Extremal Combinatorics in Finite Geometries

The Independence Number of Kneser Graphs on Flags of Projective Spaces, Implications for the Chromatic Number and a Theorem on Small Tight Sets of Polar Spaces

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Preface

In 1961 the authors Paul Erdős, Richard Rado and Chao Ko published a paper [14] titled *"Intersection theorems for systems of finite sets"*, which initiated years of mathematical research in the field of combinatorics, including this thesis. In said paper the authors considered a collection C of mutually intersecting k-subsets of a given n-set and determined, how large C can be, as well as the structure of C in the extremal case.

The examples of maximal size have a fairly simple structure: for $n < 2k$ the problem is trivial, in the special case $2k = n$ every k-subset of the n-set has a complementary k-set and one has to choose one k-set of each complementary pair, and for $2k < n$ one has $|C| \leq \binom{n-1}{k-1}$ where equality holds if and only if C is the collection of all k-subsets containing one fixed element of the n-set.

In honour of their initial research, a collection of mutually intersecting k-subsets of a given n-set is called an *Erdős-Ko-Rado set* and their initial problem as well as generalizations of it are often referred to as *Erdős-Ko-Rado problems.* An important (first) addition to their result was given in 1967 in [18] by Hilton and Milner, where an upper bound on examples of second largest cardinality was determined and - in honour of their research - results on the size of second largest examples are often referred to as *Hilton-Milner theorems.*

Now, another way to view this problem is, to view it in the graph $K(n, k)$ whose vertices are all k-subsets of a fixed n-set and in which two vertices are adjacent if and only if they are disjoint. This graph is called the Kneser graph $K(n, k)$ and any Erdős-Ko-Rado set occurs as an independent set of this graph, that is, a set of pairwise non-adjacent vertices. Thus, in the language of graph theory, Erdős, Ko and Rado originally determined the independence number of the Kneser graph $K(n, k)$.

In view of the contents of this Thesis an important generalization is the Erdős-Ko-Rado Theorem for vector spaces given in [15] by Frankl and Wilson and published in 1986. There, translated to the language of graph theory, the authors determined the independence number of the q-Kneser graph, that is, the graph whose vertices are all k-dimensional subspaces of a given n-dimensional vector space of the finite field $\mathrm{GF}(q)$, in which two vertices are adjacent if and only if the intersection of the corresponding subspaces is trivial. Note that in [7] published by Blokhuis, Brouwer, Chowdhury, Frankl, Mussche, Patkós and Szönyi in 2010, the authors determined a Hilton-Milner Theorem for the Erdős-Ko-Rado Problem in vector spaces.

In this thesis the main focus is on the study of Erdős-Ko-Rado sets in generalized q-Kneser graphs, where the vertices are flags of subspaces of a given vector space and two vertices are adjacent if and only if they are *far apart.* Here, far apart for two flags means, that the intersection of any choice of two subspaces - one from each of the two flags - has minimal dimension. In order to gain a better geometrical point of view, we study these structures in the projective space associated with the given vector space and the main results are:

- Theorem 2.2.16, where the independence number of the Kneser graph on plane solid flags in $\mathrm{PG}(6,q)$ is determined. Furthermore, in Corollary 2.2.15 an upper bound u is provided such that every independent set of this graph of size larger than u is a subset of a maximal independent set given by Examples 2.1.15 and 2.1.17.

- Theorem 2.3.20, where knowledge of the independence number of the Kneser graph on plane solid flags in $\mathrm{PG}(6,q)$ is used to derive its chromatic number. There, we also provide structural information on any colouring of minimal size.

- Theorem 2.4.51, where the independence number of the Kneser graph on line solid flags in $\mathrm{PG}(5,q)$ is determined. Furthermore, in Corollary 2.4.50 an upper bound u is provided such that every independent set of this graph of size larger than u is a subset of a maximal independent set given by Example 2.4.1.

Finally, the last chapter is on the subject of small tight sets in the hermitian polar space $H(2d,q^2)$ of even dimension. In Theorem 3.2.10 tight sets of said polar space with parameter $x \leq q$ are studied and determined to be the disjoint union of a set of $y \leq x$ disjoint generators together with an $(x-y)$-tight set which does not contain a line of $H(2d,q^2)$. In fact, if $x - y \leq \frac{q+1}{2}$, then $x - y = 0$, that is, tight sets with parameter $x \leq \frac{q+1}{2}$ are the disjoint union of x generators.

Publications and Joint Work

The content of this dissertation is in large parts based on publications of the author during his time at the mathematical institute of the Justus-Liebig-University in Gießen.

The content of Chapter 2 is split into four sections. Of these, the second section on the independence number of the Kneser graph of plane solid flags in $\mathrm{PG}(6, q)$ is based on [25] by Metsch and Werner and the third section on the chromatic number of said graph is based on a generalization by D'haeseleer, Metsch and Werner of their earlier work [12]. Note that the contents of [12] by D'haeseleer, Metsch and Werner on the chromatic number of some Kneser graphs, including the Kneser graph on line-plane flags of $\mathrm{PG}(4, q)$, are not included here and instead will be included in the thesis of Jozefien D'haeseleer.

Finally, the content of Chapter 3 on small tight sets the polar space $H(2d, q^2)$ is based on the generalization [26] by Metsch and Werner of the publication [1] by De Beule and Metsch.

Acknowledgements

Throughout my time working on this dissertation I have received a lot of support and had the opportunity to collaborate with some very interesting colleagues.

First and foremost I would like to thank my supervisor, Professor Klaus Metsch, for his confidence and trust in me. Knowing that I could always turn to him for answers, if need be, was a great help for me. I greatly appreciate his exactness and detail oriented work ethic.

Furthermore, I would like to thank my colleagues for the insightful collaborations. In particular, I would like to thank Jozefien D'haeseleer for our successful cooperation on the two papers on the Chromatic Number, one of which is the basis for Section 2.3 of this dissertation.

Lastly, I would like to thank everyone who is not mentioned here by name but without whom I could not have completed this dissertation, including, but not limited to, everyone else in the mathematical institute, my friends and family and, last but not least, my dog and our lengthy mathematically themed walks on campus.

Contents

1 Introduction

In this chapter we introduce all the necessary definitions, notations and basic results that are used in this thesis. We do not provide a completely self contained description of everything that is used, but give a decent overview. Some of the basic results, especially if the method of proof is very similar to the rest of this work, are included with proof. For the remaining required notions, which we do not prove, we refer the reader to the cited literature for more information.

Definition 1.0.1 (Sets)**.** We shall denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of all natural numbers and we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Furthermore, for any set S we let 2^S denote the set of all subsets of S.

Definition 1.0.2 (Kronecker-Delta)**.** For two arbitrary objects x and y we define the Kronecker-Delta $\delta_{x,y}$ to be 1 if $x = y$ and 0 otherwise.

1.1 Basic Algebraic Objects

In this section we introduce basic algebraic objects that we use, such as groups, division rings, fields, vector spaces etc. Furthermore, we collect some classic results, such as Wedderburn's Little Theorem, which are applied in one way or another in this work.

Definition 1.1.1 (Group)**.** A *group* $(G, \circ)$ is a set G with a binary operation $\circ : G \times G \to G$ which satisfies the following axioms:

(G1) For all $g_1, g_2, g_3 \in G$ we have $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

(G2) There is an *identity element* $e \in G$ such that for all $g \in G$ we have $e \circ g = g = g \circ e$.

(G3) For all $g \in G$ there is an *inverse element* $g' \in G$ with $g \circ g' = e = g' \circ g$.

A group $(G, \circ)$ is called *abelian* (or *commutative*) if for all $g, g' \in G$ we have $g \circ g' = g' \circ g$. Furthermore, for any group $(G, \circ)$ the cardinality of G is called the *order* of $(G, \circ)$ and if said order is finite, then we call $(G, \circ)$ *finite*.

If the operation is written multiplicatively, then the inverse element of $g \in G$ is denoted by g^{-1} and if the operation is written additively, then it is denoted by $-g$. Furthermore, if the operation is written multiplicatively, then we omit the operator $\cdot$, as is usual.

Definition 1.1.2 (Division Ring)**.** A *division ring* $(F, +, \cdot)$ is a set F with two binary operations $+ : F \times F \to F$ and $\cdot : F \times F \to F$ such that:

(F1) $(F, +)$ is an abelian group with identity element 0_F.

(F2) $(F \setminus \{0_F\}, \cdot)$ is a group with identity element 1_F.

(F3) For all $g_1, g_2, g_3 \in F$ we have $g_1 \cdot (g_2 + g_3) = g_1 \cdot g_2 + g_1 \cdot g_3$.

(F4) For all $g_1, g_2, g_3 \in F$ we have $(g_1 + g_2) \cdot g_3 = g_1 \cdot g_3 + g_2 \cdot g_3$.

A division ring $(F, +, \cdot)$ is called *field*, if the group $(F \setminus \{0_F\}, \cdot)$ is abelian and it is called *skewfield*, if the group $(F \setminus \{0_V\}, \cdot)$ is not abelian. Furthermore, for any division ring $(F, +, \cdot)$ the cardinality of the set F is called the *order* of $(F, +, \cdot)$ and if said order is finite, then we call $(F, +, \cdot)$ *finite*.

The following two theorems classify finite division rings and can, for example, be found in [22].

Theorem 1.1.3 (Wedderburn's Little Theorem). *Every finite division ring is a field.*

Theorem 1.1.4 (Finite Fields). *The order of every finite field is a* prime power q, *that is, there is a prime p and an integer $n \in \mathbb{N}$ such that $q = p^n$. Furthermore, for every prime power q all fields of order q are pairwise isomorphic.*

Therefore, for every prime power q, up to isomorphism, there is a unique finite field of order q and we denote that field by $\mathbb{F}_q$.

We conclude this section with the definition of a vector space over a division ring, which is done the very same way as the usual definition over a field.

Definition 1.1.5 ((Left) Vector Space). A *vector space* $(V, +, \cdot)$ over a division ring F is a set V with two binary operations $+ : V \times V \to V$ and $\cdot : F \times V \to V$ such that:

(V1) $(V, +)$ is an abelian group with identity element 0_V.

(V2) The multiplication $\cdot$ is called *scalar multiplication* and for all $\lambda_1, \lambda_2 \in F$ and $v_1, v_2 \in V$ it satisfies:

- $\lambda_1(\lambda_2 v_1) = (\lambda_1 \lambda_2) v_1$.
- $1_F v_1 = v_1$.
- $\lambda_1(v_1 + v_2) = \lambda_1 v_1 + \lambda_2 v_2$.
- $(\lambda_1 + \lambda_2) v_1 = \lambda_1 v_1 + \lambda_2 v_2$.

A vector space is called *finite* if the set V is finite.

Remark 1.1.6. *Basic results on vector spaces over fields can be proven such that they also hold for vector spaces over division rings. We do not include notions on vector spaces here. Instead we refer the reader to [22] for further information and assume that basic notions, such as for example* basis *and* dimension, *and basic results, like the* dimension-formula for vector spaces, *are known.*

1.2 Projective Spaces

In this thesis we work with finite projective spaces and those of interest to us stem from vector spaces over finite fields. However, we still introduce an axiomatic definition of a general projective space as point-line incidence structure first. For a more thorough introduction to projective spaces we refer the reader to [2], [11] and [9]. All omitted proofs of introductory theorems stated in this section can be found in either one of these.

1.2.1 Axiomatic Definition

Definition 1.2.1 (Point-Line Incidence Structure)**.** A *point-line incidence structure* is a triple $\mathcal{S} = (\mathcal{P}, \mathcal{L}, *)$ such that $\mathcal{P}$ and $\mathcal{L}$ are two sets and $* \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ is a symmetric relation between these two sets.

The relation $*$ is called *incidence relation* and for all $P \in \mathcal{P}$ and all $l \in \mathcal{L}$ with $(P, l) \in *$ we also write $P * l$ and say that P and l *incident*. The set $\mathcal{P}$ is called the *point-set* and its elements are called *points*, the set $\mathcal{L}$ is called the *line-set* and its elements are called *lines*.

Given $(P, l) \in *$ we use common geometric terminology such as: If $P * l$, then we call P a *point of* l, l *a line through* P and say that l *contains* P. Furthermore, any point which lies on two distinct lines is also called the *intersection* of these two lines and if $\mathcal{Q}$ is a set of points, then the points in $\mathcal{Q}$ are called *collinear*, if there is a line l such that $Q * l$ for all $Q \in \mathcal{Q}$.

Finally, the incidence structure $\mathcal{S}$ is called *finite*, if $\mathcal{P}$ and $\mathcal{L}$ are finite sets.

Given the general concept of a point-line incidence structure we now add further axioms to describe a projective space.

Definition 1.2.2 (Projective Space)**.** A *projective space* $\mathbb{P}$ is a point-line incidence structure $(\mathcal{P}, \mathcal{L}, *)$ such that

(P1) For any two distinct points P and Q there is a unique line l with $P * l$ and $Q * l$. In this situation we denote l by PQ.

(P2) If g_1 and g_2 are two distinct lines with a common point P, then for any two lines h_1 and h_2 which have common points with g_1 and g_2 but do not contain P there is a point $Q \in \mathcal{P}$ with $Q \in h_1, h_2$.

(P3) Every line contains at least three points.

A projective space $\mathbb{P}$ is called *non-degenerate* if it also satisfies

(P4) There are three points such that no line is incident with all three.

A projective space which does not satisfy (P4) is called *degenerate* and if a degenerate projective space $\mathbb{P}$ has more than one point, then it is called a *projective line*. We remark that axiom (P2) is also called *Veblen-Young* axiom.

Definition 1.2.3 (Projective Plane)**.** A *projective plane* $\mathbb{P}$ is a non-degenerate projective space that also satisfies the following axiom:

(P2') Any two lines of $\mathbb{P}$ have a point in common.

Note that (P2') implies (P2).

Since every line of a projective space $\mathbb{P}' = (\mathcal{P}, \mathcal{L}, *)$ is determined by any two distinct points of said line, we may identify every line $l \in \mathcal{L}$ with the set of its points, that is, $l = \{P \in \mathcal{P} : P * l\}$. This also explains why from now on for all points $P \in \mathcal{P}$ and all lines $l \in \mathcal{L}$ with $P * l$ we may also write $P \in l$.

Definition 1.2.4 (Subspace)**.** A set $\mathcal{P}' \subseteq \mathcal{P}$ of points of a projective space $\mathbb{P} = (\mathcal{P}, \mathcal{L}, *)$ such that $P, Q \in \mathcal{P}'$ and $R \in PQ$ implies $R \in \mathcal{P}'$ is called a *linear* subset of $\mathbb{P}$ and, given a linear subset $\mathcal{P}'$ of $\mathbb{P}$, we call $\mathbb{P}' := (\mathcal{P}', \mathcal{L}', *')$ with $\mathcal{L}' = \{PQ : P, Q \in \mathcal{P}' \wedge P \neq Q\}$ and $*' := * \cap (\mathcal{P}' \times \mathcal{L}' \cup \mathcal{L}' \times \mathcal{P}')$ a *subspace* of $\mathbb{P}$ and write $\mathbb{P}' \leq \mathbb{P}$.

Furthermore, if $\mathbb{P}_1$ and $\mathbb{P}_2$ are two subspaces of a projective space $\mathbb{P}$ such that there is no point P which lies in both $\mathbb{P}_1$ and $\mathbb{P}_2$, then we call $\mathbb{P}_1$ and $\mathbb{P}_2$ *skew*.

It is fairly simple to see that, if $\mathbb{P}$ is a projective space and $\mathbb{P}'$ is a subspace of $\mathbb{P}$, then $\mathbb{P}'$ is a projective space, too, and thus we omit a formal proof thereof.

To simplify notation later on and in view of the concept of a Buekenhout Geometry (see Remark 1.2.12) we introduce the following.

Notation 1.2.5. For any two projective spaces $\mathbb{P}_1$ and $\mathbb{P}_2$ such that one is a subspace of the other we write both $\mathbb{P}_1 * \mathbb{P}_2$ as well as $\mathbb{P}_2 * \mathbb{P}_1$.

Like we did for lines, we may identify every subspace $\mathbb{P}' = (\mathcal{P}', \mathcal{L}', *')$ of a given projective space $\mathbb{P}$ by the set of its points, that is, $\mathbb{P}' = \mathcal{P}'$. Note that if $\mathbb{P} = (\mathcal{P}, \mathcal{L}, *)$ is a projective space then $\mathbb{P}$ itself is a subspace of $\mathbb{P}$, too, and, since we identify subspaces by their point sets, we may now naturally write $P \in \mathbb{P}$ instead of $P \in \mathcal{P}$ as well as $B \subseteq \mathbb{P}$ instead of $B \subseteq \mathcal{P}$.

Every subset of a projective space is contained in at least one subspace and thus one can define the span as is usual.

Definition 1.2.6 (Span)**.** Let B be a subset of a projective space $\mathbb{P}$ and let $\mathcal{S}$ be the set of all subspaces S of $\mathbb{P}$ with $B \subseteq S$. Then

$$\langle B \rangle := \bigcap_{S \in \mathcal{S}} S$$

is called the *span* of B. Once again we use common language such as saying that B *spans* $\langle B \rangle$ and calling $\langle B \rangle$ the subspace *spanned* by B.

Again it is fairly simple to see that the span of a subset B of a projective space $\mathbb{P}$ is a subspace of $\mathbb{P}$, too, and thus we also omit the proof thereof.

Notation 1.2.7. For $s, t \in \mathbb{N}_0$, points $P_1, \ldots, P_t$ and subsets $B_1, \ldots, B_s$ of a projective space $\mathbb{P}$ we also write $\langle P_1, \ldots, P_t, B_1, \ldots, B_s \rangle$ instead of $\langle \{P_1\} \cup \cdots \cup \{P_t\} \cup B_1 \cup \cdots \cup B_s \rangle$.

Definition 1.2.8 (Linearly Independent and Basis)**.** Let $\mathbb{P}$ be a projective space. A subset B of $\mathbb{P}$ is called *linearly independent*, if and only if for any subset $B' \subset B$ and every point $P \in B \setminus B'$ we have $P \notin \langle B' \rangle$. Furthermore, a linearly independent subset B of $\mathbb{P}$ which spans $\mathbb{P}$ is called a *basis* of $\mathbb{P}$.

A basis B of $\mathbb{P}$ is called *finite* if $|B| < \infty$ and $\mathbb{P}$ is called *finitely spanned* if there is a finite basis B of $\mathbb{P}$.

Convention 1.2.9. *From now on we only consider projective spaces which are finitely spanned.*

As is expected from a basis, one can show that every basis of a projective space has the same cardinality.

Definition 1.2.10 (Rank and (Projective) Dimension)**.** Let $\mathbb{P}$ be a projective space. We call that number of elements in a basis B of $\mathbb{P}$ the *rank* of $\mathbb{P}$, denote it by $\mathrm{rk}(\mathbb{P})$ and call $\dim(\mathbb{P}) := \mathrm{rk}(\mathbb{P}) - 1$ the *(projective) dimension* of $\mathbb{P}$.

Note that we now have two different concepts of dimension, one in vector spaces and one in projective spaces. If it is clear from context which concept we use, then we simply write dimension (or dim) and if it is not then we specify by writing *vectorial* or *projective* dimension.

Remark 1.2.11. *Let $\mathbb{P}$ be a projective space. If S is a subspace of a projective space $\mathbb{P}$, then we have seen that S is a projective space, too. Hence,* $\mathrm{rk}(S)$ *and* $\dim(S)$ *is defined for every subspace S of $\mathbb{P}$. Note that this also includes the cases in which the projective space $\mathbb{P}$ is degenerate.*

In fact, a projective line has rank 2 and dimension 1, a projective space with exactly one point has rank 1 and dimension 0 and, finally, a projective space with no point, has rank 0 and dimension -1.

Remark 1.2.12 (Buekenhout Geometry)**.** *Let $\mathbb{P}$ be a projective space of dimension $2 \leq n \in \mathbb{N}$ and let X be the set of all proper subspaces of $\mathbb{P}$ without the empty set. Using Notation 1.2.5 $*$ is a symmetric and reflexive relation on X and $(X, *, \mathrm{rk})$ is a* Buekenhout geometry *over the set* $\{1, \ldots, n\}$.

Definition 1.2.13 (Hyperplane)**.** In a projective space $\mathbb{P}$ of dimension n we call any subspace $H \leq \mathbb{P}$ with $\dim(H) = n - 1$ a *hyperplane* of $\mathbb{P}$.

A very important tool when working with subspaces is the dimension-formula, that we use without proof.

Theorem 1.2.14 (Dimension Formula)**.** *For subspaces U and W of a projective space $\mathbb{P}$ we have*

$$\dim(U) + \dim(W) = \dim(\langle U, W \rangle) + \dim(U \cap W).$$

Another concept that plays a crucial role throughout this work is the dual space.

Definition 1.2.15 (Dual Space)**.** Let $\mathbb{P}$ be a non-degenerate projective space, let n be its dimension, let $\mathcal{H}$ be the set of all hyperplanes of $\mathbb{P}$ and let $\mathcal{U}$ be the set of all subspaces $U \leq \mathbb{P}$ with $\dim(U) = n - 2$. Then the point-line incidence structure $\mathbb{P}^\vee := (\mathcal{H}, \mathcal{U}, *^\vee)$, where $H \in \mathcal{H}$ and $U \in \mathcal{U}$ are adjacent if and only if $U \leq H$ in $\mathbb{P}$, is called the *dual space* of $\mathbb{P}$.

Remark 1.2.16. *Let everything be as in this definition of a dual space. One can show that, if U is a subspace of $\mathbb{P}$ and d_U is its dimension, then the set $\{H \in \mathcal{H} : U * H\}$ is a subspace of $\mathbb{P}^\vee$ and in $\mathbb{P}^\vee$ it has dimension $\dim(\mathbb{P}) - d_U - 1$. We identify $U \leq \mathbb{P}$ and the subspace $\{H \in \mathcal{H} : U * H\}$ of $\mathbb{P}^\vee$ such that each subspace $U \leq \mathbb{P}$ is also a subspace of $\mathbb{P}^\vee$. Now, if U_1 and U_2 are subspaces of $\mathbb{P}$ with $U_1 \leq U_2$, then in $\mathbb{P}^\vee$ we have $U_2 \leq U_1$. Hence, if we use the point of view of the incidence relation $*$ that we introduced in Notation 1.2.5, then we have $* = *^\vee$.*

Moreover, if we consider an n-dimensional projective space $\mathbb{P}$ as a Buekenhout Geometry $(X, *, \dim)$ over a type-set $\{0, 1 \ldots, n\}$, then the dual space has the same set X, the same incidence relation $*$ and the same type-set I and only the type-map differs: the map $\dim^\vee$ of the dual space satisfies $\dim^\vee(x) = n - \dim(x) - 1$ for all $x \in X$.

Based on the concept of a dual space one also encounters the principle of duality for projective spaces, which is used frequently in this work.

Remark 1.2.17 (Principle of Duality)**.** *The principle of duality states that, if a certain statement holds for all projective spaces, then the dual of said statement also holds for all projective spaces. Note that the dual of a given statement in a projective space $\mathbb{P}$ is the statement interpreted in the dual space $\mathbb{P}^\vee$. Furthermore, a statement is called self-dual, if its dual statement is the same. An example of a self-dual statement is the Configuration of Desargues 1.2.19 that we encounter below in the projective plane.*

Before we proceed to introduce projective spaces over division rings next, we conclude this axiomatic introduction of projective spaces with two important configurations, namely those of Desargues and Pappus, and two important theorems concerning these.

Definition 1.2.18 (Theorem/Configuration of Desargues)**.** Let $\mathbb{P}$ be a non-degenerate projective space. We say that the *Theorem of Desargues* holds in $\mathbb{P}$, if for all points P_1, P_2, P_3, Q_1, Q_2 and Q_3 of $\mathbb{P}$ such that

- there is a point S with $S \in P_iQ_i$ for all $i \in \{1, 2, 3\}$ and
- every subset B of three points of either $\{S, P_1, P_2, P_3\}$ or $\{S, Q_1, Q_2, Q_3\}$ is linearly independent,

the points $R_i := P_jP_k \cap Q_jQ_k$ for all $\{i, j, k\} = \{1, 2, 3\}$ lie on a common line.

Theorem 1.2.19 (Theorem of Desargues for Projective Spaces, see [2, Theorem 2.7.1])**.** *The Theorem of Desargues holds in every projective space $\mathbb{P}$ with* $\dim(\mathbb{P}) > 2$.

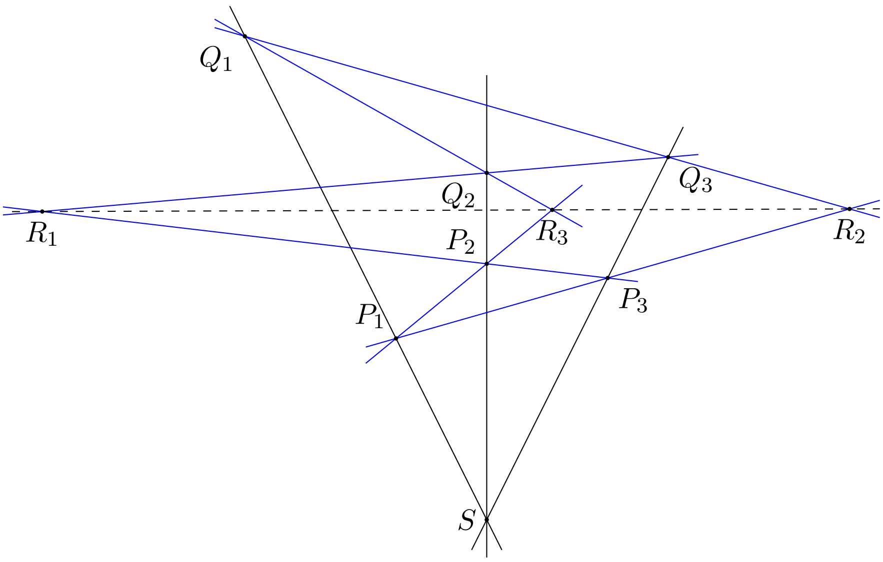

Figure 1.1: Configuration of Desargues

Definition 1.2.20 (Theorem/Configuration of Pappus)**.** Let $\mathbb{P}$ be a non-degenerate projective space. We say that the *Theorem of Pappus* holds in $\mathbb{P}$, if for any two intersection lines h and g, all distinct points $P_1, P_2, P_3 \in h \setminus g$ and all distinct points $Q_1, Q_2, Q_3 \in g \setminus h$ the points $R_i := P_jP_k \cap Q_jQ_k$ for all $\{i, j, k\} = \{1, 2, 3\}$ lie on a common line.

Theorem 1.2.21 (Hessenberg's Theorem, see [17])**.** *If the Theorem of Pappus holds in a projective space $\mathbb{P}$, then the Theorem of Desargues holds in $\mathbb{P}$, too.*

1.2.2 Projective Spaces over Vector Spaces

Definition 1.2.22. For any vector space V over a division ring F we define the point-line geometry $\mathbb{P}(V) := (\mathcal{P}, \mathcal{L}, *)$ as follows:

- $\mathcal{P} := \{U \leq V : U$ has vectorial dimension $1\}$.
- $\mathcal{L} := \{U \leq V : U$ has vectorial dimension $2\}$.
- $* := \{(U, W) \in (\mathcal{P} \times \mathcal{L} \cup \mathcal{L} \times \mathcal{P}) : U \leq W$ or $W \leq U\}$.

Theorem 1.2.23 (Projective Spaces over Vector Spaces, see [2, Theorem 2.1.1])**.** *For any vector space V over some division ring F the point-line geometry $\mathbb{P}(V)$ is a projective space.*

Remark 1.2.24. *Note that in [2] this is only shown for vector spaces of vectorial dimension at least 3. However, the remark after the proof of [2, Theorem 2.1.1] addresses this and explains, that the condition on the vectorial dimension of the vector space is only used to show that the projective space is non-degenerate. Indeed, for vector spaces V of*

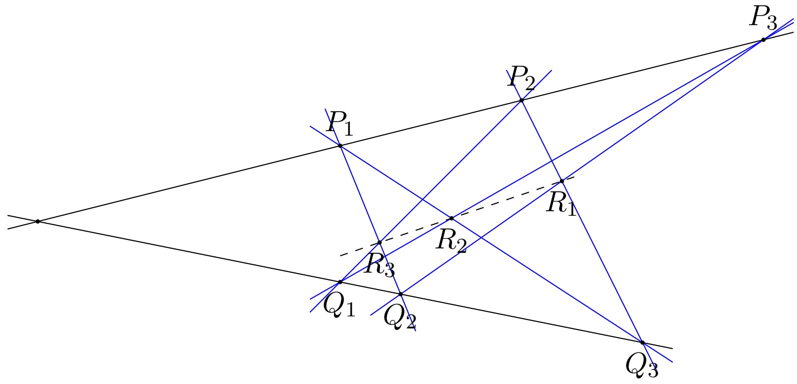

Figure 1.2: Configuration of Pappus

vectorial dimension 2 the point-line geometry $\mathbb{P}(V)$ is a projective line, for vector spaces V of vectorial dimension 1 the point-line geometry $\mathbb{P}(V)$ has only one point and even if V has vectorial dimension 0 the point-line geometry $\mathbb{P}(V)$ is defined. In particular, in all three cases the respective point-line geometry satisfies the axioms of a projective space.

Furthermore, the set of subspaces of a vector space V is in bijective correspondence with the set of subspaces of $\mathbb{P}(V)$. In fact, for every subspace U of a vector space V with vectorial dimension d_U we know that $\mathbb{P}(U)$ is a subspace of $\mathbb{P}(V)$ with projective dimension $d_U - 1$ and rank d_U.

Notation 1.2.25. In order to avoid the ambiguity between the two concepts of dimension (that is, vectorial- and projective dimension), from now on we use rank as well as rk when referring to vectorial dimension and dimension as well as dim when referring to projective dimension.

In view of 1.2.19 the following explains why projective spaces over vector spaces are of particular importance. Together with Wedderburn's Little Theorem this also explain why we only consider projective spaces which are constructed using vector spaces over fields.

Theorem 1.2.26 (Theorem of Desargues for Projective Spaces over Vector Spaces, see [2, Theorem 3.4.2])**.** *A non-degenerate projective space $\mathbb{P}$ satisfies the Theorem of Desargues, if and only if there is a vector space V of rank at least 3 over a skew field F with $\mathbb{P} = \mathbb{P}(V)$.*

Together, Theorem 1.2.19, Theorem 1.2.26 and Wedderburn's Little Theorem 1.1.3 prove the following theorem.

Theorem 1.2.27. *If $\mathbb{P}$ is a finite projective space of dimension at least 3, then $\mathbb{P}$ is isomorphic to $\mathbb{P}(\mathbb{F}_q^{\dim(\mathbb{P})+1})$.*

In view of that theorem the following definition is imperative.

Definition 1.2.28. For all $n \in \mathbb{N}$ and every prime power q set $\mathrm{PG}(n,q) := \mathbb{P}(\mathbb{F}_q^{n+1})$ for some prime power q.

We conclude this subsection with the following result regarding the Theorem of Pappus, which helps to further classify projective spaces which are not finite, and a remark.

Theorem 1.2.29 (Theorem of Pappus for Projective Spaces over Vector Spaces, [2, Theorem 2.2.2])**.** *For any vector space V over a division ring F the Theorem of Pappus holds in the projective space $\mathbb{P}(V)$ if and only if F is a field.*

Remark 1.2.30. *There exist several different projective planes which are not isomorphic to $\mathbb{P}(V)$ for all vector spaces V, for example* Moulton *planes. For a short overview of such planes we refer the reader to [11, Section 2.1].*

We have now introduced our understanding of a projective space and mentioned the important general notions for this work. From now on we only consider finite projective spaces $\mathrm{PG}(n,q)$ for some $n \in \mathbb{N}$ and some prime power q.

Furthermore, in the following we try to provide proofs to most claims, as they seem to fit thematically into this work. Only in very few instances we refer the reader to the literature.

1.2.3 Counts in Projective Spaces

Here we prepare some tools that we use to count objects in projective spaces and throughout this part we let q be a prime power. We begin with some very simple and well known facts.

Lemma 1.2.31. *For all $n \in \mathbb{N}_0 \cup \{-1\}$ we have $|\mathrm{PG}(n,q)| = \mathfrak{s}_q[n] := \frac{q^{n+1}-1}{q-1}$.*

Proof. $\mathrm{PG}(n,q)$ is the projective space constructed using a vector space of rank $n+1$ over the finite field $\mathbb{F}_q$ with q elements and as such every point of $\mathrm{PG}(n,q)$ is a rank 1 subspace of this vector space. Therefore, we only need to determine the number of rank 1 subspaces of a given rank $n+1$ vector space V over $\mathbb{F}_q$. Any $v \in V \setminus \{0_V\}$ spans a rank 1 subspace of V and, given a rank 1 subspace U of V, we know that $|U| = q$ and every vector $0_V \neq v \in U$ spans U. Therefore, there are $\frac{|V|-1}{q-1} = \frac{q^{n+1}-1}{q-1}$ such subspaces U of V. ☐

Now, using that number, we introduce a notation that is very handy and plays a crucial role in this work.

Lemma 1.2.32. *Let $\mathbb{P} := \mathrm{PG}(n,q)$ be the projective space of dimension $n \in \mathbb{N}_0 \cup \{-1\}$, let U and V be skew subspaces of $\mathbb{P}$ and set $k := \dim(U)$ as well as $l := \dim(V)$. Then, for all $d \in \mathbb{N}_0$ the cardinality of*

$$\mathfrak{S}[V,U,d,\mathbb{P}] := \{W \leq \mathbb{P} : \dim(W) = d, U \leq W, W \cap V = \emptyset\}$$

is independent of the particular choice of U and V and thus we may denote it by $\mathfrak{s}_q[l,k,d,n]$. For $k > d$ it is 0 and for $k \leq d$ it is given by

$$\mathfrak{s}_q[l,k,d,n] = \prod_{i=1}^{d-k} \frac{q^{n+1} - q^{k+l+i+1}}{q^{d+1} - q^{k+i}}. \tag{1.1}$$

Proof. First note that for $d \geq n - l$ for dimensional reasons there is no d-dimensional subspace $U \leq W \leq \mathbb{P}$ which does not intersect V and the numerator given on the right hand side of Equation (1.1) contains the factor $\left(q^{n+1} - q^{n+1}\right)$ and thus is 0. Therefore, we may assume that $d < n - l$ holds. Furthermore, if $k > d$, then there is no such subspace W and, if $d = k$, then there is only one such subspace, namely U itself, and the product given in Equation (1.1) is the empty product and as such equals 1. Thus, we may also assume that $k < d$ holds.

Now, given the subspace U and $k+1$ points $P_1, \ldots, P_{k+1} \in U$ which span U we can span any d-dimensional subspace $U \leq W \leq \mathbb{P}$ using additional points $P_{k+2}, \ldots, P_{d+1} \in \mathbb{P}$. Let m denote the number of tuples $(P_{k+2}, \ldots, P_{d+1})$ which spans a subspace of dimension d together with U. The ith entry of such a tuple must be a point of $\mathbb{P} \setminus \langle P_1, \ldots, P_{i-1}, V\rangle$, for if it was an element of $\langle P_1, \ldots, P_{i-1}\rangle$, then $\langle P_1, \ldots, P_{d+1}\rangle = \langle P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_{d+1}\rangle$ would have dimension at most $d-1$ and if it was an element of $\langle P_1, \ldots, P_{i-1}, V\rangle \setminus \langle P_1, \ldots, P_{i-1}\rangle$, then $\langle P_1, \ldots, P_i\rangle$ would intersect V non-trivially. Therefore, there are

$$\begin{aligned}\prod_{i=k+2}^{d+1}(\mathfrak{s}_q[n] - \mathfrak{s}_q[i+l-1]) &= \prod_{i=1}^{d-k}(\mathfrak{s}_q[n] - \mathfrak{s}_q[i+k+l]) \\ &= \prod_{i=1}^{d-k} \frac{q^{n+1} - q^{i+k+l+1}}{q-1}\end{aligned}$$

choices for these tuples. However, for a given d-dimensional subspace $W \leq \mathbb{P}$, any choice of such points in W spans W and thus there are

$$\prod_{i=k+2}^{d+1}((\mathfrak{s}_q[d] - \mathfrak{s}_q[i-2]) = \prod_{i=1}^{d-k}((\mathfrak{s}_q[d] - \mathfrak{s}_q[k+i-1]) = \prod_{i=1}^{d-k} \frac{q^{d+1} - q^{k+i}}{q-1}$$

choices of tuples which span the same subspace. Consequently, there are exactly

$$\frac{\prod_{i=1}^{d-k} \frac{q^{n+1}-q^{i+k+l+1}}{q-1}}{\prod_{i=1}^{d-k} \frac{q^{d+1}-q^{k+i}}{q-1}} = \prod_{i=1}^{d-k} \frac{q^{n+1} - q^{i+k+l+1}}{q^{d+1} - q^{k+i}}$$

such subspaces W and this number is independent of the specific choice of U and V. ☐

Notation 1.2.33. For all $k, d, n \in \mathbb{N}_0 \cup \{-1\}$ we set

$$\mathfrak{s}_q[k,d,n] := \mathfrak{s}_q[-1,k,d,n] \qquad \text{as well as} \qquad \mathfrak{s}_q[d,n] := \mathfrak{s}_q[-1,d,n]$$

and we note that we have $\mathfrak{s}_q[n] = \mathfrak{s}_q[0,n]$.

Remark 1.2.34. *Note that for all $d, n \in \mathbb{N}_0 \cup \{-1\}$ the number $\mathfrak{s}_q[d,n]$ of d-dimensional subspaces of an n-dimensional projective space coincides with the Gaussian coefficient*

$$\begin{bmatrix} n+1 \\ d+1 \end{bmatrix}_q = \prod_{i=0}^{d} \frac{q^{n-i+1}-1}{q^{i+1}-1} = \mathfrak{s}_q[d,n],$$

which is commonly used in the literature. Furthermore, for $n \in \mathbb{N}_0 \cup \{-1\}$ the number $\mathfrak{s}_q[n]$ of points in an n-dimensional subspace is often denoted by θ_n in the literature. Note that we use the notation θ_n in Section 2.3 and Chapter 3, too.

Lemma 1.2.35. *Let $k, d, n \in \mathbb{N}_0 \cup \{-1\}$ be such that $-1 \leq k < d < n$, then the following equations hold:*

i) For $n \geq 2$ we have $\mathfrak{s}_q[n-1, 2n-2] = \mathfrak{s}_q[0, n-1, 2n-1]$.

ii) For $j \in \mathbb{Z}$ with $k + j \geq -1$ we have $\mathfrak{s}_q[k, d, n] = \mathfrak{s}_q[k+j, d+j, n+j]$.

Proof. Using the value provided by Lemma 1.2.32 we see

$$\begin{aligned}
\mathfrak{s}_q[n-1, 2n-2] &= \prod_{i=1}^{n} \frac{q^{2n-1}-q^{i-1}}{q^n - q^{i-1}} = \prod_{i=1}^{n-1} \frac{q^{2n-1}-q^{i-1}}{q^n-q^i} \cdot \frac{q^{2n-1}-q^{n-1}}{q^n-q^0} \\
&= \prod_{i=1}^{n-1} \frac{q^{2n-1}-q^{i-1}}{q^n-q^i} \cdot q^{n-1} = \prod_{i=1}^{n-1} \frac{q^{2n}-q^i}{q^n-q^i} = \mathfrak{s}_q[0, n-1, 2n-1],
\end{aligned}$$

as well as

$$\mathfrak{s}_q[k,d,n] = \prod_{i=1}^{d-k} \frac{q^{n+1}-q^{k+i}}{q^{d+1}-q^{k+i}} = \prod_{i=1}^{d-k} \frac{q^{n+j+1}-q^{k+j+i}}{q^{d+j+1}-q^{k+j+i}} = \mathfrak{s}_q[k+j, d+j, n+j].$$

□

Lemma 1.2.36. *(a) For $n > k > 0$ and $q \geq 4$ we have*

$$\mathfrak{s}_q[1] \cdot q^{k(n-k)-1} \leq \mathfrak{s}_q[k-1, n-1] \leq (\mathfrak{s}_q[1]+1) q^{k(n-k)-1}.$$

(b) For positive integers q and c with $q > c^2 + c$ we have

$$(q^2+q+2)^c \leq (q+c+1) q^{2c-1}.$$

(c) For positive integers q and c with $q > c^2 + c$ we have $\mathfrak{s}_q[c]^c \leq (q+c+1) q^{c^2-1}$.

Proof. (a) The lower bound follows from $0 < k < n$ and for the upper bound we refer to [21, Lemma 34].

(b) This can be checked by hand for $c = 1$ and $c = 2$, so we assume that $c \geq 3$. By expansion we see that $(q^2 + q + 2)^c = \sum_{i=0}^{2c} a_i q^i$ where

$$a_{2c-i} = \sum_{j=0}^{\lfloor i/2 \rfloor} \binom{c}{j} \binom{c-j}{i-2j} 2^j,$$

since a term q^{2c-i} occurs in the expansion, if for some j with $2j \leq i$ we first choose the number 2 from j terms q^2+q+2, secondly we choose the number q from $i-2j$ of the remaining $c-j$ terms q^2+q+2, and finally we choose the number q^2 from the remaining terms q^2+q+2.

Now, we claim $a_{2c-i} \leq c^i$ for all i. Using $c \geq 3$, this can be verified for $i \leq 5$ by straight forward calculation. Thus, suppose that $i \geq 6$. Then

$$a_{2c-i} = \sum_{j=0}^{\lfloor i/2 \rfloor} \frac{c! \cdot 2^j}{(c+j-i)!(i-2j)!j!} \leq c^i \sum_{j=0}^{\lfloor i/2 \rfloor} \underbrace{\frac{2^j}{c^j (i-2j)!j!}}_{=:b_{ij}}. \tag{1.2}$$

We next show $b_{ij} \leq \frac{2}{i+2}$ for admissible i, j, that is, for i, j with $2j \leq i \leq 2c$ and $i \geq 6$. Using $i \geq 6$, this follows from direct calculation if $j \leq 3$. Otherwise $j \geq 4$ and $i \geq 8$, so $j! \geq 2^j$ and hence $b_{ij} \leq c^{-j} \leq \frac{2}{i+2}$, since $i \leq 2c$. Thus we have established the bound for b_{ij} and using it in (1.2) we find $a_{2c-i} \leq c^i$ for $i \geq 6$. Hence $a_{2c-i} \leq c^i$ for all $i \in \{0, \ldots, 2c\}$.

It follows that

$$\sum_{i=0}^{2c-2} a_i q^i \leq \sum_{i=2}^{2c} c^i q^{2c-i} = q^{2c-2} c^2 \sum_{i=0}^{2c-2} \frac{c^i}{q^i} \leq \frac{q^{2c-2} c^2}{1 - c/q} < q^{2c-1}$$

where we have used $q > c^2 + c$ in the last step. Since $a_{2c} = 1$ and $a_{2c-1} = c$, this proves the claim in (b).

(c) Since $\mathfrak{s}_q[c] \leq (q^2+q+2)q^{c-2}$ this is a corollary to the previous claim. □

1.2.4 Some specific Preparations

We conclude this introduction of projective spaces with some very specific results that we require later on but which fit better in this introduction. We let q be a prime power, we let n be a positive integer and we set $\mathbb{P} := \mathrm{PG}(n, q)$.

First, we have some results on subspaces that will be used in different settings later on. We provide this general proof here, instead of proving several lemmata in specific situations later.

Lemma 1.2.37. *Let $d \in \mathbb{N}$ and $U, U_1, U_2 \leq \mathbb{P}$ be such that $d \leq \min(\dim(U_i) - \dim(U \cap U_i) : i \in \{1, 2\})$.*

Every $(\dim(U) + d)$-dimensional subspace $\widehat{U} \leq \mathbb{P}$ with $\dim(\widehat{U} \cap U_i) = \dim(U \cap U_i) + d$ for all $i \in \{1, 2\}$ and $U \leq \widehat{U}$ is the span of U and a $(d-1)$-dimensional subspace $W \leq \langle U, U_1 \rangle \cap U_2$.

Proof. Assume that there is such a $(\dim(U) + d)$-dimensional subspace $\widehat{U}$. Due to $\dim(\widehat{U} \cap U_i) = \dim(U \cap U_i) + d$ there is a $(d-1)$-dimensional complement U_i' of $U \cap U_i$ in $\widehat{U} \cap U_i$ for all $i \in \{1,2\}$. Since $\dim(\widehat{U}) = \dim(U) + d$ the subspace U_i' is also a complement of U in $\widehat{U}$, that is, $\widehat{U} = \langle U, U_i' \rangle \leq \langle U, U_i \rangle$ for all $i \in \{1,2\}$. Therefore we have $U_2' \leq \widehat{U} \leq \langle U, U_1 \rangle$, proving the claim. □

Lemma 1.2.38. *Let U be a subspace of $\mathbb{P}$, set $d_U := \dim(U)$ and let Ξ be a non-empty set of (d_U+1)-dimensional subspaces of $\mathbb{P}$ such that for all $\xi \in \Xi$ we have $U \leq \xi$.*

There is a subset $\Xi' \subseteq \Xi$ of $d := \dim(\langle \Xi \rangle) - d_U$ pairwise distinct subspaces such that $\dim(\langle \Xi' \rangle) = d_U + |\Xi'| = \dim(\langle \Xi \rangle)$.

Proof. Let Ξ' be a maximal subset of Ξ such that $\dim(\langle \Xi' \rangle) = d_U + |\Xi'|$. Note that such a set exists, because $\Xi \neq \emptyset$ and every subspace $\xi \in \Xi$ satisfies $\dim(\xi) = d_U + 1$. Now, if $\langle \Xi \rangle \neq \langle \Xi' \rangle$, then there is a subspace $\xi \in \Xi$ such that $\xi \not\leq \langle \Xi' \rangle$. But $U \leq \xi \cap \langle \Xi' \rangle$ and therefore $U = \xi \cap \langle \Xi' \rangle$ and

$$\dim(\langle \xi, \Xi' \rangle) = 1 + \dim(\langle \Xi' \rangle) = 1 + d_U + |\Xi'| = d_U + |\{\xi\} \cup \Xi'|,$$

a contradiction to the maximal choice of Ξ'. □

Lemma 1.2.39. *Let d be an integer, let U and V be skew subspaces of $\mathbb{P}$, let d_U and d_V be their respective dimension and let $\mathcal{W}$ be a set of (d_U+d)-dimensional subspaces of $\mathbb{P}$ such that for all $W \in \mathcal{W}$ we have $U \leq W$ and such that $\dim(\langle \mathcal{W} \rangle) = d_U + |\mathcal{W}|d$.*

Then $\dim(\langle \mathcal{W}' \rangle) = d_U + |\mathcal{W}'|d$ for all $\emptyset \neq \mathcal{W}' \subseteq \mathcal{W}$ and, if $V \cap W \neq \emptyset$ for all $W \in \mathcal{W}$, then $|\mathcal{W}| - 1 \leq d_V$ and

$$V \leq \langle \mathcal{W} \rangle \qquad \Longleftrightarrow \qquad d_V = |\mathcal{W}| - 1 + \sum_{W \in \mathcal{W}} \dim(V \cap W).$$

Proof. For the first claim note that $U \leq W$ for all $W \in \mathcal{W}$ implies $U \leq \langle \mathcal{W}' \rangle$ for all $\emptyset \neq \mathcal{W}' \subseteq \mathcal{W}$. Furthermore, for any subset $\mathcal{W}'$ of $\mathcal{W}$ and any subspace $W \in \mathcal{W} \setminus \mathcal{W}'$ we have

$$\begin{aligned}\dim(\langle \mathcal{W}', W \rangle) &= \dim(\langle \mathcal{W}' \rangle) + \dim(W) - \dim(W \cap \langle \mathcal{W}' \rangle) \\ &\leq \dim(\langle \mathcal{W}' \rangle) + d_U + d - d_U = \dim(\langle W' \rangle) + d.\end{aligned}$$

Using this in the induction step of an induction on $r := |\mathcal{W}'|$ shows $\dim(\langle \mathcal{W}' \rangle) \leq |\mathcal{W}'|d + d_U$ for all $\mathcal{W}' \subseteq \mathcal{W}$.

Now, set $s := |\mathcal{W}|$ and let $\mathcal{W} = \{W_1, \ldots, W_s\}$ and assume that there is a subset $\mathcal{W}'$ of $\mathcal{W}$ such that $\dim(\langle \mathcal{W}' \rangle) < |\mathcal{W}'|d + d_U$. This implies $\mathcal{W}' \neq \mathcal{W}$ and we let $\overline{\mathcal{W}'}$ be the set $\mathcal{W} \setminus \mathcal{W}'$. Then, using $\dim(\langle \overline{\mathcal{W}'} \rangle) \leq |\overline{\mathcal{W}'}|d + d_U$, we have

$$\begin{aligned}\dim(\langle \mathcal{W} \rangle) &= \dim(\langle \mathcal{W}' \rangle) + \dim(\langle \overline{\mathcal{W}'} \rangle) - \dim(\langle \mathcal{W}' \rangle \cap \langle \overline{\mathcal{W}'} \rangle) \\ &\leq \dim(\langle \mathcal{W}' \rangle) + \dim(\langle \overline{\mathcal{W}'} \rangle) - d_U < |\mathcal{W}'|d + d_U + |\overline{\mathcal{W}'}|d + d_U - d_U \\ &= |\mathcal{W}|d + d_U,\end{aligned}$$

a contradiction, which concludes the proof of the first claim.

Now, let V be such that $P_i := V \cap W_i \neq \emptyset$ for all $i \in \{1, \ldots, s\}$. Then, obviously, $\langle P_1, \ldots, P_s \rangle \leq V$. We want to determine the dimension of $\langle P_1, \ldots, P_s \rangle$ and, in view of that, for all $i \in \{1, \ldots, s\}$, let d_i be the dimension of the subspace P_i. If there exists an index $j \in \{1, \ldots, s\}$ and a subset $J \subseteq \{1, \ldots, s\}$ with $j \in J$ but $J \neq \{j\}$, such that $R := P_j \cap \langle P_i : i \in J, i \neq j \rangle \neq \emptyset$, then $\mathcal{W}' := \{W_i : i \in J\} \subseteq \mathcal{W}$ satisfies

$$\begin{aligned} |\mathcal{W}'|d + d_U = \dim(\langle \mathcal{W}' \rangle) &\leq \dim(W_j) + \dim(\langle \mathcal{W}' \setminus \{W_j\} \rangle) - \dim(\langle U, R \rangle) \\ &= d_U + d + d_U + (|\mathcal{W}'| - 1)d - \dim(\langle U, R \rangle) \\ &= 2d_U + |\mathcal{W}'|d - \dim(\underbrace{\langle U, R \rangle}_{\neq U}) < |\mathcal{W}'|d + d_U, \end{aligned}$$

a contradiction. Therefore we have

$$s - 1 \leq \dim(P_1) + \sum_{i=2}^{s}(1 + \dim(P_i)) = \dim(\langle P_i : 1 \leq i \leq s \rangle) \leq \dim(V)$$

as well as

$$\begin{aligned} V \leq \langle \mathcal{W} \rangle \qquad &\iff \qquad \dim(V) = \dim(\langle P_1, \ldots, P_s \rangle) \\ &\iff \qquad d_V = s - 1 + \sum_{i=1}^{s} \dim(V \cap W_i). \end{aligned}$$ □

Finally, we have a technical result on point-sets, stated in Theorem 1.2.41 and prepared in the next lemma. This result will be used in Section 2.3 on the chromatic number. It is a joint work of D'haeseleer, Metsch and Werner and a generalization of a result given by the same authors in [12, Lemma 4.1].

Lemma 1.2.40. *Consider a set M of points of $\mathbb{P}$ and points $P_1, \ldots, P_{s+1} \in \mathbb{P}$, $s \geq 0$, such that $\langle P_1, \ldots, P_{s+1} \rangle$ is a subspace of dimension s with no point in M. Let μ be an upper bound on the number of lines on P_{s+1} that meet M. Let $c \in \mathbb{R}$ be positive and let $\mathcal{V}$ be a set of s-dimensional subspaces such that for all $V \in \mathcal{V}$ we have $P_1, \ldots, P_s \in V$ as well as $|V \cap M| \geq cq^s$.*

Then, for every $\gamma \in \mathbb{R}$ with $0 < \gamma < 1$, there exist at least $\frac{1-\gamma}{q}|\mathcal{V}|$ subspaces W of dimension $s+1$ satisfying $P_1, \ldots, P_{s+1} \in W$ and $|W \cap M| \geq \frac{\gamma}{\mu} c^2 q^{2s} |\mathcal{V}|$.

Proof. For $V \in \mathcal{V}$ we have $V \cap M \neq \emptyset$ and hence $P_{s+1} \notin V$. Set $x := \gamma \frac{|\mathcal{V}|cq^s}{\mu}$,

$$\begin{aligned} \mathfrak{W} &:= \{\langle V, P_{s+1} \rangle : V \in \mathcal{V}\}, \\ \mathcal{W} &:= \{W \in \mathfrak{W} : |\{V \in \mathcal{V} : V \leq W\}| > x\}, \end{aligned}$$

and $\overline{\mathcal{W}} := \mathfrak{W} \setminus \mathcal{W}$. The elements of $\mathfrak{W}$ are subspaces of dimension $s+1$. If $W \in \mathfrak{W}$, then there exists $V \in \mathcal{V}$ with $V \subseteq W$ and hence P_{s+1} lies on $|V \cap M| \geq cq^s$ lines of W which meet M. If W and W' are distinct elements of $\mathcal{W}$ and l is a line on P_{s+1} with $l \subseteq W, W'$, then $l \subseteq W \cap W' = \langle P_1, \ldots, P_{s+1} \rangle$ and thus $l \cap M = \emptyset$. Since μ is an upper bound on the number of lines on P_{s+1} which meet M, this proves that $|\mathfrak{W}| \leq \frac{\mu}{c} q^{-s}$. Since $\overline{\mathcal{W}} \subseteq \mathfrak{W}$ it follows that

$$|\{V \in \mathcal{V} : \exists W \in \overline{\mathcal{W}} \text{ with } V \leq W\}| \leq \frac{\mu}{c} q^{-s} \cdot x = \gamma |\mathcal{V}|$$

and hence $\langle V, P_{s+1} \rangle \in \mathcal{W}$ for least $(1-\gamma)|\mathcal{V}|$ elements of $\mathcal{V}$. Since every subspace $W \in \mathcal{W}$ contains at most q subspaces $V \in \mathcal{V}$, we find $|\mathcal{W}| \geq (1-\gamma)|\mathcal{V}|/q$. Since distinct elements V and V' of $\mathcal{V}$ satisfy $(V \cap V') \cap M = \langle P_1, \ldots, P_s \rangle \cap M = \emptyset$, we see that every $W \in \mathcal{W}$ satisfies

$$|W \cap M| \geq x \cdot cq^s = \frac{\gamma}{\mu} c^2 q^{2s} |\mathcal{V}|. \qquad \square$$

Theorem 1.2.41. *Suppose that M is a set of points in $\mathrm{PG}(2d, q)$ and there are $d+1$ points $P_1, P_2, \ldots P_{d+1}$ that span a d-dimensional subspace τ with $\tau \cap M = \emptyset$. Furthermore, let m, n_0 and d_0 be positive real numbers such that the following hold:*

(I) Each of the points $P_1, P_2, \ldots P_{d+1}$ lies on at most $n_0 q^d$ lines that meet M.

(II) $|M| \geq d_0 q^{d+1}$.

Then there exists a $(d+1)$-dimensional subspace U on τ with

$$|U \cap M| > (2q)^{d+1} \left(\frac{d_0}{4n_0}\right)^{2^{d+1}-1}. \tag{1.3}$$

Proof. We prove the following more general result. For each $s \in \{0, \ldots, d+1\}$, there exists a set $\mathcal{V}_s$ of s-dimensional subspaces satisfying $|\mathcal{V}_s| \geq (\frac{1}{2})^s d_0 q^{d+1-s}$ such that each $V \in \mathcal{V}_s$ satisfies

$$\{P_i \mid 1 \leq i \leq s\} \subseteq V \qquad \text{and} \qquad |V \cap M| \geq (2q)^s \left(\frac{d_0}{4n_0}\right)^{2^s-1}. \tag{1.4}$$

We use induction on s. For $s = 0$ we take $\mathcal{V}_0 = M$. For the induction step $s \to s+1$, we assume the existence of $\mathcal{V}_s$ with the desired properties. For $V \in \mathcal{V}_s$ we know from the induction hypothesis that Equation (1.4) holds and, since $\tau \cap M = \emptyset$ by hypothesis of this lemma, this also implies $V \not\leq \tau$, that is, $P_{s+1} \notin V$. Now the previous lemma, applied with $c = 2^s \left(\frac{d_0}{4n_0}\right)^{2^s-1}$, $\mathcal{V} = \mathcal{V}_s$ and $\mu = n_0 q^d$ and $\gamma = \frac{1}{2}$, proves the existence of a set $\mathcal{V}_{s+1} = \mathcal{W}$ with the desired properties.

For $s = d+1$ we find $|\mathcal{V}_{d+1}| > 0$, so each element U of $\mathcal{V}_{d+1}$ satisfies the claim of this lemma, concluding the proof. $\square$

1.3 Reguli

In preparation of Section 2.4 we define a regulus and prove some simple facts about reguli in general, all of which takes place in the projective space $\mathbb{P} := \mathrm{PG}(n, q)$ of dimension $n \geq 3$ for some prime power q. Most of these facts can be found in the literature that was already mentioned, that is, for example in [2, Section 2.4] on the hyperbolic quadric of $\mathrm{PG}(3, q)$ by Beuelspacher and Rosenbaum. However, since all but the proof of Theorem 1.3.4 (for which coordinates are used) are fairly short and seem to fit into this work, we include them here.

Definition 1.3.1 (Regulus)**.** Let h_1, h_2 and h_3 be three skew lines in a solid $S \leq \mathbb{P}$. The set $\mathcal{R}$ of all lines $l \leq \mathbb{P}$ that have non-empty intersection with the three lines h_1, h_2 and h_3 is called a *regulus* of $\mathbb{P}$. If P is a point of one of the lines of $\mathcal{R}$, then we also say that P is a *point of* $\mathcal{R}$.

Lemma 1.3.2. *Let h_1 and h_2 be two skew lines. For every point $P \in \langle h_1, h_2\rangle$ with $P \notin h_1$ and $P \notin h_2$ there is a unique line g through P that has non-empty intersection with h_1 and h_2.*

Proof. Let P be such a point. Any line g with $g \cap h_1 \neq \emptyset \neq g \cap h_2$ and $P \in g$ satisfies $g \leq \langle P, h_1\rangle$ and thus meets h_2 in a point of $Q := \langle P, h_1\rangle \cap h_2$. Since h_1 and h_2 are skew lines in a common solid $\langle h_1, h_2\rangle$ we know that Q is a point and thus g is the line $\langle P, Q\rangle$. $\square$

Corollary 1.3.3. *Every regulus $\mathcal{R}$ has cardinality $\mathfrak{s}_q[1] = q + 1$.*

Theorem 1.3.4. *Let $\mathcal{R}$ be a regulus of $\mathbb{P}$. Every line g of $\mathbb{P}$ that has non-empty intersection with three lines of $\mathcal{R}$ has non-empty intersection with all lines of $\mathcal{R}$.*

Proof. As mentioned early, this proof requires the use of coordinates and can, for example, be found in [2, Section 2.4]. $\square$

Definition 1.3.5 (Opposite Regulus)**.** Let $\mathcal{R}$ be a regulus. From Corollary 1.3.3 we have $|\mathcal{R}| \geq 3$ and from Theorem 1.3.4 we know that any line g that has non-empty intersection with three lines of $\mathcal{R}$ has non-empty intersection with all lines of $\mathcal{R}$. Hence, the set $\overline{\mathcal{R}}$ of all lines which have non-empty intersection with all lines of $\mathcal{R}$ is a regulus, too. We say that the two reguli are *opposite* and call $\overline{\mathcal{R}}$ the *opposite Regulus* of $\mathcal{R}$.

Lemma 1.3.6. *Let g_1, g_2 and g_3 be three skew lines in a solid $S \leq \mathbb{P}$. Then there is a unique regulus $\mathcal{R}$ in $\mathbb{P}$ with $g_1, g_2, g_3 \in \mathcal{R}$.*

Proof. Let $\overline{\mathcal{R}} = \{h_1, \dots, h_{q+1}\}$ be the unique regulus of lines of $\mathbb{P}$ that have non-empty intersection with all lines g_1, g_2 and g_3, let $\mathcal{R}$ be the opposite regulus of $\overline{\mathcal{R}}$ and let $\mathcal{R}'$ be an arbitrary regulus with $g_1, g_2, g_3 \in \mathcal{R}'$. Any line $h \in \overline{\mathcal{R}}$ has non-empty intersection the three lines $g_1, g_2, g_3 \in \mathcal{R}'$ and therefore, according to Theorem 1.3.4, with all lines of $\mathcal{R}'$. Thus, any line $g \in \mathcal{R}'$, has non-empty intersection with all lines $h \in \overline{\mathcal{R}}$, proving $g \in \mathcal{R}$ and thus $\mathcal{R}' \subseteq \mathcal{R}$, that is, $\mathcal{R}' = \mathcal{R}$. $\square$

Obviously, this also implies that if g_1 and g_2 are skew lines of $\mathbb{P}$ then there is more than one regulus $\mathcal{R}$ in $\mathbb{P}$ with $g_1, g_2 \in \mathcal{R}$.

Definition 1.3.7 (Tangent Plane)**.** Given two opposite reguli $\mathcal{R}$ and $\overline{\mathcal{R}}$, using Lemma 1.3.2 to every point P of $\mathcal{R}$ there are unique lines $g \in \mathcal{R}$ and $l \in \overline{\mathcal{R}}$ with $P \in g, l$ and we call the plane $\langle g, l\rangle$ the *tangent plane* of $\mathcal{R}$ in P.

Lemma 1.3.8. *Let $\mathcal{R}$ and $\overline{\mathcal{R}}$ be opposite reguli in a solid $S \leq \mathbb{P}$ and let $g \in \mathcal{R}$ be an arbitrary line. Then $\mathfrak{S}[g,2,S] = \{\langle g,h\rangle : h \in \overline{\mathcal{R}}\}$.*

Proof. Trivially the right hand side is a subset of the left hand side of this equation. Furthermore, since any two distinct lines $h,h' \in \overline{\mathcal{R}}$ are skew the corresponding planes $\langle g,h\rangle$ and $\langle g,h'\rangle$ are distinct and thus both sides of the equation have the same cardinality $|\overline{\mathcal{R}}| = \mathfrak{s}_q[1] = \mathfrak{s}_q[1,2,3]$, proving the claim. □

Corollary 1.3.9. *If $\mathcal{R}$ and $\overline{\mathcal{R}}$ are opposite reguli in a solid $S \leq \mathbb{P}$ and $E \leq S$ is a plane then E contains a line of $\mathcal{R}$ if and only if E contains a line of $\overline{\mathcal{R}}$ and thus if and only if E is a tangent plane of $\mathcal{R}$.*

Lemma 1.3.10. *If $\mathcal{R}$ and $\overline{\mathcal{R}}$ are opposite reguli, P and Q points of $\mathcal{R}$ and E_P and E_Q are tangent planes of $\mathcal{R}$ in P and Q, respectively, then we have $E_P = E_Q$ if and only if $P = Q$.*

Proof. Obviously, for $P = Q$ we have $E_P = E_Q$. Let $P \neq Q$ and let $g_P, g_Q \in \mathcal{R}$ and $h_P, h_Q \in \overline{\mathcal{R}}$ be such that $E_P = \langle g_P, h_P\rangle$ and $E_Q = \langle g_Q, h_Q\rangle$. Since $P \neq Q$ we have $g_P \neq g_Q$ or $h_P \neq h_Q$ and we may assume that $g_P \neq g_Q$. Then $g_Q \not\leq E_P$ since $g_Q \cap g_P = \emptyset$ and thus $E_P \neq E_Q$. □

We have now introduced the required basics on reguli. Additionally to that, in Section 2.4 we also need some quite specific details on distinct reguli $\mathcal{R}_1$ and $\mathcal{R}_2$ which have two lines in common. We establish these in the remainder of this section.

Lemma 1.3.11. *Let $n \geq 4$, let V be a subspace of $\mathbb{P}$ of dimension $n-4$, let S be a complement of V and let $\mathcal{U}$ be a subset of $\mathfrak{S}[V, n-2, \mathbb{P}]$ such that $\mathcal{R} := \{U \cap S : U \in \mathcal{U}\}$ is a regulus in S and let $\overline{\mathcal{R}}$ be its opposite.*

If S' is a complement of V in $\mathbb{P}$, then $\{U \cap S' : U \in \mathcal{U}\}$ is a regulus in S' and $\{\langle V, r\rangle \cap S' : r \in \overline{\mathcal{R}}\}$ is its opposite.

