The Statistical Analysis of Truncated And Censored Data Under Serial Dependence

Inaugural - Dissertation

zur Erlangung des Doktorgrades
an den Naturwissenschaftlichen Fachbereichen
(Mathematik)
der Justus-Liebig-Universität Gießen

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Gießen, Januar 2008

D-26

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Ich danke Herrn Prof. Dr. Stute für die hervorragende Betreuung der Doktorarbeit, seine Zeit und seine Hilfe.
Ich möchte mich auch bei meinen Eltern und meiner Schwester sehr herzlich für die Unterstützung bedanken.

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Chapter 1

Introduction And Main Results

1.1 Introduction

Survival analysis is the part of statistics, in which the variable of interest may often be interpreted as the time elapsed between two events. Such ”lifetimes” typically appear in a medical or an engineering context. E.g., a quantity U may denote the time between infection and the onset of a disease. In engineering, U may be the time a technical unit was on test until failure occurred. Since in each case U is a random variable one may be interested in distributional properties of U. A typical feature of such lifetime data analysis is that due to time limitations U may not always be observable. Hence the available data only provide partial information and, as a consequence, standard statistical procedures are not applicable. Maybe the most famous example is random (right) censorship where instead of U one observes $min(U, C)$ and $\delta = 1_{\{U \leq C\}}$, in which C is a censoring variable and the indicator reveals the information which of U and C was actually observed. Another important example is random truncation, in which U is observed only if $U \leq D$, where D is the associated truncating variable. In each case standard empirical estimators attaching equal weights to the observations are not recommendable and need to be replaced by others taking into account the actual structure of the data. Typically, this results is a complicated reweighting of the observations leading to estimators with distributions which are not easy to handle.
In many situations, when one observes patients over time, one may be interested in consecutive times $X_1 \leq X_2 \leq X_3 \leq ...$ denoting the beginning of different phases in the development of a disease. E.g., in HIV studies, X_1 could be the time of infection, X_2 the time when antibodies occur (seroconversion) and X_3 the time when AIDS is diagnosed. Let

$$U_1 = X_2 - X_1 \text{ and } U_2 = X_3 - X_2$$

denote the length of each period. Typically, we may expect some dependence between U_1 and U_2. Let F denote the unknown bivariate distribution function (d.f.) of (U_1, U_2):

$$F(x_1, x_2) = \mathbb{P}(U_1 \leq x_1, U_2 \leq x_2), \quad x_1, x_2 \in \mathbb{R}.$$

More generally, we may be interested in integrals

$$I = \int \varphi dF$$

w.r.t. F, where φ is a given score function. E.g., if we take $\varphi(x_1, x_2) = x_1 x_2$, we obtain an integral which is part of the covariance of U_1 and U_2. Given a sample (U_{1i}, U_{2i}), $1 \leq i \leq N$, of independent replicates of (U_1, U_2), the standard empirical estimator of I becomes

$$I_N = \frac{1}{N}\sum_{i=1}^{N} \varphi(U_{1i}, U_{2i}).$$

In a practical situation, the U's may not be all observable. For example, if E denotes the end of the study, and if we set $Z = E - X_1$, then the patient becomes part of the study only if $U_1 \leq Z$. In other words, U_1 may be truncated from the right by Z and hence gets lost if $U_1 > Z$. If no truncation occurs, both U_1 and Z will be observed. As to U_2, this variable will be available only if $U_1 + U_2 \leq Z$. Otherwise we observe $Z - U_1$. Hence given that U_1 is not truncated, U_2 is at risk of being censored. Since U_1 and U_2 may be dependent we obtain some kind of dependent censorship.
To summarize the data situation, for each person, we have three sequentially observed data $X_1 \leq X_2 \leq X_3$ giving rise to U_1 and U_2. As before, let E denote the end of the study. The following figure then displays the possible data structures depending on the location of E:

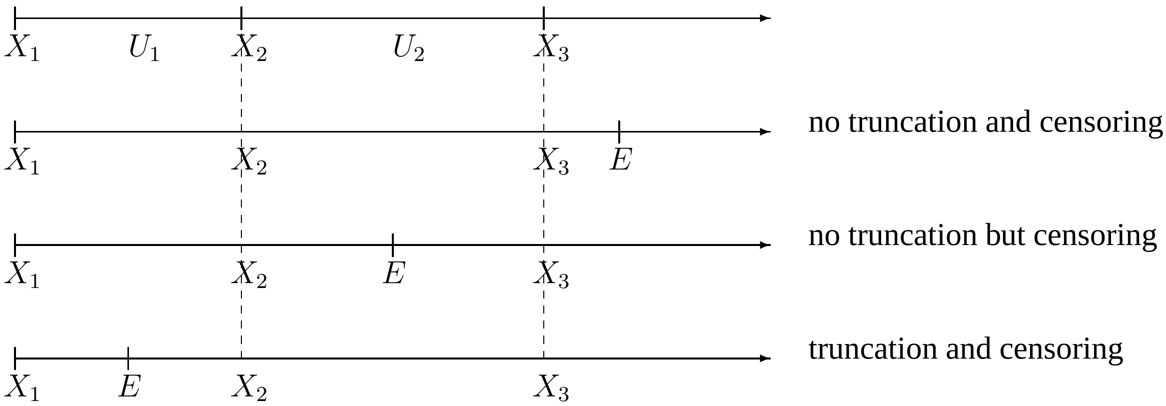

Figure 1.1: Possible Data Structures

Let N be the number of people at risk. Since, under possible truncation, N is unknown, we also have to introduce n, the number of actually observed cases. Denoting with $\alpha = \mathbb{P}(U_1 \leq Z)$ the probability of non-truncation, n is a binomial random variable with parameters α and N. Typically, for truncated data, the statistical analysis will be conditionally on a given n. In terms of n, we are given a sample $(U_{1i}, \tilde{U}_{2i})$, Z_i and δ_i, $1 \leq i \leq n$, where $\tilde{U}_{2i}$ equals U_{2i} when no censoring occurs. Otherwise,

$$\tilde{U}_{2i} = Z_i - U_{1i}.$$

Finally, $\delta_i = 1_{\{U_{2i} \leq Z_i - U_{1i}\}}$.

It is the goal of our work to derive an estimator F_n of F given the above data. As a second step we shall study estimators

$$I_n = \int \varphi dF_n$$

of I. For the computation of confidence intervals for I, one needs to compute or at least approximate the distribution of I_n. For this, we shall derive a representation of I_n as a sum of i.i.d. summands plus remainder. After that we may apply the Central Limit Theorem (CLT) to the leading term to obtain asymptotic normality of $n^{1/2}(I_n - I)$.

1.2 Main Theorem

In this section we develop an estimator I_n of I through identifiability of F. This means, that we are to find a representation of F in terms of estimable quantities.

For this, recall that $X_1 \leq X_2 \leq X_3$ are three consecutive times such that we are interested in

$$U_1 = X_2 - X_1 \text{ and } U_2 = X_3 - X_2.$$

As before denote with F the distribution function (d.f.) of (U_1, U_2):

$$F(x_1, x_2) = \mathbb{P}(U_1 \leq x_1, U_2 \leq x_2).$$

Let

$$F_1(x_1) := \mathbb{P}(U_1 \leq x_1)$$

and

$$F_2(x_2) := \mathbb{P}(U_2 \leq x_2)$$

be the associated marginal d.f.'s. Let E denote, as before, the end of the study so that

$$Z := E - X_1 \sim G$$

denotes the time elapsed between X_1 and E. It is assumed throughout that (U_1, U_2) is independent of Z and Z is observed always when U_1 is observed, whether U_2 is censored or not. Note, however, that since U_1 is observed only when $U_1 \leq Z$, truncation may cause some dependence between the actually observed U_1 and Z. As before, write

$$\alpha = \mathbb{P}(U_1 \leq Z)$$

for the probability, that (U_1, Z) can be observed. In addition to truncation, when $U_1 \leq Z$, the random variable U_2 is at risk of being censored from the right by $Z - U_1 = E - X_2$. In other words, we only have access to

$$\tilde{U}_2 = \min(U_2, Z - U_1) \tag{1.1}$$

Since in general U_1 and U_2 will be dependent and, at the same time, the observed U_1 also depends on Z, equation (1.1) incorporates a kind of dependent censorship. Along with (U_1, Z) and $\tilde{U}_2$ we also observe

$$\delta = 1_{\{U_1+U_2 \leq Z\}} = \begin{cases} 1, & \text{if } U_2 \text{ is uncensored} \\ 0, & \text{otherwise} \end{cases}$$

It is the purpose of this work to reconstruct F from a sample of independent replicates of (U_1, Z), $\tilde{U}_2$ and δ. Actually, our target will be

$$I = \int \varphi dF,$$

where φ is a given score function. In particular, when φ is the indicator of the rectangle $(-\infty, x_1] \times (-\infty, x_2]$, we are back at $I = F(x_1, x_2)$.

For identifiability of F, we also need some sub-distributions connected with F and G. Set

$$\begin{aligned} H_2^1(x,y) &= \mathbb{P}(U_1 \le x, \tilde{U_2} \le y, \delta = 1 | U_1 \le Z) \\ &= \mathbb{P}(U_1 \le x, U_2 \le y, U_1 + U_2 \le Z | U_1 \le Z) \\ &= \alpha^{-1} \mathbb{P}(U_1 \le x, U_2 \le y, U_1 + U_2 \le Z) \\ &= \alpha^{-1} \int_{-\infty}^{x} \int_{-\infty}^{y} [1 - G(x_1 + x_2)^-] F(dx_1, dx_2), \end{aligned}$$

where the last equality follows from the independence of the original (U_1, U_2) and Z.
Hence, provided that $supp(\varphi) \subset \{(x_1, x_2) : G(x_1 + x_2)^- < 1\}$, we obtain

$$I = \int \varphi dF = \int \varphi(x_1, x_2) \frac{\alpha}{1 - G(x_1 + x_2)^-} H_2^1(dx_1, dx_2).$$

Furthermore,

$$\begin{aligned} \alpha^{-1}(1 - G(x)^-) &= \frac{\mathbb{P}(Z \ge x)}{\mathbb{P}(U_1 \le Z)} = \frac{\mathbb{P}(Z \ge x, U_1 \le Z) + \mathbb{P}(Z \ge x, U_1 > Z)}{\mathbb{P}(U_1 \le Z)} \\ &= \mathbb{P}(Z \ge x | U_1 \le Z) + \alpha^{-1} \int_{[x,\infty)} [1 - F_1(y)] G(dy). \end{aligned}$$

Set

$$A(x) = \mathbb{P}(Z \ge x | U_1 \le Z)$$

and

$$B(x) = \alpha^{-1} \int_{[x,\infty)} [1 - F_1(y)] G(dy).$$

Hence

$$I = \int \varphi dF = \int \varphi(x_1, x_2) \frac{1}{A(x_1 + x_2) + B(x_1 + x_2)} H_2^1(dx_1, dx_2). \tag{1.2}$$

The function A can be easily estimated through the empirical d.f. of an observed Z-sample. In contrast, the function B contains the unknown α and the unconditional d.f.'s F_1 and G of U_1 and Z. To eliminate these terms we introduce

$$G^*(y) = \mathbb{P}(Z \le y | U_1 \le Z) = \alpha^{-1} \int_{(-\infty,y]} F_1(z) G(dz)$$

so that

$$B(x) = \int_{[x,\infty)} \frac{1 - F_1(y)}{F_1(y)} G^*(dy).$$

The function G^* is a conditional d.f. which again is easily estimable, while F_1 can by estimated through the well known Lynden-Bell estimator for truncated data.

Our statistical analysis is based on a sample of n replicates of (U_1, Z), $\tilde{U}_2$ and δ. More precisely, we assume that we are given N independent (U_{1i}, U_{2i}) random observations from the d.f. F and a sample Z_i of N independent random variables from the d.f. G such that the U-sample is also independent of the Z-sample. We only observe (U_{1i}, Z_i) if $U_{1i} \le Z_i$. Hence the actually observed number of data is

$$n = \sum_{i=1}^{N} 1_{\{U_{1i} \le Z_i\}}.$$

Note that n is a binomial random variable with parameters N and α. Throughout this work our statistical analysis will be based on a given value of n. The distribution function of the observed Z_i equals G^* and can be estimated through

$$G_n^*(y) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{Z_i \le y\}}.$$

The d.f. of an actually observed U_{1i} becomes

$$F_1^*(x_1) = \mathbb{P}(U_1 \le x_1 | U_1 \le Z)$$

which may be estimated through

$$F_{1n}^*(x_1) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{U_{1i} \le x_1\}}.$$

The empirical analogue of $A(x)$ becomes

$$A_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{Z_i \ge x\}}, \tag{1.3}$$

while $B(x)$ is estimated through

$$\begin{aligned} B_n(x) &= \int_{[x,\infty)} \frac{1 - F_{1n}(y)}{F_{1n}(y)} G_n^*(dy) \\ &= \frac{1}{n} \sum_{i=1}^{n} 1_{\{Z_i \ge x\}} \frac{1 - F_{1n}(Z_i)}{F_{1n}(Z_i)}, \end{aligned} \tag{1.4}$$

where F_{1n} is the Lynden-Bell estimator of F_1 for right-truncated data. More precisely, since

$$\begin{aligned} F_1^*(x_1) &= \mathbb{P}(U_1 \le x_1 | U_1 \le Z) \\ &= \alpha^{-1} \int_{(-\infty, x_1]} (1 - G(u^-)) F_1(du) \end{aligned}$$

we have

$$\alpha dF_1^* = (1 - G^-) dF_1$$

and therefore

$$dF_1 = \frac{dF_1^*}{\alpha^{-1}(1 - G^-)}.$$

Set

$$C(x) = \mathbb{P}(U_1 \le x \le Z | U_1 \le Z). \tag{1.5}$$

Since

$$C(x) = \alpha^{-1}\mathbb{P}(U_1 \le x \le Z) = \alpha^{-1}F_1(x)(1 - G(x^-)),$$

we obtain

$$\frac{dF_1}{F_1} = \frac{dF_1^*}{C}. \tag{1.6}$$

The cumulative hazard function associated with $\frac{dF_1}{F_1}$ is defined as

$$\Lambda(x) = \int_{[x,\infty)} \frac{F_1(du)}{F_1(u)}.$$

The product-integration formula then yields

$$F_1(t) = e^{-\Lambda^c(t)} \prod_{y>t} \left[1 + \Lambda\{y\}\right].$$

Since by (1.6)

$$\Lambda(x) = \int_{[x,\infty)} \frac{dF_1^*}{C},$$

the empirical counterparts become

$$\Lambda_n(x) := \int_{[x,\infty)} \frac{dF_{1n}^*}{C_n} = \sum_{i=1}^n \frac{1_{\{U_{1i} \ge x\}}}{nC_n(U_{1i})}$$

and, if there are no ties,

$$F_{1n}(t) = \prod_{y>t} [1 + \Lambda_n\{y\}] = \prod_{U_{1i}>t} \left[1 - \frac{1}{nC_n(U_{1i})}\right], \tag{1.7}$$

where

$$C_n(x) = \frac{1}{n} \sum_{k=1}^n 1_{\{U_{1k} \le x \le Z_k\}}.$$

Finally, the estimator of $H_2^1(x, y)$ becomes

$$H^1_{2n}(x,y) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{U_{1i}\le x, \tilde{U}_{2i}\le y, \delta_i=1\}},$$

where

$$\tilde{U}_{2i} = \min\left(U_{2i}, Z_i - U_{1i}\right),\ 1 \le i \le n.$$

We are now in the position to define our estimator of $I = \int \varphi dF$. In view of (1.2), we set

$$\begin{aligned} I_n &:= \int \varphi(x_1,x_2)\frac{1}{A_n(x_1+x_2)+B_n(x_1+x_2)} H^1_{2n}(dx_1,dx_2) \\ &= \frac{1}{n}\sum_{i=1}^{n} \varphi(U_{1i},\tilde{U}_{2i})\frac{\delta_i}{A_n(U_{1i}+\tilde{U}_{2i})+B_n(U_{1i}+\tilde{U}_{2i})}. \end{aligned}$$

As Theorem 1.1 will show, under mild integrability conditions, I_n admits a representation as a sum of i.i.d. random variables (plus remainder).

Theorem 1.1. *Under continuity of F_1, assume that*

***A1**:* $\int \frac{dF_1}{1-G_-} < \infty$

***A2**:*$\int \frac{|\varphi(x_1,x_2)|^k}{F_1^2(x_1+x_2)} F(dx_1,dx_2) < \infty$

***A3**:*$\int \frac{|\varphi(x_1,x_2)|^k}{(A+B)^2(x_1+x_2)} F(dx_1,dx_2) < \infty$

for $k = 1, 2$. Then we have

$$\begin{aligned} I_n &= \int \frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)} H^1_{2n}(dx_1,dx_2) \\ &\quad - \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{1}{C(y)}(F^*_{1n}(dy)-F^*_1(dy))G^*(dx)H^1_2(dx_1,dx_2) \\ &\quad + \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{C_n(y)-C(y)}{C^2(y)} F^*_1(dy)G^*(dx)H^1_2(dx_1,dx_2) \\ &\quad + \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)}(G^*(dx)-G^*_n(dx))H^1_2(dx_1,dx_2) + o_{\mathbb{P}}(\frac{1}{\sqrt{n}}). \end{aligned}$$

Remark 1.1. *Assumptions A1-A3 yield*

$$\int \frac{|\varphi(x_1,x_2)|}{(A+B)(x_1+x_2)F_1(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{G(dy)}{F_1(y)} F(dx_1,dx_2) < \infty$$

and

$$\int \frac{|\varphi(x_1,x_2)|^k}{(A+B)(x_1+x_2)F_1(x_1+x_2)} F(dx_1,dx_2) < \infty$$

for $k = 1, 2$.

Set $I_n = \hat{I}_n + \hat{R}_n$, where $\hat{I}_n$ is the sum of i.i.d. random variables and $\hat{R}_n$ is the remainder from Theorem 1.1. In the following Lemma we compute the variance of $\hat{I}_n$.

Lemma 1.1. *We have*

$$
\begin{aligned}
Var(\hat{I}_n) &= \frac{1}{n}\Big(\int\int\int\left[\frac{\varphi(y_1,y_2)}{(A+B)(y_1+y_2)}+\psi(y_1,y_3)\right]^2 H_3(dy_1,dy_2,dy_3)+\int\int\psi^2(y_1,y_3)\tilde{H}_2^2(dy_1,dy_3)\Big)\\
&=: \frac{\sigma^2}{n},
\end{aligned}
$$

where

$$
\begin{aligned}
&H_3(y_1,y_2,y_3) = P(U_1\le y_1, U_2\le y_2, Z\le y_3, \delta=1|U_1\le Z)\\
&\tilde{H}_2^2(y_1,y_3) = P(U_1\le y_1, Z\le y_3, \delta=0|U_1\le Z)
\end{aligned}
$$

and

$$
\begin{aligned}
\psi(y_1,y_3) &= \int\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\Big[\int_{x_1+x_2}^{\infty}\frac{1}{F_1(x)}\left\{\int_{(x,\infty)}\frac{1_{\{y_1\le y\le y_3\}}}{C^2(y)}F_1^*(dy)-\frac{1_{\{y_1>x\}}}{C(y_1)}\right\}G^*(dx)\\
&\quad -\frac{1_{\{y_3\ge x_1+x_2\}}}{F_1(y_3)}\Big]H_2^1(dx_1,dx_2).
\end{aligned}
$$

Since we can write I_n as a sum of i.i.d. r.v.'s and a remainder of the order $o_{\mathbb{P}}(\frac{1}{\sqrt{n}})$, we may apply the CLT. Actually,

$$\sqrt{n}(I_n - I)\rightarrow\mathcal{N}(0,\sigma^2). \tag{1.8}$$

If φ is the indicator of the rectangle $(-\infty,x]\times(-\infty,y]$, we have, in particular,

$$F_n(x,y) = \frac{1}{n}\sum_{i=1}^n\frac{1_{\{U_{1i}\le x,\tilde{U}_{2i}\le y\}}\delta_i}{A_n(U_{1i}+\tilde{U}_{2i})+B_n(U_{1i}+\tilde{U}_{2i})}$$

and

$$\sqrt{n}(F_n(x,y)-F(x,y))\rightarrow\mathcal{N}(0,\sigma^2(x,y)).$$

This can be used to compute confidence intervals for $F(x,y)$. For this we have to replace the unknown $\sigma^2(x,y)$ through its estimator.

To obtain an estimator of the variance of I_n, we use the Plug-In Method. This means we have to replace the unknown terms in σ^2 with their estimators. This leads us to the following lemma.

Lemma 1.2. *The Plug-In estimator of σ^2 equals*

$$\sigma_n^2 = \frac{1}{n}\sum_{i=1}^n\left[\frac{\varphi(U_{1i},\tilde{U}_{2i})}{(A_n+B_n)(U_{1i}+\tilde{U}_{2i})}+\psi_n(U_{1i},Z_i)\right]^2\delta_i+\frac{1}{n}\sum_{i=1}^n\psi_n^2(U_{1i},Z_i)(1-\delta_i),$$

where

$$\begin{aligned}\psi_n(U_{1i}, Z_i) &= \frac{1}{n}\sum_{l=1}^{n}\frac{\varphi(U_{1l}, \tilde{U}_{2l})}{(A_n+B_n)^2(U_{1l}+\tilde{U}_{2l})}\delta_l\Big[\frac{1}{n}\sum_{j=1}^{n}1_{\{Z_j\geq U_{1l}+\tilde{U}_{2l}\}}\frac{1}{F_{1n}(Z_j)}\Big\{-\frac{1_{\{U_{1i}>Z_j\}}}{C_n(U_{1i})}\\ &\quad+\frac{1}{n}\sum_{k=1}^{n}1_{\{U_{1k}>Z_j\}}\frac{1_{\{U_{1i}\leq U_{1k}\leq Z_i\}}}{C_n^2(U_{1k})}\Big\}-\frac{1_{\{Z_i\geq U_{1l}+\tilde{U}_{2l}\}}}{F_{1n}(Z_i)}\Big].\end{aligned}$$

Chapter 2

Simulations

2.1 The Marshall and Olkin Model

In this chapter we study examples for the estimator I_n of I for different sample sizes and truncation rates $1-\alpha$. In the next two sections we will deal with the estimator of a d.f., F_n, estimate variances of F_n and compute confidence intervals. In the last section we study examples of I_n for different score function φ, like estimated correlation coefficients or expectations.

The variables U_1, U_2 and Z are taken from an exponential distribution with parameters λ_{U_1}, λ_{U_2} and λ_Z, respectively. Recall that, in general, U_1 and U_2 are dependent. Because of that, we will produce dependent replicates of U_1 and U_2 using a method proposed by Marshall and Olkin. See Johnson and Kotz (1972). For this we first produce N copies of three independent vectors e_1, e_2 and e_3 from an exponential distribution with parameters λ_1, λ_2 and λ_3, respectively. While λ_1, λ_2 and the correlation between U_1 and U_2 can be chosen, the λ_3 is given by

$$\lambda_3 = \frac{Corr(U_1,U_2)(\lambda_1+\lambda_2)}{1-Corr(U_1,U_2)}.$$

Setting

$$U_1 = min(e_1, e_3)$$

and

$$U_2 = min(e_2, e_3)$$

we obtain vectors of replicates from an exponential distribution with parameters $\lambda_1+\lambda_3$ and $\lambda_2+\lambda_3$, respectively. To keep the truncation rate equal to $1-\alpha$, vector of Z's is to be produced from an exponential distribution with parameter:

$$\lambda_Z = \frac{1-\alpha}{\alpha}(\lambda_1+\lambda_3).$$

This equation is a consequence of

$$\mathbb{P}(Z \le y | U_1 \le Z) = \alpha^{-1} \int_{(-\infty,y]} F_1(z)G(dz),$$

where F_1 and G are the distribution functions of U_1 and Z, respectively. Setting $y=\infty$, we obtain

$$1 = \mathbb{P}(Z \leq \infty | U_1 \leq Z) = \alpha^{-1} \int_{(-\infty,\infty)} F_1(z) G(dz)$$

Since $F_1(z) = 1 - e^{-(\lambda_1+\lambda_3)z}$ and $G(z) = 1 - e^{-\lambda_Z z}$ for $z \geq 0$ and $F_1(z) = G(z) = 0$ for $z < 0$ we obtain

$$\alpha = \int_{(0,\infty)} (1 - e^{-(\lambda_1+\lambda_3)z}) \lambda_Z e^{-\lambda_Z z} dz = 1 - \frac{\lambda_Z}{\lambda_1 + \lambda_3 + \lambda_Z}$$

and therefore

$$\frac{\lambda_Z}{\lambda_1 + \lambda_3 + \lambda_Z} = 1 - \alpha.$$

Hence, finally, we have

$$\lambda_Z = \frac{1-\alpha}{\alpha}(\lambda_1 + \lambda_3).$$

To use the estimator from Chapter 1, we take only the n of N replicates of U_1, U_2 and Z for which the U_1's are less than or equal to Z.
The joint distribution function of the dependent (U_1, U_2) is given as

$$F(x,y) = 1 - e^{-(\lambda_1+\lambda_3)x} - e^{-(\lambda_2+\lambda_3)y} + e^{-\lambda_1 x - \lambda_2 y - \lambda_3 max(x,y)}.$$

2.2 Simulations for the estimator of a d.f. for different α and N

For our simulation study we first take $\lambda_1 = 1$, $\lambda_2 = 1$ and $Corr(U_1, U_2) = 0.5$. Then we obtain exponential U_1 and U_2 both with parameter equal to 3. The first figure shows the true distribution function:

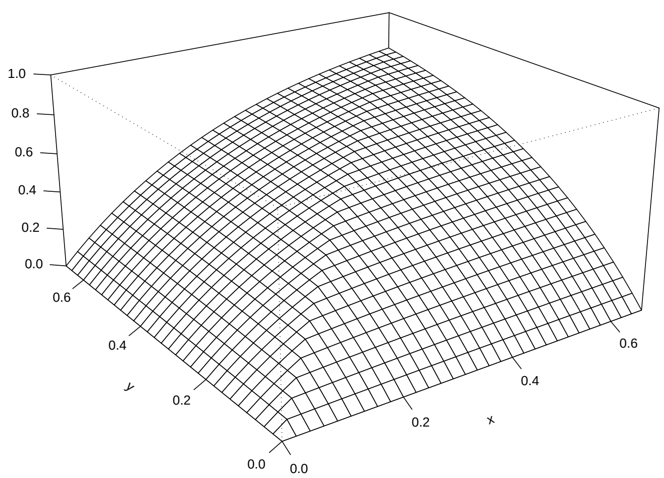

Figure 2.1: Distribution function

Figure 2.2 displays the estimator F_n for $N = 20$ data and 10% truncation. We can see that there are big differences between our estimator and the true distribution function F. This is not surprising since, after truncation, our estimator on average is based on 18 data. For the available sample the first variable can be observed, but some of U_{2i}'s are additionally censored.

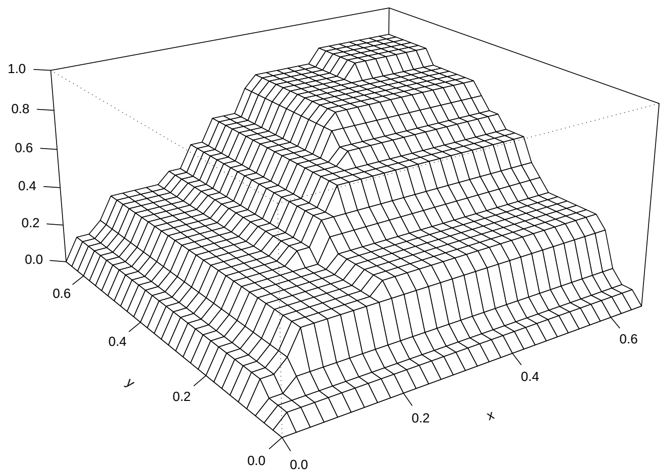

Figure 2.2: Estimator of a d.f. for $N = 20$ and $\alpha = 0.9$

The small sample size also has an influence on the estimator of the variance. We will see, that the variance is large compared to variances for 50 or 100 data. But we can observe that the estimator of the variance falls for large x and y similarly as the variance for a standard estimator of a distribution function (based on complete data). This happens because for 10% truncation the loss of information is much smaller compared with larger truncation rates, for example 40%.

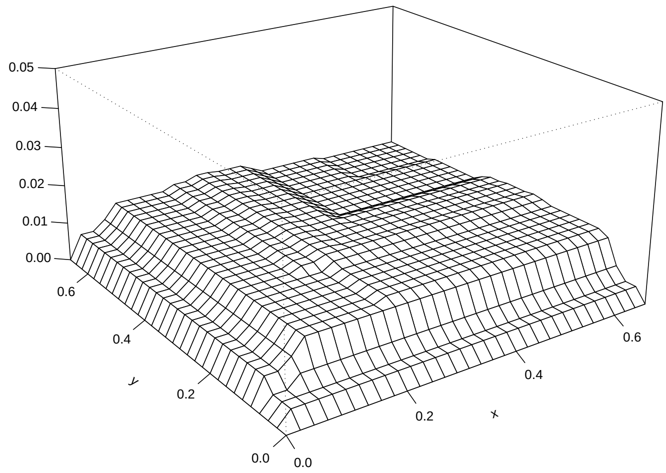

Figure 2.3: Estimator of variance for $N = 20$ and $\alpha = 0.9$

But already for $N = 50$ and 10% truncation the estimator of the distribution function is very good. As we can see, the estimator based on 45 data is not only close to the true distribution function for small (x, y) but we also haven't any overestimation for large (x, y).

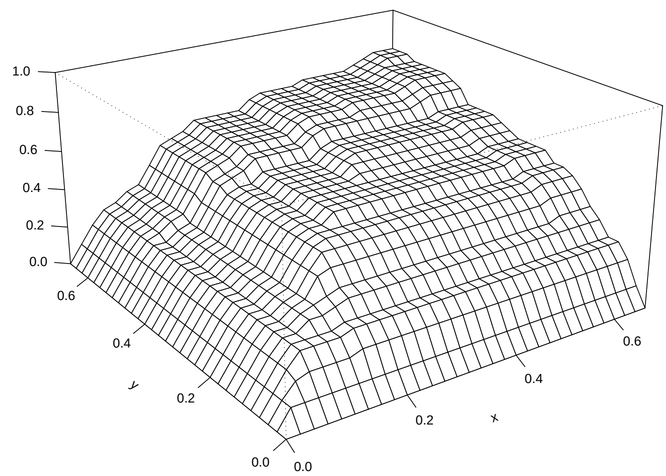

Figure 2.4: Estimator of a d.f. for $N = 50$ and $\alpha = 0.9$

For a larger data set ($N = 100$), even with a large percentage of truncation (40%) we can see that the estimator of a distribution function, for small (x, y), looks good. The truncation rate has a noticeable influence, if at all, for large (x, y). This is the effect of loss of information for large data.

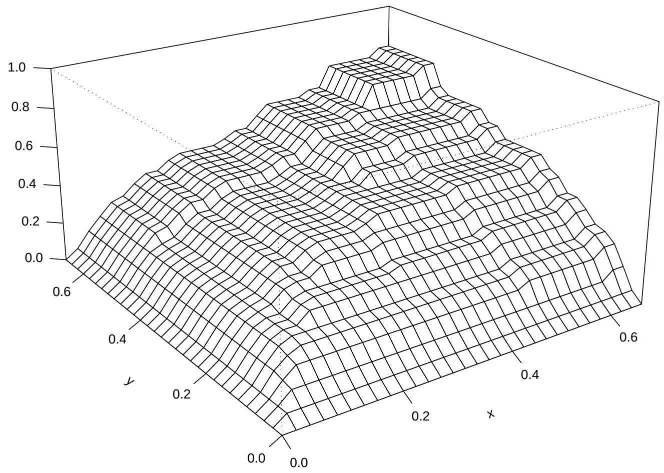

Figure 2.5: Estimator of a d.f. for $N = 100$ and $\alpha = 0.6$

As we can see on the next figure, the heavy truncation has also an influence on the estimator of variance. It increases when x and y become larger, but decreases for small (x, y).

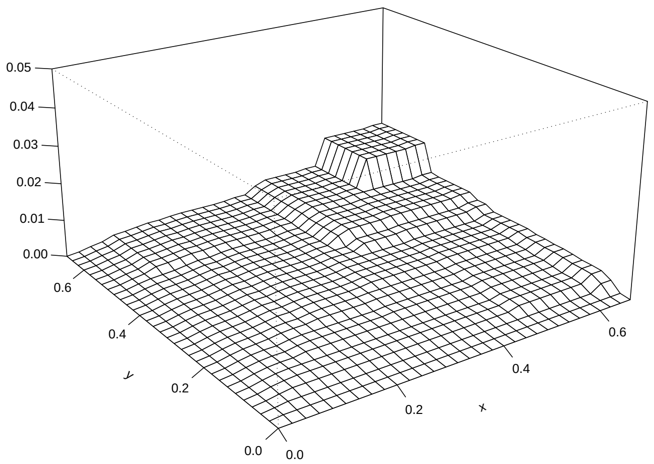

Figure 2.6: Estimator of variance for $N = 100$ and $\alpha = 0.6$

In all of our simulations we did until now, the expectations of U_1 and U_2 were both equal to $1/3$. The expectation of the truncation variable Z was equal to 3 for $\alpha = 90\%$ and 0.5 for $\alpha = 60\%$.
Next we study a more asymmetric d.f., where on avarage U_1 is much smaller that U_2. This is similar to a situation in AIDS, where the period between infection and seroconversion is much smaller than the period between seroconversion and AIDS. In such a situation, the longer second variable U_2 is more likely to be censored as in the first example.

More precisely, in this case we choose a small correlation between U_1 and U_2 namely 0.1, while $\lambda_1 = 1.5$, $\lambda_2 = 0.3$. Then we get exponential variables U_1 and U_2 with parameters 1.7 and 0.5. This gives us $EU_1 = 0.59$ and $EU_2 = 2$. In this case, the true d.f. looks as follows:

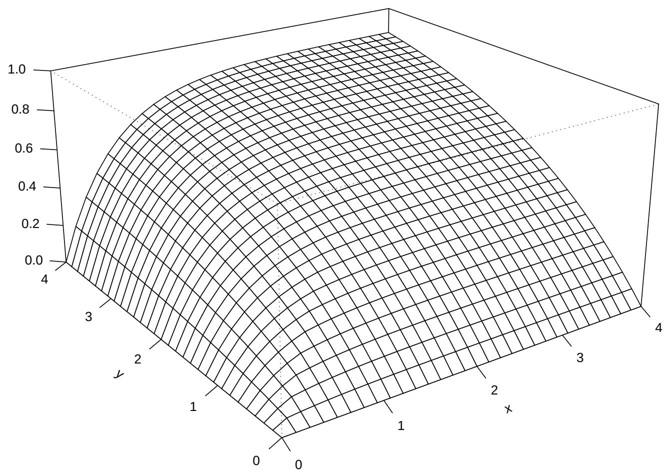

Figure 2.7: Distribution function

For heavy truncation ($\alpha = 0.7$) the expectation of Z equals 1.37. As a consequence we obtain a big censoring rate as well. Nevertheless for a large data set ($N = 200$) the estimator is pretty close to the true d.f.

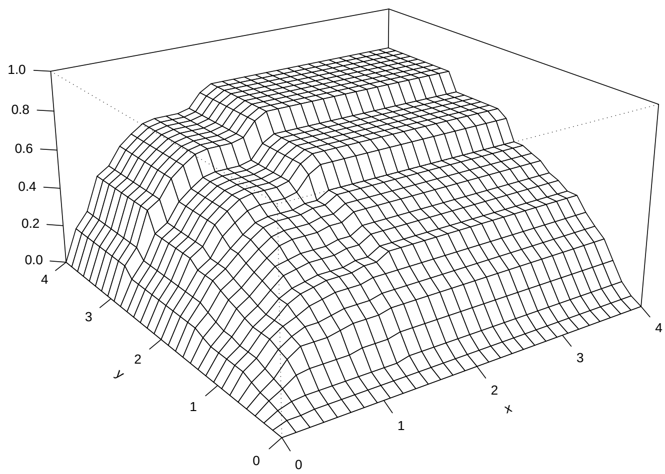

Figure 2.8: Estimator of a d.f. for $N = 200$ and $\alpha = 0.7$

Some problems occurs if we take $N = 50$.

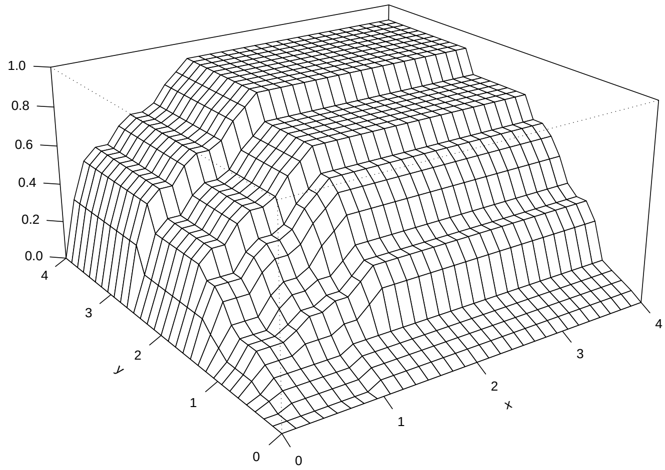

Figure 2.9: Estimator of a d.f. for $N = 50$ and $\alpha = 0.7$

This was expected, since for $\alpha = 0.7$ about 15 data are truncated, and of the remaining 35 data, about one half were additionally censored.
But if we take a more realistic case, 10% truncation, which yields an expectation of Z being equal to 5.3, we get a much better approximation of our distribution function already for a small data set ($N = 50$).

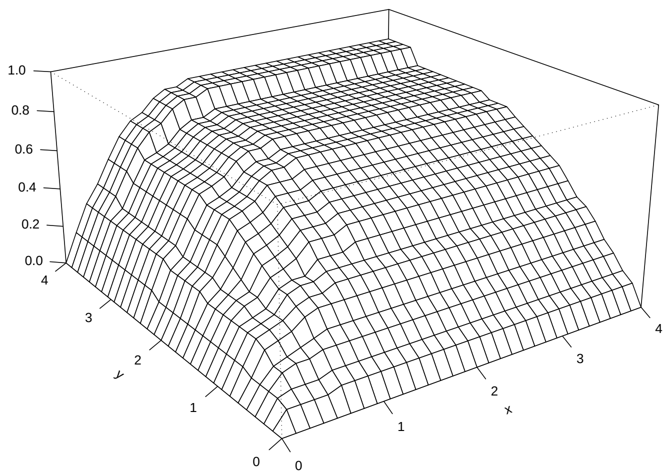

Figure 2.10: Estimator of a d.f. for $N = 50$ and $\alpha = 0.9$

Finally we make some simulations for strongly correlated data. More precisely , we choose a correlation equal 0.8, while U_1 and U_2 have parameters equal to 4.5 and 5.4, respectively. Then $EU_1 \approx 0.22$ and $EU_2 \approx 0.18$. The true distribution function is as follows:

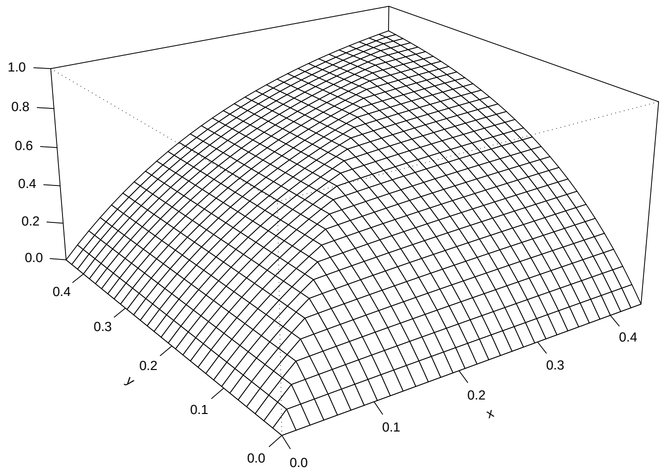

Figure 2.11: Distribution function

For $\alpha = 0.7$, the expectation of the truncation variable Z equals 0.52. For $N = 50$ we have the following estimator:

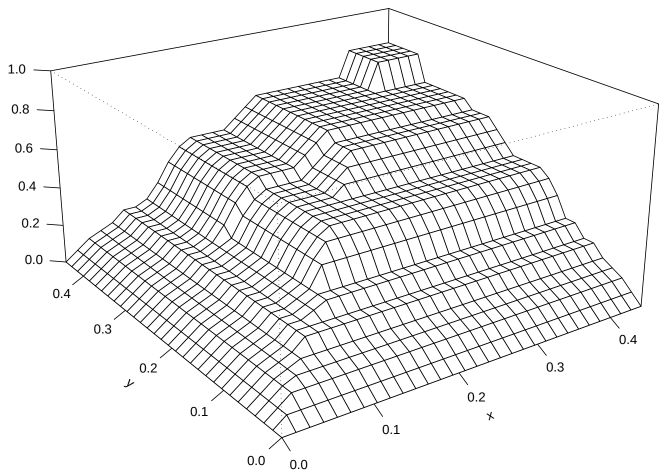

Figure 2.12: Estimator of a d.f. for $N = 50$ and $\alpha = 0.7$

If $\alpha = 0.9$, the Z expectation equals 2 and the results are much better.

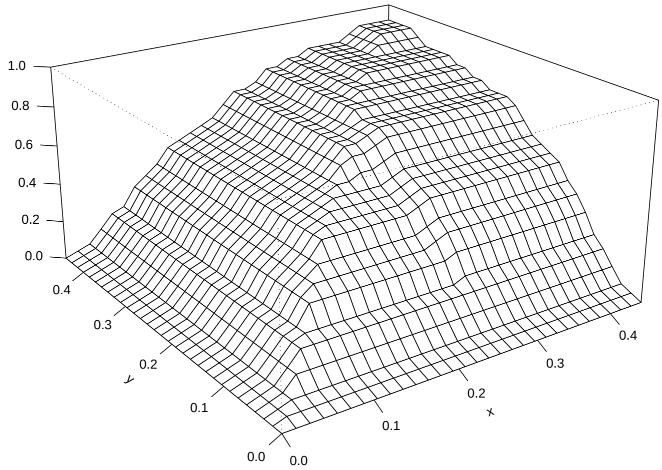

Figure 2.13: Estimator of a d.f. for $N = 50$ and $\alpha = 0.9$

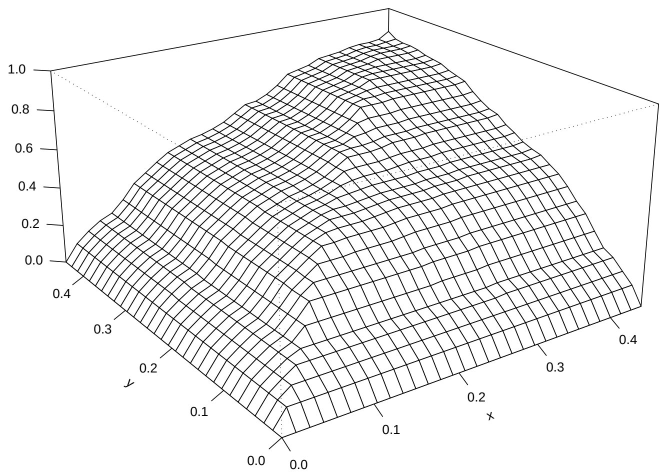

Figure 2.14: Estimator of a d.f. for $N = 200$ and $\alpha = 0.9$

In the simulations, we see that for different parameters, the truncation rate has a large influence on the estimator. This is obvious if we recall that the truncation variable Z is, at the same time, a censoring variable.

The large influence of the truncation rate $1-\alpha$ we can see better when we construct confidence intervals. We take, as in the beginning of this chapter, $\lambda_{U_1} = \lambda_{U_2} = 3$. The correlation equals 0.5. The next three figures show 95% confidence intervals at $(1,1)$. This point was chosen, because, as we could see in Figures 2.2, 2.5 and 2.6, first for large (x,y) with $x, y \geq 0.5$ occur some problems with estimation. Figure 2.15 is an example of confidence intervals for $\alpha = 70\%$ and 50 data.

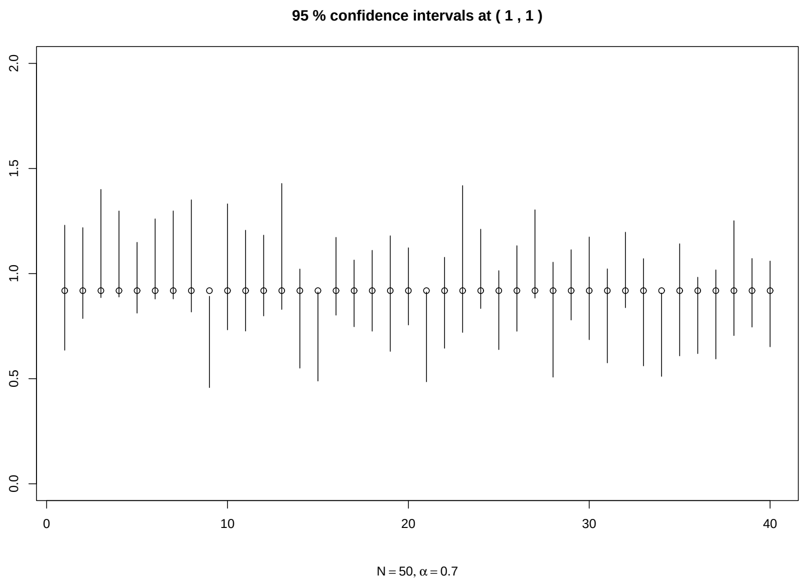

Figure 2.15: Confidence Intervals for $N = 50$ and $\alpha = 0.7$

In Figure 2.16 the intervals are smaller than in 2.15, since we take twice as many data ($N = 100$) data with α still being equal 70%.

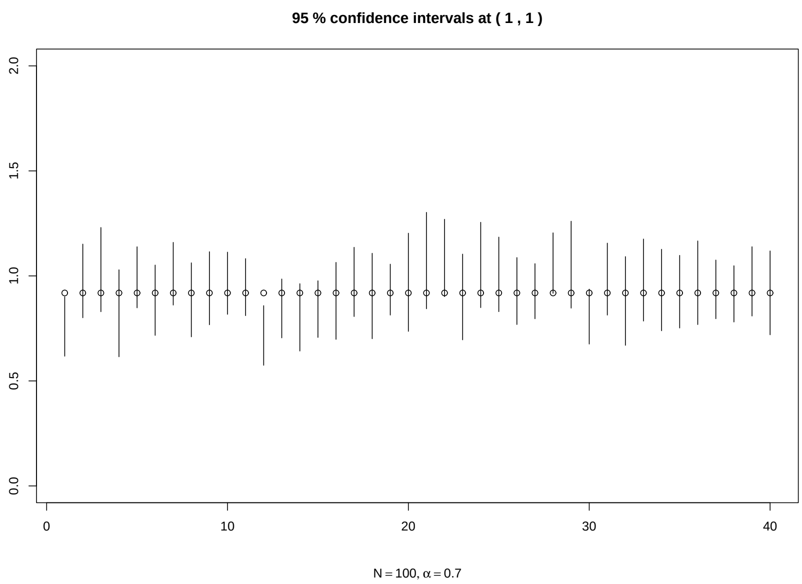

Figure 2.16: Confidence Intervals for $N = 100$ and $\alpha = 0.7$

If we take $N = 50$ and a large α equal to 90% (only 10% truncation) the confidence intervals are much smaller than in Figure 2.15. This confirms the big influence of the truncation rate.

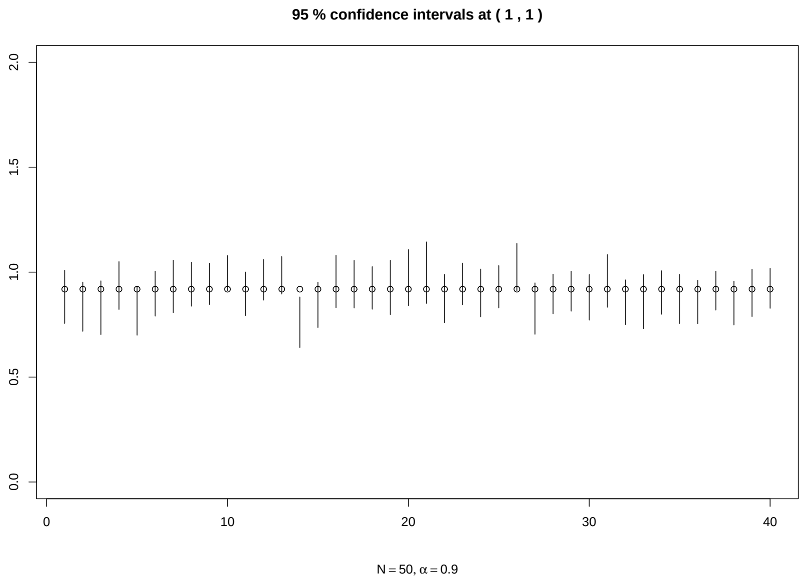

Figure 2.17: Confidence Intervals for $N = 50$ and $\alpha = 0.9$

These confidence intervals are made for only 40 replications and because of that we can use them only to say something about differences for different α's, and not about the quality of estimation.

2.3 Comparison between $F_n(x, y)$ and the standard empirical estimator

In this section we compare our estimator $F_n(x, y)$ with the standard empirical estimator $H_n(x, y)$, defined as

$$H_n(x, y) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{U_{1i} \leq x, \tilde{U}_{2i} \leq y\}}.$$

At first we compare the two empirical estimators in one point (x, y) with help of an estimated mean squared error, defined as

$$\widehat{MSE}(F_n(x, y)) = \frac{1}{m} \sum_{i=1}^{m} (F_n(x, y)^{(i)} - F(x, y))^2,$$

and estimated bias defined as

$$M(F_n(x, y), F(x, y)) := \frac{1}{m} \sum_{i=1}^{m} F_n(x, y)^{(i)} - F(x, y),$$

where m is the number of replications and $F_n(x, y)^{(i)}$ estimated d.f. based on the ith sample. Then we will compare this two dimensional functions with help of plots.

For the first comparison, we take, as in the beginning of this chapter, $\lambda_1 = \lambda_2 = 1$ and $Corr(U_1, U_2) = 0.5$. This yields $U_1, U_2 \sim exp(3)$.
In the following table we provide the estimated MSE for $m = 1000$, for different α and sample size N in point $(x, y) = (0.5, 0.5)$.

	α	$N = 20$	$N = 50$	$N = 100$	$N = 200$
$\widehat{MSE}(F_n(0.5, 0.5))$	0.7	0.0327	0.0080	0.0050	0.0043
$\widehat{MSE}(H_n(0.5, 0.5))$	0.7	0.0459	0.0170	0.0210	0.0274
$\widehat{MSE}(F_n(0.5, 0.5))$	0.9	0.0058	0.0044	0.0016	0.0007
$\widehat{MSE}(H_n(0.5, 0.5))$	0.9	0.0030	0.0089	0.0058	0.0041

Table 2.1: estimated MSE

	α	$N = 20$	$N = 50$	$N = 100$	$N = 200$
$M(F_n(0.5, 0.5), F(0.5, 0.5))$	0.7	0.1148	0.0518	0.0446	0.0127
$M(H_n(0.5, 0.5), F(0.5, 0.5))$	0.7	0.2056	0.1797	0.1569	0.1527
$M(F_n(0.5, 0.5), F(0.5, 0.5))$	0.9	-0.0381	0.0522	0.0309	0.0157
$M(H_n(0.5, 0.5), F(0.5, 0.5))$	0.9	0.0126	0.0903	0.0742	0.0623

Table 2.2: estimated bias

As we can see, both estimators are better for bigger sample size than 20, but $F_n(x, y)$ gives much better results (for MSE) than $H_n(x, y)$, when the truncation is heavy. This was expected since $H_n(x, y)$ uses the data as if they were complete and $F_n(x, y)$ takes into consideration that we have lost information. The differences can be seen even better in Figures 2.19, 2.20 and 2.21.
Recall that the true d.f. is as follows:

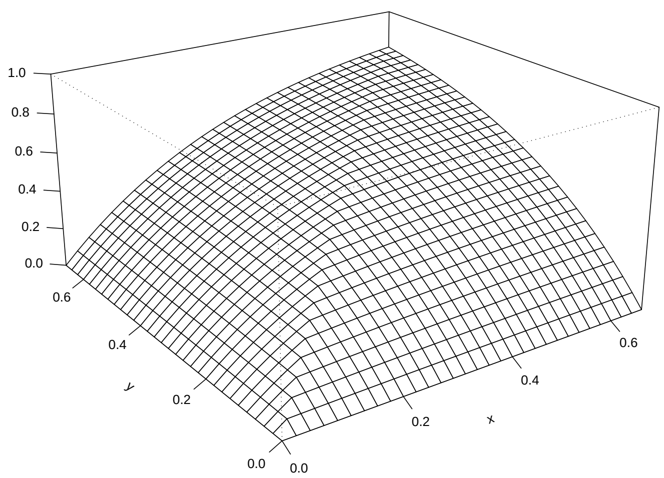

Figure 2.18: Distribution function

For $\alpha = 0.99$ (almost no lost of information) and $N = 50$ the estimators looks similar.

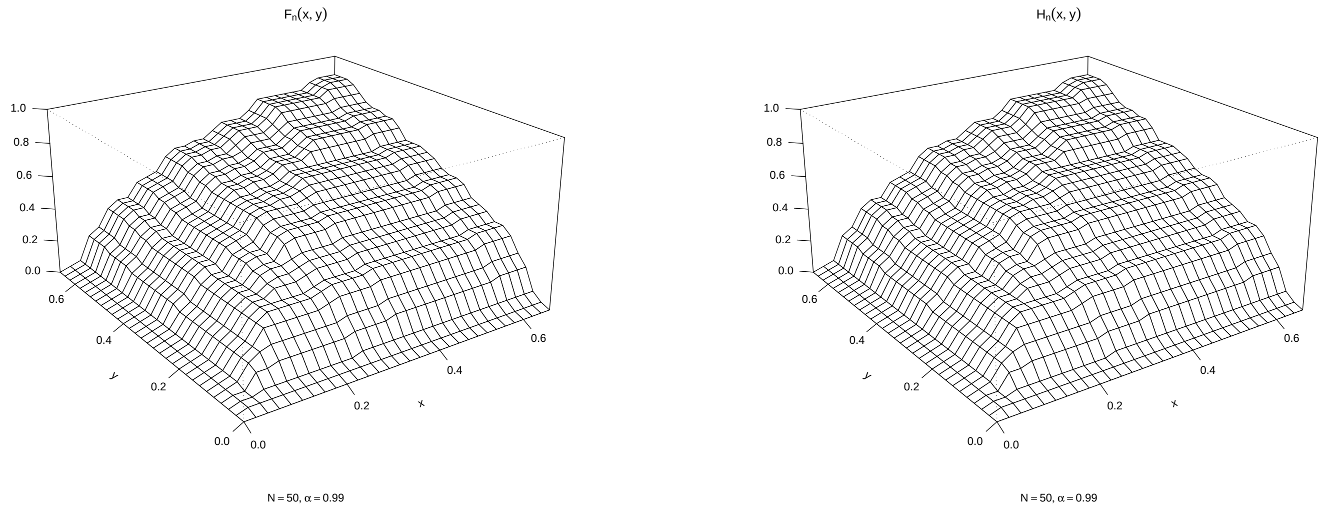

Figure 2.19: Two estimators for $N = 50$ and $\alpha = 0.99$

For $\alpha = 0.7$ and $N = 100$ we can already see some differences.

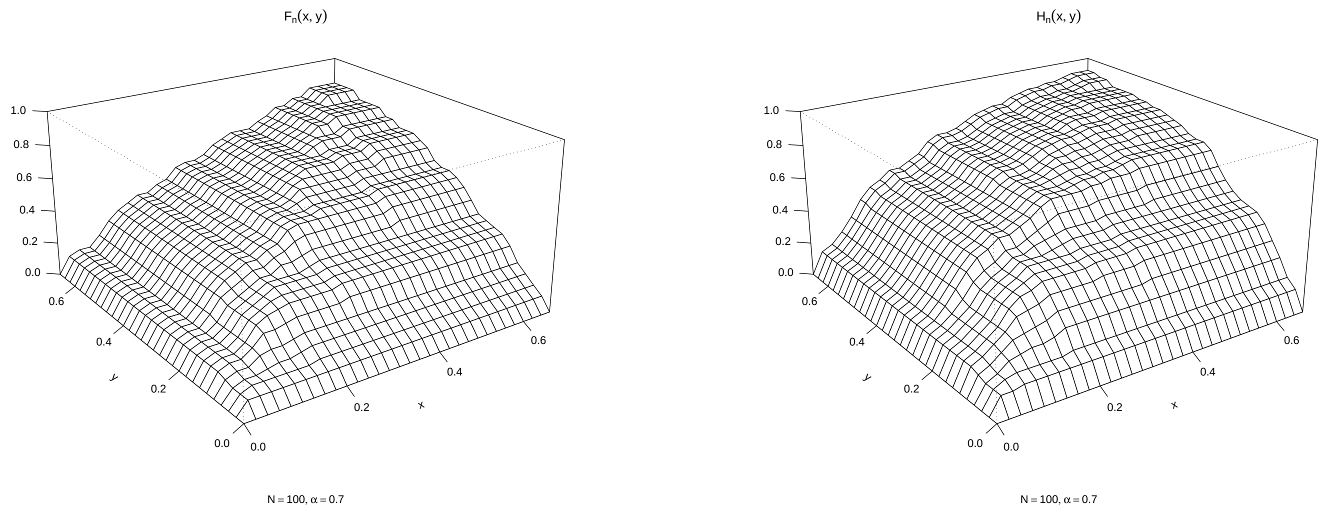

Figure 2.20: Two estimators for $N = 100$ and $\alpha = 0.7$

But for very heavy truncation ($\alpha = 0.6$) and strong censorship, when $N = 200$, the standard empirical estimator $H_n(x, y)$ strongly overestimates the true d.f. $F(x, y)$, while $F_n(x, y)$ is almost perfect.

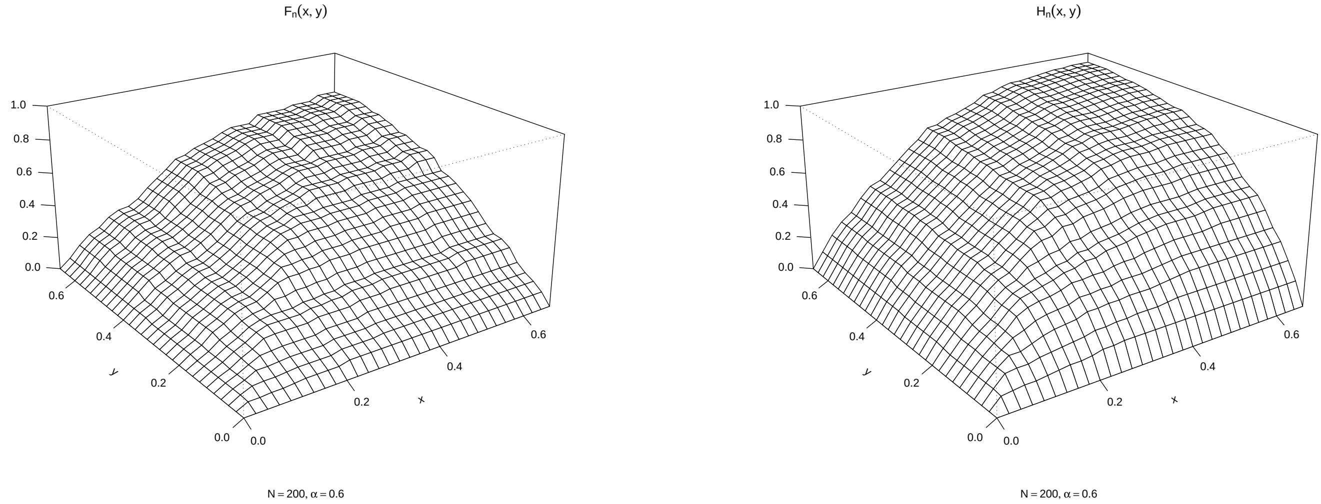

Figure 2.21: Two estimators for $N = 200$ and $\alpha = 0.6$

For the more asymmetric case, when the correlation between U_1 and U_2 equals 0.1, $U_1 \sim exp(1.7)$ and $U_2 \sim exp(0.5)$, we choose $(x, y) = (0.5, 2) \approx (EU_1, EU_2)$ and get the following results:

	α	$N = 20$	$N = 50$	$N = 100$	$N = 200$
$\widehat{MSE}(F_n(0.5, 2))$	0.6	0.0656	0.0413	0.0105	0.0061
$\widehat{MSE}(H_n(0.5, 2))$	0.6	0.1987	0.1606	0.1427	0.1312
$\widehat{MSE}(F_n(0.5, 2))$	0.9	0.0101	0.0015	0.0007	0.0006
$\widehat{MSE}(H_n(0.5, 2))$	0.9	0.0177	0.0102	0.0087	0.0048

Table 2.3: estimated MSE

	α	$N = 20$	$N = 50$	$N = 100$	$N = 200$
$M(F_n(0.5, 2), F(0.5, 2))$	0.6	0.0698	-0.0647	0.0623	0.0018
$M(H_n(0.5, 2), F(0.5, 2))$	0.6	0.3524	0.3349	0.3967	0.3609
$M(F_n(0.5, 2), F(0.5, 2))$	0.9	0.2229	-0.0371	-0.0273	-0.0183
$M(H_n(0.5, 2), F(0.5, 2))$	0.9	0.2893	0.0759	0.0685	0.0675

Table 2.4: estimated bias

As before, for heavy truncation, $F_n(x, y)$ is much better than $H_n(x, y)$ already for small sample sizes. This is a consequence not only of 40% truncation, but also of censoring, so that, in the end, we observe less than half of the complete data. On the other hand, while $\alpha = 0.9$, $F_n(x, y)$ is truly better than $H_n(x, y)$ only for bigger sample size.
Finally, when we look at the tables , we can see that the results for $H_n(x, y)$ are not changing too much for $N = 50, 100, 200$, while for our estimator, $F_n(x, y)$, we get each time much better results. This confirms the asymptotic results for this estimator.

2.4 Estimation of correlation coefficients and expectations

Since in this work we deal with estimators of $I = \int \varphi dF$ for Borel functions φ, we are able not only to compute estimators of a d.f. but also of the correlation between U_1 and U_2, EU_1 or EU_2. As in the last section, we will compare our estimator with a standard empirical estimator.
Recall

$$I_n = \int \varphi dF_n = \frac{1}{n} \sum_{i=1}^{n} \varphi(U_{1i}, \tilde{U}_{2i}) \frac{\delta_i}{A_n(U_{1i} + \tilde{U}_{2i}) + B_n(U_{1i} + \tilde{U}_{2i})},$$

and set

$$S_n = \int \varphi dH_n = \frac{1}{n} \sum_{i=1}^{n} \varphi(U_{1i}, \tilde{U}_{2i}).$$

Let $Corr_n(L)$ be the estimated correlation coefficient and $L(U_j)$ describe the estimated expectation of U_j by using the estimator L, where L is I_n or S_n. For example

$$S_n(U_1) = \frac{1}{n} \sum_{i=1}^{n} U_{1i}$$

and

$$Corr_n(S_n) = \frac{\frac{1}{n}\sum_{i=1}^{n}(U_{1i} - S_n(U_1))(\tilde{U}_{2i} - S_n(\tilde{U}_2))}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(U_{1i} - S_n(U_1))^2 \frac{1}{n}\sum_{i=1}^{n}(\tilde{U}_{2i} - S_n(\tilde{U}_2))^2}}.$$

As in the last section $\widehat{MSE}$ is the estimator of the mean squared error, defined as

$$\widehat{MSE}(Corr_n(L)) = \frac{1}{m} \sum_{i=1}^{m} (Corr_n(L)^{(i)} - Corr(U_1, U_2))^2,$$

and

$$\widehat{MSE}(L(U_j)) = \frac{1}{m} \sum_{i=1}^{m} (L(U_1)^{(i)} - EU_j)^2.$$

As before we take $\lambda_1 = \lambda_2 = 1$, $Corr(U_1, U_2) = 0.5$, whence $U_1, U_2 \sim exp(3)$. Results for $m = 1000$ are in following tables:

	α	$N = 20$	$N = 50$	$N = 100$	$N = 200$
$\widehat{MSE}(Corr_n(I_n))$	0.7	0.3054	0.0996	0.0809	0.0298
$\widehat{MSE}(Corr_n(S_n))$	0.7	0.2071	0.0645	0.0511	0.0189
$\widehat{MSE}(Corr_n(I_n))$	0.9	0.0498	0.0390	0.0313	0.0040
$\widehat{MSE}(Corr_n(S_n))$	0.9	0.0376	0.0260	0.0306	0.0044

Table 2.5: estimated MSE for correlations

	α	$N = 20$	$N = 50$	$N = 100$	$N = 200$
$\widehat{MSE}(I_n(U_1))$	0.7	0.0281	0.0155	0.0081	0.0051
$\widehat{MSE}(S_n(U_1))$	0.7	0.0172	0.0130	0.0105	0.0089
$\widehat{MSE}(I_n(U_2))$	0.7	0.0520	0.0257	0.0098	0.0045
$\widehat{MSE}(S_n(U_2))$	0.7	0.0219	0.0129	0.0135	0.0156
$\widehat{MSE}(I_n(U_1))$	0.9	0.0042	0.0039	0.0013	0.0008
$\widehat{MSE}(S_n(U_1))$	0.9	0.0032	0.0048	0.0021	0.0023
$\widehat{MSE}(I_n(U_2))$	0.9	0.0058	0.0064	0.0012	0.0014
$\widehat{MSE}(S_n(U_2))$	0.9	0.0090	0.0010	0.0023	0.0049

Table 2.6: estimated MSE for expectations

Unfortunately I_n seems to give better results only for $I_n(U_i)$. Because of that we compare the two estimators with the estimated bias as well. Recall

$$M(B_n, B) := \frac{1}{m}\sum_{i=1}^{m} B_n^{(i)} - B,$$

where m is a number of replications, $B_n^{(i)}$ is the value of estimator computed for the ith sample, and B is the true value.
For $m = 1000$ we have the following results for the correlation coefficient and expectations:

	α	$N = 20$	$N = 50$	$N = 100$	$N = 200$
$M(Corr_n(I_n), Corr(U_1, U_2))$	0.7	-0.3540	-0.1525	-0.0911	0.0071
$M(Corr_n(S_n), Corr(U_1, U_2))$	0.7	-0.3387	-0.2025	-0.1858	-0.1070
$M(Corr_n(I_n), Corr(U_1, U_2))$	0.9	0.1983	-0.0898	-0.0963	-0.0076
$M(Corr_n(S_n), Corr(U_1, U_2))$	0.9	0.1583	-0.1012	-0.1415	-0.0431

Table 2.7: estimated bias for correlations

	α	$N = 20$	$N = 50$	$N = 100$	$N = 200$
$M(I_n(U_1), EU_1)$	0.7	-0.0507	-0.0241	-0.0092	-0.0068
$M(S_n(U_1), EU_1)$	0.7	-0.1249	-0.1113	-0.1010	-0.0872
$M(I_n(U_2), EU_2)$	0.7	-0.0705	0.0174	0.0061	-0.0033
$M(S_n(U_2), EU_2)$	0.7	-0.1437	-0.1100	-0.1146	-0.1262
$M(I_n(U_1), EU_1)$	0.9	-0.0171	-0.0463	-0.0094	-0.0037
$M(S_n(U_1), EU_1)$	0.9	-0.0477	-0.0675	-0.0437	-0.0296
$M(I_n(U_2), EU_2)$	0.9	-0.0572	0.0398	-0.0002	0.0006
$M(S_n(U_2), EU_2)$	0.9	-0.0914	-0.0150	-0.0452	-0.0384

Table 2.8: estimated bias for expectations

The last two tables show that our estimator is better, and we see the positive effect for heavy truncation as well. Besides, we can see that by estimation of expectations all results for standard estimator are negative. This is the result of, already mentioned, overestimation of a d.f..

Chapter 3

Proofs

The objective of this Chapter is to prove Theorem 1.1.

First recall that

$$\begin{aligned} I_n &:= \int \varphi(x_1,x_2)\frac{1}{A_n(x_1+x_2)+B_n(x_1+x_2)}H^1_{2n}(dx_1,dx_2) \\ &= \frac{1}{n}\sum_{i=1}^n \varphi(U_{1i},\tilde{U}_{2i})\frac{\delta_i}{A_n(U_{1i}+\tilde{U}_{2i})+B_n(U_{1i}+\tilde{U}_{2i})}, \end{aligned}$$

where

$$A_n(x) = \frac{1}{n}\sum_{i=1}^n 1_{\{Z_i\ge x\}}, \tag{3.1}$$

$$\begin{aligned} B_n(x) &= \int_{[x,\infty)} \frac{1-F_{1n}(y)}{F_{1n}(y)}G^*_n(dy) \\ &= \frac{1}{n}\sum_{i=1}^n 1_{\{Z_i\ge x\}}\frac{1-F_{1n}(Z_i)}{F_{1n}(Z_i)}, \end{aligned} \tag{3.2}$$

and

$$\delta_i = 1_{\{U_{1i}+U_{2i}\le Z_i\}} = \begin{cases} 1, & \text{if } U_{2i} \text{ is uncensored} \\ 0, & \text{otherwise} \end{cases}$$

To analyze I_n it will be important to find a proper decomposition into leading terms and remainders. For this, write

$$\begin{aligned} I_n &= \int \varphi\left[\frac{1}{A_n+B_n}-\frac{1}{A+B}\right]dH^1_{2n} + \int\varphi\frac{1}{A+B}dH^1_{2n} \\ &= \int\varphi\frac{1}{A+B}dH^1_{2n} + \int\varphi\left[\frac{1}{A_n+B_n}-\frac{1}{A+B}\right](dH^1_{2n}-dH^1_2) \\ &\quad + \int\varphi\left[\frac{1}{A_n+B_n}-\frac{1}{A+B}\right]dH^1_2 = I_{1n}+I_{2n}+I_{3n}, \end{aligned} \tag{3.3}$$

say. The term I_{1n} is already a sum of i.i.d. random variables. As to I_{2n} and I_{3n}, note that

$$\frac{1}{A_n + B_n} = \frac{1}{A+B} + \frac{A+B-A_n-B_n}{(A+B)(A_n+B_n)}$$

whence

$$\begin{aligned} I_{2n} &= \int \varphi \frac{(A+B-A_n-B_n)^2}{(A+B)^2(A_n+B_n)}(dH_{2n}^1 - dH_2^1) \\ &\quad + \int \varphi \frac{A+B-A_n-B_n}{(A+B)^2}(dH_{2n}^1 - dH_2^1) \equiv J_{1n} + J_{2n} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} I_{3n} &= \int \varphi \frac{(A+B-A_n-B_n)^2}{(A+B)^2(A_n+B_n)} dH_2^1 \\ &\quad + \int \varphi \frac{A+B-A_n-B_n}{(A+B)^2} dH_2^1 \equiv J_{3n} + J_{4n}. \end{aligned} \tag{3.5}$$

Clearly,

$$J_{1n} + J_{3n} = \int \varphi \frac{(A+B-A_n-B_n)^2}{(A+B)^2(A_n+B_n)} dH_{2n}^1.$$

This term will turn out to be negligible, as will also be the case with J_{2n}. The term J_{4n} in addition to I_{1n} is the only quantity which will contribute to the leading part of I_n.

