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Dissertation

Construction of Martingales in Multi-Type Branching Processes

Ivana Tomic

Betreuer: Prof. Dr. Matthias Meiners

Für Mama, Papa und Leander

Zusammenfassung

Ein multi-typer allgemeiner (Crump-Mode-Jagers) Verzweigungsprozess ist ein mathematisches Modell zur Untersuchung dynamischer Systeme, wie etwa der Entwicklung von biologischen Populationen. Dieser stochastische Prozess liefert für eine feste Zahl $p \in \mathbb{N}$ einen Rahmen, um die Entwicklung von Populationen zu beschreiben, die p verschiedene Typen von Individuen umfassen und sich durch ihr eigenes Fortpflanzungsverhalten auszeichnen. Der Prozess startet zum Zeitpunkt $t = 0$ mit einem Individuum, dem Urahn, vom Typen $i \in \{1, ..., p\}$. Innerhalb seiner zufälligen Lebensspanne ζ^i, die Werte in $[0, \infty]$ annimmt, generiert er Nachkommen unterschiedlicher Typen $j \in \{1, ..., p\}$ (die erste Generation), deren Geburtszeiten durch einen Punktprozess $\xi^{(i,j)}$ auf $[0, \infty)$ vorgegeben sind. Wir fassen alle Punktprozesse $\xi^{(i,j)}$ in der Reproduktionsmatrix $\boldsymbol{\xi} = (\xi^{i,j})_{i,j=1,\dots,p}$ zusammen. Der Prozess setzt sich fort, indem jedes Individuum der ersten Generation ebenfalls Nachkommen verschiedener Typen generiert, die wiederum Nachkommen verschiedener Typen generieren, und so weiter.

Die vorliegende Arbeit ist motiviert durch die Ergebnisse von Alexander Iksanov, Konrad Kolesko und Matthias Meiners [Asymptotic fluctuations in supercritical Crump-Mode-Jagers processes. *Ann. Probab.* 52.4 (2024), pp. 1538–1606], die einen zentralen Grenzwertsatz für den allgemeinen Verzweigungsprozess im Fall $p = 1$ bewiesen haben. Die Grundlage für diesen Grenzwertsatz bildet eine asymptotische Entwicklung für den Erwartungswert des Prozesses sowie die Konstruktion komplexer Martingale, die in Zusammenhang mit Nermans Martingal stehen. Das Ziel dieser Arbeit ist es, diese Grundlagen auf multi-type allgemeine Verzweigungsprozesse, d. h. auf den Fall $p \in \mathbb{N}$, zu erweitern.

Wir beginnen mit der formellen Einführung von multi-typen allgemeinen Verzweigungsprozessen. Aus ihrer rekursiven Zerlegung lässt sich folgern, dass der Erwartungswert des Prozesses eine mehrdimensionale Markov-Erneuerungsgleichung löst. Daher verlagert sich unser Fokus auf die Herleitung der asymptotischen Entwicklung für allgemeine Lösungen F von multidimensionalen Markov-Erneuerungsgleichungen der Form $F(t) = f(t) + \boldsymbol{\mu} * F(t)$, wobei $f, F : \mathbb{R} \to \mathbb{R}^p$ vektorwertige Funktionen sind und $\boldsymbol{\mu} = (\mu^{i,j})_{i,j=1,\dots,p}$ eine $p \times p$ Matrix von lokal endlichen Maßen auf $[0, \infty)$. In diesem Zusammenhang diskutieren wir die Existenz und Eindeutigkeit von Lösungen F und untersuchen die charakteristische Gleichung $\det(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(\lambda)) = 0$, wobei I_p die $p \times p$ Identitätsmatrix bezeichnet und $\boldsymbol{\mathcal{L}}$ den Laplace-Operator, da das asymptotische Verhalten von F durch die komplexen Lösungen λ und deren Vielfachheiten $k_\lambda \in \mathbb{N}$ bestimmt wird. Damit leiten wir zuerst die asymptotische Entwicklung für das Erneuerungsmaß $\boldsymbol{U} := \sum_{n=0}^{\infty} \boldsymbol{\mu}^{*n}$ her und anschließend für allgemeine Lösungen F. Schließlich konstruieren wir für $q \in (1, 2]$ komplexe L^q-beschränkte Martingale $W_t(\lambda, k_\lambda)$ für den multi-typen allgemeinen Verzweigungsprozess, die mit Nermans Martingal in Zusammenhang stehen und mit den Lösungen λ der charakteristischen Gleichung sowie deren Vielfachheiten k_λ korrespondieren.

Summary

A multi-type general (Crump-Mode-Jagers) branching process is a mathematical model for studying various dynamical systems, such as the evolution of biological populations. For a fixed number $p \in \mathbb{N}$, this stochastic process provides a framework to understand the evolution of populations comprising p different types of individuals, each characterized by its own reproduction law. The process begins at time 0 with a single individual, the ancestor, who is of type $i \in \{1, ..., p\}$. Within its random lifetime ζ^i taking values in the interval $[0, \infty]$, the ancestor produces offspring of various types $j \in \{1, ..., p\}$, forming the first generation, and born at points of a reproduction point process $\xi^{(i,j)}$ on $[0, \infty)$. We collect all point processes in the reproduction matrix $\boldsymbol{\xi} = (\xi^{(i,j)})_{i,j=1,...,p}$. The process continues as each individual in the first generation produces as well offspring of various types, who in turn produce offspring of various types, and so on.

The present thesis is motivated by the work of Alexander Iksanov, Konrad Kolesko and Matthias Meiners [Asymptotic fluctuations in supercritical Crump-Mode-Jagers processes. *Ann. Probab.* 52.4 (2024), pp. 1538–1606], who established a central limit theorem for the general branching process in the case $p = 1$ (the single-type case). The foundation for this limit theorem lies in an asymptotic expansion for the mean of the process, along with the construction of complex martingales which are related to Nerman's martingale. The objective of this work is to extend this foundation in the context of multi-type general branching processes, i.e. in the case $p \in \mathbb{N}$. More specifically, we derive an asymptotic expansion for the mean of the multi-type general branching process in both, the lattice and non-lattice case, that is, when $\boldsymbol{\mu} = \mathbf{E}[\boldsymbol{\xi}]$ is concentrated on a lattice $h\mathbb{Z}$, for some $h > 0$ and when it is not concentrated on any lattice.

To achieve this, we begin by formally introducing the multi-type general branching process and its well-known recursive decomposition. From this, we infer that the mean of the process solves a multidimensional Markov renewal equation. Consequently, our focus shifts to deriving the asymptotic expansion for general solutions F to multidimensional Markov renewal equations of the form $F(t) = f(t) + \boldsymbol{\mu} * F(t)$ where $f, F : \mathbb{R} \to \mathbb{R}^p$ are vector-valued functions and $\boldsymbol{\mu} = (\mu^{i,j})_{i,j=1,...,p}$ is a $p \times p$ matrix of locally finite measures on the positive half-line $[0, \infty)$. We discuss the existence and uniqueness of F and proceed by investigating the characteristic equation $\det(\mathrm{I}_p - \boldsymbol{\mathcal{L}}\boldsymbol{\mu}(\lambda)) = 0$ where I_p denotes the $p \times p$ identity matrix and $\boldsymbol{\mathcal{L}}$ denotes the Laplace operator, since the asymptotic behavior of F is determined by its complex solutions λ and their multiplicities $k_\lambda \in \mathbb{N}$. We continue by establishing the asymptotic expansion for the renewal measure $\boldsymbol{U} := \sum_{n=0}^{\infty} \boldsymbol{\mu}^{*n}$, one of the prime examples of a solution to a Markov renewal equation, and subsequently, for general solutions F. Building on this, we construct complex martingales $W_t(\lambda, k_\lambda)$ for the multi-type general branching process, which are related to Nerman's martingale and correspond to the solutions λ of the characteristic equation and their multiplicities k_λ. Finally, we establish their convergence in L^q for $q \in (1, 2]$.

Contents

1 Introduction

Branching processes represent fundamental stochastic models used to describe various dynamical systems, such as the evolution of biological populations consisting of individuals like humans, plants or cells. The development of this interesting theory is based on demographic studies in the late 19th century by Francis Galton, who was concerned about the decay of the English peerage, and Henry W. Watson. In their work, they established the well-known Galton-Watson process, which forms the simplest possible model of a population evolving over time. It is represented as a discrete-time Markov chain $(Z_n)_{n\in\mathbb{N}_0}$ where $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ denotes the set of nonnegative integers, illustrating the population-size transitions across generations. In this framework, each individual of the population lives a unit length of time and gives independently birth to a random number $N \in \mathbb{N}_0$ of children according to the same probability distribution $\{p_k\}_{k\in\mathbb{N}_0}$ as the others. The investigation of this process naturally focuses on understanding the long-term behavior of $(Z_n)_{n\in\mathbb{N}_0}$. In this context, a comprehensive overview can be found in the introduction of [Jag75].

Building upon the Galton-Watson process, several generalizations of this model have been introduced to investigate more realistic models relevant to natural phenomena, which we generally refer to as branching processes. For instance, a natural extension is to allow the individuals to live a random lifetime leading to a continuous-time branching process $(Z_t)_{t\geq 0}$, called age-dependent branching process. In general, this process does not have the Markov property, unless the lifetimes are assumed to be independent and exponentially distributed. As one might expect, most of the theory and its associated techniques are applicable to the continuous-time case by embedding a Galton-Watson process, as discussed in [Har63] or [AN72].

During the 1960s, Kenny S. Crump, Charles J. Mode, and Peter Jagers established independently an interesting branching framework known as the general (Crump-Mode-Jagers) branching process $(\mathcal{Z}_t)_{t\geq 0}$. This model encompasses a wide range of branching frameworks, including the classical Galton-Watson process and the age-dependent branching process, and is the subject of investigation in the cited works, namely, [CM68] and [Jag75]. It describes a population, where $\mathcal{Z}_t$ represents the population size at time t, in which each individual can produce offspring at any point within their random lifetime, similar to observations in nature. Additionally, they introduced a random function φ, serving as a kind of measure for the population where each individual is assigned a measured value by φ. The corresponding size of these measured values at time t is denoted by $\mathcal{Z}_t^{\varphi}$ and the process $(\mathcal{Z}_t^{\varphi})_{t\geq 0}$ is referred to as general branching process counted with characteristic φ. In particular, $\mathcal{Z}_t^\varphi$ indeed generalizes $\mathcal{Z}_t$ by a special choice of φ. Within this framework, Olle Nerman established the law of large numbers, see [Ner79] and [Ner81], which states that, under some mild assumptions, $e^{-\alpha t}\mathcal{Z}_t^\varphi$ converges almost surely as $t \to \infty$ to cW. Here, $\alpha > 0$ is called Malthusian parameter, c is a constant and W is the limit of Nerman's martingale, which is positive on the survival event. Furthermore, under additional (second moment) assumptions, Alexander Iksanov, Konrad Kolesko and Matthias Meiners proved a central limit theorem for $(\mathcal{Z}_t^\varphi)_{t\geq 0}$, see [IKM24], which is based on an asymptotic expansion of the mean of a general branching process counted with a random characteristic and some complex martingales related to Nerman's martingale.

Besides the implementation of random lifetimes, another natural direction for generalization is to allow the individuals to belong to different types within the population. Therefore, we introduce a fixed number $p \in \mathbb{N} := \{1, 2, \ldots\}$ standing for a population comprising p different types of individuals, each characterized by its own reproduction law. Clearly, each of the mentioned examples above can be generalized by its corresponding multi-type model. In this context, the present thesis focuses on the multi-type (general) Crump-Mode-Jagers branching process $(\mathcal{Z}_t^\varphi)_{t\geq 0}$. The process begins at time 0 with a single individual, the ancestor, who is of type $i \in \{1, ..., p\}$ and possesses a random lifetime ζ^i, taking values in the interval $[0, \infty]$. Within its lifetime, the ancestor produces offspring of various types $j \in \{1, ..., p\}$, forming the first generation, and born at points of a reproduction point process $\xi^{(i,j)}$ on $[0, \infty)$. We collect all point processes in the $p \times p$ matrix $\boldsymbol{\xi} = (\xi^{(i,j)})_{i,j=1,...,p}$, referred to as reproduction matrix. The process continues now by each individual of the first generation producing as well offspring of various types, who in turn produce offspring of various types, and so on. Within this framework, the objective of the present work is to extend the results presented in [IKM24] to the multi-type case. More precisely, we aim to derive the asymptotic expansion for the mean of the multi-type (general) Crump-Mode-Jagers branching process and to construct some complex martingales related to Nerman's martingale, as these derivations form the foundational building blocks for establishing the central limit theorem.

Firstly, we formally introduce in Chapter 2 the model of multi-type (general) Crump-Mode-Jagers branching processes described above, as well as the notation and some basic facts relevant to this thesis. In particular, we discuss the concept and properties of random characteristics φ. We also present the well-known recursive decomposition of the process, which allows us to infer that the mean of the process solves a multidimensional Markov renewal equation. Given this connection, we temporarily step away from the context of multi-type branching processes and focus on the asymptotic expansion of general solutions F to multidimensional Markov renewal equations in Chapter 3. These equations take the form $F(t) = f(t) + \boldsymbol{\mu} * F(t)$ where $f, F : \mathbb{R} \to \mathbb{R}^p$ are vector-valued functions and $\boldsymbol{\mu} = (\mu^{i,j})_{i,j\in[p]}$ is a $p \times p$ matrix of locally finite measures on the positive half-line $[0, \infty)$. Although the asymptotic behavior of such equations has been previously studied, most notably by Mikhail S. Sgibnev in 2002 using Banach algebra techniques, see [Sgi02], our goal is to offer a more accessible and probabilistic approach that can be readily understood by researchers working in the field of probability theory. To this end, we begin by introducing the multidimensional Markov renewal equation and discussing the existence and uniqueness of general solutions F. We then focus on the characteristic equation $\det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\lambda)) = 0$, where I_p denotes the $p \times p$ identity matrix and $\mathcal{L}$ the Laplace operator, since the asymptotic behavior of F is determined by its complex solutions λ and their multiplicities $k_\lambda \in \mathbb{N}$. Notably, one solution of this equation is discussed in more detail, namely the Malthusian parameter, which determines the leading order term of the asymptotic expansion. After establishing the assumptions we are working with and relating them to those found in the literature, particularly to Sgibnev's assumptions, we formulate our main results, namely Theorem 3.16 and Theorem 3.20, and apply them to the mean of multi-type (general) Crump-Mode-Jagers branching processes. Finally, we turn our attention in Chapter 4 to the construction of some complex martingales $W_t(\lambda, k_\lambda)$ for the multi-type (general) Crump-Mode-Jagers branching process, which are closely related to Nerman's martingale. These martingales arise naturally from the asymptotic expansion of the mean and are associated with the solutions λ to the characteristic equation mentioned before and their multiplicities k_λ. We proceed by formulating $W_t(\lambda, k_\lambda)$ in terms of the multi-type (general) Crump-Mode-Jagers branching processes counted with some special characteristic ψ and verifying their martingale property, and close our analysis by establishing their L^q-convergence for $q \in (1, 2]$.

2 | Preliminaries

2.1 *Notation*

In the present section, we introduce the notation that will be employed consistently throughout the thesis. The objective is to establish a consistent framework that facilitates further reading and possible references. Moreover, we define and discuss the key objects relevant to this work. For ease of reference, the essential properties of these objects are detailed in the appendix, where the reader may consult them as necessary.

2.1.1 Sets and Topology

We denote by $\mathbb{N}$ the set of natural numbers $\{1, 2, 3, \ldots\}$, while the notation $\mathbb{N}_0$ refers to the set $\mathbb{N} \cup \{0\}$. The sets of integers, rational numbers, real numbers, and complex numbers are represented by $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$, respectively.

If $S \subseteq \Omega$ is a set, we write S^{c} for the complement of S in Ω, relative to the topology on Ω. The notation S^{o} refers to the interior of S, which is defined as the set of all points in S that have an open neighborhood entirely contained within S. In other words, the interior is the largest open subset of S in the given topology. The closure of S, denoted by $\overline{S}$, is the smallest closed set that contains S including all the limit points of S with respect to the topology. The boundary of S, denoted by ∂S, is given by $\overline{S} \setminus S^{\mathrm{o}}$.

For simplicity, we define $[\,n\,] := \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

2.1.2 Vectors and Matrices

By default, vectors in this paper are row vectors, so $x \in \mathbb{R}^p$ means that x is of the form $x = (x_1, \ldots, x_p)$. To emphasize this, we may write $x \in \mathbb{R}^{1\times p}$. Column vectors are transposes of row vectors, so we write x^{T} for the associated column vector. The space of p-dimensional real column vectors is sometimes denoted by $\mathbb{R}^{p\times 1}$. Here, we write $\mathsf{e}_1, \ldots, \mathsf{e}_p$ for the canonical basis of $\mathbb{R}^{1\times p}$. We use the standard Euclidean norm $|\cdot|$ and the ℓ_1-norm $|\cdot|_1$ on both spaces $\mathbb{R}^{1\times p}$ and $\mathbb{R}^{p\times 1}$, given by $|x| = |x^{\mathsf{T}}| = \sqrt{x_1^2 + \ldots + x_p^2}$ and $|x|_1 = |x^{\mathsf{T}}|_1 = |x_1| + \ldots + |x_p|$, respectively.

If M is a real $p \times n$ matrix, we write $M \in \mathbb{R}^{p\times n}$ and say that $M = (M_{j,k})_{j\in[\,p\,],k\in[\,n\,]}$ is positive if all of its entries are positive, i.e. $M_{j,k} > 0$ for all $j \in [\,p\,]$, and $k \in [\,n\,]$. Similarly, M is nonnegative if $M_{j,k} \geq 0$ for all $j \in [\,p\,]$, and $k \in [\,n\,]$. We denote this by $M > 0$ and $M \geq 0$. With this definition, we can write for two matrices $M_1, M_2 \in \mathbb{R}^{p\times p}$, $M_1 < M_2$ or $M_1 \leq M_2$ meaning that $M_2 - M_1$ is positive or nonnegative, respectively. Further, if $M \in \mathbb{R}^{p\times p}$ is nonnegative, then M is called primitive if $M^n > 0$ for some nonnegative integer $n \geq 1$.

For $M \in \mathbb{R}^{p\times n}$ we denote by $\|M\|_{\mathbb{R}}$ its matrix norm, i.e.,

$$\|M\|_{\mathbb{R}} = \sup_{x\in\mathbb{R}^{n\times 1}, |x|=1} |Mx|.$$

Similarly, if M is a complex $p \times n$ matrix, we write $\|M\|_{\mathbb{C}}$ for its complex matrix norm, i.e., $\|M\|_{\mathbb{C}} = \sup_{z\in\mathbb{C}^n, |z|=1} |Mz|$. In this context, if we consider the matrix $M = (M_{j,k})_{j\in[p], k\in[n]}$ and notice that if $M^* = (M^*_{j,k})_{j\in[p], k\in[n]}$ is a nonnegative $p \times n$ matrix with $|M_{j,k}| \leq M^*_{j,k}$ componentwise for all $j \in [p]$ and $k \in [n]$, then

$$\begin{aligned}\|M\|_{\mathbb{C}} &= \sup_{\substack{z\in\mathbb{C}^n,\\ |z|=1}} |Mz| = \sup_{\substack{z\in\mathbb{C}^n,\\ |z|=1}} \sqrt{\sum_{j=1}^{p} \left| \sum_{k=1}^{n} M_{j,k} z_k \right|^2} \leq \sup_{\substack{z\in\mathbb{C}^n,\\ |z|=1}} \sqrt{\sum_{j=1}^{p} \left(\sum_{k=1}^{n} |M_{j,k}||z_k| \right)^2} \\ &\leq \sup_{\substack{z\in\mathbb{C}^n,\\ |z|=1}} \sqrt{\sum_{j=1}^{p} \left(\sum_{k=1}^{n} M^*_{j,k} |z_k| \right)^2} = \sup_{\substack{x\in\mathbb{R}^n,\\ |x|=1}} \sqrt{\sum_{j=1}^{p} \left(\sum_{k=1}^{n} |M^*_{j,k}||x_k| \right)^2} = \|M^*\|_{\mathbb{R}}.\end{aligned}$$

2.1.3 Spectrum and Spectral Radius

Let M be a real or complex $p \times p$ matrix. An eigenvalue of M is a scalar λ such that there exists a non-zero vector x, called an eigenvector, satisfying the eigenvalue equation $Mx = \lambda x$. Equivalently, one can rearrange the eigenvalue equation into $(M - \lambda \mathrm{I}_p)x = 0$ with I_p denoting the $p \times p$ identity matrix containing on its main diagonal elements value one while the rest of the matrix elements are equal to zero. We define the spectrum σ_M as

$$\sigma_M := \sigma(M) := \{\lambda : \lambda \in \mathbb{C} \text{ is an eigenvalue of } M\}$$

which is a non-empty set containing all eigenvalues λ of M. In particular, the eigenvalues λ of M are the roots of the characteristic polynomial represented as $p_M(\lambda) := \det(M - \lambda \mathrm{I}_p)$. The multiplicity of an eigenvalue λ of M is its multiplicity as a root of the characteristic polynomial $p_M(\lambda)$ and is denoted by $k_\lambda \leq p$. Consequently, we call λ a simple root of $p_M(\lambda)$ if it is a root of multiplicity one, i.e. $p_M(\lambda) = 0$ and $p^{(1)}_M(\lambda) \neq 0$ where $p^{(1)}_M(\lambda)$ denotes the first derivative of $p_M(\lambda)$. Further, we define the spectral radius as

$$\rho_M := \rho(M) := \sup\{|\lambda| : \lambda \in \sigma_M\}$$

which represents the largest modulus eigenvalue.

2.1.4 Kronecker Product

Consider a $m \times n$ matrix A and a $k \times l$ matrix B. Following [Gra18, Chapter 2.2], the Kronecker product of the matrices A and B, denoted by $A \otimes B$, is defined as the $mk \times nl$ matrix

$$A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

In other words, the Kronecker product expands the dimensions of both matrices by replicating each element of A with the entirety of B, resulting in a block matrix.

2.1.5 Laplace Transforms

If μ is a locally finite measure on the Borel sets of $[0,\infty)$, we write $\mathcal{L}\mu$ for the associated Laplace transform, that is, the function

$$\mathcal{L}\mu : [0,\infty) \to [0,\infty], \quad \theta \mapsto \int_0^\infty e^{-\theta x}\, \mu(\mathrm{d}x)$$

where the integral is taken over the interval $(0,\infty)$. If $\mathcal{L}\mu(\theta) < \infty$ for some $\theta \geq 0$, then, by monotonicity, $\mathcal{L}\mu$ is finite on $[\theta,\infty)$. We canonically extend $\mathcal{L}\mu$ to $\mathcal{D}(\mathcal{L}\mu) = \{z \in \mathbb{C} : \mathcal{L}\mu(\mathrm{Re}(z)) < \infty\}$, where $\mathrm{Re}(z)$ denotes the real part of z and $\mathrm{Im}(z)$ the imaginary part. In particular, if $\lambda \in \mathcal{D}(\mathcal{L}\mu)$ with $\mathrm{Re}(\lambda) = \theta$, then $\|\mathcal{L}\mu(\lambda)\|_{\mathbb{C}} \leq \|\mathcal{L}\mu(\theta)\|_{\mathbb{R}}$. This notation naturally extends to vectors and matrices of measures yielding a vector-valued or matrix-valued Laplace transform, respectively. The domain of the vector- or matrix-valued Laplace transform is then defined as the intersection of the domains of its components. For instance, for the $p \times p$ - matrix of locally finite measures $\boldsymbol{\mu} = (\mu^{i,j})_{i,j\in[p]}$ on $[0,\infty)$ and its Laplace transform $\mathcal{L}\boldsymbol{\mu}(\theta) = (\mathcal{L}\mu^{i,j}(\theta))_{i,j\in[p]}$,

$$\mathcal{D}(\mathcal{L}\boldsymbol{\mu}) = \bigcap_{i,j\in[p]} \mathcal{D}(\mathcal{L}\mu^{i,j}) = \left\{ z \in \mathbb{C} : \int_0^\infty e^{-\mathrm{Re}(z)x}\, \mu^{i,j}(\mathrm{d}x) < \infty \ \forall i,j \in [p] \right\}.$$

If $f : [0,\infty) \to \mathbb{C}$ is a function, we define the Laplace transform of f at $z \in \mathbb{C}$ by

$$\mathcal{L}f(z) := \int\limits_0^\infty f(x)\, e^{-zx}\, \mathrm{d}x$$

whenever the integral converges absolutely. We write

$$\mathcal{D}(\mathcal{L}f) := \left\{ z \in \mathbb{C} : \int_0^\infty |f(x)|\, |e^{-zx}|\, \mathrm{d}x = \int_0^\infty |f(x)|\, e^{-\mathrm{Re}(z)x}\, \mathrm{d}x < \infty \right\}$$

for its domain. Notice that $\mathcal{D}(\mathcal{L}f) = I + \mathrm{i}\mathbb{R} = \{\theta + \mathrm{i}\eta \in \mathbb{C} : \theta \in I,\, \eta \in \mathbb{R}\}$ for some half-line $I \subset \mathbb{R}$.

2.1.6 Convolutions

Let $f : [0,\infty) \to \mathbb{C}$ be a function and μ a measure on $[0,\infty)$. We write

$$f * \mu(t) := \int_0^\infty f(t-x)\,\mu(\mathrm{d}x), \quad t \in \mathbb{R}$$

for the convolution of f with μ if the integral converges absolutely. If g is another measurable function, we set

$$f * g(t) := \int_0^\infty f(t-x)g(x)\,\mathrm{d}x, \quad t \in \mathbb{R}$$

whenever the integral converges absolutely.

Similarly, if $\boldsymbol{\tau} = (\tau^{i,j})_{i\in[p],j\in[m]}$ is a $p \times m$ matrix of measures and $A : [0,\infty) \to \mathbb{C}^{m\times n}$, $t \mapsto (a_{jk}(t))_{j\in[m],k\in[n]}$ is a $m \times n$ matrix with functions as entries, then we set

$$\boldsymbol{\tau} * A(t) := \left(\sum_{j=1}^{m} \tau^{i,j} * a_{jk}(t)\right)_{i\in[p],k\in[n]},$$

which is a $p \times n$ matrix of functions. Analogously, if $B = (b_{kl})_{k\in[n],l\in[p]}$ is a $n \times p$ matrix of measurable functions $b_{kl} : [0,\infty) \to \mathbb{C}$, we set

$$A * B(t) := \left(\sum_{k=1}^{n} a_{jk} * b_{kl}(t)\right)_{j\in[m],l\in[p]}.$$

2.1.7 Exponential Matrices

Similar to [IKM24, Equation (4.10)], we use a multiplicative family of matrices $\exp(\gamma, s, k)$, for $\gamma \in \mathbb{C}$, $s \in \mathbb{R}$ and $k \in \mathbb{N}$, namely, we define the following upper triangular $k \times k$ matrix

$$\exp(\gamma,s,k) \;= (\exp_{ij}(\gamma,s,k))_{i,j\in[k]} := e^{\gamma s}\begin{pmatrix} 1 & s & \frac{s^2}{2!} & \cdots & \frac{s^{k-1}}{(k-1)!} \\ 0 & 1 & s & \cdots & \frac{s^{k-2}}{(k-2)!} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & s \\ 0 & \cdots & & 0 & 1 \end{pmatrix}. \tag{2.1}$$

The (i,j)-th entry of the matrix is $\exp_{ij}(\gamma,s,k) = e^{\gamma s}\frac{1}{(j-i)!}s^{j-i}$, $1 \le i \le j \le k$ and $\exp_{ij}(\gamma,s,k) = 0$ for $i > j$. Matrices of this form will be very useful since they simplify the notation and allow us to deal with polynomial terms with relative ease. In this context, we will write $\mathsf{f}_1,\dots,\mathsf{f}_k$ for the canonical basis of $\mathbb{R}^{1\times k}$. For any $s,t \in \mathbb{R}$ and $\gamma \in \mathbb{C}$, we have

$$\exp(\gamma,s,k)\exp(\gamma,t,k) = \exp(\gamma,s+t,k).$$

To see this, the reader may notice that, since both matrices $\exp(\gamma, s, k)$ and $\exp(\gamma, t, k)$ are upper triangular matrices with only ones on the diagonal, so is their product. Now consider $1 \leq i < j \leq k$ and notice that the (i, j)-th entry of the matrix $\exp(\gamma, s, k) \exp(\gamma, t, k)$ equals

$$\begin{aligned}
\mathsf{f}_i \exp(\gamma, s, k) \exp(\gamma, t, k) \mathsf{f}_j^{\mathsf{T}} &= \sum_{l=i}^{j} e^{\gamma s} \frac{s^{l-i}}{(l-i)!} e^{\gamma t} \frac{t^{j-l}}{(j-l)!} \\
&= e^{\gamma(s+t)} \sum_{l=0}^{j-i} \frac{s^l}{l!} \frac{t^{j-i-l}}{(j-i-l)!} \\
&= e^{\gamma(s+t)} \frac{1}{(j-i)!} \sum_{l=0}^{j-i} \binom{j-i}{l} s^l t^{j-i-l} \\
&= e^{\gamma(s+t)} \frac{1}{(j-i)!} (s+t)^{j-i} \\
&= \exp_{ij}(\gamma, s+t, k).
\end{aligned}$$

With $\|\cdot\|$ denoting the operator norm and $\|\cdot\|_{\mathsf{HS}}$ denoting the Hilbert-Schmidt norm, the following (rough) estimate holds for every $\delta > 0$:

$$\|\exp(\gamma, s, k)\| \leq \|\exp(\gamma, s, k)\|_{\mathsf{HS}} \leq C'(1+|s|)^{k-1} e^{\mathrm{Re}(\gamma)s} \leq C e^{\mathrm{Re}(\gamma)s+\delta|s|} \tag{2.2}$$

for some constant $C' > 0$ depending on k only and another constant $C > 0$ depending on k and $\delta > 0$.

2.2 *The Multi-type General Branching Process*

In this section, we formally introduce the multi-type general (Crump-Mode-Jagers) branching process. The process starts with a single individual, the ancestor, born at time 0. The ancestor gets the label $\varnothing$, the empty tuple, and, like every individual in the process, has one of finitely many types $\tau(\varnothing) \in [p]$ where $p \in \mathbb{N}$ is the number of different types. The ancestor has a random lifetime $\zeta^{\tau(\varnothing)}$ which may depend on its type and takes values in $[0, \infty]$. Within its lifetime, it generates offspring of various types $j \in [p]$. If $\tau(\varnothing) = i$, i.e., if the type of the ancestor is i, then its children of type j are born at the times of the point process $\xi^{i,j}$, $j \in [p]$. Here, a point process is a random locally finite point measure, see [Res92, Chapter 3]. We write

$$\xi^{i,j} = \sum_{k=1}^{N^{i,j}} \delta_{X_k^{i,j}}$$

where here and in what follows, δ_x denotes the Dirac measure at the point $x \in \mathbb{R}$. We assume that $\xi^{i,j}$ is concentrated on $[0, \infty)$ and thus $N^{i,j} = \xi^{i,j}([0, \infty))$ is its total mass. If the ancestor's type is i, the random variables $N^{i,j}$ represent the total number of children of type j and take values in $\mathbb{N}_0 \cup \{\infty\}$. In particular, we allow that $\mathbf{P}(N^{i,j} = \infty) > 0$ meaning that $\varnothing$ could generate an infinite number of descendants of a given type. Without loss of generality, we assume that $0 \leq X_1^{i,j} \leq X_2^{i,j} \leq \dots$ holds. We write $\boldsymbol{\xi} = (\xi^{i,j})_{i,j\in[p]}$ for the $p \times p$ matrix of point processes $\xi^{i,j}$, $i, j \in [p]$ that determine the ancestor's reproduction. The $\xi^{i,j}$ are also called offspring processes and $\boldsymbol{\xi}$ offspring process matrix. Further, we write

$$\xi^i = \sum_{j=1}^{p} \xi^{i,j}\mathsf{e}_j = (\xi^{i,1}, \dots, \xi^{i,p})$$

for the vector containing all offspring processes of the ancestor if the latter has type i. Similarly, we define $N^i := \sum_{j=1}^{p} N^{i,j}\mathsf{e}_j$. With this notation, we define

$$\xi := |\xi^{\tau(\varnothing)}|_1 = \sum_{j=1}^{p} \xi^{\tau(\varnothing),j} = \sum_{k=1}^{N} \delta_{X_k}$$

where $N := |N^{\tau(\varnothing)}|_1 = \xi([0,\infty))$ is the total number of offspring of the ancestor. Accordingly, the process ξ gives the birth times of all children of the ancestor. Without loss of generality, we assume that $0 \leq X_1 \leq X_2 \leq \dots$ holds, in which case we can define all birth times as $X_k = \inf\{t \geq 0 : \xi([0,t]) \geq k\}$, $k \in \mathbb{N}$. Notice that all birth times are random variables taking values in $[0,\infty]$ satisfying $X_k \to \infty$ as $k \to \infty$. The event $\{X_k = \infty\}$ means that the k-th child is never born, so the ancestor has strictly less than k children. The multi-type general branching process is now obtained in that each child of the ancestor produce children according to independent copies of the point process $\boldsymbol{\xi}$ who in turn reproduce children according to further independent copies of $\boldsymbol{\xi}$, and so on.

To formalize this, we introduce the usual Ulam-Harris notation. Let $\mathbb{N}^0 := \{\varnothing\}$ be the set that contains only the empty tuple and $\mathbb{N}$ the set of natural numbers. Then we write $\mathcal{I} := \bigcup_{n\in\mathbb{N}_0} \mathbb{N}^n$ for the infinite Ulam-Harris tree comprising all finite tuples of natural numbers.

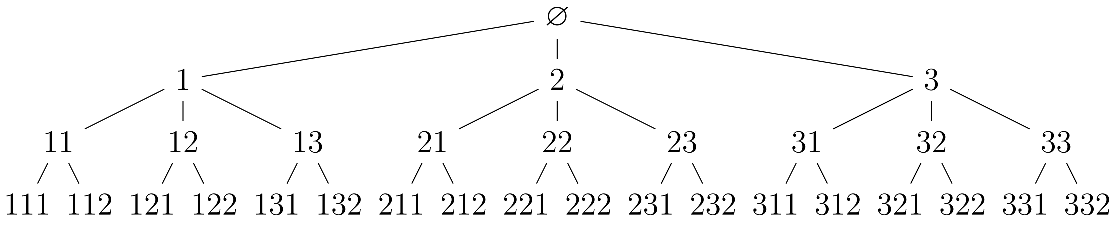

Figure 2.1: A part of the Ulam-Harris tree.

Now the individuals of the process can be embedded in the Ulam-Harris tree as follows: The ancestor corresponds to the element $\mathbb{N}^0 := \{\varnothing\}$, its children are elements in $\mathbb{N}^1$, its grandchildren are elements in $\mathbb{N}^2$, and so on. In this context, the Ulam-Harris tree can be seen as the set of all potential individuals. Each individual $u \in \mathcal{I}$ may be labelled by strings in the Ulam-Harris tree. This strings correspond to the ancestral line which is represented as

$$\varnothing \to u_1 \to u_1u_2 \to \dots \to u_1 \dots u_n = u,$$

where u_1 is the u_1-th child of the ancestor, u_1u_2 is the u_2-th child of u_1, and so on. In this situation, we write $|u| = n$, interpreting n as the generation of the individual $u \in \mathcal{I}$. If, additionally, $v = v_1 \dots v_m \in \mathcal{I}$, then we write uv for $u_1 \dots u_n v_1 \dots v_m$. For any $u \in \mathcal{I}$ and $i \in \mathbb{N}$, ui is the i-th child of u and we refer to u as the mother of ui. More generally, we write $u \preceq v$ if $uw = v$ holds for some $w \in \mathcal{I}$, and call u an ancestor of v. Conversely, in this case, v is a descendant of u. Furthermore, we write $u \prec v$ if $u \preceq v$, but $u \neq v$. We use $u|_k$ to denote, for $u = u_1 ... u_n$ and $k \leq n$, the ancestor of u in generation k. Formally, $u|_k$ arises by restricting u to its first k components, i.e.:

$$u|_k = \begin{cases} \varnothing & \text{if } k = 0, \\ u_1 ... u_k & \text{if } 1 \leq k \leq |u|, \\ u & \text{if } k > |u|. \end{cases}$$

Furthermore, it is possible to make shifts in the infinite Ulam-Harris tree and consider the trees rooted at a specific individual $u \in \mathcal{I}$. For instance, let $\boldsymbol{\xi}_u = (\xi_u^{i,j})_{i,j \in [p]}$ and $\boldsymbol{\zeta}_u = (\zeta_u^i)_{i \in [p]}$, $u \in \mathcal{I} \setminus \{\varnothing\}$ be families of i. i. d. copies of $\boldsymbol{\xi} =: \boldsymbol{\xi}_\varnothing$ and $\boldsymbol{\zeta} =: \boldsymbol{\zeta}_\varnothing$. If individual u is of type i, then the offspring process

$$\xi_u^{i,j} = \sum_{k=1}^{N_u^{i,j}} \delta_{X_{u,k}^{i,j}},$$

$i, j \in [p]$ determines the birth times of children of type j of u and, correspondingly, $N_u^{i,j}$ denotes the total number of children of type j of u. Therefore, the vector

$$\xi_u^i = \sum_{j=1}^{p} \xi_u^{i,j} \mathsf{e}_j$$

contains all offspring processes of individual u. With this notation we formally set

$$X_{u,k}^{i,j} := \inf\{t \geq 0 : \xi_u^{i,j}([0,t]) \geq k\}$$

where $\inf \varnothing := \infty$ is stipulated. Thus, $X_{u,k}^{i,j} = \infty$ means that if individual u has type i, then it has strictly less than k children of type j, respectively.

We suppose that $(\Omega, \mathcal{A}, \mathbf{P})$ is a probability space on which all $\boldsymbol{\xi}_u$, $u \in \mathcal{I}$ are defined and i. i. d. and independent of $\tau(\varnothing)$, the random variable that gives the ancestor's type. We write $\mathbf{P}^i$ if the ancestor's type is $i \in [p]$ and denote the associated expected value operators by $\mathbf{E}$ and $\mathbf{E}^i$, respectively.

