Dissertation

Finiteness properties of S-arithmetic subgroups of Chevalley groups in characteristic 0

von

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Abstract

We consider in this thesis S-arithmetic subgroups of certain algebraic matrix groups defined over $\mathbb{Q}$. The simplest example of such a group is $\Gamma = \mathrm{SL}_n(\mathbb{Z}[1/p])$. Each of these groups is of type F_∞ by a well-known result of Borel and Serre. On a formal level, this means that there is a $K(\Gamma, 1)$ complex with finite m-skeleton for every $m \in \mathbb{N}$. A nice consequence is that Γ is finitely presented. While the method of Borel and Serre is more algebraic, we give here a new, purely geometric, proof that uses Morse theory.

Doing so, we first develop the terminology of a Morse function without critical values greater than a constant $r > 0$, which is defined on the product of a Riemannian manifold and a metric space. After that, we deduce some properties from the reduction theory of S-arithmetic groups, which we translate into geometric terms to a space X, on which our group acts canonically. Finally, we construct a real-valued function on that space. We show that this is a Morse function in the sense above. From that we deduce the statement concerning the finiteness properties of the group.

Zusammenfassung

Wir betrachten in dieser Dissertation S-arithmetische Untergruppen von bestimmten algebraischen Matrixgruppen über $\mathbb{Q}$. Das einfachste Beispiel einer solchen ist $\Gamma = \mathrm{SL}_n(\mathbb{Z}[1/p])$. Nach einem bekannten Resultat von Borel und Serre ist jede dieser Gruppen vom Typ F_∞. Rein formal bedeutet das, dass es einen $K(\Gamma, 1)$-Komplex gibt, der endliches m-Skelett hat für jede natürliche Zahl $m \in \mathbb{N}$. Eine schöne Folgerung daraus ist, dass Γ endlich präsentiert ist. Während die Methode von Borel und Serre algebraisch ist, geben wir hier einen neuen, rein geometrischen Beweis, der Morse-Theorie benutzt.

Dazu entwickeln wir zunächst den Begriff einer Morse-Funktion ohne kritische Werte oberhalb einer Konstante $r > 0$, die auf dem Produkt einer Riemannschen Mannigfaltigkeit und eines metrischen Raumes definiert ist. Danach folgern wir einige Eigenschaften aus der Reduktionstheorie S-arithmetischer Gruppen und übersetzen diese in geometrische Form auf einen Raum X, auf dem unsere Gruppe kanonisch wirkt. Schließlich konstruieren wir eine reellwertige Funktion auf diesem Raum und zeigen, dass sie die Kriterien einer Morse-Funktion wie oben erfüllt. Daraus leiten wir die Aussage über die Endlichkeitseigenschaften der Gruppe ab.

Contents

1 Introduction

In geometric group theory the structure of groups is examined via geometrical methods. The approach is often to construct a geometric space on which they act with good properties. The groups we will deal with in this thesis are S-arithmetic subgroups of algebraic matrix groups defined over $\mathbb{Q}$. A typical example for this is

$$\Gamma = \mathrm{SL}_n(\mathbb{Z}[1/p])$$

for some prime p. Via the diagonal embedding one can view this as a discrete subgroup of $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_p)$. The latter acts with compact stabilizers on a contractible space $X = X_\infty \times X_p$ where $X_\infty \cong \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ is the symmetric space associated to $\mathrm{SL}_n(\mathbb{R})$ and X_p is the Bruhat-Tits building associated to $\mathrm{SL}_n(\mathbb{Q}_p)$, an example of a Euclidean building.

Now consider a group as the fundamental group of some connected CW complex C: It is an easy consequence of algebraic topology that $\pi_1(C)$ is finitely generated if C has finite 1-skeleton, and finitely presented if C has finite 2-skeleton. This can be generalized to higher dimensions in the following way: The group $\pi_1(C)$ is of type F_n if C has finite n-skeleton and the universal cover $\tilde{C}$ is contractible. Here we want to show that the group under consideration is of type F_∞, that means of type F_n for all n. One way to do so is to construct a contractible CW complex on which our group acts with finite stabilizers and with compact quotient. But in the setup described above the induced action of Γ fails to be cocompact, so one needs to modify the space X.

Raghunathan [Rag68] considered arithmetic subgroups of semisimple algebraic groups defined over $\mathbb{Q}$. Here one can think of $\mathrm{SL}_n(\mathbb{Z})$, which is a discrete subgroup of $\mathrm{SL}_n(\mathbb{R})$. His method was to cut out some part of the symmetric space making the induced action of the arithmetic group cocompact. In fact, he constructed a smooth real valued function defined on the quotient and showed that it has no critical value greater than some constant $r > 0$. That the homotopy type does not change then follows from Morse theory as developed in [Mil63].

The way of Borel and Serre [BS73] to handle the problem of cocompactness was to attach a boundary to the space X_∞, that is in some sense compatible with the action and does not change the homotopy type, but which makes the quotient compact. In [BS76] they extended their result to S-arithmetic groups:

Theorem (Borel-Serre). *Let* $\mathbf{G}$ *be a simply connected,* $\mathbb{Q}$*-simple Chevalley group over* $\mathbb{Q}$*. Then the* S*-arithmetic subgroup* $\Gamma = \mathbf{G}(\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_s}])$ *is of type* F_∞*.*

The procedure is sketched in the book of Brown [Bro89, VII 1D and 2C].

The aim of this thesis is to give a new proof of the Borel-Serre theorem about S-arithmetic groups but following the method of Raghunathan. To do so we will construct a Morse function with no critical values outside some compact interval, not only on the smooth factor X_∞ but on the whole space $X = X_\infty \times X_p$. Note that Morse theory in the classical sense like in [Mil63] only applies to smooth structure. So we need to extend it to a function defined on the product of a smooth and a non-smooth factor.

Theorem. *Let $f\colon M \times Z \to \mathbb{R}$ be a continuous function defined on the product of a smooth Riemannian manifold M and a first countable Hausdorff space Z. We assume that it fulfills the following properties for some $r > 0$:*

a) The induced map $f^z\colon M \to \mathbb{R}$ defined by $f^z(p) := f(p, z)$ is smooth for every $z \in Z$.

b) The norm of the gradient ∇f^z is uniformly bounded from below on the set $\{p \in M \mid f^z(p) > r\}$. This bound is independent of $z \in Z$.

c) The map $Z \to C^1(M, \mathbb{R})$ defined by $z \mapsto f^z$ is continuous with respect to the weak C^1 topology.

Then $f^{-1}(]-\infty, R])$ is homotopy equivalent to $M \times Z$ for all $R > r$.

Such a map f we will call a Morse function with no critical values greater than r.

The proof of the Borel-Serre theorem we will give here will be entirely geometric. As an outlook, one possible application of the methods developed in this work could be to show a conjecture stated in the article of Hartnick and Witzel [HW22]: There the authors examine finiteness properties of so called approximate subgroups. These are defined as a symmetric subset Λ of a group G that contains the identity of the group, but is only closed under multiplication up to finiteness: This means $\Lambda\Lambda \subset F\Lambda$ for a finite set $F \subset G$. Examples of such approximate subgroups can be constructed in a similar way from algebraic groups like S-arithmetic subgroups. The authors investigate finiteness properties for the case where S contains only finite places. They conjecture that if S also has Archimedean places, the corresponding approximate subgroup is also of type F_∞. It should be possible to extend our geometric proof to this, while the construction of Borel and Serre is too algebraic to apply here.

The attentive reader will have noticed that the theorem Borel and Serre originally proved is much more general. This is due to technical assumptions that we will use in our argumentation. By making some adjustments, it should also be possible to prove the result in the full generality of [BS76].

Let me outline the structure of this thesis, it is divided into two parts. Part I consists of Sections 2 to 13 and serves as an introductory part. It collects terminology and basic results that are needed later on. These include proper actions, CAT(0) spaces and Busemann functions, finiteness properties, (Riemannian) manifolds, Lie groups, root systems, algebraic matrix groups and Euclidean buildings. Those topics with which the reader is already familiar can be skipped. Specific notation or special results that are required will be clearly referenced to at the appropriate points.

In Part II we will prove the theorems stated above. Section 14 begins with the theorem about Morse functions on two factors. In Section 15 we fix an algebraic matrix group $\mathbf{G}$ defined over $\mathbb{Q}$. To work with it we will give a list of conventions that should apply to that group and introduce notation. We also define the space X the S-arithmetic group acts on. Raghunathan lists several properties about arithmetic groups that are necessary for his proof, see [Rag68, (i)-(iv) on p.326]. In Section 16 we will deduce analogous statements from reduction theory of S-arithmetic groups using the classical results stated in [PR94] and [Bor63]. These we will translate to geometric properties of the action from Γ on the space X, and describe them in terms of Busemann functions. Here we use some constructions from [HW22]. Then in Section 17 we will define a real valued function on the space $X = X_\infty \times X_p$ in a similar way Raghunathan does: It will be invariant under the action of the group Γ. Using the geometric results already mentioned we will show that it is a Morse function as described above. From this we will deduce the finiteness properties of S-arithmetic subgroups of $\mathbf{G}$ in Section 18.

Part I

Fundamentals and background material

In this part we give definitions for the terms needed later and collect some background material. Most of the time we give references for the statements needed. In some cases we need more specific results or easy consequences from the general theory, then we give short proofs.

2 Proper maps and actions

Here we collect some basic facts about proper maps and group actions. A good source is the overview article by Kramer [Kra22].

Definition 2.1. Let X, Y be Hausdorff spaces, where we additionally assume Y to be locally compact. A map $f\colon X \to Y$ is called *proper* if $f^{-1}(K) \subset X$ is compact for all compact subsets $K \subset Y$.

Remark 2.2. There are various definitions for a map to be proper. But our definition is equivalent to the one given in [Kra22] since we assumed Y to be locally compact.

We give a criterion for a map to be proper.

Lemma 2.3. *Let $f\colon X \to Y$ be a continuous map between topological Hausdorff spaces, where we assume X to be a countable union of open, relatively compact subsets. Let further f fulfill the following property:*
For every sequence $(x_n)_{n\in\mathbb{N}}$ in X with the property that the set $\{n \in \mathbb{N} \mid x_n \in C\}$ is finite for any compact subset $C \subset X$, the set $\{n \in \mathbb{N} \mid f(x_n) \in K\}$ is also finite for all compact subsets $K \subset Y$.
Then $f^{-1}(K)$ is a compact subset of X for all compact $K \subset Y$.

Proof. Let $K \subset Y$ be compact, then the preimage $f^{-1}(K) \subset X$ is closed. Suppose it were not compact. By assumption $X = \bigcup_{n\in\mathbb{N}} U_n$ for some relatively compact open subsets U_n of X. For every $n \in \mathbb{N}$ the preimage $f^{-1}(K)$ is not contained in $\bigcup_{i=1}^{n} U_i$, so we can pick some $x_n \in f^{-1}(K) \setminus (\bigcup_{i=0}^{n} U_i)$. On the other hand, for all compact subsets C of X there is an $m \in \mathbb{N}$ such that C is a subset of the finite union $\bigcup_{i=0}^{m} U_i$, which implies $x_n \notin C$ for all $n \geq m$. But this means that the set $\{n \in \mathbb{N} \mid x_n \in C\}$ is finite for any compact subset C of X, so $\{n \in \mathbb{N} \mid f(x_n) \in K\}$ must be finite, too. This is a contradiction, since every $f(x_n)$ is an element of K by definition. $\square$

Definition 2.4. The continuous action of a topological Hausdorff group G on a locally compact space X is called *proper* if the map

$$f\colon G \times X \to X \times X,\ (g, x) \mapsto (g.x, x)$$

is proper.

The following is a useful characterization of proper group actions.

Lemma 2.5. *Let G be a topological Hausdorff group acting continuously on a locally compact space X. The following are equivalent:*

a) The action is proper.

b) The set $\{g \in G \mid g.B \cap C \neq \varnothing\}$ is compact for all compact subsets $B, C \subset X$.

c) The set $\{g \in G \mid g.C \cap C \neq \varnothing\}$ is compact for all compact subsets $C \subset X$.

Proof. By [Kra22, Proposition 1.5] statements a) and b) are equivalent, and obviously b) implies c). So take $B, C \subset X$ compact, then c) implies that $\{g \in G \mid g.(B \cup C) \cap (B \cup C) \neq \varnothing\}$ is also compact. Using the fact that

$$H := \{g \in G \mid g.B \cap C \neq \varnothing\} \subset \{g \in G \mid g.(B \cup C) \cap (B \cup C) \neq \varnothing\}$$

we only have to show that H is closed in G. Take any net $(g_\lambda)_{\lambda \in \Lambda}$ in H converging to some $g \in G$, we have to show that $g \in H$. We can choose nets $(b_\lambda)_{\lambda \in \Lambda} \subset B$ and $(c_\lambda)_{\lambda \in \Lambda} \subset C$ such that $g_\lambda.b_\lambda = c_\lambda$ for all $\lambda \in \Lambda$ by definition of H. Since C is compact we get a subnet $(c_\mu)_\mu$ converging to some $c \in C$. If we now consider the corresponding net $(b_\mu)_\mu$, then we can choose another subnet $(b_\nu)_\nu$ converging to some $b \in B$ again by compactness. This leads to nets $(g_\nu)_\nu$, $(b_\nu)_\nu$ and $(c_\nu)_\nu$ converging to g, b and c. By continuity of the action we get

$$g.b = \lim_\nu g_\nu.b_\nu = \lim_\nu c_\nu = c$$

so in fact $g \in H$ holds. □

Corollary 2.6. *If G is a topological Hausdorff group acting properly on a locally compact space X, then any closed subgroup $H \subset G$ acts also properly on X. Especially this holds for all discrete subgroups.*

Proof. Follows directly from the lemma above. □

Properness of group actions also passes over to products.

Lemma 2.7. *Let $(G_i)_{i\in I}$ be a family of topological Hausdorff groups, each of them acting on a locally compact space X_i. Then the induced action of $\prod_{i\in I} G_i$ on $\prod_{i\in I} X_i$, is proper if and only if each of the actions G_i on X_i is proper.*

Proof. Follows from [Eng89, Theorem 3.7.9]. Just note that what we have called proper here is called perfect there. ☐

Finally we prove a criterion for a group action to be proper that we want to apply later. Since properness passes to closed subgroups by Corollary 2.6 this will also give a criterion for a discrete subgroup to act properly.

Lemma 2.8. *Let G be a locally compact group acting continuously and transitively on a Hausdorff space X. Suppose there is a point $x_0 \in X$ such that:*

a) The stabilizer $\mathrm{Stab}_G(x_0)$ is compact in G,

b) the continuous map $\varphi\colon G \to X,\ g \mapsto g.x_0$ is open.

Then for any compact subset $C \subset X$ the set $\{g \in G \mid g.C \cap C \neq \varnothing\}$ is compact. Consequently, if X is assumed to be locally compact, then the action is proper.

Proof. It suffices to find a compact subset $A \subset G$ to a given $C \subset X$ compact with $A.x_0 = C$, because this implies that

$$\{g \in G \mid g.C \cap C \neq \varnothing\} = AKA^{-1}$$

is compact for $K := \mathrm{Stab}_G(x_0)$. Under the given assumptions both sets indeed coincide: If $ga.x_0 = a'.x_0$ for $a, a' \in A$ and $g \in G$, then $x_0 = a^{-1}ga'$, so $g \in aKa'^{-1} \subset AKA^{-1}$. On the other hand, if $g = a'ka^{-1} \in AKA^{-1}$ then $g.(a.x_0) = a'.x_0$.
So let

$$A := \{g \in G \mid g.x_0 \in C\} = \varphi^{-1}(C)$$

then $A.x_0 = C$ by transitivity. Thus we have to show that A is compact. For any $a \in A$ choose a relatively compact open neighborhood $U_a \subset G$ by local compactness of G. Since φ is open, the set $\bigcup_{a\in A} \varphi(U_a)$ is an open cover of the compact set C, so we can find $U_1, \dots, U_n \in \{U_a \mid a \in A\}$ such that

$$C \subset \bigcup_{i=1}^{n} \varphi(U_i) = \varphi\left(\bigcup_{i=1}^{n} U_i\right) = \left(\bigcup_{i=1}^{n} U_i\right).x_0.$$

But this implies

$$A = \varphi^{-1}(C) \subset \varphi^{-1}\left(\left(\bigcup_{i=1}^{n} U_i\right).x_0\right) = \left(\bigcup_{i=1}^{n} U_i\right)\cdot K$$

since $K = \mathrm{Stab}(x_0)$. Thus, A is a subset of a compact subset of G. Furthermore, A is closed by definition, so it is indeed compact. ☐

3 Metric spaces of non-positive curvature

We give a short introduction into CAT(0) spaces. The goal of the first subsection is to define the boundary at infinity and Busemann functions and state some basic results about their interplay. In the second part we consider products of metric spaces. Main source for both parts is the book of Bridson and Haefliger [BH99].

3.1 CAT(0) spaces and Busemann functions

Throughout this whole subsection let X be a metric space with metric d.

Definition 3.1. Let $\lambda > 0$. A map $c\colon [a,b] \to X$ with $d(c(t),c(s)) = \lambda\,|t-s|$ for all $s,t \in [a,b] \subset \mathbb{R}$ is a *linearly reparametrized geodesic.* We call it a *geodesic* if $\lambda = 1$. Its image $[c(a),c(b)] := \mathrm{im}(c([a,b]))$ is a *geodesic segment* from $c(a)$ to $c(b)$.

If for any two points $x, y \in X$ there is a (unique) geodesic segment from x to y, then we call X a *(unique) geodesic space.* A *geodesic ray* is a map $[0,\infty[\to X$ such that the restriction to $[a,b]$ is a geodesic for any $[a,b] \subset [0,\infty[$. Similarly, we define a *geodesic line* as a map $\mathbb{R} \to X$ with the same property for all $[a,b] \subset \mathbb{R}$. Equivalently, one could define geodesics, geodesic rays and geodesic lines as isometric embeddings.

Two geodesic rays $c, c'\colon [0,\infty[\to X$ are called *asymptotic* if there is a constant $k > 0$ with $d(c(t),c'(t)) < k$ for all $t \geq 0$. This clearly defines an equivalence relation on the set of geodesic rays into X. Write $c(\infty)$ for the equivalence class of the ray $c\colon [0,\infty[\to X$, and define the *boundary at infinity* of X as the set

$$\partial X := \{c(\infty) \mid c\colon [0,\infty[\to X \text{ is a geodesic ray}\}.$$

In the following text we denote an element $\xi \in \partial X$ as *point at infinity.* The isometry group $\mathrm{Isom}(X)$ of a metric space X has a canonical action on ∂X since $d(\gamma.c(t),\gamma.c'(t)) = d(c(t),c'(t))$ for $\gamma \in \mathrm{Isom}(X)$. Thus, every group that acts via isometries on X also acts on its boundary.

A *geodesic triangle* in X with vertices $x, y, z \in X$ is the union

$$\Delta(x,y,z) := ([x,y] \cup [y,z] \cup [x,z]) \subset X$$

of three geodesic segments. We call a triangle $\overline{\Delta}(\overline{x},\overline{y},\overline{z}) \subset \mathbb{R}^2$ with $d(x,y) = d(\overline{x},\overline{y})$, $d(y,z) = d(\overline{y},\overline{z})$ and $d(x,z) = d(\overline{x},\overline{z})$ a *comparison triangle* for $\Delta(x,y,z)$. A *comparison point* for $p \in [x,y] \subset X$ is a point $\overline{p} \in [\overline{x},\overline{y}] \subset \mathbb{R}^2$ with $d(p,x) = d(\overline{p},\overline{x})$. A geodesic triangle $\Delta(x,y,z)$ satisfies the *CAT(0)-inequality,* if for all $p, q \in \Delta(x,y,z)$ and comparison points $\overline{p},\overline{q} \in \overline{\Delta}(x,y,z)$ the inequality $d(p,q) \leq d(\overline{p},\overline{q})$ holds.

Definition 3.2. A geodesic metric space X is a CAT(0) *space*, if for every triangle $\Delta(x, y, z) \subset X$ the CAT(0) inequality holds.

Furthermore, we say a metric space X is *locally* CAT(0) or *of non-positive curvature* if for every $x \in X$ there is an $\varepsilon >$ such that $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ is a CAT(0) space when endowed with the canonical subspace metric.

Every CAT(0) space X is in fact uniquely geodesic. Further it has the property that its *geodesics vary continuously with their endpoints*: If $c, c_n \colon [0, 1] \to X$ are linearly reparametrized geodesics with the properties $\lim_{n\to\infty} c_n(0) = c(0)$ and $\lim_{n\to\infty} c_n(1) = c(1)$, then c_n converges to c uniformly on the whole interval $[0, 1]$. For both statements see [BH99, Proposition II.1.4]. One can easily deduce:

Lemma 3.3. *Every* CAT(0) *space is contractible.*

Proof. See [BH99, Corollary II.1.5]. ☐

For any two geodesics $c, c' \colon [a, b] \to X$ the function $f \colon [a, b] \to \mathbb{R};\ t \mapsto d(c(t), c'(t))$ is convex by [BH99, Proposition II.2.2].

Example 3.4. We give some examples of CAT(0) spaces here.

a) Obviously, the Euclidean space $\mathbb{R}^n$ is a CAT(0) space.

b) Every tree endowed with the canonical metric is a CAT(0) space, compare [BH99, Example II.1.15 (4)].

c) We construct the *hyperboloid model* $\mathcal{H}^n$ for the hyperbolic n-space: For $x, y \in \mathbb{R}^{n+1}$ let $x \circ y := \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}$. Then we define

$$\mathcal{H}^n = \{x \in \mathbb{R}^{n+1} \mid x \circ x = -1 \text{ and } x_{n+1} > 0\}$$

and set

$$d_{\mathcal{H}^n}(x, y) = \operatorname{arcosh}(-x \circ y).$$

The induced map $d_{\mathcal{H}^n}$ defines a metric on $\mathcal{H}^n$, see [Rat94, Theorem 3.2.2]. One can show that the corresponding metric space is uniquely geodesic, compare [BH99, Corollary I.2.8 (1)]. We define a CAT(-1) *space* analogously to a CAT(0) space, but taking comparison triangles $\overline{\Delta}$ in $\mathcal{H}^2$ instead of $\mathbb{R}^2$. The space $\mathcal{H}^2$ canonically embeds as a subspace into $\mathcal{H}^n$ for all $n \geq 2$. For all $x, y, z \in \mathcal{H}^n$ there is an isometry $\gamma \in \operatorname{Isom}(\mathcal{H}^n)$ sending all the three points x, y, z into $\mathcal{H}^2$, this is proven in [Rat94, §3.2]. Thus, $\mathcal{H}^n$ is CAT(-1) so [BH99, Theorem II.1.12] implies that it is CAT(0).

d) A non-example is the sphere $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ endowed with the angular metric $d(u, v) := \arccos(\langle u, v\rangle)$ which is constructed in [BH99, Proposition I.2.1]. This fails to be CAT(0), because it is not contractible.

More examples will be constructed later.

Assume now the CAT(0) space X to be complete. For every $\xi \in \partial X$ and every $x \in X$ there exists a unique geodesic ray $c_x \colon [0, \infty[\to X$ with $c_x(0) = x$ and $c_x(\infty) = \xi$. See [BH99, Proposition II.8.2]. We can endow the set $\overline{X} := X \cup \partial X$ with the inverse limit topology of the system $\mathrm{pr}_r \colon \overline{B_s}(x_0) \to \overline{B_r}(x_0)$ for $s \geq r$. Here pr_r is defined in the following way: Take for any $y \in \overline{B_s}(x_0)$ the unique geodesic $c\colon [0, b] \to \overline{B_s}(x_0)$ with $c(0) = x_0$ and $c(b) = y$. Then we set $\mathrm{pr}_r(y) = c(\max\{r, b\})$. A basis of this topology is given by the open sets of X together with the sets of the form

$$U(c, r, \varepsilon) = \{x \in \overline{X} \mid d(x, c(0)) > r,\ d(\mathrm{pr}_r(x), c(r)) < \varepsilon\}$$

where $c\colon [0, \infty[\to X$ is a geodesic ray and $\varepsilon, r > 0$. The inclusion $X \hookrightarrow \overline{X}$ then becomes a homeomorphism onto its image. See [BH99, II.8.5 and 8.6] for details. Furthermore, any isometry $\gamma \in \mathrm{Isom}(X)$ extends naturally to a homeomorphism $\overline{X} \to \overline{X}$ by [BH99, Corollary II.8.9].

Definition 3.5. Let $c\colon [0, \infty[\to X$ be a geodesic ray.

a) The map $\beta_c \colon X \to \mathbb{R};\ x \mapsto \lim_{t\to\infty} t - d(x, c(t))$ is called the *Busemann function* associated to c.

This is well defined as the limit always exists by [BH99, Lemma II.8.18].

b) Let $r \in \mathbb{R}$ be any real number, the preimage $\beta_c^{-1}(]r, \infty[) \subset X$ is called a *horoball* and $\beta_c^{-1}(r)$ is a *horosphere*.

Remark 3.6. In some sources, for example [BH99], Busemann functions are defined as $x \mapsto \lim_{t\to\infty} d(x, c(t)) - t$ which is just $-\beta$ in our definition. So we can apply their results by changing the sign if necessary.

Further, a *reparametrized Busemann function* is a map of the form $x \mapsto a\beta(x)$ for some $a > 0$ and a Busemann function β .

Example 3.7. We describe easy examples of Busemann functions and the corresponding horoballs.

a) First we consider a rather trivial example: Let $c\colon \mathbb{R} \to \mathbb{R}$ be the identity, then the restriction to $[0, \infty[$ will be a geodesic ray. The corresponding Busemann function is just the identity which can be seen by

$$\beta_c(x) = \lim_{t\to\infty} t - d(c(t), x) = \lim_{t\to\infty} t - t + x = x$$

so the horoballs are $\beta_c^{-1}(]r, \infty[) =]r, \infty[$.

b) More generally, we consider the Euclidean space $X = \mathbb{R}^n$, then any geodesic ray is of the form $c(t) = b + tu$ for some $u \in \mathbb{S}^{n-1}$ and $b \in \mathbb{R}^n$. The corresponding Busemann function is

$$\begin{aligned}
\beta_c(x) &= \lim_{t\to\infty} t - \|b + tu - x\| \\
&= \lim_{t\to\infty} t - \sqrt{\langle b-x, b-x\rangle + 2t\langle b-x, u\rangle + t^2\langle u,u\rangle} \\
&= \lim_{t\to\infty} \frac{t^2 - (\langle b-x, b-x\rangle + 2t\langle b-x, u\rangle + t^2\langle u,u\rangle)}{t + \sqrt{\langle b-x, b-x\rangle + 2\langle b-x, u\rangle + t^2\langle u,u\rangle}} \\
&= \lim_{t\to\infty} -\frac{\langle b-x, b-x\rangle + 2t\langle b-x, u\rangle}{t + \sqrt{\langle b-x, b-x\rangle + 2\langle b-x, u\rangle + t^2}} \\
&= -\frac{2\langle b-x, u\rangle}{1+\sqrt{1}} \\
&= \langle x-b, u\rangle
\end{aligned}$$

So the horoballs are the affine half-spaces

$$\beta_c^{-1}(]r,\infty[) = \{x \in \mathbb{R}^n \mid \langle x-b, u\rangle > r\}.$$

This is from [BH99, Example II.8.24 (1)].

More examples can be found in [BH99]. The next lemmas state useful properties of Busemann functions.

Lemma 3.8. *Every Busemann function is distance decreasing and therefore continuous.*

Proof. Let X be a metric space, let $c\colon [0,\infty[\to X$ be a geodesic ray and let $\beta = \beta_c$ be the corresponding Busemann function. For any two points $x, y \in X$ we have

$$\begin{aligned}
|\beta(y) - \beta(x)| &= \left|\lim_{t\to\infty} t - d(y, c(t)) - t + d(x, c(t))\right| \\
&= \lim_{t\to\infty} |d(x, c(t)) - d(y, c(t))| \\
&\leq d(x,y).
\end{aligned}$$

Thus, $\lim_{n\to\infty} \beta(x_n) = \beta(x)$ if $\lim_{n\to\infty} x_n = x$. □

Consider now a complete CAT(0) space X and let $c, c'\colon [0,\infty[\to X$ be geodesic rays with $c(\infty) = c'(\infty)$. Let further $\beta_c, \beta_{c'}\colon X \to \mathbb{R}$ be the associated Busemann functions. Then $\beta_c - \beta_{c'}$ is constant as a function $X \to \mathbb{R}$ by [BH99, Corollary II.8.20]. Therefore the following definition makes sense: We call a Busemann function $\beta\colon X \to \mathbb{R}$ *centered at* $\xi \in \partial X$ if $\beta - \beta_c = \text{const}$ for any geodesic ray $c\colon [0,\infty[\to X$ with $c(\infty) = \xi$. We also call a horoball or a horosphere *centered at* ξ if the corresponding Busemann function is.

Lemma 3.9. *Let X be a complete* CAT(0) *space and let $\beta\colon X \to \mathbb{R}$ be a Busemann function centered at a point at infinity $\xi \in \partial X$. For every geodesic ray $c\colon [0,\infty[\to X$ with $c(\infty) = \xi$ we have $\beta(c(t)) = \beta(c(0)) + t$ for all $t \in [0,\infty[$. If c extends to a geodesic line $\mathbb{R} \to X$ this equation holds for all $t \in \mathbb{R}$. The Busemann function associated to c fulfills $\beta_c(c(0)) = 0$.*

Proof. For $t \geq s$ we have $d(c(t), c(s)) = t - s$ as c is an isometric embedding, so

$$\beta_c(c(s)) = \lim_{t\to\infty} t - d(c(s), c(t)) = \lim_{t\to\infty} t - t + s = s$$

holds for all $s \in \mathbb{R}$, on which c is defined. Now choose a constant $k \in \mathbb{R}$ with $\beta - \beta_c = k$, then the above implies

$$\beta(c(t)) - t = \beta(c(t)) - \beta_c(c(t)) = k.$$

Therefore, $\beta(c(0)) = k$ and thus $\beta(c(t)) = \beta(c(0)) + t$ hold for all $t \in \mathbb{R}$ on which c is defined. □

Lemma 3.10. *Let G be a group acting via isometries on X and let $\beta\colon X \to \mathbb{R}$ be a Busemann function centered at a point $\xi \in \partial X$. For every $g \in G$ the map $x \mapsto \beta(g.x)$ is a Busemann function centered at $g^{-1}.\xi$.*

Proof. Let c be a geodesic ray with $\beta = \beta_c$. Then

$$\beta_c(g.x) = \lim_{t\to\infty} t - d(c(t), g.x) = \lim_{t\to\infty} t - d(g^{-1}.c(t), x) = \beta_{g^{-1}.c}(x)$$

is a Busemann function associated to the geodesic ray $g^{-1}.c$ defined by $t \mapsto g^{-1}.c(t)$. □

At this point we also define the terminology of a proper metric space, that we will need later.

Definition 3.11. A metric space X is called *proper* if for all $x \in X$ the closed balls $\overline{B_r}(x) = \{y \in X \mid d(x,y) \leq r\}$ are compact for any $r \geq 0$.

It follows directly from the definition that a metric space X is proper if and only if any bounded and closed subset $A \subset X$ is compact.

Remark 3.12. The terminology of a proper metric space comes from the following easy fact: A metric space is proper if and only if for every $x_0 \in X$ the real valued function $d_{x_0}(x) := d(x_0, x)$ is a proper map in the sense of Definition 2.1.

Properness gives a useful criterion to check if a metric space X is CAT(0).

Proposition 3.13. *Let X be a proper, uniquely geodesic metric space of non-positive curvature. Then X is a* CAT(0) *space.*

Proof. See [BH99, Corollary I.3.13 and Proposition II.4.9]. □

3.2 Products of metric spaces

In this subsection we will state some basic facts about products of CAT(0) spaces. Recall that for metric spaces $(X_i, d_i)_{i=1,\dots,n}$ the *product metric* d on $X_1 \times \ldots \times X_n$ is defined by $d(x,y) = (\sum_{i=1}^{n} d_i(x_i,y_i)^2)^{\frac{1}{2}}$ for two points $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n)$. In the following we will state the facts only for $X = X_1 \times X_2$ for the sake of brevity in the notation. They all extend easily via induction to the case with n factors.

Lemma 3.14. *For a product $X = X_1 \times X_2$ of metric spaces the following statements hold:*

a) X is complete if and only if X_1 and X_2 are complete.

b) X is a proper metric space if and only if X_1 and X_2 are proper.

c) X is a geodesic space if and only if X_1 and X_2 are geodesic spaces.

d) A map $c\colon [0,a] \to X,\ c(t) = (c_1(t), c_2(t))$ is a linearly reparametrized geodesic if and only if c_1 and c_2 are linear reparametrized geodesics.

e) A map $c\colon [a,b] \to X$ is a geodesic, if and only if $c(t) = (c_1(\lambda_1 t), c_2(\lambda_2 t))$ for some geodesics $c_i\colon [\lambda_i a, \lambda_i b] \to X_i$ and $\lambda_i \geq 0$ with $\lambda_1^2 + \lambda_2^2 = 1$.

f) X is a CAT(0) *space if and only if X_1 and X_2 are* CAT(0).

g) If G_i are groups acting via isometries on X_i for $i = 1, 2$ then $G_1 \times G_2$ acts via isometries on X.

Proof. For a), c), d and g) see [BH99, Proposition I.5.3].
To b): The product $B_1 \times B_2$ of two closed bounded subsets $B_i \subset X_i$ is bounded and closed in X. Hence, if X is proper, then B_1 and B_2 are compact. On the other hand, if X_1 and X_2 are proper, then we can cover any bounded and closed subset B of X by two compact balls B_1 and B_2, so B is compact, too.
To e): By c) the map c is a linearly reparametrized geodesic if and only if it has the form stated above for some $\lambda_i \geq 0$. From

$$d(c(s), c(t))^2 = d_1(c_1(\lambda_1 s, \lambda_1 t))^2 + d_2(c_1(\lambda_2 s, \lambda_2 t))^2 = (\lambda_1^2 + \lambda_2^2)\,|s - t|$$

follows the statement.
To f): If X_1 and X_2 are CAT(0), then X is by [BH99, Example I.1.15 (3)], too. If on the other hand X is a CAT(0) space, then both factors are geodesic by b). Furthermore the CAT(0) inequality must hold, because they embed as isometric subspaces. $\square$

Note that e) extends to geodesic rays and lines by definition.

Corollary 3.15. *Let* $X = X_1 \times X_2$ *be a complete* CAT(0) *space. Assume* $c\colon [0,\infty[\to X$ *is a geodesic ray given by* $c(t) = (c_1(\lambda_1 t), c_2(\lambda_2 t))$ *like above and let* $x = (x_1, x_2) \in X$ *be any point. The unique ray starting at* x *asymptotic to* c *is given by* $c_x(t) = (c_{x_1}(\lambda_1 t), c_{x_2}(\lambda_2 t))$*, where* $c_{x_i}\colon [0,\infty[\to X_i$ *are geodesic rays with* $c_{x_i}(0) = x_i$ *and* $c_{x_i}(\infty) = c_i(\infty)$*.*

Proof. Follows directly from

$$d(c(t), c_x(t))^2 = d_1(c_1(\lambda_1 t), c_{x_1}(\lambda_1 t))^2 + d_2(c_2(\lambda_2 t), c_{x_2}(\lambda_2 t))^2 \leq k_1^2 + k_2^2$$

for constants k_i bounding $d_i(c_i(t), c_{x_i}(t))$. ☐

If $X = X_1 \times X_2$ is a complete CAT(0) space, then the boundary at infinity is the *spherical join* $\partial X = \partial X_1 * \partial X_2$: This is the set of equivalence classes of $\partial X_1 \times \partial X_2 \times [0, \frac{\pi}{2}]$ where two elements (ξ_1, ξ_2, θ) and $(\xi_1', \xi_2', \theta')$ define the same class if and only if ($\theta = 0$ and $\xi_2 = \xi_2'$) or ($\theta = \frac{\pi}{2}$ and $\xi_1 = \xi_1'$) holds. The ray $c(t) = (c_1(\cos(\theta)t), c_2(\sin(\theta)))$ represents

$$c(\infty) = [c_1(\infty), c_2(\infty), \theta] =: \cos(\theta)c_1(\infty) + \sin(\theta)c_2(\infty).$$

Compare [BH99, Example II.8.11 (6)]. The notation fits to the following observation about Busemann functions.

Lemma 3.16. *Let* $c\colon [0,\infty[\to X = X_1 \times X_2$ *be a geodesic ray given by* $c(t) = (c_1(\cos(\theta)t), c_2(\sin(\theta)t))$*. The corresponding Busemann function is* $\beta_c((x_1, x_2)) = \cos(\theta)\beta_{c_1}(x_1) + \sin(\theta)\beta_{c_2}(x_2)$*.*

Proof. See [BH99, Example II.8.24 (3)]. ☐

4 Finiteness properties

We begin this section by reminding the reader that a *covering space projection* is a continuous map $p\colon \tilde{X} \to X$ such that any point $x \in X$ has an open neighborhood $U \subset X$ so that $p^{-1}(U) = \coprod_{i \in I} U_i$ is a disjoint union of open subsets $U_i \subset \tilde{X}$ and p induces homeomorphisms $U_i \to U$. We call such a $\tilde{X}$ a *covering space* of X. If $f\colon Y \to X$ is a continuous map, then another map $\tilde{f}\colon Y \to \tilde{X}$ with the property $p \circ \tilde{f} = f$ is a *lift* of f. Recall further that a topological space X is *locally path connected* if for any open subset $U \subset X$ and any point $x \in U$ there is a path connected neighborhood $V \subset U$ of x. It is not hard to see that a covering space of a locally simply connected space is also locally simply connected and vice versa. Furthermore, we call X *semilocally simply-connected* if every point $x \in X$ has a neighborhood $U \subset X$ such that the canonical homomorphism $\pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion is trivial. These properties are not really harsh restrictions, for example manifolds and CW complexes are locally path connected and semilocally simply-connected: In fact both are *locally contractible* (for every $U \subset X$ open and every $x \in U$ there is a contractible neighborhood $V \subset U$ of x), since manifolds are locally Euclidean, and for CW complexes see [Hat02, Proposition A.4]).

One can show that for any path connected, locally path connected and semilocally simply-connected space X there is a covering space $\tilde{X}$ that is simply connected, see the construction in [Hat02, p.63-65]. To be semilocally simply-connected in fact is also a necessary condition for a space to have a simply connected covering space. We call such an $\tilde{X}$ the *universal cover* of X, because it fulfills the following universal property:

Proposition 4.1. *Let X be path connected, locally path connected space and let $p\colon \tilde{X} \to X$ be a covering projection, mapping a point $x_0 \in X$ to $\tilde{x_0} \in \tilde{X}$, where $\tilde{X}$ is assumed to be simply connected. Then for any other covering projection $q\colon Z \to X$ mapping z_0 to x_0 with Z path connected and locally path connected, there is a unique map $\tilde{p}\colon \tilde{X} \to Z$ with the properties $q \circ \tilde{p} = p$ and $\tilde{p}(\tilde{x_0}) = z_0$.*

Proof. This follows directly from the *lifting criterion* and the *unique lifting property*, see [Hat02, Propositions 1.33 and 1.34]. □

This implies that the universal cover (if it exists) is unique up to homeomorphism, see [Hat02, Proposition 1.37].

Further recall that a *deck transformation* of a covering space projection $p\colon \tilde{X} \to X$ is a homeomorphism $f\colon \tilde{X} \to \tilde{X}$ such that $p = p \circ f$. Obviously, the set of all deck transformations form a group with $\circ$ as group multiplication. The covering space projection is called *normal* if for every $x \in X$ and every pair $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there is a deck transformation mapping $\tilde{x}$ to

$\tilde{x}'$. This terminology comes from the fact that the covering space projection p is normal if and only if $p_*(\pi_1(\tilde{X}, \tilde{x_0})) \subset \pi_1(X, x_0)$ is a normal subgroup, where p_* is the induced homomorphism between fundamental groups. See [Hat02, Proposition 1.39 a)].

Definition 4.2. A $K(\Gamma, 1)$ *space* is a path connected, locally path connected and semilocally simply-connected space X such that

a) $\pi_1(X) \cong \Gamma$,

b) the universal cover of X is contractible.

If X is a CW complex, then we call it a $K(\Gamma, 1)$ *complex*.

Definition 4.3. A path connected space X is called n*-aspherical* if any continuous map $\mathbb{S}^k \to X$ has an extension $D^{k+1} \to X$ for all $2 \leq k \leq n$. It is called n*-connected* if the same holds for all $0 \leq k \leq n$. Further we call it *aspherical* if it is n-aspherical for any $n \in \mathbb{N}$.

Remark 4.4. In [Geo08, p.162] a $K(\Gamma, 1)$ complex is defined to be an aspherical path connected CW complex X with $\pi_1(X, x_0) = \Gamma$. But since a path connected CW complex is aspherical if and only if its universal cover is contractible by [Geo08, Proposition 7.1.3] these two definitions coincide.

Definition 4.5. Let Γ be a group.

a) We say Γ is of *type* F_n if there is a $K(\Gamma, 1)$ complex X which has finite n-skeleton.

b) We call Γ of *type* F_∞ if there is a $K(\Gamma, 1)$ complex X which has finite n-skeleton for every $n \in \mathbb{N}$.

Remark 4.6. a) One can show that Γ is of type F_1 if and only if it is finitely generated, and is of type F_2 if and only if it is finitely presented, see [Geo08, Proposition 7.2.1]. So being of type F_n is a topological generalization of of these two algebraic properties.

b) The group Γ is of type F_∞ if and only if it is of type F_n for all $n \in \mathbb{N}$, see [Geo08, Proposition 7.2.2].

Example 4.7. Any finite group is of type F_∞.

Proof. See [Geo08, Corollary 7.2.5].

This comes from the fact that a group is of type F_n if and only if some (and therefore any) of its finite index subgroups is, see [Geo08, Corollary 7.2.4]. Thus, finiteness properties are in fact invariant under commensurability.

Next we give a criterion for a group to be of type F_n. Recall that a pair $(\Lambda, \leq)$ is a *directed set* if $\leq$ is a preorder on the set Λ, such that for all $\lambda, \lambda' \in \Lambda$ there is μ with $\lambda, \lambda' \leq \mu$.

Definition 4.8. Let X be a topological space.

a) A *filtration* of X is a collection of subspaces X_λ indexed by a directed set $(\Lambda, \leq)$ such that X_λ is a subset of X_μ whenever $\lambda \leq \mu$ and such that $\bigcup_{\lambda\in\Lambda} X_\lambda = X$.

b) We call a filtration $(X_\lambda)_{\lambda\in\Lambda}$ *essentially n-connected* if for every $0 \leq i \leq n$ the following holds: For all $\lambda \in \Lambda$ there is a $\mu \in \Lambda$, such that the group homomorphism $\pi_i(X_\lambda) \to \pi_i(X_\mu)$ induced by the inclusion $X_\lambda \subset X_\mu$, is trivial.

c) Two filtrations $(X_\lambda)_{\lambda\in\Lambda}$ and $(Y_\mu)_{\mu\in M}$ of X are *equivalent*, if for all $\lambda \in \Lambda$ there is $\mu \in M$ such that X_λ is a subset of Y_μ, and if for all $\mu \in M$ there is $\lambda \in \Lambda$ with the converse subset relation.

Lemma 4.9. *Let $(X_\lambda)_{\lambda\in\Lambda}$ and $(Y_\mu)_{\mu\in M}$ be two equivalent filtrations of the same space X. One of them is essentially n-connected if and only if the other one is.*

Proof. See [HW22, Lemma 2.3]. ☐

Lemma 4.10. *Let (X, d) be a $\mathrm{CAT}(0)$ space with a point $o \in X$ and let G be a locally compact, second countable group acting on it properly and continuously via isometries. A closed subgroup Γ of G is of type F_n if and only if the filtration $(N_r(\Gamma.o))_{r>0}$ is essentially $(n-1)$-connected. Here $(N_r(\Gamma.o))$ is defined to be the set of all $x \in X$ such that $d(\gamma.o, x) < r$ for some $\gamma \in \Gamma$.*

Proof. See [HW22, Proposition 2.23]. ☐

This will be our strategy to check the finiteness properties of a group Γ: Viewing it as a discrete subgroup of a locally compact, second countable group G, which acts naturally as described in the lemma above on a space X. We will construct a map $f\colon X \to \mathbb{R}$ and show that for $X_t := f^{-1}(]-\infty, t])$ the filtration $(X_t)_{t\in\mathbb{R}}$ is essentially n-connected and equivalent to $(N_r(\Gamma.o))_{r>0}$.

5 Real and complex manifolds

Here we give an introduction into the topic of manifolds. This section is divided into two subsections: In the first we deal with smooth manifolds, the second is about the holomorphic counterpart.

5.1 Smooth manifolds, vector fields and flows

Main source here is the book of Tu [Tu11]. We also use some aspects of [Con01].

Recall that a *topological manifold* of *dimension* $m \in \mathbb{N}$ is a second countable topological Hausdorff space M, such that for any point $p \in M$ there exists an open neighborhood $U \subset M$ of p and a continuous map $\phi\colon U \to \mathbb{R}^m$ such that $\phi(U) \subset \mathbb{R}^m$ is open and $\phi\colon U \to \phi(U)$ is a homeomorphism. We will refer to the pair (U, ϕ) as a *chart* around p and we will always assume that $\phi(p) = 0$. Setting $x^j = \mathrm{pr}_j \circ \phi$ for $j = 1, \ldots, m$ we get $\phi(q) = (x^1(q), \ldots, x^m(q)) \in \mathbb{R}^m$ for $q \in U$. We call $(x^1, \ldots, x^m)$ the *coordinates* of ϕ and refer to $(U, x^1, \ldots, x^m)$ as a *local coordinate system* around p.
Define the *chart map* from a chart (U, ϕ) to another one (V, ψ) of M as $\psi \circ \phi^{-1}\colon \phi(U \cap V) \to \psi(U \cap V)$. We call it *smooth* or C^∞ if it is smooth as a map of open subsets of $\mathbb{R}^m$. An *atlas* of M is a set of charts $\mathcal{U} = \{(U_i, \phi_i) \mid i \in I\}$ such that $M = \bigcup_{i \in I} U_i$. It is called *smooth* if all of its chart maps are. It is called *maximal* if for any other atlas $\mathcal{V} = \{(V_j, \phi_j) \mid j \in J\}$ with $\mathcal{U} \subset \mathcal{V}$ we already have $\mathcal{U} = \mathcal{V}$.

Definition 5.1. A topological manifold M is called *smooth* if it has a maximal smooth atlas $\mathcal{U}$.

The existence of a maximal smooth atlas is equivalent to the existence of some smooth atlas (not necessary maximal) since any smooth atlas is contained in a unique maximal one, see [Tu11, Proposition 5.10].

We make the following convention: If we talk of a manifold, we will always mean a smooth manifold.

Example 5.2. Here are some examples of smooth manifolds.

a) Obviously, $\mathbb{R}^m$ is a smooth manifold with one chart $(\mathbb{R}^m, \mathrm{id})$, compare [Tu11, Example 5.11].

b) Let M be a smooth manifold, then any open subset $V \subset M$ is also a smooth manifold of the same dimension. An atlas on V is given by $\{(U_i \cap V, \psi_i) \mid i \in I\}$ where $\psi_i = \phi_i \mid_{U_i \cap V}$ for an atlas $\{(U_i, \phi_i) \mid i \in I\}$ for M. See [Tu11, Example 5.12].

c) Let $U \subset \mathbb{R}^m$ be open and $f\colon U \to \mathbb{R}^n$ be a smooth map, then $\phi\colon \Gamma(f) \to U,\ (x, f(x)) \mapsto x$ is a chart of the graph $\Gamma(f) = \{(x,y) \in U \times \mathbb{R}^n \mid f(x) = y\}$. So $\Gamma(f)$ is also a smooth manifold, see [Tu11, Example 5.14].
This example shows that the hyperboloid
$$\mathcal{H}^n = \{x \in \mathbb{R}^{n+1} \mid -x_{n+1}^2 + \sum_{i=1}^n x_i^2 = -1 \text{ and } x_{n+1} > 0\}$$
is a smooth manifold of dimension n, because $\mathcal{H}^n = \Gamma(f)$ for the smooth map $f\colon \mathbb{R}^n \to \mathbb{R},\ x \mapsto \sqrt{\sum_{i=1}^n x_i^2 + 1}$.

d) Using b) we see that $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ is a smooth manifold of dimension n, see [Tu11, Example 5.15]. In fact this is also the prototype example of a Lie group, as multiplication and inversion are easily seen to be smooth. We will deal with that in the section about Lie groups.

e) If M and N are smooth manifolds of dimensions m and n, then the product $M \times N$ is also a smooth manifold of dimension $m+n$: An atlas is given by $\{U_i \times V_j, (\phi_i, \psi_j) \mid (i,j) \in I \times J\}$ for atlases $\{(U_i, \phi_i) \mid i \in I\}$ for M and $\{(V_J, \psi_j) \mid j \in J\}$ for N. See [Tu11, Example 5.17 and Proposition 5.18].

Let M, N be smooth manifolds of dimension m and n. A continuous map $f\colon M \to N$ is called *smooth in a point* $p \in M$ if for some charts (U, ϕ) around p and (V, ψ) around $f(p)$ the map $(\psi \circ f \circ \phi^{-1})\colon \phi(U \cap f^{-1}(V)) \to \mathbb{R}^n$ is smooth as a real function. Further f is called *smooth* if it is smooth in any point $p \in M$.
A smooth map $f\colon M \to N$ is called a *diffeomorphism* if there is another smooth map $g\colon N \to M$ such that $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. It is called a *local diffeomorphism* if for every point $p \in M$ there is a neighborhood $U \subset M$ of p such that the restriction $f|_U\colon U \to f(U)$ is a diffeomorphism.

The definition of smoothness does not depend on the choice of the charts, see [Tu11, Proposition 6.7 and 6.8]. In the special case $N = \mathbb{R}^n$ with the atlas $\{(\mathbb{R}^n, \mathrm{id})\}$ from Example 5.2 one directly deduces: A continuous map $f\colon M \to \mathbb{R}^n$ is smooth (in p) if and only if the map $f \circ \phi^{-1}$ is smooth (in p) for any chart (U, ϕ) of M (around p). This shows that chart maps are smooth. Furthermore, one can show that the composition of two smooth maps is smooth again, see [Tu11, Proposition 6.9].

