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Preference and Similarity between Alternatives*

Abstract:
We discuss how information about choice-relevant differences between alternatives can be revealed from preference information. We provide axiomatic characterisations of two classes of Similarity Revelation Rules: one that allows for different similarity thresholds for different pairs of alternatives, and one in which the threshold is the same for all pairs of alternatives. A third result proves the necessary and sufficient condition for the characterised class of rules to yield a transitive similarity relation. The article concludes with a discussion of the limitations of the analysis and the relationship between transitivity (of preferences) and the choice-relevant similarities between options.

Keywords: Preference, Similarity, Transitivity, Freedom Ranking, Rational Choice, Social Choice.

1. Introduction

The diversity of choice options matters in many different areas of the social sciences; one is the literature on freedom rankings.[1] The main objective of the literature on freedom rankings is to compare different choice situations, depicted by sets of mutually exclusive alternatives, in terms of the freedom of choice they offer a person (Pattanaik and Xu 1990).[2] Information about the similarity between available alternatives is considered to be of crucial importance in this endeavour: a set which contains very similar options will offer less freedom than

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[1] Other areas in which diversity of choice options matters is the literature on the violation of rationality axioms due to similarity between choice options (Rubinstein 1988; Tversky 1972); studies in marketing investigating the effect the addition of a new similar product has on people's choices (Huber and Puto 1983) or contributions in political science about those differences between political parties that matter to voters (RePass 1971).

[2] For a survey of the literature, see Dowding and van Hees 2009; Barberà, Bossert and Pattanaik 2004.

a set that contains the same number of very dissimilar options. To illustrate by way of a prominent example in the literature (Pattanaik and Xu 2000), consider two choice situations, one in which a person can choose between a green and a red bus and another in which a green bus and a red car are on offer. It seems clear, so the argument goes, that the person enjoys more freedom of choice in the latter situation than in the first one because of the greater dissimilarity between the options. Alternatively, one can think of the freedom a party system offers to the citizens of a country. It seems doubtful to claim that a system in which voters can 'choose' between two communist parties on election day offers the same freedom of choice as one in which a communist party and a liberal party are up for election.

Which similarities have significance is less straightforward though. Is the similarity between a red and a blue bus equally relevant as the similarity between a bus built in 1954 and a bus built in 2009? Are dissimilar party names equally significant as dissimilarities in their proposed tax policy? More generally, which differences between alternatives do indeed matter to people and the choices they make? Contributions to the literature on freedom rankings commonly take information about the differences between alternatives as given primitives (Bavetta and Del Seta 2001; Pattanaik and Xu 2000). Pattanaik and Xu (2000), for instance, employ a given similarity relation on the alternatives in order to explore how similarity information can be incorporated in freedom rankings. They emphasise, though, that they "view similarity of options as a matter of social judgments or norms" (Pattanaik and Xu 2000, 125). In like manner, Sugden (2003) stresses the point that any account of dissimilarity between alternatives is based upon some evaluation of the world, that is, upon some judgment as to which differences are important and which ones are not. If the choice-relevant dissimilarities between options are dependent on a person's judgment as to which differences between options are choice-relevant, the question then arises as to how one can obtain information about the differences which matter to the people whose freedom is under assessment.

One important question is whose judgment about choice-relevant differences should be reflected in the similarity relation employed in freedom rankings. Should one employ the differences that matter to the person whose freedom is under assessment? Or should the similarity relation be based on a broader social judgment as to which differences matter for choice? The answer to these questions depends on the conception of freedom one is interested in and the value of freedom one aims to capture.[3] If one is interested in the specific value freedom has for an individual, that is, in the value freedom has because of what it allows a person to achieve, then it seems sensible to conjecture that (only) this person's

[3] Note that this distinction is analogous to Sen's distinction between self-evaluation and standard-evaluation in the assessment of human well-being (Sen 1987). In the case of the former a person's own values are employed to assess her standard of living, whereas in the case of the latter, values widely held in the respective society are employed for this purpose.

judgment as to which differences matter should be incorporated.[4] If, however, one is interested in freedom's non-specific value, such as the value freedom has in virtue of being a necessary condition for a person's agency, then a focus on a person's actual judgment about choice-relevant differences is too narrow.

