An invariant manifold of slowly oscillating solutions for $\dot{x}(t) = -\mu x(t) + f(x(t-1))$

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1. Introduction

The equation

$$(\mu, f) \qquad\qquad \dot{x}(t) = -\mu x(t) + f(x(t-1))$$

with $\mu > 0$ and with a smooth function $f: \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 0$ and $f'(0) < 0$, models delayed negative feedback: Sufficiently small deviations at times $t-1$ and t from equilibrium $x = 0$, e.g. $x(t-1) > 0$ and $x(t) > 0$, are followed by a motion into the opposite direction, $\dot{x}(t) < 0$.

Equation (μ, f) has applications in ecology, physiology and physics, see for example [8], [11], [18], [19], [23], [25], [31] and the references in [20]. Systems of equations of this type are used to describe neuronal nets [13], [21].

An appropriate phase space for equation (μ, f) is the space C of continuous functions $\varphi: [-1, 0] \to \mathbb{R}$, equipped with the norm $\|\varphi\| = \max_{t \in [-1,0]} |\varphi(t)|$. Real-valued solutions define phase curves with values in C by

$$x_t(s) := x(t+s) \quad \text{for} \quad -1 \leqq s \leqq 0,$$

provided $[t-1, t]$ belongs to the domain of x.

For a wide range of the parameters μ and f, the dynamics appears to be structured by periodic orbits. Existence and further results have been obtained in [2], [3], [4], [8], [12], [15], [17], [20], [22], [28]. The present paper is an attempt to understand existence and stability of periodic orbits in terms of an unstable manifold of the stationary state.

A few remarks on the linearized equation

$$(\mu, \alpha) \qquad \dot{x}(t) = -\mu x(t) - \alpha x(t-1)$$

(for $\alpha = -f'(0) > 0$) may help to put the result into perspective. Consider equation (μ, α) with μ fixed and a varying parameter $\alpha > 0$. The analysis of the characteristic equation shows that for small α, the zero solution is stable and attractive. It becomes unstable when α exceeds a critical value. In case of instability there exists a 2-dimensional subspace $L \subset C$ on which equation (μ, α) is given by a linear vectorfield of spiral source type (Section 4). Let $p: C \to C$ denote the eigenprojection associated with L. The space L attracts all phase curves of equation (μ, α) which do not belong to the complementary subspace $Q = qC$, $q := \mathrm{id} - p$.

For the nonlinear equation (μ, f), there is a 2-dimensional local invariant manifold $W_0 \subset C$, tangent to L at $0 \in C$. Phase curves of equation (μ, f) in $W_0 \setminus \{0\}$ spiral away from $0 \in W_0$ (Section 5 below). These phase curves are given by slowly oscillating solutions of equation (μ, f); i.e. by solutions whose zeros $z \neq z'$ are spaced at distances $|z - z'| > 1$.

The main result of the present paper is that for f monotone and bounded from above (or from below), the forward extension W of W_0, given by the collection of all phase curves who start in W_0, is the graph of a Lipschitz-bounded C^1-map w from an open subset $L_w \subset L$ into the complementary space Q. Furthermore, the continuation of w to the boundary bd L_w, i.e. the set

$$\mathrm{bd}'\, W := \mathrm{cl}\, W \setminus W$$

consists of a single periodic orbit which attracts all phase curves in $W \setminus \{0\}$.

The precise statements are given in Theorems 8.1 and 10.1.

The behavior of phase curves in cl W can be expressed in terms of a Poincaré-map on the closure of a one-dimensional C^1-submanifold $Y \subset W$ which connects the stationary point $0 \in W$ to a point of the periodic orbit bd$'$ W. Proposition 10.4 exhibits that the Poincaré-map on cl Y is equivalent to a monotone interval map $h: [0,1] \to [0,1]$ which is strictly above the diagonal on $(0,1)$ and has fixed points 0 and 1.

An outline of the proof for the main part of Theorem 8.1 is as follows. The space L sits in the closure $S \cup \{0\}$ of a cone S which contains all segments $x_t \in C$ of slowly oscillating solutions. The eigenprojection p does not vanish on S (or, $S \cap Q = \emptyset$). The set W consists of segments of slowly oscillating solutions (and $0 \in W$, of course), hence

$$W \subset S \cup \{0\}.$$

Consider two points φ, ψ in W; $\varphi \neq \psi$. There are solutions $x: \mathbb{R} \to \mathbb{R}$ and $y: \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_0 = \varphi$, $y_0 = \psi$. Segments x_{-t} and y_{-t} with $t > 0$ sufficiently large belong to W_0; $\lim_{t \to \infty} x_{-t} = 0 = \lim_{t \to \infty} y_{-t}$. Tangency

$$T_0 W_0 = L$$

implies much more than $x_{-t} \in S$, $y_{-t} \in S$: For $t > 0$ sufficiently large, the difference

$$x_{-t} - y_{-t} (\neq 0)$$

belongs to S, too. Monotonicity of f yields that the cone S is positively invariant for differences of phase curves. In particular,

$$\varphi - \psi = x_0 - y_0 \in S,$$

and therefore

$$p(\varphi - \psi) \neq 0, \quad p\varphi \neq p\psi;$$

p is injective on W. This means that W is given by a map from $pW \subset L$ into Q (Section 6).

The proof of Lipschitz boundedness requires a quantitative version of the result on injectivity, namely an a-priori estimate of the form

$$\|qx_t - qy_t\| \leqq \text{const} \|px_t - py_t\| \tag{1.1}$$

along phase curves in W. Such an estimate is derived in Section 7.

A special case of (1.1), an inequality of the form

$$\|qx_t\| \leqq \text{const} \|px_t\| \tag{1.2}$$

along sufficiently small slowly oscillating solutions with segments in the cone S goes back to work in [32] on equation

$$\dot{x}(t) = f(x(t-1)) \tag{f}$$

with $\xi f(\xi) < 0$ for all $\xi \neq 0$. Equation (f) is the basic differential equation for delayed negative feedback. It has, however, not as many applications as equation (μ, f).

The analysis of equation (μ, f) is considerably facilitated by a simple fact which was first used by Pesin [28] and by Hadeler and Tomiuk [8]: If x is a solution of equation (μ, f) then the function $y : t \to e^{\mu t} x(t)$ satisfies the equation

$$\dot{y}(t) = g(t, y(t-1))$$

with $g : (t, \eta) \to e^{\mu t} f(e^{-\mu(t-1)} \eta)$. The latter is a nonautonomous version of equation (f). Clearly

$$\eta g(t, \eta) < 0 \quad \text{for all} \quad \eta \neq 0, \quad t \in \mathbb{R}.$$

All proofs in this paper can be modified (in fact, simplified) for equation (f). This yields analogues of Theorems 8.1 and 10.1 for C^1-functions f which satisfy $f(0) = 0, f'(\xi) < 0$ for all $\xi \in \mathbb{R}$, $f'(0) < -\pi/2$, $\sup f < \infty$ (or, $-\infty < \inf f$).

In case $f = f_\alpha$, $f_\alpha(\xi) = -\alpha(e^\xi - 1)$ for all $\xi \in \mathbb{R}$, with $\alpha > \frac{\pi}{2}$, one obtains a result for Wright's equation [34]

$$\dot{x}(t) = -\alpha x(t-1)[1 + x(t)]$$

since solutions x with values in $(1, \infty)$ are in a one-to-one correspondence with the solutions x' of equation (f_α), via

$$x' = \log \circ (1 + x).$$

The periodic orbit in Theorem 10.1 is the limit cycle of phase curves which spiral away from zero in the 2-dimensional manifold W. This should be compared to an older result of Kaplan and Yorke [16], [17]: Certain slowly oscillating solutions x define planar curves $t \to (x(t), \dot{x}(t))$ without self-intersections; these curves spiral towards a limit cycle in $\mathbb{R}^2$, which is the $(x(t), \dot{x}(t))$-"projection" of a periodic orbit in C. The result is applicable to phase curves in W, and also to certain phase curves which are sufficiently close to W.

Another aspect of the present approach is that it has also implications on the problem of periodic solutions in cases where the nonlinearity in equation (μ, f) is not assumed to be globally monotone. In such cases the estimate (1.2) along sufficiently small slowly oscillating solutions remains valid. Estimate (1.2) can be used to give an alternative proof for a very general result of Mallet-Paret and Nussbaum, [20], Theorem 1.1, on global bifurcation of slowly oscillating periodic solutions for one-parameter-families of equations of the form (μ, f). The crucial step in the proof of [20], Theorem 1.1, is to show that instability of the linearized delay equation implies

$$\operatorname{ind}(0, P) = 0 \tag{1.3}$$

for the index of the fixed point $\varphi = 0$ of a map P which is defined by intersections of phase curves with a convex cone; $\varphi = 0$ is the extremal point of the cone. Estimate (1.2) permits a rather easy proof of (1.3). Details are as in the note [33] where the simpler equation (f) was discussed.

What is the role played by the manifold W in the full dynamics of equation (μ, f)? Note that cl W will in general not be a global attractor: There exist periodic solutions which are not slowly oscillating, for $-f'(0) > 0$ sufficiently large [22], [20]. Most likely there are also multiple slowly oscillating periodic solutions for certain monotone nonlinearities – compare the conditions for uniqueness in Nussbaum's work [26] on equation (f).

An interesting question seems to be whether the Lipschitz graph cl W continues beyond bd$'$ W to a sort of inertial manifold (see e.g. [7]) for phase curves in $S \cup \{0\}$.

Another result which guarantees that invariant sets of infinite-dimensional systems are in fact smooth graphs has recently been obtained by Poláčik [29]. This applies to classes of parabolic partial differential equations.

The subsequent sections 2–5 make use of diverse basic properties of functional differential equations. A general reference for most of these is Hale's book [19]. For calculus in Banach spaces, see [6]. For submanifolds and transversality, see [1].

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2. Notation, preliminaries

$\mathbb{N}_0$ denotes the set $\mathbb{N} \cup \{0\}$. $\mathbb{R}^+$ stands for the interval $[0, \infty)$.

Consider a subset $X \subset E$ of a real Banach space E, and a point $x \in X$. The set $T_x X$ of tangents to X at x is defined to be the set of all vectors

$$v = Dc(0)1$$

where $c : (-1, 1) \to E$ is a differentiable curve with $c(0) = x$ and $c((-1, 1)) \subset X$. Note $0 \in T_x X$. In general, $T_x X$ is *not* a vector space. For a differentiable map $f : U \to F$, $U \supset X$,

$$Df(x) T_x X \subset T_{f(x)} f(X).$$

The closure, the interior and the boundary of X are denoted by $\operatorname{cl} X$, $\operatorname{int} X$, $\operatorname{bd} X$, respectively.

Let a function $g : \mathbb{R}^2 \to \mathbb{R}$ be given. A solution of the differential delay equation

$$\dot{x}(t) = g(x(t), x(t-1)) \tag{2.1}$$

is either a differentiable function $x : \mathbb{R} \to \mathbb{R}$ so that (2.1) is satisfied for all $t \in \mathbb{R}$, or a continuous function $x : [t_0 - 1, \infty) \to \mathbb{R}$, $t_0 \in \mathbb{R}$, which is differentiable on (t_0, ∞) and satisfies (2.1) for all $t > t_0$.

Analogously one defines complex-valued solutions in case g is linear, and solutions of nonautonomous equations

$$\dot{x}(t) = g(t, x(t-1))$$

for functions $g : \mathbb{R}^2 \to \mathbb{R}$ or $g : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$, $t_0 \in \mathbb{R}$.

C and C' denote the real and complex Banach spaces of continuous functions $\varphi : [-1, 0] \to \mathbb{R}$ and $\varphi : [-1, 0] \to \mathbb{C}$, respectively. In both cases, $\|\varphi\| = \max_{t \in [-1,0]} |\varphi(t)|$.

Solutions define phase curves $t \to x_t$ with values in C or C' by

$$x_t(s) := x(t+s) \quad \text{for all} \quad s \in [-1, 0],$$

provided the interval $[t-1, t]$ belongs to the domain of x.

3. Hypotheses, basic properties of solutions

Let a C^1-function $f: \mathbb{R} \to \mathbb{R}$ be given, with

$$\text{(H1)} \qquad f(0) = 0 \quad \text{and} \quad f'(\xi) < 0 \quad \text{for all} \quad \xi \in \mathbb{R}, \quad \text{bounded from above by a constant } c_f > 0.$$

Note that f satisfies

$$\text{(NF)} \qquad \xi f(\xi) < 0 \quad \text{for all} \quad \xi \neq 0,$$

a condition which expresses negative feedback.

Let a constant $\mu > 0$ be given, and assume in addition that

$$\text{(H2)} \qquad -f'(0) > -\frac{\mu}{\cos v(\mu)}$$

where $v(\mu) \in \left(\frac{\pi}{2}, \pi\right)$ and $v(\mu) = -\mu \tan v(\mu)$. Condition (H2) will imply that the stationary solution $t \to 0$ of equation

$$(\mu, f) \qquad \dot{x}(t) = -\mu x(t) + f(x(t-1))$$

is unstable.

Existence and uniqueness of solutions $x: [-1, \infty) \to \mathbb{R}$ of the initial value problems

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)), \quad x_0 = \varphi \in C$$

follow most easily by repeated application of the variation-of-constants formulae

$$x(t) = e^{-\mu(t-n)} \left[x(n) + \int_n^t e^{\mu(s-n)} f(x(s-1))\, ds \right]$$

for $n \leqq t \leqq n+1$, $n \in \mathbb{N}_0$. The solutions obtained are denoted by x^φ.

They depend continuously on initial data in the following sense: For every $\varphi \in C$, $t \geqq 0$, $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\psi \in C$ with $\|\varphi - \psi\| \leqq \delta$,

$$|x^\psi(s) - x^\varphi(s)| < \varepsilon \quad \text{for all} \quad s \in [-1, t].$$

The phase curves $t \in \mathbb{R}^+ \to x_t^\varphi \in C$ define a continuous semiflow

$$F: (t, \varphi) \in \mathbb{R}^+ \times C \to x_t^\varphi \in C;$$

$F(1, \cdot): C \to C$ maps bounded sets into sets with compact closure. The last assertion follows from the variation-of-constants formula for x_1^φ, by means of the Theorem of Arzela-Ascoli.

The restriction of F to $(1, \infty) \times C$ is of class C^1. For $t > 1$ and $\varphi \in C$,

$$D_1 F(t, \varphi)1 = \dot{x}_t^{\varphi},$$

$\dot{x}_t^{\varphi}(s) := \dot{x}^{\varphi}(t+s)$ for $-1 \leqq s \leqq 0$. Observe that for every solution $x : [t_0 - 1, \infty) \to \mathbb{R}$ of equation (μ, f) and for all $t > t_0 + 1$, the derivative of the map

$$s \in [t_0, \infty) \to x_s \in C$$

at t exists, and that one has

$$D(s \to x_s)(t)1 = \dot{x}_t.$$

The partial derivatives $D_2 F(t, \varphi)$ exist on all of $\mathbb{R}^+ \times C$. They are given by

$$D_2 F(t, \varphi)\psi = y_t$$

where $y : [-1, \infty) \to \mathbb{R}$ is the solution of the initial value problem

$$\dot{y}(t) = -\mu y(t) + f'(x^{\varphi}(t-1))y(t-1), \quad y_0 = \psi.$$

For a solution $x : \mathbb{R} \to \mathbb{R}$ of equation (μ, f) it follows in particular that for all $t \geqq 0$,

$$D_2 F(t, x_0)\dot{x}_0 = \dot{x}_t.$$

In case $\varphi = 0$ one obtains the autonomous equation

$$(\mu, \alpha) \qquad \dot{y}(t) = -\mu y(t) - \alpha y(t-1)$$

with $\alpha := -f'(0) > 0$. The operators $T(t) := D_2 F(t, 0)$, $t \geqq 0$, form a C_0-semigroup, with

$$T(t)\varphi = x_t^{\varphi}$$

where now $x^{\varphi} : [-1, \infty) \to \mathbb{R}$ is the solution of the initial value problem given by equation (μ, α) and the initial condition $x_0 = \varphi \in C$. $T(1)$ is a compact operator.

Let $T'(t)$, $t \geqq 0$, denote the operators of the C_0-semigroup on C' which is defined by complex-valued solutions of equation (μ, α) on $[-1, \infty)$.

