# ON THE EIGENVALUES OF LINEAR AUTONOMOUS DIFFERENTIAL DELAY EQUATIONS

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1. The stability of the zero solution of the equation

$$\dot{y}(t) = L(y_t), \tag{1}$$

with $L: C[-1,0] \to R$ linear and continuous with respect to the supremum-norm and with $y_t(a) = y(t+a)$, is determined by the distribution of the eigenvalues, i.e. of the complex solutions $\lambda = u + iv$ of

$$\lambda - L(\exp(\lambda\cdot)) = 0. \tag{2}$$

We consider the case $L \neq 0$, $L(\varphi) \leqslant 0$ for $\varphi \geqslant 0$, which includes the equations $\dot{y}(t) = -\alpha y(t)$ (3)

and $\dot{y}(t) = -\alpha y(t-1)$ (4)

with $\alpha > 0$. We may write $L(\varphi) = -\alpha\int_{-1}^{0} \varphi(a)ds(a)$ with $\alpha > 0$ and $s \in S := \{\sigma: [-1,0] \to R \mid \sigma(-1) = 0,\ \sigma \text{ increasing},\ \sigma(1) = 1\}$.

Equation (2) becomes $f(\lambda,\alpha,s) := \lambda + \alpha\int_{-1}^{0} \exp(\lambda a)ds(a) = 0.$ (5)

The parameter $\alpha$ may serve as a measure of the power of the negative feedback in the system given by $\dot{y}(t) = -\alpha\int_{-1}^{0} y(t+a)ds(a)$ while the function s describes the hereditary dependence. For example, one might expect that for s concave the stability is in some way less than for s convex because the system takes longer to produce a sufficient reaction to perturbations of the equilibrium. In the extremal cases this conjecture is right in the following way. Equation (3) corresponds to the minimal (convex) function in S, and we have asymptotic stability for all $\alpha > 0$. Equation (4) comes from the maximal (concave) function in S, and for every $\alpha > \pi/2$ there is at least one eigenvalue with $u > 0$, see the paper of Wright [6]. In addition, we have

<u>Theorem 1</u>: For every $s \in S$ and for every $\alpha < \pi/2$, every eigenvalue has negative real part.

Proof: [4].

We shall see how this behaviour of the minimal and maximal function in S carries over to two classes of smooth functions in S.

First, let us state some preliminary facts.

$$\lambda + \alpha\int_{-1}^{0} \exp(\lambda a)ds(a) = 0 \iff \Big(u + \alpha\int_{-1}^{0} \exp(ua)\cos(va)ds(a) = 0 \;\wedge\; v + \alpha\int_{-1}^{0} \exp(ua)\sin(va)ds(a) = 0\Big), \tag{6}$$

$$f(\lambda,\alpha,s) = 0 \iff f(\bar{\lambda},\alpha,s) = 0, \tag{7}$$

$$f(\lambda,\alpha,s) = 0 \wedge u \geqslant 0 \implies |\lambda| \leqslant \alpha. \tag{8}$$

<u>Theorem 2</u> (Stability for all $\alpha > 0$): Let $s \in S \cap C^2[-1,0] \cap C^3(-1,0]$ and $s'(-1) = s''(-1) = 0$, $s''' \geqslant 0$, $s''' \neq 0$. Then for every

$\alpha > 0$, every eigenvalue has negative real part.

Sketch of proof: Integration by parts yields $\int_{-1}^{0}\cos(va)ds(a) > 0$ for all $v > 0$. Hence there are no eigenvalues on $iR$, by (6) and (7). Now the existence of an eigenvalue in $C^+ := R^+ + iR$ for certain $\alpha_o$ would imply the existence of an eigenvalue in $C^+$ for $\alpha = 1 < \pi/2$, by (8) and by the continuous dependence of the eigenvalues on $\alpha$. But this contradicts Theorem 1.