We denote by $C^\infty(M)$ the *set of smooth maps* $M \to \mathbb{R}$. Taking a coordinate system $(U, x^1, \dots, x^m) = (U, \phi)$ around a point $p \in M$ we define for $f \in C^\infty$

the *partial derivative* with respect to x^i as

$$\frac{\partial}{\partial x^i}|_p f := \frac{\partial f}{\partial x^i}(p) := \frac{\partial(f \circ \phi^{-1})}{\partial x_i}(\phi(p)),$$

where the latter is the usual derivative at the point $\phi(p) \in \mathbb{R}^m$ of the smooth function $f \circ \phi^{-1}$ defined on the open subset $\phi(U)$ of $\mathbb{R}^m$.

In analogy to the above we write $C^\infty(U)$ for the set of all smooth real valued functions defined on an open subset $U \subset M$. For a function $f \in C^\infty(U)$ defined on an open neighborhood U of p we define the *germ* of f as the equivalence class $[f]_p$ given by the following relation: A function $g \in C^\infty(V)$ defined on the neighborhood $V \subset M$ of p is *equivalent to* f if they coincide on an open set $W \subset U \cap V$ containing the point p. We write $C_p^\infty(M)$ for the *set of all germs* at p.
One easily verifies that addition $[f]_p + [g]_p = [f+g]_p$, multiplication $[f]_p \cdot [g]_p = [f \cdot g]_p$ and scalar multiplication $r \cdot [f_p] = [r \cdot f]_p$ are well defined, such that $C_p^\infty(M)$ becomes an $\mathbb{R}$-algebra. A *derivation at* p or a *point-derivation* of $C_p^\infty(M)$ is an $\mathbb{R}$-linear map $X_p \colon C_p^\infty(M) \to \mathbb{R}$ such that $X_p([fg]_p) = X_p([f]_p)g(p) + f(p)X_p([g]_p)$ holds. Since this definition does not depend on the choice of the representatives, we will write $X_p(f)$ instead of $X_p([f]_p)$. The set T_pM of all derivations at p is a real vector subspace of the dual space $C_p^\infty(M)^*$, as one can verify easily.

Definition 5.3. The real vector space T_pM is the *tangent space* of M at p. We will call an element $X_p \in T_pM$ a *tangent vector* at p.

Remark 5.4. For an open neighborhood $U \subset M$ of a point $p \in M$ there is a canonical identification $T_pU = T_pM$ by definition, see [Tu11, Remark 8.2].

A direct calculation shows that the map

$$\frac{\partial}{\partial x^i}|_p \colon C_p^\infty(M) \to \mathbb{R},\ [f]_p \mapsto \frac{\partial}{\partial x^i}|_p f$$

is a derivation at p. In fact, the set $\{\frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^m}|_p\}$ is a basis for the tangent space T_pM by [Tu11, Proposition 8.9].

Lemma 5.5. *Let M be a smooth manifold of dimension m. Consider the local coordinate systems $(U, x^1, \dots, x^m)$ and $(V, y^1, \dots, y^m)$ around a point $p \in M$. Then we have $\frac{\partial}{\partial x^i} = \sum_{j=1}^m \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$ on $U \cap V$.*

Proof. See [Tu11, Proposition 8.10]. □

For a smooth map $f \colon M \to N$ we define the *differential* of f at $p \in M$ as the map $f_{*,p} \colon T_pM \to T_{f(p)}N,\ X_p \mapsto f_{*,p}(X_p)$ where $f_{*,p}(X_p)(g) := X_p(g \circ f)$

for any smooth map $g\colon N \to \mathbb{R}$. This is a linear map between the two vector spaces T_pM and $T_{f(p)}N$ because of

$$f_{*,p}(X_p + rY_p)(g) = X_p(g \circ f) + rY_p(g \circ f) = f_{*,p}(X_p)(g) + rf_{*,p}(Y_p)(g).$$

As in the case of real maps $\mathbb{R}^m \to \mathbb{R}^n$ we have the chain rule $(g \circ f)_{*,p} = g_{*,f(p)} \circ f_{*,p}$ at every point $p \in M$ for smooth maps $f\colon M \to N$ and $g\colon N \to P$, see [Tu11, Theorem 8.5].

We get a more geometric intuition of the tangent space if we describe its vectors via curves: A *curve* into a smooth manifold M is a continuous map $c\colon I \to M$ where $I \subset \mathbb{R}$ is any interval. We call the curve *smooth* if there is an open interval $J \subset \mathbb{R}$ with containing I and a smooth map $\bar{c}\colon J \to M$ with $\bar{c}\mid_I = c$. We call c a *piecewise smooth curve* if $I = [a, b]$ for some $a < b$ in $\mathbb{R}$ and if there are $a = t_0 < t_1 < \ldots < t_k = b$ such that $c|_{[t_i,t_{i+1}]}$ is a smooth curve for all $i = 0, \ldots, k-1$.
For $t_0 \in I$ we define the *velocity vector* at $c(t_0) \in M$ of a smooth curve as the tangent vector $c'(t_0) := c_*(\frac{d}{dt}|_{t_0}) \in T_{c(t_0)}M$. A direct calculation shows

$$c'(t_0)(f) = c_*\left(\frac{d}{dt}|_{t_0}\right)(f) = \frac{d}{dt}|_{t_0}(f \circ c) = (f \circ c)'(t_0)$$

for all $f \in C^\infty_{c(t_0)}(M)$. Note that the latter becomes the usual derivative at t_0 of the real valued function $f \circ c$ defined on the interval I. If we pick a local coordinate system $(U, x^1, \ldots, x^m)$ around $c(t) \in M$, then describing the velocity vector via the induced basis we get $c'(t) = \sum_{i=1}^m \dot{c}^i(t)\frac{\partial}{\partial x^i}\mid_{c(t)}$ for $\dot{c}^i(t) := (x^i \circ c)'(t)$. See [Tu11, Proposition 8.15,].
On the other hand, let $p \in M$ be a point. For any tangent vector $X_p \in T_pM$ there is $\varepsilon > 0$ and a smooth curve $c\colon]-\varepsilon, \varepsilon[\to M$ with $c(0) = p$ and $c'(0) = X_p$. For this we have $X_p(f) = (f \circ c)'(0) \in \mathbb{R}$ for all $f \in C^\infty_p(M)$. More generally, if $g\colon M \to N$ is a smooth map, then $g_*(X_p) = (g \circ c)'(0) \in T_{g(p)}N$. See [Tu11, Propositions 8.16,8.17 and 8.18].

Definition 5.6. The *tangent bundle* of a smooth manifold M is defined as the set $TM := \bigcup_{p\in M}\{p\} \times T_pM$.

Define a topology on TM in the following way: For any chart $(U, \phi) = (U, x^1, \ldots, x^m)$ on M consider $TU = \bigcup_{p\in U}\{p\} \times T_pU = \bigcup_{p\in U}\{p\} \times T_pM$, see Remark 5.4. For all $p \in U$ the set $\{\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^m}|_p\}$ is a basis on T_pM. Thus, picking any vector $v \in T_pM$ there are unique scalars $a^1(v), \ldots, a^m(v) \in \mathbb{R}$ such that $v = \sum_{i=1}^m a^i(v)\frac{\partial}{\partial x^i}\mid_p$. We define a map

$$\tilde{\phi}\colon TU \to \phi(U) \times \mathbb{R}^m,\ (p, v) \mapsto (x^1(p), \ldots, x^m(p), a^1(v), \ldots, a^m(v)).$$

One can show that $\tilde{\phi}$ is a bijection and therefore we get a topology on TU defining $A \subset TU$ to be open if and only if $\tilde{\phi}(A) \subset \phi(U) \times \mathbb{R}^m$ is open. A basis for this topology on TM is given by

$$\mathcal{B} = \bigcup_{(U,\phi) \text{ chart of } M} \{A \mid A \subset TU \text{ open }\}.$$

Using this topology TM becomes a smooth manifold with an atlas $\{(TU_i, \tilde{\phi_i})\}$ for any atlas $\{(U_i, \phi_i)\}$ for M.
See [Tu11, §12.1-12.2, p.129-133] for details.

Definition 5.7. A *vector field* on a smooth manifold M is a map $X\colon M \to TM,\ p \mapsto (p, X_p)$. It is called *smooth* if it is a smooth map with respect to the manifold structure on TM defined above. We write $\mathfrak{X}(M)$ for the *set of smooth vector fields* on M.

For any tangent vector $X_p \in T_pM$ there are scalars $a^1(p), \ldots, a^m(p) \in \mathbb{R}$ such that $X_p = \sum_{i=1}^m a^i(p) \frac{\partial}{\partial x^i} \mid_p$, where $(U, x^1, \ldots, x^m)$ is a coordinate system around $p \in M$. Thus, an arbitrary vector field $X\colon M \to TM$ induces maps $a^i\colon U \to \mathbb{R}$ and locally we can write $X = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}$ via the identification $X(p) = X_p$. In fact, X defines a smooth vector field on U if and only if all the maps $a^i\colon U \to \mathbb{R}$ are smooth. See [Tu11, Lemma 14.1].

Lemma 5.8. *Let X be any vector field on a smooth manifold M. The following are equivalent:*

a) X is a smooth vector field.

b) X defines a smooth vector field on any chart $(U, x^1, \ldots, x^m)$.

c) For any $f \in C^\infty(M)$ the induced map $Xf\colon M \to \mathbb{R},\ p \mapsto X_p(f)$ is smooth.

Proof. See [Tu11, Propositions 14.2 and 14.3]. □

Remark 5.9. Point c) of the above lemma shows that we can see a smooth vector field $X \in \mathfrak{X}(M)$ also as a derivation of the $\mathbb{R}$-algebra $C^\infty(M)$ since the identities

$$X(f + rg)(p) = X_p(f + rg) = X_pf + rX_pg = (Xf)(p) + r(Xg)(p)$$

and

$$X(fg)(p) = X_p(fg) = X_p(f)g(p) + f(p)X_p(g) = (X(f) \cdot g)(p) + (f \cdot X(g))(p)$$

hold for any $p \in M,\ f, g \in C^\infty(M),\ r \in \mathbb{R}$.

Consider a vector field $X \in \mathfrak{X}(U)$ that is defined on a neighborhood $U \subset M$ of a point p. Then there is a smooth vector field $\tilde{X} \in \mathfrak{X}(M)$ that coincides with Y on a possibly smaller neighborhood V of p, see [Tu11, Proposition 14.4].

Lemma 5.10. *Let M be a smooth manifold and let $p \in M$ be a point. For any tangent vector $v \in T_pM$ there is a smooth vector field $X \in \mathfrak{X}(M)$ such that $X_p = v$.*

Proof. Choose a coordinate chart $(U, x^1, \dots, x^m)$ around p and let $a_1, \dots, a_m$ be real numbers with $v = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i} \mid_p$. We define $X_q = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i} \mid_q$ for any $q \in U$. Obviously, the induced vector field X is smooth on U, so we can extend it smoothly to the whole manifold M. $\square$

An *integral curve* or a *trajectory* of $X \in \mathfrak{X}(M)$ is a smooth curve $c\colon I \to X$ for some open interval $I \subset \mathbb{R}$ containing 0, such that $c'(t) = X_{c(t)}$ holds for all $t \in I$. We say the curve *starts* at the *initial point* $c(0) \in M$. The integral curve is called *maximal* if for any other integral curve $\gamma\colon J \to M$ with $I \subset J$ and $\gamma \mid_I = c$ we already have $I = J$.

Observe that in a chart (U, ϕ) a smooth curve $c\colon I \to U$ is an integral curve of $X = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(U)$ if and only if $\dot{c}^i(t) = a^i(c(t))$ for all $i = 1, \dots, m$, compare [Tu11, p.154]. Thus, the curve c is an integral curve for X with initial point $p \in U$ if and only if the smooth curve $(\phi \circ c)\colon I \to \phi(U) \subset \mathbb{R}^m$ is a solution of the initial value problem

$$\gamma'(t) = (a \circ \phi^{-1})(\gamma(t)), \quad \gamma(0) = \phi(p) \in \phi(U)$$

in $\mathbb{R}^m$. Here $a\colon U \to \mathbb{R}^m$ denotes the smooth function defined by $q \mapsto (a^1(q), \dots, a^m(q))$. Locally, such a curve always exists uniquely in the sense that any two of them agree on their common domain. See for example [Con01, Theorem 2.8.4 and Corollary 2.8.6.]. This extends globally to the whole manifold in the following way.

Proposition 5.11. *Let $X \in \mathfrak{X}(M)$ be a vector field of a smooth manifold M. For any $p \in M$ there is a unique maximal integral curve $c_p\colon I_p \to M$ starting at p. Furthermore, the map*

$$F\colon \Omega := \bigcup_{p \in M} I_p \times \{p\} \to M, \ (t, p) \mapsto c_p(t)$$

is smooth and Ω is an open subset of $\mathbb{R} \times M$. For any given $p \in M$ the interval I_p either contains $[0, \infty[$ or $c_p(t)$ leaves any compact set as $t \to \sup(I_p)$. The same holds for $]-\infty, 0]$ and $t \to \inf I_p$.

Proof. See [Pet, Theorem 2.2.4.]. $\square$

We call the map F in the proposition above the *flow* that is *generated* by the vector field X. In the case $\Omega = \mathbb{R} \times M$ we speak of a *global flow*.

Definition 5.12. A vector field $X \in \mathfrak{X}(M)$ is called *complete* if it generates a global flow. We denote by $\mathfrak{X}_c(M)$ the *set of complete vector fields* on M.

The global flow has the following useful property:

Lemma 5.13. *Let $X \in \mathfrak{X}_c(M)$ be a complete vector field. Its global flow $F\colon \mathbb{R} \times M \to M$ fulfills $F_t \circ F_s = F_{t+s}$ for the induced maps $p \mapsto F_t(p) = F(t,p)$ for all $t, s \in \mathbb{R}$. Furthermore, $F_0(p) = p$ holds for every $p \in M$. Each map $F_t\colon M \to M$ is invertible with $F_t^{-1} = F_{-t}$.*

Proof. Fix any $s \in \mathbb{R}$. By definition

$$F_t(F_s(p)) = F_t(c_p(s)) = c_{c_p(s)}(t)$$

so $t \mapsto F_t(F_s(p))$ is the integral curve of X with starting point $c_p(s)$. Consider the curve $t \mapsto c(t) := c_p(t+s) = F_{t+s}(p)$. For every $f \in C^\infty(M)$ we have

$$c'(t)(f) = (f \circ c)'(t) = \frac{d}{dt}|_t(f \circ c) = \frac{d}{dt}|_{t+s}(f \circ c_p) = c_p'(t+s)(f).$$

So $c'(t) = c_p'(t+s)$ holds for all $t \in \mathbb{R}$. Thus, $c(0) = c_p(0+s) = c_p(s)$ implies that c is also an integral curve of X with starting point $c_p(s)$. From uniqueness follows $F_{t+s} = F_t \circ F_s$.
By definition, $F_0(p) = c_p(0) = p$ holds for all $p \in M$. Consequently,

$$(F_{-t} \circ F_t)(p) = F_{-t+t}(p) = F_0(p) = p$$

implies $F_{-t} = F_t^{-1}$ for all $t, -t \in \mathbb{R}$. □

Recall that a smooth vector field X and a function $f \in C^\infty(M)$ define a smooth map $Xf\colon M \to \mathbb{R}$ via $p \mapsto X_p(f)$. Thus, if $Y \in \mathfrak{X}(M)$ is another vector field, the term $Y_p(Xf)$ makes sense for $p \in M$. Define the *Lie bracket* $[X,Y]$ as follows: For any point $p \in M$ and any smooth map $f \in C^\infty(M)$ we set $[X,Y]_p(f) := (X_pY - Y_pX)(f)$. This defines again a smooth vector field on M, that means $[X,Y] \in \mathfrak{X}(M)$. See [Tu11, Proposition 14.10].

Definition 5.14. Let M be a smooth manifold of dimension m. A subset $S \subset M$ is called a *regular submanifold of dimension* $k \leq m$, if for all $p \in S$ there is a coordinate system $(U, x^1, \dots, x^m)$ around p, such that $\left|\{i \in \{1, \dots, m\} \mid x^i|_{U\cap S} \neq 0\}\right| = k$. We also say that S has *codimension* $m-k$.

By permuting the coordinates if necessary, we can assume that $\phi\ |_{U\cap S} = (x^1, \dots, x^k, 0, \dots, 0)$. We call such a chart (U, ϕ) an *adapted chart* relative to S and set

$$\phi_S := (\mathrm{pr}_{\mathbb{R}^k} \circ \phi\ |_{U\cap S})\colon U \cap S \to \mathbb{R}^k.$$

The pair $(U \cap S, \phi_S)$ becomes a chart of S in the subspace topology. If $\{(U_i, \phi_i) \mid i \in I\}$ is a family of adapted charts on M such that all the chart maps are smooth and $S \subset \bigcup_{i\in I} U_i$ is a subset, then the family $\{(U_i \cap$

$S, \phi_{iS}) \mid i \in I\}$ is a smooth atlas on S. Thus, S itself becomes a smooth manifold of dimension k using the notation of the definition above. See [Tu11, Proposition 9.4].

Consider now a smooth map $f\colon M \to N$ between smooth manifolds M, N of dimension $\dim(M) = m$ and $\dim(N) = n$. We call f an *immersion at* $p \in M$ if the differential $f_{*,p}\colon T_pM \to T_{f(p)}N$ is injective, and a *submersion at* p if it is surjective. Linearity of f_* implies that $m \leq n$ ($m \geq n$) if f is an immersion (a submersion) at p. Further f is an *immersion (submersion)* if it is an immersion (a submersion) at every point. A point $p \in M$ is called a *regular point* and $f(p) \in N$ is a *regular value* of the function if $f_{*,p}\colon T_pM \to T_{f(p)}N$ is a submersion at p. Otherwise p is a *critical point* and $f(p)$ is a *critical value*. We call the preimage $f^{-1}(c)$ of a point $c \in N$ a *level set* of f. It is a *regular* if $c \in N$ is a regular value of f.

Example 5.15. Let M be a smooth manifold.

a) If $U \subset M$ is an open subset, then the inclusion $U \hookrightarrow M$ is a submersion and an immersion, see [Tu11, example on p.96]. This shows that a submersion does not need to be surjective in general.

b) If $S \subset M$ is a regular submanifold, then the inclusion $S \hookrightarrow M$ is an immersion by [Tu11, Theorem 11.14].

c) A point $p \in M$ is a critical point of a smooth function $f\colon M \to \mathbb{R}$, if and only if $\frac{\partial f}{\partial x^i}(p) = 0$ for all $i =, 1 \ldots, m$ for a chart $(U, x^1, \ldots, x^m)$ around p. See [Tu11, Proposition 8.23] for details. Using the fact that $\{\frac{\partial}{\partial x^1}|_p, \ldots \frac{\partial}{\partial x^m}|_p\}$ is a basis for T_pM this is equivalent to $\frac{\partial f}{\partial x^i}(p) = 0$ for every chart $(U, x^1, \ldots, x^m)$.

The following is called *regular level set theorem* or *implicit function theorem.*

Lemma 5.16. *Let $f\colon M \to N$ be a smooth map between smooth manifolds M, N of dimensions m and n. If $c \in N$ is a regular value of f and the preimage $f^{-1}(c)$ is not empty, then the level set $f^{-1}(c) \subset M$ is a regular submanifold of dimension $m - n$.*
In the case $N = \mathbb{R}$ the level set $f^{-1}(c) \subset M$ is a regular submanifold of dimension $m - 1$.

Proof. See [Tu11, Theorems 9.8 and 9.9]. □

5.2 Complex manifolds

The definition and much of the terminology can be developed analogous to the real case replacing "$\mathbb{R}^m$" by "$\mathbb{C}^m$" and "smooth" by "holomorphic". We will give a short introduction and some basic terminology. A good reference is [FG02, Section IV.1]. We also use [Lee02, Section 1.1].

Let M be a topological Hausdorff space. A *complex chart* or *complex coordinate system* is a pair (U, ϕ) consisting of an open subset $U \subset M$ and a homeomorphism $\phi\colon U \to \phi(U)$ for some open set $\phi(U) \subset \mathbb{C}^m$. We refer to m as the *dimension* of the complex chart. Two charts (U, ϕ) and (V, ψ) of the same dimension are *complex compatible*, if either $U \cap V = \varnothing$ or the induced map $\phi \circ \psi^{-1}\colon \psi(U \cap V) \to \phi(U \cap V)$ is holomorphic and has holomorphic inverse (this is a map between open subsets of $\mathbb{C}^m$.) A *complex atlas* $\mathcal{U}$ is a set of complex charts (U, ϕ) all of the same dimension such that the union of all U covers M and such that any two complex charts are complex compatible. An *m-dimensional complex structure* is the equivalence class of a complex atlas, where two atlases $\mathcal{U}$ and $\mathcal{V}$ are *equivalent* if all two coordinate systems $(U, \phi) \in \mathcal{U}$ and $(V, \psi) \in \mathcal{V}$ are complex compatible.

Definition 5.17. A *complex manifold of dimension m* is a first countable topological Hausdorff space M endowed with an m-dimensional complex structure.

To avoid confusion we make the following convention: If we talk of a (smooth) manifold, then we will always mean a smooth real manifold as defined in Definition 5.1. Otherwise we will always use the phrase complex as above.

Example 5.18. We give three easy examples.

a) As a trivial example, the space $\mathbb{C}^m$ is a complex manifold of dimension m, its complex structure is given by $(\mathbb{C}^m, \mathrm{id})$.

b) If M is a complex manifold with complex structure $\mathcal{U}$, then every open subset $B \subset M$ is again a complex manifold of the same dimension. Its complex structure is given by the set of all $(U \cap B, \phi|_{U \cap B})$ for $(U, \phi) \in \mathcal{U}$.

See [FG02, Examples 1. and 2. on p.155].

c) Let M and N be two complex manifolds of dimension m and n, then the product $M \times N$ carries the structure of a complex manifold of dimension $n + m$: If (U, ϕ) is a complex chart of M and (V, ψ) is a complex chart of N then $(\phi \times \psi, U \times V)$ is a complex coordinate system of $M \times N$, and the image $\phi(U) \times \psi(V)$ is an open subset of $\mathbb{C}^m \times \mathbb{C}^n$.

See the paragraph to product manifolds in [Lee02, Section 1.1].

A continuous map $f\colon M \to N$ between two complex manifolds is *holomorphic* if for some (and therefore any [FG02, Proposition IV.1.5]) charts (U, ϕ) of M and (V, ψ) of N containing $f(U)$ the induced map $(\psi \circ f \circ \phi^{-1})$ is holomorphic as a function between open subsets $\phi(U) \subset \mathbb{C}^m$ and $\psi(V) \subset \mathbb{C}^n$. The holomorphic map $f\colon M \to N$ is *biholomorphic* or an *isomorphism of complex manifolds* if it has a holomorphic inverse $f^{-1}\colon N \to M$.

Definition 5.19. A subset S of an m-dimensional complex manifold M is a *complex submanifold of codimension k*, if every point $p \in S$ has an open neighborhood $U \subset M$ and holomorphic functions $f_1, \ldots, f_k\colon U \to \mathbb{C}$ such that $S \cap U = \{q \in U \mid f_1(q) = \ldots = f_k(q) = 0\}$ and $\mathrm{rk}_p(f_1, \ldots, f_k) = k$.

Here $\mathrm{rk}(f_1, \ldots, f_k)_p := \mathrm{rk}\left(\frac{\partial(f_i \circ \psi^{-1})}{\partial z_j}(\psi(p))\right)_{\substack{i=1,\ldots,k\\ j=1,\ldots,m}}$ is the rank of the Jacobi matrix, which is independent of the choice of the chart (V, ψ) [FG02, p.161]. The submanifold $S \subset M$ canonically becomes itself a complex manifold of dimension $m - k$: In fact there are charts (U, ϕ) of M covering S with $\phi = (z^1, \ldots, z^m)$ such that $U \cap S = \{q \in U \mid z^j(q) = 0 \text{ for } j = m-k+1, \ldots, m\}$ and the coordinate systems are then given by $((\mathrm{pr}_{\mathbb{C}^{m-k}} \circ \phi|_{U \cap S}), U \cap S)$. See [FG02, Proposition IV.1.7].

Next we define the tangent space at a point p of a complex manifold M. A *germ of holomorphic functions at p* is an equivalence class of complex valued holomorphic functions defined on an open subset containing p, where two of them are *equivalent* if they coincide on a neighborhood of p that contains the intersection of their domains. The set of germs at p canonically has the structure of a $\mathbb{C}$-algebra. A *complex derivation* at p is a complex valued $\mathbb{C}$-linear map δ defined on the set of germs at p, that fulfills the identity $\delta(fg) = \delta(f)g(p)f(p)\delta(g)$ (here we write shortly f and g for their corresponding equivalence classes). Analogously to the real case, they form a complex vector space, the *tangent space* T_pM, and the derivations are therefore also called *tangent vectors*. Choosing a complex chart (U, ϕ) and setting $z^j := \mathrm{pr}_j \circ \phi$ we obtain a basis for T_pM by $\{\frac{\partial}{\partial z^1}|_p, \ldots, \frac{\partial}{\partial z^m}|_p\}$ where $\frac{\partial}{\partial z^j}|_p(f)$ is defined to be the usual complex derivative $\frac{\partial f(\phi(p))}{\partial z_j}$. The map $\mathbb{C}^m \to T_pM$ given by $(c_1, \ldots, c_m) \mapsto \sum_{j=1}^m c_j \frac{\partial}{\partial z^j}|_p$ is therefore an isomorphism of complex vector spaces. See [FG02, Proposition IV.1.14].

Via the natural identification $\mathbb{R}^2 \cong \mathbb{C}$ every complex manifold M of dimension m can be also viewed as a smooth real manifold of dimension $2m$, since under this identification every holomorphic function defined on open subsets of $\mathbb{C}^m$ induces canonically a smooth map on the corresponding open subset of $\mathbb{R}^{2m}$. We denote this smooth real manifold by $M_{\mathbb{R}}$. Rewriting the map ϕ of a chart (U, ϕ) of M as $p \mapsto (z^1(p), \ldots, z^m(p))$, every z^j induces two real coordinate maps x^j and y^j, so the corresponding real chart is given by $p \mapsto (x^1(p), y^1(p), \ldots, x^m(p), y^m(p))$. We therefore get an isomorphism of real vector spaces

$$\mathbb{R}^{2m} \to T_pM_{\mathbb{R}},\ (a_1, b_1, \ldots, a_m, b_m) \mapsto \sum_{j=1}^{m} a^j \frac{\partial}{\partial x^j} + b^j \sum_{j=1}^{m} \frac{\partial}{\partial y^j}.$$

But on the other hand, $(a_1, b_1, \ldots, a_m, b_m) \mapsto (a_1 + ib_1, \ldots, a_m + ib_m)$ is a canonical isomorphism $\mathbb{R}^{2m} \to \mathbb{C}^m$. As a complex vector space T_pM is also one over $\mathbb{R}$.

Lemma 5.20. *Let (U, ϕ) be a complex chart of M containing a point p. Let $z^j = \mathrm{pr}_j \circ \phi$ be the coordinates of the complex chart, and let $x^j, y^j \colon M \to \mathbb{R}$ be the corresponding smooth maps with $z^j = x^j + iy^j$. Then*

$$T_pM_{\mathbb{R}} \to T_pM,\ \left(\sum_{j=1}^{m} a_j \frac{\partial}{\partial x^j}|_p + \sum_{j=1}^{m} b_j \frac{\partial}{\partial y^j}|_p \right) \mapsto \sum_{j=1}^{m} (a_j + ib_j) \frac{\partial}{\partial z^j}|_p$$

is an isomorphism of real vector spaces.

Proof. See [Lee02, Proposition 1.1]. ☐

Thus we can calculate the underlying real tangent space of a complex manifold to determine complex tangent vectors T_pM.

Remark 5.21. In the sources we have given, the proof is done the other way around: There the basis for the corresponding real vector space $T_pM_{\mathbb{R}}$ is calculated first and from this the complex variant is deduced using the Cauchy-Riemann equations. See in [FG02, Section IV.1] or in [Lee02, Section 1.1] the corresponding paragraphs.

You can also see directly: If $S \subset M$ is a complex submanifold, then the isomorphism $T_pM \to T_pM_{\mathbb{R}}$ restricts to an isomorphism $T_pS \to T_pS_{\mathbb{R}}$ for $p \in S$.

Let now $f \colon M \to N$ be a holomorphic map between two complex manifolds and let $p \in M$ be a point. The *differential* of f at p is the $\mathbb{C}$-linear map $f_{*,p} \colon T_pM \to T_{f(p)}M$ defined like in the real case: For a tangent vector $X_p \in T_pM$ we define $(f_{*,p}(X_p))(g) := X_p(g \circ f)$ for holomorphic functions $g \colon V \to \mathbb{C}$ defined on an open set $V \subset N$ containing $f(p)$.

6 Lie algebras

We collect a few facts and terminology about Lie algebras. A good source is [HN12, Section 5].

Definition 6.1. A *Lie algebra* over a field k is a finite dimensional k-vector space $\mathfrak{g}$ together with a bilinear map $[.,.]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},\ (X,Y) \mapsto [X,Y]$ that has the following properties:

1) $[X,X] = 0$ for all $X \in \mathfrak{g}$,

2) $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$ for all $X,Y,Z \in \mathfrak{g}$.

This map $[.,.]$ is called the *Lie bracket* of the Lie algebra and 2) is the *Jacobi identity.*

Example 6.2. For every field k the vector space $k^{n\times n}$ of $n \times n$ matrices together with $[X,Y] = XY - YX$ for all $X,Y \in k^{n\times n}$ is a Lie algebra.
More general, for a k vector space V the set of endomorphisms $\mathrm{End}(V)$ is a Lie algebra with Lie bracket $[f,g] = (f \circ g) - (g \circ f)$.
This is [HN12, Example 5.1.3].

See [HN12, Example 5.1.6] for more examples.

A *Lie subalgebra* is a k-linear subspace $\mathfrak{h} \subset \mathfrak{g}$ which is closed under the Lie bracket, that means $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$. A k-linear map $\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ is called a *homomorphism of Lie algebras*, if $\varphi([X,Y]) = [\varphi(X),\varphi(Y)]'$ holds for all $X,Y \in \mathfrak{g}$. An *ideal* of $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ such that $[\mathfrak{g},\mathfrak{h}]$ lies in $\mathfrak{h}$. The Lie algebra is called *solvable*, if $D^n\mathfrak{g} = 0$ holds for some $n \in \mathbb{N}$, where we define $D^0\mathfrak{g} := \mathfrak{g}$ and $D^{k+1}\mathfrak{g} := [D^k\mathfrak{g}, D^k\mathfrak{g}]$ for all $k \in \mathbb{N}$. By [HN12, Proposition 5.4.3] every Lie algebra has a maximal solvable ideal $\mathfrak{rad}(\mathfrak{g})$, the *radical* of $\mathfrak{g}$.

Definition 6.3. A Lie algebra $\mathfrak{g}$ is called *semisimple*, if $\mathfrak{rad}(\mathfrak{g}) = \{0\}$. It is *simple*, if $\mathfrak{g}$ and $\{0\}$ are the only ideals.

Every simple Lie algebra is semisimple [HN12, Lemma 5.5.2].

7 Lie groups

We give the definition of a Lie group (real and complex) and collect some basic properties.

7.1 Real Lie groups

A good reference for Lie groups is the book of Hilgert and Neeb [HN12].

Definition 7.1. A *Lie group* is a group G which also carries the structure of a smooth manifold such that the multiplication $G \times G \to G,\ (g,h) \mapsto gh$ and inversion $G \to G,\ g \mapsto g^{-1}$ are smooth maps.

The tangent space at the identity is the *Lie algebra* of the Lie group G, we denote it by $\mathfrak{g} = T_1 G$. This terminology comes from the fact that it carries in a canonical way the structure of a real Lie algebra as defined in the previous section, see [HN12, Definition 9.1.7]. A *homomorphism of Lie groups* is a smooth map $G \to H$ that is also a group homomorphism. The *exponential map* of the Lie group is defined by $\exp_G \colon \mathfrak{g} \to G,\ X \mapsto \gamma_X(1)$, where $\gamma_X \colon \mathbb{R} \to G$ is the unique homomorphism of Lie groups that fulfills $\gamma_X(0) = 1$ and $\gamma_X'(0) = X$. Note that [HN12, Definition 9.2.2] coincides with ours by [HN12, Lemma 9.2.4].

Lemma 7.2 (One-parameter Group Theorem). *For every $X \in \mathfrak{g}$ the map $\mathbb{R} \to G$ defined by $t \mapsto \exp_G(tX)$ is a smooth group homomorphism. Conversely, any continuous homomorphism $\mathbb{R} \to G$ is of the form $t \mapsto$ $exp_G(tX)$ for some $X \in \mathfrak{g}$.*

Proof. See [HN12, Theorem 9.2.15]. $\square$

We call a homomorphism of Lie groups $\mathbb{R} \to G$ a *one-parameter group*. The exponential map has the following useful property.

Lemma 7.3. *If $\varphi \colon G \to H$ is a homomorphism of Lie groups, then we have $\exp_H \circ \varphi_{*,1} = \varphi \circ \exp_G$, where $\varphi_{*,1} \colon \mathfrak{g} \to \mathfrak{h}$ is the differential at $1 \in G$.*

Proof. See [HN12, Proposition 9.2.10]. $\square$

For $g \in G$ let $c_g \colon G \to G$ be the conjugation map defined by $c_g(x) = gxg^{-1}$, this is a smooth group homomorphism. We define $\mathrm{Ad}_g := (c_g)_{*,1}$ as the induced linear map of tangent spaces $\mathfrak{g} \to \mathfrak{g}$. This implies $\mathrm{Ad}_g(X) = \frac{d}{dt}|_{t=0}\, g \exp_G(tX) g^{-1}$. The map $\mathrm{Ad} \colon g \mapsto \mathrm{Ad}_g$ is a homomorphism of groups $G \to \mathrm{GL}(\mathfrak{g})$ because of $c_g \circ c_h = c_{gh}$, we call it the *adjoint representation.*

Lemma 7.4. *Let G_0 be the connected component of $1 \in G$. Then $\ker(\mathrm{Ad})$ is the center of G_0.*

Proof. See [HN12, Lemma 9.2.21]. ☐

We give an example.

Example 7.5. The group $G = \mathrm{GL}_n(\mathbb{R})$ of invertible real $n \times n$ matrices is a Lie group by [HN12, Example 9.1.4]. Its Lie algebra is $\mathfrak{g} = \mathbb{R}^{n\times n}$ with the Lie bracket $[X,Y] = XY - YX$, and the exponential map is the usual matrix exponential $\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}$. [HN12, Example 9.2.3]. The adjoint representation is given by conjugation, this follows from

$$\mathrm{Ad}_g(X) = \frac{d}{dt}|_{t=0}\, g\exp(tX)g^{-1} = g\big(\frac{d}{dt}|_{t=0}\exp(tX)\big)g^{-1} = gXg^{-1}.$$

Proposition 7.6 (Closed Subgroup Theorem)**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let H be a closed subgroup of G. Then H is a submanifold of G and the restriction of the multiplication and inversion maps to H are smooth. Its tangent space at 1 is a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and its exponential map is $\exp_H = \exp_G|_{\mathfrak{h}}$. Conversely, if a subgroup H of a Lie group G is again a Lie group with the induced topology, then it is closed.*

Proof. See [HN12, Theorem 9.3.7 and Proposition 9.3.9]. ☐

In view of this proposition we call a closed subgroup of a Lie group a *Lie subgroup*.

Corollary 7.7. *Let G be a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. Then it is a Lie group, its Lie algebra $\mathfrak{g}$ is a Lie subalgebra of $\mathbb{R}^{n\times n}$ with Lie bracket given by $[X,Y] = XY - YX$. The exponential map is the usual matrix exponential restricted to $\mathfrak{g}$ and the adjoint representation of G is given by conjugation.*

Proof. Follows directly from the above proposition, see also [HN12, Proposition 4.1.4]. ☐

We state some more facts that we will need later.

Lemma 7.8. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the subgroup generated by $\exp_G(\mathfrak{g})$ is the connected component of G that contains the identity element.*

Proof. See [HN12, Lemma 9.2.9]. ☐

Let M be a smooth manifold. We call the action of a Lie group G on M *smooth*, if the induced map $G \times M \to M$ is smooth.

Lemma 7.9. *Let G be a Lie group and $H \subset G$ be a Lie subgroup. The quotient G/H of topological groups carries a canonical manifold structure such that the quotient map is a submersion of manifolds. Moreover, the action of G on G/H via left multiplication is smooth.*

Proof. See [HN12, Theorem 10.1.10]. ☐

Lemma 7.10. *Let G be a Lie group acting smoothly on a manifold M and let $K \subset G$ be the stabilizier of a point $p \in M$. Then K is a closed subgroup of G. If additionally the action is transitive, then the canonical map $G/K \to M$ is a diffeomorphism.*

Proof. See [HN12, Lemma 10.1.5 and Corollary 10.1.17]. $\square$

An easy consequence of this is the following: If $G \to G'$ is an isomorphism of Lie groups mapping a Lie subgroup K of G to a subgroup K' of G', then the induced map $G/K \to G'/K'$ is a dieffeomorphism.

7.2 Complex Lie groups

Like in the case of smooth manifolds there is also a complex counterpart for Lie groups. We collect some facts about them. A good reference is [Lee02, Section 1.2].

Definition 7.11. A *complex Lie group* is a group G which is also a complex manifold, such that the map $G \times G \to G,\ (g, h) \mapsto g^{-1}h$ is holomorphic.

To avoid confusion we will always write complex Lie group when we mean such a structure as above. We will use the terminology Lie group only for the smooth real ones from Definition 7.1.

A *homomorphism of complex Lie groups* is a holomorphic map $G \to H$ between two complex Lie groups, that is also a group homomorphism. The *Lie algebra* $\mathfrak{g}$ of a complex Lie group G is the tangent space T_1G. This also has the canonical structure as a complex Lie algebra [Lee02, Theorem 1.7] and the differential of a homomorphism $G \to H$ is a homomorphism of the corresponding Lie algebras $\mathfrak{g} \to \mathfrak{h}$ [Lee02, Theorem 1.8]. The *adjoint representation* is the group homomorphism $G \to \mathrm{GL}(\mathfrak{g})$ given by $g \mapsto \mathrm{Ad}_g$, the latter is the differential of the conjugation automorphism $c_g(h) = ghg^{-1}$.

Since every complex manifold is also a real manifold and holomorphic maps are smooth under the canonical identification $\mathbb{R}^2 \cong \mathbb{C}$ every complex Lie group G is also a (real) Lie group which we denote by $G_{\mathbb{R}}$. If $\mathfrak{g}$ denotes the complex Lie algebra of G, then it also has the structure of a real vector space, and consequently can be viewed as a real Lie algebra. This is canonically isomorphic to the real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ that comes from the Lie group $G_{\mathbb{R}}$ under the map from Lemma 5.20, see [Lee02, Proposition 1.9]. Further we define the *exponential map* of G by $\exp_G(X) := \exp_{G_{\mathbb{R}}}(X)$ for every $X \in \mathfrak{g}$ using the identification $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}}$. Viewing this as a map $\exp\colon \mathfrak{g} \to G$ the exponential is holomorphic, see [Lee02, Theorem 1.15] (the complex vector space $\mathfrak{g}$ has a canonical structure as a complex manifold).

Example 7.12. The group $G = \mathrm{GL}_n(\mathbb{C})$ is a complex manifold as it is an open subset of $\mathbb{C}^{n\times n}$ given by $\det^{-1}(\mathbb{C}\setminus\{0\})$. Multiplication and inversion are holomorphic and its Lie algebra is $\mathfrak{g} = \mathbb{C}^{n\times n}$, the corresponding Lie bracket is $[X,Y] = XY - YX$. The exponential map is the usual matrix exponential $\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}$ and the adjoint action is given by conjugation $\mathrm{Ad}_g(X) = gXg^{-1}$ like in the real case.
See the corresponding example in [Lee02, Section 1.3].

A *complex Lie subgroup* of a complex Lie group G is a subgroup H which is also a complex submanifold of G, and that is again a complex Lie group with this structure. Then $H_{\mathbb{R}}$ is also a (real) Lie subgroup of $G_{\mathbb{R}}$ and therefore is closed by the Closed Subgroup Theorem. Furthermore, the exponential map of H is $\exp_H = \exp_G|_{\mathfrak{h}}$ where $\mathfrak{h}$ is the Lie algebra of H and Proposition 7.6 also implies that $\mathfrak{h}_{\mathbb{R}}$ is a real Lie subalgebra of $\mathfrak{g}_{\mathbb{R}}$. Consequently, $\mathfrak{h}$ must be a complex Lie subalgebra of $\mathfrak{g}$ (the isomorphism $\mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}$ of real vector spaces restricts to an isomorphism $\mathfrak{h}_{\mathbb{R}} \to \mathfrak{h}$).

Remark 7.13. In the real case to be a closed subgroup was already sufficient to be a submanifold which is again a Lie group with this structure. In the complex case, however, this is not true as the following simple example shows: The subgroup $(\mathbb{R}, +)$ of $(\mathbb{C}, +)$ is a closed and a real Lie subgroup, but surely $\mathbb{R} \subset \mathbb{C}$ is not a complex submanifold.

From Example 7.12 now directly follows:

Corollary 7.14. *Let G be a complex Lie subgroup of* $\mathrm{GL}_n(\mathbb{C})$*. Then its Lie algebra $\mathfrak{g}$ is a Lie subalgebra of $\mathbb{C}^{n\times n}$ with the Lie bracket $[X,Y] = XY - YX$, the exponential map is given by the matrix exponential* $\exp$*, and the adjoint action* Ad_g *is via conjugation.* $\square$

Example 7.15. Here are some examples of complex Lie subgroups of $\mathrm{GL}_n(\mathbb{C})$, they are taken from [Lee02, Section 1.3].

a) The group $G = \mathrm{SL}_n(\mathbb{C})$ is a complex Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ with Lie algebra $\mathfrak{g} = \{X \in \mathbb{C}^{n\times n} \mid \mathrm{tr}(X) = 0\}$. Note that it is connected.

b) The subgroup $\mathrm{B}_n(\mathbb{C}) \leq \mathrm{GL}_n(\mathbb{C})$ of upper triangular invertible matrices is a complex Lie subgroup. Its Lie algebra is the set of all upper triangular matrices in $\mathbb{C}^{n\times n}$.

c) Furthermore, the group $\mathrm{U}_n(\mathbb{C})$ of upper triangular matrices with 1 on the diagonal is also a complex Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$, which has the set of all upper triangular matrices with 0 on the diagonal as Lie algebra. Note that it is also a complex subgroup of $\mathrm{B}_n(\mathbb{C})$.

8 Riemannian manifolds

Here we introduce some additional structure on a manifold, a so called Riemannian metric. This can be used to study manifolds under lots of geometric aspects. But it also allows us to define more smooth structure like the gradient of a function.

8.1 Riemannian geometry

A good introduction to this topic is the book of do Carmo [dC92], another one is [Pet16]. We will only introduce some very basic terminology that we need in this thesis, because we want to view a Riemannian manifold as a metric space and work with the metric directly. To learn about all the other important aspects like the exponential function, curvature and so on, see the references already mentioned. The interplay between the curvature in the sense of the Riemannian manifold and in the sense of the induced metric space is analyzed in [BH99, Appendix to Chapter II.1].

Throughout the whole subsection let M be a smooth manifold of dimension m.

Definition 8.1. A *Riemannian metric* or a *Riemannian structure* on M is a map

$$\langle .,. \rangle : M \to \bigcup_{p \in M} \{T_pM \times T_pM \to \mathbb{R}\},\ p \mapsto \langle \_,\_ \rangle_p$$

such that $\langle \_,\_ \rangle_p$ is a scalar product on the real vector space T_pM and such that for all $X, Y \in \mathfrak{X}(M)$ the induced map $\langle X, Y \rangle : M \to \mathbb{R},\ p \mapsto \langle X_p, Y_p \rangle_p$ is smooth. We call a such a pair $(M, \langle .,. \rangle)$ a *Riemannian manifold.*

Every smooth manifold can be endowed with a Riemannian structure, see [dC92, Proposition 1.2.10]. Using the Riemannian structure we can define a lot of geometric terminology on the manifold M: For a point $p \in M$ and a tangent vector $v \in T_pM$ define the *norm* or *length* as $\|v\|_p = \sqrt{\langle v, v \rangle_p}$. Let $c\colon [a,b] \to M$ be a smooth curve, then the *arc length* of c is defined as $\ell(c) = \int_a^b \|c'(t)\|_{c(t)} dt$. Using this we can also define the *arc length* of a piecewise smooth curve $\gamma\colon [a,b] \to M$ by $\ell(\gamma) = \sum_{i=0}^k \ell(\gamma|_{[t_i,t_{i+1}]})$ if γ is smooth on all intervals $[t_i, t_{i+1}]$ for $i = 0, \dots, k-1$. Define the *arc length function* of γ by $s(t) = \ell(\gamma|_{[a,t]})$ for $t \in [a,b]$. Furthermore, we define the set

$$\Omega_{pg} = \{c\colon [0,1] \to M \mid c(0) = p, c(1) = q, c \text{ is piecewise smooth}\}$$

and set $d(p,q) = \inf\{\ell(c) \mid c \in \Omega_{pg}\}$ for any $p, q \in M$. This gives us a map $d\colon M \times M \to \mathbb{R}$. One can show that (M,d) is a metric space and that the topology on M induced by the metric d coincides with the original manifold topology. See [dC92, Propositions 7.2.5 and 7.2.6] for details.

Example 8.2. At this point we give some examples of Riemannian manifolds.

a) Let $M = \mathbb{R}^m$ together with the trivial chart $(\mathbb{R}^m, \mathrm{id})$. A Riemannian structure is given by $\left\langle \frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p \right\rangle_p = \delta_{ij}$ for all $p \in \mathbb{R}^m$. So for any piecewise smooth curve $c\colon [a,b] \to \mathbb{R}^m$ we have

$$\begin{aligned} \|c'(t)\|_{c(t)}^2 &= \left\langle \sum_{i=1}^m \dot{c}^i(t) \frac{\partial}{\partial x_i}|_{c(t)}, \sum_{j=1}^m \dot{c}^j(t) \frac{\partial}{\partial x_j}|_{c(t)} \right\rangle \\ &= \sum_{i,j=1}^m \dot{c}^i(t)\dot{c}^j(t) \left\langle \frac{\partial}{\partial x_i}|_{c(t)}, \frac{\partial}{\partial x_j}|_{c(t)} \right\rangle \\ &= \sum_{i=1}^m (\dot{c}^i(t))^2. \end{aligned}$$

Thus, the metric on $\mathbb{R}^m$ coming from the Riemannian structure coincides with the Euclidean metric. Compare [dC92, Example 1.2.4].

b) Let $f\colon M \to N$ be an immersion between smooth manifolds, where N is additionally assumed to be a Riemmannian manifold. Then $\langle u, v\rangle_p := \langle f_*(u), f_*(v)\rangle_{f(p)}$ also defines a Riemannian structure on M, compare [dC92, Example 1.2.5].
An application of this is the following: Consider the smooth map $h\colon \mathbb{R}^m \to \mathbb{R}$, $x \mapsto \left(\sum_{i=1}^m x_i^2\right) - 1$, then $\mathbb{S}^{m-1} = h^{-1}(0)$ is a regular submanifold by the implicit function theorem (Lemma 5.16). The inclusion $\mathbb{S}^{m-1} \hookrightarrow \mathbb{R}^m$ is an immersion by Example 5.15. So we can pull back the Riemannian structure on $\mathbb{R}^m$ via f like indicated above. The induced metric is the usual angular metric on $\mathbb{S}^{m-1}$, see [BH99, Proposition I.6.17].

c) In Example 5.2 we have seen that $\mathcal{H}^n = \Gamma(f) \subset \mathbb{R}^n \times \mathbb{R}$ is a smooth manifold for $f\colon \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \sqrt{\sum_{i=1}^n x_i^2 + 1}$ with an atlas consisting of one chart $\phi\colon \Gamma(f) \to \mathbb{R}^n$, $(x, f(x)) \mapsto x$. Thus, the graph $\Gamma(f) \subset \mathbb{R}^n \times \mathbb{R}$ is a regular submanifold of codimension 1 and the inclusion $\Gamma(f) \hookrightarrow \mathbb{R}^n \times \mathbb{R}$ is an immersion by Example 5.15. We define for $p \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ the symmetric bilinear form

$$\left\langle \frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p \right\rangle = \begin{cases} -1 & \text{if } i = n+1 = j \\ \delta_{ij} & \text{else.} \end{cases}$$

This surely is not positive-definite on $T_p(\mathbb{R}^{n+1})$ but the restriction to $T_p\mathcal{H}^n$ is for any $p \in \mathcal{H}^n$, see [BH99, p. 93] for details.
The metric induced by this Riemannian structure on $\mathcal{H}^n$ coincides with the hyperbolic metric as defined in c) of Example 3.4, see [BH99, Proposition I.6.17].

d) We also have seen that the product $M \times N$ of two smooth manifolds M, N is again a smooth manifold and one can show that the projection on each of the factors is a smooth map, see [Tu11, Example 6.12]. If both factors also have a Riemannian structure, then a Riemannian metric on the product is defined by

$$\langle u, v\rangle_{(p,q)} := \langle\, \mathrm{pr}_{M*}(u),\, \mathrm{pr}_{M*}(v)\rangle_p + \langle\, \mathrm{pr}_{N*}(u),\, \mathrm{pr}_{N*}(v)\rangle_q$$

for $(p, q) \in M \times N$ and $u, v \in T_{(p,q)}(M \times N)$. See [dC92, Example 1.2.7].

A diffeomorphism $f\colon M \to N$ between two Riemannian manifolds is called a *Riemannian isometry* if $\langle f_*(u), f_*(v)\rangle_{f(p)} = \langle u, v\rangle$ holds for all $u, v \in T_pM$. Further, f is a *local Riemannian isometry* if every point $p \in M$ has a neighborhood $U \subset M$ such that the restriction $f\mid_U$ is a Riemannian isometry. It is not hard to see that any Riemannian isometry is also an isometry in the sense of metric spaces. But the converse is also true by a theorem of Myers-Steenrod:

Proposition 8.3. *Let M, N be Riemannian manifolds and let $f\colon M \to N$ be a bijection with $d(f(p), f(q)) = d(p, q)$ for all $p, q \in M$. Then f is also a Riemannian isometry.*

Proof. See [Pet16, Theorem 5.6.15]. ☐

We call a Riemannian manifold *complete* if the induced metric space is complete. By the theorem of Hopf-Rinow a Riemannian manifold is complete if and only if the corresponding metric space is proper, see [dC92, Theorem 7.2.8].

