STABILITY OF ATTRACTIVITY REGIONS FOR AUTONOMOUS FUNCTIONAL DIFFERENTIAL EQUATIONS

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We consider attractivity regions of the zero solutions of autonomous retarded functional differential equations (FDEs). It is shown that the attractivity region remains unchanged if the FDE undergoes a small perturbation in certain classes of FDEs. Especially we obtain the openness of the set of parameters $\alpha \in (0,\pi/2)$ such that the attractivity region of the equation $\dot{x}(t) = -\alpha x(t-1)[1 + x(t)]$ is given by the set of continuous functions $\varphi: [-1,0] \to R$ with $\varphi(0) > -1$.

1. Introduction and main results.

The differential-delay equation (DDE)

$$\dot{x}(t) = -\alpha x(t-1)[1 + x(t)] \tag{1}$$

with $\alpha > 0$ has several applications, especially it has been used as a simple growth model in population biology (see [2] and the literature cited there).

Most of the present knowledge of the stability properties of the zero solution of eq. (1) is due to E.M.Wright [10]. He proved that the zero solution is unstable, if $\alpha > \pi/2$, and that it is asymptotically stable, if $\alpha < \pi/2$. (For definitions of (asymptotic) stability, see [2]). Moreover he obtained information about the size of the attractivity region $A(\alpha)$ of the zero solution of eq. (1), i.e. the set of all continuous initial functions $\varphi: [-1,0] \to R$ such that the unique continuous solution $x(\alpha,\varphi): [-1,\infty) \to R$ of the initial value problem (IVP) $\dot{x}(t) = -\alpha x(t-1)[1 + x(t)]$ for $t > 0$, $x|[-1,0] = \varphi$

tends to zero for $t \to \infty$. It is easy to show that $A(\alpha)$ is contained in the set $\widetilde{C_1} := \{\varphi \in C_1 \,|\, \varphi(0) > -1\}$ for all α in R^+. (For $r > 0$, C_r denotes the Banach space of continuous functions mapping the interval $[-r,0]$ into R, with sup-norm). Wright proved that $A(\alpha) = \widetilde{C_1}$ for $0 < \alpha \leqslant 37/24 < \pi/2$, and he suggested that this is still true for $\alpha \leqslant 1{,}567$.

The question arises if $A(\alpha) = \widetilde{C_1}$ for all $\alpha \in (0,\pi/2)$. An affirmative answer would exclude the possibility of non-trivial periodic solutions of eq. (1) for $\alpha < \pi/2$, hence give further insight into the bifurcation behaviour of periodic solutions near the bifurcation point $(\varphi,\alpha) = (0,\pi/2)$ in $C_1 \times R^+$ which has been investigated by R.D.Nussbaum [7].

To obtain a positive answer to the question posed one could try to show that the non-empty set $\{\alpha \in (0,\pi/2) \mid A(\alpha) = \widetilde{C_1}\}$ is open and closed in the interval $(0,\pi/2)$. In the present paper we prove the openness (Corollary 3 in section 3).

In other words: The attractivity region is stable under small perturbations of the DDE. Proving the assertion of Corollary 3 for the special equation (1) we noticed that the method of proof could be generalized to derive results of the same type for rather general classes of autonomous retarded FDEs. Let us indicate three of them.

First, an analogue of Corollary 3 holds for equation

$$(2) \qquad \dot{x}(t) = -\alpha \int_{-r}^{0} x(t+a)\,d\lambda(a)[1 + x(t)]$$

with $\alpha > 0$, $r > 0$, $\lambda: [-r,0] \to R$ increasing, $0 < \lambda(0) - \lambda(-r) \leqslant 1$. This generalization of eq. (1) is related to a population model of G.Dunkel[1] - compare [9].

Secondly, we consider the equations

$$(3) \qquad \dot{y}(t) = -f(y(t-1))$$

with continuous functions $f: R \to R$ having the properties

(H1) (f differentiable in 0 and $f(0) = 0$) and

(H2) ($yf(y) > 0$ for all $y \neq 0$ and $\inf f > -\infty$).

These equations were studied by R.D.Nussbaum[6] and by J.L. Kaplan and J.A.Yorke[4,5]. - Let $A(f)$ denote the attractivity region of the zero solution of eq. (3). For $w: (0,\delta_1] \to R^+$ with $\delta_1 > 0$, $\lim_{\delta \to 0} w(\delta) = 0$, we define f_w to be the set of all continuous functions $f: R \to R$ with (H1), (H2) and with the property

(H3) $(|y| < \delta \leqslant \delta_1 \Rightarrow |f(y) - f'(0)y| \leqslant w(\delta)|y|)$.

Then we have the following result on stability of global attractivity.

COROLLARY 2. *For every* $f \in f_w$ *with* $f'(0) \in (0, \pi/2)$ *and* $A(f) = C_1$, *there is an* $\varepsilon > 0$ *such that* $A(g) = C_1$ *for all* $g \in f_w$ *with* $|g'(0) - f'(0)| + \sup_R |g(y) - f(y)| \leqslant \varepsilon$.

Both results follow from our Theorem 3 (section 3) about the stability of global attractivity for the equations

$$(4) \qquad \dot{y}(t) = -\int_{-r}^{0} f(y(t+a))d\lambda(a)$$

with r, λ as before and $f \in f_w$. For every $f_1 \in f_w$ there is a " neighborhood " U in f_w and a compact set K in C_r such that for every continuous function $y: [-r, \infty) \to R$ solving eq. (4) for $t > 0$ with $f \in U$ there is a $t > 0$ with $y_t \in K$ (with $y_t(a) := y(t+a)$ for $t \geqslant 0$, $-r \leqslant a \leqslant 0$). - This property of the eqs. (4) permits one to reduce the proof of Theorem 3 to an application of Theorem 1 concerning only compact subsets of attractivity regions.

