

Marlies Ahlert

If Not Only Numbers Count: Allocation of Equal Chances*

Abstract:

It is assumed that medical guidelines specify the appropriate amount of a divisible good which each individual should receive. Individual requirements and probabilities that the treatment is successful if an appropriate amount is received differ. The same applies to the success probabilities if individuals are inadequately treated. If supply is insufficient to serve all with appropriate amounts an allocation decision is necessary. We define probabilistic allocation rules that allocate chances of successful treatment to all individuals. We analyse a specific random allocation rule that assigns maximal equal gains of chances. We characterize the equal gain rule axiomatically.

1. Introduction and Overview

The problem and model of this paper is inspired by Taurek's (1977) intriguing example of six patients competing for a fixed supply of five units of a drug. Five patients need one unit each of the medicine to survive while one needs five. If allocation is non-stochastic only three allocations seem plausible: either the five, the one or (in the name of equality) none can be treated. The quest for equality without having to forego all lives for sure raises the issue of what it means to have equal access to basic treatment or what it means to provide equal chances to all individuals (c.f. Parfit 1978, and the Taurek-Parfit discussion e.g. in Lübbe 2004). As Taurek himself suggested introducing probabilistic allocations renders it feasible to give each person an equal chance of $\frac{1}{2}$ by assigning the total amount with probability $\frac{1}{2}$ to the one who needs all of the drug's supply and with probability $\frac{1}{2}$ one unit to each of the other individuals.

The models of this paper explore and generalize Taurek's basic idea. *Section 2* lays the ground by generalizing Ahlert 2006. It is assumed that there is a fixed supply of a medical resource and a fixed set of individuals who are in need of treatment. Medical guidelines specify for each patient which amount

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of the good is appropriate to treat that individual. The allocation problem that is considered here arises, if the sum of the amounts required according to the guidelines and individual diagnoses is larger than the supply. An individual who receives the appropriate amount of the medical resource as specified by the guideline has a certain probability to become healthy again (or of survival, or of success of the treatment). Without the treatment there is another, lesser probability of success which forms the individual's component of a status quo. In view of this random allocations and random allocation rules are introduced that lead to gains in success probability for the individuals. In medical treatment problems often different guidelines or schools exist. We define how a finite number of competing medical guidelines can be combined to find the most efficient treatments. The fact that combinations of guidelines are possible implies that the model is rich enough to allow the application of axiomatic techniques known from bargaining theory. In *section 3* we calculate the chances of success under allocation procedures, first with respect to one guideline and for combinations of finitely many guidelines afterwards. We also give examples how to construct the set of feasible chance allocations for a given problem. We show that the sets of feasible chances are convex and comprehensive with respect to the status quo. They can be interpreted as special cases of bargaining sets. In *section 4* desirable properties of a random allocation rule assigning chances of success to individuals are specified. These properties are Weak Pareto-Efficiency, Symmetry, Monotonicity in Gains, and Translation Invariance. In *section 5*, by these properties we characterize the rule E that allocates equal gains in chances to all individuals. In the general case we consider in our model, gains in success probability are measured compared to a status quo that is defined by the probability of success under no treatment. The model and the solution, however, can also be applied to situations where the status quo or reference point is defined to be some other state, e.g. the worst case of probability 0 for each patient. The theorem characterizes a type of egalitarian solution on a space that is a strict subspace of the space of traditional bargaining situations for n persons. To develop the appropriate space for the theorem it is crucial that finitely many guidelines can be incorporated when the set of feasible allocations is derived. *Section 6* concludes.

2. The Model

Let q be a given quantity of a divisible good that has to be allocated to n individuals. The individuals are named $1, \dots, n$. We assume that there is a medical guideline that, given the health status of each individual, specifies that for proper treatment a well defined amount q_i of the good has to be allocated to individual i . We assume that according to the guideline there are only two possible choices: to allocate amount q_i of the good to individual i or nothing. If individual i receives the quantity q_i , the probability of success of the treatment is $0 < s_i \leq 1$. If the individual receives less than q_i or nothing the probability of

success is $0 \leq r_i \leq 1$. If the individual receives q_i this increases the probability of success such that $r_i < s_i$ holds.

