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Root Graded Groups

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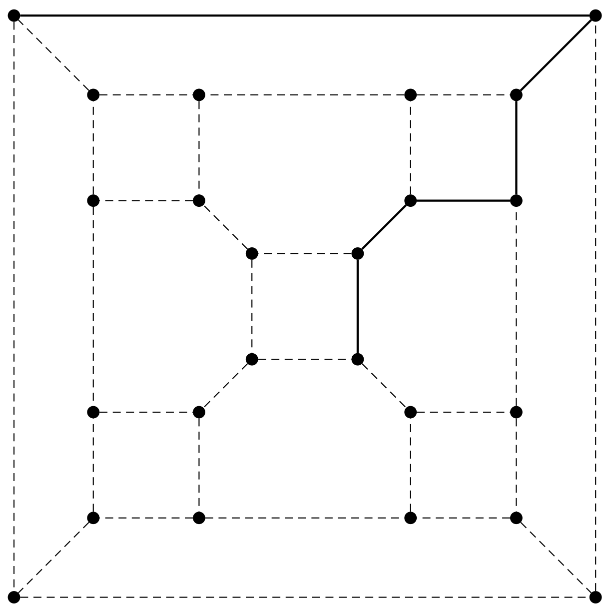

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Abstract

We define and study *root graded groups*, that is, groups graded by finite root systems. These provide a uniform framework to investigate several existing concepts in the literature, including in particular Jacques Tits' notion of RGD-systems. The most prominent examples of root graded groups are Chevalley groups over commutative associative rings. Our main result is that every root graded group of rank at least 3 is coordinatised by some algebraic structure such that a variation of the Chevalley commutator formula is satisfied. This result can be regarded as a generalisation of Tits' classification of thick irreducible spherical buildings of rank at least 3 to the case of non-division algebraic structures.

All coordinatisation results in this thesis are proven in a characteristic-free way. This is made possible by a new computational method that we call the *blueprint technique*.

Deutsche Zusammenfassung

Wir definieren und untersuchen Wurzel-graduierte Gruppen, das heißt, Gruppen mit einer Graduierung durch ein endliches Wurzelsystem. Diese liefern einen einheitlichen Rahmen, um verschiedene Konzepte aus der Literatur zu studieren, insbesondere Jacques Tits' Begriff von RGD-Systemen. Das bekannteste Beispiel von Wurzel-graduierten Gruppen sind Chevalley-Gruppen über kommutativen assoziativen Ringen. Unser Hauptresultat besagt, dass alle Wurzel-graduierten Gruppen vom Rang mindestens 3 durch eine algebraische Struktur koordinatisiert sind, sodass eine Abwandlung der Chevalley-Kommutatorformel erfüllt ist. Dieses Resultat lässt sich als Verallgemeinerung von Tits' Klassifikation der dicken irreduziblen sphärischen Gebäude vom Rang mindestens 3 auf den Fall von algebraischen Strukturen ohne Division interpretieren.

Alle Koordinatisierungsresultate in dieser Thesis werden in einer charakteristik-freien Weise bewiesen. Dies ist möglich durch eine neue Technik, die wir die *Blueprint-Methode* nennen.

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Contents

Preface

Let Φ be a finite irreducible root system. A Φ-*grading* of a group G is a family of non-trivial subgroups $(U_\alpha)_{\alpha\in\Phi}$ generating G, called the *root groups of* G, such that some commutator relations and a non-degeneracy condition are satisfied and such that so-called Weyl elements exist for each root. If the root system Φ is not specified, we will also call these objects *root gradings*, and the pair $(G, (U_\alpha)_{\alpha\in\Phi})$ is called a Φ-*graded group* or *root graded group*. Our definition of Φ-gradings makes sense for any finite root system, crystallographic or not, but all the central results of this thesis concern mainly the crystallographic root systems. We will briefly remark on the situation in the non-crystallographic case in the last part of this preface. Except for this detour, all root systems in the preface are assumed to be crystallographic.

Motivation and Examples: Chevalley Groups

The main examples of root graded groups are the Chevalley groups, which were introduced by Chevalley in his famous Tohoku paper [Che55]. By construction, every Chevalley group G is defined over a commutative associative unital ring $\mathcal{R}$: It is a matrix group with coefficients in $\mathcal{R}$, and for each root group U_α there exists a canonical isomorphism $\theta_\alpha\colon (\mathcal{R},+) \to U_\alpha$. Further, the celebrated *Chevalley commutator formula* asserts that

$$[\theta_\alpha(a), \theta_\beta(b)] = \prod_{\substack{i,j\geq 1\\ i\alpha+j\beta\in\Phi}} \theta_{i\alpha+j\beta}\big(c_{\alpha\beta ij}a^i b^j\big)$$

for all non-proportional roots α, β and all $a, b \in \mathcal{R}$ where $c_{\alpha\beta ij}$ are integral constants which do not depend on $\mathcal{R}$, a and b. In other words, commutators in G are described by the ring multiplication. For any root α, an element $w_\alpha \in U_{-\alpha}U_\alpha U_{-\alpha}$ is called an α-*Weyl element* if it is a "lift" of the reflection σ_α in the Weyl group of Φ to the group G. By this we mean that $U_\beta^{w_\alpha} = U_{\sigma_\alpha(\beta)}$ for all roots β. It turns out that an element $w_\alpha \in G$ is an α-Weyl element if and only if it has the form $w_\alpha = \theta_{-\alpha}(-a^{-1})\theta_\alpha(a)\theta_{-\alpha}(-a^{-1})$ where a is an invertible element of $\mathcal{R}$. In particular, α-Weyl elements exist because there exists at least one invertible element in $\mathcal{R}$, namely $1_{\mathcal{R}}$. We conclude that the group-theoretic structure of a Chevalley group G is intricately connected with the algebraic structure of the ring $\mathcal{R}$ over which it is defined.

On the other hand, the definition of root graded groups is purely combinatorial: it does not refer to any underlying algebraic structure. In fact, the axioms of a root grading can be seen as a list of the most important properties of Chevalley groups which can be formulated without reference to the underlying ring $\mathcal{R}$.

Goals

The Coordinatisation Problem

It is now a natural question whether the combinatorial axioms of a root graded group are enough to construct (or, in the special case of Chevalley groups, reconstruct) a ring "over which it is defined". To make this precise, we say that a root graded group $(G, (U_\alpha)_{\alpha \in \Phi})$ is *coordinatised by the ring* $\mathcal{R}$ if there exist isomorphisms $(\theta_\alpha \colon (\mathcal{R}, +) \to U_\alpha)_{\alpha \in \Phi}$ such that the Chevalley commutator formula holds. Thus the question is whether each root graded group is coordinatised by some commutative associative unital ring $\mathcal{R}$. While this question is a step in the right direction, it is too naively posed to admit a positive answer. We will explain why in the following.

First of all, the theory of root graded groups seems to be very unruly without additional assumptions in the lower-rank cases. For this reason, we will largely restrict ourselves to the case that Φ is of rank at least 3.

Secondly, we cannot hope for the ring to be commutative in all cases because, for example, (a variation of) the construction of Chevalley groups of type A_n works for non-commutative rings. There are even constructions of A_2- and C_3-graded groups from so-called alternative rings, which form a class of nonassociative rings.

Thirdly, even if we allow a more general class of rings, we make the following observation: If G is coordinatised by a single ring $\mathcal{R}$, then in particular, all root groups must be pairwise isomorphic. However, we will see that the axioms of a root grading merely imply that U_α is isomorphic to U_β if α and β are conjugate under the Weyl group of Φ. Thus in general, we have to allow the distinct orbits of root groups to be coordinatised by distinct algebraic structures. For example, we can construct B_n-graded groups in which the long root groups are coordinatised by a commutative associative unital ring $\mathcal{k}$ and the short root groups are coordinatised by a quadratic $\mathcal{k}$-module (M, q) (that is, a $\mathcal{k}$-module M with a quadratic form $q \colon M \to \mathcal{k}$). The commutator formulas in these groups look exactly like the Chevalley commutator formula, except that terms a^2 for $a \in M$ are replaced by $q(a)$ and terms $2ab$ for $a, b \in M$ are replaced by $f(a, b)$ where f denotes the linearisation of q.

In light of the previous paragraph, we make the following definition where $\mathbb{O}$ denotes the set of orbits in Φ: A *coordinatisation of* $(G, (U_\alpha)_{\alpha \in \Phi})$ consists of a family of (usually abelian) groups $(M_O)_{O \in \mathbb{O}}$ called the *coordinatising groups* or *coordinatising structures*, a family of maps between the coordinatising groups $(M_O)_{O \in \mathbb{O}}$ which "equips $(M_O)_{O \in \mathbb{O}}$ with an algebraic structure" and a family of group isomorphisms $\theta_\alpha \colon M_O \to U_\alpha$ for $O \in \mathbb{O}$ and $\alpha \in O$. We further require that a generalised version of the Chevalley commutator formula (involving the maps between $(M_O)_{O \in \mathbb{O}}$ and the isomorphisms $(\theta_\alpha)_{\alpha \in \Phi}$) is satisfied. For example, in the case of B_n-graded groups, the coordinatising structures are abelian groups $(\mathcal{k}, +)$ and $(M, +)$ and we have a map $\mathcal{k} \times \mathcal{k} \to \mathcal{k}$ which turns $\mathcal{k}$ into a commutative associative unital ring, a map $\mathcal{k} \times M \to M$ which turns M into a $\mathcal{k}$-module and a map $M \to \mathcal{k}$ which is a quadratic form on M. The connection between Weyl elements in G and invertible elements in the algebraic structure that we have seen earlier for Chevalley groups remains valid in this more general setting, though of course the precise meaning of "invertibility" depends on the specific algebraic structure. (For example, an element v in a quadratic module (M, q) is called "invertible" if $q(v)$ is invertible, and its "inverse" is then defined to be $q(v)^{-1}v$.)

In particular, the existence of Weyl elements in G yields that there exists at least one "invertible" element in each algebraic structure that we encounter. For this reason, all rings in this thesis are unital.

With these refinements in mind, we can now formulate the main goal of this work: For each irreducible crystallographic root system Φ of rank at least 3, show that there exists a class $\mathcal{C}_\Phi$ of (families of) algebraic objects such that each Φ-graded group is coordinatised by an object in $\mathcal{C}_\Phi$. This is the *coordinatisation problem*. We will discuss in a moment which results in this direction have already been obtained in the literature. In this thesis, we present a complete solution to the coordinatisation problem which does not rely on any previous work on abstract root gradings. We emphasise that our results are completely characteristic-free, whereas some previous works exclude the case of "characteristic 2".

Apart from elementary undergraduate algebra, the list of mathematical prerequisites that we need is short: We will use the basic theory of finite reflection groups and some results on Chevalley groups, though the latter could be replaced by a sequence of long but easy computations in a purely combinatorial setting. Further, we will cite a few elementary results on nonassociative rings without proof. We will also encounter several interesting algebraic structures, such as quadratic modules, composition algebras and involutions on rings, but we will derive all their basic properties in this text.