Theorem 1 is our most general result. Its statement requires additional definitions. - For $r > 0$, $n \in N$, consider an open set D in the Banach space C of continuous functions mapping the interval $[-r, 0]$ into R^n (with sup-norm). Assume $0 \in D$ and let w be given as above. We define F_w to be the set of all continuous mappings $F: D \to R^n$ with (H1) and

(H3') $(\varphi \in D \wedge |\varphi| < \delta \leqslant \delta_1 \Rightarrow |F(\varphi) - F'(0)(\varphi)| \leqslant w(\delta)|\varphi|)$

and

(H4) (For all $\varphi \in D$ there is a unique continuous solution $x: [-r, \infty) \to R^n$ of the IVP $x_0 = \varphi$,

$$(5) \qquad \dot{x}(t) = F(x_t) \quad \text{for } t > 0).$$

The solution x of the IVP is also designated by $x(F, \varphi)$. Setting $A(F) := \{\varphi \in D \mid \lim_{t \to \infty} x_t(F, \varphi) = 0\}$, we have

THEOREM 1. *Let* $F \in F_w$ *be given such that the zero solution of the linear equation* $\dot{x}(t) = F'(0)(x_t)$ *is asymptotically stable. Let* $K \subset A(F)$. *Suppose there is an open set* D' *with*

$K \subset D' \subset D$ and with $x_t(F,\varphi) \in D'$ for all $\varphi \in K$ and $t \geqslant 0$. Then there is an $\varepsilon > 0$ such that $K \subset A(G)$ for all $G \in F_W$ with $\|G'(0) - F'(0)\| + \sup_{D'}|G(\varphi) - F(\varphi)| \leqslant \varepsilon$.

$\| \ \|$ denotes the usual norm in the Banach space C' of linear continuous mappings from C into R^n.

2. Proof of Theorem 1. We need some lemmas and Theorem 2 below.

LEMMA 1. Let $L \in C'$, let $b > \max\{\operatorname{Re} z | z \in \mathbf{C}$ solution of the characteristic equation of the FDE $\dot{y}(t) = L(y_t)\}$. Then there are constants $k > 0$, $\eta > 0$ with $|x_t(L',\varphi)| \leqslant |\varphi|ke^{bt}$ for all $t \geqslant 0$, $\varphi \in C$ and $L' \in C'$ with $\|L - L'\| \leqslant \eta$.

For a definition of the characteristic equation, see [2]. Lemma 1 is a special case of Lemma 3 in [3].

THEOREM 2. Let $F \in F_W$ be given such that every solution of the characteristic equation of the FDE $\dot{y}(t) = F'(0)(y_t)$ has negative real part. Then there are positive constants η, R, k, b with

$(G \in F_W \wedge \|G'(0) - F'(0)\| \leqslant \eta \wedge |\varphi| \leqslant R \wedge t \geqslant 0) \Rightarrow |x_t(G,\varphi)| \leqslant ke^{-bt}$.

Remark: In [7], R.D.Nussbaum states a weaker version of Theorem 2 for a one-parameter-family of DDEs in his Lemma 2.7.

Proof of Theorem 2: One has to observe the constants in the estimates needed to prove a version of Theorem 18.3 [2]. Since J.K.Hale gives only an outline of the proof of that theorem, and since his hypotheses on the FDEs are different from ours, we shall indicate the essential steps of our proof. First, we need the following

LEMMA 2 (Variation of constants). Let $L \in C'$ and let a continuous function $h: R_0^+ \to R^n$ be given. Then there exists a mapping U_L, defined on the set $\{(t,s) | s \geqslant 0 \wedge t \geqslant s-r\}$ with values in the vector space M_n of real $n \times n$-matrices, such that U_L has the properties

i) $U_L(t,s) = 0$ for $t \in [s-r,s)$, $U_L(s,s) = I$ for $s \geqslant 0$,

ii) U_L continuous on $\{(t,s) | t \geqslant s \geqslant 0\}$,

iii) $t \geq s \geq 0 \Rightarrow$

$$U_L(t,s) = \int_s^t \int_{-r}^0 [d\lambda(a)] U_L(u+a,s)\,du + I$$

(where $L(\varphi) = \int_{-r}^0 [d\lambda(a)]\varphi(a)$ for all $\varphi \in C$, with $\lambda: [-r,0] \to M_n$ of bounded variation),

iv) $x(t) = x(L,\varphi)(t) + \int_0^t U_L(t,s)h(s)ds$ <u>on</u> R_0^+ <u>for the continuous solutions</u> $x: [-r,\infty) \to R^n$ <u>of the IVPs</u>

(6) $\dot{x}(t) = L(x_t) + h(t)$ for $t > 0$, $x_0 = \varphi \in C$.