Remark: Theorem 2 holds for $s: a \to (a+1)^\beta$, $\beta > 2$. The case $\beta = 2$ shows that Theorem 2 is optimal in a certain sense: $\tilde{s}: a \to (a+1)^2$ fulfills the hypotheses except of $\tilde{s}''' \neq 0$, and $f(2\pi k i, (2\pi k)^2/2, \tilde{s}) = 0$ for $k \in Z \setminus \{0\}$.

Theorem 3 (Instability): For $s \in S \cap C^1[-1,0]$ with $s(a) \geqslant a+1$, there are $\alpha > 0$ and $\lambda$ with $u > 0$ and $f(\lambda,\alpha,s) = 0$.

Remark: The hypothesis in Theorem 3 can be replaced by "s concave".

Sketch of proof: Let $s \in S$. Define a mapping

$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}: R^2 \times R^+ \to R^2$ by $F_1(u,v,\alpha) = \mathrm{Re}\, f(\lambda,\alpha,s)$, $F_2(u,v,\alpha) = \mathrm{Im}\, f(\lambda,\alpha,s)$. Suppose $iv \in iR^+$ is an eigenvalue for $s$ and $\alpha > 0$. Then $F(0,v,\alpha) = 0$, and

$$d := \det \begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} (0,v,\alpha) \geqslant \alpha^2 \left(\int_{-1}^{0} a \sin(va)ds(a)\right)^2.$$

For $d > 0$ there are neighbourhoods $U$ of $\alpha$ and $W$ of $(0,v)$ and a mapping $G: U \to W$ with $G(\alpha) = (0,v)$ and $F \circ (G, id) = 0$ on $U$, hence $G_1(\alpha') + iG_2(\alpha')$ are eigenvalues for $\alpha' \in U$, and $G_1'(\alpha) > 0$ would imply the assertion. - We have $G_1(\alpha) = v/d \int_{-1}^{0} a \sin(va)ds(a)$. Therefore we only have to find an eigenvalue $iv \in iR^+$ with $\int_{-1}^{0} a \sin(va)ds(a)$ positive. - Let $s \in S \cap C^1[-1,0]$, $s(a) \geqslant a+1$. Then $\int_{-1}^{0}\cos(\pi a)ds(a) = 1 + \pi\int_{-1}^{0}\sin(\pi a)s(a)da \leqslant 1 + \pi\int_{-1}^{0}(a+1)\sin(\pi a)da = 0$, and the function $h: t \to \int_{-1}^{0}\cos(ta)ds(a)$ has a zero $v$ in $(0,\pi]$. Obviously, $\int_{-1}^{0}\sin(va)ds(a) < 0$ and $\int_{-1}^{0} a\sin(va)ds(a) > 0$ and $f(iv, -v/\int_{-1}^{0}\sin(va)ds(a), s) = 0$.

2. For the simplest smooth function in S, $s(a) = a+1$, we can describe the location of all eigenvalues for all $\alpha > 0$.

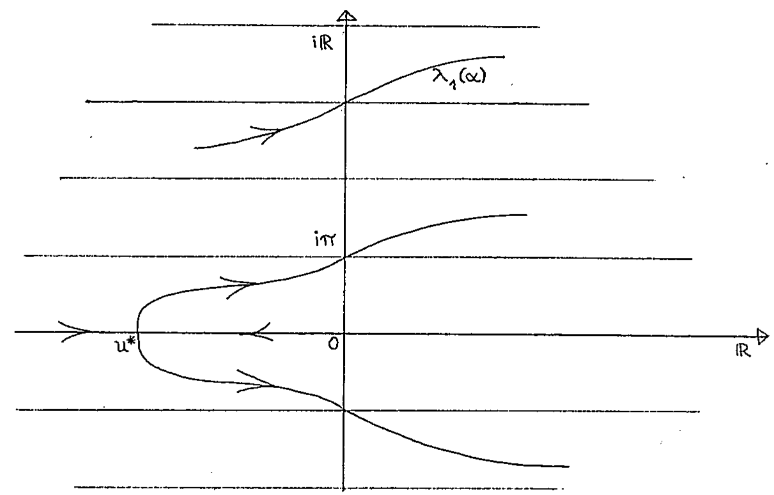
Theorem 4: Let $s(a) = a+1$ for $-1 \leqslant a \leqslant 0$.

