). Due to Marques [27] there exists a positive, smooth function $u : \partial M \times M \to \mathbb{R}$, $u(a,x) = u_a(x)$ such that

$$|\sqrt{g}_a(x,t) - 1| \le C|(x,t)|^{10} \tag{2.1}$$

for $(x,t) \in B^+_{2\rho_0}$ in ψ_a Fermi-coordinates around a with respect to the metric $g_a = u_a^2 g$. Here $\sqrt{g}_a(x,t)$ is the volume element with respect to the metric g_a. Moreover, since ∂M is compact, ρ_0 and the constant in (2.1) can be chosen to be independent of a. Furthermore the function u_a can be chosen such that $u_a(a) = 1$ for all $a \in \partial M$.

We now choose a family $(\psi_a)_{a\in\partial M}$ of Fermi-coordinates with respect to this family of metrics $(g_a)_{a\in\partial M}$. Furthermore let $\chi : \mathbb{R} \to [0,1]$ be a smooth function such that $\chi(t) = 1$ if $t \le \frac{4}{3}$ and $\chi(t) = 0$ if $t \ge \frac{5}{3}$. For $\rho > 0$ set $\chi_\rho(t) := \chi\left(\frac{t}{\rho}\right)$. Finally let $G_a(a,\cdot)$ be the normalized Green's function at $a \in \partial M$ with respect to the operator (L_{g_a}, B_{g_a}). The normalized Green's function satisfies

$$\lim_{x\to a} d_{g_a}(a,x)^2 G_a(a,x) = 1 \quad (x \in \partial M).$$

Here $d_{g_a}(\cdot,\cdot)$ is the distance with respect to the metric g_a.
For $a \in \partial M$ and $\lambda > 0$, we define the family of global test functions (bubbles) as follows

$$\hat{\varphi}_{a,\lambda}(x) := \chi_\rho\left(|\psi_a(x)|\right)\delta_\lambda\left(\psi_a(x)\right) + \left(1 - \chi_\rho\left(|\psi_a(x)|\right)\right)\frac{G_a(a,x)}{\lambda} \tag{2.2}$$

and $\varphi_{a,\lambda} := u_a \hat{\varphi}_{a,\lambda}$. Here δ_λ is the standard bubble on $\mathbb{R}^4_+$, which was defined in (1.11). Furthermore we set

$$\hat{\delta}_{a,\lambda}(x) := \chi_\rho\left(|\psi_a(x)|\right)\Big(\delta_\lambda\left(\psi_a(x)\right)\Big) \tag{2.3}$$

and $\delta_{a,\lambda} = u_a \hat{\delta}_{a,\lambda}$.

Let us remark that $M \times \partial M \times \mathbb{R}_+ \ni (x, a, \lambda) \mapsto \hat{\varphi}_{a,\lambda}(x)$ is smooth. Furthermore it holds $L_{g_a} G_a(a, x) = 0$ and $B_{g_a} G_a(a, x) = 0$ for all $x \neq a$.
From now on we will always identify $x \in M$ with $\psi_a(x) = (x, t) \in \mathbb{R}^3 \times \mathbb{R}_+ = \mathbb{R}^4_+$.

Proposition 2. *If* $2 \leq \lambda\rho$, *then*

$$\begin{aligned}|L_{g_a}\hat{\varphi}_{a,\lambda}(x)| \leq\, & C\Big(\frac{\lambda^2|\Pi(a)|}{(1+\lambda t)+\lambda|x|)^3} + \frac{\lambda}{(1+\lambda|(x,t)|)^2}\Big)1_{\{|(x,t)|\leq\rho\}} \\ & + C\left(\frac{1}{\lambda^2\rho^5} + \frac{|\Pi(a)|}{\lambda\rho^3}\right)1_{\{\rho\leq|(x,t)|\leq 2\rho\}}\end{aligned}$$

as well as

$$|B_{g_a}\hat{\varphi}_{a,\lambda} - 2\hat{\varphi}^2_{a,\lambda}| \leq C\chi_\rho \cdot \frac{1}{\lambda} + C\frac{1}{\lambda^2\rho^4}1_{\{\rho\leq d_{g_a}(a,x)\}},$$

where $|\Pi(a)|$ *is the norm of the umbilicity tensor, defined in* (1.9), *with respect to the metric* g_a.

Here, and in the rest of this work, C always represents a constant which does not depend on any variable.

Proof. Using the definition of the conformal Laplacian we get

$$\begin{aligned}L_{g_a}\hat{\varphi}_{a,\lambda} &= -\Delta_{g_a}\hat{\varphi}_{a,\lambda} + \frac{1}{6}R_{g_a}\hat{\varphi}_{a,\lambda} \\ &= -\Delta_{g_a}\chi_\rho \cdot \delta_\lambda - 2\nabla_{g_a}\chi_\rho \cdot \nabla_{g_a}\delta_\lambda - \chi_\rho\Delta_{g_a}\delta_\lambda \\ &+ \Delta_{g_a}\chi_\rho\frac{G_a(a,\cdot)}{\lambda} + 2\nabla_{g_a}\chi_\rho \cdot \nabla_{g_a}\frac{G_a(a,\cdot)}{\lambda} + \frac{1}{6}R_{g_a}\chi_\rho\delta_\lambda \\ &= -\Delta_{g_a}\chi_\rho\Big(\delta_\lambda - \frac{1}{\lambda|(x,t)|^2}\Big) - 2\nabla_{g_a}\chi_\rho \cdot \nabla_{g_a}\Big(\delta_\lambda - \frac{1}{\lambda|(x,t)|^2}\Big) \\ &- \chi_\rho\Delta_{g_a} \cdot \delta_\lambda + \Delta_{g_a}\chi_\rho\Big(\frac{G_a(a,\cdot)}{\lambda} - \frac{1}{\lambda|(x,t)|^2}\Big) \\ &+ 2\nabla_{g_a}\chi_\rho \cdot \nabla_{g_a}\Big(\frac{G_a(a,\cdot)}{\lambda} - \frac{1}{\lambda|(x,t)|^2}\Big) + \frac{1}{6}R_{g_a}\chi_\rho\delta_\lambda.\end{aligned} \tag{2.4}$$

The expansion of the Green's function, given in appendix E, yields

$$\left|\nabla^k\left(G_a\big(a,\psi_a^{-1}(x,t)\big)-\frac{1}{|(x,t)|^2}\right)\right|\le C\left(\frac{|\Pi(a)|}{|(x,t)|^{1+k}}+\frac{1}{|(x,t)|^k}\right),\quad k=0,1,2. \tag{2.5}$$

Furthermore it holds

$$\left|\nabla^k\left(\delta_\lambda-\frac{1}{\lambda|(x,t)|^2}\right)\right|\le C\frac{1}{\lambda^2}\frac{1}{|(x,t)|^{3+k}},\quad k=0,1,2;\ \frac{2}{\lambda}\le|(x,t)|. \tag{2.6}$$

Hence from (2.4) – (2.6) we infer:

$$\begin{aligned}|L_{g_a}\hat\varphi_{a,\lambda}(x)|\le& C\left(\frac{1}{\lambda^2\rho^5}+\frac{|\Pi(a)|}{\lambda\rho^3}\right)1_{\{\rho\le|(x,t)|\le2\rho\}}\\&+|\chi_\rho\Delta_{g_a}\delta_\lambda|+\left|\frac{1}{6}R_{g_a}\chi_\rho\delta_\lambda\right|.\end{aligned} \tag{2.7}$$

In ψ_a Fermi-coordinates the coefficients of the inverse metric are expanded as follows:

$$g^{ij}(x,t)=\delta_{ij}+2h_{ij}(a)t+O(|(x,t)|^2)\ 1\le i,j\le3$$

as well as $g^{i4}(x,t)=0,\ 1\le i\le3$ and $g^{44}(x,t)=1$ (see (A.1)). Here h_{ij} are the coefficients of the second fundamental form with respect to g_a. Since $h_{g_a}(a)=0$, $\Pi_{ij}(a)=h_{ij}(a)$, where Π_{ij} are the coefficients of the umbilicity tensor. Therefore a simple computation yields

$$\begin{aligned}\Delta_{g_a}\delta_\lambda=&-8\lambda^5\frac{h_{ij}(a)x_ix_jt}{((1+\lambda t)^2+\lambda^2|x|^2)^3}\\&+\partial_i\left((g^{ij}-\delta_{ij}-2h_{ij}t)\partial_j\delta_\lambda\right)+\partial_i\log(\sqrt{g_a})g^{ij}\partial_j(\delta_\lambda).\end{aligned} \tag{2.8}$$

Due to the fact that $\sqrt{g_a}(x,t)=1+O(|(x,t)|^{10})$,

$$|\Delta_{g_a}\delta_\lambda|\le C\bigg(\frac{\lambda^2|\Pi(a)|}{(1+\lambda t)+\lambda|x|)^3}+\frac{\lambda}{(1+\lambda|(x,t)|)^2}\bigg)1_{\{|(x,t)|\le2\rho\}}. \tag{2.9}$$

Finally adding (2.9) to (2.7) proves the first assertion.

In the following we prove the second inequality. First observe that

$$\begin{aligned}B_{g_a}\hat\varphi_{a,\lambda}=\chi_\rho B_{g_a}\hat\varphi_{a,\lambda}=&\chi_\rho\bigg(-\partial_t\delta_\lambda+h_{g_a}(x)\delta_\lambda\bigg)\\=&2\chi_\rho\delta_\lambda^2+\chi_\rho h_{g_a}(x)\delta_\lambda\end{aligned} \tag{2.10}$$

and hence

$$|B_{g_a}\hat\varphi_{a,\lambda}-2\chi_\rho\delta_\lambda^2|\le C\chi_\rho\cdot\frac{1}{\lambda}.$$

Since

$$|2\hat\varphi_{a,\lambda}^2-2\chi_\rho\delta_\lambda^2|\le C\frac{1}{\lambda^2\rho^4}1_{\{\rho\le d_{g_a}(a,x)\}},$$

the stated inequality follows. □

Furthermore we need estimates for the derivatives of $\hat{\varphi}_{a,\lambda}$ with respect to λ and a. First we estimate the derivative with respect to λ.

Proposition 3. *If* $2 \leq \lambda\rho$ *then*

$$\left|\lambda\frac{\partial}{\partial\lambda}L_{g_a}\hat{\varphi}_{a,\lambda}(x)\right| \leq C\Big(\frac{\lambda^2|\Pi(a)|}{(1+\lambda t)+\lambda|x|)^3}+\frac{\lambda}{(1+\lambda|(x,t)|)^2}\Big)1_{\{|(x,t)|\leq\rho\}} + C\left(\frac{1}{\lambda^2\rho^5}+\frac{|\Pi(a)|}{\lambda\rho^3}\right)1_{\{\rho\leq|(x,t)|\leq 2\rho\}}$$

and

$$\left|\lambda\frac{\partial}{\partial\lambda}B_{g_a}\hat{\varphi}_{a,\lambda}(x)-2\lambda\frac{\partial}{\partial\lambda}\hat{\varphi}^2_{a,\lambda}\right| \leq C\chi_\rho\cdot\frac{1}{\lambda}+C\frac{1}{\lambda^2\rho^4}1_{\{\rho\leq d_{g_a}(a,x)\}}.$$

Proof. First observe that

$$|\lambda\frac{\partial}{\partial\lambda}\nabla^k\delta_\lambda| \leq C\frac{\lambda^{k+1}}{(1+\lambda|(x,t)|)^{k+2}} \quad k=0,1,2 \tag{2.11}$$

and

$$\left|\lambda\frac{\partial}{\partial\lambda}\left(\frac{1}{(1+\lambda t)^2+\lambda^2|x|^2)^{\frac{m}{2}}}\right)\right| \leq \frac{C(m)}{(1+\lambda t)^2+\lambda^2|x|^2)^{\frac{m}{2}}},\ m\in\mathbb{N}. \tag{2.12}$$

In addition it holds

$$\left|\lambda\frac{\partial}{\partial\lambda}\nabla^k\Big(\delta_\lambda-\frac{1}{\lambda|(x,t)|^2}\Big)\right| \leq C\frac{1}{\lambda^2}\frac{1}{|(x,t)|^{3+k}},\quad k=0,1,2. \tag{2.13}$$

Therefore the estimates (2.11)-(2.13), combined with the expansions (2.4) and (2.8), prove the first claim. It remains to prove the estimate on the boundary. But from (2.10) and (2.11) we derive the estimate

$$\left|\lambda\frac{\partial}{\partial\lambda}B_{g_a}\hat{\varphi}_{a,\lambda}-2\chi_\rho\lambda\frac{\partial}{\partial\lambda}\delta^2_\lambda\right| \leq C\chi_\rho\cdot\frac{1}{\lambda}. \tag{2.14}$$

Since

$$\left|2\lambda\frac{\partial}{\partial\lambda}\hat{\varphi}^2_{a,\lambda}-2\chi_\rho\lambda\frac{\partial}{\partial\lambda}\delta^2_\lambda\right| \leq C\frac{1}{\lambda^2\rho^4}1_{\{\rho\leq d_{g_a}(a,x)\}}$$

the second claim follows through the previous inequality and (2.14). □

We finally need to estimate the derivative with respect to $a\in\partial M$.

Proposition 4. *If* $2 \leq \lambda\rho$ *then*

$$\left|\frac{1}{\lambda}\nabla_a L_{g_a}\hat{\varphi}_{a,\lambda}(x)\right| \leq C\left(\frac{\lambda^2|\Pi(a)|}{(1+\lambda t)+\lambda|x|)^3} + \frac{\lambda}{(1+\lambda|(x,t)|)^2}\right)1_{\{|(x,t)|\leq\rho\}}$$
$$+C\left(\frac{1}{\lambda^2\rho^5}+\frac{|\Pi(a)|}{\lambda\rho^3}\right)1_{\{\rho\leq|(x,t)|\leq 2\rho\}}$$

and

$$\left|\frac{1}{\lambda}\nabla_a B_{g_a}\hat{\varphi}_{a,\lambda}(x) - 2\frac{1}{\lambda}\nabla_a\hat{\varphi}^2_{a,\lambda}\right| \leq C\chi_\rho\cdot\frac{1}{\lambda} + C\frac{1}{\lambda^2\rho^4}1_{\{\rho\leq d_{g_a}(a,x)\}}.$$

Proof. We choose ψ_{a_0} Fermi-coordinates and want to estimate $\frac{1}{\lambda}\frac{\partial}{\partial a^m}L_{g_a}\hat{\varphi}_{a,\lambda}$ at $a_0 \in \partial M$ in this coordinates. Here we identify $a^m = \psi^m_{a_0}(a)$ for $m = 1, 2, 3$. If $x \notin B_{2\rho}(a_0)$ then $L_{g_a}\hat{\varphi}_{a,\lambda} = 0$ for a close to a_0. Otherwise we identify $x = \psi_{a_0}(x,t)$ and observe

$$\begin{aligned} L_{g_a}\hat{\varphi}_{a,\lambda}(x) &= -\Delta_{g_a}\hat{\varphi}_{a,\lambda}(x) + \frac{1}{6}R_{g_a}\hat{\varphi}_{a,\lambda}(x) \\ &= -\frac{1}{\sqrt{g_a(x,t)}}\partial_\mu\left(\sqrt{g_a(x,t)}g_a^{\mu,\nu}(x,t)\partial_\nu\hat{\varphi}_{a,\lambda}\right) + \frac{1}{6}R_{g_a}\hat{\varphi}_{a,\lambda}(x). \end{aligned}$$

Moreover

$$\begin{aligned} &\frac{1}{\sqrt{g_a(x,t)}}\partial_\mu\left(\sqrt{g_a(x,t)}g_a^{\mu,\nu}(x,t)\partial_\nu\hat{\varphi}_{a,\lambda}\right) \\ &\qquad = \partial_\mu\log(\sqrt{g_a(x,t)})g_a^{\mu,\nu}\partial_\nu\hat{\varphi}_{a,\lambda} + \partial_\mu\left(g_a^{\mu,\nu}(x,t)\partial_\nu\hat{\varphi}_{a,\lambda}\right). \end{aligned}$$

Since $g_a(x,t) = \frac{u_a^2}{u_{a_0}^2}g_{a_0}(x,t)$ and $u_a(a) = 1$ for all $a \in \partial M$ we get

$$\frac{\partial}{\partial a^m|a_0}\partial_\mu\log(\sqrt{g_a(x,t)})g_a^{\mu,\nu} = O(|x,t|^9)$$

and

$$\frac{\partial}{\partial a^m|a_0}g_a^{\mu,\nu}(x,t) = O(|(x,t)|).$$

Furthermore

$$|\left(\nabla^k\hat{\varphi}_{a,\lambda}\right)(\psi_a^{-1}(x,t))| \leq \frac{C}{\lambda}\left(\frac{1}{\frac{1}{\lambda}+|(x,t)|}\right)^{2+k}, \quad k = 1, 2$$

provided $2 \leq \lambda\rho$ and therefore

$$\frac{1}{\lambda}\frac{\partial}{\partial a^m|a_0}\Delta_{g_a}\hat{\varphi}_{a,\lambda} = \Delta_{g_{a_0}}\left(\frac{1}{\lambda}\frac{\partial}{\partial a^m|a_0}\hat{\varphi}_{a,\lambda}\right) + O(\hat{\delta}_{a_0,\lambda}).$$

Using Lemma 10 in appendix A we derive

$$\begin{aligned}\frac{1}{\lambda}\frac{\partial}{\partial a^m|a_0}\hat{\varphi}_{a,\lambda}(\psi_{a_0}(x,t)) &= \frac{1}{\lambda}\frac{\partial}{\partial b^m|0}\left(\delta_\lambda(x-b,t)\right)\\ &+\frac{\lambda(1+\lambda t)O(|(x,t)|^2)+\lambda^2 O(|(x,t)|^3)}{\left((1+\lambda t)^2+\lambda^2|x|^2\right)^2} \qquad (2.15)\end{aligned}$$

for $|(x,t)|\leq\rho$ and hence

$$\Delta_{g_{a_0}}\left(\frac{1}{\lambda}\frac{\partial}{\partial a^m|a_0}\hat{\varphi}_{a,\lambda}(\psi_{a_0}(x,t))\right)=\frac{1}{\lambda}\frac{\partial}{\partial b^m|0}\Delta_{g_{a_0}}\delta_\lambda(x-b,t)+O\left(\delta_\lambda\right).$$

Finally (2.8) yields

$$\begin{aligned}&\left|\Delta_{g_{a_0}}\left(\frac{1}{\lambda}\frac{\partial}{\partial a^m|a_0}\hat{\varphi}_{a,\lambda}(\psi_{a_0}(x,t))\right)\right|\leq\\ &C\Bigg(\frac{\lambda^2|\Pi(a)|}{(1+\lambda t)+\lambda|x|)^3}+\frac{\lambda}{(1+\lambda|(x,t)|)^2}\Bigg)\end{aligned}$$

if $|(x,t)|\leq\rho$. In the case $\rho\leq|(x,t)|\leq 2\rho$ we easily derive the estimate

$$\left|\Delta_{g_{a_0}}\left(\frac{1}{\lambda}\frac{\partial}{\partial a^m|a_0}\hat{\varphi}_{a,\lambda}(\psi_{a_0}(x,t))\right)\right|\leq\frac{C}{\lambda^2\rho^5}.$$

Therefore the first assertion is proved. It remains to prove the second inequality. From (2.10) and Lemma 10 in appendix A we derive

$$\left|\frac{1}{\lambda}\frac{\partial}{\partial a^m|a_0}B_{g_a}\hat{\varphi}_{a,\lambda}-2\frac{1}{\lambda}\frac{\partial}{\partial a^m|a_0}\chi_\rho\delta^2_{a,\lambda}\right|\leq C\left(\frac{1}{\lambda}\right)1_{\{|x|\leq 2\rho\}}.$$

Lastly we easily estimate

$$\left|\frac{1}{\lambda}\frac{\partial}{\partial a^m|a_0}\hat{\varphi}^2_{a,\lambda}-\frac{1}{\lambda}\frac{\partial}{\partial a^m|a_0}\chi_\rho\delta^2_{a,\lambda}\right|\leq\left(\frac{1}{\lambda^2\rho^4}+\frac{1}{\lambda^3\rho^5}\right)1_{\{\rho\leq d_{g_a}(a,x)\}}.$$

Adding the previous two estimates proves the second assertion. □

As already mentioned in the introduction we want to define the sets $W(p,\varepsilon)$ with the functions $\varphi_{a,\lambda}$ instead of

$$\tilde{\delta}_{a,\lambda}(x):=\chi_\rho(|\tilde{\psi}_a(x)|)\delta_\lambda(|\tilde{\psi}_a(x)),$$

where $\tilde{\psi}_a$ are Fermi-coordinates with respect to the metric g. Remember that we used Fermi-coordinates ψ_a with respect to the metric g_a in the definition of $\varphi_{a,\lambda}$. Therefore we need to prove that $||\varphi_{a_n,\lambda_n}-\tilde{\delta}_{a_n,\lambda_n}||_{H^1(M)}\to 0$ if $\lambda_n\to\infty$, which will be justified by the following Lemma.

Lemma 1. *(a)* $||\varphi_{a,\lambda} - u_a\delta_{a,\lambda}||^2_{H^1} \leq C\frac{\log(\lambda\rho)}{\lambda}$ *uniformly in* $a \in \partial M$.

(b) If $(a_n)_n \subset \partial M$ *and* $\lambda_n \to \infty$*, then* $||\tilde{\delta}_{a_n,\lambda_n} - u_{a_n}\delta_{a_n,\lambda_n}||_{H^1} \to 0$.

Proof. (a) Using the conformal covariance of (L_g, B_g) (see (1.5)) we compute

$$\begin{aligned} ||\varphi_{a,\lambda} - u_a\delta_{a,\lambda}||^2_{H^1} =& \int_M L_{g_a}\big(\hat{\varphi}_{a,\lambda} - \delta_{a,\lambda}\big)(\hat{\varphi}_{a,\lambda} - \delta_{a,\lambda})dV_{g_a} \\ &+ \int_{\partial M} B_{g_a}\big(\hat{\varphi}_{a,\lambda} - \delta_{a,\lambda}\big)(\hat{\varphi}_{a,\lambda} - \delta_{a,\lambda})d\sigma_{g_a} \\ &\leq \left(\int_M \left|L_{g_a}\big(\hat{\varphi}_{a,\lambda} - \delta_{a,\lambda}\big)\right|^{\frac{4}{3}} dV_{g_a}\right)^{\frac{3}{4}} ||\varphi_{a,\lambda} - u_a\delta_{a,\lambda}||_{H^1} \\ &+ \left(\int_{\partial M} \left|B_{g_a}\big(\hat{\varphi}_{a,\lambda} - \delta_{a,\lambda}\big)\right|^{\frac{3}{2}} d\sigma_{g_a}\right)^{\frac{2}{3}} ||\varphi_{a,\lambda} - u_a\delta_{a,\lambda}||_{H^1}. \end{aligned} \tag{2.16}$$

The computation of those integrals can be done easily by using the definition of the bubbles, which proves (a).
(b) First observe that $||\tilde{\delta}_{a_n,\lambda_n}||^2 = 2I_0 + o(1) = ||u_{a_n}\delta_{a_n,\lambda_n}||^2$, where

$$I_0 = \int_{\mathbb{R}^3} \frac{1}{(1+|x|^2)^3}dx$$

and $o(1) \to 0$ for $n \to \infty$. Hence

$$\begin{aligned} ||\tilde{\delta}_{a_n,\lambda_n} - u_{a_n}\delta_{a_n,\lambda_n}||^2_{H^1} &= 4I_0 - 2 < \tilde{\delta}_{a_n,\lambda_n}, u_{a_n}\delta_{a_n,\lambda_n} > + o(1) \\ &= 4I_0 - 4\int_{\partial M} \chi_\rho(|\tilde{\psi}_a(x)|)\delta_{\lambda_n}(\tilde{\psi}_{a_n}(x))^2 u_{a_n}\delta_{a_n,\lambda_n}d\sigma_g + o(1) \\ &= 4I_0 - 4\int_{B_\rho(a_n)} \delta_{\lambda_n}(\tilde{\psi}_{a_n}(x))^2 u_{a_n}\delta_{\lambda_n}(\psi_{a_n}(x))d\sigma_g + o(1). \end{aligned} \tag{2.17}$$

In appendix A (see page 112) we prove the smoothness of the function

$$\partial M \times M \ni (a,x) \mapsto \chi_{\rho_0}(|\psi_a(x)|)\Big(1 + \lambda^2|\psi_a(x)|^2\Big).$$

A Taylor expansion in Fermi-coordinates at a yields:

$$1 + \lambda^2|\psi_a\big(\tilde{\psi}_a^{-1}(z)\big)|^2 = 1 + \lambda^2|z|^2 + O(\lambda^2|z|^3)\ \forall\ |z| \leq 2\rho << \rho_0, \tag{2.18}$$

where we use Lemma 10 and $g_a(a) = g(a)$. Furthermore $\tilde{\psi}_{a_n}^{-1*}d\sigma_g(x) = 1 + O(|x|^2)$ and $u_{a_n}(\psi_{a_n}^{-1}(x)) = 1 + O(|x|^2)$, where the O-terms do not depend on a. These observations, combined with and (2.17), yield

$$\begin{aligned}
&||\tilde{\delta}_{a_n,\lambda_n} - u_{a_n}\delta_{a_n,\lambda_n}||^2_{H^1}\\
&\quad = 4I_0 - 4\int_{B_{\lambda_n\rho}} \left(\frac{1}{1+|x|^2}\right)^2 \left(\frac{1}{1+\lambda_n^2|\psi_{a_n}(\tilde{\psi}^{-1}_{a_n}(\frac{x}{\lambda_n}))|^2}\right) dx + o(1)\\
&\quad = 4I_0 - 4\int_{B_{\lambda_n\rho}} \left(\frac{1}{1+|x|^2}\right)^2 \left(\frac{1}{1+|x|^2 + O\left(\frac{|x|^3}{\lambda_n}\right)}\right) dx + o(1)\\
&\quad = 4I_0 - 4\int_{\mathbb{R}^3} \left(\frac{1}{1+|x|^2}\right)^3 dx + o(1) = o(1).
\end{aligned}$$

□

So far we have defined the bubbles and proved some technical estimates which will become important in the expansion of the functional and its gradient. For $p \in \mathbb{N}$ and $\varepsilon > 0$ we now set

$$\begin{aligned}
W(p,\varepsilon) :=& \Bigg\{ u \in V_{\varepsilon_0}(\Sigma^+) \;\Bigg|\; \exists a_1, \cdots, a_p \in \partial M;\ \lambda_1, \cdots, \lambda_p \in \left(\frac{1}{\varepsilon}, \infty\right) \ s.t.\\
&\left|\left| u - \frac{1}{J(u)^{\frac{3}{2}}} \sum_{i=1}^p \left(\frac{2}{K(a_i)}\right) \varphi_{a_i,\lambda_i} \right|\right| < \varepsilon;\ \varepsilon_{ij} < \varepsilon \ \forall\ i \neq j \Bigg\}. \qquad (2.19)
\end{aligned}$$

Here ε_{ij} was defined in (1.14). Due to Lemma 1, Proposition 1 holds true with $W(p,\varepsilon)$ defined in (2.19). In the next chapter we will prove a convenient parametrization for functions $u \in W(p,\varepsilon)$ and define new neighbourhoods $V(p,\varepsilon)$ of non-converging Palais-Smale sequences.

3. An appropriate representation in $W(p,\varepsilon)$ and the modified functional

3.1. Minimization in $W(p,\varepsilon)$

For $u \in W(p,\varepsilon)$ we write $u = J(u)^{-\frac{3}{2}}\sum_{i=1}^{p}\left(\frac{2}{K(a_i)}\right)\varphi_{a_i,\lambda_i} + v$ with $||v|| < \varepsilon$. Since $u \in \Sigma$

$$1 = ||u||^2 = \left|\left|J(u)^{-\frac{3}{2}}\sum_{i=1}^{p}\left(\frac{2}{K(a_i)}\right)\varphi_{a_i,\lambda_i}\right|\right|^2 + O(\varepsilon),$$

where $|O(\varepsilon)| \le C\varepsilon$ for some universal constant, which does not depend on u. Remark 4 in appendix B implies $| < \varphi_{a_i,\lambda_i}, \varphi_{a_j,\lambda_j} > | \le C\varepsilon_{ij} \le C\varepsilon$ from which we deduce

$$1 = J(u)^{-3}\sum_{i=1}^{p}\left(\frac{2}{K(a_i)}\right)^2 ||\varphi_{a_i,\lambda_i}||^2 + O(\varepsilon).$$

Furthermore, the identity $||\varphi_{a_i,\lambda_i}||^2 = 2I_0 + O(\varepsilon)$ yields

$$\left|J(u)^{-\frac{3}{2}} - \frac{1}{\sqrt{2I_0}}\left(\sum_{j=1}^{p}\left(\frac{2}{K(a_j)}\right)^2\right)^{-\frac{1}{2}}\right| \le C\varepsilon$$

as well as

$$\left|J(u)^{-\frac{3}{2}}\left(\frac{2}{K(a_i)}\right) - \frac{1}{\sqrt{2I_0}}\left(\sum_{j=1}^{p}\left(\frac{K(a_i)}{K(a_j)}\right)^2\right)^{-\frac{1}{2}}\right| \le C\varepsilon \qquad (3.1)$$

uniformly in $W(p,\varepsilon)$.

Since K is a positive, smooth function on ∂M the quantity $\frac{1}{\sqrt{2I_0}}\left(\sum_{j=1}^{p}\left(\frac{K(a_i)}{K(a_j)}\right)^2\right)^{-\frac{1}{2}}$ may be bounded from below by $0 < \frac{1}{\gamma}$ and from above by γ. With this notations we define

$$B^p_{\varepsilon,\gamma} := \left\{(\alpha, a, \lambda) \in \mathbb{R}^p_+ \times (\partial M)^p \times \mathbb{R}^p_+ \;\middle|\; \lambda_i > \frac{1}{\varepsilon}, \varepsilon_{ij} < \varepsilon, \frac{1}{2}\frac{1}{\gamma} < \alpha_i < 2\gamma \;\forall i, j\right\}.$$

From now on we choose ε small such that $\frac{1}{2}\frac{1}{\gamma} < J(u)^{-\frac{3}{2}}\left(\frac{2}{K(a_i)}\right) < 2\gamma$ for all $u \in W(p,\varepsilon)$. In this chapter we prove the following Proposition:

Proposition 5. *There exists $\varepsilon_0 > 0$ such that the minimization problem*

$$\inf_{(\alpha,a,\lambda)\in B^p_{2\varepsilon,\gamma}} \left|\left|u - \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i}\right|\right|^2$$

has, up to permutation of (α, a, λ), a unique solution, provided $u \in W(p,\varepsilon_0)$. Moreover, for the minimizer (α, a, λ) there holds

$$\left|\alpha_i - \frac{1}{\sqrt{2I_0}}\left(\sum_{j=1}^{p}\left(\frac{K(a_i)}{K(a_j)}\right)^2\right)^{-\frac{1}{2}}\right| \to 0 \text{ iniformly in } W(p,\varepsilon) \text{ if } \varepsilon \to 0.$$

We prove this Proposition in several steps. Essentially we follow the proof in [9]. First we need the following Lemma:

Lemma 2. *Let $(\alpha^n, a^n, \lambda^n), (\tilde{\alpha}^n, \tilde{a}^n, \tilde{\lambda}^n) \in \mathbb{R}^p_+ \times (\partial M)^p \times \mathbb{R}^p_+$ two sequences such that $\lambda^n_i, \tilde{\lambda}^n_i \to \infty$; $\varepsilon^n_{ij}, \tilde{\varepsilon}^n_{ij} \to 0$; $\frac{1}{C} \leq \alpha^n_i, \tilde{\alpha}^n_i \leq C$ and*

$$\left|\left|\sum_{i=1}^{p}\alpha^n_i\varphi_{a^n_i,\lambda^n_i} - \sum_{i=1}^{p}\tilde{\alpha}^n_i\varphi_{\tilde{a}^n_i,\tilde{\lambda}^n_i}\right|\right| \to 0$$

for $n \to \infty$, then (up to permutation):

$$|\alpha^n_i - \tilde{\alpha}^n_i| \to 0,\ \lambda^n_i\tilde{\lambda}^n_i d_g(a^n_i, \tilde{a}^n_i)^2 \to 0,\ \frac{\lambda^n_i}{\tilde{\lambda}^n_i} \to 1 \text{ for } i = 1, \cdots, p. \tag{3.2}$$

Proof. From now on we omit the index n and we write o(1) whenever a term tends to zero for $n \to \infty$. Since $< \varphi_{a_i,\lambda_i}, \varphi_{a_j,\lambda_j} >= O(\varepsilon_{ij})$ (see Remark 4 in appendix B), it follows

$$\begin{aligned} o(1) &= \left|\left|\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i} - \sum_{i=1}^{p}\tilde{\alpha}_i\varphi_{\tilde{a}_i,\tilde{\lambda}_i}\right|\right|^2 \\ &= \sum_{i=1}^{p}\alpha_i^2||\varphi_{a_i,\lambda_i}||^2 - 2\sum_{i,j=1}^{p}\alpha_i\tilde{\alpha}_i < \varphi_{a_i,\lambda_i}, \varphi_{\tilde{a}_j,\tilde{\lambda}_j} > + \sum_{j=1}^{p}\tilde{\alpha}_j^2||\varphi_{\tilde{a}_j,\tilde{\lambda}_j}||^2 + o(1). \end{aligned} \tag{3.3}$$

For all i exists at most one j such that $w_{ij} := \frac{\lambda_i}{\tilde{\lambda}_j} + \frac{\tilde{\lambda}_j}{\lambda_i} + \lambda_i\tilde{\lambda}_j d_g(a_i, \tilde{a}_j)^2$ is bounded. Because if there were j and k such that

$$\frac{\lambda_i}{\tilde{\lambda}_j} + \frac{\tilde{\lambda}_j}{\lambda_i} + \lambda_i\tilde{\lambda}_j d_g(a_i, \tilde{a}_j)^2 \text{ and } \frac{\lambda_i}{\tilde{\lambda}_k} + \frac{\tilde{\lambda}_k}{\lambda_i} + \lambda_i\tilde{\lambda}_k d_g(a_i, \tilde{a}_k)^2$$

would be bounded, then also $\frac{\tilde\lambda_k}{\tilde\lambda_j}+\frac{\tilde\lambda_j}{\tilde\lambda_k}+\tilde\lambda_i\tilde\lambda_j d_g(\tilde a_k,\tilde a_j)^2$ would be bounded, which is a contradiction to $\tilde\varepsilon_{jk}=o(1)$.
Set

$$M:=\{i\in\{1,\cdots,p\}\mid \exists j\ \ s.t.\ w_{ij} \text{ is bounded}\}.$$

We permute such that w_{ii} is bounded for all $i\in M$. Using (3.3) we derive

$$o(1)=\sum_{i\in M}||\alpha_i\varphi_{a_i,\lambda_i}-\tilde\alpha_i\varphi_{\tilde a_i,\tilde\lambda_i}||^2+\sum_{i\in M^c}\alpha_i^2||\varphi_{a_i,\lambda_i}||^2+\sum_{i\in M^c}\tilde\alpha_i^2||\varphi_{\tilde a_i,\tilde\lambda_i}||^2 \tag{3.4}$$

along a subsequence. Since α_i and $\tilde\alpha_i$ are bounded from below and $||\varphi_{a,\lambda}||^2=2I_0+o(1)$ we have proved $M=\{1,\cdots,p\}$ and

$$o(1)=\sum_{i=1}^p||\alpha_i\varphi_{a_i,\lambda_i}-\tilde\alpha_i\varphi_{\tilde a_i,\tilde\lambda_i}||^2. \tag{3.5}$$

Equation (3.5) implies

$$o(1)=||\alpha_i\varphi_{a_i,\lambda_i}-\tilde\alpha_i\varphi_{\tilde a_i,\tilde\lambda_i}||^2\ge\Big(\alpha_i||\varphi_{a_i,\lambda_i}||-\tilde\alpha_i||\varphi_{\tilde a_i,\tilde\lambda_i}||\Big)^2,$$

hence $\alpha_i-\tilde\alpha_i=o(1)$, because

$$\lim_{n\to\infty}||\varphi_{a_i,\lambda_i}||^2=\lim_{n\to\infty}||\varphi_{\tilde a_i,\tilde\lambda_i}||^2=2I_0.$$

Due to Lemma 1, $\varphi_{a,\lambda}=\tilde\delta_{a,\lambda}+o(1)$ in $H^1(M)$. Therefore (3.5) implies

$$o(1)=||\tilde\delta_{a_i,\lambda_i}-\tilde\delta_{\tilde a_i,\tilde\lambda_i}||^2. \tag{3.6}$$

Since $\lambda_i\tilde\lambda_i d_g(a_i,\tilde a_i)^2$ is bounded from above, $d_g(a_i,\tilde a_i)$ tends to zero. To continue the proof we need an expansion of $d_g(\tilde a_i,\tilde\psi_{a_i}^{-1}(x))^2$ for n large, where $\tilde\psi_{a_i}$ are Fermi-coordinates at a_i w.r.t. g.

Claim: For $x\in B_\rho(0)$ it holds

$$d_g(\tilde a_i,\tilde\psi_{a_i}^{-1}(x))^2=|x-\tilde\psi_{a_i}(\tilde a_i)|^2+O\left(|x-\tilde\psi_{a_i}(\tilde a_i)|^3\right). \tag{3.7}$$

Proof of the claim. In appendix A (see page 113) we prove that the function

$$\partial M\times M\ni(a,y)\mapsto w(a,y)=\chi_{2\rho}(|\tilde\psi_a(y)|)|\tilde\psi_a(y)|^2$$

is smooth. If n is large then $d_g(\tilde a_i,a_i)=|\tilde\psi_{\tilde a_i}(a_i)|<\rho$ and hence $|\tilde\psi_{\tilde a_i}(y)|<2\rho$ for $y\in B_\rho(a_i)$. Therefore a Taylor-expansion at $\tilde x=\tilde\psi_{a_i}(\tilde a_i)$ yields

$$\begin{aligned} d_g(\tilde a_i,\tilde\psi_{a_i}^{-1}(x))^2&=|\tilde\psi_{\tilde a_i}(\tilde\psi_{a_i}^{-1}(x))|^2\\ &=\sum_{k,l=1}^3<\frac{\partial}{\partial x_k|\tilde x}\tilde\psi_{\tilde a_i}(\tilde\psi_{a_i}^{-1}(x)),\frac{\partial}{\partial x_l|\tilde x}\tilde\psi_{\tilde a_i}(\tilde\psi_{a_i}^{-1}(x))>(x^k-\tilde\psi_{a_i}(\tilde a_i)^k)(x^l-\tilde\psi_{a_i}(\tilde a_i)^l)\\ &\quad+O\left(|x-\tilde\psi_{a_i}(\tilde a_i)|^3\right),\end{aligned}$$

where the O-term does not depend on n. Since Fermi-coordinates are Riemannian normal coordinates at the boundary:

$$< \frac{\partial}{\partial x_k}\tilde{\psi}_{\tilde{a}_i}(\tilde{\psi}_{a_i}^{-1}(x)), \frac{\partial}{\partial x_l}\tilde{\psi}_{\tilde{a}_i}(\tilde{\psi}_{a_i}^{-1}(x)) >= g_{|\tilde{a}_i}\left((d\tilde{\psi}_{a_i}^{-1})_{|\tilde{x}}(e_k), (d\tilde{\psi}_{a_i}^{-1})_{|\tilde{x}}(e_l)\right).$$

Let (v_1, v_2, v_3) be an orthonormal basis of $T_{a_i}\partial M$ such that

$$\psi_{a_i}^{-1}(x) = \exp_{a_i}\left(\sum_{i=1}^{3} x_i v_i\right)$$

where $\exp_{a_i}$ is the geodesic exponential map. Then

$$\begin{aligned} &g_{|\tilde{a}_i}\left(d(\tilde{\psi}_{a_i}^{-1})_{|\tilde{x}}(e_k), d(\tilde{\psi}_{a_i}^{-1})_{|\tilde{x}}(e_l)\right) \\ &= g_{|\tilde{a}_i}\left((d\exp_{a_i})_{|\exp_{a_i}(\sum_{i=1}^3 \tilde{x}_i v_i)}[v_k], (d\exp_{a_i})_{|\exp_{a_i}(\sum_{i=1}^3 \tilde{x}_i v_i)}[v_l]\right) \\ &= g_{|a_i}(v_k, v_l) = \delta_{kl}, \end{aligned}$$

where we used the Gauss's Lemma (see [17]) for the last step. Finally we have proved

$$d_g(\tilde{a}_i, \tilde{\psi}_{a_i}^{-1}(x))^2 = |x - \tilde{\psi}_{a_i}(\tilde{a}_i)|^2 + O\left(|x - \tilde{\psi}_{a_i}(\tilde{a}_i)|^3\right),$$

which proves the claim. □

To use (3.6) we need to expand the interaction $< \tilde{\delta}_{a_i,\lambda_i}, \tilde{\delta}_{\tilde{a}_i,\tilde{\lambda}_i} >$. Therefore we compute:

$$\begin{aligned} &< \tilde{\delta}_{a_i,\lambda_i}, \tilde{\delta}_{\tilde{a}_i,\tilde{\lambda}_i} >= \int_M L_g\tilde{\delta}_{a_i,\lambda_i}\tilde{\delta}_{\tilde{a}_i,\tilde{\lambda}_i} dV_g + \int_{\partial M} B_g\tilde{\delta}_{a_i,\lambda_i}\tilde{\delta}_{\tilde{a}_i,\tilde{\lambda}_i} d\sigma_g \\ &= 2\int_{\partial M} \tilde{\delta}_{a_i,\lambda_i}^2 \tilde{\delta}_{\tilde{a}_i,\tilde{\lambda}_i} d\sigma_g + o(1) \\ &= 2\int_{B_\rho(0)} \left(\frac{\lambda_i}{1+\lambda_i^2|x|^2}\right)^2 \left(\frac{\tilde{\lambda}_i}{1+\tilde{\lambda}_i^2 d_g(\tilde{a}_i, \psi_{a_i}^{-1}(x))^2}\right) dx + o(1) \\ &= 2\left(\frac{\tilde{\lambda}_i}{\lambda_i}\right)\int_{B_{\lambda_i\rho}(0)} \left(\frac{1}{1+|x|^2}\right)^2 \left(\frac{1}{1+\tilde{\lambda}_i^2 d_g(\tilde{a}_i, \psi_{a_i}^{-1}(\frac{x}{\lambda_i}))^2}\right) dx + o(1) \end{aligned} \tag{3.8}$$

Since $\frac{\tilde{\lambda}_i}{\lambda_i}$ and $\lambda_i\tilde{\lambda}_i d(a_i,\tilde{a}_i)^2 = \lambda_i\tilde{\lambda}_i|\psi_{a_i}(\tilde{a}_i)|^2$ are bounded form above we can assume

$$\frac{\tilde{\lambda}_i}{\lambda_i} \to \mu > 0 \quad \text{and} \quad \lambda_i^2\psi_{a_i}(\tilde{a}_i) \to b \in \mathbb{R}^3$$

along a subsequence. Hence expansion (3.7) implies:

$$\left(\frac{1}{1+|x|^2}\right)^2 \left(\frac{1}{1+\tilde{\lambda}_i^2 d_g(\tilde{a}_i, \psi_{a_i}^{-1}(\frac{x}{\lambda_i}))^2}\right) 1_{B_{\lambda_i\rho}(0)}(x) \xrightarrow{n\to\infty} \left(\frac{1}{1+|x|^2}\right)^2 \left(\frac{1}{1+\mu^2|x-b|^2}\right)$$

pointwise in $\mathbb{R}^3$. Since

$$\left|\left(\frac{1}{1+|x|^2}\right)^2\left(\frac{1}{1+\tilde{\lambda}_i^2 d_g(\tilde{a}_i,\psi_{a_i}^{-1}(\frac{x}{\lambda_i}))^2}\right)1_{B_{\lambda_i\rho}(0)}(x)\right|\leq\left(\frac{1}{1+|x|^2}\right)^2\in L^1(\mathbb{R}^3), \quad (3.9)$$

we can use Lebesgue's theorem and (3.8) to conclude

$$<\tilde{\delta}_{a_i,\lambda_i},\tilde{\delta}_{\tilde{a}_i,\tilde{\lambda}_i}>=2\int_{\mathbb{R}^3}\left(\frac{1}{1+|x|^2}\right)^2\left(\frac{\mu}{1+\mu^2|x-b|^2}\right)dx+o(1). \quad (3.10)$$

From (3.6) and (3.10) we derive $0=||u_{1,0}-u_{\mu,b}||^2_{D^{1,2}(\mathbb{R}^4_+)}$, where

$$u_{\lambda,b}(x,t)=\left(\frac{\lambda}{(1+\lambda t)^2+\lambda^2|x-b|^2}\right),\quad \lambda>0,\quad b\in\mathbb{R}^3$$

are the unique solutions (see [25]) to

$$\begin{cases}\Delta u=0 & \text{in } \mathbb{R}^4_+\\ \partial_t u=-2u^2 & \text{on } \partial\mathbb{R}^4_+\\ u>0.\end{cases}$$

Hence $\mu=1$ and $b=0$, which implies

$$\lambda_i\tilde{\lambda}_i d_g(a_i,\tilde{a}_i)^2\to 0 \text{ and } \frac{\tilde{\lambda}_i}{\lambda_i}\to 1$$

along a subsequence. Finally a sub-subsequence argument proves the Lemma. □

Remark 1.

(a) *From Lemma 2 we deduce the following statement: For all $\delta>0$ exists $\varepsilon>0$ such that*

$$\left|\frac{\lambda_i}{\tilde{\lambda}_i}-1\right|+|\alpha_i-\tilde{\alpha}_i|+\lambda_i\tilde{\lambda}_i d_g(a_i,\tilde{a}_i)<\delta\ \forall i$$

if $(\alpha,a,\lambda),(\tilde{\alpha},\tilde{a},\tilde{\lambda})\in B^p_{2\varepsilon,\gamma}$ and

$$\left|\left|\sum_{i=1}^p\alpha_i\varphi_{a_i,\lambda_i}-\sum_{i=1}^p\tilde{\alpha}_i\varphi_{\tilde{a}_i,\tilde{\lambda}_i}\right|\right|<2\varepsilon.$$

(b) *The following statement is a conclusion from (a) and (3.1). For all $\delta>0$ exists $\varepsilon>0$ such that*

$$(\alpha,a,\lambda)\in\overline{B^p_{2\varepsilon,\gamma}},u\in W(p,\varepsilon)\ s.t.\ ||u-\sum_{i=1}^p\alpha_i\varphi_{a_i,\lambda_i}||<2\varepsilon$$

implies

$$\left| \alpha_i - \frac{1}{\sqrt{2I_0}} \left(\sum_{j=1}^{p} \left(\frac{K(a_i)}{K(a_j)} \right)^2 \right)^{-\frac{1}{2}} \right| < \delta.$$

Now we are prepared to prove the Proposition.

Proof of Proposition 5.
(i) Existence of a minimizer:
We choose ε small such that Remark 1 (a) is true for $\delta < 1/2$. Since $u \in W(p,\varepsilon)$ there exits $a_1, \cdots, a_p \in \partial M$ and $\lambda_1, \cdots, \lambda_p > \varepsilon^{-1}$ such that

$$\left\| u - \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} \right\| < \varepsilon, \text{ where } \alpha_i = J(u)^{-\frac{3}{2}} \left(\frac{2}{K(a_i)} \right).$$

Let $(\alpha^n, a^n, \lambda^n) \in B^p_{2\varepsilon,\gamma}$ be a minimizing sequence, then

$$\left\| \sum_{i=1}^{p} \alpha_i^n \varphi_{a_i^n,\lambda_i^n} - \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} \right\| < 2\varepsilon \text{ for } n \text{ large}.$$

Hence, from Lemma 2 we deduce that the λ_i^n are bounded from above and below. Thus $(\alpha^n, a^n, \lambda^n) \to (\bar\alpha, \bar a, \bar\lambda) \in \overline{B^p_{2\varepsilon}}$ along a subsequence. Using Remark 1 (b) we can choose ε smaller, if necessary, to obtain $1/2\frac{1}{\gamma} < \bar\alpha_i < 2\gamma$.
Claim: $(\bar\alpha, \bar a, \bar\lambda) \in B^p_{\varepsilon,\gamma}$ for ε small.
If the claim was wrong we could find $u_n \in W(p,\varepsilon_n)$ with $\varepsilon_n \to 0$, $(\bar\alpha^n, \bar a^n, \bar\lambda^n) \in \overline{B^p_{2\varepsilon,\gamma}} \setminus B^p_{\varepsilon,\gamma}$ such that

$$\left\| J(u_n)^{-\frac{3}{2}} \sum_{i=1}^{p} \left(\frac{2}{K(a_i^n)} \right) \varphi_{a_i^n,\lambda_i^n} - \sum_{i=1}^{p} \bar\alpha_i^n \varphi_{\bar a_i^n,\bar\lambda_i^n} \right\| < 2\varepsilon_n.$$

But then Lemma 2 would imply

$$\frac{\lambda_i^n}{\bar\lambda_i^n} = 1 + o(1), \quad \lambda_i^n \bar\lambda_i^n d_g(a_i^n, \bar a_i^n)^2 = o(1),$$

which contradicts $(\bar\alpha^n, \bar a^n, \bar\lambda^n) \in \overline{B^p_{2\varepsilon,\gamma}} \setminus B^p_{\varepsilon,\gamma}$ for n large. This proves the claim.

So far we have proved the existence of a minimizer in $B^p_{2\varepsilon,\gamma}$ for ε small. It remains to prove uniqueness.