The preceding assumptions are not arbitrary. Quite to the contrary it will often be the case that in developing guidelines there is sufficient statistical evidence for the probabilities of success and failure. This is the strictly empirical element in specifying a guideline. But there is also a normative aspect involved since guidelines seek to fix an optimal course of action. They are setting a standard rather than making predictions etc. This standard chooses a diagnosis-quantity pair where a quantity q_i that has to be used in response to a diagnosis on the basis of the same quantitative knowledge which informs the probability estimates r_1, \dots, r_n and, in particular, s_1, \dots, s_n involved. An allocation problem of the kind scrutinized here emerges if the standard is such that the quantity of the good does not suffice to satisfy the needs of all individuals, i.e. $\sum_{i=1, \dots, n} q_i > q$. However, cases where everybody can be treated, i.e. situations with $\sum_{i=1, \dots, n} q_i \leq q$, are included, too.

Definition: Allocation Problem

A vector $(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$ of quantities with $0 < q_i \leq q$ for all $i = 1, \dots, n$, and probabilities r_1, \dots, r_n and s_1, \dots, s_n such that $0 \leq r_i < s_i \leq 1$ for all $i = 1, \dots, n$ is called a stochastic allocation problem or, for short, an allocation problem with n individuals. The set of all allocation problems with n individuals is denoted by A_n .

It may be the case that competing medical guidelines or alternative ‘diagnosis-treatment pairs’ sum up the medical evidence in different recommendations. The latter are developed by competing schools such that the same set of individuals and the same quantity q could lead to different representations of the same original problem in allocation problems $(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n) \neq (q; q'_1, \dots, q'_n; r_1, \dots, r_n; s'_1, \dots, s'_n)$. We assume that the estimates of the probabilities of success r_i without treatment are based on the same empirical evidence in all schools. They are identical but due to different normative and evaluative reasoning the suggested amounts q_i and q'_i and the probabilities s_i and s'_i , may be different. We will come back to this when we present what will be called the set of feasible chances.

For a given allocation problem, allocations that fulfil the ‘budget constraint’ are called feasible. They can be represented by n -dimensional vectors of zeros and ones indicating whether the individuals to which they refer receive the appropriate amount, i.e. are treated according to the standard, or not.

Definition: Feasible Allocation

For a given allocation problem $(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$ with n individuals a feasible allocation is a vector $x = (x_1, \dots, x_n)$ such that $x_i \in \{0, 1\}$ and $\sum_{i=1, \dots, n} x_i q_i \leq q$. x can be interpreted as follows: If $x_i = 1$, individual i receives the necessary amount q_i of the good. If $x_i = 0$, individual i receives nothing of the good. $F(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n) \subseteq \{0, 1\}^n$ —abbreviated by F —denotes the set of all feasible allocations in an allocation problem $(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$.

Among the feasible allocations there are allocations such that the amount of the good that remains under that allocation does not suffice to satisfy the need of an additional person. Such an allocation is called strongly Pareto-efficient or, for short, efficient.

Definition: Efficient Allocation

Let x be a feasible allocation for some problem $(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n) \in A_n$ that assigns the good to the individuals $j \in J \subseteq \{1, \dots, n\}$, such that for all $j \in J$ we have $x_j = 1$ and for all $k \notin J$ we have $x_k = 0$. The allocation x is called efficient, if and only if there is no $k \in \{1, \dots, n\} \setminus J$ with $\sum_{j \in J} q_j + q_k \leq q$. $E(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$ —abbreviated by E —denotes the set of all efficient allocations for the situation $(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$.

In many of the deterministic allocations in F , even in strongly Pareto-efficient allocations, some individuals may not receive any of the good though in principle adequate treatment could be provided for those individuals. This property may be regarded as undesirable since positive gains in success probabilities for all individuals whose needs could conceivably be satisfied should be deemed desirable. In response to the fact that otherwise not all ‘chances to improve chances’ would be realised we will now deal with allocation rules that incorporate the idea of chances, especially positive chances in a special way. Using the term random as in ‘random variable’ and not to prescribe any specific distribution on the set of outcomes we define random allocations as random choices of feasible allocations (cf. publications by Moulin and co-authors on random assignment problems in different contexts as well as Young 1994).