Before we come to details, we represent B_n in a way which is more convenient for the analysis of $A + B - A_n - B_n$. For this, write

$$\begin{aligned} B(x) - B_n(x) &= \int_{[x,\infty)} \frac{1-F_1}{F_1} dG^* - \int_{[x,\infty)} \frac{1-F_{1n}}{F_{1n}} dG_n^* \\ &= \int_{[x,\infty)} \frac{1-F_1}{F_1}[dG^* - dG_n^*] + \int_{[x,\infty)} \frac{F_{1n}-F_1}{F_1 F_{1n}} dG_n^*. \end{aligned}$$

Since

$$\frac{1}{F_{1n}} = \frac{1}{F_1} + \frac{F_1 - F_{1n}}{F_1 F_{1n}},$$

we obtain

$$\begin{aligned} B(x) - B_n(x) &= \int_{[x,\infty)} \frac{F_{1n}-F_1}{F_1^2} dG_n^* - \int_{[x,\infty)} \frac{(F_{1n}-F_1)^2}{F_1^2 F_{1n}} dG_n^* \\ &\quad + \int_{[x,\infty)} \left(\frac{1}{F_1} - 1\right)[dG^* - dG_n^*] \end{aligned}$$

and therefore

$$A(x) + B(x) - A_n(x) - B_n(x) = \int_{[x,\infty)} 1[dG^* - dG_n^*] + B(x) - B_n(x)$$

$$= \int_{[x,\infty)} \frac{1}{F_1}[dG^* - dG_n^*] + \int_{[x,\infty)} \frac{F_{1n} - F_1}{F_1^2} dG_n^* - \int_{[x,\infty)} \frac{(F_{1n} - F_1)^2}{F_1^2 F_{1n}} dG_n^*. \tag{3.6}$$

In the following we discuss and derive some fundamental properties of the Lynden-Bell estimator F_{1n}.

A basic role in the analysis of F_{1n} will be played by the process

$$H_n^1(t) = \frac{1}{n} \sum_{i=1}^n 1_{\{t \le U_{1i} \le Z_i\}}.$$

This process is adapted to the decreasing filtration

$$\mathcal{G}_n(t) = \sigma(\{U_{1i} < s \le Z_i\}, \{s \le U_{1i} \le Z_i\} : t \le s, 1 \le i \le n)$$

and has left-continuous sample paths. In the following lemma we derive the Doob-Meyer decomposition of H_n^1 in reverse time. For this we have to assume that G and F_1 have no jumps in common. Otherwise, separate discontinuities are allowed.

Lemma 3.1. *The process H_n^1 has (in reverse time) the innovation martingale*

$$M_n(t) = H_n^1(t) - \int_{[t,\infty)} \frac{C_n(u^+)}{F_1(u)} F_1(du),$$

where as in section 1.2

$$C_n(u) = \frac{1}{n} \sum_{i=1}^n 1_{\{U_{1i} \le u \le Z_i\}}.$$

Remark 3.1. *The function $C_n(u)$ is an unbiased estimator of the function*

$$C(u) = \mathbb{P}(U_1 \le u \le Z | U_1 \le Z) = \alpha^{-1} F_1(u)(1 - G(u^-)), \tag{3.7}$$

which plays a key role in the analysis of F_1. Note that C_n is neither right nor left-continuous. Because of that, in the compensator of H_n^1, we will obtain $C_n(u^+)$ which is the right-continuous and hence predictable version of C_n.

Proof of Lemma 3.1. Because of independence it suffices to consider the case n=1.
Fix $t < \infty$ and consider a finite grid

$$t = t_{m+1} < t_m < t_{m-1} < ... < t_1 < \infty \equiv t_0.$$

We then have

$$1_{\{t_k \le U_1 \le Z\}} = 1_{\{t_{k-1} \le U_1 \le Z\}} + 1_{\{t_k \le U_1 < t_{k-1} \le Z\}} + 1_{\{t_k \le U_1 \le Z < t_{k-1}\}}.$$

The first indicator is measurable w.r.t. $\mathcal{G}_1(t_{k-1})$. As to the second, we have

$$E(1_{\{t_k\le U_1<t_{k-1}\le Z\}}|\mathcal{G}_1(t_{k-1})) = 1_{\{U_1<t_{k-1}\le Z\}}\frac{\mathbb{P}(t_k\le U_1<t_{k-1}\le Z)}{\mathbb{P}(U_1<t_{k-1}\le Z)}$$

The last equality follows from the fact that the process of interest is Markovian and at time t_{k-1} the σ-field is generated by the partition

$$\{U_1<t_{k-1}\le Z\},\{t_{k-1}\le U_1\le Z\},\{U_1\le Z<t_{k-1}\},\{U_1>Z\}.$$

For the third summand we obtain

$$E(1_{\{t_k\le U_1\le Z<t_{k-1}\}}|\mathcal{G}_1(t_{k-1})) = 1_{\{U_1\le Z<t_{k-1}\}}\frac{\mathbb{P}(t_k\le U_1\le Z<t_{k-1})}{\mathbb{P}(U_1\le Z<t_{k-1})}.$$

Altogether we obtain

$$\begin{aligned} E(1_{\{t_k\le U_1\le Z\}}|\mathcal{G}_1(t_{k-1})) &= 1_{\{t_{k-1}\le U_1\le Z\}} + 1_{\{U_1<t_{k-1}\le Z\}}\frac{\mathbb{P}(t_k\le U_1<t_{k-1}\le Z)}{\mathbb{P}(U_1<t_{k-1}\le Z)} \\ &\quad + 1_{\{U_1\le Z<t_{k-1}\}}\frac{\mathbb{P}(t_k\le U_1\le Z<t_{k-1})}{\mathbb{P}(U_1\le Z<t_{k-1})}. \end{aligned}$$

In the Doob-Meyer decomposition in discrete time the martingale part $M_1(t_k)$ satisfies the recursion

$$\begin{aligned} M_1(t_k) &= M_1(t_{k-1}) + 1_{\{t_k\le U_1\le Z\}} - E(1_{\{t_k\le U_1\le Z\}}|\mathcal{G}_1(t_{k-1})) = M_1(t_{k-1}) + 1_{\{t_k\le U_1\le Z\}} - 1_{\{t_{k-1}\le U_1\le Z\}} \\ &\quad - 1_{\{U_1<t_{k-1}\le Z\}}\frac{\mathbb{P}(t_k\le U_1<t_{k-1}\le Z)}{\mathbb{P}(U_1<t_{k-1}\le Z)} - 1_{\{U_1\le Z<t_{k-1}\}}\frac{\mathbb{P}(t_k\le U_1\le Z<t_{k-1})}{\mathbb{P}(U_1\le Z<t_{k-1})}. \end{aligned}$$

By induction we obtain

$$\begin{aligned} M_1(t_k) = \sum_{j=0}^{k-1}(M_1(t_{j+1})-M_1(t_j)) &= 1_{\{t_k\le U_1\le Z\}} - \sum_{j=0}^{k-1} 1_{\{U_1<t_j\le Z\}}\frac{\mathbb{P}(t_{j+1}\le U_1<t_j\le Z)}{\mathbb{P}(U_1<t_j\le Z)} \\ &\quad - \sum_{j=0}^{k-1} 1_{\{U_1\le Z<t_j\}}\frac{\mathbb{P}(t_{j+1}\le U_1\le Z<t_j)}{\mathbb{P}(U_1\le Z<t_j)}. \end{aligned}$$

Setting $k=m+1$ we get $t_k=t$. Since

$$1_{\{U_1<t_j\le Z\}}\frac{\mathbb{P}(t_{j+1}\le U_1<t_j\le Z)}{\mathbb{P}(U_1<t_j\le Z)} = 1_{\{U_1<t_j\le Z\}}\frac{F_1(t_j^-)-F_1(t_{j+1}^-)}{F_1(t_j^-)}$$

the first sum converges, as the partitions get finer and finer, to

$$\int_{[t,\infty)} 1_{\{U_1\le u<Z\}}\frac{F_1(du)}{F_1(u)}.$$

The second sum converges to zero, at least when G and F_1 have no jumps in common. Conclude that in continuous time and for $n = 1$

$$M_1(t) = 1_{\{t \le U_1 \le Z\}} - \int_{[t,\infty)} 1_{\{U_1 \le u < Z\}} \frac{F_1(du)}{F_1(u)},$$

as desired.

$\boxtimes$

Next, recall

$$\Lambda(x) = \int_{[x,\infty)} \frac{dF_1^*}{C},$$

$$\Lambda_n(x) = \int_{[x,\infty)} \frac{dF_{1n}^*}{C_n} = \sum_{i=1}^{n} \frac{1_{\{U_{1i} \ge x\}}}{nC_n(U_{1i})}$$

and, see (1.7),

$$F_{1n}(t) = \prod_{y>t} [1 + \Lambda_n\{y\}] = \prod_{U_{1i}>t} \left[1 - \frac{1}{nC_n(U_{1i})}\right].$$

It is easy to see that F_{1n} satisfies the integral equation

$$-\int_{(t,\infty)} F_{1n}(y)\Lambda_n(dy) = 1 - F_{1n}(t).$$

Our next goal will be to find proper upper and lower bounds for F_{1n}/F_1.
For this define $\hat{\Lambda}_0$ through

$$\hat{\Lambda}_0(dt) = 1_{\{C_n(t^+)>0\}}\Lambda(dt),$$

and let

$$\hat{F}_0(t) := \prod_{s>t} \left[1 + \hat{\Lambda}_0\{s\}\right] e^{-\hat{\Lambda}_0^c(t)}, \tag{3.8}$$

the pertaining d.f. We are going to show that the process

$$1 - \frac{F_{1n}(t^-)}{\hat{F}_0(t^-)}$$

is a martingale in reverse time. For this the following lemma will be helpful.

Lemma 3.2 (Gill). *Let A and B be two nonincreasing, left-continuous functions satisfying*

$$A\{x\} \ge -1 \ \textit{ and } \ B\{x\} > -1 \ \textit{ for all } x \in \mathbb{R}.$$

The function

$$Z(t) = 1 - \frac{\prod_{s \geq t}(1 + A\{s\})exp(-A^c(t))}{\prod_{s \geq t}(1 + B\{s\})exp(-B^c(t))}$$

satisfies the integral equation

$$\int_{[t,\infty)} \frac{1 - Z(s^+)}{1 + B\{s\}}(B(ds) - A(ds)) = Z(t).$$

In our application $A(t)$ and $B(t)$ are the left-continuous cumulative hazard functions of F_{1n} and $\hat{F}_0$, respectively.

Lemma 3.3. *We have that the process*

$$\frac{F_{1n}(t^-)}{\hat{F}_0(t^-)}$$

is a martingale in reverse time w.r.t. $\mathcal{G}_n(t)$.

Proof.
Since

$$1 - \frac{F_{1n}(t^-)}{\hat{F}_0(t^-)} = 1 - \frac{\prod_{s \geq t}(1 + \Lambda_n\{s\})}{\prod_{s \geq t}(1 + \hat{\Lambda}_0\{s\})e^{-\hat{\Lambda}_0^c(t)}},$$

according to Lemma 3.2, the process

$$Z(t) := 1 - \frac{F_{1n}(t^-)}{\hat{F}_0(t^-)}$$

satisfies the integral equation

$$Z(t) = -\int_{[t,\infty)} \frac{1 - Z(s^+)}{1 + \hat{\Lambda}_0\{s\}} \left[\Lambda_n(ds) - \hat{\Lambda}_0(ds)\right]. \tag{3.9}$$

Next we will write $Z(t)$ as a stochastic integral w.r.t. the martingale M_n. From Lemma 3.1 we obtain that the differential of M_n satisfies the equation

$$dM_n = dH_n^1 + \frac{C_n^+}{F_1}dF_1.$$

On the set $\{s : C_n(s^+) > 0\}$ we have

$$\frac{dM_n}{C_n^+} = \frac{dH_n^1}{C_n^+} + \frac{dF_1}{F_1} = d\Lambda_n - d\Lambda,$$

since the function H_n^1 has jumps of size $-1/n$ at the U_{1i} and, because F_1 and G have no jumps in common, the function C_n satisfies $C_n(U_{1i}^+) = C_n(U_{1i})$. Since on the support of H_n^1 the function $C_n(s^+)$ is positive, we obtain

$$\frac{1_{\{C_n^+>0\}}}{C_n^+}dM_n = d\Lambda_n - 1_{\{C_n^+>0\}}d\Lambda = d\Lambda_n - d\hat{\Lambda}_0.$$

Therefore, by (3.9), we get

$$\frac{F_{1n}(t^-)}{\hat{F}_0(t^-)} = 1 + \int_{[t,\infty)} \frac{F_{1n}(s)}{(1+\hat{\Lambda}_0\{s\})\hat{F}_0(s)} \frac{1_{\{C_n(s^+)>0\}}}{C_n(s^+)} dM_n(s).$$

Furthermore, M_n is a martingale in reverse time and the function under the integral is predictable. Hence the process $F_{1n}(t^-)/\hat{F}_0(t^-)$ is a reverse martingale.

⊠

To proceed with the properties of the above process, we need to work with a stopped martingale. For this, in the next Lemmas, we will study a special time T and prove that it is a stopping time.

Lemma 3.4. *Set*

$$T = \sup\{t < \max Z_i : C_n(t^+) = 0\}.$$

Then we have

$$T = \max\left\{U_{1i} : C_n(U_{1i}^+) = \frac{1}{n}\right\}$$

Proof.
Let $t < \max Z_i$ be chosen so that $C_n(t^+) = 0$. Then $1_{\{U_{1i} \le t < Z_i\}} = 0$ for $i = 1, ..., n$. Since $t < \max Z_i$, there exists U_{1j} such that $U_{1j} > t$. For the smallest among such U_{1j}'s we have $C_n(U_{1j}^+) = \frac{1}{n}$ and therefore $T \le \max\{U_{1i} : C_n(U_{1i}^+) = \frac{1}{n}\}$.
On the other hand, let U_{1i} be the maximum of the U's such that $C_n(U_{1i}^+) = \frac{1}{n}$. Then $C_n(t^+) = 0$ for every $t \in [a, U_{1i})$, where a is the largest of Z_j's, which are smaller than U_{1i}, if such exist or $-\infty$ if not. The supremum of such t's equals U_{1i}. This means $T = U_{1i}$ and the proof is complete.

⊠

Lemma 3.5. *For all t we have*

$$\{T \ge t\} = \bigcup_{i=1}^{n} \left\{U_{1i} \ge t, 1_{\{U_{1j} < U_{1i} \le Z_j\}} = 0, j = 1, ..., n\right\} =: \bigcup_{i=1}^{n} A_i(t)$$

Proof.
According to Lemma 3.4 we have $T = \max\left\{U_{1i} : C_n(U_{1i}^+) = \frac{1}{n}\right\}$. Since by assumption the U's and the Z's have no jumps in common, we get

$$C_n(U_{1i}^+) = \frac{1}{n}\sum_{j=1}^{n} 1_{\{U_{1j} \le U_{1i} < Z_j\}} = \frac{1}{n} + \frac{1}{n}\sum_{j \ne i} 1_{\{U_{1j} \le U_{1i} < Z_j\}} = \frac{1}{n} + \frac{1}{n}\sum_{j=1}^{n} 1_{\{U_{1j} < U_{1i} \le Z_j\}}.$$

Hence

$$T = \max\left\{U_{1i} : 1_{\{U_{1j} < U_{1i} \le Z_j\}} = 0 \text{ for every } j = 1, ..., n\right\}.$$

To prove the lemma we take $\omega \in \{T \ge t\}$. Then there exists at least one $U_{1i} \ge t$ (e.g. $U_{1i} = T$) so that $1_{\{U_{1j} < U_{1i} \le Z_j\}} = 0$ for every $j = 1, ..., n$. Hence $\omega \in A_i(t)$ and therefore $\omega \in \bigcup_{i=1}^n A_i(t)$. To complete the proof we take $\omega \in \bigcup_{i=1}^n A_i(t)$. This means there exists i so that $\omega \in A_i(t)$. Hence there exists $U_{1i} \ge t$ such that $1_{\{U_{1j} < U_{1i} \le Z_j\}} = 0$ for every $j = 1, ..., n$. Since T is the maximum of such U_{1i}'s, $T(\omega) \ge t$.

⊠

Lemma 3.6. *The r.v. T is a stopping time w.r.t.*

$$\mathcal{G}_n(t) = \sigma(\{U_{1j} < s \le Z_j\}, \{s \le U_{1j} \le Z_j\} : t \le s, 1 \le j \le n)$$

Proof.
Assume w.l.o.g. that all r.v.'s are non-negative and set

$$U_{1i}^{(m)} = \sum_{k \ge 0} \frac{k}{2^m} 1_{\{\frac{k}{2^m} \le U_{1i} < \frac{k+1}{2^m}\}}$$

and

$$T^{(m)} = \max\left\{U_{1i}^{(m)} : 1_{\{U_{1j} < U_{1i}^{(m)} \le Z_j\}} = 0 \text{ for every } j = 1, ..., n\right\}.$$

As before,

$$\{T^{(m)} \ge t\} = \bigcup_{i=1}^{n} \left\{U_{1i}^{(m)} \ge t, 1_{\{U_{1j} < U_{1i}^{(m)} \le Z_j\}} = 0, j = 1, ..., n\right\} =: \bigcup_{i=1}^{n} A_i^{(m)}(t).$$

For $t \in (\frac{k}{2^m}, \frac{k+1}{2^m}]$ we have

$$\{U_{1i}^{(m)} \ge t\} = \{U_{1i} \ge \frac{k+1}{2^m}\} \in \mathcal{G}_n(\frac{k+1}{2^m})$$

and

$$\{U_{1i}^{(m)} \ge t\} = \{U_{1i}^{(m)} \ge \frac{k+1}{2^m}\} = \bigcup_{l \ge k+1} \{U_{1i}^{(m)} = \frac{l}{2^m}\}$$

Furthermore, for $t \in (\frac{k}{2^m}, \frac{k+1}{2^m}]$, we have

$$\{U_{1j} < U_{1i}^{(m)} \le Z_j\} \cap \{U_{1i}^{(m)} \ge t\} = \bigcup_{l \ge k+1} \left(\{U_{1j} < \frac{l}{2^m} \le Z_j\} \cap \{U_{1i}^{(m)} = \frac{l}{2^m}\}\right).$$

Since, according to the definition of $\mathcal{G}_n(t)$,

$$\{U_{1j} < \frac{l}{2^m} \le Z_j\} \in \mathcal{G}_n(\frac{l}{2^m})$$

we get

$$\{U_{1j} < U_{1i}^{(m)} \le Z_j\} \cap \{U_{1i}^{(m)} \ge t\} \in \mathcal{G}_n(\frac{k+1}{2^m}).$$

Altogether, for every $i = 1, ..., n$, $k \ge 0$ and $t \in (\frac{k}{2^m}, \frac{k+1}{2^m}]$, we have

$$A_i^{(m)}(t) = \left\{U_{1i}^{(m)} \ge t, 1_{\{U_{1j} < U_{1i}^{(m)} \le Z_j\}} = 0, j = 1, ..., n\right\} \in \mathcal{G}_n(\frac{k+1}{2^m}).$$

Hence, since $\mathcal{G}_n(t)$ is decreasing, we have

$$\{T^{(m)} \ge t\} = \bigcup_{i=1}^{n} A_i^{(m)}(t) \in \mathcal{G}_n(\frac{k+1}{2^m}) \subseteq \mathcal{G}_n(t).$$

Conclude that $T^{(m)}$ is a stopping time. Furthermore, we have

$$\left\{U_{1i}^{(m)} : 1_{\{U_{1j}<U_{1i}^{(m)}\leq Z_j\}} = 0 \text{ for every } j=1,...,n\right\} = \left\{U_{1i}^{(m)} : U_{1j} \geq U_{1i}^{(m)} \text{ or } U_{1i}^{(m)} > Z_j \text{ for every } j=1,...,n\right\}$$

Since $U_{1i}^{(m)} \leq U_{1i} < U_{1i}^{(m)} + \frac{1}{2^m}$, $U_{1i}^{(m)} > Z_j$ yields that $U_{1i} > Z_j$. Furthermore, since we assumed $U_{1j} \neq U_{1i}$ for $i \neq j$, there exists $M = M(n,\omega)$ such that for every $m \geq M$, $|U_{1j} - U_{1i}| > \frac{1}{2^m}$. Hence from $U_{1j} \geq U_{1i}^{(m)}$ we obtain $U_{1j} \geq U_{1i}$ for every $j \neq i$. Since $U_{1i}^{(m)} \leq U_{1i}$, we get

$$\begin{aligned} T^{(m)}(\omega) &= \max_{1\leq i\leq n}\left\{U_{1i}^{(m)} : U_{1j} \geq U_{1i}^{(m)} \text{ or } U_{1i}^{(m)} > Z_j \text{ for every } j=1,...,n\right\} \\ &\leq \max_{1\leq i\leq n}\left\{U_{1i}^{(m)} : U_{1j} \geq U_{1i} \text{ or } U_{1i} > Z_j \text{ for every } j=1,...,n\right\} \\ &\leq \max_{1\leq i\leq n}\left\{U_{1i} : U_{1j} \geq U_{1i} \text{ or } U_{1i} > Z_j \text{ for every } j=1,...,n\right\} = T(\omega), \quad \text{for } m \geq M. \end{aligned}$$

On the other hand, since $U_{1i}^{(m)} \leq U_{1i}$, from $U_{1j} \geq U_{1i}$ we obtain $U_{1j} \geq U_{1i}^{(m)}$. Furthermore, since we assumed $Z_j \neq U_{1i}$ for $i \neq j$, there exists $M_1 = M_1(n,\omega)$ such that for every $m \geq M_1$, $|Z_j - U_{1i}| > \frac{1}{2^m}$. Hence from $U_{1i} > Z_j$ it follows that $U_{1i}^{(m)} > Z_j$. So,

$$\begin{aligned} &\max_{1\leq i\leq n}\left\{U_{1i}^{(m)} : U_{1j} \geq U_{1i} \text{ or } U_{1i} > Z_j \text{ for every } j=1,...,n\right\} \\ &\leq \max_{1\leq i\leq n}\left\{U_{1i}^{(m)} : U_{1j} \geq U_{1i}^{(m)} \text{ or } U_{1i}^{(m)} > Z_j \text{ for every } j=1,...,n\right\} = T^{(m)}(\omega), \quad \text{for } m \geq M_1. \end{aligned}$$

Since $U_{1i} < U_{1i}^{(m)} + \frac{1}{2^m}$,

$$\begin{aligned} T(\omega) - \frac{1}{2^m} &= \max_{1\leq i\leq n}\left\{U_{1i} : U_{1j} \geq U_{1i} \text{ or } U_{1i} > Z_j \text{ for every } j=1,...,n\right\} \\ &\leq \max_{1\leq i\leq n}\left\{U_{1i}^{(m)} : U_{1j} \geq U_{1i}^{(m)} \text{ or } U_{1i}^{(m)} > Z_j \text{ for every } j=1,...,n\right\} = T^{(m)}(\omega), \quad \text{for } m \geq M_1. \end{aligned}$$

For each ε set $M_2 = log_2(1/\varepsilon)$, so that for every $m \geq M_2$ we have $\frac{1}{2^m} \leq \varepsilon$.
Finally $\forall\omega \in \Omega$ and $\forall\varepsilon > 0$ $\exists\tilde{M}(= max(M, M_1, M_2))$ so that $\forall m \geq \tilde{M}$ we obtain $|T^{(m)} - T| \leq \varepsilon$.
Hence $T^{(m)} \to T$ and T is, as a limit of stopping times, a stopping time as well.

$\boxtimes$

Now, set

$$a_{\hat{F}_0} = \inf\{t : \hat{F}_0(t) > 0\}$$

and recall

$$T = \sup\{t < \max_{1\leq i\leq n} Z_i : C_n(t^+) = 0\}.$$

Since

$$\hat{\Lambda}_0(ds) = 1_{\{C_n(s^+)>0\}}\Lambda(ds) = -1_{\{C_n(s^+)>0\}}\frac{F_1(ds)}{F_1(s)} \geq -\frac{F_1(ds)}{F_1(s)} = \Lambda(ds)$$

we have

$$\hat{F}_0(t) = e^{\int_{[t,\infty)} \hat{\Lambda}_0(ds)} \geq e^{\int_{[t,\infty)} \Lambda(ds)} = F_1(t).$$

Hence $a_{\hat{F}_0} \leq a_{F_1}$ and $b_{\hat{F}_0} \leq b_{F_1}$. Together with $a_{F_1} \leq \min(U_{1i}) \leq T$ we obtain $\hat{F}_0(t \vee T^-) > 0$ for every $t > a_{F_1}$.
Furthermore, according to Lemmas 3.4 and 3.6, T is a stopping time w.r.t. $\mathcal{G}_n(t)$. Set

$$\tilde{Z}(t) := 1 - Z(t \vee T) = \frac{F_{1n}(t \vee T^-)}{\hat{F}_0(t \vee T^-)}.$$

Then the process $\tilde{Z}(t)$ is, according to Lemma 3.3, a martingale in reverse time for $t > a_{F_1}$.
Therefore an application of Doob's maximal inequality yields for every $a_{F_1} < s$ and any $c > 0$:

$$\mathbb{P}\left(\sup_{s \leq t} \tilde{Z}(t) > c\right) \leq \frac{E\tilde{Z}(s)}{c} = \frac{E\tilde{Z}(b_{F_1})}{c} \leq \frac{1}{c}.$$

Letting $s \downarrow a_{F_1}$, we get

$$\mathbb{P}\left(\sup_{a_{F_1} < t < \infty} \frac{F_{1n}(t \vee T^-)}{\hat{F}_0(t \vee T^-)} > c\right) \leq \frac{1}{c}. \tag{3.10}$$

In the next lemma we show that we may replace $\hat{F}_0$ by the original F_1.

Lemma 3.7. *As $n \to \infty$, we have*

$$\sup_{a_{F_1} < t} \frac{F_{1n}(t^-)}{F_1(t^-)} = O_{\mathbb{P}}(1).$$

Proof. We shall bound the ratio separately in three different regions:

- For $a_{F_1} < t < \min_{1 \leq i \leq n} U_{1i}$, we have $C_n(t^+) = 0 = F_{1n}(t^-)$ so that the ratio vanishes there.
- For $\min_{1 \leq i \leq n} U_{1i} \leq t < \max_{1 \leq i \leq n} Z_i = T_1$, we have that, if $C_n(t^+) = 0$ for at least one $t \in [\min_{1 \leq i \leq n} U_{1i}, \max_{1 \leq i \leq n} Z_i)$, there exists at least one pair (U_{1i}, Z_i) such that $t < U_{1i} \leq Z_i$. Since by continuity we may assume that $U_{1i} \neq U_{1j}$ for $i \neq j$, then for the smallest among U_{1i}'s with $t < U_{1i} \leq Z_i$, we have $nC_n(U_{1i}) = 1$ and therefore

$$F_{1n}(t^-) = \prod_{U_{1i} \geq t} \left[1 - \frac{1}{nC_n(U_{1i})}\right] = 0.$$

  By monotonicity, $F_{1n}(s^-) = 0$ for all $s \leq t$. Hence, recalling

$$T = \sup\{t < \max_{1 \leq i \leq n} Z_i : C_n(t^+) = 0\},$$

we have $F_{1n}(s^-) = 0$ for all $s \le T$. Hence the ratio vanishes there.

Since

$$\hat{\Lambda}_0(dt) = \begin{cases} \Lambda(dt), & \text{for } t \in [T, T_1) \\ 0, & \text{for } t \ge T_1 \end{cases},$$

we have for $s \in [T, T_1)$

$$\hat{F}_0(s) = e^{-\hat{\Lambda}_0(s)} = e^{\int_{[s,\infty)} \hat{\Lambda}_0(dt)} = e^{\int_{[s,T_1)} \Lambda(dt)} = e^{\Lambda(T_1)} e^{-\Lambda(s)} = e^{\Lambda(T_1)} F_1(s)$$

and therefore

$$\frac{F_{1n}(s^-)}{F_1(s^-)} = e^{\Lambda(T_1)} \frac{F_{1n}(s^-)}{\hat{F}_0(s^-)} = e^{\Lambda(T_1)} \frac{F_{1n}(s \vee T^-)}{\hat{F}_0(s \vee T^-)} = \frac{1}{F_1((\max\limits_{1 \le i \le n} Z_i)^-)} \tilde{Z}(s).$$

Since $\tilde{Z}(s)$ is, according to (3.10), bounded in probability on $s > a_{F_1}$ and $T \ge a_{F_1}$, we get that $\tilde{Z}(s)$ is bounded in probability on $s > T$. Furthermore, $\max\limits_{1 \le i \le n} Z_i \uparrow b_{G^*}$ and $F_1(b_{G^*}^-) > 0$. Conclude that $F_1^{-1}((\max\limits_{1 \le i \le n} Z_i)^-)$ is bounded in probability. In conclusion the ratio $F_{1n}(s^-)/F_1(s^-)$ it bounded in probability on $s \in (T, T_1]$.

- For $\max\limits_{1 \le i \le n} Z_i \le t$ we have $C_n(t^+) = 0$ and $F_{1n}(t^-) = 1$. Hence

$$\frac{F_{1n}(t^-)}{F_1(t^-)} \le \frac{1}{F_1((\max\limits_{1 \le i \le n} Z_i)^-)}$$

As before $\max\limits_{1 \le i \le n} Z_i \uparrow b_{G^*}$ and $F_1(b_{G^*}^-) > 0$. This completes the proof of the lemma.

$\boxtimes$

To prove the main result of our work (Theorem 1.1) we also need to study the ratio F_1/F_{1n}. For this it will turn out that the process

$$\hat{\alpha}(t) := \frac{F_{1n}(t)(1 - G_n(t^-))}{C_n(t)} \tag{3.11}$$

plays a crucial role. Due to

$$\alpha(t) = \frac{F_1(t)(1 - G(t^-))}{C(t)} \equiv \alpha,$$

it is likely that also $\hat{\alpha}(t)$ is the same for all t. This problem has been studied in detail in He and Yang (1998). Below we give a quick and straightforward proof of their main result.

Lemma 3.8. *On the set*

$$T \le t \le \max_{1\le i\le n} Z_i,$$

we have that

$$\hat\alpha(t) \equiv \hat\alpha \;\; \textit{is a strictly positive constant.}$$

Proof. Since F_{1n}, G_n and C_n are constant between two successive data, F_{1n}, C_n are right-continuous at U_{1j} and G_n, C_n are left-continuous at Z_j, it remains to show that

$$\frac{\hat\alpha(U_{1j})}{\hat\alpha(U_{1j}^-)} = 1 = \frac{\hat\alpha(Z_j^+)}{\hat\alpha(Z_j)} \;\text{ for all }\; 1 \le j \le n.$$

As to the first equation, we have

$$\frac{\hat\alpha(U_{1j})}{\hat\alpha(U_{1j}^-)} = \frac{\prod\limits_{U_{1i}>U_{1j}}\left[1-\dfrac{1}{nC_n(U_{1i})}\right]}{C_n(U_{1j})}\frac{C_n(U_{1j}^-)}{\prod\limits_{U_{1i}\ge U_{1j}}\left[1-\dfrac{1}{nC_n(U_{1i})}\right]} = \frac{C_n(U_{1j})-\frac{1}{n}}{C_n(U_{1j})}\frac{1}{1-\frac{1}{nC_n(U_{1j})}} = 1.$$

Similarly, we obtain

$$\frac{\hat\alpha(Z_j^+)}{\hat\alpha(Z_j)} = \frac{\prod\limits_{Z_i\le Z_j}\left[1-\dfrac{1}{nC_n(Z_i)}\right]}{C_n(Z_j^+)}\frac{C_n(Z_j)}{\prod\limits_{Z_i<Z_j}\left[1-\dfrac{1}{nC_n(Z_i)}\right]} = (1-\frac{1}{nC_n(Z_j)})\frac{C_n(Z_j)}{C_n(Z_j)-\frac{1}{n}} = 1.$$

Finally $F_{1n}(t)$ and G_n are positive for $t\in[T,\max(Z_i)]$. This completes the proof of the lemma.

$\boxtimes$

Lemma 3.9. *Set*

$$\Omega_0^{(n)} := \{\omega\in\Omega : C_n(U_{1i}) = \frac{1}{n} \;\textit{ for at least one }\; U_{1i} > \min_{1\le j\le n} U_{1j}\}. \tag{3.12}$$

Then

$$\mathbb{P}(\Omega_0^{(n)}) \to 0, \;\; \textit{as} \;\; n\to\infty. \tag{3.13}$$

Proof.
Let $U_{11:n} < U_{12:n} < \ldots < U_{1n:n}$ be the order statistics of the U_{1i}'s and let $Z_{[1:n]}, Z_{[2:n]}, \ldots, Z_{[n:n]}$ be the concomitants paired with the $U_{1i:n}$'s. Since

$$nC_n(U_{1i:n}) = \sum_{k=1}^{n} 1_{\{U_{1k:n}\le U_{1i:n}\le Z_{[k:n]}\}} = \sum_{k=1}^{i-1} 1_{\{U_{1i:n}\le Z_{[k:n]}\}} + 1$$

we have that

$$\{nC_n(U_{1i:n}) = 1\} = \bigcap_{k=1}^{i-1} \{Z_{[k:n]} < U_{1i:n}\}. \tag{3.14}$$

Furthermore,

$$\begin{aligned} H_2(x, y^-) &:= \mathbb{P}(U_1 \le x, Z < y | U_1 \le Z) = \alpha^{-1} \int_0^x (G(y^-) - G(z^-)) 1_{\{y \ge z\}} F_1(dz) \\ &= \int_0^x \frac{G(y^-) - G(z^-)}{1 - G(z^-)} 1_{\{y \ge z\}} \alpha^{-1} (1 - G(z^-)) F_1(dz) = \int_0^x \frac{G(y^-) - G(z^-)}{1 - G(z^-)} 1_{\{y \ge z\}} F_1^*(dz). \end{aligned}$$

Then, according to Lemma 2.1 in Stute and Wang (1993),

$$\mathbb{P}(Z_{[k:n]} < y | U_{1k:n} = z) = \mathbb{P}(Z_1 < y | U_{11} = z) = K(z, y),$$

where

$$K(z, y) = \frac{G(y^-) - G(z^-)}{1 - G(z^-)} 1_{\{y \ge z\}}.$$

Furthermore, the $Z_{[k:n]}$ are conditionally independent, given $U_{11:n}, ..., U_{1n:n}$.
Hence, by (3.14),

$$\begin{aligned} \mathbb{P}(nC_n(U_{1i:n}) = 1 | U_{11:n}, ..., U_{1n:n}) &= \prod_{k=1}^{i-1} \mathbb{P}(Z_{[k:n]} < U_{1i:n} | U_{11:n}, ..., U_{1n:n}) = \prod_{k=1}^{i-1} \frac{G(U_{1i:n}^-) - G(U_{1k:n}^-)}{1 - G(U_{1k:n}^-)} \\ &= \prod_{k=1}^{i-1} \left[1 - \frac{1 - G(U_{1i:n}^-)}{1 - G(U_{1k:n}^-)} \right]. \end{aligned}$$

Conclude that

$$\mathbb{P}(nC_n(U_{1i:n}) = 1) = E\left(\prod_{k=1}^{i-1} \left[1 - \frac{1 - G(U_{1i:n}^-)}{1 - G(U_{1k:n}^-)} \right] \right). \tag{3.15}$$

Moreover, note that $U_{11:n}, ..., U_{1i-1:n}$ given $U_{1i:n} = z$ are, in distribution, equal to $U^*_{11:i-1}, ..., U^*_{1i-1:i-1}$, where

$$\mathbb{P}(U_1^* \le x | U_1 \le Z) = \frac{F_1^*(x)}{F_1^*(z)} 1_{\{x \le z\}}.$$

Therefore

$$\begin{aligned} & E\left(\prod_{k=1}^{i-1} \left[1 - \frac{1 - G(U_{1i:n}^-)}{1 - G(U_{1k:n}^-)} \right] | U_{1i:n} = z \right) = E\left(\prod_{k=1}^{i-1} \left[1 - \frac{1 - G(z^-)}{1 - G(U_{1k:i-1}^{*-})} \right] \right) \\ &= E\left(\prod_{k=1}^{i-1} \left[1 - \frac{1 - G(z^-)}{1 - G(U_{1k}^{*-})} \right] \right) = \prod_{k=1}^{i-1} E\left(1 - \frac{1 - G(z^-)}{1 - G(U_{1k}^{*-})} \right) \\ &= \prod_{k=1}^{i-1} \left(1 - \frac{1 - G(z^-)}{F_1^*(z)} \int_0^z \frac{F_1^*(dx)}{1 - G(x^-)} \right) = \left(1 - \frac{\alpha^{-1} F_1(z)(1 - G(z^-))}{F_1^*(z)} \right)^{i-1} =: (1 - q(z))^{i-1}, \end{aligned}$$

where

$$q(z) = \frac{F_1(z)(1-G(z^-))}{\int_0^z (1-G(x^-))F_1(dx)}.$$

By (3.15),

$$\mathbb{P}(nC_n(U_{1i:n}) = 1) = E(1-q(U_{1i:n}))^{i-1}. \tag{3.16}$$

Since $\mathbb{P}(nC_n(U_{11:n}) = 1) = 1$, we get

$$\begin{aligned}
\mathbb{P}(\bigcup_{i=2}^{n}\{nC_n(U_{1i:n}) = 1\}) &\le \sum_{i=2}^{n}\mathbb{P}(nC_n(U_{1i:n}) = 1) = \sum_{i=1}^{n}\mathbb{P}(nC_n(U_{1i:n}) = 1) - 1 \\
&= \sum_{i=1}^{n} E(1-q(U_{1i:n}))^{i-1} - 1 = \sum_{i=1}^{n} E(1-q(U_{1i}))^{R_i-1} - 1 \\
&= nE(1-q(U_{11}))^{R_1-1} - 1,
\end{aligned}$$

where $R_1 - 1 = \sum_{j=1}^{n} 1_{\{U_{1j}\le U_{11}\}} - 1 = \sum_{j=2}^{n} 1_{\{U_{1j}\le U_{11}\}}$.
Finally, since

$$\begin{aligned}
E\left((1-q(U_{11}))^{R_1-1}|U_{11} = z\right) &= E\left((1-q(U_{11}))^{\sum_{j=2}^{n} 1_{\{U_{1j}\le U_{11}\}}}|U_{11} = z\right) = E\left((1-q(z))^{\sum_{j=2}^{n} 1_{\{U_{1j}\le z\}}}\right) \\
&= E\left[(1-q(z))^{1_{\{U_{12}\le z\}}}\right]^{n-1} = \left[(1-q(z))F_1^*(z) + 1 - F_1^*(z)\right]^{n-1} \\
&= \left[1-q(z)F_1^*(z)\right]^{n-1},
\end{aligned}$$

we obtain

$$\begin{aligned}
nE(1-q(U_{11}))^{R_1-1} - 1 &= nE\left(E\left((1-q(U_{11}))^{R_1-1}|U_{11}\right)\right) - 1 \\
&= nE\left[1-q(U_{11})F_1^*(U_{11})\right]^{n-1} - 1 \\
&= n\int \left[1-q(z)F_1^*(z)\right]^{n-1} F_1^*(dz) - 1.
\end{aligned}$$

To complete the proof we need to show that the right side goes to zero, as $n \to \infty$.
For this we split the integral into three pieces. Since $q(z) \to 1$ for $z \downarrow a_{F_1}$, we may find for a given $\varepsilon \in (0,1)$, a $z_0 > a_{F_1}$ such that $q(z) \ge 1-\varepsilon$ for all $z \le z_0$. Conclude that

$$\begin{aligned}
n\int_{-\infty}^{z_0} \left[1-q(z)F_1^*(z)\right]^{n-1} F_1^*(dz) - 1 &\le n\int_{-\infty}^{z_0} \left[1-(1-\varepsilon)F_1^*(z)\right]^{n-1} F_1^*(dz) - 1 \\
&\le n\int_0^1 [1-(1-\varepsilon)u]^{n-1} du - 1 = -\frac{\varepsilon^n - 1}{1-\varepsilon} - 1 \to \frac{\varepsilon}{1-\varepsilon},
\end{aligned}$$

which can be made arbitrarily small. The integral $n\int_{z_0}^{z_1} \left[1-q(z)F_1^*(z)\right]^{n-1} F_1^*(dz)$ converges to zero geometrically fast for each $z_1 < b_G$. Just note that on (z_0, z_1) the product $q(z)F_1^*(z)$ is bounded away from zero. Finally, on $[z_1, b_{F_1}]$ we have, since $n(1-u)^{n-1} \leq u^{-2}$ for each $0 < u \leq 1$ and every $n \geq 1$:

$$n \int_{z_1}^{b_{F_1}} [1 - q(z)F_1^*(z)]^{n-1} F_1^*(dz) \leq \int_{z_1}^{b_{F_1}} \frac{F_1^*(dz)}{[q(z)F_1^*(z)]^2} \leq \int_{z_1}^{b_{F_1}} \frac{F_1(dz)}{F_1^2(z)(1 - G(z^-))}.$$

Since F_1 is bounded away from zero on $[z_1, b_{F_1}]$ it follows from A1 that the last integral can be made arbitrarily small whenever z_1 is close enough to b_{F_1}. This completes the proof.

$\boxtimes$

We are now in a position to prove the following lemma.

Lemma 3.10. *We have, as* $n \to \infty$*,*

$$\sup_{a_n \leq t} \frac{F_1(t)}{F_{1n}(t)} = O_{\mathbb{P}}(1)$$

where $a_n = \min_{1 \leq i \leq n} U_{1i}$.

Proof.
First of all we have the following equality:

$$\mathbb{P}\Big(\omega \in \Omega : \sup_{a_n \leq t} \frac{F_1(t)}{F_{1n}(t)} \geq \lambda\Big) = \mathbb{P}\Big(\omega \in \Omega_0^{(n)} : \sup_{a_n \leq t} \frac{F_1(t)}{F_{1n}(t)} \geq \lambda\Big) + \mathbb{P}\Big(\omega \in \Omega \backslash \Omega_0^{(n)} : \sup_{a_n \leq t} \frac{F_1(t)}{F_{1n}(t)} \geq \lambda\Big).$$

According to Lemma 3.9

$$\mathbb{P}(\Omega_0^{(n)}) \to 0, \text{ as } n \to \infty.$$

Hence

$$\mathbb{P}\Big(\omega \in \Omega_0^{(n)} : \sup_{a_n \leq t} \frac{F_1(t)}{F_{1n}(t)} \geq \lambda\Big) \leq \mathbb{P}(\Omega_0^{(n)}) \to 0.$$

It remains to show that

$$\mathbb{P}\Big(\omega \in \Omega \setminus \Omega_0^{(n)} : \sup_{a_n \leq t} \frac{F_1(t)}{F_{1n}(t)} \geq \lambda\Big)$$

goes to zero, as $\lambda \to \infty$. Since on the set $\Omega \setminus \Omega_0^{(n)}$ there are "no holes", i.e., $a_n = T$, from Lemma 3.8 we have

$$\frac{F_1(t)}{F_{1n}(t)} = \frac{\alpha(t)}{\hat{\alpha}(t)} \frac{C(t)}{C_n(t)} \frac{1 - G_n(t^-)}{1 - G(t^-)} \tag{3.17}$$

for all $a_n \leq t \leq \max_{1 \leq i \leq n} Z_i$. Since $F_{1n}(t) = 1$ for all $\max_{1 \leq i \leq n} U_{1i} \leq t$, it suffices to study the ratio over $a_n \leq t \leq \max_{1 \leq i \leq n} U_{1i}$. Moreover, since F_{1n} is constant between two successive order statistics of the U's and F_1 is nondecreasing and continuous, we get

$$\sup_{a_n \le t \le \max_{1\le i\le n} U_{1i}} \frac{F_1(t)}{F_{1n}(t)} \le \max_{2\le i\le n} \frac{F_1(U_{1i:n})}{F_{1n}(U_{1i:n}^-)}$$

Set $X_i = -U_{1i}$ and $Y_i = -Z_i$ so that for the observed (X_i, Y_i) we have $X_i \ge Y_i$. From equation (3.1) in Stute and Wang (2007) we have

$$\sup_{1\le i\le n} \frac{C(X_i)}{C_n(X_i)} = O_{\mathbb{P}}(1), \text{ as } n \to \infty.$$

Since $C(X_i) = C(U_{1i})$ and $C_n(X_i) = C_n(U_{1i})$ we get

$$\sup_{1\le i\le n} \frac{C(U_{1i})}{C_n(U_{1i})} = O_{\mathbb{P}}(1), \text{ as } n \to \infty. \tag{3.18}$$

On the set $\Omega \setminus \Omega_0^{(n)}$, since $U_{1i:n} \ne U_{1j:n}$ for $i \ne j$, we also have $C_n(U_{1i:n}) \ge \frac{2}{n}$ for $i = 2, 3, ..., n$. Then, according to the definition of C_n, $C_n(U_{1i:n}) \ge \frac{1}{n}$ with jumps at U_{1i}'s being of size $\frac{1}{n}$. On the other hand, $C_n(U_{1i:n}) = \frac{1}{n}$ for $i = 2, 3, ..., n$ would mean that $U_{11:n} < T = \max\left\{U_{1i} : C_n(U_{1i}^+) = \frac{1}{n}\right\}$. This is a contradiction to $T = a_n = U_{11:n}$ on $\Omega \setminus \Omega_0^{(n)}$.
This yields

$$C_n(U_{1i:n}^-) = C_n(U_{1i:n}) - \frac{1}{n} \ge \frac{1}{2} C_n(U_{1i:n})$$

and therefore

$$\sup_{2\le i\le n} \frac{C(U_{1i:n})}{C_n(U_{1i:n}^-)} = O_{\mathbb{P}}(1), \text{ as } n \to \infty. \tag{3.19}$$

From Lemma 3.7 we obtain that the ratio

$$\frac{1 - G_n(t^-)}{1 - G(t^-)} \text{ is uniformly bounded in probability on } t \le \max_{1\le i\le n} U_{1i}. \tag{3.20}$$

Finally, for any $\lambda > 0$,

$$\mathbb{P}\left(\omega \in \Omega \setminus \Omega_0^{(n)} : \max_{2\le i\le n} \frac{F_1(U_{1i:n})}{F_{1n}(U_{1i:n}^-)} > \lambda\right) = \mathbb{P}\left(\omega \in \Omega \setminus \Omega_0^{(n)} : \max_{2\le i\le n} \frac{\alpha}{\hat\alpha} \frac{C(U_{1i:n})}{C_n(U_{1i:n}^-)} \frac{1 - G_n(U_{1i:n}^-)}{1 - G(U_{1i:n}^-)} > \lambda\right),$$

The last probability is, however, less than or equal to

$$\begin{aligned}
&\mathbb{P}\left(\omega \in \Omega \setminus \Omega_0^{(n)} : \max_{2\le i\le n} \frac{\alpha}{\hat\alpha} \frac{C(U_{1i:n})}{C_n(U_{1i:n}^-)} > \sqrt{\lambda}\right) + \mathbb{P}\left(\omega \in \Omega : \max_{2\le i\le n} \frac{1 - G_n(U_{1i:n}^-)}{1 - G(U_{1i:n}^-)} > \sqrt{\lambda}\right) \\
&\le \mathbb{P}\left(\omega \in \Omega \setminus \Omega_0^{(n)} : \frac{\alpha}{\hat\alpha} > \lambda^{1/4}\right) + \mathbb{P}\left(\omega \in \Omega \setminus \Omega_0^{(n)} : \max_{2\le i\le n} \frac{C(U_{1i:n})}{C_n(U_{1i:n}^-)} > \lambda^{1/4}\right) \\
&+ \mathbb{P}\left(\omega \in \Omega : \max_{2\le i\le n} \frac{1 - G_n(U_{1i:n}^-)}{1 - G(U_{1i:n}^-)} > \sqrt{\lambda}\right).
\end{aligned}$$

By He and Yang (1998) we have $\hat{\alpha} \to \alpha$ as $n \to \infty$ and therefore the first probability converges to zero for $\lambda > 1$, as $n \to \infty$. From (3.19) and (3.20) conclude that the second and the third probability can be made as small as possible by letting $\lambda \to \infty$. This completes the proof.

$\boxtimes$

Lemma 3.11. *We have*

$$\sup_{a_{F_1} < x \leq c_n} \frac{A(x) + B(x)}{A_n(x) + B_n(x)} = O_{\mathbb{P}}(1),$$

where

$$c_n = \max_{1 \leq i \leq n} (U_{1i} + \tilde{U}_{2i}).$$

Proof. We have

$$\frac{A(x) + B(x)}{A_n(x) + B_n(x)} = \frac{\int_{[x,\infty)} \frac{1}{F_1(y)} G^*(dy)}{\int_{[x,\infty)} \frac{1}{F_{1n}(y)} G_n^*(dy)} = \frac{\int_{[x,\infty)} \frac{1}{F_1(y)} G^*(dy)}{\int_{[x,\infty)} \frac{1}{F_1(y)} \frac{F_1(y)}{F_{1n}(y)} G_n^*(dy)}.$$

By Lemma 3.7, we have for an appropriate $\tilde{K} < \infty$:

$$\frac{F_{1n}(y)}{F_1(y)} \leq \tilde{K} \text{ for all } a_{F_1} < y$$

with large probability for all $n \geq 1$. Conclude that

$$\frac{A(x) + B(x)}{A_n(x) + B_n(x)} \leq \tilde{K} \frac{\int_{[x,\infty)} \frac{1}{F_1(y)} G^*(dy)}{\int_{[x,\infty)} \frac{1}{F_1(y)} G_n^*(dy)}.$$

Next, fix $a_{F_1} < x_\varepsilon < b_G$. For $x \leq x_\varepsilon$ we have

$$\frac{\int_{[x,\infty)} \frac{1}{F_1(y)} G^*(dy)}{\int_{[x,\infty)} \frac{1}{F_1(y)} G_n^*(dy)} \leq \frac{\int_{[a_{F_1},\infty)} \frac{1}{F_1(y)} G^*(dy)}{\int_{[x_\varepsilon,\infty)} \frac{1}{F_1(y)} G_n^*(dy)}.$$

The numerator equals $\alpha^{-1}(1 - G(a_{F_1})) < \infty$, while the denominator converges to

$$\int_{[x_\varepsilon,\infty)} \frac{1}{F_1(y)} G^*(dy) = 1 - G(x_\varepsilon) > 0$$

For $x > x_\varepsilon$, we obtain

$$\frac{\int_{[x,\infty)} \frac{1}{F_1(y)} G^*(dy)}{\int_{[x,\infty)} \frac{1}{F_1(y)} G_n^*(dy)} \leq \frac{F_1^{-1}(x_\varepsilon)(1 - G^*(x^-))}{1 - G_n^*(x^-)}.$$

This ratio, however, for $x \leq \max_{1\leq i\leq n}(U_{1i}+\tilde{U}_{2i})$, is again uniformly bounded in probability. See, for example, Shorack and Wellner (1986), Chapter 10.3.

$\boxtimes$

Our next goal will be to show that

$$J_{1n} + J_{3n} \quad = \quad \int \varphi \frac{(A+B-A_n-B_n)^2}{(A+B)^2(A_n+B_n)} dH^1_{2n}. \tag{3.21}$$

is asymptotically negligible. To bound the numerator the following expansion of $F_{1n} - F_1$, which corresponds to the expansion of the Lynden-Bell estimator for left-truncated data, will be helpful.

Lemma 3.12. *We have*

$$F_{1n}(x) - F_1(x) = \int \frac{\Psi_x(y)}{C(y)}(F^*_{1n}(dy) - F^*_1(dy)) - \int \frac{C_n(y)-C(y)}{C^2(y)}\Psi_x(y)F^*_1(dy) + R_n(x)$$

where

$$\Psi_x(y) = 1_{(-\infty,x]}(y)\gamma(y) - \int_{y>z} \frac{1_{(-\infty,x]}(z)\gamma(z)}{C(z)}F^*_1(dz)$$

and

$$\gamma(z) = \exp\left\{-\int_{(z,\infty)} \frac{F^*_1(ds)}{C(s)}\right\}.$$

Furthermore, $R_n(x)$ is a remainder of the order $o_{\mathbb{P}}(n^{-1/2})$. See Section A.3 of Appendix A for details.

Proof. According to Lemma A.22 in Appendix A, $F_{1n}(x) - F_1(x)$ is a sum of i.i.d. r.v.'s and a remainder. Furthermore the leading term is a sum of four coefficients L_{n1}, L_{n2}, L_{n3} and L_{n4}, defined in Section A.3. In particular,

$$\begin{aligned}
L_{n1} + L_{n3} &= \int_{(-\infty,x]} \frac{\gamma(y)}{C(y)}(F^*_{1n}(dy) - F^*_1(dy)) - \int_{(-\infty,x]} \frac{\gamma(z)}{C(z)} \int_{(z,\infty)} \frac{1}{C(y)}(F^*_{1n}(dy) - F^*_1(dy))F^*_1(dz) \\
&= \int \frac{1}{C(y)}\left(1_{(-\infty,x]}(y)\gamma(y) - \int_{y>z} \frac{1_{(-\infty,x]}(z)\gamma(z)}{C(z)}F^*_1(dz)\right)(F^*_{1n}(dy) - F^*_1(dy)) \\
&= \int \frac{\Psi_x(y)}{C(y)}(F^*_{1n}(dy) - F^*_1(dy))
\end{aligned}$$

and

$$L_{n2} + L_{n4} = -\int \frac{C_n(y)-C(y)}{C^2(y)}\Psi_x(y)F^*_1(dy).$$

The proof is complete.

$\boxtimes$

Remark 3.2. *Since by (1.6)* $\frac{dF_1^*}{C} = \frac{dF_1}{F_1}$, $\gamma(z) = F_1(z)$ *and* $\Psi_x(y) = -1_{\{y>x\}} F_1(x)$.

Lemma 3.13. *We have*

$$\sup_{1\le i\le n} \frac{C_n(U_{1i})}{C(U_{1i})} = O_{\mathbb{P}}(1).$$

Proof.
Similar to Lemma 3.10, we split the probability of interest in two parts:

$$\mathbb{P}\Big(\omega\in\Omega : \sup_{1\le i\le n} \frac{C_n(U_{1i})}{C(U_{1i})} > \lambda\Big) = \mathbb{P}\Big(\omega\in\Omega_0^{(n)} : \sup_{1\le i\le n} \frac{C_n(U_{1i})}{C(U_{1i})} > \lambda\Big) + \mathbb{P}\Big(\omega\in\Omega\backslash\Omega_0^{(n)} : \sup_{1\le i\le n} \frac{C_n(U_{1i})}{C(U_{1i})} > \lambda\Big).$$

According to Lemma 3.9, for the first probability we have

$$\mathbb{P}(\Omega_0^{(n)}) \to 0, \text{ as } n\to\infty.$$

Hence

$$\mathbb{P}\Big(\omega\in\Omega_0^{(n)} : \sup_{1\le i\le n} \frac{C_n(U_{1i})}{C(U_{1i})} > \lambda\Big) \le \mathbb{P}(\Omega_0^{(n)}) \to 0.$$

By (3.11), for the second probability we have

$$\begin{aligned}
&\mathbb{P}\Big(\omega\in\Omega\setminus\Omega_0^{(n)} : \sup_{1\le i\le n} \frac{C_n(U_{1i})}{C(U_{1i})} > \lambda\Big) = \mathbb{P}\Big(\omega\in\Omega\setminus\Omega_0^{(n)} : \sup_{1\le i\le n} \frac{\alpha}{\hat\alpha}\frac{F_{1n}(U_{1i})}{F_1(U_{1i})}\frac{1-G_n(U_{1i}^-)}{1-G(U_{1i}^-)} > \lambda\Big) \\
&\le \mathbb{P}\Big(\omega\in\Omega\setminus\Omega_0^{(n)} \frac{\alpha}{\hat\alpha} > \lambda^{1/4}\Big) + \mathbb{P}\Big(\omega\in\Omega \sup_{1\le i\le n} \frac{F_{1n}(U_{1i})}{F_1(U_{1i})} > \lambda^{1/4}\Big) + \mathbb{P}\Big(\omega\in\Omega \sup_{1\le i\le n} \frac{1-G_n(U_{1i}^-)}{1-G(U_{1i}^-)} > \sqrt{\lambda}\Big).
\end{aligned}$$

Since the right side goes to zero, the proof is complete.

$\boxtimes$

According to Lemma A.6, for given $\varepsilon > 0$, there exist $K \ge 1$ and sequences a_n and b_n, so that the event

$$\tilde\Omega_n^0 = \Big\{\sup_{1\le i\le n} \frac{C(U_{1i})}{C_n(U_{1i})} \le K, U_{11:n} > a_n, Z_{n:n} < b_n\Big\} \cap \Omega_1^{(n)},$$

where $\Omega_1^{(n)} := \Omega\setminus\Omega_0^{(n)}$, has probability larger than or equal to $1-4\varepsilon$. Furthermore, by Lemmas 3.11, 3.7 and 3.10, for given $\varepsilon > 0$, there exist $L\ge 1$, $\tilde K \ge 1$ and $\tilde K_1 \ge 1$ so that the event

$$\tilde\Omega_n^3 = \Big\{\sup_{a_{F_1} < x \le c_n} \frac{A(x)+B(x)}{A_n(x)+B_n(x)} \le L, \sup_{a_{F_1}<t} \frac{F_{1n}(t^-)}{F_1(t^-)} \le \tilde K, \sup_{a_n \le t} \frac{F_1(t)}{F_{1n}(t)} \le \tilde K_1\Big\}$$

has probability larger than or equal to $1-3\varepsilon$. Altogether the event $\tilde\Omega_n^7 := \tilde\Omega_n^0 \cap \Omega_1^{(n)} \cap \tilde\Omega_n^3$ has probability exceeding $1-7\varepsilon$.
Now we are in the position to prove that $J_{1n} + J_{3n}$ is a remainder. In the next lemma we consider part of this term, and show that it is of the order $o(n^{-1/2})$.

Lemma 3.14. *Under A1-A3, on the set* $\tilde{\Omega}_n^7$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \frac{\varphi(U_{1i}, \tilde{U}_{2i})\delta_i}{(A+B)^3(U_{1i}+\tilde{U}_{2i})} \left[\int_{U_{1i}+\tilde{U}_{2i}}^{\infty} \frac{F_{1n}-F_1}{F_1^2} dG_{n-1}^* \right]^2 \right\} \tag{3.22}$$

goes to zero in probability.

Proof. W.l.o.g. we may assume that $\varphi \geq 0$. On the set $\tilde{\Omega}_n^7$ we may restrict integration w.r.t. G_n^* to $(-\infty, b_n)$. Therefore

$$\begin{aligned} (3.22) \quad \leq \quad & n^{-5/2} \sum_{i=1}^{n} \sum_{j \neq i} \frac{\varphi(U_{1i}, \tilde{U}_{2i})\delta_i}{(A+B)^3(U_{1i}+\tilde{U}_{2i})} 1_{\{b_n \geq Z_j \geq U_{1i}+\tilde{U}_{2i}\}} \frac{(F_{1n}(Z_j)-F_1(Z_j))^2}{F_1^4(Z_j)} & (3.23) \\ + \quad & n^{-5/2} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i,j} \frac{\varphi(U_{1i}, \tilde{U}_{2i})\delta_i}{(A+B)^3(U_{1i}+\tilde{U}_{2i})} \frac{|F_{1n}(Z_j)-F_1(Z_j)|}{F_1^2(Z_j)} 1_{\{b_n \geq Z_j \geq U_{1i}+\tilde{U}_{2i}\}} \times \\ & \frac{|F_{1n}(Z_k)-F_1(Z_k)|}{F_1^2(Z_k)} 1_{\{b_n \geq Z_k \geq U_{1i}+\tilde{U}_{2i}\}} & (3.24) \end{aligned}$$

As to (3.23), by Lemma 3.7 there exists some constant $\tilde{K}$ so that on the set $\tilde{\Omega}_n^7$

$$\frac{(F_{1n}-F_1)^2(Z_j)}{F_1^4(Z_j)} \leq \frac{1}{F_1^2(Z_j)} \left(2\frac{F_{1n}^2(Z_j)}{F_1^2(Z_j)} + 2 \right) \leq \frac{2\tilde{K}^2+2}{F_1^2(Z_j)}.$$

Hence

$$(3.23) \leq n^{-5/2} \sum_{i=1}^{n} \sum_{j \neq i} \frac{\varphi(U_{1i}, \tilde{U}_{2i})\delta_i}{(A+B)^3(U_{1i}+\tilde{U}_{2i})} 1_{\{b_n \geq Z_j \geq U_{1i}+\tilde{U}_{2i}\}} \frac{2\tilde{K}^2+2}{F_1^2(Z_j)},$$

where the expectation of the right side is less than or equal to

$$n^{-1/2} 2(\tilde{K}^2+1) \int \frac{\varphi(x_1,x_2)}{(A+B)^3(x_1+x_2)} \int_{x_1+x_2}^{b_n} \frac{G^*(dx)}{F_1^2(x)} H_2^1(dx_1, dx_2). \tag{3.25}$$

Finally, since $dH_2^1 = (A+B)dF$, $dG^* = \alpha^{-1}F_1 dG$ and $1 - G^- = \alpha(A+B)$, we get

$$(3.25) \leq n^{-1/2} 2(\tilde{K}^2+1) \int \frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)} F(dx_1, dx_2) \to 0.$$

As to (3.24), according to Lemma A.12, on $\tilde{\Omega}_n^0 \subset \tilde{\Omega}_n^7$,

$$(F_{1n-1} - F_1)^2(Z_k) \leq k_1 E_{n-2}^2(Z_k) + M_n(U_{1i}, Z_i, Z_k),$$

where E_{n-2} is defined as E_n in Lemma A.6 but doesn't contain U's and Z's with index i and k. Furthermore, k_1 is a constant. Hence

$$(3.24) \quad \leq \quad n^{-5/2}\sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\neq i,j}\frac{\varphi(U_{1i},\tilde{U}_{2i})\delta_i}{(A+B)^3(U_{1i}+\tilde{U}_{2i})}\frac{1_{\{b_n\geq Z_j, Z_k\geq U_{1i}+\tilde{U}_{2i}\}}}{F_1^2(Z_j)F_1^2(Z_k)}\sqrt{|k_1E_{n-2}^2(Z_k)+M_n(U_{1i},Z_i,Z_k)|}$$
$$\times\sqrt{|k_1E_{n-2}^2(Z_j)+M_n(U_{1i},Z_i,Z_j)|}. \tag{3.26}$$

Next we compute the expectation of the right side. Since the U's and Z's are i.i.d., we get

$$\begin{aligned} E(3.26) \quad \leq \quad & n^{1/2}E\Big(\frac{\varphi(U_{11},\tilde{U}_{21})\delta_1}{(A+B)^3(U_{11}+\tilde{U}_{21})}\frac{1_{\{b_n\geq Z_2, Z_3\geq U_{11}+\tilde{U}_{21}\}}}{F_1^2(Z_2)F_1^2(Z_3)}\sqrt{|k_1E_{n-2}^2(Z_2)+M_n(U_{11},Z_1,Z_2)|} \\ & \times\sqrt{|k_1E_{n-2}^2(Z_3)+M_n(U_{11},Z_1,Z_3)|}\Big) = E(E(...|U_{11},\tilde{U}_{21},Z_1)) \\ \leq \quad & n^{1/2}E\Big(\frac{\varphi(U_{11},\tilde{U}_{21})\delta_1}{(A+B)^3(U_{11}+\tilde{U}_{21})}\frac{1_{\{b_n\geq Z_2, Z_3\geq U_{11}+\tilde{U}_{21}\}}}{F_1^2(Z_2)F_1^2(Z_3)}\Big(E(k_1E_{n-2}^2(Z_2)+M_n(U_{11},Z_1,Z_2)|U_{11},\tilde{U}_{21},Z_1) \\ & \times E(k_1E_{n-2}^2(Z_3)+M_n(U_{11},Z_1,Z_3)|U_{11},\tilde{U}_{21},Z_1)\Big)^{1/2}\Big) \end{aligned} \tag{3.27}$$

According to A.15 in Lemma A.6 and Lemma A.13

$$\begin{aligned} &\Big(E(k_1E_{n-2}^2(Z_2)+M_n(U_{11},Z_1,Z_2)|U_{11},\tilde{U}_{21},Z_1)\Big)^{1/2} \leq\Big(kk_1\frac{K^5M^2}{cc_1\alpha}\frac{ln^3(\frac{n}{c_1\alpha})}{n}+\frac{1}{n}k_2\frac{M^5K^4}{c_1^2c^2}ln^2(\frac{n}{c_1\alpha}) \\ &\times\Big(1+\frac{\gamma^2(U_{11})}{C(U_{11})}+\frac{\gamma^2(U_{11})}{nC^2(U_{11})}\Big)\Big)^{1/2} \leq k_3n^{-1/2}\frac{M^{5/2}K^{5/2}}{c_1c_2\alpha}\Big(1+\frac{\gamma(U_{11})}{\sqrt{C(U_{11})}}+\frac{\gamma(U_{11})}{\sqrt{n}C(U_{11})}\Big), \end{aligned}$$

where k_3 is a constant. Hence

$$\begin{aligned} E(3.27) \quad \leq \quad & n^{-1/2}k_3^2\frac{M^5K^5}{c_1^2c_2^2\alpha^2}E\Big(\frac{\varphi(U_{11},\tilde{U}_{21})\delta_1}{(A+B)^3(U_{11}+\tilde{U}_{21})}\frac{1_{\{b_n\geq Z_2, Z_3\geq U_{11}+\tilde{U}_{21}\}}}{F_1^2(Z_2)F_1^2(Z_3)}\Big(1+\frac{\gamma(U_{11})}{\sqrt{C(U_{11})}}+\frac{1}{\sqrt{n}C(U_{11})}\Big)^2\Big) \\ & \leq n^{-1/2}k_3^2\frac{M^5K^5}{c_1^2c_2^2\alpha^2}\Big[2\int\frac{\varphi(x_1,x_2)}{(A+B)^3(x_1+x_2)}\left(\int_{x_1+x_2}^{b_n}\frac{G^*(dx)}{F_1^2(x)}\right)^2H_2^1(dx_1,dx_2) \\ & +4\int\frac{\varphi(x_1,x_2)}{(A+B)^3(x_1+x_2)}\frac{\alpha F_1(x_1)}{1-G(x_1^-)}\left(\int_{x_1+x_2}^{b_n}\frac{G^*(dx)}{F_1^2(x)}\right)^2H_2^1(dx_1,dx_2) \\ & +4n^{-1}\int\frac{\varphi(x_1,x_2)}{(A+B)^3(x_1+x_2)}\frac{\alpha^2}{(1-G(x_1^-))^2}\left(\int_{x_1+x_2}^{b_n}\frac{G^*(dx)}{F_1^2(x)}\right)^2H_2^1(dx_1,dx_2)\Big] \end{aligned} \tag{3.28}$$

To show that the right side goes to zero we need to consider the three integrals separately. Since $dH_2^1=(A+B)dF$, $dG^*=\alpha^{-1}F_1dG$ and $1-G^-=\alpha(A+B)$, for the first integral we obtain

$$\begin{aligned} & 2\int\frac{\varphi(x_1,x_2)}{(A+B)^3(x_1+x_2)}\left(\int_{x_1+x_2}^{b_n}\frac{G^*(dx)}{F_1^2(x)}\right)^2H_2^1(dx_1,dx_2) \\ & \leq 2\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\frac{\alpha^{-2}(1-G(x_1+x_2^-))^2}{F_1^2(x_1+x_2)}F(dx_1,dx_2) = 2\int\frac{\varphi(x_1,x_2)}{F_1^2(x_1+x_2)}F(dx_1,dx_2). \end{aligned}$$

For the second integral we have

$$
\begin{aligned}
&4\int \frac{\varphi(x_1,x_2)}{(A+B)^3(x_1+x_2)}\frac{\alpha F_1(x_1)}{1-G(x_1^-)}\left(\int_{x_1+x_2}^{b_n}\frac{G^*(dx)}{F_1^2(x)}\right)^2 H_2^1(dx_1,dx_2)\\
&\le \int \frac{\varphi(x_1,x_2)}{(A+B)^3(x_1+x_2)}\frac{\alpha F_1(x_1+x_2)}{1-G(x_1+x_2^-)}\left(\int_{x_1+x_2}^{b_n}\frac{G^*(dx)}{F_1^2(x)}\right)^2 H_2^1(dx_1,dx_2)\\
&\le \int \frac{\varphi(x_1,x_2)}{F_1(x_1+x_2)(A+B)(x_1+x_2)}F(dx_1,dx_2)
\end{aligned}
$$

To bound the third integral in (3.28) we use assumption A1. Since

$$\frac{1-F_1(x)}{1-G(x^-)}\le\int_{(x,\infty)}\frac{F_1(dy)}{1-G(y^-)}\le M$$

we get

$$\frac{1}{F_1(x)}=\frac{1-F_1(x)}{F_1(x)}+1\le\frac{M(1-G(x^-))}{F_1(x)}+1=\frac{M\alpha(A+B)}{F_1(x)}+1.$$

Therefore

$$
\begin{aligned}
&4\int \frac{\varphi(x_1,x_2)}{(A+B)^3(x_1+x_2)}\frac{\alpha^2}{(1-G(x_1^-))^2}\left(\int_{x_1+x_2}^{b_n}\frac{G^*(dx)}{F_1^2(x)}\right)^2 H_2^1(dx_1,dx_2)\\
&\le 4\int \frac{\varphi(x_1,x_2)}{(A+B)^4(x_1+x_2)}\frac{\alpha^{-2}(1-G(x_1+x_2^-))^2}{F_1^2(x_1+x_2)}F(dx_1,dx_2)=4\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\frac{F(dx_1,dx_2)}{F_1^2(x_1+x_2)}\\
&\le 4\int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\left(2\frac{M^2\alpha^2(A+B)^2(x_1+x_2)}{F_1^2(x_1+x_2)}+2\right)F(dx_1,dx_2)\\
&=8M^2\alpha^2\int\frac{\varphi(x_1,x_2)}{F_1^2(x_1+x_2)}F(dx_1,dx_2)+\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}F(dx_1,dx_2)
\end{aligned}
$$

Altogether, since by assumptions the integrals are bounded, we get

$$
\begin{aligned}
(3.28)\quad \le\quad & n^{-1/2}k_4\frac{M^7K^5}{c_1^2c_2^2\alpha^2}\Bigg(\int\frac{\varphi(x_1,x_2)}{F_1^2(x_1+x_2)}F(dx_1,dx_2)+\int\frac{\varphi(x_1,x_2)}{F_1(x_1+x_2)(A+B)(x_1+x_2)}F(dx_1,dx_2)\\
&+\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}F(dx_1,dx_2)\Bigg)\to 0.
\end{aligned}
$$

$\boxtimes$

Lemma 3.15. *Under A1-A3, on the set $\tilde\Omega_n^7$,*

$$n^{1/2}[J_{1n}+J_{3n}]\to 0,\ \ \textit{in probability.}$$

Proof. W.l.o.g. we may assume $\varphi\ge 0$. Since, by Lemma 3.11, $\frac{A+B}{A_n+B_n}\le L$, it suffices to show that

$$n^{1/2}\int\varphi\frac{[A+B-A_n-B_n]^2}{(A+B)^3}dH_{2n}^1\to 0 \tag{3.29}$$

Recall

$$(A_n + B_n)(x) = \int_{[x,\infty)} \frac{G_n^*(dz)}{F_{1n}(z)}.$$

Then the integral in (3.29) can be bounded from above by

$$\begin{aligned}
&n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\frac{\varphi(U_{1i},\tilde U_{2i})\delta_i}{(A+B)^3(U_{1i}+\tilde U_{2i})}[A+B-A_n-B_n]^2(U_{1i}+\tilde U_{2i})\\
&\le n^{1/2}\frac{2}{n}\sum_{i=1}^{n}\frac{\varphi(U_{1i},\tilde U_{2i})\delta_i}{(A+B)^3(U_{1i}+\tilde U_{2i})}[A+B-A_{n-1}-B_{n-1}]^2(U_{1i}+\tilde U_{2i}) \qquad (3.30)\\
&+n^{1/2}\frac{2}{n}\sum_{i=1}^{n}\frac{\varphi(U_{1i},\tilde U_{2i})}{(A+B)^3(U_{1i}+\tilde U_{2i})}\frac{1}{n^2}\frac{1_{\{Z_i\ge U_{1i}+\tilde U_{2i}\}}}{F_{1n}^2(Z_i)}, \qquad (3.31)
\end{aligned}$$

where

$$(A_{n-1}+B_{n-1})(U_{1i}+\tilde U_{2i}) = \frac{1}{n}\sum_{j\ne i}1_{\{Z_j\ge U_{1i}+\tilde U_{2i}\}}\frac{1}{F_{1n}(Z_j)} = \frac{n-1}{n}\int_{[U_{1i}+\tilde U_{2i},\infty)}\frac{G_{n-1}^*(dz)}{F_{1n}(z)}.$$

As to (3.31), by definition of F_{1n} and since $U_{1i}\le Z_i$, we have the following bound

$$(3.31) \le n^{-5/2}\sum_{i=1}^{n}\frac{\varphi(U_{1i},\tilde U_{2i})}{(A+B)^3(U_{1i}+\tilde U_{2i})}1_{\{Z_i\ge U_{1i}+\tilde U_{2i}\}}\sum_{j\ne i}\frac{1_{\{Z_i<U_{1j}\}}}{F_{1n}^2(U_{1j}^-)}. \qquad (3.32)$$

By Lemma 3.10, there exists $\tilde K_1$ so that

$$(3.32) \le \tilde K_1^2 n^{-5/2}\sum_{i=1}^{n}\frac{\varphi(U_{1i},\tilde U_{2i})}{(A+B)^3(U_{1i}+\tilde U_{2i})}1_{\{Z_i\ge U_{1i}+\tilde U_{2i}\}}\sum_{j\ne i}\frac{1_{\{Z_i<U_{1j}\}}}{F_1^2(U_{1j}^-)}. \qquad (3.33)$$

Since U_{1j} is independent of the Z_i and U_i's for $j\ne i$, we have

$$\begin{aligned}
E(3.33) \quad &= n^{-1/2}\tilde K_1^2 E\left(\frac{\varphi(U_{11},\tilde U_{21})}{(A+B)^3(U_{11}+\tilde U_{21})}1_{\{Z_1\ge U_{11}+\tilde U_{21}\}}E\left[\frac{1_{\{Z_1<U_{12}\}}}{F_1^2(U_{12}^-)}|U_{11},\tilde U_{21},Z_1\right]\right)\\
&= n^{-1/2}\tilde K_1^2 E\left(\frac{\varphi(U_{11},\tilde U_{21})}{(A+B)^3(U_{11}+\tilde U_{21})}1_{\{Z_1\ge U_{11}+\tilde U_{21}\}}E\left[\frac{1_{\{y<U_{12}\}}}{F_1^2(U_{12}^-)}\right](y=Z_1)\right)\\
&= n^{-1/2}\tilde K_1^2 E\left(\frac{\varphi(U_{11},\tilde U_{21})}{(A+B)^3(U_{11}+\tilde U_{21})}1_{\{Z_1\ge U_{11}+\tilde U_{21}\}}\int_{(Z_1,\infty)}\frac{F_1^*(dz)}{F_1^2(z^-)}\right)\\
&= n^{-1/2}\tilde K_1^2\int\frac{\varphi(x_1,x_2)}{(A+B)^3(x_1+x_2)}1_{\{y\ge x_1+x_2\}}\int_{(y,\infty)}\frac{\alpha^{-1}(1-G(z^-))F_1(dz)}{F_1^2(z^-)}H_3(dx_1,dx_2,dy)\\
&\le n^{-1/2}\tilde K_1^2\int\frac{\varphi(x_1,x_2)}{(A+B)^3(x_1+x_2)}\alpha^{-1}(1-G(x_1+x_2^-))\frac{1_{\{y\ge x_1+x_2\}}}{F_1(y)}H_3(dx_1,dx_2,dy). \quad (3.34)
\end{aligned}$$

Finally, since $H_3(x_1,x_2,y)=\alpha^{-1}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}(G(y)-G(y_1+y_2))F(dy_1,dy_2)$ if $1_{\{y\geq x_1+x_2\}}$ and $\alpha^{-1}(1-G(x_1+x_2^-))=(A+B)(x_1+x_2)$, we obtain

$$
\begin{aligned}
(3.34)\qquad &= n^{-1/2}\tilde{K}_1^2\int\frac{\varphi(x_1,x_2)\alpha^{-1}}{(A+B)^2(x_1+x_2)}\frac{1_{\{y\geq x_1+x_2\}}}{F_1(y)}G(dy)F(dx_1,dx_2)\\
&= n^{-1/2}\tilde{K}_1^2\int\frac{\varphi(x_1,x_2)\alpha^{-1}}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{G(dy)}{F_1(y)}F(dx_1,dx_2)\\
&\leq n^{-1/2}\tilde{K}_1^2\int\frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)}\frac{1}{F_1(x_1+x_2)}F(dx_1,dx_2).
\end{aligned}
$$

By assumption, the integral goes to zero, as $n\to\infty$.