(ii) Uniqueness of the minimizer:
The proof of the uniqueness follows essentially from the fact that the function

$B^p_{2\varepsilon,\gamma} \ni (\alpha,a,\lambda) \mapsto \left|\left|u - \sum_{i=1}^p \alpha_i\varphi_{a_i,\lambda_i}\right|\right|^2$ is locally a convex function, hence the gradient is a strictly monotone operator, which guaranties uniqueness. These arguments are hidden in the following proof, which works by contradiction.
From now on assume that the statement of the Proposition is wrong. Since we already have proved the existence of a minimizer we can find a sequence $u_n \in W(p,\varepsilon_n)$ with $\varepsilon_n \to 0$ and two minimizing sequences $(\alpha_n,a_n,\lambda_n),(\tilde\alpha_n,\tilde a_n,\tilde\lambda_n) \in B^p_{2\varepsilon_n,\gamma}$. From now on we omit the index n in our notation and we will use $o(1)$ for sequences that tend to zero for $n\to\infty$. Furthermore we define $v := u - \sum_{i=1}^p \alpha_i\varphi_{a_i,\lambda_i}$, $\tilde v := u - \sum_{i=1}^p \tilde\alpha_i\varphi_{\tilde a_i,\tilde\lambda_i}$ and

$$f(\alpha,a,\lambda) := \left|\left|u - \sum_{i=1}^p \alpha_i\varphi_{a_i,\lambda_i}\right|\right|^2.$$

Taking the derivative w.r.t. α_i at the minimizers yields

$$\begin{aligned}0 &=< v,\varphi_{a_i,\lambda_i} > - < \tilde v,\varphi_{\tilde a_i,\tilde\lambda_i} >=< v-\tilde v,\varphi_{a_i,\lambda_i} > + < \tilde v,\varphi_{a_i,\lambda_i} - \varphi_{\tilde a_i,\tilde\lambda_i} > \\ &= \sum_{j=1}^p < \tilde\alpha_j\varphi_{\tilde a_j,\tilde\lambda_j} - \alpha_j\varphi_{a_j,\lambda_j},\varphi_{a_i,\lambda_i} > - < \tilde v,\varphi_{\tilde a_i,\tilde\lambda_i} - \varphi_{a_i,\lambda_i} > . \end{aligned} \tag{3.11}$$

Since $\varepsilon_n \to 0$ Lemma 2 implies

$$||\tilde v|| + \lambda_i\tilde\lambda_i d_g(a_i,\tilde a_i)^2 + |\alpha_i - \tilde\alpha_i| = o(1) \text{ as well as } \frac{\lambda_i}{\tilde\lambda_i} = 1 + o(1), \tag{3.12}$$

hence $d_g(a_i,\tilde a_i) = o(1)$. We choose n large such that $\tilde a_i$ is in the domain of definition of Fermi-coordinates around a_i.
Next we apply Lemma 13 and 14 in appendix D to obtain

$$| < \tilde v,\varphi_{\tilde a_i,\tilde\lambda_i} - \varphi_{a_i,\lambda_i} > | \le Co(1)\left(\lambda_i d_g(a_i,\tilde a_i) + \left|1 - \frac{\tilde\lambda_i}{\lambda_i}\right|\right) \tag{3.13}$$

as well as

$$\begin{aligned}< \tilde\alpha_j\varphi_{\tilde a_j,\tilde\lambda_j} - \alpha_j\varphi_{a_j,\lambda_j},\varphi_{a_i,\lambda_i} > &= (\tilde\alpha_j - \alpha_j)(2I_0\delta_{ij} + o(1)) \\ &\quad + O\left(\sum_{j=1}^p \lambda_j^2 d_g(a_j,\tilde a_j)^2 + \left|1 - \frac{\tilde\lambda_j}{\lambda_j}\right|^2\right).\end{aligned} \tag{3.14}$$

We combine (3.11), (3.13) and (3.14) to infer that

$$\begin{aligned}&c\sum_{j=1}^p |\alpha_j - \tilde\alpha_j| \\ &\quad \le o(1)\left(\sum_{j=1}^p \lambda_j d_g(a_j,\tilde a_j) + \left|1 - \frac{\tilde\lambda_j}{\lambda_j}\right|\right) + O\left(\sum_{j=1}^p \lambda_j^2 d_g(a_j,\tilde a_j)^2 + \left|1 - \frac{\tilde\lambda_j}{\lambda_j}\right|^2\right).\end{aligned} \tag{3.15}$$

Now we take the derivative of f w.r.t. a_i at the minimizers to get

$$\begin{aligned} 0 =& < v, \frac{1}{\lambda_i}\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i} > - < \tilde{v}, \frac{1}{\tilde{\lambda}_i}\frac{\partial}{\partial a_i^m}\varphi_{\tilde{a}_i,\tilde{\lambda}_i} > \\ =& < v-\tilde{v}, \frac{1}{\lambda_i}\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i} > - < \tilde{v}, \frac{1}{\tilde{\lambda}_i}\frac{\partial}{\partial a_i^m}\varphi_{\tilde{a}_i,\tilde{\lambda}_i} - \frac{1}{\lambda_i}\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i} > \\ =& \sum_{j=1}^{p} < \tilde{\alpha}_j\varphi_{\tilde{a}_j,\tilde{\lambda}_j} - \alpha_j\varphi_{a_j,\lambda_j}, \frac{1}{\lambda_i}\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i} > - < \tilde{v}, \frac{1}{\tilde{\lambda}_i}\frac{\partial}{\partial a_i^m}\varphi_{\tilde{a}_i,\tilde{\lambda}_i} - \frac{1}{\lambda_i}\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i} > \end{aligned} \tag{3.16}$$

From Lemma 13 and 14 in appendix D and (3.16) we derive the following inequality

$$\begin{aligned} c\sum_{j=1}^{p}\lambda_j d_g(a_j,\tilde{a_j}) \le& o(1)\left(\sum_{j=1}^{p}\lambda_j d_g(a_j,\tilde{a}_j) + \left|1-\frac{\tilde{\lambda}_j}{\lambda_j}\right| + |\alpha_j-\tilde{\alpha}_j|\right) \\ &+ O\left(\sum_{j=1}^{p}\lambda_j^2 d_g(a_j,\tilde{a}_j)^2 + \left|1-\frac{\tilde{\lambda}_j}{\lambda_j}\right|^2 + |\alpha_j-\tilde{\alpha}_j|^2\right). \end{aligned} \tag{3.17}$$

In a last step, we take the derivative of f w.r.t. λ_i to compute

$$\begin{aligned} 0 =& < v, \lambda_i\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i} > - < \tilde{v}, \tilde{\lambda}_i\frac{\partial}{\partial\lambda_i}\varphi_{\tilde{a}_i,\tilde{\lambda}_i} > \\ =& \sum_{j=1}^{p} < \tilde{\alpha}_j\varphi_{\tilde{a}_j,\tilde{\lambda}_j} - \alpha_j\varphi_{a_j,\lambda_j}, \lambda_i\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i} > - < \tilde{v}, \tilde{\lambda}_i\frac{\partial}{\partial\lambda_i}\varphi_{\tilde{a}_i,\tilde{\lambda}_i} - \lambda_i\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i} > . \end{aligned} \tag{3.18}$$

Again from (3.18) and Lemma 13 and 14 in appendix D we get the inequality

$$\begin{aligned} c\sum_{j=1}^{p}\left|1-\frac{\tilde{\lambda}_j}{\lambda_j}\right| \le& o(1)\left(\sum_{j=1}^{p}\lambda_j d_g(a_j,\tilde{a}_j) + \left|1-\frac{\tilde{\lambda}_j}{\lambda_j}\right| + |\alpha_j-\tilde{\alpha}_j|\right) \\ &+ O\left(\sum_{j=1}^{p}\lambda_j^2 d_g(a_j,\tilde{a}_j)^2 + \left|1-\frac{\tilde{\lambda}_j}{\lambda_j}\right|^2 + |\alpha_j-\tilde{\alpha}_j|^2\right). \end{aligned} \tag{3.19}$$

Finally from (3.14), (3.17) and (3.19) we obtain

$$\begin{aligned} &c\left(\sum_{j=1}^{p}\lambda_j d_g(a_j,\tilde{a}_j) + \left|1-\frac{\tilde{\lambda}_j}{\lambda_j}\right| + |\alpha_j-\tilde{\alpha}_j|\right) \\ &\le o(1)\left(\sum_{j=1}^{p}\lambda_j d_g(a_j,\tilde{a}_j) + \left|1-\frac{\tilde{\lambda}_j}{\lambda_j}\right| + |\alpha_j-\tilde{\alpha}_j|\right) \\ &\quad + O\left(\sum_{j=1}^{p}\lambda_j^2 d_g(a_j,\tilde{a}_j)^2 + \left|1-\frac{\tilde{\lambda}_j}{\lambda_j}\right|^2 + |\alpha_j-\tilde{\alpha}_j|^2\right), \end{aligned}$$

which contradicts (3.12) for n large. Hence the minimizer must be unique for ε small. Thus we have proved the minimization part of the Proposition. The stated estimate is an immediate consequence of Remark 1. □

Due to Proposition 5 and Remark 1 every $u \in W(p,\varepsilon)$ has a unique presentation

$$u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v,$$

where $(\alpha, a, \lambda) \in B^p_{2\varepsilon,\gamma}$ is unique up to permutation and

$$v \in E_{(\alpha,a,\lambda)} := \left\langle \varphi_{a_i,\lambda_i}, \lambda_i \frac{\partial}{\partial \lambda_i}\varphi_{a_i,\lambda_i}, \frac{1}{\lambda_i}\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i} \;\middle|\; 1 \le i \le p,\; 1 \le m \le 3 \right\rangle^{\perp} \subset H^1(M).$$

Furthermore

$$\left| \alpha_i - \frac{1}{\sqrt{2I_0}} \left(\sum_{j=1}^{p} \left(\frac{K(a_i)}{K(a_j)} \right)^2 \right)^{-\frac{1}{2}} \right| \to 0$$

uniformly if $\varepsilon \to 0$.

We define new neighbourhoods of non-converging Palais-Smale sequences, which we will use from now on. For $p \in \mathbb{N}$ and $\varepsilon > 0$ set

$$\begin{aligned} V(p,\varepsilon) := \Bigg\{ u \in \Sigma \;\Bigg|\; & u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v \text{ s.t. } \lambda_i > \frac{1}{\varepsilon} \;\forall i,\; \varepsilon_{ij} < \varepsilon \;\forall\, i \neq j, \\ & v \in E_{(\alpha,a,\lambda)},\; \|v\| < \varepsilon,\; \left| \alpha_i - \frac{1}{\sqrt{2I_0}} \left(\sum_{j=1}^{p} \left(\frac{K(a_i)}{K(a_j)} \right)^2 \right)^{-\frac{1}{2}} \right| < \varepsilon \;\forall i \Bigg\}. \end{aligned} \tag{3.20}$$

Clearly $V(p,\varepsilon_1) \subset W(p,\varepsilon_2) \subset V(p,\varepsilon_3)$ for $\varepsilon_1 << \varepsilon_2 << \varepsilon_3$. Therefore we can work in $V(p,\varepsilon)$ instead of $W(p,\varepsilon)$.

3.2. The modified functional

The negative gradient flow of J induces a "shadow" flow with respect to the variables (α, a, λ, v) for $u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v \in V(p,\varepsilon)$. We would like to construct a pseudo-gradient, which represents the major terms of this shadow flow and which simplifies the movement of (α, a, λ, v).

Unfortunately, for our argument the functional J is not smooth enough. Nevertheless it is possible to remedy this issue by introducing the new functional:

$$I(u) := \frac{||u||^2}{\left(\int_{\partial M} K(x)u^3 d\sigma_g\right)^{\frac{2}{3}}},$$

defined on the set

$$U := \Sigma \cap \left\{ u \in H^1(M) \;\middle|\; \int_{\partial M} K(x)u^3 d\sigma_g > 0 \right\}.$$

This replacement is justified by the following Lemma:

Lemma 3. *(a) Critical points of I are critical points of J.*

(b) Palais-Smale sequences of I in U are Palais-Smale sequences of J in $V_{\varepsilon_0}(\Sigma^+)$.

Proof. First we prove (a). Let $u \in U$ be a critical point of I, then

$$0 =< u, h > - \frac{1}{\int_{\partial M} K(x)u^3 d\sigma_g} \int_{\partial M} K(x)u^2 h d\sigma_g$$

for all $h \in H^1(M)$, especially for $h = u^- = \max\{0, -u\}$, which implies

$$||u^-||^2 + \frac{1}{\int_{\partial M} K(x)u^3 d\sigma_g} \int_{\partial M} K(x)(u^-)^3 d\sigma_g = 0$$

and therefore $u \geq 0$ a.e. in M. Hence u is also a critical point of J.
(b) Let $(u_n)_n$ be a Palais-Smale sequence of I. Since I is bounded form below

$$\sup_{||h|| \leq 1} \left| < u_n, u > - I(u_n)^{\frac{3}{2}} \int_{\partial M} K(x)u_n^2 h d\sigma_g \right| = o(1).$$

$||u_n^-|| \leq 1$, which implies $||u_n^-|| \to 0$ if $n \to \infty$. Hence we derive

$$\begin{aligned}\int_{\partial M} K(x)|u_n|^3 d\sigma_g &= \int_{\partial M} K(x)\left((u_n^+)^3 + (u_n^-)^3\right) d\sigma_g = \int_{\partial M} K(x)\left((u_n^+)^3 - (u_n^-)^3\right) d\sigma_g \\ &\quad + 2\int_{\partial M} K(x)(u_n^-)^3 d\sigma_g = \int_{\partial M} K(x)u_n^3 d\sigma_g + o(1),\end{aligned}$$

which yields $J(u_n) = I(u_n) + o(1)$. Therefore

$$\begin{aligned} DJ(u_n)[h] &= (2I(u_n) + o(1)) \left(< u_n, h > - \left(I(u_n)^{\frac{3}{2}} + o(1)\right) \int_{\partial M} K(x)u_n|u_n|h \right) \\ &= 2I(u_n) \left(< u_n, h > - I(u_n)^{\frac{3}{2}} \int_{\partial M} K(x)u_n|u_n|h \right) + o(1)||h|| \\ &= 2I(u_n) \left(< u_n, h > - I(u_n)^{\frac{3}{2}} \int_{\partial M} K(x)u_n^2 h \right) + o(1)||h|| \\ &= DI(u_n)[h] + o(1)||h||. \end{aligned}$$

Thus $(u_n)_n$ is a Palais-Smale sequence of J. Furthermore $(u_n)_n \subset V_{\varepsilon_0}(\Sigma^+)$ if n is large. □

From now on and in the rest of our proof we assume that I does not have any critical point in U. This assumption yields the following Proposition.

Proposition 6. *Let $(u_n)_n \subset U$ be a Palais-Smale sequence of I. Then there exist $p \in \mathbb{N}$ and a sequence $\varepsilon_n \to 0$ such that $u_n \in V(p, \varepsilon_n)$ along a subsequence.*

Proof. The result follows through Proposition 1, Lemma 3 and the definition of $V(p,\varepsilon)$. □

From now on we again write $J(u)$ instead of $I(u)$, hence

$$J(u) = \frac{||u||^2}{\left(\int_{\partial M} K(x) u^3 d\sigma_g\right)^{\frac{2}{3}}}.$$

Since we assume that J does not have critical points, flow lines of the negative gradient flow will enter $V(p,\varepsilon)$ for some $p \geq 1$. Therefore we need to understand the behaviour of J in $V(p,\varepsilon)$. A first step is the expansion of J in $V(p,\varepsilon)$ which gives us a first understanding of the behaviour in $V(p,\varepsilon)$ with respect to the variables (α, a, λ, v). This expansion will be done in the next chapter.

4. Expansion of the functional

In this chapter we expand the functional in $V(p, \varepsilon)$ which will give us a rough idea how the functional behaves with respect to the variables (α, a, λ, v). Again let $G(\cdot, \cdot)$ be the Green's function of the operator (L_g, B_g). Moreover let $H_a(x)$ be the regular part, which appears in the expansion of the Green's function $G_a(a, \cdot)$ with respect to the operator (L_{g_a}, B_{g_a}). See appendix E for more details.

For $(\lambda_i, a_i), (\lambda_j, a_j) \in \mathbb{R}_+ \times \partial M$ we define the interaction

$$\begin{aligned} I(\varepsilon_{ij}) := & u_{a_j}(a_i)\chi_\rho(|\psi_{a_j}(a_i)|) \left(\frac{1}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j d_{g_{a_j}}(a_i, a_j)^2} \right) \\ & + \big(1 - \chi_\rho(|\psi_{a_j}(a_i)|)\big) \frac{G(a_i, a_j)}{\lambda_i \lambda_j}. \end{aligned} \tag{4.1}$$

in case $\lambda_i \geq \lambda_j$ and $I(\varepsilon_{ij}) := I(\varepsilon_{ji})$ in case $\lambda_j > \lambda_i$.

Let $|\Pi(a)|$ be the norm of the umbilicity tensor (see (1.9)) at $a \in \partial M$ with respect to g_a, then the functional can be expanded as follows:

Proposition 7. *Let $u = \sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + v \in V(p,\varepsilon)$ and $2 \le \lambda_i \rho^2$ $\forall i$, then*

$$
\begin{aligned}
J(u) = & \frac{Q(B^4,\partial B^4)\sum_{i=1}^p \alpha_i^2}{\left(\sum_{i=1}^p \alpha_i^3 K(a_i)\right)^{\frac{2}{3}}} \Bigg\{ 1 + \frac{J}{I_0 6} \sum_{i=1}^p \frac{\alpha_i^2}{\sum_{j=1}^p \alpha_j^2} \frac{|\Pi(a_i)|^2 \log(\lambda_i\rho)}{\lambda_i^2} \\
& - \sum_{i=1}^p \left(\frac{\alpha_i^2}{\sum_{j=1}^p \alpha_j^2} \frac{|S_+^3| H_{a_i}(a_i)}{I_0} + \frac{\alpha_i^3}{\sum_{j=1}^p \alpha_j^3 K(a_j) I_0} \frac{I_4}{9} \Delta K(a_i) \right) \frac{1}{\lambda_i^2} \\
& + 2I_1 \sum_{i\neq j} \left(\frac{\alpha_i\alpha_j}{\sum_{l=1}^p 2I_0\alpha_l^2} - \frac{K(a_i)\alpha_i^2\alpha_j}{\sum_{l=1}^p \alpha_l^3 K(a_l) I_0} \right) I(\varepsilon_{ij}) - \frac{2}{\sum_{j=1}^p \alpha_j^3 K(a_j) I_0} f^*(v) \\
& + \frac{1}{2\sum_{j=1}^p \alpha_j^2 I_0} \left(||v||^2 - 4\sum_{i=1}^p \frac{\alpha_i K(a_i) \sum_{j=1}^p \alpha_j^2}{\sum_{l=1}^p \alpha_l^3 K(a_l)} \int_{\partial M} \varphi_{a_i,\lambda_i} v^2 d\sigma_g \right) \Bigg\} \\
& + O\left(\sum_{i\neq j} \rho\varepsilon_{ij}\right) + o\left(\sum_{i\neq j} \varepsilon_{ij}\right) + O\left(\sum_{i=1}^p \frac{\rho}{\lambda_i^2} + \frac{|\Pi(a_i)|^2}{\lambda_i^2}\right) + o\left(\sum_{i=1}^p \frac{1}{\lambda_i^2}\right) \\
& + O\left(\sum_{i=1}^p \frac{\log(\lambda_i\rho)}{\lambda_i} + \frac{1}{\lambda_i\rho}\right) ||v||^2 + O(||v||^3).
\end{aligned}
$$

Here $f^*(v)$ is a linear map on $E_{(\alpha,a,\lambda)}$ given by

$$f^*(v) = \int_{\partial M} K(x) \left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i}\right)^2 v d\sigma_g$$

such that

$$
\begin{aligned}
& ||f^*|| \\
\le & C\left(\sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{\log(\lambda_i)^{\frac{2}{3}}}{\lambda_i^2} + \frac{1}{\lambda_i^2\rho^2} + \frac{|\Pi(a_i)|\log(\lambda_i\rho)^{\frac{3}{4}}}{\lambda_i} + \frac{\rho}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij} \log(\varepsilon_{ij}^{-1})^{\frac{2}{3}}\right).
\end{aligned}
\tag{4.2}
$$

Moreover J, I_0, I_1 are positive constants.

Proof. First we expand the nominator. Since $v \in E_{(\alpha,a,\lambda)}$

$$||u||^2 = \sum_{i=1}^p \alpha_i^2 ||\varphi_{a_i,\lambda_i}||^2 + \sum_{i\neq j} \alpha_i\alpha_j \langle \varphi_{a_i,\lambda_i}, \varphi_{a_j,\lambda_j} \rangle + ||v||^2.$$

We compute the norm of $\varphi_{a,\lambda}$, therefore we essentially follow the computations in [15, 28]. Since

$$h_{g_a}(x) = \frac{1}{2}\partial_t\big(\log(\det(g_a(x))) = O(|x|^9)$$

in ψ_a Fermi-coordinates with respect to the metric g_a, we conclude $h_{g_a}(a) = \nabla h_{g_a}(a) = 0$. This identities will become very useful in the sequel. Furthermore we often will apply the following formula:

$$\int_{S_r^{n-2}} q dS = \frac{r^2}{d(d+n-3)} \int_{S_r^{n-2}} \Delta q dS \tag{4.3}$$

for a homogeneous polynomial q of degree d.

Now we expand the norm:

$$||\varphi_{a,\lambda}||^2 = \int_M \big(|\nabla \hat{\varphi}_{a,\lambda}|^2_{g_a} + \frac{1}{6} R_{g_a} \hat{\varphi}^2_{a,\lambda}\big) dV_{g_a} + \int_{\partial M} h_{g_a} \hat{\varphi}^2_{a,\lambda} d\sigma_{g_a}. \tag{4.4}$$

In the following we identify $B_\rho^+ \subset \mathbb{R}^4_+$ with its image under ψ_a^{-1}. We begin with a local expansion of the gradient.

$$\begin{aligned}\int_{B_\rho^+} |\nabla \hat{\varphi}_{a,\lambda}|^2_{g_a} dV_{g_a} &= \int_{B_\rho^+} \big(|\nabla \hat{\varphi}_{a,\lambda}|^2 + (g^{ij} - \delta^{ij}) \partial_i \hat{\varphi}_{a,\lambda} \partial_j \hat{\varphi}_{a,\lambda}\big)\big(1 + O(|(x,t)|^{10})\big) dx dt \\ &= \int_{B_\rho^+} \big(|\nabla \hat{\varphi}_{a,\lambda}|^2 + (g^{ij} - \delta^{ij}) \partial_i \hat{\varphi}_{a,\lambda} \partial_j \hat{\varphi}_{a,\lambda}\big) dx dt + O\left(\frac{\rho^2}{\lambda^2}\right). \end{aligned} \tag{4.5}$$

First observe

$$\int_{B_\rho^+} |\nabla \hat{\varphi}_{a,\lambda}|^2 = \int_{B_\rho^+} |\nabla \delta_\lambda|^2 = 2I_0 + \int_{S^3_{\rho,+}} \partial_\nu \delta_\lambda \delta_\lambda dS + O\left(\frac{1}{(\lambda\rho)^3}\right). \tag{4.6}$$

Next we expand

$$\int_{B_\rho^+} (g^{ij} - \delta^{ij}) \partial_i \hat{\varphi}_{a,\lambda} \partial_j \hat{\varphi}_{a,\lambda} = \int_{B_\rho^+} (g^{ij} - \delta^{ij}) \partial_i \delta_\lambda \partial_j \delta_\lambda. \tag{4.7}$$

Let R_{ijkl} and $\bar{R}_{\mu,\nu,\alpha,\beta}$ be the coefficients of the curvature tensors of ∂M and M respectively and h_{ij} be the coefficients of the second fundamental form of the metric g_a in Fermi-coordinates, then (see (A.1)):

$$g^{ij} = \delta^{ij} + 2h_{ij}t + \frac{1}{3} R_{ikjl} x_k x_l + 2\partial_k h_{ij} t x_k + (\bar{R}_{ninj} + 3h_{ik}h_{kj}) t^2 + O(|(x,t)|^3). \tag{4.8}$$

We use (4.3), (4.8) as well as $\bar{R}_{nn} = -|\Pi(a)|^2$ to compute

$$\int_{B_\rho^+} (g^{ij} - \delta^{ij}) \partial_i \delta_\lambda \partial_j \delta_\lambda = \frac{8}{3} \frac{|\Pi(a)|^2}{\lambda^2} \int_{B_{\lambda\rho}^+} \frac{t^2 |x|^2}{\big((1+t)^2 + |x|^2\big)^4} + O\left(\frac{\rho}{\lambda^2}\right). \tag{4.9}$$

Furthermore

$$\begin{aligned}\frac{1}{6} \int_{B_\rho^+} R_{g_a} \hat{\varphi}^2_{a,\lambda} dV_{g_a} &= \frac{1}{6} \int_{B_\rho^+} R_{g_a} \hat{\varphi}^2_{a,\lambda} dx dt + O\left(\frac{\rho}{\lambda^2}\right) \\ &= -\frac{1}{6} \frac{|\Pi(a)|^2}{\lambda^2} \int_{B_{\lambda\rho}^+} \frac{1}{\big((1+t)^2 + |x|^2\big)^2} + O\left(\frac{\rho}{\lambda^2}\right), \end{aligned} \tag{4.10}$$

where we used that $R_{g_a}(x,t) = -|\Pi(a)|^2 + O(|(x,t)|)$ in Fermi-coordinates. Finally, since $h_{g_a}(x) = O(|x|^9)$ in Fermi coordinates,

$$\int_{B_\rho} h_g \hat{\varphi}_{a,\lambda}^2 d\sigma_{g_a} = O\left(\frac{\rho}{\lambda^2}\right). \tag{4.11}$$

From (4.4), (4.6) and (4.9)-(4.11) we derive the expansion

$$\begin{aligned}
&\int_{B_\rho^+} \left(|\nabla \hat{\varphi}_{a,\lambda}|_{g_a}^2 + \frac{1}{6} R_{g_a} \hat{\varphi}_{a,\lambda}^2\right) dV_{g_a} + \int_{B_\rho} h_{g_a} \hat{\varphi}_{a,\lambda}^2 d\sigma_{g_a} = 2I_0 + \int_{S^3_{\rho,+}} \partial_\nu \delta_\lambda \delta_\lambda dS \\
&+ \frac{8}{3} \frac{|\Pi(a)|^2}{\lambda^2} \int_{B^+_{\lambda\rho}} \frac{t^2 |x|^2}{\left((1+t)^2 + |x|^2\right)^4} - \frac{1}{6} \frac{|\Pi(a)|^2}{\lambda^2} \int_{B^+_{\lambda\rho}} \frac{1}{\left((1+t)^2 + |x|^2\right)^2} \\
&+ O\left(\frac{\rho}{\lambda^2}\right) + O\left(\frac{1}{(\lambda\rho)^3}\right).
\end{aligned} \tag{4.12}$$

It remains to compute

$$\begin{aligned}
&\int_{M \setminus B_\rho^+} \left(|\nabla \hat{\varphi}_{a,\lambda}|_{g_a}^2 + \frac{1}{6} R_{g_a} \hat{\varphi}_{a,\lambda}^2\right) dV_{g_a} + \int_{\partial M \setminus B_\rho} h_{g_a} \hat{\varphi}_{a,\lambda}^2 d\sigma_{g_a} \\
&= \int_{M \setminus B_\rho^+} L_{g_a} \hat{\varphi}_{a,\lambda} \left(\hat{\varphi}_{a,\lambda} - \frac{G_a(a,\cdot)}{\lambda}\right) dV_{g_a} + \int_{S^3_{\rho,+}} \left(\partial_\nu \hat{\varphi}_{a,\lambda} \frac{G_a(a,\cdot)}{\lambda} - \partial_\nu \frac{G_a(a,\cdot)}{\lambda} \hat{\varphi}_{a,\lambda}\right) d\sigma_{g_a} \\
&+ \int_{\partial M \setminus B_\rho} B_{g_a} \hat{\varphi}_{a,\lambda} \left(\hat{\varphi}_{a,\lambda} - \frac{G_a(a,\cdot)}{\lambda}\right) d\sigma_{g_a} + \int_{S^3_{\rho,+}} \partial_\nu \hat{\varphi}_{a,\lambda} \hat{\varphi}_{a,\lambda} d\sigma_{g_a}.
\end{aligned} \tag{4.13}$$

We use the expansion of the Green's function (see appendix E) as well as the definition of the bubble to get the estimate

$$\left|\hat{\varphi}_{a,\lambda} - \frac{G_a(a,\cdot)}{\lambda}\right| = \chi_\rho \left|\delta_\lambda - \frac{G_a(a,\cdot)}{\lambda}\right| \leq C\left(\frac{|\Pi(a)|}{\lambda\rho} + \frac{1}{\lambda} + \frac{1}{\lambda^2 \rho^3}\right) \tag{4.14}$$

provided $\lambda\rho \geq 2$ and $d_{g_a}(a,x) \geq \rho$. Furthermore, under the previous conditions, Proposition 2 yields

$$|L_{g_a} \hat{\varphi}_{a,\lambda}| \leq C\left(\frac{1}{\lambda^2 \rho^5} + \frac{|\Pi(a)|}{\lambda \rho^3}\right). \tag{4.15}$$

From (4.14) and (4.15) we derive the estimate

$$\int_M L_{g_a} \hat{\varphi}_{a,\lambda} \left(\hat{\varphi}_{a,\lambda} - \frac{G_a(a,\cdot)}{\lambda}\right) dV_{g_a} \leq C\left(\frac{|\Pi(a)|^2}{\lambda^2} + \frac{\rho^2}{\lambda^2} + \frac{1}{\lambda^3 \rho^2}\right). \tag{4.16}$$

In the following we expand the last integral in (4.13).

$$\begin{aligned}
\int_{S^3_{\rho,+}} \partial_\nu \hat{\varphi}_{a,\lambda} \hat{\varphi}_{a,\lambda} d\sigma_{g_a} &= \int_{S^3_{\rho,+}} \frac{\sqrt{g}}{\rho} g^{ij} \partial_i \delta_\lambda x_j \delta_\lambda dS \\
&= \int_{S^3_{\rho,+}} \frac{1}{\rho} g^{ij} \partial_i \delta_\lambda x_j \delta_\lambda dS + O\left(\frac{\rho^2}{\lambda^2}\right) \\
&= \int_{S^3_{\rho,+}} \partial_\nu \delta_\lambda \delta_\lambda dS + O\left(\frac{|\Pi(a)|^2}{\lambda^2} + \frac{\rho^2}{\lambda^2} + \frac{1}{\lambda^3 \rho^2}\right).
\end{aligned} \tag{4.17}$$

Furthermore we find the estimate

$$\left|\int_{\partial M\setminus B_\rho} B_{g_a}\hat{\varphi}_{a,\lambda}(\hat{\varphi}_{a,\lambda}-\frac{G_a(a,\cdot)}{\lambda})d\sigma_{g_a}\right| \le C\left(\frac{|\Pi(a)|^2}{\lambda^2}+\frac{\rho^2}{\lambda^2}+\frac{1}{\lambda^3\rho^2}\right). \tag{4.18}$$

Finally we need to expand

$$\begin{aligned}
&\int_{S^3_{\rho,+}}\left(\partial_\nu\hat{\varphi}_{a,\lambda}\frac{G_a(a,\cdot)}{\lambda}-\partial_\nu\frac{G_a(a,\cdot)}{\lambda}\hat{\varphi}_{a,\lambda}\right)d\sigma_{g_a}\\
&=\int_{S^3_{\rho,+}}\left(\partial_\nu\hat{\varphi}_{a,\lambda}\frac{G_a(a,\cdot)}{\lambda}-\partial_\nu\frac{G_a(a,\cdot)}{\lambda}\hat{\varphi}_{a,\lambda}\right)dS\\
&+\int_{S^3_{\rho,+}}(g^{ij}-\delta^{ij})\left(\partial_i\hat{\varphi}_{a,\lambda}\frac{G_a(a,\cdot)}{\lambda}-\partial_i\frac{G_a(a,\cdot)}{\lambda}\hat{\varphi}_{a,\lambda}\right)\frac{x_j}{\rho}dS+O\left(\frac{\rho^2}{\lambda^2}\right)\\
&=\int_{S^3_{\rho,+}}\left(\partial_\nu\delta_\lambda\frac{G_a(a,\cdot)}{\lambda}-\partial_\nu\frac{G_a(a,\cdot)}{\lambda}\delta_\lambda\right)dS+O\left(\frac{|\Pi(a)|^2}{\lambda^2}+\frac{\rho^2}{\lambda^2}\right)\\
&=\frac{1}{\lambda^2}\int_{S^3_{\rho,+}}\left(\partial_\nu\frac{1}{|(x,t)|^2}G_a(a,\cdot)-\partial_\nu G_a(a,\cdot)\frac{1}{|(x,t)|^2}\right)dS+O\left(\frac{|\Pi(a)|^2}{\lambda^2}+\frac{\rho}{\lambda^2}+\frac{1}{\lambda^3\rho^3}\right)\\
&=-2|S^3_+|\frac{H_a(a)}{\lambda^2}+O\left(\frac{|\Pi(a)|^2|\log(\rho)|}{\lambda^2}+\frac{\rho}{\lambda^2}+\frac{1}{\lambda^3\rho^3}\right).
\end{aligned}\tag{4.19}$$

By adding (4.12),(4.13) and (4.16) - (4.19) we have proved the following expansion if $\lambda\rho\ge 2$:

$$\begin{aligned}
||\varphi_{a,\lambda}||^2 = {}& 2I_0-2|S^3_+|\frac{H_a(a)}{\lambda^2}\\
&+\frac{8}{3}\frac{|\Pi(a)|^2}{\lambda^2}\int_{B^+_{\lambda\rho}}\frac{t^2|x|^2}{\left((1+t)^2+|x|^2\right)^4}-\frac{1}{6}\frac{|\Pi(a)|^2}{\lambda^2}\int_{B^+_{\lambda\rho}}\frac{1}{\left((1+t)^2+|x|^2\right)^2}\\
&+O\left(\frac{\rho}{\lambda^2}+\frac{1}{\lambda^3\rho^3}\right)+o\left(\frac{1}{\lambda^2}\right).
\end{aligned}\tag{4.20}$$

Furthermore, due to ([28]),

$$\int_{B^+_{\lambda\rho}}\frac{t^2|x|^2}{\left((1+t)^2+|x|^2\right)^4}=\log(\lambda\rho)|S^2|\int_0^\infty\frac{r^4}{(1+r^2)^4}dr+O(1),$$

$$\int_{B^+_{\lambda\rho}}\frac{1}{\left((1+t)^2+|x|^2\right)^2}=\log(\lambda\rho)|S^2|\int_0^\infty\frac{r^2}{(1+r^2)^2}dr+O(1).$$

Since

$$\int_0^\infty\frac{r^4}{(1+r^2)^4}dr=\frac{1}{8}\int_0^\infty\frac{r^2}{(1+r^2)^2}dr,$$

$$\begin{aligned}
||\varphi_{a,\lambda}||^2 = {}& 2I_0-2|S^3_+|\frac{H_a(a)}{\lambda^2}+\frac{1}{3}\frac{|\Pi(a)|^2\log(\lambda\rho)}{\lambda^2}J\\
&+O\left(\frac{|\Pi(a)|^2}{\lambda^2}+\frac{\rho}{\lambda^2}+\frac{1}{\lambda^3\rho^3}\right)+o\left(\frac{1}{\lambda^2}\right),
\end{aligned}\tag{4.21}$$

where

$$J = |S^2| \int_0^\infty \frac{r^2}{(1+r^2)^2} dr.$$

In Proposition 25, appendix B, we expand the interaction of two different bubbles, which is given by:

$$\langle \varphi_{a_i,\lambda_i}, \varphi_{a_j,\lambda_j} \rangle = 2I_1 I(\varepsilon_{ij}) + O(\rho\varepsilon_{ij}) + o(\varepsilon_{ij}) \quad (\varepsilon_{ij} \to 0),$$

provided $2 \le \lambda_i \rho^2, \lambda_j \rho^2$. Hence (4.21) implies

$$||u||^2 = 2I_0 \sum_{i=1}^p \alpha_i^2 - 2|S^3_+| \sum_{i=1}^p \alpha_i^2 \frac{H_{a_i}(a_i)}{\lambda_i^2} + \frac{J}{3} \sum_{i=1}^p \alpha_i^2 \frac{|\Pi(a_i)|^2 \log(\lambda_i \rho)}{\lambda_i^2} \tag{4.22}$$

$$+ 2I_1 \sum_{i \neq j} \alpha_i \alpha_j I(\varepsilon_{ij})$$

$$+ O\left(\sum_{i=1}^p \frac{|\Pi(a_i)|^2}{\lambda_i^2} + \frac{\rho}{\lambda_i^2} + \frac{1}{\lambda_i^3 \rho^3} \right) + O\left(\sum_{i \neq j} \rho \varepsilon_{ij} \right) + o\left(\sum_{i \neq j} \varepsilon_{ij} \right)$$

$$+ ||v||^2. \tag{4.23}$$

Now we turn to the expansion of the denominator:

$$\int_{\partial M} K(x) \left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + v \right)^3 d\sigma_g = \int_{\partial M} K(x) \left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} \right)^3 d\sigma_g$$

$$+ 3 \int_{\partial M} K(x) \left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} \right)^2 v d\sigma_g + 3 \int_{\partial M} K(x) \left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} \right) v^2 d\sigma_g + O(||v||^3). \tag{4.24}$$

First we compute

$$\int_{\partial M} K(x) \left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} \right)^3 d\sigma_g = \sum_{i=1}^p \alpha_i^3 \int_{\partial M} K(x) \varphi^3_{a_i,\lambda_i} d\sigma_g$$

$$+ 3 \sum_{i \neq j} \alpha_i^2 \alpha_j \int_{\partial M} K(x) \varphi^2_{a_i,\lambda_i} \varphi_{a_j,\lambda_j} d\sigma_g + O\left(\sum_{i \neq j} \int_{\partial M} \varphi^{\frac{3}{2}}_{a_i,\lambda_i} \varphi^{\frac{3}{2}}_{a_j,\lambda_j} d\sigma_g \right). \tag{4.25}$$

$$\int_{\partial M} K(x) \varphi^3_{a_i,\lambda_i} d\sigma_g = K(a_i) \int_{\partial M} \hat\varphi^3_{a_i,\lambda_i} d\sigma_{g_{a_i}} + \int_{\partial M} \big(K(x) - K(a_i)\big) \hat\varphi^3_{a_i,\lambda_i} d\sigma_{g_{a_i}}$$

and

$$\begin{aligned} \int_{\partial M} \hat\varphi^3_{a_i,\lambda_i} d\sigma_{g_{a_i}} &= \int_{B_\rho} \hat\varphi^3_{a_i,\lambda_i} d\sigma_{g_{a_i}} + \int_{\partial M \setminus B_\rho} \hat\varphi^3_{a_i,\lambda_i} d\sigma_{g_{a_i}} \\ &= I_0 + O\left(\frac{|\Pi(a_i)|^2}{\lambda_i^2} + \frac{1}{(\lambda_i \rho)^3} \right) + \int_{\partial M \setminus B_\rho} \hat\varphi^3_{a_i,\lambda_i} d\sigma_{g_{a_i}} \\ &= I_0 + O\left(\frac{|\Pi(a_i)|^2}{\lambda_i^2} + \frac{1}{(\lambda_i \rho)^3} \right). \end{aligned}$$

Moreover

$$\int_{\partial M}\big(K(x)-K(a_i)\big)\hat{\varphi}^3_{a_i,\lambda_i}d\sigma_{g_{a_i}} = \int_{B_\rho}\big(K(x)-K(a_i)\big)\hat{\varphi}^3_{a_i,\lambda_i}dx + O\left(\frac{1}{(\lambda_i\rho)^3}\right)$$

$$= \int_{B_\rho}\big(K(x)-K(a_i)\big)\delta^3_{\lambda_i}dx + O\left(\frac{1}{(\lambda_i\rho)^3}\right)$$

$$= \frac{I_4}{6}\frac{\Delta K(a_i)}{\lambda_i^2} + O\left(\frac{1}{(\lambda_i\rho)^3}\right) + o\left(\frac{1}{\lambda_i^2}\right),$$

where

$$I_4 = \int_{\mathbb{R}^3}\frac{|x|^2}{\left(1+|x|^2\right)^3}dx.$$

Hence

$$\int_{\partial M}K(x)\varphi^3_{a_i,\lambda_i}d\sigma_g = K(a_i)I_0 + \frac{I_4}{6}\frac{\Delta K(a_i)}{\lambda_i^2} + O\left(\frac{1}{(\lambda_i\rho)^3}\right) + o\left(\frac{1}{\lambda_i^2}\right). \tag{4.26}$$

Next we expand the second integral in (4.25). Proposition 2 implies

$$\int_{\partial M}\varphi^2_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}d\sigma_g = \frac{1}{2}\int_{\partial M}B_g\varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}d\sigma_g + O\left(\frac{1}{\lambda_i^2\lambda_j\rho^4}\right) + O\left(\frac{\rho}{\lambda_i\lambda_j}\right). \tag{4.27}$$

Thus

$$\int_{\partial M}\varphi^2_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}d\sigma_g = \frac{1}{2}\langle\varphi_{a_i,\lambda_i},\varphi_{a_j,\lambda_j}\rangle - \frac{1}{2}\int_M L_g\varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}dV_g$$
$$+ O\left(\frac{\rho}{\lambda_i\lambda_j}\right) + o\left(\frac{1}{\lambda_i^2}+\frac{1}{\lambda_j^2}\right)$$

Since u_a is smooth, there exist constants $C, c, \rho_0 > 0$ such that

$$cd_{g_a}(a,x)^2 \le |\psi_a(x)|^2 \le Cd_{g_a}(a,x)^2 \quad \forall a\in\partial M, \forall x\in M \ : d_{g_a}(a,x)\le 2\rho_0 \tag{4.28}$$

and

$$cd_{g_a}(a,x)^2 \le d_g(a,x)^2 \le Cd_{g_a}(a,x)^2 \ \forall a\in\partial M,\ \forall x\in M. \tag{4.29}$$

Therefore, from Proposition 2, (4.28), (4.29) and the estimate

$$\varphi_{a,\lambda} \le C\left(\frac{\lambda}{\big(1+\lambda d_g(a,x)\big)^2}\right) \ \text{ for } \ \lambda\rho\ge 2, \tag{4.30}$$

we derive

$$\int_M L_g\varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}dV_g = O\left(\int_{B^+_{2\rho}(a_i)}\left(\frac{\lambda_i^2}{(1+\lambda_i d_g(a_i,x))^3}\right)\left(\frac{\lambda_j}{(1+\lambda_j d_g(a_j,x))^2}\right)dV_g\right)$$
$$+\frac{1}{\rho}O\left(\int_{B^+_{2\rho}(a_i)}\left(\frac{\lambda_i}{(1+\lambda_i d_g(a_i,x))^2}\right)\left(\frac{\lambda_j}{(1+\lambda_j d_g(a_j,x))^2}\right)dV_g\right).$$

If $\lambda_i \geq \lambda_j$ and $\frac{\lambda_i}{\lambda_j} \geq \lambda_i\lambda_j d_g(a_i,a_j)^2$ the previous integral can be easily estimated by

$$\left|\int_M L_g\varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}dV_g\right| \leq C\rho\varepsilon_{ij}.$$

In all other cases we integrate both integrals over $B^+_{2\rho}(a_i)\cap A$ and $B^+_{2\rho}(a_i)\cap A^c$, where

$$A = \left\{x \in M \mid 2d_g(a_j,x) \leq \frac{1}{\lambda_i} + d_g(a_i,a_j)\right\} \tag{4.31}$$

to obtain

$$\left|\int_M L_g\varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}dV_g\right| \leq C\rho\varepsilon_{ij}.$$

Hence, in any case

$$\int_{\partial M} K(x)\varphi^2_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}d\sigma_g = K(a_i)I_1I(\varepsilon_{ij}) + \int_{\partial M}\big(K(x)-K(a_i)\big)\varphi^2_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}d\sigma_g$$
$$+ O(\rho\varepsilon_{ij}) + O\left(\frac{\rho}{\lambda_i\lambda_j}\right) + o\left(\frac{1}{\lambda_i^2}+\frac{1}{\lambda_j^2}\right).$$

Lastly

$$\int_{\partial M}\big(K(x)-K(a_i)\big)\varphi^2_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}d\sigma_g = \int_{B_\rho(a_i)}\big(K(x)-K(a_i)\big)\varphi^2_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}d\sigma_g$$
$$+ O\left(\frac{1}{\lambda_i^2\lambda_j\rho^4}\right)$$
$$= O\left(\left(\int_{B_\rho}|K(x)-K(a_i)|^3\delta^3_{\lambda_i}\right)^{\frac{1}{3}}\left(\int_{\partial M}\varphi^{\frac{3}{2}}_{a_i,\lambda_i}\varphi^{\frac{3}{2}}_{a_j,\lambda_j}\right)^{\frac{2}{3}}\right) + O\left(\frac{1}{\lambda_i^2\lambda_j\rho^4}\right)$$
$$= O\left(\frac{\log(\lambda_i)^{\frac{1}{3}}}{\lambda_i}\varepsilon_{ij}\log(\varepsilon_{ij}^{-1})^{\frac{2}{3}}\right) + O\left(\frac{1}{\lambda_i^2\lambda_j\rho^4}\right)$$
$$= o\left(\frac{1}{\lambda_i^2}+\frac{1}{\lambda_j^2}\right) + o(\varepsilon_{ij}).$$

Here we used the estimate

$$\int_{\partial M}\varphi^{\frac{3}{2}}_{a_i,\lambda_i}\varphi^{\frac{3}{2}}_{a_j,\lambda_j}d\sigma_g \leq C\varepsilon_{ij}^{\frac{3}{2}}\log(\varepsilon_{ij}^{-1}), \tag{4.32}$$

which can be proved by integrating over $A\cap\partial M$ and $A^c\cap\partial M$ (see also Estimate 2 in [7]). Summing up, we have shown

$$\int_{\partial M} K(x)\varphi^2_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}d\sigma_g = K(a_i)I_1I(\varepsilon_{ij}) + O(\rho\varepsilon_{ij}) + o(\varepsilon_{ij})$$
$$+ O\left(\frac{\rho}{\lambda_i\lambda_j}\right) + o\left(\frac{1}{\lambda_i^2}+\frac{1}{\lambda_j^2}\right). \tag{4.33}$$

We add (4.25), (4.26), (4.32) and (4.33) to get the expansion

$$\int_{\partial M} K(x)\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}\right)^3 d\sigma_g = \sum_{i=1}^{p}\alpha_i^3 K(a_i)I_0 + \sum_{i=1}^{p}\alpha_i^3\frac{I_4}{6}\frac{\Delta K(a_i)}{\lambda_i^2}$$
$$+ 3\sum_{i\neq j}\alpha_i^2\alpha_j K(a_i)I_1 I(\varepsilon_{ij}) + O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right) + o\left(\sum_{i\neq j}\varepsilon_{ij}\right)$$
$$+ O\left(\sum_{i=1}^{p}\frac{\rho}{\lambda_i^2} + \frac{|\Pi(a_i)|^2}{\lambda_i^2}\right) + o\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}\right). \tag{4.34}$$

The next step is to estimate the integral

$$\int_{\partial M} K(x)\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}\right)^2 v d\sigma_g = \sum_{i=1}^{p}\alpha_i^2\int_{\partial M} K(x)\varphi_{a_i,\lambda_i}^2 v d\sigma_g$$
$$+ \sum_{i\neq j}\int_{\partial M} K(x)\alpha_i\alpha_j\varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j} v d\sigma_g.$$

The second integral is bounded from above by

$$\left|\int_{\partial M} K(x)\alpha_i\alpha_j\varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j} v d\sigma_g\right| \leq C\left(\int_{\partial M}\varphi_{a_i,\lambda_i}^{\frac{3}{2}}\varphi_{a_j,\lambda_j}^{\frac{3}{2}} d\sigma_g\right)^{\frac{2}{3}} ||v|| \leq C\varepsilon_{ij}\log(\varepsilon_{ij}^{-1})^{\frac{2}{3}}||v||,$$

hence it remains to estimate the first integral.

$$\int_{\partial M} K(x)\varphi_{a_i,\lambda_i}^2 v d\sigma_g = K(a_i)\int_{\partial M}\varphi_{a_i,\lambda_i}^2 v d\sigma_g + \int_{\partial M}\big(K(x)-K(a_i)\big)\varphi_{a_i,\lambda_i}^2 v d\sigma_g$$
$$= K(a_i)\int_{\partial M}\varphi_{a_i,\lambda_i}^2 v d\sigma_g + O\left(\int_{B_\rho(a_i)}\big(K(x)-K(a_i)\big)\varphi_{a_i,\lambda_i}^2 v d\sigma_g\right) + O\left(\frac{1}{\lambda_i^2\rho^2}\right)||v||$$
$$= K(a_i)\int_{\partial M}\varphi_{a_i,\lambda_i}^2 v d\sigma_g + O\left(\int_{B_\rho(a_i)}|K(x)-K(a_i)|^{\frac{3}{2}}\delta_{\lambda_i}^3 dx\right)^{\frac{2}{3}}||v||$$
$$+ O\left(\frac{1}{\lambda_i^2\rho^2}\right)||v||$$
$$= K(a_i)\int_{\partial M}\varphi_{a_i,\lambda_i}^2 v d\sigma_g + O\left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{\log(\lambda_i)^{\frac{2}{3}}}{\lambda_i^2} + \frac{1}{\lambda_i^2\rho^2}\right)||v||.$$

Since $v \in E_{(\alpha,a,\lambda)}$

$$0 =< v, \varphi_{a_i,\lambda_i} >= \int_M L_g\varphi_{a_i,\lambda_i} v dV_g + \int_{\partial M} B_g\varphi_{a_i,\lambda_i} v d\sigma_g$$

and therefore

$$2\int_{\partial M}\varphi^2_{a_i,\lambda_i}vd\sigma_g=\int_{\partial M}B_g\varphi_{a_i,\lambda_i}vd\sigma_g+\int_{\partial M}\big(2\varphi^2_{a_i,\lambda_i}-B_g\varphi_{a_i,\lambda_i}\big)vd\sigma_g$$
$$=-\int_M L_g\varphi_{a_i,\lambda_i}vdV_g+\int_{\partial M}\big(2\varphi^2_{a_i,\lambda_i}-B_g\varphi_{a_i,\lambda_i}\big)vd\sigma_g$$
$$=O\left(\left(\int_M|L_{g_{a_i}}\hat\varphi_{a_i,\lambda_i}|^{\frac43}dV_{g_{a_i}}\right)^{\frac34}+\left(\int_{\partial M}|\big(2\hat\varphi^2_{a_i,\lambda_i}-B_{g_{a_i}}\hat\varphi_{a_i,\lambda_i}\big)|^{\frac32}d\sigma_{g_{a_i}}\right)^{\frac23}\right)||v||.$$

Thus Proposition 2 implies

$$\left|\int_{\partial M}\varphi^2_{a_i,\lambda_i}vd\sigma_g\right|\le C\left(\frac{|\Pi(a_i)|\log(\lambda_i\rho)^{\frac34}}{\lambda_i}+\frac{\rho}{\lambda_i}+\frac{1}{\lambda_i^2\rho}\right)||v||$$

and therefore

$$\left|\int_{\partial M}K(x)\left(\sum_{i=1}^p\alpha_i\varphi_{a_i,\lambda_i}\right)^2vd\sigma_g\right|\le$$
$$C\left(\sum_{i=1}^p\frac{|\nabla K(a_i)|}{\lambda_i}+\frac{\log(\lambda_i)^{\frac23}}{\lambda_i^2}+\frac{1}{\lambda_i^2\rho^2}+\frac{|\Pi(a_i)|\log(\lambda_i\rho)^{\frac34}}{\lambda_i}+\frac{\rho}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\log(\varepsilon_{ij}^{-1})^{\frac23}\right)||v||. \tag{4.35}$$

Finally we expand the last integral in (4.24):

$$\int_{\partial M}K(x)\left(\sum_{i=1}^p\alpha_i\varphi_{a_i,\lambda_i}\right)v^2d\sigma_g$$
$$=\sum_{i=1}^pK(a_i)\alpha_i\int_{\partial M}\varphi_{a_i,\lambda_i}v^2d\sigma_g+O\left(\sum_{i=1}^p\frac{\log(\lambda_i\rho)}{\lambda_i}+\frac{1}{\lambda_i\rho}\right)||v||^2. \tag{4.36}$$

We add (4.34), (4.35) and the last expansion to (4.24) to get

$$\int_{\partial M}K(x)\left(\sum_{i=1}^p\alpha_i\varphi_{a_i,\lambda_i}+v\right)^3d\sigma_g=\sum_{i=1}^p\alpha_i^3K(a_i)I_0+\sum_{i=1}^p\alpha_i^3\frac{I_4}{6}\frac{\Delta K(a_i)}{\lambda_i^2}$$
$$+3\sum_{i\neq j}\alpha_i^2\alpha_jK(a_i)I_1I(\varepsilon_{ij})+3f^*(v)+3\sum_{i=1}^pK(a_i)\alpha_i\int_{\partial M}\varphi_{a_i,\lambda_i}v^2d\sigma_g$$
$$+O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right)+o\left(\sum_{i\neq j}\varepsilon_{ij}\right)+O\left(\sum_{i=1}^p\frac{\rho}{\lambda_i^2}+\frac{|\Pi(a_i)|^2}{\lambda_i^2}\right)+o\left(\sum_{i=1}^p\frac{1}{\lambda_i^2}\right)$$
$$+O\left(\sum_{i=1}^p\frac{\log(\lambda_i\rho)}{\lambda_i}+\frac{1}{\lambda_i\rho}\right)||v||^2+O(||v||^3). \tag{4.37}$$

Lastly we use the Taylor-expansion $(1+t)^{-\frac{2}{3}} = 1 - \frac{2}{3}t + O(t^2)$ to expand

$$\left(\int_{\partial M} K(x) \left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v\right)^3 d\sigma_g\right)^{\frac{-2}{3}}$$

and multiply the expansion with (4.22), which proves the Proposition. □

Using Proposition 7 we can know have a first look into the behaviour of the functional in $V(p,\varepsilon)$. Since the expansion in Proposition 7 holds in the C^1-sense, which will be proved precisely in chapter 5 and 6, we can try to understand the behaviour of J in $V(p,\varepsilon)$ through the variables (α, a, λ, v), where

$$u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v.$$

We have to be careful because v is not independent of (α, a, λ). But for a rough understanding let us assume v to be independent. Due to the definition of $V(p,\varepsilon)$ (see (3.20)):

$$\alpha_i K(a_i) = \alpha_j K(a_j) + o(\varepsilon) \text{ in } V(p,\varepsilon) \text{ uniformly for } \varepsilon \to 0.$$

Moreover, the assumptions of our Theorems yield $|\Pi(a)| \leq C|\nabla K(a)|$ on ∂M. Let us first try to understand the behaviour with respect to (α, a, λ). The v-term will be investigated precisely in the next chapter. Therefore, by neglecting all lower order terms in Proposition 7, we get the rough expansion:

$$\begin{aligned} & J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i}\right) \\ & \approx \frac{Q(B^4, \partial B^4) \sum_{i=1}^{p} \alpha_i^2}{\left(\sum_{i=1}^{p} \alpha_i^3 K(a_i)\right)^{\frac{2}{3}}} \left\{ 1 - \sum_{i=1}^{p} \frac{\alpha_i^2}{\sum_{j=1}^{p} \alpha_j^2} \left(\frac{|S_+^3| H_{a_i}(a_i)}{I_0} + \frac{I_4}{9 I_0} \frac{\Delta K(a_i)}{K(a_i)} \right) \frac{1}{\lambda_i^2} \right. \\ & \left. - \frac{I_1}{I_0} \sum_{i \neq j} \frac{\alpha_i \alpha_j}{\sum_{l=1}^{p} \alpha_l^2} I(\varepsilon_{ij}) \right\}. \end{aligned}$$

We set

$$f(\alpha, a) = \frac{Q(B^4, \partial B^4) \sum_{i=1}^{p} \alpha_i^2}{\left(\sum_{i=1}^{p} \alpha_i^3 K(a_i)\right)^{\frac{2}{3}}}.$$

Hence, first we could move (α, a) along the flow of $\nabla f(\alpha, a)$, which is a pseudo gradient of J as long as

$$|\nabla f(\alpha, a)|^2 \geq O\left(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}\right).$$

If

$$|\nabla f(\alpha, a)|^2 \leq O\left(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}\right)$$

we have to add a vector field, which moves the λ-variables. Due to the interaction ε_{ij} we have to distiguish many cases, which makes a rough understanding complicated. For simplicity, let us assume $p = 1$. Then

$$J\big(\alpha\varphi_{a,\lambda}\big) \approx \frac{Q(B^4, \partial B^4)}{K(a)^{\frac{2}{3}}}\left(1 - \left(|S_+^3|\frac{H_a(a)}{I_0} + \frac{I_4}{9I_0}\frac{\Delta K(a)}{K(a)}\right)\frac{1}{\lambda^2}\right).$$

Since a is close to a critical point in this case, a natural pseudo gradient has so increase λ, if

$$|S_+^3|\frac{H_x(x)}{I_0} + \frac{I_4}{9I_0}\frac{\Delta K(x)}{K(x)} > 0 \tag{4.38}$$

and decrease λ, if

$$|S_+^3|\frac{H_x(x)}{I_0} + \frac{I_4}{9I_0}\frac{\Delta K(x)}{K(x)} < 0 \tag{4.39}$$

Therefore, flow lines of a negative pseudo gradient vector field, which remain in $V(1, \varepsilon(t))$ for $\varepsilon(t) \to 0$ will accumulate at critical points of f. Furthermore they have to accumulate at critical points of K such that (4.39) holds. For $p \geq 2$ it is more complicated to understand flow lines, which remain in $V(p, \varepsilon)$. A rigorous proof will be given in chapters 7 and 8 (see Proposition 19).