It is convenient to introduce stopping maps. Let a hyperplane V of C be given, $V = \ker v$ for some linear continuous map $v : C \to \mathbb{R}$. Consider $\varphi \in C$ so that the trajectory

$$0 \leqq t \to F(t, \varphi) \in C$$

passes at $t > 1$ transversally through V:

$$F(t, \varphi) \in V \quad \text{and} \quad D_1 F(t, \varphi)1 \notin V.$$

Remark 3.1. There exist an open neighborhood U of φ and a C^1-map $\tau : U \to (1, \infty)$ with the properties

$$\tau(\varphi) = t,$$

$$F(\tau(\psi), \psi) \in V \quad \text{and} \quad D_1(F(\tau(\psi), \psi)1) \notin V \quad \text{for all} \quad \psi \in U.$$

Sketch of proof. Apply the Implicit Function Theorem and solve the equation

$$0 = v(F(s, \psi))$$

close to $(s, \psi) = (t, \varphi)$. This is possible because of

$$D_1(v \circ F)(t, \varphi)1 = v(D_1 F(t, \varphi)1) \neq 0. \qquad \text{QED.}$$

For the computation of derivatives of the C^1-map

$$F \circ (\tau \times \mathrm{id}) : \psi \in U \to F(\tau(\psi), \psi) \in C$$

it is convenient to introduce projections

$$p_\chi : C \to C$$

onto V, parallel to vectors $\chi \in C \setminus V$:

$$p_\chi(\psi) := \psi - \frac{v(\psi)}{v(\chi)}\chi \quad \text{for all} \quad \psi \in C.$$

Remark 3.2. For every $\psi \in U$,

$$D(F \circ (\tau \times \mathrm{id}))(\psi) = p_\chi D_2 F(\tau(\psi), \psi)$$

with $\chi = D_1 F(\tau(\psi), \psi)1 \; (= \dot{x}^{\psi}_{\tau(\psi)})$.

Sketch of proof. Differentiation of

$$v \circ F \circ (\tau \times \mathrm{id}) = 0$$

at $\psi \in U$ yields

$$0 = D\tau(\psi)\underline{\psi}\, v(\chi) + v(D_2 F(\tau(\psi), \psi)\underline{\psi})$$

for every $\underline{\psi} \in C$. Hence

$$\begin{aligned} D(F \circ (\tau \times \mathrm{id}))(\psi)\underline{\psi} &= D_1 F(\tau(\psi), \psi) D\tau(\psi)\underline{\psi} + D_2 F(\tau(\psi), \psi)\underline{\psi} \\ &= D_2 F(\tau(\psi), \psi)\underline{\psi} - \frac{v(D_2 F(\tau(\psi), \psi)\underline{\psi})}{v(\chi)} D_1 F(\tau(\psi), \psi)1 \\ &= p_\chi(D_2 F(\tau(\psi), \psi)\underline{\psi}). \qquad \text{QED.} \end{aligned}$$

An important consequence of injectivity of f is injectivity for time-t-maps.

Remark 3.3. Each map $F(t,\cdot)$, $t \geqq 0$, and $D_2F(t,\varphi)$, $(t,\varphi) \in \mathbb{R}^+ \times C$, is injective. Any two solutions $x: \mathbb{R} \to \mathbb{R}$, $x': \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_t = x'_t$ for some $t \in \mathbb{R}$ coincide.

Sketch of proof. Consider solutions $x: \mathbb{R} \to \mathbb{R}$ and $x': \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_0 \neq x'_0$ but $x_t = x'_t$ for some $t > 0$. Then $x = x'$ on $[t-1, \infty)$, and there is a smallest $n \in \mathbb{N}$ with $x_n = x'_n$. Injectivity of f in the variation-of-constants formula for x and x' on $[n-1, n]$ yields $x_{n-1} = x'_{n-1}$, a contradiction either to minimality of n, or to $x_0 \neq x'_0$. The proof for the maps $D_2F(t,\varphi)$ is analogous. QED.

The strong monotonicity of f implies that scaled differences of solutions of equation (μ, f) satisfy nonautonomous equations of the form

$$(g) \qquad \dot{x}(t) = g(t, x(t-1))$$

where the continuous function $g: [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ or $g: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ has the negative feedback property

$$(\mathrm{NF}, t) \qquad \xi g(t, \xi) < 0 \quad \text{for all } t \quad \text{and all} \quad \xi \neq 0.$$

For example, if x' and x are solutions of equation (μ, f) which are defined on $[t_0 - 1, \infty)$, set $d := x' - x$ and

$$\underline{d}(t) := e^{\mu t} d(t) \quad \text{for} \quad t \geqq t_0 - 1.$$

It follows that for all $t > t_0$,

$$\begin{aligned}(\mu, f, x'-x) \qquad \dot{\underline{d}}(t) &= e^{\mu t}(\mu d(t) + \dot{d}(t)) = e^{\mu t}(f(x'(t-1)) - f(x(t-1))) \\ &= e^{\mu t}(f(e^{-\mu t} e^{\mu} \underline{d}(t-1) + x(t-1)) - f(x(t-1))) \\ &= e^{\mu t} \int_{x(t-1)}^{x(t-1)+e^{-\mu t}e^{\mu}\underline{d}(t-1)} f'(\xi)\, d\xi .\end{aligned}$$

In particular, if $x: [t_0 - 1, \infty) \to \mathbb{R}$ is a solution of equation (μ, f), then $\underline{x}: t \in [t_0 - 1, \infty) \to e^{\mu t} x(t) \in \mathbb{R}$ is a solution of the equation

$$(\mu, f, \underline{x}) \qquad \dot{\underline{x}}(t) = e^{\mu t} f(e^{-\mu t} e^{\mu} \underline{x}(t-1))$$

which is of type (g), with property (NF, t).

4. Eigenvalues

The spectrum of the generator of the C_0-semigroup $t \to T'(t)$ on C' consists of isolated eigenvalues with finite multiplicities, given by the zeros of the entire function

$$E: z \to z + \mu + \alpha e^{-z}$$

and their orders. For $\eta > 0$, let E_η denote the entire function

$$\zeta \to \zeta + \eta e^{-\zeta}.$$

Obviously, $E(z) = 0$ if and only if $z + \mu$ is a zero of E_η with $\eta = \alpha e^{-\mu}$. Long-known results (see e.g. [34]) on E_η imply that for $\eta > 1/e$ each double strip

$$\{\zeta \in \mathbb{C}: 2k\pi < |\mathrm{Im}\,\zeta| < 2k\pi + \pi\}, \quad k \in \mathbb{N}_0,$$

contains precisely one pair

$$\zeta_k(\eta),\ \bar{\zeta}_k(\eta) \quad \text{with} \quad 0 < \mathrm{Im}\,\zeta_k(\eta)$$

of simple zeros; there are no further zeros. For $\eta < \pi/2$, all $\mathrm{Re}\,\zeta_k(\eta)$ are negative. As α increases, the zeros move from left to right with nonzero speed. Moreover,

$$\mathrm{Re}\,\zeta_{k+1}(\eta) < \mathrm{Re}\,\zeta_k(\eta) \quad \text{for all} \quad k \in \mathbb{N}_0,$$

and

$$\mathrm{Re}\,\zeta_k(\eta) \to -\infty \quad \text{as} \quad k \to +\infty.$$

It is not hard to see that at

$$\eta(\mu) := -\frac{\mu}{\cos v(\mu)} e^{\mu} \quad \text{with} \quad \frac{\pi}{2} < v(\mu) < \pi \quad \text{and} \quad v(\mu) = -\mu \tan v(\mu),$$

we have

$$\zeta_0(\eta(\mu)) = \mu + iv(\mu).$$

For $\alpha = -f'(0)$ and $k \in \mathbb{N}_0$, set

$$\lambda_k := \zeta_k(\alpha e^{\mu}) - \mu \quad \text{and} \quad u_k := \mathrm{Re}\,\lambda_k, \quad v_k := \mathrm{Im}\,\lambda_k.$$

Then

$$E^{-1}(0) = \bigcup_{k \in \mathbb{N}_0} \{\lambda_k, \bar{\lambda}_k\},$$

each zero of E is simple,

$$u_{k+1} < u_k \quad \text{and} \quad 2k\pi < v_k < 2k\pi + \pi \quad \text{for all} \quad k \in \mathbb{N}_0,$$
$$u_k \to -\infty \quad \text{as} \quad k \to +\infty.$$

Using hypothesis (H2) one concludes that

$$0 < u_0.$$

For every eigenvalue z, the function

$$\psi_z : t \to e^{zt}$$

in C' is an eigenvector. The associated eigenprojections

$$p(z) : C' \to C', \quad z \in E^{-1}(0),$$

onto the one-dimensional generalized eigenspaces

$$G(z) = \mathbb{C}\psi_z$$

satisfy

(4.1) $$\overline{p(z)\varphi} = p(\bar{z})\varphi \quad \text{for all real-valued } \varphi.$$

Each $G(z)$ consists of the segments x_t of the solutions $x : t \in \mathbb{R} \to ce^{zt} \in \mathbb{C}$, $c \in \mathbb{C}$, of equation (α, μ).

The relation (4.1) implies that the expression

$$(p(\lambda_0) + p(\bar{\lambda}_0))\varphi$$

defines a projection $p : C \to C$ onto the subspace

$$L := \operatorname{Re} G(\lambda_0) = \operatorname{Re} G(\bar{\lambda}_0) \subset C;$$
$$\dim L = 2.$$

Set

$$Q := (\mathrm{id} - p)C.$$

Note

$$T(t)p = pT(t) \quad \text{for all} \quad t \geqq 0;$$

L and Q are invariant under $T(t)$.

The real eigenspace L consists of all segments x_t of the solutions

$$x : t \in \mathbb{R} \to e^{u_0 t}(a \cos v_0 t + b \sin v_0 t) \in \mathbb{R}, \quad a \text{ and } b \text{ in } \mathbb{R},$$

of equation (α, μ). Observe that in case $(a, b) \neq (0, 0)$, zeros of x are spaced at the distance

$$\frac{\pi}{v_0} \in (1, 2)$$

while real parts of solutions with segments in $G(\lambda_k)$, $k \in \mathbb{N}$, have zeros at distances $\frac{\pi}{v_k} < 1$. I.e., the spacing of zeros distinguishes solutions associated with the leading eigenvalues λ_0, $\bar{\lambda}_0$ from other eigensolutions.

Remark 4.1. There exists a constant $a_0 > 0$ such that for every solution $x: \mathbb{R} \to \mathbb{R}$ of equation (μ, α) with $0 \neq x_0 \in L$, there is $t \in [0, 2]$ with

$$|x(s)| \geqq a_0 \|x_0\| \quad \text{for all} \quad s \in [t-1, t].$$

Sketch of proof. Fix $\varepsilon > 0$ with $1 + 2\varepsilon < \frac{\pi}{v_0}$. Set $c := \cos v_0 \left(\frac{\pi}{2v_0} - \varepsilon \right) > 0$, and $a_0 := c e^{-u}$. Consider the case $\|x_0\| = 1$. There exist a and b in $\mathbb{R}$ with $x(t) = a e^{u_0 t} \cos(v_0 t + b)$ for all $t \in \mathbb{R}$. $\|x_0\| = 1$ yields $a \geqq 1$. Show that there exists $t \in [0, 2]$ so that $|\cos(v_0 s + b)| \geqq c$ for all $s \in [t-1, t]$. QED.

Remark 4.2. $1 < \alpha e^{\mu}$.

Proof. Otherwise,

$$1 \geqq \alpha e^{\mu} > \alpha > -\frac{\mu}{\cos v(\mu)} = \frac{v(\mu)}{\sin v(\mu)},$$

a contradiction to $\frac{\pi}{2} < v(\mu) < \pi$. QED.

5. A local invariant manifold and its extension

The derivative $T(1)$ of the map $F(1, \cdot)$ at $\varphi = 0$ induces linear continuous maps $T_L: L \to L$ and $T_Q: Q \to Q$ with spectra $\{e^{\lambda_0}, e^{\bar{\lambda}_0}\}$ and $\{0\} \cup \bigcup_{k \in \mathbb{N}} \{e^{\lambda_k}, e^{\bar{\lambda}_k}\}$, respectively. Fix $\beta > 1$ with

$$e^{u_1} < \beta < e^{u_0}.$$

There exists an equivalent norm $|\ |$ on C with

$$|T_L^{-1}| \left(= \sup_{|\varphi| = 1} |T_L^{-1} \varphi| \right) < \beta^{-1}$$

and

$$|T_Q| < \beta.$$

(Use e. g. arguments from [15], Appendix to Chapter 4.5, with a corrected definition of the norm on the complexified space. Compare also [15], Theorem 4.19, on hyperbolic isomorphisms.)

For open neighborhoods U of 0 in C, define

$$W(U) := \{\varphi \in U: \text{There exists a sequence } (\varphi_n)_{-\infty}^{0} \text{ in } C \text{ with } \varphi_0 = \varphi,$$
$$\varphi_n = F(1, \varphi_{n-1}) \text{ and } \varphi_n \beta^{-n} \in U \text{ for all } n \in -\mathbb{N}_0,$$
$$\varphi_n \beta^{-n} \to 0 \text{ as } n \to -\infty\}.$$

A modification of the proof of the Unstable Manifold Theorem from [10], [24] yields

Theorem 5.1. *There exist convex open neighborhoods L_0 of 0 in L, Q_0 of 0 in Q and a C^1-map $w_0: L_0 \to Q$ with the following properties.*

1. $w_0(L_0) \subset Q_0$, $w_0(0) = 0$, $Dw_0(0) = 0$.

2. $\{\chi + w_0(\chi) : \chi \in L_0\} = W(L_0 + Q_0)$.

3. *Every trajectory* $(\varphi_n)_{-\infty}^{0}$ *(i.e.,* $\varphi_n = F(1, \varphi_{n-1})$ *for* $n \in -\mathbb{N}_0$*) with* $\varphi_n \beta^{-n} \in L_0 + Q_0$ *for all* $n \in -\mathbb{N}_0$ *satisfies* $\varphi_n \beta^{-n} \to 0$ *as* $n \to -\infty$.

Assertion 3 implies $\varphi_0 \in W(L_0 + Q_0)$; using part 2 one finds

$$\varphi_0 = \chi + w_0(\chi) \quad \text{for some} \quad \chi \in L_0 .$$

Set

$$W_0 := \{\chi + w_0(\chi) : \chi \in L_0\} .$$

Analogues of further statements from [10], Theorem 3.1/[24], Theorem 2.7, hold also true. For example, there exists an open neighborhood U of 0 in C so that $F(1, \cdot)$ induces a C^1-diffeomorphism from $W_0 \cap U$ onto W_0, with an inverse map which is Lipschitz bounded with respect to the norm $|\ |$, with a Lipschitz constant strictly less than β^{-1}.

But this will not be needed in the sequel.

Define

$$W := F(\mathbb{R}^+ \times W_0) .$$

This forward extension of the "local unstable manifold" W_0 will turn out to be (the graph of) a Lipschitz bounded C^1-map from an open subset of L into Q. The proof requires an a-priori-estimate of the form

$$(5.*) \qquad \|q\varphi - q\psi\| \leqq \text{const}\, \|p\varphi - p\psi\| \quad \text{for} \quad \varphi \text{ and } \psi \text{ in } W .$$

Such an estimate will be derived in the subsequent sections.,

More obvious properties of W_0 and W are the following.

Proposition 5.1. *For every* $\varphi \in W_0$ *there exists a uniquely determined solution* $x : \mathbb{R} \to \mathbb{R}$ *of equation* (μ, f) *with* $x_0 = \varphi$. *There exists* $t(\varphi) \leqq 0$ *with*

$$x_t \in W_0 \quad \textit{for all} \quad t \leqq t(\varphi),$$

and

$$x_t \to 0 \quad \textit{as} \quad t \to -\infty .$$

Proof. 1. Part 2 of Theorem 5.1 yields a sequence $(\varphi_n)_{-\infty}^{0}$ as in the definition of $W(U)$, for $U = L_0 + Q_0$; $\varphi = \varphi_0$. Define $x_t := F(t, \varphi)$ for $t \geqq 0$, and $x_t := F(t-n, \varphi_n)$ for $n \in -\mathbb{N}$ and $0 \leqq t < 1$. This determines a solution $x : \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_0 = \varphi$. For uniqueness, see Remark 3.3.