As to (3.30) recall (3.6). By repeated use of $(a+b)^2\leq 2(a^2+b^2)$, it remains to show that on the set $\tilde{\Omega}_n^7$,

$$
n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\left\{\frac{\varphi(U_{1i},\tilde{U}_{2i})\delta_i}{(A+B)^3(U_{1i}+\tilde{U}_{2i})}\left[\int_{U_{1i}+\tilde{U}_{2i}}^{\infty}\frac{1}{F_1}(dG^*-dG^*_{n-1})\right]^2\right\}\to 0 \tag{3.35}
$$

$$
n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\left\{\frac{\varphi(U_{1i},\tilde{U}_{2i})\delta_i}{(A+B)^3(U_{1i}+\tilde{U}_{2i})}\left[\int_{U_{1i}+\tilde{U}_{2i}}^{\infty}\frac{F_{1n}-F_1}{F_1^2}dG^*_{n-1}\right]^2\right\}\to 0 \tag{3.36}
$$

and

$$
n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\left\{\frac{\varphi(U_{1i},\tilde{U}_{2i})\delta_i}{(A+B)^3(U_{1i}+\tilde{U}_{2i})}\left[\int_{U_{1i}+\tilde{U}_{2i}}^{\infty}\frac{(F_{1n}-F_1)^2}{F_1^2F_{1n}}dG^*_{n-1}\right]^2\right\}\to 0 \tag{3.37}
$$

For the proof of (3.36) see Lemma 3.14. Expression (3.35) equals

$$
n^{-1/2}\sum_{i=1}^{n}\frac{\varphi(U_{1i},\tilde{U}_{2i})\delta_i}{(A+B)^3(U_{1i}+\tilde{U}_{2i})}\left[\frac{1}{n}\sum_{j\neq i}\left(\int_{U_{1i}+\tilde{U}_{2i}}^{\infty}\frac{1}{F_1}dG^*-\frac{1_{\{Z_j\geq U_{1i}+\tilde{U}_{2i}\}}}{F_1(Z_j)}\right)\right]^2 \tag{3.38}
$$

Note that

$$
E\left\{\left[\frac{1}{n}\sum_{j\neq i}\left(\int_{U_{1i}+\tilde{U}_{2i}}^{\infty}\frac{1}{F_1}dG^*-\frac{1_{\{Z_j\geq U_{1i}+\tilde{U}_{2i}\}}}{F_1(Z_j)}\right)\right]^2|U_{1i},\tilde{U}_{2i}\right\}\leq\int_{U_{1i}+\tilde{U}_{2i}}^{\infty}\frac{G^*(dy)}{F_1^2(y)}
$$

Using $dG^*=\alpha^{-1}F_1dG$ and $A+B=\alpha^{-1}(1-G-)$ the expectation of (3.38) is bounded from above by

$$
\begin{aligned}
E(3.38)\qquad &\leq n^{-1/2}\int\frac{\varphi(x_1,x_2)\alpha^{-1}}{(A+B)^3(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{G(dy)}{F_1(y)}H_2^1(dx_1,dx_2)\\
&\leq n^{-1/2}\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)F_1(x_1+x_2)}H_2^1(dx_1,dx_2)\\
&= n^{-1/2}\int\frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)}F(dx_1,dx_2).
\end{aligned}
$$

Since the integral is finite the last term goes to zero, as $n \to \infty$.

As to (3.37), we can restrict the integral w.r.t. G_n^* to $(-\infty, b_n)$. Then we can bound the term (3.37) in probability from above by

$$\begin{aligned}
&\frac{n^{1/2}}{n^3} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i} \frac{\varphi(U_{1i}, \tilde{U}_{2i})\delta_i}{(A+B)^3(U_{1i}+\tilde{U}_{2i})} \left[\frac{(F_{1n}-F_1)^2(Z_j)}{F_1^3(Z_j)} 1_{\{b_n \geq Z_j \geq U_{1i}+U_{2i}\}} \right] \\
&\times \left[\frac{(F_{1n}-F_1)^2(Z_k)}{F_1^3(Z_k)} 1_{\{b_n \geq Z_k \geq U_{1i}+U_{2i}\}} \right] = n^{-5/2} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i} \frac{\varphi(U_{1i}, \tilde{U}_{2i})\delta_i}{(A+B)^3(U_{1i}+\tilde{U}_{2i})} \\
&\times \left[\frac{|F_{1n}(Z_j)-F_1(Z_j)|}{F_1^2(Z_j)} \frac{|F_{1n}(Z_k)-F_1(Z_k)|}{F_1^2(Z_k)} 1_{\{b_n \geq Z_j, Z_k \geq U_{1i}+U_{2i}\}} \right] \\
&\times \frac{|F_{1n}(Z_j)-F_1(Z_j)|}{F_1(Z_j)} \frac{|F_{1n}(Z_k)-F_1(Z_k)|}{F_1(Z_k)} \qquad (3.39)
\end{aligned}$$

By Lemma 3.7 there exists some constant $\tilde{K}$, so that

$$\frac{F_{1n}(Z_j)}{F_1(Z_j)} \leq \tilde{K},$$

whence

$$\frac{|F_{1n}(Z_j)-F_1(Z_j)|}{F_1(Z_j)} \frac{|F_{1n}(Z_k)-F_1(Z_k)|}{F_1(Z_k)} \leq (\tilde{K}+1)^2.$$

Hence the proof that (3.39) goes to zero is the same as that of (3.23) if $k = j$. In case $k \neq j$ we refer to (3.24) in Lemma 3.14. This completes the proof of the Lemma.

⊠

In the next lemma we show that

$$J_{2n} = \int \varphi \frac{A+B-A_n-B_n}{(A+B)^2} (dH_{2n}^1 - dH_2^1)$$

is negligible.

Lemma 3.16. *Under A1-A3, on the set $\tilde{\Omega}_n^7$,*

$$n^{1/2} J_{2n} \to 0, \quad \textit{in probability.}$$

Proof. Similarly to Lemma 3.15 we write J_{2n} as follows:

$$\begin{aligned}
J_{2n} \quad &= \int \varphi \frac{A+B-A_{n-1}-B_{n-1}}{(A+B)^2} (dH_{2n}^1 - dH_2^1) \\
&-n^{-2} \sum_{i=1}^{n} \left(\frac{\varphi(U_{1i}, \tilde{U}_{2i})\delta_i}{(A+B)^2(U_{1i}+\tilde{U}_{2i})F_{1n}(Z_i)} - \int \frac{\varphi(x_1,x_2)1_{\{Z_i \geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)F_{1n}(Z_i)} H_2^1(dx_1, dx_2) \right) \\
&=: J_{2n}^a + J_{2n}^b
\end{aligned}$$

and assume, w.l.o.g., that $\varphi \geq 0$.

As to J_{2n}^b, we bound its absolute value from above by the sum of two terms. According to Lemma 3.10, there exists $\tilde{K}_1$ so that

$$\begin{aligned} n^{1/2}|J_{2n}^b| \quad & \leq n^{-3/2}\sum_{i=1}^{n}\left(\frac{\varphi(U_{1i},\tilde{U}_{2i})\delta_i}{(A+B)^2(U_{1i}+\tilde{U}_{2i})F_{1n}(Z_i)}+\int\frac{\varphi(x_1,x_2)1_{\{Z_i\geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)F_{1n}(Z_i)}H_2^1(dx_1,dx_2)\right) \\ & \leq n^{-3/2}\tilde{K}_1\sum_{i=1}^{n}\left(\frac{\varphi(U_{1i},\tilde{U}_{2i})\delta_i}{(A+B)^2(U_{1i}+\tilde{U}_{2i})F_{1}(Z_i)}+\int\frac{\varphi(x_1,x_2)1_{\{Z_i\geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)F_{1}(Z_i)}H_2^1(dx_1,dx_2)\right)\end{aligned}$$

The expectation of the right hand side is bounded from above by

$$\begin{aligned} & n^{-1/2}\tilde{K}_1\int\frac{\varphi(x_1,x_2)1_{\{y\geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)F_1(y)}G(dy)F(dx_1,dx_2) \\ & +n^{-1/2}\tilde{K}_1\int\frac{\varphi(x_1,x_2)1_{\{y\geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)F_1(y)}G^*(dy)H_2^1(dx_1,dx_2) \\ & \leq n^{-1/2}2\tilde{K}_1\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{G(dy)}{F_1(y)}F(dx_1,dx_2) \\ & \leq n^{-1/2}2\tilde{K}_1\int\frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)}F(dx_1,dx_2)\end{aligned}$$

and this, according to the finiteness of the integral, goes to zero, as $n\to\infty$.

As to J_{2n}^a, by (3.6), it suffices to show that on the set $\tilde{\Omega}_n^7$,

$$n^{1/2}\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1}(dG^*-dG_{n-1}^*)(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2))\to 0 \tag{3.40}$$

$$n^{1/2}\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{F_{1n}-F_1}{F_1^2}dG_{n-1}^*(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2))\to 0 \tag{3.41}$$

and

$$n^{1/2}\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{(F_{1n}-F_1)^2}{F_1^2F_{1n}}dG_{n-1}^*(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2))\to 0 \tag{3.42}$$

As to (3.40), we will prove that

$$n^{1/2}E\left|\int\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\frac{1_{\{y\geq x_1+x_2\}}}{F_1(y)}(G_{n-1}^*(dy)-G^*(dy))(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2))\right|$$

goes to zero, as $n\to\infty$.

Set

$$\tilde{\varphi}(x_1, x_2, y) = \frac{\varphi(x_1, x_2) 1_{\{y \geq x_1 + x_2\}}}{(A+B)^2 (x_1 + x_2) F_1(y)}.$$

By Cauchy-Schwarz it is sufficient to prove

$$nE\left[\int\int \tilde{\varphi}(x_1, x_2, y)(G^*_{n-1}(dy) - G^*(dy))(H^1_{2n}(dx_1, dx_2) - H^1_2(dx_1, dx_2))\right]^2 \to 0, \qquad (3.43)$$

as $n \to \infty$.
Now set

$$\begin{aligned} \varphi^*(x_1, x_2, \delta^1, y) : &= \tilde{\varphi}(x_1, x_2, y)\delta^1 - \int \tilde{\varphi}(x_1, x_2, u)\delta^1 G^*(du) - \int \tilde{\varphi}(v_1, v_2, y) H^1_2(dv_1, dv_2) \\ &+ \int\int \tilde{\varphi}(v_1, v_2, u) G^*(du) H^1_2(dv_1, dv_2). \end{aligned}$$

Then to show (3.43), we have to prove that

$$n^{-3} \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{\substack{j=1 \\ j \neq k,i}}^{n} \sum_{\substack{l=1 \\ l \neq k,i}}^{n} E(\varphi^*(U_{1i}, \tilde{U}_{2i}, \delta_i, Z_j)\varphi^*(U_{1k}, \tilde{U}_{2k}, \delta_k, Z_l)) \to 0,$$

as $n \to \infty$.

Since the U's, δ's and Z's are independent for different indices $i \neq j \neq k \neq l$ and since the mean of φ^* equals zero, the following term vanish.

$$n\frac{n(n-1)(n-2)(n-3)}{n^4} E(\varphi^*(U_{11}, \tilde{U}_{21}, \delta_1, Z_2)\varphi^*(U_{13}, \tilde{U}_{23}, \delta_3, Z_4))$$

The terms

$$n\frac{n(n-1)(n-2)}{n^4} E(\varphi^*(U_{11}, \tilde{U}_{21}, \delta_1, Z_2)\varphi^*(U_{11}, \tilde{U}_{21}, \delta_1, Z_3)).$$

and

$$n\frac{n(n-1)(n-2)}{n^4} E(\varphi^*(U_{11}, \tilde{U}_{21}, \delta_1, Z_2)\varphi^*(U_{13}, \tilde{U}_{23}, \delta_3, Z_2))$$

can be written as expectations of conditional expectations, given $(U_{11}, \tilde{U}_{21}, \delta_1, Z_1)$ and Z_2, respectively. It can be proved that these conditional expectations equal zero.
To complete the proof of (3.40) it remains to show that

$$n^{-1} E[(\varphi^*(U_{11}, \tilde{U}_{21}, \delta_1, Z_2))^2]$$

goes to zero. Since, by A1, $\frac{1-F_1(x)}{1-G(x^-)} \leq \int_{[x,\infty)} \frac{dF_1}{1-G-} \leq M < \infty$ and hence $\frac{1}{F_1} = \frac{1-F_1}{F_1} + 1 \leq M\frac{1-G-}{F_1} + 1$, then, together with $dG^* = \alpha^{-1} F_1 dG$ and $dH^1_2 = (A+B)dF$, we obtain

$$n^{-1}E[(\varphi^*(U_{11},\tilde{U}_{21},\delta_1,Z_2))^2] \le 16n^{-1}\int \frac{\varphi^2(x_1,x_2)}{(A+B)^4(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{dG^*}{F_1^2}H_2^1(dx_1,dx_2)$$
$$\le 16n^{-1}\int \frac{\varphi^2(x_1,x_2)\alpha^{-1}}{(A+B)^2(x_1+x_2)F_1(x_1+x_2)}F(dx_1,dx_2)$$
$$\le 16Mn^{-1}\int \frac{\varphi^2(x_1,x_2)\alpha}{(A+B)(x_1+x_2)F_1(x_1+x_2)}F(dx_1,dx_2)+16n^{-1}\int \frac{\varphi^2(x_1,x_2)}{(A+B)^2(x_1+x_2)}F(dx_1,dx_2),$$

which goes to zero, because the integrals are finite by assumptions A2 and A3.

Now we will deal with (3.41). Set

$$\hat{\varphi}(x_1,x_2) := \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}.$$

Then the term in (3.41) equals

$$n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\bigg(\hat{\varphi}(U_{1i},\tilde{U}_{2i})\delta_i\frac{1}{n}\sum_{j\ne i}^{n}\frac{F_{1n}(Z_j)-F_1(Z_j)}{F_1^2(Z_j)}1_{\{U_{1i}+\tilde{U}_{2i}\le Z_j\}}$$
$$-\int \hat{\varphi}(x_1,x_2)\frac{1}{n}\sum_{j\ne i}^{n}\frac{F_{1n}(Z_j)-F_1(Z_j)}{F_1^2(Z_j)}1_{\{x_1+x_2\le Z_j\}}H_2^1(dx_1,dx_2)\bigg). \tag{3.44}$$

By Lemma 3.12,

$$F_{1n}(x)-F_1(x)=L_n(x)+R_n(x),$$

where

$$L_n(x)=\int \frac{\Psi_x(y)}{C(y)}(F_{1n}^*(dy)-F_1^*(dy))-\int \frac{C_n(y)-C(y)}{C^2(y)}\Psi_x(y)F_1^*(dy).$$

Since, according to definition of F_{1n}, we have

$$F_{1n}(Z_j)=\prod_{U_{1l}>Z_j}\left[1-\frac{1}{nC_n(U_{1l})}\right]=\prod_{U_{1l}>Z_j,l\ne j}\left[1-\frac{1}{(n-1)C_{n-1}(U_{1l})}\right]=: F_{1n-1}^{\ne j}(Z_j),$$

we obtain

$$F_{1n}(Z_j)-F_1(Z_j)=F_{1n-1}^{\ne j}(Z_j)-F_1(Z_j)=L_{n-1}^{\ne j}(Z_j)+R_{n-1}^{\ne j}(Z_j).$$

Hence

$$
\begin{aligned}
(3.44) \quad &= n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\Bigg(\hat{\varphi}(U_{1i},\tilde{U}_{2i})\delta_i\frac{1}{n}\sum_{j\neq i}^{n}\frac{L_{n-1}^{\neq j}(Z_j)+R_{n-1}^{\neq j}(Z_j)}{F_1^2(Z_j)}1_{\{U_{1i}+\tilde{U}_{2i}\leq Z_j\}} \\
&\quad -\int\hat{\varphi}(x_1,x_2)\frac{1}{n}\sum_{j\neq i}^{n}\frac{L_{n-1}^{\neq j}(Z_j)+R_{n-1}^{\neq j}(Z_j)}{F_1^2(Z_j)}1_{\{x_1+x_2\leq Z_j\}}H_2^1(dx_1,dx_2)\Bigg) \\
&= n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\Bigg(\hat{\varphi}(U_{1i},\tilde{U}_{2i})\delta_i\frac{1}{n}\sum_{j\neq i}^{n}\frac{L_{n-1}^{\neq j}(Z_j)}{F_1^2(Z_j)}1_{\{U_{1i}+\tilde{U}_{2i}\leq Z_j\}} \\
&\quad -\int\hat{\varphi}(x_1,x_2)\frac{1}{n}\sum_{j\neq i}^{n}\frac{L_{n-1}^{\neq j}(Z_j)}{F_1^2(Z_j)}1_{\{x_1+x_2\leq Z_j\}}H_2^1(dx_1,dx_2)\Bigg) \\
&\quad +n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\hat{\varphi}(U_{1i},\tilde{U}_{2i})\delta_i\frac{1}{n}\sum_{j\neq i}^{n}\frac{R_{n-1}^{\neq j}(Z_j)}{F_1^2(Z_j)}1_{\{U_{1i}+\tilde{U}_{2i}\leq Z_j\}} \\
&\quad -n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\int\hat{\varphi}(x_1,x_2)\frac{1}{n}\sum_{j\neq i}^{n}\frac{R_{n-1}^{\neq j}(Z_j)}{F_1^2(Z_j)}1_{\{x_1+x_2\leq Z_j\}}H_2^1(dx_1,dx_2)
\end{aligned}
$$

Furthermore, since F_1 is continuous, we have that $\Psi_x(y) = -1_{\{x<y\}}F_1(x)$. Hence $\Psi_{Z_j}(U_{1j}) = 0$, $1_{\{U_{1j}\leq y\leq Z_j\}}\Psi_{Z_j}(y) = 0$ and therefore

$$
\begin{aligned}
L_n(Z_j) &= \frac{n-1}{n}L_{n-1}^{\neq j}(Z_j)+\frac{\Psi_{Z_j}(U_{1j})}{nC(U_{1j})}-\frac{1}{n}\int\frac{\Psi_{Z_j}(y)}{C(y)}F_1^*(dy)-\frac{1}{n}\int\frac{1_{\{U_{1j}\leq y\leq Z_j\}}-C(y)}{C^2(y)}\Psi_{Z_j}(y)F_1^*(dy) \\
&= \frac{n-1}{n}L_{n-1}^{\neq j}(Z_j)-\frac{1}{n}\int\frac{\Psi_{Z_j}(y)}{C(y)}F_1^*(dy)+\frac{1}{n}\int\frac{\Psi_{Z_j}(y)}{C(y)}F_1^*(dy)=\frac{n-1}{n}L_{n-1}^{\neq j}(Z_j).
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
&n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\Bigg(\hat{\varphi}(U_{1i},\tilde{U}_{2i})\delta_i\frac{1}{n}\sum_{j\neq i}^{n}\frac{L_{n-1}^{\neq j}(Z_j)}{F_1^2(Z_j)}1_{\{U_{1i}+\tilde{U}_{2i}\leq Z_j\}}-\int\hat{\varphi}(x_1,x_2)\frac{1}{n}\sum_{j\neq i}^{n}\frac{L_{n-1}^{\neq j}(Z_j)}{F_1^2(Z_j)}1_{\{x_1+x_2\leq Z_j\}}H_2^1(dx_1,dx_2)\Bigg) \\
&= n^{1/2}\frac{n}{n-1}\int\hat{\varphi}(x_1,x_2)\int_{x_1+x_2}^{\infty}\frac{L_n(x)}{F_1^2(x)}G_{n-1}^*(dx)(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2)).
\end{aligned}
$$

So, to prove (3.41), we have to show that the following terms go to zero in probability:

$$
n^{1/2}\int\hat{\varphi}(x_1,x_2)\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int\frac{\Psi_x(y)}{C(y)}(F_{1n}^*(dy)-F_1^*(dy))G_{n-1}^*(dx)(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2)) \tag{3.45}
$$

$$
n^{1/2}\int\hat{\varphi}(x_1,x_2)\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int\frac{C_n(y)-C(y)}{C^2(y)}\Psi_x(y)F_1^*(dy)G_{n-1}^*(dx)(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2)) \tag{3.46}
$$

$$n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\hat{\varphi}(U_{1i},\tilde{U}_{2i})\delta_i\frac{1}{n}\sum_{j\neq i}^{n}\frac{|R_{n-1}^{\neq j}(Z_j)|}{F_1^2(Z_j)}1_{\{U_{1i}+\tilde{U}_{2i}\leq Z_j\}} \tag{3.47}$$

and

$$\int\hat{\varphi}(x_1,x_2)\frac{1}{n}\sum_{j=1}^{n}\frac{|R_{n-1}^{\neq j}(Z_j)|}{F_1^2(Z_j)}1_{\{x_1+x_2\leq Z_j\}}H_2^1(dx_1,dx_2). \tag{3.48}$$

At first we deal with (3.47). According to Lemma A.22, $|R_{n-1}^{\neq j}(Z_j)|\leq|\tilde{R}_{n-1}^{\neq j}(Z_j)|$. Since, on the set $\tilde{\Omega}_n^7$,

$$(3.47)\leq n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\frac{\hat{\varphi}(U_{1i},\tilde{U}_{2i})\delta_i}{F_1(U_{1i}+\tilde{U}_{2i})}\times\frac{1}{n}\sum_{j=1}^{n}\frac{|\tilde{R}_{n-1}^{\neq j}(Z_j)|}{F_1(Z_j)}1_{\{a_n\leq Z_j\leq b_n\}}$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\hat{\varphi}(U_{1i},\tilde{U}_{2i})\delta_i}{F_1(U_{1i}+\tilde{U}_{2i})}\to\int\frac{\hat{\varphi}(x_1,x_2)}{F_1(x_1+x_2)}H_2^1(dx_1,dx_2)=\int\frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)}F(dx_1,dx_2)<\infty$$

it remains to show that

$$n^{1/2}\frac{1}{n}\sum_{j=1}^{n}\frac{|\tilde{R}_{n-1}^{\neq j}(Z_j)|}{F_1(Z_j)}1_{\{a_n\leq Z_j\leq b_n\}}=o_{\mathbb{P}}(1).$$

By Lemma A.22

$$\begin{aligned}E\tilde{R}_n(t) \;\leq\;& k_{10}\frac{1}{n}\frac{K^3M^3}{c_1^2c^2}ln^2(\frac{n}{c_1\alpha})+\frac{2\sqrt{\alpha M}}{n}\frac{1}{\sqrt{1-G(t^-)}}\left(ln(\frac{n}{c_1\alpha})+1\right)^{1/2}+\frac{\sqrt{8M\alpha}}{n^{3/2}}\frac{1}{1-G(t^-)}\\&+\frac{2M}{\sqrt{c\alpha}}\frac{\varepsilon}{\sqrt{n}}.\end{aligned}$$

Since $\tilde{R}_{n-1}^{\neq j}(.)$ doesn't contain U's and Z's with index j, we get

$$\begin{aligned}&E\left(n^{1/2}\frac{1}{n}\sum_{j=1}^{n}\frac{|\tilde{R}_{n-1}^{\neq j}(Z_j)|}{F_1(Z_j)}1_{\{a_n\leq Z_j\leq b_n\}}\right)=E(E(...|Z_j))=E\left(n^{1/2}\frac{1}{n}\sum_{j=1}^{n}\frac{E(|\tilde{R}_{n-1}^{\neq j}(Z_j)||Z_j)}{F_1(Z_j)}1_{\{a_n\leq Z_j\leq b_n\}}\right)\\&\leq k_{10}\frac{1}{\sqrt{n}}\frac{K^3M^3}{c_1^2c^2}ln^2(\frac{n}{c_1\alpha})\int_{(a_n,b_n)}\frac{G^*(dx)}{F_1(x)}+\frac{2\sqrt{\alpha M}}{\sqrt{n}}\left(ln(\frac{n}{c_1\alpha})+1\right)^{1/2}\int_{(a_n,b_n)}\frac{G^*(dx)}{F_1(x)\sqrt{1-G(x^-)}}\\&+\frac{\sqrt{8M\alpha}}{n}\int_{(a_n,b_n)}\frac{G^*(dx)}{F_1(x)(1-G(x^-))}+\frac{2M}{\sqrt{c\alpha}}\varepsilon\int_{(a_n,b_n)}\frac{G^*(dx)}{F_1(x)}.\end{aligned}$$

Since $dG^* = \alpha^{-1} F_1 dG$ and by Remark A.1 $-ln(1-G(b_n)) \le ln(\frac{n}{c\alpha})$, for every $\varepsilon > 0$, the right side is bounded from above by

$$k_{10}\frac{1}{\sqrt{n}}\frac{K^3M^3}{c_1^2c^2}ln^2(\frac{n}{c_1\alpha}) + \frac{2\sqrt{\alpha M}}{\sqrt{n}}\left(ln(\frac{n}{c_1\alpha})+1\right)^{1/2} ln(\frac{n}{c\alpha}) + \frac{\sqrt{8M\alpha}}{n}ln(\frac{n}{c\alpha}) + \frac{2M}{\sqrt{c\alpha}}\varepsilon.$$

Since the first three coefficients go to zero as $n \to \infty$ and the last can be made as small as possible, the proof of (3.47) is completed. As to (3.48), this is bounded from above by

$$(3.48) \le n^{1/2}\int \frac{\hat{\varphi}(x_1,x_2)}{F_1(x_1+x_2)}H_2^1(dx_1,dx_2) \times \frac{1}{n}\sum_{j=1}^{n}\frac{|\tilde{R}_{n-1}^{\neq j}(Z_j)|}{F_1(Z_j)}1_{\{a_n \le Z_j \le b_n\}}$$

which, like (3.47), goes to zero.
As to (3.45) we will show that

$$n^{1/2}\int \hat{\varphi}(x_1,x_2)\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int \frac{\Psi_x(y)}{C(y)}(F_{1n}^*(dy)-F_1^*(dy))G^*(dx)(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2)) \tag{3.49}$$

$$n^{1/2}\int\left|\hat{\varphi}(x_1,x_2)\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int \frac{\Psi_x(y)}{C(y)}(F_{1n}^*(dy)-F_1^*(dy))(G_{n-1}^*(dx)-G^*(dx))\right|H_2^1(dx_1,dx_2) \tag{3.50}$$

and

$$n^{1/2}\int\left|\hat{\varphi}(x_1,x_2)\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int \frac{\Psi_x(y)}{C(y)}(F_{1n}^*(dy)-F_1^*(dy))(G_{n-1}^*(dx)-G^*(dx))\right|H_{2n}^1(dx_1,dx_2) \tag{3.51}$$

go to zero in probability.
The term (3.49) equals

$$n^{1/2}\int\int S(x_1,x_2,y)(F_{1n}^*(dy)-F_1^*(dy))(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2))$$

where

$$S(x_1,x_2,y) = \hat{\varphi}(x_1,x_2)\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\frac{\Psi_x(y)}{C(y)}G^*(dx).$$

Set

$$\begin{aligned} S^*(x_1,x_2,\delta,y) &= S(x_1,x_2,y)\delta - \int S(x_1,x_2,v)\delta F_1^*(dv) - \int S(u_1,u_2,y)H_2^1(du_1,du_2) \\ &\quad + \int\int S(u_1,u_2,v)F_1^*(dv)H_2^1(du_1,du_2). \end{aligned}$$

Then, by Cauchy-Schwarz, it remains to show that

$$n^{-3}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{\substack{j=1\\ j\neq i,k}}^{n}\sum_{\substack{l=1\\ j\neq i,k}}^{n} ES^*(U_{1i},\tilde{U}_{2i},\delta_i,U_{1j})S^*(U_{1k},\tilde{U}_{2k},\delta_k,U_{1l}) \to 0.$$

Using similar arguments as for (3.40) we only have to prove that

$$n^{-1}ES^*(U_{11},\tilde{U}_{21},\delta_1,U_{12})^2 \tag{3.52}$$

goes to zero.
Since, by continuity of F_1, $\gamma = F_1$ and therefore $\Psi_x(y) = -1_{\{y>x\}}F_1(x)$, we have

$$\begin{aligned}
(3.52) \quad &= n^{-1}ES^*(U_{11},\tilde{U}_{21},\delta_1,U_{12})^2 = n^{-1}E\Big(E\Big(S(U_{11},\tilde{U}_{21},U_{12})\delta_1 - \int S(U_{11},\tilde{U}_{21},v)\delta_1 F_1^*(dv)\\
&- \int S(u_1,u_2,U_{12})H_2^1(du_1,du_2) + \int\int S(u_1,u_2,v)F_1^*(dv)H_2^1(du_1,du_2)\Big)^2|U_{11},\tilde{U}_{21},\delta_1\Big)\\
&\le 8n^{-1}E\Big(\hat{\varphi}(U_{11},\tilde{U}_{21})\delta_1 E\Big(\Big(\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\frac{1_{\{U_{12}>x\}}F_1(x)}{C(U_{12})}G^*(dx)\Big)^2(x_1=U_{11},x_2=\tilde{U}_{21})\Big)\Big)\\
&+24n^{-1}\int\int\Big(\hat{\varphi}(u_1,u_2)\int_{u_1+u_2}^{\infty}\frac{1}{F_1^2(x)}\frac{1_{\{v>x\}}F_1(x)}{C(v)}G^*(dx)\Big)^2 F_1^*(dv)H_2^1(du_1,du_2)\\
&\le 32n^{-1}\int\hat{\varphi}^2(x_1,x_2)\frac{1}{C(z)}\Big(\int_{x_1+x_2}^{z}\alpha^{-1}dG\Big)^2 F_1^*(dz)H_2^1(dx_1,dx_2)\\
&\le 32n^{-1}\int\frac{\varphi^2(x_1,x_2)\alpha^{-2}}{(A+B)^4(x_1+x_2)}(1-G(x_1+x_2^-))^2\int_{x_1+x_2}^{\infty}\frac{F_1(dz)}{F_1^2(z)}H_2^1(dx_1,dx_2)\\
&\le 32n^{-1}\int\frac{\varphi^2(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)}F(dx_1,dx_2)
\end{aligned}$$

which goes to zero by finiteness of the integral.

As to (3.50), by Cauchy-Schwarz, we have to prove that

$$n^{1/2}\int\hat{\varphi}(x_1,x_2)\left(E\left[\int\int\frac{\Psi_x(y)1_{\{x\ge x_1+x_2\}}}{F_1^2(x)C(y)}[F_{1n}^*(dy)-F_1^*(dy)](G_{n-1}^*(dx)-G^*(dx))\right]^2\right)^{1/2}H_2^1(dx_1,dx_2)$$

goes to zero.

Set

$$\varphi_1(x,y) = \frac{\Psi_x(y)1_{\{x\ge x_1+x_2\}}}{F_1^2(x)C(y)}$$

and

$$\varphi^*(x,y) = \varphi_1(x,y) - \int \varphi_1(x,u)F_1^*(du) - \int \varphi_1(v,y)G^*(dv) + \int\int \varphi_1(v,u)G^*(dv)F_1^*(du).$$

Then we have to prove

$$n^{1/2}\int \hat{\varphi}(x_1,x_2)\left(E\left(\int\int \varphi^*(x,y)F_{1n}^*(dy)G_n^*(dx)\right)^2\right)^{1/2} H_2^1(dx_1,dx_2) \to 0, \text{ as } n\to\infty.$$

Similarly to the proof of (3.40) and by using $\Psi_x(y) = -1_{\{y>x\}}F_1(x)$ the expectation equals

$$\frac{1}{n^4}\sum_{i=1}^n\sum_{j=1}^n\sum_{\substack{k=1\\k\neq i,j}}^n\sum_{\substack{l=1\\k\neq i,j}}^n E(\varphi^*(Z_i,U_{1k})\varphi^*(Z_j,U_{1l})) = \frac{1}{n^2}E\varphi^*(Z_1,U_{12})^2 \le \frac{1}{n^2}\frac{1}{F_1^2(x_1+x_2)}\int_{x_1+x_2}^\infty \frac{dG}{F_1}\int_{x_1+x_2}^\infty \frac{dF_1}{1-G-}.$$

Finally, since by A1, $\int_{x_1+x_2}^\infty \frac{dF_1}{1-G-} \le M < \infty$, we obtain

$$\begin{aligned} E|(3.50)| &\le n^{-1/2}\int \frac{\hat{\varphi}(x_1,x_2)}{F_1(x_1+x_2)}\left(\int_{x_1+x_2}^\infty \frac{dG}{F_1}\int_{x_1+x_2}^\infty \frac{dF_1}{1-G-}\right)^{1/2} H_2^1(dx_1,dx_2) \\ &= n^{-1/2}\int \frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)}\left(\int_{x_1+x_2}^\infty \frac{dG}{F_1}\int_{x_1+x_2}^\infty \frac{dF_1}{1-G-}\right)^{1/2} F(dx_1,dx_2) \\ &\le \sqrt{M}n^{-1/2}\int \frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)}\frac{\sqrt{1-G(x_1+x_2^-)}}{\sqrt{F_1(x_1+x_2)}}F(dx_1,dx_2). \qquad (3.53) \end{aligned}$$

Furthermore, by A1, $1-F_1(x) \le M(1-G(x^-))$, we have

$$\frac{1}{F_1(x_1+x_2)} = \frac{1-F_1(x_1+x_2)}{F_1(x_1+x_2)} + 1 \le M\frac{1-G(x_1+x_2^-)}{F_1(x_1+x_2)} + 1 = M\frac{\alpha(A+B)(x_1+x_2)}{F_1(x_1+x_2)} + 1.$$

Therefore

$$\begin{aligned} (3.53) \quad &\le \quad n^{-1/2}\int \frac{\varphi(x_1,x_2)M\alpha}{F_1^{3/2}(x_1+x_2)}F(dx_1,dx_2) + n^{-1/2}\int \frac{\varphi(x_1,x_2)\alpha^{1/2}}{(A+B)^{1/2}(x_1+x_2)F_1^{1/2}(x_1+x_2)}F(dx_1,dx_2) \\ &\le \quad n^{-1/2}\int \frac{\varphi(x_1,x_2)M\alpha}{F_1^2(x_1+x_2)}F(dx_1,dx_2) + n^{-1/2}\int \frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)}F(dx_1,dx_2), \end{aligned}$$

which again goes to zero.

As to (3.51), we can bound it from above by

$$\begin{aligned} (3.51) \quad \le \quad & n^{1/2}\int\left|\hat{\varphi}(x_1,x_2)\int_{x_1+x_2}^\infty \frac{1}{F_1^2(x)}\int \frac{\Psi_x(y)}{C(y)}(F_{1n-1}^*(dy) - F_1^*(dy))(G_{n-1}^*(dx) - G^*(dx))\right| H_{2n}^1(dx_1,dx_2) \\ & + n^{-1/2}\int\left|\hat{\varphi}(x_1,x_2)\int_{x_1+x_2}^\infty \frac{1}{F_1^2(x)}\frac{\Psi_x(U_{11})}{C(U_{11})}(G_{n-1}^*(dx) - G^*(dx))\right| H_{2n}^1(dx_1,dx_2) \end{aligned}$$

The first coefficient is dealt with in the same way as (3.50). The expectation of the second is bounded from above by

$$n^{-1}\int \frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)}\left(\int_{x_1+x_2}^{\infty}\frac{dG}{F_1}\int_{x_1+x_2}^{\infty}\frac{dF_1}{1-G-}\right)^{1/2}F(dx_1,dx_2),$$

which goes to zero.

(3.46) is dealt with similarly to (3.45). We bound its absolute value as follows:

$$n^{1/2}\int\hat{\varphi}(x_1,x_2)\left|\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int\frac{C_n(y)-C(y)}{C^2(y)}\Psi_x(y)F_1^*(dy)(G_{n-1}^*(dx)-G^*(dx))\right|H_{2n}^1(dx_1,dx_2) \tag{3.54}$$

$$+n^{1/2}\int\hat{\varphi}(x_1,x_2)\left|\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int\frac{C_n(y)-C(y)}{C^2(y)}\Psi_x(y)F_1^*(dy)(G_n^*(dx)-G^*(dx))\right|H_2^1(dx_1,dx_2) \tag{3.55}$$

$$+n^{1/2}\left|\int\hat{\varphi}(x_1,x_2)\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int\frac{C_n(y)-C(y)}{C^2(y)}\Psi_x(y)F_1^*(dy)G^*(dx)(H_{2n}^1-H_2^1)(dx_1,dx_2))\right| \tag{3.56}$$

As to (3.55), its expectation equals

$$n^{1/2}\int\hat{\varphi}(x_1,x_2)E\left|\int\int 1_{\{x\geq x_1+x_2\}}\frac{1}{F_1^2(x)}\frac{C_n(y)-C(y)}{C^2(y)}\Psi_x(y)F_1^*(dy)(G_n^*(dx)-G^*(dx))\right|H_2^1(dx_1,dx_2) \tag{3.57}$$

Since $\gamma=F_1$ and $\Psi_x(y)=-1_{\{y>x\}}F_1(x)$ the expectation is, by Fubini, less than or equal to

$$n^{1/2}\int\frac{1}{C^2(y)}E\left|\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}(C_n(y)-C(y))1_{\{y>x\}}F_1(x)(G_n^*(dx)-G^*(dx))\right|F_1^*(dy) \tag{3.58}$$

Since $C_n(y)$ and $C(y)$ are independent of x, by use of Cauchy-Schwarz, we come up with an upper bound for the expectation, namely:

$$\sqrt{E(C_n(y)-C(y))^2}\sqrt{E\left(\int_{x_1+x_2}^{\infty}\frac{1_{\{y>x\}}}{F_1(x)}[G_n^*(dx)-G^*(dx)]\right)^2}$$

The first square root equals

$$\sqrt{\frac{1}{n}C(y)(1-C(y))}$$

and the second

$$\sqrt{\frac{1}{n}\left(\int_{x_1+x_2}^{\infty}\frac{1_{\{y>x\}}}{F_1^2(x)}G^*(dx)-\left[\int_{x_1+x_2}^{\infty}\frac{1_{\{y>x\}}}{F_1(x)}G^*(dx)\right]^2\right)}$$

Finally, by $dG^* = \alpha^{-1}F_1 dG$, (3.58) is bounded from above by

$$n^{-1/2}\int \frac{\alpha^{-1/2}}{C^2(y)}\sqrt{C(y)}1_{\{y\geq x_1+x_2\}}\left(\int_{x_1+x_2}^{y}\frac{dG}{F_1}\right)^{1/2}F_1^*(dy) \leq n^{-1/2}\int_{x_1+x_2}^{\infty}\frac{\alpha^{-1/2}}{C^{3/2}(y)}F_1^*(dy)\left(\int_{x_1+x_2}^{\infty}\frac{dG}{F_1}\right)^{1/2}$$
$$= n^{-1/2}\int_{x_1+x_2}^{\infty}\frac{F_1(dy)}{F_1^{3/2}(y)(1-G(y-))^{1/2}}\left(\int_{x_1+x_2}^{\infty}\frac{dG}{F_1}\right)^{1/2}$$
$$\leq n^{-1/2}\left(\int_{x_1+x_2}^{\infty}\frac{F_1(dy)}{F_1^3(y)}\right)^{1/2}\left(\int_{x_1+x_2}^{\infty}\frac{F_1(dy)}{1-G(y-)}\right)^{1/2}\left(\int_{x_1+x_2}^{\infty}\frac{dG}{F_1}\right)^{1/2}$$
$$\leq n^{-1/2}\frac{1}{F_1(x_1+x_2)}\left(\int_{x_1+x_2}^{\infty}\frac{F_1(dy)}{1-G(y-)}\right)^{1/2}\left(\int_{x_1+x_2}^{\infty}\frac{dG}{F_1}\right)^{1/2}$$

From this it follows that (3.57) is bounded from above by

$$n^{-1/2}\int\frac{\hat{\varphi}(x_1,x_2)}{F_1(x_1+x_2)}\left(\int_{x_1+x_2}^{\infty}\frac{F_1(dy)}{1-G(y-)}\right)^{1/2}\left(\int_{x_1+x_2}^{\infty}\frac{dG}{F_1}\right)^{1/2}H_2^1(dx_1,dx_2)$$
$$= n^{-1/2}\int\frac{\varphi(x_1,x_2)\alpha}{(A+B)(x_1+x_2)F_1(x_1+x_2)}\left(\int_{x_1+x_2}^{\infty}\frac{F_1(dy)}{1-G(y-)}\right)^{1/2}\left(\int_{x_1+x_2}^{\infty}\frac{dG}{F_1}\right)^{1/2}F(dx_1,dx_2).$$

Under our assumptions, (3.57) goes to zero, as $n\to\infty$. The proof of (3.54) is almost the same. As to (3.56), by $\Psi_x(y) = -1_{\{y>x\}}F_1(x)$ and Fubini we need to prove that

$$n^{1/2}\int\int\frac{1}{F_1(x)}\frac{1}{C^2(y)}E\left|\int\hat{\varphi}(x_1,x_2)1_{\{y>x\geq x_1+x_2\}}(C_n(y)-C(y))(H_{2n}^1-H_2^1)(dx_1,dx_2))\right|F_1^*(dy)G^*(dx)$$

goes to zero. Furthermore, by Cauchy Schwarz,

$$E\left|\int\hat{\varphi}(x_1,x_2)1_{\{y>x\geq x_1+x_2\}}(C_n(y)-C(y))(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2))\right|$$
$$\leq \frac{1}{n}\sqrt{C(y)}\left(\int\hat{\varphi}^2(x_1,x_2)1_{\{y>x\geq x_1+x_2\}}H_2^1(dx_1,dx_2)\right)^{1/2}.$$

Hence, since $dG^* = \alpha^{-1}F_1dG$, $dF_1^* = \alpha^{-1}(1-G^-)dF_1$, $1-G^- = \alpha(A+B)$ and $dH_2^1 = (A+B)dF$, we get

$$\begin{aligned}
E|(3.56)| &\leq n^{-1/2}\int\int\frac{1}{F_1(x)}\frac{1}{C^{3/2}(y)}\left(\int\hat{\varphi}^2(x_1,x_2)1_{\{y>x\geq x_1+x_2\}}H_2^1(dx_1,dx_2)\right)^{1/2}F_1^*(dy)G^*(dx)\\
&= n^{-1/2}\int\int\frac{\alpha^{-1}}{(1-G(y^-))^{1/2}F_1^{3/2}(y)}\left(\int\frac{\varphi^2(x_1,x_2)1_{\{y>x\geq x_1+x_2\}}}{(A+B)^3(x_1+x_2)}F(dx_1,dx_2)\right)^{1/2}F_1(dy)G(dx)\\
&\leq n^{-1/2}\int\frac{\alpha^{-1}}{(1-G(y^-))^{1/2}F_1^{3/2}(y)}\left(\int\int\frac{\varphi^2(x_1,x_2)1_{\{y>x\geq x_1+x_2\}}}{(A+B)^3(x_1+x_2)}G(dx)F(dx_1,dx_2)\right)^{1/2}F_1(dy)\\
&= n^{-1/2}\int\frac{\alpha^{-1}}{(1-G(y^-))^{1/2}F_1^{3/2}(y)}\left(\int\frac{\alpha\varphi^2(x_1,x_2)1_{\{y\geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)}F(dx_1,dx_2)\right)^{1/2}F_1(dy).
\end{aligned}$$

By Cauchy-Schwarz we get that the right side is bounded from above by

$$n^{-1/2}\left(\int \frac{\alpha^{-2}}{1-G(y^-)}F_1(dy)\right)^{1/2}\left(\int \frac{1}{F_1^3(y)}\int \frac{\alpha\varphi^2(x_1,x_2)1_{\{y\geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)}F(dx_1,dx_2)F_1(dy)\right)^{1/2}$$

$$\leq \sqrt{M}n^{-1/2}\left(\int \frac{\alpha\varphi^2(x_1,x_2)}{(A+B)^2(x_1+x_2)F_1^2(x_1+x_2)}F(dx_1,dx_2)\right)^{1/2} \tag{3.59}$$

By A1 we have that $1-F_1\leq M(1-G^-)$. Then by repeated use of $\frac{1}{F_1}\leq M\frac{1-G^-}{F_1}+1$ we get

$$\begin{aligned}(3.59) \;\leq\; & O(n^{-1/2})\Bigg(\int \frac{\alpha\varphi^2(x_1,x_2)}{F_1^2(x_1+x_2)}F(dx_1,dx_2)+\int \frac{\alpha\varphi^2(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)}F(dx_1,dx_2)\\ & +\int \frac{\alpha\varphi^2(x_1,x_2)}{(A+B)^2(x_1+x_2)}F(dx_1,dx_2)\Bigg)^{1/2}\to 0.\end{aligned}$$

To show (3.42) we prove

$$n^{1/2}\int \hat{\varphi}(x_1,x_2)\int_{x_1+x_2}^{\infty}\frac{(F_{1n}-F_1)^2}{F_1^2F_{1n}}dG_n^*H_2^1(dx_1,dx_2)\to 0 \text{ in probability} \tag{3.60}$$

and

$$n^{1/2}\int \hat{\varphi}(x_1,x_2)\int_{x_1+x_2}^{\infty}\frac{(F_{1n}-F_1)^2}{F_1^2F_{1n}}dG_{n-1}^*H_{2n}^1(dx_1,dx_2)\to 0 \text{ in probability.} \tag{3.61}$$

As to (3.60), on the set $\tilde{\Omega}_n^7$, F_1/F_{1n} is bounded from above by $\tilde{K}_1$. Hence

$$(3.60)=n^{1/2}\tilde{K}_1\int \hat{\varphi}(x_1,x_2)\frac{1}{n}\sum_{i=1}^{n}\frac{(F_{1n}-F_1)^2(Z_i)}{F_1^3(Z_i)}1_{\{Z_i\geq x_1+x_2\}}H_2^1(dx_1,dx_2)$$

Furthermore, since $U_{1i}\leq Z_i$, $(F_{1n}-F_1)^2(Z_i)=(F_{1n-1}-F_1)^2(Z_i)$ and F_{1n-1} does not include U_{1i} and Z_i. By Lemma A.6, $(F_{1n-1}-F_1)^2$ is bounded from above by E_{n-1}^2. Hence

$$(3.60)\leq n^{1/2}\int \hat{\varphi}(x_1,x_2)\frac{1}{n}\sum_{i=1}^{n}\frac{E_{n-1}^2(Z_i)}{F_1^3(Z_i)}1_{\{Z_i\geq x_1+x_2\}}H_2^1(dx_1,dx_2).$$

Finally, the expectation of the last part, according to (A.15), is bounded from above by

$$n^{1/2}\int \hat{\varphi}(x_1,x_2)\frac{1}{n}\sum_{i=1}^{n}E\left(\frac{E_{n-1}^2(Z_i)}{F_1^3(Z_i)}1_{\{Z_i\geq x_1+x_2\}}\right)H_2^1(dx_1,dx_2)$$

$$=n^{1/2}\int \hat{\varphi}(x_1,x_2)\frac{1}{n}\sum_{i=1}^{n}E\left(\frac{1_{\{Z_i\geq x_1+x_2\}}}{F_1^3(Z_i)}E\left(E_{n-1}^2(Z_i)|Z_i\right)\right)H_2^1(dx_1,dx_2)$$

$$\leq n^{1/2}\int \hat{\varphi}(x_1,x_2)\frac{1}{n}\sum_{i=1}^{n} E\left(\frac{1_{\{Z_i\geq x_1+x_2\}}}{F_1^3(Z_i)}\left(k\frac{K^5M^2}{cc_1\alpha}\frac{ln^3(\frac{n}{c_1\alpha})}{n}\right)\right)H_2^1(dx_1,dx_2)$$
$$\leq \frac{T}{\sqrt{n}}\int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{b_n}\frac{1}{F_1^2(t)}G(dt)H_2^1(dx_1,dx_2)$$
$$\leq \frac{T}{\sqrt{n}}\int \frac{\varphi(x_1,x_2)\alpha}{(A+B)(x_1+x_2)F_1(x_1+x_2)}\int_{x_1+x_2}^{b_n}\frac{1}{F_1(t)}G(dt)F(dx_1,dx_2),$$

where $T=k\frac{K^5M^2}{cc_1\alpha}ln^3(\frac{n}{c_1\alpha})$. Since by assumption the integral is bounded, the right side goes to zero. As to (3.61), the arguments are similar to (3.60). On the set $\tilde{\Omega}_n^7$, and by Lemma A.12, we have

$$\begin{aligned}
(3.61) \quad &\leq \quad \tilde{K}_1 n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\hat{\varphi}(U_{1i},\tilde{U}_{2i})\frac{1}{n}\sum_{j\neq i}\frac{(F_{1n}(Z_j)-F_1(Z_j))^2}{F_1^3(Z_j)}1_{\{U_{1i}+\tilde{U}_{2i}\leq Z_j\}}\\
&\leq \quad \tilde{K}_1 n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\hat{\varphi}(U_{1i},\tilde{U}_{2i})\frac{1}{n}\sum_{j\neq i}\frac{k_1E_{n-2}^2(Z_k)+M_n(Z_k,U_{1i},Z_i)1_{\{Z_k,Z_i\leq b_n,U_{1i}>a_n\}}}{F_1^3(Z_j)}1_{\{U_{1i}+\tilde{U}_{2i}\leq Z_j\}}.
\end{aligned}$$

The proof that

$$\tilde{K}_1 n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\hat{\varphi}(U_{1i},\tilde{U}_{2i})\frac{1}{n}\sum_{j\neq i}\frac{k_1E_{n-2}^2(Z_k)}{F_1^3(Z_j)}1_{\{U_{1i}+\tilde{U}_{2i}\leq Z_j\}}\to 0$$

follows from (3.60). Therefore it remains to show that

$$\tilde{K}_1 n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\hat{\varphi}(U_{1i},\tilde{U}_{2i})\frac{1}{n}\sum_{j\neq i}\frac{M_n(Z_k,U_{1i},Z_i)1_{\{Z_k,Z_i\leq b_n,U_{1i}>a_n\}}}{F_1^3(Z_j)}1_{\{U_{1i}+\tilde{U}_{2i}\leq Z_j\}}\to 0$$

According to Lemma A.13,

$$E(M_n(Z_k,U_{1i},Z_i)1_{\{Z_k,Z_i\leq b_n,U_{1i}>a_n\}}|Z_k,U_{1i},\tilde{U}_{2i},Z_i)\leq \frac{1}{n}k_2\frac{M^5K^4}{c_1^2c^2}ln^2(\frac{n}{c_1\alpha})\left(1+\frac{\gamma^2(U_{1i})}{C(U_{1i})}+\frac{\gamma^2(U_{1i})}{nC^2(U_{1i})}\right).$$

Hence

$$\begin{aligned}
&E\left(\tilde{K}_1 n^{1/2}\frac{1}{n}\sum_{i=1}^{n}\hat{\varphi}(U_{1i},\tilde{U}_{2i})\frac{1}{n}\sum_{j\neq i}\frac{M_n(Z_k,U_{1i},Z_i)1_{\{Z_k,Z_i\leq b_n,U_{1i}>a_n\}}}{F_1^3(Z_j)}1_{\{U_{1i}+\tilde{U}_{2i}\leq Z_j\}}\right)\\
&\leq n^{-1/2}\tilde{K}_1k_2\frac{M^5K^4}{c_1^2c^2}ln^2(\frac{n}{c_1\alpha})\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\left(1+\frac{\gamma^2(x_1)}{C(x_1)}+\frac{\gamma^2(x_1)}{nC^2(x_1)}\right)\int_{x_1+x_2}^{b_n}\frac{G^*(dx)}{F_1^3(x)}H_2^1(dx_1,dx_2)\\
&\leq n^{-1/2}\tilde{K}_1k_2\frac{M^5K^4}{c_1^2c^2}ln^2(\frac{n}{c_1\alpha})\int\frac{\alpha\varphi(x_1,x_2)}{F_1^2(x_1+x_2)}\left(1+\frac{\alpha F_1(x_1)}{1-G(x_1^-)}+\frac{\alpha^2}{n(1-G(x_1^-))^2}\right)F(dx_1,dx_2)\\
&\leq n^{-1/2}\tilde{K}_1k_2\frac{M^5K^4}{c_1^2c^2}ln^2(\frac{n}{c_1\alpha})\int\frac{\alpha\varphi(x_1,x_2)}{F_1^2(x_1+x_2)}F(dx_1,dx_2)\\
&+n^{-1/2}\tilde{K}_1k_2\frac{M^5K^4}{c_1^2c^2}ln^2(\frac{n}{c_1\alpha})\int\frac{\alpha^2\varphi(x_1,x_2)}{F_1(x_1+x_2)(1-G(x_1+x_2^-))}F(dx_1,dx_2)\\
&+n^{-3/2}\tilde{K}_1k_2\frac{M^5K^4}{c_1^2c^2}ln^2(\frac{n}{c_1\alpha})\int\frac{\alpha^3\varphi(x_1,x_2)}{F_1^2(x_1+x_2)(1-G(x_1+x_2^-))^2}F(dx_1,dx_2).
\end{aligned}$$

By assumption, the first and the second term go to zero. As to the third term, since by A1 $1 - F_1(x) \leq M(1 - G(x^-))$, we get

$$\left(\frac{1}{F_1(x)}\right)^2 \leq \left(\frac{1 - F_1(x)}{F_1(x)} + 1\right)^2 \leq 2M^2 \frac{(1 - G(x^-))^2}{F_1^2(x)} + 2.$$

Hence, since $\alpha^{-1}(1 - G-) = A + B$, we obtain

$$\begin{aligned}
& n^{-3/2}\tilde{K}_1 k_2 \frac{M^5 K^4}{c_1^2 c^2} ln^2(\frac{n}{c_1\alpha}) \int \frac{\alpha^3 \varphi(x_1, x_2)}{F_1^2(x_1 + x_2)(1 - G(x_1 + x_2^-))^2} F(dx_1, dx_2) \\
& \leq n^{-3/2}\tilde{K}_1 k_2 \frac{M^5 K^4}{c_1^2 c^2} ln^2(\frac{n}{c_1\alpha}) \int \frac{2M^2 \alpha^3 \varphi(x_1, x_2)}{F_1^2(x_1 + x_2)} F(dx_1, dx_2) \\
& + n^{-3/2}\tilde{K}_1 k_2 \frac{M^5 K^4}{c_1^2 c^2} ln^2(\frac{n}{c_1\alpha}) \int \frac{\alpha \varphi(x_1, x_2)}{(A + B)^2 (x_1 + x_2)} F(dx_1, dx_2) \to 0.
\end{aligned}$$

$\boxtimes$

To write I_n as a sum of i.i.d. random variables and a remainder, we have to prove that J_{4n} has a proper representation.

Since

$$\int_{[x,\infty)} \frac{F_{1n} - F_1}{F_1^2} dG_n^* = \int_{[x,\infty)} \frac{F_{1n} - F_1}{F_1^2} (dG_n^* - dG^*) + \int_{[x,\infty)} \frac{F_{1n} - F_1}{F_1^2} dG^*,$$

J_{4n} can be written as

$$\begin{aligned}
J_{4n} \;=\; & \int \hat{\varphi}(x_1, x_2) \int_{x_1+x_2}^{\infty} \frac{F_{1n} - F_1}{F_1^2} dG^* H_2^1(dx_1, dx_2) \\
& - \int \hat{\varphi}(x_1, x_2) \int_{x_1+x_2}^{\infty} \frac{(F_{1n} - F_1)^2}{F_1^2 F_{1n}} dG_n^* H_2^1(dx_1, dx_2) \\
& + \int \hat{\varphi}(x_1, x_2) \int_{x_1+x_2}^{\infty} \frac{F_{1n} - F_1}{F_1^2} (dG_n^* - dG^*) H_2^1(dx_1, dx_2) \\
& + \int \hat{\varphi}(x_1, x_2) \int_{x_1+x_2}^{\infty} \frac{1}{F_1} (dG^* - dG_n^*) H_2^1(dx_1, dx_2) = K_{1n} + K_{2n} + K_{3n} + K_{4n},
\end{aligned}$$

where

$$\hat{\varphi}(x_1, x_2) = \frac{\varphi(x_1, x_2)}{(A + B)^2 (x_1 + x_2)}.$$

The term K_{4n} is already a sum of i.i.d. random variables. Next we will prove that K_{2n} and K_{3n} are negligible.

Lemma 3.17. *Under A1-A3, on the set $\tilde{\Omega}_n^7$,*

$$n^{1/2} K_{2n} \to 0, \quad \textit{in probability.}$$

Proof.
The proof is identical to the proof of (3.60) in Lemma 3.16.

$\boxtimes$

Next we prove that K_{3n} is a remainder.

Lemma 3.18. *Under A1-A3, on the set* $\tilde{\Omega}_n^7$,

$$n^{1/2} K_{3n} \to 0, \quad \textit{in probability.}$$

Proof. It is sufficient to show that, on the set $\tilde{\Omega}_n^7$,

$$n^{1/2} \int \hat{\varphi}(x_1, x_2) \left| \int_{x_1+x_2}^{\infty} \frac{F_{1n} - F_1}{F_1^2} (dG_n^* - dG^*) \right| H_2^1(dx_1, dx_2) \to 0.$$

By Lemma 3.12 and use of $|a + b| \leq |a| + |b|$ it remains to show that the following expectations

$$n^{1/2} \int \hat{\varphi}(x_1, x_2) E \left| \int_{x_1+x_2}^{\infty} \frac{1}{F_1^2(x)} \int \frac{\Psi_x(y)}{C(y)} [F_{1n}^*(dy) - F_1^*(dy)](G_n^*(dx) - G^*(dx)) \right| H_2^1(dx_1, dx_2) \quad (3.62)$$

$$n^{1/2} \int \hat{\varphi}(x_1, x_2) E \left| \int_{x_1+x_2}^{\infty} \frac{1}{F_1^2(x)} \int \frac{C_n(y) - C(y)}{C^2(y)} \Psi_x(y) F_1^*(dy)(G_n^*(dx) - G^*(dx)) \right| H_2^1(dx_1, dx_2) \quad (3.63)$$

go to zero and the integral

$$n^{1/2} \int \hat{\varphi}(x_1, x_2) \left| \int_{x_1+x_2}^{\infty} \frac{R_n}{F_1^2} (dG_n^* - dG^*) \right| H_2^1(dx_1, dx_2) \quad (3.64)$$

goes to zero in probability, as $n \to \infty$.

The proof of (3.64) is similar to the proof of (3.47).

As to (3.62), by Cauchy-Schwarz, it suffices to prove that

$$n^{1/2} \int \hat{\varphi}(x_1, x_2) \left(E \left[\int \int \frac{\Psi_x(y) 1_{\{x \geq x_1 + x_2\}}}{F_1^2(x) C(y)} [F_{1n}^*(dy) - F_1^*(dy)](G_n^*(dx) - G^*(dx)) \right]^2 \right)^{1/2} H_2^1(dx_1, dx_2) \quad (3.65)$$

goes to zero, as $n \to \infty$.
Set

$$\varphi_1(x, y) = \frac{\Psi_x(y) 1_{\{x \geq x_1 + x_2\}}}{F_1^2(x) C(y)}$$

and

$$\varphi^*(x,y) = \varphi_1(x,y) - \int \varphi_1(x,u)F_1^*(du) - \int \varphi_1(v,y)G^*(dv) + \int\int \varphi_1(v,u)G^*(dv)F_1^*(du).$$

Then the expectation in (3.65) equals

$$E\left(\int\int \varphi^*(x,y)F_{1n}^*(dy)G_n^*(dx)\right)^2 = \frac{1}{n^4}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{l=1}^n E(\varphi^*(Z_i,U_{1j})\varphi^*(Z_k,U_{1l}))$$

We have that the U's and Z's are independent for different indices. Similarly to the proof of (3.40) we can show that the expectation and conditional expectation of φ^* given U_{1i} or Z_i equals zero.
Since, for continuous F_1, $\Psi_x(y) = -1_{\{y>x\}}F_1(x)$ then, similarly to the proof of (3.52), we get

$$\begin{aligned}
&\frac{1}{n^4}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{l=1}^n E(\varphi^*(Z_i,U_{1j})\varphi^*(Z_k,U_{1l})) \le \frac{1}{n^2}E(\varphi^*(Z_2,U_{11}))^2 + \frac{1}{n^3}E(\varphi^*(Z_1,U_{11}))^2\\
&\le \frac{4}{n^2}E(\varphi_1(Z_2,U_{11}))^2 + \frac{4}{n^2}E\int \varphi_1^2(Z_2,u)F_1^*(du) + \frac{4}{n^2}E\int \varphi_1^2(v,U_{11})G^*(dv)\\
&+\frac{4}{n^2}\int\int \varphi_1^2(v,u)G^*(dv)F_1^*(du) + \frac{4}{n^3}E(\varphi_1(Z_1,U_{11}))^2 + \frac{4}{n^3}E\int \varphi_1^2(Z_1,u)F_1^*(du)\\
&+\frac{4}{n^3}E\int \varphi_1^2(v,U_{11})G^*(dv) + \frac{4}{n^3}\int\int \varphi_1^2(v,u)G^*(dv)F_1^*(du)\\
&\le \frac{28}{n^2}\int\int \varphi_1(x,y)^2F_1^*(dy)G^*(dx) + \frac{4}{n^3}\int\int \varphi_1(x,y)^2dP(U_1\le y, Z\le x|U_1\le Z)\\
&= \frac{28}{n^2}\int\int \frac{1_{\{y>x\}}F_1^2(x)1_{\{x\ge x_1+x_2\}}}{F_1^4(x)C^2(y)}F_1^*(dy)G^*(dx) + \frac{1}{n^3}\int\int \varphi_1(x,y)^2 1_{\{x\ge y\}}F_1(dy)G(dx)\\
&\le \frac{28}{n^2}\frac{1}{F_1^2(x_1+x_2)}\int_{x_1+x_2}\frac{dG}{F_1}\int_{x_1+x_2}\frac{dF_1}{1-G_-}.
\end{aligned}$$

Therefore (3.65) is bounded from above by

$$\begin{aligned}
&n^{1/2}\sqrt{28}\int \hat{\varphi}(x_1,x_2)\left(\frac{1}{n^2}\frac{1}{F_1^2(x_1+x_2)}\int_{x_1+x_2}\frac{dG}{F_1}\int_{x_1+x_2}\frac{dF_1}{1-G_-}\right)^{1/2} H_2^1(dx_1,dx_2)\\
&\le n^{-1/2}\sqrt{28}\int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)F_1(x_1+x_2)}\left(\int_{x_1+x_2}\frac{dG}{F_1}\int_{x_1+x_2}\frac{dF_1}{1-G_-}\right)^{1/2} H_2^1(dx_1,dx_2)\\
&= n^{-1/2}\sqrt{28}\int \frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)}\left(\int_{x_1+x_2}\frac{dG}{F_1}\int_{x_1+x_2}\frac{dF_1}{1-G_-}\right)^{1/2} F(dx_1,dx_2)\\
&\le n^{-1/2}\sqrt{28M}\int \frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)}\left(\int_{x_1+x_2}\frac{dG}{F_1}\right)^{1/2} F(dx_1,dx_2).
\end{aligned}$$

By assumption, the integral is bounded and hence the right side goes to zero, as $n\to\infty$. This completes the proof of (3.62).