One reason why diversity of choice options increases the value freedom has for a person's agency is that it allows her to train and develop "the human faculties of perception, judgment, discriminative feeling, mental activity, and even moral preference" (Mill 1985[1859], 122). However, in order to exercise one's faculties of perception and judgment it is necessary to move beyond a person's actual judgment about choice-relevant differences. One way to do so is to focus on a broader social judgment about choice-relevant differences in order to include differences that are perceived as and considered important by some people in society, say the difference in the taste of two exquisite wines, but possibly not (yet) by others. A different reason as to why one would want to employ a social similarity relation in freedom rankings is a concern with the freedom members of a society enjoy, such as the impact of an additional political party on the freedom a party system offers to the citizens of a country, or the effect a newly founded newspaper has on the freedom of the press in a country. In this case the judgment of choice-relevant differences between options depends on a broader social judgment prevailing in the respective society. Irrespective of whether one includes an individual's similarity relation or a social similarity relation in freedom rankings, the question remains as to how one can obtain information about the choice-relevant differences that matter to individuals and how can one obtain information about the choice-relevant differences that matter to members of a group or society.

The main objective of this article is to explore how information about the choice-relevant similarities between alternatives can be revealed from preference information. This is done on the basis of a simple assumption: if a person is indifferent between two alternatives then she does not consider them to be different in a choice-relevant way. As is shown, this allows one to induce a similarity relation from the symmetric part of a person's preferences, reflecting his or her judgments as to which differences are choice-relevant. A Similarity Revelation Rule, as characterized in this article, assigns an overall or social similarity relation to each profile of similarity relations of the members of a given group or society. Two options are considered to be choice-relevantly dissimilar overall if, and only if, a similarity threshold is reached; that is, if a sufficient number of people considers the options in question to be dissimilar in a relevant way. While the class of rules characterized in the first result of this article allows for different similarity thresholds for different pairs of alternatives, the characterization of a special case, in which the similarity threshold is required to be equal for all alternatives, is provided as well.

A third result establishes the necessary and sufficient conditions for the overall similarity relation to be transitive. The article concludes with a discussion

[4] For a more detailed discussion about the difference between freedom's specific and non-specific value, see Carter 1999.

of the limitations of the main assumption underlying the analysis of this article. More specifically, cases are discussed in which indifference in a person's preference relation does not have to entail the absence of choice-relevant dissimilarities. It is shown how the formal framework developed in this article can be reinterpreted in order to investigate a person's judgment of the similarity between options along a plurality of characteristics into an overall similarity judgment that informs the formation (of the symmetric part) of her all-things-considered preferences.

The article is structured as follows. *Section 2* introduces the notation and the Similarity Revelation Rule and discusses how a similarity relation can be derived from a person's preferences. In *section 3* the conditions which are imposed on the Similarity Revelation Rule are introduced and discussed. *Section 4* contains the characterization of the class of Similarity Revelation Rules and the special case in which the similarity threshold is the same for all pairs of alternatives. In *section 5* the necessary and sufficient conditions for the Similarity Revelation Rule to yield a transitive similarity relation are proven and the relationship between the similarity of options and the transitivity of a person's preferences is discussed. Section 6 concludes the article with a short discussion of future areas of research.

2. Notation, Definitions and Assumptions

Let X be the universal non-empty set of alternatives, assumed to be finite. S is a binary relation on X ($S \subseteq X \times X$). S is *symmetric* if, and only if, for all $x, y \in X$, xSy implies ySx; S is *transitive* if, and only if, for all $x, y, z \in X$, xSy and ySz implies xSz. $\mathscr{S}$ denotes the set of all symmetric binary relations S on X. $\mathscr{S}^N$ is the n-fold Cartesian product of $\mathscr{S}$, where $N = \{1, \ldots, n\}$, and $n \in \mathbb{N}_+$ henceforth. An element $(S_1, \ldots, S_n)$ in $\mathscr{S}^N$ is called a *similarity profile*.

