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The Central Decomposition of $FD_{01}(n)$

Peter Köhler[1]

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Abstract
The paper presents a method of composing finite distributive lattices from smaller pieces and applies this to construct the finitely generated free distributive lattices from appropriate Boolean parts.

Keywords Free distributive lattices · Triple sum

1 Introduction

The free distributive lattice $FD_{01}(3)$ on three generators as drawn in Fig. 1 can be viewed as a sort of composition of four Boolean lattices layered on top of each other, with the three generators a, b, c serving as additional merging points. Another way of seeing this is via the 'central' elements $0, p, q, r, 1$, where the intervals $[0, p]$, $[p, q]$, $[q, r]$, $[r, 1]$ form the respective Boolean lattices.

In this paper we will show, that this behaviour can be found in all the finitely generated free distributive lattices. Moreover we will give a nonrecursive construction of these lattices from their Boolean lattice building blocks.

The original hope that this might provide a better way to compute their cardinalities did not materialize. As in the known recursive attempts (see e.g. [2, 11]) also this approach requires an addition of interval sizes which for larger values of n goes beyond the capacities of current computers.

2 Lattices

In the following all lattices L are finite distributive lattices with a 0-element 0_L and a 1-element 1_L. By B_n we denote the Boolean lattice with elements $0,...,2^n - 1$ and binary join and meet. In particular $0_{B_n} = 0$ and $1_{B_n} = 2^n - 1$, and the atoms of B_n are $1, 2, ..., 2^{n-1}$.

For elements a and b of a lattice L we denote by $(a]$ the principal ideal (or 'downset') $\{x | x \in L, x \leq a\}$, by $[a)$ the principal filter (or 'upset') $\{x | x \in L, x \geq a\}$ and by $[a, b]$ the interval $\{x | x \in L, a \leq x \leq b\}$.

✉ Peter Köhler
Peter.Koehler@math.uni-giessen.de; peter@koehler-weilburg.de

1 Mathematisches Institut, Justus-Liebig-Universität Giessen, Giessen, Germany

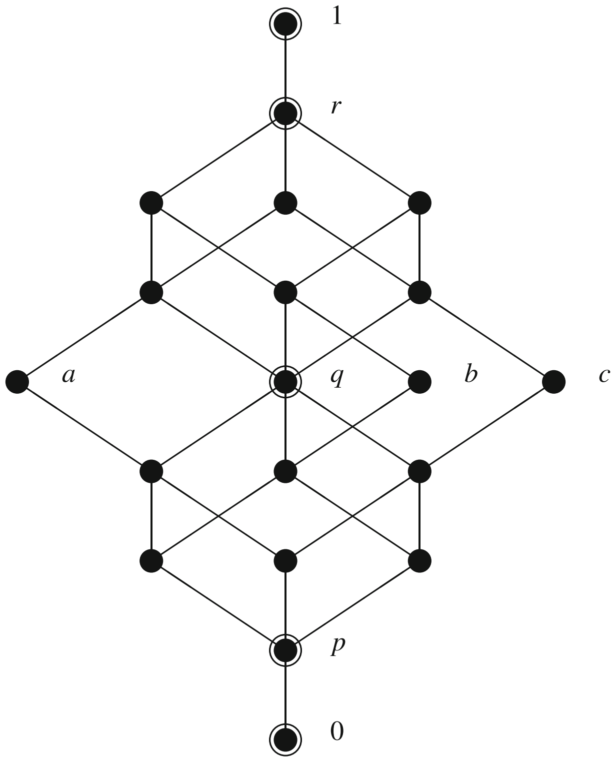

Fig. 1 $FD_{01}(3)$

We start with an easy observation belonging to the folklore of distributive lattices (see e.g. [1, 6]):

Theorem 1 *Let L be a finite distributive lattice and $a \in L$. Then*

(i) $[a)$ *is a dual ideal of L.*
(ii) *The relation Θ_a defined by*

$$x\Theta_a y =_{def} x \wedge a = y \wedge a$$

is a congruence of L.
(iii) $L/\Theta_a = (a]$.