a) For every $\alpha > 0$, every eigenvalue lies in one of the strips $R + i(-2\pi,2\pi)$ and $R \pm i(2\pi k, 2\pi k + 2\pi)$ with $k \in N$.

b) For every $\alpha > 0$ and every $k \in N$, there is exactly one eigenvalue $\lambda_k(\alpha)$ in $R + i(2\pi k, 2\pi k + 2\pi)$. We have

$$\lambda_k(\alpha) \begin{cases} \in R^- + i(2\pi k, 2\pi k + \pi) & \text{for } \alpha < \alpha_k := (2\pi k + \pi)^2/2 \\ = iv_k := i(2\pi k + \pi) & \text{for } \alpha = \alpha_k \\ \in R^+ + i(2\pi k + \pi, 2\pi k + 2\pi) & \text{for } \alpha > \alpha_k. \end{cases}$$

c) For every $\alpha > 0$, there are exactly two eigenvalues in $R + i(-2\pi,2\pi)$. Let $\alpha^* := -2u^*\exp(u^*)$ with $u^* < 0$ and $2\exp(u^*) - 2 = u^*$. For $\alpha \leqslant \alpha^*$, both eigenvalues are real. If we denote them by $u_1(\alpha)$ and $u_2(\alpha)$ with $u_1(\alpha) \leqslant u_2(\alpha)$, then $u_1(\alpha) < u^* < u_2(\alpha) < 0$ for $\alpha < \alpha^*$, $u_1(\alpha) \to -\infty$ and $u_2(\alpha) \to 0$ for $\alpha \to 0$, and $u_1(\alpha^*) = u^* = u_2(\alpha^*)$.
For $\alpha^* < \alpha < \alpha_o := \pi^2/2$, there is exactly one eigenvalue in $R^- + i(0,\pi)$, for $\alpha = \alpha_o$ $i\pi$ is an eigenvalue, and for $\alpha > \alpha_o$ there is exactly one eigenvalue in $R^+ + i(\pi,2\pi)$.




The arrows indicate the direction of increasing $\alpha$.

Remarks: 1) We see: If $\lambda$ is an eigenvalue with $u > 0$, then $|v| > \pi$. This exhibits one of the difficulties which arise if one tries to prove the existence of a nonconstant periodic solution of the nonlinear equation $\dot{x}(t) = -\alpha\int_{-1}^{0} x(t+a)da[1 + x(t)]$. - Even in the simpler case of $\dot{x}(t) = -\alpha x(t-1)[1 + x(t)]$ the existence of eigenvalues of the linearised equation with $u > 0$ and $0 < v < \pi$ is required, see the different proofs of Nussbaum [3], Grafton [2] and Chow [1].

2) A similar theorem concerning the equation $\lambda + \alpha\exp(-\lambda) = 0$ was proved by Wright [6]. He used elementary functions to derive his result. Our method is different:

Remarks on the proof of Theorem 4: Set $f(\lambda,\alpha) := f(\lambda,\alpha,\mathrm{id} + 1)$. We have

$$f(\lambda,\alpha) = 0 \iff \lambda \neq 0 \ \wedge \ (\lambda^2 + \alpha)\exp(\lambda) = \alpha. \tag{9}$$

From (9) we infer a) and

$\{(iv,\alpha) \in i\mathbb{R} \times \mathbb{R}^+ | f(iv,\alpha) = 0\} = \{((2\pi k + \pi)i, (2\pi k + \pi)^2/2) | k \in \mathbb{Z}\}$.