2. Fix $r > 0$ with $\psi \in L_0 + Q_0$ for $\|\psi\| \leqq r$. Set

$$m := 1 + e^{\mu} \max_{|\xi| \leqq r} |f'(\xi)|.$$

For $\|\psi\| \leqq r$ and $0 \leqq t \leqq 1$, the variation-of-constants formula yields

$$|x^{\psi}(t)| \leqq |\psi(0)| + e^{\mu} \max_{|\xi| \leqq r} |f'(\xi)| \|\psi\| \leqq m \|\psi\|.$$

Hence

$$\|F(t, \psi)\| \leqq m \|\psi\| \quad \text{for} \quad \|\psi\| \leqq r, \quad 0 \leqq t \leqq 1.$$

3. Fix $k \in -\mathbb{N}$ with $\|x_n\| (= \|\varphi_n\|) \leqq r$ for $n \leqq k$. For such n, and for $0 \leqq t \leqq 1$,

$$\|x_{t+n}\| = \|F(t, x_n)\| \leqq m \|x_n\|.$$

It follows that $x_t \to 0$ as $t \to -\infty$.

4. Fix $v \in -\mathbb{N}$ such that $m\beta^{v} \leqq 1$ and $\|\varphi_n \beta^{-n}\| \leqq r$ for $n \leqq v$ in $-\mathbb{N}$. Let $t < v$; $n \leqq t < n+1$ for some $n \in -\mathbb{N}$ with $n < v$. For all $j \in -\mathbb{N}_0$,

$$\|x_{t+j}\beta^{-j}\| = \|F(t-n, x_{n+j})\| \beta^{-j} \leqq m\beta^{-j} \|x_{n+j}\| = m\beta^{n} \|\varphi_{n+j}\| \beta^{-n-j} \leqq r,$$

and therefore

$$x_{t+j}\beta^{-j} \in L_0 + Q_0.$$

Clearly $x_{t+j} = F(1, x_{t+j-1})$ for all $j \in -\mathbb{N}_0$. Using part 3 of Theorem 5.1, the definition of $W(L_0 + Q_0)$ and part 2 of Theorem 5.1, one obtains

$$x_t = x_{t+0} \in W_0. \qquad \text{QED.}$$

Corollary 5.1. 1. *$F(\mathbb{R}^+ \times W) \subset W$.*

2. *$W = F(\mathbb{N}_0 \times W_0)$.*

3. *For every $\varphi \in W$ there exists a uniquely determined solution $x : \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_0 = \varphi$. For all $t \in \mathbb{R}$, $x_t \in W$. There exists $t(\varphi) \leqq 0$ with $x_t \in W_0$ for $t \leqq t(\varphi)$, and $x_t \to 0$ as $t \to -\infty$.*

Proof of $W \subset F(\mathbb{N}_0 \times W_0)$. Suppose $\varphi = F(t, \varphi')$, $t \geqq 0$, $\varphi' \in W_0$. There exist $\varphi'' \in W_0$ and $n \in \mathbb{N}$ with $n \geqq t$ so that $\varphi' = F(n-t, \varphi'')$ – see Proposition 5.1. It follows that $\varphi = F(n, \varphi'')$. QED.

6. Slowly oscillating solutions

A necessary condition for the estimate (5. $*$) to hold on the set W is

$$p\varphi \neq 0 \quad \text{for all} \quad \varphi \in W \setminus \{0\}.$$

$p\varphi \neq 0$ is obviously valid for nontrivial tangent vectors $\varphi \in T_0 W_0 = L$ of $W \supset W_0$.

The observation made in Section 4, namely that nontrivial solutions $x : \mathbb{R} \to \mathbb{R}$ of the linear equation (μ, α) with segments x_t in L have zeros spaced at distances larger than the delay 1 – while solutions associated with other eigenspaces are more rapidly oscillating – leads to a subset of C containing $L = T_0 W$ where p does not vanish (see Proposition 6.2 below):

Let S denote the set of all nonzero $\varphi \in C$ with at most one change of sign, i.e. $\varphi \in S$ if and only if

$$\begin{aligned} \varphi \neq 0, \quad &\text{and} \quad 0 \leqq \varphi \text{ on } [-1, z],\ \varphi \leqq 0 \text{ on } [z, 0] \text{ for some } z \in [-1, 0] \\ &\text{or} \quad \varphi \leqq 0 \text{ on } [-1, z],\ 0 \leqq \varphi \text{ on } [z, 0] \text{ for some } z \in [-1, 0]. \end{aligned}$$

It follows that

$$S \supset L \setminus \{0\}.$$

S contains all nonzero $\varphi \geqq 0$ and all nonzero $\varphi \leqq 0$; it is a cone $\big((0, \infty) S \subset S\big)$, but not convex. Elementary considerations yield

$$\text{(6.1)} \qquad \operatorname{cl} S = S \cup \{0\}.$$

The negative feedback property (NF, t) implies forward invariance:

Remark 6.1. 1. If x is a solution of equation (g) (with property (NF, t)), and if $x_t \in S$ for some $t \in \mathbb{R}$, then $x_s \in S$ for all $s \geqq t$.

2. $F(\mathbb{R}^+ \times S) \subset S$, $T(t) S \subset S$ for all $t \geqq 0$.

Proof. 1. Suppose $x_t \in S$. In case $0 \leqq x$ on $[t-1, t]$, $\dot{x} \leqq 0$ on $(t, t+1]$, by (NF, t). $x(s) \neq 0$ implies $\dot{x}(s+1) \neq 0$. Using this and monotonicity of x on $[t, t+1]$, one sees that all x_s, $t \leqq s \leqq t+1$, are nonzero and belong to S. Similar arguments apply to the other cases. Use induction.

2. Let $\varphi \in S$ be given. Consider the solution $x : [-1, \infty) \to \mathbb{R}$ of equation (μ, f) with $x_0 = \varphi$. Apply part 1 to the solution $\underline{x} : t \to e^{\mu t} x(t)$ of equation $(\mu, f, \underline{x})$, and conclude that all x_t, $t \geqq 0$, belong to S.

Analogously for solutions of equation (μ, α). QED.

Solutions which start in S become more regular in the following sense.

Proposition 6.1. *Let a solution x of equation (g) be given, with $x_t \in S$ for some t. Let $\varphi \in S$ be given.*

1. *There exist $s \in [t, t+4]$, $s' \in [0, 4]$ and $s'' \in [0, 4]$ so that x_s, $F(s', \varphi)$ and $T(s'')\varphi$ have no zero.*

2. *In case $0 \begin{Bmatrix} > \\ < \end{Bmatrix} x$ on $(t-1, t]$, either $\dot{x} \begin{Bmatrix} < \\ > \end{Bmatrix} 0 \begin{Bmatrix} < \\ > \end{Bmatrix} x$ on (t, ∞), or there exists a zero $z > t$ with $\dot{x} \begin{Bmatrix} < \\ > \end{Bmatrix} 0$ on $(t, z+1)$ and $\dot{x}(z+1) = 0$. All zeros of x in $J := [t-1, \infty)$ are simple, and*

(SO) $\qquad z' > z + 1 \quad$ *for all zeros* $\quad z' > z$ *of x in J.*

3. *Suppose x is strictly $\begin{Bmatrix} \text{decreasing} \\ \text{increasing} \end{Bmatrix}$ on $[t-1, t]$, with $x(t-1) \begin{Bmatrix} > \\ < \end{Bmatrix} 0$. Suppose $m > t-1$ is a local extremum of x. Then $t < m$, $x(m-1) = 0 \neq x(m)$, and x is strictly monotone on $[m-1, m]$. In case $0 \begin{Bmatrix} > \\ < \end{Bmatrix} x(m)$, either $x \begin{Bmatrix} < \\ > \end{Bmatrix} 0 \begin{Bmatrix} < \\ > \end{Bmatrix} \dot{x}$ on (m, ∞), or there is a zero $z > m$ of x such that $0 \begin{Bmatrix} < \\ > \end{Bmatrix} \dot{x}$ in $(m, z+1)$.*

Definition. Let a function $x : I \to \mathbb{R}$, $I \subset \mathbb{R}$, be given. Let $J \subset I$ be an interval. x is called slowly oscillating on J if property (SO) holds.

Proof of Proposition 6.1. 1.1. Consider first the solution x of equation (g).

1.1.1. The case $0 \leqq x_t$. By (NF, t), $\dot{x} \leqq 0$ on $(t, t+1]$. Suppose x has a zero in $[t, t+1]$. Monotonicity implies that the zeros of x in $[t, t+1]$ form an interval $[z, z'] \subsetneqq [t, t+1]$ ($x_{t+1} \neq 0$ since $x_{t+1} \in S$).

In case $t < z \leqq z' \leqq 1$, $\dot{x} < 0$ on $[t+1, z+1)$ and $\dot{x} = 0$ on $[z+1, z'+1]$. Hence $x < 0$ on $(z', z'+1]$. Continuity permits to find $s \in (z'+1, t+3)$ with $x < 0$ on $[s-1, s]$.

The case $t = z \leqq z' < t+1$ is analogous.

1.1.2. Suppose there exist t', t'' in $[t-1, t]$ with $t' < t''$, $x(t') < 0 < x(t'')$. Then there is a zero $z \in (t-1, t)$ of x with $x \leqq 0$ on $[t-1, z]$, $0 \leqq x$ on $[z, t]$. Hence $0 \leqq \dot{x}$ on $(t, z+1]$. Therefore, $0 \leqq x$ on $[z, z+1]$, and one can argue as in part 1.1.1.

1.1.3. The remaining cases are analogous.

1.2. Let $\varphi \in S$ be given. Apply the result proved above to the solution $\underline{x} : t \to e^{\mu t} x(t)$ of equation $(\mu, f, \underline{x})$, where $x : [-1, \infty) \to \mathbb{R}$ is the solution of equation (μ, f) with $x_0 = \varphi$. In the same way, one obtains the result for the linear equation (μ, α).

2. The first part follows from (NF, t).

In case $x^{-1}(0) \cap [t, \infty) \neq \emptyset$ (bounded or unbounded), use induction to derive (SO) and $\dot{x}(z) \neq 0$ for all zeros.

3. Suppose x is strictly decreasing on $[t-1, t]$, with $0 < x(t-1)$. (NF, t) implies $\dot{x} < 0$ on $(t, t+\varepsilon]$ for some $\varepsilon > 0$. Hence $t < m$; $\dot{x}(m) = 0$ and, by (NF, t), $x(m-1) = 0$.

In case $0 < x(t)$, one has $0 < x$ on $[t-1, t]$, and assertion 2 gives property (SO), with all zeros of x simple.

In case $x(t) \leqq 0$, there is a unique zero z of x in $(t-1, t]$. (NF, t) yields $\dot{x} < 0$ on $(t, z+1)$. Hence $x < 0$ on $(z, z+1]$. As before, one sees that property (SO) holds true for all zeros of x in (z, ∞). It follows that (SO) holds on $[t-1, \infty)$ also in case $x(t) \leqq 0$, with all zeros of x in (z, ∞) simple. In particular, $x(m) \neq 0$ (since $t < m$ and $\dot{x}(m) = 0$).

Suppose $t \leqq m-1$. $x(m-1) = 0$ and (SO) yield $x(s) \neq 0$ for all $s \in [m-2, m-1)$. (NF, t) implies that x is strictly monotone on $[m-1, m]$.

Suppose $m-1 < t$; $m-1$ is a zero of x in $[t-1, t)$. The discussion of the case $x(t) \leqq 0$ above shows that x is strictly decreasing on $[m-1, m]$.

The remaining part of assertion 3 is a consequence of assertion 2. QED.

Observations as in Remark 6.1 and in Proposition 6.1 have been made in nearly every paper on the equations (μ, f) and (f). They are fundamental for any investigation of the dynamics in case of negative feedback.

The proof of the next result is as in section 5 of [32].

Proposition 6.2. $0 \notin pS$.

Proof. Let $\varphi \in S$ be given. Assume $p\varphi = 0$. Consider the solution $x : [-1, \infty) \to \mathbb{R}$ of equation (μ, α) with $x_0 = \varphi$. For $\psi \in C$, set

$$p'\psi := (p(\lambda_1) + p(\bar{\lambda}_1))\psi \in C.$$

The case $p'\varphi \neq 0$: There are $\varepsilon > 0$, $c > 0$ such that for all $t \geqq 0$,

$$\|T(t)(\varphi - p\varphi - p'\varphi)\| \leqq c e^{(u_1 - \varepsilon)t} \|\varphi - p\varphi - p'\varphi\|,$$

see Chapter 7 in [9]. The solution $y : [-1, \infty) \to \mathbb{R}$ of equation (μ, α) with $y_0 = p\varphi + p'\varphi = p'\varphi$ (by assumption) has the form

$$y(t) = e^{u_1 t}(a \cos v_1 t + b \sin v_1 t)$$

with constants a, b in $\mathbb{R}$, not both equal to zero since $p'\varphi \neq 0$. The last estimate yields

$$x(t)e^{-u_1 t} - a\cos v_1 t - b\sin v_1 t \to 0 \quad \text{as} \quad t \to \infty,$$

and $2\pi < v_1$ leads to a contradiction to the fact that all x_t, $t \geqq 0$, belong to S (Remark 6.1).

The case $p'\varphi = 0$: Proposition 6.1 guarantees that there exists $s \geqq 0$ with $x(t) \neq 0$ for $s-1 \leqq t \leqq s$. By continuity of $T(s)$, there is a neighborhood U of φ with $T(s)\psi(t) \neq 0$ for $-1 \leqq t \leqq 0$, $\psi \in U$. In particular, $T(s)U \subset S$. Choose $\chi \in p'C\setminus\{0\}$ so small that $\psi := \varphi + \chi \in U$. Then $p\psi = 0$, $p'\psi = \chi \neq 0$, $T(t)\psi \in S$ for all $t \geqq s$ (Remark 6.1), and one obtains a contradiction as above. QED.

It is also possible to complement Proposition 6.2 by a necessary condition ("If $p\varphi \neq 0$ then the curve $t \to T(t)\varphi$ is absorbed into the set S"), compare section 5 in [32].

The subsequent results deal with slowly oscillating solutions of the nonlinear equation (μ, f). It is shown that the forward extension W of the local invariant manifold W_0 consists of segments x_t of such solutions (and of the stationary point $0 \in C$), and that W is given by a map from $pW \subset L$ into Q.

For $\varphi \in C$, let $\underline{\varphi}$ denote the function

$$t \in [-1, 0] \to e^{\mu t}\varphi(t) \in \mathbb{R}.$$

It is convenient to introduce the convex cone

$$K := \{\varphi \in C : \varphi \neq 0, \varphi(-1) = 0, \underline{\varphi} \text{ increasing}\},$$

compare [8], [20]. Note

$$K \subset S, \quad \mathrm{cl}\, K = K \cup \{0\}.$$

Remark 6.2. For every $\varphi \in K$ there exists $z = z(\varphi) \in [-1, 0)$ with $\varphi = 0$ on $[-1, z]$ and $0 < \varphi$ on $(z, 0]$.

The next elementary step is to exclude eventually monotone solutions, and to estimate distances of zeros and extrema of slowly oscillating solutions. Existence of zeros (in other words, oscillatory behavior) is a consequence of the inequality

$$1 < \alpha e^{\mu}$$

of Remark 4.2 which expresses sufficiently strong feedback at $\xi = 0$.

Analogous results can be found in many papers on the equations (μ, f) and (f).

Fix $\varepsilon_0 > 0$ with $1 < (\alpha - \varepsilon_0)e^{\mu}$, and $\delta_0 > 0$ with

$$(\alpha - \varepsilon_0)|\xi| \leqq |f(\xi)| \quad \text{for} \quad |\xi| \leqq \delta_0.$$

Proposition 6.3. *Let $\varphi \in S$ be given; $\varphi > 0$ or $\varphi \in K$. The solution $x : [-1, \infty) \to \mathbb{R}$ of equation (μ, f) with $x_0 = \varphi$ has the following properties.*

1. *The zeros of x in $\mathbb{R}^+$ form a sequence $(z_j)_1^\infty$ with*

$$(\mathrm{Z})\qquad \left(\begin{array}{l} 0 < z_1 \quad \text{and} \quad x(t) \neq 0 \quad \text{for} \quad 0 < t < z_1, \\[2ex] \begin{pmatrix} z_j + 1 < z_{j+1}, \\ \dot{x}(z_j) \neq 0, \end{pmatrix} \quad \text{for all} \quad j \in \mathbb{N}, \\[2ex] x_{z_j+1} \in \begin{Bmatrix} K \\ -K \end{Bmatrix} \quad \text{if} \quad \dot{x}(z_j) \begin{Bmatrix} > \\ < \end{Bmatrix} 0. \end{array}\right.$$

In case $\varphi \in K$, $z(\varphi) + 1 < z_1$.