Proof. Let S' be a complement of V in $\mathbb{P}$ and set $\mathcal{R}' := \{U \cap S' : U \in \mathcal{U}\}$ as well as $\overline{\mathcal{R}}' := \{\langle V, \overline{r}\rangle \cap S' : \overline{r} \in \overline{\mathcal{R}}\}$. Note that for all $U \in \mathcal{U}$ we have $U \cap S \in \mathcal{R}$ with $U = \langle V, U \cap S\rangle$ (since S is a complement of V) and thus $\mathcal{R}' = \{\langle V, r\rangle \cap S' : r \in \mathcal{R}\}$.

First, consider two arbitrary lines g and h of S with $g \cap h = \emptyset$. Since both S and S' are complements of V in $\mathbb{P}$ we know that both $g' := \langle V, g\rangle \cap S'$ and $h' := \langle V, h\rangle \cap S'$ are lines, too. Obviously we have $g' \cap V = \emptyset = h' \cap V$ and thus $\langle V, g\rangle = \langle V, g'\rangle$ as well as $\langle V, h\rangle = \langle V, h'\rangle$. This proves

$$\langle V, S'\rangle = \mathbb{P} = \langle V, S\rangle = \langle V, \langle g,h\rangle\rangle = \langle\langle V,g\rangle, \langle V,h\rangle\rangle = \langle\langle V,g'\rangle, \langle V,h'\rangle\rangle = \langle V, \langle g', h'\rangle\rangle$$

and together with $g', h' \leq S'$, using the fact that S' is a complement of V in $\mathbb{P}$, this implies $\langle g', h'\rangle = S'$ and thus $g' \cap h' = \emptyset$. Hence, both $\mathcal{R}'$ and $\overline{\mathcal{R}}'$ are sets of $q+1$ pairwise skew lines.

Now, let $r' \in \mathcal{R}'$ and $\overline{r}' \in \overline{\mathcal{R}}'$ be lines and let $r \in \mathcal{R}$ and $\overline{r} \in \overline{\mathcal{R}}$ be such that $r' = \langle V, r\rangle \cap S'$ and $\overline{r}' = \langle V, \overline{r}\rangle \cap S'$. Then $P := r \cap \overline{r}$ is a point of both $\langle V, r\rangle$ and $\langle V, \overline{r}\rangle$. Furthermore, since S' is a complement of $V \not\ni P$, we know that $P' := \langle V, P\rangle \cap S'$ is point, too, and P' is obviously an element of both $\langle V, r\rangle$ and $\langle V, \overline{r}\rangle$. Hence, the two lines $r' = \langle V, r\rangle \cap S'$ and $\overline{r}' = \langle V, \overline{r}\rangle \cap S'$ contain P' and thus have non-empty intersection.

Therefore, all lines of $\mathcal{R}'$ and all lines of $\overline{\mathcal{R}}'$ have pairwise non-empty intersection and, since both $\mathcal{R}'$ and $\overline{\mathcal{R}}'$ have cardinality $q+1 \geq 3$, this proves that $\mathcal{R}'$ and $\overline{\mathcal{R}}'$ are opposite reguli in S'. □

Lemma 1.3.12. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be distinct reguli of $\mathbb{P}$ with $|\mathcal{R}_1 \cap \mathcal{R}_2| \geq 2$, let $\overline{\mathcal{R}}_1$ and $\overline{\mathcal{R}}_2$ be their respective opposite reguli and set $\mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_2$.*

Then there is a solid S such that both $\mathcal{R}_1$ and $\mathcal{R}_2$ are reguli in S, we have $|\mathcal{R}_1 \cap \mathcal{R}_2| = 2$, $\xi := |\overline{\mathcal{R}}_1 \cap \overline{\mathcal{R}}_2| \leq 2$ and for all $g \in \mathcal{R}_1 \setminus \mathcal{R}$ we know that

i) g has non-empty intersection with a line $l \in \overline{\mathcal{R}}_2$ if and only if $l \in \overline{\mathcal{R}}_1$, and

ii) g has non-empty intersection with exactly ξ lines of $\mathcal{R}_2$, namely those which contain one of the ξ points of $\{g \cap l : l \in \overline{\mathcal{R}}_2\}$.

Furthermore, if $g \in \mathcal{R}_1 \setminus \mathcal{R}$ and $h \in \mathcal{R}_2$ are such that $g \cap h \neq \emptyset$ and if $l \in \overline{\mathcal{R}}_1 \cap \overline{\mathcal{R}}_2$ is the line with $g \cap h \in l$ and l' is the line with $\overline{\mathcal{R}}_1 \cap \overline{\mathcal{R}}_2 = \{l, l'\}$, then we have $l' \leq \langle g, h\rangle$ and $\xi = 1$ occurs if and only if $h \leq \langle g, l\rangle$.

Proof. Since $|\mathcal{R}| \geq 2$ we know that $S := \langle \mathcal{R}\rangle$ satisfies $\dim(S) \geq 3$ and, since for all $i \in \{1, 2\}$ the set $\mathcal{R}_i$ is a regulus, we have $\dim(\langle \mathcal{R}_i\rangle) \leq 3$. This proves $\dim(S) = 3$ and $\mathcal{R}_1$ and $\mathcal{R}_2$ are reguli in S. Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are distinct, $\overline{\mathcal{R}}_1$ and $\overline{\mathcal{R}}_2$ are distinct, too. Hence, we know from Lemma 1.3.6 that both $\mathcal{R}_1$ and $\mathcal{R}_2$, as well as their opposites, do not have more than two lines in common, that is, $|\mathcal{R}| = 2$ and $\xi \leq 2$.

From now on let g be an arbitrary but fixed line of $\mathcal{R}_1 \setminus \mathcal{R}$. Since $\mathcal{R} \neq \emptyset$ and since any line $g' \in \mathcal{R}$ satisfies $g' \in \mathcal{R}_2$ as well as $g \cap g' = \emptyset$ we have $g \notin \overline{\mathcal{R}}_2$.

Now, for any line $l \in \mathcal{R}_2$ that has non-empty intersection with g there is a line in $\overline{\mathcal{R}}_2$ through $l \cap g$, that is, a line in $\overline{\mathcal{R}}_2$ which has non-empty intersection with g. Furthermore, any line $l \in \overline{\mathcal{R}}_2$ has non-empty intersection with the two lines in $\mathcal{R}$. Therefore, any line $l \in \overline{\mathcal{R}}_2$ that has non-empty intersection with g in fact has non-empty intersection with at least three and thus, according to Theorem 1.3.4, with all lines of $\mathcal{R}_1$, proving that it is an element of $\overline{\mathcal{R}}_1$ and i).

In fact, if $\xi = 0$, then this proves that g has empty intersection with all lines of $\mathcal{R}_2$, concluding the proof of all claims given for $\xi = 0$.

Hence, from now on assume $\xi > 0$. Then there is a line $l \in \overline{\mathcal{R}}_1 \cap \overline{\mathcal{R}}_2$ and this line has non-empty intersection with g, which implies that there is a line $h \in \mathcal{R}_2$ containing $g \cap l$ and hence h satisfies $h \cap g \neq \emptyset$. Now, if there is a line $h' \in \mathcal{R}_2$ distinct from h with $g \cap h' \neq \emptyset$, then the line $l' \in \overline{\mathcal{R}}_1 \cap \overline{\mathcal{R}}_2$ with $g \cap h' \in l'$ is distinct from l (otherwise $h \cap h' = \emptyset$ implies $g = \langle g \cap h, g \cap h'\rangle = l \in \overline{\mathcal{R}}_2$, a contradiction), which proves $|\{h \in \mathcal{R}_2 : g \cap h \neq \emptyset\}| \leq |\overline{\mathcal{R}}_1 \cap \overline{\mathcal{R}}_2| = \xi$. Furthermore, for $\{P_1, P_\xi\} = \{g \cap l : l \in \overline{\mathcal{R}}_2\}$ and $i \in \{1, \xi\}$ there obviously is a line $l_i \in \mathcal{R}_2$ through P_i and, if $\xi > 1$, then $P_1 \neq P_\xi$ and $\langle P_1, P_\xi\rangle = g \notin \mathcal{R}_2$ proves $l_1 \neq l_\xi$. Together this proves ii).

Now, let l' be the line with $\overline{\mathcal{R}}_1 \cap \overline{\mathcal{R}}_2 = \{l, l'\}$. Since g is a line of $\mathcal{R}_1$ and since $E := \langle g, h\rangle$ is a plane we know from Corollary 1.3.9 that E is a tangent plane of $\mathcal{R}_1$. Thus, there is a line $e \in \overline{\mathcal{R}}_1$ with $E = \langle g, e\rangle$ and, since $h \leq E$, we have $h \cap e \neq \emptyset$ and thus $e \in \overline{\mathcal{R}}_2$. Furthermore, for any line $e' \in \overline{\mathcal{R}}_1 \setminus \{e\}$ we have $e' \cap E = e' \cap g$ and thus $e' \cap h \neq \emptyset$ if and only if $e' = l$. Hence, $l' = e \leq E = \langle g, h \rangle$, that is, $\overline{\mathcal{R}}_1 \cap \overline{\mathcal{R}}_2 = \{l, e\}$, and $\xi = 1$ occurs if and only if $l = e$, that is, if and only if $h \leq E = \langle g, l \rangle$, concluding the proof. $\square$

Lemma 1.3.13. *For two lines g and h and all points $P, Q \in \mathbb{P} \setminus \langle g, h \rangle$ we have*

$$\dim(\langle P, g \rangle \cap \langle Q, h \rangle) = \dim(g \cap h) + \begin{cases} 0 & \text{for } Q \notin \langle P, g, h \rangle, \\ 1 & \text{for } Q \in \langle P, g, h \rangle. \end{cases}$$

Proof. Set $d := \dim(\langle P, g \rangle \cap \langle Q, h \rangle)$. Since $P, Q \notin \langle g, h \rangle$ the subspaces $\langle P, g \rangle$ and $\langle Q, h \rangle$ are planes which have $g \cap h$ in common. Therefore, we have $d \geq \dim(g \cap h)$ and $d \leq 2$. Furthermore, since $P \notin \langle g, h \rangle$ we have $\langle P, g \rangle \cap h = g \cap h$. Therefore, we have $\langle P, g \rangle \cap \langle Q, h \rangle = \langle P, g \rangle \cap h = g \cap h$ and thus $d = \dim(g \cap h)$ for $Q \notin \langle P, g, h \rangle$, as well as $\dim(\langle P, g \rangle \cap \langle Q, h \rangle) = \dim(\langle P, g \rangle \cap h) + 1$ otherwise. $\square$

Lemma 1.3.14. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be distinct reguli such that $\mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_2$ has cardinality two, let S be the solid containing both $\mathcal{R}_1$ and $\mathcal{R}_2$, let P be a point of $\mathbb{P} \setminus S$ and set $U := \langle P, S \rangle$. Furthermore, for any point $Q \in U \setminus S$ set*

$$\Lambda_Q := \{(g, h) \in \mathcal{R}_1 \times \mathcal{R}_2 : \dim(\langle P, g \rangle \cap \langle Q, h \rangle) \geq 1\}$$

and for all $i \in \Omega := \{|\Lambda_Q| : Q \in U \setminus S\}$ set $\Omega_i := \{Q \in U \setminus S : |\Lambda_Q| = i\}$.

Then we have $\Omega \subseteq \{2, 3, 4, q+1, q+2, 2q\}$, $\Omega_{2q} \subseteq \{P\}$ and for all $(g, h) \in \mathcal{R}_1 \times \mathcal{R}_2$ and $Q \in U \setminus S$ we have $\dim(\langle P, g \rangle \cap \langle Q, h \rangle) = 2$ if and only if $g = h \in \mathcal{R}$ and $Q \in \langle P, g \rangle$.

Proof. Note that for lines $g \in \mathcal{R}_1$ and $h \in \mathcal{R}_2$ with $g \cap h = \emptyset$ we have $\langle g, h \rangle = S$ and thus $U = \langle P, g, h \rangle$, that is, Lemma 1.3.13 proves that $\langle P, g \rangle \cap \langle Q, h \rangle$ is a point for all $Q \in U \setminus S$. In view of Lemma 1.3.13 this implies that:

i) We have to determine $M := \{(g, h) \in \mathcal{R}_1 \times \mathcal{R}_2 : g \cap h \neq \emptyset\}$.

ii) Given lines $g, g' \in \mathcal{R}_1$ and $h, h' \in \mathcal{R}_2$ with

$$g \cap h \neq \emptyset, \ g' \cap h' \neq \emptyset \text{ and } |\{g, g', h, h'\}| \geq 3, \tag{1.5}$$

we have to determine $\langle P, g, h \rangle \cap \langle P, g', h,' \rangle$.

Lemma 1.3.12 provides all that is needed to do that and we set $\overline{\mathcal{R}} := \overline{\mathcal{R}}_1 \cap \overline{\mathcal{R}}_2$ and recall that Lemma 1.3.12 states $\xi := |\overline{\mathcal{R}}| \leq 2$.

i) Let g_1 and g_2 be the two lines in $\mathcal{R}$. Lemma 1.3.12 (part ii)) also shows that for any line $g \in \mathcal{R}_1 \setminus \mathcal{R}$ there are ξ lines $h \in \mathcal{R}_2$ with $g \cap h \neq \emptyset$ (all of which satisfy $h \notin \mathcal{R}$) and that $g \cap h$ is one of the $\xi(q+1)$ points in $\mathcal{Q} := \{Q \in l : l \in \overline{\mathcal{R}}\}$.

   Therefore, for any line $g \in \mathcal{R}_1 \setminus \mathcal{R}$ there are ξ lines $h \in \mathcal{R}_2$ with $g \cap h \neq \emptyset$ (all of which satisfy $h \notin \mathcal{R}$). Furthermore, for all $g \in \mathcal{R}$ and $h \in \mathcal{R}_2$ we obviously have $g \cap h \neq \emptyset$ if and only if $g = h$ and thus for any line $g \in \mathcal{R}$ there is a unique line $h \in \mathcal{R}_2$ with $g \cap h \neq \emptyset$ and this line satisfies $h = g \in \mathcal{R}$.

ii) Assume that $\xi > 0$. Let $g, g' \in \mathcal{R}_1$ and $h, h' \in \mathcal{R}_2$ be lines which satisfy Equation (1.5) and note that, since $P \notin S \geq \langle g, h\rangle, \langle g', h'\rangle$ it suffices to study $\langle g, h\rangle \cap \langle g', h'\rangle$. From $|\{g, g', h, h'\}| \geq 3$ we have $g \cap g' = \emptyset$ or $h \cap h' = \emptyset$ and thus $T := g \cap h \neq g' \cap h' =: T'$ as well as $g' \not\leq \langle g, h\rangle$ or $h' \not\leq \langle g, h\rangle$, that is, $\langle g, h\rangle \neq \langle g', h'\rangle$.

   Therefore, if $\langle T, T'\rangle \in \overline{\mathcal{R}}$ and l' is such that $\{\langle T, T'\rangle, l'\} = \overline{\mathcal{R}}$, then Lemma 1.3.12 shows $l' \leq \langle g, h\rangle, \langle g', h'\rangle$ and $\langle g \cap h\rangle \neq \langle g', h'\rangle$ proves $\langle g, h\rangle \cap \langle g', h'\rangle = l'$. Furthermore, if $\langle T, T'\rangle \notin \overline{\mathcal{R}}$, then the lines $l, l' \in \overline{\mathcal{R}}$ with $T \in l$ and $T' \in l'$ provided by Lemma 1.3.12 are distinct, satisfy $l \leq \langle g', h'\rangle$ as well as $l' \leq \langle g, h\rangle$ and we have $\langle g, h\rangle \cap \langle g', h'\rangle = \langle T, T'\rangle$ from $T \in l \leq \langle g', h'\rangle \ni T' \in l' \leq \langle g, h\rangle \ni T$ and $\langle g, h\rangle \neq \langle g', h'\rangle$.

Now, let $Q \in U \setminus S$. In the very beginning of this prove we have shown that $\langle P, g\rangle \cap \langle Q, h\rangle$ is a point for all $(g, h) \in \mathcal{R}_1 \times \mathcal{R}_2 \setminus M$ and for all $i \in \{1, 2\}$ we have $\dim(\langle P, g_i\rangle \cap \langle Q, g_i\rangle) \geq 1$. In the following we study the remaining pairs in M, that is, those in $M' := M \setminus \{(g_1, g_1), (g_2, g_2)\}$. We consider the three cases that may occur for ξ and for $\xi > 0$ we let l, l' be such that $\overline{\mathcal{R}} = \{l, l'\}$.

- First, let $\xi = 0$, that is, $\overline{\mathcal{R}} = \emptyset$. In this case we have $M = \{(g_i, g_i) : i \in \{1, 2\}\}$ from i) and thus $\Omega = \{2\}$ with $\Omega_2 = U \setminus S$.

- Secondly, let $\xi = 1$, that is, $l = l'$. For distinct elements (h_1, h_2) and (h_1', h_2') of $M \setminus \{(g_1, g_1), (g_2, g_2)\}$ we have

$$\langle P, h_1, h_2\rangle \cap \langle P, h_1', h_2'\rangle = \langle P, \langle h_1, h_2\rangle \cap \langle h_1', h_2'\rangle\rangle = \langle P, l\rangle$$

  from ii) and thus $\Omega = \{2, 3, q+1\}$ with $\Omega_{q+1} = \langle P, l\rangle \setminus S$, $\Omega_3 = \bigcup_{(g,h)\in M'} \langle P, g, h\rangle \setminus (S \cup \langle P, l\rangle)$ and $\Omega_2 = U \setminus (S \cup \Omega_{q+1} \cup \Omega_3)$.

- Finally, let $\xi = 2$, that is, $l \neq l'$. We set $P_1 := l \cap g_1$, $P_2 := l \cap g_2$, $P_1' := l' \cap g_1$, $P_2' := l' \cap g_2$ and we let $P_3, \ldots, P_{q+1}$ and $P_3', \ldots, P_{q+1}'$ be the remaining points of l and l', respectively. For distinct elements (h_1, h_2) and (h_1', h_2') of $M \setminus \{(g_1, g_1), (g_2, g_2)\}$ we have

$$\begin{aligned}\langle P, h_1, h_2\rangle \cap \langle P, h_1', h_2'\rangle &= \langle P, \langle h_1, h_2\rangle \cap \langle h_1', h_2'\rangle\rangle \\ &= \begin{cases} \langle P, \langle h_1 \cap h_2, h_1' \cap h_2'\rangle\rangle & \text{if } \langle h_1 \cap h_2, h_1' \cap h_2'\rangle \notin \overline{\mathcal{R}}, \\ \langle P, l\rangle & \text{if } \langle h_1 \cap h_2, h_1' \cap h_2'\rangle = l', \\ \langle P, l'\rangle & \text{if } \langle h_1 \cap h_2, h_1' \cap h_2'\rangle = l \end{cases}\end{aligned}$$

  from ii) and thus $\Omega = \{2, 3, 4, q+1, q+2, 2q\}$ with $\Omega_{2q} = \{P\}$ and

$$\Omega_{q+2} = \bigcup_{i=3}^{q+1} (\langle P, P_i\rangle \cup \langle P, P_i'\rangle) \setminus (S \cup \{P\})$$
$$\Omega_{q+1} = \bigcup_{i=1}^{2} (\langle P, P_i\rangle \cup \langle P, P_i'\rangle) \setminus (S \cup \{P\})$$

$$\Omega_4 = \bigcup_{i=3}^{q+1} \bigcup_{j=3}^{q+1} \langle P, P_i, P'_j \rangle \setminus (S \cup \langle P, P_i \rangle \cup \langle P, P'_j \rangle)$$

$$\Omega_3 = \bigcup_{i=3}^{q+1} \bigcup_{j=1}^{2} (\langle P, P_i, P'_j \rangle \cup \langle P, P_j, P'_i \rangle) \setminus \left(S \cup \bigcup_{i=1}^{2} \langle P, P_i \rangle \cup \langle P, P'_i \rangle \right),$$

$$\Omega_2 = \bigcup_{i=1}^{2} \bigcup_{j=1}^{2} \langle P, P_i, P'_j \rangle \setminus \left(S \cup \bigcup_{i=1}^{2} \langle P, P_i \rangle \cup \langle P, P'_i \rangle \right). \qquad \square$$

Corollary 1.3.15. *Let $\mathbb{P}$ have dimension 4, let $\mathcal{R}$ be a regulus in a solid $S \leq \mathbb{P}$ and let $P \notin S$ be a point of $\mathbb{P}$. For every point $Q \in \mathbb{P} \setminus S$ with $Q \notin \bigcup_{g \in \mathcal{R}} \langle P, g \rangle$ and every regulus $\mathcal{R}'$ with $|\mathcal{R}' \cap \mathcal{R}| = 2$ we have*

$$|\{(g,h) \in \mathcal{R} \times \mathcal{R}' : \dim(\langle P, g \rangle \cap \langle Q, h \rangle) \geq 1\}| \leq 4.$$

A regulus is an example of a more general concept of a geometry, a polar space, which we introduce next. This is in preparation of the last chapter of this thesis, which takes place in a Hermitian polar space.

1.4 Polar Spaces

For a detailed introduction to polar spaces we refer the reader to [11] by Cameron or, for a more comprehensive work, also to [9] by Buekenhout and Cohen. For us only one special kind of polar space is of importance — namely one of the Hermitian polar spaces — and thus we keep the general introduction short. Still, we include the abstract definition of a polar space as it was suggested in [10] by Buekenhout and Shult.

Definition 1.4.1 (Polar Space)**.** A point line incidence structure $(\mathcal{P}, \mathcal{L}, *)$ with $L \subseteq 2^{\mathcal{P}}$ and $P * l$ as well as $l * P$ if and only if $P \in l$ is called *polar space* if it satisfies the *one-or-all* axiom:

> For all $P \in \mathcal{P}$ and all $l \in \mathcal{L}$ with $P \notin l$ either one or all points of l are collinear to P.

Polar spaces were first studied by Veldkamp in [29] and his results were simplified and completed by Tits in [28]. In fact, due to their work the (thick) polar spaces of rank $n \geq 4$ are known to be classical polar spaces, one of which is the Hermitian polar space we are interested in. The complete proof of that classification is also given in [9] by Buekenhout and Cohen.

As mentioned earlier, we forgo any further abstract introduction and instead focus on the cases that are of interest here. In particular, we give a short list of the *finite classical polar spaces* of rank d over a finite field of order q as well as the quadratic, bilinear or sesquilinear form (up to transformation of coordinates) which defines it and then focus only on the Hermitian polar spaces. The finite classical polar spaces are:

- The *Hyperbolic Quadric* $Q^+(2d-1,q)$ in $\mathrm{PG}(2d-1,q)$, which is defined using the non-degenerate quadratic form $x_1x_2+\cdots+x_{2d-1}x_{2d}$.
- The *Parabolic Quadric* $Q(2d,q)$ in $\mathrm{PG}(2d,q)$, which is defined using the non-degenerate quadric form $x_1^2+x_2x_3+\cdots+x_{2d}x_{2d+1}$.
- The *Elliptic Quadric* $Q^-(2d+1,q)$ in $\mathrm{PG}(2d+1,q)$, which is defined using the non-degenerate quadratic form $f(x_1,x_2)+x_3x_4+\cdots+x_{2d+1}x_{2d+2}$, where $f(x_1,x_2)$ is an irreducible homogenous quadratic polynomial of $\mathbb{F}_q$.
- The *Hermitian Polar Space* $H(n,q)$ for $n\in\{2d-1,2d\}$ in $\mathrm{PG}(n,q)$ (with $q=p^2$ for some prime power p) which is defined using the non-degenerate sesquilinear form $x_1^py_1+\cdots+x_{n+1}^py_{n+1}$.
- The *Simplectic Polar Space* $W(2d-1,q)$ in $\mathrm{PG}(2d-1,q)$, which is defined using the non-degenerate bilinear form $x_1y_2-x_2y_1+\cdots+x_{2d-1}y_{2d}-x_{2d}y_{2d-1}$.

Note that each of these forms induces a *polarity* π on the projective space. The subspaces of the polar space given by a polarity π is the set of all subspaces U of the projective space (also referred to as the *ambient (projective) space*) with $U\subseteq\pi(U)$ and these are called *totally isotropic*. The set of these subspaces together with induced incidence from the ambient projective space compose the polar space defined by the form.

However, in view of the formal definition of a polar space that we provided above, the respective polar space is given only by the point-line incidence structure induced by the incidence relation of the ambient projective space on the set of totally isotropic points and totally isotropic lines of the projective space.

The maximal totally isotropic subspaces U are called the *generators* of the polar space. It is known that all generators of a given polar space have the same rank and said rank is the same as the rank of the polar space itself.

Finally, we note that we use the notation $U^\perp$ to denote $\pi(U)$ whenever π is known from context, for every subspace $U\leq\mathbb{P}$ the subspace $U\cap U^\perp$ is called the *radical* of both U and the intersection of U with the polar space in question and, with regard to the Hermitian polar spaces, we also remark the following:

Remark 1.4.2. *Let q be a prime power and $n\in\mathbb{N}$. Throughout this work we always consider the Hermitian polar spaces in relation to the given ambient projective space. As such we may use the notions on projective spaces that we introduced earlier, such as the relation $*$ or the dimension-formula.*

Furthermore, a subspace U of $\mathrm{PG}(n,q^2)$ is totally isotropic (with regard to $H(n,q^2)$) if and only if all of its points are totally isotropic (with regard to $H(n,q^2)$). Thus, we may understand $H(n,q^2)$ as a set of points in $\mathrm{PG}(n,q^2)$ and, given the form f as above, a point $\sum_{i=0}^n\lambda_iv_i$ belongs to $H(n,q^2)$ if and only if $\sum_{i=0}^n\lambda_i^{q+1}=0$.

Finally, note that the Hermitian polar spaces $H(2d-1,q^2)$ and $H(2d,q^2)$ of rank d are inherently distinct and here we only study Hermitian polar spaces in projective spaces of even dimension $2d$ and in particular their tight sets *(see Definition 3.0.1).*

1.5 Graphs

Since the main focus of this work is the analysis of a certain type of graph, namely a Kneser graph, we must provide some basic notions on graphs. For a more thorough introduction to graph theory we refer the reader to [30] by West. Note that here, too, q is a prime power and n is a positive integer.

Definition 1.5.1 (Graph)**.** A *graph* $\Gamma = (V, E)$ is a tuple consisting of a *vertex-set* V and an *edge-set* $E \subseteq \{W \subseteq V : |W| = 2\}$. In such a graph the elements of V are called *vertices* and the elements of E are called *edges*. Note that two vertices $v, w \in V$ are said to be *adjacent* or *neighbours* if and only if $\{v, w\}$ is an element of E. For every vertex $v \in V$ we let $N_\Gamma(v) := \{w \in V : \{v, w\} \in E\}$ denote the set of all neighbours of v in Γ.

Furthermore, if a graph Γ is given in an abstract manner, then we still want to be able to refer to its vertex- and edge-set and thus we let $\mathcal{V}(\Gamma)$ and $\mathcal{E}(\Gamma)$ denote these sets, respectively.

Remark 1.5.2. *It is also possible to define a graph on a set of vertices V using an abstract set for the edge-set E and introducing a relation $I \subseteq V \times E$ with the property that every edge is in relation with two (not necessarily distinct) vertices of V. One advantage of that alternative definition is that an edge naturally has a direction. However, in this particular work the definition given above is sufficient and also more convenient to work with.*

Now that we have a settled on a definition for a graph, we introduce the notion of a co-clique of a graph. For sake of completeness we also include the definition of a clique.

Definition 1.5.3 (Clique and Co-Clique)**.** In a graph $\Gamma = (V, E)$ a subset $C \subseteq V$ is called a *clique* (respectively *co-clique*) if we have $\{v, w\} \in E$ (respectively $\{v, w\} \notin E$) for all vertices $v, w \in C$. Note that a co-clique is also called an *independent set* and the number

$$\alpha(\Gamma) := \max\{|W| : W \text{ is a co-clique of } \Gamma\}$$

is called the *independence number* of the graph Γ. Furthermore, the number

$$\omega(\Gamma) := \max\{|W| : W \text{ is a clique of } \Gamma\}$$

is called the *clique number* of the graph Γ.

Note that cliques and the clique number do not play a role in this work. Another property of a graph that is sometimes of interest and closely related to co-cliques is the chromatic number, which we introduce next.

Definition 1.5.4 (Colourings and the Chromatic Number)**.** Let Γ be a graph. A *colouring* of Γ is a map g from $\mathcal{V}(\Gamma)$ to a set such that for all $c \in \mathcal{C}$ the set $g^{-1}(c)$ is a co-clique of Γ. If $g : \mathcal{V}(\Gamma) \to \mathcal{C}$ is a colouring, then the elements of $\mathcal{C}$ are called *colours*. Furthermore, if $\mathcal{C}$ is a set with minimal cardinality such that there exists a colouring $g : \mathcal{V}(\Gamma) \to \mathcal{C}$, then $\chi(\Gamma) := |\mathcal{C}|$ is called the *chromatic number* of Γ. $\chi(\Gamma)$ obviously satisfies

$$\chi(\Gamma) = \min \left\{ |\mathcal{C}| : \mathcal{C} \text{ is a set of co-cliques with } \bigcup_{C \in \mathcal{C}} C = \mathcal{V}(\Gamma) \right\}.$$

Indeed, knowledge of the independence number and the structure of the largest and second largest examples of independent sets is sometimes sufficient to determine the chromatic number of a graph. The main reason for that is that for every graph Γ one has the obvious relation $\chi(\Gamma) \leq \frac{|\mathcal{V}(\Gamma)|}{\alpha(\Gamma)}$. In particular, in [12] by D'haeseleer, Metsch and Werner the authors have been successful in determining the chromatic number using this type of approach in a graph that is very similar to the one that we study in this work. A generalization of that work by the same authors is the content of Section 2.3 on the chromatic number of Kneser graphs of type $(n-1, n)$ in $\mathrm{PG}(2n, q)$ later on, where the chromatic number is determined for $n = 3$ as well as for all $n \geq 4$ for which Conjecture 2.1.19 holds.

1.5.1 The Kneser Graph on Flags of a Projective Space

We now proceed to introduce the Kneser graph, the object that we study in the main part of this work.

Definition 1.5.5 (Flag)**.** Let s be some integer and let $a = (a_1, \ldots, a_s)$ be some tuple. We set $\Omega(a) := \{a_1, \ldots, a_s\}$ and let $\mathrm{len}(a) := s$ denote its *length*. If a tuple a has length 1, then we identify it by its only entry a_1, that is, $(a_1) = a_1$.

Let $U_1, \ldots, U_s$ be subspaces of a given projective space $\mathbb{P}$ such that $U_1 < U_2 < \cdots < U_s$. Then the tuple $f = (U_1, U_2, \ldots, U_s)$ is called a *flag* of *type*

$$\mathrm{type}\,(f) := (\dim (U_1), \dim (U_2), \ldots, \dim (U_s)).$$

For every type d we call a type d' a *sub-type* (of d), if $\Omega(d') \subseteq \Omega(d)$.

Now, let U be a subspace of $\mathbb{P}$ and let $f = (U_1, \ldots, U_s)$ be a flag. We define $U \cap f$ to be the unique flag f' with $\Omega(f') = \{U \cap U_i : i \in \{1, \ldots, s\}\}$. Furthermore, if $U_s \leq U$, then we say that f is *contained in* U and write $f \leq U$ and if $U \leq U_1$, then we say that f is a flag *through* / *containing* U and write $U \leq f$.

Definition 1.5.6 (Kneser graph)**.** Let $\mathbb{P}$ be the projective space $\mathrm{PG}(n, q)$, let $s \geq 1$ as well as $-1 < d_1 < d_2 < \cdots < d_s < n$ be integers and let V be the set of all flags in $\mathbb{P}$ of type $(d_1, d_2, \ldots, d_s)$. Let $E \subseteq \{W \subseteq V : |W| = 2\}$ be defined as follows: For any two flags $f = (U_1, \ldots, U_s)$ and $g = (W_1, \ldots, W_s) \in V$ we have $\{f, g\} \in E$ if and only if for all $i, j \in \{1, \ldots, s\}$ we have

$$U_i \cap W_j = \emptyset \vee \langle U_i, W_j \rangle = \mathbb{P}.. \tag{1.6}$$

The graph Γ with vertex-set V and edge-set E is called the *Kneser graph* of *type* $(d_1, \ldots, d_s)$ in $\mathbb{P}$.

Finally, let C be a vertex-set of Γ, let d' be a sub-type of d and let f' be an arbitrary flag of type d' in $\mathbb{P}$. Then f' is called *saturated* (with regard to / in C) if for all $f \in \mathcal{V}(\Gamma)$ we have $\pi_{d'}(f) = f' \Longrightarrow f \in C$.

Remark 1.5.7. *The condition given in Equation (1.6) is equivalent to each of the following*

$$\begin{aligned} \dim(U_i \cap W_j) &\leq \max\{-1, d_i + d_j - n\}, \\ \dim(\langle U_i, W_j \rangle) &\geq \min\{d_i + d_j + 1, n\}. \end{aligned} \tag{1.7}$$

note that in both of those equations a strict inequality may not occur and any flag $f = (U_1, \ldots, U_s)$ may not be adjacent to itself.

Remark 1.5.8. *This generalization of Kneser graphs as well as a generalization to Buildings and flags of sets is, for example, given in the thesis [16] by Güven. That work also provides some results on co-cliques of Kneser graphs, including the result on point-hyperplane flags, which is also given in [4] by Blokhuis, Brouwer and Güven. For other results on co-cliques and the chromatic number of these kind of Kneser graphs also consider [3] on the size of co-cliques in the Kneser graph of line-plane flags in* $\mathrm{PG}(4, q)$ *by Blokhuis and Brouwer as well as [12] by D'haeseleer, Metsch and Werner on the chromatic number therein; [5] on the chromatic number of q-Kneser graphs as well as [6] on the size of co-cliques in the Kneser graph of point-plane flags in* $\mathrm{PG}(4, q)$*, both by Blokhuis, Brouwer and Szönyi; as well as the thesis [27] by Mussche.*

Furthermore, note that the Erdős-Ko-Rado Theorem for vector spaces, given by Frankl and Wilson in [15], and the Hilton-Milner Theorem given by Blokhuis, Brouwer, Chowdhury, Frankl, Mussche, Patkós and Szönyi in [7] can also be interpreted as results on Kneser graphs on flags of length 1.

We conclude the introduction with a definition of some specific maps in Kneser graphs, that we use quite frequently later on.

Definition 1.5.9. Let C be a subset of the vertex-set of the Kneser graph Γ of type $d := (d_1, \ldots, d_s)$ in $\mathbb{P} := \mathrm{PG}(n, q)$ and let $d' := (d_{i_1}, \ldots, d_{i_r})$ be a sub-type of $(d_1, \ldots, d_s)$. For every flag $f = (U_1, \ldots, U_s) \in C$ let the *projection* of f to d' be the flag

$$\pi_{d'}(f) := (U_{i_1}, \ldots, U_{i_s})$$

and let the projection of C to d' be the set

$$\Pi_{d'}(C) := \{\pi_{d'}(g) : g \in C\}.$$

Furthermore, for every flag $f' \leq \mathbb{P}$ let the *restriction* of C to f' be the set

$$\Delta_{f'}(C) := \{g \in C : U * U', \forall U \in \Omega(g), U' \in \Omega(f')\}$$

and set $\overline{\Delta}_{f'}(C) := C \setminus \Delta_{f'}(C)$. Since we identify tuples of length 1 by their only entry, in the cases of $\mathrm{len}(d') = 1$ or $f' = (W)$ for some subspace $W \leq \mathbb{P}$, we may also write $\pi_{d_{i_1}}(f)$, $\Pi_{d_{i_1}}(C)$, $\Delta_W(C)$ and $\overline{\Delta}_W(C)$ for $\pi_{(d_{i_1})}(f)$, $\Pi_{(d_{i_1})}(C)$, $\Delta_{(W)}(C)$ and $\overline{\Delta}_{(W)}(C)$, respectively.

2 Erdős-Ko-Rado Sets in Kneser Graphs

2.1 Preparation in a more general Setting

Here we collect some lemmata in preparation for Sections 2.2 and 2.4 on plane-solid flags in $\mathrm{PG}(6,q)$ and line-solid flags in $\mathrm{PG}(5,q)$. They provide some linear structure in the graph, that is needed later on.

2.1.1 Kneser Graphs of Type $(d_1,\dots,d_s)$ in $\mathrm{PG}(n,q)$

For this first part let $\mathbb{P}$ be the projective space $\mathrm{PG}(n,q)$ for some prime power q and some dimension $2 \leq n \in \mathbb{N}$. Furthermore, let $s \in \mathbb{N}$ be at least 2, let $-1 < d_1 < \dots < d_s < n$ be integers and let Γ be the Kneser graph of type $d := (d_1,\dots,d_s)$ in $\mathbb{P}$.

Lemma 2.1.1. *Let f' be a flag of some sub-type d' of d and let g be an arbitrary flag of $\mathbb{P}$ such that $\pi_{t'}(f') \cap \pi_t(g) = \emptyset$ or $\langle \pi_{t'}(f'), \pi_t(g)\rangle = \mathbb{P}$ for all $t' \in \Omega(d')$ and $t \in \Omega(e)$ with $e := \mathrm{type}(g)$.*

Then there is a flag f of type d with $\pi_{d'}(f) = f'$ such that $\pi_{t'}(f) \cap \pi_t(g) = \emptyset$ or $\langle \pi_{t'}(f), \pi_t(g)\rangle = \mathbb{P}$ for all $t' \in \Omega(d)$ and $t \in \Omega(e)$.

Proof. Let f'' with $\pi_{d'}(f'') = f'$ be a flag of maximal length such that its type d'' satisfies $\Omega(d'') \subseteq \Omega(d)$ and such that $\pi_{t'}(f'') \cap \pi_t(g) = \emptyset$ or $\langle \pi_{t'}(f''), \pi_t(g)\rangle = \mathbb{P}$ for all $t' \in \Omega(d'')$ and $t \in \Omega(e)$.

If $d'' = d$ then there remains nothing to prove and thus we assume that $\Omega(d)\backslash\Omega(d'') \neq \emptyset$ and we let r' be an integer of this set. We now augment the flag f'' to a flag f''' of type $d''' := d'' \cup \{r'\}$ such that $\pi_{t'}(f''') \cap \pi_t(g) = \emptyset$ or $\langle \pi_{t'}(f'''), \pi_t(g)\rangle = \mathbb{P}$ for all $t' \in \Omega(d''')$ and $t \in \Omega(d)$, in contradiction to the maximal choice of f''.

Let r be the largest integer in $\Omega(e) \cup \{-1\}$ such that $r + r' \leq n-1$ and let $\overline{r}$ be the smallest integer in $\Omega(e) \cup \{n\}$ such that $r < \overline{r}$. For $r \neq -1$ let W be a complement of $\pi_r(g)$ in $\mathbb{P}$ and for $r = -1$ set $W := \mathbb{P}$. Furthermore, for $\overline{r} \neq n$ let $\overline{W}$ be a complement of $\pi_{\overline{r}}(g) \cap W$ in W and for $\overline{r} = n$ let $\overline{W}$ be the empty subspace of $\mathbb{P}$. Then we have $\dim(W) \geq r'$ and $\dim(\overline{W}) \leq r'$ and there is an r'-dimensional subspace U of W with $\overline{W} \leq U$.

Now, for all $t \in \Omega(e)$ with $t \leq r$ we have $U \cap \pi_t(g) \leq W \cap \pi_r(g) = \emptyset$ and for all $t \in \Omega(e)$ with $t > r$ we even have $t \geq \overline{r}$ and thus $\langle U, \pi_t(g)\rangle \geq \langle \overline{W}, \pi_{\overline{r}}(g)\rangle = \mathbb{P}$. Hence the unique flag f''' with $\Omega(f''') = \Omega(f'') \cup \{U\}$ satisfies $\pi_{t'}(f''') \cap \pi_t(g) = \emptyset$ or $\langle \pi_{t'}(f'''), \pi_t(g)\rangle = \mathbb{P}$ for all $t' \in \Omega(d''')$ and $t \in \Omega(d)$, in contradiction to the maximal choice of f'' and hence concluding the proof. □

Lemma 2.1.2. *Let C be a co-clique in Γ, let d' be a sub-type of d with $d' \neq d$ and let f' be a saturated flag of type d'.*

Then for all $g \in C$ there exist $t' \in \Omega(d')$ and $t \in \Omega(d)$ such that $\pi_{t'}(f') \cap \pi_t(g) \neq \emptyset$ and $\langle \pi_{t'}(f'), \pi_t(g) \rangle \neq \mathbb{P}$.

Proof. Assume that there is $g \in C$ such that for all $t' \in \Omega(d')$ and $t \in \Omega(d)$ we have $\pi_{t'}(f') \cap \pi_t(g) = \emptyset$ or $\langle \pi_{t'}(f'), \pi_t(g) \rangle = \mathbb{P}$. Then, according to Lemma 2.1.1, there is a flag $f \in \mathcal{V}(\Gamma)$ with $\pi_{d'}(f) = f'$ such that $\pi_{t'}(f) \cap \pi_t(g) = \emptyset$ or $\langle \pi_{t'}(f), \pi_t(g) \rangle = \mathbb{P}$ for all $t, t' \in \Omega(d)$, that is, a flag f which is adjacent to g. Now, since f' is saturated we have $f \in C$, in contradiction to $g \in C$. □

Lemma 2.1.3. *Let C be a maximal co-clique in Γ, let d' be a sub-type of d with $d' \neq d$ and let f' be a flag of type d'.*

Then f' is saturated in C if and only if for all $g \in C$ there exist $t' \in \Omega(d')$ and $t \in \Omega(d)$ such that $\pi_{t'}(f') \cap \pi_t(g) \neq \emptyset$ and $\langle \pi_{t'}(f'), \pi_t(g) \rangle \neq \mathbb{P}$.

Proof. If f' is saturated then Lemma 2.1.2 proves the claim and if for all $g \in C$ there exist $t' \in \Omega(d')$ and $t \in \Omega(d)$ such that $\pi_{t'}(f') \cap \pi_t(g) \neq \emptyset$ and $\langle \pi_{t'}(f'), \pi_t(g) \rangle \neq \mathbb{P}$, then for all $f \in \mathcal{V}(\Gamma)$ with $\pi_{d'}(f) = f'$ we know that f and g are not adjacent in Γ and thus, because C is maximal, $f \in C$. □

Lemma 2.1.4. *Let C be a co-clique in Γ, let d' be a sub-type of d and let f' and g' be two vertices of the Kneser graph Γ' of type d' in $\mathbb{P}$.*

If f' and g' are saturated in C, then f' and g' are not adjacent in Γ'.

Proof. Let f' and g' be saturated in C and assume that f' and g' are adjacent in Γ. Then for all $t, t' \in \Omega(d')$ we have $\pi_t(f') \cap \pi_{t'}(g') = \emptyset$ or $\langle \pi_t(f'), \pi_{t'}(g') \rangle = \mathbb{P}$. According to Lemma 2.1.1 there is a flag f of type d with $\pi_{d'}(f) = f'$ such that $\pi_t(f) \cap \pi_{t'}(g') = \emptyset$ or $\langle \pi_t(f), \pi_{t'}(g') \rangle = \mathbb{P}$ for all $t \in \Omega(d)$ and $t' \in \Omega(d')$. Furthermore, again according to Lemma 2.1.1 there is a flag g of type d with $\pi_{d'}(g) = g'$ such that $\pi_t(f) \cap \pi_{t'}(g) = \emptyset$ or $\langle \pi_t(f), \pi_{t'}(g) \rangle = \mathbb{P}$ for all $t, t' \in \Omega(d)$. Therefore, f and g are adjacent in Γ, in contradiction to f' and g' being saturated in C. □

Lemma 2.1.5. *For $2 \leq m \in \mathbb{N}$ let $f_1, \dots, f_m$ be distinct flags of a maximal co-clique C in Γ, let t be an element of $\Omega(d)$ and let d' be a sub-type of d such that $\Omega(d') = \Omega(d) \setminus \{t\}$. Furthermore, let h be a flag of type d' and let $W \leq \mathbb{P}$ be a $(t-1)$-dimensional subspace with $\pi_{d'}(f_k) = h$ and $W \leq \pi_t(f_k)$ for all $k \in \{1, \dots, m\}$.*

Then we have $f \in C$ for all $f \in \mathcal{V}(\Gamma)$ with $\pi_{d'}(f) = h$ and $W \leq \pi_t(f) \leq \langle \pi_t(f_k) : k \in \{1, \dots, m\} \rangle$.

Proof. Set $\widehat{W} := \langle \pi_t(f_i) : i \in \{1, \dots, m\} \rangle$ and $\widehat{t} := \dim(\widehat{W})$. According to Lemma 1.2.38 there exists $\Xi \subseteq \{\pi_t(f_i) : i \in \{1, \dots, m\}\}$ such that $\widehat{W} = \langle \Xi \rangle$ and $|\Xi| = \widehat{t} - t + 1$ and without loss of generality we may assume that $\Xi = \{\pi_t(f_i) : i \in \{1, \dots, m\}\}$ and thus $|\Xi| = m$ hold.

Now, in contrary to the claim, assume that there is a flag $f \in \mathcal{V}(\Gamma)$ with $f \notin C$, $\pi_{d'}(f) = h$ and $W \leq \pi_t(f) \leq \widehat{W}$. Since C is maximal this implies that there must be a flag $g \in C$ which is adjacent to f. For the remainder of this passage let $i \in \{1, \ldots, m\}$ be arbitrary but fixed. We know from $f_i \in C$ that there must be $k_i, r_i \in \Omega(d)$ such that $\pi_{k_i}(f_i) \cap \pi_{r_i}(g)$ does not satisfy the condition given in Equation (1.7). Since f and g are non-adjacent and since $\pi_{d'}(f) = h = \pi_{d'}(f_i)$ we know that $k_i \notin \Omega(d')$ and thus $k_i = t$. Furthermore, if $\pi_t(f_i) \cap \pi_{r_i}(g) \leq W \leq \pi_t(f)$, then $\pi_t(f) \cap \pi_{r_i}(g)$ would not satisfy the condition given in Equation (1.7) either, in contradiction to f and g being adjacent. In conclusion, from the arbitrary choice of $i \in \{1, \ldots, m\}$ we have

$$\forall i \in \{1, \ldots, m\} : \pi_t(f_i) \cap \pi_{r_i}(g) \not\leq W. \tag{2.1}$$

Now, set $r := \max\{r_1, \ldots, r_m\}$, let $\iota \in \{1, \ldots, m\}$ be such that $r = r_\iota$ and note that this implies $\pi_{r_i}(g) \leq \pi_r(g)$ for all $i \in \{1, \ldots, m\}$. Furthermore, for all $i \in \{1, \ldots, m\}$ set $\xi_i := \pi_t(f_i) \cap \pi_r(g)$ and let $\overline{W}$ be a complement of $\xi_\iota \cap W$ in W. Again, for the remainder of this passage let $i \in \{1, \ldots, m\}$ be arbitrary but fixed. Using Equation (2.1) and $\pi_t(f_i) \cap \pi_{r_i}(g) \leq \xi_i$ we have $\xi_i \not\leq W$. Since W is a hyperplane of $\pi_t(f_i)$ and since $\xi_i \leq \pi_t(f_i)$ we may conclude that $\xi_i \cap W$ is a hyperplane of ξ_i. Hence,

$$\xi_i \cap W = (\pi_t(f_i) \cap \pi_r(g)) \cap W = (\pi_t(f_i) \cap W) \cap \pi_r(g) = W \cap \pi_r(g)$$

and thus $\xi_i \cap W = \xi_\iota \cap W$. Since W is a hyperplane of $\pi_t(f_i)$ Equation (2.1) implies

$$\pi_t(f_i) = \langle \pi_t(f_i) \cap \pi_{r_i}(g), W \rangle = \langle \xi_i, W \rangle = \langle \xi_i, \overline{W} \rangle$$

and from the arbitrary choice of $i \in \{1, \ldots, m\}$ we now have

$$\forall \emptyset \neq I \subseteq \{1, \ldots, m\} : \langle \pi_t(f_i) : i \in I \rangle = \langle \overline{W}, \xi_i : i \in I \rangle = \langle \overline{W}, \langle \xi_i : i \in I \rangle \rangle. \tag{2.2}$$

Thus, if there was a non-empty subset $I \subseteq \{1, \ldots, m\}$ such that $\dim(\langle \xi_i : i \in I \rangle) < \dim(\xi_\iota \cap W) + |I|$, then, using Equation (2.2), it follows that

$$\begin{aligned} \dim(\langle \pi_t(f_i) : i \in I \rangle) &\leq \dim(\overline{W}) + \dim(\langle \xi_i : i \in I \rangle) + 1 \\ &< \dim(\overline{W}) + \dim(\xi_\iota \cap W) + |I| + 1 = \dim(W) + |I|, \end{aligned}$$

in contradiction to Lemma 1.2.39. Consequently $\langle \xi_i : i \in \{1, \ldots, m\} \rangle \leq \pi_r(g) \cap \widehat{W}$ has dimension $\dim(\xi_\iota \cap W) + m = \dim(\xi_\iota) - 1 + m$, $\widehat{W}$ has dimension $\widehat{t} = t - 1 + m$ and $\pi_t(f) \leq \widehat{W}$ has dimension t. Therefore,

$$\begin{aligned} \dim(\pi_t(f) \cap (\pi_r(g) \cap \widehat{W})) &= \dim(\pi_t(f)) + \dim(\pi_r(g) \cap \widehat{W}) - \dim(\langle \pi_t(f), \pi_r(g) \cap \widehat{W} \rangle) \\ &\overset{(*)}{\geq} t + \dim(\xi_\iota) - 1 + m - (t - 1 + m) = \dim(\xi_\iota), \end{aligned}$$

where the step marked with $(*)$ uses $\langle \pi_t(f), \pi_r(g) \cap \widehat{W} \rangle \leq \widehat{W}$. Now, since $\pi_t(f_\iota) \cap \pi_r(g)$ does not satisfy the condition given in Equation (1.7), neither does $\pi_t(f) \cap \pi_r(g)$. This is in contradiction to the choice of g as a neighbour of f, concluding the proof. □

Corollary 2.1.6. *Let C be a maximal co-clique in Γ, let $i \in \{1, \ldots, s\}$ be a fixed index, let W be a subspace of dimension $d_i - 1$ and let g be an arbitrary flag of the type d' defined by $\Omega(d') = \Omega(d) \setminus \{d_i\}$. Then*

$$|\{f \in \Delta_W(C) : \pi_{d'}(f) = g\}| \in \{\mathfrak{s}_q[j] : j \in \{-1, \ldots, n - d_i\}\}.$$

Proof. Set $C' := \{f \in \Delta_W(C) : \pi_{d'}(f) = g\}$. For $|C'| \in \{0, 1\} = \{\mathfrak{s}_q[-1], \mathfrak{s}_q[0]\}$ there is nothing to prove. Thus, let C' contain at least two distinct flags and let t be the dimension of the subspace $\widehat{W} := \langle \Pi_{d_i}(C') \rangle$. According to Lemma 2.1.5 every subspace $U \leq \widehat{W}$ with $W \leq U$ satisfies $(U_1, \ldots, U_{i-1}, U, U_{i+1}, \ldots, U_s) \in C'$ and the number of such subspaces U is given by

$$\mathfrak{s}_q[d_i - 1, d_i, t] \overset{1.2.35 \text{ ii)}}{=} \mathfrak{s}_q[0, t - d_i] = \mathfrak{s}_q[t - d_i].$$

Therefore, we have $|C'| = \mathfrak{s}_q[t - d_i]$ with $t \in \{d_i, \ldots, n\}$, which implies the claim. □

Lemma 2.1.7. *Let H be a subspace of $\mathbb{P}$ and set $d_H := \dim(H)$ as well as $\overline{d_H} := n - d_H$. Furthermore, set $d_0 := -1$, let r be the largest integer in $\{0, \ldots, s\}$ with $2d_r + 1 \leq d_H$ and let $\overline{d_H} \leq 2d_j + 1 - n$ hold for all $j \in \{r + 1, \ldots, s\}$. Finally, let C be a non-empty independent set of Γ such that $C' := \{H \cap f : f \in C\}$ is a set of flags of type*

$$d' := \begin{cases} (d_1, \ldots, d_r, d_{r+1} - \overline{d_H}, \ldots, d_s - \overline{d_H}) & \text{for } d_{r+1} - \overline{d_H} > d_r, \\ (d_1, \ldots, d_r, d_{r+2} - \overline{d_H}, \ldots, d_s - \overline{d_H}) & \text{otherwise.} \end{cases}$$

Then C' is an independent set of the Kneser graph Γ' of type d' in H.

Proof. First we note that for $r = s$ the claim is trivial and thus we may assume that $r < s$. We prove the claim via contradiction and thus assume that C' is not an independent set of Γ'. Then there are two adjacent flags $f_1' \neq f_2' \in C'$ of Γ' and two non-adjacent flags $f_1 \neq f_2 \in C$ of Γ with $H \cap f_1 = f_1'$ and $H \cap f_2 = f_2'$. Since f_1 and f_2 are non-adjacent flags of Γ, there exist $m_1, m_2 \in \{1, \ldots, s\}$ with $\pi_{d_{m_1}}(f_1) \cap \pi_{d_{m_2}}(f_2) \neq \emptyset$ and $\langle \pi_{d_{m_1}}(f_1), \pi_{d_{m_2}}(f_2) \rangle \neq \mathbb{P}$. Since f_1' and f_2' are adjacent and since $d_H \geq 2d_i + 1$ for all $i \in \{1, \ldots, r\}$ we know that $m_1 > r$ or $m_2 > r$ and without loss of generality we may assume that $m_1 > r$. We set $d'_{m_1} := \dim(\pi_{d_{m_1}}(f_1) \cap H)$ and $d'_{m_2} := \dim(\pi_{d_{m_2}}(f_2) \cap H)$. Since $m_1 > r$ we know that $\pi_{d_{m_1}}(f_1)$ and H span $\mathbb{P}$ and from $\langle \pi_{d_{m_1}}(f_1), \pi_{d_{m_2}}(f_2) \rangle \neq \mathbb{P}$ we have

$$H \neq \langle \pi_{d_{m_1}}(f_1) \cap H, \pi_{d_{m_2}}(f_2) \cap H \rangle = \langle \pi_{d'_{m_1}}(f_1'), \pi_{d'_{m_2}}(f_2') \rangle.$$

Now, if $m_2 \leq r$ then $\pi_{d_{m_2}}(f_2) \leq H$ and

$$\emptyset \neq \pi_{d_{m_1}}(f_1) \cap \pi_{d_{m_2}}(f_2) = \pi_{d'_{m_1}}(f_1') \cap \pi_{d'_{m_2}}(f_2') = \emptyset,$$

a contradiction. Hence, we have $m_2 > r$, too. However, $\pi_{d'_{m_1}}(f_1')$ and $\pi_{d'_{m_2}}(f_2')$ are skew and thus we have $\dim(\pi_{d_{m_1}}(f_1) \cap \pi_{d_{m_2}}(f_2)) \leq \overline{d_H} - 1 \leq 2d_{r+1} - n$, which implies

$$\dim(\langle \pi_{d_{m_1}}(f_1), \pi_{d_{m_2}}(f_2) \rangle) \geq \underbrace{d_{m_1} + d_{m_2} - 2d_{r+1}}_{\geq 0, \text{ since } m_1, m_2 \geq r+1} + n \geq n,$$

a contradiction to $\langle \pi_{d_{m_1}}(f_1), \pi_{d_{m_2}}(f_2) \rangle \neq \mathbb{P}$. □

Lemma 2.1.8. *If for all $i \in \{1, \ldots, s\}$ we have $d_i + d_{s-i+1} = n - 1$, then for all $f, g \in \mathcal{V}(\Gamma)$ we have*

$$\{f, g\} \in \mathcal{E}(\Gamma) \Longleftrightarrow \forall j \in \{1, \ldots, s\} : \pi_{d_i}(f) \cap \pi_{d_{s-i+1}}(g) = \emptyset.$$

Proof. Let $d_i + d_{s-i+1} = n - 1$ for all $i \in \{1, \ldots, s\}$. Note that for dimensional reasons this implies that $\pi_{d_i}(f)$ and $\pi_{d_j}(g)$ span $\mathbb{P}$ if and only if their intersection is empty.

First, if f and g are adjacent vertices of Γ, then we have $\pi_{d_i}(f) \cap \pi_{d_j}(g) = \emptyset$ or $\langle \pi_{d_i}(f), \pi_{d_j}(g) \rangle = \mathbb{P}$ for all $i, j \in \{1, \ldots, s\}$. Hence, for all $i \in \{1, \ldots, s\}$ we have $\pi_{d_i}(f) \cap \pi_{d_{s-i+1}}(g) = \emptyset$, as claimed.

Now, let f and g be non-adjacent vertices. Then there exist $i, j \in \{1, \ldots, s\}$ such that $\pi_{d_i}(f) \cap \pi_{d_j}(g) \neq \emptyset$ and $\langle \pi_{d_i}(f), \pi_{d_j}(g) \rangle \neq \mathbb{P}$. If $j \leq s-i+1$, then from $\pi_{d_j}(g) \leq \pi_{d_{s-i+1}}(g)$ we have

$$\emptyset \neq \pi_{d_i}(f) \cap \pi_{d_j}(g) \leq \pi_{d_i}(f) \cap \pi_{d_{s-i+1}}(g)$$

and for dimensional reasons this also implies $\mathbb{P} \neq \langle \pi_{d_i}(f), \pi_{d_{s-i+1}}(g)$. On the other hand, if $j > s - i + 1$, then from $\pi_{d_{s-i+1}}(g) \leq \pi_{d_j}(g)$ we have

$$\mathbb{P} \neq \langle \pi_{d_i}(f), \pi_{d_j}(g) \rangle \geq \langle \pi_{d_i}(f), \pi_{d_{s-i+1}}(g) \rangle$$

and again this also implies $\emptyset \neq \pi_{d_i}(f) \cap \pi_{d_{s-i+1}}(g)$, concluding the proof. □

2.1.2 Kneser Graphs of Type $(n-1, n)$ in $\mathrm{PG}(2n, q)$

For this section let $\mathbb{P}$ be the projective space $\mathrm{PG}(2n, q)$ for some prime power q and some positive integer n and let Γ be the Kneser graph of type $(n-1, n)$ in $\mathbb{P}$.

Corollary 2.1.9. *Let V be a saturated subspace of $\mathbb{P}$ of dimension n in a given co-clique C of Γ. Then for any flag $(U', V') \in C$ we have $U' \cap V \neq \emptyset$.*

Proof. Lemma 2.1.2 shows that for any flag $(U', V') \in C$ we have $U' \cap V \neq \emptyset$ or $\dim(V' \cap V) \geq 1$. However, since U' is a hyperplane of V' we know that the latter also implies $U' \cap V' \neq \emptyset$, as claimed. □

Corollary 2.1.10. *Let U be a saturated subspace of $\mathbb{P}$ of dimension $n-1$ in a given co-clique C. Then for any flag $(U', V') \in C$ we have $V' \cap U \neq \emptyset$.*

Proof. On the one hand this, too, is implied by Lemma 2.1.2 and on the other hand it is also the dual statement of Corollary 2.1.9 and the type under consideration is self-dual. □

Corollary 2.1.11. *If V and V' are two saturated n-dimensional subspaces in a given co-clique C, then there is a line l in the intersection of V and V'.*

Furthermore, if U and U' are two saturated $(n-1)$-dimensional subspaces in a given co-clique C, then they are contained in a common subspace of co-dimension 2, that is, they share a point.

Proof. In Lemma 2.1.4 we have seen, that if V and V' are saturated, then V and V' must be non-adjacent in the Kneser graph Γ' of type n in $\mathbb{P}$. Two n-dimensional subspaces of Γ' are adjacent if and only if they span the entire space $\mathbb{P}$. In this case this is equivalent to $\dim(V \cap V') = 0$ and thus there must be a line $l \leq V \cap V'$.

The second claim is the dual statement of the first claim and it follows analogously or from the fact that the type under consideration is self-dual. □

Corollary 2.1.12. *Let V be a subspace of $\mathbb{P}$ of dimension n in a given maximal co-clique C. If we have $V \cap U' \neq \emptyset$ for all $U' \in \Pi_{n-1}(C)$, then V is saturated.*

Proof. This is implied by Lemma 2.1.3. □

Lemma 2.1.13. *If C is a maximal co-clique of Γ and H is a hyperplane of $\mathbb{P}$ such that $U \in \Pi_{n-1}(C)$ implies $U \leq H$, then every n-dimensional subspace of H is saturated in C.*

Proof. Let H be a hyperplane of $\mathbb{P}$ such that $U \leq H$ for all subspaces $U \in \Pi_{n-1}(C)$. If V is an n-dimensional subspace of H, then, for dimensional reasons, we have $V \cap U \neq \emptyset$ for all $(n-1)$-dimensional subspaces U of H. Thus we have $U \cap V \neq \emptyset$ for all $U \in \Pi_{n-1}(C)$ and know from Lemma 2.1.12 that V is saturated. □

Lemma 2.1.14. *For any co-clique C and any hyperplane $H \leq \mathbb{P}$ there are at most*

$$\mathfrak{s}_q[n-1, 2n-2] \cdot q^n$$

flags $(U_{n-1}, U_n) \in C$ with $U_{n-1} \leq H$ and $U_n \not\leq H$.

Proof. According to Lemma 2.1.7 the set $\mathcal{U}$ of $(n-1)$-dimensional subspaces $U_{n-1} \leq H$ which occur in a flag $(U_{n-1}, U_n) \in C$ with $U_n \not\leq H$ forms an independent set in the Kneser graph of type d_{n-1} in H. According to [15] by Frankl and Wilson we have $|\mathcal{U}| \leq \mathfrak{s}_q[n-1, 2n-2]$ and through any one of those $(n-1)$-dimensional subspaces there are

$$\mathfrak{s}_q[n-1, n, 2n] - \mathfrak{s}_q[n-1, n, 2n-1] = q^n$$

n-dimensional subspaces which are not a subspace of H, as claimed. □

Example 2.1.15. Let H be a hyperplane of $\mathbb{P}$ and let $\mathcal{U}$ be a set of $(n-1)$-dimensional subspaces $U \leq H$ such that $\mathcal{U}$ is a maximal independent set of the Kneser graph Γ' of type $n-1$ in H. Furthermore, let C be the set of all flags $(U, V) \in \mathcal{V}(\Gamma)$ such that $V \leq H$ or $U \in \mathcal{U}$. Then C is a maximal independent set of Γ of size

$$\mathfrak{s}_q[n, 2n-1] \cdot \mathfrak{s}_q[n-1, n] + |\mathcal{U}| \cdot q^n.$$

Proof. Let $f_1 = (U_1, V_1)$ and $f_2 = (U_2, V_2)$ be two arbitrary flags of C. If $V_i \leq H$ for some $i \in \{1, 2\}$, then, for dimensional reasons, we have $V_i \cap U_{3-i} \neq \emptyset$ as well as $\langle V_i, U_{3-i} \rangle = H \neq \mathbb{P}$ and the flags are non-adjacent. If $V_1, V_2 \not\leq H$, then $U_1, U_2 \in \mathcal{U}$ are non-adjacent flags of Γ' and as such $U_1 \cap U_2 \neq \emptyset$ with $\langle U_1, U_2 \rangle \leq H \neq \mathbb{P}$ and the flags f_1

and f_2 are non-adjacent as well. Consequently C is an independent set of Γ. It remains to show that C is maximal and to determine its cardinality.

For the maximality of C let C' be an independent set of Γ with $C \subseteq C'$. According to Lemma 2.1.7 the set

$$\mathcal{U}' := \{U \in \Pi_{n-1}(C') : U \leq H, \exists V \leq \mathbb{P} \text{ with } V \not\leq H \wedge (U, V) \in C\}$$

is an independent set of the Kneser graph of type $n-1$ in H. Since U is maximal and obviously a subset of $\mathcal{U}'$ we have $\mathcal{U}' = \mathcal{U}$. Now, for every $(n-1)$-dimensional subspace $U \leq \mathbb{P}$ with $U \not\leq H$ we have $\dim(U \cap H) = n-2$ and there is an n-dimensional complement V' of $U \cap H$ in H, which by construction of C is saturated in both C and C'. According to Corollary 2.1.9 we then have $U \notin \Pi_{n-1}(C')$, which implies $C' \subseteq C$ and thus C is maximal.