One way to interpret a similarity profile is to take it to reflect those similarities between alternatives which members of a society (N) deem to be important for their choices. A similarity profile can be induced by a society's preference profile, if for all members $i \in N$ of the respective society, S_i is taken to be the symmetric part of their (complete and reflexive) preference relation.[5] The symmetric part of a person's preference relation is usually taken to reflect indifference between two alternatives. A standard account of indifference (Kreps 1988, 10) is the absence of strict preference for one alternative over the other. Dependent on the interpretation of the preferences one adopts, however, the interpretation of

[5] A preference relation R is a binary relation on X ($R \subseteq X \times X$), I denotes the symmetric part of R (for all $x, y \in X$, xIy if, and only if xRy and yRx) and P denotes the asymmetric part of R (for all $x, y \in X$, xPy if, and only if xRy and not yRx). Formally, a similarity relation can be induced by any reflexive preference relation such that for all $i \in N$, for all $x, y \in X$, xS_iy if, and only if xI_iy. The interpretation of S, as reflecting choice-relevant similarities induced from preference information is only intelligible, if preferences are assumed to be complete (for all $x, y \in X, xRy$ or yRx).

indifference may vary accordingly. Along a desire-based account of preferences for instance, indifference between two alternatives x and y can be interpreted as an equal (amount of) desire for x as for y. Along a revealed preference account, on the other hand, indifference between x and y can be interpreted as an equal disposition to choose x and y.[6]

The way described here to derive information about the similarities a person considers to be choice-relevant from his preferences is thus based on the following assumption: if a person does not strictly prefer one alternative x to another alternative y then he does not perceive a choice-relevant difference between the two options. On the other hand, if a person prefers an alternative x strictly to an alternative y, then there is a (choice-relevant) difference between the two alternatives. While the latter assumption seems uncontroversial, the first might be less straight forward. Indeed, it seems perfectly plausible for a person not to prefer one option over another, that is to be indifferent between a red and a blue bus, for instance, even though she is perfectly aware of the difference between the two. This highlights what the following framework is about: it aims at revealing the choice-relevant differences, rather than any difference a person might perceive. Thus a person might perceive a difference between two communist parties, say their name or the haircut of their first candidate, and yet can be indifferent between the two parties since she does not consider the perceived differences to be choice-relevant.

It is important to note, however, that this interpretation of indifference as the absence of choice-relevant differences between options refers to a person's overall judgment about choice-relevant differences. A person's overall judgment about choice-relevant differences between options can itself be the result of an intra-personal aggregation over choice-relevant differences along different characteristics of options. In this case, it is possible that a person perceives a number of choice-relevant differences between two choice options but happens to weight these differences such that the preference for one option along one choice-relevant difference is out-balanced by a preference for the other option along another choice-relevant difference, resulting in him ranking the two options as indifferent overall.

Consider, for instance, two parties x and y, which differ in their tax policy and their migration policy. A voter might consider both of these differences to be choice-relevant and prefer party x to party y when it comes to the proposed tax policy, but favour y to x in terms of the migration policy they propose. It might then happen that after weighting the differences between these two parties the person concludes that the reasons to favour party x over party y in terms of tax policy are exactly counterbalanced by his preference for party y in terms of

[6] Note that preferences can only be revealed from choices if choices satisfy certain consistency conditions (see e.g. Samuelson 1938; Arrow 1959; Sen 1971). Theorems 1 and 2 of this article assume the symmetric part of people's preferences to be known. If one thus adopts a revealed preference interpretation, the derived results are subject to the respective conditions revealed preference theory requires people's choices to satisfy.

migration policy, leaving him indifferent between both parties.[7] It is important to note that such cases constitute a limitation of the above assumption, i.e. that indifference reflects the absence of choice-relevant differences between options.

In the first part of this article, we shall accept this limitation and interpret the formal results as the aggregation of the individual's overall judgments of choice-relevant differences (possibly induced from information about their preferences as described in the penultimate paragraph) into a judgment as to which differences are considered to be important in a society. In *sections 5 and 6*, however, we show that the formal framework developed in this article can also be employed to analyze the limitation illustrated by the previous example: there we explore how the formal framework can be interpreted as the aggregation of an individual's judgment, about the similarities between alternatives in terms of their various characteristics, to a person's overall similarity judgment and, subsequently, the formation of (the symmetric part of) her preferences.

A Similarity Revelation Rule assigns a social similarity relation to each similarity profile. Two alternatives (x,y) are socially similar along the Similarity Revelation Rule (henceforth denoted by xSy or $xf(S_1,\ldots,S_n)y$ interchangeably) if, and only if, a sufficient number p, $1 \leq p \leq n$, of persons do not detect a relevant difference between the two alternatives.[8]

Similarity Revelation Rule

Let $f(.)$ be a mapping $f : \mathscr{D} \to \mathscr{S}$, such that $\mathscr{D} \subseteq \mathscr{S}^N$. $f(.)$ is a Similarity Revelation Rule if, and only if, for all $x,y \in X$, there is some p, $1 \leq p \leq n$, such that for all $(S_1,\ldots,S_n) \in \mathscr{D}, xf(S_1,\ldots,S_n)y$ if, and only if, $\#\{i \in N \mid xS_iy\} \geq p$.

