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Applications of Mellin-Barnes Integrals to Deconvolution Problems

by

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In this thesis we study the additive model of errors in variables, which is also known as the deconvolution problem. The objective consists particularly in the reconstruction of the distribution F associated with a random variable X, which is observable only through a sample of a blurred variable Y, due to an additive random error ε with known distribution H. Our initial considerations yield an unbiased estimator for F for various discrete and some continuous distributions. A more general approach then leads us to the symmetrized model of errors in variables. It is obtained by an additional convolution of G with the conjugate error distribution of H, thereby resulting in an error distribution of symmetric type. As a consequence the characteristic function of X can be represented as the limit of a geometric series. By truncation of this series we deduce an approximation of F, which is valid for arbitrary error distributions. This approximation, termed the deconvolution function, converges to F in many cases. To determine the corresponding rates of convergence, techniques from complex calculus and particularly Mellin-Barnes integrals turn out to be appropriate. The latter describe a special class of integrals that can be evaluated by residue analysis. The results are established in a more general setting, which makes them applicable to other Laplace-type integrals. With the aid of the deconvolution function we also construct an estimator for F. The asymptotic properties of its variance, a peculiar integral of dimension two, can be specified by virtue of our findings from the concluding chapter. These results rely on iterated Mellin-Barnes integrals.

In dieser Dissertation befassen wir uns mit dem additiven Modell der Fehler in den Variablen, auch bekannt als Dekonvolutionsproblem. Dabei geht es um die Rekonstruktion der Verteilungsfunktion F einer Zufallsvariable X, welche aufgrund eines zufälligen additiven Fehlers ε mit bekannter Verteilung H nur in Stichproben einer gestörten Größe Y beobachtbar ist. Der Einstieg zeigt, dass F im Fall diskreter und diverser stetiger Verteilungen erwartungstreu geschätzt werden kann. Ein allgemeinerer Ansatz führt dann zur Definition des symmetrisierten additiven Modells der Fehler in den Variablen. Dazu wird die gestörte Verteilung G zusätzlich mit der konjugierten Verteilung von H gefaltet, was zu einer symmetrischen Verteilung der Fehler führt. Infolgedessen lässt sich die charakteristische Funktion von X als Grenzwert einer geometrischen Reihe darstellen. Durch deren Abbruch erhalten wir für beliebige Fehlerverteilungen eine Approximation von F, bezeichnet als Dekonvolutionsfunktion. Deren Konvergenz gegen F ist für ein breites Spektrum an Verteilungen verifizierbar. Für die Berechnung der zugehörigen Konvergenzraten erweisen sich die Methoden der komplexen Analysis als am geeignetsten. Besonders zielführend ist die Verwendung von Mellin-Barnes-Integralen, spezieller komplexwertiger Integrale deren Auswertung mit dem Residuensatz erfolgt. Die präsentierten Ergebnisse können unmittelbar auf allgemeinere Laplacesche Integrale angewendet werden. Basierend auf der Dekonvolutionsfunktion konstruieren wir außerdem einen Schätzer für F. Das asymptotische Verhalten von dessen Varianz, ein eigenartiges zweidimensionales Integral, lässt sich anhand der Ergebnisse des finalen Kapitels genau charakterisieren. Deren Herleitung erfolgt mittels iterierter Mellin-Barnes-Integrale.

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Contents

Preface

Unknown quantities are of major interest in many mathematical fields. From simple equations such as $11 = x + 7$, in a generalized fashion countless advanced problems arise. For instance, the search for the zeros of a certain function, in higher dimensions the solution of equation systems as well as eigenvalue problems and integral equations in operator theory. In view of this variety, it is not surprising that unknown quantities also will be encountered in stochastics. But contrary to calculus these occur twofold, since their approximation or computation need not be an entirely numerical problem. For example, of interest might be the unknown distribution F of a random variable X. Then, if a finite sample of independent observations is available, F can be approximated by a step function. In stochastics, this procedure is rather called estimation. Due to a contamination with some random noise, however, the variable X may not always be observable. The area dedicated to problems of this kind is referred to as *measurement errors* or *errors in variables*. Noises can be described in various ways. Particularly this thesis focusses on the *additive model of errors in variables* also known as the *deconvolution problem*. In this, X is assumed to be accessible only through the error-prone surrogate variable Y, which represents the sum of X and an independent error ε. With known distributions of ε and Y, respectively denoted by H and G, parallels to calculus can be drawn, where a representation for F is already available. If, however, G is only estimable, the reconstruction of F has the calibre of a serious problem. Then, a naive application of the techniques from calculus bears additional risks.

The additive model of errors in variables was first discussed by [Stefanski and Carroll, 1990], who initially considered the model in terms of the associated characteristic functions, i.e., Fourier transforms, from which they constructed a smoothed estimator for the density of X. The approach to be presented below substantially differs, due to the admissibility of arbitrary distributions and the dissociation of artificial smoothing techniques.

After a more technical introduction to the additive model of errors in variables, Chapter 1 begins with the study of a setup, in which the range of X and ε almost surely coincides with the positive real axis. By means of a recursion it is then possible to establish a formula for the point probabilities, that eventually gives rise to the unknown distribution F. In subsequent steps, the condition of discreteness will be relaxed, and definite formulae for F will be developed, if either of the variables X or ε is of arbitrary one-sided type. A more general approach finally leads to an integral equation that is valid for any kind of distributions. Its solution will be approximated by an iterative procedure, which is known from Volterra-type equations. In a short discussion of the convergence properties of the obtained approximation it will turn out that, in special cases, the equation for the characteristic functions in the additive model of errors in variables can be rewritten as the limit of a geometric sum.

In view of these findings, in Chapter 2 the notion of *symmetrization* will be introduced. It relies on the property, that the squared modulus of a characteristic function represents the characteristic function associated with a distribution, which is symmetric with respect to the real axis. Under general assumptions, after simple manipulations of the equation for the characteristic functions in the additive model of errors in variables, this enables an application of the well-known formula for the geometric series. The truncation of the series at some $m \geq 0$ gives rise to the *deconvolution function*, which equals a sum of m single binomial sums, that involve multiple convolution products of distribution functions. The subsequent part of this work basically deals with the properties of the deconvolution function. In Chapter 2, for example, its boundedness at infinity and the conformity of its finite moments with those of F will be shown. The deconvolution function will even turn out to possesses a density, referred to as the *deconvolution density*, if F or H are absolutely continuous. The examinations continue with a discussion of the deconvolution problem from an operator theoretical perspective. In this context, it will be pointed out, why deconvolution is in come circumstances actually not an ill-posed problem. Chapter 2 is concluded with a study of the most important feature of the deconvolution function, namely its convergence as $m \to \infty$ to F at continuity points.

For applications, it does not suffice to know about the convergence to zero of the bias between the deconvolution function and its target F, but additional information on the particular rate is required. For this reason, Chapters 3 to 6 are devoted to the topic of asymptotic expansions. An important means for asymptotic investigations will be the representation of the deconvolution function via the inversion formula for characteristic functions, since the convolution products are then decomposed multiplicatively. Due to the distinguishing structure of this Fourier-type integral, an appropriate technique for its evaluation is not obvious. In Chapter 3 more or less accurate estimates for the rate of convergence will be presented, which can be achieved solely with the tools of real calculus. In order to make exact statements, however, the existing literature on asymptotic expansions suggests the necessity to employ the tools of complex calculus. Particularly a modification of the method of Mellin transforms will finally yield complete asymptotic expansions for the bias, in case of distributions with certain characteristic functions. An introduction to this approach and the derivation of auxiliary results can be found in Chapter 4. An application of the method of Mellin transforms to the bias then will be discussed in Chapter 5. As an extension of the method of Mellin transforms, in Chapter 6 the method of analytic continuation will be introduced. It enables the asymptotic evaluation of many Laplace-type integrals and particularly of the bias for additional types of distributions. Chapters 5 and 6 expose a large variety of admissible rates of convergence that range from logarithmic, across algebraic, to exponential. Even oscillatory contributions to the rate are possible.

The applicability of the deconvolution function in statistics will be discussed in Chapter 7, where an estimator for the unknown distribution F will be constructed by means of a finite sample of Y-observations. If there exist densities $f(x)dx = F(dx)$ and $h(z)dz = H(dz)$, even an estimator for f will be provided. Each estimator appears in the shape of an integral of Fourier-type, whose absolute convergence holds under mild assumptions on Φ_ε. Important quantities in the context of estimators are expectation and variance. A study of the expectation corresponds to a study of the bias. In contrast, the investigation of the variance primarily encompasses a study of the effect of the magnitude of m on the growth of a certain two-dimensional integral. Accordingly, this task is substantially more complicated than the asymptotic evaluation of the bias, yet the basic technique is analogous. This is an extension of the method of analytic continuation, to be presented in the Chapter 8, which concludes the main part of this thesis. Altogether the variance will turn out to grow no faster than m^2 as $m \to \infty$, but considerably slower in most cases or it will even remain bounded.

The appendix is meant to provide an overview on the topic of integral transforms and special functions. In Appendix A some basic properties of Fourier, Laplace and Mellin transforms and their connection will be pointed out. Most of these results can be found in existing literature, in different sources, but a few were developed during the work on this thesis. Finally, Appendix B serves as an introduction to the realm of special functions. The appendix as a whole can be considered independent from the main part, and for the understanding of Chapters 1 to 8, it suffices to consult the single references.

1. An Introduction to Errors in Variables

A common problem in applications is the interest in the probability for an unknown random quantity X not to exceed a certain threshold $\xi \in \mathbb{R}$. This is represented by the distribution function $F(\xi) := \mathbb{P}(X \leq \xi)$. Most of the time F is unknown and needs to be estimated, for instance with the aid of the *empirical distribution function*

$$F_n(\xi) := \sum_{k=1}^{n} \mathbb{1}_{\{X_k \leq \xi\}},$$

for $n \in \mathbb{N}$ and a sample $X_1, \ldots, X_n \sim F$ of independent observations. Here, $\mathbb{1}_{\{x \in A\}}$ for $A \subset \mathbb{R}$ equals one if $x \in A$ and zero otherwise. However, an X-sample is only available in particularly convenient scenarios. The actually difficult cases are those, in which F is not even estimable. A straightforward approach to describe such a setting is the *additive model of errors in variables*. In this, it is assumed that the desired quantity is tainted with a random error ε, leading to the blurred variable

$$Y = X + \varepsilon. \tag{1.0.1}$$

Throughout this thesis we require independence of X and ε, and we suppose the distribution H corresponding to ε is completely known. Moreover, for our theoretical investigations the distribution G of the surrogate variable Y is also supposed to be completely known, but it will eventually be replaced by its empirical analogue, to be estimated by an independent sample $Y_1, \ldots, Y_n \sim G$. The distributions in the additive model of errors in variables are related through the *convolution* or *Faltung*, which refers to the integral

$$G = \int_{-\infty}^{\infty} F(\cdot - z) H(dz). \tag{1.0.2}$$

More concisely we write $G = F * H$, where F and H commute. Assuming F possesses a density f with respect to the Lebesgue-measure, the density corresponding to G is given by

$$g = \int_{-\infty}^{\infty} f(\cdot - z) H(dz). \tag{1.0.3}$$

If also H has a density, then $H(dz) = h(z)dz$ and $g = f * h$ with commuting f and h. Particularly distribution functions play a pivotal role in stochastics, since their existence holds without loss of generality, whereas the existence of densities is limited. Hence, the demand for a means to reconstruct F is only natural. This is known as the *deconvolution problem*. Due to the complicated structure of convolution products, it is in fact a serious problem. A seemingly simplification can be obtained in terms of characteristic functions. Regarding the random variable X, the latter is for $t \in \mathbb{R}$ given by the complex-valued integral

$$\Phi_X(t) := \mathbb{E}e^{itX} = \int_{-\infty}^{\infty} e^{itx} F(dx), \tag{1.0.4}$$

and it thus constitutes the Fourier-Stieltjes transform of F. The integral converges absolutely for any kind of distribution, and the resulting function is unique. By virtue of inversion formulae, F and f, provided the latter exists, can be represented in terms of Φ_X. Another remarkable advantage of Fourier transforms unfolds in the context of convolutions, which thereby become multiplicative products. Accordingly, the additive model of errors in variables equivalently can be written in the form

$$\Phi_Y = \Phi_X \Phi_\varepsilon. \tag{1.0.5}$$

A discussion of the deconvolution problem from a calculus point of view can be found in §1.9 in [Tricomi, 1985], for example. Equation (1.0.5) is then barely problematic, since each of the involved functions are supposed to be known, except Φ_X, whence a representation for Φ_X immediately can be deduced upon dividing by Φ_ε. An application of the Fourier inversion formula finally yields f. The situation, however, essentially changes in stochastics, where at least one side of equation (1.0.5) is merely estimable. Particularly if only Φ_Y is estimable, it exposes devastating consequences for errors with a vanishing characteristic function. In this event, a division is dangerous, since the estimator for Φ_Y does not feature the factor Φ_ε to cancel out. Yet, in many texts the existence of a density of X is assumed, and its recovery is attempted similar to the technique from calculus. Indeed, most of the existing work on errors in variables, of which [Stefanski and Carroll, 1990] and [Fan, 1991b] are the earliest contributions, relies on equation (1.0.5). Even in later literature it is still relevant, although different methods to solve the deconvolution problem are available, see [Meister, 2009] and [Goldenshluger and Kim, 2021].

1.1. Estimation of Densities and Distributions by Means of Smoothing Kernels

Before we begin our investigations, we briefly summarize the approach due to Stefanski and Carroll. For a better understanding of its origin, we first outline the history of density estimation in error-free setups.

1.1.1. A Non-Perturbed Variable

Basically the idea of Stefanski and Carroll is a modification of a slightly older technique due to Parzen and Rosenblatt, known for estimating the density corresponding to a data sample. First, replacing F by F_n in the integral definition (1.0.4), gives rise to the *empirical characteristic function*

$$\Phi_X(t,n) := \frac{1}{n}\sum_{k=1}^{n} e^{itX_k}. \tag{1.1.1}$$

Subject to the strong law of large numbers, $\Phi_X(t,n) \to \Phi_X(t)$ almost surely as $n \to \infty$ for any $t \in \mathbb{R}$, and [Feuerverger and Mureika, 1977] verified the almost sure uniformity on any compact subset of $\mathbb{R}$. However, regardless of the behaviour of the target, the estimator is almost periodic in the sense of Bohr[1]. An essential consequence is that, even if $\Phi_X(t)$ vanishes as $t \to \pm\infty$, this property is not shared by $\Phi_X(t,n)$. This fact especially causes problems if we aim to integrate the empirical characteristic function along an infinite segment of the real axis. To solve this issue, [Rosenblatt, 1956] and [Parzen, 1962] independently proposed an estimator for f, by applying the inversion formula for densities to the product $\Phi_X\Phi_I(\lambda\cdot)$, where $\Phi_I \in L^1(\mathbb{R})$ denotes the Fourier transform of a suitable probability density f_I. For $\xi \in \mathbb{R}$ and $\lambda > 0$ this leads to

$$\hat{f}_n(\xi,\lambda) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi t}\Phi_X(t,n)\Phi_I(\lambda t)dt. \tag{1.1.2}$$

The estimator particularly matches the inversion formula for functions with a non-absolutely integrable Fourier transform, which can be found in [Lukacs, 1970], [Pinsky, 2002], [Körner, 1988] and in equation (A.1.17). The function f_I and the parameter λ are frequently referred to as the *smoothing kernel* and the *bandwidth*, respectively, where the latter specifies the smoothness degree of $\hat{f}_n(\xi,\lambda)$. For any $\xi \in \mathbb{R}$ we have

$$\hat{f}_n(\xi,\lambda) = \frac{1}{\lambda}\int_{-\infty}^{\infty} f_I\left(\frac{\xi - x}{\lambda}\right) F_n(dx).$$

It shows that the estimator is composed of the convolution of F_n with f_I, and it therefore again establishes a probability density. More precisely, if we introduce the random variable $I \sim f_I$, then (1.1.2) equals the probability density of the random variable $X + \lambda I$. In other words, we find ourselves in an artificial additive model of errors in variables. Some authors even admit kernels $f_I \in L^1(\mathbb{R})$ that do not necessarily correspond to probability measures.

Now, the integral (1.1.2) converges absolutely for any $\lambda > 0$. It does, however, no longer exist for $\lambda = 0$, although the sequence of integrals might approach a finite limit as $\lambda \downarrow 0$. This

[1] According to [Bohr, 1932], a continuous function $f(t)$, $t \in \mathbb{R}$, is almost periodic, if for any $\varepsilon > 0$ there exists $L \equiv L(\varepsilon) > 0$ such that any interval of length L contains a so-called translation number $\tau \equiv \tau(\varepsilon)$, i.e., a number τ with the property $|f(t+\tau) - f(t)| \le \varepsilon$ for all $t \in \mathbb{R}$.

is the reason, why density estimation is considered *ill-posed*, already in an error-free setting. It confirms that the empirical distribution function as a step function does not have a density with respect to the Lebesgue measure. Important questions concerning $\hat{f}_n(\xi, \lambda)$ are the rates of convergence of the bias, properties of the variance and the optimal choice of the bandwidth for a given sample size. Integration of (1.1.2) along a finite interval leads to a smoothed estimator for the distribution function F, in particular for the probability $\mathbb{P}(a < X \leq b)$ with $a < b$.

1.1.2. Errors in Variables

If only a blurred version of the target X is observable, the estimation of distribution and density substantially increases in difficulty. [Stefanski and Carroll, 1990] accessed this problem by taking advantage of the product formula (1.0.5). Thereof, by division, and with Φ_Y replaced by its empirical analogue, they deduced for Φ_X the estimator

$$\frac{\Phi_Y(\cdot, n)}{\Phi_\varepsilon}. \tag{1.1.3}$$

Unfortunately this representation comes into conflict with the assumption of a completely known Φ_ε. It basically intensifies the problems that are encountered in density estimation without errors in variables, due to the disbalance of the quotient. Since $\Phi_Y(t, n)$ is almost periodic with random zeros, the estimator (1.1.3) is not only non-integrable on any infinite segment of the real axis, but in general it is also unbounded whenever $\inf_{t \in \mathbb{R}} |\Phi_\varepsilon(t)| = 0$. To circumvent these obstacles, [Stefanski and Carroll, 1990] suggested a smoothing kernel f_I, such that the product of its Fourier transform with (1.1.3) is absolutely integrable along the real axis. Then, analogous to (1.1.2), they employed the Fourier inversion formula, to introduce for the unknown density f with $\xi \in \mathbb{R}$ and $\lambda > 0$ the estimator

$$\mathfrak{f}_n(\xi, \lambda) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi t} \frac{\Phi_Y(t, n)}{\Phi_\varepsilon(t)} \Phi_I(\lambda t) dt. \tag{1.1.4}$$

Permissible kernels need not be probability densities. The estimator $\mathfrak{f}_n(\xi, \lambda)$ is unbiased with respect to $\mathbb{E}\hat{f}_n(\xi, \lambda)$, i.e.,

$$\mathbb{E}\mathfrak{f}_n(\xi, \lambda) = \frac{1}{\lambda} \int_{-\infty}^{\infty} f_I\left(\frac{\xi - x}{\lambda}\right) f(x) dx,$$

where the right hand side finally tends to $f(\xi)$ as $\lambda \downarrow 0$, under appropriate conditions. The rate of convergence essentially depends on the involved distributions. Furthermore, special conditions on f_I are also necessary to ensure a certain asymptotic behaviour of $\mathfrak{f}_n(\xi, \lambda)$. Consistency, for instance, was first discussed by [Fan, 1991a]. To simplify the choice of an appropriate kernel f_I, it is common to assume that the characteristic function of the error distribution vanishes at infinity only or has a compact support. Many authors distinguish particularly between *ordinary* *smooth* and *super smooth* characteristic functions. This notion was suggested by [Fan, 1991b], to characterize functions with algebraic and exponential behaviour at infinity, respectively. It is, however, clearly restrictive, as it especially excludes error distributions whose characteristic functions exhibit finite zeros.

Finally, analogous to the error-free setup, integration of (1.1.4) yields a kernel estimator for the unknown distribution F. This technique was shortly reviewed by [Meister, 2009]. As an alternative for the estimator obtained in this fashion, in a non-perturbed setup we refer to the empirical distribution function. Conversely, for a setting with errors in variables we are unaware of any alternatives that do not rely on artificial smoothing kernels.

1.2. Deconvolution: The Elementary Approach

According to (1.0.1), (1.0.2) and (1.0.5), the additive model of errors in variables can be described in three different ways, i.e., as a sum of random variables, as a convolution integral or as a product of characteristic functions. While most of the existing literature accesses this topic by consideration of (1.0.5), apparently no attention is put on the plain equation (1.0.1). But the latter perfectly illustrates the convolution of two random variables, from which, due to the identity $\mathbb{P}(X \leq \xi) \equiv F(\xi)$, the transition to distributions is immediate. We therefore approach the deconvolution problem from the direction of (1.0.1), where we begin with the possibly simplest setup and successively generalize our assumptions. A major role throughout this section will be played by the elementary equation

$$\mathbb{P}\left(A \cup B\right) = \mathbb{P}\left(A\right) + \mathbb{P}\left(B\right) - \mathbb{P}\left(A \cap B\right), \tag{1.2.1}$$

which holds for arbitrary $A, B \in \mathfrak{A}$ in a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. Moreover, we denote by $\mathbb{T}_Y \subset \mathbb{R}$ the support of the indicated random variable, i.e., the set of admissible values of Y, and by $p_Y(k)$ for $k \in \mathbb{R}$ the associated probability function.

1.2.1. Some Examples for Discrete Variables

In our first scenario we assume Y and ε are discrete random variables with support $\mathbb{T}_Y = \mathbb{T}_\varepsilon = \mathbb{N}_0$. Then $\mathbb{T}_X \subset \mathbb{N}_0$ and $0 \in \mathbb{T}_X$. By discreteness, the distribution of X is known if and only if $p_X(k)$ is known for all $k \in \mathbb{T}_X$, and in these circumstances for $\xi \in \mathbb{R}$ we have

$$F(\xi) = \sum_{k=0}^{\lfloor \xi \rfloor} p_X(k). \tag{1.2.2}$$

Here $\lfloor \xi \rfloor = \max\left\{k \in \mathbb{Z} : k \leq \xi\right\}$ denotes the floor function and an empty sum equals zero. To determine the probability function $p_X(k)$, by independence of X and ε, for any $k \in \mathbb{T}_Y$ we observe:

$$\begin{aligned}
p_Y(k) &= \mathbb{P}(Y = k, \varepsilon \in \mathbb{T}_\varepsilon) \\
&= \sum_{l \in \mathbb{T}_\varepsilon} \mathbb{P}(Y = k, \varepsilon = l) \\
&= \sum_{l \in \mathbb{T}_\varepsilon} p_X(k-l) p_\varepsilon(l) \\
&= \sum_{l=0}^{k} p_X(k-l) p_\varepsilon(l) \qquad (1.2.3)
\end{aligned}$$

The last equation incorporates the non-negativity of X. The advantage of the particular scenario under consideration consists in the possibility to fix the error variable at a certain value with a positive probability. For $k = 1$ the above equation yields $p_Y(1) = p_X(1)p_\varepsilon(0) + p_X(0)p_\varepsilon(1)$. But $p_X(0)$ can be derived in turn from the equation $p_Y(0) = p_X(0)p_\varepsilon(0)$, whence

$$p_X(1) = \frac{p_Y(1)}{p_\varepsilon(0)} - \frac{p_Y(0)p_\varepsilon(1)}{\{p_\varepsilon(0)\}^2}.$$

More generally, if we know $p_X(j)$ for $0 \le j \le k-1$ and $p_\varepsilon(l)$ for $0 \le l \le k$ and also $p_Y(k)$, then $p_X(k)$ is easily computed by virtue of the equation (1.2.3), which can be rearranged to become

$$p_X(k) = \frac{p_Y(k)}{p_\varepsilon(0)} - \sum_{l=1}^{k} p_X(k-l) \frac{p_\varepsilon(l)}{p_\varepsilon(0)}. \qquad (1.2.4)$$

We can even iteratively apply (1.2.4), to represent $p_X(k)$ solely in terms of the probability functions of Y and ε. For this we denote by $\delta_{\{a\}}(x)$ the Dirac point-measure at $a \in \mathbb{R}$, taking the value one if $x = a$ and zero otherwise.

Theorem 1.2.1 (recovery of p_X in a non-negative discrete setup). *Assume that X and ε are independent with $\mathbb{T}_X \subset \mathbb{N}_0$ and $\mathbb{T}_\varepsilon = \mathbb{N}_0$. For $l \in \mathbb{N}_0$ define*

$$p(j,l) := \begin{cases} \displaystyle\sum_{\substack{z_1,\ldots,z_j \in \mathbb{N} \\ \sum_{i=1}^{j} z_i = l}} \prod_{i=1}^{j} p_\varepsilon(z_i), & \text{if } j \ge 1, \\ \delta_{\{0\}}(l), & \text{if } j = 0. \end{cases} \qquad (1.2.5)$$

Observe $p(j,l) = 0$ for $l < j$. Then, for all $k \in \mathbb{T}_X$ we have

$$p_X(k) = \sum_{l=0}^{k} p_Y(k-l) \sum_{j=0}^{l} (-1)^j \{p_\varepsilon(0)\}^{-(j+1)} p(j,l). \qquad (1.2.6)$$

Proof. The proof uses induction on k. Starting with $k = 0$ we obtain from (1.2.6) the correct result $p_X(0) = \frac{p_Y(0)}{p_\varepsilon(0)}$. Suppose now that $p_X(s)$ is given by (1.2.6) for $0 \le s \le k-1$. Then, from

(1.2.4) we obtain for $s = k$:

$$
\begin{aligned}
p_X(k) &= \frac{p_Y(k)}{p_\varepsilon(0)} - \sum_{r=1}^{k} p_X(k-r)\frac{p_\varepsilon(r)}{p_\varepsilon(0)} \\
&= \frac{p_Y(k)}{p_\varepsilon(0)} + \sum_{r=1}^{k}\sum_{l=0}^{k-r} p_Y(k-r-l)\sum_{j=0}^{l}(-1)^{j+1}\left\{p_\varepsilon(0)\right\}^{-(j+2)} p_\varepsilon(r)p(j,l)
\end{aligned}
$$

Substituting $r_2 = k-r$, $r_3 = k-r_2$ and $r_4 = r_3 + (l-1)$, finally leads to:

$$
\begin{aligned}
&\sum_{r=1}^{k}\sum_{l=0}^{k-r} p_Y((k-r)-l)\sum_{j=0}^{l}(-1)^{j+1}\left\{p_\varepsilon(0)\right\}^{-(j+2)} p_\varepsilon(k-(k-r))p(j,l) \\
&\quad= \sum_{r_2=0}^{k-1}\sum_{l=0}^{r_2} p_Y(r_2-l)\sum_{j=0}^{l}(-1)^{j+1}\left\{p_\varepsilon(0)\right\}^{-(j+2)} p_\varepsilon(k-r_2)p(j,l) \\
&\quad= \sum_{l=0}^{k-1}\sum_{r_2=l}^{k-1}\sum_{j=0}^{l}(-1)^{j+1}\left\{p_\varepsilon(0)\right\}^{-(j+2)} p_Y(r_2-l)p_\varepsilon(k-r_2)p(j,l) \\
&\quad= \sum_{l=0}^{k-1}\sum_{j=0}^{l}\sum_{r_2=l}^{k-1}(-1)^{j+1}\left\{p_\varepsilon(0)\right\}^{-(j+2)} p_Y(r_2-l)p_\varepsilon(k-r_2)p(j,l) \\
&\quad= \sum_{j=0}^{k-1}\sum_{l=j}^{k-1}\sum_{r_2=l}^{k-1}(-1)^{j+1}\left\{p_\varepsilon(0)\right\}^{-(j+2)} p_Y(k-l-(k-r_2))p_\varepsilon(k-r_2)p(j,l) \\
&\quad= \sum_{j=0}^{k-1}\sum_{l=j}^{k-1}\sum_{r_3=1}^{k-l}(-1)^{j+1}\left\{p_\varepsilon(0)\right\}^{-(j+2)} p_Y(k-l-r_3)p_\varepsilon(r_3)p(j,l) \\
&\quad= \sum_{j=1}^{k}(-1)^{j}\left\{p_\varepsilon(0)\right\}^{-(j+1)} \sum_{l=j-1}^{k-1}\sum_{r_3=1}^{k-l} p_Y(k-l-r_3)p_\varepsilon(r_3)p(j-1,l) \\
&\quad= \sum_{j=1}^{k}(-1)^{j}\left\{p_\varepsilon(0)\right\}^{-(j+1)} \sum_{l=j}^{k}\sum_{r_3=1}^{k-(l-1)} p_Y(k-(l-1)-r_3)p_\varepsilon(r_3)p(j-1,l-1) \\
&\quad= \sum_{j=1}^{k}(-1)^{j}\left\{p_\varepsilon(0)\right\}^{-(j+1)} \sum_{l=j}^{k}\sum_{r_4=l}^{k} p_Y(k-r_4)p_\varepsilon(r_4-(l-1))p(j-1,l-1) \\
&\quad= \sum_{j=1}^{k}(-1)^{j}\left\{p_\varepsilon(0)\right\}^{-(j+1)} \sum_{r_4=j}^{k}\sum_{l=j}^{r_4} p_Y(k-r_4)p_\varepsilon(r_4-(l-1))p(j-1,l-1) \\
&\quad= \sum_{r_4=1}^{k} p_Y(k-r_4)\sum_{j=1}^{r_4}(-1)^{j}\left\{p_\varepsilon(0)\right\}^{-(j+1)} \sum_{l=j}^{r_4} p_\varepsilon(r_4-(l-1))p(j-1,l-1) \\
&\quad= \sum_{r_4=1}^{k} p_Y(k-r_4)\sum_{j=1}^{r_4}(-1)^{j}\left\{p_\varepsilon(0)\right\}^{-(j+1)} p(j,r_4)
\end{aligned}
$$

For $j = 1$ the last equation is easily seen to hold. To confirm the equation for $j \geq 2$, we note that the definition of $p(j,l)$ implies:

$$\sum_{l=j}^{r_4} p_\varepsilon(r_4-(l-1))p(j-1,l-1) = \sum_{l=j}^{r_4} p_\varepsilon(r_4-(l-1)) \sum_{\substack{z_1,\dots,z_{j-1}\in\mathbb{N} \\ \sum_{i=1}^{j-1} z_i = l-1}} \prod_{i=1}^{j-1} p_\varepsilon(z_i)$$
$$= \sum_{\substack{z_1,\dots,z_j\in\mathbb{N} \\ \sum_{i=1}^{j} z_i = r_4}} \prod_{i=1}^{j} p_\varepsilon(z_i)$$

A comparison with (1.2.5) shows that the last sum equals $p(j,r_4)$, thereby verifying (1.2.6). ■

The preceding theorem eventually enables a representation of F solely in terms of the distributions of Y and ε.

Corollary 1.2.1 (recovery of F in a non-negative discrete setup). *Under the conditions of Theorem 1.2.1, for $\xi \in \mathbb{R}$ we have*

$$F(\xi) = \sum_{l=0}^{\lfloor\xi\rfloor} G(\xi-l) \sum_{j=0}^{l} (-1)^j \{p_\varepsilon(0)\}^{-(j+1)} p(j,l). \tag{1.2.7}$$

Especially notice that the validity of (1.2.7) does not require any assumptions concerning the right boundary of the support of Y.

Proof. It suffices to plug the formula for the point probabilities (1.2.6) into (1.2.2). Then some simple rearrangements of the sums yield:

$$F(\xi) = \sum_{k=0}^{\lfloor\xi\rfloor}\sum_{l=0}^{k} p_Y(k-l) \sum_{j=0}^{l} (-1)^j \{p_\varepsilon(0)\}^{-(j+1)} p(j,l)$$
$$= \sum_{l=0}^{\lfloor\xi\rfloor}\sum_{k=l}^{\lfloor\xi\rfloor} p_Y(k-l) \sum_{j=0}^{l} (-1)^j \{p_\varepsilon(0)\}^{-(j+1)} p(j,l)$$
$$= \sum_{l=0}^{\lfloor\xi\rfloor} G(\lfloor\xi\rfloor-l) \sum_{j=0}^{l} (-1)^j \{p_\varepsilon(0)\}^{-(j+1)} p(j,l)$$

But $G(\lfloor\xi\rfloor-l) = G(\xi-l)$ since $\lfloor\xi-l\rfloor = \lfloor(\xi-\lfloor\xi\rfloor)+(\lfloor\xi\rfloor-l)\rfloor = \lfloor\xi\rfloor-l$ for any $l\in\mathbb{N}_0$. ■

Although (1.2.7) provides a tool to recover the unknown distribution F only for a special and particularly simple scenario, it shows that deconvolution already then is very complicated. With these impressions we leave the purely discrete case and proceed with a slightly more general situation. More precisely, an inspection of (1.2.7) suggests that the formula could also hold if the distribution of X is arbitrary, whereas ε remains discrete.

Example 1.2.2 (arbitrary $X \geq 0$ and non-negative discrete errors). Suppose F is an arbitrary distribution with $\mathbb{T}_X \subset [0,\infty)$ and $\mathbb{T}_\varepsilon \subset \mathbb{N}_0$ with $p_\varepsilon(0) > 0$. As a consequence

$\mathbb{T}_Y \subset [0,\infty)$ and the sum variable Y is not necessarily purely discrete, whence in general $p_Y(0) = p_X(0)p_\varepsilon(0) \geq 0$. However, in any case we have $G(\xi) = F(\xi)p_\varepsilon(0)$ for $0 \leq \xi < 1$ and $G(\xi) = F(\xi)p_\varepsilon(0) + F(\xi-1)p_\varepsilon(1)$ for $1 \leq \xi < 2$, and so on. Analogous to (1.2.4), for $\xi \in \mathbb{R}$ this pattern gives rise to the recursion

$$F(\xi) = \frac{G(\xi)}{p_\varepsilon(0)} - \sum_{l=1}^{\lfloor \xi \rfloor} F(\xi - l)\frac{p_\varepsilon(l)}{p_\varepsilon(0)}. \tag{1.2.8}$$

We will now verify by induction that the formula for F, which was derived in (1.2.7), still satisfies this pattern. For $\xi < 1$ this is easy to see by comparison of (1.2.8) with (1.2.7). Assume now $F(\xi)$ for $\xi < K$ with arbitrary $K \in \mathbb{N}$ is given by (1.2.7). Then for $K \leq \xi < K+1$ we obtain from (1.2.8), upon substituting $n = i + l$, $r = n - i$, $k = j + 1$ and $s = 1 + r$, especially since $\lfloor \xi - i \rfloor = \lfloor \xi \rfloor - i$ for $i \in \mathbb{N}_0$:

$$\begin{aligned}
F(\xi) &= \frac{G(\xi)}{p_\varepsilon(0)} - \sum_{i=1}^{\lfloor \xi \rfloor} F(\xi - i)\frac{p_\varepsilon(i)}{p_\varepsilon(0)} \\
&= \frac{G(\xi)}{p_\varepsilon(0)} - \sum_{i=1}^{\lfloor \xi \rfloor} \frac{p_\varepsilon(i)}{p_\varepsilon(0)} \sum_{l=0}^{\lfloor \xi \rfloor - i} G(\xi - i - l) \sum_{j=0}^{l} (-1)^j \{p_\varepsilon(0)\}^{-(j+1)} p(j,l) \\
&= \frac{G(\xi)}{p_\varepsilon(0)} - \sum_{i=1}^{\lfloor \xi \rfloor} \frac{p_\varepsilon(i)}{p_\varepsilon(0)} \sum_{n=i}^{\lfloor \xi \rfloor} G(\xi - n) \sum_{j=0}^{n-i} (-1)^j \{p_\varepsilon(0)\}^{-(j+1)} p(j,n-i) \\
&= \frac{G(\xi)}{p_\varepsilon(0)} - \sum_{n=1}^{\lfloor \xi \rfloor} G(\xi - n) \sum_{i=1}^{n} \sum_{j=0}^{n-i} (-1)^j \{p_\varepsilon(0)\}^{-(j+2)} p(j,n-i)p_\varepsilon(i) \\
&= \frac{G(\xi)}{p_\varepsilon(0)} - \sum_{n=1}^{\lfloor \xi \rfloor} G(\xi - n) \sum_{r=0}^{n-1} \sum_{j=0}^{r} (-1)^j \{p_\varepsilon(0)\}^{-(j+2)} p(j,r)p_\varepsilon(n-r) \\
&= \frac{G(\xi)}{p_\varepsilon(0)} - \sum_{n=1}^{\lfloor \xi \rfloor} G(\xi - n) \sum_{j=0}^{n-1} \sum_{r=j}^{n-1} (-1)^j \{p_\varepsilon(0)\}^{-(j+2)} p(j,r)p_\varepsilon(n-r) \\
&= \frac{G(\xi)}{p_\varepsilon(0)} + \sum_{n=1}^{\lfloor \xi \rfloor} G(\xi - n) \sum_{k=1}^{n} \sum_{r=k-1}^{n-1} (-1)^k \{p_\varepsilon(0)\}^{-(k+1)} p(k-1,r)p_\varepsilon(n-r) \\
&= \frac{G(\xi)}{p_\varepsilon(0)} + \sum_{n=1}^{\lfloor \xi \rfloor} G(\xi - n) \sum_{k=1}^{n} (-1)^k \{p_\varepsilon(0)\}^{-(k+1)} \sum_{s=k}^{n} p(k-1,s-1)p_\varepsilon(n-(s-1))
\end{aligned}$$

In accordance with (1.2.5), the probabilities in the last sum satisy the following identity:

$$\begin{aligned}
\sum_{s=k}^{n} p(k-1,s-1)p_\varepsilon(n-(s-1)) &= \sum_{s=k}^{n} p_\varepsilon(n-(s-1)) \sum_{\substack{z_1,\ldots,z_{k-1}\in\mathbb{N} \\ \sum_{i=1}^{k-1} z_i = s-1}} \prod_{i=1}^{k-1} p_\varepsilon(z_i) \\
&= p(k,n)
\end{aligned}$$

We have thus shown that $F(\xi)$ for $K \leq \xi < K+1$ indeed has the representation (1.2.7). Since K was arbitrary, the formula is applicable to recover $F(\xi)$ for any $\xi \in \mathbb{R}$.

The arguments of the preceding example clearly become invalid if the error distribution is no longer discrete. As a clue how to handle such scenarios, we consider the case converse to Example 1.2.2.

Example 1.2.3 (discrete $X \geq 0$ and arbitrary one-sided errors). Assume $\mathbb{T}_X \subset \mathbb{N}_0$ and, for convenience, $\mathbb{T}_\varepsilon = [0, \infty)$ with $p_\varepsilon(0) = 0$ and $H(1) > 0$. The generality of the latter property is in contrast to the above examples, making the probability function of ε useless. Furthermore, again Y is a distribution of arbitrary type with $\mathbb{T}_Y = [0, \infty)$ and $p_Y(0) = p_X(0)p_\varepsilon(0) = 0$. The probability $p_X(0)$ is thus concealed. On the other hand, we also have $G(1) = p_X(0)H(1) + p_X(1)H(0) = p_X(0)H(1)$, whence $p_X(0)$ still can be recovered. For $k \in \mathbb{T}_X$ we may deduce more generally

$$p_X(k-1) = \frac{G(k)}{H(1)} - \sum_{i=0}^{k-2} p_X(i) \frac{H(k-i)}{H(1)}. \tag{1.2.9}$$

It must be emphasized that this formula becomes invalid if $p_\varepsilon(0) > 0$. The evaluation of $p_X(k)$ for some small $k \in \mathbb{N}$ gives the clue that the non-recursive formula for the point probability (1.2.9) equals

$$p_X(k) = \sum_{l=0}^{k} G(1+k-l) \sum_{j=0}^{l} \{H(1)\}^{-(1+j)} (-1)^j P(j,l), \tag{1.2.10}$$

where the probabilities $P(j,l)$ for $j, l \in \mathbb{N}_0$ are defined by

$$P(j,l) := \begin{cases} \sum\limits_{\substack{z_1,\ldots,z_j \in \mathbb{N}\setminus\{1\} \\ \sum_{i=1}^{j} z_i = j+l}} \prod\limits_{i=1}^{j} H(z_i), & \text{if } j \geq 1, \\ \delta_{\{0\}}(l), & \text{if } j = 0. \end{cases} \tag{1.2.11}$$

It is clear that $P(j,l) = 0$ for $l < j$. Furthermore, a comparison with (1.2.6) shows a slight similarity with respect to the basic structure of both sums. By induction it is readily verified that (1.2.10) indeed satisfies the recursion (1.2.9). The validity for $k = 0$ is immediate. Supposing now that the formula holds for $0, \ldots, k-1$ with arbitrary $k \in \mathbb{N}$, we obtain for the k-th probability from (1.2.9) after putting $r = k - i$:

$$\begin{aligned} p_X(k) &= \frac{G(k+1)}{H(1)} - \sum_{r=1}^{k} p_X(k-r) \frac{H(1+r)}{H(1)} \\ &= \frac{G(k+1)}{H(1)} + \sum_{r=1}^{k} \sum_{l=0}^{k-r} G(1+k-r-l) \sum_{j=0}^{l} (-1)^{1+j} \{H(1)\}^{-(j+2)} H(1+r) P(j,l) \end{aligned}$$

This last triple sum resembles that in the proof of equation (1.2.6). Analogous manipulations thus yield:

$$\begin{aligned}\sum_{r=1}^{k}\sum_{l=0}^{k-r} G(1+k-r-l)\sum_{j=0}^{l}(-1)^{1+j}\{H(1)\}^{-(j+2)} H(1+r)P(j,l)\\ = \sum_{r=1}^{k} G(1+k-r)\sum_{j=1}^{r}(-1)^{j}\{H(1)\}^{-(j+1)}\sum_{l=j}^{r} H(1+r-(l-1))P(j-1,l-1)\end{aligned}$$

On the one hand, for $j=1$ we have

$$\sum_{l=1}^{r} H(1+r-(l-1))\delta_{\{0\}}(l-1) = H(1+r) = P(1,r).$$

On the other hand, for $j \geq 2$ we find:

$$\begin{aligned}\sum_{l=j}^{r} H(1+r-(l-1))P(j-1,l-1) &= \sum_{l=j}^{r} H(1+r-(l-1)) \sum_{\substack{z_1,\ldots,z_{j-1}\in\mathbb{N}\setminus\{1\}\\ \sum_{i=1}^{j-1} z_i = j-1+l-1}} \prod_{i=1}^{j-1} H(z_i)\\ &= \sum_{\substack{z_1,\ldots,z_j\in\mathbb{N}\setminus\{1\}\\ \sum_{i=1}^{j} z_i=j+r}} \prod_{i=1}^{j} H(z_i)\\ &= P(j,r)\end{aligned}$$

Collecting the above results, we arrive at (1.2.10). The formula is thus indeed valid. Summing up, according to (1.2.2), even yields the distribution function of X. Subsequent substitution of $i=k-l$, $r=k-i$ and $s=\lfloor\xi\rfloor-i$ leads to:

$$\begin{aligned}F(\xi) &= \sum_{k=0}^{\lfloor\xi\rfloor}\sum_{l=0}^{k} G(1+k-l)\sum_{j=0}^{l}\{H(1)\}^{-(1+j)}(-1)^{j}P(j,l)\\ &= \sum_{k=0}^{\lfloor\xi\rfloor}\sum_{i=0}^{k} G(1+i)\sum_{j=0}^{k-i}\{H(1)\}^{-(1+j)}(-1)^{j}P(j,k-i)\\ &= \sum_{i=0}^{\lfloor\xi\rfloor} G(1+i)\sum_{k=i}^{\lfloor\xi\rfloor}\sum_{j=0}^{k-i}\{H(1)\}^{-(1+j)}(-1)^{j}P(j,k-i)\\ &= \sum_{i=0}^{\lfloor\xi\rfloor} G(1+i)\sum_{r=0}^{\lfloor\xi\rfloor-i}\sum_{j=0}^{r}\{H(1)\}^{-(1+j)}(-1)^{j}P(j,r)\\ &= \sum_{s=0}^{\lfloor\xi\rfloor} G(1+\lfloor\xi\rfloor-s)\sum_{r=0}^{s}\sum_{j=0}^{r}\{H(1)\}^{-(1+j)}(-1)^{j}P(j,r) \qquad (1.2.12)\end{aligned}$$

Observe that the right hand side of (1.2.12) is well-defined for any distribution F, H, provided

$H(1) > 0$. It is, however, easy to see that it only equals F if the latter is a step function. Choose for instance $0 \leq \xi < 1$ to obtain $F(\xi) = \frac{G(1)}{H(1)}$. Clearly, this is not always true. It does not even remain valid if all of the above conditions stay the same but $p_\varepsilon(0) > 0$.

To summarize our findings so far, in some circumstances the deconvolution of the unknown distribution function F is possible. For this we essentially exploited the specific structure of the assumed error distribution. Moreover, it is ascertainable that the derivation of each deconvolution formula also requires information about the structure of F and the result thus heavily depends on it. Finally, the above examples can be modified to cover setups involving random variables with more general one-sided supports. Such modifications will not be presented here, and instead we will now treat more general scenarios of errors in variables under minimal assumptions.

1.2.2. Derivation of an Integral Equation

Recall that in (1.2.6) and also in the Examples 1.2.2 and 1.2.3, the starting point was not X but the sum Y, which was decomposed into X and ε. More precisely, we fixed ε in a sense, in order to make an exact statement on the amount of X with respect to Y. The most important ingredient to this procedure besides the independence of X and ε was the formula (1.2.1). For a more general access to the problem of errors in variables we observe that, subject to the elementary rules for the arithmetic of probabilities and sets, for any $c, \xi \in \mathbb{R}$ the following holds:

$$\begin{aligned}
\mathbb{P}(Y \leq \xi + c) &= \mathbb{P}(Y \leq \xi + c, X \leq \xi, \varepsilon \leq c) + \mathbb{P}\left(Y \leq \xi + c, \{X \leq \xi, \varepsilon \leq c\}^{\mathrm{C}}\right) \\
&= \mathbb{P}(Y \leq \xi + c, X \leq \xi, \varepsilon \leq c) + \mathbb{P}(\{Y \leq \xi + c, X > \xi\} \cup \{Y \leq \xi + c, \varepsilon > c\}) \\
&= \mathbb{P}(X \leq \xi, \varepsilon \leq c) + \mathbb{P}(Y \leq \xi + c, X > \xi) + \mathbb{P}(Y \leq \xi + c, \varepsilon > c) \\
&\qquad - \mathbb{P}(Y \leq \xi + c, X > \xi, \varepsilon > c) \\
&= \mathbb{P}(X \leq \xi)\,\mathbb{P}(\varepsilon \leq c) + \mathbb{P}(\varepsilon \leq \xi + c - X, X > \xi) + \mathbb{P}(X \leq \xi + c - \varepsilon, \varepsilon > c)
\end{aligned}$$

In the first step we separated $\{Y \leq \xi + c\}$ into two particular disjoint sets and were accordingly allowed to split the probabilities. But $\{Y \leq \xi + c\} \supset \{X \leq \xi, \varepsilon \leq c\}$, and the probability involving this set could be written as a product, due to the independence of X and ε. An additional application of (1.2.1), since $\{Y \leq \xi + c, X > \xi, \varepsilon > c\} = \emptyset$, finally lead to the above result. This in turn can be rewritten in terms of integrals of distribution functions. Then, assuming $c \in \mathbb{R}$ is such that $H(c) > 0$, we may divide by $H(c)$, which leads to

$$F(\xi) = \frac{G(\xi + c)}{H(c)} - \int\limits_{(\xi,\infty)} \frac{H(\xi + c - x)}{H(c)} F(dx) - \int\limits_{(c,\infty)} \frac{F(\xi - (z - c))}{H(c)} H(dz). \tag{1.2.13}$$

By expressing, for example, the second summand in this equation as an integral with respect to H, we return to (1.0.2) with ξ replaced by $\xi + c$. Hence, both integral equations are actually equivalent. Contrary to the latter, the former is a special integral equation of the second kind that slightly resembles those of Volterra-type. However, while common Volterra-type equations involve only one integral that features the unknown function, the equation (1.2.13) consists of two such integrals. Even more distinguishing is the fact that one of the integrals is computed with respect to F. Despite these differences (1.2.13) shares one obvious similarity with integral equations of the aforementioned type. That is, the function F emerges iteratively from

$$G_c := \frac{G(\cdot + c)}{H(c)}. \tag{1.2.14}$$

It thus appears natural to adapt the standard idea for the solution of such integral equations, which is sometimes referred to as the method of successive approximations, compare [Tricomi, 1985]. For this purpose we introduce the initial function $\mathfrak{E}_c(\cdot, 0) := G_c$ and for $m \in \mathbb{N}$ the recursively defined functions

$$\begin{aligned} \mathfrak{E}_c(\xi, m) = G_c(\xi) - \int\limits_{(\xi,\infty)} \frac{H(\xi + c - x)}{H(c)} \mathfrak{E}_c(dx, m-1) \\ - \int\limits_{(c,\infty)} \frac{\mathfrak{E}_c(\xi - (z - c), m-1)}{H(c)} H(dz). \end{aligned} \tag{1.2.15}$$

To gain an insight on the functionality of (1.2.15) we compute $\mathfrak{E}_c(\cdot, 1)$. For this we recall that $\mathfrak{E}_c(\cdot, 0) = G_c$ equals the distribution function of $Y - c$ times the factor $\frac{1}{H(c)} \geq 1$. Denoting by $\varepsilon_1 \sim H$ a version of ε that is independent of Y, the second summand in (1.2.15) for $m = 1$ can be cast as follows:

$$\begin{aligned} \int\limits_{(\xi,\infty)} \frac{H(\xi + c - x)}{H(c)} \mathfrak{E}_c(dx, 0) &= \frac{1}{H(c)} \int\limits_{(\xi,\infty)} H(\xi + c - x) G_c(dx) \\ &= \frac{1}{\{H(c)\}^2} \mathbb{E}\{\mathbb{I}\{\xi < Y - c\} \mathbb{I}\{\varepsilon_1 \leq \xi + c - (Y - c)\}\} \\ &= \frac{1}{\{H(c)\}^2} \mathbb{E}\{\mathbb{I}\{\xi < Y - c \leq \xi + c - \varepsilon_1\}\} \\ &= \frac{1}{\{H(c)\}^2} \int\limits_{(-\infty,c]} G(\xi + 2c - z) - G(\xi + c) H(dz) \\ &= \int\limits_{(-\infty,c]} \frac{G_c(\xi - (z - c))}{H(c)} H(dz) - G_c(\xi) \end{aligned}$$

For $m = 1$ we thus obtain from (1.2.15):

$$\mathfrak{E}_c(\xi, 1) = G_c(\xi) - \left[\int\limits_{(-\infty,c]} \frac{G_c(\xi - (z - c))}{H(c)} H(dz) - G_c(\xi) \right] - \int\limits_{(c,\infty)} \frac{G_c(\xi - (z - c))}{H(c)} H(dz)$$

$$= 2\frac{G(\xi+c)}{H(c)} - \frac{1}{\{H(c)\}^2}\int_{-\infty}^{\infty} G(\xi+c-(z-c))H(dz)$$

For $m=2$ analogous computations yield

$$\begin{aligned}\mathfrak{E}_c(\xi,2) = 3\frac{G(\xi+c)}{H(c)} &- \frac{3}{\{H(c)\}^2}\int_{-\infty}^{\infty} G(\xi+c-(z-c))H(dz)\\ &+ \frac{1}{\{H(c)\}^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\xi+c-(z_1-c)-(z_2-c))H(dz_1)H(dz_2).\end{aligned}$$

The preceding examples of $\mathfrak{E}_c(\cdot,m)$ for $m=1,2$ already expose the pattern behind the recursion (1.2.15), which will be established in the theorem below.

Theorem 1.2.2 (a non-iterative formula). *Let $\varepsilon_1,\varepsilon_2,\ldots \sim H$ be mutually independent versions of ε and also independent of Y. Then, for any $m\in\mathbb{N}$ the function*

$$\begin{aligned}\mathfrak{E}_c(\xi,m) &= \sum_{k=0}^{m}\binom{m+1}{k+1}(-1)^k\{H(c)\}^{-(k+1)}\,\mathbb{P}\left((Y-c)+\sum_{r=1}^{k}(\varepsilon_r-c)\le\xi\right)\\ &= \sum_{l=0}^{m}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\{H(c)\}^{-(k+1)}\,\mathbb{P}\left((Y-c)+\sum_{r=1}^{k}(\varepsilon_r-c)\le\xi\right)\end{aligned} \tag{1.2.16}$$

is the result of the m-th iteration of (1.2.15). Again we use the convention that an empty sum equals zero.

Proof. We show that the function (1.2.16) for any $m\in\mathbb{N}$ results from the recursive definition (1.2.15). Therefore we apply the single sum representation and first consider the second integral in (1.2.15):

$$\begin{aligned}&-\int_{(\xi,\infty)}\frac{H(\xi+c-x)}{H(c)}\mathfrak{E}_c(dx,m-1)\\ &= \sum_{k=0}^{m-1}\binom{m}{k+1}(-1)^{k+1}\{H(c)\}^{-(k+2)}\\ &\qquad\times\mathbb{E}\left[\mathbb{I}\left\{\xi<(Y-c)+\sum_{r=1}^{k}(\varepsilon_r-c)\right\}H\left(\xi+c-(Y-c)-\sum_{r=1}^{k}(\varepsilon_r-c)\right)\right]\\ &= \sum_{k=0}^{m-1}\binom{m}{k+1}(-1)^{k+1}\{H(c)\}^{-(k+2)}\\ &\qquad\times\mathbb{E}\left[\int_{-\infty}^{\infty}\mathbb{I}\left\{\xi<(Y-c)+\sum_{r=1}^{k}(\varepsilon_r-c)\le\xi-(z-c)\right\}H(dz)\right]\end{aligned}$$

$$= \sum_{k=0}^{m-1} \binom{m}{k+1} (-1)^{k+1} \{H(c)\}^{-(k+2)} \int_{(-\infty,c]} \mathbb{P}\left((Y-c) + \sum_{r=1}^{k} (\varepsilon_r - c) \le \xi - (z-c) \right) H(dz)$$
$$+ \sum_{k=0}^{m-1} \binom{m}{k+1} (-1)^{k} \{H(c)\}^{-(k+1)} \mathbb{P}\left((Y-c) + \sum_{r=1}^{k} (\varepsilon_r - c) \le \xi \right)$$

Altogether we thus obtain for (1.2.15):

$$\begin{aligned}
\mathfrak{E}_c(\xi, m) = G_c(\xi) &+ \sum_{k=0}^{m-1} \binom{m}{k+1} (-1)^{k+1} \{H(c)\}^{-(k+2)} \\
&\qquad \times \int_{(-\infty,c]} \mathbb{P}\left((Y-c) + \sum_{r=1}^{k} (\varepsilon_r - c) \le \xi - (z-c) \right) H(dz) \\
&+ \sum_{k=0}^{m-1} \binom{m}{k+1} (-1)^{k} \{H(c)\}^{-(k+1)} \mathbb{P}\left((Y-c) + \sum_{r=1}^{k} (\varepsilon_r - c) \le \xi \right) \\
&+ \sum_{k=0}^{m-1} \binom{m}{k+1} (-1)^{k+1} \{H(c)\}^{-(k+2)} \\
&\qquad \times \int_{(c,\infty)} \mathbb{P}\left((Y-c) + \sum_{r=1}^{k} (\varepsilon_r - c) \le \xi - (z-c) \right) H(dz) \\
= \sum_{k=0}^{m} \binom{m}{k} &(-1)^{k} \{H(c)\}^{-(k+1)} \mathbb{P}\left((Y-c) + \sum_{r=1}^{k} (\varepsilon_r - c) \le \xi \right) \\
&+ \sum_{k=0}^{m-1} \binom{m}{k+1} (-1)^{k} \{H(c)\}^{-(k+1)} \mathbb{P}\left((Y-c) + \sum_{r=1}^{k} (\varepsilon_r - c) \le \xi \right) \\
= (-1)^m \{H(c)\}^{-(m+1)} &\mathbb{P}\left((Y-c) + \sum_{r=1}^{m} (\varepsilon_r - c) \le \xi \right) \\
&+ \sum_{k=0}^{m-1} \binom{m+1}{k+1} (-1)^{k} \{H(c)\}^{-(k+1)} \mathbb{P}\left((Y-c) + \sum_{r=1}^{k} (\varepsilon_r - c) \le \xi \right)
\end{aligned}$$

The last equation incorporates the binomial identity (26.3.5) from [Olver et al., 2010] and thus matches exactly the first formula in (1.2.16). Finally the second formula in (1.2.16) follows from a simple application of the additional binomial identity (26.3.7) in [Olver et al., 2010], accompanied by an interchange in the summation order. ∎

We will now see that the function (1.2.16) is closely related to our examples presented in the earlier Section 1.2.1. In before we remind the reader about the binomial theorem, which frequently will be referred to below and later on.

Example 1.2.4 (connection to Example 1.2.2). If X is associated with an arbitrary distribution F with $\mathbb{T}_X \subset [0, \infty)$ and $\varepsilon \ge 0$ corresponds to a discrete distribution with $\mathbb{T}_\varepsilon \subset \mathbb{N}_0$ and $p_\varepsilon(0) > 0$, we verified in Example 1.2.2 that F can be reconstructed by means of (1.2.7).

Regarding (1.2.13), under these assumptions we may choose $c = 0$ to obtain $H(c) = p_\varepsilon(0)$ and the second term in this integral equation then equals zero. Moreover, the third term, i.e., the integral with respect to H becomes a finite sum. Hence, the entire equation (1.2.13) is equivalent to (1.2.8). This already indicates a close relation between the right hand side of (1.2.7) and (1.2.16), which shall be confirmed by elementary manipulations of the latter. Preliminary, subject to the non-negativity of Y, the discrete structure of ε and the independence of $\varepsilon_1, \varepsilon_2, \ldots$, for $j, k \in \mathbb{N}_0$ in terms of (1.2.5) we have:

$$\begin{aligned}\mathbb{P}\left(Y \le \xi - \sum_{r=1}^{k} \varepsilon_r\right) &= \sum_{j=0}^{\lfloor \xi \rfloor} G(\xi - j)\, \mathbb{P}\left(\sum_{r=1}^{k} \varepsilon_r = j\right) \\ &= \sum_{j=0}^{\lfloor \xi \rfloor} G(\xi - j) \sum_{t=0}^{k} \binom{k}{t} \{p_\varepsilon(0)\}^t\, p(k-t, j)\end{aligned} \tag{1.2.17}$$

Particularly for $k = 0$ this sequence of equations still holds with $\mathbb{P}\left(\sum_{r=1}^{0} \varepsilon_r = j\right) = \delta_{\{0\}}(j)$. In view of these preparations, (1.2.16) for $\xi \in \mathbb{R}$ equals

$$\mathfrak{E}_0(\xi, m) = \sum_{j=0}^{\lfloor \xi \rfloor} G(\xi - j) \sum_{l=0}^{m} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \{p_\varepsilon(0)\}^{-(k+1)} \sum_{t=0}^{k} \binom{k}{t} \{p_\varepsilon(0)\}^t\, p(k-t, j).$$

Provided $m \ge \lfloor \xi \rfloor$, after the substitution $s = k - t$, the above double sum may be rewritten as follows:

$$\begin{aligned}&\sum_{l=0}^{m} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \sum_{t=0}^{k} \binom{k}{t} \{p_\varepsilon(0)\}^{-(k-t+1)}\, p(k-t, j) \\ &= \sum_{l=0}^{m} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \sum_{s=0}^{k} \binom{k}{k-s} \{p_\varepsilon(0)\}^{-(s+1)}\, p(s, j) \\ &= \sum_{l=0}^{m} \sum_{s=0}^{l} \sum_{k=s}^{l} \binom{l}{k} \binom{k}{k-s} (-1)^k \{p_\varepsilon(0)\}^{-(s+1)}\, p(s, j) \\ &= \sum_{s=0}^{m} \sum_{l=s}^{m} \sum_{k=s}^{l} \binom{l}{k} \binom{k}{k-s} (-1)^k \{p_\varepsilon(0)\}^{-(s+1)}\, p(s, j) \\ &= \sum_{s=0}^{j} (-1)^s \{p_\varepsilon(0)\}^{-(s+1)}\, p(s, j) \sum_{l=s}^{m} \sum_{k=s}^{l} \binom{l}{k} \binom{k}{k-s} (-1)^{k-s}\end{aligned} \tag{1.2.18}$$

The last equation holds, since $p(s, j) = 0$ for $s > j$ and since $m \ge \lfloor \xi \rfloor \ge j$. In addition, an application of the binomial theorem, after proper rearrangements and substitution of $r = k - s$, leads to:

$$\sum_{l=s}^{m} \sum_{k=s}^{l} \binom{l}{k} \binom{k}{k-s} (-1)^{k-s} = \sum_{l=s}^{m} \sum_{k=s}^{l} \frac{l!}{(l-k)!s!(k-s)!} (-1)^{k-s}$$

$$
\begin{aligned}
&= \sum_{l=s}^{m} \binom{l}{s} \sum_{r=0}^{l-s} \binom{l-s}{r} (-1)^r \\
&= \sum_{l=s}^{m} \binom{l}{s} (1-1)^{l-s} \\
&= 1
\end{aligned}
$$

We have thus verified that $\mathfrak{E}_0(\xi, m)$ for $m \geq \lfloor \xi \rfloor$ equivalently can be represented as the right hand side of (1.2.7), whence $\mathfrak{E}_0(\xi, m) = F(\xi)$ for $m \geq \lfloor \xi \rfloor$. This especially means $\mathfrak{E}_0(\xi, \lfloor \xi \rfloor) = F(\xi)$.

Example 1.2.5 (connection to Example 1.2.3). In Example 1.2.3 we have shown that the distribution function associated with a discrete $X \geq 0$, which is contaminated with errors $\varepsilon \geq 0$ of arbitrary type and $p_\varepsilon(0) = 0$ but $H(1) > 0$, can be recovered by means of (1.2.12). This function, however, is not a solution of the integral equation (1.2.13). It is rather the solution of a slight modification that can be obtained from (1.2.13), if X is a discrete distribution with support $\mathbb{T}_X$, and if H is an arbitrary distribution such that there exists $\xi_0 \in \mathbb{R}$ with $H(\xi) = 0$ for $\xi < \xi_0$. Then, denote $k_\varepsilon := \min\{n \in \mathbb{T}_X : H(n) > 0\}$. If we confine to Example 1.2.3, we have $k_\varepsilon = 1$ and for $k \in \mathbb{N}$ the integral equation (1.2.13) takes on the following form:

$$
\begin{aligned}
G(k) &= \int\limits_{[0, k-k_\varepsilon]} H(k-x) F(dx) \\
&= \sum_{x=0}^{k-2} H(k-x) p_X(x) + H(1) p_X(k-1) \\
&= \sum_{j=1}^{k-1} H(1+j) p_X(k-1-j) + H(1) p_X(k-1)
\end{aligned}
$$

This formula is readily confirmed to equal (1.2.9). Furthermore, since F is a step function, division and summation for $1 \leq k \leq \lfloor \xi \rfloor$ with $\xi \in \mathbb{R}$ leads to:

$$
\begin{aligned}
F(\xi) &= \sum_{k=1}^{\lfloor \xi \rfloor + 1} \frac{G(k)}{H(1)} - \sum_{k=1}^{\lfloor \xi \rfloor + 1} \sum_{j=1}^{k-1} \frac{H(1+j)}{H(1)} p_X(k-1-j) \\
&= \sum_{l=0}^{\lfloor \xi \rfloor} \frac{G(1+l)}{H(1)} - \sum_{l=0}^{\lfloor \xi \rfloor} \sum_{j=1}^{l} \frac{H(1+j)}{H(1)} p_X(l-j) \\
&= \sum_{l=0}^{\lfloor \xi \rfloor} \frac{G(1+l)}{H(1)} - \sum_{j=1}^{\lfloor \xi \rfloor} \frac{H(1+j)}{H(1)} F(\lfloor \xi \rfloor - j) \qquad (1.2.19)
\end{aligned}
$$

Hence, the function (1.2.12) is obviously a solution of (1.2.19). In addition, it is ascertainable that the latter equation becomes invalid if F and H do not satisfy the required properties, and thus especially if X is not purely discretely distributed. Despite the formula (1.2.12) is not equivalent to (1.2.16), it can be rearranged to resemble $\mathfrak{E}_c(\xi, m)$. For this we first note that, by definition of $P(j,r)$, for $m \geq r$ analogous manipulations as in (1.2.18) yield:

$$\begin{aligned}\sum_{j=0}^{r} \{H(1)\}^{-(1+j)} (-1)^j P(j,r) &= \sum_{j=0}^{m} \{H(1)\}^{-(1+j)} (-1)^j P(j,r) \\ &= \sum_{l=0}^{m}\sum_{k=0}^{l} \binom{l}{k} (-1)^k \sum_{t=0}^{k} \binom{k}{t} \{H(1)\}^{-(k-t+1)} P(k-t,r)\end{aligned}$$

Hence, (1.2.12) with $m \geq \lfloor\xi\rfloor$ equivalently can be cast in the form

$$F(\xi) = \sum_{l=0}^{m}\sum_{k=0}^{l} \binom{l}{k} (-1)^k \{H(1)\}^{-(1+k)} \sum_{s=0}^{\lfloor\xi\rfloor} G(1+\lfloor\xi\rfloor - s) \sum_{r=0}^{s}\sum_{t=0}^{k} \binom{k}{t} \{H(1)\}^t P(k-t,r).$$

According to (1.2.11), the t-sum satisfies the following identity:

$$\begin{aligned}\sum_{t=0}^{k} \binom{k}{t} \{H(1)\}^t P(k-t,r) &= \{H(1)\}^k \delta_{\{0\}}(r) + \sum_{t=0}^{k-1} \binom{k}{t} \{H(1)\}^t \sum_{\substack{z_1,\dots,z_{k-t}\in\mathbb{N}\setminus\{1\} \\ \sum_{i=1}^{k-t} z_i = k-t+r}} \prod_{i=1}^{k-t} H(z_i) \\ &= \sum_{\substack{z_1,\dots,z_k\in\mathbb{N} \\ \sum_{i=1}^{k} z_i = k+r}} \prod_{i=1}^{k} H(z_i)\end{aligned}$$

Consequently for $m \geq \lfloor\xi\rfloor$ we arrive at

$$F(\xi) = \sum_{l=0}^{m}\sum_{k=0}^{l} \binom{l}{k} (-1)^k \{H(1)\}^{-(1+k)} \sum_{s=0}^{\lfloor\xi\rfloor} G(1+\lfloor\xi\rfloor - s) \sum_{r=0}^{s} \sum_{\substack{z_1,\dots,z_k\in\mathbb{N} \\ \sum_{i=1}^{k} z_i = k+r}} \prod_{i=1}^{k} H(z_i). \tag{1.2.20}$$

This evidently resembles the formula (1.2.16) with $c = 1$. In particular, the only but essential difference consists in the presence of the sum with respect to s, where in (1.2.16) we have the probability of the convolutions. The argument $\lfloor\xi\rfloor$ indicates the former as a step function, while the latter is continuous whenever H is continuous. Hence, in general both expressions can not coincide. They are also not equivalent if we simply replace the argument ξ in (1.2.16) by $\lfloor\xi\rfloor$. This becomes clear if we assume $X \equiv 0$, in which event F indeed matches the conditions. Then, a comparison, for instance with $0 \leq \xi < 1$, of the k-th summands in (1.2.16) and in (1.2.20) for $k = 1$ shows that these can not coincide. This observation is in accordance with the fact that both functions satisfy different integral equations.

By (1.2.16) we were able to provide a solution for the integral equation (1.2.15) that turned out as an extension of our very first deconvolution formula. It remains, however, to determine the general relation between this function, the unknown distribution F and the role that is played by the parameter m. Therefore we abandon the above sum representation, due to its complicated nature as a composition of binomial coefficients and convolutions, and rather consider the associated characteristic function, defined by

$$\Phi_{\mathfrak{E}_c}(t,m) := \int_{-\infty}^{\infty} e^{itx}\mathfrak{E}_c(dx,m). \tag{1.2.21}$$

Its existence is guaranteed since $\mathfrak{E}_c(\cdot,m)$ is basically a finite sum of distribution functions. Recalling the product formula for characteristic functions, we expect (1.2.21) to take on a considerably simpler form than (1.2.16). According to our convention, $\Phi_{Y-c}(t) = e^{-ict}\Phi_Y(t)$ and $\Phi_{\varepsilon-c}(t) = e^{-ict}\Phi_\varepsilon(t)$ denote the characteristic functions of the shifted random variables $Y-c$ and $\varepsilon - c$, respectively. We thus obtain for $m \in \mathbb{N}_0$ subject to independence and by application of the binomial theorem:

$$\begin{aligned}\Phi_{\mathfrak{E}_c}(t,m) &= \sum_{l=0}^{m}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\left\{H(c)\right\}^{-(k+1)}\Phi_{Y-c}(t)\left\{\Phi_{\varepsilon-c}(t)\right\}^k \\ &= \frac{\Phi_{Y-c}(t)}{H(c)}\sum_{l=0}^{m}\left\{1-\frac{\Phi_{\varepsilon-c}(t)}{H(c)}\right\}^l\end{aligned} \tag{1.2.22}$$

The final sum is readily identified as a geometric sum with truncation index $m \in \mathbb{N}_0$. For $q \in \mathbb{C}$ it is known to have the summability properties

$$\sum_{l=0}^{m} q^l = \begin{cases}\frac{1-q^{m+1}}{1-q}, & \text{if } q \neq 1,\\ m+1, & \text{if } q = 1.\end{cases} \tag{1.2.23}$$

By virtue of this formula, since $\Phi_{\varepsilon-c}(t) = 0$ if and only if $\Phi_\varepsilon(t) = 0$, for (1.2.22) we obtain

$$\Phi_{\mathfrak{E}_c}(t,m) = \begin{cases}\Phi_X(t)\left[1-\left\{1-\frac{\Phi_{\varepsilon-c}(t)}{H(c)}\right\}^{m+1}\right], & \text{if } \Phi_\varepsilon(t)\neq 0,\\ m+1, & \text{if } \Phi_\varepsilon(t) = 0.\end{cases} \tag{1.2.24}$$

If we suppose, a necessary condition for the convergence $\lim_{m\to\infty}\mathfrak{E}_c(\xi,m)$ for $\xi \in \mathbb{R}$ is the convergence of the associated characteristic functions, according to (1.2.24), for all $t \in \mathbb{R}$ we must require

$$\left|1-\frac{\Phi_{\varepsilon-c}(t)}{H(c)}\right| < 1. \tag{1.2.25}$$

Then $\lim_{m\to\infty}\Phi_{\mathfrak{E}_c}(t,m) = \Phi_X(t)$. In addition, returning to (1.2.16) we observe:

$$\begin{aligned}\lim_{\xi\to\infty}\mathfrak{E}_c(\xi,m) &= \sum_{l=0}^{m}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\left\{H(c)\right\}^{-(k+1)}\lim_{\xi\to\infty}\mathbb{P}\left((Y-c)+\sum_{r=1}^{k}(\varepsilon_r - c)\leq\xi\right)\\ &= \frac{1}{H(c)}\sum_{l=0}^{m}\left[1-\frac{1}{H(c)}\right]^l\end{aligned}$$

$$= \left\{ 1 - \left[1 - \frac{1}{H(c)} \right]^{m+1} \right\} \tag{1.2.26}$$

Depending on $H(c) \in (0,1]$, if we now let $m \to \infty$ we either have convergence to unity, i.e., to $1 = \lim_{\xi\to\infty} F(\xi)$, or divergence. Summarizing, we conclude that admissible values for $c \in \mathbb{R}$ not only need to satisfy (1.2.25) but also $H(c) > \frac{1}{2}$, in order to expect the convergence $\lim_{m\to\infty} \mathfrak{E}_c(\xi, m) = F(\xi)$. At this point, however, we recall that Example 1.2.4 revealed the fact that special and particularly convenient cases may arise if $m = m(\xi)$ depends[2] on ξ. We emphasize that (1.2.24) is then not the characteristic function of $\mathfrak{E}_c(\xi, m(\xi))$ and also (1.2.26) becomes invalid. Rather than discussing such a dependence, or the existence of an appropriate c and further required properties of this parameter, we quit our investigation of the formula (1.2.16) and consider the problem of errors in variables from a new perspective, inspired by the above findings.

[2]As an example, for $\lambda > 0$ suppose $\varepsilon \sim \text{Poiss}(\lambda)$, implying $p_\varepsilon(0) = e^{-\lambda}$ and $\Phi_\varepsilon(t) = e^{\lambda(it-1)}$. In these circumstances there are intervals $I \subset \mathbb{R}$, where $\left|1 - \{p_\varepsilon(0)\}^{-1} \Phi_\varepsilon(t)\right| > 1$ for $t \in I$. Hence, the geometric expression (1.2.24) will not even converge for Lebesgue-almost any $t \in \mathbb{R}$ as $m \to \infty$. Alternatively, the Poisson distribution clearly matches the assumptions of Example 1.2.2 implying $F(\xi) = \mathfrak{E}_0(\xi, \lfloor \xi \rfloor)$ for any distribution F that satisfies the required conditions.

2. Symmetrized Additive Models of Errors in Variables

In our previous investigations we discovered a close connection between binomial and geometric sums and the deconvolution problem. In particular, we started with a study of the distribution of Y and arrived at a result, according to which it is reasonable to conceive the quotient

$$\Phi_X = \frac{\Phi_Y}{\Phi_\varepsilon}, \tag{2.0.1}$$

obtainable from (1.0.5) if $\Phi_\varepsilon \neq 0$, as the limit of a geometric series. Recall that the geometric series, for which the sum formula was given in (1.2.23), converges only as $m \to \infty$ if $|q| < 1$ and then equals $(1-q)^{-1}$ in the limit. A comparison with (2.0.1) shows that the reciprocal characteristic function in the present form constitutes the limit of a geometric series only if $|1 - \Phi_\varepsilon(t)| < 1$ for $t \in \mathbb{R}$. Needless to say that there are many examples violating this condition. With (1.2.25) the approach of the preceding chapter provides a different condition that is sufficient in order to attribute (2.0.1) to the limit of a geometric series. This involves the additional shift $c \in \mathbb{R}$ and the scaling parameter $H(c)$. There are, however, more convenient methods, avoiding the factor $H(c)$ but aiming for simple manipulations of the quotient, to obtain a function in the denominator whose range is the unit interval. One of them is applicable in any scenario and essentially relies on the property that the product of an arbitrary characteristic function with its complex conjugate establishes again a characteristic function, which is in fact non-negative and associated with a symmetric distribution. The second method will turn out to be, in a sense, a weaker form of the aforementioned and is only applicable in special cases.

2.1. Symmetrization by Convolution with the Conjugate Distribution

Suppose $\varepsilon, \varepsilon_2 \sim H$ are two independent random variables with distribution H. Then $-\varepsilon_2$ has the distribution $1 - H((-\zeta)-)$, $\zeta \in \mathbb{R}$, denoted as the conjugate distribution of H, where the minus sign to the right of the argument indicates the limit from the left. Its characteristic function is given by $\Phi_\varepsilon(-\cdot) = \overline{\Phi_\varepsilon}$ with the wide overline signifying the complex conjugate. Moreover, the random variable $\bar{\varepsilon} := \varepsilon - \varepsilon_2$ has the distribution

$$\bar{H}(\zeta) := \mathbb{P}\left(\bar{\varepsilon} \leq \zeta\right) = \int\limits_{-\infty}^{\infty} H(\zeta + z)H(dz), \qquad \zeta \in \mathbb{R}.$$

Since the latter is symmetric around $\zeta = 0$, we refer to $\bar{H}$ as the symmetrization of H. Evidently $-\bar{\varepsilon} \sim \bar{H}$ and $\bar{H}(\zeta) = 1 - \mathbb{P}(\bar{\varepsilon} < -\zeta)$. The most desired property of $\bar{H}$, however, is that, due to the independence of ε and ε_2, the corresponding characteristic function equals the product

$$\Phi_{\bar{\varepsilon}}(t) = \mathbb{E}\left[e^{it(\varepsilon - \varepsilon_2)}\right] = \Phi_\varepsilon(t)\overline{\Phi}_\varepsilon(t) = |\Phi_\varepsilon(t)|^2, \tag{2.1.1}$$

which satisfies $\Phi_{\bar{\varepsilon}}(t) \in [0,1]$ and is even with respect to $t \in \mathbb{R}$. Furthermore, $\Phi_{\bar{\varepsilon}}(t) = 0$ if and only if $\Phi_\varepsilon(t) = 0$ or equivalently if $\overline{\Phi}_\varepsilon(t) = \Phi_\varepsilon(-t) = 0$. Hence, the set

$$N_\varepsilon := \{t \in \mathbb{R} \cup \{\pm\infty\} : \Phi_\varepsilon(t) = 0\} \tag{2.1.2}$$

is comprised of the points at which $\Phi_{\bar{\varepsilon}}(t)$, $\Phi_\varepsilon(t)$ and $\Phi_\varepsilon(-t)$ vanish. Denoting by $\bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots \sim \bar{H}$ mutually independent versions of $\bar{\varepsilon}$, that are additionally independent of $\bar{\varepsilon}$ and Y, for $t \in \mathbb{R} \setminus N_\varepsilon$ we may rearrange (2.0.1) in the following form:

$$\begin{aligned}
\Phi_X(t) &= \frac{\Phi_Y(t)}{\Phi_\varepsilon(t)} \\
&= \Phi_Y(t)\Phi_\varepsilon(-t)\frac{1}{\Phi_{\bar{\varepsilon}}(t)} \\
&= \Phi_Y(t)\Phi_\varepsilon(-t)\sum_{l=0}^{\infty}(1 - \Phi_{\bar{\varepsilon}}(t))^l \\
&= \Phi_Y(t)\Phi_\varepsilon(-t)\sum_{l=0}^{\infty}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\{\Phi_{\bar{\varepsilon}}(t)\}^k \\
&= \sum_{l=0}^{\infty}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\Phi_Y(t)\Phi_\varepsilon(-t)\{\Phi_{\bar{\varepsilon}}(t)\}^k \\
&= \sum_{l=0}^{\infty}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\mathbb{E}\left[\exp\left\{it\left((Y - \varepsilon_2) + \sum_{r=1}^{k}\bar{\varepsilon}_r\right)\right\}\right]
\end{aligned}$$

The fourth equality holds according to the binomial theorem. As a consequence of this result, we introduce the random variable $\dot{Y} := Y - \varepsilon_2$ with distribution function

$$\dot{G}(\xi) := \int_{-\infty}^{\infty} G(\xi + z_2)H(dz_2) = \bar{H} * F(\xi). \tag{2.1.3}$$

This leads to the *symmetrized additive model of errors in variables*, defined by

$$\dot{Y} = X + \bar{\varepsilon}. \tag{2.1.4}$$

While the symmetrization in the context of errors in variables is new, in other fields of mathematics, symmetry is already known to play a major role. The most frequently encountered examples are principal value integrals and the partial sum operator in Fourier analysis as a special integral of that kind. It is not difficult to show that such integrals do not converge without symmetry. In view of the preceding observations it is reasonable to introduce the following notions.

Definition 2.1.1 (deconvolution function and sum). *For $m \in \mathbb{N}_0$ and $\xi \in \mathbb{R}$ the function*

$$\mathfrak{D}(\xi, m) := \sum_{l=0}^{m} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \mathbb{P}\left(\sum_{r=1}^{k} \bar{\varepsilon}_r \leq \xi - (Y - \varepsilon_2)\right) \tag{2.1.5}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{S}_{\bar{\varepsilon}}^{m}(\xi - (y - z)) H(dz) G(dy) \tag{2.1.6}$$

$$= \mathcal{S}_{\bar{\varepsilon}}^{m} * \bar{H} * F(\xi) \tag{2.1.7}$$

is referred to as the deconvolution function. It is composed of the convolution with the deconvolution sum, for $\zeta \in \mathbb{R}$ defined by

$$\mathcal{S}_{\bar{\varepsilon}}^{m}(\zeta) := \sum_{l=0}^{m} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \mathbb{P}\left(\sum_{r=1}^{k} \bar{\varepsilon}_r \leq \zeta\right), \tag{2.1.8}$$

with $\sum_{r=1}^{0} \bar{\varepsilon}_r = 0$. Hence, the probabilities in (2.1.8) for which the sum is empty equal the Dirac distribution function $\mathbb{1}_{\{0 \leq \zeta\}}$.

It is ascertainable from the definitions (2.1.5) and (2.1.8) that the deconvolution function and the deconvolution sum for $m \geq 1$ are associated with signed measures rather than with probability measures. An exception occurs for $m = 0$ in which event $\mathcal{S}_{\bar{\varepsilon}}^{0} = \mathbb{1}_{\{0 \leq \cdot\}}$. Moreover, if F or H are continuous, we observe that the deconvolution function inherits continuity properties. On the other hand $\mathcal{S}_{\bar{\varepsilon}}^{m}(\zeta)$ is never continuous at $\zeta = 0$, since the Dirac function with mass at the origin appears in at least one summand in (2.1.8) for any $m \in \mathbb{N}_0$.

The above sum representation for the deconvolution function is problematic for both, theoretical investigations of its properties and numerical evaluation. The presence of probabilities of multiple convolutions and binomial coefficients already for rather small numbers leads to computational inaccuracies and errors, since the limit of capacity is reached. It is therefore recommended to search for a simplification of the convolution probabilities, in order to be eventually able to simplify the sum. A useful tool in this context are integral transforms and particularly characteristic functions, in view of the product rule, uniqueness and broad applicability. Preliminary we require some frequently occuring definitions. For $t \in \mathbb{R}$ and $m \geq 0$ we refer to

$$\mathcal{P}_{\bar{\varepsilon}}(t, m) := (1 - \Phi_{\bar{\varepsilon}}(t))^{m+1} \tag{2.1.9}$$

as the *m-power*. By a simple application of the binomial and geometric sum formulae it is then easy to show from (2.1.8), that the characteristic function of the deconvolution sum is of the following form:

$$\mathcal{G}_{\bar{\varepsilon}}(t,m) := \int_{-\infty}^{\infty} e^{itz} \mathcal{S}_{\bar{\varepsilon}}^{m}(dz) \tag{2.1.10}$$

$$= \sum_{l=0}^{m} (1 - \Phi_{\bar{\varepsilon}}(t))^{l}$$

$$= \begin{cases} \frac{1-\mathcal{P}_{\bar{\varepsilon}}(t,m)}{\Phi_{\bar{\varepsilon}}(t)}, & \text{for } t \in \mathbb{R} \setminus N_{\varepsilon} \\ m+1, & \text{for } t \in N_{\varepsilon} \end{cases} \tag{2.1.11}$$

We denote $\mathcal{G}_{\bar{\varepsilon}}(t,m)$ as the *geometric sum function*, and we observe

$$\lim_{m\to\infty} \mathcal{G}_{\bar{\varepsilon}}(t,m) = \frac{1}{\Phi_{\bar{\varepsilon}}(t)}, \qquad t \in \mathbb{R} \setminus N_{\varepsilon}. \tag{2.1.12}$$

The property $0 \leq \Phi_{\bar{\varepsilon}} \leq 1$ implies $\{\Phi_{\bar{\varepsilon}}\}^{-1} \geq 1$, which shows that the characteristic function (2.1.12) never corresponds to a probability measure. Notice, however, if

$$\mathcal{S}_{\bar{\varepsilon}}^{\infty} := \lim_{m\to\infty} \mathcal{S}_{\bar{\varepsilon}}^{m}, \tag{2.1.13}$$

as long as there is no justification to interchange the order of limit and integration, we must assume $\int_{-\infty}^{\infty} e^{itz} \mathcal{S}_{\bar{\varepsilon}}^{\infty}(dz) \neq \frac{1}{\Phi_{\bar{\varepsilon}}(t)}$. In addition we also have

$$\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_{\varepsilon}(-t) = \begin{cases} \frac{1-\mathcal{P}_{\bar{\varepsilon}}(t,m)}{\Phi_{\varepsilon}(t)}, & \text{for } t \in \mathbb{R} \setminus N_{\varepsilon}, \\ 0, & \text{for } t \in N_{\varepsilon}. \end{cases} \tag{2.1.14}$$

According to (2.1.3), the equation for the characteristic functions in the symmetrized model of errors in variables is given by

$$\Phi_{\dot{Y}}(t) = \Phi_{Y}(t)\Phi_{\varepsilon}(-t) = \Phi_{X}(t)\Phi_{\bar{\varepsilon}}(t). \tag{2.1.15}$$

Hence, subject to (2.1.5) the characteristic function corresponding to the deconvolution function takes on the following form:

$$\Phi_{\mathfrak{D}}(t,m) := \int_{-\infty}^{\infty} e^{itx} \mathfrak{D}(dx,m) \tag{2.1.16}$$

$$= \Phi_{\dot{Y}}(t)\mathcal{G}_{\bar{\varepsilon}}(t,m)$$

$$= \Phi_{X}(t)(1 - \mathcal{P}_{\bar{\varepsilon}}(t,m)) \tag{2.1.17}$$

Below we briefly summarize some important properties of the recently introduced characteristic functions. These basically justify the applicability of the deconvolution function for the reconstruction of F.

Theorem 2.1.1 (properties of the deconvolution characteristic function). *The characteristic function associated with the deconvolution function has the boundedness property*

$$\sup_{t\in\mathbb{R}} |\Phi_{\mathfrak{D}}(t,m)| \leq 1 \qquad \text{for } m \geq 0, \tag{2.1.18}$$

and it exhibits the following convergence behaviour:

(1) If $t \in \mathbb{R} \setminus N_\varepsilon$ or if $t \in N_\varepsilon$ with $\Phi_X(t) = 0$, we have

$$\lim_{m\to\infty} |\Phi_{\mathfrak{D}}(t,m) - \Phi_X(t)| = 0. \tag{2.1.19}$$

The convergence is uniform on any compact interval $I \subset \mathbb{R}$ with $\Phi_X(t) = 0$ for $t \in I \cap \mathbb{N}_\varepsilon$.

(2) Provided $\Phi_X(t) = 0$ for $t \in N_\varepsilon$ and $\lim_{|t|\to\infty} \Phi_X(t) = 0$, then

$$\lim_{m\to\infty} \|\Phi_{\mathfrak{D}}(\cdot,m) - \Phi_X\|_\infty = 0. \tag{2.1.20}$$

The first condition for the uniform convergence is necessary, whereas the second is sufficient. In fact, this type of convergence is not trivial[1] and will only occur if there exists $m_0 > 0$ such that the sequence $\|\Phi_{\mathfrak{D}}(\cdot,m) - \Phi_X\|_\infty$ is bounded away from unity for $m \geq m_0$.

Proof. The uniform boundedness (2.1.18) is an immediate consequence of the representation (2.1.17), since Φ_X and $\Phi_{\bar{\varepsilon}}$ are also uniformly bounded. According to this representation, for $t \in \mathbb{R}$ we obtain

$$\Phi_X(t) - \Phi_{\mathfrak{D}}(t,m) = \Phi_X(t)\mathcal{P}_{\bar{\varepsilon}}(t,m). \tag{2.1.21}$$

Therefore $|\Phi_X(t) - \Phi_{\mathfrak{D}}(t,m)| < 1$ for $t \in \mathbb{R} \setminus N_\varepsilon$ and the modulus equals zero if $\Phi_X(t) = 0$. The monotonicity of $\mathcal{P}_{\bar{\varepsilon}}(t,m)$ with respect to $m \geq 0$ for $t \in \mathbb{R} \setminus N_\varepsilon$ thus implies the pointwise convergence (2.1.19). The uniformity on any compact subset is then merely a consequence of Dini's theorem, by continuity of (2.1.21) and by continuity of the limit function that holds under the assumption $\Phi_X(t) = 0$ for $t \in I \cap N_\varepsilon$. To eventually verify (2) we note that, since $\lim_{|t|\to\infty} \Phi_X(t) = 0$, for any $\delta > 0$ there exists $R > 0$ such that

$$\sup_{|t|>R} |\Phi_X(t)\mathcal{P}_{\bar{\varepsilon}}(t,m)| \leq \sup_{|t|>R} |\Phi_X(t)| < \delta$$

for all $m \geq 0$. In view of (1), however, the convergence on $[-R, R]$ is uniform. The proof is thus finished. ∎

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[1] Consider for instance the characteristic function $\Phi_{\bar{\varepsilon}}(t) = \frac{1}{2}\cos^2(t) + \frac{1}{2}e^{-t^2}$ associated with a mixture distribution. Then $N_\varepsilon = \emptyset$. But for $t_k := (2k+1)\frac{\pi}{2}$ with $k \in \mathbb{N}_0$ we have $\mathcal{P}_{\bar{\varepsilon}}(t_k,m) = (1 - \frac{1}{2}e^{-t_k^2})^{m+1} \to 1$ as $k \to \infty$. Hence, $\sup_{t\in\mathbb{R}} \mathcal{P}_{\bar{\varepsilon}}(t,m) = 1$ for $m \geq 0$. If in addition $\Phi_X(t) \equiv 1$, then $\|\Phi_{\mathfrak{D}}(\cdot,m) - \Phi_X\|_\infty = 1$ for $m \geq 0$ implying non-uniform convergence.

Since the deconvolution function is particularly not associated with a non-negative measure, the convergence statements from the above corollary are insufficient, to conclude the actual convergence of $\mathfrak{D}(\cdot, m)$ to F. To verify this convergence for a large class of distributions will be the task in the later sections. Before we establish some supplementary results.

2.1.1. Basic Properties of the Deconvolution Function and Sum

For $u, v \in L^1(\mathbb{R})$ the integral $w(\xi) := \int_{\mathbb{R}} u(\xi - x)v(x)dx = \int_{\mathbb{R}} v(\xi - y)u(y)dy$ describes the Lebesgue convolution of u and v and it follows from Fubini's theorem that $w \in L^1(\mathbb{R})$. The function $w(\xi)$ is thus finite for Lebesgue-almost all $\xi \in \mathbb{R}$. For brevity we occasionally write $w = u * v = v * u$. Furthermore, for a signed measure $\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{K}$, where $\mathcal{B}(\mathbb{R})$ refers to the Borel σ-algebra and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we denote by

$$|\mu|(E) = \sup \left\{ \sum_{j=1}^{n} |\mu(E_j)| : n \in \mathbb{N}, E_1, \ldots, E_n \in \mathcal{B}(\mathbb{R}) \text{ disjoint with } \bigcup_{j=1}^{n} E_j \subset E \right\} \tag{2.1.22}$$

its total variation on $E \in \mathcal{B}(\mathbb{R})$, which is a measure $|\mu| : \mathcal{B}(\mathbb{R}) \to [0, \infty]$, compare Definition 9.8 and Theorem 9.11 in [Axler, 2019]. If $|\mu|(E) < \infty$ we say μ is of finite or bounded total variation[2] on E. The sets of real- and complex-valued signed measures, respectively denoted by $M(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $M(\mathbb{C}, \mathcal{B}(\mathbb{R}))$, establish vector spaces. For $\mu_U, \mu_V \in M(\mathbb{C}, \mathcal{B}(\mathbb{R}))$ we define the convolution of μ_U and μ_V by $\mu_W(E) := \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_E(x - y)\mu_U(dx)\mu_V(dy)$ for $E \in \mathcal{B}(\mathbb{R})$. The existence and finiteness of this integral is due to the inequality $|\mu_W(E)| \leq |\mu_U|(\mathbb{R}) |\mu_V|(\mathbb{R})$, which holds uniformly with respect to $E \in \mathcal{B}(\mathbb{R})$ and thus $\mu_W \in M(\mathbb{C}, \mathcal{B}(\mathbb{R}))$. If we introduce by $U(\xi) := \mu_U((-\infty, \xi])$ a function of the variable $\xi \in \mathbb{R}$, which is uniformly bounded along the real axis, then $\int_E \mu_U(dx) = \int_E U(dx)$ and $|\mu_U|(E) = |U|(E)$ for any $E \in \mathcal{B}(\mathbb{R})$. Furthermore, $W(\xi) = \int_{\mathbb{R}} U(\xi - x)V(dx) = \int_{\mathbb{R}} V(\xi - y)U(dy)$ or, for brevity, $W = U * V = V * U$. We refer to $W(\xi)$ as the convolution of signed measures, as the Stieltjes convolution or, as in some older texts, as the Stieltjes resultant of U and V. If we speak of convolutions without prefix, it will always be clear from the context which notion we mean. In any case, the convolution of two functions establishes some kind of product. Regarding densities, the space $L^1(\mathbb{R})$ endowed with the operations addition and convolution is a ring, with the exception that it does not include a neutral element with respect to convolution. The situation, however, essentially improves if, endowed with these operations, we consider $M(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $M(\mathbb{C}, \mathcal{B}(\mathbb{R}))$. Each of these spaces especially contains the set of probability measures and particularly the measure $\delta_{\{0\}}$ with

[2] In some older texts, for instance in [Wheeden, 2015] and [Widder, 1946], the reader may encounter the notion of functions of bounded variation. In particular, a function U is of bounded variation on $[a, b] \subset \mathbb{R}$ if formally $\int_a^b |U|(dx) = \sup \sum_{i=1}^{n} |U(x_i) - U(x_{i-1})| < \infty$ with the supremum being taken over all partitions $a = x_0 < x_1 < \ldots < x_n = b$. A comparison of this definition with (2.1.22) shows the equivalence of both notions, i.e., $|\mu|(E) = \int_E |\mu|(dx)$. Moreover, any function U that is of bounded variation on $\mathbb{R}$ establishes a (real- or complex-valued) signed measure on $\mathcal{B}(\mathbb{R})$ by writing $\mu_U(E) := \int_E U(dx)$ for $E \in \mathcal{B}(\mathbb{R})$.

distribution function $\mathbb{1}_{\{0 \leq \cdot\}} = \delta_{\{0\}}((-\infty, \xi])$. But for $\mu_L \in M(\mathbb{C}, \mathcal{B}(\mathbb{R}))$ we have

$$L(\xi) = \int_{-\infty}^{\infty} \mathbb{1}_{\{0 \leq \xi - x\}} L(dx) = \left(L * \mathbb{1}_{\{0 \leq \cdot\}}\right)(\xi), \qquad \xi \in \mathbb{R}. \tag{2.1.23}$$

In other words, the Dirac distribution function with mass at the origin establishes the neutral element of Stieltjes convolution. The fact that this distribution is not absolutely continuous with respect to the Lebesgue-measure explains the non-existence of a neutral element of Lebesgue convolution.

The perception of convolution as a product, enables us to introduce convolution powers. In particular, for $k \in \mathbb{N}_0$ we write $\bar{H}^{*k}$ for the k-th convolution power of $\bar{H}$, i.e., for the k-times convolution of $\bar{H}$ with itself, where $\bar{H}^{*0} = \mathbb{1}_{\{0 \leq \zeta\}}$. Analogous to the binomial theorem for multiplicative products, for $l \geq 0$ we may then eventually cast the generating function of the second sum in (2.1.8) in the following form:

$$\begin{aligned} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \mathbb{P}\left(\sum_{r=1}^{k} \bar{\varepsilon}_r \leq \zeta\right) &= \sum_{k=0}^{l} \binom{l}{k} (-1)^k \bar{H}^{*k}(\zeta) \\ &= \left(\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}\right)^{*l}(\zeta) \end{aligned} \tag{2.1.24}$$

It is easy to see that the above notion is in accordance with the convention which we agreed for the left hand side, namely that the sum for $l = 0$ equals the Dirac distribution. Furthermore, in terms of convolution powers we can write:

$$\mathfrak{D}(\xi, m) = \sum_{l=0}^{m} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \bar{H}^{*k} * \dot{G}(\xi) \tag{2.1.25}$$

$$= \sum_{l=0}^{m} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \bar{H}^{*(k+1)} * F(\xi) \tag{2.1.26}$$

$$\mathcal{S}_{\bar{\varepsilon}}^{m}(\zeta) = \sum_{l=0}^{m} \left(\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}\right)^{*l}(\zeta) \tag{2.1.27}$$

The last equality exposes the deconvolution sum as a special Neumann partial sum. Those are of frequent occurence in functional analysis, especially in the context of integral equations of Fredholm- and Volterra-type. There, they are closely related to the so-called resolvent. Sums similar to (2.1.27) are also known from renewal theory, where they are referred to as renewal functions or renewal measures. Such sums are, however, rather composed of convolution powers of distributions, while the summands in (2.1.27) generally attain both signs. The particular Neumann partial sum (2.1.27), as a sum of convolution powers of the signed measure $\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}$, facilitates an interesting interpretation. Observe that $\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}$ is the difference of two symmetric distributions. In the standard model of errors in variables (1.0.1) the distribution G equals the convolution of F with H, whence $Y = X$ if and only if $H = \mathbb{1}_{\{0 \leq \cdot\}}$. As a consequence, the Dirac distribution with mass at the origin is not only associated with the neutral element of convolution but in the context of errors in variables it constitutes the optimal error distribution. Conversely the situation $H \neq \mathbb{1}_{\{0\leq\cdot\}}$ is rather problematic, because then G deviates from F with probability one. The expression $\mathbb{1}_{\{0\leq\cdot\}} - \bar{H}$ can be considered a measure for this deviation, and it seems reasonable to assume that those errors in variables cause less problems whose distribution H is most similar to $\mathbb{1}_{\{0\leq\cdot\}}$. We continue with some rather technical properties of the single summands in (2.1.27).

Lemma 2.1.2. *(1) Decay at infinity for $l \in \mathbb{N}$:*

$$\lim_{\zeta\to\pm\infty} \left(\mathbb{1}_{\{0\leq\cdot\}} - \bar{H}\right)^{*l}(\zeta) = 0 \tag{2.1.28}$$

(2) Provided $\bar{\varepsilon}$ possesses the required moments, the odd moments of $\left(\mathbb{1}_{\{0\leq\cdot\}} - \bar{H}\right)^{*l}(\zeta)$ vanish for $l \in \mathbb{N}_0$, and

$$\int_{-\infty}^{\infty} z^2 \left(\mathbb{1}_{\{0\leq\cdot\}} - \bar{H}\right)^{*l}(dz) = \begin{cases} -\mathbb{E}\left[\bar{\varepsilon}^2\right], & \text{if } l = 1, \\ 0, & \text{if } l \in \mathbb{N}_0 \setminus \{1\}. \end{cases} \tag{2.1.29}$$

(3) For $l \in \mathbb{N}$, if $\bar{H}$ is continuous,

$$\left(\mathbb{1}_{\{0\leq\cdot\}} - \bar{H}\right)^{*l}(-\zeta) = \begin{cases} -\left(\mathbb{1}_{\{0\leq\cdot\}} - \bar{H}\right)^{*l}(\zeta), & \text{if } \zeta \neq 0, \\ \frac{1}{2}, & \text{if } \zeta = 0. \end{cases} \tag{2.1.30}$$

Proof. The statement (1) is trivial and follows from the binomial sum representation. Regarding (2) it is clear that, due to symmetry, the odd moments equal zero. Moreover, equation (2.1.29) for $l = 0$ is a consequence of the properties of the Dirac measure with mass at the origin. To verify the result for $l \geq 1$ we observe that, according to Bienaymé's identity, for mutually independent and identically distributed $\bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots \sim \bar{H}$ the following holds:

$$\begin{aligned}
\int_{-\infty}^{\infty} z^2 \left(\mathbb{1}_{\{0\leq\cdot\}} - \bar{H}\right)^{*l}(dz) &= \sum_{k=0}^{l} \binom{l}{k} (-1)^k \mathbb{E}\left\{\left[\sum_{r=1}^{k} \bar{\varepsilon}_r\right]^2\right\} \\
&= \sum_{k=0}^{l} \binom{l}{k} (-1)^k k \mathbb{E}\left[\bar{\varepsilon}^2\right] \\
&= \mathbb{E}\left[\bar{\varepsilon}^2\right] \left[\frac{d}{dq} \sum_{k=0}^{l} \binom{l}{k} (-1)^k q^k\right]_{q=1} \\
&= \mathbb{E}\left[\bar{\varepsilon}^2\right] \left[\frac{d}{dq} (1-q)^l\right]_{q=1} \\
&= -\mathbb{E}\left[\bar{\varepsilon}^2\right] \left[l (1-q)^{l-1}\right]_{q=1}
\end{aligned}$$

For (3) we note that the convolutions $\bar{H}^{*k}$ for $k \in \mathbb{N}$ inherit the continuity and the symmetry of $\bar{H}$ with $\bar{H}^{*k}(0) = \frac{1}{2}$. On the one hand, for $\zeta \neq 0$ this leads to:

$$\begin{aligned}
\left(\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}\right)^{*l}(\zeta) &= \mathbb{1}_{\{0 \leq \zeta\}} + \sum_{k=1}^{l} \binom{l}{k} (-1)^k \bar{H}^{*k}(\zeta) \\
&= \sum_{k=0}^{l} \binom{l}{k} (-1)^k \left[1 - \bar{H}^{*k}(-\zeta)\right] \\
&= (1-1)^l - \left(\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}\right)^{*l}(-\zeta)
\end{aligned}$$

On the other hand, for $\zeta = 0$ we obtain

$$\left(\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}\right)^{*l}(0) = \mathbb{1}_{\{0 \leq 0\}} + \frac{1}{2} \sum_{k=1}^{l} \binom{l}{k} (-1)^k = \frac{1}{2} + \frac{1}{2}(1-1)^l.$$

The proof is thus complete. ∎

We now establish some elementary properties of the deconvolution function and sum, that follow immediately from their definitions and the above lemma.

Lemma 2.1.3. *(1) Finite behavior at infinity for $m \in \mathbb{N}_0$:*

$$\lim_{\xi \to \xi_0} \mathfrak{D}(\xi, m) = \lim_{\xi \to \xi_0} \mathcal{S}_{\bar{\varepsilon}}^m(\xi) = \begin{cases} 0, & \text{if } \xi_0 = -\infty \\ 1, & \text{if } \xi_0 = \infty \end{cases} \tag{2.1.31}$$

(2) For any $m \in \mathbb{N}_0$ we have $|\mathcal{S}_{\bar{\varepsilon}}^m(\xi)| \leq 2^{m+1} - 1$ and also $|\mathfrak{D}(\xi, m)| \leq 2^{m+1} - 1$, uniformly with respect to $\xi \in \mathbb{R}$.

(3) For $m \in \mathbb{N}$, provided the corresponding moments of $\bar{\varepsilon}$ exist,

$$\int_{-\infty}^{\infty} z^j \mathcal{S}_{\bar{\varepsilon}}^m(dz) = \begin{cases} 0, & \text{if } j \in 2\mathbb{N}_0 + 1, \\ -\mathbb{E}\left[\bar{\varepsilon}^2\right], & \text{if } j = 2. \end{cases} \tag{2.1.32}$$

(4) For continuous $\bar{H}$ and $m \in \mathbb{N}_0$ we have symmetry with respect to the origin:

$$\mathcal{S}_{\bar{\varepsilon}}^m(\zeta) = \begin{cases} 1 - \mathcal{S}_{\bar{\varepsilon}}^m(-\zeta), & \text{if } \zeta \neq 0 \\ \frac{m+2}{2}, & \text{if } \zeta = 0 \end{cases} \tag{2.1.33}$$

(5) For $m \in \mathbb{N}_0$ the following holds:

$$\mathcal{S}_{\bar{\varepsilon}}^m * \bar{H} = \mathbb{1}_{\{0 \leq \cdot\}} - \left(\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}\right)^{*(m+1)} \tag{2.1.34}$$

$$\mathfrak{D}(\cdot, m) = F - F * \left(\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}\right)^{*(m+1)} \tag{2.1.35}$$

The negativity of the second moment of the deconvolution sum confirms that it is not a probability measure but that it necessarily attains both signs. However, recalling (2.1.7), the necessity of this property is not surprising. This definition shows that the deconvolution function emerges from the convolution of $\mathcal{S}_{\bar{\varepsilon}}^m$ with the distribution $\dot{G}$. Now, if the deconvolution sum also would be a probability distribution it had a non-negative second moment, whence the second moment of the deconvolution function would be greater than that of F. If, in addition, the second moment of the deconvolution sum would not depend on m, then that of $\mathfrak{D}(\cdot, m)$ would be unchanged in the limit as $m \to \infty$. But the recovery of F particularly requires to remove the variance of the errors from the sum variable Y, which is thus only possible if $\mathcal{S}_{\bar{\varepsilon}}^m$ attains both signs.

Proof. The statements in (2.1.31) are immediate consequences of (2.1.5) and (2.1.8). Moreover, concerning the estimates in (2) we observe that

$$|\mathcal{S}_{\bar{\varepsilon}}^m(\xi)| \leq \sum_{l=0}^{m} \sum_{k=0}^{l} \binom{l}{k} = 2^{m+1} - 1.$$

The validity of (2.1.32) follows from Lemma 2.1.2(2). Regarding (2.1.33), by continuity of $\bar{H}$, the identity (2.1.30) implies for $\zeta \neq 0$:

$$\begin{aligned}
\mathcal{S}_{\bar{\varepsilon}}^m(\zeta) &= \mathbb{1}_{\{0 \leq \zeta\}} + \sum_{l=1}^{m} \left(\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}\right)^{*l}(\zeta) \\
&= 1 - \mathbb{1}_{\{0 \leq -\zeta\}} - \sum_{l=1}^{m} \left(\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}\right)^{*l}(-\zeta) \\
&= 1 - \mathcal{S}_{\bar{\varepsilon}}^m(-\zeta)
\end{aligned}$$

Particularly for $\zeta = 0$, by (2.1.30), this equals $\mathcal{S}_{\bar{\varepsilon}}^m(0) = \frac{m+2}{2}$. To verify (2.1.34), according to the properties of convolution products, we observe:

$$\begin{aligned}
\mathcal{S}_{\bar{\varepsilon}}^m * \bar{H} &= \mathcal{S}_{\bar{\varepsilon}}^m * \left(\mathbb{1}_{\{0 \leq \cdot\}} - \mathbb{1}_{\{0 \leq \cdot\}} + \bar{H}\right) \\
&= \sum_{l=0}^{m} \left(\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}\right)^{*l} - \sum_{l=0}^{m} \left(\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}\right)^{*(l+1)} \\
&= \mathbb{1}_{\{0 \leq \cdot\}} - \left(\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}\right)^{*(m+1)}
\end{aligned}$$

An application of this result to definition (2.1.7) finally yields (2.1.35). ■

Preliminary to our upcoming investigations on the convergence of the deconvolution function we close this section with an observation concerning discontinuities of F. For simplicity suppose $F = \mathbb{1}_{\{x_0 \leq \cdot\}}$ for a fixed $x_0 \in \mathbb{R}$. Then $\mathfrak{D}(\xi, m) = \mathcal{S}_{\bar{\varepsilon}}^m * \bar{H}(\xi - x_0)$, by (2.1.7). Moreover, assuming $\bar{H}$ is an arbitrary symmetrized continuous distribution, according to (2.1.30) and (2.1.34), for

$m \geq 0$ we have

$$\mathcal{S}_{\bar{\varepsilon}}^m * \bar{H}(0) = \mathbb{1}_{\{0 \leq 0\}} - \left(\mathbb{1}_{\{0 \leq \cdot\}} - \bar{H}\right)^{*(m+1)}(0) = \frac{1}{2}. \tag{2.1.36}$$

At $\xi = x_0$ we therefore observe the convergence

$$\lim_{m \to \infty} \left(\mathcal{S}_{\bar{\varepsilon}}^m * \bar{H}\right)(\xi - x_0) \neq \mathbb{1}_{\{x_0 \leq \xi\}} \tag{2.1.37}$$

or equivalently $\lim_{m\to\infty} \mathfrak{D}(x_0, m) \neq F(x_0)$. In other words, the function $\mathcal{S}_{\bar{\varepsilon}}^m * \bar{H}(\xi - x_0)$ does not converge to $\mathbb{1}_{\{x_0 \leq \xi\}}$ for all $\xi \in \mathbb{R}$. But in the above example the point $\xi = x_0$ is special, as it constitutes the only discontinuity of the indicator function. Furterhmore $\mathfrak{D}(\cdot, m)$ is continuous for all $m \in \mathbb{N}_0$ by continuity of $\bar{H}$. Hence, the statement (2.1.37) is basically not surprising. Indeed, if the convergence was true at $\xi = x_0$ we had found a sequence of continuous functions that converges to a discontinuous function, in particular to a step function. Instead we observe that it converges to the mean of the left and the right side limit of $F(\xi)$ at $\xi = x_0$. Such an observation is actually very common, for instance in the context of integral transforms and their inversion formulae.

2.1.2. Moments of the Deconvolution Function

We will now investigate the moments of the deconvolution function. In the previous section we already observed that the second moment of the deconvolution sum equals the negative second moment of $\bar{\varepsilon}$, if it exists. Regarding higher moments, general statements are possible without additional assumptions. In particular, we only suppose F and $\bar{H}$ have moments up to order $K_F, K_{\bar{H}} \in \mathbb{N}_0 \cup \{\infty\}$, respectively, and denote $K_0 := 1 + \min\{K_F, K_{\bar{H}}\}$. Then, $\bar{H}^{*l}$ and $F * \bar{H}^{*l}$ for $l \in \mathbb{N}_0$ have moments up to order $K_{\bar{H}}$ and $(K_0 - 1)$, respectively[3]. Furthermore, we introduce

$$\mu_*(k, l) := \int_{-\infty}^{\infty} x^k F * \bar{H}^{*l}(dx), \qquad l \geq 0,$$

which implies $\mu_*(k, 0) = \mu_X(k)$. Under the above assumptions also the function $\mathfrak{D}(\xi, m)$ has moments up to order $(K_0 - 1)$ and, according to (2.1.35), for $0 \leq k < K_0$, we have:

$$\begin{aligned}
\mu_{\mathfrak{D}}(k, m) &:= \int_{-\infty}^{\infty} x^k \mathfrak{D}(dx, m) \\
&= \int_{-\infty}^{\infty} x^k F(dx) - \sum_{l=0}^{m+1} \binom{m+1}{l} (-1)^l \int_{-\infty}^{\infty} x^k F * \bar{H}^{*l}(dx)
\end{aligned}$$

[3]This follows from the multinomial theorem by consideration of the expectations $\mathbb{E}\left\{\left[X + \sum_{r=1}^{l} \bar{\varepsilon}_r\right]^k\right\}$ and $\mathbb{E}\left\{\left[\sum_{r=1}^{l} \bar{\varepsilon}_r\right]^k\right\}$ for $0 \leq k < K_0$, $l \in \mathbb{N}_0$, where $X \sim F$ and $\bar{\varepsilon}_r \sim \bar{H}$ are mutually independent random variables.

$$= \mu_X(k) - \sum_{l=0}^{m+1} \binom{m+1}{l} (-1)^l \mu_*(k,l) \tag{2.1.38}$$

Since $\Phi_X \Phi_{\bar{\varepsilon}}^l$ constitutes the characteristic function associated with $F * \bar{H}^{*l}$, Corollary 2 to Theorem 2.3.1 in [Lukacs, 1970] tells us that this function may be differentiated $(K_0 - 1)$-times with

$$\mu_*(k,l) = i^{-k} \left[\frac{d^k}{dt^k} \Phi_X(t) \Phi_{\bar{\varepsilon}}^l(t) \right]_{t=0}, \qquad 0 \le k < K_0.$$

This result leads us from (2.1.38), upon interchanging the order of differentiation and summation, to a representation in terms of a well-known function:

$$\begin{aligned} \mu_{\mathfrak{D}}(k,m) &= \mu_X(k) - i^{-k} \sum_{l=0}^{m+1} \binom{m+1}{l} (-1)^l \left[\frac{d^k}{dt^k} \Phi_X(t) \Phi_{\bar{\varepsilon}}^l(t) \right]_{t=0} \\ &= \mu_X(k) - i^{-k} \left[\frac{d^k}{dt^k} \Phi_X(t) \sum_{l=0}^{m+1} \binom{m+1}{l} (-1)^l \Phi_{\bar{\varepsilon}}^l(t) \right]_{t=0} \\ &= \mu_X(k) - i^{-k} \left[\frac{d^k}{dt^k} \Phi_X(t) \mathcal{P}_{\bar{\varepsilon}}(t,m) \right]_{t=0} \end{aligned} \tag{2.1.39}$$

Indeed, the last equality involves the m-power which was defined in (2.1.9). Now, by Theorem 2.33 in [Lukacs, 1970], as $t \to 0$ an expansion of the form[4]

$$\Phi_{\bar{\varepsilon}}(t) = \sum_{j=0}^{\left\lfloor \frac{K_{\bar{H}}}{2} \right\rfloor} c_{2j} (it)^{2j} + o\left\{ t^{K_{\bar{H}}} \right\} \tag{2.1.40}$$

holds, where $c_{2j} = (2j!)^{-1} \mu_{\bar{\varepsilon}}(2j)$, $\mu_{\bar{\varepsilon}}(2j)$ refers to the $(2j)$-th moment of $\bar{\varepsilon}$ and $\lfloor \cdot \rfloor$ equals the floor function. Especially, as a consequence of the symmetry of $\bar{H}$, any odd moment equals zero. Rewrite (2.1.39) as

$$\begin{aligned} \mu_{\mathfrak{D}}(k,m) &= \mu_X(k) - i^{-k} \left[\frac{d^k}{dt^k} t^{2(m+1)} \rho(t,m) \right]_{t=0}, \\ \rho(t,m) &:= \Phi_X(t) \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t^{2(m+1)}}. \end{aligned} \tag{2.1.41}$$

The function $\rho(t,m)$ is $(K_0 - 1)$-times differentiable and, according to (2.1.40), it satisfies $\rho(0,m) = c_2^{m+1}$. Moreover, for $0 \le k < \min\{K_0, 2(m+1)+1\}$, as $t \to 0$, it is evident from the product rule of differentiation that

$$\frac{d^k}{dt^k} t^{2(m+1)} \rho(t,m) = \frac{(2(m+1))!}{(2(m+1)-k)!} t^{2(m+1)-k} \rho(t,m) + o\left\{ t^{2(m+1)-k} \right\}. \tag{2.1.42}$$

[4] If $K_{\bar{H}} = \infty$ this expansion is only absolutely convergent if it has a non-zero radius of convrgence. See p. 25 in [Lukacs, 1970].

This verifies that (2.1.41) cancels to

$$\mu_{\mathfrak{D}}(k,m) = \mu_X(k), \qquad \text{for } 0 \leq k < \min\{K_0, 2(m+1)\}. \tag{2.1.43}$$

At the same time it is ascertainable from (2.1.42) that this result can not be improved, since generally

$$\mu_{\mathfrak{D}}(2(m+1),m) = \mu_X(2(m+1)) - i^{-2(m+1)}(2(m+1))!c_2^{m+1} \neq 0, \tag{2.1.44}$$

provided $2(m+1) < K_0$. Equivalently we may write (2.1.43) in the form

$$\lim_{m\to\infty} \mu_{\mathfrak{D}}(k,m) = \mu_X(k), \qquad 0 \leq k < K_0. \tag{2.1.45}$$

We have thus shown that, as $m \to \infty$ any finite moment of the deconvolution function equals the corresponding moment of F. Finally we remark that this fact does not allow us to make any conclusions with respect to the moments of $\lim_{m\to\infty} \mathfrak{D}(\cdot,m)$, since these are for $k \in \mathbb{N}_0$ given by $\int_{-\infty}^{\infty} x^k \lim_{m\to\infty} \mathfrak{D}(dx,m)$. It is also not possible to deduce from (2.1.45) the convergence[5] $\lim_{m\to\infty} \mathfrak{D}(\cdot,m) = F$. Especially recall that the deconvolution function is a signed measure, whence statements for distribution functions become inapplicable.

2.1.3. Fourier-Type Integral Representations for the Deconvolution Function

We will now present some Fourier-type integral representations for the deconvolution function that will be of frequent use. As we already mentioned earlier, these integrals have the advantage of a clearer representation of $\mathfrak{D}(\cdot,m)$ and an easier computability, which is in contrast to the complicated sum formula given in (2.1.5). The derivation of these Fourier-type integrals is particularly simple as they essentially follow from the common Fourier inversion formulae for distributions. An important function in this context is the characteristic function $\Phi_{a,b}(t)$ corresponding to the continuous uniform distribution on $[a,b]$ for real numbers $a < b$, which was derived in (A.1.6). In addition, a special role is played by

$$D_F := \{\xi \in \mathbb{R} : F(\xi+) \neq F(\xi-)\}, \tag{2.1.46}$$

$$D_{\mathfrak{D}} := \left\{\xi \in \mathbb{R} : F * \bar{H}^{*j}(\xi+) \neq F * \bar{H}^{*j}(\xi-) \text{ for some } j \in \mathbb{N}\right\}, \tag{2.1.47}$$

the set of discontinuities of F and $\mathfrak{D}(\cdot,m)$ for $m \in \mathbb{N}_0$, and by

$$C_F := \mathbb{R} \setminus D_F, \tag{2.1.48}$$

$$C_{\mathfrak{D}} := \mathbb{R} \setminus D_{\mathfrak{D}}, \tag{2.1.49}$$

the associated continuity intervals. Note that continuity of F or $\bar{H}$ implies $C_{\mathfrak{D}} = \mathbb{R}$.

[5] According to Hausdorff, a distribution function is uniquely determined through its sequence of moments only if it has a compact support. See pp. 21-23 in [Körner, 1988] and p. 60 in [Widder, 1946] for details.

Theorem 2.1.2 (some Fourier-type integrals). *(1) For $\xi \in C_{\mathfrak{D}}$,*

$$\mathfrak{D}(\xi, m) = \frac{1}{2} + \frac{1}{2\pi i} \lim_{T_2\uparrow\infty} \lim_{T_1\downarrow 0} \int_{T_1}^{T_2} \frac{e^{it\xi}\Phi_{\dot{Y}}(-t) - e^{-it\xi}\Phi_{\dot{Y}}(t)}{t} \mathcal{G}_{\bar{\varepsilon}}(t, m)dt. \tag{2.1.50}$$

(2) For $a, b \in C_{\mathfrak{D}}$ with $a < b$,

$$\mathfrak{D}(b, m) - \mathfrak{D}(a, m) = \lim_{T\to\infty} \frac{b-a}{2\pi} \int_{-T}^{T} \Phi_{a,b}(-t)\Phi_{\mathfrak{D}}(t, m)dt. \tag{2.1.51}$$

The verification of the requirement $\xi, a, b, \in C_{\mathfrak{D}}$ can be difficult if $\bar{H}$ has discontinuities. It is, however, necessary for the applicability of the Fourier inversion. Although the limits of the integrals in Theorem 2.1.2 may still exist if any of these requirements is violated, they might not match the respective left hand side. An example that warned us to take special care at discontinuity points of F was already given in (2.1.37).

Proof. According to (2.1.26), for any $m \in \mathbb{N}_0$ each summand in the sum representation for the deconvolution function constitutes a distribution function. As a consequence we obtain by application of the inversion theorem A.7.12 for any continuity point $\xi \in C_{\mathfrak{D}}$ subject to the evenness of $\Phi_{\bar{\varepsilon}}(t)$ and subject to the formulae for binomial and geometric sums:

$$\begin{aligned}
\mathfrak{D}(\xi, m) &= \sum_{l=0}^{m}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\left[\frac{1}{2} + \frac{1}{2\pi}\lim_{T_2\uparrow\infty}\lim_{T_1\downarrow 0}\int_{T_1}^{T_2}\frac{e^{it\xi}\Phi_X(-t) - e^{-it\xi}\Phi_X(t)}{it}\left\{\Phi_{\bar{\varepsilon}}(t)\right\}^{k+1}dt\right] \\
&= \frac{1}{2}\sum_{l=0}^{m}(1-1)^l + \frac{1}{2\pi}\lim_{T_2\uparrow\infty}\lim_{T_1\downarrow 0}\int_{T_1}^{T_2}\frac{e^{it\xi}\Phi_X(-t) - e^{-it\xi}\Phi_X(t)}{it}\Phi_{\bar{\varepsilon}}(t)\sum_{l=0}^{m}\left(1-\Phi_{\bar{\varepsilon}}(t)\right)^l dt \\
&= \frac{1}{2} + \frac{1}{2\pi i}\lim_{T_2\uparrow\infty}\lim_{T_1\downarrow 0}\int_{T_1}^{T_2}\frac{e^{it\xi}\Phi_X(-t) - e^{-it\xi}\Phi_X(t)}{t}\Phi_{\bar{\varepsilon}}(t)\mathcal{G}_{\bar{\varepsilon}}(t, m)dt
\end{aligned}$$

The interchange in the order of limit and sum is permitted by finiteness of the sum and since the limits of the single summands exist. Writing the result in terms of the definition (2.1.15) we arrive at (2.1.50). To eventually verify (2.1.51) we simply apply the inversion theorem A.7.10 to (2.1.26), which yields for $a, b \in C_{\mathfrak{D}}$ with $a < b$:

$$\begin{aligned}
\mathfrak{D}(b, m) - \mathfrak{D}(a, m) &= \sum_{l=0}^{m}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\left\{\bar{H}^{*(k+1)} * F(b) - \bar{H}^{*(k+1)} * F(a)\right\} \\
&= \frac{1}{2\pi}\sum_{l=0}^{m}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\lim_{T\to\infty}\int_{-T}^{T}\frac{e^{-itb} - e^{-ita}}{-it}\Phi_X(t)\left\{\Phi_{\bar{\varepsilon}}(t)\right\}^{k+1}dt
\end{aligned}$$

By the same arguments as before we may interchange the order of limit and sum. Finally employing the usual sum formulae and writing the result in terms of the definitions (2.1.16) and (A.1.6) leads to (2.1.51). ∎

An important means to measure the deviation of the deconvolution function from the distribution of interest is the bias. Its Fourier-type integral representations are readily derived from the above findings. For computational convenience we distinguish between the local bias and the bias of the increments.

Corollary 2.1.4 (Fourier-type integrals for the bias). *(1) If there exists $\tau_1 > 0$ with*

$$\int_0^{\tau_1} \frac{\mathcal{P}_{\bar{\varepsilon}}(t,0)}{t} dt < \infty, \tag{2.1.52}$$

then we may write for the local bias at $\xi \in C_{\mathfrak{D}} \cap C_F$:

$$\begin{aligned} \mathrm{LB}(m,\xi) &:= \mathfrak{D}(\xi,m) - F(\xi) \\ &= \lim_{T\to\infty} \frac{1}{2\pi i} \int_{-T}^{T} \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t} e^{-it\xi} \Phi_X(t) dt \end{aligned} \tag{2.1.53}$$

If additionally there exists $\tau_2 > 0$ with

$$\int_{\tau_2}^{\infty} \frac{|\Phi_X(t)|}{t} dt < \infty, \tag{2.1.54}$$

then F is continuous and (2.1.53) holds for any $\xi \in \mathbb{R}$. Moreover, the uniform bound

$$\|\mathfrak{D}(\cdot,m) - F\|_\infty \le \mathrm{ULB}(m) \tag{2.1.55}$$

applies, where the uniform local bias is given by

$$\mathrm{ULB}(m) := \frac{1}{\pi} \int_0^{\infty} \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t} |\Phi_X(t)| \, dt. \tag{2.1.56}$$

(2) Denoting the bias of the increments for $a, b \in C_{\mathfrak{D}} \cap C_F$ with $a < b$ by

$$\mathrm{BI}(m,b,a) := (\mathfrak{D}(b,m) - \mathfrak{D}(a,m)) - (F(b) - F(a)), \tag{2.1.57}$$

we have

$$\mathrm{BI}(m,b,a) = -\lim_{T\to\infty} \frac{b-a}{2\pi} \int_{-T}^{T} \Phi_{a,b}(-t)\Phi_X(t)\mathcal{P}_{\bar{\varepsilon}}(t,m) dt. \tag{2.1.58}$$

Again if (2.1.54) holds we have continuity of F and validity of (2.1.58) for any pair of real *numbers $a < b$. The latter is then bounded by*

$$|\mathrm{BI}(m, b, a)| \leq \mathrm{ABI}(m, b-a), \tag{2.1.59}$$

where the absolute bias of increments is given by

$$\mathrm{ABI}(m, b-a) := \frac{2}{\pi} \int_0^\infty \frac{\left|\sin\left\{\frac{b-a}{2}t\right\}\right|}{t} \left|\Phi_X(t)\right| \mathcal{P}_{\bar{\varepsilon}}(t, m) dt. \tag{2.1.60}$$

Observe that the local bias given in (2.1.53) is again a Fourier transform with respect to $\xi \in \mathbb{R}$. Furthermore, besides the fact that (2.1.53) depends only on one point whereas (2.1.57) depends on two, the difference consists in the assumption concerning the integrability condition imposed on $t^{-1}(1 - \Phi_{\bar{\varepsilon}}(t))$ in a neighborhood of $t = 0$. Although this may be expected to hold for many characteristic functions since we always have $\Phi_{\bar{\varepsilon}}(0) = 1$, it is appearantly not naturally satisfied. We could, however, not find any counterexamples. A sufficient condition is that, for $b > 0$,

$$\Phi_{\bar{\varepsilon}}(t) = 1 + \mathcal{O}(t^b) \quad \text{as } t \downarrow 0. \tag{2.1.61}$$

Lemma A.7.1 in the appendix specifies this property in terms of the distribution function. Indeed, generally speaking, the behaviour of a characteristic function near the origin depends on the tail behaviour of its distribution.

Proof. We continue from (2.1.50). A separation of the difference, which appears in the geometric sum function, by Theorem A.7.12, for $\xi \in C_F$ yields

$$\mathfrak{D}(\xi, m) = F(\xi) - \frac{1}{2\pi i} \lim_{T_2 \uparrow \infty} \lim_{T_1 \downarrow 0} \int_{T_1}^{T_2} \frac{e^{it\xi}\Phi_X(-t) - e^{-it\xi}\Phi_X(t)}{t} \mathcal{P}_{\bar{\varepsilon}}(t, m) dt. \tag{2.1.62}$$

The principal value integral on the right hand side exists for any $m \geq 0$, subject to the binomial theorem. Moreover, under the above conditions for fixed $T_2 > 0$ we have absolute convergence of the following integral:

$$\begin{aligned}
&-\int_0^{T_2} \frac{e^{it\xi}\Phi_X(-t) - e^{-it\xi}\Phi_X(t)}{t} \mathcal{P}_{\bar{\varepsilon}}(t, m) dt \\
&\qquad = \int_0^{T_2} \frac{\mathcal{P}_{\bar{\varepsilon}}(t, m)}{t} e^{-it\xi}\Phi_X(t) dt - \int_0^{T_2} \frac{\mathcal{P}_{\bar{\varepsilon}}(t, m)}{t} e^{it\xi}\Phi_X(-t) dt \\
&\qquad = \int_{-T_2}^{T_2} \frac{\mathcal{P}_{\bar{\varepsilon}}(t, m)}{t} e^{-it\xi}\Phi_X(t) dt
\end{aligned}$$

This calculation verifies (2.1.53). Finally the existence of (2.1.54) implies the decay as $t \to \pm\infty$

of $\Phi_X(t)$, whence, according to Corollary 2 to Theorem 3.2.3 in [Lukacs, 1970], it must be associated with a continuous distribution. But then $F * H^{*j}$ is also continuous for any $j \in \mathbb{N}_0$, so that $C_{\mathfrak{D}} = C_F = \mathbb{R}$ and the representation (2.1.53) is applicable for all $\xi \in \mathbb{R}$. Moreover, the validity of (2.1.54) implies the absolute and with respect to $T \geq 0$ uniform convergence of (2.1.53). Then, subject to the triangle inequality, (2.1.55) holds and the integral (2.1.56) in this bound is finite. Analogous justifications yield the statements in (2) of Corollary 2.1.4. ■

Each of the above principal value integrals can equivalently be represented as the limit of a sequence of Laplace transforms by an additional application of Abel's lemma for integrals. We will occasionally make use of this equivalence for computational advantages. In fact, the presence of the additional factor $e^{-\delta|t|}$ allows for an optimal exploitation of the properties of the complex exponential function appearing in each formula. Alternatively it is, of course, also possible to express the formulae of Theorem 2.1.2 and the bias by means of the inversion formula from Theorem A.7.13, involving a smoothing function.

2.1.4. The Deconvolution Density

We already mentioned the continuity of the deconvolution function if F or H is continuous. Moreover, if one of these two distributions is absolutely continuous with respective densities f and h, it follows from Theorem 3.3.2 in [Lukacs, 1970], that each summand in the sum representation of the deconvolution function is also absolutely continuous and thus the whole sum is. As a consequence $\mathfrak{D}(\cdot, m)$ is differentiable for any $m \geq 0$ with derivative $\mathfrak{D}'(\cdot, m)$. This observation enables us to approximate the density corresponding to the distribution F of X, provided it exists. Therefore we introduce the *deconvolution density*, defined by

$$\mathfrak{d}(\xi, m) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(-\xi + y + z) \mathcal{S}_{\varepsilon}^{m}(dz) G(dy). \tag{2.1.63}$$

That (2.1.6) is indeed an antiderivative of $\mathfrak{d}(\cdot, m)$ is readily confirmed by integrating the latter along the ray $(-\infty, x]$ for $x \in \mathbb{R}$, where interchanges in the order of integration are permitted subject to absolute convergence. Conversely if we started from (2.1.6), to justify the differentiation under the integral sign would require additional restrictions on h. This was pointed out in chapter 53 of [Körner, 1988]. Although we refer to $\mathfrak{d}(\cdot, m)$ as the deconvolution density it must be emphasized that it does not constitute a density in the sense of probability theory. This follows from the fact that $\mathfrak{D}(\cdot, m)$ is a signed measure. Despite its negativity, however, it satisfies $\int_{-\infty}^{\infty} \mathfrak{d}(x, m) dx = \int_{-\infty}^{\infty} \mathfrak{D}(dx, m) = 1$. A more convenient representation for the deconvolution density can be obtained by assuming continuity of h along the real axis. Then, according to the inversion formula of Theorem A.1.3, for any $\xi \in \mathbb{R}$ we have

$$h(-\xi) = \lim_{\delta \downarrow 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\xi} \Phi_I(\delta t) \Phi_\varepsilon(t) dt. \tag{2.1.64}$$

Here the function $\Phi_I \in L^1(\mathbb{R})$ is the Fourier transform of an approximate identity that satisfies the conditions of Theorem A.1.3 and thus $\Phi_I(t)$ is especially even with respect to $t \in \mathbb{R}$. In these circumstances for fixed $\delta > 0$ the integral (2.1.64) has the following properties:

$$\begin{aligned}\left|\int_{-\infty}^{\infty} e^{it\xi}\Phi_I(\delta t)\Phi_\varepsilon(t)dt\right| &= \left|\int_{-\infty}^{\infty} h(x)\int_{-\infty}^{\infty} e^{it(\xi+x)}\Phi_I(\delta t)dtdx\right| \\ &= \delta^{-1}\left|\int_{-\infty}^{\infty} h(x) f_I\left(\frac{-\xi - x}{\delta}\right)dx\right| \\ &\leq \sup_{x\in\mathbb{R}} h(x)\int_{-\infty}^{\infty} f_I(y)dy \end{aligned} \tag{2.1.65}$$

The continuity together with the fact $\lim_{x\to\pm\infty} h(x) = 0$ imply uniform boundedness of h on the whole real axis, whence the above estimate is finite. If we therefore apply the integral representation (2.1.64) to (2.1.63) it is permitted to interchange the order of limit and integration, for any $\xi \in \mathbb{R}$, $m \geq 0$ leading to:

$$\begin{aligned}\mathfrak{d}(\xi, m) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lim_{\delta\downarrow 0}\int_{-\infty}^{\infty} e^{it(\xi-y-z)}\Phi_I(\delta t)\Phi_\varepsilon(t)dt\mathcal{S}^m_{\bar\varepsilon}(dz)G(dy) \\ &= \frac{1}{2\pi}\lim_{\delta\downarrow 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it(\xi-y-z)}\Phi_I(\delta t)\Phi_\varepsilon(t)dt\mathcal{S}^m_{\bar\varepsilon}(dz)G(dy) \\ &= \frac{1}{2\pi}\lim_{\delta\downarrow 0}\int_{-\infty}^{\infty} e^{it\xi}\Phi_I(\delta t)\Phi_Y(-t)\Phi_\varepsilon(t)\mathcal{G}_{\bar\varepsilon}(t, m)dt\end{aligned}$$

The last equality incorporates the definition (2.1.10) and the evenness of that function. After a simple change of variables we arrive at

$$\mathfrak{d}(\xi, m) = \frac{1}{2\pi}\lim_{\delta\downarrow 0}\int_{-\infty}^{\infty} e^{-it\xi}\Phi_I(\delta t)\Phi_Y(t)\Phi_\varepsilon(-t)\mathcal{G}_{\bar\varepsilon}(t, m)dt. \tag{2.1.66}$$

This is exactly the Fourier integral representation of the deconvolution density. Recall that its validity depends solely on the properties of the error density, whereas the X-distribution can be arbitrary. However, of course, the expression makes only sense if X actually possesses a density. Throughout this work we pay merely little attention to the deconvolution density and confine mostly to its integrated counterpart. Many statements concerning the former, however, are easy to verify, once they have been shown for the deconvolution function. Similarly, techniques for the evaluation of the deconvolution density can be adapted. Instead of intensifying this discussion we now proceed with a different approach to symmetrize the additive model of errors in variables (1.0.1).

2.2. Symmetrization by Centering

The idea of symmetrization can be modified by employing a constant, instead of a random, symmetrizing variable. This is possible if in (1.0.1) the error variable ε has a characteristic function that, for some fixed $\mu_\varepsilon \in \mathbb{R}$, satisfies

$$\Phi_\varepsilon(t) = e^{it\mu_\varepsilon}\Phi_{\grave{\varepsilon}}(t), \tag{2.2.1}$$

where $\Phi_{\grave{\varepsilon}}$ has its values in the closed unit interval and is even. In other words, there exists a random variable $\grave{\varepsilon} := \varepsilon - \mu_\varepsilon$ associated with the symmetric distribution $\grave{H} := H(\cdot + \mu_\varepsilon)$ whose characteristic function $\Phi_{\grave{\varepsilon}}$ is even, non-negative and satisfies (2.2.1). Note that the non-negativity is important but not necessarily satisfied by the characteristic function of any symmetric distribution. A counterexample is furnished by the uniform distribution with Fourier transform $\frac{\sin(t)}{t}$. The parameter μ_ε is referred to as the location or shift parameter. We emphasize that $\mu_\varepsilon = 0$ is permitted, in which event the error distribution itself is already symmetric with a characteristic function that exhibits the desired properties. Now, in addition we define by $\ddot{Y} := Y - \mu_\varepsilon$ a random variable with distribution $\ddot{G} := G(\cdot + \mu_\varepsilon)$ and characteristic function

$$\Phi_{\ddot{Y}}(t) := e^{-it\mu_\varepsilon}\Phi_Y(t). \tag{2.2.2}$$

Then, instead of (1.0.1) we propose the *centered additive model of errors in variables*

$$\ddot{Y} = X + \grave{\varepsilon}, \tag{2.2.3}$$

such that the associated convolution equation in terms of characteristic functions becomes

$$\Phi_{\ddot{Y}}(t) = \Phi_X(t)\Phi_{\grave{\varepsilon}}(t). \tag{2.2.4}$$

The derivation of the deconvolution function to solve (2.2.3) is analogous to the steps in the model (2.1.4). Details will therefore be omitted. Recall that $t \in \mathbb{N}_\varepsilon$ if and only if $\Phi_\varepsilon(t) = 0$ which is equivalent to $\Phi_{\grave{\varepsilon}}(t) = 0$. Thus, in accordance with (2.2.4), the definition of the geometric series yields

$$\Phi_X(t) = \Phi_{\ddot{Y}}(t)\sum_{l=0}^{\infty}\left(1 - \Phi_{\grave{\varepsilon}}(t)\right)^l, \qquad \text{for } t \in \mathbb{R} \setminus N_\varepsilon, \tag{2.2.5}$$

which is in fact absolutely convergent, since $\Phi_{\grave{\varepsilon}}$ is non-negative. This equation, for $m \in \mathbb{N}_0$ and $\xi \in \mathbb{R}$, gives rise to the definition of the *deconvolution function*

$$\begin{aligned} \mathfrak{F}(\xi, m) &:= \int_{-\infty}^{\infty} \mathcal{T}_{\grave{\varepsilon}}^m(\xi - (y - \mu_\varepsilon))G(dy) \\ &= \mathcal{T}_{\grave{\varepsilon}}^m * \grave{H} * F(\xi), \end{aligned} \tag{2.2.6}$$

where the *deconvolution sum* on the right hand side is represented by

$$\mathcal{T}_{\grave{\varepsilon}}^{m}(\zeta) := \sum_{l=0}^{m} \left(\mathbb{1}_{\{0 \le \cdot\}} - \grave{H}\right)^{*l}(\zeta). \tag{2.2.7}$$

Introducing the *m-power*

$$\mathcal{Q}_{\grave{\varepsilon}}(t,m) := (1 - \Phi_{\grave{\varepsilon}}(t))^{m+1}, \tag{2.2.8}$$

the characteristic function of (2.2.7), again referred to as a *geometric sum function*, for $t \in \mathbb{R}$, takes on the following form:

$$\begin{aligned} \mathcal{H}_{\grave{\varepsilon}}(t,m) &:= \sum_{l=0}^{m} (1 - \Phi_{\grave{\varepsilon}}(t))^{l} \\ &= \begin{cases} \frac{1-\mathcal{Q}_{\grave{\varepsilon}}(t,m)}{\Phi_{\grave{\varepsilon}}(t)}, & \text{if } t \in \mathbb{R} \setminus N_{\varepsilon} \\ m+1, & \text{if } t \in N_{\varepsilon} \end{cases} \end{aligned} \tag{2.2.9}$$

In terms of these definitions the characteristic function associated with (2.2.6), for $t \in \mathbb{R}$, is given by:

$$\begin{aligned} \Phi_{\mathfrak{F}}(t,m) &:= \Phi_{\ddot{Y}}(t)\mathcal{H}_{\grave{\varepsilon}}(t,m) \\ &= \Phi_X(t)\left(1 - \mathcal{Q}_{\grave{\varepsilon}}(t,m)\right) \end{aligned} \tag{2.2.10}$$

The latter satisfies boundedness and convergence properties analogous to those stated in Theorem 2.1.1, which shall not be repeated here. Finally, again for convenience we aim to cast (2.2.6) as a Fourier-type integral, which requires to define by

$$D_{\mathfrak{F}} := \left\{\xi \in \mathbb{R} : F * \grave{H}^{*j}(\xi+) \neq F * \grave{H}^{*j}(\xi-) \text{ for } j \in \mathbb{N}\right\}, \tag{2.2.11}$$

$$C_{\mathfrak{F}} := \mathbb{R} \setminus D_{\mathfrak{F}} \tag{2.2.12}$$

the discontinuity points and continuity intervals of $\mathfrak{F}(\cdot, m)$, respectively.

Theorem 2.2.1 (integral representations for the deconvolution function). *(1) For $\xi \in C_{\mathfrak{F}}$,*

$$\mathfrak{F}(\xi,m) = \frac{1}{2} + \frac{1}{2\pi i} \lim_{T_2\uparrow\infty} \lim_{T_1\downarrow 0} \int_{T_1}^{T_2} \frac{e^{it\xi}\Phi_{\ddot{Y}}(-t) - e^{-it\xi}\Phi_{\ddot{Y}}(t)}{t} \mathcal{H}_{\grave{\varepsilon}}(t,m)dt. \tag{2.2.13}$$

(2) For $a, b \in C_{\mathfrak{F}}$ with $a < b$,

$$\mathfrak{F}(b,m) - \mathfrak{F}(a,m) = \lim_{T\to\infty} \frac{b-a}{2\pi} \int_{-T}^{T} \Phi_{a,b}(-t)\Phi_{\mathfrak{F}}(t,m)dt, \tag{2.2.14}$$

where the characteristic function of the uniform distribution $\Phi_{a,b}$ was computed in (A.1.6).

The derivation of these integral representations resembles the proof of Theorem 2.1.2 and also shall not be repeated here. Regarding the properties of the deconvolution function (2.2.6), it is straightforward to establish analogous statements that hold for the deconvolution function (2.1.7). However, throughout this work we mostly study the latter, as it is applicable for a wider class of error distributions.

2.3. Errors in Variables and Operator Theory

The symmetrized models of errors in variables (2.1.4) and (2.2.3) are special cases of (1.0.1) with tranformed errors, respectively. It is therefore no restriction to confine our following considerations to the latter. These encompass a treatment of the deconvolution problem involving distributions from a functional analytic point of view, including the solution of the integral equation (1.0.2), some properties of the solution and their relation to the solution of the integral equation (1.2.13). Provided the density f associated with F exists, the analogous equation for the density deconvolution problem is

$$g(\xi) = \int_{-\infty}^{\infty} f(\xi - z) H(dz). \tag{2.3.1}$$

In each case we aim to recover F or f for given functions H and G or g, respectively. In other words we are looking for the inverse operator $H^{*(-1)}$.

2.3.1. Derivation of Another Integral Equation

For fixed H the right hand sides of (1.0.2) and (2.3.1) can be generalized by virtue of the linear convolution-type operator

$$S_H Q(\xi) := \int_{-\infty}^{\infty} Q(\xi - z) H(dz), \tag{2.3.2}$$

where in the first case Q is a distribution and in the second case it is a density. Observe that, contrary to usual integral transforms, the distribution H corresponding to the kernel function does not depend on the argument $\xi \in \mathbb{R}$ but is, even more distinguishing, connected with the integrating measure. The integral is thus generally not of Lebesgue- but of Stieltjes-type. If H is particularly discrete it equals a sum or a series. The Stieltjes convolution is a commuting operation if Q is of bounded total variation, meaning that we can equivalently write

$$S_H Q(\xi) = \int_{-\infty}^{\infty} H(\xi - x) Q(dx). \tag{2.3.3}$$

In the latter form the kernel H depends on ξ whereas the argument function Q determines the measure of integration. We already mentioned above that, technically speaking, in the deconvolution problem we are interested in the invertibility of S_H, to determine Q. A common criterion for this is given by the next theorem, see Theorem 3.6-2 in [Ciarlet, 2013]. In the following, for a vector space V we denote by $L(V)$ the space of continuous, or equivalently bounded, linear operators $T : V \to V$ and by Id_V the identity that maps any function of V to itself.

Theorem 2.3.1 (invertibility in Banach spaces). *Suppose $(V, \|\cdot\|)$ is a Banach space and $T \in L(V)$ with operator norm $\|T\| < 1$. Then the continuous linear operator $(Id_V - T) : V \to V$ is bijective and its inverse $(Id_V - T)^{-1} : V \to V$ is also a continuous linear operator. Besides $(Id_V - T)^{-1} = \sum_{n=0}^{\infty} T^n$ and $\left\|(Id_V - T)^{-1}\right\| \leq (1 - \|T\|)^{-1}$, where $T^n = T \circ \ldots \circ T$ stands for the n-times iteration of T. The series representation for $(Id_V - T)^{-1}$ is denoted as the resolvent.*

According to the theorem, in order to determine the invertibility of S_H we must first specify an appropriate Banach space V on which our operator shall be defined and then consider $Id_V - S_H$. The latter has an interesting integral representation that benefits from the fact that S_H is of Stieltjes-type. Assume V is an arbitrary vector space such that for any $Q \in V$ we have $|Q(\xi)| < \infty$ for Lebesgue almost all $\xi \in \mathbb{R}$. The Dirac distribution with mass at the origin then enables us to write

$$Q(\xi) = \int\limits_{\mathbb{R}} Q(\xi - z)\mathbb{1}_{\{0 \leq dz\}}, \tag{2.3.4}$$

which is again finite Lebesgue-almost everywhere. Evidently, this convolution integral represents the identity operator corresponding to the vector space V. We already made this observation in the context of equation (2.1.23) in the particular case $V = M(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. A consequence of (2.3.4) is the possibility to cast the operator $Id_V - S_H$ in the equivalent form

$$T_H Q := Q - S_H Q = \int\limits_{\mathbb{R}} Q(\cdot - z) K_H(dz) \tag{2.3.5}$$

with the nucleus or kernel function

$$K_H := \mathbb{1}_{\{0 \leq \cdot\}} - H. \tag{2.3.6}$$

Similar to S_H also T_H is linear but differs from common integral operators. It does, for instance, again not match the usual convention, according to which the kernel depends on the argument $\xi \in \mathbb{R}$. Yet $T_H Q(\xi) = \int_{\mathbb{R}} K_H(\xi - x)Q(dx)$ if Q is of bounded total variation. The operator T_H plays another important role in the context of an integral equation depending on the unknown Q, which results from setting $S_H Q = P$ for a given function P. This shares some similarities with a first kind Fredholm equation. In the general literature, where they most frequently occur as Lebesgue-type integrals, they are known to bear ill-posed problems. The common approach to solve them, especially if they involve a convolution-type operator, is by means of Fourier analysis. Regarding the integral equation $S_H Q = P$ the statement of ill-posedness, however, is too general and further distinctions are necessary. In fact, in terms of the operator T_H the integral equation $S_H Q = P$ can be rearranged to equal

$$Q = P + T_H Q, \tag{2.3.7}$$

which, in opposition to the former, rather resembles a second kind Fredholm equation. Their solvability essentially differs from those of the first kind. The standard approach is the method of succesive approximations, leading to a Neumann sum, see for instance ch. II in [Tricomi, 1985]. This procedure was already applied in the context of the integral equation (1.2.15). Consider (2.3.7) for instance in the case where Q, P are replaced by F, G, respectively, i.e., consider the deconvolution problem for distributions (1.0.2). Then, with the start function $\mathfrak{F}(\cdot, 0) := G$ we define the recursion $\mathfrak{F}(\cdot, m) := G + T_H \mathfrak{F}(\cdot, m-1)$ for $m \in \mathbb{N}$. By induction it is easy to verify, that

$$\mathfrak{F}(\cdot, m) = G * \sum_{l=0}^{m} \left(\mathbb{1}_{\{0 \le \cdot\}} - H\right)^{*l}. \tag{2.3.8}$$

Note that this equation employs the convention of the binomial theorem for convolutions, which was introduced in (2.1.24). Regarding the density deconvolution problem, the transformation of the Fredholm-type integral equation of the first kind (2.3.1) into one of the second kind involving T_H is possible only if we keep the Stieltjes integral. It is not possible if we assume the existence of a Lebesgue-density and write $H(dz) = h(z)dz$, due to the fact that $\mathbb{1}_{\{0 \le \cdot\}}$ is not absolutely continuous with respect to the Lebesgue-measure. Especially note that the associated probability function $\delta_{\{0\}}$ does not play the role of a density but actually equals zero Lebesgue-almost everywhere, whence $\int_{\mathbb{R}} \delta_{\{0\}}(x)dx = 0$. Yet, if we keep the Stieltjes-integral in (2.3.1) we can still deduce the second kind Fredholm-type integral equation (2.3.7) with f, g rather than with Q, P. Its solution can be approximated, analogous to (2.3.8), through the recursion $\mathfrak{f}(\cdot, 0) := g$ and $\mathfrak{f}(\cdot, m) := g + T_H * \mathfrak{f}(\cdot, m-1)$ for $m \in \mathbb{N}$, which leads us to the closed formula

$$\mathfrak{f}(\cdot, m) = g * \sum_{l=0}^{m} \left(\mathbb{1}_{\{0 \le \cdot\}} - H\right)^{*l}. \tag{2.3.9}$$

The sums (2.3.8) and (2.3.9) resemble the resolvent appearing in Theorem 2.3.1. The essential difference consists in the additional convolution with G and g, respectively, which will turn out to be of major importance.

We will now derive some auxiliary results corresponding to the nucleus (2.3.6) before we can finally discuss the applicability of Theorem 2.3.1. Being composed as the difference of two distributions it generates a signed measure. More precisely,

$$\mu_H(A) := T_H \mathbb{1}_A = \int\limits_A K_H(dz) \tag{2.3.10}$$

establishes a real-valued signed measure for $A \in \mathcal{B}(\mathbb{R})$, where $\mathbb{1}_A$ is the indicator function associated with the set A of the Borel σ-algebra $\mathcal{B}(\mathbb{R})$. In accordance with the Jordan decomposition theorem, compare Theorem 9.30 in [Axler, 2019], there exist unique mutually singular measures μ_H^+, μ_H^- such that $\mu_H := \mu_H^+ - \mu_H^-$. This decomposition is readily deduced by noting that on the one hand $\mu_H(A) \leq 0$ if and only if $0 \notin A \in \mathcal{B}(\mathbb{R})$. On the other hand, for $A \in \mathcal{B}(\mathbb{R})$ with $0 \in A$ the function $\mu_H(A) \geq 0$ is decreasing with respect to A. For $A \in \mathcal{B}(\mathbb{R})$ we thus conclude

$$\begin{cases} \mu_H^+(A) = \delta_{\{0\}}(A)\,(1 - p_\varepsilon(0))\,, \\ \mu_H^-(A) = \mathbb{P}\,(\varepsilon \in A \setminus \{0\})\,. \end{cases} \tag{2.3.11}$$

By means of this representation the total variation measure equals $|\mu_H| = \mu_H^+ + \mu_H^-$, and $|\mu_H|\,(\mathbb{R}) = \int_{\mathbb{R}} |K_H|\,(dz)$ is simply given by:

$$\begin{aligned} |\mu_H|\,(\mathbb{R}) &= \mu_H^+(\mathbb{R}) + \mu_H^-(\mathbb{R}) \\ &= 1 - p_\varepsilon(0) + \mathbb{P}\,(\varepsilon \neq 0) \\ &= 2(1 - p_\varepsilon(0)) \end{aligned} \tag{2.3.12}$$

Additional properties of the kernel $K_H(\zeta)$ are right continuity and, in accordance with the asymptotic behaviour of distributions, decay as $\zeta \to \pm\infty$. Furthermore, generally $\|K_H\|_\infty \leq 1$ and particularly if H is symmetric with respect to the origin we even have $\|K_H\|_\infty \leq \frac{1}{2}$, since then

$$H(0) \begin{cases} = \frac{1}{2}, & \text{if } p_\varepsilon(0) = 0, \\ > \frac{1}{2}, & \text{if } p_\varepsilon(0) > 0. \end{cases}$$

We are eventually ready to discuss the invertibility of S_H in certain Banach spaces by means of Theorem 2.3.1. Therefore, since the set of distributions and densities are both convex and especially not vector spaces, we must extend to more general sets. This necessity naturally leads us to the larger space of functions of finite total variation on $\mathbb{R}$ and to $L^1(\mathbb{R})$.

Example 2.3.1 (space of finite signed measures on $\mathcal{B}(\mathbb{R})$). Making use of the definition (2.1.22) for $\mu \in M\,(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we denote by $\|\mu\|_{TV} := |\mu|\,(\mathbb{R})$ the total variation norm, which is in fact a norm. More precisely, according to Theorem 9.18 in [Axler, 2019], the space of finite signed measures on $\mathcal{B}(\mathbb{R})$ is a Banach space if it is endowed with the total variation norm $\|\cdot\|_{TV}$. Then, if for $\mu_L \in M\,(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we again denote $L(\xi) = \mu_L((-\infty, \xi])$, also $T_H L$ generates a signed measure, i.e., $\mu_{T_H L}(A) := \int_A T_H L(dv)$ is a signed measure for $A \in \mathcal{B}(\mathbb{R})$. Regarding the norm of the operator T_H on the one hand, according to Fubini's theorem and subject to (2.3.12), the following holds:

$$
\begin{aligned}
\|T_H\| &= \sup_{\|\mu_L\|_{TV}=1} \|\mu_{T_H L}\|_{TV} \\
&= \sup_{\|\mu_L\|_{TV}=1} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} L(\cdot - z) \left(\mathbb{1}_{\{0 \leq \cdot\}} - H\right)(dz) \right| (dy) \\
&\leq \sup_{\|\mu_L\|_{TV}=1} \int_{\mathbb{R}} |L|\,(dy) \int_{\mathbb{R}} \left|\mathbb{1}_{\{0 \leq \cdot\}} - H\right|(dz) \\
&= |\mu_H|\,(\mathbb{R}) \\
&= 2(1 - p_\varepsilon(0))
\end{aligned}
$$

On the other hand, since $\delta_{\{0\}} \in M(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $\left\|\delta_{\{0\}}\right\|_{TV} = 1$ and $T_H \mathbb{1}_{\{0 \leq \cdot\}} = T_H$, we have $\left\|T_H \mathbb{1}_{\{0 \leq \cdot\}}\right\|_{TV} = \|\mu_H\|_{TV}$. Hence, the operator norm is exactly given by $\|T_H\| = |\mu_H|\,(\mathbb{R})$ and $T_H \in L(M(\mathbb{R}, \mathcal{B}(\mathbb{R})))$. But it shows that $\|T_H\| < 1$ only if $p_\varepsilon(0) > \frac{1}{2}$. For continuous H we observe $\|T_H\| = 2$ showing that the criterion of Theorem 2.3.1 is then never satisfied.

Example 2.3.2 (space of absolutely integrable functions). Consider the operator T_H defined on the Banach space $(L^1(\mathbb{R}), \|\cdot\|_1)$. It is easy to verify by means of simple estimates for $l \in L^1(\mathbb{R})$ that $\|T_H l\|_1 \leq \|l\|_1 |\mu_H|\,(\mathbb{R})$. Hence, $\|T_H\| \leq |\mu_H|\,(\mathbb{R})$ and the operator is bounded. But subject to (2.3.12) this estimate again does not always satisfy the condition of Theorem 2.3.1.

There is a remarkable difference between an operator having an inverse and being invertible, which is pointed out in the introduction to section 11.5 in [Robinson, 2020]. Indeed, an operator can have an inverse although it need not be invertible. More precisely, invertibility is a special property, which implies the continuity and the boundedness of the inverse operator. Therefore, the non-applicability of Theorem 2.3.1 merely suggests the unboundedness of the inverse operator corresponding to S_H. Despite the lack of invertibility of S_H in the sense of the cited theorem, as $m \to \infty$ the sums (2.3.8) or (2.3.9) may still converge in the considered spaces for some functions G or g. This, however, is not easily verified by simple estimates. In fact, the operator theoretical setting seems rather inappropriate to discuss their convergence properties, whence in a later chapter we will instead accomplish this task by means of Fourier analysis.

Finally, the fact that the kernel $K_H(\zeta)$ vanishes as $\zeta \to \pm\infty$ allows for another interesting consideration of the space $L^1(\mathbb{R})$ or even of $L^p(\mathbb{R})$ with $p \geq 1$.

Example 2.3.3 ($L^p(\mathbb{R})$-spaces). It is known that any function $l \in L^1(\mathbb{R})$ establishes a signed measure that is absolutely continuous with respect to the Lebesgue-measure, i.e., with $L(\xi) = \int_{-\infty}^{\xi} l(x)dx$ we have $L(dx) = l(x)dx$. In these circumstances the operator (2.3.5) can be written in the form

$$
T_H L(\xi) = \int_{\mathbb{R}} K_H(\xi - x) l(x) dx. \tag{2.3.13}
$$

In comparison to the the preceding representations, the above shares the most similarities with a classical Fredholm operator. The main exception still is the right-continuity of the kernel K_H and the non-compactness of the range of integration. Furthermore, owing to the additive argument of $K_H(\xi - z)$, we have $K_H(\xi - z) \notin L^2(\mathbb{R}^2)$. This is a general observation concerning convolution-type kernel functions making them inappropriate for Hilbert space theory. Despite, if $K_H(\zeta)$ decays not too slow as $\zeta \to \pm\infty$ the operator can be defined on some $L^p(\mathbb{R})$-spaces, for instance if $K_H \in L^1(\mathbb{R})$. Integration by parts shows that this is equivalent to $\mathbb{E}\{|\varepsilon|\} < \infty$:

$$\begin{aligned}\mathbb{E}\{|\varepsilon|\} &= \int\limits_{[0,\infty)} zH(dz) - \int\limits_{(-\infty,0)} zH(dz)\\ &= \int\limits_{[0,\infty)} (1 - H(z))dz + \int\limits_{(-\infty,0)} H(z)dz\\ &= \|K_H\|_1\end{aligned}$$

In these circumstances the mapping $T_H : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is continuous for any $p \geq 1$, because for $l \in L^p(\mathbb{R})$, subject to Hölder's inequality, the following holds:

$$\begin{aligned}\|T_H L\|_p^p &\leq \int\limits_{\mathbb{R}} \left[\int\limits_{\mathbb{R}} |K_H(\xi - x)|^{\frac{p-1}{p}} |K_H(\xi - x)|^{\frac{1}{p}} |l(x)|\, dx\right]^p d\xi\\ &\leq \int\limits_{\mathbb{R}} \left[\int\limits_{\mathbb{R}} |K_H(\xi - x)|\, dx\right]^{p-1} \int\limits_{\mathbb{R}} |K_H(\xi - y)|\, |l(y)|^p\, dy d\xi\\ &= \|K_H\|_1^p\, \|l\|_p^p\end{aligned}$$

This estimate yields the operator norm $\|T_H\| \leq \|K_H\|_1^p$, whence, according to Theorem 2.3.1, the operator S_H is invertible if $\|K_H\|_1 < 1$ or if $\mathbb{E}\{|\varepsilon|\} < 1$. The latter condition shows that the convergence requires the errors to be extremely concentrated around the origin.

We conclude our functional analytic investigations with an extension of the above operator that will ultimately reveal the link between many of the previously introduced deconvolution functions.

2.3.2. Contiguous Relations Between Deconvolution Functions

For brevity we confine our discussion to (1.0.2) and omit (2.3.1). By elementary manipulations, similar to the justification of (2.3.7), we obtain for $\xi \in \mathbb{R}$ and fixed $\nu \in \mathbb{R}$, $\lambda \in \mathbb{C}$ the following equivalent representation for the integral equation (1.0.2):

$$\lambda G(\xi + \nu) = F(\xi) - \int\limits_{\mathbb{R}} \left(\mathbb{1}_{\{0 \leq \xi - x\}} - \lambda H(\xi + \nu - x)\right) F(dx) \tag{2.3.14}$$

We therefore define $G_{\nu,\lambda} := \lambda G(\cdot + \nu)$ and $H_{\nu,\lambda} := \lambda H(\cdot + \nu)$. It is easy to see that these two functions are the scaled distributions associated with the random variables $Y - \nu$ and $\varepsilon - \nu$, respectively. Consequently, in contrast to (1.0.2) the equation (2.3.14) can not be expressed in terms of random variables. Next, defining $K_{H_{\nu,\lambda}} := \mathbb{1}_{\{0 \leq \cdot\}} - H_{\nu,\lambda}$ and the operator

$$T_{H_{\nu,\lambda}} Q := \int\limits_{\mathbb{R}} Q(\xi - z) K_{H_{\nu,\lambda}}(dz) \tag{2.3.15}$$

for some function Q of bounded total variation, we can rearrange the integral equation (2.3.14) to obtain

$$F = G_{\nu,\lambda} + T_{H_{\nu,\lambda}} F. \tag{2.3.16}$$

By means of the recursion $\mathfrak{E}_{\nu,\lambda}(\cdot, 0) := G_{\nu,\lambda}$ and $\mathfrak{E}_{\nu,\lambda}(\cdot, m) := G_{\nu,\lambda} + T_{H_{\nu,\lambda}} \mathfrak{E}_{\nu,\lambda}(\cdot, m-1)$ for $m \in \mathbb{N}$, it is straightforward to verify the closed formula

$$\mathfrak{E}_{\nu,\lambda}(\cdot, m) = G_{\nu,\lambda} * \sum_{l=0}^{m} \left\{ \mathbb{1}_{\{0 \leq \cdot\}} - \lambda H(\cdot + \nu) \right\}^{*l}. \tag{2.3.17}$$

Evidently, comparison with (2.3.8) verifies the identity $\mathfrak{E}_{0,1}(\xi, m) = \mathfrak{F}(\xi, m)$. Furthermore, $\mathfrak{E}_{c, \frac{1}{H(c)}}(\xi, m) = \mathfrak{E}_c(\xi, m)$ provided $c \in \mathbb{R}$ is such that $H(c) > 0$. This follows upon writing the probabilities in (1.2.16) in terms of convolutions of distributions accompanied by an application of the binomial convolution theorem, compare (2.1.24). The function (2.3.17) thus indeed generalizes the previously introduced deconvolution functions, except that of Example 1.2.3. In addition, we have the following asymptotic relation.

Theorem 2.3.2 (finite deconvolution). *Provided F is an arbitrary distribution with $F(\xi) = 0$ for $\xi < 0$ and $\mathbb{T}_\varepsilon \subset \mathbb{N}_0$ with $p_\varepsilon(0) > 0$, the deconvolution functions (1.2.16) and (2.3.8) satisfy the identity*

$$\lim_{m \to \infty} \mathfrak{F}(\xi, m) = \mathfrak{E}_0(\xi, \lfloor \xi \rfloor) = F(\xi), \qquad \textit{for } \xi \in \mathbb{R}. \tag{2.3.18}$$

In terms of (2.3.17) this equation is equivalent to $\mathfrak{E}_{0,1}(\xi, m) \to \mathfrak{E}_{0, \frac{1}{p_\varepsilon(0)}}(\xi, \lfloor \xi \rfloor)$ as $m \to \infty$.

Proof. For $k \geq 0$, in (1.2.17) we have shown

$$G * H^{*k}(\xi) = \sum_{j=0}^{\lfloor \xi \rfloor} G(\xi - j) \sum_{t=0}^{k} \binom{k}{t} \{p_\varepsilon(0)\}^t \, p(k - t, j).$$

Upon interchanging the order of summation, for (2.3.6) we thus obtain

$$\mathfrak{F}(\xi, m) = \sum_{j=0}^{\lfloor \xi \rfloor} G(\xi - j) \sum_{l=0}^{m} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \sum_{t=0}^{k} \binom{k}{t} \{p_\varepsilon(0)\}^t \, p(k - t, j).$$

In (1.2.18) we considered a triple sum that is very similar to the above. Bearing in mind $p(t,j)=0$ for $t>j$ by definition, analogous manipulations yield

$$\sum_{l=0}^{m}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\sum_{t=0}^{k}\binom{k}{t}\{p_\varepsilon(0)\}^t\,p(k-t,j)$$
$$=\sum_{s=0}^{m\wedge j}p(s,j)(-1)^s\sum_{l=s}^{m}\sum_{k=s}^{l}\binom{l}{k}\binom{k}{s}(-1)^{k-s}\{p_\varepsilon(0)\}^{k-s}.$$

Next, by means of the substitutions $t=k-s$, $n=l-s$, and according to the binomial theorem, we obtain:

$$\begin{aligned}\sum_{l=s}^{m}\sum_{k=s}^{l}\binom{l}{k}\binom{k}{s}(-1)^{k-s}\{p_\varepsilon(0)\}^{k-s}&=\sum_{l=s}^{m}\sum_{t=0}^{l-s}\binom{l}{s+t}\binom{s+t}{s}(-1)^t\{p_\varepsilon(0)\}^t\\&=\sum_{n=0}^{m-s}\frac{(n+s)!}{s!n!}\sum_{t=0}^{n}\binom{n}{t}(-1)^t\{p_\varepsilon(0)\}^t\\&=\sum_{n=0}^{m-s}\frac{(n+s)!}{s!n!}(1-p_\varepsilon(0))^n\end{aligned}$$

Each addend in this last sum is non-negative. If we therefore apply the integral definition of the gamma function for the factorial in the numerator, according to the monotone convergence theorem, we may interchange the order of summation and integration, leading to:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(n+s)!}{s!n!}(1-p_\varepsilon(0))^n&=\frac{1}{s!}\sum_{n=0}^{\infty}\frac{1}{n!}(1-p_\varepsilon(0))^n\int_0^\infty x^{n+s}e^{-x}dx\\&=\frac{1}{s!}\int_0^\infty x^se^{-p_\varepsilon(0)x}dx\\&=\{p_\varepsilon(0)\}^{-(1+s)}\end{aligned}$$

The proof is thus finished. ∎

Similar to the preceding subsection it is possible to discuss the applicability of Theorem 2.3.1, i.e., to discuss the invertibility of the linear operator

$$S_{H_{\nu,\lambda}}Q(\xi):=\int_{\mathbb{R}}Q(\xi-z)H_{\nu,\lambda}(dz) \tag{2.3.19}$$

on some Banach spaces. This requires us to examine the operator norm of $Id-S_{H_{\nu,\lambda}}=T_{H_{\nu,\lambda}}$. We briefly confine to the space of complex-valued signed measures of finite total variation $(M(\mathbb{C},\mathcal{B}(\mathbb{R})),\|\cdot\|_{TV})$, where $\|\mu\|_{TV}=|\mu|(\mathbb{R})$ for $\mu\in M(\mathbb{C},\mathcal{B}(\mathbb{R}))$ with the convention (2.1.22). First we denote by $\mu_{H_{\nu,\lambda}}$ the complex-valued signed measure which is generated by the kernel

$K_{H_{\nu,\lambda}}$. Then, for $A \in \mathcal{B}(\mathbb{R})$, we can write

$$\mu_{H_{\nu,\lambda}}(A) = \Re\mu_{H_{\nu,\lambda}}(A) + i\Im\mu_{H_{\nu,\lambda}}(A), \tag{2.3.20}$$

where the real and imaginary part are again signed measures, respectively given by:

$$\begin{aligned} \Re\mu_{H_{\nu,\lambda}}(A) &= \int\limits_A \left(\mathbb{1}_{\{0 \le \cdot\}} - \Re\lambda H(\cdot + \nu)\right)(dx) \qquad (2.3.21)\\ &= \delta_{\{0\}}(A)\left(1 - \Re\lambda p_\varepsilon(\nu)\right) - \Re\lambda \mathbb{P}\left(\varepsilon - \nu \in A \setminus \{0\}\right) \end{aligned}$$

$$\Im\mu_{H_{\nu,\lambda}}(A) = -\Im\lambda \int\limits_A H(\cdot + \nu)(dx) \tag{2.3.22}$$

Furthermore, according to Theorem 9.18 in [Axler, 2019], $(M(\mathbb{C}, \mathcal{B}(\mathbb{R})), \|\cdot\|_{TV})$ is a Banach space and, analogous to Example 2.3.1, the operator $T_{H_{\nu,\lambda}}$ is continuous there with norm $\|T_{H_{\nu,\lambda}}\| = |\mu_{H_{\nu,\lambda}}|(\mathbb{R})$. Now, by Theorem 2.3.1 the operator $S_{H_{\nu,\lambda}}$ is invertible if the total variation on $\mathbb{R}$ of (2.3.20) is less than one. To determine its magnitude we note that the imaginary part equals a measure times the constant $-\Im\lambda$. Furthermore, depending on the sign of $\Re\lambda$, it is ascertainable from the second line in (2.3.21) that the real part is either the difference of two measures that are singular with respect to each other, or equals a purely non-positive or a purely non-negative measure. It suffices to distinguish between two particular cases. Firstly, if $p_\varepsilon(\nu) = 0$, i.e., if ν is a continuity point of H and $0 < \Re\lambda < \infty$, or if $p_\varepsilon(\nu) > 0$ and $0 < \Re\lambda \le \{p_\varepsilon(\nu)\}^{-1}$, we obtain for the total variation of $\mu_{H_{\nu,\lambda}}$ on $\mathbb{R}$ by means of the partition $\mathbb{R} = (-\infty, 0) \cup \{0\} \cup (0, \infty)$:

$$\begin{aligned} |\mu_{H_{\nu,\lambda}}|(\mathbb{R}) &= |\mu_{H_{\nu,\lambda}}((-\infty,0))| + |\mu_{H_{\nu,\lambda}}(\{0\})| + |\mu_{H_{\nu,\lambda}}((0,\infty))| \\ &= |\lambda|\,\mathbb{P}\left(\varepsilon \ne \nu\right) + \sqrt{1 - 2\Re\lambda p_\varepsilon(\nu) + |\lambda|^2 \{p_\varepsilon(\nu)\}^2} \end{aligned}$$

Notice that for any $\Im\lambda \in \mathbb{R}$ we have:

$$|\mu_{H_{\nu,\lambda}}|(\mathbb{R}) \ge \Re\lambda\mathbb{P}\left(\varepsilon \ne \nu\right) + \sqrt{1 - 2\Re\lambda p_\varepsilon(\nu) + \{\Re\lambda\}^2 \{p_\varepsilon(\nu)\}^2} = 1 - \Re\lambda(2p_\varepsilon(\nu) - 1)$$

Hence, $\|T_{H_{\nu,\lambda}}\| < 1$ is only admissible if $p_\varepsilon(\nu) > \frac{1}{2}$. Secondly, assume $p_\varepsilon(\nu) > 0$ and $\Re\lambda > \{p_\varepsilon(\nu)\}^{-1}$, or $p_\varepsilon(\nu) \ge 0$ and $\Re\lambda \le 0$. In these circumstances the total variation is readily verified to equal

$$|\mu_{H_{\nu,\lambda}}|(\mathbb{R}) = \sqrt{\{1 - \Re\lambda\}^2 + \{\Im\lambda\}^2}.$$

But again for any $\Im\lambda \in \mathbb{R}$ we have

$$|\mu_{H_{\nu,\lambda}}|(\mathbb{R}) \ge |1 - \Re\lambda|.$$

Observe, if $p_\varepsilon(\nu) > 0$ then $\|T_{H_{\nu,\lambda}}\| < 1$ only if, in addition to $\Re\lambda > \{p_\varepsilon(\nu)\}^{-1}$, we also have

$1 < \Re\lambda < 2$. These constraints can only hold simultaneously if $p_\varepsilon(\nu) > \frac{1}{2}$, in which circumstances Theorem 2.3.1 is applicable. Furthermore, if $\Re\lambda \leq 0$ and $\lambda \neq 0$ we always have $\left\|T_{H_{\nu,\lambda}}\right\| > 1$. To summarize these findings, the arbitrariness of the parameters λ, ν yields no obvious advantage with respect to the invertibility of the convolution operator (2.3.19) in comparison to the standard case $\lambda = 1$, $\nu = 0$, which was considered in Example 2.3.1.

We close this section with a general reference to integral equations of the first kind before we eventually discuss the convergence properties of the deconvolution function in the next chapter.

2.3.3. Concluding Remark: Transformation of General First Kind Fredholm-Type Equations

For completeness we mention that the procedure applied to convert the convolution equations (1.0.2) and (2.3.1) into equations of the second kind can basically be applied to more general first kind equations. In particular, if for a given function f and a kernel $k(\cdot,\cdot)$ we have an integral equation

$$f(x) = \int_a^b k(x,t)u(t)dt, \qquad a \leq x \leq b, \tag{2.3.23}$$

for $\nu \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ it can simply be rewritten in the form

$$u(x) = \lambda f(x+\nu) + \left[u(x) - \lambda \int_a^b k(x+\nu,t)u(t)dt \right]. \tag{2.3.24}$$

In order to conceive this as a second kind equation we introduce the primitive integral

$$K_{\nu,\lambda}(x,t) := \lambda \int_a^t k(x+\nu,s)ds \tag{2.3.25}$$

and the Dirac distribution function $\mathbb{1}_{\{x\leq\cdot\}}$, allowing us to represent the identity function as an integral, so that (2.3.24) can equivalently be cast as

$$u(x) = \lambda f(x+\nu) + \int_a^b u(t)\left(\mathbb{1}_{\{x\leq\cdot\}} - K_{\nu,\lambda}(x,\cdot)\right)(dt). \tag{2.3.26}$$

The right hand side is an integral with respect to $\mathbb{1}_{\{x\leq\cdot\}} - K_{\nu,\lambda}(x,\cdot)$, which corresponds to the kernel. Note that $\mathbb{1}_{\{x\leq\xi\}} = \mathbb{1}_{\{0\leq\xi-x\}}$ for $\xi \in \mathbb{R}$, i.e., the kernel involves a convolution component. The solution of the above transformed equation again can be approximated via the method of successive approximations. Therefore, for brevity considering (2.3.24) merely for

$\lambda = 1$ and $\nu = 0$, with $m \in \mathbb{N}_0$ we denote

$$\begin{aligned} u_{0,1}(x,0) &:= f(x), \\ u_{0,1}(x,m) &:= f(x) + \left[u_{0,1}(x,m-1) - \int_a^b k(x,t) u_{0,1}(t,m-1) dt \right]. \end{aligned} \tag{2.3.27}$$

The closed form representation of $u(x,m)$ for $m \in \mathbb{N}_0$ is readily verified to equal

$$u_{0,1}(x,m) = \sum_{l=0}^{m} \binom{m+1}{l+1} (-1)^l \Xi_{0,1}(x,l), \tag{2.3.28}$$

where the coefficients are

$$\Xi_{0,1}(x,l) := \begin{cases} f(x), & \text{if } l = 0, \\ \int_a^b k(x,t) \Xi_{0,1}(t,l-1) dt, & \text{if } l \geq 1. \end{cases} \tag{2.3.29}$$

It is easy to confirm by induction that (2.3.28) suffices the recursion (2.3.27). For $m = 0$ this is obvious. Assuming the formula (2.3.28) holds for $m \in \mathbb{N}_0$, for the $(m+1)$-th iteration we obtain from (2.3.27):

$$\begin{aligned} u_{0,1}(x,m+1) &= f(x) + \sum_{l=0}^{m} \binom{m+1}{l+1} (-1)^l \Xi_{0,1}(x,l) - \int_a^b k(x,t) \sum_{l=0}^{m} \binom{m+1}{l+1} (-1)^l \Xi_{0,1}(t,l) dt \\ &= f(x) + (m+1) \Xi_{0,1}(x,0) + \sum_{l=1}^{m} \left[\binom{m+1}{l+1} + \binom{m+1}{l} \right] (-1)^l \Xi_{0,1}(x,l) \\ &\quad + (-1)^{m+1} \Xi_{0,1}(x,m+1) \end{aligned}$$

Making use of the binomial identity (26.3.5) in [Olver et al., 2010] eventually results in (2.3.28), which finishes the induction. An additional application to (2.3.28) of the binomial identity (26.3.7) in [Olver et al., 2010] yields

$$u_{0,1}(x,m) = \sum_{l=0}^{m} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \Xi_{0,1}(x,k). \tag{2.3.30}$$

Recall that the function $u_{0,1}(x,m)$ only approximates the solution of (2.3.24) for $\lambda = 1$ and $\nu = 0$. It thus especially gives rise to merely one possible solution of (2.3.23). Further candidates can be obtained for different parameter values.

2.4. Convergence Properties of the Deconvolution Function

As we mentioned earlier, the deconvolution function is associated with a signed measure, whence the convergence of the corresponding characteristic functions is insufficient to conclude the convergence of $\mathfrak{D}(\cdot, m)$ to F. The aim of this section is therefore to examine

$$\lim_{m\to\infty} \mathfrak{D}(\xi, m) = F(\xi) \tag{2.4.1}$$

in a general setup, mostly by consideration of the Fourier integral given in (2.1.50). It is then straightforward to adapt these results to deduce the convergence of the bias of the increments

$$\lim_{m\to\infty} \mathrm{BI}(m, b, a) = 0, \tag{2.4.2}$$

compare (2.1.57). The choice of the former rather than the latter has the advantage of a clearer presentation, due to its dependence on merely one local parameter. Moreover, an additional simplification occurs under the apparently mild integrability assumption (2.1.52). According to (2.1.53), for $\xi \in C_{\mathfrak{D}} \cap C_F$ and $m \geq 0$, with

$$\mathfrak{I}_T(m, \xi) := \frac{1}{2\pi i} \int_{-T}^{T} \frac{\mathcal{P}_{\bar{\varepsilon}}(t, m)}{t} e^{-it\xi} \Phi_X(t) dt \tag{2.4.3}$$

we are then allowed to write

$$\mathfrak{D}(\xi, m) - F(\xi) = \lim_{T\to\infty} \mathfrak{I}_T(m, \xi). \tag{2.4.4}$$

In any case, the convergence (2.4.1) and (2.4.2) can only be investigated by means of the provided Fourier integral representations for $\xi, a, b \in C_{\mathfrak{D}} \cap C_F$, i.e., for continuity points of $F * \bar{H}^{*j}$ for any $j \in \mathbb{N}_0$. For other points the limits of the integrals on the right hand side of (2.1.50) and (2.1.51) may still exist but will in general not match their respective left hand side. However, actually this is not a restriction since we observed in Section 2.1.1 that the pointwise convergence of the deconvolution function to F might not be guaranteed at discontinuity points of F. This is also known from limits of sequences of distribution functions and referred to as weak convergence.

A first inspection, for instance of the Fourier integral representation (2.4.3), already suggests that the convergence of the deconvolution function essentially depends on the involved characteristic functions. Roughly speaking, one can distinguish between the integral being absolutely and uniformly convergent with respect to $T > 0$ or existing merely as a limit of a sequence of integrals. While in the former case a strong kind of convergence is easy to verify, in the latter case the situation can be much more difficult. Possibly the simplest scenario occurs if F corresponds to a symmetric distribution that is continuous at $\xi = 0$. In these circumstances the bias integral (2.4.3) is immediately confirmed to equal zero at $\xi = 0$ for any $T > 0$ without further conditions. Hence, pointwise convergence at $\xi = 0$ is then for free. This general assumption, however, is already insufficient to conclude the convergence at any other $\xi \neq 0$.

2.4.1. Classes of Characteristic Functions

Before diving into our investigations on the convergence of the deconvolution function, it is important to recall the different types of pure probability distributions. Basically this is a consequence of the Lebesgue decomposition theorem, compare Theorem 1.1.3 in [Lukacs, 1970], since each distribution function can be decomposed according to its discrete, absolutely continuous and continuously singular part. In terms of characteristic functions, this allows for the following classification:

- The integral representation of a characteristic function Φ associated with a discrete distribution is a sum or a series of complex exponential functions with coefficients equal to the point probabilities. In these circumstances Φ belongs to the class of almost periodic functions in the sense of Bohr[6] and satisfies $\limsup_{|t|\to\infty} |\Phi(t)| = 1$. As a special case, according to Theorem 2.1.4 in [Lukacs, 1970], Φ is periodic if and only if the set of atoms of the associated distribution function is the subset of a sequence of equidistant points on the real axis. The latter are referred to as lattice distributions.

- A characteristic function Φ of an absolutely continuous distribution satisfies the Riemann-Lebesgue lemma, i.e., it satisfies $\lim_{|t|\to\infty} |\Phi(t)| = 0$. Note that the decay need not happen monotonically, for example in case of the rectangular or triangular distribution.

- Regarding singular distributions one can merely state $\limsup_{|t|\to\infty} |\Phi(t)| \in [0,1]$ with the exact superior limit depending on the distribution. Particularly if the superior limit equals zero, i.e., if $\Phi(t)$ vanishes as $|t| \to \infty$, this needs to happen slower than the decay of any function of the space $L^1(\mathbb{R})$. Otherwise it would be a contradiction to the inversion formula for density functions, compare Theorem A.7.11 in the appendix or Theorem 3.2.2 in [Lukacs, 1970].

In addition, a characteristic function Φ can also be a mixture or equivalently a convex combination of the above three classes, i.e., it can be decomposed in the form

$$\Phi = a_1\Phi_d + a_2\Phi_{ac} + a_3\Phi_s \text{ for } a_j \geq 0 \text{ and } \sum_{j=1}^{3} a_j = 1, \tag{2.4.5}$$

where each summand represents the characteristic function of the purely discrete, purely absolutely continuous and purely singular part, respectively. In particular, as a consequence of Lebesgue's decomposition theorem, a representation of the form (2.4.5) holds for any characteristic function and only if $a_j = 1$ for some $1 \leq j \leq 3$, then Φ corresponds to a pure distribution. Finally, according to the convolution theorem, compare Theorem 3.3.1 in [Lukacs, 1970],

[6] A continuous function $a(t)$, $t \in \mathbb{R}$, is almost periodic if for any $\varepsilon > 0$ there exists $L \equiv L(\varepsilon) > 0$ such that any interval of length L contains a so-called translation number $\tau \equiv \tau(\varepsilon)$, i.e., a number τ with the property $|a(t+\tau) - a(t)| \leq \varepsilon$ for all $t \in \mathbb{R}$. The number τ is thus almost a period of a. The almost periodicity of $\Phi(t)$ is easily a consequence of the approximability by a sequence of functions of that kind, uniformly with respect to $t \in \mathbb{R}$. Compare with the corollary corresponding to Theorem V. on p. 33 in [Bohr, 1932].

arbitrary finite products of characteristic functions yield the characteristic function of a new distribution. But, while in case of mixture distributions, possible atoms of a discrete summand determine the behaviour at infinity, the vanishing factor of a convolution is always dominant. For example, according to Theorem 3.3.2 in [Lukacs, 1970], the convolution of an arbitrary with an absolutely continuous distribution again leads to the latter type and the resulting characteristic function vanishes at infinity. On the other hand, if $a_1 > 0$ in (2.4.5), the asymptotic behaviour is completely changed to non-convergence.

2.4.2. Uniform Convergence

Summarizing the preceding subsection, there is a vast field of characteristic functions and a restriction to one class seems inappropriate. For example, a function Φ_X that vanishes at infinity should perhaps be considered an exception rather than common. Yet, it is immediately ascertainable from the Fourier integral representations (2.1.53) and (2.1.58) that such characteristic functions yield the most convenient result, provided the decay happens fast enough.

Theorem 2.4.1 (uniform convergence). *Under the conditions (2.1.52) and (2.1.54) we have*

$$\lim_{m\to\infty} \|\mathfrak{D}(\cdot,m) - F\|_\infty \in [0,\infty). \tag{2.4.6}$$

This limit equals zero if N_ε is of zero Lebesgue measure or if $\Phi_X(t) = 0$ for any $t \in N_\varepsilon$.

The condition (2.1.54) holds, for instance, if $\Phi_X(t) = \mathcal{O}\left\{\log^{-\alpha}(|t|)\right\}$ as $t \to \pm\infty$ for some $\alpha > 1$. Thus, Theorem 2.4.1 applies to absolutely continuous but possibly also to some singular distributions. Moreover, the theorem reveals the effect of an error variable with a characteristic function that has a compact support. It is then not possible to recover the distribution of the blurred variable X, except if its characteristic function also possesses a compact support. For a more general discussion of this case we refer to the next subsection. Finally, it is evident from Theorem 2.4.1 that the uniform convergence to zero neither depends on the existence of any moments nor on the support of F. This is quite remarkable in view of the properties of the deconvolution function. For instance, suppose $\bar{H}$ has moments up to order $K_{\bar{H}}$ and F has moments up to order K_F, where $K_{\bar{H}} < K_F \leq \infty$. It then follows from the properties of convolution that $\mathfrak{D}(\cdot,m)$ has moments up to order $K_{\bar{H}}$, see Subsection 2.1.2. But if $\mathfrak{D}(\cdot,m) \to F$ as $m \to \infty$, this implies that in the limit we return to a function with moments up to order K_F. Similarly if F has a finite support. Then, regardless of $\bar{H}$, the support of $\mathfrak{D}(\cdot,m)$ is either infinite or increases as m increases. In this event, if the uniform convergence to zero holds, the limit function possesses a finite support.

Proof (of Theorem 2.4.1). According to Corollary 2.1.4(1), we have:

$$\|\mathfrak{D}(\cdot,m) - F\|_\infty \leq \frac{1}{\pi} \int_0^\infty \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t} \left|\Phi_X(t)\right| dt$$

$$= \frac{1}{\pi} \int\limits_{[0,\infty)\setminus N_\varepsilon} \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t} |\Phi_X(t)|\, dt + \frac{1}{\pi} \int\limits_{[0,\infty)\cap N_\varepsilon} \frac{|\Phi_X(t)|}{t} dt$$

Observing $0 \leq \mathcal{P}_{\bar{\varepsilon}}(t,m) \leq \mathcal{P}_{\bar{\varepsilon}}(t,0)$, regarding the first integral it is clear that, under the conditions (2.1.52) and (2.1.54), the integrand is bounded by an integrable function which does not depend on m. In addition, $\lim_{m\to\infty} \mathcal{P}_{\bar{\varepsilon}}(t,m) = 0$ for all $t \in [0,\infty) \setminus N_\varepsilon$, so that Lebesgue's dominated convergence theorem yields the decay of this integral. Regarding the second integral we note that $t = 0$ is never included in N_ε since $\Phi_{\bar{\varepsilon}}(0) = 1$ and that $\Phi_{\bar{\varepsilon}}(t)$ is continuous along the real axis. Hence, the denominator in the integral is bounded away from zero and the integral is finite subject to condition (2.1.54). More precisely, it equals either zero or a finite positive constant whose magnitude depends on N_ε and on Φ_X. This finishes the proof. ∎

Theorem 2.4.2 (convergence of increments). *Given (2.1.54), for any pair of real numbers $a < b$ we have*

$$\lim_{m\to\infty} |\mathrm{BI}(m,b,a)| \in [0,\infty). \tag{2.4.7}$$

If in addition (2.1.52) is satisfied, the convergence is uniform with respect to $b-a$. Finally, the limit equals zero if N_ε is of zero Lebesgue measure or if $\Phi_X(t) = 0$ for any $t \in N_\varepsilon$.

Proof. The proof is analogous to that of Theorem 2.4.1 by virtue of the formula (2.1.58). ∎

Evidently, the convergence to zero of the uniform bias of the deconvolution function or of the bias of its increments does not require to distinguish between different local behaviour of $\Phi_{\bar{\varepsilon}}$, as long as N_ε remains a set of zero Lebesgue measure. The particular form of this characteristic function will become rather important in later chapters on the discussion of exact rates of convergence. On the other hand, the local behaviour of Φ_X is always crucial. Therefore, distributions which satisfy the conditions of the preceding theorems play an outstanding role. However, in many circumstances these may not be satisfied. For example, if F has merely one atom the Fourier integral representation for the deconvolution function is not absolutely convergent. Indeed, the peculiarity in considering a general distribution F is, that the associated characteristic function Φ_X need not contribute to the absolute convergence. Thus, Lebesgue's dominated convergence theorem may not be applied in a straightforward way, to deduce the convergence of the deconvolution function. Scenarios of such a general nature shall be treated in the subsections below.

2.4.3. Errors with a Compactly Supported Characteristic Function

The convergence behaviour of the deconvolution function is particularly simple if $\Phi_{\bar{\varepsilon}}$ possesses a compact support $I_{\bar{\varepsilon}} \subset \mathbb{R}$. We will now illustrate this situation by consideration of the integral (2.4.3), supposing validity of (2.1.52). Since $\Phi_{\bar{\varepsilon}}(t)$ is even with respect to $t \in \mathbb{R}$, the assumption of a compact support is equivalent to the existence of a finite $T_{\bar{\varepsilon}} > 0$ with $\Phi_{\bar{\varepsilon}}(t) = 0$ for $|t| > T_{\bar{\varepsilon}}$

and $I_{\bar{\varepsilon}} \equiv [-T_{\bar{\varepsilon}}, T_{\bar{\varepsilon}}]$. In these circumstances, for fixed $T > T_{\bar{\varepsilon}}$ and $\xi \in C_{\mathfrak{D}} \cap C_F$, (2.4.3) can be decomposed into three integrals

$$\mathfrak{D}(\xi, m) - F(\xi) = I_0(m, T_{\bar{\varepsilon}}, \xi) + \mathfrak{R}_0(T_{\bar{\varepsilon}}, \xi) + \lim_{T\to\infty} \mathfrak{R}(T, T_{\bar{\varepsilon}}, \xi), \tag{2.4.8}$$

where we denote:

$$I_0(m, T_{\bar{\varepsilon}}, \xi) := \frac{1}{2\pi i} \int\limits_{[-T_{\bar{\varepsilon}}, T_{\bar{\varepsilon}}]\setminus N_\varepsilon} \frac{\mathcal{P}_{\bar{\varepsilon}}(t, m)}{t} e^{-it\xi} \Phi_X(t) dt \tag{2.4.9}$$

$$\mathfrak{R}_0(T_{\bar{\varepsilon}}, \xi) := \frac{1}{2\pi i} \int\limits_{[-T_{\bar{\varepsilon}}, T_{\bar{\varepsilon}}]\cap N_\varepsilon} e^{-it\xi} \Phi_X(t) \frac{dt}{t} \tag{2.4.10}$$

$$\mathfrak{R}(T, T_{\bar{\varepsilon}}, \xi) := \frac{1}{2\pi i} \left[\int\limits_{-T}^{-T_{\bar{\varepsilon}}} + \int\limits_{T_{\bar{\varepsilon}}}^{T} \right] e^{-it\xi} \Phi_X(t) \frac{dt}{t} \tag{2.4.11}$$

The integral (2.4.9) is the only component where $\mathcal{P}_{\bar{\varepsilon}}(t, m) < 1$. Since $0 \notin N_\varepsilon$ the neighborhood of $t = 0$ is included in its range of integration. Moreover, the integrand is uniformly bounded with respect to m by $t^{-1}\mathcal{P}_{\bar{\varepsilon}}(t, 0)$, which is integrable, according to (2.1.52), by finiteness of $T_{\bar{\varepsilon}}$. Hence, Lebesgue's dominated convergence theorem implies the decay of the integral (2.4.9) as $m \to \infty$. Furthermore, the finiteness of $T_{\bar{\varepsilon}}$ combined with the continuity of the integrand implies the finiteness of the remainder integral (2.4.10). It is even absolutely and with respect to ξ uniformly convergent in any compact subset of $C_{\mathfrak{D}} \cap C_F$. If $\Phi_{\bar{\varepsilon}}(t) \neq 0$ for Lebesgue almost any $t \in (-T_{\bar{\varepsilon}}, T_{\bar{\varepsilon}})$ it equals zero. Finally it remains to examine the remainder integral (2.4.11) and its behaviour as $T \to \infty$. The following alternative representation in terms of the sine integral (B.1.1) holds:

$$\begin{aligned} \mathfrak{R}(T, T_{\bar{\varepsilon}}, \xi) &= -\int\limits_{-\infty}^{\infty} \int\limits_{T_{\bar{\varepsilon}}}^{T} \frac{e^{is(\xi-x)} - e^{-is(\xi-x)}}{i2\pi s} ds F(dx) \\ &= -\frac{1}{\pi} \int\limits_{-\infty}^{\infty} \left\{ \mathrm{Si}((\xi - x)T) - \mathrm{Si}((\xi - x)T_{\bar{\varepsilon}}) \right\} F(dx) \end{aligned} \tag{2.4.12}$$

The interchange in the order of integration is permitted due to the boundedness of the integrand. Moreover, $\sup_{z\in\mathbb{R}} |\mathrm{Si}(z)| \leq \mathrm{Si}(\pi)$, whence for arbitrary distributions F the integral (2.4.12) is uniformly bounded with respect to $T_{\bar{\varepsilon}}, T \geq 0$. In particular we deduce from the oddness of the sine integral and subject to inequality (B.1.5):

$$\mathfrak{R}(T, T_{\bar{\varepsilon}}, \xi) \leq \frac{1}{\pi} \int\limits_{-\infty}^{\infty} |\mathrm{Si}(|\xi - x|\, T) - \mathrm{Si}(|\xi - x|\, T_{\bar{\varepsilon}})|\, F(dx) \leq \frac{\mathrm{Si}(\pi)}{\pi}$$

Similar to the proof of the inversion formula from Theorem A.7.12, an application of Lebesgue's dominated convergence theorem thus yields for $\xi \in C_{\mathfrak{D}} \cap C_F$ the existence of the limit of the sequence of integrals (2.4.12), which is

$$\lim_{T\to\infty} \mathfrak{R}(T, T_{\bar{\varepsilon}}, \xi) = \frac{1}{2} - F(\xi) + \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{Si}((\xi - x)T_{\bar{\varepsilon}})F(dx).$$

To summarize our findings, for $\xi \in C_{\mathfrak{D}} \cap C_F$ we have just verified

$$\lim_{m\to\infty} \mathfrak{D}(\xi, m) = F(\xi) + \mathfrak{R}_0(T_{\bar{\varepsilon}}, \xi) + \lim_{T\to\infty} \mathfrak{R}(T, T_{\bar{\varepsilon}}, \xi). \tag{2.4.13}$$

The remainder integrals appearing in this limit are uniformly bounded with respect to ξ in any compact subset of $C_{\mathfrak{D}} \cap C_F$. Equality (2.4.13) shows that, for $\Phi_{\bar{\varepsilon}}$ with a compact support as $m \to \infty$ the bias between the deconvolution and the distribution function always converges to a finite limit. The magnitude of this limit depends on $\xi \in C_{\mathfrak{D}} \cap C_F$, on F and on the range of the support $I_{\bar{\varepsilon}}$. This promises a smaller magnitude if $T_{\bar{\varepsilon}}$ is large and $\Phi_X(t)$ decays as $t \to \pm\infty$ but not necessarily if the latter function exhibits almost periodicity.

2.4.4. Weak Convergence Properties of the m-Power

Particularly if $\Phi_{\bar{\varepsilon}}(t)$ is non-increasing with respect to $t \geq 0$ and vanishing at infinity, it becomes evident that for fixed $m \geq 0$ the graph of the m-power $\mathcal{P}_{\bar{\varepsilon}}(t, m)$ on the positive real axis resembles that of a probability distribution. In such a case it is reasonable to expect, that the convergence of the deconvolution function can be justified by weak convergence. Therefore denote $\Phi_{\bar{\varepsilon}}(\infty) := \lim_{t\to\infty} \Phi_{\bar{\varepsilon}}(t)$ and suppose $\Phi_{\bar{\varepsilon}}(\infty) \in [0, 1)$ exists[7]. Then also $\mathcal{P}_{\bar{\varepsilon}}(\infty, m) := \lim_{t\to\infty} \mathcal{P}_{\bar{\varepsilon}}(t, m)$ exists and $\mathcal{P}_{\bar{\varepsilon}}(\infty, m) \in (0, 1]$. In addition, assume $\mathcal{P}_{\bar{\varepsilon}}(t, m)$ is of finite total variation on $[0, \infty]$. Since $\Phi_{\bar{\varepsilon}}(0) = 1$, by definition of a characteristic function, for $t \in [0, \infty]$ and $m \geq 0$ we can then write

$$\mathcal{P}_{\bar{\varepsilon}}(t, m) = \int_{[0,t]} \mathcal{P}_{\bar{\varepsilon}}(dv, m). \tag{2.4.14}$$

If $\Phi_{\bar{\varepsilon}}(t)$ is once continuously differentiable on $[0, \infty]$, then (2.4.14) possesses a density with respect to the Lebesgue measure, which is given by

$$\mathcal{P}_{\bar{\varepsilon}}(dv, m) = -(m + 1)\Phi'_{\bar{\varepsilon}}(v)\mathcal{P}_{\bar{\varepsilon}}(v, m - 1)dv. \tag{2.4.15}$$

In any case, by continuity of $\Phi_{\bar{\varepsilon}}(t)$ the function (2.4.14) is also continuous with respect to $t \in [0, \infty]$ and the integral signs $\int_{[0,t]}$, $\int_{(0,t)}$ and $\int_0^t$ have the same meaning. The transition $m \to \infty$, however, requires us to employ the notion of a compact interval, especially if $\Phi_{\bar{\varepsilon}}$

[7] According to Theorem 2.1.4 and Corollary 2 to Theorem 3.2.3 in [Lukacs, 1970], we can never have $\Phi_{\bar{\varepsilon}}(\infty) = 1$ since then $\Phi_{\bar{\varepsilon}}$ was especially periodic, which would contradict the existence of the limit at infinity.

vanishes at one of the endpoints. More precisely, we observe that

$$\mathcal{P}_{\bar{\varepsilon}}(t,\infty) := \lim_{m\to\infty} \mathcal{P}_{\bar{\varepsilon}}(t,m) = \begin{cases} 1, & \text{if } t \in [0,\infty] \cap N_\varepsilon, \\ 0, & \text{if } t \in [0,\infty] \setminus N_\varepsilon. \end{cases} \tag{2.4.16}$$

Evidently, this function establishes a signed measure of discrete type and a point $t \in [0,\infty]$ has mass one if $\Phi_{\bar{\varepsilon}}(t)$ vanishes there and mass zero otherwise. In contrast to (2.4.14) the limit measure thus exhibits discontinuities. It can be expressed in terms of indicator functions. Therefore suppose the existence of a set of consecutive integers $I = \{1, 2, \dots, 2K\}$ for some $K \in \mathbb{N}$ and of a non-decreasing sequence $\{\tau_k\}_{k\in I}$ of points from the set $N_\varepsilon \cap [0,\infty]$ with the properties

$$\begin{cases} \tau_k \le \tau_{k+1} \text{ and } \Phi_{\bar{\varepsilon}}(t) = 0 \text{ for } \tau_k \le t \le \tau_{k+1}, & \text{for odd } k \in I, \\ \tau_k < \tau_{k+1}, & \text{for even } k \in I, \\ N_\varepsilon \cap [0,\infty] = \bigcup\limits_{k=1}^{K} [\tau_{2k-1}, \tau_{2k}]. \end{cases} \tag{2.4.17}$$

A segment $[\tau_k, \tau_{k+1}]$ for odd $k \in I$ is thus either an isolated point or a continuous interval of the positive real axis, where $\Phi_{\bar{\varepsilon}}$ vanishes. Furthermore, $\tau_1 > 0$ since $0 \notin N_\varepsilon$. A comparison with (2.4.16) shows for $t \in [0,\infty]$ the validity of

$$\mathcal{P}_{\bar{\varepsilon}}(t,\infty) = \sum_{k=1}^{K-1} \left[\mathbb{1}_{\{t\ge\tau_{2k-1}\}} - \mathbb{1}_{\{t>\tau_{2k}\}}\right] + \left[\mathbb{1}_{\{t\ge\tau_{2K-1}\}} - \mathbb{1}_{\{\infty\ge t>\tau_{2K}\}}\right]. \tag{2.4.18}$$

The last indicator vanishes if $\tau_{2K} = \infty$. If also $\tau_{2K-1} = \infty$, the second last indicator equals one if and only if $t = \infty$. In order to be able to derive the convergence of the deconvolution function by means of the limit statement (2.4.16), we require $\mathcal{P}_{\bar{\varepsilon}}(t,m)$ to be of uniformly bounded total variation on $[0,\infty]$, i.e., we suppose that one of the following boundedness conditions holds:

$$\begin{aligned} &\sup_{m\ge0} \int\limits_{[0,\infty]} |\mathcal{P}_{\bar{\varepsilon}}|\,(dt,m) < \infty \\ &\sup_{m\ge0} \int\limits_{[0,\infty]} (m+1)\left|\Phi'_{\bar{\varepsilon}}(t)\right| \mathcal{P}_{\bar{\varepsilon}}(t,m-1)dt < \infty \end{aligned} \tag{2.4.19}$$

These are clearly equivalent under continuous differentiability of $\Phi_{\bar{\varepsilon}}(t)$. Validity of (2.4.19) eventually permits us to employ the Helly-Bray theorem, which can be found for instance as Theorem 16.4 in ch. 1 of [Widder, 1946]. Accordingly, $\mathcal{P}_{\bar{\varepsilon}}(t,\infty)$ is of finite total variation on $[0,\infty]$, and for all with respect to $v \in [0,\infty]$ continuous functions $u(v)$ we have weak convergence, formally

$$\lim_{m\to\infty} \int\limits_{[0,\infty]} u(v)\mathcal{P}_{\bar{\varepsilon}}(dv,m) = \int\limits_{[0,\infty]} u(v)\mathcal{P}_{\bar{\varepsilon}}(dv,\infty). \tag{2.4.20}$$

To evaluate the latter integral we first require an appropriate representation for the sum (2.4.18). On the one hand for $0 \le \tau \le \infty$ it is clear that $\mathbb{1}_{\{t \ge \tau\}}$ is the distribution function associated with the Dirac measure with mass at τ, i.e., $\mathbb{1}_{\{t \ge \tau\}} = \delta_{\{\tau\}}([0,t])$. On the other hand $\mathbb{1}_{\{t > \tau\}}$ for $0 \le \tau < \infty$ corresponds to the limit of a sequence of such measures. In particular

$$\mathbb{1}_{\{t > \tau\}} = \lim_{\substack{\eta\downarrow 0\\ \eta>0}} \delta_{\{\tau+\eta\}}([0,t]).$$

In addition $\delta_{\{\tau+\eta\}}([0,\infty]) = 1$ for any $\eta > 0$. Hence, $\delta_{\{\tau+\eta\}}([0,t])$ is a sequence of functions of bounded variation on $[0,\infty]$ uniformly with respect to $\eta \ge 0$ and, again as a consequence of the Helly-Bray theorem and by continuity of $u(v)$, we thus have:

$$\int\limits_{[0,\infty]} u(v) \lim_{\substack{\eta\downarrow 0\\ \eta>0}} \delta_{\{\tau+\eta\}}(dv) = \lim_{\substack{\eta\downarrow 0\\ \eta>0}} \int\limits_{[0,\infty]} u(v)\delta_{\{\tau+\eta\}}(dv) = \lim_{\substack{\eta\downarrow 0\\ \eta>0}} u(\tau+\eta) = u(\tau) \tag{2.4.21}$$

To summarize these findings, the limit (2.4.20) can be written in the following form:

$$\begin{aligned}
\lim_{m\to\infty} \int\limits_{[0,\infty]} u(v)\mathcal{P}_{\bar{\varepsilon}}(dv,m) &= \sum_{k=1}^{K-1} \int\limits_{[0,\infty]} u(v) \left[\delta_{\{\tau_{2k-1}\}}(dv) - \lim_{\substack{\eta\downarrow 0\\ \eta>0}} \delta_{\{\tau_{2k}+\eta\}}(dv) \right] \\
&\quad + \int\limits_{[0,\infty]} u(v)\delta_{\{\tau_{2K-1}\}}(dv) - \mathbb{1}_{\{\tau_{2K} < \infty\}} \int\limits_{[0,\infty]} u(v) \lim_{\substack{\eta\downarrow 0\\ \eta>0}} \delta_{\{\tau_{2K}+\eta\}}(dv) \\
&= \sum_{k=1}^{K-1} \{u(\tau_{2k-1}) - u(\tau_{2k})\} + u(\tau_{2K-1}) - \mathbb{1}_{\{\tau_{2K} < \infty\}} u(\tau_{2K})
\end{aligned} \tag{2.4.22}$$

Before continuing with an application of this result we briefly discuss an example of a characteristic function $\Phi_{\bar{\varepsilon}}(t)$ with infinitely many isolated zeros to point out the difficulties arising in this context.

Assume $\Phi_{\bar{\varepsilon}}(t) = \frac{\sin^2(t)}{t^2}$. Then $\mathcal{P}_{\bar{\varepsilon}}(t,m)$ has a density and the total variation on $[\pi k, \pi(k+1)]$ for $k \in \mathbb{N}$ is readily computed. In particular, since the derivative $\Phi_{\bar{\varepsilon}}'(t)$ on $(\pi k, \pi(k+1))$ has only one zero, denoted by t_k, according to its monotonicity, we obtain:

$$\begin{aligned}
\int\limits_{\pi k}^{\pi(k+1)} \left|\Phi_{\bar{\varepsilon}}'(t)\right| \mathcal{P}_{\bar{\varepsilon}}(t,m-1)dt &= \int\limits_{\pi k}^{t_k} \Phi_{\bar{\varepsilon}}'(t)\mathcal{P}_{\bar{\varepsilon}}(t,m-1)dt - \int\limits_{t_k}^{\pi(k+1)} \Phi_{\bar{\varepsilon}}'(t)\mathcal{P}_{\bar{\varepsilon}}(t,m-1)dt \\
&= \left[-\frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{m+1}\right]_{\pi k}^{t_k} + \left[\frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{m+1}\right]_{t_k}^{\pi(k+1)}
\end{aligned}$$

$$= \frac{2}{m+1}(1 - \mathcal{P}_{\bar{\varepsilon}}(t_k, m))$$

Therefore the total variation on $[0,\infty]$ of an m-power which is composed of the squared sinc function is given by

$$\int_0^\infty |\mathcal{P}_{\bar{\varepsilon}}|\,(dt, m) = -(m+1)\int_0^\pi \Phi'_{\bar{\varepsilon}}(t)\mathcal{P}_{\bar{\varepsilon}}(t, m-1)dt + 2\sum_{k=1}^\infty (1 - \mathcal{P}_{\bar{\varepsilon}}(t_k, m)).$$

The series on the right hand side converges for every finite $m \geq 0$ since, subject to the asymptotic behaviour of $\Phi_{\bar{\varepsilon}}(t)$, we have $1 - \mathcal{P}_{\bar{\varepsilon}}(t, m) = \mathcal{O}\left\{t^{-2}\right\}$ as $t \to \infty$. But the summands are non-negative and $\sup_{m\geq 0}(1 - \mathcal{P}_{\bar{\varepsilon}}(t_k, m)) = 1$ for any $k \in \mathbb{N}$. Consequently the total variation of $\mathcal{P}_{\bar{\varepsilon}}(t, m)$ on $[0,\infty]$ is not uniformly bounded, thereby violating the conditions (2.4.19) for the application of the Helly-Bray theorem. This example suggests a general problem with m-powers composed of functions that vanish at an infinite countable set of points.

We close this subsection with a sufficient condition for the pointwise convergence of the deconvolution function.

Theorem 2.4.3 (pointwise convergence I). *Suppose there exists a sequence $\tau_1, \ldots, \tau_{2K}$ for $K \in \mathbb{N}$ that satisfies (2.4.17). If, in addition, one of the conditions (2.4.19) holds, for any $\xi \in C_{\mathfrak{D}} \cap C_F$ we have*

$$\lim_{m\to\infty} \mathfrak{D}(\xi, m) = F(\xi) + \sum_{k=1}^K \mathfrak{R}(\tau_{2k}, \tau_{2k-1}, \xi). \tag{2.4.23}$$

The remainder integral was introduced in (2.4.11) but can also be found in equation (2.4.25) below. It equals zero if N_ε has Lebesgue measure zero.

Note that the condition of the theorem is especially satisfied if $\Phi_{\bar{\varepsilon}}(t) \neq 0$ for any finite $t \in \mathbb{R}$ and if there exists a $t_0 > 0$ such that $\Phi_{\bar{\varepsilon}}(t)$ exhibits monotonicity on $|t| > t_0$. This emphasizes the outstanding role that is played by error distributions with monotonic characteristic functions.

Proof. Elementary manipulations of the integral (2.1.50) combined with the definitions (2.1.11) and (2.1.15) yield

$$\mathfrak{D}(\xi, m) = \frac{1}{2} + \lim_{T_2\uparrow\infty}\lim_{T_1\downarrow 0}\frac{1}{2\pi i}\int_{[T_1,T_2]} \frac{e^{it\xi}\Phi_X(-t) - e^{-it\xi}\Phi_X(t)}{t}\left\{1 - \mathcal{P}_{\bar{\varepsilon}}(t, m)\right\}dt. \tag{2.4.24}$$

In (2.4.12) for $T > S > 0$ we derived the following representation for the integral (2.4.11):

$$
\begin{aligned}
\mathfrak{R}(T, S, \xi) &= -\int\limits_S^T \frac{e^{it\xi}\Phi_X(-t) - e^{-it\xi}\Phi_X(t)}{i2\pi t} dt \\
&= -\frac{1}{\pi} \int\limits_{-\infty}^{\infty} \{\mathrm{Si}((\xi - x)T) - \mathrm{Si}((\xi - x)S)\} F(dx)
\end{aligned}
\tag{2.4.25}
$$

The boundedness of the sine integral, in particular $\sup_{z\in\mathbb{R}} |\mathrm{Si}(z)| \leq \mathrm{Si}(\pi)$, implies the uniform boundedness of (2.4.25) with respect to $T, S \geq 0$. In addition, since $\xi \in C_{\mathfrak{D}} \cap C_F$, i.e., ξ is especially a continuity point of F, a comparison with the proof of the inversion theorem A.7.12 shows:

$$
\mathfrak{R}(t, 0, \xi) := \lim_{T_1 \downarrow 0} \mathfrak{R}(t, T_1, \xi) = -\frac{1}{\pi} \int\limits_{-\infty}^{\infty} \mathrm{Si}((\xi - x)t) F(dx) \tag{2.4.26}
$$

$$
\mathfrak{R}(\infty, 0, \xi) := \lim_{T_2 \uparrow \infty} \lim_{T_1 \downarrow 0} \mathfrak{R}(T_2, T_1, \xi) = \frac{1}{2} - F(\xi) \tag{2.4.27}
$$

Now, since $\mathfrak{R}(T_1, T_1, \xi) = 0$ for any $T_1 > 0$ by continuity of the integrand, upon integrating by parts the right hand side in (2.4.24) for fixed $T_2 > T_1 > 0$ we arrive at:

$$
\begin{aligned}
&\frac{1}{2\pi i} \int\limits_{[T_1, T_2]} \{1 - \mathcal{P}_{\bar{\varepsilon}}(t, m)\} \frac{e^{it\xi}\Phi_X(-t) - e^{-it\xi}\Phi_X(t)}{t} dt \\
&\qquad = -\{1 - \mathcal{P}_{\bar{\varepsilon}}(T_2, m)\} \mathfrak{R}(T_2, T_1, \xi) - \int\limits_{[T_1, T_2]} \mathfrak{R}(t, T_1, \xi) \mathcal{P}_{\bar{\varepsilon}}(dt, m)
\end{aligned}
\tag{2.4.28}
$$

Under the theorem's conditions $\mathcal{P}_{\bar{\varepsilon}}(\infty, m)$ exists. Hence, if we combine (2.4.24), (2.4.26), (2.4.27) and (2.4.28), we obtain

$$
\mathfrak{D}(\xi, m) = \frac{1}{2} - \{1 - \mathcal{P}_{\bar{\varepsilon}}(\infty, m)\} \mathfrak{R}(\infty, 0, \xi) - \int\limits_{[0,\infty]} \mathfrak{R}(t, 0, \xi) \mathcal{P}_{\bar{\varepsilon}}(dt, m).
$$

As $m \to \infty$ the second summand either vanishes or tends to unity, depending on whether or not $\infty \in N_\varepsilon$. Moreover, it is obvious from (2.4.26) that $\mathfrak{R}(t, 0, \xi)$ is a continuous function of the variable $0 \leq t \leq \infty$. Hence, the limit result (2.4.22) applies and yields:

$$
\begin{aligned}
\lim_{m\to\infty} \mathfrak{D}(\xi, m) = \frac{1}{2} - \mathfrak{R}(\infty, 0, \xi) \mathbb{1}_{\{\infty \notin N_\varepsilon\}} - \sum_{k=1}^{K-1} \{\mathfrak{R}(\tau_{2k-1}, 0, \xi) - \mathfrak{R}(\tau_{2k}, 0, \xi)\} \\
- \mathfrak{R}(\tau_{2K-1}, 0, \xi) + \mathbb{1}_{\{\tau_{2K} < \infty\}} \mathfrak{R}(\tau_{2K}, 0, \xi)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2} - \mathfrak{R}(\infty, 0, \xi)\mathbb{1}_{\{\infty \notin N_\varepsilon\}} + \sum_{k=1}^{K-1} \mathfrak{R}(\tau_{2k}, \tau_{2k-1}, \xi) \\
&\qquad - \mathfrak{R}(\tau_{2K-1}, 0, \xi) + \mathbb{1}_{\{\tau_{2K} < \infty\}}\mathfrak{R}(\tau_{2K}, 0, \xi)
\end{aligned} \tag{2.4.29}
$$

The second equality is an immediate consequence of the definition (2.4.25). Finally, since $\infty \notin N_\varepsilon$ implies $\tau_{2K} < \infty$, according to (2.4.27), in this event the result (2.4.29) matches (2.4.23). Moreover, if $\infty \in N_\varepsilon$ then necessarily $\tau_{2K} = \infty$ and the second and the last summand in (2.4.29) both vanish. But we always have $\mathfrak{R}(\tau_{2K-1}, 0, \xi) = \mathfrak{R}(\infty, 0, \xi) - \mathfrak{R}(\infty, \tau_{2K-1}, \xi)$, which eventually validates (2.4.23) and concludes the proof. ∎

2.4.5. Test for Pointwise Convergence by Means of Alternating Sums

The essential ingredient that enabled us to establish Theorem 2.4.3 was not only the fact that the m-power is of uniform bounded variation on $[0, \infty]$ but especially the presence of the oscillatory terms combined with the decreasing behaviour of the function t^{-1}. These allowed us to make a reference to the sine integral and to verify (2.4.26) as a continuous function that is well-defined on the closed positive segment of the real axis. Generally speaking, fluctuations substantially contribute to the existence of many Fourier-type integrals, although these may fail to converge absolutely. In this subsection we present another approach to exploit this distinguishing behaviour for the derivation of a pointwise convergence statement about the deconvolution function. The oscillations of trigonometric functions are easily extracted either by integration by parts or by sophisticated partitioning of the range of integration. At this point we remind the reader of Appendix B.1, where we have shown the boundedness of the sine integral by dividing the range of integration according to the sign of the sine function. A similar technique can be applied, for instance, to the representation (2.4.3) for the bias of the deconvolution function. First, for fixed $T > 0$, $m \geq 0$ and $\xi \in C_{\mathfrak{D}} \cap C_F$ we write this as an integral along the positive real axis only:

$$
\begin{aligned}
\mathfrak{I}_T(m, \xi) &= \frac{1}{2\pi i} \int_{-T}^{T} \frac{\mathcal{P}_{\bar{\varepsilon}}(t, m)}{t} e^{-i\xi t} \Phi_X(t) dt \\
&= \frac{1}{2\pi i} \int_{0}^{T} \frac{\mathcal{P}_{\bar{\varepsilon}}(t, m)}{t} \left\{ e^{-i\xi t} \Phi_X(t) - e^{i\xi t} \Phi_X(-t) \right\} dt \\
&= \frac{1}{\pi} \Im \left\{ \int_{0}^{T} \frac{\mathcal{P}_{\bar{\varepsilon}}(t, m)}{t} e^{-i\xi t} \Phi_X(t) dt \right\}
\end{aligned} \tag{2.4.30}
$$

Regarding the contribution of Φ_X to the convergence of this integral we must distinguish between oscillatory and monotonic components. Therefore we assume the existence of functions ϕ_X and

φ_X with $\Phi_X = \phi_X \varphi_X$ such that the following holds:

$$
\text{(2.4.31)} \qquad \begin{cases} \phi_X(t) = \int_{-\infty}^{\infty} e^{itx} F_\phi(dx) \text{ for } t \in \mathbb{R}, \text{ where } F_\phi \text{ equals a step function with} \\ \qquad \text{jump points } D_{F_\phi} \text{ that is of finite total variation on } \mathbb{R}, \text{i.e., } |F_\phi|(\mathbb{R}) < \infty. \\ \frac{\varphi_X(t)}{t} \text{ is continuous, of finite total variation on } [T_0, \infty) \text{ for some } T_0 > 0 \\ \qquad \text{and vanishes as } t \to \infty. \end{cases}
$$

We emphasize that $\phi_X(t) \equiv 1$ but also $\varphi_X(t) \equiv 1$ is possible, whence Φ_X can especially be purely oscillatory or even constant. Moreover, the function F_ϕ need not be a probability distribution. For brevity we denote $a(t) := \frac{\varphi_X(t)}{t}$. Under the above assumptions we obtain for $T \geq T_0$ from integration by parts:

$$
\begin{aligned}
\int_{T_0}^{T} \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t} e^{-i\xi t} \Phi_X(t) dt &= \left[a(t) \int_{T_0}^{t} \mathcal{P}_{\bar{\varepsilon}}(s,m) e^{-i\xi s} \phi_X(s) ds \right]_{T_0}^{T} \\
&\quad - \int_{T_0}^{T} \int_{T_0}^{t} \mathcal{P}_{\bar{\varepsilon}}(s,m) e^{-i\xi s} \phi_X(s) ds a(dt) \\
&= a(T) \int_{T_0}^{T} \mathcal{P}_{\bar{\varepsilon}}(s,m) e^{-i\xi s} \phi_X(s) ds - \int_{T_0}^{T} \int_{T_0}^{t} \mathcal{P}_{\bar{\varepsilon}}(s,m) e^{-i\xi s} \phi_X(s) ds a(dt) \qquad \text{(2.4.32)}
\end{aligned}
$$

The second equality incorporates the continuity of the integrand of the primitive integral

$$
B(t,m,\xi) := \int_{T_0}^{t} \mathcal{P}_{\bar{\varepsilon}}(s,m) e^{-i\xi s} \phi_X(s) ds. \qquad \text{(2.4.33)}
$$

Clearly, for $t = T_0$ this equals zero. Furthermore, by assumption $a(T) \to 0$ as $T \to \infty$. A sufficient condition for the decay of the first summand in (2.4.32) as $T \to \infty$ is thus for fixed $m \geq 0$ and $\xi \in C_{\mathfrak{D}} \cap C_F$ the uniform boundedness with respect to $t \geq 0$ of (2.4.33). But in these circumstances, since $a(t)$ is of finite total variation on $[T_0, \infty)$, the integral with respect to $a(t)$ in (2.4.32) converges absolutely and uniformly with respect to $T \geq T_0$. Combining (2.4.4), (2.4.30) and (2.4.32) we may then write for fixed $\xi \in C_{\mathfrak{D}} \cap C_F$ and $m \geq 0$:

$$
\mathfrak{D}(\xi,m) - F(\xi) = \frac{1}{\pi} \Im \left\{ \int_{0}^{T_0} \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t} e^{-i\xi t} \Phi_X(t) dt \right\} - \frac{1}{\pi} \Im \left\{ \int_{T_0}^{\infty} B(t,m,\xi) a(dt) \right\} \qquad \text{(2.4.34)}
$$

The interesting task now consists in the investigation of the primitive integral (2.4.33) and its behaviour as a function of $m \geq 0$ and $t \geq T_0$. Of particular interest is the validity of the assumption concerning its uniform boundedness. Besides additional integration by parts, a useful tool in this context can be the procedure of Abelian summation by parts. While the former enables us to transform some non-absolutely convergent integrals to absolutely convergent ones, the latter provides a formula to rearrange non-absolutely convergent to absolutely convergent series. For $N \in \mathbb{N}_0$ and two sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$, according to §§182-183 on pp. 322-323 in [Knopp, 1976], the formula of Abelian summation by parts states

$$\sum_{n=0}^{N} a_n b_n = b_{N+1} \sum_{n=0}^{N} a_n - \sum_{j=0}^{N} (b_{j+1} - b_j) \sum_{n=0}^{j} a_n. \tag{2.4.35}$$

Since our preliminary considerations involve neither a sum nor a series, the reader may wonder how to apply this formula to (2.4.34). As we indicated in before, it is our aim to exploit the oscillatory behaviour of the complex exponential function appearing in (2.4.33). Therefore, similar to the sine integral, we could straightforwardly split the range of integration according to a half period of the function $e^{i(x-\xi)s}$ for $x \in D_{F_\phi}$, to obtain an alternating sum. Yet, this approach seems to lead us to a statement about (2.4.33) that is only applicable for m-powers with monotonicity properties. Those were, however, already covered by Theorem 2.4.3 in the preceding section. A partition in the described manner is only one example. Another possibility arises by supposing the existence of a constant $\rho > 0$ with the property that

$$\begin{cases} \text{there exists a } j_{\bar{\varepsilon}} \in \mathbb{N}_0 \text{ for which } \mathcal{P}_{\bar{\varepsilon}}(t + j\rho, m) \text{ is monotonic} \\ \qquad \text{with respect to integer } j \geq j_{\bar{\varepsilon}} \text{ for fixed } 0 \leq t \leq \rho. \end{cases} \tag{2.4.36}$$

Clearly, in these circumstances the m-power and $\Phi_{\bar{\varepsilon}}$ both involve a periodic component and we can divide the range of integration according to this period. A necessary condition to establish the pointwise convergence of the deconvolution function is then presumably that the m-power may not perturb the oscillations of the complex exponential function too much, i.e., that both fluctuations do not come into conflict. This is confirmed by the following theorem.