The following provides a construction of a complete Riemannian manifold that we will work with later.

Example 8.4. a) The set $S_n(\mathbb{R})$ of symmetric $n \times n$ matrices with entries in $\mathbb{R}$ is a real vector space of dimension $\frac{n(n+1)}{2}$, and therefore the open subset $P_n(\mathbb{R})$ of all positive-definite matrices is a manifold of the same dimension. For every $p \in P_n(\mathbb{R})$ the tangent space $T_pP_n(\mathbb{R})$ is canonically given by $S_n(\mathbb{R})$ and $\langle X, Y\rangle_p = \mathrm{tr}(p^{-1}Xp^{-1}Y)$ for $X, Y \in S_n(\mathbb{R})$ defines a Riemannian metric structure on $P_n(\mathbb{R})$. The action of $\mathrm{GL}_n(\mathbb{R})$ on $P_n(\mathbb{R})$ given by $g.p = gpg^\top$ extends canonically to $S_n(\mathbb{R})$ via $g.X = gXg^\top$. It is smooth and via Riemannian isometries. Further, it is transitive and the stabilizer of the unit matrix $\mathbb{1}_n$ is exactly $\mathrm{O}(n)$, the orthogonal group. Therefore, we have a natural identification $\mathrm{GL}_n(\mathbb{R})/\,\mathrm{O}(n) \cong P_n(\mathbb{R})$ via $g\,\mathrm{O}(n) \mapsto gg^\top$. See [BH99, II.10.31-34] for all of these statements. The subgroup $\mathrm{O}(n)$ of $\mathrm{GL}_n(\mathbb{R})$ is maximal compact [BH99, Corollary II.10.41], and the Riemannian manifold $P_n(\mathbb{R})$ is a symmetric space by [BH99, Proposition II.10.34]. As a metric space it is proper and CAT(0) [BH99, Theorem II.10.39]. The geodesic lines $c\colon \mathbb{R} \to P_n(\mathbb{R})$ with $c(0) = p$ are exactly the maps $t \mapsto g\exp(tX)g^\top$ for $g \in \mathrm{GL}_n(\mathbb{R})$ fulfilling $gg^\top = p$ and $X \in S_n(\mathbb{R})$ with $\mathrm{tr}(X^2) = 1$. If $\xi = c(\infty)$ is the corresponding point in the boundary, then its stabilizer is

$$\mathrm{Stab}(\xi) = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid \lim_{t\to\infty} \exp(-tX)g\exp(tX) \text{ exists}\}.$$

Consider the special case of the above X being a diagonal matrix, and let $\lambda_1 > \ldots > \lambda_k$ be the different entries on the diagonal, such that every λ_i shows up exactly r_i times. Then $g \in \mathrm{GL}_n(\mathbb{R})$ is an element of $\mathrm{Stab}(\xi)$ if and only if

$$g = \begin{pmatrix} a_{11} & \ldots & \ldots & a_{1k} \\ 0 & a_{22} & \ldots & a_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & a_{kk} \end{pmatrix}$$

where a_{ij} is an $r_i \times r_j$ matrix. See [BH99, Proposition II.10.64]. The corresponding Busemman functions are all smooth by [BH99, Proposition II.10.69].

b) More generally, let $G \subset \mathrm{GL}_n(\mathbb{R})$ be a closed subgroup that is also closed under matrix transposition, that means $g^\top \in G$ holds for all $g \in G$. Then G is a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$. Further we assume that the group to have the following property:

$$\forall X \in S_n(\mathbb{R})\colon\ \exp(X) \in G \Rightarrow \exp(\mathbb{R}X) \subset G. \tag{8.1}$$

The space $M := G \cap P_n(\mathbb{R})$ is a totally geodesic Riemannian submanifold of $P_n(\mathbb{R})$, which is therefore itself a complete CAT(0) symmetric space. The subgroup $K := G \cap \mathrm{O}(n)$ is maximal compact and the maps $G/K \to M$ defined by $gK \mapsto gg^\top$ and $M \times K \to G$ defined by $(m, k) \mapsto mk$ are diffeomorphisms. Especially, M is the G orbit of $\mathbb{1}_n$. See [BH99, Theorem II.10.58] for all of this. We further see that the action of G on M is proper by Lemma 2.8 as the quotient map $G \to G/K$ is open.

A criterion for the property in (8.1) is provided by [BH99, Lemma II.10.59]: If $G = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid \forall f \in F\colon f(g) = 0\}$ for a finite set of polynomials F with real coefficients in $n \times n$ variables, then the desired property is fulfilled.

8.2 Vector fields defined on Riemannian manifolds

So far we only cared about metric aspects. But the Riemannian structure also has effects on smooth vector fields and enables us to construct the gradient of a function. We deal with that in the following.

Lemma 8.5. *Let M be a complete Riemannian manifold and let $X \in \mathfrak{X}(M)$ be a smooth vector field. If the norm on X is uniformly bounded, then X is a complete vector field.*

Proof. Let $C > 0$ be a constant such that $\|X\| \leq C$. Let further $c\colon I \to M$ be any integral curve of X starting at the point $c(0) \in M$. For every $t \in I \cap [0, \infty[$ we have

$$d(c(0), c(t)) \leq \int_0^t \langle c'(s), c'(s) \rangle_{c(s)} \leq C^2 \cdot t$$

where d is the metric on M induced by the Riemannian structure. Thus, if $T = \sup(I)$ were a real number, the image $c([0, T[)$ would be a subset of $\overline{B_{C^2T}(c(0))}$, which is compact by completeness. This contradicts Proposition 5.11. The same argument shows $]-\infty, 0] \subset I$. $\square$

For every smooth function $f \in C^\infty(M)$ on the Riemannian manifold M there is a unique vector field $\nabla f \in \mathfrak{X}(M)$ such that

$$\langle X, \nabla f \rangle = X(f)$$

for every $X \in \mathfrak{X}(M)$, see [Con01, Lemma 4.2.1.].

Definition 8.6. This vector field ∇f is called the *gradient* or *gradient vector field* of f with respect to the Riemannian metric.

Let us have a further look at the gradient vector field. When we write $\nabla f = \sum_{j=1}^m d^j \frac{\partial}{\partial x^j}$ in a chart (U, ϕ), then for any other vector field $X = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(U)$ we see

$$\langle X, \nabla f \rangle = \sum_{i,j=1}^m a^i d^j g_{ij} = \sum_{i=1}^m a^i \left(\sum_{j=1}^m g_{ij} d^j \right)$$

for $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$. On the other hand,

$$X(f) = \sum_{i=1}^m a^i \frac{\partial f}{\partial x^i}.$$

Since $X \in \mathfrak{X}(M)$ was arbitrary, this leads to the system of linear equations

$$\frac{\partial f}{\partial x^i} = \sum_{j=1}^m g_{ij} d^j \qquad \text{i=1,\ldots,m}$$

so the coefficients of the gradient are given by

$$\begin{pmatrix} d^1 \\ \vdots \\ d^m \end{pmatrix} = G^{-1} \cdot \begin{pmatrix} \frac{\partial f}{\partial x^1} \\ \vdots \\ \frac{\partial f}{\partial x^m} \end{pmatrix}$$

where $G = (g_{ij})_{ij=1,\ldots,m}$.

Remark 8.7. This shows especially that the above definition of the gradient is a generalization of the standard definition of the gradient on $\mathbb{R}^m$, because in this case G is the unit matrix.

One can further deduce that the gradient of constant functions is equal to 0. This and the following observation will be useful later.

Proposition 8.8. *Let M be a Riemannian manifold and let $f_j \in C^\infty(M)$ and $\lambda_j \in \mathbb{R}$ for $j = 1, \dots, m$.*

a) If $f = \sum_{j=1}^n \lambda_j f_j$, then $\nabla f = \sum_{j=1}^n \lambda_j \nabla f_j$.

b) If $f = f_1 \cdot f_2$, then $\nabla f = \nabla f_1 \cdot f_2 + f_1 \cdot \nabla f_2$.

c) If $f \colon M \to \mathbb{R}$ and $g \colon \mathbb{R} \to \mathbb{R}$ are smooth functions, then $\nabla(g \circ f) = (g' \circ f) \cdot \nabla f$.

Proof. It suffices to prove these statements in an arbitrary chart (U, ϕ). Let $\nabla f = \sum_{i=1}^m d^i \frac{\partial}{\partial x^i}$ and $\nabla f_j = \sum_{i=1}^m d_j^i \frac{\partial}{\partial x^i}$ in this chart. Furthermore, let $G = (g_{ij})_{ij=1,\dots,m}$ for $g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$.
If $f = \sum_{j=1}^n \lambda_j f_j$, then the fact $\frac{\partial f}{\partial x^i} = \sum_{j=1}^n \lambda_j \frac{\partial f_j}{\partial x^i}$ shows

$$\begin{pmatrix} d^1 \\ \vdots \\ d^m \end{pmatrix} = G^{-1} \cdot \begin{pmatrix} \frac{\partial f}{\partial x^1} \\ \vdots \\ \frac{\partial f}{\partial x^m} \end{pmatrix} = G^{-1} \cdot \sum_{j=1}^n \lambda_j \begin{pmatrix} \frac{\partial f_j}{\partial x^1} \\ \vdots \\ \frac{\partial f_j}{\partial x^m} \end{pmatrix} = \sum_{j=1}^n \lambda_j \begin{pmatrix} d_j^1 \\ \vdots \\ d_j^m \end{pmatrix}$$

which proves a).
But in a similar way assuming $f = f_1 \cdot f_2$ the fact $\frac{\partial f}{\partial x^i} = \frac{\partial f_1}{\partial x^i} \cdot f_2 + f_1 \cdot \frac{\partial f_2}{\partial x^i}$ shows

$$\begin{aligned} \begin{pmatrix} d^1 \\ \vdots \\ d^m \end{pmatrix} &= G^{-1} \cdot \begin{pmatrix} \frac{\partial f}{\partial x^1} \\ \vdots \\ \frac{\partial f}{\partial x^m} \end{pmatrix} \\ &= G^{-1} \cdot \left(\begin{pmatrix} \frac{\partial f_1}{\partial x^1} \\ \vdots \\ \frac{\partial f_1}{\partial x^m} \end{pmatrix} \cdot f_2 + f_1 \cdot \begin{pmatrix} \frac{\partial f_2}{\partial x^1} \\ \vdots \\ \frac{\partial f_2}{\partial x^m} \end{pmatrix} \right) \\ &= \left(G^{-1} \cdot \begin{pmatrix} \frac{\partial f_1}{\partial x^1} \\ \vdots \\ \frac{\partial f_1}{\partial x^m} \end{pmatrix} \right) \cdot f_2 + f_1 \cdot \left(G^{-1} \cdot \begin{pmatrix} \frac{\partial f_2}{\partial x^1} \\ \vdots \\ \frac{\partial f_2}{\partial x^m} \end{pmatrix} \right) \\ &= \begin{pmatrix} d_1^1 \\ \vdots \\ d_1^m \end{pmatrix} \cdot f_2 + f_1 \cdot \begin{pmatrix} d_2^1 \\ \vdots \\ d_2^m \end{pmatrix} \end{aligned}$$

and this proves b).
For c) we calculate

$$G^{-1} \cdot \begin{pmatrix} \frac{\partial(g \circ f)}{\partial x^1} \\ \vdots \\ \frac{\partial(g \circ f)}{\partial x^m} \end{pmatrix} = G^{-1} \cdot \begin{pmatrix} (g' \circ f) \cdot \frac{\partial f}{\partial x^1} \\ \vdots \\ (g' \circ f) \frac{\partial f}{\partial x^m} \end{pmatrix} = (g' \circ f) \cdot G^{-1} \cdot \begin{pmatrix} \frac{\partial f}{\partial x^1} \\ \vdots \\ \frac{\partial f}{\partial x^m} \end{pmatrix}$$

which proves that the coefficient vectors of $\nabla(g \circ f)$ and $(g' \circ f) \cdot \nabla f$ coincide.
□

Remark 8.9. It is possible to prove the proposition without choosing a coordinate system, but using the defining property of the gradient. Let $f = \sum_{i=1}^n \lambda_i f_i$ be a linear combination of smooth functions as in a) then

$$\begin{aligned} \langle X, \nabla f \rangle = X(f) = X\left(\sum_{i=1}^n \lambda_i f_i\right) &= \sum_{i=1}^n \lambda_i X(f_i) = \sum_{i=1}^n \lambda_i \langle X, \nabla f_i \rangle \\ &= \left\langle X, \sum_{i=1}^n \lambda_i \nabla f_i \right\rangle \end{aligned}$$

holds for all $X \in \mathfrak{X}(M)$. We obtain $\nabla f = \sum_{i=1}^n \lambda_i \nabla f_i$ from uniqueness. The proof of part b is similar: Again, let X be any smooth vector field then

$$\begin{aligned} \langle X, \nabla(f_1 f_2) \rangle &= X(f_1 f_2) = X(f_1) f_2 + f_1 X(f_2) \\ &= \langle X, \nabla f_1 \rangle \cdot f_2 + f_1 \cdot \langle X, \nabla f_2 \rangle \\ &= \langle X, \nabla f_1 \cdot f_2 \rangle + \langle X, f_1 \cdot \nabla f_2 \rangle \\ &= \langle X, \nabla f_1 \cdot f_2 + f_1 \cdot \nabla f_2 \rangle \end{aligned}$$

implies $\nabla(f_1 f_2) = \nabla f_1 \cdot f_2 + f_1 \cdot \nabla f_2$. In c) we get the assertion from

$$\begin{aligned} \langle X, \nabla(g \circ f) \rangle &= X(g \circ f) = (g \circ f)_*(X) = (g' \circ f) \cdot f_*(X) \\ &= (g' \circ f) \cdot X(f) = (g' \circ f) \cdot \langle X, \nabla f \rangle \\ &= \left\langle X, (g' \circ f) \cdot \nabla f \right\rangle . \end{aligned}$$

Here we write $f_*(X)$ for the induced map $p \mapsto f_{*,p}(X_p)$.

9 Root systems

This paragraph collects some facts about root systems. We fix a real vector space V. An automorphism $r_\alpha \in \mathrm{Aut}(V)$ with $r_\alpha(\alpha) = -\alpha$ that fixes a hyperplane H in V pointwise is called a *reflection* with respect to a vector $\alpha \neq 0_V$. This property makes it unique.

Definition 9.1. An *abstract root system* is a finite subset ϕ of V such that the following hold:

a) The set ϕ generates V as a real vector space and does not contain 0_V.

b) For every $\alpha \in \phi$ there is a reflection r_α that stabilizes ϕ as a set.

c) For all $\alpha, \beta \in \phi$ there is an integer $n_{\beta,\alpha} \in \mathbb{Z}$ with $r_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha$.

The elements of ϕ are called *roots*. We call the root system ϕ *reduced* if for all roots α and every $a \in [-1, 1]$ from $a\alpha \in \phi$ follows $a \in \{\pm 1\}$. We make the convention every root system that we consider is reduced. The group $W = W(\phi) := \langle r_\alpha \mid \alpha \in \phi\rangle$ is the *Weyl group* of ϕ. By b) of the definition above holds $w.\phi = \phi$ for all $w \in W$.

Lemma 9.2. *We can endow V with a scalar product $\langle .,. \rangle$, such that the Weyl group acts via orthogonal transformations, that is $\langle w.x, w.y\rangle = \langle x, y\rangle$ holds for all $w \in W$ and $x, y \in V$.*

Proof. See [Mil11, Proposition III.1.9]. $\square$

We refer to such a $\langle .,. \rangle$ as *Weyl-invariant scalar product*, and this makes $(V, \langle .,. \rangle)$ a Euclidean space. A subset ψ of ϕ is *closed* if $\alpha, \beta \in \psi \Rightarrow \alpha + \beta \in \psi$ holds. A *basis* of ϕ is a subset $\Delta \subset \phi$ such that

a) Δ is a basis of V (in the sense of vector spaces),

b) every $\beta \in \phi$ is a $\mathbb{Z}$-linear combination $\beta = \sum_{\alpha\in\Delta} m_{\alpha,\beta}\alpha$ such that all $m_{\alpha,\beta}$ are either non-negative or non-positive.

One can show that a basis always exists [Mil11, Proposition III.1.10]. Once it has been chosen we refer to the elements of Δ as *simple roots*. If a root $\beta = \sum_{\alpha\in\Delta} m_{\alpha,\beta}\alpha$ has all $m_{\alpha,\beta} \geq 0$ then we call it a *positive root*. We write ϕ^+ for the *set of positive roots*. Obviously we have $\Delta \subset \phi^+ \subset \phi$.

Lemma 9.3. *With the above notation the following are equivalent for a subset $\psi \subset \phi$:*

a) ψ is a closed subset and $\psi \cup -\psi$ is a partition of ϕ.

b) There is a vector $x \in V$ such that $\psi = \{\alpha \in \phi \mid \langle \alpha, x \rangle > 0\}$.

c) There is an order on V making it an ordered vector space such that each $\alpha \in \phi$ is either positive or negative and $\psi = V^+ \cap \phi$ for the set V^+ of positive elements.

Proof. See [Bou05, Corollaries 1 and 2 in VI.1.7]. ☐

Such a subset ψ we the call *system of positive roots in ϕ*. This is also a set of positive roots in the previous sense, because the set of all $\alpha \in \psi$ with $\alpha \neq \beta + \gamma$ for all $\beta, \gamma \in \psi$ is then a basis of the root system ϕ making ψ its set of positive roots. Consequently, systems of positive roots and bases of the root system are in one to one relation.

For all roots $\alpha, \beta \in \phi$ we have

$$\langle \alpha, \beta \rangle = \langle r_\alpha(\alpha), r_\alpha(\beta) \rangle = \langle -\alpha, \beta - n_{\beta,\alpha}\alpha \rangle = -\langle \alpha, \beta \rangle + n_{\beta,\alpha} \langle \alpha, \alpha \rangle$$

and from this follows

$$2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = n_{\beta,\alpha} \in \mathbb{Z}.$$

Thus $r_\alpha(\beta) = \beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha$. Furthermore, if α and β are different simple roots, then

$$\langle \alpha, \beta \rangle \leq 0,$$

because otherwise the root $r_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha$ would be a linear combination of two simple roots with coefficients of different signs. Thus we have:

Lemma 9.4. *For all simple roots $\alpha, \beta \in \Delta$ the following hold:*

a) $\langle \alpha, \beta \rangle \leq 0$ if $\alpha \neq \beta$,

b) $2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$ is an integer. ☐

One could also start with a Euclidean vector space, and then define a *root system* via reflections with respect to the corresponding scalar product. See for example [AB08, Appendix B].

We give one more definition and state two consequences about root systems from [Rag68].

Definition 9.5. For every simple root α we define the corresponding *fundamental weight* as the unique vector $\lambda_\alpha \in V$ such that $\langle \lambda_\alpha, \beta \rangle = \delta_{\alpha,\beta}$ holds for all $\beta \in \Delta$.

To simplify the notation, we write $\Delta = \{\alpha_1, \dots, \alpha_r\}$ for the set of simple roots. The fundamental weights are denoted by λ_j for $j = 1, \dots, r$ with the property $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$.

Lemma 9.6. *Let* I *be any subset of* $\{1, \ldots, r\}$*. Writing* $\lambda_j = \sum_{i \in I} a_i \alpha_i + \sum_{i \notin I} b_i \lambda_i$ *then all the* a_i *and* b_i *are greater than or equal to* 0*.*

Proof. See [Rag68, Lemma 1.1]. □

Lemma 9.7. *Let* $\Lambda = \sum_{j=1}^{r} m_j \lambda_j$ *be any linear combination of the fundamental weights with all* $m_j > 0$*. For every subset* I *of* $\{1, \ldots, r\}$ *we have* $\Lambda = \sum_{j \in I} m_{Ij} \lambda_j + \sum_{i \notin I} n_{Ii} \alpha_i$ *with all* $m_{Ij} \geq m_j > 0$ *and all* $n_{Ii} \geq 0$*.*
Setting $\Lambda_I = \sum_{j \in I} m_{Ij} \lambda_j$ *we get that* $\Lambda_{I \cup \{i\}} - \Lambda_I = \sum_{k=1}^{r} c_{Iik} \alpha_k$ *is always a non-negative linear combination of the simple roots. For* $i \notin I$ *we obtain* $c_{Iii} > 0$*.*

Proof. See [Rag68, Lemma 3.1 and the text preceding it]. □

10 Affine algebraic varieties

We give a very short and naive introduction to the topic, as we will only need few basic results. Here we are guided by the description in [CSM95, Section III.1]. A standard reference is [Spr98, Sections 1.1-1.3]. We will only consider affine varieties over the field of complex numbers $\mathbb{C}$.

Let $A_m = \mathbb{C}[T_1, \dots, T_m]$ be the ring of complex polynomials in m variables $T_1, \dots, T_m$. For a subset $F \subset A_m$ we define

$$V(F) = \{x \in \mathbb{C}^m \mid \forall f \in F\colon f(x) = 0\}$$

and call this an *algebraic subset* of $\mathbb{C}^m$. One easily deduces $V(\varnothing) = \mathbb{C}^m$ and $V(A_m) = \varnothing$. Furthermore, for a family $(F_i)_{i \in I}$ of subsets of A_m we have $\bigcap_{i \in I} V(F_i) = V(\bigcup_{i \in I} F_i)$. For two subsets $F, G \subset A_m$ also holds $V(F) \cup V(G) = V(F \cdot G)$, see [CSM95, III.(1.1)]. Thus, the algebraic subsets of $\mathbb{C}^m$ fulfill the axioms of a family of closed subsets of a topology on $\mathbb{C}^m$.

Definition 10.1. We call the topology on $\mathbb{C}^m$ whose closed sets are exactly the algebraic subsets the *Zariski topology.* A Zariski closed subset V of $\mathbb{C}^m$ endowed with the corresponding relative topology is called an *affine algebraic variety.*

It is immediate from the definition that the Zariski topology is coarser than the usual Euclidean topology on $\mathbb{C}^m$, as every Zariski closed subset of $\mathbb{C}^m$ must also be closed in the standard topology. For any subset $S \subset \mathbb{C}^m$ we set

$$I(S) = \{f \in A_m \mid \forall x \in S\colon f(x) = 0\}.$$

This is clearly an ideal in A_m, which is finitely generated by Hilbert's basis theorem.

Lemma 10.2. *The Zariski closure of a subset $S \subset \mathbb{C}^m$ is $V(I(S))$.*

Proof. We first prove that a set S is Zariski closed if and only if $V(I(S)) = S$. If $V(I(S)) = S$ holds, then S clearly is closed in the Zariski topology. So let $S = V(F)$ be Zariski closed, we show that this implies $S = V(I(S))$. On the one hand, if x is an element of S, then $f(x) = 0$ for all $f \in I(S)$ by definition, and therefore we have $x \in V(I(S))$. On the other hand, if y lies in $V(I(S))$, we obtain $f(y) = 0$ for all $f \in F$ from $F \subset I(S)$. So in this case we have $y \in V(F) = S$.
Obviously, if S is a subset of T we have $I(T) \subset I(S)$, and therefore $V(I(S)) \subset V(I(T))$ holds by [CSM95, (1.1)]. Thus, if T is the Zariski closure of S, then we get $T \subset V(I(S)) \subset V(I(T)) = T$, which implies $T = V(I(S))$. $\square$

If $X \subset \mathbb{C}^m$ is an algebraic variety, then $\mathbb{C}[X] = A_m / I(X)$ is the *coordinate ring* of X.

Remark 10.3. The usual definition of an affine algebraic variety goes as follows, see [CSM95, p.142]: A map $\varphi\colon X \to \mathbb{C}$ is called a *regular function* if there is a polynomial $f \in A_m$ with $\varphi(x) = f(x)$ for all $x \in X$. Via pointwise addition and multiplication the set $\mathbb{C}[X]$ of regular functions on X forms a $\mathbb{C}$-algebra. Every polynomial $f \in A_m$ canonically induces a regular function φ_f on X via $\varphi_f(x) = f(x)$ and one can show that the map $f \mapsto \varphi_f$ is a homomorphism of $\mathbb{C}$-algebras $A_m \to \mathbb{C}[X]$. Its kernel is exactly $I(X)$ so we get an isomorphism $A_m/I(X) \cong \mathbb{C}[X]$. Under this identification, evaluation at $x \in X$ defined by $\mathrm{ev}_x\colon \mathbb{C}[X] \to \mathbb{C}$, $\mathrm{ev}_x(f) = f(x)$ defines a map from X to $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C})$ the set of all $\mathbb{C}$-algebra morphisms, which in fact is a bijection. The closed subsets of X are then exactly

$$V_X(F) = \{x \in X \mid \forall f \in F \in\colon f(x) = 0\}$$

for $F \subset \mathbb{C}[X]$. More generally than before one calls the pair $(X, \mathbb{C}[X])$ an *affine algebraic variety.* But for our purposes, the previous naive definition is sufficient.

We give some more terminology. A *morphism* between two affine varieties $X \subset \mathbb{C}^m$ and $Y \subset \mathbb{C}^n$ is a map $X \to Y$ given by $x \mapsto (f_1(x), \ldots, f_n(x))$ for polynomials $f_1, \ldots, f_n \in A_m$. The morphism is an *isomorphism* if it is bijective and its inverse map is also a morphism of varieties. The product $X \times Y$ is again an affine variety and its coordinate ring is $\mathbb{C}[X \times Y] \cong \mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[Y]$, see [CSM95, III.(1.9)].

Recall now that a topological space is *irreducible* if it is not the union of two proper non-empty closed subsets.

Convention. For the rest of this section let $X \subset \mathbb{C}^m$ be an affine algebraic variety that is irreducible in the Zariski topology.

Lemma 10.4. *In the corresponding Euclidean topology (the relative topology coming from $\mathbb{C}^m$ endowed with the standard topology) X is connected.*

Proof. See [PR94, Theorem 3.5, p.118]. □

By [Spr98, Proposition 1.2.5] the ring $\mathbb{C}[X]$ is an integral domain. The *dimension* of X is the transcendence degree of the quotient field of $\mathbb{C}[X]$ over $\mathbb{C}$. We write $\dim(X)$ for its dimension.

Lemma 10.5. *Let $X, Y \subset \mathbb{C}^m$ be irreducible varieties, such that Y is a proper subset of X. Then* $\dim(Y) < \dim(X)$.

Proof. See [Spr98, Proposition 1.8.2]. □

From this also follows that $\dim(X) \leq m$ holds, because of $\dim(\mathbb{C}^m) = m$. Furthermore, if $X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n = X$ is an ascending chain of closed irreducible varieties, then $n \leq \dim(X)$.

A vector $v \in \mathbb{C}^m$ is a *tangent vector at* $x \in X$ if for all $f \in I(X)$ the polynomial $h(T) := f(x+Tv)$ is contained in the ideal of $\mathbb{C}[T]$ generated by T^2. The set of all such v is the *tangent space at* x, written as T_xX. By [Sha69, §II.1.2] a vector $v \in \mathbb{C}^m$ is a tangent vector at x if and only if

$$\forall f \in I(X)\colon d_xf(v) := \sum_{i=1}^{m} v_i \frac{\partial f}{\partial T_i}|_x = 0$$

where $\frac{\partial f}{\partial T_i}|_x$ is the formal derivative of the polynomial $f \in \mathbb{C}[T_1,\dots,T_m]$ evaluated at x. Since the map $v \mapsto d_xf(v)$ is linear, this shows that the tangent space is indeed a complex vector space. In fact, by Hilbert's basis theorem we can write $T_xX = \bigcap_{i=1}^{r} \ker(d_xf_i)$ for some polynomials $f_1,\dots,f_r$ generating $\mathbb{C}[X]$.

A point x of X is called *simple* if $\dim(X) = \dim(T_xX)$. We give a criterion for that.

Lemma 10.6. *A point $x \in X$ is simple if and only if there are polynomials $f_1,\dots,f_r \in \mathbb{C}[T_1,\dots,T_m]$, where $r = m - \dim(X)$, and a Zariski open subset $U \subset \mathbb{C}^m$ with $x \in \{y \in U \mid \forall i = 1,\dots,r\colon f_i(y) = 0\}$ and such that the complex matrix a with entries $a_{ij} = (\frac{\partial f_i}{\partial T_j}|_x)$ for $i = 1,\dots,r$ and $j = 1,\dots,m$ has rank r.*

Proof. See [PR94, Proposition 2.22, p.97]. □

The irreducible variety X is called *smooth* if every point $x \in X$ is simple. In fact, as every Zariski open subset U of $\mathbb{C}^m$ is also open in the Euclidean topology, the above lemma shows that a smooth complex variety X is a complex submanifold of $\mathbb{C}^m$ of codimension r.

11 Algebraic matrix groups

In this section we define algebraic matrix groups, which are a special case of so-called linear algebraic groups. In the first subsection we define them as a Zariski closed subgroup of some $\mathrm{GL}_n(\mathbb{C})$. As such we can also view them as subvarieties of some $\mathbb{C}^m$. Later we will see that this canonically induces a functor into the category of groups.

11.1 Algebraic matrix groups as complex linear groups

Here we mainly follow the presentation in [PR94, Section 2.1], another good introduction is given by [CSM95, Chapter III]. Standard references for linear algebraic groups (although by far too general for our goals) are [Bor91] and [Spr98].

Definition 11.1. We call a subgroup $\mathbf{G} \leq \mathrm{GL}_n(\mathbb{C})$ an *algebraic matrix group* if there is a subset F of $\mathbb{C}[(T_{i,j})_{i,j=1,\ldots,n}]$, the ring of polynomials in $n \times n$ variables, such that $\mathbf{G} = \{g = (g_{ij})_{i,j=1,\ldots,n} \in \mathrm{GL}_n(\mathbb{C}) \mid \forall f \in F \colon f(g) = 0\}$.

Equivalent to this definition is to say that $\mathbf{G}$ is (relatively) closed in the Zariski topology induced on $\mathrm{GL}_n(\mathbb{C}) \subset \mathbb{C}^{n\times n}$. A trivial example is $\mathbf{GL}_n = \mathrm{GL}_n(\mathbb{C})$ (here the set F can be chosen to be empty).

We want to view $\mathbf{G}$ not only as a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$, but also as an affine variety. Therefore, we use the inclusion

$$\iota\colon \mathrm{GL}_n(\mathbb{C}) \hookrightarrow \mathrm{GL}_{n+1}(\mathbb{C}),\ g \mapsto \begin{pmatrix} g & 0_{n\times 1} \\ 0_{1\times n} & \det(g^{-1}) \end{pmatrix}$$

whose inverse is clearly

$$\rho\colon \iota(\mathrm{GL}_n(\mathbb{C})) \to \mathrm{GL}_n(\mathbb{C}),\ x \mapsto (x_{ij})_{ij=1,\ldots,n}.$$

Under this identification $\mathrm{GL}_n(\mathbb{C})$ is the intersection of the two Zariski closed subsets

$$\{x \in \mathbb{C}^{(n+1)\times(n+1)} \mid \forall i,j = 1,\ldots,n \colon x_{i,n+1} = 0 = x_{n+1,j}\}$$

and

$$\{x \in \mathbb{C}^{(n+1)\times(n+1)} \mid \det((x_{ij})_{i,j=1,\ldots,n}) \cdot x_{n+1} - 1 = 0\}$$

and so $\iota(\mathbf{G}) \subset \mathrm{GL}_{n+1}(\mathbb{C})$ must be Zariski closed, too. It follows directly that we can write $\iota(\mathbf{G})$ as the subset

$$\begin{aligned} V\bigl(F &\cup \{x_{i,n+1} \mid i = 1,\ldots,n\} \\ &\cup \{x_{n+1,j} \mid j = 1,\ldots,n\} \\ &\cup \{\det((x_{i,j})_{i,j=1,\ldots,n})x_{n+1,n+1} - 1\}\bigr) \end{aligned}$$

of $\mathbb{C}^{(n+1)\times(n+1)}$ (here we view F as a set of polynomials in $(n+1)\times(n+1)$ variables using the canonical inclusion $\mathbb{C}[T_{1,1},\dots,T_{n,n}] \subset \mathbb{C}[T_{1,1},\dots,T_{n+1,n+1}]$). Thus, $\iota\colon \mathbf{G} \to \iota(\mathbf{G})$ is a homeomorphism with respect to the induced Zariski topologies, which is inverse to $\rho\colon \iota(\mathbf{G}) \to \mathbf{G}$.

Remark 11.2. As closed subgroups of $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}_{n+1}(\mathbb{C})$ the groups $\mathbf{G}$ and $\iota(\mathbf{G})$ are in fact Lie groups, and the identification maps ι and ρ provide isomorphisms of Lie groups. The identification therefore transfers not only algebraic but also topological structures. We shall see later that $\mathbf{G}$ is even a complex Lie group holomorphic to $\rho(\mathbf{G})$.

The *coordinate ring* of an algebraic matrix group $\mathbf{G}$ is

$$I[\mathbf{G}] = \mathbb{C}[(T_{ij})_{i,j=1,\dots,n+1}]/I(\mathbf{G})$$

where

$$I(\mathbf{G}) = \{f \in \mathbb{C}[(T_{ij})_{i,j=1,\dots,n+1}] \mid f(\iota(\mathbf{G})) = 0\}.$$

A *morphism (of algebraic groups)* between two algebraic matrix groups $\mathbf{G} \subset \mathrm{GL}_n(\mathbb{C})$ and $\mathbf{H} \subset \mathrm{GL}_m(\mathbb{C})$ is a group homomorphism that is defined by polynomials, that is $g \mapsto (f_{i,j}(g))_{i,j=1,\dots,m}$ for some complex polynomials $f_{i,j}$ in $n \times n$ variables. A morphism is an *isomorphism (of algebraic groups)* if it is an isomorphism of groups and its inverse is also given by polynomials.

Remark 11.3. This also means that any (iso-)morphism between algebraic matrix groups induces an (iso-)morphism of the corresponding affine varieties. As already mentioned before, our terminology of an algebraic matrix group is a special case of a so-called linear algebraic group. The most naive definition of that would be like above, but considering isomorphism classes of algebraic matrix groups. However, we do not want to do that here, because we will later work with a concrete subgroup of some $\mathrm{GL}_n(\mathbb{C})$.
Another way to define linear algebraic groups is to start with a group whose underlying set is an affine variety and whose multiplication and inversion are morphisms of varieties. But all such groups embed into a closed subgroup of some $\mathbf{GL}_n$, see for example [Bor91, Proposition 1.10].

Let now k be a subfield of $\mathbb{C}$ (for example $k = \mathbb{Q}$.) The algebraic matrix group $\mathbf{G}$ is *defined over k* or a *k-group* if the ideal $I(\mathbf{G})$ is generated by elements in $k[(T_{i,j})_{i,j=1,\dots,n+1}]$. A morphism $\mathbf{G} \to \mathbf{H}$ between two k-groups is said to be *defined over k* or a *k-morphism* if it is given by polynomials with coefficients in k.

Example 11.4. We give some examples of algebraic matrix groups. All the examples are defined over any subfield $k \subset \mathbb{C}$. Compare [Bor91, Example 1.6] and [Spr98, Example 2.1.4 (4)].

a) As already mentioned, the general linear group $\mathbf{GL}_n = \mathrm{GL}_n(\mathbb{C})$ is an algebraic matrix group. It is clearly defined over k.

b) The special linear group

$$\mathbf{SL}_n = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid \det(g) - 1 = 0\}$$

is also an algebraic matrix group defined over k. As an abstract group it is (canonically isomorphic to) $\mathrm{SL}_n(\mathbb{C})$.

c) Let $F = \{T_{ij} \mid 1 \leq j < i \leq n\} \cup \{T_{ii} - 1 \mid 1 \leq i \leq n\}$ then

$$\mathbf{U}_n = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid \forall f \in F\colon f(g) = 0\}$$

is the group $\mathrm{U}_n(\mathbb{C})$ of upper triangular matrices with 1 on the diagonal.

d) The set $F = \{T_{ij} \mid i,j = 1,\ldots,n,\ i \neq j\}$ defines the group of all diagonal matrices $\mathbf{D}_n = \mathrm{D}_n(\mathbb{C})$.

e) For $F = \{T_{ij} \mid 1 \leq j < i \leq n\}$ we obtain the group $\mathbf{B}_n = \mathrm{B}_n(\mathbb{C})$ of upper triangular matrices.

f) Considering a special case of a) we define the multiplication group $\mathbf{G}_m = \mathrm{GL}_1(\mathbb{C})$. As an abstract group it is isomorphic to $(\mathbb{C}^\times, \cdot)$, the group of units.

g) Setting $n = 2$ in c) we obtain $\mathbf{G}_a = \mathrm{U}_2(\mathbb{C})$, which is isomorphic to the additive group $(\mathbb{C}, +)$.

h) Let

$$J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \in \mathbb{C}^{2n\times 2n}$$

then the *symplectic group*

$$\mathbf{Sp}_{2n} = \{g \in \mathrm{GL}_{2n}(\mathbb{C}) \mid gJg^\top = J\}$$

is an algebraic matrix group defined over k.

i) Let $S \in \mathrm{GL}_n(k)$ be any symmetric matrix, then the *orthogonal group*

$$\mathbf{O}(S) = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid gSg^\top = S\}$$

is also an algebraic matrix group defined over k.

Definition 11.5. A subgroup $\mathbf{H}$ of an algebraic matrix group $\mathbf{G} \leq \mathrm{GL}_n(\mathbb{C})$ is an *algebraic matrix subgroup* if it is closed under the induced Zariski topology. Therefore it is also called a *closed subgroup*. We call a closed subgroup *normal* if it is normal as an abstract subgroup.

Example 11.6. If $\mathbf{G}$ is an algebraic matrix group containing a closed subgroup $\mathbf{H}$, then its *centralizer*

$$\mathrm{Cen}_{\mathbf{G}}(\mathbf{H}) := \{g \in \mathbf{G} \mid \forall h \in \mathbf{H}\colon gh = hg\}$$

and its *normalizer*

$$\mathrm{Nor}_{\mathbf{G}}(\mathbf{H}) := \{g \in \mathbf{G} \mid g\mathbf{H}g^{-1} = \mathbf{H}\}$$

are again closed subgroups of $\mathbf{G}$. See [Bor91, 1.7].

We also make the following construction: The cartesian product of two algebraic matrix groups $\mathbf{G} \subset \mathrm{GL}_n(\mathbb{C})$ and $\mathbf{H} \subset \mathrm{GL}_m(\mathbb{C})$ embeds as an abstract group into $\mathrm{GL}_{n+m}(\mathbb{C})$ via $(g,h) \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & h \end{smallmatrix}\right)$. Therefore, we define the *product* $\mathbf{G} \times \mathbf{H}$ as the resulting algebraic matrix group in $\mathrm{GL}_{n+m}(\mathbb{C})$.

Example 11.7. The product $\mathbf{G}_m \times \mathbf{G}_m$ equals $\mathbf{D}_2$. Consequently, the n-fold product $(\mathbf{G}_m)^n := \mathbf{G}_m \times \ldots \times \mathbf{G}_m$ is $\mathbf{D}_n$.

This leads to the following important definition.

Definition 11.8. An *algebraic torus* is an algebraic matrix group $\mathbf{T}$ that is isomorphic to $(\mathbf{G}_m)^d$ for some $d \in \mathbb{N}$. We call this the *dimension* of $\mathbf{T}$ and write $d = \dim(\mathbf{T})$.
The torus is *k-split* or a *k-split torus* if $\mathbf{T}$ is a k-group and if there is an isomorphism $\mathbf{T} \cong (\mathbf{G}_m)^d$ defined over k.

Thus, the group $\mathbf{D}_n$ is a k-split torus of dimension n and any (k-split) torus of the same dimension is (k-)isomorphic to it. Closely connected to tori are the characters.

Definition 11.9. A *character* of an algebraic matrix group $\mathbf{G}$ is a morphism $\chi\colon \mathbf{G} \to \mathbf{G}_m$. If $\mathbf{G}$ is a k-group and χ is defined over k, then we call it a *k-character*. We write $X(\mathbf{G})$ for the set of characters and $X_k(\mathbf{G})$ for the set of k-characters.

Lemma 11.10. *A k-torus $\mathbf{T}$ is k-split if and only if $X(\mathbf{T}) = X_k(\mathbf{T})$.*

Proof. See [Spr98, Proposition 3.2.12]. ☐

An algebraic matrix group is *diagonalizable* if it is isomorphic to a closed subgroup of some $\mathbf{D}_n$. Its characters then form an abelian group via $(\chi + \psi)(g) := \chi(g) \cdot \psi(g)$, see [Spr98, Theorem 3.2.3]. We call an algebraic matrix group *connected* if it is connected in the Zariski topology.

Lemma 11.11. *Let $\mathbf{T}$ be a diagonalizable group. The following are equivalent:*

(i) $\mathbf{T}$ *is a torus,*

(ii) $\mathbf{T}$ *is connected,*

(iii) $X(\mathbf{T})$ *is isomorphic to some* $\mathbb{Z}^d$.

Proof. [Spr98, Corollary 3.2.7]. □

Consequently, if $\mathbf{T}$ is a torus with $X(\mathbf{T}) \cong \mathbb{Z}^d$, there are characters $\chi_1, \dots, \chi_d$ such that any $\chi \in X(\mathbf{T})$ can be written uniquely as a $\mathbb{Z}$-linear combination $\sum_{k=1}^{d} n_k \chi_k$. This also makes the dimension of the torus well-defined since every homomorphism $\mathbf{T} \to (\mathbf{G}_m)^n$ can be written as $t \mapsto (\psi_1(t), \dots, \psi_n(t))$ for characters ψ_i.

Let us take a deeper look into what the property of being connected means. In fact, every algebraic matrix group $\mathbf{G}$ has a unique irreducible component $\mathbf{G}_0$ containing the identity element, which is a finite index closed normal subgroup. This is also the connected component with $1 \in \mathbf{G}_0$. See [Spr98, Proposition 2.2.1]. Thus we get the following:

Corollary 11.12. *An algebraic matrix group* $\mathbf{G}$ *is connected if and only if it is irreducible as an algebraic variety.*
Consequently, if it is connected as an algebraic matrix group, then the following two statements hold:

a) It is also connected in the Euclidean topology.

b) There is a uniform bound on the length of chains $\mathbf{H}_0 \subsetneq \mathbf{H}_1 \subsetneq \dots \subsetneq \mathbf{H}_n = \mathbf{G}$ *of proper connected closed subgroups.*

Proof. Follows directly from Lemma 10.4 and 10.5. □

We can now define the *dimension* of a connected algebraic matrix group as the dimension of the corresponding variety. By [Bor91, Proposition 8.5] the dimension of a torus as defined above is the same as its dimension as an algebraic matrix group.

Convention: For the rest of this subsection $\mathbf{G}$ denotes a connected algebraic matrix group.

Lemma 11.13. *Let* $(\mathbf{G}_i)_{i \in I}$ *be a family of closed connected subgroups of* $\mathbf{G}$. *Then the subgroup* $\mathbf{H}$ *generated by all of them is again closed and connected. If* $\mathbf{G}$ *and all the* $\mathbf{G}_i$ *are defined over* k, *then the same holds for* $\mathbf{H}$.

Proof. See [Spr98, Corollary 2.2.7]. □

We say that an algebraic matrix group $\mathbf{G}$ is the *semidirect product* of two closed connected subgroups $\mathbf{N}$ and $\mathbf{H}$, if $\mathbf{N}$ is normal in $\mathbf{G}$, the intersection $\mathbf{N} \cap \mathbf{H}$ is trivial and $\mathbf{G} = \mathbf{N} \cdot \mathbf{H}$.

We introduce further terminology. An algebraic matrix group is *unipotent* if all of its elements are unipotent in the usual sense of linear algebra. It is *solvable* if it is solvable as an abstract group. By [Spr98, 6.4.14] there is a maximal connected solvable normal closed subgroup $R(\mathbf{G})$, the *radical*. There is also a maximal connected unipotent normal closed subgroup $R_u(\mathbf{G})$, the *unipotent radical*.

Definition 11.14. A connected algebraic matrix group $\mathbf{G}$ is *semisimple* if $R(\mathbf{G})$ is trivial. It is *reductive* if $R_u(\mathbf{G}) = \{1\}$.

Since any unipotent group is solvable by [Spr98, Corollary 2.4.13], every semisimple group is reductive.

Definition 11.15. A maximal closed connected solvable subgroup $\mathbf{B}$ of $\mathbf{G}$ is called *Borel subgroup*. A *maximal torus* in $\mathbf{G}$ is a closed subgroup $\mathbf{T}$, that is a torus and maximal with that property.

Borel subgroups and maximal tori always exist by Corollary 11.12. If $\mathbf{B}$ is a Borel subgroup of $\mathbf{G}$, then there is a $x \in \mathrm{GL}_n(\mathbb{C})$ such that $x\mathbf{B}x^{-1}$ is a subgroup of $\mathbf{B}_n$ as defined in Example 11.4, see [Spr98, Theorem 6.3.1].

Lemma 11.16. *Every connected solvable group is semidirect product of its unipotent radical and a maximal torus.*

Proof. See [Bor91, Theorem 10.6]. □

Thus we can write any Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ as $R_u(\mathbf{B}) \cdot \mathbf{T}$. In fact, as every torus of $\mathbf{G}$ is connected and abelian, it must lie in some Borel subgroup, and therefore the maximal torus of $\mathbf{B}$ is also a maximal torus of $\mathbf{G}$. The canonical inclusion $\iota \colon \mathbf{T} \hookrightarrow \mathbf{B}$ provides a natural homomorphism $X(\mathbf{B}) \to X(\mathbf{T})$ via $\chi \mapsto \chi \circ \iota$. By [Spr98, Theorem 2.4.8 (ii)] every character χ of $\mathbf{B}$ maps any $u \in R_u(\mathbf{B})$ to 1, so χ is uniquely determined by its values on $\mathbf{T}$. On the other hand, using the structure of the semidirect product, we can extend every character from $\mathbf{T}$ to $\mathbf{B}$. This makes the homomorphism bijective and we will identify $X(\mathbf{B})$ with $X(\mathbf{T})$ under this isomorphism.

Definition 11.17. A closed subgroup $\mathbf{P}$ of $\mathbf{G}$ containing a Borel subgroup is called *parabolic*. If $\mathbf{G}$ and $\mathbf{P}$ are defined over k, then we call the latter a *parabolic k-subgroup*.

It is clear from definition that every Borel subgroup is also parabolic. In fact, all parabolic subgroups are connected by [Bor91, Theorem 11.16].

However, even if $\mathbf{G}$ is a k-group, it is not always the case that it has a Borel subgroup that is also defined over k. If $\mathbf{G}$ is a reductive k-group and contains a Borel subgroup $\mathbf{B}$ defined over k, then we call it *k-quasisplit*. In this case, the unipotent radical of $\mathbf{B} = R_u(\mathbf{B}) \cdot \mathbf{T}$ is also defined over k by [Spr98, 12.1.7 (d)].

Definition 11.18. We say that a reductive k-group $\mathbf{G}$ is *k-split* or a *k-split group* if it contains a maximal k-torus, which is k-split.

To say more about the structure of Borel groups and parabolic groups, we need terminology of the Lie algebra of an algebraic matrix group. Therefore we show that it has the structure of a complex Lie group: The *tangent space* at $g \in \mathbf{G}$ is the tangent space $T_g\mathbf{G}$ viewing the algebraic matrix group as an algebraic variety. We can describe this as $T_g\mathbf{G} = \bigcap_{i=1}^{r} \ker(d_g f_i)$ for some polynomials $f_1, \dots, f_r$ generating $I(\mathbf{G})$ (see the previous section). By [Spr98, Theorem 4.3.7] the algebraic variety $\mathbf{G}$ is smooth, so Lemma 10.6 shows that it is a complex manifold. Therefore it is a complex Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$. Its tangent space as a complex Lie group coincides with the one as an algebraic variety by [PR94, Lemma 3.1, p.113]. Thus, invoking Corollary 7.14 we get:

Lemma 11.19. *The algebraic matrix group $\mathbf{G}$ is a complex Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$. Its tangent space at the identity $\mathfrak{g}$ is a complex Lie subalgebra of $\mathbb{C}^{n\times n}$ with Lie bracket $[X, Y] = XY - YX$. The map $\mathbf{G} \to \mathrm{GL}(\mathfrak{g})$ defined by $g \mapsto \mathrm{Ad}_g$ for $\mathrm{Ad}_g(X) = gXg^{-1}$ is well defined and a homomorphism of (abstract) groups.* □

Definition 11.20. We call $\mathfrak{g}$ the *Lie algebra* of the algebraic matrix group $\mathbf{G}$, and $g \mapsto [\mathrm{Ad}_g \colon X \mapsto gXg^{-1}]$ is the *adjoint representation.*

If $\mathbf{G}$ is a k-group we can choose the polynomials f defining the tangent space $T_1\mathbf{G} = \mathfrak{g}$ so that they have coefficients in k. In that case, $d_1 f$ also induces a k-linear map $k^{n\times n} \to k$ via $v \mapsto \sum_{i,j=1}^{n} v_{ij} \frac{\partial f}{\partial T_{ij}}|_1$, so the *$k$-points* of the Lie algebra $\mathfrak{g}_k := \mathfrak{g} \cap k^{n\times n}$ canonically form a k-vector space of the same dimension as the $\mathbb{C}$-vector space $\mathfrak{g}$. In fact, it is a Lie subalgebra of $k^{n\times n}$ because $[X, Y] = XY - YX$ is again an element of $k^{n\times n}$ for $X, Y \in \mathfrak{g}_k$.

Lemma 11.21. *The connected algebraic matrix group $\mathbf{G}$ is semisimple if and only if its Lie algebra is semisimple.*

Proof. See [Mil11, Theorem II.5.23]. □

Convention: From now on, in addition to the previous assumption of $\mathbf{G}$ being connected, we will also assume that it is semisimple. Fix a maximal torus $\mathbf{T}$ in $\mathbf{G}$.

It is useful to keep this fact in mind for the following: The maximal torus $\mathbf{T}$ equals its own centralizer in $\mathbf{G}$, see [Spr98, Corollary 7.6.4].