Now, for the size of C note that we have $\mathfrak{s}_q[n, 2n-1]$ choices for an n-dimensional subspace $V \leq H$ and subsequently $\mathfrak{s}_q[n-1, n]$ choices for an $(n-1)$-dimensional subspace $U \leq V$, providing $\mathfrak{s}_q[n, 2n-1] \cdot \mathfrak{s}_q[n-1, n]$ flags $(U, V) \leq H$. Finally, there are $|\mathcal{U}|$ choices for an $(n-1)$-dimensional subspace in $\mathcal{U}$ and for each of those there are $\mathfrak{s}_q[n-1, n, 2n]$ choices for an n-dimensional subspace $U \leq V$, but $\mathfrak{s}_q[n-1, n, 2n-1]$ of those are contained in H and have been counted already. □

Remark 2.1.16. *The independent set C given in Example 2.1.15 has cardinality*

$$|C| \leq \mathfrak{s}_q[n, 2n-1] \cdot \mathfrak{s}_q[n-1, n] + \mathfrak{s}_q[n-1, 2n-2] \cdot q^n$$

with equality if and only if $\mathcal{U}$ is not only a maximal independent set of Γ' but also an independent set of Γ' of maximal size. According to [15] by Frankl and Wilson those are the sets of all $(n-1)$-dimensional subspaces of H which contain a common point P of H and the sets of all $(n-1)$-dimensional subspaces which are all contained in a hyperplane H' of H.

Note that any set C that we construct using such an independent set of Γ' of maximal size was already provided in [3, Section 5.1] by Blokhuis and Brouwer. Also note that, regardless of the choice of $\mathcal{U}$, the set $\mathcal{C}$ has size

$$|C| > \mathfrak{s}_q[n, 2n-1] \cdot \mathfrak{s}_q[n-1, n].$$

Example 2.1.17. The examples given in 2.1.15 yield a second set of examples using the dual construction of the one given there. In particular, the dual C^* of each independent set C that we have described there is also a maximal independent set of the Kneser graph of type $(n-1, n)$ in $\mathbb{P}$ and the independent sets of maximal size were already given in [3] by Blokhuis and Brouwer, too.

Remark 2.1.18. *Any maximal independent set of Γ which either contains all flags in a given hyperplane or all flags through a given point is given by one of those two sets of examples.*

Conjecture 2.1.19. *For every integer $n \geq 2$ there are integers α_n and q_n such that every maximal co-clique of the Kneser graph of Type $(n-1, n)$ in $\mathrm{PG}(2n, q)$ with $q > q_n$ is given by Examples 2.1.15 and 2.1.17, or has at most $\alpha_n q^{n^2+n-2}$ elements.*

Remark 2.1.20. *This Conjecture is proven for $n = 2$ by Blokhuis and Brouwer in [3] (also consider the Appendix of [12] by D'haeseleer, Metsch and Werner) and $n = 3$ in Section 2.2.*

2.1.3 Kneser Graphs on Flags of Length 1

Here we collect some results on Kneser graphs that have been proven already and will be used in this work.

We start by considering sets of planes with pairwise 1-dimensional intersection. Note that the result given in Lemma 2.1.21 and Corollary 2.1.22 thereafter were proven (more generally) in [15] by Frankl and Wilson. However, in this special case the proof is easy and we provide it below.

Lemma 2.1.21. *Let n be an integer, let q be a prime power and let $\mathcal{E}$ be a set of planes of $\mathbb{P} := \mathrm{PG}(n, q)$ such that any two planes in $\mathcal{E}$ share a line. Then either there is a line $l \leq \mathbb{P}$ with $l \leq E$ for all $E \in \mathcal{E}$ or $|\mathcal{E}| \leq \mathfrak{s}_q[2, 3]$.*

Proof. Let there be no line $l \leq \mathbb{P}$ such that $l \leq E$ for all $E \in \mathcal{E}$ and let E_1 and E_2 be in $\mathcal{E}$. Set $l := E_1 \cap E_2$ and let E_3 be such that $l \not\leq E_3$. Then $E_3 \cap E_1 \neq E_3 \cap E_2$ and thus $E_3 = \langle E_3 \cap E_1, E_3 \cap E_2 \rangle \leq \langle E_1, E_2 \rangle$. Now, for all $E \in \mathcal{E}$ the set $\mathcal{L}_E := \{E \cap E_1, E \cap E_2, E \cap E_3\}$ satisfies $|\mathcal{L}_E| \geq 2$ and therefore we have $E = \langle \mathcal{L}_E \rangle \leq \langle E_1, E_2, E_3 \rangle = \langle E_1, E_2 \rangle$. Since $\langle E_1, E_2 \rangle$ is a solid this implies $|\mathcal{E}| \leq \mathfrak{s}_q[2, 3]$, as claimed. □

Corollary 2.1.22. *Let $n \geq 5$ be an integer, let q be a prime power and let $\mathcal{E}$ be a set of planes of $\mathbb{P} := \mathrm{PG}(n, q)$ such that any two planes in $\mathcal{E}$ share a line. Then $|\mathcal{E}| \leq \mathfrak{s}_q[n-2]$.*

Proof. Either there is a line l with $l \in E$ for all $E \in \mathcal{E}$, which implies $|\mathcal{E}| \leq \mathfrak{s}_q[1, 2, n] = \mathfrak{s}_q[n-2]$, or Lemma 2.1.21 implies $|\mathcal{E}| \leq \mathfrak{s}_q[2, 3] = \mathfrak{s}_q[3] \leq \mathfrak{s}_q[n-2]$. □

The following two results are given in [5, Theorem 3.1 & Theorem 6.1] by Blokhuis, Brouwer and Szönyi. Theorem 2.1.26 thereafter is implied by the dual statement of [7, Theorem 1.4] by Blokhuis et al.

Note that the independence number given in these theorems was already determined by Frankl and Wilson in [15] without a bound on second largest examples.

Theorem 2.1.23 ([5, Theorem 3.1])**.** *Let C be a co-clique of the Kneser graph of type $n \in \mathbb{N}$ in $\mathrm{PG}(2n + 1, q)$. If*

$$|C| > \left(1 + \frac{1}{q}\right) \cdot \mathfrak{s}_q[n-1] \cdot \mathfrak{s}_q[n]^n,$$

then there either is a point P with $P \in f$ for all $f \in C$, or a hyperplane H with $f \leq H$ for all $f \in C$.

Theorem 2.1.24 ([5, Theorem 6.1])**.** *The independence number $\alpha(\Gamma)$ of the Kneser graph Γ of type 2 in $\mathrm{PG}(5, q)$ is given by*

$$\alpha(\Gamma) = \mathfrak{s}_q[2, 4] = q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1.$$

The independent sets of maximal size are the sets $\mathfrak{S}[P, 2, \mathrm{PG}(5, q)]$ and $\mathfrak{S}[2, H]$ for points P and hyperplanes H of $\mathrm{PG}(5, q)$. Every other maximal independent set has cardinality at most $q^5 + 2q^4 + 3q^3 + 2q^2 + q + 1$.

Corollary 2.1.25. *Let Γ be the Kneser graph of type 2 in $\mathbb{P} := \mathrm{PG}(5, q)$ with $q \geq 3$ and let $v \in \mathcal{V}(\Gamma)$ be an arbitrary vertex. Furthermore, let Γ' be the graph induced by Γ on the set $N_\Gamma(v)$ of neighbours of v and let $\mathcal{E}$ be an independent set of Γ'.*

Then we have $|\mathcal{E}| \leq \max(q^6, \xi)$ with $\xi := q^5 + 2q^4 + 3q^3 + 2q^2 + q + 1$ and $|\mathcal{E}| > \xi$ occurs if and only if $\mathcal{E}$ is either a set of planes of $\mathcal{V}(\Gamma')$ through a given point $P \in \mathbb{P} \setminus v$ or a set of planes of $\mathcal{V}(\Gamma')$ in a given hyperplane $H \leq \mathbb{P}$ with $v \not\leq H$.

Proof. The set $\mathcal{E}$ is an independent set of Γ, too, and, according to Theorem 2.1.24, there is a point $P \in \mathbb{P}$ with $P \in E$ for all $E \in \mathcal{E}$ or a hyperplane $H \leq \mathbb{P}$ with $E \leq H$ for all $E \in \mathcal{E}$, or $|\mathcal{E}| \leq \xi$.

Now, if $\mathcal{E}$ is a set of planes of $\mathcal{V}(\Gamma')$ through a given point $P \in \mathbb{P}$, then either $P \notin v$ and there are at most $\mathfrak{s}_q[2, 0, 2, 5] = q^6$ planes through P which are adjacent to v (that is, they do not meet the plane v), or $P \in v$ and $\mathcal{E} = \emptyset$. Furthermore, if $\mathcal{E}$ is a set of planes of $\mathcal{V}(\Gamma')$ in a given hyperplane $H \leq \mathbb{P}$, then either $v \not\leq H$ and there are at most $\mathfrak{s}_q[1, -1, 2, 4] = q^6$ planes in H which are adjacent to v (that is, they do not meet the line $v \cap H$), or $v \leq H$ and $\mathcal{E} = \emptyset$. $\square$

Theorem 2.1.26 ([7, Theorem 1.4]). *For $q \geq 3$ the independence number $\alpha(\Gamma)$ of the Kneser graph Γ of type 3 in $\mathrm{PG}(6, q)$ is given by*

$$\alpha(\Gamma) = \mathfrak{s}_q[3, 5] = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1.$$

The independent sets of maximal size are the sets $\mathfrak{S}[3, H]$ for hyperplanes H of $\mathrm{PG}(6, q)$. Every other maximal independent set has cardinality at most $q^6 + 2q^5 + 3q^4 + 3q^3 + 2q^2 + q + 1$.

Remark 2.1.27. *Note that the theorem given in [7] by Blokhuis et al yields a Hilton-Milner type result for arbitrary dimension under given circumstances and not only for the case that is stated in Theorem 2.1.26.*

2.2 The Independence Number of Kneser Graphs of Type $(2,3)$ in $\mathrm{PG}(6,q)$

For this section let $\mathbb{P}$ be the projective space $\mathrm{PG}(6,q)$ for some prime power q and let Γ be the Kneser graph of type $(2,3)$ in $\mathbb{P}$. Recall that Lemma 2.1.8 proves that two flags (E,S) and (E',S') of Γ are adjacent if and only if $E \cap S' = \emptyset = E' \cap S$.

We show that any maximal independent set of Γ of size larger than roughly $27q^{10}$ (a more precise formulation can be found in Theorem 2.2.14) is given by Example 2.1.15 or Example 2.1.17 above and thus, for $q \geq 27$, we determine the independence number of Γ.

First, we have some general notions and then the remainder of the proof is split into three parts, where we first consider two special cases that may occur.

Remark 2.2.1. *Examples 2.1.15 and 2.1.17 provide independent sets of Γ of size at least*

$$q^{11} + 2q^{10} + 4q^9 + 6q^8 + 8q^7 + 9q^6 + 9q^5 + 8q^4 + 7q^3 + 4q^2 + 2q + 1.$$

Lemma 2.2.2. *Let C be an independent set of Γ. For all $P \in \mathbb{P}$ the set*

$$\mathcal{S}_P := \{\pi_3(f) : f \in C \text{ with } P \in \pi_3(f) \setminus \pi_2(f)\}$$

is an independent set of the Kneser graph of type 3 on $\mathbb{P}$ and we have $|\mathcal{S}_P| \leq \mathfrak{s}_q[2,4]$.

Proof. Let $P \in \mathbb{P}$ be a point and let $f = (E,S)$ and $f' = (E',S')$ be two flags such that $P \notin E, E'$ and $P \in S, S'$. If $P = S \cap S'$, then $S \cap E' = \emptyset = E \cap S'$ and f and f' are adjacent flags of Γ. Therefore, any two solids $S, S' \in \mathcal{S}_P$ satisfy $\dim(S \cap S') \geq 1$, proving that $\mathcal{S}_P$ is an independent set of the Kneser graph of type 3 on $\mathbb{P}$.

Now, in the dual space $\mathbb{P}^\vee$ the set $\mathcal{S}_P$ is an independent set of the Kneser graph of type 2 on $\mathbb{P}^\vee$. Furthermore, from $P \in S$ for all $S \in \mathcal{S}_P$ we even know that in the dual space $\mathbb{P}^\vee$ every plane $E \in \mathcal{S}_P$ is a subspace of the hyperplane P of $\mathbb{P}^\vee$, that is, $\mathcal{S}_P$ is an independent set of the Kneser graph of type 2 in the hyperplane P of $\mathbb{P}^\vee$. Finally, we may apply Theorem 2.1.24 and have $|\mathcal{S}_P| \leq \mathfrak{s}_q[2,4]$, as claimed. ☐

Lemma 2.2.3. *Let $\mathcal{C}$ be an independent set of Γ, let $\xi \in \mathbb{N}$ be such that $|\Delta_U(\mathcal{C})| \leq \xi$ for all $U \in \Pi_3(C)$ and let (E,S) be a flag of $\mathcal{C}$.*

Then there are at most $\mathfrak{s}_q[2] \cdot \mathfrak{s}_q[2,4] \cdot \xi$ flags $(E',S') \in \mathcal{C}$ with $E' \cap E = \emptyset$ and $S' \cap E \neq \emptyset$.

Proof. If there is a flag $(E',S') \in \mathcal{C}$ with $E' \cap E = \emptyset$ and $S' \cap E \neq \emptyset$, then $S' \cap E$ must be a point $P \notin E'$. According to Lemma 2.2.2 for every point $P \in E$ the set

$$\mathcal{S}_P := \{S \in \Pi_3(\Delta_P(\mathcal{C})) \mid \exists E \in \Pi_2(C) : (E,S) \in C \wedge P \notin E\}$$

has cardinality $|\mathcal{S}_P| \leq \mathfrak{s}_q[2,4]$. Furthermore, for every point $P \in E$ every solid $S \in \mathcal{S}_P$ occurs in at most ξ flags of $\mathcal{C}$. Since there are $\mathfrak{s}_q[2]$ choices for a point $P \in E$, this yields an upper bound of $\mathfrak{s}_q[2] \cdot \mathfrak{s}_q[2,4] \cdot \xi$, as claimed. ☐

As mentioned earlier, the proof of the claim is now split into three parts. These parts are such that in the ith part we only consider independent sets with the property that every plane and every solid occurs in at most $\mathfrak{s}_q[i]$ flags. Thus, the third part is in fact the general case.

2.2.1 Planes and Solids occur in at most $\mathfrak{s}_q[1]$ Flags

Throughout this part we assume that C is an independent set of Γ such that for every subspace $U \leq \mathbb{P}$ with $\dim(U) \in \{2, 3\}$ we have $|\Delta_U(C)| \leq \mathfrak{s}_q[1]$.

Lemma 2.2.4. *Let P_1, P_2 and P_3 be non-collinear points of $\mathbb{P}$.*

i) If

$$|\Delta_{P_1}(C)| \geq 6q^7 + 14q^6 + 16q^5 + 11q^4 + q^3 - 5q^2 - q + 3, \tag{2.3}$$

then there are flags $f_i = (E_i, S_i) \in C$ for $i \in \{1, 2, 3\}$ with $\dim(\langle E_1, E_2, E_3\rangle) \geq 5$, $P_2, P_3 \notin S_1, S_2, S_3$ as well as $E_i \cap E_j = P_1$ and $P_2, P_3 \notin \langle E_i, E_j\rangle$ for all distinct $i, j \in \{1, 2, 3\}$.

ii) If there are flags f_1, f_2 and f_3 with the properties stated in i) and if

$$|\Delta_{P_2}(C)| \geq 6q^7 + 14q^6 + 16q^5 + 14q^4 + 4q^3 - 5q^2 - q + 3,$$

then there are flags $f'_i = (E'_i, S'_i) \in C$ for $i \in \{1, 2, 3\}$ with $\dim(\langle E'_1, E'_2, E'_3\rangle) \geq 5$, $P_1, P_3 \notin S'_1, S'_2, S'_3$, $\dim(S_i \cap S'_j) \leq 1$ for all $i, j \in \{1, 2, 3\}$ as well as $E'_i \cap E'_j = P_2$ and $P_1, P_3 \notin \langle E'_i, E'_j\rangle$ for all distinct $i, j \in \{1, 2, 3\}$.

Proof. i) There are exactly $\mathfrak{s}_q[2, 3, 6] + 2 \cdot \mathfrak{s}_q[0, 1, 3, 6]$ solids $S \leq \mathbb{P}$ with $P_1 \in S$ and $P_i \in S$ for some $i \in \{2, 3\}$. According to the assumption of this part each of those solids occurs in at most $\mathfrak{s}_q[1]$ flags of C yielding an upper bound of at most

$$\begin{aligned}(\mathfrak{s}_q[2, 3, 6] &+ 2 \cdot \mathfrak{s}_q[0, 1, 3, 6]) \cdot \mathfrak{s}_q[1] \\ &= 2q^7 + 4q^6 + 6q^5 + 7q^4 + 6q^3 + 4q^2 + 2q + 1\end{aligned} \tag{2.4}$$

flags of C. Note that this number accounts for all flags of C through P_1 whose plane contains P_2 or P_3. Comparing this with the bound given in Equation (2.3) yields a flag $f_1 = (E_1, S_1) \in C$ with $P_1 \in E_1$ and $P_2, P_3 \notin S_1$.

However, there are only $(\mathfrak{s}_q[0, 1, 2] - 2) \cdot \mathfrak{s}_q[0, 1, 2, 6]$ planes through P_1 in $\mathbb{P}$ which meet $\langle P_1, P_2, P_3\rangle$ in a line but do not contain P_2 nor P_3, providing at most

$$(\mathfrak{s}_q[0, 1, 2] - 2) \cdot \mathfrak{s}_q[0, 1, 2, 6] \cdot \mathfrak{s}_q[1] = q^6 + q^5 - q^2 - q \tag{2.5}$$

flags $f = (E, S) \in C$ such that $P_1 \in E$, $\dim(E \cap \langle P_1, P_2, P_3\rangle) \geq 1$ and $P_2, P_3 \notin S$. The sum of this and the number given in Equation (2.4) is still smaller than the bound given in Equation (2.3) and thus we may even chose f_1 such that $E_1 \cap$

$\langle P_1, P_2, P_3\rangle = P_1$. Note that this implies $\dim(\langle E_1, P_2, P_3\rangle) = 4$ and thus $P_2 \notin \langle E_1, P_3\rangle$ as well as $P_3 \notin \langle E_1, P_2\rangle$.

Now, an upper bound on the number of flags $(E, S) \in \Delta_{P_1}(C)$ with $P_2 \in S$ or $P_3 \in S$ is already given in Equation (2.4). Furthermore, for all $i \in \{2, 3\}$ and every flag $(E, S) \in \Delta_{P_1}(C)$ with $P_i \notin E$ we have $P_i \in \langle E_1, E\rangle \iff \dim(\langle E_1, P_i\rangle \cap E) \geq 1$. There are

$$\mathfrak{s}_q[0, 1, 2] \cdot (\mathfrak{s}_q[1, 2, 6] - 3) + 1 = q^5 + 2q^4 + 2q^3 + 2q^2 - q - 1 \tag{2.6}$$

planes E through P_1 which meet E_1 in at least a line and thus satisfy $\dim(\langle E_1, P_i\rangle \cap E) \geq 1$, but do not contain P_2 nor P_3. This number also includes all planes $E \leq \langle E_1, P_i\rangle$ for some $i \in \{2, 3\}$, since all those planes meet E_1 in a line for dimensional reasons. Additionally, for all $i \in \{2, 3\}$ there are

$$\begin{aligned}(\mathfrak{s}_q[0, 1, 3] - \mathfrak{s}_q[0, 1, 2] - 1) \cdot (\mathfrak{s}_q[1, 2, 6] - \mathfrak{s}_q[1, 2, 3] - 1)&\\ = q^6 + q^5 - q^3 - 2q^2 + 1&\end{aligned} \tag{2.7}$$

planes E which meet $\langle E_1, P_i\rangle$ in exactly a line which does not lie in E_1 and satisfy $P_2, P_3 \notin E$. Thus, considering the fact that there are

$$(\mathfrak{s}_q[0, 1, 3] - \mathfrak{s}_q[0, 1, 2] - 1)^2 = q^4 - 2q^2 + 1$$

planes E which satisfy $P_2, P_3 \notin E$ and meet both $\langle E_1, P_2\rangle$ and $\langle E_1, P_3\rangle$ in lines which are not contained in E_1, this provides

$$\begin{aligned}q^5 + 2q^4 + 2q^3 + 2q^2 - q - 1 + 2(q^6 + q^5 - q^3 - 2q^2 + 1) - (q^4 - 2q^2 + 1)&\\ = 2q^6 + 3q^5 + q^4 - q&\end{aligned} \tag{2.8}$$

planes E with $P_1 = E_1 \cap E$ which meet at least one of the spaces $\langle E_1, P_2\rangle$ or $\langle E_1, P_3\rangle$ in a line and do not contain P_2 nor P_3. According to the assumption of this part each of the planes counted in Equation (2.8) occurs in at most $\mathfrak{s}_q[1]$ flags and we also need to consider the flags counted in Equation (2.4), providing a total of at most

$$4q^7 + 9q^6 + 10q^5 + 8q^4 + 6q^3 + 3q^2 + q + 1 \tag{2.9}$$

flags. Comparing this to the bound given in Equation (2.3) yields a further flag $f_2 = (E_2, S_2) \in C$ and this flag satisfies $P_2, P_3 \notin \langle E_1, E_2\rangle$, $E_1 \cap E_2 = P_1$ and $P_2, P_3 \notin S_2$.

Again, we note that even the sum of the number in Equation (2.9) and the number of flags given in Equation (2.5) is smaller than the bound given in Equation (2.3) and thus we may assume that $E_2 \cap \langle P_1, P_2, P_3\rangle = P_1$ holds. Note that this implies $\dim(\langle E_2, P_2, P_3\rangle) = 4$ and thus $P_2 \notin \langle E_2, P_3\rangle$ as well as $P_3 \notin \langle E_2, P_2\rangle$. Furthermore, we remark that

$$d := \dim(\langle E_1, P_j\rangle \cap \langle E_2, P_{5-j}\rangle) = \begin{cases} 0 & \text{if } \langle E_1, E_2, P_2, P_3\rangle = \mathbb{P}, \\ 1 & \text{otherwise,} \end{cases}$$

is independent of the choice of $j \in \{2,3\}$.

It remains to prove the existence of the flag f_3. We reuse the upper bound given in Equation (2.4). Furthermore, for all $i \in \{1,2\}$ the number of planes E through P_1 which meet E_i in at least a line and thus satisfy $\dim(\langle E_i, P_j\rangle \cap E) \geq 1$ for both $j \in \{2,3\}$, but do not contain P_2 nor P_3 is given in Equation (2.6). Subtracting the $\mathfrak{s}_q[0,1,2]^2$ planes which are being counted twice yields

$$2q^5 + 4q^4 + 4q^3 + 3q^2 - 4q - 3 \tag{2.10}$$

planes through P_1 which meet E_1 or E_2 in at least a line but do not contain P_2 nor P_3. Finally, 4 times the number given in Equation (2.7), that is

$$4q^6 + 4q^5 - 4q^3 - 8q^2 + 4,$$

serves as upper bound for the number of still uncounted planes E with $P_1 \in E$ which meet at least one of the spaces $\langle E_i, P_j\rangle$ for some $i \in \{1,2\}$ and $j \in \{2,3\}$ in exactly line, do not contain P_2 nor P_3 and do not contain a line of E_1 nor of E_2. Together with the number given in Equation (2.10) this yields

$$4q^6 + 6q^5 + 4q^4 - 5q^2 - 4q + 1$$

planes E with $P_1 \in E$ which meet at least one of the spaces $\langle E_i, P_j\rangle$ for some $i \in \{1,2\}$ and $j \in \{2,3\}$ in exactly a line and does not contain P_2 nor P_3. According to the assumption of this part every plane occurs in at most $\mathfrak{s}_q[1]$ flags of C and thus together with the number given in Equation (2.4) this yields a total of at most

$$6q^7 + 14q^6 + 16q^5 + 11q^4 + q^3 - 5q^2 - q + 2 \tag{2.11}$$

flags through P_1 in C. Since this is smaller than the bound given in Equation (2.3) in the claim, we know that there must be a further flag f_3, concluding the proof of i).

ii) For all $i \in \{1,2,3\}$ there are exactly $\mathfrak{s}_q[0,-1,2,3] = q^3$ planes E in S_i not containing P_1, each of which provides a unique solid $\langle P_2, E\rangle$ through P_2. According to the assumption of this part each of those solids occurs in at most $\mathfrak{s}_q[1]$ flags of C providing at most

$$3 \cdot \mathfrak{s}_q[1] \cdot \mathfrak{s}_q[0,-1,2,3] = 3q^4 + 3q^3 \tag{2.12}$$

flags of C. Other than that we reuse the counts given in the proof of part i). For the existence of the first flag f_1' consider the sum

$$2q^7 + 5q^6 + 7q^5 + 10q^4 + 9q^3 + 3q^2 + q + 1$$

of Equations (2.4), (2.5) and (2.12), for the existence of the second flag f_2' consider the sum

$$4q^7 + 10q^6 + 11q^5 + 11q^4 + 9q^3 + 2q^2 + 1$$

of Equations (2.9), (2.5) and (2.12) and for the existence of the third flag f_3' consider the sum

$$6q^7 + 14q^6 + 16q^5 + 14q^4 + 4q^3 - 5q^2 - q + 2$$

of Equations (2.11) and (2.12). All of those are smaller than the bound given in the claim and thus each of them proves proves the existence of the respective flag f_i', concluding the proof of part ii). $\square$

Lemma 2.2.5. *Let P_1 and P_2 be two distinct points of $\mathbb{P}$ and let E_1, E_2 and E_3 be planes such that $E_i \cap E_j = P_1$ as well as $P_2 \notin \langle E_i, E_j\rangle =: U_k$ for all $\{i,j,k\} = \{1,2,3\}$. Set $H := \langle E_1, E_2, E_3\rangle$, $d := \dim(\langle H, P_2\rangle)$ and let $\mathcal{S}$ be the set of all solids of $\mathbb{P}$ with $P_2 \in S$ and $S \cap E_i \neq \emptyset$ for all $i \in \{1,2,3\}$. Then we have*

$$|\mathcal{S}| = (d-4)q^{11-d} + 2q^6 + 3q^5 + 3q^4 + 3q^3 + 2q^2 + q + 1 \textit{ with } d \in \{5,6\}.$$

Proof. For all $i \in \{1,2,3\}$ let $\mathcal{S}_i \subseteq \mathcal{S}$ be such that for all $S \in \mathcal{S}_i$ there is an i-dimensional subspace U with $P_2 \in U \leq S$ which has non-empty intersection with all planes E_1, E_2 and E_3 and such that there is no such subspace of dimension smaller than i. Then $\mathcal{S}$ is the disjoint union of $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ and we determine the cardinalities of these subsets.

$\mathcal{S}_1$: Every solid $S \in \mathcal{S}_1$ contains a line l through P_2 which meets all three planes E_1, E_2 and E_3. From $P_2 \notin \langle E_1, E_2\rangle$ we know that any line through P_2 meets $\langle E_1, E_2\rangle$ in at most a point. Therefore any line through P_2 which meets both E_1 and E_2 in a point is the span of P_2 and an element of $E_1 \cap E_2 = P_1$, that is, it is the line $\langle P_1, P_2\rangle$. However, this line already meets all three planes and thus is the only line with that property. Therefore, every solid $S \in \mathcal{S}_1$ must have $\langle P_1, P_2\rangle$ as subspace and every solid through this line is a solid of $\mathcal{S}_1$, yielding

$$|\mathcal{S}_1| = \mathfrak{s}_q[1,3,6]. \tag{2.13}$$

$\mathcal{S}_2$: Every solid $S \in \mathcal{S}_2$ is such that S contains a plane through P_2 which meets all three planes E_1, E_2 and E_3, but S contains no line with that property. Note that from the last case we already know that the only line through P_2 which meets all three planes is the line $\langle P_1, P_2\rangle$. Thus, solids $S \in \mathcal{S}_2$ do not contain $\langle P_1, P_2\rangle$, which is equivalent to $P_1 \notin S$.

Note that for all $S \in \mathcal{S}_2$ there is at most one index $i \in \{1,2,3\}$ such that $\dim(S \cap E_i) = 1$, for if there were two distinct indices $i,j \in \{1,2,3\}$ such that $\dim(S \cap E_i) = 1 = \dim(S \cap E_j)$, then from $E_i \cap E_j = P_1 \notin S$ we would have $(S \cap E_i) \cap (S \cap E_j) = \emptyset$, which would imply

$$P_2 \in S = \langle S \cap E_i, S \cap E_j\rangle \leq \langle E_i, E_j\rangle \not\ni P_2,$$

a contradiction. Therefore, for all $S \in \mathcal{S}_2$ there are distinct $i,j \in \{1,2,3\}$ with $\dim(S \cap E_i) = 0 = \dim(S \cap E_j)$ and the only plane $E \leq S$ with $P_2 \in E$ that meets both E_i and E_j is the plane $\langle P_2, E_i \cap S, E_j \cap S\rangle$.

Furthermore, if E is a plane in a solid $S \in \mathcal{S}_2$ with $P_2 \in E$ and $E \cap E_i \neq \emptyset$ for all $i \in \{1, 2, 3\}$, then there may not be a line $l \leq E$ with $l \cap \langle E_i, E_j \rangle = \emptyset$ for some distinct $i, j \in \{1, 2, 3\}$, because otherwise $E \cap \langle E_i, E_j \rangle$ would be a point, only, and due to $E \cap E_i \neq \emptyset \neq E \cap E_j$ it would have to be the point $P_1 = E_i \cap E_j$, a contradiction.

Consequently, any plane E in a solid $S \in \mathcal{S}_2$ with $P_2 \in E$ and $E \cap E_i \neq \emptyset$ for all $i \in \{1, 2, 3\}$ is the span of a line l through P_2 which meets both E_1 and $\langle E_2, E_3 \rangle$ together with a point of E_2. Moreover, if E is such a plane, then for all $i \in \{1, 2, 3\}$ we have $\dim(E \cap E_i) = 0$ and thus in E there is a unique line l with $P_2 \in l$ and $l \cap E_1 \neq \emptyset$.

We now determine the number of such planes E. According to Lemma 1.2.37 every line l through P_2 which meets E_1 and $\langle E_2, E_3 \rangle$ is the span of P_2 and a point $P \in \langle E_1, P_2 \rangle \cap \langle E_2, E_3 \rangle$. We have $\dim(\langle E_1, P_2 \rangle \cap \langle E_2, E_3 \rangle) = 7 - d$ and from $P_2 \notin U_1$ we have $d \in \{5, 6\}$, that is, $7 - d \in \{1, 2\}$. Therefore, there are $\delta_{5,d} \cdot q^2 + q$ lines l through P_2 which meet both E_1 and $\langle E_2, E_3 \rangle$ but do not contain P_1.

Furthermore, for all $i \in \{1, 2\}$ we have $P_2 \notin \langle E_1, E_i \rangle$ and thus $\langle E_1, P_2 \rangle \cap E_i = P_1$. Therefore, for any line l through P_2 which meets E_1 as well as $\langle E_2, E_3 \rangle$ we know that $l \cap U_1$ is a point which does not lie in $E_2 \cup E_3$. Now, if l is such a line, then Lemma 1.2.37 shows that every plane E with $P_1 \notin E$, $l \leq E$ and $E \cap E_i \neq \emptyset$ for all $i \in \{1, 2, 3\}$ is the span of l and one of the q points of

$$(\langle E_2, l \rangle \cap E_3) \setminus \{P_1\} = (\langle l \cap U_1, E_2 \rangle \cap E_3) \setminus \{P_1\},$$

which provides a total of $\delta_{d,5} \cdot q^3 + q^2$ such planes.

Each of those planes occurs in $\mathfrak{s}_q[0, 2, 3, 6] = q^3 + q^2 + q$ solids which do not contain P_1 and thus we have

$$|\mathcal{S}_2| = \begin{cases} q^6 + 2q^5 + 2q^4 + q^3 & \text{for } d = 5, \\ q^5 + q^4 + q^3 & \text{for } d = 6. \end{cases} \tag{2.14}$$

Note that equality follows from the fact that every solid $S \in \mathcal{S}_2$ contains only one plane E through P_2 which meets all three planes E_1, E_2 and E_3, as well as from the fact that this plane E contains only one line l through P_2 which meets E_1, as we have seen above.

$\mathcal{S}_3$: For $S \in \mathcal{S}_3$ we have $\dim(S \cap E_i) = 0$ for all $i \in \{1, 2, 3\}$, because otherwise, if $\dim(S \cap E_i) \geq 1$ for some $i \in \{1, 2, 3\}$, then every plane in S meets E_i and thus there is a plane through P_2 in S which meets all planes E_1, E_2 and E_3. Furthermore, also because every solid in $\mathcal{S}_2$ contains no plane through P_2 which meets all planes E_1, E_2 and E_3, we know that every solid in $\mathcal{S}_3$ is the span of P_2 and three points $Q_1 \in E_1$, $Q_2 \in E_2$ and $Q_3 \in E_3$. Using this we determine $|\mathcal{S}_2|$ as follows.

There are q^2+q choices for a point $Q_1 \in E_1 \setminus \{P_1\}$ and, given the point Q_1, we set $R_1 := \langle P_2, Q_1\rangle \cap U_1$. Due to $P_2 \notin U_2, U_3$ we have $R_1 \cap E_3, R_1 \cap E_2 = \emptyset$ and, using Lemma 1.2.37, we have

$$\dim(R_1) = \begin{cases} 0 & \text{for } Q_1 \in \langle U_1, P_2\rangle, \\ -1 & \text{for } Q_1 \notin \langle U_1, P_2\rangle, \end{cases} \quad \text{with } \dim(\langle U_1, P_2\rangle \cap E_1) = \begin{cases} 2 & \text{for } d=5, \\ 1 & \text{for } d=6. \end{cases}$$

Now, $P_2 \notin U_3 \geq E_1, E_2$ and $Q_1 \in E_1$ implies $\langle E_1, \langle P_2, Q_1\rangle\rangle \cap E_2 = E_1 \cap E_2 = P_1$ and thus Lemma 1.2.37 implies $\langle P_2, Q_1, Q_2\rangle \cap E_1 = Q_1$ for all $Q_2 \in E_2 \setminus \{P_1\}$. However, note that Lemma 1.2.37 also implies that for all points $Q_2 \in \langle R_1, E_3\rangle \cap E_2$ the plane $\langle P_2, Q_1, Q_2\rangle$ has non-empty intersection with all planes E_1, E_2 and E_3 and as such these planes do not yield solids of $\mathcal{S}_3$. Consequently, we must only consider the choices for Q_2 among $E_2 \setminus \langle R_1, E_3\rangle$ and we have $\dim(\langle R_1, E_3\rangle \cap E_2) = \dim(R_1)+1$.

We let Q_2 be such a point and set $R_2 := \langle P_2, Q_2\rangle \cap U_2$. Note that, similar to above, $P_2 \notin U_1, U_3$ implies $R_2 \cap E_3, R_2 \cap E_1 = \emptyset$ and we have

$$\dim(R_2) = \begin{cases} 0 & \text{for } Q_2 \in \langle U_2, P_2\rangle, \\ -1 & \text{for } Q_2 \notin \langle U_2, P_2\rangle, \end{cases} \quad \text{with } \dim(\langle U_2, P_2\rangle \cap E_2) = \begin{cases} 2 & \text{for } d=5, \\ 1 & \text{for } d=6. \end{cases}$$

However, for $d=6$ and $\dim(R_1)=0$ the lines $\langle U_2, P_2\rangle \cap E_2 = \langle U_2, R_1\rangle \cap E_2$ and $\langle E_3, R_1\rangle \cap E_2$ coincide and as such in this case, due to $Q_2 \in E_2 \setminus \langle E_3, R_1\rangle$, the situation $\dim(R_2)=0$ may not occur.

It remains to determine the number of choices for $Q_3 \in E_3$ in the respective cases provided by the possible choices for Q_1 and Q_2. For all points $Q_3 \in \langle R_1, E_2\rangle \cap E_3$ (respectively $Q_3 \in \langle R_2, E_1\rangle \cap E_3$) the solid $\langle P_2, Q_1, Q_2, Q_3\rangle$ contains the plane $\langle P_2, Q_1, Q_3\rangle$ (respectively $\langle P_2, Q_2, Q_3\rangle$) which has non-empty intersection with all three planes E_1, E_2 and E_3 and thus the solid is not an element of $\mathcal{S}_3$. Hence, we must only consider the choices for Q_3 among the points of $E_3 \setminus (\langle R_1, E_2\rangle \cup \langle R_2, E_1\rangle)$.

Now, for all $i \in \{1,2\}$ the subspace $l_i := \langle R_i, E_{3-i}\rangle \cap E_3$ is either P_1 or a line through P_1. Furthermore, if both l_1 and l_2 are lines, then we have $l_1 \neq l_2$, since otherwise, for $X \in l_1 \setminus \{P_1\}$ both $g_1 := E_1 \cap \langle Q_1, R_2, X\rangle$ and $g_2 := E_2 \cap \langle Q_2, R_1, X\rangle$ are lines, too, and then $\langle R_1, Q_1, Q_2, X\rangle \cap E_3 = X \neq P_1$ implies $P_1 \notin g_1 \cap g_2 \leq E_1 \cap E_2 = P_1$, that is, $g_1 \cap g_2 = \emptyset$ and thus

$$\langle P_2, Q_1, Q_2, X\rangle = \langle \langle R_1, Q_2, X\rangle \cap E_2, \langle R_2, Q_1, X\rangle \cap E_1\rangle \leq U_3 \not\ni P_2,$$

a contradiction.

Altogether this proves

$$\begin{aligned} |\mathcal{S}_3| &= \begin{cases} (q^2+q)\cdot q^2 \cdot (q^2-q) & \text{for } d=5, \\ q\cdot q^2\cdot q^2 + q^2\cdot q\cdot q^2 + q^2\cdot q^2\cdot(q^2+q) & \text{for } d=6, \end{cases} \\ &= \begin{cases} q^6-q^4 & \text{for } d=5, \\ q^6+3q^5 & \text{for } d=6. \end{cases} \end{aligned} \tag{2.15}$$

Finally, the sum of the three numbers given in Equations (2.13), (2.14) and (2.15) is the cardinality given for $\mathcal{S}$ in the claim. $\square$

Lemma 2.2.6. *Let P_1 and P_2 be two distinct points of $\mathbb{P}$ and for all $k \in \{1,2,3\}$ let there be a flag $f_k = (E_k, S_k) \in \Delta_{P_1}(C)$ such that for all distinct $i, j \in \{1,2,3\}$ we have $E_i \cap E_j = P_1$ and $P_2 \notin \langle E_i, E_j\rangle$ as well as $P_2 \notin S_i$. Then*

$$\begin{aligned}|\Delta_{P_2}(C)| \le\ & 3q^8 + 9q^7 + 9q^6 + 9q^5 + 11q^4 + 9q^3 + 6q^2 + 4q + 2 \\ & + (d-4)(q^{12-d} + q^{11-d}),\end{aligned}$$

with $d := \dim(\langle E_1, E_2, E_3, P_2\rangle) \in \{5,6\}$.

Proof. Every flag in $\Delta_{P_2}(C)$ must be non-adjacent to the three flags f_1, f_2 and f_3. Therefore, for all $f = (E,S) \in \Delta_{P_2}(C)$ we have $E \cap S_i \neq \emptyset$ for some $i \in \{1,2,3\}$ or $S \cap E_i \neq \emptyset$ for all $i \in \{1,2,3\}$.

In view of that, for all $i \in \{1,2,3\}$ let $\mathcal{E}_i$ be the set of all planes through P_2 which have non-empty intersection with S_i, set $\mathcal{E} := \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ and let $\mathcal{S}$ be the set of all solids through P_2 which have non-empty intersection with all planes E_1, E_2 and E_3.

For all $i \in \{1,2,3\}$ we have

$$|\mathcal{E}_i| \le \mathfrak{s}_q[0,2,6] - \mathfrak{s}_q[3,0,2,6] = q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1. \tag{2.16}$$

and for all distinct $i, j \in \{1,2,3\}$ we have $\dim(\langle S_i, S_j, P_2\rangle) \ge 5$ as well as

$$d_{\{i,j\}} := \dim(\langle S_i, P_2\rangle \cap \langle S_j, P_2\rangle) = \begin{cases} 2 & \text{for } \dim(\langle S_i, S_j, P_2\rangle) = 6, \\ 3 & \text{for } \dim(\langle S_i, S_j, P_2\rangle) = 5. \end{cases}$$

Hence, the number of planes through P_2 which meet both S_i and S_j for given $i, j \in \{1,2,3\}$ with $i \neq j$ is exactly

$$\begin{aligned}&\mathfrak{s}_q[0,2,d_{\{i,j\}}] + \mathfrak{s}_q[0,1,d_{\{i,j\}}] \cdot \mathfrak{s}_q[d_{\{i,j\}} - 2,1,2,6] + (\mathfrak{s}_q[3] - \mathfrak{s}_q[d_{\{i,j\}} - 1])^2 \\ &\qquad = q^{d_{\{i,j\}}+3} + q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1\end{aligned}$$

and, using $|\mathcal{E}| \le |\mathcal{E}_1| + |\mathcal{E}_2| + |\mathcal{E}_3| - |\mathcal{E}_1 \cap \mathcal{E}_2| - |\mathcal{E}_1 \cap \mathcal{E}_3|$ as well as the number given in Equation (2.16), this implies

$$|\mathcal{E}| \le 3q^7 + 4q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1 - \left(q^{d_{\{1,2\}}+3} + q^{d_{\{1,3\}}+3}\right).$$

Thus, according to the assumption of this part, there are at most

$$3q^8 + 7q^7 + 6q^6 + 5q^5 + 5q^4 + 4q^3 + 3q^2 + 2q + 1 - \mathfrak{s}_q[1]\left(q^{d_{\{1,2\}}+3} + q^{d_{\{1,3\}}+3}\right) \tag{2.17}$$

flags $f \in \Delta_{P_2}(C)$ such that $\pi_2(f)$ has non-empty intersection with at least one of the solids S_1, S_2 or S_3.

Every other flag $f = (E,S) \in \Delta_{P_2}(C)$ must satisfy $S \cap E_i \neq \emptyset$ for all $i \in \{1,2,3\}$ and thus $S \in \mathcal{S}$. Using the value given in Lemma 2.2.5 for $|\mathcal{S}|$ and the fact that, according to the assumption of this part, every such solid occurs in at most $\mathfrak{s}_q[1]$ flags of C, yields at most

$$(d-4)(q^{12-d}+q^{11-d})+2q^7+5q^6+6q^5+6q^4+5q^3+3q^2+2q+1 \tag{2.18}$$

further flags.

Finally, summing up the weaker bound given by Equation (2.17) for $d_{\{1,2\}}=2=d_{\{1,3\}}$ and Equation (2.18) yields the claim. □

Lemma 2.2.7. *Let P_1, P_2 and P_3 be non-collinear points of $\mathbb{P}$ and for all $i\in\{1,2\}$ and all $r\in\{1,2,3\}$ let there be a flag $f_{i,r}=(E_{i,r},S_{i,r})\in\Delta_{P_i}(C)$ such that*

- $\forall r,s\in\{1,2,3\}:\dim(S_{1,r}\cap S_{2,s})\le 1$ *and*
- $\forall i\in\{1,2\},\forall\{r,s,t\}=\{1,2,3\}:E_{i,r}\cap E_{i,s}=P_i$ *and* $P_{3-i},P_3\notin\langle E_{i,r},E_{i,s}\rangle\cup S_{i,r}$.

Then $|\Delta_{P_3}(C)|\le 24q^7+48q^6+57q^5+57q^4+46q^3+33q^2+22q+11$.

Proof. Any flag $f=(E,S)\in\Delta_{P_3}(C)$ must be non-adjacent to the six flags $f_{i,r}$ with $i\in\{1,2\}$ and $r\in\{1,2,3\}$. Therefore, for all $i\in\{1,2\}$ and all $(E,S)\in C$ we have $S\cap E_{i,r}\neq\emptyset$ for all $r\in\{1,2,3\}$ or $E\cap S_{i,r}\neq\emptyset$ for at least one $r\in\{1,2,3\}$ and we begin by counting flags which satisfy the latter condition.

First, we let $r,s\in\{1,2,3\}$ be arbitrary but fixed and count the number of planes E through P_3 which meet both $S_{1,r}$ and $S_{2,s}$. Any such plane E either contains a line l through P_3 which already meets both $S_{1,r}$ and $S_{2,s}$, or does not, and a line l through P_3 meets both $S_{1,r}$ and $S_{2,s}$ if and only if it is a subspace of $U_{\{r,s\}}:=\langle P_3,S_{1,r}\rangle\cap\langle P_3,S_{2,s}\rangle$. Hence, a plane E contains a line l through P_3 which meets both $S_{1,r}$ and $S_{2,s}$ if and only if E contains a line of $U_{\{r,s\}}$ and $U_{\{r,s\}}$ has dimension

$$d_{\{r,s\}}=\begin{cases}2 & \text{for } \dim(\langle P_3,S_{1,r},S_{2,s}\rangle)=6,\\ 3 & \text{for } \dim(\langle P_3,S_{1,r},S_{2,s}\rangle)=5.\end{cases}$$

This implies that there are exactly

$$\begin{aligned}\mathfrak{s}_q[0,2,d_{\{r,s\}}]+\mathfrak{s}_q[0,1,d_{\{r,s\}}]\cdot\mathfrak{s}_q[d_{\{r,s\}}-2,1,2,6]+(\mathfrak{s}_q[3]-\mathfrak{s}_q[d_{\{r,s\}}-1])^2&\\ =q^{d_{\{r,s\}}+3}+q^6+2q^5+3q^4+2q^3+2q^2+q+1&\end{aligned} \tag{2.19}$$

planes through P_3 which meet both $S_{1,r}$ and $S_{2,s}$.

Now, there there are 9 choices for $r,s\in\{1,2,3\}$ and we may use the larger number given in Equation (2.19) for $d_{\{r,s\}}=3$ to receive an upper bound of at most

$$18q^6+18q^5+27q^4+18q^3+18q^2+9q+9 \tag{2.20}$$

planes through P_3 which meet $S_{1,r}$ and $S_{2,s}$ for some $r,s\in\{1,2,3\}$.

Furthermore, for all $i\in\{1,2\}$ we may use the larger number given in Lemma 2.2.5 for $d=5$ as bound for the number of solids through P_3 which meet all three planes $E_{i,1}$, $E_{i,2}$ and $E_{i,3}$. This shows that there is a total of at most

$$6q^6+6q^5+6q^4+6q^3+4q^2+2q+2 \tag{2.21}$$

solids through P_3 which meet all planes $E_{i,1}$, $E_{i,2}$ and $E_{i,3}$ for some $i \in \{1,2\}$.

Finally, according to the assumption of this part, every plane counted in Equation (2.20) and every solid counted in Equation (2.21) occurs in at most $\mathfrak{s}_q[1]$ flags of C each, which yields the claimed bound. $\square$

Lemma 2.2.8. *The cardinality of* C *is at most*

$$24q^{10} + 73q^9 + 135q^8 + 178q^7 + 179q^6 + 156q^5 + 123q^4 + 84q^3 + 45q^2 + 18q + 3.$$

Proof. Let $P_1, P_2 \in \mathbb{P}$ be distinct such that $|\Delta_{P_1}(C)|, |\Delta_{P_2}(C)| \geq |\Delta_P(C)|$ for all $P \in \mathbb{P} \setminus \{P_1\}$.

If there is no flag $f = (E,S) \in C$ with $\langle P_1, P_2\rangle \cap E = \emptyset$, then, using the assumption of this part, we have $|C| \leq (\mathfrak{s}_q[2,6] - \mathfrak{s}_q[1,-1,2,6]) \cdot \mathfrak{s}_q[1]$ and, since this is better than the claim, there remains nothing to prove.

Therefore, assume that there is a flag $f = (E,S) \in C$ with $\langle P_1, P_2\rangle \cap E = \emptyset$ and thus $\dim(S \cap \langle P_1, P_2\rangle) \leq 0$. Note that this implies that every flag $f' = (E', S') \in C$ satisfies either $E' \cap S \neq \emptyset$ or $S' \cap E \neq \emptyset = E' \cap S$ and, according to the assumption of this part and Lemma 2.2.3, there are at most

$$\mathfrak{s}_q[2] \cdot \mathfrak{s}_q[2,4] \cdot \mathfrak{s}_q[1] \tag{2.22}$$

flags $f' = (E', S') \in C$ with $S' \cap E \neq \emptyset = E' \cap S$.

Now, it only remains to determine the number of flags $f' = (E', S')$ with $E' \cap S \neq \emptyset$ and, in view of Lemma 2.2.4, we note that we either have

$$|\Delta_P(C)| \leq |\Delta_{P_2}(C)| < 6q^7 + 14q^6 + 16q^5 + 14q^4 + 4q^3 - 5q^2 - q + 3 \tag{2.23}$$

for all $P \in \mathbb{P} \setminus \langle P_1, P_2\rangle$, or

$$|\Delta_{P_1}(C)| \geq |\Delta_{P_2}(C)| \geq 6q^7 + 14q^6 + 16q^5 + 14q^4 + 4q^3 - 5q^2 - q + 3.$$

In fact, if we study the second situation more closely, then we see that in that case Lemma 2.2.4 provides the flags $f_{i,j} \in C$ for all $i \in \{1,2\}$ and all $j \in \{1,2,3\}$ required to apply Lemma 2.2.7, which proves

$$|\Delta_P(C)| \leq 24q^7 + 48q^6 + 57q^5 + 57q^4 + 46q^3 + 33q^2 + 22q + 11 \tag{2.24}$$

for all $P \in \mathbb{P} \setminus \langle P_1, P_2\rangle$. Since this bound is weaker than the bound given in Equation (2.23) it holds in either situation.

In particular, Equation (2.24) holds for all $P \in S \setminus (S \cap \langle P_1, P_2\rangle)$ and it only remains to consider points in $S \cap \langle P_1, P_2\rangle$. Recall that we chose f such that this intersection is at most a point $\widehat{P}$. Now, since P_1 and P_2 are distinct there is an index $i \in \{1,2\}$ such that $\widehat{P} \neq P_i$ and, using the flags $f_{i,1}$, $f_{i,2}$ and $f_{i,3}$, we may apply Lemma 2.2.6 (using the larger value given there for $d = 5$) and have

$$|\Delta_{\widehat{P}}(C)| \leq 3q^8 + 10q^7 + 10q^6 + 9q^5 + 11q^4 + 9q^3 + 6q^2 + 4q + 2.$$

Therefore, the total number of all flags $f' = (E', S')$ with $E' \cap S \neq \emptyset$ is at most

$$\begin{aligned}(\mathfrak{s}_q[3]-1)\cdot(24q^7+48q^6+57q^5+57q^4+46q^3+33q^2+22q+11)\\ +3q^8+10q^7+10q^6+9q^5+11q^4+9q^3+6q^2+4q+2\\ =24q^{10}+72q^9+132q^8+172q^7+170q^6+145q^5+112q^4+75q^3+39q^2+15q+2\end{aligned}$$

and together with the number given in Equation (2.22) this proves the claim. □

2.2.2 Planes and Solids occur in at most $\mathfrak{s}_q[2]$ Flags

Throughout this part we assume that C is an independent set of Γ such that for every subspace $U \leq \mathbb{P}$ with $\dim(U) \in \{2,3\}$ we have $|\Delta_U(C)| \leq \mathfrak{s}_q[2]$.

Lemma 2.2.9. *If $E \in \Pi_2(C)$ is a plane which occurs in more than $\mathfrak{s}_q[1]$ flags of C, then there are at most $\mathfrak{s}_q[2](q^6+3q^3+2q^2+q+1)$ flags $(E', S') \in C$ with $E \cap S' = \emptyset$.*

Proof. Let E be such a plane. There are at most $q+1$ solids through a given plane in a 4-dimensional subspace of $\mathbb{P}$ and thus there are solids $S_1, S_2, S_3 \in \Pi_3(\Delta_E(C))$ such that $H := \langle S_1, S_2, S_3\rangle$ has dimension 5.

Any flag $(E', S') \in C$ with $E \cap S' = \emptyset$ satisfies $E' \cap S_j \neq \emptyset$ for all $j \in \{1,2,3\}$ and together with $E' \cap E \subseteq S' \cap E = \emptyset$ this implies $S_j = \langle E, S_j \cap E'\rangle$. This shows $H = \langle S_1, S_2, S_3\rangle \leq \langle E, E'\rangle$ and for dimensional reasons E' must be a complement of E in H. Furthermore, from $E \cap S' = \emptyset$ we know that, for dimensional reasons, S' may not be contained in H.

Now, let $\mathcal{E}$ be the set of all planes which occur in such a flag and apply Lemma 2.1.7 to see that the planes in $\mathcal{E}$ form an independent set in the Kneser graph Γ' of type 2 in H. From $E' \cap E = \emptyset$ for all $E' \in \mathcal{E}$ we even know that $\mathcal{E}$ is an independent set of the graph induced by Γ' on the set $N_{\Gamma'}(E)$ and we may apply Corollary 2.1.25 to see that $|\mathcal{E}| \leq \max(q^6, q^5+2q^4+3q^3+2q^2+q+1) \leq q^6+3q^3+2q^2+q+1$ holds.

Finally, the assumption of this part shows $|\Delta_E(C)| \leq \mathfrak{s}_q[2]$, which concludes the proof. □

Lemma 2.2.10. *For all $E \in \Pi_2(C)$ there are at most $\mathfrak{s}_q[2]^2 \cdot \mathfrak{s}_q[2,4]$ flags $(E', S') \in C$ with $E' \cap E = \emptyset$ and $S' \cap E \neq \emptyset$.*

Proof. Let $E \in \Pi_2(C)$ and let C' be the set of flags to be counted. The assumption of this part shows $|\Delta_S(C')| \leq |\Delta_S(C)| \leq \mathfrak{s}_q[2]$ for all $S \in \Pi_3(C)$ and thus with $\xi := \mathfrak{s}_q[2]$ Lemma 2.2.3 proves the claim. □

Lemma 2.2.11. *The cardinality of C is at most*

$$24q^{10}+73q^9+135q^8+178q^7+181q^6+158q^5+125q^4+86q^3+47q^2+18q+3.$$

Proof. Let $\mathcal{E}$ be the set of all planes E with $|\Delta_E(C)| > \mathfrak{s}_q[1]$ and let $\mathcal{S}$ be the set of all solids S with $|\Delta_S(C)| > \mathfrak{s}_q[1]$.

First note that if both $\mathcal{E}$ and $\mathcal{S}$ have size at most $\mathfrak{s}_q[4]$, then there is a subset C' of C such that every plane and every solid lies in at most $\mathfrak{s}_q[1]$ flags of C' and such that $|C'| \geq |C| - 2\mathfrak{s}_q[4]q^2$. Thus, in this case we have $|C| \leq |C'| + 2\mathfrak{s}_q[4]q^2$ and may apply Lemma 2.2.8 to C' to find the claimed bound for $|C|$.

Hence, from now on we may assume that $\mathcal{E}$ or $\mathcal{S}$ has size greater than $\mathfrak{s}_q[4]$ and, since either one of these situations is dual to the other, we may assume that it is the set $\mathcal{E}$, that is, we may assume that $|\mathcal{E}| > \mathfrak{s}_q[4]$.

Now, according to Corollary 2.1.22, $|\mathcal{E}| > \mathfrak{s}_q[4]$ implies that there are two planes $E_1, E_2 \in \mathcal{E}$ with $\dim(E_1 \cap E_2) \leq 0$. For all $i \in \{1,2\}$ every flag (E,S) of C satisfies one of the following cases

- $E \cap E_i = \emptyset$ and $S \cap E_i \neq \emptyset$,
- $S \cap E_i = \emptyset$ and thus $E \cap S_i \neq \emptyset$ for all solids S_i with $(E_i, S_i) \in C$,
- $E \cap E_i \neq \emptyset$.

Consider $i \in \{1,2\}$ and the cases above: According to Lemma 2.2.10 there are at most $\mathfrak{s}_q[2]^2 \cdot \mathfrak{s}_q[2,4]$ flags of the first type and, using $q \geq 3$, according to Lemma 2.2.9 there are at most $\mathfrak{s}_q[2](q^6 + 3q^3 + 2q^2 + q + 1)$ flags of the second type. Hence, this provides a total of at most

$$2 \cdot \mathfrak{s}_q[2](\mathfrak{s}_q[2] \cdot \mathfrak{s}_q[2,4] + q^6 + 3q^3 + 2q^2 + q + 1) \tag{2.25}$$

flags in C which satisfy one of the first two conditions for some $i \in \{1,2\}$. Every other flag $(E,S) \in C$ satisfies $E \cap E_i \neq \emptyset$ for both $i \in \{1,2\}$. We count these flags in the following and set $d := \dim(E_1 \cap E_2) \in \{-1, 0\}$.

There are $\mathfrak{s}_q[d,-1,0,2]^2$ lines l which meet both E_1 and E_2 and satisfy $l \cap E_1 \cap E_2 = \emptyset$ and through each such line l there are $\mathfrak{s}_q[d,1,2,6]$ planes E with $E \cap E_1 \cap E_2 = \emptyset$. However, some of those planes, namely $2 \cdot \mathfrak{s}_q[d,0,1,2]$ planes through each line l, meet one of the planes E_1 or E_2 in a line. Thus, there is a total of

$$\mathfrak{s}_q[d,-1,0,2]^2 \cdot (\mathfrak{s}_q[d,1,2,6] - 2 \cdot \mathfrak{s}_q[d,0,1,2])$$

planes E which meet E_1 and E_2 in a point each and satisfy $E \cap E_1 \cap E_2 = \emptyset$.

Furthermore, every plane E with $E \cap E_1 \cap E_2 = \emptyset$ which meets one of those planes in a line l and the other plane in a point Q is the span of the line l and the point Q. Now, there are $2 \cdot \mathfrak{s}_q[d,-1,1,2]$ lines l contained in one of the planes E_1 or E_2 with $E_1 \cap E_2 \cap l = \emptyset$ and this provides a total of

$$2 \cdot \mathfrak{s}_q[d,-1,1,2] \cdot \mathfrak{s}_q[d,-1,0,2]$$

planes E with $E \cap E_1 \cap E_2 = \emptyset$ which meet E_1 or E_2 in a line and meet the other plane in a point.

Finally for $d = 0$, there are $\mathfrak{s}_q[0,2,6]$ planes E with $E \cap E_1 \cap E_2 \neq \emptyset$, yielding a total of

$$q^8+3q^7+6q^6+6q^5+5q^4+2q^3+2q^2+q+1+\delta_{0,d}(q^8+q^7-2q^5-q^4+q^3)$$
$$\le 2q^8+4q^7+6q^6+4q^5+4q^4+3q^3+2q^2+q+1$$

planes which have non-empty intersection with E_1 and E_2. Therefore, using the assumption of this part, there are at most

$$\mathfrak{s}_q[2](2q^8+4q^7+6q^6+4q^5+4q^4+3q^3+2q^2+q+1)$$
$$=2q^{10}+6q^9+12q^8+14q^7+14q^6+11q^5+9q^4+6q^3+4q^2+2q+1$$

flags of the last type. Together with the number given in Equation (2.25) this yields

$$4q^{10}+12q^9+28q^8+38q^7+46q^6+49q^5+49q^4+40q^3+26q^2+12q+5$$

as upper bound on $|C|$, concluding the proof. □

2.2.3 The General Case

Throughout this final part we assume that C is an arbitrary maximal independent set of Γ. Note that we trivially have $|\Delta_U(C)| \le \mathfrak{s}_q[3]$ for every subspace $U \le \mathbb{P}$ with $\dim(U) \in \{2,3\}$.

Lemma 2.2.12. *If there are more than $\xi := q^7+2q^6+2q^5+3q^4+2q^3+2q^2+q+1$ saturated solids in $\Pi_3(C)$, then there is a hyperplane H of $\mathbb{P}$ such that every solid $S \le H$ is saturated.*

Dually, if there are more than ξ saturated planes in $\Pi_2(C)$, then there is a point $P \in \mathbb{P}$ such that every plane through P is saturated.

Proof. Let $\mathcal{S}$ be the set of saturated solids in $\Pi_3(C)$ and let $|\mathcal{S}| > \xi$. Then, according to Corollary 2.1.11, we have $\dim(S_1 \cap S_2) \ge 1$ for all $S_1, S_2 \in \mathcal{S}$ and, according to Theorem 2.1.26 and since $\xi > q^6+2q^5+3q^4+3q^3+2q^2+q+1$, we know that all solids of $\mathcal{S}$ are contained in a hyperplane H.

Now, if there would be a flag $(E,S) \in C$ such that $E \not\le H$, then according to Corollary 2.1.9 all solids of $\mathcal{S}$ would have non-empty intersection with the line $E \cap H$ and thus there would only be $\mathfrak{s}_q[3,5] - \mathfrak{s}_q[1,-1,3,5] = \xi$ solids in $\mathcal{S}$, a contradiction. Consequently, we have $E \le H$ for all $E \in \Pi_2(C)$ and according to Lemma 2.1.13 all solids $S \le H$ are saturated. □

Corollary 2.2.13. *If C contains more than $q^7+2q^6+2q^5+3q^4+2q^3+2q^2+q+1$ saturated solids (saturated planes), then C is given by Example 2.1.15 (Example 2.1.17).*

Theorem 2.2.14. *Every maximal independent set of Γ of size larger than*

$$26q^{10}+77q^9+139q^8+184q^7+185q^6+162q^5+127q^4+88q^3+47q^2+18q+3$$

is given by Examples 2.1.15 and 2.1.17.

Proof. Note that according to Lemma 2.1.5 the set of flags of C through a given plane E is the set of all flags through E in a given subspace of $\mathbb{P}$ and the set of flags of C in a given solid S is the set of all flags in S through a given subspace of $\mathbb{P}$. Let $\mathcal{E}$ be the set of all saturated planes of C and let $\mathcal{S}$ be the set of all saturated solids, that is,

$$\mathcal{E} = \{E \in \Pi_2(C) : |\Delta_E(C| > \mathfrak{s}_q[2]\} \qquad \text{and} \qquad \mathcal{S} = \{S \in \Pi_3(C) : |\Delta_S(C| > \mathfrak{s}_q[2]\}.$$

For every plane $E \in \mathcal{E}$ let $H_E \geq E$ be an arbitrarily chosen but fixed hyperplane of $\mathbb{P}$ and for every solid $S \in \mathcal{S}$ let $P_S \in S$ be an arbitrarily chosen but fixed point. For every plane $E \in \Pi_2(C) \setminus \mathcal{E}$ we have $\langle \Pi_3(\Delta_E(C)) \rangle \neq \mathbb{P}$ and let H_E be an arbitrarily chosen but fixed hyperplane containing this subspace, for every solid $S \in \Pi_3(C) \setminus \mathcal{S}$ we have $\bigcap_{E \in \Pi_2(\Delta_S(C))} E \neq \emptyset$ and let P_S be an arbitrarily chosen but fixed point therein and set

$$C' := \{(E,S) \in C : (E \in \mathcal{E} \Rightarrow S \leq H_E) \wedge (S \in \mathcal{S} \Rightarrow P_S \in E)\}.$$

According to Lemma 2.2.12 we know that either C is given by one of the two examples and there remains nothing to prove, or we have

$$|\mathcal{E}|, |\mathcal{S}| \leq \xi := q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1.$$

Hence, we may assume the latter, which implies $|C| \leq |C'| + 2\xi q^3$. Now, the construction of C' is such that for every plane $E \in \Pi_2(C')$ all solids S with $(E,S) \in C'$ satisfy $S \leq H_E$ and for every solid $S \in \Pi_3(C')$ all planes E with $(E,S) \in C'$ satisfy $P_S \in E$. Therefore, every plane and every solid occurs in at most $\mathfrak{s}_q[2]$ flags of C'. Finally, we may apply Lemma 2.2.11 to receive an upper bound on $|C'|$ and, using $|C| \leq |C'| + 2\xi q^3$, we know that the cardinality of C is at most

$$26q^{10} + 77q^9 + 139q^8 + 184q^7 + 185q^6 + 162q^5 + 127q^4 + 88q^3 + 47q^2 + 18q + 3,$$

as claimed. □

Corollary 2.2.15. *Every independent set of* Γ *of size larger than*

$$26q^{10} + 77q^9 + 139q^8 + 184q^7 + 185q^6 + 162q^5 + 127q^4 + 88q^3 + 47q^2 + 18q + 3$$

is contained in a maximal independent set given by Examples 2.1.15 and 2.1.17.

Theorem 2.2.16. *For* $q \geq 27$ *the independence number of the Kneser graph of flags of type* $(2,3)$ *in* $\mathrm{PG}(6,q)$ *is* $\mathfrak{s}_q[3,5] \cdot \mathfrak{s}_q[2,3] + \mathfrak{s}_q[2,4]q^3$ *and the independent sets attaining this bound are precisely the four examples given in Examples 2.1.15 and 2.1.17 using independent sets* $\mathcal{U}$ *of maximal size.*

Proof. On the one hand, the referenced Examples provide independent sets of the given size and thus the independence number of Γ is at least as large as given in this claim.