What is less known is the following reverse construction, which has its origin in the general theory of 'triple sums' originally developed in [4, 5] and later extended in [7–10].

Theorem 2 *Let L, M, N be finite distributive lattices and $\phi : L \to M$ be a 1-meet-preserving mapping. Let $L \otimes^{\phi} M$ be the set of all pairs*

$$L \otimes^{\phi} M = \{(l, m) | l \in L, m \in M, m \leq \phi(l)\}.$$

Then

(i) *The composition $L \otimes^{\phi} M$ is a distributive lattice (as a sublattice of $L \times M$).*

(ii) *There is an element* $a \in L \otimes^{\phi} M$ *such that* $(a] \cong L$ *and* $[a) \cong M$.
(iii) *For each* $a \in N$ *there exists an 1-meet-preserving mapping* $\phi : (a] \to [a)$ *such that* $N \cong (a] \otimes^{\phi} [a)$.

Proof (i) is obvious. For (ii) let $a = (1_L, 0_M)$. Then $a \in L \otimes^{\phi} M$ and the mappings $x \mapsto (x, 0_M)$ and $y \mapsto (1_L, y)$ are clearly isomorphisms from L to $(a]$ and M to $[a)$.

For (iii) define ϕ by $\phi(x) = a * x$ for all $x \leq a$, where $*$ denotes the relative pseudocomplement, i.e. $a * x = \bigvee\{z | z \in L, z \wedge a \leq x\}$. Then it is well known from the theory of pseudocomplented lattices that ϕ has the required properties. □

Let us note that the theorem above could have also formulated using the notion of split exact sequences (see [9, 10]).

There are three well known special cases:

(i) If ϕ is the '1-mapping', i.e. $\phi(x) = 1_M$ for all $x \in L$, then $L \otimes^{\phi} M$ is the direct product $L \times M$.
(ii) If ϕ is the '0-mapping', i.e. $\phi(x) = 0_M$ for all $x \in L \setminus \{1_L\}$ and $\phi(1_L) = 1_M$, then $L \otimes^{\phi} M$ is the ordinal sum $L \oplus M$ with 1_L and 0_M amalgamated.
(iii) And if $L = M$ and ϕ is the identity mapping, then $L \otimes^{\phi} L$ is the 'skew square' $L \triangle L$ of L, which is used in the recursive construction of $FD_{01}(n)$ via $FD_{01}(n) \cong FD_{01}(n-1) \triangle FD_{01}(n-1)$. (see e.g. [2])

Another use of the skew square can be seen in the following easy observation:

Theorem 3 *Let* B_n *be the Boolean lattice of order* 2^n*, and let* C_3 *be the three element chain. Then* $B_n \triangle B_n \cong C_3^n$.

Proof This is obvious for $n = 1$, the rest follows by an easy induction argument, enumerating pairs of pairs in two different ways. □

Another interesting observation concerning the 'skew square' of a composition is:

Lemma 1 *Let* L, M *be distributive lattices and let* $\phi : L \to M$ *be a 1-meet-preserving map. Then*

$$(L \otimes^{\phi} M) \triangle (L \otimes^{\phi} M) \cong L \otimes^{\psi} (M \times L) \otimes^{\chi} M,$$

where the mappings $\psi : L \to M \times L$ *and* $\chi : M \times L \to M$ *are defined by:*

$$\psi(x) = (\phi(x), x) \text{ for all } x \in L$$

$$\chi((y, x)) = \phi(x) \wedge y \text{ for all } x \in L \text{ and } y \in M.$$