Definition 11.22. A *root* is a character $\alpha \in X(\mathbf{T})$ with $\alpha \neq 0$ such that the *weight space*

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid \forall t \in \mathbf{T}\colon\ \mathrm{Ad}_t(X) = \alpha(t) \cdot X\}$$

is not equal to $\{0_\mathfrak{g}\}$. We call

$$\phi := \phi(\mathbf{T}, \mathbf{G}) = \{\alpha \in X(\mathbf{T}) \mid \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0\}$$

the corresponding *root system.*

This terminology makes sense because of the following:

Lemma 11.23. *The set* $\phi = \phi(\mathbf{T}, \mathbf{G})$ *is an abstract root system in the vector space* $V = \mathbb{R} \otimes_\mathbb{Z} X(\mathbf{T})$. *The root system is reduced.*

Proof. The connected component of the center of $\mathbf{G}$ is the radical $R(\mathbf{G})$ by [Spr98, Proposition 7.3.1]. As we assumed $\mathbf{G}$ to be semisimple, the statement follows from [Bor91, Theorem 14.8]. ☐

Let $\mathfrak{t}$ be the Lie algebra of the torus $\mathbf{T}$. The Lie algebra of $\mathbf{G}$ splits as a direct sum $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \phi} \mathfrak{g}_\alpha$, see [Bor91, Theorem 13.18 (1)].

We consider now a Borel group $\mathbf{B} = \mathbf{U} \cdot \mathbf{T}$ with $\mathbf{U}$ its unipotent radical. There is an ordering on the abelian group $X(\mathbf{T})$, such that for the corresponding subset $\phi^+ \subset \phi$ of positive elements the Lie algebra of $\mathbf{B}$ is given by $\mathfrak{b} = \bigoplus_{\alpha \in \phi^+} \mathfrak{g}_\alpha \oplus \mathfrak{t}$. See [Bor91, Theorem 13.18 (5), d)+e)]. In fact, for every $\alpha \in \phi^+$ there is a unique closed connected subgroup $\mathbf{U}_\alpha$ normalized by $\mathbf{T}$, which has $\mathfrak{g}_\alpha$ as its Lie algebra [Bor91, Theorem 13.18 4) (d)]. These $\mathbf{U}_\alpha$ generate the unipotent radical $\mathbf{U}$ of $\mathbf{B}$ [Spr98, Proposition 8.2.1]. Furthermore, there is a subset Δ of ϕ^+, which is a basis of the root system ϕ and which makes ϕ^+ be the set of positive roots (in the sense of root systems). See [Bor91, Corollary 1 to Theorem 14.8].

Definition 11.24. We refer to Δ as the set of *simple roots* and ϕ^+ is the set of *positive roots* of the root system ϕ corresponding to the Borel group $\mathbf{B}$.

On the other hand, having any system of positive roots $\psi \subset \phi$ we get a Borel subgroup generated by $\mathbf{T}$ and the $\mathbf{U}_\alpha$ with $\alpha \in \psi$, see [Spr98, Proposition 8.2.4].

Convention: Assume now additionally that $\mathbf{G}$ is defined over k and $\mathbf{T}$ is a maximal k-torus, which is k-split. Fix any system of positive roots $\phi^+ \subset \phi$.

Because of $X_k(\mathbf{T}) = X(\mathbf{T})$ all we said so far also applies to the k-split case.

Lemma 11.25. *For every root $\alpha \in \phi$ there is a unique closed connected unipotent subgroup $\mathbf{U}_\alpha$ of $\mathbf{G}$ defined over k, which has $\mathfrak{g}_\alpha$ as Lie algebra. It is normalized by $\mathbf{T}$.*
More generally, if ψ is a closed subset of ϕ^+, then there is a unique closed connected unipotent k-subgroup $\mathbf{U}_\psi \leq \mathbf{G}$, such that its Lie algebra is $\bigoplus_{\alpha\in\psi}\mathfrak{g}_\alpha$. It is the subgroup generated by all the $\mathbf{U}_\alpha$ with $\alpha \in \psi$.

Proof. See [Bor91, Proposition 21.9]. □

Obviously, $\psi = \phi^+$ is a closed subset of ϕ (in the sense of root systems). The corresponding group $\mathbf{U}_{\phi^+}$ is then the unipotent radical $\mathbf{U}$ of a Borel group $\mathbf{B} = \mathbf{U} \cdot \mathbf{T}$, which is defined over k. Consequently, if an algebraic matrix group $\mathbf{G}$ is k-split, then it is also k-quasisplit.

We now use the construction in [Bor91, 21.11]: For a subset $I \subset \Delta$ we define the *standard parabolic* k-subgroup $\mathbf{P}_I$ as the semidirect product $\mathbf{T}_I \cdot \mathbf{U}_{\psi(I)}$. Here $\mathbf{T}_I \leq \mathbf{G}$ is the subgroup generated by $\mathbf{T}$ and the $\mathbf{U}_\alpha$ for which α is a linear combination of the $\beta \in I$ (including negative coefficients), and $\psi(I) = \phi^+ \smallsetminus \phi_I$ for $\phi_I = \{\sum_{\beta\in\Delta} m_\beta\beta \in \phi^+ \mid \forall\beta \notin I\colon m_\beta = 0\}$. As all the $\mathbf{U}_\alpha$ and $\mathbf{T}$ are connected and defined over k, the same holds for $\mathbf{P}_I$.
The Lie algebra of the group $\mathbf{P}_I$ is given by

$$\mathfrak{p}_I = \bigoplus_{\beta\in\phi^+}\mathfrak{g}_\beta \oplus \mathfrak{t} \oplus \bigoplus_{\beta\in\phi_I}\mathfrak{g}_{-\beta},$$

compare [Rag72, 12.4].

Definition 11.26. In the case $I = \varnothing$ we get back the Borel subgroup $\mathbf{P}_\varnothing = \mathbf{B}$, we call it a *minimal parabolic* k-subgroup. In the future we will refer to such a group as $\mathbf{P}$ instead of $\mathbf{B}$.
For every simple root $\alpha \in \Delta$ we call $\mathbf{P}_{\Delta\smallsetminus\{\alpha\}}$ the corresponding *maximal parabolic* k-subgroup of $\mathbf{G}$. To simplify notation, we will denote it by $\mathbf{P}_\alpha := \mathbf{P}_{\Delta\smallsetminus\{\alpha\}}$.

The Lie algebra of $\mathbf{P}_\alpha$ is

$$\mathfrak{p}_\alpha = \bigoplus_{\beta\in\phi^+}\mathfrak{g}_\beta \oplus \mathfrak{t} \oplus \bigoplus_{\beta\in\phi_{\Delta\smallsetminus\{\alpha\}}}\mathfrak{g}_{-\beta}.$$

In fact, the construction above provides us with a complete list of parabolic groups containing $\mathbf{P}$ by [Bor91, Proposition 21.12]. These are sometimes called *standard parabolic* k-subgroups. But as no other parabolic subgroups are considered in this thesis, we will not use that terminology.

Lemma 11.27. *Let $\mathbf{G}$ be a connected semisimple k-split group. Then the following hold:*

a) All its k-split tori are conjugate under an element of $\mathbf{G} \cap \mathrm{GL}_n(k)$.

b) All its minimal parabolic k-subgroups are conjugate under $\mathbf{G} \cap \mathrm{GL}_n(k)$.

Proof. See [Bor91, Theorem 20.9]. □

We will say something more about the structure of our semisimple group $\mathbf{G}$. It has only finitely many non-trivial closed connected normal subgroups $\mathbf{G}_1, \ldots, \mathbf{G}_r$ and is an *almost direct product* of those: The product map $\mathbf{G}_1 \times \ldots \times \mathbf{G}_r \to \mathbf{G}$ is surjective and has finite kernel. See [PR94, Proposition 2.4, p.62].

Definition 11.28. A semisimple algebraic matrix group $\tilde{\mathbf{G}}$ is *simply connected* if it has the following property: Every surjective homomorphism of algebraic groups from a connected algebraic matrix group $\mathbf{H}$ to $\tilde{\mathbf{G}}$ whose kernel is finite is already an isomorphism.

There is always a simply connected group $\tilde{\mathbf{G}}$ and a homomorphism of algebraic groups $\pi\colon \tilde{\mathbf{G}} \to \mathbf{G}$ having finite kernel $\mathbf{F}$. This π is the *universal covering* and $\mathbf{F}$ is the *fundamental group* of $\mathbf{G}$. A simply connected group is the direct product of its minimal closed connected normal subgroups $\mathbf{G}_1, \ldots, \mathbf{G}_r$. For both statements see [PR94, Theorem 2.6, p.62].

Example 11.29. As an example, $\mathbf{SL}_n$ is simply connected by [PR94, Example on p.63]. Just note that it is not true in the sense of abstract groups: $\mathrm{SL}_n(\mathbb{C})$ has its center as a proper normal subgroup. One can show that this is not connected as an algebraic matrix group.

Since our group $\mathbf{G}$ is semisimple and defined over k, it is the almost direct product of its minimal closed connected normal subgroups defined over k by [Bor91, Theorem 22.10]. Their number is finite by the above. We refer to them as the *k-simple factors* of $\mathbf{G}$.

Definition 11.30. A semisimple group defined over k is called *k-simple* if it has no non-trivial closed connected normal subgroups over k.

11.2 Algebraic matrix groups as functors

A good introduction into this is provided by the book of Waterhouse [Wat79].

To avoid confusion with the notation in the last subsection, we let $\mathbf{H} \leq \mathrm{GL}_n(\mathbb{C})$ denote a connected algebraic matrix group defined over a field $k \subset \mathbb{C}$. For any subring $B \subset \mathbb{C}$ we define the *B-points* of our group as $\mathbf{H}(B) := \mathbf{H} \cap \mathrm{GL}_n(B)$, where $\mathrm{GL}_n(B)$ denotes as usual the set of regular matrices g, such that g and g^{-1} lie in $B^{n \times n}$. Thus, we can define the $\mathbb{Z}$-points, $\mathbb{Q}$-points

and $\mathbb{R}$-points of $\mathbf{H}$, and we directly see $\mathbf{H} = \mathbf{H}(\mathbb{C})$. To generalize this, we also define $\mathbf{H}(R)$ for a k-algebra R using the following construction:
Let J_k be the ideal of all polynomials $f \in k[(T_{ij})_{i,j=1,\dots,n}]$ vanishing on $\mathbf{H}(k)$, that means $f(g) = 0$ holds for all $g \in \mathbf{H}(k)$. We define

$$\mathbf{H}_k(R) := \{g \in R^{n\times n} \mid \forall f \in J_k \colon f(g) = 0\}.$$

If $\mathbf{H}(\mathbb{C}) = \mathbf{H}_k(\mathbb{C})$ holds, then we call $\mathbf{H}$ to be *determined by its k-points.* By Lemma 10.2 this is equivalent to $\mathbf{H}(k)$ being Zariski dense in $\mathbf{H}(\mathbb{C}) = \mathbf{H}$ But that is already the case because of [Bor91, Corollary 18.3], compare also [Bro89, VII Appendix L]. Consequently, any connected group defined over k is determined by its k-points.

Example 11.31. As an example, $\mathbf{SL}_n$ is determined by its k-points.
A non-example is given by the following: Consider the group $\mathbf{H} = \{z \in \mathbb{C} \mid z^3 - 1 = 0\}$ of third roots of unity. For its rational points we get $\mathbf{H}(\mathbb{Q}) = \{x \in \mathbb{Q} \mid x^3 - 1 = 0\}$ the trivial group, so $(x-1)$ generates $J_{\mathbb{Q}}$. From this follows $\mathbf{H}_{\mathbb{Q}}(\mathbb{C}) = \{1\} \neq \mathbf{H}(\mathbb{C})$, so this group is not determined by its $\mathbb{Q}$-points.
See [Bro89, Appendix §D] for both examples.

Definition 11.32. For any k-algebra R we define *the set of R-points* of $\mathbf{H}$ as $\mathbf{H}(R) := \mathbf{H}_k(R)$.

As $\mathbf{H}$ is determined by its k-points, this definition is compatible with the former one in the case $R \subset \mathbb{C}$.

Lemma 11.33. *The mapping $R \mapsto \mathbf{H}(R)$ defines canonically a functor from the category of k-algebras to the category of groups. It can be represented by the element $k[\mathbf{H}_k] := k[(T_{ij})_{i,j=1,\dots,n}]/J_k$ via the canonical identification $R \to \operatorname{Hom}_{k-alg}(k[\mathbf{H}_k], R)$ given by $x \mapsto \operatorname{ev}_x$.
A k-morphism $f \colon \mathbf{H} \to \mathbf{K}$ between two k-groups canonically extends to a natural transformation between the induced functors, such that the maps $f_R \colon \mathbf{H}(R) \to \mathbf{K}(R)$ are homomorphisms of groups.*

Proof. The Zariski topology on $k^{n\times n}$ is induced by that on $\mathbb{C}^{n\times n}$, see [Wat79, §4.1]. The assertion now follows directly from [Wat79, §4.4]. ☐

We will therefore identify every connected algebraic matrix group $\mathbf{H}$ with its corresponding functor.

Remark 11.34. It is also possible to define algebraic groups directly as a functor from k-algebras to groups, where homomorphisms are then given by natural transformations between those functors. See [Wat79, chapter 1].
However, like in the case of varieties, this does not lead new objects, as every such group functor embeds into some $\mathbf{GL}_n$ by[Wat79, §3.4].

11.3 The Iwasawa decomposition of a real algebraic matrix group

Here we collect some facts about algebraic matrix groups that we will need later.

First we recall the *Iwasawa decomposition* of $G = \mathrm{GL}_n(\mathbb{R})$: Let $K = \mathrm{O}(n)$, let $A \subset G$ be the diagonal subgroup with all entries greater than 0, and let $N \subset G$ be the set of all upper triangular matrices with 1 on the diagonal. Then the multiplication map $N \times A \times K \to G$ is a homeomorphism, see for example [PR94, Proposition 3.12, p.129].

Lemma 11.35. *Let* $\mathbf{G}$ *be a reductive algebraic matrix group defined over* $\mathbb{R}$. *Then there is a matrix* $x \in \mathrm{GL}_n(\mathbb{R})$ *such that for the* $\mathbb{R}$*-group* $\mathbf{G}' := x\mathbf{G}x^{-1}$ *the following hold:*

a) $\mathbf{G}'$ *is self adjoint, that means* $g^{\top} \in \mathbf{G}'$ *holds for every* $g \in \mathbf{G}'$.

b) The group $\mathbf{G}'$ *contains a maximal* $\mathbb{R}$*-split torus* $\mathbf{S}$*, which lies in the group* $\mathbf{D}_n$ *of diagonal matrices.*

c) Let $\phi' = \phi(\mathbf{S}, \mathbf{G}')$ *be the corresponding set of roots. It contains a system of positive roots* $\psi = \phi'^{+}$*, whose elements are exactly those* $(\varepsilon_i - \varepsilon_j)$ *for* $1 \leq i < j \leq n$*, which appear as roots. Here* ε_i *is the character* $\mathrm{diag}(d_1 \ldots, d_n) \mapsto d_i$*. Furthermore, the corresponding unipotent* $\mathbb{R}$*-subgroup* $\mathbf{N}' = \mathbf{U}_{\psi}$ *lies in the group of upper triangular matrices* $\mathbf{U}_n$.

d) The components of the Iwasawa decomposition of $\mathrm{GL}_n(\mathbb{R})$ *of each element* $g \in \mathbf{G}'(\mathbb{R})$ *lie again in* $\mathbf{G}'(\mathbb{R})$.

Proof. See [PR94, Proposition 3.13, p.133]. □

Our point c) slightly differs from the one in the source cited above. Compare the proof of [PR94, Lemma 3.7, p.131] and recall the fact that choosing an ordering on the vector space corresponds to a choice of positive roots [Bou05, Section VI.1.7].

Lemma 11.36. *With the notation from above, let* $G' = \mathbf{G}'(\mathbb{R})$, $N' = G' \cap N$, $A' = G' \cap A$ *and* $K' = G' \cap K$*. Then the product map* $N' \times A' \times K' \to G'$ *is a homeomorphism.*

Proof. See [PR94, Theorem 3.9, p.131]. □

12 Adeles, adelic groups and S-arithmetic groups

Fix an algebraic number field k, that means $k = \mathbb{Q}$ or a finite extension of the rational numbers. We closed the last section by explaining how to define the R-points of an algebraic matrix group for a k-algebra R. Here we construct a specific ring, the ring of adeles, which leads to the definition of adelic groups and S-arithmetic subgroups of $\mathbf{G}(k)$.

12.1 Absolute values

Following [PR94, Sections 1.1 and 1.2] we give a short introduction to absolute values and recall some foundations about that topic.

An *absolute value* on k is a map $|.| : k \to \mathbb{R}_{\geq 0}$ that has the following properties:

a) The restriction to the group of units is a group homomorphism from $(k^{\times}, \cdot)$ to $(\mathbb{R}_{>0}, \cdot)$,

b) $|0| = 0$,

c) the triangle inequality $|x + y| \leq |x| + |y|$ holds for all $x, y \in k$.

It induces a metric on k and thus makes it a topological field such that the induced map $x \mapsto |x|$ is continuous and extends continuously to the completion $\hat{k}$. An absolute value is called *non-Archimedean* or *finite* if we can replace c) by the stronger property

c') $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in k$.

Otherwise, it is called *Archimedean* or *infinite.* Two absolute values $|.|_1$ and $|.|_2$ are *equivalent*, if $|.|_1 = |.|_2^a$ for some $a > 0$. Such an equivalence class is called *place* and we denote with V the set of all places of k. Obviously, equivalence preserves the property of being (non-)Archimedean, so we can say that a place is or not. We denote by V^{∞} the set of infinite places and by V^f the set of finite places of k. For every place $v \in V$ we pick one representative $|.|_v$ such that the *product formula* $\prod_{v \in V} |x|_v = 1$ holds for every $x \in k^{\times}$. We can do so by [HW22, Proposition 3.1], see also [PR94, p.12]. The corresponding completion of k is denoted by k_v. Viewing this as a topological space it becomes locally compact. Furthermore, we write $\mathcal{O}$ for the ring of integers of k and we write $\mathcal{O}_v$ for its completion with respect to $v \in V^f$. The stronger form of the triangle inequality makes it a ring; in fact we have $\mathcal{O}_v = \{x \in k_v \mid |x| \leq 1\}$ and its group of units is $\mathcal{O}_v^{\times} = \{x \in k_v \mid |x| = 1\}$.

Example 12.1. On the field $\mathbb{Q}$ of rational numbers we have the *standard absolute value*

$$|q|_\infty := \begin{cases} q & \text{if } q \geq 0 \\ -q & \text{if } q < 0 \end{cases}$$

which is Archimedean and the corresponding completion is the field of real numbers $\mathbb{R}$.
Furthermore, for every prime p we can define the *p-adic absolute value* by

$$\left|p^k \frac{a}{b}\right|_p := p^{-k} \text{ for } a, b, k \in \mathbb{Z} \text{ with } b > 0 \text{ and } p \nmid a, b.$$

The corresponding completion is the field $\mathbb{Q}_p$ of *p-adic numbers*.
It is a well known fact that these are the only places of $\mathbb{Q}$, see [PR94, Theorem 1.1]. With the notation above we have $\mathcal{O} = \mathbb{Z}$ and $\mathcal{O}_p = \mathbb{Z}_p$ for every prime p. From $|p|_q = 1$ for all primes $q \neq p$ one easily deduces the product formula for $\mathbb{Q}$.

12.2 Adeles

Here we are guided by [PR94, Section 1.2.1] and [HW22, Section 3.1].

If S is a finite subset of V with $V^\infty \subset S$ we define the *ring of S-adeles* as product of topological rings

$$\mathbb{A}_S := \prod_{v \in S} k_v \times \prod_{v \in V^f \smallsetminus S} \mathcal{O}_v.$$

Just note that the notation in [PR94] is slightly different. In the special case $S = V^\infty$ we write $\mathbb{A}_\infty$ instead of $\mathbb{A}_{V^\infty}$. If S is a subset of $T \subset V$, then $\mathbb{A}_S$ is a subspace of $\mathbb{A}_T$ and we define the *ring of adeles* as the colimit

$$\mathbb{A} := \operatorname{co}\lim_S \mathbb{A}_S.$$

Thus, an element of $\mathbb{A}$ is of the form $x = (x_v)_{v \in V}$ with $x_v \in \mathcal{O}_v$ for all but finitely many $v \in V^f$. One could also see $\mathbb{A}$ as a subset of $\prod_{v \in V} k_v$ (only as a set, not as a topological subspace!). Since every $x \in k$ lies in all but finitely many of the $\mathcal{O}_v$ we can embed k diagonally into $\mathbb{A}$ via $a \mapsto (a)_{v \in V}$. Identifying k with its image we can view it as a subset of $\mathbb{A}$ that is discrete by [PR94, Proposition 1.5, p.11]. Furthermore, $\mathbb{A}$ becomes a k algebra via left multiplication $a.x = (ax_v)_{v \in V}$. We define the *idele norm* $\|.\| := \prod_{v \in V} |\cdot|_v$ on $\mathbb{A}^\times$. This is well defined, since for any $x = (x_v)_{v \in V} \in \mathbb{A}^\times$ all but finitely many of the x_v lie in $\mathcal{O}_v$ and therefore in $\mathcal{O}_v^\times = \{a \in k_v \mid |a|_v = 1\}$. The product formula then tells us: For every $a \in k$ we have $\|a\| = \prod_{v \in V} |a|_v = 1$.

Finally, fixing a finite set $S \subset V$ containing V^∞ we define the *ring of S-integers*

$$\mathcal{O}_S := \{x \in k \mid \forall v \in V \smallsetminus S \colon x \in \mathcal{O}_v\} = k \cap \bigcap_{v \in V \smallsetminus S} \mathcal{O}_v.$$

If $S = V^\infty$ the $\mathcal{O}_S$ is just the ring of integers $\mathcal{O}$.

Example 12.2. Extending Example 12.1 we see that the ring of $\mathbb{Q}$-adeles is the set of all $x = (x_\infty, (x_p)_{p\in\mathbb{P}}) \in \mathbb{R} \times \prod_{p\in\mathbb{P}} \mathbb{Q}_p$ with all but finitely many x_p in $\mathbb{Z}_p$. Consider the set $S = \{|.|_\infty, |.|_{p_1}, \dots, |.|_{p_s}\}$ of places of $\mathbb{Q}$ for some primes $p_1, \dots, p_s \in \mathbb{Z}$. The corresponding ring of S-adeles is $\mathbb{R} \times \mathbb{Q}_{p_1} \times \ldots \times \mathbb{Q}_{p_s} \times \prod_{p\notin\{p_1,\dots,p_s\}} \mathbb{Z}_p$, the corresponding ring of S-integers is $\mathbb{Z}_S = \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_s}]$.

12.3 Adelic groups

Sources are [HW22, Section 3.2] and [PR94, Section 5.1].

Let $\mathbf{G} \leq \mathrm{GL}_n(\mathbb{C})$ be a connected algebraic matrix group defined over k, which we identify with its corresponding group functor on the category of k-algebras. We have seen above that the ring of adeles is a k algebra so we can consider the *group of adelic points* $\mathbf{G}(\mathbb{A})$. By definition, an element $g \in \mathbf{G}(\mathbb{A})$ is a matrix in $\mathrm{GL}_n(\mathbb{A})$ that fulfills the polynomial equations defining $\mathbf{G}$. The entries of g are tuples in the product $\prod_{v\in V} k_v$ (with all but finitely many of them in $\mathcal{O}_v$). Consequently, using $\mathbf{G}(\prod_{v\in V} k_v) = \prod_{v\in V} \mathbf{G}(k_v)$ we can view $\mathbf{G}(\mathbb{A})$ as a subset of $\prod_{v\in V} \mathbf{G}(k_v)$. Just note that this is not a topological embedding. Setting

$$\mathbf{G}(\mathcal{O}_v) := \mathbf{G}(k_v) \cap \mathrm{GL}_n(\mathcal{O}_v)$$

regularity of an element $g \in \mathbf{G}(\mathbb{A}) \subset \mathrm{GL}_n(\mathbb{A})$ means $\det(g) = \det((g_v)_{v\in V}) \in \mathbb{A}^\times$, that is $\det(g_v) \in \mathcal{O}_v^\times$ for almost all $v \in V^f$. Thus, there is a finite subset S of V containing V^∞ and such that g is an element of

$$\mathbf{G}(\mathbb{A}_S) := \prod_{v\in S} \mathbf{G}(k_v) \times \prod_{v\in V^f \setminus S} \mathbf{G}(\mathcal{O}_v).$$

We endow this with the product topology induced by the topological fields k_v. As in the case of $\mathbb{A}$, we can describe the group of adelic points as the colimit of topological groups

$$\mathbf{G}(\mathbb{A}) = \operatorname*{co\,lim}_{S} \mathbf{G}(\mathbb{A}_S)$$

and the corresponding topology on $\mathbf{G}(\mathbb{A})$ coincides with the one induced by $\mathbb{A}$. We also write

$$\mathbf{G}(\mathbb{A}_\infty) = \prod_{v\in V^\infty} \mathbf{G}(k_v) \times \prod_{v\in V^f} \mathbf{G}(\mathcal{O}_v).$$

Via the diagonal embedding $q \mapsto (q)_{v\in V}$ we can view $\mathbf{G}(k)$ as a subgroup of $\mathbf{G}(\mathbb{A})$, which is discrete like in the case of $k \subset \mathbb{A}$.

Lemma 12.3. *For every k-character χ holds $\|\chi(q)\| = 1$ for all $q \in \mathbf{G}(k)$.*

Proof. The character is a k-morphism $\mathbf{G} \to \mathbf{G}_m$ and induces a homomorphism of groups $\mathbf{G}(k) \to \mathbf{G}_m(k) = k^\times$. Thus, $\|\chi(q)\| = 1$ follows from the product formula. $\square$

Furthermore, for any subset $S \subset V$ we define

$$\mathbf{G}_S := \mathrm{pr}_S(\mathbf{G}(\mathbb{A}))$$

where $\mathrm{pr}_S \colon \mathbf{G}(\mathbb{A}) \to \prod_{v \in S} \mathbf{G}(k_v)$ denotes the canonical projection. We endow $\mathbf{G}_S$ with the corresponding quotient topology. If S is finite, this coincides with the product topology and we have $\mathbf{G}_S = \prod_{v\in S} \mathbf{G}(k_v)$.

Definition 12.4. The *class number* $\mathrm{cl}(\mathbf{G})$ is the number of double coset classes $\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A})/\mathbf{G}(\mathbb{A}_\infty)$.

It is always finite by [PR94, Theorem 5.1, p.251]. We want to work with groups having class number equal to one, an example is $\mathbf{G} = \mathbf{SL}_n$ by [Bor63, Proposition 2.2]. We give a criterion for this to check: An algebraic matrix group $\mathbf{G}$ has the *absolute strong approximation property* if $\mathbf{G}(k)$ is dense in $\mathbf{G}_{V\smallsetminus V^\infty}$. Every group with this property has $\mathrm{cl}(\mathbf{G}) = 1$ by [PR94, Proposition 5.4, p.251].

Lemma 12.5. *Let $\mathbf{G}$ be a simply connected and semi-simple algebraic matrix group defined over k. If $\prod_{v\in V^\infty} \mathbf{G}_i(k_v)$ is non-compact for every of the k-simple components $\mathbf{G}_i$ of $\mathbf{G}$, then it has class number $\mathrm{cl}(\mathbf{G}) = 1$.*

Proof. By [PR94, Theorem 7.12, p.427] such a group has the absolute strong approximation property.

12.4 S-arithmetic groups

Fix a finite subset S containing V^∞. We give three different descriptions of a subgroup $\mathbf{G}(\mathcal{O}_S)$ of $\mathbf{G}(k)$, all leading to the same object:

1. $\mathbf{G}(\mathcal{O}_S) := \mathbf{G}(k) \cap \bigcap_{v\in V\smallsetminus S} \mathbf{G}(\mathcal{O}_v)$ as a subgroup of $\mathbf{G}(k)$,
2. $\mathbf{G}(\mathcal{O}_S) := \mathbf{G}(k) \cap \mathbf{G}(\mathbb{A}_S)$ as a subgroup of $\mathbf{G}(\mathbb{A}_S)$,
3. $\mathbf{G}(\mathcal{O}_S) := \mathrm{pr}_S(\mathbf{G}(k) \cap \mathbf{G}(\mathbb{A}_S))$ as a subgroup of $\mathbf{G}_S$.

We identify $\mathbf{G}(\mathcal{O}_S)$ in the sense of 1 with the group in 2 via the diagonal embedding $\mathbf{G}(k) \hookrightarrow \mathbf{G}(\mathbb{A})$ and then also with the set in description 3 applying the projection pr_S which is injective on $\mathbf{G}(k)$. We will not distinguish between these three objects in the following, but will always use the appropriate description tacitly.

Definition 12.6. The group $\mathbf{G}(\mathcal{O}_S)$ defined above is called the *S-arithmetic subgroup* of $\mathbf{G}(k)$. If $S = V^\infty$ we refer to this as the *arithmetic subgroup.*

Note that in the special case $S = V^\infty$ the above construction just gives us back the group $\mathbf{G}(\mathcal{O})$. Since $\mathbf{G}(k) \subset \mathbf{G}(\mathbb{A})$ is discrete, the S-arithmetic subgroup $\mathbf{G}(\mathcal{O}_S) = \mathbf{G}(k) \cap \mathbf{G}(\mathbb{A}_S)$ becomes a discrete subset of $\mathbf{G}(\mathbb{A}_S)$. Furthermore, since $\mathbf{G}(\mathbb{A}_S)$ is the product $\mathbf{G}_S \times \prod_{v \in V \smallsetminus S} \mathbf{G}(\mathcal{O}_v)$ and all the $\mathbf{G}(\mathcal{O}_v)$ are compact, $\mathbf{G}(\mathcal{O}_S)$ is also a discrete subgroup of $\mathbf{G}_S$.

Example 12.7. Consider $\mathbf{G} = \mathbf{SL}_n$ and $k = \mathbb{Q}$. The corresponding arithmetic subgroup is

$$\mathbf{SL}_n(\mathbb{Z}) = \mathbf{SL}_n(\mathbb{Q}) \cap \bigcap_{p \in \mathbb{P}} \mathbf{SL}_n(\mathbb{Z}_p).$$

For $S = \{|.|_\infty, |.|_p\}$ we obtain

$$\mathbf{SL}_n(\mathbb{Z}[1/p]) = \mathbf{SL}_n(\mathbb{Q}) \bigcap_{q \neq p} \mathbf{SL}_n(\mathbb{Z}_q).$$

13 Euclidean buildings

We give a short introduction to Euclidean buildings as chamber complexes and as geometric objects, and state a few facts we need about them. Our approach here will be a little unusual, as we will not introduce Coxeter complexes to define them. We do it this way because we do not need the theory of Euclidean buildings in detail in this work, but only some of their properties as proper CAT(0) spaces. So we only introduce the necessary terms. A standard reference for buildings is the book of Abramenko and Brown [AB08].

13.1 Buildings as chamber complexes

An *(abstract) simplicial complex* is a non empty, partially ordered set $(\Delta, \leq)$ such that $\leq$ fulfills the following conditions:

a) For all $A, B \in \Delta$ there is a maximal lower bound in Δ.

b) For every $A \in \Delta$ the set $\Delta_{\leq A} := \{B \in \Delta \mid B \leq A\}$ is order-isomorphic to $(\mathcal{P}(\{1, \ldots, r\}), \subset)$ for an integer $r \geq 0$.

An element $A \in \Delta$ is called *simplex* and we call the number r of b the *dimension* of A. A subset B of A is a *face* and a simplex of dimension 0 is a *vertex*. We will denote by $\mathcal{V}(\Delta)$ the *set of vertices* of Δ.
A subset Δ' of Δ is called a *subcomplex* if it fulfills $A \in \Delta' \Rightarrow \Delta_{\leq A} \subset \Delta'$. Any subcomplex is again a simplicial complex. Two simplicial complexes $(\Delta_1, \leq_1)$ and $(\Delta_2, \leq_2)$ are *isomorphic* if they are isomorphic as partially ordered sets.
A *chamber complex* is a simplicial complex whose maximal simplices (these are called *chambers*) all have the same dimension $\ell \in \mathbb{N}$, and such that any two chambers c, c' are connected by a *gallery*: This means that there is a finite sequence of chambers $c = c_0, \ldots, c_n = c'$ where two consecutive ones have a common face of dimension $\ell - 1$. We call a chamber complex *thin*/*thick*, if every simplex of dimension $\ell - 1$ is a face of exactly two/at least three chambers. The chamber complex is *locally finite* if any simplex of such dimension is the face of only finitely many chambers. We call it *colorable* if there is a set I with $|I| = \ell + 1$ and a map $\varphi\colon \mathcal{V}(\Delta) \to I$ such that the restriction to $\mathcal{V}(c)$ is bijective for all chambers c.

Definition 13.1. A *thick building* is a thick chamber complex Δ together with a family $\mathcal{A}$ of thin subcomplexes $\Sigma \subset \Delta$ such that:

a) For all simplices $a, b \in \Delta$ there is a $\Sigma \in \mathcal{A}$ containing both.

b) If Σ and Σ' are both elements of $\mathcal{A}$ containing two simplices $a, b \in \Delta$, then there is an isomorphism $\Sigma \to \Sigma'$ fixing a and b.

The elements $\Sigma \in \mathcal{A}$ are called *apartments*.

By [AB08, Theorem 4.131] this definition of a thick building coincides with the usual one, as given for example in [AB08, Definition 4.1]. A thick building is always colorable and the isomorphism $\Sigma \to \Sigma'$ above can always be chosen type-preserving [AB08, Proposition 4.6].

We make the following convention: When we talk about an action of a group on a thick building, then the group acts via type-preserving simplicial automorphisms.

Definition 13.2. Let G be a group.

a) We call the action of G on a thick building Δ *strongly transitive* if the induced action on pairs (c, Σ) of chambers c and apartments Σ containing it, is transitive.

b) Let now B and N be subgroups of G such that G is generated by $B \cup N$ and such that $T := B \cap N$ is normal in N. We call (B, N) a *BN-pair* of G if there is a generating set S of $W := N/T$ fulfilling the following:

   (i) For all $s \in S$ and $w \in W$ we have $sBw \subset BswB \cup BwB$,

   (ii) for all $s \in S$ holds $sBs^{-1} \not\subset B$.

Proposition 13.3. *A group G has a BN-pair if and only if it acts strongly transitive on a thick building Δ. In this case B is the stabilizer of a chamber c and N fixes an apartment Σ containing c. This building is unique up to isomorphism.*

Proof. See [AB08, Theorem 6.56]. □

In the light of the above proposition we denote by $\Delta(G, B)$ the building associated to a group G with a BN-pair (B, N).

13.2 Euclidean buildings as geometric objects

So far we have defined buildings as abstract chamber complexes, but in fact we want to view them as geometric objects. We use a construction from [BH99].

A *geodesic n-simplex* or *geodesic simplex* of *dimension n* is the convex hull of $n+1$ linearly independent points in the Euclidean space $\mathbb{R}^m$ for some $m \geq n+1$. This means an n-simplex is the set

$$c = \left\{ \sum_{k=1}^{n+1} \lambda_k v_k \;\middle|\; \sum_{k=1}^{n+1} \lambda_k = 1 \text{ and all } \lambda_k \geq 0 \right\}$$

for $v_1, \dots, v_{n+1} \in \mathbb{R}^m$ linearly independent vectors. Let I be an index set, we define

$$X := \coprod_{i \in I} c \Big/ \sim$$

where $\sim$ is an equivalence relation, that fulfills the following two properties:

a) If $(p, i) \sim (q, j)$ for some $i, j \in I$, then $p = q$ holds for the two points $p, q \in c$.

Note that this implies: For every $i \in I$ the map $\iota_i \colon c \to X$ given by $\iota_i(p) = [(p, i)]$ is injective.

b) If $\iota_i(c) \cap \iota_j(c) \neq \varnothing$ for some $i \neq j$, then there is a proper, non-empty subset $F \subset \{1, \dots, n+1\}$ such that this intersection is given by the set $\iota_i(\{p \in c \mid p = \sum_{k \in F} \lambda_k v_k\})$.

In a canonical way, this provides us the *geometric realization* $X = |\Delta|$ (as a set, not as a topological space in the usual sense) of a simplicial complex Δ whose set of vertices is

$$\mathcal{V}(\Delta) = \{[v_k, i] \mid k = 1, \dots, n+1, \ i \in I\}.$$

This construction is a special case of a piecewise Euclidean simplicial complex as defined in [BH99, Definition I.7.2]. In [BH99, I.7.4], a metric d on X is constructed such that the canonical inclusions $\iota_i \colon c \hookrightarrow X$ given by $\iota_i(p) = (p, i)$ are isometric embeddings.

Definition 13.4. Assume that the simplicial complex Δ constructed above carries the structure of a thick building. If $|\Sigma|$ is isometric to the Euclidean space $\mathbb{R}^n$ for every $\Sigma \in \mathcal{A}$, then Δ is called a *Euclidean building.* Further we call a BN-pair of a group G *Euclidean* if the corresponding building is.

Our definition of a Euclidean building coincides with the usual one, compare [BH99, II.10A.1 + 10A.4]. We will also refer to the corresponding metric space $X = |\Delta|$ as a *Euclidean building.*

Proposition 13.5. *As a metric space, (X, d) is a complete* CAT(0) *space, which is proper if Δ is locally finite (in this case X also carries the canonical topology of the geometric realization).*

Proof. See [BH99, Theorem II.10A.4] and [Kra22, Theorem 3.7]. □

Further one can show that every type-preserving automorphism of Δ induces an isometry of X (see for example [AB08, Section 11.2]).

Lemma 13.6. *If a group G acts chamber-transitive on a Euclidean building X, then there are a point $o \in X$ and a real number $d > 0$ such that $G.B_d(o) = X$.*

Proof. By construction, every chamber is compact, so the action of G on X is cocompact. ☐

Lemma 13.7. *Let G be a topological group with a Euclidean BN-pair and let $H \leq G$ be a maximal compact subgroup. Let $X = |\Delta(G, B)|$ be the corresponding Euclidean building. Then H is the stabilizer of a vertex.*

Proof. Follows directly from [AB08, Theorem 11.34], see also the remark below it. ☐

13.3 Algebraic groups and Euclidean buildings

The standard reference for this are the articles by Bruhat and Tits [BT72] and [BT84]. But as they are written in French (and in much more generality than we need here), we refer to the survey article of Tits [Tit79].

Proposition 13.8. *Let $\mathbf{G}$ be a semisimple, simply connected algebraic matrix group defined over $k = \mathbb{Q}$. Let $v \in V^f$ be a finite place and let $G = \mathbf{G}(k_v)$. Then the following hold:*

a) G contains a Euclidean BN-pair.

Let X be the corresponding Euclidean building viewed as a metric space.

b) The action of G on X is proper.

c) X is locally compact.

Proof. For a) see [Tit79, 3.1.1], for b) and c) see [Tit79, §2.2]. ☐

Remark 13.9. In fact, Tits first gives the construction of the Euclidean building X on which G acts in [Tit79, §2], and then he shows that in the case we described above the stabilizer of a chamber and that of an apartment form a BN-pair. The terminology of Euclidean buildings that he uses is much more general (it is not a simplicial but a polysimplicial complex), but the existence of a BN-pair ensures us that it is a Euclidean building in the way we described above.

Part II

Geometric proof of the Borel-Serre theorem

In this part we will prove a version of the theorem of Borel and Serre mentioned in the introduction: An S-arithmetic subgroup of a simply connected, $\mathbb{Q}$-simple Chevalley group over $\mathbb{Q}$ is of type F_∞. First, we define the notion of a Morse function without critical values above a fixed constant, which is defined on the product of a Riemannian manifold and a non-smooth factor. Then we fix an algebraic matrix group that we want to work with and introduce some notation for it. We will show in an example that this includes the Chevalley groups mentioned above. For an S-arithmetic subgroup of this group, we deduce some properties from reduction theory, and extend them to a geometric space on which the group acts. Furthermore, we define a function similar to that of Raghunathan and show that it is a Morse function in the above sense. Finally, we will use the results to determine the finiteness properties of the S-arithmetic subgroup.

14 Morse theory on two factors

Let $f\colon M \to \mathbb{R}$ be a smooth function defined on a Riemannian manifold M. A classical result of Morse theory is the following [Mil63, Theorem I.3.1]: If $f^{-1}([a,b])$ is compact and f has no critical value in the interval $[a,b] \subset \mathbb{R}$, then $f^{-1}(]-\infty,a])$ is a strong deformation retract of $f^{-1}(]-\infty,b])$. One can easily deduce that if f is proper and has no critical value greater or equal to some $r \in \mathbb{R}$, then M is homotopy equivalent to $f^{-1}(]-\infty,r])$.

The goal of this chapter is to extend this result to a function f defined on a product of a Riemannian manifold M and a non smooth factor Z, for example a (locally finite) CW complex. This is how we want to proceed: The map f induces functions f^z on M via $p \mapsto f(p,z)$ and we want to think of these as smooth Morse functions on M parametrized by $z \in Z$. Under suitable conditions on f they will induce deformation retractions h^z on M and the idea is to uniformly glue these together, to get a well-defined map h on $M \times Z$ via $h(p,z) = (h^z(p), z)$.

Therefore we need a kind of uniformity in the behavior of the f^z for $z \in Z$. We will deal with that in the second section. Another problem is that for the applications that we have in mind, the functions f^z will not be proper, so we have to replace properness by a different condition. This is what the first section is about. In the last section we will specify the conditions that a Morse function should fulfill and construct the deformation retraction.

14.1 Morse theory in the non compact case

Let f be a real-valued smooth function defined on a Riemannian manifold M. The plan is to use the arguments from the proof of [Mil63, Theorem I.3.1] to obtain a deformation retraction onto a sublevel set $f^{-1}(]-\infty, r])$, but without the assumption of properness. Here Milnor uses the assumption of $[a, b]$ containing no critical value to define a vector field as $\frac{1}{\langle \nabla f, \nabla f\rangle} \cdot \nabla f$ on $f^{-1}([a, b])$ and which vanishes outside of a compact neighborhood of this preimage (here properness comes into play). To use his further arguments we construct a vector field X by rescaling the gradient in a similar way.

Lemma 14.1. *Let $f\colon M \to \mathbb{R}$ be a smooth function on a complete Riemannian manifold M such that $\|\nabla f_p\|_p \geq c > 0$ for all $p \in f^{-1}([r, \infty[)$. Pick some $R > r$ and choose $\varepsilon > 0$ with $R > r + \varepsilon$ and let $\tau\colon \mathbb{R} \to [0, 1]$ be smooth with $\tau \mid_{[R,\infty[}= 1$ and $\tau \mid_{]-\infty,r+\varepsilon]}= 0$. Then the vector field*

$$X = \begin{cases} \frac{\rho}{\langle \nabla f, \nabla f\rangle} \cdot \nabla f & \text{on } f^{-1}(]r, \infty[) \\ 0 & \text{on } f^{-1}(]-\infty, r+\varepsilon[) \end{cases}$$

is well-defined, smooth and complete, where $\rho = \tau \circ f$.

Proof. First note that such a τ always exists, compare the construction in [Tu11, §13.1]. For all $p \in M$ with $f(p) > r$ we see that $\langle \nabla f_p, \nabla f_p\rangle_p \geq c^2$, so the quotient $\frac{1}{\langle \nabla f_p, \nabla f_p\rangle_p}$ is defined in $\mathbb{R}$. Further we have

$$\left\| \frac{\rho(p)}{\langle \nabla f_p, \nabla f_p\rangle_p} \cdot \nabla f_p \right\|_p = \frac{\rho(p)}{\langle \nabla f_p, \nabla f_p\rangle_p} \cdot \|\nabla f_p\|_p = \frac{\rho(p)}{\|\nabla f_p\|_p} \leq \frac{\rho(p)}{c},$$

because of $\frac{1}{\|\nabla f_p\|_p} \leq \frac{1}{c}$ and $\rho(p) \geq 0$. In fact, $\rho = 0$ on $f^{-1}(]-\infty, r+\varepsilon[)$, which makes $\frac{\rho}{\langle \nabla f, \nabla f\rangle} \cdot \nabla f = 0$ on the overlap $f^{-1}(]r, r+\varepsilon[)$. Thus, X is well-defined. But this directly implies that it is smooth, because X is defined as smooth vector fields on the open subsets $f^{-1}(]r, \infty[)$ and $f^{-1}(]-\infty, r+\varepsilon[)$ of M. Furthermore, since $\rho \leq 1$, the above inequality leads to

$$\|X\| \leq \frac{1}{c}$$

on the whole manifold M, so X is complete by Lemma 8.5. □

Now we can extend Morse theory to the case where f is not proper, by assuming f to have uniformly bounded gradient. As already noted before, this follows the proof of [Mil63, Theorem I.3.1].

Theorem 14.2. *Let $f\colon M \to \mathbb{R}$ be a smooth function on a complete Riemannian manifold M, such that there exists $r \in \mathbb{R}$ with $\|\nabla f\| \geq c > 0$ on $f^{-1}([r, \infty[)$. Then for every $R > r$ the map*

$$h\colon [0,1] \times M \to M,\ (t,p) \mapsto \begin{cases} p & f(p) \leq R, \\ F_{t(R-f(p))}(p) & f(p) \geq R \end{cases}$$

is a strong deformation retraction $M \simeq f^{-1}(]-\infty, R])$, where $F\colon \mathbb{R}\times M \to M$ is the global flow generated by the complete vector field X from the previous lemma.

Proof. First note that h is well-defined since $F_{t(R-f(p))} = F_0 = \mathrm{id}_M$ holds for $f(p) = R$ and that h is continuous as it is defined as a piecewise continuous map on two closed subsets of $[0,1] \times M$. It is immediate from the definition that we have $h(t,q) = q$ for all $q \in f^{-1}(]-\infty, R])$ and $t \in [0,1]$. Pick now a $p \in M$ with $f(p) \geq R$. Then we have $h(0,p) = F_0(p) = p$. Using $\rho(p) = 1$ and setting $c(t) = F_t(p)$ we get

$$\begin{aligned}
\frac{d}{dt} f(F_t(p)) &= \frac{d}{dt} f(c(t)) \\
&= c'(t)(f) \\
&= X_{c(t)}(f) \\
&= \left\langle X_{c(t)}, \nabla f_{c(t)} \right\rangle_{c(t)} \\
&= \left\langle \frac{1}{\left\langle \nabla f_{c(t)}, \nabla f_{c(t)} \right\rangle_{c(t)}} \cdot \nabla f_{c(t)}, \nabla f_{c(t)} \right\rangle_{c(t)} \\
&= \frac{\left\langle \nabla f_{c(t)}, \nabla f_{c(t)} \right\rangle_{c(t)}}{\left\langle \nabla f_{c(t)}, \nabla f_{c(t)} \right\rangle_{c(t)}} \\
&= 1
\end{aligned}$$

as long as $f(F_t(p)) \geq R$. Using $f(F_0(p)) = f(p) \geq R$ this implies

$$f(F_t(p)) = f(p) + t$$

for all $t \in [R - f(p), \infty[$. Thus,

$$f(h(1,p)) = f(F_{R-f(p)}(p)) = f(p) + R - f(p) = R$$

holds for every $p \in f^{-1}([R, \infty[)$. ☐

The crucial point in the application of Theorem 14.2 is that the gradient of the function $f\colon M \to \mathbb{R}$ is bounded from below. We give a criterion for this that we want to use later.

Lemma 14.3. *Let $f\colon M \to \mathbb{R}$ be a smooth function where M is a Riemannian manifold. Let further two constants $C, D > 0$ be given. If there is a curve $c(t)$ through a point $p \in M$ with $(f \circ c)'(0) \geq C$ and $\|c'(0)\| \leq D$, then $\|\nabla f_p\|_p \geq \frac{C}{D}$.*

Proof. Setting $X_p = c'(0)$ we get

$$C \leq X_p(f) = \langle \nabla f_p, X_p \rangle \leq \|\nabla f_p\| \cdot \|X_p\| \leq \|\nabla f_p\| \cdot D$$

by the Cauchy-Schwarz inequality. □

14.2 Vector fields and flows

The goal of this paragraph is to show that the map

$$\varphi\colon \mathfrak{X}_c(M) \times \mathbb{R} \times M \to M,\ (X, t, p) \mapsto F_t^X(p)$$

is continuous for some suitable topology on $\mathfrak{X}_c(M)$, the set of complete vector fields on the smooth manifold M, where F^X denotes the global flow of $X \in \mathfrak{X}_c(M)$. We start with a typical local-to-global argument.

Proposition 14.4. *Assume there is any topology on $\mathfrak{X}_c(M)$ such that the following holds:*
For every pair $(X, x) \in \mathfrak{X}_c(M) \times M$ there are a neighborhood $B \subset \mathfrak{X}_c(M)$ of X and a neighborhood $W \subset M$ of x and an interval $I = [-S, S] \subset \mathbb{R}$ for some $S > 0$ such that the map

$$B \times I \times W \to M,\ (Y, q, t) \mapsto F_t^Y(q)$$

is continuous.
Then the global map φ as defined above is also continuous.

Proof. We adapt the typical proof that global flows are smooth. In concrete terms we follow the lecture notes from [Wan21, proof of Theorem 1.3].
Pick some $p \in M$, $X \in \mathfrak{X}_c(M)$ and $T \in \mathbb{R}$, we show that the map

$$\Phi\colon \mathfrak{X}_c(M) \times \mathbb{R} \times M \to \mathfrak{X}_c(M) \times M,\ (X, p, t) \mapsto (X, F_t^X(p))$$

is continuous in a neighborhood of (X, T, p). The reason why we work with Φ instead of φ is the following technical one: The maps

$$\Phi_t\colon \mathfrak{X}_c(M) \times M \to \mathfrak{X}_c(M) \times M,\ (Y, q) \mapsto (Y, F_t^Y(q))$$

are bijective with inverse $\Phi_t^{-1} = \Phi_{-t}$ because $\Phi_0(Y, q) = (Y, F_0^Y(q)) = (Y, q)$ and

$$(\Phi_t \circ \Phi_r)(Y, q) = \Phi_t(Y, F_r^Y(q)) = (Y, F_t^Y(F_r^Y(q))) = (Y, F_{t+r}^Y(q)) = \Phi_{t+r}(Y, q)$$

hold for every $t, r \in \mathbb{R}$. If we would define φ_t in a similar way this would not be true.