As to (3.63), the proof is the same as that of (3.55).

$\boxtimes$

Lemma 3.19. *Under A1-A3, on the set $\tilde{\Omega}_n^7$, we have*

$$\begin{aligned}
K_{1n} &= -\int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{1}{C(y)} (F_{1n}^*(dy) - F_1^*(dy)) G^*(dx) H_2^1(dx_1,dx_2) \\
&\quad + \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{C_n(y)-C(y)}{C^2(y)} F_1^*(dy) G^*(dx) H_2^1(dx_1,dx_2) \\
&\quad + o_{\mathbb{P}}(\frac{1}{\sqrt{n}}).
\end{aligned}$$

Proof. We assume that $\varphi \geq 0$. By Lemma 3.12 K_{1n} can be written as follow:

$$\begin{aligned}
K_{1n} &= \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1^2(x)} \int \frac{\Psi_x(y)}{C(y)} (F_{1n}^*(dy) - F_1^*(dy)) G^*(dx) H_2^1(dx_1,dx_2) \\
&\quad - \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1^2(x)} \int \frac{C_n(y)-C(y)}{C^2(y)} \Psi_x(y) F_1^*(dy) G^*(dx) H_2^1(dx_1,dx_2) \\
&\quad + \hat{\varphi}(x_1,x_2) \int_{x_1+x_2}^{\infty} \frac{R_n}{F_1^2} dG^* H_2^1(dx_1,dx_2) =: K_{1n}^a + K_{1n}^b + K_{1n}^c
\end{aligned}$$

Since $\gamma(x) = F_1(x)$, $\Psi_x(y) = -1_{\{y>x\}} F_1(x)$. Hence

$$K_{1n}^a = -\int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{1}{C(y)} (F_{1n}^*(dy) - F_1^*(dy)) G^*(dx) H_2^1(dx_1,dx_2)$$

and

$$K_{1n}^b = \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{C_n(y)-C(y)}{C^2(y)} F_1^*(dy) G^*(dx) H_2^1(dx_1,dx_2).$$

To show that $K_{1n}^c = o_{\mathbb{P}}(\frac{1}{\sqrt{n}})$, by Lemma A.22, it is sufficient to prove that

$$n^{1/2} \int \hat{\varphi}(x_1,x_2) \int_{x_1+x_2}^{\infty} \frac{|\tilde{R}_n|}{F_1^2} dG^* H_2^1(dx_1,dx_2)$$

goes to zero, where $\hat{\varphi} = \frac{\varphi}{(A+B)^2}$. The expectation of this term is, by Lemma A.22 and $dG^* = F_1 dG$, for every $\varepsilon > 0$,

$$\begin{aligned}
& n^{1/2} \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{b_n} \frac{E|\tilde{R}_n|}{F_1^2} dG^* H_2^1(dx_1,dx_2) \\
& \leq n^{-1/2} \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{b_n} \frac{1}{F_1^2(t)} \Bigg(k_{10} \frac{K^3 M^3}{c_1^2 c^2} ln^2(\frac{n}{c_1 \alpha}) + 2\sqrt{\alpha M} \frac{1}{\sqrt{1-G(t^-)}} \Bigg(ln(\frac{n}{c_1\alpha}) + 1 \Bigg)^{1/2} \\
& + \frac{\sqrt{8M\alpha}}{n^{1/2}} \frac{1}{1-G(t^-)} \Bigg) G^*(dt) H_2^1(dx_1,dx_2) + \frac{2M}{\sqrt{c\alpha}} \varepsilon \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{b_n} \frac{G^*(dt)}{F_1^2(t)} H_2^1(dx_1,dx_2)
\end{aligned}$$

$$\le n^{-1/2}k_{10}\frac{K^3M^3}{c_1^2c^2}ln^2(\frac{n}{c_1\alpha})\int\frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)}H_2^1(dx_1,dx_2)$$
$$+n^{-1/2}2\sqrt{\alpha M}\left(ln(\frac{n}{c_1\alpha})+1\right)^{1/2}\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{b_n}\frac{G(dt)}{F_1(t)\sqrt{1-G(t^-)}}H_2^1(dx_1,dx_2)$$
$$+\frac{\sqrt{8M\alpha}}{n}\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{b_n}\frac{G(dt)}{F_1(t)(1-G(t^-))}H_2^1(dx_1,dx_2)$$
$$+\frac{2M}{\sqrt{c\alpha}}\varepsilon\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{b_n}\alpha^{-1}\frac{G(dt)}{F_1(t)}H_2^1(dx_1,dx_2). \tag{3.66}$$

Since, by A1, we have

$$\int_{x_1+x_2}^{b_n}\frac{G(dt)}{F_1(t)\sqrt{(1-G(t^-))}}\le M\int_{x_1+x_2}^{b_n}\frac{G(dt)}{F_1(t)}+\int_{x_1+x_2}^{b_n}\frac{G(dt)}{\sqrt{(1-G(t^-))}}\le M\frac{\alpha(A+B)(x_1+x_2)}{F_1(x_1+x_2)}+1,$$

$$\int_{x_1+x_2}^{b_n}\frac{G(dt)}{F_1(t)(1-G(t^-))}\le M\int_{x_1+x_2}^{b_n}\frac{G(dt)}{F_1(t)}+\int_{x_1+x_2}^{b_n}\frac{G(dt)}{1-G(t^-)}\le M\frac{\alpha(A+B)(x_1+x_2)}{F_1(x_1+x_2)}+ln(\frac{n}{c\alpha})$$

and

$$\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{b_n}\alpha^{-1}\frac{G(dt)}{F_1(t)}H_2^1(dx_1,dx_2)\le\int\frac{\varphi(x_1,x_2)}{F_1(x_1+x_2)}F(dx_1,dx_2)<\infty.$$

Therefore (3.66) goes to zero if $n\to\infty$. The proof is complete.

⊠

Altogether we have

Lemma 3.20. *Under A1-A3*

$$\begin{aligned}J_{4n} &= -\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1(x)}\int_{(x,\infty)}\frac{1}{C(y)}(F_{1n}^*(dy)-F_1^*(dy))G^*(dx)H_2^1(dx_1,dx_2)\\ &\quad+\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1(x)}\int_{(x,\infty)}\frac{C_n(y)-C(y)}{C^2(y)}F_1^*(dy)G^*(dx)H_2^1(dx_1,dx_2)\\ &\quad+\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1(x)}(G^*(dx)-G_n^*(dx))H_2^1(dx_1,dx_2)+o_{\mathbb{P}}(\frac{1}{\sqrt{n}})\end{aligned}$$

Proof. To prove the Lemma we only need to use Lemma 3.17, 3.18 and 3.19.

⊠

Now we are ready to complete the proof of our main theorem.

Proof of Theorem 1.1. The proof is an immediate consequence of Lemmas 3.15, 3.16 and 3.20.

⊠

Chapter 4

A Functional Central Limit Theorem

In the last Chapter we dealt with the convergence in distribution of I_n for one φ. Now we deal with a class of φ to obtain a limit process for this class.
Recall the linearization of I_n from Theorem 1.1:

$$\begin{aligned}
I_n(\varphi) &= \int \frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)} H^1_{2n}(dx_1,dx_2) \\
&+ \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1^2(x)} \int \frac{\Psi_x(y)}{C(y)} (F^*_{1n}(dy) - F^*_1(dy)) G^*(dx) H^1_2(dx_1,dx_2) \\
&- \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1^2(x)} \int \Psi_x(y) \frac{C_n(y)-C(y)}{C^2(y)} F^*_1(dy) G^*(dx) H^1_2(dx_1,dx_2) \\
&+ \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} (G^*(dx) - G^*_n(dx)) H^1_2(dx_1,dx_2) + o_{\mathbb{P}}(\frac{1}{\sqrt{n}}) \\
&= \hat{I}_n(\varphi) + \hat{R}_n(\varphi),
\end{aligned}$$

where $\hat{R}_n(\varphi)$ is the remainder. See Lemmas 3.15, 3.16, 3.17, 3.18 and 3.19 for details.
Let

$$\varphi \in K,$$

where K is a Vapnik-Červonenkis class (VC-class) with envelope function φ_0 (that is $|\varphi| \leq \varphi_0$ for each $\varphi \in K$).

More precisely, the goal of this Chapter is to prove that the process $\{\sqrt{n}(I_n(\varphi) - I(\varphi)) : \varphi \in K\}$ converges in distribution to some Gaussian process. In the first step of the proof we deal with a class of functions $\mathcal{G}$, such that $\hat{I}_n(\varphi) = \frac{1}{n}\sum_{i=1}^n g(U_{1i}, \tilde{U}_{2i}, Z_i, \delta_i)$ for some $g \in \mathcal{G}$.
For a class $\mathcal{G}$ we prove the uniform entropy condition (2.5.1) from van der Vaart and Wellner (1996).
To be more precisely, for a class $\mathcal{G}$ with envelope function G we need to prove that the integral

$$\int_0^1 \sup_Q \sqrt{logN(\varepsilon||G||, \mathcal{G}, L_2(Q))} d\varepsilon$$

is finite, where Q is a probability measure on $\mathbb{R}^3 \times \{0,1\}$, $||G||^2 := ||G||^2_{Q,2} = \int G^2 dQ$, and

$$N(\varepsilon||G||, \mathcal{G}, L_2(Q))$$

is the minimal number of balls $\{g : ||g - f|| \leq \varepsilon||G||\}$ with radius $\varepsilon||G||$ needed to cover the set $\mathcal{G}$, called covering number. The above-mentioned uniform entropy is the logarithm of a covering number. Note, that it is sufficient to prove the entropy condition for $\varepsilon \in (0, 1)$, because for $\varepsilon \geq 1$ one ball is enough to cover the set $\mathcal{G}$ and then $logN(\varepsilon||G||, \mathcal{G}, L_2(Q)) = 0$.

Additionally, the assumption that for i.i.d. Rademacher variables $(e_1, ..., e_n)$ independent of $(U_{11}, \tilde{U}_{21}, Z_1, \delta_1), ..., (U_{1n}, \tilde{U}_{2n}, Z_n, \delta_n)$ we have that

$$((U_{11}, \tilde{U}_{21}, Z_1, \delta_1), ..., (U_{1n}, \tilde{U}_{2n}, Z_n, \delta_n)) \to \sup_{g \in \mathcal{G}} |\sum_{i=1}^{n} e_i g(U_{1i}, \tilde{U}_{2i}, Z_i, \delta_i)|$$

is a measurable function, yield the Donsker property for the class $\hat{I}_n(\varphi)$.
The second step is to prove that $\sqrt{n}\hat{R}_n(\varphi)$ converges to zero in probability uniformly in φ. This is to find in Lemma 4.6.

We assume that F_1 and F_2 are continuous and that the assumptions A2 and A3 from Chapter 1 are valid for φ_0. This means, we have

A1: $\int \frac{dF_1}{1-G_-} < \infty$

A2':$\int \frac{|\varphi_0(x_1,x_2)|^k}{F_1^2(x_1+x_2)} F(dx_1, dx_2) < \infty$

A3':$\int \frac{|\varphi_0(x_1,x_2)|^k}{(A+B)^2(x_1+x_2)} F(dx_1, dx_2) < \infty$

for $k = 1, 2$.

Theorem 4.1. *Under assumption that φ is VC-class of functions with envelope function φ_0, assumptions A1,A2' and A3' and for continuous F_1 and F_2 we have*

$$\{\sqrt{n}(I_n(\varphi) - I(\varphi)) : \varphi \in K\} \xrightarrow{d} L \circ \alpha_1 \;\; in \;\; l^\infty(K)$$

where $l^\infty(K)$ is a space of uniformly bounded functions and $L \circ \alpha_1$ a mean-zero Gaussian process with covariance

$$\begin{aligned} Cov(L \circ \alpha_1(\varphi_1), L \circ \alpha_1(\varphi_2)) \;\; &= \;\; \int_{\mathbb{R}^3} \left(\psi_1(y_1, y_3) + \frac{\varphi_1(y_1, y_2)}{(A+B)(y_1+y_2)}\right) \left(\psi_2(y_1, y_3) + \frac{\varphi_2(y_1, y_2)}{(A+B)(y_1+y_2)}\right) \\ &\quad H_3(dy_1, dy_2, dy_3) + \int_{\mathbb{R}^2} \psi_1(y_1, y_3)\psi_2(y_1, y_3)\tilde{H}_2^2(dy_1, dy_3), \end{aligned}$$

where

$$H_3(y_1, y_2, y_3) = P(U_1 \leq y_1, U_2 \leq y_2, Z \leq y_3, \delta = 1 | U_1 \leq Z)$$
$$\tilde{H}_2^2(y_1, y_3) = P(U_1 \leq y_1, Z \leq y_3, \delta = 0 | U_1 \leq Z)$$

and

$$\psi_i(y_1,y_3) = \int_{\mathbb{R}^2} \frac{\varphi_i(x_1,x_2)}{(A+B)^2(x_1+x_2)}\Big[\int_{x_1+x_2}^{\infty} \frac{1}{F_1^2(x)}\left\{\frac{\psi_x(y_1)}{C(y_1)} - \int \psi_x(y)\frac{1_{\{y_1\le y\le y_3\}}}{C^2(y)}F_1^*(dy)\right\}G^*(dx) - \frac{1_{\{y_3\ge x_1+x_2\}}}{F_1(y_3)}\Big]H_2^1(dx_1,dx_2)$$

for $i=1,2$.

Before we prove this theorem, we formulate a useful corollary.

Corollary 4.1. *For continuous* F_1 *and* F_2*, under assumptions A1,A2' and A3', we have*

$$\{\sqrt{n}(F_n(t_1,t_2)-F(t_1,t_2)) : (t_1,t_2)\in\mathbb{R}^2\} \xrightarrow{d} L\circ\alpha_1 \ \text{ in } \ l^\infty(K),$$

where $L\circ\alpha_1$ *is a mean-zero Gaussian process with covariance*

$$Cov(L\circ\alpha_1(t_1,t_2), L\circ\alpha_1(s_1,s_2)) = \int_{\mathbb{R}^3}\left(\psi_1(y_1,y_3)+\frac{1_{\{y_1\le t_1,y_2\le t_2\}}}{(A+B)(y_1+y_2)}\right)\left(\psi_2(y_1,y_3)+\frac{1_{\{y_1\le s_1,y_2\le s_2\}}}{(A+B)(y_1+y_2)}\right) H_3(dy_1,dy_2,dy_3) + \int_{\mathbb{R}^2}\psi_1(y_1,y_3)\psi_2(y_1,y_3)\tilde{H}_2^2(dy_1,dy_3),$$

where

$$\psi_1(y_1,y_3) = \int_{-\infty}^{t_1}\int_{-\infty}^{t_2}\frac{1}{(A+B)^2(x_1+x_2)}\Big[\int_{x_1+x_2}^{\infty}\frac{1}{F_1(x)}\left\{-\frac{1_{\{y_1>x\}}}{C(y_1)}+\int_{(x,\infty)}\frac{1_{\{y_1\le y\le y_3\}}}{C^2(y)}F_1^*(dy)\right\}G^*(dx) - \frac{1_{\{y_3\ge x_1+x_2\}}}{F_1(y_3)}\Big]H_2^1(dx_1,dx_2)$$

and

$$\psi_2(y_1,y_3) = \int_{-\infty}^{s_1}\int_{-\infty}^{s_2}\frac{1}{(A+B)^2(x_1+x_2)}\Big[\int_{x_1+x_2}^{\infty}\frac{1}{F_1(x)}\left\{-\frac{1_{\{y_1>x\}}}{C(y_1)}+\int_{(x,\infty)}\frac{1_{\{y_1\le y\le y_3\}}}{C^2(y)}F_1^*(dy)\right\}G^*(dx) - \frac{1_{\{y_3\ge x_1+x_2\}}}{F_1(y_3)}\Big]H_2^1(dx_1,dx_2).$$

Proof.
The class of indicator functions is a VC-class. Taking φ equal to the indicator of a rectangle $(-\infty,t_1]\times(-\infty,t_2]$ completes the proof.

$\boxtimes$

Proof of Theorem 4.1. At first we deal with $\hat{I}_n(\varphi)$. Since F_1 is continuous, $\Psi_x(y) = -F_1(x)1_{\{y>x\}}$. We set

$$
\begin{aligned}
\mathcal{G} &= \Big\{g : g(w_1, w_2, w_3, \delta) = \varphi(w_1, w_2)\frac{\delta}{(A+B)(w_1+w_2)} \\
&\quad - \int \frac{\varphi(x_1, x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \left[\frac{1_{\{w_1 > x\}}}{C(w_1)} - \int_{(x,\infty)} \frac{F_1^*(dy)}{C(y)}\right] G^*(dx) H_2^1(dx_1, dx_2) \\
&\quad + \int \frac{\varphi(x_1, x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \left[\frac{1_{\{w_1 \le y \le w_3\}}}{C^2(y)} - \frac{1}{C(y)}\right] F_1^*(dy) G^*(dx) H_2^1(dx_1, dx_2) \\
&\quad + \int \frac{\varphi(x_1, x_2)}{(A+B)^2(x_1+x_2)} \left[\int_{x_1+x_2}^{\infty} \frac{G^*(dx)}{F_1(x)} - \frac{1_{\{w_3 \ge x_1+x_2\}}}{F_1(w_3)}\right] H_2^1(dx_1, dx_2) \text{ with } \varphi \in K\Big\}.
\end{aligned}
$$

To deal with $\mathcal{G}$ we write it as follows:

$$
\begin{aligned}
\mathcal{G} &= \Big\{g : g(w_1, w_2, w_3, \delta) = \varphi(w_1, w_2)\frac{\delta}{(A+B)(w_1+w_2)} \\
&\quad - \int \frac{\varphi(x_1, x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \frac{1_{\{w_1 > x\}}}{C(w_1)} G^*(dx) H_2^1(dx_1, dx_2) \\
&\quad + \int \frac{\varphi(x_1, x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{1_{\{w_1 \le y \le w_3\}}}{C^2(y)} F_1^*(dy) G^*(dx) H_2^1(dx_1, dx_2) \\
&\quad + \int \frac{\varphi(x_1, x_2)}{(A+B)^2(x_1+x_2)} \left[\int_{x_1+x_2}^{\infty} \frac{G^*(dx)}{F_1(x)} - \frac{1_{\{w_3 \ge x_1+x_2\}}}{F_1(w_3)}\right] H_2^1(dx_1, dx_2) \text{ with } \varphi \in K\Big\}.
\end{aligned}
$$

To prove the theorem, we need to show that the class of functions $\mathcal{G}$ is a Donsker class and, according to van der Vaart and Wellner (1996), it is sufficient to prove the uniform entropy condition (2.5.1). For this we must bound the covering numbers for the class $\mathcal{G}$. For this we write $\mathcal{G}$ as a subset of four classes for which the covering numbers are easier to bound. Actually

$$
\mathcal{G} \subset \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}_4,
$$

where

$$
\begin{aligned}
\mathcal{G}_1 &= \Big\{g : g(w_1, w_2, \delta) = \varphi(w_1, w_2)\frac{\delta}{(A+B)(w_1+w_2)} \text{ with } \varphi \in K\Big\} \\
\mathcal{G}_2 &= \Big\{g : g(w_1) = -\frac{1}{C(w_1)} \int \frac{\varphi(x_1, x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1_{\{w_1 > x\}}}{F_1(x)} G^*(dx) H_2^1(dx_1, dx_2) \text{ with } \varphi \in K\Big\} \\
\mathcal{G}_3 &= \Big\{g : g(w_1, w_3) = \int \frac{\varphi(x_1, x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{1_{\{w_1 \le y \le w_3\}}}{C^2(y)} F_1^*(dy) G^*(dx) H_2^1(dx_1, dx_2) \\
&\qquad \text{with } \varphi \in K\Big\}
\end{aligned}
$$

and

$$
\mathcal{G}_4 = \Big\{g : g(w_3) = \int \frac{\varphi(x_1, x_2)}{(A+B)^2(x_1+x_2)} \left[\int_{x_1+x_2}^{\infty} \frac{G^*(dx)}{F_1(x)} - \frac{1_{\{w_3 \ge x_1+x_2\}}}{F_1(w_3)}\right] H_2^1(dx_1, dx_2) \text{ with } \varphi \in K\Big\}.
$$

In Lemmas 4.1, 4.2, 4.5 and 4.3 we prove that $\mathcal{G}_i$, for $i = 1, 2, 3, 4$, are VC-classes with envelope functions g_1, g_2, g_3 and g_4, respectively and

$$logN(\varepsilon||g_i||, \mathcal{G}_i, L_2(Q)) \leq k_i \left(\frac{1}{\varepsilon}\right)^{2-2/v_i}$$

with constants $k_i \geq 0$ and $v_i \geq 2$, for $i = 1, 2, 3, 4$.

Then for a probability measure Q on $\mathbb{R}^3 \times \{0, 1\}$ and every $\varepsilon \in (0, 1)$ we have

$$\begin{aligned}
logN(\varepsilon||G||, \mathcal{G}, L_2(Q)) &\leq logN(\varepsilon||g_1 + g_2 + g_3 + g_4||, \mathcal{G}, L_2(Q)) \\
&\leq logN(\frac{\varepsilon}{2}||g_1 + g_2||, \mathcal{G}_1 + \mathcal{G}_2, L_2(Q)) + logN(\frac{\varepsilon}{2}||g_3 + g_4||, \mathcal{G}_3 + \mathcal{G}_4, L_2(Q)) \\
&\leq logN(\frac{\varepsilon}{4}||g_1||, \mathcal{G}_1, L_2(Q)) + logN(\frac{\varepsilon}{4}||g_2||, \mathcal{G}_2, L_2(Q)) \\
&\quad + logN(\frac{\varepsilon}{4}||g_3||, \mathcal{G}_3, L_2(Q)) + logN(\frac{\varepsilon}{4}||g_4||, \mathcal{G}_4, L_2(Q)) \\
&\leq k_1 4^{2-2/v_1} \left(\frac{1}{\varepsilon}\right)^{2-2/v_1} + k_2 4^{2-2/v_2} \left(\frac{1}{\varepsilon}\right)^{2-2/v_2} + k_3 4^{2-2/v_3} \left(\frac{1}{\varepsilon}\right)^{2-2/v_3} \\
&\quad + k_4 4^{2-2/v_4} \left(\frac{1}{\varepsilon}\right)^{2-2/v_4} \\
&\leq K \left(\left(\frac{1}{\varepsilon}\right)^{2-2/v_1} + \left(\frac{1}{\varepsilon}\right)^{2-2/v_2} + \left(\frac{1}{\varepsilon}\right)^{2-2/v_3} + \left(\frac{1}{\varepsilon}\right)^{2-2/v_4} \right) \\
&\leq 4K \left(\frac{1}{\varepsilon}\right)^{v},
\end{aligned}$$

where

$$v = max\{2 - 2/v_1, 2 - 2/v_2, 2 - 2/v_3, 2 - 2/v_4\} \in [1, 2)$$

and

$$K = max\{k_1 4^{2-2/v_1}, k_2 4^{2-2/v_2}, k_3 4^{2-2/v_3}, k_4 4^{2-2/v_4}\}.$$

Hence

$$\sup_Q \sqrt{logN(\varepsilon||G||, \mathcal{G}, L_2(Q))} \leq \sqrt{4K} \varepsilon^{-v/2}$$

and, since $v/2 \in [1/2, 1)$, we get that

$$\int_0^1 \sup_Q \sqrt{logN(\varepsilon||G||, \mathcal{G}, L_2(Q))} d\varepsilon \leq \sqrt{4K} \frac{1}{-v/2 + 1}$$

is finite. This proves condition (2.5.1) from van der Vaart and Wellner (1996).

In Lemma 4.6 we prove that $\sqrt{n}\hat{R}_n(\varphi)$ goes to zero uniformly in φ.

The central limit theorem and (1.8) yield, that the limit process is a mean-zero Gaussian process, which completes the proof.

$\boxtimes$

Lemma 4.1. *For the class of functions*

$$\mathcal{G}_1 = \left\{ g : g(w_1, w_2, \delta) = \varphi(w_1, w_2) \frac{\delta}{(A+B)(w_1+w_2)} \text{ with } \varphi \in K \right\}$$

with envelope function

$$g_1(w_1, w_2, \delta) = \varphi_0(w_1, w_2) \frac{\delta}{(A+B)(w_1+w_2)}$$

we have for every discrete probability measure Q on $\mathbb{R}^2 \times \{0,1\}$ and $\varepsilon \in (0,1)$

$$logN(\varepsilon||g_1||, \mathcal{G}_1, L_2(Q)) \leq k_1 \left(\frac{1}{\varepsilon}\right)^{2-2/v_1}$$

with constants k_1 and $v_1 \geq 2$.

Proof.
Since K is a VC-class, then according to Lemma 2.6.18 vi (van der Vaart and Wellner (1996)) $\mathcal{G}_1 = \{K \cdot f\}$ with $f(w_1, w_2, \delta) = \frac{\delta}{(A+B)(w_1+w_2)}$, is also a VC-class. For each function in this class we have

$$|\varphi(w_1, w_2)| \frac{\delta}{(A+B)(w_1+w_2)} \leq \varphi_0(w_1, w_2) \frac{\delta}{(A+B)(w_1+w_2)} =: g_1(w_1, w_2, \delta).$$

Hence g_1 is the envelope function for this class. Furthermore

$$||g_1||^2 = \int g_1^2(w_1, w_2, \delta) H(dw_1, dw_2, d\delta) = \int \frac{\varphi_0^2(w_1, w_2)}{(A+B)^2(w_1+w_2)} H_2^1(dw_1, dw_2)$$

is, according to our assumption, finite.
Then Corollary 2.6.12 in van der Vaart and Wellner (1996) yields

$$logN(\varepsilon||g_1||, \mathcal{G}_1, L_2(Q)) \leq k_1 \left(\frac{1}{\varepsilon}\right)^{2-2/v_1},$$

where k_1 is a constant and v_1 is a VC-Index of class $\mathcal{G}_1$ greater than or equal to 2.

$\boxtimes$

Lemma 4.2. *For the class of functions*

$$\mathcal{G}_2 = \left\{ g : g(w_1) = -\frac{1}{C(w_1)} \int \frac{\varphi(x_1, x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1_{\{w_1 > x\}}}{F_1(x)} G^*(dx) H_2^1(dx_1, dx_2) \text{ with } \varphi \in K \right\}$$

with envelope function

$$g_2(w_1) = \frac{2}{C(w_1)} \int \frac{\varphi_0(x_1, x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1_{\{w_1 > x\}}}{F_1(x)} G^*(dx) H_2^1(dx_1, dx_2)$$

we have for every discrete probability measure Q on $\mathbb{R}$ and $\varepsilon \in (0,1)$

$$logN(\varepsilon||g_2||, \mathcal{G}_2, L_2(Q)) \leq k_2 \left(\frac{1}{\varepsilon}\right)^{2-2/v_2}$$

with constants k_2 and $v_2 \geq 2$.

Proof.
To deal with $\mathcal{G}_2$ we split the function φ into its positive and negative part. Then we have

$$\mathcal{G}_2 \subset \Big\{ g : g(w_1) = -\frac{1}{C(w_1)} \int \frac{\varphi^+(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1_{\{w_1>x\}}}{F_1(x)} G^*(dx) H_2^1(dx_1,dx_2) \text{ with } \varphi \in K \Big\}$$
$$+ \Big\{ g : g(w_1) = \frac{1}{C(w_1)} \int \frac{\varphi^-(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1_{\{w_1>x\}}}{F_1(x)} G^*(dx) H_2^1(dx_1,dx_2) \text{ with } \varphi \in K \Big\} =: \mathcal{F}_1 + \mathcal{F}_2$$

As to $\mathcal{F}_1$, we set

$$\tilde{\mathcal{F}}_1 = \Big\{ g : g(w_1) = -\int \frac{\varphi^+(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1_{\{w_1>x\}}}{F_1(x)} G^*(dx) H_2^1(dx_1,dx_2) \text{ with } \varphi \in K \Big\}$$

and

$$\mathcal{F}_1^* = -\tilde{\mathcal{F}}_1.$$

The class $\mathcal{F}_1^*$ is a class of monotone increasing functions, and because of that a VC-major class with the following bound for every function in this class:

$$\begin{aligned} &\int \frac{\varphi^+(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1_{\{w_1>x\}}}{F_1(x)} G^*(dx) H_2^1(dx_1,dx_2) \\ &\leq \int \frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1_{\{w_1>x\}} G^*(dx)}{F_1(x)} H_2^1(dx_1,dx_2) \end{aligned} \tag{4.1}$$

From Lemma 2.6.18 iv (van der Vaart and Wellner (1996)) it follows that $\tilde{\mathcal{F}}_1 = -\mathcal{F}_1^*$ is also a VC-major class.
Furthermore,

$$\mathcal{F}_1 = \left\{ \frac{1}{C(w_1)} \cdot f : f \in \tilde{\mathcal{F}}_1 \right\}.$$

Then, according to Lemma 2.6.18 vi (van der Vaart and Wellner (1996)), $\mathcal{F}_1$ is also a VC-major class. Finally, since the integral in (4.1) is bounded, we have that $\tilde{\mathcal{F}}_1$ and then also $\mathcal{F}_1$, as a bounded VC-major classes, are VC-hulls with envelope function for the class $\mathcal{F}_1$

$$\begin{aligned} f_1(w_1) &:= \frac{1}{C(w_1)} \int \frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1_{\{w_1\geq x\}}}{F_1(x)} G^*(dx) H_2^1(dx_1,dx_2) \\ &= \frac{1}{C(w_1)} \int \frac{\varphi_0(x_1,x_2)\alpha^{-1}}{(A+B)(x_1+x_2)} \int_{x_1+x_2}^{\infty} 1_{\{w_1\geq x\}} G(dx) F(dx_1,dx_2). \end{aligned}$$

By assumption A1 we have

$$\frac{1-F_1(x)}{1-G(x^-)} \leq \int_{[x,\infty)} \frac{F_1(dy)}{1-G(y^-)} \leq M < \infty.$$

Hence

$$\frac{1}{F_1(x_1+x_2)} = \frac{1-F_1(x_1+x_2)}{F_1(x_1+x_2)} + 1 \le M\frac{1-G(x_1+x_2^-)}{F_1(x_1+x_2)} + 1 = M\alpha\frac{(A+B)(x_1+x_2)}{F_1(x_1+x_2)} + 1.$$

Therefore for $||f_1||$ we have the following bound

$$\begin{aligned}
\int f_1^2(w_1)F_1^*(dw_1) &\le \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-2}}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \int_{(x,\infty)} \frac{F_1^*(dw_1)}{C^2(w_1)} G(dx)F(dx_1,dx_2) \\
&= \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-2}}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \int_{(x,\infty)} \frac{F_1(dw_1)}{(1-G(w_1^-))F_1^2(w_1)} G(dx)F(dx_1,dx_2) \\
&\le \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-2}}{(A+B)^2(x_1+x_2)F_1(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} \int_{(x_1+x_2,\infty)} \frac{F_1(dw_1)}{1-G(w_1^-)} F(dx_1,dx_2) \\
&\le M\int \frac{\varphi_0^2(x_1,x_2)\alpha^{-1}}{(A+B)(x_1+x_2)F_1(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} \int_{(x_1+x_2,\infty)} \frac{F_1(dw_1)}{1-G(w_1^-)} F(dx_1,dx_2) \\
&\quad + \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-2}}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} \int_{(x_1+x_2,\infty)} \frac{F_1(dw_1)}{1-G(w_1^-)} F(dx_1,dx_2) \\
&\le M^2 \int \frac{\varphi_0^2(x_1,x_2)}{F_1^2(x_1+x_2)} F(dx_1,dx_2) + M\int \frac{\varphi_0^2(x_1,x_2)\alpha^{-1}}{(A+B)(x_1+x_2)F_1(x_1+x_2)} F(dx_1,dx_2).
\end{aligned}$$

According to our assumptions the integrals are finite. Hence $\frac{1}{2}||f_1|| < \infty$. Corollary 2.6.12 (van der Vaart and Wellner (1996)) yields a bound for the uniform entropy:

$$logN(\frac{\varepsilon}{2}||f_1||, \mathcal{F}_1, L_2(Q)) \le c_1 2^{2-2/s_1} \left(\frac{1}{\varepsilon}\right)^{2-2/s_1},$$

where $s_1 \ge 2$ is a VC-Index of $\mathcal{F}_1$ and $c_1 = const$.

Next we deal with $\mathcal{F}_2$. Set

$$\tilde{\mathcal{F}}_2 := \left\{ g : g(w_1) = \int \frac{\varphi^-(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1_{\{w_1>x\}}}{F_1(x)} G^*(dx) H_2^1(dx_1,dx_2) \text{ with } \varphi \in K \right\}$$

and

$$\mathcal{F}_2 = \left\{ \frac{1}{C(w_1)} \cdot f : f \in \tilde{\mathcal{F}}_2 \right\}.$$

The class $\tilde{\mathcal{F}}_2$ is a class of monotone increasing functions, and because of that a VC-major class of functions. For every function from this class we have

$$\begin{aligned}
&\int \frac{\varphi^-(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1_{\{w_1>x\}}}{F_1(x)} G^*(dx) H_2^1(dx_1,dx_2) \\
&\le \int \frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1_{\{w_1>x\}}}{F_1(x)} G^*(dx) H_2^1(dx_1,dx_2) < \infty.
\end{aligned}$$

As before, according to Lemma 2.6.18 vi (van der Vaart and Wellner (1996)), classes $\tilde{\mathcal{F}}_2$ and $\mathcal{F}_2$ are VC-hulls. Since, as in the case of $\mathcal{F}_1$, the envelope function for $\mathcal{F}_2$ equals f_1, where

$$f_1(w_1) = \frac{1}{C(w_1)} \int \frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1_{\{w_1 \geq x\}}}{F_1(x)} G^*(dx) H_2^1(dx_1,dx_2),$$

the L_2-norm $\frac{1}{2}||f_1||$ is finite. Then we have the following bound for the uniform entropy:

$$logN(\frac{\varepsilon}{2}||f_1||, \mathcal{F}_2, L_2(Q)) \leq c_2 2^{2-2/s_2} \left(\frac{1}{\varepsilon}\right)^{2-2/s_2},$$

where $s_2 \geq 2$ is a VC-Index of $\mathcal{F}_2$ and $c_2 = const$.

According to Proposition 5.1.13 iv in de la Peña and Giné (1999) the class $\mathcal{G}_2$ is, as a subset of $\mathcal{F}_1 + \mathcal{F}_2$, a VC-hull with envelope function $g_2 := 2f_1$. Therefore,

$$\begin{aligned} &logN(\varepsilon||g_2||, \mathcal{G}_2, L_2(Q)) \leq logN(\frac{\varepsilon}{2}||f_1||, \mathcal{F}_1, L_2(Q)) + logN(\frac{\varepsilon}{2}||f_1||, \mathcal{F}_2, L_2(Q)) \\ &\leq c_1 2^{2-2/s_1} \left(\frac{1}{\varepsilon}\right)^{2-2/s_1} + c_2 2^{2-2/s_2} \left(\frac{1}{\varepsilon}\right)^{2-2/s_2} \leq k_2 \left(\frac{1}{\varepsilon}\right)^{2-2/v_2}, \end{aligned}$$

where

$$v_2 = max\{s_1, s_2\} \geq 2$$

and

$$k_2 = 2max\{c_1 2^{2-2/s_1}, c_2 2^{2-2/s_2}\}.$$

⊠

Lemma 4.3. *For the class of functions*

$$\mathcal{G}_4 = \left\{ g : g(w_3) = \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \left[\int_{x_1+x_2}^{\infty} \frac{G^*(dx)}{F_1(x)} - \frac{1_{\{w_3 \geq x_1+x_2\}}}{F_1(w_3)} \right] H_2^1(dx_1,dx_2) \text{ with } \varphi \in K \right\}$$

with envelope function

$$g_4(w_3) = \int \frac{\varphi_0(x_1,x_2)}{(A+B)(x_1+x_2)} H_2^1(dx_1,dx_2) + 2 \int \frac{\varphi_0(x_1,x_2) 1_{\{w_3 \geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)F_1(w_3)} H_2^1(dx_1,dx_2)$$

we have for every discrete probability measure Q on $\mathbb{R}$ with and $\varepsilon \in (0,1)$

$$logN(\varepsilon||g_4||, \mathcal{G}_1, L_2(Q)) \leq k_4 \left(\frac{1}{\varepsilon}\right)^{2-2/v_4}$$

with constants k_4 and $v_4 \geq 2$.

Proof.
As before, we write $\mathcal{G}_4$ as a subset of two classes of functions

$$\mathcal{G}_4 \subset \Big\{ g : g = \int \frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)} H_2^1(dx_1,dx_2) \ \text{ with } \ \varphi \in K \Big\}$$
$$+ \Big\{ g : g(w_3) = -\frac{1}{F_1(w_3)} \int \frac{\varphi(x_1,x_2) 1_{\{w_3 \geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)} H_2^1(dx_1,dx_2) \ \text{ with } \ \varphi \in K \Big\} =: \mathcal{F}_3 + \mathcal{F}_4$$

Since $\mathcal{F}_3$ is a class of constants and K is a VC-class, $\mathcal{F}_3$ is also VC-class bounded from above by

$$\int \frac{\varphi_0(x_1,x_2)}{(A+B)(x_1+x_2)} H_2^1(dx_1,dx_2) =: f_3.$$

The function f_3 is then the envelope function for the class $\mathcal{F}_3$. Since $||f_3||^2$ is, by the assumptions, finite we have

$$logN(\varepsilon ||f_3||, \mathcal{F}_3, L_2(Q)) \leq d_1 \left(\frac{1}{\varepsilon} \right)^{2-2/l_1},$$

where $d_1 = const$ and $l_1 \geq 2$.
To deal with $\mathcal{F}_4$ we need to write φ as a sum of φ^+ and $-\varphi^-$, where φ^+ and φ^- are the positive and negative part of φ, respectively.

Similarly to the proof of Lemma 4.2, the sets of functions

$$\mathcal{F}_4^1 = \Big\{ g : g(w_3) = -\int \frac{\varphi^+(x_1,x_2) 1_{\{w_3 \geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)} H_2^1(dx_1,dx_2) \ \text{ with } \ \varphi \in K \Big\}$$

and

$$\mathcal{F}_4^2 = \Big\{ g : g(w_3) = \int \frac{\varphi^-(x_1,x_2) 1_{\{w_3 \geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)} H_2^1(dx_1,dx_2) \ \text{ with } \ \varphi \in K \Big\}$$

are VC-major classes with the same envelope function

$$\int \frac{\varphi_0(x_1,x_2) 1_{\{w_3 \geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)} H_2^1(dx_1,dx_2).$$

Since the integral is finite then

$$\mathcal{F}_4 \subset \left\{ \frac{1}{F_1(w_3)} \cdot f : f \in \mathcal{F}_4^1 \right\} + \left\{ \frac{1}{F_1(w_3)} \cdot f : f \in \mathcal{F}_4^2 \right\}$$

is a VC-hull with envelope function

$$\begin{aligned} f_4(w_3) &:= \frac{2}{F_1(w_3)} \int \frac{\varphi_0(x_1,x_2) 1_{\{w_3 \geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)} H_2^1(dx_1,dx_2) \\ &= \frac{2}{F_1(w_3)} \int \frac{\varphi_0(x_1,x_2) 1_{\{w_3 \geq x_1+x_2\}}}{(A+B)(x_1+x_2)} F(dx_1,dx_2). \end{aligned}$$

Since $G^*(dw_3) = \alpha^{-1}F_1(w_3)G(dw_3)$,

$$\int f_4^2(w_3)G^*(dw_3) \leq 4\int \frac{\varphi_0(x_1,x_2)\alpha^{-2}}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{G(dw_1)}{F_1(w_3)}F(dx_1,dx_2)$$

is bounded. Corollary 2.6.12 in van der Vaart and Wellner (1996) yields the following bound for the entropy:

$$logN(\varepsilon||f_4||, \mathcal{F}_4, L_2(Q)) \leq d_2\left(\frac{1}{\varepsilon}\right)^{2-2/l_2},$$

where $d_2 = const$ and $l_2 \geq 2$.
Finally, according to Proposition 5.1.13 iv (de la Peña and Giné (1999)) the class $\mathcal{G}_4$ is a VC-hull with envelope function

$$g_4 = f_3 + f_4$$

and

$$\begin{aligned} logN(\varepsilon||g_4||, \mathcal{G}_4, L_2(Q)) &\leq logN(\frac{\varepsilon}{2}||f_3||, \mathcal{F}_3, L_2(Q)) + logN(\frac{\varepsilon}{2}||f_4||, \mathcal{F}_4, L_2(Q)) \\ &\leq d_1 2^{2-2/l_1}\left(\frac{1}{\varepsilon}\right)^{2-2/l_1} + d_2 2^{2-2/l_2}\left(\frac{1}{\varepsilon}\right)^{2-2/l_2} \leq k_4\left(\frac{1}{\varepsilon}\right)^{2-2/v_4}, \end{aligned}$$

where

$$k_4 = 2max\{d_1 2^{2-2/l_1}, d_2 2^{2-2/l_2}\}$$

and

$$v_4 = max\{l_1, l_2\} \geq 2.$$

$\boxtimes$

Before we may proceed with the proof for the class $\mathcal{G}_3$, we need the following Lemma.

Lemma 4.4. *Let $\tilde{F}$ be a given d.f. on the unit square, let $h : \mathbb{R}^4 \to \mathbb{R}$ be a given function and K a VC class. Then the class of functions*

$$\mathcal{G} = \left\{g : \mathbb{R}^2 \to \mathbb{R} \mid g(w_1, w_3) = \int_0^1\int_0^1 \tilde{\varphi}(x_1,x_2)h(x_1,x_2,w_1,w_3)\tilde{F}(dx_1,dx_2) \text{ where } \tilde{\varphi} \in K\right\}$$

is a VC-hull.

Proof.
According to Section 2.6.3 in van der Vaart and Wellner (1996) we need to show that every function $g \in \mathcal{G}$ is a pointwise limit of $g_m = \sum_{i=1}^m \alpha_i f_i$, where $f_i \in \mathcal{M}$, $\mathcal{M}$ is a VC class and $\sum_{i=1}^m |\alpha_i| \leq 1$.

First we will prove that the following class of functions

$$\mathcal{H} := \{f : \mathbb{R}^4 \to \mathbb{R} \mid f(x_1,x_2,w_1,w_3) = \tilde{\varphi}(x_1,x_2)h(x_1,x_2,w_1,w_3) \text{ with } \tilde{\varphi} \in K\}$$

is a VC class. For this we define the class of sets

$$\begin{aligned}\mathcal{A} &:= \{\{(t, x_1, x_2, w_1, w_3) : \tilde{\varphi}(x_1, x_2) > t\} \mid \tilde{\varphi} \in K\} = \{\{(t, x_1, x_2) : \tilde{\varphi}(x_1, x_2) > t\} \mid \tilde{\varphi} \in K\} \times \mathbb{R}^2 \\ &=: \mathcal{B} \times \mathbb{R}^2.\end{aligned}$$

Since $\mathcal{B}$ is a VC class of sets, then $\mathcal{A}$ is, according to Proposition 5.1.13 in de la Peña and Giné (1999), as well a VC class of sets. Hence the class of functions

$$\tilde{K} := \{\tilde{\tilde{\varphi}} : \mathbb{R}^4 \to \mathbb{R} \mid \tilde{\tilde{\varphi}}(x_1, x_2, w_1, w_3) = \tilde{\varphi}(x_1, x_2) \text{ with } \tilde{\varphi} \in K\}$$

is a VC class. Finally, since

$$\mathcal{H} = \tilde{K} \cdot h = \{f : \mathbb{R}^4 \to \mathbb{R} \mid f(x_1, x_2, w_1, w_3) = \tilde{\tilde{\varphi}}(x_1, x_2, w_1, w_3) h(x_1, x_2, w_1, w_3) \text{ with } \tilde{\tilde{\varphi}} \in \tilde{K}\},$$

the class of functions $\mathcal{H}$ is, according to Lemma 2.6.18 (vi) in van der Vaart and Wellner (1996), a VC class. Hence

$$\mathcal{D} := \{\{(t, x_1, x_2, w_1, w_3) : f(x_1, x_2, w_1, w_3) > t\} \mid f \in \mathcal{H}\}.$$

is a VC class of sets.
Now, for every $x_1^0, x_2^0 \in \mathbb{R}$ and $f \in \mathcal{H}$ set

$$\begin{aligned}D^f_{x_1^0, x_2^0} &:= \{(t, x_1^0, x_2^0, w_1, w_3) : f(x_1^0, x_2^0, w_1, w_3) > t\} \\ &= \{(t, x_1, x_2, w_1, w_3) : f(x_1, x_2, w_1, w_3) > t\} \cap \{\mathbb{R} \times \{x_1^0\} \times \{x_2^0\} \times \mathbb{R}^2\} \\ &=: D^f \cap E_{x_1^0, x_2^0}.\end{aligned}$$

Since

$$\{D^f : f \in \mathcal{H}\} = \mathcal{D}$$

and

$$\{E_{x_1^0, x_2^0} : x_1^0, x_2^0 \in \mathbb{R}\}$$

are VC classes of sets, then, according to Proposition 5.1.13 in de la Peña and Giné (1999), the class of sets

$$\mathcal{C} := \{D^f_{x_1^0, x_2^0} : f \in \mathcal{H}, x_1^0, x_2^0 \in \mathbb{R}\}$$

is a VC class. Hence the class of functions

$$\begin{aligned}\mathcal{M} &:= \{f(x_1^0, x_2^0, \cdot, \cdot) \mid f \in \mathcal{H} \text{ and } x_1^0, x_2^0 \in \mathbb{R}\} \\ &= \{f(x_1^0, x_2^0, \cdot, \cdot) \mid f(x_1^0, x_2^0, w_1, w_3) = \tilde{\varphi}(x_1^0, x_2^0) h(x_1^0, x_2^0, w_1, w_3), \tilde{\varphi} \in K \text{ and } x_1^0, x_2^0 \in \mathbb{R}\}\end{aligned}$$

is a VC class.

Next let $0 = x_{10} < x_{11} < ... < x_{1m_1} = 1$ and $0 = x_{20} < x_{21} < ... < x_{2m_2} = 1$ be finite grids of the interval $[0, 1]$,

$$f_{i,j}(w_1, w_3) := \tilde{\varphi}(x_{1i}, x_{2j})h(x_{1i}, x_{2j}, w_1, w_3)$$

and

$$\alpha_{ij} := \tilde{F}(x_{1i}, x_{2j}) - \tilde{F}(x_{1i}, x_{2j-1}) - \tilde{F}(x_{1i-1}, x_{2j}) + \tilde{F}(x_{1i-1}, x_{2j-1}).$$

According to the definition of $\mathcal{M}$, we have that for every i, j

$$f_{i,j}(w_1, w_3) = \tilde{\varphi}(x_{1i}, x_{2j})h(x_{1i}, x_{2j}, w_1, w_3) = f(x_{1i}, x_{2j}, w_1, w_3) \in \mathcal{M}.$$

As to α_{ij}, since $\tilde{F}$ is a two dimentional d.f., we have that $\alpha_{ij} \geq 0$. Furthermore, since $\tilde{F}(x_{10}, x_{2j}) = 0$, we obtain

$$\sum_{i=1}^{m_1} \alpha_{ij} = \tilde{F}(x_{1m_1}, x_{2j}) - \tilde{F}(x_{1m_1}, x_{2j-1}).$$

Hence

$$\sum_{j=1}^{m_2}\sum_{i=1}^{m_1} \alpha_{ij} = \sum_{j=1}^{m_2}(\tilde{F}(x_{1m_1}, x_{2j}) - \tilde{F}(x_{1m_1}, x_{2j-1})) = \tilde{F}(x_{1m_1}, x_{2m_2}) \leq 1.$$

Finally if partitions $(x_{1i})_i$ and $(x_{2j})_j$ get finer and finer

$$\sum_{j=1}^{m_2}\sum_{i=1}^{m_1} f_{i,j}(w_1, w_3)(\tilde{F}(x_{1i}, x_{2j}) - \tilde{F}(x_{1i}, x_{2j-1}) - \tilde{F}(x_{1i-1}, x_{2j}) + \tilde{F}(x_{1i-1}, x_{2j-1}))$$

goes to

$$\int_{[0,1]^2} \tilde{\varphi}(x_1, x_2)h(x_1, x_2, w_1, w_3)\tilde{F}(dx_1, dx_2) = g(w_1, w_3)$$

and this completes the proof.

$\boxtimes$

Lemma 4.5. *For the class of functions*

$$\begin{aligned}\mathcal{G}_3 = \Big\{ g : g(w_1, w_3) \;\; &= \;\; \int \frac{\varphi(x_1, x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \\ &\times \int_{(x,\infty)} \frac{1_{\{w_1 \leq y \leq w_3\}}}{C^2(y)} F_1^*(dy) G^*(dx) H_2^1(dx_1, dx_2) \;\; \textit{with} \;\; \varphi \in K \Big\}\end{aligned}$$

with envelope function

$$g_3(w_1, w_3) = \int \frac{\varphi_0(x_1, x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{1_{\{w_1 \leq y \leq w_3\}} F_1^*(dy)}{C^2(y)} G^*(dx) H_2^1(dx_1, dx_2)$$

we have for every discrete probability measure Q on $\mathbb{R}^2$ and $\varepsilon \in (0,1)$

$$logN(\varepsilon||g_3||, \mathcal{G}_3, L_2(Q)) \leq k_3 \left(\frac{1}{\varepsilon}\right)^{2-2/v_3}$$

with constants k_3 and $v_3 \geq 2$.

Proof.
Since $dH_2^1 = (A+B)dF$, we have

$$\mathcal{G}_3 = \Big\{ g : g(w_1,w_3) = \int \frac{\varphi(x_1,x_2)}{(A+B)(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \times \int_{(x,\infty)} \frac{1_{\{w_1 \leq y \leq w_3\}}}{C^2(y)} F_1^*(dy) G^*(dx) F(dx_1,dx_2) \text{ with } \varphi \in K \Big\}.$$

Set

$$h(x_1,x_2,w_1,w_3) = \frac{1}{(A+B)(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{1_{\{w_1 \leq y \leq w_3\}}}{C^2(y)} F_1^*(dy) G^*(dx).$$

Then

$$\mathcal{G}_3 = \Big\{ g : g(w_1,w_3) = \int \varphi(x_1,x_2) h(x_1,x_2,w_1,w_3) F(dx_1,dx_2) \text{ with } \varphi \in K \Big\}.$$

Moreover, since the distribution functions of U_1 and U_2, F_1 and F_2, are continuous, $F(x_1,x_2) = \tilde{F}(F_1(x_1), F_2(x_2))$, where $\tilde{F}$ is a d.f. on the unit square. Finally, by Lemma 4.4, we have that $\mathcal{G}_3$ is a VC-hull.

The envelope function for this class equals

$$\begin{aligned} g_3(w_1,w_3) &= \int \frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{1_{\{w_1 \leq y \leq w_3\}} F_1^*(dy)}{C^2(y)} G^*(dx) H_2^1(dx_1,dx_2) \\ &= \int \frac{\varphi_0(x_1,x_2)\alpha^{-1}}{(A+B)(x_1+x_2)} \int_{x_1+x_2}^{\infty} \int_{(x,\infty)} \frac{1_{\{w_1 \leq y \leq w_3\}} F_1(dy)}{(1-G(y^-))F_1^2(y)} G(dx) F(dx_1,dx_2). \end{aligned}$$

Since $H_2^2(dw_1,dw_3) = \alpha^{-1} F_1(dw_1) G(dw_3)$, we get

$$||g_3||^2 = \int g_3^2 H_2^2(dw_1,dw_3) = \alpha^{-1} \int g_3^2 F_1(dw_1) G(dw_3)$$

and , since by assumption A1, $1 - F_1(y) \leq M(1 - G(y^-))$, we obtain

$$\begin{aligned} ||g_3||^2 &\leq \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-4}}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \int_{(x,\infty)} \int_{\{w_1 \leq y \leq w_3\}} F_1(dw_1) G(dw_3) \frac{F_1(dy)}{(1-G(y^-))^2 F_1^4(y)} G(dx) F(dx_1,dx_2) \\ &\leq \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-4}}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \int_{(x,\infty)} \frac{F_1(dy)}{(1-G(y^-))F_1^3(y)} G(dx) F(dx_1,dx_2) \end{aligned}$$

$$
\begin{aligned}
&= \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-4}}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \int_{(x,\infty)} \frac{F_1(dy)}{(1-G(y^-))F_1^2(y)} \left(\frac{1-F_1(y)}{F_1(y)} + 1 \right) G(dx)F(dx_1,dx_2) \\
&\leq \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-4}}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \int_{(x,\infty)} \frac{F_1(dy)}{(1-G(y^-))F_1^2(y)} \left(\frac{M(1-G(y^-))}{F_1(y)} + 1 \right) G(dx)F(dx_1,dx_2) \\
&\leq M \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-4}}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \int_{(x,\infty)} \frac{F_1(dy)}{F_1^3(y)} G(dx)F(dx_1,dx_2) \\
&\quad + \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-4}}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \int_{(x,\infty)} \frac{F_1(dy)}{(1-G(y^-))F_1^2(y)} G(dx)F(dx_1,dx_2) \\
&\leq M \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-4}}{(A+B)^2(x_1+x_2)F_1(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} F(dx_1,dx_2) \\
&\quad + \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-4}}{(A+B)^2(x_1+x_2)F_1(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{F_1(dy)}{1-G(y^-)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} F(dx_1,dx_2) \\
&\leq M^2 \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-3}}{(A+B)(x_1+x_2)F_1(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} F(dx_1,dx_2) \\
&\quad + M \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-3}}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} F(dx_1,dx_2) \\
&\quad + M \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-3}}{(A+B)(x_1+x_2)F_1(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{F_1(dy)}{1-G(y^-)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} F(dx_1,dx_2) \\
&\quad + \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-4}}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{F_1(dy)}{1-G(y^-)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} F(dx_1,dx_2) \\
&\leq (2M^2+M) \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-2}}{F_1^2(x_1+x_2)} F(dx_1,dx_2) + M \int \frac{\varphi_0^2(x_1,x_2)\alpha^{-2}}{(A+B)(x_1+x_2)F_1(x_1+x_2)} F(dx_1,dx_2) < \infty.
\end{aligned}
$$

It follows that the entropy for $\mathcal{G}_3$ is bounded from above by

$$
logN(\varepsilon||g_3||, \mathcal{G}_3, L_2(Q)) \leq k_3 \left(\frac{1}{\varepsilon}\right)^{2-2/v_3},
$$

where

$$
k_3 = 4max\{d_5 4^{2-2/l_5}, d_6 4^{2-2/l_6}\}
$$

and

$$
v_3 = max\{l_5, l_6\} \geq 2.
$$

$\boxtimes$

Lemma 4.6. *Under assumptions A1, A2' and A3':*

$$
\sup_{\varphi \in K} |\hat{R}_n(\varphi)| = o_{\mathbb{P}}(n^{-1/2})
$$

Proof.
To prove that the remainder is $o_{\mathbb{P}}(n^{-1/2})$ uniformly in φ we consider the functions in Lemmas 3.15, 3.16, 3.17, 3.18 and a remainder in 3.19, separately.
We begin with the remainders from Lemmas 3.15, 3.17 and 3.18 because each of them can be bounded from above by functions containing φ_0, which are, in the same way as in this Lemmas, $o_{\mathbb{P}}(n^{-1/2})$.

The proof that the function in Lemma 3.16 is $o_{\mathbb{P}}(n^{-1/2})$ uniformly in φ is more complicated. For some of the terms in these function we need the theory of U-statistics, Hoeffding projection and Theorem 5.3.7 from de la Peña and Giné (1999).

Remainder from Lemma 3.15:

$$J_{1n} + J_{3n} = \int \varphi(x_1, x_2) \frac{[A + B - A_n - B_n]^2(x_1 + x_2)}{(A + B)^2(x_1 + x_2)(A_n + B_n)(x_1 + x_2)} H_{2n}^1(dx_1, dx_2).$$

Since

$$\begin{aligned}
&\sup_{\varphi \in K} \left| \int \varphi(x_1, x_2) \frac{[A + B - A_n - B_n]^2(x_1 + x_2)}{(A + B)^2(x_1 + x_2)(A_n + B_n)(x_1 + x_2)} H_{2n}^1(dx_1, dx_2) \right| \\
&\leq \int \sup_{\varphi \in K} |\varphi(x_1, x_2)| \frac{[A + B - A_n - B_n]^2(x_1 + x_2)}{(A + B)^2(x_1 + x_2)(A_n + B_n)(x_1 + x_2)} H_{2n}^1(dx_1, dx_2) \\
&\leq \int \varphi_0(x_1, x_2) \frac{[A + B - A_n - B_n]^2(x_1 + x_2)}{(A + B)^2(x_1 + x_2)(A_n + B_n)(x_1 + x_2)} H_{2n}^1(dx_1, dx_2)
\end{aligned}$$

and the assumptions in Lemma 3.15 hold for $\varphi_0(x_1, x_2)$, then the proof that $J_{1n} + J_{3n} = o_{\mathbb{P}}(n^{-1/2})$ uniformly in φ is the same as the proof of Lemma 3.15.

Remainder from Lemma 3.17:

$$K_{2n} = -\int \frac{\varphi(x_1, x_2)}{(A + B)^2(x_1 + x_2)} \int_{x_1 + x_2}^{\infty} \frac{(F_{1n} - F_1)^2}{F_1^2 F_{1n}} dG_n^* H_2^1(dx_1, dx_2)$$

We have

$$\begin{aligned}
&\sup_{\varphi \in K} \left| -\int \frac{\varphi(x_1, x_2)}{(A + B)^2(x_1 + x_2)} \int_{x_1 + x_2}^{\infty} \frac{(F_{1n} - F_1)^2}{F_1^2 F_{1n}} dG_n^* H_2^1(dx_1, dx_2) \right| \\
&\leq \int \sup_{\varphi \in K} |\varphi(x_1, x_2)| \frac{1}{(A + B)^2(x_1 + x_2)} \int_{x_1 + x_2}^{\infty} \frac{(F_{1n} - F_1)^2}{F_1^2 F_{1n}} dG_n^* H_2^1(dx_1, dx_2) \\
&\leq \int \frac{\varphi_0(x_1, x_2)}{(A + B)^2(x_1 + x_2)} \int_{x_1 + x_2}^{\infty} \frac{(F_{1n} - F_1)^2}{F_1^2 F_{1n}} dG_n^* H_2^1(dx_1, dx_2).
\end{aligned}$$

The proof of Lemma 3.17 with the assumptions for $\varphi_0(x_1, x_2)$ yields the proof for this part.

Remainder from Lemma 3.18:

$$K_{3n} = \int \hat{\varphi}(x_1, x_2) \int_{x_1 + x_2}^{\infty} \frac{F_{1n} - F_1}{F_1^2} (dG_n^* - dG^*) H_2^1(dx_1, dx_2)$$

and

$$
\begin{aligned}
&\sup_{\varphi\in K}\left|\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{F_{1n}-F_1}{F_1^2}(dG_n^*-dG^*)H_2^1(dx_1,dx_2)\right|\\
&\int\sup_{\varphi\in K}|\varphi(x_1,x_2)|\frac{1}{(A+B)^2(x_1+x_2)}\left|\int_{x_1+x_2}^{\infty}\frac{F_{1n}-F_1}{F_1^2}(dG_n^*-dG^*)\right|H_2^1(dx_1,dx_2)\\
&\int\frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)}\left|\int_{x_1+x_2}^{\infty}\frac{F_{1n}-F_1}{F_1^2}(dG_n^*-dG^*)\right|H_2^1(dx_1,dx_2).
\end{aligned}
$$

The proof for Lemma 3.18 yields that $n^{1/2}K_{3n}$ goes to zero in probability uniformly in φ.

<u>Remainder from Lemma 3.16</u>:

$$
\begin{aligned}
J_{2n} &= \int\varphi(x_1,x_2)\frac{(A+B-A_n-B_n)(x_1+x_2)}{(A+B)^2(x_1+x_2)}(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2))\\
&= \int\varphi\frac{A+B-A_{n-1}-B_{n-1}}{(A+B)^2}(dH_{2n}^1-dH_2^1)\\
&\quad -n^{-2}\sum_{i=1}^{n}\left(\frac{\varphi(U_{1i},\tilde{U}_{2i})\delta_i}{(A+B)^2(U_{1i}+\tilde{U}_{2i})F_{1n}(Z_i)}-\int\frac{\varphi(x_1,x_2)1_{\{Z_i\geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)F_{1n}(Z_i)}H_2^1(dx_1,dx_2)\right)\\
&=: J_{2n}^a(\varphi)+J_{2n}^b(\varphi).
\end{aligned}
$$

As to $J_{2n}^b(\varphi)$

$$
\begin{aligned}
\sup_{\varphi\in K}\left|J_{2n}^b(\varphi)\right| &\leq n^{-2}\sum_{i=1}^{n}\left(\frac{\varphi_0(U_{1i},\tilde{U}_{2i})\delta_i}{(A+B)^2(U_{1i}+\tilde{U}_{2i})F_{1n}(Z_i)}+\int\frac{\varphi_0(x_1,x_2)1_{\{Z_i\geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)F_{1n}(Z_i)}H_2^1(dx_1,dx_2)\right)\\
&\leq n^{-2}\tilde{K}_1\sum_{i=1}^{n}\left(\frac{\varphi_0(U_{1i},\tilde{U}_{2i})\delta_i}{(A+B)^2(U_{1i}+\tilde{U}_{2i})F_1(Z_i)}+\int\frac{\varphi_0(x_1,x_2)1_{\{Z_i\geq x_1+x_2\}}}{(A+B)^2(x_1+x_2)F_1(Z_i)}H_2^1(dx_1,dx_2)\right)
\end{aligned}
$$

and, in the same way as for J_{2n}^b in Lemma 3.16, the expectation of the right side is bounded from above by

$$
n^{-1}2\tilde{K}_1\int\frac{\varphi_0(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)}F(dx_1,dx_2).
$$

To deal with $J_{2n}^a(\varphi)$ recall (3.6)

$$
(A+B-A_n-B_n)(x_1+x_2)=\int_{x_1+x_2}^{\infty}\frac{1}{F_1}[dG^*-dG_n^*]+\int_{x_1+x_2}^{\infty}\frac{F_{1n}-F_1}{F_1^2}dG_n^*-\int_{x_1+x_2}^{\infty}\frac{(F_{1n}-F_1)^2}{F_1^2F_{1n}}dG_n^*.
$$

We will prove that

$$
\sup_{\varphi\in K}\left|\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1}(dG_{n-1}^*-dG^*)(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2))\right| \tag{4.2}
$$

$$\sup_{\varphi \in K} \left| \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{F_{1n}-F_1}{F_1^2} dG^*_{n-1}(H^1_{2n}(dx_1,dx_2) - H^1_2(dx_1,dx_2)) \right| \tag{4.3}$$

and

$$\sup_{\varphi \in K} \left| \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{(F_{1n}-F_1)^2}{F_1^2 F_{1n}} dG^*_{n-1}(H^1_{2n}(dx_1,dx_2) - H^1_2(dx_1,dx_2)) \right| \tag{4.4}$$

are $o_{\mathbb{P}}(n^{-1/2})$.

We begin with (4.4). Since

$$\begin{aligned}
&\sup_{\varphi \in K} \left| \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{(F_{1n}-F_1)^2}{F_1^2 F_{1n}} dG^*_{n-1}(H^1_{2n}(dx_1,dx_2) - H^1_2(dx_1,dx_2)) \right| \\
&\le \sup_{\varphi \in K} \left| \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{(F_{1n}-F_1)^2}{F_1^2 F_{1n}} dG^*_{n-1} H^1_{2n}(dx_1,dx_2) \right| \\
&+ \sup_{\varphi \in K} \left| \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{(F_{1n}-F_1)^2}{F_1^2 F_{1n}} dG^*_{n-1} H^1_2(dx_1,dx_2) \right| \\
&\le \int \frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{(F_{1n}-F_1)^2}{F_1^2 F_{1n}} dG^*_{n-1} H^1_{2n}(dx_1,dx_2) \\
&+ \int \frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{(F_{1n}-F_1)^2}{F_1^2 F_{1n}} dG^*_{n-1} H^1_2(dx_1,dx_2)
\end{aligned}$$

then the proofs of (3.60) and (3.61) in Lemma 3.16 yields that (4.4) is $o_{\mathbb{P}}(n^{-1/2})$.

To deal with (4.2), set

$$\begin{aligned}
M_{1n}(\varphi) \;&:=\; \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1} (dG^*_{n-1} - dG^*)(H^1_{2n}(dx_1,dx_2) - H^1_2(dx_1,dx_2)) \\
&=\; \frac{1}{n^2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \Big(\frac{\varphi(U_{1i},\tilde{U}_{2i})\delta_i 1_{\{Z_j \ge U_{1i}+\tilde{U}_{2i}\}}}{(A+B)^2(U_{1i}+\tilde{U}_{2i})F_1(Z_j)} - \int \frac{\varphi(U_{1i},\tilde{U}_{2i})\delta_i 1_{\{u \ge U_{1i}+\tilde{U}_{2i}\}}}{(A+B)^2(U_{1i}+\tilde{U}_{2i})F_1(u)} G^*(du) \\
&-\; \int \frac{\varphi(x_1,x_2) 1_{\{Z_j \ge x_1+x_2\}}}{(A+B)^2(x_1+x_2)F_1(Z_j)} H^1_2(dx_1,dx_2) + \int \frac{\varphi(x_1,x_2) 1_{\{u \ge x_1+x_2\}}}{(A+B)^2(x_1+x_2)F_1(u)} G^*(du) H^1_2(dx_1,dx_2) \Big).
\end{aligned}$$

Define

$$\tilde{H}_3(x,y,z,\tilde{\delta}) := P(U_1 \le x, U_2 \le y, Z \le z, \delta = \tilde{\delta} | U_1 \le Z)$$

a measure on $\mathbb{R}^3 \times \{0,1\}$ and

$$\tilde{\varphi}((x_1,x_2,x_3,\delta^1),(y_1,y_2,y,\delta^2)) = \frac{\varphi(x_1,x_2)\delta^1 1_{\{y \ge x_1+x_2\}}}{(A+B)^2(x_1+x_2)F_1(y)}$$

a $\tilde{H}_3 \times \tilde{H}_3$ integrable function.

Since

$$\int \tilde{\varphi}((x_1, x_2, x_3, \delta^1), (y_1, y_2, u, \delta^2)) d\tilde{H}_3(x_1, x_2, x_3, \delta^1) = \int \frac{\varphi(x_1, x_2) 1_{\{u \geq x_1 + x_2\}}}{(A+B)^2 (x_1 + x_2) F_1(u)} H_2^1(dx_1, dx_2)$$

and

$$\int \tilde{\varphi}((x_1, x_2, x_3, \delta^1), (y_1, y_2, u, \delta^2)) d\tilde{H}_3(y_1, y_2, u, \delta^2) = \frac{\varphi(x_1, x_2)\delta^1}{(A+B)^2 (x_1 + x_2)} \int_{x_1+x_2}^{\infty} \frac{G^*(du)}{F_1(u)}$$

then M_{1n} can be written as a function of $\tilde{\varphi}$ as follows:

$$\begin{aligned} M_{1n}(\tilde{\varphi}) &= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \Big(\tilde{\varphi}((U_{1i}, \tilde{U}_{2i}, Z_i, \delta_i), (U_{1j}, \tilde{U}_{2j}, Z_j, \delta_j)) \\ &\quad - \int \tilde{\varphi}((U_{1i}, \tilde{U}_{2i}, Z_i, \delta_i), (y_1, y_2, u, \delta^2)) d\tilde{H}_3(y_1, y_2, u, \delta^2) \\ &\quad - \int \tilde{\varphi}((x_1, x_2, x_3, \delta^1), (U_{1j}, \tilde{U}_{2j}, Z_j, \delta_j)) d\tilde{H}_3(x_1, x_2, x_3, \delta^1) \\ &\quad + \int \tilde{\varphi}((x_1, x_2, x_3, \delta^1), (y_1, y_2, u, \delta^2)) d\tilde{H}_3(x_1, x_2, x_3, \delta^1) d\tilde{H}_3(y_1, y_2, u, \delta^2) \Big) \\ &\equiv \frac{1}{n^2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \pi_2 \tilde{\varphi}((U_{1i}, \tilde{U}_{2i}, Z_i, \delta_i), (U_{1j}, \tilde{U}_{2j}, Z_j, \delta_j)). \end{aligned}$$

Furthermore $\pi_2\tilde{\varphi}$ is, according to de la Peña and Giné (1999) page 137, the Hoeffding projection with $k = 2$, $m = 2$ and $S = \mathbb{R}^3 \times \{0, 1\}$. Therefore

$$\frac{n}{n-1} M_{1n}(\tilde{\varphi}) = U_n^{(2)}(\pi_2 \tilde{\varphi})$$

is a U-statistic of Hoeffding's projection.
For this we want to use Theorem 5.3.7 in de la Peña and Giné (1999) with the function $f = \tilde{\varphi}$, which, according to Remark 5.3.9 (de la Peña and Giné (1999)) do not have to be symmetric since we use a symmetrized version of U-statistic (the sum is taken over all $i \neq j$ and not only over $i < j$). To prove the assumptions we define a class of functions

$$\mathcal{H} = \{\tilde{\varphi} : \varphi \in K\},$$

which can be proved to be a VC-subgraph with envelope function

$$\tilde{\varphi}_0((x_1, x_2, x_3, \delta^1), (y_1, y_2, y, \delta^2)) = \frac{\varphi_0(x_1, x_2) 1_{\{y \geq x_1 + x_2\}} \delta^1}{(A+B)^2 (x_1 + x_2) F_1(y)}$$

and

$$\begin{aligned}
||\tilde\varphi_0||^2 &= \int \tilde\varphi_0^2((x_1,x_2,x_3,\delta^1),(y_1,y_2,y,\delta^2))d\tilde H_3(x_1,x_2,x_3,\delta^1)d\tilde H_3(y_1,y_2,y,\delta^2)\\
&= \int \frac{\varphi_0^2(x_1,x_2)}{(A+B)^4(x_1+x_2)}\int_{x_1+x_2}^\infty \frac{1}{F_1^2(y)}G^*(dy)H_2^1(dx_1,dx_2)\\
&\le \int \frac{\varphi_0^2(x_1,x_2)}{(A+B)^2(x_1+x_2)F_1(x_1+x_2)}F(dx_1,dx_2). \qquad (4.5)
\end{aligned}$$

Assumption A1 yields that $1-F_1(x)\le M(1-G(x^-))$. Together with $1-G(x^-)=\alpha(A+B)(x)$ we get

$$\begin{aligned}
(4.5) &= \int \frac{\varphi_0^2(x_1,x_2)}{(A+B)^2(x_1+x_2)}\frac{1-F_1(x_1+x_2)}{F_1(x_1+x_2)}F(dx_1,dx_2)+\int \frac{\varphi_0^2(x_1,x_2)}{(A+B)^2(x_1+x_2)}F(dx_1,dx_2)\\
&\le M\int \frac{\varphi_0^2(x_1,x_2)\alpha}{(A+B)(x_1+x_2)F_1(x_1+x_2)}F(dx_1,dx_2)+\int \frac{\varphi_0^2(x_1,x_2)}{(A+B)^2(x_1+x_2)}F(dx_1,dx_2)<\infty.
\end{aligned}$$

Theorem 5.3.7 in de la Peña and Giné (1999) yields

$$\{nU_n^{(2)}(\pi_2\tilde\varphi):\tilde\varphi\in\mathcal{H}\}\xrightarrow{d}\{\sqrt{2}K_P(\pi_2\tilde\varphi):\tilde\varphi\in\mathcal{H}\}\ \text{ in } l^\infty(\mathcal{H}),$$

where K_P is a chaos process with $P=\tilde H_3$.