In this process, the generation 0 is given by $\mathcal{G}_0 := \{\varnothing\}$. Further, the children of the ancestor form the first generation $\mathcal{G}_1$ of the process, which is given by

$$\mathcal{G}_1 := \{k \in \mathbb{N} : X_k < \infty\} \subseteq \mathbb{N}.$$

Each individual $k \in \mathcal{G}_1$ has a unique type from the type space $[p]$ that we denote by $\tau(k)$. Now, recursively, suppose that we have constructed $\mathcal{G}_n$, the n-th generation of the process and that $u \in \mathcal{G}_n$. Then u has a unique type $\tau(u) \in [p]$ and we set

$$\xi_u := |\xi_u^{\tau(u)}|_1 = \sum_{k=1}^{N_u} \delta_{X_{u,k}}$$

where $X_{u,k} = \inf\{t \geq 0 : \xi_u([0,t]) \geq k\}$, $k \in \mathbb{N}$ and $N_u = \xi_u([0,\infty))$. We then define

$$\mathcal{G}_{n+1} := \{uk \in \mathcal{I} : u \in \mathcal{G}_n \text{ and } X_{u,k} < \infty\}.$$

The set of all individuals that are ever born is the family tree $\mathcal{T} := \bigcup_{n\in\mathbb{N}_0} \mathcal{G}_n$. Each individual $u \in \mathcal{T}$ has a clearly defined type $\tau(u) \in [\,p\,]$ and a time of birth $S(u)$, namely,

$$S(u) = \sum_{k=1}^{|u|} X_{u|_{k-1},u_k}$$

where $u = u_1 \dots u_{|u|}$. Notice that $S(u) = \infty$ means $u \notin \mathcal{T}$. If we define $\zeta_u := \zeta_u^{\tau(u)}$ as the lifetime of u, then u is alive at time t if $S(u) \leq t < S(u) + \zeta_u$ is fulfilled, i.e. if u is born, but not yet dead at time t. Provided that $S(u) \leq t$, the age of u at time t is $t - S(u)$. In particular, the children of u are born at the times $S(u1) = S(u) + X_{u,1}, S(u2) = S(u) + X_{u,2}, \dots S(uN_u) = S(u) + X_{u,N_u}$. For $t \geq 0$ and $j \in [\,p\,]$, define

$$\mathcal{Z}_t^j := \sum_{u\in\mathcal{T}} \mathbb{1}_{\{j\}}(\tau(u))\mathbb{1}_{[0,\zeta_u)}(t - S(u)) \tag{2.3}$$

where $\mathbb{1}(\cdot)$ denotes the indicator function, i.e., $\mathcal{Z}_t^j$ is the number of individuals of type j alive at time t. Now write

$$\mathcal{Z}_t := (\mathcal{Z}_t^1 \dots \mathcal{Z}_t^p)$$

for the associated row vector. Then $(\mathcal{Z}_t)_{t\geq 0}$ is the multi-type general branching process.

It should be noted that the process $(\mathcal{Z}_t)_{t\geq 0}$ is merely a series of counts that takes into account only one particular property of the individuals, namely whether they are alive or not at time t. In this context, we can generalize the process by replacing the indicator functions in Equation (2.3) to perform counts that are based on other aspects of the individuals. Therefore, we focus on multi-type general branching processes counted with a random characteristic. This natural generalization is defined by the following construction. We assume a product-measurable, separable random characteristic $\varphi_u = (\varphi_u^1, \dots, \varphi_u^p)^\mathsf{T} : \Omega \times \mathbb{R} \to \mathbb{R}^{p\times 1}$ (taking values in the space of p-dimensional real column vectors) such that $(\boldsymbol{\xi}_u, \boldsymbol{\zeta}_u)$ are i. i. d. for $u \in \mathcal{I}$ and the φ_u are identically distributed. Further, we assume that for every $n \in \mathbb{N}$, the φ_u, $|u| = n$ are independent and independent of the $(\boldsymbol{\xi}_u, \boldsymbol{\zeta}_u)$, $|u| < n$. These assumptions are satisfied, for instance, when the $(\boldsymbol{\xi}_u, \boldsymbol{\zeta}_u, \varphi_u)$, $u \in \mathcal{I}$ are i. i. d. However, our assumption is weaker and allows φ_u to be a function of the $(\boldsymbol{\xi}_{uv}, \boldsymbol{\zeta}_{uv})$, $v \in \mathcal{I}$. The multi-type general branching process counted with a random characteristic φ is defined via

$$\mathcal{Z}_t^\varphi := \sum_{u\in\mathcal{T}} \mathsf{e}_{\tau(u)}\varphi_u(t - S(u)), \quad t \in \mathbb{R}. \tag{2.4}$$

Clearly, any individual u is weighted at its age by the characteristic φ_u.

Example 2.2.1 (Some special characteristics)

① Set $f := \mathbb{1}_{[0,\infty)}$. An important characteristic that will appear frequently throughout this work is the characteristic $f_j := f\mathsf{e}_j^\mathsf{T}$. In this case, $\mathcal{Z}_t^{f_j}$ gives the number of all individuals of type j born up to and including time t. This quantity we denote by N_t^j and set $N_t := (N_t^1, \dots, N_t^p)$ for its corresponding row vector. Notice that we can write

$$\begin{aligned} N_t &= (N_t^1, \dots, N_t^p) \\ &= \left(\sum_{u \in \mathcal{I}} \mathsf{e}_{\tau(u)} \mathbb{1}_{[0,\infty)}(t - S(u)) \mathsf{e}_1^\mathsf{T}, \dots, \sum_{u \in \mathcal{I}} \mathsf{e}_{\tau(u)} \mathbb{1}_{[0,\infty)}(t - S(u)) \mathsf{e}_p^\mathsf{T} \right) \\ &= \sum_{u \in \mathcal{I}} \mathbb{1}_{[0,\infty)}(t - S(u)) (\mathsf{e}_{\tau(u)} \mathsf{e}_1^\mathsf{T}, \dots, \mathsf{e}_{\tau(u)} \mathsf{e}_p^\mathsf{T}) \\ &= \sum_{u \in \mathcal{I}} \mathbb{1}_{[0,\infty)}(t - S(u)) \mathsf{e}_{\tau(u)}, \end{aligned}$$

and, the function $t \to N_t$ is non-decreasing, for $t \in \mathbb{R}$.

② For $\varphi_u = \mathbb{1}_{[0,\zeta_u^i)} \mathsf{e}_j^\mathsf{T}$ we get $\mathcal{Z}_t^\varphi = \mathcal{Z}_t^j$, i.e. the number of all individuals of type j alive at time t. This example also illustrates that introducing a general score φ indeed generalizes the multi-type general branching process.

③ Fix a number $a > 0$. If we choose $\varphi_u = \mathbb{1}_{[0,a \wedge \zeta_u^j)} \mathsf{e}_j^\mathsf{T}$, then $\mathcal{Z}_t^\varphi$ gives the number of all individuals of type j alive and younger than a at time t.

As one may see, the setup covers a wide range of possible applications, as different choices of φ result in different counts.

In the subsequent stages of this work, specifically when deriving the martingales, it will be necessary to define characteristics of higher dimensions. Therefore, we will introduce an appropriate definition for such higher dimensional characteristics in order to ensure a clear and rigorous framework for their application. To this end, consider a product-measurable, separable random characteristic $\boldsymbol{\varphi}_u : \Omega \times \mathbb{R} \to \mathbb{R}^{dp \times p}$ (taking values in the space of $dp \times p$ dimensional real matrices) such that the $(\boldsymbol{\xi}_u, \boldsymbol{\zeta}_u)$, $u \in \mathcal{I}$ are i. i. d. and the $\boldsymbol{\varphi}_u$ are identically distributed. As before, we assume throughout that for every $n \in \mathbb{N}$, the $\boldsymbol{\varphi}_u$, $|u| = n$ are independent and independent of the $(\boldsymbol{\xi}_u, \boldsymbol{\zeta}_u)$, $|u| < n$. Then, the multi-type general branching process counted with a random characteristic $\boldsymbol{\varphi}$ is defined via

$$\mathcal{Z}_t^{\boldsymbol{\varphi}} := \sum_{u \in \mathcal{T}} (\mathrm{I}_d \otimes \mathsf{e}_{\tau(u)}) \boldsymbol{\varphi}_u(t - S(u)), \quad t \in \mathbb{R}, \tag{2.5}$$

where $\otimes$ denotes the Kronecker product as introduced in Section 2.1.4. Notably, $\mathcal{Z}_t^{\boldsymbol{\varphi}}$ forms a $d \times p$ matrix.

2.2.1 Intensity Measures and Their Laplace Transforms

For all types $i, j \in [\,p\,]$ in the multi-type general branching process, we define

$$\mu^{i,j}(B) := \mathbf{E}[\xi^{i,j}(B)], \quad B \subseteq [0,\infty) \text{ Borel set}$$

and $\mu^i := \sum_{j=1}^{p} \mu^{i,j}\mathsf{e}_j = \mathbf{E}[\xi^i]$ as the intensity measures of $\xi^{i,j}$ and ξ^i, respectively. In this context, we apply the expectations $\mathbf{E}[\cdot]$ and $\mathbf{E}^i[\cdot]$ componentwise to vectors and matrices such that the matrix of intensity measures is constructed by

$$\boldsymbol{\mu} := \mathbf{E}[\boldsymbol{\xi}] = \begin{pmatrix} \mathbf{E}[\xi^1] \\ \vdots \\ \mathbf{E}[\xi^p] \end{pmatrix} = \begin{pmatrix} \mu^1 \\ \vdots \\ \mu^p \end{pmatrix} = \begin{pmatrix} \mu^{1,1} & \mu^{1,2} & \cdots & \mu^{1,p} \\ \mu^{2,1} & \mu^{2,2} & \cdots & \mu^{2,p} \\ \vdots & \ddots & \ddots & \vdots \\ \mu^{p,1} & \mu^{p,2} & \cdots & \mu^{p,p} \end{pmatrix}.$$

By applying the Laplace operator $\mathcal{L}$ componentwise to vectors and matrices of intensity measures, i.e.

$$\mathcal{L}\mu^{i,j}(\theta) := \int e^{-\theta x}\, \mu^{i,j}(\mathrm{d}x) = \mathbf{E}^i\left[\sum_{k=1}^{N} \mathbb{1}_{\{\tau(k)=j\}} e^{-\theta X_k}\right],$$

we obtain the $p \times p$ -matrix $\mathcal{L}\boldsymbol{\mu}(\theta) = (\mathcal{L}\mu^{i,j}(\theta))_{i,j\in[\,p\,]}$. For $i, j \in [\,p\,]$, $\mathcal{L}\mu^{i,j}(\theta)$ gives the expected number of children born at exponentially distributed times. Notice that $\mathcal{L}\mu^{i,j}$, for $i, j \in [\,p\,]$, is logarithmic convex, since $\mu^{i,j}$ is a nonnegative measure, cf. Section A.4. With this notation,

$$\mathcal{L}\boldsymbol{\mu}(0) = (\mathcal{L}\mu^{i,j}(0))_{i,j\in[\,p\,]} = \left(\mathbf{E}^i\left[\sum_{k=1}^{N} \mathbb{1}_{\{\tau(k)=j\}}\right]\right)_{i,j\in[\,p\,]} = (\mathbf{E}^i[N^{i,j}])_{i,j\in[\,p\,]})$$

is the matrix of total masses, which may have infinite entries as well.

2.2.2 Recursive Decomposition

In the present section, we derive the basic recursive decomposition of the multi-type (general) Crump-Mode-Jagers branching process. This fundamental representation is an rearrangement of the summands in (2.4) by decomposing the process in the first generation. Since the decomposition will only be required for N_t within this study, we will derive it only for this particular case. Therefore, recall that N_t^j is the number of individuals of type j born up to and including time t and that $N_t = (N_t^1, \ldots, N_t^p)$

is its corresponding row vector. If the ancestor has type $i \in [p]$, then

$$\begin{aligned} N_t &= \sum_{u\in\mathcal{I}} \mathbb{1}_{[0,\infty)}(t - S(u))\mathsf{e}_{\tau(u)} \\ &= \mathbb{1}_{[0,\infty)}(t)\mathsf{e}_i + \sum_{k=1}^{N}\sum_{u\in\mathcal{I}} \mathbb{1}_{[0,\infty)}(t - S(ku))\mathsf{e}_{\tau(ku)} \\ &= \mathbb{1}_{[0,\infty)}(t)\mathsf{e}_i + \sum_{k=1}^{N}\sum_{u\in\mathcal{I}} \mathbb{1}_{[0,\infty)}(t - S(k) - S_k(u))\mathsf{e}_{\tau(ku)} \\ &= \mathbb{1}_{[0,\infty)}(t)\mathsf{e}_i + \sum_{k=1}^{N} N_{k,t-S(k)}. \end{aligned} \tag{2.6}$$

Here, $S_k(u) = X_{k,u_1} + X_{ku_1,u_2} + \ldots + X_{ku_1\ldots u_{n-1},u_n}$ for $u = u_1 \ldots u_n \in \mathcal{I}$, i.e., $S_k(u)$ represents the birth time of individual u but with k as its ancestor. Moreover, the random variables $(N_{k,t})_{t\geq 0}$ are independent copies of $(N_t)_{t\geq 0}$ obtained when k is the ancestor. This implies that N_t, the vector of individuals at time t, can be expressed as a sum of independent random variables, as shown in (2.6). In analogy to [IKM24, Equation (6.7)], taking expectations in (2.6) yields

$$\begin{aligned} m_t^i := \mathbf{E}^i[N_t] = \mathbf{E}^i\left[\mathbb{1}_{[0,\infty)}(t)\,\mathsf{e}_i + \sum_{k=1}^{N} N_{k,t-S(k)}\right] \\ = \mathbb{1}_{[0,\infty)}(t)\,\mathsf{e}_i + \mathbf{E}^i\left[\sum_{k=1}^{N} N_{k,t-S(k)}\right] \\ = \mathbb{1}_{[0,\infty)}(t)\,\mathsf{e}_i + \sum_{j=1}^{p}\int m_{t-x}^{j}\,\mu^{i,j}(\mathrm{d}x) \\ = \mathbb{1}_{[0,\infty)}(t)\,\mathsf{e}_i + \sum_{j=1}^{p} m^j * \mu^{i,j}(t) \end{aligned} \tag{2.7}$$

for $t \in \mathbb{R}$. It should be noted that (2.7) represents the Markov renewal equation as described in [Jag75, Equation (5.2.1)]. In particular, with $M(t) = (m_t^{i,j})_{i,j\in[p]} := (\mathbf{E}^i[N_t^j])_{i,j\in[p]}$, we can express (2.7) in matrix notation as

$$M(t) = \mathbb{1}_{[0,\infty)}(t)\mathrm{I}_p + \boldsymbol{\mu} * M(t), \quad t \in \mathbb{R}. \tag{2.8}$$

Here, our objective is to determine the asymptotic expansion of $M(t)$, i.e. the expected number of individuals born up to and including time t, as $t \to \infty$.

Since $M(t)$ solves the multidimensional Markov renewal equation, it is advantageous to temporarily step away from the context of multi-type branching processes and focus on solutions to multidimensional Markov renewal equations in general. Consequently, we will proceed by investigating the asymptotic expansion of solutions to multidimensional Markov renewal equations in a general framework, which will yield the desired result to our application.

3 | Asymptotic Expansion of Solutions to Markov Renewal Equations

This chapter focuses on deriving the asymptotic expansion for the solution F to the Markov renewal equation $F(t) = f(t) + \boldsymbol{\mu} * F(t)$ for vector-valued functions $f, F : \mathbb{R} \to \mathbb{R}^p$ and a $p \times p$ matrix $\boldsymbol{\mu} = (\mu^{i,j})_{i,j \in [p]}$ of locally finite measures on the positive half-line $[0, \infty)$.

In 2002, M. S. Sgibnev established an asymptotic expansion for the solution F to this equation, as detailed in [Sgi02]. His proof relies on a representation theorem for homomorphisms on a specific Banach algebra, which may go beyond the usual prior knowledge of readers from the probability community. Therefore, the aim of this chapter is to offer a more accessible framework for researchers in probability and to provide a more elementary proof of Sgibnev's result, primarily based on fundamental results about Laplace transforms and complex analysis.

We commence our analysis by introducing the multidimensional Markov renewal equation and formulating conditions in order to ensure the existence and uniqueness of solutions F. In this context, the spectral radius of $\boldsymbol{\mu}$ plays a crucial role, and thus we will first investigate its properties to build a foundation for the subsequent discussion. Following this, we turn our attention to the characteristic equation $\det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z)) = 0$ for $z \in \mathbb{C}$, and more precisely, to its solutions, denoted by λ. These solutions are intimately connected to the asymptotic expansion of solutions to Markov renewal equations. In many applications, a maximal real solution α exists, which is of particular interest as it determines the leading order term in the asymptotic expansion.

Subsequently, we proceed by a discussion of the (technical) assumptions we use in our work and a comparison to those presented in the literature, particularly with Sgibnev's assumptions, see [Sgi02, Theorem 3]. With this, we derive our first main result, Theorem 3.12, which provides the asymptotic expansion for the renewal measure $\boldsymbol{U} := \sum_{n=0}^{\infty} \boldsymbol{\mu}^{*n}$, one of the prime examples of a solution to a Markov renewal equation. Here, we treat the lattice and non-lattice case separately, depending on whether $\boldsymbol{\mu}$ is concentrated on a grid $h\mathbb{Z}$, for $h > 0$, or not, as different mathematical tools are required for each. Building on this, we derive our second main result, Theorem 3.16, an expansion for solutions F to Markov renewal equations in general.

Finally, we apply the asymptotic expansion to multi-type general branching processes, which, due to the classical recursive decomposition of the process in the first generation, satisfy a Markov renewal equation.

3.1 *Markov Renewal Equations*

The investigation of Markov renewal equations of the form

$$F(t) = f(t) + \boldsymbol{\mu} * F(t) = f(t) + \int_0^\infty \boldsymbol{\mu}(\mathrm{d}x) F(t-x) \tag{3.1}$$

for vector-valued functions $f, F : \mathbb{R} \to \mathbb{R}^{p \times 1}$ and a $p \times p$ matrix

$$\boldsymbol{\mu} = \begin{pmatrix} \mu^{1,1} & \mu^{1,2} & \cdots & \mu^{1,p} \\ \mu^{2,1} & \mu^{2,2} & \cdots & \mu^{2,p} \\ \vdots & \ddots & \ddots & \vdots \\ \mu^{p,1} & \mu^{p,2} & \cdots & \mu^{p,p} \end{pmatrix}$$

of locally finite measures $\mu^{i,j}$ on $[0, \infty)$, $i, j \in [p]$, was originally motivated by their natural occurrence in the theory of multi-type branching processes, as discussed in [BC63] or [Cru70a]. In this context, the function f is given while F is the solution to the Equation (3.1). Here, the convolution of the vector-valued function F and the $p \times p$ matrix of measures $\boldsymbol{\mu}$ is the column vector whose i-th entry is given by

$$(\boldsymbol{\mu} * F)_i(t) = \sum_{j=1}^{p} \int_0^\infty F_j(t-x) \mu^{i,j}(\mathrm{d}x), \quad i \in [p].$$

If we define $\boldsymbol{\mu}^{*0}(t) = \mathbb{1}_{[0,\infty)}(t)\mathrm{I}_p$, where I_p denotes the $p \times p$ identity matrix, $\boldsymbol{\mu}^{*1}(t) = \boldsymbol{\mu}(t)$ and recursively, $\boldsymbol{\mu}^{*n}(t) = \boldsymbol{\mu}^{*n-1} * \boldsymbol{\mu}(t)$, then the Markov renewal function is given by $\boldsymbol{U}(t) = \sum_{n=0}^{\infty} \boldsymbol{\mu}^{*n}(t)$. In fact, the Markov renewal function is a key example for a solution to a Markov renewal equation, namely,

$$\boldsymbol{U}(t) = \sum_{n=0}^{\infty} \boldsymbol{\mu}^{*n}(t) = \boldsymbol{\mu}^{*0}(t) + \boldsymbol{\mu} * \sum_{n=1}^{\infty} \boldsymbol{\mu}^{*n-1}(t) = \mathbb{1}_{[0,\infty)}(t)\mathrm{I}_p + \boldsymbol{\mu} * \boldsymbol{U}(t) \tag{3.2}$$

for any $t \geq 0$.

3.1.1 The Lattice Type

In renewal theory, a fundamental distinction is made based on whether the measures $\mu^{i,j}$, $i, j \in [p]$ are concentrated on a grid, i.e., on integer multiples of a fixed number, or not. This classification gives rise to separate limit theorems for the so-called lattice and non-lattice case. Similarly, in this work, we consistently distinguish between these two cases and determine the asymptotic expansions in both.

① We say that the process is lattice if, for some $h > 0$, the measure $\mu^{i,j}$ is concentrated on a set of points $h\mathbb{Z}$ for every $i, j \in [p]$.

② Otherwise, we refer to the process as non-lattice if for all $h > 0$ and $h_1, ..., h_p \in [0, h)$, $\mu^{i,j}((h_j - h_i + h\mathbb{Z})^{\mathrm{c}}) > 0$ is satisfied for some $i, j \in [\,p\,]$.

In the lattice case, we assume without loss of generality that $h = 1$ is maximal with the property that all the $\mu^{i,j}$, $i, j \in [\,p\,]$ are concentrated on $h\mathbb{Z}$. The general lattice case can be reduced to this one by scaling. In both cases, we write $\mathbb{G}$ for the minimal closed additive subgroup of $\mathbb{R}$ on which all the $\mu^{i,j}$ are concentrated. Hence, $\mathbb{G} = \mathbb{Z}$ in the lattice case and $\mathbb{G} = \mathbb{R}$ in the non-lattice case.

3.2 *Existence and Uniqueness of Solutions*

A classical result ensuring the existence and uniqueness of solutions F to the Equation (3.1) in the single-type case, i.e. in the case $p = 1$, is well-established in the literature, as detailed in [Res92, Theorem 3.5.1]. This result relies on the assumptions that $\mu(0) < 1$, f vanishes on the negative half-line, and is locally bounded, meaning it is bounded on compact intervals. For the multi-type case, existence and uniqueness are addressed in [Cru70a] for the irreducible case and are implicitly covered in [Cru70b] for the reducible case. For the sake of completeness and the reader's convenience, we will derive the basic existence and uniqueness result in the multi-type case and outline the proof below.

Here, similar conditions apply, but rather than considering $\boldsymbol{\mu}(0)$, which would be the first natural conjecture, we instead focus on its spectral radius $\rho_{\boldsymbol{\mu}(0)}$. Since $\boldsymbol{\mu}$ and its Laplace transform $\mathcal{L}\boldsymbol{\mu}$ are connected due to the identities

$$\begin{aligned} \mathcal{L}\boldsymbol{\mu}(0) &= \boldsymbol{\mu}(\infty), \\ \mathcal{L}\boldsymbol{\mu}(\infty) &= \boldsymbol{\mu}(0) + \int \lim_{\theta \to \infty} e^{-\theta x}\, \boldsymbol{\mu}(\mathrm{d}x) = \boldsymbol{\mu}(0), \end{aligned} \tag{3.3}$$

it is reasonable to begin by analyzing the spectral radius of $\mathcal{L}\boldsymbol{\mu}(z)$, for $z \in \mathbb{C}$. Recall that $\mathcal{L}\boldsymbol{\mu} = (\mathcal{L}\mu^{i,j})_{i,j \in [\,p\,]}$ meaning that the Laplace transform is applied componentwise. Whenever $\mathcal{L}\mu^{i,j}(\theta) < \infty$ for some $\theta \in \mathbb{R}$, we can define the Laplace transform $\mathcal{L}\mu^{i,j}$ in the half-space $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq \theta\}$ since for any z from this space, the integral $\mathcal{L}\mu^{i,j}(z) = \int e^{-zx}\, \mu^{i,j}(\mathrm{d}x)$ converges absolutely. Hence, we define

$$\mathcal{D}(\mathcal{L}\boldsymbol{\mu}) := \left\{ z \in \mathbb{C} : \int e^{-\mathrm{Re}(z)x}\, \mu^{i,j}(\mathrm{d}x) < \infty \ \forall i, j \in [\,p\,] \right\}$$

as the domain of $\mathcal{L}\boldsymbol{\mu}$. In general, $\mathcal{D}(\mathcal{L}\boldsymbol{\mu})$ is either empty, a half-space, or $\mathbb{C}$. Further, we denote by $\vartheta_0 \in \mathbb{R}$ the first point on the real line where the matrix $\mathcal{L}\boldsymbol{\mu}$ is finite, i.e.,

$$\vartheta_0 := \inf \left\{ \theta \in \mathbb{R} : \int e^{-\theta x}\, \mu^{i,j}(\mathrm{d}x) < \infty \ \forall i, j \in [\,p\,] \right\} \in \mathbb{R} \cup \{-\infty, \infty\}.$$

Here, the infimum of the empty set is defined to be ∞ and $\inf \mathbb{R} = -\infty$ is stipulated. We write $\mathcal{D}(\mathcal{L}\boldsymbol{\mu})^\circ$ for the interior of $\mathcal{D}(\mathcal{L}\boldsymbol{\mu})$ and infer that $\mathcal{D}(\mathcal{L}\boldsymbol{\mu})^\circ = (\vartheta_0, \infty) + \mathrm{i}\mathbb{R}$.

Throughout the thesis, we make the basic assumption that the domain of the finiteness of the Laplace transform $\mathcal{L}\boldsymbol{\mu}$ is non-empty.

Assumption (A1)

There exists a $\vartheta \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu})^\circ \cap \mathbb{R}$, i.e.

$$\mathcal{L}\mu^{i,j}(\vartheta) = \int e^{-\vartheta x}\, \mu^{i,j}(\mathrm{d}x) = \mathbf{E}\left[\sum_{k=1}^{N^{i,j}} e^{-\vartheta X_k^{i,j}}\right] < \infty \tag{3.4}$$

for all $i, j \in [p]$. In other words, $\vartheta_0 = \inf\{\mathcal{D}(\mathcal{L}\boldsymbol{\mu}) \cap \mathbb{R}\} < \infty$ and $\vartheta_0 < \vartheta$.

Returning to the spectral radius of $\mathcal{L}\boldsymbol{\mu}(z)$, for $z \in \mathbb{C}$, we may define the function

$$\varrho : \mathcal{D}(\mathcal{L}\boldsymbol{\mu}) \to [0,\infty), \quad z \mapsto \varrho(z) := \rho_{\mathcal{L}\boldsymbol{\mu}(z)}.$$

To some extent, the function ϱ plays the role that the Laplace transform of the intensity measure $\mathcal{L}\mu$ plays in the single-type case. We continue with a brief summary of properties of ϱ.

Proposition 3.1 [Properties of the spectral radius of $\mathcal{L}\boldsymbol{\mu}$]

Suppose that (A1) *holds.*

① *The function* $\varrho|_{\mathcal{D}(\mathcal{L}\boldsymbol{\mu})\cap\mathbb{R}} : \mathcal{D}(\mathcal{L}\boldsymbol{\mu}) \cap \mathbb{R} \to [0,\infty)$ *is logarithmic convex and continuous.*

② $\lim_{\theta\to\infty} \varrho(\theta) = \rho_{\boldsymbol{\mu}(0)}$.

③ *The function* $\varrho|_{\mathcal{D}(\mathcal{L}\boldsymbol{\mu})\cap\mathbb{R}} : \mathcal{D}(\mathcal{L}\boldsymbol{\mu}) \cap \mathbb{R} \to [0,\infty)$ *is strictly decreasing on the set* $\{\theta \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu}) \cap \mathbb{R} : \varrho(\theta) > \rho_{\boldsymbol{\mu}(0)}\}$.

④ *For every* $z \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu})$, *we have* $\varrho(z) \leq \varrho(\mathrm{Re}(z))$.

Proof of Proposition 3.1.

① Recall that each entry of $\mathcal{L}\boldsymbol{\mu}(\theta)$, $\theta \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu}) \cap \mathbb{R}$ is logarithmic convex since $\mu^{i,j}$ is a nonnegative measure for every $i, j \in [p]$. Consequently, the logarithmic convexity of $\varrho|_{\mathcal{D}(\mathcal{L}\boldsymbol{\mu})\cap\mathbb{R}}$ follows from the main theorem of [Kingman1961], which states that the spectral radius ρ_M of a square matrix M is logarithmic convex if each entry of M is logarithmic convex. In particular, $\varrho|_{\mathcal{D}(\mathcal{L}\boldsymbol{\mu})\cap\mathbb{R}}$ is convex since logarithmic convexity implies convexity. The continuity on the interior of the domain of $\varrho|_{\mathcal{D}(\mathcal{L}\boldsymbol{\mu})\cap\mathbb{R}}$ follows by convexity, but we claim continuity one the entire domain, which may contain the left boundary point. However, the continuity on the entire domain follows from Lemma A.7.

② As mentioned before, $\mathcal{L}\mu^{i,j}(\theta) \to \mu^{i,j}(0)$ as $\theta \to \infty$ for every $i, j \in [p]$. Hence, $\mathcal{L}\boldsymbol{\mu}(\theta) \to \boldsymbol{\mu}(0)$ as $\theta \to \infty$. We may thus extend the function ϱ to ∞ by setting $\varrho(\infty) := \rho_{\boldsymbol{\mu}(0)}$. Lemma A.7 still applies and yields that the extended ϱ is continuous on $\mathcal{D}(\mathcal{L}\boldsymbol{\mu}) \cap \mathbb{R}$. We conclude

$$\varrho(\theta) = \rho_{\mathcal{L}\boldsymbol{\mu}(\theta)} \to \rho_{\boldsymbol{\mu}(0)} \quad \text{as } \theta \to \infty.$$

③ That ϱ is decreasing again follows from Lemma A.7. Consequently, if ϱ is not strictly decreasing on an interval, it is constant there. Let $I = [i_1, i_2] \subseteq \mathbb{R}$ be an interval on which ϱ is constant. Since $\varrho(\theta) \to \rho_{\boldsymbol{\mu}(0)}$ as $\theta \to \infty$, $\varrho(\theta)$, there must exist an interval $[i_3, i_4]$ with $i_2 < i_3$ on which $\varrho(\theta)$ is strictly decreasing. However, we can exclude this case, because otherwise the function is no longer convex in the area $[i_1, i_4]$. Hence, by the first part of this proposition and convexity, ϱ must be strictly decreasing on the set $\{\theta \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu}) \cap \mathbb{R} : \varrho(\theta) > \rho_{\boldsymbol{\mu}(0)}\}$.

④ follows from Lemma A.3.

■

Having discussed some of the key properties of the function ϱ, we shall now return to our previous question, namely, under which conditions the existence and uniqueness of solutions F to the multidimensional Markov renewal equation (3.1) holds. To this end, it is essential to make the following crucial assumption:

Assumption (A2)

The spectral radius $\rho_{\boldsymbol{\mu}(0)}$ of the matrix $\boldsymbol{\mu}(0) = (\mu^{i,j}(0))_{i,j\in[p]}$ satisfies $\rho_{\boldsymbol{\mu}(0)} < 1$.

Notice that (A2) implies $\varrho(\theta) = \rho_{\mathcal{L}\boldsymbol{\mu}(\theta)} < 1$ for all sufficiently large θ by Proposition 3.1.

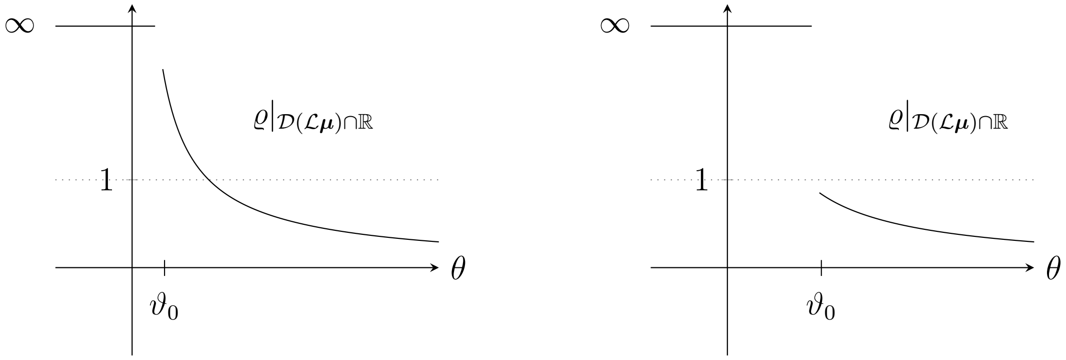

Figure 3.1: Possible behavior of the function $\varrho|_{\mathcal{D}(\mathcal{L}\boldsymbol{\mu})\cap\mathbb{R}}$ under assumptions (A1) and (A2).

Proposition 3.2 $\big[$Existence and uniqueness of solutions $F\big]$
Let $\boldsymbol{\mu} = (\mu^{i,j})_{i,j\in[p]}$ *be a* $p \times p$ *matrix of locally finite measures on* $[0,\infty)$ *satisfying* (A2). *Further, let* $\boldsymbol{U} = \sum_{n=0}^{\infty} \boldsymbol{\mu}^{*n}$ *be the associated Markov renewal measure. Then the following assertions hold true:*

① *The Markov renewal function* $\boldsymbol{U}(t)$ *is finite at every* $t \geq 0$.

② *If, additionally,* (A1) *holds, then, for any sufficiently large* $\theta \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu})$, *we have* $\rho(\mathcal{L}\boldsymbol{\mu}(\theta)) < 1$ *and for any such* θ, *it holds that* $U^{i,j}(t) = o(e^{\theta t})$ *as* $t \to \infty$ *for all* $i, j \in [p]$.

③ *Let* $f = (f_1, \ldots, f_p)^{\mathsf{T}}$ *be such that each* f_i *is a locally bounded and vanishes on the negative half-line, for* $i \in [p]$. *Then there is a unique locally finite solution* $F : \mathbb{R} \to \mathbb{R}^p$ *to the Markov renewal equation* (3.1), *namely,*

$$F(t) = \boldsymbol{U} * f(t), \quad t \in \mathbb{R}. \tag{3.5}$$

Proof of Proposition 3.2.

① The case where (A2) holds but (A1) is violated can be reduced to the case where (A1) and (A2) hold. Indeed, fix $t > 0$ and replace $\boldsymbol{\mu}(\cdot)$ by $\boldsymbol{\mu}_t(\cdot) := \boldsymbol{\mu}(\cdot \cap [0,t])$. The Laplace transform of $\boldsymbol{\mu}_t(\cdot)$ is finite everywhere and the associated Markov renewal measure coincides with $\boldsymbol{U}$ on $[0,t]$. In particular, $\boldsymbol{U}(t)$ is finite by assertion ②.

② By assumption (A1), $\mathcal{L}\boldsymbol{\mu}(\vartheta)$ has finite entries only, for some $\vartheta \in \mathbb{R}$. Recall from Proposition 3.1 that $\rho_{\mathcal{L}\boldsymbol{\mu}(\theta)} \to \rho_{\boldsymbol{\mu}(0)} < 1$ as $\theta \to \infty$, where the last inequality is guaranteed by (A2). Fix some $\theta \in \mathbb{R}$ with $\rho_{\mathcal{L}\boldsymbol{\mu}(\theta)} < 1$. Then

$$\mathcal{L}\boldsymbol{U}(\theta) = \sum_{n=0}^{\infty} (\mathcal{L}\boldsymbol{\mu})^{*n}(\theta) = \sum_{n=0}^{\infty} \mathcal{L}\boldsymbol{\mu}(\theta)^n = (\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\theta))^{-1}, \tag{3.6}$$

which is a finite matrix by Lemma A.2. Consequently, Lemma B.1 gives $\boldsymbol{U}(t) = o(e^{\theta t})$ as $t \to \infty$ (coordinatewise).

③ The assertion that F, as defined in (3.5), is a solution to (3.1) can be derived from the proof of [Res92, Theorem 3.5.1]. This follows from the associativity of matrix convolution and the Markov renewal equation for the Markov renewal measure $\boldsymbol{U}$. The local finiteness of F follows from the first part of Proposition 3.2 since the convolution of a locally finite function supported on $[0,\infty)$ with a locally finite measure is locally finite. The uniqueness can be obtained directly from the proof of [Cru70a, Theorem 2.1]. Although the cited theorem covers only the irreducible case, the proof also works without this assumption.

■

3.3 *Roots of the Characteristic Equation*

As previously mentioned, it is a reasonable approach to use Laplace transforms in order to investigate Equation (3.1) since the convolution term $\boldsymbol{\mu} * F$ can be replaced by a product using the convolution theorem for Laplace transforms, see [Mil71, Section 2.4]. In fact, applying the Laplace transform to both sides in (3.1), we obtain

$$\mathcal{L}F(z) = \mathcal{L}f(z) + \mathcal{L}\boldsymbol{\mu}(z)\mathcal{L}F(z) = (\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}\mathcal{L}f(z), \tag{3.7}$$

provided that $(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))$ is invertible, i.e. $\det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z)) \neq 0$. As discussed in the work of R. Bellman and K. L. Cooke [BC63], this observation leads to the conclusion that the asymptotic behavior of $F_i(t)$ is determined by the solutions of the characteristic equation

$$\det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\lambda)) = 0, \quad \lambda \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu}) \tag{3.8}$$

and their multiplicities $k_\lambda \in \mathbb{N}$. Under certain regularity conditions, K. S. Crump established in [Cru70a] that $F_i(t) \sim C_i e^{\alpha t}$ when $\boldsymbol{\mu}(\infty)$ is irreducible. Here, C_i is a constant, and α is the Perron-Frobenius eigenvalue such that $\rho(\mathcal{L}\boldsymbol{\mu}(\alpha)) = 1$. In [Cru70b] the irreducibility condition is dropped, resulting in $F_i(t) \sim C_i t^{r_i} e^{\alpha_i t}$, where C_i, r_i and α_i are constants.

Building on this, we extend the analysis of the solutions to the characteristic equation (3.8) since they are closely related to the asymptotic expansion of solutions $F_i(t)$ to (3.1) as $t \to \infty$. Notably, if λ satisfies (3.8), then the matrix $\mathcal{L}\boldsymbol{\mu}(\lambda)$ has eigenvalue 1, which implies $\varrho(\lambda) \geq 1$. Here and throughout our work, we denote by Λ the set of all solutions of (3.8), i.e.,

$$\Lambda := \{\lambda \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu}) : \det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\lambda)) = 0\}$$

in the non-lattice case, and

$$\Lambda := \{\lambda = \theta + \mathrm{i}\eta \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu}) : -\pi < \eta \leq \pi \text{ and } \det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\lambda)) = 0\}$$

in the lattice case. It should be noted that Λ is a closed subset of $\mathcal{D}(\mathcal{L}\boldsymbol{\mu})$ since the determinant is a continuous functional on $\mathbb{C}^{p\times p}$ equipped with the matrix norm.