Without loss of generality, we assume $T > 0$. The case $T < 0$ is analogous. The set

$$K := F^X([0,T] \times \{p\}) \subset M$$

is compact since F^X is continuous. Now we use the assumption: For every $x \in K$ there are neighborhoods $B_x \subset \mathfrak{X}_c(M)$ of X, $W_x \subset M$ of x and $S_x > 0$ such that

$$\varphi_x \colon B_x \times I_x \times W_x \to M, \ (Y,t,q) \mapsto F_t^Y(q)$$

is continuous, where $I_x = [-S_x, S_x]$. By compactness we can find $x_1, \dots, x_n \in K$ such that $W := \bigcup_{i=1}^n W_{x_i}$ covers K, so setting

$$B = \bigcap_{i=1}^n B_{x_i} \quad \text{and} \quad I = [-S,S]$$

for $S = \min\{S_{x_i} \mid i = 1, \dots, n\}$ we see that the map

$$B \times I \times W \to M, \ (Y,q,t) \mapsto F_t^Y(q)$$

is continuous. Therefore the restriction of Φ to $B \times I \times W$ given by $(Y,q,t) \mapsto (Y, F_t^Y(q))$ must be continuous, too.

Choose now $N > 0$ such that $N \cdot S > T$ and let $s := \frac{T}{N} < S$. Define further

$$U_1 = B \times W \quad \text{and} \quad U_{k+1} = (\Phi_s \mid_{B \times W})^{-1}(U_k)$$

for all $k = 1, \dots, N$. The continuity of Φ on $B \times I \times W$ implies that Φ_s is continuous on $B \times W$, so inductively $U_k \subset B \times W$ is open for every $k = 1, \dots, N+1$. Furthermore, as $\Phi_s(U_{k+1}) = \Phi_s(\Phi_s^{-1}(U_k) \cap B \times W)$ is a subset of U_k, we obtain continuous maps

$$\Phi_s \colon U_{k+1} \to U_k$$

for every $k = 1, \dots, N$. Concatenating Φ_s N times (in a row) we get a map

$$\Phi_s \circ \ldots \circ \Phi_s \colon U_{N+1} \to U_1 = B \times W$$

which is also continuous.

Next we show $\Phi(X,0,p) \in U_{N+1}$. To do so we prove

$$\Phi(X, T - ks, p) \in U_{k+1}$$

inductively for all $k = 0, \dots, N$. Starting with $k = 0$ we have

$$\Phi(X,T,p) = (X, F_T^X(p)) \in B \times K$$

which is a subset of $B \times W = U_1$. We assume now that the statement is true for $0 \leq k \leq N-1$ and deduce that it also holds for $k+1$. On the one hand, we have $T > T-(k+1)s \geq T-Ns = 0$, which implies

$$\Phi(X, T-(k+1)s, p) = (X, F^X_{T-(k+1)s}(p)) \in B \times K$$

so $\Phi(X, T-(k+1)s, p)$ is an element of $B \times W$. On the other hand, this also lies in $\Phi_s^{-1}(U_{k+1})$ because of the following: The identity $F^X_{T-(k+1)s}(p) = F^X_{-s}(F^X_{T-ks}(p))$ implies

$$\begin{aligned}\Phi(X, T-(k+1)s, p) = (X, F^X_{T-(k+1)s}(p)) &= \Phi_{-s}(X, F^X_{T-ks}(p)) \\ &= \Phi_{-s}(\Phi(X, T-ks, p)).\end{aligned}$$

By induction hypothesis $\Phi(X, T-ks, p)$ is an element of U_{k+1} which implies

$$\Phi(X, T-(k+1)s, p) \in \Phi_s^{-1}(U_{k+1})$$

using $\Phi_{-s} = \Phi_s^{-1}$. Putting both things together we obtain

$$\Phi(X, T-(k+1)s, p) \in \left(\Phi_s^{-1}(U_{k+1}) \cap (B \times W)\right) = U_{k+2}.$$

In particular, we now see $\Phi(X, 0, p) \in U_{N+1}$ because of $Ns = T$. By continuity of $\Phi \mid_{B \times I \times W}$ we get neighborhoods $I_0 \subset I$ of 0, $B_0 \subset B$ of X and $W_0 \subset W$ of p with

$$B_0 \times I_0 \times W_0 \subset (\Phi \mid_{B \times I \times W})^{-1}(U_{N+1}).$$

Thus, Φ induces a continuous map

$$B_0 \times I_0 \times W_0 \to U_{N+1}.$$

But this implies that Φ is continuous on the neighborhood $B_0 \times (T+I_0) \times W_0$ of (X, T, p), because

$$\Phi \mid_{B_0 \times (T+I_0) \times W_0} = \Phi_s \circ \ldots \circ \Phi_s \circ \Phi \mid_{B_0 \times I_0 \times W_0}$$

is a concatenation of continuous maps. □

Therefore, we want to define a topology on $\mathfrak{X}(M)$, which makes the above map φ continuous in a neighborhood of $(X, 0, p) \in \mathfrak{X}_c \times \mathbb{R} \times M$. It turns out that the right choice is the weak topology. We will deal with this in the following and give some results that we will need later.

Definition 14.5. Let M, N be smooth manifolds. We define the weak topology on $C^r(M, N)$, the set of r-times differentiable functions $M \to N$ as follows: Pick $f \in C^r(M, N)$. Then for a chart (U, ϕ) on M, a compact subset

$K \subset U$, and a chart (V, ψ) on N such that $f(K) \subset V$, and for $\varepsilon > 0$, we define

$$B^r(f, (U, \phi), K, (V, \psi), \varepsilon)$$

to be the set of all $g \in C^r(M, N)$ such that $g(K) \subset V$ and

$$\left\| D^k(\psi \circ f \circ \phi^{-1})(x) - D^k(\psi \circ g \circ \phi^{-1})(x) \right\| < \varepsilon$$

hold for all $x \in \phi(K)$ and $k = 0, \ldots, r$, where $\|.\|$ is the standard Euclidean norm. The *weak topology* on $C^r(M, N)$ then is the topology generated by the $B^r(f, (U, \phi), K, (V, \psi), \varepsilon)$, which means that these sets form a subbase of the weak topology.

For a deeper analysis of the weak topology in general see [Hir76, chapter 2]. We need the following result:

Proposition 14.6. *The weak topology on $C^r(M, N)$ is first countable.*

Proof. This follows directly from [Hir76, Theorem 2.4.4], where in fact the author proves the much stronger statement that it is metrizable. ☐

Therefore, when analyzing convergence in the weak topology, we only have to consider sequences instead of nets, which avoids some technical difficulties.

In the case $r = 0$ we see that the condition

$$\left\| (\psi \circ f \circ \phi^{-1})(x) - (\psi \circ g \circ \phi^{-1})(x) \right\| < \varepsilon \text{ for all } x \in \phi(K)$$

is equivalent to

$$\|(\psi \circ f)(p) - (\psi \circ g)(p)\| < \varepsilon \text{ for all } p \in K$$

and in fact does not depend on the chart (U, ϕ) of M. Since M is locally compact we can write any compact subset K as a finite union of some compact sets K_i all lying in charts (U_i, ϕ_i). Thus, the weak topology on $C(M, N) = C^0(M, N)$ is also generated by sets of the form

$$B(f, K, (V, \psi), \varepsilon)$$

which we define to be the set of all $g \in C(M, N)$ such that $g(K) \subset V$ and

$$\|(\psi \circ f)(p) - (\psi \circ g)(p)\| < \varepsilon \text{ for all } p \in K.$$

Proposition 14.7. *The weak topology on $C(M, N)$ is generated by the sets $B(f, K, (V, \psi), \varepsilon)$.* ☐

When $N = \mathbb{R}$ convergence in the weak C^r topology just becomes uniform convergence on compact subsets of M. We only need this for $r = 0, 1$.

Proposition 14.8. *Pick $f \in C^\infty(M, \mathbb{R})$ and a sequence $(f_n)_{n\in\mathbb{N}}$ of smooth functions $M \to \mathbb{R}$. Then the following hold:*

a) $f_n \to f$ in the weak C^0 topology if and only if $f_n \to f$ uniformly on compact subsets.

b) The sequence $(f_n)_{n\in\mathbb{N}}$ converges to f in the weak C^1 topology if and only if it converges in the weak C^0 topology and additionally $\frac{\partial f_n}{\partial x^i} \to \frac{\partial f}{\partial x^i}$ for all $i = 1, \dots, m$ uniformly on compact subsets K of M lying in a chart (U, ϕ).

Proof. Assume first $f_n \to f$ in the weak C^0 topology. Then f_n is an element of $B(f, K, \varepsilon) := B(f, K, (\mathbb{R}, \mathrm{id}), \varepsilon)$ for all but finitely many $n \in \mathbb{N}$. But this exactly means that f_n converges to f uniformly on all compact sets K.

If on the other hand, f_n converges to f uniformly on compact subsets of M, then any $B(f, K, \varepsilon)$ contains all but finitely many f_n by definition, so the same must be true for finite intersections $\bigcap_{i=1}^{\ell} B(f, K_i, \varepsilon_i)$. Thus, $f_n \to f$ in the weak topology.

Now we prove b): If f_n converges to f in the weak C^1 topology, then by definition it converges in C^0, and additionally f_n is an element of

$$B^1(f, (U, \phi), K, \varepsilon) := B^1(f, (U, \phi), K, (\mathbb{R}, \mathrm{id}), \varepsilon)$$

for all but finitely many n. This implies

$$\left\| D(f \circ \phi^{-1})(x) - D(f_n \circ \phi^{-1})(x) \right\| < \varepsilon,$$

which means

$$\sum_{i=1}^{m} \left| \frac{\partial f}{\partial x^i} \,|_{\phi^{-1}(x)} - \frac{\partial f_n}{\partial x^i} \,|_{\phi^{-1}(x)} \right| < \varepsilon$$

for all $x \in \phi(K)$ and almost all $n \in \mathbb{N}$. Thus,

$$\frac{\partial f_n}{\partial x^i} \to \frac{\partial f}{\partial x^i}$$

uniformly on K for all $i = 1, \dots, m$.

At least we assume $f_n \to f$ in the weak C^0 topology and that $\frac{\partial f_n}{\partial x^i}$ converges to $\frac{\partial f}{\partial x^i}$ uniformly on compact subsets K of M lying in a chart (U, ϕ). The latter means that for any $\varepsilon > 0$ we have

$$\sup_{p \in K} \left| \frac{\partial f}{\partial x^i} \,|_p - \frac{\partial f_n}{\partial x^i} \,|_p \right| < \frac{\varepsilon}{m}$$

for almost all n. But this implies

$$\left\| D(f \circ \phi^{-1})(x) - D(f_n \circ \phi^{-1})(x) \right\| < \varepsilon$$

for all $x \in \phi(K)$ and therefore $f_n \in B^1(f, (U, \phi), K, \varepsilon)$ for all but finitely many $n \in \mathbb{N}$. Again, this must also hold for finite intersections, so we obtain $f_n \to f$ in the weak C^1 topology. $\square$

Remark 14.9. For a compact subset K in M the following are equivalent:

a) $\frac{\partial f_n}{\partial x^i} \to \frac{\partial f}{\partial x^i}$ uniformly on K for some chart (U, ϕ) containing K,

b) $\frac{\partial f_n}{\partial x^i} \to \frac{\partial f}{\partial x^i}$ uniformly on K for any chart (U, ϕ) containing K.

This comes from the fact that chart maps are smooth and therefore Lipschitz on the compact set K.

Let us return to vector fields. Recall that we can view the tangent bundle TM as a manifold endowed with charts of the form $(TU, \tilde{\phi})$, where

$$\tilde{\phi}\colon TU \to \phi(U) \times \mathbb{R}^m, \quad \left(p, \sum_{i=1}^{m} a^i(p) \frac{\partial}{x^i} \,|_p \right) \mapsto (\phi(p), a^1(p), \dots, a^m(p))$$

for a chart (U, ϕ). Since a vector field $X\colon M \to TM$ can be viewed as a smooth map between manifolds, we can endow the set $\mathfrak{X}(M)$ with the weak $C(M, TM)$ topology. Picking a chart $(TU, \tilde{\phi})$ on TM and a compact subset $K \subset M$, we see

$$X(K) = \{(p, X_p) \mid p \in K \text{ and } X_p \in T_pM\}.$$

So $X(K)$ is a subset of $TU = \bigcup_{p \in U} \{p\} \times T_pM$ if and only if K lies in U. By definition now

$$(\tilde{\phi} \circ X)(p) = \tilde{\phi}(p, X_p) = (\phi(p), a^1(p), \dots, a^m(p))$$

where $X = \sum_{i=1}^{m} a^i \frac{\partial}{\partial x^i}$ on U. So for any other vector field $Y = \sum_{i=1}^{m} b^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(U)$ we have

$$\begin{aligned}\left\|(\tilde{\phi} \circ X)(p) - (\tilde{\phi} \circ Y)(p)\right\| &= \left\|(\phi(p), a^1(p), \dots, a^m(p)) - (\phi(p), b^1(p), \dots, b^m(p))\right\| \\ &= \left\|(a^1(p), \dots, a^m(p)) - (b^1(p), \dots, b^m(p))\right\|.\end{aligned}$$

Writing now $a(p) = (a^1(p), \dots, a^m(p))$ we deduce that

$$B(X, K, (TU, \tilde{\phi}), \varepsilon)$$

is the set of all $Y = \sum_{j=1}^{m} b^j \frac{\partial}{\partial x^i} \in \mathfrak{X}(U)$ such that $K \subset U$ and $\|a(p) - b(p)\| < \varepsilon$ for all $p \in K$. Finally, since the chart $(TU, \tilde{\phi})$ on TM is completely determined by the chart (U, ϕ) on M, we define

$$B(X, K, (U, \phi), \varepsilon) = B(X, K, (TU, \tilde{\phi}), \varepsilon).$$

Thus we obtain:

Proposition 14.10. *The weak topology on $\mathfrak{X}(M)$ is generated by the sets $B(X, K, (U, \phi), \varepsilon)$.* □

To understand convergence in the weak topology on $\mathfrak{X}(M)$ we state the following lemma and proposition.

Lemma 14.11. *Let* $(U, x^1, \ldots, x^m)$ *and* $(V, y^1, \ldots, y^m)$ *be two coordinate systems on the manifold* M *both containing a compact subset* K *and let* X *and* X_n *for* $n \in \mathbb{N}$ *be smooth vector fields. We write* X *(respectively* X_n*) as* $\sum_{i=1}^m a^i \frac{\partial}{\partial x^i}$ *(respectively* $\sum_{i=1}^m a_n^i \frac{\partial}{\partial x^i}$*) on* U *and as* $\sum_{j=1}^m b^j \frac{\partial}{\partial y^j}$ *(respectively* $\sum_{j=1}^m b_n^j \frac{\partial}{\partial y^j}$*) on* V*.*
Then $a_n \to a$ *uniformly on* K *if and only if* $b_n \to b$ *uniformly on* K *as* $n \to \infty$*.*

Proof. By Lemma 5.5 we know $\frac{\partial}{\partial y^j} = \sum_{i=1}^m \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i}$ on $U \cap V$ so

$$\begin{aligned}\sum_{i=1}^m a^i \frac{\partial}{\partial x^i} = X &= \sum_{j=1}^m b^j \frac{\partial}{\partial y^j} \\ &= \sum_{j=1}^m b^j \sum_{i=1}^m \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i} \\ &= \sum_{i=1}^m \left(\sum_{j=1}^m b^j \frac{\partial x^i}{\partial y^j} \right) \frac{\partial}{\partial x^i}\end{aligned}$$

on $U \cap V$. From the fact that $(\frac{\partial}{\partial x^1} |_p, \ldots, \frac{\partial}{\partial x^m} |_p)$ is a basis on T_pM for all $p \in U$, we obtain

$$a^i = \sum_{j=1}^m b^j \frac{\partial x^i}{\partial y^j} = \left(\frac{\partial x}{\partial y} \right) \cdot b$$

for every $i = 1, \ldots, m$, where $\left(\frac{\partial x}{\partial y}\right)$ is the matrix with entries $m_{ij} = \frac{\partial x^i}{\partial y^j}$. Therefore, assuming b_n converges to b uniformly on K we see that a_n^i converges to a^i uniformly on K for every $i = 1, \ldots, m$, since the matrix entries of $\left(\frac{\partial x}{\partial y}\right)$ are bounded on the compact set K. Thus, $a_n \to a$ uniformly on K and with the same argumentation we get the reverse implication. □

Proposition 14.12. *Let* $X \in \mathfrak{X}(M)$ *be a vector field and* $(X_n)_{n \in \mathbb{N}} \subset \mathfrak{X}(M)$ *be a sequence. The following are equivalent:*

- *(i)* $X_n \to X$ *in the weak topology,*
- *(ii) for any chart* (U, ϕ) *on* M *we have that* $a_n \to a$ *uniformly on every compact subset* $K \subset U$*, where* $X = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}$ *and* $X_n = \sum_{i=1}^m a_n^i \frac{\partial}{\partial x^i}$ *on* U*,*
- *(iii) for every point* $p \in M$ *there is a chart* (U, ϕ) *and a compact neighborhood* $C \subset U$ *of* p *such that* $a_n \to a$ *uniformly on* C*, where* $X = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}$ *and* $X_n = \sum_{i=1}^m a_n^i \frac{\partial}{\partial x^i}$ *on* U*.*

Proof. Assume first that X_n converges to X in the weak topology and take an arbitrary chart (U, ϕ) and a compact subset $K \subset U$. For every $\varepsilon > 0$ the neighborhood $B(X, K(U, \phi), \varepsilon)$ contains all but finitely many X_n, which means that $\|a(p) - a_n(p)\| < \varepsilon$ for all $p \in K$ and for almost all $n \in \mathbb{N}$. This shows (i) $\Rightarrow$ (ii).
To prove the reverse implication, we have to show that every finite intersection $B = \bigcap_{j=1}^{r} B_j$ for $B_j = B(X, K_j, (U_j, \phi_j), \varepsilon_j)$ contains all but finitely many X_n. By (ii) we know that a_{j_n} converges to uniformly to a_j on K_j where a_{j_n} and a_j are the coefficient vectors of X_n and X on (U_j, ϕ_j). Thus we know that every B_j contains almost all X_n and this implies $X_n \in B$ for all but finitely many n.
That (ii) implies (iii) is trivial, since M is locally compact. So it remains to show that statement (ii) is true under the assumption of (iii). Take an arbitrary chart (U, ϕ) of M and any compact subset $K \subset U$. For every $p \in K$ there is a chart (V_p, ψ_p) containing a compact neighborhood C_p of p, such that the coefficient vector of X_n with respect to this chart converges uniformly to that of X on C_p. By taking a finite subcover $C_1, \ldots, C_k$ and replacing every C_i by $C_i \cap K$ we obtain that K is a finite union of compact subsets C_i lying in charts (V_i, ψ_i). Thus, $C_i \subset U \cap V_i$ and by the above lemma we get $a_n \to a$ uniformly on K. $\square$

The proposition reduces convergence of vector fields in the weak topology to uniform convergence on compact subsets of charts.

We will now return to prove the continuity of φ. To find a suitable neighborhood W of $p \in M$, an obvious attempt is to choose a subset of a chart (U, ϕ). In the following, we therefore focus on vector fields defined on open subsets of the Euclidean space $\mathbb{R}^m$ and recall some results. However, we must ensure that integral curves of fixed length always exist and do not leave the chart when we vary the starting point. We will work with the following version of the *Picard-Lindelöf* theorem:

Proposition 14.13. *Let* $X \in C^\infty(U, \mathbb{R}^m)$ *be a vector field defined on an open set* $U \subset \mathbb{R}^m$ *containing a point* $x \in U$. *If we choose* $r > 0$ *such that the closed r-ball* $\overline{B_r(x)}$ *lies inside* U, *then for* $M = \sup\{\|X(y)\| \mid y \in \overline{B_r(x)}\}$ *and* $T = \frac{r}{M}$ *the unique integral curve* $c\colon [-T, T] \to \mathbb{R}^m$ *for* X *starting at* x *exists for all* $t \in [-T, T]$ *and the image* $c([-T, T])$ *is a subset of* $\overline{B_r(x)}$.

Proof. This follows from [Tes12, Theorem 2.2] since any smooth function defined on an open subset of $\mathbb{R}^m$ is locally Lipschitz. $\square$

Corollary 14.14. *Let* $X \in C^\infty(U, \mathbb{R}^m)$, $r > 0$ *and* M *be like in the above proposition. Let further* $\varepsilon > 0$ *be given and define* $S = \frac{r}{3(M+\varepsilon)}$. *For every vector field* $Y \in C^\infty(U, \mathbb{R}^m)$ *with* $\sup\{\|Y(z) - X(z)\| \mid z \in \overline{B_r(x)}\} \leq \varepsilon$ *and every point* $y \in \overline{B_{\frac{r}{3}}(x)}$, *the unique integral curve* c_y *for* Y *starting at* y *is defined at least on* $[-S, S]$ *and* $c_y([-S, S])$ *is a subset of* $\overline{B_{\frac{r}{3}}(y)} \subset \overline{B_r(x)}$.

Proof. The assumption $d(x,y) \leq \frac{r}{3}$ ensures that $\overline{B_{\frac{r}{3}(y)}}$ is a subset of U. By the previous proposition c_y is defined at least on $[-T_y, T_y]$ and $c_y([-T_y,T_y])$ is a subset of $\overline{B_{\frac{r}{3}}(y)}$ where

$$T_y = \frac{r}{3M_y} \quad \text{and } M_y = \sup\{\|Y(z)\| \mid z \in \overline{B_{\frac{r}{3}}(y)}\}.$$

The inequality $\|Y(z) - X(z)\| \leq \varepsilon$ implies that $\|Y(z)\| \leq \|X(z)\| + \varepsilon$ holds for every $z \in \overline{B_r(x)}$ so we get

$$M_y \leq M + \varepsilon$$

from the inequality $M_y \leq \sup\{\|Y(z)\| \mid z \in \overline{B_r(x)}\}$. Therefore

$$T_y = \frac{r}{3M_y} \geq \frac{r}{3(M+\varepsilon)} = S$$

and thus $c_y([-S,S]) \subset c_y([-T_y,T_y])$ is a subset of $\overline{B_{\frac{r}{3}}(y)}$. ☐

By choosing a chart (U,ϕ), we can therefore restrict ourselves to the case when $W = \overline{B_{\frac{r}{3}}(p)}$ is a compact neighborhood of a point p in the Euclidean space $\mathbb{R}^m$: The corollary guarantees that for a suitable number $S > 0$, the integral curve for each $q \in W$ always exists on $I = [-S,S]$ and stays inside this chart U. It still remains to find a suitable $B \subset \mathfrak{X}(M)_c$ containing X. We will see in the following that we can assume B to be the set of all functions Y that have uniformly bounded distance to X on $V = \overline{B_r(p)}$, this is a neighborhood of X by definition. The weak topology on this set B is exactly the topology of uniform convergence on the compact set V.

The last crucial ingredient is the next proposition, which is an application of Grönwall's lemma:

Proposition 14.15. *Let $c^X, c^Y : [-T,T] \to U$ be integral curves for vector fields $X, Y \in C^\infty(U,\mathbb{R}^m)$ with starting points $c^X(0) = x$ and $c^Y(0) = y$, where $U \subset \mathbb{R}^m$ is an open subset containing x and y. If there is a compact neighborhood $V \subset U$ containing the images $c^X([-T,T])$ and $c^Y([-T,T])$, then the inequality*

$$\left\|c^X(t) - c^Y(t)\right\| \leq \|x-y\|\, e^{L|t|} + \frac{M}{L}\cdot(e^{L|t|}-1)$$

holds for every $t \in [-T,T]$ where

$$L = \sup\left\{ \frac{\|X(z)-X(z')\|}{\|z-z'\|} \,\middle|\, z,z' \in V,\ z \neq z' \right\}$$

and

$$M = \sup\{\|X(z) - Y(z)\| \mid z \in V\}.$$

Proof. Note that M and L both exist since V is compact and X is smooth. For a proof see [Tes12, Theorem 2.8]. ☐

Now we can prove continuity of φ on $B \times I \times W$.

Proposition 14.16. *Let $X\colon U \to \mathbb{R}^m$ be a smooth map defined on an open subset $U \subset \mathbb{R}^m$, and let $x \in U$ and $r > 0$ be given such that $\overline{B_r(x)} \subset U$. For $\varepsilon > 0$ we set*

$$B(X, \overline{B_r(x)}, \varepsilon) = \{Y \in C^\infty(U, \mathbb{R}^m) \mid \|Y(z) - X(z)\| < \varepsilon \text{ for all } z \in \overline{B_r(x)}\}$$

and $S = \frac{r}{3(M+\varepsilon)}$ where $M = \sup\{\|X(y)\| \mid y \in \overline{B_r(x)}\}$. The map

$$B(X, \overline{B_r(x)}, \varepsilon) \times [-S, S] \times \overline{B_{\frac{r}{3}}(x)} \to \overline{B_r(x)}$$
$$(Y, t, y) \mapsto c_y^Y(t)$$

well-defined and continuous, where c_y^Y denotes the integral curve of Y starting at y.

Proof. The map is well-defined because $c_y^Y([-S, S])$ is a subset of $V = \overline{B_r(x)}$ for all $Y \in B(X, \overline{B_r(x)}, \varepsilon)$ and every $y \in \overline{B_{\frac{r}{3}}(x)}$ by Corollary 14.14. To prove continuity take an arbitrary sequence $(Y_n)_{n\in\mathbb{N}}$ in $B(X, \overline{B_r(x)}, \varepsilon)$ converging to some Y uniformly on $\overline{B_r(x)}$. Further, we take sequences $(y_n)_{n\in\mathbb{N}}$ converging to y in $\overline{B_{\frac{r}{3}}(x)}$ and $(t_n)_{n\in\mathbb{N}}$ converging to t in $[-S, S]$. Then we have

$$\left\|c_y^Y(t) - c_{y_n}^{Y_n}(t_n)\right\| \leq \left\|c_y^Y(t) - c_y^Y(t_n)\right\| + \left\|c_y^Y(t_n) - c_{y_n}^{Y_n}(t_n)\right\|$$

where the left summand converges to 0, since $t \mapsto c_y^Y(t)$ is a continuous map. Considering the right summand, the previous proposition tells us

$$\left\|c_y^Y(t_n) - c_{y_n}^{Y_n}(t_n)\right\| \leq \|y - y_n\| \, e^{L|t_n|} + \frac{M_n}{L} \cdot (e^{L|t_n|} - 1),$$

where

$$L = \sup\left\{ \frac{\|Y(z) - Y(z')\|}{\|z - z'\|} \,\middle|\, z, z' \in \overline{B_r(x)},\ z \neq z' \right\}$$

and

$$M_n = \sup\{\|Y(z) - Y_n(z)\| \mid z \in \overline{B_r(x)}\}.$$

We see $\lim_{n\to\infty} M_n = 0$, since $Y_n \to Y$ uniformly on $\overline{B_r(x)}$. The assertion now follows from $\|y - y_n\| \to 0$ and $e^{L|t_n|} \to e^{L|t|}$ as $n \to \infty$. ☐

Corollary 14.17. *Let M be a smooth manifold, (U, ϕ) a chart of M containing a point p and let $X \in \mathfrak{X}(U)$ be a vector field. Then for every compact neighborhood $V \subset U$ of p there are a (possibly) smaller compact neighborhood $W \subset V$ of p, an $\varepsilon > 0$ and an $S > 0$ such that the map*

$$B(X, V, (U, \phi), \varepsilon) \times [-S, S] \times W \to V,\ (Y, t, q) \mapsto F_t^Y(q)$$

is well defined and continuous.

Proof. This follows directly from the above proposition, since $t \mapsto F_t^Y(q)$ is the integral curve of Y starting at q. ☐

Theorem 14.18. *Let M be a smooth manifold, then the map*

$$\varphi\colon \mathfrak{X}_c(M) \times \mathbb{R} \times M \to M,\ (X,t,p) \to F_t^X(p)$$

is continuous.

Proof. This follows from Corollary 14.17 and Proposition 14.4. $\square$

14.3 Morse theory on products

We now want to extend the results of Section 14.1 to the case of a product of a Riemannian manifold M and a topological space Z. The following definition imposes suitable conditions on f, so that for each $z \in Z$ we obtain a strong deformation retraction h^z to a sublevel set of f^z, and these together form a homotopy equivalence $f^{-1}(-\infty,t]) \simeq M \times Z$. In the remainder of the section we will prove that statement.

Definition 14.19. Let M be a complete Riemannian manifold and let Z be a first countable topological Hausdorff space and let $r \in \mathbb{R}$. A continuous function $f\colon M \times Z \to \mathbb{R}$ is called a *Morse function without critical values in* $]r,\infty[$ if it fulfills the following properties:

a) The induced map $f^z\colon M \to \mathbb{R},\ p \mapsto f(p,z)$ is smooth for all $z \in Z$,

b) for every $z \in Z$ there is a constant $c^z > 0$ with $\|\nabla f^z\| \geq c^z$ on $(f^z)^{-1}([r,\infty[)$,

c) the map $Z \to C^1(M,\mathbb{R})$ given by $z \mapsto f^z$ is continuous with respect to the weak C^1 topology.

Remark 14.20. Point c) of the above definition is equivalent to the following property:

c') $z_n \to z$ in Z implies $f^{z_n} \to f$ and $\nabla f^{z_n} \to \nabla f^z$ in the weak topology.

This comes from the fact that the coefficient vector of ∇f in a chart (U,ϕ) is given by

$$G^{-1} \cdot \begin{pmatrix} \frac{\partial f}{\partial x^1} \\ \vdots \\ \frac{\partial f}{\partial x^m} \end{pmatrix}$$

for G be the matrix with entries $g_{i,j} = \left\langle \frac{\partial}{x^i}, \frac{\partial}{x^j} \right\rangle$. We have that

$$G^{-1} \cdot \begin{pmatrix} \frac{\partial f^{z_n}}{\partial x^1} \\ \vdots \\ \frac{\partial f^{z_n}}{\partial x^m} \end{pmatrix} \to G^{-1} \cdot \begin{pmatrix} \frac{\partial f^{z}}{\partial x^1} \\ \vdots \\ \frac{\partial f^{z}}{\partial x^m} \end{pmatrix}$$

if and only if

$$\begin{pmatrix} \frac{\partial f^{z_n}}{\partial x^1} \\ \vdots \\ \frac{\partial f^{z_n}}{\partial x^m} \end{pmatrix} \to \begin{pmatrix} \frac{\partial f^{z}}{\partial x^1} \\ \vdots \\ \frac{\partial f^{z}}{\partial x^m} \end{pmatrix}$$

uniformly on compact subsets $K \subset U$.

Lemma 14.21. *Let $f\colon M \times Z \to \mathbb{R}$ be a Morse function without critical values $> r$. Pick any $\varepsilon > 0$ and $R > r + \varepsilon > r$. The map*

$$Z \to \mathfrak{X}(M),\ z \mapsto X^z = \begin{cases} \frac{\rho^z}{\langle \nabla f^z, \nabla f^z \rangle} \cdot \nabla f^z & \textit{on } (f^z)^{-1}(]r, \infty[) \\ 0 & \textit{on } (f^z)^{-1}(]-\infty, r+\varepsilon[) \end{cases}$$

is continuous, where $\rho^z = \tau \circ f^z$ for a smooth function $\tau\colon \mathbb{R} \to [0,1]$ with $\tau\mid_{[R,\infty[} = 1$ and $\tau\mid_{]-\infty, r+\varepsilon]} = 0$.

Proof. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in Z converging to z and consider the vector fields

$$X = \begin{cases} \frac{\rho^z}{\langle \nabla f^z, \nabla f^z \rangle} \cdot \nabla f^z & \text{on } (f^z)^{-1}(]r, \infty[) \\ 0 & \text{on } (f^z)^{-1}(]-\infty, r+\varepsilon[) \end{cases}$$

and

$$X_n = \begin{cases} \frac{\rho^{z_n}}{\langle \nabla f^{z_n}, \nabla f^{z_n} \rangle} \cdot \nabla f^{z_n} & \text{on } (f^{z_n})^{-1}(]r, \infty[) \\ 0 & \text{on } (f^{z_n})^{-1}(]-\infty, r+\varepsilon[) \end{cases}$$

on M, which are smooth and well-defined by Lemma 14.1. We have to show $X_n \to X$ in the weak topology. Pick some $p \in M$.

If $f^z(p) > r$, then choose $\delta > 0$ with $f^z(p) > r + \delta$. By local compactness we can find a compact neighborhood C of p lying inside $U \cap (f^z)^{-1}(]r+\delta, \infty[)$ where (U, ϕ) is a chart around p. Since $f^{z_n} \to f^z$ uniformly on C, we have

$$\frac{\delta}{2} > |f^z - f^{z_n}| \geq f^z - f^{z_n}$$

for all but finitely many $n \in \mathbb{N}$, which implies

$$f^{z_n} > f^z - \frac{\delta}{2} > r + \frac{\delta}{2} > r$$

on C. So, without loss of generality, we can assume

$$X_n = \frac{\rho^{z_n}}{\langle \nabla f^{z_n}, \nabla f^{z_n} \rangle} \cdot \nabla f^{z_n}$$

for all $n \in \mathbb{N}$. Furthermore, when we write

$$\nabla f^z = \sum_{i=1}^m d^i \frac{\partial}{\partial x^i} \quad \text{and} \quad \nabla f^{z_n} = \sum_{i=1}^m d_n^i \frac{\partial}{\partial x^i}$$

in the chart (U, ϕ), then on C we get

$$\begin{aligned} X &= \frac{\rho^z}{\langle \nabla f^z, \nabla f^z \rangle} \cdot \nabla f^z \\ &= \left(\frac{\rho^z}{\sum_{j,k=1}^m d^j d^k g_{jk}} \right) \cdot \sum_{i=1}^m d^i \frac{\partial}{\partial x^i} \\ &= \sum_{i=1}^m (\tau \circ f^z) \cdot d^i \cdot \left(\frac{1}{\sum_{j,k=1}^m d^j d^k g_{jk}} \right) \frac{\partial}{\partial x^i} \end{aligned}$$

and analogously also

$$X_n = \sum_{i=1}^m (\tau \circ f^{z_n}) \cdot d_n^i \cdot \left(\frac{1}{\sum_{j,k=1}^m d_n^j d_n^k g_{jk}} \right) \frac{\partial}{\partial x^i}$$

where $g_{jk} = \langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \rangle$. By Remark 14.20 we know that $(\tau \circ f^{z_n}) \to (\tau \circ f^z)$ and $d_n = (d_n^1, \dots, d_n^m) \to d = (d^1, \dots, d^m)$ converge uniformly on C, so the coefficients of X_n converge to that of X uniformly on C.
But we can cover the second case $f^z(p) < r + \varepsilon$ by finding a compact neighborhood $D \subset (f^z)^{-1}(]-\infty, r+\varepsilon[)$ of p in the same way as we did above. Thus, Proposition 14.12 tells us that $X_n \to X$ in the weak topology. ☐

Corollary 14.22. *If $f\colon M \times Z \to \mathbb{R}$ is a Morse function without critical values greater than r, then the map*

$$\varphi\colon \mathbb{R} \times M \times Z \to M,\ (t, p, z) \mapsto F_t^z(p)$$

is continuous, where F^z is the flow of the vector field X^z of Lemma 14.21.

Proof. First notice that φ is well-defined, since X^z is complete for every $z \in Z$ by Lemma 14.1. Now, the map

$$\mathbb{R} \times M \times Z \to \mathbb{R} \times M \times \mathfrak{X}_c(M),\ (t, p, z) \mapsto (t, p, X^z)$$

is continuous by Lemma 14.21 and the map

$$\mathbb{R} \times M \times \mathfrak{X}_c(M) \to M,\ (t, p, X) \mapsto F_t^X(p)$$

is also continuous by Theorem 14.18. ☐

Theorem 14.23. *Let $f\colon M \times Z \to \mathbb{R}$ be a Morse function without critical values in $]r, \infty[$. For every $R > r$, there is a strong deformation retraction $M \times Z \simeq f^{-1}(]-\infty, R])$.*

Proof. From Theorem 14.2 we obtain strong deformation retractions $M \simeq (f^z)^{-1}(]-\infty, R])$ for $z \in Z$ via

$$h^z\colon [0,1] \times M \to M,\ (t, p) \mapsto \begin{cases} p & \text{if } f^z(p) \leq R, \\ F^z_{t(R - f^z(p))}(p) & \text{if } f^z(p) \geq R. \end{cases}$$

We define

$$h\colon [0,1]\times(M\times Z)\to M\times Z,\ (t,p,z)\mapsto(h^z(t,p),z).$$

By definition we have

$$h(t,p,z)=\begin{cases}(p,z) & \text{if } f(p,z)\leq R,\\ (F^z_{t(R-f^z(p))}(p),z) & \text{if } f(p,z)\geq R.\end{cases}$$

The map $(t,p,z)\mapsto F^z_{t(R-f^z(p))}(p)$ is continuous by the above corollary, so h too must be continuous, as it is defined as piecewise continuous maps on the two closed subsets $[0,1]\times f^{-1}(]-\infty,R])$ and $[0,1]\times f^{-1}([R,\infty[)$ of $[0,1]\times M\times Z$. Furthermore, because $F^z_0=\mathrm{id}_M$, we get $h(0,p,z)=(p,z)$ for all $(p,z)\in M\times Z$. We also see that $h(t,p,z)=(p,z)$ holds for all $t\in[0,1]$ if $f(p,z)\leq R$. Finally, we have

$$f(h(1,p,z))=f(h^z(1,p),z)=f^z(h^z(1,p))\leq R$$

for all $(p,z)\in M\times Z$. □

15 Conventions and notation

For the rest of this thesis we fix an algebraic matrix group $\mathbf{G} \subset \mathrm{GL}_n(\mathbb{C})$. We list some axioms that this group should fulfill and introduce notation that we want to work with from now on. Everything we assert here without proofs or references is either trivial or just repitition from Part I, mainly Sections 11 and 12.
This section is divided into three subsections. The first one deals with properties of the algebraic matrix group itself. In the second one we use $\mathbf{G}$ to define some groups that we will work with later. The third one introduces geometric objects on which these groups naturally act.

15.1 Conventions for the algebraic matrix group

The following is a list of five conventions that should apply to $\mathbf{G}$. In parallel, we introduce some notation. At the end of this subsection, we use an example to show that simply connected, k-simple Chevalley groups fulfill these conventions.

Convention 1. The algebraic matrix group $\mathbf{G}$ is semisimple, connected, and defined over the field $k = \mathbb{Q}$. Furthermore, we assume simple connectedness.

This also implies that $\mathbf{G}$ is determined by its k-points. We identify $\mathbf{G}$ with the corresponding functor from the category of k-algebras to the category of groups, that maps some R to a subgroup $\mathbf{G}(R)$ of $\mathrm{GL}_n(R)$. We do the same for every closed connected k-subgroup $\mathbf{H} \subset \mathbf{G}$.

Convention 2. $\mathbf{G}$ is k-split, that means it contains a maximal torus $\mathbf{T}$ defined over k, which is k-split. We fix such a $\mathbf{T} \subset \mathbf{G}$.

Further we denote by $X(\mathbf{T})$ the set of characters of the torus, which equals the set of k-characters $X_k(\mathbf{T})$. Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$, which is semisimple. The set

$$\phi := \phi(\mathbf{T}, \mathbf{G}) = \{\alpha \in X(\mathbf{T}) \mid \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0\}$$

is a root system in the real vector space $V = \mathbb{R} \otimes_{\mathbb{Z}} X(\mathbf{T})$, where

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall t \in \mathbf{T} \colon \ \mathrm{Ad}_t(X) = \alpha(t)X\}.$$

Fixing a Weyl-invariant scalar product $\langle .,. \rangle$ the real vector space V becomes Euclidean. To choose a set of positive roots and a basis of the root system we do the following: From the inclusion $k \subset \mathbb{R}$ follows that $\mathbf{T}$ is also a maximal $\mathbb{R}$-split torus, that is $X(\mathbf{T}) = X_{\mathbb{R}}(\mathbf{T})$. Using the fact that $\mathbf{G}$ is reductive, by Lemma 11.35 there is a matrix $x \in \mathrm{GL}_n(\mathbb{R})$ such that the following statements are fulfilled for the $\mathbb{R}$-group $\mathbf{G}' := x\mathbf{G}x^{-1}$:

a) $\mathbf{G}'$ is self adjoint, that means $g^\top \in \mathbf{G}'$ holds for every $g \in \mathbf{G}'$.

b) The group $\mathbf{G}'$ contains a maximal $\mathbb{R}$-split torus $\mathbf{S}$, which lies in the group $\mathbf{D}_n$ of diagonal matrices.

c) Let $\phi' = \phi(\mathbf{S}, \mathbf{G}')$ be the corresponding set of roots. It contains a system of positive roots $\psi = \phi'^+$, whose elements are exactly those $(\varepsilon_i - \varepsilon_j)$ for $1 \le i < j \le n$, which appear as roots. Here ε_i is the character $\mathrm{diag}(d_1 \ldots, d_n) \mapsto d_i$. Furthermore, the corresponding unipotent $\mathbb{R}$-subgroup $\mathbf{N}' = \mathbf{U}_\psi$ lies in the group of upper triangular matrices $\mathbf{U}_n$.

d) The components of the Iwasawa decomposition of $\mathrm{GL}_n(\mathbb{R})$ of each element $g \in \mathbf{G}'(\mathbb{R})$ lie again in $\mathbf{G}'(\mathbb{R})$.

Define further $\mathbf{T}' := x\mathbf{T}x^{-1}$. The canonical $\mathbb{R}$-isomorphism $\varphi_x \colon \mathbf{G} \to \mathbf{G}'$ induces isomorphisms of groups $X_\mathbb{R}(\mathbf{T}') \to X_\mathbb{R}(\mathbf{T})$ and $X(\mathbf{T}') \to X(\mathbf{T})$ via $\chi' \mapsto \chi' \circ \varphi_x$. Thus, $\mathbf{T}'$ also must be an $\mathbb{R}$-split torus of $\mathbf{G}'$. By Lemma 11.27 $\mathbf{T}'$ is conjugate to $\mathbf{S}$ by an element of $\mathbf{G}'(\mathbb{R})$. Replacing x by a suitable element of $\mathrm{GL}_n(\mathbb{R})$ we can therefore assume without loss of generality $\mathbf{T}' = \mathbf{S}$ in c) above.

The $\mathbb{R}$-isomorphism φ_x further induces an isomorphism between the Lie algebras $(\varphi_x)_* \colon \mathfrak{g} \to \mathfrak{g}'$ of $\mathbf{G}$ and $\mathbf{G}'$ (the corresponding tangent map). This is also compatible with the adjoint action by

$$(\varphi_x)_*(\mathrm{Ad}_g(X)) = xgXg^{-1}x^{-1} = \mathrm{Ad}_{xgx^{-1}}(xXx^{-1}) = \mathrm{Ad}_{\varphi_x(g)}((\varphi_x)_*(X)).$$

So the induced isomorphism of groups $X(\mathbf{T}') \cong X(\mathbf{T})$ maps $\phi(\mathbf{G}', \mathbf{T}')$ bijectively to $\phi(\mathbf{G}, \mathbf{T})$. Thus we can endow $V' = X(\mathbf{T}') \otimes_\mathbb{Z} \mathbb{R}$ with a Weyl invariant scalar product, such that the isomorphism of groups extends to an isomorphism of Euclidean vector spaces $V' \to V = X(\mathbf{T}) \otimes_\mathbb{Z} \mathbb{R}$. Let ϕ'^+ be the set of positive roots from c) above and choose $\phi^+ \subset \phi$ as its corresponding image under the induced isomorphism. If $\mathbf{N}'$ is the unipotent $\mathbb{R}$-subgroup from c), then

$$\mathbf{N} := x^{-1}\mathbf{N}'x$$

is the unique connected unipotent k-subgroup of $\mathbf{G}$ from Lemma 11.25, which corresponds to ϕ^+. We set

$$\mathbf{P} := \mathbf{N} \cdot \mathbf{T}$$

as the semidirect product. This is a Borel subgroup in $\mathbf{G}$ which is defined over k, because $\mathbf{N}$ and $\mathbf{T}$ are k-groups. We identify $X(\mathbf{P}) \cong X(\mathbf{T})$ under the isomorphism induced by the inclusion $\mathbf{T} \subset \mathbf{P}$. Pick the unique basis of the root system $\Delta \subset \phi^+$ corresponding to $\mathbf{P}$. This also corresponds to the choice of a basis Δ'. For later use, we set $\bar{\Delta} = \{1, \ldots, |\Delta|\}$, and a typical simple root is denoted by α_i with $i \in \bar{\Delta}$. For every $j \in \bar{\Delta}$ denote by λ_j the fundamental weight in V, that is defined by $\langle \lambda_j, \alpha_i \rangle = \delta_{ij}$ for all $i \in \bar{\Delta}$. We define

$$\mathbf{P}_i := \mathbf{P}_{\Delta \smallsetminus \{\alpha_i\}}$$

for $i \in \bar{\Delta}$ to be the maximal parabolic k-subgroups of $\mathbf{G}$ containing $\mathbf{P}$. Their Lie algebras are given by

$$\mathfrak{p}_i = \bigoplus_{\alpha \in \phi^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \phi_i} \mathfrak{g}_{-\alpha} \oplus \mathfrak{t}$$

where $\mathfrak{t}$ is the Lie algebra of $\mathbf{T}$ and $\phi_i = \{\sum_{j \in \bar{\Delta}} m_j \alpha_j \in \phi^+ \mid m_i = 0\}$. Just note that we can define parabolic $\mathbb{R}$-subgroups $\mathbf{P}'$ and $\mathbf{P}'_i$ of $\mathbf{G}'$ in the same way. These are then exactly given by $\mathbf{P}' = \varphi_x(\mathbf{P})$ and $\mathbf{P}'_i = \varphi_x(\mathbf{P}_i)$.

Let $\mathcal{O} = \mathbb{Z}$ be the ring of integers in $k = \mathbb{Q}$. Let further $V = V^\infty \cup V^f$ be the set of places of k, where V^∞ is the set of infinite and V^f is the set of finite places. By k_v we denote the completion of k with respect to the absolute value $|.|_v$ for $v \in V$. If v is a finite place we also write $\mathcal{O}_v$ for the completion of $\mathcal{O}$ in k_v. As usual, we set

$$\mathbf{G}(\mathcal{O}) := \mathbf{G}(k) \cap \mathrm{GL}_n(\mathcal{O}) \quad \text{and} \quad \mathbf{G}(\mathcal{O}_v) := \mathbf{G}(k_v) \cap \mathrm{GL}_n(\mathcal{O}_v).$$

Convention 3. We assume that the number $\nu(\mathbf{G}, \mathbf{P}) := |\mathbf{P}(k) \backslash \mathbf{G}(k) / \mathbf{G}(\mathcal{O})|$ is equal to 1, which means that $\mathbf{G}(k) = \mathbf{P}(k)\mathbf{G}(\mathcal{O})$.

Convention 4. For all finite places $v \in V^f$ the subgroup $\mathbf{G}(\mathcal{O}_v)$ of $\mathbf{G}(k_v)$ is maximal compact. In addition, let $\mathbf{G}(k_v) = \mathbf{P}(k_v)\mathbf{G}(\mathcal{O}_v)$.

Let $\mathbb{A}$ be the set of adeles of k. For a finite subset $S \subset V$ containing V^∞ we set $\mathbb{A}_S = \prod_{v \in S} k_v \times \prod_{v \in V \smallsetminus S} \mathcal{O}_v$. We write $\mathbb{A}_\infty = \mathbb{A}_{V^\infty}$ for the special case $S = V^\infty$. Via the diagonal embedding $k \subset \mathbb{A}$, the latter becomes a k-algebra. Viewing $\mathbf{G}$ as the corresponding group functor, we obtain the group of adelic points $\mathbf{G}(\mathbb{A})$. For $x = (x_v)_{v \in V} \in \mathbb{A}$, we write $\|x\| = \prod_{v \in V} |x_v|_v$, for the idele norm. Recall that $\|q\| = 1$ for all $q \in k$. In the same way as in the case $k \subset \mathbb{A}$, we can see $\mathbf{G}(k)$ as a discrete subgroup of $\mathbf{G}(\mathbb{A})$. As before, we write

$$\mathbf{G}(\mathbb{A}_S) = \prod_{v \in S} \mathbf{G}(k_v) \times \prod_{v \in V^f \smallsetminus S} \mathbf{G}(\mathcal{O}_v) \quad \text{and} \quad \mathbf{G}_S = \mathrm{pr}_S(\mathbf{G}(\mathbb{A}_S))$$

for every finite subset $S \subset V$ containing V^∞. Define further

$$\mathbf{G}(\mathbb{A}_\infty) := \mathbf{G}(\mathbb{A}_{V^\infty}) = \prod_{v \in V^\infty} \mathbf{G}(k_v) \times \prod_{v \in V^f} \mathbf{G}(\mathcal{O}_v).$$

Convention 5. Furthermore, we assume that the class number $\mathrm{cl}(\mathbf{G})$ is equal to 1. That is $\mathbf{G}(k)\mathbf{G}(\mathbb{A}_\infty) = \mathbf{G}(\mathbb{A})$.

Example 15.1. Conventions 1 to 5 all hold in the case of a simply connected Chevalley group $\mathbf{G}$ if we assume that it is k-simple: It is an algebraic matrix group fulfilling 1 and 2 because of [Ste16, Theorem 6 and Corollary 4]. Properties 3 and 4 are [Ste16, Theorem 18 and Corollary 2 on p. 72]. The class number is equal to 1 by Lemma 12.5 because $\mathbf{G}(\mathbb{R})$ is not compact by [Ste16, Lemma 34 a)] and $\mathbf{G}$ is k-simple. For the precise definition of a Chevalley group see [Ste16, Chapter 3]. In this source the terminology of being simply connected is stated as *universal*, but this coincides with the definition of simple connectedness of an algebraic matrix group. See also [PR94, Section 2.1.13].

More explicitly, consider the case $\mathbf{G} = \mathbf{SL}_n$. This is a simply connected Chevalley group, see [Ste16, examples on p.22 and p.30]. It is obviously stable under matrix transposition, so that we can assume $\mathbf{G} = \mathbf{G}'$ in this case. Its diagonal subgroup $\mathbf{T} = \mathbf{SL}_n \cap \mathbf{D}_n$ is a maximal k-split torus and the characters $\big(\alpha_i = (\varepsilon_i - \varepsilon_{i+1})\big)_{i=1,\ldots,n-1}$ form a base of the corresponding root system $\phi(\mathbf{T}, \mathbf{G})$. This and simple connectedness are also stated in [PR94, Example to Theorem 2.6 on p. 63]. The assertion about the class number is [Bor63].