Finally, by Cramér-Slutsky,

$$\sup_{\tilde\varphi\in\mathcal{H}}|\sqrt{n}M_{1n}(\tilde\varphi)|=\sup_{\tilde\varphi\in\mathcal{H}}|\sqrt{n}\frac{n-1}{n}U_n^{(2)}(\pi_2\tilde\varphi)|=\sqrt{n}\frac{n-1}{n^2}\sup_{\tilde\varphi\in\mathcal{H}}|nU_n^{(2)}(\pi_2\tilde\varphi)|\xrightarrow{P}0.$$

Now we will deal with (4.3). We set

$$\begin{aligned}
M_{2n}(\varphi) &= \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^\infty \frac{F_{1n}(x)-F_1(x)}{F_1^2(x)}G^*_{n-1}(dx)(H^1_{2n}(dx_1,dx_2)-H_2^1(dx_1,dx_2))\\
&= \frac{1}{n}\sum_{i=1}^n\Bigg(\hat\varphi(U_{1i},\tilde U_{2i})\delta_i\frac{1}{n}\sum_{j\neq i}^n\frac{F_{1n}(Z_j)-F_1(Z_j)}{F_1^2(Z_j)}1_{\{U_{1i}+\tilde U_{2i}\le Z_j\}}\\
&\quad -\int\hat\varphi(x_1,x_2)\frac{1}{n}\sum_{j\neq i}^n\frac{F_{1n}(Z_j)-F_1(Z_j)}{F_1^2(Z_j)}1_{\{x_1+x_2\le Z_j\}}H_2^1(dx_1,dx_2)\Bigg).
\end{aligned}$$

Using properties of F_{1n} as for (3.41) in Lemma 3.16, it is sufficient to prove that

$$\sup_{\varphi\in K}\Big|\int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^\infty\frac{1}{F_1^2(x)}\int\frac{\Psi_x(y)}{C(y)}(F^*_{1n}(dy)-F_1^*(dy))G^*_{n-1}(dx)$$
$$(H^1_{2n}(dx_1,dx_2)-H^1_2(dx_1,dx_2))\Big| \qquad (4.6)$$

$$\sup_{\varphi\in K}\Big|\int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^\infty\frac{1}{F_1^2(x)}\int\Psi_x(y)\frac{C_n(y)-C(y)}{C^2(y)}F_1^*(dy)G^*_{n-1}(dx)$$
$$(H^1_{2n}(dx_1,dx_2)-H^1_2(dx_1,dx_2))\Big| \qquad (4.7)$$

$$\sup_{\varphi\in K}\Big|\frac{1}{n}\sum_{i=1}^{n}\hat{\varphi}(U_{1i},\tilde{U}_{2i})\delta_i\frac{1}{n}\sum_{j\neq i}^{n}\frac{|R_{n-1}^{\neq j}(Z_j)|}{F_1^2(Z_j)}1_{\{U_{1i}+\tilde{U}_{2i}\leq Z_j\}}\Big| \tag{4.8}$$

and

$$\sup_{\varphi\in K}\Big|\int\hat{\varphi}(x_1,x_2)\frac{1}{n}\sum_{j=1}^{n}\frac{|R_{n-1}^{\neq j}(Z_j)|}{F_1^2(Z_j)}1_{\{x_1+x_2\leq Z_j\}}H_2^1(dx_1,dx_2)\Big| \tag{4.9}$$

are $o_{\mathbb{P}}(n^{-1/2})$, where $R_n(x)$ is the remainder in the linearization of F_{1n}.

At first we bound (4.6) by the sum of

$$\begin{aligned}&\sup_{\varphi\in K}\Big|\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int\frac{\Psi_x(y)}{C(y)}(F_{1n}^*(dy)-F_1^*(dy))\\&(G_{n-1}^*(dx)-G^*(dx))(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2))\Big|\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}&\sup_{\varphi\in K}\Big|\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int\frac{\Psi_x(y)}{C(y)}(F_{1n}^*(dy)-F_1^*(dy))G^*(dx)\\&(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2))\Big|\end{aligned} \tag{4.11}$$

The first term is bounded by

$$\begin{aligned}&\sup_{\varphi\in K}\Big|\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int\frac{\Psi_x(y)}{C(y)}(F_{1n}^*(dy)-F_1^*(dy))(G_{n-1}^*(dx)-G^*(dx))H_{2n}^1(dx_1,dx_2)\Big|\\&+\sup_{\varphi\in K}\Big|\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int\frac{\Psi_x(y)}{C(y)}(F_{1n}^*(dy)-F_1^*(dy))(G_{n-1}^*(dx)-G^*(dx))H_2^1(dx_1,dx_2)\Big|\\&\leq\int\frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)}\Big|\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int\frac{\Psi_x(y)}{C(y)}(F_{1n-1}^*(dy)-F_1^*(dy))(G_{n-1}^*(dx)-G^*(dx))\Big|H_{2n}^1(dx_1,dx_2)\\&+\int\frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)}\Big|\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int\frac{\Psi_x(y)}{C(y)}(F_{1n}^*(dy)-F_1^*(dy))(G_{n-1}^*(dx)-G^*(dx))\Big|H_2^1(dx_1,dx_2)\end{aligned}$$

and, in exactly the same way as for (3.50) and (3.51), it can be proved that the expectations of these terms are $o(n^{-1/2})$.

For (4.11) we need to use Theorem 5.3.7 in de la Peña and Giné (1999). Set

$$\begin{aligned}N_{2n}(\varphi)\;=\;&\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1^2(x)}\int\frac{\Psi_x(y)}{C(y)}(F_{1n}^*(dy)-F_1^*(dy))G^*(dx)\\&(H_{2n}^1(dx_1,dx_2)-H_2^1(dx_1,dx_2)),\end{aligned}$$

Since F_1 is continuous $\Psi_x(y) = -1_{\{y>x\}}F_1(x)$. Hence

$$\begin{aligned}
N_{2n}(\varphi) &= -\int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int \frac{1_{\{y>x\}}}{C(y)} (F_{1n}^*(dy) - F_1^*(dy)) G^*(dx) \\
&\quad (H_{2n}^1(dx_1,dx_2) - H_2^1(dx_1,dx_2)) \\
&= \int\int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)C(y)} \int_{x_1+x_2}^{\infty} \frac{-1_{\{y>x\}}}{F_1(x)} G^*(dx)(F_{1n}^*(dy) - F_1^*(dy)) \\
&\quad (H_{2n}^1(dx_1,dx_2) - H_2^1(dx_1,dx_2)) \\
&= \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \Big[-\frac{\varphi(U_{1i},\tilde U_{2i})\delta_i}{(A+B)^2(U_{1i}+\tilde U_{2i})C(U_{1j})} \int_{U_{1i}+\tilde U_{2i}}^{\infty} \frac{1_{\{x<U_{1j}\}}}{F_1(x)} G^*(dx) \\
&\quad + \int \frac{\varphi(U_{1i},\tilde U_{2i})\delta_i}{(A+B)^2(U_{1i}+\tilde U_{2i})C(y)} \int_{U_{1i}+\tilde U_{2i}}^{\infty} \frac{1_{\{x<y\}}}{F_1(x)} G^*(dx) F_1^*(dy) \\
&\quad + \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)C(U_{1j})} \int_{U_{1i}+\tilde U_{2i}}^{\infty} \frac{1_{\{x<U_{1j}\}}}{F_1(x)} G^*(dx) H_2^1(dx_1,dx_2) \\
&\quad - \int\int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)C(y)} \int_{x_1+x_2}^{\infty} \frac{1_{\{x<y\}}}{F_1(x)} G^*(dx) F_1^*(dy) H_2^1(dx_1,dx_2) \Big].
\end{aligned}$$

Furthermore, since $N_{2n}(\varphi)$ is a double sum, we split it into a part $N_{2n}^{i\neq j}(\varphi)$ for $i \neq j$ and $N_{2n}^{i=j}(\varphi)$ for $i = j$. Since

$$\sup_{\varphi\in K} |N_{2n}(\varphi)| \leq \sup_{\varphi\in K} |N_{2n}^{i\neq j}(\varphi)| + \sup_{\varphi\in K} |N_{2n}^{i=j}(\varphi)|$$

we can deal with each part separately.
To deal with $\sup_{\varphi\in K} |N_{2n}^{i\neq j}(\varphi)|$ we set

$$\varphi^1((x_1,x_2,x_3,\delta^1),(y_1,y_2,y_3,\delta^2)) = -\frac{\varphi(x_1,x_2)\delta^1}{(A+B)^2(x_1+x_2)C(y_1)} \int_{x_1+x_2}^{\infty} \frac{1_{\{x<y_1\}}}{F_1(x)} G^*(dx)$$

a $\tilde H_3 \times \tilde H_3$ integrable function, where

$$\tilde H_3(x,y,z,\tilde\delta) := P(U_1 \leq x, U_2 \leq y, Z \leq z, \delta = \tilde\delta | U_1 \leq Z).$$

The class of functions

$$\mathcal{H}^1 = \{\varphi^1 : \varphi \in K\}$$

is VC with envelope function

$$\varphi_0^1((x_1,x_2,x_3,\delta^1),(y_1,y_2,y_3,\delta^2)) = \frac{\varphi_0(x_1,x_2)\delta^1}{(A+B)^2(x_1+x_2)C(y_1)} \int_{x_1+x_2}^{\infty} \frac{1_{\{x<y_1\}}}{F_1(x)} G^*(dx).$$

Since $C(y_1) = \alpha^{-1}F_1(y_1)(1-G(y_1^-))$, $dG^* = \alpha^{-1}F_1 dG$, $dF_1^* = \alpha^{-1}(1-G^-)dF_1$ and $1 - F_1(x) \leq M(1-G(x^-))$, we get

$$||\varphi_0^1||^2 = \int (\varphi_0^1)^2 d\tilde H_3 \times d\tilde H_3$$

$$
\begin{aligned}
&\le \int \frac{\varphi_0^2(x_1,x_2)}{(A+B)^4(x_1+x_2)} \left(\int_{x_1+x_2}^{\infty} \frac{1_{\{x<y_1\}}}{C(y_1)F_1(x)} G^*(dx) \right)^2 F_1^*(dy_1) H_2^1(dx_1,dx_2) \\
&= \int \frac{\varphi_0^2(x_1,x_2)}{(A+B)^4(x_1+x_2)} \left(\int_{x_1+x_2}^{\infty} \frac{1_{\{x<y_1\}}}{F_1(y_1)(1-G(y_1^-))} G(dx) \right)^2 F_1^*(dy_1) H_2^1(dx_1,dx_2) \\
&\le \int \frac{\varphi_0^2(x_1,x_2)}{(A+B)^4(x_1+x_2)} \int \frac{1_{\{x_1+x_2<y_1\}}}{(1-G(y_1^-))^2} \left(\int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} \right)^2 F_1^*(dy_1) H_2^1(dx_1,dx_2) \\
&= \int \frac{\varphi_0^2(x_1,x_2)}{(A+B)^3(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} \int_{x_1+x_2}^{\infty} \frac{\alpha^{-1}F_1(dy_1)}{1-G(y_1^-)} F(dx_1,dx_2) \\
&\le \int \frac{\varphi_0^2(x_1,x_2)}{(A+B)^2(x_1+x_2)F_1(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} \int_{x_1+x_2}^{\infty} \frac{F_1(dy_1)}{1-G(y_1^-)} F(dx_1,dx_2) \\
&\le M\alpha \int \frac{\varphi_0^2(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} \int_{x_1+x_2}^{\infty} \frac{F_1(dy_1)}{1-G(y_1^-)} F(dx_1,dx_2) \\
&\quad + \int \frac{\varphi_0^2(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} \int_{x_1+x_2}^{\infty} \frac{F_1(dy_1)}{1-G(y_1^-)} F(dx_1,dx_2) \\
&\le M^2\alpha \int \frac{\varphi_0^2(x_1,x_2)}{F_1^2(x_1+x_2)} F(dx_1,dx_2) + M \int \frac{\varphi_0^2(x_1,x_2)}{(A+B)(x_1+x_2)F_1(x_1+x_2)} F(dx_1,dx_2)
\end{aligned}
$$

The last two integrals are, according to the assumptions, finite. Then

$$\frac{n}{n-1} N_{2n}^{i\neq j}(\varphi) = U_n^{(2)}(\pi_2\varphi^1)$$

is a U-statistic of Hoeffding's projection. Therefore, as in the proof of (4.2), Theorem 5.3.7 in de la Peña and Giné (1999) and Cramér-Slutsky yield

$$\sup_{\varphi\in K} |\sqrt{n} N_{2n}^{i\neq j}(\varphi)| = \sqrt{n}\frac{n-1}{n^2} \sup_{\varphi^1\in\mathcal{H}^1} |nU_n^{(2)}(\pi_2\varphi^1)| \xrightarrow{\mathbb{P}} 0.$$

To complete the proof of (4.6) we have to deal with $N_{2n}^{i=j}(\varphi)$. The supremum over $\varphi\in K$ is bounded from above by

$$
\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{n} \Big[\sup_{\varphi\in K} \left| \frac{\varphi(U_{1i},\tilde U_{2i})\delta_i}{(A+B)^2(U_{1i}+\tilde U_{2i})C(U_{1i})} \int_{U_{1i}+\tilde U_{2i}}^{\infty} \frac{1_{\{x<U_{1i}\}}}{F_1(x)} G^*(dx) \right| \\
&+ \sup_{\varphi\in K} \left| \int \frac{\varphi(U_{1i},\tilde U_{2i})\delta_i}{(A+B)^2(U_{1i}+\tilde U_{2i})C(y)} \int_{U_{1i}+\tilde U_{2i}}^{\infty} \frac{1_{\{x<y\}}}{F_1(x)} G^*(dx) F_1^*(dy) \right| \\
&+ \sup_{\varphi\in K} \left| \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)C(U_{1i})} \int_{U_{1i}+\tilde U_{2i}}^{\infty} \frac{1_{\{x<U_{1i}\}}}{F_1(x)} G^*(dx) H_2^1(dx_1,dx_2) \right| \\
&+ \sup_{\varphi\in K} \left| \int\int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)C(y)} \int_{x_1+x_2}^{\infty} \frac{1_{\{x<y\}}}{F_1(x)} G^*(dx) F_1^*(dy) H_2^1(dx_1,dx_2) \right| \Big].
\end{aligned}
$$

Since $U_{1i} < U_{1i} + \tilde U_{2i}$, the first term is zero and

$$E(\sup_{\varphi\in K} |\sqrt{n} N_{2n}^{i\neq j}(\varphi)|) \;\le\; \frac{3}{\sqrt{n}} \int \frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)C(y)} \int_{x_1+x_2}^{\infty} \frac{1_{\{x<y\}}}{F_1(x)} G^*(dx) F_1^*(dy) H_2^1(dx_1,dx_2)$$

$$
\begin{aligned}
&= \frac{3}{\sqrt{n}} \int \frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{F_1^*(dy)}{C(y)} G^*(dx) H_2^1(dx_1,dx_2) \\
&= \frac{3}{\sqrt{n}} \int \frac{\varphi_0(x_1,x_2)\alpha^{-1}}{(A+B)(x_1+x_2)} \int_{x_1+x_2}^{\infty} \int_{(x,\infty)} \frac{F_1(dy)}{F_1(y)} G(dx) F(dx_1,dx_2) \\
&\le \frac{3}{\sqrt{n}} \int \frac{\varphi_0(x_1,x_2)\alpha^{-1}}{(A+B)(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{G(dx)}{F_1(x)} F(dx_1,dx_2)
\end{aligned}
$$

goes to zero as $n \to \infty$. Hence

$$\sup_{\varphi \in K} |N_{2n}^{i=j}(\varphi)| = o_{\mathbb{P}}(n^{-1/2}).$$

As to (4.7), by continuity of F_1, for the inner integral we have the following equation

$$\int \Psi_x(y) \frac{C_n(y)-C(y)}{C^2(y)} F_1^*(dy) = -F_1(x) \int_{(x,\infty)} \frac{C_n(y)-C(y)}{C^2(y)} F_1^*(dy).$$

Then, in a similar way as for (4.6), we have

$$
\begin{aligned}
(4.7) &= \sup_{\varphi\in K} \Big| \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{C_n(y)-C(y)}{C^2(y)} F_1^*(dy) G_{n-1}^*(dx) \\
&\quad (H_{2n}^1(dx_1,dx_2) - H_2^1(dx_1,dx_2)) \Big| \\
&\le \sup_{\varphi\in K} \Big| \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{C_n(y)-C(y)}{C^2(y)} F_1^*(dy) \\
&\quad (G_{n-1}^*(dx) - G^*(dx)) H_{2n}^1(dx_1,dx_2) \Big| \qquad (4.12) \\
&\quad + \sup_{\varphi\in K} \Big| \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{C_n(y)-C(y)}{C^2(y)} F_1^*(dy) \\
&\quad (G_{n-1}^*(dx) - G^*(dx)) H_2^1(dx_1,dx_2) \Big| \qquad (4.13) \\
&\quad + \sup_{\varphi\in K} \Big| \int \frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)} \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{C_n(y)-C(y)}{C^2(y)} F_1^*(dy) G^*(dx) \\
&\quad (H_{2n}^1(dx_1,dx_2) - H_2^1(dx_1,dx_2)) \Big| \qquad (4.14)
\end{aligned}
$$

The term (4.12) is bounded from above by

$$\int \frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)} \left| \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{C_n(y)-C(y)}{C^2(y)} F_1^*(dy)(G_{n-1}^*(dx) - G^*(dx)) \right| H_{2n}^1(dx_1,dx_2)$$

and (4.13) by

$$\int \frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)} \left| \int_{x_1+x_2}^{\infty} \frac{1}{F_1(x)} \int_{(x,\infty)} \frac{C_n(y)-C(y)}{C^2(y)} F_1^*(dy)(G_{n-1}^*(dx) - G^*(dx)) \right| H_2^1(dx_1,dx_2),$$

As for (3.63) in Lemma 3.18 we can show that the terms above are $o_{\mathbb{P}}(n^{-1/2})$.

The integral in (4.14) we write as a U-statistic and use, as in the proof for (4.6), Theorem 5.3.7 in de la Peña and Giné (1999). We set

$$\begin{aligned} T_n(\varphi) &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Big[\frac{\varphi(U_{1i},\tilde U_{2i})\delta_i}{(A+B)^2(U_{1i}+\tilde U_{2i})}\int_{U_{1i}+\tilde U_{2i}}^{\infty}\frac{1}{F_1(x)}\int_{(x,\infty)}\frac{1_{\{U_{1j}\le y<Z_j\}}-C(y)}{C^2(y)}F_1^*(dy)G^*(dx) \\ &\quad -\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1(x)}\int_{(x,\infty)}\frac{1_{\{U_{1j}\le y<Z_j\}}-C(y)}{C^2(y)}F_1^*(dy)G^*(dx)H_2^1(dx_1,dx_2)\end{aligned}$$

and write it as $T_n^{i\neq j}+T_n^{i=j}$, where $T_n^{i\neq j}$ is a sum over different and $T_n^{i=j}$ is a sum over equal indices, respectively. We define a $\tilde H_3\times\tilde H_3$ integrable function

$$\varphi_t((x_1,x_2,x_3,\delta^1),(y_1,y_2,y_3,\delta^2)) := \frac{\varphi(x_1,x_2)\delta^1}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1(x)}\int_{(x,\infty)}\frac{1_{\{y_1\le y\le y_3\}}-C(y)}{C^2(y)}F_1^*(dy)G^*(dx).$$

Then $T_n^{i\neq j}$ can be written as follows:

$$\begin{aligned} T_n^{i\neq j} &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\Big[\varphi_t((U_{1i},\tilde U_{2i},Z_i,\delta_i),(U_{1j},\tilde U_{2j},Z_j,\delta_j)) \\ &\quad -\int\varphi_t((x_1,x_2,x_3,\delta^1),(U_{1j},\tilde U_{2j},Z_j,\delta_j))d\tilde H_3(x_1,x_2,x_3,\delta^1)\Big].\end{aligned}$$

Since $E(1_{\{U_{1j}\le y<Z_j\}})=C(y)$, we have

$$\begin{aligned} &-\int\varphi_t((U_{1i},\tilde U_{2i},Z_i,\delta_i),(y_1,y_2,y_3,\delta^2))d\tilde H_3(y_1,y_2,y_3,\delta^2) \\ &+\int\varphi_t((x_1,x_2,x_3,\delta^1),(y_1,y_2,y_3,\delta^2))d\tilde H_3(x_1,x_2,x_3,\delta^1)d\tilde H_3(y_1,y_2,y_3,\delta^2) \\ &=\int\Big[\varphi_t((x_1,x_2,x_3,\delta^1),(y_1,y_2,y_3,\delta^2))d\tilde H_3(x_1,x_2,x_3,\delta^1) \\ &-\varphi_t((U_{1i},\tilde U_{2i},Z_i,\delta_i),(y_1,y_2,y_3,\delta^2))\Big]d\tilde H_3(y_1,y_2,y_3,\delta^2) \\ =\;& E\Big(\int\int\frac{1_{\{y\ge x\}}}{F_1(x)}\frac{1_{\{U_{1j}\le y\le Z_j\}}-C(y)}{C^2(y)}\int\left[\frac{\varphi(x_1,x_2)1_{\{x\ge x_1+x_2\}}}{(A+B)^2(x_1+x_2)}-\frac{\varphi(U_{1i},\tilde U_{2i})1_{\{x\ge U_{1i}+\tilde U_{2i}\}}\delta_i}{(A+B)^2(U_{1i}+\tilde U_{2i})}\right] \\ &d\tilde H_3(x_1,x_2,x_3,\delta^1)F_1^*(dy)G^*(dx)\Big) \\ =\;& \int\int\frac{1_{\{y\ge x\}}}{F_1(x)}E\left(\frac{1_{\{U_{1j}\le y\le Z_j\}}-C(y)}{C^2(y)}\right)\int E\left[\frac{\varphi(x_1,x_2)1_{\{x\ge x_1+x_2\}}}{(A+B)^2(x_1+x_2)}-\frac{\varphi(U_{1i},\tilde U_{2i})1_{\{x\ge U_{1i}+\tilde U_{2i}\}}\delta_i}{(A+B)^2(U_{1i}+\tilde U_{2i})}\right] \\ &d\tilde H_3(x_1,x_2,x_3,\delta^1)F_1^*(dy)G^*(dx)\Big)=0.\end{aligned}$$

Hence

$$
\begin{aligned}
T_n^{i\neq j} &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\Big[\varphi_t((U_{1i},\tilde{U}_{2i},Z_i,\delta_i),(U_{1j},\tilde{U}_{2j},Z_j,\delta_j)) \\
&\quad -\int \varphi_t((U_{1i},\tilde{U}_{2i},Z_i,\delta_i),(y_1,y_2,y_3,\delta^2))d\tilde{H}_3(y_1,y_2,y_3,\delta^2) \\
&\quad +\int \varphi_t((x_1,x_2,x_3,\delta^1),(y_1,y_2,y_3,\delta^2))d\tilde{H}_3(x_1,x_2,x_3,\delta^1)d\tilde{H}_3(y_1,y_2,y_3,\delta^2) \\
&\quad -\int \varphi_t((x_1,x_2,x_3,\delta^1),(U_{1j},\tilde{U}_{2j},Z_j,\delta_j))d\tilde{H}_3(x_1,x_2,x_3,\delta^1)\Big],
\end{aligned}
$$

is a U-statistic.

Set

$$\mathcal{H}_2 = \{\varphi_t : \varphi \in K\}$$

which is a VC with envelope function

$$\varphi_t^0((x_1,x_2,x_3,\delta^1),(y_1,y_2,y_3,\delta^2)) := \frac{\varphi_0(x_1,x_2)\delta^1}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1(x)}\int_{(x,\infty)}\frac{|1_{\{y_1\le y\le y_3\}}-C(y)|}{C^2(y)}F_1^*(dy)G^*(dx).$$

Since $dF_1^* = \alpha^{-1}(1-G^-)dF_1$, $dG^* = \alpha^{-1}F_1 dG$, $A+B = \alpha^{-1}(1-G^-)$ and, by A1, $1-F_1 \le M(1-G^-)$, we have

$$
\begin{aligned}
&\varphi_t^0((x_1,x_2,x_3,\delta^1),(y_1,y_2,y_3,\delta^2)) \le \frac{\varphi_0(x_1,x_2)\delta^1\alpha}{(A+B)(x_1+x_2)}\int_{(x_1+x_2,\infty)}\frac{1_{\{y_1\le y\le y_3\}}}{F_1^2(y)(1-G(y^-))}F_1(dy) \\
&\quad +\frac{\varphi_0(x_1,x_2)\delta^1}{(A+B)(x_1+x_2)}\int_{(x_1+x_2,\infty)}\frac{1_{\{y_1\le y\le y_3\}}}{F_1(y)}F_1(dy) \le \frac{\varphi_0(x_1,x_2)\delta^1\alpha}{(A+B)(x_1+x_2)}M\int_{(x_1+x_2,\infty)}\frac{1_{\{y_1\le y\le y_3\}}}{F_1^2(y)}F_1(dy) \\
&\quad +\frac{\varphi_0(x_1,x_2)\delta^1\alpha}{(A+B)(x_1+x_2)}\int_{(x_1+x_2,\infty)}\frac{1_{\{y_1\le y\le y_3\}}}{F_1(y)(1-G(y^-))}F_1(dy) + \frac{\varphi_0(x_1,x_2)\delta^1}{(A+B)(x_1+x_2)}\frac{M(1-G(x_1+x_2^-))}{F_1(x_1+x_2)} \\
&\quad \le 3\frac{\varphi_0(x_1,x_2)\delta^1}{(A+B)(x_1+x_2)F_1(x_1+x_2)}M1_{\{x_1+x_2\le y_3\}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
||\varphi_t^0||^2 &\le \int\int(\varphi_t^0)^2 d\tilde{H}_3(x_1,x_2,x_3,\delta^1)d\tilde{H}_3(y_1,y_2,y_3,\delta^2) \\
&\le 9M^2\int\int\frac{\varphi_0^2(x_1,x_2)1_{\{x_1+x_2\le y_3\}}}{(A+B)^2(x_1+x_2)F_1^2(x_1+x_2)}G(dy_3)H_2^1(dx_1,dx_2) \\
&= 9M^2\int\frac{\varphi_0^2(x_1,x_2)\alpha}{(A+B)(x_1+x_2)F_1^2(x_1+x_2)}H_2^1(dx_1,dx_2) = 9M^2\int\frac{\varphi_0^2(x_1,x_2)\alpha}{F_1^2(x_1+x_2)}F(dx_1,dx_2).
\end{aligned}
$$

This is, by assumption A2’, finite. Then

$$\frac{n-1}{n}T_n^{i\neq j}(\varphi) = U_n^{(2)}(\pi_2\varphi_t)$$

and, with Theorem 5.3.7 in de la Peña and Giné (1999) and Cramér-Slutsky, we have that

$$\sup_{\varphi\in K}|\sqrt{n}T_n^{i\neq j}(\varphi)| = \frac{\sqrt{n}}{n-1}|nU_n^{(2)}(\pi_2\varphi_t)|$$

goes to zero in probability.

Now we deal with $\sqrt{n}T_n^{i=j}(\varphi)$. For the expectation of its absolute value we have:

$$\begin{aligned}
\sqrt{n}E(\sup_{\varphi\in K}|T_n^{i=j}(\varphi)|) \quad \le \quad & \frac{1}{\sqrt{n}}E\Big[\frac{\varphi_0(U_{11},\tilde{U}_{21})\delta_1}{(A+B)^2(U_{11}+\tilde{U}_{21})}\int_{U_{11}+\tilde{U}_{21}}^{\infty}\frac{1}{F_1(x)}\int_{(x,\infty)}\frac{|1_{\{U_{11}\le y\le Z_1\}}-C(y)|}{C^2(y)} \\
& F_1^*(dy)G^*(dx)\Big] \\
& +\frac{1}{\sqrt{n}}\int\frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1(x)}\int_{(x,\infty)}E\Big[\frac{|1_{\{U_{11}\le y\le Z_1\}}-C(y)|}{C^2(y)}\Big] \\
& F_1^*(dy)G^*(dx)H_2^1(dx_1,dx_2)
\end{aligned}$$

Since $|1_{\{U_{11}\le y\le Z_1\}}-C(y)| \le 1_{\{U_{11}\le y\le Z_1\}}+C(y)$ the right side is bounded from above by

$$\begin{aligned}
& \frac{1}{\sqrt{n}}\Big(E\Big[\frac{\varphi_0(U_{11},\tilde{U}_{21})\delta_1}{(A+B)^2(U_{11}+\tilde{U}_{21})}\int_{U_{11}+\tilde{U}_{21}}^{\infty}\frac{1}{F_1(x)}\int_{(x,\infty)}\frac{1_{\{U_{11}\le y<Z_1\}}}{C^2(y)}F_1^*(dy)G^*(dx)\Big] \\
& +3\int\frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1(x)}\int_{(x,\infty)}\frac{1}{C(y)}F_1^*(dy)G^*(dx)H_2^1(dx_1,dx_2)\Big) \\
& = \frac{1}{\sqrt{n}}\Big(\int\frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1(x)}\int_{(x,\infty)}\frac{1_{\{x_1\le y<z\}}}{C^2(y)}F_1^*(dy)G^*(dx)\alpha^{-1}G(dz)F(dx_1,dx_2) \\
& +3\int\frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{1}{F_1(x)}\int_{(x,\infty)}\frac{1}{C(y)}F_1^*(dy)G^*(dx)H_2^1(dx_1,dx_2)\Big) \\
& \le \frac{1}{\sqrt{n}}\Big(\int\frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\int_{(x,\infty)}\frac{(1-G(y^-))}{F_1^2(y)(1-G(y^-))^2}F_1^*(dy)G(dx)F(dx_1,dx_2) \\
& +3\int\frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\int_{(x,\infty)}\frac{1}{F_1(y)}F_1(dy)G(dx)H_2^1(dx_1,dx_2)\Big) \\
& \le \frac{4}{\sqrt{n}}\int\frac{\varphi_0(x_1,x_2)\alpha^{-1}}{(A+B)(x_1+x_2)F_1(x_1+x_2)}F(dx_1,dx_2).
\end{aligned}$$

Since the integral is finite, we get

$$\sqrt{n}E(\sup_{\varphi\in K}|T_n^{i=j}(\varphi)|) \to 0,$$

as $n\to\infty$.

Finally, to prove (4.8) and (4.9) we bound them from above. We have

$$(4.8) \quad \leq \quad \frac{1}{n}\sum_{i=1}^{n}\frac{\varphi_0(U_{1i},\tilde{U}_{2i})}{(A+B)^2(U_{1i}+\tilde{U}_{2i})}\delta_i\frac{1}{n}\sum_{j\neq i}^{n}\frac{|R_{n-1}^{\neq j}(Z_j)|}{F_1^2(Z_j)}1_{\{U_{1i}+\tilde{U}_{2i}\leq Z_j\}}$$

and

$$(4.9) \quad \leq \quad \int\frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)}\frac{1}{n}\sum_{j=1}^{n}\frac{|R_{n-1}^{\neq j}(Z_j)|}{F_1^2(Z_j)}1_{\{x_1+x_2\leq Z_j\}}H_2^1(dx_1,dx_2).$$

In the same way as in the proofs for (3.47) and (3.48), we can show that (4.8) and (4.9) are $o_{\mathbb{P}}(n^{-1/2})$.

<u>Remainder from Lemma 3.19</u>:

Since

$$\begin{aligned}
&\sup_{\varphi\in K}\Big|\int\frac{\varphi(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{R_n(x)}{F_1^2(x)}G^*(dx)H_2^1(dx_1,dx_2)\Big|\\
&\leq\int\frac{\varphi_0(x_1,x_2)}{(A+B)^2(x_1+x_2)}\int_{x_1+x_2}^{\infty}\frac{|R_n(x)|}{F_1^2(x)}G^*(dx)H_2^1(dx_1,dx_2)
\end{aligned}$$

the proof for Lemma 3.19 yields that the term above is $o_{\mathbb{P}}(n^{-1/2})$.

$\boxtimes$

Appendix A

Basic Properties of $F_{1n}(t)$

In this chapter we study $F_{1n}(t)$. We start with a linearization of $F_{1n}(t)$ and then provide some properties of the remainder and its second moment. These are useful for the linearization of I_n in Chapter 3. Recall that $F_1(t) = \mathbb{P}(U_1 \leq t)$ and its estimator is the Lynden-Bell estimator for right-truncated data (for details see (1.7)):

$$F_{1n}(t) = \prod_{U_{1i}>t} \left[1 - \frac{1}{nC_n(U_{1i})}\right]$$

Since $nC_n(U_{11:n}) = 1$ and

$$\prod_{j=i+1}^{n} \left[1 - \frac{1}{nC_n(U_{1j:n})}\right] + \frac{1}{nC_n(U_{1i+1:n})} \prod_{j=i+2}^{n} \left[1 - \frac{1}{nC_n(U_{1j:n})}\right] = \prod_{j=i+2}^{n} \left[1 - \frac{1}{nC_n(U_{1j:n})}\right],$$

on $\Omega_1^{(n)} := \Omega \setminus \Omega_0^{(n)}$ (under ”no holes”, for details see (3.12)), $F_{1n}(t)$ can be written as

$$\begin{aligned}
F_{1n}(t) &= \sum_{i=1}^{n} \frac{1_{\{U_{1i:n} \leq t\}}}{nC_n(U_{1i:n})} \prod_{j=i+1}^{n} \left[1 - \frac{1}{nC_n(U_{1j:n})}\right] \\
&= \int_{(-\infty,t]} \frac{1}{C_n(x)} \exp\left\{n \int_{(x,\infty)} \ln\left(1 - \frac{1}{nC_n(y)}\right) F_{1n}^*(dy)\right\} F_{1n}^*(dx) \\
&= \int_{(-\infty,t]} \frac{1}{C_n(x)} \gamma_n(x) F_{1n}^*(dx)
\end{aligned}$$

where

$$\gamma_n(x) = \exp\left\{n \int_{(x,\infty)} \ln\left(1 - \frac{1}{nC_n(y)}\right) F_{1n}^*(dy)\right\}.$$

Set

$$\gamma(x) = \exp\left\{-\int_{(x,\infty)} \frac{F_1^*(dy)}{C(y)}\right\}.$$

Note that, since by (1.6) $\frac{dF_1^*}{C} = \frac{dF_1}{F_1}$, we have $\gamma(z) = F_1(z)$.

With Taylor's formula we get

$$\gamma_n(x) = \gamma(x) + e^{\Delta_n(x)}\left[n\int_{(x,\infty)} \ln\left(1 - \frac{1}{nC_n(y)}\right)F^*_{1n}(dy) + \int_{(x,\infty)} \frac{F^*_1(dy)}{C(y)}\right],$$

where $e^{\Delta_n(x)} \in (\gamma_n(x), \gamma(x))$. Hence $\Delta_n(x) \leq 0$.
Before we may proceed with [...], for given $\varepsilon > 0$ we may choose a small c and sequences $b_n \to b_{G^*}$ so that $1 - G^*(b_n) = \frac{c}{n}$ and $P(U_{1n:n} < b_n) \geq P(Z_{n:n} \leq b_n) \geq 1 - \varepsilon$. Hence on an event $\Omega_n^{b_n}$ of probability greater than or equal to $1 - \varepsilon$ we may restrict integration w.r.t. F^*_{1n} to $(-\infty, b_n)$.

Furthermore, for given $\varepsilon > 0$ we may choose a small c_1 and sequences $a_n \to a_{F_1^*}$ so that $F^*_1(a_n) = \frac{c_1}{n}$ and $P(Z_{1:n} > a_n) \geq P(U_{11:n} > a_n) \geq 1 - \varepsilon$. Hence on an event $\Omega_n^{a_n}$ of probability greater than or equal to $1 - \varepsilon$ we may restrict integration w.r.t. F^*_{1n} to (a_n, ∞). Hence, on $\Omega^*_n = \Omega_n^{b_n} \cap \Omega_n^{a_n}$, an event of probability greater than or equal to $1 - 2\varepsilon$, F^*_{1n} is supported by (a_n, b_n).

Now, on $\Omega_1^{(n)} \cap \Omega^*_n$,

$$\begin{aligned}
[...] &= \left[n\int_{(x,b_n)} \ln\left(1 - \frac{1}{nC_n(y)}\right)F^*_{1n}(dy) + \int_{(x,\infty)} \frac{F^*_1(dy)}{C(y)}\right] \\
&= \left[n\int_{(x,b_n)} \ln\left(1 - \frac{1}{nC_n(y)}\right)F^*_{1n}(dy) + \int_{(x,b_n)} \frac{F^*_{1n}(dy)}{C_n(y)}\right] - \int_{(x,\infty)} \frac{F^*_{1n}(dy) - F^*_1(dy)}{C(y)} \\
&\quad + \int_{(x,b_n)} \frac{C_n(y) - C(y)}{C_n(y)C(y)}F^*_{1n}(dy).
\end{aligned}$$

Define, for $x \geq U_{11:n}$,

$$B_n(x) = n\int_{(x,b_n)} \ln\left(1 - \frac{1}{nC_n(y)}\right)F^*_{1n}(dy) + \int_{(x,b_n)} \frac{F^*_{1n}(dy)}{C_n(y)}$$

$$D_{n1}(x) = -\int_{(x,\infty)} \frac{F^*_{1n}(dy) - F^*_1(dy)}{C(y)}$$

$$D_{n2}(x) = \int_{(x,b_n)} \frac{C_n(y) - C(y)}{C_n(y)C(y)}F^*_{1n}(dy).$$

Then

$$\begin{aligned}
F_{1n}(t) &= \int_{(-\infty,t]} \frac{\gamma(x)}{C_n(x)}F^*_{1n}(dx) + \int_{(-\infty,t]} \frac{e^{\Delta_n(x)}}{C_n(x)}B_n(x)F^*_{1n}(dx) + \int_{(-\infty,t]} \frac{e^{\Delta_n(x)}}{C_n(x)}D_{n1}(x)F^*_{1n}(dx) \\
&\quad + \int_{(-\infty,t]} \frac{e^{\Delta_n(x)}}{C_n(x)}D_{n2}(x)F^*_{1n}(dx) = S_{n1}(t) + S_{n2}(t) + S_{n3}(t) + S_{n4}(t).
\end{aligned}$$

The term $S_{n1}(t)$ can be written as follows:

$$\begin{aligned}
S_{n1}(t) &= \int_{(-\infty,t]} \frac{\gamma(x)}{C(x)}F^*_1(dx) + \int_{(-\infty,t]} \frac{\gamma(x)}{C(x)}[F^*_{1n}(dx) - F^*_1(dx)] \\
&\quad + \int_{(-\infty,t]} \frac{\gamma(x)}{C(x)C_n(x)}(C(x) - C_n(x))F^*_{1n}(dx) = F_1(t) + L_{n1}(t) + S^b_{n1}(t).
\end{aligned}$$

Finally

$$F_{1n}(t) - F_1(t) \;=\; L_{n1}(t) + S^b_{n1}(t) + S_{n2}(t) + S_{n3}(t) + S_{n4}(t),$$

where

$$\begin{aligned}
L_{n1}(t) &= \int_{(-\infty,t]} \frac{\gamma(x)}{C(x)}(F^*_{1n}(dx) - F^*_1(dx)) \\
S^b_{n1}(t) &= \int_{(a_n,t]} \frac{\gamma(x)}{C(x)C_n(x)}(C(x) - C_n(x))F^*_{1n}(dx) \\
S_{n2}(t) &= \int_{(a_n,t]} \frac{e^{\Delta_n(x)}}{C_n(x)} B_n(x) F^*_{1n}(dx) \\
S_{n3}(t) &= \int_{(a_n,t]} \frac{e^{\Delta_n(x)}}{C_n(x)} D_{n1}(x) F^*_{1n}(dx) \\
S_{n4}(t) &= \int_{(a_n,t]} \frac{e^{\Delta_n(x)}}{C_n(x)} D_{n2}(x) F^*_{1n}(dx).
\end{aligned}$$

In the later parts of this chapter we will need the following remark.

Remark A.1. *Since $F_1(a_n) \geq \alpha F^*_1(a_n) = \frac{\alpha c_1}{n}$ and $1 - G(b_n) \geq \alpha(1 - G^*(b_n)) = \frac{\alpha c}{n}$, we have that $ln((F_1(a_n))^{-1}) \leq ln(\frac{n}{\alpha c_1})$ and $ln((1 - G(b_n))^{-1}) \leq ln(\frac{n}{\alpha c})$.*

A.1 Bounds for $(F_{1n} - F_1)^2(t)$

According to the last section, by repeated use of $(a + b)^2 \leq 2a^2 + 2b^2$, we have, on an event $\Omega^*_n \cap \Omega^{(n)}_1$ of probability greater than or equal to $1 - 3\varepsilon$,

$$(F_{1n} - F_1)^2(t) \leq 8(L^2_{n1}(t) + (S^b_{n1})^2(t) + S^2_{n2}(t) + S^2_{n3}(t) + S^2_{n4}(t)).$$

Next, we will bound each of the terms separately and compute their expectations.

According to equation (3.1) in Stute and Wang (2007) we have that for every $\varepsilon > 0$, there exist large K and n_0, so that for $n \geq n_0$

$$\mathbb{P}(\sup_{1\leq i\leq n} \frac{C(U_{1i})}{C_n(U_{1i})} \leq K) \geq 1 - \varepsilon.$$

Set $\tilde{\Omega}_n = \{\omega \in \Omega : \sup_{1\leq i\leq n} \frac{C(U_{1i})}{C_n(U_{1i})}(\omega) \leq K\}$ and $\tilde{\Omega}^0_n := \Omega^*_n \cap \Omega^{(n)}_1 \cap \tilde{\Omega}_n$. Then $\mathbb{P}(\tilde{\Omega}^0_n) \geq 1 - 4\varepsilon$.

Lemma A.1. *On the set $\tilde{\Omega}_n^0$, we have*

$$
\begin{aligned}
S_{n3}^2(t) \quad & \leq \frac{K}{n}\sum_{i=1}^n 1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma^2(U_{1i})}{C(U_{1i})}\left(\int_{(U_{1i},\infty)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right)^2 \\
& +\frac{K^2}{n^2}\sum_{i=1}^n\sum_{j\neq i}\frac{\gamma(U_{1i})}{C(U_{1i})}\frac{\gamma(U_{1j})}{C(U_{1j})}1_{\{a_n<U_{1i},U_{1j}\leq t\}}\left|\int_{(U_{1i},\infty)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right|\left|\int_{(U_{1j},\infty)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right| \\
& =: \tilde{S}_{n3}^2(t).
\end{aligned}
$$

Furthermore,

$$
E\tilde{S}_{n3}^2(t) \;\leq\; \frac{2KM\alpha}{n}ln(\frac{n}{c_1\alpha})+\frac{2K}{n^2}ln^2(\frac{n}{c_1\alpha})+18\frac{K^2\alpha M}{n}ln^2(\frac{n}{c_1\alpha})+4\frac{\sqrt{72}K^2}{n^{3/2}}\alpha^{3/2}M^{3/2}ln(\frac{n}{c_1\alpha}).
$$

Proof.
Since $C_n(U_{1i})\geq 1/n$, $C/C_n\leq K$ on $\tilde{\Omega}_n^0$ and $\exp\{\Delta_n(x)\}\leq\gamma(x)$, we have

$$
\begin{aligned}
S_{n3}^2(t) \quad & \leq \left(\int_{(a_n,t]}\frac{\gamma(x)}{C_n(x)}|D_{n1}(x)|F_{1n}^*(dx)\right)^2 \\
& = \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n 1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma(U_{1i})}{C_n(U_{1i})}1_{\{a_n<U_{1j}\leq t\}}\frac{\gamma(U_{1j})}{C_n(U_{1j})}\left|\int_{(U_{1i},\infty)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right| \\
& \times\left|\int_{(U_{1j},\infty)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right| = \frac{1}{n^2}\sum_{i=1}^n 1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma^2(U_{1i})}{C_n^2(U_{1i})}\left(\int_{(U_{1i},\infty)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right)^2 \\
& +\frac{1}{n^2}\sum_{i=1}^n\sum_{j\neq i}1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma(U_{1i})}{C_n(U_{1i})}1_{\{a_n<U_{1j}\leq t\}}\frac{\gamma(U_{1j})}{C_n(U_{1j})}\left|\int_{(U_{1i},\infty)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right| \\
& \times\left|\int_{(U_{1j},\infty)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right| \leq K\frac{1}{n}\sum_{i=1}^n 1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma^2(U_{1i})}{C(U_{1i})}\left(\int_{(U_{1i},\infty)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right)^2 \\
& +\frac{K^2}{n^2}\sum_{i=1}^n\sum_{j\neq i}\frac{\gamma(U_{1i})}{C(U_{1i})}\frac{\gamma(U_{1j})}{C(U_{1j})}1_{\{a_n<U_{1i},U_{1j}\leq t\}}\left|\int_{(U_{1i},\infty)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right|\left|\int_{(U_{1j},\infty)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right| \\
& = \tilde{S}_{n3}^2(t).
\end{aligned}
$$

Since

$$
\int_{(U_{1i},\infty)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}=\frac{n-1}{n}\int_{(U_{1i},\infty)}\frac{F_{1n-1}^{\neq i*}(dy)-F_1^*(dy)}{C(y)}-\frac{1}{n}\int_{(U_{1i},\infty)}\frac{F_1^*(dy)}{C(y)}
$$

and

$$
E\left(\int_{(x,\infty)}\frac{F_{1n-1}^*(dz)-F_1^*(dz)}{C(z)}\right)^2\leq\frac{1}{n-1}\int_{(x,\infty)}\frac{F_1^*(dz)}{C^2(z)}
$$

we have

$$
E\left(K\frac{1}{n}\sum_{i=1}^{n}1_{\{a_n<U_{1i}\le t\}}\frac{\gamma^2(U_{1i})}{C(U_{1i})}\left(\int_{(U_{1i},\infty)}\frac{F^*_{1n}(dy)-F^*_1(dy)}{C(y)}\right)^2\right)
$$
$$
=KE\left(1_{\{a_n<U_{11}\le t\}}\frac{\gamma^2(U_{11})}{C(U_{11})}\left(\int_{(U_{11},\infty)}\frac{F^*_{1n}(dy)-F^*_1(dy)}{C(y)}\right)^2\right)
$$
$$
\le 2K\frac{(n-1)^2}{n^2}E\left(1_{\{a_n<U_{11}\le t\}}\frac{\gamma^2(U_{11})}{C(U_{11})}\left(\int_{(U_{11},\infty)}\frac{F^*_{1n-1}(dy)-F^*_1(dy)}{C(y)}\right)^2\right)
$$
$$
+2K\frac{1}{n^2}E\left(1_{\{a_n<U_{11}\le t\}}\frac{\gamma^2(U_{11})}{C(U_{11})}\left(\int_{(U_{11},\infty)}\frac{F^*_1(dy)}{C(y)}\right)^2\right)\le\frac{2K}{n}\int_{(a_n,t]}\frac{\gamma^2(x)}{C(x)}\int_{(x,\infty)}\frac{F^*_1(dy)}{C^2(y)}F^*_1(dx)
$$
$$
+\frac{2K}{n^2}\int_{(a_n,t]}\frac{\gamma^2(x)}{C(x)}\left(\int_{(x,\infty)}\frac{F^*_1(dy)}{C(y)}\right)^2F^*_1(dx).
$$

To bound the right side we use $dF^*_1=\alpha^{-1}(1-G-)dF_1$, $C=\alpha^{-1}(1-G-)F_1$ and $\gamma=F_1$. Since by A1 $\int\frac{F_1(dy)}{1-G(y^-)}\le M$ and using Remark A.1, we get

$$
\frac{2K}{n}\int_{(a_n,t]}\frac{\gamma^2(x)}{C(x)}\int_{(x,\infty)}\frac{F^*_1(dy)}{C^2(y)}F^*_1(dx)=\frac{2K\alpha}{n}\int_{(a_n,t]}F_1(x)\int_{(x,\infty)}\frac{F_1(dy)}{F_1^2(y)(1-G(y^-))}F_1(dx)
$$
$$
\le\frac{2K\alpha}{n}\int_{(a_n,t]}\frac{F_1(dx)}{F_1(x)}\int_{(-\infty,\infty)}\frac{F_1(dy)}{1-G(y^-)}\le\frac{2K\alpha}{n}\int_{(a_n,\infty)}\frac{F_1(dx)}{F_1(x)}\int_{(-\infty,\infty)}\frac{F_1(dy)}{1-G(y^-)}
$$
$$
=\frac{2K\alpha}{n}ln(F_1(a_n))^{-1}\int_{(-\infty,\infty)}\frac{F_1(dy)}{1-G(y^-)}\le\frac{2KM\alpha}{n}ln(\frac{n}{c_1\alpha})
$$

and

$$
\frac{2K}{n^2}\int_{(a_n,t]}\frac{\gamma^2(x)}{C(x)}\left(\int_{(x,\infty)}\frac{F^*_1(dy)}{C(y)}\right)^2F^*_1(dx)\le\frac{2K}{n^2}ln^2(\frac{n}{c_1\alpha}).
$$

As to the second sum in $\tilde{S}^2_{n3}(t)$, first consider

$$
E\left(\left|\int_{(U_{1i},\infty)}\frac{F^*_{1n}(dz)-F^*_1(dz)}{C(z)}\right|\left|\int_{(U_{1j},\infty)}\frac{F^*_{1n}(dz)-F^*_1(dz)}{C(z)}\right||U_{1i},U_{1j}\right). \tag{A.1}
$$

It follows that

$$
E\left(\left|\int_{(U_{1i},\infty)}\frac{F^*_{1n}(dz)-F^*_1(dz)}{C(z)}\right|\left|\int_{(U_{1j},\infty)}\frac{F^*_{1n}(dz)-F^*_1(dz)}{C(z)}\right||U_{1i}=x,U_{1j}=y\right)
$$
$$
=E\left(\left|\frac{1}{n}\sum_{k=1}^{n}\left[\frac{1_{\{U_{1k}>U_{1i}\}}}{C(U_{1k})}-\int_{(U_{1i},\infty)}\frac{F^*_1(dz)}{C(z)}\right]\right|\left|\frac{1}{n}\sum_{l=1}^{n}\left[\frac{1_{\{U_{1l}>U_{1j}\}}}{C(U_{1l})}-\int_{(U_{1j},\infty)}\frac{F^*_1(dz)}{C(z)}\right]\right||U_{1i}=x,U_{1j}=y\right)
$$
$$
=E\Bigg(\left|\frac{1}{n}\sum_{k\neq i,j}^{n}\left[\frac{1_{\{U_{1k}>U_{1i}\}}}{C(U_{1k})}-\int_{(U_{1i},\infty)}\frac{F^*_1(dz)}{C(z)}\right]+\frac{1_{\{U_{1j}>U_{1i}\}}}{nC(U_{1j})}-\frac{2}{n}\int_{(U_{1i},\infty)}\frac{F^*_1(dz)}{C(z)}\right|
$$

$$\left|\frac{1}{n}\sum_{l\neq i,j}\left[\frac{1_{\{U_{1l}>U_{1j}\}}}{C(U_{1l})}-\int_{(U_{1j},\infty)}\frac{F_1^*(dz)}{C(z)}\right]+\frac{1_{\{U_{1i}>U_{1j}\}}}{nC(U_{1i})}-\frac{2}{n}\int_{(U_{1j},\infty)}\frac{F_1^*(dz)}{C(z)}\right| |U_{1i}=x,U_{1j}=y\Bigg)$$

$$= E\Bigg(\left|\frac{1}{n}\sum_{k\neq i,j}^{n}\left[\frac{1_{\{U_{1k}>x\}}}{C(U_{1k})}-\int_{(x,\infty)}\frac{F_1^*(dz)}{C(z)}\right]+\frac{1_{\{y>x\}}}{nC(y)}-\frac{2}{n}\int_{(x,\infty)}\frac{F_1^*(dz)}{C(z)}\right|$$
$$\left|\frac{1}{n}\sum_{l\neq i,j}\left[\frac{1_{\{U_{1l}>y\}}}{C(U_{1l})}-\int_{(y,\infty)}\frac{F_1^*(dz)}{C(z)}\right]+\frac{1_{\{x>y\}}}{nC(x)}-\frac{2}{n}\int_{(y,\infty)}\frac{F_1^*(dz)}{C(z)}\right|\Bigg)$$

By Cauchy-Schwarz, the right side to the power 2 is smaller than or equal to

$$E\Bigg(\frac{1}{n}\sum_{k\neq i,j}^{n}\left[\frac{1_{\{U_{1k}>x\}}}{C(U_{1k})}-\int_{(x,\infty)}\frac{F_1^*(dz)}{C(z)}\right]+\frac{1_{\{y>x\}}}{nC(y)}-\frac{2}{n}\int_{(x,\infty)}\frac{F_1^*(dz)}{C(z)}\Bigg)^2$$
$$\times E\Bigg(\frac{1}{n}\sum_{l\neq i,j}\left[\frac{1_{\{U_{1l}>y\}}}{C(U_{1l})}-\int_{(y,\infty)}\frac{F_1^*(dz)}{C(z)}\right]+\frac{1_{\{x>y\}}}{nC(x)}-\frac{2}{n}\int_{(y,\infty)}\frac{F_1^*(dz)}{C(z)}\Bigg)^2 \qquad \text{(A.2)}$$

Furthermore, by using $(a+b+c)^2 \leq 2a^2+4b^2+4c^2$ and Bienaymé,

$$E\Bigg(\frac{1}{n}\sum_{k\neq i,j}^{n}\left[\frac{1_{\{U_{1k}>x\}}}{C(U_{1k})}-\int_{(x,\infty)}\frac{F_1^*(dz)}{C(z)}\right]+\frac{1_{\{y>x\}}}{nC(y)}-\frac{2}{n}\int_{(x,\infty)}\frac{F_1^*(dz)}{C(z)}\Bigg)^2$$
$$\leq \frac{2}{n^2}\sum_{k\neq i,j}^{n}E\left[\frac{1_{\{U_{1k}>x\}}}{C(U_{1k})}-\int_{(x,\infty)}\frac{F_1^*(dz)}{C(z)}\right]^2+4\frac{1_{\{y>x\}}}{n^2C^2(y)}+\frac{16}{n^2}\left(\int_{(x,\infty)}\frac{F_1^*(dz)}{C(z)}\right)^2$$
$$\leq \frac{2(n-2)}{n^2}\int_{(x,\infty)}\frac{F_1^*(dz)}{C^2(z)}+4\frac{1_{\{y>x\}}}{n^2C^2(y)}+\frac{16}{n^2}\left(\int_{(x,\infty)}\frac{F_1^*(dz)}{C(z)}\right)^2 \leq \frac{18}{n}\int_{(x,\infty)}\frac{F_1^*(dz)}{C^2(z)}+\frac{4}{n^2}\frac{1_{\{y>x\}}}{C^2(y)}.$$

Hence, since $1_{\{y>x\}}1_{\{x>y\}}=0$,

$$(A.2) \leq \frac{18^2}{n^2}\int_{(x,\infty)}\frac{F_1^*(dz)}{C^2(z)}\int_{(y,\infty)}\frac{F_1^*(dz)}{C^2(z)}+\frac{4*18}{n^3}\frac{1_{\{y>x\}}}{C^2(y)}\int_{(y,\infty)}\frac{F_1^*(dz)}{C^2(z)}+\frac{4*18}{n^3}\frac{1_{\{x>y\}}}{C^2(x)}\int_{(x,\infty)}\frac{F_1^*(dz)}{C^2(z)}$$

Finally,

$$\begin{aligned}(A.1) \quad \leq \quad & \frac{18}{n}\sqrt{\int_{(U_{1i},\infty)}\frac{F_1^*(dz)}{C^2(z)}\int_{(U_{1j},\infty)}\frac{F_1^*(dz)}{C^2(z)}}+\frac{\sqrt{72}}{n^{3/2}}\frac{1_{\{U_{1j}>U_{1i}\}}}{C(U_{1j})}\sqrt{\int_{(U_{1j},\infty)}\frac{F_1^*(dz)}{C^2(z)}}\\ & +\frac{\sqrt{72}}{n^{3/2}}\frac{1_{\{U_{1i}>U_{1j}\}}}{C(U_{1i})}\sqrt{\int_{(U_{1i},\infty)}\frac{F_1^*(dz)}{C^2(z)}}\end{aligned}$$

Hence the expectation of the second sum in $\tilde{S}^2_{n3}(t)$ equals

$$\frac{K^2}{n^2}n(n-1)E\Bigg(\frac{\gamma(U_{11})}{C(U_{11})}\frac{\gamma(U_{12})}{C(U_{12})}1_{\{a_n<U_{11},U_{12}\leq t\}}\left|\int_{(U_{11},\infty)}\frac{F^*_{1n}(dy)-F^*_1(dy)}{C(y)}\right|\left|\int_{(U_{12},\infty)}\frac{F^*_{1n}(dy)-F^*_1(dy)}{C(y)}\right|\Bigg)$$

$$= \frac{K^2}{n}(n-1)E(E(...|U_{11},U_{12})) \le \frac{18K^2}{n}\left(\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left(\int_{(x,\infty)}\frac{F_1^*(dy)}{C^2(y)}\right)^{1/2}F_1^*(dx)\right)^2$$
$$+2\frac{\sqrt{72}K^2}{n^{3/2}}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,t)}\frac{\gamma(y)}{C^2(y)}\left(\int_{(y,\infty)}\frac{F_1^*(dz)}{C^2(z)}\right)^{1/2}F_1^*(dy)F_1^*(dx)$$

As to the first sum, since

$$\left(\int_{(x,\infty)}\frac{F_1^*(dy)}{C^2(y)}\right)^{1/2} \le \frac{\sqrt{\alpha}}{F_1(x)}\left(\int_{(x,\infty)}\frac{F_1(dy)}{1-G(y^-)}\right)^{1/2} \le \frac{\sqrt{\alpha M}}{F_1(x)},$$

we get

$$18\frac{K^2}{n}\left(\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left(\int_{(x,\infty)}\frac{F_1^*(dy)}{C^2(y)}\right)^{1/2}F_1^*(dx)\right)^2 \le 18\frac{K^2\alpha M}{n}\left(\int_{(a_n,t]}\frac{F_1(dx)}{F_1(x)}\right)^2 \le 18\frac{K^2\alpha M}{n}ln^2(\frac{n}{c_1\alpha}).$$

The second term equals

$$2\frac{\sqrt{72}K^2}{n^{3/2}}\int_{(a_n,t]}\int_{(x,t)}\frac{\alpha}{F_1(y)(1-G(y^-))}\left(\alpha\int_{(y,\infty)}\frac{F_1(dz)}{F_1^2(z)(1-G(z^-))}\right)^{1/2}F_1(dy)F_1(dx)$$
$$\le 2\frac{\sqrt{72}K^2}{n^{3/2}}\alpha^{3/2}\sqrt{M}\int_{(a_n,t]}\int_{(x,t)}\frac{F_1(dy)}{F_1^2(y)(1-G(y^-))}F_1(dx) \tag{A.3}$$

Since by A1

$$\frac{1-F_1(x)}{1-G(x^-)} \le \int_{(x,\infty)}\frac{F_1(dy)}{1-G(y^-)} \le M$$

we have

$$\frac{1}{F_1(x)} = \frac{1-F_1(x)}{F_1(x)}+1 \le \frac{M(1-G(x^-))}{F_1(x)}+1.$$

Hence

$$\begin{aligned}
(A.3) \;&\le\; 2\frac{\sqrt{72}K^2}{n^{3/2}}\alpha^{3/2}\sqrt{M}\Big[M\int_{(a_n,t]}\int_{(x,t)}\frac{F_1(dy)}{F_1^2(y)}F_1(dx)+\int_{(a_n,t]}\int_{(x,t)}\frac{F_1(dy)}{F_1(y)(1-G(y^-))}F_1(dx)\Big]\\
&\le\; 2\frac{\sqrt{72}K^2}{n^{3/2}}\alpha^{3/2}\sqrt{M}\left[Mln(\frac{n}{c_1\alpha})+\int_{(a_n,t]}\frac{1}{F_1(x)}\int_{(x,t)}\frac{F_1(dy)}{1-G(y^-)}F_1(dx)\right]\\
&\le\; 2\frac{\sqrt{72}K^2}{n^{3/2}}\alpha^{3/2}\sqrt{M}\left[Mln(\frac{n}{c_1\alpha})+\int_{(-\infty,\infty)}\frac{F_1(dy)}{1-G(y^-)}\int_{(a_n,\infty)}\frac{F_1(dx)}{F_1(x)}\right]\\
&\le\; 2\frac{\sqrt{72}K^2}{n^{3/2}}\alpha^{3/2}\sqrt{M}\left[Mln(\frac{n}{c_1\alpha})+Mln(\frac{n}{c_1\alpha})\right] \le 4\frac{\sqrt{72}K^2}{n^{3/2}}\alpha^{3/2}M^{3/2}ln(\frac{n}{c_1\alpha})
\end{aligned}$$

It follows that

$$E\tilde{S}^2_{n3}(t) \;\leq\; \frac{2KM\alpha}{n}ln(\frac{n}{c_1\alpha}) + \frac{2K}{n^2}ln(\frac{n}{c_1\alpha}) + 18\frac{K^2\alpha M}{n}ln^2(\frac{n}{c_1\alpha}) + 4\frac{\sqrt{72}K^2}{n^{3/2}}\alpha^{3/2}M^{3/2}ln(\frac{n}{c_1\alpha}).$$

The proof is complete.