Next chapter we expand the gradient of J with respect to the v-part to get a precise understanding for the movement of the v-variable for

$$\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v \in V(p, \varepsilon).$$

5. Minimization of the v-part

Let us first introduce some notation which will be used frequently in the rest of this thesis. For $p \in \mathbb{N}$ and $\varepsilon > 0$ we set:

$$B_\varepsilon^p := \Bigg\{ (\alpha, a, \lambda) \in \mathbb{R}_+^p \times \partial M^p \times \mathbb{R}_+^p \;\Bigg|\; \lambda_i > \frac{1}{\varepsilon} \ \forall i, \ \varepsilon_{ij} < \varepsilon \ \forall \ i \neq j,$$
$$\left| \alpha_i - \frac{1}{\sqrt{2I_0}} \left(\sum_{j=1}^p \left(\frac{K(a_i)}{K(a_j)} \right)^2 \right)^{-\frac{1}{2}} \right| < \varepsilon \ \forall i \Bigg\}. \tag{5.1}$$

In this chapter we prove the following Proposition:

Proposition 8. *There exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for all $(\alpha, a, \lambda) \in B_{\varepsilon_0}^p$ the minimization problem*

$$\min \left\{ J\left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + v \right) \;\Bigg|\; v \in E_{(\alpha,a,\lambda)} \cap B_{\delta_0}(0) \right\}$$

has a unique solution $\bar{v} = \bar{v}(\alpha, a, \lambda)$. Moreover it exits $C > 0$ such that

$$||\bar{v}|| \leq C||f^*||$$
$$\leq C \left(\sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{\log(\lambda_i)^{\frac{2}{3}}}{\lambda_i^2} + \frac{1}{\lambda_i^2 \rho^2} + \frac{|\Pi(a_i)| \log(\lambda_i \rho)^{\frac{3}{4}}}{\lambda_i} + \frac{\rho}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} \log(\varepsilon_{ij}^{-1})^{\frac{2}{3}} \right).$$

Crucial for the proof of this statement is the following Proposition which has been proved in [4] by using ideas of [7] and [12].

Proposition 9. *For all $p \in \mathbb{N}$ there exists $\varepsilon(p) > 0$ and $c(p) > 0$ such that*

$$||v||^2 - 4 \sum_{i=1}^p \int_{\partial M} \varphi_{a_i,\lambda_i} v^2 d\sigma_g \geq c(p)||v||^2 \ \text{ for all } \ v \in E_{(\alpha,a,\lambda)}$$

provided $(\alpha, a, \lambda) \in B_{\varepsilon(p)}^p$.

We set

$$q(v) := ||v||^2 - 4\sum_{i=1}^{p} \int_{\partial M} \varphi_{a_i,\lambda_i} v^2 d\sigma_g,$$

which is a quadratic form on $E_{(\alpha,a,\lambda)}$ due to Proposition 9.
We need a rough expansion of

$$p(\alpha,a,\lambda)\nabla J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v\right),$$

where $p(\alpha,a,\lambda)$ is the orthogonal projection (in $H^1(M)$) onto $E_{(\alpha,a,\lambda)}$. For $h \in E_{(\alpha,a,\lambda)}$ and $u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v$ we obtain

$$< \nabla J(u), h >= \frac{1}{\left(\int_{\partial M} K(x) u^3 d\sigma_g\right)^{\frac{2}{3}}} \left(2 < v, h > -2l(u) \int_{\partial M} K(x) u^2 h d\sigma_g\right).$$

Here

$$l(u) = \frac{||u||^2}{\int_{\partial M} K(x) u^3 d\sigma_g}.$$

For $f \in L^3(\partial M)$ let $B_g^{-1}(f) \in H^1(M)$ be the unique weak solution of the boundary value problem:

$$\begin{cases} L_g u = 0 & \text{in } M \\ B_g u = K(x) f & \text{on } \partial M. \end{cases}$$

Then

$$p(\alpha,a,\lambda)\nabla J(u) = \frac{1}{\left(\int_{\partial M} K(x) u^3 d\sigma_g\right)^{\frac{2}{3}}} \left(2v - 2l(u) p(\alpha,a,\lambda) B_g^{-1}\left(u^2\right)\right).$$

Hence

$$p(\alpha,a,\lambda)\nabla J(u) = 0 \Leftrightarrow v - l(u) p(\alpha,a,\lambda) B_g^{-1}\left(u^2\right) = 0.$$

Set $u = u(v) = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v$ and $F : E_{(\alpha,a,\lambda)} \to E_{(\alpha,a,\lambda)}$,

$$F(v) := v - l(u) p(\alpha,a,\lambda) B_g^{-1}\left(u^2\right)$$

as well as

$$r(u)[h] := l(u)\left(\int_{\partial M} K(x) u^2 h d\sigma_g - f^*(h) - 2\int_{\partial M} K(x)\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i}\right) h v d\sigma_g\right).$$

We easily derive the estimate

$$|r(u)[h]| \le C||h|| \cdot ||v||^2. \tag{5.2}$$

Moreover the same expansion as in (4.36) yields

$$\begin{aligned} l(u)\int_{\partial M} K(x)\left(\sum_{i=1}^p \alpha_i\varphi_{a_i,\lambda_i}\right) hvd\sigma_g &= l(u)\sum_{i=1}^p K(a_i)\alpha_i\int_{\partial M}\varphi_{a_i,\lambda_i} vhd\sigma_g \\ &\quad + O\left(\sum_{i=1}^p \frac{\log(\lambda_i)}{\lambda_i}+\frac{1}{\lambda_i\rho}\right)||v||\cdot||h|| \\ &= 2\sum_{i=1}^p\int_{\partial M}\varphi_{a_i,\lambda_i} vhd\sigma_g + o(\varepsilon)||v||\cdot||h||, \end{aligned} \tag{5.3}$$

because

$$l(u)=\frac{2\sum_{i=1}^p\alpha_i^2}{\sum_{i=1}^p\alpha_i^3K(a_i)}+o(\varepsilon).$$

Hence, the identity

$$\begin{aligned} &< F(v),h >=< v,h > \\ &- l(u(v))\left(f^*(h)+2\int_{\partial M}K(x)\left(\sum_{i=1}^p\alpha_i\varphi_{a_i,\lambda_i}\right)hvd\sigma_g\right) - r(u)[h], \end{aligned}$$

(5.2) and (5.3) implie

$$F(v) = -l(u)a + \frac{1}{2}\nabla q(v) + o(\varepsilon)||v|| + o(||v||), \tag{5.4}$$

where $a\in E_{(\alpha,a,\lambda)}$ such that $< a,h >= f^*(h)$ for all $h\in E_{(\alpha,a,\lambda)}$. Let $B: E_{(\alpha,a,\lambda)}\times E_{(\alpha,a,\lambda)}\to\mathbb{R}$,

$$B(h,w) =< h,w > -4\sum_{i=1}^p\int_{\partial M}\varphi_{a_i,\lambda_i} hwd\sigma_g$$

the bilinear form such that $q(v)=B(v,v)$. Proposition 9 yields that B is coercive, hence there exists a linear, selfadjoint operator $A: E_{(\alpha,a,\lambda)}\to E_{(\alpha,a,\lambda)}$ such that $B(h,w) =< Ah,w >$. Since $c||v||^2\le q(v)\le C||v||^2$ for all $(\alpha,a,\lambda)\in B_\varepsilon^p$, we get $cId\le A\le CId$. Thus A is an isomorphism and $||A^{-1}||\le\frac{1}{c}$. Therefore F is equal to

$$F(v) = l(u(v))a + Av + o(\varepsilon)||v|| + o(||v||). \tag{5.5}$$

We also need to expand the derivative of F.

$$\begin{aligned} < DF(v)[h],w > &= \frac{d}{dt|t=0} < F(v+th),w > \\ &=< h,w > -Dl(u(v))[h]\int_{\partial M}K(x)u(v)^2wd\sigma_g \\ &- 2l(u(v))\int_{\partial M}K(x)u(v)hwd\sigma_g. \end{aligned}$$

As in (5.3) an expansion yields

$$< h, w > -2l(u(v)) \int_{\partial M} K(x)u(v)hwd\sigma_g =< Ah, w > + < r_1(v)[h], w >,$$

where

$$||r_1(v)|| \leq C\left(o(\varepsilon) + o(||v||)\right).$$

It remains to estimate the second term

$$Dl(u(v))[h] \int_{\partial M} K(x)u(v)^2 wd\sigma_g = Dl(u(v))[h] < B_g^{-1}(u(v)^2), w >,$$

where

$$\begin{aligned} Dl(u(v))[h] &= \frac{2}{\int_{\partial M} K(x)u^3 d\sigma_g}\left(< v, h > -\frac{3}{2}l(u)\int_{\partial M} K(x)u^3 hd\sigma_g\right) \\ &= \frac{2}{\int_{\partial M} K(x)u^3 d\sigma_g}\left(\frac{3}{2}l(u)f^*(h) + O(||v||)\right)||h|| \\ &= \left(o(\varepsilon) + O(||v||)\right)||h|| \end{aligned}$$

for $h \in E_{(\alpha,a,\lambda)}$. Therefore, the previous expansions yield

$$DF(v) = A + r_1(v) + B_g^{-1}(u(v)) \cdot Dl(u(v)), \tag{5.6}$$

where

$$||r_1(v) + B_g^{-1}(u(v)) \cdot Dl(u(v))|| \leq C\left(o(\varepsilon) + O(||v||)\right).$$

Set $\tilde{F}(v) = -A^{-1}\left(l(u(v))a + o(\varepsilon||v||) + o(||v||)\right) = v - A^{-1}F(v)$. Due to Proposition 7, $||l(u(v))a|| = o(\varepsilon)$. Hence it exist $\varepsilon, \delta > 0$ such that $\tilde{F}$ maps $B_\delta(0)$ into itself, if ε is small. Moreover, form (5.6) we derive for $v_1, v_2 \in B_\delta(0)$:

$$||\tilde{F}(v_2) - \tilde{F}(v_1)|| \leq \int_0^1 ||D\tilde{F}\big(v_1 + t(v_2 - v_1)\big)||dt \cdot ||v_2 - v_1|| = \left(o(\varepsilon) + O(\delta)\right)||v_2 - v_1||.$$

Therefore, it exists $\varepsilon_0, \delta_0 > 0$ such that

$$\tilde{F} : B_{\delta_0}(0) \cap E_{(\alpha,a,\lambda)} \to B_{\delta_0}(0) \cap E_{(\alpha,a,\lambda)}$$

is a contraction for all $(\alpha, a, \lambda) \in B^p_{\varepsilon_0}$. The contraction mapping principle yields a unique fixpoint for $\tilde{F}$ and hence a unique solution of $F(v) = 0$. Summing up, we have proved the following statement:

Proposition 10. *There exists $\varepsilon_0 > 0$ such that the equation*

$$p(\alpha, a, \lambda)\nabla J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v\right) = 0$$

has exactly one solution $\bar{v} = \bar{v}(\alpha, a, \lambda) \in E_{(\alpha,a,\lambda)} \cap B_{\delta_0}(0)$ provided $(\alpha, a, \lambda) \in B^p_{\varepsilon_0}$.

In the last part of this section we show that $\bar{v}$ minimizes

$$E_{(\alpha,a,\lambda)} \cap B_{\delta_0}(0) \ni v \mapsto J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v\right)$$

for all $(\alpha, a, \lambda) \in B^p_{\varepsilon_0}$ and ε_0 small. Due to Proposition 10 it remains to show:

$$D^2 J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v\right)[h,h] \geq c_0 ||h||^2 \quad \forall h \in E_{(\alpha,a,\lambda)} \text{ and } v \in E_{(\alpha,a,\lambda)} \cap B_{\delta_0}(0). \tag{5.7}$$

For $u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v$ we compute:

$$\begin{aligned} D^2 J(u)[h,h] &=< D\nabla J(u)[h], h > \\ &= \frac{1}{\left(\int_{\partial M} K(x) u^3 d\sigma_g\right)^{\frac{2}{3}}} \Bigg[- \frac{2 \int_{\partial M} K(x) u^2 h d\sigma_g}{\left(\int_{\partial M} K(x) u^3 d\sigma_g\right)} < F(v), h > \\ &\quad + < DF(v)[h], h > \Bigg] \\ &= \frac{1}{\left(\int_{\partial M} K(x) u^3 d\sigma_g\right)^{\frac{2}{3}}} \left(2 < Ah, h > + o(\varepsilon) ||h||^2 + O(||v||) ||h||^2\right) \\ &\geq c_0 ||h||^2 \end{aligned}$$

for ε and $||v||$ small. Therefore $\bar{v}$ form Proposition 10 is a minimizer. From the equation $0 = F(\bar{v})$ (see (5.4)) we derive the estimate

$$||\bar{v}|| \leq C||a|| = C||f^*||.$$

Hence the stated estimate in Proposition 8 follows from (4.2). Therefore the proof of Proposition 8 is completed.

Remark 2. *Returning to the question of understanding the behaviour of J in $V(p, \varepsilon)$, we can now understand the behaviour with respect to v. Since*

$$v \mapsto J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v\right)$$

is a convex functional with a local minimizer, a natural negative pseudo gradient of J has to move v such that v comes closer to $\bar{v}$.

6. Expansion of the gradient

In the previous chapter we analysed the behaviour of J with respect to the v-part. This chapter is crucial for the construction of a pseudo gradient vector field which moves the variables (α, a, λ).
For any $h \in H^1(M)$ and $u \in U$ it holds

$$< \nabla J(u), h >= 2J(u)\left(< u, h > -\frac{||u||^2}{\int_{\partial M} K(x)u^3 d\sigma_g}\int_{\partial M} K(x)u^2 h d\sigma_g\right). \tag{6.1}$$

In the following $u = \sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + v \in V(p,\varepsilon)$ and h is either one of this bubbles or a derivative of this bubbles with respect to λ_i or a_i. We did the expansion of the functional in the general case. Since we assume that all critical points are umbilic points there exits a constant such that $|\Pi(a)| \leq C|\nabla K(a)|$ for all $a \in \partial M$. This estimate is true because K is a Morse-function. From now on we use this inequality in the expansion of the gradient.

6.1. Expansion of the gradient applied to a bubble

Proposition 11. *For $u = \sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + v \in V(p,\varepsilon)$ it holds:*

$$\begin{aligned} < \nabla J(u), \varphi_{a_j,\lambda_j} > &= 4I_0 J(u)\left(\alpha_j - \frac{\alpha_j^2 K(a_j)\sum_{i=1}^p \alpha_i^2}{\sum_{i=1}^p \alpha_i^3 K(a_i)}\right) \\ &\quad + O\left(\sum_{i=1}^p \frac{1}{\lambda_i^2} + \frac{|\Pi(a_i)|^2 \log(\lambda_i \rho)^2}{\lambda_i^2} + \sum_{i\neq j} \varepsilon_{ij}\right) + O(||v||^2). \end{aligned}$$

Proof. Since $u \in V(p,\varepsilon)$ and $v \in E_{(\alpha,a,\lambda)}$

$$\begin{aligned} &< \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v, \varphi_{a_j,\lambda_j} > = \alpha_j ||\varphi_{a_j,\lambda_j}||^2 + O\left(\sum_{i:i\neq j} \varepsilon_{ij}\right) \\ &\qquad = 2I_0\alpha_j + O\left(\frac{|\Pi(a_j)|^2 \log(\lambda_j\rho)}{\lambda_j^2} + \frac{1}{\lambda_j^2}\right) + O\left(\sum_{i:i\neq j} \varepsilon_{ij}\right), \end{aligned} \tag{6.2}$$

where we used (4.21). Furthermore we expand

$$\begin{aligned} \int_{\partial M} K(x)\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v\right)^2 \varphi_{a_j,\lambda_j} d\sigma_g &= \int_{\partial M} K(x)\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i}\right)^2 \varphi_{a_j,\lambda_j} d\sigma_g \\ &+ 2\int_{\partial M} K(x)\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i}\right) \varphi_{a_j,\lambda_j} v + O(||v||^2). \end{aligned} \tag{6.3}$$

The first term on the right hand side can be computed as follows

$$\begin{aligned} \int_{\partial M} K(x)\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i}\right)^2 \varphi_{a_j,\lambda_j} d\sigma_g &= \sum_{i=1}^{p} \alpha_i^2 \int_{\partial M} K(x)\varphi_{a_i,\lambda_i}^2 \varphi_{a_j,\lambda_j} d\sigma_g \\ &+ 2\sum_{i\neq l} \alpha_i\alpha_l \int_{\partial M} K(x)\varphi_{a_i,\lambda_i}\varphi_{a_l,\lambda_l}\varphi_{a_j,\lambda_j} d\sigma_g. \end{aligned}$$

In addition

$$\begin{aligned} &\sum_{i=1}^{p} \alpha_i^2 \int_{\partial M} K(x)\varphi_{a_i,\lambda_i}^2 \varphi_{a_j,\lambda_j} d\sigma_g \\ &\qquad = \alpha_j^2 K(a_j) I_0 + O\left(\frac{1}{\lambda_j^2}\right) + O\left(\sum_{i\neq j} \int_{\partial M} \varphi_{a_i,\lambda_i}^2 \varphi_{a_j,\lambda_j} d\sigma_g\right) \end{aligned}$$

and

$$\int_{\partial M} \varphi_{a_i,\lambda_i}^2 \varphi_{a_j,\lambda_j} d\sigma_g \leq C \int_{\partial M} \left(\frac{\lambda_i}{1+\lambda_i^2 d_g(a_i,x)^2}\right)^2 \left(\frac{\lambda_j}{1+\lambda_j^2 d_g(a_j,x)^2}\right) d\sigma_g.$$

In case $\lambda_i \geq \lambda_i$ and $\frac{\lambda_i}{\lambda_j} \geq \lambda_i\lambda_j d_g(a_i,a_j)^2$ we obtain

$$\begin{aligned} &\int_{\partial M} \left(\frac{\lambda_i}{1+\lambda_i^2 d_g(a_i,x)^2}\right)^2 \left(\frac{\lambda_j}{1+\lambda_j^2 d_g(a_j,x)^2}\right) d\sigma_g \\ &\leq \lambda_j \left(\int_{B_{\rho_0}(a_i)} \left(\frac{\lambda_i}{1+\lambda_i^2 d_g(a_i,x)^2}\right)^2 d\sigma_g\right) + O\left(\frac{\lambda_j}{\lambda_i^2}\right) \\ &\leq C\frac{\lambda_j}{\lambda_i} \leq C\varepsilon_{ij}. \end{aligned}$$

In all other cases we integrate over

$$A := \left\{ x \in \partial M \ : \ 2d_g(a_j, x) \le \frac{1}{\lambda_i} + d_g(a_i, a_j) \right\}$$

and A^c to get the estimate

$$\int_{\partial M} \left(\frac{\lambda_i}{1+\lambda_i^2 d_g(a_i,x)^2} \right)^2 \left(\frac{\lambda_j}{1+\lambda_j^2 d_g(a_j,x)^2} \right) d\sigma_g \le C \left(\frac{1}{\frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j d_g(a_i,a_j)^2} \right) \le C\varepsilon_{ij}.$$

Hence, in any case we obtain

$$\int_{\partial M} \varphi^2_{a_i,\lambda_i} \varphi_{a_j,\lambda_j} d\sigma_g \le C\varepsilon_{ij} \tag{6.4}$$

and therefore

$$\sum_{i=1}^p \alpha_i^2 \int_{\partial M} K(x) \varphi^2_{a_i,\lambda_i} \varphi_{a_j,\lambda_j} d\sigma_g$$
$$= \alpha_j^2 K(a_j) I_0 + O\left(\frac{1}{\lambda_j^2}\right) + O\left(\sum_{i\neq j} \varepsilon_{ij}\right). \tag{6.5}$$

Finally

$$\left| 2\sum_{i\neq l} \alpha_i \alpha_l \int_{\partial M} K(x) \varphi_{a_i,\lambda_i} \varphi_{a_l,\lambda_l} \varphi_{a_j,\lambda_j} d\sigma_g \right| \le C \sum_{k\neq l} \int_{\partial M} \varphi^2_{a_k,\lambda_k} \varphi_{a_l,\lambda_l} d\sigma_g \le C \sum_{k\neq l} \varepsilon_{kl},$$

hence

$$\int_{\partial M} K(x) \left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} \right)^2 \varphi_{a_j,\lambda_j} d\sigma_g = I_0 \alpha_j^2 K(a_j) + O\left(\frac{1}{\lambda_j^2}\right) + O\left(\sum_{k\neq l} \varepsilon_{kl}\right). \tag{6.6}$$

Therefore the first term of the right hand side of (6.3) is expanded. Now we estimate the linear term

$$\int_{\partial M} K(x) \left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} \right) \varphi_{a_j,\lambda_j} v$$
$$= \int_{\partial M} K(x) \alpha_j \varphi^2_{a_j,\lambda_j} v d\sigma_g + O\left(\sum_{i\neq j} \left(\int_{\partial M} \varphi^{\frac{3}{2}}_{a_i,\lambda_i} \varphi^{\frac{3}{2}}_{a_j,\lambda_j} d\sigma_g \right)^{\frac{2}{3}} \right) ||v||$$
$$= O\left(\frac{|\nabla K(a_j)|}{\lambda_j} + \frac{\log(\lambda_j)}{\lambda_j^2} + \frac{|\Pi(a_j)| \log(\lambda_j \rho)}{\lambda_j} + \frac{\rho}{\lambda_j} + \sum_{i\neq j} \varepsilon_{ij} \log(\varepsilon_{ij}^{-1})^{\frac{2}{3}} \right) ||v||,$$

where we used the same estimates as in the expansion of the functional. Hence we have derived the following expansion:

$$\int_{\partial M} K(x)\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v\right)^2\varphi_{a_j,\lambda_j}d\sigma_g = I_0\alpha_j^2K(a_j)+O\left(\frac{1}{\lambda_j^2}\right)+O\left(\sum_{k\neq l}\varepsilon_{kl}\right)$$
$$+O\left(\frac{|\nabla K(a_j)|}{\lambda_j}+\frac{\log(\lambda_j)}{\lambda_j^2}+\frac{|\Pi(a_j)|\log(\lambda_j\rho)}{\lambda_j}+\frac{\rho}{\lambda_j}+\sum_{i\neq j}\varepsilon_{ij}\log(\varepsilon_{ij}^{-1})^{\frac{2}{3}}\right)||v||$$
$$+O(||v||^2). \tag{6.7}$$

Finally, using the expansion of the functional, we obtain

$$\frac{||u||^2}{\int_{\partial M}K(x)u^3d\sigma_g}$$
$$=2\frac{\sum_{i=1}^p\alpha_i^2}{\sum_{i=1}^p\alpha_i^3K(a_i)}+O\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\frac{|\Pi(a_i)|^2\log(\lambda_i\rho)}{\lambda_i^2}+\sum_{i\neq j}\varepsilon_{ij}\right)+O(||v||^2). \tag{6.8}$$

From (6.2), (6.7) and (6.8) we derive

$$< u,\varphi_{a_j,\lambda_j} > - \frac{||u||^2}{\int_{\partial M}K(x)u^3d\sigma_g}\int_{\partial M}K(x)u^2\varphi_{a_j,\lambda_j}d\sigma_g$$
$$=2I_0\alpha_j-2I_0\frac{\alpha_j^2K(a_j)\sum_{i=1}^p\alpha_i^2}{\sum_{i=1}^p\alpha_i^3K(a_i)}$$
$$+O\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\frac{|\Pi(a_i)|^2\log(\lambda_i\rho)}{\lambda_i^2}+\sum_{i\neq j}\varepsilon_{ij}\right)+O(||v||^2),$$

which proves the Proposition. □

6.2. Expansion of the gradient applied to a derivative w.r.t. λ

In this section we prove a precise expansion of $< \nabla J(u), \lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j} >$. This expression will tell us how the non-compact variables λ_j will move along a suitable pseudo gradient.

Proposition 12. *If $u=\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v\in V(p,\varepsilon)$ and $2\leq\lambda_i\rho^2$ for all i, then*

$$
\begin{aligned}
&<\nabla J(u),\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}>=2J(u)\Big(2|S^3_+|\alpha_j\frac{H_{a_j}(a_j)}{\lambda_j^2}+2\frac{\sum_{i=1}^{p}\alpha_i^2}{\sum_{i=1}^{p}\alpha_i^3K(a_i)}\alpha_j^2\frac{I_4}{9}\frac{\Delta K(a_j)}{\lambda_j^2}\\
&+\sum_{i\neq j}\alpha_j 2I_1\lambda_j\frac{\partial}{\partial\lambda_j}I(\varepsilon_{ij})-2\frac{\sum_{i=1}^{p}\alpha_i^2}{\sum_{i=1}^{p}\alpha_i^3K(a_i)}\sum_{i\neq j}I_1\big(\alpha_i^2K(a_i)+\alpha_i\alpha_jK(a_j)\big)\lambda_j\frac{\partial}{\partial\lambda_j}I(\varepsilon_{ij})\Big)\\
&+O\left(\frac{\rho}{\lambda_j^2}+\frac{1}{(\lambda_j\rho)^3}\right)+O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right)+o\left(\sum_{i\neq j}\varepsilon_{ij}\right)\\
&+o\left(\frac{|\nabla K(a_j)|^2}{\lambda_j}+\frac{1}{\lambda_j^2}\right)+O(||v||^2).
\end{aligned}
$$

Proof. First we compute

$$
\begin{aligned}
&< u,\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}>\\
&=\alpha_j<\varphi_{a_j,\lambda_j},\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}>+\sum_{i\neq j}\alpha_i<\varphi_{a_i,\lambda_i},\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}>\\
&=\alpha_j2|S^3_+|\frac{H_{a_j}(a_j)}{\lambda_j^2}+O\left(\frac{\log(\rho\lambda_j)|\Pi(a_j)|^2}{\lambda_j^2}+\frac{\rho}{\lambda_j^2}+\frac{1}{(\lambda_j\rho)^3}\right)\\
&+\sum_{i\neq j}\alpha_i\left(2I_1\lambda_j\frac{\partial}{\partial\lambda_j}I(\varepsilon_{ij})+O(\rho\varepsilon_{ij})+o(\varepsilon_{ij})\right).
\end{aligned}\tag{6.9}
$$

Here we used Proposition 26 and 27 in appendix C. Furthermore we need to expand the expression

$$\int_{\partial M}K(x)\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v\right)^2\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g,$$

which will be done in the following.

$$
\begin{aligned}
&\int_{\partial M}K(x)\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v\right)^2\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g\\
&=\int_{\partial M}K(x)\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}\right)^2\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g\\
&+2\int_{\partial M}K(x)\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}\right)\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}vd\sigma_g+O(||v||^2).
\end{aligned}\tag{6.10}
$$

Moreover

$$\int_{\partial M} K(x)\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}\right)^2\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g=\sum_{i=1}^{p}\alpha_i^2\int_{\partial M}K(x)\varphi_{a_i,\lambda_i}^2\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g$$
$$+\sum_{i\neq l}\alpha_i\alpha_l\int_{\partial M}K(x)\varphi_{a_i,\lambda_i}\varphi_{a_l,\lambda_l}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g \tag{6.11}$$

and

$$\sum_{i\neq l}\alpha_i\alpha_l\int_{\partial M}K(x)\varphi_{a_i,\lambda_i}\varphi_{a_l,\lambda_l}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g$$
$$=2\sum_{i\neq j}\alpha_i\alpha_j\int_{\partial M}K(x)\varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g$$
$$+O\left(\sum_{l\neq k\neq m}\int_{\partial M}\varphi_{a_l,\lambda_l}\varphi_{a_k,\lambda_k}\varphi_{a_m,\lambda_m}d\sigma_g\right)$$
$$=2\sum_{i\neq j}\alpha_i\alpha_j\int_{\partial M}K(x)\varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g+O\left(\sum_{k\neq l}\varepsilon_{kl}^2\log(\varepsilon_{kl}^{-1})\right)$$

which we have already seen in the expansion of the functional.

$$2\int_{\partial M}K(x)\varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g=\int_{\partial M}K(x)\varphi_{a_i,\lambda_i}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}^2d\sigma_g$$
$$=K(a_j)\int_{\partial M}\varphi_{a_i,\lambda_i}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}^2d\sigma_g+O\left(\int_{\partial M}|K(x)-K(a_j)|\varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}^2d\sigma_g\right)$$
$$=K(a_j)\int_{\partial M}\varphi_{a_i,\lambda_i}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}^2d\sigma_g$$
$$+O\left(\left(\int_{\partial M}|K(x)-K(a_j)|^3\varphi_{a_j,\lambda_j}^3d\sigma_g\right)^{\frac{1}{3}}\left(\int_{\partial M}\varphi_{a_i,\lambda_i}^{\frac{3}{2}}\varphi_{a_j,\lambda_j}^{\frac{3}{2}}d\sigma_g\right)^{\frac{2}{3}}\right)$$
$$=K(a_j)\int_{\partial M}\varphi_{a_i,\lambda_i}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}^2d\sigma_g$$
$$+O\left(\left(\int_{\partial M}|K(x)-K(a_j)|^3\varphi_{a_j,\lambda_j}^3d\sigma_g\right)^{\frac{1}{3}}\varepsilon_{ij}\log(\varepsilon_{ij}^{-1})^{\frac{2}{3}}\right).$$

Furthermore

$$\int_{\partial M}|K(x)-K(a_j)|^3\varphi_{a_j,\lambda_j}^3d\sigma_g$$
$$=\int_{B_{\rho_0}(a_j)}|K(x)-K(a_j)|^3\hat{\varphi}_{a_j,\lambda_j}^3d\sigma_{g_{a_j}}+\int_{\partial M\setminus B_{\rho_0}(a_j)}|K(x)-K(a_j)|^3\hat{\varphi}_{a_j,\lambda_j}^3d\sigma_{g_{a_j}}$$
$$\leq C\left(\frac{|\nabla K(a_j)|^3\log(\lambda_j)}{\lambda_j^3}+\frac{1}{\lambda_j^3}\right),$$

hence

$$O\left(\left(\int_{\partial M}|K(x)-K(a_j)|^3\varphi^3_{a_j,\lambda_j}d\sigma_g\right)^{\frac{1}{3}}\varepsilon_{ij}\log(\varepsilon_{ij}^{-1})^{\frac{2}{3}}\right)=o\left(\frac{|\nabla K(a_j)|^2}{\lambda_j}+\frac{1}{\lambda_j^2}+\varepsilon_{ij}\right),$$

where we used Young's inequality. Therefore

$$\begin{aligned}&2\int_{\partial M}K(x)\varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g\\&\qquad=K(a_j)\int_{\partial M}\varphi_{a_i,\lambda_i}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi^2_{a_j,\lambda_j}d\sigma_g+o\left(\frac{|\nabla K(a_j)|^2}{\lambda_j}+\frac{1}{\lambda_j^2}+\varepsilon_{ij}\right).\end{aligned}\tag{6.12}$$

It is left to expand the first integral on the right hand side in the previous equation.

$$\begin{aligned}2\int_{\partial M}&\varphi_{a_i,\lambda_i}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi^2_{a_j,\lambda_j}d\sigma_g\\&=\int_{\partial M}\varphi_{a_i,\lambda_i}B_g\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g\\&+\int_{\partial M}\varphi_{a_i,\lambda_i}\left(2\lambda_j\frac{\partial}{\partial\lambda_j}\varphi^2_{a_j,\lambda_j}-B_g\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}\right)d\sigma_g\\&=<\varphi_{a_i,\lambda_i},\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}>-\int_M\varphi_{a_i,\lambda_i}L_g\left(\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}\right)dV_g\\&+\int_{\partial M}\varphi_{a_i,\lambda_i}\left(2\lambda_j\frac{\partial}{\partial\lambda_j}\varphi^2_{a_j,\lambda_j}-B_g\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}\right)d\sigma_g\\&=2I_1\lambda_j\frac{\partial}{\partial\lambda_j}I(\varepsilon_{ij})+O(\rho\varepsilon_{ij})+o(\varepsilon_{ij}),\end{aligned}$$

where we used Proposition 3, Proposition 27 and the same estimates as in the expansion of the functional. Hence the previous expansion and (6.12) yield

$$\begin{aligned}\sum_{i\neq l}&\alpha_i\alpha_l\int_{\partial M}K(x)\varphi_{a_i,\lambda_i}\varphi_{a_l,\lambda_l}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g\\&=\sum_{i\neq j}\alpha_i\alpha_jK(a_j)I_0\lambda_j\frac{\partial}{\partial\lambda_j}I(\varepsilon_{ij})+O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right)+o\left(\sum_{k\neq l}\varepsilon_{kl}\right)\\&+o\left(\frac{|\nabla K(a_j)|^2}{\lambda_j}+\frac{1}{\lambda_j^2}\right).\end{aligned}\tag{6.13}$$

In (6.11) it is left to expand the first term on the right hand side:

$$\begin{aligned}\sum_{i=1}^p&\alpha_i^2\int_{\partial M}K(x)\varphi^2_{a_i,\lambda_i}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g\\&=\alpha_j^2\int_{\partial M}K(x)\varphi^2_{a_j,\lambda_j}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g+\sum_{i\neq j}\alpha_i^2\int_{\partial M}K(x)\varphi^2_{a_i,\lambda_i}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g.\end{aligned}$$

A simple computation shows

$$\lambda_j \frac{\partial}{\partial \lambda_j} \varphi_{a_j,\lambda_j} = -\varphi_{a_j,\lambda_j} + 2u_{a_j}\chi_\rho \delta_{a_j,\lambda_j} \left(\frac{1}{1+\lambda_j^2 d_{g_{a_j}}(a_j,x)^2} \right)$$

on ∂M. A computation as in the expansion of the functional (see (4.26)) yields

$$\int_{\partial M} K(x)\varphi^3_{a_i,\lambda_i} d\sigma_g = K(a_i)I_0 + \frac{I_4}{6}\frac{\Delta K(a_i)}{\lambda_i^2} + O\left(\frac{1}{(\lambda_i \rho)^3}\right) + o\left(\frac{1}{\lambda_i^2}\right).$$

In addition

$$\begin{aligned}
&\int_{\partial M} K(x)\varphi^2_{a_i,\lambda_i} 2u_{a_j}\chi_\rho \delta_{a_j,\lambda_j} \left(\frac{1}{1+\lambda_j^2 d_{g_{a_j}}(a_j,x)^2} \right) d\sigma_g \\
&= \int_{B_\rho(a_j)} K(x)\hat{\varphi}^2_{a_i,\lambda_i} 2\chi_\rho \delta_{a_j,\lambda_j} \left(\frac{1}{1+\lambda_j^2 d_{g_{a_j}}(a_j,x)^2} \right) d\sigma_{g_{a_j}} \\
&+ \int_{\partial M \setminus B_\rho(a_j)} K(x)\hat{\varphi}^2_{a_i,\lambda_i} 2\chi_\rho \delta_{a_j,\lambda_j} \left(\frac{1}{1+\lambda_j^2 d_{g_{a_j}}(a_j,x)^2} \right) d\sigma_{g_{a_j}} \\
&= \int_{B_\rho(a_j)} K(x)\hat{\varphi}^2_{a_i,\lambda_i} 2\chi_\rho \delta_{a_j,\lambda_j} \left(\frac{1}{1+\lambda_j^2 d_{g_{a_j}}(a_j,x)^2} \right) d\sigma_{g_{a_j}} + O\left(\frac{1}{(\lambda\rho)^3}\right) \\
&= 2K(a_j)\int_{B_\rho} \left(\frac{\lambda_j^3}{\left(1+\lambda_j^2|x|^2\right)^4} \right) dx + \int_{B_\rho} D^2K(a_j)[x,x] \left(\frac{\lambda_j^3}{\left(1+\lambda_j^2|x|^2\right)^4} \right) dx \\
&+ O\left(\frac{1}{(\lambda\rho)^3}\right) \\
&= K(a_j)I_0 + \frac{1}{3}\frac{\Delta K(a_j)}{\lambda_j^2} \int_{\mathbb{R}^3} \frac{|x|^2}{\left(1+|x|^2\right)^4} dx + O\left(\frac{1}{(\lambda\rho)^3}\right) \\
&= K(a_j)I_0 + \frac{I_4}{18}\frac{\Delta K(a_j)}{\lambda_j^2} + O\left(\frac{1}{(\lambda\rho)^3}\right)
\end{aligned}$$

and therefore

$$\alpha_j^2 \int_{\partial M} K(x)\varphi^2_{a_j,\lambda_j} \lambda_j \frac{\partial}{\partial \lambda_j}\varphi_{a_j,\lambda_j} d\sigma_g = -\alpha_j^2 \frac{I_4}{9}\frac{\Delta K(a_j)}{\lambda_j^2} + o\left(\frac{1}{\lambda_j^2}\right). \tag{6.14}$$

Finally, the same computations as for (6.13) give the expansion

$$\begin{aligned}
\sum_{i\neq j} \alpha_i^2 \int_{\partial M} K(x)\varphi^2_{a_i,\lambda_i} \lambda_j \frac{\partial}{\partial \lambda_j}\varphi_{a_j,\lambda_j} d\sigma_g &= \sum_{i\neq j} \alpha_i^2 K(a_i) I_1 \lambda_j \frac{\partial}{\partial \lambda_j} I(\varepsilon_{ij}) \\
+ O\left(\sum_{i\neq j} \rho\varepsilon_{ij} \right) &+ o\left(\sum_{k\neq l} \varepsilon_{kl} \right) + o\left(\frac{|\nabla K(a_j)|^2}{\lambda_j} + \frac{1}{\lambda_j^2} \right).
\end{aligned} \tag{6.15}$$

Hence from the previous expansions we derive

$$\begin{aligned}\int_{\partial M} K(x)\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}\right)^2 \lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g &= -\alpha_j^2\frac{I_4}{9}\frac{\Delta K(a_j)}{\lambda_j^2}\\ &+\sum_{i\neq j} I_1\big(\alpha_i^2 K(a_i)+\alpha_i\alpha_j K(a_j)\big)\lambda_j\frac{\partial}{\partial\lambda_j}I(\varepsilon_{ij})+O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right)\\ &+o\left(\sum_{k\neq l}\varepsilon_{kl}\right)+o\left(\frac{|\nabla K(a_j)|^2}{\lambda_j}+\frac{1}{\lambda_j^2}\right).\end{aligned} \tag{6.16}$$

For a precise expansion of (6.10) it is left to estimate the linear term.

$$\begin{aligned}2\int_{\partial M} K(x)\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}\right)\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}v d\sigma_g &= 2\alpha_j\int_{\partial M}K(x)\varphi_{a_j,\lambda_j}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}vd\sigma_g\\ &+O\left(\sum_{i\neq j}\left(\int_{\partial M}\varphi^{\frac{3}{2}}_{a_i,\lambda_i}\varphi^{\frac{3}{2}}_{a_j,\lambda_j}d\sigma_g\right)^{\frac{2}{3}}\right)||v||\\ &= 2\alpha_j\int_{\partial M}K(x)\varphi_{a_j,\lambda_j}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}vd\sigma_g+O\left(\sum_{i\neq j}\varepsilon_{ij}\log(\varepsilon_{ij}^{-1})^{\frac{2}{3}}\right)||v||,\end{aligned}$$

because $\left|\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}\right|\leq C\varphi_{a_j,\lambda_j}$ if $2\leq\rho\lambda_j$. This inequality also implies

$$\begin{aligned}&2\int_{\partial M}K(x)\varphi_{a_j,\lambda_j}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}vd\sigma_g\\ &=2K(a_j)\int_{\partial M}\varphi_{a_j,\lambda_j}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}vd\sigma_g+O\left(\frac{|\nabla K(a_j)|}{\lambda_j}+\frac{\log(\lambda_j)}{\lambda_j^2}+\frac{1}{(\lambda_j\rho)^2}\right)||v||.\end{aligned}$$

Furthermore

$$\begin{aligned}4\int_{\partial M}\varphi_{a_j,\lambda_j}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}vd\sigma_g &= 2\int_{\partial M}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi^2_{a_j,\lambda_j}vd\sigma_g\\ &=\int_{\partial M}\left(2\lambda_j\frac{\partial}{\partial\lambda_j}\varphi^2_{a_j,\lambda_j}-B_g\left(\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}\right)\right)vd\sigma_g+\int_{\partial M}B_g\left(\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}\right)vd\sigma_g\\ &=\int_{\partial M}\left(2\lambda_j\frac{\partial}{\partial\lambda_j}\varphi^2_{a_j,\lambda_j}-B_g\left(\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}\right)\right)vd\sigma_g-\int_{M}L_g\left(\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}\right)vdV_g,\end{aligned}$$

where we used that $v\in E_{(\alpha,a,\lambda)}$. Finally, we use Proposition 3 to derive the estimate

$$\left|\int_{\partial M}\varphi_{a_j,\lambda_j}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}vd\sigma_g\right|\leq C\left(\frac{|\Pi(a_j)|\log(\lambda_j\rho)^{\frac{3}{4}}}{\lambda_j}+\frac{\rho}{\lambda_j}+\frac{1}{(\lambda_j\rho)^2}\right)||v||.$$

Therefore the linear term can be estimated by:

$$
\begin{aligned}
&2\int_{\partial M} K(x)\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}\right)\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}vd\sigma_g \\
&= O\left(\frac{|\Pi(a_j)|\log(\lambda_j\rho)^{\frac{3}{4}}}{\lambda_j}+\frac{\rho}{\lambda_j}+\frac{|\nabla K(a_j)|}{\lambda_j}+\frac{\log(\lambda_j)}{\lambda_j^2}+\sum_{i\neq j}\varepsilon_{ij}\log(\varepsilon_{ij}^{-1})^{\frac{2}{3}}\right)||v||.
\end{aligned}
\tag{6.17}
$$

Hence, adding (6.10), (6.16) and (6.17) yields the expansion

$$
\begin{aligned}
\int_{\partial M} K(x)\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v\right)^2\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}d\sigma_g &= -\alpha_j^2\frac{I_4}{9}\frac{\Delta K(a_j)}{\lambda_j^2} \\
&+\sum_{i\neq j}I_1\big(\alpha_i^2K(a_i)+\alpha_i\alpha_jK(a_j)\big)\lambda_j\frac{\partial}{\partial\lambda_j}I(\varepsilon_{ij})+O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right)+o\left(\sum_{k\neq l}\varepsilon_{kl}\right) \\
&+O\left(\frac{\rho^2}{\lambda_j^2}+\frac{|\Pi(a_j)|^2\log(\lambda_j\rho)^2}{\lambda_j^2}\right)+o\left(\frac{|\nabla K(a_j)|^2}{\lambda_j}+\frac{1}{\lambda_j^2}\right)+O(||v||^2).
\end{aligned}
\tag{6.18}
$$

Finally, multiplying (6.8) with (6.18) and adding (6.9) proves the Proposition. □

6.3. Expansion of the gradient applied to a derivative w.r.t. a

Proposition 13. *If* $u=\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v\in V(p,\varepsilon)$ *and* $2\leq\lambda_i\rho^2$ *for all* i, *then*

$$
\begin{aligned}
<\nabla J(u),-\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}\cdot\nabla K(a_j)> &= 2J(u)\left(2\frac{\sum_{i=1}^{p}\alpha_i^2}{\sum_{i=1}^{p}\alpha_i^3K(a_i)}\alpha_j^2\frac{2}{3}I_4\frac{|\nabla K(a_j)|^2}{\lambda_j}\right) \\
&+O\left(\frac{|\nabla K(a_j)|}{(\lambda_j\rho)^3}\right)+o\left(\frac{|\nabla K(a_j)|^2}{\lambda_j}+\frac{1}{\lambda_j^2}\right) \\
&+O\left(\sum_{i\neq j}\varepsilon_{ij}\right)|\nabla K(a_j)|+O(||v||^2).
\end{aligned}
$$

Proof. Due to Proposition 26 and 27 in appendix C

$$
<u,\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}\cdot\nabla K(a_j)>=O\left(\frac{|\nabla K(a_j)|}{\lambda_j^2}+\sum_{i\neq j}\varepsilon_{ij}|\nabla K(a_j)|\right). \tag{6.19}
$$

Since $\left|\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}\cdot\nabla K(a_j)\right| \leq C|\nabla K(a_j)|\varphi_{a_j,\lambda_j}$ we can use the same estimates as in the proof of the previous Proposition to obtain

$$\begin{aligned}
&\int_{\partial M} K(x)\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v\right)^2\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}\cdot\nabla K(a_j)d\sigma_g\\
&\quad=\alpha_j^2\int_{\partial M}K(x)\varphi_{a_j,\lambda_j}^2\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}\cdot\nabla K(a_j)d\sigma_g+O\left(|\nabla K(a_j)|\sum_{i\neq j}\varepsilon_{ij}\right)\\
&\quad+2\alpha_j\int_{\partial M}K(x)\varphi_{a_j,\lambda_j}\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}\cdot\nabla K(a_j)vd\sigma_g+O\left(\sum_{i\neq j}\varepsilon_{ij}\log(\varepsilon_{ij}^{-1})^{\frac{2}{3}}\right)||v||\\
&\quad+O(||v||^2).
\end{aligned}\tag{6.20}$$

Furthermore, due to Proposition 4, we can estimate the second integral on the right hand side in the previous equation as follows

$$\begin{aligned}
&2\alpha_j\int_{\partial M}K(x)\varphi_{a_j,\lambda_j}\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}\cdot\nabla K(a_j)vd\sigma_g\\
&\qquad=O\left(\frac{|\Pi(a_j)|\log(\lambda_j\rho)^{\frac{3}{4}}}{\lambda_j}+\frac{\rho}{\lambda_j}+\frac{|\nabla K(a_j)|}{\lambda_j}+\frac{\log(\lambda_j)}{\lambda_j^2}\right)|\nabla K(a_j)|\cdot||v||.
\end{aligned}\tag{6.21}$$

It is left to expand the integral

$$\int_{\partial M}K(x)\varphi_{a_j,\lambda_j}^2\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}\cdot\nabla K(a_j)d\sigma_g.$$

Since

$$\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}\cdot\nabla K(a_j)=u_{a_j}\frac{1}{\lambda_j}\nabla_{a_j}\hat{\varphi}_{a_j,\lambda_j}\cdot\nabla K(a_j)+\hat{\varphi}_{a_j,\lambda_j}\frac{1}{\lambda_j}\nabla_{a_j}u_{a_j}\cdot K(a_j)$$

and $\nabla_{a_j}u_{a_j}(a_j)=0$, due to $u_a(a)=1$ for all $a\in\partial M$, we can first estimate

$$\begin{aligned}
&\int_{\partial M}K(x)\varphi_{a_j,\lambda_j}^2\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}\cdot\nabla K(a_j)d\sigma_g\\
&\qquad=\int_{B_\rho(a_j)}K(x)\hat{\varphi}_{a_j,\lambda_j}^2\frac{1}{\lambda_j}\nabla_{a_j}\hat{\varphi}_{a_j,\lambda_j}\cdot\nabla K(a_j)d\sigma_{g_{a_j}}+O\left(\frac{|\nabla K(a_j)|}{(\lambda_j\rho)^3}\right)\\
&\qquad=\sum_{m=1}^{3}\partial_m K(a_j)\int_{B_\rho}K(x)\delta_{\lambda_j}^3\left(\frac{2\lambda_j x_m+O(\lambda_j|x|^3)}{1+\lambda_j^2|x|^2}\right)dx+O\left(\frac{|\nabla K(a_j)|}{(\lambda_j\rho)^3}\right)\\
&\qquad=\frac{2}{3}\frac{|\nabla K(a_j)|^2}{\lambda_j}\int_{\mathbb{R}^3}\frac{|x|^2}{(1+|x|^2)^4}dx+O\left(\frac{|\nabla K(a_j)|}{(\lambda_j\rho)^3}\right)\\
&\qquad=\frac{2}{3}I_5\frac{|\nabla K(a_j)|^2}{\lambda_j}+O\left(\frac{|\nabla K(a_j)|}{(\lambda_j\rho)^3}\right),
\end{aligned}$$

where

$$I_5 = \int_{\mathbb{R}^3} \frac{|x|^2}{\left(1+|x|^2\right)^4} dx.$$

We combine the previous estimates to derive the expansion

$$\begin{aligned}
&\int_{\partial M} K(x) \left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v \right)^2 \frac{1}{\lambda_j} \nabla_{a_j} \varphi_{a_j,\lambda_j} \cdot \nabla K(a_j) d\sigma_g \\
&\qquad = \alpha_j^2 \frac{2}{3} I_4 \frac{|\nabla K(a_j)|^2}{\lambda_j} + O\left(\frac{|\nabla K(a_j)|}{(\lambda_j \rho)^3} \right) + o\left(\frac{|\nabla K(a_j)|^2}{\lambda_j} + \frac{1}{\lambda_j^2} \right) \\
&\qquad + O\left(\sum_{i \neq j} \varepsilon_{ij} \right) |\nabla K(a_j)| + O(||v||^2). \qquad (6.22)
\end{aligned}$$

Finally, multiplying (6.8) with (6.22) and adding (6.19) proves the Proposition. □

7. Construction of a pseudo gradient

In this chapter we construct a pseudo gradient Z on $V(p,\varepsilon)$ such that, along the flow of $\dot{u} = Z(u)$ in $V(p,\varepsilon)$, the following inequality holds true:

$$< \nabla J(u), Z(u) > \geq c_0 \left(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|^2}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} + |\alpha - |\alpha|\bar{\alpha}|^2 + ||v - \bar{v}||^2 \right). \tag{7.1}$$

Here $u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v$, $\bar{v}$ is the minimizer from Proposition 8 and $\bar{\alpha} = \bar{\alpha}(a)$ is the unique critical point of

$$S^{p-1} \ni \alpha \mapsto \frac{|\alpha|^2}{\left(\sum_{i=1}^{p} \alpha_i^3 K(a_i)\right)^{\frac{2}{3}}}$$

such that $\bar{\alpha}_i > 0$ for all i. The vector field will be constructed in such a way that the movement of $(\alpha, a, \lambda, v - \bar{v})$ will be properly understood.