On the other hand, for $q \geq 27$ the size given here is larger than the bound given in Theorem 2.2.14 and thus said theorem shows that any independent set of size $\mathfrak{s}_q[3,5] \cdot \mathfrak{s}_q[2,3] + \mathfrak{s}_q[2,4]q^3$ is given by one of the Examples. Considering Remark 2.1.16 concludes the proof. □

2.3 The Chromatic Number of Kneser Graphs of Type $(n-1,n)$ in $\mathrm{PG}(2n,q)$

In this section we determine the chromatic number of Kneser graphs Γ of type $(n-1,n)$ in $\mathbb{P} := \mathrm{PG}(2n,q)$ with $n \geq 3$ and q very large (see Equation (2.26) below) which satisfy Conjecture 2.1.19. In fact, we even show that for every colouring $g : \mathcal{V}(\Gamma) \to C$ and every colour $c \in C$ the set $g^{-1}(c)$ is a subset of a co-clique defined in Examples 2.1.15 and 2.1.17.

Throughout this section we assume that n is such that Conjecture 2.1.19 holds and let α_n and q_n denote the values given there. Without loss of generality may assume $\alpha_n \geq 5$. Furthermore, we assume that the prime power q satisfies

$$q \geq q_n, \qquad q > \frac{3 \cdot 112^{2^{d+1}-1}}{2^{d+1}} \text{ and } \qquad q \geq \frac{3}{2}\alpha^2 + \frac{21}{2}\alpha + 17. \tag{2.26}$$

Note that there are in fact integers $n \geq 3$ which satisfy said conjecture: In Theorem 2.2.14 of the previous section we have shown that for $n = 3$ the Kneser graph Γ of type $(n-1,n) = (2,3)$ in $\mathrm{PG}(6,q)$ satisfies the conjecture with $\alpha_n = 27$ and $q_n = 78$.

We also remark that Conjecture 2.1.19 holds for $n = 2$, too, as is shown in [3] by Blokhuis and Brouwer (also consider the Appendix of [12] by D'haeseleer, Metsch and Werner). In fact, in [12] by D'haeseleer, Metsch and Werner the chromatic number was determined for $n = 2$ and many of the techniques used here for $n \geq 3$ are a generalization of the techniques used there. The generalization given here is also the result of a joint work of these three authors. Note that the contents of [12] are part of the Ph.D. thesis of D'haeseleer.

We begin with a section on examples and notation. Thereafter, we determine the chromatic number of the graph Γ in two steps. In the first step we do not consider a colouring of Γ, but instead a covering of $\mathcal{V}(\Gamma)$ by $\mathfrak{s}_q[n+1] - q$ special co-cliques. In the second step we then use the results of the first step to derive the chromatic number of Γ in Theorem 2.3.20.

Note that in this section we use θ_x to denote $\mathfrak{s}_q[x]$ for all $x \in \{-1\} \cup \mathbb{N}_0$.

2.3.1 Examples of Co-Cliques and Colourings

We recall the co-cliques of Γ defined in Examples 2.1.15 and 2.1.17 and introduce some new notation for these.

Notation 2.3.1 (Co-Cliques of Γ). i) For a hyperplane H and a maximal co-clique $\mathcal{E}$ of the Kneser graph Γ' of type $n-1$ in H we let

$$C(H,\mathcal{E}) := \{(\pi,\tau) \in \mathcal{V}(\Gamma) : \tau \leq H \vee \pi \in \mathcal{E}\}$$

denote the corresponding maximal co-clique of Γ and say that it is *based on* the hyperplane H. We call $\{(\pi,\tau) \in C(H,\mathcal{E}) : \tau \leq H\}$ the *generic part* and $\{(\pi,\tau) \in C(H,\mathcal{E}) : \tau \not\leq H\}$ the *special part* of $C(H,\mathcal{E})$.

ii) Dually, for any point P and a maximal set $\mathcal{U}$ of n-dimensional subspaces through P, such that $\dim(\tau \cap \tau') \geq 1$ for all $\tau, \tau' \in \mathcal{U}$, we let

$$C(P,\mathcal{U}) := \{(\pi,\tau) \in \mathcal{V}(\Gamma) : P \in \pi \vee \tau \in \mathcal{U}\}$$

denote the corresponding maximal co-clique of Γ. We say that $C(P,\mathcal{U})$ is *based on* the point P and call P the *base point* of $C(P,\mathcal{U})$. Again, we call $\{(\pi,\tau) \in C(P,\mathcal{E}) : P \in \pi\}$ the *generic part* and $\{(\pi,\tau) \in C(P,\mathcal{E}) : P \notin \pi\}$ the *special part* of $C(P,\mathcal{U})$.

Furthermore, if there exists a line l on P such that $\mathcal{U}$ consists of all d-dimensional subspaces τ with $l \leq \tau$, then we also denote $C(P,\mathcal{U})$ by $C(P,l)$ and say that the special part of this set is *based on* the line l.

Similarly, if there exists a hyperplane H on P such that $\mathcal{U}$ consists of all d-dimensional subspaces τ with $P \in \tau \leq H$, then we also denote $C(P,\mathcal{U})$ by $C(P,H)$ and say that the special part of this set is *based on* the hyperplane H.

Recall that the co-cliques defined in i) and ii) are dual with one another. Here we work more frequently with the co-cliques introduced in ii), which explains why we introduced more elaborate notation for these and also explains why this is the point of view in the following Lemma.

Lemma 2.3.2. *Let P be a point of $\mathbb{P}$ and let $\mathcal{U}$ be a maximal set of n-dimensional subspaces through P such that $\dim(\tau \cap \tau') \geq 1$ for all $\tau, \tau' \in \mathcal{U}$.*

(a) The generic part of $C(P,\mathcal{U})$ has cardinality $\mathfrak{s}_q[n, 2n-1] \cdot \theta_n$ and the special part of $C(P,\mathcal{U})$ has cardinality $|\mathcal{U}| \cdot q^n$.

(b) If the special part of $C(P,\mathcal{U})$ is based on a line or hyperplane, then $|\mathcal{U}| = \mathfrak{s}_q[1, n, 2n]$. Otherwise we have

$$|\mathcal{U}| < \left(1 + \frac{1}{q}\right) \cdot \theta_{n-2} \cdot \theta_{n-1}^{n-1}. \tag{2.27}$$

The dual statements to these claims hold as well.

Proof. This is mostly a corollary to Example 2.1.15 and Remark 2.1.16 thereafter. We point out that, since we are in fact the situation of Example 2.1.17, that is, the dual of Example 2.1.15, we have to consider the dual situation when reading Remark 2.1.16. Therefore, for the proof of (b), we have to consider $\mathcal{U}$ as a maximal co-clique of the Kneser graph of type $n-1$ in the quotient space $\mathbb{P}/P$. For the bound given in Equation (2.27) we then apply Theorem 2.1.23 by Blokhuis, Brouwer and Szönyi to that co-clique.

Finally, the dual statements of the claims hold, since the situation under consideration is self-dual. $\square$

Notation 2.3.3. In view of the previous lemma and in view of Conjecture 2.1.19 we set

$$\begin{aligned}
g_0 &:= \mathfrak{s}_q[n, 2n-1] \cdot \theta_n, \\
e_0 &:= \mathfrak{s}_q[n, 2n-1] \cdot \theta_n + \mathfrak{s}_q[1, n, 2n] \cdot q^n \text{ and} \\
e_1 &:= \alpha_n q^{n^2+n-2}.
\end{aligned}$$

Remark 2.3.4. *Note that the lower bound for q that we assumed in Equation (2.26) implies $g_0 > e_1$. Furthermore, note that Conjecture 2.1.19 implies that every co-clique C of Γ with $|C| > e_1$ is a subset of a co-clique given in Examples 2.1.15 and 2.1.17 which is covered by Notation 2.3.1.*

Example 2.3.5 (Coverings of $\mathcal{V}(\Gamma)$ by co-cliques)**.** Let $U \leq \mathbb{P}$ be a subspace of dimension $n+1$. Now, consider a set $W \subseteq U$, let L be the set of lines of U that meet W and suppose that there exists an injective map $\nu : L \to U \setminus W$ such that $\nu(l) \in l$ for all $l \in L$. Then

$$\{C(\nu(l), l) \mid l \in L\} \cup \{C(P, \emptyset) \mid P \in U \setminus (\nu(L) \cup W)\}$$

is a set of co-cliques of Γ whose union contains all vertices of Γ.

Finally, we provide an example of a set W with q points and a map ν satisfying these conditions. Let $l := \{P_0, \dots, P_q\} \leq U$ be a line and set $W := \{P_1, \dots, P_q\}$. For all planes $\pi \in \mathfrak{G}[l, 2, U]$ and all $P \in W$ let $l_P(\pi) \in \mathfrak{G}[P_0, 1, \pi]$ be such that $\{l_P(\pi) : P \in W\} = \mathfrak{G}[P_0, 1, \pi] \setminus \{l\}$. Then we may define ν by $\nu(l) = P_0$ and $\nu(g) := g \cap l_{l \cap g}(\langle l, g \rangle)$ for all $g \in L \setminus \{l\}$.

Remark 2.3.6. *i) We note that one can also find coverings of $\mathcal{V}(\Gamma)$ by replacing all co-cliques of coverings described in this example by their dual structure.*

ii) Since there are $\theta_{n+1} - q$ co-cliques in a covering given in this example we find $\chi(\Gamma) \leq \theta_{n+1} - q$.

2.3.2 Coverings by a Set of Special Co-Cliques

In this section we consider coverings of $\mathcal{V}(\Gamma)$ by a set $\mathcal{C}$ of co-cliques that satisfies conditions (I), (II), (III) and (IV) given below and, in fact, prove the following theorem.

Theorem 2.3.7. *Let $\mathcal{C}$ be a set of $\theta_{n+1} - q$ co-cliques of Γ whose union covers all vertices of Γ such that conditions (I), (II), (III) and (IV) given below hold. Then there is an $(n+1)$-dimensional subspace U of $\mathbb{P}$ such that every co-clique $C \in \mathcal{C}$ is point based on a point $P \in U$ and the base points all of co-cliques in $\mathcal{C}$ are pairwise distinct.*

The proof of this theorem is split into two steps. In the first step we construct an $(n+1)$-dimensional subspace U as a candidate for the subspace U given in the theorem and in the second step we conclude the proof.

Therefore, throughout this section we let $\mathcal{C} = \{C_1, \dots, C_{\theta_{n+1}-q}\}$ be a set of $\theta_{n+1} - q$ co-cliques of Γ whose union covers all vertices of Γ such that

(I) all co-cliques in $\mathcal{C}$ are distinct and non-empty,

(II) every co-clique $C \in \mathcal{C}$ with $|C| > e_1$ is maximal and thus given by Examples 2.1.15 and 2.1.17 and covered by Notation 2.3.1,

(III) the generic parts of any two distinct co-cliques with $C, C' \in \mathcal{C}$ of size larger than e_1 are distinct and

(IV) at least half of the co-cliques of $\mathcal{C}$ of size larger than e_1 is based on points.

We let I denote all $i \in \{1, \dots, \theta_{n+1} - q\}$ such that C_i is based on a point P_i.

Construction of the Subspace U

In this section we construct a subspace U of dimension $n+1$ that contains the points P_i for *many* elements $i \in I$.

Lemma 2.3.8. *For every subset $\mathcal{G}$ of $\mathcal{C}$ we have*

$$\left| \bigcup_{C \in \mathcal{G}} C \right| \geq |\mathcal{G}| e_0 - \left(q^2 + \frac{9}{2} q + 10 \right) q^{n^2+2n-3}.$$

Proof. Since $|C| \leq e_0$ for every $C \in \mathcal{C}$, it is sufficient to prove the statement in the case when $\mathcal{G} = \mathcal{C}$. Then $|\mathcal{G}| = \theta_{n+1} - q$ and $|\bigcup_{C \in \mathcal{G}} C|$ is equal to $\mathfrak{s}_q[n, 2n] \cdot \theta_n$, the number of all vertices of Γ. Using this, a direct calculation proves the statement for $n \in \{3, 4\}$. For $n \geq 5$ we use $|\mathcal{G}| \leq q\theta_n$ and find

$$\begin{aligned}
|\mathcal{G}| e_0 - \left| \bigcup_{C \in \mathcal{G}} C \right| &\leq \theta_n (\mathfrak{s}_q[n, 2n-1] \cdot \theta_n \cdot q + \mathfrak{s}_q[1, n, 2n] \cdot q^{n+1} - \mathfrak{s}_q[n, 2n]) \\
&= \theta_n \mathfrak{s}_q[1, n, 2n] \cdot \left(\frac{q^{2n} - 1}{q^{n+1} - 1} \theta_n q + q^{n+1} - \frac{(q^{2n+1} - 1)(q^n + 1)}{q^{n+1} - 1} \right) \\
&\leq \theta_n \cdot \mathfrak{s}_q[1, n, 2n] \cdot q^{2n-3} (q^2 + q + 2) \\
&\leq \theta_n (q+2)(q^2 + q + 2) q^{n^2+n-4},
\end{aligned}$$

where the second last step uses $n \geq 5$ and the last step uses Lemma 1.2.36. From the assumed lower bounds in Equation (2.26) we deduce $\theta_n \leq \left(q + \frac{3}{2}\right) q^{n-1}$. Using Equation (2.26) again as well as $n \geq 5$ we see that

$$\leq \left(q + \frac{3}{2} \right) (q+2)(q^2 + q + 2) \leq q^2 \left(q^2 + \frac{9}{2} q + 10 \right),$$

concluding this proof. □

Lemma 2.3.9. *Let U be an $(n+1)$-dimensional subspace. Denote by c_1 the number of indices $i \in I$ with $P_i \notin U$ and by c_3 the number of co-cliques $C \in \mathcal{C}$ with $|C| \leq e_1$. Then there is $x \in \{c_1, |I| - c_1\}$ with $x + 2c_3 \leq 2(q + 4 + \alpha_n) q^{n-1}$.*

Proof. From (IV) we have $|I| \geq \frac{1}{2}(|\mathcal{C}| - c_3)$ and we know that for all $i \in I$ the set C_i is based on a point P_i. We define $J := \{i \in I : P_i \notin U\}$. Then, for all $j \in J$ and all $i \in I \setminus J$ the generic parts of the sets C_i and C_j share the $\mathfrak{s}_q[n, 1, n-1, 2n] \cdot \theta_n = q^{n^2-n-2} \theta_n$ flags $(\pi, \tau) \in \mathcal{V}(\Gamma)$ with $P_i = \pi \cap U$ and $P_j \in \pi$. For given $j \in J$ it is obvious that distinct i in $I \setminus J$ yield distinct $q^{n^2-n-2} \theta_n$ such flags. Hence, we know that for all $j \in J$ the set

C_j contains at least $|I \setminus J| \cdot q^{n^2-n-2}\theta_n$ flags that are contained in C_i for some $i \in I \setminus J$. Therefore,

$$\left|\bigcup_{i\in I} C_i\right| \le |I|e_0 - \underbrace{|J||I \setminus J|}_{=c_1(|I|-c_1)}\theta_n \cdot q^{n^2-n-2}$$

and Lemma 2.3.8 applied to the set $\mathcal{G} := \{C_i \mid i \in I\} \cup \{C \in \mathcal{C} : |C| \le e_1\}$ shows

$$c_3(e_0 - e_1) + c_1(|I| - c_1)\theta_n \cdot q^{n^2-n-2} \le A := \left(q^2 + \frac{9}{2}q + 10\right) q^{n^2+2n-3}.$$

In particular, we already have $c_3(e_0 - e_1) \le A$ and we set $B := (q + 4 + \alpha_n)q^{n-1}$. Using the lower bounds for q assumed in Equation (2.26) we find $B(e_0 - e_1) > A$ and hence we have $c_3 < B$. It remains to show that one of the numbers in $\{c_1, |I| - c_1\}$ is at most $2(B - c_3)$. Suppose that this is wrong, that is, we have

$$c_1(|I| - c_1) \ge 2(B - c_3)(|I| - 2B + 2c_3).$$

Since $|I| \ge \frac{1}{2}(|\mathcal{C}| - c_3)$, it follows that $f(c_3) \le A$ where f is the polynomial in x given by

$$f := x(e_0 - e_1) + 2(B - x)\left(\frac{1}{2}(|\mathcal{C}| - x) - 2B + 2x\right)\theta_n \cdot q^{n^2-n-2}.$$

Since f has degree two with negative leading coefficient and since $0 \le c_3 < B$, we have $\min\{f(0), f(B)\} \le f(c_3)$, that is, $f(0) \le A$ or $f(B) \le A$. But $f(B) = B(e_0 - e_1)$ and we have already seen that this is larger than A, that is, we have

$$2B\left(\frac{1}{2}|\mathcal{C}| - 2B\right)\theta_n q^{n^2-n-2} = f(0) \le A.$$

Using $|\mathcal{C}| = \theta_{n+1} - q \ge (q + 1)q^n$ and $\theta_n \ge (q + 1)q^{n-1}$, it follows that

$$(q + 1)B((q + 1)q^n - 4B) \le \left(q^2 + \frac{9}{2}q + 10\right) q^{2n}.$$

and using the definition of B this shows

$$(q + 1)(q + 4 + \alpha_n)(q^2 - 3q - 16 - 4\alpha_n) \le \left(q^2 + \frac{9}{2}q + 10\right) q^2.$$

Since $5 \le \alpha_n \le q$ (via our assumptions), this inequality must also be satisfied when α_n is replaced by 5 or by q, but this contradicts the lower bounds for q that we assumed in Equation (2.26). $\square$

Lemma 2.3.10. *There exists an $(n + 1)$-dimensional subspace U such that*

$$|\{i \in I \mid P_i \notin U\}| + 2 \cdot |\{C \in \mathcal{C} : |C| \le e_1\}| \le 2(q + 4 + \alpha_n)q^{n-1}.$$

Proof. Let c_3 be the number of $C \in \mathcal{C}$ with $|C| \le g_0$ and thus $|C| \le e_1$. Then $\mathcal{C}$ contains $\beta \ge \frac{1}{2}(\theta_{n+1}-q-c_3)$ maximal co-cliques that are based on a point. Let $G_1,\dots,G_\beta$ denote these co-cliques, let $R_1,\dots,R_\beta$ be their respective base points and for all $i \in \{1,\dots,\beta\}$ set

$$g_i := \left| G_i \cap \bigcup_{j=1}^{i-1} G_j \right|$$

Then $|\cup_{j=1}^{i} G_j| \le ie_0 - \sum_{j=1}^{i} g_i$ for all $i \le \beta$. We may assume that the sequence $g_1,\dots,g_\beta$ is monotone increasing.

We first show that $j := \lceil \frac{1}{4}q^{n+1}\rceil + \theta_{n+1} + \theta_{n-1} - n$ satisfies $g_j < 5q^{n^2-2}\theta_{n+1}$. Indeed, otherwise we would have $\sum_{i=j}^{\beta} g_i \ge (\beta-j+1)5q^{n^2-2}\theta_{n+1}$ and Lemma 2.3.8 implies

$$(\beta-j+1)5q^{n^2-2}\theta_{n+1} + c_3(e_0-e_1) \le \left(q^2+\frac{9}{2}q+10\right)q^{n^2+2n-3}.$$

If we substitute the lower bound for β given above we see that the coefficient of c_3 therein is $e_0 - e_1 - \frac{5}{2}q^{n^2-2}\theta_{n+1}$ and in view of Equation (2.26) this is positive. Hence, we may assume $c_3 = 0$ and find

$$5q^{n^2-2}\theta_{n+1}\cdot\left(\frac{1}{4}q^{n+1} - \frac{1}{2}(\theta_{n+1}+q) - \theta_{n-1} + n - 1\right) \le \left(q^2+\frac{9}{2}q+10\right)q^{n^2+2n-3}$$

but this contradicts the assumption in Equation (2.26). Therefore, $g_j < 5q^{n^2-2}\theta_{n+1}$.

Now, let $Q_1,\dots,Q_{n+1} \in \{R_{j-\theta_{n-1}},\dots,R_j\}$ be such that $\tau := \langle Q_1,\dots,Q_{n+1}\rangle$ is an n-dimensional subspace and set

$$\mathcal{R} := \{R_i : i \in \{1,\dots,j-\theta_{n-1}-1\} \wedge R_i \notin \tau\}.$$

Then $|\mathcal{R}| \ge j - \theta_{n-1} - 1 - (|\tau| - n - 1) = \lceil \frac{1}{4}q^{n+1}\rceil$.

In the next step we show that for all $i \in \{1,\dots,n+1\}$ the point Q_i lies on fewer than $7q^n$ lines that meet $\mathcal{R}$. Assume that this is false and let $i \in \{1,\dots,n+1\}$ be such that Q_i lies on at least $7q^n$ lines that meet $\mathcal{R}$. Each of these lines lies in $\mathfrak{s}_q[1,n-1,2n]$ subspaces of dimension $n-1$ and two of these lines occur together in $\mathfrak{s}_q[2,n-1,2n]$ such subspaces. Hence there exist at least

$$\begin{aligned} 7q^n(\mathfrak{s}_q[1,n-1,2n] - 7q^n\mathfrak{s}_q[2,n-1,2n]) &= 7q^n\left(\frac{q^{2n-1}-1}{q^{n-2}-1} - 7q^n\right)\mathfrak{s}_q[2,n-1,2n] \\ &\ge 7q^{n^2-3}(q-7) =: z \end{aligned}$$

$(n-1)$-dimensional subspaces that contain one of the $7q^n$ lines. This shows that there exist $z\theta_{n+1}$ flags (E,S) with $Q_i \in E$ and such that E contains a point of $\mathcal{R}$. Since $Q_i = R_k$ for some $k \le j$, this implies that $z\theta_{n+1} \le g_k \le g_j < 5q^{n^2-2}\theta_{n+1}$, which is a contradiction.

Finally, we apply Proposition 1.2.41 with $d_0 = |\mathcal{R}|/q^{n+1} \ge \frac{1}{4}$, $n_0 = 7$ and $M := \mathcal{R}$ to find an $(n+1)$-dimensional subspace U satisfying Equation (1.3). Using the assumed lower bounds for q assumed in Equation (2.26) we conclude that $|U \cap R| \ge 3q^n > 2q^n + 2(4+\alpha_n)q^{n-1}$. The statement of the lemma follows now from Lemma 2.3.9. $\square$

Proof of Theorem 2.3.7

From now on we let U be the $(n+1)$-dimensional subspace provided by Lemma 2.3.10 and use the following notation.

- $\mathcal{C}_0 := \{C_i \in \mathcal{C} \mid i \in I,\ P_i \in U\}$, $c_0 := |\mathcal{C}_0|$.
- $\mathcal{C}_1 := \{C_i \in \mathcal{C} \mid i \in I,\ P_i \notin U\}$, $c_1 := |\mathcal{C}_1|$.
- $\mathcal{C}_2 := \{C_i \in \mathcal{C} \mid i \notin I, |C_i| > g_0\}$, $c_2 := |\mathcal{C}_2|$.
- $\mathcal{C}_3 := \{C_i \in \mathcal{C} \mid i \notin I, |C_i| \leq g_0\}$, $c_3 := |\mathcal{C}_3|$.
- $W := \{P \in U \mid P \neq P_i \forall i \in I\}$.
- $M := \{(\pi, \tau) \in \bigcup_{C \in \mathcal{C}} C \mid \pi \cap U \text{ is a point and } \pi \cap U \in W\}$.

Remark 2.3.11. *Using Remark 2.3.4 we know that, due to (II), all co-cliques $C \in \mathcal{C}_3$ satisfy $|C| \leq e_1$.*

Lemma 2.3.12. *(a) $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ is a partition of $\mathcal{C}$.*

(b) $c_1 + 2c_3 \leq 2(q + 4 + \alpha_n)q^{n-1}$.

(c) $|W| = \theta_{n+1} - c_0$.

(d) For all $P \in W$ there are exactly $q^{n^2-1}\theta_n$ flags (π, τ) with $\pi \cap U = P$.

(e) $|M| = |W| q^{n^2-1}\theta_n$.

(f) $|I| = c_0 + c_1 \geq \frac{1}{2}(\theta_{n+1} - q - c_3)$.

Proof. Claim (a) is obvious from the choice of $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ and the choice of U implies (b). From (III) we know that the base points P_i of the sets C_i, $i \in I$, are pairwise distinct, which proves $|W| = |U| - |\mathcal{C}_0| = \theta_{n+1} - c_0$ and thus (c). We know that each point $P \in W$ lies on $\mathfrak{s}_q[n, 0, n-1, 2n] = q^{n^2-1}$ subspaces of dimension $n-1$ that meet U only in P and each such subspace lies in $\mathfrak{s}_q[n-1, n, 2n] = \theta_n$ subspaces of dimension n. Hence, for every point P in W exactly $q^{n^2-1}\theta_n$ flags (π, τ) of M satisfy $\pi \cap U = P$, which proves (d) and (e). To see (f) we first note that our definitions imply $|I| = c_0 + c_1$ and that exactly $|\mathcal{C}| - c_3 = \theta_{n+1} - q - c_3$ elements of $\mathcal{C}$ have more than g_0 elements. Finally, we recall that (II) implies that every element of $\mathcal{C}$ with more than g_0 elements is based on a point or a hyperplane and that (IV) implies that at least half of these are based on a point. $\square$

Notation 2.3.13. Recall from Lemma 2.3.2 that the special parts of all co-cliques covered by Notation 2.3.1 — in particular of all co-cliques $C \in \mathcal{C}$ with $|C| > g_0$ — have cardinality at most Δ, where

$$\Delta := \mathfrak{s}_q[1, n, 2n] \cdot q^n \overset{1.2.35\text{ ii)}}{=} \mathfrak{s}_q[n-2, 2n-2] \cdot q^n \overset{1.2.36}{\leq} (q+2)q^{n^2-1}. \tag{2.28}$$

Lemma 2.3.14. *(a) For $C \in \mathcal{C}_0$ the generic part of C does not contain a flag of M.*

(b) For $C \in \mathcal{C}_1$ we have $|C \cap M| \le |W|q^{n^2-n-2}\theta_n + \Delta$.

(c) Any co-clique $C \in \mathcal{C}_2$ is based on a hyperplane H and we have

$$|C \cap M| \le \begin{cases} \Delta & \text{if } U \le H, \\ \Delta + |H \cap W|q^{n^2-n}\theta_{n-1} & \text{otherwise.} \end{cases}$$

Proof. (a) Consider some $C \in \mathcal{C}_0$. For all flags (π, τ) of the generic part of C we have $\dim(\pi \cap U) \ge 1$ or $\pi \cap U$ is the base point of C. Since M only contains flags (π, τ) such that π meets U in a point that is not a base point of the generic part of some $C \in \mathcal{C}$, this implies that these flags do not belong to M.

(b) Consider some $C \in \mathcal{C}_1$. Then C is based on a point P with $P \notin U$. If $Y \in W$, then the point P lies on exactly $\mathfrak{s}_q[n, 1, n-1, 2n] = q^{n^2-n-2}$ subspaces π of dimension $n-1$ satisfying $\pi \cap U = Y$. Each of these lies in $\mathfrak{s}_q[n-1, n, 2n] = \theta_n$ subspaces of dimension n. Hence, the generic part of C contains exactly $|W|q^{n^2-n-2}\theta_n$ flags of M. The special part of C contains at most Δ flags and thus at most this many flags of M.

(c) Consider some $C \in \mathcal{C}_2$. Since C is not based on a point and has cardinality greater than g_0 (II) shows that C is based on a hyperplane H. The generic part of C consists of all flags $(\pi, \tau) \in \mathcal{V}(\Gamma)$ with $\tau \le H$ and thus also $\pi \le H$. If $Y \in H \cap W$, then the number of $(n-1)$-dimensional subspaces π of H with $\pi \cap U = Y$ is $\mathfrak{s}_q[n, 0, n-1, 2n-1] = 0$ for $U \le H$ and it is $\mathfrak{s}_q[n-1, 0, n-1, 2n-1] = q^{n^2-n}$ for $U \not\le H$ (because then a complement of Y in $U \cap H$ has dimension $n-1$). Since for every $(n-1)$-dimensional subspace of H the number of n-dimensional subspaces of H containing it is $\mathfrak{s}_q[n-1, n, 2n-1] = \theta_{n-1}$ (see Lemma 1.2.35 ii)), it follows that the generic part of C contains no flag of C for $U \le H$ and exactly $|H \cap W|q^{n^2-n}\theta_{n-1}$ flags of M for $U \not\le H$. Finally, since the special part of C contains at most Δ flags, this implies the claim. □

Lemma 2.3.15. *If z is an integer such that there is at most one hyperplane of U which contains more than z points of W, then*

$$|M| \le (c_0 + c_1 + c_2)\Delta + c_1|W|q^{n^2-n-2}\theta_n + c_2 z q^{n^2-n}\theta_{n-1} + c_3 e_1 + q^{n^2-1}\theta_{n-1}\theta_n.$$

Proof. Let z be as in the claim. Lemma 2.3.12 (e) shows $|M| = |W|q^{n^2-1}\theta_n$ and, since every flag of M is covered by some $C \in \mathcal{C}$, we may apply Lemma 2.3.12 (a) to see

$$|M| = |W|q^{n^2-1}\theta_n \le \sum_{i=0}^{3} \sum_{C \in \mathcal{C}_i} |C \cap M|.$$

Now, if there exists a hyperplane of U with more than z points in W, then let z' denote the number of its points in W and otherwise set $z' := z$. Since every hyperplane of U lies in $\mathfrak{s}_q[0,n,2n-1,2n] = q^{n-1}$ hyperplanes of $\mathbb{P}$ which do not contain U, Lemma 2.3.14 (c) shows

$$\left|\bigcup_{C\in\mathcal{C}_2} C\cap M\right| \le \left(c_2 - q^{n-1}\right)(\Delta + zq^{n^2-n}\theta_{n-1}) + q^{n-1}(\Delta + z'q^{n^2-n}\theta_{n-1})$$
$$= c_2(\Delta + zq^{n^2-n}\theta_{n-1}) + \underbrace{(z'-z)}_{\le\theta_n} q^{n^2-1}\theta_{n-1}.$$

Finally, since $|C| \le e_1$ for $C \in \mathcal{C}_3$, the assertion follows from parts (a) and (b) of Lemma 2.3.14. □

Lemma 2.3.16. *Let τ_1 and τ_2 be distinct hyperplanes of U and set $W' := (\tau_1 \cup \tau_2) \cap W$. Then*

$$|W'|\theta_{n-1} \le (c_1 + c_2)q^{n-3}(2q+7) + q^{2n-3}((\alpha_n + 3)q + \alpha_n^2 + 4\alpha_n).$$

Proof. We set $M' := \{(\pi,\tau) \in M : \pi \cap U \in W'\}$. Lemma 2.3.12 (d) shows $|M'| = |W'|q^{n^2-1}\theta_n$ and according to Lemma 2.3.12 (a) each flag of M' lies in at least one of the co-cliques of $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. Hence, we have

$$|W'|q^{n^2}\theta_{n-1} \le |W'|q^{n^2-1}\theta_n = |M'| \le d_0 + d_1 + d_2 + d_3, \tag{2.29}$$

where for all $i \in \{1,\dots,4\}$ we let d_i denote the number of elements of M' that lie in some member of $\mathcal{C}_i$. We remark that we have $|W'| \le |(\tau_1 \cup \tau_2) \cap U| = q^n + \theta_n$ and determine upper bounds on the numbers $d_0,\dots,d_3$ in 4 steps.

First, we consider a co-clique $C \in \mathcal{C}_0$. Then $|C| \ge g_0$ and C is based on a point $P \in U$. We know from Lemma 2.3.14 (a) that only the special part T of C may contribute to M' and thus we study T and the three possible structures that T may have.

- First, assume that there is a line l with $P \in l$ such that T consists of all flags $(\pi,\tau) \in \mathcal{V}(\Gamma)$ with $l \le \tau$ and $P \notin \pi$. Then we have

$$|T\cap M'| = \begin{cases} |l\cap W'|\cdot\mathfrak{s}_q[n,0,n-1,2n] & \text{for } l \le U,\\ |W'|\cdot(|l|-1)\cdot\mathfrak{s}_q[n,1,n-1,2n] & \text{for } l\cap U = P\end{cases}$$
$$= \begin{cases} |l\cap W'|q^{n^2-1} & \text{for } l \le U,\\ |W'|q^{n^2-n-1} & \text{for } l\cap U = P\end{cases}$$

  and, using $|W'| \le q^n + \theta_n$ as well as the fact that $|l\cap W'|$ is at most q for $P \in \tau_1\cup\tau_2$ and at most 2 otherwise, we have

$$|T\cap M'| \le \begin{cases} q^{n^2} & \text{for } P\in\tau_1\cup\tau_2,\\ (q^n+\theta_n)q^{n^2-n-1} & \text{otherwise.}\end{cases}$$

- Secondly, assume that there is a hyperplane H with $P \in H$ such that T consists of all flags $(\pi,\tau) \in \mathcal{V}(\Gamma)$ with $P \in \tau \le H$ and $P \notin \pi$. If $U \le H$, then for each $(\pi,\tau) \in T$ we have $\dim(\pi \cap U) \ge 1$ and thus $(\pi,\tau) \notin M'$. Therefore, if $U \le H$, then $T \cap M' = \emptyset$. Now suppose that $U \not\le H$. Then we have

$$\begin{aligned}|T \cap M'| &= |H \cap W'| \cdot \mathfrak{s}_q[n-1,0,n-1,2n-1] = |H \cap W'|q^{n^2-n}\\ &\le \begin{cases}|W' \cap \tau_i|q^{n^2-n} & \text{if } H \cap U = \tau_i \text{ for some } i \in \{1,2\},\\ (q^{n-1}+\theta_{n-1})q^{n^2-n} & \text{otherwise.}\end{cases}\end{aligned}$$

Notice that $H \cap U = \tau_i$ for some $i \in \{1,2\}$ implies $P \in \tau_i$ and thus $|W' \cap \tau_i| \le \theta_n - 1 = q\theta_{n-1}$. Therefore, we have

$$|T \cap M'| \le \begin{cases}\theta_{n-1}q^{n^2-n+1} & \text{for } P \in \tau_1 \cup \tau_2,\\ (q^{n-1}+\theta_{n-1})q^{n^2-n} & \text{otherwise.}\end{cases}$$

- Finally, if the special part T is not based on a line or a hyperplane, then Lemma 2.3.2 shows

$$|T \cap M'| \le |T| \le q^n\left(1+\frac{1}{q}\right)\theta_{n-2}\theta_{n-1}^{n-1} \overset{1.2.36\ (c)}{\le} \theta_1(q+n)\theta_{n-2}q^{n^2-n-1}.$$

Using the lower bounds for q provided by our assumption in Equation (2.26) we may summarize these three upper bounds into

$$|T \cap M'| \le \begin{cases}\theta_{n-1}q^{n^2-n+1} & \text{for } P \in \tau_1 \cup \tau_2,\\ (q^n+\theta_n)q^{n^2-n-1} & \text{otherwise.}\end{cases}$$

Note that the bound given for $P \in \tau_1 \cup \tau_2$ is a weaker bound than the bound for $P \notin \tau_1 \cup \tau_2$. Now, since distinct sets $C \in \mathcal{C}_0$ are based on distinct points P (see (III)) and since $\tau_1 \cup \tau_2$ contains $q^n + \theta_n$ points, we find

$$d_0 \le c_0(q^n+\theta_n)q^{n^2-n-1} + (q^n+\theta_n)\theta_{n-1}q^{n^2-n+1} \le 12q^{n^2+n},$$

where the last step uses the trivial bounds $c_0 \le \theta_{n+1} \le 2q^{n+1}$, $\theta_n \le 2q^n$ and $\theta_{n-1} \le 2q^{n-1}$.

Secondly, for $C \in \mathcal{C}_1$ we see that C contains at most $|W'|q^{n^2-n-2}\theta_n + \Delta$ flags of M' analogously to the proof of 2.3.14 (b), which proves $d_1 \le c_1(|W'|q^{n^2-n-2}\theta_n + \Delta)$. Now we use $\Delta \le (q+2)q^{n^2-1}$ given in Equation (2.28) as well as $|W'| \le q^n + \theta_n$ and have

$$\begin{aligned}d_1 &\le c_1q^{n^2-n-2}((q^n+\theta_n)\theta_n + q^{n+2} + 2q^{n+1})\\ &\overset{1.2.36\ (a)}{\le} c_1q^{n^2-1}(2q^{n-1}+6q^{n-2}+4q^{n-3}+q+2) \overset{(2.26)}{\le} c_1q^{n^2+n-3}(2q+7).\end{aligned}$$

Thirdly, we consider $C \in \mathcal{C}_2$. Then $|C| \ge g_0$ and C is based on a hyperplane H. If $U \subseteq H$ and (π,τ) is a flag of C, then $\dim(\pi \cap U) \ge 1$ and thus $C \cap M' = \emptyset$. Therefore, we only need to study the case $U \not\leq H$, which implies $\dim(U \cap H) = n$. Then, analogously to the proof of 2.3.14 (c), we see that the number of flags of M' in the generic part of C is $|H \cap W'|q^{n^2-n}\theta_{n-1}$ and we have

$$|H \cap W'|q^{n^2-n}\theta_{n-1} \leq \begin{cases} |W' \cap \tau_i|q^{n^2-n}\theta_{n-1} & \text{if } H \cap U = \tau_i \text{ for some } i \in \{1,2\}, \\ (q^{n-1} + \theta_{n-1})q^{n^2-n}\theta_{n-1} & \text{otherwise.} \end{cases}$$

Since there are exactly q^{n-1} hyperplanes that meet U in τ_1 and as many that meet U in τ_2, it follows that the number of flags of M' that lie in the generic part of at least one co-clique of $\mathcal{C}_2$ is at most

$$c_2(q^{n-1} + \theta_{n-1})q^{n^2-n}\theta_{n-1} + q^{n^2-1}\left(|W' \cap \tau_1| + |W' \cap \tau_2|\right)\theta_{n-1}.$$

The special part of each co-clique of $\mathcal{C}_2$ has Δ flags and thus at most this many flags of M'. Using

$$|W' \cap \tau_1| + |W' \cap \tau_2| \leq |W'| + \theta_{n-1} \leq q^n + \theta_n + \theta_{n-1} = 2\theta_n \overset{1.2.36\text{ (a)}}{\leq} 2(\theta_1 + 1)q^{n-1}$$

it follows that

$$d_2 \leq c_2\Delta + c_2(q^{n-1} + \theta_{n-1})q^{n^2-n}\theta_{n-1} + 2q^{n^2+n-2}(\theta_1 + 1)\theta_{n-1}.$$

We now show that this bound implies

$$d_2 \leq c_2 q^{n^2+n-3}(2q + 7) + 2q^{n^2+2n-4}(q^2 + 4q + 4). \tag{2.30}$$

For $n = 3$, this can be easily verified for all $q > 2$ and thus for all values of q that are of interest here. For $n \geq 4$, we use $\Delta \leq (q + 2)q^{n^2-1}$ given in Equation (2.28) as well as the upper bound given in Lemma 1.2.36 (a) to find

$$d_2 \leq c_2 q^{n^2-1}(2q^{n-1} + 6q^{n-2} + 4q^{n-3} + q + 2) + 2q^{n^2+2n-4}(\theta_1 + 1)^2$$

and the lower bounds for q assumed in Equation (2.26) imply Equation (2.30).

Finally, we note that for $C \in \mathcal{C}_3$ we trivially have $|C \cap M'| \leq |C| \leq e_1$ and, using $c_3 \leq (q + 4 + \alpha_n)q^{n-1}$ from Lemma 2.3.12 (b) as well as $e_1 = \alpha_n q^{n^2+n-2}$, this shows

$$d_3 \leq c_3 e_1 \leq (\alpha_n q + \alpha_n^2 + 4\alpha_n)q^{n^2+2n-3}.$$

Now, substituting these upper bounds for $d_0, \ldots, d_3$ in Equation (2.29) and dividing by q^{n^2} yields

$$\begin{aligned} |W'|\theta_{n-1} \leq (c_1 + c_2)q^{n-3}(2q + 7) + (\alpha_n q + \alpha_n^2 + 4\alpha_n)q^{2n-3} \\ + q^{n-1}(2q^{n-1} + 8q^{n-2} + 8q^{n-3} + 12q) \end{aligned}$$

and, using the lower bounds for q assumed in (2.26), this implies the claim. □

Lemma 2.3.17. *We have* $|W| \le (\alpha_n + 3)q^{n-1}$.

Proof. Let π_1 and π_2 be hyperplanes of U such that $|\pi_1 \cap W| \ge |\pi_2 \cap W| \ge |\pi \cap W|$ for every hyperplane π of U other than π_1 and set $z := |\pi_2 \cap W|$. Furthermore, we set $\delta := c_1 + c_2 + c_3$ and use $c_0 + c_1 + c_2 + c_3 = |\mathcal{C}| = \theta_{n+1} - q$ as well as $|W| = \theta_{n+1} - c_0 = \delta + q$ from Lemma 2.3.12 (c) and $|M| = |W|q^{n^2-1}\theta_n$ from Lemma 2.3.12 (e) to see that the inequality given by Lemma 2.3.15 is equivalent to

$$\begin{aligned} 0 \le {} & (\theta_{n+1} - q)\Delta + q^{n^2-1}\theta_{n-1}\theta_n + c_3(e_1 - \Delta) \\ & + c_2 q^{n^2-n} z\theta_{n-1} + (\delta + q)(c_1 - q^{n+1})q^{n^2-n-2}\theta_n. \end{aligned}$$

We simplify this inequality in several steps. First, we consider the following four trivial simplifications:

- in the first term, since Δ is positive, we may replace $(\theta_{n+1} - q)$ by its upper bound $(q+2)q^n$ given in Lemma 1.2.36 (a);
- in the second term we use $q^{n^2-1}\theta_{n-1}\theta_n \le (q+5)q^{n^2+2n-3}$, which follows from Lemma 1.2.36 (a) and the lower bounds for q that we assumed in Equation (2.26);
- in the third term, since the coefficient $e_1 - \Delta$ of c_3 is positive (recall the definition of e_1 as well as Inequality (2.28) and use $\alpha_n \ge 5$), we may replace c_3 by its upper bound $(q + 4 + \alpha_n)q^{n-1}$ given in Lemma 2.3.12 (b);
- and last but not least, in the final term, since $c_1 - q^{n+1}$ is negative (consider the upper bound $c_1 \le 2(q + 4 + \alpha_n)q^{n-1}$ given in Lemma 2.3.12 (b)), we may replace $(\delta + q)q^{n^2-n-2}\theta_n$ by its lower bound $\delta(q+1)q^{n^2-3}$ implied by Lemma 1.2.36 (a).

This yields

$$\begin{aligned} 0 \le {} & (q+2)q^n\Delta + (q+5)q^{n^2+2n-3} + (q+4+\alpha_n)q^{n-1}(e_1 - \Delta) \\ & + c_2 q^{n^2-n} z\theta_{n-1} + \delta(q+1)q^{n^2-3}(c_1 - q^{n+1}). \end{aligned} \tag{2.31}$$

Next we want to take care of the variable z in the fourth term on the right hand side of this inequality. For that purpose we note that the preceding lemma is applicable to the set $W' := (\pi_1 \cup \pi_2) \cap W$ and that W' satisfies

$$|W'| \ge |\pi_1 \cap W| + |\pi_2 \cap W| - \theta_{n-1} \ge 2z - \theta_{n-1},$$

that is,

$$2z\theta_{n-1} \le |W'|\theta_{n-1} + \theta_{n-1}^2 \overset{1.2.36\,(a)}{\le} |W'|\theta_{n-1} + (q+2)^2 q^{2n-4} \overset{(2.26)}{\le} |W'|\theta_{n-1} + 2q^{2n-2}.$$

We use the bound given in the previous lemma (where, for convenience, we replace the 7 by an 8) as well as $c_1 + c_2 \le \delta$ to replace the first term on the right hand side. Subsequently we divide by 2 and simplify, which yields

$$z\theta_{n-1} \le \delta q^{n-3}(q+4) + \frac{q^{2n-3}}{2}((\alpha_n + 5)q + \alpha_n^2 + 4\alpha_n). \tag{2.32}$$

Now, we reconsider Inequality (2.31):

- using the lower bounds for q that we assumed in Equation (2.26) we see that the coefficient $(q^2+q-4-\alpha_n)q^{n-1}$ of Δ therein is positive and thus we may replace Δ by its upper bound $(q+2)q^{n^2-1}$ given in Inequality (2.28);
- the coefficients of c_1 and c_2 therein are non-negative and so we may substitute c_1 and c_2 by their respective upper bounds $2(q+4+\alpha_n)q^{n-1}$ and δ, the first of which is given in Lemma 2.3.12 (b) and the second is trivial;
- we use the upper bound found in Inequality (2.32);
- we substitute $e_1 = \alpha_n q^{n^2+n-2}$ and, finally, we divide by q^{n^2-3}.

This yields

$$0 \le \delta^2(q+4) + \delta q^{n-1}\left(-q^3 + \frac{\alpha_n+7}{2}q^2 + \left(\frac{\alpha_n^2}{2} + 4\alpha_n + 10\right)q + (2\alpha_n+8)\right)$$
$$+ q^{2n}\left((\alpha_n+1)q + \alpha_n^2 + 4\alpha_n + 5\right) + q^{n+1}\left(q^3 + 3q^2 - (\alpha_n+2)q - (2\alpha_n+8)\right).$$

Using the lower bounds for q that we assumed in Equation (2.26) as well as $\alpha_n \ge 5$ this inequality implies

$$0 \le \delta^2(q+4) + \delta q^{n+1}\left(\frac{\alpha_n}{2} + 4 - q\right) + q^{2n}\left((\alpha_n+1)q + \alpha_n^2 + 5\alpha_n\right) + q^{n+3}(q+3).$$

Let f denote the right hand side of this Inequality and set $\delta_1 := (\alpha_n+3)q^{n-1} - q$ as well as $\delta_2 := q^{n+1} - \left(\frac{\alpha_n}{2}+8\right)q^n$. We want to show that δ does not lie in the interval $[\delta_1, \delta_2]$. To see this it suffices to show that $f(\delta_1) < 0$ and $f(\delta_2) < 0$ hold; the reason is that f is a quadratic polynomial in δ with positive leading coefficient. Straight forward calculations show

$$\begin{aligned} f(\delta_1) &= -2q^{2n+1} + \left(\frac{3}{2}\alpha_n^2 + \frac{21}{2}\alpha_n + 12\right)q^{2n} + (\alpha_n^2 + 6\alpha_n + 9)q^{2n-1} \\ &\quad + (4\alpha_n^2 + 24\alpha_n + 36)q^{2n-2} + q^{n+4} + 4q^{n+3} - \left(\frac{\alpha_n}{2} + 4\right)q^{n+2} \\ &\quad - (2\alpha_n+6)q^{n+1} - (8\alpha_n+24)q^n + q^3 + 4q^2 \text{ as well as} \\ f(\delta_2) &= -(\alpha_n+31)q^{2n+1} + (2\alpha_n^2 + 37\alpha_n + 256)q^{2n} + q^{n+4} + 3q^{n+3}, \end{aligned}$$

and in view of the lower bounds for q that we assumed in Equation (2.26) both of these are negative. Hence, $\delta \notin [\delta_1, \delta_2]$. Finally, we have

$$\begin{aligned} \delta &= \theta_{n+1} - q - c_0 \overset{2.3.12\text{ (f)}}{\le} \frac{1}{2}(\theta_{n+1} - q) + c_1 + \frac{1}{2}c_3 \\ &\overset{1.2.36\text{ (a)}}{\le} \frac{1}{2}(q^{n+1} + (q+2)q^{n-1} - q) + c_1 + \frac{1}{2}c_3 \\ &\overset{2.3.12\text{ (b)}}{\le} \frac{1}{2}(q^{n+1} + (q+2)q^{n-1} - q) + 2(q+4+\alpha_n)q^{n-1} \\ &\overset{(2.26)}{<} q^{n+1} - \left(\frac{\alpha_n}{2} + 8\right)q^n = \delta_2 \end{aligned}$$

and, since $\delta \notin [\delta_1, \delta_2]$, we find $\delta < \delta_1$, as claimed. □

Lemma 2.3.18. *We have $\mathcal{C} = \mathcal{C}_0$, that is, Theorem 2.3.7 holds.*

Proof. Obviously, if $\mathcal{C} = \mathcal{C}_0$, then our choice of notation in this section implies that Theorem 2.3.7 holds. Therefore, we prove $\mathcal{C} = \mathcal{C}_0$. To do so we determine the contribution of co-cliques $C \in \mathcal{C}$ to the flags in M. We set $\delta := c_1 + c_2 + c_3$ and note that $c_0 + c_1 + c_2 + c_3 = \theta_{n+1} - q$ implies $\delta = \theta_{n+1} - q - c_0$. Recall that from Lemma 2.3.12 (c) we have $|W| = q + \delta$ and from Lemma 2.3.17 we have $|W| \le (\alpha_n + 3)q^{n-1}$.

For all $C \in \mathcal{C}_1 \cup \mathcal{C}_2$ (using the trivial bound $\theta_n \le q^2\theta_{n-1}$ for $C \in \mathcal{C}_1$) we have

$$\begin{aligned}|C \cap M| &\overset{2.3.14}{\le} (\alpha_n+3)q^{n^2-1}\theta_{n-1} + \Delta \overset{1.2.36\text{ (a)}}{\le} (\alpha_n+3)(q+2)q^{n^2+n-3} + \Delta \\ &\overset{(2.28)}{\le} (\alpha_n+3)(q+2)q^{n^2+n-3} + (q+2)q^{n^2-1} \overset{(2.26)}{\le} (\alpha_n+4)q^{n^2+n-2}.\end{aligned}$$

Now, since we have $|C| \le e_1 = \alpha_n q^{n^2+n-2}$ for all $C \in \mathcal{C}_3$ (see Remark 2.3.11), we know that $|C \cap M| \le (\alpha_n+4)q^{n^2+n-2}$ holds for all $C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. Therefore, the total contribution of all co-cliques in $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ to M is at most $\delta(\alpha_n+4)q^{n^2+n-2}$.

Furthermore, the generic parts of all co-cliques in $\mathcal{C}_0$ are disjoint from M. Thus, it only remains to consider the special parts T of co-cliques $C \in \mathcal{C}_0$ and we denote by

- ω_1 the number of those with T based on a line that is contained in U;
- ω_2 the number of those with T based on a line that is not contained in U;
- ω_3 the number of those with T based on a hyperplane of $\mathbb{P}$; and
- ω_4 the number of the remaining ones, which, according to Lemma 2.3.2, are those with cardinality at most $q^{n-1}\theta_1\theta_{n-2}\theta_{n-1}^{n-1}$.

Furthermore, we let

- Ω_1 be the set of lines l of U such that $C(P,l) \in \mathcal{C}_0$ for some point P of l;
- Ω_3 be the set of all point-hyperplane pairs (P,H) with $C(P,H) \in \mathcal{C}_0$ such that U is not contained in H;
- Ω_4 be the set of indices $i \in I$ such that C_i is an element of $\mathcal{C}_0$ and its special part T has cardinality at most $q^{n-1}\theta_1\theta_{n-2}\theta_{n-1}^{n-1}$; and
- $\widehat{\Omega}_4$ be the set of all flags $f \in M$ such that f is an element of the special part of some co-clique C_x with $x \in \Omega_4$.

Then we have $\omega_1 + \omega_2 + \omega_3 + \omega_4 = c_0$, $|\Omega_1| \le \omega_1$, $|\Omega_3| \le \omega_3$ and $|\Omega_4| = \omega_4$. In view of the definition of Ω_3 we remark that hyperplane based special parts T only contribute to M when the underlying hyperplane of $\mathbb{P}$ does not contain U. Using Lemma 2.3.12 (e) it follows that

$$|W|q^{n^2-1}\theta_n = |M| \le \delta(\alpha_n+4)q^{n^2+n-2} + \sum_{l \in \Omega_1} |l \cap W| \cdot \mathfrak{s}_q[n,0,n-1,2n]$$

$$
\begin{aligned}
&+ \omega_2|W|q \cdot \mathfrak{s}_q[n,1,n-1,2n] + \sum_{(P,H)\in\Omega_3} |W| \cdot \mathfrak{s}_q[n-1,0,n-1,2n-1] + |\widehat{\Omega}_4| \\
&= q^{n^2-n}\left(\delta(\alpha_n+4)q^{2n-2} + \sum_{l\in\Omega_1} |l\cap W|q^{n-1} + \omega_2\frac{|W|}{q} + \sum_{(P,H)\in\Omega_3}|W| + \frac{|\widehat{\Omega}_4|}{q^{n^2-n}}\right).
\end{aligned}
$$

We simplify this inequality and begin by replacing the first sum by an upper bound. Since the product of two consecutive integers is always non-negative we have

$$
\begin{aligned}
0 &\le \sum_{l\in\Omega_1}(|l\cap W|-1)(|l\cap W|-2) \\
&= \sum_{l\in\Omega_1}(|l\cap W|-1)|l\cap W| - 2\sum_{l\in\Omega_1}|l\cap W| + 2|\Omega_1| \\
&\le |W|(|W|-1) - 2\sum_{l\in\Omega_1}|l\cap W| + 2|\Omega_1|,
\end{aligned}
$$

where the last step holds, since any pair of distinct points of W is contained in at most one line of Ω_1. Since $|\Omega_1| \le \omega_1$ and $\omega_1+\omega_2+\omega_3+\omega_4 = c_0 = \theta_{n+1} - |W|$, it follows that

$$
\sum_{l\in\Omega_1}|l\cap W| \le \frac{1}{2}|W|(|W|-3) + \theta_{n+1} - \omega_2 - \omega_3 - \omega_4.
$$

Using this as well as $|\Omega_3| = \omega_3$ in our inequality above and dividing by q^{n^2-n} we find

$$
\begin{aligned}
|W|q^{n-1}\left(\theta_n - \frac{|W|-3}{2}\right) \le \delta(\alpha_n+4)q^{2n-2} + \theta_{n+1}q^{n-1} + \omega_2\frac{|W|-q^n}{q} \\
+ \sum_{(P,H)\in\Omega_3}(|W|-q^{n-1}) + \frac{|\widehat{\Omega}_4| - \omega_4 q^{n^2-1}}{q^{n^2-n}}.
\end{aligned}
$$

Now, since $|W| \le (\alpha_n+3)q^{n-1}$ (by Lemma 2.3.17) and in view of the lower bounds for q that we assumed in Equation (2.26), the coefficient of ω_2 in the inequality above is not positive and therefore the term with ω_2 can be omitted. Doing so and substituting $|W| = \delta + q$ we find

$$
\begin{aligned}
(\delta+q)q^{n-1}\underbrace{\left(\theta_n - \frac{\delta+q-3}{2}\right)}_{\ge\theta_1 q^{n-1}-\delta} \le \delta(\alpha_n+4)q^{2n-2} + \theta_{n+1}q^{n-1} \\
+ \sum_{(P,H)\in\Omega_3}(|W|-q^{n-1}) + \frac{|\widehat{\Omega}_4| - \omega_4 q^{n^2-1}}{q^{n^2-n}}. \qquad (2.33)
\end{aligned}
$$

If $|W| \ge q^{n-1}$ then, since $|\Omega_3| \le |\mathcal{C}_0| = \theta_{n+1} - |W|$, we have

$$
\sum_{(P,H)\in\Omega_3}(|W|-q^{n-1}) \le (\theta_{n+1}-|W|)(|W|-q^{n-1}).
$$

Therefore, if $|W| \geq q^{n-1}$, then due to the fact that the polynomial $f(x) = (\theta_{n+1}-x)(x-q^{n-1}) - (x-q+1)q^{n+1}$ obtains its maximum for $x_{\max} := \frac{1}{2}(\theta_n + q^{n-1})$ and since

$$\begin{aligned}4(q-1)^2 \cdot f(x_{\max}) = &-q^{2n-2}(3q^4 - 6q^3 + q^2 + 2q - 1)\\ &+ q^{n-1}(4q^5 - 12q^4 + 12q^3 - 6q^2 + 2q - 2) + 1\end{aligned}$$

is negative (using (2.26)), it follows that

$$\sum_{(P,H)\in\Omega_3}(|W| - q^{n-1}) \leq (|W| - q + 1)q^{n+1} = (\delta+1)q^{n+1}.$$

Clearly, if we do not have $|W| \geq q^{n-1}$, then this equation holds trivially. Using this and $\theta_{n+1} \leq (q^2+q+2)q^{n-1}$ (see Lemma 1.2.36 (a)) in Inequality (2.33) we find

$$\begin{aligned}(\delta+q)q^{n-1}(\theta_1 q^{n-1} - \delta) \leq\ &\delta(\alpha_n + 4)q^{2n-2} + (q^2+q+2)q^{2n-2}\\ &+ q^{n+1}(\delta+1) + \frac{|\widehat{\Omega}_4| - \omega_4 q^{n^2-1}}{q^{n^2-n}},\end{aligned}$$

which is equivalent to

$$\begin{aligned}0 \leq\ &\delta^2 q^{n-1} - \delta q^n(q^{n-1} - (\alpha_n+3)q^{n-2} - q - 1)\\ &+ 2q^{2n-2} + q^{n+1} + \frac{|\widehat{\Omega}_4| - \omega_4 q^{n^2-1}}{q^{n^2-n}},\end{aligned} \tag{2.34}$$

Finally, we study the cardinality of $\widehat{\Omega}_4$. For all $x \in \Omega_4$ we know from Lemma 2.3.2 that $C_x = C(P_x, \mathcal{U}_x)$ for a set $\mathcal{U}_x$ of n-dimensional subspaces with $|\mathcal{U}_x| \leq (1+q^{-1})\theta_{n-2}\theta_{n-1}^{n-1}$. Furthermore, for all $x \in \Omega_4$ every flag (π,τ) of the special part of C_x that lies in $\widehat{\Omega}_4$ satisfies $\dim(\tau \cap U) = 1$ and $\pi \cap U$ is a point of $\tau \cap W$. Motivated by that, we define

$$\forall x \in \Omega_4 : \zeta_x := \max\{|\tau \cap W| : \tau \in \mathcal{U}_x, \dim(\tau \cap U) = 1\}$$

and set $\zeta := \max\{\zeta_x : x \in \Omega_4\}$. There are two remarks to be made:

- For all $x \in \Omega_4$ and all $\tau \in \mathcal{U}_x$ with $\dim(\tau \cap U) = 1$ we have $W \not\ni P_x \in \tau \cap U$, which implies $\zeta_x \leq q$. Thus we also have $\zeta \leq q$.
- The definition of ζ implies that there is a line $l \leq U$ with $|l \cap W| = \zeta$. Note that there are at most q elements $x \in \Omega_4$ such that P_x is an element of l. For all $x \in \Omega_4$ with $P_x \notin l$ and all $\tau \in \mathcal{U}_x$ with $\dim(\tau \cap U) = 1$ we clearly have $|\tau \cap l| \leq 1$, which implies $|\tau \cap W| \leq \min\{\zeta, |W| + 1 - \zeta\}$.

This implies

$$\begin{aligned}|\widehat{\Omega}_4| &\leq (q\zeta + \max\{0, \omega_4 - q\} \cdot \min\{\zeta, |W|+1-\zeta\})q^{n-1}\left(1+q^{-1}\right)\theta_{n-2}\theta_{n-1}^{n-1}\\ &\leq (q^2 + \omega_4 \cdot \min\{\zeta, |W|+1-\zeta\})(q^3 + (n+3)q^2 + (3n+2)q + 2n)q^{n^2-5},\end{aligned}$$

where the second step uses $\zeta \leq q$, $\max\{0, \omega_4 - q\} \leq \omega_4$ and parts (a) and (c) of Lemma 1.2.36. Substituting this in Inequality (2.34) and dividing by q^{n-5} yields

$$\begin{aligned}0 &\leq \delta^2 q^4 - \delta q^5(q^{n-1} - (\alpha_n + 3)q^{n-2} - q - 1) + 2q^{n+3} + q^6 \\ &\quad + (q^2 + \omega_4 \cdot \min\{\zeta, |W| + 1 - \zeta\})(q^3 + (n+3)q^2 + (3n+2)q + 2n) - \omega_4 q^4\end{aligned}$$

and we let $f = f_{\zeta,\omega_4}(\delta)$ denote the right hand side of this inequality. Note that f is quadratic in δ and the leading coefficient is q^4 and as such positive. Thus, if the inequality is not satisfied for two values δ_1 and δ_2, then it is not satisfied for any value in the interval $[\delta_1, \delta_2]$. We set $\delta_1 := n + 4$ as well as $\delta_2 := (\alpha_n + 3)q^{n-1}$ and recall that Lemma 2.3.17 shows $\delta = |W| - q < \delta_2$.

Now, if the coefficient

$$\eta = -q^4 + \min\{\zeta, |W| + 1 - \zeta\}(q^3 + (n+3)q^2 + (3n+2)q + 2n)$$

of ω_4 in that equation is not positive, then we may substitute $\omega_4 = 0$ to see that, in view of the lower bounds for q that we assumed in Equation (2.26), this bound is not satisfied by $\delta \in \{1, \delta_2\}$ (both of the following equations are smaller than 0):

$$\begin{aligned}f_{\zeta,0}(1) &= -q^{n+3}(q - \alpha_n - 5) + q^2(2q^4 + 2q^3 + (n+4)q^2 + (3n+2)q + 2n), \\ f_{\zeta,0}(\delta_2) &= -q^{2n+2}((\alpha_n + 3)q - 2\alpha_n^2 - 12\alpha_n - 18) \\ &\quad + q^{n+3}((\alpha_n + 3)q^2 + (\alpha_n + 3)q + 2) \\ &\quad + q^2(q^4 + q^3 + (n+3)q^2 + (3n+2)q + 2n).\end{aligned}$$

Thus, for $\eta \leq 0$ we have $\delta = 0$, that is, $|\mathcal{C}_0| = \theta_{n+1} - q - \delta = \theta_{n+1} - q = |\mathcal{C}|$ and thus $\mathcal{C} = \mathcal{C}_0$, as desired.

From now on we may assume $\eta > 0$ and derive a contradiction. Since the lower bounds for q that we assumed in Equation (2.26) implies $q \geq n^2 + 2n + 7$, the inequality $\eta > 0$ is equivalent to $\min\{\zeta, |W| + 1 - \zeta\} \geq q - n - 2$. Furthermore, we already remarked above that ζ is at most q and thus $\min\{\zeta, |W| + 1 - \zeta\}$ is at most q, too. However, if we substitute q as upper bound for ζ and consequently use $\min\{\zeta, |W| + 1 - \zeta\} \leq q$, then we have $\eta > 0$ and may also substitute $q^{n+1} + 2q^n$ as upper bound on ω_4 to see that, in view of the lower bounds for q that we assumed in Equation (2.26), the bound is not satisfied for $\delta \in \{\delta_1, \delta_2\}$ (both of the following equations are smaller than 0):

$$\begin{aligned}f_{q,q^{n+1}+2q^n}(\delta_1) &\leq -q^{n+1}(q^3 - (\alpha_n n + 8n + 4\alpha_n + 22)q^2 - (8n+4)q - 4n) \\ &\quad + q^2((n+5)q^4 + (n+5)q^3 + (n^2 + 9n + 19)q^2 + (3n+2)q + 2n), \\ f_{q,q^{n+1}+2q^n}(\delta_2) &\leq -q^{2n+2}((\alpha_n + 3)q - 2\alpha_n^2 - 12\alpha_n - 18) \\ &\quad + q^{n+1}((\alpha_n + 3)q^4 + (n + \alpha_n + 6)q^3 + (5n+10)q^2 + (8n+4)q + 4n) \\ &\quad + q^2(q^4 + q^3 + (n+3)q^2 + (3n+2)q + 2n).\end{aligned}$$

Hence, from now on we may assume $\delta < \delta_1$, that is, $\delta \leq n + 3$. Finally, we note that we then have $\min\{\zeta, |W| + 1 - \zeta\} \geq q - n - 2$, which implies $|W| + 1 \geq 2(q - n - 2)$. However, we have $|W| = q + \delta \leq q + n + 3$, so we find $q \leq 3n + 8$, a contradiction to the lower bounds for q that we assumed in Equation (2.26). $\square$

2.3.3 Determination of the Chromatic Number

Lemma 2.3.19. *We let $g : \mathcal{V}(\Gamma) \to F$ be a colouring of Γ with $|F| = \chi(\Gamma)$ and we set $\mathcal{C} := \{g^{-1}(f) : f \in F\}$. Every co-clique $C \in \mathcal{C}$ contains a flag f such that for all $C' \in \mathcal{C} \setminus \{C\}$ the set $C' \cup \{f\}$ is not a co-clique.*

Proof. In contrary to the claim, suppose that there is a set $C \in \mathcal{C}$ such that for all $f \in C$ there is a set $C' \in \mathcal{C} \setminus \{C\}$ such that $C' \cup \{f\}$ is a co-clique of Γ. Then there is a map $\phi : C \to \mathcal{C} \setminus \{C\}$ such that for all $f \in C$ the set $\phi(f) \cup \{f\}$ is a co-clique of Γ. Therefore, for all $C' \in \mathcal{C} \setminus \{C\}$ the set $C' \cup \phi^{-1}(C')$ is a co-clique, too, and we set $\mathcal{C}' := \{C \cup \phi^{-1}(C') : C' \in \mathcal{C} \setminus \{C\}\}$. Now, $\mathcal{C}'$ is a partition of $\mathcal{V}(\Gamma)$ into $|\mathcal{C}| - 1 = \chi(\Gamma) - 1$ classes and defines a colouring of Γ with $|\mathcal{C}'| < \chi(\Gamma)$ colours, a contradiction. □

Theorem 2.3.20. *Let $g : \mathcal{V}(\Gamma) \to F$ be a colouring of Γ with $|F| = \chi(\Gamma)$ and set*

$$\left\{C_0^{(1)}, \dots, C_0^{(|\mathcal{C}_0|)}\right\} := \mathcal{C}_0 := \{g^{-1}(f) : f \in F\}.$$

Then $\mathcal{C}_0$ is a set of $\theta_{n+1} - q$ co-cliques such that

- *every co-clique $C \in \mathcal{C}_0$ is a subset of a maximal co-clique of Γ that is covered by Examples 2.1.15 and 2.1.17,*
- *the corresponding maximal co-cliques are either all point based or all hyperplane based, and*
- *the base points (base hyperplanes) are distinct and elements of an $(n+1)$-dimensional subspace (contain a common $(n-2)$-dimensional subspace).*

Proof. We use $\mathcal{C}_0$ to define a covering of $\mathcal{V}(\Gamma)$ that satisfies conditions (I), (II), (III) and (IV) and then apply Theorem 2.3.7 to this covering. Note that $\mathcal{C}_0$ already satisfies the first of these conditions.