Theorem 2.4.4 (pointwise convergence II). *If Φ_X satisfies the condition (2.4.31) and $\Phi_{\bar{\varepsilon}}$ satisfies the conditions (2.1.52), (2.4.36) and N_ε is of Lebesgue measure zero, the convergence (2.4.1) holds for any $\xi \in C_{\mathfrak{D}} \cap C_F$ such that $(x - \xi)\rho \notin 2\pi\mathbb{Z}$ for any $x \in D_{F_\phi}$.*

A comparison with Theorem 2.4.3 reveals, contrary to monotonic m-powers, in the presence of periodicity the pointwise convergence happens only subject to additional restrictions on the local parameter. This must especially be kept in mind if Theorem 2.4.1 does not apply, because otherwise the convergence is even uniform, which makes both of the aforementioned theorems superfluous.

Proof. Under the assumptions of Theorem 2.4.4 it is easy to see that the first integral in (2.4.34) vanishes as $m \to \infty$. It therefore suffices to show that (2.4.33) tends to zero as $m \to \infty$ for any fixed $t \geq T_0$ and that it is uniformly bounded with respect to $t \geq T_0$, $m \geq 0$. The result then follows from Lebesgue's dominated convergence theorem. The first property is readily verified by virtue of the bound

$$|B(t,m,\xi)| \leq |F_\phi|(\mathbb{R}) \int\limits_{T_0}^{t} \mathcal{P}_{\bar{\varepsilon}}(s,m)ds. \tag{2.4.37}$$

But $\mathcal{P}_{\bar{\varepsilon}}(u,m) \leq 1$ uniformly with respect to $u \in \mathbb{R}$ and $\mathcal{P}_{\bar{\varepsilon}}(u,m) \to 0$ as $m \to \infty$ for Lebesgue almost any $u \in \mathbb{R}$. Hence, according to Lebesgue's dominated convergence theorem, the upper bound (2.4.37) vanishes as $m \to \infty$ for any fixed $t \geq T_0$. To verify the uniform boundedness of (2.4.33) we first define

$$\begin{aligned} J_0 &:= \min\left\{j \in \mathbb{N}_0 : j \geq j_{\bar{\varepsilon}} \text{ and } a(t) \text{ is of finite total variation on } [j\rho, \infty]\right\}, \\ J_t &:= \max\left\{j \in \mathbb{N}_0 : j\rho \leq t\right\}, \end{aligned}$$

and choose $T_0 \equiv J_0\rho$. This implies $J_t \geq J_0$ for $t \geq T_0$. In accordance with this definition, we now divide the range of integration in (2.4.33) into a countable number of segments, subject to the periodic component of the m-power:

$$\begin{aligned} B(t,m,\xi) &= \int\limits_{J_0\rho}^{t} \mathcal{P}_{\bar{\varepsilon}}(s,m)e^{-i\xi s}\phi_X(s)ds \\ &= \sum_{j=J_0}^{J_t-1} \int\limits_{j\rho}^{(j+1)\rho} \mathcal{P}_{\bar{\varepsilon}}(s,m)e^{-i\xi s}\phi_X(s)ds + \int\limits_{J_t\rho}^{t} \mathcal{P}_{\bar{\varepsilon}}(s,m)e^{-i\xi s}\phi_X(s)ds \\ &= \int\limits_{0}^{\rho} e^{-i\xi(s+J_0\rho)}S_{J_t}(s,m,\xi)ds + \int\limits_{J_t\rho}^{t} \mathcal{P}_{\bar{\varepsilon}}(s,m)e^{-i\xi s}\phi_X(s)ds \end{aligned} \tag{2.4.38}$$

For the second equality we performed two linear substitutions and wrote the result in terms of

$$S_{J_t}(s,m,\xi) := \sum_{j=0}^{J_t-J_0-1} \mathcal{P}_{\bar{\varepsilon}}(s+(J_0+j)\rho,m)e^{-i\xi j\rho}\phi_X(s+(J_0+j)\rho). \tag{2.4.39}$$

We now apply the Abelian summation formula to separate the m-power in this sum from the oscillatory factors. For brevity we denote

$$C(n,s,\xi) := \sum_{j=0}^{n} e^{-i\xi j\rho}\phi_X(s+(J_0+j)\rho). \tag{2.4.40}$$

By means of (2.4.35), from (2.4.39) we then deduce

$$\begin{aligned} S_{J_t}(s,m,\xi) &= \mathcal{P}_{\bar{\varepsilon}}(s+J_t\rho,m)C(J_t-J_0-1,s,\xi) \\ &\quad - \sum_{n=0}^{J_t-J_0-1} \left\{\mathcal{P}_{\bar{\varepsilon}}(s+(J_0+n+1)\rho,m) - \mathcal{P}_{\bar{\varepsilon}}(t+(J_0+n)\rho,m)\right\} C(n,s,\xi). \end{aligned} \tag{2.4.41}$$

Finally, for finite $n \in \mathbb{N}_0$ the trigonometric sum (2.4.40) can easily be summed up by employing the geometric sum formula:

$$
\begin{aligned}
C(n,s,\xi) &= \int_{-\infty}^{\infty} e^{ix(s+J_0\rho)} \sum_{j=0}^{n} e^{i(x-\xi)j\rho} F_\phi(dx) \\
&= \int_{-\infty}^{\infty} e^{ix(s+J_0\rho)} \frac{1-e^{i(x-\xi)(n+1)\rho}}{1-e^{i(x-\xi)\rho}} F_\phi(dx) \\
&= \int_{-\infty}^{\infty} \frac{\sin\left\{\frac{(x-\xi)\rho(n+1)}{2}\right\}}{\sin\left\{\frac{(x-\xi)\rho}{2}\right\}} e^{ix(s+\frac{2J_0+n}{2}\rho)-i\xi\frac{\rho n}{2}} F_\phi(dx)
\end{aligned}
$$

Evidently, if $(x-\xi)\rho \notin 2\pi\mathbb{Z}$ the denominator is bounded away from zero and $|C(n,s,\xi)| \leq K_1$ for some constant $K_1 > 0$, uniformly with respect to $n \in \mathbb{N}_0$ and $0 \leq s \leq \rho$. The consequence thereof is that the sum (2.4.41) satisfies the following bound:

$$
\begin{aligned}
|S_{J_t}(s,m,\xi)| \leq{}& K_1 \mathcal{P}_{\bar{\varepsilon}}(s+J_t\rho,m) \\
&+ K_1 \sum_{n=0}^{J_t-J_0-1} |\mathcal{P}_{\bar{\varepsilon}}(s+(J_0+n+1)\rho,m) - \mathcal{P}_{\bar{\varepsilon}}(s+(J_0+n)\rho,m)| \\
={}& K_1 \mathcal{P}_{\bar{\varepsilon}}(s+J_t\rho,m) + K_1 \left|\mathcal{P}_{\bar{\varepsilon}}(s+J_t\rho,m) - \mathcal{P}_{\bar{\varepsilon}}(s+J_0\rho,m)\right| \\
\leq{}& 3K_1 \qquad (2.4.42)
\end{aligned}
$$

For the second equality we appeal to the monotonicity of the m-power with respect to J_0+n, while the last inequality follows from its uniform boundedness. It shows the absolute convergence and therefore especially the boundedness of the sequence of partial sums (2.4.41), uniformly with respect to $0 \leq s \leq \rho$, $J_t \geq J_0$ and $m \geq 0$. Moreover, for the second integral in (2.4.38) the following bound applies, uniformly with respect to $t \geq T_0$ and $m \geq 0$:

$$
\left| \int_{J_t\rho}^{t} \mathcal{P}_{\bar{\varepsilon}}(s,m) e^{-i\xi s} \phi_X(s) ds \right| \leq (t - J_t\rho) \left|F_\phi\right|(\mathbb{R}) \leq \rho \left|F_\phi\right|(\mathbb{R})
$$

The second inequality holds since $0 \leq t - J_t\rho < \rho$ by definition. To summarize these findings, uniformly with respect to $m \geq 0$ and $t \geq T_0$, the primitive integral (2.4.38) satisfies

$$
|B(t,m,\xi)| \leq 3\rho K_1 + \rho \left|F_\phi\right|(\mathbb{R}).
$$

Consequently, in (2.4.34) the limit $m \to \infty$ can be carried out under the integral sign and the limit value equals zero. The proof is thus finished. ∎

Examples for characteristic functions satisfying the conditions (2.4.36) are $\Phi_{\bar{\varepsilon}}(t) = \frac{\sin^2(t)}{t^2}$ or, besides products of monotonic and periodic functions, also mixtures of the form

$$\Phi_\varepsilon = b_d\Phi_{\varepsilon,d} + b_c\Phi_{\varepsilon,c}, \qquad \text{for } b_d > 0,\ b_c \geq 0,\ b_d + b_c = 1, \tag{2.4.43}$$

with characteristic functions $\Phi_{\varepsilon,d}$, $\Phi_{\varepsilon,c}$ of which the former is periodic while the latter is monotonic. We emphasize, however, that the presence of periodicity is required since the positivity of ρ essentially enters the proof of Theorem 2.4.4. Additional difficulties arise if $\Phi_{\bar{\varepsilon}}$ and thus also the m-power possesses an almost periodic component. In the periodic case we saw that the admissible $\xi \in C_{\mathfrak{D}} \cap C_F$ depend on the period ρ. An analogous result for m-powers that feature an almost periodic component must therefore be expected to incorporate information about the translation number, which is more complicated.

2.4.6. Summability of the Deconvolution Function

A common method to improve the convergence behaviour of Fourier series is the application of summation techniques. Since the deconvolution function is also a sum by definition, it is reasonable to discuss the compatibility of those methods. We will not go too much into details but rather conclude this chapter with a brief overview on the possibilities. First, for $M > 0$ the Césaro means of (2.1.26) are given by:

$$\begin{aligned}\mathfrak{D}_C(b,a,M) &:= \frac{1}{M+1}\sum_{m=0}^{M}\{\mathfrak{D}(b,m) - \mathfrak{D}(a,m)\}\\ &= \sum_{l=0}^{M}\left(1 - \frac{l}{M+1}\right)\sum_{k=0}^{l}\binom{l}{k}(-1)^k \bar{H}^{*(k+1)} * (F(b) - F(a))\end{aligned} \tag{2.4.44}$$

This definition is in general not a problem since it is known, if a series converges, the same holds for the Césaro means of its partial sums with matching limits. The definition of Césaro means, however, extends convergence in the sense that non-convergent series may be summable. Instead of letting $m \to \infty$ we now let $M \to \infty$. Similar statements apply for the Abel summation procedure, transferring the role of m to a parameter r with $r \uparrow 1$. More precisely, for $0 < r < 1$, the Abel means of (2.1.26) are given by:

$$\begin{aligned}\mathfrak{D}_A(b,a,r) &:= (1-r)\sum_{m=0}^{\infty} r^m \{\mathfrak{D}(b,m) - \mathfrak{D}(a,m)\}\\ &= (1-r)\sum_{l=0}^{\infty}\sum_{k=0}^{l}\binom{l}{k}(-1)^k \bar{H}^{*(k+1)} * (F(b) - F(a))\sum_{m=l}^{\infty} r^m\\ &= \sum_{l=0}^{\infty} r^l \sum_{k=0}^{l}\binom{l}{k}(-1)^k \bar{H}^{*(k+1)} * (F(b) - F(a))\end{aligned} \tag{2.4.45}$$

The third equality results from an interchange in the order of summation and an application of the formula for geometric series. According to Lemma 2.1.3(2), the series (2.4.45) converges absolutely for $0 < r < \frac{1}{2}$. Applying the inversion formula of Theorem A.7.10 with the definition (A.1.6), for $a, b \in C_{\mathfrak{D}}$ with $a < b$ we formally obtain:

$$\mathfrak{D}_A(b,a,r) = \frac{b-a}{2\pi} \sum_{l=0}^{\infty} r^l \sum_{k=0}^{l} \binom{l}{k} (-1)^k \int_{-\infty}^{\infty} \Phi_{a,b}(-t) \left\{\Phi_{\bar{\varepsilon}}(t)\right\}^{k+1} \Phi_X(t) dt$$

$$= \frac{b-a}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi_{a,b}(-t)\Phi_X(t)\Phi_{\bar{\varepsilon}}(t)}{\Phi_{\bar{\varepsilon}}(t) + (1-r)(1-\Phi_{\bar{\varepsilon}}(t))} dt \tag{2.4.46}$$

For the second equality we again refer to the geometric series formula. The interchange in the order of summation is admissible, for instance if $\Phi_{a,b}\Phi_{\bar{\varepsilon}} \in L^1(\mathbb{R})$. It is also admissible for any $0 < r < 1$ if $\Phi_{a,b}\Phi_X \in L^1(\mathbb{R})$. In these circumstances, however, the integral converges absolutely and uniformly with respect to $0 < r < 1$ and the limit $r \uparrow 1$ may eventually be performed under the integral sign. This leads to the known convergence statement of Theorem 2.4.2.

3. Approximation of the Rates of Uniform Convergence by Means of Real Analysis

Having provided some criteria for the convergence of the deconvolution function in the preceding chapter, it is obvious to proceed with the examination of the rates of convergence. For this purpose, again the four Fourier-type integrals from Corollary 2.1.4 are available, corresponding to the local bias and the bias of the increments as well as their uniform and absolute bounds, respectively. The discussion of techniques for the evaluation of the bias makes a great part of this work and is our objective throughout this and subsequent chapters. In the present chapter we mostly confine to methods of real analysis. For additional convenience we start with the investigation of the rates occuring in the process of uniform convergence, which was established in Theorem 2.4.1. For this, as in the corresponding proof, we employ the uniform local bias. It was given in equation (2.1.56) by

$$\text{ULB}(m) = \frac{1}{\pi} \int_0^\infty \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t} \, |\Phi_X(t)| \, dt. \tag{3.0.1}$$

According to (2.1.55), this provides an upper bound for the maximum deviation of $\mathfrak{D}(\cdot, m)$ from the target distribution F. Under the theorem's assumptions, which shall be satisfied throughout this chapter, the uniform local bias constitutes an absolutely and uniformly convergent integral function of the parameter $m \geq 0$ with a non-negative integrand. Consideration of (3.0.1) instead of the supremum of the absolute value of (2.1.53) has the advantage of simplicity. In particular, by applying the triangle inequality we immediately removed the local parameter and any oscillatory contributions from the integrand. However, since the oscillations especially depend on ξ this bound is presumably not too strong because for some $\xi \in \mathbb{R}$ in the local bias (2.1.53) the fluctuations of the integrand may diminish.

3.1. Nature of the Bias Integrals

Rather incidentally we observe that, apart from a factor, each of the bias integrals for the deconvolution function can be considered a generalization of the Riemann-Liouville integral of a certain function, compare (B.2.23). More importantly, each of them is associated with a larger and better known class of integral functions. This becomes evident upon writing for the m-power

$$\mathcal{P}_{\bar{\varepsilon}}(t,m) = e^{(m+1)\log\{1-\Phi_{\bar{\varepsilon}}(t)\}}, \tag{3.1.1}$$

which is always possible since $1 - \Phi_{\bar{\varepsilon}}(t)$ equals a real-valued function in the unit interval. In accordance with this notation, the reader who is familiar with the mathematical topic of asymptotic expansions will readily specify any of the bias integrals of Corollary 2.1.4 as Laplace-type. Those are in general of the form

$$I(m) := \int_{\mathcal{P}} e^{-mp(t)} q(t) dt, \tag{3.1.2}$$

where $\mathcal{P}$ is a line segment of the complex plane, p is referred to as the phase and q is denoted as the amplitude function. Laplace-type integrals have the property that, as $m \to \infty$ the main contribution to the total value of the integral comes from a neighborhood of the points where the integrand attains its maximum value. Those are particularly the points where the phase function attains its minimum value. Owing to the behaviour of the exponential function, which is the only factor showing the asymptotic parameter, at the minimum of p its value will always be larger than elsewhere. These so-called peaks become sharper as m grows, whereas the remaining area becomes relatively negligible. Integrals of Laplace-type are usually evaluated by Laplace's method or, in case of a complex-valued phase, by the method of stationary phase or by the method of steepest descent. In some texts the notion of Laplace's method is used synonymously for any of the aforementioned procedures. Their aim is the local approximation of the integrand in the neighborhoods of the minima of the phase, by the coefficients of the asymptotic expansions of phase and amplitude function there. The exact asymptotic behaviour of $I(m)$ as $m \to \infty$ thus substantially depends on the involved functions. A thorough discussion of Laplace's method, several examples and extensions can be found in [Bleistein and Handelsman, 1986] or in [Olver, 1974]. We only give a brief overview for a real-valued phase p, a complex q and an integration path of the form $\mathcal{P} \equiv (a, b)$ with real numbers $a < b$, to point out possible difficulties in applications. Suppose that in this setup the following conditions hold:

(1) The functions p, q and the endpoints a, b are independent of m.

(2) b is possibly infinite but a is finite with $p(t) > p(a)$ for all $t \in (a, b)$ and for each $c \in (a, b)$ we have $\inf_{c \leq t < b} (p(t) - p(a)) > 0$, i.e., a is the unique infimum of p along the integration path.

(3) p' and q are continuous in a neighborhood of a, except possibly at the point itself, i.e., both functions may be unbounded when approaching a.

(4) There exist constants $P, \mu, \lambda > 0$ and $Q \in \mathbb{C} \setminus \{0\}$ such that as $t \downarrow a$ we have

$$\begin{cases} p(t) - p(a) & \sim P(t-a)^{\mu}, \\ q(t) & \sim Q(t-a)^{\lambda - 1}, \end{cases} \tag{3.1.3}$$

with the first relation being differentiable. The latter condition holds for instance in case of a continuous p' with an asymptotic expansion as $t \downarrow a$, or in case of a function p that is analytic in a neighborhood of a. Given (3.1.3) we say p and q exhibit algebraic behaviour in a neighborhood of a.

(5) There exists an $m_0 > 0$ such that $I(m)$ is absolutely convergent for all $m > m_0$.

According to Theorem 7.1, ch. 3 in [Olver, 1974], in these circumstances as $m \to \infty$ the integral (3.1.2) exhibits the asymptotic behaviour

$$I(m) \sim \frac{Q}{\mu}\Gamma\left(\frac{\lambda}{\mu}\right)(Pm)^{-\frac{\lambda}{\mu}}e^{-mp(a)}. \tag{3.1.4}$$

A modification of the assumption (3.1.3) eventually yields a full asymptotic expansion rather than only the leading term, compare ch. 3, §8 in the cited book or equation (2.3.15) in [Olver et al., 2010].

Regarding the bias of the deconvolution function, subject to the representation (3.1.1) it is clear that these integrals are in fact of Laplace-type with phase function

$$p(t) \equiv -\log(1 - \Phi_{\bar{\varepsilon}}(t)). \tag{3.1.5}$$

Hence, as $m \to \infty$ the main contribution to the total value of the integrals comes from a neighborhood of the points where the integrand and particularly the m-power attains its maximum value. However, the applicability of Laplace's method is limited. First, the maxima of the m-power are often easy to localize, especially if these match the zeros of $\Phi_{\bar{\varepsilon}}$. If one of the maxima occurs at infinity, an additional substitution is required to map this maximum to a finite point. Issues may arise from the arbitrary and complicated structure of the integrands of the bias integrals. More precisely, if a denotes a minimum of $p(t)$, the involved functions may not satisfy (3.1.3) there, for instance, if either of them locally exhibits exponential behaviour. But exponential-type characteristic functions are frequently encountered. An important class associated with such functions is furnished by the family of alpha stable distributions. We thus refrain from an extensive treatment of Laplace's method but only give a few examples to point out the scope of applicability. Instead we aim to find different approaches for the evaluation of the bias.

3.2. Simple Results

Let us first discuss our expectations with respect to the rate of (3.0.1) as $m \to \infty$. As we mentioned in before, the main contribution comes from the points at which the m-power attains its maximum value. This shall be illustrated by a brief example. For fixed $T > 0$ write

$$\mathrm{ULB}(m) = \frac{1}{\pi}\left[\int\limits_{\{0\le t\le T\}} + \int\limits_{\{t>T\}}\right]\frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t}\left|\Phi_X(t)\right| dt.$$

If $T > 0$ is such that $\min_{0 \le t \le T} \Phi_{\bar{\varepsilon}}(t) > 0$, we observe that the contribution from the integral along $[0, T]$ is of exponential order. Regarding the integral for $t > T$, however, this statement becomes doubtful. In fact, the rate of this part is presumably much weaker if $(T, \infty] \cap N_\varepsilon \neq \emptyset$, since $\mathcal{P}_{\bar{\varepsilon}}(t, m) = 1$ for any $t \in N_\varepsilon$. The situation is similar for the integral

$$\int\limits_{\delta_1}^{1} (1-s)^m ds = \frac{(1-\delta_1)^{m+1}}{m+1}. \tag{3.2.1}$$

For $0 < \delta_1 < 1$ it clearly vanishes with an exponential rate whose order depends on δ_1. As $\delta_1 \downarrow 0$ the order decreases until we finally arrive at an algebraic rate of simple order for $\delta_1 = 0$. The connection between (3.2.1) and (3.0.1) stems from the fact that $\Phi_{\bar{\varepsilon}}(t)$ for $t \in \mathbb{R}$ maps to the unit interval. However, as we shall see below, it is fallacious to expect the rate of ULB(m) as $m \to \infty$ to be either of exponential or of algebraic order. Indeed, it will turn out soon that the rates can be very diverse, especially if Laplace's method (3.1.4) is inapplicable.

An immediate conclusion of the preceding observations concerns situations where the uniform local bias exhibits an exponential rate. From the monotonicity of s^m with respect to $m \ge 1$, for fixed $s \in [0, 1]$, it follows that

$$\sup_{t \in \mathbb{R}} (1 - \Phi_{\bar{\varepsilon}}(t))^m = \left[\sup_{t \in \mathbb{R}} (1 - \Phi_{\bar{\varepsilon}}(t)) \right]^m.$$

Hence, denoting by $I_X \subset \mathbb{R}$ the support of Φ_X, this is also the support of $|\Phi_X|$ and under the conditions of Theorem 2.4.1, from (3.0.1) for (2.1.55) we deduce

$$\|\mathfrak{D}(\cdot, m) - F\|_\infty \le \frac{1}{\pi} \left[\sup_{t \in [0,\infty) \cap I_X} (1 - \Phi_{\bar{\varepsilon}}(t)) \right]^m \int\limits_{[0,\infty) \cap I_X} \frac{1 - \Phi_{\bar{\varepsilon}}(t)}{t} |\Phi_X(t)| \, dt. \tag{3.2.2}$$

It shows that, provided the characteristic function $\Phi_{\bar{\varepsilon}}$ attains a non-zero infimum in the support of Φ_X, the rate is of exponential order. This is equivalent to the statement that $\Phi_{\bar{\varepsilon}}$ does not vanish in the closure of I_X. Regardless of I_X such a case occurs, for instance, if the errors have a Poisson distribution.

Somehow converse to the preceding scenario is the case, where $\Phi_{\bar{\varepsilon}}$ vanishes not only on a countable set. For example if $\Phi_{\bar{\varepsilon}}$ has a compact support $I_{\bar{\varepsilon}}$, which does not include the support of Φ_X. In this event our investigations of Section 2.4 have shown the presence of an additional non-vanishing remainder term. As an illustrative example consider the functions $\{1 - |t|\} \mathbb{1}_{\{|t| \le 1\}}$ and $\{1 + |t|\}^{-1}$, which, according to equations (4.3.4b) and (4.3.4d) in [Lukacs, 1970], are characteristic functions corresponding to absolutely continuous distributions. Assuming the former associated with the error and the latter with the X-distribution, upon integration by parts we deduce from (3.0.1):

$$\begin{aligned}
\pi\,\mathrm{ULB}(m) &= \int\limits_0^1 \frac{t^m}{1+t}dt + \int\limits_1^\infty \frac{1}{t(1+t)}dt \\
&= \frac{1}{2(m+1)} + \left[\frac{t^{m+2}}{(m+1)(m+2)}\frac{1}{(1+t)^2}\right]_0^1 \\
&\qquad + \frac{2}{(m+1)(m+2)}\int\limits_0^1 \frac{t^{m+2}}{(1+t)^3}dt + \int\limits_1^\infty \frac{1}{t} - \frac{1}{1+t}dt \\
&= \frac{1}{2}\sum_{j=0}^\infty \frac{m!j!}{(m+1+j)!}2^{-j} + \log(2) \\
&= \log(2) + \frac{1}{2(m+1)} + \mathcal{O}\left\{m^{-2}\right\}
\end{aligned}$$

The resulting series has non-negative summands and since $m!j! \leq (m+1+j)!$, it can be bounded by a geometric series. Finally, assuming on the other hand, that both characteristic functions Φ_X and $\Phi_{\bar{\varepsilon}}$ are of the given compactly supported form, we find

$$\mathrm{ULB}(m) = \frac{1}{\pi}\int\limits_0^1 t^m(1-t)dt = \frac{1}{\pi}\frac{1}{(m+1)(m+2)}.$$

Hence, the rate of uniform convergence can be quite good, even if the characteristic function of the error variable possesses a compact support.

Classical textbooks, among those the previously cited monographs, describe integration by parts and its modifications as a versatile technique for the derivation of asymptotic expansions for integrals. By means of such a modification also an estimate for the order of the bias in general settings can be deduced.

Theorem 3.2.1 (integration by parts). *In addition to the assumptions of Theorem 2.4.1, for $m_0 \geq 0$ suppose*

$$\text{(3.2.3)}\qquad \begin{cases} \lim\limits_{t\to t_0} \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m_0)}{-\Phi'_{\bar{\varepsilon}}(t)t}\left|\Phi_X(t)\right| = 0, & \textit{for } t_0 \in \{0,\infty\}, \\ \mathcal{P}_{\bar{\varepsilon}}(t,m_0)\left[\frac{|\Phi_X(t)|}{\Phi'_{\bar{\varepsilon}}(t)t}\right]' \in L^1(\mathbb{R}_0^+). \end{cases}$$

As $m \to \infty$ we then have

$$\text{(3.2.4)}\qquad \left\|\mathfrak{D}(\cdot,m) - F\right\|_\infty = o\left\{m^{-1}\right\}.$$

Proof. Under the above assumptions, upon integrating by parts, for $m > m_0$ we obtain:

$$\begin{aligned}
\mathrm{ULB}(m) &= \frac{1}{\pi}\int_0^\infty \frac{\mathcal{P}_{\bar\varepsilon}(t,m)}{t}\,|\Phi_X(t)|\,dt \\
&= \frac{1}{\pi(m+2)}\int_0^\infty \left[\frac{d}{dt}\mathcal{P}_{\bar\varepsilon}(t,m+1)\right]\frac{|\Phi_X(t)|}{-\Phi'_{\bar\varepsilon}(t)t}dt \\
&= \frac{1}{\pi(m+2)}\left[\frac{\mathcal{P}_{\bar\varepsilon}(t,m+1)}{-\Phi'_{\bar\varepsilon}(t)t}\,|\Phi_X(t)|\right]_0^\infty - \frac{1}{\pi(m+2)}\int_0^\infty \mathcal{P}_{\bar\varepsilon}(t,m+1)\left[\frac{|\Phi_X(t)|}{-\Phi'_{\bar\varepsilon}(t)t}\right]' dt \\
&= \frac{1}{\pi(m+2)}\int_0^\infty \mathcal{P}_{\bar\varepsilon}(t,m+1)\left[\frac{|\Phi_X(t)|}{\Phi'_{\bar\varepsilon}(t)t}\right]' dt
\end{aligned}$$

In accordance with the second requirement, Lebesgue's dominated convergence theorem yields the decay of the integral as $m \to \infty$. This verifies the theorem. ∎

With additional restrictions on the integrand and its higher derivatives, repeated integration by parts is possible, from which more precise statements can be obtained. These restrictions, however, become successively more complicated.

3.3. Characteristic Functions of Absolutely Continuous Distributions

Our introductory considerations suggest that we should rarely be able to provide statements concerning the rate of convergence, valid for general characteristic functions. Of particular interest are scenarios in which the bias integrals converge absolutely, which requires $\Phi_X(t)$ to decay sufficiently fast as $t \to \pm\infty$. Therefore, omitting the exotic type of singular distributions, it is necessary to distinguish between different classes of characteristic functions associated with absolutely continuous distributions. Since for any $\mu_X \in \mathbb{R}$ trivially

$$\Phi_{X-\mu_X} = e^{-i\mu_X\cdot}\Phi_X, \qquad |\Phi_{X-\mu_X}| = |\Phi_X|\,,$$

we do not consider characteristic functions of shifted X-variables, as it is always possible to combine shifts with the complex exponential function which is already present in the Fourier integral. Now, according to the common probability models, there are four classes of characteristic functions of special importance:

- Algebraic-type characteristic functions will be denoted in the form

$$|\Phi_X(t)| = \{1+\theta^\alpha\,|t|^\alpha\}^{-p}\,, \qquad \theta,\alpha,p>0, \tag{3.3.1}$$

$$\Phi_{\bar\varepsilon}(t) = \left\{1+\sigma^\beta\,|t|^\beta\right\}^{-q}, \qquad \sigma,\beta,q>0, \tag{3.3.2}$$

for the distribution of X and $\bar\varepsilon$, respectively. Expressing Φ_X in terms of its modulus has the advantage in covering a wider spectrum of characteristic functions. In fact, despite those corresponding to gamma-, χ^2- or geometric stable distributions differ, in absolute value they all equal (3.3.1). Moreover, in certain cases even $|\Phi_X|$ is a characteristic function. On the other hand, both of the above functions constitute characteristic functions merely for some parametrizations.

- Exponential-type characteristic functions, i.e., exponential functions with a monomial in the exponent, occur in the context of alpha stable distributions. Classical examples are the normal and Cauchy distribution. We write

$$\Phi_X(t) = \exp\left\{-p\theta^\alpha |t|^\alpha\right\}, \qquad \theta, \alpha, p > 0, \tag{3.3.3}$$

$$\Phi_{\bar{\varepsilon}}(t) = \exp\left\{-q\sigma^\beta |t|^\beta\right\}, \qquad \sigma, \beta, q > 0. \tag{3.3.4}$$

  According to Theorem 4.1.1 in [Lukacs, 1970], only if $0 < \alpha, \beta \leq 2$ those are actually characteristic functions. Note that stable distributions in addition may feature shift and skewness parameters, which we omitted for the sake of clarity.

- Characteristic functions with a compact support arise from distributions that possess a density with a trigonometric contribution. Examples for such density functions are $t^{-2k}\sin^{2k}(t)$ for arbitrary $k \in \mathbb{N}$.

- Furthermore, there are characteristic functions that feature trigonometric properties and vanish at infinity. Those occur for example in case of distributions with a compact support, such as beta, rectangular or triangular distibutions. The latter family will be denoted in the form

$$\Phi_X(t) = \prod_{j=1}^{p} \frac{e^{itb_j} - e^{ita_j}}{i(b_j - a_j)t}, \qquad \text{for } p \in \mathbb{N},\ a_j, b_j \in \mathbb{R} \text{ with } a_j < b_j, \tag{3.3.5}$$

  if it is associated with the distribution of X. Regarding $\bar{\varepsilon}$, this family is written as

$$\Phi_{\bar{\varepsilon}}(t) = \frac{\sin^{2q}(\sigma t)}{(\sigma t)^{2q}}, \qquad \text{for } q \in \mathbb{N} \text{ and } \sigma > 0, \tag{3.3.6}$$

  which is in accordance with the symmetrization.

The parameters appearing in the above list have a similar meaning for any of the functions. Particularly σ and θ are referred to as scaling parameters. Furthermore, for integer values, p and q indicate that the characteristic function arises from a p- or q-times convolution.

Of course, the preceding exposition is in no way exhaustive. Although there are necessary and sufficient conditions, which a function needs to satisfy in order to be the Fourier transform of a probability distribution, the whole scope is unimaginable. Neither are exponential-type characteristic functions those with the fastest decay, nor do algebraic-type functions constitute the class with the slowest decay. In fact, by means of Pólyas condition, compare Theorem 4.3.1

in [Lukacs, 1970], it was shown in Appendix A.7.4 that the logarithmic functions

$$\Phi_X(t) = \log^{-p}\left\{e + \theta^\beta |t|^\alpha\right\}, \qquad \theta, \alpha, p > 0, \tag{3.3.7}$$

$$\Phi_{\bar{\varepsilon}}(t) = \log^{-q}\left\{e + \sigma^\beta |t|^\beta\right\}, \qquad \sigma, \beta, q > 0, \tag{3.3.8}$$

establish characteristic functions if $0 < \alpha, \beta \leq 1$. Those are for no parametrization members of the space $L^1(\mathbb{R})$.

3.4. Approximations for Error Distributions with a Monotonic Characteristic Function

We will now present a fairly primitive approach to evaluate (3.0.1) for some of the characteristic functions given in equations (3.3.1) to (3.3.4). These are strictly decreasing on the positive real axis and infinitely many times continuously differentiable there. In addition (3.3.2) and (3.3.4) map the positive real axis on the unit interval. More precisely, since $\Phi_{\bar{\varepsilon}}(0) = 1$ and $\lim_{t\to\infty} \Phi_{\bar{\varepsilon}}(t) = 0$, both functions are bijections between $[0, \infty]$ and $[0, 1]$. Each of them is thus invertible, enabling in (3.0.1) the substitution $ds = \Phi'_{\bar{\varepsilon}}(t)dt$. This leads to the Riemann-Liouville-type integral

$$\mathrm{ULB}(m) = \frac{1}{\pi}\int_0^1 (1-s)^m \zeta_0(s)ds, \tag{3.4.1}$$

where the part of the integrand that does not depend on m is referred to as

$$\zeta_0(s) := \frac{(1-s)\left|\Phi_X \circ \Phi_{\bar{\varepsilon}}^{-1}(s)\right|}{-\Phi'_{\bar{\varepsilon}} \circ \Phi_{\bar{\varepsilon}}^{-1}(s)\Phi_{\bar{\varepsilon}}^{-1}(s)}. \tag{3.4.2}$$

The function $\zeta_0(s)$ is non-negative since $\Phi'_{\bar{\varepsilon}}(t) < 0$ by monotonicity, and it is also absolutely integrable on the unit interval because $\mathrm{ULB}(0) < \infty$. Moreover, under the above assumptions

$$\Phi_{\bar{\varepsilon}}^{-1}(s) = \begin{cases} \infty, & \text{for } s = 0, \\ 0, & \text{for } s = 1, \end{cases} \tag{3.4.3}$$

and the decay of $\Phi_{\bar{\varepsilon}}$ implies $\Phi'_{\bar{\varepsilon}}(\infty) = 0$. If

$$\sup_{0\leq s\leq 1} \zeta_0(s) = \sup_{0\leq s\leq 1} \frac{(1-s)\left|\Phi_X \circ \Phi_{\bar{\varepsilon}}^{-1}(s)\right|}{-\Phi'_{\bar{\varepsilon}} \circ \Phi_{\bar{\varepsilon}}^{-1}(s)\Phi_{\bar{\varepsilon}}^{-1}(s)} = \sup_{t\geq 0} \frac{|1-\Phi_{\bar{\varepsilon}}(t)|\,|\Phi_X(t)|}{-\Phi'_{\bar{\varepsilon}}(t)t}$$

is finite, then (3.4.1) immediately implies $\mathrm{ULB}(m) = \mathcal{O}\left\{m^{-1}\right\}$. A general impression how the leading behaviour of the integral (3.4.1) is affected by the behaviour of $\zeta_0(s)$ is provided by the following consideration. Assume $v : [0, 1] \to \mathbb{R}_0^+$ is a continuous integrable function that satisfies $0 < v(s) < \infty$ for $0 < s \leq 1$. Then, depending on $v(0)$, the function v attains a finite minimum or maximum on the closed unit interval. Consequently for $b_1, b_2, b_3 > 0$ we have

$$\int_0^1 (1-s)^m v(s) ds \begin{cases} \geq \frac{b_1}{m}, & \text{if } v(0) = \infty, \\ = \frac{b_2}{m}, & \text{if } 0 < v(0) < \infty, \\ \leq \frac{b_3}{m}, & \text{if } v(0) = 0. \end{cases} \tag{3.4.4}$$

To confirm these statements it suffices to bound the integral from below, from both sides and from above, respectively. It reveals that the exact rate of the integral (3.4.1) substantially depends on the behaviour of the function $\zeta_0(s)$ as $s \downarrow 0$. This observation is in accordance with the fact that (3.4.1) constitutes an integral of Laplace-type, where the integrand attains its maximum value in a left neighborhood of the origin. By (3.4.3) the origin corresponds to the point at infinity in the original integral representation (3.0.1).

Since the integral (3.4.1) looks very similar to a known representation for the beta function, compare (B.3.1), it is our aim to express $\zeta_0(s)$ in terms of algebraic functions to establish this connection. In this process we will apply estimates that do not incur too much losses with respect to accuracy of the estimated leading behaviour as $m \to \infty$. As a justification we refer to the statement (3.4.4). Finally, to present our results in terms of elementary functions, we make use of the functional equation for the gamma function combined with inequality (5.6.8) in [Olver et al., 2010], which for $m > 0$ and $b \geq 0$ yields

$$\frac{\Gamma(m+1)}{\Gamma(m+1+b)} = \frac{m\Gamma(m)}{\Gamma(m+1+b)} \leq m^{-b}. \tag{3.4.5}$$

A comparison with the approximation (B.2.30), obtainable by means of Stirling's formula, verifies the asympotic precision of this inequality.

Example 3.4.1 (algebraic-type characteristic functions). In our first example we consider Φ_X and $\Phi_{\bar{\varepsilon}}$ of the form (3.3.1) and (3.3.2), respectively. Then, upon substituting $s = \Phi_{\bar{\varepsilon}}(t)$ we obtain

$$\begin{aligned} t &= \Phi_{\bar{\varepsilon}}^{-1}(s) = \sigma^{-1} \left\{1 - s^{\frac{1}{q}}\right\}^{\frac{1}{\beta}} s^{-\frac{1}{\beta q}}, \\ \frac{ds}{dt} &= \Phi_{\bar{\varepsilon}}'(t) = -\beta\sigma^\beta t^{\beta-1} q \left\{1 + (\sigma t)^\beta\right\}^{-q-1} \\ &= -q\beta\sigma \left\{1 - s^{\frac{1}{q}}\right\}^{1-\frac{1}{\beta}} s^{1+\frac{1}{\beta q}}. \end{aligned} \tag{3.4.6}$$

We therefore deduce

$$\begin{aligned} \left|\Phi_X \circ \Phi_{\bar{\varepsilon}}^{-1}(s)\right| &= \sigma^{p\alpha} s^{\frac{\alpha p}{\beta q}} \underbrace{\left\{\sigma^\alpha s^{\frac{\alpha}{\beta q}} + \theta^\alpha \left\{1 - s^{\frac{1}{q}}\right\}^{\frac{\alpha}{\beta}}\right\}^{-p}}_{=: v_1(s)}, \\ \zeta_0(s) &= \frac{(1-s)\left|\Phi_X \circ \Phi_{\bar{\varepsilon}}^{-1}(s)\right|}{-\Phi_{\bar{\varepsilon}}' \circ \Phi_{\bar{\varepsilon}}^{-1}(s)\Phi_{\bar{\varepsilon}}^{-1}(s)} \end{aligned}$$

$$= \frac{\sigma^{p\alpha}}{q\beta} s^{\frac{\alpha p}{\beta q}-1} v_1(s) \underbrace{(1-s)\left\{1-s^{\frac{1}{q}}\right\}^{-1}}_{=:\, v_2(s)}.$$

The function $v_1(s)$ satisfies

$$v_1(s) \begin{cases} = \sigma^{-\alpha p}, & \text{for } s = 1, \\ > 0, & \text{for } s \in (0,1), \\ = \theta^{-\alpha p}, & \text{for } s = 0. \end{cases}$$

Regarding $v_2(s)$ the rule of de l'Hospital shows

$$\lim_{s \uparrow 1} v_2(s) = \lim_{s \uparrow 1} \frac{\frac{d}{ds}(1-s)}{\frac{d}{ds}(1-s^{\frac{1}{q}})} = q.$$

Hence, on the unit interval this function exhibits the behaviour

$$v_2(s) \begin{cases} = 1, & \text{for } s = 0, \\ > 0, & \text{for } s \in (0,1), \\ = q, & \text{for } s = 1. \end{cases}$$

Consequently, in accordance with (3.4.4), neither $v_1(s)$ nor $v_2(s)$ affects the rate. Moreover, by continuity both functions attain a maximum value on $[0,1]$, which yields:

$$\begin{aligned} \mathrm{ULB}(m) &= \frac{1}{\pi} \int_0^1 (1-s)^m \zeta_0(s) ds \\ &= \frac{\sigma^{p\alpha}}{\pi q \beta} \int_0^1 (1-s)^m s^{\frac{\alpha p}{\beta q}-1} v_1(s) v_2(s) ds \\ &\leq \frac{\sigma^{p\alpha}}{\pi q \beta} \mathrm{B}\left(m+1, \frac{\alpha p}{\beta q}\right) \max_{0 \leq r \leq 1} v_1(r) v_2(r) \end{aligned}$$

Upon expressing the beta function in terms of gamma functions, see identity (B.3.2), accompanied by an application of (3.4.5), it finally shows that the uniform convergence as $m \to \infty$ in a purely algebraic setup has the following order:

$$\|\mathfrak{D}(\cdot, m) - F\|_\infty = \mathcal{O}\left\{ m^{-\frac{\alpha p}{\beta q}} \frac{\sigma^{p\alpha}\Gamma\left(\frac{\alpha p}{\beta q}\right)}{\pi q \beta} \right\}$$

In other words, the rate of decay is of algebraic order with the degree depending on the ratio of the parameters $\alpha, \beta, p, q, > 0$, which determine the behaviour of the involved characteristic functions at infinity.

Example 3.4.2 (algebraic-type characteristic functions: Laplace's method). We briefly reconsider the preceding example to illustrate Laplace's method. Since the characteristic function (3.3.2) vanishes only at infinity, the m-power attains its maximum value there. A simple change of variables maps this point to the origin:

$$\mathrm{ULB}(m) = \frac{1}{\pi} \int_0^\infty \frac{\mathcal{P}_{\bar{\varepsilon}}\left(\frac{1}{s}, m\right)}{s} \left| \Phi_X \left(\frac{1}{s} \right) \right| ds$$

It is easy to see that this integral satisfies the conditions for the applicability of the estimate (3.1.4) with $m_0 = 0$. More precisely, the phase and amplitude function as $s \downarrow 0$ exhibit the local behaviour

$$\begin{gathered} -\log \left\{ 1 - \Phi_{\bar{\varepsilon}} \left(\frac{1}{s} \right) \right\} \sim \Phi_{\bar{\varepsilon}} \left(\frac{1}{s} \right) \sim \sigma^{-\beta q} s^{\beta q}, \\ \frac{\left| \Phi_X \left(\frac{1}{s} \right) \right|}{s} \sim \theta^{-\alpha p} s^{\alpha p - 1}. \end{gathered}$$

The first of these asymptotic relations follows from the series expansion of the logarithm. From (3.1.4), as $m \to \infty$ we thus conclude

$$\mathrm{ULB}(m) \sim \frac{1}{\pi \beta q} \left\{ \frac{\sigma}{\theta} \right\}^{\alpha p} \Gamma \left(\frac{\alpha p}{\beta q} \right) m^{-\frac{\alpha p}{\beta q}}.$$

Notice the closeness of the estimate from the preceding Example 3.4.1 to this exact result.

We already mentioned possible difficulties in the application of Laplace's method if exponential-type functions are involved. Yet, this is not a problem for the procedure of simple estimates, as the following two examples show.

Example 3.4.3 (exponential-type characteristic functions). Assume Φ_X and $\Phi_{\bar{\varepsilon}}$ as given in (3.3.3) and (3.3.4), respectively. Then

$$\begin{aligned} t &= \sigma^{-1} \left[\log \left\{ s^{-\frac{1}{q}} \right\} \right]^{\frac{1}{\beta}}, \\ \frac{ds}{dt} &= -\beta \sigma q s \left[\log \left\{ s^{-\frac{1}{q}} \right\} \right]^{1 - \frac{1}{\beta}}. \end{aligned} \tag{3.4.7}$$

From these identities we obtain

$$\begin{aligned} \left| \Phi_X \circ \Phi_{\bar{\varepsilon}}^{-1}(s) \right| &= \exp \left\{ -p \theta^\alpha \sigma^{-\alpha} \left[\log \left\{ s^{-\frac{1}{q}} \right\} \right]^{\frac{\alpha}{\beta}} \right\}, \\ \zeta_0(s) &= \beta^{-1} q^{-1} (1 - s) s^{-1} \left[\log \left\{ s^{-\frac{1}{q}} \right\} \right]^{-1} \exp \left\{ -p \theta^\alpha \sigma^{-\alpha} \left[\log \left\{ s^{-\frac{1}{q}} \right\} \right]^{\frac{\alpha}{\beta}} \right\}. \end{aligned}$$

For general $\alpha, \beta > 0$ these terms are essentially more complicated than the algebraic terms in Example 3.4.1. In fact, as $s \downarrow 0$, it is ascertainable that $\zeta_0(s)$ tends to zero slower or faster than algebraic, depending on whether $\alpha < \beta$ or $\alpha > \beta$. An exception occurs if $\alpha = \beta$, which yields the following integral representation:

$$\begin{aligned}
\zeta_0(s) &= \beta^{-1}q^{-1}(1-s)\left[\log\left\{s^{-\frac{1}{q}}\right\}\right]^{-1} s^{\frac{\theta^\beta p}{\sigma^\beta q}-1} \\
&= \beta^{-1}(1-s)\left[\log\left\{s^{-1}\right\}\right]^{-1} s^{\frac{\theta^\beta p}{\sigma^\beta q}-1} \\
&= \beta^{-1}\frac{s-1}{\log(s)} s^{\frac{\theta^\beta p}{\sigma^\beta q}-1} \\
&= \beta^{-1}\int_0^1 s^{x+\frac{\theta^\beta p}{\sigma^\beta q}-1}dx
\end{aligned}$$

In this event (3.4.1) can be written as a double integral. Upon interchanging the order of integration and applying (3.4.5) we arrive at:

$$\begin{aligned}
\mathrm{ULB}(m) &= \frac{1}{\pi}\int_0^1 (1-s)^m \zeta_0(s)ds \\
&= \frac{1}{\pi\beta}\int_0^1 (1-s)^m \int_0^1 s^{x+\frac{\theta^\beta p}{\sigma^\beta q}-1}dxds \\
&= \frac{1}{\pi\beta}\int_0^1 \mathrm{B}\left(m+1, \frac{\theta^\beta p}{\sigma^\beta q}+x\right)dx \\
&\le \frac{1}{\pi\beta}\int_0^1 \Gamma\left(\frac{\theta^\beta p}{\sigma^\beta q}+x\right) m^{-\frac{\theta^\beta p}{\sigma^\beta q}-x}dx \\
&\le \frac{m^{-\frac{\theta^\beta p}{\sigma^\beta q}}}{\pi\beta}\max_{0\le r\le 1}\Gamma\left(\frac{\theta^\beta p}{\sigma^\beta q}+r\right)\int_0^1 m^{-x}dx \\
&\le \frac{m^{-\frac{\theta^\beta p}{\sigma^\beta q}}}{\pi\beta}\max_{0\le r\le 1}\Gamma\left(\frac{\theta^\beta p}{\sigma^\beta q}+r\right)\frac{1}{\log(m)}
\end{aligned}$$

Observe that the maximum of the gamma function is indeed finite, since the parameter-dependent ratio in the argument is always positive. Especially $\Gamma(x) > 0$ for any $x > 0$, whence this factor does not affect the actual rate. As $m \to \infty$ we have thus established the estimate

$$\|\mathfrak{D}(\cdot, m) - F\|_\infty = \mathcal{O}\left\{\frac{m^{-\frac{\theta^\beta p}{\sigma^\beta q}}}{\pi\beta\log(m)}\max_{0\le r\le 1}\Gamma\left(\frac{\theta^\beta p}{\sigma^\beta q}+r\right)\right\}. \tag{3.4.8}$$

Recall that this only holds for $\alpha = \beta$. For some reason mentioned in before we were not able to find an appropriate estimate for the function $\zeta_0(s)$ in terms of algebraic functions for arbitrary $\alpha, \beta > 0$. This case will be thoroughly discussed in a later chapter by means of complex analysis.

Integrals involving gamma functions will be of frequent occurence in this work. If the integration path is a segment of the real axis, we refer to them as *Ramanujan-type integrals*, since he was the first to study them. See §3.3.5 in [Paris and Kaminski, 2001]. For some of them, Ramanujan deduced representations in terms of elementary functions.

Example 3.4.4 (algebraic Φ_X and exponential $\Phi_{\bar{\varepsilon}}$). For characteristic functions of the form (3.3.1) and (3.3.4) a substitution as in (3.4.7) yields

$$\left|\Phi_X \circ \Phi_{\bar{\varepsilon}}^{-1}(s)\right| = \left\{1 + \theta^\alpha \sigma^{-\alpha} \left[\log\left\{s^{-\frac{1}{q}}\right\}\right]^{\frac{\alpha}{\beta}}\right\}^{-p}.$$

This is a logarithmic function of algebraic order, expressible in terms of integrals involving algebraic functions, similar to the preceding example. First we cast $\zeta_0(s)$ as follows:

$$\begin{aligned}
\zeta_0(s) &= \beta^{-1} q^{-1} (1-s) \left[\log\left\{s^{-\frac{1}{q}}\right\}\right]^{-1} s^{-1} \left\{1 + \theta^\alpha \sigma^{-\alpha} \left[\log\left\{s^{-\frac{1}{q}}\right\}\right]^{\frac{\alpha}{\beta}}\right\}^{-p} \\
&= \beta^{-1} s^{-1} \left\{1 + \theta^\alpha \sigma^{-\alpha} \left[\log\left\{s^{-\frac{1}{q}}\right\}\right]^{\frac{\alpha}{\beta}}\right\}^{-p} \int_0^1 s^x dx \\
&= \beta^{-1} \underbrace{\left\{1 + \left[\log\left\{s^{-\frac{\theta^\beta}{\sigma^\beta q}}\right\}\right]^{\frac{\alpha}{\beta}}\right\}^{-p}}_{=:\, v_3(s)} \int_0^1 s^{x-1} dx
\end{aligned}$$

Denoting $\tau := \frac{\theta^\beta}{\sigma^\beta q}$ and $\omega := \frac{\alpha}{\beta}$, it remains to investigate the function $v_3(s)$. In order to make use of the property $1 + \log(z) = \log(ez)$, by continuity of $s = e^{-t}$, from the rule of de l'Hospital we deduce

$$\lim_{s \downarrow 0} \frac{\{1 + \log\{s^{-\tau}\}\}^\omega}{1 + [\log(s^{-\tau})]^\omega} = \lim_{t\to\infty} \frac{(1+t\tau)^\omega}{1 + \tau^\omega t^\omega} = \lim_{t\to\infty} \left\{1 + \frac{1}{t\tau}\right\}^{\omega-1} = 1.$$

We have thus verified that

$$v_3(s)\left\{1 + \log\left\{s^{-\tau}\right\}\right\}^{\omega p} \begin{cases} > 0, & \text{for } 0 < s < 1, \\ = 1, & \text{for } s \in \{0,1\}, \end{cases}$$

i.e., the product does not affect the actual rate. In addition, by continuity it has a unique finite maximum $\kappa > 0$ on $[0,1]$. Moreover, since $\tau > 0$ and $es^{-\tau} > 1$ for $0 \le s \le 1$, we may write

$$\left\{1 + \log\left\{s^{-\tau}\right\}\right\}^{-\omega p} = \left\{\log\left\{es^{-\tau}\right\}\right\}^{-\omega p} = \frac{1}{\Gamma(\omega p)} \int_0^\infty y^{\omega p - 1} s^{\tau y} e^{-y} dy.$$

To summarize our findings, the function $\zeta_0(s)$ is bounded by a double integral:

$$\zeta_0(s) = \beta^{-1} v_3(s) \int_0^1 s^{x-1} dx$$

$$
\begin{aligned}
&\leq \frac{\kappa}{\beta} \left\{1 + \log\left\{s^{-\tau}\right\}\right\}^{-\omega p} \int_0^1 s^{x-1} dx \\
&= \frac{\kappa}{\beta \Gamma(\omega p)} \int_0^\infty y^{\omega p - 1} e^{-y} \int_0^1 s^{x + \tau y - 1} dx dy
\end{aligned}
$$

Concerning (3.4.1), this estimate yields:

$$
\begin{aligned}
\mathrm{ULB}(m) &= \frac{1}{\pi} \int_0^1 (1-s)^m \zeta_0(s) ds \\
&\leq \frac{1}{\pi} \int_0^1 (1-s)^m \frac{\kappa}{\beta \Gamma(\omega p)} \int_0^\infty y^{\omega p - 1} e^{-y} \int_0^1 s^{x + \tau y - 1} dx dy ds \\
&= \frac{\kappa}{\pi \beta \Gamma(\omega p)} \int_0^\infty y^{\omega p - 1} e^{-y} \int_0^1 \mathrm{B}(m+1, x + \tau y) dx dy \\
&= \frac{\kappa}{\pi \beta \Gamma(\omega p)} \int_0^\infty y^{\omega p - 1} e^{-y} \int_0^1 \frac{\Gamma(m+1)\Gamma(x + \tau y)}{\Gamma(m + 1 + x + \tau y)} dx dy
\end{aligned}
$$

In contrast to Example 3.4.3 the above Ramanujan-type integral contains an unbounded expression. That is, the function $\Gamma(x + \tau y)$ exceeds any limit as $x + \tau y$ approaches zero. We shall see below that this behaviour essentially decreases the rate as $m \to \infty$. To show this, we split the range of integration according to the domains on the positive real axis where the gamma function increases and decreases, respectively. More precisely, since the gamma function is known to increase along $x > 0$ into the left and right direction of $c \approx 1.46163$, we split the range of integration into two parts, leading to:

$$
\begin{aligned}
\mathrm{ULB}(m) &= \frac{\kappa}{\pi \beta \Gamma(\omega p)} \int_0^\infty y^{\omega p - 1} e^{-y} \int_0^1 \frac{\Gamma(m+1)\Gamma(x + \tau y)}{\Gamma(m + 1 + x + \tau y)} \left(\mathbb{1}_{\{x + \tau y \leq c\}} + \mathbb{1}_{\{x + \tau y > c\}}\right) dx dy \\
&=: \frac{\kappa}{\pi \beta \Gamma(\omega p)} \left\{I_1(m) + I_2(m)\right\}
\end{aligned}
$$

For the first integral we obtain:

$$
\begin{aligned}
I_1(m) &= \int_0^\infty y^{\omega p - 1} e^{-y} \int_0^1 \frac{\Gamma(m+1)\Gamma(x + \tau y)}{\Gamma(m + 1 + x + \tau y)} \mathbb{1}_{\{x + \tau y \leq c\}} dx dy \\
&= \int_0^{\tau^{-1} c} y^{\omega p - 1} e^{-y} \int_0^{\min\{c - \tau y, 1\}} \frac{\Gamma(m+1)\Gamma(x + \tau y)}{\Gamma(m + 1 + x + \tau y)} dx dy
\end{aligned}
$$

$$\leq \int_0^{\tau^{-1}c} y^{\omega p-1} e^{-y} \int_0^c m^{-(x+\tau y)} \Gamma(x+\tau y) dx dy$$

The applied inequality makes use of the estimate (3.4.5). Assuming $m > 1$, we now employ the Γ-functional equation to separate the unbounded from the bounded part of the integrand. After some simple manipulations this reveals the leading behaviour:

$$\begin{aligned}
I_1(m) &\leq \int_0^{\tau^{-1}c} y^{\omega p-1} e^{-y} \int_0^c m^{-(x+\tau y)} \Gamma(x+\tau y) dx dy \\
&= \int_0^{\tau^{-1}c} y^{\omega p-1} (m^\tau e)^{-y} \int_0^c m^{-x} \frac{\Gamma(1+x+\tau y)}{x+\tau y} dx dy \\
&\leq \Gamma(1+2c) \int_0^{\tau^{-1}c} y^{\omega p-1} (m^\tau e)^{-y} \int_0^c \frac{m^{-x}}{x+\tau y} dx dy \\
&\leq \Gamma(1+2c) \int_0^{\tau^{-1}c} y^{\omega p-1} m^{-\tau y} \int_0^c \frac{m^{-x}}{x+\tau y} dx dy \\
&= \{\tau \log(m)\}^{-\omega p} \Gamma(1+2c) \int_0^{c\log(m)} y^{\omega p-1} e^{-y} \int_0^{c\log(m)} \frac{e^{-x}}{x+y} dx dy \\
&\leq \{\tau \log(m)\}^{-\omega p} \Gamma(1+2c) \int_0^\infty y^{\omega p-1} e^{-y} \int_0^\infty \frac{e^{-x}}{x+y} dx dy \\
&= \{\tau \log(m)\}^{-\omega p} \Gamma(1+2c) \int_0^\infty y^{\omega p-1} e^{-y} \int_0^\infty e^{-x} \int_0^\infty e^{-(x+y)t} dt dx dy \\
&= \{\tau \log(m)\}^{-\omega p} \Gamma(1+2c) \int_0^\infty \int_0^\infty y^{\omega p-1} e^{-(1+t)y} dy \int_0^\infty e^{-(1+t)x} dx dt \\
&= \{\tau \log(m)\}^{-\omega p} \Gamma(\omega p)\Gamma(1+2c) \int_0^\infty (1+t)^{-\omega p-1} dt \\
&= (\omega p)^{-1} \{\tau \log(m)\}^{-\omega p} \Gamma(\omega p)\Gamma(1+2c)
\end{aligned}$$

The third inequality incorporates the finiteness of the range of integration, so that the exponential function may be bounded by unity without affecting the rate. Finally, it remains to estimate the integral $I_2(m)$. This, however, is readily accomplished since $c > 1$ and the beta function decreases for arguments greater than unity:

$$I_2(m) = \int_0^\infty y^{\omega p-1} e^{-y} \int_0^1 B(m+1, x+\tau y) \mathbb{1}_{\{x+\tau y > c\}} dx dy$$

$$\leq \int_0^\infty y^{\omega p-1} e^{-y} \int_0^1 B(m+1,1) \mathbb{1}_{\{x+\tau y > c\}} dx dy$$
$$\leq (m+1)^{-1} \Gamma(\omega p)$$

Hence, $I_2(m)$ as $m \to \infty$ exhibits a faster decay than $I_1(m)$, which shows that

$$\|\mathfrak{D}(\cdot, m) - F\|_\infty = \mathcal{O}\left\{ (\pi \alpha p)^{-1} \left[\log \left\{ m^{\frac{\theta^\beta}{\sigma^\beta q}} \right\} \right]^{-\frac{\alpha p}{\beta}} \right\}. \tag{3.4.9}$$

The rate of uniform convergence is therefore of logarithmic order.

Our final example ultimately shows that the above procedure yields only insuffıcent estimates if the actual rate is of exponential order. This was already indicated by Example 3.4.3.

Example 3.4.5 (exponential Φ_X and algebraic $\Phi_{\bar{\varepsilon}}$). With characteristic functions $\Phi_{\bar{\varepsilon}}$ and Φ_X as in (3.3.2) and (3.3.3), respectively, the change of variables $s = \Phi_{\bar{\varepsilon}}(t)$ as in (3.4.6) yields

$$|\Phi_X(t)| = \exp\left\{ -p\theta^\alpha \sigma^{-\alpha} \left\{ 1 - s^{\frac{1}{q}} \right\}^{\frac{\alpha}{\beta}} s^{-\frac{\alpha}{\beta q}} \right\},$$
$$\zeta_0(s) = q^{-1} \beta^{-1} v_2(s) \underbrace{s^{-1} \exp\left\{ -p\theta^\alpha \sigma^{-\alpha} \left\{ 1 - s^{\frac{1}{q}} \right\}^{\frac{\alpha}{\beta}} s^{-\frac{\alpha}{\beta q}} \right\}}_{=:\, v_4(s)}.$$

In Example 3.4.1 it was shown that $v_2(s)$ does not affect the rate. Rather important is the behaviour of $v_4(s)$. Observing that

$$\lim_{t \to \infty} t^b e^{-t^\rho} = 0, \tag{3.4.10}$$

for any $b, \rho > 0$, it follows that:

$$\lim_{s \downarrow 0} \frac{v_4(s)}{s^b} = \lim_{s \downarrow 0} s^{-(1+b)} \left| \Phi_X \circ \Phi_{\bar{\varepsilon}}^{-1}(s) \right| = \lim_{t \to \infty} \{\Phi_{\bar{\varepsilon}}(t)\}^{-(1+b)} |\Phi_X(t)| = 0$$

This implies

$$\frac{v_4(s)}{s^b} \begin{cases} = 0, & \text{for } s = 0, \\ > 0, & \text{for } s \in (0,1), \\ = 1, & \text{for } s = 1. \end{cases}$$

Moreover, by continuity the function on the left hand side possesses a unique finite non-zero maximum on $[0,1]$. We thus obtain:

$$\mathrm{ULB}(m) = \frac{1}{\pi} \int_0^1 (1-s)^m \zeta_0(s) ds$$

$$
\begin{aligned}
&= \frac{1}{\pi q \beta} \int_0^1 (1-s)^m s^b \frac{v_4(s)}{s^b} v_2(s) ds \\
&\leq \frac{1}{\pi q \beta} \max_{0 \leq r \leq 1} v_2(r) \frac{v_4(r)}{r^b} \frac{\Gamma(b+1)\Gamma(m+1)}{\Gamma(m+b+2)}
\end{aligned}
$$

Since $b > 0$ was arbitrary, according to (3.4.5), as $m \to \infty$ we have verified

$$
\|\mathfrak{D}(\cdot, m) - F\|_\infty = o\left\{m^{-(1+b)}\right\}.
$$

It shows that, if Φ_X is of exponential and $\Phi_{\bar{\varepsilon}}$ is of algebraic type, the rate of uniform convergence is of exponential order. A special case occurs if $q = 1$ and $\alpha = \beta$. Then, by substitution we obtain

$$
\mathrm{ULB}(m) = \sigma^{\beta(m+1)} \frac{1}{\beta \pi} \int_0^\infty \frac{t^m}{(1+\sigma^\beta t)^{m+1}} e^{-\frac{\theta\beta}{p} t} dt.
$$

This integral equals a confluent hypergeometric function for which an asymptotic expansion was derived by means of Laplace's method, for instance in subsection 10.3.2 in [Temme, 2015]. Therefore, however, additional manipulations are required which are only viable because of the parameter restriction $\alpha = \beta$.

Despite we were only able to estimate the exact leading behaviour of the uniform bias function in certain cases, the preceding examples revealed the diversity of the possible results. Indeed, apparently the rate of uniform convergence can be very different, depending on the individual properties of the involved characteristic functions. A crucial factor is their local behaviour. However, particularly for scenarios in which the rate can be expected exponential we only obtained unsatisfactory results so far. It was not even ascertainable then, how the parameters affect the exact leading behaviour. These issues can be fixed if we do not confine to real but rather employ methods of complex analysis.

3.5. Chebyshev-Type Estimates for the Bias

We close this chapter with the presentation of another approximation method for the bias of the deconvolution function that does not claim accuracy but rather simplicity. Contrary to the approach of the preceding section the procedure to be presented below is applicable for arbitrary characteristic functions, regardless of their monotonicity, decay or zeros, provided the bias integral converges absolutely. It yields a concise overview on the controlling behaviour of the bias for large values of the asymptotic parameter m, dismissing subsequent terms of the full asymptotic expansion. The accuracy of the attainable estimates can be confirmed by comparison with the exact results to be derived in Chapter 5 below.

3.5.1. Background

Throughout this section we confine to the absolute bias of the increments given by (2.1.60), yet the procedure is equivalently applicable to (2.1.56). We remind the reader that the absolute convergence of this integral implies continuity of $\mathfrak{D}(\cdot, m)$ and of F. Our first step is the derivation of an integral representation for the m-power appearing in the bias integral. Therefore we first observe that, due to the properties of $\Phi_{\bar{\varepsilon}}(t)$, for any $t \in \mathbb{R}$ we may write

$$\mathcal{P}_{\bar{\varepsilon}}(t, m) = (m+1) \int_0^1 \mathbb{I}\left\{\Phi_{\bar{\varepsilon}}(t) \leq s\right\} (1-s)^m ds. \tag{3.5.1}$$

Hence, the single integral (2.1.60) can be separated into a double integral that is merely connected through an indicator:

$$\mathrm{ABI}(m, b-a) = (m+1)\frac{2}{\pi} \int_0^1 (1-s)^m \int_0^\infty \mathbb{I}\left\{\Phi_{\bar{\varepsilon}}(t) \leq s\right\} \frac{\left|\sin\left\{\frac{b-a}{2}t\right\}\right|}{t} \left|\Phi_X(t)\right| dt ds \tag{3.5.2}$$

Note that only the outer integral depends on the asymptotic parameter $m \geq 0$. In particular, this is evidently an integral of Laplace-type, so that the rate as $m \to \infty$ is determined from the neighborhood where $(1-s)^m$ attains its maximum value, which clearly corresponds to $s = 0$. Consequently, special attention needs to be put on the integral function

$$s \mapsto \int_0^\infty \mathbb{I}\left\{\Phi_{\bar{\varepsilon}}(t) \leq s\right\} \frac{\left|\sin\left\{\frac{b-a}{2}t\right\}\right|}{t} \left|\Phi_X(t)\right| dt \tag{3.5.3}$$

as $s \downarrow 0$. To assess the behaviour of this function as $s \downarrow 0$ we aim to find a precise estimate for the indicator such that the t-dependent part is separated multiplicatively from the s-dependent part. This in turn bounds (3.5.2) by a product of two single integrals with one of them being deterministic, whereas the second is a function of $m \geq 0$. Since the idea of finding an optimal bound for an indicator shares some similarities with the proof for Chebyshev's inequality, we refer to this approach as Chebyshev's method.

Under the assumption (2.1.54) it is clear, since $\mathbb{I}\left\{\Phi_{\bar{\varepsilon}}(t) \leq s\right\} \leq 1$, that (3.5.3) is uniformly bounded with respect to $s \in [0, 1]$. Furthermore, as $s \downarrow 0$ the range of integration shrinks to the set of t-values where $\Phi_{\bar{\varepsilon}}(t) = 0$. This requires to distinguish between three cases, which are $\Phi_{\bar{\varepsilon}}(t)$ vanishes nowhere or either on a discrete or on a continuous subset of $[0, \infty]$. In the first case the function (3.5.3) equals zero for some $s \geq 0$, while in the second case it vanishes only at $s = 0$. Finally, if $\Phi_{\bar{\varepsilon}}(t)$ equals zero on a non-discrete set, then (3.5.3) is positive for $s = 0$ if $\Phi_X(t) \neq 0$ there.

It is readily seen that the bound $\mathbb{I}\left\{\Phi_{\bar{\varepsilon}}(t) \leq s\right\} \leq 1$ applied to (3.5.2) does not only yield a finite bound for the absolute bias of the increments, but one that is independent of $m \geq 0$. This is, of course, undesired and leads to the question, which properties a useful bound should have. In the simple case $\inf_{t\geq 0}\Phi_{\bar{\varepsilon}}(t)\geq C$ for $C>0$, we may apply $\mathbb{I}\{\Phi_{\bar{\varepsilon}}(t)\leq s\}\leq \mathbb{I}\{C\leq s\}$ to (3.5.2) to verify an exponential rate for $\mathrm{ABI}(m,b-a)$ as $m\to\infty$. Hence, without loss of generality let $\inf_{t\geq 0}\Phi_{\bar{\varepsilon}}(t)=0$. Recalling that the main goal is the sophisticated manipulation of the inequality $\Phi_{\bar{\varepsilon}}(t)\leq s$, it is clear that equivalent inequalities can only be obtained by means of monotonic transformations. In particular, denoting by $T(x)\geq 0$ an increasing function of the variable $x\geq 0$, for $t\in[0,\infty)\setminus N_\varepsilon$ we have

$$\mathbb{I}\{\Phi_{\bar{\varepsilon}}(t)\leq s\}\leq \frac{T(s)}{T(\Phi_{\bar{\varepsilon}}(t))}. \tag{3.5.4}$$

This is true since the right hand side is always positive and especially ≥ 1 if and only if the indicator equals 1. However, not only monotonicity is crucial. For instance, of course

$$\mathbb{I}\{\Phi_{\bar{\varepsilon}}(t)\leq s\}\leq \exp\{s-\Phi_{\bar{\varepsilon}}(t)\},$$

but an application of this estimate yields another m-independent upper bound for $\mathrm{ABI}(m,b-a)$. Similar results should be expected for any transform $T(x)$ that is non-vanishing as $x\downarrow 0$. But since it was assumed that $\Phi_{\bar{\varepsilon}}(t)$ is not bounded away from zero, this in turn implies the unboundedness of the t-dependent denominator on the right hand side of (3.5.4). We therefore conclude that, if $\inf_{t\geq 0}\Phi_{\bar{\varepsilon}}(t)=0$, an appropriate transformation $T(x)$ of the functions in the indicator must reflect the maximum possible divergence of the reciprocal of $T(\Phi_{\bar{\varepsilon}}(t))$, which is compensable by the integrand of the dt-integral. This especially depends on the ratio between $\Phi_{\bar{\varepsilon}}(t)$ and $|\Phi_X(t)|$.

3.5.2. Main Result

The theorem below deals with the simplest monotonic transformations, i.e., logarithms and powers, which often yield sufficient bounds. Indeed, the subsequent discussion will show the applicability of this result to a large domain of characteristic functions. More extended estimates can be derived analogously from (3.5.2).

Theorem 3.5.1 (Chebyshev-type estimate). *Denote*

$$v(t,b-a):=\frac{2\left|\sin\left\{\frac{b-a}{2}t\right\}\right|}{\pi t}\left|\Phi_X(t)\right|.$$

If there exist $\beta_1,\beta_2\geq 0$ satisfying

$$I(b-a,\beta_1,\beta_2):=\int\limits_{[0,\infty)\setminus N_\varepsilon}\{\Phi_{\bar{\varepsilon}}(t)\}^{-\beta_1}\{-\log(\Phi_{\bar{\varepsilon}}(t)\}^{\beta_2}\,v(t,b-a)dt<\infty, \tag{3.5.5}$$

then as $m\to\infty$ we have

$$\mathrm{ABI}(m,b-a)=R(b-a)+\mathcal{O}\Big\{m^{-\beta_1}\{\log(m)\}^{-\beta_2}K(b-a,\beta_1,\beta_2)\Big\}, \tag{3.5.6}$$

where $R(b-a) := \int_{[0,\infty)\cap N_\varepsilon} v(t,b-a)dt$ *and* $K(b-a,\beta_1,\beta_2) := \Gamma(1+\beta_1)I(b-a,\beta_1,\beta_2)$.

If (3.5.5) remains true for some $\tilde{\beta}_j > \beta_j$, the big-$\mathcal{O}$ may be replaced by a small-o. Moreover, if the parameters β_1, β_2 can be chosen arbitrarily large, the bias exhibits an exponential-type behaviour for large m. In particular, the convergence then happens faster than any power of the respective term. Clearly, the admissibility of $\beta_1, \beta_2 \geq 0$ solely depends on the behaviour of $\Phi_{\bar{\varepsilon}}(t)$ and $v(t)$ at the points where the former function vanishes.

Proof. It is easy to see that $R(b-a)$ equals the remainder term of (3.5.2), i.e., the part that does not depend on m. It is therefore no restriction to assume $R(b-a) = 0$. Regarding the part that depends on m, we first observe that the left hand inequality below is equivalent to each inequality on the right hand side:

$$\Phi_{\bar{\varepsilon}}(t) \leq s \quad \Leftrightarrow \quad 1 \leq \begin{cases} \left\{\frac{s}{\Phi_{\bar{\varepsilon}}(t)}\right\}^{\beta_1}, & \text{for } \beta_1 \geq 0 \\ \left\{\frac{\log(\Phi_{\bar{\varepsilon}}(t))}{\log(s)}\right\}^{\beta_2}, & \text{for } \beta_2 \geq 0 \end{cases} \tag{3.5.7}$$

Furthermore, instead of just one indicator we can easily introduce finitely many additional indicators since $\mathbb{I}\{\Phi_{\bar{\varepsilon}}(t) \leq s\}\,\mathbb{I}\{\Phi_{\bar{\varepsilon}}(t) \leq s\} = \mathbb{I}\{\Phi_{\bar{\varepsilon}}(t) \leq s\}$. We thus obtain for $m+1 > \beta_2$ by applying each of the bounds (3.5.7) to (3.5.2):

$$\mathrm{ABI}(m,b-a) \leq I(b-a,\beta_1,\beta_2)(m+1)\int_0^1 (1-s)^m s^{\beta_1}\{-\log(s)\}^{-\beta_2}\,ds \tag{3.5.8}$$

If on the one hand $\beta_2 = 0$, the right hand side can be cast in terms of the beta function. Accompanied by an application of the functional equation for the gamma function, we then find

$$\mathrm{ABI}(m,b-a) \leq I(b-a,\beta_1,\beta_2)\frac{\Gamma(m+2)\Gamma(1+\beta_1)}{\Gamma(m+2+\beta_1)}.$$

By means of Stirling's formula, compare (B.2.30), this verifies the estimate (3.5.6). If on the other hand $\beta_2 > 0$, in (3.5.8) we express the reciprocal logarithm in terms of the gamma function and then make a reference to the beta function and its Mellin transform (4.7.19) to obtain:

$$\begin{aligned}\mathrm{ABI}(m,b-a) &\leq I(b-a,\beta_1,\beta_2)\frac{m+1}{\Gamma(\beta_2)}\int_0^\infty x^{\beta_2-1}\int_0^1 (1-s)^m s^{\beta_1+x}\,ds\,dx \\ &= I(b-a,\beta_1,\beta_2)\frac{m+1}{\Gamma(\beta_2)}\int_0^\infty x^{\beta_2-1}\frac{\Gamma(m+1)\Gamma(1+\beta_1+x)}{\Gamma(m+2+\beta_1+x)}dx \\ &= \beta_2 I(b-a,\beta_1,\beta_2)\mathcal{M}_{\mathrm{B}}(\beta_2,m,\beta_1)\end{aligned}$$

The Mellin transform of the beta function will be examined in Section 4.7 below. For the moment it suffices to know that in (4.7.54), as $m \to \infty$ it will be shown that

$$\beta_2 \mathcal{M}_{\mathrm{B}}(\beta_2, m, \beta_1) \sim \frac{\Gamma(m+2)\Gamma(1+\beta_1)}{\Gamma(m+2+\beta_1)} \left\{H_{m+1}(\beta_1)\right\}^{-\beta_2}.$$

But $H_{m+1}(a_1) \sim \log(m)$ as $m \to \infty$, according to (4.7.30) and according to the asymptotic properties of the digamma function, compare (B.2.29). An additional application of Stirling's formula (B.2.30) eventually verifies the estimate (3.5.6), which concludes the proof. ∎

3.5.3. Applied Examples

We close this section with a discussion of several examples to illustrate the applicability of the preceding theorem.

Example 3.5.1. Consider $\Phi_X(t)$ as in (3.3.1) so that $v(t, b-a) \sim \theta^{-\alpha p} t^{-1-\alpha p}$ as $t \to \infty$. If $\Phi_{\bar{\varepsilon}}(t)$ is then given by (3.3.2) it does not vanish at a finite point and satisfies $\Phi_{\bar{\varepsilon}}(t) \sim \theta^{-\beta q} t^{-\beta q}$ as $t \to \infty$. Finiteness of the integral (3.5.5) is thus assured for arbitrary $0 < \beta_1 < (\beta q)^{-1} \alpha p$ and $\beta_2 = 0$. Hence, from (3.5.6), as $m \to \infty$ we deduce

$$\text{(3.5.9)} \qquad \mathrm{ABI}(m, b-a) = o\left\{m^{-\beta_1} K(b-a, \beta_1, 0)\right\}, \qquad 0 < \beta_1 < \frac{\alpha p}{\beta q}.$$

Furthermore, if $\Phi_{\bar{\varepsilon}}(t)$ is given by (3.3.4) we choose $\beta_1 = 0$. Finiteness of (3.5.5) is then guaranteed for $0 < \beta_2 < \beta^{-1} \alpha p$, and, according to (3.5.6), as $m \to \infty$ we obtain

$$\text{(3.5.10)} \qquad \mathrm{ABI}(m, b-a) = o\left\{\{\log(m)\}^{-\beta_2} K(b-a, 0, \beta_2)\right\}, \qquad 0 < \beta_2 < \frac{\alpha p}{\beta}.$$

Finally suppose $\Phi_X(t)$ matches (3.3.3) and $\Phi_{\bar{\varepsilon}}(t)$ equals (3.3.2). It is then easy to see that (3.5.5) holds for any $\beta_1, \beta_2 \geq 0$, indicating an exponential-type rate of the form e^{-m^γ} for some $\gamma > 0$. Already in Section 3.4 we derived asymptotic estimates for the functions that were considered here. A comparison of these results with (3.5.9) and (3.5.10) shows the closeness of the respective estimates. Full and exact asymptotic expansions for the functions from this particular example will be deduced in Chapter 5 below.

Example 3.5.2 (products of two characteristic functions). Let

$$\text{(3.5.11)} \qquad \Phi_{\bar{\varepsilon}}(t) = \left\{1 + \sigma_{01}^{\beta_{01}} |t|^{\beta_{01}}\right\}^{-q_{01}} \exp\left\{-q_{02} \sigma_{02}^{\beta_{02}} |t|^{\beta_{02}}\right\}$$

for $\sigma_{01}, \sigma_{02}, \beta_{01}, \beta_{02}, q_{01}, q_{02} > 0$, i.e., $\Phi_{\bar{\varepsilon}}(t)$ equals the product of (3.3.2) and (3.3.4). Moreover, let $\Phi_X(t)$ be of algebraic type (3.3.1). Clearly, since $v(t, b-a) \sim \theta^{-\alpha p} t^{-\alpha p - 1}$ as $t \to \infty$ the condition (3.5.5) holds only if $\beta_1 = 0$ but $0 < \beta_2 < \beta_{02}^{-1} \alpha p$. Therefore we again obtain the estimate (3.5.10) with the indicated range for β_2. In accordance with these observations, we expect that the faster decay of $\Phi_{\bar{\varepsilon}}(t)$ due to the additional algebraic factor does not essentially decrease the rate compared to the case where $\Phi_{\bar{\varepsilon}}(t)$ equals a single exponential function.

Example 3.5.3 (countable number of zeros). Suppose $\Phi_{\bar{\varepsilon}}(t)$ vanishes at a countable set of points along the positive real axis, i.e., $N_\varepsilon \cap [0,\infty) = \{t_k : k \in I\}$ for some $I \subset \mathbb{N}$ either finite or infinite. More precisely suppose for $b_k, r_k > 0$ we have

$$\Phi_{\bar{\varepsilon}}(t) \sim b_k \left|t - t_k\right|^{r_k} \qquad \text{as } t \to t_k,$$

and if $\infty \in N_\varepsilon$, for $b_\infty, r_\infty > 0$, we have

$$\Phi_{\bar{\varepsilon}}(t) \sim b_\infty t^{-r_\infty} \qquad \text{as } t \to \infty.$$

Clearly, regarding the finiteness of the integral (3.5.5) we need not only incorporate the integrand's behaviour as $t \to \infty$ but especially the behaviour around the zeros of $\Phi_{\bar{\varepsilon}}(t)$. Hence, for $a_k, a_\infty \in \mathbb{R} \setminus \{0\}$, $p_k \geq 0$ and $p_\infty > 0$ assume

$$\begin{aligned} v(t, b-a) &\sim a_k \left|t - t_k\right|^{p_k} \qquad \text{as } t \to t_k, \\ v(t, b-a) &\sim a_\infty t^{-p_\infty} \qquad \text{as } t \to \infty. \end{aligned}$$

Note that the powers p_k depend on a, b since the zeros of the sine function appearing as a factor in $v(t, b-a)$ depend on $b-a$. Finally denote

$$r^* := \min_{k \in I} \left\{ \frac{1 + p_k}{r_k} \right\}.$$

Then, for $0 < \beta_1 < r^*$ and any $k \in I$ we have

$$v(t, b-a) \left\{\Phi_{\bar{\varepsilon}}(t)\right\}^{-\beta_1} \sim a_k b_k^{-\beta_1} \left|t - t_k\right|^{p_k - r_k \beta_1} \qquad \text{as } t \to t_k, \tag{3.5.12}$$

where $p_k - r_k\beta_1 > -1$. The right hand side of (3.5.12) is thus locally integrable on $[0,\infty)$. Moreover, if $\infty \in N_\varepsilon$ we have

$$v(t, b-a) \left\{\Phi_{\bar{\varepsilon}}(t)\right\}^{-\beta_1} \sim a_\infty b_\infty^{-\beta_1} t^{r_\infty \beta_1 - p_\infty - 1} \qquad \text{as } t \to \infty. \tag{3.5.13}$$

The right hand side is absolutely integrable on $[\kappa, \infty)$ for any $\kappa > 0$, provided $\beta_1 < \frac{p_\infty}{r_\infty}$. From Theorem 3.5.1 with $\beta_2 = 0$, as $m \to \infty$, we thereby conclude

$$\mathrm{ABI}(m, b-a) = o\left\{m^{-\beta_1}\right\}, \qquad 0 < \beta_1 < \min\left\{r^*, \frac{p_\infty}{r_\infty}\right\}. \tag{3.5.14}$$

It shows, if $\Phi_{\bar{\varepsilon}}(t)$ vanishes at some finite points, the rate of the absolute bias of increments can only be improved if $\Phi_X(t)$ also vanishes at these points. More precisely, a superior rate requires that $\Phi_X(t)$ vanishes at all these points. If this is violated for only one $t_k \in N_\varepsilon$, the estimate remains the same as for a characteristic function $\Phi_X(t)$ that is non-vanishing except at infinity.

4. Mellin Transforms and their Applications in Asymptotics

The preceding chapter suggests that the applications of real analysis are often insufficient to evaluate the bias integrals corresponding to the deconvolution function. Hence, we will now switch to complex analysis, which endows us with more efficient tools to handle these integrals. In particular, we shall soon appreciate the importance of Mellin transforms. These will eventually enable us to derive full expansions that describe the asymptotic behaviour of the bias for selected examples. For a brief overview on the properties of Mellin transforms we refer the reader to Appendix A.5. A more extensive treatment is provided by the monographs [Titchmarsh, 1937] and [Paris and Kaminski, 2001] of which the latter will be our more important reference. Finally the paper [Fikioris, 2006] concisely outlines the applicability of Mellin transforms for the asymptotic evaluation of integral functions.

4.1. An Introduction to the Method of Mellin Transforms

Mellin transforms constitute a very powerful tool, not only for the solution of integral equations but especially in asymptotics, as was established in the early 20th century by mathematicians like R. H. Mellin and E. W. Barnes. In comparison to Laplace's method it is thus a fairly new technique, bearing the advantage that its application is not restricted to a certain class of integrals. The basic idea relies on the fact that the asymptotic behaviour of a given integral essentially depends on the structure and particularly on the singularities of the analytic continuation corresponding to the Mellin transforms of the integrand. Therefore the first step of the method of Mellin transforms is always, to express a given integral function in terms of the eponymous transforms, which results in a special contour integral. Subsequent manipulations of the integrand and a sophisticated deformation of the integration path finally lead to an equivalent representation. This is typically a series expansion, either convergent or formal only. Such a series features, for example, descending powers of the asymptotic parameter. Generally, however, the structure of an expansion obtainable by means of the method of Mellin transforms can be much more intricate and it need not be a power series of the asymptotic parameter. Indeed, the method is especially able to generate expansions for integrals whose exact asymptotic behaviour can not be described by plain algebraic functions.

As a simple illustrative example for $\lambda > 0$ we shall consider the Laplace transform

$$I_\lambda := \int_0^\infty \frac{e^{-\lambda t}}{1+t} dt. \tag{4.1.1}$$

Assuming on the one hand we were interested in its behaviour as $\lambda \to \infty$, the main contribution to the integral comes from a neighborhood of the point $t = 0$, where the integrand attains its maximum value. If on the other hand we were interested in the behaviour of I_λ as $\lambda \downarrow 0$, the large t-values were of special importance since the integrand attains its minimum value as $t \to \infty$. The standard approach for the derivation of exact asymptotic statements about the integral I_λ is probably Laplace's method. There is, however, another approach that is even applicable for the investigation of the asymptotic behaviour of infinite sums, exploiting the possibility to write the exponential function as an inverse Mellin transform. That is in particular, according to (A.5.8), for $t > 0$ and $c > 0$ we have validity of the so-called *Cahen-Mellin representation*

$$e^{-\lambda t} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(z)(\lambda t)^{-z} dz. \tag{4.1.2}$$

In accordance with the exponential decay of the gamma function in the imaginary direction, this integral is absolutely convergent. More precisely, if we write $z = x + iy$ for $x, y \in \mathbb{R}$, subject to Stirling's formula, as $|z| \to \infty$ in the sector $|\arg(z)| < \pi$ the modulus of the integrand satisfies

$$|\Gamma(x+iy)| \, (\lambda t)^{-x} \sim \sqrt{2\pi}(x^2+y^2)^{\frac{x}{2}-\frac{1}{4}}(\lambda t)^{-x} e^{-x-y\arg(x+iy)}. \tag{4.1.3}$$

Note that for fixed $x \in \mathbb{R}$ the argument function on the right hand side tends to $\pm\frac{\pi}{2}$, respectively as $y \to \pm\infty$. If we apply (4.1.2) to the integral (4.1.1) a formal interchange in the order of integration yields

$$I_\lambda = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(z)\lambda^{-z} \int_0^\infty \frac{t^{-z}}{1+t} dt dz. \tag{4.1.4}$$

The set of admissible values for $c > 0$ depends on the local behaviour of the integrand of the interior integral as $t \downarrow 0$ and as $t \to \infty$, respectively. In the former direction it is $\mathcal{O}(t^{-x})$, whereas in the latter direction we have $\mathcal{O}(t^{-x-1})$. Hence, the widest common region where (4.1.2) holds for $x = c$ and the double integral (4.1.4) converges absolutely is the strip $0 < x < 1$. By Fubini's theorem the transition from (4.1.1) to (4.1.4) is thus valid for

$$0 < c < 1. \tag{4.1.5}$$

By inspection of (4.1.4) we immediately see that

$$I_\lambda \leq \lambda^{-c} \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Gamma(c+iy)|\, dy \int_{0}^{\infty} \frac{t^{-c}}{1+t} dt. \tag{4.1.6}$$

This yields a first estimate for the behaviour of I_λ with respect to λ and shows $I_\lambda = \mathcal{O}(\lambda^{-c})$ for arbitrary $0 < c < 1$ and $\lambda > 0$. More accurate statements require to examine the interior integral in (4.1.4) first. In the present setup it is readily identified as the beta function $B(z, 1-z)$, compare Example A.5.2, and by additional use of the identity (B.3.2) we can write

$$I_\lambda = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \{\Gamma(z)\}^2\, \Gamma(1-z) \lambda^{-z} dz. \tag{4.1.7}$$

The situation in this setup is particularly convenient and if the interior integral in (4.1.4) had a slightly different form it could possibly not be represented in terms of known special functions. Preliminary to a discussion of the asymptotic properties of I_λ it was then necessary to determine the properties of the analytic continuation associated with the interior integral. For the beta function these are already available.

In the literature the direct transition from (4.1.1) to (4.1.7) is referred to as Parseval's formula for the Mellin transform, compare eq. (3.1.4) in [Paris and Kaminski, 2001]. Moreover, integrals of the type (4.1.7) are called *Mellin-Barnes integrals*, occasionally abbreviated as MB-integrals. The notion refers to the two aforementioned pioneers in this topic. MB-integrals are special contour integrals, characterized by the presence of at least one gamma function in the integrand and by an integration path that partitions the complex plane in two regions without crossing any singularities of the integrand. If the integrand only involves gamma functions with the possible exception of an exponential term, a MB-integral is particularly of hypergeometric type. This is the case in (4.1.7). The integration path therein evidently separates the simple poles of $\Gamma(1-z)$, which are located at the positive integers, from the double poles of the squared gamma function at the non-positive integers.

Suppose now we are interested in the asymptotic behaviour of (4.1.7) as $\lambda \to \infty$. In other words, we are looking for an expansion that is descending with respect to the terms that depend on λ, i.e., an expansion in which each term tends to zero as $\lambda \to \infty$ with a slower rate than the subsequent terms. The fact that the integral (4.1.7) is $\mathcal{O}(\lambda^{-x})$ as $\lambda \to \infty$, compare (4.1.6), suggests that we should pay special attention to the region $x > 0$ since the big-$\mathcal{O}$ estimate decays faster as $\lambda \to \infty$ if the fixed value x is chosen larger. Therefore we consider a finite rectangle in the right z-half plane with edges of integer length $N_1, N_2 \geq 1$, encircled in the clockwise direction. Especially since the boundary of this rectangle does not run through any singularity of the integrand, the residue theorem and (B.2.20) yield:

$$\frac{1}{2\pi i}\left[\int\limits_{c-i\frac{N_2}{2}}^{c+i\frac{N_2}{2}} + \int\limits_{c+i\frac{N_2}{2}}^{c+N_1+i\frac{N_2}{2}} + \int\limits_{c+N_1+i\frac{N_2}{2}}^{c+N_1-i\frac{N_2}{2}} + \int\limits_{c+N_1-i\frac{N_2}{2}}^{c-i\frac{N_2}{2}}\right]\{\Gamma(z)\}^2\,\Gamma(1-z)\lambda^{-z}dz$$
$$= \sum_{n=0}^{N_1-1}\frac{(-1)^n}{n!}\{\Gamma(n+1)\}^2\,\lambda^{-1-n} \tag{4.1.8}$$

Now, for any $x \geq 0$ and $y \in \mathbb{R}$ such that $z \notin \mathbb{N}_0$, by virtue of the reflection formula (B.2.15) and inequalities (4.18.5) in [Olver et al., 2010] and (B.2.32) in the appendix we deduce the following bound:

$$\begin{aligned}\left|\{\Gamma(z)\}^2\,\Gamma(1-z)\lambda^{-z}\right| &= \pi\lambda^{-x}\left|\frac{\Gamma(z)}{\sin(\pi z)}\right| \\ &\leq \sqrt{2}\pi^{\frac{3}{2}}\frac{\lambda^{-x}}{|\sinh(\pi y)|}(x^2+y^2)^{\frac{x}{2}-\frac{1}{4}}e^{-\frac{\pi}{2}|y|+\frac{1}{6\sqrt{x^2+y^2}}}\end{aligned} \tag{4.1.9}$$

By comparison with the exact result we obtained in (4.1.3) from Stirling's formula, the reader may confirm the accuracy of the above bound. It shows that the integrand in (4.1.8) exhibits exponential decay as $y \to \pm\infty$ for fixed $x \geq 0$. Moreover, if we employ the above estimate to the integral along the upper edge of the rectangle we obtain for $\lambda > 1$:

$$\begin{aligned}\left|\int\limits_{c+i\frac{N_2}{2}}^{c+N_1+i\frac{N_2}{2}}\{\Gamma(z)\}^2\,\Gamma(1-z)\lambda^{-z}dz\right| &\leq \int\limits_{c}^{c+N_1}\left|\Gamma\left(x+i\frac{N_2}{2}\right)\right|^2\left|\Gamma\left(1-x-i\frac{N_2}{2}\right)\right|\lambda^{-x}dx \\ &\leq \frac{\sqrt{2}\pi^{\frac{3}{2}}}{\left|\sinh\left\{\pi\frac{N_2}{2}\right\}\right|}e^{-\frac{\pi}{2}N_2+\frac{1}{6c}}\int\limits_{c}^{c+N_1}(x^2+N_2^2)^{\frac{x}{2}-\frac{1}{4}}\lambda^{-x}dx \\ &\leq \frac{\sqrt{2}\pi^{\frac{3}{2}}N_1\lambda^{-c}}{\left|\sinh\left\{\pi\frac{N_2}{2}\right\}\right|}e^{-\frac{\pi}{2}N_2+\frac{1}{6c}}((c+N_1)^2+N_2^2)^{\frac{c+N_1}{2}-\frac{1}{4}}\end{aligned}$$

Hence, as $N_2 \to \infty$, this integral vanishes. A similar bound applies to the integral along the lower edge. Taking into account that $\Gamma(n+1) = n!$, from (4.1.8) we thus arrive at:

$$\frac{1}{2\pi i}\left[\int\limits_{c-i\infty}^{c+i\infty} + \int\limits_{c+N_1+i\infty}^{c+N_1-i\infty}\right]\{\Gamma(z)\}^2\,\Gamma(1-z)\lambda^{-z}dz = \sum_{n=0}^{N_1-1}(-1)^n n!\lambda^{-1-n} \tag{4.1.10}$$

Identifying on the left hand side the integral along the vertical line $x = c$ as our integral (4.1.7), we eventually obtain

$$I_\lambda = \sum_{n=0}^{N_1-1}(-1)^n n!\lambda^{-1-n} + \frac{1}{2\pi i}\int\limits_{c+N_1-i\infty}^{c+N_1+i\infty}\{\Gamma(z)\}^2\,\Gamma(1-z)\lambda^{-z}dz. \tag{4.1.11}$$

Concerning the remainder integral the following bound applies:

$$\left| \int\limits_{c+N_1-i\infty}^{c+N_1+i\infty} \{\Gamma(z)\}^2 \Gamma(1-z)\lambda^{-z}dz \right| \leq \pi\lambda^{-c-N_1} \int\limits_{-\infty}^{\infty} \frac{|\Gamma(c+N_1+iy)|}{|\sin(\pi(c+N_1+iy))|}dy \tag{4.1.12}$$

By comparison with (4.1.9) we readily confirm the finiteness of this upper bound. This shows the absolute convergence of the remainder integral and in addition that it is $\mathcal{O}(\lambda^{-c-N_1})$ as $\lambda \to \infty$. But in the preceding sum in (4.1.11) the n-th term is $\sim$ const $\times\, \lambda^{-1-n}$ for $0 \leq n \leq N_1 - 1$. Since $0 < c < 1$ we have eventually verified (4.1.11) as an asymptotic expansion as $\lambda \to \infty$ of I_λ. It is particularly of Poincaré-type, compare §1.1.2 in [Paris and Kaminski, 2001], because by arbitrariness of $N_1 \in \mathbb{N}$ the preceding findings can be summarized in the form

$$I_\lambda = \sum_{n=0}^{N_1-1} (-1)^n n! \lambda^{-1-n} + \mathcal{O}\left\{\lambda^{-N_1-1}\right\}. \tag{4.1.13}$$

Equivalently it is also common to write

$$I_\lambda \sim \sum_{n=0}^{\infty} (-1)^n n! \lambda^{-1-n}, \tag{4.1.14}$$

with equality if and only if the series on the right hand side converges absolutely. In the present example this is not the case, due to the fast growth of the factorial function. The lack of convergence is also seen from the remainder integral (4.1.11), whose integrand grows as N_1 increases. Moreover, the growth implies, if N_1 is chosen too large in relation to the magnitude of the asymptotic parameter λ, the contribution from the remainder integral in (4.1.11) becomes predominant.

By inspection of (4.1.11) we observe that the remainder integral is of similar type as the initial integral (4.1.7), the main difference being the shifted integration path. Therefore, in situations where the asymptotic behaviour of the integrand is sufficiently fast, for instance of exponential order, we omit routine and elaborate calculations. Instead the transition from (4.1.7) to (4.1.11) is then referred to as a rightward displacement of the integration path from the vertical line $x = c$ to match the line $x = c + N_1$. Alternatively, which is ascertainable from the above computations, this step can also be described as an encirclement of the poles in the negative or clockwise direction by a rectangle of infinite height.

In the same fashion the method of Mellin transforms can be employed to derive an asymptotic expansion for I_λ as $\lambda \downarrow 0$. The estimate (4.1.6), however, suggests that we should therefore investigate the possibility for a leftward displacement of the integration path in (4.1.7) rather than a movement to the right direction. Indeed, since the integral is $\mathcal{O}(\lambda^{-x})$ it shows a faster decay as $\lambda \downarrow 0$ for larger negative values of the real part of z. In each case, as $\lambda \to \infty$ or as $\lambda \downarrow 0$, the function λ^{-z} in the integral (4.1.7) plays the role of an *asymptotic scale*, compare §1.1.2

in [Paris and Kaminski, 2001]. To explain this notion assume $\phi_0(\lambda), \phi_1(\lambda), \ldots$ is a sequence of functions with respect to $\lambda \in \Lambda$, where Λ is a subregion of the complex plane. In addition, denote by λ_0 a limit point thereof, which is possibly infinite. Then the sequence $\phi_0(\lambda), \phi_1(\lambda), , \ldots$ establishes an asymptotic scale as $\lambda \to \lambda_0$ if the ratio of two subsequent terms tends to zero, i.e., if $\phi_{n+1}(\lambda) = o\{\phi_n(\lambda)\}$ for all $n \in \mathbb{N}_0$. The function λ^{-z} is of most frequent occurence to define an asymptotic scale, for instance with $z \in \mathbb{N}_0$ or $z \in -\mathbb{N}_0$, respectively as λ approaches infinity or zero. Another simple example is given by $(\log \lambda)^{-n}$ for $n \in \mathbb{N}_0$ as $\lambda \to \infty$. With the above convention it is reasonable to speak of (4.1.14) as an expansion of I_λ with respect to the asymptotic scale λ^{-n} for $n \in \mathbb{N}_0$.

Basically our approach to adopt the method of the Mellin transform for the evaluation of the bias integrals associated with the deconvolution function does not differ very much from the procedure described in the above example. Once we have chosen which of the integral representations that were given in Corollary 2.1.4 for the bias to investigate, we recast it as a MB-integral, involving the Mellin transforms of the ingredients. Those are in any case the Mellin transform corresponding to the m-power, representing the part that depends on the asymptotic parameter, and the Mellin transform of the characteristic function Φ_X or its modulus. Clearly, the method of Mellin transforms requires information about the behaviour of the ingredient functions along the real axis. Only under detailed assumptions it is possible to specify singularities and asymptotic behaviour of the analytic continuation associated with the involved Mellin transforms. Therefore we distinguish between different classes of characteristic functions. For the sake of clarity we mostly confine our first study to those of absolutely continuous distributions, in particular to the functions with simple algebraic and exponential behaviour that were presented in Section 3.3. We will see below that it is then possible to refer the corresponding Mellin transforms to known special functions. Preliminary we discuss some general properties of the Mellin transforms associated with the characteristic function of the random variable X and the m-power.

4.2. The Mellin Transform of a Characteristic Function

When working with the uniform bias function we frequently encounter the modulus of the characteristic function Φ_X. Throughout our investigations we define the corresponding Mellin transform by

$$M_X(\zeta) := \int_0^\infty t^{\zeta-1} |\Phi_X(t)| \, dt \qquad \text{for } \zeta \in S_X, \tag{4.2.1}$$

where S_X denotes the associated strip of analyticity., i.e., the region of the complex plane where the integral represents an analytic function of ζ. As a characteristic function, Φ_X is naturally continuous along the real axis, whence the uniform convergence of the integral (4.2.1) in any compact subset of a region of the complex plane suffices to conclude $S_X \neq \emptyset$, compare Theorem A.2.1. The latter happens to hold if for some $\delta_X > 0$ as $t \to \infty$ we have

$$\Phi_X(t) = \mathcal{O}\left\{t^{-\delta_X}\right\}. \tag{4.2.2}$$

Then, since $\Phi_X(0) = 1$, the integral (4.2.1) converges absolutely for $0 < \Re\zeta < \delta_X$. Moreover, if E denotes a compact subset of this strip with $a_0 := \min\{\Re\zeta : \zeta \in E\}$ and $b_0 := \max\{\Re\zeta : \zeta \in E\}$, for $\zeta \in E$, a constant $A > 0$ and a fixed $T > 1$ the following estimate applies:

$$\begin{aligned}
|M_X(\zeta)| &\leq \max_{0\leq r\leq T} |\Phi_X(r)| \int_0^T t^{\Re\zeta - 1} dt + A \int_T^\infty t^{\Re\zeta - \delta_X - 1} dt \\
&\leq \max_{0\leq r\leq T} |\Phi_X(r)| \frac{T^{\Re\zeta}}{\Re\zeta} + A \int_T^\infty t^{b_0 - \delta_X - 1} dt \\
&= \max_{0\leq r\leq T} |\Phi_X(r)| \frac{T^{b_0}}{a_0} + A \frac{T^{b_0 - \delta_X}}{\delta_X - b_0}
\end{aligned}$$

Hence, the integral (4.2.1) converges uniformly in E, and by arbitrariness we deduce

$$S_X = \{\zeta \in \mathbb{C} : 0 < \Re\zeta < \delta_X\}. \tag{4.2.3}$$

From (4.2.2) we ascertain that exponential decay of Φ_X implies that S_X matches the right ζ-half plane. Now, according to the inversion formula for Mellin transforms, compare Theorem A.5.1 in the appendix, under appropriate conditions

$$|\Phi_X(t)| = \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} t^{-z} M_X(z) dz, \qquad \text{for } t > 0,\ x_0 \in S_X. \tag{4.2.4}$$

Particularly if (4.2.2) applies, the inverse Mellin integral converges absolutely provided $M_X(x_0 + iy) \in L^1(\mathbb{R})$. In the examples to be studied below this will always be the case. Since (4.2.1) constitutes a common Mellin transform, its continuation can be characterized by known results or it can even be expressed in terms of well-studied special functions. A remarkable and possibly helpful tool in this context might be Ramanujan's master theorem which has been thoroughly discussed in [Hardy, 1937].

The characteristic function Φ_X without the modulus rather occurs in the context of the local bias. We denote the corresponding Mellin transform in the standard fashion of Appendix A.5 by $\mathcal{M}\{\Phi_X\}(\zeta)$ with strip of analyticity $S_{\mathcal{M}}\{\Phi_X\}$. If $\Phi_X = |\Phi_X|$ we write $\mathcal{M}_X = M_X$ and $S_X = S\{\Phi_X\}$. The properties of the Mellin transforms $\mathcal{M}\{\Phi_X\}(\zeta)$ and (4.2.1) mostly differ with respect to effects that are caused by possible oscillatory behaviour of Φ_X. Indeed, the above estimates also verify $\mathcal{M}\{\Phi_X\}(\zeta)$ analytic in $0 < \Re\zeta < \delta_X$ if (4.2.2) holds. Finally, contrary to its modulus the characteristic function of X can be specified in terms of its distribution function, which was investigated in Appendix A.5.2.

4.3. The Mellin Transform of the m-Power

The discussion of the Mellin transform of the m-power is intimately connected with the properties of binomial series. Therefore it is only natural to start with a short introduction to this kind of series.

4.3.1. Properties of Binomial Series

A binomial series, also known as Newton series, is generally of the form

$$S(m) := \sum_{l=0}^{\infty} \binom{m}{l} (-1)^l f(l), \qquad m \in \mathbb{C}, \tag{4.3.1}$$

where f denotes the coefficient function. Particularly for $m \in \mathbb{N}_0$ it is known that (4.3.1) cancels to a finite expression and is then referred to as a binomial sum. In their simplest form binomial sums are already encountered in school. There, usually $f(l) = t^l$ for some $t \in \mathbb{R}$ and $m = 2$, i.e. the binomial sum equals a polynomial of low degree. However, the series (4.3.1) with $f(l) = t^l$ is also absolutely convergent for $m \in \mathbb{C}$ and complex $|t| < 1$. By Satz 245 in [Knopp, 1976] it then equals the Taylor expansion

$$(1-t)^m = \sum_{l=0}^{\infty} \binom{m}{l} (-t)^l, \qquad |\arg(1-t)| < \pi. \tag{4.3.2}$$

Moreover, subject to Satz 247 in [Knopp, 1976] this series remains absolutely convergent on the boundary of the unit circle, provided $\Re m > 0$. In these circumstances, according to Abel's theorem for sums, from (4.3.2) we conclude

$$\sum_{l=0}^{\infty} \binom{m}{l} = 2^m, \qquad \Re m > 0. \tag{4.3.3}$$

Without loss of generality assume real-valuedness of the parameter m. As a consequence of (4.3.3), binomial series with $m > 0$ converge absolutely for arbitrary coefficient functions $f(l)$ which are uniformly bounded with respect to $l \in \mathbb{N}_0$. Then, however, a suitable approach for their evaluation is not obvious. In fact, there are only few known formulae yielding a finite expression for the sum of the series. Best-known among those are the aforementioned binomial theorem (4.3.2) and the Gaussian summation formula[1]. Regarding more general coefficient functions, further drawbacks of the representation (4.3.1) arise in the context of an asymptotic analysis with respect to m. First, it is not immediately possible to infer the leading behaviour

[1] Some binomial series can be cast as a hypergeometric function of which the Gauss hypergeometric function is best known. The summation formula mentioned in the text holds for the latter and can be found as equation (15.4.20) in [Olver et al., 2010].

as $m \to \infty$ since rather than descending the binomial coefficient is ascending with respect to m. Actually, as we shall see below, the large m-behaviour varies a lot and sensitively depends on the particular form of f. Both can occur, convergence or divergence. Second, when it comes to calculation the fast growth of the binomial coefficient substantially impedes a certain precision for large m. Fortunately all of these issues can be fixed by transforming the sum into a contour integral. In fact, actually a binomial series equals nothing but a sum of residues, provided f possesses an analytic counterpart. This means, there exists a function $A(-z)$ with $z \in \mathbb{C}$, which satisfies $A(l) = f(l)$ and is analytic in a region of the complex plane containing for a fixed $0 < r < 1$ the sequence of circles

$$C_{r,l} = \left\{-l + re^{i\phi} : -\pi \leq \phi \leq \pi\right\}, \qquad l \in \mathbb{N}_0. \tag{4.3.4}$$

It is especially allowed that $A(-z)$ has singularities at negative non-integer points. Similar to $f(l)$ we refer to $A(-z)$ as a coefficient function. At this point, however, we must emphasize that $A(-z)$ is not the same as an analytic continuation and is therefore not unique, since it coincides with f only on a discrete set. For instance, with $A(-z)$ also $A^*_n(-z) := A(-z)e^{i2\pi nz}$ for arbitrary $n \in \mathbb{Z}$ are coefficient functions. Although it is not wrong to employ $A^*_n(-z)$ instead of $A(-z)$, regarding further manipulations, it incurs unnecessary obstacles. Fortunately, in practice the appropriate choice is usually evident.

According to the residue theorem, upon writing the factorials in the binomial coefficient in terms of the gamma function, under the above assumptions for any $m \in \mathbb{R} \setminus -\mathbb{N}$ we may cast the l-th summand of (4.3.1) in the form

$$\frac{\Gamma(m+1)}{\Gamma(m+1-l)}\frac{(-1)^l}{l!}f(l) = \frac{1}{2\pi i}\oint\limits_{C_{r,l}} \frac{\Gamma(m+1)\Gamma(z)}{\Gamma(m+1+z)}A(-z)dz. \tag{4.3.5}$$

The integration path therein encircles the pole at $z = -l$ in the counterclockwise direction. Note that the ratio of gamma functions in the integrand equals the beta function $B(m+1, z)$, compare (B.3.2). An application of the asymptotic estimate (B.3.6), for fixed $m \in \mathbb{R}$, as $|z| \to \infty$ in $|\arg(z)| < \pi$ exposes the leading behaviour

$$\left|\frac{\Gamma(z)}{\Gamma(m+1+z)}\right| = \mathcal{O}\left\{|z|^{-m-1}\right\}. \tag{4.3.6}$$

Particularly if $m > -1$, this estimate indicates algebraic decay of the ratio of gamma functions in the cut complex z-plane. It is therefore no restriction to even permit coefficient functions with algebraic growth. More generally, assume $A(-z) = \mathcal{O}\left\{|z|^c\right\}$ as $\Re z \to \infty$ for some $c \geq 0$ in a region containing the sequence of circles $C_{r,l}$ and choose $m > c$ fix. Then, for sufficiently large $l \in \mathbb{N}$ there exists a constant $K > 0$ such that the following bound applies to the integral

(4.3.5):

$$
\begin{aligned}
\left| \oint_{C_{r,l}} \frac{\Gamma(m+1)\Gamma(z)}{\Gamma(m+1+z)} A(-z) dz \right| &\leq Kr \int_{-\pi}^{\pi} \left| -l + re^{i\phi} \right|^{c-m-1} d\phi \\
&\leq Kr \int_{-\pi}^{\pi} |r\cos(\phi) - l|^{c-m-1} d\phi \\
&\leq 2\pi r (l-r)^{c-m-1}
\end{aligned}
$$

As a consequence we have absolute convergence for any $m \in \mathbb{R} \setminus -\mathbb{N}$ with $m > c$ of the series representation

$$
S(m) = \frac{1}{2\pi i} \sum_{l=0}^{\infty} \oint_{C_{r,l}} \frac{\Gamma(m+1)\Gamma(z)}{\Gamma(m+1+z)} A(-z) dz. \tag{4.3.7}
$$

Basically the idea of transforming binomial series into complex integrals dates back to N. E. Noerlund[2]. Accordingly, in the literature expressions of the form (4.3.7) are referred to as *Noerlund-Rice integrals*. Recall that the integration paths $C_{r,l}$ constitute a sequence of closed contours, encircling the poles of $\Gamma(z)$ in the counterclockwise direction, excluding possible singularities of $A(-z)$. In accordance with Cauchy's formula, the analyticity of $A(-z)$ allows us for fixed $N \in \mathbb{N}$ to coalesce the circles $C_{r,l}$ for $0 \leq l \leq N-1$ to a single integration path, for instance to an oval $\mathcal{O}_N$ with the first N non-positive integers in its interior. This oval is encircled in the counterclockwise direction and excludes possible singularities of $A(-z)$. Subject to the absolute convergence of the series (4.3.7) for fixed $m > c$, for a given $\delta > 0$ we can choose $N \in \mathbb{N}$ large enough to guarantee

$$
\left| \frac{1}{2\pi i} \sum_{l=N}^{\infty} \oint_{C_{r,l}} \frac{\Gamma(m+1)\Gamma(z)}{\Gamma(m+1+z)} A(-z) dz \right| < \delta.
$$

At the same time, as N grows, the oval $\mathcal{O}_N$ approaches a loop. The series (4.3.7) can therefore be cast as an integral of the form

$$
S(m) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \frac{\Gamma(m+1)\Gamma(z)}{\Gamma(m+1+z)} A(-z) dz, \qquad m > c. \tag{4.3.8}
$$

This new integration path is in particular a possibly indented loop starting at $\infty e^{-i\pi}$, encircling the negative real axis at $z = 0$ counterclockwise and returning to $\infty e^{i\pi}$, separating the poles of $\Gamma(z)$ from possible singularities of $A(-z)$. Integration paths in the described form are also

[2]Noerlund worked with binomial sums in the context of finite differences. He introduced an integral representation for those sums in equation (6) on p. 199 in [Noerlund, 1924].

known as Hankel contours, see Fig. 4.1 below. The integral (4.3.8) is a special Mellin-Barnes integral and in analogy to the series representation we refer to it as a binomial integral.

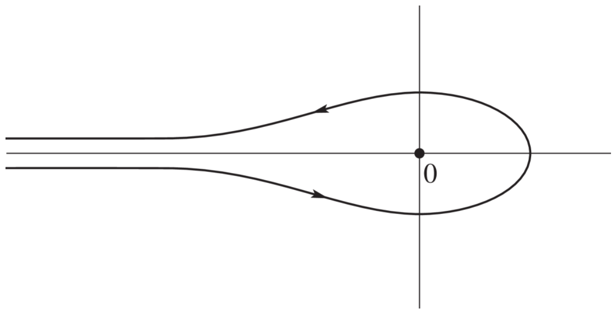

Figure 4.1.: Trace of the Hankel loop integration contour in the complex z-plane, compare fig. (5.9.1) in [Olver et al., 2010].

Before pointing out how to exploit (4.3.8) for an analysis of the large m-behaviour of the series (4.3.1) we briefly revisit the case $A(-z) = t^{-z}$ for $0 < t \leq 1$. Equality (4.3.8) then applies and immediately yields the following integral representation for the binomial series (4.3.2) with $m > 0$:

$$(4.3.9) \qquad (1-t)^m = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \frac{\Gamma(m+1)\Gamma(z)}{\Gamma(m+1+z)} t^{-z} dz$$

Since the integrand is analytic in the region to the right of the loop with sufficiently fast decay there, a convenient deformation of the integration path is viable. Therefore we consider a rectangle of height $N > 0$ and width $R + c$ for $c, R > 0$ in the complex z-plane. The rectangle is supposed to be symmetric with respect to the real axis and its right edge runs through the point $\Re z = c$. Moreover, the left edge exhibits an indentation of height 2δ in the form of a loop $\mathcal{L}_R$ for some $\frac{N}{2} > \delta > 0$, so that the simple poles of the gamma function at the non-positive integers lie to its left. According to Cauchy's theorem, the integral along this indented rectangle, if encircled in the counterclockwise direction, equals the sum of the residues in its interior. But the integrand is analytic in its interior, which implies

$$(4.3.10) \qquad 0 = \frac{1}{2\pi i} \left[\int_{\mathcal{L}_R} + \int_{-R+i\delta}^{-R+i\frac{N}{2}} + \int_{-R+i\frac{N}{2}}^{c+i\frac{N}{2}} + \int_{c+i\frac{N}{2}}^{c-i\frac{N}{2}} + \int_{c-i\frac{N}{2}}^{-R-i\frac{N}{2}} + \int_{-R-i\frac{N}{2}}^{-R-i\delta} \right] \frac{\Gamma(m+1)\Gamma(z)}{\Gamma(m+1+z)} t^{-z} dz.$$

We will now show that the contribution from all integrals vanishes as $N, R \to \infty$, except from those along the loop and the right edge of the rectangle. For this purpose we choose N, R large enough to make the estimate (4.3.6) applicable. By means of a simple substitution we then obtain for any $0 < t \leq 1$, $m > 0$ and a constant $A > 0$:

$$\left| \int_{-R+i\frac{N}{2}}^{c+i\frac{N}{2}} \frac{\Gamma(z)}{\Gamma(m+1+z)} t^{-z} dz \right| \leq \int_{-R}^{c} \frac{\left|\Gamma\left(x+i\frac{N}{2}\right)\right|}{\left|\Gamma\left(m+1+x+i\frac{N}{2}\right)\right|} t^{-x} dx$$

$$\leq At^{-c} \int_{-R}^{c} \left\{x^2 + \frac{N^2}{4}\right\}^{-\frac{m}{2}-\frac{1}{2}} dx$$

$$\leq At^{-c} 2^m N^{-m} \int_{-\infty}^{\infty} \left\{u^2+1\right\}^{-\frac{m}{2}-\frac{1}{2}} du$$

The integral on the right hand side converges absolutely and the whole upper bound vanishes as $N \to \infty$. A similar estimate can be derived for the integral along the lower edge in (4.3.10), which for $m > 0$ verifies

$$0 = \frac{1}{2\pi i} \left[\int_{\mathcal{L}_R} + \int_{-R+i\delta}^{-R+i\infty} + \int_{c+i\infty}^{c-i\infty} + \int_{-R-i\infty}^{-R-i\delta} \right] \frac{\Gamma(m+1)\Gamma(z)}{\Gamma(m+1+z)} t^{-z} dz. \tag{4.3.11}$$

Regarding the integral along the upper vertical segment of the left edge we obtain similarly from (4.3.6) for a constant $B > 0$:

$$\left| \int_{-R+i\delta}^{-R+i\infty} \frac{\Gamma(z)}{\Gamma(m+1+z)} t^{-z} dz \right| \leq t^R \int_{\delta}^{\infty} \frac{\Gamma(-R+iy)}{\Gamma(m+1-R+iy)} dy$$

$$\leq Bt^R \int_{\delta}^{\infty} \left\{R^2+y^2\right\}^{-\frac{m}{2}-\frac{1}{2}} dy$$

$$\leq Bt^R R^{-m} \int_{0}^{\infty} \left\{1+v^2\right\}^{-\frac{m}{2}-\frac{1}{2}} dy$$

Since $0 < t \leq 1$ this upper bound vanishes if we let $R \to \infty$ for $m > 0$. An analogous bound applies for the lower vertical part. Hence, as $R \to \infty$, the contribution from both vertical segments of the left edge decays, whereas $\mathcal{L}_R$ approaches a loop that runs from $\infty e^{-i\pi}$ to $\infty e^{i\pi}$, encircling the negative real axis in the positive direction. Regarding equation (4.3.11) we thus arrive at

$$0 = \frac{1}{2\pi i} \left[\int_{-\infty}^{(0+)} + \int_{c+i\infty}^{c-i\infty} \right] \frac{\Gamma(m+1)\Gamma(z)}{\Gamma(m+1+z)} t^{-z} dz. \tag{4.3.12}$$

By comparison with (4.3.9) we have eventually verified for arbitrary $c, m > 0$ and $0 < t \leq 1$

validity of

$$(1-t)^m = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \frac{\Gamma(m+1)\Gamma(z)}{\Gamma(m+1+z)} t^{-z} dz. \tag{4.3.13}$$

Note that this equality is in accordance with the Mellin inversion theorem A.5.1 by identifying the beta function in the integrand as the Mellin transform of the function $(1-t)^m \mathbb{I}\{0 < t < 1\}$. The transition from (4.3.9) to (4.3.13) can be described as a straightening of the loop to a vertical line, whereas the converse step corresponds to a bending of the line to a loop. Note that equality (4.3.10) only holds by analyticity. If there were, for instance, any poles in the interior of the indented rectangle, the value along its boundary was equal to the sum of residues.

The binomial integral (4.3.8) provides a more convenient frame for an asymptotic analysis than the initial series (4.3.1). Indeed, a first inspection of the integrand suggests that the asymptotic behaviour as m grows to infinity could be revealed by appropriate manipulations of the integration contour. While the possibilities hereof naturally depend on the structure of the coefficient function $A(-z)$, the exact location of its singularities essentially affects the result. This is indicated by the ratio of gamma functions appearing in (4.3.8), which, according to (B.3.5), possesses the leading behaviour $\sim m^{-z}$ as $m \to \infty$ for fixed $z \in \mathbb{C} \setminus -\mathbb{N}_0$. Clearly, depending on the real part of $z \in \mathbb{C}$, with respect to m, this is descending, ascending or bounded only. More precisely, three main cases may occur, namely $\Re z > 0$, $\Re z < 0$ and $\Re z = 0$. As an illustrative example we note that

$$\frac{1}{2\pi i} \int\limits_{-\infty}^{(0+)} \frac{\Gamma(m+1)\Gamma(z)}{\Gamma(m+1+z)} \frac{1}{(z-a_2)^2 + a_1^2} dz = \frac{1}{i2a_1} \left[\frac{\Gamma(m+1)\Gamma(a_2 - ia_1)}{\Gamma(m+1+a_2-ia_1)} - \frac{\Gamma(m+1)\Gamma(a_2+ia_1)}{\Gamma(m+1+a_2+ia_1)} \right]$$

for $a_1, a_2 \in \mathbb{R}$, $a_1 \neq 0$ such that each pole is of simple order. This equality holds since, for any $m > 0$ the integrand is $\mathcal{O}\left\{|z|^{-m-3}\right\}$ as $|z| \to \infty$ in $|\arg(z)| < \pi$, i.e., possesses algebraic decay in any direction of the cut complex plane. Furthermore, the loop separates the simple poles at $-\mathbb{N}_0$ from those at $a_2 \pm ia_1$. Hence, if we consider the loop as the left boundary of a closed integration path with upper, right and lower boundaries at infinity, the poles at $a_2 \pm ia_1$ are encircled in the clockwise direction. The above equality thus follows from the residue theorem since the contributions from the integrals along the boundaries at infinity vanish, subject to the fast algebraic decay of the integrand. As another example, for $a_1 = 0$, $a_2 \in \mathbb{R} \setminus -\mathbb{N}_0$ and $q \in \{1, 2\}$ it can be shown that

$$\frac{1}{2\pi i} \int\limits_{-\infty}^{(0+)} \frac{\Gamma(m+1)\Gamma(z)}{\Gamma(m+1+z)} \frac{1}{(z-a_2)^q} dz = \frac{\Gamma(m+1)\Gamma(a_2)}{\Gamma(m+1+a_2)} \left(\psi(m+1+a_2) - \psi(a_2)\right)^{q-1},$$

where ψ denotes the digamma function with $\psi(m) \sim \log(m)$ as $m \to \infty$, compare (B.2.12) and

(B.2.29) in the appendix. Hence, also the order of the poles of the coefficient function is crucial. Both examples confirm what was conjectured in before, namely that the behaviour of $S(m)$ is determined by the location of the singularities of $A(-z)$, particularly by the location of their real part. In case of poles, in the preceding examples we observed that $S(m)$ vanishes as $m \to \infty$ with a decreasing rate if $\Re a_2$ lies closer to the right of the origin, whereas $S(m)$ grows faster if $\Re a_2$ is located farer away from the origin in the left z-half plane. This suggests, in situations where $A(-z)$ has a more complicated meromorphic form it could be helpful to decompose this function according to the location of its singularities from $\Re z < 0$ to $\Re z > 0$. To conclude this introduction to binomial sums, by inspection of (4.3.8) we make some presumptions concerning the asymptotic behaviour of $S(m)$ as $m \to \infty$ for analytic coefficient functions of different types:

- For rational functions $A(-z)$ with simple poles only, the rate is purely algebraic.

- If $A(-z)$ equals a rational function with poles of order greater or equal two, the rate is in general a mixture of algebraic and logarithmic expressions since in these circumstances the residues involve derivatives of the beta function.

- If $A(-z)$ constitutes a polynomial of degree $p \in \mathbb{N}$, the integrand in (4.3.8) possesses no singularities other than simple poles at the non-positive integers. But these are completely located in the interior of the loop. Furthermore, for fixed $m > p$ the integrand vanishes fast enough as $|z| \to \infty$ in $|\arg(z)| < \pi$, enabling us to expand the loop to approach infinity in the upper, right and lower direction, where the integral eventually vanishes. As a consequence $S(m) = 0$ for any $m > p$.

- If $A(-z)$ is not a polynomial-type entire function it is transcendental. Since $A(-z)$ was allowed to have at maximum algebraic growth, the function is necessarily[3] bounded as $\Re z \to -\infty$. No specific statements are possible then, except provided $A(-z)$ remains bounded in all directions but[4] not as $\Re z \to \infty$. In this event, similar to the above example in which we verified (4.3.13), it is permitted to bend the loop to a vertical line parallel to the imaginary axis, which may be shifted by an arbitrary finite distance to the right. But this implies the rate of $S(m)$ as $m \to \infty$ is $\sim \text{const} \times m^{-z}$ for arbitrary $\Re z > 0$. In other words, $S(m)$ vanishes exponentially fast.

We are now well prepared for a treatment of the Mellin transform associated with the m-power.

[3]This follows from the fact that transcendental entire functions, according to Liouville's theorem, have an essential singularity at infinity and therefore exhibit exponential behaviour in a neighborhood of infinity.

[4]Since infinity is an essential singularity the function is naturally unbounded in at least one direction of the complex plane.

4.3.2. Representations for the Mellin Transform of the m-Power

For $m \geq 0$ the m-power was given in equation (2.1.9). The associated Mellin transform is defined as the integral

$$M_{\bar{\varepsilon}}(\zeta, m) := \int_0^\infty t^{\zeta-1} \mathcal{P}_{\bar{\varepsilon}}(t, m) dt. \tag{4.3.14}$$

Being composed by continuous functions, the m-power is continuous along the real axis for any fixed $m \geq 0$. According to Theorem A.2.1, the uniform convergence of the integral (4.3.14) in any compact subset of some region of the complex ζ-plane therefore suffices to verify analyticity of $M_{\bar{\varepsilon}}(\zeta, m)$ in this region. Following from its non-negativity and the facts that $1 - \Phi_{\bar{\varepsilon}}(0) = 0$ and $\limsup_{t\to\infty}(1 - \Phi_{\bar{\varepsilon}}(t)) > 0$, the m-power does not contribute to the absolute convergence of the integral (4.3.14), except with its behaviour at the origin. Accordingly, for $\delta_{\bar{\varepsilon}} > 0$ as $t \downarrow 0$ we suppose

$$1 - \Phi_{\bar{\varepsilon}}(t) = \mathcal{O}\left\{t^{\delta_{\bar{\varepsilon}}}\right\}. \tag{4.3.15}$$

As we shall see in our examples below, this condition is not a restriction, and it immediately implies absolute convergence of (4.3.14) for $-\delta_{\bar{\varepsilon}}(m+1) < \Re\zeta < 0$. Then, if E is an arbitrary compact subset in the indicated region with $a_0 := \min\{\Re\zeta : \zeta \in E\}$ and $b_0 := \max\{\Re\zeta : \zeta \in E\}$, for $\zeta \in E$, a number $A_m > 0$ that is constant except for its dependence on m, and a fixed $0 < T < 1$, we deduce the following bound:

$$\begin{aligned} |M_{\bar{\varepsilon}}(\zeta, m)| &\leq A_m \int_0^T t^{\Re\zeta + \delta_{\bar{\varepsilon}}(m+1)-1} dt + \int_T^\infty t^{\Re\zeta - 1} dt \\ &\leq A_m \int_0^T t^{a_0 + \delta_{\bar{\varepsilon}}(m+1)-1} dt - \frac{T^{\Re\zeta}}{\Re\zeta} \\ &\leq A_m \frac{T^{a_0+\delta_{\bar{\varepsilon}}(m+1)}}{a_0 + \delta_{\bar{\varepsilon}}(m+1)} - \frac{T^{a_0}}{b_0} \end{aligned} \tag{4.3.16}$$

For the first inequality we especially took into account the uniform boundedness of the m-power by unity. As a consequence of these findings for any $m \geq 0$ we have analyticity of (4.3.14) in the strip

$$\mathbb{S}_{m,\delta_{\bar{\varepsilon}}} := \{\zeta \in \mathbb{C} : -\delta_{\bar{\varepsilon}}(m+1) < \Re\zeta < 0\}. \tag{4.3.17}$$

The monotonicity of the m-power with respect to m implies $\mathbb{S}_{m,\delta_{\bar{\varepsilon}}} \subset \mathbb{S}_{m+1,\delta_{\bar{\varepsilon}}}$. The joint strip of analyticity for all $m \geq 0$ is therefore $\mathbb{S}_{\bar{\varepsilon}} := \mathbb{S}_{0,\delta_{\bar{\varepsilon}}}$. Furthermore, in equivalence to (4.3.17) the function $M_{\bar{\varepsilon}}(-\zeta, m)$ is holomorphic in

$$\mathbb{S}^{-}_{m,\delta_{\bar{\varepsilon}}} := \{\zeta \in \mathbb{C} : 0 < \Re\zeta < \delta_{\bar{\varepsilon}}(m+1)\}. \tag{4.3.18}$$

When considering our first example in Chapter 5, it will readily turn out that it suffices to know the integral representation for the Mellin transform of the m-power converges absolutely for $\zeta \in \mathbb{S}_{m,\delta_{\bar{\varepsilon}}}$ for any $m \geq 0$. This implies especially uniform convergence with respect to the imaginary part, whence the function is $\mathcal{O}(1)$ as $\Im\zeta \to \pm\infty$ in $\mathbb{S}_{m,\delta_{\bar{\varepsilon}}}$. No further information concerning the large ζ-behaviour of this Mellin transform are required. Instead, more important is the derivation of an asymptotic expansion for this integral as $m \to \infty$ for fixed $\zeta \in \mathbb{S}_{m,\delta_{\bar{\varepsilon}}}$. For this purpose it can be useful to express the m-power as a binomial integral, i..e, in accordance with equation (4.3.13), for $c > 0$ to write

$$\mathcal{P}_{\bar{\varepsilon}}(t,m) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{\Gamma(m+2)\Gamma(z)}{\Gamma(m+2+z)} \{\Phi_{\bar{\varepsilon}}(t)\}^{-z}\, dz, \qquad t \notin N_{\varepsilon}. \tag{4.3.19}$$

Observe, however, that $\{\Phi_{\bar{\varepsilon}}(t)\}^{-\Re z}$ for fixed $\Re z > 0$ as a function of $t \geq 0$ is unbounded if $N_{\varepsilon} \neq \emptyset$. Consequently, if we plug (4.3.19) into (4.3.14) we may not interchange the order of integration. Note that this also was not possible if we used the binomial series expansion for the m-power instead, since the first summand therein equals unity. To circumvent this obstacle we easily integrate (4.3.14) once by parts. Assuming N_{ε} is of Lebesgue-measure zero, $\Phi_{\bar{\varepsilon}}(t)$ is continuously differentiable with respect to $t > 0$ and denoting the associated derivative with a prime, for $m \geq 0$, $T_2 > T_1 > 0$ and $\zeta \in \mathbb{S}_{m,\delta_{\bar{\varepsilon}}}$ we obtain

$$\int\limits_{T_1}^{T_2} t^{\zeta-1}\mathcal{P}_{\bar{\varepsilon}}(t,m)dt = \left[\frac{t^{\zeta}}{\zeta}\mathcal{P}_{\bar{\varepsilon}}(t,m)\right]_{T_1}^{T_2} + \frac{m+1}{\zeta}\int\limits_{T_1}^{T_2} t^{\zeta}\Phi'_{\bar{\varepsilon}}(t)\mathcal{P}_{\bar{\varepsilon}}(t,m-1)dt. \tag{4.3.20}$$

Since $\Re\zeta < 0$, as $T_2 \to \infty$ the function in the first summand on the right hand side vanishes at the upper limit. Moreover, as $T_1 \downarrow 0$ the function also decays at the lower limit because $\zeta \in \mathbb{S}_{m,\delta_{\bar{\varepsilon}}}$ and $t^{\zeta}\mathcal{P}_{\bar{\varepsilon}}(t,m) = \mathcal{O}\left(t^{\zeta+\delta_{\bar{\varepsilon}}(m+1)}\right)$ as $t \downarrow 0$, compare with (4.3.15). To summarize these findings, taking the limits in (4.3.20) results in the following representation, valid for all $m \geq 0$ and $\zeta \in \mathbb{S}_{m,\delta_{\bar{\varepsilon}}}$:

$$M_{\bar{\varepsilon}}(\zeta,m) = \frac{m+1}{\zeta}\int\limits_{0}^{\infty} t^{\zeta}\Phi'_{\bar{\varepsilon}}(t)\mathcal{P}_{\bar{\varepsilon}}(t,m-1)dt \tag{4.3.21}$$

In contrast to (4.3.14) the above integral need not be absolutely convergent[5] for all $\zeta \in \mathbb{S}_{m,\delta_{\bar{\varepsilon}}}$. The complete set of admissible values depends on the behaviour of the derivative $\Phi'_{\bar{\varepsilon}}$ in a neigh-

[5]For instance if $\Phi_{\bar{\varepsilon}}$ and thus also its derivative is periodic. Then it is possible to choose $\zeta \in \mathbb{S}_{m,\delta_{\bar{\varepsilon}}}$ with $-1 < \Re\zeta < 0$ such that (4.3.21) is in fact not absolutely convergent, although (4.3.14) is, by definition of $\mathbb{S}_{m,\delta_{\bar{\varepsilon}}}$.

borhood of the origin and at infinity. For simplicity suppose for an additional parameter $\eta_{\bar{\varepsilon}} > 1$ we have

$$\begin{cases} \Phi'_{\bar{\varepsilon}}(t) = \mathcal{O}\left\{t^{\delta_{\bar{\varepsilon}}-1}\right\} & \text{as } t \downarrow 0, \\ \Phi'_{\bar{\varepsilon}}(t) = \mathcal{O}\left\{t^{-\eta_{\bar{\varepsilon}}}\right\} & \text{as } t \to \infty. \end{cases} \tag{4.3.22}$$

By means of bounds similar to (4.3.16), it is then possible to verify absolute convergence of the integral (4.3.21) in

$$\mathbb{S}'_{m,\delta_{\bar{\varepsilon}}} := \{\zeta \in \mathbb{C} : -\delta_{\bar{\varepsilon}}(m+1) < \Re\zeta < \eta_{\bar{\varepsilon}} - 1\} \tag{4.3.23}$$

and especially uniformly in any compact subset therein for all $m \geq 0$. Since (4.3.23) always contains the strip $\mathbb{S}_{m,\delta_{\bar{\varepsilon}}}$, according to Theorem A.2.1, the right hand side of (4.3.21) constitutes the analytic continuation of (4.3.14), revealing a simple pole at $\zeta = 0$. The actual benefit from the presence of the derivative $\Phi'_{\bar{\varepsilon}}$ in the integral (4.3.21), which is unavailable in (4.3.14), is that it will enable us to introduce the binomial integral representation for the m-power. Hence, denote by

$$A(\zeta, -z) := \int_0^\infty t^\zeta \Phi'_{\bar{\varepsilon}}(t) \left\{\Phi_{\bar{\varepsilon}}(t)\right\}^{-z} dt \tag{4.3.24}$$

the Mellin transform of the function $t \mapsto t\Phi'_{\bar{\varepsilon}}(t) \left\{\Phi_{\bar{\varepsilon}}(t)\right\}^{-z}$ for suitable $z \in \mathbb{C}$. The integral $A(\zeta, 0)$ is readily confirmed to be absolutely convergent for instance under the assumptions (4.3.22). In these circumstances the set $\mathbb{S}'_{\bar{\varepsilon}} := \mathbb{S}'_{0,\delta_{\bar{\varepsilon}}}$ is non-empty, implying absolute convergence of $A(\zeta, -z)$ for all $\zeta \in \mathbb{S}'_{\bar{\varepsilon}}$ and $\Re z \leq 0$, especially since $0 \leq \Phi_{\bar{\varepsilon}}(t) \leq 1$ for $t \in \mathbb{R}$. In addition, for fixed $\Re z \leq 0$ the integral can be verified holomorphic in $\mathbb{S}'_{\bar{\varepsilon}}$. Notice that $\mathbb{S}_{\bar{\varepsilon}} \subset \mathbb{S}'_{\bar{\varepsilon}}$.

It will show in our examples that it is not a restriction to suppose the existence of a proper or improper substrip $\mathbb{S}''_{\bar{\varepsilon}} \subset \mathbb{S}'_{\bar{\varepsilon}}$ and a parameter $\gamma_0 > 0$, such that $A(\zeta, -z)$ remains absolutely convergent for all $\Re z < \gamma_0$ and $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$. Then, in (4.3.21) for any fixed $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$ we can write the m-power as a MB-integral for $0 < c < \gamma_0$ by applying (4.3.19). More precisely, according to (4.3.6), for $m > 0$ we may interchange the order of integration by absolute convergence, which establishes the following binomial integral representation:

$$M_{\bar{\varepsilon}}(\zeta, m) = \frac{m+1}{\zeta 2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(m+1)\Gamma(z)}{\Gamma(m+1+z)} A(\zeta, -z) dz \tag{4.3.25}$$

This representation holds for any $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$ and by analytic continuation in larger regions of the complex ζ-plane. The integral (4.3.25) can eventually serve as a starting point for an asymptotic analysis of $M_{\bar{\varepsilon}}(\cdot, m)$ as $m \to \infty$. Therefore we first note that, given a compact subset E in

$\Re z < \gamma_0$ with $b_0 := \max\{\Re z : z \in E\}$, for fixed $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$ we have

$$|A(\zeta,-z)| \leq \int_0^\infty t^{\Re\zeta} \left|\Phi'_{\bar{\varepsilon}}(t)\right| \{\Phi_{\bar{\varepsilon}}(t)\}^{-b_0}\, dt < \infty.$$

The integral (4.3.24) thus converges uniformly in E. This enables a consideration of $A(\zeta,-z)$ as an analytic function in the half-plane $\Re z < \gamma_0$ for any fixed $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$, compare Theorem A.2.1. Its singularities necessarily lie in the half plane $\Re z \geq \gamma_0$ only. To determine their exact location, a careful analysis is required. Depending on $\Phi_{\bar{\varepsilon}}$ this can be quite elaborate since $A(\zeta,-z)$ is generally not expressible in terms of elementary functions.

As an application of the ideas presented so far, in the subsequent sections we will derive the Mellin transforms for certain classes of characteristic functions and their modulus.

4.4. Algebraic-Type Characteristic Functions

We first assume, that $|\Phi_X|$ is associated with the class of algebraic-type functions (3.3.1). The integral representation for the Mellin transform (4.2.1) is then given by

$$M_X(\zeta) = \int_0^\infty t^{\zeta-1}\{1+\theta^\alpha t^\alpha\}^{-p}\, dt. \tag{4.4.1}$$

According to our findings from Section 4.2, this establishes an analytic function in the strip

$$S_X = \{\zeta \in \mathbb{C} : 0 < \Re\zeta < \alpha p\}. \tag{4.4.2}$$

Moreover, in Example A.5.2 in the appendix it was shown that (4.4.1) can be referred to the beta function. By additional use of the identity (B.3.2) this yields:

$$\begin{aligned} M_X(\zeta) &= \alpha^{-1}\theta^{-\zeta}\,\mathrm{B}\left(\frac{\zeta}{\alpha}, p-\frac{\zeta}{\alpha}\right) \\ &= \theta^{-\zeta}\frac{\Gamma\left(\frac{\zeta}{\alpha}\right)\Gamma\left(p-\frac{\zeta}{\alpha}\right)}{\alpha\Gamma(p)} \end{aligned} \tag{4.4.3}$$

The analytic continuation associated with (4.4.1) to the whole complex plane is thus meromorphic with simple poles at $\alpha(p+\mathbb{N}_0)$ and at $-\alpha\mathbb{N}_0$. Its asymptotic behaviour as $|\zeta| \to \infty$ in $|\arg(\zeta)|, |\arg(\alpha p - \zeta)| < \pi$ is readily derived by virtue of Stirling's formula (B.2.28):

$$M_X(\zeta) \sim \frac{2\pi e^{-p}}{\alpha\Gamma(p)}\theta^{-\zeta}\left|\frac{\zeta}{\alpha}\right|^{\frac{\zeta}{\alpha}-\frac{1}{2}}\left|p-\frac{\zeta}{\alpha}\right|^{p-\frac{\zeta}{\alpha}-\frac{1}{2}} e^{i\left(\frac{\zeta}{\alpha}-\frac{1}{2}\right)\arg\left(\frac{\zeta}{\alpha}\right)+i\left(p-\frac{\zeta}{\alpha}-\frac{1}{2}\right)\arg\left(p-\frac{\zeta}{\alpha}\right)}$$

In absolute value as $|\zeta| \to \infty$ this implies

$$|M_X(\zeta)| = \mathcal{O}\left\{\theta^{-\Re\zeta}\,|\zeta|^{p-1}\,e^{-\frac{\Im\zeta}{\alpha}\{\arg(\zeta)-\arg(\alpha p-\zeta)\}}\right\}. \tag{4.4.4}$$

For appropriate arguments we can simplify (4.4.3) to finally obtain a convenient bound for this Mellin transform. Therefore we first note for $y \in \mathbb{R}$ and $J \in \mathbb{N}_0$ repeated application of the functional equation for the gamma function leads to

$$\Gamma\left(\frac{iy}{\alpha}+p+J+\frac{1}{2}\right) = \Gamma\left(\frac{iy}{\alpha}+p+\frac{1}{2}\right)\prod_{r=1}^{J}\left(p+r-\frac{1}{2}+\frac{iy}{\alpha}\right),$$
$$\Gamma\left(-\frac{iy}{\alpha}-J-\frac{1}{2}\right) = \Gamma\left(-\frac{iy}{\alpha}+\frac{1}{2}\right)\prod_{r=0}^{J}\left(-r-\frac{1}{2}-\frac{iy}{\alpha}\right)^{-1}.$$

Hence, if $x_{J+1} := \alpha\left(p+J+\frac{1}{2}\right)$ with $y \in \mathbb{R}$ and $J \in \mathbb{N}_0$ we can write for the beta function in (4.4.3):

$$\begin{aligned}\left|\mathrm{B}\left(\frac{iy+x_{J+1}}{\alpha}, p-\frac{iy+x_{J+1}}{\alpha}\right)\right| &= \frac{\left|\Gamma\left(\frac{iy}{\alpha}+p+\frac{1}{2}\right)\right|\left|\Gamma\left(-\frac{iy}{\alpha}+\frac{1}{2}\right)\right|}{\Gamma(p)\sqrt{\frac{1}{4}+\frac{y^2}{\alpha^2}}}\prod_{r=1}^{J}\left\{\frac{\left(p+r-\frac{1}{2}\right)^2+\frac{y^2}{\alpha^2}}{\left(r+\frac{1}{2}\right)^2+\frac{y^2}{\alpha^2}}\right\}^{\frac{1}{2}}\\ &= \frac{\pi^{\frac{1}{2}}\left|\Gamma\left(\frac{iy}{\alpha}+p+\frac{1}{2}\right)\right|}{\cosh^{\frac{1}{2}}\left(\frac{\pi y}{\alpha}\right)\Gamma(p)\sqrt{\frac{1}{4}+\frac{y^2}{\alpha^2}}}\prod_{r=1}^{J}\left\{\frac{\left(p+r-\frac{1}{2}\right)^2+\frac{y^2}{\alpha^2}}{\left(r+\frac{1}{2}\right)^2+\frac{y^2}{\alpha^2}}\right\}^{\frac{1}{2}}\end{aligned} \tag{4.4.5}$$

For the second equality we applied the reflection formula (B.2.17). As usually $J = 0$ implies that the product equals 1. The above is the most general factorization obtainable without restrictions on $p > 0$. Now, by elementary calculations one can show:

$$\max_{y\in\mathbb{R}}\left\{\frac{\left(p+r-\frac{1}{2}\right)^2+\frac{y^2}{\alpha^2}}{\left(r+\frac{1}{2}\right)^2+\frac{y^2}{\alpha^2}}\right\}^{\frac{1}{2}} \le \max\left\{1, \frac{\left|p+r-\frac{1}{2}\right|}{\left|r+\frac{1}{2}\right|}\right\} = \begin{cases}1, & \text{if } p \le 1\\ \frac{p+r-\frac{1}{2}}{r+\frac{1}{2}}, & \text{if } p > 1\end{cases}$$

Moreover, a sharp bound for the remaining gamma function in the numerator in (4.4.5) is given by inequality (B.2.32) and yields

$$\left|\Gamma\left(\frac{iy}{\alpha}+p+\frac{1}{2}\right)\right| \le \sqrt{2\pi}\left\{\frac{y^2}{\alpha^2}+\left(p+\frac{1}{2}\right)^2\right\}^{\frac{p}{2}} e^{-\frac{\pi|y|}{2\alpha}}\,e^{\frac{1}{3(2p+1)}}.$$

For (4.4.5) we thus arrive at the bound

$$\left|\mathrm{B}\left(\frac{iy+x_{J+1}}{\alpha}, p-\frac{iy+x_{J+1}}{\alpha}\right)\right| \le \kappa_p(J)\frac{\left\{\frac{y^2}{\alpha^2}+\left(p+\frac{1}{2}\right)^2\right\}^{\frac{p}{2}}}{\sqrt{\frac{1}{4}+\frac{y^2}{\alpha^2}}}\frac{e^{-\frac{\pi|y|}{2\alpha}}}{\cosh^{\frac{1}{2}}\left(\frac{\pi y}{\alpha}\right)}, \tag{4.4.6}$$

where we denote

$$\kappa_p(J) := \frac{\sqrt{2}\pi e^{\frac{1}{3(2p+1)}}}{\Gamma(p)} \prod_{r=1}^{J} \max\left\{1, \frac{\left|p+r-\frac{1}{2}\right|}{\left|r+\frac{1}{2}\right|}\right\}. \tag{4.4.7}$$

Clearly, the right hand side of (4.4.6) is an even function and absolutely integrable along the whole real axis for all admissible values of the parameters.

4.5. Algebraic-Type m-Powers

Suppose $\Phi_{\bar{\varepsilon}}$ is given by (3.3.2). In these circumstances an expansion of the function as $|t| \to 0$ by means of the binomial theorem reveals the local behaviour $1 - \Phi_{\bar{\varepsilon}}(t) \sim q\sigma^{\beta} |t|^{\beta}$. In accordance with our findings from Subsection 4.3.2, the integral (4.3.14) is thus for any $m \geq 0$ absolutely convergent and holomorphic in the regions

$$\mathbb{S}_{\bar{\varepsilon}} = \{\zeta \in \mathbb{C} : -\beta < \Re\zeta < 0\}, \tag{4.5.1}$$

$$\mathbb{S}_{m,\beta} = \{\zeta \in \mathbb{C} : -\beta(m+1) < \Re\zeta < 0\}. \tag{4.5.2}$$

Moreover, for $t > 0$ the function (3.3.2) is infinitely many times continuously differentiable and particularly the first derivative equals

$$\Phi'_{\bar{\varepsilon}}(t) = -q\beta\sigma^{\beta} t^{\beta-1} \left\{1 + \sigma^{\beta} t^{\beta}\right\}^{-1} \Phi_{\bar{\varepsilon}}(t). \tag{4.5.3}$$

Accordingly, we can rearrange (4.3.14) by making additional use of the definition of the m-power, which was given in (2.1.9), to find

$$M_{\bar{\varepsilon}}(\zeta, m) = -\frac{\sigma^{-\beta}}{q\beta} \int_0^{\infty} t^{\zeta-\beta} \left\{1 + \sigma^{\beta} t^{\beta}\right\} (1 - \Phi_{\bar{\varepsilon}}(t))^{m+1} \frac{\Phi'_{\bar{\varepsilon}}(t)}{\Phi_{\bar{\varepsilon}}(t)} dt. \tag{4.5.4}$$

The simple change of variables $ds = \Phi'_{\bar{\varepsilon}}(t)dt$ implies

$$t = \sigma^{-1} \left\{s^{-\frac{1}{q}} - 1\right\}^{\frac{1}{\beta}},$$

and for any $\zeta \in \mathbb{S}_{m,\beta}$ the integral (4.5.4) then takes on the shape

$$M_{\bar{\varepsilon}}(\zeta, m) = \frac{\sigma^{-\zeta}}{q\beta} \int_0^1 s^{-\frac{\zeta}{\beta q}-1} \left\{1 - s^{\frac{1}{q}}\right\}^{\frac{\zeta}{\beta}-1} (1 - s)^{m+1} ds. \tag{4.5.5}$$

For brevity we introduce the integral function

$$\mathfrak{I}_H^q(-\zeta, m) := \int_0^1 s^{-\frac{\zeta}{q}-1} \left\{1 - s^{\frac{1}{q}}\right\}^{\zeta-1} (1 - s)^{m+1} ds, \qquad \zeta \in \mathbb{S}_{m,1}. \tag{4.5.6}$$

The function in the interior of the curved brackets is continuous and non-vanishing for $0 \leq s < 1$. In addition, for $q > 0$ and $|s-1| < 1$ according to the binomial theorem, we have:

$$
\begin{aligned}
1 - s^{\frac{1}{q}} &= -\sum_{j=1}^{\infty} \binom{\frac{1}{q}}{j} (s-1)^j \\
&= (1-s) \sum_{j=0}^{\infty} \frac{\Gamma\left(1+\frac{1}{q}\right)}{\Gamma\left(\frac{1}{q}-j\right)\Gamma(2+j)} (s-1)^j
\end{aligned}
\tag{4.5.7}
$$

For integer values of $q^{-1} > 0$ the above series cancels to a polynomial of degree q^{-1}. By virtue of (4.5.7) it is readily confirmed that (4.5.6) in fact converges absolutely for any $\zeta \in \mathbb{S}_{m,1}$. Finally, in terms of the latter definition for $\zeta \in \mathbb{S}_{m,\beta}$ we can write

$$
M_{\bar{\varepsilon}}(\zeta, m) = \frac{\sigma^{-\zeta}}{q\beta} \mathfrak{I}_H^q \left(-\frac{\zeta}{\beta}, m\right). \tag{4.5.8}
$$

The integral (4.5.6) can be conceived as a generalization of Euler's integral of the first kind and for $q = 1$ we have equality. Indeed, by comparison with (B.3.1) we readily verify $\mathfrak{I}_H^1(-\zeta, m) = B(m+1+\zeta, -\zeta)$. As a consequence, for $q = 1$ by additional use of the identity (B.3.2) we obtain

$$
M_{\bar{\varepsilon}}(\zeta, m) = \sigma^{-\zeta} \frac{\Gamma\left(m+1+\frac{\zeta}{\beta}\right)\Gamma\left(-\frac{\zeta}{\beta}\right)}{\beta\Gamma(m+1)}. \tag{4.5.9}
$$

The right hand side immediately extends the integral definition (4.5.5) to the whole complex ζ-plane and reveals the location of the singularities. Even more remarkable, from (4.5.9) the asymptotic behaviour of $M_{\bar{\varepsilon}}(\zeta, m)$ with respect to both variables is easily ascertainable. On the one hand, for fixed $\zeta \in \mathbb{S}_{m,\beta}$ subject to (B.3.5), we observe a descending character as $m \to \infty$. On the other hand, Stirling's formula (B.2.28), as $|\zeta| \to \infty$ in $|\arg(\zeta)|, |\arg(\beta(m+1)-\zeta)| < \pi$ shows that

$$
\left|\Gamma\left(m+1+\frac{\zeta}{\beta}\right)\Gamma\left(-\frac{\zeta}{\beta}\right)\right| = \mathcal{O}\left\{|\zeta|^m e^{-\frac{\Im\zeta}{\beta}(\arg(\zeta)-\arg(\beta(m+1)-\zeta))}\right\}. \tag{4.5.10}
$$

Unfortunately the case $q = 1$ is apparently an exception and for other parametrizations no references to particular special functions are known.

4.5.1. Derivation of the Binomial Integral Representation

Although we will see below that the integral (4.5.5) already furnishes the appropriate setting for the derivation of an m-asymptotic expansion of $M_{\bar{\varepsilon}}(\cdot, m)$, for completeness we also deduce the binomial integral representation (4.3.25). Given (4.5.3) the coefficient function (4.3.24) becomes

$$
A(\zeta, -z) = -q\beta\sigma^\beta \int_0^\infty t^{\zeta+\beta-1} \left\{1 + \sigma^\beta t^\beta\right\}^{-1-q(1-z)} dt. \tag{4.5.11}
$$

By comparison with (4.3.22) we specify the region $\mathbb{S}'_{\bar{\varepsilon}}$, where this integral is absolutely convergent and holomorphic for any fixed $\Re z \leq 0$, as the strip

$$\mathbb{S}'_{\bar{\varepsilon}} = \{\zeta \in \mathbb{C} : -\beta < \Re\zeta < q\beta\}. \tag{4.5.12}$$

Furthermore, after a simple substitution, similar to Example A.5.2, the integral (4.5.11) is readily identified as Euler's integral of the first kind again. Accompanied by another application of (B.3.2) this yields

$$A(\zeta, -z) = -q\sigma^{-\zeta} \frac{\Gamma\left(1 + \frac{\zeta}{\beta}\right)\Gamma\left(q - qz - \frac{\zeta}{\beta}\right)}{\Gamma(1 + q - qz)}. \tag{4.5.13}$$

As a function of $\zeta \in \mathbb{C}$ with fixed $z \in \mathbb{C}$ and vice versa, the analytic continuation of (4.5.11) is thus meromorphic with order and location of poles depending on the parameters and on the second variable. The asymptotic behaviour as $|z| \to \infty$ in $|\arg(-z)| < \pi$ for fixed ζ is readily specified by virtue of (B.3.6):

$$|A(\zeta, -z)| = \mathcal{O}\left\{|qz|^{-1-\frac{\Re\zeta}{\beta}}\right\} \tag{4.5.14}$$

Now, choosing $\mathbb{S}''_{\bar{\varepsilon}} \equiv \{\zeta \in \mathbb{C} : -\beta < \Re\zeta < 0\}$ we have $\mathbb{S}''_{\bar{\varepsilon}} = \mathbb{S}_{\bar{\varepsilon}} \cap \mathbb{S}'_{\bar{\varepsilon}}$. It is then easy to see that the integral (4.5.11) for the coefficient function $A(\zeta, -z)$ converges absolutely for $\Re z < 1$ and any fixed $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$. Hence, according to (4.3.25), upon application of the functional equation for the gamma function, for $0 < c < 1$, $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$ and $m > 0$ we obtain the following hypergeometric integral representation:

$$\begin{aligned} M_{\bar{\varepsilon}}(\zeta, m) &= -q\sigma^{-\zeta} \frac{\Gamma\left(\frac{\zeta}{\beta}\right)}{\beta 2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(m+2)\Gamma(z)\Gamma\left(q - qz - \frac{\zeta}{\beta}\right)}{\Gamma(m+1+z)\Gamma(1+q-qz)} dz \\ &= \sigma^{-\zeta} \frac{\Gamma\left(\frac{\zeta}{\beta}\right)}{\beta 2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(m+2)\Gamma(z-1)\Gamma\left(q - qz - \frac{\zeta}{\beta}\right)}{\Gamma(m+1+z)\Gamma(q-qz)} dz \end{aligned} \tag{4.5.15}$$

The integrand is for fixed $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$ with respect to z meromorphic in $\mathbb{C}$ with an infinite sequence of simple poles at $-\mathbb{N}_0$ and at $1 + \frac{k-\zeta}{q\beta}$ for $k \in \mathbb{N}_0$. Note that the singularity at $z = 1$ is removable. Moreover, upon combining (4.3.6) and (4.5.14) we see that as $|z| \to \infty$ in in the cut z-plane the integrand has the behaviour

$$\left|\frac{\Gamma(z-1)\Gamma\left(q - qz - \frac{\zeta}{\beta}\right)}{\Gamma(m+1+z)\Gamma(q-qz)}\right| = \mathcal{O}\left\{|z|^{-m-1}|qz|^{-1-\frac{\Re\zeta}{\beta}}\right\}. \tag{4.5.16}$$

For any $m > 0$ it is thus permitted to displace the integration path in (4.5.15) by an arbitrary finite distance to the right or to the left without changing the value of the integral, provided we do not cross any singularities of the integrand. As a consequence, the path can be replaced by an arbitrary vertical line, separating the poles at $-\mathbb{N}_0$ from those of $\Gamma(q - qz - \beta^{-1}\zeta)$. This is the case if

$$0 < c < 1 - \frac{\Re\zeta}{q\beta}. \tag{4.5.17}$$

Note the admissibility of this condition for any $\zeta \in \mathbb{C}$ with $\Re\zeta < q\beta$. Taking into account the large z-behaviour (4.5.16), the absolute convergence of the hypergeometric integral in (4.5.15) remains valid for any fixed

$$-\beta(m+1) < \Re\zeta < q\beta \tag{4.5.18}$$

with an appropriately specified integration path. To conclude that the integral (4.5.15) for $\zeta \in \mathbb{S}_{m,\beta}$ coincides with the integral (4.5.5), it remains to verify its analyticity in that region. For the moment we only know that this is true for $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$. We do not further pursue the analyticity properties of (4.5.15) because they are irrelevant for our subsequent considerations.

4.5.2. m-Asymptotic Expansion for a Fixed Complex Argument

We proceed with the derivation of an asymptotic expansion for the integral (4.5.6). This is surprisingly simple and does not require additional supportive results. Therefore we first observe that, subject to the binomial theorem the following expansion holds for $\zeta \in \mathbb{C}$ and $0 \le s < 1$:

$$\left\{1 - s^{\frac{1}{q}}\right\}^{\zeta-1} = \sum_{k=0}^{\infty} \binom{\zeta-1}{k} (-1)^k s^{\frac{k}{q}} \tag{4.5.19}$$

This series remains absolutely convergent for $s = 1$, however, only if $\Re\zeta > 1$. We can thus not simply apply it to (4.5.6) and interchange the order of summation and integration for $\zeta \in \mathbb{S}_{m,1}$. Instead we first note, subject to the reflection formula (B.2.15) and upon exploiting the periodicity of the sine function, for $\zeta \in \mathbb{C} \setminus -\mathbb{N}_0$ and $k \in \mathbb{N}_0$ we can write:

$$\begin{aligned} \frac{\Gamma(\zeta)}{\Gamma(\zeta-k)} &= \frac{\Gamma(1-\zeta+k)\sin(\pi(1-\zeta+k))}{\Gamma(1-\zeta)\sin(\pi(1-\zeta))} \\ &= (-1)^k \frac{\Gamma(1-\zeta+k)}{\Gamma(1-\zeta)} \end{aligned} \tag{4.5.20}$$

Hence, expressing in (4.5.19) the binomial coefficient in terms of gamma functions and bearing in mind $z! = \Gamma(1+z)$ for $z \in \mathbb{C} \setminus -\mathbb{N}_0$, the alternating sign cancels out, and we obtain

$$\left\{1 - s^{\frac{1}{q}}\right\}^{\zeta-1} = \sum_{k=0}^{\infty} \frac{\Gamma(1-\zeta+k)}{\Gamma(1-\zeta)} \frac{s^{\frac{k}{q}}}{k!}. \tag{4.5.21}$$

Now, for any $m \geq 0$, $\Re\zeta < 0$ and sufficiently large $k \in \mathbb{N}$ the following statement is readily verified by comparison with definition (A.5.13) and identity (B.3.2):

$$\frac{|\Gamma(1-\zeta+k)|}{k!}\int_0^1 s^{\frac{k-\Re\zeta}{q}-1}(1-s)^{m+1}ds = \frac{|\Gamma(1-\zeta+k)|}{\Gamma(1+k)}\frac{\Gamma(m+2)\,\Gamma\left(\frac{k}{q}-\frac{\Re\zeta}{q}\right)}{\Gamma\left(m+2+\frac{k}{q}-\frac{\Re\zeta}{q}\right)}$$
$$= \mathcal{O}\left\{k^{-\Re\zeta-m-2}\right\}$$

For the last equality we refer to the bound (B.3.6). The above result justifies for $\zeta \in \mathbb{S}_{m,1}$ the following interchangeability in the order of summation and integration in (4.5.6), subject to absolute convergence:

$$\begin{aligned}\mathfrak{I}_H^q(-\zeta,m) &= \int_0^1 s^{-\frac{\zeta}{q}-1}\left\{1-s^{\frac{1}{q}}\right\}^{\zeta-1}(1-s)^{m+1}ds \\ &= \sum_{k=0}^{\infty}\frac{\Gamma(1-\zeta+k)}{\Gamma(1-\zeta)}\frac{1}{k!}\int_0^1 s^{\frac{k-\zeta}{q}-1}(1-s)^{m+1}ds \\ &= \sum_{k=0}^{\infty}\frac{\Gamma(m+2)\,\Gamma\left(\frac{k}{q}-\frac{\zeta}{q}\right)\Gamma(1-\zeta+k)}{\Gamma\left(m+2+\frac{k}{q}-\frac{\zeta}{q}\right)\Gamma(1-\zeta)}\frac{1}{k!}\end{aligned}$$

Because the coefficients in this power series are solely gamma functions, it is of hypergeometric type. Roughly speaking, a power series of the form

$$\sum_{k=0}^{\infty} c_k\frac{z^k}{k!}$$

is denoted a hypergeometric series if for each $k \in \mathbb{N}_0$ the ratio $\frac{c_{k+1}}{c_k}$ of subsequent coefficients c_k, c_{k+1} constitutes a rational function of the summation index k. According to (4.5.8), for $\zeta \in \mathbb{S}_{m,\beta}$ we have just shown

$$M_{\bar{\varepsilon}}(\zeta,m) = \frac{\sigma^{-\zeta}}{q\beta}\sum_{k=0}^{\infty}\frac{\Gamma(m+2)\Gamma\left(\frac{k}{q}-\frac{\zeta}{\beta q}\right)\Gamma\left(1-\frac{\zeta}{\beta}+k\right)}{\Gamma\left(m+2-\frac{\zeta}{\beta q}+\frac{k}{q}\right)\Gamma\left(1-\frac{\zeta}{\beta}\right)}\frac{1}{k!}. \tag{4.5.22}$$

By comparison with (B.3.5) of the m-dependent terms in the k-th summand of this series, as $m \to \infty$, we perceive

$$\frac{\Gamma(m+2)}{\Gamma\left(m+2-\frac{\zeta}{\beta q}+\frac{k}{q}\right)} \sim m^{\frac{\zeta}{\beta q}-\frac{k}{q}}. \tag{4.5.23}$$

This indicates a decreasing character of the series (4.5.22) for large values of the parameter m and fixed $\zeta \in \mathbb{S}_{m,\beta}$. It thus establishes an expansion as $m \to \infty$ of the Mellin transform $M_{\bar{\varepsilon}}(\zeta,m)$ for any fixed $\zeta \in \mathbb{S}_{m,\beta}$, where the asymptotic scale is for $k \in \mathbb{N}_0$ given by the ratio of gamma functions on the left hand side in (4.5.23), related to algebraic behaviour. Series of the above type as a function of m are termed inverse factorial expansions. In accordance with these findings, it is reasonable to cast (4.5.22) after proper standardisation in the equivalent form

$$M_{\bar{\varepsilon}}(\zeta, m) = \frac{\sigma^{-\zeta}}{q\beta} \frac{\Gamma(m+2)\Gamma\left(-\frac{\zeta}{\beta q}\right)}{\Gamma\left(m+2-\frac{\zeta}{\beta q}\right)} S_H^q\left(-\frac{\zeta}{\beta}, m\right), \tag{4.5.24}$$

where in terms of Pochhammer's symbol (B.2.11) we denote the series by

$$S_H^q\left(-\zeta, m\right) := \sum_{k=0}^{\infty} \frac{\left(-\frac{\zeta}{q}\right)_{\frac{k}{q}} (1-\zeta)_k}{\left(m+2-\frac{\zeta}{q}\right)_{\frac{k}{q}}} \frac{1}{k!}. \tag{4.5.25}$$

Clearly, according to our preceding findings, for $\zeta \in \mathbb{S}_{m,1}$ the series converges absolutely, we have $S_H^q\left(-\zeta, m\right) = \mathcal{O}(1)$ as $m \to \infty$, and if ζ is real-valued each summand in (4.5.25) is positive. In the special case $q = 1$ we identify the right hand side as the Gaussian hypergeometric function

$$S_H^1\left(-\zeta, m\right) = {}_2F_1\left[\begin{matrix}-\zeta,\ 1-\zeta \\ m+2-\zeta\end{matrix}; 1\right].$$

Owing to the unit argument we can apply Theorem 2.2.2 in [Andrews et al., 1999], which is the Gaussian summation formula. This eventually results in (4.5.9). For other values of the parameter $q = 1$ no known summability properties are available. We must then inevitably refer to the series (4.5.25) whenever an m-asymptotic expansion of $M_{\bar{\varepsilon}}(\zeta, m)$ is required.

The series expansion (4.5.22) can also be obtained from the MB-integral (4.5.15). Therefore we note that, for fixed $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$ and $m > 0$ the asymptotic behaviour (4.5.16) of the integrand indicates the possibility to shift the integration path to the right direction of the z-plane over some of the simple poles of the gamma function $\Gamma(q - qz - \beta^{-1}\zeta)$. Those are located at $1 - (\beta q)^{-1}\zeta + q^{-1}\mathbb{N}_0$, according to (B.2.20), with residues

$$\operatorname*{Res}_{z=1-\frac{\zeta}{\beta q}+\frac{k}{q}} \Gamma\left(q - qz - \frac{\zeta}{\beta}\right) = q^{-1}\frac{(-1)^{k+1}}{k!}, \qquad k \in \mathbb{N}_0.$$

Now, suppose $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$ and consider a rectangle in the complex z-plane of infinite height whose left and right edges are respectively given by the vertical lines $\Re z = c$ and $\Re z = K_{\Re\zeta}$ with $K_{\Re\zeta} := 1 - (\beta q)^{-1}\Re\zeta + q^{-1}\left(K + \frac{1}{2}\right)$ for $K \in \mathbb{N}_0$. Actually the choice of $K_{\Re\zeta}$ is arbitrary and we only need to make sure that it does not run through any pole of the integrand. Keeping in mind that we encircle the poles in the clockwise direction, the residue theorem for $m > 0$ yields

$$M_{\bar{\varepsilon}}(\zeta, m) = \frac{\sigma^{-\zeta}}{\beta q} \sum_{k=0}^{K} \frac{\Gamma(m+2)\Gamma\left(\frac{k}{q} - \frac{\zeta}{\beta q}\right)\Gamma\left(\frac{\zeta}{\beta}\right)}{\Gamma\left(m+2-\frac{\zeta}{\beta q}+\frac{k}{q}\right)\Gamma\left(\frac{\zeta}{\beta}-k\right)} \frac{(-1)^k}{k!} + R_H^q(\zeta, m, K), \tag{4.5.26}$$

with the remainder term given by

$$R_H^q(\zeta, m, K) := \sigma^{-\zeta} \frac{\Gamma\left(\frac{\zeta}{\beta}\right)}{\beta 2\pi i} \int\limits_{K_{\Re\zeta}-i\infty}^{K_{\Re\zeta}+i\infty} \frac{\Gamma(m+2)\Gamma(z-1)\Gamma\left(q-qz-\frac{\zeta}{\beta}\right)}{\Gamma(m+1+z)\Gamma(q-qz)} dz. \tag{4.5.27}$$

If we eventually let $K \to \infty$, it can be shown by means of simple estimates that this integral vanishes for any fixed $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$. By additional use of (4.5.20) this leads to (4.5.22). Moreover, appealing to the analytic continuation, this result can be extended to larger ζ-regions.

4.5.3. Uniform Bound with Respect to the Imaginary Part

We finally want to employ the integral representation (4.5.6) to derive a convenient bound for $M_{\bar{\varepsilon}}(\zeta, m)$ that is valid in its strip of analyticity. Therefore we first note that, according to (4.5.7), the function

$$f_q(s) := \frac{1-s^{\frac{1}{q}}}{1-s} \tag{4.5.28}$$

satisfies $\lim_{s\uparrow 1} f_q(s) = \frac{1}{q}$. It is thus continuous on $0 \le s \le 1$ and especially bounded away from zero. In particular, following for $0 \le s \le 1$ from $s \ge s^{\frac{1}{q}}$ if $0 < q \le 1$, we have $f_q(s) \ge 1$ then. Conversely, following for $0 \le s \le 1$ from $t \le t^{\frac{1}{q}}$ if $q \ge 1$, we have $q(1-s^{\frac{1}{q}}) = \int_s^1 t^{\frac{1}{q}-1} dt \ge \int_s^1 dt = 1-s$ then. As a consequence

$$\begin{cases} f_q(s) \ge 1, & \text{if } 0 < q \le 1, \\ f_q(s) \ge q^{-1}, & \text{if } q \ge 1. \end{cases} \tag{4.5.29}$$

In terms of (4.5.28) we can cast (4.5.6) in the following form:

$$\mathfrak{I}_H^q(-\zeta, m) = \int\limits_0^1 s^{-\frac{\zeta}{q}-1}(1-s)^{m+\zeta} \left\{f_q(s)\right\}^{\zeta-1} ds \tag{4.5.30}$$

From this representation, by virtue of simple bounds, with $\zeta = x+iy$ for $\zeta \in \mathbb{S}_{m,1}$ we deduce:

$$\begin{aligned} \left|\mathfrak{I}_H^q(-\zeta, m)\right| &\le \int\limits_0^1 s^{-\frac{x}{q}-1}(1-s)^{m+x} \left\{f_q(s)\right\}^{x-1} ds \\ &\le \max_{0\le s\le 1} \left\{f_q(s)\right\}^{x-1} \mathrm{B}\left(m+1+x, -\frac{x}{q}\right) \end{aligned}$$

Since $x < 0$ we have $\max_{0\le s\le 1}\left\{f_q(s)\right\}^{x-1} = \left\{\min_{0\le s\le 1} f_q(s)\right\}^{x-1}$. But in accordance with the previously observed boundary properties of f_q, the function attains its minimum at one of the endpoints of the unit interval. More precisely, from (4.5.29) we conclude $\min_{0\le s\le 1} f_q(s) =$

$\min\left\{1, q^{-1}\right\}$. For $\zeta \in \mathbb{S}_{m,1}$ we have thus proved

$$\left|\mathfrak{I}_H^q\left(-\zeta, m\right)\right| \leq \omega_q^{1-x} \frac{\Gamma\left(m+1+x\right)\Gamma\left(-\frac{x}{q}\right)}{\Gamma\left(m+1+x-\frac{x}{q}\right)}, \tag{4.5.31}$$

where we denote

$$\omega_q := \max\left\{1, q\right\}. \tag{4.5.32}$$

By comparison with (4.5.8) and (4.5.24) we observe that the above bound especially reflects the asymptotic behaviour of $M_{\bar{\varepsilon}}(\zeta, m)$ as $m \to \infty$ for fixed ζ in its strip of analyticity.

4.5.4. A Second m-Asymptotic Expansion for a Fixed Complex Argument

For some purposes the inverse factorial expansion (4.5.22) is inappropriate. An alternative expansion can be obtained by virtue of Watson's lemma. Therefore we introduce the function

$$g_q(t, \zeta) := e^{-2t}\left\{\frac{1-e^{-t}}{t}\right\}^{-\frac{\zeta}{q}-1}\left\{1-(1-e^{-t})^{\frac{1}{q}}\right\}^{\zeta-1}. \tag{4.5.33}$$

Then, starting with (4.5.6) we first make the change of variables $s = 1 - e^{-t}$, to cast the integral for any $\zeta \in \mathbb{S}_{m,1}$ as a Laplace transform. After proper rearrangement of the integrand we arrive at:

$$\begin{aligned}\mathfrak{I}_H^q(-\zeta, m) &= \int_0^\infty (1-e^{-t})^{-\frac{\zeta}{q}-1}\left\{1-(1-e^{-t})^{\frac{1}{q}}\right\}^{\zeta-1} e^{-(m+2)t}dt \\ &= \int_0^\infty t^{-\frac{\zeta}{q}-1}e^{-mt}g_q(t,\zeta)dt \end{aligned} \tag{4.5.34}$$

For fixed $\zeta \in \mathbb{S}_{m,1}$ the main contribution to this integral as $m \to \infty$ evidently comes from a neighborhood of the origin. There, in a circle of radius 2π the first two factors in (4.5.33), which constitute the generating function associated with the generalized Bernoulli polynomials, by analyticity possess a convergent expansion in terms of integer t-powers:

$$e^{-2t}\left\{\frac{t}{1-e^{-t}}\right\}^{\frac{\zeta}{q}+1} = \sum_{k=0}^{\infty} B_k^{(1+\frac{\zeta}{q})}(2)\frac{(-t)^k}{k!} \tag{4.5.35}$$

Here $B_k^{(\mu)}(z)$ denotes the generalized Bernoulli polynomial of degree $k \in \mathbb{N}_0$, order $\mu \in \mathbb{C}$ and argument $z \in \mathbb{C}$, see also equation (24.16.1) in [Olver et al., 2010] or §15.6 in [Temme, 2015]. It is defined as the k-th derivative of the left hand side in (4.5.35) evaluated at $t = 0$. Regarding the third factor in (4.5.33), for $t \geq 0$ and $\zeta \in \mathbb{C}$, from (4.5.21) we deduce

$$\left\{1-(1-e^{-t})^{\frac{1}{q}}\right\}^{\zeta-1} = \sum_{k=0}^{\infty} \frac{\Gamma(1-\zeta+k)}{\Gamma(1-\zeta)} \frac{t^{\frac{k}{q}}}{k!} \left\{\frac{1-e^{-t}}{t}\right\}^{\frac{k}{q}}. \tag{4.5.36}$$

The function in the curved brackets on the right hand side is again analytic in a circle of radius 2π around the origin and thus expandable in integer t-powers there. Summarizing, if we take the product of the series (4.5.35) and (4.5.36) and arrange the t-powers in descending order as $t \downarrow 0$, the first term in the resulting expansion is always the same whereas the powers of the subsequent terms depends on q. In other words, there exists an increasing sequence p_k with $k \in \mathbb{N}_0$ that grows to infinity and depends on q except $p_0 \equiv 0$, such that, as $t \downarrow 0$, the function (4.5.33) can be written in the form

$$g_q(t,\zeta) = \sum_{k=0}^{\infty} \gamma_{p_k}(\zeta) t^{p_k}. \tag{4.5.37}$$

The series converges absolutely for real $0 \leq t < 2\pi$, and the coefficients $\gamma_{p_k}(\zeta)$ are readily identified as polynomials of ζ with $\gamma_{p_0}(\zeta) \equiv 1$. By means of (4.5.37) similar to Watson's lemma, compare Theorem 3.1, ch. 3 in [Olver, 1974], for fixed $\zeta \in \mathbb{S}_{m,1}$ and $q > 0$, as $m \to \infty$, it eventually can be shown that

$$\mathfrak{I}_H^q(-\zeta, m) \sim m^{\frac{\zeta}{q}} \sum_{k=0}^{\infty} \frac{\gamma_{p_k}(\zeta)}{m^{p_k}} \Gamma\left(p_k - \frac{\zeta}{q}\right). \tag{4.5.38}$$

This establishes an expansion of the integral (4.5.34) with respect to the asymptotic scale m^{-p_k} for $k \in \mathbb{N}_0$. It must be emphasized, however, that contrary to (4.5.22) the series (4.5.38) is not absolutely convergent but merely valid in the asymptotic sense. Moreover, both have in common their pointwise validity with respect to ζ. Finally we remark, due to the relation of $\mathfrak{I}_H^q(\cdot, m)$ to the beta function if $q = 1$ the expansions (4.5.38) and (B.3.5) with $z = m$, $a = 1+\zeta$ and $b = 1$ differ only by the factor $\Gamma(-\zeta)$.

4.6. Exponential-Type Characteristic Functions

The Mellin transforms corresponding to the modulus of characteristic functions that are of exponential type (3.3.3) are particularly simple to determine. The integral representation (4.2.1) then takes on the form

$$M_X(\zeta) = \int_0^{\infty} t^{\zeta-1} \exp\{-p\theta^{\alpha} t^{\alpha}\}\, dt. \tag{4.6.1}$$

Our findings of Section 4.2 enable us to conclude analyticity of this integral in the entire right ζ-half plane, i.e., in

$$S_X = \{\zeta \in \mathbb{C} : \Re\zeta > 0\}\,. \tag{4.6.2}$$

Moreover, after a simple substitution the integral (4.6.1) can be referred to the gamma function, which immediately yields the extension to the whole complex ζ-plane in the shape of

$$M_X(\zeta) = \alpha^{-1}\left\{p^{\frac{1}{\alpha}}\theta\right\}^{-\zeta}\Gamma\left(\frac{\zeta}{\alpha}\right). \tag{4.6.3}$$

The asymptotic behaviour as $|\zeta| \to \infty$ in $|\arg(\zeta)| < \pi$ is then readily specified by means of Stirling's formula (B.2.28):

$$|M_X(\zeta)| \sim \sqrt{2\pi}\alpha^{-1}\left\{p^{\frac{1}{\alpha}}\theta\right\}^{-\Re\zeta}\left|\frac{\zeta}{\alpha}\right|^{\frac{\Re\zeta}{\alpha}-\frac{1}{2}} e^{-\frac{\Re\zeta}{\alpha}-\frac{\Im\zeta}{\alpha}\arg(\zeta)} \tag{4.6.4}$$

A useful inequality for the gamma function is provided by (B.2.32). However, in some circumstances a sharper result with respect to the imaginary part is preferable, particularly to apply Laplace's method. For this purpose in Section 2.5 in [Paris and Kaminski, 2001] an appropriate bound was deduced in (2.5.3). We adopt this approach for the derivation of a similar estimate. For $\tau \in \mathbb{R}$, sufficiently large $s > 0$ and fixed $a \in \mathbb{R}$ such that $a + s > 0$ is still large, according to Stirling's formula (B.2.28), there exists a constant $K > 0$ with the following property:

$$\begin{aligned} |\Gamma(a+s+i\tau)| &\le Ke^{-s}\,|a+s+i\tau|^{a+s-\frac{1}{2}}\,e^{-\tau\arg(a+s+i\tau)} \\ &= Ke^{-s}s^{a+s-\frac{1}{2}}\left\{1+\frac{\tau^2}{s^2}\right\}^{\frac{a+s}{2}-\frac{1}{4}} e^{-\tau\arg\left(\frac{i\tau}{s}\right)} \\ &\quad\times\left|\frac{a+s+i\tau}{s+i\tau}\right|^{a+s-\frac{1}{2}} e^{-\tau\left\{\arg(a+s+i\tau)-\arg\left(\frac{i\tau}{s}\right)\right\}} \end{aligned} \tag{4.6.5}$$

The last factor on the right hand side approaches a finite positive value as $\tau \to \pm\infty$. This is true because the distance of the argument functions tends to zero with the same rate as τ tends to infinity. In particular, subject to the rule of de l'Hospital we have for $\sigma_1, \sigma_2 > 0$:

$$\lim_{\tau\to\infty} \pm\tau\left(\arg(\sigma_1 \pm i\tau) - \arg(\sigma_2 \pm i\tau)\right) = \lim_{\tau\to\infty}\frac{\arctan\left(\frac{\tau}{\sigma_1}\right) - \arctan\left(\frac{\tau}{\sigma_2}\right)}{\tau^{-1}} = \sigma_2 - \sigma_1 \tag{4.6.6}$$

As a consequence, as $\tau \to \pm\infty$ with fixed s the last two factors in (4.6.5) approach a finite non-zero limit. Similarly as $s \to \infty$ with τ fixed. Accordingly, there exists another constant $K_2 > 0$ such that for sufficiently large $|s + i\tau|$ we arrive at

$$|\Gamma(a+s+i\tau)| \le K_2 e^{-s}s^{a+s-\frac{1}{2}}\left\{1+\frac{\tau^2}{s^2}\right\}^{\frac{a}{2}-\frac{1}{4}} e^{-s\varphi\left(\frac{\tau}{s}\right)}, \tag{4.6.7}$$

where the phase, with the argument function in terms of the arctangent, is given by

$$\varphi(v) := v \arctan(v) - \frac{1}{2} \log\left(1 + v^2\right). \tag{4.6.8}$$

A comparison of (4.6.7) with (B.2.32) shows that the main difference consists in the presence of the arctangent in the exponent instead of its asymptote $\frac{\pi}{2}$. This yields some convenient simplifications with respect to the location of the saddle point when applying Laplace's method.

4.7. Exponential-Type m-Powers

Consider a characteristic function $\Phi_{\bar{\varepsilon}}$ of exponential-type (3.3.4). From its series expansion in powers of $|t|^{\beta}$, by comparison with Subsection 4.3.2, we ascertain absolute convergence and analyticity of the integral (4.3.14) for any $m \geq 0$ in

$$\mathbb{S}_{\bar{\varepsilon}} = \{\zeta \in \mathbb{C} : -\beta < \Re\zeta < 0\}, \tag{4.7.1}$$

$$\mathbb{S}_{m,\beta} = \{\zeta \in \mathbb{C} : -\beta(m+1) < \Re\zeta < 0\}. \tag{4.7.2}$$

In addition, the first derivative of $\Phi_{\bar{\varepsilon}}(t)$ with respect to $t > 0$ is given by

$$\Phi_{\bar{\varepsilon}}'(t) = -q\beta\sigma^{\beta} t^{\beta-1} \Phi_{\bar{\varepsilon}}(t). \tag{4.7.3}$$

After plugging this into (4.3.21) and substituting $t = q^{-\frac{1}{\beta}} \sigma^{-1} s^{\frac{1}{\beta}}$ we arrive at

$$M_{\bar{\varepsilon}}(\zeta, m) = -\frac{m+1}{\zeta} \left\{q^{\frac{1}{\beta}}\sigma\right\}^{-\zeta} \int_0^{\infty} t^{\frac{\zeta}{\beta}} e^{-t} \left(1 - e^{-t}\right)^m dt. \tag{4.7.4}$$

Since the estimate (4.3.22) applies with arbitrary $\eta_{\bar{\varepsilon}} > 1$, the right-hand side of (4.7.4) constitutes the analytic continuation of (4.3.14) into the half plane $\Re\zeta > -\beta(m+1)$. It shows that the only singularity therein is a simple pole located at the origin. Evidently from (4.7.4) we have a close connection between $M_{\bar{\varepsilon}}(\cdot, m)$ and the gamma function. Indeed, for $m = 0$ this integral equals the gamma function. Besides we also recognize a similarity between the integrand in (4.7.4) and the generating function of the generalized Bernoulli polynomials, compare definition (24.16.1) in [Olver et al., 2010]. More precisely, the integral can be considered its Mellin transform.

Following from (4.3.3), the series expansion of the m-power in (4.7.4) is bounded by 2^m for any $m \geq 0$. Hence, if in (4.7.4) we apply the binomial theorem, for any $\zeta \in \mathbb{S}_{\bar{\varepsilon}}$ and $m \geq 0$ subject to absolute convergence we may interchange the order of summation and integration, which leads to:

$$M_{\bar{\varepsilon}}(\zeta, m) = -\frac{m+1}{\zeta} \left\{q^{\frac{1}{\beta}}\sigma\right\}^{-\zeta} \sum_{l=0}^{\infty} \binom{m}{l} (-1)^l \int_0^{\infty} t^{\frac{\zeta}{\beta}} e^{-(1+l)t} dt$$

$$= -\frac{m+1}{\zeta}\left\{q^{\frac{1}{\beta}}\sigma\right\}^{-\zeta}\Gamma\left(\frac{\zeta}{\beta}\right)\sum_{l=0}^{\infty}\binom{m}{l}(-1)^l(1+l)^{-1-\frac{\zeta}{\beta}} \tag{4.7.5}$$

Particularly for $m \in \mathbb{N}$ this series cancels to a finite sum. For general $m \geq 0$ it can be verified holomorphic in the half plane $\Re\zeta > -\beta(m+1)$, subject to absolute and uniform convergence in any compact subset therein.

4.7.1. Derivation of the Binomial Integral Representation

To obtain the binomial integral representation associated with (4.7.4) we first introduce the coefficient function (4.3.24). According to (4.7.3), in the present setup it has the integral definition

$$A(\zeta, -z) = -q\beta\sigma^{\beta}\int_0^{\infty} t^{\zeta+\beta-1}e^{-(1-z)q\sigma^{\beta}t^{\beta}}dt. \tag{4.7.6}$$

It is easy to see that this integral converges absolutely and is holomorphic for any fixed $z \in \mathbb{C}$ with $\Re z < 1$ in

$$\mathbb{S}'_{\bar{\varepsilon}} = \{\zeta \in \mathbb{C} : \Re\zeta > -\beta\}. \tag{4.7.7}$$

By virtue of a simple substitution we refer (4.7.6) to the gamma function, which immediately gives access to its analytic continuation to the whole complex plane::

$$A(\zeta, -z) = -\left\{q^{\frac{1}{\beta}}\sigma\right\}^{-\zeta}\Gamma\left(1+\frac{\zeta}{\beta}\right)(1-z)^{-1-\frac{\zeta}{\beta}} \tag{4.7.8}$$

Here, the argument $\zeta \in \mathbb{C}$ determines if the analytic continuation of (4.7.6) with respect to $z \in \mathbb{C}$ is entire, meromorphic or multi-valued with branch points at $z \in \{-1, \infty\}$. In the latter case we choose the branch cut to match the axis $(-\infty, -1]$, i.e., $|\arg(1-z)| < \pi$. Now, supposing $\mathbb{S}''_{\bar{\varepsilon}} \equiv \{\zeta \in \mathbb{C} : -\beta < \Re z < 0\}$ we have $\mathbb{S}''_{\bar{\varepsilon}} = \mathbb{S}_{\bar{\varepsilon}} \cap \mathbb{S}'_{\bar{\varepsilon}}$. It is then ascertainable from (4.7.6) that the integral definition of $A(\zeta, -z)$ converges absolutely for any $\Re z < 1$ and $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$. Hence, in accordance with (4.3.25), upon application of the functional equation for the gamma function, for $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$ and $m > 0$, we obtain:

$$\begin{aligned} M_{\bar{\varepsilon}}(\zeta, m) &= -\left\{q^{\frac{1}{\beta}}\sigma\right\}^{-\zeta}\Gamma\left(\frac{\zeta}{\beta}\right)\frac{m+1}{\beta 2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(m+1)\Gamma(t)}{\Gamma(m+1+t)}(1-t)^{-1-\frac{\zeta}{\beta}}dt \\ &= -\left\{q^{\frac{1}{\beta}}\sigma\right\}^{-\zeta}\frac{\Gamma\left(\frac{\zeta}{\beta}\right)}{\beta 2\pi i}\int_{c_0-i\infty}^{c_0+i\infty}\frac{\Gamma(m+2)\Gamma(1+t)}{\Gamma(m+2+t)}(-t)^{-1-\frac{\zeta}{\beta}}dt \end{aligned} \tag{4.7.9}$$

In the first equality the integration path is a vertical line with $0 < c < 1$, separating the poles at $-\mathbb{N}_0$ from the branch cut at $[1, \infty)$, whereas the second equality results from a simple substitution with $c_0 = c - 1$. By (4.3.6) there exists a constant $A > 0$, which does especially not depend on ζ, such that for sufficiently large $|t|$ in $|\arg(-t)| < \pi$ we have

$$\left|\frac{\Gamma(1+t)}{\Gamma(m+2+t)}(-t)^{-\frac{\zeta}{\beta}-1}\right| \leq A\,|t|^{-m-2-\frac{\Re\zeta}{\beta}}\,e^{\frac{\pi}{\beta}|\Im\zeta|}. \tag{4.7.10}$$

It is thus a routine step to verify absolute convergence of the integral (4.7.9) for

$$\Re\zeta > -\beta(m+1), \tag{4.7.11}$$

and in addition uniformity in any compact subset therein. Consequently, appealing to Theorem A.2.1, the representation (4.7.9) equals (4.7.4) not only for $\zeta \in \mathbb{S}''_{\bar{\varepsilon}}$ but by analytic continuation in the whole half plane (4.7.11). Finally, the integral appearing in (4.7.9) will play an important role also in later chapters, which is the reason why we denote it for $m \geq 0$ and $|\arg(-t)| < \pi$ by

$$\mathcal{M}_{\mathrm{B}}(-\zeta, m) := -\frac{\Gamma(\zeta)}{2\pi i}\int\limits_{c_0-i\infty}^{c_0+i\infty}\frac{\Gamma(m+2)\Gamma(1+t)}{\Gamma(m+2+t)}(-t)^{-\zeta-1}dt, \qquad \Re\zeta > -(m+1). \tag{4.7.12}$$

This definition enables us to write (4.7.9) for any $\zeta \in \mathbb{S}_{m,\beta}$ and $m > 0$ in the form

$$M_{\bar{\varepsilon}}(\zeta, m) = \beta^{-1}\left\{q^{\frac{1}{\beta}}\sigma\right\}^{-\zeta}\mathcal{M}_{\mathrm{B}}\left(-\frac{\zeta}{\beta}, m\right). \tag{4.7.13}$$

4.7.2. Properties of the Binomial Integral Representation

The integral (4.7.12) furnishes a good setting for a discussion of the fundamental properties of $M_{\bar{\varepsilon}}(\zeta, m)$. As we already mentioned earlier, the argument ζ determines the type of the continuation (4.7.8) associated with the coefficient function and thus of the integrand in (4.7.12). This suggests a distinction between two cases:

- The integrand constitutes a meromorphic function with respect to t if $\zeta \in \mathbb{Z}$ with $\Re\zeta > -(m+1)$. Particularly if $\zeta \in -\mathbb{N}$ there are simple poles at any integer $-(m+1) \leq t \leq -1$ and the integrand is analytic in the entire half plane to the right of the integration path. According to Cauchy's theorem, the whole integral thus equals zero. This corresponds to the phenomenon that the series appearing in (4.7.5) also equals zero for $\zeta \in -\beta\mathbb{N}$ with $\Re\zeta > -\beta(m+1)$. On the other hand, if $\zeta \in \mathbb{N}_0$ the integrand has an additional pole of order $1+\zeta$ at $t=0$. If we imagine the integration path $c_0 + i\tau$ with $\tau \in \mathbb{R}$ as the left edge of a rectangle of infinite height and width in the complex t-plane, the pole at $t=0$ is encircled in the clockwise direction. Accordingly, the residue theorem yields for $\zeta \in \mathbb{N}_0$:

$$\begin{aligned}\mathcal{M}_{\mathrm{B}}(-\zeta, m) &= \Gamma(\zeta)\operatorname*{Res}_{t=0}\frac{\Gamma(m+2)\Gamma(1+t)}{\Gamma(m+2+t)}(-t)^{-1-\zeta}\\ &= \frac{(-1)^{1+\zeta}}{\zeta}\left[\frac{d^{\zeta}}{dt^{\zeta}}\frac{\Gamma(m+2)\Gamma(1+t)}{\Gamma(m+2+t)}\right]_{t=0}\end{aligned} \tag{4.7.14}$$

The derivatives can be expressed in terms of the polygamma function $\psi^{(k)}$ with $k \in \mathbb{N}_0$, compare (B.2.12), and the first two are given by

$$\left[\frac{d}{dt}\frac{\Gamma(m+2)\Gamma(1+t)}{\Gamma(m+2+t)}\right]_{t=0} = \psi(1)-\psi(m+2),$$
$$\left[\frac{d^2}{dt^2}\frac{\Gamma(m+2)\Gamma(1+t)}{\Gamma(m+2+t)}\right]_{t=0} = \{\psi(m+2)-\psi(1)\}^2+\left\{\psi^{(1)}(1)-\psi^{(1)}(m+2)\right\}.$$

To find a closed formula for derivatives of higher order seems to be cumbersome. However, comparison of (4.7.13) and (4.7.14) with (4.7.5) gives a deeper insight about the structure of the binomial series (4.7.5). For instance with $\zeta=\beta$ and $\zeta=2\beta$, respectively, it shows

$$\psi(1)-\psi(m+2) = -(m+1)\sum_{l=0}^{\infty}\binom{m}{l}(-1)^l(1+l)^{-2},$$
$$\{\psi(m+2)-\psi(1)\}^2+\left\{\psi^{(1)}(1)-\psi^{(1)}(m+2)\right\} = 2(m+1)\sum_{l=0}^{\infty}\binom{m}{l}(-1)^l(1+l)^{-3}.$$

If in addition $m\in\mathbb{N}_0$, the polygamma functions have a nice series representation. Thus, in this case each of the above binomial series is comprised of a finite number of single series. For example the left hand side of the first equation then represents the $(m+1)$-th harmonic number:

$$\sum_{k=1}^{m+1}\frac{1}{k} = \psi(m+2)-\psi(1) = (m+1)\sum_{l=0}^{\infty}\binom{m}{l}(-1)^l(1+l)^{-2}$$

Summarizing, the binomial series (4.7.5) has a very intricate structure.

- The integrand in (4.7.12) is a multi-valued function if $\Re\zeta > -(m+1)$ and $\zeta\notin\mathbb{Z}$, exhibiting a branch cut along the positive real axis. We may then exploit the fact that the limits of the integrand at the upper and lower end of the branch cut differ. In particular, letting $m\geq 0$, $-1<\Re\zeta<0$ and $\Im\zeta=0$ we may bend the integration path to a loop, encircling the branch cut in the negative sense, and eventually collapse this loop around the positive real axis to result in an integral along the positive real axis. This is a routine procedure in complex analysis and leads to the integral representation

$$\mathcal{M}_{\mathrm{B}}(-\zeta,m) = \frac{1}{\Gamma(1-\zeta)}\int_0^{\infty}\frac{\Gamma(m+2)\Gamma(1+x)}{\Gamma(m+2+x)}x^{-1-\zeta}dx. \tag{4.7.15}$$

It converges absolutely and uniformly in any compact subset in $\mathbb{S}_{m,1}$ for $m\geq 0$ and there, by analytic continuation, it thus coincides with the integral (4.7.12). The representation (4.7.15) can also be obtained from (4.7.4) without complex analysis. To show this backwards, in (4.7.15) we first apply the functional equation for the gamma function. Then we introduce the integral representation for the beta function that was given in (A.3.8) and, subject to absolute convergence, interchange the order of integration to obtain for

$\zeta \in \mathbb{S}_{m,1}$ and $m \geq 0$:

$$\begin{aligned}\mathcal{M}_{\mathrm{B}}(-\zeta, m) &= \frac{m+1}{\Gamma(1-\zeta)} \int_0^\infty \frac{\Gamma(m+1)\Gamma(1+x)}{\Gamma(m+2+x)} x^{-1-\zeta} dx \\ &= \frac{m+1}{\Gamma(1-\zeta)} \int_0^\infty x^{-\zeta} \int_0^\infty e^{-(1+x)t}(1-e^{-t})^m dt dx \\ &= -\frac{m+1}{\zeta} \int_0^\infty t^\zeta e^{-t}(1-e^{-t})^m dt \end{aligned} \tag{4.7.16}$$

The integral (4.7.15) can be conceived as the Mellin transform of the beta function $\mathrm{B}(m+2, x)$ with respect to the second argument. We therefore denote it as the *beta Mellin transform* and refer to it as an integral of Ramanujan-type, since he was apparently the first mathematician to calculate integrals over the real-axis that involve gamma functions in the integrand, compare §3.3.5 in [Paris and Kaminski, 2001]. However, he was concerned with Fourier transforms of ratios of gamma functions rather than with their Mellin transforms. In fact, the integral (4.7.15) seems to be unknown in the literature.

The representation (4.7.15) will play an important role in our subsequent discussion on the derivation of an asymptotic expansion for $M_{\bar{\varepsilon}}(\zeta, m)$ as $m \to \infty$. In before we remark, similar to (4.7.16) or by comparison of (4.7.13) with (4.3.14) one can show that $\mathcal{M}_{\mathrm{B}}(-\zeta, m)$ basically constitutes the Mellin transform of the function $(1-e^{-t})^{m+1}$ with strip of analyticity $\mathbb{S}_{m,1}$. Hence, Theorem 4.7.2 in [Bleistein and Handelsman, 1986] for any $\eta > 0$ as $|\Im\zeta| \to \infty$ in $\mathbb{S}_{m,1}$ yields

$$\mathcal{M}_{\mathrm{B}}(-\zeta, m) = \mathcal{O}\left\{e^{-\left(\frac{\pi}{2}-\eta\right)|\Im\zeta|}\right\}. \tag{4.7.17}$$

The beta Mellin transform is therefore particularly absolutely integrable along any line that runs parallel to the imaginary axis in its strip of analyticity. It thus follows from the inversion formula for Mellin transforms for $m \geq 0$ and $-(m+1) < \delta_0 < 0$, compare Theorem A.5.1:

$$(1-e^{-t})^{m+1} = \frac{1}{2\pi i} \int_{\delta_0 - i\infty}^{\delta_0 + i\infty} t^{-z} \mathcal{M}_{\mathrm{B}}(-z, m) dz, \qquad t > 0 \tag{4.7.18}$$

This integral especially converges absolutely.

4.7.3. m-Asymptotic Expansion for a Fixed Complex Argument

Concerning the evaluation of $\mathcal{M}_{\mathrm{B}}(-\zeta, m)$ as $m \to \infty$ we refrain from an application of the standard expansion (B.3.5), approximating the ratio of gamma functions by a simple algebraic power. Instead we shall employ a different expansion for a ratio of gamma functions that involves the polygamma functions. To cover more general situations we generalize the *beta*

Mellin transform by defining for fixed $a > -1$ the integral

$$\mathcal{M}_{\mathrm{B}}(-\zeta, m, a) := \frac{1}{\Gamma(1-\zeta)} \int_0^\infty t^{-\zeta-1} e^{\varphi(t,m,a)} dt, \tag{4.7.19}$$

which features the phase function

$$\varphi(t, m, a) := \log \frac{\Gamma(m+2)\Gamma(1+a+t)}{\Gamma(m+2+a+t)}. \tag{4.7.20}$$

In accordance with (B.3.5), the integral (4.7.19) is holomorphic with respect to ζ in $\mathbb{S}_{m,1}$ for any $m \geq 0$, $a > -1$. Furthermore, a comparison with (4.7.15) shows $\mathcal{M}_{\mathrm{B}}(-\zeta, m) \equiv \mathcal{M}_{\mathrm{B}}(-\zeta, m, 0)$. The derivative of the phase is a difference of digamma functions

$$\frac{d}{dt}\varphi(t, m, a) = \psi(1+a+t) - \psi(m+2+a+t), \tag{4.7.21}$$

which, subject to (B.2.29), diverges logarithmically for fixed $t \geq 0$ and $a > -1$ as $m \to \infty$. Moreover, as a function of $t \geq 0$ it commences at $\psi(1+a) - \psi(m+2+a) < 0$ and monotonically approaches the positive real t-axis at infinity. Therefore the phase function is convex and does not possess any saddle points on $[0, \infty)$. However, it is decreasing and attains its minimum value along the integration path at $t = 0$. The asymptotic behaviour of $\mathcal{M}_{\mathrm{B}}(-\zeta, m, a)$ as $m \to \infty$ is thus determined in a neighborhood of the origin, whence the integral is of Laplace-type for any $a > -1$. To ascertain the corresponding leading term we need to expand the phase function at the origin.

4.7.3.1. An Expansion for the Beta Function

The greatest circle centered at the origin in which the phase is analytic with respect to t is controlled by the nearest singularity at $t = -1 - a$. As a consequence, by definition of the polygamma functions for $|t| < 1 + a$ we can write:

$$\begin{aligned} \varphi(t, m, a) &= \log \frac{\Gamma(1+a)}{\Gamma(m+2+a)} + \sum_{j=1}^\infty \frac{t^j}{j!} \left\{ \psi^{(j-1)}(1+a) - \psi^{(j-1)}(m+2+a) \right\} \\ &= \log \frac{\Gamma(1+a)}{\Gamma(m+2+a)} + t \left\{ \psi(1+a) - \psi(m+2+a) \right\} + t^2 r(t, m, a) \end{aligned} \tag{4.7.22}$$

Here we denote:

$$r(t, m, a) := \sum_{j=0}^\infty \frac{t^j}{(2+j)!} \left\{ \psi^{(j+1)}(1+a) - \psi^{(j+1)}(m+2+a) \right\} \tag{4.7.23}$$

$$= \sum_{j=0}^\infty \frac{(-t)^j}{2+j} \left\{ \zeta(2+j, 1+a) - \zeta(2+j, m+2+a) \right\} \tag{4.7.24}$$

For the last equality we applied the connection formula for the Hurwitz zeta and the polygamma function, compare eq. (25.11.12) in [Olver et al., 2010]:

$$\Gamma(n+1)\zeta(n+1,a) = (-1)^{n+1}\psi^{(n)}(a), \qquad \Re a > 0, n \in \mathbb{N} \tag{4.7.25}$$

Provided $\Re s > 1$ and $b \in \mathbb{C} \setminus -\mathbb{N}_0$ the Hurwitz zeta function equals the Dirichlet-type series

$$\zeta(s,b) = \sum_{n=0}^{\infty} \frac{1}{(n+b)^s}. \tag{4.7.26}$$

Clearly, $\zeta(s,b)$ is uniformly bounded with respect to $b \geq b_0$ for any $b_0 > 0$ and fixed $\Re s > 1$. Moreover, it is also uniformly bounded with respect to $\Im s \in \mathbb{R}$ and especially $\mathcal{O}(b^{-\Re s})$ as $|s| \to \infty$ in $\Re s > 1$, assuming $b > 0$ is fixed. The series (4.7.24) can therefore be transformed to the contour integral

$$r(t,m,a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{t^z \pi}{(2+z)\sin(-\pi z)} \left\{\zeta(2+z,1+a) - \zeta(2+z,m+2+a)\right\} dz, \tag{4.7.27}$$

in which the integration path is a vertical line with real part $-1 < c < 0$. This is readily verified by observing that as $|z| \to \infty$ in $\Re z > -1$ the integrand exhibits the following asymptotic behaviour for any $t \in \mathbb{C}$, $m \geq 0$, $a > -1$:

$$\frac{t^z \pi}{(2+z)\sin(-\pi z)} \left\{\zeta(2+z,1+a) - \zeta(2+z,m+2+a)\right\} = \mathcal{O}\left\{\frac{1}{|z|}\left\{\frac{|t|}{1+a}\right\}^{\Re z} e^{-|\Im z|\pi - \Im z \arg(t)}\right\}$$

For $|t| < 1+a$ with $|\arg(t)| < \pi$ it yields exponential decay, justifying a rightward displacement of the integration path in (4.7.27) to infinity in $\Re z > -1$ over the infinite sequence of simple poles of the cosecant, located at the non-negative integers. Subject to the residue theorem this returns the series (4.7.24). The $\mathbb{C}$-domain of admissible t-values for the integral (4.7.27) is much larger. To show this we choose $c = -\frac{1}{2}$, implying $|\sin(\pi\,(c+iy))| = \cosh(\pi y)$, and, according to (4.7.26), we have

$$\left|\zeta\left(\frac{3}{2}+iy, 1+a\right) - \zeta\left(\frac{3}{2}+iy, m+2+a\right)\right| \leq 2\zeta\left(\frac{3}{2}, 1+a\right), \qquad m \geq 0,\ a > -1.$$

Applying this bound to (4.7.27) after substituting $y = -i\,(z-c)$, and employing the triangle inequality leads to:

$$\begin{aligned} |r(t,m,a)| &\leq |t|^{-\frac{1}{2}}\, \zeta\left(\frac{3}{2}, 1+a\right) \int_{-\infty}^{\infty} \frac{e^{-y\arg(t)}}{\left(\frac{9}{4}+y^2\right)^{\frac{1}{2}} \cosh(\pi y)} dy \\ &= 2\,|t|^{-\frac{1}{2}}\, \zeta\left(\frac{3}{2}, 1+a\right) \int_{0}^{\infty} \frac{\cosh(y\arg(t))}{\left(\frac{9}{4}+y^2\right)^{\frac{1}{2}} \cosh(\pi y)} dy \end{aligned} \tag{4.7.28}$$

It is easily seen that the absolute convergence of this integral also requires $|\arg(t)| < \pi$, in which case the convergence is again uniform in any compact subset. In addition we observe that the bound (4.7.28) does not depend on m. Finally we note that the series representation of the exponential function implies

$$\left|e^{t^2 r(t,m,a)}\right| \leq e^{|t|^2 |r(t,m,a)|}.$$

By definition of the phase function (4.7.20) and (4.7.22), for $m \geq 0$ and $a > -1$ in $|\arg(t)| < \pi$ we have thus verified by analytic continuation

$$\frac{\Gamma(m+2)\Gamma(1+a+t)}{\Gamma(m+2+a+t)} = \frac{\Gamma(m+2)\Gamma(1+a)}{\Gamma(m+2+a)} e^{-t(\psi(m+2+a)-\psi(1+a))+t^2 r(t,m,a)}. \tag{4.7.29}$$

The remainder function $r(t,m,a)$ was given in (4.7.27) and is uniformly bounded with respect to $m \geq 0$ for any $a > -1$. For convenience we introduce the following notation:

$$H_{m+1}(a) := \psi(m+2+a) - \psi(1+a) \tag{4.7.30}$$

$$p(m,n,a) := \begin{cases} 0, & \text{for } n = 1 \\ \psi^{(n-1)}(1+a) - \psi^{(n-1)}(m+2+a), & \text{for } n \geq 2 \end{cases} \tag{4.7.31}$$

In the special case $a = 0$ we write

$$H_{m+1}(0) := H_{m+1} = \psi(m+2) + \gamma, \tag{4.7.32}$$

where γ denotes the Euler-Mascheroni constant, see (B.2.4). For $m \in \mathbb{N}_0$ definition (4.7.32) equals the $(m+1)$-th harmonic number. According to (B.2.29), the controlling behaviour of (4.7.30) as $m \to \infty$ is

$$H_{m+1}(a) \sim \log(m). \tag{4.7.33}$$

Suppose now $|t| < 1 + a$. Then we may employ the series representation (4.7.23) for $r(t,m,a)$ to expand the second exponential function in (4.7.29) in terms of Bell polynomials. In particular, by making use of the notation (4.7.31) we have:

$$\begin{aligned} \exp\left\{t^2 r(t,m,a)\right\} &= \exp\left\{\sum_{j=1}^{\infty} \frac{t^j}{j!} p(m,j,a)\right\} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(p(m,1,a), \ldots, p(m,n,a)) \end{aligned} \tag{4.7.34}$$

The expression $B_n(\dots)$ denotes the n-th complete exponential Bell polynomial with the first few given by:

$$\begin{aligned} B_0 = 1 \qquad\qquad B_1(x_1) = x_1 \qquad\qquad B_2(x_1,x_2) &= x_1^2 + x_2 \\ B_3(x_1,x_2,x_3) &= x_1^3 + 3x_1x_2 + x_3 \\ B_4(x_1,x_2,x_3,x_4) &= x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4 \end{aligned} \tag{4.7.35}$$

Combining the expansion (4.7.34) with (4.7.29), eventually for $|t| < 1+a$ in $|\arg(t)| < \pi$ and $m \geq 0$, $a > -1$ leads to

$$\frac{\Gamma(m+2)\Gamma(1+a+t)}{\Gamma(m+2+a+t)} = \frac{\Gamma(m+2)\Gamma(1+a)}{\Gamma(m+2+a)} e^{-H_{m+1}(a)t} \sum_{j=0}^{\infty} t^j g(j,m,a), \tag{4.7.36}$$

where we denote

$$g(j,m,a) := \frac{1}{j!} B_j(p(m,1,a), \dots, p(m,j,a)), \tag{4.7.37}$$

and we abbreviate $g(j,m) := g(j,m,0)$. As a consequence of our preceding observations the coefficients in (4.7.36) are also uniformly bounded with respect to $m \geq 0$. Equivalently, by comparison of (4.7.29) and (4.7.36), it is ascertainable that, in accordance with Cauchy's formula, for any $m \geq 0$, $a > -1$ and $j \in \mathbb{N}_0$ the coefficients can be written in the form

$$g(j,m,a) = \frac{\Gamma(m+2+a)}{\Gamma(1+a)} \frac{1}{2\pi i} \oint_C \frac{\Gamma(1+a+z)}{\Gamma(m+2+a+z)} e^{H_{m+1}(a)z} z^{-j-1} dz, \tag{4.7.38}$$

where the integration path C encircles the origin in the positive sense, excluding any additional pole of the integrand. As the essential difference between (4.7.36) and the Maclaurin expansion at the origin, which is rather given by

$$\frac{\Gamma(m+2)\Gamma(1+a+t)}{\Gamma(m+2+a+t)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[\frac{d^k}{dt^k} \frac{\Gamma(m+2)\Gamma(1+a+t)}{\Gamma(m+2+a+t)} \right]_{t=0}, \qquad |t| < 1+a, \tag{4.7.39}$$

it must be emphasized that the coefficients in the latter series are unbounded as $m \to \infty$ for any $a > -1$.

4.7.3.2. Application of the Expansion

To apply the preceding results for the derivation of an asymptotic expansion for $\mathcal{M}_{\mathrm{B}}(-\zeta,m,a)$, for $J \in \mathbb{N}$ and $t, m \geq 0$ and $a > -1$ we introduce the function

$$G_J(t,m,a) := \frac{\Gamma(m+2)\Gamma(1+a+t)}{\Gamma(m+2+a+t)} - \frac{\Gamma(m+2)\Gamma(1+a)}{\Gamma(m+2+a)} e^{-H_{m+1}(a)t} \sum_{j=0}^{J-1} t^j g(j,m,a). \tag{4.7.40}$$

In accordance with (4.7.36), for some constants $A, B > 0$ and $0 < a_0 < 1 + a$ it satisfies

$$|G_J(t,m,a)| \leq \begin{cases} A\frac{\Gamma(m+2)}{\Gamma(m+2+a)}t^J e^{-H_{m+1}(a)t}, & \text{for } t \in [0, a_0), \\ \frac{\Gamma(m+2)\Gamma(1+a+t)}{\Gamma(m+2+a+t)} + B\frac{\Gamma(m+2)}{\Gamma(m+2+a)}t^{J-1}e^{-H_{m+1}(a)t}, & \text{for } t \geq a_0. \end{cases} \tag{4.7.41}$$

By means of (4.7.40) it is possible to telescope $\mathcal{M}_{\mathrm{B}}(-\zeta, m, a)$ for fixed $\zeta \in \mathbb{S}_{m,1}$ in the following manner:

$$\begin{aligned} \mathcal{M}_{\mathrm{B}}(-\zeta,m,a) &= \frac{\Gamma(m+2)\Gamma(1+a)}{\Gamma(m+2+a)}\sum_{j=0}^{J-1} g(j,m,a)\frac{1}{\Gamma(1-\zeta)}\int_0^\infty t^{j-\zeta-1}e^{-H_{m+1}(a)t}dt \\ &\quad + \frac{1}{\Gamma(1-\zeta)}\int_0^\infty t^{-\zeta-1}G_J(t,m,a)dt \\ &= \frac{\Gamma(m+2)\Gamma(1+a)}{\Gamma(m+2+a)}\sum_{j=0}^{J-1} g(j,m,a)\frac{\Gamma(j-\zeta)}{\Gamma(1-\zeta)}\{H_{m+1}(a)\}^{\zeta-j} + R_J(-\zeta,m,a) \end{aligned} \tag{4.7.42}$$

The remainder term is given by

$$R_J(-\zeta,m,a) := \frac{1}{\Gamma(1-\zeta)}\int_0^\infty t^{-\zeta-1}G_J(t,m,a)dt, \tag{4.7.43}$$

and it is by definition of $G_J(t,m,a)$ absolutely convergent for $-(m+1) < \Re\zeta < J$ and readily verified uniformly convergent in any compact subset therein. Hence, the right hand side of (4.7.42) constitutes the analytic continuation of $\mathcal{M}_{\mathrm{B}}(-\zeta,m,a)$ into the strip $-(m+1) < \Re\zeta < J$ for any $m \geq 0$ and $a > -1$. To show the asymptotic validity of this expansion for large m we must now study the remainder integral.