2. $$z_1 < 2 + \max\left\{0, \frac{1}{\mu}\log\frac{x(0)}{\delta_0}\right\},$$

$$z_2 < z_1 + 3 + \max\left\{0, \frac{1}{\mu}\log\frac{-x(z_1+1)}{\delta_0}\right\}.$$

3. *For $z_1 < t < z_2$, $\frac{1}{\mu}\min_{[0,\|\varphi\|]} f \leqq x(t) < 0$. For $j \in \mathbb{N}$ even and $z_j < t < z_j + 1$, $0 < x(t) \leqq \frac{1}{\mu} c_f$.*

Remarks. Analogous results hold for $\varphi < 0$ or $\varphi \in -K$. Assertion 1 and $0 < \varphi(0)$ imply

$$0 < x \quad \text{on} \quad [0, z_1) \quad \text{and on} \quad (z_j, z_{j+1}) \quad \text{with} \quad j \in \mathbb{N} \quad \text{even},$$
$$x < 0 \quad \text{on} \quad (z_j, z_{j+1}) \quad \text{with} \quad j \in \mathbb{N} \quad \text{odd}.$$

Proof of Proposition 6.3. a. Existence of a first zero z_1 of x in $\mathbb{R}^+$, with $0 < z_1 \leqq 2 + \max\left\{0, \frac{1}{\mu}\log\frac{\varphi(0)}{\delta_0}\right\}$: Suppose $0 < x$ in $[0, t'+1]$ where

$$t' := \max\left\{0, \frac{1}{\mu}\log\frac{x(0)}{\delta_0}\right\}.$$

The solution $\underset{\sim}{x} : t \in [-1, \infty) \to e^{\mu t} x(t) \in \mathbb{R}$ of equation $(\mu, f, \underset{\sim}{x})$ decreases on $[0, t'+2]$, due to (NF, t). For $t' + 1 \leqq t \leqq t' + 2$,

$$\dot{\underset{\sim}{x}}(t) = e^{\mu t} f(x(t-1)) \leqq e^{\mu t}(-\alpha + \varepsilon_0) x(t-1)$$

$$\left(\text{since } 0 \leqq x(t-1) = e^{-\mu(t-1)} \underset{\sim}{x}(t-1) \leqq e^{-\mu t'} \underset{\sim}{x}(0) = e^{-\mu t'} x(0) \leqq \delta_0\right)$$

$$= (-\alpha + \varepsilon_0) e^{\mu} \underset{\sim}{x}(t-1),$$

and therefore

$$\underline{x}(t'+2)-\underline{x}(t'+1)=\int_{t'+1}^{t'+2}\dot{\underline{x}}(t)\,dt \leqq (-\alpha+\varepsilon_0)e^{\mu}\int_{t'}^{t'+1}\underline{x}(t)\,dt$$

$$\leqq (-\alpha+\varepsilon_0)e^{\mu}\underline{x}(t'+1)\leqq -\underline{x}(t'+1).$$

It follows that

$$x(t'+2)=e^{-\mu(t'+2)}\underline{x}(t'+2)\leqq 0.$$

b. Simplicity of z_1, $x_{z_1+1}\in -K$: In case $\varphi\in K$, recall $z=z(\varphi)$ from Remark 6.2. $\underline{x}=0$ on $[-1,z]$ and $0<\underline{x}$ on $(z,0]$ imply $\underline{x}(t)=\underline{x}(0)>0$ for $0\leqq t\leqq z+1$. Consequently, $z_1>z+1$, and $0<\underline{x}$ on (z,z_1). It follows that $\dot{\underline{x}}<0$ on $(z+1,z_1+1)$; in particular, $x_{z_1+1}\in -K$ (use $e^{\mu t}(-x(z_1+1+t))=e^{-\mu(z_1+1)}\underline{x}(z_1+1+t)$ for $-1\leqq t\leqq 0$) and $x<0$ on $(z_1,z_1+1]$. Also,

$$\dot{x}(z_1)=0+f(x(z_1-1))<0$$

since

$$x(z_1-1)=e^{-\mu(z_1-1)}\underline{x}(z_1-1)>0.$$

In case $\varphi>0$ one finds $0<\underline{x}$ on $[-1,z_1)$, $\dot{\underline{x}}<0$ on $(0,z_1+1)$, $x_{z_1+1}\in -K$, $\dot{x}(z_1)<0$ and $x<0$ on $[z_1,z_1+1)$.

c. Estimates on $[-1,z_1+1]$: $\dot{\underline{x}}\leqq 0\leqq\underline{x}$ on $(0,z_1]$ implies that the map $x: t\to e^{-\mu t}\underline{x}(t)$ is decreasing on $(0,z_1]$. Hence $|x|\leqq\|\varphi\|$ on $[-1,z_1]$. For $z_1\leqq t\leqq z_1+1$,

$$0\geqq x(t)=0+\int_{z_1}^{t}e^{-\mu(t-s)}f(x(s-1))\,ds\geqq \min_{[0,\|\varphi\|]} f\frac{1}{\mu}.$$

d. As before, one finds a first zero $z_2>z_1+1$ of x with

$$z_2-(z_1+1)\leqq 2+\max\left\{0,\frac{1}{\mu}\log\frac{-x(z_1+1)}{\delta_0}\right\}$$

and

$$0<\dot{\underline{x}}\quad\text{on}\quad[z_2,z_2+1),\ x_{z_2+1}\in K,\ 0<\dot{x}(z_2+1),\ 0<x\quad\text{on}\quad(z_2,z_2+1];$$

x is increasing on $[z_1+1,z_2]$. For $z_2\leqq t\leqq z_2+1$,

$$0\leqq x(t)=0+\int_{z_2}^{t}e^{-\mu(t-s)}f(x(s-1))\,ds\leqq c_f\frac{1}{\mu}.$$

e. It is now obvious how to construct the full sequence $(z_j)_1^{\infty}$. QED.

Set

$$r_0 := \frac{1}{\mu} \max\left\{c_f, -\min_{[0,c_f/\mu]} f\right\} > 0.$$

Proposition 6.3, the Remark thereafter and Proposition 6.1 yield a criterion for slowly oscillating solutions on $\mathbb{R}$ with arbitrarily large zeros in both directions $t > 0$ and $t < 0$:

Corollary 6.1. *Suppose $x: \mathbb{R} \to \mathbb{R}$ is a solution of equation (μ, f), and there exists a sequence $(t_n)_1^\infty$ with $t_n \to -\infty$ as $n \to \infty$, so that $(x_{t_n})_1^\infty$ is a bounded sequence in S. Then the zeros of x form a sequence $(z_j)_{-\infty}^\infty$ with property* (Z), *and*

$$|x(t)| \leqq r_0 \quad \textit{for all} \quad t \in \mathbb{R}.$$

The properties of L, notably the no-zero-statement of Remark 4.1, and tangency can be used to show that small solutions of the nonlinear equation (μ, f) which start in $W_0 \setminus \{0\} \subset W$ have segments without zeros, i.e. in S.

The next result goes even further:

Proposition 6.4. *There is an open neighborhood U_0 of 0 in C such that for all points φ and φ' in $U_0 \cap W_0$ with $\varphi \neq \varphi'$, there exists $t \in [0, 2]$ so that*

$$F(t, \varphi) - F(t, \varphi') \in C \quad \textit{has no zero}.$$

Proof. 1. An estimate. Let solutions $x: [-1, \infty) \to \mathbb{R}$ and $x': [-1, \infty) \to \mathbb{R}$ of equation (μ, f), and a solution $y: [-1, \infty) \to \mathbb{R}$ of equation (μ, α) be given. Set

$$c := 1 + \alpha e^\mu + e^\mu$$

and

$$\begin{aligned} m &:= m(x, x') := \max\{|f'(\xi) - f'(0)| : |\xi| \leqq 2\|x_0\| + 2\|x'_0\|\}; \\ d &:= x - x'. \end{aligned}$$

Proof of

$$|d(t) - y(t)| \leqq c((1 + m)\|d_0 - y_0\| + m\|y_0\|) \quad \text{for} \quad 0 \leqq t \leqq 1:$$

By the variation-of-constants formula,

$$|d(t) - y(t)| \leqq e^{-\mu t}\left(|d(0) - y(0)| + \int_0^t e^{\mu s}|f(x(s-1)) - f(x'(s-1)) + \alpha y(s-1)|\, ds\right).$$

For each $s \in [0, t]$ there exists $\Theta \in [0, 1]$ with

$$\begin{aligned}
&|f(x(s-1)) - f(x'(s-1)) + \alpha y(s-1)| \\
&= |f'(x'(s-1) + \Theta d(s-1)) d(s-1) + \alpha y(s-1)| \\
&\leqq |(f'(\ldots) + \alpha) d(s-1)| + \alpha |y(s-1) - d(s-1)| \\
&\leqq m\|d_0\| + \alpha \|d_0 - y_0\| \qquad\qquad (\text{with } \alpha = -f'(0))
\end{aligned}$$

Hence

$$e^{-\mu t}(\ldots) \leqq 1(\|d_0 - y_0\| + e^{\mu} m\|d_0 - y_0\| + e^{\mu} m\|y_0\| + \alpha e^{\mu}\|d_0 - y_0\|).$$

2. Construction of U_0. Choose

$$\begin{aligned}
&r > 0 \quad \text{with} \quad cr\|T(1)\| < \frac{a_0}{4}, \\
&r' > 0 \quad \text{with} \quad |f'(\xi) - f'(0)| \leqq r \quad \text{for} \quad |\xi| \leqq r', \\
&\delta > 0 \quad \text{with} \quad \|F(1, \psi)\| \leqq \frac{r'}{4} \quad \text{for} \quad \|\psi\| \leqq \delta, \\
&\varepsilon \in (0, 1) \quad \text{with} \quad c\varepsilon < \frac{a_0}{4} \quad \text{and} \quad (1 + r)c^2\varepsilon < \frac{a_0}{8}, \\
&\delta' \in (0, \delta) \quad \text{with} \quad |f'(\xi) - f'(0)| < \varepsilon \quad \text{for} \quad |\xi| \leqq \delta', \\
&\varepsilon' > 0 \quad \text{with} \quad c(1 + \varepsilon)\varepsilon' < \frac{a_0}{4} \quad \text{and} \quad c(1 + r)c(1 + \varepsilon)\varepsilon' < \frac{a_0}{8}.
\end{aligned}$$

$Dw_0(0) = 0$ permits to find $\delta'' \in \left(0, \frac{\delta'}{4}\right)$ such that

$$\begin{aligned}
&\|\psi\| \leqq \delta'' \quad \text{and} \quad \|\psi'\| \leqq \delta'' \quad \text{imply} \quad p\psi \in L_0, \quad p\psi' \in L_0, \\
&\|w_0(p\psi) - w_0(p\psi')\| \leqq \varepsilon'\|p\psi - p\psi'\|.
\end{aligned}$$

Set

$$U_0 := \{\psi \in C : \|\psi\| < \delta''\}.$$

3. Let φ, φ' in $U_0 \cap W_0$ be given; $\varphi \neq \varphi'$. As W_0 is the graph of w_0, $p\varphi \neq p\varphi'$. Let $x : [-1, \infty) \to \mathbb{R}$ and $x' : [-1, \infty) \to \mathbb{R}$ denote the solutions of equation (μ, f) with initial conditions $x_0 = \varphi$, $x'_0 = \varphi'$. Set $d := x - x'$, $m := m(x, x')$ as in part 1. Note $m < \varepsilon$. Consider the solution $y : [-1, \infty) \to \mathbb{R}$ of equation (μ, α) with

$$y_0 = p\varphi - p\varphi' \in L \backslash \{0\}.$$

Observe

$$d_0 - y_0 = q\varphi - q\varphi' = w_0(p\varphi) - w_0(p\varphi').$$

Part 1 and the choice of δ'', δ', ε', ε yield

$$\begin{aligned}\|d_1 - y_1\| &\leqq c(1+m)\varepsilon'\|p\varphi - p\varphi'\| + cm\|y_0\| \leqq c(1+\varepsilon)\varepsilon'\|y_0\| + c\varepsilon\|y_0\| \\ &\leqq \frac{a_0}{2}\|y_0\|.\end{aligned}$$

Set $m_2 := m(x(\cdot+1), x'(\cdot+1))$. Note $m < r$, by the choice of $\delta'' < \frac{\delta'}{4} < \frac{\delta}{4}$, of δ and of r'. A second application of part 1 yields

$$\begin{aligned}\|d_2 - y_2\| &\leqq c(1+m_2)\|d_1 - y_1\| + cm_2\|y_1\| \leqq c(1+r)\|d_1 - y_1\| + cr\|y_1\| \\ &\leqq c(1+r)c\big((1+\varepsilon)\varepsilon' + \varepsilon\big)\|y_0\| + cr\|T(1)\|\|y_0\|\end{aligned}$$

(with the estimate for $\|d_1 - y_1\|$)

$$< \frac{a_0}{2}\|y_0\|$$

(by the choice of ε', ε, r). Also,

$$\|d_0 - y_0\| = \|q\varphi - q\varphi'\| \leqq \varepsilon'\|p\varphi - p\varphi'\| = \varepsilon'\|y_0\| \leqq \frac{a_0}{2}\|y_0\|.$$

It follows that

$$|d(t) - y(t)| \leqq \frac{a_0}{2}\|y_0\| \quad \text{for all} \quad t \in [-1, 2].$$

Remark 4.2 guarantees the existence of $t \in [0, 2]$ with

$$|d(s)| \geqq |y(s)| - |d(s) - y(s)| \geqq \frac{a_0}{2}\|y_0\| > 0 \quad \text{for all} \quad s \in [t-1, t]. \qquad \text{QED.}$$

Corollary 6.2. 1. $W \subset \{\varphi \in C : \|\varphi\| \leqq r_0\}$.

2. *Let $\varphi \in W \setminus \{0\}$ be given. The zeros of the solution $x : \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_0 = \varphi$ form a sequence $(z_j)_{-\infty}^{\infty}$ with property* (Z). *In particular, $\varphi \in S$.*

3. *There exists a map $w : L_w \to Q$, with $L_w := pW \supset L_0$, such that*

$$W = \{\chi + w(\chi) : \chi \in L_w\}.$$

Proof. a. Consider $\varphi \in W \setminus \{0\}$ and the solution $x : \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_0 = \varphi$ (Corollary 5.1). Proposition 6.4, applied to $x_t \in U_0$ with $-t > 0$ sufficiently large, and to $0 \in U_0$, yields a sequence $(t_n)_1^{\infty}$ as in Corollary 6.1; this proves assertions 2 and 1.

b. Assertion 3 follows from injectivity of $p|W$. Injectivity of $p|W$ is a consequence of Corollary 5.1 and of Proposition 6.4:

Let φ and ψ in W be given, $\varphi \neq \psi$. Consider the solutions $x : \mathbb{R} \to \mathbb{R}$ and $y : \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_0 = \varphi$ and $y_0 = \psi$. It follows that for some $t < -2$,

$$x_t \in U_0, \quad y_t \in U_0, \quad x_t \neq y_t.$$

Proposition 6.4 gives $s < 0$ so that $d := x - y$ has no zero on $[s-1, s]$. In particular, the segment $\underline{d}_s$ of the solution $\underline{d} : t \to e^{\mu t} d(t)$ of equation $(\mu, f, x - y)$ belongs to S, and Remark 6.1 yields $\underline{d}_0 \in S$. It follows that $\varphi - \psi = d_0$ belongs to S.

By Corollary 6.1, $p\varphi \neq p\psi$. QED.

The map w extends w_0, of course. The next section contains an estimate along slowly oscillating solutions of equation (μ, f) which in turn will imply that w satisfies a global Lipschitz condition.