Lemma 2 is proved in section 4. Now, by the hypotheses of Theorem 2 and by Lemma 1, there are constants b, k, η such that $|x_t(L,\varphi)| \leq ke^{-bt}|\varphi|$ for all $t \geq 0$, $\varphi \in C$ and $L \in C'$ with $\|L - F'(0)\| \leq \eta$. Proceeding as in the proof of Lemma 18.1[2] we infer the existence of a constant $k_1 \geq k$ with $n|U_{L,t}(\cdot,s)| \leq k_1 \exp(-b(t-s))|\varphi|$ for all $t \geq s \geq 0$, $\varphi \in C$, $L \in C'$ with $\|L - F'(0)\| \leq \eta$. (The matrix norm is defined in section 4). By assertion iv) of Lemma 2, we obtain

(7) $$|x_t(G,\varphi)| \leq k_1 e^{-bt}|\varphi| + \int_0^t k_1 e^{-b(t-s)}|G(x_s(G,\varphi)) - G'(0)(x_s(G,\varphi))|ds$$

for all $t \geq 0$, $\varphi \in D$ and $G \in F_W$ with $\|G'(0) - F'(0)\| \leq \eta$. By the definition of F_W, there is a $\delta > 0$ with $|G(\varphi) - G'(0)(\varphi)| \leq |\varphi|k_1/2b$ for all $G \in F_W$, $|\varphi| \leq \delta$. Now we can use Gronwall's lemma (compare e.g. Theorem VII, p. 215[8]) to derive the following estimate from (7): $|x_t(G,\varphi)| \leq \delta\exp(-bt/2)$ for all $t \geq 0$, all $\varphi \in D$ with $|\varphi| \leq \delta/2k_1$ and for all $G \in F_W$ with $\|G'(0) - F'(0)\| \leq \eta$. This proves Theorem 2.

<u>LEMMA 3</u>. <u>For</u> $t \to \infty$, $x_t(F,\varphi)$ <u>tends to zero uniformly on</u> K.

Proof: By the preceding theorem, there are constants R, k, b with

(8) $|\varphi| \leq R \wedge t \geq 0 \Rightarrow |x_t(F,\varphi)| \leq ke^{-bt}$.

By an easy consequence of Theorem 5.1[2] on continuous dependence (see Lemma 4 below) and by $K \subset A(F)$, we infer:
$\forall \varphi \in K\ \exists T_\varphi \in R\ \exists U_\varphi \subset D$ open: $\varphi \in U_\varphi \wedge (\psi \in U_\varphi \Rightarrow |x_{T_\varphi}(F,\psi)| \leq R)$.
We consider autonomous FDEs, so (8) yields

$t \geqslant T_\varphi \wedge \psi \in U_\varphi \;\Rightarrow\; |x_t(F,\psi)| \leqslant k\exp(-b(t-T_\varphi))$.
K is covered by a finite number of such neighborhoods, say $U_{\varphi_1},\dots,U_{\varphi_m}$. Setting $T := \max\{T_{\varphi_1},\dots, T_{\varphi_m}\}$, we infer:
$\psi \in K \Rightarrow \psi \in U_{\varphi_\mu}$ with $\mu \in \{1,\dots,m\}$, hence
$|x_t(F,\psi)| \leqslant k\exp(-b(t-T_{\varphi_\mu})) \leqslant k\exp(bT)\exp(-bt)$ for
$t \geqslant T \geqslant T_{\varphi_\mu}$ and $\psi \in U_{\varphi_\mu}$. This proves the lemma.

LEMMA 4. Let D be an open subset of C, consider subsets K and D' of D with $K \subset D'$ and D' open. Let M be a set of continuous mappings $G: D \to R^n$ such that for all $\varphi \in D$ there is exactly one continuous solution $x: [-r,\infty) \to R^n$ of the IVP $x_0 = \varphi$, $x(t) = G(x_t)$ for $t > 0$. Let $F \in M$. Suppose $x_t(F,\varphi) \in D'$ for all $\varphi \in K$, $t \geqslant 0$. Then we have:

$$\forall \varepsilon > 0\ \forall T > 0\ \forall \varphi \in K\ \exists \delta > 0:$$
$$\Big(G \in M \wedge \sup_{D'}|G(\psi) - F(\psi)| \leqslant \delta \wedge \psi \in K \wedge |\psi - \varphi| \leqslant \delta\Big) \Rightarrow \sup_{[-r,T]}|x(G,\psi)(t) - x(F,\varphi)(t)| \leqslant \varepsilon.$$

Proof: Suppose the lemma is false. We obtain sequences (G_m) - $m \in N$ - in M and (ψ_m) in K with $|\psi_m - \varphi| \to 0$, with $\sup_{D'}|G_m(\psi) - F(\psi)| \to 0$ and with

$$m \in N \;\Rightarrow\; \sup_{[-r,T]}|x(G_m,\psi_m)(t) - x(F,\varphi)(t)| > \varepsilon$$

for certain fixed $\varphi \in K$, $T > 0$, $\varepsilon > 0$. These sequences and F and φ have the following properties:

i) $G_m(\psi) \to F(\tilde\varphi)$ for $\psi \to \tilde\varphi$, $\psi \in D'$, $\tilde\varphi \in D'$ and $m \to \infty$.
Proof: We have the estimate
$$|G_m(\psi) - F(\tilde\varphi)| \leqslant \sup_{D'}|G_m(\tilde\psi) - F(\tilde\psi)| + |F(\psi) - F(\tilde\varphi)|.$$

ii) $\psi_m \to \varphi$.