Whenever $\det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z)) \neq 0$, we can invert the $p \times p$ matrix $\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z)$ using the formula

$$(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1} = \frac{1}{\det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))} \operatorname{adj}(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z)), \tag{3.9}$$

where $\operatorname{adj}(A)$ denotes the adjoint of the matrix A. This matrix is defined as transpose of the cofactor matrix of A, where each entry of the cofactor matrix is computed as the determinant of the submatrix obtained by deleting the corresponding row and column from A. Returning to (3.9), each entry of $(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ has a pole with multiplicity at most the multiplicity of the zero of $\det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))$ at $z = \lambda$. Write $k_\lambda \in \mathbb{N}$ for the maximal multiplicity of the poles of the entries of $(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ at $\lambda \in \Lambda$. For completeness, we set $k_z = 0$ for $z \notin \Lambda$.

3.3.1 The Malthusian Parameter

In many applications, the set Λ will contain a real root and in fact even a maximal such root. More precisely, we call a root $\alpha \in \Lambda \cap \mathbb{R}$ Malthusian parameter if $\varrho(\alpha) = 1$.

If we work in the positive or irreducible case, i.e., if $\boldsymbol{\mu}$ is a positive or irreducible matrix, then Perron-Frobenius theory guarantees the existence, uniqueness and simplicity of the Malthusian parameter α, which corresponds to the Perron-Frobenius eigenvalue. In the general case, the Malthusian parameter may not necessarily exist. However, if it does, it represents the largest root in $\Lambda \cap \mathbb{R}$ and determines the leading-order term in the asymptotic expansion of $F_i(t)$ as $t \to \infty$. In the context of branching processes, this corresponds to the maximum exponential growth rate of the expected population size. Moreover, in the general case, the Malthusian parameter does not necessarily have to be a simple solution of (3.8).

In this subsection, we will discuss the relevant properties, beginning with an examination of the existence of the Malthusian parameter in the general case. Provided that the assumptions (A1) and (A2) are satisfied, the existence of a Malthusian parameter will be ensured by the following condition:

Assumption (A3)

There exists $\vartheta \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu})^\circ \cap \mathbb{R}$ with $1 \leq \varrho(\vartheta) < \infty$.

Consequently, reviewing Figure 3.1, it becomes evident that assumption (A3) excludes the second plot of $\varrho|_{\mathcal{D}(\mathcal{L}\boldsymbol{\mu})\cap\mathbb{R}}$, leading to the existence of some $\alpha \in \mathbb{R}$ with $\varrho(\alpha) = 1$.

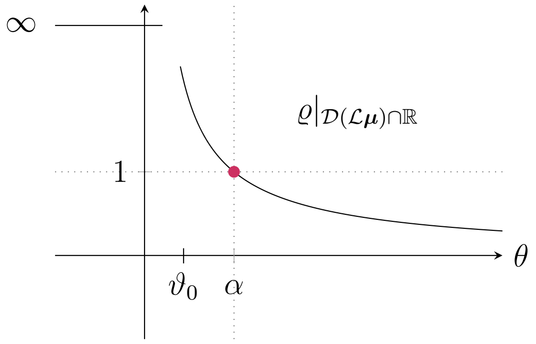

Figure 3.2: Behavior of the function $\varrho|_{\mathcal{D}(\mathcal{L}\boldsymbol{\mu})\cap\mathbb{R}}$ under assumption (A3).

In particular, the same symbol ϑ is used in (A1) and (A3) since if both assumptions hold, then we can and do assume without loss of generality that the parameters ϑ in (A1) and (A3) are identical.

Next, we formulate sufficient conditions for the existence, uniqueness and simplicity of the Malthusian parameter in the general case.

Proposition 3.3 [Existence and uniqueness of the Malthusian parameter]
Suppose that (A1) *and* (A2) *hold.*

① *There exists at most one* $\alpha \in \mathbb{R}$ *with* $\varrho(\alpha) = 1$.

② *If* (A3) *holds, then there exists a Malthusian parameter* α *and* $\mathrm{Re}(\lambda) \leq \alpha$ *for every* $\lambda \in \Lambda$.

③ *Condition* (A3) *is necessary for* Λ *to be non-empty.*

④ *In the primitive case, if* $\alpha \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu})^{\circ}$, *then* α *is a simple root of* $\det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z)) = 0$ *and* $\mathrm{Re}(\lambda) < \alpha$ *for every* $\lambda \in \Lambda \setminus \{\alpha\}$.

Proof of Proposition 3.3.
Assume that (A1) and (A2) hold, i.e., $\mathcal{L}\boldsymbol{\mu}(\vartheta)$ has finite entries only for some $\vartheta \in \mathbb{R}$ and $\rho_{\boldsymbol{\mu}(0)} < 1$.

① Suppose there is some $\alpha \in \Lambda \cap \mathbb{R}$ with $\varrho(\alpha) = 1$. By (A2) and Proposition 3.1, we have $\alpha \in \{\theta \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu}) \cap \mathbb{R} : \varrho(\theta) > \rho_{\boldsymbol{\mu}(0)}\}$ and, therefore, that $\varrho|_{\mathcal{D}(\mathcal{L}\boldsymbol{\mu})\cap\mathbb{R}}$ is strictly decreasing in a neighborhood of α. In particular, α is unique with the property $\varrho(\alpha) = 1$.

② Now suppose that additionally (A3) holds, i.e., there is $\vartheta \in \mathbb{R}$ with $1 \leq \varrho(\vartheta) < \infty$. Again, by Proposition 3.1, $\varrho|_{\mathcal{D}(\mathcal{L}\boldsymbol{\mu})\cap\mathbb{R}}$ is continuous and decreasing with $\lim_{\theta\to\infty} \varrho(\theta) = \rho_{\boldsymbol{\mu}(0)} < 1$, so by the intermediate value theorem there exists an $\alpha \geq \vartheta$ with $\varrho(\alpha) = 1$. Thus, the matrix $\mathcal{L}\boldsymbol{\mu}(\alpha)$ has eigenvalue 1 and, therefore, $\det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\alpha)) = 0$, i.e., $\alpha \in \Lambda \cap \mathbb{R}$.

Furthermore, for every $\lambda \in \Lambda \setminus \{\alpha\}$, it holds

$$\varrho(\alpha) = 1 \leq \varrho(\lambda) \leq \varrho(\mathrm{Re}(\lambda))$$

by Proposition 3.1. It follows that $\mathrm{Re}(\lambda) \in \{\theta \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu}) \cap \mathbb{R} : \varrho(\theta) > \rho_{\boldsymbol{\mu}(0)}\}$ and, that $\varrho|_{\mathcal{D}(\mathcal{L}\boldsymbol{\mu})\cap\mathbb{R}}$ is strictly decreasing in a neighborhood of $\mathrm{Re}(\lambda)$ by Proposition 3.1. Consequently, we conclude $\mathrm{Re}(\lambda) < \alpha$.

③ If the set Λ is non-empty, then there exists $\lambda \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu})$ with $\det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\lambda)) = 0$. Thus, the matrix $\mathcal{L}\boldsymbol{\mu}(\lambda)$ has eigenvalue 1 and, consequently, $\varrho(\lambda) \geq 1$. Since $\varrho(\lambda) \leq \varrho(\mathrm{Re}(\lambda))$ holds by Proposition 3.1, we can set $\vartheta = \varrho(\mathrm{Re}(\lambda))$ to find that $\varrho(\vartheta) \geq 1$ and assumption (A3) holds.

④ Suppose that $\boldsymbol{\mu}$ is primitive. Let $\theta > \alpha$. Then $\varrho(\theta) < 1$ and is the Perron-Frobenius eigenvalue of $\mathcal{L}\boldsymbol{\mu}(\theta)$. By Lemma A.1, there exists a unique eigenvector v_θ associated with the eigenvalue $\varrho(\theta)$ of $\mathcal{L}\boldsymbol{\mu}(\theta)$ with positive entries only and $|v_\theta| = 1$. Moreover, Lemma A.1 yields that $\theta \mapsto v_\theta$ is continuous. In particular, all entries of v_θ are bounded away from 0 in a right neighborhood $[\alpha, \alpha+\varepsilon]$ of α and for any given vector $y \in \mathbb{R}^d_{\geq}$, we can find a finite constant $c > 0$ such that $y \leq c v_\theta$ for every $\theta \in [\alpha, \alpha+\varepsilon]$. For $\alpha < \theta \leq \alpha + \varepsilon$, using Lemma A.2, we infer

$$\begin{aligned}(\theta-\alpha)(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(\theta))^{-1} y &= (\theta-\alpha)\sum_{n=0}^{\infty} \boldsymbol{\mathcal{L}\mu}(\theta)^n y \leq c(\theta-\alpha)\sum_{n=0}^{\infty} \boldsymbol{\mathcal{L}\mu}(\theta)^n v_\theta \\ &= c(\theta-\alpha)\sum_{n=0}^{\infty} \varrho(\theta)^n v_\theta = c\frac{\theta-\alpha}{1-\varrho(\theta)} v_\theta \leq \frac{c\varepsilon}{1-\varrho(\alpha+\varepsilon)} v_\theta\end{aligned}$$

by the convexity of ϱ, which remains bounded $\theta \downarrow \alpha$. Therefore, $(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(\theta))^{-1} y \sim C\frac{1}{\theta-\alpha} v_\theta$ and, consequently, α is a pole of order one of $(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))^{-1}$.

■

In particular, if we set

$$\Lambda_\theta := \{\lambda \in \Lambda : \mathrm{Re}(\lambda) > \theta\},$$

then Proposition 3.3 states that the Malthusian parameter, provided that it exists, represents the maximal real root in Λ, which implies that Λ_α is empty.

3.3.2 Examples

In order to enhance our comprehension of the solutions $\lambda \in \Lambda$ and, in particular, the Malthusian parameter α, we will discuss some relevant examples. First, we present an example demonstrating that we have not excluded the possibility that the solutions $\lambda \in \Lambda$ can be located on the negative half-space $\mathcal{H}_0 := \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$.

Example 3.4 Let $p = 2$ and suppose that $\xi^{1,1}$ is a Poisson point process on $[0, \infty)$ with intensity measure $g(x)\mathrm{d}x$ where

$$g(x) = \mathbb{1}_{(0,\infty)}(x)\frac{b^a x^{a-1}}{\Gamma(a)} e^{-bx}, \quad x \in \mathbb{R}, \tag{3.10}$$

is the density of the Gamma distribution with parameters $a, b > 0$ with $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\mathrm{d}x$ denoting the gamma function for any $a > 0$. In this situation, we choose $a = b = 1$. Further, $\xi^{1,2}$, $\xi^{2,1}$ and $\xi^{2,2}$ are homogeneous Poisson point processes on $[0, \infty)$ with rate $\frac{1}{2}$. We infer

$$\boldsymbol{\mu}(\mathrm{d}x) = \begin{pmatrix} \mathbb{1}_{(0,\infty)}(x)e^{-x} & \frac{1}{2}x \\ \frac{1}{2}x & \frac{1}{2}x \end{pmatrix}.$$

Since

$$\mathcal{L}\mu^{1,1}(z) = \int_0^\infty e^{-zx}\mu^{1,1}(\mathrm{d}x) = \int_0^\infty \mathbb{1}_{(0,\infty)}(x)e^{-x}e^{-zx}\mathrm{d}x = \int_0^\infty e^{-x(1+z)}\mathrm{d}x = \frac{1}{1+z},$$

and,

$$\mathcal{L}\mu^{1,2}(z) = \mathcal{L}\mu^{2,1}(z) = \mathcal{L}\mu^{2,2}(z) = \int_0^\infty \frac{1}{2}e^{-zx}\mathrm{d}x = \frac{1}{2z},$$

we conclude

$$\mathcal{L}\boldsymbol{\mu}(z) = \begin{pmatrix} \frac{1}{1+z} & \frac{1}{2z} \\ \frac{1}{2z} & \frac{1}{2z} \end{pmatrix}.$$

Thus, we derive the characteristic equation

$$\det(\mathrm{I}_2 - \mathcal{L}\boldsymbol{\mu}(z)) = \det\begin{pmatrix} 1-\frac{1}{1+z} & -\frac{1}{2z} \\ -\frac{1}{2z} & 1-\frac{1}{2z} \end{pmatrix} = \left(1-\frac{1}{1+z}\right)\left(1-\frac{1}{2z}\right) - \frac{1}{4z^2}.$$

The solutions of $\det(\mathrm{I}_2 - \mathcal{L}\boldsymbol{\mu}(z)) = 0$ are given by $\Lambda = \{-\frac{1}{4} - \frac{\sqrt{3}}{4}\mathrm{i}, -\frac{1}{4} + \frac{\sqrt{3}}{4}\mathrm{i}, 1\}$.

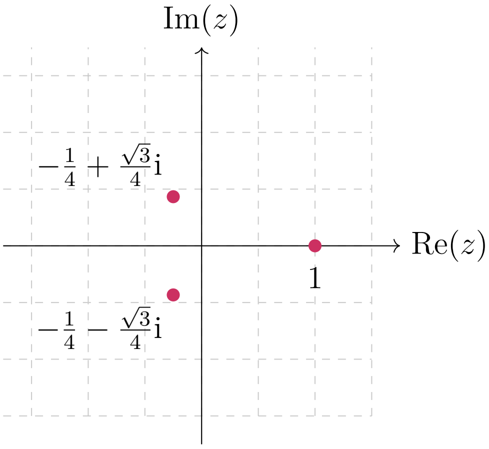

Figure 3.3: The solutions $\Lambda = \{1, -\frac{1}{4} - \frac{\sqrt{3}}{4}\mathrm{i}, -\frac{1}{4} + \frac{\sqrt{3}}{4}\mathrm{i}\}$.

Here, the Malthusian parameter α corresponds to the root 1 as it is the only solution in $\Lambda \cap \mathbb{R}$. In order to determine $\varrho|_{\mathcal{D}(\mathcal{L}\boldsymbol{\mu})\cap\mathbb{R}}$, we need to consider the spectral radius $\rho_{\mathcal{L}\boldsymbol{\mu}(\theta)}$, $\theta \in \mathbb{R}$, which corresponds to the largest solution of

$$\det(\lambda\mathrm{I}_2 - \mathcal{L}\boldsymbol{\mu}(\theta)) = \det\begin{pmatrix} \lambda-\frac{1}{1+\theta} & -\frac{1}{2\theta} \\ -\frac{1}{2\theta} & \lambda-\frac{1}{2\theta} \end{pmatrix} = \left(\lambda-\frac{1}{1+\theta}\right)\left(\lambda-\frac{1}{2\theta}\right) - \frac{1}{4\theta^2} = 0.$$

We conclude

$$\lambda_{1,2} = \frac{1}{4}\left(\frac{1}{\theta} + \frac{2}{1+\theta} \pm \frac{\sqrt{5+6\theta+5\theta^2}}{\theta(1+\theta)}\right).$$

If $\theta > 0$, then $\varrho(\theta) = \frac{1}{4}\left(\frac{1}{\theta} + \frac{2}{1+\theta} + \frac{\sqrt{5+6\theta+5\theta^2}}{\theta(1+\theta)}\right)$. In particular, $\varrho(1) = 1$ leading to $\alpha = 1$ as Malthusian parameter.

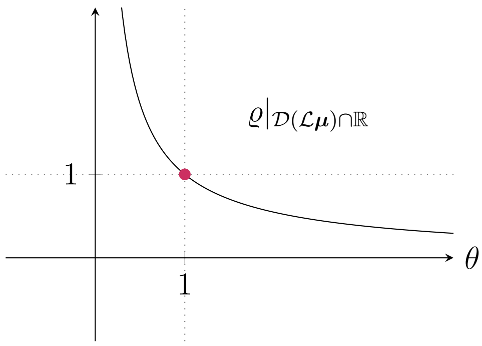

Figure 3.4: Graph of the function $\varrho|_{\mathcal{D}(\mathcal{L}\boldsymbol{\mu})\cap\mathbb{R}}$ with Malthusian parameter $\alpha = 1$.

□

In part 2.1.3 we recalled the Perron-Frobenius theorem, which provides a detailed description of the spectral radius ρ_M for a nonnegative primitive matrix M. In this case, the spectral radius ρ_M is a simple eigenvalue with $|\lambda| < \rho_M$ for all eigenvalues $\lambda \in \sigma_M \setminus \{\rho_M\}$ as shown in figure A.1. However, in the general case, where the matrix of intensity measures $\boldsymbol{\mu}$ is not a nonnegative primitive matrix, the Malthusian parameter α, if it exists, is not comparable with the properties of the Perron-Frobenius eigenvalue. Although we have shown in Proposition 3.3 that all solutions $\lambda \in \Lambda \setminus \{\alpha\}$ satisfy $\mathrm{Re}(\lambda) < \alpha$, the Malthusian parameter does not represent the eigenvalue with the largest absolute value, i.e. $|\lambda| \not< \alpha$ for all $\lambda \in \Lambda \setminus \{\alpha\}$. Moreover, the Malthusian parameter may have multiplicity > 1. To demonstrate this, consider the following both examples.

Example 3.5 Let $p = 2$ and suppose that $\xi^{1,1}$ is a Poisson point process on $[0, \infty)$ with intensity measure $g(x)\mathrm{d}x$ and $g(x)$ as in (3.10), i.e., the density of the Gamma distribution with parameters $a, b > 0$. Here, we choose $a = 4$ and $b = 1$. Further, $\xi^{1,2}$, $\xi^{2,1}$ and $\xi^{2,2}$ are homogeneous Poisson point processes on $[0, \infty)$ with rate $\frac{1}{2}$. Therefore,

$$\boldsymbol{\mu}(\mathrm{d}x) = \begin{pmatrix} \mathbb{1}_{(0,\infty)}(x)\frac{x^3}{3!}e^{-x} & \frac{1}{2}x \\ \frac{1}{2}x & \frac{1}{2}x \end{pmatrix}.$$

Since

$$\begin{aligned} \mathcal{L}\mu^{1,1}(z) &= \int_0^\infty e^{-zx}\mu^{1,1}(\mathrm{d}x) = \int_0^\infty \mathbb{1}_{(0,\infty)}(x)\frac{x^3}{3!}e^{-x}e^{-zx}\mathrm{d}x \\ &= \frac{1}{3!}\int_0^\infty x^3 e^{-x(1+z)} = \frac{1}{(1+z)^4}, \end{aligned}$$

and,

$$\mathcal{L}\mu^{1,2}(z) = \mathcal{L}\mu^{2,1}(z) = \mathcal{L}\mu^{2,2}(z) = \int_0^\infty \frac{1}{2}e^{-zx}\mathrm{d}x = \frac{1}{2z},$$

we infer

$$\mathcal{L}\boldsymbol{\mu}(z) = \begin{pmatrix} \frac{1}{(1+z)^4} & \frac{1}{2z} \\ \frac{1}{2z} & \frac{1}{2z} \end{pmatrix}.$$

Then, we derive the characteristic equation

$$\det(\mathrm{I}_2 - \mathcal{L}\boldsymbol{\mu}(z)) = \det \begin{pmatrix} 1 - \frac{1}{(1+z)^4} & -\frac{1}{2z} \\ -\frac{1}{2z} & 1 - \frac{1}{2z} \end{pmatrix} = \left(1 - \frac{1}{(1+z)^4}\right)\left(1 - \frac{1}{2z}\right) - \frac{1}{4z^2}.$$

The solutions of $\det(\mathrm{I}_2 - \mathcal{L}\boldsymbol{\mu}(z)) = 0$ are given approximatively by

$$\Lambda \approx \{-2.01, -1.02 - 1.00\mathrm{i}, -1.024 + 1.00\mathrm{i}, -0.135 - 0.23\mathrm{i}, -0.135 + 0.23\mathrm{i}, 0.83\}.$$

In this situation, the Malthusian parameter corresponds to $\alpha \approx 0.83$ and each solution $\lambda \in \Lambda \setminus \{\alpha\}$ fulfills $\mathrm{Re}(\lambda) < \alpha$, but not $|\lambda| < \alpha$ as shown in the following figure.

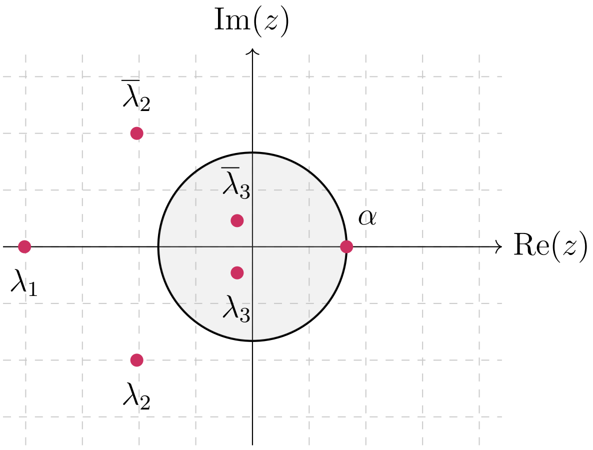

Figure 3.5: Illustration of the approximative solutions Λ with Malthusian parameter $\alpha \approx 0.83$.

□

Example 3.6 Let $p = 2$ and suppose that $\xi^{1,1}$ and $\xi^{2,2}$ are homogeneous Poisson point processes on $[0, \infty)$ with rate $a > 0$ and that $\xi^{1,2}$ is a homogeneous Poisson point process with rate $b > 0$. Finally, let $\xi^{2,1} = 0$. We conclude

$$\boldsymbol{\xi} = \begin{pmatrix} \xi^{1,1} & \xi^{1,2} \\ 0 & \xi^{1,1} \end{pmatrix} \quad \text{and,} \quad \boldsymbol{\mu}(\mathrm{d}x) = \begin{pmatrix} ax & bx \\ 0 & ax \end{pmatrix}.$$

Thus, $\boldsymbol{\mu}$ is not a primitive matrix since

$$\boldsymbol{\mu}^k(\mathrm{d}x) = \begin{pmatrix} a^k x^k & k a^{k-1} b x^k \\ 0 & a^k x^k \end{pmatrix} \not\geq 0$$

for all $k \in \mathbb{N}_0$. Furthermore,

$$\mathcal{L}\boldsymbol{\mu}(z) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \frac{1}{z}, \quad \mathrm{Re}(z) > 0.$$

Then, we derive the characteristic equation

$$\det(\mathrm{I}_2 - \mathcal{L}\boldsymbol{\mu}(z)) = \det \begin{pmatrix} 1 - \frac{a}{z} & -\frac{b}{z} \\ 0 & 1 - \frac{a}{z} \end{pmatrix} = \left(1 - \tfrac{a}{z}\right)^2.$$

Notice that the only solution of $\det(\mathrm{I}_2 - \mathcal{L}\boldsymbol{\mu}(z)) = \left(1 - \frac{a}{z}\right)^2 = 0$ is a leading to $\Lambda = \{a\}$. Since the spectral radius $\rho_{\mathcal{L}\boldsymbol{\mu}(\theta)}$, $\theta \in \mathbb{R}$ corresponds to the largest solution of

$$\det(\lambda \mathrm{I}_2 - \mathcal{L}\boldsymbol{\mu}(\theta)) = \det \begin{pmatrix} \lambda - \frac{a}{\theta} & -\frac{b}{\theta} \\ 0 & \lambda - \frac{a}{\theta} \end{pmatrix} = \left(\lambda - \tfrac{a}{\theta}\right)^2,$$

we conclude $\varrho(\theta) = \rho_{\mathcal{L}\boldsymbol{\mu}(\theta)} = \left|\frac{a}{\theta}\right|$. In this case, $\varrho(a) = 1$, such that a is the Malthusian parameter, but with multiplicity 2.

☐

The next example illustrates that the order of the pole λ in $(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ can be strictly smaller than the order of the zero λ of the determinant of $\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z)$.

Example 3.7 Let $m : [0, \infty) \to (0, \infty)$ be the Laplace transform of a locally finite measure on $[0, \infty)$ with $1 < m(\vartheta) < \infty$ for some $\vartheta \in \mathbb{R}$ and $\lim_{\theta \to \infty} m(\theta) < 1$. Consequently, this ensures the existence of some $\alpha \in \mathbb{R}$ such that $m(\alpha) = 1$. Consider

$$\mathcal{L}\boldsymbol{\mu}(z) = \begin{pmatrix} m(z) & 0 \\ 0 & m(z) \end{pmatrix}, \quad \mathrm{Re}(z) \geq \vartheta.$$

Then

$$\det(\mathrm{I}_2 - \mathcal{L}\boldsymbol{\mu}(z)) = \det \begin{pmatrix} 1 - m(z) & 0 \\ 0 & 1 - m(z) \end{pmatrix} = (1 - m(z))^2.$$

Any root of the determinant is, therefore, a root of even multiplicity. In particular, since $m'(\alpha) \in (-\infty, 0)$, the Malthusian parameter α is a zero of the determinant with multiplicity 2. On the other hand,

$$(\mathrm{I}_2 - \mathcal{L}\boldsymbol{\mu}(z))^{-1} = \begin{pmatrix} \frac{1}{1-m(z)} & 0 \\ 0 & \frac{1}{1-m(z)} \end{pmatrix} = \begin{pmatrix} \frac{1}{-m'(\alpha)} & 0 \\ 0 & \frac{1}{-m'(\alpha)} \end{pmatrix} (z - \alpha)^{-1} + \begin{pmatrix} h(z) & 0 \\ 0 & h(z) \end{pmatrix}$$

where $h(z)$ is holomorphic at $z = \alpha$. In other words, $(\mathrm{I}_2 - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ has a pole of order 1 at $z = \alpha$.

☐

3.4 *Sgibnev's Result*

In this section, we will introduce briefly the result of M. S. Sgibnev, who, under suitable assumptions, derived an asymptotic expansion for the solution F to (3.1) in the non-lattice case, see [Sgi02, Theorem 3].

Let $\varphi(x)$, $x \in \mathbb{R}$, be a semimultiplicative function, that is a finite, positive and Borel measurable function satisfying the conditions $\varphi(x) = 1$ for $x \leq 0$ and $\varphi(x+y) \leq \varphi(x) + \varphi(y)$, for $x, y \in \mathbb{R}$. Based on this, Sgibnev assumes the finiteness of a φ-moment of the $p \times p$ matrix of measures $\boldsymbol{\mu}$ for some semimultiplicative function φ such that $\varphi(x)e^{-r_+ x}$ is nondecreasing on $[0, \infty)$ for some $r_+ \geq 0$.

In addition, Sgibnev requires, in our notation, that the matrix $\mathcal{L}\boldsymbol{\mu}(0)$ is irreducible and the spectral radius of the matrix $\mathcal{L}(\boldsymbol{\mu}^{*m})_{\mathsf{s}}(r_+)$ is strictly smaller than 1 for some convolution power $m \in \mathbb{N}$. Here and throughout this section, we denote by ν_{s} the singular part of a measure ν on the Borel sets of $[0, \infty)$ and extend this canonically to matrices of measures by applying the singular part entrywise. Similarly, we use ν_{a} to refer the absolutely continuous part of a measure ν, so that $\nu = \nu_{\mathsf{a}} + \nu_{\mathsf{s}}$.

Finally, assuming the finiteness of the set $\Lambda \subseteq \mathbb{C}$, i.e. the set of solutions to the characteristic equation (3.8), and a φ-moment of the function $|f|$ for some semimultiplicative function φ, Sgibnev establishes an asymptotic expansion for F of the form

$$F(t) = \sum_{\lambda \in \Lambda} \sum_{k=0}^{k_\lambda - 1} C_{\lambda,k}\, t^k e^{\lambda t} + \Delta_f(t) \quad \text{as } t \to \infty. \tag{3.11}$$

Here, $k_\lambda \in \mathbb{N}$ is the multiplicity of the solution $\lambda \in \Lambda$, $C_{\lambda,k}$ is a suitable real $p \times p$ matrix, and $\Delta_f(t)$ is a lower-order remainder term.

3.5 *Assumptions for the Main Results*

We now focus on the assumptions that are essential for establishing our main results, the asymptotic expansion for the Markov renewal measure $\boldsymbol{U}$ and for general solutions F to (3.1) as $t \to \infty$, in both, the lattice and the non-lattice case. In addition to assumptions (A1) and (A2), which we have previously discussed, there is another (technical) assumption that is crucial for our analysis. In this section, we examine this assumption in detail and discuss its significance for our results. Furthermore, we will relate our assumptions with those presented in the literature, particularly Sgibnev's assumptions.

We commence our discussion by considering Equation (3.7), which is given by $\mathcal{L}F(z) = \mathcal{L}f(z)(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$, for $z \in \mathbb{C}$. This identity suggests the application of the Laplace inversion formula, see e.g. [Wid41, Theorem 10.1], to derive a representation for F, namely,

$$F(t) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} e^{tz} \mathcal{L}F(z)\, \mathrm{d}z.$$

However, to apply the Laplace inversion formula, it is necessary that the right-hand side of (3.7) satisfies certain integrability properties, for instance, that each entry of the matrix $(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ is integrable along vertical lines. Therefore, we work using the following assumption.

Assumption (A4)

For some $\vartheta \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu})^{\circ} \cap \mathbb{R}$ it holds

$$\sup_{|\eta|\geq\eta_0} \|(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta))^{-1}\| < \infty \tag{3.12}$$

for some $\eta_0 \geq 0$.

Here, we assume without loss of generality that ϑ in (3.12) and assumption (A1) coincide.

Besides this assumption, the literature presents several other conditions that can be associated with (A4). For instance, in the recent work by A. Iksanov, K. Kolesko, and M. Meiners [IKM24], as well as in the work by S. Janson and R. Neininger [JN08], both addressing the single-type case, the condition $\limsup_{\eta\to\infty} |\mathcal{L}\boldsymbol{\mu}(\vartheta+\mathrm{i}\eta)| < 1$ is used to ensure integrability along vertical lines. A slightly weaker (multi-type) version appears in Lemma 3.8 as Condition (C). Furthermore, we recall Sgibnev's assumption, mentioned in the previous section, which states $\rho_{\mathcal{L}(\boldsymbol{\mu}^{*m})_{\mathsf{s}}(\vartheta)} < 1$ for some $m \in \mathbb{N}$. In the following lemma, we will relate these assumptions to one another and identify equivalent conditions.

Lemma 3.8 [Comparison of conditions]

Suppose that (A1) *and* (A2) *hold. Consider the conditions,*

(A) There is an $m \in \mathbb{N}$ such that $\rho_{\mathcal{L}(\boldsymbol{\mu}^{*m})_{\mathsf{s}}(\vartheta)} < 1$.

(B) There is an $m \in \mathbb{N}$ such that $\|\mathcal{L}(\boldsymbol{\mu}^{*m})_{\mathsf{s}}(\vartheta)\| < 1$.

(C) There is an $m \in \mathbb{N}$ such that $\limsup_{\eta\to\infty} \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^m\| < 1$.

(D) There is an $m \in \mathbb{N}$ such that $\limsup_{\eta\to\infty} \sup_{\theta\geq\vartheta} \|\mathcal{L}\boldsymbol{\mu}(\theta + \mathrm{i}\eta)^m\| < 1$.

(E) There is an $\eta_0 \geq 0$ such that $\sup_{\eta\geq\eta_0} \|(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta))^{-1}\| < \infty$.

(F) There is $\eta_0 \geq 0$ such that $\sup_{\eta\geq\eta_0,\theta\geq\vartheta} \|(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\theta + \mathrm{i}\eta))^{-1}\| < \infty$.

Then (A) *is equivalent to* (B), (B) *implies* (C), (C) *is equivalent to* (D) *and, each of the conditions* (C), (D) *and* (F) *implies* (E).

If, additionally, Λ_ϑ *is finite, then* (E) *and* (F) *are equivalent. Finally,* (D) *implies that* Λ_ϑ *is finite.*

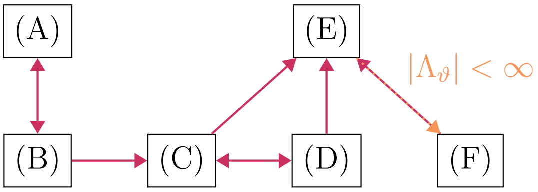

Figure 3.6: Illustration of the relationships between the conditions in Lemma 3.8.

Proof of Lemma 3.8.

① Since $\rho_{\mathcal{L}(\boldsymbol{\mu}^{*m})_{\mathsf{s}}(\vartheta)} \leq \|\mathcal{L}(\boldsymbol{\mu}^{*m})_{\mathsf{s}}(\vartheta)\|$, as noted in Lemma A.2, we conclude that (B) implies (A). To see that the converse implication also holds true, first observe that $(\boldsymbol{\nu}^{*n})_{\mathsf{s}} \leq (\boldsymbol{\nu}_{\mathsf{s}})^{*n}$ for any matrix $\boldsymbol{\nu}$ of measures on the Borel sets of $[0,\infty)$ and $n \in \mathbb{N}$. Indeed, denoting by $\boldsymbol{\nu}_{\mathsf{a}}$ the absolutely continuous part of $\boldsymbol{\nu}$, we have $\boldsymbol{\nu}^{*n} = (\boldsymbol{\nu}_{\mathsf{a}} + \boldsymbol{\nu}_{\mathsf{s}})^{*n} = (\boldsymbol{\nu}_{\mathsf{s}})^{*n} + \boldsymbol{\nu}'$, where $\boldsymbol{\nu}'$ is a sum of convolution products of length n of measures $\boldsymbol{\nu}_{\mathsf{a}}$ and $\boldsymbol{\nu}_{\mathsf{s}}$ with at least one convolution factor $\boldsymbol{\nu}_{\mathsf{a}}$. In particular, $\boldsymbol{\nu}'$ is absolutely continuous. This implies

$$(\boldsymbol{\nu}^{*n})_{\mathsf{s}} = ((\boldsymbol{\nu}_{\mathsf{s}})^{*n} + \boldsymbol{\nu}')_{\mathsf{s}} = ((\boldsymbol{\nu}_{\mathsf{s}})^{*n})_{\mathsf{s}} \leq (\boldsymbol{\nu}_{\mathsf{s}})^{*n}. \tag{3.13}$$

Let $m \in \mathbb{N}$ be such that the condition (A) holds. From Gelfand's formula, see Lemma A.1, we infer the existence of some $n \in \mathbb{N}$ with $\|(\mathcal{L}\boldsymbol{\mu}_{\mathsf{s}}^{*m}(\vartheta))^n\| < 1$. Set $\boldsymbol{\nu} = \boldsymbol{\mu}^{*m}$ in (3.13) to infer

$$((\boldsymbol{\mu}^{*m})_{\mathsf{s}})^{*n} \geq ((\boldsymbol{\mu}^{*m})^{*n})_{\mathsf{s}} = (\boldsymbol{\mu}^{*mn})_{\mathsf{s}}.$$

In particular, $\|\mathcal{L}(\boldsymbol{\mu}^{*mn})_{\mathsf{s}}(\vartheta)\| \leq \|(\mathcal{L}\boldsymbol{\mu}_{\mathsf{s}}^{*m}(\vartheta))^n\| < 1$ proving that (B) is satisfied for $mn \in \mathbb{N}$.

② Now assume that (B) holds, i.e., $\|\mathcal{L}(\boldsymbol{\mu}^{*m})_{\mathsf{s}}(\vartheta)\| < 1$ for some $m \in \mathbb{N}$. We write

$$\|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^m\| = \|\mathcal{L}\boldsymbol{\mu}^{*m}(\vartheta + \mathrm{i}\eta)\| \leq \|\mathcal{L}(\boldsymbol{\mu}^{*m})_{\mathsf{s}}(\vartheta + \mathrm{i}\eta)\| + \|\mathcal{L}(\boldsymbol{\mu}^{*m})_{\mathsf{a}}(\vartheta + \mathrm{i}\eta)\|.$$

The Riemann-Lebesgue lemma yields $\mathcal{L}(\boldsymbol{\mu}^{*m})_{\mathsf{a}}(\vartheta + \mathrm{i}\eta) \to 0$ as $|\eta| \to \infty$. Consequently, we infer

$$\limsup_{\eta \to \pm\infty} \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^m\| \leq \limsup_{\eta \to \pm\infty} \|\mathcal{L}(\boldsymbol{\mu}^{*m})_{\mathsf{s}}(\vartheta + \mathrm{i}\eta)\| \leq \|\mathcal{L}(\boldsymbol{\mu}^{*m})_{\mathsf{s}}(\vartheta)\| < 1$$

which gives condition (C).

③ To prove the equivalence of (C) and (D), it is sufficient to prove the implication from (C) to (D) since (D) is formally stronger than (C). Since the argument is somewhat more involved, it is moved to Lemma 3.10 below.

④ We continue with the proof that (C) implies (E). Notice that validity of (C) ensures the existence of $\eta_0 \geq 0$ and $0 < \delta < 1$ such that, for $|\eta| \geq \eta_0$, $\|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^m\| \leq 1 - \delta$.

For any such η, the von Neumann formula, see Remark A.9, provides

$$\|(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta))^{-1}\| = \left\| \sum_{n=0}^{\infty} \mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^n \right\|.$$

Notice that the Neumann series $\sum_{n=0}^{\infty} \mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^n$ converges, as we have

$$(\rho(\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)))^m = \rho(\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^m) \leq \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^m\| \leq 1 - \delta$$

which implies $\rho(\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)) < 1$. Further, we write the exponent in the form $n = mk + l$, where $k \in \mathbb{N}_0$ and $0 \leq l < m$. This allows us to separate the powers of the matrix $\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)$ into blocks of size m, i.e. $\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^n = \mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^{mk+l} = (\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^m)^k \mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^l$. Using this block structure, we obtain

$$\begin{aligned} \left\| \sum_{n=0}^{\infty} \mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^n \right\| &= \left\| \sum_{k=0}^{\infty} \sum_{l=0}^{m-1} (\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^m)^k \mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^l \right\| \\ &\leq \sum_{k=0}^{\infty} \sum_{l=0}^{m-1} \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^m\|^k \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)\|^l. \end{aligned}$$

Given that $\|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^m\| < 1$ for any such η, we conclude

$$\sum_{k=0}^{\infty} \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^m\|^k = \frac{1}{1 - \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^m\|} \leq \frac{1}{1 - (1 - \delta)} = \frac{1}{\delta},$$

Consequently, this yields

$$\sum_{k=0}^{\infty} \sum_{l=0}^{m-1} \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^m\|^k \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)\|^l \leq \sum_{l=0}^{m-1} \frac{1}{\delta} \|\mathcal{L}\boldsymbol{\mu}(\vartheta)\|^l < \infty,$$

proving that also condition (E) holds.