15.2 Notation about the groups

Fix a finite subset S of V containing V^∞. In the following, we define several subgroups of $\mathbf{G}_S = \prod_{v\in S} \mathbf{G}(k_v)$. To do so we use the partition $S = (V^\infty \cap S) \cup (V^f \cap S)$.

The field $k = \mathbb{Q}$ has only one Archimedean place and the corresponding absolute value is the standard absolute value which we denote by $|.|_\infty$. The completion k_∞ is given by the field of real numbers $\mathbb{R}$. We write

$$G = \mathbf{G}(\mathbb{R}) \quad \text{and} \quad P^\infty = \mathbf{P}(\mathbb{R}) \quad \text{and} \quad N^\infty = \mathbf{N}(\mathbb{R}).$$

Furthermore, let $A \subset \mathbf{T}(\mathbb{R})$ be the identity component with respect to the topology induced by $\mathbb{R}$. We observe that $\chi(a) > 0$ holds for every $\chi \in X_k(\mathbf{T})$ and $a \in A$, as the characters induce continuous maps $\mathbf{T}(\mathbb{R}) \to \mathbb{R}^\times$ mapping the identity to 1. Now we use the group $\mathbf{G}'$: The induced map $\mathbf{T}(\mathbb{R}) \to \mathbf{T}'(\mathbb{R})$ is a homeomorphism, therefore $A' = xAx^{-1}$ is the identity component in $G' := \mathbf{G}'(\mathbb{R})$. Note that every $a = \operatorname{diag}(a_1, \ldots, a_n) \in A'$ only has positive entries on the diagonal, because the maps $a \mapsto a_i$ define $\mathbb{R}$-characters. Further, let

$$K^\infty := \varphi_x^{-1}(K')$$

for $K' := G' \cap \mathrm{O}(n)$. We also see $N' := \mathbf{N}'(\mathbb{R}) = \varphi_x(N^\infty)$. The following is the *Iwasawa decomposition*:

Lemma 15.2. *The multiplication map*

$$N^\infty \times A \times K^\infty \to G,\ (u,a,k) \mapsto uak$$

is a homeomorphism.

Proof. By Lemma 11.36 the multiplication map $N' \times A' \times K' \to G'$ is a homeomorphism. The homeomorphism $\varphi_x \colon G \to G'$ maps N^∞ to N', A to A' and K^∞ to K'. From this follows the assertion. $\square$

The Iwasawa decomposition directly implies

$$G = N^\infty A K^\infty = P^\infty K^\infty.$$

The groups G, N^∞, A and K^∞ are all Lie groups, as they are closed subgroups of $\mathrm{GL}_n(\mathbb{R})$, and their exponential map at 1 is given by the matrix exponential by Corollary 7.7. From this follows that the map $(u,a,k) \mapsto uak$ from the Iwasawa decomposition is in fact smooth. The same properties hold for G' and its corresponding subgroups.

We write

$$N'_c = \{u \in N' \mid \text{ for all } 1 \le i < j \le n \colon\ |u_{ij}|_\infty \le c\}$$

and define

$$N_c^\infty := \varphi_x^{-1}(N'_c).$$

Further we set

$$A_r = \{a \in A \mid \alpha_i(a) \ge r \text{ for all } i \in \bar{\Delta}\}$$

and for later use we also note

$$A'_r := \{a' \in A' \mid \forall \alpha' \in \Delta' \colon \alpha'(a') > r\} = \varphi_x(A_r).$$

Using the terminology above we call the subset

$$S_{c,r} = N_c^\infty A_r K^\infty$$

of G a *Siegel set.*

Lemma 15.3. *There are constants* $c, r > 0$ *with* $\mathbf{G}(\mathcal{O}) S_{c,r} = G$.

Proof. Using $\nu(\mathbf{G}, \mathbf{P}) = 1$ the statement follows directly from [PR94, Theorem 4.15 b), p.224]. $\square$

In addition, we set

$$A_{r,R,i} = \{a \in A_r \mid \alpha_i(a) \ge R\} \quad \text{and} \quad S_{c,r,R,i} = N_c^\infty A_{r,R,i} K^\infty$$

for $R > r > 0$ and $i \in \bar{\Delta}$.

Now, we also use the finite places and set

$$H = \prod_{v \in V^f \cap S} \mathbf{G}(k_v).$$

For $v \in V^f \cap S$, let

$$K^f = \prod_{v \in V^f \cap S} \mathbf{G}(\mathcal{O}_v).$$

Further, we write

$$P = \prod_{v \in S} \mathbf{P}(k_v) \quad \text{and} \quad P_i = \prod_{v \in S} \mathbf{P}_i(k_v)$$

as well as

$$T = \prod_{v \in S} \mathbf{T}(k_v) \quad \text{and} \quad N = \prod_{v \in S} \mathbf{N}(k_v) \quad \text{and} \quad K = K^\infty \times K^f.$$

We notice

$$\mathbf{G}_S = G \times H = PK$$

because of 4) and the Iwasawa decomposition of G. Set $|a| := \prod_{v \in S} |a_v|_v$ for $a = (a_v)_{v \in S} \in \prod_{v \in S} k_v$.

Lemma 15.4. *Whenever $pk = p'k'$ holds for $p, p' \in P$ and $k, k' \in K$ we have $|\alpha(p)| = |\alpha(p')|$ for all $\alpha \in \Delta$.*

Proof. The group $P \cap K$ is compact with respect to the topology induced by the fields k_v. So the image of the continuous homomorphism $P \cap K \to \mathbb{R}_{>0}$ which maps p to $|\alpha(p)|$ is bounded, and therefore trivial. $\square$

Furthermore, we define the subsets

$$P^\circ = \{p \in P \mid |\alpha_i(p)| = 1 \text{ for all } i \in \bar{\Delta}\}$$

and

$$P_r = \{p \in P \mid |\alpha_i(p)| \geq r \text{ for all } i \in \bar{\Delta}\}$$

of P.

Finally, we denote by

$$\Gamma = \mathbf{G}(\mathcal{O}_S) = \mathbf{G}(k) \cap \bigcap_{v \in V \setminus S} \mathbf{G}(\mathcal{O}_v)$$

the S-arithmetic subgroup of $\mathbf{G}(k)$. Under the diagonal embedding into $\mathbf{G}(\mathbb{A})$ we also can see it as the subgroup

$$\Gamma = \mathbf{G}(k) \cap \mathbf{G}(\mathbb{A}_S).$$

Using the projection pr_S,

$$\Gamma = \mathrm{pr}_S(\mathbf{G}(k) \cap \mathbf{G}(\mathbb{A}_S))$$

becomes a discrete subgroup of $G \times H$. In the following, we will refer to all three versions of it as the group Γ and use the suitable description of it depending on the context.

Lemma 15.5. *We have* $(\Gamma \cap P) \subset P^\circ$. *In particular, this means* $\Gamma \cap P = \Gamma \cap P^\circ$.

Proof. An element $\gamma \in (\Gamma \cap P)$ may be written as $(\gamma)_{v \in S} \in \prod_{v \in S} \mathbf{G}(k_v)$ and by definition $|\alpha(\gamma)| = \prod_{v \in S} |\alpha(\gamma)|_v$. But for all $v \in V \smallsetminus S$ the element γ lies in $\mathbf{G}(\mathcal{O}_v)$, which means $|\alpha(\gamma)|_v = 1$. Thus, by the product formula we get

$$|\alpha(\gamma)| = \prod_{v \in S} |\alpha(\gamma)|_v \cdot \prod_{v \in V \smallsetminus S} |\alpha(\gamma)|_v = \|\alpha(\gamma)\| = 1. \qquad \square$$

15.3 Notation about the geometric space

Here we construct a space $X = X_\infty \times X_f$ on which $G \times H$ naturally acts.

We begin with the smooth part X_∞. As the isomorphism of Lie groups $\varphi_x \colon G \to G'$ maps K^∞ to K' we can identify the manifolds $G/K^\infty \cong G'/K'$ under the induced diffeomorphism. The group $G' = \mathbf{G}'(\mathbb{R})$ fulfills the assumptions of the group in Example 8.4 by construction: The space $P_n(\mathbb{R}) \cap G' \cong G'/K'$ is a Riemannian manifold with scalar product $\langle X, Y \rangle_p = \mathrm{tr}(p^{-1}Xp^{-1}Y)$ for $X, Y \in T_p(G' \cap P_n(\mathbb{R})) \leq S_n(\mathbb{R})$. The group G' acts smoothly by Riemannian isometries on this space via $g.p = gpg^\top$. The action is transitive and the subgroup $K' = \mathrm{O}(n) \cap G'$ of G' is maximal compact and the stabilizer of the unit matrix $\mathbb{1}_n \in P_n(\mathbb{R}) \cap G'$. Thus, the action is proper. The geodesic lines $\mathbb{R} \to P_n(\mathbb{R}) \cap G'$ are exactly the maps $t \mapsto \exp(tX)$ with $X \in T_p(P_n(\mathbb{R}) \cap G')$ and with $\mathrm{tr}(X) = 1$. We identify

$$X_\infty := G' \cap P_n(\mathbb{R}) \cong G'/K' \cong G/K^\infty$$

as Riemannian manifolds and call it the symmetric space associated to the group G (the Riemannian structure on the right side is the one induced by the identification as smooth manifolds, compare Lemma 7.10). As a metric space, X_∞ is then a proper CAT(0) space. In particulay, it is locally compact and complete. We denote by $o_\infty := \mathbb{1}_n \in X_\infty$ the *basepoint*. The canonical action of G on X_∞ is then also smooth and by Riemannian isometries, and we have

$$G.o_\infty = X_\infty \quad \text{and} \quad \mathrm{Stab}_G(o_\infty) = K^\infty.$$

Furthermore, it is also proper, as G acts with compact stabilizers.

Lemma 15.6. *The map* $N^\infty \times A \to X_\infty$ *defined by* $(u,a) \mapsto (ua).o_\infty$ *is a smooth homeomorphism.*

Proof. Using the identification $X_\infty \cong G/K^\infty$, it suffices to show that the smooth map from $N^\infty \times A$ to G/K^∞ defined by $(u,a) \mapsto uaK$ is a homeomorphism. This follows directly from the Iwasawa decomposition. $\square$

Remark 15.7. Note that we defined the group G' only as an auxiliary construction so that we could construct the space X_∞. In fact, most of the times we will only care about the abstract action of G on the Riemannian manifold with the properties stated above. In those few cases where we need the concrete Riemannian structure of X_∞ we will pass from G to G' in the following sense: For every $p \in X_\infty = G' \cap P_n(\mathbb{R})$ an element $g \in G$ acts on it via $g.p = (xgx^{-1}).p$, where on the left side we consider the action of G and on the right side that of G'.

For $v \in V^f$ let X_v be the Euclidean building associated to the group $\mathbf{G}(k_v)$. Its existence we deduce from Convention 1 and Proposition 13.8. This is also a locally compact, complete CAT(0) space on which $\mathbf{G}(k_v)$ acts properly via isometries. As we demanded in Convention 4, the subgroup $\mathbf{G}(\mathcal{O}_v)$ of $\mathbf{G}(k_v)$ is maximal compact and therefore the stabilizer of a vertex $o_v \in X_v$. We now set

$$X_f := \prod_{v \in V^f \cap S} X_v$$

and endow it with the product metric. Then X_f also becomes a complete locally compact CAT(0) space on which $H = \prod_{v \in S \cap V^f} \mathbf{G}(k_v)$ acts properly, cocompactly and via isometries. We also refer here to $o_f := (o_v)_{v \in S \cap V^f} \in X^f$ as the *basepoint*.

Finally, we define the space

$$X := X_\infty \times X_f$$

and here we take the product metric, too. Then X is also a locally compact, complete CAT(0) space on which $G \times H$ acts properly and continuously via isometries. By the diagonal embedding into $G \times H$ we also get an action of the discrete subgroup Γ on X with the same properties. Again we call $o := (o_\infty, o_f)$ the *basepoint* of X.

16 Reduction theory

After we fixed the notation in the last section, let me briefly explain what this one is about. We want to extend Raghunathan's proof in [Rag68] to the case of the S-arithmetic group Γ, but using the action on X instead of that on the group $G \times H$. To prove his statements, he lists some results from reduction theory of arithmetic groups, see [Rag68, (i) - (iv) on p.326]. The goal now is to give analogous statements that fit our case.
This section is divided into two parts: The first part serves to deduce some statements from the reduction theory of S-arithmetic groups that hold in our setting. In the second part we will give geometric versions of these statements about the space X.

16.1 Reduction theory of S-arithmetic groups

The theorems from the classical reduction theory of S-arithmentic groups like in [PR94, Chapter 5] or [Bor63]are too general for our later argumentation. To work with them we adjust some of the proofs. We also extend some results from reduction theory of arithmetic groups to the S-arithmetic case. These include [PR94, Lemma 4.7 (1), p. 189] and [Rag68, Lemma 2.1.]

In the following we will occasionally use these two easy facts, so it is useful to keep them in mind:

a) If X is an abstract group containing a subgroup Y and two subsets A, B, such that $X = A \cdot B$ and $B \subset Y$ hold, then $Y = (A \cap Y) \cdot B$.

b) Let X and Y be two abstract groups such that A is a subgroup of both. Assume further Y contains a subgroup C with $A \subset C$ and X contains a subset B. Then $(AB) \times C = A.(B \times C)$ holds where $a.(b, c) := (ab, ac)$.

Proposition 16.1. *Let $B \subset G$ be any subset with $\mathbf{G}(\mathcal{O})B = G$. Then $\Gamma.(B \times K^f) = G \times H$ holds. This implies in particular $\Gamma.(S_{c,r} \times K^f) = G \times H$ for a suitable Siegel set $S_{c,r} \subset G$.*

Proof. A more general statement is proven in [PR94, Proposition 5.11, p.267], see also [Bor63, Theorem 8.5]. We will go through the proof of the first source in detail, as it will give us the desired statement in the case that we consider. From $\mathbf{G}(\mathbb{A}) = \mathbf{G}(k)\mathbf{G}(\mathbb{A}_\infty)$ and a) follows

$$\mathbf{G}(\mathbb{A}_S) = (\mathbf{G}(k) \cap \mathbf{G}(\mathbb{A}_S)) \cdot \mathbf{G}(\mathbb{A}_\infty)$$

by $\mathbf{G}(\mathbb{A}_\infty) \subset \mathbf{G}(\mathbb{A}_S)$. Applying pr_S to both sides, this implies

$$\mathbf{G}_S = \Gamma \cdot \left(\prod_{v \in (S \cap V^\infty)} \mathbf{G}(k_v) \times \prod_{v \in (S \cap V^f)} \mathbf{G}(\mathcal{O}_v) \right)$$

as $\mathrm{pr}_S(\mathbf{G}(k) \cap \mathbf{G}(\mathbb{A}_S)) = \Gamma$. Because of $V^\infty \subset S$, we know that $G = \mathbf{G}(\mathcal{O})B$ is equal to $\prod_{v \in (S \cap V^\infty)} \mathbf{G}(k_v)$. Furthermore, using $\mathbf{G}_S = G \times H$ and $K^f = \prod_{v \in (S \cap V^f)} \mathbf{G}(\mathcal{O}_v)$, the above reduces to

$$G \times H = \Gamma . \left((\mathbf{G}(\mathcal{O}) \cdot B) \times K^f \right).$$

Now $\mathbf{G}(\mathcal{O})$ embeds diagonally into the product $K^f = \prod_{v \in S \cap V^f} \mathbf{G}(\mathcal{O}_v)$ and can thus be seen as a subgroup of this. We deduce with b)

$$(\mathbf{G}(\mathcal{O})B) \times K^f = \mathbf{G}(\mathcal{O}).(B \times K^f),$$

which implies that $G \times H = \Gamma.(B \times K^f)$, using the fact $\mathbf{G}(\mathcal{O}) \subset \Gamma$. That we can find such a Siegel set now follows from Lemma 15.3. □

Next, we want to show that $(\Gamma \cap P^\circ) \cdot (S_{c,r} \times K^f) = P_r K$ holds for a suitable Siegel set $S_{c,r} \subset G$. To do this, we first prove two lemmas.

Lemma 16.2. *We have* $G \times H = (\Gamma \cap P).(G \times K^f)$.

Proof. In the case at hand we have $\mathrm{cl}(\mathbf{G}) = 1$ by Convention 5 and $\mathbf{G}(k_v) = \mathbf{P}(k_v)\mathbf{G}(\mathcal{O}_v)$ for all $v \in V^f$ by Convention 4. Thus, [PR94, Proposition 5.10, p.259] implies that $\mathrm{cl}(\mathbf{P}) = \nu(\mathbf{G}, \mathbf{P})$. The groups $\mathbf{P}(\mathbb{A})$ and $\mathbf{P}(\mathbb{A}_\infty)$ are defined in the same way as $\mathbf{G}(\mathbb{A})$ and $\mathbf{G}(\mathbb{A}_\infty)$. Using $\nu(\mathbf{G}, \mathbf{P}) = 1$ (see Convention 3) we get

$$\mathbf{P}(\mathbb{A}) = \mathbf{P}(k)\mathbf{P}(\mathbb{A}_\infty).$$

Again, $\mathbf{G}(k_v) = \mathbf{P}(k_v)\mathbf{G}(\mathcal{O}_v)$ holds for all $v \in V^f$ and we know $G = P^\infty K^\infty$. So we obtain

$$\mathbf{G}(\mathbb{A}) = \mathbf{P}(\mathbb{A}) \cdot \left(K^\infty \times \prod_{v \in V^f} \mathbf{G}(\mathcal{O}_v) \right).$$

These two equations lead to

$$\mathbf{G}(\mathbb{A}) = \mathbf{P}(k) \left(\mathbf{P}(\mathbb{A}_\infty) \cdot \left(K^\infty \times \prod_{v \in V^f} \mathbf{G}(\mathcal{O}_v) \right) \right)$$

Furthermore, $\mathbf{P}(\mathbb{A}_\infty) \cdot (K^\infty \times \prod_{v \in V^f} \mathbf{G}(\mathcal{O}_v))$ is a subset of $\mathbf{G}(\mathbb{A}_\infty)$, so we see $\mathbf{G}(\mathbb{A}) \subset \mathbf{P}(k)\mathbf{G}(\mathbb{A}_\infty)$. From this follows

$$\mathbf{G}(\mathbb{A}) = \mathbf{P}(k)\mathbf{G}(\mathbb{A}_\infty),$$

compare [PR94, Equation (5.10) on p. 258]. Since $\mathbf{G}(\mathbb{A}_\infty)$ is a subset of $\mathbf{G}(\mathbb{A}_S)$ we obtain

$$\mathbf{G}(\mathbb{A}_S) = (\mathbf{P}(k) \cap \mathbf{G}(\mathbb{A}_S)) \cdot \mathbf{G}(\mathbb{A}_\infty)$$

using a). Applying the projection pr_S to both sides, we get

$$\mathbf{G}_S = \mathrm{pr}_S(\mathbf{P}(k) \cap \mathbf{G}(\mathbb{A}_S)) \cdot \left(\prod_{v \in V^\infty \cap S} \mathbf{G}(k_v) \times \prod_{v \in V^f \cap S} \mathbf{G}(\mathcal{O}_v) \right).$$

The assertion follows from the identities $G \times H = \mathbf{G}_S$, $\Gamma \cap P = \mathrm{pr}_S(\mathbf{P}(k) \cap \mathbf{G}(\mathbb{A}_S))$, $G = \prod_{v \in V^\infty \cap S} \mathbf{G}(k_v)$, and $K^f = \prod_{v \in V^f \cap S} \mathbf{G}(\mathcal{O}_v)$). ☐

Lemma 16.3. *The identity*

$$(\Gamma \cap P^\circ) \cdot (N^\infty A_r K^\infty \times K^f) = P_r K$$

holds for all $r > 0$.

Proof. Take some $\gamma \in (\Gamma \cap P^\circ)$, $u \in N^\infty$, $a \in A_r$, and $k \in K$. Then $\gamma.(ua, 1_H)$ lies in P_r, and thus $(\gamma.(ua, 1_H)) \cdot k$ is an element of $P_r K$.
Now, choose $pk \in P_r K$. This means $|\alpha(p)| \geq r$ for all $\alpha \in \Delta$. Recall $\Gamma \cap P = \Gamma \cap P^\circ$ from Lemma 15.5. So by the previous lemma we can find $\gamma \in (\Gamma \cap P^\circ)$ such that $\gamma.(pk) \in (G \times K^f)$. Using the Iwasawa decomposition $G = N^\infty A K^\infty$, there are $u \in N^\infty$, $a \in A$ and $k' \in K$ such that

$$\gamma.(pk) = (ua, 1_H) \cdot k' \in (G \times K^f) = (N^\infty A \times \{1_H\}) \cdot K.$$

But $\gamma \in P^\circ \subset P$ implies $(\gamma.p) \in P$ and thus $(\gamma.p)k \in PK$. So both, $(\gamma.p) \cdot k$ and $(ua, 1_H) \cdot k'$ are decompositions of the same element into $P \cdot K$. Thus, $|\alpha(\gamma.p)| = |\alpha(ua, 1_H)|$ holds by Lemma 15.4. So

$$|\alpha(\gamma.p)| = |\alpha(p)| \geq r \quad \text{and} \quad |\alpha(ua, 1_H)| = \alpha(a)$$

imply that $\alpha(a) \geq r$ so that $(\gamma.p)k \in (N^\infty A_r \times \{1_H\}) \cdot K$. This leads to $pk \in \gamma^{-1}.(N^\infty A_r K^\infty \times K^f)$. ☐

Proposition 16.4. *There is* $c > 0$ *with* $\mathbf{N}(\mathcal{O}).N^\infty_c = N^\infty$. *For every such* c, *the identity*

$$(\Gamma \cap P^\circ) \cdot (S_{c,r} \times K^f) = P_r K$$

holds for all $r > 0$.

Proof. The existence of such a $c > 0$ follows directly from [PR94, Lemma 4.7 (1), p.189]. From the fact that we can see $\mathbf{N}(\mathcal{O})$ as a subgroup of K^f via the diagonal embedding $\mathbf{G}(\mathcal{O}) \subset K^f$, using b) we obtain

$$\mathbf{N}(\mathcal{O}).(N^\infty_c \times K^f) = (\mathbf{N}(\mathcal{O}).N^\infty_c) \times K^f = N^\infty \times K^f.$$

Furthermore, $|\alpha(u)| = 1$ holds for all $u \in N$ and all $\alpha \in \Delta$, so $\mathbf{N}(\mathcal{O})$ is a subset of $\Gamma \cap P^\circ$. Therefore, using

$$S_{c,r} \times K^f = \big((N^\infty_c \times K^f) \cdot (A_r K^\infty \times \{1_H\})\big)$$

we get

$$\begin{aligned}(\Gamma \cap P^\circ).(S_{c,r} \times K^f) &= \big((\Gamma \cap P^\circ) \cdot \mathbf{N}(\mathcal{O})\big).\big((N_c^\infty \times K^f) \cdot (A_r K^\infty \times \{1_H\})\big) \\ &= (\Gamma \cap P^\circ).\big((N^\infty \times K^f) \cdot (A_r K^\infty \times \{1_H\})\big) \\ &= (\Gamma \cap P^\circ).(N^\infty A_r K^\infty \times K^f)\end{aligned}$$

so that the statement follows from the previous lemma. □

Next, we need a lemma of reduction theory of arithmetic groups stated and proven by Raghunathan. Choose a *Chevalley basis* $e_1, \ldots, e_N$ for the semisimple Lie algebra $\mathfrak{g}$, see [Ste16, Theorem 1].

Remark 16.5. In fact what we need is a basis fulfilling properties a) to c) from [Rag68, p.321]. But with our restrictions, considering a Chevalley basis is sufficient, see also [PR94, p.64-64].

Example 16.6. The Lie algebra $\mathfrak{g}$ of $\mathbf{G} = \mathbf{SL}_n$ is given by the matrices with trace 0. The matrices of the form E_{ij} and D_i form a Chevalley basis for $\mathfrak{g}$. Here E_{ij} is the $n \times n$ matrix with (i, j)-entry is 1 and all the other entries are 0, and $D_i = \operatorname{diag}(0, \ldots, 0, 1, -1, , 0, \ldots, 0)$ where the 1 on the diagonal is on the i-th coordinate. See [Ste16, Remark c) to Theorem 1].

In the next lemma, we use the following terminology: For $q \in \mathbf{G}(k)$ we refer to the *entries* of Ad_q as the entries of the matrix of the linear map Ad_q with respect to the basis $e_1, \ldots, e_N$.

Lemma 16.7 (Raghunathan). *Fix a Siegel set $S_{c,r} \subset G$ and an integer $p \in \mathbb{N}$. Then there is $R > r$ such that, for every $i \in \bar{\Delta}$, the following holds: If $q \in \mathbf{G}(k)$ is an element such that the denominators of all entries of Ad_q and $\mathrm{Ad}_{q^{-1}}$ when reduced to the simplest form divide p, and such that $q.S_{c,r,R,i} \cap S_{c,r} \neq \varnothing$, then q lies in P_i^∞.*
Furthermore, for each $r' > 0$ we can choose $R > r$ in a way that additionally the following statement is true: If $q.g = g'$ for some $g \in S_{c,r,R,i}$ and $g' = u'a'k' \in S_{c,r} = N_c^\infty A_r K^\infty$, then $\alpha(a') > r'$.

Proof. This is exactly [Rag68, Lemma 2.1.] stated in terms of left instead of right action. □

Using this we can prove a similar result for the action of Γ on the product $G \times H$.

Proposition 16.8. *For every Siegel set $S_{c,r}$ there is $R > r$ such that for every $i \in \bar{\Delta}$ the following holds: Each $\gamma \in \Gamma$ fulfills*

$$\gamma.(S_{c,r,R,i} \times K^f) \cap (S_{c,r} \times K^f) \neq \varnothing \Rightarrow \gamma \in P_i^\infty.$$

Furthermore, for every $r' > 0$ we can choose $R > r$ so that it also fulfills the following statement: If $\gamma.(g,h) = (g',h') \in S_{c,r} \times K^f$ for some $(g,h) \in S_{c,r,R,i} \times K^f$ then $\alpha_i(a') > r'$ writing $g' = u'a'k' \in N^\infty AK^\infty$.

Proof. Let $\gamma.(g,h) = (g',h')$ with $g \in S_{c,r,R,i}$, $g' \in S_{c,r}$ and $h, h' \in K^f$. Since $\gamma.h = h'$ implies $\gamma = h'h^{-1} \in K^f$, we have $\gamma \in (\Gamma \cap K^f) = \mathbf{G}(\mathcal{O})$. The adjoint action of $\mathbf{G}(k)$ on $\mathfrak{g}$ is given by conjugation, so if we consider the matrices corresponding to the linear maps $\mathrm{Ad}_\gamma \colon \mathfrak{g} \to \mathfrak{g}$ with respect to the basis $e_1, \ldots, e_N$ they will have uniformly bounded denominators for $\gamma \in \mathbf{G}(\mathcal{O})$. Now $\gamma.S_{c,r,R,i} \cap S_{c,r} \neq \varnothing$ implies $\gamma \in P_i^\infty$ by the lemma above. Writing g' as $u'a'k' \in N^\infty AK^\infty$, part two of the lemma also implies that $\alpha_i(a') > r'$. ☐

Corollary 16.9. *Let $S_{c,r} \subset G$ be a Siegel set fulfilling $(\Gamma \cap P^\circ) \cdot (S_{c,r} \times K^f) = P_r K$. The constant $R > r$ from the proposition is such that, for all $\gamma \in \Gamma$, we have*

$$\gamma.(S_{c,r,R,i} \times K^f) \cap P_r K \neq \varnothing \Rightarrow \gamma \in P_i^\infty.$$

Proof. As $(\Gamma \cap P^\circ) \cdot (S_{c,r} \times K^f) = P_r K$, the above condition implies that

$$\gamma.(S_{c,r,R,i} \times K^f) \cap \gamma'(S_{c,r} \times K^f) \neq \varnothing$$

for some $\gamma' \in (P^\circ \cap \Gamma)$. We obtain

$$(\gamma'^{-1}\gamma).(S_{c,r,R,i} \times K^f) \cap (S_{c,r} \times K^f) \neq \varnothing$$

so from the previous proposition follows the fact that $(\gamma'^{-1}\gamma) \in P_i^\infty$. We get $\gamma \in P_i^\infty$ from $\gamma' \in (\Gamma \cap P^\circ) \subset P_i^\infty$. ☐

Corollary 16.10. *Let $S_{c,r} \subset G$ be a Siegel set with $(\Gamma \cap P^\circ) \cdot (S_{c,r} \times K^f) = P_r K$ and let $r' > 0$. The constant $R > r$ from the proposition is such that for all $\gamma \in \Gamma$ the following hold: If $\gamma.(g,h) = pk \in P_r K$ for some $(g,h) \in S_{c,r,R,i} \times K^f$ then $|\alpha_i(p)| > r'$.*

Proof. By assumption there is $\gamma' \in (\Gamma \cap P^\circ)$ such that $(\gamma'.(\gamma.(g,h)) \in S_{c,r} \times K^f$. Writing the G part of this element $(\gamma'\gamma).(g,h))$ as $\gamma'\gamma g = u'a'k' \in N_c^\infty A_r K^\infty$ we get $\alpha_i(a') > r'$ by the previous proposition. Because of $(\gamma'\gamma h) \in K^f$ we have a decomposition $(u'a', 1_H) \cdot (k', \gamma'\gamma h) \in PK$. But as $\gamma' \in (\Gamma \cap P^\circ) \subset P$ we also have $(\gamma' p) \cdot k \in PK$. This leads to

$$(u'a', 1_H) \cdot (k', \gamma'\gamma h) = (\gamma'\gamma).(g,h) = (\gamma' p) \cdot k \in P \cdot K.$$

So Lemma 15.4 implies

$$\big|\alpha_i(\gamma'.p)\big| = \big|\alpha_i(u'a', 1_H)\big| = \alpha_i(a') > r'.$$

The assertion follows from $|\alpha_i(\gamma' p)| = |\alpha_i(p)|$. ☐

16.2 Geometric reduction theory

Now consider the action of $G \times H$ on the space $X = X_\infty \times X_f$ via $(g,h).x = (g.x_\infty, h.x_f)$ where $x = (x_\infty, x_f)$. Recall that the basepoint $o = (o_\infty, o_f)$ is the unique point in X with $\mathrm{Stab}(o) = K$.

We use the results of [HW22, §5]: For every $i \in \bar{\Delta}$, there is a rescaled Busemann function $\tilde{\beta}_i \colon X \to \mathbb{R}$ with the property

$$\tilde{\beta}_i(p.x) = \log(|\chi_i(p)|) + \tilde{\beta}_i(x)$$

for all $x \in X$ and $p \in P_i$, and such that $\tilde{\beta}_i(o) = 0$, see [HW22, Proposition 5.7]. So these equations especially hold for every $p \in P$ and all $i \in \bar{\Delta}$. The χ_i are characters with the property $\langle \chi_i, \alpha_j \rangle = 0$ for $i \neq j$ and $\langle \chi_i, \alpha_i \rangle > 0$ for all $i \in \bar{\Delta}$, see [HW22, Lemma 4.1]. Thus, there are constants $s_i > 0$ with $\lambda_i = \chi_i^{s_i}$ for all $i \in \bar{\Delta}$. We modify $\tilde{\beta}_i$ in a similar way as in [HW22, §8.2], but with the above choice of s_i, and set

$$\beta_i = s_i \tilde{\beta}_i$$

for all $i \in \bar{\Delta}$. Equation [HW22, (8.5) in §8.2] then leads to

$$\beta_i(p.x) = s_i \log(|\chi_i(p)|) + \beta_i(x) = \log(|\chi_i(p)^{s_i}|) + \beta_i(x) = \log(|\lambda_i(p)|) + \beta_i(x)$$

for all $p \in P$ and $x \in X$. Since $\tilde{\beta}_i(o) = 0$ we also have $\beta_i(o) = 0$ and therefore

$$\beta_i(p.o) = \log(|\lambda_i(p)|) + \beta_i(o) = \log(|\lambda_i(p)|)$$

for all $p \in P$.

Further, the authors construct in [HW22, §5] functions $\mu_i \colon X \to \mathbb{R}$ by setting

$$\mu_i = \sum_{j \in \bar{\Delta}} c_{ij} \tilde{\beta}_j,$$

where $c_{ij} \in \mathbb{R}$ are the unique coefficients fulfilling the equations $\alpha_i = \sum_{j \in \bar{\Delta}} c_{ij} \chi_j$. Now

$$\mu_i(p.x) = \log(|\alpha_i(p)|) + \mu_i(x)$$

holds for all $p \in P$ and $x \in X$ by equation (8.6) in [HW22, §8.2]. Since

$$\mu_i(o) = \sum_{j \in \bar{\Delta}} c_{ij} \tilde{\beta}_j(o) = 0$$

we also get

$$\mu_i(p.o) = \log(|\alpha_i(p)|)$$

for all $p \in P$ and $i \in \bar{\Delta}$. To express μ_i as a linear combination of the β_j, we use equations (8.7) and (8.8) from [HW22, §8.2] given by

$$\mu_i = \sum_{j \in \bar{\Delta}} \frac{c_{ij}}{s_j} \beta_j \quad \text{and} \quad \beta_j = \sum_{i \in \bar{\Delta}} s_j n_{ji} \mu_i$$

where $n_{ji} \in \mathbb{R}$ are the unique coefficients with $\chi_j = \sum_{i\in\bar{\Delta}} n_{ji}\alpha_i$. Setting

$$b_{ij} := \frac{c_{ij}}{s_j} \quad \text{and} \quad m_{ji} := s_j n_{ji}$$

we get

$$\mu_i = \sum_{j\in\bar{\Delta}} b_{ij}\beta_j \quad \text{and} \quad \beta_j = \sum_{i\in\bar{\Delta}} m_{ji}\mu_i$$

as desired. Furthermore, we obtain

$$\lambda_j = s_j\chi_j = s_j \sum_{i\in\bar{\Delta}} n_{ji}\alpha_i = \sum_{i\in\bar{\Delta}} m_{ji}\alpha_i$$

and

$$\alpha_i = \sum_{j\in\bar{\Delta}} c_{ij}\chi_j = \sum_{j\in\bar{\Delta}} \frac{c_{ij}}{s_j}\lambda_j = \sum_{j\in\bar{\Delta}} b_{ij}\lambda_j.$$

We summarize our results.

Proposition 16.11. *There are (unique) rescaled Busemann functions β_j defined on X and functions $\mu_i\colon X \to \mathbb{R}$ for all $i, j \in \bar{\Delta}$ such that*

$$\begin{aligned}
\beta_j(p.x) &= \log(|\lambda_j(p)|) + \beta_j(x)\\
\beta_j(p.o) &= \log(|\lambda_j(p)|)\\
\mu_i(p.x) &= \log(|\alpha_i(p)|) + \mu_i(x)\\
\mu_i(p.o) &= \log(|\alpha_i(p)|)
\end{aligned}$$

hold for every $p \in P$ and $x \in X$. These functions fulfill

$$\beta_j = \sum_{i\in\bar{\Delta}} m_{ji}\mu_i \quad \textit{and} \quad \mu_i = \sum_{j\in\bar{\Delta}} b_{ij}\beta_j$$

for the unique coefficients $m_{ji}, b_{ij} \in \mathbb{R}$ with

$$\lambda_j = \sum_{i\in\bar{\Delta}} m_{ji}\alpha_i \quad \textit{and} \quad \alpha_i = \sum_{j\in\bar{\Delta}} b_{ij}\lambda_j.$$

Remark 16.12. In [HW22] the Busemann functions are rescaled to get a linear combination of them with “nice” geometric properties. In our case, we rescale them in the way above, to extend Raghunathan’s proof in [Rag68] to the case $X = X_\infty \times X_f$.

The above rescaling now allows us to use statements about the interplay between the simple roots α_i and the fundamental weights λ_j.

Lemma 16.13. *For every subset* $I \subset \bar{\Delta}$ *and any* $j \in \bar{\Delta}$, *we can write*

$$\beta_j = \sum_{i \in I} c_{ji}\mu_i + \sum_{k \in \bar{\Delta} \setminus I} d_{jk}\beta_k$$

as a linear combination such that all the c_{ji} *and* d_{jk} *are non-negative.*

Proof. Follows directly from Lemma 9.6 and Proposition 16.11. □

Corollary 16.14. *For every* $j \in \bar{\Delta}$, *the rescaled Busemann function* $\beta_j = \sum_{i \in \bar{\Delta}} m_{ji}\mu_i$ *is a non-negative linear combination with* $m_{jj} > 0$.

Proof. Considering the case $I = \bar{\Delta}$ in the previous lemma we only need to check $m_{jj} > 0$. But the equality $\langle \lambda_j, \alpha_i \rangle = \delta_{ij}$ implies

$$0 < \langle \lambda_j, \lambda_j \rangle = \sum_{i \in \bar{\Delta}} m_{ji} \langle \alpha_i, \lambda_j \rangle = m_{jj}. \qquad \square$$

For later use, we also state the following lemma.

Lemma 16.15. *There is* $b > 0$ *such that* $|\mu_i(x) - \mu_i(y)| \leq b \cdot d(x,y)$ *holds for all* $x, y \in X$ *and* $i \in \bar{\Delta}$.

Proof. From $\mu_i = \sum_{j \in \bar{\Delta}} b_{ij}\beta_j$ we obtain

$$\begin{aligned} |\mu_i(x) - \mu_i(y)| &= \left| \sum_{j \in \bar{\Delta}} b_{ij}(\beta_j(x) - \beta_j(y)) \right| \\ &\leq \sum_{j \in \bar{\Delta}} |b_{ij}| \, |\beta_j(x) - \beta_j(y)| \\ &\leq \sum_{j \in \bar{\Delta}} |b_{ij}| \, a_j d(x,y) \end{aligned}$$

for some $a_j > 0$, since β_j is a rescaled Busemann function. But at least one of the b_{ij} must be $\neq 0$, as $\alpha_i \neq 0$. This implies that

$$b_i := \sum_{j \in \bar{\Delta}} |b_{ij}| \, a_j > 0.$$

Therefore $b = \min\{b_i \mid i \in \bar{\Delta}\}$ fulfills the statement. □

Next, we define subsets of the space X that will play an important role in our further argumentation.

Definition 16.16. For $s > 0$ we define the set

$$\Psi_s = \{x \in X \mid \exp(\mu_i(x)) \geq s \text{ for all } i \in \bar{\Delta}\}$$

and additionally

$$\Psi_{s,S,i} = \{x \in \Psi_s \mid \exp(\mu_i(x)) \geq S\}$$

for $S > s$ and $i \in \bar{\Delta}$. Let further

$$\Omega^d_{c,r} = (S_{c,r}.o_\infty) \times \overline{B_d}(o_f)$$

for some $c, d, r > 0$, where $S_{c,r} \subset G$ is a Siegel set and $\overline{B_d}(o_f) \subset X_f$ is the closed d-ball.

Since H acts cocompactly on X_f, we can find some $d > 0$ with $H.\overline{B_d}(o_f) = X_f$. We will need the following finiteness result.

Proposition 16.17. *For any choice of $c, d > 0$ and $r > 0$ the set*

$$\{\gamma \in \Gamma \mid (\gamma q).\Omega^d_{c,r} \cap q'.\Omega^d_{c,r} \neq \varnothing\}$$

is finite for all $q, q' \in \mathbf{G}(k)$.

Proof. Let γ be in the above set. Then we have

$$\gamma q S_{c,r} \cap q' S_{c,r} \neq \varnothing \quad \text{and} \quad \gamma q \overline{B_d}(o_f) \cap q' \overline{B_d}(o_f) \neq \varnothing.$$

The subset

$$C' = \{h \in H \mid hq\overline{B_d}(o_f) \cap q'\overline{B_d}(o_f) \neq \varnothing\}$$

is compact by properness of the action of H on X_f, so the same holds for

$$C := C' \cup C'^{-1}.$$

We now follow the argumentation in the proof of [Bor63, Lemma 8.4.]: For every finite place $v \in S \cap V^f$, the elements γ and γ^{-1} lie in the compact subspace $\mathrm{pr}_v(C)$ of $\mathbf{G}(k_v)$, where $\mathrm{pr}_v \colon H \to \mathbf{G}(k_v)$ denotes the canonical projection. Since $\mathbf{G}(k_v)$ is a subset of $k_v^{n\times n}$ we can also view $\mathrm{pr}_v(C)$ as a compact subset of $k_v^{n\times n}$. Thus, denoting by π_v a uniformizing parameter in $\mathcal{O}_v$, there is $\ell_v \in \mathbb{Z}$ with $\mathrm{pr}_v(C) \subset (\pi_v^{\ell_v}\mathcal{O}_v)^{n\times n}$: This follows from the fact that $(\pi_v^{\ell_v}\mathcal{O}_v)$ is the open and closed ball of radius $|\pi_v|_v^{\ell_v}$ in k_v. By multiplying with some $u \in \mathcal{O}_v^*$ if necessary, we can assume that π_v is an element of $\mathcal{O}$. Setting

$$s_v := \begin{cases} \pi_v^{-\ell_v} & \text{if } \ell_v \leq 0, \\ 1 & \text{if } \ell_v \geq 0, \end{cases}$$

we have $s_v \in \mathcal{O}$ and $s_v\gamma, s_v\gamma^{-1} \in (\mathcal{O}_v)^{n\times n}$. Since $\gamma, \gamma^{-1} \in \mathbf{G}(\mathcal{O}_w) \subset (\mathcal{O}_w)^{n\times n}$ for all $w \in (V^f \setminus S)$, we get $s\gamma, s\gamma^{-1} \in \mathcal{O}^{n\times n}$ for

$$s := \prod_{v \in V^f \cap S} s_v \in \mathcal{O}.$$

Thus we have shown: For every $\gamma \in \Gamma$ with

$$\gamma q\overline{B_d}(o_f) \cap q'\overline{B_d}(o_f) \neq \varnothing$$

we have $s\gamma, s\gamma^{-1} \in \mathcal{O}^{n\times n}$. By [Bor63, Lemma 3.1.] the set

$$\{\gamma \in \Gamma \mid s\gamma, s\gamma^{-1} \in \mathcal{O}^{n\times n} \text{ and } \gamma q S_{c,r} \cap q' S_{cr} \neq \varnothing\}$$

is finite, since the Siegel set $S_{c,r} \subset G$ fulfills the following property: The set

$$\{\lambda \in \mathbf{G}(\mathcal{O}) \mid \lambda q S_{c,r} \cap q' S_{c,r} \neq \varnothing\}$$

is finite by [Rag68, Theorem on p.321], see also [Bor19, Theorem 15.4]. $\square$

The main goal of this section is to prove the following theorem.

Theorem 16.18. *Let $b > 0$ fulfill Lemma 16.15 and pick $c > 0$ with $\mathbf{N}(\mathcal{O}).N_c^\infty = N^\infty$. Choose further $d > 0$ with $H.\overline{B_d}(o_f) = X_f$.*
We can find $s > 0$ such that for all $\varepsilon > 0$ with $s' := s - \varepsilon > 0$, and for $r := s'e^{-bd}$, there is $S_0 > s$ such that the following statements hold for all $S \geq S_0$:

- *a) $\Gamma.(\Omega^d_{c,r} \cap \Psi_s) = X$.*
- *b) $\Psi_{s'} \subset (P^\circ \cap \Gamma).\Omega^d_{c,r}$.*
- *c) If $\gamma.(\Omega^d_{c,r} \cap \Psi_{s',S,i}) \cap \Psi_{s'} \neq \varnothing$ for some $\gamma \in \Gamma$, then $\gamma \in P_i^\infty$.*
- *d) If $\gamma.x \in \Psi_{s'}$ for some $\gamma \in \Gamma$ and $x \in \Omega^d_{c,r} \cap \Psi_{s',S,i}$, then $\exp(\mu_i(\gamma.x)) > s$.*

Proof. Let us first sketch what will happen in the proof to get a geometric idea of what is going on. We pick some $t > 0$ such that translates of the set $\Omega^d_{c,t}$ cover X (we do not care about this constant later), then choose some s making $\Omega^d_{c,t}$ lying inside Ψ_s. If we take $r \leq t$ small enough, then it makes every $x \in \Psi_{s'}$ with $d_f(x_f, o_f) \leq d$ an element of $\Omega^d_{c,r}$. After that, we select $R > r$ so that Corollaries 16.9 and 16.10 are fulfilled. Finally we choose $S_0 > 0$ big enough such that $\mu_i(g.o_\infty, x_f) \geq S_0$ for $(g.o_\infty, x_f)$ in $\Omega^d_{c,r}$ implies that α_i of the A part in the Iwasawa decomposition of g is bigger than R.

We now prove a) in a sequence of three lemmas:

Lemma A. *There is $t > 0$ with $\Gamma.\Omega^d_{c,t} = X$.*

Proof of Lemma A. Pick an element $x = (x_\infty, x_f) \in X$. We show that there is a $\gamma \in \Gamma$ with $\gamma.x \in \Omega^d_{c,t}$, where $t > 0$ is chosen to be small enough to fulfill $\Gamma.(S_{c,t} \times K^f) = G \times H$, see Proposition 16.1. Choose $g \in G$ with

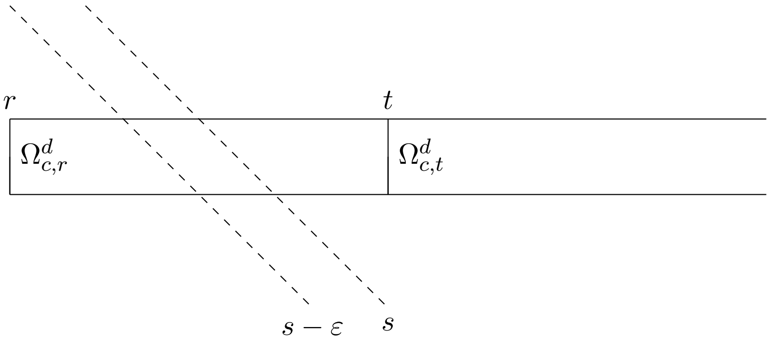

Figure 1: $\Omega^d_{c,r}$ and Ψ_s, geometric interpretation of the proof of a)

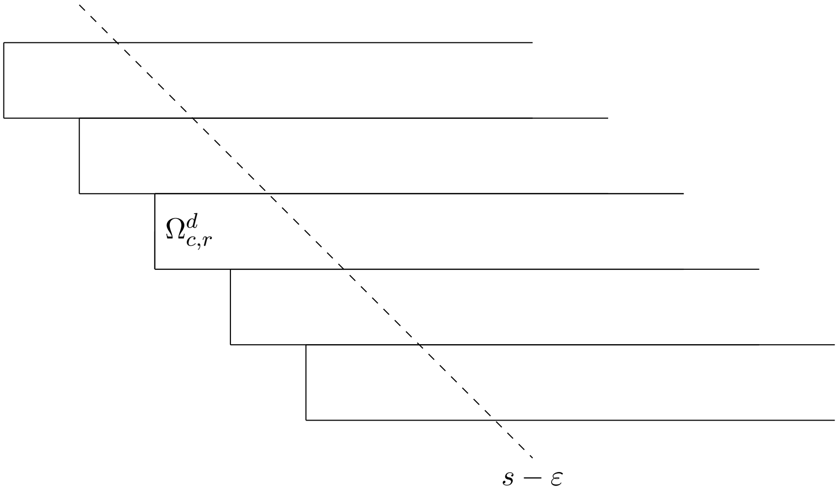

Figure 2: The $(\Gamma \cap P^\circ)$-translates of $\Omega^d_{c,r}$ cover $\Psi_{s'}$ in b)

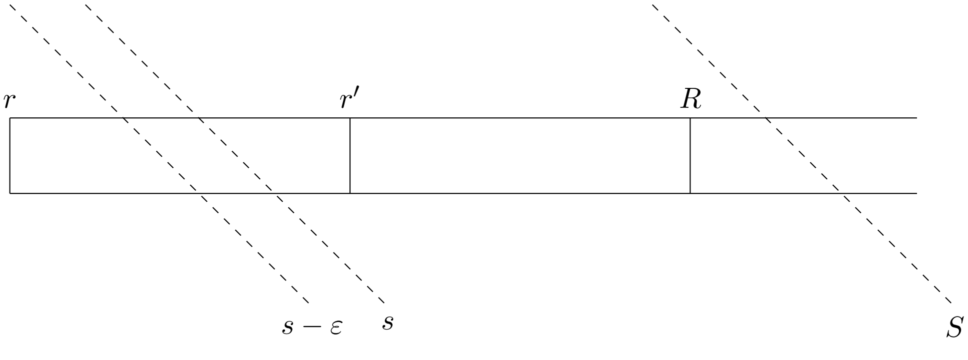

Figure 3: Geometric interpretation of c) and d) and their proofs

$g.o_\infty = x_\infty$ and choose $h \in H$ with $d(h.o_f, x_f) \leq d$. Pick further a $\gamma \in \Gamma$ with $\gamma.(g,h) \in (S_{c,t} \times K^f)$. Then $\gamma g \in S_{c,t}$ and $\gamma h \in K^f$. The former implies that $\gamma.x_\infty = (\gamma g).o_\infty \in S_{c,t}.o_\infty$ and the latter implies that

$$d(\gamma.x_f, o_f) = d(\gamma.x_f, (\gamma h).o_f) = d(x_f, h.o_f) \leq d,$$

since Γ acts via isometries on X_f. Thus, $\gamma.x = ((\gamma g).o_\infty, \gamma.x_f) \in \Omega^d_{c,t}$. ■

Lemma B. *If $t > 0$, then for every $s > 0$ with $s \leq e^{-bd}t$ we have $\Omega^d_{c,t} \subset \Psi_s$.*

Proof of Lemma B. Since $\Psi_s \subset \Psi_{s'}$ for $s' \leq s$, we only need to show the statement for the choice of $s = e^{-bd}t$. Pick an $x = (x_\infty, x_f) \in \Omega^d_{c,t}$ and choose $g \in S_{c,t}$ with $g.o_\infty = x_\infty$. Because of the inequality

$$d((g.o_\infty, o_f), x) = d((x_\infty, o_f), (x_\infty, x_f)) = d(x_f, o_f) \leq d,$$

Lemma 16.15 implies that

$$\mu_i(x) \geq \mu_i(g.o_\infty, o_f) - bd$$

for all $i \in \bar{\Delta}$. Writing g as $uak \in N^\infty_c A_t K^\infty \subset P^\infty K^\infty$, we have

$$\begin{aligned} \mu_i(g.o_\infty, o_f) &= \mu_i((ua, 1_H).o) = \log(|\alpha_i(ua, 1_H)|) \\ &= \log(\alpha_i(a)) \\ &\geq \log(t). \end{aligned}$$

Thus,

$$\exp(\mu_i(x)) \geq \exp(\mu_i(g.o_\infty, o_f) - bd) \geq \exp(\log(t)) \cdot e^{-bd} \geq s,$$

which implies the statement. ■

Lemma C. *If $t > 0$ fulfills $\Gamma.\Omega^d_{c,t} = X$, then we have $\Gamma.(\Omega^d_{c,r} \cap \Psi_s) = X$ for every $r, s \in \mathbb{R}$ with $0 < r \leq s \leq e^{-bd}t$.*

Proof of Lemma C. Because $r \leq t$, we have $\Omega^d_{c,t} \subset \Omega^d_{c,r}$. We also have $\Omega^d_{c,t} \subset \Psi_s$. Thus,

$$\Omega^d_{c,t} = (\Omega^d_{c,t} \cap \Psi_s) \subset (\Omega^d_{c,r} \cap \Psi_s),$$

which implies the assertion. ■

So pick any $t > 0$ with $\Gamma.\Omega_{c,t} = X$ and choose $s > 0$ with $s \leq e^{-bd}t$. Let $\varepsilon > 0$ such that $s' = s - \varepsilon > 0$. Then part a) follows for the choice of $r := s'e^{-bd} \leq s$.