7. An a-priori estimate

Estimates of the form

$$\mathrm{const}\,\|\varphi\| \leqq \|p\varphi\|$$

on subcones of S can be characterized by estimates involving the simpler operator $T(1)$;

$$T(1)\varphi(t) = e^{-\mu(1+t)}\varphi(0) + \int_0^{1+t} e^{-\mu(1+t-s)} f(\varphi(s-1))\,ds$$

for $\varphi \in C$ and $-1 \leqq t \leqq 0$. This is done as in Lemma 4 of [32]:

Lemma 7.1. *Let a subcone $S' \subset S$ be given. The following statements are equivalent*

1. *There exists $c^* > 0$ with $c^* \|\varphi\| \leqq \|T(1)\varphi\|$ for all $\varphi \in S'$.*

2. *There exists $c > 0$ with $c\|\varphi\| \leqq \|p\varphi\|$ for all $\varphi \in S'$.*

Proof. a. Suppose statement 1 holds true. Set $S_1' := \{\varphi \in S' : \|\varphi\| = 1\}$. cl $T(1)S_1'$ is compact. $T(1)S \subset S$ (Remark 6.1) and (6.1) imply cl $T(1)S_1' \subset$ cl $S = S \cup \{0\}$. The estimate in statement 1 gives $0 \notin$ cl $T(1)S_1'$, so that cl $T(1)S_1' \subset S$. Proposition 6.2 yields

$$0 < \inf\{\|p\varphi\| : \varphi \in \mathrm{cl}\, T(1)S_1'\} \leqq \inf\{\|pT(1)\varphi\| : \varphi \in S_1'\} = \inf\{\|T(1)p\varphi\| : \varphi \in S_1'\}$$
$$\leqq \|T(1)\|\inf\{\|p\varphi\| : \varphi \in S_1'\}.$$

Set $c := \inf\{\|p\varphi\| : \varphi \in S_1'\} > 0$ and use $(0, \infty)S' \subset S'$.

b. Assume statement 2 holds. Then $0 \notin$ cl pS_1'. The latter is a closed bounded subset of the 2-dimensional space $L = pC$. $T(1)\psi \neq 0$ for all $\psi \in L \setminus \{0\}$ implies $0 < \inf\{\|T(1)\psi\| : \psi \in \mathrm{cl}\, pS_1'\}$, and statement 1 follows by arguments as in part a of the proof. QED.

Proposition 7.1. *Let $r > 0$ be given. There exists a constant $c(r) > 0$ with the following property: If $x : [t_0 - 1, \infty) \to \mathbb{R}$ and $x' : [t_0 - 1, \infty) \to \mathbb{R}$ are solutions of equation (μ, f) with*

$$|x(t)| \leqq r \quad \text{and} \quad |x'(t)| \leqq r \quad \text{for all} \quad t \geqq t_0 - 1,$$

and if

$$x(t) - x'(t) \neq 0 \quad \text{for all} \quad t \in [t_0 - 1, t_0],$$

then

$$c(r)\|d_t\| \leqq \|pd_t\| \quad \text{for all} \quad t \geqq t_0 + 2$$

where $d := x - x'$.

This is a generalization of Lemma 5 from [32].

Proof of Proposition 7.1. 1. Let $r > 0$ be given. There exist $a = a_r \in (0, 1)$ and $b = b_r > 1 > a$ such that

$$a \leqq e^{\mu}|f'(\xi)| \leqq b \quad \text{for} \quad |\xi| \leqq r. \tag{7.1}$$

Consider solutions $x : [t_0 - 1, \infty) \to \mathbb{R}$ and $x' : [t_0 - 1, \infty) \to \mathbb{R}$ of equation (μ, f) which are bounded by r and satisfy $x(s) - x'(s) \neq 0$ for $t_0 - 1 \leqq s \leqq t_0$. Set $d := x' - x$ and $\underline{d}(s) := e^{\mu s} d(s)$ for $s \geqq t_0 - 1$. The solution $\underline{d} : [t_0 - 1, \infty) \to \mathbb{R}$ of equation $(\mu, f, x' - x)$ satisfies, for all $s > t_0$,

$$\tag{7.2} \left\{\begin{aligned} a|\underline{d}(s-1)| &= e^{\mu s} a e^{-\mu} |x'(s-1) - x(s-1)| \\ &\leqq \left| e^{\mu s} \int_{x(s-1)}^{x'(s-1)} f'(\xi)\, d\xi \right| && \text{(see (7.1))} \\ &= |\dot{\underline{d}}(s)| \\ &= \left| e^{\mu s} \int_{x(s-1)}^{x'(s-1)} f'(\xi)\, d\xi \right| \\ &\leqq e^{\mu s} b e^{-\mu} |x'(s-1) - x(s-1)| && \text{(see (7.1))} \\ &= b|\underline{d}(s-1)|. \end{aligned}\right.$$

$\underline{d}_0$ has no zero.

Property (NF, t) for equation $(\mu, f, x' - x)$ implies that either

$$0 < \underline{d}(0), \quad \text{and} \quad \dot{\underline{d}} < 0 \quad \text{on} \quad (t_0, t_0 + 1],$$

or

$$\underline{d}(0) < 0, \quad \text{and} \quad 0 < \dot{\underline{d}} \quad \text{on} \quad (t_0, t_0 + 1].$$

Let $T: C \to C$ denote the time-1-map associated with the linear equation

$$\dot{y}(s) = -\alpha e^{\mu} y(s-1),$$

i.e.

$$T(\varphi)(s) = \varphi(0) - \alpha e^{\mu} \int_{-1}^{s} \varphi(s')\,ds' \quad \text{for} \quad \varphi \in C \quad \text{and} \quad -1 \leqq s \leqq 0.$$

2. Claim: There exists a constant $c = c(a_r, b_r, \mu, \alpha) > 0$ such that

$$c\|\underline{d}_t\| \leqq \|T\underline{d}_t\| \quad \text{for all} \quad t \geqq t_0 + 2.$$

Proof: Let $t \geqq t_0 + 2\left(> t_0 + 1 + \frac{1}{2b}\right)$ be given. Note that $\underline{d}$ is differentiable at each $s \geqq t - 1 - \frac{1}{2b} > t_0$.

2.1. For $t - 1 \leqq s \leqq s' \leqq t$,

$$\alpha e^{\mu}\left|\int_{s}^{s'} \underline{d}(s'')\,ds''\right| = \alpha e^{\mu}\left|\int_{s-t}^{s'-t} \underline{d}_t(s'')\,ds''\right| = |T\underline{d}_t(s'-t) - T\underline{d}_t(s-t)| \leqq 2\|T\underline{d}_t\|.$$

2.2. For every local extremum $m > t_0$ of $\underline{d}$ let g_m denote the affine linear function with $g_m(m) = \underline{d}(m)$ and $\dot{g}_m(s) = -b\underline{d}(m)$, $s \in \mathbb{R}$. Then $g_m\left(m + \frac{1}{b}\right) = 0$.

Part 3 of Proposition 6.1 yields $m > t_0 + 1$ and $\underline{d}(m) \neq 0$.

Proof of $|\underline{d}(s)| \geqq |g_m(s)|$ for $m \leqq s \leqq m + \frac{1}{b}$: Suppose m is a local maximum. Part 3 of Proposition 6.1 says that $\underline{d}$ is increasing on $[m-1, m]$ with $\underline{d}(m-1) = 0$. For $m \leqq s \leqq m + \frac{1}{b}$ $(< m+1$, by $1 < b)$, one has $(t_0 <) m - 1 \leqq s - 1 \leqq m$, hence

$$\begin{aligned} \dot{g}_m(s) = -b\underline{d}(m) \leqq -b\underline{d}(s-1) \leqq \dot{\underline{d}}(s) & \qquad \text{(see (7.2))} \\ \leqq 0 & \qquad \text{(with (NF, } t\text{))}. \end{aligned}$$

The argument for a local minimum is analogous.

2.3. It is convenient to distinguish the following cases.

A. There is a local extremum $m \in [t-1, t]$.

A.1 $t \leqq m + \frac{1}{2b}$.

A.2 $m + \frac{1}{2b} < t$, $|\underline{d}(m)| = \|\underline{d}_t\|$.

A.3 $m + \frac{1}{2b} < t$, $|\underset{\sim}{d}(m)| < \|\underset{\sim}{d}_t\|$.

B. $\dot{\underset{\sim}{d}}(s) \neq 0$ for $t-1 \leqq s \leqq t$.

B.1 $|\underset{\sim}{d}(t)| = \|\underset{\sim}{d}_t\|$.

B.2 $|\underset{\sim}{d}(t)| < \|\underset{\sim}{d}_t\| = |\underset{\sim}{d}(t-1)|$, $|\underset{\sim}{d}(s)| \geqq |\underset{\sim}{d}(t-1)|$ for $t-1-\frac{1}{2b} \leqq s \leqq t-1$,

B.3 $|\underset{\sim}{d}(t)| < \|\underset{\sim}{d}_t\| = |\underset{\sim}{d}(t-1)|$, and there exists $s \in \left[t-1-\frac{1}{2b}, t-1\right]$ with $|\underset{\sim}{d}(s)| < |\underset{\sim}{d}(t-1)|$.

2.4. In case A.1,

$$\|T\underset{\sim}{d}_t\| \geqq |T\underset{\sim}{d}_t(-1)| = |\underset{\sim}{d}(t)| \geqq |g_m(t)| \geqq \left| g_m\left(m + \frac{1}{2b}\right)\right| = \frac{1}{2}|\underset{\sim}{d}(m)| = \frac{1}{2}\|\underset{\sim}{d}_t\|.$$

In case A.2, consider first $s \in \left[m, m + \frac{1}{2b}\right] \subset [t-1, t]$: One has

$$|\underset{\sim}{d}(s)| \geqq |g_m(s)| \geqq \frac{1}{2}|\underset{\sim}{d}(m)| = \frac{1}{2}\|\underset{\sim}{d}_t\|.$$

Part 2.1 yields

$$2\|T\underset{\sim}{d}_t\| \geqq \alpha e^{\mu} \left| \int_m^{m+\frac{1}{2b}} \underset{\sim}{d}(s)\, ds \right| \geqq \alpha e^{\mu} \frac{1}{2b} \frac{1}{2} \|\underset{\sim}{d}_t\|.$$

In case A.3, apply part 3 of Proposition 6.1 to the restriction of $\underset{\sim}{d}$ to $[t_0, \infty)$: $|\underset{\sim}{d}|$ is increasing on $[m-1, m]$, $m > t_0 + 1$, and one has the following alternative: Either

there exists a first zero $z > m$ of $\underset{\sim}{d}$

(in which case $|\underset{\sim}{d}|$ decreases to 0 on $[m, z]$, and increases on $[z, z+1]$) or

$|\underset{\sim}{d}| > 0$ is decreasing on $[m, \infty)$.

The second possibility would imply $\|\underset{\sim}{d}_t\| \leqq |\underset{\sim}{d}(m)|$, so it does not occur in case A.3. It follows that $z < t$ and $|\underset{\sim}{d}(t)| = \|\underset{\sim}{d}_t\|$. Hence $\|T\underset{\sim}{d}_t\| \geqq |T\underset{\sim}{d}_t(-1)| = |\underset{\sim}{d}(t)| = \|\underset{\sim}{d}_t\|$.

Case B.1 is trivial: Compare the preceding inequality.

2.5. Case B.2. $\underset{\sim}{d}$ is monotone on $[t-1, t]$; $|\underset{\sim}{d}|$ is decreasing on $[t-1, t]$. For $s \in \left(t - \frac{1}{2b}, t\right]$, $s - 1 \in \left(t-1-\frac{1}{2b}, t-1\right] \subset (t_0, \infty)$, and consequently

$$|\dot{\underset{\sim}{d}}(s)| \geqq a|\underset{\sim}{d}(s-1)| \geqq a|\underset{\sim}{d}(t-1)| = a\|\underset{\sim}{d}_t\| > 0.$$

It follows that

$$\left|\underset{\sim}{d}(t)-\underset{\sim}{d}\left(t-\frac{1}{2b}\right)\right| \geqq \frac{a}{2b}\|\underset{\sim}{d}_t\|,$$

hence

$$\frac{a}{4b}\|\underset{\sim}{d}_t\| \leqq |\underset{\sim}{d}(t)| \quad (\leqq \|T\underset{\sim}{d}_t\|)$$

or

$$\frac{a}{4b}\|\underset{\sim}{d}_t\| \leqq \left|\underset{\sim}{d}\left(t-\frac{1}{2b}\right)\right|.$$

In the last case, monotonicity implies that for $t-1 \leqq s \leqq t-\frac{1}{2b}$,

$$|\underset{\sim}{d}(s)| \geqq \left|\underset{\sim}{d}\left(t-\frac{1}{2b}\right)\right| \quad \left(\geqq \frac{a}{4b}\|\underset{\sim}{d}_t\|\right),$$

and part 2.1 yields

$$2\|T\underset{\sim}{d}_t\| \geqq \alpha e^{\mu}\left(1-\frac{1}{2b}\right)\left|\underset{\sim}{d}\left(t-\frac{1}{2b}\right)\right| \geqq \alpha e^{\mu}\left(1-\frac{1}{2b}\right)\frac{a}{4b}\|\underset{\sim}{d}_t\|.$$

2.6. Case B.3.

2.6.1. Proof that there exists $m \in \left(t-1-\frac{1}{2b}, t-1\right]$ with $\dot{\underset{\sim}{d}}(m)=0$: Suppose $\dot{\underset{\sim}{d}}(s)>0$ for all $s \in \left(t-1-\frac{1}{2b}, t-1\right]$.

It follows that $\dot{\underset{\sim}{d}}(s)>0$ also for all $s \in [t-1, t]$. $|\underset{\sim}{d}(t-1)| > |\underset{\sim}{d}(t)|$ yields $\underset{\sim}{d}(t-1)<0$. Furthermore, $\underset{\sim}{d}(s) \leqq \underset{\sim}{d}(t-1)<0$ for all $s \in \left(t-1-\frac{1}{2b}, t-1\right]$, a contradiction to case B.3. Analogously one excludes $\dot{\underset{\sim}{d}}<0$ on $\left(t-1-\frac{1}{2b}, t-1\right]$.

2.6.2. Note

$$m \leqq t-1 < m+\frac{1}{2b}. \tag{7.3}$$

Part 3 of Proposition 6.1 implies that either

$$|\underline{d}| \text{ decreases on } [m, \infty), \tag{7.4}$$

or

that there exists a first zero $z > m$ of $\underline{d}$

(in which case $|\underline{d}|$ decreases to 0 on $[m, z]$). In the second case, the inequalities

$$|\underline{d}| \geqq |g_m| > 0 \quad \text{on} \quad \left[m, m + \frac{1}{b}\right)$$

and (7.3) yield

$$m \leqq t - 1 < z,$$

and

$$|\underline{d}(m)| \geqq |\underline{d}(t-1)|. \tag{7.5}$$

The last inequality holds also true in case of (7.4).

For $s \in \left[t-1, t-1+\frac{1}{4b}\right] \subset \left[m, m+\frac{3}{4b}\right]$, one finds

$$\begin{aligned}|\underline{d}(s)| &\geqq |g_m(s)| \geqq \left|g_m\left(m + \frac{3}{4b}\right)\right| = \frac{1}{4}|\underline{d}(m)| \geqq \frac{1}{4}|\underline{d}(t-1)| \qquad \text{(see (7.5))}\\ &= \frac{1}{4}\|\underline{d}_t\|,\end{aligned}$$

and therefore

$$2\|T\underline{d}_t\| \geqq \alpha e^{\mu}\frac{1}{4b}\frac{1}{4}\|\underline{d}_t\|.$$

2.7. Set

$$c := \min\left\{\frac{1}{2}, \alpha e^{\mu}\frac{1}{8b}, \frac{a}{4b}, \alpha e^{\mu}\left(1 - \frac{1}{2b}\right)\frac{a}{16b}, \alpha e^{\mu}\frac{1}{32b}\right\}.$$

3. Proof of $\|T(1)d_t\| \geqq e^{-2\mu}c\|d_t\|$ for $t \geqq t_0 + 2$: Let $t \geqq t_0 + 2$ be given. For $-1 \leqq s \leqq 0$,

$$\begin{aligned}|T(1)d_t(s)| &= \left|e^{-\mu(1+s)}d_t(0) - \alpha\int_0^{1+s} e^{-\mu(1+s-s')}d_t(s'-1)\,ds'\right|\\ &= \left|e^{-\mu(1+s)}d(t) - \alpha\int_0^{1+s} e^{-\mu(t+s)}e^{\mu(t+s'-1)}d(t+s'-1)\,ds'\right|\\ &= e^{-\mu(t+1+s)}\left|e^{\mu t}d(t) - \alpha e^{\mu}\int_0^{1+s} \underline{d}(t+s'-1)\,ds'\right|\\ &= e^{-\mu(t+1+s)}|T\underline{d}_t(s)|\\ &\geqq e^{-\mu(t+1)}|T\underline{d}_t(s)|.\end{aligned}$$

Hence

$$\begin{aligned}\|T(1)d_t\| &= \max_{s\in[-1,0]} |T(1)d_t(s)| \geqq e^{-\mu(t+1)} \max_{s\in[-1,0]} |T\underline{d}_t(s)| = e^{-\mu(t+1)}\|T\underline{d}_t\| \\ &\geqq e^{-\mu(t+1)}c\|\underline{d}_t\| = e^{-\mu(t+1)}c \max_{s\in[-1,0]} |e^{\mu(t+s)}d(t+s)| \\ &\geqq e^{-\mu}c \max_{s\in[-1,0]} |e^{\mu s}d(t+s)| \\ &\geqq e^{-\mu}ce^{-\mu} \max_{s\in[-1,0]} |d(t+s)| = e^{-2\mu}c\|d_t\|.\end{aligned}$$

4. The segment d_t, $t \geqq t_0+2$, belongs to the cone

$$S' := \{\varphi\in S : \|T(1)\varphi\| \geqq e^{-2\mu}c\|\varphi\|\}$$

($d_0 \in S$ implies $d_t \in S$, Remark 6.1). Lemma 7.1 guarantees a constant $c(r) > 0$ with

$$c(r)\|\varphi\| \leqq \|p\varphi\| \quad \text{for all} \quad \varphi\in S'. \qquad \text{QED.}$$

8. Lipschitz condition and C^1-smoothness

Theorem 8.1. *The domain $L_w \subset L$ of the map $w : L_w \to Q$ with*

$$W = \{\chi + w(\chi) : \chi\in L_w\}$$

is open, and w is of class C^1. There exists a constant $l_w > 0$ such that

$$\|w(\chi) - w(\chi')\| \leqq l_w\|\chi-\chi'\| \quad \textit{for all} \quad \chi, \chi' \textit{ in } L_w.$$

Proof. 1. Set $l_w := \|q\|c(r_0)^{-1}$, with the constant from Proposition 7.1.