For all $i \in \{1, \dots, |\mathcal{C}_0|\}$ we let f_i be the flag provided by Lemma 2.3.19. Now, for all $i \in \{1, \dots, |\mathcal{C}_0|\}$ with $|C_0^{(i)}| > e_1$ we let $C_1^{(i)}$ be an arbitrary maximal co-clique of Γ with $C_0^{(i)} \subseteq C_1^{(i)}$ and for all $i \in \{1, \dots, |\mathcal{C}_0|\}$ with $|C_0^{(i)}| \le e_1$ we set $C_1^{(i)} := C_0^{(i)}$. Lemma 2.3.19 implies that the co-cliques $C_1^{(1)}, \dots, C_1^{|\mathcal{C}_0|}$ are pairwise distinct. Furthermore, we assumed that Conjecture 2.1.19 holds and thus $\mathcal{C}_1 := \{C_1^{(i)} : i \in \{1, \dots, |\mathcal{C}_0|\}\}$ is a covering of $\mathcal{V}(\Gamma)$ by co-cliques of Γ which satisfy conditions (I) and (II).

Now, for all $i \in \{1, \dots, |\mathcal{C}_0|\}$ with $|C_1^{(i)}| \le e_1$ we set $C_2^{(i)} := C_1^{(i)}$. Furthermore, for any subset $I = \{i_1, \dots, i_{|I|}\}$ of $\{1, \dots, |\mathcal{C}_0|\}$ such that for all $i \in I$ we have $|C_1^{(i)}| > e_1$ and such that I is maximal with respect to the property that the co-cliques $\{C_1^{(i)} : i \in I\}$ all have the same generic part we set $C_2^{(i_1)} := C_1^{(i_1)}$ and for all $i \in I \setminus \{i_1\}$ we let $C_2^{(i)}$ be the special part of $C_1^{(i)}$. Then the definition of f_i implies that for all $i \in \{1, \dots, |\mathcal{C}_0|\}$ we still have $f_i \in C_2^{(i)}$ and no other co-clique of $\mathcal{C}_2 := \{C_2^j : j \in \{1, \dots, |\mathcal{C}_0|\}\}$ contains f_i. Thus, $\mathcal{C}_2$ still satisfies condition (I). Furthermore, using Equation (2.28) the special part T of a co-clique $C \in \mathcal{C}_1$ clearly satisfies $|T| \leq e_1$ and thus $\mathcal{C}_2$ also satisfies condition (II). Finally, the definition of $\mathcal{C}_2$ implies that this set also satisfies condition (III).

If $\mathcal{C}_2$ already satisfies condition (IV), too, then we set $\mathcal{C}_3 := \mathcal{C}_2$ and if $\mathcal{C}_2$ does not satisfy condition (IV), then we let $\mathcal{C}_3$ be the dual of $\mathcal{C}_2$. Either way, $\mathcal{C}_3$ satisfies all four conditions. Finally, if $|\mathcal{C}_3| = \theta_{n+1} - q$, then we set $\mathcal{C}_4 := \mathcal{C}_3$ and if otherwise, if $|\mathcal{C}_3| < \theta_{n+1} - q$, then let $\mathcal{C}_4$ be the union of $\mathcal{C}_3$ and a set of $\theta_{n+1} - q - |\mathcal{C}_3|$ arbitrary co-cliques of size 1 such that $\mathcal{C}_4$ still satisfies the four conditions (this is clearly possible).

Finally, we may apply Theorem 2.3.7 to $\mathcal{C}_4$ and see that in fact $\mathcal{C}_4$ is a set of $\theta_{n+1} - q$ maximal co-cliques, all of which are point based and the $\theta_{n+1} - q$ base points are pairwise distinct points of an $(n+1)$-dimensional subspace U of $\mathbb{P}$. However, if we reconsider the construction of $\mathcal{C}_4$ from $\mathcal{C}_0$, then this implies that $\mathcal{C}_1$ already was a set of $\theta_{n+1} - q$ co-cliques that satisfied conditions (I), (II) and (III) and therefore $\mathcal{C}_0$ satisfies the claim of this theorem. $\square$

2.4 The Independence Number of Kneser Graphs of Type $(1,3)$ in $\mathrm{PG}(5,q)$

Throughout this section we let $\mathbb{P}$ be the projective space $\mathrm{PG}(5,q)$ for some prime power q and we let Γ be the Kneser graph of type $(1,3)$ in $\mathbb{P}$. We show that any maximal independent set of Γ of size larger than roughly $377q^7$ (a more precise formulation can be found in Theorem 2.4.49) is given by Example 2.4.1 below and thus, for $q \geq 376$, we determine the independence number of Γ.

We first introduce the aforementioned family of examples of independent sets of Γ. The examples are analogous to Examples 2.1.15 and 2.1.17 given above and thus we omit the proof here.

Example 2.4.1. i) Let H be a hyperplane of $\mathbb{P}$ and let $\mathcal{U}$ be a set of flags $f \leq H$ of type $(1,2)$ such that $\mathcal{U}$ is a maximal independent set of the Kneser graph Γ' of type $(1,2)$ in H. Furthermore, let C be the set of all flags $(l,S) \in \mathcal{V}(\Gamma)$ such that $S \leq H$ or $(l, S \cap H) \in \mathcal{U}$. Then C is a maximal independent set of Γ of size

$$\mathfrak{s}_q[3,4] \cdot \mathfrak{s}_q[1,3] + |\mathcal{U}| \cdot q^2.$$

ii) The structures dual to those given in i).

Remark 2.4.2. *The independent sets C given in Example 2.4.1 have cardinality*

$$|C| \leq \mathfrak{s}_q[3,4] \cdot \mathfrak{s}_q[1,3] + \mathfrak{s}_q[2](\mathfrak{s}_q[3] + q^2)q^2$$

with equality if and only if $\mathcal{U}$ is not only a maximal independent set of Γ' but also an independent set of Γ' of maximal size. According to [3, Proposition 2.1] by Blokhuis and Brouwer maximal independent sets of the Kneser graph of type $(1,2)$ in $\mathrm{PG}(4,q)$ have size at most $\mathfrak{s}_q[2](\mathfrak{s}_q[3] + q^2)$.

Note that any set C that we construct using such an independent set of Γ' of maximal size was already provided in [3, Section 5.1] by Blokhuis and Brouwer. Also note that, regardless of the choice of $\mathcal{U}$, the set C has size $|C| > \mathfrak{s}_q[3,4] \cdot \mathfrak{s}_q[1,3]$.

For the remainder of this section we let C be a maximal co-clique of Γ. We show that, if C is not given by Example 2.4.1 above, then its size is significantly smaller than the size of these examples. We give a short outline of the method of proof below:

We begin by first determining some structure on the set of flags through a given line in the maximal co-clique C. Thereafter, in Section 2.4.2, we proceed to study the structure that is provided by a set of flags of C through a given point. In Section 2.4.3 we then provide some first bounds on the number of flags of C in a given hyperplane. Finally, in Section 2.4.4 we assume that C is not given by Example 2.4.1, then consider a fixed hyperplane containing a maximal number of flags and, using the maximal choice of said hyperplane, determine an upper bound on $|C|$.

Recall that Lemma 2.1.8 proves that two flags (l,S) and (l',S') of Γ are adjacent if and only if $l \cap S' = \emptyset = l' \cap S$.

2.4.1 Structure of Flags through a given Line

Throughout this section we let h be an arbitrary but fixed line in $\mathbb{P}$ and we set

$$\mathcal{L} := \{l \leq \mathbb{P} : \exists S \leq \mathbb{P} \text{ with } S \cap h = \emptyset \text{ and } (l, S) \in C\}.$$

Lemma 2.4.3. *For all solids $S \leq \mathbb{P}$ with $h \leq S$ we have $S \in \Pi_3(\Delta_h(C))$ if and only if $S \cap l \neq \emptyset$ for all $l \in \mathcal{L}$.*

Proof. If $\mathcal{L} = \emptyset$, then for all $f = (l, S) \in C$ we have $S \cap h \neq \emptyset$ and according to Lemma 2.1.3 this is equivalent to h being saturated in C. Therefore, for $\mathcal{L} = \emptyset$ we have $S \in \Pi_3(\Delta_h(C))$ for all solids $S \leq \mathbb{P}$ with $h \leq S$ and $l \cap S \neq \emptyset$ for all $l \in \mathcal{L}$ holds trivially, proving the claim.

Now, assume that $\mathcal{L} \neq \emptyset$. For $l \in \mathcal{L}$, $\widehat{S} \in \Pi_3(\Delta_l(C))$ with $\widehat{S} \cap h = \emptyset$ and $S \leq \mathbb{P}$ with $h \leq S$ as well as $S \cap l = \emptyset$ we have $(h, S) \notin C$, proving $S \in \Pi_3(\Delta_h(C)) \implies \forall l \in \mathcal{L} : S \cap l \neq \emptyset$. Now, let $S \leq \mathbb{P}$ be a solid with $h \leq S$ and $S \cap l \neq \emptyset$ for all $l \in \mathcal{L}$. Then for all $f \in C$ we either have $\pi_1(f) \notin \mathcal{L}$, which implies $\pi_3(f) \cap h \neq \emptyset$, or $\pi_1(f) \in \mathcal{L}$, which implies $\pi_1(f) \cap S \neq \emptyset$. Therefore, for all $f \in C$ we have $\{f, (h, S)\} \notin \mathcal{E}(\Gamma)$. Since C is maximal this implies $(h, S) \in C$, concluding this proof. $\square$

Corollary 2.4.4. *The line h is saturated in C if and only if $\mathcal{L} = \emptyset$.*

Lemma 2.4.5. *Let there be a subset $L \subseteq \mathcal{L}$ such that $\langle h, L\rangle$ is a hyperplane of $\mathbb{P}$. Then there are two lines $l_1, l_2 \in L$ with $H = \langle h, l_1, l_2\rangle$.*

Proof. Let $l_1 \in L$ be arbitrary but fixed. Then $l_1 \cap h = \emptyset$, which proves that $\langle h, l_1\rangle$ is a solid. From $H = \langle h, L\rangle$ we know that there must be a line $l_2 \in L$ with $l_2 \not\leq \langle h, l_1\rangle$ and thus $\langle h, l_1, l_2\rangle = H$, as claimed. $\square$

Lemma 2.4.6. *Let there be two lines $l_1, l_2 \in \mathcal{L}$ such that $H := \langle h, l_1, l_2\rangle$ is a hyperplane of $\mathbb{P}$. Then $\langle h, l_1\rangle \cap \langle h, l_2\rangle$ is a plane and for all $S \in \Pi_3(\Delta_h(C))$ we have $\langle h, l_1\rangle \cap \langle h, l_2\rangle \leq S$ or $S \leq H$.*

Proof. Since $\langle h, l_1, l_2\rangle$ is a hyperplane and $S_i := \langle h, l_i\rangle$ is a solid for both $i \in \{1, 2\}$ we know that $S_1 \neq S_2$ and, for dimensional reasons, this implies that $E := S_1 \cap S_2$ is a plane.

Now, let $S \in \Pi_3(\Delta_h(C))$ be such that $E \not\leq S$. For all $i \in \{1, 2\}$ Lemma 2.4.3 shows $S \cap l_i \neq \emptyset$ and, since $E \not\leq S$ and $h \leq E, S$, we have $h = E \cap S$. Note that this also implies that $P_i := l_i \cap S \notin h$ is a point for both $i \in \{1, 2\}$. Finally, since $H = \langle h, l_1, l_2\rangle = \langle E, P_1, P_2\rangle$ is a hyperplane we know that $\langle h, P_1, P_2\rangle$ is a solid with $\langle h, P_1, P_2\rangle \leq S$, which proves $S = \langle h, P_1, P_2\rangle \leq H$, as claimed. $\square$

Lemma 2.4.7. *Let there be a subset $L \leq \mathcal{L}$ with $\langle h, L\rangle = \mathbb{P}$ as well as a plane E with $h \leq E$ and $l \cap E \neq \emptyset$ for all $l \in L$. Then we have $E \leq S$ for all $S \in \Pi_3(\Delta_h(C))$.*

Proof. Set $P := l \cap E$ for some $l \in L$ and note that from $l \cap h = \emptyset$ we have $E = \langle h, P\rangle$. Furthermore, let S be a subspace of $\mathbb{P}$ with $h \leq S$ and $E \not\leq S$ which meets all lines in L. Then every line $l \in L$ satisfies $l = \langle l \cap E, l \cap S\rangle$ for all $l \in L$ and thus we have

$$\mathbb{P} = \langle h, L\rangle = \langle E, L\rangle = \langle E, S\rangle = \langle P, S\rangle,$$

which implies $\dim(S) \geq 4$. Therefore, there is no solid S with $h \leq S$, $E \not\leq S$ and $S \cap l \neq \emptyset$ for all $l \in L$ and now Lemma 2.4.3 proves $E \leq S$ for all $S \in \Pi_3(\Delta_h(C))$. □

Lemma 2.4.8. *For all $l \in \mathcal{L}$ we have*

$$\Pi_3(\Delta_h(C)) \subseteq \{S \in \mathfrak{S}[h, 3, \mathbb{P}] : \dim(S \cap \langle h, l\rangle) \geq 2\}$$

and thus, if $\mathcal{L} \neq \emptyset$, then $|\Pi_3(\Delta_h(C))| \leq q^3 + 2q^2 + q + 1$. Furthermore, equality holds if and only if $\mathcal{L} \neq \emptyset$ and there is a solid $\widehat{S} \geq h$ such that $l \leq \widehat{S}$ for all $l \in \mathcal{L}$.

Proof. If $\mathcal{L} = \emptyset$ then, according to Corollary 2.4.4, h is saturated and thus equality does not hold and there remains nothing to prove.

Thus, assume that $\mathcal{L} \neq \emptyset$. Then, according to Lemma 2.4.3, for all $S \in \mathfrak{S}[h, 3, \mathbb{P}]$ we have $S \in \Pi_3(\Delta_h(C))$ if and only if for all $l \in \mathcal{L}$ we have $S \cap l \neq \emptyset$, that is, if and only if for all $l \in \mathcal{L}$ we have $\dim(S \cap \langle h, l\rangle) \geq 2$. Since $\mathcal{L} \neq \emptyset$ there is a line $l_1 \in \mathcal{L}$ and we have

$$|\Pi_3(\Delta_h(C))| \leq \mathfrak{s}_q[1, 2, 3] \cdot \mathfrak{s}_q[0, 2, 3, 5] + 1 = q^3 + 2q^2 + q + 1.$$

Now, if $l \leq \widehat{S} := \langle h, l_1\rangle$ for all $l \in \mathcal{L}$ and S is a solid through h with $\dim(S \cap \widehat{S}) \geq 2$, then S meets every line $l \leq \widehat{S}$ and thus all lines of $\mathcal{L}$, that is, Lemma 2.4.3 shows $S \in \Pi_3(\Delta_h(C))$. Hence, in this situation equality does hold.

However, if there is a line $l \in \mathcal{L}$ with $l \not\leq \widehat{S}$, then there is a plane $E \leq \widehat{S}$ with $E \cap l = \emptyset$ as well as a solid S through E with $S \cap l = \emptyset$, which proves that equality does not hold and concludes the proof. □

Lemma 2.4.9. *Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ be such that $\widehat{S}_1 := \langle h, \mathcal{L}_1\rangle$ and $\widehat{S}_2 := \langle h, \mathcal{L}_2\rangle$ are two solids with $h = \widehat{S}_1 \cap \widehat{S}_2$ and such that $l \not\leq \widehat{S}_1, \widehat{S}_2$ for all $l \in \mathcal{L}_3$. Furthermore, let $l_1 \in \mathcal{L}_1$ and $l_2 \in \mathcal{L}_2$ be arbitrary but fixed. Then*

$$\Pi_3(\Delta_h(C)) \subseteq \{\langle h, P_1, P_2\rangle : P_1 \in l_1, P_2 \in l_2\}$$

as well as $|\Pi_3(\Delta_h(C))| \leq q^2 + 2q + 1$ and for $\mathcal{L}_3 = \emptyset$ these hold with equality.

Proof. First, let $S \in \Pi_3(\Delta_h(C))$ be arbitrary and fixed. According to Lemma 2.4.8 we have $P_i := l_i \cap S \neq \emptyset$ for all $i \in \{1, 2\}$. Furthermore, for all $i \in \{1, 2\}$ we have $\widehat{S}_i = \langle h, l_i\rangle$. Therefore, if $P_i = l_i$ for some $i \in \{1, 2\}$, then $S = \widehat{S}_i$ and $S \cap \widehat{S}_{3-i} = h$, that is, $S \cap l_{3-i} = \emptyset$, a contradiction.

Hence, we know that P_1 and P_2 are points and from $P_2 \notin \widehat{S}_1 \geq \langle h, P_1\rangle$ we have $S = \langle h, P_1, P_2\rangle$, as claimed. Since there are $\mathfrak{s}_q[1]^2$ choices for points $P_1 \in l_1$ and $P_2 \in l_2$ and every solid $S \in \Pi_3(\Delta_h(C))$ is uniquely determined by two such points, we have $|\Pi_3(\Delta_h(C))| \leq \mathfrak{s}_q[1]^2 = q^2 + 2q + 1$

Now, assume that $\mathcal{L}_3 = \emptyset$, let $P_1 \in l_1$ and $P_2 \in l_2$ be arbitrary points and set $S := \langle h, P_1, P_2 \rangle$. Then, for all $i \in \{1,2\}$ we know that $\langle h, P_i \rangle$ is a plane contained in $\widehat{S}_i$ and S. Therefore, S contains a hyperplane of both $\widehat{S}_1$ and $\widehat{S}_2$ and since every line $l \in \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ is contained in one of these two subspaces, S has non-empty intersection with all lines $l \in \mathcal{L}$. Therefore, according to Lemma 2.4.3, $S \in \Pi_3(\Delta_h(C))$ and thus equality holds. $\square$

Lemma 2.4.10. *Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ be such that there is a complement U of h in $\mathbb{P}$ and a unique regulus $\overline{\mathcal{R}}$ in U with $\langle h, l \rangle \cap U \in \overline{\mathcal{R}}$ for all $l \in \mathcal{L}_1$. Furthermore, for all $l \in \mathcal{L}_2$ assume that $\langle h, l \rangle \cap U \notin \overline{\mathcal{R}}$.*

Then $\Pi_3(\Delta_h(C)) \subseteq \{\langle h, g \rangle : g \in \mathcal{R}\}$, where $\mathcal{R}$ is the unique opposite regulus of $\overline{\mathcal{R}}$ in U, we have $|\Pi_3(\Delta_h(C))| \leq q+1$ and in both equations equality holds if and only if $\mathcal{L}_2 = \emptyset$. Furthermore, if $\mathcal{L}_2 \neq \emptyset$ then $|\Pi_3(\Delta_h(C))| \in \{0,1,2\}$.

Proof. Since there is a regulus $\overline{\mathcal{R}}$ in U with $\langle h, l \rangle \cap U \in \overline{\mathcal{R}}$ for all $l \in \mathcal{L}_1$ we know that $\overline{\mathcal{S}} := \{\langle h, l \rangle : l \in \mathcal{L}_1\}$ is a set of solids such that for all distinct $\overline{S}, \overline{S}' \in \overline{\mathcal{S}}$ we have $\overline{S} \cap \overline{S}' = h$. Furthermore, since the regulus $\overline{\mathcal{R}}$ in U with $\langle h, l \rangle \cap U \in \overline{\mathcal{R}}$ for all $l \in \mathcal{L}_1$ is unique, we know that $\overline{\mathcal{S}}$ has size $r \geq 3$. We let $\mathcal{R}$ be the unique opposite regulus of $\overline{\mathcal{R}}$ in U.

Then, for all $S \in \Pi_3(\Delta_h(C))$ and all $l \in \mathcal{L}$ we have $S \cap l \neq \emptyset$ and thus $\dim(S \cap \overline{S}) \geq 2$ for all $\overline{S} \in \overline{\mathcal{S}}$. Hence, for all $S \in \Pi_3(\Delta_h(C))$ and at least three distinct lines $\overline{g}_1, \overline{g}_2, \overline{g}_3 \in \overline{\mathcal{R}}$ we have $S \cap \overline{g}_1, S \cap \overline{g}_2, S \cap \overline{g}_3 \neq \emptyset$ and thus $S \cap U$ is a line of U with non-empty intersection with at least three distinct and thus all lines of $\overline{\mathcal{R}}$, that is, $S \cap U \in \mathcal{R}$. This proves $\Pi_3(\Delta_h(C)) \subseteq \{\langle h, g \rangle : g \in \mathcal{R}\}$ as well as $|\Pi_3(\Delta_h(C))| \leq |\mathcal{R}| = q+1$.

Now, if $\mathcal{L}_2 = \emptyset$, then for all $l \in \mathcal{L} = \mathcal{L}_1$ we know that $\langle h, l \rangle \cap U$ is an element of $\overline{\mathcal{R}}$ and for all $g \in \mathcal{R}$ this proves that $g \cap \langle h, l \rangle$ is a point and $\langle h, l \rangle \cap \langle h, g \rangle$ is a plane, that is, $\langle h, g \rangle \cap l \neq \emptyset$. Hence, if $\mathcal{L}_2 = \emptyset$, then Lemma 2.4.3 proves $\langle h, g \rangle \in \Pi_3(\Delta_h(C))$ for all $g \in \mathcal{R}$ and thus $|\Pi_3(\Delta_h(C))| = |\mathcal{R}| = q+1$.

Finally, consider the case $\mathcal{L}_2 \neq \emptyset$, let $l \in \mathcal{L}_2$ be an arbitrary but fixed line and set $\overline{g} := \langle h, l \rangle \cap U$. For $S \in \Pi_3(\Delta_h(C))$ we have $S \cap l \neq \emptyset$, which proves $\dim(S \cap \langle h, l \rangle) \geq 2$ and thus also $S \cap \overline{g} \neq \emptyset$. From $l \in \mathcal{L}_2$ we have $\overline{g} \notin \overline{\mathcal{R}}$, which shows that $\overline{g}$ has non-empty intersection with at most two lines of $\mathcal{R}$. Therefore, $\{\langle h, g \rangle : g \in \mathcal{R} \wedge \overline{g} \cap g \neq \emptyset\}$ has cardinality at most 2 and is a superset of $\Pi_3(\Delta_h(C))$, which proves the last claim. $\square$

Lemma 2.4.11. *Let $H := \langle h, \mathcal{L} \rangle$ be a hyperplane of $\mathbb{P}$ and let there be a plane E with $h \leq E \leq H$ such that $l \cap E \neq \emptyset$ for all $l \in \mathcal{L}$. Then $\Pi_3(\Delta_h(C)) = \mathfrak{S}[E, 3, \mathbb{P}] \cup \mathfrak{S}[h, 3, H]$ and $|\Pi_3(\Delta_h(C))| = 2q^2 + q + 1$.*

Proof. First, note that every solid $S \leq H$ is a hyperplane of H and as such has non-empty intersection with all lines $l \in \mathcal{L}$. Furthermore, every solid S with $E \leq S$ has non-empty intersection with all lines $l \in \mathcal{L}$ because E has non empty intersection with all these lines. Therefore, according to Lemma 2.4.3, all solids S with $E \leq S$ or $S \leq H$ are elements of $\Pi_3(\Delta_h(C))$.

Now, from Lemma 2.4.5 we know that there are lines $l_1, l_2 \in \mathcal{L}$ with $\langle h, l_1, l_2 \rangle = H$ and Lemma 2.4.6 shows $\langle h, l_1 \rangle \cap \langle h, l_2 \rangle \leq S$ or $S \leq H$ for all $S \in \Pi_3(\Delta_h(C))$. Furthermore, any plane on h which meets both l_1 and l_2 is a subspace of the plane $\langle h, l_1\rangle \cap \langle h, l_2\rangle$, which proves $E = \langle h, l_1\rangle \cap \langle h, l_2\rangle$.

Finally, there are $\mathfrak{s}_q[2,3,6]$ solids through E, $\mathfrak{s}_q[1,3,5]$ solids in H and $\mathfrak{s}_q[2,3,5]$ solids through E in H, yielding a total of $\mathfrak{s}_q[2,3,6]+\mathfrak{s}_q[1,3,5]-\mathfrak{s}_q[2,3,5] = 2q^2+q+1$ solids in $\Pi_3(\Delta_h(C))$, concluding the proof. $\square$

Lemma 2.4.12. *Let $\mathcal{L}$ be such that there is a plane E with $h \leq E \leq \mathbb{P}$ for which $\mathcal{L}_1 := \{l \in \mathcal{L} : l \cap E \neq \emptyset\}$ satisfies $\langle h, \mathcal{L}_1\rangle = \mathbb{P}$. Then $\Pi_3(\Delta_h(C)) \subseteq \mathfrak{S}[E,3,\mathbb{P}]$ as well as $|\Pi_3(\Delta_h(C))| \leq q^2+q+1$ and equality holds if and only if $\mathcal{L} = \mathcal{L}_1$.*

Proof. Lemma 2.4.7 shows $\Pi_3(\Delta_h(C)) \subseteq \mathfrak{S}[E,3,\mathbb{P}]$ and thus $|\Pi_3(\Delta_h(C))| \leq \mathfrak{s}_q[2,3,5] = q^2+q+1$. Furthermore, for every solid S with $E \leq S$ we have $h \leq S$ and $l\cap S \geq l \cap E \neq \emptyset$ for all $l \in \mathcal{L}_1$, that is, if $\mathcal{L}\setminus\mathcal{L}_1 = \emptyset$, then Lemma 2.4.3 shows that equality holds. Finally, if $\mathcal{L} \neq \mathcal{L}_1$, then there is a line $l \in \mathcal{L}\setminus\mathcal{L}_1$ and a 3-dimensional complement S of l in $\mathbb{P}$ with $E \leq S$, that is $S \in \mathfrak{S}[E,3,\mathbb{P}]$, and due to $S \cap l = \emptyset$ Lemma 2.4.3 shows $S \notin \Pi_3(\Delta_h(C))$, concluding the proof. $\square$

Lemma 2.4.13. *Let E be a plane and let H be a hyperplane of $\mathbb{P}$ with $h \leq E \leq H$. We set $\mathcal{L}_1 := \Delta_H(\mathcal{L})$, $\mathcal{L}_2 := \{l \in \mathcal{L} : l \cap E \neq \emptyset\}$, $\mathcal{L}_3 := \mathcal{L} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2)$ and $U := \langle h, l \cap H : l \in \mathcal{L} \setminus \mathcal{L}_1\rangle$. Furthermore, assume that one of the following holds*

(I) $\langle h, \mathcal{L}_2\rangle = \mathbb{P}$ *and we have* $\mathcal{L}_1 \setminus \mathcal{L}_2 \neq \emptyset$*, or*

(II) $\langle h, \mathcal{L}_1\rangle = H$*, there is a plane* $E' \neq E$ *with* $h \leq E' \leq H$ *as well as* $l \cap E' \neq \emptyset$ *for all* $l \in \mathcal{L}_1$ *and we have* $\langle E', \mathcal{L}_2 \setminus \mathcal{L}_1\rangle = \mathbb{P}$*, or*

(III) $\langle h, \mathcal{L}_1\rangle = H$ *and for every plane* E' *with* $h \leq E' \leq H$ *there is a line* $l \in \mathcal{L}_1$ *such that* $l \cap E' = \emptyset$*.*

Then $\Pi_3(\Delta_h(C)) = \mathfrak{S}[U,3,H]$ *and* $\dim(U) < 2$ *may only occur if (III) holds.*

Proof. First, note that from $l \leq H$ for all $l \in \mathcal{L}_1$ and $l \cap U \neq \emptyset$ for all $l \in \mathcal{L}\setminus\mathcal{L}_1$ we have $l \cap S \neq \emptyset$ for all $l \in \mathcal{L}$ and all $S \in \mathfrak{S}[U,3,H]$. Therefore, Lemma 2.4.3 shows $\mathfrak{S}[U,3,H] \subseteq \Pi_3(\Delta_h(C))$.

Also note that in the first case $\langle h, \mathcal{L}_2\rangle = \mathbb{P} \neq H \geq \langle h, \mathcal{L}_1\rangle$ implies $\mathcal{L}_2 \setminus \mathcal{L}_1 \neq \emptyset$ and in the second case $\langle E', \mathcal{L}_2 \setminus \mathcal{L}_1\rangle = \mathbb{P}$ implies the same. However, if $\mathcal{L}_2 \setminus \mathcal{L}_1 \neq \emptyset$, then there is a line $l \in \mathcal{L}_2 \setminus \mathcal{L}_1$ with $h \not\ni l \cap H \in U$, that is, $\dim(U) \geq 2$. Therefore, $\dim(U) < 2$ may only occur in the third case.

Now, it remains to show $\Pi_3(\Delta_h(C)) \subseteq \mathfrak{S}[U,3,H]$, that is, it remains to show that every solid $S \in \Pi_3(\Delta_h(C))$ satisfies both $U \leq S$ as well as $S \leq H$. In order to prove both these claims we let $S \in \Pi_3(\Delta_h(C))$ be an arbitrary but fixed solid for the remainder of this proof.

The first of the two remaining claims, that is $U \leq S$, is fairly simple to see: According to Lemma 2.4.3 we have $S \cap l \neq \emptyset$ for all $l \in \mathcal{L}$. For $l \in \mathcal{L}\setminus\mathcal{L}_1$ we have $l \not\leq H$ and then $S \leq H$ implies that the point $l \cap H$ is an element of S. Since h is also a subspace of S this implies $U = \langle h, l \cap H : l \in \mathcal{L}\setminus\mathcal{L}_1\rangle \leq S$, as claimed.

For the second claim, that is $S \leq H$, we consider the three cases given in the claim separately:

(I) In the first case we may apply Lemma 2.4.7, which shows $E \leq S$. Since $\mathcal{L}_1 \setminus \mathcal{L}_2 \neq \emptyset$ we may let l_1 be a line therein, which implies $l_1 \cap E = \emptyset$. Then $\langle E, l_1 \rangle = H$ and according to Lemma 2.4.3 we have $S \cap l_1 \neq \emptyset$. Together with $E \leq S$, $E \leq H$ and $E \cap l_1 = \emptyset$ this proves $S = \langle E, S \cap l_1 \rangle \leq H$.

(II) In the second case, according to Lemma 2.4.5 there are two lines $l_1, l_1' \in \mathcal{L}_1$ with $\langle h, l_1, l_1' \rangle = H$ and the only plane through h in H which has non-empty intersection with all lines in $\mathcal{L}_1$ is the plane $E' := \langle h, l_1 \rangle \cap \langle h, l_1' \rangle$. According to Lemma 2.4.6 we have $E' \leq S$ or $S \leq H$. We assume $E' \leq S$ and show that $S \leq H$ holds nonetheless. Since $\langle E', \mathcal{L}_2 \setminus \mathcal{L}_1 \rangle = \mathbb{P}$ there is a line $l_2 \in \mathcal{L}_2 \setminus \mathcal{L}_1$ and this line satisfies $l_2 \cap E \neq \emptyset$. Since $h = E \cap E'$ and $l_2 \cap h = \emptyset$ as well as $l_2 \not\leq H$ and $h \leq E' \leq H$ this implies $l_2 \cap E' = \emptyset$ and thus $\langle E', l_2 \rangle$ is a hyperplane of $\mathbb{P}$. However, $\langle E', \mathcal{L}_2 \setminus \mathcal{L}_1 \rangle = \mathbb{P}$ and thus there is another line $l_2' \in \mathcal{L}_2 \setminus \mathcal{L}_1$ with $l_2' \not\leq \langle E', l_2 \rangle$ and thus $\langle E', l_2, l_2' \rangle = \mathbb{P}$. Since $l_2 \cap E \neq \emptyset \neq l_2' \cap E$ and $h \leq E$ we know that $H' := \langle h, l_2, l_2' \rangle = \langle E, l_2, l_2' \rangle$ is a hyperplane of $\mathbb{P}$. According to Lemma 2.4.6 we thus have $E \leq S$ or $S \leq H'$. Now, we have $E' \not\leq H'$ from $\langle E', l_1, l_2 \rangle = \mathbb{P} \neq H' \geq l_1, l_2$ and we assumed $E' \leq S$, which proves that $S \leq H'$ does not occur. Therefore, we have $E \leq S$ and since E and E' are distinct planes in H this shows $S = \langle E, E' \rangle \leq H$.

(III) In the third case, if $S \not\leq H$ then $E' := S \cap H$ is a plane and there is a line $l \in \mathcal{L}_1$ with $l \cap E' = \emptyset$ and since $l \leq H$ this implies $l \cap S = \emptyset$, a contradiction to $S \in \Pi_3(\Delta_h(C))$ and Lemma 2.4.3. □

Lemma 2.4.14. *Let $\mathcal{L}$ be such that $\langle h, l : l \in \mathcal{L}, l \cap E \neq \emptyset \rangle \neq \mathbb{P}$ for every plane $E \geq h$ and such that to every hyperplane $H \geq h$ of $\mathbb{P}$ there is a plane $E \geq h$ with $E \cap l \neq \emptyset$ for all $l \in \Delta_H(\mathcal{L})$. Furthermore, let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ with $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$ be such that all of the following hold*

(I) $H_1 := \langle h, \mathcal{L}_1 \rangle$ is a hyperplane of $\mathbb{P}$ with $l \not\leq H_1$ for all $l \in \mathcal{L} \setminus \mathcal{L}_1$ and there is a plane E_1 with $h \leq E_1 \leq H_1$ and $l \cap E_1 \neq \emptyset$ for all $l \in \mathcal{L}_1$.

(II) $H_2 := \langle E_1, \mathcal{L}_2 \rangle \neq H_1$ is a hyperplane of $\mathbb{P}$ and there is a plane $E_2 \neq E_1$ with $h \leq E_2 \leq H_1$ as well as $l \cap E_2 \neq \emptyset$ for all $l \in \mathcal{L}_2$.

(III) For all $i \in \{1, 2\}$ all lines $l \in \mathcal{L}_3$ satisfy $l \cap E_i = \emptyset$ as well as $l \not\leq H_i$ and we set $U_i := \langle E_i, l \cap H_{3-i} : l \in \mathcal{L}_3 \rangle$.

Then $\Pi_3(\Delta_h(C)) = \mathfrak{S}[U_1, 3, H_2] \cup \mathfrak{S}[U_2, 3, H_1]$, we have $|\Pi_3(\Delta_h(C))| = 2q + 1$ if $\mathcal{L}_3 = \emptyset$ and $|\Pi_3(\Delta_h(C))| \in \{0, 1, 2\}$ otherwise. Furthermore, if $|\Pi_3(\Delta_h(C))| = 2$, then the two distinct solids $S, S' \in \Pi_3(\Delta_h(C))$ satisfy $S \cap S' = h$.

Proof. First, let $S \in \mathfrak{S}[U_i, 3, H_{3-i}]$ be an arbitrary but fixed solid for some arbitrary but fixed index $i \in \{1, 2\}$. Then S satisfies $h \leq S$, meets every line $l \in \mathcal{L}_3$ because $l \cap S \geq l \cap U_i \neq \emptyset$, meets every line $l \in \mathcal{L}_i$ because $l \cap S \geq l \cap E_i \neq \emptyset$ and meets every line $l \in \mathcal{L}_{3-i}$ because S is a hyperplane of $H_{3-i} \geq l$. Therefore, S meets all lines $l \in \mathcal{L}$ and thus, according to Lemma 2.4.3, is an element of $\Pi_3(\Delta_h(C))$.

Now, let S be an arbitrary but fixed solid in $\Pi_3(\Delta_h(C))$. Since $\langle h, \mathcal{L}_1\rangle$ is a hyperplane we know from Lemma 2.4.5 that there are lines $l_1, g_1 \in \mathcal{L}_1$ with $\langle h, l_1, g_1\rangle = H_1$ and the only plane through h in H which may have non-empty intersection with all lines in $\mathcal{L}_1$ is the plane $\langle h, l_1\rangle \cap \langle h, g_1\rangle$. Thus, $E_1 = \langle h, l_1\rangle \cap \langle h, l_2\rangle$ and Lemma 2.4.6 shows $E_1 \leq S$ or $S \leq H_1$. Furthermore, since $H_2 = \langle E_1, \mathcal{L}_2\rangle$ is a hyperplane, we have $\mathcal{L}_2 \neq \emptyset$, that is, there is a line $l_2 \in \mathcal{L}_2$. We have $l_2 \not\leq H_1$ and, since

$$\langle h, l \in \mathcal{L} : l \cap E_1 \neq \emptyset\rangle \neq \mathbb{P} = \langle h, l_1, g_1, l_2\rangle,$$

we have $l_2 \cap E_1 = \emptyset$, which proves $H_2 = \langle E_1, l_2\rangle$. Since $l_2 \cap E_2 \neq \emptyset = l_2 \cap h$, $h \leq E_2 \leq H_1$ and $l_2 \not\leq H_1$ we have $E_2 = \langle h, l_2 \cap H_1\rangle \leq \langle E_1, l_2\rangle = H_2$. From Lemma 2.4.3 we know that $S \cap l_2 \neq \emptyset$ as well as $S \cap l_3 \neq \emptyset$ for all $l_3 \in \mathcal{L}_3$. Thus, S either satisfies $S \leq H_1$ and $S \cap l_2 \neq \emptyset$ implies $E_2 = \langle h, l_2 \cap H_1\rangle \leq S$ or $S \not\leq H_1$ and thus $E_1 \leq S$ with $S = \langle E_1, S \cap l_2\rangle \leq H_2$. Hence, there is an index $i \in \{1,2\}$ with $E_i \leq S \leq H_{3-i}$ and, since the smallest subspace of H_{3-i} through E_i that meets all lines of $\mathcal{L}_3$ is the subspace U_i, we also have $U_i \leq S$, concluding the proof of the first claim.

We proceed to determine $|\Pi_3(\Delta_h(C))|$. First, assume that $\mathcal{L}_3$ is the empty set. Then for all $i \in \{1,2\}$ we have $U_i = E_i$ and there are $\mathfrak{s}_q[2,3,4] = q+1$ solids through E_i in H_{3-i}. Since there is one solid, namely the solid $\langle E_1, E_2\rangle$, that is being counted twice, we have $|\Pi_3(\Delta_h(C))| = 2q+1$. Hence, from now on we may assume that $\mathcal{L}_3 \neq \emptyset$. Now, if there is a line $l_3 \in \mathcal{L}_3$ with $l_3 \cap \langle E_1, E_2\rangle \neq \emptyset$, then $\langle E_1, E_2\rangle \leq U_1, U_2$ and there is at most one solid in $\Pi_3(\Delta_h(C))$, namely the solid $\langle E_1, E_2\rangle$. Therefore, from now on we may also assume $l_3 \cap \langle E_1, E_2\rangle = \emptyset$ for all $l_3 \in \mathcal{L}_3$. Note that this implies $l_3 \cap H_i \notin \langle E_1, E_2\rangle = H_1 \cap H_2$ and proves $l_3 \cap H_i \notin H_{3-i}$ for all $l_3 \in \mathcal{L}_3$ and all $i \in \{1,2\}$. Since $E_i \leq U_i \leq H_{3-i}$ for all $i \in \{1,2\}$ this implies that U_1 and U_2 are subspaces of dimension at least 3 which are distinct from the solid $\langle E_1, E_2\rangle$. Thus, we have $\Pi_3(\Delta_h(C)) \subseteq \{U_1, U_2\}$ as well as $|\Pi_3(\Delta_h(C))| \leq 2$.

Finally, consider $|\Pi_3(\Delta_h(C))| = 2$, that is, $\Pi_3(\Delta_h(C)) = \{U_1, U_2\}$ with two solids $U_1 \neq U_2$. According to the above the two solids U_1 and U_2 are then distinct from $\langle E_1, E_2\rangle = H_1 \cap H_2$ and we have $U_i \cap H_i = E_i$ for all $i \in \{1,2\}$. This implies

$$U_1 \cap U_2 = (U_1 \cap H_2) \cap (U_2 \cap H_1) = (U_1 \cap H_1) \cap (U_2 \cap H_2) = E_1 \cap E_2 = h$$

and concludes the proof. □

Theorem 2.4.15. *Exactly one of the following cases occurs:*

C1 $\Pi_3(\Delta_h(C))$ *is the set of all* $\mathfrak{s}_q[1,3,5] = q^4 + q^3 + 2q^2 + q + 1$ *solids through* h*,*

C2 (a) *there is a solid* $\widehat{S} \geq h$ *such that* $\Pi_3(\Delta_h(C))$ *is the set of all* $q^3 + 2q^2 + q + 1$ *solids* S *through* h *with* $\dim(S \cap \widehat{S}) \geq 2$*,*

(b) *there are two solids* $\widehat{S}_1, \widehat{S}_2$ *with* $h = \widehat{S}_1 \cap \widehat{S_2}$ *such that* $\Pi_3(\Delta_h(C))$ *is the set of all* $q^2 + 2q + 1$ *solids* S *through* h *with* $\dim(S \cap \widehat{S}_1) = 2 = \dim(S \cap \widehat{S}_2)$*,*

(c) *there is a regulus* G *in a complement* U *of* h *in* $\mathbb{P}$ *such that* $\Pi_3(\Delta_h(C))$ *is the set of all* $q+1$ *solids* $\langle h, g\rangle$ *with* $g \in G$*,*

(d) *there are two solids* S_1 *and* S_2 *with* $h = S_1 \cap S_2$ *such that* $\Pi_3(\Delta_h(C)) = \{S_1, S_2\}$,

C3 (a) *there is a plane* $E \geq h$ *and a hyperplane* $H \geq E$ *such that* $\Pi_3(\Delta_h(C))$ *is the set of all* $2q^2 + q + 1$ *solids* S *with* $E \leq S$ *or* $h \leq S \leq H$,
(b) *there is a plane* $E \geq h$ *such that* $\Pi_3(\Delta_h(C))$ *is the set of all* $q^2 + q + 1$ *solids* S *through* E,
(c) *there is a hyperplane* $H \geq h$ *such that* $\Pi_3(\Delta_h(C))$ *is the set of all* $q^2 + q + 1$ *solids* S *through* h *in* H,
(d) *there is a plane* $E \geq h$ *and a hyperplane* $H \geq E$ *such that* $\Pi_3(\Delta_h(C))$ *is the set of all* $q + 1$ *solids* S *with* $E \leq S \leq H$,

C4 *there are two distinct planes* $E_1, E_2 \geq h$ *and two distinct hyperplanes* $H_1, H_2 \geq \langle E_1, E_2 \rangle$ *such that* $\Pi_3(\Delta_h(C))$ *is the set of all* $2q + 1$ *solids* S *with* $E_1 \leq S \leq H_2$ *or* $E_2 \leq S \leq H_1$,

C5 $\Pi_3(\Delta_h(C))$ *contains exactly one solid,*

C6 $\Pi_3(\Delta_h(C))$ *is the empty set.*

Proof. If $\mathcal{L} = \emptyset$ then according to Corollary 2.4.4 we know that C1 occurs, if $\langle h, \mathcal{L} \rangle$ is a solid, then according to Lemma 2.4.8 we know that C2 (a) occurs and if $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ such that $\langle h, \mathcal{L}_1 \rangle$ and $\langle h, \mathcal{L}_2 \rangle$ are two solids with $h = \langle h, \mathcal{L}_1 \rangle \cap \langle h, \mathcal{L}_2 \rangle$, then Lemma 2.4.9 shows that C2 (b) occurs. Thus, assume that there are three solids $\overline{S}_1$, $\overline{S}_2$ and $\overline{S}_3$ with pairwise intersection h and such that for all $i \in \{1, 2, 3\}$ there is a line $l \in \mathcal{L}$ with $\overline{S}_i = \langle h, l \rangle$. Let U be a complement of h in $\mathbb{P}$. Then $g_1 := \overline{S}_1 \cap U$, $g_2 := \overline{S}_2 \cap U$ and $g_3 := \overline{S}_3 \cap U$ are three skew lines in U and according to Lemma 1.3.6 there is a unique regulus $\overline{\mathcal{R}}$ in U which contains these three lines. Using $\mathcal{L}_1 = \{l \in \mathcal{L} : \exists \overline{g} \in \overline{\mathcal{R}} \text{ with } l \leq \langle h, \overline{g} \rangle\}$ and $\mathcal{L}_2 = \mathcal{L} \setminus \mathcal{L}_1$ Lemma 2.4.10 is applicable, showing that C2 (c) occurs if $\mathcal{L}_2 = \emptyset$ and that either C2 (d), C5 or C6 occurs if $\mathcal{L}_2 \neq \emptyset$. Consequently, from now on we may assume that

(I) there are no two solids S and S' with $h \leq S, S'$ and $l \leq S$ or $l \leq S'$ for all $l \in \mathcal{L}$,
(II) to every choice of $h_1, h_2, h_3 \in \mathcal{L}$ the solids $\langle h, h_1 \rangle$, $\langle h, h_2 \rangle$ and $\langle h, h_3 \rangle$ do not have pairwise intersection h.

Now, consider the following two situations: First, let E be a plane with $h \leq E$ and set $L := \{l \in \mathcal{L} : l \cap E \neq \emptyset\}$. If $\langle h, L \rangle = \mathbb{P}$ and $L = \mathcal{L}$ then Lemma 2.4.12 applies, proving that C3 (b) occurs; if $\langle h, L \rangle = \mathbb{P}$ and $L \neq \mathcal{L}$ then Lemma 2.4.13 (the first condition is fulfilled for $H = \langle E, l \rangle$ for some $l \in \mathcal{L} \setminus L$) applies, proving that C3 (d), C5 or C6 occurs; and if $\langle h, L \rangle \neq \mathbb{P}$ and $\mathcal{L} = L$, then Lemma 2.4.11 ($\dim(\langle h, \mathcal{L} \rangle) \geq 4$ per (I)) applies proving that C3 (a) occurs. Secondly, let H be a hyperplane of $\mathbb{P}$ with $h \leq H$ such that to every plane E' with $h \leq E' \leq H$ there is a line $l \in \mathcal{L}$ with $l \leq H$ and $l \cap E' = \emptyset$. Then, there is some line $x_1 \in \mathcal{L}$ with $x_1 \leq H$, some plane $E_{x_1} \leq H$ that is the span of h and a point of x_1 as well as a second line $x_2 \in \mathcal{L}$ with $x_2 \leq H$ and $x_2 \cap E_{x_1} = \emptyset$, that is, we have $H = \langle \delta_H(\mathcal{L}) \rangle$. Thus, Lemma 2.4.13 is applicable (the third condition is fulfilled), showing that C3 (c), C3 (d), C5 or C6 occurs. Hence, from now on we may also assume that

(III) to every plane E such that $h \leq E$ the set $L := \{l \in \mathcal{L} : l \cap E \neq \emptyset\}$ satisfies $\langle h, L\rangle \neq \mathbb{P}$ as well as $L \neq \mathcal{L}$,
(IV) to every hyperplane H of $\mathbb{P}$ with $h \leq H$ there is a plane E with $h \leq E \leq H$ and $l \cap E \neq \emptyset$ for all $l \in \Delta_H(\mathcal{L})$.

Note that (I) and (II) imply the existence of $l_1, l_1' \in \mathcal{L}$ such that $E_1 := \langle h, l_1\rangle \cap \langle h, l_1'\rangle$ is a plane and thus $H_1 := \langle h, l_1, l_1'\rangle$ is a hyperplane of $\mathbb{P}$. Furthermore, note that the only plane through h that has non-empty intersection with both l_1 and l_1' is the plane E_1 and thus (IV) implies $l \cap E_1 \neq \emptyset$ for all $l \in \mathcal{L}_1 := \Delta_{H_1}(\mathcal{L})$ (in the notation of (III) and (IV) the set $\mathcal{L}_1$ was called L). Now, (III) implies that every line $l \in \mathcal{L}$ with $l \cap E_1 \neq \emptyset$ satisfies $l \leq H_1$, that is, $\Delta_{H_1}(\mathcal{L}) = \mathcal{L}_1 = \{l \in \mathcal{L} : l \cap E_1 \neq \emptyset\}$ and (III) also implies that there is a line $l_2 \in \mathcal{L}$ with $l_2 \cap E_1 = \emptyset$ and thus $H_2 := \langle E_1, l_2\rangle$ is a hyperplane of $\mathbb{P}$. From $\mathcal{L}_1 = \{l \in \mathcal{L} : l \cap E_1 \neq \emptyset\}$ we have $l_2 \notin \mathcal{L}_1$, that is, $l_2 \not\leq H_1$, which implies $H_2 \neq H_1$. We also know that $l_2 \cap H_1$ is a point with $l_2 \cap H_1 \notin E_1$ and thus $E_2 := \langle h, l_2 \cap H_1\rangle \leq H_1 \cap H_2$ is a plane with $E_2 \neq E_1$. We set $\mathcal{L}_2 := \{l \in \mathcal{L} \setminus \mathcal{L}_1 : l \cap E_2 \neq \emptyset\}$ as well as $\mathcal{L}_3 := \mathcal{L} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2)$.

If $\langle E_1, \mathcal{L}_2\rangle = \mathbb{P}$, then Lemma 2.4.13 applies (the second condition is fulfilled for $H = H_1$, $E = E_2$ and $E' = E_1$), proving that C3 (d), C5 or C6 occurs. Thus, we may assume that $\langle E_1, \mathcal{L}_2\rangle = H_2$. If $l \not\leq H_2$ for all $l \in \mathcal{L}_3$, then Lemma 2.4.14 applies, proving that C4, C2 (d), C5 or C6 occurs. Hence, let there be a line $l_3 \in \mathcal{L}_3$ with $l_3 \leq H_2$ and let E_3 be the unique plane provided by (IV) with $h \leq E_3 \leq H_2$ and $l \cap E_3 \neq \emptyset$ for all $l \in \mathcal{L}$ with $l \leq H_2$. From $l_3 \notin \mathcal{L}_2$ we have $l_3 \cap E_2 = \emptyset$, proving $E_2 \neq E_3$ as well as $l_3 \not\leq \langle h, l_2\rangle$, that is, $H_2 = \langle h, l_2, l_3\rangle$. From $l_2 \cap H_1 = l_2 \cap E_2 \neq l_2 \cap E_3$ we know that $l_2 \cap E_3 \notin H_1$ and thus $E_3 \not\leq H_1$. Every line $l \in \mathcal{L}$ with $l \leq H_1 \cap H_2$ satisfies $l \cap E_1 \neq \emptyset \neq l \cap E_3$ and thus $l \not\leq H_1$, a contradiction, which proves $l_1, l_1' \not\leq H_2$. Now, Lemma 2.4.13 applies (the second condition is fulfilled for $H = H_2$, $E = E_1$ and $E' = E_3$ — note that the roles of $\mathcal{L}_1$ and $\mathcal{L}_2$ are swapped) and the subspace U given there satisfies $\dim(U) \geq 2$, proving that C3 (d), C5 or C6 occurs. $\square$

Remark 2.4.16. *From the proof of Theorem 2.4.15 we also gather that C1 only occurs if Corollary 2.4.4 is applicable, C2 (a) only occurs if Lemma 2.4.8 is applicable with equality, C2 (b) only occurs if Lemma 2.4.9 is applicable with $\mathcal{L}_3 = \emptyset$, C2 (c) only occurs if Lemma 2.4.10 is applicable with $\mathcal{L}_2 = \emptyset$, C3 (a) only occurs if Lemma 2.4.11 is applicable, C3 (b) only occurs if Lemma 2.4.12 is applicable with $\mathcal{L} = \mathcal{L}_1$, C3 (c) only occurs if Lemma 2.4.13 is applicable with* $\dim(U) = 1$*, that is, $\mathcal{L} = \mathcal{L}_1$, C3 (d) only occurs if Lemma 2.4.13 is applicable with* $\dim(U) = 2$ *and C4 only occurs if Lemma 2.4.14 is applicable with $\mathcal{L}_3 = \emptyset$.*

2.4.2 Structure provided by Flags through a given Point P

Throughout this section we let H be a hyperplane of $\mathbb{P}$ and we let P be a point of H. We always consider a set $\mathfrak{L}$ of lines on P which all satisfy a given subset of the cases of Theorem 2.4.15 and determine implications that the flags through lines of that set yield for the rest of C. In fact, the flags that correspond to lines in $\mathfrak{L}$ will never provide any further information on flags the contain P, but only on flags that do not contain P. One should keep that in mind when reading this section, as the results here are mostly trivially true for flags that do contain P.

We begin with a quite obvious result that will be used frequently.

Lemma 2.4.17. *If $\mathfrak{L}$ is a non-empty set of lines through P such that $l \not\leq H$ for all $l \in \mathfrak{L}$, then $|\mathfrak{L}| \leq q^{\dim(\langle \mathfrak{L} \rangle)-1}$.*

Proof. We let $\mathfrak{L}$ be as in the claim and set $d := \dim(\langle \mathfrak{L} \rangle)$. Since H is a hyperplane we know that $\langle \mathfrak{L} \rangle \cap H$ is a hyperplane of $\langle \mathfrak{L} \rangle$ and thus has dimension $d-1$. Then the claim follows from

$$|\mathfrak{L}| \leq |\mathfrak{S}[P, 1, \langle \mathfrak{L} \rangle] \setminus \mathfrak{S}[P, 1, \langle \mathfrak{L} \rangle \cap H]| = \mathfrak{s}_q[0, 1, d] - \mathfrak{s}_q[0, 1, d-1] = q^{d-1}. \qquad \square$$

The following Lemma is in regard to cases C2 (a), C3 (a) and C3 (c) of Theorem 2.4.15 and uses the existence of the solid (in case of C2 (a)) or the hyperplane (in the other two cases) that is given there. Note that, although in these cases the arguments are basically the same, the proof is quite technical and it is best to first read it for the first case and then for the other two cases. In the terminology of the proof these two situations are distinguished via $\kappa = 3$ and $\kappa = 4$.

Proposition 2.4.18. *Let $\mathfrak{L}$ be a non-empty set of lines l with $P \in l \not\leq H$ such that one of the following holds:*

(I) Every line $l \in \mathfrak{L}$ satisfies C2 (a) of Theorem 2.4.15.

(II) Every line $l \in \mathfrak{L}$ satisfies C3 (a) or C3 (c) of Theorem 2.4.15.

Furthermore, let $d \in \{1, \dots, 4\}$ and, if (I) occurs, then set $\kappa := 3$ and otherwise, if (II) occurs, then set $\kappa := 4$.

If for some integer ξ we have $|\mathfrak{L}| > \kappa\xi$ and for every subspace G of dimension d we have $|\Delta_G(\mathfrak{L})| \leq \xi$, then there is a subspace U with

$$\dim(U) \leq \begin{cases} \kappa - 1 & \text{for } d \geq \kappa, \\ \kappa & \text{otherwise} \end{cases}$$

such that for every line l with $l \not\leq U$ there is a subspace $G_l \ni P$ with $\dim(G_l) \geq d+1$ for which every solid $S \in \Pi_3(\Delta_l(C))$ contains P or a complement of P in G_l. Moreover, in the above, if there is a subspace V with $\dim(V) = d+1$ and $g \leq V$ for all $g \in \mathfrak{L}$, then $V = G_l$ for all lines l with $l \not\leq U$, and, if $\dim(U) = \kappa$, then $G_l \leq U$ for all lines l with $l \not\leq U$.

Proof. For every line $l \in \mathfrak{L}$ let W_l be the subspace with $\pi_1(f) \leq W_l$ or $\pi_3(f) \cap l \neq \emptyset$ for all $f \in C$ provided by Theorem 2.4.15 and the respective Lemma that occurs, as is listed in Remark 2.4.16. Note that this implies $\pi_1(f) \leq W_l$ or $\pi_3(f) \cap l \neq \emptyset$ for all $l \in \mathfrak{L}$ and all $f \in C$. Furthermore, we have $\dim(W_l) = \dim(W_g)$ for all $l, g \in \mathfrak{L}$, that is, we may set $\kappa(\mathfrak{L}) := \dim(W_l)$ for some arbitrary $l \in \mathfrak{L}$ and note that we have $\kappa(\mathfrak{L}) = 3$ if case (I) occurs and $\kappa(\mathfrak{L}) = 4$ if case (II) occurs, that is, we have $\kappa(\mathfrak{L}) = \kappa$.

The idea is now as follows. Recall, that our aim was to find implications that the set $\mathfrak{L}$ gives for the set of flags of C on a line l. If l contains P, then, as we mentioned earlier, one may not gather any implications for the flags on l from the set $\mathfrak{L}$ and its corresponding flags. Thus, we only consider lines l with $P \notin l$. Among those lines we choose the line l for which we may gather the smallest amount of information. This line is such, that for most lines $g \in \mathfrak{L}$ we have $l \leq W_g$ and its span with P is a plane that will be called U_2. We note that $\mathfrak{L}$ yields the same amount of information for any other line in this plane which does not contain P. Now, if the information that $\mathfrak{L}$ yields for l is already sufficient, then it is sufficient for all lines per our choice of the line. However, if it is not, then we choose another line with that property (which meets U_2 in a point) and the span of that line together with U_2 will be called U_3. We proceed as such until we finally find a line that provides enough information. We now provide the formal definition of these subspaces U_i.

For all $U \leq \mathbb{P}$ with $P \in U$ we set $\mathfrak{L}_U := \{l \in \mathfrak{L} : U \leq W_l\}$ and note that for $U, V \leq \mathbb{P}$ with $P \in U, V$ we obviously have $U \leq V \implies \mathfrak{L}_V \subseteq \mathfrak{L}_U$. Furthermore, for convenience we set $U_1 := P$ and for all $i \in \{2, \ldots, 5\}$ we let $U_i \in \mathfrak{G}[U_{i-1}, i, \mathbb{P}]$ be such that for all $U \in \mathfrak{G}[U_{i-1}, i, \mathbb{P}]$ we have $|\mathfrak{L}_{U_{i-1}} \setminus \mathfrak{L}_{U_i}| \leq |\mathfrak{L}_{U_{i-1}} \setminus \mathfrak{L}_U|$. Note that, although U_1 has dimension 0, the subspaces U_i for $i \in \{2, \ldots, 5\}$ have dimension i.

From our definition above we have $\mathfrak{L}_{U_1} = \mathfrak{L}_P = \mathfrak{L}$, we know that $\dim(U_\kappa) = \kappa = \dim(W_l)$ implies $l \leq W_l = U_\kappa$ for all $l \in \mathfrak{L}_{U_\kappa}$ and, in addition to that, for all $l \in \mathfrak{L}$ and all $j \in \{1, \ldots, 5\}$ with $j > \kappa$ we have $\dim(U_j) = j > \kappa = \dim(W_l)$ and thus $U_j \not\leq W_l$, that is, $\mathfrak{L}_{U_j} = \emptyset$. Moreover, for all $i \in \{2, \ldots, 5\}$ and every subspace U with $U_{i-1} \leq U$ and $\dim(U) \geq i$ we have

$$|\mathfrak{L}_{U_{i-1}} \setminus \mathfrak{L}_U| \geq |\mathfrak{L}_{U_{i-1}} \setminus \mathfrak{L}_{U_{\dim(U)}}| \geq |\mathfrak{L}_{U_{i-1}} \setminus \mathfrak{L}_{U_i}|. \tag{2.35}$$

Now, let ξ be an integer, let $|\mathfrak{L}_{U_1}| = |\mathfrak{L}| > \kappa\xi$ and assume that every subspace G of dimension d satisfies $|\Delta_G(\mathfrak{L})| \leq \xi$. Since $\mathfrak{L}_{U_{\kappa+1}} = \emptyset$ there is $\nu \in \{2, \ldots, \kappa+1\}$ such that $|\mathfrak{L}_{U_{\nu-1}} \setminus \mathfrak{L}_{U_\nu}| > \xi$. We choose ν minimal with that property, set $U := U_{\nu-1}$ and note that we have $\dim(U) \leq \nu - 1 \leq \kappa$. Note that the existence of an integer $j \leq \kappa$ with $|\mathfrak{L}_{U_j}| \leq (\kappa + 1 - j)\xi$ proves $\nu \leq j$ and, if $d \geq \kappa$, then, since $l \leq U_\kappa$ for all $l \in \mathfrak{L}_{U_\kappa}$, we have $|\mathfrak{L}_{U_\kappa}| \leq \xi$ and thus $\nu \leq \kappa$, which implies $\dim(U) \leq \kappa - 1$ for $d \geq \kappa$.

Note that for any line l with $P \in l$ the claim holds trivially (choosing a fitting subspace for G_l) and thus we only consider lines l with $P \notin l$ in the following. For any line l with $P \notin l \not\leq U$ we have $|\mathfrak{L}_U \setminus \mathfrak{L}_{\langle U,l\rangle}| \geq |\mathfrak{L}_U \setminus \mathfrak{L}_{U_\nu}| > \xi$ (using Equation (2.35)) and any line $g \in \overline{\mathfrak{L}}_l := \mathfrak{L}_U \setminus \mathfrak{L}_{\langle U,l\rangle}$ satisfies $l \not\leq W_g$. Hence, for all $f \in C$ we know that $\pi_1(f) \not\leq U$ implies that $\pi_3(f)$ has non-empty intersection with all lines in $\overline{\mathfrak{L}}_{\pi_1(f)}$. Let l be an arbitrary line with $P \notin l \not\leq U$. Every solid $S \in \Pi_3(\Delta_l(C))$ contains P or a complement of P in $G_l := \langle \overline{\mathfrak{L}}_l \rangle \ni P$. Since any d-dimensional subspace contains at most ξ lines of $\mathfrak{L}$ we know that G_l has dimension at least $d+1$. Obviously, if there is a subspace V of dimension $d+1$ such that $g \leq V$ for all $g \in \mathfrak{L}$, then $G_l \leq V$ and thus $G_l = V$. Finally, note that for $\dim(U) = \kappa$ we have $g \leq U_\kappa = U$ for all $g \in \mathfrak{L}_U \supseteq \overline{\mathfrak{L}}_l$, that is, $G_l \leq U$. $\square$

Cases C3 (a) and C3 (b) of Theorem 2.4.15

For this part we let $\mathfrak{L}$ be a set of lines l with $P \in l \not\leq H$ which satisfy case C3 (a) or C3 (b) of Theorem 2.4.15 (the set $\mathfrak{L}$ may contain lines of both cases). Furthermore, for every line $l \in \mathfrak{L}$ we let $E_l \geq l$ be the plane with $\pi_1(f) \cap E_l \neq \emptyset$ or $\pi_3(f) \cap l \neq \emptyset$ for all $f \in C$ provided by Theorem 2.4.15 and the respective Lemma that occurs, as is listed in Remark 2.4.16. Finally, for all lines $h \in \mathfrak{G}[P, 1, \mathbb{P}]$ we set $\mathfrak{L}_h := \{l \in \mathfrak{L} : h \leq E_l\}$, for all lines $h \in \mathcal{L} := \mathfrak{G}[P, \emptyset, 1, \mathbb{P}]$ we set $\mathfrak{L}_h := \bigcup_{Q \in h} \mathfrak{L}_{\langle Q,P \rangle}$ and for every line $h \in \mathfrak{G}[1, \mathbb{P}]$ we set $\overline{\mathfrak{L}}_h := \mathfrak{L} \setminus \mathfrak{L}_h$.