⑤ Since (F) is formally stronger than (E), we continue with the proof that (E) implies (F) under the additional assumption that the set Λ_ϑ is finite. To this end, observe that if Λ_ϑ is finite, then we can find $\eta_0 \geq 0$ such that the function $z \mapsto (\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ is holomorphic on a neighborhood of $\{\mathrm{Re}(z) \geq \vartheta, \mathrm{Im}(z) \geq \eta_0\}$. Since $\mathcal{L}\boldsymbol{\mu}(\theta + \mathrm{i}\eta_0) \to \boldsymbol{\mu}(0)$ as $\theta \to \infty$ by the dominated convergence theorem and $\rho_{\boldsymbol{\mu}(0)} < 1$, we conclude that

$$\sup_{\theta \geq \vartheta} \left\| (\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\theta + \mathrm{i}\eta_0))^{-1} \right\| < \infty.$$

Condition (E) ensures that the holomorphic function $z \mapsto (\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ is bounded on the boundary of the quadrant $\{\mathrm{Re}(z) \geq \theta, \mathrm{Im}(z) \geq \eta_0\}$. The function is therefore bounded on this set.

⑥ Finally, suppose that (D) holds true. We prove that the set Λ_ϑ is then finite. It suffices to prove that Λ_ϑ is contained in a compact rectangle, which in turn is contained in $\overline{\mathcal{H}}_\vartheta := \{z \in \mathbb{C} : \mathrm{Re}(z) \geq \vartheta\}$. Indeed, the entries of $\boldsymbol{\mathcal{L}\mu}(z)$ are holomorphic on $\overline{\mathcal{H}}_\vartheta$, and hence so is $\det(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))$. Further, $\det(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))$ is not identically zero by Proposition 3.1 since the continuity of $\varrho|_{\mathcal{D}(\boldsymbol{\mathcal{L}\mu})\cap\mathbb{R}}$ and $\varrho(\theta) \to \rho_{\boldsymbol{\mu}(0)} < 1$ as $\theta \to \infty$ ensures the existence of some sufficiently large $\theta \in \mathbb{R}$ with $\varrho(\theta) < 1$ and leads to $\det(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(\theta)) > 0$ by Lemma A.8. Hence the set of its zeroes has no accumulation point within $\overline{\mathcal{H}}_\vartheta$ and in particular not within the rectangle.

To prove the existence of such a compact rectangle, first recall that $\det(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(\lambda)) = 0$ implies that $\boldsymbol{\mathcal{L}\mu}(\lambda)$ has eigenvalue 1 and, as a consequence, $\varrho(\lambda) \geq 1$. On the other hand, $\varrho(\theta) \to \rho_{\boldsymbol{\mu}(0)} < 1$ as $\theta \to \infty$ by Proposition 3.1 and (A2). Consequently, $\varrho(z) \leq \varrho(\mathrm{Re}(z)) < 1$ for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > \theta$ and some sufficiently large θ. Further, from (D) we conclude that for some $\varepsilon > 0$ and $m \in \mathbb{N}$, we find $\eta_0 > 0$ such that $\rho_{\boldsymbol{\mathcal{L}\mu}(z)^m} \leq \|\boldsymbol{\mathcal{L}\mu}(z)^m\| \leq 1 - \varepsilon$ whenever $\mathrm{Re}(z) \geq \vartheta$ and $|\,\mathrm{Im}(z)| \geq \eta_0$. For such z, $\boldsymbol{\mathcal{L}\mu}(z)$ can not have eigenvalue 1 proving that $\Lambda_\vartheta \subseteq \{z \in \mathbb{C} : \vartheta \leq \mathrm{Re}(z) \leq \theta,\ |\,\mathrm{Im}(z)| \leq \eta_0\}$.

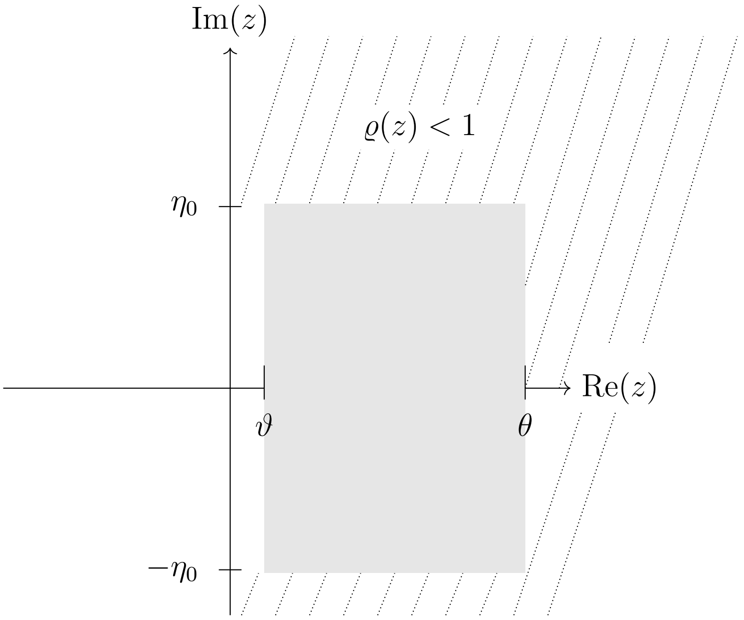

Figure 3.7: Λ_ϑ is contained in a compact rectangle in the right half-plane.

$\blacksquare$

We now turn to the task of proving that condition (C) implies condition (D). As mentioned before, in [IKM24] and [JN08], where the single-type case is treated, the assumption $\limsup_{|\eta|\to\infty} |\boldsymbol{\mathcal{L}\mu}(\vartheta + \mathrm{i}\eta)| < 1$ is used. S. Janson and R. Neininger show in [JN08, Lemma 2.1] that this condition implies $\limsup_{|\eta|\to\infty} \sup_{\theta\geq\vartheta} |\boldsymbol{\mathcal{L}\mu}(\theta + \mathrm{i}\eta)| < 1$ and, in particular, the finiteness of the set Λ_ϑ. Our result is an adapted version of this implication, and the corresponding proof extends to our setting without major obstacles. Nevertheless, we present a version of the proof adapted to the situation here for the readers' convencience.

Lemma 3.9 [Condition (C) implies (D)]
Suppose that (A1) *and* (A2) *hold. If the property*

$$\limsup_{\eta\to\pm\infty} \|\boldsymbol{\mathcal{L}\mu}(\vartheta+\mathrm{i}\eta)^m\| < 1 \tag{3.14}$$

holds for some $m\in\mathbb{N}$, *then*

$$\limsup_{\eta\to\pm\infty}\sup_{\theta\geq\vartheta} \|\boldsymbol{\mathcal{L}\mu}(\theta+\mathrm{i}\eta)^m\| < 1. \tag{3.15}$$

Proof of Lemma 3.10.
From (A1), we have $\overline{\mathcal{H}}_\vartheta = \{z\in\mathbb{C} : \mathrm{Re}(z)\geq\vartheta\}\subseteq\mathcal{D}(\boldsymbol{\mathcal{L}\mu}(\vartheta))^{\mathrm{o}}$. Now suppose that, for some $m\in\mathbb{N}$, (3.14) holds. Choose $\varepsilon>0$ with

$$\limsup\nolimits_{\eta\to\infty}\|\boldsymbol{\mathcal{L}\mu}(\vartheta+\mathrm{i}\eta)^m\| < 1-3\varepsilon$$

and $\zeta\geq 0$ with $\|\boldsymbol{\mathcal{L}\mu}(\vartheta+\mathrm{i}\eta)^m\|\leq 1-2\varepsilon$ when $\eta\geq\zeta$ and thus also when $\eta\leq-\zeta$. Define $C := \sup_{\eta\in\mathbb{R}}\|\boldsymbol{\mathcal{L}\mu}(\vartheta+\mathrm{i}\eta)^m\|$ and notice that $C<\infty$. The entries of $(\boldsymbol{\mathcal{L}\mu})^m$ are Laplace transforms of locally finite measures on $[0,\infty)$ and all bounded and continuous on $\overline{\mathcal{H}}_\vartheta$, and holomorphic, thus harmonic, on a neighborhood of $\mathcal{H}_\vartheta$. We may thus apply the Poisson transformation on the closed half-space and infer that $(\boldsymbol{\mathcal{L}\mu})^m$ is the integral of its boundary values with respect to the Poisson kernel $\mathsf{P}_x(y) := \frac{1}{\pi}\frac{x}{x^2+y^2}$ for $x,y\in\mathbb{R}$. More precisely,

$$\boldsymbol{\mathcal{L}\mu}(\theta+\mathrm{i}\eta)^m = \int_{-\infty}^{\infty}\boldsymbol{\mathcal{L}\mu}(\vartheta+\mathrm{i}y)^m\,\mathsf{P}_{\theta-\vartheta}(\eta-y)\,\mathrm{d}y, \quad \theta>\vartheta,\ \eta\in\mathbb{R}, \tag{3.16}$$

see e.g. [Gar81, Lemma 3.4]. We conclude

$$\begin{aligned}
\|\boldsymbol{\mathcal{L}\mu}(\theta+\mathrm{i}\eta)^m\| &= \left\|\int_{-\infty}^{\infty}\boldsymbol{\mathcal{L}\mu}(\vartheta+\mathrm{i}y)^m\,\mathsf{P}_{\theta-\vartheta}(\eta-y)\,\mathrm{d}y\right\| \\
&\leq \int_{-\infty}^{\infty}\left\|\boldsymbol{\mathcal{L}\mu}(\vartheta+\mathrm{i}y)^m\right\|\mathsf{P}_{\theta-\vartheta}(\eta-y)\,\mathrm{d}y \\
&= \int_{\{|y|>\zeta\}}\left\|\boldsymbol{\mathcal{L}\mu}(\vartheta+\mathrm{i}y)^m\right\|\mathsf{P}_{\theta-\vartheta}(\eta-y)\,\mathrm{d}y + \int_{-\zeta}^{\zeta}\left\|\boldsymbol{\mathcal{L}\mu}(\vartheta+\mathrm{i}y)^m\right\|\mathsf{P}_{\theta-\vartheta}(\eta-y)\,\mathrm{d}y \\
&< 1-2\varepsilon + C\int_{-\zeta}^{\zeta}\mathsf{P}_{\theta-\vartheta}(\eta-y)\,\mathrm{d}y.
\end{aligned} \tag{3.17}$$

Define $\omega(\theta+\mathrm{i}\eta) := \int_{-\zeta}^{\zeta}\mathsf{P}_{\theta-\vartheta}(\eta-y)\,\mathrm{d}y = \mathbf{P}(Y\in[\frac{\eta-\zeta}{\theta-\vartheta},\frac{\eta+\zeta}{\theta-\vartheta}])\geq 0$ where Y is a real-valued random variable with Lebesgue density $\frac{1}{\pi}\frac{1}{1+y^2}$, $y\in\mathbb{R}$. The set $\mathcal{H} := \{z\in$

$\mathcal{H}_\vartheta : \omega(z) > \frac{\varepsilon}{C}\}$ is an intersection of the right half-plane $\mathcal{H}_\vartheta$ and a circular disk, meaning it is a bounded set. Thus, $H' := \sup\{\mathrm{Im}(z) : z \in \mathcal{H}\} < \infty$. Now, if $\mathrm{Re}(z) \geq \theta$ and $|\,\mathrm{Im}(z)| > H'$, then $\omega(z) \leq \frac{\varepsilon}{C}$. Hence,

$$\|\mathcal{L}\boldsymbol{\mu}(\theta + \mathrm{i}\eta)^m\| \leq 1 - \varepsilon$$

proving that (3.15) holds.

■

After discussing related conditions from the literature, we now return to assumption (3.12) and proceed to investigate it. First, notice that (3.12) implies $\mathrm{Re}(\lambda) \neq \vartheta$ for every $\lambda \in \Lambda$. Additionally, the following observation holds.

Lemma 3.10

Suppose that (A1), (A2) *and* (A4) *hold for some* $\vartheta \in \mathbb{R}$ *and that* Λ_ϑ *is finite. Then there exists some* $\eta_0 \geq 0$ *such that*

$$\sup_{\theta \geq \vartheta, \eta \geq \eta_0} \|(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\theta + \mathrm{i}\eta))^{-1}\| < \infty.$$

Proof of Lemma 3.10.

Fix $\eta_0 \geq 0$ so large that $\mathrm{Im}(\lambda) < \eta_0$ for every $\lambda \in \Lambda_\vartheta$. Then the function $z \mapsto (\mathrm{I}_p - \mathcal{L}\mu(z))^{-1}$ is holomorphic on a neighborhood of $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq \vartheta, \mathrm{Im}(z) \geq \eta_0\}$. Since

$$\lim_{\theta \to \infty} \mathcal{L}\boldsymbol{\mu}(\theta + \mathrm{i}\eta) = \boldsymbol{\mu}(0)$$

and $\rho_{\boldsymbol{\mu}(0)} < 1$ by Proposition 3.1 and (A2), we conclude that

$$\lim_{\theta \to \infty} \det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\theta + \mathrm{i}\eta_0)) = \det(\mathrm{I}_p - \boldsymbol{\mu}(0)) > 0,$$

where the strict inequality follows from Lemma A.8. In view of the classical formula (3.9), this implies

$$\sup_{\theta \geq \vartheta} \|(\mathrm{I}_p - \mathcal{L}\mu(\theta + \mathrm{i}\eta_0))^{-1}\| < \infty$$

since the entries of the adjoint matrix are bounded on the half-line $\{\theta + \mathrm{i}\eta_0 : \theta \geq \vartheta\}$. The claim now follows from the maximum principle for holomorphic functions.

■

Remark 3.11 $\big[$Sgibnev's assumption implies (A4)$\big]$

Suppose that (A1) and (A2) hold. Then condition (A) in Lemma 3.8, i.e. $\rho(\mathcal{L}(\boldsymbol{\mu}_{\mathsf{s}}^{*m})(\vartheta)) < 1$ for some $m \in \mathbb{N}$, implies

$$\sup_{\eta \in \mathbb{R}} \|(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta))^{-1}\| < \infty$$

if there is no $\lambda \in \Lambda$ with $\mathrm{Re}(\lambda) = \vartheta$.

Proof of Remark 3.11.

Fix $m \in \mathbb{N}$ with $\rho(\mathcal{L}(\boldsymbol{\mu}_{\mathsf{s}}^{*m})(\vartheta)) < 1$, where $\boldsymbol{\mu}_{\mathsf{s}}^{*m}$ is the singular part of $\boldsymbol{\mu}^{*m}$. Write $\boldsymbol{\mu}_{\mathsf{a}}^{*m}$ for the absolutely continuous part of $\boldsymbol{\mu}^{*m}$ so that $\boldsymbol{\mu}^{*m} = \boldsymbol{\mu}_{\mathsf{a}}^{*m} + \boldsymbol{\mu}_{\mathsf{s}}^{*m}$. Clearly,

$$\|(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta))^{-1}\| = \left\| \sum_{n=0}^{\infty} \mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^n \right\| \leq \sum_{n=0}^{\infty} \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^n\|$$

if the series $\sum_{n=0}^{\infty} \mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^n$ converges, i.e. if $\varrho(\vartheta + \mathrm{i}\eta) < 1$. By writing the exponent in the form $n = mk + l$, where $k \in \mathbb{N}_0$ and $0 \leq l < m$, we separate the powers of the matrix $\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)$ into blocks of size m. Using this block structure and the submultiplicativity of the (operator) matrix norm, we infer

$$\begin{aligned}
\sum_{n=0}^{\infty} \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^n\| &\leq \sum_{k=0}^{\infty} \sum_{j=0}^{m-1} \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^{mk+l}\| \\
&\leq \sum_{k=0}^{\infty} \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^{mk}\| \sum_{l=0}^{m-1} \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)^{l}\| \\
&\leq \sum_{k=0}^{\infty} \|\mathcal{L}\boldsymbol{\mu}^{*m}(\vartheta + \mathrm{i}\eta)^{k}\| \sum_{l=0}^{m-1} \|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)\|^{l}.
\end{aligned}$$

Since $\|\mathcal{L}\boldsymbol{\mu}(\vartheta + \mathrm{i}\eta)\| \leq \|\mathcal{L}\boldsymbol{\mu}(\vartheta)\| < \infty$ by (A1), the last sum is bounded by a constant that does not depend on η. According to the distributive law, $\mathcal{L}\boldsymbol{\mu}^{*m}(\vartheta + \mathrm{i}\eta)^k = (\mathcal{L}\boldsymbol{\mu}_{\mathsf{a}}^{*m}(\vartheta + \mathrm{i}\eta) + \mathcal{L}\boldsymbol{\mu}_{\mathsf{s}}^{*m}(\vartheta + \mathrm{i}\eta))^k$ can be decomposed into $\mathcal{L}\boldsymbol{\mu}_{\mathsf{s}}^{*m}(\vartheta + \mathrm{i}\eta)^k$ plus a sum $S(k, \vartheta + \mathrm{i}\eta)$ of products consisting of k factors all of which are either $\mathcal{L}\boldsymbol{\mu}_{\mathsf{a}}^{*m}(\vartheta + \mathrm{i}\eta)$ or $\mathcal{L}\boldsymbol{\mu}_{\mathsf{s}}^{*m}(\vartheta + \mathrm{i}\eta)$ and at least one of which is $\mathcal{L}\boldsymbol{\mu}_{\mathsf{a}}^{*m}(\vartheta + \mathrm{i}\eta)$, i.e.

$$\|\mathcal{L}\boldsymbol{\mu}^{*m}(\vartheta + \mathrm{i}\eta)^k\| = \|\mathcal{L}\boldsymbol{\mu}_{\mathsf{s}}^{*m}(\vartheta + \mathrm{i}\eta)^k\| + \|S(k, \vartheta + \mathrm{i}\eta)\|.$$

We first deal with the term $\|\mathcal{L}\boldsymbol{\mu}_{\mathsf{s}}^{*m}(\vartheta + \mathrm{i}\eta)^k\|$. Pick $\varepsilon > 0$ so small that $\rho(\mathcal{L}\boldsymbol{\mu}_{\mathsf{s}}^{*m}(\vartheta)) < 1 - 3\varepsilon$ and $k \in \mathbb{N}$ so large that

$$\|\mathcal{L}\boldsymbol{\mu}_{\mathsf{s}}^{*m}(\vartheta + \mathrm{i}\eta)^k\| \leq \|\mathcal{L}\boldsymbol{\mu}_{\mathsf{s}}^{*m}(\vartheta)^k\| \leq (\rho(\mathcal{L}\boldsymbol{\mu}_{\mathsf{s}}^{*m}(\vartheta)) + \varepsilon)^k.$$

This is possible by Gelfand's formula for the spectral radius, see Lemma A.1. By the Riemann-Lebesgue lemma, for sufficiently large $|\eta| \geq \eta_0$, we can make sure that $\|\mathcal{L}\boldsymbol{\mu}_{\mathsf{a}}^{*m}(\vartheta+\mathrm{i}\eta)\| < \delta$ for $\delta > 0$ to be specified later. We can thus estimate the matrix norm of $S(k,\vartheta+\mathrm{i}\eta)$ by first using the triangular inequality and then the submultiplicativity of the matrix norm by a sum of k-fold products of the two factors $\|\mathcal{L}\boldsymbol{\mu}_{\mathsf{a}}^{*m}(\vartheta+\mathrm{i}\eta)\| \leq \|\mathcal{L}\boldsymbol{\mu}_{\mathsf{a}}^{*m}(\vartheta)\| < \delta$ and $\|\mathcal{L}\boldsymbol{\mu}_{\mathsf{s}}^{*m}(\vartheta+\mathrm{i}\eta)\| \leq \|\mathcal{L}\boldsymbol{\mu}_{\mathsf{s}}^{*m}(\vartheta)\|$. In each summand, at least one factor is bounded by δ, so we can factor it out and infer

$$\|S(k,\vartheta+\mathrm{i}\eta)\| \leq \delta C$$

where $C = C(\vartheta,k)$ is a finite constant depending only on ϑ and k. By choosing $\delta > 0$ small enough, we can guarantee $\delta C < \varepsilon$. Overall, we infer $\|\mathcal{L}\boldsymbol{\mu}^{*m}(\vartheta+\mathrm{i}\eta)^k\| < 1-\varepsilon$. Consequently,

$$\sum_{n=0}^{\infty}\|\mathcal{L}\boldsymbol{\mu}^{*m}(\vartheta+\mathrm{i}\eta)^n\| \leq \sum_{j=0}^{\infty}\sum_{l=0}^{k-1}\|\mathcal{L}\boldsymbol{\mu}^{*m}(\vartheta+\mathrm{i}\eta)^{kj+l}\| \leq \sum_{k=0}^{\infty}(1-\varepsilon)^k\sum_{l=0}^{k-1}\|\mathcal{L}\boldsymbol{\mu}^{*m}(\vartheta)\|^l < \infty.$$

We conclude that $\sup_{|\eta|\geq\eta_0}\|(\mathrm{I}_p-\mathcal{L}\boldsymbol{\mu}(\vartheta+\mathrm{i}\eta))^{-1}\| < \infty$. On the other hand, by continuity and since there is no $\lambda\in\Lambda$ with $\mathrm{Re}(\lambda)=\vartheta$, we also have $\sup_{|\eta|\leq\eta_0}\|(\mathrm{I}_p-\mathcal{L}\boldsymbol{\mu}(\vartheta+\mathrm{i}\eta))^{-1}\| < \infty$.

■

3.6 *Main Results*

Having presented the fundamental concepts necessary for our main results, we now proceed to formulate and prove the asymptotic expansion for general solutions F to the Markov renewal equation (3.1). The starting point for this investigation is the asymptotic expansion of the Markov renewal measure $\boldsymbol{U} = \sum_{n=0}^{\infty}\boldsymbol{\mu}^{*n}$, from which the result for general solutions can be derived directly, using essentially integration by parts. As usual, we will distinguish between the lattice and non-lattice case, beginning with the latter.

3.6.1 The Non-Lattice Case

First, we assume that the $p\times p$ -matrix of measures $\boldsymbol{\mu}$ is non-lattice. As previously mentioned, we begin by studying the asymptotic behavior of the Markov renewal measure $\boldsymbol{U}$. This was already done in the context of single-type general branching processes by A. Iksanov, K. Kolesko and M. Meiners in [IKM24, Lemma 6.3], under the condition $\limsup_{\eta\to\infty}|\mathcal{L}\mu(\vartheta+\mathrm{i}\eta)| < 1$. However, the proof given in the cited source can be adapted to suit the more general setting considered here, with slightly different assumptions.

In our analysis, we work under the assumptions (A1), i.e. $\vartheta_0 = \inf\{\mathcal{D}(\mathcal{L}\boldsymbol{\mu})\cap\mathbb{R}\} < \infty$, and (A2), which ensures that $\rho_{\boldsymbol{\mu}(0)} < 1$. Additionally, we require for some $\vartheta_0 < \vartheta < \mathrm{Re}(\lambda)$, $\lambda\in\Lambda$ that the set Λ_ϑ is finite and condition (A4) is satisfied, namely,

$$\sup_{|\eta|\geq\eta_0}\|(\mathrm{I}_p-\mathcal{L}\boldsymbol{\mu}(\vartheta+\mathrm{i}\eta))^{-1}\| < \infty \quad \text{for some } \eta_0 \geq 0.$$

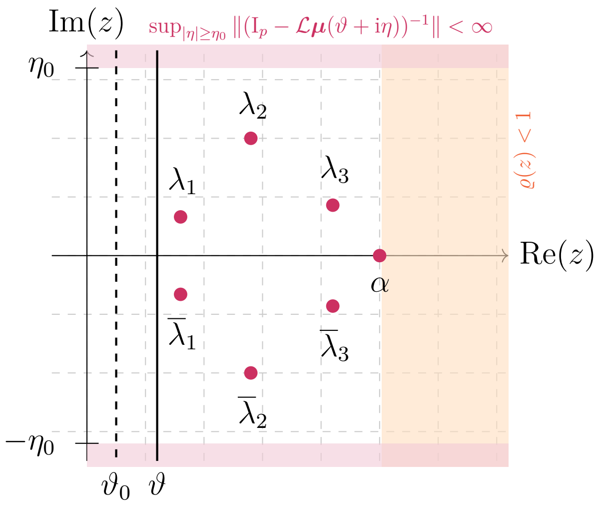

Figure 3.8: Illustration of the regions where the assumptions (A1), (A2) and (A4) hold and their impact on the structure.

With these assumptions, we are able to state our first main result.

Theorem 3.12 [Asymptotic expansion for $\boldsymbol{U}(t)$ in the non-lattice case]
Let $\boldsymbol{\mu} = (\mu^{i,j})_{i,j\in[p]}$ *be a* $p \times p$ *matrix of locally finite measures on* $[0, \infty)$ *satisfying* (A1), (A2) *and* (A4). *Further, we assume that* Λ_ϑ *is finite. Then there exist deterministic matrices* C_{0,k_0}, $C_{\lambda,k} \in \mathbb{R}^{p\times p}$ *with* $k = 0, \ldots, k_\lambda - 1$, $\lambda \in \Lambda_\vartheta$, *and* $k_\lambda \in \mathbb{N}$ *such that*

$$\boldsymbol{U}(t) = \sum_{\lambda\in\Lambda_\vartheta} e^{\lambda t} \sum_{k=0}^{k_\lambda - 1} C_{\lambda,k}\, t^k + \mathbb{1}_{\vartheta<0} C_{0,k_0} t^{k_0} + O(e^{\vartheta t}) \quad \text{as } t \to \infty, \tag{3.18}$$

where the error bound $O(te^{\vartheta t})$ *as* $t \to \infty$ *applies entrywise.*

Remark 3.13 The asymptotic expansion (3.18) can also be rewritten in terms of the exponential matrices introduced in Subsection 2.1.7, namely

$$\boldsymbol{U}(t) = \sum_{\lambda\in\Lambda_\vartheta} (\mathsf{f}_1 \exp(\lambda, t, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{C}}_\lambda + \mathbb{1}_{\vartheta<0} C_{0,k_0} t^{k_0} + O(e^{\vartheta t}) \tag{3.19}$$

as $t \to \infty$ with $\vec{\mathbf{C}}_\lambda := \sum_{k=1}^{k_\lambda} \mathsf{f}_k^\mathsf{T} \otimes C_{\lambda,k}$.

□

Proof of Theorem 3.12.
We first suppose that $\vartheta > 0$. Recall that the matrix-valued Markov renewal measure

$\boldsymbol{U}$ satisfies the Markov renewal equation

$$(U^{i,j}(t))_{i,j\in[p]} = \boldsymbol{U}(t) = \mathbb{1}_{[0,\infty)}(t)\mathrm{I}_p + \boldsymbol{\mu} * \boldsymbol{U}(t), \quad t \in \mathbb{R}. \tag{3.2}$$

As in [IKM24, Lemma 6.3], the strategy is to take Laplace transforms in (3.2), to analyze these and to use the Laplace inversion formula to infer the asymptotic expansion for the Markov renewal measure $\boldsymbol{U}$. However, the Laplace inversion theorem requires certain integrability properties which are satisfied only when using a smoothed version of the indicator $\mathbb{1}_{[0,\infty)}(t)$, $t \in \mathbb{R}$. Therefore, for every $\varepsilon > 0$, we define $g_\varepsilon(t) := (\frac{1}{\varepsilon}\mathbb{1}_{[0,\varepsilon]}) * \mathbb{1}_{[0,\infty)}(t)$, $t \in \mathbb{R}$. More explicitly,

$$\begin{aligned} g_\varepsilon(t) &= \left(\frac{1}{\varepsilon}\mathbb{1}_{[0,\varepsilon]}\right) * \mathbb{1}_{[0,\infty)}(t) = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \mathbb{1}_{[0,\varepsilon]}(t-\tau)\mathbb{1}_{[0,\infty)}(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \mathbb{1}_{[\max(0,t-\varepsilon),t]}(\tau)\,\mathrm{d}\tau = \left(\frac{t}{\varepsilon}\mathbb{1}_{[0,\varepsilon)}(t) + \mathbb{1}_{[\varepsilon,\infty)}(t)\right), \quad t \in \mathbb{R}. \end{aligned}$$

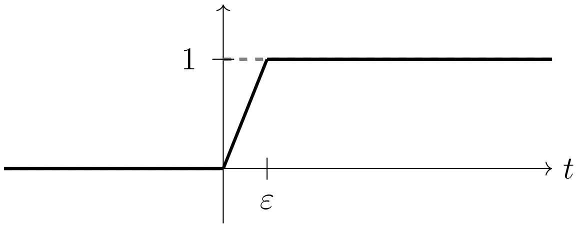

Figure 3.9: Graphs of $\mathbb{1}_{[0,\infty)}$ (dashed) and g_ε (solid).

Clearly,

$$g_\varepsilon(t) \le \mathbb{1}_{[0,\infty)}(t) \le g_\varepsilon(t+\varepsilon), \quad t \in \mathbb{R}.$$

This immediately implies

$$\boldsymbol{U}_\varepsilon(t) := g_\varepsilon \mathrm{I}_p * \boldsymbol{U}(t) \le \boldsymbol{U}(t) \le \boldsymbol{U}_\varepsilon(t+\varepsilon), \quad t \in \mathbb{R}, \tag{3.20}$$

where as usual the inequality has to be understood entrywise. According to Lemma A.8, the Laplace transform

$$\mathcal{L}\boldsymbol{U}(z) := \left(\int_0^\infty e^{-zx}\,\boldsymbol{U}(\mathrm{d}x)\right)_{i,j\in[p]} = \sum_{n=0}^{\infty} \mathcal{L}\boldsymbol{\mu}(z)^n = (\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$$

is finite for every z with $\varrho(\mathrm{Re}(z)) < 1$. For any such z, we infer

$$\begin{aligned} \mathcal{L}\boldsymbol{U}_\varepsilon(z) &= \mathcal{L}((\tfrac{1}{\varepsilon}\mathbb{1}_{[0,\varepsilon]}) * \boldsymbol{U})(z) = \mathcal{L}((\tfrac{1}{\varepsilon}\mathbb{1}_{[0,\varepsilon]}) * (\mathbb{1}_{[0,\infty)}\mathrm{I}_p + \boldsymbol{\mu} * \boldsymbol{U}))(z) \\ &= \mathcal{L}((\tfrac{1}{\varepsilon}\mathbb{1}_{[0,\varepsilon]}) * \mathbb{1}_{[0,\infty)}\mathrm{I}_p)(z) + \mathcal{L}((\tfrac{1}{\varepsilon}\mathbb{1}_{[0,\varepsilon]}) * \boldsymbol{\mu} * \boldsymbol{U})(z) \\ &= \mathcal{L}g_\varepsilon(z)\mathrm{I}_p + \mathcal{L}(\boldsymbol{\mu} * \boldsymbol{U}_\varepsilon)(z) \\ &= \mathcal{L}g_\varepsilon(z)\mathrm{I}_p + \mathcal{L}\boldsymbol{\mu}(z)\mathcal{L}\boldsymbol{U}_\varepsilon(z) \end{aligned} \tag{3.21}$$

where we have used

$$(\tfrac{1}{\varepsilon}\mathbb{1}_{[0,\varepsilon]}) * \boldsymbol{\mu} * \boldsymbol{U} = \boldsymbol{\mu} * (\tfrac{1}{\varepsilon}\mathbb{1}_{[0,\varepsilon]}) * \boldsymbol{U} = \boldsymbol{\mu} * \boldsymbol{U}_\varepsilon(z)$$

and the convolution theorem for Laplace transforms. Solving (3.21) for $\boldsymbol{\mathcal{L}U}_\varepsilon(z)$ gives

$$\boldsymbol{\mathcal{L}U}_\varepsilon(z) = (\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))^{-1}\mathcal{L}g_\varepsilon(z) \tag{3.22}$$

provided that $(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))$ is invertible, i.e. $\det(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z)) \neq 0$. Now use that

$$\begin{aligned}\mathcal{L}g_\varepsilon(z) &= \mathcal{L}(\tfrac{1}{\varepsilon}\mathbb{1}_{[0,\varepsilon]} * \mathbb{1}_{[0,\infty)})(z) = \mathcal{L}(\tfrac{1}{\varepsilon}\mathbb{1}_{[0,\varepsilon]})(z) \cdot \mathcal{L}\mathbb{1}_{[0,\infty)}(z) \\ &= \frac{1}{\varepsilon}\int_0^\varepsilon e^{-tz}\mathrm{d}t \int_0^\infty e^{-tz}\mathrm{d}t = \frac{1-e^{-\varepsilon z}}{\varepsilon z}\frac{1}{z} = \frac{1-e^{-\varepsilon z}}{\varepsilon z^2}\end{aligned}$$

to infer

$$\boldsymbol{\mathcal{L}U}_\varepsilon(z) = \frac{1-e^{-\varepsilon z}}{\varepsilon z^2}(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))^{-1} \tag{3.23}$$

for $z \in \mathcal{D}(\boldsymbol{\mathcal{L}\mu})$ with $\det(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z)) \neq 0$. The right-hand side of (3.23) defines a meromorphic extension of $\boldsymbol{\mathcal{L}U}_\varepsilon(z)$ on the half-plane $\mathrm{Re}(z) > \vartheta$. If any of the entries of the matrix on the right-hand side of (3.23) has a pole in $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \vartheta$, then $\lambda \in \Lambda_\vartheta$. Recall from Lemma 3.10 that condition (A4) implies

$$\sup_{\theta \geq \vartheta, |\eta| \geq \eta_0} \|(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(\theta + \mathrm{i}\eta))^{-1}\| < \infty \tag{3.24}$$

for some suitably large $\eta_0 > 0$. In particular, the entries of $(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))^{-1}$ are bounded on $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq \vartheta,\ |\,\mathrm{Im}(z)| \geq \eta_0\}$. Therefore, the entries of the matrix on the right-hand side of (3.23) are bounded in absolute value by a constant times $|z|^{-2}$ on $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq \vartheta,\ |\,\mathrm{Im}(z)| \geq \eta_0\}$, making them integrable along vertical lines that do not contain a root. Pick $\sigma > \vartheta$ with $\rho_{\boldsymbol{\mathcal{L}\mu}(\sigma)} < 1$ (such a σ exists by Proposition 3.1 and (A2)). Then, in particular, $\sigma > \mathrm{Re}(\lambda)$ for all $\lambda \in \Lambda_\vartheta$ by Proposition 3.1. For any such σ, the Laplace inversion formula, see e.g. [Wid41, Theorem 10.1], gives

$$\boldsymbol{U}_\varepsilon(z) = \frac{1}{2\pi\mathrm{i}}\int\limits_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z. \tag{3.25}$$

Cauchy's residue theorem, see e.g. [Ahl79, Theorem 5.17], then implies

$$\begin{aligned}2\pi\mathrm{i}\sum_{\lambda\in\Lambda_\vartheta}\mathrm{Res}_{z=\lambda}(e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)) &= \int_{\sigma-\mathrm{i}R}^{\sigma+\mathrm{i}R} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z + \int_{\sigma+\mathrm{i}R}^{\vartheta+\mathrm{i}R} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z \\ &\quad + \int_{\vartheta+\mathrm{i}R}^{\vartheta-\mathrm{i}R} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z + \int_{\vartheta-\mathrm{i}R}^{\sigma-\mathrm{i}R} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z\end{aligned} \tag{3.26}$$

for all sufficiently large $R > 0$.

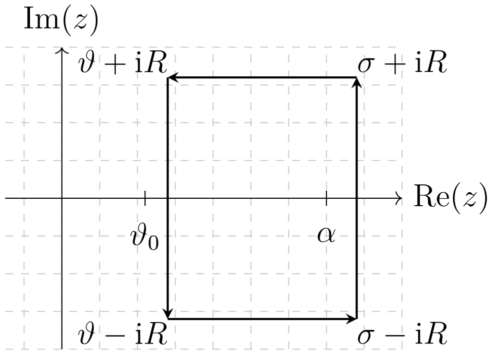

Figure 3.10: A piecewise continuously differentiable path that encloses all solutions $\lambda \in \Lambda_\vartheta$.