Proof of the Lipschitz estimate: Let χ and χ' in L_w be given, $\chi \neq \chi'$. Let $x : \mathbb{R}\to\mathbb{R}$ and $x' : \mathbb{R}\to\mathbb{R}$ denote the solutions of equation (μ, f) with $x_0 = \chi + w(\chi)$, $x_0' = \chi' + w(\chi')$. Then $x_t \neq x_t'$ for all $t\in\mathbb{R}$, and x and x' are bounded by r_0 (Corollary 6.2). Recall $x_t \to 0$ and $x_t' \to 0$ as $t\to-\infty$, and $x_t\in W_0$, $x_t'\in W_0$ for $-t > 0$ sufficiently large. Proposition 6.4 yields the existence of $t_0 \leqq -2$ so that $x - x'$ has no zero on $[t_0-1, t_0]$. The a-priori-estimate of Proposition 7.1 gives

$$c(r_0)\|x_0 - x_0'\| \leqq \|px_0 - px_0'\|,$$

hence

$$\begin{aligned}\|w(\chi) - w(\chi')\| &= \|qx_0 - qx_0'\| \leqq \|q\|\|x_0 - x_0'\| \\ &\leqq \|q\|c(r_0)^{-1}\|px_0 - px_0'\| = l_w\|\chi-\chi'\|.\end{aligned}$$

2. By Corollary 5.1, $W = F(\mathbb{N}_0 \times W_0)$. Hence $L_w = pW = \bigcup_{n \in \mathbb{N}_0} pF(\{n\} \times W_0)$, and it is sufficient to show that each set $L_n := pF(\{n\} \times W_0)$, $n \in \mathbb{N}_0$, is open in L, with $w|L_n$ of class C^1. Let $n \in \mathbb{N}_0$ be given. Part 1 implies

$$\|q\varphi\| \leqq l_w \|p\varphi\| \quad \text{for every tangent vector} \quad \varphi \in T_\psi W, \quad \psi \in W.$$

It follows that all restrictions $p|T_\psi W$, $\psi \in W$, are injective.

Using $F(\{n\} \times W_0) \subset W$ and Remark 3.3, one obtains that all derivatives of the injective map

$$p \circ F(n, \cdot) \circ \gamma \quad \text{with} \quad \gamma : \chi \in L_0 \ \rightarrow \ \chi + w_0(\chi) \in C$$

are injective. This implies that the image $pF(\{n\} \times W_0) = L_n$ is an open subset of L, and that the inverse map $(p \circ F(n, \cdot) \circ \gamma)^{-1} : L_n \rightarrow L$ is of class C^1.

3. Proof of $w|L_n = q \circ F(n, \cdot) \circ \gamma \circ (p \circ F(n, \cdot) \circ \gamma)^{-1}$: For $\chi = pF(n, \chi' + w(\chi'))$ and $\chi' \in L_0$, $F(n, \chi' + w(\chi')) \in W$ and

$$w(\chi) = qF(n, \chi' + w(\chi')) = q \circ F(n, \cdot) \circ \gamma \circ (p \circ F(n, \cdot) \circ \gamma)^{-1}(\chi).$$

4. Now it is obvious that $w|L_n$ is of class C^1. QED.

The Lipschitz map w has a unique continuation cl w to cl $L_w = L_w \cup \text{bd}\, L_w$, which is again Lipschitz continuous with Lipschitz constant l_w, and

$$\text{cl}\, W = \{\chi + \text{cl}\, w(\chi) : \chi \in \text{cl}\, L_w\}.$$

Set

$$\text{bd}'\, W := \{\chi + \text{cl}\, w(\chi) : \chi \in \text{bd}\, L_w\} = \text{cl}\, W \setminus W$$

(of course, bd$'$ $W \neq$ bd W = cl W). Then

$$\text{cl}\, W = W \cup \text{bd}'\, W, \quad W \cap \text{bd}'\, W = \emptyset, \quad 0 \notin \text{bd}'\, W.$$

cl W and bd$'$ W are compact.

Theorem 10.1 of the last section will state that bd$'$ W consists of a single periodic orbit, given by a slowly oscillating solution, which attracts all phase curves in the 2-dimensional submanifold W (except the stationary solution at 0).

This problem is not completely analogous to Poincaré-Bendixson theory for planar vectorfields in so far as the desired limit cycle will not belong to the 2-dimensional manifold W on which one has a flow; the limit cycle will only be bordering W.

Until now, not much has been derived about the set bd$'$ W = cl $W \setminus W$. Its projection $p(\text{bd}'\, W) = \text{bd}\, L_w$ is only known to be the boundary of an open subset of the space L.

The proof of Theorem 10.1 begins with some a-priori information on phase curves in bd' W.

Proposition 8.1 (Solutions through points of bd' W). *Let $\varphi \in$ bd' W be given. There exists a uniquely determined solution $x: \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_0 = \varphi$. For every $t \in \mathbb{R}$, $x_t \in$ bd' W and $p\dot{x}_t \neq 0$. The zeros of x form a sequence $(z_j)_{-\infty}^{\infty}$ with property* (Z).

Proof. 1. Let $\varphi = \chi + \mathrm{cl}\, w(\chi)$ with $\chi \in \mathrm{bd}\, L_w$ be given. Choose a sequence $(\chi_n)_1^\infty$ in L_w with $\chi_n \to \chi$ as $n \to \infty$. For $n \in \mathbb{N}$, consider the solution $x^{(n)}: \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_0^{(n)} = \chi_n + w(\chi_n)$.

Proof of $F(t, \varphi) \in$ bd' W for all $t \geqq 0$: All $x_t^{(n)}$, $n \in \mathbb{N}$ and $t \geqq 0$, belong to W. Therefore

$$F(t, \varphi) = \lim_{n \to \infty} x_t^{(n)} \in \mathrm{cl}\, W = W \cup \mathrm{bd}'\, W.$$

Suppose $\psi := F(t, \varphi) \in W$ for some $t \geqq 0$. Then $\varphi \in W$ (see Corollary 5.1), a contradiction.

2. Backward extension of φ. Let $k \in -\mathbb{N}$ be given. The sequence $(x_k^{(n)})_{n \in \mathbb{N}}$ in the compact set cl W has convergent subsequences. Consider any such subsequence, say $(x_k^{(n_\nu)})_{\nu \in \mathbb{N}}$, with limit $\psi \in \mathrm{cl}\, W$. The solution $x': [k-1, \infty) \to \mathbb{R}$ of equation (μ, f) with $x'_k = \psi$ satisfies

$$x'_0 = F(-k, \psi) = \lim_{\nu \to \infty} F(-k, x_k^{(n_\nu)}) = \lim_{\nu \to \infty} x_0^{(n_\nu)} = \varphi.$$

Using Remark 3.3 one infers that there exists a uniquely determined solution $x^{((k))}: [k-1, \infty) \to \mathbb{R}$ of equation (μ, f) with $x_0^{((k))} = \varphi$. A second application of Remark 3.3 yields

$$x^{((k))} = x^{((k-1))} | [k-1, \infty) \quad \text{for all} \quad k \in -\mathbb{N}.$$

It follows that there exists a (uniquely determined) solution $x: \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_0 = \varphi$. Part 1 guarantees $x_t \in$ bd' W for all $t \geqq 0$.

3. Note $x_k \in \mathrm{cl}\, W$ for all $k \in -\mathbb{N}$, by construction. Invariance of W under all maps $F(t, \cdot)$, $t \geqq 0$, and continuity yield

$$x_t \in \mathrm{cl}\, W = W \cup \mathrm{bd}'\, W \quad \text{for all} \quad t \in \mathbb{R}.$$

$x_t \in W$ for some $t < 0$ would imply $\varphi = F(-t, x_t) \in W$, a contradiction. Hence $x_t \in$ bd' W also for all $t < 0$.

4. It follows that x is bounded, with

$$0 \neq x_t \in \mathrm{bd}'\, W \subset \mathrm{cl}\, W \subset \mathrm{cl}\, S = S \cup \{0\} \quad \text{for all} \quad t \in \mathbb{R}.$$

Corollary 6.1 gives the assertion on the zeros of x.

5. The vectors $\dot{x}_t$, $t \in \mathbb{R}$, are tangent to the (graph of the) Lipschitz continuous map $\mathrm{cl}\, w : \mathrm{cl}\, L_w \to Q$. This yields

$$\|q\dot{x}_t\| \leqq l_w \|p\dot{x}_t\| .$$

Assume $p\dot{x}_t = 0$. By the last estimate, $\dot{x}_t = 0$, so that x is constant on $[t-1, t]$, with

$$0 = \dot{x}(t) = -\mu x(t) + f(x(t-1)) = -\mu x(t) + f(x(t)) .$$

In particular,

$$0 = x(t) = x(t-1) ,$$

a contradiction to the fact that x is slowly oscillating on $\mathbb{R}$. QED.

9. A submanifold and a return map

The next objective is to find a one-dimensional submanifold of W which is transverse to the phase curves in W, and to describe intersections of phase curves with the submanifold in terms of a smooth map.

Consider the hyperplane

$$H := \{\varphi \in C : \varphi(-1) = 0\} .$$

H is the nullspace of the evaluation map $\mathrm{ev} : \varphi \in C \to \varphi(-1) \in \mathbb{R}$. Set

$$X := W \cap H .$$

Recall that for every solution $x : \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $0 \neq x_t \in W$, $t \in \mathbb{R}$, one has

$$x_{z_j+1} \in X \quad \text{for all } j \in \mathbb{Z}$$

where $(z_j)_{-\infty}^{\infty}$ is the sequence of zeros of x (see Corollary 6.2). Moreover, $\dot{x}(z_j) \neq 0$ yields

$$\text{(9.1)} \qquad D(t \to x_t)(z_j + 1)1 = \dot{x}_{z_j+1} \in T_\varphi W \setminus H \quad \text{for all} \quad j \in \mathbb{Z};$$

that is, phase curves in $W \setminus \{0\}$ pass transversally through H.

Proposition 9.1. *X is a one-dimensional C^1-submanifold of C.*

Proof. In view of [1], Corollary 17.2, it is enough to show that the inclusion map

$$i : \varphi \in H \to \varphi \in C$$

is transversal to W (note $X = i^{-1}(W)$); i.e. that for every $\varphi \in X$, $(Di(\varphi))^{-1} T_{i(\varphi)} W = H \cap T_\varphi W$ is a closed subspace of H with a closed complementary subspace in H, and that each $Di(\varphi)H = H$, $\varphi \in X$, contains a closed subspace which is a complement of $T_{i(\varphi)} W = T_\varphi W$ in C. The case $\varphi = 0$. Then

$$T_0 W = L = \mathbb{R}\chi \oplus \mathbb{R}\psi,$$

where

$$\chi(t) = e^{u(t+s)} \cos v(t+s), \quad \psi(t) = e^{u(t+s)} \sin v(t+s)$$

for $-1 \leqq t \leqq 0$; with $s \in \mathbb{R}$ such that $v(-1+s) = \frac{\pi}{2}$. $0 \neq \chi \in H$ and $\psi \notin H$ give $\dim H \cap L = 1$. Consequently, there is a complementary subspace H' of $H \cap T_0 W = \mathbb{R}\chi$ in H. Furthermore,

$$C = H \oplus \mathbb{R}\psi = (H' \oplus \mathbb{R}\chi) \oplus \mathbb{R}\psi = H' \oplus T_0 W.$$

In case $0 \neq \varphi \in X$, consider the solution $x: \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_0 = \varphi$ and its sequence of zeros $(z_j)_{-\infty}^{\infty}$. Property (Z) gives $z_0 = 1$. By (9.1), $\dot{\varphi} = \dot{x}_{z_0+1} \in T_\varphi W \setminus H$, and therefore $\dim H \cap T_\varphi W < \dim T_\varphi W = 2$. $H \cap T_\varphi W = \{0\}$ would imply $C = H \oplus T_\varphi W$, a contradiction to $\operatorname{codim} H = 1$ and $\dim W = 2$.

Hence $\dim H \cap T_\varphi W = 1$, and one can proceed as in the previous case. QED.

Property (Z) in Corollary 6.2 implies that the open subset

$$Y := X \cap \{\varphi \in C : 0 < \varphi(0)\}$$

of X is contained in the convex cone K. In fact,

$$Y = X \cap K.$$

Proposition 6.3 defines a map $K \to K$ by

$$\varphi \to F(z_2(\varphi) + 1, \varphi)$$

where $z_2(\varphi)$ is the second zero of the solution $x^\varphi : [-1, \infty) \to \mathbb{R}$ of equation (μ, f) in $\mathbb{R}^+$ (or, the first zero in $\mathbb{R}^+$ where $\dot{x}^\varphi$ is positive). This is as in [8] and in [20]; similar return maps have been used e.g. in [2], [3], [28].

Proposition 9.2. *There exist an open neighborhood N of K in C and a continuous map $P: N \cup \{0\} \to C$ which is continuously differentiable on N and satisfies*

$$P(\varphi) = F(z_2(\varphi) + 1, \varphi) \quad \text{for all} \quad \varphi \in K,$$

$$P(0) = 0,$$

$$DP(\varphi)\psi = p_\chi D_2 F(z_2(\varphi) + 1, \varphi)\psi \quad \text{for all} \quad \varphi \in K, \quad \psi \in C$$

where $\chi = D_1 F(z_2(\varphi) + 1, \varphi) 1$.

Proof. 1. Let $\varphi \in K$ be given. Consider the solution $x = x^{\varphi}$ of equation (μ, f), and the sequence $(z_j)_1^{\infty}$ of zeros of x in $\mathbb{R}^+$ (Proposition 6.3). Set $z := z(\varphi)$ (so that $\varphi = 0$ on $[-1, z]$, $0 < \varphi$ on $(z, 0]$). There exists $\varepsilon \in \left(0, \frac{1}{2}\right)$ so small that

$$z + \varepsilon < 0, \quad z + \varepsilon + 1 < z_1 - \varepsilon, \quad z_1 + 1 + \varepsilon < z_2 - \varepsilon.$$

Continuous dependence implies that there exists an open neighborhood N_φ of φ such that for each $\varphi' \in N_\varphi$ the solution $x' := x^{\varphi'}$ of equation (μ, f) satisfies

$$0 < x' \quad \text{on} \quad [z + \varepsilon, z_1 - \varepsilon], \quad x' < 0 \quad \text{on} \quad [z_1 + \varepsilon, z_2 - \varepsilon],$$
$$0 < x' \quad \text{on} \quad [z_2 + \varepsilon, z_2 + 1].$$

An application of Proposition 6.3 to the initial value $x'_{z+\varepsilon+1} \in S$, which is strictly positive, and the Intermediate Value Theorem show that there exist a unique zero z'_1 of x' in $(z_1 - \varepsilon, z_1 + \varepsilon)$, and a unique zero z'_2 of x' in $(z_2 - \varepsilon, z_2 + \varepsilon)$, and that $0 < x'$ on $[0, z'_1)$, $x' < 0$ on (z'_1, z'_2). In case $\varphi' \in N_\varphi \cap K$, z'_1 and z'_2 coincide with the first and second zero of x' in $\mathbb{R}^+$ as given by Proposition 6.3.