Definition: Random Allocation

For a given allocation problem $(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n) \in A_n$ a random allocation is defined by a probability distribution p on the set of feasible allocations $F(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$.

Property: Efficiency of a Random Allocation

A random allocation defined by a probability distribution p on F is called efficient, if and only if all allocations in F that have positive probability under p are efficient allocations.

Definition: Random Allocation Rule

For any problem $(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n) \in A_n$ a random allocation rule g chooses a probability distribution $g(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n) = p$ on $F(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$.

Remark

It is obvious that the set of random allocations is closed under finite lotteries on the probability distributions on F .

3. Allocation of Chances

Before we model the space of chances of success we need to introduce a few additional notational conventions. Let P denote the set of all probability distributions on F . For a given probability distribution $p \in P$ the probability that $x \in F$ is chosen is $p(x)$. For such a probability distribution p we denote the probability that individual i receives the good by $w_i(p)$. Obviously $w_i(p) = \sum_{x \in F} x_i p(x)$. Vectors $(w_1(p), \dots, w_n(p))$ form the convex hull W of F .

Since in a random allocation p individual i receives the amount q_i with probability $w_i(p)$ the probability of success for individual i under random allocation p is $w_i(p)s_i + (1 - w_i(p))r_i = r_i + w_i(p)(s_i - r_i)$. We call this term the success probability for individual i under p . This ‘chance of success’ is denoted by $c_i(p)$.

Definition: Feasible Allocation of Chances of Success

For a given $p \in P$ the vector of chances $c = (c_1(p), \dots, c_n(p))$ is called a feasible allocation of chances of success. $C(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$, short C , is the set of all feasible allocations of chances of success to the n individuals given the problem $(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$. Each feasible allocation of chances $c \in C$ is generated by some distribution p on F .

Notation

Each problem in A_n generates a set of feasible allocations of chances C . B_n denotes the set of all feasible sets of allocations of chances C generated by problems in A_n .

Remarks

1. It is easy to see that the sum of the individuals’ probabilities of receiving treatment $\sum_{i=1, \dots, n} w_i(p)$ is equal to the expected number of treated individuals.
2. For any efficient random allocation p holds: $\sum_{i=1, \dots, n} w_i(p) \geq 1$.
 Proof: If p is efficient there is at least one efficient x with positive probability $p(x)$. For each efficient x there is at least one $j(x) = j$ such that $x_j = 1$. This implies $\sum_{i=1, \dots, n} w_i(p) = \sum_{x \in E} \sum_{i=1, \dots, n} x_i p(x) \geq \sum_{x \in E} x_{j(x)} p(x) = \sum_{x \in E} 1 p(x) = 1$. •
3. C is generated by convex combinations from feasible allocations in F in the following way: Given a feasible $x \in F$ this induces a vector of allocated success probabilities $((1-x_1)r_1 + x_1s_1, \dots, (1-x_n)r_n + x_ns_n) = (r_1 + x_1(s_1 - r_1), \dots, r_n + x_n(s_n - r_n))$. Iff $x_i = 1$ the success probability is s_i , iff $x_i = 0$ the success probability is r_i . We have generated C by probability distributions on F . Any probability distribution p on F leads to the convex combination $(r_1 + w_1(p) \cdot (s_1 - r_1), \dots, r_n + w_n(p) \cdot (s_n - r_n))$.

Notation

For a given set of feasible chances C there is a special chance vector $r(C)$. This chance vector is generated by $x = (0, \dots, 0) \in F$, which is the vector where nobody is treated. No treatment leads to the vector of chances $(r_1, \dots, r_n) = r(C)$ consisting of the minimal components for each individual in C .

Lemma

For each allocation problem holds, the set C is convex.