By rearranging (3.26) and changing the orientation of the integrals $\int_{\vartheta+\mathrm{i}R}^{\sigma+\mathrm{i}R} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\mathrm{d}z$ and $\int_{\vartheta-\mathrm{i}R}^{\vartheta+\mathrm{i}R} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z$, we obtain

$$\begin{aligned}\int_{\sigma-\mathrm{i}R}^{\sigma+\mathrm{i}R} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z &= 2\pi\mathrm{i}\sum_{\lambda\in\Lambda_\vartheta}\mathrm{Res}_{z=\lambda}(e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)) + \int_{\vartheta+\mathrm{i}R}^{\sigma+\mathrm{i}R} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z\\ &\quad+\int_{\vartheta-\mathrm{i}R}^{\vartheta+\mathrm{i}R} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z - \int_{\vartheta-\mathrm{i}R}^{\sigma-\mathrm{i}R} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z. \qquad (3.27)\end{aligned}$$

The first integral on the right-hand side of (3.27) can be estimated as follows

$$\begin{aligned}\left\|\int_{\vartheta+\mathrm{i}R}^{\sigma+\mathrm{i}R} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z\right\| &\le \int_{\vartheta+\mathrm{i}R}^{\sigma+\mathrm{i}R} |e^{tz}|\|\boldsymbol{\mathcal{L}U}_\varepsilon(z)\|\,\mathrm{d}z\\ &= \int_{\vartheta+\mathrm{i}R}^{\sigma+\mathrm{i}R} |e^{tz}|\|\tfrac{1-e^{-\varepsilon z}}{\varepsilon z^2}(\mathrm{I}_p-\mathcal{L}\boldsymbol{\mu}(z))^{-1}\|\,\mathrm{d}z\\ &\le \int_{\vartheta+\mathrm{i}R}^{\sigma+\mathrm{i}R} |e^{tz}||\tfrac{1-e^{-\varepsilon z}}{\varepsilon z^2}|\|(\mathrm{I}_p-\mathcal{L}\boldsymbol{\mu}(z))^{-1}\|\mathrm{d}z\\ &= \int_{\vartheta}^{\sigma} |e^{t(x+\mathrm{i}R)}||\tfrac{1-e^{-\varepsilon(x+\mathrm{i}R)}}{\varepsilon(x+\mathrm{i}R)^2}|\|(\mathrm{I}_p-\mathcal{L}\boldsymbol{\mu}(x+\mathrm{i}R))^{-1}\|\,\mathrm{d}x\\ &\le Ce^{t\sigma}\int_{\vartheta}^{\sigma}\left|\tfrac{1}{\varepsilon(x+\mathrm{i}R)^2}\right|\mathrm{d}x \to 0 \quad \text{as } R\to\infty \qquad (3.28)\end{aligned}$$

where we used representation (3.23) of $\boldsymbol{\mathcal{L}U}_\varepsilon(z)$, condition (A4) to bound $\|(\mathrm{I}_p-\mathcal{L}\boldsymbol{\mu}(z))^{-1}\|$ and a substitution of $x = z - \mathrm{i}R$. The last integral on the right-hand side of (3.27) can be estimated analogously. Taking $R \to \infty$, we thus infer

$$\begin{aligned}\boldsymbol{U}_\varepsilon(t) &= \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z\\ &= \frac{1}{2\pi\mathrm{i}}\left(2\pi\mathrm{i}\sum_{\lambda\in\Lambda_\vartheta}\mathrm{Res}_{z=\lambda}(e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)) + \int_{\vartheta-\mathrm{i}\infty}^{\vartheta+\mathrm{i}\infty} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z\right)\\ &= \sum_{\lambda\in\Lambda_\vartheta}\mathrm{Res}_{z=\lambda}(e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)) + \frac{1}{2\pi\mathrm{i}}\int_{\vartheta-\mathrm{i}\infty}^{\vartheta+\mathrm{i}\infty} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z. \qquad (3.29)\end{aligned}$$

We now focus on the sum on the right-hand side of (3.29). Here, we expand each entry of the matrix $(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ into a Laurent series around the point $z = \lambda \in \Lambda_\vartheta$. Note that λ is a root of $\det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z)) = 0$. Recall that k_λ denotes the maximal multiplicity of the root λ. Because

$$(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1} = \frac{1}{\det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))}\operatorname{adj}(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z)), \tag{3.9}$$

each entry of $(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ has a pole with multiplicity at most the multiplicity of the zero of $\det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))$ at $z = \lambda$. Let us define $a^{i,j}_{\lambda,k}$ as the coefficient in front of $(z-\lambda)^{-k}$ in the Laurent series of the (i,j)-th entry in $(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ around the point $\lambda \in \Lambda_\vartheta$. In particular, $a^{i,j}_{\lambda,k} = 0$ for $k > k_\lambda$. Summarizing, let us consider

$$A_{\lambda,k} := (a^{i,j}_{\lambda,k})_{i,j\in[p]}$$

as the matrix of all coefficients in front of $(z-\lambda)^{-k}$ for $k \le k_\lambda$, i.e.

$$(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1} = A_{\lambda,k_\lambda}(z-\lambda)^{-k_\lambda} + \ldots + A_{\lambda,1}(z-\lambda)^{-1} + H_\lambda(z) \tag{3.30}$$

for all $z \neq \lambda$ in a small disc $D(\lambda)$ centered at λ, where $H_\lambda(z)$ is holomorphic in $D(\lambda)$. Then, for $\lambda \in \Lambda_\vartheta$ with $\mathrm{Re}(\lambda) > 0$,

$$\begin{aligned}\mathrm{Res}_{z=\lambda}(e^{tz}\mathcal{L}\boldsymbol{U}_\varepsilon(z)) &= \mathrm{Res}_{z=\lambda}(e^{tz}\mathcal{L}g_\varepsilon(z)(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}) \\ &= e^{\lambda t}\sum_{\substack{n,l\ge 0,\\ n+l<k_\lambda}} \frac{t^l}{l!}\frac{1}{n!}\,\mathcal{L}g_\varepsilon^{(n)}(\lambda)A_{\lambda,n+l+1} \end{aligned}\tag{3.31}$$

where $\mathcal{L}g_\varepsilon^{(n)}$ denotes the n-th derivative of $\mathcal{L}g_\varepsilon$. Here, in order to arrive at (3.31), one has to determine the residue of $e^{tz}\mathcal{L}g_\varepsilon(z)(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ at the point $z = \lambda$. To this end, one can compute the corresponding Laurent series around $z = \lambda$, as the residue corresponds to the coefficient with index -1. This Laurent series can be obtained by multiplying the Laurent series of e^{tz}, $\mathcal{L}g_\varepsilon(z)$, and $(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ around $z = \lambda$. On one hand, we have the Taylor series

$$e^{tz} = \sum_{l=0}^{\infty} a_l(z-\lambda)^l, \text{ with coefficients } a_l = \frac{t^l e^{t\lambda}}{l!},\ l \in \mathbb{N}_0.$$

On the other hand, $(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1} = \sum_{k=-k_\lambda}^{\infty} A_{\lambda,k}(z-\lambda)^{-k}$. However, when determining the Laurent series of $\mathcal{L}g_\varepsilon(z)$, one must be cautious and distinguish between two different cases: $\vartheta > 0$, and $\vartheta \le 0$. If $\vartheta > 0$, then $\mathrm{Re}(\lambda) > 0$ for all $\lambda \in \Lambda_\vartheta$, and we can determine the Taylor series of $\mathcal{L}g_\varepsilon(z)$ by

$$\mathcal{L}g_\varepsilon(z) = \sum_{n=0}^{\infty} b_n(z-\lambda)^n, \text{ with coefficients } b_n = \frac{1}{n!}\mathcal{L}g_\varepsilon^{(n)}(\lambda),\ l \in \mathbb{N}_0.$$

The case $\vartheta \le 0$ will be handled separately and thus be excluded for the time being. The series of e^{tz}, $\mathcal{L}g_\varepsilon(z)$, and $(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ around $z = \lambda$ are multiplied, and for the residue, we need the coefficient of $(z-\lambda)^{-1}$. In particular, for $l+n \geq k_\lambda$, the coefficient matrix $A_{\lambda,l+n+1}$ is the zero matrix, which means that only summands with indices with $l+n<k_\lambda$ contribute. Write $g_\varepsilon = \mathbb{1}_{[0,\infty)} + g_\varepsilon - \mathbb{1}_{[0,\infty)}$ and use the linearity of the Laplace transform to infer $\mathcal{L}g_\varepsilon = \mathcal{L}\mathbb{1}_{[0,\infty)} + \mathcal{L}(g_\varepsilon - \mathbb{1}_{[0,\infty)})$. We can split the right-hand side of (3.31) as follows

$$\begin{aligned} e^{\lambda t} \sum_{\substack{n,l\geq 0,\\ n+l<k_\lambda}} \frac{t^l}{l!}\frac{1}{n!}\mathcal{L}g_\varepsilon^{(n)}(\lambda)A_{\lambda,n+l+1} &= e^{\lambda t} \sum_{\substack{n,l\geq 0,\\ n+l<k_\lambda}} \frac{t^l}{l!}\frac{1}{n!}(\mathcal{L}\mathbb{1}_{[0,\infty)})^{(n)}(\lambda)A_{\lambda,n+l+1} \\ &\quad + e^{\lambda t} \sum_{\substack{n,l\geq 0,\\ n+l<k_\lambda}} \frac{t^l}{l!}\frac{1}{n!}(\mathcal{L}(g_\varepsilon - \mathbb{1}_{[0,\infty)}))^{(n)}(\lambda)A_{\lambda,n+l+1}. \end{aligned} \tag{3.32}$$

We continue by estimating the coefficients $\frac{1}{n!}\,(\mathcal{L}(g_\varepsilon - \mathbb{1}_{[0,\infty)}))^{(n)}(\lambda)$, $n \in \mathbb{N}_0$ of the Laurent series of $\mathcal{L}(g_\varepsilon - \mathbb{1}_{[0,\infty)})$ around $z=\lambda$. We have

$$\begin{aligned} \mathcal{L}(\mathbb{1}_{[0,\infty)} - g_\varepsilon)(z) &= \frac{1}{z} - \frac{1-e^{-\varepsilon z}}{\varepsilon z^2} = \frac{1}{\varepsilon z^2}\sum_{k=2}^{\infty}\frac{(-\varepsilon z)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+2)!}\varepsilon^{k+1}z^k \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+2)!}\varepsilon^{k+1}(z-\lambda+\lambda)^k \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+2)!}\varepsilon^{k+1}\sum_{n=0}^{k}\binom{k}{n}\lambda^{k-n}(z-\lambda)^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}\frac{(-1)^k}{(k+2)!}\varepsilon^{k+1}\binom{k}{n}\lambda^{k-n}\right)(z-\lambda)^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{1}{(k+n+1)(k+n+2)}\frac{(-\varepsilon\lambda)^k}{k!}\right)\frac{(-1)^n\varepsilon^{n+1}}{n!}(z-\lambda)^n. \end{aligned} \tag{3.33}$$

In particular, the coefficient of $(z-\lambda)^n$ of $\mathcal{L}(\mathbb{1}_{[0,\infty)} - g_\varepsilon)(z)$ is bounded in absolute value by

$$\begin{aligned} \left|\frac{1}{n!}\,(\mathcal{L}(g_\varepsilon - \mathbb{1}_{[0,\infty)}))^{(n)}(\lambda)\right| &\leq \left|\sum_{k=0}^{\infty}\frac{1}{(k+n+1)(k+n+2)}\frac{(-\varepsilon\lambda)^k}{k!}\frac{(-1)^n\varepsilon^{n+1}}{n!}\right| \\ &\leq e^{\varepsilon|\lambda|}\frac{\varepsilon^{n+1}}{(n+2)!}. \end{aligned}$$

We conclude that

$$\begin{aligned} \left|e^{\lambda t}\sum_{\substack{l,n\geq 0,\\ l+n<k_\lambda}}\frac{t^l}{l!}\frac{1}{n!}(\mathcal{L}(g_\varepsilon - \mathbb{1}_{[0,\infty)}))^{(n)}(\lambda)a^{i,j}_{\lambda,l+n+1}\right| &\leq e^{\mathrm{Re}(\lambda)t}\sum_{\substack{l,n\geq 0,\\ l+n<k_\lambda}}\frac{t^l}{l!}e^{\varepsilon|\lambda|}\frac{\varepsilon^{n+1}}{(n+2)!}|a^{i,j}_{\lambda,l+n+1}| \\ &= \varepsilon O(e^{\mathrm{Re}(\lambda)t}t^{k_\lambda-1}) \quad \text{as } t\to\infty, \end{aligned}$$

where $O(e^{\mathrm{Re}(\lambda)t}t^{k_\lambda-1})$ is an error term independent of $\varepsilon \in (0,1]$. Summarizing,

$$\begin{aligned}\mathrm{Res}_{z=\lambda}(e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)) &= e^{\lambda t}\sum_{\substack{n,l\geq 0,\\ n+l<k_\lambda}} \frac{t^l}{l!}\frac{1}{n!}(\mathcal{L}\mathbb{1}_{[0,\infty)})^{(n)}(\lambda)A_{\lambda,n+l+1}\\ &\quad + e^{\lambda t}\sum_{\substack{n,l\geq 0,\\ n+l<k_\lambda}} \frac{t^l}{l!}\frac{1}{n!}\mathcal{L}(g_\varepsilon - \mathbb{1}_{[0,\infty)})^{(n)}(\lambda)A_{\lambda,n+l+1}\\ &= e^{\lambda t}\sum_{l=0}^{k_\lambda-1} C_{\lambda,l}\, t^l + \varepsilon O(e^{\mathrm{Re}(\lambda)t}t^{k_\lambda-1}),\end{aligned} \tag{3.34}$$

where $C_{\lambda,l}$ is a $p\times p$ matrix (depending only on the parameters λ and l) and $O(e^{\mathrm{Re}(\lambda)t}t^{k_\lambda-1})$ is, in slight abuse of notation, shorthand for a $p\times p$ matrix every entry of which is $O(e^{\mathrm{Re}(\lambda)t}t^{k_\lambda-1})$ as $t\to\infty$.

Next, let us focus on the integral on the right-hand side of (3.29), namely

$$\frac{1}{2\pi\mathrm{i}}\int_{\vartheta-\mathrm{i}\infty}^{\vartheta+\mathrm{i}\infty} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z.$$

Write $C<\infty$ for the supremum in condition (A4). Then

$$\begin{aligned}\left\|\int_{\vartheta-\mathrm{i}\infty}^{\vartheta+\mathrm{i}\infty} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z\right\| &\leq \int_{\vartheta-\mathrm{i}\infty}^{\vartheta+\mathrm{i}\infty} |e^{tz}|\|\boldsymbol{\mathcal{L}U}_\varepsilon(z)\|\,\mathrm{d}z\\ &\leq \int_{\vartheta-\mathrm{i}\infty}^{\vartheta+\mathrm{i}\infty} |e^{tz}||\tfrac{1-e^{-\varepsilon z}}{\varepsilon z^2}|\|(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}\|\,\mathrm{d}z\\ &\leq Ce^{t\vartheta}\int_{\vartheta-\mathrm{i}\infty}^{\vartheta+\mathrm{i}\infty} \tfrac{|1-e^{-\varepsilon z}|}{|\varepsilon z^2|}\,\mathrm{d}z \leq Ce^{t\vartheta}\int_{\vartheta-\mathrm{i}\infty}^{\vartheta+\mathrm{i}\infty} \tfrac{|\varepsilon z|\wedge 2}{|\varepsilon z^2|}\,\mathrm{d}z\\ &\leq 2Ce^{t\vartheta}\int_{\varepsilon\vartheta-\mathrm{i}\infty}^{\varepsilon\vartheta+\mathrm{i}\infty} |u|^{-1}\wedge|u|^{-2}\,\mathrm{d}u\\ &\leq 2Ce^{t\vartheta}\int_{-\infty}^{\infty} x^{-1}\wedge x^{-2}\wedge(\varepsilon|\vartheta|)^{-1}\,\mathrm{d}x\\ &= 2Ce^{t\vartheta}\left(\int_0^{\varepsilon\vartheta}(\varepsilon|\vartheta|)^{-1} + \int_{\varepsilon\vartheta}^1 |x|^{-1}\mathrm{d}x + \int_1^\infty |x|^{-2}\mathrm{d}x\right)\\ &= 2Ce^{t\vartheta}(2+|\log(\varepsilon)|+|\log(\vartheta)|) \leq C_\vartheta e^{t\vartheta}\,(|\log(\varepsilon)|+1),\end{aligned} \tag{3.35}$$

where we used again representation (3.23) of $\boldsymbol{\mathcal{L}U}_\varepsilon(z)$, the inequality

$$\frac{|1-e^{-\varepsilon z}|}{|\varepsilon z^2|} \leq \frac{|\varepsilon z|\wedge 2}{|\varepsilon z^2|} \leq \frac{|\varepsilon z|\wedge 2}{|\varepsilon z|\,|z|} = \frac{2\wedge|\varepsilon z|^{-1}}{|z|}$$

and, a substitution of $u=\varepsilon z$. The constant C_ϑ in the last step does not depend on either t or ε. Returning to (3.29), we conclude with (3.34) and (3.35)

$$\begin{aligned}\boldsymbol{U}_\varepsilon(t) &= \sum_{\lambda\in\Lambda_\vartheta} \operatorname{Res}_{z=\lambda}(e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)) + \frac{1}{2\pi\mathrm{i}}\int_{\vartheta-\mathrm{i}\infty}^{\vartheta+\mathrm{i}\infty} e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z)\,\mathrm{d}z \\ &\le \sum_{\lambda\in\Lambda_\vartheta} e^{\lambda t}\sum_{l=0}^{k_\lambda-1} C_{\lambda,l}\,t^l + \varepsilon O(e^{\mathrm{Re}(\lambda)t}t^{k_\lambda}) + O(e^{\vartheta t}(|\log(\varepsilon)|+1)).\end{aligned} \tag{3.36}$$

By using (3.20), we infer

$$\boldsymbol{U}(t) = \boldsymbol{U}_\varepsilon(t) + \boldsymbol{U}(t) - \boldsymbol{U}_\varepsilon(t) \le \boldsymbol{U}_\varepsilon(t) + \boldsymbol{U}_\varepsilon(t+\varepsilon) - \boldsymbol{U}_\varepsilon(t).$$

In order to estimate $\boldsymbol{U}_\varepsilon(t+\varepsilon) - \boldsymbol{U}_\varepsilon(t)$, we replace t by $t+\varepsilon$ in (3.34), and get

$$\begin{aligned}&\left\|\operatorname{Res}_{z=\lambda}(e^{(t+\varepsilon)z}\boldsymbol{\mathcal{L}U}_\varepsilon(z)) - \operatorname{Res}_{z=\lambda}(e^{tz}\boldsymbol{\mathcal{L}U}_\varepsilon(z))\right\| \\ &= e^{\mathrm{Re}(\lambda)t}\left\|e^{\lambda\varepsilon}\sum_{l=0}^{k_\lambda-1}(t+\varepsilon)^l\,C_{\lambda,l} - \sum_{l=0}^{k_\lambda-1} t^l C_{\lambda,l}\right\| + \varepsilon O(e^{\mathrm{Re}(\lambda)t}t^{k_\lambda-1}) \\ &= \varepsilon O(e^{\mathrm{Re}(\lambda)t}t^{k_\lambda-1}).\end{aligned} \tag{3.37}$$

Combining (3.35) and (3.37), we infer

$$0 \le \boldsymbol{U}_\varepsilon(t+\varepsilon) - \boldsymbol{U}_\varepsilon(t) \le \sum_{\lambda\in\Lambda_\vartheta} O(\varepsilon e^{\mathrm{Re}(\lambda)t}t^{k_\lambda} + |\log(\varepsilon)|\,e^{t\vartheta}).$$

From this, (3.20), (3.29) and with $\varepsilon := e^{-\sigma t}$, we finally deduce

$$\boldsymbol{U}(t) = \sum_{\lambda\in\Lambda_\vartheta} e^{\lambda t}\sum_{k=0}^{k_\lambda-1} C_{\lambda,k}\,t^k + O(te^{\vartheta t}).$$

It remains to remove the additional assumption that $\vartheta > 0$. We do this using an exponential tilting technique. Therefore, suppose that $\vartheta \le 0$. Then choose an auxiliary parameter $\gamma < \vartheta$ and define the exponentially tilted measures $\boldsymbol{\mu}^\gamma(\mathrm{d}x) := e^{-\gamma x}\boldsymbol{\mu}(\mathrm{d}x)$ as well as $\boldsymbol{U}^\gamma(\mathrm{d}x) := e^{-\gamma x}\boldsymbol{U}(\mathrm{d}x)$, representing the shifted versions of $\boldsymbol{\mu}(\mathrm{d}x)$ and $\boldsymbol{U}(\mathrm{d}x)$, respectively. With this, we infer

$$\boldsymbol{U}^\gamma(\mathrm{d}x) = e^{-\gamma x}\boldsymbol{U}(\mathrm{d}x) = e^{-\gamma x}\sum_{n=0}^\infty \boldsymbol{\mu}^{*n}(\mathrm{d}x) = \sum_{n=0}^\infty (e^{-\gamma x}\boldsymbol{\mu})^{*n}(\mathrm{d}x) = \sum_{n=0}^\infty (\boldsymbol{\mu}^\gamma)^{*n}(\mathrm{d}x).$$

Further,

$$\boldsymbol{\mathcal{L}\mu}^\gamma(z) = \int_0^\infty e^{-zt}\boldsymbol{\mu}^\gamma(\mathrm{d}t) = \int_0^\infty e^{-zt}e^{-\gamma t}\boldsymbol{\mu}(\mathrm{d}t) = \int_0^\infty e^{-(z+\gamma)t}\boldsymbol{\mu}(\mathrm{d}t) = \boldsymbol{\mathcal{L}\mu}(z+\gamma)$$

for all $z \in \mathbb{C}$ with $z+\gamma \in \mathcal{D}(\boldsymbol{\mathcal{L}\mu})$. In particular, $\mathcal{D}(\boldsymbol{\mathcal{L}\mu}^\gamma) = \mathcal{D}(\boldsymbol{\mathcal{L}\mu}) - \gamma$ and, similarly, $\Lambda^\gamma := \{\lambda \in \mathcal{D}(\boldsymbol{\mathcal{L}\mu}^\gamma) : \det(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}^\gamma(\lambda)) = 0\} = \Lambda - \gamma$ as well as $\Lambda^\gamma_{\vartheta-\gamma} = \Lambda_\vartheta - \gamma$. Consequently, if ϑ is a parameter for which $\boldsymbol{\mathcal{L}\mu}(\vartheta)$ has finite entries only, it follows that $\boldsymbol{\mathcal{L}\mu}^\gamma(\vartheta-\gamma)$ has finite entries as well. Moreover, $\vartheta-\gamma>0$ and therefore, we have an asymptotic expansion for $\boldsymbol{U}^\gamma$, namely,

$$\boldsymbol{U}^\gamma(t)=\sum_{\lambda\in\Lambda_\vartheta-\gamma}e^{(\lambda-\gamma)t}\sum_{k=0}^{k_\lambda-1}C^\gamma_{\lambda,k}\,t^k+O(te^{(\vartheta-\gamma)t})$$

where $k_\lambda=k^\gamma_{\lambda-\gamma}$ in obvious notation should be noticed and $C^\gamma_{\lambda,k}$ is the coefficient matrix for the term $e^{(\lambda-\gamma)t}t^k$ in the asymptotic expansion of $\boldsymbol{U}^\gamma$. Hence, by writing

$$\boldsymbol{U}(t)=\int_{[0,t]}e^{\gamma x}\,\boldsymbol{U}^\gamma(\mathrm{d}x),$$

we infer with $\int_0^x\gamma e^{\gamma y}\,\mathrm{d}y=e^{\gamma x}-1$,

$$\int_0^t e^{\gamma x}\,\boldsymbol{U}^\gamma(\mathrm{d}x)=\int_0^t\left(\int_0^x\gamma e^{\gamma y}\,\mathrm{d}y+1\right)\boldsymbol{U}^\gamma(\mathrm{d}x)=\int_0^t\int_0^x\gamma e^{\gamma y}\,\mathrm{d}y\,\boldsymbol{U}^\gamma(\mathrm{d}x)+\boldsymbol{U}^\gamma(t).$$

Hence, using integration by parts with $u(x)=\int_0^x\gamma e^{\gamma y}\,\mathrm{d}y$ and $v'(x)=\boldsymbol{U}^\gamma(\mathrm{d}x)$,

$$\begin{aligned}\int_0^t\int_0^x\gamma e^{\gamma y}\,\mathrm{d}y\,\boldsymbol{U}^\gamma(\mathrm{d}x)+\boldsymbol{U}^\gamma(t)&=\Big[(e^{\gamma x}-1)\boldsymbol{U}^\gamma(x)\Big]_0^t-\int_0^t\gamma e^{\gamma y}\boldsymbol{U}^\gamma(y)\,\mathrm{d}y+\boldsymbol{U}^\gamma(t)\\&=e^{\gamma t}\boldsymbol{U}^\gamma(t)-\int_0^t\gamma e^{\gamma y}\boldsymbol{U}^\gamma(y)\,\mathrm{d}y.\end{aligned}$$

Since we have an asymptotic expansion for $\boldsymbol{U}^\gamma(y)$, we obtain

$$\begin{aligned}\boldsymbol{U}(t)&=e^{\gamma t}\boldsymbol{U}^\gamma(t)-\int_0^t\gamma e^{\gamma y}\boldsymbol{U}^\gamma(y)\,\mathrm{d}y\\&=\sum_{\lambda\in\Lambda_\vartheta}e^{\lambda t}\sum_{k=0}^{k_\lambda-1}C^\gamma_{\lambda,k}t^k+O(te^{\vartheta})-\gamma\sum_{\lambda\in\Lambda_\vartheta}\sum_{k=0}^{k_\lambda-1}C^\gamma_{\lambda,k}\int_0^t e^{\lambda y}y^k\,\mathrm{d}y-\int_0^t O(ye^{\vartheta y})\,\mathrm{d}y\\&=\sum_{\lambda\in\Lambda_\vartheta}\sum_{k=0}^{k_\lambda-1}C^\gamma_{\lambda,k}\left(e^{\lambda t}t^k-\gamma\int_0^t e^{\lambda y}y^k\,\mathrm{d}y\right)+O(te^{\vartheta t})\\&=\sum_{\lambda\in\Lambda_\vartheta}\sum_{k=0}^{k_\lambda-1}C^\gamma_{\lambda,k}\left(e^{\lambda t}t^k-\gamma\frac{(-1)^{k+1}k!}{\lambda^{k+1}}\left(1-e^{\lambda t}\sum_{j=0}^{k}(-1)^j\frac{\lambda^jt^j}{j!}\right)\right)+O(te^{\vartheta t})\end{aligned}$$

in the case $\lambda\neq0$ with Lemma A.12. Similarly, for $\lambda=0$, we get

$$\boldsymbol{U}(t)=\sum_{\lambda\in\Lambda_\vartheta}\sum_{k=0}^{k_\lambda-1}C^\gamma_{\lambda,k}\left(e^{\lambda t}t^k-\gamma\left(\frac{1}{1+k}t^{k+1}\right)\right)+O(te^{\vartheta t})$$

where we used the elementary fact that $\int_0^t y^k\,\mathrm{d}y=\frac{1}{1+k}t^{k+1}$. However, in both cases, we can rearrange terms to infer an expansion of the form (3.18).

■

Remark 3.14 The $p \times p$ matrices of coefficients $C_{\lambda,k}$, introduced in Theorem 3.12, can be explicitly formulated. For a given $\lambda \in \Lambda$ with $\mathrm{Re}(\lambda) > 0$, we denote $k_\lambda \in \mathbb{N}$ as the maximal multiplicity of the poles of the entries of $(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))^{-1}$ at the point $\lambda \in \Lambda$. As in (3.46), we may write

$$(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))^{-1} = A_{\lambda,k_\lambda}(z-\lambda)^{-k_\lambda} + \ldots + A_{\lambda,1}(z-\lambda)^{-1} + H_\lambda(z) \tag{3.38}$$

for all $z \neq \lambda$ in a small disc $D(\lambda)$ centered at λ. Here, $H_\lambda(z)$ is holomorphic in $D(\lambda)$, while $A_{\lambda,k_\lambda}, \ldots, A_{\lambda,1}$ are complex $p \times p$ matrices with A_{λ,k_λ} having at least one non-zero entry. Then, the coefficient matrix $C_{\lambda,k}$ takes the form

$$C_{\lambda,k} = \frac{1}{k!} \sum_{n=0}^{k_\lambda - 1 - k} \frac{(-1)^n}{n! \lambda^{n+1}} A_{\lambda,n+k+1}.$$

□

Remark 3.15 The asymptotic expansion (3.18) in Theorem 3.12 can also be formulated in a compact form, for the case where $\mathrm{Re}(\lambda) > 0$. Since this representation will be of use in the subsequent chapter, we proceed with its derivation. Building upon Equation (3.34) in the previous proof, we first write

$$\begin{aligned}\boldsymbol{U}(t) &= \sum_{\lambda \in \Lambda_\vartheta} e^{\lambda t} \sum_{k=0}^{k_\lambda - 1} C_{\lambda,k}\, t^k + O(te^{\vartheta t}) = \sum_{\lambda \in \Lambda_\vartheta} e^{\lambda t} \sum_{\substack{l,n \geq 0, \\ l+n < k_\lambda}} \frac{t^l}{l!} c_n A_{\lambda,l+n+1} + O(te^{\vartheta t}) \\ &= \sum_{\lambda \in \Lambda_\vartheta} e^{\lambda t} \sum_{n=0}^{k_\lambda - 1} \sum_{l=0}^{k_\lambda - 1 - n} \frac{t^l}{l!} c_n A_{\lambda,l+n+1} + O(te^{\vartheta t}), \quad t \in \mathbb{R}\end{aligned}$$

where $c_n = \frac{1}{n!}(\mathcal{L}\mathbb{1}_{[0,\infty)})^{(n)}(\lambda)$. Recall that for $\gamma \in \mathbb{C}$, $s \in \mathbb{R}$ and $k \in \mathbb{N}$, the (i,j)-th entry of the $k \times k$ matrix $\exp(\gamma, s, k)$ corresponds to $\exp_{ij}(\gamma, s, k) = e^{\gamma s} \frac{1}{(j-i)!} s^{j-i}$, $1 \leq i \leq j \leq k$ and $\exp_{ij}(\gamma, s, k) = 0$ for $i > j$. Hence, denoting by f_l the l-th base vector in $\mathbb{R}^{1 \times k}$, we have $\mathsf{f}_i \exp(\lambda, t, k_\lambda) \mathsf{f}_j^\mathsf{T} = \exp_{ij}(\lambda, t, k_\lambda)$. Consequently,

$$\begin{aligned}\sum_{\lambda \in \Lambda_\vartheta} e^{\lambda t} \sum_{n=0}^{k_\lambda - 1} \sum_{l=0}^{k_\lambda - 1 - n} \frac{t^l}{l!} c_n A_{\lambda,l+n+1} &= \sum_{\lambda \in \Lambda_\vartheta} c_0 \sum_{l=1}^{k_\lambda} (\mathsf{f}_1 \exp(\lambda, t, k_\lambda) \mathsf{f}_l^\mathsf{T}) A_{\lambda,l} \\ &\quad + c_1 \sum_{l=1}^{k_\lambda} (\mathsf{f}_2 \exp(\lambda, t, k_\lambda) \mathsf{f}_l^\mathsf{T}) A_{\lambda,l} \\ &\quad + \ldots + c_{k_\lambda - 1} \sum_{l=1}^{k_\lambda} (\mathsf{f}_{k_\lambda} \exp(\lambda, t, k_\lambda) \mathsf{f}_l^\mathsf{T}) A_{\lambda,l}.\end{aligned}$$

Further, for each $\lambda \in \Lambda_\vartheta$, we summarize the matrices $A_{\lambda,l} = (a_{\lambda,l}^{i,j})_{i,j \in [p]}$, $l = 0, \ldots, k_\lambda - 1$, which appeared in the previous proof as coefficients of the Laurent series expansion of $(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))^{-1}$, by writing

$$\vec{\mathbf{A}}_\lambda := \sum_{l=1}^{k_\lambda} \mathsf{f}_l^\mathsf{T} \otimes A_{\lambda,l} \tag{3.39}$$

where $\otimes$ denotes the Kronecker product, as defined in Subsection 2.1.4. Notice that $\vec{\mathbf{A}}_\lambda$ represents a $pk_\lambda \times p$ matrix. Thus, we can rewrite the sum as

$$\begin{aligned}\sum_{\lambda\in\Lambda_\vartheta} c_0 \sum_{l=1}^{k_\lambda} (\mathsf{f}_1 \exp(\lambda, t, k_\lambda)\, \mathsf{f}_l^\mathsf{T}) A_{\lambda,l} + c_1 \sum_{l=1}^{k_\lambda} (\mathsf{f}_2 \exp(\lambda, t, k_\lambda)\, \mathsf{f}_l^\mathsf{T}) A_{\lambda,l} \\ + \ldots + c_{k_\lambda - 1} \sum_{l=1}^{k_\lambda} (\mathsf{f}_{k_\lambda} \exp(\lambda, t, k_\lambda)\, \mathsf{f}_l^\mathsf{T}) A_{\lambda,l} = \sum_{\lambda\in\Lambda_\vartheta} \sum_{l=1}^{k_\lambda} c_{l-1} (\mathsf{f}_l \exp(\lambda, t, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{A}}_\lambda.\end{aligned}$$

Writing the coefficients c_n in the explicit form $\frac{1}{n!}(\mathcal{L}\mathbb{1}_{[0,\infty)})^{(n)}(\lambda)$ yields

$$\begin{aligned}&\sum_{\lambda\in\Lambda_\vartheta} \int_0^\infty \mathbb{1}_{[0,\infty)}(x) \sum_{l=1}^{k_\lambda} \frac{1}{(l-1)!} e^{-\lambda x} (-x)^l (\mathsf{f}_l \exp(\lambda, t, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{A}}_\lambda \,\mathrm{d}x \\ &= \sum_{\lambda\in\Lambda_\vartheta} \int_0^\infty (\mathsf{f}_1 \exp(\lambda, t - x, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{A}}_\lambda \,\mathrm{d}x,\end{aligned}$$

and finally,

$$\boldsymbol{U}(t) = \sum_{\lambda\in\Lambda_\vartheta} \int_0^\infty (\mathsf{f}_1 \exp(\lambda, t - x, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{A}}_\lambda \,\mathrm{d}x + O(te^{\vartheta t}). \tag{3.40}$$

□

Building upon Theorem 3.12, using essentially integration by parts, we can derive an asymptotic expansion for the solutions $F(t)$ to general Markov renewal equations of the form (3.1) as $t \to \infty$. Before presenting this formulation, we must first introduce some appropriate assumption on the function f in (3.1). Therefore, we define for a function $f : \mathbb{R} \to \mathbb{R}$ the total variation function $\mathrm{V}f$ by

$$\mathrm{V}f(x) := \sup \left\{ \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| : -\infty < x_0 < \ldots < x_n \le x, n \in \mathbb{N} \right\}$$

for $x \in \mathbb{R}$. In particular, if $f = (f_1, \ldots, f_p)^\mathsf{T}$ is vector-valued, we write $\mathrm{V}f$ for the vector $(\mathrm{V}f_1, \ldots, \mathrm{V}f_p)^\mathsf{T}$.

Assumption (A5) non-lattice

Suppose that (A1) holds. Let $f : \mathbb{R} \to \mathbb{R}^{p\times 1}$ be a column vector-valued function vanishing on the negative half-line. Then, f satisfies the property

$$\int_0^\infty e^{-\vartheta x}\, \mathrm{V}f(x) \,\mathrm{d}x < \infty$$

for some $\vartheta_0 < \vartheta < \mathrm{Re}(\lambda)$, $\lambda \in \Lambda$.

Theorem 3.16 $\big[$Asymptotic expansion for $F(t)$ in the non-lattice case$\big]$
Let $\boldsymbol{\mu} = (\mu^{i,j})_{i,j\in[p]}$ *be a* $p \times p$ *matrix of locally finite measures on* $[0, \infty)$ *satisfying* (A1), (A2) *and* (A4). *Further, we assume that* $f : \mathbb{R} \to \mathbb{R}^{p\times 1}$ *is a column vector-valued function satisfying* (A5) *for some* $\vartheta_0 < \vartheta < \mathrm{Re}(\lambda)$, $\lambda \in \Lambda$ *and that the set* Λ_ϑ *is finite. Then there exist deterministic matrices* D_{0,k_0}, $D_{\lambda,k} \in \mathbb{R}^{p\times p}$ *with* $\lambda \in \Lambda_\vartheta$, $k = 0, \dots, k_\lambda - 1$ *and* $k_\lambda \in \mathbb{N}$ *such that*

$$\begin{aligned} F(t) = &\sum_{\lambda\in\Lambda_\vartheta} \int_0^t (\mathsf{f}_1 \exp(\lambda, t-x, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{D}}_\lambda\, f(x)\, \mathrm{d}x \\ &+ \mathbb{1}_{\vartheta<0}\, D_{0,k_0} \int_0^t f(x)(t-x)^{k_0-1}\, \mathrm{d}x + O(e^{\vartheta t}) \quad \text{as } t \to \infty \end{aligned} \tag{3.41}$$

where $\vec{\mathbf{D}}_\lambda := \sum_{k=1}^{k_\lambda} \mathsf{f}_k^\mathsf{T} \otimes D_{\lambda,k}$ *and the error bound* $O(e^{\vartheta t})$ *as* $t \to \infty$ *applies entrywise.*

Remark 3.17 By expanding the matrix $(\mathsf{f}_1 \exp(\lambda, t-x, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{D}}_\lambda$, we can reformulate the asymptotic expansion (3.41) in the form

$$\begin{aligned} F(t) &= \sum_{\lambda\in\Lambda_\theta} e^{\lambda t} \sum_{k=0}^{k_\lambda-1} D_{\lambda,k} \int\limits_0^t f(x)(t-x)^k e^{-\lambda x}\, \mathrm{d}x \\ &\quad + \mathbb{1}_{\vartheta<0}\, D_{0,k_0} \int_0^t f(x)(t-x)^{k_0-1}\, \mathrm{d}x + O(e^{\vartheta t}) \\ &= \sum_{\lambda\in\Lambda_\theta} e^{\lambda t} \sum_{k=0}^{k_\lambda-1} d_{\lambda,k}\, t^k + O(e^{(\theta+\varepsilon)t}) \end{aligned} \tag{3.42}$$

as $t \to \infty$, where $d_{\lambda,k} \in \mathbb{R}^p$ are deterministic vectors and the error bound $O(e^{(\theta+\varepsilon)t})$ applies componentwise. Further, $\varepsilon > 0$ can be chosen arbitrarily small, in particular, $\theta + \varepsilon < \mathrm{Re}(\lambda)$ for every $\lambda \in \Lambda_\theta$.

□

Remark 3.18 The coefficients $D_{\lambda,k}$ in the asymptotic expansion (3.41) can be given explicitly in terms of the coefficients $C_{\lambda,k}$ from the asymptotic expansion of $\boldsymbol{U}(t)$ given in Theorem 3.12 and Remark 3.14. We refrain from stating the exact expressions as they become quite involved.