7.1. Finite dimensional reduction

Let S_p be the symmetric group on p letters. The symmetric group on p letters S_p acts on B_ε^p via

$$(\alpha, a, \lambda)^\pi := \left(\alpha_{\pi(1)}, \cdots, \alpha_{\pi(p)}, a_{\pi(1)}, \cdots, a_{\pi(p)}, \lambda_{\pi(1)}, \cdots, \lambda_{\pi(p)}\right).$$

and on $T_{(\alpha,a,\lambda)} B_\varepsilon^p$ through the same definition. We call a vector field $W = (W_\alpha, W_a, W_\lambda)$ S_p-equivariant if

$$W_\alpha((\alpha,a,\lambda)^\pi)^i = W_\alpha(\alpha,a,\lambda)^{\pi(i)},\ W_a((\alpha,a,\lambda)^\pi)^i = W_a(\alpha,a,\lambda)^{\pi(i)},$$
$$W_\lambda((\alpha,a,\lambda)^\pi)^i = W_\lambda(\alpha,a,\lambda)^{\pi(i)}$$

for all $\pi \in S_p$.

In this section we assume the existence of a S_p-equivariant vector field
$W = (W_\alpha, W_a, W_\lambda)$ on B_ε^p such that, along the flow of this vector field, the following

inequality holds:

$$\frac{d}{dt}J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)\geq c_0\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}+|\alpha-|\alpha|\bar{\alpha}|^2\right). \quad (7.2)$$

Furthermore, we assume

$$|W_\alpha^i|,|\lambda_i W_a^i|,|\lambda_i^{-1}W_\lambda^i|\leq C\ \forall\ 1\leq i\leq p.$$

The existence will be proved in section 7.4.

First remember that any $u\in V(p,\varepsilon)$ can be written as

$$u=\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v,$$

where $(\alpha,a,\lambda)\in B_\varepsilon^p$ and $v\in E_{(\alpha,a,\lambda)}$. This representation is unique up to permutation of (α,a,λ). Define

$$E:=\bigcup_{(\alpha,a,\lambda)\in B_\varepsilon^p}\{(\alpha,a,\lambda)\}\times E_{(\alpha,a,\lambda)}\subset B_\varepsilon^p\times H^1(M).$$

We would like to show that E is a smooth submanifold in $B_\varepsilon^p\times H^1(M)$ of codimension $N:=\dim\left(E_{(\alpha,a,\lambda)}^\perp\right)$. Thus let $f:B_\varepsilon^p\times H^1(M)\to\mathbb{R}^N$ the smooth map defined by

$$f(\alpha,a,\lambda,v):=\begin{pmatrix}<v,\varphi_{a_i,\lambda_i}>\\ <v,\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i}>\\ <v,\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i}>\end{pmatrix}$$

and observe $E=f^{-1}(0)$. We need to show that $Df(\alpha,a,\lambda,v)$ is onto for $(\alpha,a,\lambda,v)\in E$. Observe that

$$Df(\alpha,a,\lambda,v)[0,0,0,h]=\begin{pmatrix}<h,\varphi_{a_i,\lambda_i}>\\ <h,\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i}>\\ <h,\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i}>\end{pmatrix}.$$

For $b\in\mathbb{R}^N$ we would like to solve the equation

$$Df(\alpha,a,\lambda,v)[0,0,0,h]=\begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix} \quad (7.3)$$

with the ansatz

$$h=\sum_{j=1}^{p}x_j\varphi_{a_j,\lambda_j}+\sum_{j=1}^{p}y_j\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}+\sum_{j=1}^{p}Z_j\cdot\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}$$

and

$$(x, Z_1, \cdots, Z_p, y, h) \in \mathbb{R}^p \times T_{a_1}\partial M \times \cdots \times T_{a_p}\partial M \times \mathbb{R}^p.$$

Set

$$2I_2 := \lim_{\lambda\to\infty} \left|\left|\lambda\frac{\partial}{\partial\lambda}\varphi_{a,\lambda}\right|\right|^2, \quad 2I_3 := \lim_{\lambda\to\infty} \left|\left|\frac{1}{\lambda}\frac{\partial}{\partial a^m}\varphi_{a,\lambda}\right|\right|^2,$$

then (7.3) is equivalent to

$$\Big(diag(2I_0, \cdots 2I_0, 2I_2, \cdots 2I_2, 2I_3, \cdots, 2I_3) + o(\varepsilon)\Big) \begin{pmatrix} x \\ y \\ Z \end{pmatrix} = \begin{pmatrix} b_1 \\ \lambda_i b_2 \\ \frac{1}{\lambda_i} b_3 \end{pmatrix},$$

where $o(\varepsilon) \to 0$ uniformly if $\varepsilon \to 0$. Here, we used the estimates from Lemma 11 in appendix C. Hence (7.3) has, for ε small, a unique solution with the previous ansatz. Therefore Df is onto and hence E a smooth submanifold of finite codimension. The tangent space is $T_{(\alpha,a,\lambda,v)}E = kerDf(\alpha, a, \lambda, v)$. We now define a parametrization, given by $\psi : E \to H^1(M)$:

$$\psi(\alpha, a, \lambda, v) := \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v.$$

Since ψ is smooth on $B_\varepsilon^p \times H^1(M)$ it is also smooth from E to $H^1(M)$. The main goal is to show that $\psi : \psi^{-1}(V(p, \varepsilon)) \to V(p, \varepsilon)$ is a local diffeomorphism. Therefore it is useful to show that the derivative $D\psi$ is an isomorphism form the tangent space to $H^1(M)$. Let $u_0 = \sum_{i=1}^{p} \alpha_i\varphi_{a_i,\lambda_i} + v \in V(p, \varepsilon)$, then $(\alpha, a, \lambda, v) \in \psi^{-1}(V(p, \varepsilon))$. Remember that

$$kerDf(\alpha, a, \lambda, v) = \Bigg\{ (x, Z, y, h) \in \mathbb{R}^p \times T_a\partial M^p \times \mathbb{R}^p \times H^1(M) \ :$$

$$\begin{pmatrix} < h, \varphi_{a_j,\lambda_j} > \\ < h, \frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j} > + < v, y_j\frac{\partial^2}{\partial\lambda_j^2}\varphi_{a_j,\lambda_j} + \sum_{m=1}^3 Z_j^m \frac{\partial^2}{\partial a_j^m\partial\lambda_j}\varphi_{a_j,\lambda_j} > \\ < h, \frac{\partial}{\partial a_j^m}\varphi_{a_j,\lambda_j} > + < v, \sum_{l=1}^3 \frac{\partial^2}{\partial a_j^l a_j^m}\varphi_{a_j,\lambda_j} Z_j^l + y_j\frac{\partial^2}{\partial a_j^m\partial\lambda_j}\varphi_{a_j,\lambda_j} > \end{pmatrix} = 0 \Bigg\}. \quad (7.4)$$

Let $(x, Z, y, h) \in kerDf$ and assume

$$D\psi(\alpha, a, \lambda, v)[x, Z, y, h] = 0.$$

We need to show that $x = y = Z = h = 0$. More precisely:

$$D\psi(\alpha, a, \lambda, v)[x, Z, y, h] = \sum_{i=1}^{p} x_i\varphi_{a_i,\lambda_i} + \sum_{i=1}^{p} \frac{y_i}{\lambda_i}\lambda_i\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i} + \sum_{i=1}^{p} \lambda_i Z_i \cdot \frac{1}{\lambda_i}\nabla_{a_i}\varphi_{a_i,\lambda_i} + h.$$

$D\psi(\alpha, a, \lambda, v)[x, Z, y, h] = 0$ implies:

$$< D\psi(\alpha, a, \lambda, v)[x, Z, y, h], \varphi_{a_j,\lambda_j} >= 0 =< D\psi(\alpha, a, \lambda, v)[x, Z, y, h], \lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j} >,$$

$$0 =< D\psi(\alpha, a, \lambda, v)[x, Z, y, h], \frac{1}{\lambda_j}\frac{\partial}{\partial a_j^m}\varphi_{a_j,\lambda_j} > \quad \forall\, 1 \le j \le p,\ 1 \le m \le 3. \quad (7.5)$$

Using (7.4) and the interaction estimates in appendix B, C as well as $||v|| < \varepsilon$ in $V(p,\varepsilon)$, (7.5) is equivalent to

$$\left(diag(2I_0,\cdots 2I_0, 2I_2,\cdots 2I_2, 2I_3,\cdots,2I_3)+o(\varepsilon)\right)\begin{pmatrix} x \\ \lambda_i^{-1}y \\ \lambda_i Z\end{pmatrix} = 0,$$

which implies $x = y = Z = 0$ if ε is small. But then h must be zero as well. Hence, we have proved that $D\psi(\alpha, a, \lambda, v)$ is into. It remains to show that $D\psi$ is onto.
Let $u \in H^1(M)$ and write

$$\begin{aligned} u =& \sum_{i=1}^p x_i\varphi_{a_i,\lambda_i} + \sum_{i=1}^p \frac{y_i}{\lambda_i}\lambda_i\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i} + \sum_{i=1}^p \lambda_i Z_i\cdot\frac{1}{\lambda_i}\nabla_{a_i}\varphi_{a_i,\lambda_i} \\ &+ u - \left(\sum_{i=1}^p x_i\varphi_{a_i,\lambda_i} + \sum_{i=1}^p \frac{y_i}{\lambda_i}\lambda_i\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i} + \sum_{i=1}^p \lambda_i Z_i\cdot\frac{1}{\lambda_i}\nabla_{a_i}\varphi_{a_i,\lambda_i}\right). \end{aligned}$$

We need to find x, y, Z such that

$$u - \left(\sum_{i=1}^p x_i\varphi_{a_i,\lambda_i} + \sum_{i=1}^p \frac{y_i}{\lambda_i}\lambda_i\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i} + \sum_{i=1}^p \lambda_i Z_i\cdot\frac{1}{\lambda_i}\nabla_{a_i}\varphi_{a_i,\lambda_i}\right)$$

in an element of the tangent space. Due to (7.4), this is equivalent to

$$\left(diag(2I_0,\cdots 2I_0, 2I_2,\cdots 2I_2, 2I_3,\cdots,2I_3)+o(\varepsilon)\right)\begin{pmatrix} x \\ \lambda_i^{-1}y \\ \lambda_i Z\end{pmatrix} = \begin{pmatrix} < u, \varphi_{a_j,\lambda_j} > \\ < u, \lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j} > \\ < u, \frac{1}{\lambda_j}\frac{\partial}{\partial a_j^m}\varphi_{a_j,\lambda_j} > \end{pmatrix}.$$

Provided ε is small, the previous equation has a unique solution (x, y, Z) for all $u \in H^1(M)$. Moreover, this solution depends smooth on $(\alpha, a, \lambda, v, u)$. Hence we have proved that $D\psi$ is onto as well. Therefore $D\psi(\alpha, a, \lambda, v)$ is an isomorphism provided $\sum_{i=1}^p \alpha_i\varphi_{a_i,\lambda_i} + v \in V(p,\varepsilon)$.

Now we will construct a vector field $\tilde{Z}$ on E. We assume that the vector field $W = (W_\alpha, W_a, W_\lambda)$ on B_ε^p exists. We make the following ansatz for the vector field

$$\tilde{Z}(\alpha, a, \lambda, v) = (W_\alpha + t\alpha, W_a, W_\lambda, C(v-\bar{v}) + < \nabla\bar{v}, (W_\alpha, W_a, W_\lambda) > - t\bar{v} - R),$$

where

$$R = \sum_{i=1}^p x_i\varphi_{a_i,\lambda_i} + \sum_{i=1}^p \frac{y_i}{\lambda_i}\lambda_i\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i} + \sum_{i=1}^p \lambda_i Z_i\cdot\frac{1}{\lambda_i}\nabla_{a_i}\varphi_{a_i,\lambda_i}$$

for some unknown x, y, Z and $t \in \mathbb{R}, C > 0$.The correction term $t\alpha$ is needed to make the flow preserve the norm. We first choose R, depended on $t \in \mathbb{R}$, and later t. We need to find R such that $\tilde{Z}$ is tangential to E. Later we will prove (see Lemma 5) that

$$| < D\bar{v}[W], u > | \leq C(m)||\bar{v}|| \quad \forall u \in E^{\perp}_{(\alpha,a,\lambda)},\ ||u|| \leq m.$$

Using this inequality and (7.4), $\tilde{Z}$ is tangential iff

$$\left(diag(2I_0,\cdots 2I_0,2I_2,\cdots 2I_2,2I_3,\cdots,2I_3)+o(\varepsilon)\right)\begin{pmatrix} x \\ \lambda_i^{-1}y \\ \lambda_i Z\end{pmatrix}$$

$$=\begin{pmatrix} \langle<\nabla\bar{v},(W_\alpha,W_a,W_\lambda)>,\varphi_{a_j,\lambda_j}\rangle \\ \langle<\nabla\bar{v},(W_\alpha,W_a,W_\lambda)>,\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}\rangle \\ \langle<\nabla\bar{v},(W_\alpha,W_a,W_\lambda)>,\frac{1}{\lambda_j}\frac{\partial}{\partial a_j^m}\varphi_{a_j,\lambda_j}\rangle \end{pmatrix}.$$

The previous equation has a unique solution (x,y,Z), which does not depend on t, such that $|x|,|\frac{y_i}{\lambda_i}|,||\lambda_i Z_i||=O\left(||\bar{v}||\right)$. We now define $\tilde{Z}_t(\alpha,a,\lambda,v)$ as the unique vector field tangent to E, depending on t, constructed above. Finally we would like to choose $t=t(\alpha,a,\lambda,v)$ such that

$$\left|\left|\sum_{i=1}^p\alpha_i\varphi_{a_i,\lambda_i}+v\right|\right|^2$$

is preserved under the flow of $\frac{d}{ds}(\alpha,a,\lambda,v)=\tilde{Z}_t(\alpha,a,\lambda,v)$. Hence the equation, which needs to be satisfied is:

$$\begin{aligned}0&=\frac{1}{2}\frac{d}{ds}\left|\left|\sum_{i=1}^p\alpha_i\varphi_{a_i,\lambda_i}+v\right|\right|^2\\ &=t\left|\left|\sum_{i=1}^p\alpha_i\varphi_{a_i,\lambda_i}\right|\right|^2+t<\bar{v},v>\\ &+<\sum_{i=1}^p W_\alpha^i\varphi_{a_i,\lambda_i}+\sum_{i=1}^p\alpha_i W_\lambda^i\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i}+\sum_{i=1}^p\alpha_i W_{a_i}\cdot\nabla_{a_i}\varphi_{a_i,\lambda_i},\sum_{j=1}^p\alpha_j\varphi_{a_j,\lambda_j}>\\ &+<C(v-\bar{v})+<\nabla\bar{v},(W_\alpha,W_a,W_\lambda)>-R,v>. \qquad (7.6)\end{aligned}$$

Since

$$\left|\left|\sum_{i=1}^p\alpha_i\varphi_{a_i,\lambda_i}\right|\right|^2+<\bar{v},v>=2I_0|\alpha|^2+o(\varepsilon)>0$$

for ε small, there exists exactly one $t(\alpha,a,\lambda,v)\in\mathbb{R}$ such that (7.6) is satisfied. Due to the implicit function theorem $t=t(\alpha,a,\lambda,v)$ is C^2. Furthermore t is bounded. Set $\tilde{Z}(\alpha,a,\lambda,v)=\tilde{Z}_{t(\alpha,a,\lambda,v)}(\alpha,a,\lambda,v)$, which defines a vector field on E such that the norms are preserved. Finally, in $V(p,\varepsilon)$ we define $Z(u)=D\psi(\alpha,a,\lambda,v)[\tilde{Z}(\alpha,a,\lambda,v)]$, where $(\alpha,a,\lambda,v)\in E$ is such that

$$u=\sum_{i=1}^p\alpha_i\varphi_{a_i,\lambda_i}+v.$$

Since ψ is a local diffeomorphism Z is C^2. Since the vector field W is S_p-equivarinat the definition of $Z(u)$ does not depend on the preimage of u under ψ. Hence Z is a well defined C^2-vector field in $V(p,\varepsilon)$. Our main goal is to show that Z is a pseudo gradient, provided (7.2) holds and C is large. Due to the construction of Z we have an obvious, but important Lemma, which we will need later.

Lemma 4. *If $u \in V(p, \varepsilon)$ moves along $\frac{d}{dt}u = Z(u)$ and $u(t) = \sum_{i=1}^{p} \alpha_i(t)\varphi_{a_i(t),\lambda_i(t)} + v(t)$, then (α, a, λ, v) moves along $\frac{d}{dt}(\alpha, a, \lambda, v) = \tilde{Z}(\alpha, a, \lambda, v)$.*

Proof. Let u move along $\frac{d}{dt}u = Z(u)$, then locally around $u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v$, ψ is a diffeomorphism and hence

$$\frac{d}{dt}(\alpha, a, \lambda, v) = \frac{d}{dt}\psi^{-1}(u(t)) = D\psi^{-1}(u)[D\psi(\alpha, a, \lambda, v)[\tilde{Z}(\alpha, a, \lambda, v)] = \tilde{Z}(\alpha, a, \lambda, v).$$

□

Due to Lemma 4 (a, λ) moves along the flow of (W_a, W_λ) and

$$\frac{d}{dt}||v - \bar{v}||^2 = 2 < \frac{d}{dt}(v - \bar{v}), v - \bar{v} >= 2C||v - \bar{v}||^2.$$

Using the previous observations, we are prepared to prove that Z is a pseudo gradient.

Proposition 14. *In $V(p, \varepsilon)$ there holds*

$$< \nabla J(u), Z(u) >\geq c_0 \left(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|^2}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} + |\alpha - |\alpha|\bar{\alpha}|^2 + ||v - \bar{v}||^2 \right).$$

Proof. Let u move along $\dot{u} = Z(u)$, then

$$\begin{aligned} < \nabla J(u), Z(u) > =& < \nabla J \left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v} \right), \frac{d}{dt} \left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v \right) > \\ &+ \int_0^1 D^2 J \left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v} + s(v - \bar{v}) \right) [v - \bar{v}, \frac{d}{dt} \left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v \right)]ds. \end{aligned}$$

The first term can be estimated by:

$$\begin{aligned}
&< \nabla J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right), \frac{d}{dt}\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v\right) > \\
&=< \nabla J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right), \frac{d}{dt}\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)+\frac{d}{dt}(v-\bar{v}) > \\
&\geq c_0\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}+|\alpha-|\alpha|\bar{\alpha}|^2\right) \\
&- o(\varepsilon)\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}+|\alpha-|\alpha|\bar{\alpha}|^2\right) \\
&\geq c_1\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}+|\alpha-|\alpha|\bar{\alpha}|^2\right)
\end{aligned}$$

for ε small. Since

$$\begin{aligned}
\left(D^2J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}+s(v-\bar{v})\right)-D^2J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)\right)[h,h'] \\
\leq c||v-\bar{v}||\cdot||h||\cdot||h'||,
\end{aligned}$$

we deduce

$$\begin{aligned}
&\int_0^1 D^2J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}+s(v-\bar{v})\right)[v-\bar{v},\frac{d}{dt}\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v\right)]ds \\
&\geq D^2J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)[v-\bar{v},\frac{d}{dt}\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v\right)]-c||v-\bar{v}||^2 \\
&= D^2J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)[v-\bar{v},\frac{d}{dt}\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)+\frac{d}{dt}(v-\bar{v})]-c||v-\bar{v}||^2.
\end{aligned}$$

In addition

$$\begin{aligned}
&D^2J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)[v-\bar{v},\frac{d}{dt}\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)] \\
&\quad = - < \nabla J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right), \frac{d}{dt}(v-\bar{v}) > \\
&\quad = o\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}+|\alpha-|\alpha|\bar{\alpha}|^2\right).
\end{aligned}$$

Lastly

$$D^2J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)[v-\bar{v},\frac{d}{dt}(v-\bar{v})]\geq C||v-\bar{v}||^2-c||v-\bar{v}||^2$$
$$+o\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}+|\alpha-|\alpha|\bar{\alpha}|^2\right).$$

Hence, if we choose C in the definition of $\tilde{Z}$ large enough, the previous estimates yield

$$< \nabla J(u), Z(u) >$$
$$\geq c_0\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}+|\alpha-|\alpha|\bar{\alpha}|^2+||v-\bar{v}||^2\right)$$

provided ε is small. Therefore the Proposition is proved. □

Before we can construct the vector field W on B^p_ε we need to prove some technical Lemmas. This will be done in the following two sections.

7.2. C^2 dependence of $\bar{v}$.

Let us recall Proposition 8. There exists $\varepsilon_0, \delta_0 > 0$ such that the minimization problem

$$\min_{v\in B_{\delta_0}(0)\cap E_{(\alpha,a,\lambda)}} J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+v\right)$$

has a unique solution $\bar{v}=\bar{v}(\alpha,a,\lambda)$ for all $(\alpha,a,\lambda)\in B^p_{\varepsilon_0}$.

We are now going to prove that the (α,a,λ) dependence of $\bar{v}$ is at least C^2. Therefore define

$$V_{(\alpha,a,\lambda)}:=\left\langle \varphi_{a_i,\lambda_i},\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i},\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i}\;\middle|\; i=1,\cdots,p,\ m=1,2,3\right\rangle.$$

For ε small this vector space has dimension $d=5p$. Since these vectors depend smoothly on (a,λ) we can construct an orthonormal basis $(v_1,\cdots,v_d)$ of $V_{(\alpha,a,\lambda)}$ by using the Gram-Schmidt procedure, which depends smoothly on (a,λ). Furthermore $E_{(\alpha,a,\lambda)}=V^\perp_{(\alpha,a,\lambda)}$. Fix $(\alpha_0,a_0,\lambda_0)\in B^p_\varepsilon$.

Claim: For (α,a,λ) close to (α_0,a_0,λ_0), the orthogonal projection

$$p:E_{(\alpha,a,\lambda)}\to E_{(\alpha_0,a_0,\lambda_0)}\ ,p(u)=u-\sum_{i=1}^{d}< u,v_i(a_0,\lambda_0) > v_i(a_0,\lambda_0)$$

is an isomorphism.

Proof. Define the matrix $A = (a_{ij})$ by $a_{ij}(a,\lambda) =< v_i(a,\lambda), v_j(a_0,\lambda_0) >$. Since A depends smooth on (a,λ) and $A(a_0,\lambda_0) = Id$, A is invertible for (α, a, λ) in a neighbourhood U of $(\alpha_0, a_0, \lambda_0)$.
(a) injectivity: If $p(u) = 0$, then $u = \sum_{i=1}^{d} < u, v_i(a_0,\lambda_0) > v_i(a_0,\lambda_0)$. But u is also in $E_{(\alpha,a,\lambda)}$, hence

$$0 = \sum_{i=1}^{d} < u, v_i(a_0,\lambda_0) >< v_i(a_0,\lambda_0), v_j(a,\lambda) > \text{ for } j = 1, \cdots, d.$$

This implies $< u, v_i(a_0,\lambda_0) >= 0$ for all i, because A is invertible. Therefore $u = 0$.
(b) p is surjective: For $w \in E_{(\alpha_0,a_0,\lambda_0)}$ we need to find numbers $\beta_1, \cdots, \beta_d$ such that $w + \sum_{i=1}^{d} \beta_i v_i(a_0,\lambda_0) \in E_{(\alpha,a,\lambda)}$. This is equivalent to

$$< w, v_j(a,\lambda) >= -\sum_{i=1}^{d} \beta_i < v_i(a_0,\lambda_0), v_j(a,\lambda) > \quad \text{for} \quad j = 1, \cdots, d,$$

which has a unique solution $(\beta_1, \cdots, \beta_d)$ because A is invertible. □

Using this observation we know prove the main result in this section.

Proposition 15. *The map* $(\alpha, a, \lambda) \mapsto \bar{v}(\alpha, a, \lambda)$ *is* C^2.

Proof. Let $p(\alpha, a, \lambda) : H^1(M) \to E_{(\alpha,a,\lambda)}$ the (smooth) orthogonal projection.
We set $p = p(\alpha_0, a_0, \lambda_0)$ and define $F : U \times E_{(\alpha_0,a_0,\lambda_0)} \to E_{(\alpha_0,a_0,\lambda_0)}$ by

$$F(\alpha, a, \lambda, w) = p\Bigg(p(\alpha, a, \lambda)\nabla J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + p(\alpha, a, \lambda)w\right)\Bigg).$$

Here U is a neighbourhood of $(\alpha_0, a_0, \lambda_0)$ such that $p(\alpha, a, \lambda)$ is an isomorphism on U. We have $F(\alpha_0, a_0, \lambda_0, \bar{v}(\alpha_0, a_0, \lambda_0)) = 0$. Moreover

$$D_w F(\alpha_0, a_0, \lambda_0, \bar{v}) = (p \circ D\nabla J)\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right)$$

is invertible on $E_{(\alpha_0,a_0,\lambda_0)}$, which was proved in chapter 5 (see (5.7)). The implicit function theorem implies $w = w(\alpha, a, \lambda)$ is a C^2 map locally around $(\alpha_0, a_0, \lambda_0)$. Therefore also $\bar{w}(\alpha, a, \lambda) := p(\alpha, a, \lambda)w(\alpha, a, \lambda)$ is C^2. The claim implies that $F(\alpha, a, \lambda, w(a, \alpha, \lambda)) = 0$ is locally equivalent to

$$p(\alpha, a, \lambda)\nabla J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{w}(\alpha, a, \lambda)\right) = 0.$$

Since $\bar{v}(\alpha, a, \lambda)$ is the only solution, $\bar{w} = \bar{v}$ and hence $\bar{v}$ is C^2. □

7.3. Technical Lemmas

We set

$$F_{(\alpha,a,\lambda)} := \left\langle \frac{1}{\lambda_i}\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i}, \lambda_i \frac{\partial}{\partial \lambda_i}\varphi_{a_i,\lambda_i} \;\middle|\; i=1,\cdots,p,\; m=1,\cdots,3 \right\rangle \subset H^1(M).$$

If $(\alpha,a,\lambda)\in B_\varepsilon^p$ with ε small, then there exist constants c, C, independend of (α,a,λ), such that

$$c\left(\sum_{i=1}^p\sum_{m=1}^3 |t_{im}|^2 + \sum_{i=1}^p |r_i|^2\right) \le ||\phi||^2 \le C\left(\sum_{i=1}^p\sum_{m=1}^3 |t_{im}|^2 + \sum_{i=1}^p |r_i|^2\right)$$

for all

$$\phi = \sum_{i=1}^p\sum_{m=1}^3 t_{im}\frac{1}{\lambda_i}\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i} + \sum_{i=1}^p r_i\lambda_i\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i}.$$

Lemma 5. *If* $W=(0,W_{a_1},\cdots,W_{a_p},W_{\lambda_1},\cdots,W_{\lambda_p})\in\mathbb{R}^p\times T_{a_1}\partial M\times\cdots\times T_{a_p}\partial M\times\mathbb{R}^p$ *is a tangential vector, then*

$$| < D\bar v(\alpha,a,\lambda)[W],\phi > | \le C||\bar v||\cdot|\tilde W|\cdot||\phi||$$

for $(\alpha,a,\lambda)\in B_\varepsilon^p$*. Here*

$$\tilde W = (0,\lambda_1 W_{a_1},\cdots,\lambda_p W_{a_p},\frac{1}{\lambda_1}W_{\lambda_1},\cdots,\frac{1}{\lambda_p}W_{\lambda_p}).$$

Proof. Since $\bar v\in E_{(\alpha,a,\lambda)}$ we have

$$< \frac{\partial}{\partial a_j^l}\bar v, \frac{1}{\lambda_i}\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i} > = -\delta_{ij} < \bar v, \frac{1}{\lambda_i}\frac{\partial}{\partial a_j^l}\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i} >,$$
$$< \frac{\partial}{\partial \lambda_j}\bar v, \frac{1}{\lambda_i}\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i} >= -\delta_{ij} < \bar v, \frac{1}{\lambda_i}\frac{\partial}{\partial \lambda_j}\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i} >$$

as well as

$$< \frac{\partial}{\partial a_j^l}\bar v, \lambda_i\frac{\partial}{\partial \lambda_i}\varphi_{a_i,\lambda_i} > = -\delta_{ij} < \bar v, \lambda_i\frac{\partial}{\partial a_j^l}\frac{\partial}{\partial \lambda_i}\varphi_{a_i,\lambda_i} >,$$
$$< \frac{\partial}{\partial \lambda_j}\bar v, \lambda_i\frac{\partial}{\partial \lambda_i}\varphi_{a_i,\lambda_i} >= -\delta_{ij} < \bar v, \lambda_i\frac{\partial}{\partial \lambda_j}\frac{\partial}{\partial \lambda_i}\varphi_{a_i,\lambda_i} > .$$

For

$$\phi = \sum_{i=1}^p\sum_{m=1}^3 t_{im}\frac{1}{\lambda_i}\frac{\partial}{\partial a_i^m}\varphi_{a_i,\lambda_i} + \sum_{i=1}^p r_i\lambda_i\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i}$$

these identities yield

$$\left|\left\langle \frac{\partial}{\partial a_j^l}\bar{v},\phi\right\rangle\right| \leq C\lambda_j||\bar{v}||\cdot||\phi|| \text{ and } \left|\left\langle \frac{\partial}{\partial \lambda_j}\bar{v},\phi\right\rangle\right| \leq C\frac{1}{\lambda_j}||\bar{v}||\cdot||\phi||.$$

For $(a_1,\cdots,a_p)$ we choose Riemannian normal coordinates at these points. Then the vectors W_{a_j} have the expansion $W_{a_j}=W_{a_j}^l\partial_l$ and

$$D\bar{v}(\alpha,a,\lambda)[W]=\sum_{j=1}^{p}\sum_{l=1}^{n-1}\frac{\partial}{\partial a_j^l}\bar{v}W_{a_j}^l+\sum_{j=1}^{p}\frac{\partial}{\partial\lambda_j}\bar{v}W_{\lambda_j}.$$

Hence

$$|< D\bar{v}(\alpha,a,\lambda)[W],\phi>|\leq C\left(\sum_{j=1}^{p}\lambda_j|W_{a_j}|+\frac{1}{\lambda_j}|W_{\lambda_j}|\right)||\bar{v}||\cdot||\phi||\leq C||\bar{v}||\cdot|\tilde{W}|\cdot||\phi||.$$

□

Lemma 6. *If all critical points of K are umbilic points, then there exists a constant $C>0$, independent of $(\alpha,a,\lambda)\in B_\varepsilon^p$, such that*

$$\left|<\nabla J\Big(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\Big),\phi>\right|\leq C\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}+||\bar{v}||^2\right)||\phi||$$

for all $\phi\in F_{(\alpha,a,\lambda)}$.

Proof. The proof is a direct application of Proposition 12 and 13 in chapter 6. □

For a tangential vector $Z=(Z_{\alpha_1},\cdots,Z_{\alpha_p},Z_{a_1},\cdots Z_{a_p},Z_{\lambda_1},\cdots,Z_{\lambda_p})$ at (α,a,λ) we define the vector

$$Z_0:=(0,Z_{a_1},\cdots,Z_{a_p},Z_{\lambda_1},\cdots,Z_{\lambda_p}).$$

Lemma 7. *With the notation from above we have*

$$\left|<\nabla J\Big(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\Big),D\bar{v}[Z]>\right|\leq C\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^3}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i^2}+\sum_{i\neq j}\varepsilon_{ij}^2\right)|\tilde{Z}_0|.$$

for $(\alpha,a,\lambda)\in B_\varepsilon^p$ and ε small.

Proof. Since

$$\nabla J\Big(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\Big)\in E^{\perp}_{(\alpha,a,\lambda)},$$

we conclude

$$\nabla J\Big(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\Big)=\sum_{j=1}^{p}\beta_j\varphi_{a_j,\lambda_j}+\phi\ \text{ with } \phi\in F_{(\alpha,a,\lambda)}.$$

Furthermore $0=<\frac{\partial}{\partial\alpha_i}\bar{v},\varphi_{a_j,\lambda_j}>=0=<\frac{\partial}{\partial a_i^m}\bar{v},\varphi_{a_j,\lambda_j}>=0=<\frac{\partial}{\partial\lambda_i}\bar{v},\varphi_{a_j,\lambda_j}>$ and therefore

$$\Big|<\nabla J\Big(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\Big),D\bar{v}[Z]>\Big|=|<\phi,D\bar{v}[Z_0]>|\leq C||\phi||\cdot||\bar{v}||\cdot|\tilde{Z}_0|.$$

The result follows by Proposition 8 in chapter 5 and Lemma 6. □

7.4. Construction of a vector field on B^p_ε.

This section is crucial for our argument because we construct a vector field W, which will help us to understand the behaviour of (α,a,λ) if $u\in V(p,\varepsilon)$. This construction is motivated by a construction of a pseudo gradient in [11], which goes back to [8].

In the following we have to introduce some notation which we need in the proof of the main Proposition. We define the function

$$f:\mathbb{R}^p_+\times\partial M^p\to\mathbb{R},\quad f(\alpha,a):=\frac{Q(B^4,\partial B^4)\sum_{i=1}^{p}\alpha_i^2}{\left(\sum_{i=1}^{p}\alpha_i^3K(a_i)\right)^{\frac{2}{3}}}.$$

For $a\in\partial M^p$ fixed, the function

$$\mathbb{R}^p_+\ni\alpha\mapsto f(\alpha,a)$$

has exactly one critical point $\bar{\alpha}=\bar{\alpha}(a)$ with $|\bar{\alpha}|=1$ and $\bar{\alpha}_i>0$ for all i. To be more precise:

$$\bar{\alpha}_i=\frac{1}{\sqrt{\sum_{j=1}^{p}\frac{K(a_i)^2}{K(a_j)^2}}}.$$

Proposition 16. *There exists an S_p-equivariant vector field $W = (W_\alpha, W_a, W_\lambda)$ on B^p_ε such that*

$$|W^i_\alpha|, |\lambda_i W^i_a|, |\lambda_i^{-1} W^i_\lambda| \leq C \ \forall \ 1 \leq i \leq p,$$

$$\frac{d}{dt} J\left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right) \geq c_0 \left(\sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i=1}^p \frac{|\nabla K(a_i)|^2}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij} + |\alpha - |\alpha|\bar{\alpha}|^2\right)$$

if $p \geq 2$ and

$$\frac{d}{dt} J\left(\alpha\varphi_{a,\lambda} + \bar{v}\right) \geq c_0 \left(\frac{1}{\lambda^2} + \frac{|\nabla K(a)|^2}{\lambda}\right)$$

if $p = 1$, as soon as (α, a, λ) moves along $\frac{d}{dt}(\alpha, a, \lambda) = W(\alpha, a, \lambda)$.

Moreover $\max\{\lambda_1, \cdots, \lambda_p\}$ *is decreasing along the flow, if $(\alpha, a, \lambda) \notin V_2$, where*

$$V_2 := \Bigg\{(\alpha, a, \lambda) \in B^p_\varepsilon \ : \ \frac{|\nabla K(a_i)|^2}{\lambda_i} < \gamma \frac{1}{\lambda_j^2} \ \forall \ i; \quad d_g(a_i, x_i) < \rho_0 \text{ for some } x \in \mathcal{F}_p^\infty;$$
$$|\alpha - |\alpha|\bar{\alpha}|^2 < R\left(\sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i=1}^p \frac{|\nabla K(a_i)|^2}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij}\right)\Bigg\}.$$

Here $R, \gamma > 0$ are some constants.

Before we proof this Proposition, we recall the gradient expansions, which we have proved in the previous chapter. Since we assume that all critical points of K are umbilic points, the estimate $|\Pi(a)| \leq C|\nabla K(a)|$ hold true on ∂M. Using these estimate we can state a refined expansion of the gradient:

For $u = \sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v} \in V(p, \varepsilon)$ it holds (see Proposition 11-13)

$$< \nabla J(u), \varphi_{a_j,\lambda_j} >= 2l(u) \frac{\partial}{\partial \alpha_j} f(\alpha, a) + O\left(\frac{|\nabla K(a_j)|^2}{\lambda_j} + \frac{1}{\lambda_j^2} + \sum_{i\neq j} \varepsilon_{ij}\right), \tag{7.7}$$

where

$$l(u) = \frac{1}{\left(\int_{\partial M} K(x) u^3 d\sigma_g\right)^{\frac{2}{3}}}.$$

Furthermore if

$$|\alpha - |\alpha|\bar{\alpha}|^2 \leq R\left(\sum_{i=1}^p \frac{|\nabla K(a_j)|^2}{\lambda_j} + \sum_{i=1}^p \frac{1}{\lambda_j^2} + \sum_{i\neq j} \varepsilon_{ij}\right),$$

then

$$< \nabla J(u), \lambda_j \frac{\partial}{\partial \lambda_j} \varphi_{a_j,\lambda_j} >= 2l(u) \Bigg\{ \alpha_j \left(2|S^3_+| H_{a_j}(a_j) + \frac{2I_4}{9} \frac{\Delta K(a_j)}{K(a_j)} \right) \frac{1}{\lambda_j^2}$$

$$- 2 \sum_{i:i \neq j} \alpha_i I_1 \lambda_j \frac{\partial}{\partial \lambda_j} I(\varepsilon_{ij}) + o\left(\frac{1}{\lambda_j^2} + \frac{|\nabla K(a_j)|^2}{\lambda_j} \right) + o\left(\sum_{i \neq j} \varepsilon_{ij} \right)$$

$$+ O\left(\frac{\rho^2}{\lambda_j^2} \right) + O\left(\sum_{i \neq j} \rho \varepsilon_{ij} \right) \Bigg\} \tag{7.8}$$

and

$$< -\nabla J(u), \frac{1}{\lambda_j} \nabla K(a_j) \cdot \nabla_{a_j} \varphi_{a_j,\lambda_j} >$$

$$\geq c_0 \frac{|\nabla K(a_j)|^2}{\lambda_j} + o\left(\frac{|\nabla K(a_j)|^2}{\lambda_j} + \frac{1}{\lambda_j^2} \right) + O\left(\sum_{i:i \neq j} \varepsilon_{ij} |\nabla K(a_j)| \right) + O\left(\sum_{i=1}^{p} \frac{\rho^2}{\lambda_i^2} \right). \tag{7.9}$$

Proof of Proposition 16. First we construct a vector field W in B^1_ε. Let $crit(K) = \{x_1, \cdots, x_m\}$ be the set of critical points of K on ∂M. We define the following sets:

$$V_0 := \left\{ (\alpha, a, \lambda) \in B^1_\varepsilon \ : \ \frac{|\nabla K(a)|^2}{\lambda} > \frac{1}{\lambda^2} \right\}$$

as well as

$$V_i := \left\{ (\alpha, a, \lambda) \in B^1_\varepsilon \ : \ \frac{|\nabla K(a)|^2}{\lambda} < 2\frac{1}{\lambda^2}, \ d(a, x_i) < \rho_0 \right\} \quad \text{for} \quad 1 \leq i \leq m,$$

where ρ_0 is chosen small such that $V_i \cap V_j = \emptyset$ for $i \neq j$. W.l.o.g. we assume that $x_1, \cdots, x_l$ are those critical points with $2|S^3_+| H_{x_i}(x_i) + \frac{2I_4}{9} \frac{\Delta K(x_i)}{K(x_i)} > 0$ and $x_{l+1}, \cdots, x_m$ are those critical points such that $2|S^3_+| H_{x_i}(x_i) + \frac{2I_4}{9} \frac{\Delta K(x_i)}{K(x_i)} < 0$. If necessary, we choose ρ_0 smaller such that the previous inequalities hold true in V_i.
We define the vector field

$$W_0(\alpha, a, \lambda) = (0, -\frac{\nabla K(a)}{\lambda}, 0) \in \mathbb{R} \times T_a \partial M \times \mathbb{R},$$

which yields

$$- < \nabla J(\alpha \varphi_{a,\lambda} + \bar{v}), \alpha \frac{1}{\lambda} \nabla K(a) \cdot \nabla_a \varphi_{a,\lambda} >$$

$$\geq c_0 \left(\frac{|\nabla K(a)|^2}{\lambda} - C \frac{|\nabla K(a)| \log(\lambda)}{\lambda^2} \right)$$

$$= c_0 \left(\frac{|\nabla K(a)|^2}{\lambda} + o\left(\frac{|\nabla K(a)|^2}{\lambda} \right) + o\left(\frac{1}{\lambda^2} \right) \right). \tag{7.10}$$

For $(\alpha, a, \lambda) \in V_0$, (7.10) implies the estimate:

$$-<\nabla J(\alpha\varphi_{a,\lambda}+\bar{v}), \alpha\frac{1}{\lambda}\nabla K(a)\cdot\nabla_a\varphi_{a,\lambda}>\geq \frac{c_0}{4}\left(\frac{|\nabla K(a)|^2}{\lambda}+\frac{1}{\lambda^2}\right)$$

for ε small. Hence W_0 is a pseudo gradient in V_0. For $(\alpha, a, \lambda) \in V_i$ we define

$$W_i(\alpha,a,\lambda)=(0,0,\lambda) \text{ if } 1\leq i\leq l \text{ and } W_i(\alpha,a,\lambda)=(0,0,-\lambda) \text{ otherwise.}$$

In any case we obtain on V_i:

$$<\nabla J(\alpha\varphi_{a,\lambda}+\bar{v}), W_i\frac{\partial}{\partial\lambda}\varphi_{a,\lambda}>\geq c_0\frac{1}{\lambda^2}\geq c_1\left(\frac{|\nabla K(a)|^2}{\lambda}+\frac{1}{\lambda^2}\right) \tag{7.11}$$

for ε small. Finally choose a smooth partition of unity η_i, subordinate to the cover $(V_i)_{i=0}^m$ and set

$$W=W_0+\sum_{i=1}^m\eta_iW_i,$$

which is a pseudo-gradient due to (7.10) and (7.11).

It is left to construct a vector field W in B_ε^p for $p\geq 2$. For our argument later (see Proposition 19) it is crucial that the a_i move in any case. Hence we define a first vector field on B_ε^p by

$$W_0(\alpha,a,\lambda):=\big(0,-\frac{\nabla K(a_1)}{\lambda_1},\cdots,-\frac{\nabla K(a_p)}{\lambda_p},0\big). \tag{7.12}$$

We subdivide the proof in many cases and we add different vector fields to W_0 to prove the stated inequality. For $R>0$ large, which will be chosen fixed later, we define:

$$U_\alpha:=\left\{(\alpha,a,\lambda)\in B_\varepsilon^p\ :\ |\alpha-|\alpha|\bar{\alpha}|^2<R\left(\sum_{i=1}^p\frac{1}{\lambda_i^2}+\sum_{i=1}^p\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\right)\right\}$$

and

$$U_\alpha^c:=\left\{(\alpha,a,\lambda)\in B_\varepsilon^p\ :\ |\alpha-|\alpha|\bar{\alpha}|^2>2R\left(\sum_{i=1}^p\frac{1}{\lambda_i^2}+\sum_{i=1}^p\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\right)\right\}.$$

We begin to construct a vector field on U_α^c and set

$$W(\alpha,a,\lambda)=\big(\nabla_\alpha f(\alpha,a),0,0\big)+W_0(\alpha,a,\lambda).$$

Clearly W is S_p-equivariant. Along the flow of W we obtain:

$$<\nabla J\left(\sum_{i=1}^p\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right),\frac{d}{dt}\sum_{j=1}^p\alpha_j\varphi_{a_j,\lambda_j}>$$

$$\geq c_0|\nabla_\alpha f(\alpha,a)|^2-C_0\left(\sum_{i=1}^p\frac{1}{\lambda_i^2}+\sum_{i=1}^p\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\right).$$

It remains to show $|\nabla_\alpha f(\alpha, a)| \geq c_1|\alpha - |\alpha|\bar{\alpha}|$. First we work on the sphere of radius $r := \frac{1}{\sqrt{2I_0}}$ and set $\hat{\alpha} = \frac{1}{\sqrt{2I_0}}\bar{\alpha}$. Due to Remark 1, $|\alpha - \hat{\alpha}| = o(\varepsilon)$. Hence, the difference can be chosen arbitrary small. A computation yields

$$D^2_\alpha f(\hat{\alpha}, a)[h,h] = -2\frac{f(\hat{\alpha}, a)}{|\hat{\alpha}|}|h|^2 \leq -\beta|h|^2$$

for $h \in T_{\hat{\alpha}}S^{p-1}_r$ and some constant $\beta > 0$, which does not depend on a. Furthermore, form the mean value theorem we derive the estimate

$$|D^2 f(\alpha, a)[h,h] - D^2 f(\hat{\alpha}, a)[h,h]| \leq C|\alpha - \hat{\alpha}||h|^2$$

for arbitrary h , if α is close to $\hat{\alpha}$. We choose ρ small such that $C|\alpha - \hat{\alpha}| < \frac{\beta}{4}$ for all $\alpha \in B_\rho(\hat{\alpha}) \cap S^{p-1}_r$. For α in this neighbourhood we choose a geodesic $\gamma : [0,1] \to S^{p-1}_r$ such that $\gamma(0) = \hat{\alpha}$ and $\gamma(1) = \alpha$ and compute

$$\begin{aligned}
<\nabla_\alpha f(\alpha, a), \dot{\gamma}(1) >= &\int_0^1 D^2_\alpha f(\gamma(t), a)[\dot{\gamma}(t), \dot{\gamma}(t)]dt \\
= \int_0^1 D^2_\alpha f(\hat{\alpha}, a)[\dot{\gamma}(t), \dot{\gamma}(t)]dt &+ \int_0^1 \left(D^2_\alpha f(\gamma(t), a) - D^2_\alpha f(\hat{\alpha}, a)\right)[\dot{\gamma}(t), \dot{\gamma}(t)]dt \\
\leq \int_0^1 D^2_\alpha f(\hat{\alpha}, a)[\dot{\gamma}(t), \dot{\gamma}(t)]dt &+ \frac{\beta}{4}\int_0^1 |\dot{\gamma}(t)|^2 dt.
\end{aligned}$$

Denote by p the orthogonal projection onto $T_{\hat{\alpha}}S^{p-1}_r$. Since $< \hat{\alpha} >= kern(D^2_\alpha f(\hat{\alpha}, a))$, we conclude

$$\begin{aligned}
\int_0^1 D^2_\alpha f(\hat{\alpha}, a)[\dot{\gamma}(t), \dot{\gamma}(t)]dt = \int_0^1 D^2_\alpha f(\hat{\alpha}, a)[p\dot{\gamma}(t), p\dot{\gamma}(t)]dt &\leq -\beta \int_0^1 |p\dot{\gamma}(t)|^2 dt \\
&\leq -\frac{3\beta}{4}\int_0^1 |\dot{\gamma}(t)|^2 dt
\end{aligned}$$

for ρ small, which implies

$$< \nabla_\alpha f(\alpha, a), -\dot{\gamma}(1) >\geq \frac{\beta}{2}\int_0^1 |\dot{\gamma}(t)|^2 dt = \frac{\beta}{2}|\dot{\gamma}(1)|^2$$

and finally

$$|\nabla_\alpha f(\alpha, a)| \geq \langle \nabla_\alpha f(\alpha, a), \frac{-\dot{\gamma}(1)}{|\dot{\gamma}(1)|}\rangle \geq \frac{\beta}{2} d_{S^{p-1}_r}(\hat{\alpha}, \alpha) \geq \frac{\beta}{2}|\hat{\alpha} - \alpha|. \tag{7.13}$$

If $|\alpha| \neq r$, then a scaling argument yields

$$|\nabla_\alpha f(\alpha, a)| \geq c_1|\alpha - |\alpha|\bar{\alpha}| \tag{7.14}$$

for some uniform constant c_1. Therefore the stated inequality is proved. Hence we get the estimate

$$< \nabla J\left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right), \frac{d}{dt}\sum_{j=1}^p \alpha_j \varphi_{a_j,\lambda_j} >$$
$$\geq c_0|\alpha - |\alpha|\bar{\alpha}|^2 - C_0\left(\sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i=1}^p \frac{|\nabla K(a_i)|^2}{\lambda_i} + \sum_{i\neq j}\varepsilon_{ij}\right)$$

for some uniform constants c_0, C_0. We choose R big enough such that $c_0 R > 5C_0$ to get the stated inequality:

$$< \nabla J\left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right), \frac{d}{dt}\sum_{j=1}^p \alpha_j \varphi_{a_j,\lambda_j} >$$
$$\geq c\left(|\alpha - |\alpha|\bar{\alpha}|^2 + \sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i=1}^p \frac{|\nabla K(a_i)|^2}{\lambda_i} + \sum_{i\neq j}\varepsilon_{ij}\right)$$

in U_α^c. From now on R is fixed. Observe that λ is not moving in U_α^c along the flow of W. It remains to construct a vector field in U_α.