4.7.3.3. Estimation of the Remainder Integral

If, for brevity we denote

$$S_J(-\Re\zeta,m,a) := \int_0^\infty (a_0+t)^{-\Re\zeta-1}\frac{\Gamma(m+2)\Gamma(1+a+a_0+t)}{\Gamma(m+2+a+a_0+t)}dt, \tag{4.7.44}$$

$$T_J(-\Re\zeta,m,a) := \frac{\Gamma(m+2)}{\Gamma(m+2+a)}\int_{a_0}^\infty t^{J-\Re\zeta-2}e^{-H_{m+1}(a)t}dt, \tag{4.7.45}$$

for fixed $m \geq 0$, $-(m+1) < \Re\zeta < J$ and $a > -1$ we obtain by virtue of the estimates (4.7.41):

$$|R_J(-\zeta,m,a)| \leq \frac{A\Gamma(m+2)}{\Gamma(m+2+a)\,|\Gamma(1-\zeta)|}\int_0^{a_0} t^{J-\Re\zeta-1}e^{-H_{m+1}(a)t}dt$$

$$+\frac{1}{|\Gamma(1-\zeta)|}\int_{a_0}^{\infty} t^{-\Re\zeta-1}\frac{\Gamma(m+2)\Gamma(1+a+t)}{\Gamma(m+2+a+t)}dt$$

$$+\frac{B\Gamma(m+2)}{\Gamma(m+2+a)\,|\Gamma(1-\zeta)|}\int_{a_0}^{\infty} t^{J-\Re\zeta-2}e^{-H_{m+1}(a)t}dt$$

$$\le \{H_{m+1}(a)\}^{\Re\zeta-J}\,\frac{A\Gamma(m+2)\Gamma(J-\Re\zeta)}{\Gamma(m+2+a)\,|\Gamma(1-\zeta)|} \tag{4.7.46}$$

$$+\frac{1}{|\Gamma(1-\zeta)|}\left\{S_J(-\Re\zeta,m,a)+BT_J(-\Re\zeta,m,a)\right\}$$

To determine the order of (4.7.44) is slightly more difficult than in the situation of a standard Laplace-type integral such as those covered by Watson's lemma. The reason consists in the particular form of the phase function (4.7.20), which is non-linear with respect to m. To solve these difficulties we first note for any fixed $\zeta \in \mathbb{C}$ with $\Re\zeta > -(m+1)$ and $0 < c < a_0$ with the arguments satisfying $|\arg(z)| < \pi$ and $|\arg(a_0 - z)| < \pi$ it is possible to write

$$S_J(-\Re\zeta,m,a) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\log(z)\eta(-z)dz, \tag{4.7.47}$$

where we defined

$$\eta(-z) := (a_0 - z)^{-\Re\zeta-1}\frac{\Gamma(m+2)\Gamma(1+a+a_0-z)}{\Gamma(m+2+a+a_0-z)}. \tag{4.7.48}$$

The integrand in (4.7.47) is thus a holomorphic function in the twice cut plane $|\arg(z)| < \pi$ and $|\arg(a_0 - z)| < \pi$. According to (B.3.5), therein its asymptotic behaviour as $|z| \to \infty$ for fixed $m \ge 0$ is

$$\log(z)\eta(-z) = \mathcal{O}\left\{\log(|z|)\,|z|^{-\Re\zeta-m-2}\right\}. \tag{4.7.49}$$

As a consequence the integral (4.7.47) indeed converges absolutely for any fixed $\zeta \in \mathbb{C}$ with $\Re\zeta > -(m+1)$. Now it is routine to show that, appealing to the large $|z|$-behaviour of the integrand, the vertical integration path in (4.7.47) can be bent to a Hankel loop which encircles the negative real axis in the counterclockwise direction and cuts the real axis somewhere on the interval $(0, a_0)$. Equivalently we can thus write

$$S_J(-\Re\zeta,m,a) = -\frac{1}{2\pi i}\int_{-\infty}^{(0+)}\log(z)\eta(-z)dz. \tag{4.7.50}$$

By analyticity of the integrand, it is reasonable to assume that the loop is composed of two straight line segments running with a distance of $\delta > 0$ parallel to the negative real axis and a circle γ_r of radius $r > 0$, whose center is the origin. Accordingly, we obtain

$$S_J(-\Re\zeta, m, a) = -\frac{1}{2\pi i}\left[\int\limits_{-\infty-i\delta}^{-r-i\delta} + \int\limits_{\gamma_r} + \int\limits_{-r+i\delta}^{-\infty+i\delta}\right] \log(z)\eta(-z)dz. \tag{4.7.51}$$

If, in the integrals along the lines parallel to the real axis, we make the change of variables $x = -z \mp i\delta$, respectively, we obtain for fixed $r, \delta > 0$:

$$\begin{aligned}\pm\int\limits_{-\infty\mp i\delta}^{-r\mp i\delta} \log(z)\eta(-z)dz &= \pm\int\limits_{r}^{\infty}\left\{\log\sqrt{x^2+\delta^2} + i\arg(-x\mp i\delta)\right\}\eta(x\pm i\delta)dx \\ &\to \int\limits_{r}^{\infty}\{\pm\log(x) - i\pi\}\,\eta(x)dx\end{aligned}$$

The last equality results from the limit $\delta \downarrow 0$ in accordance with the choice of the argument. Moreover, regarding the circle around the origin, for $r > 0$ we have

$$\int\limits_{\gamma_r} \log(z)\eta(-z)dz = ir\int\limits_{-\pi}^{\pi}\{\log(r) + i\phi\}\,\eta(-re^{i\phi})e^{i\phi}d\phi \to 0.$$

For the last equality we let $r \downarrow 0$. Following from the preceding findings, if in (4.7.51) we let $\delta, r \downarrow 0$ we arrive at (4.7.44). This confirms the validity of the representation (4.7.47) for any $\zeta \in \mathbb{C}$ with $\Re\zeta > -(m+1)$. Regarding the leading behaviour as $m \to \infty$ of $S_J(-\Re\zeta, m, a)$ we first note continuity of the function $\log(c+iy)(a_0-c-iy)^{-\Re\zeta-1}$ with respect to $y \in \mathbb{R}$. Moreover, for arbitrary $\varepsilon > 0$ it is $\mathcal{O}\left\{|y|^{\varepsilon-\Re\zeta-1}\right\}$ as $y \to \pm\infty$. By virtue of (B.3.6) and Corollary B.3.1, from the representation (4.7.47) as $m \to \infty$ it thus follows

$$S_J(-\Re\zeta, m, a) = \mathcal{O}\left\{m^{c-a_0-a}\right\}. \tag{4.7.52}$$

Finally, after a simple substitution the second remainder integral (4.7.45) can be referred to the upper incomplete gamma function (B.2.7) and we conclude from (8.11.2) in [Olver et al., 2010] as $m \to \infty$:

$$\begin{aligned}T_J(-\Re\zeta, m, a) &= \frac{\Gamma(m+2)}{\Gamma(m+2+a)}\left\{H_{m+1}(a)\right\}^{1+\Re\zeta-J}\Gamma(J-\Re\zeta-1, a_0H_{m+1}(a)) \\ &= \mathcal{O}\left\{m^{-a-a_0}\left\{\log(m)\right\}^{-1}\right\}\end{aligned} \tag{4.7.53}$$

The big-$\mathcal{O}$ incorporates the asymptotic statements (4.7.33) and (B.3.6). By comparison of (4.7.52) and (4.7.53) with (4.7.46) since $0 < c < a_0$ we have eventually verified as $m \to \infty$ for any $\zeta \in \mathbb{C}$ with $-(m+1) < \Re\zeta < J$ and arbitrary $J \in \mathbb{N}$:

$$|R_J(-\zeta, m, a)| = \mathcal{O}\left\{m^{-a}\left\{\log(m)\right\}^{\Re\zeta-J}\right\}$$

Observe that the term in the big-$\mathcal{O}$ has a higher order as $m \to \infty$ than the last addend in the sum (4.7.42).

4.7.3.4. Expansion for the Beta Mellin Transform

Since the above result holds for arbitrary $J \in \mathbb{N}$ and $-(m+1) < \Re\zeta < J$ with $\zeta \neq 0$, we deduce from (4.7.42) for fixed $\zeta \in \mathbb{C} \setminus \{0\}$ and $a > -1$ as $m \to \infty$:

$$\mathcal{M}_{\mathrm{B}}\left(-\zeta, m, a\right) \sim \frac{\Gamma(m+2)\Gamma(1+a)}{\Gamma(m+2+a)} \frac{\{H_{m+1}(a)\}^{\zeta}}{-\zeta} \sum_{j=0}^{\infty} \{H_{m+1}(a)\}^{-j} g(j,m,a) \left(-\zeta\right)_j \tag{4.7.54}$$

The coefficients are a mixture of polygamma functions and ζ-polynomials that are expressed in terms of the Pochhammer symbol. The right hand side of (4.7.54) represents an expansion for the integral (4.7.19) with the asymptotic scale $\{H_{m+1}(a)\}^{-j}$ for $j \in \mathbb{N}_0$, corresponding to logarithmic decay.

4.7.4. Uniform Bound with Respect to the Imaginary Part

We close this section with the derivation of some sophisticated bounds for the beta Mellin transform and thus also for $M_{\bar{\varepsilon}}(z,m)$, where $z = x + iy$. Our first result is required to hold uniformly with respect to $y \in \mathbb{S}_{m,\beta}$ in any closed subregion, and at the same time it should reflect the asymptotic behaviour as $m \to \infty$. We proceed similar to (4.7.16) with the exception that we first apply the triangle inequality, which yields for $z \in \mathbb{S}_{m,1}$ since $x < 0$:

$$\begin{aligned} |\mathcal{M}_{\mathrm{B}}(-z,m)| &\leq \frac{m+1}{|x|} \int_0^{\infty} t^x e^{-t} \left(1-e^{-t}\right)^m dt \\ &= \frac{m+1}{\Gamma\left(-x\right)|x|} \int_0^{\infty} \frac{\Gamma(m+2)\Gamma(1+s)}{\Gamma(m+2+s)} s^{-x-1} ds \\ &= \mathcal{M}_{\mathrm{B}}\left(-x,m\right) \end{aligned} \tag{4.7.55}$$

A comparison with (4.7.13) shows that we have just verified

$$|M_{\bar{\varepsilon}}(z,m)| \leq \beta^{-1} \left\{q^{\frac{1}{\beta}}\sigma\right\}^{-x} \mathcal{M}_{\mathrm{B}}\left(-\frac{x}{\beta}, m\right), \qquad z \in \mathbb{S}_{m,\beta}. \tag{4.7.56}$$

Our next aim is a bound that is absolutely integrable along $c + iy$ with respect to $y \in \mathbb{R}$ for any fixed real $c \in \mathbb{S}_{m,1}$ and also reflects the large m-asymptotic behaviour. Similar to the preceding derivation we first consider (4.7.16) for $z \in \mathbb{S}_{m,1}$, which is

$$\mathcal{M}_{\mathrm{B}}(-z,m) = -\frac{m+1}{z} \int_0^{\infty} t^z e^{-t}(1-e^{-t})^m dt. \tag{4.7.57}$$

The product of exponential functions in the integrand is not monotonic along the positive real line, whence the associated derivative has at least one zero there. A simple modification, however, fixes this issue. In fact, according to the chain rule, we have:

$$\begin{aligned} \frac{d}{dt}e^{-t}\left\{\frac{1-e^{-t}}{t}\right\}^m &= me^{-t}\left\{\frac{1-e^{-t}}{t}\right\}^{m-1}\left[\frac{e^{-t}}{t}-\frac{1-e^{-t}}{t^2}\right]-e^{-t}\left\{\frac{1-e^{-t}}{t}\right\}^m \\ &= -e^{-t}\left\{\frac{1-e^{-t}}{t}\right\}^{m-1}\left[m\frac{1-e^{-t}-te^{-t}}{t^2}+\frac{1-e^{-t}}{t}\right] \end{aligned} \tag{4.7.58}$$

But it follows from the series expansion of the exponential function that $\frac{e^t-1}{t} \geq 1$ for $t \geq 0$, implying $\frac{1-e^{-t}}{t} \geq e^{-t}$ and

$$\frac{\frac{1-e^{-t}}{t}-e^{-t}}{t} \geq 0.$$

Consequently the derivative (4.7.58) is non-positive for $t \geq 0$, $m > 0$. If we now integrate by parts the integral (4.7.57), for $m > 0$ and $z \in \mathbb{S}_{m,1}$ we arrive at:

$$\begin{aligned} \mathcal{M}_{\mathrm{B}}(-z,m) &= -\frac{m+1}{z}\int_0^\infty t^{z+m}e^{-t}\left\{\frac{1-e^{-t}}{t}\right\}^m dt \\ &= \frac{m+1}{z(m+1+z)}\int_0^\infty t^{z+m+1}\frac{d}{dt}\left[e^{-t}\left\{\frac{1-e^{-t}}{t}\right\}^m\right] dt \end{aligned} \tag{4.7.59}$$

The modulus of the derivative in the integrand equals simply the derivative with a negative sign. Hence, the triangle inequality leads to

$$|\mathcal{M}_{\mathrm{B}}(-z,m)| \leq \frac{-(m+1)}{|z|\,|m+1+z|}\int_0^\infty t^{x+m+1}\frac{d}{dt}\left[e^{-t}\left\{\frac{1-e^{-t}}{t}\right\}^m\right] dt. \tag{4.7.60}$$

Note that the upper bound depends on the imaginary part of $z = x+iy$ only through the rational function in front of the integral. The integral itself is a function of the real part and m. Finally we apply (4.7.59) to the right hand side of (4.7.60) with z replaced by x, which yields

$$|\mathcal{M}_{\mathrm{B}}(-z,m)| \leq \frac{-x(m+1+x)}{|z|\,|m+1+z|}\mathcal{M}_{\mathrm{B}}(-x,m). \tag{4.7.61}$$

Observe that simultaneously this bound is $\mathcal{O}\left\{y^{-2}\right\}$ as $y \to \pm\infty$ in $\mathbb{S}_{m,1}$, and as $m \to \infty$ it has the same asymptotic behaviour as the left hand side, compare (4.7.54). Hence, (4.7.61) satisfies the desired properties. The deduction of an upper bound with the properties of (4.7.61) but where the arguments y and m are multiplicatively separated, seems to be a very complicated task. Several attempts have been made, none of which were fruitful.

4.8. Mellin Transforms of Oscillatory Functions

This section is devoted to Mellin transforms of functions which are compositions of the complex exponential function. As the kernel of the Fourier transform the latter plays an oustanding role in the theory of characteristic functions, for example in the context of discrete distributions of which the empirical distribution is most important, or regarding the continuous uniform distribution.

4.8.1. Complex Exponential Function

The Mellin transform corresponding to the complex exponential function possesses the integral definition

$$\mathcal{M}\left\{e^{iat}\right\}(\zeta) = \int_0^\infty t^{\zeta-1}e^{iat}dt, \qquad a \in \mathbb{R}\setminus\{0\}. \tag{4.8.1}$$

Clearly, there does not exist $\zeta \in \mathbb{C}$ for which this integral converges absolutely. Yet, the strip of analyticity is non-empty. To show this, we first note that for fixed $T > 1$ integration by parts yields

$$\int_1^T t^{\zeta-1}e^{iat}dt = T^{\zeta-1}\frac{e^{iaT}}{ia} - \frac{e^{ia}}{ia} - \frac{\zeta-1}{ia}\int_1^T t^{\zeta-2}e^{iat}dt.$$

If we now let $T \to \infty$ and assume $0 < \Re\zeta < 1$, on the right hand side the first summand vanishes and the integral converges absolutely. Moreover, instead of (4.8.1) we can then write

$$\mathcal{M}\left\{e^{iat}\right\}(\zeta) = \int_0^1 t^{\zeta-1}e^{iat}dt - \frac{e^{ia}}{ia} - \frac{\zeta-1}{ia}\int_1^\infty t^{\zeta-2}e^{iat}dt. \tag{4.8.2}$$

Now, for any $0 < \Re\zeta < 1$ the following bound applies:

$$\begin{aligned}\left|\mathcal{M}\left\{e^{iat}\right\}(\zeta)\right| &\le \int_0^1 t^{\Re\zeta-1}dt + \frac{1}{|a|} + \frac{|\zeta-1|}{|a|}\int_1^\infty t^{\Re\zeta-2}dt \\ &\le \frac{1}{\Re\zeta} + \frac{1}{|a|} + \frac{|\zeta-1|}{|a|\,(1-\Re\zeta)}\end{aligned}$$

This confirms the uniform convergence of the integral (4.8.1) in any compact subset of the region $0 < \Re\zeta < 1$. By Theorem A.2.1 we have thus verified analyticity of the indicated integral in the strip

$$S\left\{e^{iat}\right\} = \{\zeta \in \mathbb{C} : 0 < \Re\zeta < 1\}. \tag{4.8.3}$$

By application of Cauchy's theorem upon exploiting the decay of the trigonometric exponential function in either the upper or lower t-half plane, depending on the sign of the constant in the exponent, it is possible to cast the analytic continuation of the integral (4.8.1) in terms of the gamma function. Therefore we first assume $a > 0$ and $0 < \zeta < 1$. This makes the function $t \mapsto t^{\zeta}$ for $t \in \mathbb{C}$ multi-valued with branch points at the origin and at infinity. Since $a > 0$, to maintain $\Re iat < 0$ for $t \in \mathbb{C}$ our integration path shall be a quarter annulus in the first quadrant. This requires us to choose the branch cut of t^{ζ} to run somewhere in the left t-half plane, for example along the negative real axis. The function $t \mapsto t^{\zeta-1}e^{ita}$ is then analytic in $\Re t > 0$ and from Cauchy's theorem we obtain for $0 < r < R$:

$$\begin{aligned}0 = &\int_r^R t^{\zeta-1}e^{iat}dt + iR\int_0^{\frac{\pi}{2}}(Re^{i\phi})^{\zeta-1}e^{iaRe^{i\phi}+i\phi}d\phi \\ &+ i\int_R^r (iy)^{\zeta-1}e^{-ay}dy + ir\int_{\frac{\pi}{2}}^0 (re^{i\phi})^{\zeta-1}e^{iare^{i\phi}+i\phi}d\phi\end{aligned} \tag{4.8.4}$$

As $r \downarrow 0$ the integral along the small arc is easily seen to tend to zero. Regarding the larger arc, since $\frac{\sin(\phi)}{\phi} \geq \frac{2}{\pi}$ for $\phi \in [0, \frac{\pi}{2}]$ we have

$$\left| R\int_0^{\frac{\pi}{2}}(Re^{i\phi})^{\zeta-1}e^{iaRe^{i\phi}+i\phi}d\phi \right| \leq R^{\zeta}\int_0^{\frac{\pi}{2}} e^{-\frac{2aR\phi}{\pi}}d\phi = \frac{R^{\zeta-1}\pi\left\{1-e^{-Ra}\right\}}{2a}.$$

The right hand side clearly vanishes as $R \to \infty$. Thus, if in (4.8.4) we take the limits and then perform the change of variables $x = ay$, we arrive at:

$$\begin{aligned}0 &= \int_0^{\infty} t^{\zeta-1}e^{iat}dt + i\int_{\infty}^0 (iy)^{\zeta-1}e^{-ay}dy \\ &= M_e(\zeta, a) - (-ia)^{-\zeta}\Gamma(\zeta)\end{aligned}$$

Observe that contour integration enabled us to represent the non-absolutely convergent integral (4.8.1) in terms of the absolutely convergent integral for the gamma function. The proof for $a < 0$ is similar and uses a quarter annulus in the fourth quadrant so that we still have $\Re iat < 0$ for $t \in \mathbb{C}$. Summarizing these findings, by analytic continuation for any $\zeta \in \mathbb{C}$ we have just shown

$$\mathcal{M}\left\{e^{ita}\right\}(\zeta) = (-ia)^{-\zeta}\Gamma(\zeta), \qquad a \in \mathbb{R} \setminus \{0\}\,. \tag{4.8.5}$$

Finally, the fact that the integral representation (4.8.1) for the Mellin transform does not converge absolutely for some arguments of its strip of analyticity bears immediate consequences regarding the inversion formula. Indeed, the inversion theorem A.5.1 is thus inapplicable. Nevertheless, an interesting question is, for which $t \in \mathbb{C}$ we may write

$$e^{iat} = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} (-iat)^{-z}\Gamma(z)dz \qquad \text{for } a \in \mathbb{R} \setminus \{0\} \text{ and } c > 0. \tag{4.8.6}$$

To check this, we note as $|z| \to \infty$ along the integration path the integrand possesses the following behaviour:

$$\begin{aligned} \left|(-iat)^{-z}\Gamma(z)\right| &= \mathcal{O}\left\{|at|^{-\Re z}\,|z|^{\Re z - \frac{1}{2}}\,e^{-\Re z + \Im z \arg(-iat) - \Im z \arg(z)}\right\} \\ &= \mathcal{O}\left\{|z|^{c-\frac{1}{2}}\,e^{\Im z \arg(-iat) - \frac{\pi|\Im z|}{2}}\right\}, \end{aligned} \tag{4.8.7}$$

The integral (4.8.6) is hence absolutely convergent only for $0 < \arg(t) < \pi$ if $a > 0$, and for $-\pi < \arg(t) < 0$ if $a < 0$. A wider range of values can be enabled by deforming the vertical line to a loop that commences at $\infty e^{-i\pi}$, encircles the origin in the positive sense and then returns to $\infty e^{i\pi}$. This representation, however, has properties that correspond to the classical series representation of the exponential function, which differ quite a lot from an integral where the integration path is a vertical line. Probably most remarkable is the drawback that the real part along the loop is not constant. Contrary to (4.8.1) the integral

$$\mathcal{M}\left\{e^{ita} - 1\right\}(\zeta) = \int\limits_0^\infty t^{\zeta-1}\left(e^{iat} - 1\right)dt, \qquad a \in \mathbb{R} \setminus \{0\},$$

certainly constitutes a Mellin transform with an absolutely convergent integral definition and it is routine to verify the associated strip of analyticity as the region $-1 < \Re\zeta < 0$. Using integration by parts and the identity (4.8.5) we obtain

$$\mathcal{M}\left\{e^{ita} - 1\right\}(\zeta) = -\frac{ia}{\zeta}\int\limits_0^\infty t^{\zeta}e^{ita}dt = (-ia)^{-\zeta}\Gamma(\zeta). \tag{4.8.8}$$

Hence, according to the inverse Mellin formula (A.5.5), the function $e^{iat} - 1$ can be represented by the right hand side of (4.8.6) but with $-1 < c < 0$. Moreover, subject to (4.8.7) with the choice $-1 < c < -\frac{1}{2}$ this inverse Mellin integral is even absolutely convergent for $0 \le \arg(t) \le \pi$ if $a > 0$ and for $-\pi \le \arg(t) \le 0$ if $a < 0$, i.e., especially for any real $t \neq 0$.

4.8.2. Sine and Cosine

As a consequence of Euler's formula the complex exponential function is intimately connected with the two well-known trigonometric functions cosine and sine. It is therefore not surprising that there is also a close connection between the corresponding Mellin transforms. In particular, for $0 < \zeta < 1$ and $a \in \mathbb{R} \setminus \{0\}$ from (4.8.1) we deduce

$$\int_0^\infty t^{\zeta-1} e^{ita} dt = \int_0^\infty t^{\zeta-1} \cos(at) dt + i \int_0^\infty t^{\zeta-1} \sin(at) dt. \tag{4.8.9}$$

A closer look exhibits a substantial difference between the first and the second integral on the right hand side. While the first is again absolutely convergent for no $\zeta \in \mathbb{C}$, the second is in fact absolutely convergent for $-1 < \Re\zeta < 0$. However, similar to (4.8.1) it can be shown that each integral converges uniformly in any compact subset of $0 < \Re\zeta < 1$ and $-1 < \Re\zeta < 1$, respectively. Moreover, if we suppose $0 < \zeta < 1$ from (4.8.5) and (4.8.9) we obtain upon comparison of the real and imaginary parts the representation of the Mellin transform of the cosine and sine, respectively in terms of the gamma function and the function itself. This leads to the following results:

$$\begin{cases} \mathcal{M}\left\{\cos(at)\right\}(\zeta) = |a|^{-\zeta} \cos\left\{\frac{\pi\zeta}{2}\right\} \Gamma(\zeta) \\ S_{\mathcal{M}}\left\{\cos(at)\right\} = \{\zeta \in \mathbb{C} : 0 < \Re\zeta < 1\} \end{cases} \tag{4.8.10}$$

$$\begin{cases} \mathcal{M}\left\{\sin(at)\right\}(\zeta) = \operatorname{sgn}(a)\, |a|^{-\zeta} \sin\left\{\frac{\pi\zeta}{2}\right\} \Gamma(\zeta) \\ S_{\mathcal{M}}\left\{\sin(at)\right\} = \{\zeta \in \mathbb{C} : -1 < \Re\zeta < 1\} \end{cases} \tag{4.8.11}$$

By analytic continuation each representation remains valid for arbitrary $\zeta \in \mathbb{C}$. Throughout this work the latter Mellin transform is of most frequent occurence. For later use it will be helpful to note that, for $a \in \mathbb{R}$ and $z \in \mathbb{C}$ as a consequence of the representation of the sine in terms of the complex exponential function we can write:

$$\begin{aligned} |\sin(az)|^2 &= \frac{1}{4}\left[\cos^2(a\Re z)\left\{e^{-a\Im z} - e^{a\Im z}\right\}^2 + \sin^2(a\Re z)\left\{e^{-a\Im z} + e^{a\Im z}\right\}^2\right] \\ &= \cos^2(a\Re z)\sinh^2(a\Im z) + \sin^2(a\Re z)\cosh^2(a\Im z) \end{aligned}$$

Accordingly, a simple bound is given by:

$$\begin{aligned} |\sin(az)|^2 &\le \cosh^2(a\Im z)\left\{\cos^2(a\Re z) + \sin^2(a\Re z)\right\} \\ &\le e^{2|a\Im z|} \end{aligned} \tag{4.8.12}$$

Moreover, subject to Stirling's formula the asymptotic behaviour of the Mellin transform (4.8.11) as $|z| \to \infty$ in $|\arg(z)| < \pi$ strongly depends on the real part of the argument:

$$\begin{aligned} \left|\Gamma(z)\sin\left\{\frac{\pi z}{2}\right\}\right| &\sim \sqrt{\frac{\pi}{2}} e^{-\Re z}\, |z|^{\Re z - \frac{1}{2}}\, e^{-\Im z \arg(z) + \frac{\pi|\Im z|}{2}} \\ &= \mathcal{O}\left\{|z|^{\Re z - \frac{1}{2}}\, e^{-\Re z}\right\} \end{aligned} \tag{4.8.13}$$

In fact, regarding $\Im z \to \pm\infty$ the line $\Re z = \frac{1}{2}$ constitutes a transition from algebraic decay to growth. Such a property is typical for Mellin transforms of oscillatory functions.

4.8.3. Characteristic Functions Associated with Convolutions of Continuous Uniform Distributions

We now investigate a special type of characteristic functions that features trigonometric properties and exhibits algebraic decay at infinity. In particular, we consider the characteristic function Φ_X that was given in (3.3.5). The associated random variable X is thus attributed for $p \in \mathbb{N}$ to a p-times convolution of uniform distributions. To derive the Mellin transform corresponding to Φ_X we first denote

$$d_p := e^{-ip\frac{\pi}{2}} \prod_{j=1}^{p} (b_j - a_j)^{-1}. \tag{4.8.14}$$

We can then write

$$\mathcal{M}\left\{\Phi_X\right\}(\zeta) = d_p \int_0^\infty t^{\zeta-p-1} \prod_{j=1}^{p} \left\{e^{itb_j} - e^{ita_j}\right\} dt. \tag{4.8.15}$$

According to Section 4.2, the strip of analyticity is given by

$$S\left\{\Phi_X\right\} = \left\{\zeta \in \mathbb{C} : 0 < \Re\zeta < p\right\}. \tag{4.8.16}$$

To determine the analytic continuation of (4.8.15) we rearrange the product as a sum. Therefore we define the set

$$D_p := \bigtimes_{j=1}^{p} \left\{a_j, b_j\right\}. \tag{4.8.17}$$

Moreover, for a p-dimensional vector $v \in D_p$ denote $s(v) := \sum_{j=1}^{p} v_j$ and suppose $n(v)$ equals the number of components in v corresponding to lower endpoints a_j for $1 \leq j \leq p$. It is then ascertainable that the following holds:

$$\begin{aligned} \Phi_X(t) &= t^{-p} d_p \prod_{j=1}^{p} \left[e^{itb_j} - e^{ita_j}\right] \\ &= t^{-p} d_p \sum_{v \in D_p} (-1)^{n(v)} e^{its(v)} \end{aligned} \tag{4.8.18}$$

This constitutes a finite sum with $|D_p| = 2^p$ addends. Observe that there is a balanced number of addends with positive and negative signs, i.e., $\sum_{v \in D_p} (-1)^{n(v)} = 0$. We may thus write:

$$\Phi_X(t) = t^{-p} d_p \sum_{v \in D_p} (-1)^{n(v)} e^{its(v)}$$

$$= t^{-p} d_p \sum_{v \in D_p} (-1)^{n(v)} \left[e^{its(v)} - 1 \right]$$
$$= t^{-p} d_p \sum_{\substack{v \in D_p \\ s(v) \neq 0}} (-1)^{n(v)} \left[e^{its(v)} - 1 \right]$$

Suppose now $\zeta \in S\{\Phi_X\}$ satisfies $-1 < \zeta - p < 0$. Then we can apply the above decomposition to (4.8.15) and subject to absolute convergence interchange the order of summation and integration. By additional use of (4.8.8) this yields:

$$\begin{aligned} \mathcal{M}\{\Phi_X\}(\zeta) &= d_p \sum_{\substack{v \in D_p \\ s(v) \neq 0}} (-1)^{n(v)} \int_0^\infty t^{\zeta - p - 1} \left[e^{its(v)} - 1 \right] dt \\ &= d_p \Gamma(\zeta - p) \sum_{\substack{v \in D_p \\ s(v) \neq 0}} (-1)^{n(v)} \{-is(v)\}^{p-\zeta} \\ &= e^{-ip\frac{\pi}{2}} \frac{\Gamma(\zeta - p)}{\prod_{j=1}^{p} (b_j - a_j)} \sum_{\substack{v \in D_p \\ s(v) \neq 0}} (-1)^{n(v)} \{-is(v)\}^{p-\zeta} \end{aligned} \tag{4.8.19}$$

By analytic continuation the result extends to arbitrary $\zeta \in \mathbb{C}$ and since the integral (4.8.15) is holomorphic in $0 < \Re\zeta < p$, especially there the right hand side is analytic. It is evident from (4.8.19) that the meromorphic structure of the analytic continuation corresponding to this Mellin transform strongly depends on the values $s(v)$. In any case, regarding $\Re\zeta \leq 0$ there is a sequence of poles located at $\zeta \in -\mathbb{N}_0$ with some of them being removable, where the sum in (4.8.19) equals zero. For $\Re\zeta > 0$ there is only one possible pole, namely at $\zeta = p$. Its presence particularly depends on the number of summands with $s(v) \neq 0$ whose sign is positive or negative. If this number is balanced the pole at $\zeta = p$ is removable whereas in case of an imbalance it is not. Note that the latter implies

$$\sum_{\substack{v \in D_p \\ s(v) \neq 0}} (-1)^{n(v)} \neq 0.$$

From this we conclude, since we always have $\sum_{v \in D_p} (-1)^{n(v)} = 0$, the presence of a constant summand in the expansion (4.8.18) of Φ_X. This, however, gives a weaker contribution to the integral definition of the associated Mellin transform than any oscillatory expression $e^{its(v)}$ for $s(v) \neq 0$.

By inspection of (4.8.19) we can immediately determine the Mellin transform $\mathcal{M}_\square(\zeta)$ with strip of analyticity $S_\square$ of a characteristic function $\Phi_\square$ corresponding to a rectangular distribution on $[a_1, b_1]$ for $a_1, b_1 \in \mathbb{R}$, which is

$$\begin{cases} \mathcal{M}_\square(\zeta) = -i\frac{\Gamma(\zeta-1)}{b_1-a_1}\left\{(-ib_1)^{1-\zeta}\mathbb{1}_{\{b_1\neq 0\}} - (-ia_1)^{1-\zeta}\mathbb{1}_{\{a_1\neq 0\}}\right\}, \\ S_\square = \{\zeta \in \mathbb{C} : 0 < \Re\zeta < 1\}. \end{cases} \tag{4.8.20}$$

4.8.4. The Modulus of the Sinc Function

In the preceding subsection, particularly in (4.8.20), we established the Mellin transform of a characteristic function $\Phi_X = \Phi_\square$ associated with a continuous uniform distribution, compare also (3.3.5) with $p = 1$. We close this chapter with the derivation of the Mellin transform corresponding to the modulus of this characteristic function and briefly point out the differences between their analyticity properties. First we observe for $a, b \in \mathbb{R}$ with $a \neq b$ in terms of the sinc function (A.1.7) we can write

$$\frac{e^{itb} - e^{ita}}{i(b-a)t} = e^{it\frac{b+a}{2}} \operatorname{si}\left\{\frac{t(b-a)}{2}\right\}. \tag{4.8.21}$$

Accordingly, $|\Phi_X(t)| = |\operatorname{si}(\vartheta t)|$ for an appropriate parameter $\vartheta > 0$. The integral definition (4.2.1) of the associated Mellin transform M_X thus has the strip of analyticity

$$S_X = \{\zeta \in \mathbb{C} : 0 < \Re\zeta < 1\}. \tag{4.8.22}$$

Moreover, upon introducing the gamma function by means of the identity (A.5.11), for fixed $\zeta \in S_X$ we find:

$$M_X(\zeta) = \int_0^\infty t^{\zeta-1}\frac{|\sin(\vartheta t)|}{\vartheta t} dt \tag{4.8.23}$$

$$= \frac{\vartheta^{-\zeta}}{\Gamma(2-\zeta)} \int_0^\infty x^{1-\zeta} \int_0^\infty e^{-xt} |\sin(t)| \, dt dx \tag{4.8.24}$$

Exploiting the periodicity of the sine permits us to express for $x > 0$ the dt-integral in terms of elementary functions:

$$\begin{aligned} \int_0^\infty e^{-xt} |\sin(t)| \, dt &= \sum_{k=0}^\infty \int_0^\pi e^{-x(t+\pi k)} |\sin(t)| \, dt \\ &= \frac{1}{1-e^{-\pi x}} \int_0^\pi e^{-xt} \sin(t) dt \\ &= \frac{1}{2i(1-e^{-\pi x})} \int_0^\pi \left[e^{-(x-i)t} - e^{-(x+i)t}\right] dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2i(1-e^{-\pi x})}\left[\frac{e^{-(x-i)\pi}-1}{-(x-i)} - \frac{e^{-(x+i)\pi}-1}{-(x+i)}\right] \\
&= \frac{1+e^{-\pi x}}{(x^2+1)(1-e^{-\pi x})} \\
&= \frac{1}{x^2+1}\coth\left\{\frac{\pi x}{2}\right\}
\end{aligned}
$$

For the last equality we applied the definition of the hyperbolic cotangent. If we eventually plug this result into (4.8.24), we verified

$$
M_X(\zeta) = \frac{\vartheta^{-\zeta}}{\Gamma(2-\zeta)} \int_0^\infty \frac{x^{1-\zeta}}{x^2+1} \coth\left\{\frac{\pi x}{2}\right\} dx. \tag{4.8.25}
$$

The hyperbolic cotangent satisfies $\mathcal{O}\left\{x^{-1}\right\}$ as $x \downarrow 0$ and $\mathcal{O}\left\{1\right\}$ as $x \to \infty$. Consequently, similar to the original integral (4.8.23) the integral (4.8.25) is also absolutely convergent for any $\zeta \in S_X$ only and can be verified holomorphic in S_X. In contrast, to determine the associated analytic continuation we may now integrate by parts or employ an asymptotic expansion. However, if in (4.8.25) we make the simple change of variables $v = \frac{1}{x}$, we obtain equivalently

$$
M_X(\zeta) = \frac{\vartheta^{-\zeta}}{\Gamma(2-\zeta)} \int_0^\infty \frac{v^{\zeta-1}}{v^2+1} \coth\left\{\frac{\pi}{2v}\right\} dv. \tag{4.8.26}
$$

From the expansion for the hyperbolic tangent at the origin, see eq. (4.33.3) in [Olver et al., 2010], we deduce the existence of a power series expansion for the integrand in (4.8.26) as $|v| \to \infty$ in $|\arg(v)| < \frac{\pi}{2}$, whose leading term is

$$
\frac{1}{v^2+1}\coth\left\{\frac{\pi}{2v}\right\} = \frac{1}{v} + \ldots. \tag{4.8.27}
$$

We thus know from Lemma 4.3.3 in [Bleistein and Handelsman, 1986] the analytic continuation to $\Re\zeta \geq 1$ of the integral (4.8.26) is meromorphic with an infinite sequence of poles. Recalling that (4.8.20) never possesses more than one pole in this region, we conclude that the meromorphic structure of M_X is the price that must be paid for considering the Mellin transform of the modulus of the oscillatory function Φ_X. Moreover, as a consequence of Lemma 4.7.2 in [Bleistein and Handelsman, 1986], for arbitrary $\varepsilon > 0$, as $\Im\zeta \to \pm\infty$ in the half plane $\Re\zeta > 0$ we observe exponential decay

$$
M_X(\zeta) = \mathcal{O}\left\{e^{-\left(\frac{\pi}{2}-\varepsilon\right)|\Im\zeta|}\right\}.
$$

5. Derivation of Asymptotic Statements About the Bias by Means of Mellin Transforms

We now want to employ the Mellin transforms that were determined in the preceding chapter, to derive asymptotic statements about some of the bias integrals from Corollary 2.1.4. Our results will provide an overview on the possible leading behaviour and on the effect of different parameter values. While our first study is confined to the uniform local bias function, in the second part of this chapter we also present examples where the local bias is considered.

5.1. The Uniform Bias Function

For $m \geq 0$, in equation (2.1.56) the uniform bias function was defined as

$$\mathrm{ULB}(m) = \frac{1}{\pi} \int_0^\infty \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t} \left|\Phi_X(t)\right| dt. \tag{5.1.1}$$

Suppose validity of the conditions (4.2.2) and (4.3.15) and for sufficiently large m the existence of $x_0 \in S_X \cap \mathbb{S}^-_{m,\delta_{\bar{\varepsilon}}}$ with $M_X(x_0 + iy) \in \mathbb{R}$, where the indicated sets were given in (4.2.3) and (4.3.18). It is then permitted to plug the inverse Mellin representation (4.2.4) into (5.1.1) and interchange the order of integration, subject to absolute convergence, which yields

$$\mathrm{ULB}(m) = \frac{1}{2\pi^2 i} \int_{x_0 - i\infty}^{x_0 + i\infty} M_X(z) M_{\bar{\varepsilon}}(-z,m) dz. \tag{5.1.2}$$

The above integral eventually furnishes the appropriate setting for a discussion of the large m-behaviour of the uniform bias function. In further steps we need to investigate the properties of the integrand with respect to $z \in \mathbb{C}$ as $m \to \infty$.

5.1.1. Two Algebraic-Type Characteristic Functions

We begin our discussion with the simple scenario of two distributions that possess algebraic-type characteristic functions, i.e., we consider (5.1.1) for $|\Phi_X|$ and $\Phi_{\bar{\varepsilon}}$ as in (3.3.1) and (3.3.2), respectively. The Mellin transform of the former function equals the beta function (4.4.3). Regarding the Mellin transform of the m-power we refer to (4.5.8), the integral representation (4.5.6) for the function appearing therein being valid in $\mathbb{S}_{m,\beta}$. Assuming m large enough such that $\beta(m+1) > \alpha p$ and selecting $0 < x_0 < \alpha p$, from (5.1.2) we deduce

$$\text{ULB}(m) = \frac{1}{\alpha q \beta 2\pi^2 i} \int_{x_0 - i\infty}^{x_0 + i\infty} \lambda^z \frac{\Gamma\left(\frac{z}{\alpha}\right)\Gamma\left(p - \frac{z}{\alpha}\right)}{\Gamma(p)} \mathfrak{I}_H^q\left(\frac{z}{\beta}, m\right) dz, \tag{5.1.3}$$

where for convenience we denote

$$\lambda := \frac{\sigma}{\theta}. \tag{5.1.4}$$

At this point we do not recommend to employ any series or integral representations for the function $\mathfrak{I}_H^q(\cdot, m)$ accompanied by an interchange in the order of integration. The reason will become clear in the subsequent steps. For the most comprehensive insight on our approach we start with the simple case $q = 1$ before admitting arbitrary $q > 0$.

5.1.1.1. The Plain Scenario $q = 1$

According to (4.5.6), in this situation $\mathfrak{I}_H^q(\cdot, m)$ equals another beta function, which is by (B.3.2) expressible in terms of gamma functions, leading to

$$\text{ULB}(m) = \frac{1}{\alpha \beta 2\pi^2 i} \int_{x_0 - i\infty}^{x_0 + i\infty} \lambda^z \frac{\Gamma\left(\frac{z}{\alpha}\right)\Gamma\left(p - \frac{z}{\alpha}\right)\Gamma\left(\frac{z}{\beta}\right)\Gamma\left(m + 1 - \frac{z}{\beta}\right)}{\Gamma(p)\Gamma(m+1)} dz. \tag{5.1.5}$$

If additionally $\alpha = \beta$ and $\sigma = \theta$, Barnes' first lemma states that this integral equals a simple ratio of gamma functions. The general evaluation procedure requires a closer inspection of the integrand, particularly of the part that features the asymptotic parameter m. This is merely a ratio of two gamma functions. Subject to (B.3.5), for fixed $z \in \mathbb{C}$, their behaviour as $m \to \infty$ can be described in the elementary form

$$\frac{\Gamma\left(m + 1 - \frac{z}{\beta}\right)}{\Gamma(m+1)} \sim m^{-\frac{z}{\beta}}, \tag{5.1.6}$$

which clearly corresponds to negative powers of m if $\Re z > 0$. Accordingly, for any sequence of complex numbers $\{z_n\}_{n \in I}$ that does not depend on m with increasing real parts and possibly finite $I \subset \mathbb{N}_0$, the sequence

$$\mathfrak{I}_H^1\left(\frac{z_n}{\beta}, m\right) = \frac{\Gamma\left(\frac{z_n}{\beta}\right)\Gamma\left(m + 1 - \frac{z_n}{\beta}\right)}{\Gamma(m+1)}$$

for $n \in I$ constitutes an asymptotic scale as $m \to \infty$. For that reason our main interest concerns the behaviour of the integrand in (5.1.5) in the region to the right of integration path, i.e. in the half plane $\Re z > x_0$. There, according to Stirling's formula, for fixed $m \geq 0$, as $|z| \to \infty$ we find algebraic growth in the real direction and exponential decay in any direction of the imaginary axis, compare (4.4.4) and (4.5.10). Consequently we have freedom in displacing the integration path by an arbitrary finite distance to the right, incorporating possibly traversed poles. However, in $\Re z > 0$ we find two sequences of simple poles. On the one hand there are those at $\alpha(p+\mathbb{N}_0)$ generated by the factor $\Gamma\left(p-\frac{z}{\alpha}\right)$ that corresponds to M_X with residues, compare (B.2.20):

$$\underset{z=\alpha(p+j)}{\operatorname{Res}} \Gamma\left(p-\frac{z}{\alpha}\right) = \alpha\frac{(-1)^{1+j}}{j!}, \qquad j \in \mathbb{N}_0 \tag{5.1.7}$$

On the other hand there is an m-dependent set of poles located at $\beta(m+1+\mathbb{N}_0)$ that moves towards the right margin of the half plane $\Re z > 0$ as $m \to \infty$. By choosing m large enough we may thus collect any finite number of poles from the first sequence without crossing any of the m-dependent poles. In accordance with the definition of $\mathbb{S}^-_{m,\beta}$, that means we expand this strip as $m \to \infty$ to the right, but for fixed m we always remain therein. Let $x_{J+1} := \alpha(p+J+\frac{1}{2})$ for a given $J \in \mathbb{N}_0$ and choose m sufficiently large to ensure $x_{J+1} \in \mathbb{S}^-_{m,\beta}$. Then we shift the integration path in (5.1.5), subject to the asymptotic behaviour of the integrand, to the right, such that the new path satisfies $\Re z = x_{J+1}$. Keeping in mind that we encircle the poles with residues (5.1.7) in the negative direction, Cauchy's theorem yields

$$\mathrm{ULB}(m) = \frac{1}{\beta\pi}\sum_{j=0}^{J} \lambda^{\alpha(p+j)} \frac{\Gamma(p+j)\,\Gamma\left(\frac{\alpha(p+j)}{\beta}\right)\Gamma\left(m+1-\frac{\alpha(p+j)}{\beta}\right)}{\Gamma(p)\Gamma(m+1)}\frac{(-1)^j}{j!} + R_J(m), \tag{5.1.8}$$

with the remainder integral given by

$$R_J(m) := \frac{1}{\alpha\beta 2\pi^2 i}\int\limits_{x_{J+1}-i\infty}^{x_{J+1}+i\infty} \lambda^z \frac{\Gamma\left(\frac{z}{\alpha}\right)\Gamma\left(p-\frac{z}{\alpha}\right)\Gamma\left(\frac{z}{\beta}\right)\Gamma\left(m+1-\frac{z}{\beta}\right)}{\Gamma(p)\Gamma(m+1)}dz. \tag{5.1.9}$$

Clearly, the sum in (5.1.8) is descending as $m \to \infty$. To verify that it indeed establishes an asymptotic expansion it thus remains to show that (5.1.9) is of faster decay than the last summand. Therefore we substitute $y = -i(z - x_{J+1})$ which transforms (5.1.9) to an integral along the real axis and enables us to apply the triangle inequality:

$$|R_J(m)| \le \frac{\lambda^{x_{J+1}}}{\alpha\beta 2\pi^2}\int\limits_{-\infty}^{\infty} \frac{\left|\Gamma\left(\frac{iy+x_{J+1}}{\alpha}\right)\Gamma\left(p-\frac{iy+x_{J+1}}{\alpha}\right)\Gamma\left(\frac{iy+x_{J+1}}{\beta}\right)\Gamma\left(m+1-\frac{iy+x_{J+1}}{\beta}\right)\right|}{\Gamma(p)\Gamma(m+1)}dy$$

Since the arguments of the third and fourth gamma function in the numerator have a positive real part we can easily bound them by virtue of (B.2.31). Furthermore, to the beta function we apply the estimate (4.4.6). Accompanied by the change of variables $y = \alpha v$ we then arrive at:

$$|R_J(m)| \le \frac{\lambda^{x_{J+1}}}{\alpha\beta 2\pi^2}\Gamma\left(\frac{x_{J+1}}{\beta}\right)\frac{\Gamma\left(m+1-\frac{x_{J+1}}{\beta}\right)}{\Gamma(m+1)}\int\limits_{-\infty}^{\infty} \frac{\left|\Gamma\left(\frac{iy+x_{J+1}}{\alpha}\right)\Gamma\left(p-\frac{iy+x_{J+1}}{\alpha}\right)\right|}{\Gamma(p)}dy$$

$$\begin{aligned}&\leq \kappa_p(J)\frac{\lambda^{\alpha\left(p+J+\frac{1}{2}\right)}}{\beta\pi^2}\Gamma\left(\frac{\alpha}{\beta}\left(p+J+\frac{1}{2}\right)\right)\frac{\Gamma\left(m+1-\frac{\alpha}{\beta}\left(p+J+\frac{1}{2}\right)\right)}{\Gamma(m+1)}\\&\times\int\limits_0^\infty \frac{\left\{v^2+\left(p+\frac{1}{2}\right)^2\right\}^{\frac{p}{2}}}{\sqrt{\frac{1}{4}+v^2}}\frac{e^{-\frac{\pi v}{2}}}{\cosh^{\frac{1}{2}}(\pi v)}dv\end{aligned} \tag{5.1.10}$$

The constant $\kappa_p(J)$ was given in (4.4.7) and the integral in the upper bound (5.1.10) is evidently absolutely convergent. A comparison with (5.1.8) confirms the higher order of the remainder and thus verifies the asymptotic validity of our expansion as $m \to \infty$. Moreover, subject to the algebraic growth of the integrand in (5.1.9) in the direction of the real axis, we expect the remainder not to vanish as $J \to \infty$. The expansion (5.1.8) is thus merely of asymptotic type. Summarizing our findings, in terms of Pochhammer's symbol we have just shown:

$$\begin{aligned}\mathrm{ULB}(m) &\sim \frac{1}{\beta\pi}(m+1)_{-\frac{\alpha p}{\beta}}\sum_{j=0}^{\infty}\left(\frac{\sigma}{\theta}\right)^{\alpha(p+j)}(p)_j\,\Gamma\left(\frac{\alpha(p+j)}{\beta}\right)\left(m+1-\frac{\alpha p}{\beta}\right)_{-\frac{\alpha j}{\beta}}\frac{(-1)^j}{j!}\\&=(m+1)_{-\frac{\alpha p}{\beta}}\sum_{j=0}^{\infty}\Omega_j^1\left(m+1-\frac{\alpha p}{\beta}\right)_{-\frac{\alpha j}{\beta}}\end{aligned} \tag{5.1.11}$$

Here we denote the associated coefficients by

$$\Omega_j^1 := \frac{1}{\beta\pi}\left(\frac{\sigma}{\theta}\right)^{\alpha(p+j)}(p)_j\,\Gamma\left(\frac{\alpha(p+j)}{\beta}\right)\frac{(-1)^j}{j!}. \tag{5.1.12}$$

5.1.1.2. The Case of Arbitrary $q > 0$

In order to get a clue how to proceed from (5.1.3), we note by comparison of (4.5.8) and (4.5.24) for $z \in \mathbb{S}^-_{m,\beta}$ we can write

$$\mathfrak{I}_H^q\left(\frac{z}{\beta},m\right)=\frac{\Gamma(m+2)\Gamma\left(\frac{z}{\beta q}\right)}{\Gamma\left(m+2+\frac{z}{\beta q}\right)}S_H^q\left(\frac{z}{\beta},m\right). \tag{5.1.13}$$

Here $S_H^q(\cdot,m)$ is given by the series (4.5.25) and accordingly as $m \to \infty$ it is $\mathcal{O}(1)$. By means of (B.3.5) from the above representation we ascertain a faster decay as $m \to \infty$ for larger fixed values of $\Re z$ in $\mathbb{S}^-_{m,\beta}$. In other words, for an appropriate sequence of numbers in this region the function $\mathfrak{I}_H^q(\cdot,m)$ can be employed to establish an asymptotic scale as $m \to \infty$. This suggests we should easily repeat the steps we performed in the case $q = 1$ to generate an expansion for the uniform bias in the case of arbitrary $q > 0$. Therefore we recall that $\mathfrak{I}_H^q(z,m)$ constitutes an analytic function in the strip $\mathbb{S}^-_{m,1}$ that is especially uniformly bounded with respect to the imaginary part of its argument there. The controlling behaviour of the integrand in (5.1.3) as $|z| \to \infty$ towards the imaginary direction of this region with fixed $m \geq 0$, according to (4.4.4), is thus of exponential order. Summarizing, again the integration path may be shifted arbitrarily inside $\mathbb{S}^-_{m,\beta}$, bearing in mind additional contributions from crossing some singularities of $\Gamma\left(p-\frac{z}{\alpha}\right)$. Formally speaking we displace the integration path in (5.1.3) to the right direction to match the vertical line that cuts the real axis at $\Re z = x_{J+1}$. Again $x_{J+1} := \alpha(p+J+\frac{1}{2})$ for a given $J \in \mathbb{N}_0$ with $x_{J+1} \in \mathbb{S}^-_{m,\beta}$ and appropriate m, and the line does not run through any singularity. Since the poles at $\alpha(p+j)$ with $0 \leq j \leq J$ are encircled in the clockwise direction, subject to (5.1.7) the residue theorem for (5.1.3) yields

$$\text{ULB}(m) = \frac{1}{q\beta\pi}\sum_{j=0}^{J}\lambda^{\alpha(p+j)}\frac{\Gamma(p+j)}{\Gamma(p)}\mathfrak{I}^q_H\left(\frac{\alpha(p+j)}{\beta}, m\right)\frac{(-1)^j}{j!} + R_J(m), \tag{5.1.14}$$

where the remainder integral is now equal to

$$R_J(m) := \frac{1}{\alpha q\beta 2\pi^2 i}\int_{x_{J+1}-i\infty}^{x_{J+1}+i\infty}\lambda^z\frac{\Gamma\left(\frac{z}{\alpha}\right)\Gamma\left(p-\frac{z}{\alpha}\right)}{\Gamma(p)}\mathfrak{I}^q_H\left(\frac{z}{\beta}, m\right)dz. \tag{5.1.15}$$

Observe, contrary to (5.1.8), the exact asymptotic structure of (5.1.14) stays hidden in the integral function $\mathfrak{I}^q_H(\cdot, m)$. However, from (5.1.13) we deduce

$$\begin{aligned}\text{ULB}(m) = \frac{1}{q\beta\pi}\sum_{j=0}^{J}\lambda^{\alpha(p+j)}&\frac{\Gamma(p+j)\Gamma\left(\frac{\alpha(p+j)}{\beta q}\right)\Gamma(m+2)}{\Gamma(p)\Gamma\left(m+2+\frac{\alpha(p+j)}{\beta q}\right)}S^q_H\left(\frac{\alpha(p+j)}{\beta}, m\right)\frac{(-1)^j}{j!}\\ &+ R_J(m).\end{aligned} \tag{5.1.16}$$

To verify that the remainder integral (5.1.15) is of higher order than the last addend in the preceding sum remains a straightforward application of the triangle inequality, inequalities (4.4.6) and (4.5.31), accompanied by two small changes of variables, similar to the derivation of the bound (5.1.10). We thus find with $\kappa_p(J)$ and ω_q given in (4.4.7) and (4.5.32), respectively:

$$\begin{aligned}|R_J(m)| &\leq \frac{\lambda^{x_{J+1}}}{\alpha q\beta\Gamma(p)2\pi^2}\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{iy+x_{J+1}}{\alpha}\right)\Gamma\left(p-\frac{iy+x_{J+1}}{\alpha}\right)\mathfrak{I}^q_H\left(\frac{iy+x_{J+1}}{\beta}, m\right)\right|dy\\ &\leq \frac{\kappa_p(J)}{q\beta\pi^2}\omega_q^{\frac{\alpha}{\beta}(p+J+\frac{1}{2})+1}\lambda^{\alpha(p+J+\frac{1}{2})}\frac{\Gamma\left(m+1-\frac{\alpha}{\beta}\left(p+J+\frac{1}{2}\right)\right)\Gamma\left(\frac{\alpha}{\beta q}\left(p+J+\frac{1}{2}\right)\right)}{\Gamma\left(m+1-\frac{\alpha(q-1)}{\beta q}\left(p+J+\frac{1}{2}\right)\right)}\\ &\qquad\times\int_0^{\infty}\frac{\left\{v^2+\left(p+\frac{1}{2}\right)^2\right\}^{\frac{p}{2}}}{\sqrt{\frac{1}{4}+v^2}}\frac{e^{-\frac{\pi v}{2}}}{\cosh^{\frac{1}{2}}(\pi v)}dv\\ &= \mathcal{O}\left\{m^{-\frac{\alpha(p+J)}{\beta q}-\frac{\alpha}{2\beta q}}\right\}\end{aligned} \tag{5.1.17}$$

The last equality is a consequence of (B.3.6) as $m \to \infty$ for any fixed $J \in \mathbb{N}_0$. But according to (B.3.5), the j-th summand in (5.1.16) for $0 \leq j \leq J$ satisfies $\sim \text{const} \times m^{-\frac{\alpha(p+j)}{\beta q}}$ as $m \to \infty$. Concluding our findings, in the purely algebraic scenario, as $m \to \infty$ the bias function ULB(m)

has the following asymptotic expansion:

$$\begin{aligned}\mathrm{ULB}(m) &\sim \frac{1}{q\beta\pi}\sum_{j=0}^{\infty}\left(\frac{\sigma}{\theta}\right)^{\alpha(p+j)}\frac{\Gamma(p+j)}{\Gamma(p)}\mathfrak{I}_H^q\left(\frac{\alpha(p+j)}{\beta},m\right)\frac{(-1)^j}{j!}\\ &= \frac{1}{(m+2)_{\frac{\alpha p}{\beta q}}}\sum_{j=0}^{\infty}\frac{\Omega_j^2}{\left(m+2+\frac{\alpha p}{\beta q}\right)_{\frac{\alpha j}{\beta q}}}\end{aligned} \tag{5.1.18}$$

The associated coefficients are given by

$$\Omega_j^2 := \frac{1}{q\beta\pi}\left(\frac{\sigma}{\theta}\right)^{\alpha(p+j)}\Gamma\left(\frac{\alpha(p+j)}{\beta q}\right)(p)_j\frac{(-1)^j}{j!}S_H^q\left(\frac{\alpha(p+j)}{\beta},m\right). \tag{5.1.19}$$

These involve the inverse factorial series (4.5.25), absolutely convergent for sufficiently large m. If, in addition, we denote

$$\Omega_{00}^2 := \frac{1}{q\beta\pi}\left(\frac{\sigma}{\theta}\right)^{\alpha p}\Gamma\left(\frac{\alpha p}{\beta q}\right), \tag{5.1.20}$$

as $m\to\infty$ we can write

$$\mathrm{ULB}(m) \sim \frac{\Omega_{00}^2}{(m+2)_{\frac{\alpha p}{\beta q}}}. \tag{5.1.21}$$

It shows that the dominating behaviour of the uniform local bias merely depends on the ratio of the degrees, that specify the algebraic decay at infinity of the characteristic function of X and $\bar{\varepsilon}$, respectively.

5.1.2. An Algebraic-Type $|\Phi_X|$ and an Exponential-Type $\Phi_{\bar{\varepsilon}}$

In this section we study a scenario of two characteristic functions that do not belong to the same class. While that of the distribution of interest is assumed to have a modulus of algebraic type, for the error distribution we postulate an exponential-type characteristic function, see (3.3.1) and (3.3.4). Thus, in comparison with the previous subsection, $\Phi_{\bar{\varepsilon}}$ possesses a much faster decay, which in turn implies that the m-power approaches unity more rapidly. The rate of ULB(m) as $m\to\infty$ therefore should be expected slower than algebraic. Nonetheless we will see that the basic procedure to generate an expansion has many steps in common with the purely algebraic case. The reason is that the Mellin transform of the m-power, subject to (4.7.13), can be cast in terms of the beta Mellin transform. But in Subsection 4.7.3 we revealed the possibility to expand the latter in powers of the digamma function whose order especially depends on the argument. In particular, in (4.7.54) for fixed $z\in\mathbb{C}\setminus\{0\}$ as $m\to\infty$ we verified

$$\mathcal{M}_{\mathrm{B}}(z,m) \sim \frac{H_{m+1}^{-z}}{z}. \tag{5.1.22}$$

Hence, especially for an appropriate sequence of numbers in the region $\mathbb{S}_{m,1}^{-}$, where the right hand side of (5.1.22) is descending, the beta Mellin transform establishes an asymptotic scale as $m \to \infty$. This property was also shared by the function $\mathfrak{I}_H^q(\cdot, m)$ in the preceding subsection, which indicates the similarity between both cases.

Now, assuming an algebraic-type modulus of Φ_X, we know from Section 4.4 that the associated Mellin transform is again given by the beta function (4.4.3). Then, if we choose m large enough to have $\beta(m+1) > \alpha p$ we can pick $0 < x_0 < \alpha p$, located in the common strip of analyticity of $M_X(z)$ and $M_{\bar{\varepsilon}}(-z, m)$. In these circumstances, if we denote

$$\lambda := \frac{q^{\frac{1}{\beta}}\sigma}{\theta}, \tag{5.1.23}$$

it follows at once from (5.1.2) that we can write

$$\mathrm{ULB}(m) = \frac{1}{\alpha\beta 2\pi^2 i} \int_{x_0 - i\infty}^{x_0 + i\infty} \lambda^z \frac{\Gamma\left(\frac{z}{\alpha}\right)\Gamma\left(p - \frac{z}{\alpha}\right)}{\Gamma(p)} \mathcal{M}_{\mathrm{B}}\left(\frac{z}{\beta}, m\right) dz. \tag{5.1.24}$$

In accordance with (4.4.4) and the uniform boundedness with respect to $\Im z$ of $\mathcal{M}_{\mathrm{B}}\left(\beta^{-1}z, m\right)$ in $\mathbb{S}_{m,\beta}^{-}$, the integrand in (5.1.24) possesses exponential decay as $\Im z \to \pm\infty$ therein for any fixed $m \geq 0$. Hence, in $\mathbb{S}_{m,\beta}^{-}$ we may move the contour arbitrarily to the right. In this process we only need to incorporate the poles of $\Gamma\left(p - \frac{z}{\alpha}\right)$ with residues (5.1.7), since the remaining integrand is analytic. Therefore, given $K \in \mathbb{N}_0$ we choose m such that $x_{K+1} \in \mathbb{S}_{m,\beta}^{-}$, where $x_{K+1} := \alpha(p + K + \frac{1}{2})$. Then we shift the original path from $\Re z = x_0$ rightwards, to match the line whose real part is $\Re z = x_{K+1}$. Recalling that we encircle the poles in the negative sense, from the residue theorem we obtain

$$\mathrm{ULB}(m) = \frac{1}{\beta\pi} \sum_{k=0}^{K} \lambda^{\alpha(p+k)} \frac{\Gamma(p+k)}{\Gamma(p)} \mathcal{M}_{\mathrm{B}}\left(\frac{\alpha(p+k)}{\beta}, m\right) \frac{(-1)^k}{k!} + R_K(m), \tag{5.1.25}$$

where the remainder term is given by

$$R_K(m) = \frac{1}{\alpha\beta 2\pi^2 i} \int_{x_{K+1} - i\infty}^{x_{K+1} + i\infty} \lambda^z \frac{\Gamma\left(\frac{z}{\alpha}\right)\Gamma\left(p - \frac{z}{\alpha}\right)}{\Gamma(p)} \mathcal{M}_{\mathrm{B}}\left(\frac{z}{\beta}, m\right) dz. \tag{5.1.26}$$

To finally verify the asymptotic character of the sum (5.1.25) as $m \to \infty$, we must find a suitable estimate for the remainder term. For this purpose, in (5.1.26) we substitute $y = -i\left(z - x_{K+1}\right)$ to arrive at a real-valued integral, and then apply the triangle inequality:

$$|R_K(m)| \leq \frac{\lambda^{x_{K+1}}}{\alpha\beta 2\pi^2} \int_{-\infty}^{\infty} \frac{\left|\Gamma\left(\frac{iy + x_{K+1}}{\alpha}\right)\right| \left|\Gamma\left(p - \frac{iy + x_{K+1}}{\alpha}\right)\right|}{\Gamma(p)} \left|\mathcal{M}_{\mathrm{B}}\left(\frac{iy}{\beta} + \frac{x_{K+1}}{\beta}, m\right)\right| dy$$

For $\mathcal{M}_{\mathrm{B}}(\cdot, m)$ it suffices to employ the simple bound (4.7.56). Furthermore, regarding the beta function in the above integral we employ the estimate (4.4.6) with $\kappa_p(K)$ denoted in (4.4.7). Upon performing the additional change in variables $\alpha v = y$, we arrive at:

$$\begin{aligned} |R_K(m)| &\leq \frac{\kappa_p(K)}{\beta\pi^2} \lambda^{x_{K+1}} \mathcal{M}_{\mathrm{B}}\left(\frac{x_{K+1}}{\beta}, m\right) \int_0^\infty \frac{\left\{v^2 + \left(p + \frac{1}{2}\right)^2\right\}^{\frac{p}{2}}}{\sqrt{\frac{1}{4} + v^2}} \frac{e^{-\frac{\pi|v|}{2}}}{\cosh^{\frac{1}{2}}(\pi v)} dv \\ &= \mathcal{O}\left\{H_{m+1}^{-\frac{\alpha(p+K)}{\beta} - \frac{\alpha}{2\beta}}\right\} \end{aligned}$$

The last bound holds in accordance with (4.7.54) as $m \to \infty$, which shows that the remainder integral in fact exhibits a faster decay than the last term in the sum (5.1.25). Making use of (5.1.23) we have thus verified that

$$\text{(5.1.27)} \qquad \mathrm{ULB}(m) \sim \frac{1}{\beta\pi} \sum_{k=0}^{\infty} \left[\frac{q^{\frac{1}{\beta}}\sigma}{\theta}\right]^{\alpha(p+k)} (p)_k \mathcal{M}_{\mathrm{B}}\left(\frac{\alpha(p+k)}{\beta}, m\right) \frac{(-1)^k}{k!}$$

constitutes an asymptotic expansion as $m \to \infty$. At this point, employing the integral definition of $\mathcal{M}_{\mathrm{B}}(\cdot, m)$ provides the highest accuracy. Instead it can also be approximated by the asymptotic expansion (4.7.54) in terms of digamma functions. As $m \to \infty$ this yields the formal series

$$\text{(5.1.28)} \qquad \mathrm{ULB}(m) \sim H_{m+1}^{-\frac{\alpha p}{\beta}} \sum_{k=0}^{\infty} H_{m+1}^{-\frac{\alpha k}{\beta}} \Xi_k,$$

in which we denote the associated coefficients by

$$\text{(5.1.29)} \qquad \Xi_k := \left[\frac{q^{\frac{1}{\beta}}\sigma}{\theta}\right]^{\alpha(p+k)} \frac{(p)_k}{\pi\alpha(p+k)} \frac{(-1)^k}{k!} \sum_{j=0}^{\infty} g(j, m) \left(\frac{\alpha(p+k)}{\beta}\right)_j H_{m+1}^{-j}.$$

The additional coefficients $g(j, m)$ are those in (4.7.38) with $a = 0$. For brevity we eventually define

$$\text{(5.1.30)} \qquad \Xi_{00} := \left[\frac{q^{\frac{1}{\beta}}\sigma}{\theta}\right]^{\alpha p} \frac{1}{\pi\alpha p},$$

enabling us to concisely describe the controlling behaviour as $m \to \infty$ in the form

$$\text{(5.1.31)} \qquad \mathrm{ULB}(m) \sim H_{m+1}^{-\frac{\alpha p}{\beta}} \Xi_{00}.$$

It shows that, in case of a characteristic function Φ_X with a modulus of algebraic-type and an exponential-type $\Phi_{\bar{\varepsilon}}$, the exact rate of uniform convergence solely depends on the ratio of the degrees α, p, β but not on q.

5.1.3. An Exponential-Type $|\Phi_X|$ and an Algebraic-Type $\Phi_{\bar{\varepsilon}}$

In our next setup we consider a distribution for X whose characteristic function is in absolute value attributed to the stable class, and a distribution for $\bar{\varepsilon}$ with a characteristic function of algebraic type, compare (3.3.3) and (3.3.2). Clearly, since the decay of the former function is faster than algebraic, our preceding findings suggest that we should expect an improved rate for the uniform bias. In particular, the rate can be expected exponential. To justify this statement we first note that the Mellin transforms are in the present setup given by (4.6.3) and (4.5.8), respectively. Hence, $\mathbb{S}^-_{m,\beta} \subset S_X$ for any $m \geq 0$, and, according to (5.1.2), for each $x_0 \in \mathbb{S}^-_{m,\beta}$ the uniform bias can be cast in the form

$$(5.1.32) \qquad \mathrm{ULB}(m) = \frac{1}{q\beta\alpha 2\pi^2 i} \int\limits_{x_0 - i\infty}^{x_0 + i\infty} \lambda^{-\frac{z}{\alpha}} \Gamma\left(\frac{z}{\alpha}\right) \mathfrak{I}^q_H\left(\frac{z}{\beta}, m\right) dz,$$

where for brevity we denote

$$(5.1.33) \qquad \lambda := \frac{p\theta^\alpha}{\sigma^\alpha}.$$

From (4.6.4) and the uniform boundedness of $\mathfrak{I}^q_H(z, m)$ with respect to $\Im z$ in the strip $\mathbb{S}^-_{m,1}$ for any $m \geq 0$, concerning the integrand in (5.1.32) we conclude exponential decay in each direction of the imaginary axis there. As a consequence, arbitrary leftward and rightward displacements of the integration path in $\mathbb{S}^-_{m,1}$ are permitted. However, in contrast to the preceding scenarios we do not encounter any singularities in this region, regardless of the magnitude of m. The reason is that the strip of analyticity of the Mellin transform of $|\Phi_X|$ matches the entire half plane $\Re z > 0$. Recalling the capability of the function $\mathfrak{I}^q_H(\cdot, m)$ to establish an asymptotic scale, the analyticity of the integrand in $\mathbb{S}^-_{m,\beta}$ corresponds to the fact that the actual rate of $\mathrm{ULB}(m)$ as $m \to \infty$ is faster than anything expressible solely in terms of this function. In particular, since $\mathfrak{I}^q_H(\cdot, m)$ possesses an expansion of algebraic-type, the actual rate need be faster than any algebraic order, i.e., it is necessarily of exponential order. This claim can readily be confirmed by a simple estimate of (5.1.32) involving (4.5.31):

$$\begin{aligned} |\mathrm{ULB}(m)| &\leq \frac{\lambda^{-\frac{x_0}{\alpha}}}{q\beta\alpha 2\pi^2} \omega_q^{1+\frac{x_0}{\beta}} \frac{\Gamma\left(m+1-\frac{x_0}{\beta}\right)\Gamma\left(\frac{x_0}{q\beta}\right)}{\Gamma\left(m+1-\frac{x_0}{\beta}+\frac{x_0}{q\beta}\right)} \int\limits_{-\infty}^{\infty} \left|\Gamma\left(\frac{x_0+iy}{\alpha}\right)\right| dy \\ &\sim \mathrm{const} \times m^{-\frac{x_0}{q\beta}} \end{aligned}$$

The constant ω_q was defined in (4.5.32) and the integral which features the gamma function is clearly absolutely convergent. Moreover, the asymptotic relation holds as $m \to \infty$ subject to (B.3.5). Recalling the arbitrariness of $x_0 \in \mathbb{S}_{m,\beta}$, we conclude the decay of $\mathrm{ULB}(m)$ is in fact faster than any reciprocal power of m. For applications the above estimate is of course unsatisfactory. The derivation of exact statements, however, is by no means simple and can not be accomplished by a straightforward displacement of the integration path. Instead it requires to introduce a sophisticated expansion for the integrand in (5.1.32).

For simplicity, we first assume $q = 1$, in which case $\mathfrak{I}_H^q(\cdot, m)$ can be referred to as the beta function. In particular, to the integral (5.1.32), we then apply for $M_{\bar{\varepsilon}}(-z, m)$ the representation (4.5.9). Accompanied by a simple change of variables, for $x_0 \in \mathbb{S}_{m,\beta}^-$, this yields

$$\text{(5.1.34)} \qquad \mathrm{ULB}(m) = \frac{1}{\beta 2\pi^2 i} \int\limits_{\frac{x_0}{\alpha}-i\infty}^{\frac{x_0}{\alpha}+i\infty} \lambda^{-z}\Gamma(z)\Gamma\left(\frac{\alpha z}{\beta}\right)\frac{\Gamma\left(m+1-\frac{\alpha z}{\beta}\right)}{\Gamma(m+1)} dz.$$

Apart from the m-dependent simple poles at $\frac{\beta}{\alpha}(m+1+\mathbb{N}_0)$ the integrand in (5.1.34) possesses always a double pole at the origin. The remaining poles are located in the left z-half plane with multiplicities depending on the parameters $\alpha, \beta > 0$. These are not relevant because we conclude from (5.1.6) that poles in $\Re z < 0$ lead to an expansion in ascending powers of m, which, however, does not reflect the behaviour of ULB(m) as $m \to \infty$. Although it is not immediately clear how to proceed from (5.1.34), the function

$$\text{(5.1.35)} \qquad \Gamma(z)\Gamma\left(\frac{\alpha z}{\beta}\right)$$

evidently constitutes the key expression in this integral, since it is the only part that is independent of m. Indeed, it turns out to be possible for large $|z|$-values to replace (5.1.35) by its so-called inverse factorial expansion. For details we refer to [Paris and Kaminski, 2001]. Roughly speaking, an inverse factorial expansion is a finite expansion with the intention to approximate the leading behaviour of a ratio of gamma functions by a sum of single gamma functions plus a remainder term. This eventually facilitates a reference of the integral (5.1.34) to the exponential function by virtue of the inversion formula for Mellin transforms. However, it will turn out that this reference can not be drawn if in (5.1.34) we retain the ratio of gamma functions that depends on m. Instead we must first employ an appropriate asymptotic expansion as $m \to \infty$ in powers of m. Such an expansion should ideally possess uniformity properties with respect to z, and it is not recommended to employ one that holds merely for fixed z, since we integrate along an infinite ray in the complex plane. We do not want to involve such an expansion and rather simply estimate the controlling m-term in the integrand by means of (B.3.5). For the derivation of the leading term of the whole integral (5.1.34) this strategy should work, since a different kind of expansion for a function only affects the terms succeeding the dominating part, but it does not affect the asymptotic behaviour of that function.

To keep the effort as low as possible, we abandon the integral (5.1.34) and rather restart our investigation of (5.1.32) for arbitrary $q > 0$. An asymptotic expansion for the integral $\mathfrak{I}_H^q(\cdot, m)$ appearing therein was given in Subsections 4.5.2 and 4.5.4, respectively in terms of gamma functions and simple algebraic powers. Following the above considerations, we employ the latter to approximate the leading term of $\mathfrak{I}_H^q(\cdot, m)$, which coincides with (B.3.5) for $q = 1$. For this, we first transform (5.1.32) by a simple change of variables, with $0 < x_0 < \beta(m+1)$, leading to

$$\mathrm{ULB}(m) = \frac{1}{q\beta 2\pi^2 i} \int_{\frac{x_0}{\alpha}-i\infty}^{\frac{x_0}{\alpha}+i\infty} \lambda^{-z}\Gamma(z)\,\mathfrak{I}_H^q\left(\frac{\alpha z}{\beta}, m\right) dz. \tag{5.1.36}$$

Then, if in (5.1.36) we approximate the controlling behaviour by virtue of (4.5.38), for sufficiently large m it follows that

$$\mathrm{ULB}(m) \sim \frac{1}{q\beta\pi}\widehat{\mathrm{ULB}}(m), \tag{5.1.37}$$

with the estimated integral defined by

$$\widehat{\mathrm{ULB}}(m) := \frac{1}{2\pi i} \int_{\frac{x_0}{\alpha}-i\infty}^{\frac{x_0}{\alpha}+i\infty} \lambda^{-z}\Gamma(z)\Gamma\left(\frac{\alpha z}{q\beta}\right) m^{-\frac{\alpha z}{q\beta}} dz. \tag{5.1.38}$$

In accordance with this approximation, we confine our discussion of the large m-behaviour of the uniform bias to a study of (5.1.38). We proceed from the latter integral and, appealing to the exponential decay of the integrand in the direction of the imaginary axis, displace the integration path towards the right, such that $|z|$ is everywhere large. More precisely, we suppose the real part of the new path satisfies $\Re z = x_m$ and it is especially permitted to depend on m. For fixed m, this yields

$$\widehat{\mathrm{ULB}}(m) = \frac{1}{2\pi i} \int_{x_m-i\infty}^{x_m+i\infty} \lambda^{-z} Q(z) m^{-\frac{\alpha z}{q\beta}} dz. \tag{5.1.39}$$

Of particular importance now is the function

$$Q(z) := \Gamma(z)\Gamma\left(\frac{\alpha z}{q\beta}\right). \tag{5.1.40}$$

In the present setup the parameters that were defined in Section 2.2 in [Paris and Kaminski, 2001] are given by

$$\begin{cases} h = \left(\frac{q\beta}{\alpha}\right)^{\frac{\alpha}{q\beta}}, \\ \vartheta = -\frac{1}{2}, \\ \kappa = 1 + \frac{\alpha}{q\beta}. \end{cases} \tag{5.1.41}$$

Since $|z|$ is large everywhere on the integration path in (5.1.39) and $\kappa > 0$, according to Lemma 2.2 in [Paris and Kaminski, 2001], the following expansion holds for arbitrary $J \in \mathbb{N}$:

$$Q(z) = 2\pi \left(h\kappa^{\kappa}\right)^{-z} \left\{ \sum_{j=0}^{J-1} (-1)^j A_j \Gamma\left(\kappa z + \vartheta - j\right) + \rho_J(z) \Gamma\left(\kappa z + \vartheta - J\right) \right\} \tag{5.1.42}$$

The remainder $\rho_J(z)$ constitutes an analytic function of z except at the points where $Q(z)$ has its poles, and it satisfies $\rho_J(z) = \mathcal{O}(1)$ as $|z| \to \infty$ uniformly in any closed interior subsector of $|\arg(z)| < \pi$. Furthermore, the coefficients A_j for $j \in \mathbb{N}_0$ depend on the parameters only with the first of them equal to

$$A_0 = \kappa \sqrt{\frac{q\beta}{2\pi\alpha}}. \tag{5.1.43}$$

The derivation of further coefficients is elaborate and applicable algorithms substantially depend on the multiplicities of the gamma functions in (5.1.40). This shall not be discussed right here and we merely employ the inverse factorial expansion for $J = 1$ to approximate the leading term of $Q(z)$. The first summand in (5.1.42) then has simple poles at $-\kappa^{-1}(\vartheta + \mathbb{N}_0)$ with corresponding residues computable by virtue of elementary calculations, see (B.2.20):

$$\operatorname*{Res}_{z=-\kappa^{-1}(\vartheta+n)} \Gamma\left(\kappa z + \vartheta\right) = \kappa^{-1} \frac{(-1)^n}{n!}, \qquad n \in \mathbb{N}_0 \tag{5.1.44}$$

Finally, plugging (5.1.42) with $J = 1$ into (5.1.39) and performing an interchange in the order of integration and summation, for fixed m, leads to

$$\widehat{\mathrm{ULB}}(m) = 2\pi \left\{A_0 I_0(m) + I_1(m)\right\}, \tag{5.1.45}$$

where, by (5.1.41), with $\vartheta = -\frac{1}{2}$ we denote

$$I_0(m) := \frac{1}{2\pi i} \int\limits_{x_m - i\infty}^{x_m + i\infty} \left(\lambda h \kappa^{\kappa}\right)^{-z} \Gamma\left(\kappa z - \frac{1}{2}\right) m^{-\frac{\alpha z}{q\beta}} dz, \tag{5.1.46}$$

$$I_1(m) := \frac{1}{2\pi i} \int\limits_{x_m - i\infty}^{x_m + i\infty} \left(\lambda h \kappa^{\kappa}\right)^{-z} \rho_1(z) \Gamma\left(\kappa z - \frac{3}{2}\right) m^{-\frac{\alpha z}{q\beta}} dz. \tag{5.1.47}$$

By inspection the first integral is readily identified as the inverse Mellin representation for the exponential function and consequently we deduce from the inversion theorem A.5.1:

$$\begin{aligned} I_0(m) &= \frac{1}{2\pi i} \int\limits_{x_m - i\infty}^{x_m + i\infty} \left(\lambda h \kappa^{\kappa}\right)^{-z} \Gamma\left(\kappa z - \frac{1}{2}\right) m^{-\frac{\alpha z}{q\beta}} dz \\ &= \kappa^{-\frac{3}{2}} \left(\lambda h\right)^{-\frac{1}{2\kappa}} m^{-\frac{\alpha}{2\kappa q\beta}} \exp\left\{-\kappa (\lambda h)^{\frac{1}{\kappa}} m^{\frac{\alpha}{\kappa q\beta}}\right\} \end{aligned} \tag{5.1.48}$$

Finally, in order to verify the asymptotic validity of (5.1.45) we must investigate the large m-behaviour of the remainder integral (5.1.47). Therefore, instead of a simple estimate, we require an additional saddle point approximation. Preliminary we make the change of variables $y = -i(z - x_m)$ and use the triangle inequality, to arrive at the bound

$$|I_1(m)| \leq (\lambda h \kappa^\kappa)^{-x_m} m^{-\frac{\alpha}{q\beta}x_m} \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |\rho_1(x_m + iy)| \left| \Gamma\left(\kappa(x_m + iy) - \frac{3}{2}\right) \right| dy. \tag{5.1.49}$$

To estimate the gamma function in this integral we employ (4.6.7) with $s = \kappa x_m$, which puts the integrand in a convenient setting for our saddle point approximation:

$$\left| \Gamma\left(\kappa x_m - \frac{3}{2} + \kappa y i\right) \right| \leq K_2 e^{-\kappa x_m} (\kappa x_m)^{\kappa x_m - 2} \left\{ 1 + \frac{y^2}{x_m^2} \right\}^{-1} e^{-\kappa x_m \varphi\left(\frac{y}{x_m}\right)}$$

The phase function φ was defined in (4.6.8). Furthermore, since x_m is assumed large, following from the properties of the remainder function, there exists an additional constant $K_1 > 0$ such that $K_2 |\rho_1(x_m + iy)| \leq K_1$. Upon combining these two bounds and making in (5.1.49) the change of variables $x_m v = y$, we obtain:

$$\begin{aligned} |I_1(m)| &\leq \frac{K_1}{2\pi} (\lambda h \kappa^\kappa)^{-x_m} e^{-\kappa x_m} (\kappa x_m)^{\kappa x_m - 2} m^{-\frac{\alpha}{q\beta}x_m} \int\limits_{-\infty}^{\infty} \left\{ 1 + \frac{y^2}{x_m^2} \right\}^{-1} e^{-\kappa x_m \varphi\left(\frac{y}{x_m}\right)} dy \\ &= \frac{K_1}{\pi} (\lambda h)^{-x_m} e^{-\kappa x_m} \kappa^{-2} x_m^{\kappa x_m - 1} m^{-\frac{\alpha}{q\beta}x_m} \int\limits_0^{\infty} \left(1 + w^2\right)^{-1} e^{-\kappa x_m \varphi(w)} dw \end{aligned} \tag{5.1.50}$$

The latter is a Laplacian-type integral in which the phase function has a saddle point at $0 = \varphi'(v) = \arctan(v)$, i.e., at $v_0 = 0$, with $\varphi''(v_0) = 1$ and $\varphi(v_0) = 0$. According to the Laplacian approximation (3.1.4), with $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, as $m \to \infty$ we thus have

$$\int\limits_0^{\infty} \left(1 + w^2\right)^{-1} e^{-\kappa x_m \varphi(w)} dw = \mathcal{O}\left\{ \frac{\sqrt{\pi}}{2} (\kappa x_m)^{-\frac{1}{2}} \right\}.$$

By virtue of this estimate, from the bound which was deduced in (5.1.50), as $m \to \infty$ we conclude

$$|I_1(m)| = \mathcal{O}\left\{ \frac{K_1}{2\sqrt{\pi}} (\lambda h)^{-x_m} e^{-\kappa x_m} \kappa^{-\frac{5}{2}} x_m^{\kappa x_m - \frac{3}{2}} m^{-\frac{\alpha}{q\beta}x_m} \right\}. \tag{5.1.51}$$

The dominating behaviour of the sequence in the big-$\mathcal{O}$ comes from the term

$$(\lambda h)^{-x_m} x_m^{\kappa x_m} m^{-\frac{\alpha}{q\beta}x_m}. \tag{5.1.52}$$

In particular, if $x_m \to \infty$ too fast as $m \to \infty$ this factor diverges faster than $e^{-\kappa x_m}$ decays. We must therefore determine, in which circumstances (5.1.52) approaches unity and is thus asymptotically bounded. This is the case if and only if $x_m \equiv (\lambda h)^{\frac{1}{\kappa}} m^{\frac{\alpha}{\kappa q\beta}}$. With that choice of sequence, from (5.1.51) we obtain

$$|I_1(m)| = \mathcal{O}\left\{\kappa^{-\frac{5}{2}}(\lambda h)^{-\frac{3}{2\kappa}} m^{-\frac{3\alpha}{2\kappa q\beta}} e^{-\kappa(\lambda h)^{\frac{1}{\kappa}} m^{\frac{\alpha}{\kappa q\beta}}}\right\}. \tag{5.1.53}$$

By comparison of this result with (5.1.48) we see, that the remainder integral indeed possesses a faster decay for increasing m than the first summand in (5.1.45), which verifies the asymptotic validity of the latter. To conclude our findings, as $m \to \infty$ we have just shown

$$\widehat{\mathrm{ULB}}(m) \sim 2\pi m^{-\frac{\alpha}{2\kappa q\beta}} e^{-\kappa(\lambda h)^{\frac{1}{\kappa}} m^{\frac{\alpha}{\kappa q\beta}}} A_0 \kappa^{-\frac{3}{2}} (\lambda h)^{-\frac{1}{2\kappa}}. \tag{5.1.54}$$

It must be emphasized that a higher order approximation of the function $Q(z)$ by means of (5.1.42) would only unlock higher terms in the asymptotic expansion of the integral (5.1.39) but, owing to the approximation (5.1.37), these terms are not necessarily the subsequent terms in the expansion of the uniform local bias. Instead, later terms in the expansion of ULB(m) would inevitably require use of a suitable expansion for the function $\mathfrak{I}_H^q(\cdot, m)$.

Finally, from (5.1.41) we recall $\kappa = (q\beta)^{-1}(\alpha + q\beta)$, and the coefficient A_0 was specified in (5.1.43). Moreover, according to (5.1.33) and (5.1.41), the constant appearing on the right hand side in (5.1.54) equals

$$(\lambda h)^{\frac{1}{\kappa}} = \left[\frac{q\beta p^{\frac{q\beta}{\alpha}} \theta^{q\beta}}{\alpha\sigma^{q\beta}}\right]^{\frac{\alpha}{\alpha+q\beta}}. \tag{5.1.55}$$

If we therefore denote

$$\Upsilon_{00} := \sqrt{\frac{2}{\pi\alpha(\alpha+q\beta)}} \left[\frac{q\beta p^{\frac{q\beta}{\alpha}} \theta^{q\beta}}{\alpha\sigma^{q\beta}}\right]^{-\frac{\alpha}{2(\alpha+q\beta)}}, \tag{5.1.56}$$

we deduce from (5.1.37) that the dominating term of the uniform bias as $m \to \infty$ is given by

$$\mathrm{ULB}(m) \sim m^{-\frac{\alpha}{2(\alpha+q\beta)}} \exp\left\{-\frac{\alpha+q\beta}{q\beta}\left[\frac{q\beta p^{\frac{q\beta}{\alpha}} \theta^{q\beta}}{\alpha\sigma^{q\beta}}\right]^{\frac{\alpha}{\alpha+q\beta}} m^{\frac{\alpha}{\alpha+q\beta}}\right\} \Upsilon_{00}. \tag{5.1.57}$$

The uniform bias thus tends to zero with a rate that exceeds any algebraic reciprocal power.

5.1.4. Two Exponential-Type Characteristic Functions

In this section we study the setup where the modulus of Φ_X and $\Phi_{\bar{\varepsilon}}$ are both of exponential type, i.e., we assume (3.3.3) and (3.3.4), respectively. Throughout the section we denote

$$\lambda := \frac{p\theta^{\alpha}}{q^{\frac{\alpha}{\beta}}\sigma^{\alpha}}. \tag{5.1.58}$$

Before examining the representation for the uniform bias as a MB-integral we briefly reconsider the original definition (5.1.1). By substitution of $t = \sigma^{-1}q^{-\frac{1}{\beta}}x^{\frac{1}{\beta}}$ this is readily transformed to:

$$\begin{aligned}
\mathrm{ULB}(m) &= \frac{1}{\pi}\int\limits_0^\infty \left\{1 - \exp\left\{-q\sigma^\beta t^\beta\right\}\right\}^{m+1} \exp\left\{-p\theta^\alpha t^\alpha\right\} \frac{dt}{t} \\
&= \frac{1}{\beta\pi}\int\limits_0^\infty \left(1 - e^{-t}\right)^{m+1} e^{-\lambda t^{\frac{\alpha}{\beta}}} \frac{dt}{t}
\end{aligned}$$

Apart from the reciprocal t-power the integral obviously equals the beta function if and only if $\alpha = \beta$. Indeed, a simple interchange in the order of integration then yields:

$$\begin{aligned}
\mathrm{ULB}(m) &= \frac{1}{\beta\pi}\int\limits_0^\infty \left(1 - e^{-t}\right)^{m+1} e^{-\lambda t}\int\limits_0^\infty e^{-xt}dxdt \\
&= \frac{1}{\beta\pi}\int\limits_0^\infty \frac{\Gamma(m+2)\Gamma\left(\lambda + x\right)}{\Gamma\left(m+2+\lambda+x\right)}dx \\
&= \frac{1}{m+2}\frac{1}{\beta\pi}\mathcal{M}_{\mathrm{B}}(1, m+1, \lambda - 1)
\end{aligned}$$

For the last equality we refer to the beta Mellin transform, which was defined in (4.7.19). As a consequence, as $m \to \infty$ we deduce from the expansion (4.7.54):

$$\mathrm{ULB}(m) \sim \frac{1}{\beta\pi}\left\{\psi\left(m+2+\lambda\right) - \psi\left(\lambda\right)\right\}^{-1}\frac{\Gamma(m+2)\Gamma\left(\lambda\right)}{\Gamma\left(m+2+\lambda\right)} \tag{5.1.59}$$

This verifies the rate of the uniform bias as a mixture of an algebraic and a logarithmic expression if $\alpha = \beta$. Clearly, in the present setup we can equivalently apply the MB-integral representation (5.1.2) for any $\alpha, \beta > 0$. The associated Mellin transforms are then given by (4.6.3) and (4.7.13), of which the latter involves the beta Mellin transform (4.7.15). Accordingly, after a simple change of variables for $0 < x_0 < \beta(m+1)$ we obtain

$$\mathrm{ULB}(m) = \frac{1}{\beta 2\pi^2 i}\int\limits_{\frac{x_0}{\alpha}-i\infty}^{\frac{x_0}{\alpha}+i\infty} \lambda^{-z}\Gamma\left(z\right)\mathcal{M}_{\mathrm{B}}\left(\frac{\alpha z}{\beta}, m\right)dz. \tag{5.1.60}$$

Following from (4.6.4) and the uniform boundedness of the beta Mellin transform with respect to the imaginary part of its argument in its strip of analyticity, the integrand in (5.1.60) exhibits exponential decay in the strip $\mathbb{S}^-_{m,\frac{\alpha}{\beta}}$. This implies the possibility of arbitrary rightward or leftward displacements of the integration path in the indicated region. However, by analyticity of the integrand there, we do not traverse any singularities. Transferring our observations from the preceding subsection, we thus conclude the actual rate of decay of $\mathrm{ULB}(m)$ as $m \to \infty$ is faster than any expression that can be represented solely by the beta Mellin transform. This is not a contradiction to (5.1.59) since the beta Mellin transform possesses an expansion in powers of the digamma function, which is known for its logarithmic behaviour. But the elementary rules of calculus imply, that a combination of a logarithmic and an exponential term may exhibit algebraic properties. Consequently, we can not immediately conclude that the integral (5.1.60) as $m \to \infty$ decays faster than any algebraic term but only that it exceeds any reciprocal logarithmic order.

To derive the exact leading behaviour of the uniform bias in the present scenario or even a few more subsequent terms of its asymptotic expansion, we proceed from (5.1.60) with a formal application of the asymptotic expansion for the beta Mellin transform, which we obtained in (4.7.54). Upon interchanging the order of summation and integration and employing the functional equation for the gamma function, for sufficiently large m and an appropriate $R \in \mathbb{N}_0$ we arrive at

(5.1.61) $$\mathrm{ULB}(m) \sim \frac{1}{\beta\pi} \sum_{r=0}^{R} H_{m+1}^{-r} g(r,m) \frac{1}{2\pi i} \int\limits_{\frac{x_0}{\alpha}-i\infty}^{\frac{x_0}{\alpha}+i\infty} \lambda^{-z} H_{m+1}^{-\frac{\alpha z}{\beta}} \frac{\Gamma(z)\,\Gamma\left(\frac{\alpha z}{\beta}+r\right)}{\Gamma\left(\frac{\alpha z}{\beta}+1\right)} dz.$$

Since the expansion (4.7.54) actually holds for fixed $\zeta \in \mathbb{C}\setminus\{0\}$ only, we expect the index R to be finite and to depend on the parameters, in order for the above sum to be valid in the asymptotic sense. This will show in the sequence. In our next step, appealing to the exponential decay of the integrand, in the r-th integral we displace the integration path as far as possible to the right to match a line with real part $\Re z = x_m$, along which $|z|$ is everywhere large. For convenience with $0 \le r \le R$ we denote

(5.1.62) $$I_r(m) := \frac{H_{m+1}^{-r}}{2\pi i} \int\limits_{x_m-i\infty}^{x_m+i\infty} \lambda^{-z} H_{m+1}^{-\frac{\alpha z}{\beta}} Q_r(z) dz,$$

(5.1.63) $$Q_r(z) := \frac{\Gamma(z)\,\Gamma\left(r+\frac{\alpha z}{\beta}\right)}{\Gamma\left(1+\frac{\alpha z}{\beta}\right)}.$$

The described movement of the integration path for fixed m then leads to

$$\mathrm{ULB}(m) \sim \frac{1}{\beta\pi}\sum_{r=0}^{R} g(r,m) I_r(m). \tag{5.1.64}$$

Regarding (5.1.62) it is our aim to introduce an inverse factorial expansion for $Q_r(z)$. The corresponding parameters are for $0 \leq r \leq R$ given by

$$\begin{cases} h = 1, \\ \vartheta \equiv \vartheta_r = -1 + r, \\ \kappa = 1. \end{cases} \tag{5.1.65}$$

According to Lemma 2.2 in [Paris and Kaminski, 2001], with $J = 1$, we can thus write

$$Q_r(z) = \Gamma(z - 1 + r) + \rho_{r1}(z)\Gamma(z - 2 + r). \tag{5.1.66}$$

For each fixed $0 \leq r \leq R$ the remainder function is analytic with respect to $z \in \mathbb{C}$, except at the points where $Q_r(z)$ has its poles. In addition, $\rho_{r1}(z) = \mathcal{O}(1)$ as $|z| \to \infty$ uniformly in any closed subsector of $|\arg(z)| < \pi$. Upon plugging (5.1.66) into (5.1.62) and separating the result in two integrals, for any $0 \leq r \leq R$ and fixed m, we arrive at:

$$\begin{aligned} I_r(m) &= H_{m+1}^{-r} \frac{1}{2\pi i} \int_{x_m - i\infty}^{x_m + i\infty} \lambda^{-z} H_{m+1}^{-\frac{\alpha z}{\beta}} \Gamma(z - 1 + r) dz + R_1(m,r) \\ &= \lambda^{r-1} H_{m+1}^{-r\left(1-\frac{\alpha}{\beta}\right)-\frac{\alpha}{\beta}} e^{-\lambda H_{m+1}^{\frac{\alpha}{\beta}}} + R_1(m,r) \end{aligned} \tag{5.1.67}$$

The last equality follows at once from the representation of the exponential function as an inverse Mellin transform, and the remainder integral is given by

$$R_1(m,r) := \frac{H_{m+1}^{-r}}{2\pi i} \int_{x_m - i\infty}^{x_m + i\infty} \lambda^{-z} H_{m+1}^{-\frac{\alpha z}{\beta}} \rho_{r1}(z) \Gamma(z - 2 + r) dz. \tag{5.1.68}$$

From Lemma 2.7 in [Paris and Kaminski, 2001], as $m \to \infty$, we ascertain that

$$|R_1(m,r)| = \mathcal{O}\left\{ H_{m+1}^{-r\left(1-\frac{\alpha}{\beta}\right)-2\frac{\alpha}{\beta}} e^{-\lambda H_{m+1}^{\frac{\alpha}{\beta}}} \right\}. \tag{5.1.69}$$

This verifies the asymptotic character of (5.1.67) as $m \to \infty$ for any fixed $0 \leq r \leq R$ and $\alpha, \beta > 0$. However, we must now take special care with respect to the validity of the sum (5.1.64) in the asymptotic sense. On the one hand we deduce from (5.1.67) that the powers of the digamma function appearing in (5.1.64) are only descending as $m \to \infty$ if $\beta > \alpha$. On the other hand, in the special case $r = 0$ from (5.1.67) and (5.1.69) we obtain

$$I_0(m) = \lambda^{-1} H_{m+1}^{-\frac{\alpha}{\beta}} e^{-\lambda H_{m+1}^{\frac{\alpha}{\beta}}} + \mathcal{O}\left\{ H_{m+1}^{-2\frac{\alpha}{\beta}} e^{-\lambda H_{m+1}^{\frac{\alpha}{\beta}}} \right\}. \tag{5.1.70}$$

Accordingly, if in the case $\beta > \alpha$ we designate

$$r(\alpha, \beta) := \max\left\{ k \in \mathbb{N}_0 : k < \frac{\alpha}{\beta - \alpha} \right\}, \tag{5.1.71}$$

we have $r(\alpha, \beta) \geq 0$. In addition, for $0 \leq r \leq R$ the dominating term of the r-th summand in the expansion (5.1.64) then shows a lower order of decay as $m \to \infty$ than the remainder integral for $r = 0$, only if $R \equiv r(\alpha, \beta)$. We thus eventually conclude, upon combining (5.1.64) and (5.1.67), the first terms in the expansion as $m \to \infty$ of the uniform bias in the case $\beta > \alpha$ are given by:

$$\begin{aligned} \mathrm{ULB}(m) &\sim \frac{1}{\beta\pi} H_{m+1}^{-\frac{\alpha}{\beta}} e^{-\lambda H_{m+1}^{\frac{\alpha}{\beta}}} \sum_{r=0}^{r(\alpha,\beta)} g(r, m) \lambda^{r-1} H_{m+1}^{-r\left(1-\frac{\alpha}{\beta}\right)} \\ &= H_{m+1}^{-\frac{\alpha}{\beta}} e^{-\lambda H_{m+1}^{\frac{\alpha}{\beta}}} \sum_{r=0}^{r(\alpha,\beta)} H_{m+1}^{-r\left(1-\frac{\alpha}{\beta}\right)} \Gamma_r \end{aligned} \tag{5.1.72}$$

The parameter λ and the coefficients $g(j, m)$ were given in (5.1.58) and (4.7.38), respectively, and we denote

$$\Gamma_r := \frac{\lambda^{r-1}}{\beta\pi} g(r, m). \tag{5.1.73}$$

To describe the controlling behaviour of (5.1.72) in just one term we introduce for brevity

$$\Gamma_{00} := \frac{q^{\frac{\alpha}{\beta}} \sigma^{\alpha}}{\beta\pi p \theta^{\alpha}}, \tag{5.1.74}$$

and as $m \to \infty$ we therefore conclude

$$\mathrm{ULB}(m) \sim H_{m+1}^{-\frac{\alpha}{\beta}} \exp\left\{ -\left[\frac{p^{\frac{\beta}{\alpha}} \theta^{\beta}}{q \sigma^{\beta}} \right]^{\frac{\alpha}{\beta}} H_{m+1}^{\frac{\alpha}{\beta}} \right\} \Gamma_{00}. \tag{5.1.75}$$

Subject to the properties of asymptotic expansions we know that a different expansion for the beta Mellin transform would not change its leading behaviour. Accordingly, the first term approximation (5.1.75) applies for arbitrary $\alpha, \beta > 0$ but only the partial expansion (5.1.72) fails to hold in the asymptotic sense if $\beta \leq \alpha$. Finally, from the asymptotic relation (4.7.33) and the elementary properties of the logarithm it follows that the exponential function on the right hand side of (5.1.75) decays slower than m^{-k} for arbitrary $k > 0$ if $\beta > \alpha$ and conversely vanishes faster if $\beta < \alpha$. Only in case of equality $\alpha = \beta$ the exponential function exhibits asymptotically a similar behaviour as m^{-k}, where the order k is determined by the ratio of the parameters which we designated in (5.1.58) by λ. This implies, although for $\alpha = \beta$ the right hand side of (5.1.75) does not coincide with (5.1.59), as $m \to \infty$, both expressions approach the limit value zero with the same rate.

5.2. The Local Bias Function

In this section we discuss how the method of Mellin transforms can be applied to derive asymptotic statements about the local bias function (2.1.53) if the Mellin transform associated with the m-power has special properties. More precisely, throughout the section we assume validity of (4.3.15), implying for any $m \geq 0$ non-voidness of the strip $\mathbb{S}^-_{m,\delta_{\bar{\varepsilon}}}$, which was defined in (4.3.18). Furthermore, suppose the Mellin transform $M_{\bar{\varepsilon}}(-z, m)$ establishes an asymptotic scale as $m \to \infty$, for any m-independent finite or infinite sequence of numbers in $\mathbb{S}^-_{m,\delta_{\bar{\varepsilon}}}$ with increasing real parts. In addition, with $m_0 \geq 0$ we assume there exists $x_0 \in \mathbb{S}^-_{m,\delta_{\bar{\varepsilon}}}$ for all $m \geq m_0$, such that we have absolute convergence for any $t > 0$ of the inverse Mellin integral

$$\mathcal{P}_{\bar{\varepsilon}}(t, m) = \frac{1}{2\pi i} \int\limits_{x_0 - i\infty}^{x_0 + i\infty} t^z M_{\bar{\varepsilon}}(-z, m) dz. \tag{5.2.1}$$

Compared with the derivation of asymptotic statements for the uniform bias, a treatment of the local bias (2.1.53) appears essentially more elaborate because the integrand is more complicated and the integral in general need not be absolutely convergent. Fortunately the additional effort will pay off, since it will turn out that the effect of oscillatory terms, represented by the complex exponential function, should not be underestimated. First, by conceiving $e^{-i\xi t}\Phi_X(t)$ as the characteristic function of the random variable $X - \xi$, we recall that in (2.4.30) for $\xi \in C_{\mathfrak{D}} \cap C_F$ and $T > 0$ we have shown

$$\mathfrak{I}_T(m, \xi) = \frac{1}{\pi} \int\limits_0^T \frac{\mathcal{P}_{\bar{\varepsilon}}(t, m)}{t} \Im\Phi_{X-\xi}(t) dt. \tag{5.2.2}$$

According to (2.1.53) and (2.4.4), the local bias function can thus be cast in the form

$$\mathrm{LB}(m, \xi) = \lim_{T\to\infty} \frac{1}{\pi} \int\limits_0^T \frac{\mathcal{P}_{\bar{\varepsilon}}(t, m)}{t} \Im\Phi_{X-\xi}(t) dt, \qquad \xi \in C_{\mathfrak{D}} \cap C_F. \tag{5.2.3}$$

For the imaginary part of the characteristic function appearing in the integrand, in terms of Φ_X we can write

$$\Im\Phi_{X-\xi}(t) = \cos(|\xi|\, t)\Im\Phi_X(t) - \operatorname{sgn}(\xi) \sin(|\xi|\, t)\Re\Phi_X(t). \tag{5.2.4}$$

It shows that, if Φ_X is even, we have $\Im\Phi_{X-\xi}(t) = -\operatorname{sgn}(\xi)\sin(|\xi|\, t)\Re\Phi_X(t)$. If in these circumstances $0 \in C_{\mathfrak{D}} \cap C_F$, the integral (5.2.3) vanishes at $\xi = 0$ for any $m \geq 0$. Now, plugging (5.2.1) into (5.2.3), upon formally interchanging the order of integration with $x_0 \in \mathbb{S}^-_{m,\delta_{\bar{\varepsilon}}}$, we arrive at

$$\text{(5.2.5)} \qquad \mathrm{LB}(m,\xi) = \frac{1}{2\pi^2 i} \int\limits_{x_0 - i\infty}^{x_0 + i\infty} M_{\bar{\varepsilon}}(-z,m) \mathcal{M}\left\{\Im \Phi_{X-\xi}\right\}(z) dz, \qquad \xi \in C_{\mathfrak{D}} \cap C_F,$$

where the right hand side features the Mellin transform

$$\text{(5.2.6)} \qquad \mathcal{M}\left\{\Im \Phi_{X-\xi}\right\}(z) = \int\limits_0^\infty t^{z-1} \Im \Phi_{X-\xi}(t) dt.$$

The interchange is permitted if the integral (5.2.6) converges absolutely for $\Re z = x_0$. This should be achievable if (4.2.2) holds, in which event the strip of analyticity corresponding to (5.2.6) matches the set (4.2.3). Moreover, F is then a continuous distribution and thus $C_{\mathfrak{D}} \cap C_F = \mathbb{R}$. As a consequence of the assumed property of $M_{\bar{\varepsilon}}(-z,m)$ to define an asymptotic scale in the strip $\mathbb{S}^-_{m,\delta_{\bar{\varepsilon}}}$, this region plays a major role in specifying the asymptotic behaviour as $m \to \infty$ of (5.2.5). Exact statements, however, do not only depend on the type of asymptotic scale but especially on the singularities of (5.2.6). Basically we can distinguish between two cases, assuming validity of (4.2.2) and provided the integrand in (5.2.5) admits appropriate displacements of the integration path in the region $\mathbb{S}^-_{m,\delta_{\bar{\varepsilon}}}$:

- The integrand in (5.2.6) is oscillatory at infinity in the sense that it changes its signs infinitely many times as $t \to \infty$. This is certainly the case if $\xi \neq 0$ and $\Phi_X(t)$ is non-oscillatory as $t \to \infty$ but depends on possible cancellations if Φ_X features trigonometric expressions. Then, according to Lemma 4.3.2 in [Bleistein and Handelsman, 1986], the analytic continuation associated with the integral (5.2.6) has no singularities in $\Re z > x_0$. This suggests exponential-type behaviour of the local bias function as $m \to \infty$. In particular, the rate as $m \to \infty$ then can be expected to be of the form $e^{-p(m)}$ with a function $p(m)$ whose real part grows to infinity and is determined by the asymptotic scale associated with $M_{\bar{\varepsilon}}(-z,m)$. Therefore, if $\Re p(m)$ grows logarithmically it can happen that its exponential $e^{-p(m)}$ still decays with an algebraic rate as $m \to \infty$.

- The integrand does not exhibit the described oscillatory behaviour. In these circumstances the rate as $m \to \infty$ of $\mathrm{LB}(m,\xi)$ can only be of exponential type if $\Im \Phi_{X-\xi}(t)$ decays faster than any algebraic term as $t \to \infty$.

We briefly refer in passing to Subsection A.5.2 in the appendix, where the analytic structure of a Mellin transform of a characteristic function and thus especially of (5.2.6) was shown to depend on the behaviour of the associated distribution function near the origin. Finally, regarding $\mathrm{LB}(m,\xi)$, subject to the preceding observations, it is reasonable to presume a weaker rate of pointwise convergence as the oscillations of the complex exponential function die out. Note that (5.2.3) as a function of $\xi \in \mathbb{R}$ equals a Fourier transform, so that this conjecture is in accordance with the Riemann-Lebesgue lemma. In the subsections below we will study the local bias in three different scenarios and, with a few exceptions, extract the leading terms of its asymptotic expansion.

5.2.1. Leading Behaviour for Two Algebraic-Type Characteristic Functions

We begin with an extension of the purely algebraic setup which was first presented in Subsection 5.1.1, i.e., we suppose $\Phi_X = |\Phi_X|$ and $\Phi_{\bar{\varepsilon}}$ to equal (3.3.1) and (3.3.2), respectively. Especially regarding the error distribution, for the sake of clarity we confine ourselves to the case $q = 1$. According to the choice of Φ_X, the distribution F is continuous, the integral (5.2.3) is absolutely convergent as $T \to \infty$ for any parametrization and $\Phi_X = \Re\Phi_X$. Thus, (5.2.3) for any $\xi \in \mathbb{R}$ becomes

$$\text{LB}(m, \xi) = -\frac{\text{sgn}(\xi)}{\pi} \int_0^\infty \frac{\mathcal{P}_{\bar{\varepsilon}}(t, m)}{t} \sin(|\xi|\, t)\Phi_X(t)dt. \tag{5.2.7}$$

Since $\text{LB}(m, \xi) = 0$, without loss of generality we assume $\xi \in \mathbb{R} \setminus \{0\}$. It is generally not recommendable in (5.2.7) to first introduce the inverse Mellin transform of the sine. The reason is that, subject to (4.8.13), we would then obtain a double integral which, due to the asymptotic behaviour of this Mellin transform, is not absolutely convergent for any $\alpha p > 0$. Instead, following from our findings of Subsections 4.4 and 4.5, we note it is possible for any $m \geq 0$ to choose $u_0 \in S_X$ and $x_0 \in \mathbb{S}^-_{m,\beta}$ such that $-1 < x_0 - u_0 < 0$. Then, in (5.2.7) we may rather represent Φ_X and the m-power in terms of their respective Mellin transforms, which are given by (4.4.3) and (4.5.9). Upon eventually interchanging the order of integration subject to absolute convergence, and applying the Mellin transform of the sine given in (4.8.11), we arrive at:

$$\begin{aligned}
\text{LB}(m, \xi) &= -\frac{\text{sgn}(\xi)}{\pi} \int_0^\infty \frac{\sin(|\xi|\, t)}{t} \frac{1}{2\pi i} \int_{u_0 - i\infty}^{u_0 + i\infty} (t\theta)^{-w} \frac{\Gamma\left(\frac{w}{\alpha}\right)\Gamma\left(p - \frac{w}{\alpha}\right)}{\alpha\Gamma(p)} dw \\
&\qquad\qquad \times \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} (t\sigma)^z \frac{\Gamma\left(\frac{z}{\beta}\right)\Gamma\left(m + 1 - \frac{z}{\beta}\right)}{\beta\Gamma(m+1)} dz dt \\
&= -\frac{\text{sgn}(\xi)}{\alpha\beta 2\pi^2 i} \int_{u_0 - i\infty}^{u_0 + i\infty} \theta^{-w} \frac{\Gamma\left(\frac{w}{\alpha}\right)\Gamma\left(p - \frac{w}{\alpha}\right)}{\Gamma(p)} \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} \sigma^z \frac{\Gamma\left(\frac{z}{\beta}\right)\Gamma\left(m + 1 - \frac{z}{\beta}\right)}{\Gamma(m+1)} \\
&\qquad\qquad \times \int_0^\infty t^{z - w - 1} \sin(|\xi|\, t) dt dz dw \\
&= -\frac{\text{sgn}(\xi)}{\alpha\beta 2\pi^2 i} \int_{u_0 - i\infty}^{u_0 + i\infty} \left\{\frac{\theta}{|\xi|}\right\}^{-w} \frac{\Gamma\left(\frac{w}{\alpha}\right)\Gamma\left(p - \frac{w}{\alpha}\right)}{\Gamma(p)} \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} \left\{\frac{|\xi|}{\sigma}\right\}^{-z} \sin\left\{\frac{\pi(z - w)}{2}\right\} \\
&\qquad\qquad \times \Gamma(z - w)\Gamma\left(\frac{z}{\beta}\right) \frac{\Gamma\left(m + 1 - \frac{z}{\beta}\right)}{\Gamma(m+1)} dz dw
\end{aligned}$$

$$= \frac{\operatorname{sgn}(\xi)}{\alpha\beta 2\pi^2 i} \int_{u_0-i\infty}^{u_0+i\infty} \lambda_2^{-w} \frac{\Gamma\left(\frac{w}{\alpha}\right)\Gamma\left(p-\frac{w}{\alpha}\right)}{\Gamma(p)} I(w,m,\xi)dw \tag{5.2.8}$$

Here, for brevity we write $\lambda_l \equiv \lambda_l(\xi)$, and we denote

$$\lambda_l(\xi) := \begin{cases} \frac{|\xi|}{\sigma}, & \text{if } l=1, \\ \frac{\theta}{|\xi|}, & \text{if } l=2, \end{cases} \tag{5.2.9}$$

$$I(w,m,\xi) := -\frac{1}{2\pi i} \int_{x_0-i\infty}^{x_0+i\infty} \lambda_1^{-z} \sin\left\{\frac{\pi(z-w)}{2}\right\} \Gamma(z-w)\Gamma\left(\frac{z}{\beta}\right) \frac{\Gamma\left(m+1-\frac{z}{\beta}\right)}{\Gamma(m+1)} dz. \tag{5.2.10}$$

Observe that the order of integration of the integrals in (5.2.8) differs from (5.2.5). This decision was made intentionally, because the order of integration in (5.2.5) appears rather helpful for general statements but inappropriate for a treatment of definite examples. The reason is, since in (5.2.5) only the exterior integral depends on m we had to analyze the interior as a function of $z \in \mathbb{C}$. The strategy for which we aim was then inapplicable. That is, we start with an approximation of (5.2.10), proceed with the evaluation of the obtained estimate for fixed w with $\Re w = u_0$ in the usual fashion known from single integrals, and then eventually apply the obtained estimate for the leading term to (5.2.8). More precisely, with $u_0 - 1 < x_0 < u_0$ for $x_0 \in \mathbb{S}^-_{m,\beta}$, $u_0 \in S_X$ and fixed m we define

$$\hat{I}(w,m,\xi) := -\frac{1}{2\pi i} \int_{x_0-i\infty}^{x_0+i\infty} \left\{m^{\frac{1}{\beta}}\lambda_1\right\}^{-z} \sin\left\{\frac{\pi(z-w)}{2}\right\} \Gamma(z-w)\Gamma\left(\frac{z}{\beta}\right) dz. \tag{5.2.11}$$

Then, by virtue of (B.3.5) as $m \to \infty$ for fixed $w \in \mathbb{C}$ with $\Re w = u_0$ we conclude

$$I(w,m,\xi) \sim \hat{I}(w,m,\xi). \tag{5.2.12}$$

The surrogate integral (5.2.11) is of standard Mellin-Barnes type. To extract its leading term we must consider the half plane $\Re z > x_0$, where the power of m is descending. The vertical line $\Re z = x_0$ was specified to satisfy $\Re w - 1 < \Re z < \Re w$ with $\Re w = u_0$. But the singularity at $z = w$ is removable, implying analyticity of the integrand in (5.2.11) in the whole half plane $\Re z > x_0$. Furthermore, subject to Stirling's formula and (4.8.13) for fixed $\Re z > 0$ and $\Re w = u_0$ the integrand decays exponentially fast as $|\Im z| \to \infty$. It is hence possible to displace the integration path by an arbitrary but, owing to the exponential growth as $\Re z \to \infty$, finite distance to the right. In this process we do not cross any singularities. This indicates that the integral (5.2.11) has exponential type behaviour as $m \to \infty$. To take this into account we introduce an inverse factorial expansion. For this, we begin with a displacement of the integration path to the right, to match a line $\Re z = x_m$ with $x_m > x_0$ such that $|z|$ is large everywhere on the new path. Moreover, after this shift of the integration path we express the sine in terms of the complex exponential function. We therefore define

$$\hat{I}^{\pm}(w,m,\xi) := \frac{1}{2\pi i}\int\limits_{x_m-i\infty}^{x_m+i\infty}\left\{m^{\frac{1}{\beta}}\lambda_1 e^{\mp\frac{i\pi}{2}}\right\}^{-z}\Gamma(z-w)\Gamma\left(\frac{z}{\beta}\right)dz. \tag{5.2.13}$$

The integral (5.2.11) for fixed m then can be decomposed in the following manner:

$$\hat{I}(w,m,\xi) = -\frac{1}{2i}\left\{e^{-\frac{i\pi w}{2}}\hat{I}^{+}(w,m,\xi) - e^{\frac{i\pi w}{2}}\hat{I}^{-}(w,m,\xi)\right\} \tag{5.2.14}$$

We are now ready to employ the aforementioned inverse factorial expansion to approximate the leading behaviour of the gamma functions in the integral (5.2.13). The parameters defined in Section 2.2 in [Paris and Kaminski, 2001] for fixed $w \in \mathbb{C}$ with $\Re w = u_0$ are now given by

$$\begin{cases} h = \beta^{\frac{1}{\beta}}, \\ \vartheta \equiv \vartheta(w) = -\frac{1}{2} - w, \\ \kappa = 1 + \frac{1}{\beta}. \end{cases} \tag{5.2.15}$$

In terms of these, from Lemma 2.2 in [Paris and Kaminski, 2001], we ascertain the first order approximation

$$\Gamma(z-w)\Gamma\left(\frac{z}{\beta}\right) = 2\pi(h\kappa^{\kappa})^{-z}\left\{A_0(w)\Gamma(\kappa z+\vartheta) + \rho_{w1}(z)\Gamma(\kappa z+\vartheta-1)\right\}. \tag{5.2.16}$$

For any fixed $w \in \mathbb{C}$ with $\Re w = u_0$ the remainder $\rho_{w1}(z)$ is an analytic function of $z \in \mathbb{C}$, except at the points where the left hand side of (5.2.16) has poles, and it satisfies $\rho_{w1}(z) = \mathcal{O}(1)$ as $|z| \to \infty$ uniformly in any closed interior sector of $|\arg(z)| < \pi$. Furthermore, the coefficient $A_0(w)$ equals

$$A_0(w) = \kappa^{1+w}\sqrt{\frac{\beta}{2\pi}}. \tag{5.2.17}$$

Especially note, since the integrals $\hat{I}^{\pm}$ differ only with respect to the sign in the complex exponential function, we can employ the inverse factorial expansion (5.2.16) for both. Accordingly, for (5.2.13) we write

$$\hat{I}^{\pm}(w,m,\xi) = 2\pi\left\{A_0(w)\hat{I}_0^{\pm}(w,m,\xi) + \hat{I}_1^{\pm}(w,m,\xi)\right\}, \tag{5.2.18}$$

where the single MB-integrals on the right hand side are given by

$$\hat{I}_0^{\pm}(w,m,\xi) := \frac{1}{2\pi i}\int\limits_{x_m-i\infty}^{x_m+i\infty}\left\{m^{\frac{1}{\beta}}\lambda_1 h\kappa^{\kappa} e^{\mp\frac{i\pi}{2}}\right\}^{-z}\Gamma(\kappa z+\vartheta)dz, \tag{5.2.19}$$

$$\hat{I}_1^{\pm}(w,m,\xi) := \frac{1}{2\pi i}\int\limits_{x_m-i\infty}^{x_m+i\infty}\left\{m^{\frac{1}{\beta}}\lambda_1 h\kappa^{\kappa}e^{\mp\frac{i\pi}{2}}\right\}^{-z}\rho_{w1}(z)\Gamma(\kappa z+\vartheta-1)dz. \tag{5.2.20}$$

Upon eventually combining (5.2.18) with (5.2.14), for fixed m we arrive at the following expansion:

$$\begin{aligned}\hat{I}(w,m,\xi) = &-\frac{\pi}{i}A_0(w)\left\{e^{-\frac{i\pi w}{2}}\hat{I}_0^{+}(w,m,\xi) - e^{\frac{i\pi w}{2}}\hat{I}_0^{-}(w,m,\xi)\right\}\\ &-\frac{\pi}{i}\left\{e^{-\frac{i\pi w}{2}}\hat{I}_1^{+}(w,m,\xi) - e^{\frac{i\pi w}{2}}\hat{I}_1^{-}(w,m,\xi)\right\}\end{aligned} \tag{5.2.21}$$

To evaluate the integrals (5.2.19) we can easily apply the inversion formula for Mellin transforms, admitting a reference to the exponential function, from which we deduce:

$$\begin{aligned}\hat{I}_0^{\pm}(w,m,\xi) &= \kappa^{\vartheta-1}\left\{m^{\frac{1}{\beta}}\lambda_1 h e^{\mp\frac{i\pi}{2}}\right\}^{\frac{\vartheta}{\kappa}}\exp\left\{-(\lambda_1 h)^{\frac{1}{\kappa}}\kappa e^{\mp\frac{i\pi}{2\kappa}}m^{\frac{1}{\beta\kappa}}\right\}\\ &= \kappa^{\vartheta-1}\left\{m^{\frac{1}{\beta}}\lambda_1 h\right\}^{\frac{\vartheta}{\kappa}}\exp\left\{\mp\frac{i\pi\vartheta}{2\kappa} - (\lambda_1 h)^{\frac{1}{\kappa}}\kappa e^{\mp\frac{i\pi}{2\kappa}}m^{\frac{1}{\beta\kappa}}\right\}\end{aligned} \tag{5.2.22}$$

The sum of the first two integrals in (5.2.21) can therefore be expressed in terms of elementary functions, where the imaginary parts cancel out:

$$\begin{aligned}\hat{I}(w,m,\xi) = &-\frac{\pi}{i}\kappa^{\vartheta-1}\left\{m^{\frac{1}{\beta}}\lambda_1 h\right\}^{\frac{\vartheta}{\kappa}}A_0(w)\Bigg\{\exp\left\{-\frac{i\pi(\vartheta+w\kappa)}{2\kappa} - (\lambda_1 h)^{\frac{1}{\kappa}}\kappa e^{-\frac{i\pi}{2\kappa}}m^{\frac{1}{\beta\kappa}}\right\}\\ &\qquad - \exp\left\{\frac{i\pi(\vartheta+w\kappa)}{2\kappa} - (\lambda_1 h)^{\frac{1}{\kappa}}\kappa e^{\frac{i\pi}{2\kappa}}m^{\frac{1}{\beta\kappa}}\right\}\Bigg\}\\ &-\frac{\pi}{i}\left\{e^{-\frac{i\pi w}{2}}\hat{I}_1^{+}(w,m,\xi) - e^{\frac{i\pi w}{2}}\hat{I}_1^{-}(w,m,\xi)\right\}\\ = &\,2\pi\kappa^{-w-\frac{3}{2}}\left\{m^{\frac{1}{\beta}}\lambda_1 h\right\}^{-\frac{w}{\kappa}-\frac{1}{2\kappa}}\exp\left\{-(\lambda_1 h)^{\frac{1}{\kappa}}\kappa m^{\frac{1}{\beta\kappa}}\cos\left\{\frac{\pi}{2\kappa}\right\}\right\}\\ &\times A_0(w)\sin\left\{\frac{\pi(\vartheta+w\kappa)}{2\kappa} - (\lambda_1 h)^{\frac{1}{\kappa}}\kappa m^{\frac{1}{\beta\kappa}}\sin\left\{\frac{\pi}{2\kappa}\right\}\right\}\\ &-\frac{\pi}{i}\left\{e^{-\frac{i\pi w}{2}}I_1^{+}(w,m,\xi) - e^{\frac{i\pi w}{2}}I_1^{-}(w,m,\xi)\right\}\end{aligned} \tag{5.2.23}$$

The second equality is an immediate consequence of the identity

$$\exp\left\{-ia-be^{-ic}\right\} - \exp\left\{ia-be^{ic}\right\} = -2i\sin(a-b\sin(c))e^{-b\cos(c)}. \tag{5.2.24}$$

Observe that $\kappa^{-1}\in(0,1)$ by definition, whence the exponent in (5.2.23) is always positive. Concerning the remainder integrals defined in (5.2.20), from equation (2.5.8) below Lemma 2.7 in [Paris and Kaminski, 2001] as $m\to\infty$ for any fixed $w\in\mathbb{C}$ with $\Re w = u_0$ we know

$$\left|\hat{I}_1^{\pm}(w,m,\xi)\right| = \mathcal{O}\left\{(\lambda_1 h)^{-\frac{1-\vartheta}{\kappa}}m^{-\frac{1-\vartheta}{\beta\kappa}}e^{-(\lambda_1 h)^{\frac{1}{\kappa}}m^{\frac{1}{\beta\kappa}}}\right\}. \tag{5.2.25}$$

As a consequence we have just established as $m \to \infty$ for fixed $w \in \mathbb{C}$ with $\Re w = u_0$ and with $\kappa = \frac{1+\beta}{\beta}$ the asymptotic validity of (5.2.23), which is in particular

$$\hat{I}(w, m, \xi) \sim B_0 \left\{ m^{\frac{1}{1+\beta}} (\lambda_1 h)^{\frac{1}{\kappa}} \right\}^{-w-\frac{1}{2}} \sin\left\{ \frac{w\pi}{2(1+\beta)} - \frac{\beta\pi}{4(1+\beta)} - \lambda_4 m^{\frac{1}{1+\beta}} \right\} e^{-\lambda_3 m^{\frac{1}{1+\beta}}}. \tag{5.2.26}$$

The constants and coefficients appearing on the right hand side are given by

$$\begin{cases} B_0 := \beta\sqrt{\frac{2\pi}{1+\beta}}, \\ \{\lambda_1(\xi)h\}^{\frac{1}{\kappa}} = \left\{\frac{|\xi|}{\sigma}\right\}^{\frac{\beta}{1+\beta}} \beta^{\frac{1}{1+\beta}}, \\ \lambda_3(\xi) := \frac{1+\beta}{\beta} \cos\left\{\frac{\pi\beta}{2(1+\beta)}\right\} \{\lambda_1(\xi)h\}^{\frac{1}{\kappa}}, \\ \lambda_4(\xi) := \frac{1+\beta}{\beta} \sin\left\{\frac{\pi\beta}{2(1+\beta)}\right\} \{\lambda_1(\xi)h\}^{\frac{1}{\kappa}}. \end{cases} \tag{5.2.27}$$

Observe that the exponential leading term in (5.2.26) does not depend on w.

We proceed with an attempt to employ the approximation (5.2.26) for a final estimation of the initial double integral. Due to its pointwise validity we expect some difficulties to arise. This guess is already confirmed by first noting as $|w| \to \infty$ in $|\arg(w)| < \pi$ and $|\arg(\alpha p - w)| < \pi$, according to (4.4.4) and (4.8.12), we have

$$\left| \Gamma\left(\frac{w}{\alpha}\right) \Gamma\left(p - \frac{w}{\alpha}\right) \sin\left\{ \frac{w\pi}{2(1+\beta)} - \frac{\beta\pi}{4(1+\beta)} - \lambda_4 m^{\frac{1}{1+\beta}} \right\} \right| = \mathcal{O}\left\{ |w|^{p-1} e^{-\frac{\pi|\Im w|}{\alpha} + \frac{\pi|\Im w|}{2(1+\beta)}} \right\}.$$

The term in the big-$\mathcal{O}$ tends to zero as $|\Im w| \to \infty$ only if $\alpha < 2(1+\beta)$. Consequently, with $u_0 \in S_X$, absolute convergence of the following integral is only guaranteed for $\alpha < 2(1+\beta)$:

$$\begin{aligned} \widehat{\mathrm{LB}}(m, \xi) := \frac{1}{2\pi i} \int_{u_0 - i\infty}^{u_0 + i\infty} \left\{ \lambda_2 m^{\frac{1}{1+\beta}} (\lambda_1 h)^{\frac{1}{\kappa}} \right\}^{-w} \frac{\Gamma\left(\frac{w}{\alpha}\right) \Gamma\left(p - \frac{w}{\alpha}\right)}{\Gamma(p)} \\ \times \sin\left\{ \frac{w\pi}{2(1+\beta)} - \frac{\beta\pi}{4(1+\beta)} - \lambda_4 m^{\frac{1}{1+\beta}} \right\} dw \end{aligned} \tag{5.2.28}$$

Appealing to the relations (5.2.12) and (5.2.26), the right hand side of (5.2.28) furnishes an estimate for the local bias integral in (5.2.8). More precisely, for $\alpha < 2(1+\beta)$ as $m \to \infty$ we conclude

$$\mathrm{LB}(m, \xi) \sim \left\{ m^{\frac{1}{1+\beta}} (\lambda_1 h)^{\frac{1}{\kappa}} \right\}^{-\frac{1}{2}} e^{-\lambda_3 m^{\frac{1}{1+\beta}}} B_0 \frac{\operatorname{sgn}(\xi)}{\alpha\beta\pi} \widehat{\mathrm{LB}}(m, \xi). \tag{5.2.29}$$

To extract the dominating term as $m \to \infty$ of $\widehat{\mathrm{LB}}(m, \xi)$, the integrand suggests to confine our study to the region to the right of the integration path $\Re w = u_0$. But from the above asymptotic estimate for the integrand we deduce algebraic growth as $\Re w \to \infty$ and exponential decay in the imaginary direction, subject to the indicated parameter restriction. This gives the permission to displace the integration path by an arbitrary finite distance rightwards, say to match the line $u_1 := \alpha\left(p+\frac{1}{2}\right)$. In this process we merely traverse the simple pole of the beta function located at $w = \alpha p$ with the associated residue given in (5.1.7), yielding

$$\begin{aligned}\widehat{\mathrm{LB}}(m,\xi) = {}& \alpha\left\{\lambda_2 m^{\frac{1}{1+\beta}}(\lambda_1 h)^{\frac{1}{\kappa}}\right\}^{-\alpha p}\sin\left\{\frac{\pi\alpha p}{2(1+\beta)} - \frac{\beta\pi}{4(1+\beta)} - \lambda_4 m^{\frac{1}{1+\beta}}\right\}\\ &+\frac{1}{2\pi i}\int\limits_{u_1-i\infty}^{u_1+i\infty}\left\{\lambda_2 m^{\frac{1}{1+\beta}}(\lambda_1 h)^{\frac{1}{\kappa}}\right\}^{-w}\frac{\Gamma\left(\frac{w}{\alpha}\right)\Gamma\left(p-\frac{w}{\alpha}\right)}{\Gamma(p)}\\ &\times\sin\left\{\frac{w\pi}{2(1+\beta)} - \frac{\beta\pi}{4(1+\beta)} - \lambda_4 m^{\frac{1}{1+\beta}}\right\}dw.\end{aligned} \tag{5.2.30}$$

The boundedness of the remainder integral is readily verified by virtue of the inequalities (4.4.6) and (4.8.12). Moreover, since $u_1 > \alpha p$ the remainder has a higher order with respect to the asymptotic parameter m. Further rightward displacements of the integration path in (5.2.30) are permitted and unlock higher order terms in the asymptotic expansion of $\widehat{\mathrm{LB}}(m,\xi)$. It must, however, be emphasized that these terms are not necessarily the subsequent terms in the expansion of $\mathrm{LB}(m,\xi)$, because of the earlier applied approximations. To summarize our findings, for $\alpha < 2(1+\beta)$ as $m \to \infty$ we have just verified

$$\mathrm{LB}(m,\xi) \sim \operatorname{sgn}(\xi)\sqrt{\frac{2}{\pi(1+\beta)}}\,\frac{\sin\left\{\frac{\pi\alpha p}{2(1+\beta)} - \frac{\beta\pi}{4(1+\beta)} - \lambda_4 m^{\frac{1}{1+\beta}}\right\}}{\lambda_2^{\alpha p}(\lambda_1 h)^{\frac{\alpha p}{\kappa}+\frac{1}{2\kappa}} m^{\frac{2\alpha p+1}{2(1+\beta)}}}e^{-\lambda_3 m^{\frac{1}{1+\beta}}}. \tag{5.2.31}$$

Before proceeding with our next example we take a short break for a comparison of this result with the dominating term we deduced for the uniform bias function in Subsection 5.1.1. There, according to (5.1.11), for the above setup as $m \to \infty$ we have shown

$$\mathrm{ULB}(m) \sim \frac{1}{\beta\pi}\left(\frac{\sigma}{\theta}\right)^{\alpha p}\Gamma\left(\frac{\alpha p}{\beta}\right)(m+1)_{-\frac{\alpha p}{\beta}}. \tag{5.2.32}$$

This clearly differs from (5.2.31) by an additional oscillatory and an exponential factor. Besides, the order of the algebraic term in (5.2.32) is slightly higher in comparison to (5.2.31). The exponential and the oscillatory terms arise from the presence of the sine in (5.2.7), as indicated in our introductory discussion. Furthermore, we observe an increasement in the parameters λ_1 and λ_3, λ_4 for larger values of $|\xi|$. This results in a faster rate and a higher frequency with respect to the oscillations. The parameter λ_2 which is decreasing with respect to ξ does not affect the order of the rate. Conversely, in (5.2.31) the rate is decreasing for smaller arguments $|\xi|$, thereby confirming our guess. At $\xi = 0$ the statement (5.2.31) does no longer hold but $\mathrm{LB}(m,0) = 0$, according to (5.2.7). As a consequence of the limitation $\xi \neq 0$, inaccuracies should especially occur if ξ is located too close to the origin and m is not large enough, to yield a reliable estimate by means of the dominating term (5.2.31). To avoid those inaccuracies, for ξ in a neighborhood of the origin we recommend use of (5.2.32).

5.2.2. Leading Behaviour for a Discretely Distributed X-Variable

We now examine the local bias integral (5.2.3) for a characteristic function Φ_X associated with a discrete distribution. In contrast to the previous scenario we then no longer have absolute convergence of the integral. More precisely, assume F defines an arbitrary discrete distribution with set of atoms $D_F = \{\xi_k : k \in \mathcal{I}\}$ for some $\mathcal{I} \subset \mathbb{Z}$. The corresponding characteristic function is then almost periodic and given by a trigonometric polynomial:

$$\Phi_X(t) = \sum_{k\in\mathcal{I}} e^{it\xi_k} F\{\xi_k\} \tag{5.2.33}$$

Denote $a_k \equiv a_k(\xi)$ with $a_k(\xi) := |\xi_k - \xi|$ for $k \in \mathcal{I}$ and $\xi \in C_{\mathfrak{D}} \cap C_F$, which implies $0 < a_k < \infty$. This enables us to write

$$\Im\Phi_{X-\xi}(t) = \sum_{k\in\mathcal{I}} \operatorname{sgn}(\xi_k - \xi) \sin(ta_k) F\{\xi_k\}. \tag{5.2.34}$$

In order to cast (5.2.3) as a MB-integral we first recall the validity of this representation for any fixed $m \geq 0$. In particular, due to the boundedness of F and $\mathfrak{D}(\cdot, m)$, the limit on the right hand side of (5.2.3) is finite for any $m \geq 0$. Hence, according to Abel's lemma A.4.1(2), we can equivalently write

$$\mathrm{LB}(m,\xi) = \lim_{\delta\downarrow 0} \mathfrak{A}_\delta(m,\xi), \tag{5.2.35}$$

where the integral function on the right hand side is given by

$$\mathfrak{A}_\delta(m,\xi) := \frac{1}{\pi}\int_0^\infty \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t} \Im\Phi_{X-\xi}(t) e^{-\delta t} dt. \tag{5.2.36}$$

For fixed $\delta > 0$ we now employ the sum representation (5.2.34) and subject to absolute convergence interchange the order of summation and integration, leading to

$$\mathfrak{A}_\delta(m,\xi) = \frac{1}{\pi} \sum_{k\in I} F\{\xi_k\} \operatorname{sgn}(\xi_k - \xi) I_k^\delta(m,\xi), \tag{5.2.37}$$

for $k \in \mathcal{I}$ with

$$I_k^\delta(m,\xi) := \int_0^\infty \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t} \sin(ta_k) e^{-\delta t} dt. \tag{5.2.38}$$

Besides the conditions imposed on the m-power in the introductory part of this section, we suppose there exists $0 < x_0 < \frac{1}{2}$ such that we have absolute convergence of (5.2.1) for all sufficiently large m. In these circumstances, again due to absolute convergence, for (5.2.38) we can write

$$I_k^\delta(m,\xi) = \frac{1}{2\pi i}\int\limits_{x_0-i\infty}^{x_0+i\infty} M_{\bar{\varepsilon}}(-z,m)\int\limits_0^\infty t^{z-1}\sin(a_k t)e^{-\delta t}dtdz. \tag{5.2.39}$$

It is a simple exercise in complex analysis to specify for fixed $\delta > 0$ in terms of known special functions the Mellin transform

$$\mathcal{M}\left\{\sin(a_k t)e^{-\delta t}\right\}(z) = \int\limits_0^\infty t^{z-1}\sin(a_k t)e^{-\delta t}dt. \tag{5.2.40}$$

The associated strip of analyticity is the entire half plane $\Re z > -1$. Assuming $0 < z < 1$, it follows upon substitution of $s = (\delta - ia_k)t$ for any $\delta > 0$:

$$\begin{aligned}
\mathcal{M}\left\{\sin(a_k t)e^{-\delta t}\right\}(z) &= \Im\int\limits_0^\infty t^{z-1}e^{-(\delta-ia_k)t}dt \\
&= \Im(\delta - ia_k)^{-z}\int\limits_0^\infty s^{z-1}e^{-s}ds \\
&= |\delta - ia_k|^{-z}\,\Gamma(z)\Im\exp\{-iz\arg(\delta - ia_k)\} \\
&= -|\delta - ia_k|^{-z}\,\Gamma(z)\sin\{z\arg(\delta - ia_k)\} \qquad (5.2.41)
\end{aligned}$$

The second equality was obtained after the described substitution, by an appropriate rotation of the integration path similar to Subsection 4.8.1, with an argument function for $s \mapsto s^z$ whose branch cut does not run through the right half plane. This implies analyticity of the whole integrand there with exponential decay. Now, the right hand side of (5.2.41) constitutes an analytic function of $z \in \mathbb{C}$, except for a countable set of poles which are a subset of $-\mathbb{N}_0$. Hence, it defines the analytic continuation to the whole complex plane of the integral (5.2.40) for any $\delta > 0$. Furthermore, regarding (5.2.39) with $0 < x_0 < \frac{1}{2}$ we can now write

$$I_k^\delta(m,\xi) = -\frac{1}{2\pi i}\int\limits_{x_0-i\infty}^{x_0+i\infty} |\delta - ia_k|^{-z}\,M_{\bar{\varepsilon}}(-z,m)\Gamma(z)\sin\{z\arg(\delta - ia_k)\}\,dz. \tag{5.2.42}$$

To eventually justify in (5.2.35) an interchange in the order of limit, summation and integration, it remains to verify the uniform boundedness of (5.2.42) with respect to $k \in \mathcal{I}$ and $\delta \geq 0$. This can be achieved by assuming $\xi \in C_{\mathfrak{D}} \cap C_F$ satisfies the weak condition

$$\inf_{k\in I} a_k > 0. \tag{5.2.43}$$

Indeed, the condition certainly holds for any $\xi \in C_{\mathfrak{D}} \cap C_F$ if D_F is nowhere dense. Then, by putting together (B.2.32) and (4.8.12) we see, for $z = x_0 + iy$ with $0 < x_0 < \frac{1}{2}$ and $y \in \mathbb{R}$:

$$
\begin{aligned}
|\delta - ia_k|^{-x_0} |\Gamma(z) \sin\{z \arg(\delta - ia_k)\}| &\le a_k^{-x_0}\sqrt{2\pi}\,|z|^{x_0-\frac{1}{2}}\, e^{-\frac{\pi}{2}|y| + \frac{1}{6|z|} + |\arg(\delta - ia_k)y|} \\
&\le a_k^{-x_0}\sqrt{2\pi} x_0^{x_0-\frac{1}{2}} e^{\frac{1}{6x_0}}
\end{aligned} \tag{5.2.44}
$$

For the second inequality notice $\arg(\delta - ia_k) \in \left(-\frac{\pi}{2}, 0\right)$ for any $\delta, a_k > 0$. Consequently, since we assumed absolute convergence of the inverse Mellin integral (5.2.1), the integral (5.2.42) is indeed uniformly bounded with respect to $k \in \mathcal{I}$ and $\delta \ge 0$ by:

$$
\left| I_k^\delta(m, \xi) \right| \le \sqrt{2\pi} x_0^{x_0 - \frac{1}{2}} e^{\frac{1}{6x_0}} \sup_{k \in I} a_k^{-x_0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} |M_{\bar{\varepsilon}}(-x_0 - iy, m)|\, dy
$$

Furthermore, $\lim_{\delta \downarrow 0} \arg(\delta - ia_k) = -\frac{\pi}{2}$, from which we conclude by virtue of Lebesgue's dominated convergence theorem, upon combining (5.2.35), (5.2.37) and (5.2.42):

$$
\mathrm{LB}(m, \xi) = \frac{1}{\pi} \sum_{k \in \mathcal{I}} F\{\xi_k\} \operatorname{sgn}(\xi_k - \xi) \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} a_k^{-z} M_{\bar{\varepsilon}}(-z, m) \Gamma(z) \sin\left\{\frac{z\pi}{2}\right\} dz \tag{5.2.45}
$$

Under the above assumptions this equals an absolutely convergent sum with respect to $k \in \mathcal{I}$ of absolutely convergent integrals. Depending on whether or not F is a finite distribution, the sum is also possibly finite. Suppose the order of each integral as $m \to \infty$ is increasing for large values of $a_k = |\xi_k - \xi|$. This means, the main contribution comes from those integrals involving a_k associated with F-atoms $\xi_k \in D_F$ which have the closest distance to the given point $\xi \in C_{\mathfrak{D}} \cap C_F$. Note that such a property solely depends on the structure of $M_{\bar{\varepsilon}}(\cdot, m)$ and recall that $a_k = 0$ is impossible by definition of $C_{\mathfrak{D}} \cap C_F$. Now, if larger values of a_k have a positive effect, it is clear that distributions F with a wider span between two consecutive atoms yield better rates of convergence as $m \to \infty$. Indeed, the difference with respect to the order between two consecutive terms in an asymptotic expansion of (5.2.45) is then greater in comparison to a situation where D_F is rather dense.

For a given $k \in \mathcal{I}$ each integral in (5.2.45) can be evaluated in the usual manner. In particular, assume $M_{\bar{\varepsilon}}(-z, m) = \mathcal{O}\left\{e^{-a|\Im z|}\right\}$ as $|\Im z| \to \infty$ in $\mathbb{S}^-_{m,\delta_{\bar{\varepsilon}}}$ for some $a > 0$ and all sufficiently large m, so that the Mellin transform of the m-power is able to overcome the algebraic growth of the Mellin transform of the sine in the region $\Re z > \frac{1}{2}$, compare (4.8.13). In this event, by increasing m arbitrary displacements of the integration path to the right direction are viable, without encountering any singularities. Hence, the integrand in each integral in (5.2.45) is analytic in any strip of finite width in $\Re z > 0$ which indicates an exponential type rate of $\mathrm{LB}(m, \xi)$ as $m \to \infty$. More precisely, the rate is then presumably of the form $e^{-p(m)}$, where the real part of the phase $p(m)$ grows to infinity as $m \to \infty$ with a rate depending on the asymptotic scale associated with

$M_{\bar{\varepsilon}}(\cdot, m)$.

5.2.2.1. Example: Discrete X Blurred by Errors with a Simple Algebraic-Type Characteristic Function

Finally, as an illustrative example we again suppose $\Phi_{\bar{\varepsilon}}$ is a member of the algebraic type function family (3.3.2) with $q = 1$. For the corresponding Mellin transform we refer to (4.5.9). Then, with an appropriate $0 < x_0 < \min\left\{\frac{1}{2}, \beta\right\}$, the MB-integral (5.2.45) for any $m \geq 0$ takes on the form

$$\text{LB}(m, \xi) = \frac{1}{\beta\pi} \sum_{k \in \mathcal{I}} F\{\xi_k\} \operatorname{sgn}(\xi_k - \xi) I_k^0(m, \xi), \tag{5.2.46}$$

where for $k \in \mathcal{I}$ we denote

$$I_k^0(m, \xi) := \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} \left\{\frac{a_k}{\sigma}\right\}^{-z} \sin\left\{\frac{z\pi}{2}\right\} \Gamma(z) \Gamma\left(\frac{z}{\beta}\right) \frac{\Gamma\left(m + 1 - \frac{z}{\beta}\right)}{\Gamma(m+1)} dz. \tag{5.2.47}$$

We already encountered the latter integral in equation (5.2.10) of the previous Subsection 5.2.1. Indeed, upon approximating by virtue of (B.3.5) the ratio of gamma functions which depends on m, by comparison with (5.2.12) and (5.2.11) we conclude

$$I_k^0(m, \xi) \sim -\hat{I}(0, m, a_k). \tag{5.2.48}$$

The dominating term of the integral on the right hand side has been established in (5.2.26). Accordingly, as $m \to \infty$ we deduce

$$\begin{aligned} I_k^0(m, \xi) \sim \beta \sqrt{\frac{2\pi}{1+\beta}} \left\{\frac{a_k}{\sigma}\right\}^{-\frac{\beta}{2(1+\beta)}} (\beta m)^{-\frac{1}{2(1+\beta)}} e^{-\lambda_3(a_k) m^{\frac{1}{1+\beta}}} & \\ \times \sin\left\{\frac{\beta\pi}{4(1+\beta)} + \lambda_4(a_k) m^{\frac{1}{1+\beta}}\right\}. & \end{aligned} \tag{5.2.49}$$

With $\kappa = 1 + \frac{1}{\beta}$ the corresponding coefficients and parameters are given by

$$\begin{cases} \lambda_3(a_k) = \frac{1+\beta}{\beta} \cos\left\{\frac{\pi\beta}{2(1+\beta)}\right\} \left\{\frac{a_k}{\sigma}\right\}^{\frac{\beta}{1+\beta}} \beta^{\frac{1}{1+\beta}}, \\ \lambda_4(a_k) = \frac{1+\beta}{\beta} \sin\left\{\frac{\pi\beta}{2(1+\beta)}\right\} \left\{\frac{a_k}{\sigma}\right\}^{\frac{\beta}{1+\beta}} \beta^{\frac{1}{1+\beta}}. \end{cases} \tag{5.2.50}$$

It is easy to see that the rate of decay of (5.2.49) is in fact lower for integrals associated with smaller a_k, in comparison to larger values. Hence, if we define

$$\mathcal{I}(\xi) := \{k \in \mathcal{I} : a_j(\xi) \geq a_k(\xi) \text{ for any } j \in \mathcal{I} \setminus \{k\}\},$$

the controlling term of (5.2.46) comes from the integral $I_k^0(m,\xi)$ associated with the indices $k \in \mathcal{I}(\xi)$. For a given $\xi \in C_{\mathfrak{D}} \cap C_F$ the set $\mathcal{I}(\xi)$ contains exactly one element denoted by k_0, except if ξ lies exactly in the middle of two subsequent F-atoms. In the former case we deduce for the local bias as $m \to \infty$:

$$\begin{aligned}\mathrm{LB}(m,\xi) \sim \sqrt{\frac{2}{\pi(1+\beta)}} F\left\{\xi_{k_0}\right\} \operatorname{sgn}(\xi_{k_0}-\xi) \left\{\frac{a_{k_0}}{\sigma}\right\}^{-\frac{\beta}{2(1+\beta)}} (\beta m)^{-\frac{1}{2(1+\beta)}} e^{-\lambda_3(a_{k_0}) m^{\frac{1}{1+\beta}}} \\ \times \sin\left\{\frac{\beta\pi}{4(1+\beta)} + \lambda_4(a_{k_0}) m^{\frac{1}{1+\beta}}\right\}\end{aligned} \tag{5.2.51}$$

If the set $\mathcal{I}(\xi)$ contains two elements, the associated points a_k coincide and we must thus multiply the right hand side of (5.2.51) by the factor two.

5.2.3. A Uniformly Distributed Continuous X-Variable

We close this chapter with the study of scenarios where the random variable X is associated with a continuous uniform distribution on the interval $[-b,b]$ for a constant $b>0$. Rather than deriving exact results we keep our discussion general and point out some interesting observations. In the present setup the characteristic function of X, subject to (3.3.5) in terms of the sinc function (A.1.7) is given by $\Phi_X(t) = \mathrm{si}(bt)$. Although the decay of this function as $t \to \pm\infty$ is of algebraic order, it will turn out that the structure of an asymptotic expansion for the local bias substantially differs from what we would obtain if Φ_X was of the purely algebraic kind (3.3.1). In fact, the presence of the sine function and its oscillatory behaviour has strong consequences. According to our specification of Φ_X, we have $C_{\mathfrak{D}} \cap C_F = \mathbb{R}$, and the local bias (5.2.3) at some point $\xi \in \mathbb{R}$ is

$$\mathrm{LB}(m,\xi) = -\frac{\operatorname{sgn}(\xi)}{\pi} \int_0^\infty \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t} \sin(|\xi|\,t)\,\mathrm{si}(bt)dt.$$

The integral converges absolutely and equals zero for $\xi = 0$. Elementary manipulations equivalently yield

$$\mathrm{LB}(m,\xi) = -\frac{\xi}{\pi} \int_0^\infty \mathcal{P}_{\bar{\varepsilon}}(t,m)\,\mathrm{si}(|\xi|\,t)\,\mathrm{si}(bt)dt. \tag{5.2.52}$$

Suppose now the Mellin transform $M_{\bar{\varepsilon}}(-z,m)$ in $\mathbb{S}^-_{m,\delta_{\bar{\varepsilon}}}$ satisfies the properties described in the introductory part of Section 5.2, and for $\xi \in \mathbb{R}\setminus\{0\}$ denote

$$M_\xi(\zeta) := \int_0^\infty t^{\zeta-1}\,\mathrm{si}(|\xi|\,t)\,\mathrm{si}(bt)dt. \tag{5.2.53}$$

Subject to absolute convergence we can then apply (5.2.1) to (5.2.52) and interchange the order of integration, to obtain for sufficiently large m with an integration path whose real part satisfies $0 < x_0 < 1$:

$$\mathrm{LB}(m,\xi) = -\frac{\xi}{2\pi^2 i}\int_{x_0-i\infty}^{x_0+i\infty} M_{\bar{\varepsilon}}(-z,m)M_\xi(z+1)dz \tag{5.2.54}$$

The integral (5.2.53) is readily identified as the Mellin transform associated with the characteristic function of a convolution of two uniformly distributed random variables, respectively on the intervals $[-|\xi|,|\xi|]$ and $[-b,b]$. It establishes a holomorphic function in the strip $0 < \Re\zeta < 2$. Its continuation was determined in Subsection 4.8.3. The set (4.8.17) is now given by

$$D_2 = \{b+|\xi|, b-|\xi|, -b+|\xi|, -b-|\xi|\}\,. \tag{5.2.55}$$

According to (4.8.19), we must distinguish between two cases, which are $|\xi| = b$ and $|\xi| \neq b$. We thus conclude

$$M_\xi(\zeta) = -\frac{\Gamma(\zeta-2)}{4|\xi|\,b} \times \begin{cases} \{-i2b\}^{2-\zeta} + \{i2b\}^{2-\zeta}\,, \\ \{-i(b+|\xi|)\}^{2-\zeta} - \{i(b-|\xi|)\}^{2-\zeta} - \{-i(b-|\xi|)\}^{2-\zeta} + \{i(b+|\xi|)\}^{2-\zeta}\,. \end{cases}$$

The upper line of the bracket applies if $\xi \in \{\pm b\}$ and the lower line otherwise. If we designate by $E_\xi(\zeta)$ an entire function of the variable $\zeta \in \mathbb{C}$, defined by

$$E_\xi(\zeta) := \begin{cases} (2b)^{2-\zeta}, & \text{if } \xi \in \{\pm b\}\,, \\ (b+|\xi|)^{2-\zeta} - |b-|\xi||^{2-\zeta}\,, & \text{otherwise,} \end{cases} \tag{5.2.56}$$

the function $M_\xi(\zeta)$ can be recast in a more appropriate form:

$$\begin{aligned} M_\xi(\zeta) &= -\frac{\Gamma(\zeta-2)}{4|\xi|\,b} \times \begin{cases} -2\cos\left\{\frac{\pi\zeta}{2}\right\}(2b)^{2-\zeta}, & \text{if } \xi \in \{\pm b\} \\ -2\cos\left\{\frac{\pi\zeta}{2}\right\}(b+|\xi|)^{2-\zeta} + 2\cos\left\{\frac{\pi\zeta}{2}\right\}|b-|\xi||^{2-\zeta}\,, & \text{otherwise} \end{cases} \\ &= \frac{\Gamma(\zeta-2)}{2|\xi|\,b}\cos\left\{\frac{\pi\zeta}{2}\right\}E_\xi(\zeta) \end{aligned} \tag{5.2.57}$$

The right hand side of (5.2.57) thus extends the integral (5.2.53) to a meromorphic function in the complex plane, whose singularities in the region $\Re\zeta > 0$ depend on the argument of the local bias ξ. More precisely, the only possible singularity therein occurs at $\zeta = 2$, where $E_\xi(2) \neq 0$ if $\xi \in \{\pm b\}$. But in this event, regarding (5.2.52) we observe cancellations. The integrand is then non-negative for any $t > 0$, which is not the case if $\xi \notin \{\pm b\}$. Now, as $|\zeta| \to \infty$ in $\Re\zeta > 0$ the function (5.2.57) exhibits a very special asymptotic behaviour. That is, as $|\zeta| \to \infty$ in

$|\arg(\zeta - 2)| < \pi$ we observe

$$\left|\Gamma(\zeta - 2)\cos\left(\frac{\pi\zeta}{2}\right)\right| = \mathcal{O}\left\{|\zeta - 2|^{\Re\zeta - \frac{5}{2}}\, e^{-\Re\zeta - \Im\zeta \arg(\zeta-2) + \frac{\pi|\Im\zeta|}{2}}\right\}. \tag{5.2.58}$$

The fact that the exponential terms involving the imaginary part cancel out as $\Im\zeta \to \pm\infty$ requires a distinction between two regions of the complex plane. In $\Re\zeta < \frac{5}{2}$ the controlling behaviour in any direction of the imaginary axis is algebraic decay. Upon traversing the line $\Re\zeta = \frac{5}{2}$ this suddenly reverses to algebraic growth in $\Re\zeta > \frac{5}{2}$. The exact order of decay or growth respectively depends on $\Re\zeta$.

With (5.2.57) we can write for (5.2.54), since $\cos(\frac{\pi}{2}(z+1)) = -\sin(\frac{\pi}{2}z)$ for any $z \in \mathbb{C}$:

$$\mathrm{LB}(m,\xi) = \frac{\operatorname{sgn}\xi}{2b\pi}\frac{1}{2\pi i}\int_{x_0 - i\infty}^{x_0 + i\infty} M_{\bar{\varepsilon}}(-z,m)\Gamma(z-1)\sin\left\{\frac{\pi z}{2}\right\}E_\xi(z+1)dz \tag{5.2.59}$$

A first look at the integrand shows in the strip $\mathbb{S}^-_{m,\delta_{\bar{\varepsilon}}}$ the presence of a simple pole at $z = 1$ if $\xi \in \{\pm b\}$ but analyticity otherwise. To overcome the indicated algebraic growth of the Mellin transform $M_\xi(1+z)$ we require exponential decay of $M_{\bar{\varepsilon}}(-z,m)$ as $\Im z \to \pm\infty$ in $\mathbb{S}^-_{m,\delta_{\bar{\varepsilon}}}$ for all sufficiently large m. It is then permitted to displace the integration path to the right to match an arbitrary line with real part $x_m > 1$, collecting the residue of the single pole at $z = 1$ if $\xi \in \{\pm b\}$. Keeping in mind that we encircle this pole in the clockwise direction, we arrive at

$$\mathrm{LB}(m,\xi) = -\frac{\operatorname{sgn}(\xi)}{2b\pi}M_{\bar{\varepsilon}}(-1,m)\mathbb{I}\{|\xi| = b\} + \frac{\operatorname{sgn}(\xi)}{2b\pi}I(m), \tag{5.2.60}$$

with the indicator being equal to zero or one, depending on ξ, and the integral in the second summand referring to

$$I(m) := \frac{1}{2\pi i}\int_{x_m - i\infty}^{x_m + i\infty} M_{\bar{\varepsilon}}(-z,m)\Gamma(z-1)\sin\left\{\frac{\pi z}{2}\right\}E_\xi(z+1)dz. \tag{5.2.61}$$

The first summand in (5.2.60) yields the leading term as $m \to \infty$ if the condition in the indicator is true, and is solely determined by the structure of $M_{\bar{\varepsilon}}(\cdot, m)$. Furthermore, the analyticity of the integrand in the new integral (5.2.61) in the subregion $\Re z > 1$ of the strip $\mathbb{S}^-_{m,\delta_{\bar{\varepsilon}}}$ suggests an additional exponential contribution. It can be obtained by separating the sine term in two exponential functions, so that $I(m) = I^+(m) + I^-(m)$ with the components

$$I^\pm(m) := \frac{1}{2i}\frac{1}{2\pi i}\int_{x_m - i\infty}^{x_m + i\infty} M_{\bar{\varepsilon}}(-z,m)\Gamma(z-1)e^{\pm\frac{i\pi z}{2}}E_\xi(z+1)dz. \tag{5.2.62}$$

An approach to evaluate either of these integrals is provided by an appropriate inverse factorial expansion, for instance.

To summarize our observations, in case of a uniformly distributed random variable X on $[-b, b]$ the dominating term in the asymptotic expansion of the local bias function depends on ξ. It can be represented by virtue of $M_{\bar{\varepsilon}}(\cdot, m)$ if ξ equals either of the endpoints, in which case the upfollowing terms in the expansion exhibit a faster decay as $m \to \infty$ than $M_{\bar{\varepsilon}}(-z, m)$ for any fixed $z \in \mathbb{S}^{-}_{m,\delta_{\bar{\varepsilon}}}$. If $\xi \in \mathbb{R} \setminus \{\pm b\}$ the last statement already holds for the leading term, i.e., the rate as $m \to \infty$ of $\mathrm{LB}(m, \xi)$ is then of the form $e^{-p(m)}$ for a function $p(m)$ that is determined by $M_{\bar{\varepsilon}}(\cdot, m)$.

Finally we briefly compare the results we obtained for the local bias with those for the uniform bias (5.1.1). In the present scenario of a random variable $X \sim U[-b, b]$, subject to our findings from Subsection 4.8.4, the strip of analyticity of the Mellin transform corresponding to the modulus of Φ_X is given by $0 < \Re\zeta < 2$. According to (5.1.2), for sufficiently large m and $0 < x_1 < 2$, the uniform bias thus takes on the form

$$\mathrm{ULB}(m) = \frac{1}{2\pi^2 i} \int\limits_{x_1 - i\infty}^{x_1 + i\infty} M_{\bar{\varepsilon}}(-z, m) M_X(z) dz. \tag{5.2.63}$$

The Mellin transform $M_X(z)$ was determined in (4.8.26). In this context it was also mentioned that the indicated integral definition can be extended to a meromorphic function in the region $\Re z > 0$, possessing an infinite sequence of poles and exhibiting exponential decay as $\Im z \to \pm\infty$ therein. We conclude that a rightward displacement of the integration path in (5.2.63) is permitted across arbitrary but finitely many poles of $M_X(z)$, leading to an asymptotic expansion of $\mathrm{ULB}(m)$ as $m \to \infty$ with asymptotic scale $M_{\bar{\varepsilon}}(-z, m)$. This is in contrast to the results we obtained for the local bias. Note that the situation essentially changes if we consider the uniform local bias, assuming a random variable X associated with a triangular distribution on $[-b, b]$. Then Φ_X is given by (3.3.5) with $p = 2$ which implies $|\Phi_X(t)| = \Phi_X(t) = (\mathrm{si}(bt))^2$. But the properties of the squared sinc function completely differ from those of its modulus. It follows that also the analytic structure of the corresponding Mellin transforms substantially differs.

5.3. Conclusion

The preceding investigations confirmed what was already indicated in Chapter 3, namely that the admissible rates of convergence occuring in the process of deconvolution of distribution functions essentially depend on the associated characteristic functions. In our examples considered so far these were predominantly monotonic, revealing as a key ingredient their behaviour at infinity. Consequently, large variations with respect to the possible results became observable, with rates of arbitrary order. Furthermore, it turned out that neither absolute integrability of Φ_X along the real axis suffices to guarantee a certain level, nor does the existence of any moments of F. Instead, this is rather prescribed by the properties of Φ_X and the m-power, which is respectively inherited to the analytic structure of their Mellin transforms.

Concerning the uniform bias, the composition of the exact rate can be described in two steps. First, the Mellin transform of $|\Phi_X|$ specifies the basic structure by the location of its poles, i.e., whether we have a power series, an exponential or a mixture type of expansion. It then depends on the asymptotic behaviour with respect to m of the Mellin transform corresponding to the m-power, if the result is algebraic or merely logarithmic, for instance. Thus, especially not only one component determines the result, but for a given X there are errors which make deconvolution more or less difficult. For example, if $\Phi_{\bar{\varepsilon}}(t)$ decays exponentially fast, the rate of uniform convergence can still exceed any reciprocal power of m if $\Phi_X(t)$ exhibits a faster decay as $t \to \infty$. Even scenarios in which we encounter an error distribution with a compact support can be quite well, provided Φ_X has a smaller support. Summarizing, information about merely one distribution involved in a deconvolution problem is insufficient. For any type of error distribution there are counterpart distributions F with the ability to weaken the effect of errors in variables in the sense that the rate of uniform convergence can still take a satisfactory order.

Regarding the pointwise convergence, the composition of the rate increases in complexity, due to the possible presence of oscillatory terms. The latter describes the behaviour of a function to infinitely many times change the sign of its real and imaginary part in the process of approaching a limit point. In our final discussion of the local bias function such a behaviour has been shown to yield an exponential improvement of the rate of convergence.

So far we strictly considered rather well-behaved characteristic functions in the sense that particularly the m-power was monotonic. Indeed, in all our examples $\Phi_{\bar{\varepsilon}}(t) \neq 0$ for $t \in \mathbb{R}$ but $\Phi_{\bar{\varepsilon}}(t) \to 0$ as $t \to \infty$. Consequently, $\mathcal{P}_{\bar{\varepsilon}}(t, m) \to 0$ as $m \to \infty$ at any $t \in \mathbb{R}$ which implies, the main contribution to the large m-behaviour of the Laplace-type integrals for the bias only comes from infinity. It should be expected that the method of the Mellin transforms is always applicable if $\Phi_{\bar{\varepsilon}}$ is of such a type and if (4.3.15) holds, since the associated Mellin transform $M_{\bar{\varepsilon}}(\cdot, m)$ then exhibits special properties. In this event, also the leading behaviour as $m \to \infty$ of the integral (4.3.14) is determined at infinity. But the argument ζ substantially controls the behaviour of the integrand there, which suggests that $M_{\bar{\varepsilon}}(\zeta, m)$ can be employed to define an asymptotic scale in the strip $\mathbb{S}_{m,\delta_{\bar{\varepsilon}}}$. Speaking formally, for any sequence of numbers z_n with $z_n \in \mathbb{S}^-_{m,\delta_{\bar{\varepsilon}}}$ and $\Re z_{n+1} > \Re z_n$ for all $n \in I$ and a finite or infinite $I \subset \mathbb{N}_0$, the sequence being independent of m, it is reasonable to presume $M_{\bar{\varepsilon}}(-z_{n+1}, m) = o\{M_{\bar{\varepsilon}}(-z_n, m)\}$ as $m \to \infty$ for any $n \in I$. The latter property will most likely become invalid if the set N_ε contains a finite point. The reason is that the kernel $t^{\zeta-1}$ of the Mellin transform (4.3.14) is continuous and non-vanishing along the positive real axis. The argument ζ thus can not weaken the effect of a finite point at which the m-power equals unity for all $m \geq 0$.

Transferring our observations from the above examples, we suppose exact statements concerning the rate of the bias in general setups to depend on the local behaviour of the integrand at the points where $\Phi_{\bar{\varepsilon}}$ vanishes. A study of such scenarios by means of MB-integrals is the objective of the next chapter. There, we shall also establish some general results concerning the asymptotic behaviour of the local bias function.

6. Asymptotic Expansions by Analytic Continuation

Due to the apparently restricted applicability of the procedure which was presented in the preceding two chapters, it seems reasonable to consider an alternative approach for the asymptotic evaluation of the bias. To discuss this topic, also with regard to more general settings, is the subject of this chapter. Indeed, rather than confining ourselves to any of the bias integrals, for $\lambda \geq 0$, we focus on the integral

$$I_\lambda := \int_{\mathcal{P}} (1 - \Psi(t))^\lambda a(t) dt, \tag{6.0.1}$$

where $\mathcal{P}$ denotes a possibly infinite segment of the real axis, the λ-power is described as the kernel and $a(t)$ is referred to as the amplitude function. At the moment it suffices to assume for $t \in \mathcal{P}$ continuity of $a(t)$ and $\Psi(t)$ with $0 \leq \Psi(t) \leq 1$ and absolute convergence of (6.0.1). Clearly, the latter constitutes an integral of Laplace-type, implying that the main contribution to the rate as $\lambda \to \infty$ comes from a neighborhood of the points, where the integrand and particularly the kernel attains its maximum value. Moreover, by means of a simple bound it is easy to verify that the rate is certainly of exponential order if $\Psi(t)$ attains a non-zero infimum along the range of integration. Conversely, if the infimum along $\mathcal{P}$ equals zero, this is no longer assured. The effect of a zero infimum on the rate as $\lambda \to \infty$ of I_λ substantially depends on the local behaviour of the involved functions Ψ and a. The method to be presented below enables us to handle scenarios in which the former exhibits arbitrary exponential or algebraic behaviour there, and at the same time a is locally of algebraic type. Despite being based on MB-integrals, it does not require information about any Mellin transform. Furthermore, the approach is versatile and leaves much space for modifications.

6.1. Basic Idea

Before diving into deeper technical details, we begin with an illustrative overview. Several contour integrals are available to represent the kernel, of which the most obvious was already introduced in equation (4.3.19), by virtue of the binomial theorem. There, for $\Psi(t) \neq 0$, it was established that

$$(6.1.1) \qquad (1-\Psi(t))^{\lambda} = \frac{1}{2\pi i} \int\limits_{\varsigma_0 - i\infty}^{\varsigma_0 + i\infty} \{\Psi(t)\}^{-z} \frac{\Gamma(\lambda+1)\Gamma(z)}{\Gamma(\lambda+1+z)} dz.$$

In general $\varsigma_0 > 0$ is arbitrary but its choice is restricted in order to plug this integral into (6.0.1) and interchange the order of integration. Evidently, the admissible values then depend on the behaviour of a at the points where Ψ vanishes. Assuming a valid $\varsigma_0 > 0$ exists, we formally obtain

$$(6.1.2) \qquad I_\lambda = \frac{1}{2\pi i} \int\limits_{\varsigma_0 - i\infty}^{\varsigma_0 + i\infty} \frac{\Gamma(\lambda+1)\Gamma(z)}{\Gamma(\lambda+1+z)} \int\limits_{\mathcal{P}} \{\Psi(t)\}^{-z} a(t) dt dz.$$

We already encountered integrals of this type in our introductory discussion on binomial series, compare Subsection 4.3.1. There we observed by means of the saddle point approximation (B.3.5) that the beta function appearing in the dz-integral is $\sim$ const $\times \lambda^{-z}$ as $\lambda \to \infty$, i.e., it exhibits a descending algebraic character with respect to λ for fixed $z \in \mathbb{C}$ with a larger real part. In other words, for a possibly finite sequence $z_1, z_2, \dots$ of complex numbers, excluding the non-positive integers, with increasing real parts which does not depend on λ, for $n = 1, 2, \dots$ the sequence

$$\frac{\Gamma(\lambda+1)\Gamma(z_n)}{\Gamma(\lambda+1+z_n)}$$

establishes an asymptotic scale as $\lambda \to \infty$. Accordingly, we conclude that the choice of ς_0 indicates the asymptotic behaviour of I_λ as $\lambda \to \infty$. Roughly speaking we must distinguish between three cases. If the interchange in the order of integration in (6.1.2) is invalid for any $\varsigma_0 > 0$, the rate of decay of I_λ is presumably slower than any power of λ, i.e., slower than algebraic. Conversely, if $\varsigma_0 > 0$ can be chosen arbitrary, it suggests a rate faster than any power of λ. Finally, if the set of admissible $\varsigma_0 > 0$ has a finite upper bound, say $\chi_0 > 0$, we must determine the analytic structure of the dt-integral in (6.1.2) in the half plane $\Re z \geq \chi_0$. In these circumstances the integral

$$(6.1.3) \qquad \mathfrak{M}(-z) := \int\limits_{\mathcal{P}} \{\Psi(t)\}^{-z} a(t) dt$$

is absolutely and uniformly convergent only in the half plane $\Re z < \chi_0$. Clearly, (6.1.3) constitutes a generalization of the Mellin transform with a different phase function, namely with $\log t$ replaced by $-\log \Psi$. Referring to the common notion for integral transforms, see equation (A.0.1) in the appendix, we refer to the factor that depends on the argument z as the kernel, and we denote $\mathfrak{M}(-z)$ as a generating function. Its analytic structure in the extended plane $\Re z \geq \chi_0$ depends on the functions Ψ and a. As an example for the classical Mellin transform, for an appropriate function f suppose absolute and uniform convergence of the integral $\int_0^\infty t^{\zeta-1} f(t)dt$ in the strip $0 < \Re\zeta < x_0$. To determine the analytic continuation to either direction of the complex plane, it is common to integrate by parts or to employ a power series expansion for f at the points where the phase function diverges for certain values $0 < \Re\zeta < x_0$. This was thoroughly described in A.5.1 in the appendix. The purpose of the subsequent sections is to adapt these ideas and to develop analogous procedures for the analytic continuation of (6.1.3).

Note that it is not a disaster if the interchange in the order of integration in (6.1.2) is inapplicable for any $\varsigma_0 > 0$, but it merely indicates the inappropriateness of the integral representation (6.1.1). Then, there might rather exist $0 < -\delta_0 < \lambda$ for which the interchange is permitted with

$$(1 - \Psi(t))^\lambda = \frac{1}{2\pi i} \int_{-\delta_0 - i\infty}^{-\delta_0 + i\infty} \{-\log\{\Psi(t)\}\}^z \, \mathcal{M}_{\mathrm{B}}(z, \lambda - 1) dz. \tag{6.1.4}$$

The validity of this representation is subject to equation (4.7.18), where $\mathcal{M}_{\mathrm{B}}(-z, \lambda - 1)$ was established as the Mellin transform corresponding to the function $(1 - e^{-t})^\lambda$. From Section 4.7.3 we recall that $\mathcal{M}_{\mathrm{B}}(z, \lambda - 1)$ is expandable in negative powers of the digamma function $\psi(\lambda + 1)$, which asymptotically exhibits a logarithmic character. As $\Re z$ attains a larger fixed value in the strip $0 < \Re z < \lambda$, these powers show a higher order. Furthermore, similar to (6.1.1) we observe that the t-dependent factor in the integral (6.1.4) becomes unbounded as t approaches a point where $\Psi(t)$ vanishes. Due to the logarithm, however, the divergence happens slower in comparison to any reciprocal power of $\Psi(t)$. With a justification analogous to that for the use of (6.1.1), it is therefore also possible to obtain an asymptotic expansion of I_λ in terms of $\mathcal{M}_{\mathrm{B}}(z, \lambda - 1)$ by employing (6.1.4). The close connection to the digamma function and to the logarithm suggests that such an expansion is of logarithmic-type.

Finally, as an alternative to (6.1.1), for completeness we mention that the kernel of I_λ can be represented in terms of the inverse Mellin transform associated with the exponential function, see (A.5.8). In particular for $\varsigma_0 > 0$ the Cahen-Mellin integral applies, which is

$$(1 - \Psi(t))^\lambda = \frac{1}{2\pi i} \int_{\varsigma_0 - i\infty}^{\varsigma_0 + i\infty} \lambda^{-z} \{-\log(1 - \Psi(t))\}^{-z} \, \Gamma(z) dz. \tag{6.1.5}$$

As a consequence of the series expansion of the logarithm, in a neighborhood of the points where Ψ is small, the function $\log(1 - \Psi)$ has the same order. It is therefore merely a matter of taste to apply (6.1.5) rather than (6.1.1), depending on whether a power series expansion of I_λ is preferred over an expansion of inverse factorial-type. Yet, the composition with the logarithm in the former integral representation might bear additional difficulties.

6.2. Functions with the Property of an Asymptotic Scale

Preliminary to our investigation of the particular integral I_λ we establish an auxiliary theorem to characterize the asymptotic behaviour of a special kind of integrals. It will be frequently referred to throughout this chapter and avoid a repeated justification of reoccuring technical arguments.

Theorem 6.2.1 (expansion with respect to an asymptotic scale). *Consider as a function of $\lambda > 0$ for fixed $\varsigma_0 \in \mathbb{R}$ the contour integral*

$$J_\lambda := \frac{1}{2\pi i} \int\limits_{\varsigma_0 - i\infty}^{\varsigma_0 + i\infty} S(z,\lambda) M(-z) dz, \tag{6.2.1}$$

where the integration path is supposed to run through the region of analyticity of the integrand. Regarding the latter we assume there exists $\epsilon > 0$ with the following properties:

(1) For fixed $\lambda > 0$ the function $S(z,\lambda)$ is holomorphic in a strip $\varsigma_0 - \epsilon < \Re z < \chi(\lambda)$, where $\chi(\lambda) \to \infty$ as $\lambda \to \infty$, and $S(z,\lambda) = \mathcal{O}\left\{z^{-2}\right\}$ as $\Im z \to \pm\infty$ there. Moreover, on the one hand, for any $k \in \mathbb{N}_0$ and fixed z_1, z_2 in $\varsigma_0 - \epsilon < \Re z < \chi(\lambda)$ with $\Re z_1 < \Re z_2$,

$$S^{(k)}(z_2,\lambda) = o\left\{S(z_1,\lambda)\right\} \qquad \text{as } \lambda \to \infty. \tag{6.2.2}$$

The index refers to the derivative with respect to z. On the other hand, for $0 \le j < k$ and fixed z contained in the above strip,

$$S^{(j)}(z,\lambda) = o\left\{S^{(k)}(z,\lambda)\right\} \qquad \text{as } \lambda \to \infty. \tag{6.2.3}$$

Finally, for $\varsigma_0 - \epsilon < x_0 < \chi(\lambda)$,

$$\int\limits_{-\infty}^{\infty} \left|S(x_0 + iy, \lambda)\right| dy = \mathcal{O}\left\{S(x_0,\lambda)\right\} \qquad \text{as } \lambda \to \infty. \tag{6.2.4}$$

The function $S(z,\lambda)$ is termed an asymptotic scale.

(2) The function $M(-z)$, independent of λ, is meromorphic in a region $\varsigma_0 - \epsilon < \Re z < \eta$, where $\varsigma_0 < \eta < \infty$, with a finite sequence of poles $p_1, \dots, p_K$ for some $K \in \mathbb{N}$ whose real parts are ascending. Moreover, $M(-z)$ is $\mathcal{O}(1)$ as $\Im z \to \pm\infty$ in the punctured region, uniformly with respect to $\Re z$ in any closed vertical substrip.

In these circumstances the contour integral (6.2.1) converges absolutely and its asymptotic behaviour as $\lambda \to \infty$ is described by

$$J_\lambda \sim -\sum_{k=1}^{K} \operatorname*{Res}_{z=p_k} S(z,\lambda) M(-z). \tag{6.2.5}$$

Proof. Without loss of generality assume $\chi(\lambda) > \eta$ and keep λ fixed for a moment. Subject to the assumed conditions on the integrand, a displacement of the integration path by an arbitrary finite distance within $\varsigma_0 - \epsilon < \Re z < \eta$ is viable for some $\epsilon > 0$. We perform a movement of the integration path to the right across the K poles located in this region, to match some line $\Re p_K < \varsigma_K < \eta$. Since these poles are encircled in the negative sense, according to the residue theorem, this leads to

$$J_\lambda = -\sum_{k=1}^{K} \operatorname*{Res}_{z=p_k} S(z,\lambda)M(-z) + \frac{1}{2\pi i}\int\limits_{\varsigma_K - i\infty}^{\varsigma_K + i\infty} S(z,\lambda)M(-z)dz. \tag{6.2.6}$$

Denoting by $a_k \in \mathbb{N}$ the order of the pole p_k, it follows from the elementary rules of complex analysis for any $1 \le k \le K$:

$$\begin{aligned}
\operatorname*{Res}_{z=p_k} S(z,\lambda)M(-z) &= \frac{1}{(a_k-1)!}\frac{d^{a_k-1}}{dz^{a_k-1}}\left\{S(z,\lambda)(z-p_k)^{a_k}M(-z)\right\}\Bigg|_{z=p_k} \\
&= \frac{1}{(a_k-1)!}\sum_{j=0}^{a_k-1} S^{(j)}(p_k,\lambda)c(p_k,j)
\end{aligned}$$

The second equality is a conclusion from the product and the chain rule with coefficients $c(p_k, j)$, which are independent of λ. From (6.2.3) for any $1 \le k \le K$ as $\lambda \to \infty$ we see that

$$\operatorname*{Res}_{z=p_k} S(z,\lambda)M(-z) \sim \frac{c(p_k, a_k-1)}{(a_k-1)!} \times S^{(a_k-1)}(p_k,\lambda). \tag{6.2.7}$$

Furthermore, from (6.2.2) combined with (6.2.3) we deduce for $2 \le k \le K$ as $\lambda \to \infty$, especially since $\Re p_{k-1} < \Re p_k$:

$$\begin{aligned}
S^{(a_k-1)}(p_k,\lambda) &= S^{(a_{k-1}-1)}(p_{k-1},\lambda)\frac{S(p_{k-1},\lambda)}{S^{(a_{k-1}-1)}(p_{k-1},\lambda)}\frac{S^{(a_k-1)}(p_k,\lambda)}{S(p_{k-1},\lambda)} \\
&= o\left\{S^{(a_{k-1}-1)}(p_{k-1},\lambda)\right\}
\end{aligned}$$

Hence, the sum of residues in (6.2.6) exhibits an asymptotic character for large λ. Finally, regarding the remainder integral, subject to (6.2.4) we observe:

$$\begin{aligned}
\left|\int\limits_{\varsigma_K - i\infty}^{\varsigma_K + i\infty} S(z,\lambda)M(-z)dz\right| &\le \max_{v\in\mathbb{R}}|M(-\varsigma_K - iv)| \int\limits_{-\infty}^{\infty} |S(\varsigma_K + iy,\lambda)|\,dy \\
&= \mathcal{O}\left\{S(\varsigma_K,\lambda)\right\}
\end{aligned}$$

The maximum in this upper bound is finite because $M(-\varsigma_K - iy)$ by analyticity is especially a continuous function of $y \in \mathbb{R}$, and by assumption it is bounded at infinity. Upon comparison with (6.2.7) for $k = K$, since $\varsigma_K > \Re p_K$ again by (6.2.2) and (6.2.3) we immediately confirm the asymptotic validity of the expansion (6.2.6), which finishes the proof. ∎

As a remarkable consequence of the preceding theorem we add an important observation.

Corollary 6.2.1 (rate exceeds the asymptotic scale). *If in the situation of Theorem 6.2.1 the function $M(-z)$ is holomorphic in the entire half plane $\Re z > \varsigma_0 - \epsilon$, for arbitrary $x_0 > \varsigma_0$ as $\lambda \to \infty$ we have*

$$J_\lambda = o\{S(x_0, \lambda)\}.$$

In other words, the exact behaviour of J_λ then can not be represented solely by the asymptotic scale $S(z, \lambda)$.

We close this section with an overview on some functions that satisfy the properties of $S(\lambda, z)$, which were required in Theorem 6.2.1(1).

6.2.1. Mellin Transform of the Exponential Function

Possibly of most frequent occurence in asymptotics is the Mellin transform associated with the exponential function $e^{-\lambda t}$. Assume $\varsigma_0 > 0$ in (6.2.1) with

$$S(z, \lambda) \equiv \lambda^{-z}\Gamma(z). \tag{6.2.8}$$

The function is evidently holomorphic in $\Re z > 0$, and it follows from the exponential decay of the gamma function that $S(z, \lambda) = \mathcal{O}\left\{z^{-2}\right\}$ as $\Im z \to \pm\infty$ for any $\lambda > 0$. Furthermore, the property (6.2.4) is a consequence of the absolute convergence of J_λ, since the modulus of the integrand is $\mathcal{O}\{\lambda^{-\varsigma_0}\}$ as $\lambda \to \infty$. Finally, for $k \in \mathbb{N}_0$ and $z_1, z_2 \in \mathbb{C}$ with $\Re z_2 > \Re z_1$ as $\lambda \to \infty$ we observe

$$\left.\frac{d^k}{dz^k}\lambda^{-z}\right|_{z=z_2} = (-\log\lambda)^k\lambda^{-z_2} = o\left\{\lambda^{-z_1}\right\}.$$

The validity of (6.2.2) and (6.2.3) is thus due to the product and the chain rule. Summarizing, the function (6.2.8) satisfies the conditions of Theorem 6.2.1(1). By virtue of the functional equation for the gamma function, it is possible to cover analogously cases of non-integer $\varsigma_0 < 0$ by appropriately rearranging the integrand in (6.2.1).

6.2.2. Beta Function

The integral (6.2.1) is of binomial type if

$$S(z, \lambda) \equiv \frac{\Gamma(\lambda+1)\Gamma(z)}{\Gamma(\lambda+1+z)}, \tag{6.2.9}$$

for instance with $\varsigma_0 > 0$. From (B.3.5) we know $S(z,\lambda) = \mathcal{O}\left\{z^{-\lambda-1}\right\}$ as $|z| \to \infty$ for any $\lambda > 0$, and in addition for any $z \in \mathbb{C}$ we have

$$\frac{d}{dz}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+z)} = -\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+z)}\psi(\lambda+1+z).$$

The function ψ refers to the digamma function, see (B.2.12). Derivatives of higher order can be obtained from the product and the chain rule in terms of the polygamma functions. Their asymptotic behaviour as $\lambda \to \infty$ implies

$$\frac{d^k}{dz^k}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+z)} \sim (-\log\lambda)^k \lambda^{-z}. \tag{6.2.10}$$

In particular, all of the polygamma functions vanish for large values of their argument, except the digamma function. With (6.2.10), by arguments similar to those which were employed for (6.2.8), it is easy to verify (6.2.2) and (6.2.3). Finally, the estimate (6.2.4) can be confirmed with the aid of Corollary B.3.1. In case of a non-integer $\varsigma_0 < 0$ in (6.2.1), analogous arguments apply after an appropriate rearrangement of the integrand by means of the functional equation for the gamma function.

6.2.3. Beta Mellin Transform

If in (6.2.1) we have $\varsigma_0 > 0$ and

$$S(z,\lambda) \equiv \mathcal{M}_{\mathrm{B}}(z,\lambda-1), \tag{6.2.11}$$

the situation becomes slightly more complicated. Firstly, we know from (4.7.17) that the function exhibits exponential decay as $\Im z \to \pm\infty$ in $0 < \Re z < \lambda$ for any fixed $\lambda > 0$. Secondly, from (4.7.54) for fixed $z \in \mathbb{C} \setminus \{0\}$, as $\lambda \to \infty$, we ascertain that

$$\mathcal{M}_{\mathrm{B}}(z,\lambda-1) \sim \frac{H_\lambda^{-z}}{z}. \tag{6.2.12}$$

This asymptotic relation is differentiable with respect to z. The reason is that the finite expansion (4.7.42), of which (6.2.12) constitutes the leading term, establishes an analytic function of z. Term by term differentiation is thus permitted and particularly regarding the remainder integral (4.7.43) we know from Theorem A.2.1 that differentiation can be performed under the integral sign. Hence, the asymptotic expansion of any derivative of the beta Mellin transform can be obtained by differentiation of the asymptotic series. From the product and the chain rule we conclude, that the dominating term as $\lambda \to \infty$ of the k-th derivative for $k \in \mathbb{N}_0$ is

$$\frac{d^k}{dz^k}\mathcal{M}_{\mathrm{B}}(z,\lambda-1) \sim (-\log H_\lambda)^k\,\frac{H_\lambda^{-z}}{z}.$$

As a consequence of the asymptotic behaviour of the digamma function, see (4.7.33), for any $k \in \mathbb{N}_0$ as $\lambda \to \infty$ we can equivalently write

$$\frac{d^k}{dz^k}\mathcal{M}_{\mathrm{B}}(z, \lambda - 1) \sim (-\log\log\lambda)^k \, \frac{\{\log\lambda\}^{-z}}{z}. \tag{6.2.13}$$

By means of this approximation it is easy to verify (6.2.2) and (6.2.3). Concerning inequality (6.2.4), some difficulties arise. For fixed $0 < x_0 < \lambda$ and $\lambda > 0$ we have

$$\int_{-\infty}^{\infty} |S(x_0 + iy, \lambda)| \, dy < \infty,$$

i.e., the integral certainly converges absolutely by (4.7.17). The difficulty consists in showing that it is $\mathcal{O}\left\{(\log\lambda)^{-x_0}\right\}$ as $\lambda \to \infty$. For this purpose the bound (4.7.61) appears to be useful. It turns out, however, that it is insufficient:

$$\begin{aligned}\int_{-\infty}^{\infty} |S(x_0 + iy, \lambda)| \, dy &\le x_0(\lambda - x_0)\mathcal{M}_{\mathrm{B}}(x_0, \lambda - 1) \int_{-\infty}^{\infty} \frac{dy}{|z|\,|\lambda - z|} \\ &= 2x_0\mathcal{M}_{\mathrm{B}}(x_0, \lambda - 1) \int_0^{\infty} \frac{\lambda - x_0}{\sqrt{x_0^2 + y^2}\sqrt{(\lambda - x_0)^2 + y^2}} dy\end{aligned} \tag{6.2.14}$$

It is a simple consequence of Beppo Levi's theorem that the integral which depends on λ, diverges as $\lambda \to \infty$. In particular, for any $a, b > 0$ we have:

$$\frac{1}{3}(a + b)^2 \le a^2 + b^2$$

By virtue of this bound, accompanied by a partial fraction decomposition we obtain for fixed x_0 and sufficiently large λ:

$$\begin{aligned}\int_0^{\infty} \frac{dy}{\sqrt{x_0^2 + y^2}\sqrt{(\lambda - x_0)^2 + y^2}} &\le 3 \int_0^{\infty} \frac{1}{(x_0 + y)(\lambda - x_0 + y)} dy \\ &= \frac{3}{\lambda - 2x_0} \int_0^{\infty} \left[\frac{1}{x_0 + y} - \frac{1}{\lambda - x_0 + y}\right] dy \\ &= \frac{3}{\lambda - 2x_0} \left\{\log(\lambda - x_0) - \log(x_0)\right\}\end{aligned}$$

Combining this with (4.7.54) and (4.7.33) shows that, as $\lambda \to \infty$, the integral on the left hand side in (6.2.14) is

$$\int_{-\infty}^{\infty} |S(x_0 + iy, \lambda)| \, dy = \mathcal{O}\left\{\{\log(\lambda)\}^{1 - x_0}\right\}.$$

Further efforts in deriving a more accurate estimate failed. Regarding the beta Mellin transform we were thus able to verify all the conditions of Theorem 6.2.1(1), except (6.2.4). We are, however, brave enough to ignore this lack and assume validity of the property.

6.3. Analytic Continuation Techniques for Generating Functions

The study of the asymptotic behaviour of I_λ is immediately connected with the determination of the analytic continuation of a certain class of integral functions. Therefore we proceed with a treatment of generating functions of the shape

$$\mathfrak{M}_0^{\pm}(-z) := \int_{\mathcal{P}} \{\varphi(t)\}^{-z}\, a(t)dt, \tag{6.3.1}$$

where $\varphi(t) \geq 0$ and $a(t)$ are continuous, $\mathcal{P}$ is a finite segment of the real axis, and the branch of the power satisfies $\arg\{\varphi(t)\} = 0$ for $t \in \mathcal{P}$. The function φ is assumed to possess exactly one zero t_0 along $\mathcal{P}$, which coincides with either the lower or the upper endpoint, indicated by a positive or a negative sign in the index of (6.3.1), respectively. At this zero, without loss of generality we suppose, the behaviour of each function can be characterized by powers of t. We can then immediately establish the following analyticity statement, which is of major importance throughout this chapter and beyond.

Lemma 6.3.1 (kernel with algebraic behaviour). *Denote by $\phi(t)$ and $c(t)$ respectively a real- and a complex-valued function which is uniformly continuous on any closed subinterval of $(0,1]$. Suppose $\inf_{\varepsilon\leq v\leq 1}\phi(v) > 0$ for all $0<\varepsilon<1$, and there exist $\beta\geq 0$, $p>0$, $\gamma\in\mathbb{R}$ for which as $t\downarrow 0$ we have $\phi(t)\sim pt^\beta$ and $c(t)=\mathcal{O}\{t^\gamma\}$, with $\gamma>-1$ if $\beta=0$. In these circumstances the integral*

$$\mathfrak{N}(-z) := \int_0^1 \{\phi(t)\}^{-z}\, c(t)dt$$

is holomorphic in its region of absolute convergence, i.e., for $\beta=0$ in the whole z-plane and for $\beta>0$ in the half plane $\Re z < \frac{\gamma+1}{\beta}$. The computation of derivatives and residues can be performed under the sign of integration.

Proof. By Theorem A.2.1 it remains to verify the uniform convergence. In order to accomplish this task we notice, by assumption,

$$M := \max_{0\leq v\leq 1} \frac{v^\beta}{\phi(v)} < \infty,$$

and we designate by E a compact subset, if $\beta=0$ of $\mathbb{C}$ and if $\beta>0$ of the half plane $\Re z < \frac{\gamma+1}{\beta}$.

From the series expansion of the exponential function for any $z \in \mathbb{C}$ and $0 < t \leq 1$ we find

$$\{\phi(t)\}^{-\Re z} \leq t^{-\beta\Re z} \sum_{j=0}^{\infty} \frac{|\Re z|^j}{j!} M^j = t^{-\beta\Re z} e^{|\Re z| M}.$$

By virtue of this estimate, with $x_- := \min\{\Re z : z \in E\}$ and $x_+ := \max\{\Re z : z \in E\}$, for any $z \in E$ we finally deduce

$$|\mathfrak{N}(-z)| \leq e^{\max\{|x_-|,|x_+|\}M} \max_{0 \leq u \leq 1} \left| \frac{c(u)}{u^\gamma} \right| \int_0^1 t^{\gamma - \beta x_+} dt.$$

By uniformity with respect to $z \in E$ the proof is finished. ■

From the theory of integral functions we know that asymptotic expansions often provide useful tools to determine the analytic continuation of integrals. This approach is particularly viable in case of (6.3.1) if the involved functions are holomorphic, and the objective of the first subsequent subsections is to establish an appropriate technique. For this purpose, however, we require an auxiliary result, since we are interested in the expansion of a function in terms of another function rather than in its ordinary power series representation.

6.3.1. A Generalized Laurent Expansion

About a century ago the Portuguese mathematician Teixeira published an article on a remarkable generalization of the Laurent expansion. See [Teixeira, 1900] for the original French article or §7.31 in [Whittaker and Watson, 1952] for an English translation. Another hundred years earlier, a generalization of Taylor's expansion was presented, which is known as the Bürmann series. For details we refer to §7.3 in [Whittaker and Watson, 1952]. It is the aim of the present subsection to combine these results for establishing a finite expansion of Laurent-type under a minimum of conditions.

Suppose we have a function $\theta(z)$ that is analytic in some region of the complex plane containing a circle $\mathcal{R}$ centered at the point z_0, at which $\theta(z)$ is assumed to possess a zero of simple order. Denote by Γ the boundary curve of $\mathcal{R}$, traversed in the positive direction and suppose that, besides z_0 there lie no additional zeros of $\theta(z)$ in $\mathcal{R}$ or on Γ. Moreover, let $g(z)$ be a function which is analytic in some region of $\mathbb{C}$ including $\mathcal{R}$, except possibly at z_0, where it is admitted to have multiplicity $\nu \in \mathbb{Z}$. In these circumstances the function

$$f(z) := \frac{g(z)}{\{\theta(z)\}^\nu} \tag{6.3.2}$$

is holomorphic in $\mathcal{R}$ and on its boundary curve Γ, particularly the pole at $z = z_0$ being removable. According to Cauchy's theorem, we can thus write for any $\xi \in \mathcal{R}$:

$$
\begin{aligned}
f(\xi) &= \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z-\xi} dz \\
&= \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)\theta'(z)}{\theta(z)-\theta(\xi)} dz + \frac{1}{2\pi i} \oint_{\Gamma} f(z) \left\{ \frac{1}{z-\xi} - \frac{\theta'(z)}{\theta(z)-\theta(\xi)} \right\} dz
\end{aligned}
\tag{6.3.3}
$$

Since the point z_0 is a zero of $\theta(z)$ of order one we have $\theta'(z_0) \neq 0$. Following from the inverse function theorem, see Theorem 5.7.13 in [Asmar and Grafakos, 2018], the region $\mathcal{R}$ can be chosen small enough to arrange for any given $\xi \in \mathcal{R}$ validity of $\theta(z) = \theta(\xi)$ if and only if $z = \xi$. A function with this property is also termed conformal, compare Theorem 7.1.2 in [Asmar and Grafakos, 2018]. Then, the second integral in (6.3.3) equals zero by analyticity of the integrand and we arrive at

$$
f(\xi) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)\theta'(z)}{\theta(z)-\theta(\xi)} dz. \tag{6.3.4}
$$

Now, by virtue of the geometric sum formula (1.2.23) for any $K \in \mathbb{N}_0$ and $\xi \in \mathcal{R}$ we can write:

$$
\begin{aligned}
f(\xi) &= \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)\theta'(z)}{\theta(z)} \frac{1}{1-\frac{\theta(\xi)}{\theta(z)}} dz \\
&= \sum_{k=0}^{K-1} \{\theta(\xi)\}^k \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)\theta'(z)}{\{\theta(z)\}^{k+1}} dz + \{\theta(\xi)\}^K \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)\theta'(z)}{\{\theta(z)\}^K} \frac{dz}{\theta(z)-\theta(\xi)}
\end{aligned}
\tag{6.3.5}
$$

For convenience we denote the K-th remainder integral by

$$
R_K(\xi) := \{\theta(\xi)\}^K \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)\theta'(z)}{\{\theta(z)\}^K} \frac{dz}{\theta(z)-\theta(\xi)}. \tag{6.3.6}
$$

Furthermore, by additional use of the definition of f in (6.3.2), for $0 \leq k \leq K-1$ we introduce the coefficients

$$
c_k := \frac{1}{2\pi i} \oint_{\Gamma} \frac{g(z)\theta'(z)}{\{\theta(z)\}^{\nu+k+1}} dz. \tag{6.3.7}
$$

According to Cauchy's theorem, there is a close connection between these coefficients and the derivatives of the involved functions:

$$
c_k = \frac{1}{k!} \frac{d^k}{dz^k} \left[g(z)\theta'(z) \frac{(z-z_0)^{k+1}}{\{\theta(z)\}^{\nu+k+1}} \right] \Bigg|_{z=z_0} \tag{6.3.8}
$$

In terms of the above quantities, for arbitrary $K \in \mathbb{N}_0$ and $\xi \in \mathcal{R}$ we arrive at the finite expansion

$$g(\xi) = \sum_{k=0}^{K-1} c_k \{\theta(\xi)\}^{k+\nu} + \{\theta(\xi)\}^{\nu} R_K(\xi). \tag{6.3.9}$$

The behaviour of the series as $K \to \infty$ primarily depends on the properties of $\theta(z)$. A convergence statement can especially be established if for any $\xi \in \mathcal{R}$ we have

$$|\theta(\xi)| < \min_{z\in\Gamma} |\theta(z)|\,. \tag{6.3.10}$$

In this event the function $\theta(z)$ is one-to-one in the indicated region, because for any $\xi \in \mathcal{R}$ the following equalities hold:

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\Gamma} \frac{\theta'(z)}{\theta(z)-\theta(\xi)}dz &= \frac{1}{2\pi i}\oint_{\Gamma} \frac{\theta'(z)}{\theta(z)}\frac{1}{1-\frac{\theta(\xi)}{\theta(z)}}dz\\ &= \sum_{k=0}^{\infty}\{\theta(\xi)\}^k \frac{1}{2\pi i}\oint_{\Gamma}\frac{\theta'(z)}{\{\theta(z)\}^{k+1}}dz\\ &= \frac{1}{2\pi i}\oint_{\Gamma}\frac{\theta'(z)}{\theta(z)}dz\end{aligned}$$

The interchange in the order of summation and integration is permitted subject to uniform convergence, whereas the third equality follows from the fact that the integrand for $k \geq 1$ possesses an antiderivative that is analytic on Γ, namely $\frac{1}{k}\{\theta(z)\}^{-k}$. According to the argument principle, the preceding exposition shows, in the interior of Γ, just like the equality $\theta(z) = 0$ also $\theta(z) - \theta(\xi) = 0$ has exactly one solution. Furthermore, if $R > 0$ denotes the radius of Γ, from (6.3.6) we deduce

$$|R_K(\xi)| \leq \left\{\frac{|\theta(\xi)|}{\min\limits_{z\in\Gamma}|\theta(z)|}\right\}^K \frac{R}{2\pi}\int_{-\pi}^{\pi}\frac{\left|f(z_0+Re^{i\phi})\theta'(z_0+Re^{i\phi})\right|}{|\theta(z_0+Re^{i\phi})-\theta(\xi)|}d\phi.$$

Hence, as $K \to \infty$ the remainder vanishes. In a similar fashion it can be shown that the preceding sum in (6.3.5) as $K \to \infty$ converges absolutely and uniformly in any compact subset of $\mathcal{R}$, which verifies validity of these properties for the series

$$f(\xi) = \sum_{k=0}^{\infty} c_k \{\theta(\xi)\}^k\,. \tag{6.3.11}$$

Finally, the bound (6.3.10) especially holds if $\theta(z) \equiv z - z_0$, taking the form $|\xi - z_0| < R$. Thus, the right hand side of (6.3.11) constitutes the Taylor expansion of f then, whose region of validity depends on the closest singularity of this function only. By additional use of the definition of f, a transformation to the Laurent-expansion of g is possible.

6.3.2. Analytic Ingredient Functions

Keeping in mind the result from the preceding subsection, we finally reconsider the generating function (6.3.1), proceeding with a complete overview on the required assumptions:

(1) Suppose $\mathcal{P}$ is the finite half open segment of the real axis which connects the zero t_0 of $\varphi(t)$ with some real point $T \neq t_0$ such that $\varphi(t) > 0$ for any $t \in \mathcal{P}$.

(2) Let $\mathcal{P}$ be contained in a neighborhood of t_0, where $\varphi(t)$ and $a(t)$ are analytic and the following power series are uniformly convergent:

$$a(t) = \sum_{j=0}^{\infty} a(j; t_0)(t - t_0)^{j+\alpha_0} \tag{6.3.12}$$

$$\varphi(t) = \sum_{j=0}^{\infty} b(j; t_0)(t - t_0)^{j+\beta_0} \tag{6.3.13}$$

Here $b(0; t_0) \neq 0$, and by positivity of $\varphi(t)$ for $t \in \mathcal{P}$ we have $\frac{\varphi(t)}{(t-t_0)^{\beta_0}} > 0$. Therefore $b(0; t_0) > 0$ by continuity. Furthermore, necessarily $\alpha_0 \in \mathbb{N}_0$ and $\beta_0 \in \mathbb{N}$, where odd β_0 can occur only if $T > t_0$.

As a consequence of these assumptions, since the zero t_0 lies isolated by analyticity, depending on whether or not at $t = t_0$ the derivative φ' also has a zero, the function

$$A_0(t) := \frac{a(t)}{\varphi'(t)} \tag{6.3.14}$$

is holomorphic in a possibly punctured neighborhood of t_0. Moreover, if $\beta_0 \geq 2$ we conclude that the point t_0 is not only a zero but a saddle point of φ of order $\beta_0 - 1$. Then β_0 paths of steepest descent emanate from t_0, i.e., lines along which $\Im\varphi$ is constant. But φ was assumed to be real-valued along the integration path, whence especially $\mathcal{P}$ is a path of steepest descent. Finally, if in the above setup for $n \in \mathbb{N}_0$ we denote

$$\chi_{0n} := \frac{\alpha_0 + 1}{\beta_0} + \frac{n}{\beta_0}, \tag{6.3.15}$$

the generating function (6.3.1) converges absolutely for any $z \in \mathbb{C}$ with $\Re z < \chi_{00}$, and is by Lemma 6.3.1 holomorphic there. To access the associated analytic continuation into $\Re z \geq \chi_{00}$, in a neighborhood of t_0 we aim to find the Laurent expansion in terms of φ of the function A_0, which will eventually facilitate a partial evaluation of the generating function by virtue of the fundamental theorem of calculus.

6.3.2.1. Preliminaries

Besides analyticity of the underlying functions, the essential requirement for the applicability of the generalized Laurent expansion is the conformality of $\varphi(t)$ at $t = t_0$. However, it is ascertainable from (6.3.13) that t_0 is a zero of simple order only if $\beta_0 = 1$. Otherwise the function $\varphi(t)$ amplifies the angle between two curves intersecting at t_0 by β_0-times, which is equivalent to a small neighborhood of t_0 being mapped on a Riemann surface with β_0 sheets. On this surface φ attains any value β_0-times. Conversely, to choose the appropriate branch of the root we must first determine on which sheet the integration path is mapped. With $-\pi < \arg(t - t_0) \leq \pi$ as the principal branch, we denote by

$$\omega_{\mathcal{P}} := \lim_{\substack{t \to t_0 \\ t \in \mathcal{P}}} \arg(t - t_0) \tag{6.3.16}$$

the slope of the integration path at the point t_0. Clearly $\omega_{\mathcal{P}} \in \{0, \pi\}$, since $\mathcal{P}$ is a segment of the real axis. Moreover, the branches of the argument associated with $b(0; t_0)$ are given by $\arg_j \{b(0; t_0)\} = 2\pi j$ for $j \in \mathbb{Z}$. In terms of these, from (6.3.13), we deduce

$$\lim_{\substack{t \to t_0 \\ t \in \mathcal{P}}} \arg_j \{\varphi(t)\} = \beta_0 \omega_{\mathcal{P}} + 2\pi j. \tag{6.3.17}$$

Based on the branch of the argument $\arg_j \{\varphi(t)\}$ with this property, and defined by continuity elsewhere along $\mathcal{P}$, we will construct any fractional powers of φ, where $j \in \mathbb{Z}$ is to be specified. Particularly for the β_0 roots we introduce the definition

$$\varphi_j(t; t_0) := |\varphi(t)|^{\frac{1}{\beta_0}} \, e^{\frac{i}{\beta_0} \arg_j \{\varphi(t)\}}, \qquad j \in \mathbb{Z}. \tag{6.3.18}$$

According to (6.3.13), these are given by

$$\varphi_j(t; t_0) = e^{i \frac{2\pi j}{\beta_0}} \{b(0; t_0)\}^{\frac{1}{\beta_0}} (t - t_0) \left\{ 1 + \sum_{j=1}^{\infty} \frac{b(j; t_0)}{b(0; t_0)} (t - t_0)^j \right\}^{\frac{1}{\beta_0}} \tag{6.3.19}$$

with the powers of the curved brackets taking their principal values. Indeed, it is easy to see that the β_0-th power of each function satisfies (6.3.17), since

$$\lim_{\substack{t \to t_0 \\ t \in \mathcal{P}}} \beta_0 \arg \{\varphi_j(t; t_0)\} = \beta_0 \omega_{\mathcal{P}} + 2\pi j, \tag{6.3.20}$$

from which, by continuity, we conclude

$$\{\varphi_j(t; t_0)\}^{\beta_0} = \varphi(t), \qquad t \in \mathcal{P}. \tag{6.3.21}$$

In other words, the functions (6.3.19) are the β_0-th roots of $\varphi(t)$. Observe that each of them has a simple zero[1] at $t = t_0$ and each derivative satisfies $\varphi_j'(t_0; t_0) \neq 0$, whence $\varphi_j(t; t_0)$ maps a neighborhood of t_0 conformally onto a subregion of the complex plane. The region of analyticity

[1] This would not be true if we defined the powers of $\varphi(t)$ by simply assuming the branch $|\arg \{\varphi(t)\}| < \pi$. With this choice of the argument, the function $\{\varphi(t)\}^{\frac{1}{\beta_0}}$ is not analytic in a neighborhood of t_0.

of (6.3.19) depends on the radius of convergence of (6.3.13) and the closest zero of φ, where the function in the second curved bracket vanishes[2]. In order to choose the appropriate branch for the functions in our generalized Laurent expansion, we note that by assumption $\varphi(t) > 0$ for $t \in \mathcal{P}$. Therefore, to confine the mapping to one Riemann sheet the modulus of the argument (6.3.17) may not exceed $\frac{\pi}{2}$, which implies

$$j \in \left\{0, -\frac{\beta_0}{2}\right\}. \tag{6.3.22}$$

The two relevant branches of the β_0-th root of φ are thus given by $\varphi_+(t;t_0) := \varphi_0(t;t_0)$ and $\varphi_-(t;t_0) := \varphi_{-\frac{\beta_0}{2}}(t;t_0)$. By comparison with (6.3.19) we see that these satisfy

$$\varphi_+(t;t_0) = \{\varphi(t)\}^{\frac{1}{\beta_0}}, \qquad t > t_0, \tag{6.3.23}$$

$$\varphi_-(t;t_0) = \{\varphi(t)\}^{\frac{1}{\beta_0}}, \qquad t < t_0, \tag{6.3.24}$$

provided the power on the right hand side is constructed in terms of the argument function $|\arg\{\varphi(t)\}| < \pi$.

6.3.2.2. Derivation of the Generalized Laurent Expansion

We are now well prepared to introduce the desired expansion, which is a series representation of $A_0(t)$ optionally in powers of $\varphi_+(t;t_0)$ or in powers of $\varphi_-(t;t_0)$. Therefore we first note that $A_0(t)$ and $\varphi_\pm(t;t_0)$ in a neighborhood of $t = t_0$ satisfy the conditions of the functions g and θ of Subsection 6.3.1, respectively. The multiplicity at $t = t_0$ of $A_0(t)$ is readily identified by comparison with (6.3.14) to be equal to $\nu \equiv \beta_0(\chi_{00} - 1)$. We thus conclude from (6.3.9), for a sufficiently small region $\mathcal{R}_0$ with center t_0, encircled counterclockwise by a curve Γ_0, and any $t \in \mathcal{R}_0$ and $N \in \mathbb{N}_0$, the following finite expansion holds:

$$A_0(t) = \sum_{n=0}^{N-1} c_\pm(n;t_0)\{\varphi_\pm(t;t_0)\}^{n+\beta_0(\chi_{00}-1)} + \{\varphi_\pm(t;t_0)\}^{\beta_0(\chi_{00}-1)} A_0^\pm(t,N) \tag{6.3.25}$$

The corresponding coefficients subject to (6.3.7) are for $0 \le n \le N-1$ given by

$$c_\pm(n;t_0) := \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{A_0(w)\varphi'_\pm(w;t_0)}{\{\varphi_\pm(w;t_0)\}^{\beta_0(\chi_{00}-1)+n+1}} dw. \tag{6.3.26}$$

Moreover, due to (6.3.6) the remainder integral equals

$$A_0^\pm(t,N) := \{\varphi_\pm(t;t_0)\}^N \frac{1}{2\pi i}\oint_{\Gamma_0} \frac{A_0(w)\varphi'_\pm(w;t_0)}{\{\varphi_\pm(w;t_0)\}^{\beta_0(\chi_{00}-1)+N}} \frac{dw}{\varphi_\pm(w;t_0) - \varphi_\pm(t;t_0)}. \tag{6.3.27}$$

[2] A necessary condition for a branch of the root of an analytic function to exist is that the function is non-zero, compare Lemma 3.6.11 in [Wegert, 2012].

According to (6.3.21) and the chain rule, the first derivative of φ and $\varphi_\pm(\cdot;t_0)$ are related through

$$\varphi'(w) = \beta_0 \frac{\varphi'_\pm(w;t_0)}{\varphi_\pm(w;t_0)} \left\{\varphi_\pm(w;t_0)\right\}^{\beta_0}. \tag{6.3.28}$$

By definition of $A_0(t)$, for $0 \le n \le N-1$ instead of (6.3.26) we can thus equivalently write:

$$\begin{aligned} c_\pm(n;t_0) &= \frac{1}{\beta_0}\frac{1}{2\pi i}\oint\limits_{\Gamma_0} \frac{a(w)}{\{\varphi_\pm(w;t_0)\}^{\alpha_0+n+1}}dw \\ &= \frac{1}{\beta_0}\frac{1}{n!}\frac{d^n}{dz^n}\left[\frac{a(z)(z-t_0)^{n+1}}{\{\varphi_\pm(z;t_0)\}^{\alpha_0+n+1}}\right]\Bigg|_{z=t_0} \end{aligned} \tag{6.3.29}$$

Regarding coefficients with different signs, for $0 \le n \le N-1$ we observe that $\varphi_-(t;t_0) = -\varphi_+(t;t_0)$ implies

$$c_-(n;t_0) = (-1)^{n+\alpha_0+1} c_+(n;t_0). \tag{6.3.30}$$

For actually computing these coefficients it may be helpful to write

$$\varphi(t) = (t-t_0)^{\beta_0}\left\{\frac{\varphi(t)}{(t-t_0)^{\beta_0}}\right\}.$$

The function in the curved brackets is then holomorphic in some neighborhood of $t = t_0$ but does not vanish there. Furthermore, the branch of the root we require for the coefficients with the positive sign in the index, for instance, can conveniently be represented in the form

$$\varphi_+(t;t_0) = (t-t_0)\left\{\frac{\varphi(t)}{(t-t_0)^{\beta_0}}\right\}^{\frac{1}{\beta_0}}, \tag{6.3.31}$$

where the power of the function in the curved brackets attains its principal value. In view of (6.3.29) we then arrive at:

$$\begin{aligned} \Theta(n;t_0) &:= c_+(n;t_0) \\ &= \frac{1}{\beta_0}\frac{1}{n!}\frac{d^n}{dz^n}\left[\frac{a(z)}{(z-t_0)^{\alpha_0}}\left\{\frac{(z-t_0)^{\beta_0}}{\varphi(z)}\right\}^{\chi_{0n}}\right]\Bigg|_{z=t_0} \end{aligned} \tag{6.3.32}$$

Finally, by virtue of (6.3.28) from (6.3.27) we deduce

$$A_0^\pm(t,N) = \{\varphi_\pm(t;t_0)\}^N \frac{1}{\beta_0}\frac{1}{2\pi i}\oint\limits_{\Gamma_0}\frac{a(w)}{\{\varphi_\pm(w;t_0)\}^{\alpha_0+N}}\frac{dw}{\varphi_\pm(w;t_0)-\varphi_\pm(t;t_0)}. \tag{6.3.33}$$

6.3.2.3. Application of the Expansion

We will now employ the expansion (6.3.25) to establish the analytic continuation of the generating function (6.3.1). Therefore we assume without loss of generality that $\mathcal{P}$ is completely located in the interior region of $\mathcal{R}_0$. Then, $\varphi'(t) = 0$ if and only if $t = t_0$, and in terms of A_0 for $\Re z < \chi_{00}$ we can write

$$\mathfrak{M}_0^{\pm}(-z) = \int\limits_{\mathcal{P}} \{\varphi(t)\}^{-z} A_0(t)\varphi'(t)dt. \tag{6.3.34}$$

Recall the analyticity of the integral in $\Re z < \chi_{00}$. Upon introducing to (6.3.34) the finite expansion (6.3.25), for $N \in \mathbb{N}_0$ we arrive at

$$\mathfrak{M}_0^{\pm}(-z) = \sum_{n=0}^{N-1} c_{\pm}(n;t_0) \int\limits_{\mathcal{P}} \{\varphi(t)\}^{-z} \{\varphi_{\pm}(t;t_0)\}^{\beta_0(\chi_{0n}-1)} \varphi'(t)dt + \mathfrak{M}_0^{\pm}(-z,N), \tag{6.3.35}$$

where the remainder integral is defined by

$$\mathfrak{M}_0^{\pm}(-z,N) := \int\limits_{\mathcal{P}} \{\varphi(t)\}^{-z} \{\varphi_{\pm}(t;t_0)\}^{\beta_0(\chi_{00}-1)} A_0^{\pm}(t,N)\varphi'(t)dt. \tag{6.3.36}$$

Depending on whether t_0 is the starting or the ending point of the integration path, we choose the expansion with the positive or the negative sign. Keeping $0 \le n \le N-1$ fixed, by Lemma 6.3.1 the integral appearing in the n-th summand in (6.3.35) is absolutely convergent and holomorphic in $\Re z < \chi_{0n}$. If we assume $T > t_0$, by employing the identity (6.3.23) followed by an application of the fundamental theorem of calculus, we obtain for fixed $\Re z < \chi_{0n}$:

$$\begin{aligned}\int\limits_{t_0}^{T} \{\varphi(t)\}^{-z} \{\varphi_{+}(t;t_0)\}^{\beta_0(\chi_{0n}-1)} \varphi'(t)dt &= \int\limits_{t_0}^{T} \{\varphi(t)\}^{\chi_{0n}-z-1} \varphi'(t)dt \\ &= \frac{\{\varphi(T)\}^{\chi_{0n}-z}}{\chi_{0n}-z}\end{aligned}$$

This constitutes the analytic continuation to the whole complex plane of the integral representation for the n-th summand. It is composed by a product of a rational and an entire function, exhibiting a simple pole at $z = \chi_{0n}$. In addition, it is evidently $\mathcal{O}(|z|^{-1})$ as $\Im z \to \pm\infty$. Conversely, if $T < t_0$ analogous steps involving the identity (6.3.24) for fixed $\Re z < \chi_{0n}$ yield

$$\int\limits_{T}^{t_0} \{\varphi(t)\}^{-z} \{\varphi_{-}(t;t_0)\}^{\beta_0(\chi_{0n}-1)} \varphi'(t)dt = -\frac{\{\varphi(T)\}^{\chi_{0n}-z}}{\chi_{0n}-z}.$$

Instead of (6.3.35), for $\Re z < \chi_{00}$ and $N \in \mathbb{N}_0$ we can thus write

$$\mathfrak{M}_0^{+}(-z) = \pm \sum_{n=0}^{N-1} c_{\pm}(n;t_0) \frac{\{\varphi(T)\}^{\chi_{0n}-z}}{\chi_{0n}-z} + \mathfrak{M}_0^{\pm}(-z,N). \tag{6.3.37}$$

To eventually determine the region of validity of this expansion we must examine the remainder integral. It was given in (6.3.36) and by virtue of (6.3.33) it can be cast in the following form:

$$\mathfrak{M}_0^{\pm}(-z,N)=\frac{1}{\beta_0}\int\limits_{\mathcal{P}}\{\varphi(t)\}^{-z}\{\varphi_{\pm}(t;t_0)\}^{\beta_0(\chi_{0N}-1)}\varphi'(t)$$
$$\times\frac{1}{2\pi i}\oint\limits_{\Gamma_0}\frac{a(w)}{\{\varphi_{\pm}(w;t_0)\}^{\alpha_0+N}}\frac{dw}{\varphi_{\pm}(w;t_0)-\varphi_{\pm}(t;t_0)}dt$$

Since $\mathcal{P}$ is supposed to run in the neighborhood $\mathcal{R}_0$, where $\varphi_{\pm}(\cdot,t_0)$ is one-to-one, we conclude that the denominator in the integral along the curve Γ_0 is bounded away from zero, uniformly with respect to $t\in\mathcal{P}$ and $w\in\Gamma_0$. Moreover, the double integral depends on z only through the power of φ, and as $t\to t_0$ along $\mathcal{P}$ the integrand is $\mathcal{O}\left\{(t-t_0)^{\beta_0(\chi_{0N}-\Re z)-1}\right\}$. Absolute convergence is thus guaranteed for $\Re z<\chi_{0N}$. Assume $0<\varphi(t)\le 1$ for all $t\in\mathcal{P}$. Then, if E denotes a compact subset of the half plane $\Re z<\chi_{0N}$ and $x_E:=\max\{\Re z: z\in E\}$, subject to the identities (6.3.23) and (6.3.24), for $N\in\mathbb{N}_0$ we obtain

$$\left|\mathfrak{M}_0^{\pm}(-z,N)\right|\le\int\limits_{\mathcal{P}}\{\varphi(t)\}^{\chi_{0N}-x_E-1}\frac{|\varphi'(t)|}{\beta_0 2\pi}\left|\oint\limits_{\Gamma_0}\frac{a(w)}{\{\varphi_{\pm}(w;t_0)\}^{\alpha_0+N}}\frac{dw}{\varphi_{\pm}(w;t_0)-\varphi_{\pm}(t;t_0)}\right|dt.$$

The upper bound is finite, which confirms the uniformity of the convergence in E. A similar bound applies for general $\varphi>0$. By arbitrariness of E and since N was an arbitrary non-negative integer, we conclude analyticity of (6.3.36) in $\Re z<\chi_{0N}$ for any $N\in\mathbb{N}_0$. Moreover, from the above estimate we deduce that for fixed N the remainder integral is $\mathcal{O}(1)$ as $\Im z\to\pm\infty$, uniformly with respect to $\Re z$ in any closed vertical substrip.

To summarize our findings, for arbitrary fixed $N\in\mathbb{N}_0$ we have just verified that (6.3.37) represents the analytic continuation of (6.3.34) to the region $\Re z<\chi_{0N}$, where it is $\mathcal{O}(1)$ as $\Im z\to\infty$ uniformly with respect to $\Re z$ in any subregion. In the indicated region the expansion is meromorphic with the n-th summand exhibiting a simple pole located at $z=\chi_{0n}$ for $0\le n\le N-1$, whose residue is given by:

$$\begin{aligned}\operatorname*{Res}_{z=\chi_{0n}}\mathfrak{M}_0^{\pm}(-z)&=\mp c_{\pm}(n;t_0)\\&=-(\pm1)^{n+\alpha_0}\Theta(n;t_0)\end{aligned}\tag{6.3.38}$$

For the second equality we took into account the identity (6.3.30) and definition (6.3.32). Observe that neither of the above properties depends on the point T. We merely know the existence of $T\in\mathcal{R}_0$. Further details require additional investigations of the involved functions, particularly of a and of $\varphi_{\pm}(\cdot;t_0)$.

6.3.3. General Ingredient Functions

The natural analogue to a generalized Laurent expansion, making use of real analysis only, would be a Taylor expansion of one function in terms of another. There is, however, a more convenient method to determine the analytic continuation of the generating function (6.3.1), that can even handle scenarios in which the ingredient functions exhibit arbitrary algebraic behaviour in a neighborhood of a point, where the function φ vanishes. This is integration by parts. A special case occurs if the ingredient functions are analytic, in which event the technique below provides an alternative to the generalized Laurent expansion.