To illustrate how a Similarity Revelation Rule works, consider the following example. Take p to be equal for all pairs of alternatives, and the preferences of members in a society as depicted in the following table.[9]

S_1	S_2	S_2		$p=1$	$p=2$	$p=3$
xSy	not (xSy)	xSy	$\xrightarrow{f(\cdot)}$	xSy	xSy	not (xSy)
not (ySz)	ySz	ySz		ySz	ySz	not (ySz)
not (xSz)	not (xSz)	xSz		xSz	not (xSz)	not (xSz)

Table 1: Similarity Revelation Rule

7 In the light of insights gained from the application of voting paradoxes to multi-criteria decision making (Nurmi and Meskanen 2000) and results that highlight the limits to intra-personal aggregation more generally (May 1954), it is not self-evident whether intra-personal aggregation of similarity judgements over various characteristics always works. For a more detailed discussion about the conditions under which a (transitive) similarity relation can be derived by intra-personal aggregation of the similarity relations over different characteristics, see Theorem 3 in *section 5* of this article.

8 Another way to derive information about the difference between alternatives from preferences is to identify the degree of difference between alternatives with the number of people that are indifferent among them (Binder 2011).

9 Note that in the employed framework two alternatives are either similar or dissimilar. The framework does not allow for individuals or society as a whole to remain undecided on whether two alternatives are similar or not.

3. Conditions

The Similarity Revelation Rule is characterised by a set of axioms which seem to enjoy great plausibility if one takes a Similarity Revelation Rule to assign a social similarity relation to each society given the similarity relations of its members.

Anonymity $f : \mathscr{D} \to \mathscr{S}$, $\mathscr{D} \subseteq \mathscr{S}^N$, is anonymous if, and only if, for any profiles $(S_1,\ldots,S_n),(S_{\phi(1)},\ldots,S_{\phi(n)})$ in the domain of $f(.)$ that are permutations of each other, $f(S_1,\ldots,S_n) = f(S_{\phi(1)},\ldots,S_{\phi(n)})$.

Anonymity rules out any 'experts' on diversity. All people are assumed to be equally competent at judging which difference between alternatives (should) matter.

Monotonicity $f : \mathscr{D} \to \mathscr{S}$, $\mathscr{D} \subseteq \mathscr{S}^N$, is monotonic if, and only if, for all $(S_1,\ldots, S_n),(S'_1,\ldots,S'_n) \in \mathscr{D}$, for all $x,y \in X$, if [for all $i \in N$, xS_iy implies xS'_iy] then [$xf(S_1,\ldots,S_n)y$ implies $xf(S'_1,\ldots,S'_n)y$].

Monotonicity rules out that the social similarity relation between two alternatives switches from similarity to dissimilarity, in the case in which nobody starts to consider a choice-irrelevant difference between the two alternatives to be relevant.

Unanimity $f : \mathscr{D} \to \mathscr{S}$, $\mathscr{D} \subseteq \mathscr{S}^N$, is unanimous if, and only if, for all $x,y \in X$, [for all $i \in N$ not (xS_iy) implies not $xf(S_1,\ldots,S_n)y$] and [for all $i \in N$,xS_iy implies $xf(S_1,\ldots,S_n)y$].

Unanimity rules out cases in which one's social similarity relation does not reflect unanimous agreement among all members of society on the existence or absence of choice-relevant differences between options. Hence it rules out externally imposed aggregation rules which declare two alternatives to be similar (or dissimilar) irrespective of the similarity relation all members of a society actually hold.

4. Characterization Results

Theorem 1 $f : \mathscr{D} \to \mathscr{S}$, $\mathscr{D} = \mathscr{S}^N$, satisfies Anonymity, Monotonicity and Unanimity if, and only if, $f(.)$ is a Similarity Revelation Rule.