Proof: Any convex combination $\lambda c + (1-\lambda)c'$ of two feasible allocations of chances of success c generated by p and c' generated by p' can be realized by choosing the probability $p'' = \lambda p + (1-\lambda)p'$. The chances generated by p'' are $c_i(p'') = r_i + (s_i - r_i) \sum_{x \in F} x_i (\lambda p(x) + (1-\lambda)p'(x)) = \lambda c_i(p) + (1-\lambda)c_i(p')$. •

Lemma

The set C is comprehensive in the sense that, if $(c_1, \dots, c_n) \in C$ then each vector of chances c' such that $r_i \leq c'_i \leq c_i$ for all $i = 1, \dots, n$ belongs to C , too.

Proof: The proof uses a construction of an appropriate probability on F (cf. Ahlert 2006). To prove that all component-wise reductions from c_i to r_i are feasible, consider a reduction of chances for any single individual, w.l.o.g. individual 1. Arbitrary reductions can then be achieved by sequential application of the result to all dimensions. Imagine a situation such that $c_1(p) > c'_1 \geq r_1$ and $c_2(p) = c'_2 \dots c_n(p) = c'_n$. Let x^1, \dots, x^m be all allocations x in F such that $p(x) > 0$ and $x_1 = 1$. Define a new random allocation p' on F by substituting each x^i by an allocation y^i such that $y_1^i = 0$ and $y_2^i = x_2^i, \dots, y_n^i = x_n^i$. y^i is feasible since x^i is feasible and the only difference between y^i and x^i is that individual 1 does not receive the good. p' leads to the chances $c_1(p') = r_1, c_2(p') = c_2(p), \dots, c_n(p') = c_n(p)$. Choose $\lambda = c'_1/c_1(p)$, then the probability distribution $p'' = \lambda p + (1-\lambda)p'$ implies the chances $c(p'') = c'$. •

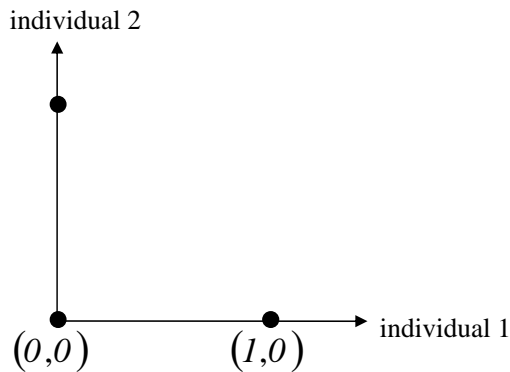
Example 1

Figure 1: feasible allocations F in example 1

This example shows the construction of C from a given problem with $n = 2$ and $q_1 + q_2 > q$, use e.g. $n = 2, q = 10, q_1 = 3, q_2 = 9; r_1 = 0.3, r_2 = 0.25; s_1 = 0.9, s_2 = 0.8$.

Figure 1 shows the set of all feasible deterministic allocations. There are three possibilities, nobody receives the good or exactly one of the individuals receives it. The case that both receive the good is not feasible. From these vectors in F we construct the set W which contains the set of all vectors of chances to receive the good (Figure 2). Receiving the good means for each person i to have a success probability of s_i , having a chance to receive the good of 0 means the success probability is r_i . The set of feasible chances of success is presented in Figure 3.

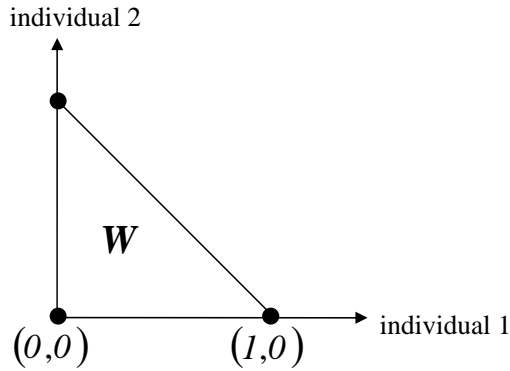


Figure 2: W is the convex hull of F generated by random allocations

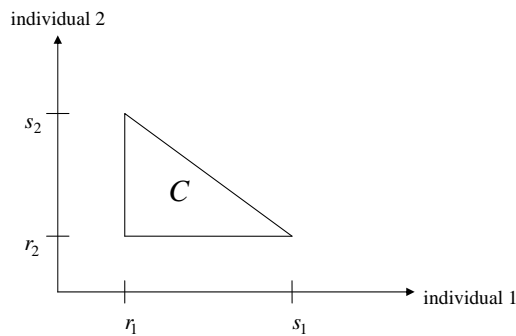


Figure 3: C is the set of chances of success in example 1

In a set of chances C it is possible to compare vectors of chances of success by the Pareto-criterion.