Proof Obviously ψ and χ are 1-meet-preserving. Now by definition $(L \otimes^{\phi} M) \triangle (L \otimes^{\phi} M) = \{((x_1, y_1), (x_2, y_2)) | y_1 \leq \phi(x_1), y_2 \leq \phi(x_2), x_2 \leq x_1, y_2 \leq y_1\}$ whereas $L \otimes^{\psi} (M \times L) \otimes^{\chi} M = \{(x_1, (y_1, x_2), y_2) | (y_1, x_2) \leq \psi(x_1), y_2 \leq \chi((y_1, x_2))\}$ and by the definition of ψ and χ these conditions coincide. □

So far we have only considered the composition of pairs of distributive lattices. Now if $L \otimes^{\phi} M$ and $M \otimes^{\psi} N$ are two such compositions, then these give rise to two combinations, namely $(L \otimes^{\phi} M) \otimes^{\psi^*} N$ and $L \otimes^{\phi^*} (M \otimes^{\psi} N)$ where ϕ^* and ψ^* are the natural extensions of ϕ and ψ defined by

$$\phi^*(l) = (\phi(l), \psi(\phi(l)) \ for \ l \in L$$
$$\psi^*(m, n) = \psi(m) \ for \ (m, n) \in M \otimes^{\psi} N.$$

Obviously both compositions amount to the same set, namely $\{(l, m, n) | l \in L, m \in M, n \in N, m \leq \phi(l), n \leq \psi(m)\}$. Therefore it makes sense to introduce the notion of a triple composition $L \otimes^{\phi} M \otimes^{\psi} N$, and more generally that of an n-fold composition

$$L_0 \otimes^{\phi_0} L_1 \otimes^{\phi_1} ... \otimes^{\phi_{n-2}} L_{n-1}.$$

And as such an (n+1)-fold composition we will construct $FD_{01}(n)$.

However, before turning to the general case we describe the construction of $FD_{01}(3)$ as a quadruple

$$B_1 \otimes^{\phi_0} B_3 \otimes^{\phi_1} B_3 \otimes^{\phi_2} B_1,$$

where ϕ_0 and ϕ_2 are the 0-mappings and $\phi_1 : B_3 \to B_3$ is defined by $\phi_1(7) = 7$, $\phi_1(6) = 4$, $\phi_1(5) = 2$, $\phi_1(3) = 1$ and $\phi_1(x) = 0$ for all other $x \in \{0, 1, ..., 7\}$.

That this really gives $FD_{01}(3)$, can be seen from its diagram in the canonical numbering as in Fig. 2 and the expression of the element numbers as 4-tuples as in Table 1, where the correspondence is given by $(c_0, c_1, c_2, c_3) \mapsto c_0 * 2^0 + c_1 * 2^1 + c_2 * 2^4 + c_3 * 2^7$:

For the general case of $n \in \mathbb{N}$ this suggests to use the Boolean lattices $B_{\binom{n}{k}}$ corresponding to the binomial coefficients $\binom{n}{k}$ for $k = 0, ..., n$ as building blocks.

Theorem 4 *Let $n \in \mathbb{N}$. For $k = 0, ..., n$ let L_k be the Boolean lattice $L_k = B_{\binom{n}{k}}$. Then there exist 1-meet-preserving mappings $\phi_k : L_k \to L_{k+1}$ for $k = 0, .., n-1$ such that*

$$FD_{01}(n) \cong L_0 \otimes^{\phi_0} L_1 \otimes^{\phi_1} ... \otimes^{\phi_{n-1}} L_n.$$