$\boxtimes$

Lemma A.2. *On the set $\tilde{\Omega}^0_n$ we have*

$$\begin{aligned} S^2_{n2}(t) \;\leq\;& \frac{K^4}{n^4}\sum_{i=1}^n 1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}\sum_{k\neq i}\sum_{l\neq i,k}\frac{1_{\{U_{1k}>U_{1i}\}}}{C(U_{1k})}\frac{1_{\{U_{1l}>U_{1i}\}}}{C(U_{1l})} + \frac{K^3}{n^3}\sum_{i=1}^n 1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}\sum_{k\neq i}\frac{1_{\{U_{1k}>U_{1i}\}}}{C(U_{1k})} \\ &+\frac{K^5}{n^5}\sum_{i=1}^n\sum_{j\neq i}1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1j}\leq t\}}\frac{\gamma(U_{1j})}{C(U_{1j})}\sum_{k\neq i,j}\sum_{l\neq i,j,k}\frac{1_{\{U_{1k}>U_{1i}\}}}{C^2(U_{1k})}\frac{1_{\{U_{1l}>U_{1j}\}}}{C(U_{1l})} \\ &+\frac{4K^4}{n^4}\sum_{i=1}^n\sum_{j\neq i}1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1j}\leq t\}}1_{\{U_{1i}>U_{1j}\}}\frac{\gamma(U_{1j})}{C(U_{1j})}\sum_{k\neq i,j}\frac{1_{\{U_{1k}>U_{1i}\}}}{C^2(U_{1k})} =: \tilde{S}^2_{n2}(t). \end{aligned}$$

Furthermore,

$$E\tilde{S}^2_{n2}(t) \leq \frac{K^4M\alpha}{n}ln^2(\frac{n}{c_1\alpha}) + \frac{K^3M\alpha}{n}ln(\frac{n}{c_1\alpha}) + \frac{K^5M\alpha}{n}ln^3(\frac{n}{c_1\alpha}) + \frac{4K^4\alpha}{n}ln(\frac{n}{c_1\alpha}).$$

Proof. To find the bound for $S^2_{n2}(t)$, recall

$$B_n(x) = n\int_{(x,b_n)}\ln\left(1-\frac{1}{nC_n(y)}\right)F^*_{1n}(dy) + \int_{(x,b_n)}\frac{F^*_{1n}(dy)}{C_n(y)}.$$

On the set $\Omega_1^{(n)}$, where there are ”no holes”, $nC_n(U_{1i}) \geq 2$ for $U_{1i} > U_{11:n}$. Since for $x \in [0, 1/2]$ we have $-x - x^2 \leq ln(1-x) \leq -x$, we obtain

$$0 \geq B_n(x) \geq -\frac{1}{n}\int_{(x,b_n)}\frac{F^*_{1n}(dy)}{C^2_n(y)}.$$

Hence

$$\begin{aligned} S^2_{n2}(t) \quad&\leq \left(\int_{(a_n,t]}\frac{\gamma(x)}{C_n(x)}\frac{1}{n}\int_{(x,b_n)}\frac{F^*_{1n}(dy)}{C^2_n(y)}F^*_{1n}(dx)\right)^2 \leq \frac{1}{n^4}\sum_{i=1}^n 1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma^2(U_{1i})}{C^2_n(U_{1i})}\left(\int_{(U_{1i},\infty)}\frac{F^*_{1n}(dy)}{C^2_n(y)}\right)^2 \\ &+\frac{1}{n^4}\sum_{i=1}^n\sum_{j\neq i}1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma(U_{1i})}{C_n(U_{1i})}1_{\{a_n<U_{1j}\leq t\}}\frac{\gamma(U_{1j})}{C_n(U_{1j})}\int_{(U_{1i},\infty)}\frac{F^*_{1n}(dy)}{C^2_n(y)}\int_{(U_{1j},\infty)}\frac{F^*_{1n}(dy)}{C^2_n(y)} \qquad \text{(A.4)} \end{aligned}$$

Since $nC_n(U_{1i}) \geq 1$ and, on the set $\tilde{\Omega}_n$, we have $C/C_n \leq K$, we obtain

$$\int_{(U_{1i},\infty)}\frac{F^*_{1n}(dy)}{C^2_n(y)}\int_{(U_{1j},\infty)}\frac{F^*_{1n}(dy)}{C^2_n(y)} = \frac{1}{n^2}\sum_{k\neq i}\sum_{l\neq j}\frac{1_{\{U_{1k}>U_{1i}\}}}{C^2_n(U_{1k})}\frac{1_{\{U_{1l}>U_{1j}\}}}{C^2_n(U_{1l})}$$

$$= \frac{1}{n^2}\sum_{k\neq i,j}\sum_{l\neq i,j,k}\frac{1_{\{U_{1k}>U_{1i}\}}}{C_n^2(U_{1k})}\frac{1_{\{U_{1l}>U_{1j}\}}}{C_n^2(U_{1l})} + \frac{1}{n^2}\sum_{k\neq i,j}\frac{1_{\{U_{1k}>U_{1i}\}}1_{\{U_{1k}>U_{1j}\}}}{C_n^4(U_{1k})} + \frac{1}{n^2}\sum_{l\neq i,j}\frac{1_{\{U_{1j}>U_{1i}\}}}{C_n^2(U_{1j})}\frac{1_{\{U_{1l}>U_{1j}\}}}{C_n^2(U_{1l})}$$
$$+\frac{1}{n^2}\sum_{k\neq i,j}\frac{1_{\{U_{1k}>U_{1i}\}}}{C_n^2(U_{1k})}\frac{1_{\{U_{1i}>U_{1j}\}}}{C_n^2(U_{1i})} \le \frac{1}{n}\sum_{k\neq i,j}\sum_{l\neq i,j,k}\frac{1_{\{U_{1k}>U_{1i}\}}}{C_n^2(U_{1k})}\frac{1_{\{U_{1l}>U_{1j}\}}}{C_n(U_{1l})} + \sum_{k\neq i,j}\frac{1_{\{U_{1k}>\max(U_{1i},U_{1j})\}}}{C_n^2(U_{1k})}$$
$$+\sum_{l\neq i,j}1_{\{U_{1j}>U_{1i}\}}\frac{1_{\{U_{1l}>U_{1j}\}}}{C_n^2(U_{1l})} + \sum_{k\neq i,j}\frac{1_{\{U_{1k}>U_{1i}\}}}{C_n^2(U_{1k})}1_{\{U_{1i}>U_{1j}\}} \le \frac{K^3}{n}\sum_{k\neq i,j}\sum_{l\neq i,j,k}\frac{1_{\{U_{1k}>U_{1i}\}}}{C^2(U_{1k})}\frac{1_{\{U_{1l}>U_{1j}\}}}{C(U_{1l})}$$
$$+K^2\sum_{k\neq i,j}\frac{1_{\{U_{1k}>max(U_{1i},U_{1j})\}}}{C^2(U_{1k})} + K^2\sum_{l\neq i,j}1_{\{U_{1j}>U_{1i}\}}\frac{1_{\{U_{1l}>U_{1j}\}}}{C^2(U_{1l})} + K^2\sum_{k\neq i,j}\frac{1_{\{U_{1k}>U_{1i}\}}}{C^2(U_{1k})}1_{\{U_{1i}>U_{1j}\}}$$

and therefore, by $C_n(U_{1k}) \geq 1/n$,

$$\begin{aligned}
\left(\int_{(U_{1i},\infty)}\frac{F_{1n}^*(dy)}{C_n^2(y)}\right)^2 &= \left(\frac{1}{n}\sum_{k=1}^n\frac{1_{\{U_{1k}>U_{1i}\}}}{C_n^2(U_{1k})}\right)^2 \le \left(\sum_{k=1}^n\frac{1_{\{U_{1k}>U_{1i}\}}}{C_n(U_{1k})}\right)^2 = \sum_{k\neq i}\sum_{l\neq i,k}\frac{1_{\{U_{1k}>U_{1i}\}}}{C_n(U_{1k})}\frac{1_{\{U_{1l}>U_{1i}\}}}{C_n(U_{1l})} \\
&\quad+\sum_{k\neq i}\frac{1_{\{U_{1k}>U_{1i}\}}}{C_n^2(U_{1k})} \le \sum_{k\neq i}\sum_{l\neq i,k}\frac{1_{\{U_{1k}>U_{1i}\}}}{C_n(U_{1k})}\frac{1_{\{U_{1l}>U_{1i}\}}}{C_n(U_{1l})} + n\sum_{k\neq i}\frac{1_{\{U_{1k}>U_{1i}\}}}{C_n(U_{1k})} \\
&\le K^2\sum_{k\neq i}\sum_{l\neq i,k}\frac{1_{\{U_{1k}>U_{1i}\}}}{C(U_{1k})}\frac{1_{\{U_{1l}>U_{1i}\}}}{C(U_{1l})} + Kn\sum_{k\neq i}\frac{1_{\{U_{1k}>U_{1i}\}}}{C(U_{1k})}.
\end{aligned}$$

Hence, for (A.4) using $C/C_n \le K$, we get

$$\begin{aligned}
(A.4) \quad \le\quad & \frac{K^4}{n^4}\sum_{i=1}^n 1_{\{a_n<U_{1i}\le t\}}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}\sum_{k\neq i}\sum_{l\neq i,k}\frac{1_{\{U_{1k}>U_{1i}\}}}{C(U_{1k})}\frac{1_{\{U_{1l}>U_{1i}\}}}{C(U_{1l})} \\
&+\frac{K^3}{n^3}\sum_{i=1}^n 1_{\{a_n<U_{1i}\le t\}}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}\sum_{k\neq i}\frac{1_{\{U_{1k}>U_{1i}\}}}{C(U_{1k})} \\
&+\frac{K^5}{n^5}\sum_{i=1}^n\sum_{j\neq i}1_{\{a_n<U_{1i}\le t\}}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1j}\le t\}}\frac{\gamma(U_{1j})}{C(U_{1j})}\sum_{k\neq i,j}\sum_{l\neq i,j,k}\frac{1_{\{U_{1k}>U_{1i}\}}}{C^2(U_{1k})}\frac{1_{\{U_{1l}>U_{1j}\}}}{C(U_{1l})} \\
&+\frac{K^4}{n^4}\sum_{i=1}^n\sum_{j\neq i}1_{\{a_n<U_{1i}\le t\}}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1j}\le t\}}\frac{\gamma(U_{1j})}{C(U_{1j})}\sum_{k\neq i,j}\frac{1_{\{U_{1k}>max(U_{1i},U_{1j})\}}}{C^2(U_{1k})} \\
&+\frac{K^4}{n^4}\sum_{i=1}^n\sum_{j\neq i}1_{\{a_n<U_{1i}\le t\}}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1j}\le t\}}\frac{\gamma(U_{1j})}{C(U_{1j})}\sum_{l\neq i,j}1_{\{U_{1j}>U_{1i}\}}\frac{1_{\{U_{1l}>U_{1j}\}}}{C^2(U_{1l})} \\
&+\frac{K^4}{n^4}\sum_{i=1}^n\sum_{j\neq i}1_{\{a_n<U_{1i}\le t\}}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1j}\le t\}}\frac{\gamma(U_{1j})}{C(U_{1j})}\sum_{k\neq i,j}\frac{1_{\{U_{1k}>U_{1i}\}}}{C^2(U_{1k})}1_{\{U_{1i}>U_{1j}\}} \qquad \text{(A.5)}
\end{aligned}$$

Now, since $1_{\{U_{1k}>max(U_{1i},U_{1j})\}} = 1_{\{U_{1k}>U_{1i}\}}1_{\{U_{1i}>U_{1j}\}} + 1_{\{U_{1k}>U_{1j}\}}1_{\{U_{1j}>U_{1i}\}}$ and using symmetry in i, j for the last three sums, we get

$$(A.5) \quad \le \quad \frac{K^4}{n^4}\sum_{i=1}^n 1_{\{a_n<U_{1i}\le t\}}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}\sum_{k\neq i}\sum_{l\neq i,k}\frac{1_{\{U_{1k}>U_{1i}\}}}{C(U_{1k})}\frac{1_{\{U_{1l}>U_{1i}\}}}{C(U_{1l})} + \frac{K^3}{n^3}\sum_{i=1}^n 1_{\{a_n<U_{1i}\le t\}}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}\sum_{k\neq i}\frac{1_{\{U_{1k}>U_{1i}\}}}{C(U_{1k})}$$

$$+\frac{K^5}{n^5}\sum_{i=1}^n\sum_{j\neq i}1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1j}\leq t\}}\frac{\gamma(U_{1j})}{C(U_{1j})}\sum_{k\neq i,j}\sum_{l\neq i,j,k}\frac{1_{\{U_{1k}>U_{1i}\}}}{C^2(U_{1k})}\frac{1_{\{U_{1l}>U_{1j}\}}}{C(U_{1l})}$$
$$+\frac{4K^4}{n^4}\sum_{i=1}^n\sum_{j\neq i}1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1j}\leq t\}}1_{\{U_{1i}>U_{1j}\}}\frac{\gamma(U_{1j})}{C(U_{1j})}\sum_{k\neq i,j}\frac{1_{\{U_{1k}>U_{1i}\}}}{C^2(U_{1k})}=\tilde{S}^2_{n2}(t)$$

Furthermore, for any m_1, m_2, we get

$$E\left(\sum_{k\neq i,j}\sum_{l\neq i,j,k}\frac{1_{\{U_{1k}>U_{1i}\}}}{C^{m_1}(U_{1k})}\frac{1_{\{U_{1l}>U_{1j}\}}}{C^{m_2}(U_{1l})}\middle|U_{1i},U_{1j}\right)\leq n^2\int_{(U_{1i},\infty)}\frac{F_1^*(dy)}{C^{m_1}(y)}\int_{(U_{1j},\infty)}\frac{F_1^*(dy)}{C^{m_2}(y)}.$$

Hence

$$\begin{aligned}E\tilde{S}^2_{n2}(t) \;\leq\; & \frac{K^4}{n}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\left(\int_{(x,\infty)}\frac{F_1^*(dy)}{C(y)}\right)^2F_1^*(dx)+\frac{K^3}{n}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\int_{(x,\infty)}\frac{F_1^*(dy)}{C(y)}F_1^*(dx)\\ &+\frac{K^5}{n}\left(\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left(\int_{(x,\infty)}\frac{F_1^*(dy)}{C^2(y)}\right)^{1/2}\left(\int_{(x,\infty)}\frac{F_1^*(dy)}{C(y)}\right)^{1/2}F_1^*(dx)\right)^2\\ &+\frac{4K^4}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,\infty)}\frac{\gamma(y)}{C(y)}\int_{(y,\infty)}\frac{F_1^*(dz)}{C^2(z)}F_1^*(dy)F_1^*(dx).\end{aligned}$$

Next, we consider each summand separately. Recall

$$dF_1^*=\alpha^{-1}(1-G-)dF_1, C=\alpha^{-1}(1-G-)F_1 \text{ and } \gamma=F_1.$$

By A1 and using Remark A.1, we get

$$\frac{K^4}{n}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\left(\int_{(x,\infty)}\frac{F_1^*(dy)}{C(y)}\right)^2F_1^*(dx)\leq\frac{K^4}{n}\int_{(a_n,t]}\frac{F_1(dx)}{1-G(x^-)}\left(\int_{(a_n,\infty)}\frac{F_1(dy)}{F_1(y)}\right)^2\leq\frac{K^4M\alpha}{n}ln^2(\frac{n}{c_1\alpha}),$$

$$\frac{K^3}{n}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\int_{(x,\infty)}\frac{F_1^*(dy)}{C(y)}F_1^*(dx)\leq\frac{K^3M\alpha}{n}ln(\frac{n}{c_1\alpha}),$$

$$\frac{K^5}{n}\left(\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left(\int_{(x,\infty)}\frac{F_1^*(dy)}{C^2(y)}\right)^{1/2}\left(\int_{(x,\infty)}\frac{F_1^*(dy)}{C(y)}\right)^{1/2}F_1^*(dx)\right)^2\leq\frac{K^5M\alpha}{n}ln^3(\frac{n}{c_1\alpha})$$

and

$$\frac{4K^4}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,\infty)}\frac{\gamma(y)}{C(y)}\int_{(y,\infty)}\frac{F_1^*(dz)}{C^2(z)}F_1^*(dy)F_1^*(dx)\leq\frac{4K^4\alpha}{n}ln(\frac{n}{c_1\alpha}).$$

The proof is complete.

⊠

Lemma A.3. *We have*

$$E(L^2_{n1}(t))\leq\frac{\alpha M}{n}$$

Proof. We have

$$L^2_{n1}(t) \;=\; \left(\int_{(-\infty,t]} \frac{\gamma(x)}{C(x)}(F^*_{1n}(dx) - F^*_1(dx))\right)^2.$$

Hence, using $dF^*_1 = \alpha^{-1}(1-G-)dF_1$, $C = \alpha^{-1}(1-G-)F_1$, $\gamma = F_1$ and A1, we obtain

$$E(L^2_{n1}(t)) \le \frac{1}{n}\int_{(-\infty,t]} \frac{\gamma^2(x)}{C^2(x)}F^*_1(dx) = \frac{\alpha}{n}\int_{(-\infty,t]} \frac{F_1(dx)}{1-G(x^-)} \le \frac{\alpha M}{n}$$

⊠

Lemma A.4. *On the set $\tilde{\Omega}^0_n$, we have*

$$\begin{aligned}
(S^b_{n1})^2(t) \;\le\;& 2K^2\frac{1}{n^2}\sum_{i=1}^n\sum_{j\ne i}\frac{\gamma(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C^2(U_{1i})}\frac{\gamma(U_{1j})1_{\{a_n<U_{1j}\le t\}}}{C^2(U_{1j})}\frac{1}{n^2}\Big|\sum_{k\ne i}\left(C(U_{1i})-1_{\{U_{1k}\le U_{1i}\le Z_k\}}\right)\\
&\sum_{l\ne j}\left(C(U_{1j})-1_{\{U_{1l}\le U_{1j}\le Z_l\}}\right)\Big| + \frac{2K}{n}\sum_{i=1}^n\frac{\gamma^2(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C^3(U_{1i})}\left(\frac{1}{n}\sum_{k\ne i}\left(C(U_{1i})-1_{\{U_{1k}\le U_{1i}\le Z_k\}}\right)\right)^2\\
&+2K^2\frac{1}{n^3}\sum_{i=1}^n\sum_{j\ne i}\frac{\gamma(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C^{3/2}(U_{1i})}\frac{\gamma(U_{1j})1_{\{a_n<U_{1j}\le t\}}}{C^{3/2}(U_{1j})} + 2\frac{1}{n^2}\sum_{i=1}^n\frac{\gamma^2(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C^2(U_{1i})} =: (\tilde{S}^b_{n1})^2(t).
\end{aligned}$$

Furthermore,

$$E(\tilde{S}^b_{n1})^2(t) \le \frac{K^2M\alpha}{n}ln^2(\frac{n}{c_1\alpha}) + \frac{4\sqrt{8}K^2}{n}\alpha M^2\frac{1}{\sqrt{c_1}} + \frac{4\sqrt{8}K^2}{n\sqrt{n}}\alpha^{3/2}M^{3/2}\sqrt{ln(\frac{n}{c_1\alpha})} + \frac{2(K+1)M\alpha}{n}$$

Proof.
We have

$$\begin{aligned}
(S^b_{n1})^2(t) \;=\;& \left(\int_{(a_n,t]}\frac{\gamma(x)}{C(x)C_n(x)}(C(x)-C_n(x))F^*_{1n}(dx)\right)^2\\
\le\;& \left(\frac{1}{n}\sum_{i=1}^n\frac{\gamma(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C(U_{1i})C_n(U_{1i})}\frac{1}{n}\left|\sum_{k=1}^n\left(C(U_{1i})-1_{\{U_{1k}\le U_{1i}\le Z_k\}}\right)\right|\right)^2\\
\le\;& 2\left(\frac{1}{n}\sum_{i=1}^n\frac{\gamma(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C(U_{1i})C_n(U_{1i})}\frac{1}{n}\left|\sum_{k\ne i}\left(C(U_{1i})-1_{\{U_{1k}\le U_{1i}\le Z_k\}}\right)\right|\right)^2\\
&+2\left(\frac{1}{n}\sum_{i=1}^n\frac{\gamma(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C(U_{1i})C_n(U_{1i})}\frac{1}{n}\left|C(U_{1i})-1\right|\right)^2\\
\le\;& 2\frac{1}{n^2}\sum_{i=1}^n\sum_{j\ne i}\frac{\gamma(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C(U_{1i})C_n(U_{1i})}\frac{\gamma(U_{1j})1_{\{a_n<U_{1j}\le t\}}}{C(U_{1j})C_n(U_{1j})}\frac{1}{n^2}\Big|\sum_{k\ne i}\sum_{l\ne j}\left(C(U_{1i})-1_{\{U_{1k}\le U_{1i}\le Z_k\}}\right)\times
\end{aligned}$$

$$\left(C(U_{1j}) - 1_{\{U_{1l}\le U_{1j}\le Z_l\}}\right)\Bigg| + \frac{2}{n^2}\sum_{i=1}^n \frac{\gamma^2(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C^2(U_{1i})C_n^2(U_{1i})}\left(\frac{1}{n}\sum_{k\neq i}\left(C(U_{1i}) - 1_{\{U_{1k}\le U_{1i}\le Z_k\}}\right)\right)^2$$
$$+2\left(\frac{1}{n^2}\sum_{i=1}^n \frac{\gamma(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C(U_{1i})C_n(U_{1i})}\right)^2. \qquad \text{(A.6)}$$

Furthermore, since $nC_n(U_{1i}) \ge 1$, for the last sum we have

$$2\left(\frac{1}{n^2}\sum_{i=1}^n \frac{\gamma(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C(U_{1i})C_n(U_{1i})}\right)^2 = \frac{2}{n^4}\sum_{i=1}^n\sum_{j=1}^n \frac{\gamma(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C(U_{1i})C_n(U_{1i})}\frac{\gamma(U_{1j})1_{\{a_n<U_{1j}\le t\}}}{C(U_{1j})C_n(U_{1j})}$$
$$\le \frac{2}{n^3}\sum_{i=1}^n\sum_{j\neq i} \frac{\gamma(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C(U_{1i})\sqrt{C_n(U_{1i})}}\frac{\gamma(U_{1j})1_{\{a_n<U_{1j}\le t\}}}{C(U_{1j})\sqrt{C_n(U_{1j})}} + \frac{2}{n^2}\sum_{i=1}^n \frac{\gamma^2(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C^2(U_{1i})}$$

Since on the set $\tilde{\Omega}_n$ we have $C/C_n \le K$ and using that $nC_n(U_{1i}) \ge 1$ for the second sum, we get

$$(A.6) \qquad \le 2K^2\frac{1}{n^2}\sum_{i=1}^n\sum_{j\neq i}\frac{\gamma(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C^2(U_{1i})}\frac{\gamma(U_{1j})1_{\{a_n<U_{1j}\le t\}}}{C^2(U_{1j})}\frac{1}{n^2}\Bigg|\sum_{k\neq i}\left(C(U_{1i}) - 1_{\{U_{1k}\le U_{1i}\le Z_k\}}\right)$$
$$\sum_{l\neq j}\left(C(U_{1j}) - 1_{\{U_{1l}\le U_{1j}\le Z_l\}}\right)\Bigg| + \frac{2K}{n}\sum_{i=1}^n \frac{\gamma^2(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C^3(U_{1i})}\left(\frac{1}{n}\sum_{k\neq i}\left(C(U_{1i}) - 1_{\{U_{1k}\le U_{1i}\le Z_k\}}\right)\right)^2$$
$$+2K^2\frac{1}{n^3}\sum_{i=1}^n\sum_{j\neq i}\frac{\gamma(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C^{3/2}(U_{1i})}\frac{\gamma(U_{1j})1_{\{a_n<U_{1j}\le t\}}}{C^{3/2}(U_{1j})} + 2\frac{1}{n^2}\sum_{i=1}^n\frac{\gamma^2(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C^2(U_{1i})} = (\tilde{S}_{n1}^b)^2(t).$$

Before we compute the expectation of $(\tilde{S}_{n1}^b)^2(t)$, for fixed i, j we bound:

$$E\left(\Bigg|\sum_{k\neq i}\left(C(U_{1i}) - 1_{\{U_{1k}\le U_{1i}\le Z_k\}}\right)\sum_{l\neq j}\left(C(U_{1j}) - 1_{\{U_{1l}\le U_{1j}\le Z_l\}}\right)\Bigg| \,|U_{1i}, U_{1j}, Z_i, Z_j\right).$$

Taking w.l.o.g. $i = 1, j = 2$, we have:

$$E\left(\Bigg|\sum_{k\neq 1}\left(C(U_{11}) - 1_{\{U_{1k}\le U_{11}\le Z_k\}}\right)\sum_{l\neq 2}\left(C(U_{12}) - 1_{\{U_{1l}\le U_{12}\le Z_l\}}\right)\Bigg| \,|U_{11} = x_1, U_{12} = x_2, Z_1 = y_1, Z_2 = y_2\right)$$
$$= E\Bigg(\Bigg|\sum_{k\neq 1,2}\left(C(U_{11}) - 1_{\{U_{1k}\le U_{11}\le Z_k\}}\right) + \left(C(U_{11}) - 1_{\{U_{12}\le U_{11}\le Z_2\}}\right)\Bigg|$$
$$\Bigg|\sum_{l\neq 1,2}\left(C(U_{12}) - 1_{\{U_{1l}\le U_{12}\le Z_l\}}\right) + \left(C(U_{12}) - 1_{\{U_{11}\le U_{12}\le Z_1\}}\right)\Bigg| \,|U_{11} = x_1, U_{12} = x_2, Z_1 = y_1, Z_2 = y_2\Bigg)$$
$$= E\Bigg(\Bigg|\sum_{k\neq 1,2}\left(C(x_1) - 1_{\{U_{1k}\le x_1\le Z_k\}}\right) + \left(C(x_1) - 1_{\{x_2\le x_1\le y_2\}}\right)\Bigg|$$
$$\Bigg|\sum_{l\neq 1,2}\left(C(x_2) - 1_{\{U_{1l}\le x_2\le Z_l\}}\right) + \left(C(x_2) - 1_{\{x_1\le x_2\le y_1\}}\right)\Bigg|\Bigg).$$

Hence, by Cauchy-Schwarz, the right side to the power 2 is less than or equal to

$$E\Big(\sum_{k\neq 1,2}\big(C(x_1)-1_{\{U_{1k}\le x_1\le Z_k\}}\big)+\big(C(x_1)-1_{\{x_2\le x_1\le y_2\}}\big)\Big)^2$$
$$\times E\Big(\sum_{l\neq 1,2}\big(C(x_2)-1_{\{U_{1l}\le x_2\le Z_l\}}\big)+\big(C(x_2)-1_{\{x_1\le x_2\le y_1\}}\big)\Big)^2.$$

Furthermore, since $C^2\le C$, we have

$$\begin{aligned}E\Big(\sum_{k\neq 1,2}\big(C(x_1)-1_{\{U_{1k}\le x_1\le Z_k\}}\big)+\big(C(x_1)-1_{\{x_2\le x_1\le y_2\}}\big)\Big)^2 &\le 2(n-2)C(x_1)+4C^2(x_1)+41_{\{x_2\le x_1\le y_2\}}\\ &\le 2nC(x_1)+41_{\{x_2\le x_1\le y_2\}}.\end{aligned}$$

Finally, since $U_{11}\neq U_{12}$,

$$\begin{aligned}&E\left(\Big|\sum_{k\neq 1}\big(C(U_{11})-1_{\{U_{1k}\le U_{11}\le Z_k\}}\big)\sum_{l\neq 2}\big(C(U_{12})-1_{\{U_{1l}\le U_{12}\le Z_l\}}\big)\Big|\,|U_{11},U_{12},Z_1,Z_2\right)\\ &\le 2n\sqrt{C(U_{11})C(U_{12})}+\sqrt{8n}\sqrt{C(U_{12})}1_{\{U_{12}\le U_{11}\le Z_2\}}+\sqrt{8n}\sqrt{C(U_{11})}1_{\{U_{11}\le U_{12}\le Z_1\}}\\ &\le 2n\sqrt{C(U_{11})C(U_{12})}+\sqrt{8n}\sqrt{C(U_{12})}1_{\{U_{12}\le U_{11}\}}+\sqrt{8n}\sqrt{C(U_{11})}1_{\{U_{11}\le U_{12}\}}.\end{aligned}$$

For the expectation of the first sum in $(\tilde S^b_{n1})^2(t)$ we have:

$$\begin{aligned}&E\Big(2K^2\frac{1}{n^2}\sum_{i=1}^n\sum_{j\neq i}\frac{\gamma(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C^2(U_{1i})}\frac{\gamma(U_{1j})1_{\{a_n<U_{1j}\le t\}}}{C^2(U_{1j})}\frac{1}{n^2}\Big|\sum_{k\neq i}\big(C(U_{1i})-1_{\{U_{1k}\le U_{1i}\le Z_k\}}\big)\times\\ &\quad\sum_{l\neq j}\big(C(U_{1j})-1_{\{U_{1l}\le U_{1j}\le Z_l\}}\big)\Big|\Big)=2K^2\frac{n(n-1)}{n^4}E\Big(\frac{\gamma(U_{11})1_{\{a_n<U_{11}\le t\}}}{C^2(U_{11})}\frac{\gamma(U_{12})1_{\{a_n<U_{12}\le t\}}}{C^2(U_{12})}\times\\ &\Big|\sum_{k\neq 1}\big(C(U_{11})-1_{\{U_{1k}\le U_{11}\le Z_k\}}\big)\sum_{l\neq 2}\big(C(U_{12})-1_{\{U_{1l}\le U_{12}\le Z_l\}}\big)\Big|\Big)=E(E(...|U_{11},U_{12},Z_1,Z_2))\\ \le\;& 4K^2\frac{n^2(n-1)}{n^4}E\Big(\frac{\gamma(U_{11})1_{\{a_n<U_{11}\le t\}}}{C^{3/2}(U_{11})}\frac{\gamma(U_{12})1_{\{a_n<U_{12}\le t\}}}{C^{3/2}(U_{12})}\Big)+4K^2\frac{n(n-1)}{n^4}\sqrt{8n}\times\\ &E\Big(\frac{\gamma(U_{11})1_{\{a_n<U_{11}\le t\}}}{C^{3/2}(U_{11})}\frac{\gamma(U_{12})1_{\{a_n<U_{12}\le t\}}}{C^2(U_{12})}1_{\{U_{11}\le U_{12}\}}\Big)\\ \le\;&\frac{4K^2}{n}\left(\int_{(a_n,t]}\frac{\gamma(x)}{C^{3/2}(x)}F_1^*(dx)\right)^2+\frac{4\sqrt{8}K^2}{n\sqrt{n}}\int_{(a_n,t]}\frac{\gamma(x)}{C^{3/2}(x)}\int_{[x,\infty)}\frac{\gamma(x)}{C^2(y)}F_1^*(dy)F_1^*(dx).\end{aligned}$$

As to the second sum in $(\tilde S^b_{n1})^2(t)$ we have

$$E\Big(\frac{2K}{n}\sum_{i=1}^n\frac{\gamma^2(U_{1i})1_{\{a_n<U_{1i}\le t\}}}{C^3(U_{1i})}\left(\frac{1}{n}\sum_{k\neq i}\big(C(U_{1i})-1_{\{U_{1k}\le U_{1i}\le Z_k\}}\big)\right)^2\Big)$$

$$= \; 2K\frac{1}{n^2}E\Bigg(\frac{\gamma^2(U_{11})1_{\{a_n<U_{11}\le t\}}}{C^3(U_{11})}\Bigg(\sum_{k\neq 1}\big(C(U_{11})-1_{\{U_{1k}\le U_{11}\le Z_k\}}\big)\Bigg)^2\Bigg) = 2K\frac{1}{n^2}E(E(...|U_{11}))$$

$$\le \; \frac{2K}{n}E\Bigg(\frac{\gamma^2(U_{11})1_{\{a_n<U_{11}\le t\}}}{C^2(U_{11})}\Bigg) = \frac{2K}{n}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}F_1^*(dx)$$

Hence

$$\begin{aligned} E((\tilde S^b_{n1})^2(t)) \;\le\; & \frac{4K^2}{n}\Bigg(\int_{(a_n,t]}\frac{\gamma(x)}{C^{3/2}(x)}F_1^*(dx)\Bigg)^2 + \frac{4\sqrt{8}K^2}{n\sqrt{n}}\int_{(a_n,t]}\frac{\gamma(x)}{C^{3/2}(x)}\int_{[x,\infty)}\frac{\gamma(x)}{C^2(y)}F_1^*(dy) \\ & +\frac{2K}{n}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}F_1^*(dx) + \frac{2K^2}{n}\Bigg(\int_{(a_n,t]}\frac{\gamma(x)}{C^{3/2}(x)}F_1^*(dx)\Bigg)^2 + \frac{2}{n}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}F_1^*(dx) \\ =\; & \frac{6K^2}{n}\Bigg(\int_{(a_n,t]}\frac{\gamma(x)}{C^{3/2}(x)}F_1^*(dx)\Bigg)^2 + \frac{4\sqrt{8}K^2}{n\sqrt{n}}\int_{(a_n,t]}\frac{\gamma(x)}{C^{3/2}(x)}\int_{[x,\infty)}\frac{\gamma(x)}{C^2(y)}F_1^*(dy)F_1^*(dx) \\ & +\frac{2(K+1)}{n}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}F_1^*(dx). \end{aligned}$$

Now we need to bound each of the terms separately. As to the first we obtain

$$\begin{aligned} & \frac{6K^2}{n}\Bigg(\int_{(a_n,t]}\frac{\gamma(x)}{C^{3/2}(x)}F_1^*(dx)\Bigg)^2 \le \frac{K^2\alpha}{n}\Bigg(\int_{(a_n,t]}\frac{1}{\sqrt{F_1(x)(1-G(x^-))}}F_1(dx)\Bigg)^2 \\ & \le \frac{K^2\alpha}{n}ln^2((F_1(a_n))^{-1})\int_{(-\infty,t]}\frac{F_1(dx)}{(1-G(x^-))} \le \frac{K^2M\alpha}{n}ln^2(\frac{n}{c_1\alpha}). \end{aligned}$$

As to the second, since by A1 $\int\frac{F_1(dx)}{1-G(x^-)}\le M$ and $1-F(x)\le M(1-G(x^-))\le M\sqrt{1-G(x^-)}$, we get

$$\begin{aligned} & \frac{4\sqrt{8}K^2}{n\sqrt{n}}\int_{(a_n,t]}\frac{\gamma(x)}{C^{3/2}(x)}\int_{[x,\infty)}\frac{\gamma(x)}{C^2(y)}F_1^*(dy)F_1^*(dx) \\ & = \frac{4\sqrt{8}K^2}{n\sqrt{n}}\alpha^{3/2}\int_{(a_n,t]}\frac{1}{\sqrt{F_1(x)(1-G(x^-))}}\int_{[x,\infty)}\frac{F_1(dy)}{F_1(y)(1-G(y^-))}F_1(dx) \\ & \le \frac{4\sqrt{8}K^2}{n\sqrt{n}}\alpha^{3/2}\int_{(a_n,t]}\frac{1}{\sqrt{F_1(x)(1-G(x^-))}}\frac{1}{F_1(x)}\int_{[x,\infty)}\frac{F_1(dy)}{1-G(y^-)}F_1(dx) \\ & \le \frac{4\sqrt{8}K^2}{n\sqrt{n}}\alpha^{3/2}M\int_{(a_n,t]}\frac{1}{\sqrt{F_1(x)(1-G(x^-))}}\Bigg(\frac{1-F_1(x)}{F_1(x)}+1\Bigg)F_1(dx) \\ & \le \frac{4\sqrt{8}K^2}{n\sqrt{n}}\alpha^{3/2}M^2\int_{(a_n,t]}\frac{1}{F_1^{3/2}(x)}F_1(dx) + \frac{4\sqrt{8}K^2}{n\sqrt{n}}\alpha^{3/2}M\int_{(a_n,t]}\frac{F_1(dx)}{\sqrt{F_1(x)(1-G(x^-))}} \\ & \le \frac{4\sqrt{8}K^2}{n\sqrt{n}}\alpha^{3/2}M^2\frac{1}{\sqrt{F_1(a_n)}} + \frac{4\sqrt{8}K^2}{n\sqrt{n}}\alpha^{3/2}M\sqrt{\int_{(a_n,t]}\frac{F_1(dx)}{F_1(x)}}\sqrt{\int_{(a_n,t]}\frac{F_1(dx)}{1-G(x^-)}}. \end{aligned}$$

According to Remark A.1, $F_1(a_n)\ge\frac{c_1\alpha}{n}$. Conclude that the right side is less than or equal to

$$\frac{4\sqrt{8}K^2}{n}\alpha M^2\frac{1}{\sqrt{c_1}} + \frac{4\sqrt{8}K^2}{n\sqrt{n}}\alpha^{3/2}M^{3/2}\sqrt{ln(\frac{n}{c_1\alpha})}.$$

As to the third one:

$$\frac{2(K+1)}{n}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}F_1^*(dx) \leq \frac{2(K+1)M\alpha}{n}.$$

The proof is complete.

$\boxtimes$

Lemma A.5. *On the set $\tilde{\Omega}_n^0$ we have*

$$\begin{aligned} S_{n4}^2(t) \quad & \leq K^2\frac{1}{n^2}\sum_{i=1}^n 1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}(\tilde{D}_{n2}(U_{1i}))^2 \\ & +K^2\frac{1}{n^2}\sum_{i=1}^n\sum_{j\neq i}1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1j}\leq t\}}\frac{\gamma(U_{1j})}{C(U_{1j})}|\hat{D}_{n2}(U_{1i})||\hat{D}_{n2}(U_{1j})| =: \tilde{S}_{n4}^2(t), \end{aligned}$$

where

$$\tilde{D}_{n2}(U_{1i}) = \frac{K}{n}\sum_{k=1}^n\sum_{l\neq k}1_{\{x<U_{1k},U_{1l}\leq b_n\}}\frac{|C_n(U_{1k})-C(U_{1k})|}{C^{3/2}(U_{1k})}\frac{|C_n(U_{1l})-C(U_{1l})|}{C^{3/2}(U_{1l})}$$

and

$$\hat{D}_{n2}(U_{1i}) = \frac{K^2}{n^2}\sum_{k=1}^n\sum_{l\neq k}1_{\{x<U_{1k}\leq b_n\}}1_{\{x<U_{1l}\leq b_n\}}\frac{|C_n(U_{1k})-C(U_{1k})|}{C^2(U_{1k})}\frac{|C_n(U_{1l})-C(U_{1l})|}{C^2(U_{1l})}.$$

Furthermore,

$$\begin{aligned} E\tilde{S}_{n4}^2(t) \quad \leq \quad & \frac{14K^2}{n}\alpha M ln(\frac{n}{c_1\alpha}) + \frac{32K^2M}{nc_1} + \frac{2K^2}{n}\alpha M ln^2(\frac{n}{c_1\alpha}) + \frac{32K^2}{n}\alpha M ln(\frac{n}{c_1\alpha})\left(\frac{M}{\sqrt{c_1}} + \sqrt{M\alpha ln(\frac{n}{c_1\alpha})}\right) \\ & +\frac{128K^2}{n}\alpha M\left(\frac{M}{\sqrt{c_1}} + \sqrt{M\alpha ln(\frac{n}{c_1\alpha})}\right)^2 + \frac{AK^2}{n}ln^2(\frac{n}{c_1\alpha}) + 2\frac{\sqrt{A}\sqrt{B}K^2}{n}2\alpha M ln(\frac{n}{c_1\alpha}) \\ & +2\frac{\sqrt{A}\sqrt{C}K^2}{n}\alpha M ln(\frac{n}{c_1\alpha})\left(ln(\frac{n}{c_1\alpha})+1\right), \end{aligned}$$

where

$$A = 28KM\alpha + \frac{32MK\alpha}{c} + 2K^2\alpha M ln(\frac{n}{c_1\alpha}) + 32K^2\alpha M\left(\sqrt{\frac{M}{c_1}} + \sqrt{\frac{1}{c}}ln(\frac{n}{c_1\alpha})\right) + 128K^2 * 4M^2\alpha$$

$$B = 18K + \frac{8K}{cc_1\alpha} + 2K^2\alpha^2\sqrt{Mln(\frac{n}{c_1\alpha})} + 16K^2\left(\sqrt{\frac{M}{c_1}} + \sqrt{\frac{1}{c}}ln(\frac{n}{c_1\alpha})\right) + \frac{128K^2}{c_1}2M$$

and

$$C = 2K^2 M\alpha^2 + \frac{32K^2 M\alpha^2}{\sqrt{c_1}}.$$

Proof.
Before we consider the term $S_{n4}^2(t)$, we derive two bounds for $D_{n2}(x)$. Since $C_n(U_{1i}) \geq 1/n$ and $C/C_n \leq K$ on $\tilde{\Omega}_n$, we get

$$\begin{aligned}
D_{n2}(x)^2 \quad &= \left(\int_{(x,b_n)} \frac{C_n(y)-C(y)}{C_n(y)C(y)} F_{1n}^*(dy)\right)^2 = \frac{1}{n^2}\sum_{k=1}^n 1_{\{x<U_{1k}\leq b_n\}} \frac{(C_n(U_{1k})-C(U_{1k}))^2}{C_n^2(U_{1k})C^2(U_{1k})} \\
&+ \frac{1}{n^2}\sum_{k=1}^n \sum_{l\neq k} 1_{\{x<U_{1k}\leq b_n\}} 1_{\{x<U_{1l}\leq b_n\}} \frac{C_n(U_{1k})-C(U_{1k})}{C_n(U_{1k})C(U_{1k})} \frac{C_n(U_{1l})-C(U_{1l})}{C_n(U_{1l})C(U_{1l})} \\
&\leq \sum_{k=1}^n 1_{\{x<U_{1k}\leq b_n\}} \frac{(C_n(U_{1k})-C(U_{1k}))^2}{n^2 C_n^2(U_{1k})C^2(U_{1k})} \\
&+ \frac{1}{n}\sum_{k=1}^n \sum_{l\neq k} 1_{\{x<U_{1k},U_{1l}\leq b_n\}} \frac{|C_n(U_{1k})-C(U_{1k})|}{\sqrt{nC_n(U_{1k})}\sqrt{C_n(U_{1k})}C(U_{1k})} \frac{|C_n(U_{1l})-C(U_{1l})|}{\sqrt{nC_n(U_{1l})}\sqrt{C_n(U_{1l})}C(U_{1l})} \\
&\leq \sum_{k=1}^n 1_{\{x<U_{1k}\leq b_n\}} \frac{(C_n(U_{1k})-C(U_{1k}))^2}{C^2(U_{1k})} \\
&+ \frac{K}{n}\sum_{k=1}^n \sum_{l\neq k} 1_{\{x<U_{1k},U_{1l}\leq b_n\}} \frac{|C_n(U_{1k})-C(U_{1k})|}{C^{3/2}(U_{1k})} \frac{|C_n(U_{1l})-C(U_{1l})|}{C^{3/2}(U_{1l})} = \tilde{D}_{n2}(x)^2
\end{aligned}$$

and

$$\begin{aligned}
D_{n2}(x)^2 \quad &= \left(\int_{(x,b_n)} \frac{C_n(y)-C(y)}{C_n(y)C(y)} F_{1n}^*(dy)\right)^2 = \frac{1}{n^2}\sum_{k=1}^n 1_{\{x<U_{1k}\leq b_n\}} \frac{(C_n(U_{1k})-C(U_{1k}))^2}{C_n^2(U_{1k})C^2(U_{1k})} \\
&+ \frac{1}{n^2}\sum_{k=1}^n \sum_{l\neq k} 1_{\{x<U_{1k}\leq b_n\}} 1_{\{x<U_{1l}\leq b_n\}} \frac{C_n(U_{1k})-C(U_{1k})}{C_n(U_{1k})C(U_{1k})} \frac{C_n(U_{1l})-C(U_{1l})}{C_n(U_{1l})C(U_{1l})} \\
&\leq \frac{K}{n}\sum_{k=1}^n 1_{\{x<U_{1k}\leq b_n\}} \frac{(C_n(U_{1k})-C(U_{1k}))^2}{C^3(U_{1k})} \\
&+ \frac{K^2}{n^2}\sum_{k=1}^n \sum_{l\neq k} 1_{\{x<U_{1k}\leq b_n\}} 1_{\{x<U_{1l}\leq b_n\}} \frac{|C_n(U_{1k})-C(U_{1k})|}{C^2(U_{1k})} \frac{|C_n(U_{1l})-C(U_{1l})|}{C^2(U_{1l})} =: \hat{D}_{n2}(x)^2.
\end{aligned}$$

Furthermore, for fixed i, j, k, l, by $(a+b+c)^2 \leq 2a^2 + 4b^2 + 4c^2$, we get

$$\begin{aligned}
E((C_n(U_{1k}) - C(U_{1k}))^2 | U_{1i}, U_{1k}, Z_i, Z_k) &\leq E\Bigg(2\left(\frac{1}{n}\sum_{m\neq i,k} [1_{\{U_{1m}\leq U_{1k}\leq Z_m\}} - C(U_{1k})]\right)^2 \\
&+ \frac{4}{n^2}(1_{\{U_{1i}\leq U_{1k}\leq Z_i\}} - C(U_{1k}))^2 + \frac{4}{n^2}(1_{\{U_{1k}\leq U_{1k}\leq Z_k\}} - C(U_{1k}))^2 | U_{1i}, U_{1k}, Z_i, Z_k\Bigg)
\end{aligned}$$

$$\leq 2\frac{n-2}{n^2}C(U_{1k}) + \frac{8}{n^2}1_{\{U_{1i}\leq U_{1k}\leq Z_i\}} + \frac{8}{n^2}C^2(U_{1k}) + \frac{8}{n^2} + \frac{8}{n^2}C^2(U_{1k})$$
$$\leq \frac{14}{n}C(U_{1k}) + \frac{16}{n^2}$$

and therefore

$$\begin{aligned}
&E(|C_n(U_{1k}) - C(U_{1k})||C_n(U_{1l}) - C(U_{1l})||U_{1i}, U_{1j}, U_{1k}, U_{1l}, Z_i, Z_j, Z_k, Z_l) \\
&= E\Bigg(\Big|\frac{1}{n}\sum_{m\neq i,j,k,l}[1_{\{U_{1m}\leq U_{1k}\leq Z_m\}} - C(U_{1k})] + \frac{1_{\{U_{1i}\leq U_{1k}\leq Z_i\}}}{n} + \frac{1_{\{U_{1j}\leq U_{1k}\leq Z_j\}}}{n} + \frac{1}{n} + \frac{1_{\{U_{1l}\leq U_{1k}\leq Z_l\}}}{n} \\
&\quad -\frac{4}{n}C(U_{1k})\Big|\Big|\frac{1}{n}\sum_{s\neq i,j,k,l}[1_{\{U_{1s}\leq U_{1l}\leq Z_s\}} - C(U_{1l})] + \frac{1_{\{U_{1i}\leq U_{1l}\leq Z_i\}}}{n} + \frac{1_{\{U_{1j}\leq U_{1l}\leq Z_j\}}}{n} + \frac{1_{\{U_{1k}\leq U_{1l}\leq Z_k\}}}{n} \\
&\quad +\frac{1}{n} - \frac{4}{n}C(U_{1l})\Big||U_{1i}, U_{1j}, U_{1k}, U_{1l}, Z_i, Z_j, Z_k, Z_l)\Bigg) \leq E\Bigg(\left\{\left|\frac{1}{n}\sum_{m\neq i,j,k,l}[1_{\{U_{1m}\leq U_{1k}\leq Z_m\}} - C(U_{1k})]\right| + \frac{8}{n}\right\} \\
&\quad \times\left\{\left|\frac{1}{n}\sum_{s\neq i,j,k,l}[1_{\{U_{1s}\leq U_{1l}\leq Z_s\}} - C(U_{1l})]\right| + \frac{8}{n}\right\}|U_{1i}, U_{1j}, U_{1k}, U_{1l}, Z_i, Z_j, Z_k, Z_l)\Bigg) \\
&\leq \sqrt{\frac{2}{n^2}(n-4)C(U_{1k}) + \frac{2*64}{n^2}}\sqrt{\frac{2}{n^2}(n-4)C(U_{1l}) + \frac{2*64}{n^2}} \\
&\leq \frac{2}{n}\sqrt{C(U_{1k})C(U_{1l})} + \frac{16}{n\sqrt{n}}\sqrt{C(U_{1k})} + \frac{16}{n\sqrt{n}}\sqrt{C(U_{1l})} + \frac{128}{n^2}.
\end{aligned} \tag{A.7}$$

Coming back to our goal, we obtain

$$\begin{aligned}
S_{n4}^2(t) \quad &\leq \left(\int_{(a_n,t]}\frac{\gamma(x)}{C_n(x)}|D_{n2}(x)|F_{1n}^*(dx)\right)^2 = \frac{1}{n^2}\sum_{i=1}^n 1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma^2(U_{1i})}{C_n^2(U_{1i})}(D_{n2}(U_{1i}))^2 \\
&+\frac{1}{n^2}\sum_{i=1}^n\sum_{j\neq i}1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma(U_{1i})}{C_n(U_{1i})}1_{\{a_n<U_{1j}\leq t\}}\frac{\gamma(U_{1j})}{C_n(U_{1j})}|D_{n2}(U_{1i})||D_{n2}(U_{1j})| \\
&\leq K^2\frac{1}{n^2}\sum_{i=1}^n 1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}(\tilde{D}_{n2}(U_{1i}))^2 \\
&+K^2\frac{1}{n^2}\sum_{i=1}^n\sum_{j\neq i}1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1j}\leq t\}}\frac{\gamma(U_{1j})}{C(U_{1j})}|\hat{D}_{n2}(U_{1i})||\hat{D}_{n2}(U_{1j})| =: \tilde{S}_{n4}^2(t).
\end{aligned}$$

To compute the expectation, for the first sum in $\tilde{S}_{n4}^2(t)$ we have

$$\begin{aligned}
&E\left(K^2\frac{1}{n^2}\sum_{i=1}^n 1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}(\tilde{D}_{n2}(U_{1i}))^2\right) = K^2\frac{1}{n}E\left(1_{\{a_n<U_{11}\leq t\}}\frac{\gamma^2(U_{11})}{C^2(U_{11})}(\tilde{D}_{n2}(U_{11}))^2\right) \\
&\leq K^2\frac{1}{n}E\left(1_{\{a_n<U_{11}\leq t\}}\frac{\gamma^2(U_{11})}{C^2(U_{11})}\sum_{k\neq 1}1_{\{U_{11}<U_{1k}\leq b_n\}}\frac{(C_n(U_{1k}) - C(U_{1k}))^2}{C^2(U_{1k})}\right) \\
&+K^2\frac{1}{n}E\left(1_{\{a_n<U_{11}\leq t\}}\frac{\gamma^2(U_{11})}{C^2(U_{11})}\frac{K}{n}\sum_{k\neq 1}\sum_{l\neq k,1}1_{\{x<U_{1k},U_{1l}\leq b_n\}}\frac{|C_n(U_{1k}) - C(U_{1k})|}{C^{3/2}(U_{1k})}\frac{|C_n(U_{1l}) - C(U_{1l})|}{C^{3/2}(U_{1l})}\right)
\end{aligned}$$

$$
\begin{aligned}
\leq\ & K^2 E\left(1_{\{a_n<U_{11}\leq t\}}\frac{\gamma^2(U_{11})}{C^2(U_{11})}1_{\{U_{11}<U_{12}\leq b_n\}}\frac{(C_n(U_{12})-C(U_{12}))^2}{C^2(U_{12})}\right) \\
& +K^3 E\left(1_{\{a_n<U_{11}\leq t\}}\frac{\gamma^2(U_{11})}{C^2(U_{11})}1_{\{x<U_{12},U_{13}\leq b_n\}}\frac{|C_n(U_{12})-C(U_{12})|}{C^{3/2}(U_{12})}\frac{|C_n(U_{13})-C(U_{13})|}{C^{3/2}(U_{13})}\right) \\
=\ & K^2 E\left(1_{\{a_n<U_{11}\leq t\}}\frac{\gamma^2(U_{11})}{C^2(U_{11})}\frac{1_{\{U_{11}<U_{12}\leq b_n\}}}{C^2(U_{12})}E\left(C_n(U_{12})-C(U_{12}))^2|U_{11},U_{12}\right)\right) \\
& +K^3 E\left(\frac{\gamma^2(U_{11})1_{\{a_n<U_{11}\leq t\}}}{C^2(U_{11})}\frac{1_{\{x<U_{12},U_{13}\leq b_n\}}}{C^{3/2}(U_{12})C^{3/2}(U_{13})}E\left(|C_n(U_{12})-C(U_{12})||C_n(U_{13})-C(U_{13})||U_{11},U_{12},U_{13}\right)\right)
\end{aligned}
$$

The first expectation is bounded from above by

$$
\begin{aligned}
& K^2 E\left(1_{\{a_n<U_{11}\leq t\}}\frac{\gamma^2(U_{11})}{C^2(U_{11})}\frac{1_{\{U_{11}<U_{12}\leq b_n\}}}{C^2(U_{12})}\left[\frac{14}{n}C(U_{12})+\frac{16}{n^2}\right]\right)\leq\frac{14}{n}K^2\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C(y)}F_1^*(dx) \\
& +\frac{16}{n^2}K^2\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}F_1^*(dx)
\end{aligned}
$$

The second expectation is bounded from above by

$$
\begin{aligned}
& K^2 E\Bigg(1_{\{a_n<U_{11}\leq t\}}\frac{\gamma^2(U_{11})}{C^2(U_{11})}\frac{1_{\{x<U_{12},U_{13}\leq b_n\}}}{C^{3/2}(U_{12})C^{3/2}(U_{13})}\Bigg[\frac{2}{n}\sqrt{C(U_{12})C(U_{13})}+\frac{16}{n\sqrt{n}}\sqrt{C(U_{12})} \\
& +\frac{16}{n\sqrt{n}}\sqrt{C(U_{13})}+\frac{128}{n^2}\Bigg]\Bigg)=\frac{2K^2}{n}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C(y)}\right)^2F_1^*(dx) \\
& +\frac{32K^2}{n\sqrt{n}}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C(y)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}F_1^*(dx)+\frac{128K^2}{n^2}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}\right)^2F_1^*(dx).
\end{aligned}
$$

Now we will deal with the second sum in $\tilde{S}_{n4}^2(t)$. Its expectation equals

$$
\begin{aligned}
& K^2\frac{1}{n^2}E\Bigg(\sum_{i=1}^{n}\sum_{j\neq i}1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1j}\leq t\}}\frac{\gamma(U_{1j})}{C(U_{1j})}|\hat{D}_{n2}(U_{1i})||\hat{D}_{n2}(U_{1j})|\Bigg) \\
& \quad\leq\ K^2 E\Bigg(1_{\{a_n<U_{11},U_{12}\leq t\}}\frac{\gamma(U_{11})}{C(U_{11})}\frac{\gamma(U_{12})}{C(U_{12})}|\hat{D}_{n2}(U_{11})||\hat{D}_{n2}(U_{12})|\Bigg) \\
& \quad=\ K^2 E\Bigg(1_{\{a_n<U_{11},U_{12}\leq t\}}\frac{\gamma(U_{11})}{C(U_{11})}\frac{\gamma(U_{12})}{C(U_{12})}E\left[|\hat{D}_{n2}(U_{11})||\hat{D}_{n2}(U_{12})|\Big|U_{11},U_{12},Z_1,Z_2\right]\Bigg).
\end{aligned}
$$

By Cauchy-Schwarz, the right side is less than or equal to

$$
K^2 E\Bigg(1_{\{a_n<U_{11},U_{12}\leq t\}}\frac{\gamma(U_{11})}{C(U_{11})}\frac{\gamma(U_{12})}{C(U_{12})}\sqrt{E\left[\hat{D}_{n2}^2(U_{11})|U_{11},U_{12},Z_1,Z_2\right]E\left[\hat{D}_{n2}^2(U_{12})|U_{11},U_{12},Z_1,Z_2\right]}\Bigg).\quad\text{(A.8)}
$$

Next, we will deal with one of the conditional expectations. We have

$$E\left[\hat{D}_{n2}^2(U_{11})|U_{11},U_{12},Z_1,Z_2\right] = E\Bigg[\frac{K}{n}\sum_{k=2}^{n} 1_{\{U_{11}<U_{1k}\le b_n\}}\frac{(C_n(U_{1k})-C(U_{1k}))^2}{C^3(U_{1k})}|U_{11},U_{12},Z_1,Z_2\Bigg]$$
$$+E\Bigg[\frac{K^2}{n^2}\sum_{k=2}^{n}\sum_{l\ne k,1} 1_{\{U_{11}<U_{1k}\le b_n\}}1_{\{U_{11}<U_{1l}\le b_n\}}\frac{|C_n(U_{1k})-C(U_{1k})|}{C^2(U_{1k})}\frac{|C_n(U_{1l})-C(U_{1l})|}{C^2(U_{1l})}$$
$$|U_{11},U_{12},Z_1,Z_2\Bigg]. \tag{A.9}$$

Furthermore,

$$E\Bigg[\frac{K}{n}\sum_{k=2}^{n} 1_{\{U_{11}<U_{1k}\le b_n\}}\frac{(C_n(U_{1k})-C(U_{1k}))^2}{C^3(U_{1k})}|U_{11}=x_1,U_{12}=x_2,Z_1=y_1,Z_2=y_2\Bigg]$$
$$= E\Bigg[\frac{K}{n}\sum_{k=3}^{n}\frac{1_{\{U_{11}<U_{1k}\le b_n\}}}{C^3(U_{1k})}\Bigg(\frac{1}{n}\sum_{i=3}^{n}[1_{\{U_{1i}\le U_{1k}\le Z_i\}}-C(U_{1k})]+\frac{1}{n}1_{\{U_{11}\le U_{1k}\le Z_1\}}+\frac{1}{n}1_{\{U_{12}\le U_{1k}\le Z_2\}}$$
$$-\frac{2}{n}C(U_{1k})\Bigg)^2+\frac{K}{n}\frac{1_{\{U_{11}<U_{12}\le b_n\}}}{C^3(U_{12})}\Bigg(\frac{1}{n}\sum_{i=3}^{n}[1_{\{U_{1i}\le U_{12}\le Z_i\}}-C(U_{12})]+\frac{1}{n}1_{\{U_{11}\le U_{12}\le Z_1\}}+\frac{1}{n}$$
$$-\frac{2}{n}C(U_{12})\Bigg)^2|U_{11}=x_1,U_{12}=x_2,Z_1=y_1,Z_2=y_2\Bigg]$$
$$\le E\Bigg[\frac{K}{n}\sum_{k=3}^{n}\frac{1_{\{x_1<U_{1k}\le b_n\}}}{C^3(U_{1k})}\Bigg[2\Bigg(\frac{1}{n}\sum_{i=3}^{n}[1_{\{U_{1i}\le U_{1k}\le Z_i\}}-C(U_{1k})]\Bigg)^2+\frac{4}{n^2}(1_{\{x_1\le U_{1k}\le y_1\}}+1_{\{x_2\le U_{1k}\le y_2\}})^2$$
$$+4\frac{4}{n^2}C^2(U_{1k})\Bigg]+\frac{K}{n}\frac{1_{\{x_1<x_2\le b_n\}}}{C^3(x_2)}\Bigg[2\Bigg(\frac{1}{n}\sum_{i=3}^{n}[1_{\{U_{1i}\le x_2\le Z_i\}}-C(x_2)]\Bigg)^2+4\frac{1}{n^2}(1_{\{x_1\le x_2\le y_1\}}+1)^2$$
$$+4\frac{4}{n^2}C^2(x_2)\Bigg]\Bigg]\le\frac{K}{n}(n-2)E\Bigg[\frac{1_{\{x_1<U_{13}\le b_n\}}}{C^3(U_{13})}\Bigg[2\Bigg(\frac{1}{n}\sum_{i=3}^{n}[1_{\{U_{1i}\le U_{13}\le Z_i\}}-C(U_{13})]\Bigg)^2+\frac{8}{n^2}+\frac{16}{n^2}C^2(U_{13})\Bigg]\Bigg]$$
$$+\frac{K}{n}\frac{1_{\{x_1<x_2\le b_n\}}}{C^3(x_2)}E\Bigg[2\Bigg(\frac{1}{n}\sum_{i=3}^{n}[1_{\{U_{1i}\le x_2\le Z_i\}}-C(x_2)]\Bigg)^2+\frac{8}{n^2}+\frac{16}{n^2}C^2(x_2)\Bigg]$$
$$\le 2KE\Bigg[\frac{1_{\{x_1<U_{13}\le b_n\}}}{C^3(U_{13})}\Bigg(\frac{1}{n}\sum_{i=4}^{n}[1_{\{U_{1i}\le U_{13}\le Z_i\}}-C(U_{13})]+\frac{1}{n}-\frac{1}{n}C(U_{13})\Bigg)^2\Bigg]+\frac{8K}{n^2}\int_{(x_1,b_n)}\frac{F_1^*(dx)}{C^3(x)}$$
$$+\frac{16K}{n^2}\int_{(x_1,b_n)}\frac{F_1^*(dx)}{C(x)}+\frac{2K}{n^2}\frac{1_{\{x_1<x_2\le b_n\}}}{C^2(x_2)}+\frac{8K}{n^3}\frac{1_{\{x_1<x_2\le b_n\}}}{C^3(x_2)}+\frac{16K}{n^3}\frac{1_{\{x_1<x_2\le b_n\}}}{C(x_2)}$$
$$= E(E(...|U_{13}))+...\le\frac{4K}{n}\int_{(x_1,b_n)}\frac{F_1^*(dx)}{C^2(x)}+\frac{8K}{n^2}\int_{(x_1,b_n)}\frac{F_1^*(dx)}{C^3(x)}+\frac{8K}{n^2}\int_{(x_1,b_n)}\frac{F_1^*(dx)}{C^2(x)}$$
$$+\frac{8K}{n^2}\int_{(x_1,b_n)}\frac{F_1^*(dx)}{C^3(x)}+\frac{16K}{n^2}\int_{(x_1,b_n)}\frac{F_1^*(dx)}{C(x)}+\frac{2K}{n^2}\frac{1_{\{x_1<x_2\le b_n\}}}{C^2(x_2)}+\frac{8K}{n^3}\frac{1_{\{x_1<x_2\le b_n\}}}{C^3(x_2)}+\frac{16K}{n^3}\frac{1_{\{x_1<x_2\le b_n\}}}{C(x_2)}$$
$$\le\frac{28K}{n}\int_{(x_1,b_n)}\frac{F_1^*(dx)}{C^2(x)}+\frac{16K}{n^2}\int_{(x_1,b_n)}\frac{F_1^*(dx)}{C^3(x)}+\frac{18K}{n^2}\frac{1_{\{x_1<x_2\le b_n\}}}{C^2(x_2)}+\frac{8K}{n^3}\frac{1_{\{x_1<x_2\le b_n\}}}{C^3(x_2)}.$$

Hence, the first sum in (A.9) equals

$$E\bigg[\frac{K}{n}\sum_{k=2}^{n}1_{\{U_{11}<U_{1k}\le b_n\}}\frac{(C_n(U_{1k})-C(U_{1k}))^2}{C^3(U_{1k})}|U_{11},U_{12},Z_1,Z_2\bigg]\le\frac{28K}{n}\int_{(U_{11},b_n)}\frac{F_1^*(dx)}{C^2(x)}$$
$$+\frac{16K}{n^2}\int_{(U_{11},b_n)}\frac{F_1^*(dx)}{C^3(x)}+\frac{18K}{n^2}\frac{1_{\{U_{11}<U_{12}\le b_n\}}}{C^2(U_{12})}+\frac{8K}{n^3}\frac{1_{\{U_{11}<U_{12}\le b_n\}}}{C^3(U_{12})}. \quad \text{(A.10)}$$

The second sum in (A.9) equals

$$E\bigg[\frac{K^2}{n^2}\sum_{k=2}^{n}\sum_{l\neq k,1}1_{\{U_{11}<U_{1k},U_{1l}\le b_n\}}\frac{|C_n(U_{1k})-C(U_{1k})|}{C^2(U_{1k})}\frac{|C_n(U_{1l})-C(U_{1l})|}{C^2(U_{1l})}|U_{11},U_{12},Z_1,Z_2\bigg]$$
$$= E\bigg[\frac{K^2}{n^2}\sum_{k=3}^{n}\sum_{l\neq k,1,2}1_{\{U_{11}<U_{1k},U_{1l}\le b_n\}}\frac{|C_n(U_{1k})-C(U_{1k})|}{C^2(U_{1k})}\frac{|C_n(U_{1l})-C(U_{1l})|}{C^2(U_{1l})}|U_{11},U_{12},Z_1,Z_2\bigg]$$
$$+E\bigg[\frac{2K^2}{n^2}\sum_{l\neq 1,2}1_{\{U_{11}<U_{12}\le b_n\}}1_{\{U_{11}<U_{1l}\le b_n\}}\frac{|C_n(U_{12})-C(U_{12})|}{C^2(U_{12})}\frac{|C_n(U_{1l})-C(U_{1l})|}{C^2(U_{1l})}|U_{11},U_{12},Z_1,Z_2\bigg]$$
$$\le K^2E\bigg[1_{\{U_{11}<U_{13},U_{14}\le b_n\}}\frac{|C_n(U_{13})-C(U_{13})|}{C^2(U_{13})}\frac{|C_n(U_{14})-C(U_{14})|}{C^2(U_{14})}|U_{11},U_{12},Z_1,Z_2\bigg]$$
$$+\frac{2K^2}{n}E\bigg[1_{\{U_{11}<U_{12},U_{13}\le b_n\}}\frac{|C_n(U_{12})-C(U_{12})|}{C^2(U_{12})}\frac{|C_n(U_{13})-C(U_{13})|}{C^2(U_{13})}|U_{11},U_{12},Z_1,Z_2\bigg]$$
$$= K^2E\bigg[\frac{1_{\{U_{11}<U_{13},U_{14}\le b_n\}}}{C^2(U_{13})C^2(U_{14})}E\Big(|C_n(U_{13})-C(U_{13})||C_n(U_{14})-C(U_{14})|\Big|\{U_{1i},Z_i,i=1,2,3,4\}\Big)$$
$$|U_{11},U_{12},Z_1,Z_2\bigg]$$
$$+\frac{2K^2}{n}E\bigg[\frac{1_{\{U_{11}<U_{12},U_{13}\le b_n\}}}{C^2(U_{12})C^2(U_{13})}E\Big(|C_n(U_{12})-C(U_{12})||C_n(U_{13})-C(U_{13})|\Big|\{U_{1i},Z_i,i=1,2,3\}\Big)$$
$$|U_{11},U_{12},Z_1,Z_2\bigg].$$

According to (A.7), for the first term we have

$$K^2E\bigg[\frac{1_{\{U_{11}<U_{13},U_{14}\le b_n\}}}{C^2(U_{13})C^2(U_{14})}E\Big(|C_n(U_{13})-C(U_{13})||C_n(U_{14})-C(U_{14})|\Big|\{U_{1i},Z_i,i=1,2,3,4\}\Big)|U_{11},U_{12},Z_1,Z_2\bigg]$$
$$\le K^2E\bigg[\frac{1_{\{U_{11}<U_{13},U_{14}\le b_n\}}}{C^2(U_{13})C^2(U_{14})}\left(\frac{2}{n}\sqrt{C(U_{13})C(U_{14})}+\frac{16\sqrt{C(U_{13})}}{n\sqrt{n}}+\frac{16\sqrt{C(U_{14})}}{n\sqrt{n}}+\frac{128}{n^2}\right)|U_{11},U_{12},Z_1,Z_2\bigg]$$
$$\le\frac{2K^2}{n}\left(\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}\right)^2+\frac{32K^2}{n\sqrt{n}}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)}+\frac{128K^2}{n^2}\left(\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)}\right)^2 \text{(A.11)}$$

and for the second term

$$\frac{2K^2}{n}E\bigg[\frac{1_{\{U_{11}<U_{12},U_{13}\le b_n\}}}{C^2(U_{12})C^2(U_{13})}E\Big(|C_n(U_{12})-C(U_{12})||C_n(U_{13})-C(U_{13})|\Big|\{U_{1i},Z_i,i=1,2,3\}\Big)|U_{11},U_{12},Z_1,Z_2\bigg]$$

$$\leq K^2 E\left[\frac{1_{\{U_{11}<U_{12},U_{13}\leq b_n\}}}{C^2(U_{12})C^2(U_{13})}\left(\frac{2}{n}\sqrt{C(U_{12})C(U_{13})}+\frac{16\sqrt{C(U_{12})}}{n\sqrt{n}}+\frac{16\sqrt{C(U_{13})}}{n\sqrt{n}}+\frac{128}{n^2}\right)|U_{11},U_{12},Z_1,Z_2\right]$$
$$\leq \frac{2K^2}{n}\frac{1_{\{U_{11}<U_{12}\leq b_n\}}}{C^{3/2}(U_{12})}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}+\frac{16K^2}{n\sqrt{n}}\frac{1_{\{U_{11}<U_{12}\leq b_n\}}}{C^{3/2}(U_{12})}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)}$$
$$+\frac{16K^2}{n\sqrt{n}}\frac{1_{\{U_{11}<U_{12}\leq b_n\}}}{C^2(U_{12})}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}+\frac{128K^2}{n^2}\frac{1_{\{U_{11}<U_{12}\leq b_n\}}}{C^2(U_{12})}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)}. \tag{A.12}$$

Altogether, $E\left[\hat{D}_{n2}^2(U_{11})|U_{11},U_{12},Z_1,Z_2\right]$ is bounded from above by the sum of (A.10), (A.11) and (A.12). Hence we get

$$E\left[\hat{D}_{n2}^2(U_{11})|U_{11},U_{12},Z_1,Z_2\right]\leq\frac{28K}{n}\int_{(U_{11},b_n)}\frac{F_1^*(dx)}{C^2(x)}+\frac{16K}{n^2}\int_{(U_{11},b_n)}\frac{F_1^*(dx)}{C^3(x)}+\frac{18K}{n^2}\frac{1_{\{U_{11}<U_{12}\leq b_n\}}}{C^2(U_{12})}$$
$$+\frac{8K}{n^3}\frac{1_{\{U_{11}<U_{12}\leq b_n\}}}{C^3(U_{12})}+\frac{2K^2}{n}\left(\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}\right)^2+\frac{32K^2}{n\sqrt{n}}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)}$$
$$+\frac{128K^2}{n^2}\left(\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)}\right)^2+\frac{2K^2}{n}\frac{1_{\{U_{11}<U_{12}\leq b_n\}}}{C^{3/2}(U_{12})}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}+\frac{16K^2}{n\sqrt{n}}\frac{1_{\{U_{11}<U_{12}\leq b_n\}}}{C^{3/2}(U_{12})}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)}$$
$$+\frac{16K^2}{n\sqrt{n}}\frac{1_{\{U_{11}<U_{12}\leq b_n\}}}{C^2(U_{12})}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}+\frac{128K^2}{n^2}\frac{1_{\{U_{11}<U_{12}\leq b_n\}}}{C^2(U_{12})}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)}. \tag{A.13}$$

Before we may deal with the second sum in $\tilde{S}_{n4}^2(t)$, we will bound

$$1_{\{a_n<U_{11}\}}E\left[\hat{D}_{n2}^2(U_{11})|U_{11},U_{12},Z_1,Z_2\right]$$

by a simpler function than the one in the above inequality. For this we consider each term of (A.13)$*1_{\{a_n<U_{11}\}}$ separately. Since $dF_1^*=\alpha^{-1}(1-G-)dF_1$ and $C=\alpha^{-1}(1-G-)F_1$, we have for the first term

$$\frac{28K}{n}1_{\{a_n<U_{11}\}}\int_{(U_{11},b_n)}\frac{F_1^*(dx)}{C^2(x)}\leq\frac{28KM\alpha}{n}\frac{1_{\{a_n<U_{11}\}}}{F_1^2(U_{11})}.$$

By Remark A.1, $1-G(b_n^-)\geq\frac{n}{c\alpha}$ and by A1 $1-F_1(x)\leq M(1-G(x^-))$. Hence the second term is bounded from above by

$$\frac{16K}{n^2}1_{\{a_n<U_{11}\}}\int_{(U_{11},b_n)}\frac{F_1^*(dx)}{C^3(x)}=\frac{16K\alpha^2}{n^2}1_{\{a_n<U_{11}\}}\int_{(U_{11},b_n)}\frac{F_1(dx)}{F_1^3(x)(1-G(x^-))^2}$$
$$\leq\frac{16K\alpha^2}{n^2}1_{\{a_n<U_{11}\}}\left(M\int_{(U_{11},b_n)}\frac{F_1(dx)}{F_1^3(x)(1-G(x^-))}+\int_{(U_{11},b_n)}\frac{F_1(dx)}{F_1^2(x)(1-G(x^-))^2}\right)$$
$$\leq\frac{16K\alpha^2}{n^2}1_{\{a_n<U_{11}\}}\frac{2M}{F_1^2(U_{11})(1-G(b_n^-))}\leq\frac{16K\alpha^2}{n^2}1_{\{a_n<U_{11}\}}\frac{2M}{F_1^2(U_{11})}\frac{n}{c\alpha}=\frac{32MK\alpha}{nc}1_{\{a_n<U_{11}\}}\frac{1}{F_1^2(U_{11})}.$$

The third summand in (A.13) is bounded from above by

$$\frac{18K}{n^2}\frac{1_{\{a_n<U_{11}<U_{12}\leq b_n\}}}{C^2(U_{12})}\leq\frac{18K}{n}\frac{1_{\{a_n<U_{11}<U_{12}\leq b_n\}}}{C^2(U_{12})}.$$

By Remark A.1, $1-G(b_n^-)\geq\frac{n}{c\alpha}$ and $F_1(a_n)\geq\frac{n}{c_1\alpha}$. For the 4th term in (A.13), we therefore get

$$\frac{8K}{n^3}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{C^3(U_{12})} \le \frac{8K}{n^3}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{C^2(U_{12})}\frac{\alpha}{F_1(a_n)(1-G(b_n^-))} \le \frac{8K}{n}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{C^2(U_{12})}\frac{1}{cc_1\alpha}.$$

Cauchy-Schwarz yields for the 5th term in (A.13)

$$\begin{aligned}&\frac{2K^2}{n}1_{\{a_n<U_{11}\}}\left(\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}\right)^2 = \frac{2K^2\alpha}{n}1_{\{a_n<U_{11}\}}\left(\int_{(U_{11},b_n)}\frac{F_1(dy)}{F_1(y)\sqrt{F_1(y)(1-G(y^-))}}\right)^2\\ &\quad\le \frac{2K^2\alpha}{n}\frac{1_{\{a_n<U_{11}\}}}{F_1^2(U_{11})}Mln(\frac{n}{c_1\alpha}).\end{aligned}$$

For the 6th term we have

$$\begin{aligned}&\frac{32K^2}{n\sqrt{n}}1_{\{a_n<U_{11}\}}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)} \le \frac{32K^2\sqrt{\alpha}}{n\sqrt{n}}\int_{(U_{11},b_n)}\frac{F_1(dy)}{F_1(y)\sqrt{F_1(y)(1-G(y^-))}}\frac{\alpha M}{F_1^2(U_{11})}\\ &\quad\le \frac{32K^2\sqrt{\alpha}}{n\sqrt{n}}\frac{\alpha M}{F_1^2(U_{11})}\int_{(U_{11},b_n)}\left(\sqrt{\frac{M(1-G(y^-))}{F_1(y)}}+1\right)\frac{1}{F_1(y)\sqrt{1-G(y^-)}}F_1(dy)\\ &\quad\le \frac{32K^2\sqrt{\alpha}}{n\sqrt{n}}\frac{\alpha M}{F_1^2(U_{11})}\left(\sqrt{M}\frac{1}{\sqrt{F_1(U_{11})}}+\frac{1}{\sqrt{1-G(b_n^-)}}ln(\frac{n}{c_1\alpha})\right)\\ &\quad\le \frac{32K^2\sqrt{\alpha}}{n\sqrt{n}}\frac{\alpha M}{F_1^2(U_{11})}\left(\sqrt{M}\sqrt{\frac{n}{c_1\alpha}}+\sqrt{\frac{n}{c\alpha}}ln(\frac{n}{c_1\alpha})\right) \le \frac{32K^2}{n}\frac{\alpha M}{F_1^2(U_{11})}\left(\sqrt{\frac{M}{c_1}}+\sqrt{\frac{1}{c}}ln(\frac{n}{c_1\alpha})\right).\end{aligned}$$

For the 7th term we get

$$\frac{128K^2}{n^2}1_{\{a_n<U_{11}\}}\left(\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)}\right)^2 \le \frac{128K^2}{n}\frac{4M^2\alpha}{F_1^2(U_{11})}1_{\{a_n<U_{11}\}}$$

By Remark A.1, $1-F_1(y)\le M(1-G(y^-)$. Therefore, by Cauchy-Schwarz, for the 8th term we obtain

$$\begin{aligned}&\frac{2K^2}{n}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{C^{3/2}(U_{12})}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)} = \frac{2K^2\sqrt{\alpha}}{n}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{C^{3/2}(U_{12})}\int_{(U_{11},b_n)}\frac{F_1(dy)}{F_1^{3/2}(y)\sqrt{1-G(y^-)}}\\ &\quad\le \frac{2K^2\sqrt{\alpha}}{n}\frac{\alpha^{3/2}1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{F_1^{3/2}(U_{12})(1-G(U_{12}^-))^{3/2}}\Big(\int_{(U_{11},b_n)}\frac{1}{F_1^{1/2}(y)\sqrt{1-G(y^-)}}\left(\frac{M(1-G(y^-))}{F_1(y)}+1\right)F_1(dy)\Big)\\ &\quad\le \frac{2K^2\sqrt{\alpha}}{n}\frac{\alpha^{3/2}1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{F_1^{3/2}(U_{12})(1-G(U_{12}^-))^{3/2}}\Big(\frac{M}{\sqrt{F_1(U_{11})}}+\sqrt{Mln(\frac{n}{c_1\alpha})}\Big)\\ &\quad\le \frac{2K^2M\alpha^2}{n}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{F_1^2(U_{11})(1-G(U_{12}^-))^2}+\frac{2K^2\alpha^2}{n}\sqrt{Mln(\frac{n}{c_1\alpha})}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{C^2(U_{12})}.\end{aligned}$$

For the 9th summand in the (A.13) we have

$$
\begin{aligned}
&\frac{16K^2}{n\sqrt{n}}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{C^{3/2}(U_{12})}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)} \le \frac{16K^2\alpha^{3/2}}{n\sqrt{n}}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{F_1^{3/2}(U_{12})(1-G(U_{12}^-))^{3/2}}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)}\\
&\quad\le \frac{16K^2\alpha^{3/2}}{n\sqrt{n}}\sqrt{\frac{n}{c_1\alpha}}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{F_1(U_{12})(1-G(U_{12}^-))^2}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)}\\
&\quad\le \frac{16K^2\alpha}{n}\sqrt{\frac{1}{c_1}}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{F_1(U_{11})(1-G(U_{12}^-))^2}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)} \le \frac{16K^2\alpha}{n}\sqrt{\frac{1}{c_1}}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{F_1(U_{11})(1-G(U_{12}^-))^2}\frac{2\alpha M}{F_1(U_{11})}\\
&\quad= \frac{32K^2M\alpha^2}{n\sqrt{c_1}}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{F_1^2(U_{11})(1-G(U_{12}^-))^2}.
\end{aligned}
$$

Using the bound for $\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}$ in 6th term, we get for the 10th term in (A.13)

$$
\begin{aligned}
&\frac{16K^2}{n\sqrt{n}}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{C^2(U_{12})}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)} \le \frac{16K^2}{n\sqrt{n}}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{C^2(U_{12})}\sqrt{\alpha}\left(\sqrt{M}\sqrt{\frac{n}{c_1\alpha}}+\sqrt{\frac{n}{c\alpha}ln(\frac{n}{c_1\alpha})}\right)\\
&\quad\le \frac{16K^2}{n}\left(\sqrt{\frac{M}{c_1}}+\sqrt{\frac{1}{c}ln(\frac{n}{c_1\alpha})}\right)\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{C^2(U_{12})}.
\end{aligned}
$$

For the 11th term we obtain

$$
\frac{128K^2}{n^2}\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{C^2(U_{12})}\int_{(U_{11},b_n)}\frac{F_1^*(dy)}{C^2(y)} \le \frac{128K^2}{nc_1}2M\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{C^2(U_{12})}.
$$

Hence

$$
E\left[\hat{D}_{n2}^2(U_{11})|U_{11},U_{12},Z_1,Z_2\right] \le \frac{1}{n}\left(A\frac{1}{F_1^2(U_{11})}+B\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{C^2(U_{12})}+C\frac{1_{\{a_n<U_{11}<U_{12}\le b_n\}}}{F_1^2(U_{11})(1-G(U_{12}^-))^2}\right),
$$

where

$$
A = 28KM\alpha+\frac{32MK\alpha}{c}+2K^2\alpha Mln(\frac{n}{c_1\alpha})+32K^2\alpha M\left(\sqrt{\frac{M}{c_1}}+\sqrt{\frac{1}{c}ln(\frac{n}{c_1\alpha})}\right)+128K^2*4M^2\alpha
$$

$$
B = 18K+\frac{8K}{cc_1\alpha}+2K^2\alpha^2\sqrt{Mln(\frac{n}{c_1\alpha})}+16K^2\left(\sqrt{\frac{M}{c_1}}+\sqrt{\frac{1}{c}ln(\frac{n}{c_1\alpha})}\right)+\frac{128K^2}{c_1}2M
$$

and

$$
C = 2K^2M\alpha^2+\frac{32K^2M\alpha^2}{\sqrt{c_1}}.
$$

Finally the second sum in $\tilde{S}_{n4}^2(t)$ is less than or equal to

$$
\begin{aligned}
(A.8) \quad &= K^2 E\left(1_{\{a_n<U_{11},U_{12}\le t\}}\frac{\gamma(U_{11})}{C(U_{11})}\frac{\gamma(U_{12})}{C(U_{12})}\sqrt{E\left[\hat{D}^2_{n2}(U_{11})|U_{11},U_{12},Z_1,Z_2\right]E\left[\hat{D}^2_{n2}(U_{12})|U_{11},U_{12},Z_1,Z_2\right]}\right)\\
&\le \frac{K^2}{n}E\left(1_{\{a_n<U_{11},U_{12}\le t\}}\frac{\gamma(U_{11})}{C(U_{11})}\frac{\gamma(U_{12})}{C(U_{12})}\left(\sqrt{A}\frac{1}{F_1(U_{11})}+\sqrt{B}\frac{1_{\{U_{11}<U_{12}\le b_n\}}}{C(U_{12})}+\sqrt{C}\frac{1_{\{U_{11}<U_{12}\le b_n\}}}{F_1(U_{11})(1-G(U_{12}^-))}\right)\right.\\
&\left.\times\left(\sqrt{A}\frac{1}{F_1(U_{12})}+\sqrt{B}\frac{1_{\{U_{12}<U_{11}\le b_n\}}}{C(U_{11})}+\sqrt{C}\frac{1_{\{U_{12}<U_{11}\le b_n\}}}{F_1(U_{12})(1-G(U_{11}^-))}\right)\right)\\
&\le \frac{K^2}{n}E\left(1_{\{a_n<U_{11},U_{12}\le t\}}\frac{\gamma(U_{11})}{C(U_{11})}\frac{\gamma(U_{12})}{C(U_{12})}\left[A\frac{1}{F_1(U_{11})F_1(U_{12})}+\sqrt{A}\sqrt{B}\frac{1_{\{U_{12}<U_{11}\le b_n\}}}{C(U_{11})F_1(U_{11})}\right.\right.\\
&+\sqrt{A}\sqrt{C}\frac{1}{F_1(U_{11})}\frac{1_{\{U_{12}<U_{11}\le b_n\}}}{F_1(U_{12})(1-G(U_{11}^-))}+\sqrt{B}\sqrt{A}\frac{1_{\{U_{11}<U_{12}\le b_n\}}}{C(U_{12})}\frac{1}{F_1(U_{12})}\\
&\left.\left.+\sqrt{C}\sqrt{A}\frac{1_{\{U_{11}<U_{12}\le b_n\}}}{F_1(U_{11})(1-G(U_{12}^-))}\frac{1}{F_1(U_{12})}\right]\right)=K^2\frac{A}{n}\left(\int_{(a_n,t]}\frac{F_1^*(dx)}{C(x)}\right)^2\\
&+2K^2\frac{\sqrt{AB}}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}F_1^*(dx)+2K^2\frac{\sqrt{AC}}{n}\int_{(a_n,t]}\frac{1}{C(x)}\int_{(x,b_n)}\frac{\gamma(y)}{C^2(y)}F_1^*(dy)F_1^*(dx)
\end{aligned}
$$

Hence

$$
\begin{aligned}
E\tilde{S}^2_{n4}(t) \quad &\le \frac{14}{n}K^2\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C(y)}F_1^*(dx)+\frac{16}{n^2}K^2\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}F_1^*(dx)\\
&+\frac{2K^2}{n}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C(y)}\right)^2F_1^*(dx)\\
&+\frac{32K^2}{n\sqrt{n}}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C(y)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}F_1^*(dx)\\
&+\frac{128K^2}{n^2}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}\right)^2F_1^*(dx)+\frac{AK^2}{n}\left(\int_{(a_n,t]}\frac{F_1^*(dx)}{C(x)}\right)^2\\
&+2\frac{\sqrt{AB}K^2}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}F_1^*(dx)+2\frac{\sqrt{AC}K^2}{n}\int_{(a_n,t]}\frac{1}{C(x)}\int_{(x,b_n)}\frac{\gamma(y)}{C^2(y)}F_1^*(dy)F_1^*(dx)
\end{aligned}
$$

Finally, similarly to the lemmas before we get

$$
\begin{aligned}
E\tilde{S}^2_{n4}(t) \quad &\le \frac{14K^2}{n}\alpha M ln(\frac{n}{c_1\alpha})+\frac{32K^2M}{nc_1}+\frac{2K^2}{n}\alpha M ln^2(\frac{n}{c_1\alpha})+\frac{32K^2}{n}\alpha M ln(\frac{n}{c_1\alpha})\left(\frac{M}{\sqrt{c_1}}+\sqrt{M\alpha ln(\frac{n}{c_1\alpha})}\right)\\
&+\frac{128K^2}{n}\alpha M\left(\frac{M}{\sqrt{c_1}}+\sqrt{M\alpha ln(\frac{n}{c_1\alpha})}\right)^2+\frac{AK^2}{n}ln^2(\frac{n}{c_1\alpha})+2\frac{\sqrt{AB}K^2}{n}2\alpha M ln(\frac{n}{c_1\alpha})\\
&+2\frac{\sqrt{AC}K^2}{n}\alpha M ln(\frac{n}{c_1\alpha})\left(ln(\frac{n}{c_1\alpha})+1\right)
\end{aligned}
$$

$\boxtimes$

Finally, by Lemmas A.1-A.5, we get the following result.