We may assume $G_1 = F$. Then:

iii) Every compact set W in D' has an open neighborhood V in D' with $G_m(\psi)$ bounded on $N \times V$. Proof: $F(W)$ is compact and has a bounded neighborhood. Hence there is an open set V with $W \subset V \subset D'$ such that $F|V$ is bounded. For $m \in N$, $\psi \in V$, we have $|G_m(\psi)| \leqslant \sup_N \sup_{D'}|G_m(\tilde\psi) - F(\tilde\psi)| + \sup_V |F(\tilde\psi)| < \infty$. - Hence we can use the following version of Theorem 5.1 [2] on continuous dependence to derive a contradiction:

"Consider an open set D in C and continuous mappings $G_m: D \to R^n$ for $m \in N$. Assume that for all $\varphi \in D$ the IVP

$x_0 = \varphi$, $\dot{x}(t) = G_m(x_t)$ for $t > 0$, has a unique continuous solution defined on the interval $[-r,\infty)$. Let sets K and D' be given with D' open and $K \subset D' \subset D$. Assume $x_t(G_1,\varphi) \in D'$ for all $\varphi \in K$, $t \geqslant 0$. Suppose that $G_m(\psi) \to G_1(\varphi)$, if $m \to \infty$ and $\psi \in D'$, $\varphi \in D'$, $\psi \to \varphi$. Suppose further that every compact set $W \subset D'$ has a neighborhood V with $\sup\limits_{N} \sup\limits_{V} |G_m(\psi)|$ finite. Then we have $\sup\limits_{[-r,T]} |x(G_m,\varphi_m)(t) - x(G_1,\varphi)(t)| \to 0$ for every $T \geqslant 0$ and for every sequence (φ_m) in D' tending to $\varphi \in K$."

This theorem can be proved exactly as Theorem 5.1 [2].

Now we can prove Theorem 1: The hypotheses of Theorem 2 are fulfilled. If R is the constant guaranteed in Theorem 2, then Lemma 3 implies the existence of $T \geqslant 0$ with

i) $\varphi \in K \Rightarrow |x_T(F,\varphi)| \leqslant R/2$.

By Lemma 4, we have

ii) $\forall \varphi \in K\ \exists\, \delta_\varphi > 0 : (G \in F_W \wedge \sup\limits_{D'} |G(\psi) - F(\psi)| \leqslant \delta_\varphi \wedge$
$\psi \in K \wedge |\psi - \varphi| \leqslant \delta_\varphi) \Rightarrow$
$|x_T(G,\psi)| \leqslant |x_T(F,\varphi)| + R/2$.

K is covered by a finite number of balls $\{\varphi \in K|\ |\psi - \varphi| \leqslant \delta_\varphi\}$, given by $\varphi_1,..,\varphi_m$. Define $\varepsilon := \min(\{\delta_{\varphi_\mu} | \mu=1,..,m\} \cup \{\eta\})$, where η is given by Theorem 2. Now consider $\psi \in K$ and $G \in F_W$ with $\|G'(0) - F'(0)\| + \sup\limits_{D'} |G(\psi) - F(\psi)| \leqslant \varepsilon$. We infer $|x_T(G,\psi)| \leqslant |x_T(F,\varphi_\mu)| + R/2$ for a certain $\mu \in \{1,..,m\}$, by ii). By i), $|x_T(F,\varphi_\mu)| \leqslant R/2$. Hence $|x_T(G,\psi)| \leqslant R$. With $\|G'(0) - F'(0)\| \leqslant \varepsilon \leqslant \eta$, Theorem 2 implies $x_T(G,\psi) \in A(G)$, hence $\psi \in A(G)$, by the autonomy of the equation.

3. Applications of Theorem 1.

We start with the equations (4). First, we have

LEMMA 5. *For all $\varphi \in C_r$, there is a continuous solution $x : [-r,\infty) \to \mathbb{R}$ of the IVP $x_0 = \varphi$,*

$$\dot{x}(t) = -\int_{-r}^{0} f(x(t+a))\, d\lambda(a) \quad \textit{for } t > 0.$$

Proof: The existence of a solution on an interval $[-r,b)$, $b > 0$, follows from Theorem 3.1 [2] by the continuity of the mapping

$$F(f): C_r \ni \varphi \to - \int_{-r}^{0} f(\varphi(a))d\lambda(a) \in R.$$

$F(f)$ maps bounded subsets of C_r into bounded subsets of R. Therefore the assertion of Lemma 5 is a consequence of Theorem 4.2 [2], if we can show that every solution x, defined on any interval $[-r,b)$ with $b > 0$, is bounded. Let $t \in (0,b)$. Then

$$x(t) = x(0) + \int_0^t \dot{x}(u)du = x(0) + \int_0^t (- \int_{-r}^{0} f(x(u+a))d\lambda(a))du$$

$\leqslant |\varphi| - bd_f$, with $d_f := \inf f$, hence $-f(x(u+a)) \geqslant \min\{-|f \circ \varphi|, \inf(-f|[0, |\varphi| - bd_f])\}$ for u+a in $[-r,b)$, therefore $x(t) \geqslant x(0) + b \min\{\ldots\}$ on $(0,b)$.

<u>LEMMA 6. A solution</u> $x: [-r,\infty) \to R$ <u>of eq.</u> (4) <u>with the set of its zeros bounded tends to zero for</u> $t \to \infty$.

Proof: If $x > 0$ on $[t,\infty)$, then $\dot{x} < 0$ on $[t+r,\infty)$. $\lim_{t\to\infty} x(t)$ exists and is $\geqslant 0$. Eq. (4) implies the existence of $\lim_{t\to\infty} \dot{x}(t) \leqslant 0$. $\lim_{t\to\infty} \dot{x}(t) < 0$ leads to a contradiction. By $\lim_{t\to\infty} \dot{x}(t) = 0$, we obtain $0 = \int_{-r}^{0} f(\lim_{t\to\infty} x(t))d\lambda(a)$, hence the assertion.