□

Proof of Theorem 3.16.
By Theorem 3.12, there exist deterministic matrices C_{0,k_0}, $C_{\lambda,k} \in \mathbb{R}^{p\times p}$ with $\lambda \in \Lambda_\vartheta$,

$k = 0, \dots, k_\lambda - 1$ and $k_\lambda \in \mathbb{N}$ such that

$$\boldsymbol{U}(t) = \sum_{\lambda \in \Lambda_\vartheta} (\mathsf{f}_1 \exp(\lambda, t, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{C}}_\lambda + \mathbb{1}_{\vartheta < 0}\, C_{0,k_0}\, t^{k_0} + O(e^{\vartheta t}) \tag{3.19}$$

as $t \to \infty$ with $\vec{\mathbf{C}}_\lambda := \sum_{k=1}^{k_\lambda} \mathsf{f}_k^\mathsf{T} \otimes C_{\lambda,k}$. Now let $f : [0, \infty) \to \mathbb{R}^{p \times 1}$ be a nonnegative, right-continuous function with non-decreasing components satisfying

$$\int_0^\infty e^{-\vartheta x} f(x)\, \mathrm{d}x < \infty$$

where this inequality has to be understood componentwise. Then we may view f as the measure-generating function of a column vector ν of locally finite measures on $[0, \infty)$ by letting

$$f(t) =: \nu([0, t]) = \int \mathbb{1}_{[x,\infty)}(t)\, \nu(\mathrm{d}x), \quad t \geq 0. \tag{3.43}$$

By Proposition 3.2, the corresponding solution F to the Markov renewal equation (3.1) takes the form

$$\begin{aligned} F(t) = \boldsymbol{U} * f(t) = \boldsymbol{U} * \left(\int \mathbb{1}_{[x,\infty)}\, \nu(\mathrm{d}x) \right)(t) &= \int \boldsymbol{U} * \mathbb{1}_{[x,\infty)}(t)\, \nu(\mathrm{d}x) \\ = \int \left(\boldsymbol{U} * \mathbb{1}_{[0,\infty)} \right)(t - x)\, \nu(\mathrm{d}x) &= \int \boldsymbol{U}(t - x)\, \nu(\mathrm{d}x). \end{aligned}$$

Since the asymptotic expansion of $\boldsymbol{U}(t)$ is already known, we replace $\boldsymbol{U}(t - x)$ by (3.18) to obtain

$$\begin{aligned} &\int_0^t \boldsymbol{U}(t - x)\, \nu(\mathrm{d}x) \\ = &\int_0^t \left(\sum_{\lambda \in \Lambda_\vartheta} (\mathsf{f}_1 \exp(\lambda, t - x, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{C}}_\lambda + \mathbb{1}_{\vartheta < 0}\, C_{0,k_0} (t - x)^{k_0} + O(e^{\vartheta (t - x)}) \right) \nu(\mathrm{d}x) \\ = &\sum_{\lambda \in \Lambda_\vartheta} \int_0^t (\mathsf{f}_1 \exp(\lambda, t - x, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{C}}_\lambda\, \nu(\mathrm{d}x) + \mathbb{1}_{\vartheta < 0}\, C_{0,k_0} \int_0^t (t - x)^{k_0}\, \nu(\mathrm{d}x) \\ + &\int_0^t O(e^{\vartheta (t - x)})\, \nu(\mathrm{d}x). \end{aligned} \tag{3.44}$$

We focus on the first integral on the right-hand side in (3.44). By applying the functional equation of the exponential matrix $\exp(\lambda, x, k_\lambda)$, one can verify that

$$\frac{\mathrm{d}}{\mathrm{d}x} \exp(\lambda, x, k_\lambda) = \exp(\lambda, x, k_\lambda) J_\lambda$$

where the $k_\lambda \times k_\lambda$ matrix J_λ is defined as

$$J_\lambda := \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{pmatrix}.$$

Hence, we infer

$$\begin{aligned}\int_0^t (\mathsf{f}_1 \exp(\lambda, t-x, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{C}}_\lambda \, \nu(\mathrm{d}x) &= \int_0^t \int (\mathsf{f}_1 \exp(\lambda, t-x, k_\lambda) J_\lambda \otimes \mathrm{I}_p)\, \vec{\mathbf{C}}_\lambda \, \mathrm{d}x \, \nu(\mathrm{d}x) \\ &= \int_0^t (\mathsf{f}_1 \exp(\lambda, t-x, k_\lambda) J_\lambda \otimes \mathrm{I}_p)\, \vec{\mathbf{C}}_\lambda \, f(x) \, \mathrm{d}x.\end{aligned}$$

With $(\mathsf{f}_1 \exp(\lambda, t-x, k_\lambda) J_\lambda \otimes \mathrm{I}_p) = (\mathsf{f}_1 \exp(\lambda, t-x, k_\lambda) \otimes \mathrm{I}_p)(J_\lambda \otimes \mathrm{I}_p)$, we write

$$(\mathsf{f}_1 \exp(\lambda, t-x, k_\lambda) J_\lambda \otimes \mathrm{I}_p) \sum_{k=1}^{k_\lambda} (\mathsf{f}_k^\mathsf{T} \otimes C_{\lambda,k}) = (\mathsf{f}_1 \exp(\lambda, t-x, k_\lambda) \otimes \mathrm{I}_p) \sum_{k=1}^{k_\lambda} (J_\lambda \mathsf{f}_k^\mathsf{T} \otimes C_{\lambda,k}).$$

Here, we define $\vec{\mathbf{D}}_\lambda := \sum_{k=1}^{k_\lambda} (J_\lambda \mathsf{f}_j^\mathsf{T} \otimes C_{\lambda,k})$ as the new $pk_\lambda \times p$ matrix of coefficients such that we conclude

$$\int_0^t (\mathsf{f}_1 \exp(\lambda, t-x, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{C}}_\lambda \, \nu(\mathrm{d}x) = \int_0^t (\mathsf{f}_1 \exp(\lambda, t-x, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{D}}_\lambda \, f(x) \, \mathrm{d}x.$$

Furthermore, the second integral in (3.44) corresponds to

$$\begin{aligned}\mathbb{1}_{\vartheta<0}\, C_{0,k_0} \int_0^t (t-x)^{k_0} \, \nu(\mathrm{d}x) &= \mathbb{1}_{\vartheta<0}\, C_{0,k_0} \int_0^t \int k_0 (t-x)^{k_0-1} \, \mathrm{d}x \, \nu(\mathrm{d}x) \\ &= \mathbb{1}_{\vartheta<0}\, B_{0,k_0} \int_0^t (t-x)^{k_0-1} f(x) \, \mathrm{d}x,\end{aligned}$$

and the remainder term to $O(e^{\vartheta t})$ as $t \to \infty$ since

$$\int e^{\vartheta(t-y)} \, \nu(\mathrm{d}y) = e^{\vartheta t} \int e^{-\vartheta y} \, \nu(\mathrm{d}y) = e^{\vartheta t} \int f(y) \vartheta e^{-\vartheta y} \, \mathrm{d}y$$

where we have used integration by parts. Summarizing,

$$\begin{aligned}F(t) = &\int_0^t (\mathsf{f}_1 \exp(\lambda, t-x, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{D}}_\lambda \, f(x) \, \mathrm{d}x \\ &+ \mathbb{1}_{\vartheta<0}\, B_{0,k_0} \int_0^t (t-x)^{k_0-1} f(x) \, \mathrm{d}x + O(e^{\vartheta t}) \quad \text{as } t \to \infty.\end{aligned}$$

Finally, let $f : \mathbb{R} \to \mathbb{R}^{p\times 1}$ be a function that satisfies condition (A5), i.e.

$$\int_0^\infty e^{-\vartheta x} \, \mathrm{V}f(x) \, \mathrm{d}x < \infty.$$

We define the positive and negative part of f by $f^+(t) := \max(f(t), 0)$ and $f^-(t) := \max(-f(t), 0)$, respectively, for $t \in \mathbb{R}$. Thus, $f(t) = f^+(t) + f^-(t)$. Both f^+ and f^- are nonnegative, right-continuous functions from $\mathbb{R}$ to $\mathbb{R}^{p\times 1}$ with non-decreasing components. Furthermore, we have $\mathrm{V}f(t) = f^+(t) + f^-(t)$, for $t \in \mathbb{R}$. Therefore, f^+ and f^- also satisfy condition (A5) and the preceding structure of the proof extends to f^+, f^-, and f as well.

■

3.6.2 The Lattice Case

In the present subsection, we assume that the $p\times p$ -matrix of measures $\boldsymbol{\mu}$ is concentrated on $\mathbb{N}_0$. In this case, we examine the discrete solution $F(\{n\}) = \boldsymbol{U} * f(\{n\})$ to the Markov renewal equation (3.1). To determine its asymptotic behavior, it is convenient to use generating functions rather than Laplace transforms. Accordingly, we define the generating function of the $p \times p$ -matrix $\boldsymbol{\mu}$ as

$$\mathcal{G}\boldsymbol{\mu}(z) := \sum_{n=0}^{\infty} \boldsymbol{\mu}(\{n\}) z^l \tag{3.45}$$

for all $z \in \mathbb{C}$ for which the series is absolutely convergent. With this, observe that

$$\mathcal{G}\boldsymbol{\mu}(e^{-z}) = \mathcal{L}\boldsymbol{\mu}(z)$$

and $\mathcal{G}\boldsymbol{\mu}(e^{-z}) < \infty$ due to assumption (A1). As a result, the entries of the series (3.45) define holomorphic functions on an open disc that contains $\{|z| \leq e^{-\vartheta}\}$.

We begin our analysis by establishing the relationship between the poles of $(\mathrm{I}_p - \mathcal{G}\boldsymbol{\mu}(z))^{-1}$ and $(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$.

Lemma 3.19 [Connection between the poles in both cases]
Suppose that (A1) *and* (A2) *hold. If* $z = \lambda$ *is a pole of* $(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ *with multiplicity* $k_\lambda \in \mathbb{N}$*, then* $z = e^{-\lambda}$ *is a pole of* $(\mathrm{I}_p - \mathcal{G}\boldsymbol{\mu}(z))^{-1}$ *again with multiplicity* k_λ.

Proof of Theorem 3.12.
Let $\lambda \in \Lambda$ with multiplicity $k_\lambda \in \mathbb{N}$ and A_{λ,k_λ} be as introduced in the proof of Theorem 3.12 or in Remark 3.14. By definition, A_{λ,k_λ} is finite and has a non-zero entry. We show that $(\mathrm{I}_p - \mathcal{G}\boldsymbol{\mu}(z))^{-1}$ also has a pole at $z = e^{-\lambda}$ with multiplicity k_λ. Indeed,

$$\begin{aligned}
\lim_{z\to e^{-\lambda}} (\mathrm{I}_p - \mathcal{G}\boldsymbol{\mu}(z))^{-1}(z - e^{-\lambda})^{k_\lambda} &= \lim_{e^{-z}\to e^{-\lambda}} (\mathrm{I}_p - \mathcal{G}\boldsymbol{\mu}(z))^{-1}(e^{-z} - e^{-\lambda})^{k_\lambda} \\
&= \lim_{z\to\lambda} (\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1}(z-\lambda)^{k_\lambda} \frac{(e^{-z} - e^{-\lambda})^{k_\lambda}}{(z-\lambda)^{k_\lambda}} \\
&= A_{\lambda,k_\lambda}(-e^{-\lambda})^{k_\lambda}.
\end{aligned}$$

■

Let us define $b^{i,j}_{\lambda,k}$ as the coefficient in front of $(z - e^{-\lambda})^{-k}$ in the Laurent series of the (i,j)-th entry in $(\mathrm{I}_p - \mathcal{G}\boldsymbol{\mu}(z))^{-1}$ around the point $z = e^{-\lambda}$. In particular, $b^{i,j}_{\lambda,k} = 0$ for $k > k_\lambda$. Summarizing, let us consider

$$B_{\lambda,k} := (b^{i,j}_{\lambda,k})_{i,j\in[p]}$$

as the matrix of all coefficients in front of $(z-e^{-\lambda})^{-k}$ for $k \leq k_\lambda$, i.e.

$$(\mathrm{I}_p - \mathcal{G}\boldsymbol{\mu}(z))^{-1} = B_{\lambda,k_\lambda}(z-e^{-\lambda})^{-k_\lambda} + \ldots + B_{\lambda,1}(z-e^{-\lambda})^{-1} + H_\lambda(z) \tag{3.46}$$

where $H_\lambda(z)$ is holomorphic in a neighborhood of $e^{-\lambda}$. We proceed by determining the asymptotic expansion of $F(n)$. To facilitate this, we first introduce a lattice version of assumption (A5).

Assumption (A6) lattice

Suppose that (A1) holds. Let $f : \mathbb{N}_0 \to \mathbb{R}^{p\times 1}$ be a column vector-valued function vanishing on the negative half-line. Then, f satisfies the property

$$\sum_{n=0}^{\infty} e^{-\vartheta x} f(n)\,\mathrm{d}x < \infty$$

for some $\vartheta_0 < \vartheta < \mathrm{Re}(\lambda)$, $\lambda \in \Lambda$.

Theorem 3.20 [Asymptotic expansion for $F(n)$ in the lattice case]

Let $\boldsymbol{\mu} = (\mu^{i,j})_{i,j\in[p]}$ be a $p\times p$ matrix of locally finite measures concentrated on $\mathbb{N}_0$ satisfying (A1) and (A2). Further, we assume that $f : \mathbb{N}_0 \to \mathbb{R}^{p\times 1}$ is a column vector-valued function satisfying (A5) for some $\vartheta_0 < \vartheta < \mathrm{Re}(\lambda)$, $\lambda \in \Lambda$ and that the set Λ_ϑ is finite. Then there exist deterministic vectors $c_{\lambda,k} \in \mathbb{R}^{p\times 1}$ with $\lambda \in \Lambda_\vartheta$, $k = 0,\ldots,k_\lambda - 1$ and $k_\lambda \in \mathbb{N}$ such that

$$F(n) = \sum_{\lambda\in\Lambda_\vartheta} e^{\lambda n} \sum_{k=0}^{k_\lambda - 1} c_{\lambda,k}\, n^k + O(e^{\vartheta n}) \quad \text{as } n\to\infty, \tag{3.47}$$

where the error bound $O(e^{\vartheta n})$ as $n \to \infty$ applies entrywise.

Proof of Theorem 3.20.

For $r > 0$, let $B_r := \{|z| < r\}$ be the open disc with radius r in $\mathbb{C}$ centered at the origin and $\partial B_r := \{|z| = r\}$ be its boundary. We first suppose that $\vartheta > 0$. Now, fix $r < e^{-\alpha}$ where α is the maximal real root in Λ_ϑ. Since $\mathcal{G}\boldsymbol{\mu}$ and $\mathcal{G}f$ have holomorphic entries on $B_{e^{-\vartheta}}$ by (A1) and (A5), we infer from Cauchy's integral formula and the convolution theorem for generating functions that

$$\boldsymbol{\mu}^{*l} * f(n) = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_r} \frac{\mathcal{G}(\boldsymbol{\mu}^{*l} * f)(z)}{z^{n+1}}\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_r} \frac{(\mathcal{G}\boldsymbol{\mu})^l(z)\mathcal{G}f(z)}{z^{n+1}}\,\mathrm{d}z. \tag{3.48}$$

In particular,

$$\begin{aligned} F(n) = \boldsymbol{U} * f(n) = \sum_{l=0}^{\infty} \boldsymbol{\mu}^{*l} * f(n) = \sum_{l=0}^{\infty} \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r} \frac{(\mathcal{G}\boldsymbol{\mu})^l(z)\mathcal{G}f(z)}{z^{n+1}} \,\mathrm{d}z \\ = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r} \frac{(\mathrm{I}_p - \mathcal{G}\boldsymbol{\mu}(z))^{-1}\mathcal{G}f(z)}{z^{n+1}} \,\mathrm{d}z \end{aligned} \tag{3.49}$$

where the last step follows from Lemma A.8 since $\rho_{\mathcal{G}\boldsymbol{\mu}(z)} < 1$ for $|z| = r < e^{-\alpha}$.

For $\lambda \in \Lambda_\vartheta$, recall that $b^{i,j}_{\lambda,n}$ is the coefficient of $(z - e^{-\lambda})^{-n}$ in the Laurent series of the (i,j)-th entry of $(\mathrm{I}_p - \mathcal{G}\boldsymbol{\mu}(z))^{-1}$ at $z = e^{-\lambda}$ and $B_{\lambda,k} = (b^{i,j}_{\lambda,k})_{i,j\in[p]}$. We write

$$\mathcal{G}f(z) = \sum_{l=0}^{\infty} f_l(e^{-\lambda})(z - e^{-\lambda})^l, \text{ with coefficients } f_l(e^{-\lambda}) = \frac{1}{l!}(\mathcal{G}f)^{(l)}(e^{-\lambda}),\ l \in \mathbb{N}_0.$$

Then, we define the $p \times p$ matrices $G(z)$ and $H(z)$ by

$$\begin{aligned} G(z) &:= \sum_{\lambda\in\Lambda_\vartheta} \sum_{k=1}^{k_\lambda} B_{\lambda,k} \sum_{l=0}^{k-1} f_l(e^{-\lambda})(z - e^{-\lambda})^{-k+l}, \\ H(z) &:= (\mathrm{I}_p - \mathcal{G}(z))^{-1}\mathcal{G}f(z) - G(z). \end{aligned}$$

Here, G is meromorphic in $\mathbb{C}$ with poles located in Λ_ϑ and H is holomorphic in a neighborhood of $B_{e^{-\vartheta}}$. Further, it holds that $(\mathrm{I}_p - \mathcal{G}(z))^{-1}\mathcal{G}f(z) = G(z) + H(z)$, $z \in B_{e^{-\vartheta}} \setminus \Lambda_\vartheta$. Hence, with (3.49), we infer

$$F(n) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r} \frac{G(z) + H(z)}{z^{n+1}} \,\mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r} \frac{G(z)}{z^{n+1}} \,\mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r} \frac{H(z)}{z^{n+1}} \,\mathrm{d}z. \tag{3.50}$$

We focus on the first integral on the right-hand side in (3.50). By the residue theorem, for any $d \in \mathbb{N}_0$, we derive

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial B_r} \frac{(z - e^{-\lambda})^{-d}}{z^{n+1}} \,\mathrm{d}z = (-1)^d e^{\lambda(d+n)} \binom{n+d-1}{d-1}.$$

Consequently,

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r} \frac{G(z)}{z^{n+1}} \mathrm{d}z &= \sum_{\lambda\in\Lambda_\vartheta} \sum_{k=1}^{k_\lambda} B_{\lambda,k} \sum_{l=0}^{k-1} f_l(e^{-\lambda})(-e^{\lambda})^{-k+l} e^{\lambda n} \binom{n+k-l-1}{k-l-1} \\ &= \sum_{\lambda\in\Lambda_\vartheta} e^{\lambda n} \sum_{k=0}^{k_\lambda-1} c_{\lambda,k}\, n^k. \end{aligned}$$

Here, the $p \times 1$ vectors $c_{\lambda,k}$ depend only on the parameters λ, k and f_k.

Now, consider the second integral on the right-hand side in (3.50). Observe that the matrix $H(z)$ has holomorphic entries on the disc $B_{e^{-\vartheta}}$. Thus, with $\vartheta_n := \vartheta + \frac{1}{n}$, we infer

$$\left\| \int_{\partial B_{e^{-r}}} \frac{H(z)}{z^{n+1}} \,\mathrm{d}z \right\| = \left\| \int_{\partial B_{e^{-\vartheta_n}}} \frac{H(z)}{z^{n+1}} \,\mathrm{d}z \right\| \le \sup_{z\in B_{e^{-\vartheta}}} \|H(z)\| \frac{1}{(e^{-\vartheta_n})^n} = O(e^{\vartheta n}) \quad \text{as } n \to \infty \tag{3.51}$$

and with (3.50),

$$F(n) = \sum_{\lambda \in \Lambda_\vartheta} e^{\lambda n} \sum_{k=0}^{k_\lambda - 1} c_{\lambda,k}\, n^k + O(e^{\vartheta n}). \tag{3.52}$$

The proof can be extended to the case $\vartheta < 0$ using a similar strategy as in the non-lattice case. Indeed, as in the non-lattice case, we may choose $\gamma < \vartheta$ and define $\boldsymbol{\mu}_\gamma(\{n\}) := e^{-\gamma n}\boldsymbol{\mu}(\{n\})$, $\boldsymbol{U}_\gamma(\{n\}) := e^{-\gamma n}\boldsymbol{U}(\{n\})$, $f_\gamma(n) := e^{-\gamma n} f(n)$ and $F_\gamma(n) := e^{-\gamma n}F(n)$, $n \in \mathbb{N}_0$. Then,

$$F_\gamma(n) = e^{-\gamma n}F(n) = e^{-\gamma n}(\boldsymbol{U} * f)(n) = \boldsymbol{U}_\gamma * f_\gamma(n).$$

In particular, we have

$$\sum_{n=0}^{\infty} e^{-(\vartheta-\gamma)n} f_\gamma(n) = \sum_{n=0}^{\infty} e^{-\vartheta n + \gamma n} e^{-\gamma n} f(n) = \sum_{n=0}^{\infty} e^{-\vartheta n} f(n) < \infty$$

which holds due to assumption (A5). Additionally,

$$\mathcal{L}\boldsymbol{\mu}_\gamma(\vartheta - \gamma) = \sum_{n=0}^{\infty} e^{-(\vartheta-\gamma)n}\boldsymbol{\mu}_\gamma(\{n\}) = \sum_{n=0}^{\infty} e^{-\vartheta n + \gamma n} e^{-\gamma n}\boldsymbol{\mu}(\{n\}) = \mathcal{L}\boldsymbol{\mu}(\vartheta) < \infty$$

by assumption (A1). As in the non-lattice case, $\Lambda^\gamma = \{\lambda \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu}_\gamma) : \det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}_\gamma(z)) = 0\} = \Lambda - \gamma$ as well as $\Lambda^\gamma_{\vartheta-\gamma} = \Lambda_\vartheta - \gamma$. Moreover, $\vartheta - \gamma > 0$ and therefore, we have an asymptotic expansion for $F_\gamma(n)$, namely,

$$F_\gamma(n) = \sum_{\lambda \in \Lambda^\gamma_{\vartheta-\gamma}} e^{\lambda n} \sum_{k=0}^{k_\lambda - 1} c^\gamma_{\lambda,k}\, n^k + O(e^{(\vartheta-\gamma)n}).$$

Consequently, we infer

$$\begin{aligned} F(n) = e^{\gamma n}F_\gamma(n) &= e^{\gamma n}\Bigg(\sum_{\lambda \in \Lambda^\gamma_{\vartheta-\gamma}} e^{\lambda n} \sum_{k=0}^{k_\lambda - 1} c^\gamma_{\lambda,k}\, n^k + O(e^{(\vartheta-\gamma)n})\Bigg) \\ &= \sum_{\lambda \in \Lambda_\vartheta} e^{\lambda n} \sum_{k=0}^{k_\lambda - 1} c^\gamma_{\lambda,k}\, n^k + O(e^{\vartheta n}). \end{aligned}$$

■

3.7 *Application to Multi-Type Branching Processes*

As outlined in Section 2.2.2, the multi-type (general) Crump-Mode-Jagers branching process satisfies a Markov renewal equation. In this context, let $\boldsymbol{\xi}$ represent the offspring process matrix and $\boldsymbol{\mu} = \mathbf{E}[\boldsymbol{\xi}]$ its intensity measure. Recall that N^j_t denotes the number of individuals of type j born up to and including time t, while $N_t = (N_t^1, \dots, N_t^p)$ is its corresponding row vector. Then, with $M(t) = (m_t^{i,j})_{i,j\in[p]} := (\mathbf{E}^i[N_t^j])_{i,j\in[p]}$ we have established

$$M(t) = \mathbb{1}_{[0,\infty)}(t)\mathrm{I}_p + M * \boldsymbol{\mu}(t), \quad t \in \mathbb{R}. \tag{2.8}$$

Building on the previous sections of this thesis, where we have thoroughly examined the asymptotic behavior of general solutions F to Markov renewal equations (3.1), we are now prepared to apply our main results, namely Theorem 3.12 and Theorem 3.16, to this particular framework. We begin by discussing the implications of our assumptions (A1) and (A2) on $M(t)$ as $t \to \infty$. Notably, these assumptions serve as the multi-type analogues to those presented in Jagers' classical textbook [Jag75], where the finiteness of the process in the single-type case in ensured.

3.7.1 No Explosion in Finite Time

A multi-type general branching process with the property that there exists $i, j \in [p]$ and $t > 0$ such that $\mathbf{P}^i(\mathcal{Z}_t^j = \infty) > 0$ is called explosive. A classical result that guarantees that the single-type general branching process is not explosive can be found in [Jag75, Theorem 6.2.2] and states that explosion does not occur if the expected number of children of the ancestor born at time 0 is strictly less than 1 and the expected number of children born up to and including time ε is finite for some $\varepsilon > 0$, i.e. $\mu(0) < 1$ and $\mu(\varepsilon) < \infty$. The following proposition forms the multi-type analogue of [Jag75, Theorem 6.2.2].

Proposition 3.21 [Non-explosive multi-type general branching processes]

① *Suppose that* (A2) *holds and that* $\|\boldsymbol{\mu}(\varepsilon)\| < \infty$ *for some* $\varepsilon > 0$. *Then, for any* $i, j \in [p]$, *we have*

$$\mathbf{P}^i(N_t^j < \infty \text{ for all } t \geq 0) = 1. \tag{3.53}$$

② *Suppose that* (A1) *and* (A2) *hold. Then there exists some* $\varepsilon > 0$ *with* $\|\boldsymbol{\mu}(\varepsilon)\| < \infty$. *In particular,* (3.53) *holds. Further, there exists some* $\theta \in \mathbb{R}$ *with* $\varrho(\theta) < 1$ *and for any such* $\theta \in \mathbb{R}$ *and for all* $i, j \in [p]$, *we have*

$$m_t^{i,j} = \mathbf{E}^i[N_t^j] = O(e^{\theta t}) \quad \text{as } t \to \infty. \tag{3.54}$$

Proof of Proposition 3.21.

① Following [Jag75, Theorem 6.2.2], the proof is structured into two steps. Initially, the claim up to a specific time s is established by embedding a multi-type Galton-

Watson process. Then, the proof continues after time t by applying the recursive decomposition introduced in Section 3.7.1.

Therefore, we may assume without loss of generality that $\boldsymbol{\mu}(t)$ has positive entries only for every $t > 0$ and, in particular, is primitive. Indeed, if this is not the case, then $\mu^{i,j}(t) = 0$ for some $i, j \in [p]$ and all sufficiently small $t \geq 0$. For each such pair (i, j), add an independent Poisson point process with intensity $c > 0$ to $\xi^{i,j}$ leading to an augmented general branching process. Then the intensity measure of the augmented process is as desired and still satisfies the assumptions. Moreover, the augmented process dominates the original, so if it satisfies the conclusion, then so does the original process. By the continuity of the map $\theta \to \rho_{\boldsymbol{\mu}(\theta)}$ on $[0, \varepsilon]$, see Lemma A.1, there exists some $s > 0$ with $\rho_{\boldsymbol{\mu}(s)} < 1$. For $u \in \mathcal{I}$, define $\boldsymbol{\eta}_u := \boldsymbol{\xi}_u(s)$. Then the $\boldsymbol{\eta}_u$ are i. i. d. random matrices with entries in $\mathbb{N}_0$ and thus define a multi-type Galton-Watson process $(Z_n)_{n \in \mathbb{N}_0}$ where $Z_n = (Z_n^1, \dots, Z_n^p)$ and Z_n^i denotes the number of individuals of type $i \in [p]$ in generation n. By standard theory, see e.g. [Har63, Equation (2.4.1)], we have $\mathbf{E}^i[Z_n] = \mathsf{e}_i \mathbf{E}[\boldsymbol{\eta}]^n$. Therefore, if one understands inequalities between vectors as componentwise,

$$m_s^i := \mathbf{E}^i[N_s] \leq \mathbf{E}^i\left[\sum_{n=0}^{\infty} Z_n\right] = \mathsf{e}_i \sum_{n=0}^{\infty} \mathbf{E}[\boldsymbol{\eta}]^n$$

and the last expression is finite in each component by Lemma A.2. In particular, $\mathbf{P}^i(N_t^j < \infty \text{ for all } 0 \leq t \leq s) = \mathbf{P}^i(|N_s|_1 < \infty) = 1$ where the penultimate step follows from the monotonicity of $\theta \to N_\theta^j$, $j \in [p]$ and $\theta \in \mathbb{R}$. This means that $q_t^i := \mathbf{P}^i(|N_t|_1 = \infty)$ is non-decreasing, $i \in [p]$, and for $q_t := (q_t^1, \dots, q_t^p)$, we have that $|q_s|_1 = 0$.

Now let $t \leq s$ be such that $|q_t|_1 = 0$. The recursive decomposition (2.6) can be rewritten as

$$N_{s+u} = \mathsf{e}_i + \sum_{k=1}^{\xi(s+u)} N_{k, s+u-S(k)}.$$

Consequently, if $|N_{s+u}|_1 = \infty$, then $|N_{k,s+u-S(k)}|_1 = \infty$ for some $k = 1, \dots, \xi(s+u)$ since $\xi(s+u) < \infty$ by the local finiteness of ξ. In particular, if $|N_{k,s+u-S(k)}|_1 = \infty$, then it follows that $S(k) \leq u$ as otherwise $|q_s|_1 = 0$. Therefore,

$$\begin{aligned} q_{s+u}^i = \mathbf{P}^i(|N_{s+u}|_1 = \infty) &= \mathbf{E}^i\left[\mathbf{P}^i\left(\bigcup_{k=1}^{\xi(s+u)} \{|N_{k,s+u-S(k)}|_1 = \infty\} \,\middle|\, \xi\right)\right] \\ &\leq \mathbf{E}^i\left[\sum_{k=1}^{\xi(u)} \mathbf{P}^{\tau(k)}(|N_{k,s+u-S(k)}|_1 = \infty \,|\, \xi)\right] \leq \mathbf{E}^i\left[\sum_{k=1}^{\xi(u)} q_{s+u-S(k)}^{\tau(k)}\right] \\ &\leq \mathbf{E}^i\left[\sum_{k=1}^{\xi(u)} q_{s+u}^{\tau(k)}\right] = \mathsf{e}_i \boldsymbol{\mu}(u) q_{s+u}, \end{aligned}$$

i.e., $q_{s+u} \leq \boldsymbol{\mu}(u) q_{s+u}$, which implies $q_{s+u} = 0$ by Lemma A.2.

② Now suppose that (A1) and (A2) hold, i.e., $\mathcal{L}\boldsymbol{\mu}(\vartheta)$ has finite entries only for some $\vartheta \in \mathbb{R}$ and $\rho_{\boldsymbol{\mu}(0)} < 1$. Recall from Proposition 3.1 that $\varrho(\theta) = \rho_{\mathcal{L}\boldsymbol{\mu}(\theta)} \to \rho_{\boldsymbol{\mu}(0)} < 1$

as $\theta \to \infty$. Consequently, we may choose $\theta \geq \vartheta$ with $\varrho(\theta) < 1$. Recall that $M_t = (m_t^{i,j})_{i,j\in[p]} = (\mathbf{E}^i[N_t^j])_{i,j\in[p]}$. Then, using integration by parts,

$$\begin{aligned}
\mathcal{L}M(\theta) &:= \int_0^\infty e^{-\theta t} M_t \,\mathrm{d}t = \int_0^\infty e^{-\theta t} \sum_{n=0}^\infty \boldsymbol{\mu}^{*n}(t)\,\mathrm{d}t = \sum_{n=0}^\infty \int_0^\infty e^{-\theta t} \boldsymbol{\mu}^{*n}(t)\,\mathrm{d}t \\
&= \sum_{n=0}^\infty \int_0^\infty e^{-\theta t} \int_{[0,t]} \boldsymbol{\mu}^{*n}(\mathrm{d}x)\,\mathrm{d}t = \sum_{n=0}^\infty \int_0^\infty \int_{[0,t]} e^{-\theta(t-x)} e^{-\theta x}\, \boldsymbol{\mu}^{*n}(\mathrm{d}x)\,\mathrm{d}t \\
&= \sum_{n=0}^\infty \int_{[0,\infty)} \int_t^\infty e^{-\theta(t-x)}\,\mathrm{d}t\, e^{-\theta x}\, \boldsymbol{\mu}^{*n}(\mathrm{d}x) \\
&= \frac{1}{\theta} \sum_{n=0}^\infty \int_{[0,\infty)} e^{-\theta x}\, \boldsymbol{\mu}^{*n}(\mathrm{d}x) = \frac{1}{\theta} \sum_{n=0}^\infty \mathcal{L}\boldsymbol{\mu}(\theta)^n,
\end{aligned}$$

which is a finite matrix since $\varrho(\theta) < 1$, see Lemma A.2. The assertion now follows from Lemma B.1 applied to the functions $t \mapsto m_t^{i,j}$, $i,j \in [p]$.

■

3.7.2 Asymptotic Expansion for the Mean

The asymptotic expansion of the mean matrix $M(t) = (m_t^{i,j})_{i,j\in[p]} := (\mathbf{E}^i[N_t^j])_{i,j\in[p]}$ can now be derived directly from Theorem 3.12.

Theorem 3.22 [Asymptotic expansion for $M(t)$ in the non-lattice case]
Consider a multi-type general branching process with matrix of intensity measures $\boldsymbol{\mu} = (\mu^{i,j})_{i,j\in[p]}$ *satisfying* (A1), (A2) *and* (A4). *Further, suppose that* Λ_ϑ *is finite. Write* $M(t) = (m_{ij}(t))_{i,j\in[p]} = (\mathbf{E}^i[N_t^j])_{i,j\in[p]}$ *where* $\mathbf{E}^i[N_t^j]$ *is the expected number of individuals of type* j *born up to and including time* t *given that the ancestor's type is* i. *Then there exist deterministic matrices* $C_{\lambda,k} \in \mathbb{R}^{p\times p}$, $k = 0,\dots,k_\lambda - 1$, $\lambda \in \Lambda_\vartheta$, C_{0,k_0} *such that*

$$M(t) = \sum_{\lambda\in\Lambda_\vartheta} e^{\lambda t} \sum_{k=0}^{k_\lambda - 1} C_{\lambda,k}\, t^k + \mathbb{1}_{\vartheta<0}\, C_{0,k_0}\, t^{k_0} + O(te^{\vartheta t}) \quad \text{as } t \to \infty, \tag{3.55}$$

where the error bound $O(te^{\vartheta t})$ *as* $t \to \infty$ *applies entrywise.*

Proof of Theorem 3.22.
The result follows immediately from Theorem 3.12 since $M(t) = \boldsymbol{U}(t) = \sum_{n=0}^\infty \boldsymbol{\mu}^{*n}(t)$ for all $t \geq 0$.

■

Remark 3.23 The asymptotic expansion (3.55) in Theorem 3.22 can also be formulated in a compact form, as shown in Remark 3.15, for the case where $\mathrm{Re}(\lambda) > 0$, namely,

$$M(t) = \sum_{\lambda \in \Lambda_\vartheta} \int_0^\infty (\mathsf{f}_1 \exp(\lambda, t-x, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{A}}_\lambda \, \mathrm{d}x + O(te^{\vartheta t}). \tag{3.56}$$

□

Example 3.24 Consider a 2-type process, in which particles of type 1 give birth to particles of type 1 according to a Poisson point process ξ_{11} with intensity $\alpha > 0$ and particles of type 2 give birth to particles of type 2 also according to a Poisson point process ξ_{22} with intensity $\alpha > 0$ where ξ_{11} and ξ_{22} are independent. Further, let $\xi_{12} = \delta_0$ and $\xi_{21} = 0$, i.e., at the time of birth, every type-1-particle immediately produces a type-2-particle. Then

$$\mathcal{L}\boldsymbol{\mu}(z) = \begin{pmatrix} \frac{\alpha}{z} & 1 \\ 0 & \frac{\alpha}{z} \end{pmatrix}, \quad \mathrm{Re}(z) > 0.$$

Further, $\det(\mathbf{I}_2 - \mathcal{L}\boldsymbol{\mu}(z)) = (1 - \frac{\alpha}{z})^2$, i.e., $\Lambda = \{\alpha\}$ and α is a zero of order two of the determinant $\det(\mathbf{I}_2 - \mathcal{L}\boldsymbol{\mu}(z))$. Moreover,

$$\begin{aligned}(\mathbf{I}_2 - \mathcal{L}\boldsymbol{\mu}(z))^{-1} &= \frac{1}{(1-\frac{\alpha}{z})^2} \begin{pmatrix} 1-\frac{\alpha}{z} & 1 \\ 0 & 1-\frac{\alpha}{z} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \alpha^2 \\ 0 & 0 \end{pmatrix} \frac{1}{(z-\alpha)^2} + \begin{pmatrix} \alpha & 2\alpha \\ 0 & \alpha \end{pmatrix} \frac{1}{z-\alpha} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{Re}(z) > 0,\ z \neq \alpha.\end{aligned}$$

This means that $(\mathbf{I}_2 - \mathcal{L}\boldsymbol{\mu}(z))^{-1}$ has a pole of order 2 at $z = \alpha$. Consequently, we infer

$$M(t) = e^{\alpha t} \begin{pmatrix} 1 & 1 + t\alpha \\ 0 & 1 \end{pmatrix}$$

with Theorem 3.22 and the coefficients from Remark 3.14.

□

Suppose that $(\mathcal{Z}_t^{\boldsymbol{\varphi}})_{t\geq 0}$ is a multi-type Crump-Mode-Jagers process counted with random characteristic $\boldsymbol{\varphi}$. Define $F_i(t) := \mathbf{E}^i[\mathcal{Z}_t^{\boldsymbol{\varphi}}(t)]$ for $i \in [p]$ and $F(t) := (F_1(t), \dots, F_p(t))^\mathsf{T}$, $t \in \mathbb{R}$. If $\boldsymbol{\varphi}$ satisfies appropriate integrability conditions to be specified later, then $F : \mathbb{R} \to \mathbb{R}^p$ vanishes on the negative halfline and satisfies a Markov renewal equation, namely, with $S_k(u) = S(ku) - X_k$ on $\{X_k < \infty\}$, by conditioning with respect to the first generation,

$$\begin{aligned}F_i(t) = \mathbf{E}^i[\mathcal{Z}_t^{\boldsymbol{\varphi}}(t)] = \mathbf{E}^i\left[\sum_{u\in\mathcal{T}} \varphi_u(t - S(u))\right] &= \mathbf{E}^i\left[\varphi^i(t) + \sum_{k=1}^N \sum_{u\in\mathcal{T}} \varphi_{ku}(t - X_k - S_k(u))\right] \\ = \mathbf{E}^i[\varphi^i(t)] + \sum_{j=1}^p \int F_j(t-x)\, \mu^{i,j}(\mathrm{d}x) &= f_i(t) + (\boldsymbol{\mu} * F(t))_i\end{aligned}$$

where $(\boldsymbol{\mu} * F(t))_i$ is the i-th entry of the vector $\boldsymbol{\mu} * F(t)$ and $f_i(t) := \mathbf{E}^i[\varphi^i(t)]$. In other words, $F = f + \boldsymbol{\mu} * F$, that is, F satisfies the Markov renewal equation (3.1) provided that $F(t)$ and $f(t)$ exist and are finite. Hence, this allows us to apply Theorem 3.16 on $F(t) = (\mathbf{E}^1[\mathcal{Z}_t^{\varphi}], \dots, \mathbf{E}^p[\mathcal{Z}_t^{\varphi}])^{\mathsf{T}}$.

Theorem 3.25 [Asymptotic expansion for $F(t)$ in the non-lattice case] *Consider a multi-type general branching process $(\mathcal{Z}_t^{\boldsymbol{\varphi}})_{t\geq 0}$ counted with random characteristic $\boldsymbol{\varphi}$. Suppose that the matrix $\boldsymbol{\mu} = (\mu^{i,j})_{i,j\in[p]}$ of intensity measures satisfies* (A1), (A2) *and* (A4)*. Further, suppose that Λ_ϑ is finite. Assume that $f(t) := (\mathbf{E}^1[\varphi^1](t), \dots, \mathbf{E}^p[\varphi^p](t))^{\mathsf{T}}$ is finite for every $t \geq 0$, right-continuous with existing left limits as a function of t and satisfying* (A5) *for some $\vartheta_0 < \vartheta <$* $\mathrm{Re}(\lambda)$*. Then there exist deterministic matrices D_{0,k_0}, $D_{\lambda,k} \in \mathbb{R}^{p\times p}$ with $\lambda \in \Lambda_\vartheta$, $k = 0, \dots, k_\lambda - 1$ and $k_\lambda \in \mathbb{N}$ such that for $F(t) = (\mathbf{E}^1[\mathcal{Z}_t^{\boldsymbol{\varphi}}], \dots, \mathbf{E}^p[\mathcal{Z}_t^{\boldsymbol{\varphi}}])^{\mathsf{T}}$, it holds that*

$$
\begin{aligned}
F(t) = &\sum_{\lambda\in\Lambda_\vartheta} \int_0^t (\mathsf{f}_1 \exp(\lambda, t-x, k_\lambda) \otimes \mathrm{I}_p)\, \vec{\mathbf{D}}_\lambda\, f(x)\, \mathrm{d}x \\
&+ \mathbb{1}_{\vartheta<0}\, D_{0,k_0} \int_0^t (t-x)^{k_0-1} f(x)\, \mathrm{d}x + O(e^{\vartheta t}) \quad \textit{as } t \to \infty, \qquad (3.57)
\end{aligned}
$$

where the error bound $O(e^{\vartheta t})$ applies componentwise.