Theorem 1 draws upon insights from the characterization of simple majority rule by May (1952). The main formal difference to May's result and other characterizations of quota rules in the literature (Brams and Fishburn 2002) is that the class of rules characterized in Theorem 1 focuses on the symmetric part of binary relations and allows the threshold p to take any value between 1 and n.[10]

Proof:
$\Leftarrow$: Anonymity: According to the definition of a permutation $\phi : N \to N$, we have for all $x, y \in X$ and for all $(S_1, \ldots, S_n), (S'_1, \ldots, S'_n) \in \mathscr{S}^N$ such that for any i, $S'_i = S_{\phi(i)}$, $\#\{i \in N \mid xS_i y\} = \#\{i \in N \mid xS'_i y\}$. This entails, according to the definition of a Similarity Revelation Rule, $xf(S_1, \ldots, S_n)y$ if, and only if, $xf(S'_1, \ldots, S'_n)y$.

Monotonicity: Take arbitrary $(S_1, \ldots, S_n), (S'_1, \ldots, S'_n) \in \mathscr{S}^N$, such that for some $x, y \in X$: for all i, $xS_i y$ implies $xS'_i y$. Since $\#\{i \in N \mid xS'_i y\} \geq \#\{i \in N \mid xS_i y\}$, $xf(S_1, \ldots, S_n)y$ implies $xf(S'_1, \ldots, S'_n)y$.

Unanimity: For all $x, y \in X$, if for all $i \in N$, not $(xS_i y)$, then $\#\{i \in N \mid xS_i y\} = 0$. Since $p \in N$ by the definition of a Similarity Revelation Rule, it is not the case that $xf(S_1, \ldots, S_n)y$.

For all $x, y \in X$, if for all $i \in N$, $xS_i y$, then $\#\{i \in N \mid xS_i y\} = n$. According to the definition of a Similarity Revelation Rule this yields $xf(S_1, \ldots, S_n)y$.

$\Rightarrow$: If $f(.)$ satisfies Unanimity, Anonymity and Monotonicity then for all $x, y \in X$ there is $p, 1 \leq p \leq n$, such that for all $(S_1, \ldots, S_n) \in \mathscr{S}^N$, $xf(S_1, \ldots, S_n)y$ if, and only if, $\#\{i \in N \mid xS_i y\} \geq p$.

Step 1: Let $x, y \in X$ and for all $i \in N$ let $S_i^{x,y}$ be by definition such that not $xS_i^{x,y}y$ and $yS_i^{x,y}w$, for all $w \neq x$. Furthermore, let for all $i \in N$, S_i^u be by definition such that for all $v, w \in X, vS_i^u w$. Consider the following sequence of profiles:

$d^0 = (S_1^{x,y}, \ldots, \ldots, \ldots, S_n^{x,y})$
$d^1 = (S_1^u, S_2^{x,y}, \ldots, \ldots, S_n^{x,y})$
$d^2 = (S_1^u, S_2^u, S_3^{x,y}, \ldots, S_n^{x,y})$
$\vdots$
$d^n = (S_1^u, \ldots, \ldots, S_n^u)$

By Unanimity we have not $xf(d^0)y$ and $xf(d^n)y$. Let p be the smallest number such that not $xf(d^{p-1})y$ and $xf(d^p)y$.

Step 2: Now we have to show that for all $(S_1, \ldots, S_n) \in \mathscr{S}^N$, $xf(S_1, \ldots, S_n)y$ if, and only if, $\#\{i \in N \mid xS_i y\} \geq p$.

Step 2a: If $\#\{i \in N \mid xS_i y\} = p$ then $xf(S_1, \ldots, S_n)y$. Consider some arbitrary profile d^* for which $\#\{i \in N \mid xS_i^* y\} = p$. Construct a profile d' such that for all $i, j \in N$, S_i^u in d' if, and only if, $xS_i^* y$; and $S_j^{x,y}$ in d' if, and only if, not $xS_j^* y$. By construction one can apply a permutation $\phi(.)$ on N to profile d^p (in step

[10] To my knowledge the result in the literature that comes closest to Theorem 1 is a characterization result for a class of quota rules derived by Dietrich and List (2007) in the framework of judgment aggregation. It can be shown that if the class of Similarity Revelation Rules characterized in this paper is represented in the judgment aggregation model, then it is a special case of the class of rules characterized by Dietrich and List. The axioms Dietrich and List employ in their characterization result differ from those employed in this paper however.