Definition: Strongly Pareto-Efficient Chance Vector

A vector of chances $(c_1, \dots, c_n) \in C$ is strongly Pareto-efficient, if and only if for every vector $(c'_1, \dots, c'_n) \in C$ such that $c'_i \geq c_i$ for all $i = 1, \dots, n$ it holds that $c = c'$.

Definition: Weakly Pareto-Efficient Chance Vector

A vector of chances $(c_1, \dots, c_n) \in C$ is weakly Pareto-efficient, if and only if there is no vector $(c'_1, \dots, c'_n) \in C$ such that $c'_i > c_i$ for all $i = 1, \dots, n$.

Remark

A vector of chances is strongly Pareto-efficient, if and only if it is generated by a probability distribution on F that gives positive weight only to allocations in E and is thus a Pareto-efficient random allocation. However example 2 shows that there may be random allocations that lead to a weakly Pareto-efficient vector c that is not strongly Pareto-efficient.

Example 2

$q = 10, n = 3, q_1 = 1, q_2 = 3, q_3 = 9, r_1 = r_2 = r_3 = 0, s_1 = s_2 = s_3 = 1$.

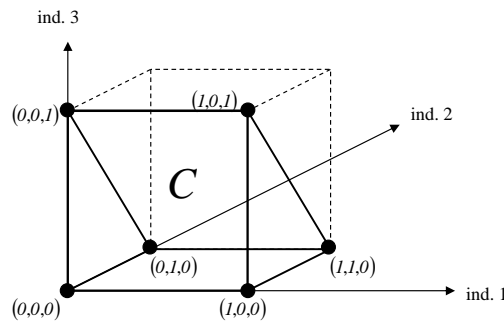


Figure 4: C is the set of chances for example 2

In example 2 all chance vectors that are in the plane with corners $(0,0,1)$, $(1,0,1)$, $(0,1,0)$, and $(1,1,0)$ are weakly Pareto-efficient, but only those on the line between $(1,0,1)$ and $(1,1,0)$ are strongly Pareto-efficient.

In order to find a solution to an allocation problem $(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$ we will condense the information given in the problem to the derived set $C(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$ of chances of success. Let g be a random allocation rule on A_n . Since we plan to model the properties of random allocations on the derived sets of chances the following property of g seems desirable.

Property: Independence of Irrelevant Information

Let g be a random allocation rule on A_n . Let $(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n) \in A_n$ and $(q'; q'_1, \dots, q'_n; r'_1, \dots, r'_n; s'_1, \dots, s'_n) \in A_n$ be two problems such that $C(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n) = C(q'; q'_1, \dots, q'_n; r'_1, \dots, r'_n; s'_1, \dots, s'_n) = C$, then $c(g(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)) = c(g(q'; q'_1, \dots, q'_n; r'_1, \dots, r'_n; s'_1, \dots, s'_n)) = c$.

This property means that the only information relevant to determine the allocated chances of success in a given problem $(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$ is the set $C(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$. If different problems lead to the same set C of feasible chances, the random allocation rule g will pick the same allocation c of chances in C . The chance vector c can in each of the two problems be generated by some probability distribution in P .

This property implies that we can now consider a random allocation rule as a mapping G that to every feasible set of chances of success C assigns a point $G(C) \in C$.

Our model tells us how to deal with one given representation of a medical treatment problem under scarcity of resources. However, we have to deal with the possibility mentioned above that different medical guidelines could map the same medical problem with n persons under the same scarcity restriction into different allocation problems in A_n . Given the same patient and diagnosis different guidelines or medical schools may specify different quantities of the resource as appropriate treatment for the patient. We can assume that each school has valid empirical data, telling the doctors if a certain patient receives the amount of the resource seen as appropriate, what the probability of success will be. However, different quantities might lead to different success probabilities. Since each guideline can be seen as locally 'optimized', where a little more of the resource would not lead to a significant improvement of the treatment, we need not consider continuous variations of quantities. Guidelines concerning diagnosis treatment pairs will specify the quantities to be assigned in case of a certain diagnosis.