The lattice case is analogous.

4 | Martingales for Multi-Type Branching Processes

The objective of the present chapter is to establish specific complex martingales, denoted by $W_t(\lambda, k_\lambda)$, for $t \in \mathbb{R}$ that are associated with the solutions λ to the characteristic equation (3.8), where $\mathrm{Re}(\lambda) > 0$, and their corresponding multiplicities $k_\lambda \in \mathbb{N}$. These martingales are crucial for determining the principal terms in the asymptotic expansion of the multi-type general branching process counted with some random characteristic $\mathcal{Z}_t^\varphi$ as $t \to \infty$, and form the foundational building block for establishing the central limit theorem. More specifically, the derivation of the principal terms in the asymptotic expansion of the multi-type general branching process $\mathcal{Z}_t^\varphi$ as $t \to \infty$ is based on the asymptotic expansion m_t^φ discussed in Section 3.7, together with the complex martingales $W_t(\lambda, k_\lambda)$, for $t \in \mathbb{R}$ constructed in this chapter. In this context, recall that, under suitable assumptions, we have obtained

$$m_t^\varphi = \sum_{\lambda \in \Lambda_\vartheta} \int_0^\infty (\mathsf{f}_1 \exp(\lambda, t - x, k_\lambda) \otimes \mathrm{I}_p) \vec{\mathbf{A}}_\lambda \mathrm{d}x + O(te^{\vartheta t}) \quad \text{as } t \to \infty, \tag{3.56}$$

for some $\vartheta > 0$ and $p \times p$ matrices of coefficients $C_{\lambda,k}$. Then, the principal terms in the asymptotic expansion of $\mathcal{Z}_t^\varphi$ are of the form

$$\sum_{\lambda \in \Lambda_\vartheta} e^{\lambda t} \sum_{k=0}^{k_\lambda - 1} W_t(\lambda, k_\lambda)\, t^k + O(te^{\vartheta t}).$$

We commence our analysis with the construction of the martingales $W_t(\lambda, k_\lambda)$, for $t \in \mathbb{R}$. Following this foundational step, we will illustrate how these martingales can be represented in terms of the multi-type general branching process $\mathcal{Z}_t^\phi$, where ϕ is a specific random characteristic. Finally, we will turn our attention to the convergence properties of these martingales and demonstrate that $W_t(\lambda, k_\lambda)$ is an L^q-bounded martingale, for $q \in (1, 2]$.

4.1 ***Preliminary Construction***

The starting point of our construction is given by Equation (3.9), namely,

$$(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))^{-1} = \frac{1}{\det(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z))} \mathrm{Adj}(\mathrm{I}_p - \mathcal{L}\boldsymbol{\mu}(z)), \quad z \in \mathcal{D}(\mathcal{L}\boldsymbol{\mu}) \setminus \Lambda.$$

Here, recall that each entry in $(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))^{-1}$ may have a pole at $\lambda \in \Lambda_\vartheta$, where k_λ denotes the maximal multiplicity of the singularity in $(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))^{-1}$. As previously mentioned, we denote by $a^{i,j}_{\lambda,k}$ the coefficient of $(z-\lambda)^{-k}$ in the Laurent series of the (i,j)-th entry in $(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))^{-1}$ around $\lambda \in \Lambda_\vartheta$, for $k = 1, \dots, k_\lambda$, where $a^{i,j}_{\lambda,k} = 0$ for $k > k_\lambda$, indicating that only finitely many coefficients $a^{i,j}_{\lambda,k} \neq 0$ exist. Summarizing this, we denote by $A_{\lambda,k} := (a^{i,j}_{\lambda,k})_{i,j\in[p]}$ the matrix comprising all coefficients corresponding to $(z-\lambda)^{-k}$.

Consequently, as in (3.38), we write

$$(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))^{-1} = A_{\lambda,k_\lambda}(z-\lambda)^{-k_\lambda} + \ldots + A_{\lambda,1}(z-\lambda)^{-1} + H_{\lambda,1}(z)$$

for all $z \neq \lambda$ in a small disc $D(\lambda)$ centered at λ. Here, $H_{\lambda,1}(z)$ is holomorphic in $D(\lambda)$, while $A_{\lambda,k_\lambda}, \dots, A_{\lambda,1}$ are complex $p \times p$ matrices. Similarly, expanding $(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z))$ around $\lambda \in \Lambda_\vartheta$, we obtain the Taylor series

$$(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(z)) = \sum_{m=0}^{k_\lambda-1} \frac{(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(\lambda))^{(m)}}{m!}(z-\lambda)^m + H_{\lambda,2}(z)(z-\lambda)^{k_\lambda}$$

with $H_{\lambda,2}(z)$ denoting another holomorphic function in $D(\lambda)$. With both series, we have the representation

$$\mathrm{I}_p = \left(\sum_{m=0}^{k_\lambda-1} \frac{(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(\lambda))^{(m)}}{m!}(z-\lambda)^m + H_{\lambda,2}(z)(z-\lambda)^{k_\lambda}\right)\left(\sum_{n=1}^{k_\lambda} A_{\lambda,n}(z-\lambda)^{-n} + H_{\lambda,1}(z)\right).$$

By expanding both series on the right-hand side and subsequently comparing the coefficients on the left- and right-hand sides, we conclude

$$\sum_{l=0}^{j} \frac{(\mathrm{I}_p - \boldsymbol{\mathcal{L}\mu}(\lambda))^{(l)}}{l!} A_{\lambda,k_\lambda-j+l} = 0 \quad \text{for } j = 0, \dots, k_\lambda - 1.$$

Rearranging this leads to the identities

$$A_{\lambda,k_\lambda-j} = \sum_{l=0}^{j} \frac{\boldsymbol{\mathcal{L}\mu}^{(l)}(\lambda)}{l!} A_{\lambda,k_\lambda-j+l} \quad \text{for } j = 0, \dots, k_\lambda - 1. \tag{4.1}$$

In order to obtain a more compact form of (4.1), we represent the identities in the matrix form

$$\begin{pmatrix} A_{\lambda,1} \\ \vdots \\ A_{\lambda,k_\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mathcal{L}\mu}(\lambda) & \boldsymbol{\mathcal{L}\mu}^{(1)}(\lambda) & \frac{\boldsymbol{\mathcal{L}\mu}^{(2)}(\lambda)}{2!} & \cdots & \frac{\boldsymbol{\mathcal{L}\mu}^{(k_\lambda-1)}(\lambda)}{(k_\lambda-1)!} \\ 0 & \boldsymbol{\mathcal{L}\mu}(\lambda) & \boldsymbol{\mathcal{L}\mu}^{(1)}(\lambda) & \cdots & \vdots \\ 0 & 0 & \boldsymbol{\mathcal{L}\mu}(\lambda) & \ddots & \frac{\boldsymbol{\mathcal{L}\mu}^{(2)}(\lambda)}{2!} \\ \vdots & \ddots & \ddots & \ddots & \boldsymbol{\mathcal{L}\mu}^{(1)}(\lambda) \\ 0 & \cdots & 0 & 0 & \boldsymbol{\mathcal{L}\mu}(\lambda) \end{pmatrix} \begin{pmatrix} A_{\lambda,1} \\ \vdots \\ A_{\lambda,k_\lambda} \end{pmatrix}. \tag{4.2}$$

In (4.2), the matrix on the left-hand side denotes a $pk_\lambda \times p$ matrix, whereas the matrix on the right-hand side which involves the derivates of $\boldsymbol{\mathcal{L}\mu}$, is of dimension $pk_\lambda \times pk_\lambda$. Furthermore, notice that for every $\lambda \in \Lambda_\vartheta$ with multiplicity $k_\lambda \in \mathbb{N}$ and $s \in \mathbb{R}$, we have

$$\boldsymbol{\mathcal{L}\mu}^{(k)}(\lambda) = \int_0^\infty (-s)^k e^{-\lambda s}\, \boldsymbol{\mu}(\mathrm{d}s),$$

for $k \leq k_\lambda - 1$. This indicates the application of the exponential $k_\lambda \times k_\lambda$ matrices $\exp(\lambda, s, k_\lambda)$, introduced in Subsection 2.1.7, to simplify the $pk_\lambda \times pk_\lambda$ matrix in (4.2). In fact, we infer

$$\begin{pmatrix} \boldsymbol{\mathcal{L}\mu}(\lambda) & \boldsymbol{\mathcal{L}\mu}^{(1)}(\lambda) & \frac{\boldsymbol{\mathcal{L}\mu}^{(2)}(\lambda)}{2!} & \cdots & \frac{\boldsymbol{\mathcal{L}\mu}^{(k_\lambda-1)}(\lambda)}{(k_\lambda-1)!} \\ 0 & \boldsymbol{\mathcal{L}\mu}(\lambda) & \boldsymbol{\mathcal{L}\mu}^{(1)}(\lambda) & \cdots & \vdots \\ 0 & 0 & \boldsymbol{\mathcal{L}\mu}(\lambda) & \ddots & \frac{\boldsymbol{\mathcal{L}\mu}^{(2)}(\lambda)}{2!} \\ \vdots & \ddots & \ddots & \ddots & \boldsymbol{\mathcal{L}\mu}^{(1)}(\lambda) \\ 0 & \cdots & 0 & 0 & \boldsymbol{\mathcal{L}\mu}(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} \int_0^\infty e^{-\lambda s}\boldsymbol{\mu}(\mathrm{d}s) & \int_0^\infty -se^{-\lambda s}\boldsymbol{\mu}(\mathrm{d}s) & \int_0^\infty \frac{s^2}{2!}e^{-\lambda s}\boldsymbol{\mu}(\mathrm{d}s) & \cdots & \int_0^\infty \frac{(-s)^{k_\lambda-1}}{(k_\lambda-1)!}e^{-\lambda s}\boldsymbol{\mu}(\mathrm{d}s) \\ 0 & \int_0^\infty e^{-\lambda s}\boldsymbol{\mu}(\mathrm{d}s) & \int_0^\infty -se^{-\lambda s}\boldsymbol{\mu}(\mathrm{d}s) & \cdots & \vdots \\ 0 & 0 & \int_0^\infty e^{-\lambda s}\boldsymbol{\mu}(\mathrm{d}s) & \ddots & \int_0^\infty \frac{s^2}{2!}e^{-\lambda s}\boldsymbol{\mu}(\mathrm{d}s) \\ \vdots & \ddots & \ddots & \ddots & \int_0^\infty -se^{-\lambda s}\boldsymbol{\mu}(\mathrm{d}s) \\ 0 & 0 & 0 & 0 & \int_0^\infty e^{-\lambda s}\, \boldsymbol{\mu}(\mathrm{d}s) \end{pmatrix}$$

$$= \int_0^\infty \exp(\lambda, -s, k_\lambda) \otimes \boldsymbol{\mu}(\mathrm{d}s) \tag{4.3}$$

where $\otimes$ denotes the Kronecker product, as discussed in Subsection 2.1.4. Consequently, given the $pk_\lambda \times p$ matrix $\vec{\mathbf{A}}_\lambda$ as defined in (4.1), i.e.

$$\vec{\mathbf{A}}_\lambda = \sum_{l=1}^{k_\lambda} \mathsf{f}_l^\mathsf{T} \otimes A_{\lambda,l} = \begin{pmatrix} A_{\lambda,1} \\ \vdots \\ A_{\lambda,k_\lambda} \end{pmatrix},$$

we can express the matrix equation (4.2) by using (4.3) in the form

$$\vec{\mathbf{A}}_\lambda = \left(\int_0^\infty \exp(\lambda, -s, k_\lambda) \otimes \boldsymbol{\mu}(\mathrm{d}s)\right)\vec{\mathbf{A}}_\lambda = \mathbf{E}\left[\int_0^\infty \exp(\lambda, -s, k_\lambda) \otimes \boldsymbol{\xi}(\mathrm{d}s)\right]\vec{\mathbf{A}}_\lambda. \tag{4.4}$$

Remark 4.1 To familiarize the reader with the computation involving Kronecker products, we will present here a detailed calculation of Equation (4.4). Hence, notice that

$$\left(\int_0^\infty \exp(\lambda,-s,k_\lambda)\otimes\boldsymbol{\mu}(\mathrm{d}s)\right)\vec{\mathbf{A}}_\lambda = \left(\int_0^\infty \exp(\lambda,-s,k_\lambda)\otimes\boldsymbol{\mu}(\mathrm{d}s)\right)\left(\sum_{l=1}^{k_\lambda}\mathsf{f}_l^\mathsf{T}\otimes A_{\lambda,l}\right)$$
$$= \sum_{l=1}^{k_\lambda}\int_0^\infty (\exp(\lambda,-s,k_\lambda)\mathsf{f}_l^\mathsf{T})\otimes(\boldsymbol{\mu}(\mathrm{d}s)A_{\lambda,l}).$$

Since the l-th column of the $k_\lambda\times k_\lambda$ matrix $\exp(\lambda,-s,k_\lambda)$ is given by

$$\exp(\lambda,-s,k_\lambda)\mathsf{f}_l^\mathsf{T} = \sum_{i=1}^{l} e^{-\lambda s}\frac{(-s)^{l-i}}{(l-i)!}\mathsf{f}_i^\mathsf{T},$$

we obtain

$$\sum_{l=1}^{k_\lambda}\int_0^\infty (\exp(\lambda,-s,k_\lambda)\mathsf{f}_l^\mathsf{T})\otimes(\boldsymbol{\mu}(\mathrm{d}s)A_{\lambda,l}) = \sum_{l=1}^{k_\lambda}\int_0^\infty\left(\sum_{i=1}^{l}\frac{e^{-\lambda s}(-s)^{l-i}}{(l-i)!}\mathsf{f}_i^\mathsf{T}\right)\otimes(\boldsymbol{\mu}(\mathrm{d}s)A_{\lambda,l})$$
$$= \sum_{l=1}^{k_\lambda}\sum_{i=1}^{l}\mathsf{f}_i^\mathsf{T}\otimes\left(\int_0^\infty\frac{e^{-\lambda s}(-s)^{l-i}}{(l-i)!}\boldsymbol{\mu}(\mathrm{d}s)A_{\lambda,l}\right)$$
$$= \sum_{l=1}^{k_\lambda}\sum_{i=1}^{l}\mathsf{f}_i^\mathsf{T}\otimes\left(\frac{\mathcal{L}\boldsymbol{\mu}^{(l-i)}(\lambda)}{(l-i)!}A_{\lambda,l}\right).$$

By rearranging both sums on the right-hand side and using (4.1), we finally deduce

$$\sum_{l=1}^{k_\lambda}\sum_{i=1}^{l}\mathsf{f}_i^\mathsf{T}\otimes\left(\frac{\mathcal{L}\boldsymbol{\mu}^{(l-i)}(\lambda)}{(l-i)!}A_{\lambda,l}\right)=\sum_{i=1}^{k_\lambda}\mathsf{f}_i^\mathsf{T}\otimes\left(\sum_{l=i}^{k_\lambda}\frac{\mathcal{L}\boldsymbol{\mu}^{(l-i)}(\lambda)}{(l-i)!}A_{\lambda,l}\right)=\sum_{i=1}^{k_\lambda}\mathsf{f}_i^\mathsf{T}\otimes A_{\lambda,i}=\vec{\mathbf{A}}_\lambda.$$

□

Returning to (4.1), we can specifically state that for any $u\in\mathcal{I}$,

$$(\mathrm{I}_{k_\lambda}\otimes\mathsf{e}_{\tau(u)})\vec{\mathbf{A}}_\lambda = (\mathrm{I}_{k_\lambda}\otimes\mathsf{e}_{\tau(u)})\mathbf{E}\left[\int\exp(\lambda,-s,k_\lambda)\otimes\boldsymbol{\xi}(\mathrm{d}s)\right]\vec{\mathbf{A}}_\lambda$$
$$= \mathbf{E}\left[\int\exp(\lambda,-s,k_\lambda)\otimes\boldsymbol{\xi}_u^{\tau(u)}(\mathrm{d}s)\right]\vec{\mathbf{A}}_\lambda.$$

In order to establish a more compact formulation of (4.4), we define for every $\lambda\in\Lambda_\vartheta$ with multiplicity $k_\lambda\in\mathbb{N}$, the random $pk_\lambda\times p$ matrix

$$Z(\lambda,k_\lambda) := \left(\int_0^\infty \exp(\lambda,-s,k_\lambda)\otimes\boldsymbol{\xi}(\mathrm{d}s)\right)\vec{\mathbf{A}}_\lambda.$$

With this, we can write (4.4) in the form $\mathbf{E}[Z(\lambda,k_\lambda)] = \vec{\mathbf{A}}_\lambda$. In this context, by defining the random $pk_\lambda\times p$ matrix $Y := Z(\lambda,k_\lambda) - \vec{\mathbf{A}}_\lambda$ and its shifted version

$Y_u := Y \circ \theta_u$ where θ_u denotes the shift operator to vertex $u \in \mathcal{I}$, we observe $\mathbf{E}[Y_u] = 0$. Further, we set $Z_u(\lambda, k_\lambda) := Z(\lambda, k_\lambda) \circ \theta_u$, i.e.

$$Z_u(\lambda, k_\lambda) = \left(\int_0^\infty \exp(\lambda, -s, k_\lambda) \otimes \boldsymbol{\xi}_u(\mathrm{d}s)\right)\vec{\mathbf{A}}_\lambda.$$

Here, notice that $Z_u(\lambda, k_\lambda)$ is $\boldsymbol{\xi}_u$-measurable. Additionally, we set

$$Z_u^{\tau(u)}(\lambda, k_\lambda) := (\mathrm{I}_{k_\lambda} \otimes \mathsf{e}_{\tau(u)}) Z_u(\lambda, k_\lambda) = \int (\exp(\lambda, -s, k_\lambda) \otimes \boldsymbol{\xi}_u^{\tau(u)}(\mathrm{d}s))\vec{\mathbf{A}}_\lambda$$

such that $\mathbf{E}[Z_u^{\tau(u)}(\lambda, k_\lambda)] = (\mathrm{I}_{k_\lambda} \otimes \mathsf{e}_{\tau(u)})\vec{\mathbf{A}}_\lambda$ holds.

4.2 *Formulation and Verification of the Martingale Property*

After introducing the fundamental construction and some necessary definitions, we turn our attention to deriving the martingales of interest, namely, $W_t(\lambda, k_\lambda)$ for $t \in \mathbb{R}$, and verifying their martingale property. To facilitate this, we define the $pk_\lambda \times p$ matrix-valued characteristic ϕ_λ which plays a central role in the analysis presented in this thesis. This characteristic enables us to represent the martingales in terms of the multi-type branching process counted with ϕ_λ, i.e. $(\mathcal{Z}_t^{\phi_\lambda})_{t\in\mathbb{R}}$. Therefore, we define

$$\phi_\lambda(t) := \mathbb{1}_{[0,\infty)}(t)\left(\int_t^\infty \exp(\lambda, t - s, k_\lambda) \otimes \boldsymbol{\xi}(\mathrm{d}s)\right)\vec{\mathbf{A}}_\lambda, \quad t \in \mathbb{R}.$$

Theorem 4.2 [Martingales]

Suppose that (A1) *and* (A2) *hold. We define* $\mathcal{G}_u := \sigma(\boldsymbol{\xi}_v : v \preceq u)$ *and* $\mathcal{F}_t^W := \sigma(\{A \cap \{S(u) \le t\} : u \in \mathcal{I} \text{ and } A \in \mathcal{G}_u\})$, $t \in \mathbb{R}$. *Further, assume* $\lambda \in \Lambda_\vartheta$ *with multiplicity* $k_\lambda \in \mathbb{N}$. *Then,*

$$W_t(\lambda, k_\lambda) := \exp(\lambda, -t, k_\lambda)\mathcal{Z}_t^{\phi_\lambda}, \quad t \in \mathbb{R}, \tag{4.5}$$

is a martingale with respect to the filtration $(\mathcal{F}_t^W)_{t\ge 0}$ *and*

$$\mathbf{E}^{\tau(\varnothing)}[W_t(\lambda, k_\lambda)] = W_0(\lambda, k_\lambda) = (\mathrm{I}_{k_\lambda} \otimes \mathsf{e}_{\tau(\varnothing)})\vec{\mathbf{A}}_\lambda.$$

In particular, we can formulate (4.5) *as*

$$W_t(\lambda, k_\lambda) = \sum_{u \in \mathcal{C}_t} (\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})\vec{\mathbf{A}}_\lambda \tag{4.6}$$

where $\mathcal{C}_t := \{uj \in \mathcal{T} : S(u) \le t < S(uj)\}$ *is the coming generation at time* t.

Proof of Theorem 4.2.
Let $\lambda \in \Lambda_\vartheta$ with multiplicity $k_\lambda \in \mathbb{N}$. First, we consider $\mathcal{Z}_t^{\phi_\lambda}$ for $t \in \mathbb{R}$, i.e.

$$\begin{aligned}\mathcal{Z}_t^{\phi_\lambda} &= \sum_{u\in\mathcal{I}} (\mathrm{I}_{k_\lambda} \otimes \mathsf{e}_{\tau(u)})\phi_\lambda(t - S(u)) \\ &= \sum_{u\in\mathcal{I}} \mathbb{1}_{\{S(u)\le t\}} \left(\int_{t-S(u)}^{\infty} \exp(\lambda, t - S(u) - s, k_\lambda) \otimes \boldsymbol{\xi}_u^{\tau(u)}(\mathrm{d}s)\right)\vec{\mathbf{A}}_\lambda.\end{aligned}$$

By using the identity $\exp(\lambda, -t, k_\lambda) = (\exp(\lambda, -t, k_\lambda) \otimes 1)$, we obtain

$$\begin{aligned}\exp(\lambda, -t, k_\lambda)\mathcal{Z}_t^{\phi_\lambda} &= \sum_{u\in\mathcal{I}} \mathbb{1}_{\{S(u)\le t\}} \left(\int_{t-S(u)}^{\infty} \exp(\lambda, -S(u) - s, k_\lambda) \otimes \boldsymbol{\xi}_u^{\tau(u)}(\mathrm{d}s)\right)\vec{\mathbf{A}}_\lambda \\ &= \sum_{u\in\mathcal{I}} \sum_{l=1}^{N_u} \mathbb{1}_{\{S(u)\le t<S(ul)\}}(\exp(\lambda, -S(ul), k_\lambda) \otimes \mathsf{e}_{\tau(ul)})\vec{\mathbf{A}}_\lambda. \qquad (4.7)\end{aligned}$$

In fact, substituting $\mathbb{1}_{\{S(u)\le t<S(ul)\}} = \mathbb{1}_{\mathcal{C}_t}(ul)$ in (4.7) directly implies (4.6). To deduce the martingale property of $\exp(\lambda, -t, k_\lambda)\mathcal{Z}_t^{\phi_\lambda}$, we continue by rewriting the indicator function appearing on the right-hand side of (4.7) as

$$\mathbb{1}_{\{S(u)\le t<S(ul)\}} = \mathbb{1}_{\{S(u)\le t\}}\mathbb{1}_{\{S(ul)>t\}} = \mathbb{1}_{\{S(u)\le t\}}\left(1 - \mathbb{1}_{\{S(ul)\le t\}}\right) = \mathbb{1}_{\{S(u)\le t\}} - \mathbb{1}_{\{S(ul)\le t\}}.$$

This holds since for any individual $u \in \mathcal{I}$, $\{S(ul) \le t\} \subseteq \{S(u) \le t\}$ leads to $\{S(ul) \le t\} \cap \{S(u) \le t\} = \{S(ul) \le t\}$, i.e., $\mathbb{1}_{\{S(u)\le t\}}\mathbb{1}_{\{S(ul)\le t\}} = \mathbb{1}_{\{S(ul)\le t\}}$ in the last step. Hence, we deduce

$$\begin{aligned}&\sum_{u\in\mathcal{I}} \sum_{l=1}^{N_u} \mathbb{1}_{\{S(u)\le t<S(ul)\}}(\exp(\lambda, -S(ul), k_\lambda) \otimes \mathsf{e}_{\tau(ul)})\vec{\mathbf{A}}_\lambda \\ &= \sum_{u\in\mathcal{I}} \sum_{l=1}^{N_u} (\mathbb{1}_{\{S(u)\le t\}} - \mathbb{1}_{\{S(ul)\le t\}})(\exp(\lambda, -S(ul), k_\lambda) \otimes \mathsf{e}_{\tau(ul)})\vec{\mathbf{A}}_\lambda.\end{aligned}$$

With Fubini's theorem, we can decompose the series in the last line and obtain

$$\begin{aligned}&\sum_{u\in\mathcal{I}} \sum_{l=1}^{N_u} (\mathbb{1}_{\{S(u)\le t\}} - \mathbb{1}_{\{S(ul)\le t\}})(\exp(\lambda, -S(ul), k_\lambda) \otimes \mathsf{e}_{\tau(ul)})\vec{\mathbf{A}}_\lambda \\ &= \sum_{u\in\mathcal{I}} \mathbb{1}_{\{S(u)\le t\}}(\exp(\lambda, -S(u), k_\lambda) \otimes 1)\left(\sum_{l=1}^{N_u} \exp(\lambda, -X_{u,l}, k_\lambda) \otimes \mathsf{e}_{\tau(ul)}\right)\vec{\mathbf{A}}_\lambda \\ &\quad - \sum_{|u|\ge 1} \mathbb{1}_{\{S(u)\le t\}}(\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})\vec{\mathbf{A}}_\lambda \qquad (4.8)\end{aligned}$$

where we substituted u_l by u in the latter sum. In fact, since $W_t(\lambda, k_\lambda)$ is integrable, i.e. $\mathbf{E}[\|W_t(\lambda, k_\lambda)\|] < \infty$, we can use Fubini's theorem and rearrange the infinite series as seen before.

In order to convince the reader of correctness, we proceed with an estimation of $\mathbf{E}[\|W_t(\lambda, k_\lambda)\|]$. Using the multiplicity of matrix norms and subsequently, the estimation provided in (2.2) for the $k_\lambda \times k_\lambda$ matrix $\exp(\lambda, -S(ul), k_\lambda)$, we infer

$$\begin{aligned}
&\mathbf{E}\Bigg[\sum_{u\in\mathcal{I}}\sum_{l=1}^{N_u}\Big\|\mathbb{1}_{\{S(u)\leq t<S(ul)\}}(\exp(\lambda,-S(ul),k_\lambda)\otimes \mathsf{e}_{\tau(u)})\vec{\mathbf{A}}_\lambda\Big\|\Bigg]\\
&\leq C\sum_{u\in\mathcal{I}}\sum_{l=1}^{N_u}\mathbf{E}\Big[\mathbb{1}_{\{S(u)\leq t<S(ul)\}}\,e^{-\vartheta(S(u)+X_{u,l})}\Big].
\end{aligned}$$

Here, $C > 0$ is a constant that depends only on k_λ and the matrices $A_{\lambda,k}$, for all $k \leq k_\lambda$. Further,

$$\begin{aligned}
C\sum_{u\in\mathcal{I}}\sum_{l=1}^{N_u}\mathbf{E}\Big[\mathbb{1}_{\{S(u)\leq t<S(ul)\}}\,e^{-\vartheta(S(u)+X_{u,l})}\Big] &\leq C\sum_{u\in\mathcal{I}}\mathbf{E}\Big[\mathbb{1}_{\{S(u)\leq t\}}e^{-\vartheta S(u)}\Big]\sum_{l=1}^{N_u}\mathbf{E}\Big[e^{-\vartheta X_{u,l}}\Big]\\
&= C\sum_{u\in\mathcal{I}}\mathbf{E}\Big[\mathbb{1}_{\{S(u)\leq t\}}e^{-\vartheta S(u)}\Big]\Big(\mathsf{e}_{\tau(u')}\boldsymbol{\mathcal{L}\mu}(\vartheta)\mathbf{1}^{\mathsf{T}}\Big)
\end{aligned}$$

where u' is the ancestor of u in the $(n-1)$-th generation, i.e. the mother of u, and $\mathbf{1} \in \mathbb{R}^{1\times p}$ denotes a row vector with all entries equal to one. Notice that the last term on the right-hand side is finite due to assumption (A1). Therefore, we continue by investigating the first series on the right-hand side. By decomposing $-\vartheta = (\alpha+\delta-\vartheta) - (\alpha+\delta)$, we obtain

$$\begin{aligned}
C\sum_{u\in\mathcal{I}}\mathbf{E}\Big[\mathbb{1}_{\{S(u)\leq t\}}e^{-(\alpha+\delta)S(u)}e^{(\alpha+\delta-\theta)S(u)}\Big] &\leq C\sum_{u\in\mathcal{I}}\mathbf{E}\Big[\mathbb{1}_{\{S(u)\leq t\}}e^{-(\alpha+\delta)S(u)}\Big]e^{(\alpha+\delta-\vartheta)t}\\
&\leq C\sum_{n\in\mathbb{N}_0}\mathbf{E}\Bigg[\sum_{|u|=n}e^{-(\alpha+\delta)S(u)}\Bigg]\\
&= C\sum_{n\in\mathbb{N}_0}\mathsf{e}_{\tau(u')}\boldsymbol{\mathcal{L}\mu}(\alpha+\delta)^n\mathbf{1}^{\mathsf{T}}.
\end{aligned}$$

Moreover,

$$\sum_{n\in\mathbb{N}_0}\|\mathsf{e}_{\tau(u')}\boldsymbol{\mathcal{L}\mu}(\alpha+\delta)^n\mathbf{1}^{\mathsf{T}}\| \leq \sqrt{p}\sum_{n\in\mathbb{N}_0}\|\boldsymbol{\mathcal{L}\mu}(\alpha+\delta)^n\|.$$

In this context, notice that for any matrix $A \in \mathbb{C}^{n\times n}$, Gelfand's formula implies $\|A^n\|^{\frac{1}{n}} \leq \rho(A) + \varepsilon$ for sufficiently large n and any $\varepsilon > 0$. Consequently, we deduce $\|A^n\| \leq (\rho(A)+\varepsilon)^n$ for large n. Recall that $\varrho(z) := \rho_{\boldsymbol{\mathcal{L}\mu}(z)}$, $z \in \mathcal{D}(\boldsymbol{\mathcal{L}\mu}(z))$. With this observation, we have

$$\sum_{n\in\mathbb{N}_0}\|\boldsymbol{\mathcal{L}\mu}(\alpha+\delta)^n\| \leq \sum_{n\in\mathbb{N}_0}(\varrho(\alpha+\delta)+\varepsilon)^n,$$

This series converges absolutely since $\varrho(\alpha+\delta) < 1$ and we can choose ε such that $\varrho(\alpha+\delta)+\varepsilon < 1$.

Returning to decomposition (4.8), we proceed by simplifying

$$\begin{aligned}
&\sum_{u\in\mathcal{I}} \mathbb{1}_{\{S(u)\leq t\}}(\exp(\lambda,-S(u),k_\lambda)\otimes 1)\left(\sum_{l=1}^{N_u}\exp(\lambda,-X_{u,l},k_\lambda)\otimes \mathsf{e}_{\tau(ul)}\right)\vec{\mathbf{A}}_\lambda\\
&-\sum_{|u|\geq 1}\mathbb{1}_{\{S(u)\leq t\}}(\exp(\lambda,-S(u),k_\lambda)\otimes \mathsf{e}_{\tau(u)})\vec{\mathbf{A}}_\lambda\\
&=\mathbb{1}_{[0,\infty)}(t)(\mathrm{I}_{k_\lambda}\otimes \mathsf{e}_{\tau(\varnothing)})\vec{\mathbf{A}}_\lambda+\sum_{u\in\mathcal{I}}\mathbb{1}_{\{S(u)\leq t\}}(\exp(\lambda,-S(u),k_\lambda)\otimes \mathsf{e}_{\tau(u)})Y_u. \qquad (4.9)
\end{aligned}$$

Thus far, we have established that the right-hand side in (4.9) corresponds precisely to $W_t(\lambda,k_\lambda)$. This representation enables us to verify both the martingale property and the $\mathcal{F}_t^W$-measurability, starting with the latter.

Therefore, it suffices to show that each summand on the right-hand side in (4.9) is $\mathcal{F}_t^W$-measurable. Notice that each summand is of the form $F_u := \mathbb{1}_{\{S(u)\leq t\}}H_u$, for $u\in\mathcal{I}$ where H_u is $\mathcal{G}_u$-measurable. Consequently, it suffices to show that for any Borel set $\mathcal{B}$ that does not contain 0, it holds $F_u(\mathcal{B})\in\mathcal{F}_t^W$. But for such Borel set $\mathcal{B}$, we have $F_u^{-1}(\mathcal{B})=\{S(u)\leq t\}\cap H_u^{-1}(\mathcal{B})\in\mathcal{F}_t^W$.

The last property that requires verification is the martingale property. To establish this, let us consider $0\leq s<t$. Then,

$$W_t(\lambda,k_\lambda)-W_s(\lambda,k_\lambda)=\sum_{u\in\mathcal{I}}\mathbb{1}_{\{s<S(u)\leq t\}}(\exp(\lambda,-S(u),k_\lambda)\otimes \mathsf{e}_{\tau(u)})Y_u.$$

Here, it suffices to show that

$$\mathbf{E}\left[\mathbb{1}_{\{s<S(u)\leq t\}}(\exp(\lambda,-S(u),k)\otimes \mathsf{e}_{\tau(u)})Y_u\middle|\mathcal{F}_s^W\right]=0 \quad \text{a.s.} \qquad (4.10)$$

Notice that $u,v\in\mathcal{I}$ with $s<S(u)$ and $s\geq S(v)$ leads to $u\not\preceq v$. For such u and v, $\mathcal{G}_v$ independent of $\boldsymbol{\xi}_u$. Hence, for any $A\in\mathcal{G}_v$,

$$\mathbf{E}[\mathbb{1}_{A\cap\{S(v)\leq s\}}\mathbb{1}_{\{s<S(u)\leq t\}}(\exp(\lambda,-S(u),k)\otimes \mathsf{e}_{\tau(u)})Y_u]=0 \quad \text{a.s.}$$

since $\mathbf{E}[Y_u]=0$, for any $u\in\mathcal{I}$. Further, we set

$$\mathcal{D}:=\{D\subseteq\Omega:\mathbf{E}[\mathbb{1}_D(\omega)\mathbb{1}_{\{s<S(u)\leq t\}}(\exp(\lambda,-S(u),k)\otimes \mathsf{e}_{\tau(u)})Y_u]=0\}$$

and observe that $\mathcal{D}$ is a Dynkinsystem. In fact,

$$\mathbf{E}[\mathbb{1}_{\{s<S(u)\leq t\}}(\exp(\lambda,-S(u),k)\otimes \mathsf{e}_{\tau(u)})Y_u]=0$$

since $\mathbb{1}_\Omega(\omega)=1$, the independence of $\mathbb{1}_{\{s<S(u)\leq t\}}(\exp(\lambda,-S(u),k)\otimes \mathsf{e}_{\tau(u)})$ and Y_u, and $\mathbf{E}[Y_u]=0$. Hence, $\Omega\in\mathcal{D}$. If $A,B\in\mathcal{D}$ with $A\subset B$, then $\mathbb{1}_{B\backslash A}(\omega)=\mathbb{1}_B(\omega)-\mathbb{1}_A(\omega)$. In this case, the linearity of the expectation yields $B\backslash A\in\mathcal{D}$. Finally, consider a disjoint sequence $(A_n)_{n\in\mathbb{N}_0}\in\mathcal{D}$. Using the fact that $\mathbb{1}_{\cup_{n\in\mathbb{N}_0}A_n}(\omega)=\sum_{n\in\mathbb{N}_0}\mathbb{1}_{A_n}(\omega)$ and the linearity of the expectation, we infer $\cup_{n\in\mathbb{N}_0}A_n\in\mathcal{D}$.

Since the set $\Pi := \{A \cap \{S(u) \leq t\} : u \in \mathcal{I} \text{ and } A \in \mathcal{G}_u\}$ forms a π-system with $\Pi \subseteq \mathcal{D}$, the π-λ-theorem implies $\mathcal{F}_t^W = \sigma(\Pi) \subseteq \mathcal{D}$, therefore (4.10). In particular, we deduce with (4.9),

$$\mathbf{E}^{\tau(\varnothing)}[W_t(\lambda, k_\lambda)] = \mathbb{1}_{[0,\infty)}(t)(\mathrm{I}_{k_\lambda} \otimes \mathsf{e}_{\tau(\varnothing)})\vec{\mathbf{A}}_\lambda,$$

completing the proof.

■

Remark 4.3 The martingale $(W_t(\lambda, k_\lambda))_{t\in\mathbb{R}}$ is a $k_\lambda \times p$ matrix and takes the form

$$\begin{aligned}
W_t(\lambda, k_\lambda) &= \sum_{u \in \mathcal{C}_t} (\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})\vec{\mathbf{A}}_\lambda \\
&= \begin{pmatrix}
W_t^{(0)}(\lambda, k_\lambda) & W_t^{(1)}(\lambda, k_\lambda) & \frac{W_t^{(2)}(\lambda,k_\lambda)}{2!} & \cdots & \frac{W_t^{(k_\lambda-1)}(\lambda,k_\lambda)}{(k_\lambda-1)!} \\
0 & W_t^{(0)}(\lambda, k_\lambda) & W_t^{(1)}(\lambda, k_\lambda) & \cdots & \vdots \\
0 & 0 & W_t^{(0)}(\lambda, k_\lambda) & \cdots & \frac{W_t^{(2)}(\lambda,k_\lambda)}{2!} \\
\vdots & \vdots & \vdots & \ddots & W_t^{(1)}(\lambda, k_\lambda) \\
0 & 0 & 0 & \cdots & W_t^{(0)}(\lambda, k_\lambda)
\end{pmatrix}
\begin{pmatrix} A_{\lambda,1} \\ \vdots \\ A_{\lambda,k_\lambda} \end{pmatrix} \\
&= \begin{pmatrix}
\sum_{l=0}^{k_\lambda-1} W_t^{(l)}(\lambda, k_\lambda) A_{\lambda,l+1} \\
\sum_{l=1}^{k_\lambda-1} W_t^{(l-1)}(\lambda, k_\lambda) A_{\lambda,l+1} \\
\cdots \\
W_t^{(0)}(\lambda, k_\lambda) A_{\lambda,k_\lambda}
\end{pmatrix}
\end{aligned} \tag{4.11}$$

where $W_t^{(j)}(\lambda, k_\lambda) = \sum_{u\in\mathcal{C}_t}(-S(u))^j e^{-\lambda S(u)} \mathsf{e}_{\tau(u)}$, $j = 1, \ldots, k_\lambda - 1$ and $t \in \mathbb{R}$.