1) such that $d^{\phi(p)} = d'$. By Anonymity we obtain $xf(d^p)y$ implies $xf(d')y$. By Monotonicity we obtain $xf(d')y$ implies $xf(d^*)y$.

Step 2b: If $\#\{i \in N \mid xS_i y\} > p$ then $xf(S_1,\ldots,S_n)y$. Consider a profile d^+ such that $\#\{i \in N \mid xS_i^+ y\} = m > p$. Let $\{1,\ldots,p\} \subset \{1,\ldots,n\}, \#\{1,\ldots,p\} = p$, such that for all $i \in \{1,\ldots,p\}, xS_i^+ y$.

Construct a profile $d^{\phi(p)}$ by applying a permutation $\phi(.)$ on N to profile d^p (in step 1) such that for all $i \in N$ if S_i^u in profile d^p then $\phi(i) \in \{1,\ldots,p\}$. By Anonymity, $xf(d^p)y$ implies $xf(d^{\phi(p)})y$. Note that for all $i \in N$, $xS_i^{\phi(p)}y$ implies $xS_i^+ y$. Hence, by Monotonicity, $xf(d^{\phi(p)})y$ implies $xf(d^+)y$.

Step 2c: If $\#\{i \in N \mid xS_i y\} < p$ then not $xf(S_1,\ldots,S_n)y$. Suppose to the contrary that there is a profile d^-, such that $xf(d^-)y$ and $\#\{i \in N \mid xS_i^- y\} = k < p$. Consider profile d^{p-1} in step 1 of the proof. Let $\{1,\ldots,k\} \subset N, \#\{1,\ldots,k\} = k, k \leq p-1$, be such that for all $i \in \{1,\ldots,k\}, xS_i^{p-1}y$ (in profile d^{p-1} in step 1 of the proof). Now construct a profile $d^{\phi(-)}$ by applying a permutation $\phi(.)$ on N to profile d^- such that for all $i \in N$, if $xS_i^- y$ then $\phi(i) \in \{1,\ldots,k\}$. By Anonymity, $xf(d^-)y$ implies $xf(d^{\phi(-)})y$. Note that, by construction, for all $i \in N$, $xS_i^{\phi(-)}y$ implies $xS_i^{p-1}y$. Hence by Monotonicity we obtain $xf(d^{p-1})y$, contradicting step 1 of the proof. **Q.E.D.**

Note that a Similarity Revelation Rule allows for different similarity thresholds for different pairs of alternatives. Depending on the purpose of the exercise and the nature of the alternatives, say whether one is concerned with the dissimilarities between political parties or between two exquisite sorts of wine, a Similarity Revelation Rule allows the setting of different thresholds for different (kinds of) alternatives. For example, the number of people who consider the difference between two sorts of exquisite wine to be relevant may be lower in order to be judged a choice-relevant difference overall than in the case where we are concerned with two parties running for election.

If one additionally requires the following Neutrality axiom to hold, the resulting Similarity Revelation Rule* requires p to be equal for all $x, y \in X$.

Neutrality $f(.)$ is neutral if, and only if, for all permutations $\phi(.)$ of X, all $(S_1,\ldots,S_n) \in \mathscr{S}^N$, for all $x, y \in X$, [for all $i \in N : xS_i y$ if, and only if, $\phi(x)S_i\phi(y)$] implies [$xf(S_1,\ldots,S_n)y$ if, and only if, $\phi(x)f(S_1,\ldots,S_n)\phi(y)$].

Similarity Revelation Rule*
Let $f(.)$ be a mapping $f : \mathscr{D} \to \mathscr{S}$, such that $\mathscr{D} \subseteq \mathscr{S}^N$. We will call $f(.)$ the Similarity Revelation Rule* if, and only if, there exists a p, $1 \leq p \leq n$, such that for all $x, y \in X$, for all $(S_1,\ldots,S_n) \in \mathscr{D}$, $xf(S_1,\ldots,S_n)y$ if, and only if, $\#\{i \in N \mid xS_i y\} \geq p$.

Theorem 2 $f : \mathscr{D} \to \mathscr{S}$, $\mathscr{D} = \mathscr{S}^N$, satisfies Anonymity, Neutrality, Unanimity and Monotonicity if, and only if, $f(.)$ is the Similarity Revelation Rule*.