For $\gamma >> 1$, which will be chosen later, we define the following sets

$$U := \left\{(\alpha, a, \lambda) \in U_\alpha \; : \; \frac{|\nabla K(a_i)|^2}{\lambda_i} < 2\gamma\frac{1}{\lambda_i^2} \; \forall\, 1 \leq i \leq p\right\},$$

$$U_1 := \left\{(\alpha, a, \lambda) \in U \; : d(a_i, a_j) > \frac{1}{4}\min_{k\neq l} d(x_l, x_k)\right\}.$$

If $(\alpha, a, \lambda) \in U_1$, then $a_1, \cdots, a_p$ are close to critical points of K. Furthermore the second condition says that different a_i are close to different critical points, hence it exist a well defined map $U_1 \ni (\alpha, a, \lambda) \xrightarrow{\beta} crit(K)^p$ which maps (α, a, λ) to $(x_1, \cdots x_p)$ iff a_i is close to x_i. Since we assume the non-degeneracy condition

$$2|S_+^3|H_x(x) + \frac{2I_4}{9}\frac{\Delta K(x)}{K(x)} \neq 0 \text{ for all } x \in crit(K),$$

the following sets are well defined ($(x_1, \cdots, x_p) = \beta(\alpha, a, \lambda)$):

$$V_1 := \left\{(\alpha, a, \lambda) \in U_1 \; : \; \exists i \in \{1, \cdots, p\} \; s.t. \; 2|S_+^3|H_{x_i}(x_i) + \frac{2I_4}{9}\frac{\Delta K(x_i)}{K(x_i)} > 0\right\},$$

$$V_2 := \left\{(\alpha, a, \lambda) \in U_1 \; : \; 2|S_+^3|H_{x_i}(x_i) + \frac{2I_4}{9}\frac{\Delta K(x_i)}{K(x_i)} < 0, \; \forall i \; \rho_1(\beta(\alpha, a, \lambda)) > 0\right\},$$

$$V_3 := \left\{(\alpha, a, \lambda) \in U_1 \; : \; 2|S_+^3|H_{x_i}(x_i) + \frac{2I_4}{9}\frac{\Delta K(x_i)}{K(x_i)} < 0 \; \forall\, i, \; \rho_1(\beta(\alpha, a, \lambda)) < 0\right\}.$$

Here ρ_1 is the least eigenvalue of the matrix $M(x)$, defined in (1.16).
We start with constructing a vector field on V_2. Therefore we define the equivariant vector field

$$W_2(\alpha, a, \lambda) := (0, 0, -\lambda_1, \cdots, -\lambda_p) \in \mathbb{R}^p \times \partial M^p \times \mathbb{R}^p$$

and observe along the flow

$$\frac{d}{dt}J\left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right) = - < \nabla J\left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right), \sum_{j=1}^p \alpha_j \lambda_j \frac{\partial}{\partial \lambda_j}\varphi_{a_j,\lambda_j} + D\bar{v}[W_2] >$$
$$= 4I_0 l(u)\left(-\sum_{j=1}^p \alpha_j^2\left(2|S^3_+|H_{a_j}(a_j) + \frac{2I_4}{9}\frac{\Delta K(a_j)}{K(a_j)}\right)\frac{1}{\lambda_j^2} - 2I_1\sum_{i\neq j}\alpha_i\alpha_j\frac{G(a_i,a_j)}{\lambda_i\lambda_j}\right)$$
$$+ o\left(\sum_{j=1}^p \frac{1}{\lambda_j^2}\right) + o\left(\sum_{i\neq j}\varepsilon_{ij}\right) + O\left(\sum_{j=1}^p \frac{\rho^2}{\lambda_j^2}\right) + O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right).$$

Moreover, from the expansion

$$l(u) = \frac{1}{\left(\sum_{l=1}^p \alpha_l^3 K(a_l) I_0\right)^{\frac{2}{3}}}\left(1 + O\left(\sum_{j=1}^p \frac{1}{\lambda_j^2}\right) + O\left(\sum_{i\neq j}\varepsilon_{ij}\right)\right)$$

we derive

$$\frac{d}{dt}J\left(\sum_{i=1}^p \alpha_i\varphi_{a_i,\lambda_i} + \bar{v}\right)$$
$$= \frac{4I_0}{\left(\sum_{l=1}^p \alpha_l^3 K(a_l) I_0\right)^{\frac{2}{3}}}\left(-\sum_{j=1}^p \alpha_j^2\left(2|S^3_+|H_{a_j}(a_j) + \frac{2I_4}{9}\frac{\Delta K(a_j)}{K(a_j)}\right)\frac{1}{\lambda_j^2}\right.$$
$$\left. - 2\sum_{i\neq j} I_1\alpha_i\alpha_j\frac{G(a_i,a_j)}{\lambda_i\lambda_j}\right)$$
$$+ o\left(\sum_{j=1}^p \frac{1}{\lambda_j^2}\right) + o\left(\sum_{i\neq j}\varepsilon_{ij}\right) + O\left(\sum_{j=1}^p \frac{\rho^2}{\lambda_j^2}\right) + O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right)$$
$$= \frac{4I_0}{\left(\sum_{l=1}^p \alpha_l^3 K(a_l) I_0\right)^{\frac{2}{3}}}\left(-\sum_{j=1}^p \alpha_j^2 K(a_j)^2\left(2|S^3_+|\frac{H_{a_j}(a_j)}{K(a_j)^2} + \frac{2I_4}{9}\frac{\Delta K(a_j)}{K(a_j)^3}\right)\frac{1}{\lambda_j^2}\right.$$
$$\left. - 2I_1\sum_{i\neq j}\alpha_i K(a_i)\alpha_j K(a_j)\frac{G(a_i,a_j)}{K(a_i)K(a_j)}\frac{1}{\lambda_i\lambda_j}\right) + o\left(\sum_{j=1}^p \frac{1}{\lambda_j^2}\right) + o\left(\sum_{i\neq j}\varepsilon_{ij}\right).$$

Since

$$|\alpha - |\alpha|\bar{\alpha}|^2 \le C\left(\sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i=1}^p \frac{|\nabla K(a_i)|^2}{\lambda_j} + \sum_{i\neq j}\varepsilon_{ij}\right),$$

$\bar{\alpha}_i K(a_i) = \bar{\alpha}_j K(a_j)$ for all i, j and $d_g(a_i, \beta(\alpha, a, \lambda)_i) \leq C\frac{1}{\sqrt{\lambda_i}}$, we get

$$\frac{d}{dt} J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right)$$
$$= \frac{4I_0 \bar{\alpha}_1^2 K(x_1)}{\left(\sum_{l=1}^{p} \bar{\alpha}_l^3 K(x_l) I_0\right)^{\frac{2}{3}}} \left(< \Lambda, M(x)\Lambda > + o\left(\sum_{j=1}^{p} \frac{1}{\lambda_j^2}\right) + o\left(\sum_{i \neq j} \varepsilon_{ij}\right)\right.$$
$$\left. + O\left(\sum_{j=1}^{p} \frac{\rho^2}{\lambda_j^2}\right) + O\left(\sum_{i \neq j} \rho \varepsilon_{ij}\right)\right),$$

where $\Lambda = \left(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_p}\right)^t$ and $M(x)$ is the matrix defined in (1.16). Since the least eigenvalue $\rho_1(x)$ is positive, we have shown:

$$\frac{d}{dt} J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right) \geq$$
$$c_1 \left(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + o\left(\sum_{j=1}^{p} \frac{1}{\lambda_j^2}\right) + o\left(\sum_{i \neq j} \varepsilon_{ij}\right) + O\left(\sum_{j=1}^{p} \frac{\rho^2}{\lambda_j^2}\right) + O\left(\sum_{i \neq j} \rho \varepsilon_{ij}\right)\right).$$

Furthermore,

$$\varepsilon_{ij} \leq C\left(\lambda_i \lambda_j d(a_i, a_j)^2\right)^{-1} \leq C\left(\frac{1}{\lambda_i^2} + \frac{1}{\lambda_j^2}\right)$$

in V_2, which implies

$$\frac{d}{dt} J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right) \geq c_0 \left(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij}\right)$$

in V_2 for ε and ρ small. Finally in V_2 we define the vector field

$$W(\alpha, a, \lambda) := M \cdot W_2(\alpha, a, \lambda) + W_0(\alpha, a, \lambda)$$

and observe for M large, but fixed,

$$\frac{d}{dt} J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right) \geq c_0 \left(\sum_{i=1}^{p} \frac{1}{\lambda_i^2} + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|^2}{\lambda_i} + \sum_{i \neq j} \varepsilon_{ij} + |\alpha - |\alpha|\bar{\alpha}|^2\right)$$

along the flow of W.

We continue with constructing a vector field W in V_3. For $x = (x_1, \cdots, x_p) \in Krit(K)^p$ such that $x_i \neq x_j$ for $i \neq j$ and $\rho_1(x) < 0$, we define the open sets

$$V_3^x := \{(\alpha, a, \lambda) \in V_3 \ : \ \beta(\alpha, a, \lambda) = x\},$$

which cover V_3 and are disjoint for different x. Since $\rho_1(x) < 0$, $M_{ii} > 0 \ \forall\ i$ and $M_{ij} < 0 \ \forall\ i \neq j$, the matrix $M(x)$ has a unique normalized eigenvector $e_1(x)$ with strictly positive components.

For $(\alpha, a, \lambda) \in V_3^x$ we define

$$W_3^x(\alpha, a, \lambda) = |\Lambda|\big(0, 0, \lambda_1^2 e_1^1(x), \cdots, \lambda_p^2 e_1^p(x)\big),$$

which defines an equivariant vector field on V_3 because $M(x^\sigma)_{ij} = M(x)_{\sigma(i)\sigma(j)}$ and hence $e_1^i(x^\sigma) = e_1^{\sigma(i)}(x)$ for all $\sigma \in S_p$. Here $\Lambda = \left(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_p}\right)^t$.

Along the flow of W_3^x, the same computations as in the previous case yield:

$$\begin{aligned}
&< \nabla J\left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right), \frac{d}{dt}\sum_{j=1}^p \alpha_j \varphi_{a_j,\lambda_j} > \\
&= 4I_0 l(u)\left(|\Lambda| \sum_{i=1}^p \alpha_j^2 \left(2|S_3^+|A_{a_j} + \frac{2I_4}{9}\frac{\Delta K(a_j)}{K(a_j)}\right)\frac{e_1^j(x)}{\lambda_j} + 2\sum_{i\neq j}\alpha_i\alpha_j \frac{I_1 G(a_i,a_j)}{\lambda_i} e_1^j(x)|\Lambda|\right) \\
&+ o\left(\sum_{j=1}^p \frac{1}{\lambda_j^2}\right) + o\left(\sum_{i\neq j}\varepsilon_{ij}\right) \\
&= \frac{4I_0\alpha_1^2 K(a_1)}{\left(\sum_{l=1}^p \alpha_l^3 K(a_l) I_0\right)^{\frac{2}{3}}}\left(-|\Lambda| < \Lambda, M(x)e_1(x) > + o\left(\sum_{j=1}^p \frac{1}{\lambda_j^2}\right) + o\left(\sum_{i\neq j}\varepsilon_{ij}\right)\right).
\end{aligned}$$

Observe that

$$\begin{aligned}
-|\Lambda| < \Lambda, M(x)e_1(x) > &= -\rho_1(x)|\Lambda| < \Lambda, e_1(x) > = |\rho_1(x)||\Lambda|^2 < \frac{\Lambda}{|\Lambda|}, e_1(x) > \\
&\geq c_0|\rho_1(x)||\Lambda|^2, \qquad (7.15)
\end{aligned}$$

where

$$c_0 = \min\{< z, e_1(x) > \ | \ z \in S_+^{p-1}\} > 0.$$

Hence, using the same arguments as for V_2, (7.15) implies

$$<\nabla J\left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right), \frac{d}{dt}\sum_{j=1}^p \alpha_j \varphi_{a_j,\lambda_j} > \geq c_0\left(\sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i=1}^p \frac{|\nabla K(a_i)|^2}{\lambda_i} + \sum_{i\neq j}\varepsilon_{ij}\right)$$

on V_3^x. Therefore, as in the previous case, the vector field

$$W(\alpha, a, \lambda) := M \cdot W_3^x(\alpha, a, \lambda) + W_0(\alpha, a, \lambda)$$

is a pseudo gradient vector field on V_3^x for M large enough. Since the sets V_3^x cover V_3 we have constructed a vector field on V_3. Moreover all λ_i are decreasing along the flow of $-W$ in V_3.

Next, we construct a vector field W_1 in V_1. For $x=(x_1,\cdots,x_p)\in Krit(K)^p$ such that $x_i\neq x_j$ for all $i\neq j$ and $2|S^3_+|H_{x_j}(x_j)+\frac{2I_4}{9}\frac{\Delta K(x_j)}{K(x_j)}>0$ for at least on x_j we define the sets

$$V_1^x := \{(\alpha,a,\lambda)\in V_1 \ : \ \beta(\alpha,a,\lambda)=x\}.$$

Since the sets V_1^x are open and cover V_1 it suffices to construct a vector field W_1^x on V_1^x. Set

$$B(x) := \left\{1\leq j\leq p \ : \ 2|S^3_+|H_{x_j}(x_j)+\frac{2I_4}{9}\frac{\Delta K(x_j)}{K(x_j)}>0\right\}$$

and

$$\tilde{W}_1(\alpha,a,\lambda)=\left(0,0,\lambda_1 1_{\{1\in B(x)\}},\cdots,\lambda_p 1_{\{p\in B(x)\}}\right),$$

which defines an equivariant vector field, because $\sigma(B(x^\sigma))=B(x)$ for all $\sigma\in S_p$.

Along this flow we get

$$\begin{aligned}
&< \nabla J\left(\sum_{i=1}^p \alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right), \sum_{j\in B(x)}\alpha_j\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}> \\
&= 4I_0 l(u)\left(\sum_{j\in B(x)}\alpha_j^2\left(2|S^3_+|H_{a_j}(a_j)+\frac{2I_4}{9}\frac{\Delta K(a_j)}{K(a_j)}\right)\frac{1}{\lambda_j^2}+2\sum_{j\in B(x)}\sum_{i\neq j}\alpha_i\alpha_j I_1\frac{G(a_i,a_j)}{\lambda_i\lambda_j}\right) \\
&+o\left(\sum_{j=1}^p\frac{1}{\lambda_j^2}\right)+o\left(\sum_{i\neq j}\varepsilon_{ij}\right)+O\left(\sum_{i=1}^p\frac{\rho^2}{\lambda_i^2}+\sum_{i\neq j}\rho\varepsilon_{ij}\right) \\
&\geq c_0\left(\sum_{j\in B(x)}\frac{1}{\lambda_j^2}+\sum_{j\in B(x)}\sum_{i\neq j}\varepsilon_{ij}\right)+o\left(\sum_{j=1}^p\frac{1}{\lambda_j^2}\right)+o\left(\sum_{i\neq j}\varepsilon_{ij}\right) \\
&+O\left(\sum_{i=1}^p\frac{\rho^2}{\lambda_i^2}+\sum_{i\neq j}\rho\varepsilon_{ij}\right).
\end{aligned}$$

To get the asserted estimate we need to add another vector field. Therefore let $A(x)=\{1,\cdots,p\}\setminus B(x)$ and define for $A\subset A(x)$:

$$V_1^x(A) := \{(\alpha,a,\lambda)\in V_1^x \ : \ \lambda_i<\frac{1}{4}\bar{\lambda}_{\min}\ \forall\, i\in A,\ \lambda_i>\frac{1}{8}\bar{\lambda}_{\min}\ \forall\, i\in\bar{A}\}.$$

Here $\bar{\lambda}_{\min}=\min\{\lambda_j \ : \ j\in B(x)\}$ and $\bar{A}=\{1,\cdots,p\}\setminus\left(B(x)\cup A\right)$. Furthermore we set

$$V_1^x(B) := \{(\alpha,a,\lambda)\in V_1^x \ : \ \lambda_i>\frac{1}{2}\bar{\lambda}_{\min}\ \forall\, i\in A(x)\}.$$

If $(\alpha,a,\lambda)\in V_1^x(B)$ then

$$\sum_{i\in B(x)}\frac{1}{\lambda_i^2}=\frac{1}{2}\sum_{i\in B(x)}\frac{1}{\lambda_i^2}+\frac{1}{2}\sum_{i\in B(x)}\frac{1}{\lambda_i^2}\geq\frac{1}{2}\sum_{i\in B(x)}\frac{1}{\lambda_i^2}+\frac{1}{8p}\sum_{i\in A(x)}\frac{1}{\lambda_i^2}\geq c_0\sum_{i=1}^p\frac{1}{\lambda_i^2},$$

which implies

$$< \nabla J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right), \sum_{j\in B(x)}\alpha_j\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j} >$$
$$\geq c_0\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\right)$$

in $V_1^x(B)$ if ε and ρ are small. Hence $\tilde{W}_1$ is pseudo-gradient on $V_1^x(B)$. It is left to construct a second vector field on $V_1^x(A)$. If the least eigenvalue $\rho_1\left((x_i)_{i\in A}\right)>0$, we set

$$\tilde{W}_1^A(\alpha,a,\lambda)=\left(0,0,-\lambda_1 1_{\{1\in A\}},\cdots,-\lambda_p 1_{\{p\in A\}}\right)$$

and if $\rho_1\left((x_i)_{i\in A}\right)<0$, we set

$$\tilde{W}_1^A(\alpha,a,\lambda)=\left(0,0,W_3\left((\alpha_i,a_i,\lambda_i)_{i\in A}\right)\right),$$

where W_3 is the vector field constructed above for $V_3\subset B_\varepsilon^q$, $q=|A|$. If $(\alpha,a,\lambda)\in V_1^x(A)$, then $(\alpha,a,\lambda)^\sigma\in V_1^{x^\sigma}(\sigma^{-1}(A))$ and hence $\tilde{W}_1^{\sigma^{-1}(A)}=\left(\tilde{W}_1^A\right)^\sigma$.

In both cases we get along the flow of $\tilde{W}_1^A$:

$$-< \nabla J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right), \sum_{j\in A}\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j} >$$
$$\geq c_0\left(\sum_{j\in A}\frac{1}{\lambda_j^2}+\sum_{i\neq j\in A}\varepsilon_{ij}\right)-C\sum_{j\in A}\sum_{i\notin A}\varepsilon_{ij},$$

where we did the same computations as for V_2 and V_3. We set

$$\tilde{W}_2^x=\sum_{A\subset B(x)}\eta_A\tilde{W}_1^A,$$

where $(\eta_A)_A$ is a smooth partition of unity, subordinate to the cover $(V_1^x(A))_A$ of V_1^x with the following property:

$$\eta_{\sigma^{-1}(A)}((\alpha,a,\lambda)^\sigma)=\eta_A(\alpha,a,\lambda)\ \ \forall\ \sigma\in S_p.$$

For $r>0$ small, we finally define

$$W_1=\tilde{W}_1^x+r\tilde{W}_2^x.$$

Along the flow of this vector field we obtain

$$
\begin{aligned}
&<\nabla J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right), \frac{d}{dt}\sum_{j=1}^{p}\alpha_j\varphi_{a_j,\lambda_j}> \\
&\geq c_0\left(\sum_{j\in B(x)}\frac{1}{\lambda_j^2}+\sum_{j\in B(x)}\sum_{i\neq j}\varepsilon_{ij}\right)+rc_0\left(\sum_{j\in A}\frac{1}{\lambda_j^2}+\sum_{i\neq j\in A}\varepsilon_{ij}\right) \\
&-Cr\sum_{j\in A}\sum_{i\notin A}\varepsilon_{ij}+o\left(\sum_{j=1}^{p}\frac{1}{\lambda_j^2}\right)+o\left(\sum_{i\neq j}\varepsilon_{ij}\right)+O\left(\sum_{i=1}^{p}\frac{\rho^2}{\lambda_i^2}\right)+O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right) \\
&\geq c_0\left(\sum_{j\in A^c}\frac{1}{\lambda_j^2}+\sum_{j\in B(x)}\sum_{i\neq j}\varepsilon_{ij}\right)+rc_0\left(\sum_{j\in A}\frac{1}{\lambda_j^2}+\sum_{i\neq j\in A}\varepsilon_{ij}\right)-Cr\sum_{i\in A^c}\frac{1}{\lambda_i^2} \\
&+o\left(\sum_{j=1}^{p}\frac{1}{\lambda_j^2}\right)+o\left(\sum_{i\neq j}\varepsilon_{ij}\right)+O\left(\sum_{i=1}^{p}\frac{\rho^2}{\lambda_i^2}\right)+O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right) \\
&\geq c_0\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}\right),
\end{aligned}
$$

if we choose r small.
Finally set $W(\alpha,a,\lambda):=MW_1(\alpha,a,\lambda)+W_0(\alpha,a,\lambda)$ on V_1 and find the estimate

$$
\frac{d}{dt}J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)\geq c_0\left(\sum_{i=1}^{p}\frac{1}{\lambda_i^2}+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}+|\alpha-|\alpha|\bar{\alpha}|^2\right)
$$

for M large. Moreover, due to the construction, $\dot{\lambda}_i\leq 0$ along the flow of $-W$ in V_1 whenever $\lambda_i=\max\{\lambda_j\ :\ 1\leq j\leq p\}$.
Hence we have constructed an equivariant vector field in V_1. Since U_1 splits into V_1, V_2 and V_3, we have constructed a vector field in U_1. Observe that $\max\{\lambda_1,\cdots,\lambda_p\}$ is decreasing along the flow of $-W$ in $V_1\cup V_3$.

We continue with the construction of a vector field on $U\setminus U_1$. Let $K=\{x_1,\cdots,x_l\}\subset crit(K)$ be a subset such that $|K|=l\leq p-1$ and define

$$
Abb(p,K):=\{r:\{1,\cdots,p\}\to K \text{ onto}\}.
$$

For $r\in Abb(p,K)$ we set

$$
V_r(K):=\{(\alpha,a,\lambda)\in U\ : d_g(a_i,r(i))<\rho_0\}\subset U \text{ open.}
$$

Thus

$$
U=U_1\cup\bigcup_{K}\bigcup_{r\in Abb(p,K)}V_r(K),
$$

where this union is disjoint. Hence it suffices to construct a vector field on each $V_r(K)$. Therefore let $K = \{x_1, \cdots, x_l\}$ be a subset as above, $r \in Abb(p, K)$ and $B_j = r^{-1}(\{x_j\})$. Furthermore we choose a smooth, non-negative, monotone increasing function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi(t) = 0$ if $t \leq \frac{1}{4}$ and $\chi(t) = 1$ if $t \geq 1$. For $l \in B_j$ we set

$$\bar{\chi}(\lambda_l) = \sum_{l \neq k \in B_j} \chi\left(\frac{\lambda_l}{\lambda_k}\right)$$

if $|B_j| \geq 2$. Otherwise we set $\bar{\chi}(\lambda_l) := 0$. We now define a first equivariant vector field as follows

$$\tilde{W}_r^K(\alpha, a, \lambda) = (0, 0, \lambda_1 \bar{\chi}(\lambda_1), \cdots, \lambda_p \bar{\chi}(\lambda_p)).$$

Along the flow we obtain

$$\begin{aligned}
&< \nabla J\left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right), \frac{d}{dt}\sum_{j=1}^p \alpha_j \varphi_{a_j,\lambda_j} > \\
&= 4I_0 l(u)\Bigg(\sum_{|B_j|\geq 2}\sum_{\bar{\chi}(\lambda_l)\neq 0} \bar{\chi}(\lambda_l)\Bigg(\alpha_l^2\left(2|S_+^3|H_{a_l}(a_l) + \frac{2I_4}{9}\frac{\Delta K(a_l)}{K(a_l)}\right)\frac{1}{\lambda_l^2} \\
&- 2\sum_{i\neq l} I_1 \alpha_l \alpha_i \lambda_l \frac{\partial}{\partial \lambda_l} I(\varepsilon_{il})\Bigg)\Bigg) \\
&+ o\left(\sum_{j=1}^p \frac{1}{\lambda_j^2}\right) + o\left(\sum_{i\neq j}\varepsilon_{ij}\right) + O\left(\sum_{i=1}^p \frac{\rho^2}{\lambda_i^2}\right) + O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right).
\end{aligned}$$

Furthermore

$$-\sum_{i\neq l} I_1 \alpha_l \alpha_i \lambda_l \frac{\partial}{\partial \lambda_l} I(\varepsilon_{il}) = -\sum_{l\neq i\in B_j} I_1 \alpha_l \alpha_i \lambda_l \frac{\partial}{\partial \lambda_l} I(\varepsilon_{il}) + \sum_{i\in B_j^c} I_1 \alpha_l \alpha_i \frac{G(a_i, a_l)}{\lambda_i \lambda_l}.$$

Since

$$-\bar{\chi}(\lambda_l)\lambda_l \frac{\partial}{\partial \lambda_l} I(\varepsilon_{il}) - \bar{\chi}(\lambda_i)\lambda_i \frac{\partial}{\partial \lambda_i} I(\varepsilon_{il}) \geq \frac{c}{2}\varepsilon_{il}$$

in any case, if is ε small and $l, i \in B_j$, we obtain

$$\begin{aligned}
&< \nabla J\left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right), \frac{d}{dt}\sum_{j=1}^p \alpha_j \varphi_{a_j,\lambda_j} > \\
&\geq c_0\left(\sum_{|B_j|\geq 2}\sum_{\bar{\chi}(\lambda_l)\neq 0}\sum_{i\neq l}\varepsilon_{il}\right) - C\sum_{|B_j|\geq 2}\sum_{\bar{\chi}(\lambda_l)\neq 0}\frac{1}{\lambda_l^2} \\
&+ o\left(\sum_{j=1}^p \frac{1}{\lambda_j^2}\right) + o\left(\sum_{i\neq j}\varepsilon_{ij}\right) + O\left(\sum_{i=1}^p \frac{\rho^2}{\lambda_i^2}\right) + O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right)
\end{aligned}$$

If $\bar\chi(\lambda_l) \neq 0$, it exists $i \in B_j$ such that $\frac{\lambda_l}{\lambda_i} \geq \frac{1}{4}$, hence

$$\frac{1}{\lambda_l^2}\frac{1}{\varepsilon_{il}} = \frac{1}{\lambda_i\lambda_j} + \frac{\lambda_i}{\lambda_l^3} + \frac{\lambda_i}{\lambda_l} d(a_i,a_l)^2 = o(1).$$

This implies $\frac{1}{\lambda_l^2} = o(\varepsilon_{il})$ and therefore

$$< \nabla J\left(\sum_{i=1}^p \alpha_i\varphi_{a_i,\lambda_i} + \bar v\right), \frac{d}{dt}\sum_{j=1}^p \alpha_j\varphi_{a_j,\lambda_j} > \geq c_0 \sum_{|B_j|\geq 2}\sum_{\bar\chi(\lambda_l)\neq 0}\left(\frac{1}{\lambda_l^2} + \sum_{i\neq l}\varepsilon_{il}\right)$$
$$+ o\left(\sum_{j=1}^p \frac{1}{\lambda_j^2}\right) + o\left(\sum_{i\neq j}\varepsilon_{ij}\right) + O\left(\sum_{i=1}^p \frac{\rho^2}{\lambda_i^2}\right) + O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right).$$

We need to add a further vector field to guarantee the stated estimate. Therefore we need to subdivide $V_r(K)$. W.l.o.g. assume $|B_1|, \cdots, |B_k| \geq 2$ for $k \leq l$ and define

$$\mathcal{M} := \{A_1 \cup \cdots \cup A_k \ : \ A_i \subset B_i, \ \ |A_i| = |B_i| - 1 \ \forall\ 1 \leq i \leq k\}.$$

For $\mathcal{A} \in \mathcal{M}$ we set

$$V_r^{\mathcal{A}}(K) := \{(\alpha,a,\lambda) \in V_r(K) \ : \ \bar\chi(\lambda_j) > 0 \ \text{ for } \ j \in \mathcal{A}\}$$

and $\underline{\lambda} = \min\{\lambda_j \ : \ j \in \mathcal{A}\}$. Furthermore, for $B \subset \{1,\cdots,p\} \setminus \mathcal{A}$ we define the sets

$$V_r^{\mathcal{A},B}(K) := \{(\alpha,a,\lambda) \in V_r^{\mathcal{A}}(K) \ : \ \lambda_i < \frac{1}{4}\underline{\lambda}\ \forall i \in B;\ \lambda_i > \frac{1}{8}\underline{\lambda}\ \forall i \in \bar B\}$$

and

$$V_r^{\mathcal{A},C}(K) := \left\{(\alpha,a,\lambda) \in V_r^{\mathcal{A}}(K) \ : \ \lambda_i > \frac{1}{8}\underline{\lambda}\ \forall i \in \ \{1,\cdots,p\}\setminus\mathcal{A}\right\}.$$

Here $\bar B = \{1,\cdots,p\} \setminus (\mathcal{A} \cup B)$. We remark that

$$(\alpha,a,\lambda) \in V_r^{\mathcal{A},B}(K) \Leftrightarrow (\alpha,a,\lambda)^\sigma \in V_{r\circ\sigma}^{\sigma^{-1}(\mathcal{A}),\sigma^{-1}(B)}(K).$$

Now we are prepared to construct a second vector field. Observe that

$$< \nabla J\left(\sum_{i=1}^p \alpha_i\varphi_{a_i,\lambda_i} + \bar v\right), \frac{d}{dt}\sum_{j=1}^p \alpha_j\varphi_{a_j,\lambda_j} >$$
$$\geq c_0 \sum_{|B_j|\geq 2}\sum_{\bar\chi(\lambda_l)\neq 0}\left(\frac{1}{\lambda_l^2} + \sum_{i\neq l}\varepsilon_{il}\right) + o\left(\sum_{i\neq j}\varepsilon_{ij}\right) + O\left(\sum_{i=1}^p \frac{\rho^2}{\lambda_i^2}\right) + O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right)$$
$$\geq c_0\left(\sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i=1}^p \frac{|\nabla K(a_i)|^2}{\lambda_i} + \sum_{i\neq j}\varepsilon_{ij}\right)$$

for ε small along the flow of $\tilde{W}_r^K$ in $V_r^{\mathcal{A},C}(K)$. Hence it is not needed to construct a second vector field in the previous situation.
If $(\alpha, a, \lambda) \in V_r^{\mathcal{A},B}(K)$, then $d(a_i, a_j) \geq \rho_0$ for all $i, j \in B$ because they are located at different critical points. Therefore the point $(\alpha_i, a_i, \lambda_i)_{i\in B} \in U_1 \subset B_\varepsilon^q$, where $q = |B|$. Hence we choose as second vector field

$$W_r^{\mathcal{A},B}(\alpha, a, \lambda) := (0, 0, W_{U_1}((\alpha_i, a_i, \lambda_i)_{i\in B})),$$

where W_{U_1} is the vector field on U_1, which we have already constructed. Along the flow of $W_r^{\mathcal{A},B}$ we get in $V_r^{\mathcal{A},B}(K)$:

$$< \nabla J \left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v} \right), \frac{d}{dt} \sum_{j=1}^p \alpha_j \varphi_{a_j,\lambda_j} >\geq c_0 \left(\sum_{i\in B} \frac{1}{\lambda_i^2} + \sum_{i\neq j \in B} \varepsilon_{ij} \right) - C \sum_{i\in B, j\in B^c} \varepsilon_{ij}.$$

Therefore, if (α, a, λ) moves along the flow of $\tilde{W}_r^K + \mu W_r^{\mathcal{A},B}$ and μ is small, we obtain

$$< \nabla J \left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v} \right), \frac{d}{dt} \sum_{j=1}^p \alpha_j \varphi_{a_j,\lambda_j} >\geq c_0 \left(\sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i=1}^p \frac{|\nabla K(a_i)|^2}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij} \right)$$

in $V_r^{\mathcal{A},B}(K)$. Finally we choose a smooth partition of unity $\eta_{\mathcal{A},B}$, subordinate to the cover $\left(V_r^{\mathcal{A},B}\right)_{\mathcal{A},B}(K)$ of $V_r(K)$ with the property

$$\eta_{\sigma-1(\mathcal{A}),\sigma^{-1}(B)}((\alpha, a, \lambda)^\sigma) = \eta_{\mathcal{A},B}(\alpha, a, \lambda) \;\forall\; \sigma \in S_p \tag{7.16}$$

and define

$$W_r^K := \tilde{W}_r^K + \mu \sum_{\mathcal{A},B} \eta_{\mathcal{A},B} W_r^{\mathcal{A},B}$$

which defines an equivariant vector field on $V_r(K)$. Finally we set

$$W(\alpha, a, \lambda) := M W_r^K(\alpha, a, \lambda) + W_0(\alpha, a, \lambda),$$

which defines a vector field on $V_r(K)$ with the property

$$\frac{d}{dt} J \left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v} \right) \geq c_0 \left(\sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i=1}^p \frac{|\nabla K(a_i)|^2}{\lambda_i} + \sum_{i\neq j} \varepsilon_{ij} + |\alpha - |\alpha|\bar{\alpha}|^2 \right)$$

for β small and M large. Moreover if $\lambda_i = \max\{\lambda_1, \cdots, \lambda_p\}$, then $\dot{\lambda}_i \leq 0$ if (α, a, λ) moves along the flow of $-W$ in $V_r(K)$. Hence we have constructed a vector field on $U \setminus U_1$.

It is left to construct a vector field on U^c. For $\emptyset \neq A \subset \{1, \cdots, p\}$ we define

$$O_A := \left\{ (\alpha, a, \lambda) \in U_\alpha \;:\; \frac{|\nabla K(a_i)|^2}{\lambda_i} > \gamma \frac{1}{\lambda_i^2} \text{ for } i \in A;\; \frac{|\nabla K(a_i)|^2}{\lambda_i} < 2\gamma \frac{1}{\lambda_i^2} \text{ for } \; i \in A^c \right\}.$$

Due to (7.9) we obtain along the flow of W_0:

$$< -\nabla J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right),\sum_{j=1}^{p}\frac{1}{\lambda_j}\nabla K(a_j)\cdot\nabla_{a_j}\varphi_{a_j,\lambda_j} >$$

$$\geq c_0\sum_{j=1}^{p}\frac{|\nabla K(a_j)|^2}{\lambda_i}+O\left(\sum_{j=1}^{p}\frac{\rho}{\lambda_j^2}\right)-c_2\sum_{j\in A}\sum_{i\neq j}\varepsilon_{ij}+o\left(\sum_{j\in A^c}\sum_{i\neq j}\varepsilon_{ij}\right)$$

$$\geq c_0\left(\sum_{j\in A}\frac{|\nabla K(a_j)|^2}{\lambda_i}+\sum_{j\in A}\frac{\gamma}{p}\frac{1}{\lambda_j^2}\right)+\sum_{j\in A^c}\frac{|\nabla K(a_j)|^2}{\lambda_j}-c_2\sum_{j\in A}\sum_{i\neq j}\varepsilon_{ij}$$

$$+O\left(\sum_{j\in A^c}^{p}\frac{\rho}{\lambda_j^2}\right)+o\left(\sum_{j\in A^c}\sum_{i\neq j}\varepsilon_{ij}\right)$$

if ε is small.

We subdivide B_ε^p into the sets

$$O_\sigma:=\left\{(\alpha,a,\lambda)\in B_\varepsilon^p\ :\ \lambda_{\sigma(1)}<2\lambda_{\sigma(2)}<\cdots<2^p\lambda_{\sigma(p)}\right\}$$

for $\sigma\in S_p$. Furthermore we set $\bar{\lambda}=\min\{\lambda_i\ :\ i\in A\}$ and for $B\subset A^c$:

$$O_A^B:=\left\{(\alpha,a,\lambda)\in O_A\ :\ \lambda_i<\frac{1}{4}\bar{\lambda}\text{ for }i\in A^c\setminus B;\ \lambda_i>\frac{1}{2}\bar{\lambda}\text{ for }i\in B\right\}.$$

On $O_{id}\cap O_A^B$ we define the vector field

$$W_{id}^A:=(0,0,2\lambda_1 1_{\{1\in A\cup B\}},\cdots,2^p\lambda_p 1_{\{p\in A\cup B\}})$$

and observe along the flow

$$< \nabla J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right),\frac{d}{dt}\sum_{j=1}^{p}\alpha_j\varphi_{a_j,\lambda_j} >$$

$$=< \nabla J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)\sum_{j\in A\cup B}^{p}2^j\alpha_j\lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j} >$$

$$\geq -c_0\sum_{j\in A\cup B}2^j\sum_{i\neq j}\alpha_i\alpha_j I_1\lambda_j\frac{\partial}{\partial\lambda_j}I(\varepsilon_{ij})-c\sum_{j\in A\cup B}\frac{1}{\lambda_j^2}-c_1\rho\sum_{j\in A\cup B}\sum_{i\neq j}\varepsilon_{ij}+o\left(\sum_{i\neq j}\varepsilon_{ij}\right)$$

$$\geq c_2\sum_{j\in A\cup B}\sum_{i\neq j}\varepsilon_{ij}-c\sum_{j\in A\cup B}\frac{1}{\lambda_j^2}+o\left(\sum_{i\neq j}\varepsilon_{ij}\right)$$

if ρ and ε are small. On $O_\pi\cap O_A^B$ we define the vector field

$$W_\pi^A(\alpha,a,\lambda):=(0,0,2^{\pi^{-1}(1)}\lambda_1 1_{\{1\in A\cup B\}},\cdots,2^{\pi^{-1}(p)}\lambda_p 1_{\{p\in A\cup B\}})$$

and observe along the flow in $O_\pi \cap O_A^B$:

$$< \nabla J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right), \frac{d}{dt}\sum_{j=1}^{p}\alpha_j\varphi_{a_j,\lambda_j} >$$

$$\geq c_2\sum_{j\in A\cup B}\sum_{i\neq j}\varepsilon_{ij} - c\sum_{j\in A\cup B}\frac{1}{\lambda_j^2} + o\left(\sum_{i\neq j}\varepsilon_{ij}\right).$$

For $M>0$ we define on $O_\pi \cap O_A^B$

$$\tilde{W} := W_0 + MW_\pi^A$$

and get along the flow

$$< \nabla J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right), \frac{d}{dt}\sum_{j=1}^{p}\alpha_j\varphi_{a_j,\lambda_j} >$$

$$\geq c_0\left(\sum_{j\in A}\frac{|\nabla K(a_j)|^2}{\lambda_i} + \sum_{j\in A\cup B}\frac{\gamma}{p}\frac{1}{\lambda_j^2}\right) + \sum_{j\in A^c}\frac{|\nabla K(a_j)|^2}{\lambda_j} - c_2\sum_{j\in A}\sum_{i\neq j}\varepsilon_{ij}$$

$$+ O\left(\sum_{j\in A^c}^{p}\frac{\rho}{\lambda_j^2}\right) + o\left(\sum_{j\in A^c}\sum_{i\neq j}\varepsilon_{ij}\right)$$

$$+ c_1M\sum_{j\in A\cup B}\sum_{i\neq j}\varepsilon_{ij} - cM\sum_{j\in A\cup B}\frac{1}{\lambda_j^2} + o\left(\sum_{i\neq j}\varepsilon_{ij}\right).$$

First we choose M large such that $c_1M > 2c_2$ and then γ large such that $\gamma > 2pcM$, then

$$< \nabla J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right), \frac{d}{dt}\sum_{j=1}^{p}\alpha_j\varphi_{a_j,\lambda_j} >$$

$$\geq c\left(\sum_{j=1}^{p}\frac{|\nabla K(a_j)|^2}{\lambda_i} + \sum_{j\in A\cup B}\frac{1}{\lambda_j^2} + \sum_{j\in A\cup B}\sum_{i\neq j}\varepsilon_{ij}\right) + O\left(\sum_{j\in A^c}^{p}\frac{\rho}{\lambda_j^2}\right) + o\left(\sum_{i\neq j}\varepsilon_{ij}\right).$$

Since this is still not the estimate we need, we have to add a third vector field on O_A^B.

Let W_U be the vector field, previously constructed in $U \subset B_\varepsilon^q, q = |A^c \setminus B|$. Set

$$W_3^A(\alpha, a, \lambda) := (0, 0, W_U((\alpha_i, a_i, \lambda_i)_{i\in A^c\setminus B}).$$

Along the flow of W_3^A we get

$$< \nabla J\left(\sum_{i=1}^p \alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right), \frac{d}{dt}\sum_{j=1}^p \alpha_j\varphi_{a_j,\lambda_j} >$$

$$\geq c_0\left(\sum_{i\in A^c\setminus B}\frac{1}{\lambda_i^2}+\sum_{i\in A^c\setminus B}\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j\in A^c\setminus B}\varepsilon_{ij}\right)$$

$$-c_1\sum_{i\in A^c\setminus B}\sum_{j\in B^c}\varepsilon_{ij}+O\left(\sum_{i=1}^p\frac{\rho}{\lambda_i^2}\right)+O\left(\sum_{i\neq j}\rho\varepsilon_{ij}\right).$$

Finally we set on $O_A^B\cap O_\pi$ $W_\pi^{A,B}:=\tilde{W}+\delta W_3^A$ and observe along the flow

$$< \nabla J\left(\sum_{i=1}^p \alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right), \frac{d}{dt}\sum_{j=1}^p \alpha_j\varphi_{a_j,\lambda_j} >$$

$$\geq c_0\left(\sum_{i=1}^p\frac{1}{\lambda_i^2}+\sum_{i=1}^p\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}+|\alpha-|\alpha|\bar{\alpha}|^2\right)$$

for δ,ρ and ε small.

We choose a partition of unity (ω_σ) subordinate to the cover (O_σ) such that $\omega_\pi((\alpha,a,\lambda)^\sigma)=\omega_{\sigma\circ\pi}(\alpha,a,\lambda)$ and a partition of unity $\eta_{A,B}$ with the invariance property as in (7.16) and define

$$W=\sum_{\sigma\in S_n}\sum_{A,B}\omega_\sigma\eta_{A,B}W_\sigma^{A,B},$$

which defines an equivariant vector field on $\bigcup_A O_A$. Finally we glue the vector fields in $\bigcup_A O_A$ and U along an S_p-invariant partition of unity to obtain an equivariant vector field on U_α. As an important fact of this construction $\max\{\lambda_1,\cdots,\lambda_p\}$ is decreasing as long as $(\alpha,a,\lambda)\notin V_2$. Thus we have constructed an equivariant vector field W on B_ε^p such that

$$< \nabla J\left(\sum_{i=1}^p \alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right), \frac{d}{dt}\sum_{j=1}^p \alpha_j\varphi_{a_j,\lambda_j} >$$

$$\geq c_0\left(\sum_{i=1}^p\frac{1}{\lambda_i^2}+\sum_{i=1}^p\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}+|\alpha-|\alpha|\bar{\alpha}|^2\right)$$

in any case. Due to Lemma 7 we also get the asserted estimate

$$\frac{d}{dt}J\left(\sum_{i=1}^p \alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)\geq c_0\left(\sum_{i=1}^p\frac{1}{\lambda_i^2}+\sum_{i=1}^p\frac{|\nabla K(a_i)|^2}{\lambda_i}+\sum_{i\neq j}\varepsilon_{ij}+|\alpha-|\alpha|\bar{\alpha}|^2\right)$$

for ε small. Hence the proof is finished. □

7.5. The global pseudo gradient

In the previous sections we constructed a pseudo gradient vector field $Z = Z_p$ in $V(p, \varepsilon_p)$ for ε_p small. For $p \in \mathbb{N}$ we choose local Lipschitz functions $\eta_p : H^1(M) \to [0,1]$ such that $\eta_p(u) = 1$ for $u \in V(p, \varepsilon_p/2)$ and $\eta_p(u) = 0$ in $H^1(M) \setminus V(p, \varepsilon_p)$. We define our global pseudo gradient X in U as follows:

$$X(u) := \sum_{p=1}^{\infty} \eta_p(u) Z_p(u) + \left(1 - \sum_{p=1}^{\infty} \eta_p(u)\right) \nabla J(u). \tag{7.17}$$

Since $V(p,\varepsilon) \cap V(q,\varepsilon) = \emptyset$ for $p \neq q$ and ε small, $X(u) = Z_p(u)$ in $V(p, \varepsilon_p/2)$. Since we assume that J does not have critical points in U, Proposition 7.1 implies $\nabla J(u) \cdot X(u) > 0$ in U. Therefore X is a global pseudo gradient.

In the next chapter we prove some general facts about this pseudo gradient which will allow us to proof Theorem 1 and 2.

8. The topological argument and proof of the Theorems

8.1. General facts about the flow

As in chapter 3 defined, let

$$U := \left\{ u \in \Sigma \ : \ \int_{\partial M} K(x) u^3 > 0 \right\}.$$

Proposition 17. *Let $\Delta \subset \mathbb{R} \times U$ and $\Phi : \Delta \to U$ be the flow of $\dot{u} = -X(u)$, where X is defined in* (7.17)*. Then all maximal solutions exist on* $[0, \infty)$.

Proof. Let $u_0 \in U$ and $u : [0, T^+) \to U$ be the maximal solution of:

$$\begin{cases} \dot{u} = -X(u) \\ u(0) = u_0. \end{cases}$$

Then u is well defined, because X is locally Lipschitz continuous. We need to show that $T^+ = \infty$ which will follow if we show that $||X||$ is bounded along the flow. First observe that

$$\frac{d}{dt} \int_{\partial M} K(x) u^3 d\sigma_g > 0,$$

because J decreases along the flow and $||u(t)||$ remains constant along the flow. Hence $\int_{\partial M} K(x) u^3 d\sigma_g$ is bounded from above and below along u. Since

$$\nabla J(u) = \frac{2}{\left(\int_{\partial M} K(x) u^3 d\sigma_g \right)^{\frac{2}{3}}} \left(u - l(u) B_g^{-1} \left(u^2 \right) \right),$$

we conclude $||\nabla J(u)|| \leq C(u_0)$ along the flow. If $u \in V(p, \varepsilon)$ the expansion of J implies

$$J(u) = \frac{Q(B^4, \partial B^4) \sum_{i=1}^p \alpha_i^2}{\left(\sum_{i=1}^p K(a_i) \alpha_i^3 \right)^{\frac{2}{3}}} \left(1 + O(\varepsilon) \right).$$

Furthermore $|\alpha - |\alpha|\bar{\alpha}| = o(\varepsilon)$, which yields

$$J(u) = \frac{Q(B^4, \partial B^4)\sum_{i=1}^{p}\bar{\alpha}_i^2}{(\sum_{i=1}^{p} K(a_i)\bar{\alpha}_i 3)^{\frac{2}{3}}}\,(1 + O(\varepsilon))\,.$$

Let $\bar{\alpha}_m := \min\{\bar{\alpha}_1, \cdots, \bar{\alpha}_p\}$, then

$$J(u) \geq \frac{Q(B^4, \partial B^4)\left(p\bar{\alpha}_m^2\right)^{\frac{1}{3}}}{(K(a_m)\bar{\alpha}_m)^{\frac{2}{3}}}\,(1 + O(\varepsilon)) \geq \frac{Q(B^4, \partial B^4)}{\max(K)^{\frac{2}{3}}} p^{\frac{1}{3}}\,(1 + O(\varepsilon)) \to \infty\ (p \to \infty).$$

Therefore $u(t)$ intersects only finitely many $V(p, \varepsilon)$ along the flow. Hence the proof will be completed once we will have shown that $||X(u)|| \leq C(p)$ for all $u \in V(p, \varepsilon)$. Since $X(u) = \eta_p(u)Z_p(u) + (1 - \eta_p)\nabla J(u)$ in $V(p, \varepsilon)$ it suffices to show that $Z = Z_p$ is bounded in $V(p, \varepsilon)$. $Z(u) = D\psi(\alpha, a, \lambda, v)[\tilde{Z}(\alpha, a, \lambda, v)]$, therefore

$$\begin{aligned} Z(u) = &\sum_{i=1}^{p}\left(W_\alpha^i + t(\alpha, a, \lambda, v)\alpha_i\right)\varphi_{a_i,\lambda_i} + \sum_{i=1}^{p} W_\lambda^i \frac{\partial}{\partial \lambda_i}\varphi_{a_i,\lambda_i} + \sum_{i=1}^{p} W_a^i \cdot \nabla_{a_i}\varphi_{a_i,\lambda_i} \\ &+ C(v - \bar{v}) + D\bar{v}[W] + t(\alpha, a, \lambda, v)\bar{v} - R. \end{aligned}$$

Due to Proposition 16 $W_\alpha^i, \lambda_i^{-1}W_\lambda^i, \lambda_i W_a^i$ are bounded. Moreover $||\bar{v}||$ is bounded, hence Lemma 5 implies that $D\bar{v}[W]$ is bounded as well. Furthermore R, t and v are also bounded. Hence Z is bounded in $V(p, \varepsilon)$ which proves the Proposition. □

Proposition 18. *Assume J does not have any critical point in U. Let $u_0 \in U$ and $u : [0, \infty) \to U$ be the solution of*

$$\begin{cases} \dot{u} = -X(u) \\ u(0) = u_0, \end{cases}$$

then there exists $p \in \mathbb{N}$ such that for all $\varepsilon > 0$: $u(t) \in V(p, \varepsilon)$, provided $t \geq t(\varepsilon)$ for some $t(\varepsilon) \geq 0$.
In other words, it exists $p \in \mathbb{N}$ and $\varepsilon(t) \searrow 0$ $(t \to \infty)$ such that $u(t) \in V(p, \varepsilon(t))$, if t is large.

Proof. As we have seen in the proof of Proposition 17 there exists p_0 such that $u(t) \notin V(p, \varepsilon_p)$ for all $p \geq p_0$, provided ε_p is small. From now on we assume that
$\varepsilon < \min\{\varepsilon_p\ :\ 1 \leq p \leq p_0\}$.
Claim 1: There exists $t_n \to \infty$ such that

$$u(t_n) \in \bigcup_{p \leq p_0} V(p, \frac{\varepsilon}{8}).$$

Proof of the claim. Assume the claim is wrong. Then there exists $t_1 \geq 0$ such that $u(t) \notin \bigcup_{p \leq p_0} V(p, \frac{\varepsilon}{8})$ for all $t \geq t_1$. Since we assume that J does not have any critical point

$$\inf \left\{ |\nabla J(u)|^2 \;\middle|\; u \in U \setminus \bigcup_{p \in \mathbb{N}} V(p, \frac{\varepsilon}{8});\; J(u) \leq J(u_0) \right\} = c_0 > 0.$$

Furthermore

$$\inf \left\{ < \nabla J(u), X(u) > \;\middle|\; u \in \bigcup_{p \leq p_0} \Big(V(p, \varepsilon) \setminus V(p, \frac{\varepsilon}{8}) \Big) \right\} = c_1 > 0,$$

which implies $< \nabla J(u(t)), X(u(t)) > \geq \min\{c_0, c_1\}$ for all $t \geq t_1$. Hence

$$J(u(t)) - J(u(t_1)) = - \int_{t_1}^{t} < \nabla J(u(s)), X(u(s)) > ds \leq - \min\{c_0, c_1\}(t - t_1) \to -\infty$$

for $t \to \infty$ which contradicts the fact that J is bounded from below. Hence claim 1 is true. □

Due to claim 1 there exists a sequence $t_n \to \infty$ and $p \in \{1, \cdots, p_0\}$ such that $u(t_n) \in V(p, \frac{\varepsilon}{8})$ for all $n \in \mathbb{N}$. Assume there exists a sequence $s_n \to \infty$ such that $u(s_n) \notin V(p, \frac{\varepsilon}{4})$ for all $n \in \mathbb{N}$. After passing to sub-sequences we can assume that $...t_n < s_n < t_{n+1}...$ for all $n \in \mathbb{N}$. Set

$$\hat{t}_n = \inf\{t \geq t_n \;: u([t, s_n]) \subset V(p, \frac{\varepsilon}{8})^c\}.$$

Then $u([\hat{t}_n, s_n]) \subset V(p, \frac{\varepsilon}{8})^c$. Since u is continuous, we can assume that $u([\hat{t}_n, s_n]) \subset V(p, \frac{\varepsilon}{2})$. From the estimate above we derive

$$J(u(s_n)) - J(u(\hat{t}_n)) \leq -c_1 |s_n - \hat{t}_n|.$$

Furthermore

$$0 < d_0 := dist\Big(V(p, \frac{\varepsilon}{4})^c, V(p, \frac{\varepsilon}{8})\Big) \leq ||u(s_n) - u(\hat{t}_n)|| \leq \int_{\hat{t}_n}^{s_n} ||X(s)|| ds \leq C|s_n - \hat{t}_n|,$$

which yields

$$J(u(s_n)) - J(u(\hat{t}_n)) \leq -c_1 d_0,$$

and hence $J(u(s_n)) \to -\infty$, thus a contradiction. Therefore the assumption is wrong and such a sequence $(s_n)_n$ does not exist. Thus $u(t) \in V(p, \varepsilon)$ for t large. □

8.2. Critical points at infinity

Let $u : [0,\infty) \to U$ be a flow line of $-X$. From the previous section we know that $u(t) \in V(p,\varepsilon)$ for some $p \in \mathbb{N}$ and t large. Therefore, for t large we can write

$$u(t) = \sum_{i=1}^{p} \alpha_i(t)\varphi_{a_i(t),\lambda_i(t)} + v(t)$$

where $(\alpha, a, \lambda, v) \in E$ move along $-\tilde{Z}$ (see Lemma 4).