□

4.3 *L^q-convergence for* $1 < q \leq 2$

In this section, we briefly discuss the convergence properties of the martingale $(W_t(\lambda, k_\lambda))_{t\in\mathbb{R}}$. More precisely, we derive results that establish L^q-convergence, for $1 < q \leq 2$. To this end, we consider the martingale in the form

$$W_t(\lambda, k_\lambda) = \mathbb{1}_{[0,\infty)}(t)(\mathrm{I}_{k_\lambda} \otimes \mathsf{e}_{\tau(\varnothing)})\vec{\mathbf{A}}_\lambda + \sum_{u\in\mathcal{I}} \mathbb{1}_{\{S(u)\leq t\}}(\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})Y_u.$$

To show that $(W_t(\lambda, k_\lambda))_{t\in\mathbb{R}}$ is an L^q-bounded martingale for $1 < q \leq 2$, it suffices to demonstrate that sum on the right-hand side is an L^q-bounded martingale for $1 < q \leq 2$. To obtain this result, we require a q-th moment assumption on Y_u ensuring that $\mathbf{E}[\|Y_u\|^q] < \infty$. We begin by deriving a sufficient condition for this.

Using the multiplicity of matrix norms and subsequently, the boundedness of the coefficient matrix $\vec{\mathbf{A}}_\lambda$, we obtain

$$\begin{aligned}\mathbf{E}[\|Y_u\|_{\mathsf{HS}}^q] &= \mathbf{E}\left[\left\|\left(\int \exp(\lambda,-s,k_\lambda)\otimes \boldsymbol{\xi}_u(\mathrm{d}s) - (\mathrm{I}_{k_\lambda}\otimes \mathrm{I}_p)\right)\vec{\mathbf{A}}_\lambda\right\|_{\mathsf{HS}}^q\right] \\ &\le \mathbf{E}\left[\left\|\left(\int \exp(\lambda,-s,k_\lambda)\otimes \boldsymbol{\xi}_u(\mathrm{d}s) - (\mathrm{I}_{k_\lambda}\otimes \mathrm{I}_p)\right)\right\|_{\mathsf{HS}}^q \|\vec{\mathbf{A}}_\lambda\|_{\mathsf{HS}}^q\right] \\ &\le C\mathbf{E}\left[\left\|\left(\int \exp(\lambda,-s,k_\lambda)\otimes \boldsymbol{\xi}_u(\mathrm{d}s) - (\mathrm{I}_{k_\lambda}\otimes \mathrm{I}_p)\right)\right\|_{\mathsf{HS}}^q\right].\end{aligned}$$

Since for any $p\times p$ matrix A, it holds $\|A-\mathrm{I}_p\|_{\mathsf{HS}}^q \le (\|A\|_{\mathsf{HS}}+\|\mathrm{I}_p\|_{\mathsf{HS}})^q \le C_p(\|A\|_{\mathsf{HS}}^q+1)$, we conclude

$$\begin{aligned}&C\mathbf{E}\left[\left\|\left(\int \exp(\lambda,-s,k_\lambda)\otimes \boldsymbol{\xi}_u(\mathrm{d}s) - (\mathrm{I}_{k_\lambda}\otimes \mathrm{I}_p)\right)\right\|_{\mathsf{HS}}^q\right] \\ &\le C\left(1+\mathbf{E}\left[\left\|\left(\int \exp(\lambda,-s,k_\lambda)\otimes \boldsymbol{\xi}_u(\mathrm{d}s)\right)\right\|_{\mathsf{HS}}^q\right]\right) \\ &\le C\left(1+\sum_{l=0}^{k_\lambda-1}\sum_{i,j=1}^{p}\mathbf{E}\left[\left(\int e^{-\operatorname{Re}(\lambda)s}s^l\,\xi^{i,j}(\mathrm{d}s)\right)^q\right]\right).\end{aligned}$$

In order to bound the final term, we note that $\operatorname{Re}(\lambda) > \vartheta$ and therefore, assume $\mathbf{E}[(\int e^{-\vartheta s}\,\xi^{i,j}(\mathrm{d}s))^q] < \infty$ for each $i,j\in[p]$. This requirement is formally stated in the following assumption:

Assumption (A6)

Suppose that (A1) holds. Then, the offspring process matrix $\boldsymbol{\xi}$ satisfies the condition

$$\mathbf{E}\left[\left\|\int e^{-\vartheta s}\,\boldsymbol{\xi}(\mathrm{d}s)\right\|^q\right] < \infty$$

for some $\vartheta > 0$ and $q \in (1,2]$.

With this condition, we can now establish the L^q-convergence for $q\in(1,2]$.

Lemma 4.4 $[L^q$-convergence for $q\in(1,2]]$

Suppose that (A1), (A2) *and* (A6) *hold. Assume* $1<q\le 2$ *and* $\lambda\in\Lambda_\vartheta$ *with multiplicity* $k_\lambda\in\mathbb{N}$ *and* $\operatorname{Re}(\lambda) > \frac{\alpha}{q}$*. Then* $(W_t(\lambda,k_\lambda))_{t\in\mathbb{R}}$ *is an* L^q*-bounded martingale.*

Proof of Lemma 4.4.
As mentioned before, it suffices to show that $\sum_{u\in\mathcal{I}} \mathbb{1}_{\{S(u)\leq t\}}(\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})Y_u$ is an L^q-bounded martingale, for $1 < q \leq 2$. Let $x \in \mathbb{R}^{1\times k}$ and $y \in \mathbb{R}^{p\times 1}$ be fixed vectors. Then, $(x \otimes 1)$ and $(1 \otimes y)$ are vectors in the space $\mathbb{R}^{1\times k} \otimes \mathbb{R}$ and $\mathbb{R} \otimes \mathbb{R}^{p\times 1}$, respectively. Without loss of generality, we may assume $\|x\|, \|y\| \leq 1$. With this, we focus on showing that

$$\begin{aligned}&(x \otimes 1) \sum_{n\in\mathbb{N}_0} \sum_{|u|=n} \mathbb{1}_{\{S(u)\leq t\}}(\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})Y_u(1 \otimes y)\\ &= \sum_{n\in\mathbb{N}_0} \sum_{|u|=n} \mathbb{1}_{\{S(u)\leq t\}}(x\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})Y_u y\end{aligned}$$

is an L^q-bounded martingale, for $1 < q \leq 2$. Therefore, we define

$$M_n(\lambda, k_\lambda) := \sum_{|u|\leq n} \mathbb{1}_{\{S(u)\leq t\}}(x\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})Y_u y$$

and notice that its increments, namely,

$$M_n(\lambda, k_\lambda) - M_{n-1}(\lambda, k_\lambda) = \sum_{|u|=n} \mathbb{1}_{\{S(u)\leq t\}}(x\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})Y_u y$$

form a martingale with respect to the filtration $(\mathcal{G}_n)_{n\in\mathbb{N}_0}$. To this end, we can use the Bahr-Esseen inequality, see [BE65, Theorem 1], to obtain

$$\begin{aligned}&\mathbf{E}\left[\left|\sum_{n\in\mathbb{N}_0} \sum_{|u|=n} \mathbb{1}_{\{S(u)\leq t\}}(x\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})Y_u y\right|^q\right]\\ &\leq C_{n,q} \sum_{n\in\mathbb{N}_0} \mathbf{E}\left[\left|\sum_{|u|=n} \mathbb{1}_{\{S(u)\leq t\}}(x\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})Y_u y\right|^q\right].\end{aligned}$$

Further, by applying the conditional version of the Bahr-Esseen inequality with respect to $\mathcal{G}_{n-1}$, we get

$$\begin{aligned}&\mathbf{E}\left[\left|\sum_{|u|=n} \mathbb{1}_{\{S(u)\leq t\}}(x\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})Y_u y\right|^q \,\middle|\, \mathcal{G}_{n-1}\right]\\ &\leq C_{n,q}\,\mathbf{E}\left[\sum_{|u|=n} \mathbb{1}_{\{S(u)\leq t\}}|(x\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})Y_u y|^q \,\middle|\, \mathcal{G}_{n-1}\right].\end{aligned}$$

Hence, we infer

$$\begin{aligned}&\mathbf{E}\left[\left|\sum_{u\in\mathcal{I}} \mathbb{1}_{\{S(u)\leq t\}}(x\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})Y_u y\right|^q\right]\\ &\leq C_{n,q} \sum_{u\in\mathcal{I}} \mathbf{E}[|\mathbb{1}_{\{S(u)\leq t\}}(x\exp(\lambda, -S(u), k_\lambda) \otimes \mathsf{e}_{\tau(u)})Y_u y|^q].\end{aligned}$$

Then, using $\|x\|, \|y\| \leq 1$ and the independence of $\mathbb{1}_{\{S(u)\leq t\}}(\exp(\lambda, -S(u), k_\lambda)\otimes \mathsf{e}_{\tau(u)})$ and Y_u yields

$$\begin{aligned}&\sum_{u\in\mathcal{I}} \mathbf{E}[|\mathbb{1}_{\{S(u)\leq t\}}(x\exp(\lambda, -S(u), k_\lambda)\otimes \mathsf{e}_{\tau(u)})Y_u y|^q] \\ &\leq C\sum_{n\in\mathbb{N}_0} \mathbf{E}\Bigg[\sum_{|u|=n} \|\exp(\lambda, -S(u), k_\lambda)\|^q\Bigg]\mathbf{E}[\|Y_u\|^q].\end{aligned}$$

By employing assumption (A6), the condition $\frac{\alpha}{q} < \operatorname{Re}(\lambda)$ and a similar argument used in the proof of Theorem 4.2 regarding the integrability of the martingale $W_t(\lambda, k_\lambda)$, we deduce

$$\begin{aligned}\sum_{n\in\mathbb{N}_0} \mathbf{E}\Bigg[\sum_{|u|=n} \|\exp(\lambda, -S(u), k_\lambda)\|^q\Bigg]\mathbf{E}[\|Y_u\|^q] &\leq C\sum_{n\in\mathbb{N}_0} \mathbf{E}\Bigg[\sum_{|u|=n} e^{-q\operatorname{Re}(\lambda)S(u)}\Bigg] \\ &\leq C\sum_{n\in\mathbb{N}_0} \mathbf{E}\Bigg[\sum_{|u|=n} e^{-q\vartheta S(u)}\Bigg] \\ &\leq C\sum_{n\in\mathbb{N}_0} \mathbf{E}\Bigg[\sum_{|u|=n} e^{-(\alpha+q\delta)S(u)}\Bigg] \\ &= C\sum_{n\in\mathbb{N}_0} \mathsf{e}_{\tau(u')}\boldsymbol{\mathcal{L}\mu}(\alpha+q\delta)^n\mathbf{1}^\mathsf{T}\end{aligned}$$

where u' is the ancestor of u in the $(n-1)$-th generation, i.e. the mother of u, and $\mathbf{1} \in \mathbb{R}^{1\times p}$ denotes a row vector with all entries equal to one. The series converges absolutely since $\varrho(\alpha + q\delta) < 1$.

■

A | Appendix

In this chapter, we gather fundamental facts about mathematical concepts that are required at various stages throughout this work. These results are well-established within the field and do not constitute original contributions. For the convenience of the reader, we provide either a reference or a sketch of a proof for each fact.

A.1 *Spectral Radius*

Let A be a $p \times p$ matrix. Then, we define the spectrum σ_A as

$$\sigma_A := \sigma(A) := \{\lambda : \lambda \in \mathbb{C} \text{ is an eigenvalue of } A\},$$

and the spectral radius as

$$\rho_A := \rho(A) := \sup\{|\lambda| : \lambda \in \sigma_A\}$$

which represents the largest modulus eigenvalue. Alternatively, the spectral radius ρ_A can be expressed by using the matrix norm of A, i.e.,

$$\|A\| = \sup_{x \in \mathbb{C}^p, |x|=1} |Ax|,$$

known as Gelfand's formula.

Lemma A.1 [Gelfand formula]
Let $\|\cdot\|$ *be a matrix norm on* $\mathbb{C}^{p\times p}$ *and* $A \in \mathbb{C}^{p\times p}$. *Then*

$$\rho_A = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

Proof of Lemma A.1.
This is [HJ13, Corollary 5.6.14].

■

In particular, the matrix norm of A provides an upper bound for the spectral radius ρ_A, as stated in the following result.

Lemma A.2

Let $\|\cdot\|$ be a matrix norm on $\mathbb{C}^{p\times p}$, $A \in \mathbb{C}^{p\times p}$, and $\lambda \in \sigma_A$. Then it holds:

① $|\lambda| \leq \rho_A \leq \|A\|$.

② If A is invertible, we have $\rho_A \geq |\lambda| \geq \frac{1}{\|A^{-1}\|}$.

Proof of Lemma A.2.
This is [HJ13, Theorem 5.6.9].

■

In order to simplify the notation, we write for each complex matrix $A = (a_{ij})_{i,j\in[p]}$ and vector $x = (x_1, \ldots, x_p)^\mathsf{T} \in \mathbb{C}^p$, $A_+ := (|a_{ij}|)_{i,j\in[p]}$ and $x_+ := (|x_1|, \ldots, |x_p|)^\mathsf{T}$. Using this notation, an inequality for the spectral radius can be derived.

Lemma A.3

Let $A, B \in \mathbb{C}^{p\times p}$ and suppose that B is nonnegative. Then, $A_+ \leq B$ implies $\rho_A \leq \rho_B$.

Proof of Lemma A.3.
This is [HJ13, Theorem 8.1.18].

■

Next, we consider the relationship between the spectral radius of a nonnegative matrix A and that of its principal submatrices, which are square submatrices obtained by removing certain rows and columns of A.

Lemma A.4

Let A be a nonnegative $p \times p$ matrix. If $\tilde{A}$ is a principal submatrix of A, then $\rho(\tilde{A}) < \rho(A)$.

Proof of Lemma A.4.
This is [HJ13, Corollary 8.1.20].

■

Example A.5 Let $A, B \in \mathbb{C}^{p\times p}$. In general, the spectral radius is not subadditive, i.e. $\rho_{A+B} \not\leq \rho_A + \rho_B$, as one can see in [HJ13, Example 2.4.8.3]. In fact, when considering the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

one can observe $\rho_{A+B} = 1$, while $\rho_A = \rho_B = 0$.

□

If A is a nonnegative matrix, the Perron-Frobenius theory, see [HJ13, Theorem 8.3.1], tells us that ρ_A is not only the absolute value of an eigenvalue λ, but an eigenvalue of A itself. Moreover, there is a nonnegative eigenvector x such that $Ax = \rho_A x$ and every eigenvalue λ of A lies in the closed bounded disk $\{z \in \mathbb{C} : |z| \leq \rho_A\}$. Additionally, if A is primitive, i.e. a nonnegative matrix with A^n has positive entries only for some nonnegative integer n, the Perron-Frobenius theory provides even more detailed information about the spectral radius, as discussed in [HJ13, Chapter 8.5].

Theorem A.6 [Perron-Frobenius Theorem]

Let $M \geq 0$ be a primitive matrix.

① $0 < \rho_A \in \sigma_A$ *is simple.*

② $|\lambda| < \rho_A$ for all other eigenvalues $\lambda \in \sigma_A \setminus \{\rho_A\}$.

③ *There exist unique left and right eigenvectors x and y^T of A, both positive, such that $xy^\mathsf{T} = 1$.*

Here, the spectral radius ρ_A is called Perron-Frobenius eigenvalue.

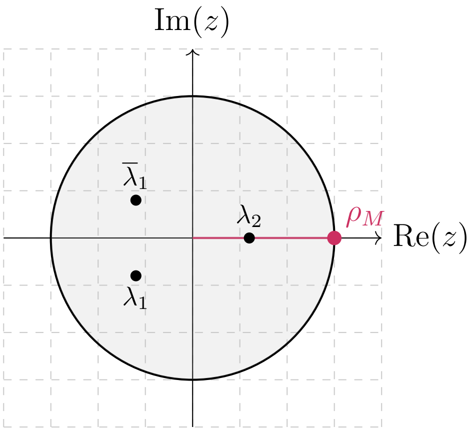

Figure A.1: Illustration of the Perron-Frobenius eigenvalue ρ_A on the complex plane.

Lemma A.7

Let $(A_s)_{s\in I}$ be a family of nonnegative $p\times p$ matrices with decreasing and continuous entries $a_s^{ij}: I\to[0,\infty)$, $i,j\in[p]$, where $I\subseteq\mathbb{R}$ is a non-empty interval. Write $\rho_s := \rho(A_s)$ for the modulus of the spectral radius of A_s.

① *The function $s\mapsto\rho_s$ is decreasing and continuous on I.*

② *If the A_s, $s\in I$ are primitive, then for every $s\in I$, there exists a unique normalized eigenvector v_s associated with the Perron-Frobenius eigenvalue ρ_s of A_s with positive entries only. The function $s\mapsto v_s$ is continuous on I.*

③ *If the A_s, $s\in I$ are primitive and $\sum_{i,j=1}^p a_s^{ij}$ is strictly decreasing, then $s\mapsto\rho_s$ is strictly decreasing.*

Proof of Lemma A.7.

In this proof, we will repeatedly use the fact that if $p_\varepsilon(z)$ is a polynomial of degree $\le p-1$ for $\varepsilon\ge 0$ and the coefficients of $p_\varepsilon(z)$ converge to those of $p_0(z)$ as $\varepsilon\downarrow 0$, then the zeros of $z^p+p_\varepsilon(z)$ converge to those of $z^p+p_0(z)$ as $\varepsilon\downarrow 0$, see e.g. [Mar49, Theorem 1.4] (a corollary of Rouché's theorem).

① We start by showing that $s\mapsto\rho_s$ is continuous. To this end, notice that ρ_s is the largest root (in absolute value) of the characteristic polynomial $\chi_s(z)=\det(z\mathrm{I}_p-A_s)$. The characteristic polynomial is of the form $\chi_s(z)=z^p+p_s(z)$ where $p_s(z)$ is a polynomial of degree $\le p-1$ with coefficients that are continuous in s. We conclude that $s\mapsto\rho_s$ is continuous. In the next step, we prove the claimed monotonicity statements. To this end, first suppose that the A_s, $s\in I$ are primitive. Then ρ_s is the Perron-Frobenius eigenvalue of A_s and $s\mapsto\rho_s$ is non-increasing by the Perron-Frobenius theorem [HJ13, Theorem 1.1]. The latter also gives that ρ_s is strictly decreasing if the sum of all entries of A_s is strictly decreasing. The next item of business is to remove the additional assumption that the A_s, $s\in I$ are primitive. In the general case, we may write A_s^ε for the matrix A_s after adding ε to every single entry of the matrix A_s where $\varepsilon\ge 0$ is a parameter. Then $A_s=A_s^0$. Write $\rho_s^\varepsilon := \rho(A_s^\varepsilon)$. Since A_s^ε is primitive, ρ_s^ε is (strictly) decreasing in s by what we have already shown. On the other hand, again by [Mar49, Theorem 1.4], we infer $\rho_s^\varepsilon\to\rho_s$ as $\varepsilon\downarrow 0$. Consequently, $s\mapsto\rho_s$ is decreasing as the pointwise limit of a sequence of decreasing functions.

② Suppose the A_s, $s\in I$ are primitive. The existence of a unique normalized eigenvector v_s with positive entries only associated with the Perron-Frobenius eigenvalue ρ_s of A_s then directly follows from the Perron-Frobenius theorem [HJ13, Theorem 1.1]. It remains to show the continuity of $s\mapsto v_s$ on I. First, again by the Perron-Frobenius theorem, there exists a unique vector $w_s=(w_s^1,\dots,w_s^p)^\mathsf{T}\in\mathbb{R}^p$ with

$A_s w_s = \rho_s w_s$ and $w_s^1 = 1$. Then $v_s = w_s/|w_s|$. This means that

$$\begin{pmatrix} a_s^{21} \\ \vdots \\ a_s^{p1} \end{pmatrix} + \begin{pmatrix} a_s^{22} & \dots & a_s^{1p} \\ \vdots & \ddots & \vdots \\ a_s^{p2} & \dots & a_s^{pp} \end{pmatrix} \begin{pmatrix} w_s^2 \\ \vdots \\ w_s^p \end{pmatrix} = \rho_s \begin{pmatrix} w_s^2 \\ \vdots \\ w_s^p \end{pmatrix}.$$

We write this identity as $a_s' + A_s' w_s' = \rho_s w_s'$. Notice that ρ_s can not be an eigenvalue of A_s' due to Lemma A.4. Hence, the matrix $(\rho_s \mathbf{I}_{p-1} - A_s')$ is invertible. We infer

$$w_s' = (\rho_s \mathbf{I}_{p-1} - A_s')^{-1} a_s'.$$

The right-hand side is a continuous function of $s \in I$ since ρ_s, A_s' and a_s' are continuous. Therefore, w_s' and thus also w_s and $v_s = w_s/|w_s|$ are continuous functions of s.

③ Follows from part ① .

■

In order to proceed with the next result, it is necessary to define a specific form of matrices, namely the Jordan normal form. These matrices are diagonal matrices augmented with blocks representing the eigenvalues and their corresponding generalized eigenvectors. In this context, let A be a complex $p \times p$ matrix with eigenvalues λ_j, $j \in [p]$, whose multiplicities are given by k_j. For each eigenvalue λ_j, define the $k_j \times k_j$ matrix

$$J_j := \begin{pmatrix} \lambda_j & 1 & 0 & \dots & 0 \\ 0 & \lambda_j & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda_j & 1 \\ 0 & \dots & \dots & 0 & \lambda_j \end{pmatrix}.$$

Thus, the Jordan normal form J of the matrix A is given by

$$J := \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_d \end{pmatrix}.$$

We further define the complex $p \times p$ matrix $P = (v_1^\top, \dots, v_p^\top)$ where $v_j^\top$, $j \in [p]$ are eigenvectors corresponding to λ_j such that P is an invertible matrix. This allows the matrix A to be decomposed as $A = P^{-1} J P$, known as the Jordan decomposition.

Lemma A.8

Let A be a nonnegative $p \times p$ matrix with $\rho(A) < 1$. Then the following assertions hold true:

① *The determinant* $\det(\mathrm{I}_p - A)$ *is positive.*

② *If $x \in \mathbb{R}^p_{\geq}$ is such that $x \leq Ax$, then $x = 0$.*

③ *The series $\sum_{n=0}^{\infty} A^n$ converges and equals $(\mathrm{I}_p - A)^{-1}$.*

Remark A.9 The series $\sum_{n=0}^{\infty} A^n$ in Lemma A.8 is a matrix analogue of the geometric series, referred to as Neumann series, and is discussed, for instance, in [Mey00, Chapter 3]. Moreover, the series converges when $\|A\| < 1$ due to Lemma A.2, namely $\rho(A) < \|A\|$ for all matrix norms. It is important to note that the existence of $(\mathrm{I}_p - A)^{-1}$ does not directly ensure the convergence of the Neumann series. For the existence of $(\mathrm{I}_p - A)^{-1}$ it is only required that $1 \notin \sigma(A)$, while for the convergence, the stricter condition $\rho(A) < 1$ is needed.

Proof of Lemma A.8.

① Let $\chi(z) = \det(z\mathrm{I}_p - A)$ denote the characteristic polynomial of A. We have to show that $\chi(1) > 0$. First, we can observe that $\chi(1) \neq 0$. If $\lambda \in \mathbb{C}$ is such that $\chi(\lambda) = 0$, then λ is an eigenvalue of A, and, according to our assumption $|\lambda| \leq \rho_A < 1$. In this case, 1 is not an eigenvalue of A such that we conclude $\chi(1) \neq 0$. On the other hand, χ is of the form

$$\chi(z) = \prod_{k=1}^{p} (z - a_{kk}) + p(z), \quad z \in \mathbb{C}$$

where p is a polynomial of degree $\leq p - 1$ and $A = (a_{ij})_{i,j \in [p]}$. Thus,

$$\lim_{x \to +\infty} \chi(x) = \infty.$$

This excludes the possibility that $\chi(1) < 0$ since this in combination with the intermediate value theorem implies the existence of an $x > 1$ with $\chi(x) = 0$, but each eigenvalue satisfies $|\lambda| \leq \rho_A < 1$.

② If $x \leq Ax$, then $x \leq A^n x$ for every $n \in \mathbb{N}_0$. Write $A = P^{-1}JP$ where P is a complex invertible $p \times p$ matrix and J is the Jordan normal form of A. Since $\rho(A) < 1$, every diagonal entry λ of the Jordan matrix J satisfies $|\lambda| < 1$. We thus conclude that $x \leq A^n x = P^{-1}J^n P x \to 0$ as $n \to \infty$.

③ This is the proof of [Mey00, (7.10.11)].

■

Let A be an $m \times n$ matrix and B a $k \times l$ matrix. Recall that the Kronecker product of the matrices A and B is defined as the $mk \times nl$ matrix $A \otimes B$. The fundamental properties and rules of the Kronecker product are discussed in [Gra18, Chapter 2.3]. Notably, the Kronecker product is not commutative, as one can see in the following example:

$$\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \neq \begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}.$$

However, the Kronecker product is distributive with respect to addition, associative, and preserves the identity matrix. These properties, along with others, are summarized in the Lemma A.10 below.

Lemma A.10 [Properties of the Kronecker product]

If $a \in \mathbb{R}$ is a scalar and A, B, C and D are $p \times p$ matrices, then the following properties hold:

$(A + B) \otimes C = A \otimes C + B \otimes C$

$A \otimes (B + C) = A \otimes B + A \otimes C$

$(aA) \otimes B = a(A \otimes B) = A \otimes (aB)$

$(A \otimes B)(C \otimes D) = AC \otimes BD$ ('Mixed Product Rule')

$\|A \otimes B\| = \|A\|\|B\|$.

Proof of Lemma A.10.
The details are presented in [Gra18, Chapter 2.3].

■

A.2 *Laurent Series, Singularities and Residue*

Let $0 \leq r_1 < r_2$ and $z_0 \in \mathbb{C}$. If $f(z)$ is an analytic function in the annulus $\{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$, then $f(z)$ can be represented as a series of the form

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{(z - z_0)^k} + \sum_{k=0}^{\infty} b_k (z - z_0)^k$$

where the coefficients a_k and b_k are complex numbers. This series is known as the Laurent series of f around z_0. For a comprehensive discussion of its properties and applications, see [Ahl79, Chapter 5.1.3]. The former sum,

$$\sum_{k=1}^{\infty} \frac{a_k}{(z - z_0)^k},$$

is referred to as principal part of the Laurent series, and the coefficient a_{-1} is termed the residue of f at z_0.

A function $f(z)$ is said to have an isolated singularity at z_0 if $f(z)$ is analytic on $\{z \in \mathbb{C} : 0 < |z - z_0| < r_2\}$ but is not analytic at z_0. Furthermore, each isolated singularity z_0 can be classified as removable, pole, or essential depending on the coefficients a_k of the principal part of the Laurent series at z_0. In this context, we call z_0 removable if $a_k = 0$ for all k. In this case, $f(z)$ can be analytically continued to z_0. The singularity z_0 is a pole of order n if $a_n \neq 0$ and $a_k = 0$ for all $k > n$. In particular, we call z_0 simple pole if z_0 is a pole of order 1. In this situation, $f(z)$ behaves asymptotically like $C\frac{1}{z-z_0}$, where $C \neq 0$ is a constant. Conversely, if $a_k \neq 0$ for infinitely many k, then we classify z_0 as an essential singularity.

A.3 *Laplace Transforms*

Recall that for a function $f : [0, \infty) \to \mathbb{C}$, the Laplace transform of f at $z \in \mathbb{C}$ is defined as

$$\mathcal{L}f(z) := \int_0^\infty f(x)\, e^{-zx}\, \mathrm{d}x$$

whenever the integral converges absolutely, and its domain by

$$\mathcal{D}(\mathcal{L}f) := \left\{ z \in \mathbb{C} : \int_0^\infty |f(x)|\, |e^{-zx}|\, \mathrm{d}x = \int_0^\infty |f(x)|\, e^{-\operatorname{Re}(z)x}\, \mathrm{d}x < \infty \right\}.$$

The following lemma is probably well known but we could not provide a suitable source in the literature. For the reader's convenience, we include a short proof.

Lemma A.11

Let $f : [0, \infty) \to [0, \infty)$ be a nondecreasing function with finite Laplace transform $\mathcal{L}f$ for every $\theta > 0$, i.e.,

$$\mathcal{L}f(\theta) = \int_0^\infty e^{-\theta x} f(x)\, \mathrm{d}x < \infty.$$

Then $f(t) = o(e^{\theta t})$ as $t \to \infty$.

Proof of Lemma A.11.
For any $t > 0$, we have

$$\begin{aligned} f(t) &= \theta \int_0^\infty e^{-\theta x} f(t)\,\mathrm{d}x = \theta e^{\theta t} \int_0^t e^{-\theta(x+t)} f(t)\,\mathrm{d}x + \theta \int_t^\infty e^{-\theta x} f(t)\,\mathrm{d}x \\ &\leq \theta e^{\theta t} \int_t^{2t} e^{-\theta x} f(x)\,\mathrm{d}x + \theta \int_t^\infty e^{-\theta x} f(x)\,\mathrm{d}x = o(e^{\theta t}) \quad \text{as } t \to \infty. \end{aligned}$$

■

A.4 *(Logarithmic) Convex Functions*

A function $f : \mathbb{R}^p \to \mathbb{R}$ is convex, if for every $x, y \in \mathbb{R}^p$ and $\lambda \in [0,1]$ the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)y$$

holds, i.e. the straight line joining any two different points on its graph lies entirely above the function's curve.

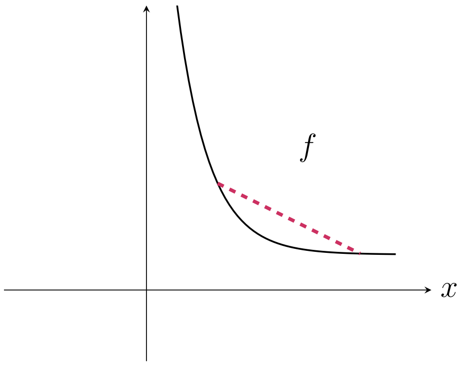

Figure A.2: A graphical representation of a convex function.

A function $f \geq 0$ is called logarithmic convex if $\log(f)$ is convex. Indeed, every logarithmic convex function is also convex. This is easily seen by defining $g := \log(f)$. If g is convex, then $\exp(g) := f$ is convex by composition rules. Further, an interesting class of logarithmic convex functions is formed by the Laplace transform of positive functions. Let $f \geq 0$, then with Hölder inequality

$$\begin{aligned} \log \mathcal{L}f(\lambda x + (1-\lambda)y) &= \log\left(\int_0^\infty e^{\lambda x + (1-\lambda)y} f(t)\mathrm{d}t\right) \\ &= \log\left(\int_0^\infty (e^{-xt} f(t))^\lambda (e^{-yt} f(t))^{1-\lambda}\mathrm{d}t\right) \\ &= \lambda \log \mathcal{L}(x) + (1-\lambda) \log \mathcal{L}(y), \end{aligned}$$

i.e. $\mathcal{L}f(t)$ is logarithmic convex, $t \in \mathbb{R}$. Similarly, if μ is a nonnegative measure, then $\mathcal{L}\mu(t)$ is logarithmic convex on $\mathcal{D}(\mathcal{L}\mu)$.

A.5 *Integrals*

In this section, we present auxiliary results related to integrals, focusing in particular on the computation of specific integrals that are required in the course of our proofs.

Lemma A.12

For $\lambda \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{N}_0$ and $t \geq 0$, we have

$$\int_0^t x^k e^{\lambda x} \,\mathrm{d}x = \frac{(-1)^{k+1} k!}{\lambda^{k+1}} \left(1 - e^{\lambda t} \sum_{j=0}^{k} (-1)^j \frac{\lambda^j t^j}{j!}\right).$$

Proof of Lemma A.12.

First consider $\lambda = -\theta$ for some $\theta > 0$ and write S_{k+1} for a random variable with distribution $\Gamma(k+1, \theta)$, which is the $(k+1)$-fold convolution power of $\mathrm{Exp}(\theta)$. Since the increments in a homogeneous Poisson process $(N_t)_{t\geq 0}$ with intensity $\theta > 0$ are independent exponentials, we have $\mathbf{P}(N_t \geq k+1) = \mathbf{P}(S_{k+1} \leq t)$. Hence,

$$\begin{aligned}\int_0^t x^k e^{-\theta x} \,\mathrm{d}x &= \frac{k!}{\theta^{k+1}} \int_0^t \frac{\theta^{k+1} x^k}{k!} e^{-\theta x} \,\mathrm{d}x = \frac{k!}{\theta^{k+1}} \mathbf{P}(S_{k+1} \leq t) = \frac{k!}{\theta^{k+1}} (1 - \mathbf{P}(N_t \leq k)) \\ &= \frac{k!}{\theta^{k+1}} \left(1 - e^{-\theta t} \sum_{j=0}^{k} \frac{(\theta t)^j}{j!}\right).\end{aligned}$$

This is (A.12) in the special case $\lambda = -\theta < 0$. Since both sides of (A.12) are holomorphic functions in $\lambda \in \mathbb{C} \setminus \{0\}$, the assertion follows from the uniqueness theorem for holomorphic functions.

■

Remark A.13 Classically, the integral in Lemma A.12 can also be determined through repeated integration by parts, yielding the alternative form

$$\int_0^t x^k e^{\lambda x} \,\mathrm{d}x = e^{\lambda t} \sum_{n=0}^{k} t^{k-n} \frac{(-1)^n k!}{\lambda^{n+1} (k-n)!} - \frac{(-1)^{k+1} k!}{\lambda^{k+1}}.$$

Bibliography

[Ahl79] Lars V. Ahlhors. *Complex Analysis.* 3rd ed., McGraw-Hill, 1979.

[AN72] K. B. Athreya and P. E. Ney. *Branching processes.* Die Grundlehren der Mathematischen Wissenschaften, Bd. 196. Springer-Verlag, New York-Heidelberg, 1972.

[BC63] R. E. Bellman and K. L. Cooke. *Differential-Difference Equations.* Santa Monica, CA: RAND Corporation, 1963.

[BE65] Bengt von Bahr and Carl-Gustav Esseen. "Inequalities for the rth Absolute Moment of a Sum of Random Variables, $1 \leqq r \leqq 2$". In: *Ann. Math. Statist.* 36.6 (1965), pp. 299–303.

[CM68] K. S. Crump and C. J. Mode. *A general age-dependent branching process. I, II.* J. Math. Anal. Appl. 24 (1968), 494-508; ibid., 25:8–17, 1968.

[Cru70a] Kenny S. Crump. "On systems of renewal equations". In: *J. Math. Anal. Appl.* 30 (1970), pp. 425–434. ISSN: 0022-247X.

[Cru70b] Kenny S. Crump. "On systems of renewal equations: The reducible case". In: *J. Math. Anal. Appl.* 31 (1970), pp. 517–528. ISSN: 0022-247X.

[Gar81] J. B. Garnett. *Bounded Analytic Functions.* Academic Press, New York, 1981.

[Gra18] A. Graham. *Kronecker Products and Matrix Calculus with Applications.* Courier Dover Publications, 2018.

[Har63] T. E. Harris. *The theory of branching processes.* Die Grundlehren der Mathematischen Wissenschaften, Bd. 119. Springer-Verlag, Berlin; Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963.

[HJ13] R. A. Horn and C. R. Johnson. *Matrix Analysis.* Cambridge: Cambridge University Press, 2013.

[IKM24] Alexander Iksanov, Konrad Kolesko, and Matthias Meiners. "Asymptotic fluctuations in supercritical Crump-Mode-Jagers processes". In: *Ann. Probab.* 52.4 (2024), pp. 1538–1606. ISSN: 0091-1798,2168-894X.

[Jag75] P. Jagers. *Branching processes with biological applications.* London: Wiley-Interscience [John Wiley & Sons], London-New York-Sydney, 1975.

[JN08] S. Janson and R. Neininger. *The size of random fragmentation trees.* Probab. Theory Related Fields 142 399–442., 2008.

[Mar49] Morris Marden. *The Geometry of the Zeros of a Polynomial in a Complex Variable.* Providence, RI: American Mathematical Society, 1949.

[Mey00] Carl D. Meyer. *Matrix analysis and applied linear algebra.* USA: Society for Industrial and Applied Mathematics, 2000. ISBN: 0898714540.

[Mil71] John W. Miles. *Integral Transforms in Applied Mathematics.* Cambridge University Press, 1971.

[Ner79] O. Nerman. *On the Convergence of Supercritical General Branching Processes.* PhD thesis, Chalmers University of Technology and the University of Göteborg, 1979.

[Ner81] Olle Nerman. “On the convergence of supercritical general (C-M-J) branching processes”. In: *Z. Wahrsch. Verw. Gebiete* 57.3 (1981), pp. 365–395. ISSN: 0044-3719.

[Res92] S. I. Resnick. *Adventures in stochastic processes.* CHE: Birkhauser Verlag, 1992. ISBN: 0817635912.

[Sgi02] M. S. Sgibnev. *Semimultiplicative estimates for the solution of the multidimensional renewal equation.* Izvestiya: Mathematics, Volume 66, Issue 3, 595–610, 2002.

[Wid41] David V. Widder. *The Laplace Transform.* Princeton Mathematical Series, vol. 6. Prince- ton University Press, Princeton, N. J., 1941.

Eidesstattliche Erklärung

Ich erkläre: Ich habe die vorgelegte Dissertation selbstständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt, die ich in der Dissertation angegeben habe. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben, die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht. Ich stimme einer evtl. Überprüfung meiner Dissertation durch eine Antiplagiat-Software zu. Bei den von mir durchgeführten und in der Dissertation erwähnten Untersuchungen habe ich die Grundsätze guter wissenschaftlicher Praxis, wie sie in der „Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis“ niedergelegt sind, eingehalten.

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