Proof:
The result follows from Theorem 1 and from the definition of Neutrality. In Theorem 1 it was proven that if the imposed conditions are satisfied then for

any pair of alternatives x and y there exists some p. Neutrality demands p to be the same for all pairs of alternatives since it requires the social similarity relation to remain the same after the alternatives have been permuted. This, however, is only the case if p is equal for all pairs of alternatives. **Q.E.D.**

5. Similarity and Transitivity

Theorems 1 and 2 show that one can derive a symmetric similarity relation on X from each profile of weak orderings. Note, however, that unless the similarity relations contained in the similarity profile are required to satisfy transitivity and $n = 1$ or $p = n$, the derived similarity relations will often violate transitivity. Having a closer look at the example in section 2 again, it becomes obvious that this can even be the case if one requires the similarity relations in the profile to be transitive. The following result shows that the derived similarity relation will be transitive if, and only if, the underlying similarity profile satisfies Similarity Restriction.

Similarity Restriction $f(.)$ satisfies Similarity Restriction if, and only if, $\mathcal{D} = \{d \in \mathcal{S}^N \mid$ for all $x, y, z \in X$ there exists $\{x,y\},\{y,z\} \subseteq \{x,y,z\}$ such that $\#\{i \in N \mid xS_iy\} = \#\{i \in N \mid yS_iz\} \leq \#\{i \in N \mid xS_iz\}\}$.

Theorem 3 The Similarity Revelation Rule* assigns a transitive similarity relation to all $(S_1,\ldots,S_n) \in \mathcal{D}$, for all $p, 1 \leq p \leq n$, if, and only if, Similarity Restriction is satisfied.

Proof:
⇒: In a first step it is proven that if the similarity relation assigned by the Similarity Revelation Rule* is transitive for all $p, n \geq p \geq 1$, then Similarity Restriction is satisfied. If S satisfies transitivity then for all $x, y, z \in X, xf(S_1,\ldots,S_n)y$ and $yf(S_1,\ldots,S_n)z$ implies $xf(S_1,\ldots,S_n)z$ for all p, $n \geq p \geq 1$. Hence $\#\{i \in N \mid xS_iy\} \geq p$ and $\#\{i \in N \mid yS_iz\} \geq p$ implies $\#\{i \in N \mid xS_iz\} \geq p$ for all p.

Step 1a: It is shown that if S satisfies transitivity for all $p, n \geq p \geq 1$, then for all $x, y, z \in X$, there exists $\{x,y\},\{y,z\} \subseteq \{x,y,z\}$ such that $\#\{i \in N \mid xS_iy\} = \#\{i \in N \mid yS_iz\}$. Assume to the contrary, $\#\{i \in N \mid xS_iy\} \neq \#\{i \in N \mid yS_iz\} \neq \#\{i \in N \mid xS_iz\}$. One can then identify a value of p, such that for some $\{x,y\},\{y,z\} \subseteq \{x,y,z\}$, $\#\{i \in N \mid xS_iy\} \geq p$ and $\#\{i \in N \mid yS_iz\} \geq p$ and $\#\{i \in N \mid xS_iz\} < p$. According to the definition of the Similarity Revelation Rule* this implies $xf(S_1,\ldots,S_n)y$ and $yf(S_1,\ldots,S_n)z$ and not $xf(S_1,\ldots,S_n)z$, violating transitivity.

Step 1b: It is shown that if S satisfies transitivity for all $p, 1 \leq p \leq n$, for all $x, y, z \in X$ if for two pairs $\{x,y\},\{y,z\} \subseteq \{x,y,z\}$, $\#\{i \in N \mid xS_iy\} = \#\{i \in N \mid yS_iz\}$ then $\#\{i \in N \mid xS_iy\} \leq \#\{i \in N \mid xS_iz\}$. Assume to the contrary, $\#\{i \in N \mid xS_iy\} = \#\{i \in N \mid yS_iz\}$ and $\#\{i \in N \mid xS_iy\} > \#\{i \in N \mid xS_iz\}$. One can then identify a value of p, such that $\#\{i \in N \mid xS_iy\} \geq p$ and $\#\{i \in N \mid yS_iz\} \geq p$ and $\#\{i \in N \mid xS_iz\} < p$. According to the definition of the Similarity Revelation Rule* this implies

$xf(S_1,\ldots,S_n)y$ and $yf(S_1,\ldots,S_n)z$ and not $xf(S_1,\ldots,S_n)z$, violating transitivity. 1a and 1b conclude the proof of step 1.