Solutions with an unbounded set of zeros are called oscillating. - The assumptions on the functions $f \in f_w$ do not guarantee uniqueness for the IVP of eq. (4). We set $f_{w1} := \{f \in f_w |$ The continuous solution of the IVP of eq. (4) is unique$\}$. If $f \in f_{w1}$, the solution of the IVP of eq. (4) is denoted by $x(f,\varphi)$. $A(f)$ is the attractivity region of the zero solution of eq. (4).

We have the following result on stability of global attractivity:

<u>THEOREM 3. If</u> $f \in f_{w1}$, $A(f) = C_r$ <u>and if</u> $\sup\{\mathrm{Re}\, z | z + \int_{-r}^{0} f'(0)e^{za}d\lambda(a) = 0\}$ <u>is negative, then there are a compact interval</u> I_f <u>and a positive number</u> ε <u>such that</u> $A(g) = C_r$ <u>for all</u> $g \in f_{w1}$ <u>with</u> $|d_g - d_f| \leqslant 1$ <u>and</u> $|g'(0) - f'(0)| + \sup_{I_f} |g(x) - f(x)| \leqslant \varepsilon$.

Proof: We may assume $r = 1$.

i) If $|d_g - d_f| \leqslant 1$ and $|g(x) - f(x)| \leqslant 1$ on $[0, 1-d_f]$, then every oscillating solution $(x_t(g,\varphi))_{t\geqslant 0}$ reaches the set $B := \{\varphi \in C_1 \mid \varphi(0) = 0 \wedge \varphi(a) \in I_1$ for all $a \in [-1,0]\}$, where $I_1 := [-1 - \sup(f|[0,1-d_f]), 1 - d_f]$. - Proof: If $x := x(g,\varphi)$ has a local maximum in $m \geqslant 1$, then $x(z) = 0$ for a certain $z \in [m-1,m]$ and $x(m) = \int_z^m \dot{x}(t)dt =$

$$= \int_z^m (-\int_{-1}^0 g(x(t+a))d\lambda(a))dt \leqslant -d_g \leqslant 1 - d_f, \text{ hence } x \leqslant 1 - d_f$$

on $[m_1,\infty)$, if m_1 is the first maximum point in $(1,\infty)$. For every minimum point $\tilde{m} \in [m_1 + 2,\infty)$, we obtain with a certain $z \in [\tilde{m} - 1,\tilde{m}]$: $x(\tilde{m}) = \int_z^{\tilde{m}} (-\int_{-1}^0 g(x(t+a))d\lambda(a))dt \geqslant$

$\geqslant - \sup(g|[0,1-d_f]) \geqslant -1 - \sup(f|[0,1-d_f])$, therefore $x \geqslant -1 - \sup(f|[0,1-d_f])$ on $[\tilde{m}_1,\infty)$, if $\tilde{m}_1$ is the first minimum point in $(m_1 + 2,\infty)$.

ii) Define $U := \{g \in f_{W1} \mid |d_g - d_f| \leqslant 1 \wedge |g(x) - f(x)| \leqslant 1$ for $x \in I_1\}$. The set $K := \text{closure}\{x_1(g,\varphi) | g \in U, \varphi \in B\}$ is compact. - Proof: Let $\varphi \in B$, $g \in U$, $t \in [0,1]$. Set $x := x(g,\varphi)$. We proceed as in the proof of i) and obtain $x(t) \in I_1$ for $t \in [0,1]$, since $\varphi \in B$ and $\varphi(0) = 0$. Estimate of $\dot{x}$ on $[0,1]$: $\dot{x}(t) = -\int_{-1}^0 g(x(t+a))d\lambda(a) \in$

$\in [\inf(-g|I_1), \sup(-g|I_1)] \subset [1 - \sup(f|I_1), 1 - \inf(f|I_1)]$. Ascoli's theorem implies assertion ii).

iii) Lemma 6 and the fact that every oscillating solution $x_t(g,\varphi)_{t\geqslant 0}$ with $g \in U$ reaches the compact set K yield the implication $(g \in U \wedge K \subset A(g) \Rightarrow A(g) = C_1)$.

iv) Define $I_2 := [-1 - 2\sup(f|[0,1-2d_f]), 1 - 2d_f]$. Obviously, $I_1 \subset I_2$. We have $x(f,\varphi)(t) \in I_2$ for all $\varphi \in K$, $t \geqslant -1$. Proof: By the proof of ii), $x(f,\varphi)(t) \in I_1 \subset I_2$ for t in $[-1,0]$. Consider $t \in [0,1]$, $\varphi \in K$ and $x := x(f,\varphi)$. We have

$$x(t) = \varphi(0) + \int_0^t \dot{x}(u)du = \varphi(0) + \int_0^t (-\int_{-1}^0 f(x(u+a))d\lambda(a))du \leqslant$$

$\leqslant 1 - d_f - d_f$. $x \leqslant 1 - 2d_f$ on $[-1,0] \cup [0,1]$ implies

$$x(t) = \varphi(0) + \int_0^t (-\int_0^{-1} f(x(u+a))d\lambda(a))du \geqslant$$

$\geqslant -1 - \sup(f|[0,1-d_f]) - \sup(f|[0,1-2d_f]) \geqslant$

$\geqslant -1 - 2\sup(f|[0,1-2d_f])$ for all $t \in [0,1]$. Therefore $x(t) \in I_2$ for $t \in [0,1]$. -

Local maxima of x on $(1,\infty)$ are $\leqslant -d_f \leqslant 1 - 2d_f$, see ii).