6.3.3.1. Algebraic Local Behaviour

For simplicity, without loss of generality, we suppose the integration path $\mathcal{P}$ in (6.3.1) commences at t_0, i.e., the generating function is of the form

$$\mathfrak{M}_0^+(-z) = \int_{t_0}^{T} \{\varphi(t)\}^{-z}\, a(t)dt. \tag{6.3.39}$$

In the technical part of this section we occasionally write

$$\mathfrak{M}_{00}(-z) := \mathfrak{M}_0^+(-z). \tag{6.3.40}$$

More precisely, concerning the integration path we assume a finite segment of the real axis, starting at some point $t_0 \in \mathbb{R}$ where $\lim_{t\downarrow t_0} \varphi(t) = 0$, and ending at $T > t_0$ such that $\varphi(t) > 0$ for any $t \in (t_0, T]$. In addition we suppose $\varphi(t)$ and $a(t)$ for $N \in \mathbb{N}$ are N-times continuously differentiable on $(t_0, T]$, and as $t \downarrow t_0$ let

$$a(t) \sim a_{00}(t - t_0)^{\alpha_0}, \tag{6.3.41}$$

$$\varphi(t) \sim b_{00}(t - t_0)^{\beta_0}, \tag{6.3.42}$$

with constants $a_{00} \in \mathbb{R} \setminus \{0\}$, $b_{00} > 0$. The parameters $\alpha_0 > -1$, $\beta_0 > 0$ need not be integers but fractional numbers are admissible. Hence, t_0 only corresponds to a zero in the usual notion if β_0 is indeed an integer. Denoting

$$\chi_{00} := \frac{\alpha_0 + 1}{\beta_0}, \tag{6.3.43}$$

by Lemma 6.3.1 under the above conditions the generating function (6.3.39) is absolutely convergent and holomorphic in $\Re z < \chi_{00}$. To determine the analytic continuation to the right direction, we introduce the functions

$$B(t) := \log\left\{\frac{(t - t_0)^{\beta_0}}{\varphi(t)}\right\}, \tag{6.3.44}$$

(6.3.45) $$A_{00}(t,z) := e^{zB(t)}\frac{a(t)}{(t-t_0)^{\alpha_0}}.$$

By assumption the first of these is N-times continuously differentiable on $(t_0, T]$ and the limit $\lim_{t\downarrow t_0} B(t)$ is finite and non-zero. As a consequence also (6.3.45) is N-times continuously differentiable there with existing non-zero limit $\lim_{t\downarrow t_0} A_{00}(t,z)$, which, however, depends on $z \in \mathbb{C}$. Particularly denote $A_{00}(t_0, \chi_{00}) := \lim_{t\downarrow t_0} A_{00}(t, \chi_{00})$. It is easy to see that there exist functions $p_{00}(t;t_0)$, $p_{01}(t;t_0)$ for which the first derivative of (6.3.45) can be arranged in the form

(6.3.46) $$A_{00}'(t,z) = e^{zB(t)}\left\{zp_{01}(t;t_0) + p_{00}(t;t_0)\right\}.$$

Throughout this section a prime always indicates the derivative with respect to t. The coefficients $p_{00}(t;t_0)$ and $p_{01}(t;t_0)$ of the polynomial in (6.3.46) are composed of $B'(t)$ and of $(t-t_0)^{-\alpha_0}a(t)$ and its first derivative. For fixed $z \in \mathbb{C}$ the function $A_{00}'(t,z)$ is thus continuous with respect to $t \in (t_0, T]$. In terms of (6.3.45) the integral (6.3.39) with $\Re z < \chi_{00}$ reads

(6.3.47) $$\mathfrak{M}_0^+(-z) = \int_{t_0}^{T} (t-t_0)^{\beta_0(\chi_{00}-z)-1} A_{00}(t,z)dt,$$

and it evidently constitutes a common incomplete Mellin-type integral. If we integrate once by parts in the usual fashion, we arrive at:

$$\mathfrak{M}_0^+(-z) = \left[\frac{(t-t_0)^{\beta_0(\chi_{00}-z)}}{\beta_0(\chi_{00}-z)}A_{00}(t,z)\right]_{t_0}^{T} - \frac{1}{\beta_0(\chi_{00}-z)}\int_{t_0}^{T}(t-t_0)^{\beta_0(\chi_{00}-z)}A_{00}'(t,z)dt$$

(6.3.48) $$= -\frac{(T-t_0)^{\beta_0(\chi_{00}-z)}}{\beta_0(z-\chi_{00})}A_{00}(T,z) + \frac{1}{\beta_0(z-\chi_{00})}\int_{t_0}^{T}(t-t_0)^{\beta_0(\chi_{00}-z)}A_{00}'(t,z)dt$$

For the second equality we note that the integrand vanishes at $t = t_0$ since $\Re z < \chi_{00}$. The first function in (6.3.48) is thus holomorphic in $\mathbb{C}$ with the exception of a simple pole at $z = \chi_{00}$. To specify the region of analyticity of the second summand, we suppose there exists $\alpha_{01} > -1$ such that as $t \downarrow t_0$ we have

(6.3.49) $$p_{0k}(t;t_0) = \mathcal{O}\left\{(t-t_0)^{\alpha_{01}}\right\}, \qquad k \in \{0,1\}.$$

It is then easy to confirm absolute convergence of the integral in (6.3.48) for $\Re z < \chi_{00} + \frac{\alpha_{01}+1}{\beta_0}$. To show analyticity in this region, we first define

(6.3.50) $$\mathfrak{M}_{01}(-z) := \frac{1}{\beta_0(z-\chi_{00})}\int_{t_0}^{T}(t-t_0)^{\beta_0(\chi_{00}-z)}A_{00}'(t,z)dt.$$

Then, in terms of the identity (6.3.46) we can write

$$\mathfrak{M}_{01}(-z) = \frac{1}{\beta_0(z-\chi_{00})} \int_{t_0}^{T} (t-t_0)^{\beta_0(\chi_{00}-z)} e^{zB(t)} \left\{ z p_{01}(t;t_0) + p_{00}(t;t_0) \right\} dt. \tag{6.3.51}$$

By continuity and subject to the asymptotic behaviour of $\varphi(t)$ as $t \downarrow t_0$, the function B assumes a finite supremum along the range of integration, which we denote by

$$M_B := \sup_{t_0 < r \leq T} |B(r)| .$$

It is thus a simple consequence of the series representation for the exponential function, that

$$e^{\Re z B(t)} \leq e^{|\Re z| M_B}.$$

Besides, following from (6.3.49) and by continuity of the coefficients of the polynomial (6.3.46),

$$|z p_{01}(t;t_0) + p_{00}(t;t_0)| \leq (t-t_0)^{\alpha_{01}} \left\{ |z| \sup_{t_0<r\leq T} \left| \frac{p_{01}(r;t_0)}{(r-t_0)^{\alpha_{01}}} \right| + \sup_{t_0<r\leq T} \left| \frac{p_{00}(r;t_0)}{(r-t_0)^{\alpha_{01}}} \right| \right\}.$$

If we denote the factor in the curved brackets by $P(z)$, we eventually deduce from (6.3.51) for fixed $\Re z < \chi_{00} + \frac{\alpha_{01}+1}{\beta_0}$ with $z \neq \chi_{00}$:

$$\begin{aligned} |\mathfrak{M}_{01}(-z)| &\leq e^{|\Re z| M_B} \frac{P(z)}{\beta_0 |z-\chi_{00}|} \int_{t_0}^{T} (t-t_0)^{\beta_0(\chi_{00}-\Re z)+\alpha_{01}} dt \\ &= e^{|\Re z| M_B} \frac{P(z)}{\beta_0 |z-\chi_{00}|} \frac{(T-t_0)^{\beta_0(\chi_{00}-\Re z)+\alpha_{01}+1}}{\beta_0(\chi_{00}-\Re z)+\alpha_{01}+1} \end{aligned} \tag{6.3.52}$$

The integrals (6.3.50) thus converge uniformly in any compact subset of the indicated region that does not contain the point χ_{00}. We therefore conclude from Theorem A.2.1, that the expansion (6.3.48) establishes the analytic continuation of the generating function (6.3.39) to the extended half plane $\Re z < \chi_{00} + \frac{\alpha_{01}+1}{\beta_0}$, exhibiting a simple pole at $z = \chi_{00}$. The corresponding residue is readily computed from the fundamental theorem of calculus:

$$\begin{aligned} \Theta(0;t_0) &:= -\operatorname*{Res}_{z=\chi_{00}} \mathfrak{M}_0^{+}(-z) \\ &= \frac{1}{\beta_0} A_{00}(T,\chi_{00}) - \frac{1}{\beta_0} \int_{t_0}^{T} A_{00}'(t,\chi_{00}) dt \\ &= \frac{1}{\beta_0} A_{00}(t_0,\chi_{00}) \end{aligned} \tag{6.3.53}$$

Finally, by taking into account the inequality (6.3.52), it is easy to see that the expansion (6.3.48) is $\mathcal{O}(1)$ as $\Im z \to \pm\infty$, uniformly with respect to $\Re z$ in any closed vertical substrip of its region of validity.

If the assumptions on the ingredient functions hold for $N \geq 2$, under certain conditions the integration by parts procedure can be repeated to eventually unlock the continuation of the generating function to a wider half plane. Therefore we replace (6.3.49) by a local relation to guarantee the integral (6.3.50) to be of similar type as the initial integral (6.3.39). First, for $1 \leq n \leq N$ and a parameter $\alpha_{0n} > -1$ we introduce the recursively defined functions

$$A_{0n}(t,z) := (t - t_0)^{-\alpha_{0n}} A'_{0(n-1)}(t,z). \tag{6.3.54}$$

These are $(N-n)$-times continuously differentiable on $(t_0, T]$. By induction it is easy to verify, starting from (6.3.46) and applying the product and the chain rule, for appropriate functions $p_{(n-1)k}(t;t_0)$ the derivative on the right hand side in (6.3.54) can always be arranged in the form

$$A'_{0(n-1)}(t,z) = e^{zB(t)} \sum_{k=0}^{n} z^k p_{(n-1)k}(t;t_0). \tag{6.3.55}$$

Evidently, this sum constitutes a polynomial whose coefficients are composed of derivatives of a and of B up to order n. According to our assumptions, the functions $p_{(n-1)k}(t;t_0)$ are thus continuous on $(t_0, T]$ for all $1 \leq n \leq N$ and $0 \leq k \leq n$. Moreover, $A'_{0(n-1)}(t,z)$ for $1 \leq n \leq N$ is also continuous there for any fixed $z \in \mathbb{C}$. To achieve continuity of $A_{0n}(t,z)$ on $[t_0, T]$, in (6.3.54) for $1 \leq n \leq N-1$ we specify the parameter $\alpha_{0n} > -1$ such that for $A \neq 0$ as $t \downarrow t_0$ we have

$$\begin{aligned} p_{(n-1)k}(t;t_0) &\sim A(t-t_0)^{\alpha_{0n}}, \\ p_{(n-1)k}(t;t_0) &\sim \mathcal{O}\left\{(t-t_0)^{\alpha_{0n}}\right\}, \end{aligned} \tag{6.3.56}$$

where the first statement holds for one while the big-$\mathcal{O}$ estimate holds for all $0 \leq k \leq n$, and the constants depend on n, k. Finally, for $n = N$ we suppose $\alpha_{0N} > -1$ is such that as $t \downarrow t_0$ for all $0 \leq k \leq N$ the coefficients satisfy

$$p_{(N-1)k}(t;t_0) \sim \mathcal{O}\left\{(t-t_0)^{\alpha_{0N}}\right\} \tag{6.3.57}$$

Then, denoting

$$\chi_{0n} := \chi_{00} + \sum_{j=1}^{n} \frac{\alpha_{0j}+1}{\beta_0}, \tag{6.3.58}$$

for any $1 \leq n \leq N$ the limit

$$A_{0n}(t_0, \chi_{0n}) := \lim_{t \downarrow t_0} A_{0n}(t, \chi_{0n}) \tag{6.3.59}$$

certainly exists. Under the above assumptions for $1 \leq n \leq N$ the integral appearing in

$$\mathfrak{M}_{0n}(-z) := \frac{1}{\beta_0^n \prod\limits_{l=0}^{n-1}(z-\chi_{0l})} \int\limits_{t_0}^{T} (t-t_0)^{\beta_0(\chi_{0(n-1)}-z)} A'_{0(n-1)}(t,z)dt \tag{6.3.60}$$

converges uniformly in $\Re z < \chi_{0n}$, and the whole right hand side of (6.3.60) is $\mathcal{O}(1)$ as $\Im z \to \pm\infty$ there, uniformly with respect to $\Re z$ in any closed vertical substrip. This is readily verified by application of (6.3.55) and (6.3.56), analogous to (6.3.52). Following from its uniform convergence and the properties of the integrand, the function (6.3.60) is analytic in the indicated half-plane, except for a sequence of simple poles at $z = \chi_l$ for $0 \leq l \leq n-1$. After a proper rearrangement of the integrand, partial integration is viable for $1 \leq n \leq N-1$, yielding:

$$\begin{aligned}\mathfrak{M}_{0n}(-z) &= \frac{1}{\beta_0^n \prod\limits_{l=0}^{n-1}(z-\chi_{0l})} \int\limits_{t_0}^{T} (t-t_0)^{\beta_0(\chi_{0n}-z)-1} A_{0n}(t,z)dt \\ &= -\frac{(T-t_0)^{\beta_0(\chi_{0n}-z)}}{\beta_0^{n+1}\prod\limits_{l=0}^{n}(z-\chi_{0l})} A_{0n}(T,z) + \frac{1}{\beta_0^{n+1}\prod\limits_{l=0}^{n}(z-\chi_{0l})} \int\limits_{t_0}^{T} (t-t_0)^{\beta_0(\chi_{0n}-z)} A'_{0n}(t,z)dt\end{aligned} \tag{6.3.61}$$

In accordance with its absolute and uniform convergence, the resulting integral exhibits analyticity in $\Re z < \chi_{0(n+1)}$ and hence establishes the analytic continuation of the original representation with simple poles located at $z = \chi_{0l}$, for $0 \leq l \leq n$. Definition (6.3.60) enables us to cast (6.3.61) in the form

$$\mathfrak{M}_{0n}(-z) = -\frac{(T-t_0)^{\beta_0(\chi_{0n}-z)}}{\beta_0^{n+1}\prod\limits_{l=0}^{n}(z-\chi_{0l})} A_{0n}(T,z) + \mathfrak{M}_{0(n+1)}(-z), \qquad 1 \leq n \leq N-1. \tag{6.3.62}$$

Combining this in turn with (6.3.40) and (6.3.48) is equivalent to repeatedly integrating the generating function $\mathfrak{M}_0^+(-z)$ by parts in the previously described manner $(n+1)$-times with $0 \leq n \leq N-1$, leading to:

$$\begin{aligned}\mathfrak{M}_0^+(-z) &= -\frac{(T-t_0)^{\beta_0(\chi_{00}-z)}}{\beta_0(z-\chi_{00})} A_{00}(T,z) + \mathfrak{M}_{01}(-z) \\ &= -\sum_{k=0}^{n} \frac{(T-t_0)^{\beta_0(\chi_{0k}-z)}}{\beta_0^{k+1}\prod\limits_{l=0}^{k}(z-\chi_{0l})} A_{0k}(T,z) + \mathfrak{M}_{0(n+1)}(-z)\end{aligned} \tag{6.3.63}$$

This function is analytic in $\Re z < \chi_{0(n+1)}$ with a not necessarily equidistant sequence of simple poles. Particularly in the strip $\chi_{0(n-1)} < \Re z < \chi_{0(n+1)}$ we encounter merely a single pole, which is of simple order. The corresponding residue is given by:

$$\begin{aligned}\Theta(n;t_0) &:= -\operatorname*{Res}_{z=\chi_{0n}} \mathfrak{M}_0^+(-z) \\ &= \frac{A_{0n}(T,\chi_{0n})}{\beta_0^{n+1}\prod\limits_{l=0}^{n-1}(\chi_{0n}-\chi_{0l})} - \frac{1}{\beta_0^{n+1}\prod\limits_{l=0}^{n-1}(\chi_{0n}-\chi_{0l})}\int_{t_0}^{T} A'_{0n}(t,\chi_{0n})dt \\ &= \frac{A_{0n}(t_0,\chi_{0n})}{\beta_0\prod\limits_{l=0}^{n-1}\left(n-l+\sum_{j=l+1}^{n}\alpha_{0l}\right)}\end{aligned} \tag{6.3.64}$$

The last equation incorporates the definition of χ_{0l} for $0 \le l \le n$. Finally, due to the properties of (6.3.60), the expansion (6.3.63) is $\mathcal{O}(1)$ as $\Im z \to \pm\infty$, uniformly with respect to $\Re z$ in any closed vertical substrip of the half plane $\Re z < \chi_{0(n+1)}$.

6.4. Derivation of Asymptotic Statements on I_λ

We will now discuss how to employ our preceding findings for the derivation of an asymptotic expansion for the integral I_λ defined in (6.0.1). Without loss of generality we assume that the integration path $\mathcal{P}$ is the half open finite segment of the real axis that connects a point t_0 with another point $T \neq t_0$ such that, if we define

$$\Psi(t_0) := \lim_{\substack{t\to t_0 \\ t\in\mathcal{P}}} \Psi(t), \tag{6.4.1}$$

we have $1 \ge \Psi(t) > \Psi(t_0) \ge 0$ for any $t \in \mathcal{P}$. The function $\Psi(t)$ thus attains its infimum value if and only if $t = t_0$. Moreover, the functions a and Ψ are supposed to be continous on any closed subinterval of $\mathcal{P}$. Depending on the local behaviour of Ψ at t_0, we will specify φ such that t_0 constitutes a zero of finite order. It will turn out below that one particular asymptotic scale even yields a useful statement if $\Psi(t_0) > 0$.

6.4.1. An Inverse Factorial Expansion

Our first approach to evaluate the integral I_λ relies on the binomial integral representation for the kernel. In order to apply our results from the earlier sections, we suppose $\Psi(t_0) = 0$. In addition, besides the above conditions we assume on a subsegment $\mathcal{P}_2 \subseteq \mathcal{P}$ that does not depend on λ, the functions $\varphi \equiv \Psi$ and a satisfy the conditions (1) and (2) of Subsection 6.3.2, or the conditions of §6.3.3.1 for some $N \in \mathbb{N}$. In these circumstances a and φ especially meet the requirement (6.3.41) and (6.3.42) as $t \to t_0$ along $\mathcal{P}$, respectively. Hence, the transition from (6.0.1) to (6.1.2) is permitted. In particular, the integral representation (6.3.1) for the associated generating function is then absolutely convergent and holomorphic in $\Re z < \chi_{00}$, enabling for

$0 < \varsigma_0 < \chi_{00}$ the following interchange in the order of integration:

$$I_\lambda = \frac{1}{2\pi i} \int_{\varsigma_0 - i\infty}^{\varsigma_0 + i\infty} \frac{\Gamma(\lambda+1)\Gamma(z)}{\Gamma(\lambda+1+z)} \int_{\mathcal{P}} \{\Psi(t)\}^{-z} a(t) dt dz$$

$$= \frac{1}{2\pi i} \int_{\varsigma_0 - i\infty}^{\varsigma_0 + i\infty} \frac{\Gamma(\lambda+1)\Gamma(z)}{\Gamma(\lambda+1+z)} \mathfrak{M}_0^{\pm}(-z) dz \tag{6.4.2}$$

The parameter χ_{00} was defined in (6.3.15) or in (6.3.43), respectively, and is accordingly the same, regardless in which of the two assumed scenarios we find ourselves. Now, for any closed subinterval $\mathcal{P}_3 \subset \mathcal{P}$ we can write

$$\mathfrak{M}_0^{\pm}(-z) = \int_{\mathcal{P} \setminus \mathcal{P}_3} \{\varphi(t)\}^{-z} a(t) dt + \int_{\mathcal{P}_3} \{\varphi(t)\}^{-z} a(t) dt. \tag{6.4.3}$$

Since $\varphi(t) \neq 0$ for any $t \in \mathcal{P}_3$, it is easy to see that the integral along the segment $\mathcal{P}_3$ constitutes an entire function of $z \in \mathbb{C}$, whereas the restriction $\Re z < \chi_{00}$ for the validity of the above representation stems solely from the integral along the first segment. But we are allowed to choose $\mathcal{P}_3$ large enough to achieve $\mathcal{P}_2 = \mathcal{P} \setminus \mathcal{P}_3$. The analytic continuation of the integral along the segment $\mathcal{P}_2$ in (6.4.3) can then be determined by virtue of the method from Subsection 6.3.2 or §6.3.3.1. In any case it is possible to continue the generating function to a meromorphic function in the region $\Re z < \chi_{0N}$, where the parameter χ_{0N} was specified in (6.3.15) and (6.3.58), respectively. In this extended half plane it was shown to possess simple poles with residues (6.3.38) or (6.3.64), and it was verified to be $\mathcal{O}(1)$ as $\Im z \to \pm\infty$, uniformly with respect to $\Re z$ in any closed vertical substrip. From Theorem 6.2.1, in view of the arguments from Subsection 6.2.2, as $\lambda \to \infty$ we thus deduce

$$I_\lambda \sim \sum_{n=0}^{N-1} \frac{\Gamma(\lambda+1)\Gamma(\chi_{0n})}{\Gamma(\lambda+1+\chi_{0n})} (\pm 1)^{n+\alpha_0} \Theta(n; t_0). \tag{6.4.4}$$

On the one hand, in the situation of Subsection 6.3.2 the positive or negative sign indicates if the integration path $\mathcal{P}$ starts or ends at $t = t_0$, respectively. Furthermore, the parameter χ_{0n} was given in (6.3.15), the coefficients can be found in (6.3.32) and the integer $N \in \mathbb{N}$ can be chosen arbitrary. On the other hand, in the situation of §6.3.3.1 the positive sign in the expansion applies, and for the parameters χ_{0n} we refer to (6.3.58) while the coefficients $\Theta(n; t_0)$ were established in (6.3.64). The index N is then specified by the assumptions on the ingredient functions.

6.4.2. A Power Series Expansion

We shall now proceed with the evaluation of (6.0.1) by making use of the Cahen-Mellin representation for the kernel. Therefore we first observe that it is always possible to rearrange I_λ to take on the form

$$I_\lambda = (1-\Psi(t_0))^\lambda \int_{\mathcal{P}} e^{-\lambda\varphi(t)} a(t)dt, \tag{6.4.5}$$

where

$$\varphi(t) \equiv \log\left\{\frac{1-\Psi(t_0)}{1-\Psi(t)}\right\}. \tag{6.4.6}$$

In comparison with the original representation for I_λ, the supremum along $\mathcal{P}$ of the kernel in the revised integral (6.4.5) equals one, or equivalently the infimum of φ equals zero. Then, if on a subsegment $\mathcal{P}_2 \subseteq \mathcal{P}$, independent from λ, the functions φ and a suffice the conditions (1) and (2) of Subsection 6.3.2 or those of §6.3.3.1 for a given $N \in \mathbb{N}$, to the integral (6.4.5) we are allowed introduce the Cahen-Mellin representation (A.5.8). For $0 < \varsigma_0 < \chi_{00}$ this results in

$$I_\lambda = (1-\Psi(t_0))^\lambda \frac{1}{2\pi i} \int_{\varsigma_0 - i\infty}^{\varsigma_0 + i\infty} \lambda^{-z}\Gamma(z)\mathfrak{M}_0^{\pm}(-z)dz.$$

The corresponding generating function now exactly matches (6.3.1). Accordingly, it can be extended to a meromorphic function in $\Re z < \chi_{0N}$ for a suitable $N \in \mathbb{N}$ and is $\mathcal{O}(1)$ as $\Im z \to \infty$ in this half plane, uniformly with respect to $\Re z$ in any closed vertical substrip. Again by Theorem 6.2.1 and subject to Subsection 6.2.1 we can thus establish that, as $\lambda \to \infty$, the expansion of I_λ up to N leading terms is given by

$$I_\lambda \sim (1-\Psi(t_0))^\lambda \sum_{n=0}^{N-1} \lambda^{-\chi_{0n}}\Gamma(\chi_{0n})(\pm 1)^{n+\alpha_0}\Theta(n;t_0). \tag{6.4.7}$$

The sign indicates if the path runs to the right or to the left of the point t_0. Regarding the parameters χ_{0n}, the coefficients $\Theta(n;t_0)$ and the magnitude of the index N, the applicant is advised to consult Subsection 6.3.2 or §6.3.3.1, respectively.

6.4.3. A Logarithmic-Type Expansion

We are finally concerned with situations in which an appropriate representation for the kernel of I_λ is given by (6.1.4) in terms of the beta Mellin transform. This is the case if $\Psi(t_0) = 0$, and on a subsegment $\mathcal{P}_2 \subseteq \mathcal{P}$ that does not depend on λ, the functions $\varphi \equiv \{-\log\Psi\}^{-1}$ and a match the requirements (1) and (2) of Subsection 6.3.2, or the conditions of §6.3.3.1 for some $N \in \mathbb{N}$. By arguments similar to those of Subsection 6.4.1, appealing to Theorem 6.2.1 and Subsection 6.2.3, it is then reasonable as $\lambda \to \infty$ to conclude

$$I_\lambda \sim \sum_{n=0}^{N-1} \mathcal{M}_{\mathrm{B}}(\chi_{0n}, \lambda-1)(\pm 1)^{n+\alpha_0}\Theta(n;t_0). \tag{6.4.8}$$

The upper and lower sign refers to the slope of $\mathcal{P}$ again, and for the parameters and coefficients we refer respectively to Subsection 6.3.2 and §6.3.3.1. In the situation of the former paragraph the index $N \in \mathbb{N}$ is arbitrary, whereas it is determined by the conditions on the ingredients in the latter paragraph.

6.4.4. A First Order Approximation

For some applications merely the leading term of I_λ is required. Hence, the purpose of this subsection is to provide such a first order estimate. Therefore we note that, a sufficient condition for the validity of (6.3.49) is the existence of $\alpha_1, \beta_1 > -1$ for which the following holds as $t \downarrow t_0$:

$$\frac{a'(t)}{a(t)} = \frac{\alpha_0}{t-t_0} + \mathcal{O}\left\{(t-t_0)^{\alpha_1}\right\} \tag{6.4.9}$$

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{\beta_0}{t-t_0} + \mathcal{O}\left\{(t-t_0)^{\beta_1}\right\} \tag{6.4.10}$$

This allows us to establish a theorem to characterizes the dominating behaviour of I_λ for a fairly large class of ingredient functions.

Theorem 6.4.1 (first order term). *Suppose a and Ψ are continuous on any closed subinterval of $(t_0, T]$ and $1 \geq \Psi(t) > \Psi(t_0) \geq 0$ for any $t_0 < t \leq T$. Furthermore, for a and an appropriately specified φ assume validity of (6.3.41), (6.4.9) and of (6.3.42), (6.4.10), respectively.*

(1) If φ is given by (6.4.6), then as $\lambda \to \infty$ we have

$$I_\lambda \sim \frac{a_{00}}{\beta_0}(b_{00}\lambda)^{-\frac{\alpha_0+1}{\beta_0}}(1-\Psi(t_0))^\lambda \Gamma\left(\frac{\alpha_0+1}{\beta_0}\right). \tag{6.4.11}$$

(2) If $\Psi(t_0) = 0$ and $\varphi \equiv \{-\log\Psi\}^{-1}$, then as $\lambda \to \infty$ we have

$$I_\lambda \sim \frac{a_{00}}{\beta_0}(b_{00})^{-\frac{\alpha_0+1}{\beta_0}}\mathcal{M}_{\mathrm{B}}\left(\frac{\alpha_0+1}{\beta_0}, \lambda - 1\right). \tag{6.4.12}$$

Proof. As a consequence of the product and the chain rule, for the first derivative of the function (6.3.45) we compute:

$$\begin{aligned} A'_{00}(t,z) &= \frac{a(t)}{(t-t_0)^{\alpha_0}} z e^{zB(t)} B'(t) + e^{zB(t)}\left[-\frac{\alpha_0 a(t)}{(t-t_0)^{\alpha_0+1}} + \frac{a'(t)}{(t-t_0)^{\alpha_0}}\right] \\ &= A_{00}(t,z)\left\{z\left[\frac{\beta_0}{t-t_0} - \frac{\varphi'(t)}{\varphi(t)}\right] - \frac{\alpha_0}{t-t_0} + \frac{a'(t)}{a(t)}\right\} \end{aligned}$$

Hence, $A'_{00}(t,z) = \mathcal{O}\left\{(t-t_0)^{\min\{\beta_1,\alpha_1\}}\right\}$ as $t \downarrow t_0$. Finally, for the coefficient (6.3.53) we deduce

$$\Theta(0;t_0) = \frac{1}{\beta_0}\lim_{t\downarrow t_0}\left\{\frac{(t-t_0)^{\beta_0}}{\varphi(t)}\right\}^{\chi_{00}} \frac{a(t)}{(t-t_0)^{\alpha_0}} = \frac{1}{\beta_0}\frac{a_{00}}{(b_{00})^{\chi_{00}}}.$$

By definition of χ_{00} in (6.3.43) the asymptotic statements (6.4.11) and (6.4.12) immediately follow with $N = 1$ from (6.4.7) and (6.4.8), respectively. ■

6.5. Examples

We shall now present a few examples to illustrate the applicability of the described procedures for the evaluation of the uniform bias function.

Example 6.5.1 (simple algebraic functions). In Appendix A.7.4 it was shown that the function (A.7.23), given by

$$\Phi_{\bar{\varepsilon}}(t) = \frac{(t^2-1)^2}{(1+t^2)^3}, \tag{6.5.1}$$

corresponds to an absolutely continuous symmetric distribution. Letting Φ_X equal to (3.3.1) with $\alpha, p \in \mathbb{N}$ and $\theta > 0$, the uniform bias function (2.1.56) takes on the following form:

$$\begin{aligned}\mathrm{ULB}(m) &= \frac{1}{\pi}\int_0^\infty \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t(1+\theta^\alpha t^\alpha)^p}dt \\ &= \frac{1}{\pi}\int_0^{\frac{1}{2}} \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t(1+\theta^\alpha t^\alpha)^p}dt + \frac{1}{\pi}\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t(1+\theta^\alpha t^\alpha)^p}dt + \frac{1}{\pi}\int_0^{\frac{2}{3}} \frac{\mathcal{P}_{\bar{\varepsilon}}(\frac{1}{s},m)}{(s^\alpha+\theta^\alpha)^p}s^{\alpha p-1}ds\end{aligned} \tag{6.5.2}$$

For the second equality we divided the range of integration and made a change of variables in the integral along the interval $[\frac{3}{2},\infty)$, to arrive at an integral along a finite segment. The separation of the range of integration is more or less arbitrary, provided the closure of the integration path does not contain more than one zero of the function $\Phi_{\bar{\varepsilon}}$. Furthermore, the amplitude function must be continuous along each segment that contains a zero. The zeros are in particular located at $t = 1$ and at $t = \infty$, both of second order, of which the latter in the last integral in (6.5.2) was mapped to the point $s = 0$. It is our aim to establish a complete expansion of the above uniform bias in powers of the asymptotic parameter.

Regarding the integral along the segment $[\frac{1}{2},\frac{3}{2}]$ we introduce the kernel

$$\varphi(t;1) := -\log\{1-\Phi_{\bar{\varepsilon}}(t)\}$$

and the amplitude function

$$a_1(t) := \frac{1}{t(1+\theta^\alpha t^\alpha)^p}.$$

Upon considering the intervals $[\frac{1}{2},1]$ and $[1,\frac{3}{2}]$ separately, we see that the functions $\varphi(\cdot;1)$ and a_1 respectively satisfy the conditions (1) and (2) of Subsection 6.3.2. More precisely, in a neighborhood of $t = 1$ they are both holomorphic and the latter attains a regular non-zero value.

We conclude $\alpha_1 = 0$, $\beta_1 = 2$ whence $\chi_{1n} = \frac{1+n}{2}$ for $n \in \mathbb{N}_0$, where the parameter χ_{1n} corresponds to definition (6.3.15). We thus immediately deduce from Subsection 6.4.2, as $m \to \infty$, by adding the expansions for the upper and lower segment of the range of integration:

$$\int\limits_{\frac{1}{2}}^{\frac{3}{2}} \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t(1+\theta^\alpha t^\alpha)} dt \sim (m+1)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)}{(m+1)^{\frac{n}{2}}} \left\{(-1)^n + 1\right\} \Theta(n;1)$$

$$(6.5.3) \qquad = (m+1)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}+n\right)}{(m+1)^n} \Theta(2n;1)$$

According to (6.3.32), the corresponding coefficients in terms of (6.5.1) are

$$(6.5.4) \qquad \Theta(2n;1) = \frac{1}{2}\frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}} \left[\frac{1}{z(1+\theta^\alpha z^\alpha)^p} \left\{ \frac{(z-1)^2}{-\log\left(1 - \frac{(z^2-1)^2}{(1+z^2)^3}\right)} \right\}^{n+\frac{1}{2}} \right] \Bigg|_{z=1}.$$

Furthermore, concerning the expansion of the last integral in (6.5.2), we first denote

$$\varphi(s;0) := -\log\left\{1 - \Phi_{\bar{\varepsilon}}\left(\frac{1}{s}\right)\right\},$$
$$a_0(s) := \frac{s^{\alpha p - 1}}{(s^\alpha + \theta^\alpha)^p}.$$

The amplitude function at $s = 0$ then has the multiplicity $\alpha_0 = \alpha p - 1$ and $\chi_{0n} = \frac{\alpha p + n}{2}$, for $n \in \mathbb{N}_0$, in accordance with (6.3.15). Again, each of the functions satisfies the analyticity properties that were required in Subsection 6.3.2. Because the point $s = 0$ is the lower endpoint of the integration path, as $m \to \infty$, from (6.4.7) we eventually deduce the expansion

$$(6.5.5) \qquad \int\limits_0^{\frac{2}{3}} \frac{\mathcal{P}_{\bar{\varepsilon}}(\frac{1}{s},m)}{(s^\alpha+\theta^\alpha)^p} s^{\alpha p-1} ds \sim (m+1)^{-\frac{\alpha p}{2}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\alpha p}{2}+\frac{n}{2}\right)}{(m+1)^{\frac{n}{2}}} \Theta(n;0).$$

The associated coefficients are ascertainable from (6.3.32), whence

$$(6.5.6) \qquad \Theta(n;0) = \frac{1}{2}\frac{1}{n!}\frac{d^n}{dz^n} \left[\frac{1}{(z^\alpha+\sigma^\alpha)^p} \left\{ \frac{z^2}{-\log\left(1 - z^2\frac{(1-z^2)^2}{(1+z^2)^3}\right)} \right\}^{\frac{\alpha p}{2}+\frac{n}{2}} \right] \Bigg|_{z=0}.$$

Since the m-power is easily seen to be bounded away from unity in the first integral in (6.5.2), its contribution is exponentially small. If we thus collect (6.5.3) and (6.5.5), as $m \to \infty$ we finally arrive at

$$\mathrm{ULB}(m) \sim (m+1)^{-\frac{1}{2}} \sum_{n_1=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}+n_1\right)}{(m+1)^{n_1}} \Theta(2n_1;1) + (m+1)^{-\frac{\alpha p}{2}} \sum_{n_2=0}^{\infty} \frac{\Gamma\left(\frac{\alpha p}{2}+\frac{n_2}{2}\right)}{(m+1)^{\frac{n_2}{2}}} \Theta(n_2;0).$$

Observe that the leading term in general depends on $\alpha, p \in \mathbb{N}$, which requires a distinction between the cases $\alpha p > 1$ and $\alpha p = 1$. In the first case the main contribution to the asymptotic behaviour of ULB(m) comes from the zero at $t = 1$, whereas if $\alpha p = 1$ both points yield an equal contribution.

Example 6.5.2 (a convolution of a normal and a Cauchy distribution). Our second example illustrates a logarithmic-type expansion, which can be obtained if $\Phi_{\bar{\varepsilon}}$ is associated with a convolution of two important stable distributions. In particular, let

$$\Phi_{\bar{\varepsilon}}(t) = e^{-\sigma_t|t|-\sigma_2^2t^2}, \qquad \sigma_1, \sigma_2 > 0, \tag{6.5.7}$$

and let Φ_X possess a zero at infinity of order $\alpha \in \mathbb{N}$. Then, in the integral definition of the uniform bias function (2.1.56) it makes sense to introduce a change of variables, leading to

$$\text{ULB}(m) = \frac{1}{\pi}\int_0^1 \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t}\,|\Phi_X(t)|\,dt + \frac{1}{\pi}\int_0^1 \frac{\mathcal{P}_{\bar{\varepsilon}}(\frac{1}{s},m)}{s}\left|\Phi_X\left(\frac{1}{s}\right)\right| ds. \tag{6.5.8}$$

On the one hand, the dominating behaviour of the first integral is readily verified to be of exponential order. On the other hand, in the second integral we observe that the kernel attains the value one if and only if $s = 0$, which is an essential singularity of $\Phi_{\bar{\varepsilon}}(\frac{1}{s})$. We therefore denote

$$\begin{aligned}\varphi(s;0) &:= \left\{-\log\Phi_{\bar{\varepsilon}}\left(\frac{1}{s}\right)\right\}^{-1}, \\ &= s^2\left\{s\sigma_1 + \sigma_2^2\right\}^{-1}.\end{aligned} \tag{6.5.9}$$

The latter function exhibits a double zero at $s = 0$. Furthermore, we define by $a(s)$ the function which is analytic in some neighborhood of $s = 0$ and coincides with $\left|\Phi_X\left(\frac{1}{s}\right)\right| s^{-1}$ on the segment of the positive real axis, located in this neighborhood. The multiplicities of $\varphi(s;0)$ and of $a(s)$ at $s = 0$ are then of order $\beta_0 = 2$ and $\alpha_0 = \alpha - 1$, respectively. As a consequence $\chi_{0n} = \frac{\alpha+n}{2}$ for $n \in \mathbb{N}_0$, and as $m \to \infty$ we deduce from (6.4.8) the expansion

$$\text{ULB}(m) \sim \frac{1}{\pi}\sum_{n=0}^{\infty}\mathcal{M}_{\text{B}}\left(\frac{\alpha}{2} + \frac{n}{2}, m\right)\Theta(n;0). \tag{6.5.10}$$

Subject to (6.3.32) for $n \in \mathbb{N}_0$ the coefficients are given by

$$\Theta(n;0) = \frac{1}{2}\frac{1}{n!}\frac{d^n}{ds^n}\left[\frac{a(s)}{s^{\alpha-1}}(\sigma_1 s + \sigma_2^2)^{\frac{\alpha}{2}+\frac{n}{2}}\right]\Bigg|_{s=0}. \tag{6.5.11}$$

Example 6.5.3 (errors composed by an exp. and an alg. factor). Note that the validity of (6.3.41) and (6.3.42) does not exclude the presence of logarithmic higher order terms. These possibly become dominant after a few iterations of integration by parts so that the procedure of §6.3.3.1 is at least able to reveal the leading terms. An an illustrative example we shall now evaluate the uniform bias function (2.1.56) for an error distribution whose characteristic function for $\beta, q > 0$ is

$$\Phi_{\bar{\varepsilon}}(t) = \{\cosh(t)\}^{-1} \left\{1 + \sigma^\beta |t|^\beta\right\}^{-q}. \tag{6.5.12}$$

That particularly the reciprocal of the hyperbolic cosine is in fact a characteristic function was established on p. 88 in [Lukacs, 1970]. It confirms $\Phi_{\bar{\varepsilon}}$ as the product of two characteristic functions. Regarding the X-distribution we assume $\Phi_X(t) = \mathcal{O}\left\{t^{-\alpha}\right\}$ as $t \to \infty$ for some $\alpha > 0$, which implies absolute convergence of the integral (2.1.56) for the uniform bias. In addition, we suppose the modulus of Φ_X is once continously differentiable on $[t_0, \infty)$ for some $t_0 > 0$, and the function

$$a(s) = \frac{1}{s}\left|\Phi_X\left(\frac{1}{s}\right)\right| \tag{6.5.13}$$

satisfies (6.4.9). In the described setting the uniform bias integral can be arranged in the following form:

$$\mathrm{ULB}(m) = \frac{1}{\pi}\int_0^{t_0} \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t}\left|\Phi_X(t)\right| dt + \frac{1}{\pi}\int_0^{\frac{1}{t_0}} \mathcal{P}_{\bar{\varepsilon}}\left(\frac{1}{s}, m\right) a(s) ds \tag{6.5.14}$$

More precisely, we splitted the range of integration into two parts, and in the integral along the segment $[\frac{1}{t_0}, \infty)$ we made the change of variables $t = \frac{1}{s}$ to map the point at infinity to the origin. It is readily confirmed that the m-power assumes the value one only in this second integral. Since $\Phi_{\bar{\varepsilon}}$ decays exponentially fast in comparison with $|\Phi_X|$, it is reasonable to consider:

$$\begin{aligned} \varphi(s;0) &:= \left\{-\log \Phi_{\bar{\varepsilon}}\left(\frac{1}{s}\right)\right\}^{-1} \\ &= s\left\{1 + s\log\left\{\frac{1+e^{-\frac{2}{s}}}{2}\right\} - sq\beta\log(s) + sq\log\left(s^\beta + \sigma^\beta\right)\right\}^{-1} \end{aligned}$$

This function shows algebraic behaviour as $s \downarrow 0$, in particular it is $\mathcal{O}(s)$. Thus, for brevity we define

$$\varphi_2(s) := \left\{1 + s\log\left\{\frac{1+e^{-\frac{2}{s}}}{2}\right\} - sq\beta\log(s) + sq\log\left(s^\beta + \sigma^\beta\right)\right\}^{-1},$$

which is $\mathcal{O}(1)$ as $s \downarrow 0$. Elementary calculations for $s > 0$ yield

$$\begin{aligned} \varphi_2'(s) = -\left\{\varphi_2(s)\right\}^2 \Bigg\{ & \log\left\{\frac{1+e^{-\frac{2}{s}}}{2}\right\} - \frac{4e^{-\frac{s}{2}}}{1+e^{-\frac{s}{2}}} - q\beta \\ & - q\beta\log(s) + q\log\left(s^\beta + \sigma^\beta\right) + \frac{q\beta s^\beta}{s^\beta + \sigma^\beta}\Bigg\}. \end{aligned} \tag{6.5.15}$$

Hence, in terms of $\varphi_2(s)$ the first derivative of $\varphi(s;0)$ for $s > 0$ is given by:

$$\varphi'(s;0) = \varphi_2(s) + s\varphi_2'(s) = \frac{\varphi(s;0)}{s} + s\varphi_2'(s) = \frac{\varphi(s;0)}{s} + \mathcal{O}\left\{-s\log(s)\right\}$$

The second equality holds as $s \downarrow 0$, since $\varphi_2'(s) = \mathcal{O}\left\{-\log(s)\right\}$ there. By comparison with (6.4.10) we immediately confirm validity of the condition with $\beta_0 = 1$ and $\beta_1 = -\delta$ for arbitrary $\delta > 0$. From Theorem 6.4.1(2) with $b_{00} = 1$, as $m \to \infty$ we therefore deduce

$$\int_0^{\frac{1}{t_0}} \mathcal{P}_{\bar{\varepsilon}}\left(\frac{1}{s}, m\right) a(s)ds \sim \mathcal{M}_{\mathrm{B}}\left(\alpha_0 + 1, m\right) a_{00}. \tag{6.5.16}$$

By comparison of this result with (6.5.14) we see that the first integral in the indicated decomposition of the uniform bias is exponentially small. Accordingly, as $m \to \infty$ we conclude

$$\mathrm{ULB}(m) \sim \mathcal{M}_{\mathrm{B}}\left(\alpha_0 + 1, m\right) a_{00}.$$

To see if further terms in this expansion can be unlocked in the fashion of §6.3.3.1, we must investigate the derivative (6.3.46) as $s \downarrow 0$. But the behaviour there depends not only on the derivative of $a(s)$ but especially on the derivative of $B(s)$ defined in (6.3.44), which is now

$$B'(s) = \frac{d}{ds}\log\left\{\frac{s}{\varphi(s;0)}\right\} = -\frac{\varphi_2'(s)}{\varphi_2(s)}.$$

In accordance with our calculations in (6.5.15), the function $B'(s)$ thus diverges logarithmically as $s \downarrow 0$, thereby violating the assumptions for a repeated application of the integration by parts procedure.

We close this chapter with two examples of kernel functions that exhibit an infinite sequence of maxima. In these circumstances special caution is required and a straightforward application of the statements from Section 6.4 is in general not possible.

Example 6.5.4 (triangular error distribution with a Cauchy-type factor). Consider a scenario in which the error distribution features a component of a symmetric triangular distribution, i.e., in which its characteristic function involves the squared sinc function (3.3.6). In addition, it is assumed to possibly feature a Cauchy-type factor with scaling parameter $\sigma \geq 0$. Moreover, suppose X corresponds to a Cauchy distribution with scale $\theta > 0$. The uniform bias (2.1.56) is then

$$\mathrm{ULB}(m, \sigma) = \frac{1}{\pi}\int_0^\infty \left\{1 - e^{-\sigma t}\frac{\sin^2(t)}{t^2}\right\}^{m+1} \frac{e^{-\theta t}}{t}dt. \tag{6.5.17}$$

Contrary to any of the preceding examples, the function $\Phi_{\bar{\varepsilon}}$ thus not only decays exponentially fast at infinity if $\sigma > 0$, but it evidently possesses an infinite sequence of zeros on the ray

$t \geq 0$, located at the equidistant points $\pi\mathbb{N}$. Accordingly, it is tempting to divide the range of integration, to obtain

$$\text{ULB}(m,\sigma) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{\mathcal{P}_{\bar{\varepsilon}}(t,m)}{t} e^{-\theta t} dt + \frac{1}{\pi}\sum_{k=1}^{\infty} \text{ULB}_k(m,\sigma), \tag{6.5.18}$$

where for brevity with $k \in \mathbb{N}$ we denote

$$\text{ULB}_k(m,\sigma) := \int_{\pi\left(k-\frac{1}{2}\right)}^{\pi\left(k+\frac{1}{2}\right)} \left\{1 - e^{-\sigma t}\frac{\sin^2(t)}{t^2}\right\}^{m+1} \frac{e^{-\theta t}}{t} dt. \tag{6.5.19}$$

In the integral in (6.5.18) along the first segment, the kernel is bounded away from unity, whence its contribution to the rate of the uniform bias is of exponential order. We may thus confine our investigations to the integrals $\text{ULB}_k(m,\sigma)$. The amplitude function therein is evidently holomorphic in a neighborhood of each zero πk for $k \in \mathbb{N}$ with multiplicity $\alpha_k \equiv 0$. Furthermore, we have

$$\varphi(t;\pi k) = -\log\left\{1 - e^{-\sigma t}\frac{\sin^2(t)}{t^2}\right\}, \tag{6.5.20}$$

which is entire with respect to $t \in \mathbb{C}$. Besides, subject to the series expansion of the squared sinc, the multiplicity at $t = \pi k$ equals $\beta_k = 2$ for any $k \in \mathbb{N}$. By applying the estimate (6.4.7) to the segment to the left and to the right of the point $t = \pi k$, respectively, and adding both expansions, as $m \to \infty$ we thus arrive at

$$\text{ULB}_k(m,\sigma) \sim (m+1)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}+n\right)}{(m+1)^n} \Theta(2n;\pi k). \tag{6.5.21}$$

According to (6.3.32), the associated coefficients are

$$\Theta(2n;\pi k) = \frac{1}{2}\frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}\left[\frac{e^{-\theta z}}{z}\left\{\frac{(z-\pi k)^2}{-\log\left(1 - e^{-\sigma z}\frac{\sin^2(z)}{z^2}\right)}\right\}^{\frac{1}{2}+n}\right]\Bigg|_{z=\pi k}. \tag{6.5.22}$$

Finally, a straightforward attempt to establish a complete expansion of $\text{ULB}(m,\sigma)$ as $m \to \infty$ consists in adding the contributions of the single integrals in (6.5.21). The present scenario is particularly convenient since the coefficients possess a finite representation. We restrict ourselves to the dominating term as $m \to \infty$ of (6.5.21), which is for each $k \in \mathbb{N}$ and $\sigma \geq 0$ given by

$$\text{ULB}_k(m,\sigma) \sim (m+1)^{-\frac{1}{2}}\sqrt{\frac{\pi}{2}}e^{-(\theta-\sigma)\pi k}. \tag{6.5.23}$$

This result especially incorporates the facts that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and

$$-\frac{\log\left(1 - e^{-\sigma z \frac{\sin^2(z)}{z^2}}\right)}{(z - \pi k)^2}\Bigg|_{z=\pi k} = \frac{1}{2(\pi k)^2} e^{-\sigma \pi k}.$$

It is easy to see that the exponential function in (6.5.23) is only decreasing with respect to $k \in \mathbb{N}$ if $\sigma < \theta$. If in these circumstances we sum up by means of the formula for the geometric series the contribution from all the zeros of $\Phi_{\bar{\varepsilon}}$ in the uniform bias integral (6.5.18), as $m \to \infty$ we deduce:

$$\begin{aligned} \mathrm{ULB}(m, \sigma) &\sim (m+1)^{-\frac{1}{2}} \sqrt{\frac{\pi}{2}} \sum_{k=1}^{\infty} e^{-(\theta - \sigma)\pi k} \\ &= (m+1)^{-\frac{1}{2}} \sqrt{\frac{\pi}{2}} \frac{1}{e^{(\theta-\sigma)\pi} - 1} \end{aligned} \tag{6.5.24}$$

Clearly, for $\sigma \geq \theta$ the geometric series fails to converge. Before we outline, why the above result is actually legit, we remark that a similar setup has been treated in the article [Paris, 2020] for the special case $\sigma = 0$. More precisely, while (6.5.24) corresponds to the uniform bias of the deconvolution function, the integral (3.1) in the cited article can be conceived the density analogue. By comparison of both results we observe that the actual rate of decay as $m \to \infty$ remains the same but the additional factor $\frac{1}{t}$ in the above integral merely contributes to the coefficient. The reason obviously is that $\frac{1}{t}$ does not vanish at any of the finite points, at which the rate of the uniform bias is determined. Finally, in the cited article, possible inaccuracies were mentioned that occur if the parameter θ in relation to m is too small. This should also be kept in mind when using (6.5.24).

It is reasonable to question the validity of the final asymptotic statement (6.5.24). First we observe that the approximation for the partial integrals, which we established in (6.5.23), is valid for all values of the parameters, since the corresponding integrand for fixed $k \in \mathbb{N}$ locally satisfies the conditions of Subsection 6.3.2. The problematic step is the summation of the single results, which requires a further discussion. Denote the series of integrals in (6.5.18) by

$$\mathrm{ULB}_\infty(m, \sigma) := \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\infty} \left\{1 - e^{-\sigma t \frac{\sin^2(t)}{t^2}}\right\}^{m+1} \frac{e^{-\theta t}}{t} dt. \tag{6.5.25}$$

We aim to write this as a MB-integral by means of the binomial integral representation for the kernel, which enables us most easily to factorize the function $\Phi_{\bar{\varepsilon}}$. With the power taking its principal value, the integrand of the generating function

$$\mathfrak{M}(-z) := \int_{\frac{\pi}{2}}^{\infty} \left\{\sin^2(t)\right\}^{-z} t^{2z-1} e^{-(\theta - \sigma z)t} dt \tag{6.5.26}$$

is locally integrable on $t \geq \frac{\pi}{2}$ for $\Re z < \frac{1}{2}$, due to the zeros of the sine, and the whole integral converges absolutely at infinity for $\Re z < \frac{\theta}{\sigma}$, for any $\sigma \geq 0$. It is thus routine to verify analyticity of the integral in the half plane $\Re z < \min\left\{\frac{1}{2}, \frac{\theta}{\sigma}\right\}$. Especially subject to absolute convergence, for $0 < x_0 < \min\left\{\frac{1}{2}, \frac{\theta}{\sigma}\right\}$ the following representation is admissible:

$$\text{ULB}_\infty(m,\sigma) = \frac{1}{2\pi i}\int\limits_{x_0-i\infty}^{x_0+i\infty} \frac{\Gamma(m+2)\Gamma(z)}{\Gamma(m+2+z)}\mathfrak{M}(-z)dz \tag{6.5.27}$$

In accordance with our preceding findings, to deduce an expansion of $\text{ULB}_\infty(m,\sigma)$ as $m \to \infty$ we must specify the analytic continuation of the corresponding generating function. This is particularly simple if $\sigma = 0$, since in this event we have analyticity in $\Re z < \frac{1}{2}$, and this region is determined by the zeros of the sine function. An expansion of $\text{ULB}_\infty(m)$ then can be obtained by confining to each finite zero and eventually summing these terms. This is the reason, why the approach from the first part of the present example works and (6.5.24) in fact holds. If $\sigma > 0$ is small enough to have $2\theta > \sigma$, the described procedure remains valid merely for the derivation of a finite number of terms in the expansion of $\text{ULB}_\infty(m)$. It eventually fails if $\sigma \geq 2\theta$, in which circumstances the generating function is analytic in $\Re z < \frac{\theta}{\sigma}$ and this limited region has its origin in the behaviour of the integrand at infinity. We must then investigate the properties of $\mathfrak{M}(-z)$ before we are able to establish an asymptotic statement. For this purpose we first observe that it is not possible to specify a neighborhood of infinity in which the integrand is non-vanishing, except at infinity itself. The reason is that the zeros of the sine lie dense in any neighborhood of infinity. To circumvent this obstacle we again exploit the periodicity, which for any $\theta > 0$, $\sigma \geq 0$ leads to

$$\mathfrak{M}(-z) = \sum_{k=1}^{\infty}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{\sin^2(t)\right\}^{-z}(t+\pi k)^{2z-1}e^{-(\theta - z\sigma)(t+\pi k)}dt.$$

Conceiving the series as a Stieltjes integral, we identify the above representation as an iterated integral. Of these two, the convergence of the integral along $(-\frac{\pi}{2}, \frac{\pi}{2})$ requires $\Re z < \frac{1}{2}$, whereas for the convergence of the series we need $\Re z < \frac{\theta}{\sigma}$. Put differently, regarding the convergence behaviour of the generating function we were able to separate the effect of the finite zeros from the contribution of the exponential factor. The above series finally can be replaced by an integral upon introducing for $\Re z < \min\left\{\frac{1}{2}, \frac{\theta}{\sigma}\right\}$ the identity (A.5.11), from which subject to absolute convergence we arrive at:

$$\mathfrak{M}(-z) = \frac{1}{\Gamma(1-2z)}\sum_{k=1}^{\infty}\int\limits_0^{\infty} u^{-2z}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\sin^2(t)\right\}^{-z}e^{-(\theta - z\sigma + u)(t+\pi k)}dtdu$$

$$= \frac{1}{\Gamma(1-2z)} \int_0^\infty u^{-2z} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{\sin^2(t)\right\}^{-z} \frac{e^{-(\theta - z\sigma + u)t}}{e^{\pi(\theta - z\sigma + u)} - 1} dt du$$

$$= \frac{2}{\Gamma(1-2z)} \int_0^\infty \frac{u^{-2z}}{e^{\pi(\theta - z\sigma + u)} - 1} \int_0^{\frac{\pi}{2}} \{\sin(s)\}^{-2z} \cosh\left\{(\theta - z\sigma + u)s\right\} ds du$$

For the last equality we made the change of variables $s = -t$. The resulting representation eventually furnishes a more appropriate setting for a study of the analytic continuation of the generating function, yet, it is still an elaborate task. We shall not present further details. Instead we close this section with a similar example, where the above procedure is applicable but does not yield a double integral. It will reveal a very unique structure of the singularities associated with the generating function, due to the correlation between the oscillatory and the exponential factor.

Example 6.5.5 (discrete uniform errors with a Cauchy-type factor). In our final example we study for parameters $\sigma, \theta > 0$ the large λ-behaviour of the integral

(6.5.28)
$$C_\lambda := \int_0^\infty \left\{1 - e^{-\sigma t} \cos^2(t)\right\}^\lambda e^{-\theta t} dt.$$

This can be conceived as the uniform bias in the deconvolution problem of densities, involving an X-distribution of Cauchy-type and an error distribution that is composed as the convolution of a Cauchy and a discrete uniform distribution. First observe, choosing the principal branch of the power, the generating function

(6.5.29)
$$\mathfrak{M}(-z) := \int_0^\infty \left\{\cos^2(t)\right\}^{-z} e^{-(\theta - \sigma z)t} dt$$

converges absolutely for $\Re z < \min\left\{\frac{1}{2}, \frac{\theta}{\sigma}\right\}$ and the convergence is uniform in any compact subset therein, whence it establishes a holomorphic function of z there. The reason for this restriction is that the cosine term has an infinite sequence of zeros of second order at $\pi(\frac{1}{2} + \mathbb{N}_0)$ and particularly as $t \to \infty$ the integrand is $\sim$ const $\times\, e^{-(\theta - \sigma \Re z)t}$. Hence, local integrability on $t \geq 0$ holds if $\Re z < \frac{1}{2}$, while absolute convergence along the entire axis requires $\Re z < \frac{\theta}{\sigma}$. In accordance with these findings, for

(6.5.30)
$$0 < x_0 < \min\left\{\frac{1}{2}, \frac{\theta}{\sigma}\right\}$$

we can introduce the binomial integral representation to (6.5.28), which leads to

$$C_\lambda = \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} \frac{\Gamma(\lambda+1)\Gamma(z)}{\Gamma(\lambda+1+z)} \mathfrak{M}(-z) dz. \tag{6.5.31}$$

By elementary manipulations, referring to the formula for the geometric series, accompanied by the change in variables $u = \pi - t$, we can write for fixed $\Re z < \min\left\{\frac{1}{2}, \frac{\theta}{\sigma}\right\}$:

$$\begin{aligned}
\mathfrak{M}(-z) &= \int_0^\infty \left\{\cos^2(t)\right\}^{-z} e^{-(\theta-\sigma z)t} dt \\
&= \sum_{k=0}^\infty \int_{\pi k}^{\pi(k+1)} \left\{\cos^2(t)\right\}^{-z} e^{-(\theta-\sigma z)t} dt \\
&= \sum_{k=0}^\infty \int_0^\pi \left\{\cos^2(t)\right\}^{-z} e^{-(\theta-\sigma z)(t+\pi k)} dt \\
&= \frac{1}{1-e^{-(\theta-\sigma z)\pi}} \int_0^\pi \left\{\cos^2(t)\right\}^{-z} e^{-(\theta-\sigma z)t} dt \\
&= \frac{e^{(\theta-\sigma z)\frac{\pi}{2}}}{2\sinh\left\{(\theta-\sigma z)\frac{\pi}{2}\right\}} \left[\int_0^{\frac{\pi}{2}} \left\{\cos^2(t)\right\}^{-z} e^{-(\theta-\sigma z)t} dt + \int_0^{\frac{\pi}{2}} \left\{\cos^2(u)\right\}^{-z} e^{-(\theta-\sigma z)(\pi-u)} du \right] \\
&= \frac{1}{\sinh\left\{(\theta-\sigma z)\frac{\pi}{2}\right\}} \int_0^{\frac{\pi}{2}} \left\{\cos(u)\right\}^{-2z} \cosh\left\{(\theta-\sigma z)\left(\frac{\pi}{2}-u\right)\right\} du \qquad (6.5.32)
\end{aligned}$$

Observe that the prefactor is a meromorphic function of z with an infinite sequence of poles with simple order which lie at $z_k := \frac{\theta + i2k}{\sigma}$ for $k \in \mathbb{Z}$. Furthermore, the integral along the segment $[0, \frac{\pi}{2}]$, which we define for convenience by

$$\mathfrak{N}(-z) := \int_0^{\frac{\pi}{2}} \left\{\cos(u)\right\}^{-2z} \cosh\left\{(\theta-\sigma z)\left(\frac{\pi}{2}-u\right)\right\} du, \tag{6.5.33}$$

is holomorphic in $\Re z < \frac{1}{2}$. The whole representation (6.5.32) thus constitutes a meromorphic function in $\Re z < \frac{1}{2}$, and if $\frac{\theta}{\sigma} < \frac{1}{2}$ it yields the continuation of (6.5.29). Moreover, it is $\mathcal{O}(1)$ as $\Im z \to \infty$, uniformly with respect to $\Re z$ in any closed vertical substrip excluding the line $\Re z = \frac{\theta}{\sigma}$. As a consequence, in (6.5.31) the integrand is $\mathcal{O}\left\{|z|^{-\lambda-1}\right\}$ as $\Im z \to \pm\infty$, and for any $\lambda > 0$ in the case $\frac{\theta}{\sigma} < \frac{1}{2}$ it is admitted to displace the integration path to the right over the infinite sequence of poles located at z_k for $k \in \mathbb{Z}$, to match a line $\frac{\theta}{\sigma} < \Re z < \frac{1}{2}$. It is readily verified that the following holds:

$$\begin{aligned}\operatorname*{Res}_{z=z_k} \frac{1}{\sinh\left\{(\theta-\sigma z)\frac{\pi}{2}\right\}} &= \lim_{z\to z_k} \frac{z-z_k}{\sinh\left\{(\theta-\sigma z)\frac{\pi}{2}\right\}} \\ &= \frac{2}{\sigma\pi}(-1)^{k+1}\end{aligned}$$

Hence, as $\lambda \to \infty$ we conclude if $\sigma > 2\theta$:

$$\begin{aligned}C_\lambda &= \frac{2}{\sigma\pi}\sum_{k=-\infty}^{\infty}\frac{\Gamma(\lambda+1)\Gamma(z_k)}{\Gamma(\lambda+1+z_k)}\mathfrak{N}(-z_k)(-1)^k + \mathcal{O}\left\{\lambda^{-x_1}\right\} \\ &\sim \frac{2}{\sigma\pi}\frac{\Gamma(\lambda+1)\,\Gamma\left(\frac{\theta}{\sigma}\right)}{\Gamma\left(\lambda+1+\frac{\theta}{\sigma}\right)}\left\{\Gamma\left(\frac{\theta}{\sigma}\right)\mathfrak{N}\left(-\frac{\theta}{\sigma}\right) + 2\sum_{k=1}^{\infty}\Re\frac{\Gamma\left(\lambda+1+\frac{\theta}{\sigma}\right)\Gamma\left(\frac{\theta}{\sigma}+\frac{i2k}{\sigma}\right)}{\Gamma\left(\lambda+1+\frac{\theta}{\sigma}+\frac{i2k}{\sigma}\right)}\mathfrak{N}(-z_k)(-1)^k\right\}\end{aligned}$$

The representation in terms of the real part is especially true because z_k lies in the range of validity of the integral definition of the beta function and of $\mathfrak{N}(-z)$. Subsequent terms in the above expansion can be unlocked by determining the analytic continuation of the integral $\mathfrak{N}(-z)$. Finally, regarding further parametrizations a distinction between the cases $\frac{\theta}{\sigma} = \frac{1}{2}$ and $\frac{\theta}{\sigma} > \frac{1}{2}$ is required, which we shall not discuss right here. Particularly in the former case, the presence of at least one pole of order two must be expected.

6.6. Asymptotics of Special Fourier-Type Integrals

In this final section of the chapter we want to provide not a quantitative but a qualitative statement about the large λ-behaviour of the sequence of integrals

$$I(\lambda,\xi) := \lim_{\delta\downarrow 0} J_\delta(\lambda,\xi), \tag{6.6.1}$$

where for $\tau > 0$ and $\xi \in \mathbb{R}$ we denote

$$J_\delta(\lambda,\xi) := \int_\tau^\infty (1-\Psi(t))^\lambda a(t) e^{-(\delta+i\xi)t}\,dt. \tag{6.6.2}$$

It is easy to see that, for example, by an application of Abel's lemma the pointwise bias of the deconvolution function can be cast as an integral of the above type. If in (6.6.2) for $\xi = 0$ it is permitted to perform the limit under the sign of integration, we return to (6.0.1). This, however, will not be assumed, as we will confine to the more general case that $a(t)$ is possibly even unbounded as $t \to \infty$ but grows no faster than algebraically. With respect to $\zeta := \delta + i\xi$ the integral (6.6.2) constitutes a Laplace transform whose abscissa of convergence matches at least the half-plane $\Re\zeta > 0$. Therefore, taking the limit as $\delta \downarrow 0$ is equivalent to approaching the imaginary ζ-axis from the right. In the preceding sections we have seen that the asymptotic behaviour of $J_\delta(\lambda,\xi)$ depends on local behaviour at the points, where $\Psi(t)$ vanishes. Throughout this section, the function is supposed not to vanish at any finite point along the range of integration but at $t = \infty$, and we agree

$$\varphi(t) := -\log(1 - \Psi(t)).$$

Under the assumptions below, we will then show that the asymptotic behaviour of (6.6.1) essentially benefits from the presence of the complex exponential term.

(1) $a(t)$ is infinitely many times continuously differentiable on $t \geq \tau$, $a(t) = \mathcal{O}\{t^{-\alpha}\}$ for a fixed $\alpha \in \mathbb{R}$ and $a^{(j)}(t) = \mathcal{O}\left\{t^{-\alpha-j}\right\}$ as $t \to \infty$ for $j \in \mathbb{N}$.

(2) $\Psi(t)$ is infinitely many times continuously differentiable on $t \geq \tau$ and $0 < \Psi(t) < 1$ for any finite $t \geq \tau$. In addition, $\varphi(t) = o(1)$ and $\frac{d^{j-1}}{dt^{j-1}} \frac{\varphi'(t)}{\varphi(t)} = \mathcal{O}\left\{t^{-j}\right\}$ for $j \in \mathbb{N}$ as $t \to \infty$, but there exists a $\beta \geq 0$ such that $t^q \varphi(t) \to \infty$ for any $q > \beta$. Note that $\frac{\varphi'(t)}{\varphi(t)} = \frac{d}{dt} \log \varphi(t)$.

As a consequence of the series expansion of the logarithm, $\varphi(t) \sim \Psi(t)$ as $t \to \infty$. The above conditions are sufficient to verify the following statement.

Theorem 6.6.1. *Provided (1) and (2) hold and $\rho > 0$, then as $\lambda \to \infty$ for arbitrary $x_0 > 0$, uniformly with respect to $|\xi| \geq \rho$, we have*

$$I(\lambda, \xi) = \mathcal{O}\left\{\lambda^{-x_0}\right\}. \tag{6.6.3}$$

In other words, due to the complex exponential function, which for $\xi \neq 0$ exhibits oscillatory behaviour at infinity, the sequence of integrals $I(\lambda, \xi)$ decays faster than any algebraic power as $\lambda \to \infty$. Notice that the function $a(t)$ can not feature additional oscillatory factors, since its derivatives then would not satisfy the required conditions. To see that the theorem might be invalid for $\xi = 0$, suppose $a(t) \geq 0$ and $\xi = 0$. In these circumstances the integrand of $J_\delta(\lambda, 0)$ is non-negative and monotonic with respect to $\delta \geq 0$ for fixed $\lambda > 0$. Hence, the monotone convergence theorem applies, yielding the divergence of the integral already in the limit $\delta \downarrow 0$, if $a(t)$ is non-integrable on the ray $t \geq \tau$.

Proof. First, for fixed $\Re\zeta > 0$ and $s \geq \tau$ we introduce the generating function

$$M_s(\zeta, -z) := \int_s^\infty \{\varphi(t)\}^{-z} a(t) e^{-\zeta t} dt. \tag{6.6.4}$$

By assumption, if $\Re z \leq 0$ we have $\{\varphi(t)\}^{-\Re z} = \mathcal{O}(1)$ as $t \to \infty$, whereas if $\Re z \geq 0$ for any $q > \beta$ and hence for arbitrary $\nu > 0$, we observe

$$\{\varphi(t)\}^{-\Re z} = \{t^q \varphi(t)\}^{-\Re z} t^{q\Re z} = \mathcal{O}\left\{t^{\Re z\beta+\nu}\right\}. \tag{6.6.5}$$

Without incorporating the exponential function, the integrand in (6.6.4) thus exhibits at worst algebraic growth. This implies for fixed $s \geq \tau$, $\Re\zeta > 0$ and $\Re z \in \mathbb{R}$ the absolute and uniform convergence with respect to $\Im\zeta, \Im z \in \mathbb{R}$ of the indicated integral. Accordingly, we may employ the Cahen-Mellin representation (6.1.5) and interchange the order of integration for fixed $\delta > 0$ and arbitrary $x_0 > 0$, leading to

$$J_\delta(\lambda, \xi) = \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} \lambda^{-z} \Gamma(z) M_\tau(\delta + i\xi, -z) dz. \tag{6.6.6}$$

Furthermore, by absolute convergence $J_\delta(\lambda, \xi) = \mathcal{O}(\lambda^{-x_0})$ as $\lambda \to \infty$ for fixed $\delta > 0$, uniformly with respect to $\xi \in \mathbb{R}$. The aim of the proof is to confirm that this statement holds uniformly with respect to $\delta \geq 0$ and $|\xi| \geq \rho$. Indeed, generally the absolute convergence of $M_s(\zeta, -z)$ cannot be expected for $\Re\zeta = 0$, particularly if this line is the boundary of the ζ-region of analyticity. Now, under the above conditions N-times integration by parts of (6.6.4) for $N \in \mathbb{N}$, $s = \tau$, $z \in \mathbb{C}$ and $\Re\zeta > 0$ is permitted, and this leads to

$$\begin{aligned} M_\tau(\zeta, -z) = \sum_{n=0}^{N-1} \left[-\frac{e^{-\zeta t}}{\zeta^{1+n}} \frac{d^n}{dt^n} \left\{ \{\varphi(t)\}^{-z} a(t) \right\} \right]_{t=\tau}^{\infty} \\ + \frac{1}{\zeta^N} \int_\tau^\infty e^{-\zeta t} \frac{d^N}{dt^N} \left\{ \{\varphi(t)\}^{-z} a(t) \right\} dt. \end{aligned} \tag{6.6.7}$$

To analyze the asymptotic behaviour of the involved derivatives, we first apply the Leibniz rule, compare eq. (1.4.12) in [Olver et al., 2010], which for $0 \leq n \leq N$ brings us

$$\frac{d^n}{dt^n} \left\{ \{\varphi(t)\}^{-z} a(t) \right\} = \sum_{k=0}^{n} \binom{n}{k} \left[\frac{d^k}{dt^k} \{\varphi(t)\}^{-z} \right] a^{(n-k)}(t). \tag{6.6.8}$$

Regarding the derivatives of the kernel, we must employ the extended chain rule, better known as the formula of Faà di Bruno. It is given in eq. (1.4.13) in [Olver et al., 2010]. A sophisticated choice of the inner and outer derivative for $0 \leq k \leq n$ yields

$$\begin{aligned} \frac{d^k}{dt^k} \{\varphi(t)\}^{-z} &= \{\varphi(t)\}^{-z} P_k(t, z), \\ P_k(t, z) &:= \sum \frac{k!}{m_1! \ldots m_k!} (-z)^{\sum_{r=1}^k m_r} \prod_{r=1}^{k} \left(\frac{1}{r!} \frac{d^{r-1}}{dt^{r-1}} \frac{\varphi'(t)}{\varphi(t)} \right)^{m_r}. \end{aligned} \tag{6.6.9}$$

The sum must be taken over all $m_1, \ldots, m_k \in \mathbb{N}_0$ with $\sum_{r=1}^k r m_r = k$. Evidently, the function $P_k(t, z)$ for $0 \leq k \leq n$ is a polynomial of z with degree k. Each of its coefficients is a continuous function on $t \geq \tau$ that, due to the conditions imposed on the derivatives of $\log \varphi(t)$ but also due to the properties of the summation indices, exhibits the asymptotic behaviour $\mathcal{O}\left\{t^{-k}\right\}$ as $t \to \infty$. Combined with (6.6.5) this implies that the k-th derivative of $\{\varphi(t)\}^{-z}$ is $\mathcal{O}\left\{t^{\beta \Re z + \nu - k}\right\}$

as $t \to \infty$ for arbitrary $\nu > 0$ and $\Re z \geq 0$. For brevity, instead of (6.6.8), we write

$$\frac{d^n}{dt^n}\left\{\{\varphi(t)\}^{-z}\, a(t)\right\} = \{\varphi(t)\}^{-z} \sum_{k=0}^{n} \binom{n}{k} P_k(t,z) a^{(n-k)}(t). \tag{6.6.10}$$

To summarize our findings, for $\Re z \geq 0$ the n-th derivative in (6.6.10) is continuous on $t \geq \tau$ and $\mathcal{O}\left\{t^{\beta \Re z + \nu - \alpha - n}\right\}$ as $t \to \infty$ with arbitrary $\nu > 0$, i.e., it grows no faster than algebraically. In (6.6.7) at $t = \infty$, for $\Re\zeta > 0$ and $\Re z \geq 0$ the derivatives are thus dominated by the exponential function. However, choosing $N \in \mathbb{N}$ for a given fixed $z \in \mathbb{C}$ with $\Re z \geq 0$ to satisfy $N > \beta \Re z - \alpha + 1$, the N-th derivative does not only decay but is especially absolutely integrable on $[\tau, \infty)$. In terms of (6.6.10), the expansion (6.6.7) becomes

$$\begin{aligned} M_\tau(\zeta, -z) = \sum_{n=0}^{N-1} \frac{e^{-\zeta\tau}}{\zeta^{n+1}} \{\varphi(\tau)\}^{-z} \sum_{k=0}^{n} \binom{n}{k} P_k(\tau,z) a^{(n-k)}(\tau) \\ + \frac{1}{\zeta^N} \sum_{k=0}^{N} \binom{N}{k} \int_\tau^\infty e^{-\zeta t} \{\varphi(t)\}^{-z} P_k(t,z) a^{(N-k)}(t) dt. \end{aligned} \tag{6.6.11}$$

For $\Re z \geq 0$ and appropriate N the integral converges absolutely and uniformly with respect to ζ in $\Re\zeta \geq 0$. Hence, the right hand side then defines $M_\tau(\zeta, -z)$ for any $\Re\zeta > 0$ and even for $\Re\zeta = 0$ with $\Im\zeta \neq 0$. The point $\zeta = 0$ is thus a singularity. Finally, with $\zeta := \delta + i\xi$ for fixed but arbitrary $\delta > 0$, $\Re z \geq 0$, $|\xi| \geq \rho$ and $N > \beta\Re z - \alpha + 1$, the triangle inequality yields

$$\begin{aligned} |M_\tau(\delta + i\xi, -z)| \leq \sum_{n=0}^{N-1} \frac{\{\varphi(\tau)\}^{-\Re z}}{\rho^{n+1}} \sum_{k=0}^{n} \binom{n}{k} |P_k(\tau,z)| \left|a^{(n-k)}(\tau)\right| \\ + \frac{1}{\rho^N} \sum_{k=0}^{N} \binom{N}{k} \int_\tau^\infty \{\varphi(t)\}^{-\Re z} |P_k(t,z)| \left|a^{(N-k)}(t)\right| dt. \end{aligned} \tag{6.6.12}$$

In accordance with the above considerations, this upper bound holds uniformly with respect to $\delta \geq 0$ and $|\xi| \geq \rho > 0$, with the integral appearing therein being absolutely convergent. Recall that the function $P_k(t,z)$ equals a polynomial of z with degree k, whence the same holds for the product $P_k(t,z)a^{(n-k)}(t)$. The coefficients of the product are continuous functions on $[\tau, \infty)$ and for $n = N$ absolutely integrable there. Consequently, for any $0 \leq n \leq N$ there exist further coefficients $c_j(t,n) \geq 0$ such that

$$\sum_{k=0}^{n} \binom{n}{k} \left|P_k(t,z) a^{(n-k)}(t)\right| \leq \sum_{j=0}^{n} |z|^j\, c_j(t,n). \tag{6.6.13}$$

The coefficients $c_j(t,n)$ depend on the variable t only. They are continuous on $t \geq \tau$, and as $t \to \infty$ they satisfy $\mathcal{O}\left\{t^{\beta\Re z + \nu - \alpha - n}\right\}$ for $\Re z \geq 0$ and arbitrary $\nu > 0$. Estimating (6.6.12) by means of (6.6.13) for $\Re z \geq 0$, we deduce the existence of coefficients $d_l \geq 0$ with the property

$$|M_\tau(\delta + i\xi, -z)| \leq \sum_{l=0}^{N} |z|^l \, d_l. \tag{6.6.14}$$

The coefficients d_l are uniformly bounded with respect to $\Im z \in \mathbb{R}$, $\delta \geq 0$ and $|\xi| \geq \rho$, whence the whole right hand side of (6.6.14) is uniformly bounded with respect to $\delta \geq 0$ and $|\xi| \geq \rho$. In addition, it is $\mathcal{O}\left\{|z|^N\right\}$ as $\Im z \to \pm\infty$. We have therefore established an absolutely integrable uniform bound for the generating function in (6.6.6), from which we conclude $J_\delta(\lambda, \xi) = \mathcal{O}\left\{\lambda^{-x_0}\right\}$ as $\lambda \to \infty$, uniformly with respect to $\delta \geq 0$ and $|\xi| \geq \rho$. By uniformity, this statement remains true for $I(\lambda, \xi)$, thereby concluding the proof. ∎

It is reasonable to expect that the above statement remains true, if $e^{-qt} = o\left\{\Psi(t)\right\}$ as $t \to \infty$ for any $q > 0$, or if $a(t) = o\left\{e^{pt}\right\}$ as $t \to \infty$ for any $p > 0$. In each case (6.6.6) still applies with arbitrary $x_0 > 0$. We were, however, not able to cover these more general cases with the above procedure, because of the distinguishing properties of the exponential function. Instead, it was merely possible to establish a weaker statement, if rather than (2) we suppose:

(2') $\Psi(t)$ is infinitely many times continuously differentiable on $t \geq \tau$ and $0 < \Psi(t) < 1$ for any finite $t \geq \tau$. In addition, $\varphi(t) = o(1)$ and there exists $\varepsilon < 1$ with $\frac{d^{j-1}}{dt^{j-1}} \frac{\varphi'(t)}{\varphi(t)} = \mathcal{O}\left\{t^{\varepsilon - j}\right\}$ for $j \in \mathbb{N}$ as $t \to \infty$ but $e^{t^q}\varphi(t) \to \infty$ for some $0 < q < 1$.

One can show that, for instance $\Psi(t) = e^{-t^{\frac{1}{2}}}$ matches (2') but not (2). The asymptotic estimate for $I(\lambda, \xi)$ under the modified assumption completes this section.

Theorem 6.6.2. *If (1) and (2') hold and $\rho > 0$, as $\lambda \to \infty$, uniformly with respect to $|\xi| \geq \rho$, we have*

$$I(\lambda, \xi) = \mathcal{O}(1).$$

Proof. By (2'), as $t \to \infty$ for some $0 < q < 1$ in $\Re z \geq 0$ we observe

$$\left\{\varphi(t)\right\}^{-z} = \mathcal{O}\left\{e^{zt^q}\right\}. \tag{6.6.15}$$

Just like in (6.6.6), for fixed $\delta > 0$ we may therefore again introduce the Cahen-Mellin representation with arbitrary $x_0 > 0$. Now, for fixed $\Re\zeta > 0$ and $t \geq \tau$ we integrate by parts N-times for $N \in \mathbb{N}$ the generating function (6.6.4) with $z = 0$, which yields

$$M_t(\zeta, 0) = \sum_{n=0}^{N-1} \frac{e^{-\zeta t}}{\zeta^{n+1}} a^{(n)}(t) + \frac{1}{\zeta^N} \int_t^\infty a^{(N)}(s) e^{-\zeta s} ds. \tag{6.6.16}$$

If we choose $\alpha + N > 0$, the integral on the right hand side can be estimated by $\mathcal{O}\left\{t^{-\alpha-N} e^{-\Re\zeta t}\right\}$ as $t \to \infty$, uniformly with respect to $\Im\zeta \in \mathbb{R}$. For fixed $z \in \mathbb{C}$, $\xi \in \mathbb{R}$ and $\delta > 0$, combined with

(6.6.15), this shows the absolute convergence of the iterated integral

$$H(\delta + i\xi, -z) := \int_{\tau}^{\infty} \{\varphi(t)\}^{-z} \frac{\varphi'(t)}{\varphi(t)} M_t(\delta + i\xi, 0)dt. \tag{6.6.17}$$

As a consequence of the above results, upon once integrating by parts the generating function (6.6.4) for fixed $\delta, \Re z > 0$ and $\xi \in \mathbb{R}$, we arrive at:

$$\begin{aligned} M_\tau(\delta + i\xi, -z) &= \left[-\{\varphi(t)\}^{-z} \int_t^{\infty} a(s)e^{-(\delta+i\xi)s}ds \right]_\tau^\infty - z \int_\tau^\infty \{\varphi(t)\}^{-z} \frac{\varphi'(t)}{\varphi(t)} \int_t^\infty a(s)e^{-(\delta+i\xi)s}dsdt \\ &= \{\varphi(\tau)\}^{-z} M_\tau(\delta + i\xi, 0) - zH(\delta + i\xi, -z) \end{aligned} \tag{6.6.18}$$

If we apply this expansion to the MB-integral (6.6.6), by definition of the Cahen-Mellin integral and according to the functional equation for the gamma function, for arbitrary $x_0 > 0$ we obtain

$$J_\delta(\lambda, \xi) = e^{-\lambda\varphi(\tau)} M_\tau(\delta + i\xi, 0) - \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} \lambda^{-z}\Gamma(z + 1)H(\delta + i\xi, -z)dz. \tag{6.6.19}$$

Observe that the simple pole at $z = 0$ of the gamma function in the MB-integral cancels with the zero of the second addend in (6.6.18) there. As $\lambda \to \infty$ the first term in (6.6.19) clearly vanishes. Moreover, an expansion for $M_\tau(\delta + i\xi, 0)$ is available by (6.6.16), which for $N > 1 - \alpha$ yields the existence of the limit as $\delta \downarrow 0$ uniformly with respect to $|\xi| \geq \rho$. Next, for fixed $\delta > 0$ we examine the integral $H(\delta + i\xi, -z)$ as a function of z. It does not only converge absolutely for all $z \in \mathbb{C}$, but the convergence is uniform in any compact subset of the z-plane. By Theorem A.2.1 this verifies $H(\delta + i\xi, -z)$ as an entire function for any fixed $\delta > 0$ and $\xi \in \mathbb{R}$. In addition, since the convergence is even uniform in any closed vertical substrip, the function is $\mathcal{O}(1)$ as $\Im z \to \pm\infty$, uniformly with respect to $\Re z$ in any closed vertical substrip. The integrand of the MB-integral in (6.6.19) is thus holomorphic in $\Re z > -1$ with exponential decay as $\Im z \to \pm\infty$ there. This enables us to shift the integration path to the left, to match the line $\Re z = 0$, which leads to

$$J_\delta(\lambda, \xi) = -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \lambda^{-z}\Gamma(z + 1)H(\delta + i\xi, -z)dz + e^{-\lambda\varphi(\tau)} M_\tau(\delta + i\xi, 0). \tag{6.6.20}$$

The benefit from the choice $\Re z = 0$ is, that the kernel of $H(\delta + i\xi, -z)$ does no longer grow as $t \to \infty$ but is $\mathcal{O}(1)$. For $\Re\zeta = 0$ the whole integrand therefore at worst grows with an algebraic order. By absolute convergence, for fixed $\delta > 0$ uniformly with respect to $\xi \in \mathbb{R}$, the MB-integral in (6.6.20) is $\mathcal{O}(1)$ as $\lambda \to \infty$. Similar to the proof of the preceding theorem, an integration by parts procedure shows the uniformity with respect to $\delta \geq 0$ and $|\xi| \geq \rho$ of this statement. In particular, by means of the expansion (6.6.16), for arbitrary $N \in \mathbb{N}$ and $z \in \mathbb{C}$ we deduce

$$H(\delta + i\xi, -z) = \sum_{n=0}^{N-1} \frac{H_n(\delta + i\xi, -z)}{(\delta + i\xi)^{n+1}} + \frac{H_N(\delta + i\xi, -z)}{(\delta + i\xi)^N}, \tag{6.6.21}$$

where we denote

$$H_n(\delta + i\xi, -z) := \int_\tau^\infty \{\varphi(t)\}^{-z} \frac{\varphi'(t)}{\varphi(t)} a^{(n)}(t) e^{-(\delta+i\xi)t} dt, \tag{6.6.22}$$

$$H_N(\delta + i\xi, -z) := \int_\tau^\infty \{\varphi(t)\}^{-z} \frac{\varphi'(t)}{\varphi(t)} \int_t^\infty a^{(N)}(s) e^{-(\delta+i\xi)s} ds dt. \tag{6.6.23}$$

For $\Re z = 0$ and $0 \leq n \leq N-1$, the integrals (6.6.22) must be integrated by parts in the fashion of the proof of Theorem 6.6.1, to verify their uniform boundedness with respect to $\delta \geq 0$ and $|\xi| \geq \rho$. This especially works because $\frac{\varphi'(t)}{\varphi(t)}$ and all its derivatives are decreasing by assumption, for sufficiently large t. Finally, regarding (6.6.23), with $\Re z = 0$ we choose N sufficiently large, to have $N > 1 - \alpha + \max\{0, \varepsilon\}$. In these circumstances the iterated integral converges absolutely and uniformly with respect to $\delta \geq 0$ and $\xi \in \mathbb{R}$. Altogether this yields an absolutely integrable bound for (6.6.21) with $\Re z = 0$, whose validity is uniform for $\delta \geq 0$ and $|\xi| \geq \rho$, from which by (6.6.20) we conclude $J_\delta(\lambda, \xi) = \mathcal{O}(1)$ as $\lambda \to \infty$, uniformly with respect to $\delta \geq 0$ and $|\xi| \geq \rho$. This finishes the proof of Theorem 6.6.2. During the last steps it becomes clear, that an integration path in (6.6.20) that runs in the half plane $\Re z > 0$, would not lead to a stronger result, because the derivatives of a and $\log \varphi$ are unable to dominate the possibly exponential growth of the kernel of $H_n(\delta + i\xi, -z)$ for $0 \leq n \leq N$. ■

6.7. Conclusion

To summarize this chapter, the method of analytic continuation enables us to evaluate a huge class of integrals, represented by I_λ. The technique can be concisely described in three steps:

(1) Find an appropriate representation for the kernel in I_λ.

(2) Expand the generating function by the order of its singularities, ascending with respect to the real part of its argument.

(3) Deduce the asymptotic expansion of I_λ in terms of the asymptotic scale that is determined by the integral representation for the kernel.

With the tools that were provided in this chapter, for some scenarios a full asymptotic expansion can be established, whereas for others only a few leading terms can be extracted. The latter is particularly the case if higher order derivatives of the ingredient functions involve non-algebraic expressions. Those terms indicate the presence of singularities other than simple poles. Indeed, it should then be expected that the analytic continuation associated with the generating function exhibits poles of order greater than one or even branch points. Neither the approach of the generalized Laurent expansion, nor the described integration by parts technique are then applicable. Instead we require an appropriate modification. This was also necessary if already the leading term of any of the ingredient functions involves non-algebraic expressions.

By comparison of (6.4.7) with the statement of Watson's lemma or of Laplace's method, see Theorem 3.1 or Theorem 7.1 in ch. 3 in [Olver, 1974], respectively, we ascertain a close similarity between the first and the latter two expansions. In fact, those which we derived in the present chapter can be conceived as generalizations of Laplace's method. They were obtained by employing different representations for the kernel function in I_λ, and particularly the representation (6.1.5) then resulted in an expansion in powers of λ, which can also be established by means of Laplace's method, provided the ingredient functions possess an appropriate power series expansion. In general Laplace's method is restricted to functions that are expandable in integer powers. It does not cover scenarios where an expansion involves arbitrary fractional powers. In these circumstances, however, the integration by parts technique remains relevant. Although its applicability is by far more complicated to determine a full expansion, the scope is much larger. Furthermore, compared to Laplace's method, the method of analytic continuation is even capable to deduce an asymptotic expansion for the integral I_λ, if its decay is slower than any power of the asympotic parameter λ.

In some circumstances the generalized Laurent expansion and the integration by parts procedure may both be applicable. The preference then depends on the purpose. While the former might be more convenient to construct a whole expansion, since the derivation of the coefficients is considerably less elaborate, the latter suffices to quantify the leading behaviour by virtue of Theorem 6.4.1. Finally, it will turn out in later chapters that the method of analytic continuation is adaptable to evaluate iterated integrals of Laplace-type in an analogous fashion.

7. Estimation with Errors in Variables

Throughout this chapter we denote by $Y_1, \ldots, Y_n$ for $n \in \mathbb{N}$ a random sample of independent observations associated with the distribution function G. Then, there are two possibilities to estimate the deconvolution function in the centered and in the symmetrized additive model of errors in variables, respectively. The first is a simple plug-in estimator, obtained by replacing in (2.1.6) or (2.2.6) the G-integral with G_n, i.e., with the empirical distribution function. Due to the extremely complicated structure of the deconvolution sum in terms of a Neumann partial sum, however, this approach does not seem to be feasible for applications. One further drawback of the plug-in estimator consists in the fast growth of the binomial coefficient, making it very difficult to preserve a certain accuracy. Secondly we have the Fourier-type integral representations of Theorems 2.1.2 and 2.2.1, which give access to a more convenient estimator by replacing the characteristic function Φ_Y by its empirical analogue. Before constructing any of these estimators and discussing their properties, we therefore begin with an introduction to the empirical characteristic function.

7.1. The Empirical Characteristic Function and its Deconvolution Analogues

The empirical characteristic function corresponding to a given n-sized Y-sample is obtained from replacing in the integral definition of Φ_Y the G-integral by G_n, i.e., from estimating the characteristic function by an integral with respect to the empirical distribution function. This step results in the mean of a sum of complex-valued exponential functions:

(7.1.1)
$$\begin{aligned}\Phi_Y(t,n) &:= \int\limits_{-\infty}^{\infty} e^{ity} G_n(dy) \\ &= \frac{1}{n}\sum_{j=1}^{n} e^{itY_j}\end{aligned}$$

According to the fact that G_n establishes a discrete probability distribution, $\Phi_Y(t,n)$ represents a characteristic function of almost periodic type due to [Bohr, 1932]. Roughly speaking an almost periodic function approaches any value in its range infinitely many times arbitrarily close. This property especially implies $\limsup_{t\to\pm\infty} |\Phi_Y(t,n)| = 1$, regardless of the actual behaviour of $\Phi_Y(t)$ at infinity. Hence, even if G possesses a density, the Riemann-Lebesgue lemma does no longer apply to $\Phi_Y(t,n)$. As a consequence, any integral along an infinite segment of the real axis involving (7.1.1) may converge only if additional factors contribute. On the other hand, evidently

$$\mathbb{E}\left\{\Phi_Y(t,n)\right\} = \Phi_Y(t), \qquad \text{for all } t \in \mathbb{R}, \tag{7.1.2}$$

and the law of large numbers states $\Phi_Y(t,n) \to \Phi_Y(t)$ almost surely as $n \to \infty$. Further basic properties were proved in [Feuerverger and Mureika, 1977]. These include that as $n \to \infty$ the empirical characteristic function converges to Φ_Y uniformly on any compact subset of $\mathbb{R}$. Along the entire real axis, however, the convergence is uniform only if Φ_Y is indeed almost periodic, i.e., if G represents a discrete distribution. The convergence behaviour is not in contradiction to the behaviour for finite sample sizes, since $\Phi_Y(\cdot,n)$ for fixed $n \in \mathbb{N}$ still constitutes the superposition of finitely many sine and cosine functions of different frequencies, which merge to a non-trigonometric function only in the limit $n \to \infty$. Finally, due to the almost periodicity of $\Phi_Y(\cdot,n)$, the function may exhibit finite zeros whose location, however, depends on the random Y-sample. To summarize our observations so far, according to (7.1.2), the empirical characteristic function is an unbiased estimator for the target function, but special care must be taken whenever it is considered in the context of integration.

An important means to measure the dispersion of the estimator (7.1.1) is the covariance function. Keeping in mind that any characteristic function Φ is complex-valued with

$$\overline{\Phi}(t) = \Phi(-t),$$

where the overline indicates the complex conjugate, for $s,t \in \mathbb{R}$ we compute:

$$\begin{aligned}
\operatorname{Cov}\left\{\Phi_Y(s,n),\Phi_Y(t,n)\right\} &= \mathbb{E}\left\{(\Phi_Y(s,n)-\Phi_Y(s))\,\overline{(\Phi_Y(t,n)-\Phi_Y(t))}\right\} \\
&= \mathbb{E}\left\{\Phi_Y(s,n)\Phi_Y(-t,n)\right\} - \Phi_Y(s)\Phi_Y(-t) \\
&= \frac{1}{n^2}\sum_{j=1}^{n}\mathbb{E}\left\{e^{i(s-t)Y_j}\right\} + \frac{1}{n^2}\sum_{\substack{1\le j,k\le n \\ j\ne k}}\mathbb{E}\left\{e^{isY_j - itY_k}\right\} - \Phi_Y(s)\Phi_Y(-t)
\end{aligned}$$

Since the observations $Y_1,\dots,Y_n$ are independent and identically distributed, it was just verified that

$$\operatorname{Cov}\left\{\Phi_Y(s,n),\Phi_Y(t,n)\right\} = \frac{1}{n}\left\{\Phi_Y(s-t) - \Phi_Y(s)\Phi_Y(-t)\right\}. \tag{7.1.3}$$

The latter naturally corresponds to the class of covariance functions, implying positive definiteness. In particular, if μ denotes an arbitrary finite measure on $\mathbb{R}$ and $u \in L^1(\mu)$, then:

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}}\Phi_Y(s-t)u(t)\mu(dt)\overline{u}(s)\mu(ds) &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-iyt}u(t)\mu(dt)\int_{\mathbb{R}} e^{iys}\overline{u}(s)\mu(ds)G(dy) \\
&= \int_{\mathbb{R}}\left|\int_{\mathbb{R}} e^{iys}\overline{u}(s)\mu(ds)\right|^2 G(dy)\int_{\mathbb{R}} G(dy)
\end{aligned}$$

$$
\begin{aligned}
&\geq \left| \int_{\mathbb{R}} \int_{\mathbb{R}} e^{iys} \overline{u}(s) \mu(ds) G(dy) \right|^2 \\
&= \left| \int_{\mathbb{R}} \Phi_Y(s) \overline{u}(s) \mu(ds) \right|^2 \\
&\geq 0
\end{aligned}
$$

In the third step we applied the Cauchy-Schwarz inequality. The preceding result shows that not only (7.1.3) is positive definite but even solely the function $\Phi_Y(s-t)$ is. If we therefore define for an arbitrary finite measure μ on $\mathbb{R}$ an integral operator by

$$
T\{f\}(s) := \int_{\mathbb{R}} \Phi_Y(s-t) u(t) \mu(dt), \tag{7.1.4}
$$

according to Mercer's theorem, cf. [Rasmussen and Williams, 2006], there exists a so-called *spectral decomposition* of the form

$$
\Phi_Y(s-t) = \sum_{j=1}^{\infty} \lambda_j \phi_j(s) \overline{\phi}_j(t). \tag{7.1.5}
$$

The functions ϕ_j are orthonormal eigenfunctions of the operator with eigenvalues λ_j, i.e., the j-th summand is a solution of the equation $T\{\phi_j\}(s) = \lambda_j \phi_j(s)$. Most of the literature classifies solely $\Phi_Y(s-t)$ as a covariance kernel and confines to the treatment of integral operators of the form (7.1.4). However, this is not a restriction since elementary manipulations show

$$
\Phi_Y(s-t) - \Phi_Y(s)\Phi_Y(-t) = \sum_{j=1}^{\infty} \lambda_j \left\{\phi_j(s) - \phi_j(0)\Phi_Y(s)\right\} \overline{\left\{\phi_j(t) - \phi_j(0)\Phi_Y(t)\right\}},
$$

whence a similar decomposition for the covariance function (7.1.3) follows immediately. Depending on the particular form of Φ_Y and the measure μ, the determination of (7.1.5) can be very elaborate. In addition, the choice of the measure essentially affects the properties of the components. A common approach is to first approximate the integral (7.1.4) by a sum and then to study the convergence of the solutions. For an additional concise overview on the topic of covariance functions we refer to chapter 4 in [Rasmussen and Williams, 2006].

Based on (2.1.15) and (2.1.16), in the symmetrized model (2.1.4) reasonable estimators for the characteristic function of $\dot{Y}$ and of $\mathfrak{D}(\cdot, m)$ can be introduced by

$$
\Phi_{\dot{Y}}(t,n) := \Phi_{\varepsilon}(-t)\Phi_Y(t,n), \tag{7.1.6}
$$

$$
\Phi_{\mathfrak{D}}(t,m,n) := \Phi_{\dot{Y}}(t,n)\mathcal{G}_{\bar{\varepsilon}}(t,m), \tag{7.1.7}
$$

which features the geometric sum function (2.1.11). Moreover, in the centered model (2.2.3) an estimator for the characteristic function of $\ddot{Y}$ and of the m-th deconvolution function, respectively, is of the form

$$\Phi_{\ddot{Y}}(t,n) := e^{-it\mu_\varepsilon}\Phi_Y(t,n), \tag{7.1.8}$$

$$\Phi_{\mathfrak{F}}(t,m,n) := \Phi_{\ddot{Y}}(t,n)\mathcal{H}_{\hat{\varepsilon}}(t,m), \tag{7.1.9}$$

where the geometric sum function $\mathcal{H}_{\hat{\varepsilon}}(t,m)$ was given in (2.2.9). The estimators (7.1.6) and (7.1.8) are both uniformly bounded by unity with respect to $t \in \mathbb{R}$. However, there are substantial differences regarding their particular behaviour along the real axis. Like the empirical characteristic function associated with the Y-sample also $\Phi_{\ddot{Y}}(\cdot,n)$ is almost periodic, while the properties of $\Phi_{\dot{Y}}(\cdot,n)$ depend on Φ_ε. This implies that $\Phi_{\dot{Y}}(\cdot,n)$ equals zero whenever Φ_ε equals zero and therefore especially vanishes at infinity if Φ_ε does. In other words, the left hand side of (7.1.6) is almost periodic if and only if the right hand side is, which is the case if and only if H is a discrete distribution. Regarding (7.1.7) and (7.1.9), the geometric sum functions yield no additional benefits with respect to the asymptotic behaviour as $t \to \pm\infty$. Since the latter are uniformly bounded with respect to $t \in \mathbb{R}$ by $m+1$, they assure that the same remains true for (7.1.7) and (7.1.9). Summarizing, it may be possible that the characteristic function corresponding to the deconvolution function in the symmetrized model is integrable on infinite segments of the real axis, whereas its analogue in the centered model is only uniformly bounded there. Besides the wider applicability of the symmetrization, the latter is perhaps the most distinguishing feature. While in the domain of characteristic functions centering merely leads to an additional complex exponential factor, the consequence of symmetrization by convolution with the conjugate distribution is the presence of a factor whose properties depend on the error distribution. Thus, the former approach is not only weaker in the positive sense that it preserves the variance $\operatorname{Var}\ddot{Y} = \operatorname{Var}Y$ but also in the negative sense, viz that the properties of the estimator for the characteristic function of the deconvolution function remain the same as those of $\Phi_Y(\cdot,n)$.

Letting first $n \to \infty$, according to the strong law of large numbers, the function $\Phi_{\mathfrak{D}}(t,m,n)$ for any $t \in \mathbb{R}$ and $m \geq 0$ converges G-almost surely to its expectation (2.1.17). Then letting $m \to \infty$, the limit equals $\{\Phi_\varepsilon(t)\}^{-1}\Phi_Y(t) = \Phi_X(t)$. We conclude that $\Phi_{\mathfrak{D}}(\cdot,m,n)$ plays the role of an estimator for Φ_X, and it may thus be compared with the function

$$\Upsilon(t,\lambda,n) := \{\Phi_\varepsilon(t)\}^{-1}\Phi_Y(t,n)\Phi_I(\lambda t),$$

which is known from the kernel approach, see equation (1.1.4). There, the Fourier transform Φ_I of the kernel is not only a means to keep $\{\Phi_\varepsilon\}^{-1}\Phi_Y(\cdot,n)$ absolutely integrable along the real axis, but especially to prevent the reciprocal factor from becoming unbounded for fixed $\lambda > 0$. Conversely, in the symmetrized model, according to our preceding observations, the function $\Phi_{\mathfrak{D}}(\cdot,m,n)$ is always bounded for fixed finite $m \geq 0$. Moreover, it features the factor Φ_ε in the numerator, which results in an estimator that vanishes whenever Φ_ε vanishes. If particularly the decay at infinity happens fast enough, the estimator is even absolutely integrable along the real axis. The number of summands m of the geometric sum function $\mathcal{G}_{\bar{\varepsilon}}(\cdot, m)$ therefore corresponds to the bandwidth λ, although it is of completely different nature. Despite these parallels, there is a substantial difference between $\Phi_{\mathfrak{D}}(\cdot, m, n)$ and $\Upsilon(\cdot, \lambda, n)$, which is the fact that the Fourier transform Φ_I of the kernel is more or less artificial and arbitrary. In addition, the standard kernel approach is usually restricted to errors whose characteristic function Φ_ε does not vanish along the real axis except at infinity. All of these issues are avoided by $\Phi_{\mathfrak{D}}(\cdot, m, n)$, since this function is made of components only that are immediately determined by the error distribution.

We proceed with the derivation of unbiased estimators for the deconvolution function, respectively, in the centered and in the symmetrized additive model of errors in variables.

7.2. Estimation in the Centered Additive Model of Errors in Variables

In the centered model (2.2.3), the random Y-observations are transformed to $\ddot{Y}_j := Y_j - \mu_\varepsilon$ for $1 \leq j \leq n$ with empirical distribution $\ddot{G}_n := G_n(\cdot + \mu_\varepsilon)$. As we mentioned in the introduction, the corresponding straightforward estimator for the deconvolution function is of plug-in type, and it is for $\xi \in \mathbb{R}$ obtained from (2.2.6) by replacing $G(dy)$ by $G_n(dy)$, resulting in

$$\mathfrak{F}_n(\xi, m) := \frac{1}{n} \sum_{j=1}^{n} \mathcal{T}_{\dot{\varepsilon}}^{m}(\xi - \ddot{Y}_j). \tag{7.2.1}$$

We denote $\mathfrak{F}_n(\xi, m)$ as the *empirical deconvolution function in the centered model.* Since the Y-observations are independent and identically distributed, for any $m \geq 0$, $n \in \mathbb{N}$ and $\xi \in \mathbb{R}$ on the one hand we have

$$\mathbb{E}\left\{\mathfrak{F}_n(\xi, m)\right\} = \mathfrak{F}(\xi, m), \tag{7.2.2}$$

and on the other hand with the aid of Bienaymé's identity we find

$$\operatorname{Var}\left\{\mathfrak{F}_n(\xi, m)\right\} = \frac{1}{n}\mathbb{E}\left\{\left[\mathcal{T}_{\dot{\varepsilon}}^{m}(\xi - \ddot{Y})\right]^2\right\} - \frac{1}{n}\left\{\mathfrak{F}(\xi, m)\right\}^2. \tag{7.2.3}$$

Equation (7.2.2) confirms the unbiasedness of the above estimator with respect to the deconvolution function $\mathfrak{F}(\cdot, m)$. Now, it was also emphasized earlier that we should rather employ a Fourier-type estimator, in order to avoid the Neumann partial sum representation for the deconvolution sum (2.2.7). Like for any Fourier inversion formula we must then take into account discontinuities. For $\xi \in \mathbb{R}$ we therefore define

$$\mathbb{P}_{\mathfrak{F}_n}(\xi) := \mathbb{P}\left(\grave{H}^{*j}\{\xi + \mu_\varepsilon - Y\} = 0 \text{ for all } j \in \mathbb{N}_0\right). \tag{7.2.4}$$

Clearly, in view of (2.2.7) the right hand side describes the probability for ξ to be a continuity point of the empirical deconvolution function $\mathfrak{F}_n(\xi, m)$ for all $m \geq 0$, and for each kind of distribution H we have $\mathbb{P}_{\mathfrak{F}_n}(0) \geq \mathbb{P}(Y \neq \mu_\varepsilon)$. Furthermore, $\mathbb{P}_{\mathfrak{F}_n}(\xi) = 1$ for all $\xi \in \mathbb{R}$ if F or H is continuous. Finally, similar to (2.1.50) one can show for all $\xi \in \mathbb{R}$ with $\mathbb{P}_{\mathfrak{F}_n}(\xi) = 1$ validity of

$$\mathfrak{F}_n(\xi, m) = \frac{1}{2} + \frac{1}{2\pi i} \lim_{T_2 \uparrow \infty} \lim_{T_1 \downarrow 0} \int_{T_1}^{T_2} \frac{e^{it\xi}\Phi_{\ddot{Y}}(-t, n) - e^{-it\xi}\Phi_{\ddot{Y}}(t, n)}{t} \mathcal{H}_{\grave{\varepsilon}}(t, m) dt. \tag{7.2.5}$$

The latter integral is, however, rather inappropriate for estimation, particularly due to the fact that it involves two limits. Hence, we rather consider the Fourier-type integral for increments. If, for $m \geq 0$ and $n \in \mathbb{N}$ on the right hand side of (2.2.14) we replace $\Phi_{\mathfrak{F}}(\cdot, m)$ by $\Phi_{\mathfrak{F}}(\cdot, m, n)$ in terms of (7.1.9), for any $a, b \in \mathbb{R}$ with $a < b$ and $\mathbb{P}_{\mathfrak{F}_n}(a) = \mathbb{P}_{\mathfrak{F}_n}(b) = 1$ we find

$$\mathfrak{F}_n(b, m) - \mathfrak{F}_n(a, m) = \lim_{T \to \infty} \frac{b-a}{2\pi} \int_{-T}^{T} \Phi_{a,b}(-t)\Phi_{\ddot{Y}}(t, n)\mathcal{H}_{\grave{\varepsilon}}(t, m) dt. \tag{7.2.6}$$

The function $\Phi_{a,b}$ refers to the Fourier transform of the indicator corresponding to the interval $[a, b]$, defined in (A.1.6). It must be pointed out that neither of the integrals (7.2.5) and (7.2.6) converges absolutely, but each of them exists merely as a limit of a sequence. By comparison with (7.2.1) we immediately confirm that the estimator (7.2.6) is unbiased with respect to the increments of the deconvolution function for each admissible pair $a, b \in \mathbb{R}$, since

$$\begin{aligned} \mu_{\mathfrak{F}}(m, b, a) :=& \mathbb{E}\left\{\mathfrak{F}_n(b, m) - \mathfrak{F}_n(a, m)\right\} \\ =& \mathfrak{F}(b, m) - \mathfrak{F}(a, m). \end{aligned} \tag{7.2.7}$$

Regarding the variance we observe real-valuedness of (7.2.6), whence it is irrelevant for the final result whether or not we consider the square of this function in absolute value. Then, if we write the square as a product, for $a, b \in \mathbb{R}$ with $a < b$ and $\mathbb{P}_{\mathfrak{F}_n}(a) = \mathbb{P}_{\mathfrak{F}_n}(b) = 1$ we obtain:

$$\begin{aligned} \sigma^2_{\mathfrak{F}}(m, n, b, a) :=& \operatorname{Var}\left\{\mathfrak{F}_n(b, m) - \mathfrak{F}_n(a, m)\right\} \\ =& \mathbb{E}\left\{\left[(\mathfrak{F}_n(b, m) - \mathfrak{F}_n(a, m)) - (\mathfrak{F}(b, m) - \mathfrak{F}(a, m))\right]^2\right\} \\ =& \mathbb{E}\left\{\left[\frac{b-a}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} \Phi_{a,b}(-t)\mathcal{H}_{\grave{\varepsilon}}(t, m)\left[\Phi_{\ddot{Y}}(t, n) - \Phi_{\ddot{Y}}(t)\right] dt\right]^2\right\} \\ =& \frac{(b-a)^2}{4\pi^2}\mathbb{E}\left\{\lim_{S \to \infty} \int_{-S}^{S} \Phi_{a,b}(-s)\mathcal{H}_{\grave{\varepsilon}}(s, m)\left[\Phi_{\ddot{Y}}(s, n) - \Phi_{\ddot{Y}}(s)\right] ds\right. \\ & \left. \times \lim_{T \to \infty} \int_{-T}^{T} \Phi_{a,b}(-t)\mathcal{H}_{\grave{\varepsilon}}(t, m)\left[\Phi_{\ddot{Y}}(t, n) - \Phi_{\ddot{Y}}(t)\right] dt\right\} \end{aligned}$$

Due to the lack of absolute convergence of the above integrals, special care is required with respect to interchanges in the order of limit and integration. To verify this step, to (7.2.6) we apply the triangle inequality accompanied by the binomial and geometric sum formulae. An additional reference to definition (A.1.6) in common with a reintroduction of the integral definitions of the involved Fourier transforms then yields:

$$\begin{aligned}
|\mathfrak{F}_n(b,m) - \mathfrak{F}_n(a,m)| &\le \frac{1}{2\pi}\sum_{l=0}^{m}\sum_{k=0}^{l}\binom{l}{k}\left|\int_{-S}^{S}\frac{e^{-ibs}-e^{-ias}}{s}\left\{\Phi_{\grave{\varepsilon}}(s)\right\}^k \Phi_{\ddot{Y}}(s,n)ds\right| \\
&\le \frac{1}{2\pi}\sum_{l=0}^{m}\sum_{k=0}^{l}\binom{l}{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\int_{-S}^{S}\frac{e^{-is(b-y-z)}-e^{-is(a-y-z)}}{s}ds\right|\grave{H}^k(dz)G_n(dy) \\
&\le \frac{2}{\pi}\operatorname{Si}(\pi)(2^{m+1}-1) \qquad (7.2.8)
\end{aligned}$$

The final bound results from the well known inequality (B.1.4) for the sine integral. An analogous bound applies for $\mathfrak{F}(b,m) - \mathfrak{F}(a,m)$. According to Lebesgue's dominated convergence theorem, both of the limits appearing in the variance thus can be written in front of the expectation, no matter which order, leading to:

$$\begin{aligned}
\sigma^2_{\mathfrak{F}}(m,n,b,a) &= \frac{(b-a)^2}{4\pi^2}\lim_{S,T\to\infty}\mathbb{E}\left\{\int_{-S}^{S}\Phi_{a,b}(-s)\mathcal{H}_{\grave{\varepsilon}}(s,m)\left[\Phi_{\ddot{Y}}(s,n)-\Phi_{\ddot{Y}}(s)\right]ds\right. \\
&\qquad\qquad \left.\times\int_{-T}^{T}\Phi_{a,b}(-t)\mathcal{H}_{\grave{\varepsilon}}(t,m)\left[\Phi_{\ddot{Y}}(t,n)-\Phi_{\ddot{Y}}(t)\right]dt\right\} \\
&= \frac{(b-a)^2}{4\pi^2}\lim_{S,T\to\infty}\int_{-T}^{T}\Phi_{a,b}(-t)\mathcal{H}_{\grave{\varepsilon}}(t,m)\int_{-S}^{S}\Phi_{a,b}(-s)\mathcal{H}_{\grave{\varepsilon}}(s,m) \qquad (7.2.9)\\
&\qquad\qquad \times\operatorname{Cov}\left\{\Phi_{\ddot{Y}}(s,n),\Phi_{\ddot{Y}}(-t,n)\right\}dsdt
\end{aligned}$$

At this point we remind the reader of the complex conjugate, occuring in the definition of the covariance, due to which from (7.1.3) for $p,q\in\mathbb{C}$ we obtain

$$\operatorname{Cov}\left\{p\Phi_Y(s,n),q\Phi_Y(t,n)\right\} = \frac{p\overline{q}}{n}\left\{\Phi_Y(s-t)-\Phi_Y(s)\Phi_Y(-t)\right\}. \qquad (7.2.10)$$

Consequently, for real-valued $a<b$ with $\mathbb{P}_{\mathfrak{F}_n}(a) = \mathbb{P}_{\mathfrak{F}_n}(b) = 1$ we eventually arrive at

$$\begin{aligned}
\sigma^2_{\mathfrak{F}}(m,n,b,a) &= \frac{(b-a)^2}{n4\pi^2}\lim_{S,T\to\infty}\int_{-T}^{T}\Phi_{a,b}(-t)\mathcal{H}_{\grave{\varepsilon}}(t,m)\int_{-S}^{S}\Phi_{a,b}(-s)\mathcal{H}_{\grave{\varepsilon}}(s,m) \qquad (7.2.11)\\
&\qquad\qquad \times e^{-i\mu_\varepsilon(s+t)}\left\{\Phi_Y(s+t)-\Phi_Y(s)\Phi_Y(t)\right\}dsdt.
\end{aligned}$$

Observe that the parameters m, n are separated from each other. To be exact, while the sample size n appears in the denominator of the prefactor, the double integral depends on m only. Throughout our investigations neither of the above estimators will play a major role, since they are associated with the centered model of errors in variables. Yet, they were included in our presentation to point out the differences in comparison with the estimators below, which will be applicable in the symmetrized model.

7.3. Estimation in the Symmetrized Additive Model of Errors in Variables

We now proceed with the estimation of the deconvolution function in the symmetrized model (2.1.4). For this purpose we denote by $\varepsilon_{21}, \ldots, \varepsilon_{2n} \sim H$ a sequence of n independent random variables, which is especially independent of Y. Then, in the indicated model for each $1 \leq j \leq n$ the j-th date of the Y-sample is transformed to the random variable $\dot{Y}_j := Y_j - \varepsilon_{2j}$. Since, contrary to F and G, the distribution H is known by assumption, the empirical distribution associated with the independent random sequence $\dot{Y}_1, \ldots, \dot{Y}_n$ is not necessarily a step function but is of the following form:

$$\begin{aligned} \dot{G}_n(\xi) &:= \int_{-\infty}^{\infty} G_n(\xi + z_2) H(dz_2) \\ &= \frac{1}{n} \sum_{j=1}^{n} \{1 - H((Y_j - \xi)-)\} \end{aligned} \tag{7.3.1}$$

In fact, $\dot{G}_n$ is continuous if and only if H is. Regarding the deconvolution function, a unilateral plug-in estimator is obtained from (2.1.6) by

$$\mathfrak{D}_n(\xi, m) := \frac{1}{n} \sum_{j=1}^{n} \int_{-\infty}^{\infty} \mathcal{S}_{\bar{\varepsilon}}^{m}(\xi - (Y_j - z_2)) H(dz_2), \tag{7.3.2}$$

referred to as the *empirical deconvolution function in the symmetrized model.* Due to the random character of the Y-sample, the above estimator is composed by a sum of independent and identically distributed random variables. Hence, the corresponding expectation equals

$$\mathbb{E}\left\{\mathfrak{D}_n(\xi, m)\right\} = \mathfrak{D}(\xi, m), \tag{7.3.3}$$

whereas for the variance, again subject to Bienaymé's identity, we compute

$$\operatorname{Var}\left\{\mathfrak{D}_n(\xi, m)\right\} = \frac{1}{n} \mathbb{E}\left\{\left[\mathcal{S}_{\bar{\varepsilon}}^{m}(\xi - \dot{Y})\right]^2\right\} - \frac{1}{n}\left\{\mathfrak{D}(\xi, m)\right\}^2. \tag{7.3.4}$$

From (7.3.3) we ascertain that also the plug-in estimator (7.3.2) is unbiased with respect to $\mathfrak{D}(\cdot, m)$. To abandon the complicated structure of the deconvolution sum according to (2.1.27), the two subsequent paragraphs are dedicated to the derivation of a convenient Fourier-type representation for the empirical deconvolution function and even for its possibly existing density.

7.3.1. The Empirical Deconvolution Function for Increments

First of all, for $\xi \in \mathbb{R}$, we define by

$$\mathbb{P}_{\mathfrak{D}_n}(\xi) := \mathbb{P}\left(H * \bar{H}^{*j}\{\xi - Y\} = 0 \text{ for all } j \in \mathbb{N}_0\right) \tag{7.3.5}$$

the probability for ξ to be a point of continuity of the empirical deconvolution function $\mathfrak{D}_n(\xi, m)$ for all $m \geq 0$. Note, $\mathbb{P}_{\mathfrak{D}_n}(\xi) = 1$ for all $\xi \in \mathbb{R}$ if F or H is continuous. Thereby, for $m \geq 0$ and $n \in \mathbb{N}$, analogous to (2.1.51) one verifies that the increment of $\mathfrak{D}_n(\cdot, m)$ for $a, b \in \mathbb{R}$ with $a < b$ and $\mathbb{P}_{\mathfrak{D}_n}(a) = \mathbb{P}_{\mathfrak{D}_n}(b) = 1$ can be represented by

$$\mathfrak{D}_n(b, m) - \mathfrak{D}_n(a, m) = \lim_{T \to \infty} \frac{b-a}{2\pi} \int_{-T}^{T} \Phi_{a,b}(-t)\Phi_{\dot{Y}}(t, n)\mathcal{G}_{\bar{\varepsilon}}(t, m)dt. \tag{7.3.6}$$

Again for the definition of $\Phi_{a,b}$ we refer to (A.1.6). The alert reader will immediately notice that, contrary to each estimator in the centered model, the integral on the right hand side certainly converges absolutely and with respect to $T > 0$ uniformly if

$$\Phi_{a,b}\Phi_{\varepsilon} \in L^1(\mathbb{R}). \tag{7.3.7}$$

This statement is especially true if the error distribution has a characteristic function with a compact support, i.e., if $\mathbb{R} \setminus N_{\varepsilon}$ is finite. In any case the above increment estimator is unbiased with respect to the target function, i.e., for all $a, b \in \mathbb{R}$ with $a < b$ and $\mathbb{P}_{\mathfrak{D}_n}(a) = \mathbb{P}_{\mathfrak{D}_n}(b) = 1$ it satisfies

$$\begin{aligned} \mu_{\mathfrak{D}}(m, b, a) :=& \,\mathbb{E}\left\{\mathfrak{D}_n(b, m) - \mathfrak{D}_n(a, m)\right\} \\ =& \,\mathfrak{D}(b, m) - \mathfrak{D}(a, m). \end{aligned} \tag{7.3.8}$$

Furthermore, by arguments similar to the derivation of the variance (7.2.11) with the aid of (7.2.10), for all $a, b \in \mathbb{R}$ with $a < b$ and $\mathbb{P}_{\mathfrak{D}_n}(a) = \mathbb{P}_{\mathfrak{D}_n}(b) = 1$ we readily confirm:

$$\begin{aligned} \sigma^2_{\mathfrak{D}}(m, n, b, a) :=& \operatorname{Var}\left\{\mathfrak{D}_n(b, m) - \mathfrak{D}_n(a, m)\right\} \\ =& \frac{(b-a)^2}{4\pi^2} \lim_{S,T \to \infty} \int_{-T}^{T} \Phi_{a,b}(-t)\mathcal{G}_{\bar{\varepsilon}}(t, m) \int_{-S}^{S} \Phi_{a,b}(-s)\mathcal{G}_{\bar{\varepsilon}}(s, m) \\ & \qquad \times \operatorname{Cov}\left\{\Phi_{\dot{Y}}(s, n), \Phi_{\dot{Y}}(-t, n)\right\} dsdt \end{aligned}$$

$$= \frac{(b-a)^2}{n4\pi^2} \lim_{S,T\to\infty} \int_{-T}^{T} \Phi_{a,b}(-t)\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_{\varepsilon}(-t) \int_{-S}^{S} \Phi_{a,b}(-s)\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_{\varepsilon}(-s)$$
$$\times \left\{\Phi_Y(s+t) - \Phi_Y(s)\Phi_Y(t)\right\} dsdt$$

If we define the squared expectation of the increment of the empirical deconvolution function for $n = 1$ by

$$\mathrm{M}_{\mathfrak{D}}(m,b,a) := 4\pi^2\mathbb{E}\left\{(\mathfrak{D}_1(b,m) - \mathfrak{D}_1(a,m))^2\right\}, \tag{7.3.9}$$

it is easy to verify for each $a, b \in \mathbb{R}$ with $a < b$ and $\mathbb{P}_{\mathfrak{D}_n}(a) = \mathbb{P}_{\mathfrak{D}_n}(b) = 1$ the Fourier-type integral representation

$$\begin{aligned}\mathrm{M}_{\mathfrak{D}}(m,b,a) := (b-a)^2 \lim_{S,T\to\infty} \int_{-T}^{T} \Phi_{a,b}(-t)\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_{\varepsilon}(-t) \\ \times \int_{-S}^{S} \Phi_{a,b}(-s)\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_{\varepsilon}(-s)\Phi_Y(s+t)dsdt.\end{aligned} \tag{7.3.10}$$

Hence, upon separating the integral appearing in the variance above, by additional comparison with (2.1.51) for $a, b \in C_{\mathfrak{D}}$ with $a < b$ and $\mathbb{P}_{\mathfrak{D}_n}(a) = \mathbb{P}_{\mathfrak{D}_n}(b) = 1$ we arrive at

$$\sigma^2_{\mathfrak{D}}(m,n,b,a) = \frac{1}{n4\pi^2}\mathrm{M}_{\mathfrak{D}}(m,b,a) - \frac{1}{n}\left\{\mu_{\mathfrak{D}}(m,b,a)\right\}^2. \tag{7.3.11}$$

Obviously the parameters m and n again occur in different factors. We refer to (7.3.10) as an *iterated integral of convolution-type*, since its two single components are connected solely through the additive argument of the function $\Phi_Y(s+t)$. The latter is substantially more complicated than the integral which we computed for the variance in the centered model. The reason is the presence of the additional factor $\Phi_{\varepsilon}(-s)$ corresponding to $e^{-i\mu_{\varepsilon}s}$ in the other model. Both functions are generally of completely different type. Finally, if we prefer a smoothed version of the above principal value integral (7.3.10), we can repeatedly apply Abel's lemma A.4.1(2), bearing in mind that $e^{-\delta|t|}$ represents the Fourier transform of a Cauchy distribution. Accordingly, for any $a, b \in \mathbb{R}$ with $a < b$ and $\mathbb{P}_{\mathfrak{D}_n}(a) = \mathbb{P}_{\mathfrak{D}_n}(b) = 1$ we can write

$$\begin{aligned}\mathrm{M}_{\mathfrak{D}}(m,b,a) = (b-a)^2 \lim_{\delta_1,\delta_2\downarrow 0} \int_{-\infty}^{\infty} e^{-\delta_2|t|}\Phi_{a,b}(-t)\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_{\varepsilon}(-t) \\ \times \int_{-\infty}^{\infty} e^{-\delta_1|s|}\Phi_{a,b}(-s)\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_{\varepsilon}(-s)\Phi_Y(s+t)dsdt,\end{aligned} \tag{7.3.12}$$

where the order of the limits is arbitrary. Compared with (7.3.10) the integral representation (7.3.12) will prove more useful to study the effects of oscillations of the integrand on the large-m

behaviour of the whole integral.

7.3.2. The Empirical Deconvolution Density

In Section 2.1.4 we already verified, that a sufficient condition for the existence of the deconvolution density (2.1.63) is the existence of a density h with $H(dz) = h(z)dz$. In this event, also the estimator for the deconvolution function (7.3.2) possesses a density, to which we refer as the *empirical deconvolution density*, denoted by

$$\mathfrak{d}_n(\xi, m) := \frac{1}{n} \sum_{j=1}^{n} \int_{-\infty}^{\infty} h(-\xi + Y_j + z) \mathcal{S}_{\bar{\varepsilon}}^{m}(dz), \qquad \xi \in \mathbb{R}. \tag{7.3.13}$$

It matches definition (2.1.63), except for the integral being computed with respect to G_n instead of G. Accordingly, it is easy to see that this function for all $\xi \in \mathbb{R}$ indeed satisfies

$$\mathfrak{D}_n(\xi, m) = \int_{-\infty}^{\xi} \mathfrak{d}_n(x, m) dx, \tag{7.3.14}$$

and that it is even unbiased with respect to the target function $\mathfrak{d}(\xi, m)$ for any $\xi \in \mathbb{R}$, since

$$\mathbb{E}\left\{\mathfrak{d}_n(\xi, m)\right\} = \mathfrak{d}(\xi, m). \tag{7.3.15}$$

This observation becomes particularly interesting if X has a density f. In these circumstances it enables us to construct an estimator for the unknown X-density if only a data set of Y-observations is available. If in addition we required continuity of h along the whole real axis, we were able to establish with (2.1.66) for $\mathfrak{d}(\xi, m)$ an integral representation of Fourier-type. Replacing therein Φ_Y by its empirical analogue, for any $\xi \in \mathbb{R}$ gives rise to

$$\mathfrak{d}_n(\xi, m) = \frac{1}{2\pi} \lim_{\delta \downarrow 0} \int_{-\infty}^{\infty} e^{-it\xi} \Phi_I(\delta t) \Phi_Y(t, n) \Phi_\varepsilon(-t) \mathcal{G}_{\bar{\varepsilon}}(t, m) dt, \tag{7.3.16}$$

where the smoothing function Φ_I satisfies the conditions of Theorem A.1.3. Observe that the above integral converges absolutely and uniformly with respect to $\delta \geq 0$ if $\Phi_\varepsilon \in L^1(\mathbb{R})$, in which event the limit may be carried out under the integral sign, for $\xi \in \mathbb{R}$ yielding

$$\mathfrak{d}_n(\xi, m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\xi} \Phi_Y(t, n) \Phi_\varepsilon(-t) \mathcal{G}_{\bar{\varepsilon}}(t, m) dt. \tag{7.3.17}$$

Concerning the variance of $\mathfrak{d}_n(\xi, m)$ at some point $\xi \in \mathbb{R}$, with the aid of (2.1.65) an interchange in the order of limit and integration can be justified, leading to:

$$\sigma_{\mathfrak{d}}^2(m, n, \xi) := \operatorname{Var}\left\{\mathfrak{d}_n(\xi, m)\right\}$$

$$=\frac{1}{4\pi^2}\mathbb{E}\left\{\left[\lim_{\delta\downarrow 0}\int_{-\infty}^{\infty} e^{-it\xi}\Phi_I(\delta t)\Phi_\varepsilon(-t)\mathcal{G}_{\bar{\varepsilon}}(t,m)\left\{\Phi_Y(t,n)-\Phi_Y(t)\right\}dt\right]^2\right\}$$

$$=\frac{1}{4\pi^2}\lim_{\delta_1,\delta_2\downarrow 0}\mathbb{E}\Bigg\{\int_{-\infty}^{\infty} e^{-it\xi}\Phi_I(\delta_2 t)\Phi_\varepsilon(-t)\mathcal{G}_{\bar{\varepsilon}}(t,m)\left\{\Phi_Y(t,n)-\Phi_Y(t)\right\}dt$$
$$\times\int_{-\infty}^{\infty} e^{-is\xi}\Phi_I(\delta_1 s)\Phi_\varepsilon(-s)\mathcal{G}_{\bar{\varepsilon}}(s,m)\left\{\Phi_Y(s,n)-\Phi_Y(s)\right\}ds\Bigg\}$$

$$=\frac{1}{4\pi^2}\lim_{\delta_1,\delta_2\downarrow 0}\int_{-\infty}^{\infty} e^{-it\xi}\Phi_I(\delta_2 t)\Phi_\varepsilon(-t)\mathcal{G}_{\bar{\varepsilon}}(t,m)\int_{-\infty}^{\infty} e^{-is\xi}\Phi_I(\delta_1 s)\Phi_\varepsilon(-s)\mathcal{G}_{\bar{\varepsilon}}(s,m)$$
$$\times\operatorname{Cov}\left\{\Phi_Y(s,n),\Phi_Y(-t,n)\right\}dsdt \tag{7.3.18}$$

For $\xi\in\mathbb{R}$ and $m\geq 0$ we denote the squared expectation of the empirical deconvolution density for $n=1$ by

$$\mathrm{M}_{\mathfrak{d}}(m,\xi):=4\pi^2\mathbb{E}\left\{(\mathfrak{d}_1(\xi,m))^2\right\}$$
$$=\lim_{\delta_1,\delta_2\downarrow 0}\int_{-\infty}^{\infty} e^{-it\xi}\Phi_I(\delta_2 t)\Phi_\varepsilon(-t)\mathcal{G}_{\bar{\varepsilon}}(t,m)$$
$$\times\int_{-\infty}^{\infty} e^{-is\xi}\Phi_I(\delta_1 s)\Phi_\varepsilon(-s)\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_Y(s+t)dsdt, \tag{7.3.19}$$

again with an arbitrary order of the limits. Then, if we finally apply the identity (7.1.3) to (7.3.18) and separate the double integral, for any $\xi\in\mathbb{R}$ we arrive at

$$\sigma^2_{\mathfrak{d}}(m,n,\xi)=\frac{1}{n4\pi^2}\mathrm{M}_{\mathfrak{d}}(m,\xi)-\frac{1}{n}\left\{\mathfrak{d}(\xi,m)\right\}^2. \tag{7.3.20}$$

7.4. Nature of Integrals Involving Geometric Sum Functions

Each of the integrals encountered in the context of the deconvolution function and its estimators exhibits a fairly distinguishing form, due to the occurrence of the geometric sum functions $\mathcal{H}_{\grave{\varepsilon}}(\cdot,m)$ and $\mathcal{G}_{\bar{\varepsilon}}(\cdot,m)$. If we confine ourselves to the symmetrized model, these integrals can be classified as special cases of the integral

$$J(m):=\int_{\mathcal{P}}\mathcal{G}_{\bar{\varepsilon}}(t,m)a(t)dt, \tag{7.4.1}$$

taken along a closed interval $\mathcal{P}\subset\mathbb{R}\cup\{\pm\infty\}$ for a continuous function $a(t)$. Ultimately the variance is an iterated version of $J(m)$. It is therefore natural to start with a study of single integrals of the above type before continuing with their two-dimensional analogues. Without loss of generality we suppose $\mathcal{P}\cap N_\varepsilon\neq\emptyset$. Moreover, we remind the reader of equation (2.1.10), according to which

$$\mathcal{G}_{\bar{\varepsilon}}(t,m) = \sum_{l=0}^{m} (1 - \Phi_{\bar{\varepsilon}}(t))^l, \qquad t \in \mathbb{R}.$$

A first inspection of $J(m)$ suggests that the integrand, whenever being non-zero at some point $t \in \mathcal{P}$, contrary to any of the bias integrals, as $m \to \infty$ tends to a non-zero function. More precisely, with respect to $t \in \mathcal{P}$ the limit is either uniformly bounded or unbounded. We therefore expect the sequence of integrals $J(m)$ to approach a finite non-zero value or to exceed any limits as $m \to \infty$. The particular result depends on the involved functions and major contributions come from the points where $\Phi_{\bar{\varepsilon}}(t)$ vanishes, i.e., from $\mathcal{P} \cap N_\varepsilon$. It must, however, be remarked that, even if the integrand of $J(m)$ becomes unbounded as $m \to \infty$ the sequence of integrals can still converge to a finite limit, due to possible cancellations in a neighborhood of some single points $t_0 \in N_\varepsilon$. This phenomenon is discussed in Appendix A.4 and may occur, for instance, similar to principal value integrals or due to oscillatory behaviour.

The importance of the points $\mathcal{P} \cap N_\varepsilon$ immediately follows from the definition of $\mathcal{G}_{\bar{\varepsilon}}(t,m)$. Accordingly,

$$1 \le \mathcal{G}_{\bar{\varepsilon}}(t,m) \le m+1$$

uniformly with respect to $t \in \mathbb{R}$ and $\mathcal{G}_{\bar{\varepsilon}}(t_0,m) = m+1$ for any $t_0 \in N_\varepsilon$. As a consequence, if $a(t)$ is absolutely integrable along $\mathcal{P}$, and if $\mathcal{P} \cap N_\varepsilon$ is a subinterval of the real axis along which $a(t)$ is non-zero, we immediately derive $J(m) \sim \text{const} \times m$ as $m \to \infty$. In each other case we can only estimate $J(m) = \mathcal{O}(m)$, making further investigations inevitable. In particular, although $\mathcal{G}_{\bar{\varepsilon}}(t_0,m) = m+1$ for any $t_0 \in N_\varepsilon$ regardless of the structure of $\Phi_{\bar{\varepsilon}}(t)$, the behaviour as $t \to t_0$ indeed depends on the structure. In order to establish exact statements we can employ our earlier findings. First, upon displacing the integration path to the left across the pole at $z = 0$, for $t \in \mathbb{R} \setminus N_\varepsilon$ we deduce from (4.3.19) the integral representation

$$1 - \mathcal{P}_{\bar{\varepsilon}}(t,m) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(m+2)\Gamma(z)}{\Gamma(m+2+z)} \{\Phi_{\bar{\varepsilon}}(t)\}^{-z} \, dz, \qquad -1 < c < 0. \tag{7.4.2}$$

Hence, in view of equation (2.1.11), again for $t \in \mathbb{R} \setminus N_\varepsilon$ and $-1 < c < 0$ we can write

$$\mathcal{G}_{\bar{\varepsilon}}(t,m) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(m+2)\Gamma(z)}{\Gamma(m+2+z)} \{\Phi_{\bar{\varepsilon}}(t)\}^{-z-1} \, dz. \tag{7.4.3}$$

Observe that the right hand side becomes unbounded as $t \to t_0$ in $\mathbb{R} \setminus N_\varepsilon$ with a rate of divergence that depends on the local behaviour of $\Phi_{\bar{\varepsilon}}(t)$. Contour integrals of the above type and their application in asymptotics have been discussed in Chapter 6. Assume $\mathcal{P}$ denotes a half open interval and $t_0 \in N_\varepsilon$ such that $\mathcal{P} \cup \{t_0\}$ is closed and $\Phi_{\bar{\varepsilon}}(t) \neq 0$ for each $t \in \mathcal{P}$. If we then apply

(7.4.3) to $J(m)$ accompanied by a formal interchange in the order of integration, we arrive at

$$J(m) = -\frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \frac{\Gamma(m+2)\Gamma(z)}{\Gamma(m+2+z)} \int\limits_{\mathcal{P}} \{\Phi_{\bar{\varepsilon}}(t)\}^{-z-1} a(t) dt dz. \tag{7.4.4}$$

From Chapter 6 we know that the values $-1 < c < 0$ for which the interchange is actually permitted, essentially depend on the behaviour of $a(t)$ as $t \to t_0$. Moreover, we pointed out there, that the magnitude of the parameter c indicates the large-m behaviour of $J(m)$ through the beta function in the MB-integral. Finally, we also presented the method of analytic continuation as an efficient technique to evaluate integrals of a type similar to (7.4.4). We confined particularly to integrals of Laplace-type that arise from $J(m)$ if $a(t)$ features the factor $\Phi_{\bar{\varepsilon}}(t)$, which then cancels with the denominator of $\mathcal{G}_{\bar{\varepsilon}}(t,m)$. It is therefore clearly a matter of an appropriate modification only, to apply the method of analytic continuation to the integral (7.4.4). Accordingly, to expose the controlling asymptotic behaviour of this integral, we must first determine the analyticity properties of the generating function in some z-region, and then perform appropriate manipulations of the path of the MB-integral.

In the symmetrized model of errors in variables the geometric sum function frequently occurs as a product with the characteristic function $\Phi_\varepsilon(-t)$. The latter downweights the negative property of the former in a neighborhood of any $t_0 \in N_\varepsilon$, since $\Phi_\varepsilon(-t) \to 0$ as $t \to t_0$. To be exact, since $\mathcal{G}_{\bar{\varepsilon}}(t_0, m) = m+1$ for each $t_0 \in N_\varepsilon$ and fixed $m \geq 0$ is a finite constant value, the product $\mathcal{G}_{\bar{\varepsilon}}(t_0, m)\Phi_\varepsilon(-t_0)$ necessarily vanishes. The downweighting of $\Phi_\varepsilon(-t)$ is, however, limited and the product is not uniformly bounded[1] with respect to $m \geq 0$ as $t \to t_0$ for $t_0 \in N_\varepsilon$. This behaviour becomes obvious by applying the triangle inequality accompanied by the estimate (B.3.7) to the binomial integral representation (7.4.3), which yields for $t \in \mathbb{R} \setminus N_\varepsilon$, $-1 < x_0 < 0$ and $m \geq 0$:

$$\begin{aligned} |\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_\varepsilon(-t)| &\leq \{\Phi_{\bar{\varepsilon}}(t)\}^{-x_0-\frac{1}{2}} \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\Gamma(m+2)\,|\Gamma(x_0+iy)|}{|\Gamma(m+2+x_0+iy)|} dy \\ &\leq \frac{\Gamma(m+2)\Gamma(x_0+2)}{\Gamma(m+2+x_0)} \{\Phi_{\bar{\varepsilon}}(t)\}^{-x_0-\frac{1}{2}} \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \left\{(x_0+1)^2+y^2\right\}^{-\frac{1}{2}} \left\{x_0^2+y^2\right\}^{-\frac{1}{2}} dy \end{aligned} \tag{7.4.5}$$

At this point we recall $|\Phi_\varepsilon(t)|^2 = \Phi_{\bar{\varepsilon}}(t)$. If and only if we choose $x_0 = -\frac{1}{2}$ the above bound becomes independent of t, leading to

$$|\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_\varepsilon(-t)| \leq \frac{\Gamma(m+2)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)} \frac{4}{\pi} \int\limits_0^\infty \frac{1}{1+4y^2} dy = \frac{\sqrt{\pi}}{2} \frac{\Gamma(m+2)}{\Gamma\left(m+\frac{3}{2}\right)}.$$

[1] An example in terms of simpler functions is furnished by $f(t) := \frac{1-e^{-mt^2}}{t}$, which equals zero at $t=0$ but is not uniformly bounded with respect to $m \geq 0$ as $t \to 0$.

For the last equality the integral was referred to the arctangent function with $\arctan(t) \to \frac{\pi}{2}$ as $t \to \infty$, and we incorporated $\Gamma(\frac{3}{2}) = \sqrt{\pi}$. Summarizing, since $\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_\varepsilon(-t) = 0$ for $t \in N_\varepsilon$, we have just established the inequality

$$\sup_{t\in\mathbb{R}} |\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_\varepsilon(-t)| \leq \frac{\sqrt{\pi}}{2}\frac{\Gamma(m+2)}{\Gamma\left(m+\frac{3}{2}\right)}. \tag{7.4.6}$$

According to Stirling's formula, the right hand side is $\sim \text{const} \times m^{\frac{1}{2}}$ as $m \to \infty$ and thus clearly exhibits a slower growth than solely $\mathcal{G}_{\bar{\varepsilon}}(t,m)$ for any $t \in N_\varepsilon$.

In Chapter 6 we introduced an alternative to the binomial integral in the shape of the Cahen-Mellin representation (6.1.5). A leftward displacement of the path in the indicated contour integral across the pole at $z = 0$ gives rise to

$$1 - \mathcal{P}_{\bar{\varepsilon}}(t,m) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(z)\left\{-(m+1)\log(1-\Phi_{\bar{\varepsilon}}(t))\right\}^{-z} dz, \qquad -1 < c < 0. \tag{7.4.7}$$

This integral can be employed to characterize the asymptotic behaviour of $J(m)$ in powers of m. Furthermore, in some circumstances neither the binomial nor the Cahen-Mellin representation may be applicable to (7.4.1), especially if the decay of $\Phi_{\bar{\varepsilon}}(t)$ as $t \to t_0$ happens too fast. To make a statement about the large m-behaviour of $J(m)$ is then slightly more complicated. For $t \in \mathbb{R}$ it can be helpful to note

$$\mathcal{G}_{\bar{\varepsilon}}(t,m) = (m+1)\int_0^1 (1-u\Phi_{\bar{\varepsilon}}(t))^m du. \tag{7.4.8}$$

If, for $0 < \delta < m+1$ to the right hand side we apply the inverse Mellin transform (4.7.18), we obtain

$$\mathcal{G}_{\bar{\varepsilon}}(t,m) = \frac{m+1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \mathcal{M}_{\mathrm{B}}(z,m)\int_0^1 \left\{-\log u - \log\Phi_{\bar{\varepsilon}}(t)\right\}^z du dz.$$

After a simple change of variables the du-integral is readily identified as an upper incomplete gamma function, which yields

$$\mathcal{G}_{\bar{\varepsilon}}(t,m) = \frac{1}{\Phi_{\bar{\varepsilon}}(t)}\frac{m+1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \mathcal{M}_{\mathrm{B}}(z,m)\Gamma(1+z,-\log\Phi_{\bar{\varepsilon}}(t))dz. \tag{7.4.9}$$

From equation (8.11.2) in [Olver et al., 2010] as $\Phi_{\bar{\varepsilon}}(t) \to 0$ we know

$$\frac{1}{\Phi_{\bar{\varepsilon}}(t)}\Gamma(1+z,-\log\Phi_{\bar{\varepsilon}}(t)) \sim \left\{-\log\Phi_{\bar{\varepsilon}}(t)\right\}^z.$$

Hence, compared with (7.4.3) and (7.4.7) we observe that the integrand of (7.4.9) as a function of t exhibits a slower growth as $t \to t_0$ for each $t_0 \in N_\varepsilon$. In addition, in Section 4.7.3 we have shown that, in the strip $0 < \Re z < m+1$ the function $\mathcal{M}_{\mathrm{B}}(z,m)$ is expandable in descending powers of the digamma function, whose order increases for greater values of $\Re z > 0$. Their logarithmic decay as $m \to \infty$, however, is always dominated by the growth of the factor $m+1$ in front of the integral (7.4.9). An asymptotic expansion of $J(m)$ that is derived with the aid of (7.4.9) is therefore of logarithmic type with an algebraically divergent leading term as $m \to \infty$.

In accordance with the above observations, we identify the convolution-type double integrals (7.2.11), (7.3.12) and (7.3.19), which appear in the variances of the deconvolution estimators, as iterations of two integrals of the type (7.4.1). Additional complications, however, arise in fact due to their iterated structure. These will be pointed out in Chapter 8 below, where we will also present definite asymptotic statements.

7.5. Properties of the Increment Estimator

Preliminary to our study of the variance we begin with a short discussion of the increment estimator (7.3.6). For $y \in \mathbb{R}$ we denote

$$\mathfrak{X}_T(y,b,a,m) := \int\limits_{[-T,T]\setminus N_\varepsilon} \{1 - \mathcal{P}_{\bar{\varepsilon}}(t,m)\}\, e^{iyt} \frac{\Phi_{a,b}(-t)}{\Phi_\varepsilon(t)} dt, \tag{7.5.1}$$

and without actually requiring absolute convergence we write

$$\mathfrak{X}_\infty(y,b,a,m) := \lim_{T\to\infty} \mathfrak{X}_T(y,b,a,m) = \int\limits_{\mathbb{R}\setminus N_\varepsilon} \{1 - \mathcal{P}_{\bar{\varepsilon}}(t,m)\}\, e^{iyt} \frac{\Phi_{a,b}(-t)}{\Phi_\varepsilon(t)} dt. \tag{7.5.2}$$

Then, if we apply the definitions (2.1.14), (7.1.1) and (7.1.6) to the empirical deconvolution function (7.3.6), for real-valued $a < b$ with $\mathbb{P}_{\mathfrak{D}_n}(a) = \mathbb{P}_{\mathfrak{D}_n}(b) = 1$ and $m \geq 0$ we obtain

$$\mathfrak{D}_n(b,m) - \mathfrak{D}_n(a,m) = \frac{b-a}{n2\pi} \sum_{k=1}^{n} \mathfrak{X}_\infty(Y_k,b,a,m). \tag{7.5.3}$$

It appeared useful to introduce the sum representation of $\Phi_Y(\cdot,n)$ and to interchange the order of summation and integration. Apart from the presence of possible zeros we do not benefit from the closed form of $\Phi_Y(\cdot,n)$, especially since it is non-vanishing at infinity. The zeros, however, depend on the random Y-observations, whence their contribution to compensate possible zeros of the denominator is not reliable. In addition, in the shape of (7.5.3) the estimator is easier to compute. By (7.3.7) the integral (7.5.2) converges absolutely for all finite $m \geq 0$ if $\Phi_\varepsilon \Phi_{a,b} \in L^1(\mathbb{R})$, in which case H must be continuous and thus $\mathbb{P}_{\mathfrak{D}_n}(\xi) = 1$ for all $\xi \in \mathbb{R}$. For brevity we define

$$q(t, y, b, a) := e^{iyt} \frac{\Phi_{a,b}(-t)}{\Phi_\varepsilon(t)} + e^{-iyt} \frac{\Phi_{a,b}(t)}{\Phi_\varepsilon(-t)}, \qquad t \geq 0,\ y \in \mathbb{R}, \tag{7.5.4}$$

which equivalently equals twice the real part of one addend. The arguments y, a, b obviously specify the oscillatory behaviour of $q(t, y, b, a)$. Regarding the properties of this function, either of the following three statements applies:

(1) $N_\varepsilon \cap \mathbb{R} = \emptyset$ and $q(t, y, b, a)$ is bounded at infinity. The function is then uniformly bounded with respect to $t \geq 0$ for any $y, a, b \in \mathbb{R}$, and if it vanishes at infinity, this happens slower than t^{-1}.

(2) $N_\varepsilon \cap \mathbb{R} \neq \emptyset$ and $q(t, y, b, a)$ is bounded at infinity. Then $q(t, y, b, a)$ may still be uniformly bounded with respect to $t \geq 0$ due to cancellations, however, possibly only for some $y, a, b \in \mathbb{R}$. A reason for cancellations can be, for instance, the sine function appearing as a factor in $\Phi_{a,b}$.

(3) $q(t, y, b, a)$ diverges as $t \to \infty$.

As a consequence, it seems natural to conclude that particularly in the last case the sequence of integrals (7.5.3) certainly diverges as $m \to \infty$. The discussion of Appendix A.4, however, warns us against the defectiveness of this conclusion, which shall be confirmed in the next paragraph.

7.5.1. Unbiasedness of the Limit

Throughout this paragraph the sample size $n \in \mathbb{N}$ is assumed to be fixed. We are then interested in the convergence behaviour of the empirical deconvolution function as $m \to \infty$ and especially in a sufficient condition for the validity of

$$\mathbb{E}\left\{\mathfrak{D}_n(b, \infty) - \mathfrak{D}_n(a, \infty)\right\} = F(b) - F(a), \tag{7.5.5}$$

where for brevity we write

$$\mathfrak{D}_n(b, \infty) - \mathfrak{D}_n(a, \infty) := \lim_{m \to \infty} \left\{\mathfrak{D}_n(b, m) - \mathfrak{D}_n(a, m)\right\}. \tag{7.5.6}$$

In other words, we want to know if the possibly existing limit establishes an unbiased estimator with respect to the target distribution F. This leads us to a statement of which the first part actually has nothing to do with the deconvolution function.

Theorem 7.5.1 (convergence of the increment estimator). *For $y, a, b \in \mathbb{R}$ with $a \neq b$ we denote the antiderivative of $q(t, y, b, a)$ on $t \geq 0$ by*

$$Q_T(y, b, a) := \int_0^T q(t, y, b, a)\, dt, \tag{7.5.7}$$

assuming finiteness for any $T > 0$.

(1) Suppose $\lim_{T\to\infty} Q_T(Y, b, a)$ exists G-almost surely and

$$\mathbb{E}\left\{\sup_{\tau\geq 0} |Q_\tau(Y, b, a)|\right\} < \infty. \tag{7.5.8}$$

In these circumstances the limits on the right hand side of

$$\hat{F}_n(b) - \hat{F}_n(a) := \frac{b-a}{n2\pi}\sum_{k=1}^{n} \lim_{T\to\infty} Q_T(Y_k, b, a) \tag{7.5.9}$$

exist G-almost surely and exhibit unbiasedness with respect to $F(b) - F(a)$ for any pair of continuity points $a < b$ of F.

(2) If, in addition to (1), $N_\varepsilon = \{\pm\infty\}$ and the m-power $\mathcal{P}_{\bar{\varepsilon}}(\cdot, m)$ satisfies one of the conditions in (2.4.19), then G-almost surely

$$\mathfrak{D}_n(b, \infty) - \mathfrak{D}_n(a, \infty) = \hat{F}_n(b) - \hat{F}_n(a). \tag{7.5.10}$$

The existence of the limit of the sequence of integrals $Q_T(y, b, a)$ as $T \to \infty$ essentially depends on the oscillations of its integrand.

Proof. The first statement is basically an application of Lebesgue's dominated convergence theorem, compare Theorem 3.31 in [Axler, 2019]. Since $|Q_T(y, b, a)| \leq \sup_{\tau\geq 0} |Q_\tau(y, b, a)|$ for any $y \in \mathbb{R}$ and $T \geq 0$, the validity of (7.5.8) gives a G-integrable bound, whereas the G-almost sure existence of the limit of $Q_T(Y, b, a)$ as $T \to \infty$ holds by assumption. Hence, the following interchange in the order of limit and expectation is permissible:

$$\begin{aligned}
\mathbb{E}\left\{\hat{F}_n(b) - \hat{F}_n(a)\right\} &= \frac{b-a}{n2\pi}\lim_{T\to\infty}\sum_{k=1}^{n}\mathbb{E}\left\{Q_T(Y_k, b, a)\right\} \\
&= \frac{b-a}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}\Phi_{a,b}(-s)\Phi_X(s)ds
\end{aligned}$$

In view of the inversion formula (A.7.27) we eventually conclude unbiasedness with respect to F. To verify statement (2) of Theorem 7.5.1, we rearrange the integral (7.5.2), which for each $1 \leq k \leq n$ and fixed $m \geq 0$ yields

$$\mathfrak{X}_\infty(Y_k, b, a, m) = \int_0^\infty \left\{1 - \mathcal{P}_{\bar{\varepsilon}}(t, m)\right\} q(t, Y_k, b, a)dt,$$

with the integral on the right hand side being not necessarily absolutely convergent. Under the current assumptions the m-power clearly satisfies the properties of the function $K_\lambda(t)$ in Theorem A.4.1. Moreover, $Q_t(Y, b, a)$ is finite for any $t \geq 0$ and the limit as $t \to \infty$ exists

G-almost surely. The G-almost certain statement (7.5.10) thus follows at once from Theorem A.4.1(2). ■

We close this section with an example for the applicability of the preceding theorem.

Example 7.5.1 (geometric stable errors). Assume the X-distribution is unspecified whereas $\Phi_\varepsilon(t) = (1+|t|^\beta)^{-1}$ for $0 < \beta < 1$, i.e., we find ourselves in a setup with geometrically stable distributed errors. Then, G is an absolutely continuous distribution and the function $q(t,y,b,a)$ due to (A.1.6) can be cast as follows:

$$\begin{aligned} q(t,y,b,a) &= 2(1+|t|^\beta)\Re\left\{e^{iyt}\Phi_{a,b}(-t)\right\} \\ &= \frac{2(1+|t|^\beta)}{b-a}\left[\frac{\sin\{(b-y)t\}}{t} - \frac{\sin\{(a-y)t\}}{t}\right] \end{aligned}$$

In terms of the sine integral (B.1.2) and by straightforward manipulations we thus obtain

$$\begin{aligned} \frac{b-a}{2}Q_\tau(y,b,a) = \operatorname{Si}((b-y)\tau) - \operatorname{Si}((a-y)\tau) &+ \operatorname{sgn}(b-y)\,|b-y|^{-\beta}\int_0^{\tau|b-y|} t^{\beta-1}\sin(t)dt \\ &- \operatorname{sgn}(a-y)\,|a-y|^{-\beta}\int_0^{\tau|a-y|} t^{\beta-1}\sin(t)dt. \end{aligned}$$

Clearly, the first two summands on the right hand side are uniformly bounded with respect to $\tau \geq 0$ and also with respect to $y \in \mathbb{R}$. Furthermore, the integrals in the last two summands exist for any $\tau \geq 0$ and $y \in \mathbb{R}$. Since $0 < \beta < 1$ they even remain finite as $\tau \to \infty$, whence $\sup_{\tau \geq 0}\left|\int_0^\tau t^{\beta-1}\sin(t)dt\right| < \infty$. By continuity of G the points a, b are of G-measure zero, from which we eventually infer the G-almost sure existence of $\lim_{T\to\infty} Q_T(Y,b,a)$. Now, suppose $a, b \in \mathbb{R}$ are continuity points of F and for each $\xi \in \{a,b\}$ we have

$$\mathbb{E}\left\{|\xi - Y|^{-\beta}\right\} = \int_{\mathbb{R}} \frac{g(y)}{|\xi - y|^\beta}dy < \infty,$$

which holds, for instance, if the density g associated with G is continuous in a neighborhood of ξ. In this event Theorem 7.5.1 applies. By taking into account Theorem A.7.10 and the Mellin transform of the sine function (4.8.11), from (7.5.9) we deduce G-almost surely

$$\begin{aligned} \hat{F}_n(b) - \hat{F}_n(a) = G_n(b) - G_n(a) &+ \sin\left\{\frac{\beta\pi}{2}\right\}\frac{\Gamma(\beta)}{n\pi}\sum_{k=1}^n \operatorname{sgn}(b-Y_k)\,|b-Y_k|^{-\beta} \\ &- \sin\left\{\frac{\beta\pi}{2}\right\}\frac{\Gamma(\beta)}{n\pi}\sum_{k=1}^n \operatorname{sgn}(a-Y_k)\,|a-Y_k|^{-\beta}, \end{aligned} \tag{7.5.11}$$

and this represents an unbiased estimator for the probability $\mathbb{P}(a < X \leq b)$. Moreover, G-almost surely $\mathfrak{D}_n(b,\infty) - \mathfrak{D}_n(a,\infty)$ equals (7.5.11). Finally we observe that it is even admitted in (7.5.11) to let $a \to -\infty$, to arrive at an unbiased estimator for $F(b)$.

It should be expected and will be confirmed by the following example, that the unbiasedness of the m-limit becomes invalid if the function $q(\cdot, y, b, a)$ is merely bounded but non-vanishing at infinity.

Example 7.5.2 (exponentially distributed errors). Consider a scenario in which an unknown X-distribution is blurred by an error variable $\varepsilon \sim \mathrm{Exp}(1)$. In these circumstances $\Phi_\varepsilon(t) = (1-it)^{-1}$ and $\Phi_{\bar{\varepsilon}}(t) = (1+t^2)^{-1}$. Moreover, the distribution G is then again absolutely continuous but $q(t, y, b, a)$ does not decay at infinity. According to (A.1.6), it is particularly given by:

$$\begin{aligned} q(t, y, b, a) &= 2\Re\left\{e^{iyt}\Phi_{a,b}(-t)(1-it)\right\} \\ &= 2\Re\left\{e^{iyt}\Phi_{a,b}(-t)\right\} + \frac{2}{b-a}\left\{\cos\left\{(b-y)t\right\} - \cos\left\{(a-y)t\right\}\right\} \end{aligned}$$

Since $1 - \mathcal{P}_{\bar{\varepsilon}}(t, m) = \mathcal{O}(t^{-2})$ as $t \to \pm\infty$, the integral functions (7.5.2) appearing in the Fourier-type estimator (7.5.3) are absolutely convergent. By means of the above identity it can be separated, to become

$$\begin{aligned} \mathfrak{D}_n(b, m) - \mathfrak{D}_n(a, m) = \frac{b-a}{n\pi} \sum_{k=1}^{n} \Re \int_0^\infty \left\{1 - \mathcal{P}_{\bar{\varepsilon}}(t, m)\right\} e^{iY_k t}\Phi_{a,b}(-t)dt \\ + \frac{1}{n\pi}\sum_{k=1}^{n}\int_0^\infty \left\{1 - \mathcal{P}_{\bar{\varepsilon}}(t, m)\right\}\cos\left\{(b - Y_k)t\right\}dt \\ - \frac{1}{n\pi}\sum_{k=1}^{n}\int_0^\infty \left\{1 - \mathcal{P}_{\bar{\varepsilon}}(t, m)\right\}\cos\left\{(a - Y_k)t\right\}dt. \end{aligned} \tag{7.5.12}$$

On the one hand, under the present assumptions by Theorem 2.4.3 for all continuity points $a < b$ of F we know

$$\lim_{m\to\infty} \mathbb{E}\left\{\mathfrak{D}_n(b, m) - \mathfrak{D}_n(a, m)\right\} = F(b) - F(a). \tag{7.5.13}$$

On the other hand, Lebesgue's dominated convergence theorem enables us to write

$$\int_0^\infty \left\{1 - \mathcal{P}_{\bar{\varepsilon}}(t, m)\right\}\cos(\xi t)dt = \lim_{\delta\downarrow 0}\int_0^\infty \left\{1 - \mathcal{P}_{\bar{\varepsilon}}(t, m)\right\}\cos(\xi t)e^{-\delta t}dt.$$

Upon separating the function in the curved brackets, for fixed $\delta > 0$ we obtain:

$$\int_0^\infty \left\{1 - \mathcal{P}_{\bar{\varepsilon}}(t, m)\right\}\cos(\xi t)e^{-\delta t}dt = \Re\int_0^\infty e^{-(\delta - i\xi)t}dt - \int_0^\infty \mathcal{P}_{\bar{\varepsilon}}(t, m)\cos(\xi t)e^{-\delta t}dt$$

$$= \frac{\delta}{\delta^2 + \xi^2} - \int_0^\infty \mathcal{P}_{\bar{\varepsilon}}(t, m) \cos(\xi t) e^{-\delta t} dt$$

As $\delta \downarrow 0$ the first summand vanishes, whereas the limit of the sequence of integrals in the second summand was shown to exist for any $\xi \in \mathbb{R} \setminus \{0\}$. More precisely, subject to Theorem 6.6.1 this limit exists and equals a function that eventually vanishes as $m \to \infty$. Furthermore, any $\xi \in \mathbb{R}$ is of zero G-measure, and the m-power $\mathcal{P}_{\bar{\varepsilon}}(\cdot, m)$ is readily verified to satisfy the conditions of the function $K_\lambda(t)$ from Theorem A.4.1 in the appendix. Hence, if in (7.5.12) we let $m \to \infty$, G-almost surely an additional application of Theorem A.4.1(2) brings us:

$$\begin{aligned} \mathfrak{D}_n(b, \infty) - \mathfrak{D}_n(a, \infty) &= \frac{b-a}{n\pi} \lim_{T \to \infty} \sum_{k=1}^n \Re \int_0^T e^{iY_k t} \Phi_{a,b}(-t) dt \\ &= G_n(b) - G_n(a) \end{aligned}$$

The last equation is a consequence from the inversion theorem A.7.10, and it reveals

$$\mathbb{E}\left\{\mathfrak{D}_n(b, \infty) - \mathfrak{D}_n(a, \infty)\right\} = G(b) - G(a).$$

In view of (7.5.13) this result tells us that the expectation of the G-almost certain limit of the empirical deconvolution function does not match the limit of its expectation as $m \to \infty$.

To summarize these findings, the empirical deconvolution function is in general represented by (7.5.3) for any $m \geq 0$. In special circumstances, however, for fixed n we may let $m \to \infty$ to arrive at a G-almost sure limit which exhibits unbiasedness with respect to the target distribution.

7.6. A First Treatment of the Variance

The variance (7.3.11) associated with the increment estimator for the deconvolution function is basically composed of two terms. These are respectively the iterated convolution-type integral (7.3.12) and the squared increment of the deconvolution function $\mathfrak{D}(b, m) - \mathfrak{D}(a, m)$. The latter is readily identified as the expectation of the estimator, compare (7.3.8), and in terms of the bias (2.1.57) it satisfies the identity

$$\mu_{\mathfrak{D}}(m, b, a) = \mathrm{BI}(m, b, a) + (F(b) - F(a)).$$

Hence, if we assume $\mathrm{BI}(m, b, a) \to 0$ as $m \to \infty$, for any $n \in \mathbb{N}$ as $m \to \infty$ we may conclude

$$\sigma^2_{\mathfrak{D}}(m, n, b, a) = \frac{1}{n4\pi^2} \mathrm{M}_{\mathfrak{D}}(m, b, a) - \frac{1}{n}\left\{(F(b) - F(a))^2 + o(1)\right\}. \tag{7.6.1}$$

The exact asymptotic behaviour of the bias can be determined by virtue of our findings from Chapter 6. Alternatively one may exploit the non-negativity of the squared term on the right hand side in (7.3.11), to conclude

$$\sigma^2_{\mathfrak{D}}(m,n,b,a) \leq \frac{1}{n4\pi^2}\,\mathrm{M}_{\mathfrak{D}}(m,b,a). \tag{7.6.2}$$

This estimate is particularly helpful for situations in which it suffices to know that in view of (7.6.1) the neglected squared term approaches a finite limit as $m \to \infty$ and finally vanishes as $n \to \infty$. The actual challenging part about the variance is, to make a statement on the m-asymptotic behaviour of the iterated integral (7.3.12). Evidently, its "soul" is furnished by the characteristic function associated with the blurred variable Y. If this function was a polynomial, which can in fact happen only if it possesses a compact support or if trivially $\Phi_Y \equiv 1$, the iterated integral could be rearranged to become a sum whose addends equal a product of two single integrals. Although this case is barely interesting, a similar simplification may occur in a purely discrete setup. To be exact, if X and ε are both discretely distributed, the function Φ_Y is almost periodic and thus a trigonometric polynomial. In other words, it represents a sum or a series of complex exponential functions. In this event the integral (7.3.11) will certainly never converge absolutely for $\delta_1 = \delta_2 = 0$, because Φ_ε is non-vanishing at infinity. By employing the integral definition of Φ_Y, the indicated double integral equivalently can be cast in the form

$$\begin{aligned}\mathrm{M}_{\mathfrak{D}}(m,b,a) = (b-a)^2 \lim_{\delta_1,\delta_2 \downarrow 0} \mathbb{E}\Bigg\{ &\int_{-\infty}^{\infty} e^{-\delta_2|t|+iYt}\Phi_{a,b}(-t)\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_\varepsilon(-t)dt \\ &\times \int_{-\infty}^{\infty} e^{-\delta_1|s|+iYs}\Phi_{a,b}(-s)\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_\varepsilon(-s)ds \Bigg\},\end{aligned}$$

where the expectation equals a finite or an infinite sum. An attempt to investigate the asymptotics for large m of the right hand side can be to confine the analysis to the single integrals that depend on m. Notice that in a purely discrete setup the Fourier integral representation has a restricted validity for $a, b \in \mathbb{R}$, due to discontinuities.

If X or ε is a continuous random variable, the above factorization is still valid but the resulting continuity of G will most likely cause difficulties if Y lies in a neighborhood of those points at which the oscillations of the complex exponential function diminish. We then return to a consideration of the iterated representation (7.3.12). To avoid additional difficulties we assume absolute and with respect to $\delta_1, \delta_2 \geq 0$ uniform convergence of this integral. A first inspection shows, according to the uniform boundedness of $\Phi_Y(t)$ with respect to $t \in \mathbb{R}$, that these convergence properties can be achieved if

$$\Phi_\varepsilon \Phi_{a,b} \in L^1(\mathbb{R}). \tag{7.6.3}$$

From (7.3.7) we recall, in these circumstances even the increment estimator converges absolutely and with respect to $T > 0$ uniformly. Finally it is clear that (7.6.3) can only hold if Φ_ε vanishes at infinity, and this is the case if and only if H is a continuous distribution. But for continuous

H we remember the admissibility of the Fourier-type integral representation of the estimator and its variance for any $a, b \in \mathbb{R}$ with $\mathbb{P}_{\mathfrak{D}_n}(a) = \mathbb{P}_{\mathfrak{D}_n}(b) = 1$, i.e., for all $a, b \in \mathbb{R}$. More exact criteria than (7.6.3) to assess the convergence properties of iterated integrals of the indicated type, which take into account the local behaviour of the integrand, will be established in the next chapter.

Now, validity of (7.6.3) enables us in (7.3.12) to interchange the order of limit and integration, which leads to

$$\begin{aligned} \mathrm{M}_{\mathfrak{D}}(m,b,a) = (b-a)^2 \int_{-\infty}^{\infty} \Phi_{a,b}(-t)\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_{\varepsilon}(-t) \\ \times \int_{-\infty}^{\infty} \Phi_{a,b}(-s)\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_{\varepsilon}(-s)\Phi_Y(s+t)dsdt. \end{aligned} \tag{7.6.4}$$

According to our findings from Section 7.4, we expect this integral either to approach a finite limit or to diverge as $m \to \infty$, which essentially depends on the set of points N_ε at which Φ_ε vanishes and on the behaviour of the integrand there. Yet, by taking into account the uniform boundedness of the geometric sum function a very general statement is possible.

Theorem 7.6.1 (quadratic divergence). *If $\Phi_{a,b}\Phi_\varepsilon \in L^1(\mathbb{R})$, then $\sigma^2_{\mathfrak{D}}(m,n,b,a) = \mathcal{O}\left\{n^{-1}m^2\right\}$ as $m \to \infty$ for any $a, b \in \mathbb{R}$.*

In other words, a sufficient condition for the variance to grow no faster than quadratic as $m \to \infty$ is the validity of (7.6.3). This property essentially distinguishes the variance from the bias, where no general statements about the slowest admissible rate are possible.

Proof. We continue from (7.6.4) with a simple application of the triangle inequality, incorporating $\sup_{t\in\mathbb{R}} |\Phi_Y(t)| \leq 1$ and definition (A.1.6), to arrive at

$$|\mathrm{M}_{\mathfrak{D}}(m,b,a)| \leq 16 \left[\int_0^\infty \frac{\left|\sin\left\{\frac{b-a}{2}t\right\}\right|}{t} \left|\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_\varepsilon(-t)\right| dt \right]^2. \tag{7.6.5}$$

Since $|\Phi_\varepsilon(t)|^2 = \Phi_{\bar{\varepsilon}}(t)$ and $\mathcal{G}_{\bar{\varepsilon}}(t,m) \leq m+1$ uniformly with respect to $t \in \mathbb{R}$, we deduce

$$|\mathrm{M}_{\mathfrak{D}}(m,b,a)| \leq 16(m+1)^2 \left[\int_0^\infty \frac{\left|\sin\left\{\frac{b-a}{2}t\right\}\right|}{t} \left\{\Phi_{\varepsilon}(t)\right\}^{\frac{1}{2}} dt \right]^2.$$

But under the assumption on the integrand the latter integral converges absolutely, which by (7.6.2) concludes the proof. ∎

It is needless to say that the bounds of the preceding proof certainly incur too much losses. In fact, a thorough investigation of the variance inevitably encompasses the analysis of the nasty double integral (7.6.4). A detailed look suggests a special role to be played by the segments along which one variable tends to positive infinity, while the second variable tends to negative infinity. Indeed, if $\Phi_Y(t)$ vanishes as $t \to \pm\infty$, the consequences of converse signs in the argument of $\Phi_Y(\pm s \pm t)$ combined with an infinite range of integration should not be underestimated. While elementary calculations show that it suffices for a double integral $\int_\tau^\infty \int_\tau^\infty a(s+t)dsdt$ with $\tau > 0$ to converge absolutely if $a(v) = \mathcal{O}\{v^{-\alpha}\}$ as $v \to \infty$ for $\alpha > 2$, this condition becomes insufficient if $a(s+t)$ is replaced by $a(s-t)$. We can, however, circumvent the difficulties with the latter function by an appropriate decomposition of the integral (7.6.4), to be conducted below.

Having specified by Theorem 7.6.1 the worst case scenario, it is reasonable to demand in which event the desired case of uniform boundedness of the variance with respect to $m \geq 0$ occurs. Unfortunately, at the moment we are not yet able to provide an adequate answer. A naive approach again consists in a simple application of the triangle inequality. If we then bound the numerator of $\mathcal{G}_{\bar{\varepsilon}}(t, m)$ by unity, we arrive at

$$|\mathrm{M}_{\mathfrak{D}}(m, b, a)| \leq 8 \int_0^\infty \frac{\left|\sin\left\{\frac{b-a}{2}t\right\}\right|}{t\left\{\Phi_{\bar{\varepsilon}}(t)\right\}^{\frac{1}{2}}} \int_0^\infty \frac{\left|\sin\left\{\frac{b-a}{2}s\right\}\right|}{s\left\{\Phi_{\bar{\varepsilon}}(s)\right\}^{\frac{1}{2}}} \left\{|\Phi_Y(s+t)| + |\Phi_Y(s-t)|\right\} dsdt.$$

An additional application of the trivial bound $|\Phi_Y| \leq 1$ would be devastating, since the resulting integral was in any case divergent. Sufficient conditions for the absolute convergence of the integral on the right hand side therefore can not be imposed merely on the basis of simple convergence criteria for single integrals. Actually, in order to make any satisfactory statements on the m-asymptotic behaviour of the iterated integral (7.6.4), including its uniform boundedness with respect to $m \geq 0$, we require auxiliary tools. These will also teach us the appropriate use of the Cahen-Mellin representation (7.4.7) for the geometric sum function, which we proposed in Section 7.4 as a means to describe the asymptotics of integrals of this particular type. An application accompanied by a formal interchange in the order of integration for some $-1 < u_0, x_0 < 0$ yields

$$\begin{aligned} \mathrm{M}_{\mathfrak{D}}(m, b, a) = \frac{1}{(2\pi i)^2} \int_{x_0 - i\infty}^{x_0 + i\infty} (m+1)^{-z}\Gamma(z) \int_{u_0 - i\infty}^{u_0 + i\infty} (m+1)^{-w}\Gamma(w) & \\ \times \int_{\mathbb{R}\setminus N_\varepsilon} \frac{e^{-ibt} - e^{-iat}}{t\Phi_\varepsilon(t)} \left\{-\log(1 - \Phi_{\bar{\varepsilon}}(t))\right\}^{-z} \int_{\mathbb{R}\setminus N_\varepsilon} \frac{e^{-ibs} - e^{-ias}}{s\Phi_\varepsilon(s)} & \\ \times \left\{-\log(1 - \Phi_{\bar{\varepsilon}}(s))\right\}^{-w} \Phi_Y(s+t) dsdtdwdz. & \end{aligned}$$

The exact procedure to evaluate this double MB-integral is as complicated as its representation suggests it to be. It requires exact convergence and analyticity criteria for the $dsdt$-integral as a function of w for fixed z and additional results. These will thoroughly be discussed in the next chapter. For the moment we confine to a simple estimate, which can be established with the aid of the binomial integral representation.

Theorem 7.6.2. *If there exists* $-1 < x_0 < 0$ *with*

$$\Phi_{a,b}(t)\left\{\Phi_{\bar{\varepsilon}}(t)\right\}^{-x_0-\frac{1}{2}} \in L^1(\mathbb{R}),$$

then $\sigma^2_{\mathfrak{D}}(m,n,b,a) = \mathcal{O}\left\{n^{-1}m^{-2x_0}\right\}$ *as* $m \to \infty$ *for any* $a, b \in \mathbb{R}$.

Proof. The proof of this theorem is similar to that of Theorem 7.6.1. Indeed, it only differs by the fact that we apply the bound (7.4.5) for the geometric sum function to the estimate (7.6.5). The big-$\mathcal{O}$ estimate then can be obtained from Stirling's formula or from (B.3.6). ■

We conclude this section with the aforementioned decomposition which, however, we carry out for the iterated integral (7.3.12).

7.6.1. Decomposition for the Variance of the Empirical Deconvolution Function

For fixed $\delta_1, \delta_2 \geq 0$ we denote the integral appearing in (7.3.12) by

$$\begin{aligned} \mathrm{M}_{\mathfrak{D}}\left[m; \begin{matrix}\delta_1, \delta_2\\ b, a\end{matrix}\right] := (b-a)^2 \int_{-\infty}^{\infty} & e^{-\delta_2|t|}\Phi_{a,b}(-t)\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_{\varepsilon}(-t) \\ & \times \int_{-\infty}^{\infty} e^{-\delta_1|s|}\Phi_{a,b}(-s)\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_{\varepsilon}(-s)\Phi_Y(s+t)\,ds\,dt. \end{aligned} \tag{7.6.6}$$

It can be separated into two double integrals along the positive real axis, to obtain

$$\begin{aligned} (b-a)^{-2}\,\mathrm{M}_{\mathfrak{D}}\left[m; \begin{matrix}\delta_1, \delta_2\\ b, a\end{matrix}\right] = {} & 2\Re\Bigg[\int_0^{\infty} e^{-\delta_2 t}\Phi_{a,b}(-t)\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_{\varepsilon}(-t) \\ & \qquad \times \int_0^{\infty} e^{-\delta_1 s}\Phi_{a,b}(-s)\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_{\varepsilon}(-s)\Phi_Y(s+t)\,ds\,dt\Bigg] \\ & + 2\Re\Bigg[\int_0^{\infty} e^{-\delta_2 t}\Phi_{a,b}(t)\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_{\varepsilon}(t) \\ & \qquad \times \int_0^{\infty} e^{-\delta_1 s}\Phi_{a,b}(-s)\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_{\varepsilon}(-s)\Phi_Y(s-t)\,ds\,dt\Bigg]. \end{aligned} \tag{7.6.7}$$

Moreover, if we split the range of integration of the last integral at $s = t$, we find

$$\begin{aligned} & \int_0^{\infty} e^{-\delta_2 t}\Phi_{a,b}(t)\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_{\varepsilon}(t) \int_0^{\infty} e^{-\delta_1 s}\Phi_{a,b}(-s)\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_{\varepsilon}(-s)\Phi_Y(s-t)\,ds\,dt \\ & \quad = \int_0^{\infty} e^{-\delta_1 s}\Phi_{a,b}(-s)\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_{\varepsilon}(-s) \int_s^{\infty} e^{-\delta_2 t}\Phi_{a,b}(t)\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_{\varepsilon}(t)\Phi_Y(s-t)\,dt\,ds \end{aligned}$$

$$+\int_0^\infty e^{-\delta_2 t}\Phi_{a,b}(t)\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_\varepsilon(t)\int_t^\infty e^{-\delta_1 s}\Phi_{a,b}(-s)\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_\varepsilon(-s)\Phi_Y(s-t)dsdt.$$

Subject to absolute convergence we interchanged the order of integration in the first double integral on the right hand side. Evidently, except for the order of δ_1 and δ_2 the first integral equals the complex conjugate of the second. For $\delta_1, \delta_2 > 0$ we therefore define

$$\text{(7.6.8)}\quad \begin{aligned} \mathrm{A}\left[m;\begin{matrix}\delta_1,\delta_2\\ b,a\end{matrix}\right] &:= \int_0^\infty e^{-\delta_2 t}\frac{e^{-ibt}-e^{-iat}}{t}\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_\varepsilon(-t)\\ &\quad\times\int_t^\infty e^{-\delta_1 s}\frac{e^{ibs}-e^{ias}}{s}\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_\varepsilon(s)\Phi_Y(t-s)dsdt, \end{aligned}$$

$$\text{(7.6.9)}\quad \begin{aligned} \Sigma\left[m;\begin{matrix}\delta_1,\delta_2\\ b,a\end{matrix}\right] &:= \int_0^\infty e^{-\delta_2 t}\frac{e^{-ibt}-e^{-iat}}{t}\mathcal{G}_{\bar{\varepsilon}}(t,m)\Phi_\varepsilon(-t)\\ &\quad\times\int_0^\infty e^{-\delta_1 s}\frac{e^{-ibs}-e^{-ias}}{s}\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_\varepsilon(-s)\Phi_Y(s+t)dsdt. \end{aligned}$$

Then, by means of the definition (A.1.6), and since $\overline{\Phi}(t) = \Phi(-t)$ for any characteristic function and $\Re z = \Re\overline{z}$ for all $z \in \mathbb{C}$, from (7.6.7) we deduce

$$\text{(7.6.10)}\quad \mathrm{M}_{\mathfrak{D}}\left[m;\begin{matrix}\delta_1,\delta_2\\ b,a\end{matrix}\right] = 2\Re\,\mathrm{A}\left[m;\begin{matrix}\delta_1,\delta_2\\ b,a\end{matrix}\right] + 2\Re\,\mathrm{A}\left[m;\begin{matrix}\delta_2,\delta_1\\ b,a\end{matrix}\right] - 2\Re\Sigma\left[m;\begin{matrix}\delta_1,\delta_2\\ b,a\end{matrix}\right].$$

In order to finally abandon the function $\Phi_Y(t-s)$ we observe that a simple change of variables results in

$$\text{(7.6.11)}\quad \begin{aligned} \mathrm{A}\left[m;\begin{matrix}\delta_1,\delta_2\\ b,a\end{matrix}\right] &= \int_0^\infty e^{-\delta_1 t}\Phi_Y(-t)\int_0^\infty e^{-(\delta_1+\delta_2)s}\frac{e^{-ibs}-e^{-ias}}{s}\mathcal{G}_{\bar{\varepsilon}}(s,m)\Phi_\varepsilon(-s)\\ &\quad\times\frac{e^{ib(s+t)}-e^{ia(s+t)}}{s+t}\mathcal{G}_{\bar{\varepsilon}}(s+t,m)\Phi_\varepsilon(s+t)dsdt. \end{aligned}$$

If we describe by symmetry of a function $f(s,t)$ the property $f(s,t) = f(t,s)$ for each $s, t \geq 0$, it is easy to see that the integrand of (7.6.9) perfectly fits this description. Contrary, the integrand of (7.6.11) is highly asymmetric. Finally, for brevity we define

$$\text{(7.6.12)}\quad \rho(t;b,a) := e^{ibt} - e^{iat},$$

which is $\mathcal{O}(t)$ as $t \to 0$ and satisfies $\overline{\rho}(t; b, a) = \rho(-t; b, a)$ for all $t \in \mathbb{R}$. Moreover, for $0 \leq \sigma < S \leq \infty$ and $0 \leq \tau < T \leq \infty$ we introduce the integrals

$$\mathfrak{Ai}\left[m; \begin{matrix} a, \sigma, S \\ b, \tau, T \end{matrix}\right] := \int_\tau^T \Phi_Y(-t) \int_\sigma^S \frac{\overline{\rho}(s; b, a)}{s} \frac{1 - \mathcal{P}_{\bar{\varepsilon}}(s, m)}{\Phi_\varepsilon(s)} \times \frac{\rho(s+t; b, a)}{s+t} \frac{1 - \mathcal{P}_{\bar{\varepsilon}}(s+t, m)}{\overline{\Phi}_\varepsilon(s+t)} ds dt, \tag{7.6.13}$$

$$\mathfrak{Si}\left[m; \begin{matrix} a, \sigma, S \\ b, \tau, T \end{matrix}\right] := \int_\tau^T \frac{\overline{\rho}(t; b, a)}{t} \frac{1 - \mathcal{P}_{\bar{\varepsilon}}(t, m)}{\Phi_\varepsilon(t)} \times \int_\sigma^S \frac{\overline{\rho}(s; b, a)}{s} \frac{1 - \mathcal{P}_{\bar{\varepsilon}}(s, m)}{\Phi_\varepsilon(s)} \Phi_Y(s+t) ds dt. \tag{7.6.14}$$

If we suppose absolute and with respect to $\delta_1, \delta_2 \geq 0$ uniform convergence of the integrals (7.6.9) and (7.6.11), we may let $\delta_1, \delta_2 \downarrow 0$ under the signs of integration. According to (7.6.10), from (7.3.12) we then obtain the decomposition

$$\mathrm{M}_{\mathfrak{D}}(m, b, a) = 4\Re\mathfrak{Ai}\left[m; \begin{matrix} a, 0, \infty \\ b, 0, \infty \end{matrix}\right] - 2\Re\mathfrak{Si}\left[m; \begin{matrix} a, 0, \infty \\ b, 0, \infty \end{matrix}\right]. \tag{7.6.15}$$

7.6.2. Decomposition of the Variance of the Empirical Deconvolution Density

Regarding the variance (7.3.20) of the empirical deconvolution density, it is easy to repeat the preceding steps for the integral (7.3.19). Supposing its absolute and with respect to $\delta_1, \delta_2 \geq 0$ uniform convergence, the limits $\delta_1, \delta_2 \downarrow 0$ can be taken under the signs of integration. Then, if for $\xi \in \mathbb{R}$ and $m \geq 0$ we define

$$\mathfrak{ai}\begin{bmatrix} m, \sigma, S \\ \xi, \tau, T \end{bmatrix} := \int_\tau^T e^{i\xi t} \Phi_Y(-t) \int_\sigma^S \frac{1 - \mathcal{P}_{\bar{\varepsilon}}(s, m)}{\Phi_\varepsilon(s)} \frac{1 - \mathcal{P}_{\bar{\varepsilon}}(s+t, m)}{\Phi_\varepsilon(-s-t)} ds dt, \tag{7.6.16}$$

$$\mathfrak{si}\begin{bmatrix} m, \sigma, S \\ \xi, \tau, T \end{bmatrix} := \int_\tau^T e^{-i\xi t} \frac{1 - \mathcal{P}_{\bar{\varepsilon}}(t, m)}{\Phi_\varepsilon(t)} \int_\sigma^S e^{-i\xi s} \frac{1 - \mathcal{P}_{\bar{\varepsilon}}(s, m)}{\Phi_\varepsilon(s)} \Phi_Y(s+t) ds dt, \tag{7.6.17}$$

we arrive at

$$\mathrm{M}_{\mathfrak{d}}(m, \xi) = 4\Re\mathfrak{ai}\begin{bmatrix} m, 0, \infty \\ \xi, 0, \infty \end{bmatrix} + 2\Re\mathfrak{si}\begin{bmatrix} m, 0, \infty \\ \xi, 0, \infty \end{bmatrix}. \tag{7.6.18}$$

7.7. Example: Estimators for Cauchy Distributed Errors

In Section 7.5 it turned out, once we have specified the error variable ε, equivalent representations for the empirical deconvolution function can be found by elementary manipulations. Another particularly convenient example is furnished by a setup with Cauchy-distributed errors, to which this section is devoted. Rather than the estimator for the unknown distribution F, however, we first consider the empirical deconvolution density without actually supposing the existence of a density $f(x)dx = F(dx)$. This approach does not violate any assumptions, since $\mathfrak{d}_n(\cdot, m)$ and $\mathfrak{d}(\cdot, m)$ in fact exist by absolute continuity of H. Now, the occurence of Cauchy-distributed errors implies uniform continuity of the density h on the whole real axis and $\Phi_\varepsilon(t) = e^{it\mu_\varepsilon - \frac{\sigma|t|}{2}}$ for some $\mu_\varepsilon \in \mathbb{R}$ and $\sigma > 0$. Hence, the empirical deconvolution density possesses the absolutely convergent Fourier-type integral representation (7.3.17). Moreover, we deduce $\Phi_{\bar{\varepsilon}}(t) = e^{-\sigma|t|}$. In addition, if in (7.3.16) we also choose $\Phi_I(t) = e^{-|t|}$ and employ the definition of $\mathcal{G}_{\bar{\varepsilon}}(t, m)$, the estimator can be rearranged to become:

$$\begin{aligned}\mathfrak{d}_n(\xi, m) &= \frac{1}{n2\pi} \sum_{k=1}^{n} \lim_{\delta \downarrow 0} \int_{-\infty}^{\infty} e^{-(\delta + \frac{\sigma}{2})|t| - it(\xi + \mu_\varepsilon - Y_k)} \mathcal{G}_{\bar{\varepsilon}}(t, m) dt \\ &= \frac{1}{n\pi} \sum_{k=1}^{n} \lim_{\delta \downarrow 0} \Re \int_{0}^{\infty} e^{-(\delta - \frac{\sigma}{2})t - it(\xi + \mu_\varepsilon - Y_k)} \left\{1 - (1 - e^{-\sigma t})^{m+1}\right\} dt \qquad (7.7.1)\end{aligned}$$

For fixed $\delta > \frac{\sigma}{2}$ on the one hand

$$\int_{0}^{\infty} e^{-(\delta - \frac{\sigma}{2})t - it(\xi + \mu_\varepsilon - Y_k)} dt = \frac{1}{\delta - \frac{\sigma}{2} + i(\xi + \mu_\varepsilon - Y_k)},$$

whereas on the other hand, by definition of the beta function,

$$\int_{0}^{\infty} e^{-(\delta - \frac{\sigma}{2})t - it(\xi + \mu_\varepsilon - Y_k)} (1 - e^{-\sigma t})^{m+1} dt = \sigma^{-1} \frac{\Gamma(m+2)\Gamma\left(\frac{\delta}{\sigma} - \frac{1}{2} + \frac{i}{\sigma}(\xi + \mu_\varepsilon - Y_k)\right)}{\Gamma\left(m + \frac{3}{2} + \frac{\delta}{\sigma} + \frac{i}{\sigma}(\xi + \mu_\varepsilon - Y_k)\right)}.$$

In terms of the digamma function, in the circle $|z| < 1$ we know there exists a power series expansion of the form

$$\frac{1}{z} - \frac{\Gamma(m+2)\Gamma(z)}{\Gamma(m+2+z)} = (\psi(m+2) + \gamma) + \mathcal{O}(z),$$

from which we conclude continuity of the left hand side with respect to $\Re z \geq -\frac{1}{2}$ for any fixed $\Im z \in \mathbb{R}$. Hence, as $\delta \downarrow 0$ from (7.7.1) we eventually obtain

$$\mathfrak{d}_n(\xi, m) = -\frac{1}{n\sigma\pi} \sum_{k=1}^{n} \left\{ \frac{2\sigma^2}{\sigma^2 + 4(\xi + \mu_\varepsilon - Y_k)^2} + \Re \frac{\Gamma(m+2)\Gamma\left(-\frac{1}{2} + \frac{i}{\sigma}(\xi + \mu_\varepsilon - Y_k)\right)}{\Gamma\left(m + \frac{3}{2} + \frac{i}{\sigma}(\xi + \mu_\varepsilon - Y_k)\right)} \right\}. \qquad (7.7.2)$$

An application of the asymptotic statement (B.3.5) shows, as $m \to \infty$, the right hand side diverges with the rate $m^{\frac{1}{2}}$.

Conversely, as $\xi \to \infty$, the function satisfies $\mathcal{O}\left\{|\xi|^{-m-2}\right\}$ as $\xi \to \pm\infty$, which is clearly absolutely integrable on the whole real axis. We therefore obtain the empirical deconvolution function by computing for $\xi \in \mathbb{R}$ the integral (7.3.14) with the above representation (7.7.2) for

$\mathfrak{d}_n(\cdot, m)$. While the first addend in the indicated sum is readily identified as the derivative of the arctangent, the second has no known elementary representation. Hence, accompanied by a simple change of variables for any $\xi \in \mathbb{R}$ we write

$$\mathfrak{D}_n(\xi, m) = -\frac{1}{n\pi}\sum_{k=1}^{n} \arctan\left\{\frac{2(\xi+\mu_\varepsilon - Y_k)}{\sigma}\right\} - \frac{1}{n\sigma\pi}\Re\sum_{k=1}^{n}\int_{-\infty}^{0} \upsilon(x;\xi, Y_k)dx, \tag{7.7.3}$$

where the function

$$\upsilon(z;\xi,Y_k) := \frac{\Gamma(m+2)\Gamma\left(-\frac{1}{2}+\frac{i}{\sigma}(z+\xi+\mu_\varepsilon - Y_k)\right)}{\Gamma\left(m+\frac{3}{2}+\frac{i}{\sigma}(z+\xi+\mu_\varepsilon - Y_k)\right)}$$

for each $1 \leq k \leq n$ is meromorphic in $\mathbb{C}$ with poles of simple order at $z = Z_{j,k}$ for random $Z_{j,k} := Y_k - \mu_\varepsilon - \xi + i\sigma(j - \frac{1}{2})$ with $j \in \mathbb{N}_0$. These poles lie in the upper z-half plane except the pole $z = Z_{0,k}$, which is located at a random point of the line $\Im z = -\frac{1}{2}$.

8. Asymptotics of Iterated Convolution-Type Integrals by Analytic Continuation

The aim of this chapter is the adaption of the method of analytic continuation from Chapter 6 to special iterated integrals. We are particularly interested in the dominating m-asymptotic properties of the variance integrals that occur in the context of the empirical deconvolution function and density. These can be generalized as follows. On the one hand, we have an integral whose integrand is symmetric with respect to the terms that depend on m, viz

$$\operatorname{Si}\left[m; \begin{matrix}\sigma, S\\ \tau, T\end{matrix}\right] := \int\limits_{\mathcal{P}_2} \left\{1 - (1-\Psi(t))^{m+1}\right\} \int\limits_{\mathcal{P}_1} \left\{1 - (1-\Psi(s))^{m+1}\right\} q_{\mathrm{Si}}(s,t)dsdt, \tag{8.0.1}$$

where each of the paths $\mathcal{P}_1$ and $\mathcal{P}_2$ is supposed to be a half open segment of the positive real axis with endpoints $0 \leq \sigma < S \leq \infty$ and $0 \leq \tau < T \leq \infty$, respectively. The functions $0 \leq \Psi \leq 1$ and q_{Si} are continuous. Notice that the integrand is symmetric with respect to s and t if and only if $q_{\mathrm{Si}}(s,t)$ is. On the other hand, for a continuous function q_{Ai}, we have the asymmetric counterpart

$$\operatorname{Ai}\left[m; \begin{matrix}\sigma, S\\ \tau, T\end{matrix}\right] := \int\limits_{\mathcal{P}_2}\int\limits_{\mathcal{P}_1} \left\{1 - (1-\Psi(s))^{m+1}\right\} \left\{1 - (1-\Psi(s+t))^{m+1}\right\} q_{\mathrm{Ai}}(s,t)dsdt. \tag{8.0.2}$$

Both of the above integrals are clearly not of Laplacian-type. In fact, it was already pointed out in Section 7.4, that the behaviour as $m \to \infty$ of their one-dimensional analogue completely differs from what is common for Laplace-type integrals. Yet, a joint property is that this behaviour depends on the suprema of the m-power, and particularly on the points at which the function Ψ vanishes. Regarding iterated Laplace-type integrals, contrary to single versions, a recipe for a general treatment apparently does not exist. This was also pointed out in [Paris and Kaminski, 2001], where several references were provided that illustrate the various efforts. A possible reason for the difficulties in a general treatment of iterated integrals consists in the incapability to provide criteria for their convergence. It must in fact be expected that the convergence of the single components of an iterated integral is insufficient to conclude the convergence as a whole. Additional conditions are essentially determined by the behaviour along the integration paths of the factors that depend on more than a single variable of integration. Analogous statements hold for the conditions for the analyticity of iterated integrals. To avoid these inconveniences, bearing in mind our application field, we therefore confine to amplitude functions $q_{\text{Si}}(s,t)$ and $q_{\text{Ai}}(s,t)$ that admit a factorization in terms of functions of the variables s, t and $s+t$. The integrals (8.0.1) and (8.0.2) are then of convolution-type and our results will immediately be applicable to either of the variance integrals.

8.1. Preliminaries

Before we come to the technical part we briefly outline our strategy. In view of our introductory comments, we first develop elementary rules for calculus of iterated convolution-type integrals. These will finally enable us to impose sufficient conditions for the absolute convergence of (8.0.1) and (8.0.2), bearing in mind that, according to the binomial theorem, for any $m \geq 0$, as $\Psi(u) \to 0$, we have

$$1 - (1 - \Psi(u))^{m+1} \sim (m+1)\Psi(u). \tag{8.1.1}$$

The next step consists in a transformation to an iterated MB-integral. From Chapter 6 we recall that this is possible by employing a suitable integral representation for the m-power. However, contrary to integrals that feature a single m-power, an appropriate choice for integrals that feature two m-powers is no longer solely a matter of the local behaviour of the integrand. Actually, a wrong choice will incur substantial computational inconveniences due to the nature of the associated asymptotic scale. As an illustrative example, for $0 < x_0 < 1$ and $0 < u_0 < 1 - x_0$ we shall consider the iterated MB-integral

$$I_\lambda := \frac{1}{(2\pi i)^2} \int\limits_{x_0 - i\infty}^{x_0 + i\infty} \frac{\Gamma(\lambda+1)\Gamma(z)}{\Gamma(\lambda+1+z)} \int\limits_{u_0 - i\infty}^{u_0 + i\infty} \frac{\Gamma(\lambda+1)\Gamma(w)}{\Gamma(\lambda+1+w)} \frac{1}{1-w-z} dw dz.$$

By inspection of the interior integral, to which we refer as

$$J_\lambda(z) := \frac{1}{2\pi i} \int\limits_{u_0 - i\infty}^{u_0 + i\infty} \frac{\Gamma(\lambda+1)\Gamma(w)}{\Gamma(\lambda+1+w)} \frac{1}{1-w-z} dw,$$

we see that the integrand is $\sim$ const $\times\, \lambda^{-\Re w}$ as $\lambda \to \infty$. In other words, the power of the asymptotic parameter λ is descending as $\Re w$ increases in $\Re w > u_0$. According to our earlier findings, the behaviour as $\lambda \to \infty$ of $J_\lambda(z)$ thus can be exposed by a rightward displacement of the integration path. Appealing to the algebraic decay of the integrand for large $|w|$, we move the integration path across the only pole at $w = 1 - z$ and further to infinity in $\Re w > 0$, to deduce

$$J_\lambda(z) = \frac{\Gamma(\lambda+1)\Gamma(1-z)}{\Gamma(\lambda+2-z)}. \tag{8.1.2}$$

Upon plugging this identity into I_λ, we arrive at the hypergeometric integral

$$I_\lambda = \frac{1}{2\pi i}\int\limits_{x_0-i\infty}^{x_0+i\infty} \frac{\Gamma(\lambda+1)\Gamma(z)}{\Gamma(\lambda+1+z)}\frac{\Gamma(\lambda+1)\Gamma(1-z)}{\Gamma(\lambda+2-z)}dz. \tag{8.1.3}$$

It is obvious that as $\lambda\to\infty$ the integrand is $\sim \text{const}\times\lambda^{-1}$ for each $z\in\mathbb{C}\backslash\mathbb{Z}$. An asymptotic expansion of I_λ therefore can *not* be generated by an appropriate displacement of the line $\Re z = x_0$. Instead an asymptotic evaluation of (8.1.3) can be conducted, for instance, by virtue of a saddle point approximation. A similar example was treated in §5.3.3 in [Paris and Kaminski, 2001]. A more convenient way to evaluate I_λ is, to abandon (8.1.2) and to identify the function $\frac{1}{1-w-z}$ rather as a single Laplace transform of the argument $w+z$. An additional application of the definition of the binomial integral (6.1.1) then for any $\lambda>0$ yields:

$$\begin{aligned}
I_\lambda &= \frac{1}{(2\pi i)^2}\int\limits_{x_0-i\infty}^{x_0+i\infty}\frac{\Gamma(\lambda+1)\Gamma(z)}{\Gamma(\lambda+1+z)}\int\limits_{u_0-i\infty}^{u_0+i\infty}\frac{\Gamma(\lambda+1)\Gamma(w)}{\Gamma(\lambda+1+w)}\int\limits_0^\infty e^{-(1-w-z)t}dtdwdz\\
&= \int\limits_0^\infty e^{-t}(1-e^{-t})^{2\lambda}dt\\
&= \frac{1}{2\lambda+1}
\end{aligned}$$

For the last equality we wrote the beta function in terms of the gamma function and employed the functional equation of the latter. Of course these computations fail, if the generating function of the iterated MB-integral I_λ does not constitute a Laplace transform. Closely related to I_λ is

$$K_\lambda := \frac{1}{(2\pi i)^2}\int\limits_{x_0-i\infty}^{x_0+i\infty}\lambda^{-z}\Gamma(z)\int\limits_{u_0-i\infty}^{u_0+i\infty}\lambda^{-w}\Gamma(w)\frac{1}{1-w-z}dwdz, \tag{8.1.4}$$

again with $0<x_0<1$ and $0<u_0<1-x_0$. Clearly, K_λ differs from I_λ by the asymptotic scale in the integrand. If we denote by $L_\lambda(z)$ the interior MB-integral, for fixed $z\in\mathbb{C}$ with $\Re z = x_0$ a rightward displacment of the path across the only pole at $w=1-x_0$ for an arbitrary finite $u_1>1-x_0$ yields

$$L_\lambda(z) = \lambda^{z-1}\Gamma(1-z) + \frac{1}{2\pi i}\int\limits_{u_1-i\infty}^{u_1+i\infty}\lambda^{-w}\Gamma(w)\frac{1}{1-w-z}dw.$$

Due to the fact that the asymptotic scale rather than a factorial function is a simple power, cancellations occur upon plugging $L_\lambda(z)$ into (8.1.4). For an arbitrary $\varsigma>1$ as $\lambda\to\infty$ we then find

$$K_\lambda = \frac{1}{\lambda}\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} \Gamma(z)\Gamma(1-z)dz + \mathcal{O}\left\{\lambda^{-\varsigma}\right\} \sim \frac{1}{2\lambda}.$$

For the final result we took into account the inverse Mellin transform of the function $\frac{1}{1+t}$. Although the above example is somehow artificial, it warns us not to employ the binomial integral representation for the m-powers in each of the integrals (8.0.1) and (8.0.2), since it might result in additional difficulties that probably can be circumvented with the Cahen-Mellin representation.

In view of the above observations, we confine ourselves to the Cahen-Mellin representation (6.1.5) for the m-power and avoid the integrals of binomial and logarithmic type (6.1.1) and (6.1.4). Furthermore, for simplicity, we confine to functions Ψ with plain algebraic behaviour and throughout this chapter we write

$$\varphi(t) := -\log(1-\Psi(t)). \tag{8.1.5}$$

First, upon displacing the path in (6.1.5) to the left across the pole at $z = 0$, it is easy to confirm for any $t \in \mathbb{R}$ with $0 < \Psi(t) < 1$ validity of the equivalent representation

$$1-(1-\Psi(t))^{m+1} = -\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} \Gamma(z)\left\{(m+1)\varphi(t)\right\}^{-z} dz, \qquad -1 < x_0 < 0. \tag{8.1.6}$$

Then, if in either of the iterated integrals (8.0.1) or (8.0.2) we cast each m-power in terms of this contour integral, for appropriately specified integration paths $-1 < x_{01}, x_{02} < 0$, we obtain integrals of the form

$$J(m) := \frac{1}{(2\pi i)^2}\int_{x_{02}-i\infty}^{x_{02}+i\infty} (m+1)^{-z_2}\Gamma(z_2)\int_{x_{01}-i\infty}^{x_{01}+i\infty} (m+1)^{-z_1}\Gamma(z_1)\mathfrak{M}(z_1,z_2)dz_1dz_2, \tag{8.1.7}$$

where the generating function $\mathfrak{M}(z_1, z_2)$ represents an iterated integral of convolution-type. To be exact, it coincides with (8.0.1) or (8.0.2), except for the factor that involves the asymptotic parameter, which is replaced by the corresponding kernel, for instance by $\{\varphi(t)\}^{-z_2}$. As in the situation of a single integral, the admissible paths depend on the region of absolute convergence of the integral representation for $\mathfrak{M}(z_1, z_2)$. Particularly the interior path $\Re z_1 = x_{01}$ is permitted to depend on the exterior. Although integrals of the above type look much more complicated in comparison with their single analogues, their evaluation is accomplished in the same fashion.

Having conducted the transformation to an iterated MB-integral, the next step encompasses a thorough study of $\mathfrak{M}(z_1, z_2)$ as a function of one variable with the second fixed. If the fixed variable is z_2 with $\Re z_2 = x_{02}$, of particular interest are the analyticity properties and the location of the singularity that lies closest to the right of the line $\Re z_1 = x_{01}$, in which direction $m^{-\Re z_1}$ is descending. This singularity is indicated by the simple pole of $\Gamma(z_1)$ at $z_1 = 0$ and by the z_1-abscissa of convergence of the integral representation for $\mathfrak{M}(z_1, z_2)$. Of course a dependence on the fixed variable z_2 is possible. The determination of the closest singularity usually requires to compute the analytic continuation of $\mathfrak{M}(z_1, z_2)$ into a region that lies to the right of the z_1-abscissa of convergence and includes it. The information about the closest singularity and a simple bound for the asymptotic order of $\mathfrak{M}(z_1, z_2)$ into both imaginary directions of its z_1-region of analyticity are sufficient, to ascertain the permission for a displacement of the integration path in the interior MB-integral of $J(m)$, for brevity denoted by

$$K(m, z_2, x_{01}) := \frac{1}{2\pi i} \int\limits_{x_{01}-i\infty}^{x_{01}+i\infty} (m+1)^{-z_1} \Gamma(z_1) \mathfrak{M}(z_1, z_2) dz_1. \tag{8.1.8}$$

In contrast to the single integrals from Chapter 6, under rather mild assumptions, poles of second order will occur. For a more technical discussion, for fixed $z_2 \in \mathbb{C}$ with $\Re z_2 = x_{02}$, we assume the function

$$\Gamma(z_1) \mathfrak{M}(z_1, z_2) \tag{8.1.9}$$

is meromorphic in a finite strip $-1 < \Re z_1 < \chi$ for $\chi > -1$, and therein, to the right of the line $\Re z_1 = x_{01}$, it exhibits a sequence of poles $z_{11}, \ldots, z_{1K}$ for $K \in \mathbb{N}$, whose real parts are ascending. The boundary χ and the poles may depend on z_2. If, in addition for fixed z_2 and a constant $p \in \mathbb{R}$ we have

$$\mathfrak{M}(z_1, z_2) = \mathcal{O}\left\{ |z_1|^p \right\} \tag{8.1.10}$$

as $\Im z_1 \to \pm\infty$ in the indicated strip, uniformly with respect to $\Re z_1$ in any closed vertical substrip, the contour integral (8.1.8) converges absolutely along any vertical line $\Re z = x_{03}$, that does not run through any of the poles z_{1k}. Furthermore, since the algebraic behaviour of the generating function in (8.1.8) is always dominated by the exponential decay of the gamma function in the direction of the imaginary axis, a displacement of the integration path is easily viable. Upon moving the path to the right across the K poles of the function (8.1.9), to match a vertical line $\Re z_1 = x_{03}$ with $\Re z_{1K} < x_{03} < \chi$, for $1 \le k \le K$, we need to take into account the corresponding residues

$$\mathfrak{R}_k(z_2, m) := \operatorname*{Res}_{z_1 = z_{1k}} \frac{\Gamma(z_1)}{(m+1)^{z_1}} \mathfrak{M}(z_1, z_2). \tag{8.1.11}$$

Since a rightward displacement of the path means, that the poles are encircled in the clockwise direction, we incur a negative sign of these residues, which eventually leads to

$$K(m, z_2, x_{01}) = -\sum_{k=1}^{K} \mathfrak{R}_k(z_2, m) + K(m, z_2, x_{03}). \tag{8.1.12}$$

The remainder term is still given by the integral (8.1.8) but with x_{01} replaced by the new path $\Re z_1 = x_{03}$. If there exist further parameters $p_1, p_2 \in \mathbb{R}$ and a constant $M > 0$ for which the bound

$$\mathfrak{M}(z_1, z_2) \leq M\, |z_1|^{p_1}\, |z_2|^{p_2} \tag{8.1.13}$$

applies with $\Re z_1 = x_{03}$ and $\Re z_2 = x_{02}$, the remainder MB-integral for $\Re z_2 = x_{02}$, uniformly with respect to $\Im z_2 \in \mathbb{R}$, satisfies

$$K(m, z_2, x_{03}) = \mathcal{O}\left\{m^{-x_{03}}\, |z_2|^{p_2}\right\}. \tag{8.1.14}$$

The evaluation of the sum of residues on the right hand side of (8.1.12) is an easy exercise in complex calculus, particularly if each pole is of simple order. In any other case the actual difficulty consists in computing the derivatives of the generating function $\mathfrak{M}(z_1, z_2)$. These are required to specify its Laurent expansion in a neighborhood of $z_1 = z_{1k}$. Fortunately, throughout this chapter no poles of order greater than two will be encountered, whence for fixed z_2 as $z_1 \to z_{1k}$ we always have an expansion of the form

$$\mathfrak{M}(z_1, z_2) = \frac{\mathcal{M}_{-2}(z_2)}{(z_1 - z_{1k})^2} + \frac{\mathcal{M}_{-1}(z_2)}{z_1 - z_{1k}} + \mathcal{M}_0(z_2) + \mathcal{O}(z_1 - z_{1k}), \tag{8.1.15}$$

where we assume $\mathcal{M}_{-n}(z_2) \neq 0$ for at least one index $n \in \{0, 1, 2\}$. Notice that this last condition is not a restriction if z_2 is a fixed but arbitrary point in a region, in which $\mathcal{M}_{-n}(z_2)$ is holomorphic, since the zeros of holomorphic functions occur isolated. Once we have determined the coefficients in the above expansion and specified the location of the pole z_{1k}, the residue (8.1.11) readily can be obtained by reference to Theorem B.2.1(2).

Finally, if the interior of the iterated MB-integral (8.1.7) is replaced by the expansion (8.1.12), an additional estimate of the remainder integral by virtue of (8.1.14) as $m \to \infty$ shows

$$J(m) = -\sum_{k=1}^{K} \frac{1}{2\pi i} \int_{x_{02}-i\infty}^{x_{02}+i\infty} (m+1)^{-z_2} \Gamma(z_2) \mathfrak{R}_k(z_2, m)\, dz_2 + \mathcal{O}\left\{m^{-x_{03}-x_{02}}\right\}. \tag{8.1.16}$$

To complete the evaluation, each residue $\mathfrak{R}_k(z_2, m)$ must be studied as function of z_2, before the paths of the single integrals eventually can be moved. At this point it is important to note that there are two possible kinds of poles z_{1k}. On the one hand, the location of z_{1k} may depend on the fixed second variable z_2. In this event it is reasonable to assume without loss of generality that z_{1k} is of first order, and the residue $\mathfrak{R}_k(z_2, m)$ is readily confirmed to feature an asymptotic scale. Hence, the integrand of the k-th term in (8.1.16) involves a product of two asymptotic scales. Except if the sum $z_{1k} + z_2$ does not depend on z_2, we must then always distinguish between positivity and negativity of the real part of $z_{1k} + z_2$, in order to perform appropriate displacements of the integration path. If, however, in $z_{1k} + z_2$ the variable z_2 cancels out, no further steps are required but the k-th integral in (8.1.16) represents a coefficient in the asymptotic expansion of $J(m)$ as $m \to \infty$. On the other hand, if the location of z_{1k} does not depend on z_2, the residue $\mathfrak{R}_k(z_2, m)$ still depends on m but it does not establish an asymptotic scale as $m \to \infty$. More precisely, the behaviour of the residue as $m \to \infty$ is then not affected by z_2. In these circumstances, the k-th summand in (8.1.16) is evaluated by a rightward displacement of the integration path. The asymptotic evaluation of $J(m)$ as a whole is eventually completed, if either of the single MB-integrals produces a term that is dominating as $m \to \infty$.

8.2. Auxiliary Results

The objective of the present section is to examine the convergence behaviour and the analyticity properties of iterated generating functions of the type

$$\mathcal{M}\begin{bmatrix} -\zeta, \sigma, S \\ \tau, T \end{bmatrix} := \int_{\mathcal{P}_2} e(t) \int_{\mathcal{P}_1} \{\varphi(s)\}^{-\zeta}\, d(s) k(s+t) ds dt, \tag{8.2.1}$$

$$\mathcal{S}\left[-\zeta; \begin{matrix} \sigma, S \\ \tau, T \end{matrix}\right] := \int_{\mathcal{P}_2} e(t) \int_{\mathcal{P}_1} d(s)\, \{\varphi(s+t)\}^{-\zeta}\, k(s+t) ds dt. \tag{8.2.2}$$

Again $\mathcal{P}_1, \mathcal{P}_2$ are half-open subintervals of the positive real axis with respective endpoints $0 \leq \sigma < S \leq \infty$ and $0 \leq \tau < T \leq \infty$ along which the integrand is continuous, and $\varphi(r), \varphi(r_1+r_2) > 0$ for $r \in \mathcal{P}_1$, $r_j \in \mathcal{P}_j$, $j \in \{1, 2\}$, with the power functions taking their principal values. We denote $\{\varphi(s)\}^{-\zeta}$ as a *kernel of the first kind* and $\{\varphi(s+t)\}^{-\zeta}$ as a *kernel of the second kind*. Accordingly, (8.2.1) and (8.2.2) are referred to as *generating functions of the first* and *of the second kind*, respectively.

As a warm-up exercise we extend our findings that were established in Chapter 6 for single integral transforms. Preliminary we introduce some important notions and definitions, which will be of frequent use throughout this chapter.

8.2.1. Ingredient Functions with Algebraic Behaviour

First of all we adopt the term *ingredient functions* from one-dimensional integral transforms to describe a few or all of the functions φ, d, e, k appearing in the generating functions (8.2.1) and (8.2.2). Furthermore, if for a function $k(t)$ and constants $k_0 \in \mathbb{C} \setminus \{0\}$, $\kappa_0 \in \mathbb{C}$ as $t \downarrow \tau$ we have

$$k(t) \sim k_0 (t-\tau)^{\kappa_0}, \tag{8.2.3}$$

we say the function $k(t)$ is *algebraic as* $t \downarrow \tau$ *with coefficient* k_0 *and parameter* κ_0. Similarly, if $k(t)$ as $t \to \infty$ for constants $k_0 \in \mathbb{C} \setminus \{0\}$, $\kappa_0 \in \mathbb{C}$ satisfies

$$k(t) \sim k_0 t^{-\kappa_0}, \tag{8.2.4}$$

we refer to $k(t)$ as *algebraic at infinity with coefficient* k_0 *and parameter* κ_0. In both cases the requirement $k_0 \neq 0$ is essential.

Now, if the function $\varphi(t)$ is positive and continuous on the above interval $(\tau, T]$ and algebraic as $t \downarrow \tau$ for a coefficient $b_0 > 0$ and a parameter $\beta_0 \geq 0$, we denote by $B_\tau(t)$ the *normalized phase function*, formally

$$B_\tau(t) := \log\left\{\frac{(t-\tau)^{\beta_0}}{\varphi(t)}\right\}. \tag{8.2.5}$$

The index shows the correspondence to the endpoint τ. In an analogous fashion, if $k(t)$ is also continuous on the indicated interval and algebraic as $t \downarrow \tau$ with coefficient $k_0 \in \mathbb{C} \setminus \{0\}$ and parameter $\kappa_0 \in \mathbb{C}$, or if merely $k(t) = \mathcal{O}\left\{(t-\tau)^{\Re\kappa_0}\right\}$, we refer to

$$K_\tau(t) := \frac{k(t)}{(t-\tau)^{\kappa_0}} \tag{8.2.6}$$

as the *normalized amplitude*. In both definitions the term normalized basically means, the function denoted by the capital letter is continuous on $(\tau, T]$, and possesses a finite limit when approaching τ, which is especially non-zero if the local behaviour is not only described by a big-$\mathcal{O}$ estimate. In the latter case we agree, for instance

$$K_\tau(\tau) := \lim_{t \downarrow \tau} K_\tau(t) = k_0. \tag{8.2.7}$$

The above convention will also be employed if $\varphi(t)$, $k(t)$ are continuous on any closed subinterval of the ray $t \geq T$ with a finite $T > 0$, where the first of these functions is even positive, and if $\varphi(t) \sim b_0 t^{-\beta_0}$ and $k(t) \sim k_0 t^{-\kappa_0}$ as $t \to \infty$, or only $k(t) = \mathcal{O}\left\{t^{-\Re\kappa_0}\right\}$. In this event we define

$$B(t) := -\log\left\{t^{\beta_0}\varphi(t)\right\}, \tag{8.2.8}$$

$$K(t) := t^{\kappa_0} k(t). \tag{8.2.9}$$

Finally, if in addition $\varphi(t)$ is once continuously differentiable on $(\tau, T]$ it possesses a *normalized phase* $B_\tau(t)$ *with a derivative of order* $\beta_1 > -1$ *as* $t \downarrow \tau$, if and only if

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{\beta_0}{t-\tau} + \mathcal{O}\left\{(t-\tau)^{\beta_1}\right\}. \tag{8.2.10}$$

Moreover, at infinity the normalized phase $B(t)$ is said to have a *derivative of order* $\beta_1 > 1$, if and only if

$$\frac{\varphi'(t)}{\varphi(t)} = -\frac{\beta_0}{t} + \mathcal{O}\left\{t^{-\beta_1}\right\}. \tag{8.2.11}$$

Exactly the same notions will be employed for either of the amplitude functions. For example, $k(t)$ has a *normalized counterpart with a derivative of order* $\kappa_1 \in \mathbb{C}$ *as* $t \downarrow \tau$ *for* $\Re\kappa_1 > -1$, if and only if

$$\frac{k'(t)}{k(t)} = \frac{\kappa_0}{t-\tau} + \mathcal{O}\left\{(t-\tau)^{\Re\kappa_1}\right\}, \tag{8.2.12}$$

and it possesses a *normalized counterpart with a derivative of order* $\kappa_1 \in \mathbb{C}$ *as* $t \to \infty$ *for* $\Re\kappa_1 > 1$, if and only if

$$\frac{k'(t)}{k(t)} = -\frac{\kappa_0}{t} + \mathcal{O}\left\{t^{-\Re\kappa_1}\right\}. \tag{8.2.13}$$

The origin of the last definitions is clearly self-explaining. In fact, in case of (8.2.10) and (8.2.12) as $t \downarrow \tau$ it is easy to see:

$$B_\tau^{(1)}(t) = \frac{\beta_0}{t-\tau} - \frac{\varphi'(t)}{\varphi(t)} = \mathcal{O}\left\{(t-\tau)^{\beta_1}\right\} \tag{8.2.14}$$

$$K_\tau^{(1)}(t) = -\kappa_0 \frac{k(t)}{(t-\tau)^{\kappa_0+1}} + \frac{k'(t)}{(t-\tau)^{\kappa_0}} = \mathcal{O}\left\{(t-\tau)^{\Re\kappa_1}\right\} \tag{8.2.15}$$

Analogously in the situation of (8.2.11) and (8.2.13) as $t \to \infty$ we observe:

$$B^{(1)}(t) = -\frac{\beta_0}{t} - \frac{\varphi'(t)}{\varphi(t)} = \mathcal{O}\left\{t^{-\beta_1}\right\} \tag{8.2.16}$$

$$K^{(1)}(t) = \kappa_0 t^{\kappa_0-1} k(t) + t^{\kappa_0} k'(t) = \mathcal{O}\left\{t^{-\Re\kappa_1}\right\} \tag{8.2.17}$$

Special conventions will be employed for functions of the additive argument $s+t$ ranging along finite intervals $\sigma < s \leq S$ and $\tau < t \leq T$. In particular, if $\varphi(u) > 0$ and $k(u)$ are continuous at any point $u \in (\sigma+\tau, S+T]$ and algebraic as $u \downarrow \sigma+\tau$ with coefficients $b_0 > 0$, $k_0 \in \mathbb{C}\setminus\{0\}$ and parameters $\beta_0 > 0$, $\kappa_0 \in \mathbb{C}$, we denote the *normalized phase and amplitude* respectively by

$$B_{\sigma,\tau}(s+t) := \log\left\{\frac{(s+t-\sigma-\tau)^{\beta_0}}{\varphi(s+t)}\right\}, \tag{8.2.18}$$

$$K_{\sigma,\tau}(s+t) := \frac{k(s+t)}{(s+t-\sigma-\tau)^{\kappa_0}}. \tag{8.2.19}$$

Besides, assuming $\sigma \leq s_0 \leq S$ fixed, if $\varphi(s_0+t) \sim b_0(\tau, s_0)(t-\tau)^{\beta_0(\tau,s_0)}$ for $\beta_0(\tau, s_0), b_0(\tau, s_0) > 0$, and if $k(s_0+t) \sim k_0(\tau, s_0)(t-\tau)^{\kappa_0(\tau,s_0)}$ for $\kappa_0(\tau, s_0), k_0(\tau, s_0) \in \mathbb{C}$ with $k_0(\tau, s_0) \neq 0$, we write

$$B(t; s_0) := \log \left\{ \frac{(t-\tau)^{\beta_0(\tau,s_0)}}{\varphi(s_0+t)} \right\}, \tag{8.2.20}$$

$$K(t; s_0) := \frac{k(s_0+t)}{(t-\tau)^{\kappa_0(\tau,s_0)}}, \tag{8.2.21}$$

and it is reasonable to refer to $B(t; s_0)$ and $K(t; s_0)$ as the normalized phase and amplitude of $\varphi(s_0+t)$ and $k(s_0+t)$ on $(\tau, T]$, respectively. It is then easy to see, if $B_{\sigma,\tau}(u)$ and $K_{\sigma,\tau}(u)$ possess a first derivative of order $\beta_1, \Re\kappa_1 > -1$ as $u \downarrow \sigma+\tau$, the first derivatives of $B(t; \sigma)$ and $K(t; \sigma)$ share this property as $t \downarrow \tau$.