Different schools of medical thought may endorse different value judgements concerning the relative merits of lesser or greater doses. Though the competing guidelines all specify a necessary and sufficient dose as adequate treatment the optimization that singles out a specific dose from a continuum of possible doses may lead to different diagnosis treatment pairs. If the guidelines fulfil the same evidence standards (as specified by EBM) and no additional evidence is available it seems reasonable to accept the different treatment proposals and the predicted success probabilities as empirically equally sound.

In view of the preceding an external observer has good reason to attribute differences in allocations to different value rather than factual judgements (as broadly understood). Such an observer can and should coherently accept the probabilistic predictions of the several guidelines as dependent on the quantities assigned (assuming that the r_i accruing to no treatment all coincide). It makes sense then to look for the most efficient allocation of chances as derived from combinations of guidelines. All stochastic combinations of feasible deterministic

allocations as proposed in any guideline based on acceptable evidence should be taken into account. This implies that convex combinations of sets of chances are formed.

Assume that for a given medical problem with resource restrictions two different allocation problems emerge. The first, $(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$ stems from guideline 1 and the second $(q; q'_1, \dots, q'_n; r_1, \dots, r_n; s'_1, \dots, s'_n)$ arises under guideline 2 in A_n . The sets of chances are then $C = C(q; q_1, \dots, q_n; r_1, \dots, r_n; s_1, \dots, s_n)$ and $C' = C(q; q'_1, \dots, q'_n; r_1, \dots, r_n; s'_1, \dots, s'_n)$ respectively. C and C' are coherent insofar as $r(C) = r(C')$. We now construct the convex combination C'' of C and C' . C'' is convex, $r(C'') = r(C) = r(C')$ and C'' is comprehensive, since C and C' are comprehensive.

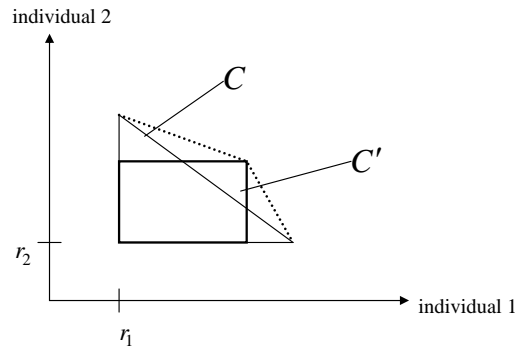


Figure 5: convex combination of two sets of chances C and C'

Figure 5 shows the set of convex combinations of C (triangle) and C' (rectangle), which are all points on and below the dotted line that are component wise larger or equal to $r = (r_1, r_2)$.

From the point of view of an external observer who intends to include all information available it makes sense to allocate 'across guidelines' by including convex combinations into the space of considered situations. We define B_n^* as the set of all situations that are convex combinations of finitely many problems $C, C', \dots \in B_n$ such that their status quo is identical i.e. $r(C) = r(C') = \dots$. This means that B_n^* is closed under convex combinations of finitely many problems with identical status quo.

4. Properties of Chance Allocation Rules

Definition: A Chance Allocation Rule

A chance allocation rule is a mapping G defined on B_n^* , such that for each $C \in B_n^*$ we have $G(C) \in C$.

Property: Efficiency of a Chance Allocation Rule G

A chance allocation rule G is weakly (strongly) efficient, if and only if for every feasible set of chances $C \in B_n^*$ the image $G(C)$ is weakly (strongly) Pareto-efficient in C .

This property means that if we apply a weakly efficient allocation rule it is never possible to increase the chance of success for all persons simultaneously. It might be possible, however, to increase the gains of some individuals given the gains of the others.

Property: Anonymity of a Chance Allocation Rule

Let G be a chance allocation rule. Consider any permutation π of the numbers $\{1, \dots, n\}$ and an allocation problem $C \in B_n^*$. The problem $\pi(C)$ defined such that $c = (c_1, \dots, c_n) \in C \Leftrightarrow \pi(c) = (c_{\pi(1)}, \dots, c_{\pi(n)}) \in C'$ is in B_n^* , too. Then $\pi(G(C)) = G(\pi(C))$ holds.