We call a tupel $(\bar{\alpha}(x), x) \in S^{p-1} \times \partial M^p$ **critical point at infinity** if $x = (x_1, \cdots, x_p)$ is contained in the following set

$$\mathcal{F}_p^\infty := \Bigg\{ (y_1, \cdots, y_p) \in crit(K)^p \;\Bigg|\; y_i \neq y_j \text{ for } i \neq j, \\ 2|S^3_+|H_{y_i}(y_i) + \frac{2I_4}{9}\frac{\Delta K(y_i)}{K(y_i)} < 0 \;\forall i, \rho_1(y_1, \cdots, y_p) > 0 \Bigg\}.$$

Here ρ_1 is the least eigenvalue, defined in (1.16). We are now able state the main Proposition in this section.

Proposition 19. *Let*

$$u(t) = \sum_{i=1}^{p} \alpha_i(t)\varphi_{a_i(t),\lambda_i(t)} + v(t)$$

be a solution of $\dot{u} = -X(u)$, which remains in $V(p,\varepsilon)$ for all $t \geq 0$, then

$$a(t) = (a_1(t), \cdots, a_p(t)) \to (x_1, \cdots, x_p) \in \mathcal{F}_p^\infty;\; \lambda_i \to \infty \;\forall i;\; \varepsilon_{ij} \to 0 \;\forall i \neq j; \\ |\alpha - |\alpha|\bar{\alpha}| \to 0;\; |\alpha| \to (2I_0)^{-\frac{1}{2}};\; ||v - \bar{v}|| \to 0$$

for $(t \to \infty)$.

Proof. From Proposition 18 we immediately infer

$$\lambda_i \to \infty, \varepsilon_{ij} \to 0, |\alpha - |\alpha|\bar{\alpha}| \to 0 \text{ and } ||v - \bar{v}|| \to 0$$

and therefore also $||v(t)|| \to 0$.
Since

$$\begin{aligned} 1 = ||u(t)||^2 &= \sum_{i=1}^{p} \alpha_i(t)^2 ||\varphi_{a_i,\lambda_i}(t)||^2 + O\left(\sum_{i\neq j} \varepsilon_{ij}(t)\right) + ||v(t)||^2 \\ &= 2\sum_{i=1}^{p} \alpha_i(t)^2 I_0 + O\left(\sum_{i=1}^{p} \frac{1}{\lambda_i(t)^2}\right) + O\left(\sum_{i\neq j} \varepsilon_{ij}(t)\right) + ||v(t)||^2, \end{aligned}$$

we derive $2I_0|\alpha(t)|^2 \to 1$. It remains to prove the statement for $a(t)$. Since (α, a, λ, v) moves along $-\tilde{Z}$, a_i moves along $-W_a^i$. Hence

$$\dot{a}_i = \frac{\nabla K(a_i)}{\lambda_i}.$$

Let $\beta_i : [0, T^+) \to [0, \infty)$ be the solution of $\dot{\beta}_i(s) = \lambda_i(\beta_i(s)), \beta_i(0) = 0$, then $b_i : [0, T^+) \to \partial M$, $b_i(s) := a_i(\beta_i(s))$ solves $\dot{b}_i(s) = \nabla K(b_i(t)); b_i(0) = a_i(0)$. If $T^+ < \infty$ then $a_i(t) \to b_i(T^+) =: a_\infty$ for $t \to \infty$. Since a_∞ is not a critical point in this case,

$$\frac{\nabla K(a_i(t))}{\lambda_i(t)} > 2\gamma \frac{1}{\lambda_i(t)^2}$$

for t large. Due to the construction of W, $\max\{\lambda_1(t), \cdots, \lambda_p(t)\}$ is decreasing for t large and therefore λ_i is bounded, which is a contradiction. Hence $T^* = \infty$ and a_∞ is a critical point of K. Therefore

$$(a_1(t), \cdots, a_p(t)) \to (x_1, \cdots, x_p) \in crit(K)^p \quad (t \to \infty).$$

If $(x_1, \cdots, x_p) \notin \mathcal{F}_p^\infty$, then $(\alpha, a, \lambda)(t) \notin V_2$ for t large. Thus, due to to the construction of W, $\max\{\lambda_1(t), \cdots, \lambda_p(t)\}$ is decreasing for t large, which is a contradiction, because J is bounded from below. Therefore $(a_1(t), \cdots, a_p(t)) \to (x_1, \cdots, x_p) \in \mathcal{F}_p^\infty$ which finishes the proof. □

Remark 3. *If $p > |crit(K)|$ then $\mathcal{F}_p^\infty = \emptyset$. Therefore flow lines of $\dot{u} = -X(u)$ are contained in*

$$\bigcup_{p \le |crit(K)|} V(p, \varepsilon)$$

for t large.

For $x \in \bigcup_{p \le |crit(K)|} \mathcal{F}_p^\infty$ define

$$c(x) := Q(B^4, \partial B^4) \left(\sum_{i=1}^{p} \frac{1}{K(x_i)^2} \right)^{\frac{1}{3}}.$$

Let $u : [0, \infty) \to U$ be any flow line of $\dot{u} = -X(u)$. Since J does not have any critical point, Proposition 7 and Proposition 19 yield $J(u(t)) \to c(x)$ $(t \to \infty)$ for some $x \in \bigcup_{p \le |crit(K)|} \mathcal{F}_p^\infty$. Especially

$$\inf\{J(u) \mid u \in U\} = \min \left\{ c(x) \mid x \in \bigcup_{p \le |crit(K)|} \mathcal{F}_p^\infty \right\},$$

hence

$$\inf\{J(u) \mid u \in U\} \ge Q(B^4, \partial B^4) \big(\max K \big)^{-\frac{2}{3}} \tag{8.1}$$

We are now able to prove Theorem 1.

Proof of Theorem 1. Let $x \in \partial M$ such that $K(x) = \max K$ and $2|S^3_+|H_x(x) + \frac{2I_4}{9}\frac{\Delta K(x)}{K(x)} > 0$. The expansion of the functional (see Proposition 7) yields

$$J(\varphi_{x,\lambda}) < Q(B^4, \partial B^4)\big(\max K\big)^{-\frac{2}{3}}$$

for λ large. This contradicts (8.1). Therefore J must have a critical point in this case. □

8.3. A Morse lemma at infinity

Let $\hat{r} = \frac{1}{\sqrt{2I_0}}$ and $x = (x_1, \cdots, x_p) \in \mathcal{F}^\infty_p$. Furthermore let

$$\varphi : B_{r_0}(0) \times B_{r_0}(0)^p \to S^{p-1}_{\hat{r}} \times \partial M^p$$

be a Morse-chart around the critical point $(\hat{r}\bar{\alpha}(x), x)$ of the function

$$f(\alpha, a) = \frac{Q(B^4, \partial B^4)|\alpha|^2}{\left(\sum_{i=1}^p \alpha_i^3 K(a_i)\right)^{\frac{2}{3}}},$$

defined on $S^{p-1}_{\hat{r}} \times \partial M^p$. Hence, for $(\alpha, a) = \varphi(h, y)$ it holds

$$f(\alpha, a) = c(x) - |h|^2 + \sum_{i=1}^p \left(-|y_i^-|^2 + |y_i^+|^2\right).$$

Here $(y_i^-, y_i^+) \in \mathbb{R}^3$ are the coordinates of unstable and stable manifold of $-\nabla\frac{1}{K}$ at x_i. Under the identification $(\alpha, a, \lambda) = (\varphi(h, y), \lambda)$ Proposition 7 yields the following expansion:

$$\begin{aligned} J\left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right) &= c(x) - |h|^2 + \sum_{i=1}^p \left(-|y_i^-|^2 + |y_i^+|^2\right) + c^*(x) < M(x)\Lambda, \Lambda > \\ &+ O\left(|h| + |y|\right)|\Lambda|^2 + O\left(\rho|\Lambda|^2\right) + O\left(|y^-|^4 + |y^+|^4\right) + O\left(|\Lambda|^3\right). \end{aligned}$$

if (α, a) are close to $(\hat{r}\bar{\alpha}(x), x)$, where

$$c^*(x) = c(x)\frac{1}{2I_0}\left(\sum_{i=1}^p \frac{1}{K(x_i)}\right)^{-1}$$

and $\Lambda = \left(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_p}\right)^t$. Furthermore we choose $r << r_0$ and define the function

$$g : \mathbb{R}^{p-1} \times (\mathbb{R}^3)^p \times \mathbb{R}^p_+ \to \mathbb{R},$$

$$g(h, y, \lambda) = c(x) - |h|^2 + \sum_{i=1}^p \left(-|y_i^-|^2 + |y_i^+|^2\right) + c^*(x) < M(x)\Lambda, \Lambda > .$$

and the vector field $\tilde{V}(h,y,\lambda) = (D\varphi^{-1}[W_\alpha, W_a], W_\lambda)$. Let $\tilde{\Phi}(h,y,\lambda,t)$ be the flow of this vector field. We would like to show

$$\frac{d}{dt}g\big(\tilde{\Phi}(h,y,\lambda,t)\big) \geq c_0 \left(|\alpha - |\alpha|\bar{\alpha}|^2 + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|^2}{\lambda_i} + |\Lambda|^2 + \sum_{i\neq j} \varepsilon_{ij} \right) \tag{8.2}$$

for all $(h,y,\lambda) \in B_r(0) \times B_r(0)^p \times \mathbb{R}^p_\varepsilon$, where

$$\mathbb{R}^p_\varepsilon := \{ y \in \mathbb{R}^p \mid y_i > \varepsilon \ \forall \ i \ \}.$$

Since

$$(\alpha(t), a(t), \lambda(t)) = (\varphi(h(t), y(t)), \lambda(t)) \in U^c_\alpha \cup V_2 \bigcup_{\emptyset \neq A \subset \{1,\cdots,p\}} \mathcal{O}_A$$

along the flow of $\tilde{V}$, we have to check the inequality in each set separately (see page 74,76 and 85 for the definition of the previous sets). In U^c_α it holds

$$\begin{aligned}
\frac{d}{dt}g\big(\tilde{\Phi}(h,y,\lambda,t)\big) &= \frac{d}{dt}\left(c(x) - |h|^2 + \sum_{i=1}^{p} \left(-|y_i^-|^2 + |y_i^+|^2\right) \right) = \frac{d}{dt}f(\alpha, a) \\
&= |\nabla_\alpha f(\alpha,a)|^2 + \sum_{i=1}^{p} \nabla_{a_i} f(\alpha,a) \cdot W_{a_i} \\
&\geq c \left(|\alpha - |\alpha|\bar{\alpha}|^2 + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|^2}{\lambda_i} \right) \\
&\geq c \left(|\alpha - |\alpha|\bar{\alpha}|^2 + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|^2}{\lambda_i} + |\Lambda|^2 + \sum_{i\neq j} \varepsilon_{ij} \right).
\end{aligned}$$

Next we check the inequality in V_2.

$$\begin{aligned}
\frac{d}{dt}g\big(\tilde{\Phi}(h,y,\lambda,t)\big) &= \frac{d}{dt}\left(c(x) - |h|^2 + \sum_{i=1}^{p} \left(-|y_i^-|^2 + |y_i^+|^2\right) \right) + \frac{d}{dt}c^*(x) < M(x)\Lambda, \Lambda > \\
&= \frac{d}{dt}f(\alpha,a) + 2c^*(x) < M(x)\Lambda, \Lambda > \\
&\geq c \left(\sum_{i=1}^{p} \frac{|\nabla K(a_i)|^2}{\lambda_i} + |\Lambda|^2 \right) \\
&\geq c \left(|\alpha - |\alpha|\bar{\alpha}|^2 + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|^2}{\lambda_i} + |\Lambda|^2 + \sum_{i\neq j} \varepsilon_{ij} \right).
\end{aligned}$$

Finally we prove the inequality in some $\mathcal{O}_A$. $\mathcal{O}_A$ is subdivided in the sets $O^B_A \cap O_\pi$. Fore the precise definitions see page 86. In $O^B_A \cap O_\pi$ we have

$$\dot{\lambda}_i = M2^{\pi^{-1}(i)}\lambda_i \ \ i \in \{A \cup B\}, \quad \dot{\lambda}_i = -\delta\lambda_i \ \ i \in A^c \setminus B,$$

thus

$$\begin{aligned}\frac{d}{dt}g\big(\tilde{\Phi}(h,y,\lambda,t)\big) &= \frac{d}{dt}\left(c(x)-|h|^2+\sum_{i=1}^{p}\left(-|y_i^-|^2+|y_i^+|^2\right)\right)+\frac{d}{dt}c^*(x) < M(x)\Lambda,\Lambda > \\ &\geq c\left(\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i}+|\Lambda|^2\right) \\ &\geq c\left(|\alpha-|\alpha|\bar{\alpha}|^2+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i}+|\Lambda|^2+\sum_{i\neq j}\varepsilon_{ij}\right).\end{aligned}$$

Therefore, the stated inequality is profed. Finally define

$$V(h,y,\lambda):=\frac{\tilde{V}(h,y,\lambda)}{|\alpha-|\alpha|\bar{\alpha}|^2+\sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i}+|\Lambda|^2+\sum_{i\neq j}\varepsilon_{ij}}$$

We choose r,ε small such that

$$c(x)-c_0 < J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right), g(h,y,\lambda) < c(x)+c_0,$$

if $(h,y)\in B_r(0)\times B_r(0)^p$ and $\lambda_i > \frac{1}{\varepsilon}\ \forall i$. If necessary, we choose r smaller such that the flow $\Phi(h,y,\lambda,t)$ of V, defined on $\in B_{r_0}(0)\times B_{r_0}(0)^p\times\mathbb{R}_+^p$, exists at least for $t\in[-2,2]$, if $(h,y,\lambda)\in B_r(0)\times B_r(0)^p\times\mathbb{R}_\varepsilon^p$.

Due to (8.2), $t\mapsto g(\Phi(h,y,\lambda,t))$ is strictly increasing with

$$g(\Phi(h,y,\lambda,2))\geq g(\Phi(h,y,\lambda,0))+2c_0 > c(x)+c_0$$

and

$$g(\Phi(h,y,\lambda,-2))\leq g(\Phi(h,y,\lambda,0))-2c_0 < c(x)-c_0.$$

Due to the mean value theorem, there exists exactly one $t=t(h,y,\lambda)\in(-2,2)$ such that

$$J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)=g\big(\Phi(h,y,\lambda,t(h,y,\lambda))\big).$$

By the implicit function theorem t depends smooth on (h,y,λ). We define the (smooth) map

$$w(h,y,\lambda):=\Phi(h,y,\lambda,t(\alpha,a,\lambda)).$$

Since

$$\frac{d}{dt}J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)>0,$$

if (h,y,λ) move along the flow of V, w is into. We would like to show that w is a diffeomorphism. Therefore we construct an inverse map. Since

$$\frac{d}{dt}J\left(\sum_{i=1}^{p}\tilde{\alpha}_i\varphi_{\tilde{a}_i,\tilde{\lambda}_i}+\bar{v}\right)\geq c_1,$$

if $(\tilde{h},\tilde{y},\tilde{\lambda})=(\varphi^{-1}(\tilde{\alpha},\tilde{a}),\tilde{\lambda})$ move along the flow of V, there exists exactly one $\tilde{t}=\tilde{t}(\tilde{h},\tilde{y},\tilde{\lambda})$ such that

$$J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)=g(\tilde{h},\tilde{y},\tilde{\lambda}),$$

where $(\alpha,a,\lambda)=(\varphi(h,y),\lambda)$ ans $(h,y,\lambda)=\Phi(\tilde{h},\tilde{y},\tilde{\lambda},\tilde{t}(\tilde{h},\tilde{y},\tilde{\lambda})))$.

We set $\tilde{w}(\tilde{h},\tilde{y},\tilde{\lambda})=\Phi(\tilde{h},\tilde{y},\tilde{\lambda},\tilde{t})$, which defines a smooth map as well.

For r,ε small, the map $\tilde{w}\circ w$ is well defined on $B_r(0)\times B_r(0)^p\times\mathbb{R}^p_\varepsilon$. Furthermore observe $w(\tilde{w}(h,y,\lambda))=\Phi(t,\Phi(\tilde{t},h,y,\lambda))=\Phi(t+\tilde{t},h,y,\lambda)$ as well as

$$g(h,y,\lambda)=g(\Phi(t+\tilde{t},h,y,\lambda)).$$

Thus, (8.2) implies $t+\tilde{t}=0$ and therefore $w\circ\tilde{w}=id$. The same arguments show $\tilde{w}\circ w=id$. Hence w is a diffeomorphism.

Summing up, we have proved the following Morse lemma at infinity:

Proposition 20. *Let $x=(x_1,\cdots,x_p)\in\mathcal{F}_p^\infty, u=\sum_{i=1}^p\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\in V(p,\varepsilon)$, where $d_g(a_i,x_i)<r$, $|\alpha|=\hat{r}$, $|\alpha-\bar{\alpha}(x)\hat{r}|<r$ and ε,r are small, then it exists a change of variables $\mathbb{R}^{p-1}\times\left(\mathbb{R}^3\right)^p\times\mathbb{R}^p\supset B_r\times B_r^p\times\mathbb{R}^p_\varepsilon\ni(h,y,\lambda')\longleftrightarrow(\alpha,a,\lambda)$ such that*

$$J\left(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\lambda_i}+\bar{v}\right)=c(x)-|h|^2-\sum_{i=1}^{p}|y_i^-|^2+\sum_{i=1}^{p}|y_i^+|^2+c^*(x)<M(x)\Lambda',\Lambda'>.$$

Moreover the map can be chosen such that $\frac{1}{2}\lambda_i'\leq\lambda_i\leq 2\lambda_i'\ \forall\ i$.
Furthermore $(y_i^-,y_i^+)\in V_i^-\oplus V_i^+=\mathbb{R}^3$, where $dim(V_i^-)=ind(\frac{1}{K},x_i)=3-ind(K,x_i)$.

8.4. The topological argument

First observe that the set

$$C:=\left\{c(x)\;\middle|\;x\in\bigcup_{p\leq|crit(K)|}\mathcal{F}_p^\infty\right\}\tag{8.3}$$

is finite. We set $4\mu_0 := \min_{a,b\in C}|a-b|$ and for $x \in \mathcal{F}_p^\infty$, ε, δ small we define

$$U_{\varepsilon,\delta} := \Big\{ u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v \in V(p, \frac{\varepsilon}{4}) \;\Big|\; d_g(a_i, x_i) < \delta,\ |\frac{\alpha}{|\alpha|} - \bar{\alpha}(x)| < \delta,$$
$$||v - \bar{v}|| < \frac{\varepsilon}{8},\ \lambda_i > \frac{100}{\varepsilon} \Big\} \subset V(p, \frac{\varepsilon}{4}).$$

For $u \in U_{\varepsilon,\delta}$ there exist a unique map $q : U_{\varepsilon,\delta} \to B_\varepsilon^p$, $q(u) = (\alpha(u), a(u), \lambda(u))$ such that $d_g(a_i(u), x_i) < \delta$ and $u = \psi(q(u), v)$ for some $v \in E_{q(u)}$, where $\psi : E \to H^1(M)$:

$$\psi(\alpha, a, \lambda, v) := \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v.$$

Since ψ is a local diffeomorphism, q is smooth.

Choose an open neighbourhood $U \subset S_{\hat{r}}^{p-1} \times \partial M^p$ around $(\hat{r}\bar{\alpha}(x), x)$ and a diffeomorphism $\phi : U \times \mathbb{R}_\varepsilon^p \to V \times \mathbb{R}_\varepsilon^p$ as in Proposition 20 such that

$$U \subset \{(\alpha, a) \in S_{\hat{r}}^{p-1} \times \partial M^p \mid |\alpha - \hat{r}\bar{\alpha}(x)| < \frac{\delta}{4}, d(a_i, x_i) < \frac{\delta}{4}\}.$$

Here $V = B_{2r}(0) \times B_{2r}(0)^p \subset \mathbb{R}^{p-1} \times (\mathbb{R}^3)^p$. Then in $V \times \mathbb{R}_\varepsilon^p$ holds:

$$J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right) = c(x) - |h|^2 + \sum_{i=1}^{p} -|y_i^-|^2 + |y_i^+|^2 + c^*(x) < M(x)\Lambda', \Lambda' >,$$

where we use the same notations as in the previous section.
Clearly $(\alpha, a, \lambda) = \phi^{-1}(h, y, \lambda')$.

Furthermore we define $\eta_\delta \in C^\infty(\mathbb{R})$ such that

$$\eta_\delta(t) = 1, t \le \delta,\ \eta_\delta(t) = 0, \quad t \ge 2\delta,\ -2\delta^{-1} \le \eta'_\delta \le 0.$$

For $u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v \in U_{\varepsilon,\delta}$ and $\mu > 0$ we define the map

$$g(u) = \mu \cdot \eta_\delta\left(|h|^2 + |y|^2 + |\Lambda'|^2 + ||v - \bar{v}||^2\right),$$

where $(h, y, \lambda') = \phi\left(\frac{\alpha}{|\alpha|}\hat{r}, a, \lambda\right)$. Outside of $U_{\varepsilon,\delta}$ we set $g(u) = 0$. Then g is smooth in $H^1(M)$.

Moreover if $g(u) > 0$, then

$$\begin{aligned} J(u) &= J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right) \\ &+ \frac{1}{2}\int_0^1 D^2 J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v} + t(v-\bar{v})\right)[(v-\bar{v}),(v-\bar{v})] \\ &\le c(x) - |h|^2 - \sum_{i=1}^{p} |y_i^-|^2 + \sum_{i=1}^{p} |y_i^+|^2 + c^*(x) < M(x)\Lambda', \Lambda' > + C||v-\bar{v}||^2 \\ &\le c(x) + C\delta < c(x) + \mu_0 \end{aligned}$$

if δ is small.

We set $F(u) = J(u) - g(u)$.

Lemma 8. *It holds*

$$\nabla F(u) \cdot X(u) > 0,$$

if μ, δ are small.

Proof. If $g(u) = 0$ the statement follows from(7.1). Therefore let $g(u) > 0$. If u moves along the flow of X, then $(\hat{r}\frac{\alpha}{|\alpha|}, a, \lambda)$ move along the flow of W. Hence

$$\left|\frac{d}{dt}\hat{r}\frac{\alpha}{|\alpha|}\right| \le \left|\nabla_\alpha f\left(\hat{r}\frac{\alpha}{|\alpha|}, a\right)\right|, \ \frac{d}{dt}a_i = \frac{\nabla K(a_i)}{\lambda_i}, \ \left|\frac{d}{dt}|\Lambda|^2\right| \le C|\Lambda|^2.$$

Moreover

$$\frac{d}{dt}||v-\bar{v}||^2 = C||v-\bar{v}||^2.$$

Since (h, y, λ') move along the flow of $\tilde{V}$:

$$\left|\frac{d}{dt}|h|^2\right| \le C|h|^2, \ \left|\frac{d}{dt}|y_i|^2\right| \le C\frac{|y_i|^2}{\lambda_i}, \ \left|\frac{d}{dt}|\Lambda'|^2\right| \le C|\Lambda'|^2.$$

Thus:

$$\begin{aligned} |\nabla g(u)\cdot X(u)| &= \left|\frac{d}{dt}g(u)\right| \\ &\le \frac{C\mu}{\delta}\frac{d}{dt}\left(|h|^2 + |y|^2 + |\Lambda'|^2 + ||v-\bar{v}||^2\right) \\ &\le \frac{C\mu}{\delta}\left(|h|^2 + \sum_{i=1}^{p}\frac{|y_i|^2}{\lambda_i} + |\Lambda'|^2 + ||v-\bar{v}||^2\right) \\ &\le \frac{C\mu}{\delta}\left(|\alpha - |\alpha|\bar{\alpha}|^2 + \sum_{i=1}^{p}\frac{|\nabla K(a_i)|^2}{\lambda_i} + |\Lambda|^2 + ||v-\bar{v}||^2\right). \end{aligned}$$

If $\frac{\mu}{\delta}$ is small (7.1) implies

$$\nabla F(u) \cdot X(u) \geq c \left(|\alpha - |\alpha|\bar{\alpha}|^2 + \sum_{i=1}^{p} \frac{|\nabla K(a_i)|^2}{\lambda_i} + |\Lambda|^2 + ||v - \bar{v}||^2 \right) > 0.$$

Therefore the Lemma is proved. □

For every $x \in \mathcal{F}^\infty$ we can construct a function g_x as above. Then Lemma 8 still holds for

$$F(u) = J(u) - \sum_{x \in \mathcal{F}^\infty} g_x(u). \tag{8.4}$$

Remember that $\mathcal{F}^\infty$ was defined in the introduction. Using F we now prove the deformation Lemma

Lemma 9. *Choose F from* (8.4)*, then*

$$J^{c(x)+\mu_0} = F^{c(x)+\mu_0} \cong F^{c(x)-\frac{\mu}{2}}.$$

Proof. The first identity is clear, we only need to proof the homotopy equivalence. Therefore let

$$\Phi : [0, \infty) \times F^{c(x)+\mu_0} \to F^{c(x)+\mu_0}$$

be the flow of $\dot{u} = -X(u)$. Due to Proposition 19 we know that $J(u(t)) \to c(z)$ for some $z \in F^\infty$. Either $c(z) < c(x) - \frac{\mu}{2}$ or $c(z) = c(x)$. In the second case $g(u(t)) \to -\mu$ and hence $F(u(t)) < c(x) - \frac{\mu}{2}$ in any case for t large. Since $\nabla F(u) \cdot X(u) > 0$ there exists exactly one $t = t(u)$, which depends C^1 on u, such that $F(\Phi(t(u), u)) = c(x) - \frac{\mu}{2}$. We set

$$H : [0, 1] \times F^{c(x)+\mu_0} \to F^{c(x)+\mu_0}, \ H(s, u) = \Phi(s \cdot t(u), u),$$

which is the stated homotopy equivalence. □

The main goal in this section is to compute the relative homology

$$H_*(J^{c(x)+\mu_0}, J^{c(x)-\frac{\mu}{2}}; \mathbb{F}_2) = H_*(J^{c(x)+\mu_0}, J^{c(x)-\frac{\mu}{2}}),$$

where $\mathbb{F}_2$ is the field with two elements. For simplicity we assume that there are no other critical tupels $y \in \mathcal{F}^\infty$ at the level $c(x)$. Set $A := \overline{F^{c(x)-\frac{\mu}{2}} \setminus J^{c(x)-\frac{\mu}{2}}}$, then $J^{c(x)+\mu_0} \cong J^{c-\frac{\mu}{2}} \cup A$ and hence

$$H_*(J^{c(x)+\mu_0}, J^{c(x)-\frac{\mu}{2}}) = H_*(J^{c(x)-\frac{\mu}{2}} \cup A, J^{c(x)-\frac{\mu}{2}}).$$

From the construction of g we derive

$$A \subset \left\{ \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v \in U_{\varepsilon,\delta} \mid |h|^2 + |y|^2 + |\Lambda'|^2 + ||v - \bar{v}||^2 < 2\delta \right\}.$$

Furthermore we define a bigger open set

$$B := \left\{ \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v \in U_{\varepsilon,\delta} \mid |h|^2 + |y|^2 + |\Lambda'|^2 + ||v - \bar{v}||^2 < 2\delta' \right\}.$$

where δ' is slightly bigger than δ.
Claim 1:

$$H_*(J^{c(x)-\frac{\mu}{2}} \cup A, J^{c(x)-\frac{\mu}{2}}) = H_*(J^{c(x)-\frac{\mu}{2}} \cap B \cup A, J^{c(x)-\frac{\mu}{2}} \cap B).$$

Proof of claim 1. We would like to apply the excision axiom for homology. Therefore we need to cut out $J^{c(x)-\frac{\mu}{2}} \setminus B \subset J^{c^*(x)-\frac{\mu}{2}}$, which can be done if

$$\overline{J^{c(x)-\frac{\mu}{2}} \setminus B} \subset int\left(J^{c(x)-\frac{\mu}{2}}\right),$$

where we need to take the closure and interior w.r.t. $J^{c(x)-\frac{\mu}{2}} \cup A$.

Since $J^{c(x)-\frac{\mu}{2}} \setminus B \subset J^{c(x)-\frac{\mu}{2}} \cup A$ is closed it remains to show

$$J^{c(x)-\frac{\mu}{2}} \setminus B \subset int\left(J^{c(x)-\frac{\mu}{2}}\right).$$

If $u \in J^{c(x)-\frac{\mu}{2}} \setminus B$, then $u \notin B$. Hence there exists $r > 0$ such that $B_r(u) \cap A = \emptyset$ and thus

$$B_r(u) \cap \left(J^{c(x)-\frac{\mu}{2}} \cup A\right) = B_r(u) \cap J^{c(x)-\frac{\mu}{2}} \subset J^{c(x)-\frac{\mu}{2}} \cup A$$

is open. Therefore $u \in int\left(J^{c(x)-\frac{\mu}{2}}\right)$, which proves the claim. □

For all $u \in B$ exists exactly one $t = t(u) > 0$ such that $|\alpha(u)t(u)| = \hat{r}$. We define the homeomorphism $\psi : B \to \overline{B} := \psi(B), \psi(u) = t(u)u$, which maps $J^{c(x)-\frac{\mu}{2}} \cap B \cup A$ to $J^{c(x)-\frac{\mu}{2}} \cap \overline{B} \cup \overline{A}$ and $J^{c(x)-\frac{\mu}{2}} \cap B$ to $J^{c(x)-\frac{\mu}{2}} \cap \overline{B}$. Here $\bar{A} = \psi(A)$.
Hence

$$H_*(J^{c(x)-\frac{\mu}{2}} \cap B \cup A, J^{c(x)-\frac{\mu}{2}} \cap B) = H_*(J^{c(x)-\frac{\mu}{2}} \cap \overline{B} \cup \overline{A}, J^{c(x)-\frac{\mu}{2}} \cap \overline{B}).$$

Moreover for $u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + v \in J^{c(x)-\frac{\mu}{2}} \cap \overline{B} \cup \overline{A}$ we define

$$u_t := \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v} + (1-t)(v - \bar{v}).$$

Claim 2: $u_t \in J^{c^*-\frac{\mu}{2}} \cap \overline{B} \cup \overline{A}$ for all $t \in [0, 1]$.

Proof of Claim 2. We show $\frac{u_t}{||u_t||} \in J^{c(x)-\frac{\mu}{2}} \cap B \cup A$, which proves the claim. Therefore we need to show $\frac{u_t}{||u_t||} \in B$ and $F(\frac{u_t}{||u_t||}) \leq c^*(x) - \frac{\mu}{2}$ for all t. Since J is homogeneous of degree zero and $\bar{v}$ is a minimizer:

$$J\left(\frac{u_t}{||u_t||}\right) = J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v} + (1-t)(v-\bar{v})\right)$$

is decreasing. Moreover

$$\bar{v}\left(\frac{\alpha}{||u_t||}, a, \lambda\right) = \frac{\bar{v}(\alpha, a, \lambda)}{||u_t||}$$

and $t \mapsto \eta_\delta(t)$ is decreasing. Therefore it remains to show that

$$(1-t)^2 \frac{||\bar{v}-v||^2}{||u_t||^2}$$

is monotone decreasing. But

$$\frac{d}{dt}\frac{(1-t)^2}{||u_t||^2} = -\frac{2(1-t)}{||u_t||^2} - 2\frac{(1-t)^2}{||u_t||^4} < \dot{u}_t, u_t > \leq 0 \Leftrightarrow -||u_t||^2 \leq (1-t) < \dot{u}_t, u_t > .$$

In addition

$$||u_t||^2 = ||\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i}||^2 + ||\bar{v}||^2 + 2(1-t) < \bar{v}, v-\bar{v} > + (1-t)^2 ||v-\bar{v}||^2$$

and

$$< \dot{u}_t, u_t > = -(1-t)||\bar{v}-v||^2 - < \bar{v}, v-\bar{v} >,$$

thus

$$\frac{d}{dt}\frac{(1-t)^2}{||u_t||^2} \leq 0 \Leftrightarrow -||\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i}||^2 - ||\bar{v}||^2 \leq (1-t) < \bar{v}, v-\bar{v} > .$$

The last inequality is true if $||v-\bar{v}||^2 \leq ||\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i}||^2$, which is satisfied for δ and ε small. Therefore the claim is true. □

Define

$$C_1 := \left\{ u \in J^{c^*-\frac{\mu}{2}} \cap \overline{B} \cup \overline{A} \;\middle|\; u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v} \right\}$$
$$\bigcup \left\{ u \in J^{c^*-\frac{\mu}{2}} \cap \overline{B} \cup \overline{A} \mid J(u) \leq c^* - \frac{\mu}{2} \right\}$$

and

$$C_2 = J^{c^*-\frac{\mu}{2}} \cap \overline{B} \cup \overline{A} \setminus C_1.$$

For $u \in C_2$ we set
$$\tau(u) = \sup \left\{ t \in [0,1] \;\middle|\; J(u_t) > c^* - \frac{\mu}{2} \right\}.$$

Claim 3: $\tau : C_2 \to (0,1]$ is continuous.

Proof of claim 3. If $t(u) < 1$, the claim follows by the implicit function theorem, because $\partial_t J(u_t) = < \nabla J(u_t), \bar{v} - v > \neq 0$ in this case. It remains to prove the claim if $\tau(u) = 1$. Let $u^n \in C_2$ converge to u, then $J(u_1) \geq c(x) - \frac{\mu}{2}$ and

$$\begin{aligned}
J(u_t^n) &= J\left(\sum_{i=1}^p \alpha_i^n \varphi_{a_i^n,\lambda_i^n} + \bar{v}^n + (1-t)(v^n - \bar{v}^n)\right) \\
&= J\left(\sum_{i=1}^p \alpha_i^n \varphi_{a_i^n,\lambda_i^n} + \bar{v}^n\right) \\
&+ (1-t)^2 \int_0^1 D^2 J\left(\sum_{i=1}^p \alpha_i^n \varphi_{a_i^n,\lambda_i^n} + \bar{v}^n + s(1-t)(v^n - \bar{v}^n)\right)[v^n - \bar{v}^n, v^n - \bar{v}^n] ds \\
&\geq J(u_1) + o(1) \\
&+ (1-t)^2 \int_0^1 D^2 J\left(\sum_{i=1}^p \alpha_i^n \varphi_{a_i^n,\lambda_i^n} + \bar{v}^n + s(1-t)(v^n - \bar{v}^n)\right)[v^n - \bar{v}^n, v^n - \bar{v}^n] ds \\
&\geq c(x) - \frac{\mu}{2} + o(1) + c_0(1-t)^2 ||v^n - \bar{v}^n||^2 = c^* - \frac{\mu}{2} + o(1) + c_0(1-t)^2 ||v - \bar{v}||^2.
\end{aligned}$$

Furthermore
$$c(x) - \frac{\mu}{2} + o(1) + c_0(1-t)^2 ||v - \bar{v}||^2 > c(x) - \frac{\mu}{2}$$

if and only if
$$(1-t)^2 > -o(1),$$

which is true if $|1-t| > o(1)$. Hence $J(u_t^n) > c(x) - \frac{\mu}{2}$ if $|1-t| > o(1)$ which implies that $\tau(u^n) \to 1$. Thus the claim is proved. □

Claim 4: Let $(u_n)_n \subset C_2$ then $\tau(u^n) \to 0$ if $u^n \to u \in C_1$ such that
$$J(u) = J\left(\sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + v\right) \leq c(x) - \frac{\mu}{2}$$

and $v \neq \bar{v}$.

Proof of claim 4. Clearly $J(u) = c(x) - \frac{\mu}{2}$, $J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}\right) < c(x) - \frac{\mu}{2}$ and

$$J(u_t^n) = J(u_t) + o(1).$$

Since $\partial_t J(u_t) < 0$ for $t \in [0,1)$, $J(u_t) < c(x) - \frac{\mu}{2}$ for all $t > \varepsilon$ and therefore $J(u_t^n) < c(x) - \frac{\mu}{2}$ for n large and $t > \varepsilon$, which implies $\tau(u_n) \leq \varepsilon$ for n large and hence $\tau(u_n) \to 0$. □

With the previous two claims we are prepared to define a homotopy which retracts the v-part. Set

$$H : [0,1] \times J^{c(x)-\frac{\mu}{2}} \cap \overline{B} \cup \overline{A} \to J^{c(x)-\frac{\mu}{2}} \cap \overline{B} \cup \overline{A},\ H(t,u) := \begin{cases} u & u \in C_1 \\ u_{t\tau(u)} & u \in C_2. \end{cases}$$

H is well defined. It remains to show that H is continuous. Let $(t^n, u^n) \to (t,u)$. First let $u \in C_1$. If $u^n \in C_1$ the convergence is clear. Let us assume $u^n \in C_2$. If u is as in claim 4, then $\tau(u^n) \to 0$ and therefore $u^n_{t^n\tau(u^n)} \to u = H(t,u)$. If $u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}$, then

$$H(t^n, u_n) = \sum_{i=1}^{p} \alpha_i^n \varphi_{a_i^n, \lambda_i^n} + \bar{v}^n + (1 - \tau(u^n) t_n)(v^n - \bar{v}^n) \to \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v} = H(t,u),$$

because $||v^n - \bar{v}^n|| \to 0$.
If $u \in C_2$ then $u^n \in C_2$ for n large and hence $H(t^n, u^n) = u_{t^n\tau(u^n)} \to u_{t\tau(u)} = H(t,u)$. We used that τ is continuous. Therefore H is continuous.

Set

$$\hat{A} := \left\{ u \in \bar{A} \;\middle|\; u = \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v} \right\},$$

then H retracts $J^{c^*-\frac{\mu}{2}} \cap \overline{B} \cup \overline{A}$ onto $J^{c^*-\frac{\mu}{2}} \cap \overline{B} \cup \hat{A}$ by deformation, which implies

$$H_*(J^{c(x)-\frac{\mu}{2}} \cap \overline{B} \cup \overline{A}, J^{c(x)-\frac{\mu}{2}} \cap \overline{B}) = H_*(J^{c(x)-\frac{\mu}{2}} \cap \overline{B} \cup \hat{A}, J^{c(x)-\frac{\mu}{2}} \cap \overline{B}).$$

Set

$$\hat{B} := \left\{ \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda_i} + \bar{v} \mid |h|^2 + |y|^2 + |\Lambda'|^2 < 2\delta' \right\} \subset \overline{B}.$$

Then the homotopy $h : [0,1] \times J^{c^*-\frac{\mu}{2}} \cap \overline{B} \cup \hat{A} \to J^{c^*-\frac{\mu}{2}} \cap \overline{B} \cup \hat{A}$, $h(t,u) = u_t$ retracts $J^{c^*-\frac{\mu}{2}} \cap \overline{B} \cup \hat{A}$ onto $J^{c^*-\frac{\mu}{2}} \cap \hat{B} \cup \hat{A}$ by deformation, which yields

$$H_*(J^{c^*-\frac{\mu}{2}} \cap \overline{B} \cup \hat{A}, J^{c^*(x)-\frac{\mu}{2}} \cap \overline{B}) = H_*(J^{c^*-\frac{\mu}{2}} \cap \hat{B} \cup \hat{A}, J^{c^*(x)-\frac{\mu}{2}} \cap \hat{B}).$$

Let $(h, y, \lambda') = \phi(\alpha, a, \lambda)$. Define

$$\hat{U} := \left\{ (h, y, \lambda') \mid |h|^2 + |y|^2 + |\Lambda'|^2 < 2\delta' \right\}$$

and the homeomorphism $\bar{\psi}: \hat{U} \to \hat{B}, \bar{\psi}(h, y, \lambda') := \sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda_i} + \bar{v}$. In $\hat{U}$ the following expansion holds true:

$$J\left(\bar{\psi}(h, y, \lambda')\right) = c(x) - |h|^2 + \sum_{i=1}^p -|y_i^-|^2 + |y_i^+|^2 + c^*(x) < M(x)\Lambda', \Lambda' > .$$

We set $\hat{\mathcal{A}} := \bar{\psi}^{-1}(\hat{A})$ as well as $\tilde{\mathcal{A}} := \{(h, y^-, 0, \lambda') \in \hat{\mathcal{A}} \mid J(\psi(h, y, \lambda')) \geq c^* - \frac{\mu}{2}\}$, where we split $y = (y^-, y^+)$. Since $\bar{\psi}$ is a homeomorphism

$$H_*(J^{c(x)-\frac{\mu}{2}} \cap \hat{B} \cup \hat{A}, J^{c(x)-\frac{\mu}{2}} \cap \hat{B}) = H_*((J \circ \bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \hat{U} \cup \hat{\mathcal{A}}, (J \circ \bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \hat{U}).$$

Furthermore a retraction yields

$$\begin{aligned} H_*((J \circ \bar{\psi})^{c(x)-\frac{\mu}{2}} &\cap \hat{U} \cup \hat{\mathcal{A}}, (J \circ \bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \hat{U}) \\ &= H_*((J \circ \bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \hat{U} \cup \tilde{\mathcal{A}}, (J \circ \bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \hat{U}). \end{aligned}$$

The retraction can be defined similarly to H, defined on page 105, to retract the v-part. Moreover

$$\begin{aligned} H_*((J \circ \bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \hat{U} &\cup \tilde{\mathcal{A}}, (J \circ \bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \hat{U}) \\ &= H_*((J \circ \bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \tilde{U} \cup \tilde{\mathcal{A}}, (J \circ \bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \tilde{U}), \end{aligned}$$

where

$$\tilde{U} = \{(h, y^-, 0, \lambda') \mid |h|^2 + |y^-|^2 + |\Lambda'|^2 < 2\delta'\}.$$

Here we used the homotopy $\tilde{h}(t, h, y^-, y^+, \lambda') = (h, y^-, (1-t)y^+, \lambda')$.

Next we set

$$\tilde{\mathcal{A}}_\mu := \left\{(h, y^-, 0, \lambda') \in \tilde{\mathcal{A}} \;\middle|\; \sum_{i=1}^p \frac{1}{\lambda_i'^2} \leq \mu\right\}.$$

Claim 5:

$$\begin{aligned} H_*((J \circ \bar{\psi})^{c(x)-\frac{\mu}{2}} &\cap \tilde{U} \cup \tilde{\mathcal{A}}, (J \circ \bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \hat{U}) \\ &= H_*((J \circ \bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \tilde{U} \cup \tilde{\mathcal{A}}_\mu, (J \circ \bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \tilde{U}). \end{aligned}$$

Proof of claim 5. Define

$$\tilde{\mathcal{A}}_1 := \left\{(h, y^-, 0, \lambda') \in \tilde{\mathcal{A}} \mid |\Lambda'|^2 > \mu,\; J(\psi(h, y^-, 0, \lambda')) > c(x) - \frac{\mu}{2}\right\}$$

and $\tau : \tilde{\mathcal{A}}_1 \to (1, \infty)$, where

$$\tau(h, y, \lambda') = \sup\left\{t \geq 1 \;:\; |\Lambda'|^2 \frac{1}{t^2} > \mu,\; J(\psi(h, y^-, 0, \lambda' t)) > c(x) - \frac{\mu}{2}\right\}.$$

τ is continuous because $\frac{d}{dt}J(\bar{\psi}(h,y^-,0,\lambda' t)) < 0$ and $\frac{d}{dt}\sum_{i=1}^p \frac{1}{\lambda_i'^2}\frac{1}{t^2} < 0$. We define the homotopy $H : [0,1]\times (J\circ\bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \tilde{U} \cup \tilde{\mathcal{A}} \to (J\circ\bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \tilde{U} \cup \tilde{\mathcal{A}}$

$$H(t,h,y,\lambda') = \begin{cases} (h,y,\tau t\lambda') & (h,y,\lambda') \in \tilde{\mathcal{A}}_1 \\ (h,y,\lambda') & \text{else.} \end{cases}$$

Since $\tau \to 1$ if $(h,y,\lambda') \to (h_0,y_0,\lambda_0) \in \tilde{\mathcal{A}}_1^c$, H is continuous. □

Claim 6:

$$\tilde{\mathcal{A}}_\mu = \left\{ (h,y^-,0,\lambda') \ : \ |h|^2 + |y^-|^2 \le \frac{\mu}{2} + c^*(x) < M(x)\Lambda',\Lambda' >, \ \sum_{i=1}^p \frac{1}{\lambda_i'^2} \le \mu \right\}.$$

Proof of claim 6. Clearly, if $(h,y,\lambda') \in \tilde{\mathcal{A}}_\mu$, then $J(\bar{\psi}(h,y,\lambda')) \ge c(x) - \frac{\mu}{2}$, which is equivalent to

$$|h|^2 + |y^-|^2 \le \frac{\mu}{2} + c^*(x) < M(x)\Lambda',\Lambda' > .$$

Therefore the first inclusion is proved. To prove the second inclusion it is left to show that $F(\bar{\psi}(h,y,\lambda')) \le c(x) - \frac{\mu}{2}$. But the conditions yield

$$|h|^2 + |y^-|^2 + |\Lambda'|^2 \le \frac{\mu}{2} + C\mu < \delta.$$

Hence $g(\bar{\psi}(h,y,\lambda')) = \mu$ and $J(\bar{\psi}(h,y,\lambda')) \le c(x) + \frac{\mu}{2}$, which proves the claim. □

Next we set $\tilde{U}_\mu := \{(h,y^-,0,\lambda') \in \tilde{U} \ : \ |\Lambda'|^2 \le \mu\}$, then homotopy arguments, familiar by now, show

$$H_*((J\circ\bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \tilde{U} \cup \tilde{\mathcal{A}}_\mu, (J\circ\bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \hat{U}) \\ = H_*((J\circ\bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \tilde{U}_\mu \cup \tilde{\mathcal{A}}_\mu, (J\circ\bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \tilde{U}_\mu).$$

Furthermore, due to claim 6, the pair $((J\circ\bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \tilde{U}_\mu \cup \tilde{\mathcal{A}}_\mu, (J\circ\bar{\psi})^{c(x)-\frac{\mu}{2}} \cap \tilde{U}_\mu)$ is homotopy equivalent to

$$(X,A) = \Bigg(\left\{ (h,y^-,\lambda') \mid |h|^2 + |y^-|^2 \le \frac{\mu}{2} + c^*(x) < M(x)\Lambda',\Lambda' >, \ |\Lambda'|^2 \le \mu \right\}, \\ \left\{ (h,y^-,\lambda') \mid |h|^2 + |y^-|^2 = \frac{\mu}{2} + c^*(x) < M(x)\Lambda',\Lambda' >, \ |\Lambda'|^2 \le \mu \right\} \Bigg).$$

Define $k = k(x) = p - 1 + 3p - \sum_{i=1}^p ind(x_i,K) = 4p - \sum_{i=1}^p ind(x_i,K) - 1$,

$$Y := \left\{ (z,\lambda') \in D^k \times \mathbb{R}_+^p \mid |\Lambda'|^2 \le \mu \right\}, \ B := \left\{ (z,\lambda') \in S^{k-1} \times \mathbb{R}_+^p \mid |\Lambda'|^2 \le \mu \right\}$$

and the homeomorphism

$$\phi : (Y,B) \to (X,A),\ \psi(z,\lambda') = \left(\frac{z}{\sqrt{\frac{\mu}{2} + c^*(x) < M(x)\Lambda', \Lambda' >}}, \lambda' \right).$$

Here D^k is the unit ball in $\mathbb{R}^k$ and $S^{k-1} = \partial D^k$. Therefore

$$\begin{aligned} H_*(X,A) = H_*(Y,B) &= H_*(D^k \times \mathbb{R}^p_+, S^{k-1} \times \mathbb{R}^p_+) \\ &= H_*(D^k, S^{k-1}; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & * = 0, k \\ \{0\} & \text{else.} \end{cases} \end{aligned}$$

For $x \in \mathcal{F}_p^\infty$ we set $k(x) = 4p - \sum_{i=1}^p ind(x_i, K) - 1$. The previous homotopy arguments have shown:

Proposition 21. *Let $x \in \mathcal{F}^\infty$ then*

$$H_*(J^{c(x)+\mu_0}, J^{c(x)-\frac{\mu}{2}}) \cong \sum_{y \in \mathcal{F}^\infty\ :\ c(y)=c(x)} H_*(D^{k(y)}, S^{k(y)-1}).$$

Using Proposition 21,we can now prove Theorem 2:

Proof of Theorem 2. Let $C = \{c_1, \dots, c_m\}$, where $c_1 < c_2 < \dots < c_m$ and C is defined in (8.3). Furthermore let $Y_0 := \emptyset$ and $Y_k = J^{c_k+\varepsilon}$, where ε is chosen very small, such that Proposition 21 holds true:

$$H_*(J^{c_k+\varepsilon}, J^{c_k-\varepsilon}) \cong \sum_{y \in \mathcal{F}^\infty\ \ c(y)=c_k} H_*(D^{k(y)}, S^{k(y)-1}).$$

Since we assume that J does not have any critical point in U, $J(u(t)) \xrightarrow{t\to\infty} b \in C$ for flow lines of $\dot{u} = -X(u)$. Thus Y_m is a strong deformation retract of U.
Let $\mathbb{F}_2$ be the field with two elements, (X,A) a pair of topological spaces and

$$P(X,A;t) = \sum_{n=0}^{\infty} \dim(H_n(X,A;\mathbb{F}_2))t^n,$$

be the Poincaré polinomial, then (see [10]):

$$P(Y_m, Y_0, -1) = \sum_{k=1}^{m} P(Y_k, Y_{k-1}, -1) = \sum_{x \in \mathcal{F}^\infty} (-1)^{\sum_{i=1}^p ind(x_i,K)+1}. \tag{8.5}$$

Since U is contractible

$$1 = \chi(U) = \chi(Y_m) = P(Y_m, Y_0, -1),$$

where χ is the Euler-characteristic. Therefore, we have shown:

$$1 = \sum_{x \in \mathcal{F}^\infty} (-1)^{\sum_{i=1}^{p} ind(x_i,K)+1}.$$

Thus, if $1 \neq \sum_{x \in \mathcal{F}^\infty} (-1)^{\sum_{i=1}^{p} ind(x_i,K)+1}$, then J must have a critical point in U. Hence, (1.1) must have a solution in this case, which proves Theorem 2.

□

A. Conformal Fermi-Coordinates

A.1. Definition and Existence

Let (M, g) be a n-dimensional manifold with boundary. First we need to define Fermi-coordinates around a point $a \in \partial M$ with respect to g. Therefore we choose an orthonormal basis $(e_1, \cdots, e_{n-1})$ of $T_a\partial M$ and define for $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$ close to $(0,0)$ $\psi_a^{-1}(x,t) := \gamma_x(t)$, where $\gamma_x(t)$ is the unique geodesic such that

$$\gamma_x(0) = \exp_a\left(\sum_{i=1}^{n-1} x_i e_i\right) \quad \text{and} \quad \dot{\gamma_x}(0) = -\nu\left(\exp_a\left(\sum_{i=1}^{n-1} x_i e_i\right)\right),$$

where ν is the unit outer normal vector field on ∂M and $\exp_a$ is the exponential map of ∂M at a. The map $\psi_a : U \to \mathbb{R}^{n-1} \times \mathbb{R}_+ = \mathbb{R}^n_+$, defined on a neighbourhood of $a \in M$, is called a chart in Fermi-coordinates around a and ψ_a^{-1} is its parametrization.
For the previous definition we chose an orthonormal basis of $T_a\partial M$, hence
Fermi-coordinates at a point $a \in \partial M$ are unique up to $O\big(n-1\big)-$ transformations. More precisely, two different Fermi-coordinate parametrisations ψ_a^{-1} and $\overline{\psi}_a^{-1}$ are related by $\psi_a^{-1}(x,t) = \overline{\psi}_a^{-1}(Ax,t)$, where $A \in O\big(n-1\big)$.
The existence of Fermi coordinates will be proved in Proposition 22 below. First we would like to present the following useful expansion in Fermi-coordinates (see [28]):

Let ψ_a^{-1} be Fermi-coordinates around $a \in \partial M$ and $g^{ij}(x,t)$ the coefficients of the inverse metric in this coordinates then $g^{nn}(x,t) = 1$, $g^{in}(x,t) = 0$ and

$$g^{ij}(x,t) = \delta_{ij} + 2h_{ij}t + \frac{1}{3}R_{ikjl}x_k x_l + 2h_{ij,k}tx_k + (\bar{R}_{ninj} + 3h_{ik}h_{kj})t^2 + O\left(|(x,t)|^3\right) \tag{A.1}$$

for all $1 \leq i,j \leq n-1$. Here $h_{ij}, R_{ikjl}, \bar{R}_{ninj}$ are the coefficients of the second fundamental form and the Riemannian curvature tensors of ∂M and M at the point a.