We briefly summarise the specific coordinatisation results that we obtain for each root system. For all $n \in \mathbb{N}_{\geq 2}$, we show that each A_n-graded group is coordinatised by a (possibly nonassociative noncommutative) unital ring which must be associative if $n \geq 3$ (Theorem 5.7.14). Now let $n \in \mathbb{N}_{\geq 3}$. Root graded groups of types D_n or E_n are coordinatised by commutative unital rings (again Theorem 5.7.14). Root graded groups of type B_n are coordinatised by pairs $(M, \Bbbk)$ where $\Bbbk$ is a commutative associative unital ring and M is a quadratic $\Bbbk$-module (Theorem 7.11.21). Root graded groups of type F_4 are coordinatised by pairs $(\mathcal{A}, \Bbbk)$ where $\Bbbk$ is a commutative associative unital ring and $\mathcal{A}$ is a multiplicative conic alternative algebra over $\Bbbk$ in the sense of [GPR, Sections 16, 17] (Theorem 10.7.8). The latter objects should be regarded as a generalisation of composition algebras. If $\Bbbk$ is a field of characteristic not 2 and a certain norm function on $\mathcal{A}$ is non-degenerate, then $\mathcal{A}$ is a composition algebra in the classical sense.

The results for the root systems C_n and BC_n are more technical. We will show that every root graded group of type C_n or BC_n is coordinatised by a pair $(J, \mathcal{R})$ where $\mathcal{R}$ is an alternative ring with nuclear involution and M is a Jordan module over $(\mathcal{R}, \sigma)$ which must be abelian if the root system is C_n (Theorem 9.10.26). The notion of Jordan modules is new and specifically tailored to be exactly the kind of algebraic structure which coordinatises $(B)C_n$-graded groups. As important special cases, the class of abelian Jordan modules contains the class of involutory sets which are known from the classification of Moufang quadrangles while the class of non-abelian Jordan modules contains the groups T which are constructed in [TW02, (11.24)] from pseudo-quadratic modules over $\mathcal{R}$. Here an involutory set over an alternative ring $\mathcal{R}$ with nuclear involution σ is a certain subgroup of the set of fixed points of σ. If $2_{\mathcal{R}}$ is invertible, then we can prove the following statement in a purely algebraic manner: Every abelian Jordan module over $\mathcal{R}$ is the direct product of an involutory set with an $\mathcal{R}$-module and every non-abelian Jordan module can be constructed from a pseudo-quadratic module over $\mathcal{R}$ as in [TW02, (11.24)].

The Existence Problem

As a second goal, it is of course desirable to show that the class $\mathcal{C}_\Phi$ in our solution of the coordinatisation problem is chosen optimally. In other words, for every object X in $\mathcal{C}_\Phi$ there should exist a Φ-graded group which is coordinatised by X. This is the *existence problem* or *construction problem*. In most cases, the objects in $\mathcal{C}_\Phi$ are related to associative rings and we can easily construct Φ-graded groups from these objects by writing down some (generalised) matrices. However, the classes $\mathcal{C}_\Phi$ for $\Phi \in \{ C_3, BC_3, F_4 \}$ involve alternative rings. For the subclass of $\mathcal{C}_\Phi$ consisting only of the "associative objects", a construction in terms of matrices is usually still possible, but the general construction problem is much more difficult.

For F_4-graded groups, no general construction is known. However, if the desired coordinatising ring $\mathcal{R}$ (which is equipped with the structure of a multiplicative conic algebra) is not only associative but also commutative, then we know that there exists an E_6-graded group G which is coordinatised by $\mathcal{R}$ (for example, a Chevalley group of type E_6). Using the general mechanism of *foldings of root graded groups*, we obtain an F_4-graded group which is coordinatised by the conic algebra $\mathcal{R}$. This provides a partial solution of the existence problem for this special case.

Now consider BC_3-graded groups. There are strong indications that, while alternative rings $\mathcal{R}$ can technically appear in this setting, only "the associative part of $\mathcal{R}$ is relevant" and thus we can restrict ourselves to the case of associative rings with good conscience. In this situation, we provide a construction which works for all objects in $\mathcal{C}_{BC_3}$ for which 2 is invertible (and also for a large subclass of objects in which 2 is not invertible).

For C_3-graded groups, Zhang provides a strategy to solve the existence problem for a certain subclass of $\mathcal{C}_{C_3}$ in [Zha14, Section 4.2]. We are confident that his construction can be adapted to the general case, and we plan to address this in future work.

The class $\mathcal{C}_{A_2}$ involves non-associative rings as well. By a construction of Faulkner (see [Fau83, Section 3] and the appendix of [Fau89]), this class is known to contain all alternative rings, but it is unclear whether it contains more general rings. Just like Zhang's construction of C_3-graded groups, Faulkner's construction realises the desired group as a group of automorphism of the Tits-Kantor-Koecher algebra of a certain Jordan pair.

Root gradings, RGD-systems and Moufang Buildings

We now turn to the history of the subject. As already said, Chevalley groups were introduced [Che55]. It should be noted, however, that the list of Chevalley groups contains several classical groups which had been studied before, such as the special linear group over a field. These are the earliest examples of root graded groups that have been considered, albeit without the abstract combinatorial framework of root gradings.

The first and most important example of an abstract structure that is similar to (but less general than) root gradings is Tits' notion of *RGD-systems*[1]. These

[1]The letters "RGD" stand for "Root Groups Data". Tits remarks in [Tit92, p. 258] that he chose this name "for lack of a better idea (or rather, because all appropriate names that I can think of seem to be already taken!)". It goes without saying that the same could be said about the name "root grading", which carries essentially the same information as "root group data".

objects were introduced under this name in [Tit92, p. 258], but predecessors of the RGD-axioms can be found as early as in [Tit62, p. 140]. A definition which is very similar to the one in [Tit92] is the one of *données radicielles* in [BT72, (6.1.1)]. Further, RGD-systems are essentially the same thing as the *groups with Steinberg relations* in [Fau77].

The crucial property that differentiates RGD-systems from root gradings is that in an RGD-system, *every* non-trivial root group element $b_\alpha \in U_\alpha \setminus \{1_G\}$ can be "extended" to a Weyl element $w_\alpha = a_{-\alpha} b_\alpha c_{-\alpha}$. In a root grading, we only require that *some* element $b_\alpha \in U_\alpha \setminus \{1_G\}$ has this property. Thus by the correspondence between Weyl elements and invertible elements, any RGD-system that is coordinatised by an algebraic structure must be coordinatised by a "division structure". In fact, the coordinatisation problem for RGD-systems is already solved by the work of Tits-Weiss in [TW02], but this fact is usually not stated in the group-theoretic language of RGD-systems. In order to explain this, we first have to make a brief detour through the theory of spherical buildings.

Spherical buildings were introduced by Tits "in an attempt to give a systematic procedure for the geometric interpretation of the semi-simple Lie groups and, in particular, the exceptional groups" (quotation from [Tit74, p. V]). More generally, they even provide a geometric framework to study simple algebraic groups over arbitrary fields. There are several different ways to define spherical buildings, but the technical details are not relevant for our purposes. Each building has an associated type which is a Coxeter system (or equivalently, a root system). The rank of a spherical building is by definition the rank of its type, and it is called irreducible if its type is irreducible. There also exist buildings which are not spherical (which means that their type is not spherical), but they are not relevant in this context. Further, there exist a notion of thickness of buildings and of the so-called *Moufang property*.[2] A spherical building which satisfies the Moufang property is called a Moufang building. It is a highly non-trivial fact, proven in [Tit77, 3.5], that every irreducible spherical building of rank at least 3 is automatically Moufang. The notion of *generalised polygons* is equivalent to that of spherical buildings of rank 2, and generalised polygons which satisfy the Moufang property are called *Moufang polygons*.

Thick irreducible spherical buildings of rank at least 3 have been classified by Tits in [Tit74], and this classification has later been extended to Moufang polygons by Tits-Weiss in [TW02]. In fact, the classification of Moufang polygons is completely independent of [Tit74], and the classification of higher-rank spherical buildings can be deduced from the classification of Moufang polygons (see [TW02, Chapter 40]). The statement of these classification results is that each thick irreducible spherical Moufang building of rank at least 2 is "coordinatised" by some algebraic division structure.

The interest in Moufang buildings for the theory of root graded groups stems from the following fact: Modulo technical details, the geometrical notion of thick Moufang buildings of type Φ is essentially equivalent to the group-theoretic notion of RGD-systems of type Φ (see [AB08, Section 7.8] for details). More precisely, the Moufang condition ensures the existence of an RGD-system in the automorphism group of such a spherical building, and a spherical building can

[2] This condition is named after Ruth Moufang, who is known for her work on the class of projective planes which today are called *Moufang planes*. In fact, a projective plane is the same thing as a thick spherical building of type A_2, and a Moufang plane is a thick spherical building of type A_2 with the Moufang property.

be constructed abstractly from an RGD-system. Thus the classification results for Moufang buildings that we have cited above provide a complete solution of the coordinatisation problem for RGD-systems of rank at least 3. In this sense, the work of this thesis can be seen as a generalisation of the classification of thick irreducible spherical buildings of rank at least 3 to the case of "non-division structures". It is noteworthy that the arguments in [TW02] are very similar in style to our work: At first Tits-Weiss construct a *root group sequence* from any Moufang polygon, which is essentially its RGD-systems, and then they classify root group sequences. The proofs in [Tit74], on the other hand, have a more geometric flavour.

Known Results about Root Gradings

We now turn to root gradings which are not necessarily RGD-systems. The terminology of "Φ-graded groups" appears for the first time in the paper [Shi93] by Shi. As in our definition, Shi only requires the existence of Weyl elements and not the stronger "division axiom" that appears in the definition of RGD-systems. In fact, Shi's definition of Φ-graded groups $(G, (U_\alpha)_{\alpha\in\Phi})$ is essentially equivalent to our definition except for one additional axiom: that G contains a homomorphic image of the Steinberg group of type Φ over some commutative associative unital ring. Here the Steinberg group is a certain group defined by generators and relations which mimic the commutator relations in Chevalley groups. We will explain the motivation behind this axiom in the following paragraph. One of the main results of [Shi93] is the solution of the coordinatisation problem for all root graded groups (in Shi's more restrictive sense) of type A, D or E and of rank at least 3. That is, Shi coordinatises root graded groups for all simply-laced types of rank at least 3. He arrives at the same conclusion that we do with our more general definition of root gradings. In fact, a non-trivial corollary of our coordinatisation result is that any root graded group in our sense is also a root graded group in Shi's sense.