Anonymity of a rule G means that the chosen allocation of chances to individuals that need certain appropriate amounts in a given problem does not depend on the index numbers of the individuals. If index numbers are permuted the chances assigned by G in the respectively permuted set are the same as the chances assigned by G in the original problem to the permuted individuals.

With the same formal arguments like e.g. in the literature on cooperative bargaining solutions the property of anonymity implies symmetry of a chance allocation rule. In the proof it is used that for symmetric sets C and symmetric vectors c for all permutations π on $\{1, \dots, n\}$ $\pi(C) = C$ and $\pi(c) = c$ holds, and vice versa.

Property: Symmetry of a Chance Allocation Rule

Let G be a chance allocation rule and let $C \in B_n^*$ be a symmetric set of feasible chances. Then $G(C)$ is symmetric, i.e. $G_1(C) = \dots = G_n(C)$.

This property means that if from the representation in the space of feasible chances one cannot distinguish between the individuals then the allocation of chances should not distinguish between them, too.

Property: Monotonicity in Gains

Let C and C' be two feasible sets of chances in B_n^* such that $r(C) = r(C')$ and $C \subseteq C'$. Then $G(C) \leq G(C')$ (component wise) should hold.

Here we compare two situations with identical 'status quo' in chances without treatment. If in situation C' all vectors of chances are feasible that belong to C , but some more (better) vectors in addition, then the chance allocation chosen by the rule G should not be worse in C' than in C for anybody. Note, that new chance vectors are pulling the Pareto frontier outwards, since C and C' are con-

vex and comprehensive with the same minimum Vector r . The property means that better possibilities should not be to the disadvantage for anybody.

In the motivation above we have already talked about gains in chances compared to some status quo r . We will make use of the fact that we are interested in the individual gains in chances and not absolute chances and formulate the following property of invariance.

Property: Translation Invariance

Let C be a set from B_n^* and let $t = (t_1, \dots, t_n) \in [0, 1]^n$ such that $C - t \in B_n^*$ with $C - t := \{x | \exists c \in C \text{ such that } x = c - t\}$. Then $G(C - t) = G(C) - t$.

Since in this model only the comparison between differences of success chances after the allocation of treatment and without treatment count the translation of the set C should not change the allocation of gains. If translations of C are made by any vector $t \leq r(C)$ the translated situation will again belong to B_n^* . t_i can be interpreted as a shift in the probability of success with and without treatment for individual i .

5. Characterization of the Equal Gain Rule

On B_n^* we define a chance allocation rule E such that for each $C \in B_n^*$ rule E chooses an allocation of chances with equal gains in chances for each individual.

Definition: Equal Gain rule E

Let $C \in B_n^*$ be given. $E(C)$ is the maximal point in C such that $E_1(C) - r_1(C) = \dots = E_n(C) - r_n(C)$.

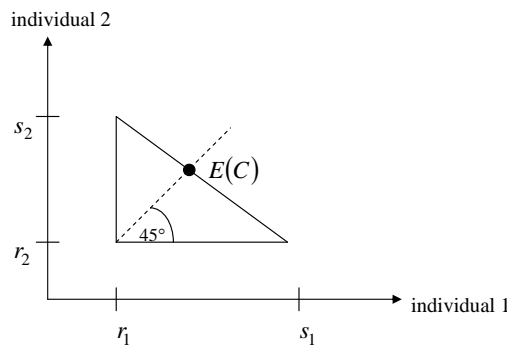


Figure 6: graphical representation of E for $n = 2$

Figure 6 shows that the chosen allocation lies on the point where the main diagonal starting at $r(C)$ intersects the boundary of C .

Remark

E has the following properties which are easy to verify:
 E is weakly Pareto-efficient. This follows from the comprehensiveness of C .
 E is anonymous and therefore symmetric.
 E is monotonic in gains.
 E is translation invariant.

Theorem: Characterization of the Equal Gain Rule

E is the only rule on B_n^* that satisfies Weak Pareto-Efficiency, Symmetry, Monotonicity in Gains and Translation Invariance.