$\Leftarrow$: It is proven that if Similarity Restriction is satisfied then the Similarity Revelation Rule* assigns a transitive similarity relation. Suppose transitivity is violated, thus there is $x,y,z \in X$ such that $xf(S_1,\ldots,S_n)y$ and $yf(S_1,\ldots,S_n)z$ but not $xf(S_1,\ldots,S_n)z$. According to the definition of the Similarity Revelation Rule* we have $\#\{i \in N \mid xS_iy\} > \#\{i \in N \mid xS_iz\}$ and $\#\{i \in N \mid yS_iz\} > \#\{i \in N \mid xS_iz\}$, violating Similarity Restriction. Steps 1 and 2 conclude the proof of Theorem 3. **Q.E.D.**

How plausible is it to demand the overall similarity relation to be transitive? The answer to this question will depend on the problem addressed and the respective interpretation adopted. If the objective is to obtain information about those choice-relevant similarities which are significant to the members of a society in order to use this similarity information in freedom rankings, then the absence of transitivity does not seem too troublesome. Indeed, in the literature on freedom rankings it has been argued and shown (Pattanaik and Xu 2000) that transitivity of similarity relations is neither a plausible nor a formally necessary condition in order to use them in freedom rankings.

Transitivity of the overall similarity relation becomes a more important requirement, however, if one adopts a different interpretation of the framework introduced. Instead of focusing on the aggregation of individual judgments about choice-relevant similarities into a social similarity relation, one can interpret the aggregation exercise as the formation of (the symmetric part of) a person's preferences on the basis of her similarity judgments over alternatives in terms of their different characteristics. In this case the absence of transitivity in the resulting similarity relation seems more troublesome since it would indicate the absence of I-transitivity of a person's preference relation, as widely used in rational choice theory.

6. Conclusion

In this article we have shown how information about choice-relevant similarities between alternatives can be revealed from preference information. The characterized Similarity Revelation Rule allows one to derive those dissimilarities which matter to members of a society from the (symmetric part of the) preferences of its members. To illustrate, consider the example raised in the introduction again: whether the availability of a green bus (in addition to a red bus) does indeed increase the diversity of choice options will depend on whether a sufficient number of people in (the respective) society consider the difference to be choice-relevant and rank the green bus strictly above the red bus (or vice versa). Thus judgments about relevant differences between alternatives will depend on the society in question. Similarly, differences considered to be choice-relevant within one society might change over time. A choice-irrelevant difference, say the colour between two buses one is boarding in Iran, might become relevant after green has become the colour of the newly formed resistance movement.

Furthermore, we have shown under what conditions the Similarity Revelation Rule yields a transitive similarity relation. Even though the absence of transitivity does not seem to be a major problem if one is concerned with a similarity relation in the assessment of a person's freedom, it seems more troublesome in a case in which one interprets the formal framework as reflecting the formation of a person's all-things-considered preference on the basis of the similarity between alternatives in terms of their various characteristics. If one takes a person to be indifferent between two alternatives if, and only if, there is no overall choice-relevant difference between the two—whereby the overall choice-relevant differences are derived from the choice-relevant similarities between the various characteristics of the options by means of the Similarity Revelation Rule—then her resulting preference ordering fails to satisfy I-transitivity.

Despite the prominence of I-transitivity in the literature, it is not yet clear whether this (interpretation of the) result is as troublesome as it might seem at first sight. The Similarity Revelation Rule is just one of many possible ways in which a person can identify the overall (choice-relevant) similarities between options on the basis of the (choice-relevant) similarities between their various characteristics. Indeed, it does not even seem to be the most plausible way. One can question, for instance, whether Anonymity is a plausible condition along this interpretation: after all it seems quite plausible that a person assigns a different importance to (the similarity between) various characteristics of alternatives. Another aggregation rule which enjoys quite some prominence in an intra-personal setting is a lexicographic procedure (Rubinstein 1988). The options are assessed step by step in terms of the characteristics ordered in accordance with their relevance. If two options are considered to be dissimilar in terms of the first characteristic then the two will be considered to be choice-relevantly dissimilar overall. If not, one moves on to judge their similarity in terms of the second characteristic. Under what conditions such a lexicographic procedure, or more generally any procedure satisfying a number of plausible conditions, yields a transitive similarity relation is one of many lines of future research on transitivity of preferences and similarity between options.

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