Finally, for arbitrary $z_1, \ldots, z_n \in \mathbb{C}$ with $n \in \mathbb{N}$ and $\beta > 0$ we denote

$$\chi_\beta(z_1, \ldots, z_n) := \frac{\min\{\Re z_1, \ldots, \Re z_n\} + 1}{\beta}, \tag{8.2.22}$$

$$\eta_\beta(z_1, \ldots, z_n) := \frac{\min\{\Re z_1, \ldots, \Re z_n\} - 1}{\beta}. \tag{8.2.23}$$

Both functions are evidently real valued and non-negative, respectively if $\Re z_k > -1$ and $\Re z_k > 1$ for any $1 \leq k \leq n$.

8.2.2. Single Generating Functions with Convolution-Type Ingredients

In Section 6.3 we considered a special type of generating functions and discussed two techniques to determine their analytic continuation. In the present subsection we shall extend these findings to the generating function

$$\mathcal{N}\begin{bmatrix} -z, t \\ \sigma, S \end{bmatrix} := \int_\sigma^S \{\varphi(s)\}^{-z}\, d(s) k(s+t) ds, \tag{8.2.24}$$

for a fixed but arbitrary $t \geq 0$ and $0 \leq \sigma < S \leq \infty$. Owing to the convolution-type argument of the amplitude $k(s+t)$, if (8.2.24) represents the interior of an iterated integral it appears reasonable to treat the situation of a finite and an infinite range of integration separately. The aim of this subsection is in particular, to point out the different roles played by $k(s+t)$ in case of a finite and an infinite endpoint, to specify the respective analytic continuation of the integral by partial integration, and to provide the required quantities for calculating with fixed $m \geq 0$ the residue of the product

$$(m+1)^{-z}\Gamma(z)\mathcal{N}\begin{bmatrix} -z, t \\ \sigma, S \end{bmatrix} \tag{8.2.25}$$

at $z = 0$ if the latter point is a second order pole. The actual computation of this residue is postponed to a later section.

8.2.2.1. A Finite Range of Integration and a Kernel of the First Kind

Concerning the ingredient functions $\varphi(s)$ and $d(s)$, we suppose once continuous differentiability and $\varphi(s) > 0$ on $(\sigma_0, S]$. Besides, $\varphi(s)$ shows algebraic behaviour as $s \downarrow \sigma_0$ for a parameter $\beta_0 > 0$ and a coefficient $b_0 > 0$, and the first derivative of the normalized phase is of order $\beta_1 > -1$ as $s \downarrow \sigma_0$. Moreover, we assume there exist parameters $\delta_0, \delta_1 \in \mathbb{C}$ with $\Re\delta_1 > -1$, for which as $s \downarrow \sigma_0$ we have

$$\begin{aligned} d(s) &= \mathcal{O}\left\{(s-\sigma_0)^{\delta_0}\right\}, \\ D'(s) &= \mathcal{O}\left\{(s-\sigma_0)^{\delta_1}\right\}, \end{aligned} \tag{8.2.26}$$

where $D(s) := (s-\sigma_0)^{-\delta_0} d(s)$ is the normalized amplitude as in (8.2.6), for brevity without the index. We then define

$$d_0 := \lim_{s\downarrow\sigma_0} D(s) \tag{8.2.27}$$

and observe $d_0 \in \mathbb{C}$. Finally, the amplitude function $k(s+t)$ is also once continuous differentiable with respect to s on $(\sigma_0, S]$, and as $s \downarrow \sigma_0$, for $k_0(\sigma_0, t) \in \mathbb{C} \setminus \{0\}$ and $\kappa_0(\sigma_0, t), \kappa_1(\sigma_0, t) \in \mathbb{C}$ with $\Re\kappa_1(\sigma_0, t) > -1$, we have

$$\begin{cases} k(s+t) \sim k_0(\sigma_0, t)(s-\sigma_0)^{\kappa_0(\sigma_0,t)}, \\ \frac{k'(s+t)}{k(s+t)} = \frac{\kappa_0(\sigma_0,t)}{s-\sigma_0} + \mathcal{O}\left\{(s-\sigma_0)^{\kappa_1(\sigma_0,t)}\right\}. \end{cases} \tag{8.2.28}$$

By inspection of (8.2.24) with $\sigma = \sigma_0$ it is easy to see that, in these circumstances the function $k(s+t)$ only contributes to the region of absolute convergence of the indicated integral, if t is a fixed point and $\Re\kappa_0(\sigma_0, t) \neq 0$. Conversely, if t is a fixed but arbitrary point from an interval $[a, b]$ for real numbers $0 \leq a < b$, and $k(u) \neq 0$ for some $u \in (\sigma_0 + a, S + b]$, it is reasonable to assume $\kappa_0(\sigma_0, t) \equiv 0$. In any case, by Lemma 6.3.1, the region of absolute convergence and analyticity of (8.2.24) coincides with the half plane $\Re z < \Re\chi_0 + \frac{\Re\kappa_0(\sigma_0,t)}{\beta_0}$, where

$$\chi_0 := \frac{1+\delta_0}{\beta_0}. \tag{8.2.29}$$

Notice, however, that the exact region of analyticity may even be larger, since the parameter δ_0 need not be unique. Denote by $B(s)$ the normalized phase as in (8.2.5), also for brevity without the index σ_0, and refer to $K(s;t)$ as the normalized amplitude on $(\sigma_0, S]$, analogous to (8.2.21). In the fashion of (6.3.48), it is then easy to deduce by partial integration for fixed $z \in \mathbb{C}$ with

$\Re z < \Re\chi_0 + \frac{\Re\kappa_0(\sigma_0,t)}{\beta_0}$:

$$\mathcal{N}\begin{bmatrix} -z, t \\ \sigma_0, S \end{bmatrix} = -\frac{(S-\sigma_0)^{\beta_0(\chi_0 - z)+\kappa_0(\sigma_0,t)}}{\beta_0(z-\chi_0)-\kappa_0(\sigma_0,t)} e^{zB(S)} D(S) K(S;t)$$

$$+ \frac{1}{\beta_0(z-\chi_0)-\kappa_0(\sigma_0,t)} \int_{\sigma_0}^{S} (s-\sigma_0)^{\beta_0(\chi_0-z)+\kappa_0(\sigma_0,t)} \tag{8.2.30}$$

$$\times \frac{d}{ds}\left\{e^{zB(s)} D(s) K(s;t)\right\} ds$$

$$= -\frac{(S-\sigma_0)^{\beta_0(\chi_0 - z)+\kappa_0(\sigma_0,t)}}{\beta_0(z-\chi_0)-\kappa_0(\sigma_0,t)} e^{zB(S)} D(S) K(S;t) \tag{8.2.31}$$

$$+ \frac{z}{\beta_0(z-\chi_0)-\kappa_0(\sigma_0,t)} \int_{\sigma_0}^{S} (s-\sigma_0)^{\beta_0(\chi_0-z)+\kappa_0(\sigma_0,t)} e^{zB(s)} B'(s) D(s) K(s;t) ds$$

$$+ \frac{1}{\beta_0(z-\chi_0)-\kappa_0(\sigma_0,t)} \int_{\sigma_0}^{S} (s-\sigma_0)^{\beta_0(\chi_0-z)+\kappa_0(\sigma_0,t)} e^{zB(s)} D'(s) K(s;t) ds$$

$$+ \frac{1}{\beta_0(z-\chi_0)-\kappa_0(\sigma_0,t)} \int_{\sigma_0}^{S} (s-\sigma_0)^{\beta_0(\chi_0-z)+\kappa_0(\sigma_0,t)} e^{zB(s)} D(s) K'(s;t) ds$$

According to the definition of $K(s;t)$, we can equivalently write

$$K'(s;t) = \frac{k'(s+t)}{(s-\sigma_0)^{\kappa_0(\sigma_0,t)}} - \kappa_0(\sigma_0,t)\frac{k(s+t)}{(s-\sigma_0)^{\kappa_0(\sigma_0,t)+1}}.$$

Following from our assumptions on the ingredient functions, the expansion (8.2.31) thus establishes the analytic continuation of (8.2.24) into the half plane

$$\Re z < \Re\chi_0 + \frac{\Re\kappa_0(\sigma_0,t)}{\beta_0} + \chi_{\beta_0}(\beta_1, \delta_1, \kappa_1(\sigma_0,t)). \tag{8.2.32}$$

The latter especially includes the vertical line $\Re z = \Re\chi_0 + \frac{\Re\kappa_0(\sigma_0,t)}{\beta_0}$, which we recall as the abscissa of convergence of the original representation (8.2.24). Along this line the analytic continuation indicates the presence of a simple pole as the only singularity in the extended region. According to (8.2.30) and subject to the fundamental theorem of calculus, the associated residue is given by:

$$\operatorname*{Res}_{z=\chi_0+\frac{\kappa_0(\sigma_0,t)}{\beta_0}} \mathcal{N}\begin{bmatrix} -z, t \\ \sigma_0, S \end{bmatrix} = -\frac{1}{\beta_0} e^{\left(\chi_0 + \frac{\kappa_0(\sigma_0,t)}{\beta_0}\right)B(S)} D(S) K(S;t)$$

$$+ \frac{1}{\beta_0}\int_{\sigma_0}^{S} \frac{d}{ds}\left\{e^{\left(\chi_0+\frac{\kappa_0(\sigma_0,t)}{\beta_0}\right)B(s)} D(s) K(s;t)\right\} ds$$

$$= -\frac{d_0}{\beta_0}\left\{b_0\right\}^{-\chi_0 - \frac{\kappa_0(\sigma_0,t)}{\beta_0}} K(\sigma_0;t) \tag{8.2.33}$$

Here, upon taking into account the first requirement in (8.2.28) and concluding $k_0(\sigma_0, t) = k(\sigma_0 + t)$ in the case $\kappa_0(\sigma_0, t) = 0$, by definition of $K(s; t)$, we have

$$K(\sigma_0; t) = \begin{cases} k(\sigma_0 + t), & \text{if } \kappa_0(\sigma_0, t) = 0, \\ k_0(\sigma_0, t), & \text{otherwise.} \end{cases} \tag{8.2.34}$$

It must be emphasized that, if and only if $d_0 = 0$, the above residue equals zero and at $z = \chi_0 + \frac{\kappa_0(\sigma_0,t)}{\beta_0}$ the principal part of the Laurent expansion of $\mathcal{N}[\ldots]$ vanishes. In this event, the function is holomorphic in the whole half plane (8.2.32).

In the situation $\chi_0 = -\frac{\kappa_0(\sigma_0,t)}{\beta_0}$ the above point matches the origin of the complex z-plane. There, the product (8.2.25) may then exhibit a pole of order no greater than two. To characterize the corresponding residue we note, in a neighborhood of $z = 0$ the generating function possesses a Laurent expansion of the form

$$\mathcal{N}\begin{bmatrix} -z, t \\ \sigma_0, S \end{bmatrix} = -\frac{1}{z}\frac{d_0}{\beta_0} K(\sigma_0; t) + \nu_0(t; \sigma_0, S) + \mathcal{O}(z), \tag{8.2.35}$$

in which the residue was computed from (8.2.33) with $\chi_0 + \frac{\kappa_0(\sigma_0,t)}{\beta_0} = 0$. Regarding the coefficient $\nu_0(t; \sigma_0, S)$ we know from Taylor's theorem:

$$\begin{aligned} \nu_0(t; \sigma_0, S) :=& \frac{d}{dz} z\mathcal{N}\begin{bmatrix} -z, t \\ \sigma_0, S \end{bmatrix}\Bigg|_{z=0} \\ =& -\frac{1}{\beta_0}\frac{d}{dz}\left\{(S - \sigma_0)^{-\beta_0 z} e^{zB(S)} D(S) K(S; t)\right\}\Bigg|_{z=0} \\ &+ \frac{1}{\beta_0}\frac{d}{dz}\int_{\sigma_0}^{S} (s - \sigma_0)^{-\beta_0 z} \frac{d}{ds}\left\{e^{zB(s)} D(s) K(s; t)\right\} ds\Bigg|_{z=0} \end{aligned}$$

Arbitrary derivatives of the integral function on the right hand side can be determined by differentiation under the integral sign, subject to uniform convergence with respect to z in any compact subset of the region (8.2.32). See also Lemma 6.3.1. But by (8.2.31) and by definition of $B(s)$ in (8.2.5) we deduce

$$\begin{aligned} \int_{\sigma_0}^{S} (s - \sigma_0)^{-\beta_0 z} \frac{d}{ds}\left\{e^{zB(s)} D(s) K(s; t)\right\} ds =& z\int_{\sigma_0}^{S} \{\varphi(s)\}^{-z} B'(s) D(s) K(s; t) ds \\ &+ \int_{\sigma_0}^{S} \{\varphi(s)\}^{-z} \left\{D'(s) K(s; t) + D(s) K'(s; t)\right\} ds. \end{aligned}$$

We thus eventually arrive at

$$\begin{aligned}\nu_0(t;\sigma_0,S) = \frac{D(S)}{\beta_0}\log\{\varphi(S)\}\,K(S;t) + \frac{1}{\beta_0}\int\limits_{\sigma_0}^{S} B'(s)D(s)K(s;t)ds \\ - \frac{1}{\beta_0}\int\limits_{\sigma_0}^{S}\log\{\varphi(s)\}\left\{D'(s)K(s;t)+D(s)K'(s;t)\right\}ds.\end{aligned} \tag{8.2.36}$$

8.2.2.2. An Infinite Range of Integration and a Kernel of the First Kind

The contribution of the amplitude $k(s+t)$ to the absolute convergence of the integral (8.2.24) becomes substantially more important if the upper endpoint is infinite, i.e., if

$$\mathcal{N}\begin{bmatrix}-z,t\\ S,\infty\end{bmatrix} = \int\limits_{S}^{\infty}\{\varphi(s)\}^{-z}\,d(s)k(s+t)ds, \tag{8.2.37}$$

where $\varphi(u)$, $d(u)$ and $k(u)$ are once continuously differentiable on $u \geq S > 0$. Furthermore, $\varphi(u) > 0$, and at infinity each function exhibits algebraic behaviour for parameters $\beta_0 > 0$, $\delta_0, \kappa_0 \in \mathbb{C}$ and constants $b_0 > 0$, $d_0, k_0 \in \mathbb{C}\setminus\{0\}$. Besides, each function is assumed to have a normalized counterpart with a derivative of order $\beta_1, \Re\delta_1, \Re\kappa_1 > 1$ at infinity. Notice that the parameter κ_0 describes not only the behaviour of $k(u)$ at infinity but especially of $k(s+t)$ as $s\to\infty$ for a fixed but arbitrary $t \geq 0$. This is in contrast to the last paragraph, where the local parameter of $k(s+t)$ was seen to depend on t. As a consequence, in the present situation for any fixed $t \geq 0$ both functions $f(s,t) := s^{-\kappa_0}k(s+t)$ and the normalized amplitude $K(s+t)$ are $\mathcal{O}(1)$ as $s\to\infty$ for any fixed $t \geq 0$. For later investigations of (8.2.37) as the interior of an iterated integral we recommend use of $K(s+t)$, since this function itself and all of its derivatives again depend on $s+t$, whereas $f(s,t)$ actually possesses two arguments s and $s+t$. The choice of $K(s+t)$, however, makes the integration by parts procedure for analytically continuing the above integral slightly more elaborate.

In the described setup the integral (8.2.37) for arbitrary $t \geq 0$ converges absolutely and uniformly in any compact subset of $\Re z < \Re\eta_0$, where we now denote

$$\begin{cases}\chi_0 := \frac{\delta_0-1}{\beta_0},\\ \eta_0 := \frac{\delta_0-1}{\beta_0}+\frac{\kappa_0}{\beta_0},\end{cases} \tag{8.2.38}$$

and it is thus holomorphic there. Representing the ingredient functions φ, d and k in terms of their normalized counterparts B, D and K, see (8.2.8) and (8.2.9), upon integrating by parts we obtain for $\Re z < \Re\eta_0$ and fixed $t \geq 0$:

$$\mathcal{N}\begin{bmatrix}-z,t\\ S,\infty\end{bmatrix} = \int\limits_{S}^{\infty} s^{\beta_0(z-\eta_0)-1}e^{zB(s)}D(s)\frac{K(s+t)}{(1+\frac{t}{s})^{\kappa_0}}ds$$

$$= -\frac{S^{\beta_0(z-\eta_0)}}{\beta_0(z-\eta_0)} e^{zB(S)} D(S) \frac{K(S+t)}{(1+\frac{t}{S})^{\kappa_0}} - \frac{1}{\beta_0(z-\eta_0)} \int_S^\infty s^{\beta_0(z-\eta_0)} \frac{d}{ds} \left\{ e^{zB(s)} D(s) \frac{K(s+t)}{(1+\frac{t}{s})^{\kappa_0}} \right\} ds \tag{8.2.39}$$

In the first equality we also factorized the integrand into a simple power of s and a function that is bounded and non-vanishing at infinity. By virtue of the product and the chain rule we finally arrive at

$$\begin{aligned} \mathcal{N}\begin{bmatrix} -z, t \\ S, \infty \end{bmatrix} = &-\frac{S^{\beta_0(z-\chi_0)}}{\beta_0(z-\eta_0)} e^{zB(S)} D(S) \frac{K(S+t)}{(S+t)^{\kappa_0}} \\ &- \frac{z}{\beta_0(z-\eta_0)} \int_S^\infty s^{\beta_0(z-\chi_0)} e^{zB(s)} B'(s) D(s) \frac{K(s+t)}{(s+t)^{\kappa_0}} ds \\ &- \frac{1}{\beta_0(z-\eta_0)} \int_S^\infty s^{\beta_0(z-\chi_0)} e^{zB(s)} D'(s) \frac{K(s+t)}{(s+t)^{\kappa_0}} ds \\ &- \frac{1}{\beta_0(z-\eta_0)} \int_S^\infty s^{\beta_0(z-\chi_0)} e^{zB(s)} D(s) \frac{K'(s+t)}{(s+t)^{\kappa_0}} ds \\ &- \frac{t\kappa_0}{\beta_0(z-\eta_0)} \int_S^\infty s^{\beta_0(z-\chi_0)-1} e^{zB(s)} D(s) \frac{K(s+t)}{(s+t)^{1+\kappa_0}} ds. \end{aligned} \tag{8.2.40}$$

Since the derivatives of B, D and K at infinity are of order β_1, $\Re\delta_1$ and $\Re\kappa_1$, respectively, it becomes clear that all integrals in (8.2.40) converge absolutely for $z \in \mathbb{C}$ with

$$\Re z < \Re\eta_0 + \eta_{\beta_0}(\beta_1, \delta_1, \kappa_1, 2). \tag{8.2.41}$$

Due to the additional uniform convergence of each integral, the expansion (8.2.40) establishes the analytic continuation of the initial integral (8.2.37) into the wider half plane (8.2.41) for any fixed $t \geq 0$. Observe that the integrals in the above expansion are of the same type as the initial representation, which was not the case if $f(s,t)$ rather than $K(s+t)$ was employed. Now, the extended half plane (8.2.41) contains the abscissa of convergence of (8.2.37). There, particularly at the point $z = \eta_{00}$, the continuation (8.2.40) exhibits a simple pole. The associated residue is readily computed from (8.2.39) with the aid of the fundamental theorem of calculus:

$$\begin{aligned} \operatorname*{Res}_{z=\eta_0} \mathcal{N}\begin{bmatrix} -z, t \\ S, \infty \end{bmatrix} &= -\frac{1}{\beta_0} e^{\eta_0 B(S)} D(S) \frac{K(S+t)}{(1+\frac{t}{S})^{\kappa_0}} \\ &\quad - \frac{1}{\beta_0} \int_S^\infty \frac{d}{ds} \left\{ e^{\eta_0 B(s)} D(s) \frac{K(s+t)}{(1+\frac{t}{s})^{\kappa_0}} \right\} ds \\ &= -\frac{d_0 k_0}{\beta_0} \{b_0\}^{-\eta_0} \end{aligned} \tag{8.2.42}$$

In the exceptional case $\eta_0 = 0$, in (8.2.25) the latter singularity merges with the simple pole of the gamma function at $z = 0$ to a pole of second order. To determine the residue of the indicated product then, as $z \to 0$ we conclude from the elementary rules of complex calculus

$$\mathcal{N}\begin{bmatrix} -z, t \\ S, \infty \end{bmatrix} = -\frac{1}{z}\frac{d_0 k_0}{\beta_0} + \nu_0(t;S) + \mathcal{O}(z), \tag{8.2.43}$$

where the residue was again obtained from (8.2.42) with $z = \eta_0$, while Taylor's theorem yields:

$$\begin{aligned} \nu_0(t;S) :=& \frac{d}{dz} z\mathcal{N}\begin{bmatrix} -z, t \\ S, \infty \end{bmatrix}\Bigg|_{z=0} \\ =& -\frac{1}{\beta_0}\frac{d}{dz}\left\{ S^{\beta_0 z} e^{zB(S)} D(S) \frac{K(S+t)}{(1+\frac{t}{S})^{\kappa_0}} \right\}\Bigg|_{z=0} \\ & - \frac{1}{\beta_0}\frac{d}{dz}\int_S^\infty s^{\beta_0 z}\frac{d}{ds}\left\{ e^{zB(s)} D(s) \frac{K(s+t)}{(1+\frac{t}{s})^{\kappa_0}} \right\} ds\Bigg|_{z=0} \end{aligned} \tag{8.2.44}$$

According to the uniform convergence of each integral in the expansion (8.2.40), differentiation under the integral sign is admitted. By taking into account the definition of the normalized phase, with $\chi_0 = -\frac{\kappa_0}{\beta_0}$ this leads to

$$\begin{aligned} \nu_0(t;S) =& \frac{S^{\kappa_0}}{\beta_0}\log\{\varphi(S)\} D(S)\frac{K(S+t)}{(S+t)^{\kappa_0}} \\ & - \frac{1}{\beta_0}\int_S^\infty s^{\kappa_0}\left\{B'(s)D(s) - \log\{\varphi(s)\} D'(s)\right\}\frac{K(s+t)}{(s+t)^{\kappa_0}} ds \\ & + \frac{1}{\beta_0}\int_S^\infty s^{\kappa_0}\log\{\varphi(s)\} D(s)\frac{K'(s+t)}{(s+t)^{\kappa_0}} ds \\ & + t\frac{\kappa_0}{\beta_0}\int_S^\infty s^{\kappa_0 - 1}\log\{\varphi(s)\} D(s)\frac{K(s+t)}{(s+t)^{1+\kappa_0}} ds. \end{aligned} \tag{8.2.45}$$

8.2.2.3. An Infinite Range of Integration and a Kernel of the Second Kind

A slight modification of the described technique is required if, under the assumptions from the last paragraph, for fixed $t \geq 0$ we consider a generating function of the form

$$\mathfrak{Z}\begin{bmatrix} -z, t \\ S \end{bmatrix} := \int_S^\infty \{\varphi(s+t)\}^{-z} d(s)k(s+t)ds. \tag{8.2.46}$$

In these circumstances the integral on the right hand side for any $t \geq 0$ still converges absolutely and is holomorphic in the half plane $\Re z < \Re\eta_0$ with the parameter η_0 that was defined in (8.2.38). To determine an expansion of its analytic continuation in terms of integrals of the above type, however, we must now also take into account the dependence of the kernel on the second variable.

By appropriately integrating by parts we then obtain for $\Re z < \eta_0$ and $t \geq 0$, similar to (8.2.39) by definition of η_0:

$$\mathfrak{Z}\begin{bmatrix} -z, t \\ S \end{bmatrix} = -\frac{S^{\beta_0(z-\eta_0)}}{\beta_0(z-\eta_0)} D(S) \frac{K(S+t)}{(1+\frac{t}{S})^{\kappa_0-\beta_0 z}} e^{zB(S+t)}$$

$$- \frac{1}{\beta_0(z-\eta_0)} \int\limits_S^\infty s^{\beta_0(z-\eta_0)} \frac{d}{ds} \left\{ D(s) \frac{K(s+t)}{(1+\frac{t}{s})^{\kappa_0-\beta_0 z}} e^{zB(s+t)} \right\} ds \tag{8.2.47}$$

$$= -\frac{S^{1-\delta_0}}{\beta_0(z-\eta_0)} D(S) \frac{K(S+t)}{(S+t)^{\kappa_0-\beta_0 z}} e^{zB(S+t)} \tag{8.2.48}$$

$$- \frac{1}{\beta_0(z-\eta_0)} \int\limits_S^\infty s^{1-\delta_0} D'(s) \frac{K(s+t)}{(s+t)^{\kappa_0-\beta_0 z}} e^{zB(s+t)} ds$$

$$- \frac{z}{\beta_0(z-\eta_0)} \int\limits_S^\infty s^{1-\delta_0} D(s) B'(s+t) \frac{K(s+t)}{(s+t)^{\kappa_0-\beta_0 z}} e^{zB(s+t)} ds$$

$$- \frac{1}{\beta_0(z-\eta_0)} \int\limits_S^\infty s^{1-\delta_0} D(s) \frac{K'(s+t)}{(s+t)^{\kappa_0-\beta_0 z}} e^{zB(s+t)} ds$$

$$+ \frac{t(\beta_0 z-\kappa_0)}{\beta_0(z-\eta_0)} \int\limits_S^\infty s^{-\delta_0} D(s) \frac{K(s+t)}{(s+t)^{1+\kappa_0-\beta_0 z}} e^{zB(s+t)} ds$$

In accordance with the order of the derivatives of B, D and K at infinity, absolute convergence of the above integrals holds for $z \in \mathbb{C}$ with

$$\Re z < \Re \eta_0 + \eta_{\beta_0}(\beta_1, \delta_1, \kappa_1, 2). \tag{8.2.49}$$

Since the convergence of each integral is uniform in any compact subset therein, the expansion (8.2.48) for fixed $t \geq 0$ analytically extends (8.2.46) to the region (8.2.49), where it shows a pole of simple order at $z = \eta_0$. The computation of the residue from (8.2.47) is a routine task and leads to

$$\operatorname*{Res}_{z=\eta_0} \mathfrak{Z}\begin{bmatrix} -z, t \\ S \end{bmatrix} = -\frac{d_0 k_0}{\beta_0} \{b_0\}^{-\eta_0}. \tag{8.2.50}$$

For $\eta_0 = 0$ this last pole lies at the origin, thereby leading to a second order pole of the product (8.2.25) there. Then, in an annulus around $z = 0$ we find a Laurent expansion of the generating function (8.2.46) with controlling terms

$$\mathfrak{Z}\begin{bmatrix} -z, t \\ S \end{bmatrix} = -\frac{1}{z}\frac{d_0 k_0}{\beta_0} + \zeta_0(t; S) + \mathcal{O}(z), \tag{8.2.51}$$

in which the coefficient of the constant summand can be computed by termwise differentiation of the expansion (8.2.48), in particular by differentiation under the integral sign. Since $\eta_0 = 0$

implies $1 - \delta_0 = \kappa_0$, this eventually brings us:

$$
\begin{aligned}
\zeta_0(t;S) :=& \frac{d}{dz} z\mathfrak{Z}\begin{bmatrix} -z, t \\ S \end{bmatrix}\Bigg|_{z=0} \\
=& -\frac{1}{\beta_0}\frac{d}{dz} S^{\beta_0 z} D(S) \frac{K(S+t)}{(1+\frac{t}{S})^{\kappa_0 - \beta_0 z}} e^{zB(S+t)}\Bigg|_{z=0} \\
& - \frac{1}{\beta_0}\frac{d}{dz}\int_S^\infty s^{\beta_0 z}\frac{d}{ds}\left\{D(s)\frac{K(s+t)}{(1+\frac{t}{s})^{\kappa_0-\beta_0 z}} e^{zB(s+t)}\right\} ds\Bigg|_{z=0} \\
=& \frac{S^{\kappa_0}}{\beta_0} D(S)\frac{K(S+t)}{(S+t)^{\kappa_0}} \log\{\varphi(S+t)\} \\
& + \frac{1}{\beta_0}\int_S^\infty s^{\kappa_0} D'(s)\frac{K(s+t)}{(s+t)^{\kappa_0}}\log\{\varphi(s+t)\}\, ds \\
& - \frac{1}{\beta_0}\int_S^\infty \frac{s^{\kappa_0}}{(s+t)^{\kappa_0}} D(s)\left\{B'(s+t)K(s+t) - K'(s+t)\log\{\varphi(s+t)\}\right\} ds \\
& + \frac{t}{\beta_0}\int_S^\infty s^{\kappa_0 - 1} D(s)\frac{K(s+t)}{(s+t)^{1+\kappa_0}}\{\beta_0 + \kappa_0 \log\{\varphi(s+t)\}\}\, ds
\end{aligned} \tag{8.2.52}
$$

8.2.3. Convergence Tests for Iterated Convolution-Type Integrals

To characterize the convergence behaviour of iterated integrals of convolution type is slightly more complicated in comparison with single integrals. This statement is readily confirmed if, for $p_1, p_2, q \in \mathbb{R}$, we consider the iterated Mellin-type integral

$$\int_0^\infty t^{p_2}\int_0^\infty s^{p_1}(s+t)^q\, ds dt.$$

From single integrals of this type we know, that there are two critical areas, where the integrand vanishes or diverges, respectively near the origin and near infinity. Regarding the above iterated integral, however, we notice four pairs of critical segments, viz as $s, t \downarrow 0$, as $s, t \to \infty$ and as $s \downarrow 0, t \to \infty$ as well as the converse. It is therefore reasonable to separate the integral, for a distinction between three different pairs of integration paths. After elementary manipulations, in each case definite conditions can be established by virtue of the criteria that are known for the convergence of single integrals.

Lemma 8.2.1 (two zero endpoints). *With $T_1, T_2 > 0$ the integral*

$$\int_0^{T_2} t^{p_2}\int_0^{T_1} s^{p_1}(s+t)^q\, ds dt$$

converges absolutely if and only if $p_1, p_2 > -1$ and $p_1 + p_2 + q > -2$.

Observe that the condition $p_j > -1$ for each $j \in \{1,2\}$ is also required for the absolute convergence of $\int_0^{T_j} u^{p_j}(u+v)^q du$ for an arbitrary fixed $v > 0$. Accordingly, we describe it as the *condition for the convergence of the single component* of the above iterated integral. The additional restriction $p_1 + p_2 + q > -2$ is irrelevant for the absolute convergence of each single component, but it is due to the iteration

Proof. From substitution and upon formally interchanging the order of integration we obtain:

$$\begin{aligned}
\int_0^{T_2} t^{p_2} \int_0^{T_1} s^{p_1}(s+t)^q ds dt &= \int_0^{T_2} t^{p_1+p_2+q+1} \int_0^{\frac{T_1}{t}} u^{p_1}(1+u)^q du dt \\
&= \int_0^{\infty} u^{p_1}(1+u)^q \int_0^{\min\left\{\frac{T_1}{u}, T_2\right\}} t^{p_1+p_2+q+1} dt du \\
&= \int_0^{\frac{T_1}{T_2}} u^{p_1}(1+u)^q du \int_0^{T_2} t^{p_1+p_2+q+1} dt + \int_{\frac{T_1}{T_2}}^{\infty} u^{p_1}(1+u)^q \int_0^{\frac{T_1}{u}} t^{p_1+p_2+q+1} dt du \\
&= \frac{T_2^{p_1+p_2+q+2}}{p_1+p_2+q+2} \int_0^{\frac{T_1}{T_2}} u^{p_1}(1+u)^q du \\
&\qquad + \frac{T_1^{p_1+p_2+q+2}}{p_1+p_2+q+2} \int_{\frac{T_1}{T_2}}^{\infty} u^{-p_2-q-2}(1+u)^q du
\end{aligned}$$

Clearly, by absolute convergence these manipulations are permitted if and only if $p_1, p_2 > -1$ and $p_1 + p_2 + q > -2$. ∎

From the case of two finite integration paths we now deduce a convergence condition if both paths emerge from a finite point but end at infinity.

Corollary 8.2.2 (two infinite endpoints). *With $T_1, T_2 > 0$ the integral*

$$\int_{T_2}^{\infty} t^{p_2} \int_{T_1}^{\infty} s^{p_1}(s+t)^q ds dt$$

converges absolutely if and only if $p_1 + q < -1$, $p_2 + q < -1$ and $p_1 + p_2 + q < -2$.

Proof. A simple change of variables in each integral maps the point at infinity to the origin, which yields

$$\int_{T_2}^{\infty} t^{p_2} \int_{T_1}^{\infty} s^{p_1}(s+t)^q ds dt = \int_0^{\frac{1}{T_2}} v^{-p_2-q-2} \int_0^{\frac{1}{T_1}} u^{-p_1-q-2}(u+v)^q du dv.$$

The absolute convergence thus follows immediately from Lemma 8.2.1. ∎

The last important case can not be derived solely by means of a substitution.

Lemma 8.2.3 (a zero endpoint and an infinite endpoint). *With $T_1, T_2 > 0$ the integral*

$$\int_{T_2}^{\infty} t^{p_2} \int_0^{T_1} s^{p_1}(s+t)^q ds dt$$

converges absolutely if and only if $p_1 > -1$ and $p_2 + q < -1$.

Contrary to the preceding two criteria we notice independence of the parameters p_1 and p_2+q. In other words, the iterated integral in Lemma 8.2.3 converges absolutely if and only if each of its single components does so. The reason is that there are three areas where the integrand exhibits critical behaviour, respectively as $s \downarrow 0$, as $t \to \infty$ and as $(s,t) \to \infty$. But $(s,t) \to \infty$ if and only if $t \to \infty$. Conversely, for instance in Lemma 8.2.1 there are also three different critical areas, namely as $s \downarrow 0$, as $t \downarrow 0$ and as $(s,t) \downarrow 0$. However, in these circumstances $(s,t) \downarrow 0$ does not follow if either $s \downarrow 0$ or $t \downarrow 0$. Similarly in Corollary 8.2.2 the critical areas are as $s \to \infty$, as $t \to \infty$ and as $(s,t) \to \infty$. But $(s,t) \to \infty$ happens if either of the single variables or even both of them simultaneously tend to infinity. This leads to the respective extra restrictions $p_1 + p_2 + q > -2$ and $p_1 + p_2 + q < -2$, to which we occasionally refer as the *supplementary conditions for the absolute convergence of the iteration.*

Proof. Again we make a change of variables, accompanied by a formal interchange in the order of integration, which brings us:

$$\begin{aligned}
\int_{T_2}^{\infty} t^{p_2} \int_0^{T_1} s^{p_1}(s+t)^q ds dt &= \int_{T_2}^{\infty} t^{p_1+p_2+q+1} \int_0^{\frac{T_1}{t}} u^{p_1}(1+u)^q du dt \\
&= \int_0^{\frac{T_1}{T_2}} u^{p_1}(1+u)^q \int_{T_2}^{\frac{T_1}{u}} t^{p_1+p_2+q+1} dt du \\
&= \frac{1}{p_1+p_2+q+2} \Bigg\{ T_1^{p_1+p_2+q+2} \int_0^{\frac{T_1}{T_2}} u^{-p_2-q-2}(1+u)^q du \\
&\qquad - T_2^{p_1+p_2+q+2} \int_0^{\frac{T_1}{T_2}} u^{p_1}(1+u)^q du \Bigg\}
\end{aligned}$$

The restrictions $p_1 > -1$ and $p_2+q < -1$ are clearly necessary to guarantee absolute convergence of the last two single integrals, in which circumstances the full sequence of equations remains valid. The proof is thus finished. ∎

8.2.4. Analyticity of Iterated Generating Functions

We will now employ the previously derived tests for absolute convergence, to deduce analyticity of certain iterated generating functions. Throughout this section we denote by $q(s,t)$ a function with the following properties:

(Q) $q(s,t)$ is uniformly continuous on any closed subset of $(0,1]^2$, and for $Q>0$ and $\kappa_1,\kappa_2,\kappa_3 \in \mathbb{R}$, uniformly with respect to $(s,t) \in (0,1]^2$, it satisfies

$$q(s,t) \le Q s^{\kappa_1}(s+t)^{\kappa_2} t^{\kappa_3}. \tag{8.2.53}$$

This function enables us, to confine to iterated integrals along the unit interval. It is easy to see, that in (8.2.1) and in (8.2.2) the integration paths always can be mapped to the unit interval by an appropriate substitution. The function $k(s+t)$ then changes to a function of the type $q(s,t)$. For example, if $\sigma=1$, $S=\infty$, $\tau=0$ and $T=1$, and as $u\to\infty$ for a parameter $\kappa\in\mathbb{C}$ we have $k(u)=\mathcal{O}(u^{-\Re\kappa})$. In this event, the change of variables $s=\frac{1}{u}$ maps the infinite segment to the unit interval, and uniformly with respect to $(u,t)\in(0,1]^2$ we find

$$\left|k(\tfrac{1}{u}+t)\right| \le u^{\Re\kappa}(1+ut)^{-\Re\kappa}\max_{r\ge1}\frac{|k(r)|}{r^{\Re\kappa}} \le u^{\Re\kappa}\max_{r\ge1}\frac{|k(r)|}{r^{\Re\kappa}}.$$

Hence, (8.2.53) holds. Both of the analyticity statements to be established below basically rely on the next lemma.

Lemma 8.2.4. *If we denote the integral in Lemma 8.2.1 by $I(T_1,T_2)$ and assume its absolute convergence, then $I(T_1,T_2)\to 0$ as $T_j\downarrow 0$ with fixed $T_k>0$ for each $j,k\in\{1,2\}$ with $j\ne k$.*

Proof. From the proof of the lemma we ascertain the identity

$$I(T_1,T_2) = \frac{T_2^{p_1+p_2+q+2}}{p_1+p_2+q+2}\int_0^{\frac{T_1}{T_2}} u^{p_1}(1+u)^q du + \frac{T_1^{p_1+p_2+q+2}}{p_1+p_2+q+2}\int_{\frac{T_1}{T_2}}^{\infty} u^{-p_2-q-2}(1+u)^q du.$$

If in the first integral we make the change of variables $v=\frac{T_2}{T_1}u$, and if in the second by means of the binomial theorem for $T_1<\frac{T_2}{2}$ we expand the function $(1+u)^q$, we obtain

$$I(T_1,T_2) = \frac{T_1^{p_1+1}T_2^{p_2+q+1}}{p_1+p_2+q+2}\int_0^1 v^{p_1}(1+\tfrac{T_1}{T_2}v)^q dv + \frac{T_1^{p_1+p_2+q+2}}{p_1+p_2+q+2}\int_{\frac{1}{2}}^{\infty} u^{-p_2-q-2}(1+u)^q du$$

$$+\frac{T_1^{p_1+p_2+q+2}}{p_1+p_2+q+2}\sum_{k=0}^{\infty}\binom{q}{k}\frac{1}{k-p_2-q-1}\left\{2^{p_2+q+1-k}-(\tfrac{T_1}{T_2})^{k-p_2-q-1}\right\}.$$

With the parameters p_1, p_2, q being subject to the conditions of Lemma 8.2.1, it is easy to see that $I(T_1, T_2) \to 0$ as $T_1 \downarrow 0$ for any fixed $T_2 > 0$. The proof of the converse statement is analogous. ■

We can now easily verify the following lemmas.

Lemma 8.2.5 (analyticity of first and second kind iterated generating functions). *In addition to the condition (Q), denote by $\psi(s,t)$ a function that is uniformly continuous with respect to $(s,t) \in I^2$ for each closed $I \subset (0,1]$. Furthermore, $\inf_{(u,v)\in I^2}\psi(u,v) > 0$, and there exist $\beta_1, \beta_3 \geq 0$, $\beta_2 \geq -\beta_1-\beta_3$ and $p > 0$ such that, as $(s,t) \downarrow 0$ we have*

$$\psi(s,t) \sim p s^{\beta_1}(s+t)^{\beta_2}t^{\beta_3}. \tag{8.2.54}$$

Concerning the involved parameters, we require

$$\begin{cases} \kappa_1 > -1, & \text{if } \beta_1 = 0,\\ \kappa_1+\kappa_2+\kappa_3 > -2, & \text{if } \beta_1+\beta_2+\beta_3 = 0,\\ \kappa_3 > -1, & \text{if } \beta_3 = 0.\end{cases}$$

The integral transform

$$\mathfrak{M}(\zeta) := \int_0^1\int_0^1 \{\psi(s,t)\}^{-\zeta} q(s,t)dsdt \tag{8.2.55}$$

is then holomorphic in its region of absolute convergence, viz for $\beta_1 = \beta_2 = \beta_3 = 0$ in the whole complex plane and otherwise in the greatest common of the half planes

$$\Re\zeta < \begin{cases} \frac{\kappa_1+1}{\beta_1}, & \beta_1 > 0,\\ \frac{\kappa_1+\kappa_2+\kappa_3+2}{\beta_1+\beta_2+\beta_3}, & \beta_1+\beta_2+\beta_3 > 0,\\ \frac{\kappa_3+1}{\beta_3}, & \beta_3 > 0.\end{cases} \tag{8.2.56}$$

Its derivatives of arbitrary order and therefore especially residues can be calculated by differentiating under the sign of integration.

According to the conditions on $\beta_1, \beta_2, \beta_3$, the function $\psi(s,t)$ can be a function of a single variable only. Therefore, (8.2.55) generalizes the first and second kind generating functions (8.2.1) and (8.2.2). As a particular example, in (8.2.2) we choose $\sigma = 1$, $S = \infty$, $\tau = 0$ and $T = 1$, and as $u \to \infty$ we assume $\varphi(u) \sim b_0 u^{-\beta}$ for $b_0, \beta > 0$. The substitution $s = \frac{1}{u}$ then results in an integral in the shape of (8.2.55) with $\psi(u,t) \equiv \varphi(\frac{1}{u}+t)$. As $(u,t) \downarrow 0$ we then conclude

$$\varphi(\tfrac{1}{u}+t) \sim b_0 u^{\beta}(1+ut)^{-\beta} \sim b_0 u^{\beta},$$

which shows that $\psi(u,t)$ satisfies the condition (8.2.54).

Proof. If we define the function

$$P(s,t) := \frac{s^{\beta_1}(s+t)^{\beta_2}t^{\beta_3}}{\psi(s,t)},$$

then $P(s,t) > 0$ for all $(s,t) \in (0,1]^2$, and we obtain

$$\mathfrak{M}(\zeta) = \int_0^1 t^{-\beta_3\zeta} \int_0^1 \frac{s^{-\beta_1\zeta}}{(s+t)^{\beta_2\zeta}} e^{\zeta P(s,t)} q(s,t)dsdt.$$

Due to the assumptions on $\psi(s,t)$, the function $P(s,t)$ is uniformly continuous on $[0,1]^2$, whence

$$M_P := \max_{(u,v)\in[0,1]^2} P(u,v)$$

is finite, which for any $\zeta \in \mathbb{C}$ yields

$$e^{\Re\zeta P(s,t)} \leq \sum_{j=0}^{\infty} \frac{|\Re\zeta|^j}{j!} |P(s,t)|^j \leq e^{|\Re\zeta| M_P}.$$

By virtue of (8.2.53), we therefore deduce the bound

$$|\mathfrak{M}(\zeta)| \leq Q e^{|\Re\zeta| M_P} \int_0^1 t^{\kappa_3-\beta_3\Re\zeta} \int_0^1 \frac{s^{\kappa_1-\beta_1\Re\zeta}}{(s+t)^{\beta_2\Re\zeta-\kappa_2}} dsdt,$$

where by Lemma 8.2.1 the integral on the right hand side converges absolutely for all $\zeta \in \mathbb{C}$ if $\beta_1 = \beta_2 = \beta_3 = 0$ and for those $\zeta \in \mathbb{C}$ subject to (8.2.56) otherwise. To verify analyticity of $\mathfrak{M}(\zeta)$, we must examine analyticity and as $n \to \infty$ the convergence behaviour of the sequence of integrals

$$\mathfrak{M}_n(\zeta) := \int_{\frac{1}{n}}^1 \int_{\frac{1}{n}}^1 \{\psi(s,t)\}^{-\zeta} q(s,t)dsdt. \tag{8.2.57}$$

For fixed n we conclude absolute convergence for all $\zeta \in \mathbb{C}$, by continuity of $\psi(s,t)$ and $q(s,t)$ with respect to $(s,t) \in [\frac{1}{n},1]^2$. Moreover, the integrand of $\mathfrak{M}_n(\zeta)$ for fixed $\frac{1}{n} \leq t \leq 1$ and $\zeta \in \mathbb{C}$ constitutes a continuous function of $\frac{1}{n} \leq s \leq 1$, but also a holomorphic function of $\zeta \in \mathbb{C}$ for fixed $\frac{1}{n} \leq s,t \leq 1$. By Theorem 5.6.1 in [Wegert, 2012], the interior integral, which we denote by

$$\mathfrak{N}_n(\zeta,t) := \int_{\frac{1}{n}}^{1} \{\psi(s,t)\}^{-\zeta}\, q(s,t)ds, \tag{8.2.58}$$

thus furnishes an entire function of ζ for fixed $\frac{1}{n} \le t \le 1$. Furthermore,

$$|\mathfrak{N}_n(\zeta,t)| \le \max_{(u,v)\in[\frac{1}{n},1]^2} |q(u,v)|\, \psi(u,v)^{-\Re\zeta}. \tag{8.2.59}$$

Since this upper bound is finite for any $\zeta \in \mathbb{C}$, Lebesgue's dominated convergence theorem implies continuity of $\mathfrak{N}_n(\zeta,t)$ with respect to $\frac{1}{n} \le t \le 1$ for each fixed $\zeta \in \mathbb{C}$. To combine these findings, again by Theorem 5.6.1 [Wegert, 2012], the iterated integral $\mathfrak{M}_n(\zeta)$ establishes an entire function of ζ for any $n \in \mathbb{N}$. Next, we denote by $\mathcal{Z}$ a compact subset of $\mathbb{C}$ for $\beta_1 = \beta_2 = \beta_3 = 0$, but of the half plane (8.2.56) otherwise. By Theorem 5.1.3 in [Wegert, 2012], it then remains to show the convergence of the sequence $\mathfrak{M}_n(\zeta)$ as $n \to \infty$ to $\mathfrak{M}(\zeta)$ uniformly in $\mathcal{Z}$. For this, we proceed with a simple application of the triangle inequality, which for any $\zeta \in \mathcal{Z}$ yields

$$|\mathfrak{M}(\zeta) - \mathfrak{M}_n(\zeta)| \le Q e^{|\Re\zeta| M_P} \left[\int_0^{\frac{1}{n}} \int_0^{1} + \int_0^{1} \int_0^{\frac{1}{n}} \right] \frac{t^{\kappa_3-\beta_3\Re\zeta} s^{\kappa_1-\beta_1\Re\zeta}}{(s+t)^{\beta_2\Re\zeta-\kappa_2}} dsdt, \tag{8.2.60}$$

with the iterated integrals on the right hand side being absolutely convergent. In order to apply a uniform bound, in the first iterated integral we separate the interior range of integration according to the segments along which $s+t$ is smaller or greater than one, to obtain

$$\int_0^{\frac{1}{n}} \int_0^{1} \frac{t^{\kappa_3-\beta_3\Re\zeta} s^{\kappa_1-\beta_1\Re\zeta}}{(s+t)^{\beta_2\Re\zeta-\kappa_2}} dsdt = \int_0^{\frac{1}{n}} \int_0^{1-t} \frac{t^{\kappa_3-\beta_3\Re\zeta} s^{\kappa_1-\beta_1\Re\zeta}}{(s+t)^{\beta_2\Re\zeta-\kappa_2}} dsdt + \int_0^{\frac{1}{n}} \int_{1-t}^{1} \frac{t^{\kappa_3-\beta_3\Re\zeta} s^{\kappa_1-\beta_1\Re\zeta}}{(s+t)^{\beta_2\Re\zeta-\kappa_2}} dsdt.$$

Then, with $x_- := \min\{\Re\zeta : \zeta \in \mathcal{Z}\}$ and $x_+ := \max\{\Re\zeta : \zeta \in \mathcal{Z}\}$, uniformly with respect to $\zeta \in \mathcal{Z}$ for $n \ge 2$ we find:

$$\begin{aligned}
\int_0^{\frac{1}{n}} \int_0^{1} \frac{t^{\kappa_3-\beta_3\Re\zeta} s^{\kappa_1-\beta_1\Re\zeta}}{(s+t)^{\beta_2\Re\zeta-\kappa_2}} dsdt &\le \int_0^{\frac{1}{n}} \int_0^{1-t} \frac{t^{\kappa_3-\beta_3 x_+} s^{\kappa_1-\beta_1 x_+}}{(s+t)^{\beta_2 x_+-\kappa_2}} dsdt + \int_0^{\frac{1}{n}} \int_{1-t}^{1} \frac{t^{\kappa_3-\beta_3 x_+} s^{\kappa_1-\beta_1 x_+}}{(s+t)^{\beta_2 x_--\kappa_2}} dsdt \\
&\le \int_0^{\frac{1}{n}} \int_0^{1} \frac{t^{\kappa_3-\beta_3 x_+} s^{\kappa_1-\beta_1 x_+}}{(s+t)^{\beta_2 x_+-\kappa_2}} dsdt \\
&\qquad + \frac{n^{\beta_3 x_+-\kappa_3-1}}{1+\kappa_3-\beta_3 x_+} \max_{0\le v\le\frac{1}{2}} \int_{\frac{1}{2}}^{1} \frac{s^{\kappa_1-\beta_1 x_+}}{(s+v)^{\beta_2 x_--\kappa_2}} ds
\end{aligned}$$

By Lemma 8.2.4, the first summand vanishes as $n \to \infty$, whereas the decay of the second summand is obvious. A similar bound can be deduced for the second iterated integral in (8.2.60). To summarize these findings, Theorem 5.1.3 in [Wegert, 2012] yields analyticity of $\mathfrak{M}(\zeta)$ in its region of absolute convergence. Finally, a repeated application of the second statement from Theorem 5.6.1 for arbitrary $k, n \in \mathbb{N}$ shows

$$\frac{d^k}{d\zeta^k}\mathfrak{M}_n(\zeta) = \int_{\frac{1}{n}}^{1}\int_{\frac{1}{n}}^{1} (-\log\psi(s,t))^k \{\psi(s,t)\}^{-\zeta} q(s,t)dsdt.$$

But Theorem 5.1.3 establishes for any $k \in \mathbb{N}_0$ as $n \to \infty$ the convergence

$$\frac{d^k}{d\zeta^k}\mathfrak{M}_n(\zeta) \to \frac{d^k}{d\zeta^k}\mathfrak{M}(\zeta), \tag{8.2.61}$$

uniformly in any compact subset within the region of analyticity of $\mathfrak{M}(\zeta)$. Accordingly, arbitrary derivatives of the limit generating function can be determined by differentiation under the integral sign. It must be emphasized, that the obtained integral representation for each derivative again converges absolutely and is a holomorphic function of ζ in the same region, where $\mathfrak{M}(\zeta)$ is holomorphic. ■

A third kind of iterated generating functions to be encountered below, is covered by the final lemma of this section.

Lemma 8.2.6 (analyticity of third kind iterated generating functions). *Provided condition (Q) holds and*

$$\kappa_1 > \max\{0, -\kappa_2\} - \kappa_3,$$

the integral transform

$$\mathfrak{L}(\zeta) := \int_0^1 t^{\zeta-1}\int_0^1 s^{-\zeta-1} q(s,t)dsdt \tag{8.2.62}$$

is absolutely convergent and holomorphic in the strip

$$-\kappa_3 < \Re\zeta < \kappa_1, \tag{8.2.63}$$

and the computation of derivatives of arbitrary order and particularly of residues is permissible under the sign of integration.

Proof. It is clear that for each $n \in \mathbb{N}$ the integral

$$\mathfrak{L}_n(\zeta) := \int_{\frac{1}{n}}^1 t^{\zeta-1}\int_{\frac{1}{n}}^1 s^{-\zeta-1} q(s,t)dsdt \tag{8.2.64}$$

represents an entire function of ζ. The proof of its uniform convergence to $\mathfrak{L}(\zeta)$ in any compact subset of the strip (8.2.63) is easily conducted by means of the bound

$$|\mathfrak{L}(\zeta) - \mathfrak{L}_n(\zeta)| \le Q \left[\int_0^1 \int_0^{\frac{1}{n}} + \int_0^{\frac{1}{n}} \int_0^1 \right] \frac{t^{\kappa_3 + \Re\zeta - 1} s^{\kappa_1 - \Re\zeta - 1}}{(s+t)^{-\kappa_2}} ds dt, \tag{8.2.65}$$

valid for any $\zeta \in \mathbb{C}$ from the indicated strip. An immdiate application of a uniform bound with respect to ζ, however, may violate the conditions for absolute convergence. Instead, one must first employ the identity from the proof of Lemma 8.2.1, to decompose each of the iterated integrals in (8.2.65) to a sum of two single integrals. These integrals provide a better foundation for a uniform bound. Their convergence to zero then can be shown in analogy to the proof of Lemma 8.2.4, from which analyticity of $\mathfrak{L}(\zeta)$ in the strip (8.2.63) follows by Theorem 5.1.3 in [Wegert, 2012]. Moreover, again similar to the proof of Lemma 8.2.5 one readily confirms the permission to differentiate under the integral sign. ∎

8.2.5. Region of Analyticity and Order of Integration

In this short section we point out, how the region of analyticity of an iterated generating function may depend on its order of integration. As an example we consider the integral

$$\mathfrak{L}(-\zeta) := \int_1^\infty e(t) \int_0^1 \{\varphi(s)\}^{-\zeta} k(s+t) ds dt \tag{8.2.66}$$

for functions $\varphi(s) > 0$, $e(t)$ and $k(s+t)$, which are uniformly continuous on any closed subset of $(0, 1]$, $[1, \infty)$ and $(0, 1] \times [1, \infty)$, respectively. Moreover, $\varphi(s)$ exhibits algebraic behaviour as $s \downarrow 0$ for a parameter $\beta_0 > 0$ and a coefficient $b_0 > 0$, whereas $e(t)$ and $k(t)$ are algebraic at infinity for parameters $\varepsilon_0, \kappa_0 \in \mathbb{C}$ with $\Re\varepsilon_0 + \Re\kappa_0 > 1$ and coefficients $e_0, k_0 \in \mathbb{C} \setminus \{0\}$. From Lemma 8.2.5 we know that absolute convergence and analyticity of the above integral then holds for any $\zeta \in \mathbb{C}$ with $\Re\zeta < \frac{1}{\beta_0}$. Particularly by absolute convergence we can write equivalently

$$\mathfrak{L}(-\zeta) = \int_0^1 \{\varphi(s)\}^{-\zeta} \int_1^\infty e(t) k(s+t) dt ds, \tag{8.2.67}$$

which of course still represents $\mathfrak{L}(-\zeta)$ in the half plane $\Re\zeta < \frac{1}{\beta_0}$. If, however, we define $d(s) := \int_1^\infty e(t)k(s+t)dt$, this last integral becomes

$$\mathfrak{L}(-\zeta) = \int_0^1 \{\varphi(s)\}^{-\zeta} d(s) ds. \tag{8.2.68}$$

By Lebesgue's dominated convergence theorem, $d(s) = \mathcal{O}(1)$ as $s \downarrow 0$. If even $d(s) = o(1)$ as $s \downarrow 0$, and if the decay is sufficiently fast, the representations (8.2.67) and (8.2.68) will be valid in a wider region than (8.2.66). Hence, a consideration of $\mathfrak{L}(-\zeta)$ as an iterated integral may confine its region of analyticity. In other words, the modulus of the integrand of the iterated integral will in general yield a smaller region of convergence than the modulus of the integrand of the related single integral (8.2.68). Although this is actually not surprising, it must be kept in mind throughout our investigations of iterated generating functions. We also mention that the point $\zeta = \frac{1}{\beta_0}$, which the representation (8.2.66) suggests to be a singularity, is actually a point of analyticity, if (8.2.68) represents $\mathfrak{L}(-\zeta)$ in a region that contains the half plane $\Re\zeta < \frac{1}{\beta_0}$.

8.2.6. Analytic Continuation of Iterated Generating Functions

Having specified absolute convergence and analyticity criteria for iterated generating functions of convolution-type, it is obvious to think about methods for determining their analytic continuation. Regarding single integral transforms we know about the possibility to employ an appropriate asymptotic expansion for the amplitude function or, even more versatile, partial integration. In view of iterated transforms it seems clear that these techniques can not immediately be adopted but require some modifications. In an introductory paragraph of this subsection we shall therefore provide an overview of the applicable modifications with the aid of a simple iterated Mellin-type integral, before we actually transfer these techniques to general transforms.

8.2.6.1. An Introductory Example: Iterated Mellin-Type Integrals

For $0 \leq \sigma < S \leq \infty$ and $0 \leq \tau < T \leq \infty$ we consider the iterated Mellin-type integral

$$\text{(8.2.69)} \qquad \mathcal{M}_0\begin{bmatrix}\xi, \sigma, S\\ \zeta, \tau, T\end{bmatrix} := \int_\tau^T t^{\zeta-1} \int_\sigma^S \frac{s^{\xi-1}}{1+s+t} ds dt.$$

From Subsection 8.2.3 we know that its absolute convergence essentially depends on the choice of the endpoints. Given (8.2.69) and assuming for instance ζ fixed, it is reasonable to study the integral as a function of the variable ξ, which by Lemma 8.2.5 is then even holomorphic in its region of absolute convergence. A special case occurs if $\sigma = \tau = 0$ and $S = T = \infty$. In this event (8.2.69) constitutes an iterated Mellin transform. Upon writing

$$\text{(8.2.70)} \qquad \mathcal{M}_0\begin{bmatrix}\xi, 0, \infty\\ \zeta, 0, \infty\end{bmatrix} = \left[\int_0^1\int_0^1 + \int_1^\infty\int_0^1 + \int_0^1\int_1^\infty + \int_1^\infty\int_1^\infty\right] \frac{t^{\zeta-1}s^{\xi-1}}{1+s+t} ds dt,$$

and applying Lemma 8.2.5 to each iterated integral, we deduce absolute convergence and analyticity in $0 < \Re\xi < \min\{1, 1-\Re\zeta\}$ for fixed $0 < \Re\zeta < 1$. To specify the analytic continuation into a wider ξ-half plane is then particularly simple and, appealing to formula (A.6.4), for any fixed

$\xi, \zeta \in \mathbb{C}$ for which (8.2.70) converges absolutely, we obtain by means of elementary manipulations

$$\mathcal{M}_0 \begin{bmatrix} \xi, 0, \infty \\ \zeta, 0, \infty \end{bmatrix} = \Gamma(\xi)\Gamma(\zeta)\Gamma\left(1 - \xi - \zeta\right). \tag{8.2.71}$$

For the latter equality we especially identified the single Mellin transform of the rational function $\frac{1}{1+u}$ as a beta function, compare Example A.5.2 in the appendix. Besides the simplicity of the above representation, the most important benefit is that it provides the analytic continuation to the whole complex ξ-plane as a meromorphic function, revealing the location of the singularities. Conversely, the situation essentially increases in difficulty if the formula (A.6.4) is unavailable. This is always the case if in (8.2.69) both integration paths do not coincide with the positive real axis. As an example how to proceed in such a situation we shall now examine

$$\mathcal{M}_0 \begin{bmatrix} \xi, 1, \infty \\ \zeta, 1, \infty \end{bmatrix} = \int_1^\infty t^{\zeta-1} \int_1^\infty \frac{s^{\xi-1}}{1+s+t} ds dt. \tag{8.2.72}$$

By Corollary 8.2.2 this integral converges absolutely for $\xi, \zeta \in \mathbb{C}$ with

$$\begin{cases} \Re\zeta < 1, \\ \Re\xi < \min\left\{1, 1 - \Re\zeta\right\}. \end{cases} \tag{8.2.73}$$

For fixed ζ it is even holomorphic in the indicated ξ-region due to its uniform convergence in any compact subset. Speaking of absolute convergence and analyticity with respect to ξ, we obviously need to incorporate the contribution of the amplitude $a(s+t) := \frac{1}{1+s+t}$ and of the integral function

$$f(s, \zeta) := \int_1^\infty \frac{t^{\zeta-1}}{1+s+t} dt. \tag{8.2.74}$$

On the one hand, the condition $\Re\xi < 1$ is prescribed by the asymptotic behaviour of the amplitude $a(s+t)$, and it is required to guarantee for fixed $t \geq 1$ the convergence of the single integral

$$\int_1^\infty \frac{s^{\xi-1}}{1+s+t} ds.$$

On the other hand, the restriction $\Re\xi < 1 - \Re\zeta$ corresponds to the supplementary condition for the absolute convergence of the iterated integral. Roughly speaking it originates in the asymptotic behaviour of $f(s, \zeta)$ as $s \to \infty$, which is substantially characterized by the fixed variable ζ. The minimum structure of the abscissa of convergence in (8.2.73) must especially be taken into account when calculating the analytic continuation. If we suppose for a moment $\Re\zeta < 0$ fixed, we have analyticity of (8.2.72) in $\Re\xi < 1$. If we then integrate once by parts in the usual fashion the interior integral for fixed $\Re\xi < 1$ and $t \geq 1$, we obtain:

$$\begin{aligned}\int_1^\infty \frac{s^{\xi-1}}{1+s+t}ds &= \left[\frac{s^{\xi-1}}{\xi-1}\frac{s}{1+s+t}\right]_1^\infty - \frac{1}{\xi-1}\int_1^\infty s^{\xi-1}\left[\frac{d}{ds}\frac{s}{1+s+t}\right]ds \\ &= -\frac{1}{\xi-1}\frac{1}{2+t} - \frac{1}{\xi-1}\int_1^\infty s^{\xi-1}\frac{1+t}{(1+s+t)^2}ds\end{aligned}$$

It is easy to see that the right hand side establishes the analytic continuation of the integral on the left hand side into the region $\Re\xi < 2$ for any fixed $t \geq 1$. An application of the above expansion to (8.2.72) yields

$$\mathcal{M}_0\begin{bmatrix}\xi, 1, \infty \\ \zeta, 1, \infty\end{bmatrix} = -\frac{1}{\xi-1}\int_1^\infty \frac{t^{\zeta-1}}{2+t}dt - \frac{1}{\xi-1}\int_1^\infty t^{\zeta-1}\int_1^\infty s^{\xi-1}\frac{1+t}{(1+s+t)^2}dsdt. \tag{8.2.75}$$

Observe that the integrand of the double integral is $\mathcal{O}\left\{t^\zeta s^{\xi-1}(s+t)^{-2}\right\}$ as $s,t \to \infty$, therefore by Lemma 8.2.5 implying absolute convergence and analyticity for fixed $\Re\zeta < 1$ in the half plane $\Re\xi < \min\{2, 1-\Re\zeta\}$. Consequently (8.2.75) represents the ξ-analytic continuation of (8.2.72).

If $\Re\zeta < -1$ we can repeat the standard integration by parts procedure to access a more extended half plane. If, however, $-1 \leq \Re\zeta < 0$ it is inappropriate, because the right boundary of the wider region $\Re\xi < \min\{2, 1-\Re\zeta\} = 1-\Re\zeta$ originates in the supplementary condition for the convergence of the iterated integral. To show how to treat such a situation, we revisit the integral (8.2.72) for arbitrary $\Re\zeta < 1$, which implies absolute convergence and analyticity in the half plane $\Re\xi < \min\{1, 1-\Re\zeta\}$. In order to overcome this minimum-type boundary we propose an approach by virtue of integral transforms. Therefore we first note, according to the Mellin inversion formula, for $s,t > 0$, $\Re a > 0$ and $0 < c < \Re a$ we have

$$(s+t)^{-a} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z)\Gamma(a-z)}{\Gamma(a)} s^{z-a}t^{-z}dz. \tag{8.2.76}$$

With $a = 1$ we see, the function on the left hand side reflects the asymptotic behaviour of the amplitude function in (8.2.72) as $s+t \to \infty$. To be exact, upon rearranging the indicated integral in the form

$$\mathcal{M}_0\begin{bmatrix}\xi, 1, \infty \\ \zeta, 1, \infty\end{bmatrix} = \int_1^\infty t^{\zeta-1}\int_1^\infty \frac{s^{\xi-1}}{s+t}\frac{s+t}{1+s+t}dsdt, \tag{8.2.77}$$

we have $\frac{s+t}{1+s+t} \to 1$ as $s+t \to \infty$. Now, the factorization of the integrand enables us to introduce the integral (8.2.76) and to formally interchange the order of integration for $c \equiv c_0(\xi,\zeta)$, leading to

$$\mathcal{M}_0\begin{bmatrix}\xi,1,\infty\\ \zeta,1,\infty\end{bmatrix}=\frac{1}{2\pi i}\int\limits_{c_0(\xi,\zeta)-i\infty}^{c_0(\xi,\zeta)+i\infty}\Gamma(z)\Gamma(1-z)\mathcal{L}(z,\xi,\zeta)dz, \tag{8.2.78}$$

with the function in the integrand for brevity denoted by

$$\mathcal{L}(z,\xi,\zeta):=\int\limits_1^\infty t^{\zeta-z-1}\int\limits_1^\infty s^{\xi+z-2}\frac{s+t}{1+s+t}dsdt. \tag{8.2.79}$$

By Lemma 8.2.6, for fixed $\xi,\zeta\in\mathbb{C}$ with $\Re(\xi+\zeta)<1$ the integral $\mathcal{L}(z,\xi,\zeta)$ as a function of z converges absolutely for and is holomorphic in

$$\Re\zeta<\Re z<1-\Re\xi. \tag{8.2.80}$$

As a consequence, with $\Re\zeta<1$ and $\Re\xi<\min\{1,1-\Re\zeta\}$ by taking into account the conditions for the validity of (8.2.76), in (8.2.78) the interchange in the order of integration is permitted for

$$\max\{0,\Re\zeta\}<c_0(\xi,\zeta)<\min\{1,1-\Re\xi\}. \tag{8.2.81}$$

Such a parameter clearly always exists. The representation (8.2.78) will now serve to determine the analytic continuation with respect to ξ for fixed ζ.

By inspection we observe that the evaluation of (8.2.78) for greater values of $\Re\xi$ is closely connected with the admissibility of greater values of $\Re\xi$ for (8.2.79). The latter is in turn equivalent to the admissibility of greater values of $\Re z$, which are, however, denied by the presence of the right boundary in (8.2.80). The boundary can be overcome by deriving the analytic continuation of $\mathcal{L}(z,\xi,\zeta)$ towards the right direction of the z-plane. Assuming $\xi,\zeta\in\mathbb{C}$ with $\Re\zeta<1$ and $\Re\xi<\min\{1,1-\Re\zeta\}$ fixed, because the restriction of $\Re z$ into this direction stems from the interior integral in (8.2.79), we partially integrate this particular integral. Bearing in mind

$$\frac{d}{ds}\frac{s+t}{1+s+t}=\frac{1}{(1+s+t)^2},$$

this yields for fixed $t\geq1$:

$$\mathcal{N}(\xi+z-1,t):=\int\limits_1^\infty s^{\xi+z-2}\frac{s+t}{1+s+t}ds \tag{8.2.82}$$

$$=-\frac{1}{\xi+z-1}\frac{1+t}{2+t}-\frac{1}{\xi+z-1}\int\limits_1^\infty\frac{s^{\xi+z-1}}{(1+s+t)^2}ds \tag{8.2.83}$$

The right hand side constitutes the analytic continuation into the half plane $\Re z < 2 - \Re\xi$, where the integral converges absolutely. The only singularity therein is a simple pole with

$$\operatorname*{Res}_{z=1-\xi} \mathcal{N}(\xi + z - 1, t) = -\frac{1+t}{2+t} - \int_1^\infty \frac{ds}{(1+s+t)^2} = -1. \tag{8.2.84}$$

In terms of (8.2.82), instead of (8.2.79), we can write

$$\mathcal{L}(z, \xi, \zeta) = \int_1^\infty t^{\zeta - z - 1} \mathcal{N}(\xi + z - 1, t) dt. \tag{8.2.85}$$

According to the above findings, the interior integral function can be continued meromorphically by virtue of the expansion (8.2.83), which yields

$$\mathcal{L}(z, \xi, \zeta) = -\frac{1}{\xi + z - 1} \left\{ \int_1^\infty t^{\zeta - z - 1} \frac{1+t}{2+t} dt + \int_1^\infty t^{\zeta - z - 1} \int_1^\infty \frac{s^{\xi + z - 1}}{(1+s+t)^2} ds dt \right\}. \tag{8.2.86}$$

The integrals in this sum converge absolutely for $\Re\zeta < \Re z$ and $\Re\zeta - 2 < \Re z < 2 - \Re\xi$ with $\Re(\xi + \zeta) < 2$, respectively. Each of them represents a holomorphic function of z in its region of absolute convergence. Hence, (8.2.86) establishes the analytic continuation of (8.2.79) to the region

$$\Re\zeta < \Re z < 2 - \Re\xi \tag{8.2.87}$$

for any fixed $\Re\zeta < 1$ and $\Re\xi < \min\{1, 1 - \Re\zeta\}$, exhibiting a simple pole at $z = 1 - \xi$. For the associated residue we obtain from the fundamental theorem of calculus:

$$\begin{aligned} \operatorname*{Res}_{z=1-\xi} \mathcal{L}(z, \xi, \zeta) &= -\int_1^\infty t^{\xi + \zeta - 2} \frac{1+t}{2+t} dt - \int_1^\infty t^{\xi + \zeta - 2} \int_1^\infty \frac{1}{(1+s+t)^2} ds dt \\ &= \frac{1}{\xi + \zeta - 1} \end{aligned} \tag{8.2.88}$$

Furthermore, by absolute and with respect to $\Im z \in \mathbb{R}$ uniform convergence of each integral in (8.2.86), we see that $\mathcal{L}(z, \xi, \zeta)$ in the region (8.2.87) is $\mathcal{O}(1)$ as $\Im z \to \pm\infty$, uniformly with respect to $\Re z$ in any closed vertical substrip whose left and right boundary is still contained therein. The uniformity with respect to $\Re z$ can be shown similar to the uniformity of convergence in any compact subset.

We now reconsider the MB-integral representation (8.2.78), yet still with $\Re\zeta < 1$ and $\Re\xi < \min\{1, 1 - \Re\zeta\}$ fixed. It is in fact easy to see by definition of $c_0(\xi, \zeta)$ that a wider range of ξ-values is still denied. To change this we first note, according to the properties of the function $\mathcal{L}(z, \xi, \zeta)$, that the closest singularity to the right of the vertical line $\Re z = c_0(\xi, \zeta)$ especially depends on ξ. If we assume without loss of generality $0 < \Re\xi < \min\{1, 1 - \Re\zeta\}$, it is a pole of simple order at $z = 1 - \xi$. Moreover, since $\mathcal{L}(z, \xi, \zeta)$ is $\mathcal{O}(1)$, the integrand of (8.2.78) shows exponential decay as $\Im z \to \pm\infty$ in $\Re\zeta < \Re z < 2 - \Re\xi$, due to the gamma functions. It is thus permitted to displace the integration path rightwards across the indicated pole to match a new vertical line with real part $\Re z = c_1(\xi, \zeta)$, where

$$\max\{0, \Re\zeta, 1 - \Re\xi\} < c_1(\xi, \zeta) < \min\{1, 2 - \Re\xi\}. \tag{8.2.89}$$

Incorporating the fact that we encircle the pole in the clockwise direction, thereby incuring a negative sign of the residue, by (8.2.88) from (8.2.78) we deduce

$$\mathcal{M}_0\begin{bmatrix}\xi, 1, \infty\\ \zeta, 1, \infty\end{bmatrix} = -\frac{\Gamma(\xi)\Gamma(1-\xi)}{\xi + \zeta - 1} + \mathcal{M}_1(\xi, \zeta), \tag{8.2.90}$$

where the second addend refers to the MB-integral

$$\mathcal{M}_1(\xi, \zeta) := \frac{1}{2\pi i} \int\limits_{c_1(\xi,\zeta)-i\infty}^{c_1(\xi,\zeta)+i\infty} \Gamma(z)\Gamma(1-z)\mathcal{L}(z, \xi, \zeta)dz. \tag{8.2.91}$$

At this point we finally return to a discussion of (8.2.90) as a function of $\xi \in \mathbb{C}$ for fixed $\Re\zeta < 1$. Regarding the first addend in this expansion, which equals a meromorphic function of the variable under consideration, we notice the presence of an infinite sequence of poles at the integers, accompanied by a pole at $\xi = 1 - \zeta$. Each pole is of simple order if $\zeta \notin \mathbb{Z}$. Moreover, concerning the MB-integral (8.2.91), we ascertain that the conditions imposed on the integration path in (8.2.89) are admissible for $\Re\zeta < 1$ and

$$0 < \Re\xi < \min\{2, 2 - \Re\zeta\}. \tag{8.2.92}$$

Due to the exponential decay of the gamma functions towards the imaginary direction, the integral is then especially absolutely convergent. Now, for a given $\zeta \in \mathbb{C}$ with $\Re\zeta < 1$ we may choose $\max\{0, \Re\zeta\} < \varepsilon < 1$ and denote by $\mathbb{X}_\varepsilon$ the strip

$$\mathbb{X}_\varepsilon := \{\xi \in \mathbb{C} : 1 - \varepsilon < \Re\xi < 2 - \varepsilon\}. \tag{8.2.93}$$

Then, $\varepsilon > 1 - \Re\xi$ and $\varepsilon < 2 - \Re\xi$ for each $\xi \in \mathbb{X}_\varepsilon$. By (8.2.89) we can thus pick $c_1(\xi, \zeta) \equiv \varepsilon$ for all $\xi \in \mathbb{X}_\varepsilon$, and we can write

$$\mathcal{M}_1(\xi, \zeta) = \frac{1}{2\pi i} \int\limits_{\varepsilon-i\infty}^{\varepsilon+i\infty} \Gamma(z)\Gamma(1-z)\mathcal{L}(z, \xi, \zeta)dz. \tag{8.2.94}$$

To verify the above integral holomorphic in the strip $\mathbb{X}_\varepsilon$, for $n \in \mathbb{N}$ we introduce the sequence of integrals

$$\mathcal{M}_1^n(\xi,\zeta) := \frac{1}{2\pi i}\int\limits_{\varepsilon-in}^{\varepsilon+in} \Gamma(z)\Gamma(1-z)\mathcal{L}(z,\xi,\zeta)dz. \tag{8.2.95}$$

Regarding the integrand we note that the integrals appearing in the representation (8.2.86) for the function $\mathcal{L}(\varepsilon+iy,\xi,\zeta)$ converge absolutely for all $y \in \mathbb{R}$ and

$$\Re\xi < 2-\varepsilon. \tag{8.2.96}$$

By uniform convergence, the function $\mathcal{L}(\varepsilon+iy,\xi,\zeta)$ is even holomorphic in this ξ-half plane with the exception of the point $\xi = 1-\varepsilon$. We thereby conclude, with $z = \varepsilon+iy$ as a function of $-n \le y \le n$ the integrand of (8.2.95) is continuous for any fixed $\xi \in \mathbb{X}_\varepsilon$, particularly because $\mathbb{X}_\varepsilon$ is contained in (8.2.96) and $1-\varepsilon \notin \mathbb{X}_\varepsilon$. In addition, for fixed $-n \le y \le n$ as a function of $\xi \in \mathbb{X}_\varepsilon$ the integrand is holomorphic. Hence, by Theorem 5.6.1 in [Wegert, 2012] the integral (8.2.95) is also holomorphic in $\mathbb{X}_\varepsilon$ for each $n \in \mathbb{N}$. Finally we denote by X a compact subset of the strip $\mathbb{X}_\varepsilon$ with

$$\begin{cases} x_- := \min\{\Re\xi : \xi \in X\}, \\ x_+ := \max\{\Re\xi : \xi \in X\}. \end{cases}$$

From (8.2.86), for any $y \in \mathbb{R}$ and $\xi \in X$, it is then easy to confirm that

$$|\mathcal{L}(\varepsilon+iy,\xi,\zeta)| \le \frac{1}{x_-+\varepsilon-1}\left\{\int\limits_1^\infty t^{\Re\zeta-\varepsilon-1}\frac{1+t}{2+t}dt + \int\limits_1^\infty t^{\Re\zeta-\varepsilon-1}\int\limits_1^\infty \frac{s^{x_++\varepsilon-1}}{(1+s+t)^2}dsdt\right\}.$$

Especially since $x_+ < 2-\varepsilon$ the preceding two integrals converge absolutely and yield a uniform bound with respect to $\xi \in X$ and $y \in \mathbb{R}$. Denoting that bound by the constant $K > 0$, for all $\xi \in X$ we eventually arrive at

$$|\mathcal{M}_1(\xi,\zeta) - \mathcal{M}_1^n(\xi,\zeta)| \le \frac{K}{2\pi}\left[\int\limits_{-\infty}^{-n} + \int\limits_n^\infty\right] |\Gamma(\varepsilon+iy)|\,|\Gamma(1-\varepsilon-iy)|\,dy.$$

By Lebesgue's dominated convergence theorem, the right hand side tends to zero as $n \to \infty$, uniformly with respect to $\xi \in X$. By Theorem 5.1.3 in [Wegert, 2012] this confirms the integral (8.2.94) as a holomorphic function of the variable ξ in the strip $\mathbb{X}_\varepsilon$. But the latter can be managed to overlap an arbitrary portion of the region $0 < \Re\xi < \min\{1, 1-\Re\zeta\}$ by appropriately choosing ε, compare (8.2.93). We therefore conclude that the expansion (8.2.90) establishes the analytic continuation of the iterated Mellin-type integral (8.2.72) into the strip

$$0 < \Re\xi < \min\{2, 2 - \Re\zeta\} \tag{8.2.97}$$

for arbitrary fixed $\Re\zeta < 1$. Therein the first summand represents a meromorphic function with two poles of simple order at $\xi \in \{1, 1-\zeta\}$ for non-zero $-1 < \Re\zeta < 1$, or with a single first or second order pole at $\xi = 1$, respectively if $\Re\zeta \leq -1$ or if $\zeta = 0$. The second summand is in each case analytic.

8.2.6.2. Two Infinite Paths and a Kernel of the First Kind

We shall now adopt the technique of the preceding paragraph to determine the analytic continuation with respect to ζ of an iterated generating function of the first kind (8.2.1) with $\mathcal{P}_j = [T_j, \infty)$ for each $j \in \{1, 2\}$, where $T_j > 0$. The ingredient functions $\varphi(r) > 0$, $d(r), e(r), k(r)$ are assumed once continuously differentiable on $r \geq \min\{T_1, T_2\}$ and algebraic at infinity with coefficients $b_0 > 0$, $d_0, e_0, k_0 \in \mathbb{C} \setminus \{0\}$ and parameters $\beta_0 > 0$, $\delta_0, \varepsilon_0, \kappa_0 \in \mathbb{C}$. Furthermore, the first derivatives of their normalized counterparts are supposed to be of respective order $\beta_1, \delta_1, \varepsilon_1, \kappa_1 \in \mathbb{C}$ at infinity with $\beta_1, \Re\delta_1, \Re\varepsilon_1, \Re\kappa_1 > 1$. We then introduce the complex-valued parameters

$$\begin{cases} \chi_0 := \frac{\delta_0 - 1}{\beta_0}, \\ \varsigma_0 := \frac{\varepsilon_0 - 1}{\beta_0}, \end{cases} \tag{8.2.98}$$

and we require

$$-\frac{\Re\kappa_0}{\beta_0} < \Re\varsigma_0 \leq 0. \tag{8.2.99}$$

By making use of the normalized ingredients according to (8.2.8) and (8.2.9), in the above setup the iterated generating function (8.2.1) can easily be rearranged in the following manner:

$$\mathcal{M}\begin{bmatrix} -\zeta, T_1, \infty \\ T_2, \infty \end{bmatrix} = \int_{T_2}^{\infty} t^{-\beta_0\varsigma_0 - 1} E(t) \int_{T_1}^{\infty} \frac{s^{\beta_0(\zeta - \chi_0) - 1}}{(s+t)^{\kappa_0}} e^{\zeta B(s)} D(s) K(s+t)\, ds dt \tag{8.2.100}$$

A quick application of Lemma 8.2.5 then immediately verifies absolute convergence and analyticity for

$$\Re\zeta < \Re(\chi_0 + \varsigma_0) + \frac{\Re\kappa_0}{\beta_0}. \tag{8.2.101}$$

The right boundary of this half plane originates in the supplementary condition for the convergence of the iterated integral. Our findings from the preceding paragraph thus suggest to determine the analytic continuation by means of the method of integral transforms, for which purpose we define

$$\mathcal{L}(w,\zeta) := \int\limits_{T_2}^{\infty} t^{-\beta_0\varsigma_0-w-1}E(t)\int\limits_{T_1}^{\infty} s^{\beta_0(\zeta-\chi_0)+w-\kappa_0-1}e^{\zeta B(s)}D(s)K(s+t)dsdt. \tag{8.2.102}$$

By Corollary 8.2.2 for fixed $\Re\zeta < \Re(\chi_0+\varsigma_0)+\frac{\Re\kappa_0}{\beta_0}$ this integral converges absolutely, provided

$$-\beta_0\Re\varsigma_0 < \Re w < \Re(\kappa_0+\beta_0(\chi_0-\zeta)). \tag{8.2.103}$$

Moreover, in accordance with Lemma 8.2.6, we have w-analyticity of (8.2.102) in its region of absolute convergence. Now, subject to (8.2.76), with the choice

$$-\beta_0\Re\varsigma_0 < u_0 < \Re\kappa_0+\min\left\{0,\beta_0\Re(\chi_0-\zeta)\right\}, \tag{8.2.104}$$

for any fixed $\zeta\in\mathbb{C}$ satisfying (8.2.101) the iterated generating function (8.2.100) can be cast in the form

$$\mathcal{M}\begin{bmatrix}-\zeta, T_1, \infty\\ T_2, \infty\end{bmatrix} = \frac{1}{2\pi i}\int\limits_{u_0-i\infty}^{u_0+i\infty}\frac{\Gamma(w)\Gamma(\kappa_0-w)}{\Gamma(\kappa_0)}\mathcal{L}(w,\zeta)dw. \tag{8.2.105}$$

It is easy to see that such a parameter u_0 indeed always exists under the required conditions on ζ and on ς_0. Concerning the arguments ζ and w of $\mathcal{L}(w,\zeta)$, we notice a relation similar to the preceding paragraph, from which we conclude the necessity to overcome the right boundary of the region (8.2.103). Therefore we perform one step of integration by parts of the interior integral, which yields:

$$\begin{aligned}
\mathcal{L}(w,\zeta) &= -\frac{T_1^{\beta_0(\zeta-\chi_0)+w-\kappa_0}}{\beta_0(\zeta-\chi_0)+w-\kappa_0}e^{\zeta B(T_1)}D(T_1)\int\limits_{T_2}^{\infty} t^{-\beta_0\varsigma_0-w-1}E(t)K(T_1+t)dt\\
&\qquad -\frac{1}{\beta_0(\zeta-\chi_0)+w-\kappa_0}\int\limits_{T_2}^{\infty} t^{-\beta_0\varsigma_0-w-1}E(t)\\
&\qquad\qquad\times\int\limits_{T_1}^{\infty} s^{\beta_0(\zeta-\chi_0)+w-\kappa_0}\frac{d}{ds}\left\{e^{\zeta B(s)}D(s)K(s+t)\right\}dsdt\\
&= -\frac{T_1^{\beta_0(\zeta-\chi_0)+w-\kappa_0}}{\beta_0(\zeta-\chi_0)+w-\kappa_0}e^{\zeta B(T_1)}D(T_1)\int\limits_{T_2}^{\infty} t^{-\beta_0\varsigma_0-w-1}E(t)K(T_1+t)dt\\
&\qquad -\frac{1}{\beta_0(\zeta-\chi_0)+w-\kappa_0}\sum_{\substack{n_1,n_2,n_3\in\{0,1\}\\ n_1+n_2+n_3=1}}\int\limits_{T_2}^{\infty} t^{-\beta_0\varsigma_0-w-1}E(t)
\end{aligned} \tag{8.2.106}$$

$$\times \int_{T_1}^{\infty} s^{\beta_0(\zeta-\chi_0)+w-\kappa_0} e^{\zeta B(s)} \left\{\zeta B'(s)\right\}^{n_1} D^{(n_2)}(s) K^{(n_3)}(s+t) ds dt$$

By taking into account the order of the derivatives of $B(s)$, $C(s)$ and $K(s+t)$ as their arguments tend to infinity, the absolute convergence and analyticity of the involved integrals follows from Lemma 8.2.6 and was specified in Table 8.1.

$\neq 0$	**single**	**iterated**
$-$	$-\beta_0 \Re \varsigma_0 < \Re w$	$-$
n_1	$-\beta_0 \Re \varsigma_0 < \Re w < \Re(\kappa_0 + \beta_0(\chi_0 - \zeta)) + \beta_1 - 1$	$\Re\zeta < \Re(\chi_0 + \varsigma_0 + \frac{\kappa_0}{\beta_0}) + \frac{\beta_1 - 1}{\beta_0}$
n_2	$-\beta_0 \Re \varsigma_0 < \Re w < \Re(\kappa_0 + \beta_0(\chi_0 - \zeta) + \delta_1) - 1$	$\Re\zeta < \Re(\chi_0 + \varsigma_0 + \frac{\kappa_0}{\beta_0} + \frac{\delta_1 - 1}{\beta_0})$
n_3	$-\Re(\beta_0 \varsigma_0 + \kappa_1) < \Re w < \Re(\kappa_0 + \beta_0(\chi_0 - \zeta) + \kappa_1) - 1$	$\Re\zeta < \Re(\chi_0 + \varsigma_0 + \frac{\kappa_0}{\beta_0} + \frac{\kappa_1 - 1}{\beta_0})$

Table 8.1.: Table of absolute convergence for the integrals in (8.2.106). The first column refers to the non-zero index n_i for $i \in \{1, 2, 3\}$, whereas the second and third columns describe the necessary conditions for the convergence of each single integral and of the iterated integral as a whole.

To summarize the content of Table 8.1 in terms of (8.2.23), we can establish, for fixed $\zeta \in \mathbb{C}$ subject to

$$\Re\zeta < \Re(\chi_0 + \varsigma_0) + \frac{\Re\kappa_0}{\beta_0} + \eta_{\beta_0}(\beta_1, \delta_1, \kappa_1), \tag{8.2.107}$$

the expansion (8.2.106) represents a meromorphic function of w in the strip

$$-\beta_0 \Re\varsigma_0 < \Re w < \Re\kappa_0 + \beta_0 \Re(\chi_0 - \zeta + \eta_{\beta_0}(\beta_1, \delta_1, \kappa_1)). \tag{8.2.108}$$

Since $\eta_{\beta_0}(\beta_1, \delta_1, \kappa_1) > 0$, this especially verifies the expansion for fixed $\zeta \in \mathbb{C}$ satisfying (8.2.101) as the analytic continuation of (8.2.102). In the extended region (8.2.108) we encounter a simple pole, which is located at $w = \kappa_0 + \beta_0(\chi_0 - \zeta)$. To specify the associated residue we denote

$$\Lambda(\zeta) := \int_{T_2}^{\infty} t^{\beta_0(\zeta - \chi_0 - \varsigma_0) - \kappa_0 - 1} E(t) dt, \tag{8.2.109}$$

where the integral on the right hand side converges absolutely for $\Re\zeta < \Re(\chi_0 + \varsigma_0) + \frac{\Re\kappa_0}{\beta_0}$. We then obtain, according to absolute and uniform convergence:

$$\begin{aligned}\operatorname*{Res}_{w=\kappa_0+\beta_0(\chi_0-\zeta)} \mathcal{L}(w,\zeta) = {} & -e^{\zeta B(T_1)} D(T_1) \int_{T_2}^{\infty} t^{\beta_0(\zeta-\chi_0-\varsigma_0)-\kappa_0-1} E(t) K(T_1+t) dt \\ & - \int_{T_2}^{\infty} t^{\beta_0(\zeta-\chi_0-\varsigma_0)-\kappa_0-1} E(t) \int_{T_1}^{\infty} \frac{d}{ds}\left\{e^{\zeta B(s)} D(s) K(s+t)\right\} ds dt\end{aligned}$$

$$= -d_0 k_0 \{b_0\}^{-\zeta} \int_{T_2}^{\infty} t^{\beta_0(\zeta-\chi_0-\varsigma_0)-\kappa_0-1} E(t) dt$$

$$= -d_0 k_0 \{b_0\}^{-\zeta} \Lambda(\zeta) \tag{8.2.110}$$

The second equality follows from the fundamental theorem of calculus and involves the coefficients b_0, d_0 and k_0, appearing in the dominating term of the ingredient functions φ, d and k at infinity.

We eventually return to a study of the MB-integral (8.2.105). As we have seen in the preceding paragraph, the pole of the expansion (8.2.106) plays a major role. If we confine the admissible values of ζ to the substrip of (8.2.101) given by

$$\Re\chi_0 < \Re\zeta < \Re(\chi_0 + \varsigma_0) + \frac{\Re\kappa_0}{\beta_0}, \tag{8.2.111}$$

the restrictions in (8.2.104) change to

$$-\beta_0 \Re\varsigma_0 < u_0 < \Re(\kappa_0 + \beta_0(\chi_0 - \zeta)). \tag{8.2.112}$$

Moreover, in these circumstances the pole at $w = \kappa_0 + \beta_0(\chi_0 - \zeta)$ lies to the left of the pole of the gamma function $\Gamma(\kappa_0 - w)$, and it is the singularity that lies closest to the right of the line $\Re w = u_0$. It remains to examine the asymptotic behaviour of the integrand in (8.2.105) for large $|w|$, in order to justify a rightward displacement of the integration path across the indicated pole. On the one hand, due to the absolute and thus with respect to $\Im w \in \mathbb{R}$ uniform convergence of each integral in the expansion (8.2.106), the function $\mathcal{L}(w, \zeta)$ is $\mathcal{O}(1)$ as $\Im w \to \pm\infty$, uniformly with respect to $\Re w$ in any closed vertical substrip of (8.2.108). On the other hand, the gamma functions in (8.2.105) decay exponentially fast as $\Im w \to \pm\infty$, uniformly with respect to $\Re w$ in any closed vertical substrip of the complex plane. The whole integrand therefore vanishes exponentially fast in the imaginary direction of the strip (8.2.108). As a consequence, arbitrary displacements of the integration path in (8.2.105) are viable in the indicated strip. We decide to move the path to the right to match a line $\Re w = u_1$ with

$$\begin{cases} u_1 > \max\{-\beta_0\Re\varsigma_0, \Re(\kappa_0 + \beta_0(\chi_0 - \zeta))\}, \\ u_1 < \Re\kappa_0 + \min\{0, \beta_0\Re(\chi_0 - \zeta + \eta_{\beta_0}(\beta_1, \delta_1, \kappa_1))\}. \end{cases} \tag{8.2.113}$$

This strip is especially non-empty for all admissible values of ς_0 and $\zeta \in \mathbb{C}$ satisfying (8.2.111). In the process of the descibed displacement we only encounter the simple pole at $w = \kappa_0 + \beta_0(\chi_0 - \zeta)$, which is traversed in the clockwise direction, thereby leading to a negative sign of the residue. We thus deduce from (8.2.105) and (8.2.110) for any $\zeta \in \mathbb{C}$ subject to (8.2.111) the expansion

$$\mathcal{M}\begin{bmatrix} -\zeta, T_1, \infty \\ T_2, \infty \end{bmatrix} = d_0 k_0 \frac{\Gamma(\kappa_0 + \beta_0(\chi_0 - \zeta))\Gamma(\beta_0(\zeta - \chi_0))}{\{b_0\}^{\zeta}\,\Gamma(\kappa_0)} \Lambda(\zeta) + \mathcal{M}_1(-\zeta; T_1, T_2), \tag{8.2.114}$$

in which the integral function in the second summand refers to

$$\mathcal{M}_1(-\zeta; T_1, T_2) := \frac{1}{2\pi i} \int\limits_{u_1 - i\infty}^{u_1 + i\infty} \frac{\Gamma(w)\Gamma(\kappa_0 - w)}{\Gamma(\kappa_0)} \mathcal{L}(w, \zeta) dw. \tag{8.2.115}$$

We can now discuss the analyticity properties of (8.2.114) with respect to ζ. Concerning the leading term we first examine the properties of the integral $\Lambda(\zeta)$. It was defined in (8.2.109) and is absolutely convergent and holomorphic in

$$\Re\zeta < \Re(\chi_0 + \varsigma_0) + \frac{\Re\kappa_0}{\beta_0}, \tag{8.2.116}$$

and this half plane is readily seen to coincide with (8.2.101) still. It is routine to derive by partial integration the analytic continuation. If we define

$$\lambda(\zeta) := T_2^{\beta_0(\zeta - \chi_0 - \varsigma_0) - \kappa_0} E(T_2) + \int\limits_{T_2}^{\infty} t^{\beta_0(\zeta - \chi_0 - \varsigma_0) - \kappa_0} E'(t) dt, \tag{8.2.117}$$

this yields

$$\Lambda(\zeta) = -\frac{\lambda(\zeta)}{\beta_0(\zeta - \chi_0 - \varsigma_0 - \frac{\kappa_0}{\beta_0})}. \tag{8.2.118}$$

The corresponding region of absolute convergence and analyticity contains (8.2.116) and matches in particular the half plane

$$\Re\zeta < \Re(\chi_0 + \varsigma_0) + \frac{\Re\kappa_0}{\beta_0} + \eta_{\beta_0}(\varepsilon_1). \tag{8.2.119}$$

Hence, the right hand side of (8.2.118) establishes the analytic continuation of the initial integral definition (8.2.109). In the extended region the only singularity is a simple pole at $\zeta = \frac{\kappa_0}{\beta_0} + \chi_0 + \varsigma_0$. The computation of the associated residue is postponed to a later paragraph. From the fundamental theorem of calculus we get

$$\lambda\left(\chi_0 + \varsigma_0 + \frac{\kappa_0}{\beta_0}\right) = e_0. \tag{8.2.120}$$

In addition, arbitrary derivatives of $\lambda(\zeta)$ can be calculated by interchanging differentiation and integration, appealing to Lemma 6.3.1. For $k \in \mathbb{N}$ we therefore obtain

$$\begin{aligned} \lambda_k &:= \lambda^{(k)}\left(\chi_0 + \varsigma_0 + \frac{\kappa_0}{\beta_0}\right), \\ &= \{\beta_0 \log(T_2)\}^k E(T_2) + \int\limits_{T_2}^{\infty} \{\beta_0 \log(t)\}^k E'(t) dt. \end{aligned} \tag{8.2.121}$$

Notice for each k the absolute convergence of these integrals. Finally, for $\zeta \in \mathbb{C}$ subject to (8.2.111) the definition of $\lambda(\zeta)$ enables us instead of (8.2.114) to write

$$\mathcal{M}\begin{bmatrix} -\zeta, T_1, \infty \\ T_2, \infty \end{bmatrix} = -\frac{d_0 k_0}{\beta_0 \Gamma(\kappa_0)} \frac{\Gamma(\kappa_0 + \beta_0(\chi_0 - \zeta))\Gamma(\beta_0(\zeta - \chi_0))}{\{b_0\}^{\zeta}\left(\zeta - \chi_0 - \varsigma_0 - \frac{\kappa_0}{\beta_0}\right)} \lambda(\zeta) + \mathcal{M}_1(-\zeta; T_1, T_2). \tag{8.2.122}$$

The first summand in the latter expansion then constitutes a meromorphic function of ζ in the region (8.2.119). The poles lying therein form an infinite sequence and are all of simple order with a few possible exceptions occuring for special parametrizations. To eventually characterize the analyticity properties of the MB-integral (8.2.115) we first recall that the vertical line $\Re w = u_1$ satisfies (8.2.113). Accordingly, the restrictions for admissible arguments $\zeta \in \mathbb{C}$ are

$$\Re\chi_0 < \Re\zeta < \Re(\chi_0 + \varsigma_0) + \frac{\Re\kappa_0}{\beta_0} + \eta_{\beta_0}(\beta_1, \delta_1, \kappa_1). \tag{8.2.123}$$

In other words, for $\zeta \in \mathbb{C}$ satisfying the above conditions we can find $u_1 \in \mathbb{R}$ with the property (8.2.113), and the MB-integral (8.2.115) then converges absolutely. Following from the assumptions on the involved parameters, see also (8.2.99), it is always possible to pick a fixed but arbitrary

$$0 \le -\Re\varsigma_0 < \varepsilon < \frac{\Re\kappa_0}{\beta_0}. \tag{8.2.124}$$

If we then create a substrip $\mathbb{Z}_\varepsilon$ of (8.2.123) by

$$\mathbb{Z}_\varepsilon := \left\{ z \in \mathbb{C} : \Re\chi_0 + \frac{\Re\kappa_0}{\beta_0} - \varepsilon < \Re z < \Re\chi_0 + \frac{\Re\kappa_0}{\beta_0} + \eta_{\beta_0}(\beta_1, \delta_1, \kappa_1) - \varepsilon \right\}, \tag{8.2.125}$$

by comparison with (8.2.113) it is easy to confirm validity of the choice $u_1 \equiv \beta_0\varepsilon$ for all $\zeta \in \mathbb{Z}_\varepsilon$. This yields for (8.2.115) the representation

$$\mathcal{M}_1(-\zeta; T_1, T_2) = \frac{1}{2\pi i} \int_{\beta_0\varepsilon - i\infty}^{\beta_0\varepsilon + i\infty} \frac{\Gamma(w)\Gamma(\kappa_0 - w)}{\Gamma(\kappa_0)} \mathcal{L}(w, \zeta) dw. \tag{8.2.126}$$

The gamma functions appearing in the integrand are holomorphic for $0 < \Re w < \Re\kappa_0$ and thus especially continuous with respect to $v \in \mathbb{R}$ for $w = \beta_0\varepsilon + iv$. In addition, for fixed $\zeta \in \mathbb{Z}_\varepsilon$ the function $\mathcal{L}(\beta_0\varepsilon + iv, \zeta)$ is also continuous with respect to $v \in \mathbb{R}$, particularly due to the absolute and with respect to $v \in \mathbb{R}$ uniform convergence of each integral in the expansion (8.2.106) and since $\chi_0 + \frac{\kappa_0}{\beta_0} - \varepsilon \notin \mathbb{Z}_\varepsilon$. Finally, $\mathcal{L}(\beta_0\varepsilon + iv, \zeta)$ for fixed $v \in \mathbb{R}$ is holomorphic with respect to $\zeta \in \mathbb{Z}_\varepsilon$. We therefore conclude from Theorem 5.6.1 in [Wegert, 2012] for each $n \in \mathbb{N}$ analyticity in $\mathbb{Z}_\varepsilon$ of the integral

$$\mathcal{M}_n(-\zeta) := \frac{1}{2\pi i} \int\limits_{\beta_0\varepsilon - in}^{\beta_0\varepsilon + in} \frac{\Gamma(w)\Gamma(\kappa_0 - w)}{\Gamma(\kappa_0)} \mathcal{L}(w, \zeta) dw.$$

Next we denote by Z a closed vertical substrip of $\mathbb{Z}_\varepsilon$. With $w = \beta_0\varepsilon + iv$, we then ascertain from (8.2.106) the existence of constants $L_1, L_2 > 0$, which are uniformly bounded with respect to $\zeta \in Z$ and $v \in \mathbb{R}$, such that

$$|\mathcal{L}(w, \zeta)| \leq \frac{L_1 + |\zeta| L_2}{\Re(\beta_0(\zeta - \chi_0) - \kappa_0) + \beta_0\varepsilon}. \tag{8.2.127}$$

By virtue of this bound it is easy to confirm the convergence of $\mathcal{M}_n(-\zeta)$ as $n \to \infty$ to (8.2.126), uniformly with respect to ζ in any compact subset of Z, which suffices to verify analyticity of the latter MB-integral. More precisely, by Theorem 5.1.3 in [Wegert, 2012] it shows, that (8.2.126) is a holomorphic function of ζ in $\mathbb{Z}_\varepsilon$, and by arbitrariness of ε within the range (8.2.124) we eventually have analyticity in the strip (8.2.123). Summarizing, the expansion (8.2.122) represents the sum of two functions which are meromorphic in (8.2.119) and analytic in (8.2.123), respectively. But since each of these regions at least overlaps with the half plane (8.2.101), the indicated expansion represents the analytic continuation of the iterated generating function (8.2.100) to the region

$$\Re\chi_0 < \Re\zeta < \Re(\chi_0 + \varsigma_0) + \frac{\Re\kappa_0}{\beta_0} + \eta_{\beta_0}(\beta_1, \delta_1, \varepsilon_1, \kappa_1). \tag{8.2.128}$$

Therein, appealing to the exponential decay of the first summand in (8.2.122) and to the bound (8.2.127), we conclude, as $\Im\zeta \to \pm\infty$, uniformly with respect to $\Re\zeta$ in any closed vertical substrip, the continuation is

$$\mathcal{M}\begin{bmatrix} -\zeta, T_1, \infty \\ T_2, \infty \end{bmatrix} = \mathcal{O}(\zeta). \tag{8.2.129}$$

Finally, the singularities occuring in the region (8.2.128) are generated merely by the first summand of the expansion (8.2.122), to be exact by the rational function and by the gamma function which depends on κ_0 and on χ_0. In fact, the gamma function with the sole argument $\zeta - \chi_0$ has all of its singularities to the left of the boundary line $\Re\zeta = \Re\chi_0$. Following from these observations, singularities of the function (8.2.122) in the strip (8.2.128) are poles only whose location and thus also their order depends on the parameters. Of particular interest is the singularity that lies on the abscissa of absolute convergence of the initial integral representation (8.2.100), i.e., on the line $\Re\zeta = \Re(\chi_0 + \varsigma_0) + \frac{\Re\kappa_0}{\beta_0}$. There, we find a pole of order $1 \leq J \leq 2$ located at $\zeta = \chi_0 + \varsigma_0 + \frac{\kappa_0}{\beta_0}$. Accordingly, in a neighborhood of this point the analytic continuation of the iterated generating function possesses a Laurent expansion of the form

$$\mathcal{M}\begin{bmatrix} -\zeta, T_1, \infty \\ T_2, \infty \end{bmatrix} = \sum_{j=0}^{J} \frac{\mu_{-j}\left(\chi_0 + \frac{\kappa_0}{\beta_0}, \varsigma_0\right)}{(\zeta - \chi_0 - \varsigma_0 - \frac{\kappa_0}{\beta_0})^j} + \mathcal{O}\left(\zeta - \chi_0 - \varsigma_0 - \frac{\kappa_0}{\beta_0}\right), \tag{8.2.130}$$

where the coefficient associated with the index $j = 1$ equals the residue. Two cases require special attention. Those are $J = 2$ or $\chi_0 + \frac{\kappa_0}{\beta_0} = -\varsigma_0$, in which event the indicated pole is of second order or matches the origin of the ζ-plane. If $J = 2$ we will compute the coefficients for $j \in \{1, 2\}$, whereas if $\chi_0 + \frac{\kappa_0}{\beta_0} = -\varsigma_0$ also the coefficient for $j = 0$ will be determined. To accomplish these tasks the formula

$$\mu_{-j}\left(\chi_0 + \tfrac{\kappa_0}{\beta_0}, \varsigma_0\right) = \frac{1}{(J-j)!} \lim_{\zeta \to \chi_0 + \varsigma_0 + \frac{\kappa_0}{\beta_0}} \frac{d^{J-j}}{d\zeta^{J-j}} \left(\zeta - \varsigma_0 - \chi_0 - \tfrac{\kappa_0}{\beta_0}\right)^J \mathcal{M}\begin{bmatrix} -\zeta, T_1, \infty \\ T_2, \infty \end{bmatrix} \tag{8.2.131}$$

will be helpful. It is a consequence of Taylor's theorem for holomorphic functions.

8.2.6.2.1. Coefficients for $\varsigma_0 \neq 0$ and $\chi_0 + \frac{\kappa_0}{\beta_0} \neq -\varsigma_0$. The pole at $\zeta = \chi_0 + \varsigma_0 + \frac{\kappa_0}{\beta_0}$ is then of simple order, whence $J = 1$, and, according to (8.2.120), the associated residue is given by

$$\mu_{-1}\left(\chi_0 + \tfrac{\kappa_0}{\beta_0}, \varsigma_0\right) = -\frac{d_0 e_0 k_0}{\beta_0 \Gamma(\kappa_0)} \{b_0\}^{-\chi_0 - \varsigma_0 - \frac{\kappa_0}{\beta_0}} \Gamma(-\beta_0 \varsigma_0)\Gamma(\kappa_0 + \beta_0 \varsigma_0). \tag{8.2.132}$$

The indicated pole especially does not match the origin.

8.2.6.2.2. Coefficients for $\varsigma_0 \neq 0$ with $\chi_0 + \frac{\kappa_0}{\beta_0} = -\varsigma_0$. In these circumstances again $J = 1$ and the residue is the same as in the preceding subparagraph but with $\kappa_0 = -\beta_0(\chi_0 + \varsigma_0)$. In contrast, however, the indicated pole coincides with the origin. To compute the coefficient corresponding to the constant term in (8.2.130), for (8.2.122) we write

$$\mathcal{M}\begin{bmatrix} -\zeta, T_1, \infty \\ T_2, \infty \end{bmatrix} = -\frac{d_0 k_0}{\beta_0 \Gamma(-\beta_0(\chi_0 + \varsigma_0))} e^{f(\zeta; -\varsigma_0, \chi_0)} \frac{\lambda(\zeta)}{\zeta} + \mathcal{M}_1(-\zeta; T_1, T_2), \tag{8.2.133}$$

where we defined for $a, b \in \mathbb{C}$, with the logarithms taking their principal values,

$$f(\zeta; a, b) := \log \Gamma(\beta_0(a - \zeta)) + \log \Gamma(\beta_0(\zeta - b)) - \zeta \log \{b_0\}. \tag{8.2.134}$$

The identity (8.2.131) combined with the product and the chain rule then yields:

$$\begin{aligned} \mu_0(-\varsigma_0, \varsigma_0) &= -\frac{d_0 k_0}{\beta_0 \Gamma(-\beta_0(\chi_0 + \varsigma_0))} \frac{d}{d\zeta} \left\{ e^{f(\zeta; -\varsigma_0, \chi_0)} \lambda(\zeta) \right\} \Bigg|_{\zeta=0} + \mathcal{M}_1(0; T_1, T_2) \\ &= -\frac{d_0 e_0 k_0}{\beta_0} \frac{\Gamma(-\beta_0 \varsigma_0)\Gamma(-\beta_0 \chi_0)}{\Gamma(-\beta_0(\chi_0 + \varsigma_0))} \left\{ f'(0; -\varsigma_0, \chi_0) + \frac{\lambda_1}{e_0} \right\} + \mathcal{M}_1(0; T_1, T_2) \end{aligned} \tag{8.2.135}$$

For the last equality we took into account (8.2.120), (8.2.121), and in terms of the digamma function we obtain

$$f'(0;a,b) = -\beta_0\psi(\beta_0 a) + \beta_0\psi(-\beta_0 b) - \log\{b_0\}\,. \tag{8.2.136}$$

8.2.6.2.3. Coefficients for $\varsigma_0 = 0$ with $\chi_0 + \frac{\kappa_0}{\beta_0} \neq 0$. For this parametrization the expansion (8.2.122) shows a second order pole at $\zeta = \chi_0 + \varsigma_0 + \frac{\kappa_0}{\beta_0}$, i.e., $J = 2$, and this point does not coincide with the origin. Upon once applying the functional equation for the gamma function we can write

$$\mathcal{M}\begin{bmatrix} -\zeta, T_1, \infty \\ T_2, \infty \end{bmatrix} = \frac{d_0 k_0}{\beta_0^2\Gamma(\kappa_0)} \frac{e^{g\left(\zeta;\chi_0+\frac{\kappa_0}{\beta_0},\chi_0\right)}\lambda(\zeta)}{(\zeta-\chi_0-\frac{\kappa_0}{\beta_0})^2} + \mathcal{M}_1(-\zeta;T_1,T_2), \tag{8.2.137}$$

where for $a, b \in \mathbb{C}$ we employed the definition

$$g(\zeta;a,b) := \log\Gamma(1+\beta_0(a-\zeta)) + \log\Gamma(\beta_0(\zeta-b)) - \zeta\log\{b_0\}\,, \tag{8.2.138}$$

assuming the principal branch of the logarithm. We then deduce from (8.2.131) by incorporating (8.2.120) and (8.2.121), since the second summand in (8.2.137) is holomorphic at $\zeta = \chi_0 + \frac{\kappa_0}{\beta_0}$:

$$\begin{aligned}\mu_{-1}\left(\chi_0 + \tfrac{\kappa_0}{\beta_0}, 0\right) &= \frac{d_0 k_0}{\beta_0^2\Gamma(\kappa_0)}\frac{d}{d\zeta}\left\{e^{g\left(\zeta;\chi_0+\frac{\kappa_0}{\beta_0},\chi_0\right)}\lambda(\zeta)\right\}\Bigg|_{\zeta=\chi_0+\frac{\kappa_0}{\beta_0}} \\ &= \frac{d_0 e_0 k_0}{\beta_0^2}\{b_0\}^{-\chi_0-\frac{\kappa_0}{\beta_0}}\left\{g'\left(\chi_0+\tfrac{\kappa_0}{\beta_0};\chi_0+\tfrac{\kappa_0}{\beta_0},\chi_0\right)+\frac{\lambda_1}{e_0}\right\}\end{aligned} \tag{8.2.139}$$

The first derivative of $g(\cdot;a,b)$ is equal to

$$g'(\zeta;a,b) := -\beta_0\psi(1+\beta_0(a-\zeta)) + \beta_0\psi(\beta_0(\zeta-b)) - \log\{b_0\}\,, \tag{8.2.140}$$

from which subject to (B.2.13) in terms of the Euler-Mascheroni constant we obtain

$$g'(a;a,b) = \beta_0\gamma + \beta_0\psi(\beta_0(a-b)) - \log\{b_0\}\,. \tag{8.2.141}$$

Finally, again with the aid of (8.2.131) we compute

$$\mu_{-2}\left(\chi_0 + \tfrac{\kappa_0}{\beta_0}, 0\right) = \frac{d_0 e_0 k_0}{\beta_0^2}\{b_0\}^{-\chi_0-\frac{\kappa_0}{\beta_0}}\,. \tag{8.2.142}$$

8.2.6.2.4. Coefficients for $\varsigma_0 = \chi_0 + \frac{\kappa_0}{\beta_0} = 0$. In case of a parametrization of the present subparagraph, in (8.2.122) we observe the same coalescence to a second order pole as in the preceding subparagraph, but the resulting pole matches the origin of the ζ-plane. Accordingly, $J = 2$ and the coefficients appearing in the Laurent expansion (8.2.130) are for $j \in \{2, 1\}$ again given by (8.2.141) and (8.2.142) but with $\chi_0 + \frac{\kappa_0}{\beta_0} = 0$. Regarding the coefficient for $j = 0$, by

(8.2.131) from (8.2.137) with $\chi_0 + \frac{\kappa_0}{\beta_0} = 0$ we find:

$$\mu_0(0,0) = \frac{d_0 k_0}{\beta_0^2 \Gamma(\kappa_0)} \frac{1}{2} \frac{d^2}{d\zeta^2} \left\{ e^{g(\zeta;0,\chi_0)} \lambda(\zeta) \right\} \Bigg|_{\zeta=0} + \frac{1}{2} \frac{d^2}{d\zeta^2} \left\{ \zeta^2 \mathcal{M}_1(-\zeta; T_1, T_2) \right\} \Bigg|_{\zeta=0}$$
$$= \frac{1}{2} \frac{d_0 k_0}{\beta_0^2 \Gamma(\kappa_0)} e^{g(0;0,\chi_0)} \Big\{ \left(g'(0;0,\chi_0) \right)^2 \lambda(0)$$
$$+ 2g'(0;0,\chi_0)\lambda'(0) + g''(0;0,\chi_0)\lambda(0) + \lambda''(0) \Big\} + \mathcal{M}_1(0; T_1, T_2)$$

The first derivative of $g(\cdot; a, b)$ was calculated in (8.2.140) and in terms of the trigamma function (B.2.12) we obtain for its second

$$g''(\zeta; 0, b) = \beta_0^2 \psi'(1 - \beta_0 \zeta) + \beta_0^2 \psi'(\beta_0(\zeta - b)). \tag{8.2.143}$$

By (B.2.14) we conclude

$$g''(0;0,\chi_0) = \beta_0^2 \frac{\pi^2}{6} + \beta_0^2 \psi'(-\beta_0 \chi_0). \tag{8.2.144}$$

Furthermore, the first and the second derivative of $\lambda(\zeta)$ at $\zeta = \chi_0 + \frac{\kappa_0}{\beta_0} + \varsigma_0 = 0$ was evaluated in (8.2.121). With $\lambda(0) = e_0$ we eventually arrive at

$$\mu_0(0,0) = \frac{1}{2} \frac{d_0 e_0 k_0}{\beta_0^2} \left\{ \left(g'(0;0,\chi_0) \right)^2 + 2g'(0;0,\chi_0) \frac{\lambda_1}{e_0} + g''(0;0,\chi_0) + \frac{\lambda_2}{e_0} \right\} + \mathcal{M}_1(0; T_1, T_2). \tag{8.2.145}$$

8.2.6.3. Two Infinite Paths and a Kernel of the Second Kind

The method of integral transforms can also be employed to derive the analytic continuation of an iterated generating function of the second kind (8.2.2) with two infinite paths. This shall be discussed below, again for ingredient functions satisfying the conditions of §8.2.6.2, with the parameters defined in (8.2.98), however, being subject to

$$\Re(\chi_0 + \varsigma_0) \le \Re\chi_0 < \Re\varsigma_0. \tag{8.2.146}$$

In terms of the normalized phase $B(r)$ and amplitude functions $D(r)$, $E(r)$ and $K(r)$, for the integral (8.2.2) we can write

$$\mathcal{S}\left[-\zeta; \begin{matrix} T_1, \infty \\ T_2, \infty \end{matrix}\right] = \int_{T_2}^{\infty} t^{-\beta_0 \varsigma_0 - 1} E(t) \int_{T_1}^{\infty} \frac{s^{-\beta_0 \chi_0 - 1}}{(s+t)^{\kappa_0 - \beta_0 \zeta}} D(s) e^{\zeta B(s+t)} K(s+t) \, ds dt. \tag{8.2.147}$$

With the aid of Lemma 8.2.5 we readily confirm that the region of absolute convergence and analyticity matches the half plane

$$\Re\zeta < \frac{\Re\kappa_0}{\beta_0} + \Re(\chi_0 + \varsigma_0). \tag{8.2.148}$$

Observe that the boundary of the region results from the supplementary condition for the convergence of the iterated integral. Therefore, similar to the preceding paragraphs we employ the integral transform (8.2.76) to unlock the analytic continuation. To show the validity of this step we first note, condition (8.2.146) implies

$$\Re\chi_0 < 0,$$
$$\Re\varsigma_0 \leq 0,$$

whence $\Re(\chi_0 + \varsigma_0) < 0$. Consequently, each ζ that lies in the half plane (8.2.148) especially satisfies $\Re(\kappa_0 - \beta_0\zeta) > 0$. For fixed ζ we may thus indeed identify $a \equiv \kappa_0 - \beta_0\zeta$ and apply formula (8.2.76). Define by

$$\mathcal{J}(w,\zeta) := \int_{T_2}^{\infty} t^{-\beta_0\varsigma_0 - w - 1} E(t) \int_{T_1}^{\infty} s^{\beta_0(\zeta-\chi_0)+w-\kappa_0-1} D(s) e^{\zeta B(s+t)} K(s+t) ds dt \tag{8.2.149}$$

a function which is for fixed $\zeta \in \mathbb{C}$ subject to (8.2.148) absolutely convergent and holomorphic with respect to $w \in \mathbb{C}$ in

$$-\beta_0\Re\varsigma_0 < \Re w < \Re\kappa_0 + \beta_0\Re(\chi_0 - \zeta). \tag{8.2.150}$$

From (8.2.147) for any $\zeta \in \mathbb{C}$ subject to (8.2.148) we then obtain the MB-representation

$$\mathcal{S}\left[-\zeta; \begin{matrix} T_1, \infty \\ T_2, \infty \end{matrix}\right] = \frac{1}{2\pi i} \int_{u_0 - i\infty}^{u_0 + i\infty} \frac{\Gamma(w)\Gamma(\kappa_0 - \beta_0\zeta - w)}{\Gamma(\kappa_0 - \beta_0\zeta)} \mathcal{J}(w,\zeta) dw, \tag{8.2.151}$$

where the integration path is a vertical line $\Re w = u_0$. Absolute convergence by (8.2.76) and by (8.2.150) requires

$$-\beta_0\Re\varsigma_0 < u_0 < \Re\kappa_0 + \beta_0\Re(\chi_0 - \zeta). \tag{8.2.152}$$

It easily follows from (8.2.148) that this latter strip is non-empty for all permitted values of the argument ζ. Note that its upper limit originates in the region of convergence of the integral (8.2.149) only, but it is not determined by definition of the transform (8.2.76) since $\Re\chi_0 < 0$. Now, by inspection of (8.2.149) we observe, similar to the preceding two paragraphs there is again a close connection between the admissibility of arguments ζ and w with a greater real part. These can be achieved by computing the analytic continuation towards the right direction of the complex $(w+\zeta)$-plane. Upon integrating by parts for $\zeta, w \in \mathbb{C}$ respectively subject to (8.2.148) and (8.2.150), we find:

$$\begin{aligned}\mathcal{J}(w,\zeta) = &-\frac{T_1^{\beta_0(\zeta-\chi_0)+w-\kappa_0} D(T_1)}{\beta_0(\zeta-\chi_0)+w-\kappa_0} \int_{T_2}^{\infty} t^{-\beta_0\varsigma_0-w-1} E(t) e^{\zeta B(T_1+t)} K(T_1+t)dt \\ &- \frac{1}{\beta_0(\zeta-\chi_0)+w-\kappa_0} \sum_{\substack{n_1,n_2,n_3\in\{0,1\} \\ n_1+n_2+n_3=1}} \int_{T_2}^{\infty} t^{-\beta_0\varsigma_0-w-1} E(t) \\ &\times \int_{T_1}^{\infty} s^{\beta_0(\zeta-\chi_0)+w-\kappa_0} D^{(n_1)}(s) e^{\zeta B(s+t)} \left\{\zeta B'(s+t)\right\}^{n_2} K^{(n_3)}(s+t) ds dt\end{aligned} \tag{8.2.153}$$

The integrals in this expansion converge absolutely for the arguments w,ζ specified in Table 8.2, and for fixed ζ each integral even represents a holomorphic function in its w-region of absolute convergence.

$\neq 0$	**single**	**iterated**
$-$	$-\beta_0\Re\varsigma_0 < \Re w$	$-$
n_1	$-\beta_0\Re\varsigma_0 < \Re w < \Re(\kappa_0+\beta_0(\chi_0-\zeta)+\delta_1)-1$	$\Re\zeta < \Re(\chi_0+\varsigma_0+\frac{\kappa_0}{\beta_0}+\frac{\delta_1-1}{\beta_0})$
n_2	$-\beta_0\Re\varsigma_0-\beta_1 < \Re w < \Re(\kappa_0+\beta_0(\chi_0-\zeta))+\beta_1-1$	$\Re\zeta < \Re(\chi_0+\varsigma_0+\frac{\kappa_0}{\beta_0})+\frac{\beta_1-1}{\beta_0}$
n_3	$-\Re(\beta_0\varsigma_0+\kappa_1) < \Re w < \Re(\kappa_0+\beta_0(\chi_0-\zeta)+\kappa_1)-1$	$\Re\zeta < \Re(\chi_0+\varsigma_0+\frac{\kappa_0}{\beta_0}+\frac{\kappa_1-1}{\beta_0})$

Table 8.2.: Table of absolute convergence for the integrals in (8.2.153). The first column refers to the non-zero index n_i for $i \in \{1,2,3\}$, whereas the second and third columns describe the necessary conditions for the convergence of each single integral and of the iterated integral as a whole.

Accordingly, the expansion (8.2.153) establishes the analytic continuation of the integral (8.2.149) to the strip

$$-\beta_0\Re\varsigma_0 < \Re w < \Re\kappa_0 + \beta_0\Re(\chi_0-\zeta+\eta_{\beta_0}(\beta_1,\delta_1,\kappa_1)), \tag{8.2.154}$$

for any fixed $\zeta \in \mathbb{C}$ that satisfies (8.2.148). Therein it exhibits a pole of simple order at $w = \kappa_0 + \beta_0(\chi_0 - \zeta)$. Again in terms of the integral function (8.2.109) the associated residue is readily computed by means of the fundamental theorem of calculus:

$$\begin{aligned}\operatorname*{Res}_{w=\kappa_0+\beta_0(\chi_0-\zeta)} \mathcal{J}(w,\zeta) &= -D(T_1)\int_{T_2}^{\infty} t^{\beta_0(\zeta-\chi_0-\varsigma_0)-\kappa_0-1} E(t) e^{\zeta B(T_1+t)} K(T_1+t)dt \\ &\quad - \int_{T_2}^{\infty} t^{\beta_0(\zeta-\chi_0-\varsigma_0)-\kappa_0-1} E(t) \int_{T_1}^{\infty} \frac{d}{ds}\left\{D(s)e^{\zeta B(s+t)} K(s+t)\right\} ds dt \\ &= -d_0 k_0 \{b_0\}^{-\zeta} \Lambda(\zeta)\end{aligned} \tag{8.2.155}$$

Having unlocked the continuation of $\mathcal{J}(w,\zeta)$ with respect to w, we shall now reconsider the MB-representation (8.2.151) for the iterated generating function and discuss the possibility for a displacement of the integration path. We conclude from the preceding paragraph that this operation will enable us to enter a wider range of ζ-values. According to the above findings, for fixed ζ subject to (8.2.148) the integrand in (8.2.151) is a meromorphic function in the w-region (8.2.154) with the singularity that lies closest to the right of the line $\Re w = u_0$ being given by the simple pole at $w = \kappa_0 + \beta_0(\chi_0 - \zeta)$, since $\chi_0 < 0$. Observe that this pole was of double order if $\chi_0 = 0$ was admitted, with a substantial increasement in difficulty for computing its residue. Due to our assumption $\chi_0 < 0$, however, this is a routine task. To collect the residue of the indicated pole we ascertain from the expansion (8.2.153) that the function $\mathcal{J}(w,\zeta)$ is $\mathcal{O}(1)$ as $\Im w \to \pm\infty$, uniformly with respect to $\Re w$ in any closed vertical substrip of (8.2.154). We may therefore appeal to the exponential decay of the gamma functions in the MB-integral (8.2.151), and displace the integration path to the right over the pole at $w = \kappa_0 + \beta_0(\chi_0 - \zeta)$, to match a line $\Re w = u_1$ with

$$(8.2.156)\qquad \begin{cases} u_1 > \max\{-\beta_0\Re\varsigma_0, \Re(\kappa_0 + \beta_0(\chi_0 - \zeta))\}, \\ u_1 < \Re(\kappa_0 - \beta_0\zeta) + \min\{0, \beta_0(\Re\chi_0 + \eta_{\beta_0}(\beta_1,\delta_1,\kappa_1))\}. \end{cases}$$

By taking into account the residue (8.2.155) and incorporating the fact that the pole is encircled in the clockwise direction, for any $\zeta \in \mathbb{C}$ subject to (8.2.148) we eventually arrive at

$$(8.2.157)\qquad \mathcal{S}\left[-\zeta; \begin{matrix} T_1, \infty \\ T_2, \infty \end{matrix}\right] = d_0 k_0 \{b_0\}^{-\zeta} \frac{\Gamma(\kappa_0 + \beta_0(\chi_0 - \zeta))\Gamma(-\beta_0\chi_0)}{\Gamma(\kappa_0 - \beta_0\zeta)} \Lambda(\zeta) + \mathcal{S}_1(-\zeta; T_1, T_2),$$

where the second summand features the MB-integral

$$(8.2.158)\qquad \mathcal{S}_1(-\zeta; T_1, T_2) := \frac{1}{2\pi i} \int_{u_1 - i\infty}^{u_1 + i\infty} \frac{\Gamma(w)\Gamma(\kappa_0 - \beta_0\zeta - w)}{\Gamma(\kappa_0 - \beta_0\zeta)} \mathcal{J}(w,\zeta)\, dw.$$

To specify the analyticity properties of the representation (8.2.157) with respect to ζ we first recall the integral function denoted by $\Lambda(\zeta)$ converges absolutely and is holomorphic in $\Re\zeta < \Re(\chi_0 + \varsigma_0) + \frac{\Re\kappa_0}{\beta_0}$, compare (8.2.109). Its continuation to the half plane

$$(8.2.159)\qquad \Re\zeta < \Re(\chi_0 + \varsigma_0) + \frac{\Re\kappa_0}{\beta_0} + \eta_{\beta_0}(\varepsilon_1)$$

is furnished by the representation (8.2.118) in terms of the function $\lambda(\zeta)$ that was defined in (8.2.117). Accordingly, if instead of (8.2.157) we write

$$(8.2.160)\qquad \mathcal{S}\left[-\zeta; \begin{matrix} T_1, \infty \\ T_2, \infty \end{matrix}\right] = -\frac{d_0 k_0}{\beta_0 \{b_0\}^{\zeta}} \frac{\Gamma(\kappa_0 + \beta_0(\chi_0 - \zeta))\Gamma(-\beta_0\chi_0)}{(\zeta - \chi_0 - \varsigma_0 - \frac{\kappa_0}{\beta_0})\Gamma(\kappa_0 - \beta_0\zeta)} \lambda(\zeta) + \mathcal{S}_1(-\zeta; T_1, T_2),$$

the first summand is meromorphic with respect to ζ in the half plane (8.2.159). Candidate poles therein are generated by the first gamma function in the numerator and by the rational function. Their location depends on the parameter values with possible coalescences up to second order. Regarding the MB-integral (8.2.158) we first note that an integration path $\Re w = u_1$ that satisfies (8.2.156) can be found for any $\zeta \in \mathbb{C}$ subject to

$$\Re\zeta < \frac{\Re\kappa_0}{\beta_0} + \Re\varsigma_0 + \min\{0, \Re\chi_0 + \eta_{\beta_0}(\beta_1, \delta_1, \kappa_1)\}, \tag{8.2.161}$$

for which we then have absolute convergence. Since $\Re\chi_0 < 0$ and $\eta_{\beta_0}(\beta_1, \delta_1, \kappa_1) > 0$ the above half plane always contains (8.2.148). Now, in the present setup by (8.2.146) it is always possible to choose a constant

$$\varepsilon > -\Re\varsigma_0 \geq 0. \tag{8.2.162}$$

The region defined by

$$\mathbb{Z}_\varepsilon := \left\{ z \in \mathbb{C} : \Re\chi_0 + \frac{\Re\kappa_0}{\beta_0} < \Re z + \varepsilon < \frac{\Re\kappa_0}{\beta_0} + \min\{0, \Re\chi_0 + \eta_{\beta_0}(\beta_1, \delta_1, \kappa_1)\} \right\}, \tag{8.2.163}$$

is then a substrip of (8.2.161), and by inspection of (8.2.156) we readily confirm the admissibility of $u_1 \equiv \beta_0\varepsilon$ for all $\zeta \in \mathbb{Z}_\varepsilon$. To verify analyticity of (8.2.158) in the strip $\mathbb{Z}_\varepsilon$, in comparison to the integral (8.2.115) from the preceding paragraph, slightly different arguments are required since there is a dependence of the gamma functions in the integrand of the above MB-integral on the variable ζ. Yet, fairly similar for $n \in \mathbb{N}$ we introduce the sequence of integrals

$$\mathcal{S}_n(-\zeta) := \frac{1}{2\pi i} \int_{\beta_0\varepsilon - in}^{\beta_0\varepsilon + in} \frac{\Gamma(w)\Gamma(\kappa_0 - \beta_0\zeta - w)}{\Gamma(\kappa_0 - \beta_0\zeta)} \mathcal{J}(w, \zeta) dw. \tag{8.2.164}$$

For fixed $\zeta \in \mathbb{Z}_\varepsilon$ we observe analyticity of the gamma functions in the strip $0 < \Re w < \Re(\kappa_0 - \beta_0\zeta)$. But the line $w = \beta_0\varepsilon + iv$ with $v \in \mathbb{R}$ runs therein, implying continuity with respect to $v \in \mathbb{R}$. Furthermore, also $\mathcal{J}(\beta_0\varepsilon + iv, \zeta)$ is a continuous function of $v \in \mathbb{R}$ for fixed $\zeta \in \mathbb{Z}_\varepsilon$ due to absolute and with respect to $v \in \mathbb{R}$ uniform convergence of the expansion (8.2.153) and since $\Re\zeta > \frac{\Re\kappa_0}{\beta_0} + \Re\chi_0 - \varepsilon$. In addition, for fixed $v \in \mathbb{R}$ the latter function is holomorphic at any $\zeta \in \mathbb{Z}_\varepsilon$. Appealing to Theorem 5.6.1 in [Wegert, 2012] we therefore conclude analyticity of (8.2.164) for any $n \in \mathbb{N}$. To verify its uniform convergence as $n \to \infty$ in any compact subset of $\mathbb{Z}_\varepsilon$, for an arbitrary subset Z we denote

$$\begin{cases} x_- := \min\{\Re\zeta : \zeta \in Z\}, \\ x_+ := \max\{\Re\zeta : \zeta \in Z\}, \\ y_- := \min\{\Im\zeta : \zeta \in Z\}, \\ y_+ := \max\{\Im\zeta : \zeta \in Z\}. \end{cases}$$

If Z is compact, each of these constants is finite. If Z denotes a closed vertical substrip, however, $y_- = -\infty$, $y_+ = \infty$ and the constants x_- and x_+ respectively describe the left and right boundary line. Moreover, for a clearer presentation we introduce the functions

$$\begin{cases} \rho(\zeta) := \Re(\kappa_0 - \beta_0\zeta), \\ \iota(\zeta) := \Im(\kappa_0 - \beta_0\zeta). \end{cases}$$

Then, for all $\zeta \in \mathbb{Z}_\varepsilon$ the first function is positive whereas the second attains both signs. Suppose for a moment Z is a closed vertical substrip of $\mathbb{Z}_\varepsilon$. With these conventions, according to the functional equation for the gamma function and the integral definition of the beta function (A.3.8), for $w = \beta_0\varepsilon + iv$ and $\zeta \in Z$ the following bound applies:

$$\begin{aligned} \text{(8.2.165)} \qquad \left|\frac{\Gamma(w)\Gamma(\kappa_0 - \beta_0\zeta - w)}{\Gamma(\kappa_0 - \beta_0\zeta)}\right| &= \frac{|\kappa_0 - \beta_0\zeta|}{|\kappa_0 - \beta_0\zeta - w|}\left|\frac{\Gamma(w)\Gamma(1 + \kappa_0 - \beta_0\zeta - w)}{\Gamma(1 + \kappa_0 - \beta_0\zeta)}\right| \\ &\le \frac{|\kappa_0 - \beta_0\zeta|}{|\kappa_0 - \beta_0(\zeta + \varepsilon) - iv|}\int_0^\infty e^{-\beta_0\varepsilon t}(1 - e^{-t})^{\Re\kappa_0 - \beta_0(x_+ + \varepsilon)}dt \\ &= \frac{|\kappa_0 - \beta_0\zeta|\,\Gamma(\beta_0\varepsilon)\Gamma(1 + \rho(x_+ + \varepsilon))}{\Gamma(1 + \rho(x_+))\sqrt{(\rho(\zeta + \varepsilon))^2 + (\iota(\zeta) - v)^2}} \\ &\le \frac{|\kappa_0 - \beta_0\zeta|\,\Gamma(\beta_0\varepsilon)\Gamma(1 + \rho(x_+ + \varepsilon))}{\Gamma(1 + \rho(x_+))\sqrt{(\rho(x_+ + \varepsilon))^2 + (\iota(\zeta) - v)^2}} \end{aligned}$$

For the last inequality we observe that $\rho(\zeta+\varepsilon)$ is decreasing as $\Re\zeta$ increases in $\mathbb{Z}_\varepsilon$. In the sequence we write $r_+ := \rho(x_+ + \varepsilon)$, where $r_+ > 0$. Next, by inspection of the expansion (8.2.153) we ascertain the existence of constants $J_1, J_2 > 0$ that depend on Z and are uniformly bounded with respect to $\zeta \in Z$, for which we have with $w = \beta_0\varepsilon + iv$ and $\zeta \in Z$:

$$\begin{aligned} \text{(8.2.166)} \qquad |\mathcal{J}(w, \zeta)| &\le \frac{J_1 + |\zeta|\,J_2}{\sqrt{(\rho(\zeta - \chi_0 + \varepsilon))^2 + (\iota(\zeta - \chi_0) - v)^2}} \\ &\le \frac{J_1 + |\zeta|\,J_2}{\sqrt{(\rho(x_- - \chi_0 + \varepsilon))^2 + (\iota(\zeta - \chi_0) - v)^2}} \end{aligned}$$

The last inequality takes into account the decreasing character of $(\rho(\zeta-\chi_0+\varepsilon))^2$ as $\Re\zeta$ decreases in $\mathbb{Z}_\varepsilon$. Writing for brevity $r_- := \rho(x_- - \chi_0 + \varepsilon)$ we have $r_- < 0$. For all $\zeta \in Z$ this yields

$$\begin{aligned} |\mathcal{S}_1(-\zeta; T_1, T_2) - \mathcal{S}_n(-\zeta)| &\le \frac{1}{2\pi}\frac{\Gamma(\beta_0\varepsilon)\Gamma(1 + r_+)}{\Gamma(1 + \rho(x_+))}\{J_1 + |\zeta|\,J_2\}\,|\kappa_0 - \beta_0\zeta| \\ \text{(8.2.167)} \qquad &\times \left[\int_{-\infty}^{-n} + \int_n^\infty\right]\frac{dv}{\sqrt{r_+^2 + (\iota(\zeta) - v)^2}\sqrt{r_-^2 + (\iota(\zeta - \chi_0) - v)^2}}. \end{aligned}$$

It is easy to see that the latter bound still holds, of course, with different constants J_1, J_2, if rather than a closed vertical substrip Z denotes an arbitrary compact subset of $\mathbb{Z}_\varepsilon$. Since the functions $-\iota(\zeta)$ and $-\iota(\zeta - \chi_0)$ are decreasing as $\Im\zeta \to -\infty$, the integral along the ray $v \ge n$ is then bounded by

$$\int_n^\infty \frac{dv}{\sqrt{r_+^2 + (\iota(y_-) - v)^2}\sqrt{r_-^2 + (\iota(y_- - \chi_0) - v)^2}}.$$

As $n \to \infty$ the right hand side vanishes uniformly with respect to $\zeta \in Z$. Analogous inequalities apply for the integral along the segment $v \leq -n$. Since the additional prefactors in (8.2.167) constitute polynomials of ζ in absolute value, they are continuous and thus also uniformly bounded with respect to $\zeta \in Z$. Hence, the sequence of integrals (8.2.164) as $n \to \infty$ converges uniformly with respect to $\zeta \in Z$ to the integral (8.2.158) with $u_1 \equiv \beta_0 \varepsilon$. By arbitrariness of the compact set Z, in accordance with Theorem 5.1.3 in [Wegert, 2012], we deduce analyticity of the limit function in $\mathbb{Z}_\varepsilon$. But for each $\varepsilon > -\Re\varsigma_0$ the region $\mathbb{Z}_\varepsilon$ is a substrip of the half plane (8.2.161), whence the MB-integral (8.2.158) is especially analytic there. To conclude these findings, the expansion (8.2.160) represents the sum of a meromorphic and a holomorphic function, respectively in the half plane (8.2.159) and (8.2.161). Each of them contains the original region (8.2.148), from which we infer validity of the indicated expansion by analytic continuation in the greatest common half plane

$$\Re\zeta < \frac{\Re\kappa_0}{\beta_0} + \Re\varsigma_0 + \min\left\{0, \Re\chi_0 + \eta_{\beta_0}(\beta_1, \delta_1, \varepsilon_1, \kappa_1)\right\}. \tag{8.2.168}$$

The only singularities that can be found therein are some poles generated by the first summand of the expansion. At the moment we are not interested in the associated residues. Instead we close our investigations with the derivation of a simple estimate for the asymptotic behaviour towards the imaginary direction. For this purpose we again describe by Z an arbitrary vertical substrip contained in $\mathbb{Z}_\varepsilon$ with left and right boundaries x_- and x_+. On the one hand, regarding the ratio of gamma functions in the first summand of (8.2.160) we obtain by virtue of the functional equation for a constant $L \in \mathbb{N}_0$ such that $\Re\zeta < \Re\chi_0 + \frac{\Re\kappa_0 + L}{\beta_0}$ for all $\zeta \in \mathbb{Z}_\varepsilon$:

$$\begin{aligned}
\left|\frac{\Gamma(\kappa_0 + \beta_0(\chi_0 - \zeta))}{\Gamma(\kappa_0 - \beta_0\zeta)}\right| &= \left|\frac{\Gamma(L + \kappa_0 + \beta_0(\chi_0 - \zeta))}{\Gamma(L + \kappa_0 - \beta_0\zeta)}\right| \left|\prod_{l=0}^{L-1} \frac{l + \kappa_0 - \beta_0\zeta}{l + \kappa_0 + \beta_0(\chi_0 - \zeta)}\right| \\
&\leq \frac{1}{|\Gamma(-\beta_0\chi_0)|} \int_0^\infty e^{\beta_0 \Re\chi_0 t}(1 - e^{-t})^{L + \rho(\zeta - \chi_0) - 1} dt \prod_{l=0}^{L-1} \frac{|l + \kappa_0 - \beta_0\zeta|}{|l + \kappa_0 + \beta_0(\chi_0 - \zeta)|} \\
&= \mathcal{O}(1)
\end{aligned}$$

For the inequality we introduced a well-known representation for the beta function, which is especially applicable because $\Re\chi_0 < 0$. The estimate in the big-$\mathcal{O}$ then holds as $\Im\zeta \to \pm\infty$ uniformly with respect to $\Re\zeta$ in Z. On the other hand, with the aid of the estimates (8.2.165) and (8.2.166) we deduce the following bound for (8.2.158) with $u_1 \equiv \beta_0\varepsilon$:

$$\begin{aligned}
|\mathcal{S}_1(-\zeta; T_1, T_2)| &\leq \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{|\Gamma(\beta_0\varepsilon + iv)|\,|\Gamma(\kappa_0 - \beta_0\zeta - \beta_0\varepsilon - iv)|}{|\Gamma(\kappa_0 - \beta_0\zeta)|} \,|\mathcal{J}(\beta_0\varepsilon + iv, \zeta)|\,dv \\
&\leq \frac{1}{2\pi}\frac{\Gamma(\beta_0\varepsilon)\Gamma(1+r_+)}{\Gamma(1+\rho(x_+))} \int\limits_{-\infty}^{\infty} \frac{\{J_1 + |\zeta|\,J_2\}\,|\kappa_0 - \beta_0\zeta|}{\sqrt{r_+^2 + (\iota(\zeta) - v)^2}\sqrt{r_-^2 + (\iota(\zeta - \chi_0) - v)^2}} dv \\
&= \frac{1}{2\pi}\frac{\Gamma(\beta_0\varepsilon)\Gamma(1+r_+)}{\Gamma(1+\rho(x_+))} \int\limits_{-\infty}^{\infty} \frac{\{J_1 + |\zeta|\,J_2\}\,|\kappa_0 - \beta_0\zeta|}{\sqrt{r_+^2 + y^2}\sqrt{r_-^2 + (y + \beta_0\chi_0)^2}} dv
\end{aligned}$$

For the last equality we performed the change of variables $y = \iota(\zeta) - v$. We eventually conclude that, as $\Im\zeta \to \pm\infty$, uniformly with respect to $\Re\zeta$ in each closed vertical substrip of $\mathbb{Z}_\varepsilon$ and thus of the half plane (8.2.168), the function satisfies

$$\mathcal{S}\left[-\zeta; \begin{matrix} T_1, \infty \\ T_2, \infty \end{matrix}\right] = \mathcal{O}\left\{|\zeta|^2\right\}. \tag{8.2.169}$$

8.2.6.4. Two Finite Paths and a Kernel of the First Kind

The technique which we presented in §8.2.6.2 can be applied analogously if the first kind iterated generating function (8.2.1) features two finite paths. Formally, for each $j \in \{1, 2\}$ we assume $\mathcal{P}_j = (\tau_j, T_j]$ with $0 \leq \tau_j < T_j < \infty$ such that $\varphi(s)$ is continuous and positive for $s \in \mathcal{P}_1$ with algebraic behaviour as $s \downarrow \tau_1$ for a coefficient $b_0 > 0$ and a parameter $\beta_0 > 0$. In addition, also $d(u), e(u), k(u)$ are supposed to be continuous along the range of their arguments with algebraic behaviour when approaching the respective lower endpoint. The associated parameters are denoted by $\delta_0, \varepsilon_0, \kappa_0 \in \mathbb{C}$, and the coefficients are referred to as $d_0, e_0, k_0 \in \mathbb{C} \setminus \{0\}$, where

$$-1 < \Re\varepsilon_0 \leq -1 - \Re\kappa_0, \tag{8.2.170}$$

which especially implies $\Re\kappa_0 \leq -1 - \Re\varepsilon_0 < 0$. Furthermore, each ingredient function possesses a normalized counterpart according to the definitions (8.2.5), (8.2.6) and (8.2.19). For brevity we refer to these as $B(u)$, $D(u)$, $E(u)$ and $K(u)$, omitting the indices for the endpoints. Their first derivative is assumed to be of respective order $\beta_1, \Re\delta_1, \Re\varepsilon_1, \Re\kappa_1 > -1$ when approaching the lower endpoint of the range of their arguments. If we denote

$$\begin{cases} \chi_1 := \frac{\delta_0 + 1}{\beta_0}, \\ \chi_2 := \frac{\varepsilon_0 + 1}{\beta_0}, \end{cases} \tag{8.2.171}$$

from (8.2.170) we conclude

$$0 < \Re\chi_2 \leq -\frac{\Re\kappa_0}{\beta_0}. \tag{8.2.172}$$

In the above setup the iterated integral (8.2.1) is readily rearranged to take on the form

$$\mathcal{M}\begin{bmatrix} -\zeta, \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} = \int_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0\chi_2 - 1} E(t) \times \int_{\tau_1}^{T_1} \frac{(s-\tau_1)^{\beta_0(\chi_1-\zeta)-1}}{(s+t-\tau_1-\tau_2)^{-\kappa_0}} D(s) e^{\zeta B(s)} K(s+t) ds dt. \tag{8.2.173}$$

By Lemma 8.2.5, due to (8.2.170), absolute convergence and analyticity holds for all $\zeta \in \mathbb{C}$ with

$$\Re\zeta < \Re(\chi_1 + \chi_2) + \frac{\Re\kappa_0}{\beta_0}, \tag{8.2.174}$$

and this abscissa of convergence results from the supplementary criterion for the convergence of the iterated integral. From $\Re\kappa_0 < 0$ we conclude the applicability of the formula (8.2.76) with $a \equiv -\kappa_0$ and $s \mapsto s - \tau_1$ and $t \mapsto t - \tau_2$. For an appropriately specified vertical line $\Re w = u_0$ this leads to

$$\mathcal{M}\begin{bmatrix} -\zeta, \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} = \frac{1}{2\pi i} \int_{u_0 - i\infty}^{u_0 + i\infty} \frac{\Gamma(w)\Gamma(-\kappa_0 - w)}{\Gamma(-\kappa_0)} \mathcal{K}(w, \zeta) dw, \tag{8.2.175}$$

involving the function

$$\mathcal{K}(w,\zeta) := \int_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0\chi_2 + w + \kappa_0 - 1} E(t) \times \int_{\tau_1}^{T_1} (s-\tau_1)^{\beta_0(\chi_1-\zeta)-w-1} D(s) e^{\zeta B(s)} K(s+t) ds dt. \tag{8.2.176}$$

The latter integral is, by Lemma 8.2.1, absolutely convergent for $\zeta, w \in \mathbb{C}$ subject to (8.2.174) and with

$$-\beta_0\Re\chi_2 - \Re\kappa_0 < \Re w < \beta_0\Re(\chi_1 - \zeta). \tag{8.2.177}$$

Taking into account the requirement for formula (8.2.76), for general $\zeta \in \mathbb{C}$ subject to (8.2.174) the integration path in (8.2.175) must thus satisfy

$$-\beta_0\Re\chi_2 - \Re\kappa_0 < u_0 < \min\left\{-\Re\kappa_0, \beta_0\Re(\chi_1 - \zeta)\right\}. \tag{8.2.178}$$

A comparison with (8.2.172) and (8.2.174) shows that this strip is non-empty. Moreover, by inspection of (8.2.176) we note that the variables w and ζ in the interior integral show matching signs, as in the preceding paragraphs, which is the reason why we easily repeat the steps that were described there. For fixed ζ the integral (8.2.176) represents a holomorphic function of w

in the strip (8.2.177), and for those values of the argument we obtain from partial integration:

$$\mathcal{K}(w,\zeta) = -\frac{(T_1-\tau_1)^{\beta_0(\chi_1-\zeta)-w}}{w-\beta_0(\chi_1-\zeta)} D(T_1)e^{\zeta B(T_1)} \int_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0\chi_2+w+\kappa_0-1} E(t)K(T_1+t)dt$$

$$+\frac{1}{w-\beta_0(\chi_1-\zeta)} \sum_{\substack{n_1,n_2,n_3\in\{0,1\}\\ n_1+n_2+n_3=1}} \int_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0\chi_2+w+\kappa_0-1} E(t) \tag{8.2.179}$$

$$\times \int_{\tau_1}^{T_1} (s-\tau_1)^{\beta_0(\chi_1-\zeta)-w} e^{\zeta B(s)} \left\{\zeta B'(s)\right\}^{n_1} D^{(n_2)}(s) K^{(n_3)}(s+t) ds dt$$

By Lemma 8.2.1 we conclude absolute convergence of each integral in the expansion for $\zeta, w \in \mathbb{C}$ with

$$\begin{cases} \Re\zeta < \Re(\chi_1+\chi_2) + \frac{\Re\kappa_0}{\beta_0} + \chi_{\beta_0}(\beta_1,\delta_1,0), \\ -\beta_0\Re\chi_2 - \Re\kappa_0 < \Re w < \beta_0 \Re(\chi_1-\zeta+\chi_{\beta_0}(\beta_1,\delta_1,0)). \end{cases} \tag{8.2.180}$$

Appealing to the uniform convergence of each integral in this w-strip, the expansion (8.2.179) analytically continues the original integral (8.2.176) therein, for fixed $\zeta \in \mathbb{C}$ subject to (8.2.174). In this wider region we notice the presence of a simple pole at $w = \beta_0(\chi_1-\zeta)$. If

$$\Phi(\zeta) := \int_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0(\chi_1+\chi_2-\zeta)+\kappa_0-1} E(t)K(\tau_1+t)dt, \tag{8.2.181}$$

which converges absolutely for $\Re\zeta < \Re(\chi_1+\chi_2) + \frac{\Re\kappa_0}{\beta_0}$, we compute:

$$\operatorname*{Res}_{w=\beta_0(\chi_1-\zeta)} \mathcal{K}(w,\zeta) = -D(T_1)e^{\zeta B(T_1)} \int_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0(\chi_1+\chi_2-\zeta)+\kappa_0-1} E(t)K(T_1+t)dt$$

$$+\int_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0(\chi_1+\chi_2-\zeta)+\kappa_0-1} E(t) \int_{\tau_1}^{T_1} \frac{d}{ds}\left\{e^{\zeta B(s)} D(s)K(s+t)\right\} ds dt$$

$$= -d_0 \left\{b_0\right\}^{-\zeta} \Phi(\zeta) \tag{8.2.182}$$

The integral (8.2.181) does not only converge absolutely but is even holomorphic in the half plane $\Re\zeta < \Re(\chi_1+\chi_2) + \frac{\Re\kappa_0}{\beta_0}$. Furthermore, one step of integration by parts suffices to verify for fixed ζ validity of

$$\Phi(\zeta) = -\frac{\phi(\zeta)}{\beta_0(\zeta-\chi_1-\chi_2-\frac{\kappa_0}{\beta_0})}, \tag{8.2.183}$$

where the function on the right hand side is equal to

$$\begin{aligned}\phi(\zeta) :=& (T_2-\tau_2)^{\beta_0(\chi_1+\chi_2-\zeta)+\kappa_0} E(T_2)K(\tau_1+T_2) \\ &- \int_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0(\chi_1+\chi_2-\zeta)+\kappa_0} \left\{E'(t)K(\tau_1+t)+E(t)K'(\tau_1+t)\right\} dt.\end{aligned} \tag{8.2.184}$$

By analytic continuation the representation (8.2.183) remains true in the extended region

$$\Re\zeta < \Re(\chi_1+\chi_2) + \frac{\Re\kappa_0}{\beta_0} + \chi_{\beta_0}(\varepsilon_1,\kappa_1). \tag{8.2.185}$$

For fixed ζ subject to (8.2.174), instead of (8.2.182) we may thus equivalently write

$$\operatorname*{Res}_{w=\beta_0(\chi_1-\zeta)} \mathcal{K}(w,\zeta) = \frac{d_0}{\beta_0}\{b_0\}^{-\zeta} \frac{\phi(\zeta)}{\zeta-\chi_1-\chi_2-\frac{\kappa_0}{\beta_0}}. \tag{8.2.186}$$

We shall now revisit the MB-integral (8.2.175) in which we recall that the integration path satisfies the conditions (8.2.178). If we impose the additional restriction

$$\frac{\Re\kappa_0}{\beta_0} + \Re\chi_1 < \Re\zeta < \Re(\chi_1+\chi_2) + \frac{\Re\kappa_0}{\beta_0}, \tag{8.2.187}$$

the conditions on u_0 become

$$-\beta_0\Re\chi_2 - \Re\kappa_0 < u_0 < \beta_0\Re(\chi_1-\zeta), \tag{8.2.188}$$

and the singularity that lies closest to the right of the line $\Re w = u_0$ is the simple pole generated by the expansion (8.2.179). Conversely, this pole lies to the left of the pole of the gamma function $\Gamma(-\kappa_0-w)$ at $w=-\kappa_0$. Appealing to the exponential decay of the integrand, in (8.2.175) we may thus perform a displacement of the integration path to the right over the simple pole at $w=\beta_0(\chi_1-\zeta)$ only, to match some vertical line $\Re w = u_1$ with

$$\begin{cases} u_1 > \max\left\{-\Re(\beta_0\chi_2+\kappa_0), \beta_0\Re(\chi_1-\zeta)\right\}, \\ u_1 < \min\left\{-\Re\kappa_0, \beta_0\Re(\chi_1-\zeta+\chi_{\beta_0}(\beta_1,\delta_1,0))\right\}. \end{cases} \tag{8.2.189}$$

In this process we collect the residue (8.2.186) with a negative sign, since the pole is encircled clockwisely, to eventually arrive at the expansion

$$\begin{aligned}\mathcal{M}\begin{bmatrix}-\zeta,\tau_1,T_1\\ \tau_2,T_2\end{bmatrix} =& -\frac{d_0}{\beta_0}\{b_0\}^{-\zeta} \frac{\Gamma(\beta_0(\chi_1-\zeta))\Gamma(\beta_0(\zeta-\chi_1)-\kappa_0)}{\Gamma(-\kappa_0)(\zeta-\chi_1-\chi_2-\frac{\kappa_0}{\beta_0})}\phi(\zeta) \\ &+ \mathcal{M}_1\begin{bmatrix}-\zeta,\tau_1,T_1\\ \tau_2,T_2\end{bmatrix},\end{aligned} \tag{8.2.190}$$

in which the function in the second summand is given by the MB-integral

$$\mathcal{M}_1\begin{bmatrix} -\zeta, \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} := \frac{1}{2\pi i} \int_{u_1-i\infty}^{u_1+i\infty} \frac{\Gamma(w)\Gamma(-\kappa_0 - w)}{\Gamma(-\kappa_0)} \mathcal{K}(w,\zeta) dw. \tag{8.2.191}$$

Conceived as a function of ζ, the first summand of this expansion is obviously meromorphic in the half plane (8.2.185). Moreover, an integration path $\Re w = u_1$ can be found for each $\zeta \in \mathbb{C}$ that satisfies

$$\Re\chi_1 + \frac{\Re\kappa_0}{\beta_0} < \Re\zeta < \Re(\chi_1 + \chi_2) + \frac{\Re\kappa_0}{\beta_0} + \chi_{\beta_0}(\beta_1, \delta_1, 0). \tag{8.2.192}$$

To verify analyticity of the MB-integral (8.2.191) in this strip, similar to §8.2.6.2 we note, by assumption we can always specify a constant ε according to

$$0 \le -\frac{\Re\kappa_0}{\beta_0} - \Re\chi_2 < \varepsilon < -\frac{\Re\kappa_0}{\beta_0}, \tag{8.2.193}$$

for which we then have non-voidness of the strip

$$\mathbb{Z}_\varepsilon := \{\zeta \in \mathbb{C} : \Re\chi_1 - \varepsilon < \Re\zeta < \Re\chi_1 + \chi_{\beta_0}(\beta_1, \delta_1, 0) - \varepsilon\}. \tag{8.2.194}$$

In this event, for all $\zeta \in \mathbb{Z}_\varepsilon$ it is permitted in (8.2.191) to pick $u_1 \equiv \beta_0\varepsilon$, and by the same arguments as in §8.2.6.2 it is straightforward to show analyticity of the indicated MB-integral with respect to $\zeta \in \mathbb{Z}_\varepsilon$. Since $\mathbb{Z}_\varepsilon$ lies for any ε in the strip (8.2.192), by arbitrariness of ε in the range (8.2.193) we conclude analyticity of the MB-integral in this strip. If we combine all these findings, we can establish the expansion (8.2.190) as a meromorphic function in the common regions (8.2.185) and (8.2.192), which exactly coincides with the half plane

$$\Re\chi_1 + \frac{\Re\kappa_0}{\beta_0} < \Re\zeta < \Re(\chi_1 + \chi_2) + \frac{\Re\kappa_0}{\beta_0} + \chi_{\beta_0}(\beta_1, \delta_1, \varepsilon_1, \kappa_1, 0). \tag{8.2.195}$$

Since this region in turn overlaps with (8.2.174), we eventually identify the expansion (8.2.190) as the analytic continuation of the original integral (8.2.173). Finally, again by arguments similar to §8.2.6.2 one readily verifies that the expansion as $\Im\zeta \to \pm\infty$, uniformly with respect to $\Re\zeta$ in any closed vertical substrip of (8.2.195), satisfies

$$\mathcal{M}\begin{bmatrix} -\zeta, \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} = \mathcal{O}(\zeta). \tag{8.2.196}$$

8.2.6.5. Two Finite Paths and a Kernel of the Second Kind

To complete this subsection, it remains to discuss the case of a second kind iterated generating function with two finite integration paths. Therefore we consider (8.2.2) with $\mathcal{P}_j = (\tau_j, T_j]$ and $0 \le \tau_j < T_j < \infty$ for each $j \in \{1, 2\}$. Suppose the function φ is positive and continuous along the interval $(\tau_1+\tau_2, T_1+T_2]$ with algebraic behaviour at the lower endpoint for a parameter $\beta_0>0$ and a coefficient $b_0>0$, and the first derivative of the normalized phase is of order $\beta_1>-1$ there. Furthermore, concerning the amplitude functions $d(u), e(u), k(u)$ and the first derivatives of their normalized analogues, we adopt the conditions from the preceding paragraph, however, with the restriction (8.2.170) replaced by

$$\Re\delta_0, \Re\varepsilon_0 > -1. \tag{8.2.197}$$

As a consequence, in terms of (8.2.171), we have $\Re\chi_1, \Re\chi_2 > 0$. Under the present assumptions the iterated generating function (8.2.1) can be recast to become

$$\begin{aligned} \mathcal{S}\left[-\zeta; \begin{matrix}\tau_1, T_1\\ \tau_2, T_2\end{matrix}\right] &= \int\limits_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0\chi_2-1} E(t) \\ &\times \int\limits_{\tau_1}^{T_1} \frac{(s-\tau_1)^{\beta_0\chi_1-1}}{(s+t-\tau_1-\tau_2)^{\beta_0\zeta-\kappa_0}} D(s) e^{\zeta B(s+t)} K(s+t) ds dt. \end{aligned} \tag{8.2.198}$$

Appealing to Lemma 8.2.5 we readily verify its absolute convergence and analyticity for any $\zeta\in\mathbb{C}$ with

$$\Re\zeta < \Re(\chi_1+\chi_2) + \frac{\Re\kappa_0}{\beta_0}. \tag{8.2.199}$$

The right boundary of that region is due to the supplementary condition for the convergence of the iterated integral. To determine the analytic continuation of (8.2.198) we first confine the argument ζ to the strip

$$\frac{\Re\kappa_0}{\beta_0} < \Re\zeta < \Re(\chi_1+\chi_2) + \frac{\Re\kappa_0}{\beta_0}. \tag{8.2.200}$$

This restriction immediately implies $\Re(\beta_0\zeta-\kappa_0)>0$ and thereby enables us to employ formula (8.2.76) with $a\equiv\beta_0\zeta-\kappa_0$, $s\mapsto s-\tau_1$ and $t\mapsto t-\tau_2$. For a suitable parameter u_0 this yields

$$\mathcal{S}\left[-\zeta; \begin{matrix}\tau_1, T_1\\ \tau_2, T_2\end{matrix}\right] = \frac{1}{2\pi i} \int\limits_{u_0-i\infty}^{u_0+i\infty} \frac{\Gamma(w)\Gamma(\beta_0\zeta-\kappa_0-w)}{\Gamma(\beta_0\zeta-\kappa_0)} \mathcal{H}(w,\zeta) dw, \tag{8.2.201}$$

with the integral function appearing therein represented by

$$\begin{aligned} \mathcal{H}(w,\zeta) &:= \int\limits_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0(\chi_2-\zeta)+w+\kappa_0-1} E(t) \\ &\times \int\limits_{\tau_1}^{T_1} (s-\tau_1)^{\beta_0\chi_1-w-1} D(s) e^{\zeta B(s+t)} K(s+t) ds dt. \end{aligned} \tag{8.2.202}$$

By Lemma 8.2.1, the latter converges absolutely for $\zeta, w \in \mathbb{C}$ subject to (8.2.199) and

$$\beta_0 \Re(\zeta - \chi_2) - \Re\kappa_0 < \Re w < \beta_0 \Re\chi_1, \tag{8.2.203}$$

establishing a holomorphic function of w in this region. We therefore conclude the validity of (8.2.201) for a path $\Re w = u_0$ that satisfies

$$\max\{0, \beta_0\Re(\zeta - \chi_2) - \Re\kappa_0\} < u_0 < \min\{\beta_0\Re\chi_1, \Re(\beta_0\zeta - \kappa_0)\}. \tag{8.2.204}$$

By comparison with the analogue functions (8.2.102), (8.2.149) and (8.2.176) from the preceding paragraphs, the converse sign of the variables w and ζ in the integral (8.2.202) becomes evident. This can not be circumvented due to the choice of the parameter a, for which formula (8.2.76) was applied. Yet, it does not make a difference for the further procedure. Indeed, ignoring this observation and integrating by parts for fixed $\zeta, w \in \mathbb{C}$ subject to (8.2.199) and (8.2.203) leads to:

$$\begin{aligned}\mathcal{H}(w,\zeta) = &-\frac{(T_1-\tau_1)^{\beta_0\chi_1 - w}}{w - \beta_0\chi_1} D(T_1) \int\limits_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0(\chi_2-\zeta)+w+\kappa_0-1} E(t) e^{\zeta B(T_1+t)} K(T_1+t)\,dt \\ &+ \frac{1}{w-\beta_0\chi_1} \sum_{\substack{n_1,n_2,n_3\in\{0,1\}\\ n_1+n_2+n_3=1}} \int\limits_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0(\chi_2-\zeta)+w+\kappa_0-1} E(t) \\ &\times \int\limits_{\tau_1}^{T_1} (s-\tau_1)^{\beta_0\chi_1 - w} e^{\zeta B(s+t)} D^{(n_1)}(s) \left\{\zeta B'(s+t)\right\}^{n_2} K^{(n_3)}(s+t)\,ds\,dt\end{aligned} \tag{8.2.205}$$

With the aid of Lemma 8.2.5 it is easy to confirm absolute convergence of all integrals in the above expansion for $\zeta, w \in \mathbb{C}$ with

$$\begin{cases}\Re\zeta < \Re(\chi_1+\chi_2) + \frac{\Re\kappa_0}{\beta_0} + \chi_{\beta_0}(\beta_1,\delta_1,\kappa_1), \\ \beta_0\Re(\zeta-\chi_2) - \Re\kappa_0 < \Re w < \beta_0(\Re\chi_1 + \chi_{\beta_0}(\delta_1, 0)).\end{cases} \tag{8.2.206}$$

Moreover, for fixed ζ we have analyticity in the indicated w-region. As a consequence the whole expansion (8.2.205) is meromorphic there, with a pole of simple order at $w = \beta_0\chi_1$. Particularly for fixed ζ subject to (8.2.200) it establishes the analytic continuation with respect to w of (8.2.202). To specify the residue corresponding to the pole at $w = \beta_0\chi_1$, we define

$$\Theta(\zeta) := \int\limits_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0(\chi_1+\chi_2-\zeta)+\kappa_0-1} e^{\zeta B(\tau_1+t)} E(t) K(\tau_1+t)\,dt, \tag{8.2.207}$$

the latter integral being absolutely convergent and holomorphic in $\Re\zeta < \Re(\chi_1+\chi_2)+\frac{\Re\kappa_0}{\beta_0}$. From (8.2.205) the fundamental theorem of calculus then yields

$$\operatorname*{Res}_{w=\beta_0\chi_1} \mathcal{H}(w,\zeta) = -d_0\Theta(\zeta). \tag{8.2.208}$$

The integral (8.2.207) can be continued to a meromorphic function in the half plane

$$\Re\zeta < \Re(\chi_1+\chi_2) + \frac{\Re\kappa_0}{\beta_0} + \chi_{\beta_0}(\beta_1,\varepsilon_1,\kappa_1) \tag{8.2.209}$$

via partial integration, leading to the identity

$$\Theta(\zeta) = -\frac{\theta(\zeta)}{\beta_0(\zeta-\chi_1-\chi_2-\frac{\kappa_0}{\beta_0})}, \tag{8.2.210}$$

where the expansion on the right hand side is for brevity denoted by

$$\begin{aligned}\theta(\zeta) := {} & (T_2-\tau_2)^{\beta_0(\chi_1+\chi_2-\zeta)+\kappa_0} e^{\zeta B(\tau_1+T_2)} E(T_2)K(\tau_1+T_2) \\ & - \zeta \int_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0(\chi_1+\chi_2-\zeta)+\kappa_0} e^{\zeta B(\tau_1+t)} B'(\tau_1+t)E(t)K(\tau_1+t)dt \\ & - \int_{\tau_2}^{T_2} (t-\tau_2)^{\beta_0(\chi_1+\chi_2-\zeta)+\kappa_0} e^{\zeta B(\tau_1+t)} \left\{E'(t)K(\tau_1+t)+E(t)K'(\tau_1+t)\right\} dt.\end{aligned} \tag{8.2.211}$$

Accordingly, for fixed ζ, we can write

$$\operatorname*{Res}_{w=\beta_0\chi_1} \mathcal{H}(w,\zeta) = \frac{d_0}{\beta_0}\frac{\theta(\zeta)}{\zeta-\chi_1-\chi_2-\frac{\kappa_0}{\beta_0}}. \tag{8.2.212}$$

If we confine the set of admissible arguments ζ from (8.2.200) to the smaller strip

$$\Re\chi_1 + \frac{\Re\kappa_0}{\beta_0} < \Re\zeta < \Re(\chi_1+\chi_2) + \frac{\Re\kappa_0}{\beta_0}, \tag{8.2.213}$$

the right boundary of admissible values for u_0 is due to the simple pole of the analytic continuation of $\mathcal{H}(w,\zeta)$ at $w=\beta_0\chi_1$. Hence, the minimum in (8.2.204) is unique, and we obtain

$$\max\{0, \beta_0\Re(\zeta-\chi_2)-\Re\kappa_0\} < u_0 < \beta_0\Re\chi_1. \tag{8.2.214}$$

In (8.2.201) we may then move the integration path to the right over the indicated pole, to match a line $\Re w = u_1$ that runs in the strip

$$\begin{cases} u_1 > \max\{\beta_0\Re(\zeta-\chi_2)-\Re\kappa_0, \beta_0\Re\chi_1\}, \\ u_1 < \min\{\beta_0(\Re\chi_1+\chi_{\beta_0}(\delta_1,0)), \Re(\beta_0\zeta-\kappa_0)\}. \end{cases} \tag{8.2.215}$$

Since the pole is encircled in the clockwise direction, bearing in mind (8.2.212) the residue theorem brings us the expansion

$$\mathcal{S}\left[-\zeta;\begin{matrix}\tau_1,T_1\\ \tau_2,T_2\end{matrix}\right]=-\frac{d_0}{\beta_0}\frac{\Gamma(\beta_0\chi_1)\Gamma(\beta_0(\zeta-\chi_1)-\kappa_0)}{(\zeta-\chi_1-\chi_2-\frac{\kappa_0}{\beta_0})\Gamma(\beta_0\zeta-\kappa_0)}\theta(\zeta)+\mathcal{S}_1\left[-\zeta;\begin{matrix}\tau_1,T_1\\ \tau_2,T_2\end{matrix}\right], \tag{8.2.216}$$

whose second term features the initial MB-integral (8.2.201) but with a shifted integration path, i.e., we have

$$\mathcal{S}_1\left[-\zeta;\begin{matrix}\tau_1,T_1\\ \tau_2,T_2\end{matrix}\right]:=\frac{1}{2\pi i}\int\limits_{u_1-i\infty}^{u_1+i\infty}\frac{\Gamma(w)\Gamma(\beta_0\zeta-\kappa_0-w)}{\Gamma(\beta_0\zeta-\kappa_0)}\mathcal{H}(w,\zeta)dw. \tag{8.2.217}$$

If we conceive the result (8.2.216) as a function of ζ, the first term is obviously meromorphic in (8.2.209). Concerning the second summand we note, an integration path that satisfies (8.2.215) can be found for any $\zeta\in\mathbb{C}$ with

$$\Re\chi_1+\frac{\Re\kappa_0}{\beta_0}<\Re\zeta<\Re(\chi_1+\chi_2)+\frac{\Re\kappa_0}{\beta_0}+\chi_{\beta_0}(\delta_1,0). \tag{8.2.218}$$

To verify analyticity of the MB-integral (8.2.217) in this strip, we first choose a fixed but arbitrary

$$0<\Re\chi_1<\varepsilon<\Re\chi_1+\chi_{\beta_0}(\delta_1,0). \tag{8.2.219}$$

It is easy to see that this is always possible. With the aid of ε we then construct a strip that lies in the interior of (8.2.218) by

$$\mathbb{Z}_\varepsilon:=\left\{\zeta\in\mathbb{C}:\frac{\Re\kappa_0}{\beta_0}+\varepsilon<\Re\zeta<\Re\chi_2+\frac{\Re\kappa_0}{\beta_0}+\varepsilon\right\}. \tag{8.2.220}$$

Notice the slightly different structure of this set, compared with the sets which we defined in the preceding paragraphs. Yet, according to (8.2.215), the line $u_1\equiv\beta_0\varepsilon$ is a permissible choice for the path of the MB-integral (8.2.217) for all $\zeta\in\mathbb{Z}_\varepsilon$. Similar to §8.2.6.3 we can then verify analyticity with respect to $\zeta\in\mathbb{Z}_\varepsilon$ of the latter MB-integral by elaborate estimates, to finally infer validity of this statement in the strip (8.2.218). Since the first summand in the expansion (8.2.216) is meromorphic in (8.2.209), if we take the greatest region in common with (8.2.218), it follows that the whole expansion is meromorphic in the strip

$$\Re\chi_1+\frac{\Re\kappa_0}{\beta_0}<\Re\zeta<\Re(\chi_1+\chi_2)+\frac{\Re\kappa_0}{\beta_0}+\chi_{\beta_0}(\beta_1,\delta_1,\varepsilon_1,\kappa_1,0). \tag{8.2.221}$$

Therein it constitutes the analytic continuation of the integral (8.2.198) with exactly one pole, that one being of simple order. It is generated by the rational function in the first summand. Finally, also analogous to (8.2.169), one can show that, as $\Im\zeta\to\pm\infty$, uniformly with respect to $\Re\zeta$ in any closed vertical substrip of (8.2.221), the continuation satisfies

$$\mathcal{S}\left[-\zeta; \begin{matrix}\tau_1, T_1\\ \tau_2, T_2\end{matrix}\right] = \mathcal{O}\left\{|\zeta|^2\right\}. \tag{8.2.222}$$

8.3. Two Finite Paths in a Symmetric-Type Iterated Integral

Our first case treats the integral (8.0.1) for $\mathcal{P}_j = (\tau_j, T_j]$ with $0 \leq \tau_j \leq T_j < \infty$ for each $j \in \{1,2\}$, which we assume to be of the particular form

$$\mathrm{Si}\left[m; \begin{matrix}\tau_1, T_1\\ \tau_2, T_2\end{matrix}\right] = \int_{\tau_2}^{T_2} \left\{1 - (1-\Psi(t))^{m+1}\right\} c(t) \int_{\tau_1}^{T_1} \left\{1 - (1-\Psi(s))^{m+1}\right\} c(s) a(s+t) ds dt. \tag{8.3.1}$$

If we denote by φ the function (8.1.5), we require the ingredients to satisfy:

(S1) φ and c are continuous on $(\tau_1, T_1] \cup (\tau_2, T_2]$ with $\varphi > 0$, and a is continuous on $[\tau_1 + \tau_2, T_1 + T_2]$.

(S2) The functions $\varphi(u)$, $c(u)$ and $a(v)$ are algebraic as $u \downarrow \tau_1$, as $u \downarrow \tau_2$ and as $v \downarrow \tau_1 + \tau_2$ for some parameters

$$\beta_{10}, \beta_{20} \geq 0,$$
$$\gamma_{10}, \gamma_{20} \in \mathbb{R},$$
$$\alpha_0(1,2) \geq 0$$

and with coefficients $b_{j0} > 0$, $c_{j0}, a_0(1,2) \in \mathbb{C} \setminus \{0\}$ for $j \in \{1,2\}$, where the index refers to the endpoint. Especially

$$\gamma_{j0} + \beta_{j0} > -1, \qquad \text{for } j \in \{1,2\}. \tag{8.3.2}$$

(S3) For each $j \in \{1,2\}$ with $\beta_{j0} > 0$, the functions φ and c are once continuously differentiable on $(\tau_j, T_j]$. Furthermore, the normalized phase $B_j(u)$ has a first derivative of order $\beta_{j1} > -1$ as $u \downarrow \tau_j$, and that of $C_j(u)$ is of order $\gamma_{j1} > -1$ there. The latter two functions were introduced in (8.2.5) and (8.2.6), respectively.

(S4) If there exists $j \in \{1,2\}$ with $\beta_{j0} > 0$, the function $a(u)$ is once continuously differentiable on $(\tau_1 + \tau_2, T_1 + T_2]$, and the normalized amplitude $A_{1,2}(u)$ as in (8.2.19) has a first derivative of order $\alpha_1(1,2) > -1$ as $u \downarrow \tau_1 + \tau_2$.

(S5) If $\beta_{10}, \beta_{20} > 0$ and $\gamma_{10} = -1$, the function $a(u)$ is even twice continuously differentiable on $(\tau_1 + \tau_2, T_1 + T_2]$. If $\alpha_0(1,2) > 0$, as $u \downarrow \tau_1 + \tau_2$ its second derivative satisfies

$$a''(u) \sim a_0(1,2)\alpha_0(1,2)(\alpha_0(1,2) - 1)(u - \tau_1 - \tau_2)^{\alpha_0(1,2)-2}. \tag{8.3.3}$$

Conversely, if $\alpha_0(1,2) = 0$, its k-th derivative for $k \in \{1,2\}$, a constant $a_1(1,2) \in \mathbb{C} \setminus \{0\}$ and $\alpha_1(1,2) > -1$ as in (S4), as $u \downarrow \tau_1 + \tau_2$ shows the behaviour

$$a^{(k)}(u) \sim a_1(1,2)\frac{\Gamma(\alpha_1(1,2)+1)}{\Gamma(\alpha_1(1,2)+2-k)}(u-\tau_1-\tau_2)^{\alpha_1(1,2)+1-k}. \tag{8.3.4}$$

Throughout our investigations, in some Laurent expansions, quantities will occur, particularly constants defined by integrals, which may equal zero in rare cases. Therefore some poles may actually have a lower order or even turn out as removable singularities. We will not mention this each time, but describe our computations only for the non-zero case. The final statement then remains unchanged, if it is not of the form $\mathrm{Si}[\ldots] \sim 0$.

First of all, under the above assumptions, for $j \in \{1,2\}$ with $\beta_{j0} > 0$ we introduce the parameters

$$\begin{cases} \chi_{j0} := \frac{1+\gamma_{j0}}{\beta_{j0}}, \\ \eta_{j0} := \frac{1+\gamma_{j0}}{\beta_{j0}} + \frac{\alpha_0(1,2)}{\beta_{j0}}. \end{cases} \tag{8.3.5}$$

By (8.3.2) and since $\alpha_0(1,2) \geq 0$, these satisfy

$$\eta_{j0} \geq \chi_{j0} > -1.$$

In the described setting, for non-identical $j,k \in \{1,2\}$, we begin with a consideration of the iterated generating function

$$\mathcal{M}_0\begin{bmatrix} -z_j, \tau_j, T_j \\ -z_k, \tau_k, T_k \end{bmatrix} := \int_{\tau_k}^{T_k} \{\varphi(t)\}^{-z_k} c(t) \int_{\tau_j}^{T_j} \{\varphi(s)\}^{-z_j} c(s)a(s+t)dsdt. \tag{8.3.6}$$

Notice the symmetry of the integral with respect to j,k. By Lemma 8.2.5, if $\beta_{10} = \beta_{20} = 0$, it is an entire function of z_j for arbitrary fixed $z_k \in \mathbb{C}$. If $\beta_{j0} > 0$ and $\beta_{k0} \geq 0$, we have absolute convergence for $z_j, z_k \in \mathbb{C}$ with

$$\begin{cases} \Re z_k < \infty, & \text{if } \beta_{k0} = 0, \\ \Re z_k < \chi_{k0}, & \text{if } \beta_{k0} > 0, \\ \Re z_j < \chi_{j0} + \min\left\{0, \frac{\alpha_0(1,2)}{\beta_{j0}} + \frac{1+\gamma_{k0}-\beta_{k0}\Re z_k}{\beta_{j0}}\right\}, \end{cases} \tag{8.3.7}$$

and analyticity with respect to z_j there. Again because $\alpha_0(1,2) \geq 0$ by assumption, for $\beta_{j0} > 0$ it shows, for any admissible $z_k \in \mathbb{C}$ we have absolute convergence in

$$\Re z_j < \chi_{j0}. \tag{8.3.8}$$

8.3.1. Transformation to an Iterated MB-Integral

With the aid of (8.1.1) and Lemma 8.2.1, due to the parameter restrictions (8.3.2), it is easy to confirm absolute convergence of the integral (8.3.1) for any fixed $m \geq 0$. Our first step now consists in an application of the Cahen-Mellin representation (8.1.6) for each of the m-powers in (8.3.1). Again by Lemma 8.2.1, for any $j, k \in \{1, 2\}$ with $j \neq k$, we can write

$$\begin{aligned}\operatorname{Si}\left[m; \begin{matrix}\tau_1, T_1\\ \tau_2, T_2\end{matrix}\right] = \frac{1}{2\pi i}\int\limits_{x_{k0}-i\infty}^{x_{k0}+i\infty}\frac{\Gamma(z_k)}{(m+1)^{z_k}}\int\limits_{\tau_k}^{T_k}\{\varphi(t)\}^{-z_k}\,c(t)\\ \times\int\limits_{\tau_j}^{T_j}\left\{1-(1-\Psi(s))^{m+1}\right\}c(s)a(s+t)dsdtdz_k,\end{aligned} \tag{8.3.9}$$

where the integration path satisfies

$$-1 < x_{k0} < \begin{cases}0, & \text{if } \beta_{k0} = 0,\\ \min\{0, \chi_{k0}\}, & \text{if } \beta_{k0} > 0.\end{cases} \tag{8.3.10}$$

A second application of the indicated Cahen-Mellin representation, for

$$-1 < x_{j0} < \begin{cases}0, & \text{if } \beta_{j0} = 0,\\ \min\{0, \chi_{j0}\}, & \text{if } \beta_{j0} > 0,\end{cases} \tag{8.3.11}$$

in terms of (8.3.6) leads to

$$\operatorname{Si}\left[m; \begin{matrix}\tau_1, T_1\\ \tau_2, T_2\end{matrix}\right] = \frac{1}{(2\pi i)^2}\int\limits_{x_{k0}-i\infty}^{x_{k0}+i\infty}\frac{\Gamma(z_k)}{(m+1)^{z_k}}\int\limits_{x_{j0}-i\infty}^{x_{j0}+i\infty}\frac{\Gamma(z_j)}{(m+1)^{z_j}}\mathcal{M}_0\left[\begin{matrix}-z_j, \tau_j, T_j\\ -z_k, \tau_k, T_k\end{matrix}\right]dz_jdz_k. \tag{8.3.12}$$

For the moment we leave j, k unspecified. Of particular interest is in any case the singularity of $\mathcal{M}_0[\ldots]$, that lies on its z_j-abscissa of convergence for fixed z_k. Since the right boundary of the half plane $\Re z_j < \chi_{j0}$ evidently stems from the condition for the convergence of a single component of the integral representation of $\mathcal{M}_0[\ldots]$, the z_j-analytic continuation can be computed via the standard integration by parts procedure.

8.3.2. An Interior Generating Function with a Kernel of the First Kind and a Finite Path

For $j, k \in \{1, 2\}$ with $j \neq k$ and fixed $\tau_k \leq t \leq T_k$, we define

$$\mathcal{L}_0\left[\begin{matrix}-z_j, t\\ \tau_j, T_j\end{matrix}\right] := \int\limits_{\tau_j}^{T_j}\{\varphi(s)\}^{-z_j}\,c(s)a(s+t)ds. \tag{8.3.13}$$

If $\beta_{j0} = 0$ this clearly represents an entire function of z_j. Hence, without loss of generality we assume $\beta_{j0} > 0$. The region of absolute convergence then essentially depends on the behaviour of $a(s+t)$ as $s \downarrow \tau_j$. If $\tau_k < t \le T_k$ is an arbitrary fixed point, by continuity we simply suppose $a(s+t) = \mathcal{O}(1)$ as $s \downarrow \tau_j$. If $t = \tau_k$, however, by assumption we certainly have algebraic behaviour of $a(s+\tau_k)$ as $s \downarrow \tau_j$ for a parameter $\alpha_0(1,2) \ge 0$. Altogether, in the case $\beta_{j0} > 0$ the integral (8.3.13) therefore converges absolutely and is analytic for $z_j \in \mathbb{C}$ with $\Re z_j < \chi_t$, for

$$\chi_t := \begin{cases} \chi_{j0}, & \text{if } \tau_k < t \le T_k, \\ \eta_{j0}, & \text{if } t = \tau_k. \end{cases} \tag{8.3.14}$$

With $q_1 \in \{0,1\}$ and $q_2 \in \{0,1,2\}$, in addition, we define the integral

$$\Lambda_0\begin{bmatrix} \chi_t - z_j, t \\ q_1, q_2, z_j, \tau_j, T_j \end{bmatrix} := \int_{\tau_j}^{T_j} (s-\tau_j)^{\beta_{j0}(\chi_t - z_j)-1} \frac{d^{q_1}}{ds^{q_1}} \left\{ e^{z_j B_j(s)} C_j(s) \right\} A^{(q_2)}(s;t) ds, \tag{8.3.15}$$

where q_2 refers to the derivative with respect to s, and in the fashion of (8.2.21) we write

$$A(s;t) := \begin{cases} a(s+t), & \text{if } \tau_k < t \le T_k, \\ \frac{a(s+\tau_k)}{(s-\tau_j)^{\alpha_0(1,2)}}, & \text{if } t = \tau_k. \end{cases} \tag{8.3.16}$$

Then, upon casting in (8.3.13) the integrand in terms of the normalized ingredients, for $\beta_{j0} > 0$ and $\tau_k \le t \le T_k$, we observe

$$\Lambda_0\begin{bmatrix} \chi_t - z_j, t \\ 0, 0, z_j, \tau_j, T_j \end{bmatrix} = \mathcal{L}_0\begin{bmatrix} -z_j, t \\ \tau_j, T_j \end{bmatrix}. \tag{8.3.17}$$

Considered as a function of $z_j \in \mathbb{C}$, the integral $\Lambda_0[\ldots]$ establishes a holomorphic function in different regions of the complex plane. In accordance with the above discussion on $a(s+t)$, with similar statements applying to its first and second derivative, these regions depend on q_1, on q_2 and on t.

Assume $q_1 = 0$, either with $q_2 = 0$ and $t = \tau_k$ or $q_2 \in \{0,1\}$ and $\tau_k < t \le T_k$. In these circumstances we have analyticity of (8.3.15) in the half plane $\Re z_j < \chi_t$. We then integrate by parts, or readily refer to our findings from §8.2.2.1, to access the analytic continuation into a wider domain. By identifying

$$\begin{cases} d(s) & \equiv c(s), \\ k(s+t) & \equiv a^{(q)}(s+t), \end{cases}$$

for fixed $\Re z_j < \chi_t$, an application of (8.2.31) leads to

(8.3.18)
$$\begin{aligned}\Lambda_0\begin{bmatrix}\chi_t - z_j, t\\ 0, q, z_j, \tau_j, T_j\end{bmatrix} &= -\frac{(T_j-\tau_j)^{\beta_{j0}(\chi_t - z_j)}}{\beta_{j0}\,(z_j-\chi_t)} e^{z_j B_j(T_j)} C_j(T_j) A^{(q)}(T_j;t)\\ &\quad + \frac{1}{\beta_{j0}\,(z_j-\chi_t)} \sum_{\substack{n_1,n_2\in\{0,1\}\\ n_1+n_2=1}} \Lambda_0\begin{bmatrix}\frac{1}{\beta_{j0}}+\chi_t - z_j, t\\ n_1, q+n_2, z_j, \tau_j, T_j\end{bmatrix}.\end{aligned}$$

From the order of the involved derivatives of the normalized phase and amplitude, we conclude analyticity of the integral for $n_1 = 1$ in $\Re z_j < \chi_t + \chi_{\beta_{j0}}(\beta_{j1}, \gamma_{j1})$, making use of (8.2.22). Regarding the integral for $n_2 = 1$, by continuity it follows for arbitrary t, that the functions $a'(s+t)$ and $a''(s+t)$ are $\mathcal{O}(1)$ as $s \downarrow \tau_j$. Hence, the second integral then exhibits analyticity in $\Re z_j < \chi_t + \frac{1}{\beta_{j0}}$. Since $\beta_{j1}, \gamma_{j1} > -1$, these regions include the half plane $\Re z_j < \chi_t$. Thus, introducing the parameter

(8.3.19)
$$\chi_{j1} := \chi_{j0} + \chi_{\beta_{j0}}(\beta_{j1}, \gamma_{j1}, 0),$$

for $q \in \{0,1\}$ and $\tau_k < t \le T_k$ the expansion (8.3.18) is valid in the extended region $\Re z_j < \chi_{j1}$, therein representing the analytic continuation of the original integral. If $t = \tau_k$ and $q = 0$, the last statement applies for the region $\Re z_j < \eta_{j1}$, where we denote

(8.3.20)
$$\eta_{j1} := \eta_{j0} + \chi_{\beta_{j0}}(\alpha_1(1,2), \beta_{j1}, \gamma_{j1}).$$

In each case, in the extended region, the expansion (8.3.18) evidently exhibits a simple pole at $z_j = \chi_t$. According to the behaviour of $B_j(s)$ and $C_j(s)$ as $s \downarrow \tau_j$, from (8.2.33) we immediately obtain

(8.3.21)
$$\operatorname*{Res}_{z_j=\chi_t} \Lambda_0\begin{bmatrix}\chi_t - z_j, t\\ 0, q, z_j, \tau_j, T_j\end{bmatrix} = -\frac{c_{j0}}{\beta_{j0}} \{b_{j0}\}^{-\chi_t} A^{(q)}(\tau_j;t),$$

by definition (8.3.16) with

(8.3.22)
$$A^{(q)}(\tau_j;t) = \begin{cases} a^{(q)}(\tau_j + t), & \text{if } \tau_k < t \le T_k \text{ and } q \in \{0,1\},\\ a_0(1,2), & \text{if } t = \tau_k \text{ and } q = 0.\end{cases}$$

Finally, for $j \in \{1,2\}$ and $\zeta_j \in \mathbb{C}$ we introduce

(8.3.23)
$$f_j(u,\zeta_j) := B_j'(u) C_j(u) - \log\{\varphi(u)\}\left\{C_j'(u) - \zeta_j C_j(u)\right\},$$

in terms of which we define

$$\begin{aligned}\lambda_0^q(t;\tau_j,T_j) := \log\{\varphi(T_j)\}\,C_j(T_j)A^{(q)}(T_j;t) + \int_{\tau_j}^{T_j} f_j(s,0)A^{(q)}(s;t)ds \\ - \int_{\tau_j}^{T_j} \log\{\varphi(s)\}\,C_j(s)A^{(q+1)}(s;t)ds.\end{aligned} \tag{8.3.24}$$

In the case $\chi_t = 0$, the pole at $z_j = \chi_t$ clearly lies at the origin of the z_j-plane, and from (8.2.35) and (8.2.36), for $t = \tau_k$ with $q = 0$ or $\tau_k < t \le T_k$ with $q \in \{0,1\}$, as $z_j \to 0$ we conclude

$$\Lambda_0\begin{bmatrix}\chi_t - z_j, t\\ 0, q, z_j, \tau_j, T_j\end{bmatrix} = -\frac{1}{z_j}\frac{c_{j0}}{\beta_{j0}}A^{(q)}(\tau_j;t) + \frac{1}{\beta_{j0}}\lambda_0^q(t;\tau_j,T_j) + \mathcal{O}(z_j). \tag{8.3.25}$$

The above integral representation for $\lambda_0^q(t;\tau_j,T_j)$ then especially converges absolutely.

8.3.3. z_j-Analytic Continuation of the Iterated Generating Function for Fixed z_k

In the preceding subsection we have seen, that the interior integral of (8.3.6) for $\beta_{j0} > 0$, which is given by (8.3.15), can be extended meromorphically across the boundary of the half plane $\Re z_j < \chi_{j0}$ by virtue of the expansion (8.3.18). In order to show that this in turn immediately gives access to the analytic continuation of (8.3.6), we introduce the expansion

$$\begin{aligned}\mathcal{M}_1\begin{bmatrix}-z_j, \tau_j, T_j\\ -z_k, \tau_k, T_k\end{bmatrix} :=& (T_j-\tau_j)^{\beta_{j0}(\chi_{j0}-z_j)}e^{z_jB_j(T_j)}C_j(T_j)\int_{\tau_k}^{T_k}\{\varphi(t)\}^{-z_k}\,c(t)a(T_j+t)dt \\ &- \sum_{\substack{n_1,n_2\in\{0,1\}\\ n_1+n_2=1}}\int_{\tau_k}^{T_k}\{\varphi(t)\}^{-z_k}\,c(t)\Lambda_0\begin{bmatrix}\frac{1}{\beta_{j0}} + \chi_{j0} - z_j, t\\ n_1, n_2, z_j, \tau_j, T_j\end{bmatrix}dt.\end{aligned} \tag{8.3.26}$$

Since $a(T_j+\tau_k) = \mathcal{O}(1)$ by continuity, the single integral on the right hand side is absolutely convergent and holomorphic for $z_k \in \mathbb{C}$ either arbitrary or with $\Re z < \chi_{k0}$, respectively if $\beta_{k0} = 0$ or $\beta_{k0} > 0$. To determine the region of convergence of the iterated integrals, by assumption (S4) on the order of $A_{1,2}'(u)$, particularly by rearranging (8.2.12), as $u \downarrow \tau_1 + \tau_2$, we note that

$$\begin{aligned}a'(u) &= \frac{\alpha_0(1,2)}{u-\tau_1-\tau_2}a(u) + \mathcal{O}\left\{(u-\tau_1-\tau_2)^{\alpha_0(1,2)+\alpha_1(1,2)}\right\} \\ &= \begin{cases}a_0(1,2)\alpha_0(1,2)(u-\tau_1-\tau_2)^{\alpha_0(1,2)-1} + o(1), & \text{if } \alpha_0(1,2) > 0,\\ \mathcal{O}\left\{(u-\tau_1-\tau_2)^{\alpha_1(1,2)}\right\}, & \text{if } \alpha_0(1,2) = 0.\end{cases}\end{aligned} \tag{8.3.27}$$

By taking these properties into account, it is easy to verify with the aid of Lemma 8.2.1 absolute convergence of the integral in (8.3.26) for $n_1 = 1$, if

$$\Re z_j < \begin{cases} \chi_{j0} + \chi_{\beta_{j0}}(\beta_{j1}, \gamma_{j1}), \\ \eta_{j0} + \chi_{\beta_{j0}}(\beta_{j1}, \gamma_{j1}) + \frac{1+\gamma_{k0}-\beta_{k0}\Re z_k}{\beta_{j0}}, \end{cases}$$

and of the integral for $n_2 = 1$, if

$$\Re z_j < \begin{cases} \chi_{j0} + \frac{1}{\beta_{j0}}, \\ \eta_{j0} + \frac{1+\gamma_{k0}-\beta_{k0}\Re z_k}{\beta_{j0}}, & \text{if } \alpha_0(1,2) > 0, \\ \chi_{j0} + \frac{\alpha_1(1,2)+1}{\beta_{j0}} + \frac{1+\gamma_{k0}-\beta_{k0}\Re z_k}{\beta_{j0}}, & \text{if } \alpha_0(1,2) = 0, \end{cases}$$

respectively for fixed $z_k \in \mathbb{C}$ with $\Re z_k < \chi_{k0}$ if $\beta_{k0} > 0$ and arbitrary if $\beta_{k0} = 0$. Upon collecting these findings, for any fixed admissible $z_k \in \mathbb{C}$, the expansion (8.3.26) establishes an analytic function in the half plane

$$\Re z_j < \eta_{(1,2)}(z_k), \tag{8.3.28}$$

where, in terms of (8.3.19), we define

$$\eta_{(1,2)}(z_k) := \begin{cases} \min\left\{\chi_{j1}, \eta_{j0} + \frac{1+\gamma_{k0}-\beta_{k0}z_k}{\beta_{j0}}\right\}, & \text{if } \alpha_0(1,2) > 0, \\ \min\left\{\chi_{j1}, \chi_{j0} + \frac{\alpha_1(1,2)+1}{\beta_{j0}} + \frac{1+\gamma_{k0}-\beta_{k0}z_k}{\beta_{j0}}\right\}, & \text{if } \alpha_0(1,2) = 0. \end{cases} \tag{8.3.29}$$

Notice, if the minimum attains the second value, it originates in the supplementary condition for the convergence of either of the iterated integrals. A repeated integration by parts will then presumably not give access to a larger region of analyticity. Now, if we apply the identity (8.3.17) accompanied by the expansion (8.3.18) to (8.3.6), we arrive at

$$\mathcal{M}_0\begin{bmatrix} -z_j, \tau_j, T_j \\ -z_k, \tau_k, T_k \end{bmatrix} = -\frac{1}{\beta_{j0}\,(z_j - \chi_{j0})}\mathcal{M}_1\begin{bmatrix} -z_j, \tau_j, T_j \\ -z_k, \tau_k, T_k \end{bmatrix}. \tag{8.3.30}$$

For fixed $z_k \in \mathbb{C}$ with $\Re z_k < \chi_{k0}$ if $\beta_{k0} > 0$ and arbitrary if $\beta_{k0} = 0$, the right hand side meromorphically extends the integral (8.3.6) into the z_j-region $\Re z_j < \eta_{(1,2)}(z_k)$. Therein, it exposes a simple pole at $z_j = \chi_{j0}$ only. In terms of (8.3.13), from (8.3.21) we compute

$$\begin{aligned} \operatorname*{Res}_{z_j=\chi_{j0}} \mathcal{M}_0\begin{bmatrix} -z_j, \tau_j, T_j \\ -z_k, \tau_k, T_k \end{bmatrix} &= \int_{\tau_k}^{T_k} \{\varphi(t)\}^{-z_k}\, c(t) \operatorname*{Res}_{\chi_j=z_{j0}} \Lambda_0\begin{bmatrix} \chi_{j0} - z_j, t \\ 0, 0, z_j, \tau_j, T_j \end{bmatrix} dt \\ &= -\frac{c_{j0}}{\beta_{j0}}\,\{b_{j0}\}^{-\chi_{j0}}\,\mathcal{L}_0\begin{bmatrix} -z_k, \tau_j \\ \tau_k, T_k \end{bmatrix}. \end{aligned} \tag{8.3.31}$$

Finally, to specify the Laurent expansion near $z_j = 0$ in the case $\chi_{j0} = 0$, we introduce

$$\mathcal{X}_0\begin{bmatrix} -z_k, \tau_k, T_k \\ \tau_j, T_j \end{bmatrix} := \log\{\varphi(T_j)\}\, C_j(T_j) \int_{\tau_k}^{T_k} \{\varphi(t)\}^{-z_k}\, c(t) a(T_j + t) dt \\ + \int_{\tau_k}^{T_k} \{\varphi(t)\}^{-z_k}\, c(t) \int_{\tau_j}^{T_j} f_j(s,0) a(s+t) ds dt \\ - \int_{\tau_k}^{T_k} \{\varphi(t)\}^{-z_k}\, c(t) \int_{\tau_j}^{T_j} \log\{\varphi(s)\}\, C_j(s) a'(s+t) ds dt, \tag{8.3.32}$$

where $f_j(s,0)$ was defined in (8.3.23). Then, if $\chi_{j0} = 0$, from (8.3.24) and (8.3.25), in an annulus around $z_j = 0$ with fixed z_k, we find

$$\mathcal{M}_0\begin{bmatrix} -z_j, \tau_j, T_j \\ -z_k, \tau_k, T_k \end{bmatrix} = -\frac{1}{z_j}\frac{c_{j0}}{\beta_{j0}}\mathcal{L}_0\begin{bmatrix} -z_k, \tau_j \\ \tau_k, T_k \end{bmatrix} + \frac{1}{\beta_{j0}}\mathcal{X}_0\begin{bmatrix} -z_k, \tau_k, T_k \\ \tau_j, T_j \end{bmatrix} + \mathcal{O}(z_j). \tag{8.3.33}$$

8.3.4. Evaluation of the Interior MB-Integral

For fixed $z_k \in \mathbb{C}$ with $\Re z_k = x_{k0}$, we define the interior of the iterated MB-integral (8.3.12) by

$$\mathrm{I}\begin{bmatrix} m; & \tau_j, T_j \\ & z_k, \tau_k, T_k \end{bmatrix} := \frac{1}{2\pi i} \int_{x_{j0}-i\infty}^{x_{j0}+i\infty} \frac{\Gamma(z_j)}{(m+1)^{z_j}} \mathcal{M}_0\begin{bmatrix} -z_j, \tau_j, T_j \\ -z_k, \tau_k, T_k \end{bmatrix} dz_j. \tag{8.3.34}$$

According to our preceding findings, $\mathcal{M}_0[\ldots]$ for fixed $z_k \in \mathbb{C}$ with $\Re z_k = x_{k0}$ represents an entire function with respect to z_j if $\beta_{j0} = 0$. The only singularity to the right of the line $\Re z_j = x_{j0}$ is then the simple pole at $z_j = 0$ of $\Gamma(z_j)$. If $\beta_{j0} > 0$, the integral definition of $\mathcal{M}_0[\ldots]$ can be extended to the region $\Re z_j < \eta_{(1,2)}(x_{k0})$ as a meromorphic function. Therein, the only singularity is a pole of simple order on the line $\Re z_j = \chi_{j0}$. Hence, in this event the closest singularity to the right of the path $\Re z_j = x_{j0}$ depends on χ_{j0} but is a simple pole, except if $\chi_{j0} = 0$, in which case we observe a coalescence to a pole of order two at $z_j = 0$. In each case $\mathcal{M}_0[\ldots]$ or its analytic continuation is represented by a sum of rational functions and integrals. These integrals converge absolutely and uniformly in any compact subset of their region of validity. Due to their uniform convergence with respect to $\Im z_k, \Im z_j \in \mathbb{R}$, it can even be shown that $\mathcal{M}_0[\ldots]$ for fixed $z_k \in \mathbb{C}$ with $\Re z_k = x_{k0}$ is $\mathcal{O}(1)$ as $\Im z_j \to \pm\infty$ in $\mathbb{C}$ if $\beta_{j0} = 0$ or in $\Re z_j < \eta_{(1,2)}(x_{k0})$ if $\beta_{j0} > 0$, uniformly with respect to $\Re z_j$ in any closed vertical substrip. Hence, the integrand in (8.3.34) decays exponentially fast as $\Im z_j \to \pm\infty$ there.

These findings enable a displacement of the integration path to the right over the pole at $z_j = 0$ if $\beta_{j0} = 0$, over the poles at $z_j \in \{0, \chi_{j0}\}$ if $\chi_{j0} > 0$, or merely over the pole at $z_j = \chi_{j0}$ if that point does not lie in the right z_j-half plane. Each pole is then encircled clockwisely. Analogous to (8.1.12), with the aid of Theorem B.2.1(2), by taking into account (8.3.31) and

(8.3.33), for a suitable vertical line with $\Re z_j = x_{j1}$, this yields

$$\begin{aligned}
\mathrm{I}\left[m; \begin{matrix} \tau_j, T_j \\ z_k, \tau_k, T_k \end{matrix}\right] &= -\mathcal{M}_0\left[\begin{matrix} 0, \tau_j, T_j \\ -z_k, \tau_k, T_k \end{matrix}\right] \mathbb{1}_{\{\beta_{j0}=0 \vee \chi_{j0}>0\}} \\
&+ \frac{c_{j0}}{\beta_{j0}} \frac{\Gamma(\chi_{j0})}{\{b_{j0}(m+1)\}^{\chi_{j0}}} \mathcal{L}_0\left[\begin{matrix} -z_k, \tau_j \\ \tau_k, T_k \end{matrix}\right] \mathbb{1}_{\{\chi_{j0}\neq 0\}} \\
&- \frac{1}{\beta_{j0}}\left\{(\log(m+1)+\gamma) c_{j0} \mathcal{L}_0\left[\begin{matrix} -z_k, \tau_j \\ \tau_k, T_k \end{matrix}\right] + \mathcal{X}_0\left[\begin{matrix} -z_k, \tau_k, T_k \\ \tau_j, T_j \end{matrix}\right]\right\} \mathbb{1}_{\{\chi_{j0}=0\}} \\
&+ \frac{1}{2\pi i} \int_{x_{j1}-i\infty}^{x_{j1}+i\infty} \frac{\Gamma(z_j)}{(m+1)^{z_j}} \mathcal{M}_0\left[\begin{matrix} -z_j, \tau_j, T_j \\ -z_k, \tau_k, T_k \end{matrix}\right] dz_j.
\end{aligned} \tag{8.3.35}$$

If $\beta_{j0} = 0$, the integration path $\Re z_j = x_{j1}$ is subject to the restrictions

$$0 < x_{j1} < \infty, \tag{8.3.36}$$

whereas for $\beta_{j0} > 0$ the singularity that lies closest to its left is the pole at $z_j = \chi_{j0}$, i.e.,

$$\chi_{j0} < x_{j1} < \begin{cases} \eta_{(1,2)}(x_{k0}), & \text{if } \chi_{j0} \geq 0, \\ \min\left\{0, \eta_{(1,2)}(x_{k0})\right\}, & \text{if } \chi_{j0} < 0. \end{cases} \tag{8.3.37}$$

Finally, for an integration path $\Re z_2 = x_{20}$ that satisfies (8.3.10) for $k = 2$, we denote

$$\mathrm{L}(m) := \frac{1}{2\pi i} \int_{x_{20}-i\infty}^{x_{20}+i\infty} \frac{\Gamma(z_2)}{(m+1)^{z_2}} \mathcal{L}_0\left[\begin{matrix} -z_2, \tau_1 \\ \tau_2, T_2 \end{matrix}\right] dz_2, \tag{8.3.38}$$

$$\mathrm{X}(m) := \frac{1}{2\pi i} \int_{x_{20}-i\infty}^{x_{20}+i\infty} \frac{\Gamma(z_2)}{(m+1)^{z_2}} \mathcal{X}_0\left[\begin{matrix} -z_2, \tau_2, T_2 \\ \tau_1, T_1 \end{matrix}\right] dz_2. \tag{8.3.39}$$

Then, if we eventually plug (8.3.35) into the iterated MB-integral (8.3.12) with $j = 1$ and $k = 2$, by means of a simple bound for the remainder integral, as $m \to \infty$ it is easy to see:

$$\begin{aligned}
\mathrm{Si}\left[m; \begin{matrix} \tau_1, T_1 \\ \tau_2, T_2 \end{matrix}\right] &= -\mathrm{I}\left[m; \begin{matrix} \tau_2, T_2 \\ 0, \tau_1, T_1 \end{matrix}\right] \mathbb{1}_{\{\beta_{10}=0 \vee \chi_{10}>0\}} \\
&+ \frac{c_{10}}{\beta_{10}} \frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}} \mathrm{L}(m) \mathbb{1}_{\{\chi_{10}\neq 0\}} \\
&- \frac{1}{\beta_{10}} \{(\log(m+1)+\gamma) c_{10} \mathrm{L}(m) + \mathrm{X}(m)\} \mathbb{1}_{\{\chi_{10}=0\}} \\
&+ \mathcal{O}\left\{m^{-x_{20}-x_{11}}\right\}
\end{aligned} \tag{8.3.40}$$

For the asymptotics of the first MB-integral on the right hand side, we may again refer to (8.3.35) with $j = 2$, $k = 1$ and $z_1 = 0$. We therefore proceed with a study of the single MB-integrals $\mathrm{L}(m)$ and $\mathrm{X}(m)$.

8.3.5. A Single MB-Integral for the Residue at $z_1 = \chi_{10}$

The residue of the iterated generating function at $z_1 = \chi_{10}$ produces the MB-integral (8.3.38), whose generating function (8.3.13) was studied in §8.3.2. While the indicated integral definition of $\mathcal{L}_0[\ldots]$ is entire for $\beta_{20} = 0$, it is holomorphic in the half plane $\Re z_2 < \eta_{20}$ for $\beta_{20} > 0$. As a consequence of its absolute convergence, in each case $\mathcal{L}_0[\ldots]$ is $\mathcal{O}(1)$ as $\Im z_2 \to \pm\infty$ in its region of analyticity, whence the integration path in (8.3.38) can be replaced by an arbitrary vertical line $x_{20} \equiv l_0$ that satisfies

$$-1 < l_0 < \begin{cases} 0, & \text{if } \beta_{20} = 0, \\ \min\{0, \eta_{20}\}, & \text{if } \beta_{20} > 0. \end{cases}$$

Notice $\eta_{20} \geq \chi_{20}$. In the case $\beta_{20} > 0$, the generating function of (8.3.38) satisfies the identity (8.3.17), and it can be extended analytically to the wider half plane $\Re z_2 < \eta_{21}$. Therein, it exhibits solely a pole of simple order at $z_2 = \eta_{20}$, whose residue by (8.3.21) and (8.3.22) equals

$$\operatorname*{Res}_{z_2=\eta_{20}} \mathcal{L}_0\begin{bmatrix} -z_2, \tau_1 \\ \tau_2, T_2 \end{bmatrix} = -\frac{a_0(1,2)c_{20}}{\beta_{20}} \{b_{20}\}^{-\eta_{20}}. \tag{8.3.41}$$

Moreover, if $\eta_{20} = 0$ we ascertain from (8.3.25), that the first two terms of the Laurent expansion at $z_2 = 0$ are given by

$$\mathcal{L}_0\begin{bmatrix} -z_2, \tau_1 \\ \tau_2, T_2 \end{bmatrix} = -\frac{1}{z_2}\frac{a_0(1,2)c_{20}}{\beta_{20}} + \frac{1}{\beta_{20}}\lambda_0^0(\tau_1; \tau_2, T_2) + \mathcal{O}(z_2), \tag{8.3.42}$$

where the coefficient associated with the constant term features the function (8.3.24).

Finally, also the analytic continuation of $\mathcal{L}_0[\ldots]$ is readily confirmed $\mathcal{O}(1)$ as $\Im z_2 \to \pm\infty$ in $\Re z_2 < \eta_{21}$, uniformly with respect to $\Re z_2$ in any closed vertical substrip. In view of the exponential decay of the gamma function, in (8.3.38) we are therefore allowed to displace the integration path rightwards, to match a vertical line that cuts the real axis at $\Re z_2 = l_1$, with

$$\begin{aligned} &0 < l_1 < \infty, && \text{if } \beta_{20} = 0, \\ &\eta_{20} < l_1 < \begin{cases} \eta_{21}, & \text{if } \eta_{20} \geq 0, \\ \min\{0, \eta_{21}\}, & \text{if } \eta_{20} < 0. \end{cases} \end{aligned} \tag{8.3.43}$$

Thereby we encircle in the clockwise direction the pole at $z_2 = \eta_{20}$ and also the pole at $z_2 = 0$ if $\eta_{20} \geq 0$. Upon taking into account (8.3.41) and (8.3.42), as $m \to \infty$ we arrive at

$$\begin{aligned} \mathrm{L}(m) = {} & -\mathcal{L}\begin{bmatrix} 0, \tau_1 \\ \tau_2, T_2 \end{bmatrix} \mathbb{1}_{\{\beta_{20}=0 \,\vee\, \eta_{20}>0\}} \\ & + \frac{c_{20}}{\beta_{20}} \frac{a_0(1,2)\Gamma(\eta_{20})}{\{b_{20}(m+1)\}^{\eta_{20}}} \mathbb{1}_{\{\eta_{20} \neq 0\}} \\ & - \frac{1}{\beta_{20}} \left\{ (\log(m+1) + \gamma) c_{20} a_0(1,2) + \lambda_0^0(\tau_1; \tau_2, T_2) \right\} \mathbb{1}_{\{\eta_{20}=0\}} \end{aligned} \tag{8.3.44}$$

$$+\,\mathcal{O}\left\{m^{-l_1}\right\}.$$

The big-$\mathcal{O}$ estimate holds by absolute convergence of the remainder integral.

8.3.6. A Single MB-Integral for the Residue at $z_1 = 0$ if $\chi_{10} = 0$

If $\chi_{10} = 0$, we encounter the additional MB-integral (8.3.39) with generating function (8.3.32). In order to extract its dominating behaviour as $m \to \infty$, we must first specify the analytic continuation of this last generating function into the right direction of the z_2-plane.

8.3.6.1. Analytic Continuation of the Generating Function

By definition (8.3.23), as $u \downarrow \tau_j$ the function $f_j(u,0)$ for $j \in \{1,2\}$ and arbitrary $\varepsilon > 0$ shows the behaviour

$$f_j(u,0) = \mathcal{O}\left\{(u-\tau_j)^{\min\{\beta_{j1},\gamma_{j1}-\varepsilon\}}\right\}.$$

Hence, the presence of the logarithm in $f_1(s,0)$ does not affect the z_2-region of absolute convergence and analyticity of (8.3.32) for $k = 2$. This is the whole complex plane if $\beta_{20} = 0$. If $\beta_{20} > 0$, by Lemma 8.2.5 and (8.3.27), we detect these regions for each integral in the expansion (8.3.32) as those $z_2 \in \mathbb{C}$ with

$$\Re z_2 < \begin{cases} \chi_{20} & \text{if } \alpha_0(1,2) \geq 0, \\ \chi_{20} + \min\left\{0, \frac{\alpha_0(1,2)}{\beta_{20}} + \chi_{\beta_{20}}(\beta_{11},\gamma_{11})\right\} & \text{if } \alpha_0(1,2) \geq 0, \\ \chi_{20} + \min\left\{0, \frac{\alpha_0(1,2)}{\beta_{20}}\right\}, & \text{if } \alpha_0(1,2) > 0, \\ \chi_{20} + \min\left\{0, \frac{\alpha_1(1,2)+1}{\beta_{20}}\right\}, & \text{if } \alpha_0(1,2) = 0. \end{cases}$$

By assumption on β_{11}, γ_{11} and $\alpha_1(1,2)$, each of the above minima uniquely equals zero. Altogether we conclude, (8.3.32) for $\beta_{20} > 0$ represents a holomorphic function in the region

$$\Re z_2 < \chi_{20}. \tag{8.3.45}$$

Its right boundary is due to the condition for the convergence of the single integral in (8.3.32) and of a single component of each of the iterated integrals. Now, in terms of (8.3.15) we obtain

$$\begin{aligned}\mathcal{X}_0\begin{bmatrix}-z_2,\tau_2,T_2\\ \tau_1,T_1\end{bmatrix} &= \log\{\varphi(T_1)\}\,C_1(T_1)\Lambda_0\begin{bmatrix}\chi_{20}-z_2,T_1\\ 0,0,z_2,\tau_2,T_2\end{bmatrix}\\ &\quad+\int_{\tau_1}^{T_1} f_1(s,0)\Lambda_0\begin{bmatrix}\chi_{20}-z_2,s\\ 0,0,z_2,\tau_2,T_2\end{bmatrix}ds\\ &\quad-\int_{\tau_1}^{T_1}\log\{\varphi(s)\}\,C_1(s)\Lambda_0\begin{bmatrix}\chi_{20}-z_2,s\\ 0,1,z_2,\tau_2,T_2\end{bmatrix}ds.\end{aligned} \tag{8.3.46}$$

The analytic continuation of the integral functions appearing in this representation was already established by virtue of partial integration in the form of the expansion (8.3.18), from which for $z_2 \in \mathbb{C}$ with $\Re z_2 < \chi_{20}$ we obtain:

$$
\begin{aligned}
\mathcal{X}_0\begin{bmatrix} -z_2, \tau_2, T_2 \\ \tau_1, T_1 \end{bmatrix} = &-\frac{\log\{\varphi(T_1)\}\,C_1(T_1)}{\beta_{20}(z_2-\chi_{20})}(T_2-\tau_2)^{\beta_{20}(\chi_{20}-z_2)}e^{z_2 B_2(T_2)}C_2(T_2)a(T_1+T_2) \\
&+\frac{\log\{\varphi(T_1)\}\,C_1(T_1)}{\beta_{20}(z_2-\chi_{20})}\sum_{\substack{n_1,n_2\in\{0,1\}\\ n_1+n_2=1}}\Lambda_0\begin{bmatrix} \frac{1}{\beta_{20}}+\chi_{20}-z_2, T_1 \\ n_1, n_2, z_2, \tau_2, T_2 \end{bmatrix} \\
&-\frac{(T_2-\tau_2)^{\beta_{20}(\chi_{20}-z_2)}}{\beta_{20}(z_2-\chi_{20})}e^{z_2 B_2(T_2)}C_2(T_2)\int_{\tau_1}^{T_1} f_1(s,0)a(s+T_2)ds \\
&+\frac{1}{\beta_{20}(z_2-\chi_{20})}\sum_{\substack{n_3,n_4\in\{0,1\}\\ n_3+n_4=1}}\int_{\tau_1}^{T_1} f_1(s)\Lambda_0\begin{bmatrix} \frac{1}{\beta_{20}}+\chi_{20}-z_2, s \\ n_3, n_4, z_2, \tau_2, T_2 \end{bmatrix}ds \\
&+\frac{(T_2-\tau_2)^{\beta_{20}(\chi_{20}-z_2)}}{\beta_{20}(z_2-\chi_{20})}e^{z_2 B_2(T_2)}C_2(T_2)\int_{\tau_1}^{T_1}\log\{\varphi(s)\}\,C_1(s)a'(s+T_2)ds \\
&-\frac{1}{\beta_{20}(z_2-\chi_{20})}\sum_{\substack{n_5,n_6\in\{0,1\}\\ n_5+n_6=1}}\int_{\tau_1}^{T_1}\log\{\varphi(s)\}\,C_1(s) \\
&\qquad\qquad\times\Lambda_0\begin{bmatrix} \frac{1}{\beta_{20}}+\chi_{20}-z_2, s \\ n_5, 1+n_6, z_2, \tau_2, T_2 \end{bmatrix}ds
\end{aligned}
\tag{8.3.47}
$$

The absolute convergence of the constant integrals is obvious. Furthermore, with the aid of Lemma 8.2.5, (8.3.3), (8.3.4) and (8.3.27) we confirm absolute convergence and analyticity of the first, second and third sum of parameter integrals, respectively for $z_2 \in \mathbb{C}$ subject to

$$
\Re z_2 < \chi_{20} + \begin{cases}
\chi_{\beta_{20}}(\beta_{21}, \gamma_{21}, 0), & \text{if } \alpha_0(1,2) \geq 0, \\
\chi_{\beta_{20}}(\beta_{21}, \gamma_{21}, 0, \alpha_0(1,2)+\beta_{11}, \alpha_0(1,2)+\gamma_{11}), & \text{if } \alpha_0(1,2) > 0, \\
\chi_{\beta_{20}}(\beta_{21}, \gamma_{21}, 0, \alpha_1(1,2)+\beta_{11}+1, \alpha_1(1,2)+\gamma_{11}+1), & \text{if } \alpha_0(1,2) = 0, \\
\min\{\alpha_0(1,2), \chi_{\beta_{20}}(\beta_{21}, \gamma_{21}, 0)\}, & \text{if } \alpha_0(1,2) > 0, \\
\chi_{\beta_{20}}(\alpha_1(1,2), \beta_{21}, \gamma_{21}, 0), & \text{if } \alpha_0(1,2) = 0.
\end{cases}
$$

Since these regions contain the half plane $\Re z_2 < \chi_{20}$, the expansion (8.3.47) represents the analytic continuation of $\mathcal{X}_0[\ldots]$ to the region

$$\Re z_2 < \chi_1(1,2), \tag{8.3.48}$$

where we denote

$$\chi_1(1,2) := \chi_{20} + \begin{cases} \min\left\{\alpha_0(1,2), \chi_{\beta_{20}}\left(\beta_{21}, \gamma_{21}, 0\right)\right\}, & \text{if } \alpha_0(1,2) > 0, \\ \chi_{\beta_{20}}\left(\alpha_1(1,2), \beta_{21}, \gamma_{21}, 0\right), & \text{if } \alpha_0(1,2) = 0. \end{cases} \tag{8.3.49}$$

The above expansion is only required, to verify the existence of a continuation of $\mathcal{X}_0[\ldots]$. To determine the residue of the pole at $z_2 = \chi_{20}$, we rather employ (8.3.46). From (8.3.21) we then readily deduce

$$\operatorname*{Res}_{z_2=\chi_{20}} \mathcal{X}_0\begin{bmatrix} -z_2, \tau_2, T_2 \\ \tau_1, T_1 \end{bmatrix} = -\frac{c_{20}}{\beta_{20}} \left\{b_{20}\right\}^{-\chi_{20}} \xi_{-1}(\tau_2; \tau_1, T_1), \tag{8.3.50}$$

which features the constant

$$\begin{aligned} \xi_{-1}(\tau_2; \tau_1, T_1) := \log\left\{\varphi(T_1)\right\} C_1(T_1) a(T_1 + \tau_2) + \int_{\tau_1}^{T_1} f_1(s, 0) a(s + \tau_2) ds \\ - \int_{\tau_1}^{T_1} \log\left\{\varphi(s)\right\} C_1(s) a'(s + \tau_2) ds. \end{aligned} \tag{8.3.51}$$

Furthermore, if $\chi_{20} = 0$, by (8.3.25), all of the interior integrals in (8.3.46) can be expanded as a Laurent series in an annulus around the origin. Upon collecting equal powers of z_2, we obtain

$$\mathcal{X}_0\begin{bmatrix} -z, \tau_2, T_2 \\ \tau_1, T_1 \end{bmatrix} = -\frac{1}{z_2}\frac{c_{20}}{\beta_{20}} \xi_{-1}(\tau_2; \tau_1, T_1) + \frac{1}{\beta_{20}} \xi_0\begin{bmatrix} \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} + \mathcal{O}(z_2), \tag{8.3.52}$$

with the function in the second summand, by (8.3.24), denoted as

$$\begin{aligned} \xi_0\begin{bmatrix} \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} := C_1(T_1) \log\left\{\varphi(T_1)\right\} \lambda_0^0(T_1; \tau_2, T_2) + \int_{\tau_1}^{T_1} f_1(s, 0) \lambda_0^0(s; \tau_2, T_2) ds \\ - \int_{\tau_1}^{T_1} \log\left\{\varphi(s)\right\} C_1(s) \lambda_0^1(s; \tau_2, T_2) ds. \end{aligned} \tag{8.3.53}$$

8.3.6.2. Evaluation of the MB-Integral

In view of our findings from the preceding paragraph, the only singularities of the integrand of the MB-integral (8.3.39), which lie to the right of the line $\Re z_2 = x_{20}$, in $\mathbb{C}$ if $\beta_{20} = 0$ or in the half plane $\Re z_2 < \chi_1(1,2)$ if $\beta_{20} > 0$, are either one or two poles of simple order or a single pole of second order. Since, in addition, we observe exponential decay of the integrand in (8.3.39) towards any imaginary direction of the respective region of analyticity, we can move the integration path to the right, to match a line with real part $\Re z_2 = k_1$, for

$$\begin{aligned} &0 < k_1 < \infty, \qquad \text{if } \beta_{20} = 0, \\ &\chi_{20} < k_1 < \begin{cases} \chi_1(1,2), & \text{if } \chi_{20} \geq 0, \\ \min\{0, \chi_1(1,2)\}, & \text{if } \chi_{20} < 0. \end{cases} \end{aligned} \tag{8.3.54}$$

In this process we clockwisely encircle the pole at $z_2 = 0$ if $\beta_{20} = 0$, at $z_2 = \chi_{20}$ if $\chi_{20} \leq 0$, or the poles at $z_2 \in \{0, \chi_{20}\}$ if $\chi_{20} > 0$. According to Theorem B.2.1(2), by means of (8.3.50) and (8.3.52), as $m \to \infty$ this yields

$$\begin{aligned} \mathrm{X}(m) = &-\mathcal{X}_0\begin{bmatrix} 0, \tau_2, T_2 \\ \tau_1, T_1 \end{bmatrix} \mathbb{1}_{\{\beta_{20} = 0 \vee \chi_{20} > 0\}} \\ &+ \frac{c_{20}}{\beta_{20}} \frac{\Gamma(\chi_{20})}{\{b_{20}(m+1)\}^{\chi_{20}}} \xi_{-1}(\tau_2; \tau_1, T_1) \mathbb{1}_{\{\chi_{20} \neq 0\}} \\ &- \frac{1}{\beta_{20}} \left\{ (\log(m+1) + \gamma) c_{20} \xi_{-1}(\tau_2; \tau_1, T_1) + \xi_0 \begin{bmatrix} \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} \right\} \mathbb{1}_{\{\chi_{20} = 0\}} \\ &+ \mathcal{O}\left\{ m^{-k_1} \right\}. \end{aligned} \tag{8.3.55}$$

The estimate for the remainder term was deduced by absolute convergence of its representation as a MB-integral.