To explain the method of our proof let us try to show that there are exactly two zeros of $f(\cdot,\pi^2/2)$ in $G := \mathbb{R} + i(-2\pi,2\pi)$. We know that there are exactly two zeros in $G \cap i\mathbb{R}$, namely $\pm i\pi$, and that $iv \in G$ and $f(iv,\alpha) = 0$ imply $v = \pm i\pi$, $\alpha = \pi^2/2$.

i) Suppose there is another zero in G, with $u > 0$. Then there exist $\alpha < \pi^2/2$ and $\lambda \in \mathbb{R}^+ + i(-2\pi,2\pi)$ with $f(\lambda,\alpha) = 0$, too. For $\alpha' \in [1,\alpha]$, every zero with $\lambda \in G$ and $u > 0$ lies in the bounded open set $B := (0,\alpha+1) + i(-2\pi,2\pi)$ (because of $\alpha' \neq \pi^2/2$ there is no zero of $f(\cdot,\alpha')$ on $i\mathbb{R} \cap \partial B$). Hence $f(\lambda,1) = 0$ with $u > 0$ in contradiction to Theorem 1.

ii) Suppose there is a zero in G with $u < 0$. Then there are $\alpha > \pi^2/2$ and $\lambda \in G$ with $u < 0$ and $f(\lambda,\alpha) = 0$. We need

Proposition 1: $\forall\ \alpha > 0\ \exists\ T < 0$: $\alpha' \geq \alpha \wedge \lambda \in G \wedge f(\lambda,\alpha) = 0 \Rightarrow T < u$.

(Proof: (9) implies $((u^2 + 4\pi^2)/\alpha + 1) \geq |\lambda^2/\alpha' + 1| \geq \exp(-u)$, hence $u^2/\alpha \geq \exp(-u) - 1 - 4\pi^2/\alpha$.)

As above, a continuity argument now yields the existence of eigenvalues in G with $u < 0$ for every $\alpha > \pi^2/2$. - But on the other hand we have

Proposition 2: $\exists\ \alpha^* > \pi^2/2$: $\lambda \in G \wedge f(\lambda,\alpha^*) = 0 \Rightarrow u > 0$.

3. There is another fact which expresses an increase of stability if the maximal (step-) function in S is replaced by a smaller, smooth function: The branches of eigenvalues in the right half-plane become bounded. Such a branch is a maximal connected subset of the set $P := \{\lambda \in \mathbb{C} | u > 0 \wedge (\exists\ \alpha > 0: f(\lambda,\alpha,s) = 0\}$.

For $s(a) = 1$ on $(-1,0]$, there are unbounded branches: Choose $v \in (\pi/2,\pi)$, set $u_v := -v\cos(v)/\sin(v)$ and $\alpha_v := -u_v\exp(u_v)/\cos(v)$. Then $f(u_v + iv,\alpha_v,s) = 0$, and $\{u_v + iv | \pi/2 < v < \pi\}$ is an unbounded connected subset of P.

On the other hand, we have

Theorem 5: For $s \in S \cap C^3[-1,0]$ with $s'(-1) > 0$ and $s'(0) > 0$, every connected subset of P is bounded.

For $s: a \to a+1$, the proof is simple: From the preceding theorem we know that every connected subset Q of P has bounded imaginary part $\mathrm{Im}\ Q := \{\mathrm{Im}\ \lambda | \lambda \in Q\}$. For $\lambda \in Q$ and suitable $\alpha > 0$, (9) gives $\lambda^2/\alpha + 1 = \exp(-\lambda)$, $(u^2 - v^2)/\alpha + 1 = \exp(-u)\cos(v)$. For sequences $\lambda_n$, $\alpha_n$ with $\lambda_n \in Q$ and $u_n \to \infty$ we infer $1 \leq \lim_{n\to\infty} (u_n^2/\alpha_n + 1) = \lim_{n\to\infty} (v_n^2/\alpha_n + \exp(-u_n)\cos(v_n)) = 0$, contradiction.

4. The proofs of Theorems 2 - 5 can be found in [5].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