Shi introduced his notion of root graded groups as a group-theoretic analogue of *root graded Lie algebras*. These Lie algebras were introduced in [BM92a] by Berman and Moody, of whom Shi is a student. A complete classification of root graded Lie algebras is available by the combined work of several authors, see [BM92a; BZ96; Neh96; ABG00; ABG02; BS03; BGP09]. It plays an important role in the classification of semisimple Lie algebras over algebraically closed fields of characteristic at least 5. One of the axioms of a Φ-graded Lie algebra L is that it contains a subalgebra L' which is split simple with root system Φ. Clearly this axiom motivated the additional axiom in Shi's definition which we discussed in the previous paragraph. A consequence of this additional axiom is that a certain *sign problem* which appears in the coordinatisation of root graded groups can be solved very efficiently in Shi's setting. This problem is related to a similar problem about the signs appearing in the Chevalley commutator formula. In order to solve this problem in our situation, we introduce a machinery which is independent of the root system Φ and which builds upon the solution of the word problem in Coxeter groups. The main result of this machinery is the *parametrisation theorem* that we will discuss in a moment. See also Remark 4.1.23 for more details on the sign problem.

The only work known to us which considers root gradings for root systems which are not simply-laced is the PhD thesis [Zha14] of Zhang, a student of

Zelmanov. Using essentially the same definition for root gradings as Shi, Zhang proves some partial results concerning C_3-gradings. Namely, he shows that every C_3-graded group satisfying some additional conditions is coordinatised by a nonassociative ring $\mathcal{R}$ with involution. One of these additional conditions on the group implies that $2_{\mathcal{R}}$ is invertible in $\mathcal{R}$, so it essentially excludes the case of "characteristic 2". The general case is significantly more difficult, so this is a serious restriction. Further, Zhang constructs a C_3-graded group from every alternative ring with involution in which 2 is invertible. The assumption on the alternativity of the ring is necessary because, as we will show in the main part of this work, the ring which appears in the coordinatisation of C_3-graded groups must actually be alternative. We prove this using a new method which we call the *blueprint technique*.

Finally, there are contributions of Faulkner on A_2-gradings that are important from the historical perspective. In [Fau14, Section 13.3], he defines *groups of Steinberg type*, which are essentially the same thing as A_2-graded groups in our sense. Faulkner proceeds to show that every such group is coordinatised by a nonassociative ring. It is unlikely that arbitrary nonassociative rings can appear here, so this solution of the coordinatisation problem is incomplete. The most general known construction of A_2-graded groups, given in [Fau83] and in the appendix of [Fau89], starts from an alternative ring.

We can thus summarise the previous state of the literature for root gradings of rank at least 3 as follows. For RGD-systems, the coordinatisation and existence problems are completely solved. For root gradings, all known coordinatisation results use Shi's more restrictive definition of root gradings. With this caveat, the coordinatisation problem for the simply-laced case is completely solved by [Shi93] and the existence problem is easy in this case. For C_n-gradings, there exist partial results by [Zha14]. For gradings of type B, BC and F_4, there are no previous results.

Another noteworthy work in the context of root gradings is [LN19], although its interest does not lie in the coordinatisation and existence problems in our sense. Still, Loos-Neher define *groups with* Φ*-commutator relations* (where Φ can be a root system, but also any subset of a free abelian group satisfying some axioms), citing [Fau77] as their inspiration (see [LN19, p. 72]). Further, they also define the notion of Weyl elements. A Φ-graded group in our sense is essentially the same thing as a group with Φ-commutator relations in which Weyl elements exist and which satisfies a non-degeneracy condition.

A Uniform Approach to Root Graded Groups

Since the class $\mathcal{C}_\Phi$ of coordinatising algebraic structures depends on the root system Φ, it is clear that a certain amount of case-by-case analysis is necessary in the study of root graded groups. However, we will encounter two root-system-independent tools which form the cornerstones of our work, and which are both new: the *parametrisation theorem* and the *blueprint technique*. The parametrisation theorem is our solution to a very delicate and technical sign problem, and it streamlines the coordinatisation process of root graded groups. The blueprint technique lies at the heart of our work, and it is pivotal for the determination of the class $\mathcal{C}_\Phi$ for each root system Φ. In particular, a characteristic-free solution to the coordinatisation problem would not have been possible without the blueprint technique. Together, these two tools provide a conceptual approach to root gradings which is as uniform as is possible.

The Parametrisation Theorem

Before we delve into the technicalities of the parametrisation theorem, we should clarify our conventions regarding the notions of "coordinatisations" and "parametrisations". Let G be a Φ-graded group and denote the set of all orbits in Φ under the Weyl group by $\mathbb{O}$. Under a *parametrisation of* G, we understand a family $(M_O)_{O \in \mathbb{O}}$ of groups and for each $O \in \mathbb{O}$ a family $(\theta_\alpha \colon M_O \to U_\alpha)_{\alpha \in O}$ of isomorphisms satisfying a certain consistency condition. Thus a *coordinatisation of* G in the sense that we have defined earlier is simply a parametrisation of G together with a family of maps between the parametrising groups satisfying two conditions: Firstly, these maps describe the commutator relations between the root groups, and secondly, they equip the parametrising groups with some algebraic structure. Hence essentially, a coordinatisation is a parametrisation by an algebraic structure, and we will use this distinction between "coordinatisations" and "parametrisations" throughout this thesis.

Our approach to the coordinatisation problem consists of first finding a parametrisation of G and then turning this parametrisation into a coordinatisation. The parametrisation theorem is the one and only criterion that we will use to establish the existence of a parametrisation of G. The main difficulty in the proof of the parametrisation theorem is a certain sign problem which is related to the consistency condition in the definition of parametrisations. This problem could be solved efficiently if the additional axiom in Shi's definition of root gradings were satisfied. In this sense, the parametrisation theorem is the solution of the sign problem in our more general setting.

The requirements of the parametrisation theorem say that, in order to find a parametrisation, we have to do three things. Firstly, we have to understand the action of w_δ^2 on each root group where δ is an arbitrary simple root (with respect to some fixed root base Δ) and w_δ is an arbitrary δ-Weyl element. Secondly, we have to show that Weyl elements satisfy the *braid relations*. These first two properties correspond to the homotopy moves which appear in the solution of the word problem in Coxeter groups, and the parametrisation theorem is proven by performing the same kind of moves on the level of Weyl elements. It should be noted that the braid relations for Weyl elements can be verified in a uniform way for all root systems, using the beautiful argument of Tits-Weiss in [TW02, (6.9)]. Thirdly, we have to a find a suitable "system of signs" η which is "consistent". Technically, η is a map from $\Phi \times \Delta$ to a finite abelian group A of exponent 2 which acts on each root group, and "consistency" refers to a set of combinatorial properties that it should satisfy.

Observe that the system of signs η in the previous paragraph is a map between finite sets. Thus the necessary consistency properties could, in theory, be checked by a long sequence of easy computations. However, there exists a more elegant way of doing this: By verifying that η appears as the system of signs in *some* example of a "sufficiently generic" root graded group. For the simply-laced root systems, the Chevalley groups are sufficient for this purpose. In the general case, a complete solution of the existence problem always produces a sufficiently generic example as a by-product. Even in the cases where no complete solution of the existence problem is available, we are still able to construct a group which is "sufficiently generic" to produce a suitable map η.

After these three problems are tackled, we can apply the parametrisation theorem to obtain a parametrisation of G. In order to turn it into a coordinatisation, we have to equip the parametrising groups with an algebraic structure. This is where the blueprint technique comes into play.

The Blueprint Technique

The blueprint technique is a powerful tool which reduces the problem of determining the commutator relations in root graded groups with a parametrisation to a straightforward computation involving certain rewriting rules. It is inspired by the work of Ronan-Tits on the construction of buildings in [RT87] and the key to our characteristic-free approach to root gradings.

The basic idea of the blueprint technique is as follows. We begin with a reduced representation f of the longest word ρ in the Weyl group W of Φ. Then there exists a sequence $h_1, \ldots, h_r$ of elementary homotopy moves which transforms f into itself and which, speaking very loosely in terms of algebraic topology, "moves around the hole in the Cayley graph of W". For example, the Cayley graph of the Weyl group of A_3 can be drawn on the 2-sphere with the identity element 1_W at the north pole and the longest element ρ at the south pole, and there is a way to move f (which is a path from 1_W to ρ) once around the sphere. Further, we take a word $\tilde{f}$ "corresponding to f" whose letters are arbitrary elements (in some sense, "indeterminates") of the parametrising groups of G.

The blueprint technique associates each elementary homotopy move h_i on f to a rewriting rule $\tilde{h}_i$ on $\tilde{f}$. This rule $\tilde{h}_i$ involves the maps appearing in the commutator relations of G. Now the *blueprint computation* consists of iteratively determining all words which are obtained from $\tilde{f}$ by applying the rewriting rules $\tilde{h}_1, \ldots, \tilde{h}_r$. This computation is purely mechanical, albeit lenghty, and perfectly suited to be executed on a computer. For abstract reasons, the end result of the blueprint computation must be the same as the initial word $\tilde{f}$, so we obtain a sequence of identities (one for each letter in $\tilde{f}$) which are valid in the parametrising structures. These identities allow us to explicitly compute the commutator relations in G, and the maps involved in these formulas equip the parametrising groups of G with an algebraic structure.

Organisation of the Thesis

In chapter 1, we recall some basic preliminaries, mainly from the theory of finite root systems. In chapter 2, we introduce the language of root graded groups and prove some general results: the bijectivity of certain product maps and the braid relations for Weyl elements. Chapter 3 is devoted to a brief summary of the theory of Chevalley groups. We prove the parametrisation theorem in chapter 4. Afterwards, we begin our investigation of Φ-graded groups for all root systems Φ of rank at least 3. As explained in the previous section, the general strategy is independent of Φ: We first verify that the requirements of the parametrisation theorem are satisfied, then we apply it and finally, we use the blueprint technique to compute the commutator relations. The only exception to this rule are the simply-laced root systems, which are covered in chapter 5, because the commutator relations in this case are so simple that we do not need to use the blueprint technique. In chapter 6, we formally introduce the blueprint technique. Afterwards, we study root graded groups of type B in chapter 7, of type BC (which contains type C as a special case) in chapter 9 and of type F_4 in chapter 10. Chapter 8 is a prelude to chapter 9 which introduces Jordan modules, the algebraic structure that coordinatises root gradings of type BC.

Remarks on Related Topics

We now turn back to geometrical aspects of the theory. Recall that Moufang polygons are geometric objects which correspond to RGD-systems of rank 2. In [MW22a], Mühlherr-Weiss have introduced *Tits polygons*, which generalise Moufang polygons. In [MW19], they show that Tits polygons correspond bijectively to root gradings of rank 2 which satisfy the so-called *stability condition*. On the algebraic side of coordinate systems, this condition corresponds to the notion of *rings of stable rank 2*, sometimes simply called *stable rings*.

The concept of stability originally stems from K-theory, but it was realised by Veldkamp in [Vel81] that stable rings are precisely those rings over which projective planes can be defined in a meaningful way. He gives combinatorial axioms for such planes, which today are called *Veldkamp planes*, and proceeds to show that a Veldkamp plane is the projective plane of a stable ring if and only if it is Desarguesian. See also [Vel95] for additional information.