Lemma A.6. *For given $\varepsilon > 0$, there exist $K \geq 1$ and sequences a_n and b_n, so that the event*

$$\tilde{\Omega}_n^0 = \left\{ \sup_{1\leq i\leq n} \frac{C(U_{1i})}{C_n(U_{1i})} \leq K, U_{11:n} > a_n, Z_{n:n} < b_n \right\} \cap \Omega_1^{(n)},$$

has probability bigger than or equal to $1 - 4\varepsilon$. On the set $\tilde{\Omega}_n^0$ we have

$$(F_{1n} - F_1)^2(t) \leq 8(L_{n1}^2(t) + (\tilde{S}_{n1}^b)^2(t) + \tilde{S}_{n2}^2(t) + \tilde{S}_{n3}^2(t) + \tilde{S}_{n4}^2(t)) =: 8E_n^2(t), \tag{A.14}$$

where $E(E_n^2(t))$ is the sum of expectations in Lemmas A.1-A.5. Furthermore, since w.l.o.g. $M \geq 1$, $c \leq 1$ and $c_1 \leq \min(\frac{1}{e\alpha}, 1)$, we have

$$E(E_n^2(t)) \leq k\frac{K^5 M^2}{cc_1\alpha}\frac{ln^3(\frac{n}{c_1\alpha})}{n}, \tag{A.15}$$

where k is a constant.

A.2 Bounds for $(F_{1n} - F_1)^2(Z_k)$

In this section, for fixed indices i and k and $U_{1i} \leq Z_k$, we bound $(F_{1n} - F_1)^2(Z_k)$ by functions which don't contain U's and Z's with indices i and k. Let L_{n-11}^2, $(S_{n-11}^b)^2$, S_{n-12}^2, S_{n-13}^2 and S_{n-14}^2 be the functions as defined in Chapter A.1, which don't contain index k. To find a bound for $(F_{1n} - F_1)^2(Z_k)$, we need the following lemmas.

Lemma A.7.

$$L_{n-11}^2(Z_k) \leq 4L_{n-21}^2(Z_k) + \frac{4}{(n-1)^2}F_1^2(Z_k) + 2\frac{\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})}$$

Proof.
We have

$$\begin{aligned}
L_{n-11}^2(Z_k) &= \left(\int_{(-\infty,Z_k]} \frac{\gamma(x)}{C(x)}(F_{1n-1}^*(dx) - F_1^*(dx))\right)^2 \leq 2\Bigg(\frac{n-2}{n-1}\int_{(-\infty,Z_k]} \frac{\gamma(x)}{C(x)}(F_{1n-2}^*(dx) - F_1^*(dx)) \\
&\quad -\frac{1}{n-1}\int_{(-\infty,Z_k]} \frac{\gamma(x)}{C(x)}F_1^*(dx)\Bigg)^2 + 2\frac{\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})} \\
&\leq 4\left(\frac{n-2}{n-1}\int_{(-\infty,Z_k]} \frac{\gamma(x)}{C(x)}(F_{1n-2}^*(dx) - F_1^*(dx))\right)^2 + \frac{4}{(n-1)^2}\left(\int_{(-\infty,Z_k]} \frac{\gamma(x)}{C(x)}F_1^*(dx)\right)^2 \\
&\quad +2\frac{\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})} \leq 4L_{n-21}^2(Z_k) + \frac{4}{(n-1)^2}F_1^2(Z_k) + 2\frac{\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})}
\end{aligned}$$

$\boxtimes$

Lemma A.8. *On the set $\tilde{\Omega}_n^0$, we have*

$$(S^b_{n-11})^2(Z_k) \;\leq\; 4(\tilde{S}^b_{n-21})^2(Z_k)+\frac{4K}{n-1}\bigg(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C^{3/2}(x)}F^*_{1n-2}(dx)\bigg)^2+4\frac{\gamma^2(U_{1i})(C(U_{1i})-C_{n-2}(U_{1i}))^2}{C^2(U_{1i})}$$
$$+\frac{4}{(n-1)^2}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}$$

Proof.
We have

$$\begin{aligned}
(S^b_{n-11})^2(Z_k) &= \bigg(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C(x)C_{n-1}(x)}(C(x)-C_{n-1}(x))F^*_{1n-1}(dx)\bigg)^2\\
&\leq 2\left(\frac{n-2}{n-1}\int_{(a_n,Z_k]}\frac{\gamma(x)}{C(x)C_{n-1}(x)}(C(x)-C_{n-1}(x))F^*_{1n-2}(dx)\right)^2\\
&\quad +2\frac{\gamma^2(U_{1i})(C(U_{1i})-C_{n-1}(U_{1i}))^2}{(n-1)^2C^2(U_{1i})C^2_{n-1}(U_{1i})}\leq 2\bigg(\frac{(n-2)^2}{(n-1)^2}\int_{(a_n,Z_k]}\frac{\gamma(x)(C(x)-C_{n-2}(x))}{C(x)C_{n-1}(x)}F^*_{1n-2}(dx)\\
&\quad +\frac{n-2}{(n-1)^2}\int_{(a_n,Z_k]}\frac{\gamma(x)}{C(x)C_{n-1}(x)}(C(x)-1_{\{U_{1i}\leq x\leq Z_i\}})F^*_{1n-2}(dx)\bigg)^2\\
&\quad +4\frac{(n-2)^2}{(n-1)^2}\frac{\gamma^2(U_{1i})(C(U_{1i})-C_{n-2}(U_{1i}))^2}{(n-1)^2C^2(U_{1i})C^2_{n-1}(U_{1i})}+\frac{4}{(n-1)^2}\frac{\gamma^2(U_{1i})(C(U_{1i})-1)^2}{(n-1)^2C^2(U_{1i})C^2_{n-1}(U_{1i})}\\
&\leq 4\bigg(\frac{(n-2)^2}{(n-1)^2}\int_{(a_n,Z_k]}\frac{\gamma(x)(C(x)-C_{n-2}(x))}{C(x)C_{n-1}(x)}F^*_{1n-2}(dx)\bigg)^2\\
&\quad +4\frac{(n-2)^2}{(n-1)^4}\bigg(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C(x)C_{n-1}(x)}(C(x)-1_{\{U_{1i}\leq x\leq Z_i\}})F^*_{1n-2}(dx)\bigg)^2\\
&\quad +4\frac{(n-2)^2}{(n-1)^2}\frac{\gamma^2(U_{1i})(C(U_{1i})-C_{n-2}(U_{1i}))^2}{(n-1)^2C^2(U_{1i})C^2_{n-1}(U_{1i})}+\frac{4}{(n-1)^2}\frac{\gamma^2(U_{1i})(C(U_{1i})-1)^2}{(n-1)^2C^2(U_{1i})C^2_{n-1}(U_{1i})}.
\end{aligned}$$

Since $(n-1)C_{n-1}(x)\geq(n-2)C_{n-2}(x)$, and according to Lemma A.4, the first sum is bounded from above by

$$4\bigg(\int_{(a_n,Z_k]}\frac{\gamma(x)|C(x)-C_{n-2}(x)|}{C(x)C_{n-2}(x)}F^*_{1n-2}(dx)\bigg)^2\leq 4(\tilde{S}^b_{n-21})^2(Z_k).$$

Furthermore, since on the set $\tilde{\Omega}_n^0$ we have $C(U_{1j})/C^{\neq k}_{n-1}(U_{1j})\leq K$ for $j\neq k$ and by $(n-1)C_{n-1}(U_{1i})\geq 1$, we obtain

$$4\frac{(n-2)^2}{(n-1)^4}\bigg(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C(x)C_{n-1}(x)}(C(x)-1_{\{U_{1i}\leq x\leq Z_i\}})F^*_{1n-2}(dx)\bigg)^2\leq\frac{4K}{n-1}\bigg(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C^{3/2}(x)}F^*_{1n-2}(dx)\bigg)^2$$

and

$$\begin{aligned}
&4\frac{(n-2)^2}{(n-1)^2}\frac{\gamma^2(U_{1i})(C(U_{1i})-C_{n-2}(U_{1i}))^2}{(n-1)^2C^2(U_{1i})C^2_{n-1}(U_{1i})}+\frac{4}{(n-1)^2}\frac{\gamma^2(U_{1i})(C(U_{1i})-1)^2}{(n-1)^2C^2(U_{1i})C^2_{n-1}(U_{1i})}\\
&\leq 4\frac{\gamma^2(U_{1i})(C(U_{1i})-C_{n-2}(U_{1i}))^2}{C^2(U_{1i})}+\frac{4}{(n-1)^2}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}.
\end{aligned}$$

The proof is complete.

$\boxtimes$

Lemma A.9. *On the set $\tilde{\Omega}_n^0$, we have*

$$
\begin{aligned}
S_{n-12}^2(Z_k) \;\; &\leq \;\; \tilde{S}_{n-22}^2(Z_k) + \frac{K^3 1_{\{b_n > U_{1i} > a_n\}}}{(n-1)C(U_{1i})} \left(\int_{(a_n, U_{1i}]} \frac{\gamma(x)}{C(x)} F_{1n-2}^*(dx) \right)^2 \\
&\quad + \frac{2K^4 \gamma^2(U_{1i})}{(n-1)^2 C^2(U_{1i})} \left(\int_{(U_{1i}, b_n)} \frac{F_{1n-1}^*(dy)}{C(y)} \right)^2 .
\end{aligned}
$$

Proof.
We have

$$
\begin{aligned}
S_{n-12}^2(Z_k) \quad &\leq \left(\int_{(a_n,t]} \frac{\gamma(x)}{C_{n-1}(x)} \frac{1}{n-1} \int_{(x,b_n)} \frac{F_{1n-1}^*(dy)}{C_{n-1}^2(y)} F_{1n-1}^*(dx) \right)^2 \\
&\leq 2 \left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-1}(x)} \frac{n-2}{(n-1)^2} \int_{(x,b_n)} \frac{F_{1n-1}^*(dy)}{C_{n-1}^2(y)} F_{1n-2}^*(dx) \right)^2 + 2 \frac{\gamma^2(U_{1i}) 1_{\{U_{1i} \leq Z_k\}}}{(n-1)^4 C_{n-1}^2(U_{1i})} \\
&\times \left(\int_{(U_{1i},b_n)} \frac{F_{1n-1}^*(dy)}{C_{n-1}^2(y)} \right)^2 \leq 4 \left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-1}(x)} \frac{(n-2)^2}{(n-1)^3} \int_{(x,b_n)} \frac{F_{1n-2}^*(dy)}{C_{n-1}^2(y)} F_{1n-2}^*(dx) \right)^2 \\
&+ 4 \left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-1}(x)} \frac{n-2}{(n-1)^3} \frac{1_{\{b_n > U_{1i} > x\}}}{C_{n-1}^2(U_{1i})} F_{1n-2}^*(dx) \right)^2 + 2 \frac{\gamma^2(U_{1i})}{(n-1)^4 C_{n-1}^2(U_{1i})} \\
&\times \left(\int_{(U_{1i},b_n)} \frac{F_{1n-1}^*(dy)}{C_{n-1}^2(y)} \right)^2 .
\end{aligned}
$$

Since $(n-1)C_{n-1}(x) \geq (n-2)C_{n-2}(x)$, the first sum on the right side is bounded from above by

$$
4 \left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-2}(x)} \frac{1}{n-2} \int_{(x,b_n)} \frac{F_{1n-2}^*(dy)}{C_{n-2}^2(y)} F_{1n-2}^*(dx) \right)^2 \leq 4 \tilde{S}_{n-22}^2(Z_k)
$$

As to the second and third term, since on the set $\tilde{\Omega}_n^0$ we have $C(U_{1j})/C_{n-1}^{\neq k}(U_{1j}) \leq K$ for $j \neq k$ and $(n-1)C_{n-1}(U_{1i}) \geq 1$, we obtain

$$
4 \left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-1}(x)} \frac{n-2}{(n-1)^3} \frac{1_{\{b_n > U_{1i} > x\}}}{C_{n-1}^2(U_{1i})} F_{1n-2}^*(dx) \right)^2 \leq \frac{K^3}{n-1} \frac{1_{\{b_n > U_{1i} > a_n\}}}{C(U_{1i})} \left(\int_{(a_n,U_{1i}]} \frac{\gamma(x)}{C(x)} F_{1n-2}^*(dx) \right)^2
$$

and

$$
2 \frac{\gamma^2(U_{1i})}{(n-1)^4 C_{n-1}^2(U_{1i})} \left(\int_{(U_{1i},b_n)} \frac{F_{1n-1}^*(dy)}{C_{n-1}^2(y)} \right)^2 \leq 2K^4 \frac{\gamma^2(U_{1i})}{(n-1)^2 C^2(U_{1i})} \left(\frac{n-2}{n-1} \int_{(U_{1i},b_n)} \frac{F_{1n-2}^*(dy)}{C(y)} \right)^2
$$

$\boxtimes$

Lemma A.10. *On the set $\tilde{\Omega}_n^0$, we have*

$$\begin{aligned} S_{n-13}^2(Z_k) &\le \tilde{S}_{n-23}^2(Z_k) + \frac{8K^2 ln(\frac{n}{c_1\alpha})}{(n-1)^2}\left(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C(x)}F_{1n-2}^*(dx)\right)^2 + 4\frac{ln(\frac{n}{c_1\alpha})}{(n-1)^2}\gamma^2(U_{1i}) \\ &\quad +8K^2\frac{1_{\{b_n>U_{1i}>a_n\}}}{(n-1)^2C^2(U_{1i})}\left(\int_{(a_n,U_{1i}]}\frac{\gamma(x)}{C(x)}F_{1n-2}^*(dx)\right)^2 + 4\gamma^2(U_{1i})D_{n-21}^2(U_{1i}). \end{aligned}$$

Proof.
We have

$$\begin{aligned} S_{n-13}^2(Z_k) &\le \left(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C_{n-1}(x)}|D_{n-11}(x)|F_{1n-1}^*(dx)\right)^2 \le 2\left(\frac{n-2}{n-1}\int_{(a_n,Z_k]}\frac{\gamma(x)}{C_{n-1}(x)}|D_{n-11}(x)|F_{1n-2}^*(dx)\right)^2 \\ &\quad +2\frac{\gamma^2(U_{1i})D_{n-11}^2(U_{1i})}{(n-1)^2C_{n-1}^2(U_{1i})}1_{\{U_{1i}>a_n\}} \end{aligned} \tag{A.16}$$

Furthermore

$$D_{n-11}(x) = -\int_{(x,\infty)}\frac{F_{1n-1}^*(dy)-F_1^*(dy)}{C(y)} = \frac{n-2}{n-1}D_{n-21}(x) + \frac{1}{n-1}\left(\int_{(x,\infty)}\frac{F_1^*(dy)}{C(y)} - \frac{1_{\{U_{1i}>x\}}}{C(U_{1i})}\right)$$

and

$$D_{n-11}(U_{1i}) = \frac{n-2}{n-1}D_{n-21}(U_{1i}) + \frac{1}{n-1}\int_{(U_{1i},\infty)}\frac{F_1^*(dy)}{C(y)}.$$

Hence

$$\begin{aligned} (A.16) &\le 4\left(\frac{n-2}{n-1}\int_{(a_n,Z_k]}\frac{\gamma(x)}{C_{n-1}(x)}|D_{n-21}(x)|F_{1n-2}^*(dx)\right)^2 \\ &\quad +\frac{8}{(n-1)^2}\left(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C_{n-1}(x)}\int_{(x,\infty)}\frac{F_1^*(dy)}{C(y)}F_{1n-2}^*(dx)\right)^2 \\ &\quad +8\frac{1}{(n-1)^2}\left(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C_{n-1}(x)}\frac{1_{\{U_{1i}>x\}}}{C(U_{1i})}F_{1n-2}^*(dx)\right)^2 + 4\frac{(n-2)^2}{(n-1)^2}\frac{\gamma^2(U_{1i})D_{n-21}^2(U_{1i})}{(n-1)^2C_{n-1}^2(U_{1i})} \\ &\quad +4\frac{1}{(n-1)^2}\frac{\gamma^2(U_{1i})}{(n-1)^2C_{n-1}^2(U_{1i})}\left(\int_{(U_{1i},\infty)}\frac{F_1^*(dy)}{C(y)}\right)^2. \end{aligned} \tag{A.17}$$

Furthermore, since $(n-1)C_{n-1}(x) \ge (n-2)C_{n-2}(x)$,

$$4\left(\frac{n-2}{n-1}\int_{(a_n,Z_k]}\frac{\gamma(x)}{C_{n-1}(x)}|D_{n-21}(x)|F_{1n-2}^*(dx)\right)^2 \le 4\left(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C_{n-2}(x)}|D_{n-21}(x)|F_{1n-2}^*(dx)\right)^2 \le \tilde{S}_{n-23}^2(Z_k).$$

Recall that $\frac{dF_1^*}{C} = \frac{dF_1}{F_1}$ and, by Remark A.1, on the set $\tilde{\Omega}_n^0$ we have

$$\int_{(x,\infty)} \frac{F_1^*(dy)}{C(y)} 1_{\{x>a_n\}} \leq ln(\frac{n}{c_1\alpha}).$$

Hence, since $C(U_{1j})/C_{n-1}(U_{1j}) \leq K$, for the second term in (A.17) we obtain

$$\frac{8}{(n-1)^2}\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-1}(x)} \int_{(x,\infty)} \frac{F_1^*(dy)}{C(y)} F_{1n-2}^*(dx)\right)^2 \leq \frac{8K^2 ln(\frac{n}{c_1\alpha})}{(n-1)^2}\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C(x)} F_{1n-2}^*(dx)\right)^2.$$

From $(n-1)C_{n-1}(U_{1j}) \geq 1$ for the last term we get

$$4\frac{1}{(n-1)^2}\frac{\gamma^2(U_{1i})}{(n-1)^2C_{n-1}^2(U_{1i})}\left(\int_{(U_{1i},\infty)} \frac{F_1^*(dy)}{C(y)}\right)^2 \leq 4\frac{ln(\frac{n}{c_1\alpha})}{(n-1)^2}\gamma^2(U_{1i})$$

Finally, for the third and fourth term, we have on the set $\tilde{\Omega}_n^0$

$$8\frac{1}{(n-1)^2}\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-1}(x)} \frac{1_{\{U_{1i}>x\}}}{C(U_{1i})} F_{1n-2}^*(dx)\right)^2 \leq 8K^2\frac{1_{\{b_n>U_{1i}>a_n\}}}{(n-1)^2C^2(U_{1i})}\left(\int_{(a_n,U_{1i}]} \frac{\gamma(x)}{C(x)} F_{1n-2}^*(dx)\right)^2$$

and, by $(n-1)^2C_{n-1}^2(U_{1i}) \geq 1$

$$4\frac{(n-2)^2}{(n-1)^2}\frac{\gamma^2(U_{1i})D_{n-21}^2(U_{1i})}{(n-1)^2C_{n-1}^2(U_{1i})} \leq 4\gamma^2(U_{1i})D_{n-21}^2(U_{1i}).$$

⊠

Lemma A.11. *On the set* $\tilde{\Omega}_n^0$*, we have*

$$\begin{aligned}
S_{n-14}^2(Z_k) \quad \leq \quad & \tilde{S}_{n-14}^2(Z_k) + \frac{8K^4}{(n-1)^2}\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C(x)} \int_{(x,b_n)} \frac{|1_{\{U_{1i}\leq y\leq Z_i\}} - C(y)|}{C^2(y)} F_{1n-2}^*(dy) F_{1n-2}^*(dx)\right)^2 \\
& + \frac{8K^2(K^2+2K+2)}{(n-1)^2C^2(U_{1i})} 1_{\{a_n<U_{1i}<b_n\}}\left(\int_{(a_n,U_{1i}]} \frac{\gamma(x)}{C(x)} F_{1n-2}^*(dx)\right)^2 \\
& + \frac{4K^2(K^2+1)}{(n-1)^2}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}\left(\int_{(U_{1i},b_n)} \frac{F_{1n-2}^*(dy)}{C(y)}\right)^2 + 4K^2\frac{\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})}\int_{(U_{1i},b_n)} \frac{F_{1n-2}^*(dy)}{C(y)}
\end{aligned}$$

Proof.
We have

$$\begin{aligned}
S_{n-14}^2(Z_k) \quad & \leq \left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-1}(x)} |D_{n-12}(x)| F_{1n-1}^*(dx)\right)^2 \leq 2\left(\frac{n-2}{n-1}\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-1}(x)} |D_{n-12}(x)| F_{1n-2}^*(dx)\right)^2 \\
& +2\frac{\gamma^2(U_{1i})}{(n-1)^2C_{n-1}^2(U_{1i})} D_{n-12}^2(U_{1i})
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
D_{n-12}(x) &= \int_{(x,b_n)} \frac{C_{n-1}(y)-C(y)}{C_{n-1}(y)C(y)} F^*_{1n-1}(dy) = \frac{n-2}{n-1}\int_{(x,b_n)} \frac{C_{n-1}(y)-C(y)}{C_{n-1}(y)C(y)} F^*_{1n-2}(dy) \\
&+\frac{1}{n-1}\frac{C_{n-1}(U_{1i})-C(U_{1i})}{C_{n-1}(U_{1i})C(U_{1i})} 1_{\{x<U_{1i}<b_n\}} = \frac{(n-2)^2}{(n-1)^2}\int_{(x,b_n)} \frac{C_{n-2}(y)-C(y)}{C_{n-1}(y)C(y)} F^*_{1n-2}(dy) \\
&+\frac{n-2}{(n-1)^2}\int_{(x,b_n)} \frac{1_{\{U_{1i}\le y\le Z_i\}}-C(y)}{C_{n-1}(y)C(y)} F^*_{1n-2}(dy) + \frac{n-2}{(n-1)^2}\frac{C_{n-2}(U_{1i})-C(U_{1i})}{C_{n-1}(U_{1i})C(U_{1i})} 1_{\{x<U_{1i}<b_n\}} \\
&+\frac{1}{(n-1)^2}\frac{1-C(U_{1i})}{C_{n-1}(U_{1i})C(U_{1i})} 1_{\{x<U_{1i}<b_n\}}.
\end{aligned}
$$

Since $(n-1)C_{n-1} \ge (n-2)C_{n-2}$, we obtain

$$
\begin{aligned}
|D_{n-12}(x)| &\le |D_{n-22}(x)| + \frac{n-2}{(n-1)^2}\int_{(x,b_n)} \frac{|1_{\{U_{1i}\le y\le Z_i\}}-C(y)|}{C_{n-1}(y)C(y)} F^*_{1n-2}(dy) \\
&+\frac{n-2}{(n-1)^2}\frac{|C_{n-2}(U_{1i})-C(U_{1i})|}{C_{n-1}(U_{1i})C(U_{1i})} 1_{\{x<U_{1i}<b_n\}} + \frac{1}{(n-1)^2}\frac{1-C(U_{1i})}{C_{n-1}(U_{1i})C(U_{1i})} 1_{\{x<U_{1i}<b_n\}}
\end{aligned}
$$

and

$$
|D_{n-12}(U_{1i})| \le |D_{n-22}(U_{1i})| + \frac{n-2}{(n-1)^2}\int_{(U_{1i},b_n)} \frac{|1_{\{U_{1i}\le y\le Z_i\}}-C(y)|}{C_{n-1}(y)C(y)} F^*_{1n-2}(dy).
$$

Therefore

$$
\begin{aligned}
S^2_{n-14}(Z_k) &\le 8\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-2}(x)} |D_{n-22}(x)| F^*_{1n-2}(dx)\right)^2 \\
&+8\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-1}(x)}\frac{n-2}{(n-1)^2}\int_{(x,b_n)} \frac{|1_{\{U_{1i}\le y\le Z_i\}}-C(y)|}{C_{n-1}(y)C(y)} F^*_{1n-2}(dy) F^*_{1n-2}(dx)\right)^2 \\
&+8\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-1}(x)}\frac{n-2}{(n-1)^2}\frac{|C_{n-2}(U_{1i})-C(U_{1i})|}{C_{n-1}(U_{1i})C(U_{1i})} 1_{\{x<U_{1i}<b_n\}} F^*_{1n-2}(dx)\right)^2 \\
&+8\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-1}(x)}\frac{1}{(n-1)^2}\frac{1-C(U_{1i})}{C_{n-1}(U_{1i})C(U_{1i})} 1_{\{x<U_{1i}<b_n\}} F^*_{1n-2}(dx)\right)^2 \\
&+4\frac{\gamma^2(U_{1i})}{(n-1)^2 C^2_{n-1}(U_{1i})} D^2_{n-22}(U_{1i}) \\
&+4\frac{\gamma^2(U_{1i})}{(n-1)^2 C^2_{n-1}(U_{1i})}\frac{n-2}{(n-1)^2}\int_{(U_{1i},b_n)} \frac{|1_{\{U_{1i}\le y\le Z_i\}}-C(y)|}{C_{n-1}(y)C(y)} F^*_{1n-2}(dy). \qquad \text{(A.18)}
\end{aligned}
$$

Furthermore, by Lemma A.5, we have

$$
8\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-2}(x)} |D_{n-22}(x)| F^*_{1n-2}(dx)\right)^2 \le \tilde{S}^2_{n-14}(Z_k).
$$

Since on the set $\tilde{\Omega}_n^0$ we have $C(U_{1j})/C_{n-1}(U_{1j}) \leq K$ and $(n-1)C_{n-1}(U_{1j}) \geq 1$, for the first summand in (A.18) we obtain

$$\begin{aligned}
&8\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-1}(x)} \frac{n-2}{(n-1)^2} \int_{(x,b_n)} \frac{|1_{\{U_{1i}\leq y\leq Z_i\}} - C(y)|}{C_{n-1}(y)C(y)} F^*_{1n-2}(dy) F^*_{1n-2}(dx)\right)^2 \\
&\leq \frac{8K^4}{(n-1)^2}\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C(x)} \int_{(x,b_n)} \frac{|1_{\{U_{1i}\leq y\leq Z_i\}} - C(y)|}{C^2(y)} F^*_{1n-2}(dy) F^*_{1n-2}(dx)\right)^2.
\end{aligned}$$

For the second we have

$$\begin{aligned}
&8\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-1}(x)} \frac{n-2}{(n-1)^2} \frac{|C_{n-2}(U_{1i}) - C(U_{1i})|}{C_{n-1}(U_{1i})C(U_{1i})} 1_{\{x<U_{1i}<b_n\}} F^*_{1n-2}(dx)\right)^2 \\
&\leq 8K^2\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C(x)} \frac{n-2}{n-1}\left(\frac{C_{n-2}(U_{1i})}{(n-1)C_{n-1}(U_{1i})C(U_{1i})} + \frac{1}{(n-1)C_{n-1}(U_{1i})}\right) 1_{\{x<U_{1i}<b_n\}} F^*_{1n-2}(dx)\right)^2 \\
&\leq \frac{8K^2(K+1)^2}{(n-1)^2C^2(U_{1i})} 1_{\{a_n<U_{1i}<b_n\}}\left(\int_{(a_n,U_{1i}]} \frac{\gamma(x)}{C(x)} F^*_{1n-2}(dx)\right)^2
\end{aligned}$$

For the third summand in (A.18) we obtain

$$\begin{aligned}
&8\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C_{n-1}(x)} \frac{1}{(n-1)^2} \frac{1 - C(U_{1i})}{C_{n-1}(U_{1i})C(U_{1i})} 1_{\{x<U_{1i}<b_n\}} F^*_{1n-2}(dx)\right)^2 \\
&\leq \frac{8K^2}{(n-1)^2C^2(U_{1i})} 1_{\{a_n<U_{1i}<b_n\}}\left(\int_{(a_n,U_{1i}]} \frac{\gamma(x)}{C(x)} F^*_{1n-2}(dx)\right)^2
\end{aligned}$$

Since $D^2_{n-22}(x) \leq \left(\int_{(x,b_n)} \frac{F^*_{1n-2}(dy)}{C(y)}\right)^2 (K^2+1)$, for the 4th term we get

$$4\frac{\gamma^2(U_{1i})}{(n-1)^2C^2_{n-1}(U_{1i})} D^2_{n-22}(U_{1i}) \leq 4K^2(K^2+1)\frac{\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})}\left(\int_{(U_{1i},b_n)} \frac{F^*_{1n-2}(dy)}{C(y)}\right)^2$$

For the 5th summand in (A.18) we obtain

$$\begin{aligned}
&4\frac{\gamma^2(U_{1i})}{(n-1)^2C^2_{n-1}(U_{1i})} \frac{n-2}{(n-1)^2} \int_{(U_{1i},b_n)} \frac{|1_{\{U_{1i}\leq y\leq Z_i\}} - C(y)|}{C_{n-1}(y)C(y)} F^*_{1n-2}(dy) \\
&\leq 4K^2\frac{\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})} \int_{(U_{1i},b_n)} \frac{F^*_{1n-2}(dy)}{C(y)}
\end{aligned}$$

This completes the proof of the lemma.

$\boxtimes$

Now we are in the position to formulate the main Lemma.

Lemma A.12. *On the set $\tilde{\Omega}_n^0$, for fixed i and k with $U_{1i} \leq Z_k$, we have*

$$(F_{1n} - F_1)^2(Z_k) \leq k_1 E_{n-2}^2(Z_k) + M_n(Z_k, U_{1i}, Z_i) 1_{\{Z_k, Z_i \leq b_n, U_{1i} > a_n\}},$$

where E_{n-2} is defined as E_n in Lemma A.6 but doesn't contain U's and Z's with index i and k. Furthermore,

$$\begin{aligned} M_n(Z_k, U_{1i}, Z_i) &= \frac{8}{(n-1)^2} F_1^2(Z_k) ln(\frac{n}{c_1\alpha}) + 6\frac{\gamma^2(U_{1i})}{(n-1)^2 C^2(U_{1i})} + \frac{4K}{n-1}\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C^{3/2}(x)} F_{1n-2}^*(dx)\right)^2 \\ &+4\frac{\gamma^2(U_{1i})(C(U_{1i}) - C_{n-2}(U_{1i}))^2}{C^2(U_{1i})} + \frac{K^3 1_{\{b_n > U_{1i} > a_n\}}}{(n-1)C(U_{1i})}\left(\int_{(a_n,U_{1i}]} \frac{\gamma(x)}{C(x)} F_{1n-2}^*(dx)\right)^2 \\ &+10K^4 \frac{\gamma^2(U_{1i})}{(n-1)^2 C^2(U_{1i})}\left(\int_{(U_{1i},b_n)} \frac{F_{1n-2}^*(dy)}{C(y)}\right)^2 + \frac{8K^2 ln(\frac{n}{c_1\alpha})}{(n-1)^2}\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C(x)} F_{1n-2}^*(dx)\right)^2 \\ &+48K^4 \frac{1_{\{b_n > U_{1i} > a_n\}}}{(n-1)^2 C^2(U_{1i})}\left(\int_{(a_n,U_{1i}]} \frac{\gamma(x)}{C(x)} F_{1n-2}^*(dx)\right)^2 + 4\gamma^2(U_{1i}) D_{n-21}^2(U_{1i}) \\ &+\frac{8K^4}{(n-1)^2}\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C(x)} \int_{(x,b_n)} \frac{|1_{\{U_{1i} \leq y \leq Z_i\}} - C(y)|}{C^2(y)} F_{1n-2}^*(dy) F_{1n-2}^*(dx)\right)^2 \\ &+\frac{4K^2}{(n-1)^2} \frac{\gamma^2(U_{1i})}{(n-1)^2 C^2(U_{1i})} \int_{(U_{1i},b_n)} \frac{F_{1n-2}^*(dy)}{C(y)}. \end{aligned}$$

Proof. Since $U_{1k} \leq Z_k$ and $1_{\{U_{1k} \leq U_{1j} \leq Z_k\}} = 0$ if $U_{1j} > Z_k$, we get

$$F_{1n}(Z_k) = \prod_{U_{1j} > Z_k} \left[1 - \frac{1}{nC_n(U_{1j})}\right] = \prod_{U_{1j} > Z_k, j \neq k} \left[1 - \frac{1}{(n-1)C_{n-1}(U_{1j})}\right] =: F_{1n-1}(Z_k).$$

According to the last section:

$$(F_{1n-1} - F_1)^2(Z_k) \leq 8(L_{n-11}^2(Z_k) + (S_{n-11}^b)^2(Z_k) + S_{n-12}^2(Z_k) + S_{n-13}^2(Z_k) + S_{n-14}^2(Z_k)),$$

where the functions don't include the variables with index k.
By Lemmas A.7-A.11, the right side is bounded from above by

$$(F_{1n-1} - F_1)^2(Z_k) \leq k_1(L_{n-21}^2(Z_k) + (\tilde{S}_{n-21}^b)^2(Z_k) + \tilde{S}_{n-22}^2(Z_k) + \tilde{S}_{n-23}^2(Z_k) + \tilde{S}_{n-24}^2(Z_k)) + \tilde{M}_n(Z_k, U_{1i}, Z_i),$$

where the functions from $L_{n-21}^2(.)$ until $\tilde{S}_{n-24}^2(.)$ on the right side don't include index k and i.
Furthermore

$$L_{n-21}^2(Z_k) + (\tilde{S}_{n-21}^b)^2(Z_k) + \tilde{S}_{n-22}^2(Z_k) + \tilde{S}_{n-23}^2(Z_k) + \tilde{S}_{n-24}^2(Z_k) = E_{n-2}^2(Z_k)$$

and

$$
\begin{aligned}
\tilde{M}_n(Z_k,U_{1i},Z_i) &= \frac{4}{(n-1)^2}F_1^2(Z_k)+2\frac{\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})}+\frac{4K}{n-1}\left(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C^{3/2}(x)}F^*_{1n-2}(dx)\right)^2\\
&+4\frac{\gamma^2(U_{1i})(C(U_{1i})-C_{n-2}(U_{1i}))^2}{C^2(U_{1i})}+\frac{4}{(n-1)^2}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}\\
&+\frac{K^3 1_{\{b_n>U_{1i}>a_n\}}}{(n-1)C(U_{1i})}\left(\int_{(a_n,U_{1i}]}\frac{\gamma(x)}{C(x)}F^*_{1n-2}(dx)\right)^2+\frac{2K^4\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})}\left(\int_{(U_{1i},b_n)}\frac{F^*_{1n-2}(dy)}{C(y)}\right)^2\\
&+\frac{8K^2 ln(\frac{n}{c_1\alpha})}{(n-1)^2}\left(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C(x)}F^*_{1n-2}(dx)\right)^2+4\frac{ln(\frac{n}{c_1\alpha})}{(n-1)^2}\gamma^2(U_{1i})\\
&+8K^2\frac{1_{\{b_n>U_{1i}>a_n\}}}{(n-1)^2C^2(U_{1i})}\left(\int_{(a_n,U_{1i}]}\frac{\gamma(x)}{C(x)}F^*_{1n-2}(dx)\right)^2+4\gamma^2(U_{1i})D^2_{n-21}(U_{1i})\\
&+\frac{8K^4}{(n-1)^2}\left(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{|1_{\{U_{1i}\le y\le Z_i\}}-C(y)|}{C^2(y)}F^*_{1n-2}(dy)F^*_{1n-2}(dx)\right)^2\\
&+\frac{8K^2(K^2+2K+2)}{(n-1)^2C^2(U_{1i})}1_{\{a_n<U_{1i}<b_n\}}\left(\int_{(a_n,U_{1i}]}\frac{\gamma(x)}{C(x)}F^*_{1n-2}(dx)\right)^2\\
&+\frac{4K^2(K^2+1)}{(n-1)^2}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}\left(\int_{(U_{1i},b_n)}\frac{F^*_{1n-2}(dy)}{C(y)}\right)^2+\frac{4K^2\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})}\int_{(U_{1i},b_n)}\frac{F^*_{1n-2}(dy)}{C(y)}.
\end{aligned}
$$

Since $\gamma=F_1$, $K\ge 1$ and $ln(\frac{n}{c_1\alpha})\ge 1$, we get

$$
\begin{aligned}
\tilde{M}_n(Z_k,U_{1i},Z_i) &\le \frac{8}{(n-1)^2}F_1^2(Z_k)ln(\frac{n}{c_1\alpha})+6\frac{\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})}+\frac{4K}{n-1}\left(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C^{3/2}(x)}F^*_{1n-2}(dx)\right)^2\\
&+4\frac{\gamma^2(U_{1i})(C(U_{1i})-C_{n-2}(U_{1i}))^2}{C^2(U_{1i})}+\frac{K^3 1_{\{b_n>U_{1i}>a_n\}}}{(n-1)C(U_{1i})}\left(\int_{(a_n,U_{1i}]}\frac{\gamma(x)}{C(x)}F^*_{1n-2}(dx)\right)^2\\
&+\frac{(6K^4+4K^2)\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})}\left(\int_{(U_{1i},b_n)}\frac{F^*_{1n-2}(dy)}{C(y)}\right)^2+\frac{8K^2 ln(\frac{n}{c_1\alpha})}{(n-1)^2}\left(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C(x)}F^*_{1n-2}(dx)\right)^2\\
&+48K^4\frac{1_{\{b_n>U_{1i}>a_n\}}}{(n-1)^2C^2(U_{1i})}\left(\int_{(a_n,U_{1i}]}\frac{\gamma(x)}{C(x)}F^*_{1n-2}(dx)\right)^2+4\gamma^2(U_{1i})D^2_{n-21}(U_{1i})\\
&+\frac{8K^4}{(n-1)^2}\left(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{|1_{\{U_{1i}\le y\le Z_i\}}-C(y)|}{C^2(y)}F^*_{1n-2}(dy)F^*_{1n-2}(dx)\right)^2\\
&+\frac{4K^2\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})}\int_{(U_{1i},b_n)}\frac{F^*_{1n-2}(dy)}{C(y)} =: M_n(Z_k,U_{1i},Z_i).
\end{aligned}
$$

Finally, on the set $\tilde{\Omega}^0_n$, $M_n(Z_k,U_{1i},Z_i)=M_n(Z_k,U_{1i},Z_i)1_{\{Z_k,Z_i\le b_n,U_{1i}>a_n\}}$.

$\boxtimes$

Next, we compute the conditional expectation of $M_n(Z_k,U_{1i},Z_i)$ given $Z_k,U_{1i},\tilde{U}_{2i},Z_i$. We have the following result.

Lemma A.13. *For fixed index i and k we have*

$$
E(M_n(Z_k,U_{1i},Z_i)1_{\{Z_k,Z_i\le b_n,U_{1i}>a_n\}}|Z_k,U_{1i},\tilde{U}_{2i},Z_i)\le \frac{1}{n}k_2\frac{M^5K^4}{c_1^2c^2}ln^2(\frac{n}{c_1\alpha})\left(1+\frac{\gamma^2(U_{1i})}{C(U_{1i})}+\frac{\gamma^2(U_{1i})}{nC^2(U_{1i})}\right),
$$

where k_5 is a constant.

Proof.
According to Lemma A.12

$$
\begin{aligned}
M_n(Z_k, U_{1i}, Z_i) \quad = \quad & \frac{8}{(n-1)^2} F_1^2(Z_k) ln(\frac{n}{c_1\alpha}) + 6\frac{\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})} + \frac{4K}{n-1}\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C^{3/2}(x)} F^*_{1n-2}(dx)\right)^2 \\
& +4\frac{\gamma^2(U_{1i})(C(U_{1i}) - C_{n-2}(U_{1i}))^2}{C^2(U_{1i})} + \frac{K^3 1_{\{b_n > U_{1i} > a_n\}}}{(n-1)C(U_{1i})}\left(\int_{(a_n,U_{1i}]} \frac{\gamma(x)}{C(x)} F^*_{1n-2}(dx)\right)^2 \\
& +\frac{10K^4\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})}\left(\int_{(U_{1i},b_n)} \frac{F^*_{1n-2}(dy)}{C(y)}\right)^2 + \frac{8K^2 ln(\frac{n}{c_1\alpha})}{(n-1)^2}\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C(x)} F^*_{1n-2}(dx)\right)^2 \\
& +48K^4\frac{1_{\{b_n > U_{1i} > a_n\}}}{(n-1)^2C^2(U_{1i})}\left(\int_{(a_n,U_{1i}]} \frac{\gamma(x)}{C(x)} F^*_{1n-2}(dx)\right)^2 + 4\gamma^2(U_{1i}) D^2_{n-21}(U_{1i}) \\
& +\frac{8K^4}{(n-1)^2}\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C(x)} \int_{(x,b_n)} \frac{|1_{\{U_{1i}\le y \le Z_i\}} - C(y)|}{C^2(y)} F^*_{1n-2}(dy) F^*_{1n-2}(dx)\right)^2 \\
& +\frac{4K^2}{(n-1)^2}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}\int_{(U_{1i},b_n)} \frac{F^*_{1n-2}(dy)}{C(y)}.
\end{aligned}
$$

To compute the conditional expectation of $M_n(Z_k, U_{1i}, Z_i)$, given $Z_k, U_{1i}, \tilde{U}_{2i}, Z_i$, we will deal with the terms separately. Recall that $\gamma = F_1$, $C = \alpha^{-1}(1-G-)F_1$ and $dF_1^* = \alpha^{-1}(1-G-)dF_1$. By A1 we have $1 - F_1(x) \le M(1 - G(x^-))$, so that

$$
\frac{1}{F_1(x)} \le \frac{M(1-G(x^-))}{F_1(x)} + 1.
$$

Therefore, by Remark A.1, we have

$$
\begin{aligned}
& E\left(\left(1_{\{Z_k,Z_i\le b_n\}}\int_{(a_n,Z_k]} \frac{\gamma(x)}{C^{3/2}(x)} F^*_{1n-2}(dx)\right)^2 | Z_k, U_{1i}, \tilde{U}_{2i}, Z_i\right) = \frac{1_{\{Z_k,Z_i\le b_n\}}}{n-2}\int_{(a_n,Z_k]} \frac{\gamma^2(x)}{C^3(x)} F^*_1(dx) \\
& +1_{\{Z_k,Z_i\le b_n\}}\left(\int_{(a_n,Z_k]} \frac{\gamma(x)}{C^{3/2}(x)} F^*_1(dx)\right)^2 \le \frac{1}{n-2}\left(\alpha^2 M\frac{n}{c\alpha} ln(\frac{n}{c_1\alpha}) + \alpha^2 M\frac{n}{c\alpha}\right) + \alpha M ln(\frac{n}{c_1\alpha}) \\
& \le 3\frac{1}{c}\alpha M\left(ln(\frac{n}{c_1\alpha}) + 1\right) + \alpha M ln(\frac{n}{c_1\alpha}) \le 7M\frac{ln(\frac{n}{c_1\alpha})}{c}.
\end{aligned}
$$

Next, since $E(C_n - C)^2 = \frac{1}{n}C(1-C)$, we get

$$
E\left((C_{n-2}(U_{1i}) - C(U_{1i}))^2 | Z_k, U_{1i}, Z_i\right) \le \frac{1}{n-2}C(U_{1i}).
$$

By A1, we have

$$
E\left(\left(\int_{(a_n,U_{1i}]} \frac{\gamma(x)}{C(x)} F^*_{1n-2}(dx)\right)^2 | Z_k, U_{1i}, \tilde{U}_{2i}, Z_i\right)
$$

$$= E\left(\frac{1}{(n-2)^2}\sum_{j\neq i,k}\sum_{l\neq i,k}\frac{\gamma(U_{1j})}{C(U_{1j})}\frac{\gamma(U_{1l})}{C(U_{1l})}1_{\{a_n<U_{1j},U_{1l}\leq U_{1i}\}}|Z_k,U_{1i},\tilde{U}_{2i},Z_i\right)$$
$$\leq \frac{1}{n-2}\int_{(a_n,U_{1i}]}\frac{\gamma^2(x)}{C^2(x)}F_1^*(dx)+\left(\int_{(a_n,U_{1i}]}\frac{\gamma(x)}{C(x)}F_1^*(dx)\right)^2\leq\frac{\alpha M}{n-2}+F_1^2(U_{1i}).$$

Similarly as above and by Remark A.1, we obtain

$$E\left(1_{\{U_{1i}>a_n\}}\left(\int_{(U_{1i},b_n)}\frac{F_{1n-2}^*(dy)}{C(y)}\right)^2|Z_k,U_{1i},\tilde{U}_{2i},Z_i\right)\leq\frac{1_{\{U_{1i}>a_n\}}}{n-2}\int_{(U_{1i},b_n)}\frac{F_1^*(dy)}{C^2(y)}$$
$$+1_{\{U_{1i}>a_n\}}\left(\int_{(U_{1i},b_n)}\frac{F_1^*(dy)}{C(y)}\right)^2\leq\frac{1_{\{U_{1i}>a_n\}}}{n-2}\frac{2M\alpha}{F_1(U_{1i})}+ln^2(\frac{n}{c_1\alpha})\leq\frac{2M}{c_1}\frac{n}{n-2}+ln^2(\frac{n}{c_1\alpha})\leq\frac{7M}{c_1}ln^2(\frac{n}{c_1\alpha}).$$

Since $D_{n1}(U_{1i})=-\int_{(x,\infty)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}$ and $\gamma=F_1$, we get

$$E\left(1_{\{U_{1i}>a_n\}}\gamma^2(U_{1i})D_{n-21}^2(U_{1i})|Z_k,U_{1i},\tilde{U}_{2i},Z_i\right)\leq F_1^2(U_{1i})\frac{1}{n-2}\int_{(U_{1i},\infty)}\frac{F_1^*(dy)}{C^2(y)}$$
$$=F_1^2(U_{1i})\frac{\alpha}{n-2}\int_{(U_{1i},\infty)}\frac{F_1(dy)}{F_1^2(y)(1-G(y^-))}\leq\frac{\alpha M}{n-2}$$

Finally, we need to consider

$$E\left(\left(\int_{(a_n,Z_k]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{|1_{\{U_{1i}\leq y\leq Z_i\}}-C(y)|}{C^2(y)}F_{1n-2}^*(dy)F_{1n-2}^*(dx)\right)^2|Z_k,U_{1i},\tilde{U}_{2i},Z_i\right)$$
$$\leq E\left(\int_{(a_n,Z_k]}\frac{\gamma^2(x)}{C^2(x)}\left(\int_{(x,b_n)}\frac{1}{C^2(y)}F_{1n-2}^*(dy)\right)^2F_{1n-2}^*(dx)|Z_k,U_{1i},\tilde{U}_{2i},Z_i\right). \qquad \text{(A.19)}$$

By repeated use of $1-F_1(x)\leq M(1-G(x^-))$ we have

$$\frac{1}{C^2(y)}\leq\frac{3\alpha^2M^2}{F_1^2(y)}+\frac{3\alpha^2M}{(1-G(y^-))^2}.$$

Hence (A.19) is bounded from above by

$$2*9\alpha^4M^4E\left(\int_{(a_n,Z_k]}\frac{\gamma^2(x)}{C^2(x)}\left(\int_{(x,b_n)}\frac{1}{F_1^2(y)}F_{1n-2}^*(dy)\right)^2F_{1n-2}^*(dx)|Z_k,U_{1i},\tilde{U}_{2i},Z_i\right)$$
$$+2*9\alpha^4M^2E\left(\int_{(a_n,Z_k]}\frac{\gamma^2(x)}{C^2(x)}\left(\int_{(x,b_n)}\frac{1}{(1-G(y^-))^2}F_{1n-2}^*(dy)\right)^2F_{1n-2}^*(dx)|Z_k,U_{1i},\tilde{U}_{2i},Z_i\right)$$
$$\leq\frac{18\alpha^4M^4}{n-2}\int_{(a_n,Z_k]}\frac{\gamma^2(x)}{C^2(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{F_1^4(y)}F_1^*(dx)+18\alpha^4M^4\int_{(a_n,Z_k]}\frac{\gamma^2(x)}{C^2(x)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{F_1^2(y)}\right)^2F_1^*(dx)$$
$$+\frac{18\alpha^4M^2}{n-2}\int_{(a_n,Z_k]}\frac{\gamma^2(x)}{C^2(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{(1-G(y^-))^4}F_1^*(dx)$$
$$+18\alpha^4M^2\int_{(a_n,Z_k]}\frac{\gamma^2(x)}{C^2(x)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{(1-G(y^-))^2}\right)^2F_1^*(dx).$$

Furthermore, since $dF_1^* = \alpha^{-1}(1-G-)dF_1$, $1-G(y^-) \le 1-G(x^-)$ for $y > x$, by Remark A.1 and since $\frac{n}{n-2} \le 3$ for $n \ge 3$, we get

$$
\begin{aligned}
(A.20) \quad &\le \quad \frac{18\alpha^4 M^4}{n-2}\frac{1}{F_1^2(a_n)} + 18\alpha^4 M^4 \frac{1}{F_1(a_n)} + \frac{18\alpha^4 M^2}{n-2}\frac{M^2}{(1-G(b_n^-))^2} + 18\alpha^3 M^5 \\
&\le \quad \frac{18\alpha^4 M^4}{n-2}\frac{n^2}{c_1^2\alpha^2} + 18\alpha^4 M^4 \frac{n}{c_1\alpha} + \frac{18\alpha^4 M^4}{n-2}\frac{n^2}{c^2\alpha^2} + 18\alpha^3 M^5 \le \frac{144 M^5 n}{c_1^2 c^2}.
\end{aligned}
$$

Therefore, for the conditional expectation of $M_n(Z_k, U_{1i}, Z_i)1_{\{Z_k, Z_i \le b_n, U_{1i} > a_n\}}$, we obtain

$$
\begin{aligned}
&E(M_n(Z_k,U_{1i},Z_i)1_{\{Z_k,Z_i\le b_n,U_{1i}>a_n\}}|Z_k,U_{1i},\tilde{U}_{2i},Z_i) \le \frac{8}{(n-1)^2}F_1^2(Z_k)ln(\frac{n}{c_1\alpha}) + 6\frac{\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})} \\
&+\frac{28KM}{n-1}\frac{ln(\frac{n}{c_1\alpha})}{c} + 4\frac{\gamma^2(U_{1i})}{(n-2)C(U_{1i})} + \frac{K^3 1_{\{b_n>U_{1i}>a_n\}}}{(n-1)C(U_{1i})}\left(\frac{\alpha M}{n-2} + F_1^2(U_{1i})\right) \\
&+10K^4\frac{\gamma^2(U_{1i})}{(n-2)^2C^2(U_{1i})}\frac{7M}{c_1}ln^2(\frac{n}{c_1\alpha}) + \frac{8K^2 ln(\frac{n}{c_1\alpha})}{(n-1)^2}\left(\frac{\alpha M}{n-2} + F_1^2(U_{1i})\right) \\
&+48K^4\frac{1_{\{b_n>U_{1i}>a_n\}}}{(n-1)^2C^2(U_{1i})}\left(\frac{\alpha M}{n-2} + F_1^2(U_{1i})\right) + 4\frac{\alpha M}{n-2} + \frac{8K^4}{(n-1)^2}\frac{144M^5 n}{c_1^2c^2} \\
&+\frac{4K^2}{(n-1)^2}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}ln(\frac{n}{c_1\alpha}). \qquad \text{(A.20)}
\end{aligned}
$$

Since $\gamma = F_1 \le 1$, $M, K \ge 1$, we have

$$
\begin{aligned}
(A.20) \quad \le \quad &\frac{\tilde{k}}{n-1}\frac{K^2M^5}{c_1^2c^2}ln(\frac{n}{c_1\alpha}) + \frac{6\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})} + \frac{4\gamma^2(U_{1i})}{(n-2)C(U_{1i})} + \frac{K^3 1_{\{b_n>U_{1i}>a_n\}}}{(n-1)C(U_{1i})}\left(\frac{\alpha M}{n-2} + \gamma^2(U_{1i})\right) \\
&+10K^4\frac{\gamma^2(U_{1i})}{(n-2)^2C^2(U_{1i})}\frac{7M}{c_1}ln^2(\frac{n}{c_1\alpha}) + 48K^4\frac{1_{\{b_n>U_{1i}>a_n\}}}{(n-1)^2C^2(U_{1i})}\left(\frac{\alpha M}{n-2} + \gamma^2(U_{1i})\right) \\
&+\frac{4K^2}{(n-1)^2}\frac{\gamma^2(U_{1i})}{C^2(U_{1i})}ln(\frac{n}{c_1\alpha}), \qquad \text{(A.21)}
\end{aligned}
$$

where $\tilde{k}$ is a constant.
Furthermore, since $1 - F_1(x) \le M(1-G(x^-))$ we get

$$
\frac{1_{\{b_n>U_{1i}>a_n\}}}{(n-1)^2C^2(U_{1i})} \le \alpha^2\frac{1}{(n-1)^2}\left(\frac{3M^2}{F_1^2(a_n)} + \frac{3M}{(1-G(b_n^-))^2}\right) \le 6\frac{M^2}{c_1^2c^2}
$$

and

$$
\frac{1_{\{b_n>U_{1i}>a_n\}}}{(n-1)C(U_{1i})} \le \alpha\frac{1}{n-1}\left(\frac{M}{F_1(a_n)} + \frac{1}{1-G(b_n^-)}\right) \le \frac{2M}{c_1c}.
$$

Hence

$$
48K^4\frac{1_{\{b_n>U_{1i}>a_n\}}}{(n-1)^2C^2(U_{1i})}\left(\frac{\alpha M}{n-2} + \gamma^2(U_{1i})\right) \le 48K^4\frac{\alpha M}{n-2}6\frac{M^2}{c_1^2c^2} + 48K^4\frac{\gamma^2(U_{1i})}{(n-1)^2C^2(U_{1i})}.
$$

and

$$\frac{K^3 1_{\{b_n > U_{1i} > a_n\}}}{(n-1)C(U_{1i})}\left(\frac{\alpha M}{n-2} + \gamma^2(U_{1i})\right) \leq \frac{K^3 \alpha M}{n-2}\frac{2M}{c_1 c} + \frac{K^3\gamma^2(U_{1i})}{(n-1)C(U_{1i})}.$$

Therefore, we obtain

$$(A.21) \quad \leq \quad \frac{1}{n}k_2\frac{M^5K^4}{c_1^2c^2}ln^2(\frac{n}{c_1\alpha})\left(1 + \frac{\gamma^2(U_{1i})}{C(U_{1i})} + \frac{\gamma^2(U_{1i})}{nC^2(U_{1i})}\right),$$

where k_2 is a constant.

$\boxtimes$

A.3 Linearization of F_{1n}

In this section we will write $F_{1n}(t)$ as a sum of leading term and remainder and proof some properties of the remainder. Recall

$$\begin{aligned} F_{1n}(t) \quad = \quad & \int_{(-\infty,t]}\frac{\gamma(x)}{C_n(x)}F^*_{1n}(dx) + \int_{(-\infty,t]}\frac{e^{\Delta_n(x)}}{C_n(x)}B_n(x)F^*_{1n}(dx) + \int_{(-\infty,t]}\frac{e^{\Delta_n(x)}}{C_n(x)}D_{n1}(x)F^*_{1n}(dx) \\ & + \int_{(-\infty,t]}\frac{e^{\Delta_n(x)}}{C_n(x)}D_{n2}(x)F^*_{1n}(dx) = S_{n1}(t) + S_{n2}(t) + S_{n3}(t) + S_{n4}(t) \end{aligned}$$

where

$$B_n(x) = n\int_{(x,\infty)}\ln\left(1 - \frac{1}{nC_n(y)}\right)F^*_{1n}(dy) + \int_{(x,\infty)}\frac{F^*_{1n}(dy)}{C_n(y)}$$

$$D_{n1}(x) = -\int_{(x,\infty)}\frac{F^*_{1n}(dy) - F^*_1(dy)}{C(y)}$$

$$D_{n2}(x) = \int_{(x,\infty)}\frac{C_n(y) - C(y)}{C_n(y)C(y)}F^*_{1n}(dy)$$

and $S_{n1}(t)$ can be written as follows

$$\begin{aligned} S_{n1}(t) \quad = \quad & \int_{(-\infty,t]}\frac{\gamma(x)}{C(x)}F^*_1(dx) + \int_{(-\infty,t]}\frac{\gamma(x)}{C(x)}(F^*_{1n}(dx) - F^*_1(dx)) \\ & + \int_{(-\infty,t]}\frac{\gamma(x)}{C(x)C_n(x)}(C(x) - C_n(x))F^*_{1n}(dx) = F_1(t) + L_{n1}(t) + S^b_{n1}(t) \end{aligned}$$

Furthermore,

$$\begin{aligned} S^b_{n1}(t) \quad = \quad & \int_{(-\infty,t]}\frac{\gamma(x)}{C^2(x)}(C(x) - C_n(x))F^*_1(dx) + \int_{(-\infty,t]}\frac{\gamma(x)}{C^2(x)}(C(x) - C_n(x))(F^*_{1n}(dx) - F^*_1(dx)) \\ & + \int_{(-\infty,t]}\frac{\gamma(x)}{C^2(x)C_n(x)}(C(x) - C_n(x))^2F^*_1(dx) = L_{n2}(t) + R_{n1}(t) + R_{n2}(t) \end{aligned}$$

$$
\begin{aligned}
S_{n3}(t) &= \int_{(-\infty,t]} \frac{\gamma(x)}{C_n(x)} \left(e^{\Delta_n(x)+\int_{(x,\infty)} \frac{F_1^*(dy)}{C(y)}} - 1 \right) D_{n1}(x) F_{1n}^*(dx) + \int_{(-\infty,t]} \frac{\gamma(x)}{C_n(x)} D_{n1}(x) F_{1n}^*(dx) \\
&= II_n(t) + A_n(t)
\end{aligned}
$$

and

$$
\begin{aligned}
A_n(t) &= \int_{(-\infty,t]} \frac{\gamma(x)}{C(x)} \int_{(x,\infty)} \frac{F_1^*(dy) - F_{1n}^*(dy)}{C(y)} F_1^*(dx) + \int_{(-\infty,t]} \frac{\gamma(x)}{C(x)} D_{n1}(x)(F_{1n}^*(dx) - F_1^*(dx)) \\
&+ \int_{(-\infty,t]} \frac{\gamma(x)}{C(x)C_n(x)} D_{n1}(x)(C(x) - C_n(x)) F_{1n}^*(dx) = L_{n3}(t) + R_{n4}(t) + R_{n5}(t)
\end{aligned}
$$

and

$$
\begin{aligned}
S_{n4}(t) &= \int_{(-\infty,t]} \frac{e^{\Delta_n(x)}}{C_n(x)} \int_{(x,\infty)} \frac{C_n(y) - C(y)}{C^2(y)} F_{1n}^*(dy) F_{1n}^*(dx) \\
&+ \int_{(-\infty,t]} \frac{e^{\Delta_n(x)}}{C_n(x)} \int_{(x,\infty)} \frac{(C_n(y) - C(y))^2}{C^2(y) C_n(y)} F_{1n}^*(dy) F_{1n}^*(dx) = A_{2n}(t) + R_{n6}(t)
\end{aligned}
$$

where

$$
A_{2n}(t) = \int_{(-\infty,t]} \frac{\gamma(x)}{C(x)} \int_{(x,\infty)} \frac{C_n(y) - C(y)}{C^2(y)} F_1^*(dy) F_1^*(dx) + R_{n7}(t) = L_{n4}(t) + R_{n7}(t)
$$

Summarizing:
The $L_{n1}(t)$, $L_{n2}(t)$, $L_{n3}(t)$ and $L_{n4}(t)$ are leading terms.
The $S_{n2}(t)$, $II_n(t)$, $R_{n1}(t)$, $R_{n2}(t)$, $R_{n4}(t)$, $R_{n5}(t)$, $R_{n6}(t)$ and $R_{n7}(t)$ are remainders.

Our goal is to bound from above the absolute value of each of the remainders by functions, which expectations are bounded. As in section A.1, on $\Omega_n^* = \Omega_n^{b_n} \cap \Omega_n^{a_n}$, an event of probability greater than or equal to $1 - 2\varepsilon$, we can restrict integration w.r.t F_{1n}^* and G_n^* to $[a_n, b_n]$.