Lemma 2.4.19. *Let $\xi \geq q$ be an integer and $g \in \mathfrak{G}[P, 1, \mathbb{P}]$ be a line with*

$$|\mathfrak{L}_g| > \begin{cases} \xi + q^2 & \text{for } \xi \geq q^2, \\ 3\xi & \text{otherwise.} \end{cases}$$

If $\xi \geq q^2$ then all lines $l \in \mathfrak{G}[g, \emptyset, 1, \mathbb{P}]$ satisfy $|\overline{\mathfrak{L}}_l| > \xi$, and if $\xi < q^2$ then there is a solid $\widehat{S} \geq g$ such that all lines $l \in \mathfrak{G}[g, \emptyset, 1, \mathbb{P}]$ with $l \not\leq \widehat{S}$ satisfy $|\overline{\mathfrak{L}}_l| > \xi$.

Proof. Let $h \in \mathfrak{G}[g, \emptyset, 1, \mathbb{P}]$ be such that

$$\forall l \in \mathfrak{G}[g, \emptyset, 1, \mathbb{P}] : |\mathfrak{L}_g \cap \mathfrak{L}_h| \geq |\mathfrak{L}_g \cap \mathfrak{L}_l|.$$

For any line $l \in \mathfrak{L}_g \cap \mathfrak{L}_h$ we have $g \leq E_l$ as well as $E_l \cap h \neq \emptyset$ and, since $g \cap h = \emptyset$ and E_l is a plane, this implies $E_l = \langle g, E_l \cap h \rangle \leq \langle g, h \rangle$ and may only occur for $l \leq \langle g, h \rangle$. This proves $\mathfrak{L}_g \cap \mathfrak{L}_h \subseteq \mathfrak{G}[P, 1, \langle h, g \rangle]$. Now, for all $l \in \mathfrak{G}[g, \emptyset, 1, \mathbb{P}]$ we have $\mathfrak{L}_g \setminus \mathfrak{L}_l \subseteq \overline{\mathfrak{L}}_l$ and thus if $|\mathfrak{L}_g \cap \mathfrak{L}_h| < |\mathfrak{L}_g| - \xi$, then

$$|\overline{\mathfrak{L}}_l| \geq |\mathfrak{L}_g \setminus \mathfrak{L}_l| = |\mathfrak{L}_g| - |\mathfrak{L}_g \cap \mathfrak{L}_l| \geq |\mathfrak{L}_g| - |\mathfrak{L}_g \cap \mathfrak{L}_h| > |\mathfrak{L}_g| - (|\mathfrak{L}_g| - \xi) = \xi$$

and there remains nothing to prove. Indeed, if $\xi \geq q^2$, then we may use Lemma 2.4.17 to see that this situation occurs:

$$|\mathfrak{L}_g \cap \mathfrak{L}_h| \overset{2.4.17}{\leq} q^2 = (\xi + q^2) - \xi < |\mathfrak{L}_g| - \xi.$$

Therefore, we may assume $q \leq \xi < q^2$ with $|\mathfrak{L}_g| > 3\xi$ as well as $|\mathfrak{L}_g \cap \mathfrak{L}_h| > 2\xi$. We set $\widehat{S} := \langle h, g \rangle$ and let l be a line with $l \not\leq \widehat{S}$ and $l \cap g = \emptyset$. If $l \cap \widehat{S} = \emptyset$, then the fact that for all $l' \in \mathfrak{L}_g \cap \mathfrak{L}_h$ we have $E_{l'} \leq \widehat{S}$ and thus $E_{l'} \cap l \leq \widehat{S} \cap l = \emptyset$ implies $|\overline{\mathfrak{L}}_l| \geq |\mathfrak{L}_g \cap \mathfrak{L}_h| > 2\xi$ and there remains nothing to prove. Thus, assume that $l \cap \widehat{S}$ is a point. Then there is only one plane in $\mathfrak{G}[g, 2, \widehat{S}]$ which meets l, namely the plane $E := \langle g, l \cap \widehat{S} \rangle$. Any line $l' \in \mathfrak{L}_h \cap \mathfrak{L}_g$ with $E_{l'} \cap l \neq \emptyset$ satisfies $l' \leq E_{l'} = E$. Furthermore, in E there are at most q lines through P which do not lie in H, which proves $|\mathfrak{L}_g \cap \mathfrak{L}_h \cap \mathfrak{L}_l| \leq q$ and thus the claim is implied by

$$|\overline{\mathfrak{L}}_l| \geq |(\mathfrak{L}_g \cap \mathfrak{L}_h) \setminus \mathfrak{L}_l| = |\mathfrak{L}_g \cap \mathfrak{L}_h| - |\mathfrak{L}_g \cap \mathfrak{L}_h \cap \mathfrak{L}_l| > 2\xi - q \geq \xi.$$ □

Lemma 2.4.20. *Let $\xi \geq q$ be an integer and let $|\mathfrak{L}| > 2\xi + \alpha_\xi + \beta_\xi$ with*

$$\alpha_\xi := \begin{cases} \xi + q^2 & \text{for } \xi \geq q^2, \\ 3\xi & \text{otherwise} \end{cases} \qquad \text{and} \qquad \beta_\xi := \begin{cases} q^2 & \text{for } \xi \geq q^2, \\ \alpha_\xi + \xi & \text{otherwise.} \end{cases}$$

Then one of the following occurs:

i) There is a line g (and a solid S for $\xi < q^2$) with $P \in g$ (and $g \leq S$ for $\xi < q^2$) such that for all $h \in \mathcal{L}$ with $h \cap g = \emptyset$ (and $h \not\leq S$ for $\xi < q^2$) we have $|\overline{\mathfrak{L}}_h| > \xi$.

ii) There is a plane $E_1 \ni P$ such that for all $h \in \mathcal{L}$ with $h \not\leq E_1$ we have $|\overline{\mathfrak{L}}_h| > \xi$.

iii) $\xi < q^3$ and there are two planes $E_1 \ni P$ and $E_2 \ni P$ such that for all $h \in \mathcal{L}$ with $h \not\leq E_1$ and $h \not\leq E_2$ we have $|\overline{\mathfrak{L}}_h| > \xi$.

iv) $\xi < q^2$ and there is a subspace H' of dimension at most 4 such that for all $h \in \mathcal{L}$ with $h \not\leq H'$ we have $|\overline{\mathfrak{L}}_h| > \xi$.

Proof. Let $h_1 \in \mathcal{L}$ be such that $|\mathfrak{L}_{h_1}| \geq |\mathfrak{L}_h|$ for all $h \in \mathcal{L}$ and set $E_1 := \langle P, h_1 \rangle$. If $|\mathfrak{L}_{h_1}| < |\mathfrak{L}| - \xi$, then for all $l \in \mathcal{L}$ we have

$$|\overline{\mathfrak{L}}_l| = |\mathfrak{L}| - |\mathfrak{L}_l| \geq |\mathfrak{L}| - |\mathfrak{L}_h| > |\mathfrak{L}| - (|\mathfrak{L}| - \xi) = \xi,$$

and, choosing arbitrary subspaces in the claims, all of these hold. Furthermore, if there is a line $g \in \mathfrak{S}[P,1,\mathbb{P}]$ with $|\mathfrak{L}_g| > \alpha_\xi$, then Lemma 2.4.19 proves that i) holds. Therefore, we assume

$$|\mathfrak{L}_{h_1}| \geq |\mathfrak{L}| - \xi > \xi + \alpha_\xi + \beta_\xi \tag{2.36}$$

as well as

$$\forall g \in \mathfrak{S}[P,1,\mathbb{P}] : |\mathfrak{L}_g| \leq \alpha_\xi \tag{2.37}$$

and let $h_2 \in \{l \in \mathcal{L} : l \not\leq E_1\}$ be such that

$$\forall h \in \{l \in \mathcal{L} : l \not\leq E_1\} : |\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}| \geq |\mathfrak{L}_{h_1} \cap \mathfrak{L}_h|. \tag{2.38}$$

Again, if $|\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}| < |\mathfrak{L}_{h_1}| - \xi$, then for all $l \in \mathcal{L}$ with $l \not\leq E_1$ we have

$$\begin{aligned} |\overline{\mathfrak{L}}_l| &\geq |\mathfrak{L}_{h_1} \setminus \mathfrak{L}_l| = |\mathfrak{L}_{h_1}| - |\mathfrak{L}_{h_1} \cap \mathfrak{L}_l| \overset{(2.38)}{\geq} |\mathfrak{L}_{h_1}| - |\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}| \\ &> |\mathfrak{L}_{h_1}| - (|\mathfrak{L}_{h_1}| - \xi) = \xi, \end{aligned} \tag{2.39}$$

and ii) holds.

Now, any line $l \in \mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}$ satisfies $E_l \cap h_i \neq \emptyset$ for all $i \in \{1,2\}$ and thus either $\langle P, E_l \cap h_1 \rangle = \langle P, E_l \cap h_2 \rangle$, or

$$l \leq E_l = \langle P, E_l \cap h_1, E_l \cap h_2 \rangle \leq \langle P, h_1, h_2 \rangle =: H'. \tag{2.40}$$

Note that $\langle P, E_l \cap h_1 \rangle = \langle P, E_l \cap h_2 \rangle$ for some $l \in \mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}$ may only occur, if E_1 and $E_2 := \langle P, h_2 \rangle$ have a line in common. Therefore, we have

$$\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2} \subseteq \begin{cases} \mathfrak{S}[P,1,H'] & \text{for } P = E_1 \cap E_2, \\ \mathfrak{S}[P,1,H'] \cup \mathfrak{L}_{E_1 \cap E_2} & \text{for } \dim(E_1 \cap E_2) = 1 \end{cases} \tag{2.41}$$

and, if we apply Lemma 2.4.17 and use Equation (2.37), then we see that

$$|\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}| \le \begin{cases} q^3 & \text{for } P = E_1 \cap E_2, \\ q^2 + \alpha_\xi & \text{for } \dim(E_1 \cap E_2) = 1. \end{cases}$$

Hence, if $P = E_1 \cap E_2$ and $\xi \ge q^3 - q^2 \ge q^2$, then we have $\alpha_\xi = \xi + q^2$ as well as

$$|\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}| \le q^3 \le \xi + q^2 = \alpha_\xi \overset{(2.36)}{<} |\mathfrak{L}_{h_1}| - \xi,$$

and if $\dim(E_1 \cap E_2) = 1$ and $\xi \ge q^2$, then we have

$$|\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}| \le \alpha_\xi + q^2 = \alpha_\xi + \beta_\xi \overset{(2.36)}{<} |\mathfrak{L}_{h_1}| - \xi$$

and either way Equation (2.39) proves ii). Since this covers all cases with $\xi \ge q^3$ this also implies that for $\xi \ge q^3$ one of the first two cases occurs, as claimed.

Therefore, from now on we may assume that

$$|\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}| \ge |\mathfrak{L}_{h_1}| - \xi \overset{(2.36)}{>} \alpha_\xi + \beta_\xi \tag{2.42}$$

and either $E_1 \cap E_2 = P$ with $\xi < q^3 - q^2$, or $\dim(E_1 \cap E_2) = 1$ with $\xi < q^2$. We study the two situations $\xi \ge q^2$ and $\xi < q^2$ separately.

First, assume that $\xi \ge q^2$ and thus $E_1 \cap E_2 = P$ occurs. Then H' is a hyperplane of $\mathbb{P}$, Equation (2.41) shows $\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2} \subseteq \mathfrak{S}[P, 1, H']$ and we have $\alpha_\xi = q^2 + \xi$ as well as $\beta_\xi = q^2$. Let $h \in \mathcal{L}$ be an arbitrary but fixed line with $h \not\le E_1, E_2$. We determine the number of planes through P which meet h_1, h_2 as well as h and, using that number, we determine the cardinality of $\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2} \cap \mathfrak{L}_h$. In view of that we recall that any plane E which contains P and meets both h_1 and h_2 is a subspace of H', as we have seen earlier in Equation (2.40). Now, since H' is a hyperplane of $\mathbb{P}$ we know that $Q := h \cap H'$ is a point or the line h itself, which leaves us with two cases to consider:

- Let Q be a point. Since $E_1 \cap E_2 = P \notin h$ we know that there is some index $i \in \{1, 2\}$ with $Q \notin E_i$. Therefore, any plane E with $P, Q \in E$ and $E \cap h_i \ne \emptyset$ satisfies $E \le \langle E_i, Q \rangle$. Now, for all $l \in \mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2} \cap \mathfrak{L}_h$ this implies $l \le E_l \le \langle E_i, Q \rangle$ and, using Lemma 2.4.17 and the fact that $\langle E_i, Q \rangle$ has dimension 3, this shows

$$|\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2} \cap \mathfrak{L}_h| \le q^2. \tag{2.43}$$

- Let $Q = h$, that is, $h \le H'$. Every plane that meets h_1 and lies in H' lies in one of the solids of $\mathfrak{S}[E_1, 3, H']$. Since $h \not\le E_1, E_2$, there is at most one solid $S \in \mathfrak{S}[E_1, 3, H']$ with $h \le S$. Furthermore, if $h \cap E_2$ is a point, then, since $E_1 \cap E_2 = P \notin h$ and thus $h \cap E_2 \notin E_1$, there is at most one solid $S \in \mathfrak{S}[E_1, 3, H']$ with $h \cap E_2 \in S$. Moreover, if $h \cap E_2$ is a point and there are solids $S, S' \in \mathfrak{S}[E_1, 3, H']$ such that $h \le S$ and $h \cap E_2 \in S'$, then obviously $S = S'$. Hence, we may let $\{S_1, \dots, S_{q+1}\} = \mathfrak{S}[E_1, 3, H']$ be such that $P_i := S_i \cap h$ is a point for all $i \in \{1, \dots, q + 1\}$ and such that, if $h \cap E_2 \ne \emptyset$, then $h \cap E_2 \in S_1$.

Now, for all $i \in \{2,\dots,q+1\}$ we know that P_i is a point which does not lie in E_2 and the only plane in S_i through P which meets h_1, h_2 and h is the plane $U_i := \langle P_i, S_i \cap E_2\rangle$. Therefore, if we set $U_1 := S_1$, then for all $l \in \mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2} \cap \mathfrak{L}_h$ we have $l \le E_l \le U_i$ for some $i \in \{1,\dots,q+1\}$ and this proves

$$|\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2} \cap \mathfrak{L}_h| \le \sum_{i=1}^{q+1} |\mathfrak{S}[P,1,U_i]| - |\mathfrak{S}[P,1,U_i \cap H]| = 2q^2. \tag{2.44}$$

Now, using Equations (2.43) and (2.44) in the step marked with $(*)$, we have

$$|\overline{\mathfrak{L}}_h| \ge |\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}| - |\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2} \cap \mathfrak{L}_h| \overset{(*)}{\ge} |\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}| - 2q^2 \overset{(2.42)}{>} \alpha_\xi + \beta_\xi - 2q^2 = \xi$$

and, due to the arbitrary choice of $h \in \mathcal{L}$ with $h \not\le E_1, E_2$, this proves iii). Since this covers all cases with $\xi \ge q^2$ this also implies that for $\xi \ge q^2$ one of the first three cases occurs, as claimed.

Finally, consider $\xi < q^2$, which implies $\beta_\xi = \alpha_\xi + \xi$, and let $h \in \mathcal{L}$ be a line with $h \not\le H'$. Recall that H' does not necessarily have to be a hyperplane of $\mathbb{P}$ — it has smaller dimension if and only if $\dim(E_1 \cap E_2) > 0$. For any line $l \in \mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}$ we either have $\dim(E_1 \cap E_2) = 1$ and $l \in \mathfrak{L}_{E_1 \cap E_2}$, or we have $l \le E_l \le H'$. Note that, if $l \in \mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}$ satisfies $l \le E_l \le H'$, then we have $l \in \mathfrak{L}_h$ if and only if $h \cap H' \ne \emptyset$ and $h \cap H' \in E_l$, that is, if and only if $h \cap H' \ne \emptyset$ and $l \in \mathfrak{L}_{\langle P, h\cap H'\rangle}$. Now, using Equation (2.37) we have $|\mathfrak{L}_{\langle P,h\cap H'\rangle}| \le \alpha_\xi$ for $h \cap H' \ne \emptyset$ as well as $|\mathfrak{L}_{E_1\cap E_2}| \le \alpha_\xi$ for $\dim(E_1 \cap E_2) = 1$. This proves $|\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2} \cap \mathfrak{L}_h| \le 2\alpha_\xi$ and thus we have

$$|\overline{\mathfrak{L}}_h| \ge |\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}| - |\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2} \cap \mathfrak{L}_h| > \alpha_\xi + \beta_\xi - 2\alpha_\xi \overset{\xi \le q^2}{=} \xi,$$

and, using the arbitrary choice of h, this proves iv) and concludes the proof. □

Proposition 2.4.21. *i) If $|\mathfrak{L}| > 5q^2$, then one of the following occurs:*

- *a) There is a line $g \ni P$ and for any line $l \in \mathcal{L}$ with $l \cap g = \emptyset$ there is a hyperplane $H_l \ni P$ such that any solid $S \in \Pi_3(\Delta_l(C))$ contains P or a complement of P in H_l.*
- *b) There are two planes E_1 and E_2 through P and for any line $l \in \mathcal{L}$ with $E_1 \not\ge l \not\le E_2$ there is a hyperplane $H_l \ni P$ such that any solid $S \in \Pi_3(\Delta_l(C))$ contains P or a complement of P in H_l.*

ii) If either $|\mathfrak{L}| > 9q$ and $\langle \mathfrak{L}\rangle$ is a solid, or if $|\mathfrak{L}| > 81q$ and any subset $\mathfrak{L}' \subseteq \mathfrak{L}$ of more than $9q$ lines spans at least a hyperplane, then there is a solid $\widehat{S}'$ on P such that one of the following occurs:

- *a) There is a line g and a solid $\widehat{S}$ with $P \in g \le \widehat{S}$ and for any line $l \in \mathcal{L}$ with $l \cap g = \emptyset$ and $l \not\le \widehat{S}, \widehat{S}'$ there is a hyperplane $H_l \ni P$ such that any solid $S \in \Pi_3(\Delta_l(C))$ contains P or a complement of P in H_l.*
- *b) There is a subspace H' of dimension at most 4 and for any line $l \in \mathcal{L}$ with $l \not\le H'$ and $l \not\le \widehat{S}'$ there is a hyperplane $H_l \ni P$ such that any solid $S \in \Pi_3(\Delta_l(C))$ contains P or a complement of P in H_l.*

Proof. We prove the claims by applying Lemma 2.4.20, we do so simultaneously and the strategy is as follows: For the proof of i) we apply Lemma 2.4.20 with $\xi = q^2$. For the proof of ii), if $|\mathfrak{L}| > 9q$ and $\langle \mathfrak{L} \rangle$ is a solid, then we set $\widehat{S}' := \langle \mathfrak{L} \rangle$ and apply Lemma 2.4.20 with $\xi = q$ and if $|\mathfrak{L}| > 81q$ such that any subset $\mathfrak{L}' \subseteq \mathfrak{L}$ of more than $9q$ lines spans at least a hyperplane, then we let $\widehat{S}'$ be an arbitrary solid on P and apply Lemma 2.4.20 with $\xi = 9q$. Furthermore, in the following we use the notation that is provided by Lemma 2.4.20. Moreover, we remark that for the proof of i), if 2.4.20 ii) occurs, then in view of 2.4.20 iii) we may let E_2 be an arbitrary plane on P and for the proof of ii), if 2.4.20 ii) or 2.4.20 iii) occurs, then in view of 2.4.20 iv) we may let H' be an arbitrary subspace of dimension at most 4 containing E_1 or E_1 and E_2. Hence, for the proof of i) we may assume that either 2.4.20 i) or 2.4.20 iii) occurs and for the proof of ii) we may assume that either 2.4.20 i) or 2.4.20 iv) occurs. Note that, for the proof of ii), if 2.4.20 i) occurs, then we denote the solid S mentioned there for $\xi < q^2$ by $\widehat{S}$ in the equation below. Thus, using $\widehat{S} := \widehat{S}' := \emptyset$ for the proof of i), we set

$$\mathcal{L}' := \begin{cases} \{l \in \mathcal{L} : l \cap g = \emptyset, l \not\leq \widehat{S}, \widehat{S}'\} & \text{if 2.4.20 i) occurs,} \\ \{l \in \mathcal{L} : l \not\leq E_1, E_2, \widehat{S}'\} & \text{if 2.4.20 iii) occurs,} \\ \{l \in \mathcal{L} : l \not\leq \widehat{S}', H'\} & \text{if 2.4.20 iv) occurs} \end{cases}$$

Furthermore, we let l be an arbitrary but fixed line of $\mathcal{L}'$ such that, as we explained above, Lemma 2.4.20 proves $|\overline{\mathfrak{L}}_l| > \xi$, using the respective value of $\xi \in \{q^2, 9q, q\}$.

Now, consider i) and the part of ii) with $|\mathfrak{L}| > 81q$ such that any subset $\mathfrak{L}' \subseteq \mathfrak{L}$ of more than $9q$ lines spans at least a hyperplane of $\mathbb{P}$. Recall that for the former we use $\xi = q^2$ and for the latter we use $\xi = 9q$. Note that every line $l' \in \mathfrak{L}$ satisfies $P \in l' \not\leq H$ and thus Lemma 2.4.17 shows that any solid on P may contain at most q^2 lines of $\overline{\mathfrak{L}}_l \subseteq \mathfrak{L}$. Hence, in both situations the lines in $\overline{\mathfrak{L}}_l$ span a subspace of $\mathbb{P}$ of dimension at least 4. Therefore, we find lines $l_1, \dots, l_4 \in \overline{\mathfrak{L}}_l$ which span a hyperplane H_l on P. Any solid $S \in \Pi_3(\Delta_l(C))$ satisfies $S \cap l_i \neq \emptyset$ for all $i \in \{1, \dots, 4\}$ and we have $\langle S \cap l_1, \dots, S \cap l_4 \rangle \leq H_l \cap S$. Now, if $P \notin S$, then for all $i \in \{1, \dots, 4\}$ we have $l_i = \langle P, S \cap l_i \rangle \leq \langle l_1, \dots, l_4 \rangle = H_l$, which implies $H_l = \langle P, S \cap H_l \rangle$ and S contains a complement of P in H_l, as claimed

Finally, consider the case $|\mathfrak{L}| > 9q$ such that $\widehat{S}' = \langle \mathfrak{L} \rangle$ is a solid. Recall that in this situation we set $\xi = q$. Since every line $l' \in \mathfrak{L}$ satisfies $P \in l' \not\leq H$ we may apply Lemma 2.4.17 and see that any plane on P may contain at most q lines of $\overline{\mathfrak{L}}_l \subseteq \mathfrak{L}$. Thus we find lines $l_1, l_2, l_3 \in \overline{\mathfrak{L}}_l$ which span a solid on P, that is, they span $\widehat{S}'$. We set $H_l := \langle l, \widehat{S}' \rangle$ and assume that there is a solid $S \in \Pi_3(\Delta_l(C))$, for otherwise the claim holds for some arbitrary hyperplane and there remains nothing to prove. Then S satisfies $S \cap l_i \neq \emptyset$ for all $i \in \{1, 2, 3\}$ and we have $\langle S \cap l_1, S \cap l_2, S \cap l_3 \rangle \leq \widehat{S}'$. Now, if $P \notin S$, then for all $i \in \{1, 2, 3\}$ we have $l_i = \langle P, S \cap l_i \rangle \leq \langle l_1, l_2, l_3 \rangle = \widehat{S}'$, which implies $\widehat{S}' = \langle P, S \cap \widehat{S}' \rangle$ and S contains the complement $S \cap \widehat{S}'$ of P in $\widehat{S}'$. Therefore, if $P \notin S$, then $S \cap \widehat{S}'$ is a plane and, since S is a solid that also contains l, this proves $\emptyset \neq l \cap \widehat{S}' \in S \cap \widehat{S}'$ and thus $\langle l, S \cap \widehat{S}' \rangle$ is a complement of P in $H_l = \langle l, \widehat{S}' \rangle$, too, which concludes the proof. $\square$

Case C2 (b) of Theorem 2.4.15

For this part we let $\mathfrak{L}$ be a set of lines l with $P \in l \not\leq H$ which satisfy case C2 (b) of Theorem 2.4.15. For every line $l \in \mathfrak{L}$ we know from Remark 2.4.16 that Lemma 2.4.9 is applicable with $\mathcal{L}_3 = \emptyset$ and we let S_l^1 and S_l^{-1} with $S_l^1 \cap S_l^{-1} = l$ be the two solids provided there, for which we know that for all $f \in C$ we have $\pi_1(f) \leq S_l^1$, $\pi_1(f) \leq S_l^{-1}$, or $\pi_3(f) \cap l \neq \emptyset$. Furthermore, we set $\mathfrak{L}_0 := \mathfrak{L}$, for every subspace U we set

$$\mathfrak{L}_U := \{l \in \mathfrak{L} : U \leq S_l^1 \vee U \leq S_l^{-1}\}$$

and for all $g \in \mathfrak{G}[1,\mathbb{P}]$ we set

$$\overline{\mathfrak{L}}_g := \mathfrak{L} \setminus \mathfrak{L}_g = \{l \in \mathfrak{L} : g \not\leq S_l^{-1} \wedge g \not\leq S_l^1\}.$$

Note that to every subspace U and every line $l \in \mathfrak{L}_U$ with $U \not\leq l = S_l^1 \cap S_l^{-1}$ (in particular, for every subspace U with $\dim(U) \geq 1$ and $P \notin U$ or $\dim(U) \geq 2$) there is a unique index $\epsilon \in \{-1,1\}$ with $U \leq S_l^\epsilon$ and in this case we denote this index by $\epsilon_U(l)$. We first construct two planes and a solid in three steps, as follows:

- Let $g_1 \in \mathfrak{G}[P,\emptyset,1,\mathbb{P}]$ be such that

$$\forall g \in \mathfrak{G}[P,\emptyset,1,\mathbb{P}] : |\mathfrak{L} \setminus \mathfrak{L}_{g_1}| \leq |\mathfrak{L} \setminus \mathfrak{L}_g| \tag{2.45}$$

  and set $U_1 := \langle P, g_1\rangle$ as well as $\mathfrak{L}_1 := \mathfrak{L}_{U_1} = \mathfrak{L}_{g_1}$.

- Let $g_2 \in \mathfrak{G}[P,\emptyset,1,\mathbb{P}]$ with $g_2 \not\leq U_1$ be such that

$$\forall g \in \mathfrak{G}[P,\emptyset,1,\mathbb{P}] \text{ with } g \not\leq U_1 : |\mathfrak{L}_1 \setminus \mathfrak{L}_{g_2}| \leq |\mathfrak{L}_1 \setminus \mathfrak{L}_g| \tag{2.46}$$

  and set $\mathfrak{L}_2 := \mathfrak{L}_1 \cap \mathfrak{L}_{U_2} \subseteq \mathfrak{L}_1 \cap \mathfrak{L}_{g_2}$ with

$$U_2 := \begin{cases} \langle P, g_2\rangle & \text{for } g_2 \cap U_1 = \emptyset,\\ \langle U_1, g_2\rangle & \text{otherwise.}\end{cases}$$

- Let $g_3 \in \mathfrak{G}[P,\emptyset,1,\mathbb{P}]$ with $g_3 \not\leq U_1, U_2$ be such that

$$\forall g \in \mathfrak{G}[P,\emptyset,1,\mathbb{P}] \text{ with } g \not\leq U_1, U_2 : |\mathfrak{L}_2 \setminus \mathfrak{L}_{g_3}| \leq |\mathfrak{L}_2 \setminus \mathfrak{L}_g| \tag{2.47}$$

  and set $\mathfrak{L}_3 := \mathfrak{L}_2 \cap \mathfrak{L}_{U_3} \subseteq \mathfrak{L}_3 \cap \mathfrak{L}_{g_3}$ with

$$U_3 := \begin{cases} \langle P, g_3\rangle & \text{for } g_2 \cap U_1 \neq \emptyset,\\ \langle U_1, g_3\rangle & \text{for } g_2 \cap U_1 = \emptyset \wedge U_1 \cap g_3 \neq \emptyset,\\ \langle U_2, g_3\rangle & \text{otherwise.}\end{cases}$$

Note that in view of Proposition 2.4.24 below we may assume that we have $\mathfrak{L}_0 \neq \emptyset$ and then our choice above implies that the sets $\mathfrak{L}_1$, $\mathfrak{L}_2$ and $\mathfrak{L}_3$ may not be empty sets, either. Therefore, if U_1 and U_2 are planes (and thus satisfy $U_1 \cap U_2 = P$), then g_3 satisfies

$g_3 \cap U_i \neq \emptyset$ for some $i \in \{1,2\}$, since otherwise $\mathcal{L}_3$ would be the empty set. Thus, we may denote the subspaces U_1, U_2 and U_3 (not necessarily in that order) by $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{S}$ with $\mathcal{E}' \leq \mathcal{S}$, that is, $\{\mathcal{E}, \mathcal{E}', \mathcal{S}\} = \{U_1, U_2, U_3\}$, where $\mathcal{E}$ and $\mathcal{E}'$ are planes and $\mathcal{S}$ is a solid. Furthermore, we let $\{\sigma, \tau, \tau'\} = \{1,2,3\}$ be such that $\mathcal{S} = U_\sigma = \langle U_{\tau'}, g_\sigma \rangle$ and finally, we let $g_4 \in \mathfrak{G}[P, \emptyset, 1, \mathbb{P}]$ with $g_4 \not\leq \mathcal{E}, \mathcal{S}$, that is $g_4 \not\leq U_1, U_2, U_3$, be such that

$$\forall g \in \mathfrak{G}[P, \emptyset, 1, \mathbb{P}] \text{ with } g \not\leq \mathcal{E}, \mathcal{S} : |\mathfrak{L}_3 \setminus \mathfrak{L}_{g_4}| \leq |\mathfrak{L}_3 \setminus \mathfrak{L}_g| \tag{2.48}$$

and set $\mathfrak{L}_4 := \mathfrak{L}_3 \cap \mathfrak{L}_{g_4}$.

Lemma 2.4.22. *For all $\kappa \in \{1,2,3\}$ with $U_\kappa = \langle P, g_\kappa \rangle$ we have $\mathfrak{L}_\kappa = \mathfrak{L}_{\kappa-1} \cap \mathfrak{L}_{g_\kappa}$ and, if $\kappa > 1$, then $\epsilon_{g_\kappa}(l) = \epsilon_{U_\kappa}(l) = -\epsilon_{U_{\kappa-1}}(l)$ for all $l \in \mathfrak{L}_\kappa$.*

Proof. Let $\kappa \in \{1,2,3\}$ be such that $U_\kappa = \langle P, g_\kappa \rangle$. For all $l \in \mathfrak{L}_{g_\kappa}$ we have $P \in l \leq S_l^{\epsilon_{g_\kappa}(l)}$, which proves $U_\kappa \leq S_l^{\epsilon_{g_\kappa}(l)}$ and thus $\mathfrak{L}_{g_\kappa} = \mathfrak{L}_{U_\kappa}$, that is, $\mathfrak{L}_\kappa = \mathfrak{L}_{\kappa-1} \cap \mathfrak{L}_{g_\kappa}$. Moreover, for $l \in \mathfrak{L}_\kappa$ and $\kappa > 1$ we have $g_\kappa \not\leq S_l^{\epsilon_{U_{\kappa-1}(l)}}$ from

$$\begin{cases} g_\kappa \cap U_{\kappa-1} = \emptyset & \text{if } \kappa = 2, \\ S_l^{\epsilon_{U_{\kappa-1}}(l)} = U_{\kappa-1} = U_2 \not\geq g_3 = g_\kappa & \text{if } \kappa = 3 \end{cases}$$

and that implies

$$\forall l \in \mathfrak{L}_\kappa : \epsilon_{g_\kappa}(l) = \epsilon_{U_\kappa}(l) = -\epsilon_{U_{\kappa-1}}(l). \qquad \square$$

Lemma 2.4.23. *We have $|\mathfrak{L}_\sigma| \leq q^2$. Furthermore, if $\sigma = 2$, then we have $|(\mathfrak{L}_1 \cap \mathfrak{L}_{g_2}) \setminus \mathfrak{L}_2| \leq 1$ and finally, if $\sigma = 3$, then we have*

$$|\mathfrak{L}_2 \cap \mathfrak{L}_{g_3}| \leq \begin{cases} 2 & \text{for } g_3 \cap U_1 \neq \emptyset \neq g_3 \cap U_2, \\ 0 & \text{for } g_3 \cap U_1 = \emptyset = g_3 \cap U_2 \end{cases} \tag{2.49}$$

as well as $\mathfrak{L}_2 \cap \mathfrak{L}_{g_3} = \mathfrak{L}_3$.

Proof. For all $l \in \mathfrak{L}_\sigma$ we have $l \leq S_l^{\epsilon_{U_\sigma}(l)} = U_\sigma$ and Lemma 2.4.17 proves $|\mathfrak{L}_\sigma| \leq q^2$. We also recall that we have $S_l^1 \cap S_l^{-1} = l$ for all $l \in \mathfrak{L}$.

We consider $\sigma = 2$. For all $l \in \mathfrak{L}_1 \cap \mathfrak{L}_{g_2}$ with $l \notin \mathfrak{L}_2$, that is $l \notin \mathfrak{L}_{U_2}$, we have $g_2 \leq S_l^{\epsilon_{g_2}(l)} \not\geq U_2 = \langle U_1, g_2 \rangle$ and thus $U_1 \not\leq S_l^{\epsilon_{g_2}(l)}$, that is, $\epsilon_{U_1}(l) \neq \epsilon_{g_2}(l)$. Therefore, for all $l \in \mathfrak{L}_1 \cap \mathfrak{L}_{g_2}$ with $l \notin \mathfrak{L}_2$ the following equation holds for $i = 1$ and proves $l = \langle P, g_2 \cap U_1 \rangle$ and thus the claim:

$$P \neq g_\sigma \cap U_i \in S_l^{\epsilon_{g_\sigma}(l)} \cap S_l^{\epsilon_{U_i}(l)} = S_l^1 \cap S_l^{-1} = l \ni P. \tag{2.50}$$

Finally, we assume that $\sigma = 3$. Then we know that Lemma 2.4.22 applies with $\kappa = 2$ and thus for all $l \in \mathfrak{L}_2 \cap \mathfrak{L}_{g_3}$ we have $\epsilon_{U_1}(l) \neq \epsilon_{U_2}(l)$. Therefore, if $g_3 \cap U_1 \neq \emptyset \neq g_3 \cap U_2$ and $l \in \mathfrak{L}_2 \cap \mathfrak{L}_{g_3}$, then there is some $i \in \{1,2\}$ with $\epsilon_{U_i}(l) \neq \epsilon_{g_3}(l)$ and Equation (2.50) holds for this index i and $\sigma = 3$, which shows $l = \langle P, g_3 \cap U_i \rangle$. Since the index $i \in \{1,2\}$

was dependent on the line l this proves that there are at most two such lines, as claimed. Furthermore, for all $l \in \mathfrak{L}_2$ and all $i \in \{1,2\}$ we have $g_3 \cap U_i = \emptyset \implies g_3 \not\leq S_l^{\epsilon_{U_i}(l)}$. Hence, if $g_3 \cap U_i = \emptyset$ for both $i \in \{1,2\}$, then $\epsilon_{U_1}(l) \neq \epsilon_{U_2}(l)$ implies $l \notin \mathfrak{L}_{g_3}$ for all $l \in \mathfrak{L}_2$ and thus $\mathfrak{L}_2 \cap \mathfrak{L}_{g_3} = \emptyset$. Finally, if there is a unique index $i \in \{1,2\}$ with $g_3 \cap U_i \neq \emptyset$ and we have a line $l \in \mathfrak{L}_2 \cap \mathfrak{L}_{g_3}$, then $g_3 \cap U_{3-i} = \emptyset$ implies $g_3 \not\leq S_l^{\epsilon_{U_{3-i}}(l)}$ and therefore $\epsilon_{U_1}(l) \neq \epsilon_{U_2}(l)$ implies $g_3 \leq S_l^{\epsilon_{U_i}(l)}$ and thus $U_3 = \langle g_3, U_i \rangle \leq S_l^{\epsilon_{U_i}(l)}$. This proves $\mathfrak{L}_2 \cap \mathfrak{L}_{g_3} \subseteq \mathfrak{L}_2 \cap \mathfrak{L}_{U_3} = \mathfrak{L}_3$ and, since we trivially have $\mathfrak{L}_3 \subseteq \mathfrak{L}_2 \cap \mathfrak{L}_{g_3}$, that concludes this proof. □

Proposition 2.4.24. *If for some $\xi \in \mathbb{N}$ we have $|\mathfrak{L}| > 4\xi + 2$, then the two subspaces $\mathcal{E}$ and $\mathcal{S}$ with $P \in \mathcal{E} \cap \mathcal{S}$, $\dim(\mathcal{E} \cap \mathcal{S}) \leq 1$, $\dim(\mathcal{E}) \leq 2$ and $\dim(\mathcal{S}) \leq 3$ are such, that for all $g \in \mathfrak{G}[P, \emptyset, 1, \mathbb{P}]$ with $g \not\leq \mathcal{E}, \mathcal{S}$ there are more than ξ lines in $\mathfrak{L}$ that meet every solid $S \in \Pi_3(\Delta_g(C))$.*

Proof. For all $(g,S) \in C$ we have $S \cap l \neq \emptyset$ for all $l \in \overline{\mathfrak{L}}_g$ from our definition of $\mathfrak{L}_g$. Hence, we shall provide a lower bound on $\overline{\mathfrak{L}}_g$ for all lines g in question and, due to Equation (2.48), we may do so by considering the line g_4.

For all $l \in \mathfrak{L}_4 = \mathfrak{L}_3 \cap \mathfrak{L}_{g_4}$ we have $g_4 \not\leq \mathcal{S} = S_l^{\epsilon_{\mathcal{S}}(l)} \geq \mathcal{E}'$ and thus $g_4 \leq S_l^{\epsilon_{\mathcal{E}}(l)}$ as well as $S_l^{\epsilon_{\mathcal{E}}(l)} \geq \mathcal{E}$, which proves $S_l^{\epsilon_{\mathcal{E}}(l)} = \langle \mathcal{E}, g_4 \rangle$ and thus $l = \mathcal{S} \cap \langle \mathcal{E}, g_4 \rangle$, that is, $|\mathfrak{L}_4| \leq 1$. Hence, if $|\mathfrak{L}_0| > 4\xi + 2$, then

$$\exists i \in \{1, \ldots, 4\} : |\mathfrak{L}_{i-1} \setminus \mathfrak{L}_i| > \begin{cases} \xi + 1 & \text{for } i = \sigma, \\ \xi & \text{otherwise.} \end{cases} \tag{2.51}$$

We let ρ be the smallest integer in $\{1, \ldots, 4\}$ for which this equation holds and set $\overline{\mathfrak{L}} := \mathfrak{L}_{\rho-1} \setminus \mathfrak{L}_{g_\rho} \subseteq \overline{\mathfrak{L}}_{g_\rho}$. Note that our minimal choice of g_ρ per Equations (2.45), (2.46), (2.47) or (2.48) now proves $|\overline{\mathfrak{L}}_g| \geq |\overline{\mathfrak{L}}|$ for all $g \in \mathfrak{G}[P, \emptyset, 1, \mathbb{P}]$ with $g \not\leq U_i$ for all $i < \rho$ and therefore, to prove the claim it suffices to show $|\overline{\mathfrak{L}}| > \xi$. Note that the way of proof also allows to determine upper bounds on ρ and we collect these in a remark after this proof.

If $\sigma \neq \rho$, then $\mathfrak{L}_\rho = \mathfrak{L}_{\rho-1} \cap \mathfrak{L}_{g_\rho}$ (this is the definition for $\rho = 4$ and for $\rho \leq 3$ it follows from Lemma 2.4.22) and thus we have

$$\mathfrak{L}_{\rho-1} \setminus \mathfrak{L}_\rho = \mathfrak{L}_{\rho-1} \setminus \mathfrak{L}_{g_\rho} \tag{2.52}$$

and Equation (2.51) implies $|\overline{\mathfrak{L}}| > \xi$. Furthermore, if $\rho = \sigma = 2$ then Lemma 2.4.23 applies, that is, we have $|(\mathfrak{L}_1 \cap \mathfrak{L}_{g_2}) \setminus \mathfrak{L}_2| \leq 1$ and together with Equation (2.51) this implies $|\overline{\mathfrak{L}}| > \xi$. Finally, if $\rho = \sigma = 3$, then the minimal choice of ρ implies $|\mathfrak{L}_{\rho-1}| = |\mathfrak{L}_2| \geq 2\xi + 2$ and Lemma 2.4.23 applies, that is, either one of the cases given in Equation (2.49) occurs, which proves $|\overline{\mathfrak{L}}| > 2\xi$, or $\mathfrak{L}_\rho = \mathfrak{L}_{\rho-1} \cap \mathfrak{L}_{g_\rho}$ and then, too, Equation (2.52) holds and again Equation (2.51) implies $|\overline{\mathfrak{L}}| > \xi$. □

Remark 2.4.25. *Using the same notation as in the proof of Proposition 2.4.24 we also have the following bounds on ρ:*

i) If $\dim(\mathcal{E} \cap \mathcal{E}') = 1$*, then* $\rho \leq 3$ *and if also* $\xi \geq q^2$*, then* $\rho \leq 2$.

Proof. Let $\dim(\mathcal{E} \cap \mathcal{E}') = 1$ and note that per our construction this may only occur if $\sigma = 2$, in which case U_3 is the second plane that we have constructed above. Any line $l \in \mathfrak{L}_3$ then satisfies

$$\mathcal{E} \cap \mathcal{E}' \leq S_l^{\epsilon_{\mathcal{E}}(l)} \cap S_l^{\epsilon_{\mathcal{E}'}(l)} = S_l^{-1} \cap S_l^1 = l,$$

that is, $l = \mathcal{E} \cap \mathcal{E}'$. Hence, we have $|\mathfrak{L}_3| \leq 1$, which proves $\rho \leq 3$. Furthermore, we additionally have $|\mathfrak{L}_2| \leq q^2$ from Lemma 2.4.23 and thus, given $\xi \geq q^2$, we may deduce that Equation (2.51) holds for some $i \in \{1, 2\}$, which proves $\rho \leq 2$. □

ii) If $\xi \geq q^2$*, then, using Lemma 2.4.23, we have* $|\mathfrak{L}_3| \leq |\mathfrak{L}_\sigma| \leq q^2$*, which proves* $\rho \leq 3$.

Case C2 (c) of Theorem 2.4.15

For this part we let $\mathfrak{L}$ be a set of lines l with $P \in l \not\leq H$ which satisfy case C2 (c) of Theorem 2.4.15. Note that for all $l \in \mathfrak{L}$ we have $|\Pi_3(\Delta_l(C))| = q + 1$ and may let $S_l^1, \dots, S_l^{q+1}$ be the solids in $\Pi_3(\Delta_l(C))$. Furthermore, for every subspace U we set

$$\mathfrak{L}_U := \{l \in \mathfrak{L} : \exists i \in \{1, \dots, q+1\} \text{ with } U \leq S_l^i\},$$

and for all $g \in \mathfrak{G}[1, \mathbb{P}]$ we set

$$\overline{\mathfrak{L}}_g := \mathfrak{L} \setminus \mathfrak{L}_g = \{l \in \mathfrak{L} : \forall i \in \{1, \dots, q+1\} \text{ we have } g \not\leq S_l^i\}.$$

We recall that the solids of $\Pi_3(\Delta_l(C))$ have pairwise intersection l and thus to every subspace U and every line $l \in \mathfrak{L}_U$ with $U \not\leq l$ (in particular, for every subspace U with $\dim(U) \geq 1$ and $P \notin U$ or $\dim(U) \geq 2$) there is a unique index $\epsilon \in \{1, \dots, q+1\}$ with $U \leq S_l^\epsilon$ and we denote this index by $\epsilon_U(l)$. Furthermore, we let $\mathfrak{L}_0$ denote $\mathfrak{L}$ and for all $i \in \{1, \dots, 5\}$ we let $g_i \in \mathfrak{G}[P, \emptyset, 1, \mathbb{P}]$ be such that, setting $U_i := \langle P, g_i \rangle$ and $\mathfrak{L}_i := \mathfrak{L}_{U_i} \cap \mathfrak{L}_{i-1}$, the planes $U_1, \dots, U_5$ are pairwise distinct and all

$$g \in \mathcal{L}_i := \{h \in \mathfrak{G}[P, \emptyset, 1, \mathbb{P}] : \forall j \in \{1, \dots, i-1\} \text{ we have } h \not\leq U_j\} \tag{2.53}$$

satisfy

$$|\mathfrak{L}_{i-1} \setminus \mathfrak{L}_{g_i}| \leq |\mathfrak{L}_{i-1} \setminus \mathfrak{L}_g|. \tag{2.54}$$

Finally, we note that $P \in l$ for all $l \in \mathfrak{L}$ implies $\mathfrak{L}_{U_i} = \mathfrak{L}_{g_i}$ and thus $\mathfrak{L}_i = \mathfrak{L}_{g_i} \cap \mathfrak{L}_{i-1}$.

Lemma 2.4.26. *For every line* $l \in \mathfrak{L}$ *and every complement* S *of* l *in* $\mathbb{P}$ *there are opposite reguli* $\mathcal{R}$ *and* $\overline{\mathcal{R}}$ *in* S *with* $\Pi_3(\Delta_l(C)) = \{\langle l, r \rangle : r \in \mathcal{R}\}$ *and such that every flag* $f \in C$ *satisfies* $\pi_3(f) \cap l \neq \emptyset$ *or* $\pi_1(f) \leq \langle l, \overline{r} \rangle$ *for some* $\overline{r} \in \overline{\mathcal{R}}$.

Proof. Let l be a line of $\mathfrak{L}$. According to Remark 2.4.16 case C2 (c) of Theorem 2.4.15 occurs if and only if Lemma 2.4.10 is applicable with $\mathcal{L}_2 = \emptyset$. Therefore, there is a complement S' of l in $\mathbb{P}$ and there are unique opposite reguli $\mathcal{R}'$ and $\overline{\mathcal{R}}'$ in S' such that $\Pi_3(\Delta_l(C)) = \{\langle l, r'\rangle : r' \in \mathcal{R}'\}$ and such that every flag $f \in C$ satisfies $\pi_3(f) \cap l \neq \emptyset$ or $\pi_1(f) \leq \langle l, \overline{r}'\rangle$ for some $\overline{r}' \in \overline{\mathcal{R}}'$.

Now, let S be another complement of l in $\mathbb{P}$. According to Lemma 1.3.11 the existence of the regulus $\mathcal{R}' = \{U \cap S' : U \in \Pi_3(\Delta_l(C))\}$ in S' proves that $\mathcal{R} := \{S \cap \langle l, r\rangle : r \in \mathcal{R}'\}$ and $\overline{\mathcal{R}} = \{S \cap \langle l, \overline{r}'\rangle : \overline{r}' \in \overline{\mathcal{R}}'\}$ are opposite reguli in S. Furthermore, for every line $r' \in \mathcal{R}'$ we have

$$\langle l, r'\rangle = \langle l, \langle l, r'\rangle \cap S\rangle \in \{\langle l, r\rangle : r \in \mathcal{R}\},$$

which proves

$$\Pi_3(\Delta_l(C)) = \{\langle l, r'\rangle : r' \in \mathcal{R}'\} \subseteq \{\langle l, r\rangle : r \in \mathcal{R}\} \tag{2.55}$$

and since both sides of Equation (2.55) have cardinality $q+1$ the equation must hold with equality.

Finally, for all $\overline{r}' \in \overline{\mathcal{R}}'$ the line $\overline{r} := S \cap \langle l, \overline{r}'\rangle \in \overline{\mathcal{R}}$ satisfies $\langle l, \overline{r}'\rangle = \langle l, \overline{r}\rangle$. Therefore, and since any flag $f \in C$ with $\pi_3(f) \cap l = \emptyset$ satisfies $\pi_1(f) \leq \langle l, \overline{r}'\rangle$ for some $\overline{r}' \in \overline{\mathcal{R}}'$, any such flag also satisfies $\pi_1(f) \leq \langle l, \overline{r}\rangle$ for some $\overline{r} \in \overline{\mathcal{R}}$. $\square$

Lemma 2.4.27. *If $h_1, h_2 \in \mathfrak{S}[P, \emptyset, 1, \mathbb{P}]$ are such that $\langle P, h_1\rangle \cap h_2$ is a point, then $|\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}| \leq q^2$.*

Proof. Let $Q := \langle P, h_1\rangle \cap h_2$ be a point and note that it may not be the point P and that thus $\langle P, h_1, h_2\rangle$ is a solid. Assume that there is a line $l \in \mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}$. On the one hand, if $\epsilon_{h_1}(l) \neq \epsilon_{h_2}(l)$, then $\langle P, Q\rangle \leq S_l^{\epsilon_{h_1}(l)} \cap S_l^{\epsilon_{h_2}(l)} = l$ and thus $l = \langle P, Q\rangle \leq \langle P, h_1, h_2\rangle$. On the other hand, if $\epsilon_{h_1}(l) = \epsilon_{h_2}(l)$, then $l \leq S_l^{\epsilon_{h_1}(l)} = \langle P, h_1, h_2\rangle$. Therefore, we have $|\mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}| \leq |\mathfrak{S}[P, 1, \langle P, h_1, h_2\rangle]|$ and Lemma 2.4.17 proves the claim. $\square$

Lemma 2.4.28. *If $h_1, h_2 \in \mathfrak{S}[P, \emptyset, 1, \mathbb{P}]$ are such that $\langle P, h_1\rangle \cap h_2 = \emptyset$, then $S := \langle h_1, h_2\rangle$ is a solid with $P \notin S$, we have $\langle P, h_1\rangle \cap \langle P, h_2\rangle = P$ and for all $l \in \mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}$ we have $\epsilon_{h_1}(l) \neq \epsilon_{h_2}(l)$.*

Proof. We have $P \notin S$ for otherwise $\langle P, h_1\rangle$ is a hyperplane of S and as such as has non-empty intersection with h_2, a contradiction. Furthermore, for all $l \in \mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}$, if $\epsilon_{h_1}(l) = \epsilon_{h_2}(l)$, then $h_2 \leq S_l^{\epsilon_{h_2}(l)} = S_l^{\epsilon_{h_1}(l)} \geq \langle P, h_1\rangle$ and again $h_2 \cap \langle P, h_1\rangle \neq \emptyset$, a contradiction. Finally, from $P \notin S \geq h_1, h_2$ and since $h_1 \cap h_2 = \emptyset$ we have $S = \langle h_1, h_2\rangle$ as well as $\langle P, h_1\rangle \cap \langle P, h_2\rangle = P$. $\square$

Lemma 2.4.29. *If $h_1, h_2 \in \mathfrak{S}[P, \emptyset, 1, \mathbb{P}]$ are skew lines with $P \notin \langle h_1, h_2\rangle$ and we have $l \in \mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}$, then $Q := l \cap \langle h_1, h_2\rangle \neq \emptyset$ occurs only if $Q \in h_1$ or $Q \in h_2$.*

Proof. Assume that h_1 and h_2 are skew lines with $P \notin \langle h_1, h_2 \rangle =: S$ and assume that $l \in \mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2}$ is a line with $Q := l \cap S \neq \emptyset$ and $Q \not\leq h_1, h_2$. Since $l \ni P \notin S$ we know that Q is a point. Thus, for dimensional reasons, $\langle Q, h_1 \rangle \cap \langle Q, h_2 \rangle$ is a line in S and $P \notin S$ shows that $\langle P, Q, h_1 \rangle \cap \langle P, Q, h_2 \rangle$ is the plane $E = \langle P, \langle Q, h_1 \rangle \cap \langle Q, h_2 \rangle \rangle$. Now, E is a subspace of $S_l^{\epsilon_{h_1}(l)} \cap S_l^{\epsilon_{h_2}(l)}$ and this yields a contradiction, since Lemma 2.4.28 shows $\epsilon_{h_1}(l) \neq \epsilon_{h_2}(l)$, that is, $S_l^{\epsilon_{h_1}(l)} \cap S_l^{\epsilon_{h_2}(l)} = l$. $\square$

Lemma 2.4.30. *If $g_i \cap U_j \neq \emptyset$ for some distinct $i, j \in \{1, \ldots, 5\}$ or $g_3 \not\leq \langle P, g_1, g_2 \rangle$, then $|\mathfrak{L}_5| \leq 5q^2 + 9q + 1$.*

Proof. If $U_i \cap g_j \neq \emptyset$ for some distinct $i, j \in \{1, \ldots, 5\}$, then Lemma 2.4.27 implies the claim. Hence, assume that $U_i \cap g_j = \emptyset$ for all distinct $i, j \in \{1, \ldots, 5\}$. Note that Lemma 2.4.28 then shows $P \notin \langle g_i, g_j \rangle$, $U_i \cap U_j = P$ and $\epsilon_{g_i}(l) \neq \epsilon_{g_j}(l)$ for all distinct $i, j \in \{1, \ldots, 5\}$ and all $l \in \mathfrak{L}_5$. Furthermore, in view of the claim, assume that $g_3 \not\leq \langle P, g_1, g_2 \rangle$ and set $S := \langle g_1, g_2 \rangle$ as well as

$$\mathfrak{L}_5' := \{l \in \mathfrak{L}_5 : \forall i \in \{1, \ldots, 5\} \text{ we have } l \cap g_i = \emptyset\}.$$

Note that, using Lemma 2.4.17, we then have $|\mathfrak{L}_5| \leq |\mathfrak{L}_5'| + 5q$ and thus it suffices to prove $|\mathfrak{L}_5'| \leq 5q^2 + 4q + 1$.

For all $l \in \mathfrak{L}_5'$ we have $l \cap g_i = \emptyset$ for all $i \in \{1, \ldots, 5\}$ and thus Lemma 2.4.29 is applicable and shows $l \cap \langle g_i, g_j \rangle = \emptyset$ for all $i, j \in \{1, \ldots, 5\}$. In particular, for all $l \in \mathfrak{L}_5'$ the solid S is a complement of l and thus Lemma 2.4.26 yields opposite reguli $\mathcal{R}_l$ and $\overline{\mathcal{R}}_l$ in S such that $\Pi_3(\Delta_l(C)) = \{\langle l, r \rangle : r \in \mathcal{R}_l\}$ (which is equivalent to $\mathcal{R}_l = \{S_l^i \cap S : i \in \{1, \ldots, q+1\}\}$) and such that every flag $f \in C$ satisfies $\pi_3(f) \cap l \neq \emptyset$ or $\pi_1(f) \leq \langle l, \overline{r} \rangle$ for some $\overline{r} \in \overline{\mathcal{R}}_l$. For all $i \in \{1, \ldots, 5\}$ and all $l \in \mathfrak{L}_5'$ we set $h_i^l := \langle l, g_i \rangle \cap S \in \mathcal{R}_l$ and note that $g_1, g_2 \leq S$ proves $g_1 = h_1^l$ as well as $g_2 = h_2^l$ and motivates $h_1 := g_1$ as well as $h_2 := g_2$.

Now, for all $l \in \mathfrak{L}_5'$ we know that $\langle l, g_3 \rangle$ is a solid containing P, h_3^l and g_3 and thus $\langle P, h_3^l, g_3 \rangle$ has dimension at most 3. Furthermore, for all $l \in \mathfrak{L}_5'$ the span $\langle P, h_3^l \rangle$ is a plane contained in $\langle P, S \rangle$ and, since g_3 does not lie in $\langle P, S \rangle$, this implies that in fact $\langle P, h_3^l, g_3 \rangle$ has dimension 3, that is, we have

$$\forall l \in \mathfrak{L}_5' : \langle P, h_3^l, g_3 \rangle = \langle l, g_3 \rangle. \tag{2.56}$$

Furthermore, for all distinct $i, j \in \{1, \ldots, 5\}$ and all $l \in \mathfrak{L}_5'$ the fact that we have $\epsilon_{g_i}(l) \neq \epsilon_{g_j}(l)$ shows that h_i^l and h_j^l are distinct lines of $\mathcal{R}_l$. Moreover, if l and l' are distinct lines in $\mathfrak{L}_5'$ with $l' \not\leq \langle l, g_3 \rangle$, then we have

$$\langle P, h_3^l, g_3 \rangle \overset{(2.56)}{=} \langle l, g_3 \rangle \neq \langle l', g_3 \rangle \geq \langle P, g_3 \rangle = U_3 \quad \Longrightarrow \quad h_3^l \not\leq \langle l', g_3 \rangle \geq \langle P, h_3^{l'} \rangle \geq h_3^{l'}$$

which proves $\langle P, h_3^l \rangle \neq \langle P, h_3^{l'} \rangle$ as well as $h_3^l \neq h_3^{l'}$. Finally, we obviously have

$$\forall l, l' \in \mathfrak{L}_5', \forall i \in \{1, \ldots, 5\} : U_i = \langle P, g_i \rangle \leq \langle l, g_i \rangle \cap \langle l', g_i \rangle = \langle l, h_i^l \rangle \cap \langle l', h_i^{l'} \rangle. \tag{2.57}$$

Indeed, for all lines $l, l' \in \mathfrak{L}_5'$ for which there is an index $i \in \{3,4,5\}$ with $h_i^l \neq h_i^{l'}$ the intersection of the lines h_i^l and $h_i^{l'}$ still contains the point $U_i \cap S$ and thus the corresponding reguli $\mathcal{R}_l$ and $\mathcal{R}_{l'}$ are distinct, that is, we have $|\mathcal{R}_l \cap \mathcal{R}_{l'}| = 2$ and $h_j^l \cap h_j^{l'}$ is in fact the point $P_j := U_j \cap S$ for all $j \in \{3,4,5\}$.

Now, we consider the quotient space $\mathbb{P}/P$. The above translates into the following: $\widehat{S} := \langle P, S\rangle$ is a solid and a complement of any point $l \in \mathfrak{L}_5'$ in $\mathbb{P}/P$ (we have seen above that in $\mathbb{P}$ the solid S and every line $l \in \mathfrak{L}_5'$ satisfy $l \cap S = \emptyset$). Furthermore, for all $l \in \mathfrak{L}_5'$, since $\mathcal{R}_l$ is a regulus in $\mathbb{P}$ (in particular, in the solid S of $\mathbb{P}$), we know that $\widehat{\mathcal{R}}_l := \{\langle P, r\rangle : r \in \mathcal{R}_l\}$ is a regulus in the quotient space $\mathbb{P}/P$ (in particular, in the solid $\widehat{S}$ of $\mathbb{P}/P$) and it contains five distinct lines of $\mathbb{P}/P$, namely $\widehat{h}_i^l := \langle P, h_i^l\rangle$ for $i \in \{1,\dots,5\}$. Moreover, for all $l, l' \in \mathfrak{L}_5'$ the reguli $\widehat{\mathcal{R}}_l$ and $\widehat{\mathcal{R}}_{l'}$ have the two lines $\langle P, h_1\rangle$ and $\langle P, h_2\rangle$ in common and from Equation (2.57) we already know that the set

$$\{(\widehat{r}, \widehat{r}') \in \widehat{\mathcal{R}}_l \times \widehat{\mathcal{R}}_{l'} : \dim_{\mathbb{P}/P}(\langle l, \widehat{r}\rangle \cap \langle l', \widehat{r}'\rangle) \geq 1\}$$

contains $(\widehat{h}_i^l, \widehat{h}_i^{l'})$ for all $i \in \{1,\dots,5\}$ and thus it has cardinality at least 5. Finally, for all $l, l' \in \mathfrak{L}_5'$ with $l' \not\leq \langle l, g_3\rangle$ we already have seen $\langle P, h_3^l\rangle \neq \langle P, h_3^{l'}\rangle$ as well as $\langle P, P_3\rangle \leq \langle P, h_3^l\rangle \cap \langle P, h_3^{l'}\rangle$, that is, a line of $\widehat{\mathcal{R}}_l$ and a line of $\widehat{\mathcal{R}}_{l'}$ meet in a point, the two reguli must thus be distinct and we may apply Corollary 1.3.15 to see that in this case in $\mathbb{P}/P$ the point l' must be an element of $\bigcup_{r \in \widehat{\mathcal{R}}_l} \langle l, r\rangle$.

Given all that we may prove the claim as follows. If there is a line $l \in \mathfrak{L}_5'$ such that $\mathfrak{L}_5' \subseteq \mathfrak{S}[P, 1, \langle l, g_3\rangle]$, then Lemma 2.4.17 shows $|\mathfrak{L}_5'| \leq q^2$ and there remains nothing to prove. Hence, we may assume that there are two lines l_1 and l_2 in $\mathfrak{L}_5'$ with $l_2 \not\leq \langle l_1, g_3\rangle$ and we have seen that this implies $h_3^{l'} \neq h_3^l$. Then, in $\mathbb{P}/P$, any point $l \in \mathfrak{L}_5'$ with $l \notin \langle l_1, g_3\rangle \cup \langle l_2, g_3\rangle$ satisfies

$$l \in \bigcup_{r_1 \in \widehat{\mathcal{R}}_{l_1}} \langle l_1, r_1\rangle \cap \bigcup_{r_2 \in \widehat{\mathcal{R}}_{l_2}} \langle l_2, r_2\rangle = \bigcup_{\substack{r_1 \in \widehat{\mathcal{R}}_{l_1} \\ r_2 \in \widehat{\mathcal{R}}_{l_2}}} \langle l_1, r_1\rangle \cap \langle l_2, r_2\rangle. \tag{2.58}$$

Since $l_1 \neq l_2$ Lemma 1.3.14 shows (in $\mathbb{P}/P$) that among the intersections in the union on the right hand side of this equation occur at most $q+2$ subspaces of dimension ≥ 1, at most one of these subspaces has dimension 2 and the remaining intersections have dimension 0 (all dimensions in $\mathbb{P}/P$). Thus, in $\mathbb{P}/P$ the number of points in the union on the right hand side of Equation (2.58) is at most

$$|\widehat{\mathcal{R}}_{l_1} \times \widehat{\mathcal{R}}_{l_2}| + (q+2) \cdot q + q^2 = 3q^2 + 4q + 1.$$

Furthermore, using Lemma 2.4.17, we know that in $\mathbb{P}$ the subspaces $\langle l_1, g_3\rangle$ and $\langle l_2, g_3\rangle$ contain at most q^2 lines of $\mathfrak{L}_5'$, each, and thus we have the upper bound $|\mathfrak{L}_5'| \leq 5q^2 + 4q + 1$, as required. □

Lemma 2.4.31. *Assume that g_1, g_2 and g_3 are such that $g_i \cap U_j = \emptyset$ for all distinct $i, j \in \{1,2,3\}$ and such that $g_3 \leq \langle P, g_1, g_2\rangle$.*

Then there is a unique regulus $\mathcal{R}$ with $g_1, g_2, U_3 \cap \langle g_1, g_2\rangle \in \mathcal{R}$ and for all lines g we have $\mathfrak{L}_3 \subseteq \mathfrak{L}_g$ if there is a line $r \in \mathcal{R}$ with $g \leq \langle P, r\rangle$ and $|\mathfrak{L}_3 \cap \mathfrak{L}_g| \leq q^2$ otherwise.

Proof. Set $h_1 := g_1$, $h_2 := g_2$, $S := \langle h_1, h_2\rangle$ and $h_3 := U_3 \cap S$. Since $g_1 \cap U_2 = \emptyset$ Lemma 2.4.28 shows $P \notin S$. Hence, $U_3 = \langle P, g_3\rangle = \langle P, h_3\rangle$ and $P \in l$ for all $l \in \mathfrak{L}$ proves $\mathfrak{L}_{g_3} = \mathfrak{L}_{U_3} = \mathfrak{L}_{h_3}$. Indeed, for all distinct $i, j \in \{1, 2, 3\}$ we even have $g_i \cap U_j = \emptyset$ and Lemma 2.4.28 shows $U_i \cap U_j = P$, that is, $h_i \cap h_j = \emptyset$ as well as $\langle h_i, h_j\rangle = S$. Therefore, according to Lemma 1.3.6, there is a unique regulus $\mathcal{R}$ in S with $h_1, h_2, h_3 \in \mathcal{R}$. Let l be an arbitrary line of $\mathfrak{L}_3 = \mathfrak{L}_{h_1} \cap \mathfrak{L}_{h_2} \cap \mathfrak{L}_{h_3}$.

We have $Q := l \cap S = \emptyset$, for otherwise, if $Q \neq \emptyset$, then from $l \ni P \notin S$ we know that Q is a point and, since h_1, h_2 and h_3 skew lines, there are distinct $i, j \in \{1, 2, 3\}$ with $Q \notin h_i, h_j$ and $S = \langle h_i, h_j\rangle$ such that Lemma 2.4.29 yields a contradiction. Hence, S is a complement of l in $\mathbb{P}$ and Lemma 2.4.26 implies that $\{S \cap S_l^i : i \in \{1, \dots, q+1\}\}$ is a regulus in S and, since it contains the three distinct lines h_1, h_2 and h_3, it is the regulus $\mathcal{R}$.