Hence $x \leqslant 1 - 2d_f$ for $t \geqslant -1$, if x oscillates. Then the local minima of an oscillating solution x for $t > 1$ are greater than $-\sup(f|[0,1-2d_f]) \geqslant -1 - 2\sup(f|[0,1-2d_f])$, compare proof of i). Therefore $x \geqslant -1 - 2\sup(f|[0,1-2d_f])$ for $t \geqslant -1$, hence $x(t) \in I_2$ for $t \geqslant -1$.

If x does not oscillate there are two cases to be considered:

a) x without zeros on $(0,\infty)$. Then $|x|$ decreases for $t > 1$, and we obtain $x(t) \in I_2$ for $t \geqslant -1$.

b) There is a greatest zero $z \in R^+$ of x. Then $|x|$ decreases on $[z + 1,\infty)$, and we have to show that $x(t) \in I_2$ on $[-1,z + 1]$. This can be done as above by estimating the local extrema on $(1,z + 1]$. They are all in I_2. Since x has a greatest local extremum point m in $[z,z + 1]$ (given by $\inf\{t > z | \dot{x}(t) < 0\}$ in case $x|(z,\infty) > 0$) and since $|x|$ decays on $[m,\infty)$, the assertion follows.

v) Define $I_f := (\inf I_2 - 1, \sup I_2 + 1)$. We apply Theorem 1: Setting $D := C_1$, we have $F(g) \in F_W$ for all $g \in f_{W1}$ and $K \subset A(f) = A(F(f))$. The set $D' := \{\varphi \in C_1 | \varphi(t) \in I_f$ for all $t \in [-1,0]\}$ is open, and we have $x_t(F(f),\varphi) = x_t(f,\varphi) \in D'$ for all $\varphi \in K$ and $t \geqslant 0$. Now Theorem 1 implies the existence of an $\varepsilon > 0$ with $K \subset A(G)$ for all $G \in F_W$ with
$\|G'(0) - F(f)'(0)\| + \sup_{D'}|G(\psi) - F(f)(\psi)| \leqslant \varepsilon.$

Consider the functions $g \in f_{W1}$ with $|d_g - d_f| \leqslant 1$ and
$|g'(0) - f'(0)| + \sup_{I_f}|g(x) - f(x)| \leqslant \varepsilon \leqslant 1.$

We obtain $\|F(g)'(0) - F(f)'(0)\| + \sup_{D'}|F(g)(\psi) - F(f)(\psi)| =$
$\sup_{\|\psi\|=1}|\int_{-1}^{0}\psi(a)d\lambda(a)||g'(0) - f'(0)| + \sup\{|\int_{-1}^{0}(g\circ\psi(a) -$
$f\circ\psi(a))d\lambda(a)| \,|\, \psi(t) \in I_f$ for $t \in [-1,0]\} \leqslant$

$\leqslant |g'(0) - f'(0)| + \sup_{I_f}|g(x) - f(x)| \leqslant \varepsilon$, hence $K \subset A(F(g))$
$= A(g)$. Since $g \in U$, we conclude $C_1 \subset A(g)$ (see iii)) for these functions g, and Theorem 3 is proved.

We apply Theorem 3 to the equations (2), (3) and (1).

<u>LEMMA 7</u>. <u>For every</u> $\varphi \in C_r$ <u>with</u> $\varphi(0) \geqslant -1$ <u>there is a unique continuous function</u> $x(\alpha,\varphi): [-r,\infty) \to R$ <u>satisfying eq.</u> (2) <u>for</u> $t > 0$ <u>and</u> $x_0 = \varphi$. <u>We have</u>

i) $(\varphi(0) > -1 \Rightarrow x(\alpha,\varphi)|[0,\infty) > -1)$ and

ii) $(\varphi(0) \leqslant -1 \Rightarrow x(\alpha,\varphi)(t) \leqslant -1$ for all $t \geqslant 0$ in its maximal interval of definition).

Proof: The map $C_r \ni \varphi \to -\alpha\int_{-r}^{0}\varphi(a)d\lambda(a) \in R$ is Lipschitz-continuous on bounded sets, and it maps bounded sets into bounded sets. Theorem 5.2[2] implies that for every $\varphi \in C_r$ there is a unique continuous function $x(\alpha,\varphi):[-r,b) \to R$, with $0 < b \leqslant \infty$, satisfying eq. (2) on $(0,b)$ and $x_0(\alpha,\varphi) = \varphi$. Furthermore, if $\tilde{b} > 0$ is given and if $\tilde{x}:[-r,\tilde{b}) \to R$ is a solution of eq. (2) on $(0,\tilde{b})$ with initial value $\tilde{x}_0 = \varphi$, then it is unique.

i) $\varphi(0) = -1 \Rightarrow x(\alpha,\varphi) = -1$ for all $t \geqslant 0$ (obvious from uniqueness and from eq. (2)).

ii) Suppose $\varphi(0) > -1$ and $x(\alpha,\varphi):[-r,b) \to R$ noncontinuable with $b > 0$. Then $\varphi(0) > -1$ implies $x := x(\alpha,\varphi) > -1$ for some maximal interval $[0,\delta) \subset [0,b)$ and

$$\log(1 + x)^{\bullet}(t) = \dot{x}(t)/[1 + x(t)] = -\alpha\int_{-r}^{0}x(t+a)d\lambda(a) \text{ on } (0,\delta).$$

$\delta < b$ gives

$$(9) \qquad x(t) + 1 = (\varphi(0) + 1)\exp(-\alpha\int_{0}^{t}\int_{-r}^{0}x(u+a)d\lambda(a)du)$$

for all $t \in [0,\delta]$ (by a continuity argument), hence $x(\delta) + 1 > 0$, contradicting the maximality of $[0,\delta)$. We infer that (9) holds for all $t \in [0,b)$. Therefore x is bounded on $[-r,b)$. We can use Theorem 4.2[2] to reach a contradiction to $b < \infty$.

iii) If $x(\alpha,\varphi) = -1$ for some $t > 0$ in its maximal interval of definition, then there is a first $t' \geqslant 0$ with this property. We infer $x(\alpha,\varphi)|[t',\infty) = -1$, by i) and by the autonomy of eq. (2).

iv) The assertions i) and ii) of Lemma 7 are now obvious from parts i), ii) and iii) of our proof.