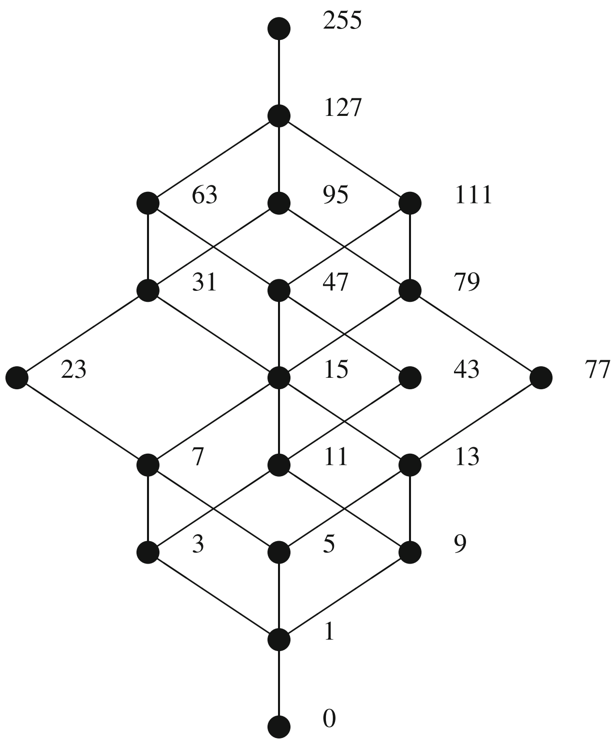

Fig. 2 The canonical numbering of $FD_{01}(3)$

Table 1 Numbers as 4-tuples

0	1	3	5	7
(0,0,0,0)	(1,0,0,0)	(1,1,0,0)	(1,2,0,0)	(1,3,0,0)
9	11	13	15	23
(1,4,0,0)	(1,5,0,0)	(1,6,0,0)	(1,7,0,0)	(1,3,1,0)
43	77	31	47	63
(1,5,2,0)	(1,6,4,0)	(1,7,1,0)	(1,7,2,0)	(1,7,3,0)
79	95	111	127	255
(1,7,4,0)	(1,7,5,0)	(1,7,6,0)	(1,7,7,0)	(1,7,7,1)

The key to the proof is the following generalization of Lemma 1:

Lemma 2 *Let $n \in \mathbb{N}$, let $L_0, L_1, ..., L_n$ be distributive lattices and for $0 \leq i < n$ let $\phi_i : L_i \to L_{i+1}$ be 1-meet-preserving maps.*
Then

$$(L_0 \otimes^{\phi_0} ... \otimes^{\phi_{n-1}} L_n) \triangle (L_0 \otimes^{\phi_0} ... \otimes^{\phi_{n-1}} L_n) \cong$$
$$L_0 \otimes^{\psi_0} (L_1 \times L_0) \otimes^{\psi_1} (L_2 \times L_1) \otimes^{\psi_2} ... \otimes^{\psi_{n-1}} (L_n \times L_{n-1}) \otimes^{\psi_n} L_n,$$

where the mappings $\psi_1 : L_0 \to L_1 \times L_0$, $\psi_i : (L_i \times L_{i-1}) \to (L_{i+1} \times L_i)$ for $i = 1, , , n-1$ and $\psi_n : (L_n \times L_{n-1}) \to L_n)$ are defined by:

$$\psi_0(x_0) = (\phi_0(x_0), x_0) \textit{ for all } x_0 \in L_0$$
$$\psi_i((y_{i-1}, x_i)) = (\phi_i(x_i), x_i \wedge \phi_{i-1}(y_{i-1})) \textit{ for all } x_i \in L_i, y_{i-1} \in L_{i-1}, 0 < i < n$$
$$\psi_n((y_{n-1}, x_n)) = x_n \wedge \phi_{n-1}(y_{n-1}) \textit{ for all } x_n \in L_n, y_{n-1} \in L_{n-1}.$$

To prove this, obviously a similar argument as in Lemma 1 shows that the conditions for the elements $((x_0, ..., x_n), (y_0, ..., y_n))$ on the left hand side and $((x_0, (y_0, x_1), ..., (y_{n-1,} x_n), y_n))$ on the right hand side coincide:

Proof of Theorem 4 The result is immediate for $n = 1$ with $FD_{0,1}(1) \cong C_3 \cong B_1 \otimes^{\phi_0} B_1$, where ϕ_0 is the 0-map. Now assume that the result holds for $n \geq 1$. As in Example (iii) on page 3 we have that $FD_{01}(n + 1) \cong FD_{01}(n) \triangle FD_{01}(n)$. By the induction hypothesis and Lemma 2 we get $FD_{01}(n + 1) \cong B_{\binom{n}{0}} \otimes^{\psi_0} (B_{\binom{n}{1}} \times B_{\binom{n}{0}}) \otimes^{\psi_1} (B_{\binom{n}{2}} \times B_{\binom{n}{1}}) \otimes^{\psi_2} ... \otimes^{\psi_{n-1}} (B_{\binom{n}{n}} \times B_{\binom{n}{n-1}}) \otimes^{\psi_n} B_{\binom{n}{n}}$ Now the fact that $B_i \times B_j \cong B_{i+j}$ for all $i, j \in \mathbb{N}$ and the addition rules for the binomial coefficients show that the statement of the theorem holds also for $n + 1$. □

In this proof the crucial mappings $\psi_0, ..., \psi_n$ are defined recursively. It is, however, possible to give a direct definition. We defer this to the next section.

3 Posets

An element x of a lattice L is called meet irreducible, if it cannot be expressed as a meet of greater elements, i.e. $x = y \wedge z$ implies $x = z$ or $x = y$. In particular, 1_L is not meet irreducible. The poset of meet irreducible elements of L is denoted by $\mathscr{M}(L)$.

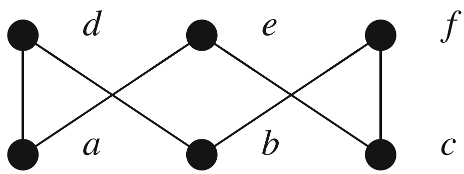

Fig. 3 Sum of two antichains

A subset I of a poset P is called an ideal, if it it "downward closed", i.e. $p \in I$ and $q \leq p$ implies $q \in I$. In particular, $\emptyset$ and P are ideals of P. By $\mathscr{I}(P)$ we denote the set (lattice) of ideals of P.

We start this section with the poset counterpart of the triple construction for lattices:

Theorem 5 *Let P, Q be finite posets and $\alpha : Q \to \mathscr{I}(P)$ be an order preserving mapping. Then the set*

$$P \oplus^\alpha Q = P \dot{\cup} Q$$

equipped with the relation $\leq$ defined by

$$x \leq y \; =_{def} \begin{cases} x \leq y & \text{if } x,y \in P \\ x \leq y & \text{if } x,y \in Q \\ x \in \alpha(y) & \text{if } x \in P \text{ and } y \in Q \end{cases}$$

is a poset.

Proof Clearly $\leq$ is reflexive and antisymmetric. To show that it is transitive too, it suffices to consider three elements x, y, z with $x \leq y$ and $y \leq z$ and the two nontrivial cases (i) $x \in P, y \in Q, z \in Q$ and (ii) $x \in P, y \in P, z \in Q$. Now for (i) transitivity comes from the fact that α is order preserving, and for (ii) from the fact that $\alpha(z)$ is an ideal. □

To illustrate this consider two 3-element antichains $P = \{a, b, c\}$, $Q = \{d, e, f\}$ and define the mapping $\alpha : Q \to \mathscr{I}(P)$ by $\alpha(d) = \{a, b\}$, $\alpha(e) = \{a, c\}$, $\alpha(e) = \{b, c\}$. Then the poset $P \oplus^\alpha Q$ has the diagram of Figure 3:

The following Lemma, taken from [8] paves the way for the next result connecting the triple constructions for lattices and posets:

Lemma 3 *Let L, M be finite distributive lattices and $\phi : L \to M$ be 1-meet-preserving. Then $(x, y) \in L \otimes^\phi M$ is meet irreducible if and only if either x is meet irreducible in L and $y = \phi(x)$ or $x = 1$ and y is meet irreducible in M.*