In [MW22b, 2.8], Mühlherr-Weiss define the notion of *Veldkamp n-gons*, or simply *Veldkamp polygons*. A Veldkamp plane is exactly the same thing as a Veldkamp triangle (that is, a Veldkamp 3-gon). In this language, a Tits polygon is a Veldkamp polygon which satisfies the Moufang condition. Higher-rank analogues of Veldkamp polygons can be defined, which leads to the notion of *Veldkamp buildings (of rank at least 3)*. It seems likely that every Veldkamp building of rank at least 3 has an associated stable root graded group (of rank at least 3), and thus the results of this thesis yield a classification of (or at least strong coordinatisation results for) Veldkamp buildings of rank at least 3. In fact, this relation to Veldkamp buildings is a strong motivation for our work. The absence of stability conditions in this thesis is simply due to the fact that they are not necessary to build a meaningful theory on the group-theoretic side, at least in the higher-rank situation.

It is possible to define an analogue of Veldkamp buildings without the stability condition, the so-called *Faulkner buildings*. Since the stability condition has proven crucial in the context of projective geometry, it is not at all clear whether Faulkner buildings actually describe a meaningful geometry. However, they are the natural class of spaces which correspond to general root graded groups.

We end with a few words on the non-crystallographic root systems H_3 and H_4. It is known from [Tit77, Hauptsatz] that there exist no thick buildings (and thus no RGD-systems) of these types, so one might expect the same to be true for root gradings. However, this is wrong: Starting from a D_6-graded group $(G, (U_\alpha)_{\alpha \in D_6})$, one can fold $(U_\alpha)_{\alpha \in D_6}$ to obtain an H_3-grading $(V_\beta)_{\beta \in H_3}$ of the same group G in which each root group V_β equals the product of two commuting root groups in $(U_\alpha)_{\alpha \in D_6}$. In particular, if $(U_\alpha)_{\alpha \in D_6}$ is coordinatised by a commutative associative ring $\Bbbk$, then $(V_\beta)_{\beta \in H_3}$ is coordinatised by the ring $\Bbbk \times \Bbbk$. A similar construction for H_4-graded groups is possible. In collaboration with Lennart Berg, we have shown in the joint and yet unpublished work [BW] that every Φ-graded group for $\Phi \in \{ H_3, H_4 \}$ is of this form. Incidentally, this provides a new proof of the known fact that RGD-systems of these types cannot exist: The ring $\Bbbk \times \Bbbk$ is never a field, even when $\Bbbk$ is a field.

Chapter 1

Preliminaries

In this chapter, we will recall some standard definitions and facts which will be needed throughout this thesis. The first section collects a variety of definitions and results from various areas which have no logical place elsewhere. The following sections are devoted to the theory of finite root systems and their Weyl groups, which plays a major role in the theory of root graded groups. None of the material in this chapter is new.

1.1 Elementary Notation and Group-theoretic Facts

1.1.1 Notation. We denote the sets of natural numbers (including 0), positive integers, integers, real numbers and complex numbers by $\mathbb{N}_0$, $\mathbb{N}_+$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively.

We will often consider sets of integers of the form $\{ n, \ldots, m \}$ where $n \leq m$. Since intervals of real numbers are completely absent from this thesis, the following notation is not ambiguous.

1.1.2 Notation (Integer intervals). For all $n, m \in \mathbb{Z}$, we set

$$[n, m] := \{ i \in \mathbb{Z} \mid n \leq i \leq m \} = \begin{cases} \varnothing & \text{if } n > m, \\ \{ n, \ldots, m \} & \text{if } n \leq m. \end{cases}$$

Further, we put

$$\langle n, m \rangle := \begin{cases} [n, m] & \text{if } n \leq m, \\ [m, n] & \text{if } n > m. \end{cases}$$

Note that the statement "$i \in \langle n, m \rangle$" can be interpreted as "i lies between n and m", independent of whether $n < m$ or $m < n$.

As we have already seen in the preface, we have to adopt a very general definition of rings in this work.

1.1.3 Convention. A ring is always understood to be unital, but the multiplication is not assumed to be associative or commutative. All the necessary background on these objects will be introduced in section 5.1. However, it suffices to know only the definition of these objects until we reach section 5.1.

1.1.4 Note. Of course, a ring in our sense can be commutative without being associative. (Linear) Jordan algebras provide examples of such objects. In this thesis, however, we will never encounter a situation in which a ring is assumed to be commutative but not assumed to be associative.

1.1.5 Notation (Matrices). For any associative ring $\mathcal{R}$ and for any positive integers n, m, we denote by $M_{nm}(\mathcal{R})$ or by $\mathcal{R}^{n\times m}$ the set of matrices with n rows and m columns. Further, we put $M_n(\mathcal{R}) := M_{nn}(\mathcal{R})$.

Note the assumption on the ring $\mathcal{R}$ to be associative in the definition of matrices. We could, of course, define matrices over arbitrary rings in the same way, but then the matrix multiplication is not necessarily associative. In particular, we cannot construct groups from such matrices, which makes them useless for our purposes.

1.1.6 Definition (Words). Let S be a set. A *word over S* is a tuple $(s_1, \ldots, s_k)$ for some $k \in \mathbb{N}_0$ such that $s_1, \ldots, s_k$ lie in S. The *free monoid over S* is the set $\mathcal{L}(S)$ of words over S together with the multiplication $(s_1, \ldots, s_k) \cdot (t_1, \ldots, t_l) := (s_1, \ldots, s_k, t_1, \ldots, t_l)$. Its neutral element, the empty tuple, is called the *empty word* and denoted by $\varnothing$. When there is no danger of confusion, we will also denote a word $(s_1, \ldots, s_k)$ by $s_1 \cdots s_k$. The *free group over S* is denoted by $F(S)$.

1.1.7 Notation. Let M be a monoid, let $m \in \mathbb{N}_0$ and let $x, y \in M$. We denote by $P_m(x,y)$ the word $\prod_{i=1}^{m} z_i$ where $z_i := x$ for all odd $i \in [1,m]$ and $z_i := y$ for all even $i \in [1,m]$. In particular, $P_0(x,y) = 1_M$. If x, y lie in a set S which is not equipped with a multiplication, we denote by $P_m(x,y)$ the word $P_m\big((x),(y)\big) = (x, y, x, \ldots)$ in the free monoid over S.

1.1.8 Notation (Group-theoretic notions). Let G be a group. For any subset U of G, we denote by $\langle U \rangle$ the subgroup of G which is *generated by U*. For all $g, h \in G$, the *commutator of g and h* is $[g,h] := g^{-1}h^{-1}gh$ and the *conjugate of g by h* is $g^h := h^{-1}gh$.

1.1.9 Note. Some authors define the commutator by $[g,h]' := ghg^{-1}h^{-1}$ and conjugation by ${}^h g := hgh^{-1}$. These slightly different conventions are related to our definitions by the simple relations $[g,h]' = [g^{-1}, h^{-1}]$ and ${}^h g = g^{h^{-1}}$. Thus every statement in this thesis can easily be translated to a statement using any other set of conventions, and vice versa. This problem is of practical importance for our purposes: We will cite several statements about the constants which appear in the Chevalley commutator formula, and some of the cited references use a different convention than we do. See Remark 3.2.19 for more details.

1.1.10 Definition. Let G be a group and let H be a subset of G. For any $g \in G$, we say that *g acts on H by inversion* if $h^g = h^{-1}$ for all $h \in H$ (where h^{-1} is not assumed to lie in H) and we say that *g acts trivially on H* if $h^g = h$ for all $h \in H$. Further, for any $a, b \in G$, we say that *a and b act identically on H* if $h^a = h^b$ for all $h \in G$.

We end this section with some relations which hold in arbitrary groups. The relations in 1.1.11 form the backbone of many computations, and we will often use them without specifically saying so.

1.1.11 Relations. Let G be a group and let $g, g_1, g_2, h, h_1, h_2 \in G$. Then the following relations hold:

(i) $g^h = g[g,h]$.
(ii) $g^h = [h, g^{-1}]g$.
(iii) $[g_1, g_2]^h = [g_1^h, g_2^h]$ and $(g_1 g_2)^h = g_1^h g_2^h$.
(iv) $gh = hg^h$.

(v) $gh = hg[g,h] = [g^{-1}, h^{-1}]hg$.
(vi) $[g,h]^{-1} = [h,g]$.
(vii) $[g_1g_2, h] = [g_1,h]^{g_2}[g_2,h]$ and $[g, h_1h_2] = [g,h_2][g,h_1]^{h_2}$.
(viii) $[g_1g_2, h] = [g_2,h]$ if $[g_1,h] = 1_G$ and $[g,h_1h_2] = [g,h_2]$ if $[g,h_1] = 1_G$.

Further, note that $[g,h] = 1_G$ is equivalent to each of the following conditions: $gh = hg$, $g^h = g$, $[h,g] = 1_G$, $[g^{-1},h] = 1_G$ and $[g,h^{-1}] = 1_G$.

Another relation which holds in arbitrary groups is the Hall-Witt identity. As a special case, it says that under certain conditions on the group elements, the commutator map is associative. It is used in this specific form in [Shi93, (2.23)] to show that for any simply-laced root system Φ of rank at least 3, any ring which coordinatises a Φ-graded group is associative. Further, the general form of the Hall-Witt identity given in Lemma 1.1.12 plays an important role in the computation of the commutator relations in C_n-graded groups (for $n \geq 3$) in [Zha14, 3.4.15, 3.4.16, 3.4.18]. Thus the Hall-Witt identity holds an important place in the theory of root graded groups. It is therefore surprising that all results in this thesis can actually be proven without invoking the Hall-Witt identity. This is made possible by the blueprint technique, a new tool which we will introduce in chapter 6.