Let $u : \partial M \times M \to (0,\infty)$ be a smooth function. Then $g_a(x) := u(a,x)^{\frac{4}{n-2}}g(x)$ is smooth family of conformal metrics on M. We call $\psi_a : U \to \mathbb{R}^n_+$ a conformal Fermi-coordinate chart at $a \in \partial M$ if ψ_a is a Fermi-coordinate chart at $a \in \partial M$ with respect to the metric

g_a. In the following we prove the existence of conformal Fermi-coordinates as well as some properties. Therefore we need to introduce some more notations. For $\rho > 0$, define

$$B_\rho^+ := \left\{(x,t) \in \mathbb{R}_+^n \ : \ |(x,t)| < \rho\right\}, \quad B_\rho := B_\rho(0) := \{x \in \mathbb{R}^{n-1} \ : \ |x| < \rho\}.$$

Proposition 22. *Let $u : \partial M \times M \to (0,\infty)$ be a smooth function and $g_a = u_a^{\frac{4}{n-2}} g$ be a smooth family of metrics on M then:*

(a) Conformal Fermi-coordinates exist.

(b) There exists $\rho_0 > 0$, independent of $a \in \partial M$, such that all parametrisations in conformal Fermi-coordinates ψ_a^{-1} are defined on $B_{\rho_0}^+$. Furthermore they are diffeomorphisms onto its image in M.

Moreover let $g_{ij} : B_{\rho_0}^+ \to \mathbb{R}$ be the coefficients of the metric g in arbitrary conformal Fermi-coordinates around a, then for $N \in \mathbb{N}$ exists $C = C(N) > 0$, independent of a, such that $|\partial^\alpha g_{ij}(x,t)| \le C$ for all $1 \le i,j \le n$, $|\alpha| \le N$ and $(x,t) \in B_{\rho_0/2}^+$.

Proof. (a) Let $\varphi : U \to B_{4r}^+$ be a chart of M around $a \in \partial M$ with $\varphi(a) = 0$. For $y, x \in B_{4r}$ let $\Gamma_{ij}^k(y,x)$ denote the Christoffel symbols on ∂M with respect to $g_{\varphi^{-1}(y,0)}$ at the point $\varphi^{-1}(x,0) \in \partial M$. They depend smoothly on y and x. Locally around 0 let $(e_1(y), \cdots, e_{n-1}(y))$ be a smooth family of vector fields which are an orthonormal frame at $\varphi^{-1}(y,0)$ w.r.t. $g_{\varphi^{-1}(y,0)}$. For $y \in B_{4r}$ and $v \in \mathbb{R}^{n-1}$ we define the geodesic initial value problem on B_{4r}:

$$\begin{cases} \ddot{\gamma}^k(t) + \Gamma_{ij}^k(y,\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t) = 0 \\ \gamma(0) = y, \quad \dot{\gamma}(0) = \sum_{i=1}^{n-1} v_i d\varphi[e_i(y)]. \end{cases}$$

We denote by $\gamma(y,v,t)$ the maximal solution of the previous initial value problem. Since the coefficients are smooth it exists $t_0 > 0$ such that $\gamma(y,v,\cdot)$ is defined on $[-t_0,t_0]$ for all $y \in B_{3r}$ and $v \in B_1$. Since $\gamma(y,v,t) = \gamma(y,tv,1)$,

$$\overline{\gamma} : B_{3r} \times B_{t_0} \to B_{4r}, \ \overline{\gamma}(y,v) := \gamma(y,v,1)$$

is well defined and smooth. For $(y,v) \in B_{3r} \times B_{t_0}$ the map $c(t) = \varphi^{-1}(\gamma(y,v,t),0)$ is a geodesic on ∂M w.r.t. $g_{\varphi^{-1}(y,0)}$ and initial conditions $c(0) = \varphi^{-1}(y,0)$ and $\dot{c}(0) = \sum_{i=1}^{n-1} e_i(y)v_i$. Hence for $y \in B_{3r}$ the maps

$$B_{t_0} \ni v \mapsto \varphi^{-1}\left(\overline{\gamma}(y,v),0\right)$$

are normal coordinates on ∂M around $\varphi^{-1}(y,0)$ w.r.t. $g_{\varphi^{-1}(y,0)}$ if they are diffeomorphisms. This will be justified in the following. Therefore set

$$\Phi : B_{3r} \times B_{t_0} \to B_{4r} \times \mathbb{R}^{n-1}, \ \Phi(y,v) = (\gamma(y,v,1), \dot{\gamma}(y,v,1)).$$

Since

$$D\Phi(0,0) = \begin{pmatrix} id & id \\ 0 & id \end{pmatrix},$$

Φ is a local diffeomorphism around $(0,0)$ which proves that the map

$$B_{r_0} \ni v \mapsto \varphi^{-1}\left(\overline{\gamma}(y,v),0\right) \in \partial M$$

is a diffeomorphism for all $y \in B_{r_0}$ as long as r_0 is small enough.

Now define $\overline{\Gamma}_{ij}^k(y,x,t)$ to be the Christoffel symbols of $g_{\varphi^{-1}(y,0)}$ on M at the point $\varphi^{-1}(x,t)$. For $(y,v) \in B_{r_0} \times B_{r_0}$ we define the geodesic initial value problem on B_r^+

$$\begin{cases} \ddot{c}^k(t) + \overline{\Gamma}_{ij}^k(y,c(t))\dot{c}^i(t)\dot{c}^j(t) = 0 \\ c(0) = \overline{\gamma}(y,v), \quad \dot{c}(0) = -d\varphi\left(\nu\left(\overline{\gamma}(y,v)\right)\right), \end{cases}$$

where $\nu(x)$ is the outer normal vector at $\varphi^{-1}(x,0)$. Let $c(y,v,t)$ be the maximal solution. The theory of ordinary differential equations implies that $c : B_{r_1} \times B_{r_1} \times [0,\varepsilon) \to B_{4r}^+$ is well defined and smooth for ε and $r_1 < r_0$ small . A similar argument as above shows that we can find $r_2 > 0$ small, such that the map

$$B_{r_2}^+ \ni (v,t) \mapsto c(y,v,t) \in B_{4r}^+$$

is a diffeomorphism onto its image for all $y \in B_{r_2}$. Now, for $b \in \partial M$ close to a, we set

$$\psi_b^{-1}(v,t) := \varphi^{-1}\left(c(y,v,t)\right) \quad \text{where} \quad (y,0) = \varphi(b).$$

Then ψ_b^{-1} are conformal Fermi-coordinates by construction, which proves the existence.

(b) follows immediately from the proof of (a) since we have already constructed conformal Fermi-coordinates, which are defined on $B_{r_2}^+$ for all $b \in \partial M$ close to $a \in \partial M$.

Since ∂M is compact it suffices to find bounds locally around a point $a \in \partial M$. But this is guaranteed for the special construction in (a). Finally, different Fermi-coordinates around a special point differ by an action of the compact group $O(n-1)$ which proves global bounds for all conformal Fermi-coordinates at ∂M w.r.t. g_a. □

We state a very important result which is due to [27]. A familiar result for Riemannian normal coordinates was proved by [24].

Proposition 23. *For $3 \leq N \in \mathbb{N}$ there exists a smooth positive function $u : \partial M \times M \to \mathbb{R}$ and a conformal family of metrics $g_a = u(a,\cdot)^{\frac{4}{n-2}} g$ such that in conformal Fermi-coordinates*

$$\left(\psi_a^{-1}\right)^* dV_{g_a}(x,t) = \left(1 + O\left(|(x,t)|^N\right)\right) dxdt, \tag{A.2}$$

where dV_{g_a} is the volume form on M w.r.t. g_a and $dxdt$ is the standard volume form on $\mathbb{R}^n_+$.
Furthermore it holds

$$h(a) = \nabla h(a) = Ric(a) = 0, \quad R_{g_a}(a) = -|\Pi(a)|^2.$$

Here h is the mean curvature, Ric the Ricci-curvature of ∂M, R_{g_a} the scalar curvature and $\Pi_{ij} = h_{ij} - hg_{ij}$ the umbilicity tensor with respect to g_a.

From now on we use this function $u_a = u(a, \cdot)$ stated in Proposition 23 for N large. We choose ρ_0 as in Proposition 22 such that all conformal Fermi-coordinates are defined at least on $B^+_{8\rho_0}$. Furthermore let $\chi \in C^\infty(\mathbb{R})$ such that $\chi = 1$ if $t \leq 3\rho_0$ and $\chi = 0$ if $t \geq 4\rho_0$. Set

$$g : \partial M \times M \to \mathbb{R}, \quad g(a,x) = \chi\left(|\psi_a(x)|\right)|\psi_a(x)|^2,$$

where ψ_a are conformal Fermi-coordinates around a. First observe that g does not depend on the choice of Fermi-coordinates. We would like to prove that g is smooth. Therefore let $(a_0, x_0) \in \partial M \times M$. The only not trivial case is $|\psi_{a_0}(x_0)| \leq 4\rho_0$, which we assume from now on. Then $|\psi_a(x_0)| < 5\rho_0$ for a close to a_0. Let $c(y,x,t)$ be the solution of the initial value problem from Proposition 22 in ψ_{a_0}-coordinates. Hence the chart φ is replaced by ψ_{a_0}. Set $\Phi(y,x,t) = \psi_{a_0}^{-1}(c(y,x,t))$, then $\Phi(0,x,t) = \psi_{a_0}^{-1}(x,t)$. Furthermore define

$$F : B_r \times B_{6\rho_0} \times B_{8\rho_0} \to \mathbb{R}^n, \quad F(y,x,t,z) = c(y,x,t) - z. \tag{A.3}$$

Let (x_0, t_0, z_0) be given such that $F(0, x_0, t_0, z_0) = 0$. Since $(x,t) \mapsto \Phi(0,x,t)$ is a diffeomorphism, also $(x,t) \mapsto c(0,x,t)$ is a diffeomorphism and hence $D_2F(0, x_0, t_0, z_0)$ an isomorphism. Locally around $(0, x_0, t_0, z_0)$ the implicit function theorem yields smooth functions $x(y,z), t(y,z)$ such that $F(y, x(y,z), t(y,z), z) = 0$. For a chart ψ_a we set $t_a(x) = \psi_a^n(x)$ and $\overline{\psi}_a(x) = (\psi_a^1(x), \cdots, \psi_a^{n-1}(x))$. Then $t_a(x)$ and $|\overline{\psi}_a(x)|^2$ are independent of the choice of Fermi-coordinates w.r.t. g_a. With the notations from above, we see

$$t_{\Phi(y,0)}\left(\psi_{a_0}^{-1}(z)\right) = t(y,z), \quad |\overline{\psi}_{\Phi(y,0)}\left(\psi_{a_0}^{-1}(z)\right)|^2 = |x(y,z)|^2$$

which are smooth functions locally around $(0, \psi_{a_0}(x_0))$. This finally proves that g is smooth.

A.2. General properties

As above we set $\psi_a(x) = (\bar{\psi}_a(x), t_a(x))$ where $\overline{\psi}_a(x) = (\psi_a^1(x), \cdots, \psi_a^{n-1}(x))$.

Lemma 10. *For $a_0 \in \partial M$ choose conformal Fermi-coordinates around a_0. In these coordinates we obtain*

(a) $\frac{\partial}{\partial a_i|a_0} \left|\overline{\psi}_a(x)\right|^2 = -2\psi_{a_0}(x)^i + O(d_{g_a}(x,a)^3)$,

(b) $\frac{\partial^2}{\partial a_j \partial a_i|a_0} \left|\overline{\psi}_a(x)\right|^2 = -2\delta_{ij} + O(d_{g_a}(x,a)^2)$,

(c) $\frac{\partial}{\partial a_i|a_0} t_a(x) = O(d_{g_a}(x,a)^2)$,

(d) $\frac{\partial^2}{\partial a_j \partial a_i|a_0} t_a(x) = O(d_{g_a}(x,a))$.

Proof. To prove this Lemma we need the Function F in (A.3) and some of its derivatives. $F(y,x,t,z) = c(y,x,t) - z$, where $c(y,x,t)$ is the solution of the initial value problem in the proof of Proposition 22. First we compute the first derivatives at zero.
Claim:

1. $D_1F(0,0,0,0)[h] = (h,0)$ for $h \in \mathbb{R}^{n-1}$,

2. $D_2F(0,0,0,0)[h] = (h,0)$ for $h \in \mathbb{R}^{n-1}$,

3. $\frac{d}{dt|t=0} F(0,0,t,0) = e_n$,

4. $D_4F(0,0,0,0)[h] = -h$ for $h \in \mathbb{R}^n$.

Proof of the claim. $F(y,0,0,0) = (y,0)$ which proves 1. Furthermore $F(0,th,0,0) = (\gamma(0,th,1),0) = (\gamma(0,h,t),0)$ which proves 2. The equations in 3. and 4. are obvious from the definition of F. □

We also need the second derivatives of F. Since $F(y,0,0,0) = y$, it follows that $D_1^2F(0,0,0,0) = 0$. Observe that $F(0, se_i + re_j, 0, 0)$ is $\gamma(s,r,1)$, where $\gamma(s,r,\cdot)$ is the solution of

$$\begin{cases} \ddot{\gamma}^k(s,r,t) + \Gamma_{ij}^k(\gamma(s,r,t))\dot{\gamma}^i(s,r,t)\dot{\gamma}^j(s,r,t) = 0 \\ \gamma(s,r,0) = y, \quad \dot{\gamma}(s,r,0) = se_i + re_j \end{cases}.$$

Set $\beta(t) = \frac{\partial^2}{\partial r \partial s}\gamma(0,0,t)$, then β solves the initial value problem

$$\ddot{\beta}(t) = 0; \quad \beta(0) = 0, \ \dot{\beta}(0) = 0$$

since $\gamma(0,0,t) = 0$ and $\Gamma_{ij}^k(0) = 0$ in Fermi-coordinates. Hence $\beta = 0$ which proves $D_2^2F(0,0,0,0) = 0$.
Now we compute the second derivative w.r.t. t. We have $F(0,0,t,0) = c(0,0,t)$. From the initial value problem, which is solved by c, we infer $\ddot{c}^k(0) = -\overline{\Gamma}_{nn}^k(0) = 0$. This implies $\frac{d^2}{dt^2|t=0} F(0,0,t,0) = 0$.

In the following we show that also the mixed second derivatives are zero. We begin with $D_1 D_2 F(0,0,0,0)$.

$$D_2 F(ty,0,0,0)[x] = \frac{d}{dt|t=0} \sum_{i=1}^{n-1} e_i(ty) x_i,$$

where $(e_1(y), \cdots, e_{n-1}(y))$ is a local g-orthonormal frame. Since Fermi coordinates are normal coordinates at the boundary, the metric vanishes up to second order and therefore $D_1 D_2 F(0,0,0,0) = 0$. Now $F(y,0,t,0) = c(y,0,t)$, where $c(y,0,\cdot)$ solves the ODE

$$\begin{cases} \ddot{c}^k(y,0,t) + \overline{\Gamma}^k_{ij}(y, c(y,0,t)) \dot{c}^i(y,0,t) \dot{c}^j(y,0,t) = 0 \\ c(y,0,0) = y, \quad \dot{c}(y,0,0) = e_n. \end{cases}$$

Thus $D_1 D_3 F(0,0,0,0) = 0$. With the same arguments as above we get
$D_3 F(0,x,0,0) = e_n$. Hence $D_2 D_3 F(0,0,0,0) = 0$. Altogether we have proved that the second derivative of F vanishes at $(0,0,0,0)$.
Taking the derivative of $0 = F(y, x(y,z), t(y,t), z)$ at $(0,0,0,0)$ yields

$$\begin{aligned} D_y x(0,0)[e_i] = -e_i,\ D_y t(0,0) = 0,\ D_z x(0,0)[e_i] = e_i,\ D_z t(0,0)[e_i] = 0 \text{ if } i < n; \\ D_z x(0,0)[e_n] = 0,\ D_z t(0,0)[e_n] = 1. \end{aligned} \tag{A.4}$$

Furthermore, taking the second derivatives at $(0,0,0,0)$ gives

$$\begin{aligned} D_y^2 x(0,0) = 0, D_z^2 x(0,0) = 0, D_y^2 t(0,0) = 0, D_z^2 t(0,0) = 0, \\ D_y D_z x(0,0) = 0,\ D_y D_z t(0,0). \end{aligned} \tag{A.5}$$

We use (A.4), (A.5) and Taylor expansion to derive

$$\begin{aligned} \frac{\partial}{\partial a_i|a_0} \left|\overline{\psi}_a \left(\psi_{a_0}(z)\right)\right|^2 &= \frac{\partial}{\partial y_i|0} \left|\overline{\psi}_{\Phi(y,0)} \left(\psi_{a_0}^{-1}(z)\right)\right|^2 = \frac{\partial}{\partial y_i|0} |x(y,z)|^2, \\ &= -2z_i + O(|z|^3) \end{aligned}$$

$$\frac{\partial^2}{\partial a_j \partial a_i|a_0} \left|\overline{\psi}_a \left(\psi_{a_0}(z)\right)\right|^2 = -2\delta_{ij} + O(|z|^2),$$

$$\frac{\partial}{\partial a_i|a_0} t_a\big(\psi_{a_0}^{-1}(z)\big) = \frac{\partial}{\partial y_i|0} t(y,z) = O(|z|^2)$$

as well as

$$\frac{\partial^2}{\partial a_j \partial a_i|a_0} t_a\big(\psi_{a_0}^{-1}(z)\big) = O(|z|).$$

Therefore the Lemma is proved. □

B. Interaction Estimates

In this chapter we expand the scalar product of two different bubbles under the assumption

$$(A) \qquad \varepsilon_{ij} := \left(\frac{1}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j d_g(a_i, a_j)^2} \right) \to 0 \text{ and } \lambda_i, \lambda_j \to \infty.$$

Proposition 24. *Under (A) it holds*

$$\int_{\partial M} \left(u_{a_i} \hat{\delta}_{a_i,\lambda_i} \right)^2 u_{a_j} \hat{\delta}_{a_j,\lambda_j} d\sigma_g = I_1 \chi_\rho \left(|\psi_{a_i}(a_j)| \right) u_{a_j}(a_i) \left(\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j d_{g_{a_j}}(a_i, a_j)^2} \right) + o(\varepsilon_{ij})$$

provided $\lambda_i \geq \lambda_j$. Here

$$I_1 = \int_{\mathbb{R}^3} \left(\frac{1}{1 + |x|^2} \right)^2 dx.$$

Proof. From now on we assume $\lambda_i \geq \lambda_j$. We would like to expand the integral

$$\int_{\partial M} \left(u_{a_i} \hat{\delta}_{a_i,\lambda_i} \right)^2 u_{a_j} \hat{\delta}_{a_j,\lambda_j} d\sigma_g = \int_{\partial M} \frac{u_{a_j}}{u_{a_i}} \hat{\delta}^2_{a_i,\lambda_i} \hat{\delta}_{a_j,\lambda_j} d\sigma_{g_{a_i}}. \tag{B.1}$$

Therefore, we need an expansion of $\hat{\delta}_{a_j,\lambda_j}$ in conformal Fermi-coordinates at a_i. Since $(a, x) \mapsto |\psi_a(x)|^2$ is smooth for $d(a, x) < 8\rho_0$, we get the following expansion if $|z| \leq 4\rho_0$:

$$|\psi_{a_j}(\psi_{a_i}^{-1}(z))|^2 = d_{g_{a_j}}(a_j, a_i)^2 + \sum_{i=1}^{3} \frac{\partial}{\partial z_i |0} |\psi_{a_j}(\psi_{a_i}^{-1}(z))|^2 z_i + O(|z|^2), \tag{B.2}$$

in which

$$|\nabla |\psi_{a_j}(\psi_{a_i}^{-1}(0))||^2 = |\nabla_{a_i} |\psi_{a_j}(a_i)|^2| \leq C d_{g_{a_j}}(a_j, a_i).$$

Under the assumption $|x| \leq \varepsilon \frac{\lambda_i}{\lambda_j}$ or $|x| \leq \varepsilon \sqrt{\lambda_i \lambda_j d_{a_j}(a_i, a_j)}$ the following expansion is due to (B.2):

$$\begin{aligned}\frac{1}{\lambda_i}\hat{\delta}_{a_j,\lambda_j}(\psi_{a_i}^{-1}(\frac{x}{\lambda_i})) &= \chi_\rho\left(|\psi_{a_j}(\psi_{a_i}^{-1}(\frac{x}{\lambda_i}))|\right)\left(\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_j\lambda_i|\psi_{a_j}(\psi_{a_i}^{-1}(\frac{x}{\lambda_i}))|^2}\right)\\ &= \chi_\rho(d_{g_{a_j}}(a_j,a_i))\left(\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_j\lambda_i d_{g_{a_j}}(a_j,a_i)^2}\right)\left(1-\frac{\lambda_j\nabla|\psi_{a_j}(\psi_{a_i}^{-1}(0))|^2\cdot x}{\frac{\lambda_i}{\lambda_j}+\lambda_j\lambda_i d_{g_{a_j}}(a_j,a_i)^2}\right)\\ &+\frac{\frac{\lambda_j}{\lambda_i}O(|x|^2)}{\left(\frac{\lambda_i}{\lambda_j}+\lambda_j\lambda_i d_{g_{a_j}}(a_j,a_i)^2\right)^2}. \end{aligned} \tag{B.3}$$

We now expand (B.1) under the assumption $\varepsilon_{ij}\to 0$. Since $\lambda_i\geq\lambda_j$ either

$$\lambda_i\lambda_j d(a_i,a_j)^2\leq\frac{\lambda_i}{\lambda_j}\to\infty \quad\text{or}\quad \frac{\lambda_i}{\lambda_j}\leq\lambda_i\lambda_j d(a_i,a_j)^2\to\infty.$$

We first assume $\lambda_i\lambda_j d(a_i,a_j)^2\leq\frac{\lambda_i}{\lambda_j}$. In this case a first expansion yields

$$\begin{aligned}&\int_{B_{\frac{\varepsilon}{\lambda_j}}(a_i)}\frac{u_{a_j}}{u_{a_i}}\hat{\delta}^2_{a_i,\lambda_i}\hat{\delta}_{a_j,\lambda_j}d\sigma_{g_{a_i}}\\ &=\int_{|x|\leq\varepsilon\frac{\lambda_i}{\lambda_j}}\left(\frac{1}{1+|x|^2}\right)^2\left(u_{a_j}(a_i)+O(\frac{|x|}{\lambda_i})\right)\frac{1}{\lambda_i}\hat{\delta}_{a_j,\lambda_j}(\psi_{a_i}^{-1}(\frac{x}{\lambda_i}))(1+O(\frac{|x|^{10}}{\lambda_i^{10}}))dx\\ &=I_1u_{a_j}(a_i)\chi_\rho(d_{g_{a_j}}(a_j,a_i))\left(\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_j\lambda_i d_{g_{a_j}}(a_j,a_i)^2}\right)+o(\varepsilon_{ij}). \end{aligned}\tag{B.4}$$

Since $\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j d_{g_{a_j}}(x,a_j)^2\geq\frac{1}{2}\left(\frac{\lambda_i}{\lambda_j}+\lambda_i\lambda_j d_{g_{a_j}}(a_i,a_j)^2\right)$ in this case, we have the following estimate:

$$\int_{B_{2\rho}(a_i)\setminus B_{\frac{\varepsilon}{\lambda_j}}(a_i)}\frac{u_{a_j}}{u_{a_i}}\hat{\delta}^2_{a_i,\lambda_i}\hat{\delta}_{a_j,\lambda_j}d\sigma_{g_{a_i}}\leq C\varepsilon_{ij}\int_{|x|\geq\varepsilon\frac{\lambda_i}{\lambda_j}}\left(\frac{1}{1+|x|^2}\right)^2dx=o(\varepsilon_{ij}). \tag{B.5}$$

Hence form (B.4) and (B.5) we derive the expansion

$$\begin{aligned}\int_{\partial M}\left(u_{a_i}\hat{\delta}_{a_i,\lambda_i}\right)^2u_{a_j}\hat{\delta}_{a_j,\lambda_j}d\sigma_g=I_1u_{a_j}(a_i)\chi_\rho(d_{g_{a_j}}(a_j,a_i))\left(\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_j\lambda_i d_{g_{a_j}}(a_j,a_i)^2}\right)\\ +o(\varepsilon_{ij}).\end{aligned}$$

From now on we assume $\frac{\lambda_i}{\lambda_j} \leq \lambda_i\lambda_j d(a_i,a_j)^2 \to \infty$. We first expand

$$
\begin{aligned}
&\int_{B_{\varepsilon\frac{\sqrt{\lambda_j}}{\sqrt{\lambda_j}}d(a_i,a_j)}(a_i)} \frac{u_{a_j}}{u_{a_i}}\hat{\delta}^2_{a_i,\lambda_i}\hat{\delta}_{a_j,\lambda_j}d\sigma_{g_{a_i}} \\
&= \int_{|x|\leq\varepsilon\sqrt{\lambda_i\lambda_j}d(a_i,a_j)} \left(\frac{1}{1+|x|^2}\right)^2 \left(u_{a_j}(a_i) + O(\frac{|x|}{\lambda_i})\right)\frac{1}{\lambda_i}\delta_{a_j,\lambda_j}(\psi^{-1}_{a_i}(\frac{x}{\lambda_i}))dx + o(\varepsilon_{ij}) \\
&= I_1 u_{a_j}(a_i)\chi_\rho(d_{g_{a_j}}(a_j,a_i))\left(\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_j\lambda_i d_{g_{a_j}}(a_j,a_i)^2}\right) + o(\varepsilon_{ij}), \qquad \text{(B.6)}
\end{aligned}
$$

where we used (B.3) again. Set $B := B_{\frac{1}{10}}(a_j)$. Then it holds $d(x,a_i) \geq \frac{9}{10}d(a_i,a_j)$ on B, which implies

$$
\begin{aligned}
\int_B \frac{u_{a_j}}{u_{a_i}}\hat{\delta}^2_{a_i,\lambda_i}\hat{\delta}_{a_j,\lambda_j}d\sigma_{g_{a_i}} \leq& C\left(\frac{1}{1+\lambda_i^2 d(a_i,a_j)^2}\right)^2\left(\frac{\lambda_j}{\lambda_i}\right)\int_{|x|\leq\lambda_i d(a_i,a_j)}\left(\frac{1}{1+\frac{\lambda_j^2}{\lambda_i^2}|x|^2}\right)dx \\
\leq& C\frac{\varepsilon_{ij}}{\lambda_i d(a_i,a_j)} = o(\varepsilon_{ij}).
\end{aligned}
$$

Finally on $C = (B \cup B_{\varepsilon\frac{\sqrt{\lambda_j}}{\sqrt{\lambda_j}}d(a_i,a_j)}(a_i))^c$ we have the estimate

$$
\begin{aligned}
&\int_C \frac{u_{a_j}}{u_{a_i}}\hat{\delta}^2_{a_i,\lambda_i}\hat{\delta}_{a_j,\lambda_j}d\sigma_{g_{a_i}} \\
&\leq C\left(\frac{1}{\frac{\lambda_i}{\lambda_j}+\lambda_j\lambda_i d_{g_{a_j}}(a_j,a_i)^2}\right)\int_{|x|\geq\varepsilon\sqrt{\lambda_i\lambda_j}d(a_i,a_j)}\left(\frac{1}{1+|x|^2}\right)^2 dx = o(\varepsilon_{ij}).
\end{aligned}
$$

Hence the proof is completed. □

With the help of Proposition 24 we now proof an expansion of the interaction between two different bubbles.

Proposition 25. *If* $2 \leq \lambda_i\rho, \lambda_j\rho$ *and* $\lambda_j \leq \lambda_i$, *then*

$$
\begin{aligned}
< \varphi_{a_i,\lambda_i}, \varphi_{a_j,\lambda_j} > =& I_1 u_{a_j}(a_i)\chi_\rho(|\psi_{a_j}(a_i)|)\left(\frac{1}{\frac{\lambda_i}{\lambda_j}+\frac{\lambda_j}{\lambda_i}+\lambda_j\lambda_i d_{g_{a_j}}(a_j,a_i)^2}\right) + o(\varepsilon_{ij}) \\
&+ I_1\left((1-\chi_\rho(|\psi_{a_j}(a_i)|))\frac{G(a_j,a_i)}{\lambda_j\lambda_i}\right) + O\left(\frac{1}{\lambda_i^2\lambda_j\rho^3}\right) + O(\rho\varepsilon_{ij}).
\end{aligned}
$$

Proof.

$$< \varphi_{a_i,\lambda_i}, \varphi_{a_j,\lambda_j} > = \int_M L_g \varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j} + \int_{\partial M} B_g \varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j} d\sigma_g$$

$$\int_{\partial M} B_g \varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j} d\sigma_g = \int_{\partial M} \frac{u_{a_j}}{u_{a_i}} B_{g_{a_i}} \hat{\varphi}_{a_i,\lambda_i}\hat{\varphi}_{a_j,\lambda_j} d\sigma_{g_{a_i}}$$
$$= \int_{\partial M} \frac{u_{a_j}}{u_{a_i}} \Big(2\chi_\rho \hat{\delta}^2_{a_i,\lambda_i} + h_{g_{a_i}} \chi_\rho \hat{\delta}_{a_i,\lambda_i} \Big) \Big(\chi_\rho \hat{\delta}_{a_j,\lambda_j} + (1-\chi_\rho) \frac{G_{a_j}(a_j,\cdot)}{\lambda_j} \Big) d\sigma_{g_{a_i}}. \tag{B.7}$$

Due to Proposition 24 we have

$$\int_{\partial M} \frac{u_{a_j}}{u_{a_i}} \chi_\rho \hat{\delta}^2_{a_i,\lambda_i} \Big(\chi_\rho \hat{\delta}_{a_j,\lambda_j} + (1-\chi_\rho) \frac{G_{a_j}(a_j,\cdot)}{\lambda_j} \Big) d\sigma_{g_{a_i}}$$
$$= I_1 u_{a_j}(a_i) \chi_\rho(|\psi_{a_j}(a_i)|) \left(\frac{1}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_j \lambda_i d_{g_{a_j}}(a_j,a_i)^2} \right) + o(\varepsilon_{ij})$$
$$+ \int_{\partial M} \frac{1}{u_{a_i}} \chi_\rho \hat{\delta}^2_{a_i,\lambda_i} \big((1-\chi_\rho) \frac{G(a_j,\cdot)}{\lambda_j} \big) d\sigma_{g_{a_i}}.$$

Furthermore a Taylor expansion yields

$$\frac{1}{u_{a_i}} (\psi^{-1}_{a_i}(z)) \Big((1-\chi_\rho) \frac{G(a_j, \psi^{-1}_{a_i}(z))}{\lambda_j} \Big)$$
$$= \Big((1-\chi_\rho(|\psi_{a_j}(a_i)|)) \frac{G(a_j,a_i)}{\lambda_j} \Big) + IN + O\left(\frac{|z|^2}{\rho^4 \lambda_j} \right),$$

where IN is a linear term (in z). The O-term does neither depend on ρ nor on a_j. Therefore

$$\int_{\partial M} \frac{1}{u_{a_i}} \chi_\rho \hat{\delta}^2_{a_i,\lambda_i} \big((1-\chi_\rho) \frac{G(a_j,\cdot)}{\lambda_j} \big) d\sigma_{g_{a_i}}$$
$$= \int_{B_\rho(a_i)} \frac{1}{u_{a_i}} \hat{\delta}^2_{a_i,\lambda_i} \big((1-\chi_\rho) \frac{G(a_j,\cdot)}{\lambda_j} \big) dx + O\left(\frac{1}{\lambda_i^2 \lambda_j \rho^3} \right)$$
$$= \int_{B_{\lambda_i \rho}} \left(\frac{1}{1+|x|^2} \right)^2 \frac{1}{u_{a_i}} (\psi^{-1}_{a_i}(\frac{x}{\lambda_i})) \Big((1-\chi_\rho) \frac{G(a_j, \psi^{-1}_{a_i}(\frac{x}{\lambda_i}))}{\lambda_j \lambda_i} \Big) dx + O\left(\frac{1}{\lambda_i^2 \lambda_j \rho^3} \right)$$
$$= I_1 \Big((1-\chi_\rho(|\psi_{a_j}(a_i)|)) \frac{G(a_j,a_i)}{\lambda_j \lambda_i} \Big) + O\left(\frac{1}{\lambda_i^2 \lambda_j \rho^3} \right). \tag{B.8}$$

Hence

$$\int_{\partial M} \frac{u_{a_j}}{u_{a_i}} \chi_\rho \hat{\delta}^2_{a_i,\lambda_i} \Big(\chi_\rho \hat{\delta}_{a_j,\lambda_j} + (1-\chi_\rho) \frac{G_{a_j}(a_j,\cdot)}{\lambda_j} \Big) d\sigma_{g_{a_i}}$$
$$= I_1 u_{a_j}(a_i) \chi_\rho(|\psi_{a_j}(a_i)|) \left(\frac{1}{\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_j \lambda_i d_{g_{a_j}}(a_j,a_i)^2} \right) + o(\varepsilon_{ij})$$
$$+ I_1 \Big((1-\chi_\rho(|\psi_{a_j}(a_i)|)) \frac{G(a_j,a_i)}{\lambda_j \lambda_i} \Big) + O\left(\frac{1}{\lambda_i^2 \lambda_j \rho^3} \right).$$

Moreover observe that

$$|\hat{\varphi}_{a_j,\lambda_j}(x)| \leq C\left(\frac{\lambda_j}{1+\lambda_j^2 d_{g_{a_j}}(a_j,x)^2}\right)$$

if $2 \leq \rho\lambda_j$. Since

$$cd_{g_a}(x,y) \leq d_g(x,y) \leq Cd_{g_a}(x,y)$$

for $a \in \partial M, x, y \in M$ and two universal constants c, C we have

$$|\hat{\varphi}_{a_j,\lambda_j}(x)| \leq C\left(\frac{\lambda_j}{1+\lambda_j^2 d_{g_{a_i}}(a_j,x)^2}\right).$$

This estimate implies

$$\begin{aligned}&\left|\int_{\partial M}\frac{u_{a_j}}{u_{a_i}}h_{g_{a_i}}\chi_\rho\hat{\delta}_{a_i,\lambda_i}\hat{\varphi}_{a_j,\lambda_j}d\sigma_{g_{a_i}}\right|\\ &\qquad\leq C\int_{B_{2\rho}(a_i)}\left(\frac{\lambda_i}{1+\lambda_i^2 d_{g_{a_i}}(a_j,x)^2}\right)\left(\frac{\lambda_j}{1+\lambda_j^2 d_{g_{a_i}}(a_j,x)^2}\right)d\sigma_{g_{a_i}} \leq C\rho\varepsilon_{ij},\end{aligned}$$

provided $2 \leq \rho\lambda_j$. Here we integrated over the sets $B_{2\rho} \cap A$ and $B_{2\rho} \cap A^c$, where

$$A := \left\{x \in \partial M \mid 2d_{a_i}(a_j,x) \leq \frac{1}{\lambda_i} + d_{g_{a_i}}(a_i,a_j)\right\}.$$

Finally we have to estimate the integral in M. From Proposition 2 we derive the estimate

$$|L_{g_{a_i}}\hat{\varphi}_{a_i,\lambda_i}| \leq C\left(\frac{\lambda_i^2}{(1+\lambda_i d_{g_{a_i}}(a_i,x))^3} + \frac{1}{\rho}\left(\frac{\lambda_i}{1+\lambda_i^2 d_{g_{a_i}}(a_i,x)^2}\right)\right)1_{d_{g_{a_i}}(a_i,x)\leq 2\rho}.$$

Set

$$A := \left\{x \in M \mid 2d_{a_i}(a_j,x) \leq \frac{1}{\lambda_i} + d_{g_{a_i}}(a_i,a_j)\right\}.$$

As above, integration over $B_{2\rho} \cap A$ and $B_{2\rho} \cap A^c$ yields

$$\left|\int_M L_g\varphi_{a_i,\lambda_i}\varphi_{a_j,\lambda_j}dV_g\right| \leq C\rho\varepsilon_{ij}.$$

We add the previous expansions and estimates which proves the Proposition. □

Remark 4. *Since $cd_g(a,x) \leq d_{g_a}(a,x) \leq Cd_g(a,x)$ uniformly for $a, x \in \partial M$, we easily derive the estimate*

$$| < \varphi_{a_i,\lambda_i}, \varphi_{a_j,\lambda_j} > | \leq C\varepsilon_{ij}$$

from the previous Proposition.

C. Interaction with the derivatives

C.1. Selfinteractions

In this section we expand the scalar product of a bubble with its derivatives w.r.t. λ and a.

Proposition 26. *It holds*

(a)

$$< \varphi_{a,\lambda}, \lambda \frac{\partial}{\partial \lambda} \varphi_{a,\lambda} >= 2|S^3_+| \frac{H_a(a)}{\lambda^2} + O\left(\frac{|\Pi(a)|^2 \log(\lambda\rho)}{\lambda^2} + \frac{\rho}{\lambda^2} + \frac{1}{(\lambda\rho)^3} \right),$$

(b)

$$\left| < \varphi_{a,\lambda}, \frac{1}{\lambda} \frac{\partial}{\partial a^m} \varphi_{a,\lambda} > \right| \leq C \frac{1}{\lambda^2}.$$

Proof. We first prove (a):
Since

$$w_{a,\lambda} := \lambda \frac{\partial}{\partial \lambda} \hat{\varphi}_{a,\lambda} + \hat{\varphi}_{a,\lambda} = 2\chi_\rho \delta_{a,\lambda} \left(\frac{(1+\lambda t)}{(1+\lambda t)^2 + \lambda^2 |x|^2} \right) \tag{C.1}$$

the scalar product is given by

$$\begin{aligned} < \varphi_{a,\lambda}, \lambda \frac{\partial}{\partial \lambda} \varphi_{a,\lambda} > = -||\varphi_{a,\lambda}||^2 + \int_M \nabla \hat{\varphi}_{a,\lambda} \cdot \nabla w_{a,\lambda} + \frac{1}{6} R_{g_a} \hat{\varphi}_{a,\lambda} w_{a,\lambda} dV_{g_a} \\ + \int_{\partial M} h_{g_a} \hat{\varphi}_{a,\lambda} w_{a,\lambda} d\sigma_{g_a}. \end{aligned} \tag{C.2}$$

Moreover

$$\begin{aligned}\int_{B_\rho^+}\nabla\hat\varphi_{a,\lambda}\cdot\nabla w_{a,\lambda}dV_{g_a} &= \int_{B_\rho^+}\nabla\hat\varphi_{a,\lambda}\cdot\nabla w_{a,\lambda}dxdt+O\left(\frac{1}{\lambda^3}\right)\\ &= \int_{B_\rho^+}\nabla\delta_\lambda\cdot\nabla w_\lambda dxdt+\int_{B_\rho^+}(g^{ij}-\delta_{ij})\partial_i\hat\varphi_{a,\lambda}\partial_j w_\lambda dxdt+O\left(\frac{1}{\lambda^3}\right)\\ &= \int_{B_\rho^+}\nabla\delta_\lambda\cdot\nabla w_\lambda dxdt+O\left(\frac{|\Pi(a)|^2}{\lambda^2}+\frac{\log(\lambda\rho)}{\lambda^3}\right)\\ &= 2\int_{B_\rho}\delta_\lambda^2 w_\lambda dx+\int_{S^3_{\rho,+}}\partial_\nu\delta_\lambda w_\lambda dS+O\left(\frac{|\Pi(a)|^2}{\lambda^2}+\frac{\log(\lambda\rho)}{\lambda^3}\right)\\ &= 4\int_{B_\rho}\frac{\lambda^3}{(1+\lambda^2|x|^2)^4}dx+O\left(\frac{|\Pi(a)|^2}{\lambda^2}+\frac{\log(\lambda\rho)}{\lambda^3}+\frac{1}{(\lambda\rho)^3}\right)\\ &= 2I_0+O\left(\frac{|\Pi(a)|^2}{\lambda^2}+\frac{\log(\lambda\rho)}{\lambda^3}+\frac{1}{(\lambda\rho)^3}\right)\end{aligned}$$

and

$$\frac{1}{6}\int_{B_\rho^+}R_{g_a}\delta_\lambda w_\lambda dV_{g_a}+\int_{B_\rho}h_{g_a}\delta_\lambda w_\lambda d\sigma_{g_a}=O\left(\frac{|\Pi(a)|^2}{\lambda^2}+\frac{\log(\lambda\rho)}{\lambda^3}\right).$$

Hence

$$\begin{aligned}&\int_{B_\rho^+}\nabla\hat\varphi_{a,\lambda}\cdot\nabla w_{a,\lambda}+\frac{1}{6}\int_{B_\rho^+}R_{g_a}\hat\varphi_{a,\lambda}w_\lambda dV_{g_a}+\int_{B_\rho}h_{g_a}\hat\varphi_{a,\lambda}w_\lambda d\sigma_{g_a}\\ &\quad= 2I_0+O\left(\frac{|\Pi(a)|^2}{\lambda^2}+\frac{\log(\lambda\rho)}{\lambda^3}+\frac{1}{(\lambda\rho)^3}\right).\end{aligned}\tag{C.3}$$

Furthermore, partial integration yields

$$\begin{aligned}&\int_{M\setminus B_\rho^+}\nabla\hat\varphi_{a,\lambda}\cdot\nabla w_{a,\lambda}+\frac{1}{6}\int_{B_\rho^+}R_{g_a}\hat\varphi_{a,\lambda}w_\lambda dV_{g_a}+\int_{\partial M\setminus B_\rho}h_{g_a}\hat\varphi_{a,\lambda}w_\lambda d\sigma_{g_a}\\ &\quad= \int_{M\setminus B_\rho^+}L_{g_a}\hat\varphi_{a,\lambda}w_{a,\lambda}dV_{g_a}+\int_{\partial M\setminus B_\rho}B_{g_a}\hat\varphi_{a,\lambda}w_\lambda d\sigma_{g_a}-\int_{S_{\rho,+}}\partial_\nu\hat\varphi_{a,\lambda}w_{a,\lambda}dS_{g_a}\\ &\quad= O\left(\frac{1}{(\lambda\rho)^3}\right)\end{aligned}\tag{C.4}$$

due to Proposition 2. Therefore (a) is proved by adding (C.2), (C.3), (C.4) and the expansion of $||\hat\varphi_{a,\lambda}||^2$.

Proof of (b): We take the derivative at a with respect to ψ_a Fermi-coordinates. In Lemma 10 we proved the expansions

$$\frac{\partial}{\partial a^m}|\bar\psi_a(x)|^2=-2\psi_a(x)^m+O(d_{g_a}(a,x)^3),\quad \frac{\partial}{\partial a^m}t_a(x)=O(d_{g_a}(a,x)^2).$$

In general

$$\frac{1}{\lambda}\frac{\partial}{\partial a^m}\varphi_{a,\lambda} = \frac{1}{\lambda}\frac{\partial}{\partial a^m}u_a\chi_\rho\delta_{a,\lambda} + u_a\frac{1}{\lambda}\frac{\partial}{\partial a^m}\left(\delta_{a,\lambda} + (1-\chi_\rho)\frac{G(a,\cdot)}{\lambda}\right), \tag{C.5}$$

form which we derive the inequality

$$\left|\frac{1}{\lambda}\frac{\partial}{\partial a^m}\varphi_{a,\lambda}\right| \leq C\varphi_{a,\lambda}$$

provided $2 \leq \lambda\rho$. Furthermore, the previous expansions yield on B^+_ρ

$$\begin{aligned}
&\frac{1}{\lambda}\frac{\partial}{\partial a^m}\varphi_{a,\lambda}\left(\psi_a^{-1}(x,t)\right) \\
&\quad = \frac{1}{\lambda}\frac{\partial}{\partial a^m}u_a\chi_\rho\delta_{a,\lambda} + u_a\frac{4\lambda^2 x^m + \lambda^2 O(|x,t|^3) + O\left(1+\lambda t)|(x,t)|^2\lambda\right)}{\left((1+\lambda t)^2 + \lambda^2|x|^2\right)^2}.
\end{aligned}$$

We compute

$$\begin{aligned}
< \varphi_{a,\lambda}, \frac{1}{\lambda}\frac{\partial}{\partial a^m}\varphi_{a,\lambda} > &= \int_M L_g\varphi_{a,\lambda}\frac{1}{\lambda}\frac{\partial}{\partial a^m}\varphi_{a,\lambda}dV_g + \int_{\partial M} B_g\varphi_{a,\lambda}\frac{1}{\lambda}\frac{\partial}{\partial a^m}\varphi_{a,\lambda}d\sigma_g \\
&= \int_M L_{g_a}\hat{\varphi}_{a,\lambda}\frac{1}{u_a}\frac{1}{\lambda}\frac{\partial}{\partial a^m}\varphi_{a,\lambda}dV_{g_a} + \int_{\partial M} B_{g_a}\hat{\varphi}_{a,\lambda}\frac{1}{u_a}\frac{1}{\lambda}\frac{\partial}{\partial a^m}\varphi_{a,\lambda}d\sigma_{g_a}.
\end{aligned}$$

First observe

$$\begin{aligned}
&\int_{M\setminus B^+_\rho} L_{g_a}\hat{\varphi}_{a,\lambda}\frac{1}{u_a}\frac{1}{\lambda}\frac{\partial}{\partial a^m}\varphi_{a,\lambda}dV_{g_a} + \int_{\partial M\setminus B_\rho} B_{g_a}\hat{\varphi}_{a,\lambda}\frac{1}{u_a}\frac{1}{\lambda}\frac{\partial}{\partial a^m}\varphi_{a,\lambda}d\sigma_{g_a} \\
&\qquad = O\left(\frac{|\Pi(a)|}{\lambda^2\rho} + \frac{\rho}{\lambda^2} + \frac{1}{(\lambda\rho)^3}\right).
\end{aligned}$$

Moreover

$$\begin{aligned}
&\int_{B^+_\rho} L_{g_a}\hat{\varphi}_{a,\lambda}\frac{1}{u_a}\frac{1}{\lambda}\frac{\partial}{\partial a^m}\varphi_{a,\lambda}dV_{g_a} + \int_{B_\rho} B_{g_a}\hat{\varphi}_{a,\lambda}\frac{1}{u_a}\frac{1}{\lambda}\frac{\partial}{\partial a^m}\varphi_{a,\lambda}d\sigma_{g_a} \\
&\qquad = \int_{B^+_\rho} L_{g_a}\hat{\varphi}_{a,\lambda}\frac{1}{\lambda}\frac{\partial}{\partial a^m}\hat{\varphi}_{a,\lambda}dV_{g_a} + \int_{B_\rho} B_{g_a}\hat{\varphi}_{a,\lambda}\frac{1}{\lambda}\frac{\partial}{\partial a^m}\hat{\varphi}_{a,\lambda}d\sigma_{g_a} \\
&\qquad + O\left(\frac{|\Pi(a)|^2}{\lambda^2} + \frac{\log(\lambda\rho)}{\lambda^3}\right) \\
&\qquad = O\left(\frac{1}{\lambda^2}\right),
\end{aligned}$$

which finally proves the Proposition. □

C.2. Interaction with the derivatives

For the expansion of the gradient we also need the interaction between a bubble and the derivative of a different bubble.

Proposition 27. *It holds*

(a)

$$< \varphi_{a_i,\lambda_i}, \lambda_j \frac{\partial}{\partial \lambda_j} \varphi_{a_j,\lambda_j} >= 2I_1 \lambda_j \frac{\partial}{\partial \lambda_j} I(\varepsilon_{ij}) + O(\rho\varepsilon_{ij}) + o(\varepsilon_{ij})$$

(b)

$$\left| < \varphi_{a_i,\lambda_i}, \frac{1}{\lambda_j} \nabla_{a_j} \varphi_{a_j,\lambda_j} \cdot \nabla K(a_j) > \right| \leq C |\nabla K(a_j)| \varepsilon_{ij}.$$

Proof. Proof of (a). We first assume that $\lambda_j \leq \lambda_i$ in which case we compute

$$\begin{aligned} < \varphi_{a_i,\lambda_i}, \lambda_j \frac{\partial}{\partial \lambda_j} \varphi_{a_j,\lambda_j} > &= \int_M L_g \varphi_{a_i,\lambda_i} \lambda_j \frac{\partial}{\partial \lambda_j} \varphi_{a_j,\lambda_j} dV_g + \int_{\partial M} B_g \varphi_{a_i,\lambda_i} \lambda_j \frac{\partial}{\partial \lambda_j} \varphi_{a_j,\lambda_j} d\sigma_g \\ &= \int_{\partial M} B_g \varphi_{a_i,\lambda_i} \lambda_j \frac{\partial}{\partial \lambda_j} \varphi_{a_j,\lambda_j} d\sigma_g + O\left(\rho\varepsilon_{ij}\right), \end{aligned}$$

where we estimate the first integral like in Proposition 25 using the estimate $\left| \lambda_j \frac{\partial}{\partial \lambda_j} \varphi_{a_j,\lambda_j} \right| \leq C \varphi_{a_j,\lambda_j}$. Furthermore, the derivative on ∂M is given by:

$$\lambda_j \frac{\partial}{\partial \lambda_j} \varphi_{a_j,\lambda_j}(x) = u_{a_j} \chi_\rho \hat{\hat{\delta}}_{a_j,\lambda_j} \left(\frac{1 - \lambda_j^2 |\psi_{a_j}(x)|^2}{1 + \lambda_j^2 |\psi_{a_j}(x)|^2} \right) - (1 - \chi_\rho) \frac{G(a_j, \cdot)}{\lambda_j}.$$

Integration as in proof of Proposition 24 and 25 yields

$$\int_{\partial M} B_g \varphi_{a_i,\lambda_i} \lambda_j \frac{\partial}{\partial \lambda_j} \varphi_{a_j,\lambda_j} d\sigma_g = 2I_1 \lambda_j \frac{\partial}{\partial \lambda_j} I(\varepsilon_{ij}) + O(\rho\varepsilon_{ij}) + o(\varepsilon_{ij}).$$

Therefore (a) is proved under the assumption $\lambda_j \leq \lambda_i$. From now on we assume $\lambda_i \leq \lambda_j$. In this case

$$\begin{aligned} &< \varphi_{a_i,\lambda_i}, \lambda_j \frac{\partial}{\partial \lambda_j} \varphi_{a_j,\lambda_j} > \\ &= \int_M L_g \left(\lambda_j \frac{\partial}{\partial \lambda_j} \varphi_{a_j,\lambda_j} \right) \varphi_{a_i,\lambda_i} dV_g + \int_{\partial M} B_g \left(\lambda_j \frac{\partial}{\partial \lambda_j} \varphi_{a_j,\lambda_j} \right) \varphi_{a_i,\lambda_i} d\sigma_g \\ &= \int_M \lambda_j \frac{\partial}{\partial \lambda_j} L_g \left(\varphi_{a_j,\lambda_j} \right) \varphi_{a_i,\lambda_i} dV_g + \int_{\partial M} \lambda_j \frac{\partial}{\partial \lambda_j} B_g \left(\varphi_{a_j,\lambda_j} \right) \varphi_{a_i,\lambda_i} d\sigma_g \end{aligned}$$

Due to Proposition 2 and 3, $\lambda_j \frac{\partial}{\partial \lambda_j} L_g\left(\varphi_{a_j,\lambda_j}\right)$ satisfies the same estimate as $L_g \varphi_{a_j,\lambda_j}$. Hence the following inequality holds true:

$$\left|\int_M \lambda_j \frac{\partial}{\partial \lambda_j} L_g\left(\varphi_{a_j,\lambda_j}\right) \varphi_{a_i,\lambda_i} dV_g\right| \leq C\rho\varepsilon_{ij}.$$

Furthermore,

$$\frac{\partial}{\partial \lambda_j} B_g\left(\varphi_{a_j,\lambda_j}\right) = u_{a_j}^2 \chi_\rho \left(2\frac{\partial}{\partial \lambda_j}\delta^2_{a_j,\lambda_j} + h_{g_{a_j}}\frac{\partial}{\partial \lambda_j}\delta_{a_j,\lambda_j}\right),$$

which implies

$$\begin{aligned}\int_{\partial M} \lambda_j \frac{\partial}{\partial \lambda_j} B_g\left(\varphi_{a_j,\lambda_j}\right) \varphi_{a_i,\lambda_i} d\sigma_g &= \int_{\partial M} u_{a_j}^2 \chi_\rho \left(2\lambda_j \frac{\partial}{\partial \lambda_j}\delta^2_{a_j,\lambda_j}\right) \varphi_{a_i,\lambda_i} d\sigma_g + O\left(\rho\varepsilon_{ij}\right)\\ &= 2I_1 \lambda_j \frac{\partial}{\partial \lambda_j} I(\varepsilon_{ij}) + O(\rho\varepsilon_{ij}) + o(\varepsilon_{ij}).\end{aligned}$$

Here we integrated like in Proposition 24 and 25. Therefore part (a) is proved.
Proof of (b): Since

$$\left|\frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}\cdot \nabla K(a_j)\right| \leq C|\nabla K(a_j)|\varphi_{a_j,\lambda_j},$$

we deduce

$$\begin{aligned}&\left| < \varphi_{a_i,\lambda_i}, \frac{1}{\lambda_j}\nabla_{a_j}\varphi_{a_j,\lambda_j}\cdot \nabla K(a_j) > \right|\\ &\quad\leq C|\nabla K(a_j)| \left(\int_M |L_g \varphi_{a_i,\lambda_i}| \varphi_{a_j,\lambda_j} dV_g + \int_{\partial M} |B_g \varphi_{a_i,\lambda_i}| \varphi_{a_j,\lambda_j} d\sigma_g\right)\\ &\quad\leq C|\nabla K(a_j)| \left(\int_{B_{2\rho}(a_i)} \hat{\delta}^2_{a_i,\lambda_i} \left(\frac{\lambda_j}{1+\lambda_j^2 d_{g_{a_i}}(a_i,a_j)^2}\right) d\sigma_{g_{a_i}} + C\rho\varepsilon_{ij}\right)\\ &\quad\leq C|\nabla K(a_j)|\varepsilon_{ij},\end{aligned}$$

where we integrate over the set $A \cap B_{2\rho}(a_i)$ and $A^c \cap B_{2\rho}(a_i)$ to estimate the last integral. Here

$$A := \left\{x \in \partial M \mid 2d_{g_{a_i}}(a_j,x) \leq \frac{1}{\lambda_i} + d_{g_{a_i}}(a_i,a_j)\right\}.$$

Therefore the proof is completed. □

C.3. Further estimates

We need more expansions and estimates which we mainly use in chapter 7.