Theorem 6 *Let L, M be finite distributive lattices and $\phi : L \to M$ be 1-meet-preserving and let $P = \mathscr{M}(L)$ and $Q = \mathscr{M}(M)$ be their posets of meet irreducible elements. Then the mapping $\alpha : Q \to \mathscr{I}(P)$ defined by*

$$\alpha(y) = \{x | x \in P, \phi(x) \leq y\} \; y \in Q$$

is order preserving and

$$\mathscr{M}(L \otimes^\phi M) \cong P \oplus^\alpha Q.$$

Proof Obviously $\alpha(y)$ is an ideal for each $y \in Q$. Moreover the fact that ϕ preserves order immediately implies that α is order preserving too. Now by Lemma 3 we conclude that $\mathscr{M}(L \otimes^\phi M) = \{(1, y) | y \in \mathscr{M}(M)\} \cup \{(x, \phi(x)) | x \in \mathscr{M}(L)\}$. Clearly the union is disjoint, so it remains to show that $(x, \phi(x)) \leq (1, y)$ if and only if $\phi(x) \in \alpha(y)$, but this is just the definition of α. □

Theorem 7 *Let P, Q be finite posets and $\alpha : Q \to \mathscr{I}(P)$ be an order preserving map. Then the mapping $\phi : \mathscr{I}(P) \to \mathscr{I}(Q)$ defined by*

$$\phi(X) = \{q | q \in Q, \alpha(q) \subseteq X\}$$

is 1-meet-preserving and

$$\mathscr{I}(P \oplus^{\alpha} Q) \cong \mathscr{I}(P) \otimes^{\phi} \mathscr{I}(Q).$$

Proof Clearly $\phi(X)$ is an ideal of Q for every $X \in \mathscr{I}(P)$, so ϕ is a mapping. It is 1-meet-preserving as well. We now observe that for any $(X, Y) \in \mathscr{I}(P) \otimes^{\phi} \mathscr{I}(Q)$ the set $X \dot{\cup} Y$ is an ideal of $P \oplus^{\alpha} Q$. In fact let $y \in X \dot{\cup} Y$ and $x \leq y$. In order to show that $x \in X \dot{\cup} Y$ too, is suffices to consider the case $x \in P$ and $y \in Q$. But then we have $x \in \alpha(y)$ and hence $x \in X$.

This implies we can define a mapping $\chi : \mathscr{I}(P) \otimes^{\phi} \mathscr{I}(Q) \to \mathscr{I}(P \oplus^{\alpha} Q)$ by $\chi(X, Y) = X \dot{\cup} Y$. Its inverse is given by $Z \mapsto (Z \cap P) \dot{\cup} (Z \cap Q)$ and since both are order preserving they are lattice isomorphisms too. □

As already indicated, we will apply this result to obtain a nonrecursive definition of the composition mappings ϕ_k of Theorem 2. In order to facilitate this we introduce some notation:

For $n \in \mathbb{N}$ let

$$P_n = \mathscr{P}(\{0, 1, ..., n-1\})$$

be the (Boolean) poset of all subsets of $\{0, 1, ..., n-1\}$ with set inclusion as ordering. More generally, for any set X let

$$\mathscr{P}_n(X) = \{Y | Y \subseteq X, |Y| = n\}$$

be the set of all n-element subsets of X.

For $n \in \mathbb{N}$ and $k = 0, ..., n$ let

$$S_{n,k} = \mathscr{P}_k(\{0, 1, ..., n-1\})$$

be the set of all k-element subsets of P_n. Then P_n can be decomposed into antichain layers as

$$P_n = S_{n,0} \dot{\cup} S_{n,1} \dot{\cup} ... \dot{\cup} S_{n,n}.$$

With the mappings $\alpha_k : S_{n,k+1} \to \mathscr{I}(S_{n,k})$ defined by

$$\alpha_k(X) = \{Y | Y \in S_{n,k}, Y \subseteq X\}$$

we can even generalize the composition to

$$P_n = S_{n,0} \oplus^{\alpha_0} S_{n,1} \oplus^{\alpha_1} ... \oplus^{\alpha_{n-1}} S_{n,n},$$

where we tacitly extend the poset triple sum to an n-fold sum.