1.1.12 Lemma (Hall-Witt identity). *Let G be a group. Then for all $x,y,z \in G$, we have*

$$\big[[x,y],z^x\big]\big[[z,x],y^z\big]\big[[y,z],x^y\big] = 1_G.$$

Proof. Let $x,y,z \in G$ be arbitrary. We have

$$\begin{aligned}\big[[x,y],z^x\big] &= [x^{-1}y^{-1}xy, x^{-1}zx] = (y^{-1}x^{-1}yx)(x^{-1}z^{-1}x)(x^{-1}y^{-1}xy)(x^{-1}zx)\\ &= y^{-1}x^{-1}yz^{-1}y^{-1}xyx^{-1}zx.\end{aligned}$$

Since x,y,z are arbitrary, it follows that

$$\begin{aligned}\big[[z,x],y^z\big] &= x^{-1}z^{-1}xy^{-1}x^{-1}zxz^{-1}yz \qquad \text{and}\\ \big[[y,z],x^y\big] &= z^{-1}y^{-1}zx^{-1}z^{-1}yzy^{-1}xy\end{aligned}$$

hold as well. Thus

$$\begin{aligned}\big[[x,y],z^x\big]\big[[z,x],y^z\big] &= (y^{-1}x^{-1}yz^{-1}y^{-1}xyx^{-1}zx)(x^{-1}z^{-1}xy^{-1}x^{-1}zxz^{-1}yz)\\ &= y^{-1}x^{-1}yz^{-1}y^{-1}(xyx^{-1}zxx^{-1}z^{-1}xy^{-1}x^{-1})zxz^{-1}yz\\ &= y^{-1}x^{-1}yz^{-1}y^{-1}zxz^{-1}yz = \big[[y,z],x^y\big]^{-1}.\end{aligned}$$

This finishes the proof. □

1.1.13 Lemma (Special case of the Hall-Witt identity). *Let G be a group and let $x,z,y \in G$ such that $[x,z] = 1_G$ and $\big[[y,x^{-1}],[y,z]\big] = 1_G$. Then*

$$\big[[x,y],z\big] = \big[x,[y,z]\big].$$

Proof. By the Hall-Witt identity, we have $\big[[x,y],z\big]\big[[y,z],x^y\big] = 1_G$. Applying Relation 1.1.11 (ii) and 1.1.11 (vii), we infer that

$$\begin{aligned}\big[[x,y],z\big] &= \big[[y,z],x^y\big]^{-1} = \big[x^y,[y,z]\big] = \big[[y,x^{-1}]x,[y,z]\big]\\ &= \big[[y,x^{-1}],[y,z]\big]^x\big[x,[y,z]\big] = \big[x,[y,z]\big],\end{aligned}$$

as desired. □

1.2 Root Systems

In this section and the following one, we collect all the necessary definitions and facts about finite root systems and their Weyl groups that will be needed later. We assume that the reader is already familiar with the basics of this theory. It should be noted, however, that all standard references include either the property of being reduced or the property of being crystallographic (or even both) in their definition of root systems. This is a reasonable decision: Either the length of each root is not considered to be important, in which case there is no reason to consider non-reduced root systems, or it is considered to be important, and then the crystallographic condition is indispensable.

While we are ultimately only interested in crystallographic root systems in this thesis, we want to proof the parametrisation theorem in chapter 4 in full generality. The reason for this is that we will also use it in [BW] in the study of root graded groups of types H_3 and H_4, which are non-crystallographic root systems. Thus it is practical for us to work in the framework of general (finite) root systems. Since this is an unusual approach, we will explicitly state all definitions and results that we will need throughout this thesis. Further, we will also state some classical results in slightly higher generality than usual. None of these generalisations are difficult, but we will still spell them out for clarity.

A standard reference for the material presented in this section and the following is [Hum90], which considers only reduced root systems. Most statements from [Hum90] transfer to the non-reduced setting by using Construction 1.2.16. Further, we will refer to [Bou81] and [Hum72] for statements which are specific to crystallographic root systems. It should be noted that root systems in [Hum72] are assumed to be reduced and crystallographic.

1.2.1 Notation for this section. Starting from subsection 1.2.B, we assume that Φ is a root system in some Euclidean space $(V, \cdot)$.

1.2.A The Definition of Root Systems

1.2.2 Definition (Euclidean space). A *Euclidean space* is a pair $(V, \cdot)$ consisting of a finite-dimensional real vector space V and an inner product $\cdot\colon V \times V \to \mathbb{R}$. That is, $\cdot$ is a symmetric bilinear form on V with the property that $v \cdot v > 0$ for all $v \in V \setminus \{0\}$. For any $v \in V$, we denote by $v^\perp := \{ w \in V \mid v \cdot w = 0 \}$ the set of vectors which are *orthogonal* to v. For any $v \in V$, the *length of* v is $\|v\| := \sqrt{v \cdot v}$.

1.2.3 Convention. In the context of root systems, we use the the convention that automorphisms of a Euclidean space $(V, \cdot)$ act from the right-hand side. Thus the image of $v \in V$ under an automorphism φ is denoted by v^φ, and the composition of two automorphisms φ, ψ is the map $\varphi\psi$ which sends v to $(v^\varphi)^\psi$. The reasoning behind this somewhat unusual convention will be explained in Note 1.2.17.

1.2.4 Definition (Orthogonal group). Let $(V, \cdot)$ be a Euclidean space. The *orthogonal group of* $(V, \cdot)$, denoted by $\mathrm{O}(V, \cdot)$, is the group of all automorphisms φ of V such that $x^\varphi \cdot y^\varphi = x \cdot y$ for all $x, y \in V$.

1.2.5 Definition (Reflection). Let $(V, \cdot)$ be a Euclidean space. Then for all $v \in V \setminus \{0_V\}$, the map

$$\sigma_v\colon V \to V, x \mapsto x^{\sigma_v} := x - 2\frac{x \cdot v}{v \cdot v} v$$

is called the *reflection associated to v* or the *reflection along $v^\perp$*. We will also denote it by $\sigma(v)$. Further, for any word $\bar{v} = (v_1, \dots, v_m)$ over $V \setminus \{0_V\}$, we define $\sigma_{\bar{v}} := \sigma_{v_1 \cdots v_m} := \sigma_{v_1} \cdots \sigma_{v_m}$, which is the map $x \mapsto ((x^{\sigma_{v_1}})^{\cdots})^{\sigma_{v_m}}$ and which we also denote by $\sigma(\bar{v})$ or $\sigma(v_1 \cdots v_m)$.

1.2.6 Remark (Computation rules for reflections). Let $v \in V \setminus \{0_V\}$. Then we have $\sigma_v^{-1} = \sigma_v$, $\sigma_{\lambda v} = \sigma_v$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $\sigma(v)^\varphi = \sigma(v^\varphi)$ for all $\varphi \in \mathrm{O}(V, \cdot)$. (The validity of the last statement relies on the fact that reflections act on V from the right side: If they act from the left, we have $\sigma(v)^\varphi = \sigma(v^{\varphi^{-1}})$.) Further, σ_v lies in the orthogonal group $\mathrm{O}(V, \cdot)$, so in particular, $\sigma(v)^{\sigma(w)} = \sigma(v^{\sigma(w)})$ for all $w \in V \setminus \{0_V\}$.

1.2.7 Note. In Definitions 7.1.16 and 7.1.20, we will see that the concepts of orthogonal groups and reflections can be generalised to modules over commutative associative rings which are equipped with a quadratic form. By taking the canonical quadratic form $v \mapsto v \cdot v$ on the Euclidean space $(V, \cdot)$, we obtain the notions in this section as special cases. However, this observation is irrelevant for the theory of root systems.

1.2.8 Definition (Root system). Let $(V, \cdot)$ be a Euclidean space. A *root system in $(V, \cdot)$* is a finite non-empty subset Φ of $V \setminus \{0_V\}$ such that $\beta^{\sigma(\alpha)} \in \Phi$ for all $\alpha, \beta \in \Phi$. The elements of Φ are called *roots (of Φ)* and the dimension of the vector space generated by Φ is called the *rank of* Φ. For any root α, its *opposite root* is $-\alpha$.

1.2.9 Definition (Reduced root system). A root system Φ is called *reduced* if $\mathbb{R}\alpha \cap \Phi = \{\, \alpha, -\alpha \,\}$ for all $\alpha \in \Phi$.

1.2.10 Definition (Root subsystem). Let Φ be a root system. A subset Φ' of Φ is called a *root subsystem of* Φ if $\sigma_\alpha(\Phi') = \Phi'$ for all $\alpha \in \Phi'$.

1.2.11 Note. There exists no standard terminology for subsystems of root systems. Definition 1.2.10 is the most general notion that one can think of, but it will not actually be useful to us in this generality. The reason for this is that root subsystems Φ' in this general sense do not give rise to Φ'-graded subgroups of Φ-graded groups. This is only true if Φ' is a closed root subsystem, a notion that we will introduce in Definition 1.2.42. Most of the time, the root subsystems that we consider will even have the stronger property of being parabolic, which we also introduce in Definition 1.2.42.

1.2.B Basic Notions

1.2.12 Notation for this section. From now on, we assume that Φ is a root system in some Euclidean space $(V, \cdot)$.

1.2.13 Definition (Weyl group). The *Weyl group of* Φ or the *Coxeter group of* Φ is the group

$$\mathrm{Weyl}(\Phi) := \langle \sigma_\alpha \mid \alpha \in \Phi \rangle \leq O(V, \cdot).$$

We will study the Weyl group and its action on Φ in more detail in section 1.3.

1.2.14 Definition (Divisible roots). A root $\alpha \in \Phi$ is called *indivisible (in Φ)* if $\lambda\alpha$ is not a root (in Φ) for all $0 < \lambda < 1$, and it is called *divisible (in Φ)* otherwise. For any set Ψ of roots, we denote the set of all indivisible roots in Ψ by Ψ^{indiv}. Further, a subset Ψ of Φ is called *reduced* if all roots in Ψ are indivisible in Φ.

1.2.15 Remark. For any root system Φ, the set Φ^{indiv} is a reduced root subsystem of Φ (which has the same Weyl group). However, it is in general not closed in the sense of Definition 1.2.39.

1.2.16 Construction (of root systems from reduced root systems). Let O be an orbit of the Weyl group in Φ and let $\lambda \in \mathbb{R} \setminus \{0\}$. Then $\Phi \cup \lambda O$ is a root system as well. Since for any root system $\bar{\Phi}$ the subset $\bar{\Phi}^{\mathrm{indiv}}$ is a reduced root system which contains a scalar multiple of each root in $\bar{\Phi}$, every root system can be constructed from a reduced root system in this way in a finite number of steps. For this reason, many statements about reduced root systems can easily be generalised to arbitrary root systems.

1.2.17 Note (on Convention 1.2.3). One of the defining properties of Φ-graded groups $(G, (U_\alpha)_{\alpha\in\Phi})$ will be that there exist so-called Weyl elements $w_\alpha \in G$ such that $U_\beta^{w_\alpha} = U_{\beta^{\sigma(\alpha)}}$ for all $\alpha, \beta \in \Phi$. Thus we have a connection between the action of a reflection σ_α on $(V, \cdot)$ and the conjugation action in G. Since it is customary to write conjugation in groups from the right, we will do the same for reflections. This explains Convention 1.2.3. However, as the distinction between left and right actions is only relevant when we compose two reflections, we will still sometimes write $\sigma_v(w)$ instead of w^{σ_v} for $v, w \in V$, but we will never write $\sigma_{\bar{v}}(w)$ in place of $w^{\sigma_{\bar{v}}}$ when $\bar{v}$ is a word over V. In fact, we will usually write $w^{\sigma(\bar{v})}$ in this situation to avoid excessive nesting of superscripts and subscripts.

1.2.18 Definition (Total ordering, [Hum90, p. 7]). Let W be a finite-dimensional real vector space. A *total ordering of* W is a relation $\lhd$ on W with the following properties.

(i) $\lhd$ is transitive.

(ii) $\lhd$ is total. That is, for all $v, w \in W$, exactly one of the statements $v \lhd w$, $v = w$, $w \lhd v$ holds.

(iii) $\lhd$ is compatible with addition: For all $u, v, w \in W$ with $u \lhd v$, we have $u + w \lhd v + w$.