2. $F(z_2 + 1, \varphi) \in H$ and $\dot{x}(z_2) \neq 0$ (so that $D_1 F(z_2 + 1, \varphi)1 = \dot{x}_{z_2+1} \notin H$) imply that there are an open neighborhood $U_\varphi \subset N_\varphi$ of φ, $\operatorname{diam} U_\varphi \leqq 1$, and a C^1-map $\tau_\varphi : U_\varphi \to (1, \infty)$ with

$$\tau_\varphi(\varphi) = z_2 + 1, \quad \tau_\varphi(U_\varphi) \subset (z_2 + 1 - \varepsilon, z_2 + 1 + \varepsilon)$$

so that for all $\varphi' \in U_\varphi$,

$$F(\tau_\varphi(\varphi'), \varphi') \in H \qquad \text{(Remark 3.1)}$$

or equivalently,

$$x'(\tau_\varphi(\varphi') - 1) = 0 \quad \text{for the solution} \quad x' := x^{\varphi'} \quad \text{of equation } (\mu, f).$$

It follows that

$$\tau_\varphi(\varphi') - 1$$

coincides with the zero z'_2 constructed in part 1 above. In particular,

$$F(\tau_\varphi(\varphi'), \varphi') = F(z_2(\varphi') + 1, \varphi') \quad \text{for all} \quad \varphi' \in U_\varphi \cap K.$$

3. Set

$$N := \bigcup_{\varphi \in K} U_\varphi, \quad P(\varphi') := F(z'_2 + 1, \varphi') \quad \text{for} \quad \varphi' \in U_\varphi \quad \text{and} \quad \varphi \in K,$$
$$P(0) := 0.$$

Recall Remark 3.2.

4. Continuity at zero: Proposition 6.3 implies that the set of all zeros $z_2(\varphi)$ where $\varphi \in K$ and $\|\varphi\| \leqq 2$ is bounded by some constant $c > 0$.

diam $U_\varphi \leqq 1$ shows that each $\varphi' \in N$ with $\|\varphi'\| \leqq 1$ belongs to some set U_φ with $\|\varphi\| \leqq 2$. It follows that

$$z_2' < c + \frac{1}{2} \quad \text{for all such} \quad \varphi'.$$

Continuity of P at 0 is now a consequence of the fact that restrictions of solutions to the interval $\left[-1, c + \frac{1}{2} + 1\right]$ depend continuously on initial data. QED.

Proposition 9.3. *Y is connected, and P maps Y diffeomorphically onto Y.*

Proof. 1. $P(Y) = Y$ is an easy consequence of Corollary 6.2.

2. Connectedness: There are an open neighborhood U of 0 in C, a constant $r > 0$ and a C^1-diffeomorphism $h: (-r, r) \to X \cap U$, with $h(0) = 0$. Property (Z) in Corollary 6.2 implies

$$h(s)(0) > 0 \quad \text{if and only if} \quad h(s) \in K,$$
$$h(s)(0) < 0 \quad \text{if and only if} \quad h(s) \in -K.$$

The map $s \to \operatorname{sign} h(s)(0)$ is nonzero and constant on $(-r, 0)$, and on $(0, r)$ – otherwise, $h(s)(0) = 0$ for some $s \neq 0$, a contradiction to $0 \neq h(s) \in X$ and $X \subset \{0\} \cup K \cup (-K)$.

Corollary 6.2 and $x_t \to 0$ as $t \to -\infty$ for each solution $x: \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_0 \in W$ imply that $X \cap U$ contains elements of K, and of $-K$. It follows that either

$$h(s)(0) > 0 \quad \text{on} \quad (-r, 0) \quad \text{and} \quad h(s)(0) < 0 \quad \text{on} \quad (0, r),$$

or

$$h(s)(0) < 0 \quad \text{on} \quad (-r, 0) \quad \text{and} \quad h(s)(0) > 0 \quad \text{on} \quad (0, r).$$

Using $Y = X \cap K$, one infers

$$h((-r, 0)) = Y \cap U \quad \text{and} \quad h((0, r)) = X \cap U \cap (-K)$$

or

$$h((-r, 0)) = X \cap U \cap (-K) \quad \text{and} \quad h((0, r)) = Y \cap U.$$

Now let φ and φ' in Y be given. Consider the solutions $x: \mathbb{R} \to \mathbb{R}$ and $x': \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_0 = \varphi$, $x_0' = \varphi'$, and let z_j, z_j', for $j \in \mathbb{Z}$, denote their zeros, ordered according to Corollary 6.2. Then

$$z_0 = -1 = z_0'.$$

Because of $x_t \to 0$ and $x'_t \to 0$ as $t \to -\infty$ there exists an integer $n > 0$ so that $x_{z_{-2n+1}}$ and $x'_{z'_{-2n+1}}$ belong to $Y \cap U$. Connect both points by a continuous path in $Y \cap U$ and apply P^n. The resulting path connects φ in Y to φ'.

3. Injectivity of P: Let φ and φ' in Y be given. Consider the corresponding solutions x and x' as in part 2, and their ordered zeros z_j, z'_j. Let $y : \mathbb{R} \to \mathbb{R}$ denote the solution of equation (μ, f) with $y_0 = P(\varphi)$, and let $\underline{z}_j$, $j \in \mathbb{Z}$, denote the zeros of y, as in Corollary 6.2. Then $\underline{z}_0 = -1$. The equation

$$y = x(\cdot + z_2 + 1)$$

implies

$$y < 0 \text{ on } (z_1 - (z_2 + 1), -1), \quad 0 < y \text{ on } \big(-1 - (z_2 + 1), z_1 - (z_2 + 1)\big),$$
$$y(-1 - (z_2 + 1)) = 0.$$

It follows that

$$\underline{z}_{-2} = -1 - (z_2 + 1).$$

Analogously one finds for the solution $y' : \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $y'_0 = P(\varphi')$ and for the ordered zeros $\underline{z}'_j$, $j \in \mathbb{Z}$, of y' that

$$y' = x'(\cdot + z'_2 + 1) \quad \text{and} \quad \underline{z}'_{-2} = -1 - (z'_2 + 1).$$

In case $P\varphi = P\varphi'$, uniqueness (Remark 3.3) implies $y = y'$. Therefore $\underline{z}_{-2} = \underline{z}'_{-2}$, and consequently $z_2 = z'_2$. Hence $x = x'$; in particular, $\varphi = \varphi'$.

4. In view of $P(Y) = Y$ and $\dim Y = 1$, it remains to show that each map $DP(\varphi)|T_\varphi Y$, $\varphi \in Y$, is nonzero. Let $\varphi \in Y$ be given. Choose $\psi \in C$ such that $T_\varphi Y = \mathbb{R}\psi$. Recall $DP(\varphi)\psi = p_\chi D_2 F(z_2(\varphi) + 1, \varphi)\psi$ with $\chi = D_1 F(z_2(\varphi) + 1, \varphi)1$ (Proposition 9.2). The vectors $\psi \in T_\varphi Y \subset H$ and $\dot{\varphi} = \dot{x}_{z_0+1} \notin H$ (where $x : \mathbb{R} \to \mathbb{R}$ is the solution of equation (μ, f) with $x_0 = \varphi$, and $z_0 = -1$, see (9.1)) are linearly independent. $D_2 F(z_2(\varphi) + 1, \varphi)$ is injective (Remark 3.3) and maps $\dot{\varphi}$ onto χ. It follows that χ and $D_2 F(z_2(\varphi) + 1, \varphi)\psi$ are linearly independent; the projection p_χ parallel to χ onto H does not annihilate $D_2 F(z_2(\varphi) + 1, \varphi)\psi$, and one has

$$DP(\varphi)\psi \neq 0. \quad \text{QED.}$$

10. Attraction to a periodic orbit

The essential step towards Theorem 10.1 is to construct a homeomorphism from a compact interval onto cl Y. This requires suitable parametrizations of a piece of cl Y containing $0 \in C$, and of a piece of cl Y containing a point of bd$'$ W. The second one is more difficult to obtain as it is not a-priori known how Y and cl Y look close to bd$'$ W.

The construction begins down in L_w.

Choose $\chi \in \operatorname{bd} L_w$ with minimal norm:

$$\|\chi\| = \min_{\chi' \in \operatorname{bd}' L_w} \|\chi'\|.$$

$0 \in L_w$ implies $\|\chi\| > 0$ and

$$\{\chi' \in L : \|\chi'\| < \|\chi\|\} \subset L_w. \tag{10.1}$$

Let $x^{(s)} : \mathbb{R} \to \mathbb{R}$, $0 < s \leqq 1$, denote the solution of equation (μ, f) with

$$x_0^{(s)} = s\chi + \operatorname{cl} w(s\chi) \in \operatorname{cl} W.$$

It follows that all $x_t^{(1)}$ belong to $\operatorname{bd}' W$ while for $0 < s < 1$, $x_t^{(s)} \in W$ for all $t \in \mathbb{R}$ (with Proposition 8.1, (10.1) and Corollary 5.1).

The projections of these phase curves pass transversally through the ray $(0, \infty)\chi$. More precisely (and weaker),

Proposition 10.1. *There exist $s_1 \in (0,1)$ and $t_1 > 0$ with*

$$p x_t^{(s)} \notin \mathbb{R}\chi \quad \textit{for} \quad s_1 \leqq s \leqq 1 \quad \textit{and} \quad 0 < |t| \leqq t_1.$$

Proof. 1. By Proposition 8.1, $p\dot{x}_0^{(1)} \neq 0$. Proof of $p\dot{x}_0^{(1)} \notin \mathbb{R}\chi$: Assume $0 \neq p\dot{x}_0^{(1)} \in \mathbb{R}\chi$. $\mathbb{R}\chi$ intersects at χ transversally with the circle of radius $\|\chi\|$ and center 0. This fact, the assumption and (10.1) together give $p x_t^{(1)} \in L_w$ for some $t \neq 0$, a contradiction to $x_t^{(1)} \in \operatorname{bd}' W$.

2. Continuity of the map $s \to x_{-2}^{(s)}$ at $s = 1$: Let points $s_n \in (0, 1]$, $n \in \mathbb{N}$, be given with $s_n \to 1$ as $n \to \infty$. The points $\varphi_n := x_{-2}^{(s_n)}$ belong to a compact set; every subsequence $(\varphi_{n_k})_1^\infty =: (\varphi_k')_1^\infty$ has a further subsequence $(\varphi_{k_j}')_1^\infty$ which converges to some $\varphi' \in C$. By continuity,

$$F(2, \varphi') = \lim_{j \to \infty} F(2, \varphi_{k_j}') = \lim_{n \to \infty} x_0^{(s_n)} = x_0^{(1)}.$$

By injectivity,

$$\varphi' = x_{-2}^{(1)}.$$

This yields the assertion.

3. The relation

$$p D_1 F(2, x_{-2}^{(1)}) 1 = p\dot{x}_0^{(1)} \notin \mathbb{R}\chi,$$

part 2 and continuity of $D_1 F$ on $(1, \infty) \times C$ imply that there exist $s_1 \in (0, 1)$, $t_1 > 0$ so that for $s_1 \leqq s \leqq 1$ and $|t| \leqq t_1$,

$$\|p D_1 F(2 + t, x_{-2}^{(s)}) 1 - p D_1 F(2, x_{-2}^{(1)}) 1\| < \frac{1}{2} \operatorname{dist}(\mathbb{R}\chi, p D_1 F(2, x_{-2}^{(1)}) 1).$$

For such s and for $0 < |t| \leqq t_1$,

$$px_t^{(s)} - px_0^{(s)} = pF(2+t, x_{-2}^{(s)}) - pF(2, x_{-2}^{(s)})$$

$$= \int_0^t pD_1 F(2+t', x_{-2}^{(s)})1\,dt'$$

$$= tpD_1 F(2, x_{-2}^{(1)})1 + t\psi$$

where

$$\|\psi\| \leqq \frac{1}{2} \operatorname{dist}\left(\mathbb{R}\chi, pD_1 F(2, x_{-2}^{(1)})1\right),$$

so that the last sum does not belong to $\mathbb{R}\chi$. Using $px_0^{(s)} \in \mathbb{R}\chi$, one obtains $px_t^{(s)} \notin \mathbb{R}\chi$. QED.

The initial values $x_0^{(s)}$, $s_1 \leqq s \leqq 1$, in the graph cl W above $[s_1, 1]\chi \subset L$ may not belong to cl $Y \subset H$. Therefore one follows the semiflow until it reaches the hyperplane H: Choose a zero $z > 0$ of $x^{(1)}$ with $0 < \dot{x}^{(1)}(z)$ and $x_{z+1}^{(1)} \in K \subset H$ (Proposition 8.1). In particular,

$$(D_1 F(z+1, x_0^{(1)})1)(-1) = \dot{x}_{z+1}^{(1)}(-1)$$

$$= \dot{x}^{(1)}(z)$$

$$\neq 0,$$

and

$$D_1 F(z+1, x_0^{(1)})1 \notin H.$$

Using the Implicit Function Theorem one obtains an open neighborhood U^* of $x_0^{(1)}$ in C and a C^1-map $\tau^* : U^* \to (1, \infty)$ with

$$\tau^*(x_0^{(1)}) = z + 1,$$

$$\tau^*(U^*) \subset \left(z + 1 - \frac{1}{2}t_1, z + 1 + \frac{1}{2}t_1\right),$$

$$F(\tau^*(\varphi), \varphi) \in H \quad \text{and} \quad F(\tau^*(\varphi), \varphi)(0) > 0 \quad \text{for all} \quad \varphi \in U^*.$$

(The last inequality is achieved by $x_{z+1}^{(1)} \in K$; $x_{z+1}^{(1)}(0) > 0$.) Choose $s_2 \in (s_1, 1)$ with $x_0^{(s)} \in U^*$ for $s_2 \leqq s \leqq 1$. The desired parametrization is the continuous map

$$\sigma : s \in [s_2, 1] \to F\big(\tau^*(s\chi + \text{cl}\, w(s\chi)), s\chi + \text{cl}\, w(s\chi)\big) \in C.$$