Repeatedly applying Theorem 4 we arrive at:

Theorem 8 *For $n \in \mathbb{N}$*

$$FD_{01}(n) \cong \mathscr{P}(S_{n,0}) \otimes^{\phi_{n,0}} \mathscr{P}(S_{n,1}) \otimes^{\phi_{n,1}} ... \otimes^{\phi_{n,n-1}} \mathscr{P}(S_{n,n}),$$

where for $k = 0, ..., n-1$ the 1-meet-preserving mappings $\phi_{n,k} : \mathscr{P}(S_{n,k}) \to \mathscr{P}(S_{n,k+1})$ are defined by

$$\phi_{n,k}(X) = \{Y | Y \in S_{n,k+1}, \mathscr{P}_k(Y) \subseteq X\}.$$

Proof It is well known that $FD_{01}(n) \cong \mathscr{I}(P_n)$ (see e.g. [3]). Moreover, as $S_{n,k}$ is an antichain, it is clear that $\mathscr{I}(S_{n,k}) = \mathscr{P}(S_{n,k})$. So the only thing that remains to be shown, is that the formula given for $\phi_{n,k}$ is equivalent to the one obtained from Theorem 7 - but that is obvious too. □

To illustrate the definition of $\phi_{n,k}$ we list some values for $n = 4$ in Table 2, where we restrict ourselves to list the mapping values for the topmost elements, i.e. the 1-element and the dual atoms:

4 Computations

Even though Theorem 8 gives a direct, nonrecursive construction, its application to determine the cardinalities for larger values of n fails with respect to the slowness of computing the 'downsets' of the partial compositions.

To see this in some more detail let us recall that the definition of the composition $L \otimes^{\phi} M$ implies that for any $(x, y) \in L \otimes^{\phi} M$ we have

$$|((x, \phi(x))]| = \sum_{a \in L, a \leq x} |(\phi(a)]|$$

and in particular

$$|L \otimes^{\phi} M| = |[(1_L, \phi(1_L))]| = \sum_{x \in L} |(\phi(x)]|.$$

Applying this repeatedly to the formula of Thereom 2 we end up with

$$|FD_{01}(n)| = \sum_{i_0 \in L_0} \sum_{i_1 \in L_1, i_1 \leq \phi(i_0)} \dots \sum_{i_n \in L_n} 2^{i_n}$$

Table 2 Mappings for $n = 4$

$S_{4,0}$	$\{\emptyset\}$	
$S_{4,1}$	$\{\{0\}, \{1\}, \{2\}, \{3\}\}$	
$S_{4,2}$	$\{\{0,1\}, \{0,2\}, \{1,2\}, \{0,3\}, \{1,3\}, \{2,3\}\}$	
$S_{4,3}$	$\{\{0,1,2\}, \{0,1,3\}, \{0,2,3\}, \{1,2,3\}\}$	
$S_{4,4}$	$\{\{0,1,2,3\}\}$	
ϕ_0	$S_{4,0}$	$S_{4,1}$
ϕ_1	$S_{4,1}$	$S_{4,2}$
	$\{\{0\}, \{1\}, \{2\}\}$	$\{\{0,1\}, \{0,2\}, \{1,2\}\}$
	$\{\{0\}, \{1\}, \{3\}\}$	$\{\{0,1\}, \{0,3\}, \{1,3\}\}$
	$\{\{0\}, \{2\}, \{3\}\}$	$\{\{0,2\}, \{0,3\}, \{2,3\}\}$
	$\{\{1\}, \{2\}, \{3\}\}$	$\{\{1,2\}, \{1,3\}, \{2,3\}\}$
ϕ_2	$S_{4,2}$	$S_{4,3}$
	$\{\{0,1\}, \{0,2\}, \{1,2\}, \{0,3\}, \{1,3\}\}$	$\{\{0,1,2\}, \{0,1,3\}\}$
	$\{\{0,1\}, \{0,2\}, \{1,2\}, \{0,3\}, \{2,3\}\}$	$\{\{0,1,2\}, \{0,2,3\}\}$
	$\{\{0,1\}, \{0,2\}, \{1,2\}, \{1,3\}, \{2,3\}\}$	$\{\{0,1,2\}, \{1,2,3\}\}$
	$\{\{0,1\}, \{0,2\}, \{0,3\}, \{1,3\}, \{2,3\}\}$	$\{\{0,1,3\}, \{0,2,3\}\}$
	$\{\{0,1\}, \{1,2\}, \{0,3\}, \{1,3\}, \{2,3\}\}$	$\{\{0,1,3\}, \{1,2,3\}\}$
	$\{\{0,2\}, \{1,2\}, \{0,3\}, \{1,3\}, \{2,3\}\}$	$\{\{0,2,3\}, \{1,2,3\}\}$
ϕ_3	$S_{4,3}$	$S_{4,4}$