Now, for all $r \in \mathcal{R}$ and every line $g \leq \langle P, r\rangle$ we have $g \leq S_l^{\epsilon_r(l)}$, that is, $l \in \mathfrak{L}_g$ and thus, due to the arbitrary choice of l, we have $\mathfrak{L}_3 \subseteq \mathfrak{L}_g$. Furthermore, for any line g with $l \in \mathfrak{L}_g$ there is a line $r \in \mathcal{R}$ such that we have $g \leq S_l^{\epsilon_r(l)}$ and, for dimensional reasons, g then meets $\langle P, r\rangle$ in at least a point. Hence, if g is a line with $g \not\leq \langle P, r\rangle$ for all $r \in \mathcal{R}$, then either $g \cap \langle P, r\rangle = \emptyset$ for all $r \in \mathcal{R}$ and we have $\mathfrak{L}_3 \cap \mathfrak{L}_g = \emptyset$, or $g \cap \langle P, r\rangle \neq \emptyset$ for some $r \in \mathcal{R}$ and we have $\mathfrak{L}_g \cap \mathfrak{L}_3 \subseteq \mathfrak{L}_g \cap \mathfrak{L}_r$ and Lemma 2.4.27 shows $|\mathfrak{L}_g \cap \mathfrak{L}_r| \leq q^2$, concluding the proof. $\square$

Proposition 2.4.32. *If $\xi \in \mathbb{N}$ is such that $|\mathfrak{L}| > 5q^2 + 9q + 1 + 5\xi$, then there is a set $\mathcal{E}$ of at most $q+1$ planes such that for all $g \in \mathfrak{S}[P, \emptyset, 1, \mathbb{P}]$ with $g \not\leq E$ for all $E \in \mathcal{E}$ there is a set $\overline{\mathfrak{L}}_g \subseteq \mathfrak{L}$ of more than ξ lines such that every solid $S \in \Pi_3(\Delta_g(C))$ satisfies $S \cap l \neq \emptyset$ for all $l \in \overline{\mathfrak{L}}_g$.*

Proof. If there is an index $i \in \{1, \dots, 5\}$ with $|\mathfrak{L}_{i-1} \setminus \mathfrak{L}_i| > \xi$, then we let ι be minimal in $\{1, \dots, 5\}$ with this property, set $\mathcal{E} := \{U_i : i \in \{1, \dots, \iota - 1\}\}$ and, considering the choice of g_ι per Equation (2.54), we then have

$$|\overline{\mathfrak{L}}_g| = |\mathfrak{L} \setminus \mathfrak{L}_g| \geq |\mathfrak{L}_{\iota-1} \setminus \mathfrak{L}_g| \overset{(2.54)}{\geq} |\mathfrak{L}_{\iota-1} \setminus \mathfrak{L}_\iota| > \xi$$

for all lines

$$g \in \mathcal{L}_\iota \overset{(2.53)}{=} \{h \in \mathfrak{S}[P, \emptyset, 1, \mathbb{P}] : g \not\leq E \text{ for all } E \in \mathcal{E}\}.$$

Thus, if there is such an index, then there remains nothing to prove. Furthermore, we remark that if $g_i \cap U_j \neq \emptyset$ for some distinct $i, j \in \{1, \dots, 5\}$, or if $g_3 \not\leq \langle P, g_1, g_2\rangle$, then we have $|\mathfrak{L}_5| \leq 5q^2 + 9q + 1$ from Lemma 2.4.30 and thus in this case there must be such an index.

Now, if there is no such index i, then $|\mathfrak{L}_3| > 5q^2 + 9q + 1 + 2\xi$ and we have $g_i \cap U_j = \emptyset$ for all distinct $i, j \in \{1, 2, 3\}$ as well as $g_3 \leq \langle P, g_1, g_2\rangle$. In this case let $\mathcal{R}$ be the regulus in $\langle g_1, g_2\rangle$ provided by Lemma 2.4.31, set $\mathcal{E} := \{\langle P, r\rangle : r \in \mathcal{R}\}$ as well as

$$\mathcal{L} := \{h \in \mathfrak{S}[P, \emptyset, 1, \mathbb{P}] : h \not\leq E \text{ for all } E \in \mathcal{E}\}$$

and note that Lemma 2.4.31 proves $|\mathfrak{L}_3 \cap \mathfrak{L}_g| \le q^2$ and thus

$$|\overline{\mathfrak{L}}_g| = |\mathfrak{L} \setminus \mathfrak{L}_g| \ge |\mathfrak{L}_3 \setminus \mathfrak{L}_g| \ge |\mathfrak{L}_3| - q^2 \ge 5q^2 + 9q + 1 + 2\xi - q^2 > \xi$$

for all $g \in \mathcal{L}$.

Finally, regardless of the case that occurred above, according to Lemma 2.4.10 every flag $f \in C$ with $\pi_1(f) \not\le E$ for all $E \in \mathcal{E}$ satisfies $\pi_3(f) \cap l \neq \emptyset$ for all $l \in \overline{\mathfrak{L}}_{\pi_1(f)}$, concluding the proof. $\square$

2.4.3 A first Approach to Bounds on the Number of Flags in a given Hyperplane

Lemma 2.4.33. *Let H be a hyperplane of $\mathbb{P}$. If there is a line $h \not\le H$ which occurs in at least one flag of C, then*

$$|\Delta_H(C)| \le 2q^7 + 4q^6 + 5q^5 + 6q^4 + 5q^3 + 4q^2 + 2q + 1.$$

Proof. Let h be such a line and let $f \in C$ be such that $\pi_1(f) = h$. Then $P := \pi_1(f) \cap H$ is a point and $E := \pi_3(f) \cap H$ is a plane. Now, every flag $f' \in \Delta_H(C)$ satisfies $\pi_1(f') \cap E \neq \emptyset$ or $P \in \pi_3(f')$ and there are

$$\begin{aligned}\mathfrak{s}_q[0,3,4] \cdot \mathfrak{s}_q[1,3] + (\mathfrak{s}_q[1,4] - \mathfrak{s}_q[2,-1,1,4] - \mathfrak{s}_q[0,1,4]) \cdot \mathfrak{s}_q[0,1,3,4] \\ = 2q^7 + 4q^6 + 5q^5 + 6q^4 + 5q^3 + 4q^2 + 2q + 1\end{aligned}$$

such flags. $\square$

Lemma 2.4.34. *Let H be a hyperplane of $\mathbb{P}$, let h be a line of $\mathbb{P}$ with $h \not\le H$ and set $\overline{C} := \{f \in \Delta_H(C) : h \cap H \notin \pi_3(f)\}$. Depending on which case of Theorem 2.4.15 occurs for the line h, we have*

$$|\overline{C}| \le \begin{cases} 0 & \text{for C1,} \\ q^4 & \text{for C2 (a),} \\ 2q^4 & \text{for C2 (b),} \\ q^5 + q^4 & \text{for C2 (c),} \end{cases} \qquad |\overline{C}| \le \begin{cases} q^5 + q^4 & \text{for C3 (a),} \\ q^6 + q^5 + q^4 & \text{for C3 (b) or (c),} \\ 2q^6 + q^5 + q^4 & \text{for C3 (d),} \\ 2q^5 + q^4 & \text{for C4.} \end{cases}$$

Proof. In Remark 2.4.16 we already noted that a given case of Theorem 2.4.15 only occurs if the respective Lemma is applicable. Hence,

- for C1 the solid of every flag of $\mathcal{C}$ has non-empty intersection with h and thus $|\overline{C}| = 0$;
- for C2 (a) there is a solid $\widehat{S}_1 \ge h \not\le H$ such that $\pi_1(f) \le \widehat{S}_1$ for all $f \in \overline{C}$, which proves

$$|\overline{C}| \le \mathfrak{s}_q[0,-1,1,2] \cdot \mathfrak{s}_q[0,1,3,4] = q^4;$$

- for C2 (b) there are two solids $\widehat{S}_1, \widehat{S}_2 \ge h \not\le H$ such that for all $f \in \overline{C}$ we have $\pi_1(f) \le \widehat{S}_i$ for some $i \in \{1, 2\}$, which proves $|\overline{C}| \le 2q^4$;

- for C2 (c) there are $q+1$ solids with $\widehat{S}_i \geq h \not\leq H$ for all $i \in \{1, \ldots, q+1\}$ such that for all $f \in \overline{C}$ we have $\pi_1(f) \leq \widehat{S}_i$ for some $i \in \{1, \ldots, q+1\}$, which proves $|\overline{C}| \leq (q+1)q^4$;
- for C3 (a) there is a plane E and a hyperplane H' with $h \leq E \leq H'$ such that for all $f \in \overline{C}$ we have $\pi_1(f) \leq H'$ and $\pi_1(f) \cap E \neq \emptyset$, which proves

$$|\overline{C}| \leq (\mathfrak{s}_q[0,-1,1,3] - \mathfrak{s}_q[1,-1,1,3]) \cdot \mathfrak{s}_q[0,1,3,4] = q^5 + q^4;$$

- for C3 (b) there is a plane E with $h \leq E$ such that for all $f \in \overline{C}$ we have $\pi_1(f) \cap E \neq \emptyset$, which proves

$$|\overline{C}| \leq (\mathfrak{s}_q[0,-1,1,4] - \mathfrak{s}_q[1,-1,1,4]) \cdot \mathfrak{s}_q[0,1,3,4] = q^6 + q^5 + q^4;$$

- for C3 (c) there is a hyperplane H' with $h \leq H'$ such that for all $f \in \overline{C}$ we have $\pi_1(f) \leq H'$, which proves

$$|\overline{C}| \leq \mathfrak{s}_q[0,-1,1,3] \cdot \mathfrak{s}_q[0,1,3,4] = q^6 + q^5 + q^4;$$

- for C3 (d) there is a plane E and a hyperplane H' with $h \leq E \leq H'$ such that for all $f \in \overline{C}$ we have $\pi_1(f) \leq H'$ or $\pi_1(f) \cap E \neq \emptyset$, which proves

$$|\overline{C}| \leq (\mathfrak{s}_q[0,-1,1,4] - \mathfrak{s}_q[1,-1,1,4] + \mathfrak{s}_q[1,-1,1,3]) \cdot \mathfrak{s}_q[0,1,3,4] = 2q^6 + q^5 + q^4;$$

- for C4 there are two distinct planes E_1 and E_2 and two distinct hyperplanes H_1 and H_2 with $h \leq E_1, E_2 \leq H_1, H_2$ such that for all $f \in \overline{C}$ there is $i \in \{1,2\}$ with $\pi_1(f) \cap E_i \neq \emptyset$ and $\pi_1(f) \leq H_i$, which proves

$$|\overline{C}| \leq (2(\mathfrak{s}_q[0,-1,1,3] - \mathfrak{s}_q[1,-1,1,3]) - \mathfrak{s}_q[0,-1,0,1]^2) \cdot \mathfrak{s}_q[0,1,3,4] = 2q^5 + q^4.$$

□

Lemma 2.4.35. *Let P be a point, let H be a hyperplane of $\mathbb{P}$ with $P \notin H$, set*

$$\mathfrak{L}_1 := \{h \in \mathfrak{S}[P,1,\mathbb{P}] : \textit{the first case Theorem 2.4.15 occurs for } h\}$$

and for all $i \in \{2,3\}$ and all $j \in \{a,b,c\}$ set

$$\mathfrak{L}_{i,(j)} := \{h \in \mathfrak{S}[P,1,\mathbb{P}] : \textit{case } i \textit{ part (j) of Theorem 2.4.15 occurs for } h\}.$$

If $\dim(\langle \mathfrak{L}_1 \rangle) \geq 3$, $\dim(\langle \mathfrak{L}_{2,(a)} \rangle) \geq 4$, $\dim(\langle \mathfrak{L}_{2,(b)} \rangle) = 5$ or $\dim(\langle \mathfrak{L}_{3,(j)} \rangle) = 5$ for some $j \in \{a,b,c\}$, then $|\Delta_H(C)| \leq 5q^5 + 20q^4 + 30q^3 + 25q^2 + 15q + 5$.

Proof. Let $\dim(\langle \mathfrak{L}_1 \rangle) \geq 3$, $\dim(\langle \mathfrak{L}_{2,(a)} \rangle) \geq 4$, $\dim(\langle \mathfrak{L}_{2,(b)} \rangle) = 5$ or $\dim(\langle \mathfrak{L}_{3,(j)} \rangle) = 5$ for some $j \in \{a,b,c\}$ and let $\mathfrak{L}$ be one of the sets for which the respective condition is fulfilled. Furthermore, if $\mathfrak{L} = \mathfrak{L}_1$, then set $d := 3$, if $\mathfrak{L} = \mathfrak{L}_{2,(a)}$, then set $d := 4$ and otherwise set $d := 5$.

Since $\dim(\langle\mathfrak{L}\rangle) \geq d$ there are lines $l_1,\dots,l_d \in \mathfrak{L}$ with $\langle l_1,\dots,l_d\rangle = \langle\mathfrak{L}\rangle$. For all $i \in \{1,\dots,d\}$ we have $l_i \ni P \notin H$ and thus $l_i \not\leq H$ and $Q_i := l_i \cap H$ is a point. Furthermore, we have $\dim(\langle Q_1,\dots,Q_d\rangle) = d-1$ and thus there are at most

$$\mathfrak{s}_q[d-1,3,4]\cdot\mathfrak{s}_q[1,3] = \begin{cases} q^5+2q^4+3q^3+3q^2+2q+1 & d=3,\\ q^4+q^3+2q^2+q+1 & d=4,\\ 0 & d=5\end{cases} \tag{2.59}$$

flags $f \in \Delta_H(C)$ with $Q_i \in \pi_3(f)$ for all $i \in \{1,\dots,d\}$. For all other flags $f \in \Delta_H(C)$ there is an index $i \in \{1,\dots,d\}$ with $Q_i \notin \pi_3(f)$.

Now, for all $i \in \{1,\dots,d\}$ Lemma 2.4.34 provides an upper bound on the set of flags $f \in \Delta_H(C)$ for which the point Q_i is not an element of $\pi_3(f)$, which proves the claim in the following cases:

$$|\Delta_H(C)| \leq \begin{cases} q^5+2q^4+3q^3+3q^2+2q+1 & \text{for } \mathfrak{L}=\mathfrak{L}_1,\\ 5q^4+q^3+2q^2+q+1 & \text{for } \mathfrak{L}=\mathfrak{L}_{2,(a)},\\ 10q^4 & \text{for } \mathfrak{L}=\mathfrak{L}_{2,(b)},\\ 5q^5+5q^4 & \text{for } \mathfrak{L}=\mathfrak{L}_{3,(a)}.\end{cases}$$

It remains to consider the cases $\mathfrak{L}=\mathfrak{L}_{3,(b)}$ and $\mathfrak{L}=\mathfrak{L}_{3,(c)}$. In these cases we let h be an arbitrary chosen but fixed line in $\mathfrak{L}$ and we determine a better bound on the number of flags $f \in \Delta_H(C)$ with $h \cap H \notin \pi_3(f)$ as follows:

- For $\mathfrak{L}=\mathfrak{L}_{3,(b)}$ according to Remark 2.4.16 Lemma 2.4.12 is applicable to lines of $\mathfrak{L}$ (with $\mathcal{L}=\mathcal{L}_1$), proving that to every line $l \in \mathfrak{L}$ there is a plane $E_l \geq l$ with $\pi_3(f)\cap l \neq \emptyset$ or $\pi_1(f)\cap E_l \neq \emptyset$ for all $f \in C$. Thus, we determine an upper bound on the number of flags $f \in \Delta_H(C)$ with $\pi_1(f)\cap E_h \neq \emptyset$. Since $\langle\mathfrak{L}\rangle = \mathbb{P}$ there is a line $h_1 \in \mathfrak{L}$ with $h_1 \not\leq E_h$ as well as a line $h_2 \in \mathfrak{L}$ with $h_2 \not\leq \langle E_h, h_1\rangle$ and for all $j \in \{1,2\}$ and every flag $f \in \Delta_H(C)$ through a point of $E_h \cap H$ either satisfies $\pi_1(f)\cap(E_{h_j}\cap H) \neq \emptyset$ or $h_j \cap H \in \pi_3(f)$. We let g, g_1 and g_2 denote the lines $E_h \cap H$, $E_{h_1}\cap H$ and $E_{h_2}\cap H$, respectively.

  Now, on the one hand, if for some $j \in \{1,2\}$ the line g is disjoint from the line g_j, then there are at most

  $$\begin{aligned}|g|\cdot(|g_j|\cdot\mathfrak{s}_q[1,3,4]+(\mathfrak{s}_q[0,1,4]-|g_j|)\cdot\mathfrak{s}_q[2,3,4])&\\ = q^5+4q^4+6q^3+5q^2+3q+1&\end{aligned} \tag{2.60}$$

  flags in question. On the other hand, if g has a point in common with g_j for all $j \in \{1,2\}$, then either $R_1 := g\cap g_1$ and $R_2 := g \cap g_2$ coincide (i.e. $R := R_1 = R_2$) and there are at most

  $$\begin{aligned}|\mathfrak{S}_q[R,1,H]|\cdot\mathfrak{s}_q[1,3,4]+(|g|-1)\cdot(\overbrace{((|g_1|+|g_2|+\mathfrak{s}_q[0,1,2]-4)}^{(*_1)}\cdot\overbrace{\mathfrak{s}_q[2,3,4]}^{(*_2)}&\\ +(\mathfrak{s}_q[0,1,4]-(|g_1|+|g_2|+\mathfrak{s}_q[0,1,2]-3)))&\end{aligned}$$

$$= q^5 + 3q^4 + 7q^3 + 3q^2 + 2q + 1$$

such flags, where

- the term marked $(*_1)$ is an upper bound on the number of lines l through a point X of $g \setminus \{R\}$ that meet g_1 or g_2 or lie in the plane $\langle X, h_1 \cap H, h_2 \cap H\rangle$,
- the term marked $(*_2)$ counts the number of solids in H on the unique plane $E \leq H$ through the chosen line l such that for all $j \in \{1,2\}$ we have $h_j \cap H \in E$ or $l \cap g_j \neq \emptyset$, and
- the second line first counts the number of lines l through a point X of g that we have not yet counted, that is, those which do not meet g_1 nor g_2 and for which there is a unique solid $S \geq l$ which contains both $h_1 \cap H$ and $h_2 \cap H$, namely $S = \langle l, h_1 \cap H, h_2 \cap H\rangle$;

or $R_1 \neq R_2$ and similar counting arguments yield $q^4 + 9q^3 + q^2 + 3q + 1$ as upper bound on the number of flags in question.

- For $\mathfrak{L} = \mathfrak{L}_{3,(c)}$, according to Remark 2.4.16, Lemma 2.4.13 is applicable with $\dim(U) = 1$ and thus $\mathcal{L} = \mathcal{L}_1$, which proves that to every line $l \in \mathfrak{L}$ there is a hyperplane $H_l \geq l$ of $\mathbb{P}$ with $\pi_3(f) \cap l \neq \emptyset$ or $\pi_1(f) \leq H_l$ for all $f \in C$. Thus, we determine an upper bound on the number of flags $f \in \Delta_H(C)$ with $\pi_1(f) \leq H_h$. Since $\langle \mathfrak{L}\rangle = \mathbb{P}$ there is a line $h' \in \mathfrak{L}$ with $h' \not\leq H_h$ and every flag $f \in \Delta_H(C)$ with $\pi_1(f) \leq H_h$ either satisfies $\pi_1(f) \leq H_{h'}$ or $h' \cap H \in \pi_3(f)$. Note that the hyperplanes H_h and $H_{h'}$ meet H in distinct solids. Hence, there are at most

$$\mathfrak{s}_q[1,2] \cdot \mathfrak{s}_q[1,3,4] + (\mathfrak{s}_q[1,3] - \mathfrak{s}_q[1,2]) \cdot \mathfrak{s}_q[2,3,4] = q^5 + 3q^4 + 4q^3 + 4q^2 + 2q + 1$$

flags in question.

Therefore, for all $i \in \{1, \dots, 5\}$ Equation (2.60) serves as bound on the number of flags $f \in \Delta_H(C)$ with $Q_i \notin \pi_3(f)$ for both $\mathfrak{L} = \mathfrak{L}_{3,(b)}$ and $\mathfrak{L} = \mathfrak{L}_{3,(c)}$. Together with the count given in Equation (2.59) this proves

$$|\Delta_H(C)| \leq 5q^5 + 20q^4 + 30q^3 + 25q^2 + 15q + 5,$$

as claimed. □

Corollary 2.4.36. *If there is a hyperplane H of $\mathbb{P}$ with $|\Delta_H(C)| > 5q^5 + 20q^4 + 30q^3 + 25q^2 + 15q + 5$, then for every point $P \in \mathbb{P} \setminus H$ we have*

$$|\Delta_P(C)| \leq 9q^5 + 10q^4 + 10q^3 + 7q^2 - 2q + 1.$$

Proof. Let $\mathfrak{L}_1$ and $\mathfrak{L}_{i,(j)}$ for all $i \in \{2,3\}$ and all $j \in \{a,b,c\}$ be the sets defined in Lemma 2.4.35 and let $\mathfrak{L}$ be the set of all lines h through P which are not contained in one of these sets. Then, according to Lemma 2.4.35, we have $\dim(\langle \mathfrak{L}_1\rangle) \leq 2$, $\dim(\langle \mathfrak{L}_{2,(a)}\rangle) \leq 3$, $\dim(\langle \mathfrak{L}_{2,(b)}\rangle) \leq 4$ and $\dim(\langle \mathfrak{L}_{3,(j)}\rangle) \leq 4$ for all $j \in \{a,b,c\}$. Furthermore, Theorem 2.4.15

provides the cardinality of $\Delta_h(C)$ for all lines h in such a set and shows $|\Delta_h(C)| \le 2q+1$ for all lines $h \in \mathfrak{L}$. Together this shows

$$\begin{aligned}|\Delta_P(C)| \le\ & \mathfrak{s}_q[0,1,2]\cdot\mathfrak{s}_q[1,3,5] + \mathfrak{s}_q[0,1,3]\cdot(q^3+2q^2+q+1)\\ &+ \mathfrak{s}_q[0,1,4]\cdot((2q^2+q+1)+2\cdot(q^2+q+1)+(q^2+2q+1))\\ &+ (\mathfrak{s}_q[0,1,5]-\mathfrak{s}_q[0,1,2]-\mathfrak{s}_q[0,1,3]-4\cdot\mathfrak{s}_q[0,1,4])\cdot(2q+1)\\ =\ & 9q^5+10q^4+10q^3+7q^2-2q+1,\end{aligned}$$

as claimed. □

2.4.4 Proof of the Theorem

Lemma 2.4.37. *Let H be a hyperplane of $\mathbb{P}$, let $\mathfrak{L}$ be the set of saturated lines l with $l \not\le H$ and set $\mathcal{P} := \{l \cap H : l \in \mathfrak{L}\}$.*

Then $\langle\mathcal{P}\rangle \le \pi_3(f)$ for all $f \in \Delta_H(C)$ and furthermore, if $|\mathcal{P}| > 1$, then $\dim(\langle\mathfrak{L}\rangle) = \dim(\langle\mathcal{P}\rangle)+1$ and, if $\dim(\langle\mathcal{P}\rangle) > 1$, then there is a point $Q \notin H$ with $\mathfrak{L} = \{\langle P, Q\rangle : P \in \mathcal{P}\}$.

Proof. For $\mathfrak{L} = \emptyset$ there is nothing to prove and thus assume $\mathfrak{L} \ne \emptyset$. Lemma 2.1.3 shows $\langle\mathcal{P}\rangle \le \pi_3(f)$ for all $f \in \Delta_H(C)$. Furthermore, according to Lemma 2.1.4, $\mathfrak{L}$ is an independent set of the Kneser graph of type 1 on $\mathbb{P}$ and as such $\langle\mathfrak{L}\rangle$ has dimension at most 2 or there is a point $Q \in \langle\mathfrak{L}\rangle$ with $Q \in l$ for all $l \in \mathfrak{L}$. For $|\mathcal{P}| = 1$ there is nothing more to prove and thus assume that $|\mathcal{P}| > 1$.

If there is a point $Q \in \langle\mathfrak{L}\rangle$ with $Q \in l$ for all $l \in \mathfrak{L}$, then both $|\mathcal{P}| > 1$ and $Q \in l \not\le H$ for all $l \in \mathfrak{L}$ together prove $Q \notin H$. Thus, in this case, for all $l \in \mathfrak{L}$ we have $l = \langle l \cap H, Q\rangle$ and thus $\langle\mathcal{P}\rangle$ is a complement of Q in $\langle\mathfrak{L}\rangle$, which proves $\dim(\langle\mathfrak{L}\rangle) = \dim(\langle\mathcal{P}\rangle)+1$. Furthermore, if there is no such point Q, then we have $\dim(\langle\mathfrak{L}\rangle) = 2$ and, since $l \not\le H$ for all $l \in \mathfrak{L}$, we have $\dim(\langle\mathfrak{L}\rangle) > \dim(\langle\mathcal{P}\rangle) \ge 1$, which proves the claim in this case and concludes the proof. □

Notation 2.4.38. From now on assume that C is not given by Example 2.4.1. Since the type under consideration is self-dual we may assume that

$$\max\{|\Delta_H(C)| : H \in \mathfrak{G}[4,\mathbb{P}]\} \ge \max\{|\Delta_P(C)| : P \in \mathbb{P}\}$$

and we let $H \in \mathfrak{G}[4,\mathbb{P}]$ be such that $|\Delta_H(C)| \ge |\Delta_{H'}(C)|$ for all $H' \in \mathfrak{G}[4,\mathbb{P}]$. Note that our choice of C shows that there is a flag $f \in C$ with $\pi_1(f) \not\le H$ and thus, in view of Lemma 2.4.33, our choice of H shows

$$\forall G \in \mathfrak{G}[4,\mathbb{P}] : |\Delta_G(C)| \le 2q^7+4q^6+5q^5+6q^4+5q^3+4q^2+2q+1. \tag{2.61}$$

Lemma 2.4.39. *The set $\mathfrak{L}$ of all saturated lines l with $l \not\le H$ satisfies $|\mathfrak{L}| < 2q^2(q+1)$.*

Proof. We set $\mathcal{P} := \{l \cap H : l \in \mathfrak{L}\}$ and, in view of Lemma 2.4.37, we consider two cases. First, if $|\mathcal{P}| = 1$, then according to the choice of both C and H the point $Q \in \mathcal{P}$ satisfies

$$|\Delta_Q(\mathfrak{L})|\cdot\mathfrak{s}_q[1,3,5] \le |\Delta_Q(C)| \le 2q^7+4q^6+5q^5+6q^4+5q^3+4q^2+2q+1$$

$$= 2q^2(q+1)\cdot \mathfrak{s}_q[1,3,5] - \underbrace{(q^5 - q^3 - 2q^2 - 2q - 1)}_{>0 \text{ for } q\geq 2} \tag{2.62}$$

which proves $|\mathfrak{L}| = |\Delta_Q(\mathfrak{L})| < 2q^2(q+1)$, as claimed. Hence, let $|P| > 1$ and assume that $|\mathfrak{L}| > \mathfrak{s}_q[2]$. Then $\dim(\langle\mathfrak{L}\rangle) > 2$ and Lemma 2.4.37 proves $\dim(\langle\mathcal{P}\rangle) > 1$ as well as the existence of a point $Q \notin H$ such that $\mathfrak{L} = \{\langle P, Q\rangle\} : P \in \mathcal{P}\}$ with $\mathcal{P} := \{l \cap H : l \in \mathfrak{L}\}$. Finally, Equation (2.62) then holds for that point Q, too, and we again have $|\mathfrak{L}| = |\Delta_Q(\mathfrak{L})| < 2q^2(q+1)$, as claimed. ☐

Lemma 2.4.40. *The set $\mathfrak{L}$ of all lines l with $l \not\leq H$ which satisfy case C2 (a) of Theorem 2.4.15 has cardinality at most $27 \cdot \mathfrak{s}_q[4]$.*

Proof. We consider Proposition 2.4.18 and its implications for different values of d. Note that the set $\mathfrak{L}$ itself (and thus also any subset of $\mathfrak{L}$) obviously satisfies condition (I) of this proposition. Therefore, we always consider the claims given there for $\kappa = 3$.

First, assume that there is a point $P \in H$ and an integer ξ with $|\Delta_P(\mathfrak{L})| > 3\xi$ such that every solid S satisfies $|\Delta_S(\Delta_P(\mathfrak{L}))| \leq \xi$. Then we may apply Proposition 2.4.18 with $d = 3$ to $\Delta_P(\mathfrak{L})$, which shows that there is a subspace U with $\dim(U) \leq 2$ such that for every line l with $l \not\leq U$ there is a hyperplane $H_l \ni P$ (if the subspace given by Proposition 2.4.18 is not a hyperplane but the entire space $\mathbb{P}$ instead, then we may let G_l be an arbitrary hyperplane on P) such that every solid $S \in \Pi_3(\Delta_l(C))$ contains P or a complement of P in H_l. A complement of P in H_l is a solid and thus any line l with $l \not\leq U$ and $P \notin l$ satisfies

$$\begin{aligned} |\Pi_3(\Delta_l(C))| &\leq |\mathfrak{S}[\langle P, l\rangle, 3, \mathbb{P}]| + |\mathfrak{S}[P, l, 3, H_l]| \\ &= \mathfrak{s}_q[2,3,5] + \mathfrak{s}_q[0,1,3,4] < q^3 + q^2 + q + 1 \end{aligned} \tag{2.63}$$

and as such $l \notin \mathfrak{L}$. Therefore, and since $l \not\leq H$ for all $l \in \mathfrak{L}$ and thus $|\Delta_P(\mathfrak{L})| \leq q^4$, in this situation we have

$$|\mathfrak{L}| \leq |\Delta_P(\mathfrak{L})| + |\mathfrak{S}[1, U]| \leq q^4 + q^2 + q + 1. \tag{2.64}$$

Secondly, assume that there is a point $P \in H$, an integer ξ and a subset $\mathfrak{L}' \subseteq \Delta_P(\mathfrak{L})$ with $\dim(\langle\mathfrak{L}'\rangle) = 3$, $|\mathfrak{L}'| > 3\xi$ and such that any plane E of $\mathbb{P}$ satisfies $|\Delta_E(\mathfrak{L}')| \leq \xi$. Then we may apply Proposition 2.4.18 with $d = 2$ to $\mathfrak{L}'$, which shows that there is a subspace U with $\dim(U) \leq 3$ such that for all flags $f \in C$ we have $\pi_1(f) \leq U$ or $\pi_3(f)$ contains P or a complement of P in the solid $\langle\mathfrak{L}'\rangle$ and, if $\dim(U) = 3$, then $U = \langle\mathfrak{L}'\rangle$. Thus, if $l \in \mathfrak{S}[P, \emptyset, 1, \mathbb{P}]$ is a line with $l \not\leq U$ and $l \not\leq \langle\mathfrak{L}'\rangle$, then every solid $S \in \Pi_3(\Delta_l(C))$ satisfies $P \in S$ or S contains a complement of P in $\langle\mathfrak{L}'\rangle$, that is, a plane $E \leq \langle\mathfrak{L}'\rangle$ and, since $l \not\leq \langle\mathfrak{L}'\rangle$, this implies that $l \cap \langle\mathfrak{L}'\rangle$ is a point as well as $S = \langle l, E\rangle \leq \langle l, \mathfrak{L}'\rangle$. Hence, for all $l \in \mathfrak{S}[P, \emptyset, 1, \mathbb{P}]$ with $l \not\leq U$ and $l \not\leq \langle\mathfrak{L}'\rangle$, using the hyperplane $H_l := \langle l, \mathfrak{L}'\rangle$, Equation (2.63) holds here, too, and implies $l \notin \mathfrak{L}$. Therefore, and since again $|\Delta_P(\mathfrak{L})| \leq q^4$, in this situation we have

$$\begin{aligned} |\mathfrak{L}| &\leq |\Delta_P(\mathfrak{L})| + |\mathfrak{S}[1, U] \cup \mathfrak{S}[1, \langle\mathfrak{L}'\rangle]| \\ &\overset{(*)}{\leq} q^4 + \mathfrak{s}_q[1,3] + \mathfrak{s}_q[1,2] = 2q^4 + q^3 + 3q^2 + 2q + 2, \end{aligned} \tag{2.65}$$

where in the step marked with $(*)$ we also use the fact that, if $\dim(U)=3$, then $U=\langle \mathfrak{L}'\rangle$.

Before we proceed we remark that, if there is a point $P\in H$ such that $|\Delta_P(\mathfrak{L})|>9q$, then one of these two situations occurs, which already proves the claim in this case as follows:

- either there is a solid S such that $|\Delta_S(\Delta_P(\mathfrak{L}))|>3q$ and we have shown above that Equation (2.65) holds, or
- every solid S satisfies $|\Delta_S(\Delta_P(\mathfrak{L}))|\le 3q$ and we have shown above that Equation (2.64) holds.

Similarly, if there is a point $P\in H$ such that $|\Delta_P(\mathfrak{L})|>27$, then there is a plane E with $|\Delta_E(\Delta_P(\mathfrak{L}))|>3$, for otherwise:

- either there is a solid S such that $|\Delta_S(\Delta_P(\mathfrak{L}))|>9$ and we have shown above that Equation (2.65) holds, or
- every solid S satisfies $|\Delta_S(\Delta_P(\mathfrak{L}))|\le 9$ and we have shown above that Equation (2.64) holds

and either way this implies the claim.

Hence, from now on we may assume that $|\Delta_P(\mathfrak{L})|\le 9q$ for all $P\in H$. Furthermore, we may assume that to every point $P\in H$ with $|\Delta_P(\mathfrak{L})|>27$ there is a plane E with $|\Delta_E(\Delta_P(\mathfrak{L}))|>3$ and, since this plane contains lines of $\mathfrak{L}$, it satisfies $P\in E\not\le H$. Moreover, we may assume that there indeed is a point $P\in H$ with $|\Delta_P(\mathfrak{L})|>27$, for otherwise we have $|\mathfrak{L}|\le |H|\cdot 27=27\cdot\mathfrak{s}_q[4]$, which is the claim.

Now, let $P\in H$ be a point with $|\Delta_P(\mathfrak{L})|>27$ and let E be the plane such that $\mathfrak{L}':=\Delta_E(\Delta_P(\mathfrak{L}))$ satisfies $|\mathfrak{L}'|>3$. Then we may apply Proposition 2.4.18 with $d=1$ to $\mathfrak{L}'$, which shows that there is a subspace U with $\dim(U)\le 3$ such that for all flags $f\in C$ we have $\pi_1(f)\le U$ or $\pi_3(f)$ contains P or a complement of P in the plane $\langle\mathfrak{L}'\rangle=E$. Thus, if l is a line with $l\not\le U$ and $l\cap E=\emptyset$, then every solid $S\in\Pi_3(\Delta_l(C))$ contains P or a complement of P in the plane E and, since $l\cap E=\emptyset$, the latter proves $S=\langle l,S\cap E\rangle\le\langle l,\mathfrak{L}'\rangle$. Hence, for all $l\in\mathfrak{G}[P,\emptyset,1,\mathbb{P}]$ with $l\not\le U$ and $l\cap E=\emptyset$, using the hyperplane $H_l:=\langle l,\mathfrak{L}'\rangle$, Equation (2.63) holds here, too, and implies $l\notin\mathfrak{L}$. This implies

$$\begin{aligned}|\mathfrak{L}|&\le \underbrace{|\Delta_P(\mathfrak{L})|}_{\le 9q}+|\mathfrak{G}[1,U]|+|\underbrace{\{l\in\mathfrak{L}:l\cap E\ne\emptyset\}}_{=:L}|\\&\le q^4+q^3+2q^2+10q+1+|L|\end{aligned}\tag{2.66}$$

and it remains to determine an upper bound on the size of L. For that purpose, note that through any line $l\in L$ there are q^3+2q^2+q+1 distinct flags in C and, if l and l' are distinct lines of L and f and f' are flags through l and l' respectively, then f and f' are obviously distinct, too. Therefore, if for some point $Q\notin H$ we have $|\Delta_Q(L)|>9q^2-8q+17$, then we have

$$|\Delta_Q(C)|>9q^5+10q^4+10q^3+35q^2+9q+17>9q^5+10q^4+10q^3+7q^2-2q+1$$

a contradiction to our choice of H and the bound given in Corollary 2.4.36. Hence, we have $|\Delta_Q(L)| \le 9q^2 - 8q + 17$ and since $E \not\le H$ this implies

$$|L| \le |E \cap H| \cdot 9q + |E \setminus H| \cdot (9q^2 - 8q + 17) = 9q^4 - 8q^3 + 26q^2 + 9q.$$

Combining this with Equation (2.66) concludes the proof. □

Lemma 2.4.41. *The set $\mathfrak{L}$ of all lines l with $l \not\le H$ which satisfy case C2 (b) of Theorem 2.4.15 has cardinality at most $(16q + 10) \cdot \mathfrak{s}_q[4]$.*

Proof. We assume $\mathfrak{L} \neq \emptyset$. We begin the proof by considering an arbitrary line $h \in \mathfrak{L}$ and remarking the following. According to Theorem 2.4.15 there are two solids $\widehat{S}_1$ and $\widehat{S}_2$ such that $h = \widehat{S}_1 \cap \widehat{S}_2$ and such that $\Pi_3(\Delta_h(C))$ is the set of all solids which meet $\widehat{S}_1$ and $\widehat{S}_2$ in a plane each. Now, for all $i \in \{1, 2\}$ there exist two distinct planes E_i and E_i' with $h \le E_i, E_i' \le \widehat{S}_i$ and the solids $S := \langle E_1, E_2\rangle$ and $S' := \langle E_1', E_2'\rangle$ satisfy $\langle S, S'\rangle = \mathbb{P}$ as well as $(h, S) \in C$ and $(h, S') \in C$. This shows that through any line of $\mathfrak{L}$ there are two flags in C such that the solids of these flags span $\mathbb{P}$.

Now, let $P \in H$ be such that $|\Delta_P(\mathfrak{L})| \ge |\Delta_Q(\mathfrak{L})|$ for all $Q \in H$. If $|\Delta_P(\mathfrak{L})| \le 4(4q + 2) + 2$, then from our choice of P we have $|\Delta_Q(\mathfrak{L})| \le 16q + 10$ for all $Q \in H$ and since every line $l \in \mathfrak{L}$ has non-empty intersection with H this implies $|\mathfrak{L}| \le |H| \cdot (16q+10)$, as claimed. Therefore, we may assume that $|\Delta_P(\mathfrak{L})| > 4(4q + 2) + 2$ and we consider two cases.

First, assume that for every solid $\widehat{S}$ on P we have $|\Delta_{\widehat{S}}(\Delta_P(\mathfrak{L}))| \le 4q + 2$. Then, let $\mathcal{E}$ and $\mathcal{S}$ with $\dim(\mathcal{E}) \le 2$ and $\dim(\mathcal{S}) \le 3$ be the subspaces provided by applying Proposition 2.4.24 with $\xi = 4q + 2$ and let $g \in \mathfrak{G}[P, \emptyset, 1, \mathbb{P}]$ be such that $g \not\le \mathcal{E}, \mathcal{S}$. Proposition 2.4.24 proves that there is a subset $\overline{\mathfrak{L}}_g \subseteq \Delta_P(\mathfrak{L})$ of more than $\xi = 4q + 2$ lines such that every solid $S \in \Pi_3(\Delta_g(C))$ satisfies $S \cap l \neq \emptyset$ for all $l \in \overline{\mathfrak{L}}_g$. The assumption of this case implies that in $\overline{\mathfrak{L}}_g$ there are lines which span a hyperplane H' of $\mathbb{P}$ and any solid $S \in \Pi_3(\Delta_g(C))$ contains P or a complement of P in H'. Since any complement of P in H' is a solid, this implies that any solid $S \in \Pi_3(\Delta_g(C))$ satisfies $S \le H'$ or $P \in S$. Now, let $\widehat{S}_1$ and $\widehat{S}_2$ be arbitrary solids through g with $\langle \widehat{S}_1, \widehat{S}_2\rangle = \mathbb{P}$. Then there is some index $i \in \{1, 2\}$ with $\widehat{S}_i \not\le H'$, that is, in $\widehat{S}_i$ there is a plane E on g with $E \not\le H'$ as well as $P \notin E$ and there is a solid S with $E \le S$ that meets $\widehat{S}_{3-i}$ in a plane. This solid S satisfies $\dim(S \cap \widehat{S}_1) = 2 = \dim(S \cap \widehat{S}_2)$ but $P \notin S \not\le H'$ and thus $S \notin \Pi_3(\Delta_g(C))$. The arbitrary choice of $\widehat{S}_1$ and $\widehat{S}_2$ through g with $\langle \widehat{S}_1, \widehat{S}_2\rangle = \mathbb{P}$ and the remark in the beginning of this proof shows $g \notin \mathfrak{L}$. Hence, using $|\Delta_P(\mathfrak{L})| \le q^4$, in this case we have

$$|\mathfrak{L}| \le |\Delta_P(\mathfrak{L})| + |\mathfrak{G}[1, \mathcal{E}]| + |\mathfrak{G}[1, \mathcal{S}]| \le q^4 + q^3 + 2q^2 + 2q + 2.$$

Secondly, assume that there is a solid $\widehat{S}$ on P with $|\Delta_{\widehat{S}}(\Delta_P(\mathfrak{L}))| > 4q + 2$. Let $\mathcal{E}$ and $\mathcal{S}$ with $\dim(\mathcal{E}) \le 2$ and $\dim(\mathcal{S}) \le 3$ be the subspaces provided by applying Proposition 2.4.24 to $\Delta_{\widehat{S}}(\Delta_P(\mathfrak{L}))$ with $\xi = q$ and let $g \in \mathfrak{G}[P, \emptyset, 1, \mathbb{P}]$ be such that $g \not\le \mathcal{E}, \mathcal{S}$. Proposition 2.4.24 proves that there is a subset $\overline{\mathfrak{L}}_g \subseteq \Delta_{\widehat{S}}(\Delta_P(\mathfrak{L}))$ of more than $\xi = q$ lines such that every solid $S \in \Pi_3(\Delta_g(C))$ satisfies $S \cap l \neq \emptyset$ for all $l \in \overline{\mathfrak{L}}_g$. Since any plane through P contains at most q lines which do not lie in H, we know that in $\overline{\mathfrak{L}}_g$ there are lines which span the solid $\widehat{S}$ and any solid $S \in \Pi_3(\Delta_g(C))$ thus contains P or a complement of P in $\widehat{S}$. We study the case $g \not\leq \widehat{S}$ more thoroughly.

If we even have $g \cap \widehat{S} = \emptyset$, then there is no solid through g which contains a complement of P in $\widehat{S}$, that is, in this situation all solids of $\Pi_3(\Delta_g(C))$ share the plane $\langle g, P\rangle$ and we have $g \notin \mathfrak{L}$.

Hence, assume that $g \cap \widehat{S}$ is a point and let $\widehat{S}_1$ and $\widehat{S}_2$ be two arbitrary solids with $\widehat{S}_1 \cap \widehat{S}_2 = g$. Then both these solids are distinct from $\widehat{S}$ and at most one of these two solids meets $\widehat{S}$ in plane, for otherwise $(\widehat{S}_1 \cap \widehat{S}) \cap (\widehat{S}_2 \cap \widehat{S}) =: U$ would have dimension at least 1 and we would have $\langle U, g\rangle \leq \widehat{S}_1 \cap \widehat{S}_2$ with $\dim(\langle U, g\rangle) \geq 2$, a contradiction. Without loss of generality we may thus assume $\dim(\widehat{S}_1 \cap \widehat{S}) \leq 1$, which implies that there is a plane $E_1 \in \mathfrak{G}[g, 2, \widehat{S}_1]$ such that $E_1 \cap \widehat{S}$ is a point, namely the point $g \cap \widehat{S} \neq P$. Now, for dimensional reasons no solid through E_1 may contain a complement of P in $\widehat{S}$ and there is only one solid through E_1 which contains P. Therefore, there is a plane $E_2 \in \mathfrak{G}[g, 2, \widehat{S}_2]$ such that the solid $S := \langle E_1, E_2\rangle$ does not contain P nor a complement of P in $\widehat{S}$ and as such is not an element of $\Pi_3(\Delta_g(C))$. Again, the arbitrary choice of $\widehat{S}_1$ and $\widehat{S}_2$ through g with $\langle \widehat{S}_1, \widehat{S}_2\rangle = \mathbb{P}$ and the remark in the beginning of this proof show $g \notin \mathfrak{L}$. Thus, again using $|\Delta_P(\mathfrak{L})| \leq q^4$, in this case we have

$$|\mathfrak{L}| \leq |\Delta_P(\mathfrak{L})| + |\mathfrak{G}[1, \mathcal{E}]| + |\mathfrak{G}[1, \mathcal{S}]| + |\mathfrak{G}[1, \widehat{S}]| \leq q^4 + 2q^3 + 3q^2 + 3q + 3. \qquad \square$$

Lemma 2.4.42. *The set $\mathfrak{L}$ of all lines l with $l \not\leq H$ which satisfy case C2 (c) of Theorem 2.4.15 has cardinality at most $(10q^2 + 9q + 1) \cdot \mathfrak{s}_q[4]$.*

Proof. We let $P \in H$ be such that $|\Delta_P(\mathfrak{L})| \geq |\Delta_Q(\mathfrak{L})|$ for all $Q \in H$ and we note that for every line $h \in \mathfrak{L}$ and any two distinct solids $S, S' \in \Pi_3(\Delta_h(C))$ we have $S \cap S' = h$.

If $|\Delta_P(\mathfrak{L})| \leq 10q^2 + 9q + 1$, then from our choice of P we have $|\Delta_Q(\mathfrak{L})| \leq 10q^2 + 9q + 1$ for all $Q \in H$ and since every line $l \in \mathfrak{L}$ has non-empty intersection with H this implies $|\mathfrak{L}| \leq |H| \cdot (10q^2 + 9q + 1)$, as claimed. Therefore, we assume that $|\Delta_P(\mathfrak{L})| > 10q^2 + 9q + 1$.

Let $\mathcal{E}$ be the set of at most $q+1$ planes provided by applying Proposition 2.4.32 with $\xi = q^2$ and let $g \in \mathfrak{G}[P, \emptyset, 1, \mathbb{P}]$ be such that $g \not\leq E$ for all $E \in \mathcal{E}$. Proposition 2.4.32 proves that we have a subset $\overline{\mathfrak{L}}_g \subseteq \Delta_P(\mathfrak{L})$ of more than $\xi = q^2$ lines such that every solid $S \in \Pi_3(\Delta_g(C))$ satisfies $S \cap l \neq \emptyset$ for all $l \in \overline{\mathfrak{L}}_g$. Since any solid through P contains at most q^2 lines which do not lie in H, we know that in $\overline{\mathfrak{L}}_g$ there are lines which span a hyperplane H' of $\mathbb{P}$ and any solid $S \in \Pi_3(\Delta_g(C))$ is either a complement of P in H' or contains P. Hence, if there are three distinct solids in $\Pi_3(\Delta_g(C))$, then two of those either have the plane $\langle P, g\rangle$ in common or do not span $\mathbb{P}$. Therefore, g may not satisfy case C2 (c) of Theorem 2.4.15, that is, any line $l \in \mathfrak{L}$ satisfies $l \in \Delta_P(\mathfrak{L})$ or $l \leq E$ for some $E \in \mathcal{E}$ and, again using $|\Delta_P(\mathfrak{L})| \leq q^4$, we have

$$|\mathfrak{L}| \leq |\Delta_P(\mathfrak{L})| + (q+1) \cdot \mathfrak{s}_q[1, 2] \leq q^4 + q^3 + 2q^2 + 2q + 1. \qquad \square$$

Lemma 2.4.43. *Let $d \in \{1, 2\}$ and let $\mathfrak{L}$ be a set of lines l with $|\Delta_l(C)| \geq \mathfrak{s}_q[d]$.*

i) For $P \notin H$ we have $|\Delta_P(\mathfrak{L})| \leq (9q^2 + q + 9)q^{3-d}$.

ii) For every hyperplane H' of $\mathbb{P}$ with $|\Delta_{H'}(C)| \leq \mathfrak{s}_q[d] \cdot \xi$ for some $\xi \in \mathbb{N}$ we have $|\Delta_{H'}(\mathfrak{L})| \leq \xi + q^5 + 3q^4 + 4q^3 + 4q^2 + 2q + 1$

Proof. If l and l' are distinct lines of $\mathfrak{L}$, then the flags in $\Delta_l(C)$ and $\Delta_{l'}(C)$ are distinct, too. Thus, using the maximal choice of H, the first claim is implied by the bound $|\Delta_P(C)| \leq 9q^5 + 10q^4 + 10q^3 + 7q^2 - 2q + 1$ given in Corollary 2.4.36.

Now, let H' be a hyperplane of $\mathbb{P}$ and let $\xi \in \mathbb{N}$ be such that $|\Delta_{H'}(C)| \leq \mathfrak{s}_q[d] \cdot \xi$. We set $\mathfrak{L}_1 := \{l \in \Delta_{H'}(\mathfrak{L}) : f \in \Delta_l(C) \implies f \leq H'\}$ and $\mathfrak{L}_2 := \Delta_{H'}(\mathfrak{L}) \setminus \mathfrak{L}_1$ as well as $C_1 := \{f \in C : \pi_1(f) \in \mathfrak{L}_1\}$ and

$$C_2 := \{f \in C : \pi_1(f) \in \mathfrak{L}_2 \text{ and } f \not\leq H'\}.$$

Then obviously $|\mathfrak{L}_1| \leq \xi$. Furthermore, according to Lemma 2.1.7 the set $C_2' := \{f \cap H' : f \in C_2\}$ is an independent set of the Kneser graph of type $(1, 2)$ in H' and, as mentioned earlier, [3, Proposition 2.1] by Blokhuis and Brouwer provides an upper bound on its cardinality. Since to every line $l \in \mathfrak{L}_2$ there is a solid $S \not\leq H'$ with $(l, S) \in C$ we have

$$|\mathfrak{L}_2| = |\Pi_1(C_2)| = |\Pi_1(C_2')| \leq |C_2'| \overset{[3]}{\leq} q^5 + 3q^4 + 4q^3 + 4q^2 + 2q + 1$$

and together with $|\mathfrak{L}_1| \leq \xi$ this proves the second claim. □

Lemma 2.4.44. *The set $\mathfrak{L}$ of all lines l with $l \not\leq H$ which satisfy case C3 (a) or C3 (b) of Theorem 2.4.15 has cardinality at most $81q^2 \cdot \mathfrak{s}_q[3] + q^4$.*

Proof. First, assume that P is a point of H such that $|\Delta_P(\mathfrak{L})| > 5q^2$. Then, according to Proposition 2.4.21 i), two situations may occur (with regard to $\Delta_P(\mathfrak{L})$ and P):

- If 2.4.21 i) a) occurs we denote the line given there by g_P, for any line $g \in \mathfrak{S}[g_P, \emptyset, 1, \mathbb{P}]$ we denote the hyperplane given there by H_P^g and we set $\mathcal{L}_P := \mathfrak{S}[1, \mathbb{P}] \setminus \mathfrak{S}[g_P, \emptyset, 1, \mathbb{P}]$.
- If 2.4.21 i) b) occurs we denote the planes given there by E_P^1 and E_P^2, for any line g with $g \not\leq E_P^1, E_P^2$ we denote the hyperplane given there by H_P^g and we set $\mathcal{L}_P := \mathfrak{S}[1, E_P^1] \cup \mathfrak{S}[1, E_P^2]$.

In the first of these two situations we have $|\mathcal{L}_P| = \mathfrak{s}_q[1, 5] - \mathfrak{s}_q[1, -1, 1, 5] = q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$ and in the second we have $|\mathcal{L}_P| \leq 2 \cdot \mathfrak{s}_q[2]$ and thus, either way, we have

$$|\mathcal{L}_P| \leq q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1. \tag{2.67}$$

Furthermore, we remark that, according to Proposition 2.4.21, for every line $g \notin \mathcal{L}_P$ and for every solid $S \in \Pi_3(\Delta_g(C))$ we know that S contains P or a complement of P in H_P^g, regardless of which case occurs above.

Secondly, assume that P is a point of H such that $81q < |\Delta_P(\mathfrak{L})| \leq 5q^2$. Then, either there is a subset $\mathfrak{L}_P'$ of $\Delta_P(\mathfrak{L})$ of $9q + 1$ lines which lie in a common solid, or any subset of $\Delta_P(\mathfrak{L})$ containing more than $9q$ lines spans at least a hyperplane of $\mathbb{P}$. In the latter situation set $\mathfrak{L}_P' := \Delta_P(\mathfrak{L})$. Then, either way, we may apply Proposition 2.4.21 ii) to see that we have a solid $\widehat{S}_P$ on P and two situations may occur (with regard to $\mathfrak{L}_P'$ and P):

– If 2.4.21 ii) a) occurs, then we denote the line given there by g_P, we denote the solid given there by $\widehat{S}'_P$, for any line $g \in \mathfrak{S}[g_P,\emptyset,1,\mathbb{P}]$ with $g \not\leq \widehat{S}_P, \widehat{S}'_P$ we denote the hyperplane given there by H_P^g and we set

$$\mathcal{L}_P := (\mathfrak{S}[1,\mathbb{P}] \setminus \mathfrak{S}[g_P,\emptyset,1,\mathbb{P}]) \cup \mathfrak{S}[1,\widehat{S}_P] \cup \mathfrak{S}[1,\widehat{S}'_P].$$

– If 2.4.21 ii) b) occurs, then we denote the subspace of dimension at most 4 given there by H'_P, for any line g with $P \notin g \not\leq H'_P$ and $g \not\leq \widehat{S}_P$ we denote the hyperplane given there by H_P^g and we set $\mathcal{L}_P := \mathfrak{S}[1,H'_P] \cup \mathfrak{S}[1,\widehat{S}_P]$.

In the first of these two situations we have

$$|\mathcal{L}_P| \leq \mathfrak{s}_q[1,5] - \mathfrak{s}_q[1,-1,1,5] + 2 \cdot \mathfrak{s}_q[1,3] = q^5 + 4q^4 + 4q^3 + 6q^2 + 3q + 3.$$

However, in the second situation the trivial bound will not be sufficient and instead, in that situation, we determine an upper bound on $\mathfrak{L} \cap \mathcal{L}_P$. We consider a hyperplane S'_P of H'_P with $S'_P \leq H$ and, for reasons that will become more clear later on, when determining that bound we assume that there is at most one point $Q \in S'_P$ with $|\Delta_Q(\mathfrak{L})| > 5q^2$. Now, we know that every line $l \in \mathfrak{L} \cap \mathcal{L}_P$ meets S'_P in a point and thus, in these circumstances, we have

$$|\mathfrak{L} \cap \mathcal{L}_P| \leq (|S'_P| - 1) \cdot 5q^2 + q^4 = 5q^5 + 6q^4 + 5q^3. \tag{2.68}$$

Note that this bound is weaker than the one given if the first situation occurs. We remark again, that, according to Proposition 2.4.21, for every line $g \notin \mathcal{L}_P$ and for every solid $S \in \Pi_3(\Delta_g(C))$ we know that S contains P or a complement of P in H_P^g, regardless of which situation occurs above.

Now, let P_1 and P_2 be distinct points of H such that

$$\forall P \in H \setminus \{P_1\} : |\Delta_{P_1}(\mathfrak{L})| \geq |\Delta_{P_2}(\mathfrak{L})| \geq |\Delta_P(\mathfrak{L})|.$$

If $|\Delta_{P_2}(\mathfrak{L})| \leq 81q$, then, since every line in $\mathfrak{L}$ meets H, we have

$$|\mathfrak{L}| \leq (|H| - 1) \cdot 81q + |\Delta_{P_1}(\mathfrak{L})| \leq (\mathfrak{s}_q[4] - 1) \cdot 81q + q^4 = 81q^2 \cdot \mathfrak{s}_q[3] + q^4,$$

as claimed. Thus, we assume that $|\Delta_{P_2}(\mathfrak{L})| > 81q$, set $\mathcal{L}_0 := \mathfrak{S}[1,\mathbb{P}] \setminus \mathfrak{S}[\langle P_1,P_2\rangle,\emptyset,1,\mathbb{P}]$ and note that through any point of the line $\langle P_1,P_2\rangle$ there are at most q^4 lines of $\mathfrak{L}$, which proves $|\mathfrak{L} \cap \mathcal{L}_0| \leq \mathfrak{s}_q[1] \cdot q^4 = q^5 + q^4$. Furthermore, using the notation we introduced above, we set $\mathcal{L} := \mathcal{L}_0 \cup \mathcal{L}_{P_1} \cup \mathcal{L}_{P_2}$. Now, for every line $g \notin \mathcal{L}$, for all $i \in \{1,2\}$ and for every solid $S \in \Pi_3(\Delta_g(C))$ we know that S contains P_i or a complement of P_i in $H_{P_i}^g$.

Moreover, we have $\mathfrak{L} \subseteq \mathcal{L}$, for otherwise, assume that there is a line $g \in \mathfrak{L} \setminus \mathcal{L}$. Then Theorem 2.4.15 shows that there is a plane $E \geq g$ with $\mathfrak{S}[E,3,\mathbb{P}] \subseteq \Pi_3(\Delta_g(C))$. However, since $g \cap \langle P_1,P_2\rangle = \emptyset$ we know that there is some $i \in \{1,2\}$ with $P_i \notin E$ and thus there is a solid S through E with $P_i \notin S \not\leq H_i^g$, that is, $S \notin \Pi_3(\Delta_g(C))$, a contradiction. Hence, we have $|\mathfrak{L}| \leq |\mathfrak{L} \cap \mathcal{L}_0| + |\mathfrak{L} \cap \mathcal{L}_{P_1}| + |\mathfrak{L} \cap \mathcal{L}_{P_2}|$.

Therefore, if $|\Delta_{P_2}(\mathfrak{L})| > 5q^2$, then, using the upper bound for $|\mathcal{L}_{P_1}|$ and $|\mathcal{L}_{P_2}|$ given in Equation (2.67) as well as $|\mathfrak{L} \cap \mathcal{L}_0| \leq q^5 + q^4$, we have

$$|\mathfrak{L}| \leq |\mathfrak{L} \cap \mathcal{L}_0| + |\mathfrak{L} \cap \mathcal{L}_{P_1}| + |\mathfrak{L} \cap \mathcal{L}_{P_2}| \leq 3q^5 + 5q^4 + 4q^3 + 4q^2 + 2q + 2,$$

which implies the claim. Furthermore, if $81q < |\Delta_{P_2}(C)| \leq 5q^2$, then there is at most one point $Q \in \mathcal{S}'_{P_2}$ with $|\Delta_Q(\mathfrak{L})| > 5q^2$ (namely the point P_1) and thus we have $|\mathfrak{L} \cap \mathcal{L}_{P_2}| \leq 5q^5 + 6q^4 + 5q^3$ per Equation (2.68). In fact, since that bound is weaker than the one given for $|\mathcal{L}_{P_1}|$ if $|\Delta_{P_1}(\mathfrak{L})| > 5q^2$, we know that it also holds for $|\mathfrak{L} \cap \mathcal{L}_{P_1}|$, regardless of whether or not $|\Delta_{P_1}(\mathfrak{L})| \leq 5q^2$. Together with $|\mathcal{L}_0| \leq q^5 + q^4$ this shows

$$|\mathfrak{L}| \leq |\mathfrak{L} \cap \mathcal{L}_0| + |\mathfrak{L} \cap \mathcal{L}_{P_1}| + |\mathfrak{L} \cap \mathcal{L}_{P_2}| \leq 11q^5 + 13q^4 + 10q^3$$

and concludes this proof. □

Lemma 2.4.45. *The set $\mathfrak{L}$ of all lines l with $l \not\leq H$ which satisfy case C3 (a) or C3 (c) of Theorem 2.4.15 has cardinality at most $64q^5 + 80q^4 + 16q^3 + 16q^2$.*

Proof. We first note that for all $l \in \mathfrak{L}$ the hyperplane H_l provided by Theorem 2.4.15 satisfies $\mathfrak{G}[l, 3, H_l] \subseteq \Pi_3(\Delta_l(C))$. Therefore, if P is a point and l a line with $P \notin l$ such that every solid $S \in \Pi_3(\Delta_l(C))$ satisfies $P \in S$, then to any hyperplane $H' \geq l$ there is a solid $S \leq H'$ with $P \notin S$ which proves $l \notin \mathfrak{L}$. Furthermore, we remark that Lemma 2.4.43 ii), the bound given in Equation (2.61) and the fact that

$$2q^7 + 4q^6 + 5q^5 + 6q^4 + 5q^3 + 4q^2 + 2q + 1 \leq \mathfrak{s}_q[2] \cdot (2q^5 + 2q^4 + q^3 + 3q^2 + q + 1)$$

together prove

$$\forall H' \in \mathfrak{G}[4, \mathbb{P}] : |\Delta_{H'}(\mathfrak{L})| \leq 3q^5 + 5q^4 + 5q^3 + 7q^2 + 3q + 2. \tag{2.69}$$

We keep that in mind and now consider Proposition 2.4.18 and its implications for different values of d. Note that any non-empty subset of $\mathfrak{L}$ obviously satisfies condition (II) of this Proposition. Therefore, we always consider the claims given there for $\kappa = 4$.

First, assume that there is a point $P \in H$ and an integer ξ with $|\Delta_P(\mathfrak{L})| > 4\xi$ such that any hyperplane G of $\mathbb{P}$ satisfies $|\Delta_G(\Delta_P(\mathfrak{L}))| \leq \xi$. Then we may apply Proposition 2.4.18 with $d = 4$ to $\Delta_P(\mathfrak{L})$, which shows that there is a subspace U with $\dim(U) \leq 3$ such that for all flags $f \in C$ we have $\pi_1(f) \leq U$ or $P \in \pi_3(f)$ (note that the second case given in Proposition 2.4.18 may not occur, because, for dimensional reasons, a solid may not contain a complement of P in the $d + 1 = 5$ dimensional subspace G_l given there). Hence, if $l \in \mathfrak{G}[P, \emptyset, 1, \mathbb{P}]$ is a line with $l \not\leq U$, then any solid $S \in \Pi_3(\Delta_l(C))$ satisfies $P \in S$ and we have already seen that this implies $l \notin \mathfrak{L}$. Therefore, and since $|\Delta_P(\mathfrak{L})| \leq q^4$, we have

$$|\mathfrak{L}| \leq |\Delta_P(\mathfrak{L})| + |\mathfrak{s}_q[1, U]| \leq q^4 + \mathfrak{s}_q[1, 3] = 2q^4 + q^3 + 2q^2 + q + 1. \tag{2.70}$$

Secondly, assume that there is a point $P \in H$, an integer ξ and a subset $\mathfrak{L}' \subseteq \Delta_P(\mathfrak{L})$ with $\dim(\langle \mathfrak{L}' \rangle) = 4$, $|\mathfrak{L}'| > 4\xi$ and such that any solid S of $\mathbb{P}$ satisfies $|\Delta_S(\mathfrak{L}')| \leq \xi$. Then we may apply Proposition 2.4.18 with $d = 3$ to $\mathfrak{L}'$, which shows that there is a subspace U with $\dim(U) \le 4$ such that for all flags $f \in C$ we have $\pi_1(f) \le U$, or $\pi_3(f)$ contains P or a complement of P in the hyperplane $\langle \mathfrak{L}' \rangle$ and, if $\dim(U) = 4$, then $U = \langle \mathfrak{L}' \rangle$. Thus, if $l \in \mathfrak{G}[P, \emptyset, 1, \mathbb{P}]$ is a line with $l \not\le U$ and $l \not\le \langle \mathfrak{L}' \rangle$, then any solid $S \in \Pi_3(\Delta_l(C))$ satisfies $P \in S$ and we have already seen that this implies $l \notin \mathfrak{L}$. Therefore, we have

$$\begin{aligned} |\mathfrak{L}| \le |\Delta_P(\mathfrak{L})| + |\Delta_U(\mathfrak{L}) \cup \Delta_{\langle \mathfrak{L}' \rangle}(\mathfrak{L})| &\overset{(*)}{\le} q^4 + \mathfrak{s}_q[1,3] + |\Delta_{\langle \mathfrak{L}' \rangle}(\mathfrak{L})| \\ &\overset{(2.69)}{=} 3q^5 + 7q^4 + 6q^3 + 9q^2 + 4q + 3, \end{aligned} \tag{2.71}$$

where in the step marked with $(*)$ we used the fact that $U = \langle \mathfrak{L}' \rangle$ if $\dim(U) = 4$ as well as $|\Delta_P(\mathfrak{L})| \le q^4$.

Before we proceed we remark that, if there is a point $P \in H$ such that $|\Delta_P(\mathfrak{L})| > 16q^2$, then one of these two situations occurs, which already proves the claim in this case as follows:

- either there is a hyperplane H' of $\mathbb{P}$ such that $|\Delta_{H'}(\Delta_P(\mathfrak{L}))| > 4q^2$ and we have shown above that Equation (2.71) holds, or
- every hyperplane H' of $\mathbb{P}$ satisfies $|\Delta_{H'}(\Delta_P(\mathfrak{L}))| \le 4q^2$ and we have shown above that Equation (2.70) holds.