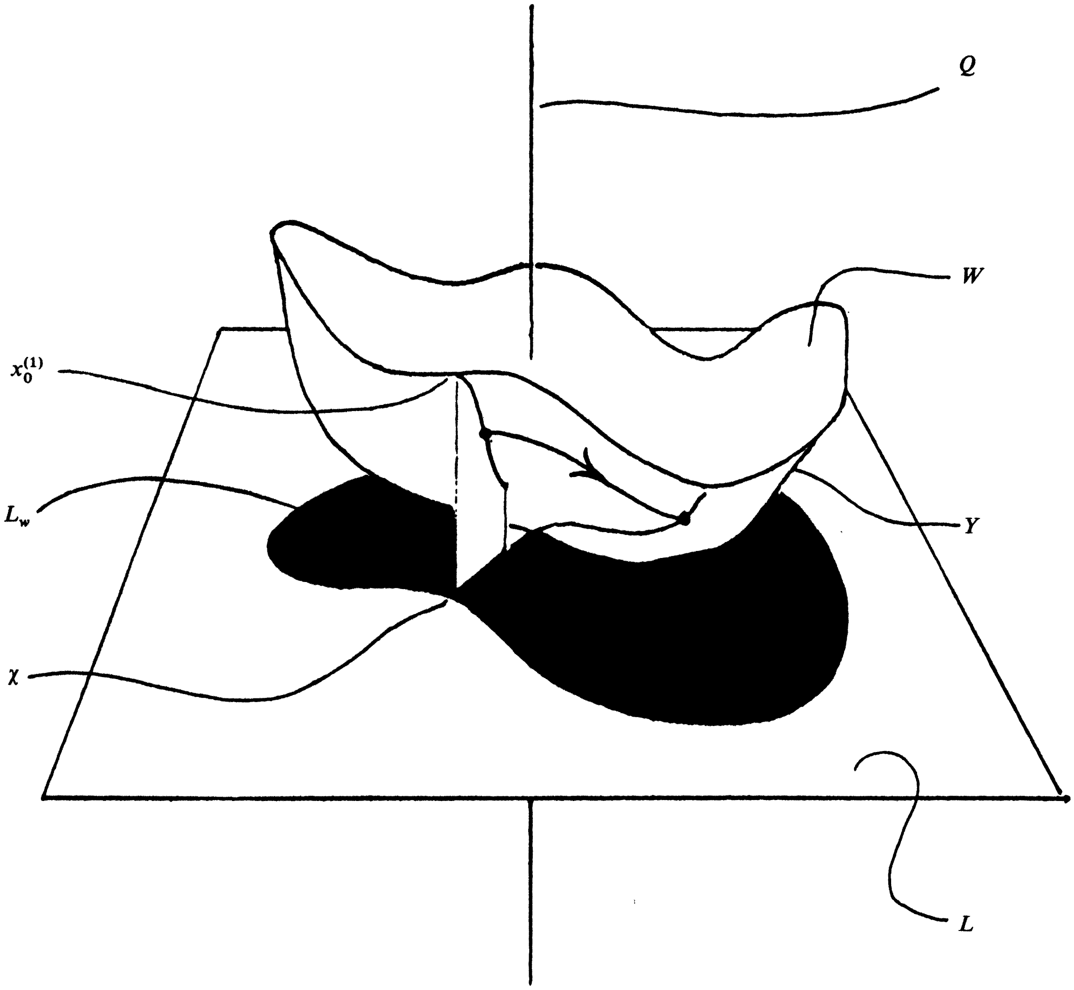

Figure 10.1

Note

$$\sigma(1) = x_{z+1}^{(1)} \in \mathrm{bd}' \, W \cap K \tag{10.2}$$

and

$$\sigma(s) \in W \cap K = Y \quad \text{for} \quad s_2 \leqq s < 1 \tag{10.3}$$

(recall $\sigma(s) = F\big(\tau^*(s\chi + \mathrm{cl}\, w(s\chi)), s\chi + \mathrm{cl}\, w(s\chi)\big) \in H$, $\sigma(s)(0) = F(\ldots,\ldots)(0) > 0$, and property (Z) for $x^{(s)}$). In particular,

$$\sigma(1) \in \mathrm{cl}\, Y \setminus W \subset \mathrm{cl}\, Y \setminus Y, \tag{10.4}$$

$$\sigma(1) \neq 0. \tag{10.5}$$

It is convenient to set

$$\Sigma := [s_2, 1].$$

Proposition 10.2. *σ is injective and maps* int Σ *homeomorphically onto an open subset of Y.*

Proof. 1. Injectivity: Suppose $\sigma(s) = \sigma(s')$. With $\varphi := x_0^{(s)}$ and $\varphi' := x_0^{(s')}$, $F(\tau^*(\varphi), \varphi) = F(\tau^*(\varphi'), \varphi')$. Without loss of generality, $\tau^*(\varphi) \leqq \tau^*(\varphi')$. By Remark 3.3,

$$\varphi = F(\tau^*(\varphi') - \tau^*(\varphi), \varphi'). \tag{10.6}$$

Assume $\tau^*(\varphi') > \tau^*(\varphi)$. Then $0 < \tau^*(\varphi') - \tau^*(\varphi) < t_1$, and Proposition 10.1 implies $pF(\tau^*(\varphi') - \tau^*(\varphi), \varphi') \notin \mathbb{R}\chi$, a contradiction to (10.6) and $p\varphi \in \mathbb{R}\chi$. $\tau^*(\varphi') = \tau^*(\varphi)$ gives $\varphi = \varphi'$, hence $s = s'$.

2. Continuity of σ, $\sigma(\text{int}\,\Sigma) \subset Y$, $\dim Y = 1$ and injectivity imply the remaining part of the assertion. QED.

Remark. One can also show that all $D\sigma(s)$, $s \in \text{int}\,\Sigma$, are nontrivial – which implies that σ maps int Σ diffeomorphically onto an open subset of Y.

Proposition 10.3. *We have*

$$\text{cl}\, Y = Y \cup \{0, \sigma(1)\},$$

and there exists a homeomorphism $h : [0, 1] \to \text{cl}\, Y$ with

$$h(0) = 0, \quad h((0, 1)) = Y, \quad h(1) = \sigma(1) = x_{z+1}^{(1)} \in \text{bd}'_{-} W.$$

Proof. 1. Y is a one-dimensional C^1-submanifold of C without boundary, which is not compact $\left(\text{as the points } \sigma\left(1 - \frac{1}{n}\right),\ n \in \mathbb{N} \text{ large, converge to } \sigma(1) \notin Y\right)$. It follows that there exists a homeomorphism h_0, defined on an open interval $(a, b) \subset \mathbb{R}$, onto Y (see e.g. [30], Ch. VI, 23.19). We shall show that h_0 has limits 0 at a and $\sigma(1)$ at b (or vice versa); this will imply the assertion. The proof of Proposition 9.3 shows that there are an open neighborhood U of 0 in C, a constant $r_1 > 0$ and a continuous map $h_1 : [-r_1, r_1) \to X$ which defines a C^1-diffeomorphism of $(-r_1, r_1)$ onto $X \cap U$, with $h_1(-r_1) \in Y$, $h_1(0) = 0$ and $h_1((-r_1, 0)) = Y \cap U$.

Consider the open subinterval $(c, d) := h_0^{-1}(h_1((-r_1, 0)))$ of (a, b).

2. Exclusion of the case $a < c < d < b$: The values $h_0\left(c + \frac{1}{n}\right)$ and $h_0\left(d - \frac{1}{n}\right)$ in $h_1((-r_1, 0))$, for $n \in \mathbb{N}$ sufficiently large, converge to $h_0(c) \in Y \backslash h_1((-r_1, 0))$ and to $h_0(d) \in Y \backslash h_1((-r_1, 0))$, respectively. By $0 \notin Y$, $h_0(c) \neq 0 \neq h_0(d)$. Both $h_0(c)$ and $h_0(d)$ belong to $\text{cl}\, h_1((-r_1, 0)) = \{h_1(-r_1)\} \cup h_1((-r_1, 0)) \cup \{0\}$. We obtain

$$h_0(c) = h_1(-r_1) = h_0(d),$$

a contradiction.

3. Part 2 and $h_1((-r_1, 0)) \subsetneqq Y$ imply that either $a = c$ and $d < b$, or $a < c$ and $d = b$. Assume the last case. (The proof in the other case is analogous.)

4. Proof of $\lim_{t \to b} h_0(t) = 0$: The map $h^* := h_1^{-1} \circ (h_0 | (c, b))$ is a homeomorphism onto $(-r_1, 0)$. Monotonicity implies that, given a sequence (t_n) in (c, b) with $t_n \to b$, we have $h^*(t_n) \to -r_1$ as $n \to \infty$, or $h^*(t_n) \to 0$ as $n \to \infty$. In the first case,

$$h_0(t_n) = h_1(h^*(t_n)) \to h_1(-r_1) \in Y,$$

therefore

$$t_n \to h_0^{-1}(h_1(-r_1)) \in h_0^{-1}(Y) \subset (a, b),$$

a contradiction. Hence

$$h_0(t_n) = h_1(h^*(t_n)) \to h_1(0) = 0.$$

5. Consider the open subinterval $(k_-, k) := h_0^{-1}(\sigma(\operatorname{int} \Sigma))$ of (a, b). We show $k_- = a$ and $k < b$: In case $a < k_-$, we have $h_0(k_-) \in Y$, $h_0(k_-) \notin \sigma(\operatorname{int} \Sigma)$, $h_0(k_-) \in \operatorname{cl} \sigma(\operatorname{int} \Sigma)$, and therefore $h_0(k_-) = \sigma(s_2)$.

Similarly, $k < b$ leads to $h_0(k) = \sigma(s_2)$. Injectivity of h_0 now implies that either $a < k_-$ and $k = b$, or $a = k_-$ and $k < b$. In the first case, part 4 yields $0 \in \operatorname{cl} \sigma(\operatorname{int} \Sigma) \subset \sigma(\Sigma)$, a contradiction.

6. Proof of $\lim_{t \to a} h_0(t) = \sigma(1)$: Consider a sequence (t_n) in (a, b), $t_n \to a$ as $n \to \infty$. Compactness of $\operatorname{cl} W \supset h_0((a, b))$ implies that there is a convergent subsequence $(h_0(t_{n_k}))$. Let l denote its limit; $l \in \operatorname{cl} Y$. We have to show $l = \sigma(1)$. Note $l \notin Y$ (otherwise, $t_{n_k} = h_0^{-1}(h_0(t_{n_k})) \to h_0^{-1}(l) \in (a, b)$, a contradiction). Part 5 implies $h_0(t_{n_k}) \in \sigma(\operatorname{int} \Sigma)$ for all sufficiently large k. Therefore

$$l \in \operatorname{cl} \sigma(\operatorname{int} \Sigma) \subset Y \cup \{\sigma(1)\}. \qquad \text{QED.}$$

Note $h(1) = x_{z+1}^{(1)} \in K$. Hence $\operatorname{cl} Y \subset K \cup \{0\}$. Continuity of P and $P(Y) = Y$ (Proposition 9.3) give $P(\operatorname{cl} Y) \subset \operatorname{cl} Y$. One obtains a continuous map

$$P_h : \xi \in [0,1] \to h^{-1}(P(h(\xi))) \in [0,1].$$

Proposition 10.4. *P_h is strictly monotonic increasing and satisfies*

$$P_h(0) = 0, \quad \xi < P_h(\xi) \quad \textit{for} \quad 0 < \xi < 1, \quad P_h(1) = 1.$$

For every $\xi \in (0,1)$, the iterates $P_h^n(\xi)$, $n \in \mathbb{N}$, are strictly increasing and converge to 1 as $n \to \infty$.

Proof. 1. $P_h(0) = 0$ is obvious. Propositions 9.3 and 10.3 guarantee that P_h is a bijection from $(0,1)$ onto $(0,1)$. For $\xi = 1$, one has

$$P(h(1)) = P(x_{z+1}^{(1)}) = F(\ldots, x_{z+1}^{(1)}) \in \operatorname{bd}' W;$$

in particular,

$$h(0) = 0 \neq P(h(1)) \notin Y = h((0,1)).$$

Using $\mathrm{cl}\, Y \setminus Y = \{0, h(1)\}$, one finds

$$P(h(1)) = h(1), \quad \text{or} \quad P_h(1) = 1.$$

It follows also that P_h is bijective. Continuity and $P_h(1) = 1 > 0 = P_h(0)$ imply that P_h is strictly monotonic increasing.

2. For $\xi \in (0,1)$, let $x: \mathbb{R} \to \mathbb{R}$ denote the solution of equation (μ, f) with

$$x_0 = h(\xi) \in Y \subset K.$$

The monotonicity property of P_h implies that the sequence $(P_h^n(\xi))_0^\infty$ is strictly increasing, constant or strictly decreasing. Let $z_j, j \in \mathbb{Z}$, denote the zeros of x, as in Corollary 6.2. Then

$$Y = W \cap K \ni x_{z_{2j}+1} \to 0 \quad \text{as} \quad j \to -\infty,$$

and for some $j \in -\mathbb{N}$,

$$\xi' := h^{-1}(x_{z_{2j}+1}) \in (0, \xi).$$

It follows that

$$\begin{aligned} P^{-j}(x_{z_{2j}+1}) &= x_0, \\ P_h^{-j}(\xi') &= \xi > \xi', \\ P_h^{-j}(\xi) &> \xi \end{aligned}$$

which excludes that the sequence $(P_h^n(\xi))_0^\infty$ is constant or decreasing. In particular,

$$\xi < P_h(\xi),$$

and $(P_h^n(\xi))_0^\infty$ converges to some $\xi'' \in (0,1]$ which is a fixed point of P_h – i.e., $\xi'' = 1$. QED.

Theorem 10.1. *There exists a periodic solution $y: \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with*

$$\mathrm{bd}' W = \{y_t : t \in \mathbb{R}\};$$

y_0 belongs to K, and the minimal period of y is $z_2(y_0) + 1$. For every solution $x: \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $0 \neq x_0 \in W$,

$$\mathrm{dist}(x_t, \mathrm{bd}' W) \to 0 \quad \textit{as} \quad t \to \infty.$$

Proof. 1. $P_h(1) = 1$ implies that the solution $y: \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with

$$y_0 = h(1) \in \mathrm{bd}' W \cap K$$

satisfies

$$F(z_2(y_0)+1, y_0) = P(h(1)) = h(1) = y_0.$$

By Remark 3.3,

$$y = y(\cdot + z_2(y_0)+1) \qquad \text{(periodicity).}$$

Proposition 8.1 gives

$$y_t \in \mathrm{bd}' W \quad \text{for all} \quad t.$$

2. Proof of $\mathrm{bd}' W \subset \{y_t : t \in \mathbb{R}\}$: Let $\varphi \in \mathrm{bd}' W$ be given. The solution $x : \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $x_0 = \varphi$ has a zero $z_n > 1$ with $\dot{x}(z_n) > 0$ and $x(z_n + 1) > 0$. Remark 3.1 yields an open neighborhood U of φ and a C^1-map $\tau : U \to (1, \infty)$ with

$$\tau(\varphi) = z_n + 1,$$
$$F(\tau(\psi), \psi) \in H \quad \text{and} \quad F(\tau(\psi), \psi)(0) > 0 \quad \text{for all} \quad \psi \in U.$$

There exists a sequence of points $\chi_j \in W \cap U$, $j \in \mathbb{N}$, with

$$\chi_j \to \varphi \quad \text{as} \quad j \to \infty.$$

The points $F(\tau(\chi_j), \chi_j)$, $j \in \mathbb{N}$, belong to $W \cap K = Y$ and converge to $F(\tau(\varphi), \varphi) = x_{z_n+1}$ as $j \to \infty$. Therefore $x_{z_n+1} \in \mathrm{cl}\, Y$. $\varphi \in \mathrm{bd}' W$ implies $x_{z_n+1} \in \mathrm{bd}' W$, so that $x_{z_n+1} \notin Y (\subset W)$. By $\mathrm{cl}\, Y \setminus Y = \{0, h(1)\}$ (and by $0 \notin \mathrm{bd}' W$),

$$x_{z_n+1} = h(1) = y_0,$$

and Remark 3.3 gives

$$\varphi = y_{-z_n-1}.$$

3. Attraction to $\mathrm{bd}' W$. Let a solution $x : \mathbb{R} \to \mathbb{R}$ of equation (μ, f) with $0 \neq x_0 \in W$ be given. For some zero z_n of x,

$$x_{z_n+1} \in W \cap K = Y.$$

It follows that

$$x_{z_n+2j+1} = P^j(x_{z_n+1}) \to y_0 \quad \text{as} \quad j \to \infty$$

(with Proposition 10.4). Boundedness of W and Proposition 6.3 yield a constant $c > 0$ such that

$$z_2(P^j(x_{z_n+1})) + 1 \leqq c \quad \text{for all} \quad j \in \mathbb{N}_0.$$

Let $\varepsilon > 0$ be given. Continuous dependence on initial data permits to find $j_\varepsilon \in \mathbb{N}$ such that for all integers $j \geqq j_\varepsilon$ and for all $t \in [0, c]$,

$$\|F(t, P^j(x_{z_n+1})) - F(t, y_0)\| \leqq \varepsilon.$$

Consider $t \geqq z_{n+2j_\varepsilon} + 1$. For some $j \geqq j_\varepsilon$,

$$\begin{aligned} z_{n+2j} + 1 \leqq t &\leqq z_{n+2(j+1)} + 1 \\ &= z_{n+2j} + 1 + z_2\big(P^j(x_{z_n+1})\big) + 1 \\ &\leqq z_{n+2j} + 1 + c\,. \end{aligned}$$

Hence

$$\begin{aligned} \operatorname{dist}(x_t, \mathrm{bd}'\, W) &\leqq \| x_t - y_{t-(z_{n+2j}+1)} \| \\ &= \| F\big(t - (z_{n+2j} + 1), x_{z_{n+2j}+1}\big) - F(\ldots, y_0) \| \\ &= \| F\big(\ldots, P^j(x_{z_n+1})\big) - F(\ldots, y_0) \| \\ &\leqq \varepsilon\,. \qquad \text{QED.} \end{aligned}$$

Remark. A proof of additional stability properties for the periodic orbit in bd′ W, such as exponential attraction, or the existence of an asymptotic phase, seems not possible without more hypotheses on f. Compare for example the result in [5].

Remark. All results proven here have analogues for nonlinearities which satisfy the variant of (H1) where "bounded from above" is replaced by "bounded from below".

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