Table 3 $\phi_{4,k}, k = 0, ..., 3$

$\phi_{4,0}$		$\phi_{4,1}$		$\phi_{4,2}$				$\phi_{4,3}$	
1	15	15	63	63	15	46	4	15	1
		14	52	62	12	43	4		
		13	42	61	10	42	4		
		12	32	60	8	39	1		
		11	25	59	6	31	3		
		10	16	58	4	29	2		
		9	8	57	2	27	2		
		7	7	55	9	25	2		
		6	4	54	8	23	1		
		5	2	53	8	15	1		
		3	1	52	8	7	1		
				47	5				

(since $L_n = B_1$ and therefore i_n has only the two choices $i_n = 0$ and $i_n = 1$).

Taking into account the number of necessary summations, which alone for the largest component is $2^{\binom{n}{n/2}}$, it is clear that this computation can be carried out only up to $n = 6$ using currently available computers.

But to be more precise:

We define for each $n \in \mathbb{N}$ a sequence of functions $c_0 : L_0 \to \mathbb{N}, ..., c_n : L_n \to \mathbb{N}$ recursively by:

$$c_n(0) = 1, c_n(1) = 2$$

$$c_{k-1}(x) = \sum_{y \in L_{k-1}, y \leq x} c_k(\phi_{k-1}(y)) \; for \; k = n, ..., 1$$

and the the formulas above finally yield

$$c_0(1) = |FD_{01}(n)|.$$

Table 4 c-values for n = 4

B_1			B_4			B_6			B_4			B_1		
1	168	1	15	167	1	63	114	1	15	17	1	1	2	1
0	1	1	7	19	4	31	41	6	7	8	4	0	1	1
			3	5	6	15	18	12	3	4	6			
			1	2	4	30	16	3	1	2	4			
			0	1	1	7	9	4	0	1	1			
						11	8	16						
						3	4	15						
						1	2	6						
						0	1	1						
		2			16			64			16			2

We have carried out a computer calculation of these sequences up to $n = 6$. Tables 3 and 4 list the values of the mappings $\phi_{n,k}$ and the respective c-values for $n = 4$. Note that in Table 3 the columns contain the nonzero function values and in Table 4 the three columns for each of the Boolean lattices contain representative elements, their c-value and the number of elements with the same value.

Concluding remarks It might be worthwhile to try to use some insight into the known structure of the Boolean lattices L_0,..., L_n to speed up the computation.

An easy result in that direction is that

$$c_{n-1}(2^{n-1}) = 2^{n+1}$$

$$c_{n-1}(x) = 2^{bitsize(x)} \; for \; x = 0, ..., 2^{n-1} - 1,$$

which is simply due to the fact that ϕ_{n-1} is the 0-mapping.

Another speedup approach would be the use of the induced action of the symmetric group S_n on the lattices L_1, ...L_{n-1}, as this was successfully done in [11].

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Declarations

Conflict of Interests The author declares that he has no conflict of interest.

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