(iv) For all $v, w \in W$ with $v \lhd w$ and all $\lambda \in \mathbb{R} \setminus \{0\}$, we have $\lambda v \lhd \lambda w$ if $\lambda > 0$ and $\lambda w \lhd \lambda v$ if $\lambda < 0$.

Given a total ordering $\lhd$, an element $v \in W$ is said to be *positive* if $0_W \lhd v$, and it is called *negative* if $v \lhd 0_W$.

1.2.19 Definition (Positive system). A subset Π of Φ is called a *positive system in* Φ if there exists a total order $\lhd$ on V such that

$$\Pi = \{\, \alpha \in \Phi \mid 0_V \lhd v \,\}.$$

We then say that Π is *the positive system induced by* $\lhd$.

1.2.20 Remark. For any basis of a finite-dimensional real vector space, there exists a total ordering with respect to which all basis vectors are positive.

1.2.21 Remark. Let Π be a subset of Φ. Then Π is a positive system in Φ if and only if there exists a non-zero linear form $f\colon V \to \mathbb{R}$ with $f(\alpha) \neq 0$ for all roots α such that

$$\Pi = \{\, \alpha \in \Phi \mid f(\alpha) > 0 \,\}.$$

Note that the kernel of f is a hyperplane in V which divides V into two half-spaces. The positive system Π is exactly the set of roots which lie in one of these half-spaces.

1.2.22 Definition (Root base, [Bou81, Proposition VI.1.20, Corollaire 3]). A *root base of* Φ is a subset Δ of Φ satisfying the following conditions:

(i) Δ is a basis of the $\mathbb{R}$-vector space spanned by Φ.

(ii) If $\delta_1, \dots, \delta_l$ are the pairwise distinct elements of Δ and $\lambda_1, \dots, \lambda_l \in \mathbb{R}$ are such that $\sum_{i=1}^{l} \lambda_i \delta_i$ is a root, then $\lambda_1, \dots, \lambda_l$ are all non-negative or all non-positive.

(iii) All the roots $\delta_1, \dots, \delta_l$ are indivisible.

Further, an *ordered root base of* Φ is a tuple $(\delta_1, \dots, \delta_l)$ such that $\delta_1, \dots, \delta_l$ are pairwise distinct and $\{ \delta_1, \dots, \delta_l \}$ is a root base of Φ. A *rescaled root base* is a set Δ for which there exist positive real numbers $(\lambda_\delta)_{\delta \in \Delta}$ such that $\{ \lambda_\delta \delta \mid \delta \in \Delta \}$ is a root base. A root base Δ will sometimes be called a *proper root base* to emphasise that it is not merely a rescaled root base. If a root base Δ is fixed, its elements are called *simple roots* and the corresponding reflections are called *simple reflections*.

1.2.23 Note. If the root system Φ is reduced, then every rescaled root base of Φ is in fact a root base. Thus the distinction between root bases and rescaled root bases is only relevant in the non-reduced case. Hence by the classification result that we will state in Theorem 1.2.53, it is only relevant for root systems of type BC. We will explain in Remark 4.1.31 why the slightly more general notion of rescaled root bases will be needed for root systems of this type. For the moment, the distinction between proper root bases and rescaled root bases is not relevant, and it suffices to notice that some well-known results about root bases (specifically, Propositions 1.2.24, 1.3.12 and 1.3.16) remain valid in this more general situation.

1.2.24 Proposition. *Every rescaled root base Δ of Φ lies in a unique positive system $\Pi(\Delta)$ (namely, $\Pi(\Delta) := \Phi \cap \{ \sum_{\delta \in \Delta} \lambda_\delta \delta \mid \lambda_\delta \geq 0$ for all $\delta \in \Delta \}$), and every positive system Π contains a unique root basis $\Delta(\Pi)$ (namely, $\Delta(\Pi)$ is the set of roots in Π which cannot be expressed as an $\mathbb{R}_{>0}$-linear combinations of two or more roots in Π). In particular, root bases exist.*

Proof. This is proven in [Hum90, Theorem 1.3] for reduced root systems and non-rescaled root bases. The general assertion follows easily from this fact by considering the reduced root system Φ^{indiv}. $\square$

1.2.25 Definition. Let Δ be a rescaled root base. The positive system $\Pi(\Delta)$ in Proposition 1.2.24 is called the *positive system corresponding to* Δ. Its elements are called the *positive roots with respect to* Δ *(or* $\Pi(\Delta)$*)*, and the elements of $-\Pi(\Delta)$ are called the *negative roots with respect to* Δ *(or* $\Pi(\Delta)$*)*.

1.2.C Crystallographic Root Systems

We now turn to the definition of crystallographic root systems.

1.2.26 Definition (Cartan number). Let α, β be two elements of a V. The number $\langle \alpha | \beta \rangle := 2\frac{\alpha \cdot \beta}{\beta \cdot \beta} \in \mathbb{R}$ is called the *Cartan number for* (α, β). For any root base Δ, the matrix $(\langle \alpha | \beta \rangle)_{\alpha, \beta \in \Delta}$ is called the *Cartan matrix of* Φ *(with respect to* Δ*)*.

Using Proposition 1.3.6, it is not difficult to show that the Cartan matrix does not depend on the choice of Δ up to a permutation of the rows and columns.

1.2.27 Definition (Crystallographic root system). The root system Φ is called *crystallographic* if for all $\alpha, \beta \in \Phi$, the Cartan number $\langle \alpha | \beta \rangle$ is an integer, which is then called the *Cartan integer for* (α, β).

The following properties are specific to crystallographic root system and not true in general.

1.2.28 Lemma. *Assume that Φ is crystallographic. Let α be a root and let $\lambda > 0$ such that $\lambda\alpha$ is also a root. Then $\lambda \in \{1/2, 1, 2\}$.*

Proof. Since Φ is crystallographic, the numbers

$$\langle \alpha | \lambda\alpha \rangle = 2\frac{\alpha \cdot \lambda\alpha}{\lambda\alpha \cdot \lambda\alpha} = \frac{2}{\lambda} \quad \text{and} \quad \langle \lambda\alpha | \alpha \rangle = 2\frac{\lambda\alpha \cdot \alpha}{\alpha \cdot \alpha} = 2\lambda$$

must be integers. The latter fact implies that $\lambda = m/2$ for some $m \in \mathbb{N}_+$, and then the former fact yields that $4/m$ is an integer. Thus $m \in \{1, 2, 4\}$, so $\lambda \in \{1/2, 1, 2\}$. □

1.2.29 Lemma. *Let Δ be a root base of Φ and let α be a positive root with respect to Δ. Assume that Φ is crystallographic. Then there exist $n \in \mathbb{N}_+$ and $\delta_1, \ldots, \delta_n \in \Delta$ such that $\alpha = \sum_{i=1}^{n} \delta_i$ and such that for each $m \in [1, n]$, the partial sum $\sum_{i=1}^{m} \delta_i$ is a root. In particular, the coefficients of any root with respect to Δ are always integral.*

Proof. For reduced root systems Φ, this is exactly the statement of [Hum72, Corollary of Lemma 10.2.A]. The general assertion is proven in [Bou81], but the other way around: It is first proven in [Bou81, Théorème VI.1.3, p. 156] that the coefficients of any root with respect to Δ are always integral. This implies that every positive root α can be written as a sum $\sum_{i=1}^{n} \delta_i$ of simple roots $\delta_1, \ldots, \delta_n \in \Delta$ and then by [Bou81, Proposition VI.1.19, p. 159], there exists a permutation π of $[1, n]$ such that each partial sum $\sum_{i=1}^{m} \delta_{\pi(i)}$ for $m \in [1, n]$ is a root. □

1.2.D Root Intervals and Root Subsystems

The following definition introduces root intervals. This notion will be crucial for the definition of groups with Φ-commutator relations. It is a special case of the notion of commutator sets (of subsets of Φ) that we will introduce in Definition 2.1.21.

1.2.30 Definition (Root intervals). Let α, β be non-proportional roots. The *open Coxeter root interval for* (α, β) is

$$]\alpha, \beta[_{\mathrm{Cox}} := \{ \lambda\alpha + \mu\beta \mid \lambda, \mu \in \mathbb{R}_{>0} \} \cap \Phi$$

and the *open crystallographic root interval for* (α, β) is

$$]\alpha, \beta[:= \{ \lambda\alpha + \mu\beta \mid \lambda, \mu \in \mathbb{N}_+ \} \cap \Phi \subseteq]\alpha, \beta[_{\mathrm{Cox}}.$$

The roots α and β are called *adjacent* if $]\alpha, \beta[_{\mathrm{Cox}}$ is empty, and they are called *crystallographically adjacent* if $]\alpha, \beta[$ is empty. Further, the sets

$$[\alpha, \beta]_{\mathrm{Cox}} := \{ \alpha, \beta \} \cup]\alpha, \beta[_{\mathrm{Cox}} \quad \text{and} \quad [\alpha, \beta] := \{ \alpha, \beta \} \cup]\alpha, \beta[$$

are called the *closed Coxeter root interval for* (α, β) and the *closed crystallographic root interval for* (α, β), respectively.

1.2.31 Note. In [LN19, (1.5.2), (1.6.3)], root intervals are defined as follows for all $\alpha, \beta \in \Phi$:

$$[\alpha, \beta] := \Phi \cap \{ m\alpha + n\beta \mid \alpha, \beta \in \mathbb{N}_0, m + n > 0 \},$$
$$]\alpha, \beta[:= \Phi \cap \{ m\alpha + n\beta \mid \alpha, \beta \in \mathbb{N}_+ \}.$$

These definitions differ from ours in two ways. Firstly, there is no assumption on α and β to be non-proportional. For example, we have $[\alpha, \alpha] = \Phi \cap \mathbb{N}_+\alpha$

and $]\alpha, \alpha[\, = \Phi \cap \mathbb{N}_{\geq 2}\alpha$. Secondly, the closed root interval $[\alpha, \beta]$ in the sense of Loos-Neher contains, in addition to the closed root interval in our sense, all the roots in $]\alpha, \alpha[\, \cup \,]\beta, \beta[\, = \Phi \cap (\mathbb{N}_{\geq 2}\alpha \cup \mathbb{N}_{\geq 2}\beta)$.

1.2.32 Note. The open interval between α and β consists of the roots which lie "between α and β". In the crystallographic setting, it is natural to consider only those roots which can be written as integral linear combinations of α and β. As explained in the preface of this section, we will only consider crystallographic root gradings in the main part of this work, which is why we simply write $]\alpha, \beta[$ for the crystallographic root interval (and not, say, $]\alpha, \beta[_{\mathrm{cry}}$). Still, many results and proofs remain valid for Coxeter root intervals. See also Note 2.3.25.

1.2.33 Note. Let α, β be non-proportional roots and assume that Φ is irreducible and crystallographic. Clearly, the crystallographic root interval is a subset of the Coxeter root interval. (In particular, adjacent roots are automatically crystallographically adjacent.) In the general case, it can be smaller. Using the classification of root systems (Theorem 1.2.53), we can prove the following statements by an inspection of the irreducible root systems Φ:

(a) If Φ is simply-laced (that is, of type A, D or E), then the Coxeter and crystallographic intervals are the same for all pairs of roots.