Proof: With the remark above we only have to prove the direction that there is no other rule G having the four properties. We follow the ideas of Thomson and Lensberg's proof (1989) of the characterization of the egalitarian bargaining solution. Let $C \in B_n^*$ be given and let the solution be $G(C)$. We construct C' by translating C with $t = r(C)$ such that the status quo $r(C')$ is now the 0-vector. By translation invariance the solution of C' is $G(C') = G(C) - r(C)$. $E(C')$ is a symmetric point and is weakly Pareto optimal in C' . We define C'' as the comprehensive closure of $E(C')$ in $[0, 1]^n$, i.e. $C'' = \{x \in [0, 1]^n \mid x \leq E(C')\}$. C'' is symmetric, therefore $E(C'') = G(C'') = E(C')$. It holds that $r(C'') = r(C') = (0, \dots, 0)$ and $C'' \subseteq C'$ because of the comprehensiveness of C' . We can apply monotonicity in gains and receive (i) $E(C') = E(C'') = G(C'') \leq G(C')$. If $E(C')$ is strongly Pareto-optimal in C' this implies $E(C') = G(C')$ and by translation invariance $E(C) = G(C)$. If $E(C')$ is not strongly Pareto-optimal we need a construction similar to Thomson and Lensberg's proof: For $\varepsilon > 0$ we define $x_\varepsilon = (1 + \varepsilon)E(C')$. The comprehensive closure of x_ε is defined by the n -dimensional rectangular $\{(y_1, \dots, y_n) \mid \forall i = 1, \dots, n \quad r_i \leq y_i \leq x_{\varepsilon i}\}$ and having the status quo r it is an element of B_n^* . We construct C_ε as the convex comprehensive closure of C' and the comprehensive closure of x_ε . C_ε is an element of B_n^* , too. x_ε is strongly Pareto-optimal in C_ε and therefore, by (i) $G(C_\varepsilon) = x_\varepsilon$. Since $C' \subseteq C_\varepsilon$ monotonicity in gains implies $G(C') \leq G(C_\varepsilon) = x_\varepsilon$. If $\varepsilon \rightarrow 0$ then $x_\varepsilon \rightarrow E(C')$. This implies (ii) $G(C') \leq E(C')$. Together with (i) this means $G(C') = E(C')$ and by translation invariance $G(C) = E(C)$. •

6. Concluding Remarks

The preceding proof exploits a structural analogy between the abstract models of probabilistic resource allocation and abstract models of bargaining theory. But this paper is not intended as a contribution to bargaining theory—not even of the kind that is used in moral theory. It is not envisioned that those who stand to gain or lose from different allocation rules have to find a kind of compromise among themselves. The perspective is rather that of the committee of medical doctors who seek to specify guidelines for situations of resource scarcity that potentially involve tragic choices. If such doctors as well as the society of which they are a part accept stochastic allocation rules at all it should count

as an argument that the specific rule characterized here has certain intuitively appealing properties and that it is the only one that has them.

Of course, how compelling the result is depends on how compelling the value judgments underlying properties like symmetry or anonymity seem. Though this would require a different and more extended discussion that would go beyond the limits of this paper it seems that the properties used in the characterization are quite firmly rooted in values that are rather widely shared. The more precarious premises of the argument presumably concern stochastic allocation as such. Throwing dice is polemically associated with something like gambling. On the other hand, we should not forget that putting statistical lives at risk is in general much more acceptable than intentionally harming a specific and known individual. Taurek himself exploits this effect in his own suggested solution of throwing a coin allocating equal chances to each individual. This solution of the numbers' problem relies on creating 'randomness' artificially. Such randomness is much less acceptable than a natural stochastic mechanism to most people (including medical doctors). However, admitting so much, it has to be asked whether there are viable alternatives that are not based on deception (the withholding of care or its basically stochastic nature is camouflaged) or require the intentional and non-stochastic withholding of care for specific persons out of resource scarcity. If we want to have honest dealings with resource scarcity creating randomness artificially and fairly may be the best moral solution all things considered. If so, the rule proposed and characterized here should have some moral appeal and maybe expected to play a crucial role.

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