Lemma 11. *It holds*

(a) $< \lambda\frac{\partial}{\partial\lambda}\varphi_{a,\lambda}, \lambda\frac{\partial}{\partial\lambda}\varphi_{a,\lambda} >= I_2 + O\left(\frac{1}{\lambda}\right),$

(b) $< \lambda\frac{\partial}{\partial\lambda}\varphi_{a,\lambda}, \frac{1}{\lambda}\frac{\partial}{\partial a_i}\varphi_{a,\lambda} >= O\left(\frac{\log(\lambda)}{\lambda}\right),$

(c) $< \frac{1}{\lambda}\frac{\partial}{\partial a_j}\varphi_{a,\lambda}, \frac{1}{\lambda}\frac{\partial}{\partial a_i}\varphi_{a,\lambda} >= I_3\delta_{ij} + O\left(\frac{1}{\lambda}\right),$

(d) $\left|< \frac{1}{\lambda_j}\frac{\partial}{\partial a_j^m}\varphi_{a_j,\lambda_j}, \frac{1}{\lambda_i}\frac{\partial}{\partial a_i^k}\varphi_{a_i,\lambda_i} >\right| + \left|< \frac{1}{\lambda_j}\frac{\partial}{\partial a_j^m}\varphi_{a_j,\lambda_j}, \lambda_i\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i} >\right| = O(\varepsilon_{ij}),$

(e) $\left|< \lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j}, \lambda_i\frac{\partial}{\partial\lambda_i}\varphi_{a_i,\lambda_i} >\right| = O(\varepsilon_{ij})$

where

$$I_2 = 4\int_{\mathbb{R}^3}\left(\frac{1}{1+|x|^2}\right)^3\left(\frac{1-|x|^2}{1+|x|^2}\right)^2 dx$$

and

$$I_3 = \frac{16}{3}\int_{\mathbb{R}^3}\frac{|x|^2}{(1+|x|^2)^5}dx.$$

Proof. (a)

$$\begin{aligned}&< \lambda\frac{\partial}{\partial\lambda}\varphi_{a,\lambda}, \lambda\frac{\partial}{\partial\lambda}\varphi_{a,\lambda} > \\ &\quad= \int_M\left(\lambda\frac{\partial}{\partial\lambda}L_{g_a}\hat{\varphi}_{a,\lambda}\right)\lambda\frac{\partial}{\partial\lambda}\hat{\varphi}_{a,\lambda}dV_{g_a} + \int_{\partial M}\left(\lambda\frac{\partial}{\partial\lambda}B_{g_a}\hat{\varphi}_{a,\lambda}\right)\lambda\frac{\partial}{\partial\lambda}\hat{\varphi}_{a,\lambda}d\sigma_{g_a}.\end{aligned}$$

Due to Proposition 3

$$\left|\int_M\left(\lambda\frac{\partial}{\partial\lambda}L_{g_a}\hat{\varphi}_{a,\lambda}\right)\lambda\frac{\partial}{\partial\lambda}\hat{\varphi}_{a,\lambda}dV_{g_a}\right| \le C\frac{1}{\lambda}.$$

Furthermore

$$\begin{aligned}\int_{\partial M}\left(\lambda\frac{\partial}{\partial\lambda}B_{g_a}\hat{\varphi}_{a,\lambda}\right)\lambda\frac{\partial}{\partial\lambda}\hat{\varphi}_{a,\lambda}d\sigma_{g_a} &= 4\int_{B_\rho}\delta_\lambda^3\left(\frac{1-\lambda^2|x|^2}{1+\lambda^2|x|^2}\right)^2 dx + O\left(\frac{1}{\lambda^3}\right) \\ &= I_2 + O\left(\frac{1}{\lambda^3}\right).\end{aligned}$$

Hence (a) is proved.
(b) In this case we expand as follows

$$\begin{aligned}&< \lambda\frac{\partial}{\partial\lambda}\varphi_{a,\lambda}, \frac{1}{\lambda}\frac{\partial}{\partial a_i}\varphi_{a,\lambda} > \\ &\quad= \int_M \lambda\frac{\partial}{\partial\lambda}L_{g_a}\hat{\varphi}_{a,\lambda}\frac{1}{\lambda}\frac{\partial}{\partial a_i}\hat{\varphi}_{a,\lambda}dV_{g_a} + \int_{\partial M}\lambda\frac{\partial}{\partial\lambda}B_{g_a}\hat{\varphi}_{a,\lambda}\frac{1}{\lambda}\frac{\partial}{\partial a_i}\hat{\varphi}_{a,\lambda}d\sigma_{g_a}.\end{aligned}$$

Like in the previous case the interior integral is bounded by $\frac{C}{\lambda}$. On the boundary we get

$$\begin{aligned}
&\int_{\partial M} \lambda \frac{\partial}{\partial \lambda} B_{g_a} \hat{\varphi}_{a,\lambda} \frac{1}{\lambda} \frac{\partial}{\partial a_i} \hat{\varphi}_{a,\lambda} d\sigma_{g_a} \\
&\qquad = 8 \int_{B_\rho} \delta_\lambda^3 \left(\frac{1 - \lambda^2 |x|^2}{1 + \lambda^2 |x|^2} \right) \left(\frac{2\lambda x_i + O(\lambda |x|^3)}{1 + \lambda^2 |x|^2} \right) dx + O\left(\frac{\log(\lambda)}{\lambda} \right) \\
&\qquad = O\left(\frac{\log(\lambda)}{\lambda} \right),
\end{aligned}$$

which proves (b).
(c) Due to Proposition 4 the same estimates as in (a) yield

$$\left| \int_M \frac{1}{\lambda} \frac{\partial}{\partial a_j} L_g \varphi_{a,\lambda} \frac{1}{\lambda} \frac{\partial}{\partial a_i} \varphi_{a,\lambda} dV_g \right| \leq C \frac{1}{\lambda}.$$

We are left to expand the boundary integral

$$\begin{aligned}
&\int_{\partial M} \frac{1}{\lambda} \frac{\partial}{\partial a_j} B_g \varphi_{a,\lambda} \frac{1}{\lambda} \frac{\partial}{\partial a_i} \varphi_{a,\lambda} d\sigma_g \\
&\quad = 2 \int_{B_\rho} \frac{1}{\lambda} \frac{\partial}{\partial a_j} \left(u_a^2 \delta_{a,\lambda}^2 \right) \frac{1}{\lambda} \frac{\partial}{\partial a_i} \left(u_a \delta_{a,\lambda} \right) d\sigma_g + O\left(\frac{\log(\lambda)}{\lambda} \right) \\
&\quad = 2 \int_{B_\rho} \frac{1}{\lambda} \frac{\partial}{\partial a_j} \delta_{a,\lambda}^2 \frac{1}{\lambda} \frac{\partial}{\partial a_i} \delta_{a,\lambda} dx + O\left(\frac{\log(\lambda)}{\lambda} \right) \\
&\quad = 16 \int_{B_\rho} \delta_\lambda^3 \left(\frac{\lambda x_j + O(\lambda |x|^3)}{1 + \lambda^2 |x|^2} \right) \left(\frac{\lambda x_i + O(\lambda |x|^3)}{1 + \lambda^2 |x|^2} \right) dx + O\left(\frac{\log(\lambda)}{\lambda} \right) \\
&\quad = I_3 \delta_{ij} + O\left(\frac{\log(\lambda)}{\lambda} \right).
\end{aligned}$$

Hence (c) is proved.
(d)+(e) follow easily by using the same arguments as in the proof of Proposition 27 (b). □

D. More Estimates

Lemma 12. *There exists a constant $C > 0$, independent of a, such that*

(a) $|\varphi_{a,\lambda}|, |\lambda \frac{\partial}{\partial\lambda}\varphi_{a,\lambda}|, |\frac{1}{\lambda}\nabla_a \varphi_{a,\lambda}| \leq C \left(\frac{\lambda}{1+\lambda^2 d_g(a,x)^2}\right)$,

(b) $|\lambda^2 \frac{\partial^2}{\partial\lambda^2}\varphi_{a,\lambda}|, |\frac{1}{\lambda^2}\nabla_a^2 \varphi_{a,\lambda}|, |\frac{\partial}{\partial\lambda}\nabla_a \varphi_{a,\lambda}| \leq C \left(\frac{\lambda}{1+\lambda^2 d_g(a,x)^2}\right)$.

Proof. If $d_g(a,x) \leq 2\delta$ and $k \in \mathbb{N}$ a computation yields

$$\begin{aligned}&\frac{\partial}{\partial a_i}\big((1+\lambda t_a(x))^2 + \lambda^2|\overline{\psi}_a(x)|^2\big)^{-\frac{k}{2}} \\ &= -\frac{k}{2}\big((1+\lambda t_a(x))^2 + \lambda^2|\overline{\psi}_a(x)|^2\big)^{-\frac{k}{2}-1}\left(2(1+\lambda t_a(x))\lambda\frac{\partial}{\partial a_i}t_a(x) + \lambda^2\frac{\partial}{\partial a_i}|\overline{\psi}_a(x)|^2\right),\end{aligned} \tag{D.1}$$

hence Lemma 10 implies

$$\left|\frac{\partial}{\partial a_i}\big((1+\lambda t_a(x))^2 + \lambda^2|\overline{\psi}_a(x)|^2\big)^{-\frac{k}{2}}\right| \leq C\lambda\big((1+\lambda t_a(x))^2 + \lambda^2|\overline{\psi}_a(x)|^2\big)^{-\frac{k}{2}}. \tag{D.2}$$

Furthermore, with the use of (D.1) and Lemma 10 we estimate

$$\left|\frac{\partial^2}{\partial a_i \partial a_j}\big((1+\lambda t_a(x))^2 + \lambda^2|\overline{\psi}_a(x)|^2\big)^{-\frac{k}{2}}\right| \leq C\lambda^2\big((1+\lambda t_a(x))^2 + \lambda^2|\overline{\psi}_a(x)|^2\big)^{-\frac{k}{2}} \tag{D.3}$$

as well as

$$\left|\frac{\partial}{\partial\lambda}\frac{\partial}{\partial a_i}\big((1+\lambda t_a(x))^2 + \lambda^2|\overline{\psi}_a(x)|^2\big)^{-\frac{k}{2}}\right| \leq C\big((1+\lambda t_a(x))^2 + \lambda^2|\overline{\psi}_a(x)|^2\big)^{-\frac{k}{2}}. \tag{D.4}$$

In addition, there holds

$$|\nabla_a G(a,x)| \leq C\frac{1}{d_g(a,x)^3}, \quad |\nabla_a^2 G(a,x)| \leq C\frac{1}{d_g(a,x)^4}.$$

Since $(1+\lambda t_a(x))^2 + \lambda^2|\overline{\psi}_a(x)|^2 \geq (1+\lambda^2|\psi_a(x)|^2) \geq c\left(1+\lambda^2 d_g(a,x)^2\right)$, if $d_g(a,x) \leq 2\rho$, the Lemma follows from the previous estimates. $\square$

Lemma 13. *Let $a, b \in \partial M$ such that $d_{g_a}(a,b) \le \rho_0$ and $v \in H^1(M)$. If $\frac{1}{2} \le \frac{\lambda}{\mu} \le 2$ then*

(a) $| < v, \varphi_{b,\mu} - \varphi_{a,\lambda} > | \le C||v|| \left(\lambda d_g(a,b) + \left|1 - \frac{\mu}{\lambda}\right|\right),$

(b) $| < \varphi_{a,\lambda}, \varphi_{b,\mu} - \varphi_{a,\lambda} > | \le C \left(\lambda^2 d_g(a,b)^2 + \left|1 - \frac{\mu}{\lambda}\right|^2\right),$

(c) $| < v, \mu\frac{\partial}{\partial \mu}\varphi_{b,\mu} - \lambda\frac{\partial}{\partial \lambda}\varphi_{a,\lambda} > | \le C||v|| \left(\lambda d_g(a,b) + \left|1 - \frac{\mu}{\lambda}\right|\right),$

(d) $| < \varphi_{b,\mu} - \varphi_{a,\lambda}, \lambda\frac{\partial}{\partial \lambda}\varphi_{a,\lambda} > | \le C \left(\lambda^2 d_g(a,b)^2 + \left|1 - \frac{\mu}{\lambda}\right|^2\right),$

(e) $| < v, \frac{1}{\mu}\frac{\partial}{\partial a_i}\varphi_{b,\mu} - \frac{1}{\lambda}\frac{\partial}{\partial a_i}\varphi_{a,\lambda} > | \le C||v|| \left(\lambda d_g(a,b) + \left|1 - \frac{\mu}{\lambda}\right|\right),$

(f) $| < \varphi_{b,\mu} - \varphi_{a,\lambda}, \frac{1}{\lambda}\frac{\partial}{\partial a_i}\varphi_{a,\lambda} > | \le C \left(\lambda^2 d_g(a,b)^2 + \left|1 - \frac{\mu}{\lambda}\right|^2\right).$

Proof. Since $L_g\varphi_{a,\lambda} = u_a^3 L_{g_a}\hat{\varphi}_{a,\lambda}$ and $B_g\varphi_{a,\lambda} = u_a^2 B_{g_a}\hat{\varphi}_{a,\lambda}$, the following estimates are an immediate consequence of Lemma 10 and the proof of Lemma 12.

$$\left|\left|\frac{1}{\lambda}\nabla_a L_g\varphi_{a,\lambda}\right|\right|_{L^{\frac{4}{3}}(M)} + \left|\left|\frac{1}{\lambda^2}\nabla_a^2 L_g\varphi_{a,\lambda}\right|\right|_{L^{\frac{4}{3}}(M)} \le C, \tag{D.5}$$

$$\left|\left|\frac{1}{\lambda}\nabla_a B_g\varphi_{a,\lambda}\right|\right|_{L^{\frac{3}{2}}(\partial M)} + \left|\left|\frac{1}{\lambda^2}\nabla_a^2 B_g\varphi_{a,\lambda}\right|\right|_{L^{\frac{3}{2}}(\partial M)} \le C, \tag{D.6}$$

$$\left|\left|\lambda\frac{\partial}{\partial\lambda} L_g\varphi_{a,\lambda}\right|\right|_{L^{\frac{4}{3}}(M)} + \left|\left|\lambda^2\frac{\partial^2}{\partial\lambda^2} L_g\varphi_{a,\lambda}\right|\right|_{L^{\frac{4}{3}}(M)} \le C, \tag{D.7}$$

$$\left|\left|\lambda\frac{\partial}{\partial\lambda} B_g\varphi_{a,\lambda}\right|\right|_{L^{\frac{3}{2}}(\partial M)} + \left|\left|\lambda^2\frac{\partial^2}{\partial\lambda^2} B_g\varphi_{a,\lambda}\right|\right|_{L^{\frac{3}{2}}(\partial M)} \le C, \tag{D.8}$$

$$\left|\left|\nabla_a\frac{\partial}{\partial\lambda} L_g\varphi_{a,\lambda}\right|\right|_{L^{\frac{4}{3}}(M)} + \left|\left|\nabla_a\frac{\partial}{\partial\lambda} B_g\varphi_{a,\lambda}\right|\right|_{L^{\frac{3}{2}}(\partial M)} \le C. \tag{D.9}$$

With the help of these estimates we can now proof the assertions.
(a)

$$< v, \varphi_{b,\mu} - \varphi_{a,\lambda} >= \int_M L_g\left(\varphi_{b,\mu} - \varphi_{a,\lambda}\right) v dV_g + \int_{\partial M} B_g\left(\varphi_{b,\mu} - \varphi_{a,\lambda}\right) v d\sigma_g.$$

In conformal Fermi coordinates at a let $x = \psi_a^{-1}(y)$. We compute

$$L_g\left(\varphi_{b,\mu} - \varphi_{a,\lambda}\right)(x) = \int_0^1 \left(\frac{\partial}{\partial y_i|} L_g\varphi_{y(t),\lambda(t)}(x)\psi_a(b)^i + \frac{\partial}{\partial\lambda} L_g\varphi_{y(t),\lambda(t)}(x)(\mu - \lambda)\right) dt,$$

where $(y(t),\lambda(t))=(t\psi_a(b),\lambda+t(\mu-\lambda))$. We use Hoelder inequality for $p=\frac{4}{3}$ to obtain

$$\begin{aligned}|L_g\left(\varphi_{b,\mu}-\varphi_{a,\lambda}\right)(x)|^p \leq & C_p\int_0^1|\nabla_y L_g\varphi_{y(t),\lambda(t)}(x)|^p dt\ |\psi_a(b)^i|^p\\ &+C_p\int_0^1|\frac{\partial}{\partial\lambda}L_g\varphi_{y(t),\lambda(t)}(x)|^p dt\ |\mu-\lambda|^p.\end{aligned}$$

Hence integration over the manifold and switching the integration, combined with (D.5) and (D.7), yields

$$||L_g\left(\varphi_{b,\mu}-\varphi_{a,\lambda}\right)||_{L^{\frac{4}{3}}(M)}\leq C\left(\lambda|\psi_a(b)|+|\frac{\mu}{\lambda}-1|\right)\leq C\left(\lambda d_g(a,b)+|\frac{\mu}{\lambda}-1|\right).$$

If we use the same arguments as above we can also prove that

$$||B_g\left(\varphi_{b,\mu}-\varphi_{a,\lambda}\right)||_{L^{\frac{3}{2}}(\partial M)}\leq C\left(\lambda d_g(a,b)+|\frac{\mu}{\lambda}-1|\right).$$

Claim (a) follows from the last two estimates.
(b) We again choose conformal Fermi-coordinates around a. In this coordinates $b=\psi_a^{-1}(z)$. By Taylor expansion we obtain

$$\begin{aligned}&\varphi_{b,\mu}(x)-\varphi_{a,\lambda}(x)=\frac{\partial}{\partial\lambda}\varphi_{a,\lambda}(x)(\mu-\lambda)+\frac{\partial}{\partial a_i}\varphi_{a,\lambda}(x)z^i+\\ &\int_0^1(1-t)\left(\frac{\partial^2}{\partial\lambda^2}\varphi_{a(t),\lambda(t)}(\mu-\lambda)^2+\frac{\partial^2}{\partial\lambda\partial a_i}\varphi_{a(t),\lambda(t)}(\mu-\lambda)z^i+\frac{\partial^2}{\partial a_i\partial a_j}\varphi_{a(t),\lambda(t)}z^iz^j\right)dt\end{aligned}\tag{D.10}$$

where

$$(a(t),\lambda(t))=(tz,\lambda+t(\mu-\lambda))$$

in coordinates. We expand

$$<\varphi_{a,\lambda},\varphi_{b,\mu}-\varphi_{a,\lambda}>=\int_M L_g\varphi_{a,\lambda}\left(\varphi_{b,\mu}-\varphi_{a,\lambda}\right)dV_g+\int_{\partial M}B_g\varphi_{a,\lambda}\left(\varphi_{b,\mu}-\varphi_{a,\lambda}\right)d\sigma_g.$$

As an example we estimate:

$$\begin{aligned}&\left|\int_M L_g\varphi_{a,\lambda}\int_0^1(1-t)\lambda(t)^2\frac{\partial^2}{\partial\lambda^2}\varphi_{a(t),\lambda(t)}dtdV_g\right|\\ &=\left|\int_0^1(1-t)\int_M L_g\varphi_{a,\lambda}\lambda(t)^2\frac{\partial^2}{\partial\lambda^2}\varphi_{a(t),\lambda(t)}dV_g dt\right|\\ &\leq C\int_0^1||L_g\varphi_{a,\lambda}||_{L^{\frac{4}{3}}(M)}||\lambda(t)^2\frac{\partial^2}{\partial\lambda^2}\varphi_{a(t),\lambda(t)}||_{L^4(M)}dt\leq C\end{aligned}$$

due to Lemma 12. The same estimate also holds for the other integrals in (D.10). Hence

$$
\begin{aligned}
< \varphi_{a,\lambda}, \varphi_{b,\mu} - \varphi_{a,\lambda} > =& < \varphi_{a,\lambda}, \lambda \frac{\partial}{\partial \lambda} \varphi_{a,\lambda} > (\frac{\mu}{\lambda} - 1) + < \varphi_{a,\lambda}, \frac{1}{\lambda} \frac{\partial}{\partial a_i} \varphi_{a,\lambda} > \lambda z^i \\
&+ O\left(\left| \frac{\mu}{\lambda} - 1 \right|^2 + \lambda^2 |z|^2 \right) \\
=& < \varphi_{a,\lambda}, \lambda \frac{\partial}{\partial \lambda} \varphi_{a,\lambda} > (\frac{\mu}{\lambda} - 1) + < \varphi_{a,\lambda}, \frac{1}{\lambda} \frac{\partial}{\partial a_i} \varphi_{a,\lambda} > \lambda z^i \\
&+ O\left(\left| \frac{\mu}{\lambda} - 1 \right|^2 + \lambda^2 d_g(a,b)^2 \right) \\
=& O\left(\frac{1}{\lambda^2} \right) \left(\left| \frac{\mu}{\lambda} - 1 \right| + \lambda d_g(a,b) \right) + O\left(\left| \frac{\mu}{\lambda} - 1 \right|^2 + \lambda^2 d_g(a,b)^2 \right)
\end{aligned}
$$

where we used Proposition 26. Therefore (b) is proved.
(c) We write

$$
\begin{aligned}
& < v, \mu \frac{\partial}{\partial \mu} \varphi_{b,\mu} - \lambda \frac{\partial}{\partial \lambda} \varphi_{a,\lambda} > \\
&= \int_M L_g \left(\mu \frac{\partial}{\partial \mu} \varphi_{b,\mu} - \lambda \frac{\partial}{\partial \lambda} \varphi_{a,\lambda} \right) v dV_g + \int_{\partial M} B_g \left(\mu \frac{\partial}{\partial \mu} \varphi_{b,\mu} - \lambda \frac{\partial}{\partial \lambda} \varphi_{a,\lambda} \right) v d\sigma_g \\
&= \int_M L_g \left(\mu \frac{\partial}{\partial \mu} \varphi_{b,\mu} - \mu \frac{\partial}{\partial \lambda} \varphi_{a,\lambda} \right) v dV_g + \int_{\partial M} B_g \left(\mu \frac{\partial}{\partial \mu} \varphi_{b,\mu} - \mu \frac{\partial}{\partial \lambda} \varphi_{a,\lambda} \right) v d\sigma_g \\
&+ (\frac{\mu}{\lambda} - 1) \int_M \lambda \frac{\partial}{\partial \lambda} L_g \varphi_{a,\lambda} v dV_g + (\frac{\mu}{\lambda} - 1) \int_{\partial M} \lambda \frac{\partial}{\partial \lambda} B_g \varphi_{a,\lambda} v d\sigma_g \\
&= \int_M L_g \left(\mu \frac{\partial}{\partial \mu} \varphi_{b,\mu} - \mu \frac{\partial}{\partial \lambda} \varphi_{a,\lambda} \right) v dV_g + \int_{\partial M} B_g \left(\mu \frac{\partial}{\partial \mu} \varphi_{b,\mu} - \mu \frac{\partial}{\partial \lambda} \varphi_{a,\lambda} \right) v d\sigma_g \\
&+ O\left(\left| \frac{\mu}{\lambda} - 1 \right| ||v|| \right).
\end{aligned}
$$

With the notations of above we expand

$$
\mu \frac{\partial}{\partial \mu} \varphi_{b,\mu} - \mu \frac{\partial}{\partial \lambda} \varphi_{a,\lambda} = \mu \int_0^1 \frac{\partial^2}{\partial \lambda^2} \varphi_{a(t),\lambda(t)} (\lambda - \mu) + \frac{\partial^2}{\partial \lambda \partial a_i} \varphi_{a(t),\lambda(t)} z^i dt.
$$

Hence the same method as in (a) implies

$$
\begin{aligned}
& \int_M L_g \left(\mu \frac{\partial}{\partial \mu} \varphi_{b,\mu} - \mu \frac{\partial}{\partial \lambda} \varphi_{a,\lambda} \right) v dV_g + \int_{\partial M} B_g \left(\mu \frac{\partial}{\partial \mu} \varphi_{b,\mu} - \mu \frac{\partial}{\partial \lambda} \varphi_{a,\lambda} \right) v d\sigma_g \\
& \quad = O\left(\left| \frac{\mu}{\lambda} - 1 \right| + \lambda d_g(a,b) \right) ||v||,
\end{aligned}
$$

which proves (c).
(d) From the same Taylor expansion of $\varphi_{b,\mu} - \varphi_{a,\lambda}$ as in (b) we derive the following

estimate

$$
\begin{aligned}
&< \varphi_{b,\mu} - \varphi_{a,\lambda}, \lambda\frac{\partial}{\partial\lambda}\varphi_{a,\lambda} > \\
&\quad =< \lambda\frac{\partial}{\partial\lambda}\varphi_{a,\lambda}, \lambda\frac{\partial}{\partial\lambda}\varphi_{a,\lambda} > \left(\frac{\mu}{\lambda}-1\right) + < \frac{1}{\lambda}\frac{\partial}{\partial a_i}\varphi_{a,\lambda}, \lambda\frac{\partial}{\partial\lambda}\varphi_{a,\lambda} > \lambda z^i \\
&\quad + O\left(\left|\frac{\mu}{\lambda}-1\right|^2 + \lambda^2 d_g(a,b)^2\right) \\
&\quad = I_2\left(\frac{\mu}{\lambda}-1\right) + O\left(\frac{\log(\lambda)}{\lambda}\right)\left(\left|\frac{\mu}{\lambda}-1\right| + \lambda d_g(a,b)\right) + O\left(\left|\frac{\mu}{\lambda}-1\right|^2 + \lambda^2 d_g(a,b)^2\right).
\end{aligned}
$$

The last expansion follows by Lemma 11.
(e) Similar to (c) we get

$$
\begin{aligned}
&< v, \frac{1}{\mu}\frac{\partial}{\partial a_i}\varphi_{b,\mu} - \frac{1}{\lambda}\frac{\partial}{\partial a_i}\varphi_{a,\lambda} > \\
&= \int_M \frac{1}{\mu}\left(\frac{\partial}{\partial a_i}L_g\varphi_{b,\mu} - \frac{\partial}{\partial a_i}L_g\varphi_{a,\lambda}\right) v dV_g + \int_{\partial M}\frac{1}{\mu}\left(\frac{\partial}{\partial a_i}B_g\varphi_{b,\mu} - \frac{\partial}{\partial a_i}B_g\varphi_{a,\lambda}\right) v d\sigma_g \\
&+ \left(\frac{\lambda}{\mu}-1\right)\int_M \frac{1}{\lambda}\frac{\partial}{\partial a_i}\varphi_{a,\lambda} v dV_g + \left(\frac{\lambda}{\mu}-1\right)\int_{\partial M}\frac{1}{\lambda}\frac{\partial}{\partial a_i}\varphi_{a,\lambda} v d\sigma_g \\
&= O\left(\left|\frac{\mu}{\lambda}-1\right| + \lambda d_g(a,b)\right)||v||,
\end{aligned}
$$

which proves (e).
(f) The same ideas as in (b) and (d) yield

$$
\begin{aligned}
&< \varphi_{b,\mu} - \varphi_{a,\lambda}, \frac{1}{\lambda}\frac{\partial}{\partial a_i}\varphi_{a,\lambda} > \\
&\quad =< \lambda\frac{\partial}{\partial\lambda}\varphi_{a,\lambda}, \frac{1}{\lambda}\frac{\partial}{\partial a_i}\varphi_{a,\lambda} > \left(\frac{\mu}{\lambda}-1\right) + < \frac{1}{\lambda}\frac{\partial}{\partial a_j}\varphi_{a,\lambda}, \frac{1}{\lambda}\frac{\partial}{\partial a_i}\varphi_{a,\lambda} > \lambda z^j \\
&\quad + O\left(\left|\frac{\mu}{\lambda}-1\right|^2 + \lambda^2 d_g(a,b)^2\right) \\
&\quad = I_3\lambda\psi_a(b)^i + O\left(\frac{\log(\lambda)}{\lambda}\right)\left(\left|\frac{\mu}{\lambda}-1\right| + \lambda d_g(a,b)\right) + O\left(\left|\frac{\mu}{\lambda}-1\right|^2 + \lambda^2 d_g(a,b)^2\right),
\end{aligned}
$$

which proves (f). □

Lemma 14. *If* $\tilde{\lambda}_i\lambda_i d_g(\tilde{a}_j, a_i) \to 0$, $\frac{\tilde{\lambda}_i}{\lambda_i} \to 1$ *then*

(a) $< \varphi_{\tilde{a}_i,\tilde{\lambda}_i} - \varphi_{a_i,\lambda_i}, \varphi_{a_j,\lambda_j} >= O(\varepsilon_{ij})\left(\lambda_i d_g(\tilde{a}_j, a_i) + \left|\frac{\tilde{\lambda}_i}{\lambda_i}-1\right|\right)$

(b) $< \varphi_{\tilde{a}_i,\tilde{\lambda}_i} - \varphi_{a_i,\lambda_i}, \lambda_j\frac{\partial}{\partial\lambda_j}\varphi_{a_j,\lambda_j} >= O(\varepsilon_{ij})\left(\lambda_i d_g(\tilde{a}_j, a_i) + \left|\frac{\tilde{\lambda}_i}{\lambda_i}-1\right|\right)$

(c) $<\varphi_{\tilde{a}_i,\tilde{\lambda}_i}-\varphi_{a_i,\lambda_i}, \frac{1}{\lambda_j}\frac{\partial}{\partial a_j^m}\varphi_{a_j,\lambda_j}>= O(\varepsilon_{ij})\left(\lambda_i d_g(\tilde{a}_j,a_i)+\left|\frac{\tilde{\lambda}_i}{\lambda_i}-1\right|\right)$

Proof. (a) Since $d_g(\tilde{a}_j,a_i)\to 0$, we choose ψ_{a_i} Fermi coordinates and write $\psi_{a_i}(\tilde{a}_i)=z$. Furthermore we set

$$(z(t),\lambda_i(t))=(tz,\lambda_i+t(\tilde{\lambda}_i-\lambda_i));\quad a_i(t)=\psi_{a_i}^{-1}(tz)$$

and obtain

$$\varphi_{\tilde{a}_i,\tilde{\lambda}_i}(x)-\varphi_{a_i,\lambda_i}(x)=\int_0^1\frac{\partial}{\partial\lambda_i}\varphi_{a_i(t),\lambda_i(t)}(x)\left(\tilde{\lambda}_i-\lambda_i\right)+\frac{\partial}{\partial a_i^m}\varphi_{a_i(t),\lambda_i(t)}(x)z^m dt.$$

Hence

$$\begin{aligned}
&<\varphi_{\tilde{a}_i,\tilde{\lambda}_i}-\varphi_{a_i,\lambda_i},\varphi_{a_j,\lambda_j}>\\
&=\int_M L_g\varphi_{a_j,\lambda_j}\left(\varphi_{\tilde{a}_i,\tilde{\lambda}_i}-\varphi_{a_i,\lambda_i}\right)dV_g+\int_{\partial M}B_g\varphi_{a_j,\lambda_j}\left(\varphi_{\tilde{a}_i,\tilde{\lambda}_i}-\varphi_{a_i,\lambda_i}\right)d\sigma_g\\
&=\int_0^1<\varphi_{a_j,\lambda_j},\lambda_i(t)\frac{\partial}{\partial\lambda_i}\varphi_{a_i(t),\lambda_i(t)}>\frac{\left(\tilde{\lambda}_i-\lambda_i\right)}{\lambda(t)}dt\\
&+\int_0^1<\varphi_{a_j,\lambda_j},\frac{1}{\lambda_i(t)}\frac{\partial}{\partial a_i^m}\varphi_{a_i(t),\lambda_i(t)}>\lambda_i(t)z^m dt\\
&=\int_0^1 O(\varepsilon_{ij}(t))dt\left(\lambda_i d_g(\tilde{a}_j,a_i)+\left|\frac{\tilde{\lambda}_i}{\lambda_i}-1\right|\right)\le C\varepsilon_{ij}\left(\lambda_i d_g(\tilde{a}_j,a_i)+\left|\frac{\tilde{\lambda}_i}{\lambda_i}-1\right|\right),
\end{aligned}$$

which proves the assertion. (b)-(c) follow similarly. □

E. Expansion of the Green's function

Let ψ_a be conformal Fermi-coordinates around $a \in \partial M$. In what follows (M,g) is a four-dimensional compact Riemannian manifold with boundary and positive Sobolev-quotient $Q(M,\partial M,[g])$. In this chapter we prove an appropriate expansion of the Green's function $G_a(\cdot)$ at a with respect to the conformal operator (L_{g_a}, B_{g_a}). We will use that

$$\det g_a(x,t) = 1 + O(|(x,t)|^{10}) \tag{E.1}$$

in these coordinates, which implies

$$h_{g_a}(x) = O(|x|^9). \tag{E.2}$$

We would like to expand G_a as follows

$$G_a(x) = \Gamma(\psi_a(x)) + H_a(x),$$

where

$$\Gamma(x,t) = \frac{1}{|(x,t)|^2}\big(1+\psi(x,t)\big)$$

with some appropriate function $\psi \in C^\infty(\mathbb{R}^4 \setminus \{0\})$ and $H_a \in C^{2,\alpha}(M)$. In the following we always write g instead of g_a. Locally around $0 \in \mathbb{R}^4_+$ we define the operators

$$Ku := \partial_i\big((g^{ij}-\delta^{ij})\partial_j u\big), \quad Lu = -|(x,t)|^2\Delta u + 4 < \nabla u, (x,t) > .$$

Then, due to (E.1)

$$\begin{aligned} |(x,t)|^4 L_g\Gamma = L\psi - |(x,t)|^4 K\left(\frac{\psi}{|(x,t)|^2}\right) - |(x,t)|^4 K\left(\frac{1}{|(x,t)|^2}\right) + \frac{1}{6}R_g|(x,t)|^2\big(1+\psi\big) \\ + O\left(|(x,t)|^{10}|D^2\psi|\right). \end{aligned} \tag{E.3}$$

Here and in the following $g(y) = O(f(y))$ means

$$|\nabla^k g(y)| \le C(k)|\nabla^k f(y)|, \quad k = 0,1,2.$$

Furthermore

$$B_g\Gamma = -\frac{\partial_t\psi}{|(x,t)|^2} + h_g\Gamma. \tag{E.4}$$

We need to find ψ such that $L_g\Gamma \in C^{0,\alpha}$ and $\partial_t\psi = 0$. Then $B_g\Gamma \in C^{1,\alpha}$, which is crucial for our argument. We will successively remove the singularities in (E.3) by using homogeneous function of increasing degree.
Let H_k be the space of (smooth) homogeneous functions on $\mathbb{R}^4_+$ and C^l the space of functions $u \in C^\infty(\mathbb{R}^4_+ \setminus \{0\})$ such that $u(x,t) = O\left(|(x,t)|^l\right)$. Now we state a first Lemma.

Lemma 15. *If $\psi \in H_k,\ k \leq 2$, then*

(a) $\partial^\alpha\psi \in H_{k-|\alpha|}$ for all multiindizes α

(b) $-|(x,t)|^4 K\left(\frac{1}{|(x,t)|^2}\right) + \frac{1}{6}R_g|(x,t)|^2\big(1+\psi\big) \in \sum_{l=k+1}^4 H_l + C^5$.

Since we work with homogeneous functions on $\mathbb{R}^4_+$, boundary value problems on $\mathbb{R}^4_+$ can be reduced to boundary value problems on the upper half sphere S^4_+. Therefore we need the following Propositions.

Proposition 28. *Let $\bar{f} \in C^\infty(S^{n-1}_+)$ and $\bar{\psi} \in C^\infty(S^{n-1}_+)$ be a solution of the boundary value problem*

$$\begin{cases} -\Delta_{S^{n-1}_+}\bar{\psi} + k(n-2-k)\bar{\psi} = \bar{f} & S^{n-1}_+ \\ \partial_\nu\bar{\psi} = 0 & \partial S^{n-1}_+, \end{cases} \tag{E.5}$$

then $\psi(x,t) := |(x,t)|^k\bar{\psi}\left(\frac{(x,t)}{|(x,t)|}\right)$ is a homogeneous function of degree k, which solves the boundary value problem

$$\begin{cases} L_n\psi = f & \mathbb{R}^n_+ \setminus \{0\} \\ \partial_t\psi = 0 & \partial\mathbb{R}^n_+ \setminus \{0\}. \end{cases} \tag{E.6}$$

Here $f(x,t) := |(x,t)|^k\bar{f}\left(\frac{(x,t)}{|(x,t)|}\right)$ and $L_n u = -|(x,t)|^2\Delta u + 2(n-2) < \nabla u, (x,t) >$.

Proof. With n-dimensional polar-coordinates the Laplacian is given by

$$\Delta u = \frac{1}{r^{n-1}}\big(\frac{d}{dr}\big(r^{n-1}\frac{d}{dr}u\big)\big) + \frac{1}{r^2}\Delta_{S^{n-1}_+}u.$$

If we write $(x,t) = r\omega$ with $\omega \in S^{n-1}_+$, then

$$\Delta\psi(x,t) = k(k+n-2)r^{k-2}\bar{\psi}(\omega) + r^{k-2}\Delta_{S^{n-1}_+}\bar{\psi}(\omega).$$

Furthermore, since ψ is homogeneous of degree k we have the equality $<\nabla\psi,(x,t)>= k\psi$. The previous identities yield

$$\begin{aligned} L_n\psi(x,t) &= -r^k\Delta_{S^{n-1}_+}\bar{\psi}(\omega) - k(k+n-2)r^k\bar{\psi}(\omega) + 2(n-2)kr^k\bar{\psi}(\omega) \\ &= r^k\bar{f}(\omega) = f(x,t). \end{aligned}$$

The boundary equation follows easily. Therefore the proof is completed. □

If we want to solve (E.5) we need to have knowledge about the sectrum $\sigma(-\Delta_{S^{n-1}_+})$ of the Laplacian on S^{n-1}_+ with respect to Neumann boundary conditions.

Proposition 29. *For $\bar{f}\in C^\infty(S^{n-1}_+)$ the boundary value problem*

$$\begin{cases} -\Delta_{S^{n-1}_+}\bar{\psi} - \lambda\bar{\psi} = \bar{f} & S^{n-1}_+ \\ \partial_\nu\bar{\psi} = 0 & \partial S^{n-1}_+ \end{cases} \tag{E.7}$$

has a solution iff $\bar{f}\in ker\big(-\Delta_{S^{n-1}_+}-\lambda\big)^\perp$. Moreover

$$\sigma(-\Delta_{S^{n-1}_+}) = \{l(l+n-2)\ :\ l\in\mathbb{N}_0\}.$$

Furthermore if $\lambda\notin\sigma(-\Delta_{S^{n-1}_+})$, then

$$||\bar{\psi}||_{C^{2,\alpha}} \leq C||\bar{f}||_{C^{0,\alpha}}.$$

Proof. The statement follows from standard elliptic theory (see [26, 33]) and the identity

$$\sigma(-\Delta_{S^{n-1}_+}) = \sigma(-\Delta_{S^{n-1}}).$$

□

Now we are prepared to begin the expansion of the Green's function. For $\psi=0$, (E.3) yields

$$-|(x,t)|^4 K\left(\frac{1}{|(x,t)|^2}\right) + \frac{1}{6}R_g|(x,t)|^2 = \sum_{k=1}^4 g_k + C^5,$$

where g_k are homogeneous function of degree k on $\mathbb{R}^4_+\setminus\{0\}$. More precisely

$$g_1(x,t) = -16\frac{th_{ij}x_ix_j}{|(x,t)|^2},$$

where h_{ij} are the coefficients of the umibilicity tensor Π at $a \in \partial M$ in ψ_a Fermi-coordinates. We set $\bar{g}_1 = g_1|_{S^3_+}$ and observe that the boundary value problem

$$\begin{cases} -\Delta_{S^3_+}\bar{\psi} + \bar{\psi} = -\bar{g}_1 & S^3_+ \\ \partial_\nu \bar{\psi} = 0 & \partial S^3_+, \end{cases}$$

has a unique solution $\bar{\psi}_1$. Proposition 28 implies that the function

$$\psi_1(x,t) := |(x,t)|\bar{\psi}_1\left(\frac{(x,t)}{|(x,t)|}\right)$$

solves $L\psi_1 = -g_1$ and $\partial_t\psi_1 = 0$. Furthermore Proposition 29 implies

$$|\nabla^k\psi_1| \le C|\Pi(a)||(x,t)|^{1-k}$$

for $k = 0, 1, 2$. In addition

$$\int_{S^3_+} \psi_1 dS = 0,$$

because $h_g(a) = 0$.

We set $\Gamma_1(x,t) = \frac{1}{|(x,t)|^2}\big(1 + \psi_1(x,t)\big)$ and observe by (E.3) and Lemma 15 that

$$|(x,t)|^4 L_g\Gamma_1 = \sum_{l=2}^{4} b^1_l + C^5,$$

where $b^1_l \in H_l$. Now we want to remove the function b^1_2. This can be done with the ansatz

$$\Gamma_2(x,t) = \frac{1}{|(x,t)|^2}\big(1 + \psi_1(x,t) + \psi_2(x,t)\big)$$

if the unknown function ψ_2 solves the equation $L\psi_2 = -b^1_2$ and $\partial_t\psi_2(x,0) = 0$. This is equivalent to

$$\begin{cases} -\Delta_{S^3_+}\bar{\psi} = -\bar{b}^2_1 & S^3_+ \\ \partial_\nu \bar{\psi} = 0 & \partial S^3_+, \end{cases}$$

which is not solvable unless

$$\int_{S^3_+} \bar{b}^2_1 dS = 0,$$

Since this identity is wrong in general we need a further Proposition to continue the expansion.

Proposition 30. *If $k \in \{2,3,4\}, m \in \mathbb{N}_0$ and $p \in H_k$, then there exist $p_0, \cdots, p_{m+1} \in H_k$ such that $\partial_t p_i = 0$ on $\partial\mathbb{R}^4_+$ and*

$$L\left(\sum_{i=0}^{m+1} p_i \log(|(x,t)|)^i\right) = p\log(|(x,t)|)^m.$$

Proof. We prove this statement by induction on m. Therefore we frequently use the formula

$$L(q\log(r)^m) = Lq\log(r)^m + m(2-2k)q\log(r)^{m-1} - m(m-1)q\log(r)^{m-2} \tag{E.8}$$

for $m \geq 0$, $q \in H_k$ and $r = |(x,t)|$.
$m = 0$: Let $< \bar{e}_1, \cdots, \bar{e}_l >= kern\big(-\Delta_{S^3_+} + k(2-k)id\big)$. Furthermore we write

$$\bar{p} = \sum_{j=1}^{l} \langle \bar{p}, \bar{e}_j\rangle_{L^2(S^3_+)}\bar{e}_j + \left(\bar{p} - \sum_{j=1}^{l}\langle \bar{p}, \bar{e}_j\rangle_{L^2(S^3_+)}\bar{e}_j\right) =: \bar{p}' + \bar{p}''.$$

Then there exists $\bar{q}''$ such that

$$\begin{cases} -\Delta_{S^3_+}\bar{q}'' + k(2-k)\bar{q}'' = \bar{p}'' & S^3_+ \\ \partial_\nu \bar{q}'' = 0 & \partial S^3_+. \end{cases}$$

Let $p, p', p'', q'' \in H_k$ be the homogeneous extensions of the previous functions then

$$p = p' + p'', \quad L(p') = 0, \quad L(q'') = p''$$

and (E.8) implies

$$L\left(q'' + \frac{1}{2-2k}p'\log(r)\right) = p'' + p' = p.$$

Hence the case $m = 0$ is proved.

$m - 1 \mapsto m$. As above we write $p\log(r)^m = p'\log(r)^m + p''\log(r)^m$, where $L(p') = 0$, $L(q'') = p''$ for some $q'' \in H_k$. Again (E.8) yields

$$\begin{aligned} &L\left(q''\log(r) + \frac{p'}{(m+1)(2-2k)}\log(r)^{m+1}\right) - p'\log(r)^m - p''\log(r)^m \\ &= m(4-2k)q''\log(r)^{m-1} - m(m+1)p'\log(r)^{m-1} - m(m-1)q''\log(r)^{m-2}. \end{aligned}$$

By induction we can solve the last equation, which proves the Proposition. □

Now we return to the expansion of the Green's function. Since $ker(-\Delta_{S^3_+}) = \mathbb{R}$, $L(|(x,t)|^2) = 0$. The proof of the previous Proposition in case $m = 0$ implies the existence of $p \in H_2$ and $c \in \mathbb{R}$ such that

$$L\left(p + c|(x,t)|^2\log(r)\right) = -b^1_2.$$

Moreover

$$\int_{S^3_+} pdS = 0 \quad \text{and} \quad c = -\frac{1}{2|S^3_+|}\int_{S^3_+} b_1^2 dS.$$

Furthermore, due to (A.1),

$$c = \frac{1}{2|S^3_+|}\int_{S^3_+} b_1^2 dS = \beta|\Pi(a)|^2$$

for some constant $\beta > 0$. We set $\psi_2 := p + c|(x,t)|^2\log(r)$ and observe

$$|(x,t)|^4 L_g\Gamma_2(x,t) = \sum_{l=3}^{4} b_l^2 + c_1|(x,t)|^4\log(r) + C^5(1+\log(r)), \quad c_1 \in \mathbb{R}$$

with $b_l^2 \in H_l$. Finally, due to Proposition 30 we find $p_1, p_2 \in H_3$ and $q_1, q_2, q_3 \in H_4$ such that

$$|(x,t)|^4 L_g\Gamma_4(x,t) = C^5(1+\log(r)^2),$$

if we set

$$\Gamma_4(x,t) = \frac{1}{|(x,t)|^2}\left(1 + \sum_{i=1}^{4}\psi_i(x,t)\right)$$

and

$$\psi_3 = p_1 + p_2\log(r), \quad \psi_4 = q_1 + q_2\log(r) + q_3\log(r)^2.$$

Furthermore, due to our construction $\partial_t\Gamma_4(x,0) = 0$. Finally on M, we define $\Gamma(x) = \chi_\rho(|\psi_a(x)|)\Gamma_4(\psi_a(x))$ which is in $C^\infty(M \setminus \{a\})$. More precisely, our construction and assumption (1.2) yield

$$\begin{cases} L_g\Gamma \in C^{0,\frac{1}{2}}(M), \\ B_g\Gamma \in C^{1,\frac{1}{2}}(\partial M) \end{cases} \tag{E.9}$$

as well as $\Gamma, |\nabla\Gamma| \in L^1(M)$. Using the Green formulas, we easily deduce the following Proposition.

Proposition 31. *For $\varphi \in C^2(M)$ the following identity holds true:*

$$\begin{aligned} &2|S^3_+|\varphi(a) \\ &\quad = \int_M \Gamma(x)L_g\varphi dV_g + \int_{\partial M}\Gamma B_g\varphi d\sigma_g - \int_M L_g\Gamma\varphi dV_g - \int_{\partial M} B_g\Gamma\varphi d\sigma_g. \end{aligned}$$

Since $Q(M,\partial M,[g]) > 0$, the operator

$$(L_g, B_g) : C^{2,\alpha}(M) \to \big(C^{0,\alpha}(M), C^{1,\alpha}(\partial M)\big), \ u \mapsto (L_g u, B_g u)$$

is an isomorphism. Due to (E.9), there exits exactly one function $H_a \in C^{2,\frac{1}{2}}(M)$ such that

$$L_g H_a = -L_g \Gamma, \quad B_g H_a = -B_g \Gamma.$$

We set

$$G_a(x) := \Gamma(x) + H_a(x)$$

and observe

$$2|S^3_+|\varphi(a) = \int_M G_a(x) L_g \varphi dV_g + \int_{\partial M} G_a(x) B_g \varphi d\sigma_g \quad \forall \varphi \in C^2(M),$$

hence G_a is the normalized Green function at a.

Summing up we have proved the following expansion of the normalized Green's function G_a in conformal Fermi-coordinates.

Proposition 32. *Let (M, g) be a four-dimensional compact Riemannian manifold with boundary and positive Sobolev-quotient. If g is a metric such that (E.1) holds, then*

$$G_a(x) = \Gamma(\psi_a(x)) + H_a(x),$$

where Γ is singular at 0 and $H_a \in C^{2,\frac{1}{2}}(M)$. More precisely:

$$G_a(\psi_a^{-1}(x,t)) = \frac{1}{|(x,t)|^2}\big(1 + \psi_1(x,t) + \psi_2(x,t)\big) + c|\Pi(a)|^2 \log(r) + H_a(a) + O(r\log(r)),$$

where ψ_i are homogeneous functions of degree i such that

$$\int_{S^3_+} \psi_i dS = 0$$

and $\Pi(a)$ is the umbilicity tensor at a.

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