8.3.7. Evaluation of the Iterated MB-Integral

We will now collect our findings from the preceding subsections, to establish definite statements on the m-asymptotic behaviour of the integral Si[...]. This essentially requires, to verify the neglibility of the remainder terms that occur, when the single expansions for I[...], L(m) and X(m) are plugged into (8.3.40). To avoid difficulties with the remainder term that appears in the big-$\mathcal{O}$ in (8.3.40), on the one hand, we assume

$$\beta_{20} \geq \beta_{10} \geq 0. \tag{8.3.56}$$

On the other hand, with the parameter χ_{11} defined in (8.3.19), for $\beta_{10} > 0$ we suppose

$$\chi_{10} + \tfrac{\alpha_0(1,2)}{\beta_{20}} < \begin{cases} \chi_{11}, & \text{if } \chi_{10} \geq 0, \\ \min\{0, \chi_{11}\}, & \text{if } \chi_{10} < 0. \end{cases} \tag{8.3.57}$$

Note that the last condition always applies if $\alpha_0(1,2) = 0$. By (8.3.5), from (8.3.56) for $\beta_{10} > 0$ we deduce

$$\chi_{10} + \eta_{20} \leq \eta_{10} + \chi_{20}. \tag{8.3.58}$$

Moreover, according to the conditions on the path $\Re z_2 = x_{20}$, compare (8.3.10) with $k = 2$, the assumption (8.3.56) also implies

$$\chi_{10} + \frac{\alpha_0(1,2)}{\beta_{20}} < \begin{cases} \eta_{10} + \frac{1+\gamma_{20}-\beta_{20}x_{20}}{\beta_{10}}, & \text{if } \alpha_0(1,2) > 0, \\ \chi_{10} + \frac{\alpha_1(1,2)+1}{\beta_{10}} + \frac{1+\gamma_{20}-\beta_{20}x_{20}}{\beta_{10}}, & \text{if } \alpha_0(1,2) = 0. \end{cases} \tag{8.3.59}$$

Hence, due to (8.3.57) and (8.3.59), instead of (8.3.37) the path of the remainder integral in (8.3.40) can be assumed to satisfy

$$\chi_{10} + \frac{\alpha_0(1,2)}{\beta_{20}} < x_{11} < \begin{cases} \eta_{(1,2)}(x_{20}), & \text{if } \chi_{10} \geq 0, \\ \min\left\{0, \eta_{(1,2)}(x_{20})\right\}, & \text{if } \chi_{10} < 0. \end{cases} \tag{8.3.60}$$

Next, we aim to express the parameters x_{20} and x_{11} in terms of the right and the left boundary of their respective range. For this, we first note, by (8.3.10), in the case $\beta_{20} > 0$ for an arbitrary $\varepsilon_2 \in (0, \min\{0, \chi_{20}\} + 1)$, we have $x_{20} = \min\{0, \chi_{20}\} - \varepsilon_2$. Secondly, if $\beta_{10} > 0$ by (8.3.60) for an arbitrary

$$\begin{cases} \varepsilon_1 \in \left(0, \eta_{(1,2)}(x_{20}) - \chi_{10} - \frac{\alpha_0(1,2)}{\beta_{20}}\right), & \text{if } \chi_{10} \geq 0, \\ \varepsilon_1 \in \left(0, \min\left\{0, \eta_{(1,2)}(x_{20})\right\} - \chi_{10} - \frac{\alpha_0(1,2)}{\beta_{20}}\right), & \text{if } \chi_{10} < 0, \end{cases}$$

equivalently $x_{11} = \chi_{10} + \frac{\alpha_0(1,2)}{\beta_{20}} + \varepsilon_1$. If we assume without loss of generality $\varepsilon := \varepsilon_1 - \varepsilon_2 > 0$, then the big-$\mathcal{O}$ in (8.3.40) as $m \to \infty$ satisfies

$$\mathcal{O}\left\{m^{-\chi_{10}-\frac{\alpha_0(1,2)}{\beta_{20}}-\min\{0,\chi_{20}\}-\varepsilon}\right\} = o\left\{m^{-\chi_{10}-\frac{\alpha_0(1,2)}{\beta_{20}}-\min\{0,\chi_{20}\}}\right\}. \tag{8.3.61}$$

In addition, the path of the remainder MB-integral in (8.3.35) for $j = 2$, $k = 1$ and $z_1 = 0$, by (8.3.37), can be managed to satisfy $x_{21} = \chi_{20} + \varepsilon$, provided x_{21} and $\varepsilon > 0$ are chosen appropriately. Similarly, at the same time $l_1 = \eta_{20} + \varepsilon$ and $k_1 = \chi_{20} + \varepsilon$ are admissible, compare (8.3.43) and (8.3.54). If $\beta_{10} = 0$ or $\beta_{20} = 0$, the corresponding remainder terms in each of the above expansions vanishes faster than any negative power of m as $m \to \infty$, i.e., it satisfies $\mathcal{O}\{m^{-q}\}$ for an arbitrary $q > 0$ and is therefore exponentially small. With these preliminaries, we are eventually ready to expose the asymptotic behaviour of (8.3.40). For a clearer presentation we distinguish between different parametrizations.

8.3.7.1. The case $\beta_{10} = 0$ or $\chi_{10} > 0$

For the assumed range of the parameters, the first summand in (8.3.40) is non-zero, whereas the third equals zero. Finally, the second summand vanishes if and only if $\beta_{10} = 0$. With $\beta_{20} \geq \beta_{10}$, upon collecting (8.3.35) and (8.3.44), as $m \to \infty$ we therefore deduce:

$$\mathrm{Si}\left[m; \begin{matrix} \tau_1, T_1 \\ \tau_2, T_2 \end{matrix}\right] = \mathcal{M}_0\begin{bmatrix} 0, \tau_1, T_1 \\ 0, \tau_2, T_2 \end{bmatrix} \mathbb{1}_{\{\beta_{20}=0 \vee \chi_{20}>0\}}$$

$$
\begin{aligned}
&- \frac{\Gamma(\chi_{20})}{\{b_{20}(m+1)\}^{\chi_{20}}} \frac{c_{20}}{\beta_{20}} \mathcal{L}\begin{bmatrix} 0, \tau_2 \\ \tau_1, T_1 \end{bmatrix} \mathbb{1}_{\{\chi_{20} \neq 0\}} \\
&+ \frac{1}{\beta_{20}} \left\{ (\log(m+1) + \gamma) c_{20} \mathcal{L}\begin{bmatrix} 0, \tau_2 \\ \tau_1, T_1 \end{bmatrix} + \mathcal{X}_0 \begin{bmatrix} 0, \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} \right\} \mathbb{1}_{\{\chi_{20} = 0\}} \\
&- \frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}} \frac{c_{10}}{\beta_{10}} \mathcal{L}_0 \begin{bmatrix} 0, \tau_1 \\ \tau_2, T_2 \end{bmatrix} \mathbb{1}_{\{\beta_{10} > 0,\, \eta_{20} > 0\}} \\
&+ \frac{\Gamma(\chi_{10})\Gamma(\eta_{20})}{\{b_{10}(m+1)\}^{\chi_{10}} \{b_{20}(m+1)\}^{\eta_{20}}} \frac{c_{10} a_0(1,2) c_{20}}{\beta_{10}\beta_{20}} \mathbb{1}_{\{\beta_{10} > 0,\, \eta_{20} \neq 0\}} \\
&- \frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}} (\log(m+1) + \gamma) \frac{c_{10} a_0(1,2) c_{20}}{\beta_{10}\beta_{20}} \mathbb{1}_{\{\beta_{10} > 0,\, \eta_{20} = 0\}} \\
&- \frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}} \frac{c_{10}}{\beta_{10}\beta_{20}} \lambda_0^0(\tau_1; \tau_2, T_2) \mathbb{1}_{\{\beta_{10} > 0,\, \eta_{20} = 0\}} \\
&+ \mathbb{1}_{\{\beta_{20} = 0\}} \mathcal{O}\left\{m^{-q}\right\} \\
&+ \mathbb{1}_{\{\beta_{20} > 0\}} \mathcal{O}\left\{m^{-\chi_{20} - \varepsilon}\right\} \\
&+ \mathbb{1}_{\{\beta_{20} \geq \beta_{10} > 0\}} \mathcal{O}\left\{m^{-\chi_{10} - \eta_{20} - \varepsilon}\right\} \\
&+ \mathbb{1}_{\{\beta_{20} \geq \beta_{10} > 0\}} \mathcal{O}\left\{m^{-\chi_{10} - \frac{\alpha_0(1,2)}{\beta_{20}} - \min\{0, \chi_{20}\} - \varepsilon}\right\}
\end{aligned}
$$

By inspection of the above expansion, the reader readily confirms the two statements below.

Theorem 8.3.1. *For $\beta_{10} = 0$ or $\chi_{10} > 0$, assume validity of the conditions (S1) to (S4) as well as (8.3.56) and (8.3.57). Then, provided at least one term on the right hand side is non-zero, as $m \to \infty$,*

(1) if $\beta_{20} = 0$ or $\chi_{20} > 0$, we have

$$
\begin{aligned}
\mathrm{Si}\left[m; \begin{matrix} \tau_1, T_1 \\ \tau_2, T_2 \end{matrix}\right] \sim{} & \mathcal{M}_0 \begin{bmatrix} 0, \tau_1, T_1 \\ 0, \tau_2, T_2 \end{bmatrix} \\
&- \frac{\Gamma(\chi_{20})}{\{b_{20}(m+1)\}^{\chi_{20}}} \frac{c_{20}}{\beta_{20}} \mathcal{L}\begin{bmatrix} 0, \tau_2 \\ \tau_1, T_1 \end{bmatrix} \left\{ \mathbb{1}_{\{\beta_{10} = 0,\, \chi_{20} > 0\}} + \mathbb{1}_{\{\chi_{20} \leq \chi_{10}\}} \right\} \\
&- \frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}} \frac{c_{10}}{\beta_{10}} \mathcal{L}_0 \begin{bmatrix} 0, \tau_1 \\ \tau_2, T_2 \end{bmatrix} \mathbb{1}_{\{\chi_{10} \leq \chi_{20}\}}.
\end{aligned}
$$

(2) if $\chi_{20} \leq 0$, we have

$$
\begin{aligned}
\mathrm{Si}\left[m; \begin{matrix} \tau_1, T_1 \\ \tau_2, T_2 \end{matrix}\right] \sim{} & - \frac{\Gamma(\chi_{20})}{\{b_{20}(m+1)\}^{\chi_{20}}} \frac{c_{20}}{\beta_{20}} \mathcal{L}\begin{bmatrix} 0, \tau_2 \\ \tau_1, T_1 \end{bmatrix} \mathbb{1}_{\{\chi_{20} < 0\}} \\
&+ \frac{1}{\beta_{20}} \left\{ (\log(m+1) + \gamma) c_{20} \mathcal{L}\begin{bmatrix} 0, \tau_2 \\ \tau_1, T_1 \end{bmatrix} + \mathcal{X}_0 \begin{bmatrix} 0, \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} \right\} \mathbb{1}_{\{\chi_{20} = 0\}}.
\end{aligned}
$$

The constants in each expansion were defined in (8.3.6), (8.3.13) and (8.3.32).

8.3.7.2. The case $\chi_{10} = 0$

In the present case, the first and the second summand in (8.3.40) both vanish. Upon representing the third summand in terms of the expansions (8.3.44) and (8.3.55), as $m \to \infty$ we obtain:

$$\begin{aligned}
\mathrm{Si}\left[m; \begin{matrix}\tau_1, T_1\\ \tau_2, T_2\end{matrix}\right] &= (\log(m+1)+\gamma)\frac{c_{10}}{\beta_{10}}\mathcal{L}_0\left[\begin{matrix}0, \tau_1\\ \tau_2, T_2\end{matrix}\right]\mathbb{1}_{\{\eta_{20}>0\}}\\
&- (\log(m+1)+\gamma)\frac{\Gamma(\eta_{20})}{\{b_{20}(m+1)\}^{\eta_{20}}}\frac{c_{10}a_0(1,2)c_{20}}{\beta_{10}\beta_{20}}\mathbb{1}_{\{\eta_{20}\neq 0\}}\\
&+ (\log(m+1)+\gamma)^2\frac{c_{10}a_0(1,2)c_{20}}{\beta_{10}\beta_{20}}\mathbb{1}_{\{\eta_{20}=0\}}\\
&+ (\log(m+1)+\gamma)\frac{c_{10}}{\beta_{10}\beta_{20}}\lambda_0^0(\tau_1;\tau_2,T_2)\mathbb{1}_{\{\eta_{20}=0\}}\\
&+ \frac{1}{\beta_{10}}\mathcal{X}_0\left[\begin{matrix}0, \tau_2, T_2\\ \tau_1, T_1\end{matrix}\right]\mathbb{1}_{\{\chi_{20}>0\}}\\
&- \frac{\Gamma(\chi_{20})}{\{b_{20}(m+1)\}^{\chi_{20}}}\frac{c_{20}}{\beta_{10}\beta_{20}}\xi_{-1}(\tau_2;\tau_1,T_1)\mathbb{1}_{\{\chi_{20}\neq 0\}}\\
&+ \frac{1}{\beta_{10}\beta_{20}}\left\{(\log(m+1)+\gamma)c_{20}\xi_{-1}(\tau_2;\tau_1,T_1) + \xi_0\left[\begin{matrix}\tau_1, T_1\\ \tau_2, T_2\end{matrix}\right]\right\}\mathbb{1}_{\{\chi_{20}=0\}}\\
&+ \mathcal{O}\left\{\log(m)m^{-\eta_{20}-\varepsilon}\right\}\\
&+ \mathcal{O}\left\{m^{-\chi_{20}-\varepsilon}\right\}\\
&+ \mathcal{O}\left\{m^{-\min\{0,\chi_{20}\}-\frac{\alpha_0(1,2)}{\beta_{20}}-\varepsilon}\right\}
\end{aligned}$$

Accordingly, Si[...] diverges at least logarithmically but no faster than algebraically. In particular, we can establish the following theorem.

Theorem 8.3.2. *For $\chi_{10} = 0$, assume validity of the conditions (S1) to (S5) as well as (8.3.56) and (8.3.57). Then, provided at least one term on the right hand side is non-zero, as $m \to \infty$,*

(1) if $\chi_{20} > 0$, we have

$$\mathrm{Si}\left[m; \begin{matrix}\tau_1, T_1\\ \tau_2, T_2\end{matrix}\right] \sim (\log(m+1)+\gamma)\frac{c_{10}}{\beta_{10}}\mathcal{L}_0\left[\begin{matrix}0, \tau_1\\ \tau_2, T_2\end{matrix}\right] + \frac{1}{\beta_{10}}\mathcal{X}_0\left[\begin{matrix}0, \tau_2, T_2\\ \tau_1, T_1\end{matrix}\right].$$

(2) if $\chi_{20} = 0$, we have

$$\begin{aligned}
\mathrm{Si}\left[m; \begin{matrix}\tau_1, T_1\\ \tau_2, T_2\end{matrix}\right] &\sim (\log(m+1)+\gamma)\frac{c_{10}}{\beta_{10}}\mathcal{L}_0\left[\begin{matrix}0, \tau_1\\ \tau_2, T_2\end{matrix}\right]\mathbb{1}_{\{\alpha_0(1,2)>0\}}\\
&+ (\log(m+1)+\gamma)^2\frac{c_{10}a_0(1,2)c_{20}}{\beta_{10}\beta_{20}}\mathbb{1}_{\{\alpha_0(1,2)=0\}}\\
&+ (\log(m+1)+\gamma)\frac{c_{10}}{\beta_{10}\beta_{20}}\lambda_0^0(\tau_1;\tau_2,T_2)\mathbb{1}_{\{\alpha_0(1,2)=0\}}\\
&+ \frac{1}{\beta_{10}\beta_{20}}\left\{(\log(m+1)+\gamma)c_{20}\xi_{-1}(\tau_2;\tau_1,T_1) + \xi_0\left[\begin{matrix}\tau_1, T_1\\ \tau_2, T_2\end{matrix}\right]\right\}.
\end{aligned}$$

(3) if $\chi_{20} < 0$*, we have*

$$\mathrm{Si}\left[m; \begin{matrix} \tau_1, T_1 \\ \tau_2, T_2 \end{matrix}\right] \sim -(\log(m+1)+\gamma)\frac{\Gamma(\eta_{20})}{\{b_{20}(m+1)\}^{\eta_{20}}}\frac{c_{10}a_0(1,2)c_{20}}{\beta_{10}\beta_{20}}\mathbb{1}_{\{\alpha_0(1,2)=0\}}$$
$$-\frac{\Gamma(\chi_{20})}{\{b_{20}(m+1)\}^{\chi_{20}}}\frac{c_{20}}{\beta_{10}\beta_{20}}\xi_{-1}(\tau_2;\tau_1,T_1).$$

The constants can be found in (8.3.13), (8.3.24), (8.3.32), (8.3.51) and (8.3.53).

8.3.7.3. The case $\chi_{10} < 0$

In this last case, the first and the third term in (8.3.40) are equal to zero. If we plug in the expansion (8.3.44), as $m \to \infty$ we arrive at:

$$\begin{aligned}
\mathrm{Si}\left[m; \begin{matrix} \tau_1, T_1 \\ \tau_2, T_2 \end{matrix}\right] = &-\frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}}\frac{c_{10}}{\beta_{10}}\mathcal{L}_0\left[\begin{matrix} 0, \tau_1 \\ \tau_2, T_2 \end{matrix}\right]\mathbb{1}_{\{\eta_{20}>0\}} \\
&+\frac{\Gamma(\chi_{10})\Gamma(\eta_{20})}{\{b_{10}(m+1)\}^{\chi_{10}}\{b_{20}(m+1)\}^{\eta_{20}}}\frac{c_{10}a_0(1,2)c_{20}}{\beta_{10}\beta_{20}}\mathbb{1}_{\{\eta_{20}\neq 0\}} \\
&-\frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}}(\log(m+1)+\gamma)\frac{c_{10}a_0(1,2)c_{20}}{\beta_{10}\beta_{20}}\mathbb{1}_{\{\eta_{20}=0\}} \\
&-\frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}}\frac{c_{10}}{\beta_{10}\beta_{20}}\lambda_0^0(\tau_1;\tau_2,T_2)\mathbb{1}_{\{\eta_{20}=0\}} \\
&+\mathcal{O}\left\{m^{-\chi_{10}-\eta_{20}-\varepsilon}\right\} \\
&+\mathcal{O}\left\{m^{-\chi_{10}-\frac{\alpha_0(1,2)}{\beta_{20}}-\min\{0,\chi_{20}\}-\varepsilon}\right\}
\end{aligned}$$

For each admissible value of $\eta_{20} \geq \chi_{20} > -1$ it is easy to specify the corresponding controlling term, which at least features algebraic growth. This gives rise to the last theorem of this section.

Theorem 8.3.3. *For* $\chi_{10} < 0$*, assume validity of the conditions (S1) to (S4) as well as (8.3.56) and (8.3.57). Then, provided at least one term on the right hand side is non-zero, as* $m \to \infty$*, we have*

$$\begin{aligned}
\mathrm{Si}\left[m; \begin{matrix} \tau_1, T_1 \\ \tau_2, T_2 \end{matrix}\right] \sim &-\frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}}\frac{c_{10}}{\beta_{10}}\mathcal{L}_0\left[\begin{matrix} 0, \tau_1 \\ \tau_2, T_2 \end{matrix}\right]\mathbb{1}_{\{\eta_{20}>0\}} \\
&+\frac{\Gamma(\chi_{10})\Gamma(\eta_{20})}{\{b_{10}(m+1)\}^{\chi_{10}}\{b_{20}(m+1)\}^{\eta_{20}}}\frac{c_{10}a_0(1,2)c_{20}}{\beta_{10}\beta_{20}}\mathbb{1}_{\{\eta_{20}<0\}} \\
&-\frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}}(\log(m+1)+\gamma)\frac{c_{10}a_0(1,2)c_{20}}{\beta_{10}\beta_{20}}\mathbb{1}_{\{\eta_{20}=0\}} \\
&-\frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}}\frac{c_{10}}{\beta_{10}\beta_{20}}\lambda_0^0(\tau_1;\tau_2,T_2)\mathbb{1}_{\{\eta_{20}=0\}}.
\end{aligned}$$

For the constants in the first and in the last term, we refer to (8.3.13) and (8.3.24).

8.4. An Infinite Interior Path in a Symmetric-Type Iterated Integral

In our second scenario we study (8.0.1) for an infinite interior path and an amplitude that admits oscillatory behaviour. For this purpose we introduce an additional argument and denote

$$\begin{aligned}\mathrm{Si}\left[m; \begin{matrix} i\xi_1, T_1, \infty \\ i\xi_2, \tau, T \end{matrix}\right] := \int_{\mathcal{P}_2} \left\{1-(1-\Psi(t))^{m+1}\right\} e^{-i\xi_2 t} c(t;p_2) \\ \times \int_{T_1}^{\infty} \left\{1-(1-\Psi(s))^{m+1}\right\} e^{-i\xi_1 s} \mathfrak{c}(s)\mathfrak{a}(s+t) ds dt,\end{aligned} \tag{8.4.1}$$

for $\xi_1, \xi_2 \in \mathbb{R}$, $T_1 > 0$ and an arbitrary half-open path $\mathcal{P}_2$ with endpoints $0 \leq \tau < T \leq \infty$. In particular, we assume either $\mathcal{P}_2 = (\tau_2, T_2]$ for $0 \leq \tau_2 < T_2 < \infty$ or $\mathcal{P}_2 = [T_2, \infty)$ for $T_2 \geq T_1$. The parameter $p_j \in \{\tau_j, \infty\}$ for $j \in \{1, 2\}$ then indicates the endpoint for which $\mathcal{P}_j \cup \{p_j\}$ is closed, and

$$c(t;p_j) := \begin{cases} c(t), & \text{if } p_j = \tau_j, \\ \mathfrak{c}(t), & \text{if } p_j = \infty. \end{cases} \tag{8.4.2}$$

In view of Section 6.6, we expect that the oscillatory terms yield special contributions at infinity. Again with φ as per definition (8.1.5), we assume:

(S6) The functions φ, $\mathfrak{a}$ and $\mathfrak{c}$ are once continuously differentiable on $[T_1, \infty)$ with $\varphi > 0$. Moreover, at infinity φ, $\mathfrak{a}$ and $\mathfrak{c}$ are of algebraic type with parameters $\beta_{00}, \alpha_{00} > 0$, $\gamma_{00} \in \mathbb{R}$ and coefficients $b_{00} > 0$, $a_{00}, c_{00} \in \mathbb{C} \setminus \{0\}$, which satisfy

$$\gamma_{00} + \beta_{00} > 1. \tag{8.4.3}$$

(S7) If $\xi_1 = 0$, or if $\mathcal{P}_2 = [T_2, \infty)$ with $\xi_2 = 0$, there exist $\beta_{01}, \alpha_{01}, \gamma_{01} > 1$, describing the order of the first derivative of $B(u)$, $\mathfrak{A}(u)$ and $\mathfrak{C}(u)$ as $u \to \infty$, where the capital letters denote the normalized ingredients, according to (8.2.8) and (8.2.9).

(S8) If $\xi_1 \neq 0$, we suppose:

- (a) $\Psi(t)$ is infinitely many times continuously differentiable on $t \geq T_1$ with $\frac{d^{j-1}}{dt^{j-1}} \frac{\varphi'(t)}{\varphi(t)} = \mathcal{O}\left\{t^{-j}\right\}$ for $j \in \mathbb{N}$ as $t \to \infty$.
- (b) $\mathfrak{a}(u)$ and $\mathfrak{c}(u)$ are infinitely many times continuously differentiable on $u \geq T_1$, and for any $j \in \mathbb{N}$ as $s \to \infty$ they satisfy $\mathfrak{a}^{(j)}(s) = \mathcal{O}\left\{s^{-\alpha_{00}-j}\right\}$ and $\mathfrak{c}^{(j)}(s) = \mathcal{O}\left\{s^{-\gamma_{00}-j}\right\}$.

(S9) If $\mathcal{P}_2 = (\tau_2, T_2]$, then φ and c are continuous there with $\varphi > 0$. Furthermore, each function shows algebraic behaviour as $t \downarrow \tau_2$ for parameters $\beta_{20} \geq 0$, $\gamma_{20} \in \mathbb{R}$ and coefficients $b_{20} > 0$, $c_{20} \in \mathbb{C} \setminus \{0\}$, where

$$\gamma_{20} + \beta_{20} > -1. \tag{8.4.4}$$

If $\beta_{20} > 0$, we have once continuous differentiability on $(\tau_2, T_2]$ and the normalized functions $B_2(t)$, $C_2(t)$ possess a first derivative of order $\beta_{21}, \gamma_{21} > -1$ as $t \downarrow \tau_2$.

(S10) If $\mathcal{P}_2 = (\tau_2, T_2]$ with $\beta_{20} > 0$, $\xi_1 = 0$ and $\alpha_{00} + \gamma_{00} = 1$, the function $\mathfrak{a}(u)$ is twice continuously differentiable on $u \geq T_1$, and as $u \to \infty$ it satisfies

$$\mathfrak{a}''(u) \sim a_{00}\alpha_{00}(\alpha_{00}+1)u^{-\alpha_{00}-2}. \tag{8.4.5}$$

Again we point out the occurence of possible zero quantities in some Laurent expansions below. Instead of mentioning this each time, we describe our steps for the more common non-zero cases and bear in mind that the final result will not become invalid, unless a statement of the form $\mathrm{Si}[\ldots] \sim 0$ is obtained.

From the assumption (S7), particularly by rearranging (8.2.13), as $u \to \infty$ it is easy to see that

$$\mathfrak{a}'(u) \sim -a_{00}\alpha_{00}u^{-\alpha_{00}-1}. \tag{8.4.6}$$

Of frequent use throughout this and later sections will be the parameter χ_{20} from (8.3.5) and the parameters

$$\begin{cases} \chi_{00} := \frac{\gamma_{00}-1}{\beta_{00}}, \\ \eta_{00} := \frac{\gamma_{00}-1}{\beta_{00}} + \frac{\alpha_{00}}{\beta_{00}}, \end{cases} \tag{8.4.7}$$

which by (8.4.3) and (8.4.4) clearly satisfy

$$\eta_{00} > \chi_{00} > -1, \tag{8.4.8}$$

$$\chi_{20} > -1. \tag{8.4.9}$$

Moreover, for $\delta_j \geq 0$ and $\xi_j \in \mathbb{R}$ with $j \in \{1, 2\}$, where $\delta_j > 0$ if and only if $\mathcal{P}_j$ is infinite with $\xi_j \neq 0$, we agree

$$\zeta_j := \delta_j + i\xi_j. \tag{8.4.10}$$

Then, if $\mathrm{sgn}(x) \in \{0, \pm 1\}$ indicates the sign of $x \in \mathbb{R}$, for $j \in \{1, 2\}$ we denote

$$\theta_j := \begin{cases} \mathrm{sgn}(\beta_{j0}), & \text{if } \mathcal{P}_j = (\tau_j, T_j], \\ \mathrm{sgn}(\delta_j), & \text{if } \mathcal{P}_j = [T_j, \infty), \end{cases} \tag{8.4.11}$$

and in case of an infinite $\mathcal{P}_j$ we conclude, $\theta_j = 1$ if and only if $\xi_j \neq 0$. In addition, we introduce the vector

$$\vec{p}_2 := (p_2, \theta_2). \tag{8.4.12}$$

To avoid duplicate cases during our examination of the integral (8.4.1), we assume

$$\vec{p}_2 \in \begin{cases} \{(\tau_2, 0), (\tau_2, 1), (\infty, 1), (\infty, 0)\}\,, & \text{if } \theta_1 = 1, \\ \{(\tau_2, 0), (\tau_2, 1), (\infty, 0)\}\,, & \text{if } \theta_1 = 0. \end{cases} \tag{8.4.13}$$

Finally, we first consider the iterated generating function

$$\mathcal{M}_0\begin{bmatrix} -w, \zeta_1, T_1, \infty \\ -z, \zeta_2, \tau, T \end{bmatrix} := \int\limits_{\mathcal{P}_2} \{\varphi(t)\}^{-z}\, e^{-\zeta_2 t} c(t; p_2) \int\limits_{T_1}^{\infty} \{\varphi(s)\}^{-w}\, e^{-\zeta_1 s} \mathfrak{c}(s)\mathfrak{a}(s+t)\,ds dt. \tag{8.4.14}$$

Regarding its convergence, we must distinguish between two situations. On the one hand, if $\mathcal{P}_2 = (\tau_2, T_2]$, or if $\mathcal{P}_2 = [T_2, \infty)$ but $\theta_1 = 1$, by Corollary 8.2.2 and Lemma 8.2.3, the above representation converges if and only if each of its single components does. This is the case for $w, z \in \mathbb{C}$ with

$$\Re z < \begin{cases} \infty, & \text{if } \vec{p}_2 \in \{(\tau_2, 0), (\infty, 1)\}\,, \\ \chi_{20}, & \text{if } \vec{p}_2 = (\tau_2, 1), \\ \eta_{00}, & \text{if } \vec{p}_2 = (\infty, 0) \wedge \theta_1 = 1, \end{cases} \tag{8.4.15}$$

$$\Re w < \begin{cases} \infty, & \text{if } \theta_1 = 1, \\ \eta_{00}, & \text{if } \theta_1 = 0 \wedge \vec{p}_2 \in \{(\tau_2, 0), (\tau_2, 1)\}\,. \end{cases} \tag{8.4.16}$$

On the other hand, if $\vec{p}_2 = (\infty, 0)$ and $\theta_1 = 0$, with the aid of Corollary 8.2.2, we readily verify absolute convergence for $w, z \in \mathbb{C}$ with

$$\begin{cases} \Re z < \eta_{00}, \\ \Re w < \eta_{00} + \min\{0, \chi_{00} - \Re z\}\,. \end{cases} \tag{8.4.17}$$

By Lemma 8.2.5, for fixed z the integral (8.4.14) especially is an analytic function in its w-region of convergence and vice versa.

8.4.1. Transformation to an Iterated MB-Integral

According to (8.1.1) and the criteria from Subsection 8.2.3, bearing in mind (8.4.3) and (8.4.4), we have absolute convergence of (8.4.1) for any fixed $m \geq 0$ and $\xi_1, \xi_2 \in \mathbb{R}$. Similarly, with $\zeta_j = \delta_j + i\xi_j$, compare (8.4.10), for $m \geq 0$ absolute and with respect to $\delta_1, \delta_2 \geq 0$ even uniform convergence also holds for

$$\mathcal{S}i\left[m;\begin{matrix}\zeta_1,T_1,\infty\\ \zeta_2,\tau,T\end{matrix}\right]:=\int\limits_{\mathcal{P}_2}\left\{1-(1-\Psi(t))^{m+1}\right\}e^{-\zeta_2 t}c(t;p_2) \times\int\limits_{T_1}^{\infty}\left\{1-(1-\Psi(s))^{m+1}\right\}e^{-\zeta_1 s}\mathfrak{c}(s)\mathfrak{a}(s+t)dsdt. \tag{8.4.18}$$

By Lebesgue's dominated convergence theorem,

$$\mathrm{Si}\left[m;\begin{matrix}\xi_1,T_1,\infty\\ \xi_2,\tau,T\end{matrix}\right]=\lim_{\delta_1,\delta_2\downarrow 0}\mathcal{S}i\left[m;\begin{matrix}\delta_1+i\xi_1,T_1,\infty\\ \delta_2+i\xi_2,\tau,T\end{matrix}\right]. \tag{8.4.19}$$

We first derive an asymptotic expansion for (8.4.18) with fixed arguments $\delta_j\geq 0$ for $j\in\{1,2\}$. For this, again we begin with an application of the Cahen-Mellin representation (8.1.6) for the m-power in the exterior integral. By Subsection 8.2.3 it is easy to verify, for

$$-1<x_0(\vec{p}_2)<\begin{cases}0, & \text{if } \vec{p}_2\in\{(\tau_2,0),(\infty,1)\},\\ \min\{0,\chi_{20}\}, & \text{if } \vec{p}_2=(\tau_2,1),\\ \min\{0,\eta_{00}\}, & \text{if } \vec{p}_2=(\infty,0),\end{cases} \tag{8.4.20}$$

by absolute convergence we can write

$$\mathcal{S}i\left[m;\begin{matrix}\zeta_1,T_1,\infty\\ \zeta_2,\tau,T\end{matrix}\right]=\frac{1}{2\pi i}\int\limits_{x_0(\vec{p}_2)-i\infty}^{x_0(\vec{p}_2)+i\infty}\frac{\Gamma(z)}{(m+1)^z}\int\limits_{\mathcal{P}_2}\{\varphi(t)\}^{-z}e^{-\zeta_2 t}c(t;p_2) \times\int\limits_{T_1}^{\infty}\left\{1-(1-\Psi(s))^{m+1}\right\}e^{-\zeta_1 s}\mathfrak{c}(s)\mathfrak{a}(s+t)dsdtdz. \tag{8.4.21}$$

The parameters χ_{20} and η_{00} evidently specify the abscissa of convergence of the integral transform, established by the $dsdt$-integral as a function of z for fixed $m\geq 0$ and $\vec{p}_2\in\{(\tau_2,1),(\infty,0)\}$, respectively.

In the particular situation $\vec{p}_2=(\infty,0)$ with $\theta_1=0$, we recall from (8.4.17) the minimum structure of the w-abscissa of convergence of the integral (8.4.14) for fixed z. In order that this minimum attains a unique value we note, since $\chi_{00}<\eta_{00}$ and due to (8.4.8), that it is possible instead of (8.4.20) to agree

$$-1<x_0(\infty,0)<\min\{0,\chi_{00}\}. \tag{8.4.22}$$

If we therefore introduce the parameter

$$\varsigma_0(\vec{p}_2) := \begin{cases} \chi_{20}, & \text{if } \vec{p}_2 = (\tau_2, 1), \\ \chi_{00}, & \text{if } \vec{p}_2 = (\infty, 0), \end{cases} \tag{8.4.23}$$

we may require the integration path in (8.4.21) to satisfy

$$-1 < x_0(\vec{p}_2) < \begin{cases} 0, & \text{if } \vec{p}_2 \in \{(\tau_2, 0), (\infty, 1)\}, \\ \min\{0, \varsigma_0(\vec{p}_2)\}, & \text{if } \vec{p}_2 \in \{(\tau_2, 1), (\infty, 0)\}. \end{cases} \tag{8.4.24}$$

A second application of the Cahen-Mellin representation, for

$$-1 < u_0(\theta_1) < \begin{cases} 0, & \text{if } \theta_1 = 1, \\ \min\{0, \eta_{00}\}, & \text{if } \theta_1 = 0, \end{cases} \tag{8.4.25}$$

then leads to

$$\begin{aligned} \mathcal{S}i\left[m; \begin{matrix} \zeta_1, T_1, \infty \\ \zeta_2, \tau, T \end{matrix}\right] &= \frac{1}{(2\pi i)^2} \int\limits_{x_0(\vec{p}_2)-i\infty}^{x_0(\vec{p}_2)+i\infty} \frac{\Gamma(z)}{(m+1)^z} \\ &\quad \times \int\limits_{u_0(\theta_1)-i\infty}^{u_0(\theta_1)+i\infty} \frac{\Gamma(w)}{(m+1)^w} \mathcal{M}_0\left[\begin{matrix} -w, \zeta_1, T_1, \infty \\ -z, \zeta_2, \tau, T \end{matrix}\right] dwdz. \end{aligned} \tag{8.4.26}$$

In our next step we compute the w-analytic continuation of the generating function $\mathcal{M}_0[\ldots]$ of this iterated MB-integral into a region that contains its w-abscissa of convergence. By (8.4.25), in the case $\Re z = x_0(\vec{p}_2)$ this is the line $\Re w = \eta_{00}$.

8.4.2. An Interior Generating Function with a Kernel of the First Kind, an Infinite Path and an Exponential Term

Preliminary we discuss some properties of the interior of the iterated integral (8.4.14). With fixed $n, q \in \mathbb{N}_0$, $S \geq T_1$, $t \geq 0$ and $\zeta \in \mathbb{C}$, we introduce

$$\mathcal{N}_0\left[\begin{matrix} -w, t, \zeta \\ n, q, S \end{matrix}\right] = \sum_{k=0}^{n} \int\limits_{S}^{\infty} \{\varphi(s)\}^{-w} e^{-\zeta s} S_{k,n}(s, w) \mathfrak{a}^{(q+k)}(s+t) ds \tag{8.4.27}$$

where in terms of (6.6.9) we define

$$S_{k,n}(s, w) := \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} P_j(s, w) \mathfrak{c}^{(n-k-j)}(s), \tag{8.4.28}$$

which satisfies $S_{0,0}(s,w) = \mathfrak{c}(s)$. Clearly, $\mathcal{N}_0[\dots]$ for $n = q = \zeta = 0$, by assumption, converges absolutely and is analytic in the half plane $\Re w < \eta_{00}$. Below, we will then specify the associated analytic continuation. Conversely, $\mathcal{N}_0[\dots]$ for $\Re\zeta > 0$ and $n = 0$ represents an entire function with respect to w. In these circumstances for $\Im\zeta \neq 0$ we will derive an expansion, that is uniformly bounded with respect to $\Re\zeta \geq 0$ and shows the convergence as $\Re\zeta \downarrow 0$ to a function of w, which is holomorphic in an arbitrary subregion of $\mathbb{C}$.

8.4.2.1. A Non-Exponential Amplitude

For a treatment of the case $\zeta = 0$, with $q_1, q_2 \in \{0,1\}$ and fixed $t \geq 0$, we additionally define

$$\mathrm{N}_0\begin{bmatrix} w-\chi, t \\ \eta, q_1, q_2, w, S \end{bmatrix} := \int_S^\infty \frac{s^{\beta_{00}(w-\chi)-1}}{(s+t)^{1-\eta}} \frac{d^{q_1}}{ds^{q_1}} \left\{ e^{wB(s)} \mathfrak{C}(s) \right\} \mathfrak{A}^{(q_2)}(s+t) ds. \tag{8.4.29}$$

With respect to $w \in \mathbb{C}$ for fixed $\eta \in \mathbb{C}$, $\chi \in \mathbb{R}$ and $q_1 = q_2 = 0$, this integral is holomorphic in $\Re w < \chi + \frac{1-\Re\eta}{\beta_{00}}$. Furthermore, it satisfies the identity

$$\mathcal{N}_0\begin{bmatrix} -w, t, 0 \\ 0, 0, S \end{bmatrix} = \mathrm{N}_0\begin{bmatrix} w-\chi_{00}, t \\ 1-\alpha_{00}, 0, 0, w, S \end{bmatrix}. \tag{8.4.30}$$

If we suppose validity of (S7), the continuation of $\mathrm{N}_0[\dots]$ can be specified by reference to §8.2.2.2. Accordingly, by (8.2.40), for $q_1 = q_2 = 0$ partial integration yields

$$\begin{aligned} \mathrm{N}_0\begin{bmatrix} w-\chi, t \\ \eta, 0, 0, w, S \end{bmatrix} &= -\frac{S^{\beta_{00}(w-\chi)}}{\beta_{00}\left(w-\chi-\frac{1-\eta}{\beta_{00}}\right)} e^{wB(S)} \mathfrak{C}(S) \frac{\mathfrak{A}(S+t)}{(S+t)^{1-\eta}} \\ &\quad - \frac{1}{\beta_{00}\left(w-\chi-\frac{1-\eta}{\beta_{00}}\right)} \Bigg\{ \sum_{\substack{n_1,n_2\in\{0,1\} \\ n_1+n_2=1}} \mathrm{N}_0\begin{bmatrix} w-\chi+\frac{1}{\beta_{00}}, t \\ \eta, n_1, n_2, w, S \end{bmatrix} \\ &\qquad + t(1-\eta)\,\mathrm{N}_0\begin{bmatrix} w-\chi, t \\ \eta-1, 0, 0, w, S \end{bmatrix} \Bigg\}. \end{aligned} \tag{8.4.31}$$

By (8.2.41), the expansion on the right hand side converges absolutely and represents a meromorphic function in the half plane

$$\Re w < \chi + \frac{1-\Re\eta}{\beta_{00}} + \eta_{\beta_{00}}(\alpha_{01}, \beta_{01}, \gamma_{01}, 2). \tag{8.4.32}$$

For $\chi = \chi_{00}$ and $\eta = 1-\alpha_{00}$, this coincides with the region $\Re w < \eta_{01}$, where we denote

$$\eta_{01} := \eta_{00} + \eta_{\beta_{00}}(\alpha_{01}, \beta_{01}, \gamma_{01}, 2). \tag{8.4.33}$$

The function $\mathrm{N}_0[\ldots]$ then exhibits a singularity only at $w = \eta_{00}$, which is a pole of simple order. According to (8.2.42), its residue is given by

$$\operatorname*{Res}_{w=\eta_{00}} \mathrm{N}_0 \begin{bmatrix} w - \chi_{00}, t \\ 1 - \alpha_{00}, 0, 0, w, S \end{bmatrix} = -\frac{a_{00} c_{00}}{\beta_{00}} \{b_{00}\}^{-\eta_{00}}. \tag{8.4.34}$$

Furthermore, if $\chi_{00} = -\frac{\alpha_{00}}{\beta_{00}}$ or equivalently $\alpha_{00} + \gamma_{00} = 1$, there exists a Laurent expansion in an annulus around the origin of the w-plane, which is by (8.2.43) of the form

$$\mathrm{N}_0 \begin{bmatrix} w - \chi_{00}, t \\ 1 - \alpha_{00}, 0, 0, w, S \end{bmatrix} = -\frac{1}{w} \frac{a_{00} c_{00}}{\beta_{00}} + \frac{1}{\beta_{00}} \{\nu_1(t; S) + t\nu_2(t; S)\} + \mathcal{O}(w), \tag{8.4.35}$$

where, according to (8.2.45) and by definition of the normalized ingredients, we have

$$\begin{aligned} \nu_1(t; S) := {} & S^{\alpha_{00}} \log\{\varphi(S)\} \mathfrak{C}(S) \mathfrak{a}(S + t) \\ & - \int_S^\infty s^{\alpha_{00}} \left\{B'(s)\mathfrak{C}(s) - \log\{\varphi(s)\} \mathfrak{C}'(s)\right\} \mathfrak{a}(s + t) ds \\ & + \int_S^\infty s^{\alpha_{00}} \log\{\varphi(s)\} \mathfrak{C}(s) \frac{\mathfrak{A}'(s + t)}{(s + t)^{\alpha_{00}}} ds, \end{aligned} \tag{8.4.36}$$

$$\nu_2(t; S) := \alpha_{00} \int_S^\infty s^{\alpha_{00}-1} \log\{\varphi(s)\} \mathfrak{C}(s) \frac{\mathfrak{a}(s + t)}{s + t} ds. \tag{8.4.37}$$

These functions are, by Lebesgue's dominated convergence theorem, continuous with respect to $t \geq 0$. By means of $\nu_2(t; S)$, we will now briefly explain why they are even once differentiable, particularly under the sign of integration. For this we first note, for fixed $s \geq S$ as a function of $t \geq 0$ the integrand of $\nu_2(t; S)$, by assumption, is once continuously differentiable with

$$s^{\alpha_{00}-1} \log\{\varphi(s)\} \mathfrak{C}(s) \frac{d}{dt} \frac{\mathfrak{a}(s + t)}{s + t} = s^{\alpha_{00}-1} \log\{\varphi(s)\} \mathfrak{C}(s) \left\{\frac{\mathfrak{a}'(s + t)}{s + t} - \frac{\mathfrak{a}(s + t)}{(s + t)^2}\right\}.$$

Hence, due to the logarithmic divergence of $\log\{\varphi(s)\}$ at infinity and by (8.4.6), for $t \geq 0$ and arbitrary $\varepsilon > 0$, as $s \to \infty$ we obtain the following estimate:

$$\begin{aligned} \left| s^{\alpha_{00}-1} \log\{\varphi(s)\} \mathfrak{C}(s) \frac{d}{dt} \frac{\mathfrak{a}(s + t)}{s + t} \right| & \leq \max_{u \geq S} \left| \log\{\varphi(u)\} \frac{\mathfrak{C}(u)}{u^\varepsilon} \right| \\ & \qquad \times \max_{v \geq S} \left| \frac{\mathfrak{a}'(v)}{v^{-\alpha_{00}-1}} - \frac{\mathfrak{a}(v)}{v^{-\alpha_{00}}} \right| \frac{s^{\varepsilon + \alpha_{00} - 1}}{(s + t)^{\alpha_{00}+2}} \\ & \leq \text{const} \times s^{\varepsilon - 3} \end{aligned}$$

This bound holds uniformly with respect to $t \geq 0$ and is absolutely integrable on the ray $s \geq S$. Similar arguments apply for the integrand of $\nu_1(t; S)$, provided $a''(u)$ is continuous and

$\mathcal{O}\left\{u^{-\alpha_{00}-2}\right\}$ as $u \to \infty$, in which case we have

$$\begin{aligned} \mathfrak{A}'(u) &= u^{\alpha_{00}}\left\{\mathfrak{a}'(u) + \alpha_{00}\frac{\mathfrak{a}(u)}{u}\right\} = \mathcal{O}\left\{u^{-\alpha_{01}}\right\}, \\ \mathfrak{A}''(u) &= u^{\alpha_{00}}\left\{2\alpha_{00}\frac{\mathfrak{a}'(u)}{u} + \alpha_{00}(\alpha_{00}-1)\frac{\mathfrak{a}(u)}{u^2} + a''(u)\right\} = \mathcal{O}\left\{u^{-2}\right\}. \end{aligned} \tag{8.4.38}$$

By Theorem 11.62 in [Körner, 2004], this confirms the computability of the first derivative of $\nu_j(t;S)$ for $j \in \{1,2\}$ by differentiation under the integral sign, leading to

$$\begin{aligned} \nu_1'(t;S) = {} & S^{\alpha_{00}} \log\{\varphi(S)\}\,\mathfrak{C}(S)\mathfrak{a}'(S+t) \\ & - \int_S^\infty s^{\alpha_{00}}\left\{B'(s)\mathfrak{C}(s) - \log\{\varphi(s)\}\,\mathfrak{C}'(s)\right\}\mathfrak{a}'(s+t)ds \\ & + \int_S^\infty s^{\alpha_{00}} \log\{\varphi(s)\}\,\mathfrak{C}(s)\left\{\frac{\mathfrak{A}''(s+t)}{(s+t)^{\alpha_{00}}} - \alpha_{00}\frac{\mathfrak{A}'(s+t)}{(s+t)^{\alpha_{00}+1}}\right\}ds, \end{aligned} \tag{8.4.39}$$

$$\nu_2'(t;S) = \alpha_{00}\int_S^\infty s^{\alpha_{00}-1}\log\{\varphi(s)\}\,\mathfrak{C}(s)\left\{\frac{\mathfrak{a}'(s+t)}{s+t} - \frac{\mathfrak{a}(s+t)}{(s+t)^2}\right\}ds. \tag{8.4.40}$$

Evidently, again by Lebesgue's dominated convergence theorem, these derivatives are continuous with respect to $t \geq 0$.

8.4.2.2. An Exponential Amplitude

In order to verify, under the assumption (S8), that the integral (8.4.27) for $n = 0$ in the case $\Im\zeta \neq 0$ approaches a finite limit as $\Re\zeta \downarrow 0$, we first note that a repeated application of the Leibniz rule for $n, q \in \mathbb{N}_0$, analogous to (6.6.10), in terms of (6.6.9) and (8.4.28), yields

$$\frac{d^n}{ds^n}\left[\{\varphi(s)\}^{-w}\,\mathfrak{c}(s)\mathfrak{a}^{(q)}(s+t)\right] = \{\varphi(s)\}^{-w}\sum_{k=0}^{n}\mathfrak{a}^{(q+k)}(s+t)S_{k,n}(s,w). \tag{8.4.41}$$

By definition of $P_j(s,w)$ in (6.6.9), the functions $S_{k,n}(s,w)$ are polynomials of $w \in \mathbb{C}$ with degree $n-k$. Their coefficients depend on the variable s only, they are continuous on $s \geq S$ and, according to our assumptions, they satisfy $\mathcal{O}\left\{s^{-\gamma_{00}+k-n}\right\}$ as $s \to \infty$. The k-sum in (8.4.41) is thus again a polynomial of $w \in \mathbb{C}$ with degree n, and its coefficients exhibit the asymptotic behaviour $\mathcal{O}\left\{s^{-\alpha_{00}-\gamma_{00}-n-q}\right\}$ as $s \to \infty$ for any fixed $t \geq 0$. Furthermore, since $\varphi(s) \sim b_{00}s^{-\beta_{00}}$, for $w \in \mathbb{C}$ and $0 \leq k \leq n$, as $s \to \infty$ for fixed $t \geq 0$ or as $t \to \infty$ for fixed $s \geq S$, we conclude

$$\{\varphi(s)\}^{-w}\,\mathfrak{a}^{(q+k)}(s+t)S_{k,n}(s,w) = \mathcal{O}\left\{\frac{s^{\beta_{00}\Re w-\gamma_{00}+k-n}}{(s+t)^{\alpha_{00}+q+k}}\right\}. \tag{8.4.42}$$

Therefore, upon choosing a fixed but arbitrary $\psi \in \mathbb{R}$, and

$$n_0 > 1 + \psi\beta_{00} - \alpha_{00} - \gamma_{00} - q, \tag{8.4.43}$$

the n-th derivative (8.4.41) for $n \geq n_0$ is absolutely integrable on the ray $s \geq S$ for any $w \in \mathbb{C}$ with $\Re w < \psi$ and $t \geq 0$. As a consequence, the integral (8.4.27) for $n \geq n_0$ and $\Re w < \psi$ converges absolutely and uniformly with respect to $\zeta \in \mathbb{C}$ in $\Re\zeta \geq 0$ for any $q \in \mathbb{N}_0$.

Now, from N-times integration by parts, for $q \in \mathbb{N}_0$, $t \geq 0$, $N \in \mathbb{N}$, $\Re w < \psi$ and $\Re\zeta > 0$, we obtain

$$\mathcal{N}_0\begin{bmatrix} -w, t, \zeta \\ 0, q, S \end{bmatrix} = \sum_{n=0}^{N-1} \frac{e^{-\zeta S}}{\zeta^{1+n}} \{\varphi(S)\}^{-w} \sum_{k=0}^{n} S_{k,n}(S,w)\mathfrak{a}^{(q+k)}(S+t) + \frac{1}{\zeta^N}\mathcal{N}_0\begin{bmatrix} -w, t, \zeta \\ N, q, S \end{bmatrix}. \tag{8.4.44}$$

According to the above observations, if $N \geq n_0$ and $\Im\zeta \in \mathbb{R}\backslash\{0\}$, this expansion, and particularly its remainder integral, is uniformly bounded with respect to $\Re\zeta \geq 0$. Moreover, it is easy to see that the limit as $\Re\zeta \downarrow 0$ of each summand then exists. Since the limit in the remainder integral may be carried out under the sign of integration, it shows

$$\begin{aligned} \lim_{\Re\zeta\downarrow 0} \mathcal{N}_0\begin{bmatrix} -w, t, \zeta \\ 0, q, S \end{bmatrix} = \sum_{n=0}^{N-1} \frac{e^{-i\Im\zeta S}}{(i\Im\zeta)^{1+n}} \{\varphi(S)\}^{-w} \sum_{k=0}^{n} S_{k,n}(S,w)\mathfrak{a}^{(q+k)}(S+t) \\ + \frac{1}{(i\Im\zeta)^N}\mathcal{N}_0\begin{bmatrix} -w, t, i\Im\zeta \\ N, q, S \end{bmatrix}. \end{aligned} \tag{8.4.45}$$

Clearly, each term in the preceding sum is an entire function of w. Moreover, with $N \geq n_0$ the remainder integral can be verified to converge absolutely and uniformly in any compact subset of the half plane $\Re w < \psi$, because

$$\psi < \eta_{00} + \frac{q+n_0}{\beta_{00}}. \tag{8.4.46}$$

By arbitrariness of ψ we conclude that the right hand side of (8.4.45) represents a function which is holomorphic in any subregion of the w-plane.

8.4.3. w-Analytic Continuation of the Iterated Generating Function

Recall that $\xi_1 = 0$ implies $\theta_1 = 0$. By comparison with (8.4.16) and (8.4.17) we see that then, except if $\vec{p}_2 = (\infty, 0)$ and $\chi_{00} \leq \Re z < \eta_{00}$, the w-abscissa of convergence of the iterated integral (8.4.14) originates in the condition for the convergence of a single component. Accordingly, if in terms of (8.4.29) we write

$$\mathcal{M}_0\begin{bmatrix} -w, 0, T_1, \infty \\ -z, \zeta_2, \tau, T \end{bmatrix} = \int_{\mathcal{P}_2} \{\varphi(t)\}^{-z} e^{-\zeta_2 t} c(t;p_2)\, \mathrm{N}_0\begin{bmatrix} w-\chi_{00}, t \\ 1-\alpha_{00}, 0, 0, w, T_1 \end{bmatrix} dt, \tag{8.4.47}$$

the w-analytic continuation for fixed z can be obtained by a simple application of the expansion (8.4.31). However, in view of the convergence criteria for iterated integrals, we must distinguish between a finite and an infinite exterior path. Furthermore, despite the case of an infinite path $\mathcal{P}_2$ with $\theta_1 = 0$, by (8.4.13), implies $\theta_2 = 0$, for later purposes we also specify the analytic continuation of the above integral for $\theta_2 = 1$

8.4.3.1. A Finite Path $\mathcal{P}_2$, or an Infinite Path $\mathcal{P}_2$ with $\xi_2 \neq 0$

It was pointed out in the introductory part of this section, that integrals of the form (8.4.47) in case of a finite $\mathcal{P}_2$ or in case of an infinite $\mathcal{P}_2$ with $\xi_2 \neq 0$, converge absolutely, if and only if each of their single components does. Their convergence is then even uniform in any compact subset of their region of absolute convergence, whence they are holomorphic with respect to one variable with the second variable fixed. But if we plug (8.4.31) into (8.4.47), we arrive at an expansion in terms of a finite number of integrals, of which each iterated integral again is of the same type as (8.4.47) itself. From §8.4.2.1 we therefore conclude, that (8.4.47) via partial integration for fixed $z \in \mathbb{C}$, arbitrary if $\vec{p}_2 \in \{(\tau_2, 0), (\infty, 1)\}$ or with $\Re z < \chi_{20}$ if $\vec{p}_2 = (\tau_2, 1)$, can be extended to the half plane

$$\Re w < \eta_{01}. \tag{8.4.48}$$

Therein it is analytic with the exception of a simple pole at $w = \eta_{00}$. Define for a half open path $\mathcal{P}_2$ with endpoints $0 \leq \tau < T \leq \infty$, and with $c(t; p_2)$ and $\nu_j(t; T_1)$ according to (8.4.2), (8.4.36) and (8.4.37), the integral transforms

$$\mathcal{P}_0 \begin{bmatrix} -z, \zeta_2 \\ \tau, T \end{bmatrix} := \int_{\mathcal{P}_2} \{\varphi(t)\}^{-z} e^{-\zeta_2 t} c(t; p_2) dt, \tag{8.4.49}$$

$$\mathcal{Q}_0 \begin{bmatrix} -z, \zeta_2 \\ \tau, T, T_1 \end{bmatrix} := \int_{\mathcal{P}_2} \{\varphi(t)\}^{-z} e^{-\zeta_2 t} c(t; p_2) \{\nu_1(t; T_1) + t\nu_2(t; T_1)\} dt. \tag{8.4.50}$$

Then, by virtue of (8.4.34), from (8.4.47) we compute

$$\operatorname*{Res}_{w=\eta_{00}} \mathcal{M}_0 \begin{bmatrix} -w, 0, T_1, \infty \\ -z, \zeta_2, \tau, T \end{bmatrix} = -\frac{a_{00} c_{00}}{\beta_{00}} \{b_{00}\}^{-\eta_{00}} \mathcal{P}_0 \begin{bmatrix} -z, \zeta_2 \\ \tau, T \end{bmatrix}. \tag{8.4.51}$$

If $\eta_{00} = 0$, the point $w = \eta_{00}$ coincides with the origin of the w-plane. In these circumstances, by (8.4.35), the Laurent expansion as $w \to 0$ can most concisely be described in the form

$$\mathcal{M}_0 \begin{bmatrix} -w, 0, T_1, \infty \\ -z, \zeta_2, \tau, T \end{bmatrix} = -\frac{1}{w} \frac{a_{00} c_{00}}{\beta_{00}} \mathcal{P}_0 \begin{bmatrix} -z, \zeta_2 \\ \tau, T \end{bmatrix} + \frac{1}{\beta_{00}} \mathcal{Q}_0 \begin{bmatrix} -z, \zeta_2 \\ \tau, T, T_1 \end{bmatrix} + \mathcal{O}(w). \tag{8.4.52}$$

8.4.3.2. An Infinite Path $\mathcal{P}_2$ with $\xi_2 = 0$ and Fixed $\Re z < \chi_{00}$

From (8.4.17) we ascertain, for $\vec{p}_2 = (\infty, 0)$, $\theta_1 = 0$ and fixed $\Re z < \chi_{00}$, that the w-region of analyticity of (8.4.47) coincides with the half plane $\Re w < \eta_{00}$, whose right boundary is prescribed by the behaviour at infinity of the amplitude $\mathfrak{a}(s+t)$. Hence, to determine the associated analytic continuation, we can rely on §8.4.2.1 again. For brevity we define

$$\begin{aligned}(8.4.53)\qquad \mathcal{M}_1\begin{bmatrix}-w, T_1\\ -z, T_2\end{bmatrix} :=& T_1^{\beta_{00}(w-\chi_{00})} e^{wB(T_1)}\mathfrak{C}(T_1)\int\limits_{T_2}^{\infty}\{\varphi(t)\}^{-z}\,\mathfrak{c}(t)\frac{\mathfrak{A}(T_1+t)}{(T_1+t)^{\alpha_{00}}}dt\\ &+\sum_{\substack{n_1,n_2\in\{0,1\}\\ n_1+n_2=1}}\int\limits_{T_2}^{\infty}\{\varphi(t)\}^{-z}\,\mathfrak{c}(t)\,\mathrm{N}_0\begin{bmatrix}w+\frac{1}{\beta_{00}}-\chi_{00}, t\\ 1-\alpha_{00}, n_1, n_2, w, T_1\end{bmatrix}dt\\ &+\alpha_{00}\int\limits_{T_2}^{\infty}\{\varphi(t)\}^{-z}\,t\mathfrak{c}(t)\,\mathrm{N}_0\begin{bmatrix}w-\chi_{00}, t\\ -\alpha_{00}, 0, 0, w, T_1\end{bmatrix}dt.\end{aligned}$$

By Lemma 8.2.5, we then have absolute convergence and analyticity of each integral in the above sum, if the variables satisfy the conditions of Table 8.3. In the first column, we enumerated the four summands of the expansion in their subsequent order. The second column shows the sufficient condition for the absolute convergence of each exterior integral for fixed arbitrary $s \geq T_1$. Furthermore, the third column provides these conditions for each interior integral for fixed arbitrary $t \geq T_2$. Finally, the fourth column shows the supplementary condition for the convergence of the iteration.

#	**exterior**	**interior**	**iterated**
1	$\Re z < \eta_{00}$	–	–
2	$\Re z < \eta_{00}$	$\Re w < \eta_{00} + \eta_{\beta_{00}}(\beta_{01}, \gamma_{01})$	$\Re w < \eta_{00} + \chi_{00} - \Re z + \eta_{\beta_{00}}(\beta_{01}, \gamma_{01})$
3	$\Re z < \eta_{00} + \frac{\alpha_{01}}{\beta_{00}}$	$\Re w < \eta_{00} + \eta_{\beta_{00}}(\alpha_{01})$	$\Re w < \eta_{00} + \chi_{00} - \Re z + \eta_{\beta_{00}}(\alpha_{01})$
4	$\Re z < \eta_{00}$	$\Re w < \eta_{00} + \frac{1}{\beta_{00}}$	$\Re w < \eta_{00} + \chi_{00} - \Re z$

Table 8.3.: Table of absolute convergence for the integrals in (8.4.53). Each row corresponds to one of the summands in the expansion.

Therefore, upon applying the expansion (8.4.31) to the integral (8.4.47), for fixed $\Re z < \chi_{00}$ it follows, that

$$(8.4.54)\qquad \mathcal{M}_0\begin{bmatrix}-w, 0, T_1, \infty\\ -z, 0, T_2, \infty\end{bmatrix} = -\frac{1}{\beta_{00}(w-\eta_{00})}\mathcal{M}_1\begin{bmatrix}-w, T_1\\ -z, T_2\end{bmatrix}$$

represents the analytic continuation of $\mathcal{M}_0[\ldots]$ into the wider half plane

$$(8.4.55)\qquad \Re w < \min\{\eta_{01}, \eta_{00} + \chi_{00} - \Re z\},$$

where η_{01} was defined in (8.4.33). The only singularity, which the function reveals in the extended region, is a simple pole at $w=\eta_{00}$. In terms of (8.4.49), according to (8.4.34), we have

$$\operatorname*{Res}_{w=\eta_{00}} \mathcal{M}_0\begin{bmatrix}-w,0,T_1,\infty\\-z,0,T_2,\infty\end{bmatrix} = -\frac{a_{00}c_{00}}{\beta_{00}}\{b_{00}\}^{-\eta_{00}}\,\mathcal{P}_0\begin{bmatrix}-z,0\\T_2,\infty\end{bmatrix}. \tag{8.4.56}$$

If $\eta_{00}=0$, this point lies at $w=0$. In this event, for later applications it is helpful to note, that the expansion (8.4.35) as $w\to 0$ shows

$$\mathcal{M}_0\begin{bmatrix}-w,0,T_1,\infty\\-z,0,T_2,\infty\end{bmatrix} = -\frac{1}{w}\frac{a_{00}c_{00}}{\beta_{00}}\mathcal{P}_0\begin{bmatrix}-z,0\\T_2,\infty\end{bmatrix} + \frac{1}{\beta_{00}}\mathcal{Q}_0\begin{bmatrix}-z,0\\T_2,\infty,T_1\end{bmatrix} + \mathcal{O}(w), \tag{8.4.57}$$

where in terms of (8.4.36) and (8.4.37) the function in the constant term was defined in (8.4.50).

8.4.3.3. An Infinite Path $\mathcal{P}_2$ with $\xi_2 = 0$ and Fixed $\chi_{00} \le \Re z < \eta_{00}$

For completeness we conclude this subsection with the derivation of the analytic continuation of the generating function (8.4.14) for $\vec{p}_2=(\infty,0)$ and fixed $z\in\mathbb{C}$ with $\chi_{00}\le\Re z<\eta_{00}$. In this case, we conclude from (8.4.17) absolute convergence and analyticity of the indicated integral in the half plane

$$\Re w < \eta_{00}+\chi_{00}-\Re z, \tag{8.4.58}$$

which furnishes a proper or improper subregion of $\Re w<\eta_{00}$. By Corollary 8.2.2, the abscissa of convergence is due to the supplementary condition for the convergence of the iterated integral. To access a wider region, we must therefore rely on our findings from §8.2.6.2. For this purpose we identify

$$\begin{cases} d(s) & \equiv \mathfrak{c}(s),\\ k(s+t) & \equiv \mathfrak{a}(s+t),\\ e(t) & \equiv \{\varphi(t)\}^{-z}\,\mathfrak{c}(t),\end{cases}$$

and, by comparison with (8.2.98), we infer $\beta_0\equiv\beta_{00}$, $\chi_0\equiv\chi_{00}$, $\varsigma_0\equiv\chi_{00}-z$ and $\kappa_0\equiv\alpha_{00}$. Note, for fixed $z\in\mathbb{C}$ subject to $\chi_{00}\le\Re z<\eta_{00}$, these parameters indeed satisfy (8.2.99), which immediately enables us to employ the result from the indicated paragraph. According to (8.2.128), the analytic continuation of the iterated generating function (8.4.14) to the strip

$$\chi_{00}<\Re w<\eta_{00}+\chi_{00}-\Re z+\eta_{\beta_{00}}(\alpha_{01},\beta_{01},\gamma_{01}), \tag{8.4.59}$$

by (8.2.122), is thus represented by the expansion

$$\begin{aligned}\mathcal{M}_0\begin{bmatrix}-w,0,T_1,\infty\\-z,0,T_2,\infty\end{bmatrix} &= -\frac{a_{00}c_{00}}{\beta_{00}}\frac{\Gamma(\beta_{00}(\eta_{00}-w))\Gamma(\beta_{00}(w-\chi_{00}))}{\{b_{00}\}^{w}\,\Gamma(\alpha_{00})(w+z-\eta_{00}-\chi_{00})}\Pi_1\begin{bmatrix}w+z-\chi_{00}-\eta_{00}\\z,T_2\end{bmatrix}\\ &\quad + \mathrm{M}_1\begin{bmatrix}-w,T_1\\-z,T_2\end{bmatrix}.\end{aligned} \tag{8.4.60}$$

By (8.2.117), the integral function in the first summand is given by

$$\Pi_1\left[\begin{matrix}\xi\\ z, T_2\end{matrix}\right] := T_2^{\beta_{00}\xi} e^{zB(T_2)}\mathfrak{C}(T_2) + \int_{T_2}^{\infty} t^{\beta_{00}\xi} e^{zB(t)} \left\{zB'(t)\mathfrak{C}(t) + \mathfrak{C}'(t)\right\} dt. \tag{8.4.61}$$

With $\xi = w + z - \eta_{00}$, the latter converges absolutely and establishes a holomorphic function in the region

$$\Re w < \eta_{00} + \chi_{00} - \Re z + \begin{cases}\eta_{\beta_{00}}(\beta_{01}, \gamma_{01}), & \text{if } z \neq 0,\\ \frac{\gamma_{01}-1}{\beta_{00}}, & \text{if } z = 0,\end{cases} \tag{8.4.62}$$

where as $\Im w \to \pm\infty$, uniformly with respect to $\Re w$ in any closed vertical substrip, and if $z \neq 0$ uniformly with respect to $\Im z \in \mathbb{R}$, we observe

$$\Pi_1\left[\begin{matrix}w + z - \chi_{00} - \eta_{00}\\ z, T_2\end{matrix}\right] = \begin{cases}\mathcal{O}\left\{|z|\right\}, & \text{if } z \neq 0,\\ \mathcal{O}(1), & \text{if } z = 0.\end{cases} \tag{8.4.63}$$

In addition, by (8.2.115) the MB-integral in the second summand in (8.4.60) equals

$$\mathrm{M}_1\left[\begin{matrix}-w, T_1\\ -z, T_2\end{matrix}\right] := \frac{1}{2\pi i}\int_{q-i\infty}^{q+i\infty} \frac{\Gamma(\zeta)\Gamma(\alpha_{00} - \zeta)}{\Gamma(\alpha_{00})}\mathcal{L}\left[\zeta; \begin{matrix}w, T_1\\ z, T_2\end{matrix}\right] d\zeta. \tag{8.4.64}$$

Its integration path is a vertical line, whose real part $\Re\zeta = q$ satisfies

$$\begin{cases} q > \max\left\{\beta_{00}(\Re z - \chi_{00}), \beta_{00}(\eta_{00} - \Re w)\right\},\\ q < \min\left\{\alpha_{00}, \beta_{00}(\eta_{00} - \Re w + \eta_{\beta_{00}}(\beta_{01}, \gamma_{01}, \alpha_{01}))\right\}.\end{cases} \tag{8.4.65}$$

Finally, according to (8.2.106), the integrand of (8.4.64), in terms of (8.4.29), features the integral function

$$\begin{aligned}\mathcal{L}\left[\zeta; \begin{matrix}w, T_1\\ z, T_2\end{matrix}\right] := &-\frac{T_1^{\beta_{00}(w-\eta_{00})+\zeta}}{\beta_{00}(w - \eta_{00}) + \zeta} e^{wB(T_1)}\mathfrak{C}(T_1)\int_{T_2}^{\infty} t^{\beta_{00}(z-\chi_{00})-\zeta-1} e^{zB(t)}\mathfrak{C}(t)\mathfrak{A}(T_1 + t)dt\\ &-\frac{1}{\beta_{00}(w - \eta_{00}) + \zeta}\sum_{\substack{n_1,n_2\in\{0,1\}\\ n_1+n_2=1}}\int_{T_2}^{\infty} t^{\beta_{00}(z-\chi_{00})-\zeta-1} e^{zB(t)}\mathfrak{C}(t)\\ &\qquad\times \mathrm{N}_0\left[\begin{matrix}w - \eta_{00} + \frac{1+\zeta}{\beta_{00}}, t\\ 1, n_1, n_2, w, T_1\end{matrix}\right] dt.\end{aligned} \tag{8.4.66}$$

In §8.2.6.2 it was pointed out that $\mathcal{M}_0[\ldots] = \mathcal{O}(w)$ as $\Im w \to \pm\infty$ for fixed z, uniformly with respect to $\Re w$ in any closed vertical substrip of the region (8.4.59). Furthermore, from (8.4.66) it is easy to confirm the existence of constants $L_1, L_2 > 0$ such that for $\Re\zeta = q$, uniformly with respect to $\Im w, \Im z, \Im\zeta \in \mathbb{R}$, we have

$$\left|\mathcal{L}\left[\zeta; \begin{matrix} w, T_1 \\ z, T_2 \end{matrix}\right]\right| \leq L_1 + L_2 \left|w\right|. \tag{8.4.67}$$

With the aid of this bound, from (8.4.64) we deduce, uniformly with respect to $\Im w, \Im z \in \mathbb{R}$:

$$\mathrm{M}_1 \begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} \leq \frac{L_1 + |w|\, L_2}{\Gamma(\alpha_{00}) 2\pi} \int\limits_{-\infty}^{\infty} \left|\Gamma(q + iy)\right| \left|\Gamma(\alpha_{00} - q - iy)\right| dy \tag{8.4.68}$$

The integral on the right hand side converges absolutely. Moreover, according to the well-known integral representation (A.5.14) for the beta function, for $w \in \mathbb{C}$ of the strip (8.4.59) we have

$$\frac{\left|\Gamma(\beta_{00}(\eta_{00} - w))\Gamma(\beta_{00}(w - \chi_{00}))\right|}{\Gamma(\alpha_{00})} \leq \int\limits_0^{\infty} \frac{u^{\beta_{00}(\Re w - \chi_{00}) - 1}}{(1 + u)^{\alpha_{00}}} du.$$

By taking into account (8.4.63), it follows the existence of further constants $K_1, K_2, K_3 > 0$, such that for fixed $\chi_{00} \leq \Re z < \eta_{00}$ and any $w \in \mathbb{C}$ from the strip (8.4.59), uniformly with respect to $\Im w, \Im z \in \mathbb{R}$, we have

$$\left|\mathcal{M}_0 \begin{bmatrix} -w, 0, T_1, \infty \\ -z, 0, T_2, \infty \end{bmatrix}\right| \leq K_1 \left|z\right| + K_2 + K_3 \left|w\right|. \tag{8.4.69}$$

Regarding the singularity of the analytic continuation of $\mathcal{M}_0[\ldots]$ on the line $\Re w = \eta_{00} + \chi_{00} - \Re z$, we first notice that, whenever $z \neq \eta_{00} + \chi_{00}$, this line does not run through the origin of the w-plane. If, in addition, $z \neq \chi_{00}$, we find a pole of simple order there, and in analogy to (8.2.132) we obtain for the residue

$$\operatorname*{Res}_{w = \eta_{00} + \chi_{00} - z} \mathcal{M}_0 \begin{bmatrix} -w, 0, T_1, \infty \\ -z, 0, T_2, \infty \end{bmatrix} = -\frac{a_{00} \left\{c_{00}\right\}^2}{\beta_{00} \Gamma(\alpha_{00})} \frac{\Gamma(\beta_{00}(z - \chi_{00}))\Gamma(\beta_{00}(\eta_{00} - z))}{\left\{b_{00}\right\}^{\eta_{00} + \chi_{00}}}. \tag{8.4.70}$$

We conclude this section by calculating the first two coefficients in the Laurent expansion at $w = \eta_{00} + \chi_{00} - z$ for $z = 0$ in case of some special parametrizations.

8.4.3.3.1. Laurent expansion for $z = 0$ and $\chi_{00} < 0$ with $\eta_{00} + \chi_{00} = 0$. If $\eta_{00} + \chi_{00} = 0$, the indicated pole is again of simple order but lies in fact at the origin. By §8.2.6.2.2, as $w \to 0$ we then find

$$\begin{aligned} \mathcal{M}_0 \begin{bmatrix} -w, 0, T_1, \infty \\ 0, 0, T_2, \infty \end{bmatrix} &= -\frac{a_{00} \left\{c_{00}\right\}^2}{\beta_{00} \Gamma(\alpha_{00})} \left\{\Gamma(1 - \gamma_{00})\right\}^2 \left\{\frac{1}{w} + \mu_{\chi_{00}}(T_2)\right\} \\ &\qquad + \mathrm{M}_1 \begin{bmatrix} 0, T_1 \\ 0, T_2 \end{bmatrix} + \mathcal{O}(w), \end{aligned} \tag{8.4.71}$$

where the constant in the second term denotes

$$\mu_{\chi_{00}}(T_2) := -\log\{b_{00}\} + \frac{\beta_{00}}{c_{00}}\log(T_2)\mathfrak{C}(T_2) + \frac{\beta_{00}}{c_{00}}\int\limits_{T_2}^{\infty}\log(s)\mathfrak{C}'(s)ds. \tag{8.4.72}$$

8.4.3.3.2. Laurent expansion for $z = 0$ and $\chi_{00} = 0$. In these circumstances the pole of the rational function in (8.4.60) merges with that of one of the gamma functions. By §8.2.6.2.3, at $w = \eta_{00}$ we have

$$\mathcal{M}_0\begin{bmatrix} -w, 0, T_1, \infty \\ 0, 0, T_2, \infty \end{bmatrix} = \frac{a_{00}\{c_{00}\}^2}{\{\beta_{00}\}^2\{b_{00}\}^{\eta_{00}}}\left\{\frac{1}{(w-\eta_{00})^2} + \frac{\mu_0(T_2)}{w-\eta_{00}} + \mathcal{O}(1)\right\}, \tag{8.4.73}$$

with the constant in the residue given by

$$\mu_0(T_2) := \beta_{00}\gamma + \beta_{00}\psi(\alpha_{00}) - \log\{b_{00}\} + \frac{\beta_{00}}{c_{00}}\log(T_2)\mathfrak{C}(T_2) + \frac{\beta_{00}}{c_{00}}\int\limits_{T_2}^{\infty}\log(s)\mathfrak{C}'(s)ds. \tag{8.4.74}$$

Here, γ refers to the Euler-Mascheroni constant.

8.4.4. Evaluation of the Interior MB-Integral

For fixed $z \in \mathbb{C}$ with $\Re z = x_0(\vec{p}_2)$, we refer to the interior of the iterated MB-integral (8.4.26) as

$$\mathrm{I}\left[m; \begin{matrix} \theta_1 \\ z, \vec{p}_2 \end{matrix}\right] := \frac{1}{2\pi i}\int\limits_{u_0(\theta_1)-i\infty}^{u_0(\theta_1)+i\infty} \frac{\Gamma(w)}{(m+1)^w}\mathcal{M}_0\begin{bmatrix} -w, \zeta_1, T_1, \infty \\ -z, \zeta_2, \tau, T \end{bmatrix} dw. \tag{8.4.75}$$

In accordance with the above considerations, the original definition of the generating function (8.4.14) for fixed $z \in \mathbb{C}$ with $\Re z = x_0(\vec{p}_2)$ is entire with respect to w if $\theta_1 = 1$, but for $\theta_1 = 0$ it exhibits analyticity only in the half plane $\Re w < \eta_{00}$. In the last case, it can be extended meromorphically to the wider region

$$\Re w < \eta_1(\vec{p}_2), \tag{8.4.76}$$

where, according to (8.4.48) and (8.4.55), we denote

$$\eta_1(\vec{p}_2) := \begin{cases} \eta_{01}, & \text{if } \vec{p}_2 \in \{(\tau_2, 0), (\tau_2, 1)\}, \\ \min\{\eta_{01}, \eta_{00} + \chi_{00} - x_0(\vec{p}_2)\}, & \text{if } \vec{p}_2 = (\infty, 0). \end{cases} \tag{8.4.77}$$

Moreover, our study has revealed, that the only singularity of the analytic continuation in the extended region in each case is a pole of simple order on the line $\Re w = \eta_{00}$. Hence, the singularity in (8.4.75), which lies closest to the right of the integration path $\Re w = u_0(\theta_1)$, is a pole of first or second order. It results from the gamma function $\Gamma(w)$ if $\theta_1 = 1$ or if $\eta_{00} > 0$, from the generating function if $\eta_{00} < 0$ or from both if $\eta_{00} = 0$, where we observe a coalescence to a second order pole. The case $\vec{p}_2 = (\infty, 0)$ with $\theta_1 = 0$ and $\eta_{00} \le 0$ is omitted for the moment and solved in a more convenient fashion in the last subsection. The reason is that, in this case the presence of a second order pole at $w = 0$ for $\eta_{00} = 0$ can be avoided.

In order to justify a displacement of the path in (8.4.75), we recall that the original definition of the iterated generating function (8.4.14) constitutes an absolutely convergent integral, which converges uniformly with respect to $\Im w, \Im z \in \mathbb{R}$. Consequently, for $\theta_1 = 1$ it is easy to verify the function $\mathcal{O}(1)$ as $\Im w \to \pm\infty$, uniformly with respect to $\Re w$ in any closed vertical substrip of $\mathbb{C}$. Furthermore, for $\theta_1 = 0$ the existence of the respective analytic continuation and its computability by partial integration was verified in §§8.4.3.1 and 8.4.3.2. In each case, it represents an expansion in terms of absolutely convergent integrals and simple rational functions. We thus conclude, for fixed $z \in \mathbb{C}$ with $\Re z = x_0(\vec{p}_2)$, each continuation is $\mathcal{O}(1)$ as $\Im w \to \pm\infty$ in $\Re w < \eta_1(\vec{p}_2)$, uniformly with respect to $\Re w$ in any closed vertical substrip. Hence, the integrand in (8.4.75) decays exponentially fast as $\Im w \to \pm\infty$, in $\mathbb{C}$ if $\theta_1 = 1$, or in the half plane $\Re w < \eta_1(\vec{p}_2)$ if $\theta_1 = 0$. We thereby deduce the permission, to displace the integration path to the right over the pole at $w = 0$ if $\theta_1 = 1$, and for $\theta_1 = 0$ over the poles at $w \in \{0, \eta_{00}\}$ if $\eta_{00} > 0$, or merely over the pole at $w = \eta_{00}$ if that point does not lie in the right w-half plane. Bearing in mind that, in the case $\vec{p}_2 = (\infty, 0)$ with $\theta_1 = 0$ we suppose $\eta_{00} > 0$, and that each pole is encircled clockwisely, for a suitable vertical line with $\Re w = u_1(\theta_1, \vec{p}_2)$ we then arrive at:

$$
\begin{aligned}
\mathrm{I}\begin{bmatrix} m; & \theta_1 \\ & z, \vec{p}_2 \end{bmatrix} &= -\mathcal{M}_0\begin{bmatrix} 0, \zeta_1, T_1, \infty \\ -z, \zeta_2, \tau, T \end{bmatrix} \mathbb{1}_{\{\theta_1 = 1 \vee \eta_{00} > 0\}} \\
&\quad + \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{a_{00}c_{00}}{\beta_{00}} \mathcal{P}_0\begin{bmatrix} -z, \zeta_2 \\ \tau, T \end{bmatrix} \mathbb{1}_{\{\theta_1 = 0,\, \eta_{00} \neq 0\}} \\
&\quad - (\log(m+1) + \gamma) \frac{a_{00}c_{00}}{\beta_{00}} \mathcal{P}_0\begin{bmatrix} -z, \zeta_2 \\ \tau_2, T_2 \end{bmatrix} \mathbb{1}_{\{\theta_1 = 0,\, p_2 = \tau_2,\, \eta_{00} = 0\}} \\
&\quad - \frac{1}{\beta_{00}} \mathcal{Q}_0\begin{bmatrix} -z, \zeta_2 \\ \tau_2, T_2, T_1 \end{bmatrix} \mathbb{1}_{\{\theta_1 = 0,\, p_2 = \tau_2,\, \eta_{00} = 0\}} \\
&\quad + \frac{1}{2\pi i} \int_{u_1(\theta_1, \vec{p}_2) - i\infty}^{u_1(\theta_1, \vec{p}_2) + i\infty} \frac{\Gamma(w)}{(m+1)^w} \mathcal{M}_0\begin{bmatrix} -w, \zeta_1, T_1, \infty \\ -z, \zeta_2, \tau, T \end{bmatrix} dw
\end{aligned}
\tag{8.4.78}
$$

This expansion was obtained similar to (8.1.12), by virtue of Theorem B.2.1(2). The coefficients appearing therein were taken from (8.4.51), (8.4.52) and (8.4.56). Moreover, the integration path in the remainder integral is a line with arbitrary positive real part if $\theta_1 = 1$, and for $\theta_1 = 0$ the closest singularity to its left is the pole at $w = \eta_{00}$, i.e.,

$$
\begin{aligned}
&0 < u_1(1, \vec{p}_2) < \infty, \\
&\eta_{00} < u_1(0, \vec{p}_2) < \begin{cases} \eta_1(\vec{p}_2), & \text{if } \eta_{00} \ge 0, \\ \min\{0, \eta_1(\vec{p}_2)\}, & \text{if } \eta_{00} < 0. \end{cases}
\end{aligned}
\tag{8.4.79}
$$

With $x_0(\vec{p}_2)$ as in (8.4.24), we introduce the single MB-integrals

$$\text{(8.4.80)} \qquad \mathrm{J}(m;\theta_1,\vec{p}_2) := \frac{1}{2\pi i}\int\limits_{x_0(\vec{p}_2)-i\infty}^{x_0(\vec{p}_2)+i\infty} \frac{\Gamma(z)}{(m+1)^z}\mathcal{M}_0\begin{bmatrix}0,\zeta_1,T_1,\infty\\ -z,\zeta_2,\tau,T\end{bmatrix}dz,$$

$$\text{(8.4.81)} \qquad \mathrm{K}\begin{bmatrix}m; & \theta_2\\ & \tau,T\end{bmatrix} := \frac{1}{2\pi i}\int\limits_{x_0(\vec{p}_2)-i\infty}^{x_0(\vec{p}_2)+i\infty} \frac{\Gamma(z)}{(m+1)^z}\mathcal{P}_0\begin{bmatrix}-z,\zeta_2\\ \tau,T\end{bmatrix}dz,$$

$$\text{(8.4.82)} \qquad \mathrm{Q}(m;\theta_2) := \frac{1}{2\pi i}\int\limits_{x_0(\vec{p}_2)-i\infty}^{x_0(\vec{p}_2)+i\infty} \frac{\Gamma(z)}{(m+1)^z}\mathcal{Q}_0\begin{bmatrix}-z,\zeta_2\\ \tau_2,T_2,T_1\end{bmatrix}dz.$$

An application of (8.4.78) to the iterated MB-integral (8.4.26), accompanied by a simple bound for the remainder integral, as $m\to\infty$ then shows

$$\begin{aligned}\text{(8.4.83)} \quad \mathcal{S}i\begin{bmatrix}m; & \zeta_1,T_1,\infty\\ & \zeta_2,\tau,T\end{bmatrix} &= -\,\mathrm{J}(m;\theta_1,\vec{p}_2)\mathbb{1}_{\{\theta_1\,=\,1\,\vee\,\eta_{00}\,>\,0\}}\\ &\quad+\frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}}\frac{a_{00}c_{00}}{\beta_{00}}\,\mathrm{K}\begin{bmatrix}m; & \theta_2\\ & \tau,T\end{bmatrix}\mathbb{1}_{\{\theta_1\,=\,0,\,\eta_{00}\,\neq\,0\}}\\ &\quad-(\log(m+1)+\gamma)\frac{a_{00}c_{00}}{\beta_{00}}\,\mathrm{K}\begin{bmatrix}m; & \theta_2\\ & \tau,T\end{bmatrix}\mathbb{1}_{\{\theta_1\,=\,0,\,p_2\,=\,\tau_2,\,\eta_{00}\,=\,0\}}\\ &\quad-\frac{1}{\beta_{00}}\,\mathrm{Q}(m;\theta_2)\mathbb{1}_{\{\theta_1\,=\,0,\,p_2\,=\,\tau_2,\,\eta_{00}\,=\,0\}}\\ &\quad+\mathcal{O}\left\{m^{-x_0(\vec{p}_2)-u_1(\theta_1,\vec{p}_2)}\right\}.\end{aligned}$$

To conclude the asymptotic evaluation, a careful analysis of the single integrals in the above expansion remains to be accomplished.

8.4.5. A Single MB-Integral for the Residue at $w=0$ if $\xi_1\neq 0$, or if $\eta_{00}>0$

We are now concerned with the large m-behaviour of the MB-integral $\mathrm{J}(m;\theta_1,\vec{p}_2)$, which was defined in (8.4.80) and occurs for $\theta_1=1$ or for $\eta_{00}>0$. Its generating function equals the iterated integral (8.4.14) evaluated at $w=0$. According to (8.4.15) and (8.4.17), the latter is absolutely convergent and holomorphic in the whole z-plane for $\vec{p}_2\in\{(\tau_2,0),(\infty,1)\}$ and otherwise in the half plane $\Re z<\iota_0(\theta_1,\vec{p}_2)$, for

$$\text{(8.4.84)} \qquad \iota_0(\theta_1,\vec{p}_2) := \begin{cases}\chi_{20}, & \text{if } \vec{p}_2=(\tau_2,1),\\ \eta_{00}, & \text{if } \vec{p}_2=(\infty,0)\wedge\theta_1=1,\\ \eta_{00}+\min\{0,\chi_{00}\}, & \text{if } \vec{p}_2=(\infty,0)\wedge\theta_1=0.\end{cases}$$

Since, in the present subsection, $\xi_1=0$, i.e., $\theta_1=0$, implies $\eta_{00}>0$, it is easy to see that

$$\iota_0(\theta_1,\vec{p}_2)\geq\varsigma_0(\vec{p}_2).$$

In addition, the integral definition of the generating function of $\mathrm{J}(m;\theta_1,\vec{p}_2)$ is $\mathcal{O}(1)$ as $\Im z \to \pm\infty$, uniformly with respect to $\Re z$ in any closed vertical vertical substrip of its region of analyticity, whence the conditions (8.4.24) on the integration path $x_0(\vec{p}_2) \equiv j_0(\theta_1,\vec{p}_2)$ can be replaced by

$$-1 < j_0(\theta_1,\vec{p}_2) < \begin{cases} 0, & \text{if } \vec{p}_2 \in \{(\tau_2,0),(\infty,1)\}, \\ \min\{0,\iota_0(\theta_1,\vec{p}_2)\}, & \text{if } \vec{p}_2 \in \{(\tau_2,1),(\infty,0)\}. \end{cases} \tag{8.4.85}$$

First of all, for different values of the parameter $\iota_0(\theta_1,\vec{p}_2)$, we will determine the analytic continuation into a wider region that contains the line $\Re w = \iota_0(\theta_1,\vec{p}_2)$, and calculate for some $n \in \mathbb{Z}$ the coefficients in the Laurent expansion

$$\mathcal{M}_0\begin{bmatrix} 0,\zeta_1,T_1,\infty \\ -z,\zeta_2,\tau,T \end{bmatrix} = \sum_{n\in\mathbb{Z}} \mu_{-n}\begin{bmatrix} \theta_1,\vec{p}_2 \\ \chi(\theta_1,\vec{p}_2) \end{bmatrix}(z-\iota_0(\theta_1,\vec{p}_2))^n, \tag{8.4.86}$$

for brevity, with the parameter

$$\chi(\theta_1,\vec{p}_2) := \begin{cases} (\iota_0(\theta_1,\vec{p}_2),\min\{0,\chi_{00}\}), & \text{if } \vec{p}_2 = (\infty,0) \wedge \theta_1 = 0, \\ \iota_0(\theta_1,\vec{p}_2), & \text{otherwise}. \end{cases} \tag{8.4.87}$$

For this, in (8.4.14) a distinction between a finite and an infinite exterior path must be made.

8.4.5.1. z-Analytic Continuation for $w = 0$ and a Finite Path $\mathcal{P}_2$

In the case $\vec{p}_2 = (\tau_2, 1)$, bearing in mind $c(t;\tau_2) = c(t)$, in terms of (8.4.27) we write

$$\mathcal{M}_0\begin{bmatrix} 0,\zeta_1,T_1,\infty \\ -z,\zeta_2,\tau_2,T_2 \end{bmatrix} = \int_{\tau_2}^{T_2} \{\varphi(t)\}^{-z} e^{-\zeta_2 t} c(t)\mathcal{N}_0\begin{bmatrix} 0,t,\zeta_1 \\ 0,0,T_1 \end{bmatrix} dt. \tag{8.4.88}$$

The integral $\mathcal{N}_0[\ldots]$ with the above arguments is then a continuous function of $\tau_2 \le t \le T_2$ that is $\mathcal{O}(1)$ as $t \downarrow \tau_2$. Moreover, by uniform convergence with respect to $\tau_2 \le t \le T_2$, its first derivative can be found by differentiation under the sign of integration, and it is again $\mathcal{O}(1)$ as $t \downarrow \tau_2$. We may therefore still employ the standard integration by parts procedure from §8.2.2.1, for

$$\begin{cases} d(t) & \equiv e^{-\zeta_2 t}c(t)\mathcal{N}_0\left[\begin{smallmatrix} 0,t,\zeta_1 \\ 0,0,T_1 \end{smallmatrix}\right], \\ k(s+t) & \equiv 1. \end{cases}$$

This shows, that (8.4.88) can be extended to a meromorphic function in the half plane

$$\Re z < \chi_{21}, \tag{8.4.89}$$

where χ_{21} was defined in (8.3.19). The only singularity therein is a simple pole at $z = \chi_{20}$, whence in (8.4.86) each coefficient for $n \le -2$ equals zero. According to (8.2.33), for the residue we obtain

$$\mu_{-1}\begin{bmatrix}\theta_1,(\tau_2,1)\\ \chi_{20}\end{bmatrix} = -\frac{c_{20}}{\beta_{20}}\{b_{20}\}^{-\chi_{20}} e^{-\zeta_2\tau_2}\mathcal{N}_0\begin{bmatrix}0,\tau_2,\zeta_1\\ 0,0,T_1\end{bmatrix}. \tag{8.4.90}$$

If $\chi_{20} = 0$, the point $z = \chi_{20}$ clearly lies at the origin of the complex z-plane. From (8.2.35) and (8.2.36) we conclude, that the constant addend in the expansion (8.4.86), in terms of (8.3.23), is then given by

$$\begin{aligned}\mu_0\begin{bmatrix}\theta_1,(\tau_2,1)\\ 0\end{bmatrix} = \frac{C_2(T_2)}{\beta_{20}}\log\{\varphi(T_2)\}\, e^{-\zeta_2 T_2}\int_{T_1}^{\infty} e^{-\zeta_1 s}\mathfrak{c}(s)\mathfrak{a}(s+T_2)ds \\ + \frac{1}{\beta_{20}}\int_{\tau_2}^{T_2} e^{-\zeta_2 t} f_2(t,\zeta_2)\int_{T_1}^{\infty} e^{-\zeta_1 s}\mathfrak{c}(s)\mathfrak{a}(s+t)dsdt \\ - \frac{1}{\beta_{20}}\int_{\tau_2}^{T_2}\log\{\varphi(t)\}\, e^{-\zeta_2 t}C_2(t)\int_{T_1}^{\infty} e^{-\zeta_1 s}\mathfrak{c}(s)\mathfrak{a}'(s+t)dsdt.\end{aligned} \tag{8.4.91}$$

8.4.5.2. z-Analytic Continuation for $w = 0$ and an Infinite Path $\mathcal{P}_2$

To specify the z-analytic continuation of (8.4.14) with $\vec{p}_2 = (\infty, 0)$ for $w = 0$, we note that $c(t;\infty) = \mathfrak{c}(t)$. By comparison with the indicated definition, it is therefore easy to confirm the identity

$$\mathcal{M}_0\begin{bmatrix}0,\zeta_1,T_1,\infty\\ -z,0,T_2,\infty\end{bmatrix} = \mathcal{M}_0\begin{bmatrix}-z,0,T_2,\infty\\ 0,\zeta_1,T_1,\infty\end{bmatrix}. \tag{8.4.92}$$

But the analytic continuation of this function has already been derived in Subsection 8.4.3. Particularly in §§8.4.3.1 and 8.4.3.2, where we discussed the case $\theta_1 = 1$ and the case $\theta_1 = 0$ with $\chi_{00} > 0$, this was possible via integration by parts, which eventually brought us

$$\mu_{-n}\begin{bmatrix}\theta_1,(\infty,0)\\ \chi(\theta_1,(\infty,0))\end{bmatrix} = \begin{cases} -\frac{a_{00}c_{00}}{\beta_{00}\{b_{00}\}^{\eta_{00}}}\mathcal{P}_0\begin{bmatrix}0,\zeta_1\\ T_1,\infty\end{bmatrix}, & \text{if } n = 1,\\ \frac{1}{\beta_{00}}\mathcal{Q}_0\begin{bmatrix}0,\zeta_1\\ T_1,\infty,T_2\end{bmatrix}, & \text{if } n = 0 \wedge \eta_{00} = 0.\end{cases} \tag{8.4.93}$$

In each of these cases, as well as in the next case, the point $w = \iota_0(\theta_1,(\infty,0))$ is a pole of simple order, whence in (8.4.86) all coefficients for $n \le -2$ vanish. However, in the case $\theta_1 = 0$ with $\chi_{00} \le 0$ instead of integration by parts we had to employ a different expansion, compare

§8.4.3.3. In view of (8.4.92), by (8.4.70) and (8.4.71) this yields

(8.4.94)

$$\mu_{-n}\begin{bmatrix} 0,(\infty,0) \\ (\eta_{00}+\chi_{00},\chi_{00}) \end{bmatrix} = \begin{cases} -\frac{a_{00}\{c_{00}\}^2}{\beta_{00}\Gamma(\alpha_{00})}\frac{\Gamma(1-\gamma_{00})\Gamma(\alpha_{00}+\gamma_{00}-1)}{\{b_{00}\}^{\eta_{00}+\chi_{00}}}, & \text{if } n=1\wedge\chi_{00}<0, \\ -\frac{a_{00}\{c_{00}\}^2}{\beta_{00}\Gamma(\alpha_{00})}\{\Gamma(1-\gamma_{00})\}^2\mu_{\chi_{00}}(T_1) & \\ \qquad\qquad +\mathrm{M}_1\begin{bmatrix} 0,T_2 \\ 0,T_1 \end{bmatrix}, & \text{if } n=0\wedge\eta_{00}=-\chi_{00}. \end{cases}$$

Notice, since $\theta_1=\theta_2=0$ implies $\eta_{00}>0$, we have $\chi_{00}=-\eta_{00}$ if and only if $\chi_{00}<0$. Finally, in the case $\chi_{00}=0$, at $z=\iota_0(0,(\infty,0))$ we certainly find a pole of order two. Accordingly, the coefficients in (8.4.86) for $n\leq-3$ vanish and by §8.4.3.3.2 we obtain

$$\mu_{-n}\begin{bmatrix} 0,(\infty,0) \\ (\eta_{00},0) \end{bmatrix} = \frac{a_{00}\{c_{00}\}^2}{\{\beta_{00}\}^2\{b_{00}\}^{\eta_{00}}}\times\begin{cases} 1, & \text{if } n=2, \\ \mu_0(T_1), & \text{if } n=1. \end{cases} \tag{8.4.95}$$

No additional coefficients are required in this last case, since $z=\eta_{00}$ then clearly lies somewhere in the right z-half plane.

8.4.5.3. Evaluation of the MB-Integral

To summarize these results, for $\vec{p}_2\in\{(\tau_2,1),(\infty,0)\}$ the integral (8.4.14) with $w=0$ can be continued analytically into the region $\Re z<\iota_1(\theta_1,\vec{p}_2)$, for

$$\iota_1(\theta_1,\vec{p}_2):=\begin{cases} \chi_{21}, & \text{if } \vec{p}_2=(\tau_2,1), \\ \eta_{01}, & \text{if } \vec{p}_2=(\infty,0)\wedge\theta_1=1, \\ \min\{\eta_{01},\eta_{00}+\chi_{00}\}, & \text{if } \vec{p}_2=(\infty,0)\wedge\theta_1=0\wedge\chi_{00}>0, \\ \eta_{00}+\chi_{00}+\eta_{\beta_{00}}(\alpha_{01},\beta_{01},\gamma_{01}), & \text{if } \vec{p}_2=(\infty,0)\wedge\theta_1=0\wedge\chi_{00}\leq 0. \end{cases} \tag{8.4.96}$$

According to §§8.4.3.1, 8.4.3.2 and 8.4.5.1, the continuation can be computed via partial integration, and each representation that can be obtained in this fashion, holds in the whole region $\Re w<\iota_1(\theta_1,\vec{p}_2)$, where its only singularity is a pole of simple order at $w=\iota_0(\theta_1,\vec{p}_2)$. Conversely, the expansion (8.4.60) from §8.4.3.3, which is applicable if $\vec{p}_2=(\infty,0)$ with $\theta_1=0$ and $\chi_{00}\leq 0$, merely covers some substrip of the above half plane. There, it was seen to exhibit a pole of simple order at $z=\iota_0(0,(\infty,0))$, except if $\chi_{00}=0$, in which circumstances a second order pole will be found at this point. Since $\eta_{00}>0$ then, in this special case the indicated pole lies somewhere in the right z-half plane. Additional poles may only show up to the right of the line $\Re z=\iota_0(0,(\infty,0))$. If we therefore incorporate the gamma function $\Gamma(z)$, the integrand of the MB-integral (8.4.80) at $z=\iota_0(\theta_1,\vec{p}_2)$ in each case only shows poles of order less or equal two. Furthermore, upon exploiting the exponential decay of this gamma function, a rightward displacement of the integration path can be justified, to match a line $\Re z = j_1(\theta_1, \vec{p}_2)$, for

$$0 < j_1(\theta_1, \vec{p}_2) < \infty, \qquad \text{if } \vec{p}_2 \in \{(\tau_2, 0), (\infty, 1)\}, \tag{8.4.97}$$

for $j_1(\theta_1, \vec{p}_2) > \iota_0(\theta_1, \vec{p}_2)$, with

$$j_1(0,(\infty,0)) < \begin{cases} \min\{\eta_{00}, \iota_1(0,(\infty,0))\}, & \text{if } \iota_0(0,(\infty,0)) \geq 0 \wedge \chi_{00} < 0, \\ \min\{0, \iota_1(0,(\infty,0))\}, & \text{if } \iota_0(0,(\infty,0)) < 0 \wedge \chi_{00} < 0, \\ \min\left\{\eta_{00} + \frac{1}{\beta_{00}}, \iota_1(0,(\infty,0))\right\}, & \text{if } \iota_0(0,(\infty,0)) > 0 \wedge \chi_{00} = 0. \end{cases} \tag{8.4.98}$$

and otherwise with

$$j_1(\theta_1, \vec{p}_2) < \begin{cases} \iota_1(\theta_1, \vec{p}_2), & \text{if } \iota_0(\theta_1, \vec{p}_2) \geq 0, \\ \min\{0, \iota_1(\theta_1, \vec{p}_2)\}, & \text{if } \iota_0(\theta_1, \vec{p}_2) < 0. \end{cases} \tag{8.4.99}$$

In the process of moving the path rightwards from $\Re z = j_0(\theta_1, \vec{p}_2)$ to $\Re z = j_1(\theta_1, \vec{p}_2)$, we clockwisely encircle the pole at $z = \iota_0(\theta_1, \vec{p}_2)$, and in addition the pole at $z = 0$ if $\iota_0(\theta_1, \vec{p}_2) > 0$. Appealing to Theorem B.2.1(2), as $m \to \infty$ the residue theorem brings us:

$$\begin{aligned} \mathrm{J}(m; \vec{p}_2) = & -\mathcal{M}_0 \begin{bmatrix} 0, \zeta_1, T_1, \infty \\ 0, \zeta_2, \tau, T \end{bmatrix} \mathbb{1}_{\{p_2 \in \{(\tau_2,0),(\infty,1)\} \vee \iota_0(\theta_1,\vec{p}_2) > 0\}} \\ & + \frac{\Gamma(\iota_0(\theta_1,\vec{p}_2))}{(m+1)^{\iota_0(\theta_1,\vec{p}_2)}} \{\log(m+1) - \psi(\iota_0(\theta_1,\vec{p}_2))\} \mu_{-2} \begin{bmatrix} \theta_1, \vec{p}_2 \\ \chi(\theta_1,\vec{p}_2) \end{bmatrix} \mathbb{1}_{\{\iota_0(\theta_1,\vec{p}_2) \neq 0\}} \\ & - \frac{\Gamma(\iota_0(\theta_1,\vec{p}_2))}{(m+1)^{\iota_0(\theta_1,\vec{p}_2)}} \mu_{-1} \begin{bmatrix} \theta_1, \vec{p}_2 \\ \chi(\theta_1,\vec{p}_2) \end{bmatrix} \mathbb{1}_{\{\iota_0(\theta_1,\vec{p}_2) \neq 0\}} \\ & + \left\{ (\log(m+1) + \gamma) \mu_{-1} \begin{bmatrix} \theta_1, \vec{p}_2 \\ \chi(\theta_1,\vec{p}_2) \end{bmatrix} - \mu_0 \begin{bmatrix} \theta_1, \vec{p}_2 \\ \chi(\theta_1,\vec{p}_2) \end{bmatrix} \right\} \mathbb{1}_{\{\iota_0(\theta_1,\vec{p}_2) = 0\}} \\ & + \mathcal{O}\left\{ m^{-j_1(\theta_1,\vec{p}_2)} \right\} \end{aligned} \tag{8.4.100}$$

The estimate for the remainder integral in the big-$\mathcal{O}$ holds by absolute convergence.

8.4.6. A Single MB-Integral for the Residue at $w = \eta_{00}$

The MB-integral K$[m; \ldots]$, compare (8.4.81), only appears if $\theta_1 = 0$. By (8.4.13), it is therefore sufficient to consider the cases

$$\vec{p}_2 \in \{(\tau_2, 0), (\tau_2, 1), (\infty, 0)\}.$$

Its generating function (8.4.49) is readily verified to be absolutely convergent and holomorphic in the whole complex plane if $\vec{p}_2 = (\tau_2, 0)$, or in the half plane $\Re z < \varsigma_0(\vec{p}_2)$ if $\vec{p}_2 \in \{(\tau_2, 1), (\infty, 0)\}$. To specify the analytic continuation is a routine step, which can be accomplished by a quick reference to §§8.2.2.1 and 8.2.2.2, respectively for $\vec{p}_2 = (\tau_2, 1)$ and $\vec{p}_2 = (\infty, 0)$. Accordingly,

$\mathcal{P}_0[\ldots]$ can be continued to a meromorphic function in the half plane $\Re z < \varsigma_1(\vec{p}_2)$, with

$$\varsigma_1(\vec{p}_2) := \begin{cases} \chi_{20} + \chi_{\beta_{20}}(\beta_{21}, \gamma_{21}), & \text{if } p_2 = \tau_2 \wedge \zeta_2 = 0, \\ \chi_{20} + \chi_{\beta_{20}}(\beta_{21}, \gamma_{21}, 0), & \text{if } p_2 = \tau_2 \wedge \zeta_2 \neq 0, \\ \chi_{00} + \eta_{\beta_{00}}(\beta_{01}, \gamma_{01}), & \text{if } \vec{p}_2 = (\infty, 0) \wedge \zeta_2 = 0. \end{cases} \tag{8.4.101}$$

The only singularity therein is a pole of simple order at $z = \varsigma_0(\vec{p}_2)$. Hence, in an annulus around this point, we find an expansion with dominating terms

$$\mathcal{P}_0\begin{bmatrix} -z, \zeta_2 \\ \tau, T \end{bmatrix} = \frac{\pi_{-1}(\vec{p}_2)}{z - \varsigma_0(\vec{p}_2)} + \pi_0(\theta_2; \tau, T) + \mathcal{O}\left\{z - \varsigma_0(\vec{p}_2)\right\}, \tag{8.4.102}$$

in which the residue equals

$$\pi_{-1}(\vec{p}_2) = \begin{cases} -\frac{c_{20}}{\beta_{20}} \left\{b_{20}\right\}^{-\chi_{20}} e^{-\zeta_2 \tau_2}, & \text{if } \vec{p}_2 = (\tau_2, 1), \\ -\frac{c_{00}}{\beta_{00}} \left\{b_{00}\right\}^{-\chi_{00}}, & \text{if } \vec{p}_2 = (\infty, 0), \end{cases} \tag{8.4.103}$$

whereas the coefficients of the constant term in the cases $\varsigma_0(\vec{p}_2) = 0$ are respectively given by

$$\pi_0(1; \tau_2, T_2) = \frac{C_2(T_2)}{\beta_{20}} \log\left\{\varphi(T_2)\right\} e^{-\zeta_2 T_2} + \frac{1}{\beta_{20}} \int_{\tau_2}^{T_2} B'(t) e^{-\zeta_2 t} C_2(t) dt - \frac{1}{\beta_{20}} \int_{\tau_2}^{T_2} \log\left\{\varphi(t)\right\} e^{-\zeta_2 t} \left\{C_2'(t) - \zeta_2 C_2(t)\right\} dt, \tag{8.4.104}$$

$$\pi_0(0; T_2, \infty) = \frac{\mathfrak{C}(T_2)}{\beta_{00}} \log\left\{\varphi(T_2)\right\} - \frac{1}{\beta_{00}} \int_{T_2}^{\infty} \left\{B'(t)\mathfrak{C}(t) - \log\left\{\varphi(t)\right\} \mathfrak{C}'(t)\right\} dt. \tag{8.4.105}$$

Similar to the integral definition of $\mathcal{P}_0[\ldots]$, in each case its analytic continuation is readily confirmed $\mathcal{O}(1)$ as $\Im z \to \pm\infty$, uniformly with respect to $\Re z$ in any closed vertical substrip of its half plane of analyticity. Hence, there the whole integrand of (8.4.81), due to the gamma function, vanishes exponentially fast towards the imaginary direction. We are therefore allowed, to replace the integration path by a vertical line $\Re z = s_1(\vec{p}_2)$, for

$$\begin{aligned} &0 < s_1(\tau_2, 0) < \infty, \\ &\varsigma_0(\vec{p}_2) < s_1(\vec{p}_2) < \begin{cases} \varsigma_1(\vec{p}_2), & \text{if } \varsigma_0(\vec{p}_2) \geq 0, \\ \min\left\{0, \varsigma_1(\vec{p}_2)\right\}, & \text{if } \varsigma_0(\vec{p}_2) < 0. \end{cases} \end{aligned} \tag{8.4.106}$$

According to the residue theorem, we must then incorporate the clockwisely encircled pole at $z = \varsigma_0(\vec{p}_2)$, and possibly also the pole at $z = 0$ if $\varsigma_0(\vec{p}_2) > 0$, which as $m \to \infty$ eventually yields

$$\mathrm{K}\left[m; \begin{matrix} \theta_2 \\ \tau, T \end{matrix}\right] = -\mathcal{P}_0\begin{bmatrix} 0, \zeta_2 \\ \tau, T \end{bmatrix} \mathbb{1}_{\{\vec{p}_2 = (\tau_2, 0) \vee \varsigma_0(\vec{p}_2) > 0\}} \tag{8.4.107}$$

$$
\begin{aligned}
&- \frac{\Gamma(\varsigma_0(\vec{p}_2))}{(m+1)^{\varsigma_0(\vec{p}_2)}} \pi_{-1}(\vec{p}_2) \mathbb{1}_{\{\varsigma_0(\vec{p}_2) \neq 0\}} \\
&+ \{(\log(m+1) + \gamma)\pi_{-1}(\vec{p}_2) - \pi_0(\theta_2; \tau, T)\} \mathbb{1}_{\{\varsigma_0(\vec{p}_2) = 0\}} \\
&+ \mathcal{O}\left\{m^{-s_1(\vec{p}_2)}\right\}.
\end{aligned}
$$

The big-$\mathcal{O}$ estimate again holds subject to absolute convergence of the remainder integral.

8.4.7. A Single MB-Integral for the Residue at $w = 0$ if $\eta_{00} = 0$

The MB-integral (8.4.82) only occurs if $p_2 = \tau_2$ with $\theta_1 = 0$ and $\eta_{00} = 0$, in which case $\zeta_2 = i\xi_2$. Its generating function $\mathcal{Q}_0[\ldots]$ was specified in (8.4.50), and for $\mathcal{P}_2 = (\tau_2, T_2]$ this becomes

$$
\mathcal{Q}_0\begin{bmatrix} -z, i\xi_2 \\ \tau_2, T_2, T_1 \end{bmatrix} = \int_{\tau_2}^{T_2} \{\varphi(t)\}^{-z} e^{-i\xi_2 t} c(t) \{\nu_1(t; T_1) + t\nu_2(t; T_1)\} \, dt. \tag{8.4.108}
$$

According to our findings from §8.4.2.1, the functions $\nu_j(t; T_1)$, which for $j \in \{1, 2\}$ were defined in (8.4.36) and (8.4.37), are $\mathcal{O}(1)$ as $t \downarrow \tau_2$. It is therefore easy, to confirm absolute convergence and analyticity of the above representation for any $z \in \mathbb{C}$, if $\beta_{20} = 0$, or for $\Re z < \chi_{20}$, if $\beta_{20} > 0$. We remind the reader, however, of Subsection 8.2.5, where it was pointed out that the exact region of analyticity certainly will be larger, if the term in the curved brackets vanishes as $t \downarrow \tau_2$. Yet, we may still proceed in the fashion of §8.2.2.1. Upon identifying

$$
\begin{cases} d(t) & \equiv e^{-i\xi_2 t} c(t) \{\nu_1(t; T_1) + t\nu_2(t; T_1)\}, \\ k(s+t) & \equiv 1, \end{cases}
$$

this immediately shows, that (8.4.108) can be expanded via partial integration. Thereby, we obtain a representation, which is valid by analytic continuation in the half plane

$$
\Re z < \chi_{21}, \tag{8.4.109}
$$

with χ_{21} as in (8.3.19). Therein its only singularity is a pole of simple order at $z = \chi_{20}$, for which, from (8.2.33), we compute

$$
\operatorname*{Res}_{z=\chi_{20}} \mathcal{Q}_0\begin{bmatrix} -z, i\xi_2 \\ \tau_2, T_2, T_1 \end{bmatrix} = -\frac{c_{20}}{\beta_{20}} \{b_{20}\}^{-\chi_{20}} e^{-i\xi_2\tau_2} \{\nu_1(\tau_2; T_1) + \tau_2\nu_2(\tau_2; T_1)\}. \tag{8.4.110}
$$

Finally, if $\chi_{20} = 0$ the first two terms in the Laurent expansion around $z = 0$ of (8.4.108) can be obtained from (8.2.35), which yields

$$
\begin{aligned}
\mathcal{Q}_0\begin{bmatrix} -z, i\xi_2 \\ \tau_2, T_2, T_1 \end{bmatrix} &= -\frac{1}{z}\frac{c_{20}}{\beta_{20}} e^{-i\xi_2\tau_2} \{\nu_1(\tau_2; T_1) + \tau_2\nu_2(\tau_2; T_1)\} \\
&\qquad + \frac{1}{\beta_{20}} \omega_0\begin{bmatrix} i\xi_2 \\ \tau_2, T_2, T_1 \end{bmatrix} + \mathcal{O}(z),
\end{aligned} \tag{8.4.111}
$$

where the constant appearing in the second coefficient, by (8.2.36), features the sum of integrals

$$\begin{aligned}
\omega_0\begin{bmatrix} i\xi_2 \\ \tau_2, T_2, T_1 \end{bmatrix} := {}& \log\{\varphi(T_2)\}\, e^{-i\xi_2 T_2} C_2(T_2)\{\nu_1(T_2;T_1) + T_2\nu_2(T_2;T_1)\} \\
& + \int_{\tau_2}^{T_2} B'(t) e^{-i\xi_2 t} C_2(t)\{\nu_1(t;T_1) + t\nu_2(t;T_1)\}\, dt \\
& - \int_{\tau_2}^{T_2} \log\{\varphi(t)\}\, e^{-i\xi_2 t} C_2'(t)\{\nu_1(t;T_1) + t\nu_2(t;T_1)\}\, dt \\
& + i\xi_2 \int_{\tau_2}^{T_2} \log\{\varphi(t)\}\, e^{-i\xi_2 t} C_2(t)\{\nu_1(t;T_1) + t\nu_2(t;T_1)\}\, dt \\
& - \int_{\tau_2}^{T_2} \log\{\varphi(t)\}\, e^{-i\xi_2 t} C_2(t)\left\{\nu_1'(t;T_1) + \nu_2(t;T_1) + t\nu_2'(t;T_1)\right\} dt.
\end{aligned} \tag{8.4.112}$$

Since the integral definition of $\mathcal{Q}_0[\ldots]$ as well as its analytic continuation are $\mathcal{O}(1)$ as $\Im z \to \pm\infty$, uniformly with respect to $\Re z$ in any closed vertical substrip of $\mathbb{C}$ if $\beta_{20} = 0$ or of $\Re z < \chi_{21}$ if $\beta_{20} > 0$, we observe exponential decay of the integrand of (8.4.82) there, towards any imaginary direction. In view of these properties, we are allowed to move the integration path to the right, to match a line with real part $\Re z = x_2$, for

$$\begin{aligned}
& 0 < x_2 < \infty, && \text{if } \beta_{20} = 0, \\
& \chi_{20} < x_2 < \begin{cases} \chi_{21}, & \text{if } \chi_{20} \geq 0, \\ \min\{0, \chi_{21}\}, & \text{if } \chi_{20} < 0. \end{cases}
\end{aligned} \tag{8.4.113}$$

Thereby, we merely traverse the pole at $z = 0$ if $\beta_{20} = 0$ or at $z = \chi_{20}$ if $\chi_{20} \leq 0$, or the poles at $z \in \{0, \chi_{20}\}$ if $\chi_{20} > 0$. According to Theorem B.2.1(2), as $m \to \infty$ this yields

$$\begin{aligned}
\mathrm{Q}(m;\theta_2) = {}& -\mathcal{Q}_0\begin{bmatrix} 0, i\xi_2 \\ \tau_2, T_2, T_1 \end{bmatrix} \mathbb{1}_{\{\beta_{20} = 0 \,\vee\, \chi_{20} > 0\}} \\
& + \frac{\Gamma(\chi_{20})}{\{b_{20}(m+1)\}^{\chi_{20}}} \frac{c_{20}}{\beta_{20}} e^{-i\xi_2\tau_2}\{\nu_1(\tau_2;T_1) + \tau_2\nu_2(\tau_2;T_1)\}\, \mathbb{1}_{\{\chi_{20} \neq 0\}} \\
& - (\log(m+1) + \gamma)\frac{c_{20}}{\beta_{20}} e^{-i\xi_2\tau_2}\{\nu_1(\tau_2;T_1) + \tau_2\nu_2(\tau_2;T_1)\}\, \mathbb{1}_{\{\chi_{20} = 0\}} \\
& - \frac{1}{\beta_{20}}\omega_0\begin{bmatrix} i\xi_2 \\ \tau_2, T_2, T_1 \end{bmatrix} \mathbb{1}_{\{\chi_{20} = 0\}} \\
& + \mathcal{O}\left\{m^{-x_2}\right\}.
\end{aligned} \tag{8.4.114}$$

The estimate for the remainder holds by absolute convergence of its integral representation.

8.4.8. Evaluation of the Iterated MB-Integral

We will now collect our findings from the preceding paragraphs, to establish definite statements on the m-asymptotic behaviour of (8.4.1). For this, we recall (8.4.19), from which in the case $\delta_1 = \delta_2 = 0$ we deduce

$$\mathrm{Si}\left[m; \begin{matrix} 0, T_1, \infty \\ i\xi_2, \tau, T \end{matrix}\right] = \mathcal{S}i\left[m; \begin{matrix} 0, T_1, \infty \\ i\xi_2, \tau, T \end{matrix}\right], \tag{8.4.115}$$

since $\delta_j = 0$ for an infinite path $\mathcal{P}_j$ implies $\xi_j = 0$, by convention. According to this identity, we continue with the expansion (8.4.83), whose remainder terms, contrary to the case of two finite paths, will turn out negligible without additional assumptions. For a better overview we again distinguish between different parameter values, such that at least one summand in the indicated expansion vanishes.

First, since the parameter $x_0(\vec{p}_2)$ satisfies (8.4.24), if $\vec{p}_2 \in \{(\tau_2, 1), (\infty, 0)\}$, for an arbitrary $\varepsilon_2 \in (0, \min\{0, \varsigma_0(\vec{p}_2)\} + 1)$ we can write $x_0(\vec{p}_2) = \min\{0, \varsigma_0(\vec{p}_2)\} - \varepsilon_2$. Conversely, $x_0(\vec{p}_2) = -\varepsilon_2$ for $\varepsilon_2 \in (0,1)$, if $\vec{p}_2 \in \{(\tau_2, 0), (\infty, 1)\}$. In addition, the integration path $\Re w = u_1(\theta_1, \vec{p}_2)$ of the remainder term in (8.4.83) for $\theta_1 = 0$ is supposed to satisfy (8.4.79), i.e., for another arbitrary

$$\begin{cases} \varepsilon_1 \in (0, \eta_1(\vec{p}_2) - \eta_{00}), & \text{if } \eta_{00} \geq 0, \\ \varepsilon_1 \in (0, \min\{0, \eta_1(\vec{p}_2)\} - \eta_{00}), & \text{if } \eta_{00} < 0, \end{cases}$$

we have $u_1(0, \vec{p}_2) = \eta_{00} + \varepsilon_1$. If we require $\varepsilon := \varepsilon_1 - \varepsilon_2 > 0$, a statement analogous to (8.3.61) can be achieved. Moreover, by choosing $j_1(\theta_1, \vec{p}_2)$, $s_1(\vec{p}_2)$ and x_2 appropriately and $\varepsilon_1, \varepsilon_2$ sufficiently small, provided $\vec{p}_2 \in \{(\tau_2, 1), (\infty, 0)\}$, we can write $j_1(\theta_1, \vec{p}_2) = \iota_0(\theta_1, \vec{p}_2) + \varepsilon$, $s_1(\vec{p}_2) = \varsigma_0(\vec{p}_2) + \varepsilon$ and $x_2 = \chi_{20} + \varepsilon$. Finally, the remainder term of the expansion (8.4.83) for $\theta_1 = 1$, that of (8.4.100) for $\vec{p}_2 \in \{(\tau_2, 0), (\infty, 1)\}$, as well as each remainder of the expansions (8.4.107) and (8.4.114) for $\vec{p}_2 = (\tau_2, 0)$, is $\mathcal{O}\{m^{-q}\}$ for an arbitrary $q > 0$.

8.4.8.1. Two Infinite Paths with $\xi_1 = \xi_2 = 0$ and $\eta_{00} > 0$

We begin with the case of two infinite paths and non-oscillatory amplitudes, supposing $\eta_{00} > 0$. Formally $\vec{p}_2 = (\infty, 0)$ and $\theta_1 = 0$, which implies $\iota_0(0, (\infty, 0)) = \eta_{00} + \min\{0, \chi_{00}\}$ and $\varsigma_0(\infty, 0) = \chi_{00}$, by definition. Hence, from (8.4.83), (8.4.100) and (8.4.107), by (8.4.115), we obtain

$$\begin{aligned} \mathrm{Si}\left[m; \begin{matrix} 0, T_1, \infty \\ 0, T_2, \infty \end{matrix}\right] = \mathcal{M}_0\left[\begin{matrix} 0, 0, T_1, \infty \\ 0, 0, T_2, \infty \end{matrix}\right] & \mathbb{1}_{\{\iota_0(\theta_1, \vec{p}_2) > 0\}} \\ & - \frac{\Gamma(\iota_0(\theta_1, \vec{p}_2))}{(m+1)^{\iota_0(\theta_1, \vec{p}_2)}} \{\log(m+1) - \psi(\iota_0(\theta_1, \vec{p}_2))\} \\ & \qquad \times \mu_{-2}\left[\begin{matrix} \theta_1, \vec{p}_2 \\ \chi(\theta_1, \vec{p}_2) \end{matrix}\right] \mathbb{1}_{\{\iota_0(\theta_1, \vec{p}_2) \neq 0\}} \\ & + \frac{\Gamma(\iota_0(\theta_1, \vec{p}_2))}{(m+1)^{\iota_0(\theta_1, \vec{p}_2)}} \mu_{-1}\left[\begin{matrix} \theta_1, \vec{p}_2 \\ \chi(\theta_1, \vec{p}_2) \end{matrix}\right] \mathbb{1}_{\{\iota_0(\theta_1, \vec{p}_2) \neq 0\}} \end{aligned}$$

$$
\begin{aligned}
&- \left\{ (\log(m+1)+\gamma)\mu_{-1}\begin{bmatrix} \theta_1, \vec{p}_2 \\ \chi(\theta_1, \vec{p}_2) \end{bmatrix} - \mu_0 \begin{bmatrix} \theta_1, \vec{p}_2 \\ \chi(\theta_1, \vec{p}_2) \end{bmatrix} \right\} \mathbb{1}_{\{\iota_0(\theta_1, \vec{p}_2) = 0\}} \\
&- \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{a_{00}c_{00}}{\beta_{00}} \mathcal{P}_0 \begin{bmatrix} 0, 0 \\ T_2, \infty \end{bmatrix} \mathbb{1}_{\{\chi_{00} > 0\}} \\
&+ \frac{\Gamma(\eta_{00})\Gamma(\chi_{00})}{\{b_{00}(m+1)\}^{\eta_{00}+\chi_{00}}} \frac{a_{00}\{c_{00}\}^2}{\{\beta_{00}\}^2} \mathbb{1}_{\{\chi_{00} \neq 0\}} \\
&- \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \left\{ (\log(m+1)+\gamma) \frac{c_{00}}{\beta_{00}} + \pi_0(0; T_2, \infty) \right\} \\
&\qquad \times \frac{a_{00}c_{00}}{\beta_{00}} \mathbb{1}_{\{\chi_{00} = 0\}} \\
&+ \mathcal{O}\left\{ m^{-\iota_0(\theta_1, \vec{p}_2) - \varepsilon} \right\}.
\end{aligned}
$$

By inspection we readily verify the next theorem.

Theorem 8.4.1. *For $\eta_{00} > 0$, assume validity of the conditions (S6) and (S7). Then, provided at least one term on the right hand side is non-zero, as $m \to \infty$,*

(1) if $\chi_{00} > 0$, we have

$$
\begin{aligned}
\mathrm{Si}\left[m; \begin{matrix} 0, T_1, \infty \\ 0, T_2, \infty \end{matrix} \right] &\sim \mathcal{M}_0 \begin{bmatrix} 0, 0, T_1, \infty \\ 0, 0, T_2, \infty \end{bmatrix} \\
&\quad - \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{a_{00}c_{00}}{\beta_{00}} \left\{ \mathcal{P}_0 \begin{bmatrix} 0, 0 \\ T_1, \infty \end{bmatrix} + \mathcal{P}_0 \begin{bmatrix} 0, 0 \\ T_2, \infty \end{bmatrix} \right\}.
\end{aligned}
$$

The second term features the integral (8.4.49).

(2) if $\chi_{00} = 0$, we have

$$
\begin{aligned}
\mathrm{Si}\left[m; \begin{matrix} 0, T_1, \infty \\ 0, T_2, \infty \end{matrix} \right] &\sim \mathcal{M}_0 \begin{bmatrix} 0, 0, T_1, \infty \\ 0, 0, T_2, \infty \end{bmatrix} \\
&\quad - \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \{\log(m+1) - \psi(\eta_{00}) - \mu_0(T_1)\} \frac{a_{00}\{c_{00}\}^2}{\{\beta_{00}\}^2} \\
&\quad - \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} (\log(m+1)+\gamma) \frac{a_{00}\{c_{00}\}^2}{\{\beta_{00}\}^2} \\
&\quad - \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{a_{00}c_{00}}{\beta_{00}} \pi_0(0; T_2, \infty).
\end{aligned}
$$

The constants in the second and fourth term were defined in (8.4.74) and (8.4.105), respectively.

(3) if $\chi_{00} < 0$, we have

$$
\begin{aligned}
\mathrm{Si}\left[m; \begin{matrix} 0, T_1, \infty \\ 0, T_2, \infty \end{matrix} \right] &\sim \mathcal{M}_0 \begin{bmatrix} 0, 0, T_1, \infty \\ 0, 0, T_2, \infty \end{bmatrix} \mathbb{1}_{\{\eta_{00} > -\chi_{00}\}} \\
&\quad - \frac{\Gamma(\eta_{00}+\chi_{00})}{\{b_{00}(m+1)\}^{\eta_{00}+\chi_{00}}} \frac{a_{00}\{c_{00}\}^2}{\beta_{00}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{\Gamma(1-\gamma_{00})\Gamma(\alpha_{00}+\gamma_{00}-1)}{\Gamma(\alpha_{00})}\mathbb{1}_{\{\eta_{00}\neq-\chi_{00}\}}\\
&+(\log(m+1)+\gamma)\frac{a_{00}\left\{c_{00}\right\}^2}{\beta_{00}}\frac{\left\{\Gamma(1-\gamma_{00})\right\}^2}{\Gamma(\alpha_{00})}\mathbb{1}_{\{\eta_{00}=-\chi_{00}\}}\\
&+\mu_0\begin{bmatrix}0,(\infty,0)\\(0,0)\end{bmatrix}\mathbb{1}_{\{\eta_{00}=-\chi_{00}\}}\\
&+\frac{\Gamma(\eta_{00})\Gamma(\chi_{00})}{\left\{b_{00}(m+1)\right\}^{\eta_{00}+\chi_{00}}}\frac{a_{00}\left\{c_{00}\right\}^2}{\left\{\beta_{00}\right\}^2}.
\end{aligned}
$$

The coefficient of the fourth term refers to (8.4.94).

8.4.8.2. A Finite Path $\mathcal{P}_2$ and $\xi_1 = 0$

In these circumstances $\vec{p}_2 \in \{(\tau_2,0),(\tau_2,1)\}$ and $\theta_1 = 0$ with $\iota_0(0,(\tau_2,1)) = \varsigma_0(\tau_2,1) = \chi_{20}$. Depending on η_{00}, we distinguish between two cases.

8.4.8.2.1. The case $\eta_{00} \neq 0$. Then, in (8.4.83) the third and the fourth summand vanishes. Accordingly, by additional use of (8.4.100) and (8.4.107), as $m \to \infty$ we obtain:

$$
\begin{aligned}
\mathrm{Si}\left[m;\begin{matrix}0,T_1,\infty\\ i\xi_2,\tau_2,T_2\end{matrix}\right] &= \mathcal{M}_0\begin{bmatrix}0,0,T_1,\infty\\0,i\xi_2,\tau_2,T_2\end{bmatrix}\mathbb{1}_{\{\eta_{00}>0\}}\mathbb{1}_{\{\beta_{20}=0\,\vee\,\chi_{20}>0\}}\\
&-\frac{\Gamma(\chi_{20})}{\left\{b_{20}(m+1)\right\}^{\chi_{20}}}\frac{c_{20}}{\beta_{20}}e^{-i\xi_2\tau_2}\mathcal{N}_0\begin{bmatrix}0,\tau_2,i\xi_2\\0,0,T_1\end{bmatrix}\mathbb{1}_{\{\eta_{00}>0,\,\chi_{20}\neq0\}}\\
&+(\log(m+1)+\gamma)\frac{c_{20}}{\beta_{20}}e^{-i\xi_2\tau_2}\mathcal{N}_0\begin{bmatrix}0,\tau_2,i\xi_2\\0,0,T_1\end{bmatrix}\mathbb{1}_{\{\eta_{00}>0,\,\chi_{20}=0\}}\\
&+\mu_0\begin{bmatrix}0,(\tau_2,1)\\0\end{bmatrix}\mathbb{1}_{\{\eta_{00}>0\}}\mathbb{1}_{\{\chi_{20}=0\}}\\
&-\frac{\Gamma(\eta_{00})}{\left\{b_{00}(m+1)\right\}^{\eta_{00}}}\frac{a_{00}c_{00}}{\beta_{00}}\mathcal{P}_0\begin{bmatrix}0,i\xi_2\\\tau_2,T_2\end{bmatrix}\mathbb{1}_{\{\beta_{20}=0\,\vee\,\chi_{20}>0\}}\\
&+\frac{\Gamma(\eta_{00})\Gamma(\chi_{20})}{\left\{b_{00}(m+1)\right\}^{\eta_{00}}\left\{b_{20}(m+1)\right\}^{\chi_{20}}}\frac{c_{20}a_{00}c_{00}}{\beta_{20}\beta_{00}}e^{-i\xi_2\tau_2}\mathbb{1}_{\{\chi_{20}\neq0\}}\\
&-\frac{\Gamma(\eta_{00})}{\left\{b_{00}(m+1)\right\}^{\eta_{00}}}(\log(m+1)+\gamma)\frac{c_{20}a_{00}c_{00}}{\beta_{20}\beta_{00}}e^{-i\xi_2\tau_2}\mathbb{1}_{\{\chi_{20}=0\}}\\
&-\frac{\Gamma(\eta_{00})}{\left\{b_{00}(m+1)\right\}^{\eta_{00}}}\frac{a_{00}c_{00}}{\beta_{00}}\pi_0(1;\tau_2,T_2)\mathbb{1}_{\{\chi_{20}=0\}}\\
&+\mathbb{1}_{\{\beta_{20}=0\}}\mathcal{O}\left\{m^{-\eta_{00}-\varepsilon}\right\}\\
&+\mathbb{1}_{\{\beta_{20}>0,\,\eta_{00}>0\}}\mathcal{O}\left\{m^{-\chi_{20}-\varepsilon}\right\}\\
&+\mathbb{1}_{\{\beta_{20}>0\}}\mathcal{O}\left\{m^{-\eta_{00}-\min\{0,\chi_{20}\}-\varepsilon}\right\}
\end{aligned}
$$

This yields the following theorem.

Theorem 8.4.2. *For $\eta_{00} \neq 0$, assume validity of the conditions (S6), (S7) and (S9). Then, provided at least one term on the right hand side is non-zero, for any $\xi_2 \in \mathbb{R}$ as $m \to \infty$,*

(1) *if* $\eta_{00} > 0$, *with either* $\beta_{20} = 0$ *or* $\chi_{20} > 0$, *we have*

$$\begin{aligned}\mathrm{Si}\left[m; \begin{matrix} 0, T_1, \infty \\ i\xi_2, \tau_2, T_2 \end{matrix}\right] \sim \mathcal{M}_0\left[\begin{matrix} 0, 0, T_1, \infty \\ 0, i\xi_2, \tau_2, T_2 \end{matrix}\right] \\ - \frac{\Gamma(\chi_{20})}{\{b_{20}(m+1)\}^{\chi_{20}}} \frac{c_{20}}{\beta_{20}} e^{-i\xi_2\tau_2} \mathcal{N}_0\left[\begin{matrix} 0, \tau_2, i\xi_2 \\ 0, 0, T_1 \end{matrix}\right] \mathbb{1}_{\{\chi_{20} \le \eta_{00}\}} \\ - \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{a_{00}c_{00}}{\beta_{00}} \mathcal{P}_0\left[\begin{matrix} 0, i\xi_2 \\ \tau_2, T_2 \end{matrix}\right] \mathbb{1}_{\{\beta_{20} = 0 \vee \eta_{00} \le \chi_{20}\}}.\end{aligned}$$

(2) *if* $\eta_{00} > 0$ *and* $\chi_{20} \le 0$ *we have*

$$\begin{aligned}\mathrm{Si}\left[m; \begin{matrix} 0, T_1, \infty \\ i\xi_2, \tau_2, T_2 \end{matrix}\right] \sim (\log(m+1) + \gamma) \frac{c_{20}}{\beta_{20}} e^{-i\xi_2\tau_2} \mathcal{N}_0\left[\begin{matrix} 0, \tau_2, i\xi_2 \\ 0, 0, T_1 \end{matrix}\right] \mathbb{1}_{\{\chi_{20} = 0\}} \\ + \mu_0\left[\begin{matrix} 0, (\tau_2, 1) \\ 0 \end{matrix}\right] \mathbb{1}_{\{\chi_{20} = 0\}} \\ - \frac{\Gamma(\chi_{20})}{\{b_{20}(m+1)\}^{\chi_{20}}} \frac{c_{20}}{\beta_{20}} e^{-i\xi_2\tau_2} \mathcal{N}_0\left[\begin{matrix} 0, \tau_2, i\xi_2 \\ 0, 0, T_1 \end{matrix}\right] \mathbb{1}_{\{\chi_{20} < 0\}}.\end{aligned}$$

(3) *if* $\eta_{00} < 0$, *with either* $\beta_{20} = 0$ *or* $\chi_{20} > 0$, *we have*

$$\mathrm{Si}\left[m; \begin{matrix} 0, T_1, \infty \\ i\xi_2, \tau_2, T_2 \end{matrix}\right] \sim -\frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{a_{00}c_{00}}{\beta_{00}} \mathcal{P}_0\left[\begin{matrix} 0, i\xi_2 \\ \tau_2, T_2 \end{matrix}\right].$$

(4) *if* $\eta_{00} < 0$ *and* $\chi_{20} \le 0$, *we have*

$$\begin{aligned}\mathrm{Si}\left[m; \begin{matrix} 0, T_1, \infty \\ i\xi_2, \tau_2, T_2 \end{matrix}\right] \sim \frac{\Gamma(\eta_{00})\Gamma(\chi_{20})}{\{b_{00}(m+1)\}^{\eta_{00}} \{b_{20}(m+1)\}^{\chi_{20}}} \frac{c_{20}a_{00}c_{00}}{\beta_{20}\beta_{00}} e^{-i\xi_2\tau_2} \mathbb{1}_{\{\chi_{20} < 0\}} \\ - \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} (\log(m+1) + \gamma) \frac{c_{20}a_{00}c_{00}}{\beta_{20}\beta_{00}} e^{-i\xi_2\tau_2} \mathbb{1}_{\{\chi_{20} = 0\}} \\ - \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{a_{00}c_{00}}{\beta_{20}\beta_{00}} \pi_0(1; \tau_2, T_2) \mathbb{1}_{\{\chi_{20} = 0\}}.\end{aligned}$$

8.4.8.2.2. The case $\eta_{00} = 0$. Now, in (8.4.83) the first and the second summand vanishes, and from (8.4.107) and (8.4.114) as $m \to \infty$ we therefore deduce:

$$\begin{aligned}\mathrm{Si}\left[m; \begin{matrix} 0, T_1, \infty \\ i\xi_2, \tau_2, T_2 \end{matrix}\right] = (\log(m+1) + \gamma) \frac{a_{00}c_{00}}{\beta_{00}} \mathcal{P}_0\left[\begin{matrix} 0, i\xi_2 \\ \tau_2, T_2 \end{matrix}\right] \mathbb{1}_{\{\beta_{20} = 0 \vee \chi_{20} > 0\}} \\ - (\log(m+1) + \gamma) \frac{\Gamma(\chi_{20})}{\{b_{20}(m+1)\}^{\chi_{20}}} \frac{c_{20}a_{00}c_{00}}{\beta_{20}\beta_{00}} e^{-i\xi_2\tau_2} \mathbb{1}_{\{\chi_{20} \ne 0\}} \\ + (\log(m+1) + \gamma)^2 \frac{c_{20}a_{00}c_{00}}{\beta_{20}\beta_{00}} e^{-i\xi_2\tau_2} \mathbb{1}_{\{\chi_{20} = 0\}} \\ + (\log(m+1) + \gamma) \frac{a_{00}c_{00}}{\beta_{00}} \pi_0(1; \tau_2, T_2) \mathbb{1}_{\{\chi_{20} = 0\}} \\ + \frac{1}{\beta_{00}} \mathcal{Q}_0\left[\begin{matrix} 0, i\xi_2 \\ \tau_2, T_2, T_1 \end{matrix}\right] \mathbb{1}_{\{\beta_{20} = 0 \vee \chi_{20} > 0\}} \\ - \frac{\Gamma(\chi_{20})}{\{b_{20}(m+1)\}^{\chi_{20}}} \frac{c_{20}}{\beta_{20}\beta_{00}} e^{-i\xi_2\tau_2} \{\nu_1(\tau_2; T_1) + \tau_2\nu_2(\tau_2; T_1)\} \mathbb{1}_{\{\chi_{20} \ne 0\}}\end{aligned}$$

$$
\begin{aligned}
&+ (\log(m+1)+\gamma)\frac{c_{20}}{\beta_{20}\beta_{00}} e^{-i\xi_2\tau_2}\left\{\nu_1(\tau_2;T_1)+\tau_2\nu_2(\tau_2;T_1)\right\}\mathbb{1}_{\{\chi_{20}=0\}}\\
&+\frac{1}{\beta_{20}\beta_{00}}\omega_0\begin{bmatrix} i\xi_2 \\ \tau_2, T_2, T_1\end{bmatrix}\mathbb{1}_{\{\chi_{20}=0\}}\\
&+\mathbb{1}_{\{\beta_{20}=0\}}\mathcal{O}\left\{m^{-\varepsilon}\right\}\\
&+\mathbb{1}_{\{\beta_{20}>0\}}\mathcal{O}\left\{\log(m)m^{-\chi_{20}-\varepsilon}\right\}\\
&+\mathbb{1}_{\{\beta_{20}>0\}}\mathcal{O}\left\{m^{-\min\{0,\chi_{20}\}-\varepsilon}\right\}
\end{aligned}
$$

For $\chi_{20} \geq 0$ the leading asymptotic behaviour is logarithmic growth as $m \to \infty$, which eventually increases to logarithmic-algebraic divergence for $\chi_{20} < 0$.

Theorem 8.4.3. *For $\eta_{00} = 0$, assume validity of the conditions (S6), (S7), (S9) and (S10). Then, provided at least one term on the right hand side is non-zero, for any $\xi_2 \in \mathbb{R}$ as $m \to \infty$,*

(1) if $\beta_{20} = 0$ or $\chi_{20} > 0$, we have

$$
\mathrm{Si}\begin{bmatrix} m; \begin{matrix} 0, T_1, \infty \\ i\xi_2, \tau_2, T_2 \end{matrix}\end{bmatrix} \sim (\log(m+1)+\gamma)\frac{a_{00}c_{00}}{\beta_{00}}\mathcal{P}_0\begin{bmatrix} 0, i\xi_2 \\ \tau_2, T_2\end{bmatrix} + \frac{1}{\beta_{00}}\mathcal{Q}_0\begin{bmatrix} 0, i\xi_2 \\ \tau_2, T_2, T_1\end{bmatrix}.
$$

(2) if $\chi_{20} = 0$, we have

$$
\begin{aligned}
\mathrm{Si}\begin{bmatrix} m; \begin{matrix} 0, T_1, \infty \\ i\xi_2, \tau_2, T_2 \end{matrix}\end{bmatrix} \sim\; & (\log(m+1)+\gamma)^2\frac{c_{20}a_{00}c_{00}}{\beta_{20}\beta_{00}} e^{-i\xi_2\tau_2}\\
&+ (\log(m+1)+\gamma)\frac{a_{00}c_{00}}{\beta_{00}}\pi_0(1;\tau_2,T_2)\\
&+ (\log(m+1)+\gamma)\frac{c_{20}}{\beta_{20}\beta_{00}} e^{-i\xi_2\tau_2}\left\{\nu_1(\tau_2;T_1)+\tau_2\nu_2(\tau_2;T_1)\right\}\\
&+\frac{1}{\beta_{20}\beta_{00}}\omega_0\begin{bmatrix} i\xi_2 \\ \tau_2, T_2, T_1\end{bmatrix}.
\end{aligned}
$$

(3) if $\chi_{20} < 0$, we have

$$
\begin{aligned}
\mathrm{Si}\begin{bmatrix} m; \begin{matrix} 0, T_1, \infty \\ i\xi_2, \tau_2, T_2 \end{matrix}\end{bmatrix} \sim\; & -(\log(m+1)+\gamma)\frac{\Gamma(\chi_{20})}{\{b_{20}(m+1)\}^{\chi_{20}}}\frac{c_{20}a_{00}c_{00}}{\beta_{20}\beta_{00}} e^{-i\xi_2\tau_2}\\
&- \frac{\Gamma(\chi_{20})}{\{b_{20}(m+1)\}^{\chi_{20}}}\frac{c_{20}}{\beta_{20}\beta_{00}} e^{-i\xi_2\tau_2}\left\{\nu_1(\tau_2;T_1)+\tau_2\nu_2(\tau_2;T_1)\right\}.
\end{aligned}
$$

8.4.8.3. An Oscillatory Amplitude with $\xi_1 \in \mathbb{R}\setminus\{0\}$

The derivation of m-asymptotic statements on Si[...] in the case $\xi_1 \in \mathbb{R}\setminus\{0\}$ requires additional arguments. By convention this means $\theta_1 = 1$ and, in view of (8.4.13), we must therefore consider the cases

$$
\vec{p}_2 \in \{(\tau_2, 0), (\tau_2, 1), (\infty, 1), (\infty, 0)\}.
$$

Furthermore, with $\theta_1 = 1$ in (8.4.83) all but the first and the remainder term vanish, where the latter is exponentially small. Hence, with the aid of (8.4.100), as $m \to \infty$ we can write

$$\begin{aligned}
\mathcal{S}i\begin{bmatrix} m; & \zeta_1, T_1, \infty \\ & \zeta_2, \tau, T \end{bmatrix} &= \mathcal{M}_0\begin{bmatrix} 0, \zeta_1, T_1, \infty \\ 0, \zeta_2, \tau, T \end{bmatrix} \mathbb{1}_{\{\vec{p}_2 \in \{(\tau_2,0),(\infty,1)\} \vee \iota_0(1,\vec{p}_2)>0\}} \\
&+ \frac{\Gamma(\iota_0(1,\vec{p}_2))}{(m+1)^{\iota_0(1,\vec{p}_2)}}\mu_{-1}\begin{bmatrix} 1, \vec{p}_2 \\ \chi(1,\vec{p}_2) \end{bmatrix} \mathbb{1}_{\{\iota_0(1,\vec{p}_2)\neq 0\}} \\
&- \left\{ (\log(m+1)+\gamma)\mu_{-1}\begin{bmatrix} 1, \vec{p}_2 \\ \chi(1,\vec{p}_2) \end{bmatrix} - \mu_0\begin{bmatrix} 1, \vec{p}_2 \\ \chi(1,\vec{p}_2) \end{bmatrix} \right\} \mathbb{1}_{\{\iota_0(1,\vec{p}_2)=0\}} \\
&+ \mathbb{1}_{\{\vec{p}_2 \in \{(\tau_1,0),(\infty,1)\}\}} \mathcal{O}\left\{m^{-q}\right\} \\
&+ \mathbb{1}_{\{\vec{p}_2 \in \{(\tau_1,1),(\infty,0)\}\}} \mathcal{O}\left\{m^{-\iota_0(1,\vec{p}_2)-\varepsilon}\right\},
\end{aligned} \tag{8.4.116}$$

for an arbitrary $q > 0$ and an appropriate $\varepsilon > 0$. Recall that $\iota_0(1,\vec{p}_2) \in \{\chi_{20}, \eta_{00}\}$ for $\vec{p}_2 \in \{(\tau_2,1),(\infty,0)\}$. Below we will show, that the dominating term of this expansion for fixed $\delta_j > 0$ with $j \in \{1,2\}$ remains the controlling term after the transition $\delta_j \downarrow 0$, and that, for infinite $\mathcal{P}_j$, this asymptotic statement holds uniformly with respect to ξ_j in any subinterval of the real axis, whose closure does not contain the origin.

Theorem 8.4.4. *Assume validity of (S6) to (S9). Then, provided at least one term on the right hand side is non-zero, for an arbitrary $\rho > 0$, as $m \to \infty$,*

(1) uniformly with respect to $|\xi_1| \geq \rho$ and $\xi_2 \in \mathbb{R}$, we have

$$\begin{aligned}
\mathrm{Si}\begin{bmatrix} m; & i\xi_1, T_1, \infty \\ & i\xi_2, \tau_2, T_2 \end{bmatrix} &\sim \lim_{\delta_1\downarrow 0}\mathcal{M}_0\begin{bmatrix} 0, \delta_1+i\xi_1, T_1, \infty \\ 0, i\xi_2, \tau_2, T_2 \end{bmatrix} \mathbb{1}_{\{\beta_{20}=0 \vee \chi_{20}>0\}} \\
&- \frac{\Gamma(\chi_{20})}{\{b_{20}(m+1)\}^{\chi_{20}}}\frac{c_{20}}{\beta_{20}}e^{-i\xi_2\tau_2}\lim_{\delta_1\downarrow 0}\mathcal{N}_0\begin{bmatrix} 0, \tau_2, \delta_1+i\xi_1 \\ 0, 0, T_1 \end{bmatrix} \mathbb{1}_{\{\chi_{20}\neq 0\}} \\
&+ (\log(m+1)+\gamma)\frac{c_{20}}{\beta_{20}}e^{-i\xi_2\tau_2}\lim_{\delta_1\downarrow 0}\mathcal{N}_0\begin{bmatrix} 0, \tau_2, \delta_1+i\xi_1 \\ 0, 0, T_1 \end{bmatrix} \mathbb{1}_{\{\chi_{20}=0\}} \\
&+ \lim_{\delta_1\downarrow 0}\mu_0\begin{bmatrix} 1, (\tau_2, 1) \\ 0 \end{bmatrix} \mathbb{1}_{\{\chi_{20}=0\}}.
\end{aligned}$$

The right hand side features the integrals (8.4.27) and (8.4.91).

(2) uniformly with respect to $|\xi_1| \geq \rho$, we have

$$\begin{aligned}
\mathrm{Si}\begin{bmatrix} m; & i\xi_1, T_1, \infty \\ & 0, T_2, \infty \end{bmatrix} &\sim \lim_{\delta_1\downarrow 0}\mathcal{M}_0\begin{bmatrix} 0, \delta_1+i\xi_1, T_1, \infty \\ 0, 0, T_2, \infty \end{bmatrix} \mathbb{1}_{\{\eta_{00}>0\}} \\
&- \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}}\frac{a_{00}c_{00}}{\beta_{00}}\lim_{\delta_1\downarrow 0}\mathcal{P}_0\begin{bmatrix} 0, \delta_1+i\xi_1 \\ T_1, \infty \end{bmatrix} \mathbb{1}_{\{\eta_{00}\neq 0\}} \\
&+ (\log(m+1)+\gamma)\frac{a_{00}c_{00}}{\beta_{00}}\lim_{\delta_1\downarrow 0}\mathcal{P}_0\begin{bmatrix} 0, \delta_1+i\xi_1 \\ T_1, \infty \end{bmatrix} \mathbb{1}_{\{\eta_{00}=0\}} \\
&+ \frac{1}{\beta_{00}}\lim_{\delta_1\downarrow 0}\mathcal{Q}_0\begin{bmatrix} 0, \delta_1+i\xi_1 \\ T_1, \infty, T_2 \end{bmatrix} \mathbb{1}_{\{\eta_{00}=0\}}.
\end{aligned}$$

For the integral functions on the right hand side, we refer to (8.4.49) and (8.4.50).

(3) uniformly with respect to $|\xi_1|, |\xi_2| \geq \rho$, we have

$$\mathrm{Si}\left[m; \begin{matrix} i\xi_1, T_1, \infty \\ i\xi_2, T_2, \infty \end{matrix}\right] \sim \lim_{\delta_1,\delta_2 \downarrow 0} \mathcal{M}_0\left[\begin{matrix} 0, \delta_1 + i\xi_1, T_1, \infty \\ 0, \delta_2 + i\xi_2, T_2, \infty \end{matrix}\right].$$

Proof. In its complete form, without big-$\mathcal{O}$ estimates, the expansion (8.4.116) features two remainder integrals. Appealing to definition (8.4.75), the first is obtained from plugging (8.4.78) into (8.4.26), which yields

$$\mathrm{R}_1(m; \vec{p}_2) := \frac{1}{(2\pi i)^2} \int\limits_{x_0(\vec{p}_2)-i\infty}^{x_0(\vec{p}_2)+i\infty} \frac{\Gamma(z)}{(m+1)^z} \int\limits_{u_1-i\infty}^{u_1+i\infty} \frac{\Gamma(w)}{(m+1)^w} \mathcal{M}_0\left[\begin{matrix} -w, \zeta_1, T_1, \infty \\ -z, \zeta_2, \tau, T \end{matrix}\right] dwdz, \tag{8.4.117}$$

where the interior path $\Re w = u_1$, by (8.4.79) with $u_1(1, \vec{p}_2) \equiv u_1$, is a line with arbitrary positive real part, whereas the exterior path $\Re z = x_0(\vec{p}_2)$ satisfies (8.4.24). The second remainder term in (8.4.116) results from the expansion (8.4.100) for the integral (8.3.38), and it is of the form

$$\mathrm{R}_2(m; \vec{p}_2) := \frac{1}{2\pi i} \int\limits_{j_1(1,\vec{p}_2)-i\infty}^{j_1(1,\vec{p}_2)+i\infty} \frac{\Gamma(z)}{(m+1)^z} \mathcal{M}_0\left[\begin{matrix} 0, \zeta_1, T_1, \infty \\ -z, \zeta_2, \tau, T \end{matrix}\right] dz, \tag{8.4.118}$$

for a path $\Re z = j_1(1, \vec{p}_2)$, which satisfies (8.4.97) or (8.4.99). Roughly speaking, $j_1(1, \vec{p}_2) > 0$ is arbitrary if $\vec{p}_2 \in \{(\tau_2, 0), (\infty, 1)\}$ or $j_1(1, \vec{p}_2) = \iota_0(1, \vec{p}_2) + \varepsilon$ otherwise, for an appropriate $\varepsilon > 0$. The generating function in (8.4.117) refers to the integral (8.4.14), whereas in (8.4.118) it possibly refers to its z-analytic continuation for $w = 0$.

We will now show that, under the assumption (S8), the generating function (8.4.14) attains a finite limit as $\delta_1 \downarrow 0$, as well as the existence of a uniform bound with respect to $\delta_1 \geq 0$ for (8.4.117). For this, in terms of (8.4.27), we write

$$\mathcal{M}_0\left[\begin{matrix} -w, \zeta_1, T_1, \infty \\ -z, \zeta_2, \tau, T \end{matrix}\right] = \int\limits_{\mathcal{P}_2} \{\varphi(t)\}^{-z} e^{-\zeta_2 t} c(t; p_2) \mathcal{N}_0\left[\begin{matrix} -w, t, \zeta_1 \\ 0, 0, T_1 \end{matrix}\right] dt. \tag{8.4.119}$$

The expansion (8.4.44) has revealed that the interior function has a finite limit as $\delta_1 \downarrow 0$ for $\xi_1 \neq 0$. Upon plugging this expansion into (8.4.119), we arrive at

$$\begin{aligned} \mathcal{M}_0\left[\begin{matrix} -w, \zeta_1, T_1, \infty \\ -z, \zeta_2, \tau, T \end{matrix}\right] &= \sum_{n_1=0}^{N_1-1} \frac{e^{-\zeta_1 T_1}}{\zeta_1^{1+n_1}} \{\varphi(T_1)\}^{-w} \sum_{k_1=0}^{n_1} S_{k_1,n_1}(T_1, w) \\ &\quad \times \int\limits_{\mathcal{P}_2} \{\varphi(t)\}^{-z} e^{-\zeta_2 t} c(t; p_2) a^{(k_1)}(T_1 + t) dt \\ &\quad + \frac{1}{\zeta_1^{N_1}} \int\limits_{\mathcal{P}_2} \{\varphi(t)\}^{-z} e^{-\zeta_2 t} c(t; p_2) \mathcal{N}_0\left[\begin{matrix} -w, t, \zeta_1 \\ N_1, 0, T_1 \end{matrix}\right] dt. \end{aligned} \tag{8.4.120}$$

We first assume $\vec{p}_2 \in \{(\tau_1, 0), (\tau_1, 1), (\infty, 0)\}$, i.e., $\zeta_2 = i\xi_2$ with $\xi_2 \neq 0$ only if $p_2 = \tau_2$. Then, absolute convergence of the single integrals in (8.4.120) for $\Re z = x_0(\vec{p}_2)$ follows immediately. Furthermore, according to Corollary 8.2.2, Lemma 8.2.3 and due to (8.4.42), for $\Re z = x_0(\vec{p}_2)$ and $\Re w = j_1(1, \vec{p}_2)$, we have absolute and with respect to $\delta_1 \geq 0$ uniform convergence of the iterated integral, if we choose

$$N_1 > j_1(1, \vec{p}_2)\beta_{00} - \alpha_{00} - \gamma_{00} + 1.$$

In these circumstances, the whole right hand side of (8.4.120) is uniformly bounded with respect to $\delta_1 \geq 0$ if $\xi_1 \neq 0$, and in the case $p_2 = \tau_2$ even with respect to $\xi_2 \in \mathbb{R}$. The limits as $\delta_1 \downarrow 0$ thus exist and may be computed under the signs of integration. Moreover, in §8.4.2.2 it was pointed out that, by definition of $S_{k_1,n_1}(s, w)$, there exist coefficients $f_l(s, k_1, n_1) \geq 0$ for $0 \leq l \leq n_1 - k_1$, which are continuous functions of the variable s, exhibiting the behaviour $f_l(s, k_1, n_1) = \mathcal{O}\left\{s^{-\gamma_{00}+k_1-n_1}\right\}$ as $s \to \infty$, such that

$$|S_{k_1,n_1}(s, w)| \leq \sum_{l=0}^{n_1-k_1} |w|^l f_l(s, k_1, n_1). \tag{8.4.121}$$

This estimate yields for (8.4.120) a bound similar to (6.6.14). In particular, there exist coefficients $F_l(\vec{p}_2) \geq 0$, which are uniformly bounded with respect to $\delta_1 \geq 0$, $|\xi_1| \geq \rho$, $\Im w, \Im z \in \mathbb{R}$, and if $p_2 = \tau_2$ also uniformly with respect to $\xi_2 \in \mathbb{R}$, such that for $\Re w = j_1(1, \vec{p}_2)$ and $\Re z = x_0(\vec{p}_2)$ we have

$$\left|\mathcal{M}_0\begin{bmatrix} -w, \zeta_1, T_1, \infty \\ -z, \zeta_2, \tau, T \end{bmatrix}\right| \leq \sum_{l=0}^{N_1} |w|^l F_l(\vec{p}_2). \tag{8.4.122}$$

An application of this bound to (8.4.117) shows, that this last MB-integral possesses a limit as $\delta_1 \downarrow 0$, which can be carried out under its signs of integration. Analogous arguments can be employed for the integral $\mathrm{R}_2(m; \vec{p}_2)$, by means of an appropriate representation for its generating function, and for the proof of the actual existence of the limit of each term in the expansions from Theorem 8.4.4(1) and (2). The validity of these expansions then follows immediately from the identity (8.4.19).

It remains to verify the statement of Theorem 8.4.4(3), i.e., to treat the case $\vec{p}_2 = (\infty, 1)$. Then, the single integrals in the expansion (8.4.120) still converge absolutely for $\delta_2 > 0$ and $\Re z = x_0(\vec{p}_2)$ but they need not be uniformly convergent with respect to $\delta_2 \geq 0$. Furthermore, with N_1 as above, also the integral in the last addend converges absolutely and with respect to $\delta_1 \geq 0$ uniformly merely for fixed $\delta_2 > 0$. A uniformity statement with respect to $\delta_2 \geq 0$ obviously can not be obtained by increasing N_1. Instead we interchange the order of integration and cast (8.4.120) in terms of the definition (8.4.27), which yields

$$\mathcal{M}_0\begin{bmatrix} -w, \zeta_1, T_1, \infty \\ -z, \zeta_2, T_2, \infty \end{bmatrix} = \sum_{n_1=0}^{N_1-1} \frac{e^{-\zeta_1 T_1}}{\zeta_1^{1+n_1}} \{\varphi(T_1)\}^{-w} \sum_{k_1=0}^{n_1} S_{k_1,n_1}(T_1, w) \mathcal{N}_0\begin{bmatrix} -z, T_1, \zeta_2 \\ 0, k_1, T_2 \end{bmatrix}$$

$$+\frac{1}{\zeta_1^{N_1}}\sum_{k_1=0}^{N_1}\int_{T_1}^{\infty}\{\varphi(s)\}^{-w}\,e^{-\zeta_1 s}S_{k_1,N_1}(s,w)\mathcal{N}_0\begin{bmatrix}-z,\,s,\,\zeta_2\\0,\,k_1,\,T_2\end{bmatrix}ds. \tag{8.4.123}$$

Then, a second application of the expansion (8.4.44) with $N = N_2$, for sufficiently large N_2, leads to an expansion for (8.4.123), whose limit as $\delta_1, \delta_2 \downarrow 0$ exists. Moreover, analogous to (8.4.122), it can be shown that the resulting expansion is uniformly bounded with respect to $\delta_1, \delta_2 \geq 0$ for $|\xi_1|, |\xi_2| \geq \rho$ by a polynomial of $|w|$ and $|z|$. ∎

8.4.8.4. Two Infinite Paths with $\xi_1 = \xi_2 = 0$ and $\eta_{00} \leq 0$

We close this section with a treatment of (8.4.1) in the special case of two infinite integration paths with non-oscillatory amplitudes and $\eta_{00} \leq 0$, which implies $\zeta_1 = \zeta_2 = 0$ and $\chi_{00} < 0$. This case has been omitted in before, and it is characterized by the possibility to induce in the iterated MB-integral (8.4.26) a dependence of the interior path on the real part of the exterior path. In particular, since $\chi_{00} > -1$ by (8.4.8), the integration path of the exterior MB-integral, instead of (8.4.20), with $x_0 \equiv x_0(\infty, 0)$ may satisfy

$$\chi_{00} < x_0 < \eta_{00}. \tag{8.4.124}$$

As a consequence, for $\zeta_1 = \zeta_2 = 0$ and fixed $z \in \mathbb{C}$ with $\Re z = x_0$, by (8.4.17), the integral definition (8.4.14) of the iterated generating function with respect to $w \in \mathbb{C}$ is absolutely convergent and holomorphic in the half plane

$$\Re w < \eta_{00} + \chi_{00} - x_0, \tag{8.4.125}$$

wherein the origin is not contained. The representation (8.4.26) is thus admissible for $u_0 \equiv u_0(0)$ subject to

$$-1 < u_0 < \eta_{00} + \chi_{00} - x_0. \tag{8.4.126}$$

Now, according to §8.4.3.3, the analytic continuation of (8.4.14) is furnished by the expansion (8.4.60), which is valid in the strip

$$\chi_{00} < \Re w < \eta_{00} + \chi_{00} - x_0 + \eta_{\beta_{00}}(\alpha_{01}, \beta_{01}, \gamma_{01}). \tag{8.4.127}$$

The only singularities we find therein are poles, of which the one at $w = \eta_{00} + \chi_{00} - z$ lies closest to the right of the line $\Re w = u_0$. Since $x_0 > \chi_{00}$, it is of simple order and lies to the left of the point $z = \eta_{00}$, where a subsequent pole will be found, whenever being contained in the strip (8.4.127). We can therefore perform a displacement of the integration path to the right, to match a line $\Re w = u_1$ with

$$\eta_{00} + \chi_{00} - x_0 < u_1 < \eta_{00} + \min\{0, \chi_{00} - x_0 + \eta_{\beta_{00}}(\alpha_{01}, \beta_{01}, \gamma_{01})\}. \tag{8.4.128}$$

The shift of the integration path is permitted due to the exponential decay of the integrand in each imaginary direction of the half plane

$$\Re w < \eta_{00} + \chi_{00} - x_0 + \eta_{\beta_{00}}(\alpha_{01}, \beta_{01}, \gamma_{01}),$$

since it was pointed out in §8.4.3.3, that the analytic continuation is $\mathcal{O}(w)$ as $\Im w \to \pm\infty$, uniformly with respect to $\Re w$ in its strip of validity. Taking into account the encirclement of the pole at $w = \eta_{00} + \chi_{00} - z$ in the negative direction and the residue (8.4.70), for fixed $z \in \mathbb{C}$ with $\Re z = x_0$, from (8.4.75) we eventually obtain

$$\mathrm{I}\left[m; \begin{matrix} 0 \\ z, (\infty, 0) \end{matrix}\right] = \frac{a_{00}\left\{c_{00}\right\}^2}{\beta_{00}\Gamma(\alpha_{00})} \frac{\Gamma(\eta_{00}+\chi_{00}-z)\Gamma(\beta_{00}(z-\chi_{00}))\Gamma(\beta_{00}(\eta_{00}-z))}{\{b_{00}(m+1)\}^{\chi_{00}+\eta_{00}}}(m+1)^z$$
$$+ \frac{1}{2\pi i}\int\limits_{u_1-i\infty}^{u_1+i\infty} (m+1)^{-w}\Gamma(w)\mathcal{M}_0\left[\begin{matrix} -w, 0, T_1, \infty \\ -z, 0, T_2, \infty \end{matrix}\right] dw.$$

Upon finally plugging this expansion into (8.4.26), we observe cancellations in the power of the asymptotic parameter. An additional use of the identity (8.4.115) accompanied by an application of the bound (8.4.69) to the remainder integral, as $m \to \infty$ shows

$$\text{(8.4.129)} \qquad \mathrm{Si}\left[m; \begin{matrix} 0, T_1, \infty \\ 0, T_2, \infty \end{matrix}\right] = \frac{\Pi(\alpha_{00}, \beta_{00}, \gamma_{00})}{\{b_{00}(m+1)\}^{\chi_{00}+\eta_{00}}} \frac{a_{00}\left\{c_{00}\right\}^2}{\beta_{00}\Gamma(\alpha_{00})} + \mathcal{O}\left\{m^{-x_0-u_1}\right\},$$

where for brevity, with $\chi_{00} < x_0 < \eta_{00} \leq 0$, we denote

$$\text{(8.4.130)} \quad \Pi(\alpha_{00}, \beta_{00}, \gamma_{00}) := \frac{1}{2\pi i}\int\limits_{x_0-i\infty}^{x_0+i\infty} \Gamma(z)\Gamma(\eta_{00}+\chi_{00}-z)\Gamma(\beta_{00}(z-\chi_{00}))\Gamma(\beta_{00}(\eta_{00}-z))dz.$$

In view of (8.4.128) the estimate in the big-$\mathcal{O}$ clearly is of higher order than the first term in the expansion (8.4.129), thereby verifying our final theorem.

Theorem 8.4.5. *For $\eta_{00} \leq 0$, under the assumptions (S6) and (S7), provided (8.4.130) is non-zero, as $m \to \infty$ we have*

$$\mathrm{Si}\left[m; \begin{matrix} 0, T_1, \infty \\ 0, T_2, \infty \end{matrix}\right] \sim \frac{\Pi(\alpha_{00}, \beta_{00}, \gamma_{00})}{\{b_{00}(m+1)\}^{\chi_{00}+\eta_{00}}} \frac{a_{00}\left\{c_{00}\right\}^2}{\beta_{00}\Gamma(\alpha_{00})}.$$

8.5. An Infinite Path of the First Kind in an Asymmetric-Type Iterated Integral

We now investigate the m-asymptotic behaviour of the asymmetric-type iterated integral (8.0.2). Particularly in the case of two integration paths of the same kind, compared to the setup with an integrand of symmetric type, several inconveniences will arise, due to the odd structure. For instance, the order in which we introduce the MB-representation for the m-powers is crucial.

A sophisticated choice of their order may lead to simplifications. Moreover, the abscissa of convergence of the associated generating function will substantially depend on, whether it is conceived as a function of the first variable with the second fixed or vice versa. Converse to the treatment of symmetric-type integrals, we first examine setups with infinite paths, whereas we conclude the chapter with the case of two finite paths.

Consider the integral (8.0.2) with $\mathcal{P}_j = [T_j, \infty)$ for $j \in \{1, 2\}$, where $T_1 > 0$ and $T_2 \geq 0$, in the form

(8.5.1)
$$\mathrm{Ai}\left[m; \begin{matrix} T_1 \\ T_2 \end{matrix}\right] := \int\limits_{T_2}^{\infty} \overline{\mathfrak{a}}(t) \int\limits_{T_1}^{\infty} \mathfrak{c}(s) \left\{1 - (1 - \Psi(s))^{m+1}\right\} \times \left\{1 - (1 - \Psi(s+t))^{m+1}\right\} \overline{\mathfrak{c}}(s+t) ds dt.$$

The overline indicates the complex conjugate of the functions $\mathfrak{a}$ and $\mathfrak{c}$. With φ as per (8.1.5), we make the following assumptions:

(A1) With respect to $r \geq \min\{T_1, T_2\}$ the functions $\varphi(r)$ and $\mathfrak{c}(r)$ are once continuously differentiable and $\varphi(r) > 0$. As $r \to \infty$ each of them is algebraic with parameters $\beta_{00} > 0$, $\gamma_{00} \in \mathbb{R}$ and coefficients $b_{00} > 0$, $c_{00} \in \mathbb{C} \setminus \{0\}$, where

(8.5.2)
$$\beta_{00} + \gamma_{00} > 1.$$

In addition, the first derivative of the normalized phase $B(r)$ and of the normalized amplitude $\mathfrak{C}(r)$ is of respective order $\beta_{01}, \gamma_{01} > 1$ as $r \to \infty$.

(A2) Either of the two properties below holds:

(a) $T_2 > 0$, and $\mathfrak{a}(t)$ is algebraic as $t \to \infty$ with parameter $\alpha_{00} > 0$ and coefficient $a_{00} \in \mathbb{C} \setminus \{0\}$. Moreover, the function is once continuously differentiable on $t \geq T_2$, and at infinity its normalized counterpart is of order $\alpha_{01} > 1$.

(b) $T_2 \geq 0$, and the function $\mathfrak{a}(t)$ is piecewise continuous on $t \geq T_2$ with a finite number of jump discontinuities, and $\mathfrak{a}(t) = \mathcal{O}\left\{t^{-\alpha_{00}}\right\}$ as $t \to \infty$ for all $\alpha_{00} > 0$.

Notice that (A2b) especially includes functions of the form $\mathfrak{a}(t) = f(t)\mathbb{1}_{\{\tau \leq t \leq T\}}$ for a continuous f and $0 \leq \tau < T < \infty$. For brevity, frequent use will be made of the parameters (8.4.7) and of the parameter

(8.5.3)
$$\nu_{00} := \frac{\gamma_{00} - 1}{\beta_{00}} + \frac{\gamma_{00}}{\beta_{00}},$$

where the last satisfies the identity

(8.5.4)
$$2\chi_{00} + \frac{1}{\beta_{00}} = \nu_{00}.$$

We remind the reader of our comments from the preceding two sections, that some of the coefficients in the Laurent expansions below may actually be zero, which we keep in mind but will not be mentioned each time. Now, under the above conditions, by Corollary 8.2.2, the integral (8.5.1) converges absolutely for any $m \geq 0$ and the corresponding iterated generating function will be denoted by

$$\mathcal{S}_0\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} := \int_{T_2}^{\infty} \bar{\mathfrak{a}}(t) \int_{T_1}^{\infty} \{\varphi(s)\}^{-w}\, \mathfrak{c}(s)\, \{\varphi(s+t)\}^{-z}\, \bar{\mathfrak{c}}(s+t) ds dt. \tag{8.5.5}$$

Again by Corollary 8.2.2, this double integral converges absolutely for

$$\begin{cases} \Re z < \eta_{00}, \\ \Re w < 2\chi_{00} - \Re z + \frac{1}{\beta_{00}} \min\{1, \alpha_{00}\}. \end{cases} \tag{8.5.6}$$

These conditions equivalently can be rewritten in the form

$$\begin{cases} w \in \mathbb{C}, \\ \Re z < \min\{\nu_{00} - \Re w, \eta_{00}, \eta_{00} + \chi_{00} - \Re w\}. \end{cases} \tag{8.5.7}$$

It shows that a distinction between different parametrizations is clearer and easier, if we first consider $\mathcal{S}_0[\ldots]$ for fixed z as a function of w.

8.5.1. Transformation to an Iterated MB-Integral

To proceed in the desired manner, in (8.5.1) we must first introduce the Cahen-Mellin representation (8.1.6) for the m-power of the variable $s+t$, which leads to

$$\begin{aligned} \mathrm{Ai}\left[m; \begin{matrix} T_1 \\ T_2 \end{matrix}\right] = \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} (m+1)^{-z}\Gamma(z) \int_{T_2}^{\infty} \bar{\mathfrak{a}}(t) \int_{T_1}^{\infty} \left\{1 - (1-\Psi(s))^{m+1}\right\} \\ \times\, \mathfrak{c}(s)\, \{\varphi(s+t)\}^{-z}\, \bar{\mathfrak{c}}(s+t) ds dt dz, \end{aligned} \tag{8.5.8}$$

with an integration path, by definition of the complex integral (8.1.6) and subject to absolute convergence, that satisfies

$$-1 < x_0 < \psi_\infty, \tag{8.5.9}$$

where we define

$$\psi_\infty := \min\{0, \nu_{00} + 1, \eta_{00}, \eta_{00} + \chi_{00} + 1\}. \tag{8.5.10}$$

In the case of (A2b), without loss of generality we assume $\psi_\infty = \min\{0, \nu_{00}+1\}$, since $\alpha_{00} > 0$ then can be chosen arbitrarily large. According to (8.5.6), the w-abscissa of convergence for fixed $z \in \mathbb{C}$ with $\Re z = x_0$ of (8.5.5) is given by

$$\begin{aligned} \nu_0(\alpha_{00}) &:= 2\chi_{00} - x_0 + \frac{1}{\beta_{00}} \min\{1, \alpha_{00}\} \\ &= \min\{\nu_{00}, \eta_{00} + \chi_{00}\} - x_0. \end{aligned} \tag{8.5.11}$$

Hence, for

$$-1 < u_0 < \min\{0, \nu_0(\alpha_{00})\}, \tag{8.5.12}$$

we may also represent the second m-power by virtue of the Cahen-Mellin integral, from which for (8.5.8) we finally deduce the iterated MB-integral

$$\mathrm{Ai}\left[m; \begin{matrix} T_1 \\ T_2 \end{matrix}\right] = \frac{1}{2\pi i} \int\limits_{x_0 - i\infty}^{x_0 + i\infty} (m+1)^{-z} \Gamma(z) \, \mathrm{J}(m,z) dz, \tag{8.5.13}$$

whose interior, in terms of (8.5.5), is equal to

$$\mathrm{J}(m,z) := \frac{1}{2\pi i} \int\limits_{u_0 - i\infty}^{u_0 + i\infty} (m+1)^{-w} \Gamma(w) \mathcal{S}_0 \left[\begin{matrix} -w, T_1 \\ -z, T_2 \end{matrix}\right] dw. \tag{8.5.14}$$

Due to the special structure of the iterated generating function, in each case the path $\Re w = u_0$ of the interior MB-integral depends on the exterior. We now specify the w-analytic continuation of $\mathcal{S}_0[\ldots]$ into a region that includes the line $\Re w = \nu_0(\alpha_{00})$. This suggests the necessity, to distinguish between $\alpha_{00} > 1$ and $\alpha_{00} \leq 1$.

8.5.2. An Interior Generating Function with an Infinite Path and a Kernel of the First and of the Second Kind

With $w, z \in \mathbb{C}$, $n_1, n_2 \in \{0, 1\}$ and $t \geq 0$, we define a generalized version of the interior of the iterated generating function (8.5.5) by

$$\begin{aligned} \mathrm{Q}_0 \left[\begin{matrix} w - \chi_{00}, z - \chi_{00} \\ t, n_1, n_2, w, z, T_1 \end{matrix}\right] &:= \int\limits_{T_1}^{\infty} \frac{s^{\beta_{00}(w - \chi_{00}) - 1}}{(s+t)^{1 - \beta_{00}(z - \chi_{00})}} \frac{d^{n_1}}{ds^{n_1}} \left\{ e^{wB(s)} \mathfrak{C}(s) \right\} \\ &\qquad \times \frac{d^{n_2}}{ds^{n_2}} \left\{ e^{zB(s+t)} \overline{\mathfrak{C}}(s+t) \right\} ds. \end{aligned} \tag{8.5.15}$$

For $n_1 = n_2 = 0$ and fixed $t \geq 0$, this integral converges absolutely, provided $\Re(w + z) < \nu_{00}$, and it establishes a holomorphic function with respect to w for fixed $z \in \mathbb{C}$ and vice versa. Its w-analytic continuation for fixed z was computed in §8.2.2.2, whereas its z-analytic continuation for fixed w can be derived by reference to §8.2.2.3. Accordingly, by integration by parts it is easy to show, that

$$\mathrm{Q}_0\begin{bmatrix} w-\chi_{00}, z-\chi_{00} \\ t,0,0,w,z,T_1 \end{bmatrix} = -\frac{T_1^{\beta_{00}(w-\chi_{00})}}{\beta_{00}\,(w+z-\nu_{00})}\frac{\mathfrak{C}(T_1)\overline{\mathfrak{C}}(T_1+t)}{(T_1+t)^{1-\beta_{00}(z-\chi_{00})}}e^{wB(T_1)+zB(T_1+t)}$$

$$(8.5.16)\qquad -\frac{1}{\beta_{00}\,(w+z-\nu_{00})}\Bigg\{\sum_{\substack{n_1,n_2\in\{0,1\}\\ n_1+n_2=1}}\mathrm{Q}_0\begin{bmatrix} \frac{1}{\beta_{00}}+w-\chi_{00}, z-\chi_{00} \\ t,n_1,n_2,w,z,T_1 \end{bmatrix}$$

$$+\,t\,(\gamma_{00}-\beta_{00}z)\,\mathrm{Q}_0\begin{bmatrix} w-\chi_{00}, z-\chi_{00}-\frac{1}{\beta_{00}} \\ t,0,0,w,z,T_1 \end{bmatrix}\Bigg\}$$

represents the analytic continuation with respect to w or z of (8.5.15) for $n_1 = n_2 = 0$ into the half plane

$$(8.5.17)\qquad \Re(w+z) < \nu_{01},$$

where we denote for brevity

$$(8.5.18)\qquad \nu_{01} := \nu_{00} + \eta_{\beta_{00}}(\beta_{01}, \gamma_{01}, 2).$$

As a function of one variable with either w or z fixed, it has exactly one pole, whose residue by (8.2.42) and by (8.2.50) equals

$$(8.5.19)\qquad \operatorname*{Res}_{w=\nu_{00}-z}\mathrm{Q}_0\begin{bmatrix} w-\chi_{00}, z-\chi_{00} \\ t,0,0,w,z,T_1 \end{bmatrix} = \operatorname*{Res}_{z=\nu_{00}-w}\mathrm{Q}_0\begin{bmatrix} w-\chi_{00}, z-\chi_{00} \\ t,0,0,w,z,T_1 \end{bmatrix} = -\frac{|c_{00}|^2}{\beta_{00}}\{b_{00}\}^{-\nu_{00}}.$$

If we finally conceive (8.5.15) for $n_1 = n_2 = 0$ and $w = 0$ as a function of z, in the case $\nu_{00} = 0$ the point $z = \nu_{00}$ will certainly match the origin of the z-plane, i.e., if $\gamma_{00} = \frac{1}{2}$. With the aid of §8.2.2.3 we can then immediately specify the first two terms in its Laurent expansion at $z = 0$, which are

$$(8.5.20)\qquad \mathrm{Q}_0\begin{bmatrix} -\chi_{00}, z-\chi_{00} \\ t,0,0,0,z,T_1 \end{bmatrix} = -\frac{1}{z}\frac{|c_{00}|^2}{\beta_{00}} + \frac{1}{\beta_{00}}\{q_1(t;T_1) + tq_2(t;T_1)\} + \mathcal{O}(z),$$

where by (8.2.51) and (8.2.52) the coefficients appearing in the second summand are given by the absolutely convergent integrals

$$(8.5.21)\qquad q_1(t;T_1) := T_1\mathfrak{c}(T_1)\overline{\mathfrak{c}}(T_1+t)\log\{\varphi(T_1+t)\} + \int_{T_1}^{\infty} s^{\frac{1}{2}}\mathfrak{C}'(s)\overline{\mathfrak{c}}(s+t)\log\{\varphi(s+t)\}\,ds$$

$$- \int_{T_1}^{\infty} s\mathfrak{c}(s) B'(s+t) \overline{\mathfrak{c}}(s+t) ds$$
$$+ \int_{T_1}^{\infty} s\mathfrak{c}(s) \frac{\overline{\mathfrak{C}}'(s+t)}{\sqrt{s+t}} \log \{\varphi(s+t)\} \, ds,$$

$$q_2(t; T_1) := \int_{T_1}^{\infty} \mathfrak{c}(s) \frac{\overline{\mathfrak{c}}(s+t)}{s+t} \left\{ \beta_{00} + \frac{1}{2} \log \{\varphi(s+t)\} \right\} ds. \tag{8.5.22}$$

8.5.3. w-Analytic Continuation for Fixed $\Re z < \eta_{00}$ if $\alpha_{00} > 1$; and z-Analytic Continuation for Fixed $w \in \mathbb{C}$ with $\Re w > \nu_{00} - \eta_{00}$ if $\alpha_{00} > 1$

With the assumed parametrization, from (8.5.6) and (8.5.7) we ascertain absolute convergence of the integral representation (8.5.5) for the iterated generating function in the region

$$\Re(w+z) < \nu_{00}, \tag{8.5.23}$$

where the function is even analytic with respect to each variable with the other fixed. Clearly, in each case the abscissa of convergence is due to the condition for the convergence of a single component. Accordingly, if in terms of (8.5.15) we write

$$\mathcal{S}_0 \begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} = \int_{T_2}^{\infty} \overline{\mathfrak{a}}(t) \, \mathrm{Q}_0 \begin{bmatrix} w - \chi_{00}, z - \chi_{00} \\ t, 0, 0, w, z, T_1 \end{bmatrix} dt, \tag{8.5.24}$$

to determine the respective analytic continuation to an extended region, an application of the expansion (8.5.16) suffices. For this, we first define

$$\begin{aligned} \mathcal{S}_1 \begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} :=& T_1^{\beta_{00}(w-\chi_{00})} \mathfrak{C}(T_1) e^{wB(T_1)} \int_{T_2}^{\infty} \frac{\overline{\mathfrak{a}}(t) \overline{\mathfrak{C}}(T_1+t)}{(T_1+t)^{1-\beta_{00}(z-\chi_{00})}} e^{zB(T_1+t)} dt \\ &+ \sum_{\substack{n_1, n_2 \in \{0,1\} \\ n_1+n_2=1}} \int_{T_2}^{\infty} \overline{\mathfrak{a}}(t) \, \mathrm{Q}_0 \begin{bmatrix} \frac{1}{\beta_{00}} + w - \chi_{00}, z - \chi_{00} \\ t, n_1, n_2, w, z, T_1 \end{bmatrix} dt \\ &+ (\gamma_{00} - \beta_{00} z) \int_{T_2}^{\infty} t \overline{\mathfrak{a}}(t) \, \mathrm{Q}_0 \begin{bmatrix} w - \chi_{00}, z - \chi_{00} - \frac{1}{\beta_{00}} \\ t, 0, 0, w, z, T_1 \end{bmatrix} dt. \end{aligned} \tag{8.5.25}$$

By Corollary 8.2.2 we readily confirm absolute convergence of each integral on the right hand side for $w, z \in \mathbb{C}$ with

$$\Re z < \begin{cases} \eta_{00}, \\ \eta_{00} + \frac{1}{\beta_{00}} \min\{\beta_{01}, \gamma_{01}\}, \end{cases}$$

$$\Re(w+z) < \begin{cases} \min\{\nu_{00}, \eta_{00} + \chi_{00}\} + \eta_{\beta_{00}}(\beta_{01}, \gamma_{01}), \\ \min\{\nu_{00}, \eta_{00} + \chi_{00}\} + \eta_{\beta_{00}}(\beta_{01}, \gamma_{01}), \\ \min\left\{\nu_{00} + \frac{1}{\beta_{00}}, \eta_{00} + \chi_{00}\right\}, \end{cases}$$

and eventually also analyticity with respect to one variable with the second fixed. Now, if for the interior integral in (8.5.24) we employ the expansion (8.5.16), we obtain

$$\mathcal{S}_0\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} = -\frac{1}{\beta_{00}\,(w+z-\nu_{00})}\mathcal{S}_1\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix}. \tag{8.5.26}$$

On the one hand, for fixed $z \in \mathbb{C}$ with $\Re z < \eta_{00}$, in the case $\alpha_{00} > 1$, this represents the w-analytic continuation into the half plane

$$\Re w < \min\{\nu_{01}, \eta_{00} + \chi_{00}\} - \Re z. \tag{8.5.27}$$

On the other hand, for fixed $w \in \mathbb{C}$ with $\Re w > \nu_{00} - \eta_{00}$, again in the case $\alpha_{00} > 1$, it furnishes the z-analytic continuation into the region

$$\Re z < \min\{\eta_{00}, \nu_{01} - \Re w, \eta_{00} + \chi_{00} - \Re w\}. \tag{8.5.28}$$

In each extended region the function exhibits only a single singularity, and this is a simple pole. For $\zeta = 0$ and $\alpha_{00} > 1$ we have absolute convergence of the integral

$$\mathcal{A}^0(\zeta; T_2) := \int_{T_2}^{\infty} e^{-\zeta t}\overline{\mathfrak{a}}(t)dt, \tag{8.5.29}$$

in terms of which from (8.5.19), for the residue of the indicated pole, we deduce

$$\begin{aligned} \operatorname*{Res}_{w=\nu_{00}-z} \mathcal{S}_0\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} &= \operatorname*{Res}_{z=\nu_{00}-w} \mathcal{S}_0\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} \\ &= -\frac{|c_{00}|^2}{\beta_{00}}\{b_{00}\}^{-\nu_{00}}\mathcal{A}^0(0; T_2). \end{aligned} \tag{8.5.30}$$

Notice the independence from the fixed variable. Finally, in a later part of this section we will require the Laurent expansion around $z = 0$ in the case $w = 0$ with $\nu_{00} = 0$. According to (8.5.20), this is given by

$$\mathcal{S}_0\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} = -\frac{1}{z}\frac{|c_{00}|^2}{\beta_{00}}\mathcal{A}^0(0; T_2) + \frac{1}{\beta_{00}}\mathcal{A}^1\begin{bmatrix} 0 \\ T_1, T_2 \end{bmatrix} + \mathcal{O}(z), \tag{8.5.31}$$

where we define

$$\mathcal{A}^1\begin{bmatrix} \zeta \\ T_1, T_2 \end{bmatrix} := \int_{T_2}^{\infty} e^{-\zeta t}\overline{\mathfrak{a}}(t)\{q_1(t; T_1) + tq_2(t; T_1)\}\,dt. \tag{8.5.32}$$

8.5.4. w-Analytic Continuation for Fixed $\Re z < \eta_{00}$ in the Case $\alpha_{00} \leq 1$

If $\alpha_{00} \leq 1$ and $\Re z < \eta_{00}$, the iterated generating function (8.5.5) is by (8.5.6) holomorphic for $w \in \mathbb{C}$ with

$$\Re w < \eta_{00} + \chi_{00} - \Re z.$$

This region is due to the supplementary condition for the convergence of the iterated integral, and if $\alpha_{00} = 1$ it also arises from the condition for the convergence of one of its single components. In any case, concerning the w-analytic continuation, we can appeal to §8.2.6.2, for fixed $z \in \mathbb{C}$ with $\Re z < \eta_{00}$, upon identifying

$$\begin{cases} d(s) & \equiv \mathfrak{c}(s), \\ k(s+t) & \equiv \overline{\mathfrak{c}}(s+t)\left\{\varphi(s+t)\right\}^{-z}, \\ e(t) & \equiv \overline{\mathfrak{a}}(t). \end{cases}$$

With $\beta_0 \equiv \beta_{00}$ this implies $\chi_0 \equiv \chi_{00}$, $\varsigma_0 \equiv \frac{\alpha_{00}-1}{\beta_{00}}$ and finally $\kappa_0 \equiv \gamma_{00} - \beta_{00} z$, and these variables satisfy (8.2.99) since $\alpha_{00} \leq 1$ and $\Re z < \eta_{00}$. Especially note that $\Re z < \frac{\gamma_{00}}{\beta_{00}}$. Due to the validity of the indicated restrictions, it follows immediately from (8.2.122) and (8.2.128) that for all $z \in \mathbb{C}$ with $\Re z < \eta_{00}$ the analytic continuation of the iterated generating function (8.5.5) to the strip

$$\chi_{00} < \Re w < \eta_{00} + \chi_{00} - \Re z + \eta_{\beta_{00}}(\alpha_{01}, \beta_{01}, \gamma_{01}) \tag{8.5.33}$$

is represented by the expansion

$$\begin{aligned} \mathcal{S}_0\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} = -\frac{|c_{00}|^2}{\beta_{00}\left\{b_{00}\right\}^{w+z}} \frac{\Gamma(\beta_{00}(\nu_{00} - z - w))\Gamma\left(\beta_{00}(w - \chi_{00})\right)}{(w + z - \eta_{00} - \chi_{00})\Gamma\left(\gamma_{00} - \beta_{00} z\right)} \mathrm{A}_1(w+z; T_2) \\ + \Sigma_1\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix}. \end{aligned} \tag{8.5.34}$$

In the first summand we denote

$$\mathrm{A}_1(\xi; T_2) := T_2^{\beta_{00}(\xi - \chi_{00} - \eta_{00})}\overline{\mathfrak{A}}(T_2) + \int_{T_2}^{\infty} t^{\beta_{00}(\xi - \chi_{00} - \eta_{00})}\overline{\mathfrak{A}}'(t)dt, \tag{8.5.35}$$

whereas the second term is given by the MB-integral

$$\Sigma_1\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} := \frac{1}{2\pi i}\int_{q-i\infty}^{q+i\infty} \frac{\Gamma(\zeta)\Gamma\left(\gamma_{00} - \beta_{00} z - \zeta\right)}{\Gamma\left(\gamma_{00} - \beta_{00} z\right)} \mathcal{L}\left[\zeta; \begin{matrix} w, T_1 \\ z, T_2 \end{matrix}\right] d\zeta. \tag{8.5.36}$$

The integration path in this latter MB-integral is the vertical line $\Re\zeta = q$, where

$$\begin{cases} q > \max\left\{1-\alpha_{00}, \beta_{00}(\nu_{00}-w-z)\right\}, \\ q < \min\left\{\gamma_{00}-\beta_{00}\Re z, \beta_{00}(\nu_{00}-\Re(w+z)+\eta_{\beta_{00}}(\beta_{01},\gamma_{01}))\right\}, \end{cases} \tag{8.5.37}$$

and, in terms of (8.5.15), the integrand features the integral function

$$\begin{aligned} \mathcal{L}\left[\zeta; \begin{matrix} w, T_1 \\ z, T_2 \end{matrix}\right] &= -\frac{T_1^{\beta_{00}(w+z-\nu_{00})+\zeta}}{\zeta+\beta_{00}(w+z-\nu_{00})} \mathfrak{C}(T_1) e^{wB(T_1)} \int_{T_2}^{\infty} t^{-\alpha_{00}-\zeta} \overline{\mathfrak{A}}(t) \overline{\mathfrak{C}}(T_1+t) e^{zB(T_1+t)} dt \\ &\quad - \frac{1}{\zeta+\beta_{00}(w+z-\nu_{00})} \left\{ \int_{T_2}^{\infty} t^{-\alpha_{00}-\zeta} \overline{\mathfrak{A}}(t)\, \mathrm{Q}_0 \left[\begin{matrix} w+z-\nu_{00}+\frac{\zeta+1}{\beta_{00}}, \frac{1}{\beta_{00}} \\ t, 1, 0, w, z, T_1 \end{matrix}\right] dt \right. \\ &\quad \left. + \int_{T_2}^{\infty} t^{-\alpha_{00}-\zeta} \overline{\mathfrak{A}}(t)\, \mathrm{Q}_0 \left[\begin{matrix} w+z-\nu_{00}+\frac{\zeta+1}{\beta_{00}}, \frac{1}{\beta_{00}} \\ t, 0, 1, w, z, T_1 \end{matrix}\right] dt \right\}. \end{aligned} \tag{8.5.38}$$

According to (8.2.129), the expansion (8.5.34) is $\mathcal{O}(w)$ as $\Im w \to \pm\infty$, uniformly with respect to $\Re w$ in any closed vertical substrip of (8.5.33), for each fixed $z \in \mathbb{C}$ with $\Re z < \eta_{00}$. Moreover, in accordance with the absolute convergence of the integrals in (8.5.38), there exist constants $L_1, L_2, L_3 > 0$ for which, uniformly with respect to $y, \Im w, \Im z \in \mathbb{R}$, we have validity of

$$\left| \mathcal{L}\left[q+iy; \begin{matrix} w, T_1 \\ z, T_2 \end{matrix}\right] \right| \leq \frac{L_1+|w|\,L_2+|z|\,L_3}{q+\beta_{00}(\Re(w+z)-\nu_{00})}. \tag{8.5.39}$$

In addition, a simple application of the functional equation for the gamma function accompanied by a rough estimate for the beta function, whose integral representation is admissible since $0 < q < \gamma_{00}-\beta_{00}\Re z$, leads to

$$\begin{aligned} &\left| \frac{\Gamma(q+iy)\Gamma\left(\gamma_{00}-\beta_{00}z-q-iy\right)}{\Gamma\left(\gamma_{00}-\beta_{00}z\right)} \right| \\ &\qquad \leq \left| \frac{(\gamma_{00}-\beta_{00}z)(1+\gamma_{00}-\beta_{00}z)}{(q+iy)(1+q+iy)} \right| \frac{\Gamma(2+q)\Gamma\left(\gamma_{00}-\beta_{00}\Re z-q\right)}{\Gamma\left(2+\gamma_{00}-\beta_{00}\Re z\right)}. \end{aligned} \tag{8.5.40}$$

Observe that this bound with respect to y is absolutely integrable along the real axis. If we eventually apply a similar estimate to the first summand in the expansion (8.5.34) and employ the bounds (8.5.39) and (8.5.40) for the MB-integral in the second summand, for any $z \in \mathbb{C}$ with $\Re z < \eta_{00}$ and $w \in \mathbb{C}$ subject to (8.5.33), we arrive at the statement

$$\left| \mathcal{S}_0\left[\begin{matrix} -w, T_1 \\ -z, T_2 \end{matrix}\right] \right| \leq \sum_{j=0}^{3} (P_j+Q_j\,|w|)\,|z|^j, \tag{8.5.41}$$

for appropriate constants $P_j, Q_j \geq 0$ that are uniformly bounded with respect to $\Im w, \Im z \in \mathbb{R}$. Concerning the pole of the analytic continuation (8.5.34) on the line $\Re w = \eta_{00}+\chi_{00}-\Re z$, due to the dependence on the fixed but arbitrary variable z, it is reasonable to assume without loss of generality $z \neq \eta_{00} + \chi_{00}$, thereby omitting the possibility for this pole to lie at $w = 0$. Then, if $\alpha_{00} < 1$, the pole at $w = \eta_{00} + \chi_{00} - z$ is of simple order, and by incorporating

$$\mathrm{A}_1(\eta_{00} + \chi_{00}; T_2) = \overline{a}_{00},$$

for the associated residue we obtain

$$\underset{w=\eta_{00}+\chi_{00}-z}{\mathrm{Res}} \mathcal{S}_0 \begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} = -\frac{\overline{a}_{00} \left| c_{00} \right|^2}{\beta_{00} \left\{ b_{00} \right\}^{\eta_{00}+\chi_{00}}} \frac{\Gamma(1-\alpha_{00})\Gamma\left(\beta_{00}(\eta_{00}-z)\right)}{\Gamma\left(\gamma_{00}-\beta_{00}z\right)}. \tag{8.5.42}$$

An inevitable coalescence will happen if $\alpha_{00} = 1$, which implies $\nu_{00} = \eta_{00} + \chi_{00}$, and the latter pole merges with the pole of the first gamma function in the numerator of (8.5.34). In these circumstances, by §8.2.6.2.3 with $\chi_0 + \frac{\kappa_0}{\beta_0} \equiv \nu_{00} - z$, in an annulus around the indicated pole, the function shows a Laurent expansion of the form

$$\mathcal{S}_0 \begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} = \frac{\overline{a}_{00} \left| c_{00} \right|^2}{\left\{ \beta_{00} \right\}^2 \left\{ b_{00} \right\}^{\nu_{00}}} \left\{ \frac{1}{(w+z-\nu_{00})^2} + \frac{\sigma_0(z;T_2)}{w+z-\nu_{00}} + \mathcal{O}(1) \right\}, \tag{8.5.43}$$

in which the constant in the second summand is given by

$$\begin{aligned} \sigma_0(z;T_2) := \beta_{00}\gamma + \beta_{00}\psi(\gamma_{00}-\beta_{00}z) - \log\left\{b_{00}\right\} \\ + \frac{\beta_{00}}{\overline{a}_{00}} \log(T_2)\overline{\mathfrak{A}}(T_2) + \frac{\beta_{00}}{\overline{a}_{00}} \int_{T_2}^{\infty} \log(t)\overline{\mathfrak{A}}'(t)dt. \end{aligned} \tag{8.5.44}$$

We close this subsection with the derivation of the Laurent expansion for a particularly special parametrization, which will be required later.

8.5.4.1. Laurent expansion for $z = 0$ and $\alpha_{00} = \gamma_{00} = \frac{2}{3}$

In this case $\eta_{00} = \nu_{00} = \frac{1}{3\beta_{00}}$ and $\nu_{00} + \chi_{00} = 0$. The pole at $w = \nu_{00} + \chi_{00}$ is therefore of simple order, located at the origin of the w-plane, and, according to §8.2.6.2.2, as $w \to 0$ we have

$$\mathcal{S}_0 \begin{bmatrix} -w, T_1 \\ 0, T_2 \end{bmatrix} = -\frac{\overline{a}_{00} \left| c_{00} \right|^2}{\beta_{00}} \frac{\left\{\Gamma\left(\frac{1}{3}\right)\right\}^2}{\Gamma\left(\frac{2}{3}\right)} \left\{ \frac{1}{w} + \sigma_{\frac{2}{3}}(T_2) \right\} + \Sigma_1 \begin{bmatrix} 0, T_1 \\ 0, T_2 \end{bmatrix} + \mathcal{O}(w), \tag{8.5.45}$$

with the constant in the curved brackets being equal to

$$\sigma_{\frac{2}{3}}(T_2) := -\log\left\{b_0\right\} + \frac{\beta_{00}}{\overline{a}_{00}} \log(T_2)\overline{\mathfrak{A}}(T_2) + \frac{\beta_{00}}{\overline{a}_{00}} \int_{T_2}^{\infty} \log(t)\overline{\mathfrak{A}}'(t)dt. \tag{8.5.46}$$

8.5.5. Evaluation of the Interior MB-Integral

We continue with the evaluation of the interior MB-integral (8.5.14) for fixed $z \in \mathbb{C}$ with $\Re z = x_0$. The associated generating function (8.5.5) is holomorphic in the region $\Re w < \nu_0(\alpha_{00})$

and, according to the preceding two subsections, it can be extended into the wider half plane $\Re w < \nu_1(\alpha_{00})$, for

$$\nu_1(\alpha_{00}) := \begin{cases} \min\{\nu_{01}, \eta_{00} + \chi_{00}\} - x_0, & \text{if } \alpha_{00} > 1, \\ \eta_{00} + \chi_{00} - x_0 + \eta_{\beta_{00}}(\alpha_{01}, \beta_{01}, \gamma_{01}), & \text{if } 0 < \alpha_{00} \le 1. \end{cases} \tag{8.5.47}$$

There, it is respectively represented by (8.5.26) and (8.5.34). The first has been established via partial integration. It exposes a meromorphic structure in the indicated half plane with exactly one pole at $w = \nu_0(\alpha_{00})$, and that one is simple. On the other hand, (8.5.34) is valid merely in some strip, in which we generally find at least one pole, again at $w = \nu_0(\alpha_{00})$. It is of simple order if $0 < \alpha_{00} < 1$ but can merge to a second order pole for $\alpha_{00} = 1$. To summarize these statements, in each case the generating function of the MB-integral (8.5.14) for fixed $z \in \mathbb{C}$ with $\Re z = x_0$ can be extended meromorphically into the wider half plane $\Re w < \nu_1(\alpha_{00})$.

In order to make out the singularity which in (8.5.14) lies closest rightwards to the line $\Re w = u_0$, we must get rid of the minimum structure of the right boundary in (8.5.12). For this, a careful distinction between different parameter values is required. First, to (8.5.9) we add the assumptions

$$\begin{cases} -1 < x_0 < \psi_\infty, & \text{if } \nu_{00}, \eta_{00} + \chi_{00} \ge \psi_\infty, \\ \max\{-1, \nu_{00}\} < x_0 < \psi_\infty, & \text{if } \nu_{00} < \psi_\infty \le \eta_{00} + \chi_{00}, \\ \max\{-1, \eta_{00} + \chi_{00}\} < x_0 < \psi_\infty, & \text{if } \eta_{00} + \chi_{00} < \psi_\infty \le \nu_{00}, \\ \max\{-1, \nu_{00}, \eta_{00} + \chi_{00}\} < x_0 < \psi_\infty, & \text{if } \nu_{00}, \eta_{00} + \chi_{00} < \psi_\infty. \end{cases} \tag{8.5.48}$$

As a consequence, from (8.5.12) we obtain

$$-1 < u_0 < \begin{cases} 0, & \text{if } \nu_{00}, \eta_{00} + \chi_{00} \ge \psi_\infty, \\ \nu_{00} - x_0, & \text{if } \nu_{00} < \min\{\eta_{00} + \chi_{00}, \psi_\infty\}, \\ \eta_{00} + \chi_{00} - x_0, & \text{if } \eta_{00} + \chi_{00} < \psi_\infty \le \nu_{00} \text{ or } \eta_{00} + \chi_{00} \le \nu_{00} < \psi_\infty, \end{cases} \tag{8.5.49}$$

or more concisely

$$-1 < u_0 < \begin{cases} 0, & \text{if } \nu_{00}, \eta_{00} + \chi_{00} \ge \psi_\infty, \\ \nu_0(\alpha_{00}), & \text{otherwise.} \end{cases} \tag{8.5.50}$$

We therefore conclude, that the closest singularity to the right of the line $\Re w = u_0$ is either the simple pole of the gamma function $\Gamma(w)$ at $w = 0$ or the pole of the analytic continuation associated with the generating function. Due to the dependence on the location of x_0 we have $\nu_0(\alpha_{00}) \ne 0$. Coalescences of the poles of these two functions are therefore impossible. In other words, at $w = \nu_0(\alpha_{00})$ the integrand of (8.5.14) always exhibits a pole of simple order, unless $\alpha_{00} = 1$. To collect the residue of the indicated pole we note, that the integral representation

(8.5.5) and its respective analytic continuation are $\mathcal{O}(w)$ as $\Im w \to \pm\infty$, uniformly with respect to $\Re w$ in any closed vertical substrip of its region of validity. The exponential decay of the gamma function in (8.5.14) thus justifies a rightward displacement of the integration path across the pole of the integrand at $w = \nu_0(\alpha_{00})$ and also across that at $w = 0$ if $\nu_0(\alpha_{00}) > 0$. To be exact, suppose

$$\nu_0(\alpha_{00}) < u_1 < \begin{cases} \min\{\nu_{00} - x_0, \nu_1(\alpha_{00})\}, & \text{if } \nu_0(\alpha_{00}) > 0 \wedge \alpha_{00} < 1, \\ \min\{0, \nu_{00} - x_0, \nu_1(\alpha_{00})\}, & \text{if } \nu_0(\alpha_{00}) < 0 \wedge \alpha_{00} < 1, \\ \min\left\{\nu_{00} + \frac{1}{\beta_{00}} - x_0, \nu_1(1)\right\}, & \text{if } \nu_0(1) > 0 \wedge \alpha_{00} = 1, \\ \min\left\{0, \nu_{00} + \frac{1}{\beta_{00}} - x_0, \nu_1(1)\right\}, & \text{if } \nu_0(1) < 0 \wedge \alpha_{00} = 1, \\ \nu_1(\alpha_{00}), & \text{if } \nu_0(\alpha_{00}) > 0 \wedge \alpha_{00} > 1, \\ \min\{0, \nu_1(\alpha_{00})\}, & \text{if } \nu_0(\alpha_{00}) < 0 \wedge \alpha_{00} > 1. \end{cases} \tag{8.5.51}$$

If in (8.5.14) we then displace the integration path far enough to the right, such that the new path satisfies $\Re w = u_1$, we gather exactly the residues corresponding to the indicated singularities, which we encircle in the clockwise direction. By incorporating the results (8.5.30), (8.5.42) and (8.5.43), as well as the definitions of ν_{00} and of $\eta_{00} + \chi_{00}$, by virtue of (B.2.21), we compute

$$\begin{aligned} \mathrm{J}(m,z) = {} & -\mathcal{S}_0\begin{bmatrix} 0, T_1 \\ -z, T_2 \end{bmatrix} \mathbb{1}_{\{\nu_{00},\, \eta_{00} + \chi_{00} \geq \psi_\infty\}} \\ & + \frac{\Gamma(\nu_{00} - z)}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{|c_{00}|^2}{\beta_{00}} \mathcal{A}^0(0; T_2)(m+1)^z \mathbb{1}_{\{\alpha_{00} > 1\}} \\ & + \frac{\Gamma(\nu_{00} - z)}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{\overline{a}_{00}\, |c_{00}|^2}{\{\beta_{00}\}^2} \{\log(m+1) - \psi(\nu_{00} - z) - \sigma_0(z; T_2)\} \\ & \qquad \times (m+1)^z \mathbb{1}_{\{\alpha_{00} = 1\}} \\ & + \frac{\Gamma(\eta_{00} + \chi_{00} - z)}{\{b_{00}(m+1)\}^{\eta_{00} + \chi_{00}}} \frac{\overline{a}_{00}\, |c_{00}|^2}{\beta_{00}} \frac{\Gamma(1 - \alpha_{00})\Gamma(\beta_{00}(\eta_{00} - z))}{\Gamma(\gamma_{00} - \beta_{00} z)} \\ & \qquad \times (m+1)^z \mathbb{1}_{\{\alpha_{00} < 1\}} \\ & + \frac{1}{2\pi i} \int_{u_1 - i\infty}^{u_1 + i\infty} (m+1)^{-w} \Gamma(w) \mathcal{S}_0\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} dw. \end{aligned} \tag{8.5.52}$$

Observe the occurence of the factor $(m+1)^z$ in each but the last term of this expansion. Consequently, cancellations will happen, when it is applied to the iterated MB-integral (8.5.13), with the benefit that we merely have to evaluate one additional MB-integral that depends on m, which for x_0 subject to (8.5.48) equals

$$\mathrm{I}(m) := \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} (m+1)^{-z} \Gamma(z) \mathcal{S}_0\begin{bmatrix} 0, T_1 \\ -z, T_2 \end{bmatrix} dz. \tag{8.5.53}$$

Moreover, by the moment we plug the expansion (8.5.52) into the iterated MB-integral (8.5.13), again with x_0 as in (8.5.48), the following hypergeometric integrals will appear:

$$\Omega(\nu_{00}) := \frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} \Gamma(z)\Gamma(\nu_{00}-z)dz \tag{8.5.54}$$

$$\Xi(\nu_{00},T_2) := \frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} \Gamma(z)\Gamma(\nu_{00}-z)\left\{\psi(\nu_{00}-z)+\sigma_0(z;T_2)\right\}dz \tag{8.5.55}$$

$$\Upsilon(\alpha_{00},\beta_{00},\gamma_{00}) := \frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} \frac{\Gamma(z)\Gamma(\eta_{00}+\chi_{00}-z)\Gamma\left(\beta_{00}(\eta_{00}-z)\right)}{\Gamma\left(\gamma_{00}-\beta_{00}z\right)}dz \tag{8.5.56}$$

The function $\sigma_0(z;T_2)$ was introduced in (8.5.44). In terms of the latter definitions, by a suitable bound for the remainder integral, for instance (8.5.41) in the case $\alpha_{00}\le 1$, as $m\to\infty$ we can eventually verify

$$\begin{aligned}\operatorname{Ai}\left[m;\begin{matrix}T_1\\T_2\end{matrix}\right] ={}& -\mathrm{I}(m)\mathbb{1}_{\{\nu_{00},\,\eta_{00}+\chi_{00}\ge\psi_\infty\}}\\ &+\frac{\mathcal{A}^0(0;T_2)}{\{b_{00}(m+1)\}^{\nu_{00}}}\frac{|c_{00}|^2}{\beta_{00}}\Omega(\nu_{00})\mathbb{1}_{\{\alpha_{00}>1\}}\\ &+\frac{\log(m+1)}{\{b_{00}(m+1)\}^{\nu_{00}}}\frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2}\Omega(\nu_{00})\mathbb{1}_{\{\alpha_{00}=1\}}\\ &-\frac{\Xi(\nu_{00},T_2)}{\{b_{00}(m+1)\}^{\nu_{00}}}\frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2}\mathbb{1}_{\{\alpha_{00}=1\}}\\ &+\frac{\Gamma(1-\alpha_{00})}{\{b_{00}(m+1)\}^{\eta_{00}+\chi_{00}}}\frac{\overline{a}_{00}\,|c_{00}|^2}{\beta_{00}}\Upsilon(\alpha_{00},\beta_{00},\gamma_{00})\mathbb{1}_{\{\alpha_{00}<1\}}\\ &+\mathcal{O}\left\{m^{-u_1-x_0}\right\}.\end{aligned} \tag{8.5.57}$$

Concerning the big-$\mathcal{O}$, by definition of u_1 we know about the existence of $\varepsilon_1>0$ with $u_1=\nu_0(\alpha_{00})+\varepsilon_1$. Hence, from (8.5.11) we conclude

$$u_1+x_0=\min\{\nu_{00},\eta_{00}+\chi_{00}\}+\varepsilon_1. \tag{8.5.58}$$

As we already mentioned, there is merely one integral whose asymptotics remains to be discussed, before we can make an ultimate statement on (8.5.57). This is the concern of the next subsections and requires a consideration of (8.5.5) as a function of z. Furthermore, below a finite representation will be established for the hypergeometric integral (8.5.54).