(b) If Φ is not of type G_2, then $]\alpha, \beta[_{\mathrm{Cox}}$ has at most two elements, and if it has two elements, it equals $]\alpha, \beta[$.

Thus if Φ is not of type G_2, then the only difference between Coxeter and crystallographic root intervals is that for some pairs of roots, $]\alpha, \beta[$ is empty but $]\alpha, \beta[_{\mathrm{Cox}}$ has exactly one element. Specifically, using the standard representations, this happens exactly for the pairs of roots $\big(\varepsilon(e_i - e_j), \varepsilon(e_i + e_j)\big)$ in B_n and the pairs $(\varepsilon 2e_i, \varepsilon' 2e_j)$ in $(B)C_n$ (where $i, j \in [1, n]$ are distinct and $\varepsilon, \varepsilon' \in \{\pm 1\}$ are signs).

Since the interval between two roots lies in a subspace of dimension 2, it is geometrically clear that there exist two "natural" ways to put an order on the roots in this interval. These two natural orders will be called *interval orderings*.

1.2.34 Definition (Interval ordering). Let Φ be root system and let S be a subset of Φ^{indiv} which does not contain a pair of opposite roots. Denote by k the cardinality of S. An *interval ordering of S* is a tuple $(\alpha_1, \dots, \alpha_k)$ such that $S = \{\, \alpha_1, \dots, \alpha_k \,\}$ and $]\alpha_i, \alpha_j[^{\mathrm{indiv}}_{\mathrm{Cox}} = \{\, \alpha_r \mid i < r < j \,\}$ for all $i < j \in [1, k]$.

1.2.35 Remark. Let S be a subset of a root system Φ^{indiv} and assume that there exists an interval ordering $(\alpha_1, \dots, \alpha_k)$ of S. Then we have

$$S = \{\, \alpha_1, \dots, \alpha_k \,\} = \{\, \alpha_1, \alpha_k \,\} \cup \,]\alpha_1, \alpha_k[^{\mathrm{indiv}}_{\mathrm{Cox}} = [\alpha_1, \alpha_k]^{\mathrm{indiv}}_{\mathrm{Cox}}.$$

Thus if there exists an interval ordering of S, then S is the set of indivisible roots in a (closed) root interval. Conversely, any set of the form $S = [\alpha, \beta]^{\mathrm{indiv}}_{\mathrm{Cox}}$ for $\alpha, \beta \in \Phi^{\mathrm{indiv}}$ has an interval ordering $(\alpha_1, \dots, \alpha_k)$, and the only other interval ordering of S is $(\alpha_k, \dots, \alpha_1)$.

1.2.36 Note. We could also define "crystallographic interval orderings" by requiring that $]\alpha_i, \alpha_j[^{\mathrm{indiv}} = \{\, \alpha_r \mid i < r < j \,\}$ for all $i < j \in [1, k]$. However, there is no need to do this: We will only use interval orderings of root intervals of length 2, and for these there is no difference by Note 1.2.33 in our situation.

Using interval orderings, we can introduce a practical way of indexing the roots of Φ in the rank-2 case, which depends only on the choice of a root base.

1.2.37 Remark (Rank-2 root systems, [TW02, (4.14), (4.15)]). Assume that Φ is reduced and of rank 2, put $n := |\Phi|/2$ and choose a root base (α, β). Denote by $(\alpha_1, \dots, \alpha_n)$ the unique interval ordering of $[\alpha, \beta]_{\mathrm{Cox}}$ with $\alpha_1 = \alpha$ and $\alpha_n = \beta$ (see Remark 1.2.35). Further, put $\alpha_{i+n} := -\alpha_i$ for all $i \in [1, n]$. In this way, we have identified Φ with the set $[1, 2n]$. It is practical to define $\alpha_z := \alpha_{z'}$ for all $z \in \mathbb{Z}$ where z' is the unique element of $[1, 2n]$ which is congruent to z modulo $2n$. Then for all $i < j \in \mathbb{Z}$ with $j - i \le n - 1$, the tuple $(\alpha_i, \dots, \alpha_j)$ is the unique interval ordering of $]\alpha_i, \alpha_j[_{\mathrm{Cox}}$ which starts with α_i. Observe that for all $z \in \mathbb{Z}$, the pair $\Delta' = (\alpha_z, \alpha_{z+n-1})$ is a root base, and the labeling which is induced by Δ' is given by $\alpha'_i = \alpha_{z-1+i}$ for all $i \in \mathbb{Z}$. Further, we have

$$\alpha_i^{\sigma(\alpha_j)} = \alpha_{2j+n-i}$$

for all $i, j \in \mathbb{Z}$.

1.2.38 Definition (Rank-2 labeling). Assume that Φ is reduced of rank 2 and let (α, β) be a root base. The *rank-2 labeling of Φ induced by (α, β)* is the map $\mathbb{Z} \to \Phi, i \mapsto \alpha_i$ which we constructed in Remark 1.2.37. A *rank-2 labeling of Φ* is the rank-2 labeling induced by some root base.

The notion of root intervals leads in a natural way to the notion of closed sets of roots. We will study this property more closely in section 2.3.

1.2.39 Definition (Closed set of roots). Let Ψ be a subset of Φ. Then Ψ is called *closed* if $]\alpha, \beta[_{\mathrm{Cox}} \subseteq \Psi$ for all non-proportional $\alpha, \beta \in \Psi$, and it is called *crystallographically closed* if $]\alpha, \beta[\subseteq \Psi$ for all $\alpha, \beta \in \Psi$.

1.2.40 Remark. Every closed set is also crystallographically closed, but the converse is not true.

We now refine our notion of root subsystems from Definition 1.2.10.

1.2.41 Lemma. *Let V' be a subspace of the Euclidean space V surrounding Φ and put $\Phi' := V' \cap \Phi$. Assume that Φ' is non-empty. Then the following hold:*

- *(a) Φ' is a root system in V' which is closed in Φ. If Φ is reduced and/or crystallographic, then Φ' has the same properties.*
- *(b) Let Δ' be a root base of Φ'. Then there exists a root base Δ of Φ such that $\Delta \cap \Phi' = \Delta'$.*
- *(c) Let Π' be a positive system in Φ'. Then there exists a positive system Π in Φ such that $\Pi \cap \Phi' = \Pi'$.*

Proof. The first assertion is easily verified. Now let Δ' be a root base of Φ'. It is proven in [Bou81, Proposition VI.1.24] for crystallographic Φ that there exists a root base Δ of Φ containing Δ', and the same proof applies in our more general situation. Thus Δ' is contained in $\Delta \cap \Phi'$. Assume that there exists a root $(\delta \in \Delta \cap \Phi') \setminus \Delta'$. Then $\Delta' \cup \{\delta\}$ is a linearly independent subset of Φ' of higher cardinality than Δ. This contradicts the fact that Δ' is an $\mathbb{R}$-basis of $\langle \Phi' \rangle_{\mathbb{R}}$, so we must have $\Delta \cap \Phi' = \Delta'$. Thus the second assertion holds. The third assertion follows from the second one by the correspondence between root bases and positive systems (see Proposition 1.2.24). □

1.2.42 Definition (Closed and parabolic root subsystems). Let Φ' be a non-empty subset of Φ. Then Φ' is called a *(crystallographically) closed root subsystem of Φ* if it is (crystallographically) closed in the sense of Definition 1.2.39 and a root subsystem in the sense of Definition 1.2.10. It is called a *parabolic root subsystem of* Φ if there exists a subspace W of V such that $\Phi' = \Phi \cap W$. For any non-empty subset A of Φ, the *root subsystem spanned by* A is $\Phi \cap \langle A \rangle_{\mathbb{R}}$.

1.2.43 Note. By Lemma 1.2.41, every parabolic root subsystem is a closed root subsystem, and clearly every closed root subsystem is a root subsystem. We will show in Proposition 2.5.11 that closed root subsystems Φ' of Φ give rise to Φ'-graded subgroups of Φ-graded groups. Most of the time, we will consider root subsystems spanned by a subset (often of cardinality 2) of Φ, which are parabolic by definition.

1.2.44 Remark. Every root subsystem Φ' of Φ is contained in the parabolic subsystem $\Phi'' := \langle \Phi' \rangle_{\mathbb{R}} \cap \Phi$. Note that the ranks of Φ' and Φ'' are identical because $\langle \Phi' \rangle_{\mathbb{R}} = \langle \Phi'' \rangle_{\mathbb{R}}$.

1.2.45 Lemma. *Let Δ be a root base of Φ and let Δ' be a subset of Δ. Then Δ' is a root base of the subsystem that it spans.*

Proof. It is clear that Δ' satisfies the axioms of Definition 1.2.22. □

1.2.46 Remark. The statement of Lemma 1.2.41 (c) remains true for closed root subsystems which are not parabolic: If Π' is a positive system of a (crystallographically) closed root subsystem Φ' in Φ, then there exists a hyperplane H' in $\langle \Phi' \rangle_{\mathbb{R}}$ which does not meet Φ' such that all roots in Π' lie on one side of H' and all roots in $-\Pi'$ lie on the other side of H. We can extend H' to a hyperplane H in V which does not meet Φ. Now all roots in Π' lie on one side of H and all roots in $-\Pi'$ lie on the other side. Thus H defines a positive system Π in Φ such that $\Pi = \Pi' \cap \Phi$.

1.2.E The Classification of Root Systems

We now turn to the classification of crystallographic root systems. We will also make some remarks on the non-crystallographic case.

1.2.47 Definition (Irreducible root system). A root system Φ is called *reducible* if there exists a non-empty proper subset I of Φ which is orthogonal to $\Phi \setminus I$ (which means that $\alpha \cdot \beta = 0$ for all $\alpha \in I$ and $\beta \in \Phi \setminus I$). It is called *irreducible* otherwise. An *irreducible component of* Φ is a root subsystem Φ' of Φ which is irreducible and orthogonal to $\Phi \setminus \Phi'$.

It should be noted that the properties "reduced" and "reducible" have nothing to do with each other despite sounding similar.

1.2.48 Remark. Using induction, it is not difficult to see that every root system can be written as a disjoint union of irreducible components. This decomposition is, in fact, unique. Further, any irreducible component Φ' of Φ is a parabolic root subsystem of Φ because it is orthogonal to $\Phi \setminus \Phi'$.

In order to state a precise classification theorem, we need a proper notion of isomorphism between root systems.

1.2.49 Definition (Isomorphic root systems, [Hum72, Section 9.2], [Bou81, Proposition VI.4.1]). Let Φ' be another root system in some Euclidean space V'. We say that Φ and Φ' are *isomorphic* if there exists a vector space isomorphism $f\colon \langle \Phi \rangle_{\mathbb{R}} \to \langle \Phi' \rangle_{\mathbb{R}}$ (which is not necessarily an isometry) such that $f(\Phi) = \Phi'$ and such that $\langle f(\alpha) | f(\beta) \rangle = \langle \alpha | \beta \rangle$

1.2.50 Example. For any $\lambda \in \mathbb{R} \setminus \{\pm 1\}$, the map $f\colon V \to V, v \mapsto \lambda v$ is an isomorphism between Φ and $\lambda\Phi$, but it is not an isometry.