Lemma 7 implies that the attractivity region $\hat{A}(\alpha)$ of the zero solution of eq. (2) is contained in the set $\tilde{C}_r := \{\varphi \in C_r | \varphi(0) > -1\}$ for all positive α.

<u>COROLLARY 1</u>. <u>For every</u> $\alpha_1 > 0$ <u>with</u> $\hat{A}(\alpha_1) = \tilde{C}_r$ <u>and</u> $\max\{\mathrm{Re}\, z | z + \alpha\int_{-r}^{0}e^{za}d\lambda(a) = 0\} < 0$, <u>there is an</u> $\varepsilon > 0$ <u>with</u>

$(\alpha > 0 \wedge |\alpha - \alpha_1| \leqslant \varepsilon \Rightarrow \hat{A}(\alpha) = \tilde{C}_r)$.

Proof: By assertion i) of Lemma 7, we have only to show that there is an $\varepsilon > 0$ with $(\alpha > 0 \wedge |\alpha - \alpha_1| \leqslant \varepsilon \Rightarrow \hat{A}(\alpha) = \{\varphi \in C_r | \varphi > -1\})$. The transformation $S: y \to e^y - 1$ is a bijective mapping of the set of solutions of the equation

$$(4') \qquad \dot{y}(t) = -\int_{-r}^{0} \alpha(e^{y(t+a)} - 1)d\lambda(a)$$

onto the set of solutions of eq. (2) which are > -1 for $t \geqslant -r$, and we have $(\lim_{t\to\infty} y(t) \Longleftrightarrow \lim_{t\to\infty} S(y)(t) = 0)$. Set $f_\alpha(x) := \alpha(e^x - 1)$. We must find a number $\varepsilon > 0$ with $C_r = A(f_\alpha)$ for $|\alpha - \alpha_1| \leqslant \varepsilon$. We define f_w by $w(\delta) := (\alpha_1 + 1)(((e^\delta - 1)/\delta) - 1)$. Then $f_\alpha \in f_{w1}$, if $\alpha > 0$ and $|\alpha - \alpha_1| \leqslant 1$. By Theorem 3, there are an $\varepsilon > 0$ and a compact interval I (depending on α_1) with $A(g) = C_r$ for all $g \in f_{w1}$ with $|d_g - d_{f_{\alpha_1}}| \leqslant 1$ and

$|g'(0) - f_{\alpha_1}'(0)| + \sup_I |g(x) - f_{\alpha_1}(x)| \leqslant \varepsilon$. Now, $\alpha > 0$ and $|\alpha - \alpha_1| \leqslant \varepsilon \leqslant 1$ imply $f_\alpha \in f_{w1}$, $|d_{f_\alpha} - d_{f_{\alpha_1}}| = |\alpha - \alpha_1| \leqslant 1$, $|f_\alpha'(0) - f_{\alpha_1}'(0)| = |\alpha - \alpha_1| \leqslant \varepsilon$ and $\sup_I |f_\alpha(x) - f_{\alpha_1}(x)| \leqslant \varepsilon \cdot \sup_I |e^x - 1|$. Hence $A(f_\alpha) = C_r$, if $|\alpha - \alpha_1| \leqslant \varepsilon'$ for some $\varepsilon' > 0$ depending on α_1.

In the case $r = 1$, $\lambda(-1) = 0$, $\lambda|(-1,0] = 1$, eqs. (4) and (2) reduce to eqs. (3) and (1).

Proof of Corollary 2: We can apply Theorem 3, for the solutions y of the IVPs $y_0 = \varphi \in C_1$, $\dot{y}(t) = -f(y(t-1))$ on R^+, exist and are uniquely determined by φ for all $f \in f_w$, see e.g.[6], and $0 < f'(0) < \pi/2$ implies $\max\{\mathrm{Re}\, z |\, z + f'(0)e^{-z} = 0\} < 0$, see [10].

<u>COROLLARY 3</u>. <u>The</u> <u>set</u> $\{\alpha \in (0,\pi/2) | A(\alpha) = \tilde{C}_1\}$ <u>is</u> <u>open</u>.

Proof: Follows from Corollary 1 and from $\max\{\mathrm{Re}\, z | z + \alpha e^{-z} = 0\} < 0$ for $0 < \alpha < \pi/2$.

Let us point out that Corollary 3 can not be derived from Corollary 2 using the transformation S since smallness of $|\alpha - \alpha_1|$ does not imply smallness of $\sup_{\mathbb{R}} |f_\alpha(x) - f_{\alpha_1}(x)|$.