Similarly, if there is a point $P \in H$ such that $|\Delta_P(\mathfrak{L})| > 64q$, then we may assume that there is a solid S with $|\Delta_S(\Delta_P(\mathfrak{L}))| > 4q$, for otherwise:

- either there is a hyperplane H' of $\mathbb{P}$ such that $|\Delta_{H'}(\Delta_P(\mathfrak{L}))| > 16q$ and we have shown above that Equation (2.71) holds, or
- every hyperplane H' of $\mathbb{P}$ satisfies $|\Delta_{H'}(\Delta_P(\mathfrak{L}))| \le 16q$ and we have shown above that Equation (2.70) holds

and either way this implies the claim.

Hence, from now on we may assume that $|\Delta_P(\mathfrak{L})| \le 16q^2$ for all $P \in H$. Furthermore, we may assume that to every point $P \in H$ with $|\Delta_P(\mathfrak{L})| > 64q$ there is a solid S with $|\Delta_S(\Delta_P(\mathfrak{L}))| > 4q$ and, since this solid contains lines of $\mathfrak{L}$, it satisfies $P \in S \not\le H$. Moreover, we may assume that there indeed is a point $P \in H$ with $|\Delta_P(\mathfrak{L})| > 64q$, for otherwise we have $|\mathfrak{L}| \le |H| \cdot 64q = 64q \cdot \mathfrak{s}_q[4]$, which implies the claim.

Now, let $P_1 \in H$ be a point with $|\Delta_{P_1}(\mathfrak{L})| > 64q$ and let S_1 be a solid such that $\mathfrak{L}_1 := \Delta_{S_1}(\Delta_{P_1}(\mathfrak{L}))$ satisfies $|\mathfrak{L}_1| > 4q$. Then we may apply Proposition 2.4.18 with $d = 2$ to $\mathfrak{L}_1$, which shows that there is a subspace U_1 with $\dim(U_1) \le 4$ such that for all flags $f \in C$ we have $\pi_1(f) \le U_1$ or $\pi_3(f)$ contains P_1 or a complement of P_1 in S_1.

However, if every point $P \in H \setminus S_1$ satisfies $|\Delta_P(\mathfrak{L})| \le 64q$, then we have

$$\begin{aligned} |\mathfrak{L}| &\le \sum_{P \in S_1} |\Delta_P(\mathfrak{L})| + \sum_{P \in H \setminus S_1} |\Delta_P(\mathfrak{L})| \le |S_1 \cap H| \cdot 16q^2 + |H \setminus S_1| \cdot 64q \\ &= 64q^5 + 80q^4 + 16q^3 + 16q^2, \end{aligned}$$

as claimed. Therefore, we may also assume that there is a point $P_2 \in H \setminus S_1$ with $|\Delta_{P_2}(\mathfrak{L})| > 64q$ and we let S_2 be a solid such that $\mathfrak{L}_2 := \Delta_{S_2}(\Delta_{P_2}(\mathfrak{L}))$ satisfies $|\mathfrak{L}_2| > 4q$. Again we may apply Proposition 2.4.18 with $d = 2$, now to $\mathfrak{L}_2$, which shows that there is a second subspace U_2 with $\dim(U_2) \leq 4$ such that for all flags $f \in C$ we have $\pi_1(f) \leq U_2$ or $\pi_3(f)$ contains P_2 or a complement of P_2 in S_2.

Now, if l is a line for which there is an index $i \in \{1, 2\}$ with $l \cap S_i = \emptyset$ and $l \not\leq U_i$, then, for dimensional reasons, there is no solid $S \geq l$ which contains a complement of P_i in S_i, that is, every solid $S \in \Pi_3(\Delta_l(C))$ satisfies $P_i \in S$ and we have already seen that this implies $l \notin \mathfrak{L}$.

Furthermore, every line $l \in \mathfrak{G}[S_1 \cap S_2, \emptyset, 1, \mathbb{P}]$ with $l \cap S_1 \neq \emptyset \neq l \cap S_2$ satisfies $l = \langle l \cap S_1, l \cap S_2 \rangle \leq \langle S_1, S_2 \rangle$ and, if $S \geq l$ is a solid for which $E_1 := S \cap S_1$ and $E_2 := S \cap S_2$ are planes, then, for dimensional reasons, the planes E_1 and E_2 meet in a line $g \leq S_1 \cap S_2$. Since for every line $l \in \mathfrak{G}[S_1 \cap S_2, \emptyset, 1, \mathbb{P}]$ with $P_1 \neq l \cap S_1 \neq \emptyset \neq l \cap S_2 \neq P_2$ and $l \not\leq U_1, U_2$ every solid $S \in \Pi_3(\Delta_l(C))$ with $P_1, P_2 \notin S$ contains a complement of P_i in S_i for both $i \in \{1, 2\}$, this proves that S is the span of l and a line $g \leq S_1 \cap S_2$. In particular, if $S_1 \cap S_2$ is a line, then every line $l \in \mathfrak{G}[S_1 \cap S_2, \emptyset, 1, \mathbb{P}]$ with $P_1 \neq l \cap S_1 \neq \emptyset \neq l \cap S_2 \neq P_2$ and $l \not\leq U_1, U_2$ is such that every solid $S \in \Pi_3(\Delta_l(C))$ satisfies $P_i \in S$ for some $i \in \{1, 2\}$, or $S = \langle l, S_1 \cap S_2 \rangle$ and, again, this implies $l \notin \mathfrak{L}$.

Now, for all $i \in \{1, 2\}$ we set $\mathcal{L}_i := \Delta_{P_i}(\mathfrak{L}) \cup \Delta_{U_i}(\mathfrak{L})$ as well as

$$\mathcal{L}_0 := \{l \in \mathfrak{L} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2) : l \cap S_1 \cap S_2 \neq \emptyset\} \text{ and}$$
$$\mathcal{L}_0' := \{l \in \mathfrak{L} \setminus (\mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_2) : l \cap S_1 \neq \emptyset \neq l \cap S_2\}.$$

Above we have seen that this implies $\mathfrak{L} = \mathcal{L}_0 \cup \mathcal{L}_0' \cup \mathcal{L}_1 \cup \mathcal{L}_2$ and, since $S_1 \not\ni P_2 \in S_2$, we know that $S_1 \cap S_2$ is at most a plane and thus Lemma 2.4.43 i) implies

$$|\mathcal{L}_0| \leq \mathfrak{s}_q[1] \cdot 16q^2 + q^2 \cdot (9q^2 + q + 9)q.$$

Furthermore, either $\langle S_1, S_2 \rangle$ is a hyperplane of $\mathbb{P}$, in which case Equation (2.69) shows

$$|\mathcal{L}_0'| \leq 3q^5 + 5q^4 + 5q^3 + 7q^2 + 3q + 2,$$

or $\langle S_1 \cap S_2 \rangle = \mathbb{P}$, in which case $S_1 \cap S_2$ is a line and we have seen above that $\mathcal{L}_0' = \emptyset$. Finally, Equation (2.69) shows

$$\forall i \in \{1, 2\} : |\mathcal{L}_i| \leq 16q^2 + 3q^5 + 5q^4 + 5q^3 + 7q^2 + 3q + 2$$

and altogether this shows

$$|\mathfrak{L}| \leq 18q^5 + 16q^4 + 40q^3 + 69q^2 + 9q + 6$$

and concludes the proof. □

Lemma 2.4.46. *Let $\mathfrak{L}$ be the set of lines $l \in \Pi_1(C)$ with $l \not\leq H$. Then*

$$|\{P \in H : \Delta_P(\mathfrak{L}) > 15q^3\}| \leq 16 \cdot \mathfrak{s}_q[2].$$

Proof. For every point $P \in \mathcal{P} := \{P \in H : \Delta_P(\mathfrak{L}) > 15q^3\}$ every hyperplane H' satisfies $|\Delta_{H'}(\Delta_P(\mathfrak{L}))| \leq |\mathfrak{S}[P, 1, H']| < 15q^3$, which proves $\langle \Delta_P(\mathfrak{L}) \rangle = \mathbb{P}$. Now, if $\dim(\langle \mathcal{P} \rangle) \leq 1$, then there is nothing to prove and thus we may assume that there are points P_1, P_2 and P_3 in $\mathcal{P}$ which span a plane. For all $i \in \{1, 2, 3\}$ let $l_1^i, \dots, l_5^i$ be lines in $\Delta_{P_i}(\mathfrak{L})$ with $\langle l_1^i, \dots, l_5^i \rangle = \mathbb{P}$. Furthermore, for all $i \in \{1, 2, 3\}$ and all $j \in \{1, \dots, 5\}$ let S_j^i be an arbitrary solid with $(l_j^i, S_j^i) \in C$ and set $\mathcal{P}' := \bigcup_{i=1}^{3} \bigcup_{j=1}^{5} S_j^i \cap H$, which implies $|P'| \leq 15 \cdot \mathfrak{s}_q[2]$.

Now, if $f \in C$ is a flag and $i \in \{1, 2, 3\}$ and $j \in \{1, \dots, 5\}$ are such that $\pi_1(f) \cap S_j^i = \emptyset$, then $\pi_3(f) \cap l_j^i \neq \emptyset$ and thus, if $i \in \{1, 2, 3\}$ is such that for all $j \in \{1, \dots, 5\}$ we have $\pi_1(f) \cap S_j^i = \emptyset$, then $\pi_3(f)$ has non-empty intersection with all five lines $l_1^j, \dots, l_5^j$, which is only possible for $P_i \in \pi_3(f)$. Hence, if $f \in C$ is a flag such that for all $i \in \{1, 2, 3\}$ and all $j \in \{1, \dots, 5\}$ we have $\pi_1(f) \cap S_j^i = \emptyset$, then $E := \langle P_1, P_2, P_3 \rangle \leq \pi_3(f)$.

Let $P \in H \setminus (\mathcal{P}' \cup E)$ be an arbitrary point and let l be an arbitrary line with $P \in l \not\leq H$ and such that for all $i \in \{1, 2, 3\}$ and all $j \in \{1, \dots, 5\}$ we have $l \cap S_j^i = \emptyset$. Since $l \cap H = P \notin E \leq H$ we have $l \cap E = \emptyset$ and thus there is no solid through l that contains E, which proves $\Delta_l(C) = \emptyset$. Therefore, every line $l \in \Delta_P(\mathfrak{L})$ satisfies $l \cap S_j^i \neq \emptyset$ for some $i \in \{1, 2, 3\}$ and some $j \in \{1, \dots, 5\}$, which proves

$$|\Delta_P(\mathfrak{L})| \leq \sum_{i=1}^{3} \sum_{j=1}^{5} |S_j^i \setminus H| = 15q^3,$$

that is, $P \notin \mathcal{P}$. This proves $\mathcal{P} \subseteq \mathcal{P}' \cup E$ and in view of the definition of $\mathcal{P}'$ this implies $|\mathcal{P}| \leq 15 \cdot \mathfrak{s}_q[2] + |E| = 16 \cdot \mathfrak{s}_q[2]$, as claimed. □

Corollary 2.4.47. *The set $\mathfrak{L}$ of all lines $l \in \Pi_1(C)$ with $l \not\leq H$ satisfies*

$$|\mathfrak{L}| \leq 15q^7 + 31q^6 - 209q^5 - 209q^4 - 225q^3.$$

Proof. Lemma 2.4.46 proves

$$|\mathfrak{L}| \leq 16 \cdot \mathfrak{s}_q[2] \cdot q^4 + (|H| - 16 \cdot \mathfrak{s}_q[2]) \cdot 15q^3 = 15q^7 + 31q^6 - 209q^5 - 209q^4 - 225q^3.$$

□

Lemma 2.4.48. *The set $\mathfrak{L}$ of all lines l with $l \not\leq H$ which satisfy case C3 (d) or C4 of Theorem 2.4.15 has cardinality at most* $72q^6 + 77q^5 + 111q^4 + 100q^3 + 40q^2 + 20q + 10$.

Proof. For every line $l \in \mathfrak{L}$, according to Theorem 2.4.15, Remark 2.4.16 and the respective applicable Lemma given therein (2.4.13 is applicable with $\dim(U) = 2$ if l satisfies case C3 (d) and 2.4.14 is applicable with $\mathcal{L}_3 = \emptyset$ if l satisfies case C4), there is a plane E_l and a hyperplane H_l of $\mathbb{P}$ with $l \leq E_l \leq H_l$ such that any flag $f \in C$ satisfies $\pi_1(f) \cap E_l \neq \emptyset$, $\pi_1(f) \leq H_l$, or $\pi_3(f) \cap l \neq \emptyset$.

Now, if every point $P \in H$ satisfies $|\Delta_P(\mathfrak{L})| \leq q^2$, then $|\mathfrak{L}| \leq |H| \cdot q^2 = q^2 \cdot \mathfrak{s}_q[4]$ and there is nothing to prove. Hence, assume the contrary and let $P_1 \in H$ be such that $|\Delta_{P_1}(\mathfrak{L})| > q^2$. Since $l \not\leq H$ for all $l \in \mathfrak{L}$ any solid S satisfies $|\Delta_S(\Delta_{P_1}(\mathfrak{L}))| \leq q^2$ and thus there are lines $l_1^1, \dots, l_4^1 \in \Delta_{P_1}(\mathfrak{L})$ which span a hyperplane $H_1 \neq H$ of $\mathbb{P}$. Furthermore, if every point $P \in H \setminus H_1$ satisfies $|\Delta_P(\mathfrak{L})| \leq q^2$, then, using Lemma 2.4.46, we have

$$|\mathfrak{L}| \leq |H| \cdot q^2 + |H_1 \cap H| \cdot 15q^3 + 16 \cdot \mathfrak{s}_q[2] \cdot q^4 < 32q^6 + 32q^5 + 32q^4 + 16q^3 + q^2$$

and, again, there remains nothing to prove. Hence, again assume the contrary, let $P_2 \in H \setminus H_1$ be such that $|\Delta_{P_2}(\mathfrak{L})| > q^2$ and let $l_1^2, \dots, l_4^2 \in \Delta_{P_2}(\mathfrak{L})$ be lines which span a hyperplane $H_2 \neq H$ of $\mathbb{P}$.

Then, for all $i \in \{1, 2\}$, using the line l_1^i, Lemma 2.4.34 shows that there are only $2q^6 + q^5 + q^4$ flags $f \in \Delta_H(C)$ with $P_i \notin \pi_3(f)$, which implies

$$\begin{aligned} |\Delta_H(C)| &\leq \mathfrak{s}_q[1,3,4] \cdot \mathfrak{s}_q[1,3] + 2(2q^6 + q^5 + q^4) \\ &= 5q^6 + 4q^5 + 6q^4 + 4q^3 + 4q^2 + 2q + 1 \leq 5(q^5 + q^3) \cdot \mathfrak{s}_q[1]. \end{aligned}$$

Hence, for all $i \in \{1, 2\}$ and $j \in \{1, \dots, 4\}$ we may apply Lemma 2.4.43 with $d = 1$ to see that $|\Delta_{H_i}(\mathfrak{L})|, |\Delta_{H_{l_j^i}}(\mathfrak{L})| \leq 6q^5+3q^4+10q^3+4q^2+2q+1$ as well as $|\Delta_P(\mathfrak{L})| \leq (9q^2+q+9)q^2$ for all $P \in E_{l_j^i} \setminus H$. This shows that the set $\mathcal{L}_i$ of all lines $l \in \mathfrak{L}$ for which there exists $i \in \{1, 2\}$ such that there is a solid S with $(l, S) \in C$ and $P_i \notin S$ satisfies

$$\begin{aligned} |\mathcal{L}_i| \leq\ & 4(6q^5 + 3q^4 + 10q^3 + 4q^2 + 2q + 1 + q^2(9q^2 + q + 9)q^2 + (q+1)q^4) \\ & + 6q^5 + 3q^4 + 10q^3 + 4q^2 + 2q + 1 \\ =\ & 36q^6 + 38q^5 + 55q^4 + 50q^3 + 20q^2 + 10q + 5. \end{aligned}$$

Furthermore, if l is a line with $l \notin \mathcal{L}_1 \cup \mathcal{L}_2$ and $l \cap H \notin \langle P_1, P_2 \rangle$, then every solid $S \in \Pi_3(\Delta_l(C))$ contains $\langle P_1, P_2 \rangle$ and thus satisfies $S = \langle l, P_1, P_2 \rangle$, which implies $l \notin \mathfrak{L}$ and proves

$$|\mathfrak{L}| \leq |\mathcal{L}_1| + |\mathcal{L}_2| + |\langle P_1, P_2 \rangle| q^4 \leq 72q^6 + 77q^5 + 111q^4 + 100q^3 + 40q^2 + 20q + 10. \qquad \square$$

Theorem 2.4.49. *The set C satisfies*

$$|C| < 376q^7 + 771q^6 + 537q^5 + 540q^4 + 212q^3 + 409q^2 + 153q + 49.$$

Proof. We let $\mathfrak{L}$ be the set of all lines l with $l \not\leq H$ and note that Theorem 2.4.15 gives a list of cases which may occur for any line $l \in \mathfrak{L}$. We let $\mathfrak{L}_1$ be the set of all saturated lines with $l \not\leq H$ and we let C_1 denote all flags $f \in C$ with $\pi_1(f) \in \mathfrak{L}_1$. Likewise, for $i \in \{2, 3\}$ and $j \in \{a, b, c\}$ we let $\mathfrak{L}_{i,(j)}$ be the set of all lines $l \not\leq H$ which satisfy case i part (j) of Theorem 2.4.15 and we let $C_{i,(j)}$ denote all flags $f \in C$ with $\pi_1(f) \in \mathfrak{L}_{i,(j)}$ as well as for $i \in \{4, 5\}$ we let $\mathfrak{L}_i$ be the set of all lines $l \not\leq H$ which satisfy case i of Theorem 2.4.15 and we let C_i denote all flags $f \in C$ with $\pi_1(f) \in \mathfrak{L}_i$.

Then, Lemma 2.4.39 shows $|\mathfrak{L}_1| \leq 2q^2(q+1)$, Lemma 2.4.40 shows $|\mathfrak{L}_{2,(a)}| \leq 27 \cdot \mathfrak{s}_q[4]$, Lemma 2.4.41 shows $|\mathfrak{L}_{2,(b)}| \leq (16q + 10) \cdot \mathfrak{s}_q[4]$, Lemma 2.4.42 shows $|\mathfrak{L}_{2,(c)}| \leq (10q^2 + 9q + 1) \cdot \mathfrak{s}_q[4]$, Lemma 2.4.44 shows $|\mathfrak{L}_{3,(a)} \cup \mathfrak{L}_{3,(b)}| \leq 81q^2 \cdot \mathfrak{s}_q[3] + q^4$, Lemma 2.4.45 shows $|\mathfrak{L}_{3,(a)} \cup \mathfrak{L}_{3,(c)}| \leq 64q^5 + 80q^4 + 16q^3 + 16q^2$, Lemma 2.4.48 shows $|\mathfrak{L}_{3,(d)} \cup \mathfrak{L}_4| \leq$

$72q^6+67q^5+81q^4$ and Corollary 2.4.47 shows $|\mathfrak{L}| \leq 15q^7+31q^6-209q^5-209q^4-225q^3$. This shows

$$\begin{aligned}
|C_1| &< 2q^7+4q^6+6q^5+6q^4+4q^3+2q^2,\\
|C_{2,(a)}| &\leq 27q^7+81q^6+108q^5+135q^4+135q^3+108q^2+54q+27,\\
|C_{2,(b)}| &\leq 16q^7+58q^6+94q^5+104q^4+104q^3+88q^2+46q+10,\\
|C_{2,(c)}| &\leq 10q^7+29q^6+39q^5+40q^4+40q^3+30q^2+11q+1,\\
|C_{2,(d)} \cup C_5| &\leq 30q^7+62q^6-418q^5-418q^4-450q^3,\\
|C_{3,(a)} \cup C_{3,(b)} \cup C_{3,(c)}| &\leq |\mathfrak{L}_{3,(a)}|(\mathfrak{s}_q[2]+q^2)+(81q^2\cdot\mathfrak{s}_q[3]+q^4-|\mathfrak{L}_{3,(a)}|)\cdot\mathfrak{s}_q[2]\\
&\quad+(64q^5+80q^4+16q^3+16q^2-|\mathfrak{L}_{3,(a)}|)\cdot\mathfrak{s}_q[2]\\
&=(145q^5+162q^4+97q^3+97q^2)\cdot\mathfrak{s}_q[2]-|\mathfrak{L}_{3,(a)}|(q+1)\\
&\leq 145q^7+307q^6+404q^5+356q^4+194q^3+97q^2,\\
|C_{3,(d)} \cup C_4| &\leq 144q^7+226q^6+299q^5+311q^4+180q^3+80q^2+40q+10
\end{aligned}$$

and together with the bound on $|\Delta_H(C)|$ given in Equation (2.61) this proves the claim. □

Corollary 2.4.50. *Every independent set of* Γ *of size larger than*

$$376q^7+771q^6+537q^5+540q^4+212q^3+409q^2+153q+49$$

is contained in a maximal independent set of Γ *given by Example 2.4.1.*

Proof. In view of Notation 2.4.38, this is a direct corollary to Theorem 2.4.49. □

Theorem 2.4.51. *For* $q \geq 376$ *the independence number of the Kneser graph of flags of type* $(1,3)$ *in* $\mathrm{PG}(5,q)$ *is* $\mathfrak{s}_q[3,4]\cdot\mathfrak{s}_q[1,3]+\mathfrak{s}_q[2](\mathfrak{s}_q[3]+q^2)q^2$ *and the independent sets attaining this bound are those given in Example 2.4.1 using an independent set* $\mathcal{U}$ *of the Kneser graph on line plane flags in* $\mathrm{PG}(4,q)$ *of maximal size.*

Proof. This is implied by Corollary 2.4.50, since $\mathfrak{s}_q[3,4]\cdot\mathfrak{s}_q[1,3]+\mathfrak{s}_q[2](\mathfrak{s}_q[3]+q^2)q^2$ is smaller than the bound given there for $q \geq 376$. □

3 Tight Sets

Throughout this chapter we let d be a positive integer and we let q be a prime power. Since in this chapter the number of points and hyperplanes of a given projective space plays a crucial role, we will use the notation $\theta_d(q)$ to denote $\mathfrak{s}_q[d]$. Furthermore, since we mainly work over the field with q^2 elements, we abbreviate the notation and simply write θ_d instead of $\theta_d(q^2)$. Finally, we use $\perp$ to denote the polarity associated with the given Hermitian polar space, as explained in the introduction.

We first introduce the object that we aim to study in this chapter.

Definition 3.0.1 (Tight Set)**.** A *tight set* of the Hermitian polar space $H(2d, q^2)$ is a subset T of its point-set such that there exists an integer $x \geq 0$ with the property

$$\forall P \in H(2d, q^2) : |P^{\perp} \cap T| = \begin{cases} \theta_{d-1} + (x-1)\theta_{d-2} & \text{for } P \in T, \\ x\theta_{d-2} & \text{for } P \notin T. \end{cases}$$

The integer x is called the *parameter* of the tight set and a tight set with parameter x is called an *x-tight set*.

An immediate consequence of this definition is the fact that for all $x \in \mathbb{N}_0$ the union of x mutually skew generators of $H(2d, q^2)$ is an x-tight set thereof. In [1] by De Beule and Metsch it was conjectured that in fact every tight set of $H(2d, q^2)$ with parameter $x < q + 1$ is the disjoint union of generators. In this chapter we take one step towards proving said conjecture.

First, though, we note that the statement given in the conjecture is best possible. That is because there exist x-tight sets of $H(2d, q^2)$ with parameter $x = q + 1$ which are not the union of disjoint generators. Two such examples are

- the embedding of the parabolic polar space $Q(2d, q)$ in $H(2d, q^2)$ and
- the embedding the symplectic polar space $W(2d-1, q)$ in $H(2d-1, q^2)$, which in turn can be embedded in $H(2d, q^2)$.

In both of these cases the set of points of the given embedding is a tight set of $H(2d, q^2)$ with parameter $x = q + 1$ and neither of the two contains a generator. Note that for $d = 2$ and $q \in \{2, 3\}$ further examples were constructed by Cossidente and Pavese in [13, Remark 4.12].

On the other hand, the conjecture has already been established for $d = 2$ in [1]. There it has also been proven under the stronger assumption $x < q + 1 - \sqrt{2q}$ for $d = 3$. For $d \geq 4$, it was shown under a much stronger assumption in [24] by Metsch. The main result of this chapter is Theorem 3.2.10, which considerably improves the corresponding result of [24]. It was first published by the author of this thesis together with Metsch in [26].

One of the main difficulties in the proof is to show that a non-empty tight set with sufficiently small parameter x contains a line and we can prove this only for $x \leq \frac{1}{2}(q+1)$. Once a line is found the condition $x \leq q$ is sufficient to find a generator in the tight set.

3.1 Preliminaries

For this section let $\mathbb{P} := \mathrm{PG}(d, q^2)$ and let $\mathcal{H} := H(d, q^2) \subseteq \mathbb{P}$ be the Hermitian polar space therein and note that, since $\perp$ is a polarity, we have $\dim(U) + \dim(U^\perp) = d - 1$ for every subspace U of $\mathbb{P}$. A subspace $U \leq \mathbb{P}$ is contained in $\mathcal{H}$ if and only if U is totally isotropic, which is equivalent to $U \subseteq U^\perp$. Totally isotropic subspaces $U \leq \mathbb{P}$ are also called subspaces of $\mathcal{H}$.

In this section we will collect some well-known facts on Hermitian polar spaces that we require in the final section of this thesis.

Lemma 3.1.1 ([11, Theorem 6.5.2])**.** $|\mathcal{H}| = \dfrac{(q^d - (-1)^d)(q^{d+1} + (-1)^d)}{q^2 - 1}$.

Corollary 3.1.2. *The points of $\mathcal{H}$ span $\mathbb{P}$.*

Proof. Using the value given in Lemma 3.1.1 we see that there are more points in $\mathcal{H}$ than in a hyperplane of $\mathbb{P}$. □

Lemma 3.1.3. *Let U be a subspace of $\mathbb{P}$, set $R := U \cap U^\perp$ and let C be a complement of R in U.*

i) If $R = U$, then $U \subseteq \mathcal{H}$.

ii) If $\dim(R) = \dim(U) - 1$, then $R = U \cap \mathcal{H}$.

iii) If $r := \dim(R) \leq \dim(U) - 2$, then $C \cap \mathcal{H}$ is a Hermitian polar space $H(c, q^2)$ with $c := \dim(U) - 1 - r$ and $U \cap \mathcal{H}$ is the union of the subspaces $\langle R, P\rangle$ of $\mathbb{P}$ with $P \in C \cap \mathcal{H}$. Furthermore, the points of U that lie in $\mathcal{H}$ span U as a subspace of $\mathbb{P}$.

Proof. First, note that we have $R \subseteq U$, which implies $U^\perp \subseteq R^\perp$, and, since $R \subseteq U^\perp$, this shows $R \subseteq R^\perp$. Therefore, R is totally isotropic and thus contained in $\mathcal{H}$. Note that this already proves i).

ii) We suppose that $\dim(R) = \dim(U) - 1$ and consider a point point $P \in U \setminus R$. Then $U = \langle R, P\rangle$, hence $U^\perp = R^\perp \cap P^\perp$ and thus

$$R = U \cap U^\perp = U \cap R^\perp \cap P^\perp = U \cap P^\perp.$$

Therefore, $P \notin P^\perp$, that is, $P \notin H(d, q^2)$, which shows $U \cap H(d, q^2) = R$.

iii) Now, suppose that $\dim(R) \leq \dim(U) - 2$ and let C be a complement of R in U. Then $\dim(C) = \dim(U) - 1 - r \geq 1$ and from $U^\perp \cap U = R$ as well as $U^\perp = R^\perp \cap C^\perp$ we have $C^\perp \cap C = \emptyset$. Thus, the restriction of the form defining $\mathcal{H}$ to C is non-degenerate and $\mathcal{H} \cap C$ is a Hermitian polar space $H(c, q^2)$. Now, a point $P \in U \setminus R$ belongs to $\mathcal{H}$ if and only if $\langle R, P\rangle$ is totally isotropic and this is the case if and only if the point $\langle R, P\rangle \cap C$ belongs to $\mathcal{H}$. Finally, Corollary 3.1.2 shows that $H(s - r - 1, q^2)$ spans C and hence $U \cap \mathcal{H}$ spans U. $\square$

Definition 3.1.4 (Cone)**.** If U is a subspace of $\mathbb{P}$ which satisfies iii) of the previous lemma and notations are as given there, then $U \cap \mathcal{H}$ is called a *cone* with *vertex* R over a Hermitian polar space $H(c, q^2)$.

Note that this definition of a cone also includes the situation where the radical of the cone is empty. In particular, every non-degenerate Hermitian polar space will be called a cone, too, as a opposed to totally isotropic subspaces, which will not be called cones. Thus, if U is a subspace of $\mathbb{P}$, then $U \cap \mathcal{H}$ is either a subspace (and thus U or a hyperplane of U), or a cone.

Corollary 3.1.5. *The number of hyperplanes H of $\mathbb{P}$ which are not spanned by $H \cap \mathcal{H}$ is $q^3 + 1$ if $d = 2$ and 0 if $d \geq 3$.*

Proof. If $d = 2$, then $H(2, q^2)$ is a hermitian curve and its $q^3 + 1$ tangent lines are the hyperplanes in question. Hence, suppose that $d \geq 3$ and let H be a hyperplane of $\mathbb{P}$. Then $\dim(H \cap H^\perp) \leq 0$, since the form defining $\mathcal{H}$ is non-degenerate. Since $d - 2 \geq 0 \geq \dim(H \cap H^\perp)$ we know that part iii) of Lemma 3.1.3 applies to H. Thus, $H \cap \mathcal{H}$ is a cone and spans H. $\square$

Lemma 3.1.6. *Let U be a subspace of $\mathbb{P}$ such that $U \cap \mathcal{H}$ is a cone, set $u := \dim(U)$ and let r be the dimension of its vertex. Then $r \leq u - 2$ and we have:*

i) $|U \cap \mathcal{H}| = \theta_r + q^{2(r+1)}|H(u - 1 - r, q^2)| > q^{2u-1}$.

ii) The number of hyperplanes H of U for which $H \cap \mathcal{H}$ is not a cone is $q^2 + 1$ if $r = u - 2$, $q^3 + 1$ if $r = u - 3$ and 0 otherwise. In particular, if $u \geq 2$, then at least θ_{u-1} hyperplanes H of U have the property that $H \cap \mathcal{H}$ is a cone.

Proof. Let R be the vertex of the cone $U \cap \mathcal{H}$ and let C be a complement of R in U. In Lemma 3.1.3 we have seen that $C \cap \mathcal{H}$ is a Hermitian polar space $H(u - r - 1, q^2)$.

i) From Lemma 3.1.3 we also know that $U \cap \mathcal{H}$ is the union of the $|H(u - r - 1, q^2)|$ subspaces of dimension $r + 1$ which are spanned by R and a point of $C \cap \mathcal{H}$. Since all these subspaces contain R as well as $q^{2(r+1)}$ additional points, we find the cardinality of $U \cap \mathcal{H}$ as stated in the claim. The inequality given in the claim is implied by the fact that the cardinality of $H(u - r - 1, q^2)$ as stated in Lemma 3.1.1 is obviously greater than $q^{2u-2r-3}$ for $u - r - 1 \geq 1$.

ii) First, consider a hyperplane H of U with $R \subseteq H$. Then each complement of $R \cap H$ in H is also a complement of R in U and hence meets $\mathcal{H}$ in a Hermitian polar space $H(u-r-1, q^2)$. Therefore, in this case $H \cap \mathcal{H}$ is not a subspace but a cone.

Now, consider a hyperplane H of U with $R \subseteq H$. Then H is spanned by R and a hyperplane of C. Since $U \cap \mathcal{H}$ is the union of the spaces $\langle R, P\rangle$ spanned by R and a point $P \in C$, we know that H meets $\mathcal{H}$ in a subspace of and only if $H \cap C$ meets $\mathcal{H}$ in a subspace. Corollary 3.1.5 shows that this is never the case for $u-r-1 \geq 3$ and that this happens exactly q^3+1 times for $u-r-1=2$. Finally, for $u-r-1=1$ we know that $H \cap C$ is a point and thus always a subspace, that is, in this case it occurs for all q^2+1 hyperplanes of U which contain R.

This proves the first part of the claim in ii). However, the total number of hyperplanes of U is $\theta_u = q^{2u} + \theta_{u-1}$ and for $u \geq 2$ this implies the second claim. □

3.2 Small Tight Sets of $H(2d, q^2)$

In this section we work in the projective space $\mathbb{P} := \mathrm{PG}(2d, q^2)$, let $\mathcal{H} := H(2d, q^2)$ be the Hermitian polar space therein and assume that $d > 1$. Furthermore, we let T be a tight set of $\mathcal{H}$ and we let x be its parameter. In the following lemmata we investigate properties of T. We are particularly interested in subspaces U of $\mathbb{P}$ with $U \cap \mathcal{H} \subseteq T$.

In the spirit of trying to keep this work as complete as possible we also include the proofs of some well known facts, such as the following lemma.

Lemma 3.2.1. *If T is an x-tight set of $\mathcal{H}$, then $|T| = x\theta_{d-1}$.*

Proof. We count pairs (P, Q) consisting of points $P \in T$ and $Q \in \mathcal{H}$ with $P \in Q^\perp$ twice.

On one hand we have $|T|$ choices for the point P and, since $P \in Q^\perp \iff Q \in P^\perp$ for all $Q \in \mathcal{H}$, we subsequently have $|P^\perp \cap \mathcal{H}|$ choices for the point Q. Note that for $P \in T$ we have $P \in \mathcal{H}$, that is, $P^\perp \cap \mathcal{H}$ is a cone with 0-dimensional vertex $P = P \cap P^\perp$ over a Hermitian polar space $H(2d-2, q^2)$ and thus the cardinality of $\mathcal{P}^\perp \cap \mathcal{H}$ is independent of the particular choice of $P \in T$ and given by $(1 + q^2|H(2d-2, q^2)|)$.

On the other hand we have $|\mathcal{H}| = \frac{(q^{2d+1}+1)(q^{2d}-1)}{q^2-1}$ choices for Q, all of which are incident with either $x\theta_{d-2}$ or $q^{d-1} + x\theta_{d-2}$ points of T, depending on whether or not Q is an element of T. This yields

$$|T|(1 + q^2|H(2d-2, q^2)|) = \frac{(q^{2d+1}+1)(q^{2d}-1)}{q^2-1} x\theta_{d-2} + |T|q^{2(d-1)}$$

and simplifications show $|T| = x\theta_{d-1}$, as claimed. □

The following is also known and given as Lemma 2.1 in [24] by Metsch, but we still include a proof here.

Lemma 3.2.2. *Let $U \leq \mathbb{P}$ be such that $U \subseteq T$. Then*

$$|U^\perp \cap T| = \theta_{d-1} + (x-1)\theta_{d-\dim(U)-2}.$$

Proof. The proof is by induction on $u := \dim(U)$ and the case $u = 0$ is trivial by the definition of a tight set. Thus, let $u > 0$ and assume that the statement holds for subspaces of smaller dimension.

There are θ_u subspaces $U_1, \ldots, U_{\theta_u}$ of dimension $u-1$ contained in U, any two of which span the whole space U. Furthermore, for any point $x \in T$ we know that $x^\perp$ is a hyperplane of $\mathbb{P}$, which implies that $x^\perp \cap U$ has dimension $\geq u-1$ and thus contains an $u-1$ dimensional subspace U' of U. Hence, $U' \subseteq x^\perp$ and thus $x \in U'^\perp$, that is, $T \subseteq \bigcup_{i=1}^{\theta_u} U_i^\perp$. Moreover, for all distinct $i, j \in \{1, \ldots, \theta_u\}$ and all $P \in U_i^\perp \cap U_j^\perp \cap T$ we have $U = \langle U_i, U_j \rangle \subseteq P^\perp$ and thus $P \in U^\perp \cap T$. Together this implies

$$x\theta_{d-1} = |T| = \left| \bigcup_{i=1}^{\theta_u} \left(U_i^\perp \cap T \right) \right| = \left| U^\perp \cap T \right| + \sum_{i=1}^{\theta_u} \left(\left| U_i^\perp \cap T \right| - \left| U^\perp \cap T \right| \right).$$

Using the induction hypothesis, stating $|U_i^\perp \cap T| = \theta_{d-1} + (x-1)\theta_{d-(u-1)-2}$, we have

$$x\theta_{d-1} = \theta_u \left(\theta_{d-1} + (x-1)\theta_{d-u-1} \right) - (\theta_u - 1) \left| U^\perp \cap T \right|,$$

which implies

$$\left| U^\perp \cap T \right| = \frac{\theta_u \left(\theta_{d-1} + (x-1)\theta_{d-u-1} \right) - x\theta_{d-1}}{\theta_u - 1}.$$

Now, consider the right hand side in two parts. First we notice

$$\theta_u \frac{\theta_{d-1} - \theta_{d-u-1}}{\theta_u - 1} = \theta_u \frac{q^{2(d-u)}\theta_{u-1}}{q^2\theta_{u-1}} = \theta_u q^{2(d-u-1)} = \theta_{d-1} - \theta_{d-u-2}$$

and then study the rest and see

$$\begin{aligned} x\frac{(\theta_u\theta_{d-u-1} - \theta_{d-1})}{\theta_u - 1} &= x\frac{(q^{2(u+1)} - 1)(q^{2(d-u)} - 1) - (q^2 - 1)(q^{2d} - 1)}{(q^2-1)q^2(q^{2u} - 1)} \\ &= x\frac{q^{2(d-1)} - q^{2(d-u-1)} - q^{2u} + 1}{(q^2-1)(q^{2u}-1)} = x\frac{q^{2(d-u-1)} - 1}{q^2 - 1} = x\theta_{d-u-2}. \end{aligned}$$

Together this proves the claim. □

Note that the case $\dim(U) = 1$ of the following lemma has already been covered in [1, Lemma 3.1] by De Beule and Metsch.

Lemma 3.2.3. *Let $U \leq \mathbb{P}$ be such that $U \cap \mathcal{H}$ is a cone contained in T. Then*

$$|U^\perp \cap T| \geq (q+1-x)q^{2d-2\dim(U)-1}\theta_{\dim(U)-1} + x\theta_{d-\dim(U)-1}.$$

Proof. The proof is, again, by induction on $u := \dim(U)$ and first we consider the case $u \leq 1$. Since $U \cap \mathcal{H}$ is not a subspace we have $u = 1$ and $U \cap \mathcal{H}$ consists of $q+1$

points $P_1, \dots, P_{q+1}$ of the line U. Now, for all distinct $i, j \in \{1, \dots, q+1\}$ we have $P_i^\perp \cap P_j^\perp = U^\perp$ and thus

$$\begin{aligned} x\theta_{d-1} = |T| &\geq \left| \bigcup_{i=1}^{q+1} (P_i^\perp \cap T) \right| = \sum_{i=1}^{q+1} \left| P_i^\perp \cap T \right| - q \left| U^\perp \cap T \right| \\ &= (q+1)(q^{2(d-1)} + x\theta_{d-2}) - q \left| U^\perp \cap T \right| \\ \implies \quad \left| U^\perp \cap T \right| &\geq \frac{(q+1)(q^{2(d-1)} + x\theta_{d-2}) - x\theta_{d-1}}{q} = (q+1-x)q^{2d-3} + x\theta_{d-2}, \end{aligned}$$

as claimed.

Now, suppose that $u \geq 2$ and that the assertion is true for subspaces of smaller dimension. Lemma 3.1.6 shows that U has at least θ_{u-1} distinct hyperplanes $H_1, \dots, H_{\theta_{u-1}}$ which intersect $\mathcal{H}$ in a cone. Since any two distinct hyperplanes of U span U we know that every point which lies in $H_i^\perp$ and $H_j^\perp$ for some distinct $i, j \in \{1, \dots, \theta_{u-1}\}$, also lies in $U^\perp$ and thus in $H_i^\perp$ for all $i \in \{1, \dots, \theta_{u-1}\}$. This implies

$$x\theta_{d-1} = |T| \geq \left| \bigcup_{i=1}^{\theta_{u-1}} \left(H_i^\perp \cap T \right) \right| = \sum_{i=1}^{\theta_{u-1}} \left| H_i^\perp \cap T \right| + (1 - \theta_{u-1}) \left| U^\perp \cap T \right|.$$

Now, we may use the induction hypothesis to see

$$x\theta_{d-1} \geq \theta_{u-1} \left((q+1-x)q^{2d+1-2u}\theta_{u-2} + x\theta_{d-u} \right) + (1 - \theta_{u-1}) \left| U^\perp \cap T \right|,$$

which implies

$$\begin{aligned} \left| U^\perp \cap T \right| &\geq \frac{\theta_{u-1} \left((q+1-x)q^{2d+1-2u}\theta_{u-2} + x\theta_{d-u} \right) - x\theta_{d-1}}{\theta_{u-1} - 1} \\ &= \frac{\theta_{u-1}(q+1-x)q^{2d+1-2u}\theta_{u-2}}{q^2\theta_{u-2}} + x\frac{\theta_{u-1}\theta_{d-u} - \theta_{d-1}}{q^2\theta_{u-2}} \\ &= \theta_{u-1}(q+1-x)q^{2d-1-2u} + \xi, \end{aligned}$$

with $\xi := x\frac{\theta_{u-1}\theta_{d-u} - \theta_{d-1}}{q^2\theta_{u-2}}$. Finally, we simplify ξ, which shows the claim

$$\begin{aligned} \xi = x&\frac{(q^{2u}-1)(q^{2(d-u+1)}-1) - (q^{2d}-1)(q^2-1)}{(q^2-1)q^2(q^{2(u-1)}-1)} \\ &= x\frac{q^{2(d-1)} - q^{2(d-u)} - q^{2(u-1)} + 1}{(q^2-1)(q^{2(u-1)}-1)} = x\frac{q^{2(d-u)}-1}{q^2-1} = x\theta_{d-u-1}. \end{aligned}$$ □

Lemma 3.2.4. *Let U_1 and U_2 be subspaces of $\mathbb{P}$ with $U_1 \cap U_2 \cap \mathcal{H} \neq \emptyset$ and*

$$\left| U_1^\perp \cap T \right| + \left| U_2^\perp \cap T \right| > \theta_{d-1} + (2x-1)\theta_{d-2}. \tag{3.1}$$

Then $\langle U_1, U_2 \rangle \cap \mathcal{H} \subseteq T$.

Proof. Set $U := \langle U_1, U_2 \rangle$ and let Q be a point of $U_1 \cap U_2 \cap \mathcal{H}$. Then $U_1^\perp, U_2^\perp \subseteq Q^\perp$ and from $U_1^\perp \cap U_2^\perp = U^\perp$ we have

$$|Q^\perp \cap T| \geq |U_1^\perp \cap T| + |U_2^\perp \cap T| - |U^\perp \cap T|.$$

From the definition of a tight set we have $|Q^\perp \cap T| \leq \theta_{d-1} + (x-1)\theta_{d-2}$. Hence, if Equation (3.1) holds, then $|U^\perp \cap T| > x\theta_{d-2}$, that is, $|P^\perp \cap T| > x\theta_{d-2}$ for all $P \in U$ and the definition of a tight set implies that $U \cap \mathcal{H} \subseteq T$. □

Lemma 3.2.5. *Let U_1 and U_2 be subspaces of $\mathbb{P}$ with $U_1 \cap U_2 \cap \mathcal{H} \neq \emptyset$. Furthermore, suppose that $U_i \cap \mathcal{H}$ is contained in T and spans U_i for $i \in \{1, 2\}$. Then $\langle U_1, U_2 \rangle \cap \mathcal{H} \subseteq T$ in each of the following cases.*

i) $2x \leq q^2 + 1$ *and U_1 and U_2 are contained $\mathcal{H}$.*

ii) $x < q$ *and U_1 is contained $\mathcal{H}$.*

iii) $x < q$ *and $U_1 \cap \mathcal{H}$ contains a line.*

iv) $2x \leq q + 1$.

Proof. First, note that by hypothesis $U_1 \cap \mathcal{H}$ is non-empty and contained in T. Therefore, $T \neq \emptyset$ and, since $|T| = x\theta_{d-1}$, this implies $x > 0$, that is, $x \geq 1$. Furthermore, if one of the subspaces U_1 or U_2 is a point the claim is trivial and thus in the following we may assume the contrary.

i) Lemma 3.2.2 shows $|U_i^\perp \cap T| \geq \theta_{d-1}$ for both $i \in \{1, 2\}$ and since

$$\theta_{d-1} > q^2\theta_{d-2} \geq (2x-1)\theta_{d-2},$$

the previous lemma proves the claim.

ii) In view of i) we may assume that U_2 is not contained in $\mathcal{H}$ and set $u_2 := \dim(U_2)$. Since U_2 is not contained in $\mathcal{H}$ and since $U_2 \cap \mathcal{H}$ spans U_2 we have $u_2 \geq 1$ and $U_2 \cap \mathcal{H}$ is a cone. Now, subsequently using Lemma 3.2.3, the fact that $u_2 \geq 1$ implies $\theta_{u_2-1} \geq q^{2(u_2-1)}$ as well as $x \leq q-1$ shows

$$\begin{aligned} |U_2^\perp \cap T| &\geq (q+1-x)q^{2d-2u_2-1}\theta_{u_2-1} \geq (q+1-x)q^{2d-3} \geq 2q^{2d-3} \\ &\geq 2(q-1)\theta_{d-2} \geq 2x\theta_{d-2} > (2x-1)\theta_{d-2}. \end{aligned}$$

Finally, Lemma 3.2.2 shows $|U_1^\perp \cap T| \geq \theta_{d-1}$ and the previous lemma proves the claim.

iii) Let P be a point of $U_1 \cap U_2 \cap \mathcal{H}$. Since $U_1 \cap \mathcal{H}$ contains a line, it even contains a line l with $P \in l$ (if h is a line of $U_1 \cap \mathcal{H}$, then either $P \in h$, or P lies on a line l of $\mathcal{H}$ that meets h and this line is contained in U_1) and ii) implies $\langle U_2, l \rangle \cap \mathcal{H} \subseteq T$. Let U be a subspace of maximal dimension subject to the properties $\langle U_2, l \rangle \subseteq U \subseteq \langle U_1, U_2 \rangle$ and $U \cap \mathcal{H} \subseteq T$. We have to show that $U = \langle U_1, U_2 \rangle$ and we assume the contrary.

Now, U_1 is not a subset of U and, since $U_1 \cap \mathcal{H}$ spans U_1, there exists a point Q in $U_1 \cap \mathcal{H}$ that does not lie in U. Then $Q \notin l$ and $\mathcal{H}$ contains a line h with $Q \in h$ that meets l. Since $l \subseteq U_1$ and $Q \in U_1$ we have $h \subseteq U_1$ and thus $h \subseteq U_1 \cap \mathcal{H} \subseteq T$. However, ii) implies $\langle U, h\rangle \cap \mathcal{H} \subseteq T$, a contradiction to the maximality of U.

iv) In view of iii) we may assume that neither U_1 nor U_2 contains a line of $\mathcal{H}$ and we remark that this also implies that neither of the two is a subspace of $\mathcal{H}$.

Let $i \in \{1, 2\}$ be arbitrarily chosen and fixed. Since $U_i \cap \mathcal{H}$ spans U_i, this implies that $U_i \cap \mathcal{H}$ is a Hermitian line $H(1, q^2)$ or a Hermitian curve $H(2, q^2)$. If $U_i \cap \mathcal{H}$ is a Hermitian line, then Lemma 3.2.3 shows

$$|U_i^{\perp} \cap T| \geq (q + 1 - x)q^{2d-3} + x\theta_{d-2}$$

and if $U_i \cap \mathcal{H}$ is a Hermitian curve, then Lemma 3.2.3 shows

$$|U_i^{\perp} \cap T| \geq (q + 1 - x)q^{2d-5}\theta_1 + x\theta_{d-3}.$$

Since $x \geq 1$ the first of these two bounds is stronger than the second one. In view of the previous lemma the claim thus is implied by

$$\begin{aligned} 2(q + 1 - x)q^{2d-5}\theta_1 + 2x\theta_{d-3} &\overset{(*)}{\geq} \left(q + \frac{2x-1}{q}\right) q^{2d-5}\theta_1 + 2x\theta_{d-3} \\ &> q^{2d-4}\theta_1 + (2x - 1)q^{2d-4} + 2x\theta_{d-3} = \theta_{d-1} + (2x - 1)\theta_{d-2}, \end{aligned}$$

where the first step marked $(*)$ used the fact that $2x \leq q + 1$ implies

$$2(q + 1 - x) \geq q + 1 \geq q + \frac{2x-1}{q}. \qquad \square$$

Lemma 3.2.6. *Suppose that $x \leq q - 1$ and that U is a subspace of $\mathbb{P}$ with $\dim(U) \geq d - 1$ and $U \cap \mathcal{H} \subseteq T$. Then $U \cap \mathcal{H}$ is a subspace and either $\dim(U) = d - 1$ with $\dim(U \cap \mathcal{H}) \geq d - 2$, or $\dim(U) = d$ with $\dim(U \cap \mathcal{H}) = d - 1$.*

Proof. We set $u := \dim(U) \geq d - 1$ and first assume that $U \cap \mathcal{H}$ is not a subspace.

From Lemma 3.1.3 we then know that $U \cap \mathcal{H}$ is a cone and Lemma 3.1.6 implies $|U \cap \mathcal{H}| > (q - 1)\theta_{u-1}$. Since $|T| = x\theta_{d-1}$ and $x \leq q - 1$ this shows $u \leq d - 1$, that is, $u = d - 1$ and $|U \cap \mathcal{H}| > (q - 1)\theta_{d-2}$.

Now, for all $P \in U^{\perp} \cap \mathcal{H}$ we have $U \subseteq P^{\perp}$ and in view of the definition of a tight set the fact that $|P^{\perp} \cap T| \geq |U \cap \mathcal{H}| \geq (q - 1)\theta_{d-2}$ implies $P \in T$, that is, we have $U^{\perp} \cap \mathcal{H} \subseteq T$ with $\dim(U^{\perp}) = d$. Note that, if $U^{\perp} \cap \mathcal{H}$ would be a cone, then Lemma 3.1.6 would imply $|T| \geq |U^{\perp} \cap \mathcal{H}| > (q - 1)\theta_{n-1}$, a contradiction to $|T| = x\theta_{d-1}$ and $x \leq q - 1$. Hence, $U^{\perp} \cap \mathcal{H}$ is a subspace S. Furthermore, since $\mathcal{H}$ does not contain subspaces of dimension d, Lemma 3.1.3 shows $\dim(S) = d - 1$.

Finally, consider a point $Q \in U^{\perp} \setminus S$. Then $Q \in S^{\perp}$, since $U^{\perp} \cap \mathcal{H} = S$, and $S \subseteq S^{\perp}$, since S is a subspace of $\mathcal{H}$. Furthermore, from $U^{\perp} = \langle S, Q\rangle$ we have $U^{\perp} \subseteq S^{\perp}$. Hence, $S \subseteq U$ and, since S and U have the same dimension, we even have $S = U$. However, this is a contradiction to the fact that $U \cap \mathcal{H}$ is a cone and hence we conclude that $U \cap \mathcal{H}$ must be a subspace V.

Now, Lemma 3.1.3 shows that V is either U or a hyperplane of U and, since every subspace contained in $\mathcal{H}$ has dimension at most $d-1$, we have $\dim(V) \leq d-1$. However, since $u \geq d-1$, this only leaves two possibilities: Either $u = d-1$ and U is a subspace of $\mathcal{H}$, or $u \in \{d-1, d\}$ and $V = U \cap \mathcal{H}$ is a subspace of dimension $u-1$. □

Lemma 3.2.7. *Suppose that $x \leq q-1$ and that T contains two subspaces U_1 and U_2 such that $U_1 \cap U_2 \neq \emptyset$ and $\dim(\langle U_1, U_2 \rangle) \geq d-1$. Then $\langle U_1, U_2 \rangle$ has dimension $d-1$ and is contained in T.*

Proof. We set $U := \langle U_1, U_2 \rangle$ as well as $S := U_1 \cap U_2$ and note that Lemma 3.2.5 ii) already shows $U \cap \mathcal{H} \subseteq T$. Now, Lemma 3.2.6 shows that $U \cap \mathcal{H}$ is a subspace and, since U is spanned by U_1 and U_2 and since $U_i \subseteq T \subseteq \mathcal{H}$, we have $U \subseteq \mathcal{H}$. However, this implies $\dim(U) \leq d-1$ and thus $\dim(U) = d-1$, as claimed. □

Lemma 3.2.8. *Let U be a subspace of $\mathbb{P}$ contained in T, set $u := \dim(U)$ and suppose that one of the following two conditions holds:*

$$1 \leq u \leq d-2 \wedge x \leq q-1 \qquad \textit{or} \qquad u = 0 \wedge 2x \leq q+1.$$

Then U is contained in at most θ_{d-u-3} subspaces W of dimension $u+1$ such that $W \cap \mathcal{H}$ is a cone contained in T.

Proof. For $u = d-2$ we have $\theta_{d-u-3} = \theta_{-1} = 0$ and the statement is implied by Lemma 3.2.6. Therefore, we may assume that $0 \leq u \leq d-3$ and we let $W_1, \ldots, W_s$ be the pairwise distinct subspaces in question, that is, for all $i \in \{1, \ldots, s\}$ we have $\dim(W_i) = u+1$, $U \leq W_i$ and $W_i \cap \mathcal{H}$ is a cone contained in T.

We now prove by induction on $i \in \{1, \ldots, s\}$ that $\overline{W}_i := \langle W_1, \ldots, W_i \rangle$ meets $\mathcal{H}$ in a cone contained in T. For $i = 1$ this is obviously trivial and thus we may suppose that the statement holds for some i with $1 \leq i < s$. Now, for $u = 0$ condition iv) and for $u \geq 1$ condition iii) of Lemma 3.2.5 is satisfied and thus we may apply said Lemma to the subspaces $\overline{W}_i$ and W_{i+1} to see that $\overline{W}_{i+1} \cap \mathcal{H}$ is contained in T, too. Since $W_1 \cap \mathcal{H}$ is a cone with $W_1 \cap \mathcal{H} \subseteq \overline{W}_{i+1} \cap \mathcal{H}$ we know that $\overline{W}_{i+1} \cap \mathcal{H}$ may not be a subspace, that is, it is also a cone, as claimed.

Now, we know that $\overline{W}_s \cap \mathcal{H}$ is a cone contained in T and thus Lemma 3.2.6 implies $\dim(\overline{W}_s) \leq d-2$. Therefore, the number of $(u+1)$-dimensional subspaces of $\overline{W}_s$ which contain U is at most $\mathfrak{s}_{q^2}[u, u+1, d-2] = \theta_{d-u-3}$, that is, we have $s \leq \theta_{d-u-3}$, as claimed. □

Lemma 3.2.9. *Let U be a subspace that is contained in T and maximal with respect to this property, and let u denote its dimension. If $1 \leq u \leq d-2$, then we have $x \geq q$, and if $u = 0$, then we have $2x \geq q+2$.*

Proof. We remark that the condition $2x \leq q+1$ for $u = 0$ will only be used when we apply Lemma 3.2.8. Furthermore, note that we prove both parts of the claim at once and do so in three steps, leading the assumption that the claim does not hold to a contradiction. Finally, also note that for $u < 0$ there is nothing to prove, that is, we may assume $u \geq 0$ and then $\emptyset \neq U \subseteq T$ implies $x \geq 1$.

Now, we assume that either $1 \leq u \leq d-2$ and $x \leq q-1$, or $u = 0$ and $2x \leq q+1$ holds and count pairs $(Q, R) \in (U^\perp \cap T) \times (T \setminus U^\perp)$ with $Q \in R^\perp$ in two different ways. In the first step we first choose the point Q of the pair (Q, R) and determine a lower bound on the number m of these pairs. In the second step we first choose the point R of the pair (Q, R) and determine an upper bound on m. In the third step we then compare these two bounds, yielding a contradiction and concluding the proof.

Thus, we first count pairs by first choosing a point Q of $U^\perp \cap T$. Any such point Q occurs in

$$\xi_Q := |Q^\perp \cap T| - |Q^\perp \cap U^\perp \cap T|$$

pairs and for $Q \in U$ we have $U^\perp \subseteq Q^\perp$ and Lemma 3.2.2 shows

$$\xi_Q = |Q^\perp \cap T| - |U^\perp \cap T| = (x-1)(\theta_{d-2} - \theta_{d-u-2}).$$

Hence, it remains to study $Q \notin U$. In this case $\langle U, Q\rangle$ is $(u+1)$-dimensional and, since $Q \in U^\perp$, it is totally isotropic. Furthermore, from the maximal choice of U we know that $\langle U, Q\rangle$ is not contained in T, that is, there exists a totally isotropic point $P \in \langle U, Q\rangle \setminus T$. Now, $U^\perp \cap Q^\perp = \langle U, Q\rangle^\perp$ is contained in $P^\perp$ and, since T is a tight set, we have $|U^\perp \cap Q^\perp \cap T| \leq |P^\perp \cap T| \leq x\theta_{n-2}$. Therefore, in this case we find

$$\xi_Q \geq |Q^\perp \cap T| - x\theta_{d-2} = q^{2(d-1)}.$$

Consequently, the total number m of pairs (Q, R) under consideration satisfies

$$m \geq |U|(x-1)(\theta_{d-2} - \theta_{d-u-2}) + (|U^\perp \cap T| - |U|)q^{2(d-1)}$$

and, using $|U| = \theta_u$ as well as $|U^\perp \cap T| = \theta_{d-1} + (x-1)\theta_{d-u-2}$ given in Lemma 3.2.2, we find

$$m \geq (x-1)\theta_u(\theta_{d-2} - \theta_{d-u-2}) + (\theta_{d-1} + (x-1)\theta_{d-u-2} - \theta_u)q^{2(d-1)}. \tag{3.2}$$

Secondly, we count pairs (Q, R) in question by first choosing $R \in T \setminus U^\perp$. For each such point R, the subspace $W := \langle U, R\rangle$ has dimension $u+1$ and the number of pairs in which R occurs is $|W^\perp \cap T|$. Since R is not contained in $U^\perp$ the set $W \cap \mathcal{H}$ is not a subspace and thus it is a cone. Obviously, if this cone is not contained in T, that is if $W \cap \mathcal{H}$ contains a point P that is not contained in T, then we have $|W^\perp \cap T| \leq |P^\perp \cap T| = x\theta_{n-2}$. But also if the cone $W \cap \mathcal{H}$ is contained in T we find a bound, because then we may use the fact that $W^\perp \cap \mathcal{H}$ is a subset of $U^\perp \cap \mathcal{H}$ which does not contain the q^{2u} points of $U \setminus (R^\perp \cap U)$. Therefore, in this case R occurs in at most $|U^\perp \cap T| - q^{2u}$ pairs. Note that the number of choices for R is $|T| - |U^\perp \cap T|$. Hence, if ν is the number of points $R \in T \setminus U^\perp$ for which $\langle U, R\rangle$ meets $\mathcal{H}$ in a cone that is contained in T, then we find

$$m \leq (|T| - |U^\perp \cap T| - \nu)x\theta_{d-2} + \nu(|U^\perp \cap T| - q^{2u}). \tag{3.3}$$

Now, Lemma 3.2.2 implies $|U^\perp \cap T| \geq \theta_{d-1}$ and, since $u \leq d-2$ and $x \leq q-1$, this shows that the coefficient of ν in Equation (3.3) is non-negative. Therefore, Equation (3.3) remains true if we replace ν by an upper bound for ν. Such an upper bound is provided by Lemma 3.2.8, which shows that at most θ_{n-u-3} subspaces W of dimension $u+1$ on U meet $\mathcal{H}$ in a cone contained in T. Note that such a cone is a cone with vertex $U \cap R^\perp$ of dimension $u-1$ over a Hermitian line $H(1, q^2)$ and thus has $\theta_{u-1} + (q+1)q^{2u}$ points in $\mathcal{H}$ of which q^{2u+1} do not lie in U. Hence, we have $\nu \leq \theta_{d-u-3}q^{2u+1}$, which implies

$$m \leq (|T| - |U^\perp \cap T|)x\theta_{d-2} + \theta_{d-u-3}q^{2u+1}(|U^\perp \cap T| - q^{2u} - x\theta_{d-2})$$

and, using $|T| = x\theta_{d-1}$ as well as $|U^\perp \cap T| = \theta_{d-1} + (x-1)\theta_{d-u-2}$ from Lemma 3.2.2, we find

$$\begin{aligned} m \leq\ & (x-1)(\theta_{d-1} - \theta_{d-u-2})x\theta_{d-2} \\ & + \theta_{d-u-3}q^{2u+1}(\theta_{d-1} + (x-1)\theta_{d-u-2} - q^{2u} - x\theta_{d-2}). \end{aligned} \tag{3.4}$$

Finally, we compare the lower bound α for m in Equation (3.2) and the upper bound β for m in Equation (3.4) and find a new bound of the form $0 \leq \beta - \alpha = f(x)$. Since β is quadratic in x with positive coefficient of x^2 and since α is linear in x, the polynomial f has degree two in x with positive leading coefficient. Since $0 \leq f(x)$ and $1 \leq x \leq q-1$, it follows that $0 \leq f(1)$ or $0 \leq f(q)$. We derive the desired contradiction by showing that this is not true. We have

$$\begin{aligned} f(1) &= \theta_{d-u-3}q^{2u+1}(\theta_{d-1} - q^{2u} - \theta_{d-2}) - (\theta_{d-1} - \theta_u)q^{2(d-1)} \\ &= \theta_{d-u-3}q^{2u+1}(q^{2(d-1)} - q^{2u}) - \theta_{d-u-2}q^{2(d+u)} \\ &\leq \theta_{d-u-3}q^{2u+1}(q^{2(d-1)} - q^{2u}) - \theta_{d-u-3}q^{2(d+u+1)} < 0 \end{aligned}$$

and straightforward calculations show $f(q)(q^2-1)^2 = A + B + C$ with

$$\begin{aligned} A &= -q^{4d-2u-5}(q^4 - 2q^2 + 1) - q^{2d-1}(q^3 - 3q^2 + 3q - 1), \\ B &= -q^{2d+2u-3}(2q^4 - q^3 + q^2 - 1) + q^{2d-2u-2}(q^2 - 2q + 1) \text{ and} \\ C &= -q^{4d-4}(q^3 - q^2 - q + 1) + q^{2d+2u+2} + q^{4u+1}(q^2 - 1). \end{aligned}$$

Of these A and B are obviously negative and since $u \leq d-2$ we have $C < 0$ except when $u = d-2$ and $q = 2$. However, $u = d-2$ may only occur if $d \geq 2$ and then we have $B + C < 0$. Hence, we have $f(q) < 0$ either way, concluding the proof. □

Theorem 3.2.10. *For every x-tight set T of $H(2d, q^2)$ with $x \leq q$ there is some $y \in \mathbb{N}_0$ with $y \leq x$ such that T is the disjoint union of y generators and an $(x-y)$-tight set T' such that T' does not contain a line of $H(2d, q^2)$. Furthermore, if $x - y \leq \frac{q+1}{2}$, then $x - y = 0$.*

Proof. The proof of the claim is by induction on x and the case $x = 0$ with $T = \emptyset$ is trivial. For the induction step we assume $x > 0$ and that x is such that the claim holds for all tight sets of $\mathcal{H}$ with parameter smaller than x. Let T be an x-tight set of $\mathcal{H}$ and let U be a subspace that is contained in T and maximal with respect to that property, that is, let U be such that any subspace U' that is contained in T has dimension at most $\dim(U)$. Since $x > 0$ we know that U is not the empty space.

If U is not a generator, then we know from $x \leq q$ and Lemma 3.2.9 that U must be a point and we have $x \geq \frac{q+2}{2}$. Hence, in this case the claim is satisfied for $y = 0$.

Now, assume that U is a generator. Then it is immediate from the definition of T, that $T \setminus U$ is a tight set of $\mathcal{H}$ and has parameter $x - 1$. From the induction hypothesis we know that there is some $y \in \mathbb{N}_0$ with $y \leq x - 1$ such that $T \setminus U$ is the disjoint union of y generators and an $(x - 1 - y)$-tight set T' which does not contain a line of $\mathcal{H}$. Furthermore, we also know that, if $(x - 1 - y) \leq \frac{q+1}{2}$, then $x - 1 - y = 0$. Hence, T is the disjoint union of $y + 1$ generators and an $(x - (y + 1))$-tight set T' that does not contain a line of $\mathcal{H}$ and, if $x - (y + 1) \leq \frac{q+1}{2}$, then $x - (y + 1) = 0$, as claimed. □

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