8.5.6. An Interior Generating Function with an Infinite Path and a Kernel of the Second Kind

An interchange in the order of integration in (8.5.5) results in an interior integral, which is a special version of the integral below, that is for $\eta \in \mathbb{R}$, $n_1, n_2 \in \{0, 1\}$ and $s \geq 0$ defined by

$$\mathrm{R}_0 \begin{bmatrix} \eta, z - \chi_{00} \\ s, n_1, n_2, z, T_2 \end{bmatrix} := \int_{T_2}^{\infty} \frac{t^{\beta_{00}\eta - 1}}{(s+t)^{1-\beta_{00}(z-\chi_{00})}} \overline{\mathfrak{A}}^{(n_1)}(t) \frac{d^{n_2}}{dt^{n_2}} \left\{ \overline{\mathfrak{C}}(s+t) e^{zB(s+t)} \right\} dt. \tag{8.5.59}$$

With $n_1 = n_2 = 0$, this integral converges absolutely for $\Re z < \chi_{00} - \eta + \frac{1}{\beta_{00}}$ and is a holomorphic function there. According to §8.2.2.3, integration by parts yields

$$\begin{aligned} \mathrm{R}_0 \begin{bmatrix} \eta, z - \chi_{00} \\ s, 0, 0, z, T_2 \end{bmatrix} = &- \frac{T_2^{\beta_{00}\eta} \overline{\mathfrak{A}}(T_2)}{\beta_{00} \left(z + \eta - \chi_{00} - \frac{1}{\beta_{00}} \right)} \frac{\overline{\mathfrak{C}}(s+T_2)}{(s+T_2)^{1-(z-\chi_{00})\beta_{00}}} e^{zB(s+T_2)} \\ &- \frac{1}{\beta_{00} \left(z + \eta - \chi_{00} - \frac{1}{\beta_{00}} \right)} \Bigg\{ \sum_{\substack{n_1, n_2 \in \{0,1\} \\ n_1 + n_2 = 1}} \mathrm{R}_0 \begin{bmatrix} \eta + \frac{1}{\beta_{00}}, z - \chi_{00} \\ s, n_1, n_2, z, T_2 \end{bmatrix} \\ &+ s \left(\gamma_{00} - \beta_{00} z \right) \mathrm{R}_0 \begin{bmatrix} \eta, z - \frac{1}{\beta_{00}} - \chi_{00} \\ s, 0, 0, z, T_2 \end{bmatrix} \Bigg\}. \end{aligned} \tag{8.5.60}$$

For $\eta = \frac{1-\alpha_{00}}{\beta_{00}}$ this represents the analytic continuation of (8.5.59) with $n_1 = n_2 = 0$ into the half plane

$$\Re z < \eta_{01}. \tag{8.5.61}$$

The parameter η_{01} was specified in (8.4.33) and is thus denoted by

$$\eta_{01} = \eta_{00} + \eta_{\beta_{00}}(\alpha_{01}, \beta_{01}, \gamma_{01}, 2).$$

In the region $\Re z < \eta_{01}$, the expansion shows exactly one pole. It is of simple order, and by (8.2.50) we find

$$\operatorname*{Res}_{z=\eta_{00}} \mathrm{R}_0 \begin{bmatrix} \frac{1-\alpha_{00}}{\beta_{00}}, z - \chi_{00} \\ s, 0, 0, z, T_2 \end{bmatrix} = - \frac{\overline{a}_{00} \overline{c}_{00}}{\beta_{00}} \left\{ b_{00} \right\}^{-\eta_{00}}. \tag{8.5.62}$$

If $\eta_{00} = 0$, i.e., for $\alpha_{00} + \gamma_{00} = 1$, this pole meets the origin of the z-plane. For such a parametrization we have absolute convergence for all $s \geq 0$ of the integrals

$$\begin{aligned} \rho_1(s; T_2) := &T_2 \overline{\mathfrak{a}}(T_2) \overline{\mathfrak{c}}(s+T_2) \log \left\{ \varphi(s+T_2) \right\} \\ &+ \int_{T_2}^{\infty} t^{\gamma_{00}} \overline{\mathfrak{A}}'(t) \overline{\mathfrak{c}}(s+t) \log \left\{ \varphi(s+t) \right\} dt \end{aligned} \tag{8.5.63}$$

$$- \int_{T_2}^{\infty} t\overline{\mathfrak{a}}(t)B'(s+t)\overline{\mathfrak{c}}(s+t)dt$$

$$+ \int_{T_2}^{\infty} t\overline{\mathfrak{a}}(t)\frac{\overline{\mathfrak{C}}'(s+t)}{(s+t)^{\gamma_{00}}} \log\{\varphi(s+t)\}\, dt,$$

$$\rho_2(s;T_2) := \int_{T_2}^{\infty} \overline{\mathfrak{a}}(t)\frac{\overline{\mathfrak{c}}(s+t)}{s+t}\{\beta_{00}+\gamma_{00}\log\{\varphi(s+t)\}\}\, dt. \tag{8.5.64}$$

By (8.2.52), in an annulus around the origin, we then immediately conclude the validity of the Laurent expansion

$$\mathrm{R}_0\left[\begin{matrix}\frac{1-\alpha_{00}}{\beta_{00}},\, z-\chi_{00}\\ s,0,0,z,T_2\end{matrix}\right] = -\frac{1}{z}\frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} + \frac{1}{\beta_{00}}\{\rho_1(s;T_2)+s\rho_2(s;T_2)\} + \mathcal{O}(z). \tag{8.5.65}$$

8.5.7. z-Analytic Continuation for Fixed $\Re w < \chi_{00} + \frac{1}{\beta_{00}}\min\{0, 1-\alpha_{00}\}$

If in (8.5.5) we perform the indicated interchange in the order of integration, in terms of (8.5.59), we arrive at

$$\mathcal{S}_0\left[\begin{matrix}-w,T_1\\ -z,T_2\end{matrix}\right] = \int_{T_1}^{\infty}\{\varphi(s)\}^{-w}\,\mathfrak{c}(s)\,\mathrm{R}_0\left[\begin{matrix}\frac{1-\alpha_{00}}{\beta_{00}},\, z-\chi_{00}\\ s,0,0,z,T_2\end{matrix}\right]ds. \tag{8.5.66}$$

The assumed specifications of the argument w can be split into the inequalities $\Re w < \chi_{00}$ and $\Re w < \nu_{00} - \eta_{00}$. By (8.5.7), this shows that the above representation converges absolutely and is analytic for $\Re z < \eta_{00}$, where the abscissa of convergence stems from the criterion for the convergence of $\mathrm{R}_0[\ldots]$. The corresponding analytic continuation is therefore readily determined by employing the expansion (8.5.60). For this purpose we denote

$$\begin{aligned}\mathcal{K}_1\left[\begin{matrix}-w,T_1\\ -z,T_2\end{matrix}\right] :=\, & T_2^{1-\alpha_{00}}\overline{\mathfrak{A}}(T_2)\int_{T_1}^{\infty}\{\varphi(s)\}^{-w}\,\mathfrak{c}(s)\frac{\overline{\mathfrak{C}}(s+T_2)}{(s+T_2)^{1-\beta_{00}(z-\chi_{00})}}e^{zB(s+T_2)}ds\\ & + \sum_{\substack{n_1,n_2\in\{0,1\}\\ n_1+n_2=1}}\int_{T_1}^{\infty}\{\varphi(s)\}^{-w}\,\mathfrak{c}(s)\,\mathrm{R}_0\left[\begin{matrix}\frac{2-\alpha_{00}}{\beta_{00}},\, z-\chi_{00}\\ s,n_1,n_2,z,T_2\end{matrix}\right]ds\\ & + (\gamma_{00}-\beta_{00}z)\int_{T_1}^{\infty}\{\varphi(s)\}^{-w}\,s\mathfrak{c}(s)\,\mathrm{R}_0\left[\begin{matrix}\frac{1-\alpha_{00}}{\beta_{00}},\, z-\frac{1}{\beta_{00}}-\chi_{00}\\ s,0,0,z,T_2\end{matrix}\right]ds.\end{aligned} \tag{8.5.67}$$

Each of these integrals converges absolutely for $w, z \in \mathbb{C}$ with

$$\Re z < \begin{cases} \eta_{00} + \frac{\alpha_{01}-1}{\beta_{00}}, \\ \eta_{00} + \eta_{\beta_{00}}(\beta_{01}, \gamma_{01}), \\ \eta_{00} + \frac{1}{\beta_{00}}, \end{cases} \quad \Re(w+z) < \begin{cases} \nu_{00}, \\ \min\left\{\nu_{00}, \eta_{00} + \chi_{00} + \frac{\alpha_{01}-1}{\beta_{00}}\right\}, \\ \min\left\{\nu_{00} + \frac{1}{\beta_{00}}, \eta_{00} + \chi_{00}\right\} + \eta_{\beta_{00}}(\beta_{01}, \gamma_{01}), \\ \min\left\{\nu_{00}, \eta_{00} + \chi_{00}\right\}, \end{cases}$$

and therein it is a holomorphic function of z for fixed w. As a consequence, writing

$$\mathcal{S}_0\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} = -\frac{1}{\beta_{00}\,(z-\eta_{00})}\mathcal{K}_1\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix}, \tag{8.5.68}$$

the right hand side establishes the z-analytic continuation of (8.5.66) into the half plane

$$\Re z < \min\left\{\nu_{00} - \Re w, \eta_{01}, \eta_{00} + \chi_{00} - \Re w\right\}, \tag{8.5.69}$$

since, in view of the conditions on $w \in \mathbb{C}$ of the present paragraph, this indeed includes the region $\Re z < \eta_{00}$. Of special interest is again the simple pole at $z = \eta_{00}$, which is the only singularity in the extended half plane. The corresponding residue is readily derived from (8.5.62) and (8.5.66), and, in terms of the integral function (8.4.49), we find

$$\operatorname*{Res}_{z=\eta_{00}} \mathcal{S}_0\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} = -\frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}}\left\{b_{00}\right\}^{-\eta_{00}}\mathcal{P}_0\begin{bmatrix} -w, 0 \\ T_1, \infty \end{bmatrix}. \tag{8.5.70}$$

To specify the first two dominating terms of the Laurent expansion near $z = 0$ in the case $\eta_{00} = 0$, by means of the functions $\rho_j(s; T_2)$ defined in (8.5.63) and (8.5.64), for an half open path $\mathcal{P}_1$ with endpoints $0 \le \sigma < S \le \infty$ and $\zeta \in \mathbb{C}$, we introduce the integral transform

$$\mathcal{Y}_0\begin{bmatrix} -w, \zeta \\ \sigma, S, T_2 \end{bmatrix} := \int\limits_{\mathcal{P}_1} \left\{\varphi(s)\right\}^{-w} e^{-\zeta s} c(s; p_1)\left\{\rho_1(s; T_2) + s\rho_2(s; T_2)\right\} ds, \tag{8.5.71}$$

where $\mathcal{P}_1 \cup \{p_1\}$ is closed and $c(s; p_1)$ was defined in (8.4.2). Then, according to (8.5.65), if $\eta_{00} = 0$, as $z \to 0$ we readily derive

$$\mathcal{S}_0\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix} = -\frac{1}{z}\frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}}\mathcal{P}_0\begin{bmatrix} -w, 0 \\ T_1, \infty \end{bmatrix} + \frac{1}{\beta_{00}}\mathcal{Y}_0\begin{bmatrix} -w, 0 \\ T_1, \infty, T_2 \end{bmatrix} + \mathcal{O}(z). \tag{8.5.72}$$

8.5.8. A Single MB-Integral for the Residue at $w = 0$

It is easy to see that the first term in the expansion (8.5.57) is non-zero, if and only if $\nu_{00}, \eta_{00} + \chi_{00} \ge \psi_\infty$. By definition of ψ_∞, this implies $\psi_\infty = \min\{0, \eta_{00}\}$ and thus

$$\nu_{00}, \eta_{00} + \chi_{00} \ge \min\left\{0, \eta_{00}\right\}. \tag{8.5.73}$$

Furthermore, the generating function corresponding to the MB-integral (8.5.53), which appears in this first term, equals the original integral (8.5.5) with $w = 0$. By (8.5.7) it is analytic in the half plane

$$\Re z < \psi_0(\alpha_{00}), \tag{8.5.74}$$

whose right boundary is defined in terms of the parameter

$$\psi_0(\alpha_{00}) := \min\{\nu_{00}, \eta_{00}, \eta_{00} + \chi_{00}\}. \tag{8.5.75}$$

Therein the function is $\mathcal{O}(1)$ as $\Im z \to \pm\infty$, uniformly with respect to $\Re z$ in any closed vertical substrip. As a consequence, the condition imposed in (8.5.48) on the integration path x_0 of the integral (8.5.53) can be replaced by

$$-1 < x_0 < \min\{0, \psi_0(\alpha_{00})\}. \tag{8.5.76}$$

Regarding the analytic continuation of the generating function, a distinction between different parameter values is obviously inevitable. Throughout this subsection we assume

$$\alpha_{00} \neq \gamma_{00},$$

since the case of equality incurs substantial difficulties. The reason is, that the z-abscissa of convergence of (8.5.5) with $w = 0$ is then determined by the criterion for the convergence of both single components, thereby complicating the required calculations. Fortunately this can be circumvented, if in (8.5.1) we introduce the MB-representation for the m-powers in a different order, which will be discussed in Section 8.6 below.

8.5.8.1. Evaluation of the MB-Integral in the case $\eta_{00} > \frac{1}{\beta_{00}} \min\{\alpha_{00}, \gamma_{00}\}$ with $\alpha_{00} \neq \gamma_{00}$

If $\eta_{00} + \chi_{00} > \min\{\nu_{00}, \eta_{00}\}$, we obtain

$$\psi_0(\alpha_{00}) = \min\{\nu_{00}, \eta_{00}\}, \tag{8.5.77}$$

and the z-abscissa of convergence of (8.5.5) with $w = 0$ is hence due to the criterion for the convergence of either of the single components. The associated z-analytic continuation into the region $\Re z < \psi_1(\alpha_{00})$, where

$$\psi_1(\alpha_{00}) := \begin{cases} \min\{\nu_{01}, \eta_{00}, \eta_{00} + \chi_{00}\}, & \text{if } \nu_{00} < \eta_{00} + \min\{0, \chi_{00}\}, \\ \min\{\nu_{00}, \eta_{01}, \eta_{00} + \chi_{00}\}, & \text{if } \eta_{00} < \min\{\nu_{00}, \eta_{00} + \chi_{00}\}, \end{cases} \tag{8.5.78}$$

has been computed in Subsections 8.5.3 and 8.5.7 via integration by parts. It is easy to see that, in each case the function is $\mathcal{O}(1)$ as $\Im z \to \pm\infty$, uniformly with respect to $\Re z$ in any closed vertical substrip of the indicated half plane. Accordingly, the integrand of (8.5.53) exhibits exponential decay in any imaginary direction. Moreover, to the right of the line $\Re z = x_0$ in $\Re z < \psi_1(\alpha_{00})$, we find no more than two poles, which lie at $z \in \{0, \psi_0(\alpha_{00})\}$. Hence, upon moving the integration path to the right, to match a line $\Re z = x_1$, for

$$\psi_0(\alpha_{00}) < x_1 < \begin{cases} \psi_1(\alpha_{00}), & \text{if } \psi_0(\alpha_{00}) \geq 0, \\ \min\{0, \psi_1(\alpha_{00})\}, & \text{if } \psi_0(\alpha_{00}) < 0, \end{cases} \tag{8.5.79}$$

we encounter one pole, except if $\psi_0(\alpha_{00}) > 0$, in which case we traverse two poles. The order of these poles is simple if $\psi_0(\alpha_{00}) \neq 0$ and otherwise it is two. The associated residues and Laurent coefficients have been calculated in (8.5.30), (8.5.31), (8.5.70) and (8.5.72). With the aid of Theorem B.2.1(2), due to the absolute convergence of the remainder integral, as $m \to \infty$ this leads to:

$$\begin{aligned} \mathrm{I}(m) = &-\mathcal{S}_0\begin{bmatrix} 0, T_1 \\ 0, T_2 \end{bmatrix} \mathbb{1}_{\{\min\{\eta_{00}, \nu_{00}\} > 0\}} \\ &+ \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{|c_{00}|^2}{\beta_{00}} \mathcal{A}^0(0; T_2) \mathbb{1}_{\{0 \neq \nu_{00} < \eta_{00}\}} \\ &- \left\{ (\log(m+1) + \gamma) \frac{|c_{00}|^2}{\beta_{00}} \mathcal{A}^0(0; T_2) + \frac{1}{\beta_{00}} \mathcal{A}^1\begin{bmatrix} 0 \\ T_1, T_2 \end{bmatrix} \right\} \mathbb{1}_{\{0 = \nu_{00} < \eta_{00}\}} \\ &+ \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \mathcal{P}_0\begin{bmatrix} 0, 0 \\ T_1, \infty \end{bmatrix} \mathbb{1}_{\{0 \neq \eta_{00} < \nu_{00}\}} \\ &- \left\{ (\log(m+1) + \gamma) \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \mathcal{P}_0\begin{bmatrix} 0, 0 \\ T_1, \infty \end{bmatrix} + \frac{1}{\beta_{00}} \mathcal{Y}_0\begin{bmatrix} 0, 0 \\ T_1, \infty, T_2 \end{bmatrix} \right\} \mathbb{1}_{\{0 = \eta_{00} < \nu_{00}\}} \\ &+ \mathcal{O}\left\{m^{-x_1}\right\} \end{aligned} \tag{8.5.80}$$

8.5.8.2. z-Analytic Continuation for $w = 0$ if $\eta_{00} \leq \frac{1}{\beta_{00}} \min\{\alpha_{00}, \gamma_{00}\}$ with $\alpha_{00} \neq \gamma_{00}$

Under the present assumptions the integral transform (8.5.5) for $w = 0$, according to (8.5.7), establishes a holomorphic function in the half plane

$$\Re z < \eta_{00} + \chi_{00}.$$

The right boundary of this region originates in the supplementary condition for the convergence of the iterated integral. In order to unlock the z-analytic continuation for $w = 0$ of (8.5.5) to a wider region, we aim for a reference to §8.2.6.3. Notice that the parametrization $\eta_{00} + \chi_{00} = \eta_{00} = \nu_{00}$ also can be treated by virtue of this method with additional elaborate calculations. Yet, it will be examined separately in the next section.

From the restrictions on η_{00} we conclude $\alpha_{00}, \gamma_{00} \leq 1$ and thus $\chi_{00} \leq 0$. Moreover, under the assumptions of the present paragraph, the two parameters α_{00}, γ_{00} are unequal, implying their minimum is unique and

$$h := \frac{\alpha_{00}}{\gamma_{00}} \tag{8.5.81}$$

is either smaller or greater than one. By comparison with the criterion (8.2.146) it becomes evident that we can not immediately apply our findings from §8.2.6.3 without specifying h. To avoid the necessity to establish twice a similar statement, we first define the parameters

$$\chi_h := \begin{cases} \frac{\gamma_{00}-1}{\beta_{00}}, & \text{if } h > 1, \\ \frac{\alpha_{00}-1}{\beta_{00}}, & \text{if } h < 1, \end{cases} \tag{8.5.82}$$

$$\varsigma_h := \begin{cases} \frac{\alpha_{00}-1}{\beta_{00}}, & \text{if } h > 1, \\ \frac{\gamma_{00}-1}{\beta_{00}}, & \text{if } h < 1. \end{cases} \tag{8.5.83}$$

These satisfy several helpful identities such as

$$\begin{cases} \chi_h + \varsigma_h = \eta_{00} - \frac{1}{\beta_{00}}, \\ \chi_h + \varsigma_h + \frac{\gamma_{00}}{\beta_{00}} = \chi_{00} + \eta_{00}. \end{cases} \tag{8.5.84}$$

In addition we introduce the functions

$$d_h(u) := \begin{cases} \mathfrak{c}(u), & \text{if } h > 1, \\ \bar{\mathfrak{a}}(u), & \text{if } h < 1, \end{cases} \tag{8.5.85}$$

$$e_h(u) := \begin{cases} \bar{\mathfrak{a}}(u), & \text{if } h > 1, \\ \mathfrak{c}(u), & \text{if } h < 1. \end{cases} \tag{8.5.86}$$

Their normalized analogues will be referred to as $D_h(u)$ and $E_h(u)$, respectively, with coefficients at infinity denoted by d_{h0} and e_{h0}, i.e., we agree

$$d_{h0} := \begin{cases} c_{00}, & \text{if } h > 1, \\ \bar{a}_{00}, & \text{if } h < 1, \end{cases} \tag{8.5.87}$$

$$e_{h0} := \begin{cases} \bar{a}_{00}, & \text{if } h > 1, \\ c_{00}, & \text{if } h < 1. \end{cases} \tag{8.5.88}$$

If we then identify $\beta_0 \equiv \beta_{00}$ and also

$$\begin{cases} d(s) & \equiv d_h(s), \\ k(s+t) & \equiv \bar{\mathfrak{c}}(s+t), \\ e(t) & \equiv e_h(t), \end{cases} \tag{8.5.89}$$

in terms of the parameters of §8.2.6.3, we obtain $\chi_0 \equiv \chi_h$, $\varsigma_0 \equiv \varsigma_h$ and $\kappa_0 \equiv \gamma_{00}$, and for any positive $h \neq 1$ we have validity of

$$\chi_h < \varsigma_h \leq 0. \tag{8.5.90}$$

Hence, the condition (8.2.146) holds, and we conclude the applicability of the statements of §8.2.6.3 on the z-analytic continuation of the integral (8.5.5) for $w = 0$. Preliminary we introduce some important components, of which the first is given by the parameters $S_h \in \{T_1, T_2\} \setminus \{T_h\}$ and

$$T_h = \begin{cases} T_2, & \text{if } h > 1, \\ T_1, & \text{if } h < 1. \end{cases} \tag{8.5.91}$$

We then define the integral

$$\Lambda_h(z) := T_h^{\beta_{00}(z-\eta_{00}-\chi_{00})} E_h(T_h) + \int\limits_{T_h}^{\infty} t^{\beta_{00}(z-\eta_{00}-\chi_{00})} E_h'(t)dt. \tag{8.5.92}$$

Appealing to the integral functions (8.4.61) and (8.5.35), it satisfies the identity

$$\Lambda_h(z) = \begin{cases} \mathrm{A}_1(z; T_2), & \text{if } h > 1, \\ \Pi_1 \left[{z-\eta_{00}-\chi_{00} \atop 0,\, T_1} \right], & \text{if } h < 1. \end{cases} \tag{8.5.93}$$

As a consequence,

$$\Lambda_h(\eta_{00} + \chi_{00}) = e_{h0}. \tag{8.5.94}$$

The integral (8.5.92) converges absolutely for $z \in \mathbb{C}$ with $\Re z < \eta_{00} + \chi_{00}$ and even uniformly in any compact subset of this half plane. According to Theorem A.2.1, it is thus a holomorphic function whose derivatives can be computed by differentiation under the integral sign, which yields:

$$\begin{aligned} \lambda_{h,j} &:= \frac{d^j}{dz^j} \Lambda_h(z) \bigg|_{z=\eta_{00}+\chi_{00}} \\ &= \{\beta_{00} \log(T_h)\}^j \, E_h(T_h) + \int\limits_{T_h}^{\infty} \{\beta_{00} \log(t)\}^j \, E_h'(t)dt \end{aligned} \tag{8.5.95}$$

Moreover, similar to (8.2.153) we define the sum of iterated integrals

$$\mathcal{J}_h(\zeta, z) = -\frac{S_h^{\beta_{00}(z-\chi_h)+\zeta-\gamma_{00}} D_h(S_h)}{\beta_{00}(z-\chi_h) + \zeta - \gamma_{00}} \int\limits_{T_h}^{\infty} t^{-\beta_{00}\varsigma_h-\zeta-1} E_h(t) e^{zB(S_h+t)} \overline{\mathfrak{C}}(S_h + t)dt \tag{8.5.96}$$

$$- \frac{1}{\beta_{00}(z-\chi_h)+\zeta-\gamma_{00}} \sum_{\substack{n_1,n_2,n_3\in\{0,1\} \\ n_1+n_2+n_3=1}} \int_{T_h}^{\infty} t^{-\beta_{00}\varsigma_h-\zeta-1} E_h(t)$$
$$\times \int_{S_h}^{\infty} s^{\beta_{00}(z-\chi_h)+\zeta-\gamma_{00}}$$
$$\times D_h^{(n_1)}(s) e^{zB(s+t)} \left\{zB'(s+t)\right\}^{n_2} \overline{\mathfrak{C}}^{(n_3)}(s+t)dsdt.$$

Finally, subject to (8.2.158) we introduce the MB-integral

$$\Sigma_1(-z;T_1,T_2) := \frac{1}{2\pi i} \int_{q-i\infty}^{q+i\infty} \frac{\Gamma(\zeta)\Gamma(\gamma_{00}-\beta_{00}z-\zeta)}{\Gamma(\gamma_{00}-\beta_{00}z)} \mathcal{J}_h(\zeta,z)d\zeta, \tag{8.5.97}$$

in which the integration path is a vertical line $\Re\zeta = q$ that runs for $h > 1$ in the strip

$$\max\{1-\alpha_{00}, \beta_{00}(\nu_{00}-\Re z)\} < q < -\beta_{00}\Re z + \min\{\gamma_{00}, \beta_{00}(\nu_{00}+\eta_{\beta_{00}}(\beta_{01},\gamma_{01}))\}. \tag{8.5.98}$$

Conversely, for $0 < h < 1$ the line $\Re\zeta = q$ runs in the strip

$$\begin{cases} q > \max\{1-\gamma_{00}, \beta_{00}(\eta_{00}-\Re z)\}, \\ q < -\beta_{00}\Re z + \min\{\gamma_{00}, \beta_{00}(\eta_{00}+\eta_{\beta_{00}}(\alpha_{01},\beta_{01},\gamma_{01}))\}. \end{cases} \tag{8.5.99}$$

In terms of these functions, according to (8.2.160) and (8.2.168) with $\chi_0 + \varsigma_0 + \frac{\kappa_0}{\beta_0} = \eta_{00} + \chi_{00}$, the expansion

$$\mathcal{S}_0\begin{bmatrix} 0, T_1 \\ -z, T_2 \end{bmatrix} = -\frac{d_{h0}\overline{c}_{00}}{\beta_{00}\{b_{00}\}^z} \frac{\Gamma(\gamma_{00}+\beta_{00}(\chi_h-z))\Gamma(-\beta_{00}\chi_h)}{(z-\chi_{00}-\eta_{00})\Gamma(\gamma_{00}-\beta_{00}z)} \Lambda_h(z) + \Sigma_1(-z;T_1,T_2) \tag{8.5.100}$$

constitutes the holomorphic continuation of the integral (8.5.5) for $w = 0$ into the wider half plane

$$\Re z < \begin{cases} \eta_{00} + \min\{0, \chi_{00} + \eta_{\beta_{00}}(\alpha_{01},\beta_{01},\gamma_{01})\}, & \text{if } h > 1, \\ \nu_{00} + \min\left\{0, \frac{\alpha_{00}-1}{\beta_{00}} + \eta_{\beta_{00}}(\alpha_{01},\beta_{01},\gamma_{01})\right\}, & \text{if } h < 1. \end{cases} \tag{8.5.101}$$

Concerning its asymptotic behaviour, in (8.2.169) we pointed out that, uniformly with respect to $\Re z$ in each closed vertical substrip of the region (8.5.101), as $\Im z \to \pm\infty$ the function satisfies

$$\mathcal{S}_0\begin{bmatrix} 0, T_1 \\ -z, T_2 \end{bmatrix} = \mathcal{O}\left\{|z|^2\right\}. \tag{8.5.102}$$

Now, by inspection of (8.5.100), the point $z = \chi_{00}+\eta_{00}$, that lies on the z-abscissa of convergence of the original integral representation (8.5.5) for $w = 0$, is easily identified as a pole of order

$1 \leq J \leq 2$. Hence, in an annulus around this point we find a Laurent expansion with

$$\mathcal{S}_0\begin{bmatrix} 0, T_1 \\ -z, T_2 \end{bmatrix} = \sum_{j=0}^{J} \frac{\sigma_{-j}\left(\chi_h + \frac{\gamma_{00}}{\beta_{00}}, \varsigma_h\right)}{(z - \chi_{00} - \eta_{00})^j} + \mathcal{O}(z - \chi_{00} - \eta_{00}). \tag{8.5.103}$$

Particularly the coefficient for $j = 1$ equals the residue. A recipe to calculate the coefficients for $0 \leq j \leq J$ is given by Taylor's theorem with

$$\sigma_{-j}\left(\chi_h + \frac{\gamma_{00}}{\beta_{00}}, \varsigma_h\right) = \frac{1}{(J-j)!} \lim_{z \to \chi_{00}+\eta_{00}} \frac{d^{J-j}}{dz^{J-j}} (z - \chi_{00} - \eta_{00})^J \, \mathcal{S}_0\begin{bmatrix} 0, T_1 \\ -z, T_2 \end{bmatrix}. \tag{8.5.104}$$

We close this paragraph by computing the coefficients of the principal part if $\chi_{00} + \eta_{00} \neq 0$, and in addition the coefficient for $j = 0$ if this parameter equals zero.

8.5.8.2.1. Coefficients for $\varsigma_h \neq 0$. The point $z = \chi_{00} + \eta_{00}$ is then a pole of simple order. From (8.5.100), by taking into account (8.5.84) and (8.5.94), for all positive $h \neq 1$ we obtain:

$$\begin{aligned} \sigma_{-1}\left(\chi_h + \frac{\gamma_{00}}{\beta_{00}}, \varsigma_h\right) &= -\frac{d_{h0} e_{h0} \bar{c}_{00}}{\beta_{00} \{b_{00}\}^{\chi_{00}+\eta_{00}}} \frac{\Gamma(-\beta_{00}\varsigma_h)\Gamma(-\beta_{00}\chi_h)}{\Gamma(-\beta_{00}(\chi_h + \varsigma_h))} \\ &= -\frac{\bar{a}_{00} \, |c_{00}|^2}{\beta_{00} \{b_{00}\}^{\chi_{00}+\eta_{00}}} \frac{\Gamma(1-\alpha_{00})\Gamma(1-\gamma_{00})}{\Gamma(2-\alpha_{00}-\gamma_{00})} \end{aligned} \tag{8.5.105}$$

If $\chi_{00} + \eta_{00} = 0$, i.e., $\alpha_{00} + 2\gamma_{00} = 2$ and $\chi_h + \frac{\gamma_{00}}{\beta_{00}} = -\varsigma_h$, the indicated pole meets the origin of the z-plane. Then, by (8.5.105),

$$\sigma_{-1}(-\varsigma_h, \varsigma_h) = -\frac{\bar{a}_{00} \, |c_{00}|^2}{\beta_{00}} \frac{\Gamma(1-\alpha_{00})\Gamma(1-\gamma_{00})}{\Gamma(\gamma_{00})}. \tag{8.5.106}$$

To calculate in this case the coefficient for $j = 0$, with the logarithm taking its principal value, we define

$$u(z) := \log\Gamma(-\beta_{00}(\varsigma_h + z)) - \log\Gamma(\gamma_{00} - \beta_{00}z) - z\log\{b_{00}\}. \tag{8.5.107}$$

This enables us for (8.5.100) to write equivalently

$$\mathcal{S}_0\begin{bmatrix} 0, T_1 \\ -z, T_2 \end{bmatrix} = -\frac{d_{h0}\bar{c}_{00}}{\beta_{00}} \Gamma(-\beta_{00}\chi_h) e^{u(z)} \frac{\Lambda_h(z)}{z} + \Sigma_1(-z; T_1, T_2). \tag{8.5.108}$$

The function $u(z)$ is analytic in a neighborhood of $z = 0$ if $\varsigma_h \neq 0$, where

$$u'(0) = -\beta_{00}\psi(-\beta_{00}\varsigma_h) + \beta_{00}\psi(\gamma_{00}) - \log\{b_{00}\}. \tag{8.5.109}$$

With the aid of (8.5.104) we thus deduce:

(8.5.110)
$$\begin{aligned}\sigma_0\left(-\varsigma_h,\varsigma_h\right) &= -\frac{d_{h0}\overline{c}_{00}}{\beta_{00}}\Gamma(-\beta_{00}\chi_h)\frac{d}{dz}\left\{e^{u(z)}\Lambda_h(z)\right\}\bigg|_{z=0} + \Sigma_1(0;T_1,T_2)\\ &= -\frac{\overline{a}_{00}\left|c_{00}\right|^2}{\beta_{00}}\frac{\Gamma(-\beta_{00}\varsigma_h)\Gamma(-\beta_{00}\chi_h)}{\Gamma(\gamma_{00})}\left\{u'(0)+\frac{\lambda_{h,1}}{e_{h0}}\right\} + \Sigma_1(0;T_1,T_2)\end{aligned}$$

The constant $\lambda_{h,1}$ was defined in (8.5.95).

8.5.8.2.2. Coefficients for $\varsigma_h = 0$. Then $\chi_h + \frac{\gamma_{00}}{\beta_{00}} = \eta_{00} + \chi_{00}$. For parameters that satisfy the conditions of the present paragraph, we therefore observe a coalescence to a second order pole at $z = \chi_{00} + \eta_{00}$, i.e., $J = 2$. In terms of the function

(8.5.111)
$$v(z) := \log\Gamma(1+\gamma_{00}+\beta_{00}(\chi_h - z)) - \log\Gamma(\gamma_{00}-\beta_{00}z) - z\log\left\{b_{00}\right\},$$

assuming the principal value of the logarithm, upon employing the functional equation for the gamma function, we can write

(8.5.112)
$$\mathcal{S}_0\begin{bmatrix}0, T_1\\ -z, T_2\end{bmatrix} = \frac{d_{h0}\overline{c}_{00}}{\left\{\beta_{00}\right\}^2}\Gamma(-\beta_{00}\chi_h)\frac{e^{v(z)}\Lambda_h(z)}{(z-\chi_{00}-\eta_{00})^2} + \Sigma_1(-z;T_1,T_2).$$

Hence, for the coefficient of the dominating term in (8.5.103), we compute

(8.5.113)
$$\sigma_{-2}\left(\chi_h + \tfrac{\gamma_{00}}{\beta_{00}}, 0\right) = \frac{\overline{a}_{00}\left|c_{00}\right|^2}{\left\{\beta_{00}\right\}^2\left\{b_{00}\right\}^{\chi_{00}+\eta_{00}}}.$$

Furthermore, by incorporating (8.5.94) and (8.5.95), we obtain

(8.5.114)
$$\sigma_{-1}\left(\chi_h + \tfrac{\gamma_{00}}{\beta_{00}}, 0\right) = \frac{\overline{a}_{00}\left|c_{00}\right|^2}{\left\{\beta_{00}\right\}^2\left\{b_{00}\right\}^{\chi_{00}+\eta_{00}}}\left\{v'\left(\chi_h + \tfrac{\gamma_{00}}{\beta_{00}}\right) + \frac{\lambda_{h,1}}{e_{h0}}\right\}.$$

The first derivative of the function $v(z)$ equals

(8.5.115)
$$v'(z) = -\beta_{00}\psi(1+\gamma_{00}+\beta_{00}(\chi_h - z)) + \beta_{00}\psi(\gamma_{00}-\beta_{00}z) - \log\left\{b_{00}\right\},$$

from which by (B.2.13) we calculate

(8.5.116)
$$v'\left(\chi_h + \tfrac{\gamma_{00}}{\beta_{00}}\right) = \beta_{00}\gamma + \beta_{00}\psi(-\beta_{00}\chi_h) - \log\left\{b_{00}\right\}.$$

If $\chi_{00}+\eta_{00} = 0$, then $\chi_h + \frac{\gamma_{00}}{\beta_{00}} = 0$ and the pole at $z = \chi_{00}+\eta_{00}$ lies at the origin of the z-plane. To specify the coefficient associated with the constant term in (8.5.103), according to (8.5.104), we must differentiate twice (8.5.112), leading to:

$$\sigma_0\left(0,0\right) = \frac{d_{h0}\overline{c}_{00}}{\left\{\beta_{00}\right\}^2}\Gamma(-\beta_{00}\chi_h)\frac{1}{2}\frac{d^2}{dz^2}\left\{e^{v(z)}\Lambda_h(z)\right\}\bigg|_{z=0} + \Sigma_1(0;T_1,T_2)$$

$$= \frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2}\frac{1}{2}\left\{(v'(0))^2 + 2v'(0)\frac{\lambda_{h,1}}{e_{h0}} + v''(0) + \frac{\lambda_{h,2}}{e_{h0}}\right\} + \Sigma_1(0;T_1,T_2) \tag{8.5.117}$$

For the second derivative of $v(z)$, by (B.2.14), we obtain

$$v''(0) = \{\beta_{00}\}^2\frac{\pi^2}{6} - \{\beta_{00}\}^2\,\psi'(\gamma_{00}). \tag{8.5.118}$$

8.5.8.3. Evaluation of the MB-Integral in the case $\eta_{00} \le \frac{1}{\beta_{00}}\min\{\alpha_{00},\gamma_{00}\}$ with $\alpha_{00} \neq \gamma_{00}$

According to the preceding paragraph, the general properties of the expansion (8.5.100) remain the same for each admissible value of h. That is, in particular, denoting

$$\psi_1(\alpha_{00}) := \begin{cases} \eta_{00} + \min\{0, \chi_{00} + \eta_{\beta_{00}}(\alpha_{01},\beta_{01},\gamma_{01})\}, & \text{if } \eta_{00}+\chi_{00} \le \nu_{00} < \eta_{00}, \\ \nu_{00} + \min\left\{0, \frac{\alpha_{00}-1}{\beta_{00}} + \eta_{\beta_{00}}(\alpha_{01},\beta_{01},\gamma_{01})\right\}, & \text{if } \eta_{00}+\chi_{00} \le \eta_{00} < \nu_{00}, \end{cases} \tag{8.5.119}$$

in the half plane $\Re z < \psi_1(\alpha_{00})$ we observe the presence of poles only, whose location and order is determined by the parameters. The pole with the smallest real part lies always at $z = \psi_0(\alpha_{00})$, i.e., at $z = \eta_{00} + \chi_{00}$. It will be of first or second order, depending on whether $\psi_0(\alpha_{00}) \le \min\{\eta_{00},\nu_{00}\}$. As a consequence, the integrand of the MB-integral (8.5.53) may show a third order pole at $z = \psi_0(\alpha_{00})$ if this point coincides with the origin of the z-plane. Furthermore, regarding the asymptotic behaviour of the integrand, from (8.5.102) we conclude exponential decay into any imaginary direction of $\Re z < \psi_1(\alpha_{00})$. Suppose $x_1 > \psi_0(\alpha_{00})$ with

$$x_1 < \begin{cases} \min\{\eta_{00},\nu_{00},\psi_1(\alpha_{00})\}, & \text{if } 0 \le \psi_0(\alpha_{00}) < \min\{\eta_{00},\nu_{00}\} \\ \min\{0,\eta_{00},\nu_{00},\psi_1(\alpha_{00})\}, & \text{if } \psi_0(\alpha_{00}) < \min\{0,\eta_{00},\nu_{00}\} \\ \min\left\{\eta_{00}+\frac{1}{\beta_{00}}, \nu_{00}+\frac{1}{\beta_{00}}, \psi_1(\alpha_{00})\right\}, & \text{if } 0 \le \psi_0(\alpha_{00}) = \min\{\eta_{00},\nu_{00}\} \\ \min\left\{0, \eta_{00}+\frac{1}{\beta_{00}}, \nu_{00}+\frac{1}{\beta_{00}}, \psi_1(\alpha_{00})\right\}, & \text{if } 0 > \psi_0(\alpha_{00}) = \min\{\eta_{00},\nu_{00}\} \end{cases} \tag{8.5.120}$$

According to the residue theorem, the integral along a clockwisely traversed rectangle of infinite height with respective left and right edges equal to $\Re z = x_0$ and $\Re z = x_1$, coincides with the sum of the residues of the poles in its interior, multiplied by a negative sign. This is solely the pole at $z = \psi_0(\alpha_{00})$, and also the pole at $z = 0$ in case of positivity of $\psi_0(\alpha_{00})$. The required quantities for the generating function were provided in (8.5.105), (8.5.106), (8.5.110), (8.5.113), (8.5.114) and (8.5.117). With the aid of Theorem B.2.1(2), as $m \to \infty$ we therefore obtain:

$$\begin{aligned} \mathrm{I}(m) = &-\mathcal{S}_0\begin{bmatrix}0, T_1\\ 0, T_2\end{bmatrix}\mathbb{1}_{\{\eta_{00}+\chi_{00}>0\}} \\ &+ \frac{\Gamma(\eta_{00}+\chi_{00})}{\{b_{00}(m+1)\}^{\chi_{00}+\eta_{00}}}\frac{\overline{a}_{00}\,|c_{00}|^2}{\beta_{00}}\frac{\Gamma(1-\alpha_{00})\Gamma(1-\gamma_{00})}{\Gamma(2-\alpha_{00}-\gamma_{00})} \\ &\qquad\times \mathbb{1}_{\{\eta_{00}+\chi_{00}\notin\{0,\min\{\eta_{00},\nu_{00}\}\}\}} \end{aligned} \tag{8.5.121}$$

$$
\begin{aligned}
&+ \log(m+1)\frac{\Gamma(\eta_{00}+\chi_{00})}{\{b_{00}(m+1)\}^{\chi_{00}+\eta_{00}}}\frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2}\mathbb{1}_{\{0\neq\eta_{00}+\chi_{00}=\min\{\eta_{00},\nu_{00}\}\}}\\
&-\frac{\Gamma(\eta_{00}+\chi_{00})}{\{b_{00}(m+1)\}^{\chi_{00}+\eta_{00}}}\frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2}\left\{\psi(\eta_{00}+\chi_{00})+v'\left(\chi_h+\tfrac{\gamma_{00}}{\beta_{00}}\right)+\frac{\lambda_{h,1}}{e_{h0}}\right\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\mathbb{1}_{\{0\neq\eta_{00}+\chi_{00}=\min\{\eta_{00},\nu_{00}\}\}}\\
&-\{\log(m+1)+\gamma\}\frac{\overline{a}_{00}\,|c_{00}|^2}{\beta_{00}}\frac{\Gamma(1-\alpha_{00})\Gamma(1-\gamma_{00})}{\Gamma(\gamma_{00})}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\mathbb{1}_{\{0=\eta_{00}+\chi_{00}<\min\{\eta_{00},\nu_{00}\}\}}\\
&+\frac{\overline{a}_{00}\,|c_{00}|^2}{\beta_{00}}\frac{\Gamma(1-\alpha_{00})\Gamma(1-\gamma_{00})}{\Gamma(\gamma_{00})}\left\{u'(0)+\frac{\lambda_{h,1}}{e_{h0}}\right\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\mathbb{1}_{\{0=\eta_{00}+\chi_{00}<\min\{\eta_{00},\nu_{00}\}\}}\\
&-\Sigma_1(0;T_1,T_2)\mathbb{1}_{\{0=\eta_{00}+\chi_{00}\leq\min\{\eta_{00},\nu_{00}\}\}}\\
&-\left\{(\log(m+1)+\gamma)^2+\frac{\pi^2}{6}\right\}\frac{\overline{a}_{00}\,|c_{00}|^2}{2\{\beta_{00}\}^2}\mathbb{1}_{\{0=\eta_{00}+\chi_{00}=\min\{\eta_{00},\nu_{00}\}\}}\\
&+(\log(m+1)+\gamma)\frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2}\left\{v'(0)+\frac{\lambda_{h,1}}{e_{h0}}\right\}\\
&\qquad\qquad\qquad\qquad\qquad\times\mathbb{1}_{\{0=\eta_{00}+\chi_{00}=\min\{\eta_{00},\nu_{00}\}\}}\\
&-\frac{\overline{a}_{00}\,|c_{00}|^2}{2\{\beta_{00}\}^2}\left\{(v'(0))^2+2v'(0)\frac{\lambda_{h,1}}{e_{h0}}+v''(0)+\frac{\lambda_{h,2}}{e_{h0}}\right\}\\
&\qquad\qquad\qquad\qquad\qquad\times\mathbb{1}_{\{0=\eta_{00}+\chi_{00}=\min\{\eta_{00},\nu_{00}\}\}}\\
&+\mathcal{O}\left\{m^{-x_1}\right\}
\end{aligned}
$$

The order of the remainder integral can be concluded from the absolute convergence of its representation as a MB-integral.

8.5.9. Computation of Some Hypergeometric Integrals

This subsection is devoted to the hypergeometric integrals (8.5.54) and (8.5.55). By comparison with the expansion (8.5.57), we ascertain their appearance in the case $\alpha_{00} \geq 1$. Following thereof,

$$\eta_{00} + \chi_{00} \geq \nu_{00}.$$

In addition, $\eta_{00} < 0$ then implies $\gamma_{00} < 1 - \alpha_{00} \leq 0$ and therefore $\nu_{00} < \eta_{00}$. Hence, if $\eta_{00} + \chi_{00} \geq \nu_{00} \geq \psi_\infty$, we may always conclude $\psi_\infty = 0$. Upon taking these observations into account, according to (8.5.48), the integration path in (8.5.54) and in (8.5.55) is supposed to satisfy

$$\begin{cases} -1 < x_0 < 0, & \text{if } \eta_{00} + \chi_{00} \geq \nu_{00} \geq 0, \\ \max\{-1, \nu_{00}\} < x_0 < \psi_\infty, & \text{if } \nu_{00} < \psi_\infty \leq \eta_{00} + \chi_{00}, \\ \max\{-1, \nu_{00}, \eta_{00} + \chi_{00}\} < x_0 < \psi_\infty, & \text{if } \nu_{00} \leq \eta_{00} + \chi_{00} < \psi_\infty. \end{cases}$$

Concerning the last of these two integrals, by definition (8.5.44), in terms of (8.5.54), it is easy to confirm

$$\Xi(\nu_{00}, T_2) = \Omega^1(\nu_{00}) + \beta_{00}\,\mathrm{S}(\beta_{00}, \gamma_{00}) + \mathrm{s}(\beta_{00}, T_2)\Omega(\nu_{00}), \tag{8.5.122}$$

where, for brevity, we denote

$$\Omega^1(\nu_{00}) := \frac{1}{2\pi i}\int\limits_{x_0 - i\infty}^{x_0 + i\infty} \Gamma(z)\Gamma(\nu_{00} - z)\psi(\nu_{00} - z)dz, \tag{8.5.123}$$

$$\mathrm{S}(\beta_{00}, \gamma_{00}) := \frac{1}{2\pi i}\int\limits_{x_0 - i\infty}^{x_0 + i\infty} \Gamma(z)\Gamma(\nu_{00} - z)\psi(\gamma_{00} - \beta_{00} z)dz, \tag{8.5.124}$$

$$\mathrm{s}(\beta_{00}, T_2) := \beta_{00}\gamma - \log\{b_{00}\} + \frac{\beta_{00}}{\overline{a}_{00}}\log(T_2)\overline{\mathfrak{A}}(T_2) + \frac{\beta_{00}}{\overline{a}_{00}}\int\limits_{T_2}^{\infty}\log(t)\overline{\mathfrak{A}}'(t)dt. \tag{8.5.125}$$

The integrand in each of the integrals $\Omega(\nu_{00})$ and $\Omega^1(\nu_{00})$ decays exponentially fast towards the imaginary direction of the complex z-plane and exhibits poles at $z \in -\mathbb{N}_0$ and at $z \in \nu_{00} + \mathbb{N}_0$. Since $\psi_\infty \leq \min\{0, \nu_{00} + 1\}$ by definition, these properties enable us, to replace the path $\Re z = x_0$ in each of the indicated integrals by an arbitrary line which satisfies

$$\begin{cases} -1 < x_0 < 0, & \text{if } \nu_{00} \geq 0, \\ \max\{-1, \nu_{00}\} < x_0 < \min\{0, \nu_{00} + 1\}, & \text{if } \nu_{00} < 0. \end{cases} \tag{8.5.126}$$

It is easy to see that such a path can always be found if $\nu_{00} > -2$. We will now derive a representation in terms of simpler functions for each of the integrals $\Omega(\nu_{00})$ and $\Omega^1(\nu_{00})$. For this purpose, for arbitrary $q > -2$ we essentially verify a connection to the MB-integral

$$f(q) := \frac{1}{2\pi i}\int\limits_{\varsigma_0 - i\infty}^{\varsigma_0 + i\infty} \Gamma(z)\Gamma(q - z)dz, \tag{8.5.127}$$

whose integration path is supposed to satisfy

$$-2 < \varsigma_0 < \min\{-1, q\}. \tag{8.5.128}$$

Subject to Lebesgue's dominated convergence theorem, particularly due to absolute and with respect to $1 \geq r \geq 0$ uniform convergence, we can write

$$f(q) = \lim_{r \uparrow 1} \frac{1}{2\pi i} \int\limits_{\varsigma_0 - i\infty}^{\varsigma_0 + i\infty} \Gamma(z)\Gamma(q-z) r^{-z} dz. \tag{8.5.129}$$

Now, consider this integral for fixed $0 < r < 1$. In view of (8.5.128) the singularities to the left of the integration path are given by the sequence of simple poles at $z \in -2 - \mathbb{N}_0$. Furthermore, since $0 < r < 1$, the integrand of $f(q)$ decays exponentially fast in any direction of the left z-half plane $|\arg(-z)| \leq \frac{\pi}{2}$. We are therefore allowed to displace the integration path by an infinite distance to the left over the indicated sequence of poles. These are encircled in the counterclockwise direction, which yields the absolutely convergent series representation

$$\frac{1}{2\pi i} \int\limits_{\varsigma_0 - i\infty}^{\varsigma_0 + i\infty} \Gamma(z)\Gamma(q-z) r^{-z} dz = \sum_{k=2}^{\infty} \frac{(-r)^k}{k!} \Gamma(q+k).$$

For non-integer $q > -2$ the series can be rearranged by means of the reflection formula for the gamma function (B.2.15). Upon exploiting the periodicity of the sine, this enables us to express the sum according to the binomial theorem:

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{(-r)^k}{k!} \Gamma(q+k) &= \frac{\pi}{\sin(\pi q)} \sum_{k=2}^{\infty} \frac{r^k}{\Gamma(1-q-k)k!} \\ &= \Gamma(q) \sum_{k=2}^{\infty} \binom{-q}{k} r^k \\ &= \Gamma(q) \left\{ (1+r)^{-q} - 1 + q \right\} \end{aligned}$$

If we eventually let $r \uparrow 1$, we have just verified

$$f(q) = \Gamma(q) \left\{ 2^{-q} - 1 + q \right\}, \tag{8.5.130}$$

where the extension to arbitrary $q > -2$ holds by continuity. To be exact, the original representation of $f(q)$ as a MB-integral and the right hand side of (8.5.130) are both continuous on $q > -2$. In our next step we suppose $[a, b] \subset (-2, \infty)$ and pick $-2 < \varsigma_0 < \min\{-1, a\}$. Then, ς_0 does not depend on $q \in [a, b]$ and $\varsigma_0 < q$ for all $q \in [a, b]$. We may therefore apply the triangle inequality to Euler's integral of the second kind, for $z \in \mathbb{C}$ with $\Re z = \varsigma_0$, to deduce

$$|\Gamma(q-z)| \leq \int\limits_0^1 t^{a-\varsigma_0-1} dt + \int\limits_1^{\infty} t^{b-\varsigma_0-1} e^{-t} dt,$$

which clearly holds uniformly with respect to $q \in [a, b]$. A uniform bound for the digamma function $\psi(q-z)$ can be found by virtue of the integral representation (5.9.12) in [Olver et al., 2010].

To summarize these findings, the function

$$\Gamma(z)\frac{d}{dq}\Gamma(q-z) = \Gamma(z)\Gamma(q-z)\psi(q-z)$$

is uniformly bounded with respect to $q \in [a,b]$ by $\Gamma(\varsigma_0 + iy) \in L^1(\mathbb{R})$. From Theorem 11.62 in [Körner, 2004] we infer the permission to compute the first derivative of $f(q)$ by differentiation under the integral sign, to find

$$f'(q) = \frac{1}{2\pi i}\int_{\varsigma_0 - i\infty}^{\varsigma_0 + i\infty} \Gamma(z)\Gamma(q-z)\psi(q-z)dz. \tag{8.5.131}$$

By arbitrariness of $[a,b]$, this equality holds for all $q > -2$. Conversely, differentiation of (8.5.130) for $q > -2$ brings us

$$f'(q) = \Gamma(q)\psi(q)\left\{2^{-q} - 1 + q\right\} + \Gamma(q)(1 - 2^{-q}\log(2)). \tag{8.5.132}$$

In order to finally deduce an equivalent representation for the integrals $\Omega(\nu_{00})$ and $\Omega^1(\nu_{00})$, in the MB-representation for each of the functions $f(q)$ and $f'(q)$, we displace the integration path rightwards to obtain a sum of residues plus a remainder integral along a vertical line $\Re z = \varsigma_1$ with

$$\begin{cases} -1 < \varsigma_1 < 0, & \text{if } q \geq 0, \\ \max\{-1, q\} < \varsigma_1 < \min\{0, q+1\}, & \text{if } -2 < q < 0. \end{cases} \tag{8.5.133}$$

By comparison with (8.5.126) we see, upon identifying $q = \nu_{00}$, the remainder integral of $f(q)$ and of $f'(q)$ respectively equals $\Omega(\nu_{00})$ and $\Omega^1(\nu_{00})$. The singularities which we encounter in the process of the described movement include the point $z = -1$ and possibly $z = q$, depending on q. At these points the integrand of (8.5.127) always exhibits poles of simple order except if $q = -1$, in which circumstances at $z = -1$ a second order pole emerges. To specify the associated residue, we require the derivative

$$\frac{d}{dz}\frac{\Gamma(z+2)\Gamma(-z)}{z} = \frac{\Gamma(z+2)\Gamma(-z)}{z}\left\{\psi(z+2) - \psi(-z) - \frac{1}{z}\right\}.$$

Thus, since each pole is traversed clockwisely, with ς_1 as in (8.5.133), from (8.5.127) we easily obtain

$$\begin{aligned} f(q) = \Gamma(q+1)\mathbb{1}_{\{q>-2\,\wedge\, q\neq -1\}} - \Gamma(q)\mathbb{1}_{\{-2<q<0\,\wedge\, q\neq -1\}} \\ - \mathbb{1}_{\{q=-1\}} + \frac{1}{2\pi i}\int_{\varsigma_1 - i\infty}^{\varsigma_1 + i\infty} \Gamma(z)\Gamma(q-z)dz. \end{aligned}$$

In terms of the integral definition (8.5.54), with $q = \nu_{00}$, a reference to (8.5.130) for any $\nu_{00} > -2$ yields

$$\Omega(\nu_{00}) = \Gamma(\nu_{00})\left\{2^{-\nu_{00}} - 1 + \nu_{00}\right\} - \Gamma(\nu_{00}+1)\mathbb{1}_{\{\nu_{00} > -2 \wedge \nu_{00} \neq -1\}} + \Gamma(\nu_{00})\mathbb{1}_{\{-2 < \nu_{00} < 0 \wedge \nu_{00} \neq -1\}} + \mathbb{1}_{\{\nu_{00} = -1\}}.$$

Moreover, concerning (8.5.131) no additional poles will be encountered upon displacing the path from $\Re z = \varsigma_0$ to $\Re z = \varsigma_1$. However, due to the digamma function, the only possible pole of simple order lies at $z = -1$ if $q \neq -1$. In fact, if on the one hand $-2 < q < 0$ but $q \neq -1$, the pole at $z = q$ is of order two, and with the aid of the functional equation for the digamma function we find:

$$\begin{aligned}\operatorname*{Res}_{z=q} \Gamma(z)\Gamma(q-z)\psi(q-z) &= -\operatorname*{Res}_{z=q} \frac{\Gamma(z)\Gamma(1+q-z)}{z-q}\left\{\psi(1+q-z) + \frac{1}{z-q}\right\} \\ &= -\psi(1)\Gamma(q) - \frac{d}{dz}\left\{\Gamma(z)\Gamma(1+q-z)\right\}\Big|_{z=q} \\ &= -\psi(q)\Gamma(q)\end{aligned}$$

If on the other hand $q = -1$, a coalescence to a third order pole happens at $z = -1$, and with (B.2.13) and (B.2.14) we compute:

$$\begin{aligned}\operatorname*{Res}_{z=-1} \Gamma(z)\Gamma(-1-z)\psi(-1-z) &= -\operatorname*{Res}_{z=-1} \frac{\Gamma(z+2)\Gamma(-z)}{(z+1)^2 z}\left\{\psi(-z) + \frac{1}{z+1}\right\} \\ &= -\frac{d}{dz}\frac{\Gamma(z+2)\Gamma(-z)}{z}\psi(-z)\Bigg|_{z=-1} - \frac{d^2}{dz^2}\frac{\Gamma(z+2)\Gamma(-z)}{z}\Bigg|_{z=-1} \\ &= 2 + \tfrac{\pi^2}{6} - \gamma\end{aligned}$$

Again bearing in mind the fact that the poles are encircled clockwisely, an application of the residue theorem leads to

$$f'(q) = \Gamma(q+1)\psi(1+q)\mathbb{1}_{\{q > -2 \wedge q \neq -1\}} + \Gamma(q)\psi(q)\mathbb{1}_{\{-2 < q < 0 \wedge q \neq -1\}} - \left(2 + \tfrac{\pi^2}{6} - \gamma\right)\mathbb{1}_{\{q=-1\}} + \frac{1}{2\pi i}\int_{\varsigma_1 - i\infty}^{\varsigma_1 + i\infty} \Gamma(z)\Gamma(q-z)\psi(q-z)dz.$$

Appealing to definition (8.5.123) and identity (8.5.132) with $q = \nu_{00}$, for any $\nu_{00} > -2$ we conclude

$$\begin{aligned}\Omega^1(\nu_{00}) = \Gamma(\nu_{00})\psi(\nu_{00})\left\{2^{-\nu_{00}} - 1 + \nu_{00}\right\} + \Gamma(\nu_{00})(1 - 2^{-\nu_{00}}\log(2)) \\ - \Gamma(\nu_{00}+1)\psi(1+\nu_{00})\mathbb{1}_{\{\nu_{00} > -2 \wedge \nu_{00} \neq -1\}} \\ - \Gamma(\nu_{00})\psi(\nu_{00})\mathbb{1}_{\{-2 < \nu_{00} < 0 \wedge \nu_{00} \neq -1\}} \\ + \left(2 + \tfrac{\pi^2}{6} - \gamma\right)\mathbb{1}_{\{\nu_{00} = -1\}}.\end{aligned}$$

If we rewrite the above results by elementary manipulations, for any $\nu_{00} > -2$ we eventually arrive at

$$\begin{aligned}\Omega(\nu_{00}) = \Gamma(\nu_{00}) 2^{-\nu_{00}} \mathbb{1}_{\{\nu_{00} > -2 \,\wedge\, \nu_{00} \notin \{-1,0\}\}} - \Gamma(\nu_{00}) \mathbb{1}_{\{\nu_{00} > 0\}} \\ - \log(2) \mathbb{1}_{\{\nu_{00} = 0\}} + \log(4) \mathbb{1}_{\{\nu_{00} = -1\}},\end{aligned} \tag{8.5.134}$$

and in addition at

$$\begin{aligned}\Omega^1(\nu_{00}) = \Gamma(\nu_{00}) \left\{2^{-\nu_{00}} (\psi(\nu_{00}) - \log(2)) - \psi(\nu_{00})\right\} \mathbb{1}_{\{\nu_{00} > -2 \,\wedge\, \nu_{00} \notin \{-1,0\}\}} \\ - \Gamma(\nu_{00}) \psi(\nu_{00}) \mathbb{1}_{\{-2 < \nu_{00} < 0 \,\wedge\, \nu_{00} \neq -1\}} \\ + \tfrac{1}{2} \log(2)(2\gamma + \log(2)) \mathbb{1}_{\{\nu_{00} = 0\}} \\ + \left(1 + \tfrac{\pi^2}{6} - \log^2(2) - (\gamma - 1)\log(4)\right) \mathbb{1}_{\{\nu_{00} = -1\}}.\end{aligned} \tag{8.5.135}$$

Upon plugging the preceding two equations into (8.5.122) a more convenient representation for the constant $\Xi(\nu_{00}, T_2)$ can be obtained.

8.5.10. Evaluation of the Iterated MB-Integral

We are finally ready to gather the above findings, to characterize the leading terms in the expansion (8.5.57). For a clearer presentation, we distinguish between three cases, depending on the presence of the first term and on the range of η_{00}.

Theorem 8.5.1. *Assume validity of the conditions (A1) and (A2) with $\alpha_{00} \neq \gamma_{00}$ and*

$$\begin{aligned}\eta_{00} \geq \min\{\nu_{00}, \eta_{00} + \chi_{00}\}, \\ \nu_{00}, \eta_{00} + \chi_{00} \geq \min\{0, \nu_{00} + 1, \eta_{00}, \eta_{00} + \chi_{00} + 1\}.\end{aligned}$$

Then, provided at least one term on the right hand side is non-zero, as $m \to \infty$,

(1) if $\alpha_{00} > 1$, we have

$$\begin{aligned}\mathrm{Ai}\begin{bmatrix} m; \begin{matrix} T_1 \\ T_2 \end{matrix} \end{bmatrix} \sim \mathcal{S}_0 \begin{bmatrix} 0, T_1 \\ 0, T_2 \end{bmatrix} \mathbb{1}_{\{\min\{\eta_{00}, \nu_{00}\} > 0\}} \\ - \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{|c_{00}|^2}{\beta_{00}} \mathcal{A}^0(0; T_2) \mathbb{1}_{\{0 \neq \nu_{00} < \eta_{00}\}} \\ + \left\{ (\log(m+1) + \gamma) \frac{|c_{00}|^2}{\beta_{00}} \mathcal{A}^0(0; T_2) + \frac{1}{\beta_{00}} \mathcal{A}^1 \begin{bmatrix} 0 \\ T_1, T_2 \end{bmatrix} \right\} \mathbb{1}_{\{0 = \nu_{00} < \eta_{00}\}} \\ - \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{\overline{a}_{00} \overline{c}_{00}}{\beta_{00}} \mathcal{P}_0 \begin{bmatrix} 0, 0 \\ T_1, \infty \end{bmatrix} \mathbb{1}_{\{0 \neq \eta_{00} < \nu_{00}\}} \\ + \left\{ (\log(m+1) + \gamma) \frac{\overline{a}_{00} \overline{c}_{00}}{\beta_{00}} \mathcal{P}_0 \begin{bmatrix} 0, 0 \\ T_1, \infty \end{bmatrix} + \frac{1}{\beta_{00}} \mathcal{Y}_0 \begin{bmatrix} 0, 0 \\ T_1, \infty, T_2 \end{bmatrix} \right\} \\ \times \mathbb{1}_{\{0 = \eta_{00} < \nu_{00}\}}\end{aligned}$$

$$+\frac{\mathcal{A}^0(0;T_2)}{\{b_{00}(m+1)\}^{\nu_{00}}}\frac{|c_{00}|^2}{\beta_{00}}\Omega(\nu_{00}).$$

The coefficients have been defined in (8.4.49), (8.5.29), (8.5.32) and (8.5.71). Furthermore, the coefficient $\Omega(\nu_{00})$ *was computed in (8.5.134).*

(2) if $0<\alpha_{00}\le 1$*, we have*

$$\begin{aligned}
\mathrm{Ai}\left[m;\begin{matrix}T_1\\T_2\end{matrix}\right] \sim \mathcal{S}_0&\left[\begin{matrix}0,T_1\\0,T_2\end{matrix}\right]\mathbb{1}_{\{\eta_{00}+\chi_{00}>0\}}\\
&-\frac{\Gamma(\eta_{00}+\chi_{00})}{\{b_{00}(m+1)\}^{\chi_{00}+\eta_{00}}}\frac{\overline{a}_{00}\,|c_{00}|^2}{\beta_{00}}\frac{\Gamma(1-\alpha_{00})\Gamma(1-\gamma_{00})}{\Gamma(2-\alpha_{00}-\gamma_{00})}\\
&\qquad\times\mathbb{1}_{\{\eta_{00}+\chi_{00}\notin\{0,\min\{\eta_{00},\nu_{00}\}\}\}}\\
&-\log(m+1)\frac{\Gamma(\eta_{00}+\chi_{00})}{\{b_{00}(m+1)\}^{\chi_{00}+\eta_{00}}}\frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2}\mathbb{1}_{\{0\neq\eta_{00}+\chi_{00}=\min\{\eta_{00},\nu_{00}\}\}}\\
&+\frac{\Gamma(\eta_{00}+\chi_{00})}{\{b_{00}(m+1)\}^{\chi_{00}+\eta_{00}}}\left\{\psi(\eta_{00}+\chi_{00})+v'\left(\chi_h+\tfrac{\gamma_{00}}{\beta_{00}}\right)+\frac{\lambda_{h,1}}{e_{h0}}\right\}\\
&\qquad\times\frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2}\mathbb{1}_{\{0\neq\eta_{00}+\chi_{00}=\min\{\eta_{00},\nu_{00}\}\}}\\
&+\{\log(m+1)+\gamma\}\frac{\overline{a}_{00}\,|c_{00}|^2}{\beta_{00}}\frac{\Gamma(1-\alpha_{00})\Gamma(1-\gamma_{00})}{\Gamma(\gamma_{00})}\\
&\qquad\times\mathbb{1}_{\{0=\eta_{00}+\chi_{00}<\min\{\eta_{00},\nu_{00}\}\}}\\
&-\frac{\overline{a}_{00}\,|c_{00}|^2}{\beta_{00}}\frac{\Gamma(1-\alpha_{00})\Gamma(1-\gamma_{00})}{\Gamma(\gamma_{00})}\left\{u'(0)+\frac{\lambda_{h,1}}{e_{h0}}\right\}\\
&\qquad\times\mathbb{1}_{\{0=\eta_{00}+\chi_{00}<\min\{\eta_{00},\nu_{00}\}\}}\\
&+\Sigma_1(0;T_1,T_2)\mathbb{1}_{\{0=\eta_{00}+\chi_{00}\le\min\{\eta_{00},\nu_{00}\}\}}\\
&+\left\{(\log(m+1)+\gamma)^2+\frac{\pi^2}{6}\right\}\frac{\overline{a}_{00}\,|c_{00}|^2}{2\,\{\beta_{00}\}^2}\mathbb{1}_{\{0=\eta_{00}+\chi_{00}=\min\{\eta_{00},\nu_{00}\}\}}\\
&-(\log(m+1)+\gamma)\frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2}\left\{v'(0)+\frac{\lambda_{h,1}}{e_{h0}}\right\}\\
&\qquad\times\mathbb{1}_{\{0=\eta_{00}+\chi_{00}=\min\{\eta_{00},\nu_{00}\}\}}\\
&+\frac{\overline{a}_{00}\,|c_{00}|^2}{2\,\{\beta_{00}\}^2}\left\{(v'(0))^2+2v'(0)\frac{\lambda_{h,1}}{e_{h0}}+v''(0)+\frac{\lambda_{h,2}}{e_{h0}}\right\}\\
&\qquad\times\mathbb{1}_{\{0=\eta_{00}+\chi_{00}=\min\{\eta_{00},\nu_{00}\}\}}\\
&+\frac{\log(m+1)}{\{b_{00}(m+1)\}^{\nu_{00}}}\frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2}\Omega(\nu_{00})\mathbb{1}_{\{\alpha_{00}=1\}}\\
&-\frac{\Xi(\nu_{00},T_2)}{\{b_{00}(m+1)\}^{\nu_{00}}}\frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2}\mathbb{1}_{\{\alpha_{00}=1\}}\\
&+\frac{\Gamma(1-\alpha_{00})}{\{b_{00}(m+1)\}^{\eta_{00}+\chi_{00}}}\frac{\overline{a}_{00}\,|c_{00}|^2}{\beta_{00}}\Upsilon(\alpha_{00},\beta_{00},\gamma_{00})\mathbb{1}_{\{\alpha_{00}<1\}}.
\end{aligned}$$

The coefficients were specified in §8.5.8.2 and in equations (8.5.56), (8.5.122) and (8.5.134).

Proof. Under the assumptions of the theorem, the first term in the expansion (8.5.57) is non-zero, and from (8.5.73) we deduce

$$\psi_\infty = \min\{0, \eta_{00}\}.$$

Morever, a comparison of (8.5.11) and (8.5.75) shows that

$$\psi_0(\alpha_{00}) = \nu_0(\alpha_{00}) + x_0.$$

As a consequence, the order as $m \to \infty$ of the non-zero terms appearing in each of the expansions (8.5.57), (8.5.80) and (8.5.121) differs at most by logarithmic factors, whereas the remainder terms are of higher algebraic order as $m \to \infty$. Indeed, there exist u_1 and x_1 as specified in (8.5.51), (8.5.79) and (8.5.120) as well as $\varepsilon_1 > 0$, such that $u_1 + x_0 = x_1 = \nu_0(\alpha_{00}) + x_0 + \varepsilon_1$. We therefore conclude neglibility of the terms in each big-$\mathcal{O}$. ∎

Theorem 8.5.2. *Assume validity of the conditions (A1) and (A2) with $\alpha_{00} \neq \gamma_{00}$ and*

$$\eta_{00} < \min\{\nu_{00}, \eta_{00} + \chi_{00}\}.$$

Then, provided at least one term on the right hand side is non-zero, as $m \to \infty$ we have

$$\begin{aligned}
\mathrm{Ai}\begin{bmatrix} m; \begin{matrix} T_1 \\ T_2 \end{matrix} \end{bmatrix} \sim{}& \mathcal{S}_0\begin{bmatrix} 0, T_1 \\ 0, T_2 \end{bmatrix} \mathbb{1}_{\{\eta_{00} > 0\}} \\
&- \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \mathcal{P}_0\begin{bmatrix} 0, 0 \\ T_1, \infty \end{bmatrix} \mathbb{1}_{\{\eta_{00} \neq 0\}} \\
&+ \left\{ (\log(m+1) + \gamma) \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \mathcal{P}_0\begin{bmatrix} 0, 0 \\ T_1, \infty \end{bmatrix} + \frac{1}{\beta_{00}} \mathcal{Y}_0\begin{bmatrix} 0, 0 \\ T_1, \infty, T_2 \end{bmatrix} \right\} \mathbb{1}_{\{\eta_{00} = 0\}},
\end{aligned}$$

with the coefficients given in (8.4.49) and (8.5.71).

Proof. According to the conditions of the present theorem, $\nu_{00} > \eta_{00} > \chi_{00} > 0$ and $\eta_{00} + \chi_{00} > 0$, implying $\psi_\infty = 0$ and, by (8.5.75),

$$\psi_0(\alpha_{00}) = \eta_{00} < \min\{\nu_{00}, \eta_{00} + \chi_{00}\} = \nu_0(\alpha_{00}) + x_0.$$

Hence, the non-zero terms in (8.5.57) and the remainder therein, as well as the remainder in (8.5.80) are of higher algebraic order than the non-zero terms in the last expansion. ∎

We conclude this section with the simplest case, occuring if the first term in the expansion (8.5.57) vanishes. Then, from (8.5.58) the following theorem immediately becomes obvious.

Theorem 8.5.3. *Assume validity of the conditions (A1) and (A2) with*

$$\min\{\nu_{00}, \eta_{00} + \chi_{00}\} < \min\{0, \nu_{00} + 1, \eta_{00}, \eta_{00} + \chi_{00} + 1\}.$$

Then, provided at least one term on the right hand side is non-zero, as $m \to \infty$ we have

$$
\begin{aligned}
\mathrm{Ai}\left[m; \begin{matrix} T_1 \\ T_2 \end{matrix}\right] \sim\ & \frac{\mathcal{A}^0(0;T_2)}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{|c_{00}|^2}{\beta_{00}} \Omega(\nu_{00}) \mathbb{1}_{\{\alpha_{00}>1\}} \\
& + \frac{\log(m+1)}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2} \Omega(\nu_{00}) \mathbb{1}_{\{\alpha_{00}=1\}} \\
& - \frac{\Xi(\nu_{00},T_2)}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2} \mathbb{1}_{\{\alpha_{00}=1\}} \\
& + \frac{\Gamma(1-\alpha_{00})}{\{b_{00}(m+1)\}^{\eta_{00}+\chi_{00}}} \frac{\overline{a}_{00}\,|c_{00}|^2}{\beta_{00}} \Upsilon(\alpha_{00},\beta_{00},\gamma_{00}) \mathbb{1}_{\{\alpha_{00}<1\}}
\end{aligned}
$$

For the coefficients we refer to (8.5.56), (8.5.122) and (8.5.134).

Notice that the conditions on the parameters from the last two theorems can not hold simultaneously.

8.6. An Infinite Path of the Second Kind in an Asymmetric-Type Iterated Integral

We shall now treat scenarios, in which of (8.0.2) only the exterior path is certainly infinite, whereas the interior may be finite or infinite. Moreover, the amplitude is supposed to admit oscillatory properties. In particular, we will now study the m-asymptotic behaviour of

(8.6.1)
$$
\begin{aligned}
\mathrm{Ai}\left[m; \begin{matrix} i\xi_1, \sigma, S \\ i\xi_2, T_2 \end{matrix}\right] := & \int_{\mathcal{P}_1} \left\{1-(1-\Psi(s))^{m+1}\right\} e^{-i\xi_1 s} c(s;p_1) \\
& \times \int_{T_2}^{\infty} e^{-i\xi_2 t} \overline{\mathfrak{a}}(t) \left\{1-(1-\Psi(s+t))^{m+1}\right\} \overline{\mathfrak{c}}(s+t) dt ds,
\end{aligned}
$$

for $\xi_1, \xi_2 \in \mathbb{R}$, $T_2 > 0$ and a half open path $\mathcal{P}_1$ with endpoints $0 \le \sigma < S \le \infty$, either $\mathcal{P}_1 = (\tau_1, T_1]$ for $0 \le \tau_1 < T_1 < \infty$ or $\mathcal{P}_1 = [T_1, \infty)$ for $T_1 > 0$. In addition, the parameter p_1 and the function $c(s;p_1)$ were specified in Section 8.4, and accordingly the first refers to the endpoint, for which $\mathcal{P}_1 \cup \{p_1\}$ is closed, while the second was introduced in (8.4.2). Finally, the ingredients, of which φ was defined in (8.1.5), are supposed to satisfy the following conditions:

(A3) $\varphi(u)$ and $\mathfrak{c}(u)$ show algebraic behaviour as $u \to \infty$ for respective parameters $\beta_{00} > 0$, $\gamma_{00} \in \mathbb{C}$ and coefficients $b_{00} > 0$, $c_{00} \in \mathbb{C} \setminus \{0\}$, where

(8.6.2)
$$\beta_{00} + \gamma_{00} > 1.$$

(A4) For each $k \in \{1,2\}$ with $\mathcal{P}_k = [T_k, \infty)$ and $\xi_k = 0$, the functions $\varphi(u)$ and $\mathfrak{c}(u)$ are once continuously differentiable on $u \ge T_k$ with $\varphi(u) > 0$, and the first derivatives of their normalized analogues $B(u)$ and $\mathfrak{C}(u)$ are of order $\beta_{01}, \gamma_{01} > 1$ as $u \to \infty$.

(A5) For each $k \in \{1, 2\}$ with $\mathcal{P}_k = [T_k, \infty)$ and $\xi_k \neq 0$, we suppose:

(a) $\varphi(u)$ is infinitely many times continuously differentiable on $u \geq T_k$ with $\frac{d^{j-1}}{dt^{j-1}} \frac{\varphi'(u)}{\varphi(u)} = \mathcal{O}\left\{u^{-j}\right\}$ as $t \to \infty$ for any $j \in \mathbb{N}$.

(b) $\mathfrak{c}(u)$ is infinitely many times continuously differentiable on $u \geq T_k$ with $\mathfrak{c}^{(j)}(u) = \mathcal{O}\left\{u^{-\gamma_{00}-j}\right\}$ as $u \to \infty$ for any $j \in \mathbb{N}$.

(A6) If $\xi_2 = 0$, $\mathfrak{a}(t)$ is algebraic at infinity for a parameter $\alpha_{00} > 0$ and a coefficient $a_{00} \in \mathbb{C}\setminus\{0\}$. Moreover, it is once continuously differentiable on $t \geq T_2$, and the normalized amplitude $\mathfrak{A}'(t)$ is of order $\alpha_{01} > 1$ as $t \to \infty$.

(A7) If $\xi_2 \neq 0$, then $\mathfrak{a}(t)$ is infinitely many times continuously differentiable on $t \geq T_2$, and there exists $\alpha_{00} > 0$, such that for any $j \in \mathbb{N}_0$ as $t \to \infty$ we have $\mathfrak{a}^{(j)}(t) = \mathcal{O}\left\{t^{-\alpha_{00}-j}\right\}$.

(A8) If $\mathcal{P}_1 = (\tau_1, T_1]$, we have continuity of φ and c there, with $\varphi > 0$ and each function shows algebraic behaviour as $s \downarrow \tau_1$ for parameters $\beta_{10} \geq 0$, $\gamma_{10} \in \mathbb{R}$ and coefficients $b_{10} > 0$, $c_{10} \in \mathbb{C} \setminus \{0\}$, where

$$\gamma_{10} + \beta_{10} > -1. \tag{8.6.3}$$

If $\beta_{10} > 0$, the normalized functions $B_1(s)$ and $C_1(s)$ on $(\tau_1, T_1]$ possess a continuous first derivative of order $\beta_{11}, \gamma_{11} > -1$ as $s \downarrow \tau_1$.

(A9) If $\mathcal{P}_1 = (\tau_1, T_1]$ with $\beta_{10} > 0$, $\xi_2 = 0$ and $\alpha_{00} + \gamma_{00} = 1$, the functions $\varphi(u)$ and $\mathfrak{c}(u)$ are, in addition to (A3) and (A4), twice continuously differentiable on $u \geq T_2$, and with a constant $c_{01} \in \mathbb{C} \setminus \{0\}$ as $u \to \infty$ each of them satisfies

$$\varphi''(u) \sim b_{00}\beta_{00}(\beta_{00}+1)u^{-\beta_{00}-2}, \tag{8.6.4}$$

$$\mathfrak{c}''(u) \sim \begin{cases} c_{00}\gamma_{00}(\gamma_{00}+1)u^{-\gamma_{00}-2}, & \text{if } \gamma_{00} \neq 0, \\ -c_{01}\gamma_{01}u^{-\gamma_{01}-1}, & \text{if } \gamma_{00} = 0. \end{cases} \tag{8.6.5}$$

Observe that $\mathfrak{c}(u) = \mathfrak{C}(u)$ if $\gamma_{00} = 0$, whence condition (A4) implies $\mathfrak{c}'(u) = \mathcal{O}\left\{u^{-\gamma_{01}}\right\}$ as $u \to \infty$. But this is especially true if $\mathfrak{c}'(u) \sim c_{01}u^{-\gamma_{01}}$, and if this relation is once differentiable, validity of (8.6.5) follows. Hence, the conditions (A4) and (A9), particularly for $\gamma_{00} = 0$, do not contradict each other. Finally, we leave it to this only remark in the introductory part, that the reader should bear in mind, that some coefficients in the Laurent expansions below actually can be zero.

Throughout this section we reemploy some definitions from Section 8.4. For instance, as in (8.4.10), we denote

$$\zeta_j := \delta_j + i\xi_j, \tag{8.6.6}$$

with $\delta_j > 0$ if and only if $\mathcal{P}_j$ is infinite with $\xi_j \neq 0$, and $\delta_j = 0$ otherwise. Furthermore, we refer to θ_1 and θ_2 as the parameters (8.4.11) and, analogously to (8.4.12), we define

$$\vec{p}_1 := (p_1, \theta_1). \tag{8.6.7}$$

We are then interested in the m-asymptotic behaviour of (8.6.1) for all admissible pairs of the parameters $\vec{p}_1$ and θ_2, i.e., for

$$\begin{cases} \vec{p}_1 \in \{(\tau_1, 0), (\tau_1, 1), (\infty, 1), (\infty, 0)\}, \\ \theta_2 \in \{0, 1\}, \end{cases} \tag{8.6.8}$$

where $\vec{p}_1 = (\infty, 0)$ with $\theta_2 = 0$ is assumed to imply

$$\alpha_{00} = \gamma_{00} \neq \tfrac{1}{2}, \tag{8.6.9}$$

or equivalently $\eta_{00} = \nu_{00} \neq 0$ with ν_{00} from (8.5.3). Indeed, in the case $\alpha_{00} = \gamma_{00} = \frac{1}{2}$, the reader easily confirms the applicability of Theorem 8.5.3. Now, due to the assumptions $\alpha_{00} > 0$ and (8.6.2), from (8.6.9) we conclude

$$\gamma_{00} > 0 \text{ and } \eta_{00}, \nu_{00} > \chi_{00} > -1, \qquad \text{for } \vec{p}_1 = (\infty, 0) \wedge \theta_2 = 0. \tag{8.6.10}$$

With the exception of the last case, we have absolute convergence of

$$\mathcal{K}_0\begin{bmatrix} -w, \zeta_1, \sigma, S \\ -z, \zeta_2, T_2 \end{bmatrix} := \int_{\sigma}^{S} \{\varphi(s)\}^{-w} e^{-\zeta_1 s} c(s; p_1) \int_{T_2}^{\infty} \{\varphi(s+t)\}^{-z} e^{-\zeta_2 t} \mathfrak{a}(t) \mathfrak{c}(s+t) dt ds, \tag{8.6.11}$$

if and only if this statement applies to each single component of this iterated integral, i.e., for $w, z \in \mathbb{C}$ with

$$\Re w < \begin{cases} \infty, & \text{if } \vec{p}_1 \in \{(\tau_1, 0), (\infty, 1)\}, \\ \chi_{10}, & \text{if } \vec{p}_1 = (\tau_1, 1), \\ \nu_{00} - \Re z, & \text{if } \vec{p}_1 = (\infty, 0) \wedge \theta_2 = 1, \end{cases} \tag{8.6.12}$$

$$\Re z < \begin{cases} \infty, & \text{if } \theta_2 = 1, \\ \eta_{00}, & \text{if } \theta_2 = 0 \wedge \vec{p}_1 \neq (\infty, 0). \end{cases} \tag{8.6.13}$$

The parameter χ_{10} was defined in (8.3.5). Finally, in the indicated special case $\vec{p}_1 = (\infty, 0)$ with $\theta_2 = 0$ and $\gamma_{00} = \alpha_{00} \neq \frac{1}{2}$, by comparison with (8.5.5) we obtain

$$\mathcal{K}_0\begin{bmatrix} -w, 0, T_1, \infty \\ -z, 0, T_2 \end{bmatrix} = \mathcal{S}_0\begin{bmatrix} -w, T_1 \\ -z, T_2 \end{bmatrix}. \tag{8.6.14}$$

Hence, by (8.5.7), absolute convergence of (8.6.11) holds for $w, z \in \mathbb{C}$ with

$$\begin{cases} \Re w < \infty, \\ \Re z < \eta_{00} + \min\{-\Re w, 0, \chi_{00} - \Re w\}. \end{cases} \tag{8.6.15}$$

To avoid the dependence of the z-abscissa of convergence from the fixed variable w, we then agree

$$\begin{cases} \Re w < \min\{0, \chi_{00}\}, \\ \Re z < \eta_{00}. \end{cases} \tag{8.6.16}$$

The z-abscissa of convergence of $\mathcal{K}_0[\ldots]$ in the case $\theta_2 = 0$ is therefore always the line $\Re z = \eta_{00}$.

8.6.1. Transformation to an Iterated MB-Integral

Under the above assumptions, the integral (8.6.1) converges absolutely for any $m \geq 0$ and $\xi_1, \xi_2 \in \mathbb{R}$, and this statement, even uniformly with respect to $\delta_1, \delta_2 \geq 0$, also applies to the related integral

$$\begin{aligned} \mathcal{A}i\left[m; \begin{matrix} \zeta_1, \sigma, S \\ \zeta_2, T_2 \end{matrix}\right] := \int\limits_{\mathcal{P}_1} & \left\{1 - (1 - \Psi(s))^{m+1}\right\} e^{-\zeta_1 s} c(s; p_1) \\ & \times \int\limits_{T_2}^{\infty} e^{-\zeta_2 t} \bar{\mathfrak{a}}(t) \left\{1 - (1 - \Psi(s+t))^{m+1}\right\} \bar{\mathfrak{c}}(s+t) dt ds. \end{aligned} \tag{8.6.17}$$

Accordingly, by Lebesgue's dominated convergence theorem, we have the permission to write

$$\mathrm{Ai}\left[m; \begin{matrix} i\xi_1, \sigma, S \\ i\xi_2, T_2 \end{matrix}\right] = \lim_{\delta_1, \delta_2 \downarrow 0} \mathcal{A}i\left[m; \begin{matrix} \zeta_1, \sigma, S \\ \zeta_2, T_2 \end{matrix}\right]. \tag{8.6.18}$$

Again we first begin with a discussion with the m-asymptotic behaviour of $\mathcal{A}i[\ldots]$ for fixed $\delta_j \geq 0$. An application of the Cahen-Mellin representation (8.1.6) for the m-power of the variable s, according to Corollary 8.2.2 and Lemma 8.2.3, bearing in mind (8.6.10), is permissible for

$$-1 < u_0(\theta_2, \vec{p}_1) < \begin{cases} 0, & \text{if } \vec{p}_1 \in \{(\tau_2, 0), (\infty, 1)\}, \\ \min\{0, \chi_{10}\}, & \text{if } \vec{p}_1 = (\tau_1, 1), \\ \min\{0, \nu_{00} + 1\}, & \text{if } \vec{p}_1 = (\infty, 0) \wedge \theta_2 = 1, \\ \min\{0, \chi_{00}\}, & \text{if } \vec{p}_1 = (\infty, 0) \wedge \theta_2 = 0, \end{cases} \tag{8.6.19}$$

and this leads to

$$\mathcal{A}i\left[m; \begin{matrix}\zeta_1, \sigma, S\\ \zeta_2, T_2\end{matrix}\right] = \frac{1}{2\pi i}\int\limits_{u_0(\theta_2,\vec{p}_1)-i\infty}^{u_0(\theta_2,\vec{p}_1)+i\infty} (m+1)^{-w}\Gamma(w)\int\limits_{\mathcal{P}_1}\{\varphi(s)\}^{-w}e^{-\zeta_1 s}c(s;p_1)$$
(8.6.20)
$$\times\int\limits_{T_2}^{\infty} e^{-\zeta_2 t}\bar{\mathfrak{a}}(t)\left\{1-(1-\Psi(s+t))^{m+1}\right\}\bar{\mathfrak{c}}(s+t)dtdsdw.$$

Then, for another parameter

$$-1 < x_0(\theta_2) < \begin{cases} 0, & \text{if } \theta_2 = 1,\\ \min\{0, \eta_{00}\}, & \text{if } \theta_2 = 0,\end{cases} \tag{8.6.21}$$

an application of the Cahen-Mellin representation for the m-power of the variable $s+t$ yields

$$\mathcal{A}i\left[m; \begin{matrix}\zeta_1, \sigma, S\\ \zeta_2, T_2\end{matrix}\right] = \frac{1}{2\pi i}\int\limits_{u_0(\theta_2,\vec{p}_1)-i\infty}^{u_0(\theta_2,\vec{p}_1)+i\infty} (m+1)^{-w}\Gamma(w)\,\mathrm{H}\left[m; \begin{matrix}\theta_2\\ w, \vec{p}_1\end{matrix}\right]dw, \tag{8.6.22}$$

where the right hand side, in terms of (8.6.11), involves the MB-integral

$$\mathrm{H}\left[m; \begin{matrix}\theta_2\\ w, \vec{p}_1\end{matrix}\right] := \frac{1}{2\pi i}\int\limits_{x_0(\theta_2)-i\infty}^{x_0(\theta_2)+i\infty} (m+1)^{-z}\Gamma(z)\mathcal{K}_0\left[\begin{matrix}-w, \zeta_1, \sigma, S\\ -z, \zeta_2, T_2\end{matrix}\right]dz. \tag{8.6.23}$$

8.6.2. z-Analytic Continuation of the Iterated Generating Function

The procedure for determining for fixed w and $\theta_2 = 0$ the z-analytic continuation of the iterated generating function (8.6.11) across the boundary line $\Re z = \eta_{00}$ is routine. First, by virtue of (8.5.59), we write

$$\mathcal{K}_0\left[\begin{matrix}-w, \zeta_1, \sigma, S\\ -z, \zeta_2, T_2\end{matrix}\right] = \int\limits_{\mathcal{P}_1}\{\varphi(s)\}^{-w}e^{-\zeta_1 s}c(s;p_1)\,\mathrm{R}_0\left[\begin{matrix}\frac{1-\alpha_{00}}{\beta_{00}}, z-\chi_{00}\\ s, 0, 0, z, T_2\end{matrix}\right]ds. \tag{8.6.24}$$

Now, if $\vec{p}_1 \in \{(\tau_1, 0), (\tau_1, 1), (\infty, 1)\}$, as we mentioned in before, we are in the convenient situation that this iterated integral converges absolutely if and only if each of its single components does. It is then particularly analytic with respect to each of the variables w and z. But by Subsection 8.5.6, the interior integral, and therefore especially the iterated integral, by means of the expansion (8.5.60), can be continued to a meromorphic function in the region

$$\Re z < \eta_{00} + \eta_{\beta_{00}}(\alpha_{01}, \beta_{01}, \gamma_{01}, 2), \tag{8.6.25}$$

for any fixed admissible $w \in \mathbb{C}$. There, it shows only a single singularity, which is a pole of simple order at $z = \eta_{00}$. From (8.5.62), with the aid of the function (8.4.49), we find

$$\operatorname*{Res}_{z=\eta_{00}} \mathcal{K}_0\begin{bmatrix} -w, \zeta_1, \sigma, S \\ -z, \zeta_2, T_2 \end{bmatrix} = -\frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \{b_{00}\}^{-\eta_{00}} \mathcal{P}_0\begin{bmatrix} -w, \zeta_1 \\ \sigma, S \end{bmatrix}. \tag{8.6.26}$$

If $\eta_{00} = 0$, this pole lies at the origin of the z-plane, and by (8.5.65) as $z \to 0$ we obtain the Laurent expansion

$$\mathcal{K}_0\begin{bmatrix} -w, \zeta_1, \sigma, S \\ -z, \zeta_2, T_2 \end{bmatrix} = -\frac{1}{z}\frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \mathcal{P}_0\begin{bmatrix} -w, \zeta_1 \\ \sigma, S \end{bmatrix} + \frac{1}{\beta_{00}}\mathcal{Y}_0\begin{bmatrix} -w, \zeta_1 \\ \sigma, S, T_2 \end{bmatrix} + \mathcal{O}(z), \tag{8.6.27}$$

where the integral transform in the second summand, in terms of (8.5.63) and (8.5.64), is given by (8.5.71).

Finally, if $\vec{p}_1 = (\infty, 0)$, since $\theta_2 = 0$, by assumption $\alpha_{00} = \gamma_{00} \neq \frac{1}{2}$. Hence, due to the identity (8.6.14) and, according to Subsection 8.5.7, in this case the function (8.5.68) for fixed $w \in \mathbb{C}$ with $\Re w < \min\{0, \chi_{00}\}$ represents the z-analytic continuation of (8.6.24) into the half plane

$$\Re z < \min\{\nu_{00} - \Re w, \eta_{01}, \nu_{00} + \chi_{00} - \Re w\}. \tag{8.6.28}$$

Therein it exhibits a pole at $z = \nu_{00}$ only, which is always of simple order and does not lie at the origin of the z-plane. Its residue was specified in (8.5.70).

8.6.3. Evaluation of the Interior MB-Integral

According to the preceding subsection, for $\theta_2 = 0$ and fixed $w \in \mathbb{C}$ with $\Re w = u_0(0, \vec{p}_1)$, the z-analytic continuation of the generating function (8.6.11) into the half plane $\Re z < \eta_1(\vec{p}_1)$, where

$$\eta_1(\vec{p}_1) := \begin{cases} \min\{\nu_{00} - u_0(0,(\infty,0)), \eta_{01}, \nu_{00} + \chi_{00} - u_0(0,(\infty,0))\}, & \text{if } \vec{p}_1 = (\infty, 0), \\ \eta_{00} + \eta_{\beta_{00}}(\alpha_{01}, \beta_{01}, \gamma_{01}, 2), & \text{otherwise}, \end{cases} \tag{8.6.29}$$

can be computed via integration by parts. In each case, the only singularity in the wider region turned out to be a simple pole, lying on the line $\Re z = \eta_{00}$. Furthermore, it is easy to confirm that $\mathcal{K}_0[\ldots]$ for fixed $w \in \mathbb{C}$ with $\Re w = u_0(\theta_2, \vec{p}_1)$ is $\mathcal{O}(1)$ as $\Im z \to \pm\infty$, uniformly with respect to $\Re z$ in any closed vertical substrip of $\mathbb{C}$, if $\theta_2 = 1$, and otherwise of the half plane $\Re z < \eta_1(\vec{p}_1)$. The dominating behaviour of the integrand in (8.6.23) into each imaginary direction of the indicated regions is therefore exponential decay, owing to the presence of the gamma function. This enables a movement of the integration path to the right, across $z = 0$ or across the indicated simple pole, which will merge to a pole of second order if $\vec{p}_1 \in \{(\tau_1, 0), (\tau_1, 1), (\infty, 1)\}$ with $\theta_2 = 0$ and $\eta_{00} = 0$. Depending on whether or not the point $z = \eta_{00}$ lies in the left half plane, we decide to displace the path rightwards, to match a line $\Re z = x_1(\theta_2)$, for

$$0 < x_1(1) < \infty, \tag{8.6.30}$$
$$\eta_{00} < x_1(0) < \begin{cases} \eta_1(\vec{p}_1), & \text{if } \eta_{00} \geq 0, \\ \min\{0, \eta_1(\vec{p}_1)\}, & \text{if } \eta_{00} < 0. \end{cases}$$

In this process, we clockwisely traverse the pole at $z = \eta_{00}$ if $\theta_2 = 0$, the pole of $\Gamma(z)$ at $z = 0$ if $\theta_2 = 1$ or both if $\theta_2 = 0$ with $\eta_{00} \geq 0$. By taking into account the identity (8.6.14) and (8.5.70), but also (8.6.26) and (8.6.27), with the aid of Theorem B.2.1(2), for fixed $w \in \mathbb{C}$ with $\Re w = u_0(\theta_2, \vec{p}_1)$ we arrive at:

$$\begin{aligned}
\mathrm{H}\left[m; \begin{matrix} \theta_2 \\ w, \vec{p}_1 \end{matrix}\right] &= -\mathcal{K}_0\left[\begin{matrix} -w, \zeta_1, \sigma, S \\ 0, \zeta_2, T_2 \end{matrix}\right] \mathbb{1}_{\{\theta_2 = 1 \vee \eta_{00} > 0\}} \\
&+ \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \mathcal{P}_0\left[\begin{matrix} -w, \zeta_1 \\ \sigma, S \end{matrix}\right] \mathbb{1}_{\{\theta_2 = 0 \wedge \eta_{00} \neq 0\}} \\
&- \left\{ (\log(m+1) + \gamma) \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \mathcal{P}_0\left[\begin{matrix} -w, \zeta_1 \\ \sigma, S \end{matrix}\right] + \frac{1}{\beta_{00}} \mathcal{Y}_0\left[\begin{matrix} -w, \zeta_1 \\ \sigma, S, T_2 \end{matrix}\right] \right\} \\
&\qquad \times \mathbb{1}_{\{\theta_2 = 0 \wedge \vec{p}_1 \neq (\infty, 0) \wedge \eta_{00} = 0\}} \\
&+ \frac{1}{2\pi i} \int\limits_{x_1(\theta_2) - i\infty}^{x_1(\theta_2) + i\infty} (m+1)^{-z} \Gamma(z) \mathcal{K}_0\left[\begin{matrix} -w, \zeta_1, \sigma, S \\ -z, \zeta_2, T_2 \end{matrix}\right] dz
\end{aligned} \tag{8.6.31}$$

For brevity we define the following single MB-integrals, whose integration paths satisfy (8.6.19):

$$\mathrm{F}(m; \theta_2, \vec{p}_1) := \frac{1}{2\pi i} \int\limits_{u_0(\theta_2, \vec{p}_1) - i\infty}^{u_0(\theta_2, \vec{p}_1) + i\infty} (m+1)^{-w} \Gamma(w) \mathcal{K}_0\left[\begin{matrix} -w, \zeta_1, \sigma, S \\ 0, \zeta_2, T_2 \end{matrix}\right] dw \tag{8.6.32}$$

$$\mathrm{G}\left[m; \begin{matrix} \theta_1 \\ \sigma, S \end{matrix}\right] := \frac{1}{2\pi i} \int\limits_{u_0(\theta_2, \vec{p}_1) - i\infty}^{u_0(\theta_2, \vec{p}_1) + i\infty} (m+1)^{-w} \Gamma(w) \mathcal{P}_0\left[\begin{matrix} -w, \zeta_1 \\ \sigma, S \end{matrix}\right] dw \tag{8.6.33}$$

$$\mathrm{Y}(m; \vec{p}_1) := \frac{1}{2\pi i} \int\limits_{u_0(\theta_2, \vec{p}_1) - i\infty}^{u_0(\theta_2, \vec{p}_1) + i\infty} (m+1)^{-w} \Gamma(w) \mathcal{Y}_0\left[\begin{matrix} -w, \zeta_1 \\ \tau_1, T_1, T_2 \end{matrix}\right] dw \tag{8.6.34}$$

Then, upon plugging the expansion (8.6.31) into the iterated MB-integral (8.6.22), accompanied by a suitable bound for the remainder term, as $m \to \infty$ we deduce:

$$\begin{aligned}
\mathcal{A}i\left[m; \begin{matrix} \zeta_1, \sigma, S \\ \zeta_2, T_2 \end{matrix}\right] &= -\mathrm{F}(m; \theta_2, \vec{p}_1) \mathbb{1}_{\{\theta_2 = 1 \vee \eta_{00} > 0\}} \\
&+ \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \mathrm{G}\left[m; \begin{matrix} \theta_1 \\ \sigma, S \end{matrix}\right] \mathbb{1}_{\{\theta_2 = 0 \wedge \eta_{00} \neq 0\}} \\
&- \left\{ (\log(m+1) + \gamma) \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \mathrm{G}\left[m; \begin{matrix} \theta_1 \\ \sigma, S \end{matrix}\right] + \frac{1}{\beta_{00}} \mathrm{Y}(m; \vec{p}_1) \right\}
\end{aligned} \tag{8.6.35}$$

$$\times \mathbb{1}_{\{\theta_2 = 0 \wedge \vec{p}_1 \neq (\infty,0) \wedge \eta_{00} = 0\}}$$
$$+ \mathcal{O}\left\{m^{-u_0(\theta_2,\vec{p}_1)-x_1(\theta_2)}\right\}$$

We proceed with a study of the single MB-integrals, which appear in this expansion.

8.6.4. A Single MB-Integral for the Residue at $z = 0$ if $\xi_2 \neq 0$ or if $\eta_{00} > 0$

The MB-integral $\mathrm{F}(m; \theta_2, \vec{p}_1)$ appears in the above expansion, if $\theta_2 = 1$ or if $\eta_{00} > 0$, and its generating function coincides with the integral (8.6.11) for $z = 0$. Accordingly, it is an entire function of w, if $\vec{p}_1 \in \{(\tau_1, 0), (\infty, 1)\}$. In the special case $\vec{p}_1 = (\infty, 0)$ with $\theta_2 = 0$, where we assumed $\alpha_{00} = \gamma_{00} \neq \frac{1}{2}$, from (8.6.14) and (8.5.6), we conclude analyticity in

$$\Re w < 2\chi_{00} + \frac{1}{\beta_{00}} \min\{1, \gamma_{00}\}. \tag{8.6.36}$$

By definition of ν_{00}, this inequality can be rewritten in the form

$$\Re w < \nu_{00} + \min\{0, \chi_{00}\}. \tag{8.6.37}$$

Altogether, for $z = 0$ the integral (8.6.11) thus constitutes a holomorphic function of w in the region $\Re w < \phi_0(\theta_2, \vec{p}_1)$, with

$$\phi_0(\theta_2, \vec{p}_1) := \begin{cases} \chi_{10}, & \text{if } \vec{p}_1 = (\tau_1, 1), \\ \nu_{00}, & \text{if } \vec{p}_1 = (\infty, 0) \wedge \theta_2 = 1, \\ \nu_{00} + \min\{0, \chi_{00}\}, & \text{if } \vec{p}_1 = (\infty, 0) \wedge \theta_2 = 0. \end{cases} \tag{8.6.38}$$

There, it is $\mathcal{O}(1)$ as $\Im w \to \pm\infty$, uniformly with respect to $\Re w$ in any closed vertical substrip, implying that the path of the MB-integral (8.6.32) can be replaced by an arbitrary line with real part $u_0(\theta_2, \vec{p}_1) \equiv f_0(\theta_2, \vec{p}_1)$, for

$$-1 < f_0(\theta_2, \vec{p}_1) < \begin{cases} 0, & \text{if } \vec{p}_1 \in \{(\tau_1, 0), (\infty, 1)\}, \\ \min\{0, \phi_0(\theta_2, \vec{p}_1)\}, & \text{otherwise.} \end{cases} \tag{8.6.39}$$

The computation of the w-analytic continuation is an easy exercise for $\vec{p}_1 = (\tau_1, 1)$ and for $\vec{p}_1 = (\infty, 0)$ with $\theta_2 = 1$. Additional considerations are required only in the final case $\vec{p}_1 = (\infty, 0)$ with $\theta_2 = 0$.

8.6.4.1. w-Analytic Continuation for $z=0$ and a Finite Path $\mathcal{P}_1$, or an Infinite Path $\mathcal{P}_1$ with $\xi_2 \neq 0$

If the exterior path in (8.6.11) is finite, or if it is infinite but $\xi_2 \neq 0$, i.e., $\theta_2 = 1$, we first we define

$$\mathcal{R}(s;\zeta_2,T_2) := \int_{T_2}^{\infty} e^{-\zeta_2 t}\overline{\mathfrak{a}}(t)\overline{\mathfrak{c}}(s+t)dt. \tag{8.6.40}$$

It is then easy to confirm, that this integral converges absolutely and uniformly with respect to $\tau_1 \leq s \leq T_1$, if $\theta_2 = 0$ and $\eta_{00} > 0$, and that this statement remains true if $\mathfrak{c}$ is replaced by the derivative $\mathfrak{c}'$. In case of an infinite $\mathcal{P}_1$ with $\theta_2 = 1$, however, it depends on γ_{00} if the convergence is uniform with respect to $s \geq T_1$. Consequently, in general differentiation under the sign of integration is only admitted in the first case. This is the reason, why the w-analytic continuation of (8.6.11) for $z = 0$ not in each case can be computed by the method for single integral transforms. Hence, instead we interchange the order of integration, to obtain

$$\mathcal{K}_0\begin{bmatrix}-w,\zeta_1,\sigma,S\\0,\zeta_2,T_2\end{bmatrix} = \int_{T_2}^{\infty} e^{-\zeta_2 t}\overline{\mathfrak{a}}(t)\int_{\mathcal{P}_1}\{\varphi(s)\}^{-w}\, e^{-\zeta_1 s}c(s;p_1)\overline{\mathfrak{c}}(s+t)dsdt. \tag{8.6.41}$$

Under the assumptions of the present paragraph, the w-region of absolute convergence and analyticity of this iterated integral is the same as for the interior integral. But by reference to §§8.2.2.1 and 8.2.2.2, via integration by parts, an expansion for the interior integral can be derived, by means of which (8.6.41) can be extended to a meromorphic function in the half plane

$$\Re w < \begin{cases}\chi_{10} + \chi_{\beta_{10}}(\beta_{11},\gamma_{11},0), & \text{if } \vec{p}_1 = (\tau_1,1),\\ \nu_{00} + \eta_{\beta_{00}}(\beta_{01},\gamma_{01},2), & \text{if } \vec{p}_1 = (\infty,0)\wedge\theta_2 = 1.\end{cases} \tag{8.6.42}$$

The only singularity therein is a simple pole at $w = \phi_0(\theta_2,\vec{p}_1)$. Therefore, as $w \to \phi_0(\theta_2,\vec{p}_1)$ we find a Laurent expansion of the form

$$\mathcal{K}_0\begin{bmatrix}-w,\zeta_1,\sigma,S\\0,\zeta_2,T_2\end{bmatrix} = \frac{\kappa_{-1}(\theta_2,\vec{p}_1)}{w-\phi_0(\theta_2,\vec{p}_1)} + \kappa_0\begin{bmatrix}\theta_2,\theta_1\\\sigma,S\end{bmatrix} + \mathcal{O}(w-\phi_0(\theta_2,\vec{p}_1)), \tag{8.6.43}$$

whose coefficients are specified below. Particularly the dominating coefficient is the residue. Keeping in mind that $\zeta_1 = i\xi_1$ if $p_1 = \tau_1$, by (8.2.33) and (8.2.42), in terms of (8.6.40) and (8.5.29), this is equal to

$$\kappa_{-1}(\theta_2,\vec{p}_1) = \begin{cases}-\frac{c_{10}}{\beta_{10}}\{b_{10}\}^{-\chi_{10}}\, e^{-i\xi_1\tau_1}\mathcal{R}(\tau_1;\zeta_2,T_2), & \text{if } \vec{p}_1 = (\tau_1,1),\\ -\frac{|c_{00}|^2}{\beta_{00}}\{b_{00}\}^{-\nu_{00}}\,\mathcal{A}^0(\zeta_2;T_2), & \text{if } \vec{p}_1 = (\infty,0)\wedge\theta_2 = 1.\end{cases} \tag{8.6.44}$$

Moreover, according to (8.2.36) and (8.2.45), for the second coefficient in the above expansion, strictly in the case $\phi_0(\theta_2, \vec{p}_1) = 0$, we obtain

$$\begin{aligned}
(8.6.45)\quad \kappa_0\begin{bmatrix}\theta_2, 1\\ \tau_1, T_1\end{bmatrix} &= \frac{C_1(T_1)}{\beta_{10}} e^{-i\xi_1 T_1} \log\{\varphi(T_1)\} \int\limits_{T_2}^{\infty} e^{-\zeta_2 t}\bar{\mathfrak{a}}(t)\bar{\mathfrak{c}}(T_1+t)dt \\
&+ \frac{1}{\beta_{10}} \int\limits_{T_2}^{\infty} e^{-\zeta_2 t}\bar{\mathfrak{a}}(t) \int\limits_{\tau_1}^{T_1} B_1'(s) e^{-i\xi_1 s} C_1(s)\bar{\mathfrak{c}}(s+t)dsdt \\
&- \frac{1}{\beta_{10}} \int\limits_{T_2}^{\infty} e^{-\zeta_2 t}\bar{\mathfrak{a}}(t) \int\limits_{\tau_1}^{T_1} \log\{\varphi(s)\} e^{-i\xi_1 s} \\
&\qquad \times \left\{(C_1'(s) - i\xi_1 C_1(s))\bar{\mathfrak{c}}(s+t) + C_1(s)\bar{\mathfrak{c}}'(s+t)\right\} dsdt,
\end{aligned}$$

$$\begin{aligned}
(8.6.46)\quad \kappa_0\begin{bmatrix}1, 0\\ T_1, \infty\end{bmatrix} &= \frac{T_1^{\gamma_{00}}}{\beta_{00}} \log\{\varphi(T_1)\} \mathfrak{C}(T_1) \int\limits_{T_2}^{\infty} e^{-\zeta_2 t}\bar{\mathfrak{a}}(t)\bar{\mathfrak{c}}(T_1+t)dt \\
&- \frac{1}{\beta_{00}} \int\limits_{T_2}^{\infty} e^{-\zeta_2 t}\bar{\mathfrak{a}}(t) \int\limits_{T_1}^{\infty} s^{\gamma_{00}} \left\{B'(s)\mathfrak{C}(s) - \log\{\varphi(s)\} \mathfrak{C}'(s)\right\} \bar{\mathfrak{c}}(s+t)dsdt \\
&+ \frac{1}{\beta_{00}} \int\limits_{T_2}^{\infty} e^{-\zeta_2 t}\bar{\mathfrak{a}}(t) \int\limits_{T_1}^{\infty} s^{\gamma_{00}} \log\{\varphi(s)\} \mathfrak{C}(s) \frac{\overline{\mathfrak{C}}'(s+t)}{(s+t)^{\gamma_{00}}} dsdt \\
&+ \frac{\gamma_{00}}{\beta_{00}} \int\limits_{T_2}^{\infty} e^{-\zeta_2 t} t\bar{\mathfrak{a}}(t) \int\limits_{T_1}^{\infty} s^{\gamma_{00}-1} \log\{\varphi(s)\} \mathfrak{C}(s) \frac{\bar{\mathfrak{c}}(s+t)}{s+t} dsdt.
\end{aligned}$$

8.6.4.2. w-Analytic Continuation for $z = 0$ and an Infinite Path $\mathcal{P}_1$ with $\xi_2 = 0$, $\eta_{00} > 0$ and $\alpha_{00} = \gamma_{00}$

From the assumptions on the parameters in the present case, we conclude

$$(8.6.47)\qquad \gamma_{00} > \frac{1}{2}.$$

Speaking of the w-analytic continuation of the integral (8.6.11) for $z = 0$, in accordance with the minimum-type boundary of the abscissa of convergence $\phi_0(0, (\infty, 0))$, we must distinguish between two cases. In each situation, appealing to the identity (8.6.14), a reference to Section 8.5 is possible.

If $\gamma_{00} > 1$, the required w-analytic continuation was already established in Subsection 8.5.3 in the shape of equation (8.5.26) with $z = 0$. The latter represents a meromorphic function in the half plane

$$(8.6.48)\qquad \Re w < \min\{\nu_{01}, \nu_{00} + \chi_{00}\},$$

and therein it shows a pole of simple order at $w = \nu_{00}$. This pole always lies in the right w-half plane. The associated residue was computed in (8.5.30).

Conversely, in the case $\frac{1}{2} < \gamma_{00} \leq 1$, we can refer to our findings from Subsection 8.5.4, according to which the expansion (8.5.34) with $z = 0$ represents the w-analytic continuation into the strip

$$\chi_{00} < \Re w < \nu_{00} + \chi_{00} + \eta_{\beta_{00}}(\alpha_{01}, \beta_{01}, \gamma_{01}). \tag{8.6.49}$$

If the pole at $w = \nu_{00} + \chi_{00}$ is of simple order, the associated residue equals (8.5.42) with $z = 0$. Particularly if $\nu_{00} + \chi_{00} = 0$, i.e., if $\gamma_{00} = \frac{2}{3}$, this pole lies at the origin of the w-plane, and the first two terms of the corresponding Laurent expansion were provided in §8.5.4.1. Finally, the pole will be of second order if and only if $\chi_{00} = 0$, i.e., if $\gamma_{00} = 1$. The associated Laurent expansion near $w = \nu_{00}$ was established in (8.5.43), and this pole lies somewhere in the right w-half plane.

8.6.4.3. Evaluation of the MB-Integral

In the preceding paragraphs it was pointed out that the generating function (8.6.11) for $z = 0$ and $\vec{p}_1 \notin \{(\tau_1, 0), (\infty, 1)\}$ can be extended to a meromorphic function in the half plane $\Re w < \phi_1(\theta_2, \vec{p}_1)$, with

$$\phi_1(\theta_2, \vec{p}_1) := \begin{cases} \chi_{10} + \chi_{\beta_{10}}(\beta_{11}, \gamma_{11}, 0), & \text{if } \vec{p}_1 = (\tau_1, 1), \\ \nu_{00} + \eta_{\beta_{00}}(\beta_{01}, \gamma_{01}, 2), & \text{if } \vec{p}_1 = (\infty, 0) \wedge \theta_2 = 1, \\ \min\{\nu_{01}, \nu_{00} + \chi_{00}\}, & \text{if } \vec{p}_1 = (\infty, 0) \wedge \theta_2 = 0 \wedge \gamma_{00} > 1, \\ \nu_{00} + \chi_{00} + \eta_{\beta_{00}}(\alpha_{01}, \beta_{01}, \gamma_{01}), & \text{if } \vec{p}_1 = (\infty, 0) \wedge \theta_2 = 0 \wedge \frac{1}{2} < \gamma_{00} \leq 1. \end{cases} \tag{8.6.50}$$

These continuations turned out to be valid in the whole half plane $\Re w < \phi_1(\theta_2, \vec{p}_1)$ or merely in some substrip. In the first three cases, their existence was verified via integration by parts, which implies the presence of a single singularity in the wider region, namely of a simple pole at $w = \phi_0(\theta_2, \vec{p}_1)$. Furthermore, in the particular case $\vec{p}_1 = (\infty, 0)$ with $\theta_2 = 0$ and $\gamma_{00} \leq 1$, the w-analytic continuation is given by the expansion (8.5.34), which is admissible in the strip $\chi_{00} < \Re w < \phi_1(0, (\infty, 0))$. Therein it shows a pole of order one or two at $w = \nu_{00} + \chi_{00}$, respectively if $\gamma_{00} < 1$ or $\gamma_{00} = 1$. Additional poles may only occur to the right of this pole. In each case it can be shown that the integrand of the MB-integral (8.6.32) vanishes exponentially fast in the imaginary direction of the half plane $\Re w < \phi_1(\theta_2, \vec{p}_1)$ or of the whole complex w-plane if $\vec{p}_1 \in \{(\tau_1, 0), (\infty, 1)\}$, thereby enabling a rightward displacement of the integration path across a selected number of poles. Since it is our aim to collect the residue of the pole at $w = 0$, if $\vec{p}_1 \in \{(\tau_1, 0), (\infty, 1)\}$, in this case we let

$$0 < f_1(\theta_2, \vec{p}_1) < \infty. \tag{8.6.51}$$

In each other case, our interest concerns the pole at $w = \phi_0(\theta_2, \vec{p}_1)$, whence for $\vec{p}_1 = (\tau_1, 1)$, for $\vec{p}_1 = (\infty, 0)$ with $\theta_2 = 1$ and for $\vec{p}_1 = (\infty, 0)$ with $\theta_2 = 0$ and $\gamma_{00} > 1$, we choose

$$\phi_0(\theta_2, \vec{p}_1) < f_1(\theta_2, \vec{p}_1) < \begin{cases} \phi_1(\theta_2, \vec{p}_1), & \text{if } \phi_0(\theta_2, \vec{p}_1) \geq 0, \\ \min\{0, \phi_1(\theta_2, \vec{p}_1)\}, & \text{if } \phi_0(\theta_2, \vec{\tau}_1) < 0. \end{cases} \tag{8.6.52}$$

For $\vec{p}_1 = (\infty, 0)$ with $\theta_2 = 0$ and $\frac{1}{2} < \gamma_{00} \leq 1$, we agree $f_1(0, (\infty, 0)) > \phi_0(0, (\infty, 0))$ and

$$f_1(0, (\infty, 0)) < \begin{cases} \min\{\nu_{00}, \phi_1(0, (\infty, 0))\}, & \text{if } \frac{2}{3} \leq \gamma_{00} < 1, \\ \min\{0, \nu_{00}, \phi_1(0, (\infty, 0))\}, & \text{if } \frac{1}{2} < \gamma_{00} < \frac{2}{3}, \\ \min\left\{\nu_{00} + \frac{1}{\beta_{00}}, \phi_1(0, (\infty, 0))\right\}, & \text{if } \gamma_{00} = 1. \end{cases} \tag{8.6.53}$$

Then, a rectangle of infinite height, with left and right edges respectively equal to $\Re w = f_0(\theta_2, \vec{p}_1)$ and $\Re w = f_1(\theta_2, \vec{p}_1)$, encloses exactly the poles at $w \in \{0, \phi_0(\theta_2, \vec{p}_1)\}$ if $\phi_0(\theta_2, \vec{p}_1) > 0$ and either of these two poles otherwise. Due to coalescences, they are up to second order. To compute the associated residues, we refer to Theorem B.2.1(2). Assuming the described rectangle is traversed in the negative direction, i.e., clockwisely, except in the case $\vec{p}_1 = (\infty, 0)$ with $\theta_2 = 0$, according to the residue theorem and (8.6.43), as $m \to \infty$ we deduce:

$$\begin{aligned} \mathrm{F}(m; \theta_2, \vec{p}_1) = &- \mathcal{K}_0 \begin{bmatrix} 0, \zeta_1, \sigma, S \\ 0, \zeta_2, T_2 \end{bmatrix} \left\{ \mathbb{1}_{\{\vec{p}_1 \in \{(\tau_1, 0), (\infty, 1)\}\}} + \mathbb{1}_{\{\phi_0(\theta_2, \vec{p}_1) > 0)\}} \right\} \\ &- \frac{\Gamma(\phi_0(\theta_2, \vec{p}_1))}{(m+1)^{\phi_0(\theta_2, \vec{p}_1)}} \kappa_{-1}(\theta_2, \vec{p}_1) \mathbb{1}_{\{\phi_0(\theta_2, \vec{p}_1) \neq 0\}} \\ &+ \left\{ (\log(m+1) + \gamma) \kappa_{-1}(\theta_2, \vec{p}_1) - \kappa_0 \begin{bmatrix} \theta_2, \theta_1 \\ \sigma, S \end{bmatrix} \right\} \mathbb{1}_{\{\phi_0(\theta_2, \vec{p}_1) = 0\}} \\ &+ \mathcal{O}\left\{ m^{-f_1(\theta_2, \vec{p}_1)} \right\} \end{aligned} \tag{8.6.54}$$

Finally, for the indicated special case, upon taking into account (8.5.30), (8.5.42), (8.5.43), (8.5.45) and the identity (8.6.14), as $m \to \infty$ we arrive at:

$$\begin{aligned} \mathrm{F}(m; 0, (\infty, 0)) = &- \mathcal{S}_0 \begin{bmatrix} 0, T_1 \\ 0, T_2 \end{bmatrix} \mathbb{1}_{\left\{\gamma_{00} > \frac{2}{3}\right\}} \\ &+ \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{|c_{00}|^2}{\beta_{00}} \mathcal{A}^0(0; T_2) \mathbb{1}_{\{\gamma_{00} > 1\}} \\ &+ \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{\overline{a}_{00}\, |c_{00}|^2}{\{\beta_{00}\}^2} \\ &\qquad \times \{\log(m+1) - \psi(\nu_{00}) - \sigma_0(0; T_2)\} \mathbb{1}_{\{\gamma_{00} = 1\}} \\ &+ \frac{\Gamma(\nu_{00} + \chi_{00})}{\{b_{00}(m+1)\}^{\nu_{00} + \chi_{00}}} \frac{\overline{a}_{00}\, |c_{00}|^2}{\beta_{00}} \\ &\qquad\qquad \times \frac{\Gamma(1 - \gamma_{00}) \Gamma(2\gamma_{00} - 1)}{\Gamma(\gamma_{00})} \mathbb{1}_{\left\{\gamma_{00} \in \left(\frac{1}{2}, 1\right) \setminus \left\{\frac{2}{3}\right\}\right\}} \end{aligned} \tag{8.6.55}$$

$$
\begin{aligned}
&- \left\{\log(m+1) + \gamma - \sigma_{\frac{2}{3}}(T_2)\right\} \frac{\overline{a}_{00}\, |c_{00}|^2}{\beta_{00}} \frac{\left\{\Gamma\left(\frac{1}{3}\right)\right\}^2}{\Gamma\left(\frac{2}{3}\right)} \mathbb{1}_{\left\{\gamma_{00} = \frac{2}{3}\right\}} \\
&- \Sigma_1 \begin{bmatrix} 0, T_1 \\ 0, T_2 \end{bmatrix} \mathbb{1}_{\left\{\gamma_{00} = \frac{2}{3}\right\}} \\
&+ \mathcal{O}\left\{m^{-f_1(0,(\infty,0))}\right\}
\end{aligned}
$$

The order of the above remainder terms was concluded by absolute convergence of the corresponding MB-integrals along the vertical line $\Re w = f_1(\theta_2, \vec{p}_1)$.

8.6.5. A Single MB-Integral for the Residue at $z = \eta_{00}$

The MB-integral (8.6.33) appears if $\xi_2 = 0$. Its generating function $\mathcal{P}_0[\ldots]$ was defined in (8.4.49). This is an entire function of w, if $\vec{p}_1 \in \{(\tau_1, 0), (\infty, 1)\}$, whereas otherwise it is holomorphic in the half plane $\Re w < \varsigma_0(\vec{p}_1)$, analogous to (8.4.23) with

(8.6.56)
$$
\varsigma_0(\vec{p}_1) = \begin{cases} \chi_{10}, & \text{if } \vec{p}_1 = (\tau_1, 1), \\ \chi_{00}, & \text{if } \vec{p}_1 = (\infty, 0). \end{cases}
$$

Furthermore, refer to $\varsigma_1(\vec{p}_1)$ as the parameter (8.4.101) with the quantities corresponding to $\mathcal{P}_1$ instead of those for $\mathcal{P}_2$. By comparison of (8.4.81) and (8.6.33), it is easy to confirm the identity

$$
\mathrm{G}\left[m; \begin{matrix} \theta_1 \\ \sigma, S \end{matrix}\right] = \mathrm{K}\left[m; \begin{matrix} \theta_1 \\ \sigma, S \end{matrix}\right].
$$

Appealing to (8.4.106) and (8.4.107), for $0 < g_1(\vec{p}_1) < \infty$, if $\vec{p}_1 \in \{(\tau_1, 0), (\infty, 1)\}$, and otherwise for

(8.6.57)
$$
\varsigma_0(\vec{p}_1) < g_1(\vec{p}_1) < \begin{cases} \varsigma_1(\vec{p}_1), & \text{if } \varsigma_0(\vec{p}_1) \geq 0, \\ \min\{0, \varsigma_1(\vec{p}_1)\}, & \text{if } \varsigma_0(\vec{p}_1) < 0, \end{cases}
$$

as $m \to \infty$ we therefore conclude:

(8.6.58)
$$
\begin{aligned}
\mathrm{G}\left[m; \begin{matrix} \theta_1 \\ \sigma, S \end{matrix}\right] = -\mathcal{P}_0 \begin{bmatrix} 0, \zeta_1 \\ \sigma, S \end{bmatrix} &\left\{\mathbb{1}_{\{\vec{p}_1 \in \{(\tau_1,0),\, (\infty,1)\}\}} + \mathbb{1}_{\{\varsigma_0(\vec{p}_1) > 0\}}\right\} \\
&- \frac{\Gamma(\varsigma_0(\vec{p}_1))}{(m+1)^{\varsigma_0(\vec{p}_1)}} \pi_{-1}(\vec{p}_1) \mathbb{1}_{\{\varsigma_0(\vec{p}_1) \neq 0\}} \\
&+ \{(\log(m+1) + \gamma)\pi_{-1}(\vec{p}_1) - \pi_0(\theta_1; \sigma, S)\}\, \mathbb{1}_{\{\varsigma_0(\vec{p}_1) = 0\}} \\
&+ \mathcal{O}\left\{m^{-g_1(\vec{p}_1)}\right\}
\end{aligned}
$$

The constants on the right hand side are those defined in (8.4.103), (8.4.104) and (8.4.105) with the path $\mathcal{P}_2$ replaced by $\mathcal{P}_1$ and its associated quantities.

8.6.6. A Single MB-Integral for the Residue at $z = 0$ if $\eta_{00} = 0$

The MB-integral (8.6.34) obviously occurs if and only if $\eta_{00} = 0$ and $\xi_2 = 0$ but $\vec{p}_1 \neq (\infty, 0)$. The associated generating function (8.5.71) is therefore entire with respect to w, except if $\vec{p}_1 = (\tau_1, 1)$. In this last case, with $\zeta_1 = i\xi_1$, it takes on the form

$$\mathcal{Y}_0\begin{bmatrix} -w, i\xi_1 \\ \tau_1, T_1, T_2 \end{bmatrix} = \int_{\tau_1}^{T_1} \{\varphi(s)\}^{-w} e^{-i\xi_1 s} c(s) \{\rho_1(s;T_2) + s\rho_2(s;T_2)\} \, ds, \tag{8.6.59}$$

where the amplitude features the functions (8.5.63) and (8.5.64). By Lebesgue's dominated convergence theorem, their integral representations can be verified uniformly continuous with respect to $\tau_1 \le s \le T_1$, from which we conclude $\rho_j(s;T_2) = \mathcal{O}(1)$ as $s \downarrow \tau_1$, for each $j \in \{1,2\}$. Hence, analyticity of (8.6.59) holds in the half plane $\Re w < \chi_{10}$. In order to specify the analytic continuation, we aim for a reference to §8.2.2.1. For this, we must first justify the differentiability of $\rho_j(s;T_2)$ for $j \in \{1,2\}$ under the sign of integration. We begin with the function for $j = 2$ which, by assumption, for any fixed $t \ge T_2$ possesses a once continuously differentiable integrand on $\tau_1 \le s \le T_1$ with

$$\begin{aligned} &\frac{d}{ds}\left\{\frac{\bar{\mathfrak{c}}(s+t)}{s+t}\{\beta_{00} + \gamma_{00}\log\{\varphi(s+t)\}\}\right\} \\ &\qquad = \left\{\frac{\bar{\mathfrak{c}}'(s+t)}{s+t} - \frac{\bar{\mathfrak{c}}(s+t)}{(s+t)^2}\right\}\{\beta_{00} + \gamma_{00}\log\{\varphi(s+t)\}\} + \gamma_{00}\frac{\bar{\mathfrak{c}}(s+t)}{s+t}\frac{\varphi'(s+t)}{\varphi(s+t)}. \end{aligned}$$

From (8.2.16) we recall, $B'(u)$ is of order $\beta_{01} > 1$ as $u \to \infty$, if and only if

$$\frac{\varphi'(u)}{\varphi(u)} = -\frac{\beta_{00}}{u} + \mathcal{O}\left\{u^{-\beta_{01}}\right\}. \tag{8.6.60}$$

Furthermore, from the condition on $\mathfrak{C}(u)$ to be of order $\gamma_{01} > 1$ at infinity, we conclude

$$\bar{\mathfrak{c}}'(u) \begin{cases} \sim -\frac{\gamma_{00}}{u}\bar{\mathfrak{c}}(u), & \text{if } \gamma_{00} \neq 0, \\ = \mathcal{O}\left\{u^{-\gamma_{01}}\right\}, & \text{if } \gamma_{00} = 0. \end{cases} \tag{8.6.61}$$

Consequently, for fixed $\tau_1 \le s \le T_1$ and arbitrary small $\varepsilon > 0$, as $t \to \infty$ we obtain, since $\alpha_{00} > 0$, and because $\eta_{00} = 0$ and $\gamma_{00} + \alpha_{00} = 1$ are equivalent:

$$\begin{aligned} \bar{\mathfrak{a}}(t)\frac{d}{ds}\left\{\frac{\bar{\mathfrak{c}}(s+t)}{s+t}\{\beta_{00} + \gamma_{00}\log\varphi(s+t)\}\right\} &= \mathcal{O}\left\{\frac{(s+t)^{\alpha_{00}}}{t^{\alpha_{00}}}(s+t)^{\varepsilon-\alpha_{00}-\gamma_{00}-2}\right\} \\ &= \mathcal{O}\left\{\frac{(T_1+t)^{\alpha_{00}}}{t^{\alpha_{00}}}(\tau_1+t)^{\varepsilon-3}\right\} \end{aligned}$$

The derivative on the left hand side is thus uniformly bounded with respect to $\tau_1 \le s \le T_1$ by a function, which is absolutely integrable along the ray $t \ge T_2$. A quick application of Theorem 11.62 in [Körner, 2004] therefore yields the differentiability of $\rho_2(s;T_2)$ under the sign of integration. Again subject to Lebesgue's theorem, this derivative is also uniformly continuous on $[\tau_1, T_1]$. Similar arguments apply for the function $\rho_1(s;T_2)$, where for $B'(s)$ and $\overline{\mathfrak{C}}(u)$, due to (8.6.5), estimates analogous to (8.4.38) can be employed if $\gamma_{00} \neq 0$. If $\gamma_{00} = 0$, concerning $\overline{\mathfrak{c}}''(u)$ a reference to the second assumption in (8.6.5) suffices. Altogether, we can therefore at once apply our findings from §8.2.2.1 to the integral (8.6.59). Accordingly, upon identifying

$$\begin{cases} d(s) & \equiv \mathfrak{c}(s)e^{-i\xi_1 s}\left\{\rho_1(s;T_2) + s\rho_2(s;T_2)\right\}, \\ k(s+t) & \equiv 1, \end{cases}$$

it shows that the indicated integral can be continued meromorphically into the half plane

$$\Re w < \chi_{10} + \chi_{\beta_{10}}(\gamma_{11}, 0). \tag{8.6.62}$$

There, we encounter merely a simple pole at $w = \chi_{10}$, subject to (8.2.33), with residue

$$\operatorname*{Res}_{w=\chi_{10}} \mathcal{Y}_0\begin{bmatrix} -w, i\xi_1 \\ \tau_1, T_1, T_2 \end{bmatrix} = -\frac{c_{10}}{\beta_{10}}\left\{b_{10}\right\}^{-\chi_{10}} e^{-i\xi_1\tau_1}\left\{\rho_1(\tau_1;T_2) + \tau_1\rho_2(\tau_1;T_2)\right\}. \tag{8.6.63}$$

In the case $\chi_{10} = 0$, i.e., if $\gamma_{10} = -1$, this pole lies at the origin of the w-plane. To specify the first two terms in the associated Laurent expansion, we define

$$\begin{aligned} \upsilon_0\begin{bmatrix} i\xi_1 \\ \tau_1, T_1, T_2 \end{bmatrix} :=\; & C_1(T_1)e^{-i\xi_1 T_1}\left\{\rho_1(T_1;T_2) + T_1\rho_2(T_1;T_2)\right\}\log\left\{\varphi(T_1)\right\} \\ & + \int_{\tau_1}^{T_1} B_1'(s)C_1(s)e^{-i\xi_1 s}\left\{\rho_1(s;T_2) + s\rho_2(s;T_2)\right\} ds \\ & - \int_{\tau_1}^{T_1} \log\left\{\varphi(s)\right\} C_1'(s)e^{-i\xi_1 s}\left\{\rho_1(s;T_2) + s\rho_2(s;T_2)\right\} ds \\ & + i\xi_1\int_{\tau_1}^{T_1} \log\left\{\varphi(s)\right\} C_1(s)e^{-i\xi_1 s}\left\{\rho_1(s;T_2) + s\rho_2(s;T_2)\right\} ds \\ & - \int_{\tau_1}^{T_1} \log\left\{\varphi(s)\right\} C_1(s)e^{-i\xi_1 s}\left\{\rho_1'(s;T_2) + \rho_2(s;T_2) + s\rho_2'(s;T_2)\right\} ds. \end{aligned} \tag{8.6.64}$$

Equation (8.2.35) then tells us that, in an annulus around $w = 0$, the integral transform (8.6.59) exhibits the expansion

$$\begin{aligned} \mathcal{Y}_0\begin{bmatrix} -w, i\xi_1 \\ \tau_1, T_1, T_2 \end{bmatrix} = & -\frac{1}{w}\frac{c_{10}}{\beta_{10}}e^{-i\xi_1\tau_1}\left\{\rho_1(\tau_1;T_2) + \tau_1\rho_2(\tau_1;T_2)\right\} \\ & + \frac{1}{\beta_{10}}\upsilon_0\begin{bmatrix} i\xi_1 \\ \tau_1, T_1, T_2 \end{bmatrix} + \mathcal{O}(1). \end{aligned} \tag{8.6.65}$$

Finally, since the generating function (8.5.71), or the corresponding analytic continuation, for $\vec{p}_1 \in \{(\tau_1, 0), (\tau_1, 1), (\infty, 1)\}$ is $\mathcal{O}(1)$ as $\Im w \to \pm\infty$, uniformly with respect to $\Re w$ in any closed vertical substrip of its region of validity, a rightward movement of the integration path in the MB-integral (8.6.34) is admissible. Choosing $\Re w = y_1(\vec{p}_1)$, with

$$
\begin{aligned}
&0 < y_1(\vec{p}_1) < \infty, \qquad \text{if } \vec{p}_1 \in \{(\tau_1, 0), (\infty, 1)\}\,, \\
&\chi_{10} < y_1(\tau_1, 1) < \begin{cases} \chi_{10} + \chi_{\beta_{10}}(\gamma_{11}, 0), & \text{if } \chi_{10} \geq 0, \\ \min\{0, \chi_{10} + \chi_{\beta_{10}}(\gamma_{11}, 0)\}\,, & \text{if } \chi_{10} < 0, \end{cases}
\end{aligned}
\tag{8.6.66}
$$

by incorporating (8.6.63) and (8.6.65), as $m \to \infty$ this leads to:

$$
\begin{aligned}
\Upsilon(m; \vec{p}_1) = &-\mathcal{Y}_0\begin{bmatrix} 0, \zeta_1 \\ \sigma, S, T_2 \end{bmatrix} \mathbb{1}_{\{\vec{p}_1 \in \{(\tau_1, 0), (\infty, 1)\} \vee \chi_{10} > 0\}} \\
&+ \frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}} \frac{c_{10}}{\beta_{10}} e^{-i\xi_1\tau_1} \{\rho_1(\tau_1; T_2) + \tau_1\rho_2(\tau_1; T_2)\} \mathbb{1}_{\{\chi_{10} \neq 0\}} \\
&- (\log(m+1) + \gamma) \frac{c_{10}}{\beta_{10}} e^{-i\xi_1\tau_1} \{\rho_1(\tau_1; T_2) + \tau_1\rho_2(\tau_1; T_2)\} \mathbb{1}_{\{\chi_{10} = 0\}} \\
&- \frac{1}{\beta_{10}} \upsilon_0 \begin{bmatrix} i\xi_1 \\ \tau_1, T_1, T_2 \end{bmatrix} \mathbb{1}_{\{\chi_{10} = 0\}} \\
&+ \mathcal{O}\left\{m^{-y_1(\vec{p}_1)}\right\}
\end{aligned}
\tag{8.6.67}
$$

The order of the big-$\mathcal{O}$ was again obtained by absolute convergence of the remainder integral.

8.6.7. Evaluation of the Iterated MB-Integral

With the findings we obtained so far, we can now reveal the m-asymptotic behaviour of the expansion (8.6.35) which, in view of (8.6.18), will eventually characterize the asymptotics of (8.6.1). Particularly for $\delta_1 = \delta_2 = 0$, the last identity brings us

$$
\mathrm{Ai}\left[m; \begin{matrix} i\xi_1, \sigma, S \\ 0, T_2 \end{matrix}\right] = \mathcal{A}i\left[m; \begin{matrix} i\xi_1, \sigma, S \\ 0, T_2 \end{matrix}\right],
\tag{8.6.68}
$$

since, by definition, in case of an infinite path $\mathcal{P}_j$, from $\delta_j = 0$ we may conclude $\xi_j = 0$.

8.6.7.1. A Finite Path $\mathcal{P}_1$ and $\xi_2 = 0$

We commence with the scenario $\mathcal{P}_1 = (\tau_1, T_1]$ with $\theta_2 = 0$, which widely resembles §8.4.8.2. To specify the order of each remainder in the above expansions, from (8.6.30) we ascertain the existence of a parameter $\varepsilon_2 > 0$ with $x_1(0) = \eta_{00} + \varepsilon_2$. Moreover, by (8.6.19) we can find a second parameter $0 < \varepsilon_1 < \varepsilon_2$, for which $u_0(0, (\tau_1, 0)) = -\varepsilon_1$ and $u_0(0, (\tau_1, 1)) = \min\{0, \chi_{10}\} - \varepsilon_1$, respectively. Defining $\varepsilon := \varepsilon_2 - \varepsilon_1$, we observe $\varepsilon > 0$. A suitable choice of $\varepsilon_1, \varepsilon_2$ and of the parameters $f_1(0, (\tau_1, 1))$, $g_1(\tau_1, 1)$ and $y_1(\tau_1, 1)$ then enables us to write $f_1(0, (\tau_1, 1)) = g_1(\tau_1, 1) = y_1(\tau_1, 1) = \chi_{10} + \varepsilon$. If $\beta_{10} = 0$, i.e., if $\theta_1 = 0$, by definition, the last three parameters are supposed to be arbitrary positive numbers, thereby verifying an exponential order of the remainder terms in (8.6.54), (8.6.58) and (8.6.67). An application of these expansions to (8.6.35) finally gives rise to two theorems.

Theorem 8.6.1. *For $\eta_{00} \neq 0$, assume validity of (A3), (A4), (A6) and (A8). Then, provided at least one term on the right hand side is non-zero, for any $\xi_1 \in \mathbb{R}$ as $m \to \infty$,*

(1) if $\eta_{00} > 0$, with either $\beta_{10} = 0$ or $\chi_{10} > 0$, we have

$$\mathrm{Ai}\left[m; \begin{matrix} i\xi_1, \tau_1, T_1 \\ 0, T_2 \end{matrix}\right] \sim \mathcal{K}_0\left[\begin{matrix} 0, i\xi_1, \tau_1, T_1 \\ 0, 0, T_2 \end{matrix}\right] - \frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}} \frac{c_{10}}{\beta_{10}} e^{-i\xi_1\tau_1} \mathcal{R}(\tau_1; 0, T_2) \mathbb{1}_{\{\chi_{10} \leq \eta_{00}\}} - \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \mathcal{P}_0\left[\begin{matrix} 0, i\xi_1 \\ \tau_1, T_1 \end{matrix}\right] \mathbb{1}_{\{\beta_{10} = 0 \,\vee\, \eta_{00} \leq \chi_{10}\}}.$$

The coefficients were defined in (8.4.49) and (8.6.40).

(2) if $\eta_{00} > 0$ and $\chi_{10} \leq 0$, we have

$$\mathrm{Ai}\left[m; \begin{matrix} i\xi_1, \tau_1, T_1 \\ 0, T_2 \end{matrix}\right] \sim -\frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}} \frac{c_{10}}{\beta_{10}} e^{-i\xi_1\tau_1} \mathcal{R}(\tau_1; 0, T_2) \mathbb{1}_{\{\chi_{10} < 0\}} + (\log(m+1) + \gamma) \frac{c_{10}}{\beta_{10}} e^{-i\xi_1\tau_1} \mathcal{R}(\tau_1; 0, T_2) \mathbb{1}_{\{\chi_{10} = 0\}} + \kappa_0\left[\begin{matrix} 0, 1 \\ \tau_1, T_1 \end{matrix}\right] \mathbb{1}_{\{\chi_{10} = 0\}}.$$

For the coefficients we refer to (8.6.40) and (8.6.45).

(3) if $\eta_{00} < 0$, with either $\beta_{10} = 0$ or $\chi_{10} > 0$, we have

$$\mathrm{Ai}\left[m; \begin{matrix} i\xi_1, \tau_1, T_1 \\ 0, T_2 \end{matrix}\right] \sim -\frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \mathcal{P}_0\left[\begin{matrix} 0, i\xi_1 \\ \tau_1, T_1 \end{matrix}\right].$$

The coefficient $\mathcal{P}_0[\ldots]$ was specified in (8.4.49).

(4) if $\eta_{00} < 0$ and $\chi_{10} \leq 0$, we have

$$\mathrm{Ai}\left[m; \begin{matrix} i\xi_1, \tau_1, T_1 \\ 0, T_2 \end{matrix}\right] \sim \frac{\Gamma(\eta_{00})\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}} \{b_{00}(m+1)\}^{\eta_{00}}} \frac{c_{10}\overline{a}_{00}\overline{c}_{00}}{\beta_{10}\beta_{00}} e^{-i\xi_1\tau_1} \mathbb{1}_{\{\chi_{10} < 0\}} - \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} (\log(m+1) + \gamma) \frac{c_{10}\overline{a}_{00}\overline{c}_{00}}{\beta_{10}\beta_{00}} e^{-i\xi_1\tau_1} \mathbb{1}_{\{\chi_{10} = 0\}} - \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{10}\beta_{00}} \pi_0(1; \tau_1, T_1) \mathbb{1}_{\{\chi_{10} = 0\}}.$$

The right hand side features the coefficient (8.4.104).

Theorem 8.6.2. *For $\eta_{00} = 0$, assume validity of (A3), (A4), (A6), (A8) and (A9). Then, provided at least one term on the right hand side is non-zero, for any $\xi_1 \in \mathbb{R}$ as $m \to \infty$,* *(1) if $\beta_{10} = 0$ or $\chi_{10} > 0$, we have*

$$\mathrm{Ai}\left[m; \begin{matrix} i\xi_1, \tau_1, T_1 \\ 0, T_2 \end{matrix}\right] \sim (\log(m+1)+\gamma)\frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}}\mathcal{P}_0\left[\begin{matrix} 0, i\xi_1 \\ \tau_1, T_1 \end{matrix}\right] + \frac{1}{\beta_{00}}\mathcal{Y}_0\left[\begin{matrix} 0, i\xi_1 \\ \tau_1, T_1, T_2 \end{matrix}\right],$$

where $\mathcal{P}_0[\ldots]$ and $\mathcal{Y}_0[\ldots]$ were defined in (8.4.49) and (8.5.71).

(2) if $\chi_{10} = 0$, we have

$$\begin{aligned} \mathrm{Ai}\left[m; \begin{matrix} i\xi_1, \tau_1, T_1 \\ 0, T_2 \end{matrix}\right] \sim\ & (\log(m+1)+\gamma)^2 \frac{c_{10}\overline{a}_{00}\overline{c}_{00}}{\beta_{10}\beta_{00}} e^{-i\xi_1\tau_1} \\ & + (\log(m+1)+\gamma)\frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{10}\beta_{00}}\pi_0(1;\tau_1,T_1) \\ & + (\log(m+1)+\gamma)\frac{c_{10}}{\beta_{00}\beta_{10}} e^{-i\xi_1\tau_1}\left\{\rho_1(\tau_1;T_2) + \tau_1\rho_2(\tau_1;T_2)\right\} \\ & + \frac{1}{\beta_{00}\beta_{10}}\upsilon_0\left[\begin{matrix} i\xi_1 \\ \tau_1, T_1, T_2 \end{matrix}\right]. \end{aligned}$$

The coefficients were specified in (8.4.104), (8.5.63), (8.5.64) and (8.6.64).

(3) if $\chi_{10} < 0$, we have

$$\begin{aligned} \mathrm{Ai}\left[m; \begin{matrix} i\xi_1, \tau_1, T_1 \\ 0, T_2 \end{matrix}\right] \sim & -(\log(m+1)+\gamma)\frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}}\frac{c_{10}\overline{a}_{00}\overline{c}_{00}}{\beta_{10}\beta_{00}} e^{-i\xi_1\tau_1} \\ & - \frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}}\frac{c_{10}}{\beta_{00}\beta_{10}} e^{-i\xi_1\tau_1}\left\{\rho_1(\tau_1;T_2) + \tau_1\rho_2(\tau_1;T_2)\right\}, \end{aligned}$$

again with (8.5.63) and (8.5.64).

8.6.7.2. Two Infinite Paths with $\xi_1 = \xi_2 = 0$ and $\alpha_{00} = \gamma_{00} \neq \frac{1}{2}$

In this paragraph we complete our treatment of integrals of the type (8.5.1) with non-oscillatory amplitudes, which encompasses the case $\alpha_{00} = \gamma_{00} \neq \frac{1}{2}$. By comparison of (8.5.1) with (8.6.1), it is easy to see that

$$\mathrm{Ai}\left[m; \begin{matrix} 0, T_1, \infty \\ 0, T_2 \end{matrix}\right] = \mathrm{Ai}\left[m; \begin{matrix} T_1 \\ T_2 \end{matrix}\right].$$

Now, analogous to the preceding paragraph, due to (8.6.19) and (8.6.30), there exist $\varepsilon_2 > \varepsilon_1 > 0$ such that $u_0(0,(\infty,0)) = \min\{0,\chi_{00}\} - \varepsilon_1$ and $x_0(0) = \nu_{00} + \varepsilon_2$. Without loss of generality we assume $\varepsilon_1, \varepsilon_2$ small enough and appropriately chosen $f_1(0,(\infty,0))$ and $g_1(\infty,0)$ with $f_1(0,(\infty,0)) = \nu_{00} + \min\{0,\chi_{00}\} + \varepsilon$ and $g_1(\infty,0) = \chi_{00} + \varepsilon$, again for $\varepsilon := \varepsilon_2 - \varepsilon_1$. From (8.6.35), (8.6.55) and (8.6.58), according to (8.6.68), as $m \to \infty$ we then obtain:

$$\begin{aligned} \mathrm{Ai}\left[m; \begin{matrix} T_1 \\ T_2 \end{matrix}\right] = & \mathcal{S}_0\left[\begin{matrix} 0, T_1 \\ 0, T_2 \end{matrix}\right]\mathbb{1}_{\left\{\gamma_{00} > \frac{2}{3}\right\}} \\ & - \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}}\frac{|c_{00}|^2}{\beta_{00}}\mathcal{A}^0(0;T_2)\mathbb{1}_{\{\gamma_{00} > 1\}} \end{aligned}$$

$$
\begin{aligned}
&- \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2} \{\log(m+1) - \psi(\nu_{00}) - \sigma_0(0;T_2)\}\, \mathbb{1}_{\{\gamma_{00}=1\}} \\
&- \frac{\Gamma(\nu_{00}+\chi_{00})}{\{b_{00}(m+1)\}^{\nu_{00}+\chi_{00}}} \frac{\overline{a}_{00}\,|c_{00}|^2}{\beta_{00}} \frac{\Gamma(1-\gamma_{00})\Gamma(2\gamma_{00}-1)}{\Gamma(\gamma_{00})} \mathbb{1}_{\left\{\gamma_{00}\in\left(\frac{1}{2},1\right)\setminus\left\{\frac{2}{3}\right\}\right\}} \\
&+ \left\{\log(m+1) + \gamma - \sigma_{\frac{2}{3}}(T_2)\right\} \frac{\overline{a}_{00}\,|c_{00}|^2}{\beta_{00}} \frac{\left\{\Gamma\left(\frac{1}{3}\right)\right\}^2}{\Gamma\left(\frac{2}{3}\right)} \mathbb{1}_{\left\{\gamma_{00}=\frac{2}{3}\right\}} \\
&+ \Sigma_1\begin{bmatrix}0, T_1\\ 0, T_2\end{bmatrix} \mathbb{1}_{\left\{\gamma_{00}=\frac{2}{3}\right\}} \\
&- \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \mathcal{P}_0\begin{bmatrix}0, 0\\ T_1, \infty\end{bmatrix} \mathbb{1}_{\{\chi_{00}>0\}} \\
&- \frac{\Gamma(\nu_{00})\Gamma(\chi_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}(m+1)^{\chi_{00}}} \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \pi_{-1}(\infty,0)\mathbb{1}_{\{\chi_{00}\neq 0\}} \\
&+ \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}} (\log(m+1)+\gamma) \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \pi_{-1}(\infty,0)\mathbb{1}_{\{\chi_{00}=0\}} \\
&- \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \pi_0(0;T_1,\infty)\mathbb{1}_{\{\chi_{00}=0\}} \\
&+ \mathcal{O}\left\{m^{-\nu_{00}-\min\{0,\chi_{00}\}-\varepsilon}\right\}
\end{aligned}
$$

A careful inspection of this expansion for different values of the parameter γ_{00} gives rise to the next theorem, which also incorporates the definitions (8.4.103) and (8.4.105).

Theorem 8.6.3. *For $\alpha_{00} = \gamma_{00} \neq \frac{1}{2}$, assume validity of (A3), (A4) and (A6). Then, provided at least one term on the right hand side is non-zero, as $m \to \infty$,*

(1) if $\gamma_{00} \geq 1$, i.e., if $\nu_{00} > \chi_{00} \geq 0$, we have

$$
\begin{aligned}
\mathrm{Ai}\begin{bmatrix}m; \begin{matrix}T_1\\ T_2\end{matrix}\end{bmatrix} \sim \mathcal{S}_0&\begin{bmatrix}0, T_1\\ 0, T_2\end{bmatrix} \\
&- \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{|c_{00}|^2}{\beta_{00}} \mathcal{A}^0(0;T_2)\mathbb{1}_{\{\gamma_{00}>1\}} \\
&- \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \mathcal{P}_0\begin{bmatrix}0, 0\\ T_1, \infty\end{bmatrix} \mathbb{1}_{\{\gamma_{00}>1\}} \\
&- \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}} \{\log(m+1) - \psi(\nu_{00}) - \sigma_0(0;T_2)\} \frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2} \mathbb{1}_{\{\gamma_{00}=1\}} \\
&- \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}} (\log(m+1)+\gamma) \frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2} \mathbb{1}_{\{\gamma_{00}=1\}} \\
&- \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{\overline{a}_{00}\overline{c}_{00}}{\{\beta_{00}\}^2} \pi_0(0;T_1,\infty)\mathbb{1}_{\{\gamma_{00}=1\}}.
\end{aligned}
$$

The coefficients were defined in (8.4.49), (8.4.105), (8.5.29) and (8.5.44).

(2) if $\frac{1}{2} < \gamma_{00} < 1$, i.e., if $\nu_{00} > 0 > \chi_{00}$, we have

$$
\mathrm{Ai}\begin{bmatrix}m; \begin{matrix}T_1\\ T_2\end{matrix}\end{bmatrix} \sim \mathcal{S}_0\begin{bmatrix}0, T_1\\ 0, T_2\end{bmatrix} \mathbb{1}_{\left\{\gamma_{00}>\frac{2}{3}\right\}}
$$

$$
\begin{aligned}
& - \frac{\Gamma(\nu_{00}+\chi_{00})}{\{b_{00}(m+1)\}^{\nu_{00}+\chi_{00}}} \frac{\overline{a}_{00}\,|c_{00}|^2}{\beta_{00}} \frac{\Gamma(1-\gamma_{00})\Gamma\left(2\gamma_{00}-1\right)}{\Gamma\left(\gamma_{00}\right)} \mathbb{1}_{\left\{\gamma_{00} \neq \frac{2}{3}\right\}} \\
& + \left\{\log(m+1)+\gamma-\sigma_{\frac{2}{3}}(T_2)\right\} \frac{\overline{a}_{00}\,|c_{00}|^2}{\beta_{00}} \frac{\left\{\Gamma\left(\frac{1}{3}\right)\right\}^2}{\Gamma\left(\frac{2}{3}\right)} \mathbb{1}_{\left\{\gamma_{00} = \frac{2}{3}\right\}} \\
& + \Sigma_1 \begin{bmatrix} 0, T_1 \\ 0, T_2 \end{bmatrix} \mathbb{1}_{\left\{\gamma_{00} = \frac{2}{3}\right\}} \\
& + \frac{\Gamma(\nu_{00})\Gamma(\chi_{00})}{\{b_{00}(m+1)\}^{\nu_{00}+\chi_{00}}} \frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2}.
\end{aligned}
$$

For the coefficients we refer to (8.5.36) and (8.5.46).

(3) *if* $0 < \gamma_{00} < \frac{1}{2}$, *i.e., if* $\chi_{00} < \nu_{00} < 0$, *we have*

$$
\mathrm{Ai}\begin{bmatrix} m; \begin{matrix} T_1 \\ T_2 \end{matrix} \end{bmatrix} \sim \frac{\Gamma(\nu_{00})\Gamma(\chi_{00})}{\{b_{00}(m+1)\}^{\nu_{00}+\chi_{00}}} \frac{\overline{a}_{00}\,|c_{00}|^2}{\{\beta_{00}\}^2}.
$$

8.6.7.3. An Oscillatory Amplitude

We will now examine the effect of an oscillatory amplitude on the m-asymptotic behaviour of (8.6.1). Owing to the odd structure of this integral, a distinction between two cases is required. In each case we proceed with the expansion (8.6.35), which holds for fixed $\delta_1, \delta_2 > 0$. By arguments analogous to the proof of Theorem 8.4.4, one can then justify the existence of the termwise limits as $\delta_1, \delta_2 \downarrow 0$ and especially of the limit of each remainder integral.

Theorem 8.6.4. *Assume validity of (A3) to (A6). Then, provided at least one term on the right hand side is non-zero, for an arbitrary $\rho > 0$, as $m \to \infty$, uniformly with respect to $|\xi_1| \geq \rho$, we have*

$$
\begin{aligned}
\mathrm{Ai}\begin{bmatrix} m; \begin{matrix} i\xi_1, T_1, \infty \\ 0, T_2 \end{matrix} \end{bmatrix} \sim\ & \lim_{\delta_1 \downarrow 0} \mathcal{K}_0 \begin{bmatrix} 0, \delta_1 + i\xi_1, T_1, \infty \\ 0, 0, T_2 \end{bmatrix} \mathbb{1}_{\{\eta_{00} > 0\}} \\
& - \frac{\Gamma(\eta_{00})}{\{b_{00}(m+1)\}^{\eta_{00}}} \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \lim_{\delta_1 \downarrow 0} \mathcal{P}_0 \begin{bmatrix} 0, \delta_1 + i\xi_1 \\ T_1, \infty \end{bmatrix} \mathbb{1}_{\{\eta_{00} \neq 0\}} \\
& + (\log(m+1)+\gamma) \frac{\overline{a}_{00}\overline{c}_{00}}{\beta_{00}} \lim_{\delta_1 \downarrow 0} \mathcal{P}_0 \begin{bmatrix} 0, \delta_1 + i\xi_1 \\ T_1, \infty \end{bmatrix} \mathbb{1}_{\{\eta_{00} = 0\}} \\
& + \frac{1}{\beta_{00}} \lim_{\delta_1 \downarrow 0} \mathcal{Y}_0 \begin{bmatrix} 0, \delta_1 + i\xi_1 \\ T_1, \infty, T_2 \end{bmatrix} \mathbb{1}_{\{\eta_{00} = 0\}}.
\end{aligned}
$$

The coefficients have been defined in (8.4.49) and (8.5.71).

Proof. Since both paths of (8.6.1) are infinite with $\xi_2 = 0$ and $\xi_1 \in \mathbb{R} \setminus \{0\}$, we conclude $\theta_2 = 0$ and $\theta_1 = 1$. In view of the identity (8.6.18), it suffices to show the termwise existence of each limit as $\delta_1 \downarrow 0$ in the expansion (8.6.35), uniformly with respect to $|\xi_1| \geq \rho$. The non-zero terms appearing therein depend on the parameter η_{00} only, whereas the remainder is $\mathcal{O}\left\{m^{-\eta_{00}-\varepsilon}\right\}$ for fixed $\delta_1 > 0$ and an appropriate $\varepsilon > 0$. Moreover, with $\vec{p}_1 = (\infty, 1)$, in each of the expansions (8.6.54), (8.6.58) and (8.6.67) only the first and the remainder term differs from zero. For brevity, we confine to a discussion of the expansion which occurs only if $\eta_{00} > 0$. With $\theta_2 = 0$ and $\vec{p}_1 = (\infty, 1)$, it is of the form

$$F(m; 0, (\infty, 1)) = -\mathcal{K}_0\begin{bmatrix} 0, \zeta_1, T_1, \infty \\ 0, 0, T_2 \end{bmatrix} + \frac{1}{2\pi i}\int_{u_1 - i\infty}^{u_1 + i\infty} (m+1)^{-w}\Gamma(w)\mathcal{K}_0\begin{bmatrix} -w, \zeta_1, T_1, \infty \\ 0, 0, T_2 \end{bmatrix} dw, \tag{8.6.69}$$

where $u_1 \equiv f_1(0, (\infty, 1))$ is an arbitrary positive number, by (8.6.51), and the generating function equals (8.6.11) with $z = 0$, i.e.,

$$\mathcal{K}_0\begin{bmatrix} -w, \zeta_1, T_1, \infty \\ 0, 0, T_2 \end{bmatrix} = \int_{T_2}^{\infty} \bar{\mathfrak{a}}(t) \int_{T_1}^{\infty} \{\varphi(s)\}^{-w} e^{-\zeta_1 s}\mathfrak{c}(s)\bar{\mathfrak{c}}(s+t) ds dt. \tag{8.6.70}$$

For $\eta_{00} > 0$, the above representation of $\mathcal{K}_0[\ldots]$ with $\Re w = u_1$ converges absolutely for fixed $\delta_1 > 0$. We will now show that it approaches a finite limit as $\delta_1 \downarrow 0$, uniformly with respect to $|\xi_1| \geq \rho$. Moreover, we will explain the existence of a uniform bound, such that the remainder in (8.6.69) converges absolutely and uniformly with respect to $\delta_1 \geq 0$ and $|\xi_1| \geq \rho$. For this, in terms of (6.6.9), with $k \in \mathbb{N}_0$ we define

$$T_k(u, z) := \sum_{j=0}^{k} \binom{k}{j} P_j(u, z)\bar{\mathfrak{c}}^{(k-j)}(u), \tag{8.6.71}$$

which satisfies $T_0(u, z) = \bar{\mathfrak{c}}(u)$ and

$$\frac{d^k}{du^k}\left\{\{\varphi(u)\}^{-z}\,\bar{\mathfrak{c}}(u)\right\} = \{\varphi(u)\}^{-z}\,T_k(u, z).$$

Then, by virtue of (8.4.28), for $t \geq 0$, $k \in \mathbb{N}_0$ and $n_1 \in \mathbb{N}$, according to the Leibniz rule, we compute:

$$\begin{aligned} &\frac{d^{n_1}}{ds^{n_1}}\left\{\{\varphi(s)\}^{-w}\,\mathfrak{c}(s)\,\{\varphi(s+t)\}^{-z}\,T_k(s+t, z)\right\} \\ &\quad = \sum_{k_1=0}^{n_1}\binom{n_1}{k_1}\frac{d^{k_1}}{ds^{k_1}}\left\{\{\varphi(s+t)\}^{-z}\,T_k(s+t, z)\right\}\frac{d^{n_1-k_1}}{ds^{n_1-k_1}}\left\{\{\varphi(s)\}^{-w}\,\mathfrak{c}(s)\right\} \\ &\quad = \{\varphi(s)\}^{-w}\,\{\varphi(s+t)\}^{-z}\sum_{k_1=0}^{n_1} T_{k+k_1}(s+t, z)S_{k_1,n_1}(s, w) \end{aligned}$$

The sum on the right hand side constitutes a polynomial of the variables w and z with degree n_1. The coefficients are functions of the variables $s+t$ and s, which are continuous on $[T_1, \infty) \times [T_2, \infty)$. Furthermore, for any $k \in \mathbb{N}_0$, $0 \leq k_1 \leq n_1$ and $w, z \in \mathbb{C}$, as $s \to \infty$ with fixed $t \geq 0$ or as $t \to \infty$ with fixed $s \geq 0$, we observe the asymptotic behaviour

$$\{\varphi(s)\}^{-w}\{\varphi(s+t)\}^{-z} T_{k+k_1}(s+t,z) S_{k_1,n_1}(s,w) = \mathcal{O}\left\{\frac{s^{\beta_{00}\Re w - \gamma_{00}+k_1-n_1}}{(s+t)^{\gamma_{00}+k+k_1-\beta_{00}\Re z}}\right\}.$$

Hence, for sufficiently large n_1 and fixed $t \geq 0$, the n_1-th derivative is absolutely integrable on $s \geq T_1$, and in addition with $\eta_{00} > 0$, the product

$$\bar{\mathfrak{a}}(t)\frac{d^{n_1}}{ds^{n_1}}\left\{\{\varphi(s)\}^{-w}\mathfrak{c}(s)\bar{\mathfrak{c}}(s+t)\right\}$$

is absolutely integrable on $[T_1,\infty) \times [T_2,\infty)$. Accordingly, repeated integration by parts of the interior integral in (8.6.70) leads to an expansion that approaches a finite limit as $\delta_1 \downarrow 0$, uniformly with respect to $|\xi_1| \geq \rho$. By means of the triangle inequality, one can verify the uniform boundedness with respect to $\delta_1 \geq 0$ and $|\xi_1| \geq \rho$ by a polynomial of $|w|$. Consequently, the remainder in (8.6.69) is $\mathcal{O}\{m^{-u_1}\}$ as $m \to \infty$, and this statement remains true as $\delta_1 \downarrow 0$, uniformly with respect to $|\xi_1| \geq \rho$. Analogous arguments apply for the expansions (8.6.58) and (8.6.67) and for the remainder term in the expansion (8.6.35). ■

In the case $\xi_2 \in \mathbb{R} \setminus \{0\}$, i.e., $\theta_2 = 1$, it is easy to see, that merely the first and the remainder term in the expansion (8.6.35) do not vanish, of which the last is exponentially small. The dominating terms as $m \to \infty$ are therefore produced by the expansion (8.6.54). By virtue of the arguments from the proof of the preceding theorem, analogous to Theorem 8.4.4, it is therefore possible to verify our concluding statements below.

Theorem 8.6.5. *Assume validity of (A3) to (A8). Then, provided at least one term on the right hand side is non-zero, for an arbitrary $\rho > 0$, as $m \to \infty$,*

(1) uniformly with respect to $|\xi_2| \geq \rho$ and $\xi_1 \in \mathbb{R}$, we have

$$\begin{aligned}
\mathrm{Ai}\left[m; \begin{matrix} i\xi_1, \tau_1, T_1 \\ i\xi_2, T_2 \end{matrix}\right] \sim \lim_{\delta_2 \downarrow 0} \mathcal{K}_0 \left[\begin{matrix} 0, i\xi_1, \tau_1, T_1 \\ 0, \delta_2 + i\xi_2, T_2 \end{matrix}\right] \mathbb{1}_{\{\beta_{10}=0 \vee \chi_{10}>0)\}} \\
- \frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}} \frac{c_{10}}{\beta_{10}} e^{-i\xi_1\tau_1} \lim_{\delta_2 \downarrow 0} \mathcal{R}(\tau_1; \delta_2 + i\xi_2, T_2) \mathbb{1}_{\{\chi_{10} \neq 0\}} \\
+ (\log(m+1) + \gamma) \frac{c_{10}}{\beta_{10}} e^{-i\xi_1\tau_1} \lim_{\delta_2 \downarrow 0} \mathcal{R}(\tau_1; \delta_2 + i\xi_2, T_2) \mathbb{1}_{\{\chi_{10}=0\}} \\
+ \lim_{\delta_2 \downarrow 0} \kappa_0 \left[\begin{matrix} 1, 1 \\ \tau_1, T_1 \end{matrix}\right] \mathbb{1}_{\{\chi_{10}=0\}}.
\end{aligned}$$

The coefficients can be found in (8.6.40) and (8.6.45).

(2) uniformly with respect to $|\xi_2| \geq \rho$, we have

$$\begin{aligned}
\mathrm{Ai}\left[m; \begin{matrix} 0, T_1, \infty \\ i\xi_2, T_2 \end{matrix}\right] \sim \lim_{\delta_2 \downarrow 0} \mathcal{K}_0 \left[\begin{matrix} 0, 0, T_1, \infty \\ 0, \delta_2 + i\xi_2, T_2 \end{matrix}\right] \mathbb{1}_{\{\nu_{00}>0\}} \\
- \frac{\Gamma(\nu_{00})}{\{b_{00}(m+1)\}^{\nu_{00}}} \frac{|c_{00}|^2}{\beta_{00}} \lim_{\delta_2 \downarrow 0} \mathcal{A}^0(\delta_2 + i\xi_2; T_2) \mathbb{1}_{\{\nu_{00} \neq 0\}}
\end{aligned}$$

$$+ (\log(m+1)+\gamma)\frac{|c_{00}|^2}{\beta_{00}} \lim_{\delta_2\downarrow 0} \mathcal{A}^0(\delta_2+i\xi_2;T_2)\mathbb{1}_{\{\nu_{00}=0\}}$$
$$+ \lim_{\delta_2\downarrow 0} \kappa_0 \begin{bmatrix} 1,0 \\ T_1,\infty \end{bmatrix} \mathbb{1}_{\{\nu_{00}=0\}}.$$

The right hand side features (8.5.29) and (8.6.46).

(3) uniformly with respect to $|\xi_1|,|\xi_2| \geq \rho$, *we have*

$$\mathrm{Ai}\begin{bmatrix} m; & i\xi_1, T_1, \infty \\ & i\xi_2, T_2 \end{bmatrix} \sim \lim_{\delta_1,\delta_2\downarrow 0} \mathcal{K}_0 \begin{bmatrix} 0,\delta_1+i\xi_1,T_1,\infty \\ 0,\delta_2+i\xi_2,T_2 \end{bmatrix}.$$

8.7. Two Finite Paths in an Asymmetric-Type Iterated Integral

We are finally concerned with scenarios in which both integration paths of the iterated integral (8.0.2) are finite, in particular $\mathcal{P}_j = (\tau_j, T_j]$ with $0 \leq \tau_j < T_j < \infty$ for each $j \in \{1,2\}$. The integral is then assumed to take on the form

$$\mathrm{Ai}\begin{bmatrix} m; & \tau_1, T_1 \\ & \tau_2, T_2 \end{bmatrix} = \int_{\tau_2}^{T_2} \overline{a}(t) \int_{\tau_1}^{T_1} c(s)\left\{1-(1-\Psi(s))^{m+1}\right\} \times \left\{1-(1-\Psi(s+t))^{m+1}\right\}\overline{c}(s+t)dsdt, \tag{8.7.1}$$

where the overline indicates the complex conjugate and, with φ as per definition (8.1.5), we suppose:

(A10) φ and c are continuous on any closed subset of $(\tau_1+\tau_2, T_1+T_2]$ and of $(\tau_1, T_1]$ with $\varphi > 0$. Moreover, a is continuous on $[\tau_2, T_2]$.

(A11) $\varphi(u)$, $c(u)$ and $a(v)$ are algebraic as $u \downarrow \tau_1$, as $u \downarrow \tau_1+\tau_2$ and as $v \downarrow \tau_2$ for parameters

$$\beta_{10}, \beta_0(1,2) \geq 0,$$
$$\gamma_{10}, \gamma_0(1,2) \in \mathbb{R},$$
$$\alpha_{20} \geq 0$$

and coefficients $b_{10}, b_0(1,2) > 0$ and $c_{10}, c_0(1,2), a_{20} \in \mathbb{C}\setminus\{0\}$. Especially

$$\begin{cases} \gamma_{10}+\beta_{10} > -1, \\ \gamma_0(1,2)+\beta_0(1,2) > -1. \end{cases} \tag{8.7.2}$$

(A12) If $\beta_{10} > 0$ or $\beta_0(1,2) > 0$, the normalized phases $B_1(s)$ and $B_{1,2}(u)$ as well as the normalized amplitudes $C_1(s)$ and $C_{1,2}(u)$ possess a first derivative, which is continuous with respect to $\tau_1 < s \leq T_1$ and $\tau_1+\tau_2 < u \leq T_1+T_2$ and of order $\beta_{11}, \gamma_{11}, \beta_1(1,2), \gamma_1(1,2) > -1$ as $s \downarrow \tau_1$ and as $u \downarrow \tau_1+\tau_2$. See also definitions (8.2.5), (8.2.6), (8.2.18) and (8.2.19).

(A13) If $\beta_0(1,2) > 0$, the normalized amplitude $A_2(t)$ has a continuous first derivative on $(\tau_2, T_2]$ with order $\alpha_{21} > -1$ as $t \downarrow \tau_2$.

Again we do not point out each situation in which some coefficients in the Laurent expansions below may equal zero. For the sake of clarity, if $\beta_{10} > 0$, frequent use will be made of the parameter χ_{10}, which was defined in (8.3.5), and we also write

$$\begin{cases} \varsigma_{10} := \frac{1+\alpha_{20}}{\beta_{10}}, \\ \kappa_{10} := \chi_{10} + \varsigma_{10} + \frac{\gamma_0(1,2)}{\beta_{10}}. \end{cases} \tag{8.7.3}$$

Besides, if $\beta_0(1,2) > 0$, we denote

$$\begin{cases} \chi_0(1,2) := \frac{1+\gamma_{10}}{\beta_0(1,2)}, \\ \varsigma_0(1,2) := \frac{1+\alpha_{20}}{\beta_0(1,2)}, \\ \kappa_0(1,2) := \chi_0(1,2) + \varsigma_0(1,2) + \frac{\gamma_0(1,2)}{\beta_0(1,2)}. \end{cases} \tag{8.7.4}$$

and we also introduce the parameter

$$k := \frac{\beta_{10}}{\beta_0(1,2)}. \tag{8.7.5}$$

Then, $\chi_{10}, \chi_0(1,2) > -1$, $\varsigma_{10}, \varsigma_0(1,2) > 0$ and especially

$$\kappa_0(1,2) > -1 - k.$$

In the described setup, preliminary to a transformation of (8.7.1) to an iterated MB-integral, we specify the convergence behaviour of the iterated generating function

$$\mathcal{S}_0\begin{bmatrix} -w, \tau_1, T_1 \\ -z, \tau_2, T_2 \end{bmatrix} := \int_{\tau_2}^{T_2} \overline{a}(t) \int_{\tau_1}^{T_1} \{\varphi(s)\}^{-w} c(s) \{\varphi(s+t)\}^{-z} \overline{c}(s+t) ds dt. \tag{8.7.6}$$

Since $\alpha_{20} \geq 0$, this integral converges absolutely, by Lemma 8.2.1, for all $w, z \in \mathbb{C}$ with

$$\begin{cases} \gamma_{10} - \beta_{10}\Re w > -1, \\ \alpha_{20} + \gamma_{10} - \beta_{10}\Re w + \gamma_0(1,2) - \beta_0(1,2)\Re z > -2. \end{cases} \tag{8.7.7}$$

The z-region of analyticity certainly depends on w, if $\beta_{10} > 0$, which then enables us to choose the location of its right boundary. It thus appears beneficial, to begin with a consideration of (8.7.6) as a function of the variable z for fixed w. Now, on the one hand, if $\beta_0(1,2) = 0$, this constitutes an entire function of z. On the other hand, if $\beta_0(1,2) > 0$, we rearrange (8.7.7), to obtain

$$\begin{cases} \gamma_{10} + 1 > \beta_{10}\Re w, \\ \Re z < \kappa_0(1,2) - k\Re w, \end{cases} \tag{8.7.8}$$

and the integral (8.7.6) is especially holomorphic in this z-half plane for fixed w.

8.7.1. Transformation to an Iterated MB-Integral

According to Lemma 8.2.1, under the above assumptions absolute convergence of the integral (8.7.1) holds for any $m \geq 0$. In view of our aim to start with a study of (8.7.6) as a function of z for fixed w, for a constant

$$\begin{cases} -1 < u_0 < 0, & \text{if } \beta_{10} = 0, \\ -1 < u_0 < \min\{0, \chi_{10}\}, & \text{if } \beta_{10} > 0, \end{cases} \tag{8.7.9}$$

we first introduce the Cahen-Mellin representation (8.1.6) for the m-power of the variable s, which leads to

$$\begin{aligned} \mathrm{Ai}\left[m; \begin{matrix} \tau_1, T_1 \\ \tau_2, T_2 \end{matrix}\right] = \frac{1}{2\pi i} \int\limits_{u_0 - i\infty}^{u_0 + i\infty} (m+1)^{-w}\Gamma(w) \int\limits_{\tau_2}^{T_2} \overline{a}(t) \int\limits_{\tau_1}^{T_1} \{\varphi(s)\}^{-w} \\ \times c(s)\left\{1 - (1 - \Psi(s+t))^{m+1}\right\} \overline{c}(s+t) ds dt dw. \end{aligned} \tag{8.7.10}$$

If, in addition, we introduce the Cahen-Mellin representation for the m-power with the argument $s+t$, in terms of (8.7.6), we arrive at the iterated MB-integral

$$\begin{aligned} \mathrm{Ai}\left[m; \begin{matrix} \tau_1, T_1 \\ \tau_2, T_2 \end{matrix}\right] = \frac{1}{(2\pi i)^2} \int\limits_{u_0 - i\infty}^{u_0 + i\infty} (m+1)^{-w}\Gamma(w) \\ \times \int\limits_{x_0 - i\infty}^{x_0 + i\infty} (m+1)^{-z}\Gamma(z)\mathcal{S}_0\left[\begin{matrix} -w, \tau_1, T_1 \\ -z, \tau_2, T_2 \end{matrix}\right] dz dw, \end{aligned} \tag{8.7.11}$$

whose integration path $\Re z = x_0$, by (8.1.6) and by Lemma 8.2.1, must satisfy

$$\begin{cases} -1 < x_0 < 0, & \text{if } \beta_0(1,2) = 0, \\ -1 < x_0 < \min\{0, \kappa_0(1,2) - ku_0\}, & \text{if } \beta_0(1,2) > 0. \end{cases} \tag{8.7.12}$$

Notice that such integration paths always exist. For $\beta_0(1,2) > 0$ we will now compute the z-analytic continuation of $\mathcal{S}_0[\ldots]$ into a region which contains the line $\Re z = \kappa_0(1,2) - k\Re w$.

8.7.2. z-Analytic Continuation of the Iterated Generating Function

Since the z-abscissa of convergence of (8.7.6) is due to the supplementary condition for the convergence of this iterated integral, its z-analytic continuation can not be obtained by simply integrating by parts. However, for fixed $w \in \mathbb{C}$ with $\gamma_{10} + 1 > \beta_{10}\Re w$ as a function of z, it is readily confirmed that (8.7.6) satisfies the conditions of §8.2.6.5. There we have shown that, upon identifying $\beta_0 \equiv \beta_0(1,2)$ and

$$\begin{cases} d(s) & \equiv \{\varphi(s)\}^{-w} c(s), \\ e(t) & \equiv \overline{a}(t), \\ k(s+t) & \equiv \overline{c}(s+t), \end{cases} \tag{8.7.13}$$

which implies $\chi_1 \equiv \chi_0(1,2) - kw$ and $\chi_2 \equiv \varsigma_0(1,2)$, for $w \neq 0$, according to (8.2.221), the integral (8.7.6) can be extended to a meromorphic function in the strip

$$\begin{cases} \Re z > \chi_0(1,2) + \frac{\gamma_0(1,2)}{\beta_0(1,2)} - k\Re w, \\ \Re z < \kappa_0(1,2) - k\Re w \\ \qquad\qquad +\chi_{\beta_0(1,2)}(\beta_1(1,2), \beta_{11}, \gamma_{11}, \alpha_{21}, \gamma_0(1,2), 0). \end{cases} \tag{8.7.14}$$

Note that, due to the dependence on w of the function $d(s)$ from (8.7.13), the z-analytic continuation for $w = 0$ may have a different region of validity. Yet, it suffices to assume $w \neq 0$. To specify the corresponding z-analytic continuation, we first define the integral function

$$\begin{aligned} \Omega\begin{bmatrix} z, kw \\ \tau_1, \tau_2 \end{bmatrix} := {} & (T_2 - \tau_2)^{\beta_0(1,2)(\kappa_0(1,2) - kw - z)} e^{zB_{1,2}(\tau_1 + T_2)} \overline{A}_2(T_2) \overline{C}_{1,2}(\tau_1 + T_2) \\ & - z \int_{\tau_2}^{T_2} (t - \tau_2)^{\beta_0(1,2)(\kappa_0(1,2) - kw - z)} e^{zB_{1,2}(\tau_1 + t)} \\ & \qquad \times B'_{1,2}(\tau_1 + t) \overline{A}_2(t) \overline{C}_{1,2}(\tau_1 + t) dt \\ & - \int_{\tau_2}^{T_2} (t - \tau_2)^{\beta_0(1,2)(\kappa_0(1,2) - kw - z)} e^{zB_{1,2}(\tau_1 + t)} \\ & \qquad \times \left\{ \overline{A}'_2(t) \overline{C}_{1,2}(\tau_1 + t) + \overline{A}_2(t) \overline{C}'_{1,2}(\tau_1 + t) \right\} dt. \end{aligned} \tag{8.7.15}$$

It converges absolutely and is holomorphic for $z \in \mathbb{C}$ with

$$\Re z < \kappa_0(1,2) - k\Re w + \chi_{\beta_0(1,2)}(\beta_1(1,2), \alpha_{21}, \gamma_1(1,2)).$$

Furthermore, following from the fundamental theorem of calculus, for arbitrary values of the parameter $\kappa_0(1,2)$ and $w \in \mathbb{C}$ with $\gamma_{10} + 1 > \beta_{10}\Re w$, it satisfies

$$\Omega\begin{bmatrix}\kappa_0(1,2) - kw, kw\\ \tau_1, \tau_2\end{bmatrix} = \overline{a}_{20}\overline{c}_0(1,2)\left\{b_0(1,2)\right\}^{kw-\kappa_0(1,2)}. \tag{8.7.16}$$

Moreover, we introduce the expansion

$$\begin{aligned}\mathcal{H}\left[\zeta;\begin{matrix}w, \tau_1, T_1\\ z, \tau_2, T_2\end{matrix}\right] := &-\frac{(T_1-\tau_1)^{\beta_0(1,2)(\chi_0(1,2)-kw)-\zeta}}{\zeta-\beta_0(1,2)(\chi_0(1,2)-kw)}e^{wB_1(T_1)}C_1(T_1)\\ &\times\int\limits_{\tau_2}^{T_2}(t-\tau_2)^{\beta_0(1,2)(\varsigma_0(1,2)-z)+\zeta+\gamma_0(1,2)-1}\overline{A}_2(t)e^{zB_{1,2}(T_1+t)}\overline{C}_{1,2}(T_1+t)dt\\ &+\frac{1}{\zeta-\beta_0(1,2)(\chi_0(1,2)-kw)}\\ &\quad\times\sum_{\substack{n_1,n_2\in\{0,1\}\\ n_1+n_2=1}}\int\limits_{\tau_2}^{T_2}(t-\tau_2)^{\beta_0(1,2)(\varsigma_0(1,2)-z)+\zeta+\gamma_0(1,2)-1}\overline{A}_2(t)\\ &\qquad\times\int\limits_{\tau_1}^{T_1}(s-\tau_1)^{\beta_0(1,2)(\chi_0(1,2)-kw)-\zeta}\frac{d^{n_1}}{ds^{n_1}}\left\{e^{wB_1(s)}C_1(s)\right\}\\ &\qquad\quad\times\frac{d^{n_2}}{ds^{n_2}}\left\{e^{zB_{1,2}(s+t)}\overline{C}_{1,2}(s+t)\right\}dsdt.\end{aligned}\tag{8.7.17}$$

Finally, with the aid of the last definition, we write

$$\Sigma_1^2\begin{bmatrix}-w, \tau_1, T_1\\ -z, \tau_2, T_2\end{bmatrix} := \frac{1}{2\pi i}\int\limits_{q-i\infty}^{q+i\infty}\frac{\Gamma(\zeta)\Gamma(\beta_0(1,2)z-\gamma_0(1,2)-\zeta)}{\Gamma(\beta_0(1,2)z-\gamma_0(1,2))}\mathcal{H}\left[\zeta;\begin{matrix}w, \tau_1, T_1\\ z, \tau_2, T_2\end{matrix}\right]d\zeta, \tag{8.7.18}$$

in which the integration path satisfies

$$\begin{cases}q > \max\left\{\beta_0(1,2)(\Re z - \varsigma_0(1,2)) - \gamma_0(1,2), \beta_0(1,2)(\chi_0(1,2) - k\Re w)\right\},\\ q < \min\left\{\beta_0(1,2)(\chi_0(1,2) - k\Re w + \chi_{\beta_0(1,2)}(\beta_{11}, \gamma_{11}, 0)), \beta_0(1,2)\Re z - \gamma_0(1,2)\right\}.\end{cases}\tag{8.7.19}$$

Then, according to (8.2.216), for fixed $w \in \mathbb{C}$ with $\gamma_{10} + 1 > \beta_{10}\Re w$, in the strip (8.7.14) the z-analytic continuation of the generating function (8.7.6) is represented by the expansion

$$\begin{aligned}\mathcal{S}_0\begin{bmatrix}-w, \tau_1, T_1\\ -z, \tau_2, T_2\end{bmatrix} = &-\frac{c_{10}}{\beta_0(1,2)\left\{b_{10}\right\}^w}\frac{\Gamma(\beta_0(1,2)(\chi_0(1,2)-kw))}{\Gamma(\beta_0(1,2)z-\gamma_0(1,2))}\\ &\times\frac{\Gamma(\beta_0(1,2)(z+kw-\chi_0(1,2))-\gamma_0(1,2))}{z-\kappa_0(1,2)+kw}\Omega\begin{bmatrix}z, kw\\ \tau_1, \tau_2\end{bmatrix} + \Sigma_1^2\begin{bmatrix}-w, \tau_1, T_1\\ -z, \tau_2, T_2\end{bmatrix}.\end{aligned}\tag{8.7.20}$$

In §8.2.6.5 it was mentioned, that the analytic continuation is $\mathcal{O}(z^2)$ as $\Im z \to \pm\infty$, uniformly with respect to $\Re z$ in any closed vertical substrip of (8.7.14). It was also pointed out that the extended region contains exactly one pole. In the above case, it depends on the fixed variable

w and its order is never greater than one. Assuming $kw \neq \kappa_0(1,2)$, the pole does not lie at the origin, and by virtue of (8.7.16) from (8.7.20) we deduce

$$\operatorname*{Res}_{z=\kappa_0(1,2)-kw} \mathcal{S}_0\begin{bmatrix} -w, \tau_1, T_1 \\ -z, \tau_2, T_2 \end{bmatrix} = -\frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\beta_0(1,2)\,\{b_{10}\}^w\,\{b_0(1,2)\}^{\kappa_0(1,2)-kw}} \times \frac{\Gamma(1+\alpha_{20})\Gamma(1+\gamma_{10}-\beta_{10}w)}{\Gamma(2+\gamma_{10}+\alpha_{20}-\beta_{10}w)}. \tag{8.7.21}$$

We conclude this paragraph with a special case, in which this pole inevitably lies at the origin of the z-plane for any admissible value of w.

8.7.2.1. Laurent Expansion for $\beta_{10} = 0$ with $\kappa_0(1,2) = 0$

The case $\beta_{10} = 0$ implies $k = 0$. Hence, $\kappa_0(1,2) - kw = 0$ and for fixed w, as $z \to 0$ there exists a Laurent expansion of the form

$$\mathcal{S}_0\begin{bmatrix} -w, \tau_1, T_1 \\ -z, \tau_2, T_2 \end{bmatrix} = \frac{\mathrm{s}^2_{-1}}{z} + \mathrm{s}^2_0 + \mathcal{O}(z),$$

with coefficients s^2_{-1} and s^2_0. From (8.7.21) we immediately deduce

$$\mathrm{s}^2_{-1} = -\frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\beta_0(1,2)\,\{b_{10}\}^w} \frac{\Gamma(1+\alpha_{20})\Gamma(1+\gamma_{10})}{\Gamma(2+\gamma_{10}+\alpha_{20})}. \tag{8.7.22}$$

Furthermore, according to the elementary rules of complex calculus, we know

$$\mathrm{s}^2_0 = \frac{d}{dz} z \mathcal{S}_0\begin{bmatrix} -w, \tau_1, T_1 \\ -z, \tau_2, T_2 \end{bmatrix}\Bigg|_{z=0}. \tag{8.7.23}$$

By Theorem A.2.1, arbitrary derivatives of (8.7.15) may be computed by differentiation under the integral sign. Taking into account

$$B_{1,2}(\tau_1 + t) = \log\left\{\frac{(t-\tau_2)^{\beta_0(1,2)}}{\varphi(\tau_1+t)}\right\},$$

valid by definition (8.2.18), we thus obtain:

$$\begin{aligned} \omega_1(\tau_1;\tau_2,T_2) := {} & \frac{d}{dz}\Omega\begin{bmatrix} z, 0 \\ \tau_1, \tau_2 \end{bmatrix}\Bigg|_{z=0} \\ = {} & -\log\{\varphi(\tau_1+T_2)\}\,\overline{A}_2(T_2)\overline{C}_{1,2}(\tau_1+T_2) \\ & - \int_{\tau_2}^{T_2} B'_{1,2}(\tau_1+t)\overline{A}_2(t)\overline{C}_{1,2}(\tau_1+t)\,dt \\ & + \int_{\tau_2}^{T_2} \log\{\varphi(\tau_1+t)\}\left\{\overline{A}'_2(t)\overline{C}_{1,2}(\tau_1+t) + \overline{A}_2(t)\overline{C}'_{1,2}(\tau_1+t)\right\}dt \end{aligned} \tag{8.7.24}$$

If, for brevity, we introduce the constant

$$\sigma_0^2(\tau_1;\tau_2,T_2) := \beta_0(1,2)\left\{\psi(1+\alpha_{20}) - \psi(-\gamma_0(1,2))\right\} + \frac{\omega_1(\tau_1;\tau_2,T_2)}{\overline{a}_{20}\overline{c}_0(1,2)}, \tag{8.7.25}$$

by routine calculations, it can be shown that

$$\frac{d}{dz}\left\{\frac{\Gamma(\beta_0(1,2)(z+\varsigma_0(1,2)))}{\Gamma(\beta_0(1,2)z-\gamma_0(1,2))}\Omega\begin{bmatrix} z,0 \\ \tau_1,\tau_2\end{bmatrix}\right\}\Bigg|_{z=0} = \frac{\Gamma(1+\alpha_{20})}{\Gamma(-\gamma_0(1,2))}\overline{a}_{20}\overline{c}_0(1,2)\sigma_0^2(\tau_1;\tau_2,T_2).$$

Therefore,

$$\mathrm{s}_0^2 = -\frac{\overline{a}_{20}c_{10}\overline{c}_0(1,2)}{\beta_0(1,2)\left\{b_{10}\right\}^w}\frac{\Gamma(1+\alpha_{20})\Gamma(1+\gamma_{10})}{\Gamma(2+\alpha_{20}+\gamma_{10})}\sigma_0^2(\tau_1;\tau_2,T_2) + \Sigma_1^2\begin{bmatrix}-w,\tau_1,T_1\\ 0,\tau_2,T_2\end{bmatrix},$$

and accordingly the dominating terms in the Laurent expansion at $z=0$ are

$$\begin{aligned}\mathcal{S}_0\begin{bmatrix}-w,\tau_1,T_1\\ -z,\tau_2,T_2\end{bmatrix} = -\frac{\overline{a}_{20}c_{10}\overline{c}_0(1,2)}{\beta_0(1,2)\left\{b_{10}\right\}^w}\frac{\Gamma(1+\alpha_{20})\Gamma(1+\gamma_{10})}{\Gamma(2+\alpha_{20}+\gamma_{10})}\left\{\frac{1}{z}+\sigma_0^2(\tau_1;\tau_2,T_2)\right\}\\ +\Sigma_1^2\begin{bmatrix}-w,\tau_1,T_1\\ 0,\tau_2,T_2\end{bmatrix} + \mathcal{O}(z).\end{aligned} \tag{8.7.26}$$

8.7.3. Evaluation of the Interior MB-Integral

We proceed with the evaluation of the interior of the iterated MB-integral (8.7.11), for fixed $w\in\mathbb{C}$ with $\Re w = u_0$, denoted by

$$\mathrm{H}(m;w) := \frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty}(m+1)^{-z}\Gamma(z)\mathcal{S}_0\begin{bmatrix}-w,\tau_1,T_1\\ -z,\tau_2,T_2\end{bmatrix}dz. \tag{8.7.27}$$

It is easy to see from (8.7.12), that the right boundary of the region of admissible values for x_0, except in the case $\beta_0(1,2)=0$, depends on the parameter $\kappa_0(1,2)$, on the factor k, and if $k>0$, i.e., if $\beta_{10}>0$, also on the location of the integration path of the exterior MB-integral. Appealing to the relation

$$k\kappa_{10} = \kappa_0(1,2), \tag{8.7.28}$$

compare (8.7.3) and (8.7.4), it is thus reasonable instead of (8.7.9) to agree

$$\begin{cases}-1<u_0<0, & \text{if } \beta_{10}=0,\\ \max\left\{-1,\kappa_{10}\right\}<u_0<\min\left\{0,\chi_{10}\right\}, & \text{if } \beta_{10}>0\wedge\kappa_{10}<\min\left\{0,\chi_{10}\right\},\\ -1<u_0<\min\left\{0,\chi_{10}\right\}, & \text{if } \beta_{10}>0\wedge\kappa_{10}\geq\min\left\{0,\chi_{10}\right\}.\end{cases} \tag{8.7.29}$$

As a consequence, the inequalities (8.7.12) become

$$\begin{cases} -1 < x_0 < \kappa_0(1,2), & \text{if } \beta_0(1,2) > 0 \wedge \beta_{10} = 0 \wedge \kappa_0(1,2) < 0, \\ -1 < x_0 < \kappa_0(1,2) - ku_0, & \text{if } \beta_0(1,2), \beta_{10} > 0 \wedge \kappa_{10} < \min\{0, \chi_{10}\}, \\ -1 < x_0 < 0, & \text{otherwise.} \end{cases} \tag{8.7.30}$$

Observe that the first two conditions on the parameters in (8.7.30) apply if and only if

$$\kappa_0(1,2) < ku_0.$$

Now, it was mentioned that the generating function (8.7.6) of $\mathrm{H}(m;w)$ is entire with respect to z for fixed $w \in \mathbb{C}$ with $\Re w = u_0$, if $\beta_0(1,2) = 0$. Furthermore, if $\beta_0(1,2) > 0$, according to Subsection 8.7.2, by virtue of the expansion (8.7.20) it can be extended to a meromorphic function in the half plane

$$\Re z < \kappa_0(1,2) - ku_0 + \chi_{\beta_0(1,2)}(\beta_1(1,2), \beta_{11}, \gamma_{11}, \alpha_{21}, \gamma_0(1,2), 0). \tag{8.7.31}$$

This expansion was seen to hold in the strip (8.7.14), where the only singularity is a pole of simple order at $z = \kappa_0(1,2) - kw$. Hence, in the first two cases in (8.7.30), the singularity that lies closest to the right of the integration path of $\mathrm{H}(m;w)$ is exactly this pole. The simple pole of $\Gamma(z)$ at $z = 0$ then lies to its right. In the third case, there are two possibilities. On the one hand, the closest singularity may be the simple pole of the indicated gamma function, whereas the generating function is entire or has its pole to the right of $z = 0$. On the other hand, and this happens if and only if $\beta_0(1,2) > 0$ and $\beta_{10} = \kappa_0(1,2) = 0$, these two poles will merge, thereby incuring a second order pole.

To expose the asymptotic behaviour as $m \to \infty$ of $\mathrm{H}(m;w)$, we perform a rightward displacement of the integration path across the closest singularity. This step is clearly permitted, since the generating function (8.7.6) and its analytic continuation in their respective regions of analyticity are $\mathcal{O}(1)$ and $\mathcal{O}(z^2)$ as $\Im z \to \pm\infty$, uniformly with respect to $\Re z$ in any closed vertical substrip. The integrand in (8.7.27) therefore vanishes exponentially fast towards the imaginary direction of the z-plane if $\beta_0(1,2) = 0$, or of the half plane (8.7.31) if $\beta_0(1,2) > 0$. If $\beta_0(1,2) = 0$, we move the integration path across the pole at $z = 0$, to match a line $\Re z = x_1$, for

$$x_1 > 0. \tag{8.7.32}$$

If $\beta_0(1,2) > 0$ and $\kappa_0(1,2) \geq ku_0$, we make a displacement across the points $z \in \{0, \kappa_0(1,2) - kw\}$, where in the case of equality this set actually contains only a single point, to obtain the sum of residues plus an integral along the line $\Re z = x_1$, for

$$\kappa_0(1,2) - ku_0 < x_1 < \kappa_0(1,2) - ku_0 + \chi_{\beta_0(1,2)}(\beta_1(1,2), \beta_{11}, \gamma_{11}, \alpha_{21}, \gamma_0(1,2), 0). \tag{8.7.33}$$

Finally, if $\beta_0(1,2) > 0$ and $\kappa_0(1,2) < ku_0$, the line $\Re z = x_1$ is supposed to satisfy

$$\begin{cases} x_1 > \kappa_0(1,2) - ku_0, \\ x_1 < \min\left\{0, \kappa_0(1,2) - ku_0 + \chi_{\beta_0(1,2)}(\beta_1(1,2), \beta_{11}, \gamma_{11}, \alpha_{21}, \gamma_0(1,2), 0)\right\}, \end{cases} \tag{8.7.34}$$

and we then solely collect the residue of the pole at $z = \kappa_0(1,2) - kw$. Incorporating the fact that each pole is encircled clockwisely, with the aid of Theorem B.2.1(2), according to (8.7.21) and (8.7.26), we arrive at:

$$\begin{aligned}
\mathrm{H}(m;w) = &-\mathcal{S}_0\begin{bmatrix} -w, \tau_1, T_1 \\ 0, \tau_2, T_2 \end{bmatrix} \mathbb{1}_{\{\beta_0(1,2)=0 \vee \kappa_0(1,2) > ku_0\}} \\
&+ \frac{\Gamma(\kappa_0(1,2) - kw)}{\{b_0(1,2)(m+1)\}^{\kappa_0(1,2)-kw}} \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\beta_0(1,2)\{b_{10}\}^w} \\
&\qquad\qquad \times \frac{\Gamma(1+\alpha_{20})\Gamma(1+\gamma_{10}-\beta_{10}w)}{\Gamma(2+\gamma_{10}+\alpha_{20}-\beta_{10}w)} \mathbb{1}_{\{\kappa_0(1,2) \neq ku_0\}} \\
&- (\log(m+1) + \gamma - \sigma_0^2(\tau_1;\tau_2,T_2)) \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\beta_0(1,2)\{b_{10}\}^w} \\
&\qquad\qquad \times \frac{\Gamma(1+\alpha_{20})\Gamma(1+\gamma_{10})}{\Gamma(2+\alpha_{20}+\gamma_{10})} \mathbb{1}_{\{\beta_{10}=\kappa_0(1,2)=0\}} \\
&- \Sigma_1^2\begin{bmatrix} -w, \tau_1, T_1 \\ 0, \tau_2, T_2 \end{bmatrix} \mathbb{1}_{\{\beta_{10}=\kappa_0(1,2)=0\}} \\
&+ \frac{1}{2\pi i} \int_{x_1-i\infty}^{x_1+i\infty} (m+1)^{-z} \Gamma(z) \mathcal{S}_0\begin{bmatrix} -w, \tau_1, T_1 \\ -z, \tau_2, T_2 \end{bmatrix} dz
\end{aligned} \tag{8.7.35}$$

Based on the above generating functions, we introduce the single MB-integrals

$$\mathrm{F}(m) := \frac{1}{2\pi i} \int_{u_0-i\infty}^{u_0+i\infty} (m+1)^{-w} \Gamma(w) \mathcal{S}_0\begin{bmatrix} -w, \tau_1, T_1 \\ 0, \tau_2, T_2 \end{bmatrix} dw, \tag{8.7.36}$$

$$\begin{aligned}
\mathrm{G}(m;\beta_{10}) := \frac{\Gamma(1+\alpha_{20})}{2\pi i} \int_{u_0-i\infty}^{u_0+i\infty} & \frac{\Gamma(w)\Gamma(\kappa_0(1,2)-kw)\Gamma(1+\gamma_{10}-\beta_{10}w)}{\Gamma(2+\gamma_{10}+\alpha_{20}-\beta_{10}w)} \\
& \times \{b_0(1,2)(m+1)\}^{kw-\kappa_0(1,2)} \{b_{10}(m+1)\}^{-w} dw,
\end{aligned} \tag{8.7.37}$$

$$\mathrm{Z}_1(m) := \frac{1}{2\pi i} \int_{u_0-i\infty}^{u_0+i\infty} \{b_{10}(m+1)\}^{-w} \Gamma(w) dw, \tag{8.7.38}$$

$$\mathrm{Z}_2(m) := \frac{1}{2\pi i} \int_{u_0-i\infty}^{u_0+i\infty} (m+1)^{-w} \Gamma(w) \Sigma_1^2\begin{bmatrix} -w, \tau_1, T_1 \\ 0, \tau_2, T_2 \end{bmatrix} dw. \tag{8.7.39}$$

It is not as trivial as in the case $\beta_0(1,2) = 0$ but still not difficult in the case $\beta_0(1,2) > 0$, from (8.7.20) to derive with respect to $\Im w$ and $\Im z$ a bound for $\mathcal{S}_0[\ldots]$, which holds for fixed $\Re w$ and $\Re z$. Then, upon plugging the expansion (8.7.35) into the iterated MB-integral (8.7.11), as

$m \to \infty$ it shows that

$$\operatorname{Ai}\left[m; \begin{matrix}\tau_1, T_1 \\ \tau_2, T_2\end{matrix}\right] = -\mathrm{F}(m)\mathbb{1}_{\{\beta_0(1,2)=0 \vee \kappa_0(1,2)>ku_0\}} \\ + \frac{\overline{a}_{20}c_{10}\overline{c}_0(1,2)}{\beta_0(1,2)}\mathrm{G}(m;\beta_{10})\mathbb{1}_{\{\kappa_0(1,2)\neq ku_0\}} \\ - (\log(m+1)+\gamma-\sigma_0^2(\tau_1;\tau_2,T_2))\frac{\overline{a}_{20}c_{10}\overline{c}_0(1,2)}{\beta_0(1,2)} \\ \times\frac{\Gamma(1+\alpha_{20})\Gamma(1+\gamma_{10})}{\Gamma(2+\alpha_{20}+\gamma_{10})}\mathrm{Z}_1(m)\mathbb{1}_{\{\beta_{10}=\kappa_0(1,2)=0\}} \\ - \mathrm{Z}_2(m)\mathbb{1}_{\{\beta_{10}=\kappa_0(1,2)=0\}} \\ + \mathcal{O}\left\{m^{-x_1-u_0}\right\}. \tag{8.7.40}$$

From the iterated integral with a symmetric-type integrand and two finite paths, see Section 8.3, we remember that definite statements were only possible under additional assumptions. Below we will see, that even more assumptions are required, to verify the asymptotic validity of the expansion (8.7.40). We shall therefore confine to the simpler cases and briefly outline, how to proceed in the more difficult ones. For this, we first discuss the asymptotics of the single MB-integrals (8.7.36) to (8.7.39).

8.7.4. A Single MB-Integral for the Residue at $z = 0$

The MB-integral (8.7.36) only occurs if $\beta_0(1,2) = 0$ or if $\kappa_0(1,2) > ku_0$. The last inequality trivially applies if $\beta_{10} = 0$ with $\kappa_0(1,2) > 0$, or if $\beta_{10} > 0$ with $\kappa_{10} \geq \min\{0, \chi_{10}\}$, by (8.7.29). The associated generating function is given by the iterated integral (8.7.6) with $z = 0$. For $\beta_{10} = 0$ this clearly constitutes an entire function of w, whereas for $\beta_{10} > 0$, by (8.7.7), it is holomorphic only in

$$\Re w < \min\{\chi_{10}, \kappa_{10}\}. \tag{8.7.41}$$

According to the minimum structure of this abscissa of convergence, we distinguish between two parametrizations.

8.7.4.1. w-Analytic Continuation for $z = 0$ and $\chi_{10} < \kappa_{10}$

Under the present assumptions on the parameters

$$\varsigma_{10} + \tfrac{\gamma_0(1,2)}{\beta_{10}} > 0, \tag{8.7.42}$$

and the w-region of analyticity of the integral transform (8.7.6) for $z = 0$ coincides with the half plane $\Re w < \chi_{10}$. Its right boundary originates in the condition for the convergence, for

$n_1 = n_2 = 0$, of the single integral

$$\Theta_0\begin{bmatrix}\chi_{10}-w, t\\ n_1, n_2, w, \tau_1, T_1\end{bmatrix} := \int_{\tau_1}^{T_1} (s-\tau_1)^{\beta_{10}(\chi_{10}-w)-1} \frac{d^{n_1}}{ds^{n_1}} \left\{ e^{wB_1(s)} C_1(s) \right\} \overline{c}^{(n_2)}(s+t) ds. \tag{8.7.43}$$

More precisely, $c(s+t) = \mathcal{O}(1)$ as $s \downarrow \tau_1$ for arbitrary fixed $\tau_2 < t \leq T_2$, by continuity, whence for $n_1 = n_2 = 0$ and fixed $\tau_2 < t \leq T_2$ the integral (8.7.43) is holomorphic in $\Re w < \chi_{10}$, and we can write

$$\mathcal{S}_0\begin{bmatrix}-w, \tau_1, T_1\\ 0, \tau_2, T_2\end{bmatrix} = \int_{\tau_2}^{T_2} \overline{a}(t)\Theta_0\begin{bmatrix}\chi_{10}-w, t\\ 0, 0, w, \tau_1, T_1\end{bmatrix} ds. \tag{8.7.44}$$

To access the analytic continuation, integration by parts or a quick reference to §8.2.2.1 suffices. For this, we introduce the expansion

$$\begin{aligned}\mathcal{S}_1\begin{bmatrix}-w, \tau_1, T_1\\ \tau_2, T_2\end{bmatrix} &:= (T_1-\tau_1)^{\beta_{10}(\chi_{10}-w)} e^{wB_1(T_1)} C_1(T_1) \int_{\tau_2}^{T_2} \overline{a}(t)\overline{c}(T_1+t) dt \\ &\quad - \sum_{\substack{n_1, n_2 \in \{0,1\} \\ n_1+n_2=1}} \int_{\tau_2}^{T_2} \overline{a}(t)\Theta_0\begin{bmatrix}\frac{1}{\beta_{10}}+\chi_{10}-w, t\\ n_1, n_2, \tau_1, T_1\end{bmatrix} dt.\end{aligned} \tag{8.7.45}$$

If we then identify

$$\begin{cases} d(s) & \equiv c(s), \\ k(s+t) & \equiv \overline{c}(s+t), \end{cases}$$

equation (8.2.31) immediately yields the representation

$$\mathcal{S}_0\begin{bmatrix}-w, \tau_1, T_1\\ 0, \tau_2, T_2\end{bmatrix} = -\frac{1}{\beta_{10}(w-\chi_{10})} \mathcal{S}_1\begin{bmatrix}-w, \tau_1, T_1\\ \tau_2, T_2\end{bmatrix}, \tag{8.7.46}$$

valid by analytic continuation for all $w \in \mathbb{C} \setminus \{\chi_{10}\}$ which lie in the region of analyticity of the expansion (8.7.45). According to Lemma 8.2.5, this is exactly the half plane

$$\begin{cases} \Re w < \chi_{10} + \min\left\{ \chi_{\beta_{10}}(\beta_{11}, \gamma_{11}, 0), \varsigma_{10} + \frac{\gamma_0(1,2)}{\beta_{10}} \right\}, & \text{if } \gamma_0(1,2) \neq 0, \\ \Re w < \chi_{10} + \min\left\{ \chi_{\beta_{10}}(\beta_{11}, \gamma_{11}, 0), \varsigma_{10} + \frac{\gamma_1(1,2)+1}{\beta_{10}} \right\}, & \text{if } \gamma_0(1,2) = 0. \end{cases} \tag{8.7.47}$$

To verify this statement, it is helpful to note that, by assumption, as $u \downarrow \tau_1 + \tau_2$ we have

$$\overline{c}'(u) \begin{cases} \sim \overline{c}_0(1,2)\gamma_0(1,2)(u-\tau_1-\tau_2)^{\gamma_0(1,2)-1}, & \text{if } \gamma_0(1,2) \neq 0, \\ = \mathcal{O}\left\{ (u-\tau_1-\tau_2)^{\gamma_1(1,2)} \right\}, & \text{if } \gamma_0(1,2) = 0. \end{cases}$$

The only singularity, which the analytic continuation shows in the wider region (8.7.47) is a pole of simple order at $w = \chi_{10}$. The corresponding residue for the interior of the integral (8.7.44) can be obtained from (8.2.33), which brings us

$$\operatorname*{Res}_{w=\chi_{10}} \mathcal{S}_0 \begin{bmatrix} -w, \tau_1, T_1 \\ 0, \tau_2, T_2 \end{bmatrix} = -\frac{c_{10}}{\beta_{10}} \{b_{10}\}^{-\chi_{10}} \mathfrak{s}_{-1}(\tau_1; \tau_2, T_2), \tag{8.7.48}$$

where we denote

$$\mathfrak{s}_{-1}(\tau_1; \tau_2, T_2) := \int_{\tau_2}^{T_2} \overline{a}(t)\overline{c}(\tau_1 + t)dt, \tag{8.7.49}$$

and this integral converges absolutely, since (8.7.42) is equivalent to $\alpha_{20} + \gamma_0(1,2) > -1$. In the particular case $\chi_{10} = 0$, the indicated pole matches the origin of the w-plane. The derivation of the Laurent expansion for the interior integral in (8.7.44) is then easily accomplished by virtue of (8.2.35) and (8.2.36). In an annulus around the origin this ultimately yields

$$\mathcal{S}_0 \begin{bmatrix} -w, \tau_1, T_1 \\ 0, \tau_2, T_2 \end{bmatrix} = -\frac{1}{w}\frac{c_{10}}{\beta_{10}} \mathfrak{s}_{-1}(\tau_1; \tau_2, T_2) + \frac{1}{\beta_{10}} \mathfrak{s}_0 \begin{bmatrix} \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} + \mathcal{O}(w), \tag{8.7.50}$$

with the coefficient associated with the constant term being equal to

$$\begin{aligned} \mathfrak{s}_0 \begin{bmatrix} \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} := & C_1(T_1) \log\{\varphi(T_1)\} \int_{\tau_2}^{T_2} \overline{a}(t)\overline{c}(T_1 + t)dt + \int_{\tau_2}^{T_2} \overline{a}(t) \int_{\tau_1}^{T_1} B_1'(s) C_1(s) \overline{c}(s+t) ds dt \\ & - \int_{\tau_2}^{T_2} \overline{a}(t) \int_{\tau_1}^{T_1} \log\{\varphi(s)\} \left\{ C_1'(s)\overline{c}(s+t) + C_1(s)\overline{c}'(s+t) \right\} ds dt. \end{aligned} \tag{8.7.51}$$

8.7.4.2. w-Analytic Continuation for $z = 0$ and $\kappa_{10} \leq \chi_{10}$

Following from our assumptions on the parameters α_{20} and $\gamma_0(1,2)$, non-positivity of the difference $\kappa_{10} - \chi_{10}$, i.e.,

$$\varsigma_{10} + \tfrac{\gamma_0(1,2)}{\beta_{10}} \leq 0,$$

can only happen if $\beta_0(1,2) > 0$. In these circumstances, by (8.7.7), the w-region of analyticity of the iterated generating function (8.7.6) with $z = 0$ is given by all $w \in \mathbb{C}$ with

$$\Re w < \kappa_{10}. \tag{8.7.52}$$

The right boundary of that region is especially determined by the supplementary condition for the convergence of the iterated integral. The analytic continuation thus can be specified by reference to §8.2.6.4. More precisely, if we identify $\beta_0 \equiv \beta_{10}$ and

$$\begin{cases} d(s) & \equiv c(s), \\ e(t) & \equiv \overline{a}(t), \\ k(s+t) & \equiv \overline{c}(s+t), \end{cases}$$

we have validity of (8.2.170), and from (8.2.171) we ascertain that $\chi_1 \equiv \chi_{10}$ and $\chi_2 \equiv \varsigma_{10}$. By (8.2.195), the integral (8.7.6) with $z = 0$ therefore can be extended analytically to the strip

$$\kappa_{10} - \varsigma_{10} < \Re w < \kappa_{10} + \chi_{\beta_{10}}(\beta_{11}, \gamma_{11}, \alpha_{21}, \gamma_1(1,2), 0). \tag{8.7.53}$$

The corresponding representation requires some auxiliary functions. First of all, for $w \in \mathbb{C}$ with $\Re w < \kappa_{10} + \chi_{\beta_{10}}(a_{21}, \gamma_1(1,2))$, we define the absolutely convergent integral

$$\begin{aligned} \Phi\begin{bmatrix} w \\ \tau_1, \tau_2 \end{bmatrix} :=& (T_2 - \tau_2)^{\beta_{10}(\kappa_{10}-w)} \overline{A}_2(T_2) \overline{C}_{1,2}(\tau_1 + T_2) \\ & - \int_{\tau_2}^{T_2} (t - \tau_2)^{\beta_{10}(\kappa_{10}-w)} \left\{ \overline{A}_2'(t) \overline{C}_{1,2}(\tau_1 + t) + \overline{A}_2(t) \overline{C}_{1,2}'(\tau_1 + t) \right\} dt. \end{aligned} \tag{8.7.54}$$

Due to the fundamental theorem of calculus,

$$\Phi\begin{bmatrix} \kappa_{10} \\ \tau_1, \tau_2 \end{bmatrix} = \overline{a}_{20} \overline{c}_0(1,2). \tag{8.7.55}$$

Furthermore, by analyticity, arbitrary derivatives of (8.7.54) can be calculated by differentiation under the sign of integration. For $j \in \mathbb{N}$ this yields:

$$\begin{aligned} \phi_j(\tau_1; \tau_2, T_2) :=& \frac{d^j}{dw^j} \Phi\begin{bmatrix} w \\ \tau_1, \tau_2 \end{bmatrix} \Bigg|_{w=\kappa_{10}} \\ =& \{-\beta_{10} \log(T_2 - \tau_2)\}^j \, \overline{A}_2(T_2) \overline{C}_{1,2}(\tau_1 + T_2) \\ & - \int_{\tau_2}^{T_2} \{-\beta_{10} \log(t - \tau_2)\}^j \left\{ \overline{A}_2'(t) \overline{C}_{1,2}(\tau_1 + t) + \overline{A}_2(t) \overline{C}_{1,2}'(\tau_1 + t) \right\} dt \end{aligned} \tag{8.7.56}$$

In addition we introduce the expansion

$$\begin{aligned} \mathcal{K}\left[\zeta; \begin{matrix} w, \tau_1, T_1 \\ \tau_2, T_2 \end{matrix}\right] :=& - \frac{(T_1 - \tau_1)^{\beta_{10}(\chi_{10}-w)-\zeta}}{\zeta - \beta_{10}(\chi_{10} - w)} C_1(T_1) e^{w B_1(T_1)} \\ & \times \int_{\tau_2}^{T_2} (t - \tau_2)^{\beta_{10}\varsigma_{10}+\zeta+\gamma_0(1,2)-1} \overline{A}_2(t) \overline{C}_{1,2}(T_1 + t) dt \end{aligned} \tag{8.7.57}$$

$$+\frac{1}{\zeta-\beta_{10}(\chi_{10}-w)}\sum_{\substack{n_1,n_2\in\{0,1\}\\ n_1+n_2=1}}\int\limits_{\tau_2}^{T_2}(t-\tau_2)^{\beta_{10}\varsigma_{10}+\zeta+\gamma_0(1,2)-1}\overline{A}_2(t)$$
$$\times\int\limits_{\tau_1}^{T_1}(s-\tau_1)^{\beta_{10}(\chi_{10}-w)-\zeta}\frac{d^{n_1}}{ds^{n_1}}\left\{e^{wB_1(s)}C_1(s)\right\}\overline{C}_{1,2}^{(n_2)}(d+t)dsdt,$$

which furnishes a substantial part of the MB-integral

$$\Sigma_1^1\begin{bmatrix}-w,\tau_1,T_1\\ \tau_2,T_2\end{bmatrix}:=\frac{1}{2\pi i}\int\limits_{q-i\infty}^{q+i\infty}\frac{\Gamma(\zeta)\Gamma(-\gamma_0(1,2)-\zeta)}{\Gamma(-\gamma_0(1,2))}\mathcal{K}\begin{bmatrix}\zeta;\ \begin{matrix}w,\tau_1,T_1\\ \tau_2,T_2\end{matrix}\end{bmatrix}d\zeta, \tag{8.7.58}$$

in which the integration path is a vertical line $\Re\zeta=q$, that satisfies

$$\begin{cases}q>\max\left\{-\beta_{10}\varsigma_{10}-\gamma_0(1,2),\beta_{10}(\chi_{10}-\Re w)\right\},\\ q<\min\left\{-\gamma_0(1,2),\beta_{10}(\chi_{10}-\Re w+\chi_{\beta_{10}}(\beta_{11},\gamma_{11},0))\right\}.\end{cases} \tag{8.7.59}$$

In terms of these quantities, for $z=0$ the w-analytic continuation of the integral transform (8.7.6) into the strip (8.7.53), according to (8.2.190), is established by

$$\begin{aligned}\mathcal{S}_0\begin{bmatrix}-w,\tau_1,T_1\\ 0,\tau_2,T_2\end{bmatrix}=-\frac{c_{10}}{\beta_{10}}\frac{\Gamma(\beta_{10}(\chi_{10}-w))\Gamma(\beta_{10}(w-\chi_{10})-\gamma_0(1,2))}{\{b_{10}\}^w\,\Gamma(-\gamma_0(1,2))\,(w-\kappa_{10})}\Phi\begin{bmatrix}w\\ \tau_1,\tau_2\end{bmatrix}\\ +\Sigma_1^1\begin{bmatrix}-w,\tau_1,T_1\\ \tau_2,T_2\end{bmatrix}.\end{aligned} \tag{8.7.60}$$

Therein it is $\mathcal{O}(w)$ as $\Im w\to\pm\infty$, uniformly with respect to $\Re w$ in any closed vertical substrip, and it shows at least one pole, which is located at $w=\kappa_{10}$. Depending on whether κ_{10} is smaller than or equal to χ_{10}, its order is $1\le J\le 2$. Hence, as $w\to\kappa_{10}$ we have

$$\mathcal{S}_0\begin{bmatrix}-w,\tau_1,T_1\\ 0,\tau_2,T_2\end{bmatrix}=\sum_{j=0}^{J}\frac{\mathrm{s}_{-j}^1\left(\chi_{10},\varsigma_{10}+\frac{\gamma_0(1,2)}{\beta_{10}}\right)}{(w-\kappa_{10})^j}+\mathcal{O}(w-\kappa_{10}), \tag{8.7.61}$$

according to Taylor's theorem, with

$$\mathrm{s}_{-j}^1\left(\chi_{10},\varsigma_{10}+\tfrac{\gamma_0(1,2)}{\beta_{10}}\right)=\frac{1}{(J-j)!}\lim_{w\to\kappa_{10}}\frac{d^{J-j}}{dw^{J-j}}(w-\kappa_{10})^J\,\mathcal{S}_0\begin{bmatrix}-w,\tau_1,T_1\\ 0,\tau_2,T_2\end{bmatrix}. \tag{8.7.62}$$

We close this paragraph with the computation of the principal part of the above Laurent expansion for different parametrizations unless $\kappa_{10}=0$, for which case we also provide the coefficient of the constant term.

8.7.4.2.1. Coefficients for $\kappa_{10} < \chi_{10}$. In these circumstances the indicated pole is of simple order and by means of (8.7.55) and (8.7.60) we find

$$\mathrm{s}^1_{-1}\left(\chi_{10}, \varsigma_{10} + \tfrac{\gamma_0(1,2)}{\beta_{10}}\right) = -\frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\beta_{10}\left\{b_{10}\right\}^{\kappa_{10}}} \frac{\Gamma(-\gamma_0(1,2) - 1 - \alpha_{20})\Gamma(1+\alpha_{20})}{\Gamma(-\gamma_0(1,2))}. \tag{8.7.63}$$

If $\kappa_{10} = 0$ or equivalently $2 + \alpha_{10} + \gamma_{10} = -\gamma_0(1,2)$, the pole lies at $w = 0$ and

$$\mathrm{s}^1_{-1}\left(\chi_{10}, -\chi_{10}\right) = -\frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\beta_{10}} \frac{\Gamma(1+\gamma_{10})\Gamma(1+\alpha_{20})}{\Gamma(2+\gamma_{10}+\alpha_{20})}. \tag{8.7.64}$$

To deduce in the case $\kappa_{10} = 0$ the coefficient for $j = 0$, we recast (8.7.60) in terms of the function (8.2.134) with $\beta_0 \equiv \beta_{10}$ and $b_0 \equiv b_{10}$, which yields

$$\mathcal{S}_0\begin{bmatrix} -w, \tau_1, T_1 \\ 0, \tau_2, T_2 \end{bmatrix} = -\frac{c_{10}}{\beta_{10}\Gamma(2+\gamma_{10}+\alpha_{20})} \frac{1}{w} e^{f(w;\chi_{10},-\varsigma_{10})} \Phi\begin{bmatrix} w \\ \tau_1, \tau_2 \end{bmatrix} + \Sigma^1_1\begin{bmatrix} -w, \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix}. \tag{8.7.65}$$

The first derivative of $f(\cdot; a, b)$ was obtained in (8.2.136). Defining

$$\sigma^1_0(\tau_1; \tau_2, T_2) := -\beta_{10}\left\{\psi(1+\gamma_{10}) - \psi(1+\alpha_{20})\right\} - \log\left\{b_{10}\right\} + \frac{\phi_1(\tau_1; \tau_2, T_2)}{\overline{a}_{20}\overline{c}_0(1,2)}, \tag{8.7.66}$$

by (8.7.62), we thus find

$$\mathrm{s}^1_0\left(\chi_{10}, -\chi_{10}\right) = -\frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\beta_{10}} \frac{\Gamma(1+\gamma_{10})\Gamma(1+\alpha_{20})}{\Gamma(2+\gamma_{10}+\alpha_{20})} \sigma^1_0(\tau_1; \tau_2, T_2) + \Sigma^1_1\begin{bmatrix} 0, \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix}. \tag{8.7.67}$$

8.7.4.2.2. Coefficients for $\kappa_{10} = \chi_{10}$. The pole at $w = \kappa_{10}$ then merges with the pole of the gamma function at $w = \chi_{10}$ to a singularity of second order. Hence, $J = 2$ and we first rearrange (8.7.60), to become

$$\begin{aligned} \mathcal{S}_0\begin{bmatrix} -w, \tau_1, T_1 \\ 0, \tau_2, T_2 \end{bmatrix} &= \frac{c_{10}}{\left\{\beta_{10}\right\}^2 \Gamma(-\gamma_0(1,2))} \frac{1}{(w-\chi_{10})^2} e^{g\left(w;\chi_{10},\chi_{10}+\frac{\gamma_0(1,2)}{\beta_{10}}\right)} \Phi\begin{bmatrix} w \\ \tau_1, \tau_2 \end{bmatrix} \\ &\quad + \Sigma^1_1\begin{bmatrix} -w, \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix}. \end{aligned} \tag{8.7.68}$$

The function $g(\cdot; 0, b)$ and its first and second derivative, with $\beta_0 \equiv \beta_{10}$ and $b_0 \equiv b_{10}$, can be found in (8.2.138), (8.2.141) and (8.2.143). In accordance with (8.7.62), for $j = 2$ it is now easy to verify, that

$$\mathrm{s}^1_{-2}\left(\chi_{10}, 0\right) = \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\left\{\beta_{10}\right\}^2 \left\{b_{10}\right\}^{\chi_{10}}}. \tag{8.7.69}$$

Moreover, for $j = 1$ we obtain

$$\mathrm{s}^1_{-1}\left(\chi_{10}, 0\right) = \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\left\{\beta_{10}\right\}^2 \left\{b_{10}\right\}^{\chi_{10}}} \zeta_{-1}(\tau_1; \tau_2, T_2), \tag{8.7.70}$$

where

$$\zeta_{-1}(\tau_1;\tau_2,T_2) := \beta_{10}\left\{\gamma+\psi(1+\alpha_{20})\right\} - \log\left\{b_{10}\right\} + \frac{\phi_1(\tau_1;\tau_2,T_2)}{\overline{a}_{20}\overline{c}_0(1,2)}. \tag{8.7.71}$$

If $\chi_{10}=0$, the above pole meets the origin of the w-plane. Concerning the coefficient for $j=0$, equations (8.7.62) and (8.7.68) yield

$$\begin{aligned}\mathrm{s}_0^1(0,0) = \frac{1}{2}\frac{\overline{a}_{20}c_{10}\overline{c}_0(1,2)}{\{\beta_{10}\}^2}\bigg\{ &\left(g\left(0;0,-\varsigma_{10}\right)\right)^2 + 2g'\left(0;0,-\varsigma_{10}\right)\frac{\phi_1(\tau_1;\tau_2,T_2)}{\overline{a}_{20}\overline{c}_0(1,2)} \\ &+ g''\left(0;0,-\varsigma_{10}\right) + \frac{\phi_2(\tau_1;\tau_2,T_2)}{\overline{a}_{20}\overline{c}_0(1,2)}\bigg\} + \Sigma_1^1\begin{bmatrix}0,\tau_1,T_1\\ \tau_2,T_2\end{bmatrix}.\end{aligned}$$

If we denote

$$\begin{aligned}\zeta_0(\tau_1;\tau_2,T_2) := {}&\left\{\log\Gamma(1+\alpha_{20})\right\}^2 + 2\left\{\beta_{10}\left(\gamma+\psi(1+\alpha_{20})\right)-\log\left\{b_{10}\right\}\right\}\frac{\phi_1(\tau_1;\tau_2,T_2)}{\overline{a}_{20}\overline{c}_0(1,2)} \\ &+ \left\{\beta_{10}\right\}^2\frac{\pi^2}{6} + \left\{\beta_{10}\right\}^2\psi'(1+\alpha_{20}) + \frac{\phi_2(\tau_1;\tau_2,T_2)}{\overline{a}_{20}\overline{c}_0(1,2)},\end{aligned} \tag{8.7.72}$$

we can write more concisely

$$\mathrm{s}_0^1(0,0) = \frac{\overline{a}_{20}c_{10}\overline{c}_0(1,2)}{2\left\{\beta_{10}\right\}^2}\zeta_0(\tau_1;\tau_2,T_2) + \Sigma_1^1\begin{bmatrix}0,\tau_1,T_1\\ \tau_2,T_2\end{bmatrix}. \tag{8.7.73}$$

8.7.4.3. Evaluation of the MB-Integral

To summarize the last two paragraphs, if we define

$$\delta := \begin{cases}\min\left\{\chi_{\beta_{10}}(\beta_{11},\gamma_{11},0), \varsigma_{10}+\frac{\gamma_0(1,2)}{\beta_{10}}\right\}, & \text{if } \chi_{10}<\kappa_{10}\wedge\gamma_0(1,2)\neq 0,\\ \min\left\{\chi_{\beta_{10}}(\beta_{11},\gamma_{11},0), \varsigma_{10}+\frac{\gamma_1(1,2)+1}{\beta_{10}}\right\}, & \text{if } \chi_{10}<\kappa_{10}\wedge\gamma_0(1,2)= 0,\\ \chi_{\beta_{10}}(\beta_{11},\gamma_{11},\alpha_{21},\gamma_1(1,2),0), & \text{if } \kappa_{10}\leq\chi_{10},\end{cases} \tag{8.7.74}$$

it has shown that the generating function of the MB-integral $\mathrm{F}(m)$, see (8.7.36), can be extended to a meromorphic function in the half plane $\Re w < \min\{\chi_{10},\kappa_{10}\}+\delta$. Therein, its analytic structure essentially depends on the ratio of the parameters χ_{10} and κ_{10}. Firstly, if $\chi_{10}<\kappa_{10}$ the only singularity in the wider region is a simple pole at $w=\chi_{10}$. Secondly, if $\kappa_{10}\leq\chi_{10}$ the analytic continuation has a pole at $z=\kappa_{10}$, which, however, is of first order if and only if $\kappa_{10}\neq\chi_{10}$. Additional poles may be found to the right of this pole, causing a coalescence in case of equality. Hence, in either case the closest singularity to the right of the integration path of $\mathrm{F}(m)$ is a pole of order no greater than three.