4. Proof of Lemma 2. Let M_n denote the vector space of real $n\times n$-matrices $A = (a_{ij})_{1\leqslant i,j\leqslant n}$ with norm $|A| = \max|a_{ij}|$. There is a mapping $\lambda = (\lambda_{ij})_{1\leqslant i,j\leqslant n}: [-r,0] \to M_n$ of finite variation $V(\lambda) = \sup\{|\lambda(a') - \lambda(a)|\,|a,a' \in [-r,0]\}$ with $L(\varphi) = (\Sigma_{j=1}^{n} \int_{-r}^{0} \varphi_j(a)d\lambda_{ij})_{1\leqslant i\leqslant n} =: \int_{-r}^{0}[d\lambda(a)]\varphi(a)$.

i) There is a unique mapping $U: [-r,\infty) \to M_n$ which is continuous for $t \geqslant 0$ and satisfies the IVP $U|[-r,0) = 0$,

(10) $$U(t) = \int_0^t \int_{-r}^{0} [d\lambda(a)]U(u+a)du + I \text{ for } t \geqslant 0,$$

where I is the unit matrix. - Proof: Let $t_1 > 0$ with $nt_1V(\lambda) < 1$ be given. Consider the Banach space X_1 of continuous mappings from the interval $[0,t_1]$ into M_n, with sup-norm. Set $\tilde{\Phi}|[0,t_1] := \Phi$, $\tilde{\Phi}|[-r,0) := 0$ for all $\Phi \in X_1$. The functions $(a,u) \to \tilde{\Phi}_{ij}(u+a)$ are bounded and semicontinuous on the product $[-r,0]\times[0,t]$ for $t \in [0,t_1]$, hence integrable relative to the product measures $d\lambda_{ij}\otimes \mu_t$ (μ_t the Lebesgue measure on $[0,t]$), and Fubini's theorem is applicable. Therefore the equation

$$K_1(\Phi)(t) = \int_0^t \int_{-r}^{0} [d\lambda(a)]\tilde{\Phi}(u+a)du + I \text{ for } t \in [0,t_1]$$

defines an operator $K_1: X_1 \to X_1$. K_1 has an unique fixed point U_1, for we have

(11) $$|K_1(\Phi) - K_1(\Psi)| \leqslant nt_1V(\lambda)|\Phi - \Psi| \text{ for all } \Phi,\Psi \in X_1.$$

$\tilde{U}_1$ solves (10) on $[0,t_1]$. - Now consider the Banach space X_2 of continuous mappings from $[t_1,2t_1]$ into M_n, with sup-norm. Set $\tilde{\Phi}|[-r,t_1) := \tilde{U}_1$, $\tilde{\Phi}|[t_1,2t_1] := \Phi$ for all $\Phi \in X_2$. We obtain an unique fixed point U_2 of the operator K_2 from X_2 into X_2 which is defined analogue to K_1. $\tilde{U}_2$ is a solution of eq. (10) on $[0,2t_1]$. - The existence of U now follows by an induction argument. Uniqueness is a consequence of the estimate (11).

ii) Setting $U_L(t,s) := U(t-s)$ for $s \geqslant 0$ and $t \geqslant s-r$, we obtain the assertions i), ii) and iii) of Lemma 2.

iii) Every continuous solution x of eq. (6) satisfies $x(t) = x(L,\varphi)(t) + x(0)(t)$ for $t \geqslant 0$, where

(12) $x(0)(t) = \int_0^t L(x_s(0))ds + \int_0^t h(s)ds$ for $t > 0$

and $x_0(0) = 0$. The solution of eq. (12) is unique in the set of continuous functions mapping $[-r,\infty)$ into R^n. Hence assertion iv) of Lemma 2 is proved if we can show that the function $\tilde{x}: [-r,\infty) \to R^n$ given by $\tilde{x}(t) = \int_0^t U_L(t,s)h(s)ds$ for $t > 0$ and by $\tilde{x}_0 = 0$ is continuous and solves eq. (12). The continuity follows from the continuity properties of U_L and h. Eq. (12) is verified by a calculation as in the proof of Theorem 16.3[2]: Assertion iii) of Lemma 2 yields

$$\tilde{x}(t) = \int_0^t \int_s^t \int_{-r}^0 [d\lambda(a)]U_L(u+a,s)du\ h(s)ds + \int_0^t h(s)ds$$

for $t \geqslant 0$. The first term on the right equals

$$\int_0^t \int_u^t \int_{-r}^0 [d\lambda(a)]U_L(s+a,u)h(u)ds\ du =: I.$$

$u \in [0,t]$, $s \in [0,u)$ and $a \in [-r,0]$ imply $s+a < u$, hence $U_L(s+\cdot,u) = 0$ on $[-r,0]$, and

$\int_{-r}^0 [d\lambda(a)]U_L(s+a,u)h(u) = 0$ for all $s \in [0,u)$. Therefore

$$I = \int_0^t \int_0^t \int_{-r}^0 [d\lambda(a)]U_L(s+a,u)h(u)ds\ du =$$

$$= \int_0^t \left(\int_{-r}^0 [d\lambda(a)]\left(\int_0^t U_L(s+a,u)h(u)du\right)\right)ds.$$ If $u > s+a \geqslant 0$, then $U_L(s+a,u) = 0$, and the inner integral can be replaced by $\int_0^{s+a} U_L(s+a,u)h(u)du = \tilde{x}(s+a)$. If $s+a < 0$, we have $\tilde{x}(s+a) = 0 = \int_0^t U_L(s+a,u)h(u)du$. Now eq. (12) is obvious.

Notation. The natural, real and complex numbers are denoted by N, R, $\mathbb{C}$ respectively. R^+ and R_0^+ stand for the positive and non-negative real numbers. A dot - as in $\dot{x}$ - or a prime - as in F' - indicates (Frechet-) differentiation.

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