Lemma A.14. *On the set $\tilde{\Omega}_n^0$, we have*

$$
|S_{n2}(t)| \le K^2 \frac{1}{n} \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \int_{(x,b_n)} \frac{F_{1n}^*(dy)}{C^2(y)} F_{1n}^*(dx) =: \tilde{R}_{n1}(t)
$$

Furthermore,

$$
E\tilde{R}_{n1}(t) \le \frac{K^2 2M\alpha}{n} ln(\frac{n}{c_1\alpha})
$$

Proof.
As in Section A.1

$$
|B_n(x)| \le \frac{1}{n} \int_{(x,b_n)} \frac{F_{1n}^*(dy)}{C_n^2(y)}
$$

Since, on the set $\tilde{\Omega}_n^0$, we have $C/C_n \leq K$, we obtain

$$|S_{n2}(t)| \leq K^2 \frac{1}{n} \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \int_{(x,b_n)} \frac{F_{1n}^*(dy)}{C^2(y)} F_{1n}^*(dx) = \tilde{R}_{n1}(t)$$

and

$$E\tilde{R}_{n1}(t) = K^2 \frac{1}{n} \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \int_{(x,b_n)} \frac{F_1^*(dy)}{C^2(y)} F_1^*(dx) = K^2 \alpha \frac{1}{n} \int_{(a_n,t]} \int_{(x,b_n)} \frac{F_1(dy)}{F_1^2(y)(1-G(y^-))} F_1(dx)$$

Since by A1, $\int \frac{dF_1}{1-G-} \leq M$ and then $1 - F_1 \leq M(1-G-)$, we have

$$\int_{(x,b_n)} \frac{F_1(dy)}{F_1^2(y)(1-G(y^-))} \leq M \int_{(x,b_n)} \frac{F_1(dy)}{F_1^2(y)} + \int_{(x,b_n)} \frac{F_1(dy)}{F_1(y)(1-G(y^-))} \leq \frac{2M}{F_1(x)}.$$

Therefore, by Remark A.1,

$$E\tilde{R}_{n1}(t) \leq \frac{K^2 2M\alpha}{n} ln(\frac{n}{c_1\alpha})$$

$\boxtimes$

Lemma A.15. *On the set $\tilde{\Omega}_n^0$, we have*

$$R_{n2}(t) \leq K \int_{(-\infty,t]} \frac{\gamma(x)}{C^3(x)} (C(x) - C_n(x))^2 F_1^*(dx) =: \tilde{R}_{n2}(t)$$

Furthermore,

$$E\tilde{R}_{n2}(t) \leq \frac{4Kc_1}{n} \frac{1}{1-G(a_n^-)} + \frac{KM\alpha}{n} \left(ln(\frac{n}{c_1\alpha}) + 1 \right)$$

Proof.
We have

$$R_{n2}(t) \leq K \int_{(-\infty,t]} \frac{\gamma(x)}{C^3(x)} (C(x) - C_n(x))^2 F_1^*(dx) = \tilde{R}_{n2}(t)$$

and

$$\tilde{R}_{n2}(t) \leq 2K \int_{(-\infty,a_n]} \frac{\gamma(x)}{C^3(x)} (C^2(x) + C_n^2(x)) F_1^*(dx) + K \int_{(a_n,t]} \frac{\gamma(x)}{C^3(x)} (C(x) - C_n(x))^2 F_1^*(dx).$$

Therefore, by Remark A.1,

$$E\tilde{R}_{n2}(t) \le 4K\int_{(-\infty,a_n]}\frac{\gamma(x)}{C(x)}F_1^*(dx)\frac{K}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}F_1^*(dx) \le 4K\frac{c_1}{n}\frac{1}{1-G(a_n^-)} + \frac{K}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}F_1^*(dx).$$

Since

$$\begin{aligned}
\frac{K}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}F_1^*(dx) &= \frac{K\alpha}{n}\int_{(a_n,t]}\frac{F_1(dx)}{F_1(x)(1-G(x^-))} \le \frac{K\alpha}{n}\left(\int_{(a_n,t]} M\frac{F_1(dx)}{F_1(x)} + \int_{(a_n,t]}\frac{F_1(dx)}{1-G(x^-)}\right)\\
&\le \frac{K\alpha}{n}\left(Mln(\frac{n}{c_1\alpha}) + M\right)
\end{aligned}$$

the proof is complete. ⊠

Lemma A.16. *On the set $\tilde{\Omega}_n^0$, we have*

$$R_{n6}(t) \le K^2\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{(C_n(y)-C(y))^2}{C^3(y)}F_{1n}^*(dy)F_{1n}^*(dx) =: \tilde{R}_{n6}(t)$$

Furthermore,

$$E\tilde{R}_{n6}(t) \le 4MK^2\alpha\frac{1}{n}ln(\frac{n}{c_1\alpha}) + \frac{1}{n}\frac{44M^2K^2}{c_1 c}$$

Proof.

$$R_{n6}(t) \le K^2\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{(C_n(y)-C(y))^2}{C^3(y)}F_{1n}^*(dy)F_{1n}^*(dx) = \tilde{R}_{n6}(t)$$

and

$$\begin{aligned}
E\tilde{R}_{n6}(t) &\le \frac{2K^2}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{1}{C^2(y)}F_1^*(dy)F_1^*(dx) + \frac{4K^2}{n^2}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{1}{C^3(y)}F_1^*(dy)F_1^*(dx)\\
&= \frac{2K^2\alpha}{n}\int_{(a_n,t]}\int_{(x,b_n)}\frac{F_1(dy)}{F_1^2(y)(1-G(y^-))}F_1(dx) + \frac{4K^2\alpha^2}{n^2}\int_{(a_n,t]}\int_{(x,b_n)}\frac{F_1(dy)}{F_1^3(y)(1-G(y^-))^2}F_1(dx)
\end{aligned}$$

Since

$$2K^2\alpha\frac{1}{n}\int_{(a_n,t]}\int_{(x,b_n)}\frac{F_1(dy)}{F_1^2(y)(1-G(y^-))}F_1(dx) \le 2K^2\alpha\frac{1}{n}\int_{(a_n,t]}\frac{2M}{F_1(x)}F_1(dx) \le 4MK^2\alpha\frac{1}{n}ln(\frac{n}{c_1\alpha})$$

and

$$\begin{aligned}
&4K^2\alpha^2\frac{1}{n^2}\int_{(a_n,t]}\int_{(x,b_n)}\frac{F_1(dy)}{F_1^3(y)(1-G(y^-))^2}F_1(dx) \le 4K^2\alpha^2\frac{1}{n^2}\int_{(a_n,t]}\int_{(x,b_n)}6M^2\frac{F_1(dy)}{F_1^3(y)}F_1(dx)\\
&+4K^2\alpha^2\frac{1}{n^2}\int_{(a_n,t]}\int_{(x,b_n)}5M\frac{F_1(dy)}{(1-G(y^-))^2}F_1(dx) \le \frac{6M^2 4K^2\alpha^2}{n^2}\frac{1}{F_1(a_n)} + \frac{5M^2 4K^2\alpha^2}{n^2}\frac{1}{1-G(b_n^-)}\\
&\le 6M^2 4K^2\alpha\frac{1}{c_1 n} + 5M^2 4K^2\alpha\frac{1}{cn} \le \frac{1}{n}\frac{44M^2K^2}{c_1 c}
\end{aligned}$$

the proof is complete.

$\boxtimes$

Lemma A.17. *We have*

$$E|R_{n1}(t)| \leq \frac{2\sqrt{\alpha M}}{n}\frac{1}{\sqrt{1-G(t^-)}}\left(ln(\frac{n}{c_1\alpha})+1\right)^{1/2}+\frac{\sqrt{8}\alpha}{n}\left(Mln(\frac{n}{c_1\alpha})+M\right)+\frac{M^{3/2}}{n\sqrt{c_1}}+\frac{\sqrt{8M\alpha}}{n^{3/2}}\frac{1}{1-G(t^-)}$$

Proof.
The expectation of $(R_{n1}(t))^2$ equals

$$\begin{aligned} E(R_{n1}(t))^2 &= E\left(\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}(C(x)-C_n(x))(F^*_{1n}(dx)-F^*_1(dx))\right)^2 \\ &= \frac{1}{n^4}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{l=1}^n E\Bigg(\frac{\gamma(U_{1i})1_{\{a_n<U_{1i}\leq t\}}}{C^2(U_{1i})}(1_{\{U_{1j}\leq U_{1i}\leq Z_j\}}-C(U_{1i})) \\ &\quad -\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}(1_{\{U_{1j}\leq x\leq Z_j\}}-C(x))F^*_1(dx)\Bigg) \\ &\quad \times\Bigg(\frac{\gamma(U_{1l})}{C^2(U_{1l})}1_{\{a_n<U_{1l}\leq t\}}(1_{\{U_{1k}\leq U_{1l}\leq Z_k\}}-C(U_{1l})) \\ &\quad -\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}(1_{\{U_{1k}\leq x\leq Z_k\}}-C(x))F^*_1(dx)\Bigg) \end{aligned}$$

Now, if only two indices are equal or $i\neq j\neq k\neq l$, it is easy to see that the expectation equals zero. So it is remains to deal with $i=l\neq j=k$, $k=l\neq j=i$, $k=i\neq j=l$ and $k=i=j=l$. For the first one we have

$$\begin{aligned} &\frac{1}{n^4}n(n-1)E\left(\frac{\gamma(U_{11})1_{\{a_n<U_{11}\leq t\}}}{C^2(U_{11})}(1_{\{U_{12}\leq U_{11}\leq Z_2\}}-C(U_{11}))-\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}(1_{\{U_{12}\leq x\leq Z_2\}}-C(x))F^*_1(dx)\right)^2 \\ &\leq\frac{1}{n^2}4\int_{(a_n,t]}\frac{\gamma^2(x)}{C^3(x)}F^*_1(dx)=\frac{4\alpha}{n^2}\int_{(a_n,t]}\frac{1}{F_1(x)(1-G(x^-))^2}F_1(dx)\leq\frac{4\alpha}{n^2}\frac{1}{1-G(t^-)}\left(Mln(\frac{n}{c_1\alpha})+M\right). \end{aligned}$$

The second one, for $k=l\neq j=i$, equals

$$\begin{aligned} &\frac{1}{n^4}n(n-1)E\Bigg(\frac{\gamma(U_{11})1_{\{a_n<U_{11}\leq t\}}}{C^2(U_{11})}(1-C(U_{11}))-\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}(1_{\{U_{11}\leq x\leq Z_1\}}-C(x))F^*_1(dx)\Bigg) \\ &\times\Bigg(\frac{\gamma(U_{12})1_{\{a_n<U_{12}\leq t\}}}{C^2(U_{12})}(1-C(U_{12}))-\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}(1_{\{U_{12}\leq x\leq Z_2\}}-C(x))F^*_1(dx)\Bigg) \\ &\leq 4\frac{1}{n^2}\left(\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}F^*_1(dx)\right)^2\leq 4\alpha^2\frac{1}{n^2}\left(Mln(\frac{n}{c_1\alpha})+M\right)^2 \end{aligned}$$

If $k=i\neq j=l$, we obtain the same bound as above. Finally, if $k=i=j=l$, we get

$$
\begin{aligned}
&\frac{1}{n^4}nE\Bigg(\frac{\gamma(U_{11})1_{\{a_n<U_{11}\le t\}}}{C^2(U_{11})}(1-C(U_{11}))-\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}(1_{\{U_{11}\le x\le Z_1\}}-C(x))F_1^*(dx)\Bigg)^2\\
&\le \frac{4}{n^3}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^4(x)}F_1^*(dx)=\frac{4\alpha^3}{n^3}\int_{(a_n,t]}\frac{1}{F_1^2(x)(1-G(x^-))^3}F_1(dx)\\
&\le \frac{4\alpha}{n^3}\int_{(a_n,t]}\left(\frac{M(1-G(x^-))}{F_1(x)}+1\right)^2\frac{1}{(1-G(x^-))^3}F_1(dx)\\
&\le \frac{4\alpha}{n^3}\int_{(a_n,t]}\frac{2M^2}{F_1^2(x)(1-G(x^-))}F_1(dx)+\frac{4\alpha}{n^3}\int_{(a_n,t]}\frac{2}{(1-G(x^-))^3}F_1(dx)\\
&\le \frac{4\alpha}{n^3}\int_{(a_n,t]}\frac{2M^3}{F_1^2(x)}F_1(dx)+\frac{4\alpha}{n^3}\int_{(a_n,t]}\frac{2M^2}{F_1(x)(1-G(x^-))}F_1(dx)+\frac{4\alpha}{n^3}\int_{(a_n,t]}\frac{2}{(1-G(x^-))^3}F_1(dx)\\
&\le \frac{16M^3\alpha}{n^3}\frac{1}{F_1(a_n)}+\frac{8M\alpha}{n^3}\frac{1}{(1-G(t^-))^2}\le\frac{16M^3}{n^2c_1}+\frac{8M\alpha}{n^3}\frac{1}{(1-G(t^-))^2}
\end{aligned}
$$

and, by Cauchy-Schwarz, we get

$$
E|R_{n1}(t)|\le\frac{2\sqrt{\alpha M}}{n}\frac{1}{\sqrt{1-G(t^-)}}\left(ln(\frac{n}{c_1\alpha})+1\right)^{1/2}+\frac{\sqrt{8}\alpha}{n}\left(Mln(\frac{n}{c_1\alpha})+M\right)+\frac{M^{3/2}}{n\sqrt{c_1}}+\frac{\sqrt{8M\alpha}}{n^{3/2}}\frac{1}{1-G(t^-)}
$$

$\boxtimes$

Lemma A.18. *We have*

$$
E|R_{n4}(t)|\le\frac{\sqrt{52}}{n}\alpha M\left(\sqrt{ln(\frac{n}{c_1\alpha})}+1\right).
$$

Proof.
We have

$$
\begin{aligned}
E(R_{n4}(t))^2 &= E\Big(\int\int\frac{\gamma(x)}{C(x)}1_{\{x\le t\}}1_{\{y>x\}}\frac{1}{C(y)}(F_{1n}^*(dy)-F_1^*(dy))(F_{1n}^*(dx)-F_1^*(dx))\Big)^2\\
&= \frac{1}{n^4}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{l=1}^n E(h(U_{1i},U_{1j})h(U_{1k},U_{1l}))
\end{aligned}
$$

with

$$
h(x,y)=h_1(x,y)-\int h_1(x,v)F_1^*(dv)-\int h_1(u,y)F_1^*(du)+\int h_1(u,v)F_1^*(du)F_1^*(dv)
$$

and

$$
h_1(x,y)=\frac{\gamma(x)}{C(x)C(y)}1_{\{x\le t\}}1_{\{y>x\}}
$$

Furthermore, $E(h(U_{1i},U_{1j})h(U_{1k},U_{1l}))=0$ if three or fourth indices are different. Since additionally

$$E(h(U_{11},U_{11})h(U_{12},U_{11}))=E(h(U_{11},U_{11})E(h(U_{12},x))(x=U_{11}))=0,$$

we get

$$E(R_{n4}(t))^2 \quad = \quad \frac{1}{n^3}E[(h(U_{11},U_{11})^2]+\frac{1}{n^2}E[(h(U_{12},U_{11})^2].$$

Since $h_1(U_{11},U_{11})=0$,

$$\frac{1}{n^3}E[(h(U_{11},U_{11})^2]\leq\frac{10}{n^3}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}F_1^*(dx).$$

Furthermore,

$$\frac{1}{n^3}E[(h(U_{12},U_{11})^2]\leq\frac{16}{n^2}\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}F_1^*(dx).$$

Since, by A1 and Remark A.1,

$$\int_{(a_n,t]}\frac{\gamma^2(x)}{C^2(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}F_1^*(dx)\leq\int_{(a_n,t]}\frac{\alpha}{1-G(x^-)}\frac{2\alpha M}{F_1(x)}F_1(dx)\leq 2\alpha^2M\left(Mln(\frac{n}{c_1\alpha})+M\right)$$

we get

$$E(R_{n4}(t))^2 \quad \leq \quad \frac{26}{n^2}2\alpha^2M\left(Mln(\frac{n}{c_1\alpha})+M\right).$$

Finally

$$E|R_{n4}(t)| \quad \leq \quad \frac{\sqrt{52}}{n}\alpha M\left(\sqrt{ln(\frac{n}{c_1\alpha})}+1\right)$$

$\boxtimes$

Lemma A.19. *On the set $\tilde{\Omega}_n^0$, we have*

$$|R_{n5}(t)|\leq K\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}\left|\int_{(x,b_n)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right||C(x)-C_n(x)|F_{1n}^*(dx)=:\tilde{R}_{n5}(t)$$

Furthermore,

$$E\tilde{R}_{n5}(t)\leq\frac{2KM\alpha}{n}ln(\frac{n}{c_1\alpha})(ln(\frac{n}{c_1\alpha})+1)+\frac{\sqrt{2}K}{n}\alpha M\left(M^{1/2}ln(\frac{n}{c_1\alpha})+1\right)+\frac{2KM}{n}\frac{\sqrt{\alpha}}{\sqrt{c_1}}.$$

Proof.
We have

$$|R_{n5}(t)| \leq K \int_{(a_n,t]} \frac{\gamma(x)}{C^2(x)} \left| \int_{(x,b_n)} \frac{F_{1n}^*(dy) - F_1^*(dy)}{C(y)} \right| |C(x) - C_n(x)| F_{1n}^*(dx) = \tilde{R}_{n5}(t)$$

and

$$\begin{aligned}
\tilde{R}_{n5}(t) \quad \leq \quad & K \int_{(a_n,t]} \frac{\gamma(x)}{C^2(x)} \left\{ \frac{n-1}{n} \left| \int_{(x,b_n)} \frac{F_{1n-1}^*(dy) - F_1^*(dy)}{C(y)} \right| + \frac{1}{n} \int_{(x,b_n)} \frac{F_1^*(dy)}{C(y)} \right\} |C(x) - C_n(x)| F_{1n}^*(dx) \\
& \leq K \int_{(a_n,t]} \frac{\gamma(x)}{C^2(x)} \left| \int_{(x,b_n)} \frac{F_{1n-1}^*(dy) - F_1^*(dy)}{C(y)} \right| |C(x) - C_n(x)| F_{1n}^*(dx) \\
& + \frac{2K}{n} \int_{(a_n,t]} \frac{\gamma(x)}{C^2(x)} \int_{(x,b_n)} \frac{F_1^*(dy)}{C(y)} F_{1n}^*(dx) =: \tilde{R}_{n5}^a(t) + \tilde{R}_{n5}^b(t)
\end{aligned}$$

The expectation of the second term equals

$$\begin{aligned}
E\tilde{R}_{n5}^b(t) \quad = \quad & \frac{2K}{n} \int_{(a_n,t]} \frac{\gamma(x)}{C^2(x)} \int_{(x,b_n)} \frac{F_1^*(dy)}{C(y)} F_1^*(dx) \leq \frac{2K}{n} ln(\frac{n}{c_1 \alpha}) \alpha \int_{(a_n,t]} \frac{F_1(dx)}{F_1(x)(1 - G(x^-))} \\
& \leq \frac{2KM\alpha}{n} ln(\frac{n}{c_1\alpha})(ln(\frac{n}{c_1\alpha}) + 1),
\end{aligned}$$

while for the first one we have

$$\begin{aligned}
\tilde{R}_{n5}^a(t) \quad \leq \quad & K \int_{(a_n,t]} \frac{\gamma(x)}{C^2(x)} \left| \int_{(x,b_n)} \frac{F_{1n-1}^*(dy) - F_1^*(dy)}{C(y)} \right| \left[\frac{n-1}{n} |C(x) - C_{n-1}(x)| + \frac{1}{n} |C(x) - 1| \right] F_{1n}^*(dx) \\
\leq \quad & K \int_{(a_n,t]} \frac{\gamma(x)}{C^2(x)} \left| \int_{(x,b_n)} \frac{F_{1n-1}^*(dy) - F_1^*(dy)}{C(y)} \right| \left[|C(x) - C_{n-1}(x)| + \frac{1}{n} \right] F_{1n}^*(dx)
\end{aligned}$$

By Cauchy-Schwarz we obtain

$$E \left| \int_{(x,b_n)} \frac{F_{1n-1}^*(dy) - F_1^*(dy)}{C(y)} \right| |C(x) - C_{n-1}(x)| \leq \frac{1}{n} \left(\int_{(x,b_n)} \frac{F_1^*(dy)}{C^2(y)} \right)^{1/2} C^{1/2}(x)$$

and

$$E \left| \int_{(x,b_n)} \frac{F_{1n-1}^*(dy) - F_1^*(dy)}{C(y)} \right| \leq \frac{1}{n} \left(\int_{(x,b_n)} \frac{F_1^*(dy)}{C^2(y)} \right)^{1/2}.$$

Therefore,

$$E\tilde{R}_{n5}^a(t) \quad \leq \quad \frac{K}{n} \int_{(a_n,t]} \frac{\gamma(x)}{C^{3/2}(x)} \left(\int_{(x,b_n)} \frac{F_1^*(dy)}{C^2(y)} \right)^{1/2} F_1^*(dx) + \frac{K}{n\sqrt{n}} \int_{(a_n,t]} \frac{\gamma(x)}{C^2(x)} \left(\int_{(x,b_n)} \frac{F_1^*(dy)}{C^2(y)} \right)^{1/2} F_1^*(dx).$$

Furthermore, for the integrals on the right side, we have

$$\frac{K}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C^{3/2}(x)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}\right)^{1/2}F_1^*(dx) \le \frac{\sqrt{2}K}{n}\alpha M\left(M^{1/2}ln(\frac{n}{c_1\alpha})+1\right)$$

and

$$\frac{K}{n\sqrt{n}}\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}\right)^{1/2}F_1^*(dx) \le \frac{2KM\alpha}{n\sqrt{n}}\frac{1}{\sqrt{F_1(a_n)}} \le \frac{2KM}{n}\frac{\sqrt{\alpha}}{\sqrt{c_1}}.$$

Hence

$$E\tilde{R}_{n5}(t) \le \frac{2KM\alpha}{n}ln(\frac{n}{c_1\alpha})(ln(\frac{n}{c_1\alpha})+1)+\frac{\sqrt{2}K}{n}\alpha M\left(M^{1/2}ln(\frac{n}{c_1\alpha})+1\right)+\frac{2KM}{n}\frac{\sqrt{\alpha}}{\sqrt{c_1}}.$$

⊠

Lemma A.20. *On the set $\tilde{\Omega}_n^0$, we have*

$$\begin{aligned}|II_n(t)| &\le \frac{K^3}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{F_{1n}^*(dy)}{C^2(y)}|D_{n1}(x)|F_{1n}^*(dx)+K\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}D_{n1}^2(x)F_{1n}^*(dx)\\ &\quad +\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}(\tilde{L}_{n3}^a(x)+L_{n3}^b(x))F_{1n}^*(dx) =: \tilde{II}_n^*(t),\end{aligned}$$

where

$$\begin{aligned}\tilde{L}_{n3}^a(x) &= \frac{K}{n}\sum_{i=1}^n\frac{1_{\{x<U_{1i}<b_n\}}}{C^3(U_{1i})}(C_{n-1}(U_{1i})-C(U_{1i}))^2\\ &\quad +\frac{K^2}{n^2}\sum_{i=1}^n\sum_{j\ne i}\frac{1_{\{x<U_{1i}<b_n\}}}{C^2(U_{1i})}\frac{1_{\{x<U_{1j}<b_n\}}}{C^2(U_{1j})}|C_{n-1}(U_{1i})-C(U_{1i})||C_{n-1}(U_{1i})-C(U_{1i})|\end{aligned}$$

and

$$L_{n3}^b(x) = \frac{K}{n}\int_{(x,b_n)}\frac{F_{1n}^*(dy)}{C^2(y)}\left|\int_{(x,b_n)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right|.$$

Furthermore,

$$E\tilde{II}_n^*(t) \le k_6\frac{K^3M^2}{n}\frac{ln^2(\frac{n}{c_1\alpha})}{c_1c},$$

where k_6 is a constant.

Proof.
We have

$$|II_n(t)| \;\leq\; \int_{(-\infty,t]} \frac{\gamma(x)}{C_n(x)} \left|e^{z_n(x)}-1\right| |D_{n1}(x)| F^*_{1n}(dx)$$

with

$$z_n(x) = \Delta_n(x) + \int_{(x,\infty)} \frac{F^*_1(dy)}{C(y)}$$

By Taylor

$$\left|e^{z_n(x)}-1\right| = |z_n(x)| e^{\tilde z_n(x)}, \quad \tilde z_n(x) \in (z_n(x), 0)$$

and since

$$\Delta_n \in \left(n \int_{(x,\infty)} \ln\left(1 - \frac{1}{nC_n(y)}\right) F^*_{1n}(dy), - \int_{(x,\infty)} \frac{F^*_1(dy)}{C(y)} \right)$$

$$(B_n + D_{n1} + D_{n2})(t) \leq z_n(x) \leq 0.$$

Furthermore, on the set $\tilde\Omega^0_n$, we have $C/C_n \leq K$. Hence

$$\begin{aligned}
|II_n(t)| \;&\leq\; K \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} |(B_{n1} + D_{n1} + D_{n2})(t)| |D_{n1}(x)| F^*_{1n}(dx) \\
&\leq\; \frac{K^3}{n} \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \int_{(x,b_n)} \frac{F^*_{1n}(dy)}{C^2(y)} |D_{n1}(x)| F^*_{1n}(dx) + K \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} D^2_{n1}(x) F^*_{1n}(dx) \\
&\quad + K^2 \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \int_{(x,b_n)} \frac{|C_n(y) - C(y)|}{C(y)C_n(y)} F^*_{1n}(dy) |D_{n1}(x)| F^*_{1n}(dx) \\
&\quad =: \tilde L_{1n}(t) + \tilde L_{2n}(t) + \tilde L_{3n}(t) =: \tilde{II}_n(t).
\end{aligned}$$

To deal with $\tilde L_{1n}$, note that

$$|D_{n1}(U_{1i})| \leq \frac{n-1}{n} |D_{n-11}(U_{1i})| + \frac{1}{n} \int_{(U_{1i},b_n)} \frac{F^*_1(dy)}{C(y)},$$

where $|D_{n-11}(.)|$ does not include U_{1i}. Hence

$$\begin{aligned}
\tilde L_{1n}(t) \;&=\; \frac{K^3}{n} \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \frac{(n-1)^2}{n^2} \int_{(x,b_n)} \frac{F^*_{1n-1}(dy)}{C^2(y)} |D_{n-11}(x)| F^*_{1n}(dx) \\
&\quad + \frac{K^3}{n^2} \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \frac{n-1}{n} \int_{(x,b_n)} \frac{F^*_{1n-1}(dy)}{C^2(y)} \int_{(x,b_n)} \frac{F^*_1(dy)}{C(y)} F^*_{1n}(dx).
\end{aligned}$$

Furthermore, by Cauchy-Schwarz, we have

$$E\left(\int_{(x,b_n)}\frac{F_{1n}^*(dy)}{C^2(y)}|D_{n-11}(x)|\right)=E\left(\int_{(x,b_n)}\frac{F_{1n}^*(dy)}{C^2(y)}\int_{(x,b_n)}\left|\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right|\right)$$
$$\leq\frac{1}{n}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^4(y)}\right)^{1/2}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}\right)^{1/2}+\frac{1}{\sqrt{n}}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}\right)^{1/2} \tag{A.22}$$

Hence

$$\begin{aligned}E\tilde{L}_{1n}(t) &= \frac{K^3}{n^2}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^4(y)}\right)^{1/2}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}\right)^{1/2}F_1^*(dx)\\ &\quad+\frac{K^3}{n}\frac{1}{\sqrt{n}}\int_{(-\infty,t]}\frac{\gamma(x)}{C(x)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}\right)^{1/2}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}F_1^*(dx)\\ &\quad+\frac{K^3}{n^2}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C(y)}F_1^*(dx).\end{aligned}$$

Next, we consider the inner integrals separately. By repeated use of $1-F_1(x)\leq M(1-G(x^-))$ and since w.l.o.g. $M\geq 1$, we get

$$\begin{aligned}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^4(y)}\right)^{1/2} &\leq \alpha^{3/2}\left(\int_{(x,b_n)}\frac{20M^3}{F_1^4(y)}F_1(dy)+\int_{(x,b_n)}\frac{15M^2}{(1-G(y^-))^3}F_1(dy)\right)^{1/2}\\ &\leq \frac{\alpha^{3/2}\sqrt{20}M^{3/2}}{F_1^{3/2}(x)}+\frac{\alpha^{3/2}\sqrt{15}M^{3/2}}{1-G(b_n^-)}\leq\frac{\alpha^{3/2}\sqrt{20}M^{3/2}}{F_1^{3/2}(x)}+\alpha^{1/2}\sqrt{15}M^{3/2}\frac{n}{c}.\end{aligned}$$

and

$$\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}\leq\frac{2M\alpha}{F_1(x)}.$$

Therefore, since $\int_{(x,b_n)}\frac{F_1^*(dy)}{C(y)}\leq ln(\frac{n}{c_1\alpha})$ for $x>a_n$ and $\frac{\gamma}{C}dF_1^*=dF_1$, we obtain

$$\begin{aligned}E\tilde{L}_{1n}(t) &\leq \frac{K^3}{n^2}\int_{(a_n,t]}\frac{\alpha^{3/2}\sqrt{20}M^{3/2}}{F_1^{3/2}(x)}\frac{\sqrt{2M\alpha}}{\sqrt{F_1(x)}}F_1(dx)+\frac{K^3}{n^2}\int_{(a_n,t]}\alpha^{1/2}\sqrt{15}M^{3/2}\frac{n}{c}\frac{\sqrt{2M\alpha}}{\sqrt{F_1(x)}}F_1(dx)\\ &\quad+\frac{K^3}{n}\frac{1}{\sqrt{n}}\int_{(-\infty,t]}\frac{(2M\alpha)^{3/2}}{F_1^{3/2}(x)}F_1(dx)+\frac{K^3}{n^2}ln(\frac{n}{c_1\alpha})\int_{(a_n,t]}\frac{2M\alpha}{F_1(x)}F_1(dx)\\ &\leq\frac{K^3}{n}\frac{\sqrt{40}M^2\alpha}{c_1}+\frac{K^3}{n}\frac{\sqrt{30}M^2\alpha}{c}+\frac{K^3}{n}\frac{(2M)^{3/2}\alpha}{\sqrt{c_1}}+\frac{K^3}{n^2}2M\alpha ln^2(\frac{n}{c_1\alpha}).\end{aligned}$$

As to $\tilde{L}_{2n}(t)$, since

$$ED_{n1}^2(x)\leq\frac{1}{n}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)},$$

we get

$$E\tilde{L}_{2n}(t) \le K\frac{1}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}F_1^*(dx) \le \frac{2KM\alpha}{n}ln(\frac{n}{c_1\alpha}).$$

As to $\tilde{L}_{3n}(t)$, the inner integral can be written as

$$\begin{aligned}
&\int_{(x,b_n)}\frac{|C_n(y)-C(y)|}{C^2(y)}F_{1n}^*(dy)|D_{n1}(x)| \le \frac{n-1}{n}\int_{(x,b_n)}\frac{|C_{n-1}(y)-C(y)|}{C(y)C_n(y)}F_{1n}^*(dy)\\
&\times\left|\int_{(x,b_n)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right| + \frac{K}{n}\int_{(x,b_n)}|1-C(y)|\frac{F_{1n}^*(dy)}{C^2(y)}\left|\int_{(x,b_n)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right|\\
&\le \int_{(x,b_n)}\frac{|C_{n-1}(y)-C(y)|}{C(y)C_n(y)}F_{1n}^*(dy)\left|\int_{(x,b_n)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right|\\
&+\frac{K}{n}\int_{(x,b_n)}\frac{F_{1n}^*(dy)}{C^2(y)}\left|\int_{(x,b_n)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right| = L_{n3}^a(x)+L_{n3}^b(x)
\end{aligned}$$

As to $L_{n3}^a(x)$, we have

$$(L_{n3}^a(x))^2 = \left(\int_{(x,b_n)}\frac{|C_{n-1}(y)-C(y)|}{C(y)C_n(y)}F_{1n}^*(dy)\right)^2\left(\int_{(x,b_n)}\frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)}\right)^2.$$

For the first summand, since $C/C_n \le K$ on $\tilde{\Omega}_n^0$ and $C_n(U_{1i}) \ge 1/n$, we obtain

$$\begin{aligned}
&\left(\int_{(x,b_n)}\frac{|C_{n-1}(y)-C(y)|}{C(y)C_n(y)}F_{1n}^*(dy)\right)^2\\
&= \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n\frac{1_{\{x<U_{1i}<b_n\}}}{C(U_{1i})C_n(U_{1i})}\frac{1_{\{x<U_{1j}<b_n\}}}{C(U_{1j})C_n(U_{1j})}|C_{n-1}(U_{1i})-C(U_{1i})||C_{n-1}(U_{1i})-C(U_{1i})|\\
&= \frac{1}{n^2}\sum_{i=1}^n\frac{1_{\{x<U_{1i}<b_n\}}}{C^2(U_{1i})C_n^2(U_{1i})}(C_{n-1}(U_{1i})-C(U_{1i}))^2\\
&+\frac{1}{n^2}\sum_{i=1}^n\sum_{j\ne i}\frac{1_{\{x<U_{1i}<b_n\}}}{C(U_{1i})C_n(U_{1i})}\frac{1_{\{x<U_{1j}<b_n\}}}{C(U_{1j})C_n(U_{1j})}|C_{n-1}(U_{1i})-C(U_{1i})||C_{n-1}(U_{1i})-C(U_{1i})|\\
&\le \frac{K}{n}\sum_{i=1}^n\frac{1_{\{x<U_{1i}<b_n\}}}{C^3(U_{1i})}(C_{n-1}(U_{1i})-C(U_{1i}))^2\\
&+\frac{K^2}{n^2}\sum_{i=1}^n\sum_{j\ne i}\frac{1_{\{x<U_{1i}<b_n\}}}{C^2(U_{1i})}\frac{1_{\{x<U_{1j}<b_n\}}}{C^2(U_{1j})}|C_{n-1}(U_{1i})-C(U_{1i})||C_{n-1}(U_{1i})-C(U_{1i})| =: \tilde{L}_{n3}^a(x).
\end{aligned}$$

Furthermore,

$$E\left(\frac{1}{n}\sum_{i=1}^n\frac{1_{\{x<U_{1i}<b_n\}}}{C^3(U_{1i})}(C_{n-1}(U_{1i})-C(U_{1i}))^2\right) = \frac{1}{n}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}$$

and

$$E\left(\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j\neq i}\frac{1_{\{x<U_{1i}<b_n\}}}{C^2(U_{1i})}\frac{1_{\{x<U_{1j}<b_n\}}}{C^2(U_{1j})}|C_{n-1}(U_{1i})-C(U_{1i})||C_{n-1}(U_{1j})-C(U_{1j})|\right)\leq\frac{1}{n}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}\right)^2.$$

Hence, for $x>a_n$

$$\begin{aligned}E\tilde{L}^a_{n3}(x) &\leq \frac{1}{n}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}+\frac{1}{n}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}\right)^2\leq\frac{2\alpha M}{n}\frac{1}{F_1(x)}+\frac{1}{n}\frac{\alpha M^2}{F_1(x)}+\frac{\alpha M}{n}ln(\frac{n}{c_1\alpha})\\ &\leq \frac{3\alpha M^2}{n}\frac{1}{F_1(x)}+\frac{\alpha M}{n}ln(\frac{n}{c_1\alpha}).\end{aligned}$$

Therefore,

$$E\left(\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\tilde{L}^a_{n3}(x)F^*_{1n}(dx)\right)\leq\frac{1}{n}3\alpha M^2 ln(\frac{n}{c_1\alpha})+\frac{\alpha M}{n}ln(\frac{n}{c_1\alpha}).$$

Furthermore, since,

$$L^b_{n3}(x)=\frac{K}{n}\int_{(x,b_n)}\frac{F^*_{1n}(dy)}{C^2(y)}\left|\int_{(x,b_n)}\frac{F^*_{1n}(dy)-F^*_1(dy)}{C(y)}\right|,$$

according to (A.22),

$$EL^b_{n3}(x)\leq\frac{1}{n^2}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^4(y)}\right)^{1/2}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}\right)^{1/2}+\frac{1}{n\sqrt{n}}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}\right)^{1/2}$$

and, similarly as for the first term in $\tilde{L}_{1n}$, we obtain

$$\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}L^b_{n3}(x)F^*_{1n}(dx)\leq\frac{K}{n}\frac{\sqrt{40}M^2\alpha}{c_1}+\frac{K}{n}\frac{\sqrt{30}M^2\alpha}{c}+\frac{K}{n}\frac{(2M)^{3/2}\alpha}{\sqrt{c_1}}$$

Altogether,

$$\tilde{II}_n(t)\leq\tilde{L}_{1n}(t)+\tilde{L}_{2n}(t)+\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}(\tilde{L}^a_{n3}(x)+L^b_{n3}(x))F^*_{1n}(dx)=:\tilde{II}^*_n(t)$$

and

$$\begin{aligned}E\tilde{II}^*_n(t) &\leq \frac{K^3}{n}\frac{\sqrt{40}M^2\alpha}{c_1}+\frac{K^3}{n}\frac{\sqrt{30}M^2\alpha}{c}+\frac{K^3}{n}\frac{(2M)^{3/2}\alpha}{\sqrt{c_1}}+\frac{K^3}{n^2}2M\alpha ln^2(\frac{n}{c_1\alpha})+\frac{2KM\alpha}{n}ln(\frac{n}{c_1\alpha})\\ &\quad+\frac{1}{n}3\alpha M^2 ln(\frac{n}{c_1\alpha})+\frac{\alpha M}{n}ln(\frac{n}{c_1\alpha})+\frac{K}{n}\frac{\sqrt{40}M^2\alpha}{c_1}+\frac{K}{n}\frac{\sqrt{30}M^2\alpha}{c}+\frac{K}{n}\frac{(2M)^{3/2}\alpha}{\sqrt{c_1}}.\end{aligned}$$

Since we may choose $K\geq 1$, $M\geq 1$, $c\leq 1$ and $c_1\leq\min(\frac{1}{e\alpha},1)$, we have

$$E\tilde{II}^*_n(t) \;\leq\; k_6\frac{K^3M^2}{n}\frac{ln^2(\frac{n}{c_1\alpha})}{c_1c},$$

where k_6 is a constant.

Lemma A.21. *On the set $\tilde{\Omega}_n^0$, we have*

$$\begin{aligned}
R_{n7}(t) \;\leq\; & \frac{K^{5/2}}{\sqrt{n}} \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \int_{(x,b_n)} \frac{F_{1n}^*(dy)}{C^{3/2}(y)} \int_{(x,b_n)} \frac{|C_n(y)-C(y)|}{C^2(y)} F_{1n}^*(dy) F_{1n}^*(dx) \\
& + K \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \left| \int_{(x,b_n)} \frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)} \right| \left| \int_{(x,b_n)} \frac{C_n(y)-C(y)}{C^2(y)} F_{1n}^*(dy) \right| F_{1n}^*(dx) \\
& + K \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \left(\int_{(x,b_n)} \frac{|C_n(y)-C(y)|}{C(y)C_n(y)} F_{1n}^*(dy) \right) \left(\int_{(x,b_n)} \frac{|C_n(y)-C(y)|}{C^2(y)} F_{1n}^*(dy) \right) F_{1n}^*(dx) \\
& + \int_{(a_n,t]} \frac{\gamma(x)}{C^2(x)} |C(x)-C_n(x)| \int_{(x,b_n)} \frac{|C_n(y)-C(y)|}{C^2(y)} F_{1n}^*(dy) F_{1n}^*(dx) \\
& + \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \left| \int_{(x,b_n)} \frac{C_n(y)-C(y)}{C^2(y)} (F_{1n}^*(dy)-F_1^*(dy)) \right| F_{1n}^*(dx) \\
& + \left| \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \int_{(x,b_n)} \frac{C_n(y)-C(y)}{C^2(y)} F_1^*(dy)(F_{1n}^*(dx)-F_1^*(dx)) \right| =: \tilde{R}_{n7}(t)
\end{aligned}$$

Furthermore,

$$E\tilde{R}_{n7}(t) \leq k_9 \frac{1}{n} \frac{K^3 M^3}{c_1^2 c^2} ln^2\left(\frac{n}{c_1 \alpha}\right) + \frac{2\sqrt{M}}{n} \int_{(a_n,t]} \left(\int_{(x,b_n)} \frac{F_1(dy)}{(1-G(y^-))^2} \right)^{1/2} F_1(dx)$$

Proof.
We have

$$\begin{aligned}
R_{n7}(t) \;=\; & \int_{(a_n,t]} \frac{\gamma(x)}{C_n(x)} \left(e^{z_n(x)}-1\right) \int_{(x,b_n)} \frac{C_n(y)-C(y)}{C^2(y)} F_{1n}^*(dy) F_{1n}^*(dx) \\
& + \int_{(a_n,t]} \frac{\gamma(x)}{C_n(x)C(x)} (C(x)-C_n(x)) \int_{(x,b_n)} \frac{C_n(y)-C(y)}{C^2(y)} F_{1n}^*(dy) F_{1n}^*(dx) \\
& + \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \int_{(x,b_n)} \frac{C_n(y)-C(y)}{C^2(y)} (F_{1n}^*(dy)-F_1^*(dy)) F_{1n}^*(dx) \\
& + \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \int_{(x,b_n)} \frac{C_n(y)-C(y)}{C^2(y)} F_1^*(dy)(F_{1n}^*(dx)-F_1^*(dx)) \\
& = R_{n7}^a(t) + R_{n7}^b(t) + R_{n7}^c(t) + R_{n7}^d(t)
\end{aligned}$$

As to $R_{n7}^a(t)$, similarly as in the proof of $II_n(t)$, $\left|e^{z_n(x)}-1\right| \leq |B_n(x)| + |D_{n1}(x)| + |D_{n2}(x)|$. Since on the set $\tilde{\Omega}_n^0$, we may replace C_n with C and $nC_n \geq 1$, we get

$$B_n(x) \leq \frac{1}{n} \int_{(x,b_n)} \frac{F_{1n}^*(dy)}{C_n^2(y)} \leq \frac{K^{3/2}}{\sqrt{n}} \int_{(x,b_n)} \frac{F_{1n}^*(dy)}{C^{3/2}(y)}$$

$$\begin{aligned}
|R_{n7}^a(t)| \;\leq\; & \frac{K^{5/2}}{\sqrt{n}} \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \int_{(x,b_n)} \frac{F_{1n}^*(dy)}{C^{3/2}(y)} \int_{(x,b_n)} \frac{|C_n(y)-C(y)|}{C^2(y)} F_{1n}^*(dy) F_{1n}^*(dx) \\
& + K \int_{(a_n,t]} \frac{\gamma(x)}{C(x)} \left| \int_{(x,b_n)} \frac{F_{1n}^*(dy)-F_1^*(dy)}{C(y)} \right| \left| \int_{(x,b_n)} \frac{C_n(y)-C(y)}{C^2(y)} F_{1n}^*(dy) \right| F_{1n}^*(dx)
\end{aligned}$$

$$+K\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left(\int_{(x,b_n)}\frac{|C_n(y)-C(y)|}{C(y)C_n(y)}F_{1n}^*(dy)\right)\left(\int_{(x,b_n)}\frac{|C_n(y)-C(y)|}{C^2(y)}F_{1n}^*(dy)\right)F_{1n}^*(dx)$$
$$=: M_{n1}+M_{n2}+M_{n3} =: \tilde{R}_{n7}^a(t)$$

in probability.
Furthermore, $M_{2n}(t)=\tilde{L}_{2n}(t)$, so that

$$EM_{2n}(t)\le K\frac{1}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}F_1^*(dx)\le\frac{2KM\alpha}{n}ln(\frac{n}{c_1\alpha}).$$

Next, we deal with expectations of M_{n1} and M_{n3}. As to M_{n3}, we have

$$\begin{aligned}
M_{n3}(t) &\le K^2\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\int_{(x,b_n)}\frac{|C_n(y)-C(y)|}{C^2(y)}\frac{|C_n(z)-C(z)|}{C^2(y)}F_{1n}^*(dz)F_{1n}^*(dy)F_{1n}^*(dx)\\
&= K^2\frac{1}{n^3}\sum_{i=1}^n\sum_{j\neq i}\sum_{k\neq i}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1i}\le t\}}\frac{|C_n(U_{1j})-C(U_{1j})|}{C^2(U_{1j})}\frac{|C_n(U_{1k})-C(U_{1k})|}{C^2(U_{1k})}1_{\{x<U_{1j},U_{1k}\le b_n\}}\\
&= K^2\frac{1}{n^3}\sum_{i=1}^n\sum_{j\neq i}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1i}\le t\}}\frac{(C_n(U_{1j})-C(U_{1j}))^2}{C^4(U_{1j})}1_{\{x<U_{1j}\le b_n\}}\\
&\quad +K^2\frac{1}{n^3}\sum_{i=1}^n\sum_{j\neq i}\sum_{k\neq i,j}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1i}\le t\}}\frac{|C_n(U_{1j})-C(U_{1j})|}{C^2(U_{1j})}\frac{|C_n(U_{1k})-C(U_{1k})|}{C^2(U_{1k})}1_{\{x<U_{1j},U_{1k}\le b_n\}}.
\end{aligned}$$

Hence, using conditional Cauchy-Schwarz for the second summand, we obtain

$$\begin{aligned}
EM_{n3}(t) &\le \frac{K^2}{n}E\left(\frac{\gamma(U_{11})}{C(U_{11})}1_{\{a_n<U_{11}\le t\}}\frac{(C_n(U_{12})-C(U_{12}))^2}{C^4(U_{12})}1_{\{x<U_{12}\le b_n\}}\right)\\
&\quad +K^2E\left(\frac{\gamma(U_{11})}{C(U_{11})}1_{\{a_n<U_{11}\le t\}}\frac{|C_n(U_{12})-C(U_{12})|}{C^2(U_{12})}\frac{|C_n(U_{13})-C(U_{13})|}{C^2(U_{13})}1_{\{x<U_{12},U_{13}\le b_n\}}\right)\\
&\le \frac{K^2}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^3(y)}F_1^*(dx)+K^2E\left(\frac{\gamma(U_{11})}{C(U_{11})}1_{\{a_n<U_{11}\le t\}}\frac{1_{\{x<U_{12},U_{13}\le b_n\}}}{C^2(U_{12})C^2(U_{13})}\right.\\
&\quad\times\sqrt{E((C_n(U_{12})-C(U_{12}))^2|U_{11},Z_1,U_{12},Z_2,U_{13},Z_3)}\\
&\quad\left.\times\sqrt{E((C_n(U_{13})-C(U_{13}))^2|U_{11},Z_1,U_{12},Z_2,U_{13},Z_3)}\right).
\end{aligned}$$

Since

$$\begin{aligned}
&E((C_n(U_{12})-C(U_{12}))^2|U_{11},Z_1,U_{12},Z_2,U_{13},Z_3)\le 2\frac{(n-3)^2}{n^2}E((C_{n-3}(U_{12})-C(U_{12}))^2|U_{12},Z_2)\\
&+\frac{2}{n^2}[(1_{\{U_{11}\le U_{12}\le Z_1\}}-C(U_{12})+(1-C(U_{12}))+(1_{\{U_{13}\le U_{12}\le Z_3\}}-C(U_{12})]^2\\
&\le\frac{2}{n}C(U_{12})+\frac{18}{n^2}, \qquad\qquad (A.23)
\end{aligned}$$

then

$$
\begin{aligned}
EM_{n3}(t) \;\; \leq \;\; & \frac{K^2}{n^2}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^3(y)}F_1^*(dx)+2\frac{K^2}{n}E\bigg(\frac{\gamma(U_{11})}{C(U_{11})}1_{\{a_n<U_{11}\leq t\}}\frac{1_{\{x<U_{12},U_{13}\leq b_n\}}}{C^{3/2}(U_{12})C^{3/2}(U_{13})}\bigg) \\
& +12\frac{K^2}{n\sqrt{n}}E\bigg(\frac{\gamma(U_{11})}{C(U_{11})}1_{\{a_n<U_{11}\leq t\}}\frac{1_{\{x<U_{12},U_{13}\leq b_n\}}}{C^{3/2}(U_{12})C^{2}(U_{13})}\bigg) \\
& +18\frac{K^2}{n^2}E\bigg(\frac{\gamma(U_{11})}{C(U_{11})}1_{\{a_n<U_{11}\leq t\}}\frac{1_{\{x<U_{12},U_{13}\leq b_n\}}}{C^{2}(U_{12})C^{2}(U_{13})}\bigg) \\
= \;\; & \frac{K^2}{n^2}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^3(y)}F_1^*(dx)+2\frac{K^2}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}\right)^2F_1^*(dx) \\
& +12\frac{K^2}{n\sqrt{n}}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}F_1^*(dx) \\
& +18\frac{K^2}{n^2}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}\right)^2F_1^*(dx)
\end{aligned}
$$

As in the Lemmas before, it can be shown, that

$$
EM_{n3}(t) \;\; \leq \;\; \frac{K^2M^2\alpha}{n}\left(\frac{6}{c_1}+\frac{4}{c}\right)+2\frac{K^2(M^2+M)\alpha}{n}ln(\frac{n}{c_1\alpha})+24\frac{K^2M\sqrt{2(M^2+M)}\alpha}{n}\frac{1}{\sqrt{c_1}}+36\frac{K^2\alpha M^2}{n}\frac{1}{c_1}
$$

As to M_{1n}, we have

$$
\begin{aligned}
M_{n1}(t) \;\; \leq \;\; & \frac{K^{5/2}}{\sqrt{n}}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{F_{1n}^*(dy)}{C^{3/2}(y)}\int_{(x,b_n)}\frac{|C_n(y)-C(y)|}{C^2(y)}F_{1n}^*(dy)F_{1n}^*(dx) \\
= \;\; & \frac{K^{5/2}}{\sqrt{n}}\frac{1}{n^3}\sum_{i=1}^{n}\sum_{j\neq i}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1i}\leq t\}}\frac{|C_n(U_{1j})-C(U_{1j})|}{C^{7/2}(U_{1j})}1_{\{x<U_{1j}\leq b_n\}} \\
& +\frac{K^{5/2}}{\sqrt{n}}\frac{1}{n^3}\sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\neq i,j}\frac{\gamma(U_{1i})}{C(U_{1i})}1_{\{a_n<U_{1i}\leq t\}}\frac{|C_n(U_{1j})-C(U_{1j})|}{C^2(U_{1j})}\frac{1}{C^{3/2}(U_{1k})}1_{\{x<U_{1j},U_{1k}\leq b_n\}}.
\end{aligned}
$$

Then, by (A.23)

$$
\begin{aligned}
EM_{n1}(t) \;\; \leq \;\; & \sqrt{2}\frac{K^{5/2}}{n^2}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^3(y)}F_1^*(dx)+\sqrt{18}\frac{K^{5/2}}{n^2\sqrt{n}}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^{7/2}(y)}F_1^*(dx) \\
& +\sqrt{2}K^{5/2}\frac{1}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left(\int_{(x,b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}\right)^2F_1^*(dx) \\
& +\sqrt{18}K^{5/2}\frac{1}{n\sqrt{n}}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}F_1^*(dx)\leq k_7\frac{K^{5/2}}{n}\frac{M^3}{(c_1c)^{3/2}}ln^2(\frac{n}{c_1\alpha}),
\end{aligned}
$$

where k_7 is a constant.
For $R_{n7}^b(t)$ we have

$$
|R_{n7}^b(t)| \qquad \leq K\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}|C(x)-C_n(x)|\int_{(x,b_n)}\frac{|C_n(y)-C(y)|}{C^2(y)}F_{1n}^*(dy)F_{1n}^*(dx)=:\tilde{R}_{n7}^b(t)
$$

Furthermore,

$$
\begin{aligned}
\tilde{R}^b_{n7}(t) &\leq K\frac{1}{n}\sum_{i=1}^n 1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma(U_{1i})}{C^2(U_{1i})}\left(\Big|\frac{1}{n}\sum_{j\neq i}\left(C(U_{1i})-1_{\{U_{1j}\leq U_{1i}\leq Z_j\}}\right)\Big|+\frac{1}{n}(1-C(U_{1i}))\right)\\
&\quad\times\frac{1}{n}\sum_{k\neq i}1_{\{U_{1i}<U_{1k}<b_n\}}\frac{\Big|\frac{1}{n}\sum_{l\neq i}\left(1_{\{U_{1l}\leq U_{1k}\leq Z_l\}}-C(U_{1k})\right)\Big|+\frac{1}{n}(1-C(U_{1k}))}{C^2(U_{1k})}\\
&\leq K\frac{1}{n}\sum_{i=1}^n 1_{\{a_n<U_{1i}\leq t\}}\frac{\gamma(U_{1i})}{C^2(U_{1i})}\frac{1}{n}\sum_{k\neq i}1_{\{U_{1i}<U_{1k}<b_n\}}\frac{1}{C^2(U_{1k})}\left(\Big|\frac{1}{n}\sum_{l\neq i,k}\left(1_{\{U_{1l}\leq U_{1k}\leq Z_l\}}-C(U_{1k})\right)\Big|+\frac{2}{n}\right)\\
&\quad\times\left(\Big|\frac{1}{n}\sum_{j\neq i,k}\left(C(U_{1i})-1_{\{U_{1j}\leq U_{1i}\leq Z_j\}}\right)\Big|+\frac{2}{n}\right).
\end{aligned}
$$

Since, for fixed i and k, by Cauchy-Schwarz

$$
\begin{aligned}
&E\left[\left(\Big|\frac{1}{n}\sum_{l\neq i,k}\left(1_{\{U_{1l}\leq U_{1k}\leq Z_l\}}-C(U_{1k})\right)\Big|+\frac{2}{n}\right)\left(\Big|\frac{1}{n}\sum_{j\neq i,k}\left(C(U_{1i})-1_{\{U_{1j}\leq U_{1i}\leq Z_j\}}\right)\Big|+\frac{2}{n}\right)\Big|U_{1i},U_{1k}\right]\\
&\leq\left(\frac{\sqrt{n-1}}{n}\sqrt{C(U_{1i})}+\frac{2}{n}\right)\left(\frac{\sqrt{n-1}}{n}\sqrt{C(U_{1k})}+\frac{2}{n}\right)=\frac{n-1}{n^2}\sqrt{C(U_{1i})}\sqrt{C(U_{1k})}\\
&+\frac{2}{n}\frac{\sqrt{n-1}}{n}\sqrt{C(U_{1i})}+\frac{2}{n}\frac{\sqrt{n-1}}{n}\sqrt{C(U_{1k})}+\frac{4}{n^2},
\end{aligned}
$$

we get

$$
\begin{aligned}
E\tilde{R}^b_{n7}(t) &\leq \frac{K}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C^{3/2}(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}F_1^*(dx)+\frac{2K}{n\sqrt{n}}\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^{3/2}(y)}F_1^*(dx)\\
&\quad+\frac{2K}{n\sqrt{n}}\int_{(a_n,t]}\frac{\gamma(x)}{C^{3/2}(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}F_1^*(dx)+\frac{4K}{n^2}\int_{(a_n,t]}\frac{\gamma(x)}{C^2(x)}\int_{(x,b_n)}\frac{F_1^*(dy)}{C^2(y)}F_1^*(dx).
\end{aligned}
$$

Therefore, since $K,M\geq 1$ and $c,c_1\leq 1$, by Remark A.1 and since by A1 $\frac{1}{F_1}\leq\frac{M(1-G-)}{F_1}+1$, we obtain

$$
E\tilde{R}^b_{n7}(t) \leq \frac{K}{n}\frac{M^2}{c_1}ln(\frac{n}{c_1\alpha})k_8,
$$

where k_8 is a constant.
Next, we deal with $\tilde{R}^d_{n7}(t)$. By Fubini we get

$$
R^d_{n7}(t)=\int\frac{1}{C^2(y)}(C_n(y)-C(y))\int 1_{\{a_n<x\leq t\}}1_{\{x<y<b_n\}}\frac{\gamma(x)}{C(x)}(F^*_{1n}(dx)-F^*_1(dx))F^*_1(dy).
$$

Since

$$E\left|(C_n(y)-C(y))\int 1_{\{a_n<x\leq t\}}1_{\{x<y<b_n\}}\frac{\gamma(x)}{C(x)}(F^*_{1n}(dx)-F^*_1(dx))\right|$$
$$\leq \frac{1}{n}\sqrt{C(y)}\sqrt{\int 1_{\{a_n<x\leq t\}}1_{\{x<y<b_n\}}\frac{\gamma^2(x)}{C^2(x)}F^*_1(dx)}$$

Furthermore, since $\gamma = F_1$ and

$$\sqrt{\int 1_{\{a_n<x\leq t\}}1_{\{x<y<b_n\}}\frac{\gamma^2(x)}{C^2(x)}F^*_1(dx)} \leq \sqrt{F_1(y)\alpha}1_{\{a_n<y<b_n\}}\int_{(a_n,t]}\frac{F_1(dx)}{F_1(x)(1-G(x^-))}$$
$$\leq \sqrt{F_1(y)\alpha}1_{\{a_n<y<b_n\}}\sqrt{Mln(\frac{n}{c_1\alpha})+M},$$

we obtain

$$\begin{aligned}|R^d_{n7}(t)| &\leq \frac{1}{n}\sqrt{Mln(\frac{n}{c_1\alpha})+M}\int_{(a_n,b_n)}\frac{\sqrt{F_1(y)\alpha}}{C^{3/2}(y)}F^*_1(dy) = \frac{1}{n}\sqrt{2Mln(\frac{n}{c_1\alpha})}\int_{(a_n,b_n)}\frac{F_1(dy)}{F_1(y)\sqrt{1-G(y^-)}} \\ &\leq \frac{1}{n}\sqrt{2Mln(\frac{n}{c_1\alpha})}\left(Mln(\frac{n}{c_1\alpha})+\sqrt{M}\right) \leq \frac{4}{n}M^2ln(\frac{n}{c_1\alpha}).\end{aligned}$$

As to $R^c_{n7}(t)$

$$\begin{aligned}|R^c_{n7}(t)| &\leq \frac{n-1}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left|\int_{(x,b_n)}\frac{C_n(y)-C(y)}{C^2(y)}(F^*_{1n-1}(dy)-F^*_1(dy))\right|F^*_{1n}(dx) \\ &\quad +\frac{1}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left|\int_{(x,b_n)}\frac{C_n(y)-C(y)}{C^2(y)}F^*_1(dy)\right|F^*_{1n}(dx) \\ &\leq \frac{n-1}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left|\int_{(x,b_n)}\frac{C_{n-1}(y)-C(y)}{C^2(y)}(F^*_{1n-1}(dy)-F^*_1(dy))\right|F^*_{1n}(dx) \\ &\quad +\frac{1}{n}\frac{n-1}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left|\int_{(x,b_n)}\frac{1-C(y)}{C^2(y)}(F^*_{1n-1}(dy)-F^*_1(dy))\right|F^*_{1n}(dx) \\ &\quad +\frac{1}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left|\int_{(x,b_n)}\frac{C_n(y)-C(y)}{C^2(y)}F^*_1(dy)\right|F^*_{1n}(dx).\end{aligned} \tag{A.24}$$

The expectation of the second coefficient is bounded from above by

$$\frac{2}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{1}{C^2(y)}F^*_1(dy)F^*_1(dx) = \frac{2}{n}\int_{(a_n,t]}\int_{(x,b_n)}\frac{F_1(dy)}{F^2_1(y)(1-G(y^-))}F_1(dx) \leq \frac{4}{n}Mln(\frac{n}{c_1\alpha}).$$

The expectation of the third term in (A.24) is bounded from above by

$$\frac{1}{n}\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\int_{(x,b_n)}\frac{2}{C^2(y)}F^*_1(dy)F^*_1(dx) \leq \frac{4}{n}Mln(\frac{n}{c_1\alpha}).$$

As to the first term in (A.24), we need to deal first with

$$E\left|\int_{(x,b_n)}\frac{C_n(y)-C(y)}{C^2(y)}(F^*_{1n}(dy)-F^*_1(dy))\right|.$$

We have

$$\begin{aligned}
&E\left|\int_{(x,b_n)}\frac{C_n(y)-C(y)}{C^2(y)}(F^*_{1n}(dy)-F^*_1(dy))\right|\\
&\leq\frac{1}{n^2}E\left|\sum_{i=1}^n\sum_{k\neq i}\left(1_{\{x<U_{1i}<b_n\}}\frac{1_{\{U_{1k}\leq U_{1i}\leq Z_k\}}-C(U_{1i})}{C^2(U_{1i})}-\int_{(x,b_n)}\frac{1_{\{U_{1k}\leq y\leq Z_k\}}-C(y)}{C^2(y)}F^*_1(dy)\right)\right|\\
&+\frac{1}{n}E\left(1_{\{x<U_{1i}<b_n\}}\frac{1-C(U_{1i})}{C^2(U_{1i})}+\int_{(x,b_n)}\frac{1-C(y)}{C^2(y)}F^*_1(dy)\right).
\end{aligned}$$

The second term is bounded from above by

$$\frac{2}{n}\int_{(x,b_n)}\frac{1}{C^2(y)}F^*_1(dy)\leq\frac{2\alpha}{n}\left(\int_{(x,b_n)}\frac{M}{F_1^2(y)}F_1(dy)+\int_{(x,b_n)}\frac{1}{F_1(y)(1-G(y^-))}F_1(dy)\right)\leq\frac{4\alpha}{n}\frac{1}{F_1(x)}.$$

As to the second term, we compute the second moment

$$\begin{aligned}
&\frac{1}{n^4}E\left(\sum_{i=1}^n\sum_{k\neq i}\left(1_{\{x<U_{1i}<b_n\}}\frac{1_{\{U_{1k}\leq U_{1i}\leq Z_k\}}-C(U_{1i})}{C^2(U_{1i})}-\int_{(x,b_n)}\frac{1_{\{U_{1k}\leq y\leq Z_k\}}-C(y)}{C^2(y)}F^*_1(dy)\right)\right)^2\\
&=\frac{1}{n^4}\sum_{i=1}^n\sum_{j=1}^n\sum_{k\neq i}\sum_{l\neq j}E\left(1_{\{x<U_{1i}<b_n\}}\frac{1_{\{U_{1k}\leq U_{1i}\leq Z_k\}}-C(U_{1i})}{C^2(U_{1i})}-\int_{(x,b_n)}\frac{1_{\{U_{1k}\leq y\leq Z_k\}}-C(y)}{C^2(y)}F^*_1(dy)\right)\\
&\times\left(1_{\{x<U_{1j}<b_n\}}\frac{1_{\{U_{1l}\leq U_{1j}\leq Z_l\}}-C(U_{1j})}{C^2(U_{1j})}-\int_{(x,b_n)}\frac{1_{\{U_{1l}\leq y\leq Z_l\}}-C(y)}{C^2(y)}F^*_1(dy)\right).
\end{aligned}$$

If one, three or four indices are different, the above expectation equals zero. So we need to consider summands for $i=j\neq k=l$ and $i=l\neq k=j$.
For $i=j\neq k=l$

$$\begin{aligned}
&\frac{1}{n^4}n(n-1)E\left(1_{\{x<U_{11}<b_n\}}\frac{1_{\{U_{12}\leq U_{11}\leq Z_2\}}-C(U_{11})}{C^2(U_{11})}-\int_{(x,b_n)}\frac{1_{\{U_{12}\leq y\leq Z_2\}}-C(y)}{C^2(y)}F^*_1(dy)\right)^2\\
&\leq\frac{1}{n^2}\int_{(x,b_n)}\frac{1}{C^3(y)}F^*_1(dy)\frac{1}{n^2}\left(\int_{(x,b_n)}\frac{1}{C^2(y)}F^*_1(dy)\right)^2\leq\frac{1}{n^2}\alpha^2\int_{(x,b_n)}\left(\frac{6M^2}{F_1^3(y)}+\frac{4M}{(1-G(y^-))^2}\right)F_1(dy)\\
&+\frac{1}{n^2}\frac{4M^2\alpha^2}{F_1^2(x)}=\frac{1}{n^2}\frac{10M^2\alpha^2}{F_1^2(x)}+\frac{1}{n^2}4M\int_{(x,b_n)}\frac{1}{(1-G(y^-))^2}F_1(dy)
\end{aligned}$$

If $i=l\neq k=j$ we obtain

$$\frac{1}{n^4}n(n-1)E\left(1_{\{x<U_{11}<b_n\}}\frac{1_{\{U_{12}\le U_{11}\le Z_2\}}-C(U_{11})}{C^2(U_{11})}-\int_{(x,b_n)}\frac{1_{\{U_{12}\le y\le Z_2\}}-C(y)}{C^2(y)}F_1^*(dy)\right)$$
$$\times\left(1_{\{x<U_{12}<b_n\}}\frac{1_{\{U_{11}\le U_{12}\le Z_1\}}-C(U_{12})}{C^2(U_{12})}-\int_{(x,b_n)}\frac{1_{\{U_{11}\le y\le Z_1\}}-C(y)}{C^2(y)}F_1^*(dy)\right)$$
$$\le\frac{4}{n^2}\left(\int_{(x,b_n)}\frac{1}{C^2(y)}F_1^*(dy)\right)^2\le\frac{4}{n^2}\frac{4M^2\alpha^2}{F_1^2(x)}$$

Hence

$$E\left|\int_{(x,b_n)}\frac{C_n(y)-C(y)}{C^2(y)}(F_{1n}^*(dy)-F_1^*(dy))\right|\le\frac{4\alpha(1+4M)}{nF_1(x)}+\frac{1}{n}\frac{\sqrt{10}M\alpha}{F_1(x)}+\frac{2\sqrt{M}}{n}\left(\int_{(x,b_n)}\frac{F_1(dy)}{(1-G(y^-))^2}\right)^{1/2}.$$

Therefore, for the expectation of the first term in (A.24), we obtain

$$E\left(\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left|\int_{(x,b_n)}\frac{C_{n-1}(y)-C(y)}{C^2(y)}(F_{1n-1}^*(dy)-F_1^*(dy))\right|F_{1n}^*(dx)\right)$$
$$\le\int_{(a_n,t]}\frac{\gamma(x)}{C(x)}\left(\frac{4\alpha(1+4M)}{n}\frac{1}{F_1(x)}+\frac{1}{n}\frac{\sqrt{10}M\alpha}{F_1(x)}+\frac{2\sqrt{M}}{n}\left(\int_{(x,b_n)}\frac{F_1(dy)}{(1-G(y^-))^2}\right)^{1/2}\right)F_1^*(dx)$$
$$\le\frac{1}{n}24M\alpha ln(\frac{n}{c_1\alpha})+\frac{2\sqrt{M}}{n}\int_{(a_n,t]}\left(\int_{(x,b_n)}\frac{F_1(dy)}{(1-G(y^-))^2}\right)^{1/2}F_1(dx)$$

Hence

$$\tilde{R}_{n7}(t)=\tilde{R}_{n7}^a(t)+\tilde{R}_{n7}^b(t)+|R_{n7}^c(t)|+|R_{n7}^d(t)|$$

and since $M,K\ge 1$, $c,c_1\le 1$ and $\frac{n}{c_1\alpha}\ge 1$

$$E\tilde{R}_{n7}(t)\le k_9\frac{1}{n}\frac{K^3M^3}{c_1^2c^2}ln^2(\frac{n}{c_1\alpha})+\frac{2\sqrt{M}}{n}\int_{(a_n,t]}\left(\int_{(x,b_n)}\frac{F_1(dy)}{(1-G(y^-))^2}\right)^{1/2}F_1(dx)$$

where k_9 is a constant.

Furthermore, according to assumption A1, $\int\frac{dF_1}{1-G-}<\infty$. Hence for $\varepsilon>0$, there exists x_0, so that for every $x\ge x_0$ we have $\int_{(x,\infty)}\frac{dF_1}{1-G-}\le\varepsilon^2$. Therefore, for $x\ge x_0$, according to definition of b_n,

$$\left(\int_{(x,b_n)}\frac{F_1(dy)}{(1-G(y^-))^2}\right)^{1/2}\le\frac{1}{\sqrt{1-G(b_n^-)}}\left(\int_{(x_0,\infty)}\frac{F_1(dy)}{1-G(y^-)}\right)^{1/2}\le\frac{\sqrt{n}}{\sqrt{c\alpha}}\varepsilon.$$

For $x<x_0$, we have

$$\left(\int_{(x,b_n)}\frac{F_1(dy)}{(1-G(y^-))^2}\right)^{1/2}\le\left(\int_{(x,x_0)}\frac{F_1(dy)}{(1-G(y^-))^2}\right)^{1/2}+\left(\int_{(x_0,b_n)}\frac{F_1(dy)}{(1-G(y^-))^2}\right)^{1/2}$$
$$\le d+\frac{1}{\sqrt{1-G(b_n^-)}}\left(\int_{(x_0,\infty)}\frac{F_1(dy)}{1-G(y^-)}\right)^{1/2}\le d+\frac{\sqrt{n}}{\sqrt{c\alpha}}\varepsilon,$$

where d is a constant.
Finally

$$E\tilde{R}_{n7}(t) \leq k_9 \frac{1}{n} \frac{K^3 M^3}{c_1^2 c^2} ln^2(\frac{n}{c_1 \alpha}) + \frac{2\sqrt{M}}{n} d + \frac{2\sqrt{M}}{\sqrt{c\alpha}} \frac{\varepsilon}{\sqrt{n}}.$$

⊠

Summarizing:

Lemma A.22.

$$F_{1n}(t) - F_1(t) = \sum_{i=1}^{4} L_{ni} + R_n(t)$$

where on the set $\tilde{\Omega}_n^0$, we have

$$\begin{aligned} |R_n(t)| \quad & \leq \tilde{R}_{n1}(t) + \tilde{R}_{n2}(t) + \tilde{R}_{n6}(t) + |R_{n1}(t)| + |R_{n4}(t)| + \tilde{R}_{n5}(t) + \tilde{I}I_n(t) + \tilde{R}_{n7}^a(t) + \tilde{R}_{n7}^b(t) \\ & + |R_{n7}^c(t)| + |R_{n7}^d(t)| =: \tilde{R}_n(t). \end{aligned}$$

Furthermore,

$$\begin{aligned} E\tilde{R}_n(t) \quad \leq \quad & k_{10} \frac{1}{n} \frac{K^3 M^3}{c_1^2 c^2} ln^2(\frac{n}{c_1 \alpha}) + \frac{2\sqrt{\alpha M}}{n} \frac{1}{\sqrt{1 - G(t^-)}} \left(ln(\frac{n}{c_1 \alpha}) + 1 \right)^{1/2} + \frac{\sqrt{8M\alpha}}{n^{3/2}} \frac{1}{1 - G(t^-)} \\ & + \frac{2M}{\sqrt{c\alpha}} \frac{\varepsilon}{\sqrt{n}}, \end{aligned}$$

where k_{10} is a constant.
Proof.
Lemmas (A.14)-(A.21).

⊠

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Erklärung:

Hiermit erkläre ich, dass ich die Arbeit selbständig verfasst und nur die angegebenen Hilfsmittel verwendet habe.

Gießen, den 08.01.2008

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