Upon taking into account the absolute convergence of the original integral definition, by means of (8.7.46) and (8.7.60), it can be verified that the generating function of the MB-integral (8.7.36) and its analytic continuation for $\chi_{10}<\kappa_{10}$ are $\mathcal{O}(1)$ as $\Im w\to\pm\infty$, uniformly with respect to $\Re w$ in any closed vertical substrip of their regions of validity. Conversely, in the case $\chi_{10}\geq\kappa_{10}$ the continuation is $\mathcal{O}(w)$. As a consequence, the integrand of (8.7.36) always exhibits exponential decay as $\Im w \to \pm\infty$, thereby enabling a rightward displacement of the integration path. If $\beta_{10} = 0$, we move the path, to match a line $\Re w = u_1$ that runs somewhere to the right of the simple pole at $w = 0$, i.e., $u_1 > 0$. Furthermore, if $\beta_{10} > 0$, we choose a new integration path $\Re w = u_1$ with $u_1 \equiv \min\{\chi_{10}, \kappa_{10}\} + \varepsilon$, for

$$(8.7.75) \qquad 0 < \varepsilon < \begin{cases} \delta - \chi_{10}, & \text{if } 0 \le \chi_{10} < \kappa_{10}, \\ \min\{0, \delta\} - \chi_{10}, & \text{if } \chi_{10} < \min\{0, \kappa_{10}\}, \\ \min\{\delta, \chi_{10}\} - \kappa_{10}, & \text{if } 0 \le \kappa_{10} < \chi_{10}, \\ \min\{0, \delta, \chi_{10}\} - \kappa_{10}, & \text{if } \kappa_{10} < \min\{0, \chi_{10}\}, \\ \min\left\{\chi_{10} + \frac{1}{\beta_{10}}, \delta\right\} - \kappa_{10}, & \text{if } 0 \le \kappa_{10} = \chi_{10}, \\ \min\left\{0, \chi_{10} + \frac{1}{\beta_{10}}, \delta\right\} - \kappa_{10}, & \text{if } \kappa_{10} = \chi_{10} < 0. \end{cases}$$

Then, if $\min\{\chi_{10}, \kappa_{10}\} \ge 0$, we traverse the poles at $w \in \{0, \min\{\chi_{10}, \kappa_{10}\}\}$, but merely one of them otherwise. By virtue of Theorem B.2.1(2), multiplying each residue by a negative sign, since the encountered poles are encircled in the negative direction, as $m \to \infty$ this leads to:

$$(8.7.76) \quad \begin{aligned} \mathrm{F}(m) = {} & -\mathcal{S}_0\begin{bmatrix} 0, \tau_1, T_1 \\ 0, \tau_2, T_2 \end{bmatrix} \mathbb{1}_{\{\beta_{10} = 0 \vee \min\{\chi_{10}, \kappa_{10}\} > 0\}} \\ & + \frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}} \frac{c_{10}}{\beta_{10}} \mathfrak{s}_{-1}(\tau_1; \tau_2, T_2) \mathbb{1}_{\{0 \ne \chi_{10} < \kappa_{10}\}} \\ & - \left\{ (\log(m+1) + \gamma) \frac{c_{10}}{\beta_{10}} \mathfrak{s}_{-1}(\tau_1; \tau_2, T_2) + \frac{1}{\beta_{10}} \mathfrak{s}_0 \begin{bmatrix} \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} \right\} \mathbb{1}_{\{0 = \chi_{10} < \kappa_{10}\}} \\ & + \frac{\Gamma(\kappa_{10})}{\{b_{10}(m+1)\}^{\kappa_{10}}} \frac{\Gamma(-\gamma_0(1,2) - 1 - \alpha_{20}) \Gamma(1 + \alpha_{20})}{\Gamma(-\gamma_0(1,2))} \\ & \qquad \times \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\beta_{10}} \mathbb{1}_{\{0 \ne \kappa_{10} < \chi_{10}\}} \\ & - \left\{ \log(m+1) + \gamma - \sigma_0^1(\tau_1; \tau_2, T_2) \right\} \frac{\Gamma(1 + \gamma_{10}) \Gamma(1 + \alpha_{20})}{\Gamma(2 + \gamma_{10} + \alpha_{20})} \\ & \qquad \times \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\beta_{10}} \mathbb{1}_{\{0 = \kappa_{10} < \chi_{10}\}} \\ & - \Sigma_1^1 \begin{bmatrix} 0, \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} \mathbb{1}_{\{0 = \kappa_{10} \le \chi_{10}\}} \\ & + \frac{\log(m+1) \Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}} \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\{\beta_{10}\}^2} \mathbb{1}_{\{0 \ne \kappa_{10} = \chi_{10}\}} \\ & - \frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}} \left\{ \psi(\chi_{10}) + \zeta_{-1}(\tau_1; \tau_2, T_2) \right\} \\ & \qquad \times \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\{\beta_{10}\}^2} \mathbb{1}_{\{0 \ne \kappa_{10} = \chi_{10}\}} \\ & - \left\{ (\log(m+1) + \gamma)^2 + \frac{\pi^2}{6} \right\} \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{2\{\beta_{10}\}^2} \mathbb{1}_{\{0 = \kappa_{10} = \chi_{10}\}} \end{aligned}$$

$$
\begin{aligned}
&+ (\log(m+1)+\gamma)\frac{\overline{a}_{20}c_{10}\overline{c}_0(1,2)}{\{\beta_{10}\}^2}\zeta_{-1}(\tau_1;\tau_2,T_2)\mathbb{1}_{\{0=\kappa_{10}=\chi_{10}\}} \\
&- \frac{\overline{a}_{20}c_{10}\overline{c}_0(1,2)}{2\{\beta_{10}\}^2}\zeta_0(\tau_1;\tau_2,T_2)\mathbb{1}_{\{0=\kappa_{10}=\chi_{10}\}} \\
&+ \mathcal{O}\left\{m^{-u_1}\right\}
\end{aligned}
$$

The estimate in the big-$\mathcal{O}$ is due to the absolute convergence of the remainder MB-integral.

8.7.5. A Single MB-Integral for the Residue at $z = \kappa_0(1,2) - kw$ if $\kappa_0(1,2) \neq ku_0$

The MB-integral $\mathrm{G}(m;\beta_{10})$, see (8.7.37), looks fairly distinguishing in comparison to those which we encountered in earlier sections. It is of hypergeometric type and occurs for $\kappa_0(1,2) \neq ku_0$. If $\beta_{10} = 0$, by (8.7.29), the integration path satisfies $-1 < u_0 < 0$. In these circumstances (8.7.37) is readily identified as the Cahen-Mellin integral, for which we know the finite representation

$$
\mathrm{G}(m;0) = -\frac{\Gamma(\kappa_0(1,2))}{\{b_0(1,2)(m+1)\}^{\kappa_0(1,2)}}\frac{\Gamma(1+\alpha_{20})\Gamma(1+\gamma_{10})}{\Gamma(2+\gamma_{10}+\alpha_{20})}\left\{1-e^{-b_{10}(m+1)}\right\}. \tag{8.7.77}
$$

Concerning general $\beta_{10} > 0$, subject to (8.7.29), the integration path of $\mathrm{G}(m;\beta_{10})$ is supposed to satisfy

$$
\begin{cases}
\max\{-1,\kappa_{10}\} < u_0 < \min\{0,\chi_{10}\}, & \text{if } \kappa_{10} < \min\{0,\chi_{10}\}, \\
-1 < u_0 < \min\{0,\chi_{10}\}, & \text{if } \kappa_{10} \geq \min\{0,\chi_{10}\}.
\end{cases} \tag{8.7.78}
$$

Moreover, in terms of the parameters (8.3.5) and (8.7.3), it takes on the form

$$
\begin{aligned}
\mathrm{G}(m;\beta_{10}) = \frac{\Gamma(\beta_{10}\varsigma_{10})}{2\pi i}\int_{u_0-i\infty}^{u_0+i\infty} &\frac{\Gamma(w)\Gamma(k(\kappa_{10}-w))\Gamma(\beta_{10}(\chi_{10}-w))}{\Gamma(\beta_{10}(\chi_{10}+\varsigma_{10}-w))} \\
&\times \{b_0(1,2)(m+1)\}^{k(w-\kappa_{10})}\{b_{10}(m+1)\}^{-w}\,dw.
\end{aligned} \tag{8.7.79}
$$

According to Stirling's formula, the integral converges absolutely and the integrand exhibits exponential decay in any imaginary direction of the complex w-plane. As a consequence, leftward and rightward displacements of the integration path are permitted across an arbitrary but finite number of poles. However, by inspection it is easy to see that the modulus of the integrand as $m \to \infty$ satisfies

$$
\sim \text{const} \times m^{-\kappa_0(1,2)+(k-1)\Re w}.
$$

It therefore depends on the magnitude of the parameter k, if the integrand is descending with respect to m as $\Re w$ increases or decreases. Regardless of the actual parametrization, in the special case $\beta_{10} = \beta_0(1,2)$, i.e., $k = 1$, no further computations are required. Upon denoting

$$\phi\begin{bmatrix} \alpha_{20}, \beta_{10} \\ \gamma_{10}, \gamma_0(1,2) \end{bmatrix} := \frac{\Gamma(1+\alpha_{20})}{\Gamma(\kappa_{10})} \frac{1}{2\pi i} \int\limits_{u_0-i\infty}^{u_0+i\infty} \left\{ \frac{b_{10}}{b_0(1,2)} \right\}^{-w} \times \frac{\Gamma(w)\Gamma(\kappa_{10}-w)\Gamma(1+\gamma_{10}-\beta_{10}w)}{\Gamma(2+\gamma_{10}+\alpha_{20}-\beta_{10}w)} dw, \tag{8.7.80}$$

we then immediately obtain

$$\mathrm{G}(m;\beta_0(1,2)) = \frac{\Gamma(\kappa_{10})}{\{b_0(1,2)(m+1)\}^{\kappa_{10}}} \phi\begin{bmatrix} \alpha_{20}, \beta_{10} \\ \gamma_{10}, \gamma_0(1,2) \end{bmatrix}. \tag{8.7.81}$$

Besides a distinction between values of k being smaller or greater than one, in view of (8.7.78) an additional distinction with respect to the range of χ_{10} and κ_{10} is necessary. Once we have specified all these quantities plus the remaining parameters, the exact meromorphic structure of the integrand in (8.7.79) is obvious. A complete asymptotic expansion as $m \to \infty$ of this hypergeometric integral then can be deduced by displacing the integration path towards the appropriate direction of the complex plane. The resulting expansion features descending powers of the asymptotic parameter up to arbitrary large order. However, due to the many possible parametrizations, a more detailed discussion will be omitted.

8.7.6. A Single MB-Integral for the Residue at $z = 0$ if $\beta_{10} = 0$ and $\kappa_0(1,2) = 0$

It is easy to see from (8.7.40) that the integrals $\mathrm{Z}_j(m)$, which for $j \in \{1,2\}$ were given in (8.7.38) and (8.7.39), are only relevant if $\beta_{10} = \kappa_0(1,2) = 0$. Hence, according to (8.7.29), their integration path is a vertical line $\Re w = u_0$ with

$$-1 < u_0 < 0.$$

The evaluation of both integrals is actually very simple. Regarding $\mathrm{Z}_1(m)$ we may easily rely on the Mellin inversion theorem, in view of which it is readily identified as the Cahen-Mellin integral. Taking into account that the path runs to the left of $w = 0$ but to the right of the point $w = -1$, for any $m > 0$ we conclude

$$\mathrm{Z}_1(m) = -1 + e^{-b_{10}(m+1)}. \tag{8.7.82}$$

In order to compute the MB-integral $\mathrm{Z}_2(m)$, we must first ascertain analyticity properties and asymptotic behaviour of its generating function, which is for $z = 0$ given by the integral (8.7.18). This is in turn again a MB-integral whose integration path $\Re\zeta = q$ was specified in (8.7.19).

With $\beta_{10} = 0$, i.e., $k = 0$, and also with $z = 0$, these conditions take on the form

$$\begin{cases} q > \max\left\{-1 - \alpha_{20} - \gamma_0(1,2), \beta_0(1,2)\chi_0(1,2)\right\}, \\ q < \min\left\{\beta_0(1,2)(\chi_0(1,2) + \chi_{\beta_0(1,2)}(\beta_{11}, \gamma_{11}, 0)), -\gamma_0(1,2)\right\}. \end{cases}$$

The path does therefore not depend on the argument w. Moreover, by inspection of the integral (8.7.17) with $z = 0$ and $\zeta = q + iy$ for $y \in \mathbb{R}$, we identify the latter as an entire function of w. By virtue of simple estimates one can thus easily confirm the entireness of the whole MB-integral (8.7.18) with respect to w for $z = 0$ in the special case $\beta_{10} = 0$. Finally, additional estimates can be employed to expose that the integral is $\mathcal{O}(w)$ as $\Im w \to \pm\infty$, uniformly with respect to $\Re w$ in any closed vertical vertical substrip of the complex w-plane. Consequently, in the integral (8.7.39) it is permitted to move the integration path by an arbitrary but finite distance to the right. We decide to perform a displacement of the path across the simple pole at the origin. Incorporating the fact that the pole is encircled clockwisely, for an arbitrary $u_1 > 0$, this leads to

$$\text{(8.7.83)} \qquad \mathrm{Z}_2(m) = -\Sigma_1^2\begin{bmatrix}0, \tau_1, T_1 \\ 0, \tau_2, T_2\end{bmatrix} + \frac{1}{2\pi i}\int\limits_{u_1 - i\infty}^{u_1 + i\infty} (m+1)^{-w}\Gamma(w)\Sigma_1^2\begin{bmatrix}-w, \tau_1, T_1 \\ 0, \tau_2, T_2\end{bmatrix} dw.$$

Appealing to absolute convergence and arbitrariness of $u_1 > 0$, we infer an exponentially small order as $m \to \infty$ of the remainder on the right hand side.

8.7.7. Evaluation of the Iterated MB-Integral

We conclude this section with a collection of m-asymptotic statements on the integral (8.7.1) for special parametrizations, continuing with the expansion (8.7.40). The case of non-zero $\beta_0(1,2) \neq \beta_{10}$ requires further investigations and will be omitted. Indeed, it then needs to be shown that the remainder term in the expansion is actually negligible.

First, $\beta_0(1,2) > 0$ with $\beta_{10} = 0$ implies $k = 0$ and for appropriately specified u_0 and x_1 we can then find $\varepsilon_1 > 0$, for which the exponent in the big-$\mathcal{O}$ in (8.7.40) satisfies $u_0 + x_1 = \kappa_0(1,2) + \varepsilon_1$. Furthermore, in the expansion (8.7.76) each but the first and the remainder term equals zero, of which the latter asymptotically decays faster than any power of m. By (8.7.77), (8.7.82) and (8.7.83), we therefore arrive at the first theorem of this section.

Theorem 8.7.1. *For $\beta_0(1,2) > 0$ and $\beta_{10} = 0$, suppose validity of the conditions (A10) to (A13). As $m \to \infty$ we then have*

$$\begin{aligned} \mathrm{Ai}\left[m; \begin{matrix}\tau_1, T_1 \\ \tau_2, T_2\end{matrix}\right] \sim \mathcal{S}_0\begin{bmatrix}0, \tau_1, T_1 \\ 0, \tau_2, T_2\end{bmatrix}&\mathbb{1}_{\{\kappa_0(1,2) > 0\}} \\ &- \frac{\Gamma(\kappa_0(1,2))}{\{b_0(1,2)(m+1)\}^{\kappa_0(1,2)}}\frac{\Gamma(1+\alpha_{20})\Gamma(1+\gamma_{10})}{\Gamma(2+\gamma_{10}+\alpha_{20})}\frac{\overline{a}_{20}c_{10}\overline{c}_0(1,2)}{\beta_0(1,2)}\mathbb{1}_{\{\kappa_0(1,2) \neq 0\}} \\ &+ (\log(m+1) + \gamma - \sigma_0^2(\tau_1; \tau_2, T_2))\frac{\Gamma(1+\alpha_{20})\Gamma(1+\gamma_{10})}{\Gamma(2+\alpha_{20}+\gamma_{10})} \end{aligned}$$

$$\times \frac{\overline{a}_{20}c_{10}\overline{c}_0(1,2)}{\beta_0(1,2)}\mathbb{1}_{\{\kappa_0(1,2)=0\}}$$
$$+ \Sigma_1^2\begin{bmatrix}0,\tau_1,T_1\\0,\tau_2,T_2\end{bmatrix}\mathbb{1}_{\{\kappa_0(1,2)=0\}},$$

where the constants were defined in (8.7.18) and (8.7.25). Observe that, by assumption, the right hand side can never be completely zero.

Conversely, if $\beta_0(1,2) = 0$, in (8.7.40) only the first term is non-zero, and as $m \to \infty$ the estimate in the big-$\mathcal{O}$ vanishes faster than any power of m. Furthermore, for any $\beta_{10} \geq 0$, in the expansion (8.7.76) the remainder term is obviously of higher algebraic order than each of the preceding non-zero terms. We combine this result with the next case.

If $\beta_0(1,2) = \beta_{10} > 0$, i.e., $k = 1$, then $\kappa_0(1,2) = \kappa_{10}$ and the third and fourth term in (8.7.40) certainly will be zero. The exponent in the big-$\mathcal{O}$, due to cancellations, for any admissible u_0 and x_1 and for suitable $\varepsilon_1 > 0$ then satisfies

$$u_0 + x_1 = \kappa_0(1,2) + \varepsilon_1.$$

According to (8.7.29), we distinguish whether the ratio of κ_{10} and $\min\{0,\chi_{10}\}$ is smaller than one or not. If $\kappa_{10} \geq \min\{0,\chi_{10}\}$, the first term in (8.7.40) is non-zero and we must also specify the ratio of κ_{10} and χ_{10}. On the one hand, if $\kappa_{10} \geq \min\{0,\chi_{10}\}$ and $\kappa_{10} > \chi_{10}$, the non-zero terms in (8.7.40) are negligible in comparison to the non-zero terms of the expansion (8.7.76). On the other hand, if $\chi_{10} \geq \kappa_{10} \geq \min\{0,\chi_{10}\}$ additional contributions come from (8.7.76) and (8.7.81). This leads us to the next theorem.

Theorem 8.7.2. *Assume validity of the conditions (A10) to (A13), with either* $\beta_0(1,2) = 0$ *or*

$$\begin{cases}\beta_0(1,2) = \beta_{10} > 0,\\ \kappa_{10} \geq \min\{0,\chi_{10}\}.\end{cases}$$

Then, provided at least one term on the right hand side is non-zero, as $m \to \infty$,

(1) if $\beta_{10} = 0$ *or* $\kappa_{10} > \chi_{10}$*, we have*

$$\begin{aligned}\mathrm{Ai}\begin{bmatrix}m;\begin{matrix}\tau_1,T_1\\\tau_2,T_2\end{matrix}\end{bmatrix} \sim{}& \mathcal{S}_0\begin{bmatrix}0,\tau_1,T_1\\0,\tau_2,T_2\end{bmatrix}\mathbb{1}_{\{\beta_{10}=0\,\vee\,\chi_{10}>0\}}\\ &- \frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}}\frac{c_{10}}{\beta_{10}}\mathfrak{s}_{-1}(\tau_1;\tau_2,T_2)\mathbb{1}_{\{\chi_{10}\neq 0\}}\\ &+ \left\{(\log(m+1)+\gamma)\frac{c_{10}}{\beta_{10}}\mathfrak{s}_{-1}(\tau_1;\tau_2,T_2) + \frac{1}{\beta_{10}}\mathfrak{s}_0\begin{bmatrix}\tau_1,T_1\\\tau_2,T_2\end{bmatrix}\right\}\mathbb{1}_{\{\chi_{10}=0\}}.\end{aligned}$$

(2) if $\chi_{10} > \kappa_{10}$*, we have*

$$\mathrm{Ai}\begin{bmatrix}m;\begin{matrix}\tau_1,T_1\\\tau_2,T_2\end{matrix}\end{bmatrix} \sim \mathcal{S}_0\begin{bmatrix}0,\tau_1,T_1\\0,\tau_2,T_2\end{bmatrix}\mathbb{1}_{\{\kappa_{10}>0\}}$$

$$
\begin{aligned}
&- \frac{\Gamma(\kappa_{10})}{\{b_{10}(m+1)\}^{\kappa_{10}}} \frac{\Gamma(-\gamma_0(1,2)-1-\alpha_{20})\Gamma(1+\alpha_{20})}{\Gamma(-\gamma_0(1,2))} \\
&\qquad\qquad\qquad\qquad \times \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\beta_{10}} \mathbb{1}_{\{\kappa_{10} \neq 0\}} \\
&+ \left\{ \log(m+1) + \gamma - \sigma_0^1(\tau_1; \tau_2, T_2) \right\} \frac{\Gamma(1+\gamma_{10})\Gamma(1+\alpha_{20})}{\Gamma(2+\gamma_{10}+\alpha_{20})} \\
&\qquad\qquad\qquad\qquad \times \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\beta_{10}} \mathbb{1}_{\{\kappa_{10} = 0\}} \\
&+ \Sigma_1^1 \begin{bmatrix} 0, \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} \mathbb{1}_{\{\kappa_{10} = 0\}} \\
&+ \frac{\Gamma(\kappa_{10})}{\{b_0(1,2)(m+1)\}^{\kappa_{10}}} \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\beta_{10}} \phi \begin{bmatrix} \alpha_{20}, \beta_{10} \\ \gamma_{10}, \gamma_0(1,2) \end{bmatrix} \mathbb{1}_{\{\beta_0(1,2) > 0\}}.
\end{aligned}
$$

(3) if $\kappa_{10} = \chi_{10}$, we have

$$
\begin{aligned}
\mathrm{Ai} \begin{bmatrix} m; \begin{matrix} \tau_1, T_1 \\ \tau_2, T_2 \end{matrix} \end{bmatrix} &\sim \mathcal{S}_0 \begin{bmatrix} 0, \tau_1, T_1 \\ 0, \tau_2, T_2 \end{bmatrix} \mathbb{1}_{\{\chi_{10} > 0\}} \\
&- \frac{\log(m+1)\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}} \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\{\beta_{10}\}^2} \mathbb{1}_{\{\chi_{10} \neq 0\}} \\
&+ \frac{\Gamma(\chi_{10})}{\{b_{10}(m+1)\}^{\chi_{10}}} \{\psi(\chi_{10}) + \zeta_{-1}(\tau_1; \tau_2, T_2)\} \\
&\qquad\qquad\qquad\qquad \times \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\{\beta_{10}\}^2} \mathbb{1}_{\{\chi_{10} \neq 0\}} \\
&+ \left\{ (\log(m+1) + \gamma)^2 + \frac{\pi^2}{6} \right\} \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{2\{\beta_{10}\}^2} \mathbb{1}_{\{\chi_{10} = 0\}} \\
&- (\log(m+1) + \gamma) \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\{\beta_{10}\}^2} \zeta_{-1}(\tau_1; \tau_2, T_2) \mathbb{1}_{\{\chi_{10} = 0\}} \\
&+ \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{2\{\beta_{10}\}^2} \zeta_0(\tau_1; \tau_2, T_2) \mathbb{1}_{\{\chi_{10} = 0\}} \\
&+ \Sigma_1^1 \begin{bmatrix} 0, \tau_1, T_1 \\ \tau_2, T_2 \end{bmatrix} \mathbb{1}_{\{\chi_{10} = 0\}} \\
&+ \frac{\Gamma(\chi_{10})}{\{b_0(1,2)(m+1)\}^{\chi_{10}}} \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\beta_{10}} \phi \begin{bmatrix} \alpha_{20}, \beta_{10} \\ \gamma_{10}, \gamma_0(1,2) \end{bmatrix} \mathbb{1}_{\{\beta_0(1,2) > 0\}}.
\end{aligned}
$$

The coefficients of the above expansions can be found in (8.7.49), (8.7.51), (8.7.58), (8.7.66), (8.7.71), (8.7.72) and (8.7.80).

If $\kappa_{10} < \min\{0, \chi_{10}\}$, also the first term in (8.7.40) vanishes, and (8.7.81) immediately yields our final theorem of this chapter.

Theorem 8.7.3. *For $\beta_0(1,2) = \beta_{10} > 0$ and $\kappa_{10} < \min\{0, \chi_{10}\}$, assume validity of the conditions (A10) to (A13). As $m \to \infty$ we then have*

$$
\mathrm{Ai} \begin{bmatrix} m; \begin{matrix} \tau_1, T_1 \\ \tau_2, T_2 \end{matrix} \end{bmatrix} \sim \frac{\Gamma(\kappa_{10})}{\{b_0(1,2)(m+1)\}^{\kappa_{10}}} \frac{\overline{a}_{20} c_{10} \overline{c}_0(1,2)}{\beta_{10}} \phi \begin{bmatrix} \alpha_{20}, \beta_{10} \\ \gamma_{10}, \gamma_0(1,2) \end{bmatrix},
$$

provided the integral $\phi[\ldots]$, which was defined in (8.7.80), is non-zero.

8.8. Conclusion

Our findings from this chapter exposed a large diversity with respect to the asymptotic behaviour of the integrals (8.0.1) and (8.0.2) for convolution-type amplitude functions. However, despite the different shape of these integrals, their dominating terms turned out to be fairly similar in some special cases. Moreover, for each integral parallels between different kinds of paths can be drawn. This was not obvious in the beginning but it transpired in the process of our investigations. As a consequence, contrary to our initial doubts, it should indeed be possible to confine alternatively to a study of each iterated integral for two finite paths. Then, in order to enable a reference to further scenarios, one must consider amplitude functions of another type that depend on s and on t but not necessarily on $s + t$. It is, however, not clear if such a generalization is actually less elaborate than our approach, since additional difficulties may occur that are overlooked at the first glance.

The advantage of our approach certainly consists in a straightforward applicability of the obtained results, which is in accordance with our intention. In fact, the above formulae immediately can be employed to assess the m-asymptotic behaviour of the variance integrals in the deconvolution problem, no matter if the characteristic functions of X and ε have a finite support, finite zeros or algebraic decay at infinity. Even distributions F with an exponential-type characteristic function are admitted.

A. Integral Transforms

In this chapter we are concerned with the basic properties of some important integral transforms, which arise as special cases of the Laplace-type integral

$$I(t) := \int_{\mathcal{P}} e^{tp(x)} p'(x) F(dx), \qquad t \in D, \tag{A.0.1}$$

where $\mathcal{P}$ is a segment of the real axis, $p(x)$ equals the *phase function* with derivative $p'(x)$, $F : \mathbb{R} \to \mathbb{C}$ is the *determining function* and $D \subset \mathbb{C}$ is a subset of the complex plane. For a fixed determining function the choice of the phase substantially affects the structure of the *kernel* $e^{tp(x)}$ and thus of the resulting transform. Particularly important phases are $p(x) = x$ and $p(x) = \log(x)$ with the respective integration paths $\mathcal{P}$ being the whole real axis or merely its positive segment. Many applications confine to determining functions of the form $F(dx) = f(x)dx$. Then f is referred to as the *amplitude function*. Moreover, the integral transform $I(t)$ is in some texts denoted as the *generating function*, a notion we will adopt. Compare for instance p. 37 in [Widder, 1946] where an explanation for this terminology is provided.

Many monographs are dedicated to integral transforms, especially to Fourier transforms, mostly in a real-valued setting. Among those [Körner, 1988] and [Pinsky, 2002] are particularly well-written. It is, however, worthwile not to confine to real-valuedness but to extend integral transforms to complex variables, as the textbooks of [Titchmarsh, 1937] and [Widder, 1946] show. Elementary knowledge on complex calculus can be acquired by [Asmar and Grafakos, 2018], [Wegert, 2012], [Fischer and Lieb, 2005] and [Behnke and Sommer, 1965].

A.1. The Fourier Transform of One Real Variable

We start our discussion by considering the generalized transform (A.0.1) with the phase $p(x) = x$ integrated along $\mathcal{P} = \mathbb{R}$ and with a purely imaginary argument. In addition we assume $F(dx) = f(x)dx$. This leads us to the definition of the most frequently occuring integral transform, which is the *Fourier transform*. It is for $t \in \mathbb{R}$ given by

$$\mathcal{F}\{f\}(t) := \int_{-\infty}^{\infty} e^{itx} f(x) dx. \tag{A.1.1}$$

Note that in the literature this definition very often differs by constants. Rather than by $\mathcal{F}$ we shall occasionally denote (A.1.1) by the greek letter Φ, sometimes with an additional index. The

Fourier transform generally results in a complex-valued function, except for even functions, i.e., if $f(x) = f(-x)$, in which case the transform is real-valued. Furthermore, as a consequence of the fact that the modulus of e^{ix} equals one for all $x \in \mathbb{R}$, the assumption $f \in L^1(\mathbb{R})$ is sufficient to guarantee absolute convergence of the integral (A.1.1). Then, many general statements are possible. One of the stronger kind is verified by a simple estimate with fixed $\delta \in \mathbb{R}$:

$$\begin{aligned}\sup_{t\in\mathbb{R}} \left|\mathcal{F}\{f\}(t+\delta) - \mathcal{F}\{f\}(t)\right| &\leq \sup_{t\in\mathbb{R}} \int\limits_{-\infty}^{\infty} \left|e^{ix\delta} - 1\right| \left|e^{ixt}\right| |f(x)|\, dx \\ &= \int\limits_{-\infty}^{\infty} \left|e^{ix\delta} - 1\right| |f(x)|\, dx\end{aligned}$$

Letting $\delta \to 0$, by Lebesgue's dominated convergence theorem, the limit equals zero, from which we conclude uniform continuity of the Fourier transform of any function $f \in L^1(\mathbb{R})$. Similarly we can show that $\mathcal{F}\{f\}(\cdot)$ is uniformly bounded along the entire real axis, formally

$$\|\mathcal{F}\{f\}(\cdot)\|_\infty \leq \|f\|_1 . \tag{A.1.2}$$

Little more effort is required to verify the convolution property of the Fourier transform. For $f_1, f_2 \in L^1(\mathbb{R})$ the *convolution*, also known as the *Faltung*, is defined as

$$f_1 * f_2 := \int_{-\infty}^{\infty} f_1(\cdot - x) f_2(x) dx. \tag{A.1.3}$$

Note that $f_1 * f_2 \in L^1(\mathbb{R})$. Integrals of this type in general have a very complicated structure, which simplifies in the Fourier domain. Indeed, subject to the functional equation of the exponential function and the translation invariance of the Lebesgue measure, we have

$$\mathcal{F}\{f_1 * f_2\}(t) = \mathcal{F}\{f_1\}(t)\, \mathcal{F}\{f_2\}(t), \tag{A.1.4}$$

i.e., the convolution becomes a product. Conversely, the amplitude function corresponding to a product of two Fourier transforms is given by the convolution of the respective amplitude functions. Finally, although the integrand in (A.1.1) does not decay as $t \to \pm\infty$, under certain conditions, integrals of Fourier-type vanish as their argument runs along the real axis to infinity. This leads us to a well-known statement that may not be missing in any treatment of Fourier analysis, compare for instance Theorem 4.1 in ch. 3 in [Olver, 1974] and Theorem 2.2.4 in [Pinsky, 2002]. For this we agree, a function is said to have a jump point at some $x_0 \in \mathbb{R}$, if the left-sided and right-sided limits of $f(x)$ at this point exist but these limits do not match. Furthermore, a point x_0 is referred to as a removable discontinuity of f, if the left-sided and right-sided limits of the function at x_0 match but do not coincide with the value $f(x_0)$, which is possibly even undefined.

Lemma A.1.1 (Riemann-Lebesgue). *Assume for a function $f : I \mapsto \mathbb{C}$ one of the following* *conditions holds:*

(1) The interval $I = [a, b]$ with $a, b \in \mathbb{R}$ is compact and f is piecewise continuous with a finite number of removable discontinuities or jump points there.

(2) $I = (a, b)$ with $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{\infty\}$ and f is continuous on I with the exception of a finite number of points. Moreover, $\int_a^b e^{itx} f(x)dx$ converges uniformly for all $t \geq t_0$.

(3) $I = (a, b)$ with $b = -a = \infty$ and $f \in L^1(\mathbb{R})$.

Then

$$\lim_{t \to \pm\infty} \int_a^b e^{itx} f(x)dx = 0. \tag{A.1.5}$$

The Riemann-Lebesgue lemma does not provide a quantitative statement on the rate of decay of a Fourier transform. In fact this is determined by the detailed properties of the amplitude function and can be arbitrarily slow or fast. On the other hand we can roughly characterize the decay in terms of derivatives from integration by parts. Therefore it is necessary to split the range of integration according to the intervals where f is differentiable. A special case occurs if we have differentiability along the entire real line. Then Proposition 2.2.5 in [Pinsky, 2002] applies.

Theorem A.1.1 (rate of decay). *Suppose $f(x)$ is N-times differentiable at any $x \in \mathbb{R}$ and $f^{(n)} \in L^1(\mathbb{R})$ for any $1 \leq n \leq N$, then the n-th derivative of f possesses the Fourier transform $(-it)^n \mathcal{F}\{f\}(t)$ and we have $\mathcal{F}\{f\}(t) = o\left\{t^{-N}\right\}$ as $t \to \pm\infty$.*

Next we present a few selected examples for functions and their Fourier-counterparts.

Example A.1.2 (indicator). Of special importance in Fourier analysis are indicator functions $\mathbb{I}\{a \leq \cdot \leq b\}$ for finite $a < b$. For instance, with $t \in \mathbb{R}$ we have:

$$\begin{aligned} \Phi_{a,b}(t) :=& \frac{1}{b-a} \int_{-\infty}^{\infty} e^{itx} \mathbb{I}\{a \leq x \leq b\} dx \\ =& \frac{e^{itb} - e^{ita}}{(b-a)it} \\ =& e^{it\frac{b+a}{2}} \frac{2 \sin\left\{\frac{b-a}{2} t\right\}}{(b-a)t} \end{aligned} \tag{A.1.6}$$

From a stochastic point of view, $\Phi_{a,b}(t)$ clearly constitutes the characteristic function of the uniform distribution on $[a, b]$. Choosing $b = -a = 1$, we arrive at the *sinc function*

$$\operatorname{si}(t) := \frac{\sin(t)}{t}. \tag{A.1.7}$$

According to the power series expansion of the sine, the singularity of the sinc function at $t = 0$ is removable, making $\mathrm{si}(t)$ an entire function of $t \in \mathbb{C}$. Moreover, it is even with respect to $t \in \mathbb{R}$ and attains its maximal value at $t = 0$. Although $\mathrm{si}(t)$ is not absolutely integrable along the whole real axis, we will show in Appendix B.1 that the primitive integral

$$\mathrm{Si}(\xi) := \int_0^{\xi} \mathrm{si}(t)dt, \tag{A.1.8}$$

referred to as the *sine integral*, has some fairly nice properties.

The computation of the preceding example only required elementary manipulations. In some cases, however, the tools of real analysis are insufficient. The Fourier transform of f may then only be computable by means of complex analysis.

Example A.1.3 (Gauss). For $\sigma > 0$ the amplitude function

$$\varphi(x) := \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \tag{A.1.9}$$

is known as the centered Gaussian density with variance σ^2. The corresponding Fourier transform can be derived by a simple application of the binomial theorem:

$$\begin{aligned}
\mathcal{F}\{\varphi\}(t) &= \frac{e^{-\frac{\sigma^2}{2}t^2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}\sigma} - i\frac{\sigma}{\sqrt{2}}t\right)^2} dx \\
&= \frac{1}{\sqrt{\pi}} e^{-\frac{\sigma^2}{2}t^2} \int_{-\infty - i\frac{\sigma}{\sqrt{2}}t}^{\infty - i\frac{\sigma}{\sqrt{2}}t} e^{-u^2} du \\
&= \frac{1}{\sqrt{\pi}} e^{-\frac{\sigma^2}{2}t^2} \int_{-\infty}^{\infty} e^{-u^2} du
\end{aligned}$$

The last equation is a consequence of Cauchy's theorem and the decay of e^{-z^2} as $\Re z \to \pm\infty$ in $\mathbb{C}$ for any fixed $\Im z \in \mathbb{R}$. Taking into account that the integral in the last equation has the numerical value $\sqrt{\pi}$, which can be verified by referring the integral to the gamma function and applying the identity (B.2.15), we eventually arrive at

$$\mathcal{F}\{\varphi\}(t) = e^{-\frac{\sigma^2}{2}t^2}. \tag{A.1.10}$$

For $\sigma = 1$ the preceding equality (A.1.10) reveals that φ is an eigenfunction of the Fourier transform considered as an operator, compare §2.4.4 in [Pinsky, 2002].

Example A.1.4 (Fejér's kernel). Another function of special importance in Fourier analysis is *Fejér's kernel*, defined by

$$\mathcal{K}(x) := \frac{1-\cos(x)}{\pi x^2}. \tag{A.1.11}$$

Evidently, $\mathcal{K} \in L^1(\mathbb{R})$ and the function is non-negative and even. The power series expansion of the cosine shows the removability of the singularity at the origin by $\mathcal{K}(0) = 1$, making (A.1.11) an entire function with

$$\lim_{M\to\infty} M\mathcal{K}(Mx) = \begin{cases} 0, & \text{if } x \in \mathbb{R}\setminus\{0\}, \\ \infty, & \text{if } x = 0. \end{cases}$$

In order to derive the associated Fourier transform, it suffices to confine to the case $t > 0$. By definition of the cosine

$$e^{itx}\{1-\cos(x)\} = \frac{2e^{itx} - e^{i(1+t)x} - 1}{2} + \frac{1-e^{i(t-1)x}}{2}.$$

We then compute each of the following two integrals separately by means of complex integration:

$$\begin{aligned} \Phi_{\mathcal{K}}(t) &:= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{itx}\frac{1-\cos(x)}{x^2}dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2e^{itx}-e^{i(1+t)x}-1}{x^2}dx + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-e^{i(t-1)x}}{x^2}dx \end{aligned} \tag{A.1.12}$$

The procedure is similar to the calculation of the sine integral, compare Appendix B.1, and we therefore confine to a brief overview. For each of the two integrals we first consider an integration contour that has the shape of a half annulus, located either in the upper or in the lower half plane with edges running along the real axis. The half annulus is traversed in the counterclockwise direction and is supposed to be symmetric with respect to the imaginary axis through the origin. Its inner and outer radii are respectively denoted by $0 < r < R < \infty$. Since the integrand in the first integral for $t > 0$ exhibits exponential growth as $\Im x \to -\infty$ and algebraic decay as $\Im x \to +\infty$, the half annulus shall be located in the upper x-half plane. Letting $R \to \infty$ the contribution from the large arc vanishes and it remains the contribution from the segment along the real axis and along the small arc γ_r. More precisely, from Cauchy's theorem we obtain:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{2e^{itx}-e^{i(1+t)x}-1}{x^2}dx &= -\lim_{r\to 0}\int_{\gamma_r}\frac{2e^{itz}-e^{i(1+t)z}-1}{z^2}dz \\ &= -\lim_{r\to 0}\sum_{j=1}^{\infty}\frac{i^j(2t^j-(1+t)^j)}{j!}\int_{\gamma_r} z^{j-2}dz \\ &= \pi(1-t) \end{aligned}$$

Finally, in the last integral in (A.1.12) we must distinguish for $t > 0$ between different signs of the argument of the complex exponential function. In particular, if on the one hand $t > 1$ the integration path shall match the half annulus that was constructed for the first integral. If on the other hand $0 < t < 1$ we choose the same integration path but reflected with respect to the real axis. In each case we obtain

$$\int_{-\infty}^{\infty} \frac{1 - e^{i(t-1)x}}{x^2} dx = \begin{cases} -\pi(1-t), & \text{if } t > 1, \\ \pi(1-t), & \text{if } 0 < t < 1. \end{cases}$$

By evenness we have thus shown

$$\Phi_{\mathcal{K}}(t) = \begin{cases} 0, & \text{if } |t| > 1, \\ 1 - |t|, & \text{if } 0 < |t| < 1. \end{cases}$$

Moreover, since the above function is a Fourier transform it is uniformly continuous, whence $\Phi_{\mathcal{K}}(0) = 1$ and $\Phi_{\mathcal{K}}(1) = 0$. This finally shows that

$$\Phi_{\mathcal{K}}(t) = (1 - |t|)\mathbb{I}\{-1 < t < 1\}, \qquad t \in \mathbb{R}. \tag{A.1.13}$$

We close our short list of Fourier examples with the observation that we evidently confined to Fourier pairs that can be expressed in terms of elementary functions, i.e., in terms of monomials and exponential functions. In many circumstances, however, for the Fourier transform associated with a given amplitude only integral or series representations are available. A simple example known from probability theory is furnished by the beta distribution.

The popularity of Fourier transforms is not solely justified by the convolution property but especially by its uniqueness and invertibility, which enables the unique reconstruction of the amplitude function under certain assumptions. This is the topic of the next section.

A.1.1. Inversion Formulae

There is a large amount of inversion formulae for Fourier transforms that mainly differ with respect to the conditions imposed on the amplitude function. Basically these formulae can be distinguished between two types of which, for the sake of clarity, we picked only a few to present below. In addition we will discuss the connection between these types. First, analogous to Fourier series, for $f \in L^1(\mathbb{R})$ with Fourier transform $\mathcal{F}\{f\}(t)$, $M > 0$ and $\xi \in \mathbb{R}$, we define the *partial sum operator* by

$$S_M f(\xi) := \frac{1}{2\pi} \int_{-M}^{M} e^{-it\xi} \mathcal{F}\{f\}(t)\, dt. \tag{A.1.14}$$

According to the uniform boundedness of $\mathcal{F}\{f\}(t)$, this integral is absolutely convergent for any $M > 0$. By Fubini's theorem we obtain:

$$S_M f(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-M}^{M} e^{-it(\xi-x)} dt f(x) dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(M(\xi - x))}{\xi - x} f(x) dx \tag{A.1.15}$$

Hence, the partial sum operator equals the convolution of f with the sinc function. Regarding its convergence behaviour, we can for instance make the following statement, see, e.g., Theorem 2.3.7 in [Pinsky, 2002].

Theorem A.1.2 (a convergence test for the partial sum operator). *If $f \in L^1(\mathbb{R})$ is of finite total variation on the real axis, then $\lim_{M\to\infty} S_M f(\xi) = \frac{f(\xi+)+f(\xi-)}{2}$ for all $\xi \in \mathbb{R}$. This is especially true if f is once differentiable with $f' \in L^1(\mathbb{R})$.*

Clearly, the convergence of the partial sum operator (A.1.14) is equivalent to the existence of a certain Cauchy-type principal value integral.

Proof. Since the sinc function is even, (A.1.15) has the following form:

$$S_M f(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(M(\xi - x))}{\xi - x} f(x) dx$$

$$= \frac{1}{\pi} \int_{0}^{\infty} (f(\xi + x) + f(\xi - x)) \frac{\sin(Mx)}{x} dx$$

By assumption $\tilde{f}(x) := f(\xi+x)+f(\xi-x)$ is of finite total variation on $\mathbb{R}$. Therefore, integration by parts for fixed $R > r > 0$ yields

$$\int_r^R \tilde{f}(x) \frac{\sin(Mx)}{x} dx = \left[\tilde{f}(x) \int_r^x \frac{\sin(Mt)}{t} dt \right]_r^R - \int_r^R \int_r^x \frac{\sin(Mt)}{t} dt \tilde{f}(dx).$$

The primitive integral appearing therein is readily identified as the sine integral (A.1.8). According to (B.1.6) and since $f \in L^1(\mathbb{R})$, in the limits $r \downarrow 0$ and $R \to \infty$ the first of the above two summands vanishes. Moreover, since $\tilde{f}$ is of finite total variation, in the second summand, by Lebesgue's dominated convergence theorem, we may interchange the order of limit and integration. This leads to

$$S_M f(\xi) = -\frac{1}{\pi} \int_{(0,\infty)} \mathrm{Si}(Mx) \tilde{f}(dx).$$

Another application of Lebesgue's dominated convergence theorem eventually shows

$$\lim_{M\to\infty} S_M f(\xi) = -\frac{1}{2}\left[\lim_{R\to\infty}\tilde{f}(R) - \lim_{r\downarrow 0}\tilde{f}(r)\right] = \frac{\tilde{f}(0-)}{2},$$

which finishes the proof. ∎

In the previous proof, the symmetry of the partial sum operator with respect to the limit $M > 0$ is essential as the following example from §2.3.2 in [Pinsky, 2002] shows.

Example A.1.5 (asymmetric partial sum operator). Similar to (A.1.14), for $M, N > 0$ and $\xi \in \mathbb{R}$ the asymmetric partial sum operator is given by

$$S_{M,N} f(\xi) := \frac{1}{2\pi}\int_{-N}^{M} e^{-i\xi t}\mathcal{F}\{f\}(t)\,dt.$$

To illustrate the drawbacks of this definition in comparison with its symmetric counterpart, we assume $f = \mathbb{I}\{a < \cdot < b\}$ for finite real numbers $a < b$. Then $\mathcal{F}\{f\}(t) = (b-a)\Phi_{a,b}(t)$, according to (A.1.6), and for the integral along the segment $[0, M]$ we obtain:

$$\int_0^M \frac{e^{it(b-\xi)} - e^{it(a-\xi)}}{it}dt = \int_0^M \frac{\sin(t(b-\xi)) - \sin(t(a-\xi))}{t}dt - i\int_0^M \frac{\cos(t(b-\xi)) - \cos(t(a-\xi))}{t}dt$$

The real part of this expression is readily identified as the sum of two sine integrals and thus converges as $M \to \infty$ for any $\xi \in \mathbb{R}$. On the other hand we immediately note that the imaginary part for $\xi \in \{a, b\}$ can not converge, since the integrand as $t \to \infty$ is then non-oscillatory and slowly decreasing. To verify this, we suppose $\xi \notin \{a, b\}$, so that Fubini's theorem for $M > 0$ yields:

$$\begin{aligned}
\int_0^M \frac{\cos(t(b-\xi)) - \cos(t(a-\xi))}{t}dt &= \int_0^\infty\int_0^M \left(\cos(t(b-\xi)) - \cos(t(a-\xi))\right)e^{-tz}dtdz \\
&= \int_0^\infty \Re\int_0^M e^{t(i(b-\xi)-z)} - e^{t(i(a-\xi)-z))}dtdz \\
&= \int_0^\infty \Re\left[\frac{1-e^{M(i(b-\xi)-z)}}{z-i(b-\xi)} - \frac{1-e^{M(i(a-\xi)-z)}}{z-i(a-\xi)}\right]dz
\end{aligned}$$

The real part thereof can be cast in the following form:

$$\begin{aligned}
\Re\frac{1-e^{M(i(b-\xi)-z)}}{z-i(b-\xi)} &= \Re\frac{\left[1-\cos(M(b-\xi))e^{-Mz} - i\sin(M(b-\xi))e^{-Mz}\right](z+i(b-\xi))}{z^2+(b-\xi)^2} \\
&= \frac{z}{z^2+(b-\xi)^2} - \cos(M(b-\xi))\frac{ze^{-Mz}}{z^2+(b-\xi)^2}
\end{aligned}$$

$$+ (b-\xi)\sin(M(b-\xi))\frac{e^{-Mz}}{z^2+(b-\xi)^2}$$

Each of these functions except the first has an integrable bound. Since, however, in the above integral we consider the difference of two such functions, we still obtain overall integrability. Furthermore, all but the first summands vanish as $M \to \infty$. Hence, Lebesgue's dominated convergence theorem eventually yields:

$$\begin{aligned}
\lim_{M\to\infty}\int_0^M \frac{\cos(t(b-\xi))-\cos(t(a-\xi))}{t}dt &= \lim_{M\to\infty}\int_0^\infty \Re\left[\frac{1-e^{M(i(b-\xi)-z)}}{z-i(b-\xi)} - \frac{1-e^{M(i(a-\xi)-z)}}{z-i(a-\xi)}\right]dz \\
&= \int_0^\infty \frac{z}{z^2+(b-\xi)^2} - \frac{z}{z^2+(a-\xi)^2}dz \\
&= \frac{1}{2}\left[\log\left((b-\xi)^2+z^2\right) - \log\left((a-\xi)^2+z^2\right)\right]_0^\infty \\
&= \log\left\{\frac{|a-\xi|}{|b-\xi|}\right\}
\end{aligned}$$

This confirms the logarithmic divergence of the integral for $\xi \in \{a, b\}$, thereby justifying the consideration of the symmetric partial sum operator. Then, $N = M$ and the imaginary part vanishes, due to symmetry:

$$\begin{aligned}
\lim_{M\to\infty} S_M f(\xi) &= \lim_{M\to\infty}\frac{1}{2\pi}\left[\int_{-M}^0 + \int_0^M\right]\frac{e^{it(b-\xi)}-e^{it(a-\xi)}}{it}dt \\
&= \lim_{M\to\infty}\frac{1}{\pi}\left(\mathrm{Si}(M(b-\xi)) - \mathrm{Si}(M(a-\xi))\right) \\
&= \begin{cases} 0, & \xi \notin [a,b] \\ 1, & \xi \in (a,b) \\ \frac{1}{2}, & \xi \in \{a,b\} \end{cases}
\end{aligned}$$

The preceding observation is not surprising, since the partial sum operator constitutes a special principal value integral, a class of integrals for which symmetry is an essential ingredient.

In §2.3.3 in [Pinsky, 2002] it was pointed out that special care must be taken when applying the partial sum operator to a discontinuous function. Consider for instance $f = \mathbb{I}\{a < \cdot < b\}$ for $a < b$. According to Example A.1.5, the corresponding partial sum operator converges to unity, pointwise for $\xi \in (a, b)$. If, however, we choose the null sequence $\xi_M := a + \frac{\pi}{M}$ for $M > 0$ such that $\xi_M \in (a, b)$, subject to the oddness of the sine integral we obtain:

$$\lim_{M\to\infty} S_M\mathbb{I}\{a<\cdot<b\}(\xi_M) = \lim_{M\to\infty}\frac{1}{\pi}\left(\mathrm{Si}(M(b-a)-\pi)) + \mathrm{Si}(\pi)\right) = \frac{1}{2} + \frac{\mathrm{Si}(\pi)}{\pi} \approx 1.09$$

This observation is usually referred to as the Gibbs-Wilbraham phenomenon and the constant on the right hand side is denoted the Gibbs overshoot. An attempt to reduce or even avoid this undesired effect but also to establish Fourier inversion under weaker assumptions is summation. More precisely, analogous to Cesàro summability for series, it is possible to consider the Cesàro-means of the partial sum operator (A.1.14). For this we average $S_M f$ with respect to the index M, which means integration and normalization of (A.1.15):

$$\begin{aligned}\frac{1}{M}\int_0^M S_m f(\xi)dm &= \frac{1}{M\pi}\int_{-\infty}^{\infty}\int_0^M \frac{\sin(m(\xi-x))}{\xi-x}dm f(x)dx \\ &= M\int_{-\infty}^{\infty}\frac{1-\cos(M(\xi-x))}{\pi M^2(\xi-x)^2}f(x)dx \end{aligned} \tag{A.1.16}$$

The interchange in the order of integration is admissible by absolute convergence. The integral (A.1.16) is again of convolution-type and involves Fejér's kernel (A.1.11). The latter, however, is a function of the space $L^1(\mathbb{R})$, contrary to the sinc function that appears in the partial sum operator (A.1.15). The convergence of (A.1.16) can be described in a more general frame, which is a generalization of Corollary 3 to Theorem 3.3.2 in [Lukacs, 1970].

Theorem A.1.3 (inversion by means of an approximate identity). *Suppose f_I is an even non-negative function with $f_I(v) = \mathcal{O}\left\{v^{-2}\right\}$ as $v \to \pm\infty$ and $\int_{\mathbb{R}} f_I(v)dv = 1$. Moreover, suppose the derivative f_I' and the Fourier transform Φ_I are contained in $L^1(\mathbb{R})$. Then, for any $\xi \in \mathbb{R}$ such that $f(\xi\pm)$ exist, we have*

$$\frac{f(\xi+)+f(\xi-)}{2} = \lim_{\lambda\downarrow 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi t}\Phi_I(\lambda t)\mathcal{F}\left\{f\right\}(t)\,dt. \tag{A.1.17}$$

The function f_I in the above theorem is sometimes termed an *approximate identity*, and its Fourier transform is referred to as a *smoothing kernel*. A measure theoretical explanation of the former notion will follow the proof below. A rather functional analytic clarification can be found in [Pinsky, 2002]. A typical example for an approximate identity is Fejér's kernel, which corresponds to the Cesàro means of the partial sum operator, according to (A.1.16).

Proof. Under the above assumptions the partial sum operator $S_M f(\xi)$ converges uniformly with respect to $\xi \in \mathbb{R}$ as $M \to \infty$, which implies the uniform continuity of the limit function. But since $f_I' \in L^1(\mathbb{R})$, according to Theorem A.1.2, the limit equals

$$f_I(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi t}\Phi_I(t)dt.$$

In terms of this representation, by Fubini's theorem and by means of some simple substitutions, for fixed $\lambda > 0$ we obtain:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi t}\Phi_I(\lambda t)\mathcal{F}\{f\}(t)\,dt = \lambda^{-1}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\frac{\xi-x}{\lambda}t}\Phi_I(t)dtf(x)dx$$
$$= \lambda^{-1}\int_{-\infty}^{\infty} f_I\left(\frac{\xi-x}{\lambda}\right)f(x)dx$$
$$= \int_{-\infty}^{\infty} f_I(v)f(\xi-\lambda v)\,dv \tag{A.1.18}$$

Notice the similarity between this convolution-type integral and (A.1.16). To examine the behaviour of the former as $\lambda \downarrow 0$, we distinguish between the positive and the negative segment of the real axis. More precisely, we first note that, since f_I is even we have $\int_0^\infty f_I(v)dv = \frac{1}{2}$, which for $a > 0$ implies:

$$\left|\int_{-\infty}^{0} f(\xi-\lambda v)f_I(v)dv - \frac{f(\xi+)}{2}\right| = \left|\int_{-\infty}^{0}\left(f(\xi-\lambda v)-f(\xi+)\right)f_I(v)dv\right|$$
$$\leq \int_{-\infty}^{-a}|f(\xi-\lambda v)|\,f_I(v)dv + \int_{-\infty}^{-a}|f(\xi+)|\,f_I(v)dv$$
$$+\int_{-a}^{0}|f(\xi-\lambda v)-f(\xi+)|\,f_I(v)dv$$

By assumption there exists a constant $A > 0$ such that $|f_I(v)| \leq Av^{-2}$ as $v \to \pm\infty$. Hence, for sufficiently large $a > 0$ we obtain:

$$\int_{-\infty}^{-a}|f(\xi-\lambda v)|\,f_I(v)dv \leq A\int_{-\infty}^{-a}|f(\xi-\lambda v)|\,\frac{dv}{v^2} = A\lambda\int_{-\infty}^{-a\lambda}|f(\xi-z)|\,\frac{dz}{z^2}$$
$$\leq \frac{A}{a^2\lambda}\int_{-\infty}^{-a\lambda}|f(\xi-z)|\,dz \leq \frac{A\,\|f\|_1}{a^2\lambda}$$
$$\int_{-\infty}^{-a}|f(\xi+)|\,f_I(v)dv \leq |f(\xi+)|\,A\int_{-\infty}^{-a}v^{-2}dv = \frac{A\,|f(\xi+)|}{a}$$

Finally also

$$\int_{-a}^{0}|f(\xi-\lambda v)-f(\xi+)|\,f_I(v)dv \leq \sup_{a\lambda\leq x<0}|f(\xi+x)-f(\xi+)|\,.$$

Summarizing, for small λ and $a = \lambda^{-\frac{2}{3}}$ we arrive at

$$\left| \int_{-\infty}^{0} f(\xi - \lambda v) f_I(v) dv - \frac{f(\xi+)}{2} \right| \leq A\lambda^{\frac{1}{3}} \|f\|_1 + A\,|f(\xi+)|\,\lambda^{\frac{2}{3}} + \sup_{\lambda^{\frac{1}{3}} \leq x < 0} |f(\xi + x) - f(\xi+)|\,.$$

But the right hand side vanishes as $\lambda \downarrow 0$. Analogously one can show

$$\lim_{\lambda \downarrow 0} \left| \int_{0}^{\infty} f(\xi - \lambda v) f_I(v) dv - \frac{f(\xi-)}{2} \right| = 0.$$

Therefore, as $\lambda \downarrow 0$ the segments $v < 0$ and $v > 0$ of the integral (A.1.18) converge to the right and left side limit of f at the point ξ, respectively, which concludes the proof. ∎

To understand why f_I is referred to as an approximate identity we first observe that the family of functions defined by $\lambda^{-1} f_I(\lambda^{-1} v)$ vanishes Lebesgue-almost everywhere as $\lambda \downarrow 0$. More precisely, it vanishes for any $v \in \mathbb{R}$ except at $v = 0$, where it exceeds any limit as $\lambda \downarrow 0$. On the other hand the sequence of corresponding Fourier transforms $\Phi_I(\lambda t)$ converges to unity, i.e., to $1 = e^{i0}$. But this is the characteristic function of a certain degenerate distribution, namely of that whose mass is concentrated at the origin. Consequently, according to the continuity theorem, compare Theorem 3.6.1 in [Lukacs, 1970], we have $\lim_{\lambda \downarrow 0} \int_{-\infty}^{\lambda^{-1} v} f_I(u) du = \mathbb{I}\{0 \leq v\}$ weakly, i.e., the distribution function of $\lambda^{-1} f_I(\lambda^{-1} v)$ converges weakly to the identity of the convolution product. This kind of convergence, however, requires acceptance of the concept of weak convergence and distribution functions or, generally speaking, of signed measures. It is not possible, to describe this convergence behaviour by arguments from L^1-theory. Actually the function $\mathbb{I}\{0 \leq v\}$ does not fit in this setting, since it does not possess a density but is defined in terms of the probability function $\delta_{\{0\}}(v)$, which equals one for $v = 0$ and zero otherwise. An approach to explain weak convergence without measure theoretical aspects was presented by [Lighthill, 1958], who introduced the class of generalized functions in the center of which the function $\delta_{\{0\}}$ stands. Roughly speaking, generalized functions are defined as limits of convolution integrals involving sequences of functions. This notion, however, is not compatible with measure theory since $\delta_{\{0\}}$ is treated in an improper fashion.

Besides Fejér's kernel other important approximate identities are the density of the Gauss and the Cauchy distribution. In the latter case $\Phi_I(t) = e^{-|t|}$ and the integral on the right hand side of (A.1.17) can be conceived in the sense of Abel summability for integrals, compare Lemma A.4.1 below or Proposition 2.7.4 in [Pinsky, 2002].

It is immediate from its definition, that the partial sum operator converges if $\mathcal{F}\{f\}(\cdot)$ is absolutely integrable on $\mathbb{R}$. The preceding theorem enables us to finally identify the associated limit.

Theorem A.1.4. *If $f \in L^1(\mathbb{R})$ has the Fourier transform $\mathcal{F}\{f\}(\cdot) \in L^1(\mathbb{R})$ the following* *integral representation converges absolutely and uniformly with respect to* $\xi \in \mathbb{R}$*:*

$$f(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi t} \mathcal{F}\{f\}(t)\, dt \tag{A.1.19}$$

Then $f(\xi)$ *is especially uniformly continuous with respect to* $\xi \in \mathbb{R}$*.*

In other words if f and $\mathcal{F}\{f\}(\cdot)$ are absolutely integrable on $\mathbb{R}$, except for a constant and for the sign of the argument, each function is the Fourier transform of the other function.

Proof. Since $\mathcal{F}\{f\}(\cdot) \in L^1(\mathbb{R})$ the integral (A.1.17) converges absolutely and uniformly with respect to $\lambda > 0$ and $\xi \in \mathbb{R}$. Hence, the limit function is uniformly continuous and we are allowed to interchange the order of limit and integration, resulting in (A.1.19). ∎

By additional averaging of the averaged integral (A.1.16) we obtain a different statement about the convergence of the partial sum operator. It requires less stronger assumptions than Theorem A.1.2 and can be found as Theorems 60.3(i) and 61.1 in [Körner, 1988].

Theorem A.1.5 (a second convergence test for the partial sum operator). *If* $f \in L^1(\mathbb{R})$ *and* $\mathcal{F}\{f\}(t) = \mathcal{O}(t^{-1})$ *as* $t \to \pm\infty$ *then* $\lim_{M\to\infty} S_M f(\xi) = \frac{f(\xi+)+f(\xi-)}{2}$.

The proof of the theorem concludes this subsection.

Proof. If in (A.1.16) we express the Fejér kernel in terms of its inverse Fourier transform, which was derived in equation (A.1.13), an interchange in the order of integration for $x \in \mathbb{R}$ and $M > 0$ leads to:

$$\begin{aligned}
\Delta_M(x) &:= \frac{1}{M} \int_0^M S_m f(x) dm \\
&= \frac{1}{2\pi} \int_{-M}^{M} e^{-ixt} \left(1 - \frac{|t|}{M}\right) \mathcal{F}\{f\}(t)\, dt
\end{aligned}$$

In terms of this function for $N > M > 0$ we consider:

$$\begin{aligned}
\Delta_M^N(x) &:= \frac{N\Delta_N(x) - M\Delta_M(x)}{N - M} \\
&= \frac{1}{2\pi} \int_{-N}^{N} e^{-ixt} \frac{N - |t|}{N - M} \mathcal{F}\{f\}(t)\, dt - \frac{1}{2\pi} \int_{-M}^{M} e^{-ixt} \frac{M - |t|}{N - M} \mathcal{F}\{f\}(t)\, dt \\
&= \frac{1}{2\pi} \left[\int_{-N}^{-M} + \int_{M}^{N} \right] e^{-ixt} \frac{N - |t|}{N - M} \mathcal{F}\{f\}(t)\, dt + S_M f(x)
\end{aligned}$$

Denote $S(x) := \frac{f(x+)+f(x-)}{2}$ and let $0 < \kappa < 1$, implying $\kappa M < M < (1+\kappa)M$. A straightforward application of the triangle inequality then yields

$$|S(x) - S_M f(x)| \leq \left| S(x) - \Delta_{\kappa M}^{(1+\kappa)M}(x) \right| + \left| \Delta_{\kappa M}^{(1+\kappa)M}(x) - S_M f(x) \right| . \tag{A.1.20}$$

By assumption, for sufficiently large M there exists a constant $A > 0$ such that $|\Phi(t)| \leq A\,|t|^{-1}$ for $|t| > \kappa M$. Thus, on the one hand, for the second modulus on the right hand side of (A.1.20) we obtain:

$$\begin{aligned}
\left| \Delta_{\kappa M}^{(1+\kappa)M}(x) - S_M f(x) \right| &= \frac{1}{2\pi} \Bigg| \left[\int\limits_{-(1+\kappa)M}^{-\kappa M} + \int\limits_{\kappa M}^{(1+\kappa)M} \right] e^{-ixt} \frac{(1+\kappa)M - |t|}{M} \mathcal{F}\{f\}(t)\, dt \\
&\qquad - \left[\int\limits_{-M}^{-\kappa M} + \int\limits_{\kappa M}^{M} \right] e^{-ixt} \mathcal{F}\{f\}(t)\, dt \Bigg| \\
&\leq \frac{1}{\pi} \left[\int\limits_{-(1+\kappa)M}^{-\kappa M} + \int\limits_{\kappa M}^{(1+\kappa)M} \right] |\mathcal{F}\{f\}(t)|\, dt \\
&\leq \frac{2A}{\pi} \int\limits_{\kappa M}^{(1+\kappa)M} t^{-1} dt \\
&= \frac{2A}{\pi} \log \left\{ 1 + \frac{1}{\kappa M} \right\}
\end{aligned}$$

As $M \to \infty$ this vanishes. On the other hand, if we apply Theorem A.1.3 with Fejér's kernel to the first modulus on the right hand side in (A.1.20), it shows that:

$$\begin{aligned}
\lim_{M\to\infty} \left| S(x) - \Delta_{\kappa M}^{(1+\kappa)M}(x) \right| &= \lim_{M\to\infty} \left| S(x) - (1+\kappa)\Delta_{(1+\kappa)M}(x) + \kappa \Delta_{\kappa M}(x) \right| \\
&= \left| S(x) - (1+\kappa)\frac{f(x+)+f(x-)}{2} + \kappa \frac{f(x+)+f(x-)}{2} \right| \\
&= 0
\end{aligned}$$

Consequently (A.1.20) decays as $M \to \infty$ and the proof is finished. ■

A.2. The Fourier Transform of One Complex Variable

Under mild conditions on the ingredients many integral transforms that can be cast in the form (A.0.1) establish analytic functions in some region of the complex plane. A criterion is provided by the following theorem.

Theorem A.2.1 (analyticity of generating functions). *Suppose the functions $\varphi(t)$ and $a(t)$, respectively real- and complex-valued, on each closed $I \subset (0,1]$ are uniformly continuous and*

$\inf_{t\in I}\varphi(t) > 0$. *Then, if the integral*

$$\mathfrak{M}(-z) := \int_0^1 \{\varphi(t)\}^{-z}\, a(t)dt$$

converges for any z of some region $\mathcal{R}$ of the complex plane, and if the convergence is uniform in any compact $D \subset \mathcal{R}$, it represents a holomorphic function in $\mathcal{R}$. In this event the derivatives of arbitrary order can be computed by differentiation under the sign of integration. Especially if $f(z)$ is analytic in a subregion $\mathcal{S} \subset \mathcal{R}$ with an isolated singularity at $z = z_0$, the corresponding residue of the product $f(z)\mathfrak{M}(-z)$ equals

$$\int_0^1 \operatorname*{Res}_{z=z_0} \{\varphi(t)\}^{-z} f(z)a(t)dt.$$

Proof. Under the present assumptions we conclude uniform continuity of $\log\varphi(t)$ on any closed subinterval of $(0, 1]$, whence Theorem 5.6.1 in [Wegert, 2012] implies for all $n \in \mathbb{N}$ entireness of

$$\mathfrak{M}_n(-z) := \int_{\frac{1}{n}}^1 \{\varphi(t)\}^{-z}\, a(t)dt.$$

A repeated application of this theorem tells us that for $k \in \mathbb{N}$ the k-th derivative can be found by differentiation under integral sign, yielding

$$\mathfrak{M}_n^{(k)}(-z) = \int_{\frac{1}{n}}^1 \{-\log\varphi(t)\}^k \{\varphi(t)\}^{-z}\, a(t)dt.$$

But the sequence $\mathfrak{M}_n(-z)$ was supposed to converge to $\mathfrak{M}(-z)$, uniformly in any compact subset of $\mathcal{R}$, from which by Theorem 5.1.3 in [Wegert, 2012] we deduce analyticity of the limit function there. Moreover, according to the indicated theorem, even $\mathfrak{M}_n^{(k)}(-z)$ for any $k \in \mathbb{N}$ converges uniformly to $\mathfrak{M}^{(k)}(-z)$. Finally, by definition, the residue is exactly the coefficient in the Laurent expansion of $f(z)\mathfrak{M}(-z)$ associated with the term $\frac{1}{z-z_0}$. Since the coefficients in the series expansion of $\mathfrak{M}(-z)$ result from differentiation, the proof is completed. ■

The above theorem especially applies to the Fourier transform if the amplitude function $f(x)$ is continuous along the real axis, except for a finite number of discontinuities either removable or of jump-type, and if it decays sufficiently fast as $x \to \pm\infty$. The reason is that $e^{itx} = e^{ix\Re t - x\Im t}$, i.e., for complex arguments the exponential function contains an additional factor that is unbounded with respect to $x \in \mathbb{R}$. The growth of this factor can be overcome if there exist real numbers

$-a < b$ with the property

$$\begin{cases} f(x) = \mathcal{O}\left\{e^{bx}\right\} & \text{as } x \to -\infty, \\ f(x) = \mathcal{O}\left\{e^{-ax}\right\} & \text{as } x \to \infty. \end{cases} \tag{A.2.1}$$

Under these conditions the Fourier integral representation (A.1.1) converges absolutely in the strip

$$S_{\mathcal{F}}\{f\} := \{t \in \mathbb{C} : -a < \Im t < b\}. \tag{A.2.2}$$

It is then particularly simple to verify its uniform convergence in any compact subset therein. For brevity let f be continuous along the whole real axis. Supposing $E \subset S_{\mathcal{F}}\{f\}$ compact and defining $a_0 := \min\{\Im t : t \in E\}$ and $b_0 := \max\{\Im t : t \in E\}$, for $-X_1, X_2 > 0$ there exist constants $B, A > 0$ such that we have for any $t \in E$:

$$\begin{aligned}
|\mathcal{F}\{f\}(t)| &\leq \int_{-\infty}^{X_1} e^{-\Im tx} |f(x)|\, dx + \int_{X_1}^{X_2} e^{-\Im tx} |f(x)|\, dx + \int_{X_2}^{\infty} e^{-\Im tx} |f(x)|\, dx \\
&\leq B \int_{-\infty}^{X_1} e^{(b-\Im t)x} dx + \max_{X_1 \leq x_1 \leq X_2} |f(x_1)| \int_{X_1}^{X_2} e^{-\Im tx} dx + A \int_{X_2}^{\infty} e^{-(\Im t + a)x} dx \\
&= B \frac{e^{(b-\Im t)X_1}}{b - \Im t} + \max_{X_1 \leq x_1 \leq X_2} |f(x_1)| \frac{e^{-\Im tX_1} - e^{-\Im tX_2}}{\Im t} + A \frac{e^{-(\Im t + a)X_2}}{\Im t + a} \\
&\leq B \frac{e^{(b-b_0)X_1}}{b - b_0} + \max_{X_1 \leq x_1 \leq X_2} |f(x_1)| \max_{a_0 \leq \Im t \leq b_0} \frac{e^{-\Im tX_1} - e^{-\Im tX_2}}{\Im t} + A \frac{e^{-(a_0 + a)X_2}}{a_0 + a}
\end{aligned}$$

Identifying the function in the second summand as the hyperbolic sine function, which is especially continuous with respect to $\Im t \in \mathbb{R}$, we see that this upper bound equals a finite constant that is independent of $t \in E$. Hence, the convergence in E of the integral (A.1.1) is indeed uniform. Since this was an arbitrary compact subset of $S_{\mathcal{F}}\{f\}$, according to Theorem A.2.1, the integral (A.1.1) defines an analytic function in the indicated region. For piecewise continuous f the justification is similar with the exception that a distinction between the particular continuity intervals is necessary. The set $S_{\mathcal{F}}\{f\}$ is referred to as the *strip of analyticity*, because it constitutes an infinite strip in the complex plane, running parallel to the real axis. Only if $-a < 0 < b$ the real axis is contained. The criterion that was applied to verify the uniform convergence was the simplest and can be considered an analogue of the Weierstrass M-test for integrals. Further tests are presented, for instance, in §5.52 in [Copson, 1970]. In case of a more general determining function $f(x)dx = F(dx)$ appropriate criteria for analyticity can be deduced similar to §§5.5 and 5.51 in [Copson, 1970].

An immediate consequence of (A.2.1) is that piecewise continuous amplitude functions with a finite support result in entire Fourier transforms. Some were already given in examples A.1.2

and A.1.4. Finally, in (A.1.1) a separation of real and imaginary part of the complex variable t yields

$$\mathcal{F}\{f\}(t) = \int_{-\infty}^{\infty} e^{i\Re tx - \Im tx} f(x)dx. \tag{A.2.3}$$

The Fourier transform of f with argument $t \in \mathbb{C}$ thus equals for fixed $\Im t \in \mathbb{R}$ the Fourier transform of $e^{-\Im tx}f(x)$ with argument $\Re t \in \mathbb{R}$. This enables us to derive properties for the transform of a complex variable from those which were deduced for a real argument. For instance, if the amplitude function $e^{-\Im tx}f(x)$ for fixed $\Im t \in \mathbb{R}$ suffices the conditions of the Riemann-Lebesgue lemma A.1.1, it follows that (A.2.3) vanishes as $\Re t \to \pm\infty$. This especially implies, given (A.2.1) the Fourier transform of a complex variable decays towards each direction of the real axis in the interior of its strip of analyticity. Furthermore, we can easily adopt the inversion theorem from the Fourier transform of a real variable.

Theorem A.2.2 (complex Fourier inversion). *Suppose there exists a purely imaginary number $i\tau_0 \in S_{\mathcal{F}}\{f\}$ such that $e^{-\tau_0 x}f(x)$ is in $L^1(\mathbb{R})$ and satisfies at $x \in \mathbb{R}$ the conditions for the convergence of the partial sum operator (A.1.14), for instance the conditions of Theorem A.1.2. Then, for any $x \in \mathbb{R}$ we have*

$$f(x) = \frac{1}{2\pi} \lim_{M\to\infty} \int_{-M+i\tau_0}^{M+i\tau_0} e^{-ixt}\mathcal{F}\{f\}(t)\,dt. \tag{A.2.4}$$

Moreover, if $e^{-\tau_0 x}f(x), \mathcal{F}\{f\}(\cdot + i\tau_0) \in L^1(\mathbb{R})$ we immediately have validity of (A.2.4), the integral being absolutely convergent.

The integral in this inversion formula is a contour integral and accordingly in some circumstances it can be evaluated by means of the residue theorem.

Proof. According to (A.2.3), the Fourier transform $\mathcal{F}\{f\}(s + i\tau_0)$ for $s \in \mathbb{R}$ is readily verified as the transform corresponding to $e^{-\tau_0 x}f(x)$. But the latter satisfies the conditions for the applicability of the inversion formula. We thus deduce from (A.1.14) by means of a simple substitution for $x \in \mathbb{R}$:

$$\begin{aligned} f(x) &= \frac{e^{\tau_0 x}}{2\pi} \lim_{M\to\infty} \int_{-M}^{M} e^{-ixs}\mathcal{F}\{f\}(s + i\tau_0)\,ds \\ &= \frac{1}{2\pi} \lim_{M\to\infty} \int_{-M+i\tau_0}^{M+i\tau_0} e^{-ixt}\mathcal{F}\{f\}(t)\,dt \end{aligned}$$

The second statement of Theorem A.2.2 follows from Theorem A.1.4. It must, however, be emphasized that the uniformity of the latter theorem does not apply for the integral (A.2.4), since the theorem was employed for the product $e^{-\tau_0 x} f(x)$ but not for $f(x)$ only. This finishes the proof. ∎

We close this section with another example for an analytic Fourier transform.

Example A.2.1 (Laplacian density). Assume the amplitude function is given by the density of the Laplace distribution, i.e., $f(x) = e^{-|x|}$. This is evidently continuous with respect to $x \in \mathbb{R}$ and we readily confirm the validity of (A.2.1) with $a = b = 1$. According to our preceding findings, the Fourier integral representation (A.1.1), which is now given by

$$\mathcal{F}\{f\}(t) = \int_{-\infty}^{\infty} e^{itx-|x|} dx, \tag{A.2.5}$$

thus establishes an analytic function in the region $-1 < \Im t < 1$. To investigate its behaviour on the boundary $\Im t = -1$, $\Im t = 1$ or beyond, we must determine the corresponding analytic continuation. This basically means, we need to find a function that is analytic in a larger region of $\mathbb{C}$ but coincides with (A.2.5) in a subregion of $-1 < \Im t < 1$. In our present example this is particularly simple, since the integral (A.2.5) can even be expressed in terms of elementary functions. Therefore we observe for fixed $t \in \mathbb{C}$ with $-1 < \Im t < 1$:

$$\begin{aligned} \mathcal{F}\{f\}(t) &= \int_{0}^{\infty} \left\{ e^{itx} + e^{-itx} \right\} e^{-x} dx \\ &= \frac{1}{1+t^2} \end{aligned} \tag{A.2.6}$$

The function on the right hand side constitutes a meromorphic function of $t \in \mathbb{C}$ with two simple poles located at $t \in \{-i, i\}$, respectively. But the strip $-1 < \Im t < 1$ is contained in the larger region $\mathbb{C} \setminus \{-i, i\}$. Hence, the rational function (A.2.6) equals the desired continuation of the integral (A.2.5). Observe that the two singularities of the analytic continuation are exactly the purely imaginary points on the boundary of the strip of analyticity $-1 < \Im t < 1$ of (A.2.5). This is typical for analytic Fourier transforms of monotonic functions, a statement that was presented by [Widder, 1946] for an integral transform of similar type. Finally it is easy to see that the conditions for the absolute convergence of the inversion formula (A.2.4) apply for all $-1 < \tau_0 < 1$, and for any $x \in \mathbb{R}$ we therefore obtain

$$e^{-|x|} = \frac{1}{2\pi} \int_{-\infty+i\tau_0}^{\infty+i\tau_0} \frac{e^{-ixt}}{1+t^2} dt. \tag{A.2.7}$$

This equality can also be verified by means of the residue theorem. For this purpose, depending on whether $x \geq 0$ or $x \leq 0$, we consider a semicircle of radius $R > 0$ that is symmetric with respect to the imaginary t-axis and whose edge runs parallel to the real t-axis and cuts the imaginary t-axis at the point $t = i\tau_0$. The radius is supposed to be large enough for the semicircle to contain the pole at $t = i$ or at $t = -i$, respectively if $x \leq 0$ or $x \geq 0$. According to the residue theorem, apart from a constant the integral along this semicircle then equals e^x if $x \leq 0$ or e^{-x} if $x \geq 0$. Letting the radius R tend to infinity, the contribution from the integral along the arc of the semicircle vanishes, whereas the integral along the edge converges to the right hand side of (A.2.7), which overall results in the indicated equality.

A.3. The Laplace Transform

Closely related to the Fourier transform is the *bilateral Laplace transform*, which is for $\zeta \in \mathbb{C}$ defined by

$$\mathcal{BL}\left\{f\right\}(\zeta) := \int_{-\infty}^{\infty} e^{-\zeta x} f(x) dx. \tag{A.3.1}$$

This is readily identified as the Fourier transform of f with the complex argument t in (A.1.1) equal to $i\zeta$. According to our findings from the preceding subsection, it thus only exists for amplitude functions with a certain exponential decay. An important special case of (A.3.1) is the *(unilateral) Laplace transform*

$$\mathcal{L}\left\{f\right\}(\zeta) := \int_{0}^{\infty} e^{-\zeta x} f(x) dx, \tag{A.3.2}$$

for which we can write

$$\mathcal{L}\left\{f\right\}(\zeta) = \int_{-\infty}^{\infty} e^{-i\Im\zeta x - \Re\zeta x} f(x) \mathbb{I}\left\{x \geq 0\right\} dx. \tag{A.3.3}$$

Clearly, this is the Fourier transform corresponding to the amplitude function $f(x)\mathbb{I}\left\{x \geq 0\right\}$ with the complex argument t in (A.1.1) replaced by $i\zeta$. For a thorough treatment of Laplace transforms, we refer the reader to [Widder, 1946].

The restriction to the positive real axis in (A.3.2) bears the advantage that, rather than the bilateral Laplace transform its unilateral counterpart exhibits analyticity under fairly weak conditions on the amplitude function. For example one can show, provided $f(x)$ is piecewise continuous with respect to $x \geq 0$, the only discontinuities being either removable or of jump-type, and if

$$f(x) = \mathcal{O}\left\{e^{-ax}\right\} \qquad \text{as } x \to \infty, \tag{A.3.4}$$

for some $a \in \mathbb{R}$, that (A.3.2) establishes a holomorphic function in

$$S_{\mathcal{L}}\left\{f\right\} := \left\{\zeta \in \mathbb{C} : \Re\zeta > -a\right\}. \tag{A.3.5}$$

In contrast to the Fourier transform this is no longer a strip but a half plane. The line $\Re\zeta = -a$ is referred to as the *abscissa of convergence*. Moreover, the Laplace also differs from the Fourier transform by the fact that the condition (A.3.4) on the amplitude function even admits growth when approaching infinity. A property which both transforms have in common is invertibility.

Theorem A.3.1 (inverse Laplace transform). *Suppose the function $e^{\sigma_0 y} f(-y)\mathbb{I}\{y \leq 0\}$ for $\sigma_0 > -a$ satisfies the conditions of Theorem A.1.2. Then, for $x \geq 0$ the amplitude function can be represented through the Bromwich integral*

$$f(x) = \lim_{M\to\infty} \frac{1}{2\pi i} \int\limits_{\sigma_0 - iM}^{\sigma_0 + iM} e^{xt} \mathcal{L}\{f\}(t)\, dt. \tag{A.3.6}$$

Proof. By comparison with (A.3.3) we identify $\mathcal{L}\{f\}(\sigma_0 + i\tau)$ with $\tau \in \mathbb{R}$ as the Fourier transform of the real variable τ corresponding to the amplitude function $e^{\sigma_0 y} f(-y)\mathbb{I}\{y \leq 0\}$. According to Theorem A.1.2, we thus obtain from (A.1.14) for $y \in \mathbb{R}$, after a simple change of variables:

$$\begin{aligned} f(-y)\mathbb{I}\{y \leq 0\} &= e^{-\sigma_0 y} \lim_{M\to\infty} \frac{1}{2\pi} \int\limits_{-M}^{M} e^{-iy\tau} \mathcal{L}\{f\}(\sigma_0 + i\tau)\, d\tau \\ &= \lim_{M\to\infty} \frac{1}{2\pi} \int\limits_{\sigma_0 - iM}^{\sigma_0 + iM} e^{-yt} \mathcal{L}\{f\}(t)\, dt \end{aligned}$$

The proof is finished if we write $x = -y$. ■

In the present work the Laplace transform mainly occurs in the context of Abel summability of integrals, a technique to be discussed in the next section, as a means to maximally exploit the oscillations of Fourier-type integrals. We shall therefore spend no more time on this particular transform and close this section with a few examples.

A few Laplace transforms can be written in terms of elementary functions, for instance $(1 - it)^{-1}$, which is associated with the amplitude function e^{ix}. Among those that do not admit an elementary representation, the *gamma* and the *beta function* are possibly most studied. For fixed $\Re q > 0$ they are given by

$$\int\limits_0^\infty e^{-\zeta t} t^{q-1} dt =: \zeta^{-q}\Gamma(q), \tag{A.3.7}$$

$$\int\limits_0^\infty e^{-\zeta t}(1 - e^{-t})^{q-1} =: \mathrm{B}(\zeta, q). \tag{A.3.8}$$

Their abscissa of convergence is evidently $\Re\zeta = 0$. The gamma and beta function are of frequent occurence in this work with their important properties to be discussed in Appendix B.2 and

B.3 below. The analytic continuation with respect to their argument is readily derived since these particular examples satisfy some remarkble identities, which are easy to verify by partial integration. This is an exception and in general the determination of the analytic continuation of a Laplace transform can be much more complicated. This already applies to the innocent looking integral

$$\mathcal{L}\left\{f_q\right\}(\zeta) = \int_0^\infty (1+x)^{q-1} e^{-\zeta x} dx, \tag{A.3.9}$$

which represents the Laplace transform of $f_q(x) := (1+x)^{q-1}$ for fixed $q \in \mathbb{R}$. Clearly, $f_q(x)$ is a rational function for $q \in \mathbb{Z}$ but otherwise it is multi-valued. The integral (A.3.9) exhibits analyticity in the half plane $S_{\mathcal{L}}\left\{f_q\right\} = \{\zeta \in \mathbb{C} : \Re\zeta > 0\}$ and remains absolutely convergent along the line $\Re\zeta = 0$ if $q < 0$. Since $f_q(x)$ is monotonic, Theorem 5b in ch. II in [Widder, 1946] tells us that the purely real part on the abscissa of convergence, i.e., $\zeta = 0$, is a singular point. The nature of this singularity depends on q. It is either a pole or a branch point. To show this we make a simple change of variables:

$$\mathcal{L}\left\{f_q\right\}(\zeta) = \zeta^{-q} e^{\zeta} \int_\zeta^\infty t^{q-1} e^{-t} dt \tag{A.3.10}$$

This representation exposes what type of function the Laplace transform (A.3.9) actually is, namely the antiderivative of the function $t^{q-1}e^{-t}$. With respect to $t \in \mathbb{C}$ the latter is rational for integer q and otherwise multi-valued. In case of multi-valuedness we choose the argument function subject to $|\arg(t)| < \pi$. The Laplace transform under consideration is then obtained up to an additional factor by integrating $t^{q-1}e^{-t}$ along an arbitrary piecewise continuous path that runs through the region of analyticity, connecting the point $\zeta \in \mathbb{C}$ with the point infinity in $\Re\zeta > 0$. This region is either $\mathbb{C} \setminus (-\infty, 0]$ or $\mathbb{C} \setminus \{0\}$, depending on q. By admitting $\Re\zeta < 0$ the described procedure eventually gives rise to the analytic continuation of the integral (A.3.9). The integral appearing in (A.3.10) is in complex calculus referred to as the upper incomplete gamma function.

A.4. Boundary Behaviour of Analytic Functions, Summability of Divergent Integrals and Weak Convergence

In this section we point out the connection between the behaviour of an analytic function on its boundaries and the notions of summability of integrals and weak convergence. Therefore we consider an illustrative example, furnished for fixed $p > 0$ by the simple Laplace transform

$$\mathcal{L}\left\{t^{p-1}\right\}(z) = \int_0^\infty t^{p-1} e^{-zt} dt, \tag{A.4.1}$$

which is of course expressible in terms of the well-known gamma function. As a function of $z \in \mathbb{C}$, by Theorem A.2.1, the integral exhibits analyticity in the half-plane $\Re z > 0$. Assuming for a moment $z > 0$ real, a change of variables leads to

$$\mathcal{L}\left\{t^{p-1}\right\}(z) = z^{-p}\Gamma(p). \tag{A.4.2}$$

This result remains true by analytic continuation for $z \in \mathbb{C} \setminus \{0\}$ if $p \in \mathbb{N}$ and for $z \in \mathbb{C} \setminus (-\infty, 0]$ if $p > 0$ is non-integer with $|\arg(z)| < \pi$. Suppose now, for $\xi \in \mathbb{R} \setminus \{0\}$, we are interested in the limit as $\lambda \downarrow 0$ of the sequence of integrals

$$\mathcal{L}\left\{t^{p-1}\right\}(\lambda + i\xi) = \int_0^\infty t^{p-1} e^{-(\lambda+i\xi)t} dt. \tag{A.4.3}$$

Separating the integration path into two segments, for instance into the intervals $(0, 1)$ and $(1, \infty)$, we see that the convergence of the integral is uniform with respect to $\lambda \geq 0$ on the first segment for any $p > 0$. Regarding the interval $(1, \infty)$, however, the behaviour of the integrand as $\lambda \downarrow 0$ especially for arbitrary $p \geq 1$ suggests the sequence of integrals will certainly diverge. At least in the case $0 < p < 1$ one could expect the existence of a limit because the integral for $\lambda = 0$ exists in the improper sense. More precisely, for any $T > 1$, from partial integration we obtain

$$\int_1^T t^{p-1} e^{-i\xi t} dt = -T^{p-1} \frac{e^{-i\xi T}}{i\xi} + \frac{e^{-i\xi}}{i\xi} + \frac{p-1}{i\xi} \int_1^T t^{p-2} e^{-i\xi t} dt. \tag{A.4.4}$$

Letting $T \to \infty$ the first summand vanishes, whereas the integral in the last summand converges even absolutely since $0 < p < 1$. Consequently the integral (A.4.3) for $\lambda = 0$ but $\xi \in \mathbb{R} \setminus \{0\}$ indeed converges if $0 < p < 1$ in the sense that the improper integral

$$\kappa := \lim_{T \to \infty} \int_0^T t^{p-1} e^{-i\xi t} dt \tag{A.4.5}$$

exists and equals a finite constant that depends on ξ, p. Yet, we will see below that the sequence of integrals (A.4.3) as $\lambda \downarrow 0$ approaches a finite limit for all $p > 0$. Since (A.4.1) is particularly a Laplace transform, taking the limit in (A.4.3) is equivalent to approaching the point $i\xi$ from the right z-half plane. Because $i\xi$ is located on the abscissa of convergence this requires information about the analytic continuation corresponding to the above integral representation. But the latter is exactly given by (A.4.2). According to the rules for complex-valued power functions, with the branch $|\arg(\lambda + i\xi)| < \pi$ this leads to:

$$\lim_{\lambda \downarrow 0} \mathcal{L}\left\{t^{p-1}\right\}(\lambda + i\xi) = \lim_{\lambda \downarrow 0} (\lambda + i\xi)^{-p}\Gamma(p)$$

$$= \lim_{\lambda \downarrow 0} |\lambda + i\xi|^{-p} e^{-p \arg(\lambda + i\xi)} \Gamma(p)$$

$$= |\xi|^{-p} e^{\mp ip\frac{\pi}{2}} \Gamma(p) \tag{A.4.6}$$

The phase of the complex exponential function indicates the argument of $-i\xi$. According to (A.4.6), the sequence of integrals (A.4.1) thus indeed possesses a finite limit for any $\xi \neq 0$ and especially for any $p > 0$. Even for $p \geq 1$ in which event the integral $\int_0^\infty t^{p-1} e^{-i\xi t} dt$ diverges. The reason is that, for any fixed $p > 0$ the analytic continuation of (A.4.1) exhibits on the line $\Re z = 0$ only a singularity at $\Im z = 0$.

Assuming $0 < p < 1$ in (A.4.3), the existence of the limit as $\lambda \downarrow 0$ can also be justified by Abel's lemma. This leads us to the topic of summability of integrals, a technique which is usually applied to non-absolutely convergent integrals. In particular, summation methods aim for a simplified computation or specification of a limit of an improper integral or of an integral whose actual convergence behaviour is unclear. An example is given by (A.4.5). Summability theorems are frequently encountered in the context of Fourier analysis and can accordingly be found in [Pinsky, 2002], [Titchmarsh, 1937] and [Widder, 1946]. For a more general discussion of this topic we refer to [Hardy, 1949]. As an example, below we present Abel's lemma in the version of Proposition 2.7.4 in [Pinsky, 2002].

Lemma A.4.1 (Abel). *(1) If $f(t)$ is locally integrable on $t \geq 0$ and the limit $\lim_{t\to\infty} f(t) = L$ exists, then*

$$\lim_{\lambda \downarrow 0} \lambda \int_0^\infty e^{-\lambda t} f(t) dt = L. \tag{A.4.7}$$

(2) If $g(s)$ is locally integrable on $s \geq 0$ and the limit $\lim_{T\to\infty} \int_0^T g(s)ds = L$ equals a finite constant, then

$$\lim_{\lambda \downarrow 0} \int_0^\infty e^{-\lambda s} g(s) ds = L. \tag{A.4.8}$$

Applying the second statement from this lemma to (A.4.5) with $g(s) = s^{p-1} e^{-i\xi s}$ for $0 < p < 1$ we arrive at the integral (A.4.3). Again, by referring to the gamma function the limit as $\lambda \downarrow 0$ is readily confirmed to equal (A.4.6).

Finally we mention that sequences of integrals also occur in measure theory. From this perspective the expression (A.4.3) as $\lambda \downarrow 0$ can be considered as a limit of an integral with respect to a sequence of measures whose density equals the function $t \mapsto e^{-\lambda t}$, an interpretation with pitfalls. To point these out, for $t \geq 0$, we introduce the function

$$E_\lambda(t) := \int_0^t e^{-\lambda s} ds = \frac{1 - e^{-\lambda t}}{\lambda}. \tag{A.4.9}$$

Clearly, $E_\lambda(dt) = e^{-\lambda t} dt$. Moreover, it is easy to see that (A.4.9) for fixed $\lambda > 0$ belongs to the class of functions of finite total variation on $[0, \infty]$, in the older literature referred to as functions of bounded variation. Their convergence behaviour can be described as weak convergence. With (A.4.9), instead of (A.4.3) we can write

$$\mathcal{L}\left\{t^{p-1}\right\}(\lambda + i\xi) = \int_0^\infty t^{p-1} e^{-i\xi t} E_\lambda(dt). \tag{A.4.10}$$

Convergence properties of integrals with respect to functions of finite total variation can be specified according to the Helly-Bray theorem. It is most frequently cited in the version for probability distributions, compare for instance Theorem 3.5.2 in [Lukacs, 1970]. A more general version for functions of finite total variation can be found as Theorem 16.4 in ch. 1 in [Widder, 1946], which shall be applied below. In before we agree that, a function $f(t)$ is said to be continuous on $[0, \infty]$, if it is continuous at any point of the interval and if $f(\infty) := \lim_{t \to \infty} f(t)$ exists.

Theorem A.4.1 (special version of the Helly-Bray theorem). *For a function $K_\lambda(t)$ of finite total variation on $[0, \infty]$ uniformly with respect to $\lambda \geq 0$ suppose $K_\lambda(0) = 0$ and $K := K_\lambda(\infty)$ does not depend on $\lambda > 0$, and for $0 \leq t < \infty$ we have*

$$\lim_{\lambda \downarrow 0} K_\lambda(t) = 0. \tag{A.4.11}$$

Then, the following statements apply:

(1) If $f(t)$ is continuous on $[0, \infty]$, we have

$$\lim_{\lambda \downarrow 0} \int_0^\infty f(s) K_\lambda(ds) = K f(\infty). \tag{A.4.12}$$

(2) If $g(s)$ is locally integrable on $s \geq 0$ and the limit $\lim_{T \to \infty} \int_0^T g(s) ds = L$ equals a finite constant, we have

$$\lim_{\lambda \downarrow 0} \int_0^\infty \{K - K_\lambda(s)\} g(s) ds = LK. \tag{A.4.13}$$

The integral on the left hand side need not converge absolutely.

Proof. Under the assumptions of the theorem we can write $K_\lambda(t) = \int_0^t K_\lambda(du)$ and we observe

$$\lim_{\lambda \downarrow 0} K_\lambda(t) = \begin{cases} 0, & \text{if } 0 \leq t < \infty, \\ K, & \text{if } t = \infty. \end{cases} \tag{A.4.14}$$

Hence, the left hand side converges weakly to the function $K\mathbb{I}\{\infty \leq t\}$, which equals zero for any $t \geq 0$ and K as $t \to \infty$. The statement of Theorem A.4.1(1) is thus an immediate consequence of the general Helly-Bray theorem, which can be found as Theorem 16.4 in ch. 1 in [Widder, 1946], leading to

$$\lim_{\lambda \downarrow 0} \int_0^\infty f(s) K_\lambda(ds) = \int_0^\infty f(s) \delta_{\{\infty\}}(ds) = K f(\infty).$$

Regarding the result in Theorem A.4.1(2), for fixed $T, \lambda > 0$ we first obtain from partial integration:

$$\int_0^T \{K - K_\lambda(s)\} g(s) ds = \{K - K_\lambda(T)\} \int_0^T g(s) ds + \int_0^T \int_0^t g(s) ds K_\lambda(dt)$$

If we eventually let $T \to \infty$, the first summand vanishes by assumption and we arrive at

$$\int_0^\infty \{K - K_\lambda(s)\} g(s) ds = \int_0^\infty \int_0^t g(s) ds K_\lambda(dt). \tag{A.4.15}$$

Since g is locally integrable the integral $\int_0^t g(s)ds$ defines a function which is continuous on $[0, \infty)$. Moreover, by assumption it approaches a finite limit as $t \to \infty$. Consequently, the integral $\int_0^t g(s)ds$ even denotes a continuous function on $[0, \infty]$. Letting $\lambda \downarrow 0$ we can again apply the Helly-Bray theorem to the right hand side of equation (A.4.15) to eventually arrive at (A.4.13). ■

With $K_\lambda(t) \equiv 1 - e^{-\lambda t}$ we exactly obtain the statement of the Abelian lemma A.4.1, i.e., of the cited Proposition 2.7.4 in [Pinsky, 2002]. To apply the preceding theorem to the integral (A.4.10), we first observe $\lim_{t\to\infty} E_\lambda(t) = \frac{1}{\lambda}$. The function is thus not of finite total variation on $[0, \infty]$ uniformly with respect to $\lambda \geq 0$. However, the function

$$F_\lambda(t) := \lambda \int_0^t e^{-\lambda s} ds = 1 - e^{-\lambda t} \tag{A.4.16}$$

is of finite total variation on $[0, \infty]$ uniformly with respect to $\lambda \geq 0$ and $F_\lambda(\infty) = 1$. In stochastics, (A.4.16) is precisely associated with the exponential distribution with parameter

$\lambda > 0$. Moreover,

$$h(t) := \int_0^t s^{p-1} e^{-i\xi s} ds \tag{A.4.17}$$

is a continuous function at any $t \in [0, \infty]$, especially since the limit as $t \to \infty$ exists with $h(\infty) = \kappa$, according to equation (A.4.5). As a consequence of these considerations, Theorem A.4.1(2) applies and we immediately obtain

$$\lim_{\lambda \downarrow 0} \int_0^\infty \{1 - F_\lambda(t)\} t^{p-1} e^{-i\xi t} dt = \kappa. \tag{A.4.18}$$

To summarize the findings of this section, summability and weak convergence describe similar concepts for the convergence of integrals by the tools of different topics of mathematics. But beyond this scope sequences of integrals may still approach a finite limit by arguments of complex calculus.

A.5. The Mellin Transform

If in (A.0.1) we choose $p(x) = \log(x)$ and $\mathcal{P} = (0, \infty)$ we arrive at the *Mellin transform.* This is in particular the integral

$$\mathcal{M}\{f\}(\zeta) := \int_0^\infty x^{\zeta-1} f(x) dx. \tag{A.5.1}$$

Necessary conditions for a piecewise continuous function f to possess a Mellin transform and further basic properties can immediately be derived from the Fourier transform. Indeed, if in (A.5.1) we make the change of variables $u = \log(x)$ and separate real and imaginary part of the argument, we arrive at

$$\mathcal{M}\{f\}(\zeta) = \int_{-\infty}^\infty e^{i\Im\zeta u + \Re\zeta u} f(e^u) du. \tag{A.5.2}$$

This is readily identified as the Fourier transform of a complex variable associated with the function $f(e^u)$. Note, however, that now $-i\zeta$ corresponds to the variable t in (A.1.1). As a consequence of (A.5.2), similar to the Fourier transform the presence of at least finitely many removable or jump discontinuities and the existence of constants $-a < b$ with the property

$$\begin{cases} f(x) = \mathcal{O}\{x^a\} & \text{as } x \to 0, \\ f(x) = \mathcal{O}\left\{x^{-b}\right\} & \text{as } x \to \infty, \end{cases} \tag{A.5.3}$$

suffice to establish analyticity of the integral (A.5.1) in the strip

$$S_{\mathcal{M}}\{f\} := \{\zeta \in \mathbb{C} : -a < \Re\zeta < b\}\,. \tag{A.5.4}$$

More precisely, in these circumstances the integral representation (A.5.1) converges absolutely and uniformly in any compact subset of the region (A.5.4). The latter is again referred to as the *strip of analyticity* of the Mellin transform but runs now parallel to the imaginary axis. According to the requirement $-a < b$, the Mellin transform of constants or polynomials does not exist, except if they are restricted to a finite subset of the positive real axis.

Another simple conclusion from the representation as a Fourier transform is the Riemann-Lebesgue lemma, implying the decay of $\mathcal{M}\{f\}(\zeta)$ as $\Im\zeta \to \pm\infty$ in $S_{\mathcal{M}}\{f\}$, provided the integral representation converges absolutely. Regarding the asymptotic behaviour of a Mellin transform in the direction of the real axis, general statements are only possible if $a = \infty$ or $b = \infty$, in which event the integral representation immediately shows its growth in the respective direction of the endpoint. For example if $b = \infty$ then $\mathcal{M}\{f\}(\zeta) \sim \int_1^\infty t^{\zeta-1} f(x)dx \to \infty$ as $\Re\zeta \to \infty$ if $f(x) \neq 0$ for some $x > 1$. Detailed asymptotic statements substantially depend on continuity and possible analyticity of the amplitude function. These can, for instance, be deduced by employing Theorem 4.7.2 in [Bleistein and Handelsman, 1986]. Finally, a reference to the Fourier transform is also helpful to establish an inversion formula for the Mellin transform.

Theorem A.5.1 (inverse Mellin transform). *If there exists a real number $\sigma_0 \in S_{\mathcal{M}}\{f\}$ for which $e^{\sigma_0 u} f(e^u)$ is in $L^1(\mathbb{R})$ and satisfies the further condition of Theorem A.1.2, the function f can be written as a complex-valued integral. In particular, for any $t > 0$ we then have*

$$f(t) = \lim_{M\to\infty} \frac{1}{2\pi i} \int_{\sigma_0 - iM}^{\sigma_0 + iM} \mathcal{M}\{f\}(z)\, t^{-z} dz. \tag{A.5.5}$$

If $e^{\sigma_0 u} f(e^u), \mathcal{M}\{f\}(\sigma_0 + iy) \in L^1(\mathbb{R})$ the validity of (A.5.5) is immediate and the integral converges absolutely.

Proof. In accordance with (A.5.2), the Mellin transform with argument $\zeta \equiv \sigma_0 + i\tau$ for $\tau \in \mathbb{R}$ is exactly the Fourier transform of the variable τ of $e^{\sigma_0 u} f(e^u)$. The Fourier inversion theorem A.1.2 therefore yields for $u \in \mathbb{R}$:

$$\begin{aligned} f(e^u) &= e^{-\sigma_0 u} \lim_{M\to\infty} \frac{1}{2\pi} \int_{-M}^{M} e^{-iu\tau} \mathcal{M}\{f\}(\sigma_0 + i\tau)\, d\tau \\ &= \lim_{M\to\infty} \frac{1}{2\pi i} \int_{\sigma_0 - iM}^{\sigma_0 + iM} e^{-uz} \mathcal{M}\{f\}(z)\, dz \end{aligned}$$

The second equality follows from a substitution. With $t = e^u$ for $u \in \mathbb{R}$ we arrive at (A.5.5).

Finally, the second claim of Theorem A.5.1 holds subject to Theorem A.1.4. Again it must be remarked that the uniformity statement of the cited theorem does not apply. ■

We proceed with a quick overview on two important Mellin transforms before we discuss some methods to expose the analytic structure of these transforms in wider regions of the complex plane.

Example A.5.1 (gamma function). Possibly the most frequently occuring Mellin transform is that of the exponential function, better known as the gamma function. For $\Re\zeta > 0$ it is denoted by:

$$\begin{aligned}\Gamma(\zeta) &:= \mathcal{M}\left\{e^{-t}\right\}(\zeta)\\ &= \int_0^\infty t^{\zeta-1}e^{-t}dt \end{aligned} \tag{A.5.6}$$

According to our preceding findings, this integral establishes an analytic function in $\Re\zeta > 0$. We take this opportunity to outline an application of the Mellin inversion formula. A more detailed discussion of the analyticity properties of the gamma function is postponed to Appendix B.2. For the moment it suffices to know that the integral (A.5.6) can be extended to a meromorphic function in the whole complex plane with an infinite sequence of simple poles at the non-positive integers. For $k \in \mathbb{N}_0$ the corresponding residues are given by

$$\operatorname*{Res}_{\zeta=-k}\Gamma(\zeta) = \frac{(-1)^k}{k!}. \tag{A.5.7}$$

Moreover, $\Gamma(\zeta)$ exhibits exponential decay into any direction of the imaginary axis for fixed $\Re\zeta \in \mathbb{R}$. Subject to the inversion theorem A.5.1 this enables us to represent the exponential function for $\sigma_0, t > 0$ as the absolutely convergent complex-valued integral

$$e^{-t} = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\Gamma(z)t^{-z}dz, \tag{A.5.8}$$

which is known as the *Cahen-Mellin integral.* By analytic continuation its validity can be verified for arbitrary $t \in \mathbb{C}$ with $\Re t > 0$. The reader who is unfamiliar with special functions might be insecure how to evaluate the integral (A.5.8) or how to confirm the whole equation without making a reference to Mellin transforms. This requires to employ the residue theorem. First we consider for fixed $t > 0$ a semicircle of radius $R > 0$ that is symmetric with respect to the real z-axis and whose edge cuts the positive real z-axis at $z = \sigma_0$. The radius is supposed be sufficiently large so that the semicircle contains exactly the first $N-1$ negative integers including zero. At the same time the arc Γ_R may not cross the N-th pole. By encircling the semicircle in the counterclockwise direction, according to the residue theorem, we obtain

$$\frac{1}{2\pi i}\int_{\sigma_0-iR}^{\sigma_0+iR}\Gamma(z)t^{-z}dz+\frac{1}{2\pi i}\int_{\Gamma_R}\Gamma(z)t^{-z}dz=\sum_{n=0}^{N-1}\frac{(-t)^n}{n!}. \tag{A.5.9}$$

If we eventually let $R\to\infty$ it can be shown by means of Stirling's formula that the contribution of the integral along the arc Γ_R vanishes, whereas on the right hand side of equation (A.5.9) residues from the poles at subsequent negative integers must be added. This eventually leads to

$$\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\Gamma(z)t^{-z}dz=\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}. \tag{A.5.10}$$

The right hand side equals exactly the well-known series representation for the complex exponential function. It converges absolutely and uniformly in any compact subset of the complex plane. This confirms that (A.5.8) holds for arbitrary $t\in\mathbb{C}$ with $\Re t>0$, since this is the largest common subregion of the complex plane where the integral and the series exhibit analyticity. Equation (A.5.10) reveals a close connection between Mellin transforms and power series. In fact, many power series possess an equivalent representation in terms of a complex-valued integral.

We close this example by illustrating an outstanding property of the gamma function as a Mellin transform. For $\Re\zeta>0$ and any $\sigma>0$, we have

$$\sigma^{-\zeta}=\frac{1}{\Gamma(\zeta)}\int_0^\infty t^{\zeta-1}e^{-\sigma t}dt. \tag{A.5.11}$$

Observe that this equation remains valid by analytic continuation for arbitrary $\sigma\in\mathbb{C}$ with $\Re\sigma>0$. It thus enables us to introduce an integral representation for reciprocals. A standard application is the derivation of an integral representation from the series for Riemann's zeta function. It can, however, also be employed to establish a connection between the Laplace and the Mellin transform of a function f, provided the latter integral converges absolutely for some $\Re\zeta<1$. Then we may apply (A.5.11) to the integral representation for the Mellin transform and, subject to absolute convergence, interchange the order of integration, to find:

$$\begin{aligned}\mathcal{M}\{f\}(\zeta)&=\int_0^\infty t^{\zeta-1}f(t)dt\\&=\frac{1}{\Gamma(1-\zeta)}\int_0^\infty x^{-\zeta}\int_0^\infty e^{-xt}f(t)dtdx\\&=\frac{1}{\Gamma(1-\zeta)}\mathcal{M}\{\mathcal{L}\{f\}\}(1-\zeta)\end{aligned} \tag{A.5.12}$$

The first part of the preceding example made it obvious that the appropriate application of Mellin transforms requires a fundamental knowledge of complex calculus. Besides the gamma function, the beta function is also important and of frequent occurence.

Example A.5.2 (beta function). We already encountered Euler's beta function in equation (A.3.8) as an example for a Laplace transform. After a simple change of variables we obtain

$$\mathrm{B}(\zeta, q) = \int_0^1 s^{\zeta-1}(1-s)^{q-1} ds, \qquad \Re\zeta, \Re q > 0. \tag{A.5.13}$$

For fixed $\Re q > 0$ this evidently constitutes the Mellin transform corresponding to the function $f(s) = (1-s)^{q-1}\mathbb{I}\{0 < s < 1\}$. The associated strip of analyticity matches the entire right ζ-half plane. However, since there exist many equivalent representations, at the same time for fixed $\alpha, p, \theta > 0$ the Mellin transform associated with

$$g(t) = (1 + \theta^\alpha t^\alpha)^{-p}$$

can also be cast in terms of the beta function. Therefore we first consider the corresponding integral definition

$$\mathcal{M}\{g\}(\zeta) = \int_0^\infty t^{\zeta-1}(1 + \theta^\alpha t^\alpha)^{-p} dt. \tag{A.5.14}$$

From the behaviour of the integrand we readily see that the strip of analyticity is

$$S_{\mathcal{M}}\{g\} = \{\zeta \in \mathbb{C} : 0 < \Re\zeta < \alpha p\}. \tag{A.5.15}$$

Making in (A.5.14) the changes of variables $\theta^\alpha t^\alpha = \frac{s}{1-s}$, we arrive at:

$$\begin{aligned} \mathcal{M}\{g\}(\zeta) &= \theta^{-\zeta}\alpha^{-1} \int_0^1 s^{\frac{\zeta}{\alpha}-1}(1-s)^{p-\frac{\zeta}{\alpha}-1} ds \\ &= \theta^{-\zeta}\alpha^{-1}\,\mathrm{B}\left(\frac{\zeta}{\alpha}, p - \frac{\zeta}{\alpha}\right) \end{aligned} \tag{A.5.16}$$

Hence, also the Mellin transform of (A.5.14) can be cast in terms of the beta function (A.5.13).

A.5.1. Analytic Continuation of Mellin Transforms

A common technique to determine the analytic continuation of a Mellin transform is, to employ an asymptotic expansion for the amplitude function. We shall illustrate this procedure for the function $f(x) = \frac{1}{1+x}$ with

$$\mathcal{M}\{f\}(\zeta) = \int_0^\infty \frac{x^{\zeta-1}}{1+x}dx. \tag{A.5.17}$$

According to our introductory discussion on Mellin transforms, this establishes a holomorphic function in the strip $0 < \Re\zeta < 1$. Assume we were interested in the continuation beyond the boundary line $\Re\zeta = 1$. This boundary is due to the behaviour of the amplitude function at infinity. In particular, the amplitude function $f(x)$ as $x \to \infty$ can be expanded by the formula for the geometric series, according to which we have for $|x| > 1$ absolute convergence of

$$f(x) = \frac{1}{x}\sum_{k=0}^{\infty}(-1)^k x^{-k}. \tag{A.5.18}$$

If we now separate the range of integration at some $x_1 > 0$ and rearrange (A.5.17), for $0 < \Re\zeta < 1$, we obtain:

$$\begin{aligned}\mathcal{M}\{f\}(\zeta) &= \int_0^{x_1} \frac{x^{\zeta-1}}{1+x}dx + \int_{x_1}^{\infty} x^{\zeta-2}dx + \int_{x_1}^{\infty} x^{\zeta-1}\left\{\frac{1}{1+x} - \frac{1}{x}\right\}dx \\ &= \int_0^{x_1} \frac{x^{\zeta-1}}{1+x}dx + \frac{x_1^{\zeta-1}}{1-\zeta} + \int_{x_1}^{\infty} x^{\zeta-1}\left\{\frac{1}{1+x} - \frac{1}{x}\right\}dx\end{aligned} \tag{A.5.19}$$

By comparison with the criterion (A.5.3) we readily confirm analyticity of the first integral in $\Re\zeta > 0$. Furthermore, the integrand in the second integral is especially continuous on $x \geq x_1$ and from (A.5.18), as $x \to \infty$, we get

$$\frac{1}{1+x} - \frac{1}{x} \sim -\frac{1}{x^2}.$$

By (A.5.3) the integral thus constitutes an analytic function in the half plane $\Re\zeta < 2$. Finally the second summand in (A.5.19) equals a meromorphic function in the complex plane. To summarize these observations, the expansion (A.5.19) establishes a meromorphic function in the strip $0 < \Re\zeta < 2$, which is the greatest common region of analyticity of all three functions. The only non-analytic point therein is a simple pole at $\zeta = 1$. The corresponding residue equals the residue of the second summand there, since the remaining summands are analytic at $\zeta = 1$. According to the basic rules of complex calculus,

$$\operatorname*{Res}_{\zeta=1} \mathcal{M}\{f\}(\zeta) = \operatorname*{Res}_{\zeta=1} \frac{x_1^{\zeta-1}}{1-\zeta} = -1. \tag{A.5.20}$$

Since the strip $0 < \Re\zeta < 1$ is contained in $0 < \Re\zeta < 2$, the expansion (A.5.19) extends the integral (A.5.17) to a meromorphic function in the indicated region. The continuation evidently does not depend on $x_1 > 0$. However, it becomes invalid for $x_1 = 0$.

Finally, the above procedure can be repeated by adding an arbitrary finite number of terms of the expansion (A.5.18). Only if $x_1 > 1$ we may take the whole series. The technique is even applicable for more general expansions involving logarithmic or exponential terms, compare §4.3 in [Bleistein and Handelsman, 1986]. In any case it must be emphasized that the asymptotic expansion for the amplitude function need not be convergent. This is also seen in the present example if we assume $x_1 \leq 1$, in which event the geometric series (A.5.18) is in fact divergent but none of the arguments for the validity of (A.5.19) is violated.

Alternatively the preceding result can be achieved by partial integration. This is slightly more elaborate but bears the advantage that the technique can be adapted to more general integral transforms of the form (A.0.1) with different phases. For this purpose we again divide the range of integration at some $x_1 > 0$ and rearrange the integrand:

$$\mathcal{M}\left\{f\right\}(\zeta) = \int_0^{x_1} \frac{x^{\zeta-1}}{1+x}dx + \int_{x_1}^{\infty} x^{\zeta-2}\frac{x}{1+x}dx \tag{A.5.21}$$

Observe that we incorporated the asymptotic behaviour of the amplitude as $x \to \infty$. Contrary to the former approach we multiplicatively separated the leading factor from the expansion (A.5.18). The new amplitude function $f_2(x) := xf(x)$ is now constant but non-vanishing at infinity. In particular, $f_2(x) \to 1$ as $x \to \infty$. But the derivative

$$f_2'(x) = \frac{1}{(1+x)^2}$$

vanishes at infinity. More precisely, $f_2'(x) \sim x^{-2}$ as $x \to \infty$. This can also be obtained by differentiation of (A.5.18). If we thus integrate by parts once the second integral in (A.5.21), for $0 < \Re\zeta < 1$ we arrive at:

$$\begin{aligned}\mathcal{M}\left\{f\right\}(\zeta) &= \int_0^{x_1} \frac{x^{\zeta-1}}{1+x}dx + \left[\frac{x^{\zeta-1}}{\zeta-1}\frac{x}{1+x}\right]_{x_1}^{\infty} - \frac{1}{\zeta-1}\int_{x_1}^{\infty}\frac{x^{\zeta-1}}{(1+x)^2}dx \\ &= \int_0^{x_1} \frac{x^{\zeta-1}}{1+x}dx - \frac{1}{\zeta-1}\frac{x_1^{\zeta}}{1+x_1} - \frac{1}{\zeta-1}\int_{x_1}^{\infty}\frac{x^{\zeta-1}}{(1+x)^2}dx \end{aligned} \tag{A.5.22}$$

Since the integrand in the last integral is a continuous function on $x \geq x_1$, by means of standard estimates, we can readily verify analyticity of the integral in $\Re\zeta < 2$. The whole expansion on the right hand side of (A.5.22) is therefore again meromorphic in $0 < \Re\zeta < 2$ with a simple pole at $\zeta = 1$. The corresponding residue can be computed by means of the fundamental theorem of calculus. By analyticity of the first summand, it is given by:

$$\operatorname*{Res}_{\zeta=1} \mathcal{M}\{f\}(\zeta) = -\frac{x_1}{1+x_1} - \int\limits_{x_1}^{\infty} \frac{1}{(1+x)^2} dx$$

$$= -1 \tag{A.5.23}$$

Of course this coincides with the residue we obtained from the procedure of analytic continuation by means of an asymptotic expansion, compare (A.5.20). This is not surprising since, subject to the identity principle, the functions (A.5.19) and (A.5.22) are equal for $0 < \Re\zeta < 2$. To conclude these findings, (A.5.22) provides another representation for the analytic continuation of the integral (A.5.17).

Rather uncommon is the observation that, in the present setup, integration by parts is also viable with $x_1 = 0$ since $f_2(x) \sim x$ as $x \downarrow 0$. Then, instead of (A.5.22) for $0 < \Re\zeta < 1$, we obtain

$$\mathcal{M}\{f\}(\zeta) = -\frac{1}{\zeta - 1} \int\limits_{0}^{\infty} \frac{x^{\zeta-1}}{(1+x)^2} dx. \tag{A.5.24}$$

This integral is also meromorphic in the strip $0 < \Re\zeta < 2$ and thereby furnishes a third representation for the continuation of the integral (A.5.17).

Obviously the described integration by parts procedure does not exploit any convergence properties of the asymptotic expansion for the amplitude function. Indeed, it suffices for $f(x)$ to be continuously differentiable on $x \geq x_2$ for some $x_2 > 0$ and to possess a differentiable asymptotic expansion of purely algebraic type. This immediately indicates a drawback of the integration by parts method, in comparison with the method of an asymptotic expansion, to be kept in mind. To be exact, if the asymptotic expansion of f involves non-algebraic expressions, for instance logarithms, we can make use of the second approach but the first will most likely not be effective without further modifications.

The strategy to determine the analytic continuation to the left direction beyond the boundary $\Re\zeta = 0$ is analogous and requires to study the behaviour of the integrand as $x \downarrow 0$.

A.5.2. Mellin Transforms of Fourier Transforms

We now derive a remarkable identity for the Mellin transform of a Fourier transform Φ associated with a real-valued[1] function F of finite total variation[2]. Since each function of finite total variation can be written as the difference of two monotonic functions, it is sufficient to assume F monotonic. Moreover, we assume $F(\infty) = 1$, whence F equals a probability distribution and Φ corresponds to the class of characteristic functions, compare Section A.7 below. In this event

[1] If F is complex-valued, consider real and imaginary part separately.

[2] Recall that F is especially of finite total variation if $F(dx) = f(x)dx$ for $f \in L^1(\mathbb{R})$.

$\Phi(0) = 1$, so that the existence of the integral

$$\mathcal{M}\{\Phi\}(\zeta) = \int_0^\infty t^{\zeta-1}\Phi(t)dt \tag{A.5.25}$$

for some $\zeta \in \mathbb{C}$ implies $\Re\zeta > 0$. Suppose that this is in fact the case, i.e., there is exists $b > 0$ such that (A.5.25) is finite for any real $0 < \zeta < b$. Then, according to Abel's lemma A.4.1, we have

$$\mathcal{M}\{\Phi\}(\zeta) = \lim_{\delta\downarrow 0} \int_0^\infty t^{\zeta-1}e^{-\delta t}\Phi(t)dt. \tag{A.5.26}$$

For fixed $\delta > 0$ and $0 < \zeta < \min\{1, b\}$ we may introduce the integral definition of Φ, interchange the order of integration and employ the property (A.5.11), to find:

$$\begin{aligned}\int_0^\infty t^{\zeta-1}e^{-\delta t}\Phi(t)dt &= \int_{-\infty}^\infty \int_0^\infty t^{\zeta-1}e^{-(\delta-ix)t}dtF(dx)\\ &= \Gamma(\zeta)\int_{-\infty}^\infty (\delta - ix)^{-\zeta}F(dx), \qquad |\arg(\delta - ix)| < \pi\end{aligned} \tag{A.5.27}$$

For fixed $\delta > 0$ the argument function maps to $(-\frac{\pi}{2}, \frac{\pi}{2})$. As $\delta \downarrow 0$, depending on positivity or negativity of the integration variable, it approaches the angle between the positive real and the negative imaginary or positive imaginary axis, respectively. More precisely, according to the rules for complex-valued power functions, for $x \in \mathbb{R}$ we obtain:

$$\begin{aligned}\lim_{\delta\downarrow 0}(\delta - ix)^{-\zeta} &= \lim_{\delta\downarrow 0} |\delta - ix|^{-\zeta} e^{-\zeta\arg(\delta-ix)}\\ &= |x|^{-\zeta} \times \begin{cases} e^{i\zeta\frac{\pi}{2}}, & \text{if } x > 0\\ e^{-i\zeta\frac{\pi}{2}}, & \text{if } x < 0\end{cases}\end{aligned}$$

Suppose that $b > 0$ may be chosen such that not only (A.5.25) exists but at the same time for $0 < \zeta < b$ we have

$$\int_{-\infty}^\infty |x|^{-\zeta} F(dx) < \infty. \tag{A.5.28}$$

Observe that this excludes $F\{0\} = 0$. In fact the characteristic functions of many known distributions with $F\{0\} \neq 0$ such as Bernoulli or Poisson do not possess a Mellin transform. Then at best only $\Phi(t) - F\{0\}$ has a Mellin transform. Or $e^{it\mu}\Phi(t)$ for arbitrary $\mu \in \mathbb{R} \setminus \{0\}$. Finally, plugging (A.5.27) into (A.5.26), for $0 < \zeta < b$ Lebesgue's dominated convergence theorem applies. Since the branch point at the origin has F-measure zero, we arrive at:

$$\begin{aligned}\mathcal{M}\left\{\Phi\right\}(\zeta) &= \Gamma(\zeta)\lim_{\delta\downarrow 0}\left[\int_{-\infty}^{0} + \int_{0}^{\infty}\right](\delta - ix)^{-\zeta}F(dx)\\ &= \Gamma(\zeta)\left\{e^{i\frac{\pi}{2}\zeta}\int_{0}^{\infty} x^{-\zeta}F(dx) + e^{-i\frac{\pi}{2}\zeta}\int_{-\infty}^{0}(-x)^{-\zeta}F(dx)\right\} \end{aligned}\tag{A.5.29}$$

Not only does this latter representation offer some computational advantages when the F-integral is easier to evaluate than (A.5.25). It also endowes us with an identity to determine the analytic continuation and permitts us to describe the properties of $\mathcal{M}\left\{\Phi\right\}(\zeta)$ in terms of F itself rather than in terms of its characteristic function Φ. This is what we shall discuss below. For convenience we denote the sum of the two F-integrals in (A.5.29) by

$$\eta(\zeta) := e^{i\frac{\pi}{2}\zeta}\int_{0}^{\infty} x^{-\zeta}F(dx) + e^{-i\frac{\pi}{2}\zeta}\int_{-\infty}^{0}(-x)^{-\zeta}F(dx). \tag{A.5.30}$$

From a viewpoint of integral transforms each integral constitutes a Mellin-Stieltjes transform of the distribution function F. We adopt this notion from [Widder, 1946] who similarly describes Laplace transforms of integrals with respect to functions of bounded variation, i.e., of finite total variation, rather than with respect to the Lebesgue measure. The origin of this definition evidently is that we consider an integral of Stieltjes-type. Except for a prefactor the sum of integrals (A.5.30) coincides with the k-th moment of F if $\zeta = -k$ for $k \in \mathbb{N}$, provided this moment exists. However, we shall observe below that, despite the integral representations (A.5.30) may not converge for negative integers, the analytic continuation of $\eta(\zeta)$ can still be well-defined for such values. We proceed with an investigation of the properties of the above Mellin-Stieltjes transforms for different types of functions F. To establish definite statements on the analyticity of Mellin-Stieltjes transforms, we advise the reader to modify the criteria of §§5.5 and 5.51 in [Copson, 1970]. Finally, since we may decompose any distribution function according to the Lebesgue decomposition theorem into the sum of its discrete, absolutely continuous and singular part, it is no restriction to study each kind separately.

Omitting singular distributions we first assume F is discrete, i.e., a step function with jump points $D_F \subset \mathbb{R}$. In this situation each integral in (A.5.30) becomes a finite or infinite sum. If the F-atoms are located on equidistant points along each segment of the real axis the series is particularly of Dirichlet-type. If the set of atoms does not lie dense in a neighborhood of the origin, i.e., especially if $\xi_0 := \inf\left\{|\xi| : \xi \in D_F\right\} > 0$, each of the two integrals in (A.5.30) is easily seen to converge absolutely and uniformly in any compact subset of the half plane $\Re\zeta > 0$ since we have $\int_{-\infty}^{\infty} F(dx) = 1$ and thus

$$|\eta(\zeta)| \leq \xi_0^{-\Re\zeta}.$$

In this event we expect analyticity of $\eta(\zeta)$ in the right ζ-half plane and therefore, if the Mellin transform (A.5.25) establishes an analytic function, also of the associated analytic continuation (A.5.29). This coincides with the statement of Lemma 4.3.2 in [Bleistein and Handelsman, 1986], according to which the Mellin transform $\mathcal{M}\{g\}(\zeta)$ of an oscillatory function g is holomorphic in $\Re\zeta > 0$. But characteristic functions associated with discrete F are almost periodic and hence especially exhibit oscillatory behaviour at infinity[3].

Assume now $F(dx) = f(x)dx$, i.e., F is absolutely continuous. Then, after a simple change of variables in (A.5.30) we arrive at

$$\eta(\zeta) = e^{i\frac{\pi}{2}\zeta}\int_0^\infty x^{-\zeta} f(x)dx + e^{-i\frac{\pi}{2}\zeta}\int_0^\infty x^{-\zeta} f(-x)dx. \tag{A.5.31}$$

These integrals constitute the Mellin transforms with argument $1-\zeta$ of f and $f(-\cdot)$, respectively. Consequently, in contrast to the common Mellin transform, regarding the singularities of $\eta(\zeta)$ possible cancellations may occur. As an example let $f(x) = \pi^{-1}(1+x^2)^{-1}$. In this event each of the integrals in (A.5.31) is analytic in $-1 < \Re\zeta < 1$. After a simple substitution these are readily referred to the beta function:

$$\begin{aligned}\eta(\zeta) &= \frac{1}{\pi}\left\{e^{i\frac{\pi}{2}\zeta} + e^{-i\frac{\pi}{2}\zeta}\right\}\int_0^\infty \frac{x^{-\zeta}}{1+x^2}dx \\ &= \frac{\sin\left\{\frac{\pi}{2}(\zeta+1)\right\}}{\pi}\Gamma\left(\frac{1-\zeta}{2}\right)\Gamma\left(\frac{1+\zeta}{2}\right) \\ &= 1\end{aligned}$$

For the second equality we refer to (B.3.2), whereas the last equality employs the reflection formula for the gamma function (B.2.17). It shows that, despite f does not have any finite moments the function $\eta(\zeta)$, particularly its analytic continuation, is still well-defined for negative integer arguments. The reason for this is the presence of the complex unit in the integral definition of $\eta(\zeta)$, generating zeros that cancel with the poles of the beta function, which corresponds to the moment integral.

To determine the analytic structure of $\eta(\zeta)$ more generally, in particular to characterize the behaviour of each Mellin transform in (A.5.31) we could simply refer to chapter 4.3 in [Bleistein and Handelsman, 1986]. Instead, however, it seems appropriate to elaborate these properties separately. For this purpose we first write (A.5.31) for $r > 0$ in the following form:

$$\eta(\zeta) = \int_0^r x^{-\zeta}\left\{e^{i\frac{\pi}{2}\zeta} f(x) + e^{-i\frac{\pi}{2}\zeta} f(-x)\right\}dx + \int_{|x|>r} (-ix)^{-\zeta} f(x)dx \tag{A.5.32}$$

[3] Such a behaviour is possibly not observable if $F\{0\} \neq 0$. Then a constant term appears in the characteristic function, since $\int_{\{0\}} e^{itx}F(dx) = F\{0\}$ for any $t \in \mathbb{R}$.

Since the integral along the segment $|x| > r$ is absolutely convergent for $\Re\zeta > 0$ and certainly uniformly convergent in any compact subset therein, the partition (A.5.32) exhibits that the analytic behaviour of $\eta(\zeta)$ in $\Re\zeta > 0$ is determined solely by the behaviour of the density f in a neighborhood of the origin[4]. The decomposition (A.5.32) immediately implies if f vanishes in a neighborhood of the origin we can choose $r > 0$ small enough such that the first integral equals zero and the above representation of $\eta(\zeta)$ converges absolutely for all $\Re\zeta > 0$. But also if f approaches the origin from each side exponentially fast the first integral in (A.5.32) remains absolutely convergent for arbitrary $\Re\zeta > 0$. If this happens to hold only for one side, the above integrals in $\eta(\zeta)$ can be absolutely convergent for any $\Re\zeta > 0$ only if f vanishes in the opposite one-sided neighborhood. Since f is assumed to be a density function, it is non-negative and therefore oscillatory behaviour in a neighborhood of the origin with infinitely many changes in sign are impossible. Finally if f is algebraic in a neighborhood of the origin it is easy to construct examples such that (A.5.32) can be continued to a meromorphic function in $\Re\zeta > 0$. An exception occurs if for some $\rho > 0$ the density f possesses an absolutely convergent power series expansion

$$f(x) = \sum_{k=0}^{\infty} c_k x^k, \qquad \text{for } |x| < \rho. \tag{A.5.33}$$

Then, upon choosing $r < \rho$, plugging this into the first integral in (A.5.32), interchanging the order of summation and integration, we arrive at:

$$\begin{aligned}\int_0^r x^{-\zeta}\left\{e^{i\frac{\pi}{2}\zeta} f(x) + e^{-i\frac{\pi}{2}\zeta} f(-x)\right\} dx &= \sum_{k=0}^{\infty} c_k \left\{e^{i\frac{\pi}{2}\zeta} + e^{-i\frac{\pi}{2}\zeta}(-1)^k\right\} \int_0^r x^{-\zeta+k} dx \\ &= 2\sum_{k=0}^{\infty} c_k \frac{\cos\left\{\frac{\pi}{2}(\zeta-k)\right\}}{1-\zeta+k} e^{i\frac{\pi}{2}k} r^{1-\zeta+k}\end{aligned} \tag{A.5.34}$$

The series converges absolutely for arbitrary $\zeta \in \mathbb{C}$ and fixed $r < \rho$. Particularly at $\zeta = k_0 + 1$ for $k_0 \in \mathbb{N}_0$, where the k_0-th denominator vanishes, the corresponding numerator equals zero. This indicates possible analyticity of $\eta(\zeta)$ and of (A.5.29) in the half-plane $\Re\zeta > 0$.

Summarizing, in case of an absolutely continuous F the chances are good for the Mellin transform $\mathcal{M}\{\Phi\}(\zeta)$ to be analytic in $\Re\zeta > 0$ or to be analytically continuable therein if $f(x)$ approaches the origin exponentially fast, vanishes in a neighborhood of the origin or shows either the former or the latter behaviour in each neighbourhood. Alternatively, analyticity of f in a neighborhood of the origin also has a positive effect.

[4]This is in accordance with the method of stationary phase, compare for instance [Olver, 1974], stating the main contribution to the asymptotic behaviour of the Fourier $\Phi(t)$ as $t \to \infty$ comes from the neighborhood of the origin, where e^{itx} oscillates lowest. But the large t-behaviour of $\Phi(t)$ in turn determines the analytic structure of the associated Mellin transform.

A.6. Iterated Mellin-Type Integrals of Convolution-Type Amplitudes

Analogous to the one-dimensional Mellin transform, the *two-dimensional*, *double* or *iterated Mellin transform* associated with a function $f(s,t)$ of two variables $s,t \geq 0$ has the integral definition

$$\mathcal{M}\{f\}(\xi,\zeta) := \int_0^\infty t^{\zeta-1} \int_0^\infty s^{\xi-1} f(s,t)dsdt. \tag{A.6.1}$$

Depending on $f(s,t)$, the evaluation of this integral can be more or less difficult. Our particular interest is confined to the amplitude functions

$$f_1(s,t) := g(s+t), \tag{A.6.2}$$

$$f_2(s,t) := \mathbb{I}\{s>t\}\, g(s-t), \tag{A.6.3}$$

for a complex-valued function $g(v)$ of one variable $v \geq 0$. Moreover, without loss of generality we assume g possesses a one-dimensional Mellin transform $\mathcal{M}\{g\}(\zeta)$ with an absolutely convergent integral representation for all $\zeta \in S_{\mathcal{M}}\{g\}$. For these functions some considerable simplifications occur in determining the iterated Mellin transform and its region of absolute convergence, respectively denoted by $S^j_{\mathcal{M}} \subset \mathbb{C}^2$ for $j \in \{1,2\}$. To ascertain this set for the iterated Mellin transform of the function (A.6.2) we make two substitutions and formally interchange the order of integration, which leads to:

$$\begin{aligned}
\mathcal{M}_1(\xi,\zeta) :=& \int_0^\infty t^{\zeta-1} \int_0^\infty s^{\xi-1} g\left(t\left(\frac{s}{t}+1\right)\right) dsdt \\
=& \int_0^\infty t^{\xi+\zeta-1} \int_0^\infty u^{\xi-1} g\left(t\left(u+1\right)\right) dudt \\
=& \int_0^\infty u^{\xi-1} \int_0^\infty t^{\xi+\zeta-1} g\left(t\left(u+1\right)\right) dtdu \\
=& \int_0^\infty u^{\xi-1} (1+u)^{-(\xi+\zeta)} du \int_0^\infty v^{\xi+\zeta-1} g(v)dv \\
=& \frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\xi+\zeta)} \mathcal{M}\{g\}(\xi+\zeta)
\end{aligned}$$

For the last equation we referred to the beta function in terms of the identity (B.3.2) and to the one-dimensional Mellin transform of g. For $\Re\xi, \Re\zeta > 0$ with $\xi+\zeta \in S_{\mathcal{M}}\{g\}$ the integrals on the right hand side are absolutely convergent and all interchanges in the order of integration are permitted by Fubini's theorem. Hence, the initial double integral is absolutely convergent if and only if each of the obtained single integrals is, implying that the region of absolute convergence of the iterated Mellin transform coincides with the intersection of the indicated three regions.

Summarizing we have verified

$$\begin{aligned}\mathcal{M}_1(\xi,\zeta) &= \frac{\Gamma(\xi)\Gamma(\zeta)}{\Gamma(\xi+\zeta)}\mathcal{M}\left\{g\right\}(\xi+\zeta)\,,\\ S^1_{\mathcal{M}} &= \left\{(\xi,\zeta)\in\mathbb{C}^2 : \Re\xi,\Re\zeta>0 \text{ and } \xi+\zeta\in S_{\mathcal{M}}\left\{g\right\}\right\}.\end{aligned}\tag{A.6.4}$$

Similarly, concerning the iterated Mellin transform corresponding to (A.6.3), upon substituting and formally interchanging the order of integration, we derive:

$$\begin{aligned}\mathcal{M}_2(\xi,\zeta) &:= \int_0^\infty t^{\zeta-1}\int_t^\infty s^{\xi-1} g\left(t\left(\frac{s}{t}-1\right)\right)dsdt\\ &= \int_0^\infty t^{\xi+\zeta-1}\int_1^\infty u^{\xi-1} g\left(t\left(u-1\right)\right)dudt\\ &= \int_1^\infty u^{\xi-1}\int_0^\infty t^{\xi+\zeta-1} g\left(t\left(u-1\right)\right)dtdu\\ &= \int_1^\infty u^{\xi-1}(u-1)^{-(\xi+\zeta)}du\int_0^\infty v^{\xi+\zeta-1} g\left(v\right)dv\\ &= \frac{\Gamma(\zeta)\Gamma(1-\xi-\zeta)}{\Gamma(1-\xi)}\mathcal{M}\left\{g\right\}(\xi+\zeta)\end{aligned}$$

Again for the last equality we introduced the beta function in terms of (B.3.2) and the one-dimensional Mellin transform of g. The corresponding integral representations appearing in the above equation are absolutely convergent if $\Re\zeta > 0$, $\Re(\xi+\zeta) < 1$ and $\xi+\zeta \in S_{\mathcal{M}}\{g\}$. Consequently we have

$$\begin{aligned}\mathcal{M}_2(\xi,\zeta) &= \frac{\Gamma(\zeta)\Gamma(1-\xi-\zeta)}{\Gamma(1-\xi)}\mathcal{M}\left\{g\right\}(\xi+\zeta)\,,\\ S^2_{\mathcal{M}} &= \{(\xi,\zeta)\in\mathbb{C}^2 : \Re\zeta>0 \text{ and } \xi+\zeta\in S_{\mathcal{M}}\left\{g\right\} \text{ with } \Re(\xi+\zeta)<1\}.\end{aligned}\tag{A.6.5}$$

Observe that the function $f_1(s,t)$ is symmetric with respect to $s,t\in\mathbb{R}$ which is also reflected by the associated iterated Mellin transform. On the other hand, the function $f_2(s,t)$ is highly asymmetric and its Mellin transform exhibits a similar property. If we eventually want to consider the iterated transforms as functions of $\xi\in\mathbb{C}$ for fixed $\zeta\in\mathbb{C}$ or vice versa, additional calculations are necessary to verify, for instance, their analyticity.

The above formulae essentially simplify the derivation of the iterated Mellin transforms by referring to the one-dimensional counterpart. Concerning an inversion formula for (A.6.1), such a reference is most likely not possible. The integral

$$f(s,t) = \frac{1}{(2\pi i)^2} \int_{x_0 - i\infty}^{x_0 + i\infty} \int_{u_0 - i\infty}^{u_0 + i\infty} \mathcal{M}\left\{f\right\}(w,z)\, s^{-w} t^{-z} dw dz, \tag{A.6.6}$$

with appropriately specified $u_0, x_0 \in \mathbb{R}$, was already examined in a concise article by [Reed, 1944], appearing to be one of the first treatments of iterated Mellin transforms. However, the proof of theorem II in this article, which we would require for our applications, is not reasonable. There the author gives sufficient conditions for the absolute convergence of (A.6.6), including piecewise continuity of $f(s,t)$. The function

$$f(s,t) = \mathbb{I}\left\{0 < s < 1\right\} \mathbb{I}\left\{0 < t < 1\right\}$$

satisfies all of these conditions but since $\mathcal{M}\left\{f\right\}(\xi,\zeta) = (\xi\zeta)^{-1}$ it is easy to see that (A.6.6) is certainly not absolutely convergent. In fact, the proof of Theorem II in [Reed, 1944] is actually incomplete, leaving as a conclusion the reference to the inversion formula for the two-dimensional Fourier-transform but without a source.

A.7. Characteristic Functions

This section is devoted to the Fourier transforms of probability distribution functions. The latter, often simply referred to as *distributions*, denotes a class of functions defined on the real axis, of which each member F satisfies the following properties:

- F is non-decreasing: For any $a \leq b$ we have $F(a) \leq F(b)$.
- F is right-continuous: $F(\xi+) = F(\xi)$
- F is bounded at infinity:

$$\lim_{\xi \to \xi_0} F(\xi) = \begin{cases} 0, & \xi_0 = -\infty \\ 1, & \xi_0 = \infty \end{cases}$$

As a consequence $\int_{\mathbb{R}} |F|(dx) = \int_{\mathbb{R}} F(dx) = 1$, i.e., each distribution F is of bounded variation or equivalently of finite total variation on $\mathbb{R}$. This implies that $\int_E F(dx) \in [0,1]$ for any $E \subset \mathbb{R}$, showing that F in fact establishes a probability measure. We begin this part of the appendix with a brief overview on the important class of distributions before proceeding with their Fourier transforms. For a more extensive treatment of the topic of characteristic functions we refer to [Lukacs, 1970].

A.7.1. Distributions

According to the above properties, a distribution function is always bounded and its discontinuities can only be of jump-type, i.e., $1 \geq F(\xi) - F(\xi-) \geq 0$. We refer to $F\{\xi\} := F(\xi) - F(\xi-)$ as the point probability of $\xi \in \mathbb{R}$. Then $F\{\xi\} = 0$ if and only if F is continuous at ξ. If $F\{\xi\} > 0$ we denote ξ as a saltus or an F-atom and note that this set is at most countable, compare Theorem 1.1.1 in [Lukacs, 1970]. This fact enables a separation of F into three ingredients with a different nature.

Theorem A.7.1 (decomposition of distributions). *Any distribution function F can be decomposed for $a, b, b_1, b_2 \geq 0$ with $a + b = 1$ and $b_1 + b_2 = b$ in the following form:*

$$\begin{aligned} F(\xi) &= aF_d(\xi) + bF_c(\xi) \\ &= aF_d(\xi) + b_1 F_{ac}(\xi) + b_2 F_s(\xi) \end{aligned} \tag{A.7.1}$$

Here, F_d represents the discrete part, whereas F_c, F_{ac} and F_s refer to the continuous part, each of them being distributions on their own. More precisely, F_{ac} is absolutely continuous and F_s is singular.

For a proof we refer to Theorems 1.1.2 and 1.1.3 in [Lukacs, 1970]. The preceding theorem gives rise to the following classification of distributions:

- A distribution F is of purely discrete type if $a = 1$, which implies the existence of a countable set $D \subset \mathbb{R}$ with $\int_D F(dx) = 1$. Possibly the most important member of this class is the Dirac distribution with mass at $\xi_0 \in \mathbb{R}$, also known as the degenerate distribution. It has the distribution function

  $$\mathbb{I}\{\xi_0 \leq \xi\} = \begin{cases} 0, & \text{if } \xi < \xi_0, \\ 1, & \text{if } \xi \geq \xi_0. \end{cases}$$

  This enables us to write arbitrary discrete distributions F with jump points ξ_j and point probabilities $F\{\xi_j\}$ for $j \in J \subset \mathbb{Z}$ in the form

  $$F(\xi) = \sum_{j \in J} F\{\xi_j\} \mathbb{I}\{\xi_j \leq \xi\}, \tag{A.7.2}$$

  where $\sum_{j \in J} F\{\xi_j\} = 1$. A discrete distribution is thus a step function that is uniquely determined by its jump points. Special discrete distributions are those with mass on a set of equidistant points, referred to as lattice distributions. The jump points are then also denoted as lattice points and can be written in the form $a + jd$ for integer values j, constant $a \in \mathbb{R}$ and span $d > 0$.

- If $b_1 = 1$ the distribution F is of purely absolutely continuous type. This is equivalent to the existence of a function f such that F can be represented as a Lebesgue integral in terms of its derivative $F' = f$. More precisely, $F(\xi) = \int_{-\infty}^{\xi} f(x)dx$ for any $\xi \in \mathbb{R}$, i.e., $F(dx) = f(x)dx$. The function f is called the Lebesgue density of F. As a consequence of this integral representation, we have $\int_N F(dx) = 0$ for any set N of Lebesgue measure zero. Conversely, any function $f \geq 0$ with $\int_{\mathbb{R}} f(x)dx = 1$ establishes the density of a probability distribution.

- For $b_2 = 1$ we refer to F as a purely singular distribution. In this case it is a continuous function whose derivative equals zero Lebesgue almost everywhere. Moreover, there exists a set N of Lebesgue measure zero with $\int_N F(dx) = 1$.

If none of the parameters a, b_1, b_2 equals 1, F is said to be a mixture distribution. In this event the function, for instance, can be continuously increasing with jump points. Throughout this thesis we mostly confine to distributions of pure type, particularly to absolutely continuous and discrete families.

It follows from the basic properties that new distributions can be created by means of linear combinations of the form $\sum_{j=1}^{k} a_j F_j(\xi)$ for $k \in \mathbb{N}$, distributions F_j and coefficients $a_j > 0$ that sum to one. In calculus such sums are referred to as convex combinations. If we conceive this operation as the addition, the multiplication analogue is given for distributions F_1, F_2 by the integral

$$F_1 * F_2(\xi) := \int_{-\infty}^{\infty} F_1(\xi - x) F_2(dx). \tag{A.7.3}$$

This is the *convolution* or *Faltung* of F_1 and F_2, the existence of the integral being an immediate consequence of the boundedness of F_1. Compare also with the definition (A.1.3) for densities.

Distributions can be described more vividly by the notion of random variables. A *(real) random variable* X is a real-valued function of the argument $\omega \in \Omega$, i.e., $X \equiv X(\omega)$. Contrary to functions from calculus the argument ω is unknown, whence the particular value of X is random. By means of a probability measure $\mathbb{P}$, however, it is possible to make statements about the probability for X to lie in a certain range. Hence, according to the definition of distributions, each F is related to a random variable X through the identity

$$F(\xi) = \mathbb{P}\left(X \leq \xi\right). \tag{A.7.4}$$

In other words, $F(\xi)$ equals the probability for a random variable X not to exceed the threshold $\xi \in \mathbb{R}$. We thus refer to F as the distribution corresponding to X and write $X \sim F$. Random variables are versatile. For instance, for a complex-valued function $T(x)$ of the real argument $x \in \mathbb{R}$, they enable us to write

$$\mathbb{E}\left\{T(X)\right\} = \int_{\mathbb{R}} T(x) F(dx). \tag{A.7.5}$$

Even the distribution itself possesses a representation of the above form with $T(X) = \mathbb{I}\{X \leq \xi\}$, which in fact yields (A.7.4).

A.7.2. Integral Transforms of Distributions

We now give a few examples for frequently occuring functions $T(x)$. The simplest examples result in the *k-th moment* and the *k-th absolute moment*, for $k \in N_0$, respectively denoted by

$$\mu_X(k) := \mathbb{E}\left[X^k\right], \tag{A.7.6}$$

$$\mu_{|X|}(k) := \mathbb{E}\left[|X|^k\right]. \tag{A.7.7}$$

The reader who is familiar with integral transforms will readily identify the first of the above as a bilateral Mellin transform with non-negative integer arguments. According to Theorem 1.4.2 in [Lukacs, 1970], the existence of $\mu_{|X|}(k_0)$ for $k_0 \in \mathbb{N}$ suffices, to conclude the existence of (A.7.6) and (A.7.7) for any $0 \leq k \leq k_0$. Depending on the tail behaviour of F, both integrals will only exist for $k = 0$, for a finite number or for infinitely many $k \in \mathbb{N}_0$. This is particularly described by the following theorem, compare p. 71 in [Cramér, 1999].

Theorem A.7.2 (Cramér). *For a distribution function F the behaviour*

$$1 - F(\xi) + F(-\xi) = \mathcal{O}(\xi^{-k_0})$$

as $\xi \to \infty$ for $k_0 \in \mathbb{N}$ implies the existence of all moments of order $k < k_0$.

In applications, moments play an important role to describe distributions. Yet, this characterization is incomplete, since a sequence of moments does not uniquely determine a distribution. A more appropriate transform is given by the well-known Fourier-type integral

$$\Phi_X(t) := \mathbb{E}\left[e^{itX}\right]. \tag{A.7.8}$$

The absolute and with respect to $t \in \mathbb{R}$ uniform convergence of this integral for any distribution F holds by finiteness of the total variation. Generally speaking, (A.7.8) constitutes the Fourier-Stieltjes transform of F. In the probabilistic context we rather refer to Φ_X as the *characteristic function* of F. Moreover, instead of Φ_X we occassionally write Φ_F, to indicate the connection to F, or solely Φ if the attribution is obvious.

A.7.3. Elementary Properties of Characteristic Functions

As an immediate consequence of its integral definition (A.7.8) the characteristic function associated with the distribution F naturally exhibits certain properties. Indicating the complex conjugate by an overline, these are in particular:

$$\text{(A.7.9)}\quad \begin{cases} \Phi_X(0) = 1 \\ |\Phi_X(t)| \le 1 \\ \overline{\Phi}_X(t) = \Phi_X(-t) \\ \Phi_X(t) \text{ is a uniformly continuous function of } t \in \mathbb{R}. \\ \text{For real } a \neq 0 \text{ and } b \in \mathbb{R} \text{ the characteristic function of } aX+b \text{ equals } e^{itb}\Phi_X(at). \end{cases}$$

Furthermore, since Φ_X is defined as an integral with respect to F, which, according to Lebesgue's decomposition theorem A.7.1, can be a composition of up to three different types of functions, for $a, b_1, b_2 \ge 0$ with $a + b_1 + b_2 = 1$ we can always write

$$\text{(A.7.10)}\qquad \Phi_X(t) = a\Phi_d(t) + b_1\Phi_{ac}(t) + b_2\Phi_s(t).$$

Accordingly, each of the summands Φ_d, Φ_{ac} and Φ_s is associated with a discrete, absolutely continuous and singular distribution, respectively. Each component possesses special characteristics, which are pointed out in the main part of this work, particularly in Subsection 2.4.1. Although most interest is confined to absolutely continuous distributions we briefly mention a remarkable property of those of discrete type. More precisely, of lattice distributions. The theorem below allows to infer additional general properties of characteristic functions.

Theorem A.7.3 (lattice distributions). *The characteristic function Φ_X corresponds to a lattice distribution if and only if it attains the value one for a non-zero real-argument, i.e., if there exists $t_0 \in \mathbb{R} \setminus \{0\}$ with $|\Phi_X(t_0)| = 1$.*

This statement can be found as Theorem 2.1.4 in [Lukacs, 1970]. As a consequence, $|\Phi_X(t)| < 1$ almost everywhere if Φ_X is associated with a non-degenerate distribution. Moreover, if Φ_X and $\frac{1}{\Phi_X}$ both establish characteristic functions we necessarily have $\Phi_X(t) = e^{it\xi_0}$ for some $\xi_0 \in \mathbb{R}$.

Proof. Suppose F constitutes a lattice distribution with lattice points $a + jd$ for $a \in \mathbb{R}$, $d > 0$ and $j \in J$ for a discrete subset $J \subset \mathbb{Z}$. According to (A.7.2), the associated characteristic function is then given by

$$\Phi_X(t) = e^{ita} \sum_{j \in J} F\{a + jd\}\, e^{itjd}.$$

Evidently, $\left|\Phi_X(2\pi d^{-1})\right| = 1$. Conversely assume Φ_X satisfies $|\Phi_X(t_0)| = 1$ for $t_0 \in \mathbb{R} \setminus \{0\}$. We can therefore write $\Phi_X(t_0) = e^{it_0 c}$ for an appropriate $c \in \mathbb{R}$, implying

$$\int_{-\infty}^{\infty} e^{it_0(x-c)} F(dx) = 1.$$

By comparison of real and imaginary parts we conclude

$$\int_{-\infty}^{\infty} \{1 - \cos(t_0(x - c))\} F(dx) = 0.$$

But the function in the integrand is non-negative and continuous. This shows, that the discontinuities of F must be contained in the set of zeros of the sine function, i.e., F has jump points at $c + 2\pi t_0^{-1} j$ for some integers j. Hence, F is of lattice type. ∎

Before we present the properties of characteristic functions, we briefly outline the link to the moments of the associated distribution. There is in fact a very close connection, which is extensively discussed in §2.3 in [Lukacs, 1970].

Theorem A.7.4. *For $k \in \mathbb{N}_0$ the following holds:*

(1) k-times differentiability of $\Phi_X(t)$ at $t = 0$ implies the existence of all moments up to order k if k is even, but only up to order $k - 1$ if k is odd.

(2) If the k-th absolute moment of F exists the corresponding characteristic function has derivatives up to order k, which can be computed by differentiating under the integral sign:

$$\Phi_X^{(j)}(t) = i^j \int_{-\infty}^{\infty} x^j e^{itx} F(dx)$$

The first statement can not be improved since there are indeed distributions with infinite moments of all order, although the corresponding characteristic function is once continuously differentiable on the entire real axis. Moreover, it must be emphasized that the existence of moments especially depends on the differentiability at the origin only. Many distributions with a finite sequence of moments have a characteristic function that can be differentiated infinitely many times but merely on $\mathbb{R} \setminus \{0\}$.

We proceed with Theorem 2.3.3 in [Lukacs, 1970] which gives conditions for the local approximability of characteristic functions.

Theorem A.7.5. *Provided the n-th absolute moment of F exists, for appropriate coefficients c_j the associated characteristic function can be expanded as $t \to 0$ in the form*

$$\Phi_X(t) = 1 + \sum_{j=1}^{n} c_j (it)^j + o(t^n). \tag{A.7.11}$$

Conversely, validity of a representation of the form (A.7.11) implies the existence of all moments of F up to order n or $n-1$, respectively if this is an even or an odd integer. In these circumstances $c_j = \frac{\mu_X(j)}{j!}$ for $1 \leq j \leq n - 1$ and also for $j = n$ if this is even.

The existence of all moments is not sufficient to conclude the convergence of the power series

$$\Phi_X(t) = \sum_{k=0}^{\infty} \frac{\mu_X(j)}{j!}(it)^j. \tag{A.7.12}$$

Instead it is additionally required to verify the finiteness of its radius of convergence, i.e., to show that

$$\limsup_{k\to\infty} \left(\frac{|\mu_X(k)|}{k!}\right)^{\frac{1}{k}} = L < \infty.$$

Only if this is guaranteed, $\Phi_X(t)$ constitutes an analytic function of t in a neighborhood of the origin and the radius of convergence then equals L^{-1}. Finally, a noteworthy consequence of Theorem A.7.5 is presented with Theorem 4.1.1 in [Lukacs, 1970], which states that the only characteristic function of the form $\Phi_X(t) = 1 + o(t^2)$ as $t \to 0$ is given by $\Phi_X(t) = 1$.

We proceed with a result that relates the behaviour of the distribution F at infinity to that of Φ_X in a neighborhood of the origin. This connection is already suggested by the preceding statements on the existence of moments.

Lemma A.7.1. *If $F(-x) + 1 - F(x) = \mathcal{O}\{x^{-\alpha}\}$ as $x \to \infty$ for $\alpha > 1$ then $\Phi_X(t) = 1 + \mathcal{O}(t)$ as $t \to 0$.*

Proof. Under the above conditions there exists a constant $A > 0$ such that $F(-x) \le Ax^{-\alpha}$ and $1 - F(x) \le Ax^{-\alpha}$ as $x \to \infty$. Hence, upon integrating by parts we obtain:

$$\begin{aligned}
\Phi_X(t) &= \left[\int\limits_{(-\infty,0)} + \int\limits_{[0,\infty)}\right] e^{itx} F(dx) \\
&= \left[e^{itx}F(x)\right]_{(-\infty,0)} - it \int\limits_{(-\infty,0)} e^{itx}F(x)dx \\
&\qquad + \left[e^{itx}(F(x)-1)\right]_{[0,\infty)} - it \int\limits_{[0,\infty)} e^{itx}(F(x)-1)dx \\
&= 1 - it \int\limits_{(-\infty,0)} e^{itx}F(x)dx + it \int\limits_{[0,\infty)} e^{itx}(1-F(x))dx
\end{aligned}$$

Each of the two resulting integrals converges absolutely and uniformly with respect to $t \in \mathbb{R}$, thus completing the proof. ∎

The following theorem, see Theorem 3.1.1 in [Lukacs, 1970], ultimately justifies the importance of characteristic functions.

Theorem A.7.6 (uniqueness). *Two distributions coincide if and only if their characteristic functions match.*

The proof of this elementary theorem relies on the approximability of Φ_X by trigonometric polynomials. Its validity, however, can be verified more illustrative by means of inversion formulae, to be discussed in Subsection A.7.5 below.

Given an arbitrary distribution F, the function $\bar{F}(\xi) := 1 - F(-\xi-)$ constitutes a new distribution, referred to as the *conjugate distribution*. The corresponding random variable differs from X by a negative sign, i.e., it is $-X$. By comparison with (A.7.8), this gives rise to the characteristic function

$$\overline{\Phi}_X(t) = \Phi_X(-t) = \Phi_{-X}(t). \tag{A.7.13}$$

A distribution is then referred to as *symmetric* if $F = \bar{F}$. The associated finite moments of odd order are all equal to zero. Throughout this work, symmetric distributions play a pivotal role. According to Theorem 3.1.2 in [Lukacs, 1970], they can be characterized uniquely by a simple property.

Theorem A.7.7 (symmetric distributions). *The symmetric distributions are exactly those with a real-valued and even characteristic function.*

Before we provide an overview on the scope of the class of characteristic functions, we present an analogue of the convolution property (A.1.4) for Fourier transforms.

Theorem A.7.8 (convolution theorem / product rule). *A distribution equals the convolution product of two distributions if and only if its characteristic function is composed of the multiplication of two characteristic functions.*

Proof. This statement is most easily verified by means of expectations, by exploiting the independence of random variables. Indeed, for independent $X_1 \sim F_1$ and $X_2 \sim F_2$ we have

$$\mathbb{E}\left[e^{it(X_1+X_2)}\right] = \mathbb{E}\left[e^{itX_1}\right]\mathbb{E}\left[e^{itX_2}\right].$$

But $X_1 + X_2 \sim F_1 * F_2$. ∎

A.7.4. Examples for Characteristic Functions

Similar to the class of Fourier transforms the class of characteristic functions is vast. A standard example is the Gauss distribution, which was already introduced in (A.1.9) and (A.1.10). Besides, the Lévy and the Cauchy distribution are further examples of families whose characteristic functions exhibit exponential decay. With shift parameter $\mu \in \mathbb{R}$ and scale $\sigma > 0$ the latter is given by the pair

$$\begin{aligned} f(x) &= \frac{1}{\pi}\frac{b}{(\xi-\mu)^2+\sigma^2}, \\ \Phi(t) &= e^{i\mu t - \sigma|t|}. \end{aligned} \tag{A.7.14}$$

Those three distributions constitute special members of the class of alpha-stable distributions with the additional similarity that their densities can be represented in terms of elementary functions, which is an exception in this class. Somehow converse to (A.7.14) is the Laplace distribution with mean $\mu \in \mathbb{R}$ and scale $\sigma > 0$:

$$\begin{aligned} f(x) &= \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{2\sigma}} \\ \Phi(t) &= \frac{e^{i\mu t}}{1+\sigma^2 t^2} \end{aligned} \tag{A.7.15}$$

This was already discussed in Example A.2.1. Further examples of families with algebraically decreasing characteristic functions are the gamma distribution and the geometric stable distributions. In their simplest parametrization, the latter are for $\sigma > 0$, $p = 1$ and $0 < \beta \leq 2$ of the form

$$\Phi(t) = \left\{1 + |\sigma t|^\beta\right\}^{-p}. \tag{A.7.16}$$

Clearly, also (A.7.15) is a member of this class. The verification of (A.7.16) as a characteristic function was accomplished in Theorem 4.5.3 in [Lukacs, 1970]. Contrary to the aforementioned distributions, this is a rather difficult task, since elementary representations of the corresponding density or distribution function are unavailable and therefore supportive results are required. In accordance with the product rule, compare Theorem A.7.8, the result (A.7.16) easily extends to arbitrary $p \in \mathbb{N}$. In passing we note that geometric and alpha stable distributions are related through the identity

$$\frac{1}{\Gamma(p)} \int_0^\infty x^{p-1} e^{-(1+|\sigma t|^\beta)x} dx = \left\{1 + |\sigma t|^\beta\right\}^{-p}. \tag{A.7.17}$$

Finally, further typical examples for characteristic functions are given by convolutions of the uniform distribution. This is a continuous distribution concentrated on a compact subset $[a, b] \subset \mathbb{R}$ for real numbers $a < b$. Also known as the rectangular distribution, the associated characteristic function was already derived in equation (A.1.6). So far we only mentioned characteristic functions corresponding to absolutely continuous distributions. For completeness but also to illustrate the scope we give examples for two discrete and a singular distribution. The characteristic function

$$\Phi(t) = e^{\lambda(e^{it}-1)} \tag{A.7.18}$$

corresponds to the Poisson distribution with parameter $\lambda > 0$. It is one of the examples in which the integral definition of Φ equals an infinite sum that possesses a finite representation. The situation is different for the discrete distribution associated with the Weierstrass function

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{e^{it5^k}}{2^{k+1}}, \tag{A.7.19}$$

compare p. 23 in [Lukacs, 1970], which is in addition nowhere differentiable. Finally, examples for singular distributions are always exotic. The characteristic function below is of this type and was constructed on pp. 28-29 in [Esseen, 1945] for a non-decreasing sequence of numbers $2 < \lambda_1 \leq \lambda_2 \leq \ldots$ that grows to infinity and satisfies $\prod_{r=1}^{\infty} \left(1 - \frac{1}{\lambda_r}\right) = 0$:

$$\Phi(t) = \prod_{r=1}^{\infty} \left\{1 - \frac{1}{\lambda_r} + \frac{1}{\lambda_r} e^{it2^{-r}}\right\} \tag{A.7.20}$$

A possible reason why standard examples for characteristic functions of absolutely continuous distributions mostly encompass Gaussian, Cauchy and rectangular distribution is that density and Fourier transform are both known and expressible in terms of elementary functions. This is, however, misleading since readers that are unfamiliar with the topic of characteristic functions might assume the asymptotic behaviour is always of a similar type. It is in fact not difficult to find counterexamples. By means of Polya's condition for instance, compare Theorem 4.3.1 in [Lukacs, 1970], one can not only show that (A.7.16) is associated with a probability distribution for arbitrary $p > 0$ and $0 < \beta \leq 1$, but it also enables us to verify many more as characteristic functions:

- The logarithmically decaying functions

  $$\Phi(t) = \log^{-p}\left\{e + \sigma^\beta |t|^\beta\right\} \tag{A.7.21}$$

  for $p, \sigma > 0$ and $0 < \beta \leq 1$ are characteristic functions, neither of them being absolutely integrable on an infinite segment of the real axis. According to the product and the chain rule, for $t > 0$ we have:

  $$\begin{aligned}
  \Phi'(t) &= -p \log^{-p-1}(e + \sigma^\beta t^\beta)(e + \sigma^\beta t^\beta)^{-1} \beta \sigma^\beta t^{\beta-1} \\
  \Phi''(t) &= (\beta - 1) t^{-1} \Phi'(t) \\
  &\qquad - \Phi'(t)(p+1) \log^{-1}(e + \sigma^\beta t^\beta)(e + \sigma^\beta t^\beta)^{-1} \beta \sigma^\beta t^{\beta-1} \\
  &\qquad - \Phi'(t)(e + \sigma^\beta t^\beta)^{-1} \sigma^\beta \beta t^{\beta-1}
  \end{aligned}$$

  The function is therefore strictly convex for the indicated parameterization, implying validity of Pólya's condition.

- Characteristic functions that decay faster than any member of the alpha stable class are given by:

$$\Phi(t) = e^{1-e^{|t|}}$$
$$\Phi(t) = \exp\left\{e - e^{e^{|t|}}\right\}$$

We close this section with an example of a distribution with known density and an even characteristic function that possesses one zero on the positive real axis.

Example A.7.2 (a characteristic function with a finite zero). Most known characteristic functions possess either none or infinitely many zeros on the finite real axis. There exist, of course, also examples with a finite number of zeros, such as the characteristic function associated with the density

$$f(x) = \frac{1}{4}e^{-|x|}(x^2 - |x| + 1). \tag{A.7.22}$$

In fact, f is obviously absolutely integrable on $\mathbb{R}$ and, since $f(0) = \frac{1}{4}$ and the polynomial $x^2 - x + 1$ has no zeros on the positive real axis, it is non-negative. Moreover, the corresponding characteristic function is readily derived from elementary calculations:

$$\begin{aligned}
\Phi(t) &= \int_{-\infty}^{\infty} e^{itx} f(x)dx \\
&= \frac{1}{2}\Re \int_{0}^{\infty} e^{-(1-it)x}(x^2 - x + 1)dx \\
&= \frac{1}{2}\Re\left\{\frac{2}{(1-it)^3} - \frac{1}{(1-it)^2} + \frac{1}{1-it}\right\} \\
&= \frac{1}{2}\left\{\frac{2-6t^2}{(1+t^2)^3} - \frac{1-t^2}{(1+t^2)^2} + \frac{1}{1+t^2}\right\}
\end{aligned}$$

We have thus shown

$$\Phi(t) = \frac{(t^2-1)^2}{(1+t^2)^3}. \tag{A.7.23}$$

This characteristic function is even and non-negative. It has zeros of double order at $t \in \{\pm 1\}$. Moreover, as a function of a complex variable it is meromorphic, exhibiting triple poles at $\pm i$ and it satisfies $\Phi(t) \sim t^{-2}$ as $|t| \to \infty$ in $\mathbb{C}$.

A.7.5. Inversion Formulae

In this subsection we present a collection of formulae, to recover the distribution corresponding to a given characteristic function. For inversion formulae concerning density functions, we refer the reader to Subsection A.1.1, yet one additional result will be derived below. As an appropriate framework for our presentation, analogous to the aforementioned section, we first extend the partial sum operator given in (A.1.14) and (A.1.15), by defining for $T > 0$ and $\xi \in \mathbb{R}$:

$$\begin{aligned} S_T F(\xi) &:= \frac{1}{2\pi} \int_{-T}^{T} e^{-it\xi} \Phi_X(t) dt \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(T(\xi - x))}{\xi - x} F(dx) \end{aligned} \tag{A.7.24}$$

This definition, however, bears some risk since the sequence of integrals will certainly diverge as $T \to \infty$ if ξ is a saltus of F. The situation can not be improved by averaging with respect to T, as in the context of (A.1.16). An attempt to fix the issue consists in an appropriate normalization of the integrals (A.7.24), which essentially leads us to Theorem 3.2.3 in [Lukacs, 1970].

Theorem A.7.9 (inversion formula for atoms). *For any characteristic function Φ_X the limit*

$$F\{\xi\} = \lim_{T\to\infty} \frac{\pi}{T} S_T F(\xi) \tag{A.7.25}$$

exists and equals the saltus of F at $\xi \in \mathbb{R}$. Moreover, $F(\xi)$ is continuous at ξ if and only if the limit equals zero.

Proof. According to (A.7.24), for any $T > 0$ we have

$$\frac{1}{2T} \int_{-T}^{T} e^{-it\xi} \Phi_X(t) dt = \int_{\{x=\xi\}} F(dx) + \int_{\{x\neq\xi\}} \frac{\sin(T(\xi - x))}{T(\xi - x)} F(dx).$$

Due to the boundedness of the sinc function and its decay at infinity, Lebesgue's dominated convergence theorem yields the asserted limit as $T \to \infty$. The statement about the continuity of F is evident. ■

Clearly, the result of the preceding theorem is only useful if F possesses jump points. By summation with respect to these jump points it can then be possible to specify the distribution function. This is no longer viable if F is not a step function. An inversion formula that is applicable for any kind of distribution can be obtained by cumulation of the partial sum operator (A.7.24) with respect to ξ. That is, more precisely, by integration with respect to ξ along a finite segment of the real axis $[a, b] \subset \mathbb{R}$, which leads to

$$\int_{a}^{b} S_T F(\xi) d\xi = \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-itb} - e^{-ita}}{-it} \Phi_X(t) dt. \tag{A.7.26}$$

A convergence statement as $T \to \infty$ corresponding to this formula can be found in many textbooks on Fourier methods in probability theory, for instance as Theorem 3.2.1 in [Lukacs, 1970] or, for absolutely continuous distributions, as Theorem 2.3.11 in [Pinsky, 2002].

Theorem A.7.10 (inversion formula for increments). *For any distribution F and $a, b \in \mathbb{R}$ with $a \neq b$ we have*

$$F(b) - F(a) + \frac{F\{a\} - F\{b\}}{2} = \lim_{T\to\infty} \frac{\operatorname{sgn}(b-a)}{2\pi} \int_{-T}^{T} \frac{e^{-itb} - e^{-ita}}{-it} \Phi_X(t)dt. \tag{A.7.27}$$

If $F\{a\} = F\{b\} = 0$, i.e., if F is continuous at a and at b, the left hand side equals the increment $F(b) - F(a)$.

Proof. According to the integral definition of $\Phi_X(t)$, for any fixed $T > 0$ we have

$$\frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-itb} - e^{-ita}}{-it} \Phi_X(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-T}^{T} \frac{e^{-it(b-x)} - e^{-it(a-x)}}{-it} dtF(dx). \tag{A.7.28}$$

The interchange in the order of integration is permitted, since

$$\left| \frac{e^{-it(b-x)} - e^{-it(a-x)}}{-it} \right| = \left| \int_{a-x}^{b-x} e^{-itz} dz \right| \leq |a - b| .$$

Moreover, by definition of the sine function in terms of the complex exponential function, for fixed $x \in \mathbb{R} \setminus \{a, b\}$, we obtain by substitution:

$$\begin{aligned} \int_{-T}^{T} \frac{e^{-it(b-x)} - e^{-it(a-x)}}{-it} dt &= 2 \int_{0}^{T} \frac{\sin(t(b-x)) - \sin(t(a-x))}{t} dt \\ &= 2\operatorname{Si}(T(b-x)) - 2\operatorname{Si}(T(a-x)) \end{aligned}$$

For the last equality we referred to the sine integral (B.1.1). Observe that this representation remains true for any $x \in \mathbb{R}$. It follows from (B.1.4) and (B.1.6) that the modulus of the preceding difference is bounded, and for any two non-identical $a, b \in \mathbb{R}$ we have

$$\lim_{T\to\infty} (\operatorname{Si}(T(b-x)) - \operatorname{Si}(T(a-x))) = \operatorname{sgn}(b-a) \begin{cases} 0, & \text{if } x \notin [a, b], \\ \frac{\pi}{2}, & \text{if } x \in \{a, b\}, \\ \pi, & \text{if } x \in (a, b). \end{cases}$$

Therefore, by Lebesgue's dominated convergence theorem, the integral (A.7.28) exhibits the following convergence behaviour:

$$\lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-itb} - e^{-ita}}{-it} \Phi_X(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-it(b-x)} - e^{-it(a-x)}}{-it} dtF(dx)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \lim_{T \to \infty} \mathrm{Si}(T(b-x)) - \mathrm{Si}(T(a-x)) F(dx)$$

$$= \mathrm{sgn}(b-a) \left[\frac{F\{a\}}{2} + \int_{(a,b)} F(dx) + \frac{F\{b\}}{2} \right]$$

This concludes the proof ■

From the last theorem we can deduce a useful statement about distributions with a density, compare Theorem 3.2.2 in [Lukacs, 1970].

Theorem A.7.11 (inversion formula for densities). *If $\Phi_X \in L^1(\mathbb{R})$ the distribution F is absolutely continuous with density $F' = f$, given by*

(A.7.29) $$f(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\xi} \Phi_X(t) dt.$$

In these circumstances f is uniformly continuous and bounded with $\|f\|_\infty \leq \|\Phi_X\|_1$.

Proof. The continuity of F is an immediate consequence of Theorem A.7.9, according to the integrability of Φ_X, due to which for any $\xi \in \mathbb{R}$ we have:

$$|F\{\xi\}| = \lim_{T \to \infty} \frac{1}{2T} \left| \int_{-T}^{T} e^{-it\xi} \Phi_X(t) dt \right| \leq \lim_{T \to \infty} \frac{\|\Phi_X\|_1}{2T} = 0$$

Moreover, with $a = \xi - \delta$ and $b = \xi + \delta$ for some $\delta > 0$ formula (A.7.27) yields:

$$\begin{aligned} \frac{F(\xi+\delta) - F(\xi-\delta)}{2\delta} &= \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-it(\xi+\delta)} - e^{-it(\xi-\delta)}}{-i2\delta t} \Phi_X(t) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi t} \frac{\sin(\delta t)}{\delta t} \Phi_X(t) dt \end{aligned}$$

The above integral converges absolutely and uniformly with respect to $\delta \geq 0$. According to the convergence properties of the sinc function, from Lebesgue's dominated convergence theorem we thus deduce

$$\lim_{\delta \downarrow 0} \frac{F(\xi+\delta) - F(\xi-\delta)}{2\delta} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\xi} \Phi_X(t) dt.$$

This shows the existence of the centered difference quotient on the left hand side for any $\xi \in \mathbb{R}$, implying the differentiability of F with the derivative having the integral representation given on the right hand side. In addition, the absolute convergence of the latter verifies $F'(\xi) = f(\xi)$ as the Fourier transform of the function $\Phi_X(t)$. The former is thus naturally bounded and uniformly continuous with respect to $\xi \in \mathbb{R}$. Note especially that the above justifications became invalid if we considered the non-centered difference quotient of F due to problems with the underlying inversion formula (A.7.27). ■

The finiteness of a and of b is essential for the applicability of Theorem A.7.10. Indeed, the integrals in (A.7.26) do not exist if either of the endpoints is infinite. A theorem that covers such cases was presented in [Gil-Pelaez, 1951] and is apparently little known. The reason is possibly that, in comparison to the formula for increments of Theorem A.7.10, the formula for infinite segments is more complicated since it involves two limits.

Theorem A.7.12 (inversion formula for infinite rays). *For any distribution F and $\xi \in \mathbb{R}$,*

$$\text{(A.7.30)} \qquad \frac{F(\xi) + F(\xi-)}{2} = \frac{1}{2} + \lim_{T_2 \uparrow \infty} \lim_{T_1 \downarrow 0} \frac{1}{2\pi} \int_{T_1}^{T_2} \frac{e^{it\xi}\Phi_X(-t) - e^{-it\xi}\Phi_X(t)}{it} dt.$$

Proof. First, for arbitrary $T_2 > T_1 > 0$ the following holds:

$$\begin{aligned}
\int_{T_1}^{T_2} \frac{e^{it\xi}\Phi_X(-t) - e^{-it\xi}\Phi_X(t)}{it} dt &= 2 \int_{T_1}^{T_2} \int_{\mathbb{R}\setminus\{\xi\}} \frac{\sin(t(\xi - x))}{t} F(dx) dt \\
&= 2 \int_{\mathbb{R}\setminus\{\xi\}} \int_{T_1}^{T_2} \frac{\sin(t(\xi - x))}{t} dt F(dx)
\end{aligned}$$

The interchange in the order of integration is permitted, since for $0 < T_1 \leq t \leq T_2$ we have

$$\left| \frac{\sin(t(\xi - x))}{t} \right| \leq \frac{1}{|t|} \leq \frac{1}{T_1}.$$

Furthermore, by substitution for fixed $x \in \mathbb{R} \setminus \{\xi\}$ we obtain

$$\int_{T_1}^{T_2} \frac{\sin(t(\xi - x))}{t} dt = \int_{(\xi-x)T_1}^{(\xi-x)T_2} \frac{\sin(z)}{z} dz.$$

The integral on the left hand side is therefore bounded by $2\,\mathrm{Si}(\pi)$ uniformly with respect to $\xi - x$. Hence, according to the convergence properties of the sine integral, Lebesgue's dominated convergence theorem again yields:

$$\lim_{T_2 \uparrow \infty} \lim_{T_1 \downarrow 0} \frac{1}{2\pi} \int_{T_1}^{T_2} \frac{e^{it\xi}\Phi_X(-t) - e^{-it\xi}\Phi_X(t)}{it} dt = \lim_{T_2 \uparrow \infty} \lim_{T_1 \downarrow 0} \frac{1}{\pi} \int_{\mathbb{R}\setminus\{\xi\}} \int_{(\xi-x)T_1}^{(\xi-x)T_2} \frac{\sin(z)}{z} dz F(dx)$$

$$= \frac{1}{\pi}\left[\int\limits_{\{x<\xi\}} + \int\limits_{\{x>\xi\}}\right] \lim_{T_2\uparrow\infty}\lim_{T_1\downarrow 0} \int\limits_{(\xi-x)T_1}^{(\xi-x)T_2} \frac{\sin(z)}{z} dz F(dx)$$
$$= \frac{1}{2}\left(F(\xi-) - (1-F(\xi))\right)$$

We have thus eventually verified (A.7.30). ■

Similar to (A.1.16) the convergence behaviour of the inversion formula (A.7.27) can be improved by averaging (A.7.26). A more extended result is provided by the theorem below, which concludes this subsection. It is an immediate analogue of Theorem A.1.3 for distributions. A formula for the multivariate case with the Gaussian smoothing kernel can be found in Proposition 5.2.4 in [Pinsky, 2002].

Theorem A.7.13 (inversion by means of an approximate identity). *Suppose I is a random variable associated with an absolutely continuous distribution F_I whose density f_I satisfies the conditions of Theorem A.1.3. Then, for any $a, b \in \mathbb{R}$ with $a \neq b$,*

$$\text{(A.7.31)} \qquad F(b) - F(a) + \frac{F\{a\} - F\{b\}}{2} = \lim_{\lambda\downarrow 0} \frac{\operatorname{sgn}(b-a)}{2\pi} \int\limits_{-\infty}^{\infty} \frac{e^{-itb} - e^{-ita}}{-it} \Phi_X(t)\Phi_I(\lambda t) dt.$$

Proof. For $\lambda > 0$ the distribution of the convoluted random variable $X + \lambda I$ is given by

$$\mathbb{P}(a < X + \lambda I \leq b) = \left[\int\limits_{-\infty}^{0} + \int\limits_{0}^{\infty}\right] (F(b-\lambda v) - F(a - \lambda v)) F_I(dv).$$

Observe the boundedness of the integrand. Moreover, for b and analogously for a, the right-continuity of F yields

$$\lim_{\lambda\downarrow 0} F(b - \lambda v) = \begin{cases} F(b-), & \text{if } v > 0, \\ F(b), & \text{if } v \leq 0. \end{cases}$$

Hence, in accordance with the symmetry of F_I, implying $F_I(0) = \frac{1}{2}$, from Lebesgue's dominated convergence theorem we conclude:

$$\lim_{\lambda\downarrow 0} \mathbb{P}(a < X + \lambda I \leq b) = \frac{F(b) - F(a)}{2} + \frac{F(b-) - F(a-)}{2}$$
$$= F(b) - F(a) + \frac{F\{a\} - F\{b\}}{2}$$

However, since $\Phi_I \in L^1(\mathbb{R})$ by Fubini's theorem and by Theorem A.7.10 for fixed $\lambda > 0$ we also have:

$$\begin{aligned}
\mathbb{P}\left(a < X + \lambda I \leq b\right) &= \int\limits_{-\infty}^{\infty} F_I\left(\frac{b-x}{\lambda}\right) - F_I\left(\frac{a-x}{\lambda}\right) F(dx) \\
&= \frac{\operatorname{sgn}(b-a)}{2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \frac{e^{-it\frac{b-x}{\lambda}} - e^{-it\frac{a-x}{\lambda}}}{-it} \Phi_I(t) dt F(dx) \\
&= \frac{\operatorname{sgn}(b-a)}{2\pi} \int\limits_{-\infty}^{\infty} \frac{e^{-it\frac{b}{\lambda}} - e^{-it\frac{a}{\lambda}}}{-it} \Phi_I(t) \Phi_X\left(\frac{t}{\lambda}\right) dt \\
&= \frac{\operatorname{sgn}(b-a)}{2\pi} \int\limits_{-\infty}^{\infty} \frac{e^{-itb} - e^{-ita}}{-it} \Phi_I(\lambda t) \Phi_X(t) dt \qquad \text{(A.7.32)}
\end{aligned}$$

The proof of formula (A.7.31) is thus complete. ∎

B. Special Functions

With this chapter we provide a concise overview on the properties of some frequently occuring special functions. Contrary to elementary functions, which encompass finite compositions of powers, exponential functions and logarithms, special functions can only be represented as series or integrals. An extensive encyclopedia about special functions and their properties is available in possibly one of the less handy books, yet titled the handbook of mathematical functions by [Olver et al., 2010]. Standard references for a detailed discussion including proofs of important theorems and identities are [Andrews et al., 1999], [Copson, 1970], [Olver, 1974] and [Whittaker and Watson, 1952].

B.1. The Sine Integral

The *sine integral* is defined as the primitive integral of the sinc function (A.1.7), viz

$$\mathrm{Si}(\xi) := \int_0^{\xi} \frac{\sin(t)}{t} dt, \qquad \xi \in \mathbb{R}. \tag{B.1.1}$$

The boundedness of the integrand implies $\mathrm{Si}(0) = 0$. Moreover, for $\xi > 0$ and $a \in \mathbb{R} \setminus \{0\}$,

$$\mathrm{Si}(a\xi) = \int_0^{\xi} \frac{\sin(at)}{t} dt = \mathrm{sgn}(a)\,\mathrm{Si}(|a|\,\xi). \tag{B.1.2}$$

This verifies the oddness of the sine integral as a function of $a \in \mathbb{R}$. The uniform boundedness of (B.1.1) can be shown by means of the following decomposition for $T = 2\pi(J(T) + 1)$ with $J(T) \in \mathbb{N}$:

$$\begin{aligned} \mathrm{Si}(T) &= \sum_{j=0}^{2J(T)+1} \int_{j\pi}^{(j+1)\pi} \frac{\sin(t)}{t} dt \\ &= \int_0^{\pi} \sin(t) \sum_{j=0}^{2J(T)+1} \frac{(-1)^j}{t + j\pi} dt \\ &\leq \mathrm{Si}(\pi) \end{aligned} \tag{B.1.3}$$

The bound is legit, because $\sin(t) \geq 0$ for $0 \leq t \leq \pi$, and since the sum is alternating with decreasing summands. We have thus verified

$$\sup_{\xi \in \mathbb{R}} |\mathrm{Si}(\xi)| \leq \mathrm{Si}(\pi), \tag{B.1.4}$$

$$0 \leq \mathrm{Si}(\xi) \leq \mathrm{Si}(\pi), \qquad \text{for } \xi \geq 0. \tag{B.1.5}$$

In the above decomposition of the range of integration we chose the segments according to the sign of the sine function. If, alternatively, we had chosen a decomposition into intervals of the length of one period, i.e., of length 2π, for $T = 2\pi J(T)$ we had obtained

$$\mathrm{Si}(T) = \int_0^{2\pi} \sin(t) \sum_{j=0}^{J(T)} \frac{1}{t + 2\pi j} dt.$$

Evidently, this sum is non-alternating with non-negative summands. It looks similar to the harmonic series, which is divergent as $T \to \infty$. To show the uniform boundedness of the sine integral, it is thus particularly important to incorporate the alternating sign of the sine function. Finally, the sum appearing in (B.1.3) already indicates the existence of the limit as $T \to \infty$. Its exact numerical value is usually computed by means of contour integration or by referring (B.1.1) to other known absolutely convergent integrals.

Example B.1.1 (contour integration). The evaluation of the sine integral is a routine exercise in complex analysis, exploiting the exponential decay of the complex exponential function in the upper half plane. It that can be found in many textbooks on this topic, for instance in Example 5.4.9 of [Asmar and Grafakos, 2018]. We will thus only sketch the proof without further details. Consider the function $z^{-1}e^{iz}$ integrated along an annulus $C_{r,R}$ in the upper half plane with inner and outer radii $0 < r < R$, respectively. The annulus is supposed to be symmetric with respect to the origin such that its lower edges coincide with the segments $[-R+r, -r]$ and $[r, R-r]$ of the real axis. Traversing this integration path in the positive direction, by analyticity of the integrand, according to Cauchy's theorem, the following holds:

$$\begin{aligned} 0 &= \oint_{C_{r,R}} \frac{e^{iz}}{z} dz \\ &= 2i \int_r^R \frac{\sin(t)}{t} dt + i \int_0^\pi e^{iR\cos(\theta)} e^{-R\sin(\theta)} d\theta - i \int_0^\pi e^{ir\cos(\theta)} e^{-r\sin(\theta)} d\theta \end{aligned}$$

The second equality results from parametrizing the integration contour. It shows the applicability of Lebesgue's dominated convergence theorem as $r \downarrow 0$, yielding the limit value π for the last integral, whereas the second summand vanishes as $R \to \infty$. Hence,

$$\lim_{T \to \infty} \mathrm{Si}(T) = \frac{\pi}{2}. \tag{B.1.6}$$

Example B.1.2 (methods of real analysis). Techniques to compute the sine integral by means of real analysis mostly aim to find a connection to an absolutely convergent integral. This can be done in different ways. First, for fixed $T > 0$ we deduce from (B.1.1):

$$\begin{aligned}\mathrm{Si}(T) &= \int_0^T \int_0^\infty \sin(t) e^{-zt} dz dt = \int_0^\infty \mathfrak{Im} \int_0^T e^{(i-z)t} dt dz \\ &= \int_0^\infty \frac{1}{z^2+1} - \cos(T)\frac{e^{-zT}}{z^2+1} - \sin(T)\frac{ze^{-zT}}{z^2+1} dz\end{aligned}$$

For a given $T_0 > 0$ the obtained integral converges absolutely and uniformly with respect to $T \geq T_0$. Lebesgue's dominated convergence theorem thus yields

$$\lim_{T\to\infty} \int_0^T \frac{\sin(t)}{t} dt = \int_0^\infty \frac{1}{z^2+1} dz = \frac{\pi}{2}.$$

For the last equality we noticed the rational function in the integrand as the derivative of the arctangent function. This result also can be derived by noting for $y > 0$:

$$\int_0^\infty e^{-yt}\frac{\sin(t)}{t} dt = \int_0^\infty \int_y^\infty e^{-xt} dx \sin(t) dt = \int_y^\infty \mathfrak{Im} \int_0^\infty e^{(i-x)t} dt dx = \int_y^\infty \frac{1}{1+x^2} dx$$

From the sum representation in (B.1.3) we conclude the existence of the limit of the sine integral as $T \to \infty$. Hence, Abel's summation theorem for integrals applies, leading to:

$$\lim_{T\to\infty} \int_0^T \frac{\sin(t)}{t} dt = \lim_{y\downarrow 0} \int_0^\infty e^{-yt}\frac{\sin(t)}{t} dt = \int_0^\infty \frac{1}{1+x^2} dx = \frac{\pi}{2}$$

The third method of real analysis to determine the limit value of (B.1.1) is by simply integrating by parts once for fixed $T > 0$ or, equivalently, by writing:

$$\begin{aligned}\mathrm{Si}(T) &= \int_0^T \sin(t) \int_t^\infty y^{-2} dy dt = \int_0^\infty \frac{1-\cos(y \wedge T)}{y^2} dy \\ &= \int_0^T \frac{1-\cos(y)}{y^2} dy + \int_T^\infty \frac{1-\cos(T)}{y^2} dy\end{aligned}$$

By absolute convergence it is easy to see, that the interchange in the order of integration is permitted, which shows

$$\lim_{T\to\infty} \mathrm{Si}(T) = \int_0^\infty \frac{1-\cos(t)}{t^2} dt = \frac{\pi}{2}.$$

Note, however, that the numerical value of the last integral is also a result of complex integration. We notice the function in the integrand as Fejér's kernel (A.1.11).

B.2. The Gamma Function

Probably the best known special function is Euler's gamma function. It was discovered by him in the early 18th century as a solution to the problem of extending the factorial function to arbitrary complex numbers. For convenience we introduce the *Pochhammer symbol*

$$(a)_k := \prod_{j=0}^{k-1}(a+j), \qquad \text{for } a \in \mathbb{C} \text{ and } k \in \mathbb{N}, \tag{B.2.1}$$

also known as the *rising factorials*. Using this notation, Euler's original representation of the *gamma function* is in terms of the infinite product

$$\Gamma(z) := \lim_{k\to\infty} \frac{k!k^{z-1}}{(z)_k}, \qquad \text{for } z \in \mathbb{C} \setminus -\mathbb{N}_0. \tag{B.2.2}$$

Alternatively the gamma function can be expressed as a Weierstrass product

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}, \qquad \text{for } z \in \mathbb{C} \setminus -\mathbb{N}_0, \tag{B.2.3}$$

where γ denotes the *Euler-Mascheroni constant*, that is

$$\gamma := \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \frac{1}{n} - \log(N) \right\}. \tag{B.2.4}$$

In applications, however, the multiplicative forms are rarely useful and instead one prefers the so called *Eulerian integral of the second kind*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \qquad \text{for } z \in \mathbb{C} \text{ with } \Re z > 0. \tag{B.2.5}$$

It can be derived from (B.2.2) by proper rearrangement of the product, see for instance ch. 12.2 in [Whittaker and Watson, 1952]. Clearly, (B.2.5) is the Mellin transform of the exponential function with strip of analyticity $\Re z > 0$. A drawback in comparison to (B.2.2) and (B.2.3) is that the integral only exists in the right half plane, since its lower endpoint is the origin. On the other hand, a separation of the integration path for $a > 0$ yields

$$\Gamma(z) = \gamma(z, a) + \Gamma(z, a), \tag{B.2.6}$$

where we denote by

$$\gamma(z,a) := \int_0^a t^{z-1}e^{-t}dt,$$
$$\Gamma(z,a) := \int_a^\infty t^{z-1}e^{-t}dt, \tag{B.2.7}$$

respectively the *lower* and the *upper incomplete gamma function.* In the above shape, with respect to the first argument for fixed $a > 0$ the function $\Gamma(z,a)$ is entire while $\gamma(z,a)$ is analytic in $\Re z > 0$. The series

$$\gamma(z,a) = \sum_{k=0}^\infty \frac{(-1)^k}{k+z}\frac{a^{k+z}}{k!},$$

which can be obtained from the series expansion of the exponential function, then furnishes the analytic continuation of $\gamma(z,a)$ to a meromorphic function in $\mathbb{C}$ with simple poles at $-\mathbb{N}_0$ and residues

$$\operatorname*{Res}_{z=-k} \gamma(z,a) = \frac{(-1)^k}{k!}, \qquad \text{for } k \in \mathbb{N}_0. \tag{B.2.8}$$

Hence, the decomposition (B.2.6) in terms of a series plus an integral extends the integral definition of $\Gamma(z)$ to the whole complex plane. As a consequence we immediately determine that also the continuation of (B.2.5) is meromorphic in $\mathbb{C}$ with simple poles at the non-positive integers and

$$\operatorname*{Res}_{z=-k} \Gamma(z) = \frac{(-1)^k}{k!}, \qquad \text{for } k \in \mathbb{N}_0. \tag{B.2.9}$$

Alternatively, partial integration subject to the restriction $\Re z > 0$ readily yields a full integral representation for the analytic continuation of (B.2.5), formally:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt = \frac{1}{z}\int_0^\infty t^z e^{-t}dt = \frac{\Gamma(1+z)}{z}$$

The integral on the right hand side exhibits analyticity in $\Re z > -1$ and coincides for $\Re z > 0$ with the left hand side. Since the latter is a subregion of the former half plane, the right hand side constitutes the analytic continuation of $\Gamma(z)$ into $\Re z > -1$ with a simple pole at $z = 0$. Upon repeating this procedure it is possible to analytically extend the gamma function arbitrarily into the left half plane. The result

$$\Gamma(1+z) = z\Gamma(z), \qquad \text{for } z \in \mathbb{C}, \tag{B.2.10}$$

is called the *functional equation for the gamma function.* It shows that $\Gamma(z)$ indeed is an extension of the factorial and that it equals the factorial function for non-negative integers, since $\Gamma(1) = 1$ and $\Gamma(n+1) = n!$ for $n \in \mathbb{N}_0$. The functional equation also generalizes the Pochhammer symbol (B.2.1) in terms of a ratio of gamma functions, viz

$$(a)_p = \frac{\Gamma(a+p)}{\Gamma(a)}, \qquad \text{for } a, p \in \mathbb{C} \text{ with } a \neq 0. \tag{B.2.11}$$

Not only for several applications the property (B.2.10) is useful but it almost uniquely defines the gamma function. In particular, according to the Bohr-Mollerup theorem, compare §1.9 in [Andrews et al., 1999], given a function $g : (0, \infty) \to (0, \infty)$ such that $g(1) = 1$, $g(1+x) = xg(x)$ and $x \mapsto \log g(x)$ is convex, it follows that $g(x) = \Gamma(x)$ for $x > 0$.

Derivatives of the gamma function are usually represented in terms of the *polygamma functions*, which are defined by

$$\psi^{(k-1)}(z) := \frac{d^k}{dz^k} \log \Gamma(z), \qquad \text{for } k \in \mathbb{N} \text{ and } z \in \mathbb{C}. \tag{B.2.12}$$

For $k \in \{1, 2\}$ it is common to write $\psi(z) \equiv \psi^{(0)}(z)$ and $\psi'(z) \equiv \psi^{(1)}(z)$, and to refer to those as the *digamma* and the *trigamma function*, respectively. Many expansions and integral representations are known for the polygamma functions. Two particularly important identities are

$$\psi(1) = -\gamma, \tag{B.2.13}$$

$$\psi'(1) = \frac{\pi^2}{6}, \tag{B.2.14}$$

with γ referring to the Euler-Mascheroni constant (B.2.4).

B.2.1. Elementary Properties

We will now provide an overview on the essential properties of the gamma function. Besides the functional equation (B.2.10), it is important to know the *reflection formula* due to Euler (Ger.: Eulersche Ergänzungsformel), which relates the gamma function to trigonometric and hyperbolic functions. It is given by

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \qquad \text{for } z \in \mathbb{C}. \tag{B.2.15}$$

The proof is an exercise in complex calculus. Let $0 < x < 1$ be such that use of (B.2.5) is permitted. A simple substitution, similar to the proof of Theorem B.3.1 below, yields

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{t^{x-1}}{1+t} dt.$$

This integral can be evaluated by contour integration, by choosing the branch $0 < \arg(t) < 2\pi$ and as integration path the boundary of an appropriately cut-out annulus, that encircles the simple pole at $t = -1$. If we eventually expand the outer boundary and collapse the interior boundary of the annulus, according to the residue theorem, we arrive at

$$\left[1 - e^{(x-1)2\pi i}\right] \int_0^\infty \frac{t^{x-1}}{1+t} dt = 2\pi i e^{(x-1)\pi i},$$

which is equivalent to (B.2.15) for $z = x$ with $0 < x < 1$. Appealing to the identity principle for analytic functions, equality also holds for $0 < \Re z < 1$, and it can be extended to arbitrary $z \in \mathbb{C} \setminus \mathbb{Z}$ via the functional equation (B.2.10), upon exploiting the periodicity of the sine function.

We proceed with further properties. For $z \in \mathbb{C}$ write $z = x + iy$ with $x, y \in \mathbb{R}$. As a consequence of the integral representation (B.2.5), for $x > 0$ we have

$$|\Gamma(z)|^2 = \Gamma(z)\Gamma(\overline{z}), \tag{B.2.16}$$

with $\overline{z}$ denoting the complex conjugate of z. According to the functional equation (B.2.10), this equality remains true for all $z \in \mathbb{C} \setminus -\mathbb{N}_0$. Particularly with $x = \frac{1}{2}$, by additional use of (B.2.15), we obtain:

$$\left|\Gamma\left(\frac{1}{2} + yi\right)\right|^2 = \Gamma\left(\frac{1}{2} + yi\right)\Gamma\left(\frac{1}{2} - yi\right) = \frac{\pi}{\sin\left(\pi\left(\frac{1}{2} + yi\right)\right)} = \frac{\pi}{\cosh(\pi y)} \tag{B.2.17}$$

Moreover, for $x = 0$ the equality (B.2.16) combined with (B.2.10) and the reflection formula (B.2.15) yields:

$$|\Gamma(yi)|^2 = \frac{\Gamma(1+yi)\Gamma(-yi)}{yi} = \frac{\pi}{-yi\sin(\pi yi)} = \frac{\pi}{y\sinh(\pi y)} = \frac{\pi}{|y|\sinh(\pi|y|)} \tag{B.2.18}$$

Sometimes it is important to relate gamma functions of different argument multiples. This is possible by virtue of the *Gauss multiplication formula*, however, only in special cases. In particular, for $m \in \mathbb{N}$ and $a \in \mathbb{C}$ we have

$$\Gamma(ma)(2\pi)^{\frac{m-1}{2}} = m^{ma-\frac{1}{2}} \prod_{k=1}^{m-1} \Gamma\left(a + \frac{k}{m}\right), \tag{B.2.19}$$

provided none of the gamma functions is singular. For $m = 2$, the above identity is known as the *Legendre duplication formula.* A proof can be found in §1.5 in [Andrews et al., 1999]. Next we compute some frequently required residues involving the gamma function.

Theorem B.2.1 (residues). *(1) For $k \in \mathbb{N}_0$, $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$ with $az + b \notin \mathbb{N}_0$,*

$$\operatorname*{Res}_{z=-\frac{k+b}{a}} \Gamma(az+b) = a^{-1}\frac{(-1)^k}{k!}. \tag{B.2.20}$$

(2) Suppose $\lambda > 0$ and $f(z)$ is analytic in a punctured neighborhood of the point $z_0 \in \mathbb{C}$ with $f(z) = \sum_{j=-2}^{\infty}(z-z_0)^j f_j$ as $z \to z_0$, for $f_j \in \mathbb{C}$. Then, if $z_0 \in \mathbb{C} \setminus -\mathbb{N}_0$,

$$\operatorname*{Res}_{z=z_0} \lambda^{-z}\Gamma(z)f(z) = -\frac{\Gamma(z_0)}{\lambda^{z_0}}\left\{(\log\lambda - \psi(z_0))f_{-2} - f_{-1}\right\}, \tag{B.2.21}$$

with the digamma function ψ, see (B.2.12). Moreover, if $z_0 = 0$,

$$\operatorname*{Res}_{z=0} \lambda^{-z}\Gamma(z)f(z) = \left\{(\log\lambda+\gamma)^2 + \frac{\pi^2}{6}\right\}\frac{f_{-2}}{2} - (\log\lambda+\gamma)f_{-1} + f_0, \tag{B.2.22}$$

where γ refers to the Euler-Mascheroni constant (B.2.4).

Proof. The first proof is easily accomplished by means of the reflection formula (B.2.15) and de l'Hospital's rule, from which we deduce:

$$\begin{aligned}\operatorname*{Res}_{z=-\frac{k+b}{a}} \Gamma(az+b) &= \operatorname*{Res}_{z=-\frac{k+b}{a}} \frac{\pi}{\Gamma(1-az-b)\sin(\pi(az+b))}\\ &= \frac{1}{k!}\lim_{z\to-\frac{k+b}{a}} \frac{\pi\left(z+\frac{b+k}{a}\right)}{\sin(\pi(az+b))}\end{aligned}$$

For the second proof we observe that, in terms of the polygamma functions (B.2.12), in a neighborhood of $z_0 \in \mathbb{C} \setminus -\mathbb{N}_0$ we find

$$\lambda^{-z}\Gamma(z) = \frac{\Gamma(z_0)}{\lambda^{z_0}}\left\{1 - (\log\lambda - \psi(z_0))(z-z_0) + \mathcal{O}\left\{(z-z_0)^2\right\}\right\}.$$

Conversely, upon taking into account (B.2.13) and (B.2.14), at the particular point $z = 0$ instead of a Taylor we have the Laurent expansion

$$\lambda^{-z}\Gamma(z) = \frac{1}{z} - (\log\lambda+\gamma) + \frac{1}{2}\left\{(\log\lambda+\gamma)^2 + \frac{\pi^2}{6}\right\}z + \mathcal{O}\left\{z^2\right\}.$$

By multiplying with the expansion of f and putting the powers of $z - z_0$ in ascending order, the definition of a residue yields (B.2.21) and (B.2.22). ∎

Finally, the gamma function enables an extended notion of antiderivatives in the sense of the fundamental theorem of calculus. For a function f, continuous on $[a,b]$, the first is given by $D^{-1}f(x_1) = \int_a^{x_1} f(x)dx$. In an analogous fashion, for $\nu \in \mathbb{N}$, one obtains through

$$D^{-\nu}f(y) = \frac{1}{\Gamma(\nu)}\int_a^y (y-x)^{\nu-1}f(x)dx, \qquad \text{for } y \in [a,b], \tag{B.2.23}$$

the ν-th antiderivative. However, it is also permitted to choose $\nu > 0$ arbitrary, resulting in the ν-th fractional integral. In the literature those are also referred to as *Riemann-Liouville integrals*. Observe that, particularly for $y = 0$ and $a = -\infty$, (B.2.23) coincides with the Mellin transform of the function $f(-x)$, compare Appendix A.5.

B.2.2. Asymptotic Behaviour

The consideration of the logarithm in connection with the gamma function is not random but arises naturally from its representation in terms of an infinite product, which is thus transformed to an infinite sum. While the gamma function as a generalization of the factorial grows incredibly fast, its logarithm is much easier to handle and more appropriate, for example, for the derivation of asymptotic statements. A common representation for the logarithmic gamma function is given by

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log(z) - z + \frac{1}{2}\log(2\pi) + \Omega(z), \qquad \text{for } z \in \mathbb{C}\setminus(-\infty, 0], \tag{B.2.24}$$

where log refers to the principal branch of the logarithm, i.e., $\log(w) = \log|w| + i\arg(w)$ with $|\arg(w)| < \pi$, and

$$\Omega(z) := \sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)} z^{1-2j} - \frac{1}{2m}\int_0^\infty \frac{B_{2m}(t - \lfloor t \rfloor)}{(t+z)^{2m}}dt, \qquad \text{for } m \in \mathbb{N}. \tag{B.2.25}$$

Moreover, $B_{2j}(x)$ signifies the $2j$-th *Bernoulli polynomial*[1] such that $B_{2j}(0) \equiv B_{2j}$ is the corresponding *Bernoulli number*, where $\lfloor t \rfloor$ is the integer part of $t > 0$. The derivation of (B.2.24) is elaborate and extensively presented in Theorem D.3.2 in [Andrews et al., 1999]. We shall not discuss any details right here. Rather important for our purposes is the observation from (B.2.24) that as $|z| \to \infty$ in $|\arg(z)| < \pi$, uniformly with respect to $\arg(z)$ in any closed interior sector, we have

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log(z) - z + \frac{1}{2}\log(2\pi) + \sum_{j=1}^{m-1} \frac{B_{2j}}{2j(2j-1)} z^{1-2j} + \mathcal{O}\left\{|z|^{1-2m}\right\}. \tag{B.2.26}$$

This statement can be shown by first writing (B.2.25) in the form

$$\Omega(z) = \sum_{j=1}^{m-1} \frac{B_{2j}}{2j(2j-1)} z^{1-2j} + \frac{1}{2m}\int_0^\infty \frac{B_{2m} - B_{2m}(t - \lfloor t \rfloor)}{(t+z)^{2m}}dt. \tag{B.2.27}$$

Upon setting $\theta \equiv \arg(z)$ and exploiting several trigonometric identities, for $|\arg(z)| < \pi$ we get:

$$|t+z|^2 = (t + |z|\cos(\theta))^2 + |z|^2\{\sin(\theta)\}^2$$

[1] For $k \in \mathbb{N}_0$ the k-th Bernoulli polynomial $B_k(x)$ is defined as the k-th derivatives with respect to t of $\frac{te^{xt}}{e^t - 1}$ evaluated at $t = 0$.

$$
\begin{aligned}
&= (t+|z|)^2 - 2t\,|z|\,(1-\cos(\theta)) \\
&= (t+|z|)^2 - 4t\,|z| \left\{ \sin\left(\frac{\theta}{2}\right) \right\}^2 \\
&= (t+|z|)^2 \left\{ \cos\left(\frac{\theta}{2}\right) \right\}^2 + \left\{ \sin\left(\frac{\theta}{2}\right) \right\}^2 (t^2 + |z|^2 - 2t\,|z|) \\
&= (t+|z|)^2 \left\{ \cos\left(\frac{\theta}{2}\right) \right\}^2 + \left\{ \sin\left(\frac{\theta}{2}\right) \right\}^2 (t-|z|)^2 \\
&\geq (t+|z|)^2 \left\{ \cos\left(\frac{\theta}{2}\right) \right\}^2
\end{aligned}
$$

Finally, since $B_{2m} - B_{2m}(t - \lfloor t \rfloor)$ consitutes a continuous and periodic function of t with period 1, we obtain:

$$
\begin{aligned}
\frac{1}{2m} \left| \int_0^\infty \frac{B_{2m} - B_{2m}\left(t - \lfloor t \rfloor\right)}{(t+z)^{2m}} dt \right| &\leq \max_{0 \leq x \leq 1} \left| B_{2m} - B_{2m}\left(x - \lfloor x \rfloor\right) \right| \frac{\left\{\sec\left(\frac{\theta}{2}\right)\right\}^{2m}}{2m} \int_0^\infty \frac{1}{(t+|z|)^{2m}} dt \\
&= \max_{0 \leq x \leq 1} \left| B_{2m} - B_{2m}\left(x - \lfloor x \rfloor\right) \right| \frac{\left\{\sec\left(\frac{\theta}{2}\right)\right\}^{2m}}{2m(2m-1)} |z|^{1-2m}
\end{aligned}
$$

The uniformity with respect to θ follows from the observation that the secant function is bounded in $|\theta| \leq \pi - \delta$ for any $\delta > 0$. This completes the verification of (B.2.26).

Upon combining the preceding findings with the series expansion for the exponential function we obtain *Stirling's formula*, according to which as $|z| \to \infty$ in $|\arg(z)| < \pi$ we have

(B.2.28) $$\Gamma(z) = \sqrt{2\pi} z^{z - \frac{1}{2}} e^{-z} \left\{ 1 + \mathcal{O}\left\{ |z|^{-1} \right\} \right\}.$$

Many more useful asymptotic relations and even expansions can be deduced from (B.2.26). For instance, since differentiation is permitted, as $|z| \to \infty$ in $|\arg(z)| < \pi$ it shows that the behaviour of the digamma function is

(B.2.29) $$\psi(z) \sim \log z - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k z^{2k}}.$$

Furthermore, for $m, \alpha > 0$ we have

(B.2.30) $$\lim_{m \to \infty} \frac{\Gamma(m+\alpha)}{\Gamma(m) m^\alpha} = 1.$$

The validity of this statement will be verified below for a more general parametrization.

B.2.3. Inequalities

In this subsection we provide a few bounds for the gamma function. The simplest estimate follows immediately from the Eulerian integral of the second kind (B.2.5), upon application of the triangle inequality, which yields

$$|\Gamma(x+iy)| \leq \Gamma(x), \qquad \text{for } x > 0. \tag{B.2.31}$$

This bound, however, cancels the exponential decay in the imaginary direction. Alternatively, by means of Stirling's formula, in §2.1.3 of [Paris and Kaminski, 2001], it was shown that

$$|\Gamma(z)| \leq \sqrt{2\pi}\, |z|^{x-\frac{1}{2}}\, e^{-\frac{\pi}{2}|y|+\frac{1}{6|z|}}, \qquad \text{for } z = x+iy \text{ with } x \geq 0. \tag{B.2.32}$$

Finally, inequality (5.6.7) in [Olver et al., 2010] provides a lower bound for the gamma function, according to which

$$|\Gamma(x+iy)| \geq \frac{\Gamma(x)}{\cosh^{\frac{1}{2}}(\pi y)}, \qquad \text{for } x \geq \frac{1}{2}. \tag{B.2.33}$$

Its proof is based on the Weierstrass product (B.2.3) and exploits that

$$\left\{1 + \frac{y^2}{(n+x)^2}\right\}^{-1} \geq \left\{1 + \frac{y^2}{(n+\frac{1}{2})^2}\right\}^{-1}, \qquad \text{for } x \geq \frac{1}{2}.$$

Consequently we find:

$$\begin{aligned}
\left|\frac{\Gamma(x+iy)}{\Gamma(x)}\right| &= \frac{x}{(x^2+y^2)^{\frac{1}{2}}} \prod_{n=1}^{\infty} \frac{n+x}{((n+x)^2+y^2)^{\frac{1}{2}}} \\
&= \left\{1 + \frac{y^2}{x^2}\right\}^{-\frac{1}{2}} \prod_{n=1}^{\infty} \left\{1 + \frac{y^2}{(n+x)^2}\right\}^{-\frac{1}{2}} \\
&\geq \left\{1 + 4y^2\right\}^{-\frac{1}{2}} \prod_{n=1}^{\infty} \left\{1 + \frac{y^2}{(n+\frac{1}{2})^2}\right\}^{-\frac{1}{2}} \\
&= \frac{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|}{\Gamma\left(\frac{1}{2}\right)}
\end{aligned}$$

In accordance with (B.2.17), with $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, this yields the validity of (B.2.33).

B.3. The Beta Function

It was already mentioned above that the gamma function is sometimes referred to as the second Eulerian integral. The *Eulerian integral of the first kind* is given by

$$\mathrm{B}(a,b) := \int_0^1 s^{a-1}(1-s)^{b-1} ds, \qquad \Re a, \Re b > 0. \tag{B.3.1}$$

It is better known as the *beta function*. Similar to the gamma function, its analytic continuation can be determined from integration by parts. The easier way, however, is to apply the following formula.

Theorem B.3.1. *For $a, b \in \mathbb{C} \setminus -\mathbb{N}_0$ we have*

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{B.3.2}$$

Equation (B.3.2) immediately reveals that the beta function is generally of meromorphic type. For computational convenience we mostly employ the above identity rather than writing $\mathrm{B}(a,b)$.

Proof (due to Poisson (1823), cf. [Andrews et al., 1999].). Assume first $a, b > 0$ real-valued in order to be able to make use of the integral representation for the beta and the gamma function. Substituting $s + t = v$, i.e., $dt = dv$, interchanging the order of integration and then substituting $s = uv$, i.e., $ds = vdu$, we eventually arrive at:

$$\begin{aligned}
\Gamma(a)\Gamma(b) &= \int_0^\infty \int_0^\infty s^{a-1} t^{b-1} e^{-(s+t)} dt ds \\
&= \int_0^\infty \int_s^\infty s^{a-1} (v-s)^{b-1} e^{-v} dv ds \\
&= \int_0^\infty e^{-v} \int_0^\infty \mathbb{I}\{s \le v\}\, s^{a-1} (v-s)^{b-1} ds dv \\
&= \int_0^\infty e^{-v} \int_0^\infty \mathbb{I}\{uv \le v\}\, (uv)^{a-1} (v-uv)^{b-1} v du dv \\
&= \int_0^\infty v^{a+b-1} e^{-v} dv \int_0^1 u^{a-1} (1-u)^{b-1} du \\
&= \Gamma(a+b)\, \mathrm{B}(a,b)
\end{aligned}$$

The extension to $a, b \in \mathbb{C}$ by analytic continuation is straightforward. ∎

B.3.1. Asymptotic Behaviour of a Ratio of Two Gamma Functions

The integral representation for the beta function can be employed to derive an asymptotic expansion for the ratio of two gamma functions. Although this is also possible by applying Stirling's formula to numerator and denominator, respectively, a more convenient expansion can be obtained by considering, for $z, a, b \in \mathbb{C}$ with $\Re(b-a) > 0$ and $\Re z > 0$, the integral

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = \frac{1}{\Gamma(b-a)} \int_0^\infty t^{b-a-1} e^{-zt} f(t) dt. \tag{B.3.3}$$

The validity of this representation is readily confirmed by comparison of (B.3.2) with (A.3.8), and we denote

$$f(t) := e^{-at}\left\{\frac{1-e^{-t}}{t}\right\}^{b-a-1}. \tag{B.3.4}$$

Roughly speaking, the integral (B.3.3) is of Laplace-type and as $|z| \to \infty$ the main contribution comes from a neighborhood of the origin. There, the function $f(t)$ possesses a power series expansion. Since the additional conditions of Watson's lemma are satisfied, compare Theorem 3.2 in [Olver, 1974], we have validity of the asymptotic expansion

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}\left\{1+\frac{(a-b)(a+b-1)}{2z}+\dots\right\}. \tag{B.3.5}$$

Despite the initial conditions on the parameters, the above result holds for arbitrary $a, b \in \mathbb{C}$ as $|z| \to \infty$ in the sector $|\arg(z)| < \pi$. This extension is justified by Theorem 3.3 in [Olver, 1974] and by analytic continuation. For details we refer to §5 in ch. 4 in [Olver, 1974]. Regarding (B.3.5) the pointwise validity of the expansion must be emphasized and is particularly reflected by the coefficients, which are polynomials of a, b. Finally, for sufficiently large $|z|$ in $|\arg(z)| < \pi$ and fixed $a, b \in \mathbb{C}$, we immediately conclude

$$\left|\frac{\Gamma(z+a)}{\Gamma(z+b)}\right| = \mathcal{O}\left\{|z|^{\Re(a-b)}\right\}. \tag{B.3.6}$$

B.3.2. Derivation of an Asymptotic Estimate for the First Argument of the Beta Function that is Integrable on $\mathbb{R}$ with Respect to the Second Argument

The beta function is of frequent occurence in the context of binomial sums and Mellin-Barnes integrals, as a means to measure the asymptotic behaviour. To establish simple order estimates for integrals of this kind, we need an upper bound for the beta function that is absolutely integrable along any infinite line, which runs parallel to the imaginary axis in the right half plane corresponding to the first argument. At the same time the bound must reflect the asymptotic behaviour as the second argument tends to infinity. For $K \in \mathbb{N}$ let $\lambda > K-1$ and $z = x+iy$ with $x > -K$, $y \in \mathbb{R}$ and especially $z \notin -\mathbb{N}_0$. In these circumstances, after K-times application of the functional equation for the gamma function the following integral representation for the beta function applies:

$$\begin{aligned}\frac{\Gamma(\lambda+1)\Gamma(z)}{\Gamma(\lambda+1+z)} &= \left[\prod_{k=0}^{K-1}\frac{\lambda-k}{z+k}\right]\frac{\Gamma(\lambda+1-K)\Gamma(z+K)}{\Gamma(\lambda+1+z)} \\ &= \left[\prod_{k=0}^{K-1}\frac{\lambda-k}{z+k}\right]\int_0^1 s^{\lambda-K}(1-s)^{z+K-1}ds\end{aligned}$$

To get rid of the imaginary part in the integral we employ the triangle inequality and then again the functional equation for the gamma function, which leads to:

$$\begin{aligned}\left|\frac{\Gamma(\lambda+1)\Gamma(z)}{\Gamma(\lambda+1+z)}\right| &\leq \frac{\Gamma(\lambda+1-K)\Gamma(x+K)}{\Gamma(\lambda+1+x)}\prod_{k=0}^{K-1}\frac{\lambda-k}{|z+k|} \\ &= \frac{\Gamma(\lambda+1)\Gamma(x+K)}{\Gamma(\lambda+1+x)}\prod_{k=0}^{K-1}\frac{1}{\sqrt{(x+k)^2+y^2}}\end{aligned} \tag{B.3.7}$$

Observe that the right hand side simultaneously satisfies $\mathcal{O}\left\{y^{-K}\right\}$ as $y \to \pm\infty$ in $x > -K$ and as $\lambda \to \infty$ we have $\mathcal{O}\left\{\lambda^{-x}\right\}$, compare (B.3.5). This matches the desired properties if $K \geq 2$.

Corollary B.3.1 (asymptotic behaviour of binomial integrals). *For $c \in \mathbb{R}\backslash -\mathbb{N}_0$ suppose $f(c+iy)$ is continuous with respect to $y \in \mathbb{R}$ and $\mathcal{O}\left\{|y|^\alpha\right\}$ as $y \to \pm\infty$ for some $\alpha \in \mathbb{R}$. As $\lambda \to \infty$ we then have*

$$\mathrm{BI}(\lambda, c) := \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{\Gamma(\lambda+1)\Gamma(z)}{\Gamma(\lambda+1+z)}f(z)dz = \mathcal{O}\left\{\lambda^{-c}\right\}.$$

Proof. An application of the triangle inequality to $\mathrm{BI}(\lambda, c)$ accompanied by the estimate (B.3.7), for fixed $K > \max\{-c, \alpha+1\}$ and $\lambda > K-1$, yields

$$|\mathrm{BI}(\lambda, c)| \leq \frac{\Gamma(\lambda+1)\Gamma(c+K)}{2\pi\Gamma(\lambda+1+c)}\int\limits_{-\infty}^{\infty}\left[\prod_{k=0}^{K-1}\frac{1}{\sqrt{(c+k)^2+y^2}}\right]|f(c+iy)|\,dy.$$

Under the above conditions the integral on the right hand side converges absolutely. Furthermore, an application of the expansion (B.3.5) exposes the asymptotic behaviour with respect to λ, which finishes the proof. ∎

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Ich erkläre: Ich habe die vorgelegte Dissertation selbstständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt, die ich in der Dissertation angegeben habe. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben, die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht. Ich stimme einer evtl. Überprüfung meiner Dissertation durch eine Antiplagiat-Software zu. Bei den von mir durchgeführten und in der Dissertation erwähnten Untersuchungen habe ich die Grundsätze guter wissenschaftlicher Praxis, wie sie in der "Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis" niedergelegt sind, eingehalten.

Giessen, April 2023