To prove b), pick an $x = (x_\infty, x_f) \in \Psi_{s'}$ and choose $g \in G$ with $g.o_\infty = x_\infty$ and $h \in H$ with $d(h.o_f, x_f) \leq d$. Using $d((g,h).o, x) = d(h.o_f, x_f) \leq d$, we get

$$\mu_i((g,h).o) \geq \mu_i(x) - bd$$

by Lemma 16.15 for all $i \in \bar{\Delta}$. Rewriting the group element as $(g,h) = pk \in PK$ and using Proposition 16.11, this implies

$$|\alpha_i(p)| = \exp(\mu_i(p.o)) = \exp(\mu_i((g,h).o)) \geq \exp(\mu_i(x)) \cdot e^{-bd} \geq s' e^{-bd} = r.$$

Thus, $(g,h) \in P_r K$ and we can find $\gamma \in (\Gamma \cap P^\circ)$ with $\gamma.(g,h) \in (S_{c,r} \times K^f)$ by Proposition 16.4. From $(\gamma h).o_f = o_f$ follows

$$d(\gamma.x_f, o_f) = d(\gamma.x_f, (\gamma h).o_f) = d(x_f, h.o_f) \leq d.$$

Hence we have $\gamma.x \in \Omega^d_{c,r}$, and thus $x \in (P^\circ \cap \Gamma).\Omega^d_{c,r}$.
Next we show a technical lemma which applies to both remaining parts, c) and d).

Lemma D. *Let $R > r$ and $S \geq S_0 := Re^{bd}$ be given and pick $x = (x_\infty, x_f) \in (\Omega^d_{c,r} \cap \Psi_{s',S,i})$ for some $i \in \bar{\Delta}$. Then, for $g \in G$ with $g.o_\infty = x_\infty$ we have $g \in S_{c,r,R,i}$.*
If additionally $\gamma.x \in \Psi_{s'}$ holds for some $\gamma \in \Gamma$, then $\gamma.(g, 1_H) \in P_r K$.

Proof of Lemma D. We can rewrite g as $uak \in N^\infty_c A_r K^\infty$ by assumption, then

$$\log(\alpha_i(a)) = \log(|\alpha_i(ua, 1_H)|) = \mu_i((ua, 1_H).o) = \mu_i(g.o_\infty, o_f) = \mu_i(x_\infty, o_f)$$

applying Proposition 16.11 in the second equation. On the other hand,

$$\mu_i(x_\infty, o_f) \geq \mu_i(x) - bd$$

holds by Lemma 16.15 because $d((x_\infty, o_f), x) = d(o_f, x_f) \leq d$. Thus, $\exp(\mu_i(x)) \geq S$ implies

$$\alpha_i(a) \geq Se^{-bd} \geq R,$$

so $g \in S_{c,r,R,i}$. Furthermore, using $d(\gamma.(x_\infty, o_f), \gamma.x) = d((x_\infty, o_f), x) \leq d$, Lemma 16.15 also implies

$$\mu_j(\gamma.(x_\infty, o_f)) \geq \mu_j(\gamma.x) - bd$$

for every $j \in \bar{\Delta}$. Assume additionally $\gamma.x \in \Psi_{s'}$. Then we obtain

$$\mu_j((\gamma.(g, 1_H)).o) \geq \log(s') - bd$$

remember $g.o_\infty = x_\infty$. Rewriting $\gamma.(g, 1_H) = pk \in PK$, then

$$|\alpha_j(p)| = \exp(\mu_j(p.o)) = \exp(\mu_j((\gamma.(g, 1_H)).o)) \geq s' \cdot e^{-bd} = r$$

for all $j \in \bar{\Delta}$, and thus $\gamma.(g, 1_H) = pk \in P_r K$. ■

Recall that our choice of $c > 0$ is such that $(\Gamma \cap P^\circ) \cdot (S_{c,r} \times K^f)$ by Proposition 16.4. Let now $R > r$ fulfill Corollary 16.9, that means for $\gamma \in \Gamma$ and $i \in \bar{\Delta}$ holds

$$\gamma.(S_{c,r,R,i} \times K^f) \cap P_r K \neq \varnothing \Rightarrow \gamma \in P_i^\infty.$$

Let further $S \geq S_0 := e^{bd}R$ as above and assume that there is $\gamma \in \Gamma$ such that $\gamma.x \in \Psi_{s'}$ for some $x = (x_\infty, x_f) \in (\Omega^d_{c,r} \cap \Psi_{s',S,i})$. The lemma above implies that for $g \in G$ with $g.o_\infty = x_\infty$, we have $g \in S_{c,r,R,i}$ and $\gamma.(g, 1_H) \in P_r K$. Thus, $\gamma \in P_i^\infty$ and part c) is proven.
To show d), define

$$r' := se^{bd}.$$

This is clearly bigger than 0. We additionally assume $R > r$ to fulfill the following statement for all $\gamma \in \Gamma$:

$$\Big[\gamma.(g, h) = pk \in P_r K \text{ for some } (g, h) \in (S_{c,r,R,i} \times K^f)\Big] \Rightarrow |\alpha_i(p)| > r',$$

we can do so by Corollary 16.10 and increasing R if necessary (of course $S_0 = Re^{bd}$ possibly changes, too). As before, we consider the situation $\gamma.x \in \Psi_{s'}$ for some $x = (x_\infty, x_f) \in (\Omega^d_{c,r} \cap \Psi_{s',S,i})$. Again for $g \in G$ with $g.o_\infty = x_\infty$ we have $g \in S_{c,r,R,i}$ and $\gamma.(g, 1_H) \in P_r K$ by the lemma. Rewriting the latter as $\gamma.(g, 1_H) = pk \in P_r K$ we get $|\alpha_i(p)| > r'$. Since

$$d(\gamma.(x_\infty, o_f), \gamma.x) = d((x_\infty, o_f), x) = d(o_f, x_f) \leq d$$

Lemma 16.15 implies

$$\mu_i(\gamma.x) \geq \mu_i(\gamma.(x_\infty, o_f)) - bd.$$

Using $\gamma.(x_\infty, o_f) = (\gamma.(g, 1_H)).o = p.o$ we get

$$\mu_i(\gamma.(x_\infty, o_f)) = \mu_i(p.o) = \log(|\alpha_i(p)|)$$

and from this follows

$$\exp(\mu_i(\gamma.x)) \geq |\alpha_i(p)| \cdot e^{-bd} > r' e^{-bd}.$$

By the choice of $r' = se^{bd}$ we finally get

$$\exp(\mu_i(\gamma.x)) > s.$$

Thus, the theorem is proven. □

17 Raghunathan's function on the space X

In the following we construct a real-valued function f on the space $X = X_\infty \times X_f$ and show that it is a Morse function in the sense of Definition 14.19. This function will be Γ-invariant and we will see that the corresponding action on a sublevel set $f^{-1}(]-\infty, r])$ is cocompact for all $r \in \mathbb{R}$. From this, we will deduce later that Γ is of type F_∞. We will deal with that in the next section.

Most ideas of this section are taken from [Rag68] and extended to the space X. Doing so, we have to face several difficulties: Firstly, Raghunathan works with the Lie group G itself instead of the geometric space it acts on. Secondly, he defines his function on the quotient G/Γ so that the application of Morse theory is much easier than in our case due to compactness. And thirdly of course, we work with a product of a manifold and a simplicial complex instead of just one smooth factor.

17.1 Construction of the function

For this section, we use the following notation: Let $b > 0$ fulfill the inequality $|\mu_i(x) - \mu_i(y)| \leq bd(x, y)$ for all $x, y \in X$ and $i \in \bar{\Delta}$ and choose $c > 0$ with $\mathbf{N}(\mathcal{O}).N_c^\infty = N^\infty$. Fix any $d > 0$ such that $H.\overline{B_d(o_f)} = X_f$.

Now select $s > 0$ and $\varepsilon > 0$ such that for $s' = s - \varepsilon > 0$ and $r = s'e^{-bd}$ we can find $S_0 > s$ to fulfill Theorem 16.18. Pick any $S \geq S_0$.

With these choices let $\varphi\colon \mathbb{R} \to [0, 1]$ be a smooth function with $\varphi'(t) \geq 0$ for all $t \in \mathbb{R}$ and with

$$\varphi(t) = \begin{cases} 1 & \text{if } t \geq S, \\ 0 & \text{if } t \leq S - \varepsilon. \end{cases}$$

The existence of such a function one can deduce easily from the construction in [Tu11, §13.1]. Additionally we define $\psi\colon \mathbb{R} \to [0, 1]$ by

$$\psi(t) = \begin{cases} 1 - \varphi(t) & \text{if } t \geq s, \\ \varphi(t - s + S) & \text{if } t \leq s. \end{cases}$$

For any subset $I \subset \bar{\Delta}$ we define $\Phi_I\colon X \to \mathbb{R}$ by

$$\Phi_I(x) = \prod_{i \in I} \varphi(\exp(\mu_i(x))) \cdot \prod_{i \in \bar{\Delta} \smallsetminus I} \psi(\exp(\mu_i(x))).$$

We observe $\Phi_I(x) = 0$ for all $I \subset \bar{\Delta}$ unless $x \in \Psi_{s'}$, because $\varphi(t) = 0 = \psi(t)$ except when $t \geq s - \varepsilon = s'$. Now, let $B = \sum_{j \in \bar{\Delta}} m_j \beta_j$ be any linear combination with $m_j > 0$ for all j. Remember the fact that the β_j are linear combinations of the μ_i in exactly the same way as the λ_j are linear

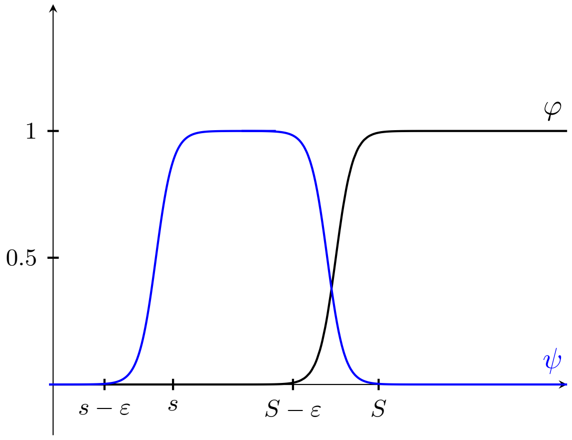

Figure 4: Sketches of φ and ψ

combinations of the α_i by Proposition 16.11. Thus, we can use the statements about the interaction of the fundamental weights and the simple roots as in Lemma 9.7 and apply them to our B playing the role of the Λ there: We get

$$B = \sum_{j \in I} m_{Ij} \beta_j + \sum_{i \in \bar{\Delta} \setminus I} n_{Ii} \mu_i$$

with $m_{Ij} \geq m_j > 0$ for all subsets $I \subset \bar{\Delta}$. Let

$$B_I = \sum_{j \in \bar{\Delta}} m_{Ij} \beta_j$$

using the notation above. We state this (and more) in a lemma for later reference.

Lemma 17.1. *For any subset $I \subset \bar{\Delta}$ we have*

$$B_I = \sum_{j \in I} m_{Ij} \beta_j$$

with $m_{Ij} > 0$ for all $j \in I$. Furthermore, $(B_{I \cup \{i\}} - B_I) = \sum_{k \in \bar{\Delta}} c_{Iik} \mu_k$ is a non-negative linear combination of the μ_k for which $c_{Iii} > 0$ holds if $i \notin I$.

Proof. Follows directly from Lemma 9.7 and the fact that β_j and μ_i are linear combinations of each other in the same way as λ_j and α_i. $\square$

One easily deduces another statement about the interplay of B_I and the μ_i that we will also need later on.

Corollary 17.2. *For all subsets I of $\bar{\Delta}$ the function $B_I = \sum_{i \in \bar{\Delta}} a_i \mu_i$ is a non-negative linear combination with $a_i > 0$ for every $i \in I$.*

Proof. By Corollary 16.14 $\beta_j = \sum_{i\in\bar{\Delta}} m_{ji}\mu_i$ is a non-negative linear combination with $m_{jj} > 0$, so the previous lemma implies that

$$B_I = \sum_{j\in I} m_{Ij}\beta_j = \sum_{i\in\bar{\Delta}} \left(\sum_{j\in I} m_{Ij}m_{ji} \right) \mu_i$$

with all $m_{Ij}m_{ji} \geq 0$. In the case $i \in I$, we have $m_{Ii} > 0$ and $m_{ii} > 0$. □

Now we can define the function.

Definition 17.3. Let $f\colon X \to \mathbb{R}$ be defined by

$$f(x) = \sum_{I\subset\bar{\Delta}} \sum_{\bar{\gamma}\in(\Gamma\cap P^\circ\backslash\Gamma)} \Phi_I(\gamma.x)B_I(\gamma.x).$$

We need to show that this is well-defined, which will follow from the next two lemmas. The sum formally makes sense by

Lemma 17.4. *If two elements γ and γ' of Γ define the same right coset modulo $\Gamma\cap P^\circ$, then $\Phi_I(\gamma.x) = \Phi_I(\gamma'.x)$ and $B_I(\gamma.x) = B_I(\gamma'.x)$ hold for all $x \in X$.*

Proof. Choose $p \in \Gamma \cap P^\circ$ with $p \cdot \gamma = \gamma'$. Then, for all $x \in X$, we get

$$\mu_i(\gamma'.x) = \mu_i(p.(\gamma.x)) = \log(|\alpha_i(p)|) + \mu_i(\gamma.x) = \mu_i(\gamma.x).$$

Thus, $\Phi_I(\gamma.x) = \Phi_I(\gamma'.x)$. But the statement with B_I follows easily, since every β_j is a linear combination of the μ_i. □

The sum is locally finite by

Lemma 17.5. *For every $x \in X$, there is a neighborhood $\Omega \subset X$ containing $\Omega^d_{c,r}$ and a finite subset $\Gamma' \subset (\Gamma \cap P^\circ\backslash\Gamma)$ (depending on Ω) such that the restriction of the function f to Ω is uniformly given by the finite sum*

$$f|_\Omega = \sum_{I\subset\bar{\Delta}} \sum_{\bar{\gamma}\in\Gamma'} \Phi_I(\gamma.\_)B_I(\gamma.\_).$$

Proof. Choose any $x = (x_\infty, x_f) \in X = X_\infty \times X_f$. By transitivity of the action of G on X_∞, we can find $g \in G$ with $g.o_\infty = x_\infty$. Choose any Siegel set $S_{c',t} \subset G$ such that g lies in the interior of that set. Then x_∞ lies in the interior of $S_{c',t}.o_\infty$. See Lemma 15.6. We also choose d' with $d_f(x_f, o_f) < d'$ so that x lies in the interior of the set

$$\Omega^{d'}_{c',t} = (S_{c',t}.o_\infty) \times \overline{B_{d'}}(o_f).$$

Without loss of generality, we can assume $\Omega^{d}_{c,r} \subset \Omega^{d'}_{c',t}$ by decreasing $t > 0$ and increasing c' and d' if necessary. We show that f is uniformly given by the sum above on the whole neighborhood $\Omega = \Omega^{d'}_{c',t}$ of x.
Let Γ' be the image under the projection $\Gamma \to (\Gamma \cap P^{\circ} \backslash \Gamma)$ of the set

$$\{\gamma \in \Gamma \mid \gamma.\Omega \cap (\Gamma \cap P^{\circ}).\Omega \neq \varnothing\}.$$

For every coset in Γ', there is a representing element $\gamma \in \Gamma$ with

$$\gamma.\Omega \cap \Omega \neq \varnothing.$$

But there are only finitely many $\gamma \in \Gamma$ with that property by Proposition 16.17, so the set Γ' must be finite, too.
Now let $y \in \Omega$ be arbitrary and pick any $\gamma \in \Gamma$. We have $\Phi_I(\gamma.y) = 0$ for every subset I of $\bar{\Delta}$ unless $\gamma.y \in \Psi_{s'}$. According to b) of Theorem 16.18, the set $\Psi_{s'}$ lies inside $(P^{\circ} \cap \Gamma).\Omega^{d}_{c,r}$, which implies $\Psi_{s'} \subset (P^{\circ} \cap \Gamma).\Omega$. So we obtain

$$\Phi_I(\gamma.y) \neq 0 \text{ for some } I \subset \bar{\Delta} \Rightarrow \gamma.\Omega \cap (P^{\circ} \cap \Gamma).\Omega \neq \varnothing.$$

Thus, for every $y \in \Omega$ holds $\Phi_I(\gamma.y) = 0$ for all $I \subset \bar{\Delta}$ unless $\bar{\gamma} \in \Gamma'$ so

$$f(y) = \sum_{I \subset \bar{\Delta}} \sum_{\bar{\gamma} \in \Gamma'} \Phi_I(\gamma.y) B_I(\gamma.y)$$

is a finite sum. □

Corollary 17.6. *The function f is continuous on the whole space X and smooth in the factor X_∞. Furthermore, it is invariant under the left action of Γ.*

Proof. Invariance under the action of Γ is clear by definition. For continuity, recall first that every μ_i is continuous as a linear combination of Busemann functions and that φ and ψ were chosen to be continuous. This implies that the maps $x \mapsto \Phi_I(x)$ and $x \mapsto B_I(x)$ are all continuous. Thus, f is locally a finite sum of continuous functions, since the action of Γ on X is continuous, too.
But with a similar argument, f is smooth in the first factor: Every Busemann function on the symmetric space X_∞ is smooth by Example 8.4, and every Busemann function β on $X = X_\infty \times X_f$ is of the form

$$\beta(x) = \cos(\theta)\beta_\infty(x_\infty) + \sin(\theta)\beta_f(x_f)$$

by Lemma 3.16 for Busemann functions β_∞ on X_∞ and β_f on X_f. Since the functions φ and ψ in fact were chosen to be smooth, and the map $x_\infty \mapsto g.x_\infty$ is also smooth for every $g \in G$, we deduce that f in the X_∞ factor is locally a finite sum of smooth functions. □

17.2 The function is Morse

The goal of this section is to prove the following

Theorem. *The function $f\colon X \to \mathbb{R}$ is a Morse function in the sense of Definition 14.19 for some $r_0 \in \mathbb{R}$. It is invariant under the left action of Γ and the induced action on $f^{-1}(]-\infty, t])$ is cocompact for every $t \in \mathbb{R}$.*

That f is Γ-invariant, continuous on X, and smooth in the first factor follows directly from the construction of the function as we saw in the previous section. Convergence in the weak C^1 topology will also be an easy consequence of the fact that f is locally "nearly" a linear combination of Busemann functions. The hardest part is to prove that the gradient is bounded, and we will show cocompactness of the induced Γ-action as a byproduct.

Proposition 17.7. *The restriction of the function f to $\Omega^d_{c,r}$ is given by a finite sum, and every summand is uniformly bounded from below. Furthermore, the function f is uniformly bounded from below on the whole space X.*

Proof. Recall first that $\Gamma.\Omega^d_{c,r} = X$ by a) of Theorem 16.18 and that f is invariant under the left action of Γ on X. Thus, in order to prove the second statement we only need to show that the restriction $f|_{\Omega^d_{c,r}}$ is uniformly bounded from below. This is given by the finite sum

$$f(y) = \sum_{I \subset \bar{\Delta}} \sum_{\bar{\gamma} \in \Gamma'} \Phi_I(\gamma.y) B_I(\gamma.y)$$

for all $y \in \Omega^d_{c,r}$ using Lemma 17.5.
Fix a subset I of $\bar{\Delta}$ and choose some $\bar{\gamma} \in \Gamma'$ and $x \in \Omega^d_{c,r}$. Assume $\Phi_I(\gamma.x) \neq 0$, which implies that $\exp(\mu_i(\gamma.x)) > s'$ for every $i \in \bar{\Delta}$. By Corollary 17.2 we can write $B_I = \sum_{i \in \bar{\Delta}} a_{Ii}\mu_i$ as a non-negative linear combination, so

$$B_I(\gamma.x) = \sum_{i \in \bar{\Delta}} a_{Ii}\mu_i(\gamma.x) \geq \sum_{i \in \bar{\Delta}} a_{Ii} \log(s').$$

Now, because of $\Phi_I(\gamma.x) \in [0,1]$:

$$\Phi_I(\gamma.x) \cdot B_I(\gamma.x) \geq \min\left\{0, \sum_{i \in \bar{\Delta}} a_{Ii} \log(s')\right\} =: c_I$$

independently from the choice of $x \in \Omega^d_{c,r}$ and $\bar{\gamma} \in \Gamma'$. Thus, every single summand, and also

$$f|_{\Omega^d_{c,r}} \geq \sum_{I \subset \bar{\Delta}} \sum_{\bar{\gamma} \in \Gamma'} c_I$$

is uniformly bounded from below. □

Next, we will show that the restriction of f to the subset $\Omega^d_{c,r} \cap \Psi_s$ is a proper function. First, we need two lemmas.

Lemma 17.8. *The subset*

$$C = \Omega^d_{c,r} \cap \bigcap_{i \in \bar{\Delta}} \mu_i^{-1}([s,T])$$

of $\Omega^d_{c,r} \cap \Psi_s$ is compact for all $T \in \mathbb{R}$.

Proof. We can write every $x \in C$ as $x = (x_\infty, x_f)$ with $x_f \in \overline{B_d}(o_f)$ and $x_\infty = g.o_\infty$ for some $g = uak \in S_{c,r} = N^\infty_c A_r K^\infty$. Then for all $i \in \bar{\Delta}$ we have

$$\begin{aligned}\mu_i(x) = \mu_i((g.o_\infty, x_f)) &= \mu_i((na, 1_H).(o_\infty, x_f)) \\ &= \log(|\alpha_i(na, 1_H)|) + \mu_i(o_\infty, x_f)\end{aligned}$$

applying Proposition 16.11 in the last equation. This leads to

$$\mu_i(x) = \log(\alpha_i(a)) + \mu_i(o_\infty, x_f)$$

as $|\alpha_i(na, 1_H)| = \alpha_i(a)$. Because $d(x_f, o_f) \le d$ and $\mu_i(o) = 0$, Lemma 16.15 implies

$$\mu_i(o_\infty, x_f) = \mu_i(o_\infty, x_f) - \mu_i(o) \ge -bd.$$

Using $T \ge \mu_i(x)$ this gives us

$$T + bd \ge \log(\alpha_i(a)).$$

So from $a \in A_r$, it follows that a is an element of

$$\tilde{A} := \{\tilde{a} \in A \mid r \le \alpha_i(\tilde{a}) \le e^{T+bd} \text{ for every } i \in \bar{\Delta}\}.$$

Thus every $x \in C$ must lie in the set

$$C' = ((N^\infty_c \times \tilde{A}).o_\infty) \times \overline{B_d}(o_f).$$

The set $\tilde{A}$ as defined above is relatively compact: Choose k-characters χ_j that define the isomorphism $\mathbf{T} \to \prod_j \mathbf{GL}_1$ of algebraic groups by $t \mapsto (\chi_j(t))_j$. This induces a homeomorphism $\mathbf{T}(\mathbb{R}) \to (\mathbb{R}^\times \times \ldots \times \mathbb{R}^\times)$. As the simple roots span the real vector space $V = X_k(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$, we can write every χ_j as a linear combination $\sum_{i \in \bar{\Delta}} m_{ji}\alpha_i$ with $m_{ji} \in \mathbb{R}$. But this implies that $\tilde{A}$ is homeomorphic to a bounded subset of $\mathbb{R}^\times \times \ldots \times \mathbb{R}^\times$ and is therefore relatively compact. Now C' must also be relatively compact, as $\tilde{A}$ and $N^\infty_c \subset N^\infty$ are and the map $N^\infty \times A \to X_\infty$, $(u, a) \mapsto (ua).o_\infty$ is continuous.
Finally, we show that C is a closed subset of X, which then gives us the statement by $C \subset C'$. On the one hand, $\overline{B_d}(o_f) \subset X_f$ is compact. On the other hand, $S_{c,r}.o_\infty$ is a closed subset of X_∞. The latter is true because $N^\infty \times A \to X_\infty$, $(u, a) \mapsto (ua).o_\infty$ is a homeomorphism by Lemma 15.6 and $N^\infty_c \subset N^\infty$ is compact and A_r is a closed subset of A. Thus, $\Omega^d_{c,r}$ is a closed subset of X, so C must be closed, too. □

Lemma 17.9. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\Omega^d_{c,r} \cap \Psi_s$ with the following property:*
For any compact subset C of $\Omega^d_{c,r} \cap \Psi_s$ the set $\{n \in \mathbb{N} \mid x_n \in C\}$ is finite.
Then there is a finite number of subsequences of $(x_n)_{n\in\mathbb{N}}$ such that:

(i) for every $n \in \mathbb{N}$ the element x_n is contained in at least one of the subsequences,

(ii) all of these subsequences $(\mu_{n_k})_{k\in\mathbb{N}}$ fulfill $\lim_{k\to\infty} \mu_i(x_{n_k}) = \infty$ for at least one $i \in \bar{\Delta}$.

Proof. For every $i \in \bar{\Delta}$, we define sets $N^i_k \subset \mathbb{N}$ indexed by $k \in \mathbb{Z}$ as follows: If $k < s$ then we set $N^i_k = \varnothing$, and for all $k \in \mathbb{Z}$ we define inductively

$$N^i_{k+1} := N^i_k \cup \left\{ n \in \mathbb{N} \,\middle|\, x_n \in \bigcap_{j\in\bar{\Delta}} \mu_j^{-1}([s, k+1]) \text{ and } \mu_i(x_n) \in]k, k+1] \right\}.$$

These sets N^i_k are all finite: The set $C = \Omega^d_{c,r} \cap \bigcap_{j\in\bar{\Delta}} \mu_j^{-1}([s,T])$ is compact for all $T \in \mathbb{R}$, so there are only finitely many $n \in \mathbb{N}$ such that $x_n \in \bigcap_{j\in\bar{\Delta}} \mu_j^{-1}([s, k+1])$. We now set

$$N^i = \bigcup_{k\in\mathbb{Z}} N^i_k.$$

For those $i \in \bar{\Delta}$ for which the set N^i is infinite, the subsequence $(\mu_i(x_n))_{n\in N^i}$ converges to ∞, since the sets $\{n \in N^i \mid \mu_i(x_n) \le k\} = \bigcup_{\ell\le k} N^i_\ell$ are all finite; remember $N^i_\ell = \varnothing$ for $\ell < s$. Furthermore, every $n \in \mathbb{N}$ lies in at least one of the N^i: Pick $i \in \bar{\Delta}$ with $\mu_i(x_n)$ to be maximal and choose $k \in \mathbb{Z}$ with $k \ge \mu_i(x_n) > k - 1$. Then $k \ge \mu_j(x_n)$ for all $j \in \bar{\Delta}$ and therefore $n \in N^i_k$. Thus, we have constructed finitely many subsets $(N^i)_{i\in\bar{\Delta}}$ of $\mathbb{N}$ such that

(i) $\bigcup_{i\in\bar{\Delta}} N^i = \mathbb{N}$,

(ii) if $|N^i| = \infty$, then the subsequence $(\mu_i(x_n))_{n\in N^i}$ converges to infinity.

From this follows the statement. □

Proposition 17.10. *The restriction of f to the subset $\Omega^d_{c,r} \cap \Psi_s$ is a proper function.*

Proof. Recall first that $\Omega^d_{c,r}$ is a closed subset of X, as $\overline{B_d}(o_f)$ is compact and $S_{c,r}.o_\infty$ is the image of a closed set under the homeomorphism $N^\infty \times A \to X_\infty$, $(u,a) \mapsto (ua).o_\infty$. Furthermore, the set $\Psi_s = \bigcap_{i\in\bar{\Delta}} \mu_i^{-1}([s,\infty[)$ is closed as the μ_i are continuous, so the intersection $\Omega^d_{c,r} \cap \Psi_s$ must also be

closed. Since the metric space X is proper we can write it as a countable union of open, relatively compact subsets. The same must be true for the closed subset $\Omega^d_{c,r} \cap \Psi_s$. Thus, using Lemma 2.3, it is enough to show the following statement for the properness of f restricted to $\Omega^d_{c,r} \cap \Psi_s$:
If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\Omega^d_{c,r} \cap \Psi_s$ such that, for all compact subsets C, the set $\{n \in \mathbb{N} \mid x_n \in C\}$ is finite, then the set $\{n \in \mathbb{N} \mid f(x_n) \in [-T, T]\}$ is also finite for all $T > 0$.
As f is bounded from below, this is equivalent to: $\lim_{n\to\infty} f(x_n) = \infty$ holds for every sequence $(x_n)_{n\in\mathbb{N}}$ in $\Omega^d_{c,r} \cap \Psi_s$ leaving any compact set. Therefore, thanks to Lemma 17.9, we only have to show the following:

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\Omega^d_{c,r} \cap \Psi_s$ with $\mu_{i_0}(x_n) \to \infty$ for some $i_0 \in \bar{\Delta}$, then also $f(x_n) \to \infty$ as $n \to \infty$.

So we pick such a sequence $(x_n)_{n\in\mathbb{N}}$ and fix a suitable $i_0 \in \bar{\Delta}$. Again, Lemma 17.5 implies

$$f(x) = \sum_{I\subset\bar{\Delta}} \sum_{\bar{\gamma}\in\Gamma'} \Phi_I(\gamma.x) B_I(\gamma.x)$$

uniformly for all $x \in \Omega^d_{c,r} \cap \Psi_s$ for some finite set Γ'. Since any of these summands is uniformly bounded from below by Proposition 17.7, it suffices to show that for $\gamma = 1$ and a fixed subset $I_0 \subset \bar{\Delta}$ we have

$$\Phi_{I_0}(x_n) B_{I_0}(x_n) \to \infty$$

as $n \to \infty$. To prove this, we set

$$I_0 = \{i \in \bar{\Delta} \mid \varphi(\exp(\mu_i(x_n))) > \frac{1}{2} \text{ for all } n \geq N\}$$

where $N \in \mathbb{N}$ is chosen to fulfill $\exp(\mu_{i_0}(x_n)) \geq S$ for all $n \geq N$; remember $\mu_{i_0}(x_n) \to \infty$. By definition $i_0 \in I_0$, so this set is not empty. Furthermore, we have

$$\varphi(\exp(\mu_i(x_n))) \begin{cases} > \frac{1}{2} & \text{if } i \in I_0, \\ \leq \frac{1}{2} & \text{if } i \in \bar{\Delta} \setminus I_0 \end{cases}$$

assuming $n \geq N$. But every x_n was chosen to be an element of Ψ_s and we have $\psi(t) = 1 - \varphi(t)$ for all $t \geq s$ by definition, so the latter implies

$$\psi(\exp(\mu_i(x_n))) = 1 - \varphi(\exp(\mu_i(x_n))) \geq \frac{1}{2}$$

for all $i \notin I_0$ and $n \geq N$. Thus,

$$\begin{aligned}\Phi_{I_0}(x_n) &= \prod_{i \in I_0} \varphi(\exp(\mu_i(x_n))) \cdot \prod_{i \in \bar{\Delta} \smallsetminus I_0} \psi(\exp(\mu_i(x_n))) \\ &\geq \prod_{i \in I_0} \frac{1}{2} \cdot \prod_{i \in \bar{\Delta} \smallsetminus I_0} \frac{1}{2} \\ &= \left(\frac{1}{2}\right)^{|\Delta|}\end{aligned}$$

is uniformly bounded from below for $n \in \mathbb{N}$ large enough, so it suffices to show

$$\lim_{n \to \infty} B_{I_0}(x_n) = \infty.$$

But we know $B_{I_0} = a_{i_0}\mu_{i_0} + \sum_{i \in \bar{\Delta} \smallsetminus \{i_0\}} a_i \mu_i$ with $a_{i_0} > 0$ and every $a_i \geq 0$ by Corollary 17.2, and therefore

$$B_{I_0}(x_n) \geq a_{i_0}\mu_{i_0}(x_n) + \sum_{i \in \bar{\Delta} \smallsetminus \{i_0\}} a_i \log(s)$$

holds because of $x_n \in \Psi_s$ for every $n \in \mathbb{N}$. Thus, $\mu_{i_0}(x_n) \to \infty$ implies that $B_{I_0}(x_n) \to \infty$ as $n \to \infty$. ☐

Corollary 17.11. *For every $t \in \mathbb{R}$ the induced action of Γ on $f^{-1}(]-\infty, t])$ is well-defined and cocompact.*

Proof. The function f is invariant under the action of Γ from the left, so the action on sublevel sets is well-defined. Let $x \in X$ be any point with $f(x) \leq t$, then by a) of Theorem 16.18 there is $\gamma \in \Gamma$ such that $\gamma.x \in \Omega^d_{c,r} \cap \Psi_s$. Using the invariance of f under the Γ-action, this implies that

$$\Gamma.((\Omega^d_{c,r} \cap \Psi_s) \cap f^{-1}(]-\infty, t])) = f^{-1}(]-\infty, t]).$$

So, for the cocompactness, it suffices to show that the intersection $(\Omega^d_{c,r} \cap \Psi_s) \cap f^{-1}(]-\infty, t])$ is compact. Since f is bounded from below, there is $u \in \mathbb{R}$ with $f^{-1}(]-\infty, t]) = f^{-1}([u, t])$. Thus,

$$(\Omega^d_{c,r} \cap \Psi_s) \cap f^{-1}(]-\infty, t]) = (\Omega^d_{c,r} \cap \Psi_s) \cap f^{-1}([u, t]) = \left(f|_{(\Omega^d_{c,r} \cap \Psi_s)}\right)^{-1}([u, t])$$

and this is compact by properness. ☐

Next we want to show that the gradient of the function $x_\infty \mapsto f(x_\infty, x_f)$ is uniformly bounded from below if $x = (x_\infty, x_f)$ is an element of $f^{-1}(]r_0, \infty[)$ for some fixed $r_0 > 0$. The idea is to construct a smooth curve $c(t)$ in X_∞ through x_∞ with the properties $\frac{d}{dt}|_{t=0} f(c(t), x_f) \geq C$ and $\|c'(0)\| \leq D$, so that we can invoke Lemma 14.3. We will deal with each boundary condition in a separate proposition, but first we need to construct the curve.

Proposition 17.12. *Pick* $i \in \bar{\Delta}$ *and* $t \in \mathbb{R}$*. There is a unique element* $a_i(t) \in A$ *with the property* $\alpha_j(a_i(t)) = e^{t\delta_{ij}}$ *for all* $j \in \bar{\Delta}$*. This element commutes with every* $k \in K^\infty \cap P_i^\infty$*. Furthermore, the induced map from* $\mathbb{R}$ *to* A *defined by* $t \mapsto a_i(t)$ *is a smooth curve through* $a_i(0) = 1_A$*.*

Proof. Let $\chi_l \in X(\mathbf{T})$ be characters that define an isomorphism $\mathbf{T} \cong \prod_l \mathbf{GL}_1$ via $t \mapsto (\chi_l(t))_l$. This induces a homeomorphism $\mathbf{T}(\mathbb{R}) \to \mathbb{R}^\times \times \ldots \times \mathbb{R}^\times$ which we denote by ξ. Every character can be written uniquely as a $\mathbb{Z}$-linear combination of the χ_l by Lemma 11.11. On the other hand, $(\alpha_j)_{j\in\bar{\Delta}}$ is a basis for $V = \mathbb{R} \otimes_{\mathbb{Z}} X_k(\mathbf{T})$. Thus, there are real valued matrices $N = (n_{jl})_{jl}$ and $M = (m_{li})_{li}$ with $\alpha_j = \sum_l n_{jl}\chi_l$ and $\chi_l = \sum_i m_{li}\alpha_i$ and such that $N = M^{-1}$. For every $t \in \mathbb{R}$ define $a_i(t) \in A$ as the unique element with $\xi(a_i(t)) = (e^{tm_{li}})_l$, that is $a_i(t) = \xi^{-1}((e^{tm_{li}})_l)$. As ξ^{-1} is a homeomorphism that is defined by polynomials it is also smooth, and therefore the map $t \mapsto a_i(t)$ must be smooth, too. Using the fact that χ_l induces the projection of ξ to the l-th factor in $\mathbb{R}^\times \ldots \times \mathbb{R}^\times$ we get $\chi_l(a_i(t)) = e^{tm_{li}}$. This implies that

$$\alpha_j(a_i(t)) = \prod_l \chi_l(a_i(t))^{n_{jl}} = \prod_l (e^{tm_{li}})^{n_{jl}} = e^{t\sum_l n_{jl}m_{li}}$$

and from $\sum_l n_{jl}m_{li} = (N \cdot M)_{ij} = \delta_{ij}$, we obtain

$$\alpha_j(a_i(t)) = e^{t\delta_{ij}}.$$

Uniqueness of $a_i(t)$ follows from the fact that on A the values of the χ_l are uniquely determined by those of the α_j.

Lemma A. *Let* $\mathfrak{a}$ *be the Lie algebra of* A*. There is an element* $h_i \in \mathfrak{a}$ *with* $a_i(t) = \exp(th_i)$ *for all* $t \in \mathbb{R}$*, where* exp *denotes the exponential map.*

Proof of Lemma A. Every k-character is given by polynomials. Therefore, every simple root α_j induces a smooth group homomorphism $A \to \mathbb{R}^\times$. Pick any $t, s \in \mathbb{R}$. Then for all $j \in \bar{\Delta}$ we have

$$\alpha_j(a_i(t)a_i(s)) = \alpha_j(a_i(t)) \cdot \alpha_j(a_i(s)) = e^{t\delta_{ij}} \cdot e^{s\delta_{ij}} = e^{(t+s)\delta_{ij}}$$

so we get $a_i(t)a_i(s) = a_i(t+s)$ by uniqueness. Thus, the map $\mathbb{R} \to A$ defined by $t \mapsto a_i(t)$ is in fact a continuous group homomorphism. The statement follows directly from Lemma 7.2. ■

From this follows $a_i(0) = 1_A$. It remains to show that $a_i(t)$ commutes with every $k \in K^\infty \cap P_i^\infty$.
Recall the isomorphism of Lie groups $G \to G'$, given by $g \mapsto xgx^{-1}$, that identifies $A \cong A'$. Let $\mathfrak{a}'$ be the Lie algebra of the Lie group A'. We can view it as a subspace of $T_{o_\infty}X_\infty$ since by definition A' lies inside $X_\infty = P_n(\mathbb{R}) \cap G'$.

Define $x_i := \frac{xh_ix^{-1}}{\operatorname{tr}(h_i)^2} \in \mathfrak{a}'$. Because $\operatorname{tr}(x_i^2) = \frac{\operatorname{tr}(h_i^2)}{\operatorname{tr}(h_i^2)} = 1$, the map $c_i \colon t \mapsto \exp(tx_i)$ is a geodesic line in X_∞ with $c_i(0) = o_\infty$. Let $\xi_i := c_i(\infty) \in \partial X_\infty$ be the corresponding point at infinity. We directly see that

$$xa_i(t)x^{-1} = \exp(txh_ix^{-1}) = \exp(\operatorname{tr}(h_i^2)tx_i) = c_i(\operatorname{tr}(h_i^2)t)$$

holds for every $t \in \mathbb{R}$.

Lemma B. *The group P_i^∞ fixes the point ξ_i at infinity.*

Proof of Lemma B. According to Example 8.4 the stabilizer is

$$\operatorname{Stab}_{G'}(\xi_i) = \{g' \in G' \mid \lim_{t\to\infty} \exp(-tx_i)g' \exp(tx_i) \text{ exists}\}.$$

Passing from G' to $G = x^{-1}G'x$, we obtain

$$\operatorname{Stab}_G(\xi_i) = \{g \in G \mid \lim_{t\to\infty} \exp(-tx_i)xgx^{-1} \exp(tx_i) \text{ exists}\}.$$

Because $\operatorname{tr}(h_i^2) > 0$ and $a_i(t) = x^{-1}\exp(t \operatorname{tr}(h_i^2)x_i)x$, the limit displayed in the set above equals

$$\lim_{t\to\infty} xa_i(-t)ga_i(t)x^{-1} = x\left(\lim_{t\to\infty} a_i(-t)ga_i(t)\right)x^{-1}.$$

So, we need to prove the existence of

$$\lim_{t\to\infty} a_i(-t)ga_i(t)$$

for all $g \in P_i^\infty$. In fact, we will show that it exists for all $g \in \mathbf{P}_i(\mathbb{C})$. The algebraic group $\mathbf{P}_i$ is connected and therefore $\mathbf{P}_i(\mathbb{C})$ is also connected with respect to the topology induced by the field $\mathbb{C}$, see Corollary 11.12. Using Lemma 7.8 the Lie group $\mathbf{P}_i(\mathbb{C})$ is generated by the image of its Lie algebra $\mathfrak{p}_i$ under the exponential map. Thus, without loss of generality, we can assume that there is $y \in \mathfrak{p}_i$ with $\exp(y) = g$. Therefore

$$a_i(-t)ga_i(t) = \exp(\operatorname{Ad}_{a_i(-t)}(y))$$

holds by Lemma 7.3 because of $a_i(-t) = \exp(-th_i) = a_i(t)^{-1}$. Recall that the Lie algebra of $\mathbf{P}_i$ splits as a direct sum

$$\mathfrak{p}_i = \bigoplus_{\alpha\in\phi^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in\phi^+\smallsetminus\phi_i} \mathfrak{g}_{-\alpha} \oplus \mathfrak{t}$$

Here $\mathfrak{g}_\alpha$ are the weight spaces, $\phi_i = \{\sum_{j\in\bar{\Delta}} m_j\alpha_j \in \phi^+ \mid m_i = 0\}$, and $\mathfrak{t}$ is the Lie algebra of $\mathbf{T}$. By linearity of the adjoint action, we can restrict to the cases induced by the direct sum decomposition. On $\mathfrak{t}$, the adjoint action of A is trivial by Lemma 7.4. For $\alpha = \sum_{j\in\bar{\Delta}} m_j\alpha_j \in \phi^+$ and $z \in \mathfrak{g}_{\pm\alpha}$, we have

$$\mathrm{Ad}_{a_i(-t)}(z) = \alpha(a_i(-t))^{\pm 1}\cdot z = \left(\prod_{j\in\bar{\Delta}} \alpha_j(a_i(-t))^{\pm m_j}\right)\cdot z = e^{\mp m_i t} z$$

as $\alpha_j(a_i(-t)) = e^{-t\delta_{ij}}$. If α is in ϕ_i, then $m_i = 0$ implies $\mathrm{Ad}_{a_i(-t)}(z) = z$. If, on the other hand, α is not an element of ϕ_i, then the only case we need to consider is $z \in \mathfrak{g}_\alpha$, and we get $\mathrm{Ad}_{a_i(-t)}(z) = e^{-m_i t}z \to 0$ as $t\to\infty$ by $m_i > 0$. This means that for every $y \in \mathfrak{p}_i$ the limit $\lim_{t\to\infty}\mathrm{Ad}_{a_i(-t)}(y)$ exists. Hence we have proven the existence of

$$\lim_{t\to\infty} \exp(-tx_i)g\exp(tx_i)$$

for all $g \in \mathbf{P}_i(\mathbb{C})$, so we get that $P_i^\infty = \mathbf{P}_i(\mathbb{C})\cap G$ is a subset of $\mathrm{Stab}_G(\xi_i)$. ■

Now the statement of the proposition follows directly from the next lemma by $a_i(t) = x^{-1}c_i(\mathrm{tr}(h_i^2)t)x$. □

Lemma 17.13. *Let $\xi \in \partial X_\infty$ be a point at infinity and let $c\colon \mathbb{R}\to X_\infty$ be the unique geodesic line with $c(0) = o_\infty$ and $c(\infty) = \xi$. For every $k \in (K^\infty \cap \mathrm{Stab}_G(\xi))$ and every $t\in\mathbb{R}$, the elements $x^{-1}c(t)x$ and k commute if we view them as elements of the group G.*

Proof. As k stabilizes both o_∞ and ξ, and acts via isometries on the CAT(0) space X_∞, it must also fix every point on the unique geodesic line through these two. So we deduce $k.c(t) = c(t)$ for all $t\in\mathbb{R}$. Passing from K^∞ to $K' = xK^\infty x^{-1}$ we obtain $(xkx^{-1}).c(t) = c(t)$. Using the definition of the action of G' this implies $(xkx^{-1})c(t)(xkx^{-1})^\top = c(t)$. But $(xkx^{-1})^\top = (xkx^{-1})^{-1}$ holds by $K' \subset O(n)$. This implies $(xkx^{-1})c(t) = c(t)(xkx^{-1})$, from which the statement is immediate. □

Example 17.14. In the case of $\mathbf{G} = \mathbf{SL}_n$ and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, we have

$$a_i(t) = \exp\left(t\cdot \mathrm{diag}\left(\frac{n-i}{n},\dots,\frac{n-i}{n},\frac{-i}{n},\dots,\frac{-i}{n}\right)\right),$$

where $\frac{n-i}{n}$ occurs i times and $\frac{-i}{n}$ occurs $(n-i)$ times on the diagonal. Define

$$h_i := \mathrm{diag}\left(\frac{\sqrt{n-i}}{\sqrt{n\cdot i}},\dots,\frac{\sqrt{n-i}}{\sqrt{n\cdot i}},\frac{-\sqrt{i}}{\sqrt{n\cdot(n-i)}},\dots,\frac{-\sqrt{i}}{\sqrt{n\cdot(n-i)}}\right),$$

where $\frac{\sqrt{n-i}}{\sqrt{n\cdot i}}$ occurs i times and $\frac{-\sqrt{i}}{\sqrt{n\cdot(n-i)}}$ occurs $n-i$ times on the diagonal. The map $c_i\colon \mathbb{R}\to X_\infty$ defined by $t\mapsto \exp(th_i)$ is a geodesic line as $\mathrm{tr}(h_i^2) =$

1. The stabilizer of $\xi_i := c_i(\infty)$ is exactly given by

$$P_i^\infty = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathrm{SL}_n(\mathbb{R}) \middle| A \in \mathbb{R}^{i\times i},\ B \in \mathbb{R}^{i\times(n-i)},\ D \in \mathbb{R}^{(n-i)\times(n-i)} \right\},$$

see Example 8.4 in the case $G = \mathrm{SL}_n(\mathbb{R})$. So Lemma 17.13 implies that every $k \in K^\infty \cap P_i^\infty$ commutes with $c_i(t)$ for all $t \in \mathbb{R}$. One easily verifies $a_i(t) = c_i\big(\frac{\sqrt{i}\cdot\sqrt{n-i}}{\sqrt{n}}\cdot t\big)$, from which we deduce that $a_i(t)$ and k must commute, too.

From Proposition 17.12, we get that $t \mapsto (a_i(t).x_\infty, x_f)$ is a continuous curve through the point $x = (x_\infty, x_f)$, which is smooth in the first factor.

Proposition 17.15. *There is a constant $C > 0$ such that, for all $i \in \bar{\Delta}$ and every $x = (x_\infty, x_f) \in \Omega^d_{c,r} \cap \Psi_{s,S,i}$, the derivative $\frac{d}{dt}|_{t=0} f(a_i(t).x_\infty, x_f) \geq C$ is uniformly bounded from below.*

Proof. Pick any $i \in \bar{\Delta}$ and choose $x = (x_\infty, x_f) \in \Omega^d_{c,r} \cap \Psi_{s,S,i}$. We know that f reduces to a finite sum

$$f(\_) = \sum_{I\subset\bar{\Delta}} \sum_{\bar{\gamma}\in\Gamma'} \Phi_I(\gamma.\_)B_I(\gamma.\_)$$

uniformly on a neighborhood Ω of x by Lemma 17.5. Thus,

$$f(a_i(t).x_\infty, x_f) = \sum_{I\subset\bar{\Delta}} \sum_{\bar{\gamma}\in\Gamma'} \Phi_I(\gamma.(a_i(t).x_\infty, x_f))B_I(\gamma.(a_i(t).x_\infty, x_f))$$

for small values of $|t|$, so by the product rule we get

$$\begin{aligned} \frac{d}{dt}|_{t=0} f(a_i(t).x_\infty, x_f) = & \sum_{I,\bar{\gamma}} \frac{d}{dt}|_{t=0} \Phi_I(\gamma.(a_i(t).x_\infty, x_f))B_I(\gamma.x) \\ & + \sum_{I,\bar{\gamma}} \Phi_I(\gamma.x)\frac{d}{dt}|_{t=0} B_I(\gamma.(a_i(t).x_\infty, x_f)). \end{aligned}$$

The proof will follow from a sequence of four lemmas, labeled from A to D. In Lemma B and Lemma C we show that the two sums above are always ≥ 0. In fact, Lemma C is a little stronger: Here we show that every single summand of the second sum is non-negative, while Lemma D proves that at least one summand is greater than or equal to some constant $C > 0$ which is independent of the choice of $x \in \Omega^d_{c,r} \cap \Psi_{s,S,i}$. Lemma A will apply to the other three Lemmas.

Lemma A. *If a $\gamma \in \Gamma$ is also an element of P_i^∞, then for some $a \in A$ independent of t and $y_f := \gamma.x_f \in X_f$, we have*

$$\mu_j(\gamma.(a_i(t).x_\infty, x_f)) = t\delta_{ij} + \log(\alpha_j(a)) + \mu_j(o_\infty, y_f).$$

Proof of Lemma A. Writing $\gamma \in \Gamma \cap P_i^\infty$ as $u_\gamma a_\gamma k_\gamma \in N^\infty A K^\infty$, we get $k_\gamma \in P_i^\infty$ by $N^\infty A \subset P^\infty \subset P_i^\infty$. Because of Proposition 17.12, $a_i(t)$ and k_γ commute. We pick some $g = u_g a_g k_g \in N^\infty A K^\infty$ with $g.o_\infty = x_\infty$. Then

$$\gamma a_i(t) g = u_\gamma a_\gamma k_\gamma a_i(t) u_g a_g k_g = u_\gamma a_i(t) a_\gamma k_\gamma u_g a_g k_g$$

remembering that A is abelian. Setting

$$a_\gamma k_\gamma u_g a_g k_g =: uak \in N^\infty A K^\infty$$

and

$$u_\gamma a_i(t) u a_i(t)^{-1} =: u_t \in N^\infty$$

(recall that A normalizes N^∞), we get

$$\gamma a_i(t) g = u_\gamma a_i(t) uak = u_t a_i(t) ak,$$

where $a \in A$ and $k \in K^\infty$ are independent of $t \in \mathbb{R}$. From this follows that

$$\begin{aligned}\gamma.(a_i(t).x_\infty, x_f) &= ((\gamma a_i(t) g).o_\infty, \gamma.x_f)\\ &= ((u_t a_i(t) ak).o_\infty, \gamma.x_f)\\ &= ((u_t a_i(t) a).o_\infty, y_f)\end{aligned}$$

for $y_f = \gamma.x_f$. Furthermore, for all $j \in \bar{\Delta}$ we obtain

$$\begin{aligned}\mu_j(\gamma.(a_i(t).x_\infty, x_f)) &= \mu_j((u_t a_i(t) a).o_\infty, y_f)\\ &= \log(|\alpha_j(u_t a_i(t) a, 1_H)|) + \mu_j(o_\infty, y_f)\\ &= \log(\alpha_j(a_i(t))) + \log(\alpha_j(a)) + \mu_j(o_\infty, y_f)\\ &= t\delta_{ij} + \log(\alpha_j(a)) + \mu_j(o_\infty, y_f),\end{aligned}$$

where we applied $|\alpha_j(u_t a_i(t) a, 1_H)| = \alpha_j(u_t a_i(t) a) = \alpha_j(a_i(t) a)$ in the third and $\alpha_j(a_i(t)) = e^{t\delta_{ij}}$ in the last equation. ■

As mentioned before, we will use Lemma A to prove the other statements. So that we can apply it, we use c) of Theorem 16.18.

Lemma B. *For all $\gamma \in \Gamma$ the sum*

$$\sum_{I \subset \bar{\Delta}} \frac{d}{dt}|_{t=0} \Phi_I(\gamma.(a_i(t).x_\infty, x_f)) \cdot B_I(\gamma.x)$$

is non-negative.

Proof of Lemma B. Fix a $\gamma \in \Gamma$. We have

$$\frac{d}{dt}|_{t=0} \Phi_I(\gamma.(a_i(t.x_\infty, x_f))) = 0$$

for all $I \subset \bar{\Delta}$, except when $\gamma.(a_i(t)x_\infty, x_f) \in \Psi_{s'}$ for small values of $|t|$. This means, in particular, unless

$$\gamma.x \in \Psi_{s'}.$$

Without loss of generality, we therefore assume that the latter is true. But then holds

$$\gamma.(\Omega^d_{c,r} \cap \Psi_{s',S,i}) \cap \Psi_{s'} \neq \varnothing,$$

so $\gamma \in P_i^\infty$ by c) of Theorem 16.18. Lemma A then implies that

$$\begin{aligned}\frac{d}{dt}\exp(\mu_j(\gamma.(a_i(t).x_\infty, x_f))) &= \frac{d}{dt}\left(e^{t\delta_{ij}} \cdot \alpha_j(a) \cdot \exp(\mu_j(o_\infty, y_f))\right) \\ &= \begin{cases} 0 & \text{if } j \neq i, \\ e^t \alpha_i(a)\exp(\mu_i(o_\infty, y_f)) & \text{if } j = i. \end{cases}\end{aligned}$$

So using the chain rule we get

$$\begin{aligned}&\frac{d}{dt}|_{t=0}\varphi(\exp(\mu_i(\gamma.(a_i(t).x_\infty, x_f)))) \\ &\qquad = \begin{cases} 0 & \text{if } j \neq i, \\ \varphi'(\exp(\mu_j(\gamma.x))) \cdot \alpha_i(a) \cdot \exp(\mu_i(o_\infty, y_f)) & \text{if } j = i \end{cases}\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dt}|_{t=0}\psi(\exp(\mu_i(\gamma.(a_i(t).x_\infty, x_f)))) \\ &\qquad = \begin{cases} 0 & \text{if } j \neq i, \\ \psi'(\exp(\mu_j(\gamma.x))) \cdot \alpha_i(a) \cdot \exp(\mu_i(o_\infty, y_f)) & \text{if } j = i. \end{cases}\end{aligned}$$

When we apply the product rule to $\frac{d}{dt}|_{t=0}\Phi_I(\gamma.(a_i(t).x_\infty, x_f))$ for $I \subset \bar{\Delta}$, then our above calculations lead to the following:
If $i \in I$, then $\frac{d}{dt}|_{t=0}\Phi_I(\gamma.(a_i(t).x_\infty, x_f))$ is a sum in which every summand is equal to 0 but

$$\begin{aligned}&\varphi'(\exp(\mu_i(\gamma.x))) \cdot \alpha_i(a) \cdot \exp(\mu_i(o_\infty, y_f)) \\ &\cdot \prod_{j \in I \setminus \{i\}} \varphi(\exp(\mu_j(\gamma.x))) \cdot \prod_{j \in \bar{\Delta} \setminus I} \psi(\exp(\mu_j(\gamma.x))).\end{aligned}$$

If, on the other hand, $i \notin I$, then $\frac{d}{dt}|_{t=0}\Phi_I(\gamma.(a_i(t).x_\infty, x_f))$ is a sum in which every summand is equal to 0 but

$$\begin{aligned}&\psi'(\exp(\mu_i(\gamma.x))) \cdot \alpha_i(a) \cdot \exp(\mu_i(o_\infty, y_f)) \\ &\cdot \prod_{j \in I} \varphi(\exp(\mu_j(\gamma.x))) \cdot \prod_{j \in \bar{\Delta} \setminus (I \cup \{i\})} \psi(\exp(\mu_j(\gamma.x))).\end{aligned}$$

We note for later that in the case $i \in I$, we always have

$$\frac{d}{dt}|_{t=0}\Phi_I(\gamma.(a_i(t).x_\infty, x_f)) \geq 0$$

as φ, ψ, φ' and the restriction of α_i to A are all non-negative.
Recall now that $\gamma.x$ is an element of $\Psi_{s'}$, but we also have assumed $x \in (\Omega^d_{c,r} \cap \Psi_{s,S,i})$, so from Theorem 16.18 d) we obtain

$$\exp(\mu_i(\gamma.x)) > s.$$

On the interval $[s, \infty[$ we have $\psi' = -\varphi'$ because of $\varphi + \psi = 1$, so for any $I \subset (\bar{\Delta} \smallsetminus \{i\})$ the derivative $\frac{d}{dt}|_{t=0}\Phi_I(\gamma.(a_i(t).x_\infty, x_f))$ is equal to

$$\begin{aligned}&-\varphi'(\exp(\mu_i(\gamma.x))) \cdot \alpha_i(a) \cdot \exp(\mu_i(o_\infty, y_f))\\ &\cdot \prod_{j \in I} \varphi(\exp(\mu_j(\gamma.x))) \cdot \prod_{j \in \bar{\Delta} \smallsetminus (I \cup \{i\})} \psi(\exp(\mu_j(\gamma.x))).\end{aligned}$$

Thus, whenever I is a subset of $\bar{\Delta}$ with $i \notin I$, we have

$$\frac{d}{dt}|_{t=0}\Phi_I(\gamma.(a_i(t).x_\infty, x_f)) = -\frac{d}{dt}|_{t=0}\Phi_{I\cup\{i\}}(\gamma.(a_i(t).x_\infty, x_f)). \tag{17.1}$$

Now we come back to looking at the whole sum

$$\sum_{I \subset \bar{\Delta}} \frac{d}{dt}|_{t=0}\Phi_I(\gamma.(a_i(t).x_\infty, x_f))B_I(\gamma.x).$$

Setting

$$\mathcal{E} = \{I \subset \bar{\Delta} \mid i \notin I\}$$

we can rewrite it as

$$\begin{aligned}&\sum_{I \in \mathcal{E}} \frac{d}{dt}|_{t=0}\Phi_I(\gamma.(a_i(t).x_\infty, x_f))B_I(\gamma.x)\\ &+\sum_{I \in \mathcal{E}} \frac{d}{dt}|_{t=0}\Phi_{I\cup\{i\}}(\gamma.(a_i(t).x_\infty, x_f))B_{I\cup\{i\}}(\gamma.x).\end{aligned}$$

But that implies

$$\begin{aligned}&\sum_{I \subset \bar{\Delta}} \frac{d}{dt}|_{t=0}\Phi_I(\gamma.(a_i(t).x_\infty, x_f))B_I(\gamma.x)\\ =&\sum_{I \in \mathcal{E}} \frac{d}{dt}|_{t=0}\Phi_{I\cup\{i\}}(\gamma.(a_i(t).x_\infty, x_f)) \cdot (B_{I\cup\{i\}}(\gamma.x) - B_I(\gamma.x))\end{aligned}$$

because of Equation (17.1). Since $i \in (I \cup \{i\})$ is always fulfilled, we have

$$\frac{d}{dt}|_{t=0}\Phi_{I\cup\{i\}}(\gamma.(a_i(t).x_\infty, x_f)) \geq 0,$$

so we just have to show $(B_{I\cup\{i\}}(\gamma.x) - B_I(\gamma.x)) \geq 0$ for all $I \in \mathcal{E}$.
Now, for any subset $I \subset \bar{\Delta}$, the derivative $\frac{d}{dt}|_{t=0}\Phi_I(\gamma.(a_i(t).x_\infty, x_f))$ equals 0, except when $\varphi'(\exp(\mu_i(\gamma.x))) \neq 0$, so the whole sum becomes 0 unless $\exp(\mu_i(\gamma.x)) \in [S-\varepsilon, S]$. Thus we additionally assume

$$\mu_i(\gamma.x) \geq \log(S-\varepsilon).$$

By Lemma 17.1 we have

$$B_{I\cup\{i\}} - B_I = \sum_{j\in\bar{\Delta}} c_{Iij}\mu_j$$

for some scalars $c_{Iij} \geq 0$ and $c_{Iii} > 0$ and every $I \in \mathcal{E}$, so from $\gamma.x \in \Psi_{s'}$ we deduce

$$\begin{aligned} B_{I\cup\{i\}}(\gamma.x) - B_I(\gamma.x) &= \sum_{j\in\bar{\Delta}} c_{Iij}\mu_j(\gamma.x) \\ &= c_{Iii}\mu_i(\gamma.x) + \sum_{j\in\bar{\Delta}\smallsetminus\{i\}} c_{Iij}\mu_j(\gamma.x) \\ &\geq c_{Iii}\log(S-\varepsilon) + \sum_{j\in\bar{\Delta}\smallsetminus\{i\}} c_{Iij}\log(s-\varepsilon). \end{aligned}$$

Since Theorem 16.18 holds for all $S \geq S_0$, we can additionally assume, by enlarging it if necessary, that $S > 0$ fulfills

$$c_{Iii}\log(S-\varepsilon) + \sum_{j\in\bar{\Delta}\smallsetminus\{i\}} c_{Iij}\log(s-\varepsilon) > 0,$$

from which we get

$$B_{I\cup\{i\}}(\gamma.x) - B_I(\gamma.x) > 0$$

for every $I \in \mathcal{E}$. Now the sum

$$\sum_{I\subset\bar{\Delta}} \frac{d}{dt}|_{t=0}\Phi_I(\gamma.(a_i(t).x_\infty, x_f)) \cdot B_I(\gamma.x)$$

is always bigger than or equal to 0. ■

This proves that the first sum is non-negative. We now take care of the second sum.

Lemma C. *For all $\gamma \in \Gamma$ and $I \subset \bar{\Delta}$, we have*

$$\Phi_I(\gamma.x) \cdot \frac{d}{dt}|_{t=0}B_I(\gamma.(a_i(t).x_\infty, x_f)) \geq 0.$$

Proof of Lemma C. Fixing a $\gamma \in \Gamma$ again, we see $\Phi_I(\gamma.x) = 0$ for all subsets I of $\bar{\Delta}$ unless $\gamma.x \in \Psi_{s'}$, so as in the proof of Lemma B we will assume that $\gamma.x$ is an element of $\Psi_{s'}$. For the same reason we deduce that $\gamma \in P_i^\infty$ by applying Theorem 16.18. Therefore we obtain

$$\mu_j(\gamma.(a_i(t).x_\infty, x_f)) = t\delta_{ij} + \log(\alpha_j(a)) + \mu_j(o_\infty, y_f)$$

for every $j \in \bar{\Delta}$ by Lemma A. We further know by Corollary 17.2 that $B_I = \sum_{j\in\bar{\Delta}} a_{Ij}\mu_j$ is a non-negative linear combination for all $I \subset \bar{\Delta}$, so we deduce

$$\begin{aligned} B_I(\gamma.(a_i(t).x_\infty, x_f)) &= \sum_{j\in\bar{\Delta}} a_{Ij}\mu_j(\gamma.(a_i(t).x_\infty, x_f)) \\ &= \sum_{j\in\bar{\Delta}} a_{Ij}(t\delta_{ij} + \log(\alpha_j(a)) + \mu_j(o_\infty, y_f)) \end{aligned}$$

which implies that

$$\frac{d}{dt}|_{t=0} B_I(\gamma.(a_i(t).x_\infty, x_f)) = a_{Ii} \geq 0.$$

Since Φ_I is always greater than or equal to 0 we obtain

$$\Phi_I(\gamma.x)\frac{d}{dt}|_{t=0} B_I(\gamma.(a_i(t).x_\infty, x_f)) \geq 0$$

for every subset I of $\bar{\Delta}$. ■

Hence every single summand in

$$\sum_{I,\bar{\gamma}} \Phi_I(\gamma.x) \cdot \frac{d}{dt}|_{t=0} B_I(\gamma.(a_i(t).x_\infty, x_f))$$

is greater than or equal to 0. Finally, we show that at least one of them with $\gamma = 1$ is bounded from below by a constant > 0.

Lemma D. *There is a constant $C > 0$ such that for all $x = (x_\infty, x_f) \in (\Omega^d_{c,r} \cap \Psi_{s,S,i})$ there is a subset I of $\bar{\Delta}$ with*

$$\Phi_I(x) \cdot \frac{d}{dt}|_{t=0} B_I(a_i(t).x_\infty, x_f) \geq C.$$

Proof of Lemma D. Pick any $x \in (\Omega^d_{c,r} \cap \Psi_{s,S,i})$ and set

$$I = \left\{ j \in \bar{\Delta} \,\middle|\, \varphi(\exp(\mu_j(x))) > \frac{1}{2} \right\}.$$

With this choice we have $i \in I$ as $\exp(\mu_i(x)) \geq S$. Since $\gamma = 1$ is an element of P_i^∞, we have again

$$B_I((a_i(t).x_\infty, x_f)) = \sum_{j \in \bar{\Delta}} a_{Ij}(t\delta_{ij} + \log(\alpha_j(a)) + \mu_j(o_\infty, y_f))$$

by Lemma A so that

$$\frac{d}{dt}|_{t=0} B_I((a_i(t).x_\infty, x_f)) = a_{Ii}$$

for the non-negative linear combination $B_I = \sum_{j \in \bar{\Delta}} a_{Ij}\mu_j$. In fact, as i is an element of I, Corollary 17.2 means that a_{Ii} is strictly positive, which implies

$$\frac{d}{dt}|_{t=0} B_I((a_i(t).x_\infty, x_f)) = a_{Ii} > 0.$$

On the other hand, we know that $\exp(\mu_j(x)) \geq s$ holds for all $j \in \bar{\Delta}$, so we get $\psi(\exp(\mu_j(x))) = 1 - \varphi(\exp(\mu_j(x)))$. By definition

$$\varphi(\exp(\mu_j(x))) \begin{cases} > \frac{1}{2} & \text{if } j \in I, \\ \leq \frac{1}{2} & \text{if } j \notin I \end{cases}$$

and we deduce $\psi(\exp(\mu_j(x))) \geq \frac{1}{2}$ in the second case. Thus,

$$\begin{aligned} \Phi_I(x) &= \prod_{j \in I} \varphi(\exp(\mu_j(x))) \cdot \prod_{j \in \bar{\Delta} \setminus I} \psi(\exp(\mu_j)) \\ &\geq \prod_{j \in I} \frac{1}{2} \cdot \prod_{j \in \bar{\Delta} \setminus I} \frac{1}{2} \end{aligned}$$

so we get

$$\Phi_I(x) \cdot \frac{d}{dt}|_{t=0} B_I((a_i(t).x_\infty, x_f)) \geq \left(\frac{1}{2}\right)^{|\Delta|} \cdot a_{Ii} > 0.$$

Therefore

$$C := \min \left\{ a_{Ii} \left(\frac{1}{2}\right)^{|\Delta|} \middle| I \subset \bar{\Delta} \text{ and } i \in I \right\} > 0$$

is the desired constant. ■

We have thus proven the proposition. □

Proposition 17.16. *There is a constant $D > 0$ such that for every $x_\infty \in (S_{c,r}.o_\infty) \subset X_\infty$ the norm $\|c_i'(0)\| \leq D$ is uniformly bounded from above for the curve $c_i(t) = a_i(t).x_\infty$.*

Proof. Recall that we defined $X_\infty = P_n(\mathbb{R}) \cap G'$ with basepoint $o_\infty = \mathbb{1}_n$ and that the action of G' is given by conjugation. The identification $G \cong G'$ also identifies the subsets $A_r \cong A_r'$ and $N_c^\infty \cong N_c'$. Thus, we can write an arbitrary point $p = x_\infty \in S_{c,r}.o_\infty$ as

$$p = (ua).\mathbb{1}_n = ua^2u^\top$$

for some $u = (u_{ij})_{i,j=1,\dots,n} \in N_c'$ and $a = \operatorname{diag}(a_1, \dots, a_n) \in A_r'$, that is such that $|u_{ij}| \leq c$ for $1 \leq i < j \leq n$ and $\alpha'(a) > r$ for all $\alpha' \in \Delta'$. Further, the groups A and A' are identified under $G \cong G'$ and we denote by $a_k(t)' \in A'$ the element to which $a_k(t)$ is mapped. Passing from A to A' we can rewrite the curve $c_k(t) = a_k(t).p$ as

$$c_k(t) = a_k(t)'.p = a_k(t)'pa_k(t)'.$$

For this curve, we have to prove that the norm

$$\left\|c_k'(0)\right\|_{c_k(0)} = \operatorname{tr}(p^{-1}c_k'(0)p^{-1}c_k'(0))$$

is uniformly bounded (here $c_k'(0)$ denotes the derivative of the curve). Our plan is to show that the entries of

$$p^{-1}c_k'(0)$$

are uniformly bounded, from which the assertion follows. First, we compute the entries of $p^{-1}c_k'(0)$ for an arbitrary regular matrix $p \in \operatorname{GL}_n(\mathbb{R})$, and then we show that the entries are uniformly bounded when we write $p = ua^2u^\top$ as above.

If $d(t) = \operatorname{diag}(d_1(t), \dots, d_n(t))$ is any curve in A' through $\mathbb{1}_n$, then $c(t) := d(t).p$ is a curve through p with entries $c(t)_{i,j} = d_i(t)d_j(t)p_{i,j}$. Thus, the entries of the derivative are $c'(0)_{i,j} = (d_i'(0) + d_j'(0))p_{i,j}$. Applying this to the curve $c_k(t)$ we obtain that the entries are

$$c_k'(0)_{i,j} = \varepsilon_{k,i,j} \cdot p_{i,j}$$

for some $\varepsilon_{k,i,j} \in \mathbb{R}$.
On the other hand, by Cramer's rule, the inverse of p has entries

$$(p^{-1})_{i,j} = \frac{1}{\det(p)} \cdot \sum_{\substack{\rho \in \operatorname{Sym}(n)\\ \rho(j)=i}} \operatorname{sgn}(\rho) \prod_{\ell \neq j} p_{\ell,\rho(\ell)}.$$

Bringing these two things together, we see

$$\begin{aligned}(p^{-1}\cdot c_k'(0))_{i,j} &= \sum_{\nu=1}^{n}(p^{-1})_{i\nu}\cdot(c_k'(0))_{\nu,j}\\ &= \sum_{\nu=1}^{n}\left(\sum_{\substack{\rho\in\mathrm{Sym}(n)\\ \rho(\nu)=i}}\frac{1}{\det(p)}\cdot\mathrm{sgn}(\rho)\prod_{\ell\neq\nu}p_{\ell,\rho(\ell)}\right)\cdot\varepsilon_{k,\nu,j}p_{\nu,j}\\ &= \sum_{\nu=1}^{n}\varepsilon_{k,\nu,j}\sum_{\substack{\rho\in\mathrm{Sym}(n)\\ \rho(\nu)=i}}\mathrm{sgn}(\rho)\cdot\frac{1}{\det(p)}\cdot\left(\prod_{\ell\neq\nu}p_{\ell,\rho(\ell)}\right)\cdot p_{\nu,j}.\end{aligned}$$

Hence, we only need to show that products of the form

$$\frac{1}{\det(p)}\cdot\left(\prod_{\ell\neq\nu}p_{\ell,\rho(\ell)}\right)\cdot p_{\nu,j} \tag{17.2}$$

are uniformly bounded for all $i,j,\nu\in\{1,\dots,n\}$ and $\rho\in\mathrm{Sym}(n)$ with $\rho(\nu)=i$.

When we compute the entries of $p=ua^2u^\top$ with $u=(u_{i,j})_{i,j}\in N'$ and $a=\mathrm{diag}(a_1,\dots,a_n)\in A'$ we have

$$p_{i,j}=(ua^2u^\top)_{i,j}=\sum_{k=1}^{n}(ua^2)_{i,k}(u^\top)_{k,j}=\sum_{k=1}^{n}u_{i,k}u_{j,k}a_k^2$$

because of $(ua^2)_{i,k}=u_{i,k}a_k^2$. Using now

$$p_{\ell,\rho(\ell)}=\sum_{t=1}^{n}u_{\ell,t}u_{\rho(\ell),t}a_t^2 \quad\text{and}\quad p_{\nu,j}=\sum_{t=1}^{n}u_{\nu,t}u_{j,t}a_t^2,$$

we get

$$\left(\prod_{\ell\neq\nu}p_{\ell,\rho(\ell)}\right)\cdot p_{\nu,j}=\left(\prod_{\ell\neq\nu}\sum_{t=1}^{n}u_{\ell,t}u_{\rho(\ell),t}a_t^2\right)\cdot\sum_{t=1}^{n}u_{\nu,t}u_{j,t}a_t^2.$$

A basic combinatorial argument shows $\prod_{i=1}^{n}\sum_{j=1}^{m}x_{i,j}=\sum_{j_1,\dots,j_n=1}^{m}\prod_{i=1}^{n}x_{i,j}$ so we can rewrite the right side above as

$$\sum_{t_1,\dots,t_n=1}^{n}\left(\prod_{\ell\neq\nu}u_{\ell,t_\ell}u_{\rho(\ell),t_\ell}a_{t_\ell}^2\right)\cdot u_{\nu,t_\nu}u_{j,t_\nu}a_{t_\nu}^2.$$

Now $u_{\ell,t_\ell} = 0$ if $\ell > t_\ell$ and $u_{\nu,t_\nu} = 0$ if $\nu > t_\nu$ so this sum is equal to

$$\sum_{\substack{t_1,\ldots,t_n=1,\ldots,n \\ t_\ell \geq \ell \text{ for all } \ell=1,\ldots,n}} \left(\prod_{\ell \neq \nu} u_{\ell,t_\ell} u_{\rho(\ell),t_\ell} \right) \cdot u_{\nu,t_\nu} u_{j,t_\nu} \cdot \prod_{\ell=1}^{n} a_{t_\ell}^2.$$

All of the $|u_{ij}|$ are uniformly bounded by a constant $c > 0$, and $\det(p) = \prod_{s=1}^n a_s^2$. So, coming back to the product (17.2), we only need to show the following: Products of the form

$$\prod_{s=1}^{n} a_{t_s} a_s^{-1}$$

with $n \geq t_s \geq s$ for every $s = 1, \ldots, n$ are uniformly bounded for all $a = \operatorname{diag}(a_1, \ldots, a_n) \in A'_r$.
Thus, we just need to show that whenever we have a product $a_j \cdot a_i^{-1}$ with $j > i$, then we can rewrite this as a product of the form $\prod_{\alpha' \in \Delta'} \alpha'(a)^{-k_{\alpha'}}$ for some $k_{\alpha'} \geq 0$ since the α' are all uniformly bounded from below by r on A'_r. We conclude with $a_j a_i^{-1} = \frac{\varepsilon_j(a)}{\varepsilon_i(a)}$ and the fact that all the $\varepsilon_i - \varepsilon_j$ are non-negative linear combinations of the simple roots Δ'. □

The two previous propositions give us the desired statement about the norm.

Proposition 17.17. *There is $r_0 > 0$ such that for every $x = (x_\infty, x_f) \in f^{-1}(]r_0, \infty[)$ the norm $\left\|\nabla f_{x_\infty}^{x_f}\right\|$ is uniformly bounded from below.*

Proof. In light of Lemma 14.3, it suffices to show that there are $r_0 > 0$ and constants $C, D > 0$ fulfilling the following property: For every $x \in f^{-1}(]r_0, \infty[)$ there is a smooth curve $c(t)$ in X_∞ through $c(0) = x_\infty$ and with $\frac{d}{dt}|_{t=0} f(c(t), x_f) \geq C$ and $\|c'(0)\|_{c(0)} \leq D$.
Recall some facts:

(i) $\Gamma.(\Omega^d_{c,r} \cap \Psi_s) = X$,

(ii) $f(\gamma.x) = f(x)$ for all $\gamma \in \Gamma$,

(iii) Γ acts on X_∞ via Riemannian isometries.

Therefore, the above is reduced to finding $r_0 > 0$ and $C, D > 0$ such that, for every $x \in (f|_{\Omega^d_{c,r} \cap \Psi_s})^{-1}(]r_0, \infty[)$, there is a curve $c(t)$ in X_∞ with $c(0) = x_\infty$, $\frac{d}{dt}|_{t=0} f(c(t), x_f) \geq C$ and $\|c'(0)\| \leq D$.

Elementary set operations give us

$$\begin{aligned}(\Omega^d_{c,r} \cap \Psi_s) \setminus \bigcup_{i\in\bar{\Delta}} (\Omega^d_{c,r} \cap \Psi_{s,S,i}) &= \bigcap_{i\in\bar{\Delta}} \Big((\Omega^d_{c,r} \cap \Psi_s) \setminus (\Omega^d_{c,r} \cap \Psi_{s,S,i}) \Big) \\ &= \bigcap_{i\in\bar{\Delta}} \Big(\Omega^d_{c,r} \cap (\Psi_s \setminus \Psi_{s,S,i}) \Big) \\ &= \Omega^d_{c,r} \cap \bigcap_{i\in\bar{\Delta}} (\Psi_s \setminus \Psi_{s,S,i}).\end{aligned}$$

Now

$$\Psi_s \setminus \Psi_{s,S,i} = \{x \in X \mid \mu_j(x) \geq s \text{ for all } j \in \bar{\Delta} \text{ and } \mu_i(x) < S\},$$

so

$$\bigcap_{i\in\bar{\Delta}} (\Psi_s \setminus \Psi_{s,S,i}) = \bigcap_{i\in\bar{\Delta}} \mu_i^{-1}([s, S[),$$

and this implies that

$$\Psi := \left((\Omega^d_{c,r} \cap \Psi_s) \setminus \bigcup_{i\in\bar{\Delta}} (\Omega^d_{c,r} \cap \Psi_{s,S,i}) \right) = \Omega^d_{c,r} \cap \bigcap_{i\in\bar{\Delta}} \mu_i^{-1}([s, S[)$$

is relatively compact by Lemma 17.8 as it is a subset of $\Omega^d_{c,r} \cap \bigcap_{i\in\bar{\Delta}} \mu_i^{-1}([s, S])$. Thus, there is $r_0 > 0$ such that $f(\Psi)$ lies inside $]-\infty, r_0]$, which means that

$$\Psi \subset (f|_{\Omega^d_{c,r}\cap\Psi_s})^{-1}(]-\infty, r_0]).$$

But in the two previous propositions, we showed that for $x \in (\Omega^d_{c,r} \cap \Psi_{s,S,i})$ the curve $c_i(t) = a_i(t).x_\infty$ fulfills $\frac{d}{dt}|_{t=0} f(c_i(t), x_f) \geq C$ and $\|c_i'(0)\| \leq D$. The statement now follows from the fact that

$$(f|_{\Omega^d_{c,r}\cap\Psi_s})^{-1}(]r_0, \infty[) = (\Omega^d_{c,r} \cap \Psi_s) \setminus \Big((f|_{\Omega^d_{c,r}\cap\Psi_s})^{-1}(]-\infty, r_0]) \Big)$$

is a subset of

$$(\Omega^d_{c,r} \cap \Psi_s) \setminus \Psi = \bigcup_{i\in\bar{\Delta}} (\Omega^d_{c,r} \cap \Psi_{s,S,i}). \qquad \square$$

To prove the theorem, we are only left with convergence in the weak C^1 topology. We need this lemma.

Lemma 17.18. *Let M be a complete Riemannian manifold and let Z be a complete metric space. We assume both to be* CAT(0)*. If a function $g\colon M \times Z \to \mathbb{R}$ is given by a linear combination of Busemann functions, then $g_{z_n} \to g_z$ converges uniformly on the whole manifold M, where $g_z\colon M \to \mathbb{R}$ is defined by $g_z(p) = g(p, z)$. If we additionally assume that all Busemann functions on M are smooth, then $\nabla g_{z'} = \nabla g_z$ holds for all $z, z' \in Z$.*

Proof. Every Busemann function β on $M \times Z$ is of the form

$$\beta(p, z) = \cos(\theta)\beta^M(p) + \sin(\theta)\beta^Z(z)$$

for Busemann functions β^M on M and β^Z on Z, see Lemma 3.16. We write g as $g(p, z) = \sum_{i=1}^n a_i\beta_i(p, z)$. If $(z_n)_{n\in\mathbb{N}}$ is a sequence in Z converging to some z, then

$$|g_{z_n}(p) - g_z(p)| \leq \sum_{i=1}^n |a_i| \, |\beta_i(p, z_n) - \beta_i(p, z)|$$

and

$$\begin{aligned} &|\beta_i(p, z_n) - \beta_i(p, z)| \\ &= \left|\cos(\theta_i)\beta_i^M(p) + \sin(\theta_i)\beta_i^Z(z_n) - (\cos(\theta_i)\beta_i^M(p) + \sin(\theta_i)\beta_i^Z(z))\right| \\ &= |\sin(\theta_i)| \cdot \left|\beta_i^Z(z_n) - \beta_i^Z(z)\right| \end{aligned}$$

imply that $g_{z_n} \to g_z$ uniformly on the whole manifold M.
Assume now that every β^M is smooth. By Proposition 8.8, the gradient is linear and it vanishes on constant functions. So we see

$$\nabla\beta_i(\_, z) = \cos(\theta_i)\nabla\beta_i^M + \sin(\theta_i)\beta_i^Z(z)\nabla 1 = \cos(\theta_i)\nabla\beta_i^M.$$

Thus,

$$\nabla(\beta_i(\_, z')) - \nabla(\beta_i(\_, z)) = 0$$

holds for all $z, z' \in Z$. Applying linearity of ∇ to the function g_z we also have $\nabla g_z = \sum_{i=1}^n a_i\nabla\beta_i(\_, z)$, and therefore

$$\begin{aligned} \nabla g_{z'} - \nabla g_z &= \sum_{i=1}^n a_i\nabla\beta_i(\_, z') - \sum_{i=1}^n a_i\nabla\beta_i(\_, z) \\ &= \sum_{i=1}^n a_i(\nabla\beta_i(\_, z') - \nabla\beta_i(\_, z)) \\ &= 0. \end{aligned}$$

□

The following is a trivial observation.

Lemma 17.19. *For every $n \in \mathbb{N}$, let h_n and g_n be continuous real-valued functions defined on a topological space Z such that the sequences $(h_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ converge uniformly on compact subsets of Z to functions $h, g\colon Z \to \mathbb{R}$. Then the following hold:*

a) The sequence $(h_n \cdot g_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets of Z to $h \cdot g$.

b) If further $\phi\colon \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then $(\phi \circ g_n) \to (\phi \circ g)$ also converges uniformly on compact subsets $C \subset Z$.

Proof. For a) it suffices to observe

$$|h_n g_n - hg| \leq |g_n| \cdot |h_n - h| + |h| \cdot |g_n - g|$$

and recall that $|g_n|$ and $|h|$ are uniformly bounded on compact sets. To prove b), let $\varepsilon > 0$. Then there is $\delta > 0$ such that

$$\left|t - t'\right| < \delta \Rightarrow \left|\phi(t) - \phi(t')\right| < \varepsilon$$

holds for all $t, t' \in \mathbb{R}$. Pick any compact subset C of Z. For the above choice of δ we take $N \in \mathbb{N}$ with $|g_n(z) - g(z)| < \delta$, which implies that

$$|(\phi \circ g_n)(z) - (\phi \circ g)(z)| < \varepsilon$$

for all $n \geq N$ and any $z \in C$. □

In the following proposition we consider the two factors of the product $X = X_\infty \times X_f$ separately. To simplify notation, we will denote a point of X_∞ by p and a point of X_f by z.

Proposition 17.20. *If $(z_n)_{n\in\mathbb{N}}$ is a sequence in X_f converging to a point z, then $f^{z_n} \to f^z$ in the weak C^1 topology.*

Proof. In light of Proposition 14.8, we have to show that $f^{z_n} \to f^z$ converges uniformly on compact subsets of X_∞ as $n \to \infty$, and that in any chart (U, ϕ) of X_∞ the coefficient vector of ∇f^{z_n} converges to that of ∇f^z uniformly on compact subsets $K \subset U$. Recall that f is locally given as a finite sum

$$f = \sum_{I \subset \bar{\Delta}} \sum_{\bar{\gamma} \in \Gamma'} \Phi_I(\gamma.\_) \cdot B_I(\gamma.\_)$$

by Lemma 17.5. Since each compact subset is covered by finitely many such neighborhoods, we can assume without loss of generality that f has the above form. The maps $\varphi \circ \exp$ and $\psi \circ \exp$ are both uniformly continuous, as they are smooth and constant outside of a compact interval. Every μ_i is a linear combination of Busemann functions, and any map of the form $x \mapsto \beta(\gamma.x)$ is again a Busemann function, as Γ acts via isometries on X. Setting

$$\mu^z_{i,\gamma}\colon X_\infty \to \mathbb{R},\ p \mapsto \mu_i(\gamma.(p, z)),$$

we see by Lemmas 17.18 and 17.19 that the sequence $\left((\varphi \circ \exp) \circ \mu^{z_n}_{i,\gamma}\right)_{n\in\mathbb{N}}$ converges uniformly on compact subsets to $(\varphi \circ \exp) \circ \mu^z_{i,\gamma}$ for all $i \in \bar{\Delta}$. The analogous statement with ψ holds for the same reason. Defining further

$$\Phi^z_{I,\gamma} := \prod_{i \in I} \left((\varphi \circ \exp) \circ \mu^z_{i,\gamma}\right) \cdot \prod_{i \notin I} \left((\psi \circ \exp) \circ \mu^z_{i,\gamma}\right),$$

we now obtain that $\Phi^{z_n}_{I,\gamma}$ converges to $\Phi^{z}_{I,\gamma}$ uniformly on compact subsets of X_∞ for all $I \subset \bar{\Delta}$ and $\gamma \in \Gamma$. Since every B_I is also a linear combination of Busemann functions, setting

$$B^z_{I,\gamma}(p) := B_I(\gamma.(p,z)),$$

the sequence $\left(B^{z_n}_{I,\gamma}\right)_{n\in\mathbb{N}}$ also converges uniformly on compact sets to $B^z_{I,\gamma}$. As f^z is locally given by a finite sum of $B^z_{I,\gamma} \cdot \Phi^z_{I,\gamma}$, we conclude that $f^{z_n} \to f^z$ uniformly on compact sets as $n \to \infty$.

Let us now have a look at the gradient. Locally, we can write it as the finite sum

$$\nabla f^z = \sum_{I\subset\bar{\Delta}} \sum_{\bar{\gamma}\in\Gamma'} \left((\nabla \Phi^z_{I,\gamma}) \cdot B^z_{I,\gamma} + \Phi^z_{I,\gamma} \cdot (\nabla B^z_{I,\gamma}) \right)$$

by Proposition 8.8. We fix some $\gamma \in \Gamma$ and a subset $I \subset \bar{\Delta}$.

We first take a closer look at $(\nabla B^z_{I,\gamma})$. Lemma 17.18 implies that $\nabla(B^{z_n}_{I,\gamma}) = \nabla(B^z_{I,\gamma})$ for all $n \in \mathbb{N}$ and from our considerations above, we already know that $\Phi^{z_n}_{I,\gamma} \to \Phi^z_{I,\gamma}$ uniformly on compact sets. Thus, the coefficient vector of the sequence $\left(\Phi^{z_n}_{I,\gamma} \cdot (\nabla B^{z_n}_{I,\gamma})\right)_{n\in\mathbb{N}}$ converges to that of $\Phi^z_{I,\gamma} \cdot (\nabla B^z_{I,\gamma})$ uniformly on compact subsets of any chart in X_∞.

Next, we consider $(\nabla \Phi^z_{I,\gamma}) \cdot B^z_{I,\gamma}$. Again, by Proposition 8.8, the gradient of $\Phi^z_{I,\gamma}$ is given by

$$\begin{aligned} &\nabla \left(\prod_{i\in I} (\varphi \circ \exp) \circ \mu^z_{i,\gamma} \right) \cdot \left(\prod_{i\notin I} (\psi \circ \exp) \circ \mu^z_{i,\gamma} \right) \\ &+ \left(\prod_{i\in I} (\varphi \circ \exp) \circ \mu^z_{i,\gamma} \right) \cdot \nabla \left(\prod_{i\notin I} (\psi \circ \exp) \circ \mu^z_{i,\gamma} \right). \end{aligned}$$

Furthermore, we have

$$\nabla \left(\prod_{i\in I} (\varphi \circ \exp) \circ \mu^z_{i,\gamma} \right) = \sum_{i\in I} \left((\nabla(\varphi \circ \exp) \circ \mu^z_{i,\gamma}) \cdot \prod_{j\in I\smallsetminus\{i\}} (\varphi \circ \exp) \circ \mu^z_{j,\gamma} \right)$$

and

$$\nabla\big((\varphi \circ \exp) \circ \mu^z_{i,\gamma}\big) = \big((\varphi \circ \exp)' \circ \mu^z_{i,\gamma}\big) \cdot \nabla \mu^z_{i,\gamma},$$

once more because of Proposition 8.8. But the smooth function $(\varphi \circ \exp)'$ is uniformly continuous as it has compact support, therefore $\big((\varphi \circ \exp)' \circ \mu^{z_n}_{i,\gamma}\big)$ converges uniformly on compact sets to $\big((\varphi \circ \exp)' \circ \mu^z_{i,\gamma}\big)$ as $n \to \infty$ using Lemma 17.19. Thus, applying Lemma 17.18 to $\mu^z_{i,\gamma}$ we see $\nabla \mu^{z_n}_{i,\gamma} = \nabla \mu^z_{i,\gamma}$, so that the coefficient vector of $\big(\nabla\big((\varphi \circ \exp) \circ \mu^{z_n}_{i,\gamma}\big)\big)_{n\in\mathbb{N}}$ converges to that of $\nabla\big((\varphi \circ \exp) \circ \mu^z_{i,\gamma}\big)$ uniformly on all compact subsets in any chart of X_∞.

From the compact convergence

$$\lim_{n\to\infty} \big((\varphi \circ \exp) \circ \mu^{z_n}_{i,\gamma}\big) = \big((\varphi \circ \exp) \circ \mu^z_{i,\gamma}\big)$$

already shown above we get the following uniform convergence of the coefficient vectors with respect to any chart:

$$\nabla\left(\prod_{i\in I}(\varphi\circ\exp)\circ\mu_{i,\gamma}^{z_n}\right)\to\nabla\left(\prod_{i\in I}(\varphi\circ\exp)\circ\mu_{i,\gamma}^{z}\right)$$

as $n\to\infty$. Exchanging the roles of φ and ψ and those of I and $\bar{\Delta}\setminus I$ the analogous statement holds by the same argument.
From all this follows that the coefficient vector of $\left(\nabla\Phi_{I,\gamma}^{z_n}\right)_{n\in\mathbb{N}}$ converges to that of $\nabla\Phi_{I,\gamma}^{z}$ uniformly on compact subsets in any chart. Thus, from the compact convergence $\lim_{n\to\infty}B_{I,\gamma}^{z_n}=B_{I,\gamma}$ by Lemma 17.18, we obtain that the coefficient vector of $(\nabla\Phi_{I,\gamma}^{z_n})\cdot B_{I,\gamma}^{z_n}$ converges to that of $(\nabla\Phi_{I,\gamma}^{z})\cdot B_{I,\gamma}^{z}$ uniformly on all compact subsets in any chart.
Putting all things together we get that $f^{z_n}\to f^{z}$ in the weak C^1 topology as $n\to\infty$. ☐

Thus, we have proven the theorem:

Theorem 17.21. *The* Γ*-invariant function* $f\colon X\to\mathbb{R}$ *is a Morse function without critical values in* $]r_0,\infty[$. *For every* $t>r_0$ *the sublevel set* $f^{-1}(]-\infty,t])\simeq X$ *is cocompact under the induced action of* Γ.

Proof. Continuity on the whole space X and smoothness in the X_∞-factor is Corollary 17.6, uniform boundedness of the gradient is Proposition 17.17 and convergence in the weak C^1 topology is the previous proposition. Therefore, the sublevel sets $f^{-1}(]-\infty,t])$ are homotopy equivalent to X by Theorem 14.23. Cocompactness is Corollary 17.11. ☐

18 Finiteness properties of the S-arithmetic group

Applying the results from the previous chapter, we deduce the finiteness properties of the group Γ. Our goal is to prove

Theorem 18.1. *The group* Γ *is of type* F_∞.

Before we come to the proof, recall that

a) X is a CAT(0) space,

b) $G \times H$ is locally compact and second-countable,

c) the action of $G \times H$ on the space X is continuous, proper, and via isometries.

Thus, using the notation

$$N_r := N_r(\Gamma.o) = \{x \in X \mid d(\gamma.o, x) < r \text{ for some } \gamma \in \Gamma\},$$

we get the following criterion for finiteness properties:

Lemma 18.2. *The discrete subgroup* $\Gamma \leq (G \times H)$ *is of type* F_n *if and only if the filtration* $(N_r)_{r>0}$ *is essentially* $(n-1)$*-connected.*

Proof. This follows directly from Lemma 4.10. □

Let

$$X_t := f^{-1}(]-\infty, t])$$

for every $t \in \mathbb{R}$. The next lemma ensures that we can replace the filtration $(N_r)_{r>0}$ by $(X_t)_{t\in\mathbb{R}}$.

Lemma 18.3. *The two filtrations* $(N_r)_{r>0}$ *and* $(X_t)_{t\in\mathbb{R}}$ *are equivalent. Consequently,* Γ *is of type* F_n *if and only if the filtration* $(X_t)_{t\in\mathbb{R}}$ *is essentially* $(n-1)$*-connected.*

Proof. Let $r > 0$ be arbitrary. Since Γ acts via isometries on X, we have

$$N_r = \Gamma.B_r(o) \subset \Gamma.\overline{B_r(o)}$$

and by properness, the subset $\overline{B_r(o)}$ of X is compact. But $\bigcup_{t\in\mathbb{R}} f^{-1}(]-\infty, t[)$ is an open cover of X, so there must be $t \in \mathbb{R}$ with $\overline{B_r(o)} \subset X_t$. The map f was constructed to be Γ-invariant, so $\Gamma.X_t$ is equal to X_t. This means that $\Gamma.\overline{B_r(o)}$ lies inside X_t and, consequently, N_r is also a subset of X_t.
Let, on the other hand, $t \in \mathbb{R}$ be given. By cocompactness of the action of Γ on X_t, we know that there is a compact subset $K \subset X_t$ such that $\Gamma.K = X_t$. But using compactness again, we can find $r > 0$ such that K lies inside $B_r(o)$, which implies that $X_t = \Gamma.K$ is a subset of $N_r = \Gamma.B_r(o)$.
Therefore, using Lemma 4.9, we can replace $(N_r)_{r>0}$ by $(X_t)_{t\in\mathbb{R}}$ in the above lemma. □

Proof of Theorem 18.1. We show that Γ is of type F_n for every $n \in \mathbb{N}$. By the preceding lemma, it suffices to prove:
For every $t \in \mathbb{R}$, there is $s > t$ such that the group homomorphism $\pi_i(X_t) \to \pi_i(X_s)$ induced by inclusion $X_t \hookrightarrow X_s$ is trivial for every $0 \leq i \leq n-1$. Choose $s = \max\{t+1, r_0+1\}$ for the constant $r_0 > 0$ of Theorem 17.21. Then $X_s \simeq X$ is contractible and, consequently, all $\pi_i(X_s)$ are trivial. □

References

[AB08] Peter Abramenko and Kenneth S. Brown. *Buildings*, volume 248 of *Graduate Texts in Mathematics*. Springer, New York, 2008. Theory and applications.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[Bor63] Armand Borel. Some finiteness properties of adele groups over number fields. *Inst. Hautes Études Sci. Publ. Math.*, (16):5–30, 1963.

[Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[Bor19] Armand Borel. *Introduction to arithmetic groups*, volume 73 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2019. Translated from the 1969 French original [MR0244260] by Lam Laurent Pham, Edited and with a preface by Dave Witte Morris.

[Bou05] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 7–9*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley.

[Bro89] Kenneth S. Brown. *Buildings*. Springer-Verlag, New York, 1989.

[BS73] A. Borel and J.-P. Serre. Corners and arithmetic groups. *Comment. Math. Helv.*, 48:436–491, 1973.

[BS76] A. Borel and J.-P. Serre. Cohomologie d'immeubles et de groupes S-arithmétiques. *Topology*, 15(3):211–232, 1976.

[BT72] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. *Inst. Hautes Études Sci. Publ. Math.*, (41):5–251, 1972.

[BT84] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée. *Inst. Hautes Études Sci. Publ. Math.*, (60):197–376, 1984.

[Con01] Lawrence Conlon. *Differentiable manifolds*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel

Textbooks]. Birkhäuser Boston, Inc., Boston, MA, second edition, 2001.

[CSM95] Roger Carter, Graeme Segal, and Ian Macdonald. *Lectures on Lie groups and Lie algebras*, volume 32 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 1995. With a foreword by Martin Taylor.

[dC92] Manfredo Perdigão do Carmo. *Riemannian geometry.* Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[Eng89] Ryszard Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics.* Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

[FG02] Klaus Fritzsche and Hans Grauert. *From holomorphic functions to complex manifolds*, volume 213 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2002.

[Geo08] Ross Geoghegan. *Topological methods in group theory*, volume 243 of *Graduate Texts in Mathematics.* Springer, New York, 2008.

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[Hir76] Morris W. Hirsch. *Differential topology.* Graduate Texts in Mathematics, No. 33. Springer-Verlag, New York-Heidelberg, 1976.

[HN12] Joachim Hilgert and Karl-Hermann Neeb. *Structure and geometry of Lie groups.* Springer Monographs in Mathematics. Springer, New York, 2012.

[HW22] Tobias Hartnick and Stefan Witzel. Higher finiteness properties of arithmetic approximate lattices: The rank theorem for number fields, 2022.

[Kra22] Linus Kramer. Some remarks on proper actions, proper metric spaces, and buildings. *Adv. Geom.*, 22(4):541–559, 2022.

[Lee02] Dong Hoon Lee. *The structure of complex Lie groups*, volume 429 of *Chapman & Hall/CRC Research Notes in Mathematics.* Chapman & Hall/CRC, Boca Raton, FL, 2002.

[Mil63] J. Milnor. *Morse theory.* Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963. Based on lecture notes by M. Spivak and R. Wells.

[Mil11] James S. Milne. Algebraic groups, lie groups, and their arithmetic subgroups, 2011. Available at www.jmilne.org/math/.

[Pet] Peter Petersen. Manifold theory. `https://www.math.ucla.edu/~petersen/manifolds.pdf`. (accessed April 20, 2023).

[Pet16] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, Cham, third edition, 2016.

[PR94] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[Rag72] M. S. Raghunathan. *Discrete subgroups of Lie groups*, volume Band 68 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, New York-Heidelberg, 1972.

[Rag68] M. S. Raghunathan. A note on quotients of real algebraic groups by arithmetic subgroups. *Invent. Math.*, 4:318–335, 1967/68.

[Rat94] John G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[Sha69] I. R. Shafarevich. Foundations of algebraic geometry. *Uspehi Mat. Nauk*, 24(6(150)):3–184, 1969.

[Spr98] T. A. Springer. *Linear algebraic groups*, volume 9 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 1998.

[Ste16] Robert Steinberg. *Lectures on Chevalley groups*, volume 66 of *University Lecture Series*. American Mathematical Society, Providence, RI, corrected edition, 2016. Notes prepared by John Faulkner and Robert Wilson, With a foreword by Robert R. Snapp.

[Tes12] Gerald Teschl. *Ordinary differential equations and dynamical systems*, volume 140 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[Tit79] J. Tits. Reductive groups over local fields. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, volume XXXIII of *Proc. Sympos. Pure Math.*, pages 29–69. Amer. Math. Soc., Providence, RI, 1979.

[Tu11] Loring W. Tu. *An introduction to manifolds.* Universitext. Springer, New York, second edition, 2011.

[Wan21] Zuoqin Wang. Manifolds (lecture notes). `http://staff.ustc.edu.cn/~wangzuoq/Courses/21F-Manifolds/Notes/Lec15.pdf`, 2021. (accessed January 24, 2023).

[Wat79] William C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1979.

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