1.2.51 Remark. Example 1.2.50 shows that the length of a root is not invariant under isomorphisms. However, the ratio of root lengths is: For all roots α, β, we have $\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle^{-1} = \frac{\alpha\cdot\alpha}{\beta\cdot\beta}$. In particular, the knowledge of the Cartan numbers is enough to decide whether α is shorter or longer than β. Further, all root intervals from Definition 1.2.30 are invariant under isomorphisms. The same holds for the properties of orthogonality and opposition. Since these are all the properties of root systems that we will use, we are thus free to treat isomorphic root systems as "the same".

1.2.52 Remark (List of crystallographic root systems). We now give a list of all irreducible crystallographic root systems up to isomorphism. For the moment, we introduce only their names. Precise definitions can be found in [Hum72, Section 12.1], [Hum90, Section 2.10] and [Bou81, Section VI.4]. First of all, there are five infinite families of such root systems: $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 2}$, $(C_n)_{n\geq 2}$, $(BC_n)_{n\geq 1}$ and $(D_n)_{n\geq 4}$. Further, there exist the root systems F_4, E_6, E_7, E_8 and G_2 which are called exceptional because they do not belong to an infinite family. In each case, the subscript denotes the rank of the root system. All these root systems are crystallographic. Further, they are pairwise not isomorphic except for the pair $B_2 \cong C_2$ and they are reduced except for BC_n, which is not reduced. We will provide an explicit description of the root systems of types A, D, B, BC, C and F_4 in Remarks 5.2.3, 5.2.4, 7.2.2, 9.1.2, 9.1.3 and 10.3.1, respectively.

1.2.53 Theorem (Classification of root systems). *Let Φ be an irreducible crystallographic root system and denote its rank by n. Then Φ is isomorphic to one of the root systems in Remark 1.2.52.*

Proof. For reduced Φ, this is proven in [Hum72, Theorem 11.4] and [Bou81, Theorème VI.4.3]. The extension to the non-reduced case and can be found in [Bou81, Section VI.4.14]. □

1.2.54 Remark. By computing the Cartan integers of the root systems in Remark 1.2.52, one can show that a reduced root system is uniquely determined by its Cartan matrix. This cedes to be true for non-reduced root systems, however: B_n and BC_n have the same Cartan matrix. In fact, we will see in Remarks 7.2.2 and 9.1.2 that their standard root bases in the standard representation are exactly the same.

1.2.55 Remark (Coxeter and Dynkin diagrams). Let Δ be a root base of Φ. For all $\delta, \delta' \in \Delta$ we denote by $m_{\delta\delta'} \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ the order of $\sigma_\delta\sigma_{\delta'}$ in the Weyl group. The matrix $(m_{\delta\delta'})_{\delta,\delta'\in\Delta}$ is called the *Coxeter matrix of* Φ. The Coxeter matrix is independent of the choice of Δ and we will see in Remark 1.3.13 that it determines the Weyl group of Φ up do isomorphism. However, it does not determine Φ up to isomorphism: The root systems B_n and C_n have the same Coxeter matrix and the same Weyl group.

Another way of expressing the Coxeter matrix of Φ is by its *Coxeter diagram (with respect to* Δ*)*: This is the edge-labelled graph whose vertex set is Δ and where two vertices $\delta \neq \delta'$ are connected by an edge if and only if $m_{\delta\delta'} \geq 3$. If δ, δ' are connected by an edge, then this edge is labelled by $m_{\delta,\delta'}$ if $m_{\delta,\delta'} \geq 3$, and otherwise it is not labelled. Edges without labels are also called *simple edges*. If Φ is crystallographic, then the *Dynkin diagram of* Φ is essentially the Coxeter diagram of Φ with some additional "decorations" which tell us for every pair (α, β) of roots in Δ whether $\|\alpha\| < \|\beta\|$, $\|\alpha\| > \|\beta\|$ or $\|\alpha\| = \|\beta\|$. This additional information allows us to reconstruct the Cartan matrix of Φ and thus the Dynkin diagram determines the root system up to isomorphism.

1.2.56 Remark. For any parabolic root subsystem Φ' of Φ, the Dynkin diagram of Φ' is a subgraph of the Dynkin diagram of Φ. This is a consequence of Lemma 1.2.41 (b).

1.2.57 Remark (Non-crystallographic root systems). A classification of reduced root systems which are not assumed to be crystallographic can be found in [Hum90, Chapter 2]. Note that the reducedness assumption is harmless by Construction 1.2.16. The only non-crystallographic root systems of rank at least 3 which appear in this classification are H_3 and H_4. We will not study them and their associated root graded groups in this work, but we will sometimes remark on them in informal discussions.

The following terminology will be used often.

1.2.58 Definition (Type of a root system). Let Φ be a crystallographic root system and let $X \in \{ A, B, BC, C, D, E, F, G \}$. We say that Φ *is of type* X if it is isomorphic to X_n for some $n \in \mathbb{N}_+$.

1.2.59 Note. We will also refer to H_3 and H_4 as the "root systems of type H", though we have not defined the notion of isomorphism for non-crystallographic root systems.

1.2.60 Notation (Direct products of root systems). Let $\Phi_1, \dots, \Phi_n$ be the irreducible components of Φ and let $X_1, \dots, X_n$ denote their respective types. Then the *type of* Φ is $X_1 \times \cdots \times X_n$.

Using the classification of crystallographic root systems, we can easily state the definition of simply-laced root systems. We will give some more equivalent characterisations of these root systems in Remark 5.2.2.

1.2.61 Definition (Simply-laced root system). A crystallographic root system Φ is called *simply-laced* if each irreducible component of Φ is of type A, D or E. (We do not assume that all irreducible components have the same type.)

1.3 Weyl Groups

1.3.1 Notation for this section. We denote by Φ an arbitrary root system in some Euclidean space $(V, \cdot)$.

In this section, we study properties of the Weyl group of Φ that we have introduced in Definition 1.2.13.

1.3.2 Remark. Recall from Remark 1.2.15 that the set Φ^{indiv} of indivisible roots in Φ is a reduced root system. Since $\sigma_{\lambda\alpha} = \sigma_\alpha$ for all $\alpha \in \Phi$ and $\lambda \in \mathbb{R} \setminus \{0\}$, it has the same Weyl group as Φ. As a consequence, most statements about Weyl groups of reduced root systems can easily be generalised to arbitrary root systems.

1.3.A Basic Properties

We begin with some properties of the action of the Weyl group on Φ. This comprises several transitivity properties.

1.3.3 Lemma. *The Weyl group* $\mathrm{Weyl}(\Phi)$ *acts faithfully on the subset* Φ *of* V. *In particular, it is a finite group.*

Proof. This is proven in [Hum90, p. 7] for reduced root systems, which implies the general assertion. □

1.3.4 Proposition. *Let* α *be an indivisible root and let* Δ *be a root base of* Φ. *Then there exist* $\delta \in \Delta$ *and* $w \in \mathrm{Weyl}(\Phi)$ *such that* $\alpha = \delta^w$.

Proof. This is proven for reduced root systems in [Hum90, Corollary 1.5] and for crystallographic root systems in [Bou81, Proposition VI.1.15, p. 154]. The general assertion follows from the one for reduced root systems because both α and Δ are contained in Φ^{indiv} (by Axiom 1.2.22 (iii) in the latter case). □

1.3.5 Proposition (Corollary of 1.3.4 and 1.3.12). *Let* α *be any root and let* Δ *be a rescaled root base of* Φ. *Then there exist* $\delta, \delta_1, \dots, \delta_n \in \Delta$ *and* $\lambda \in \mathbb{R}_{>0}$ *and a word* $\bar{\delta}$ *over* Δ *such that* $\lambda\alpha = \delta^{\sigma(\delta_1 \cdots \delta_m)}$. *In particular,* $\sigma_\alpha = \sigma_{\bar{\rho}}$ *where* $\bar{\rho} := (-\delta_n, \dots, -\delta_1, \delta, \delta_1, \dots, \delta_n)$. *If* Φ *is reduced, then we have* $\lambda = 1$.

1.3.6 Proposition. *The Weyl group of* Φ *acts simply transitively on the set of root bases and positive systems of* Φ.

Proof. This is proven in [Hum90, Theorem 1.4, 1.8] for reduced root systems. Since any root base of Φ is contained in Φ^{indiv} by Axiom 1.2.22 (iii), the general assertion follows. □

1.3.7 Lemma (Corollary of 1.3.6). *Let* Φ' *be a parabolic root subsystem of* Φ, *let* Δ' *be a root base of* Φ' *and let* Δ *be a root base of* Φ. *Then there exists* $w \in \mathrm{Weyl}(\Phi)$ *such that* Δ' *is a subset of* Δ^w.

Proof. By Lemma 1.2.41 (b), there exists a root base Δ_2 of Φ containing Δ'. Using Proposition 1.3.6, we find $w \in \mathrm{Weyl}(\Phi)$ such that $\Delta_2 = \Delta^w$, and the assertion follows. □

In crystallographic root systems, "all roots of the same length are essentially the same". The following result makes this statement precise.

1.3.8 Proposition. *Assume that* Φ *is crystallographic and irreducible and let* $\alpha \in \Phi$. *Then the orbit of* α *under the Weyl group is* $\{ \beta \in \Phi \mid \|\alpha\| = \|\beta\| \}$, *the set of roots which have the same length as* α.

Proof. This is proven in [Hum72, Lemma 10.4.C] for reduced Φ. If Φ is crystallographic, irreducible and not reduced, then it is of type BC. It is not difficult to verify that the assertion is true in this case as well. □

In light of Proposition 1.3.8, we introduce the following notation.

1.3.9 Notation (Orbits). Assume that Φ is irreducible and either crystallographic or of type H. We put $\mathrm{Orb}(\Phi) := (O_1, \dots, O_k)$ where $O_1, \dots, O_k$ are the orbits of Φ under the Weyl group, ordered by ascending length of roots. This means that for all $i < j \in [1, k]$ and all $\alpha \in O_i$, $\beta \in O_j$, we have $\|\alpha\| < \|\beta\|$. Further, we denote by $\mathrm{Orb}^{\mathrm{red}}(\Phi)$ the sub-tuple of $\mathrm{Orb}(\Phi)$ which contains only the indivisible orbits, that is, the orbits consisting of indivisible roots.

1.3.10 Note. The point of Notation 1.3.9 is that, in Definition 4.3.4 of parametrisations of root graded groups, we will need to fix some ordering on the set of orbits. The ordering of ascending root length is simply a practical way to introduce such an ordering for all irreducible root systems. This definition makes sense for crystallographic root systems by Proposition 1.3.8 and it also makes sense for root systems of type H because these root systems have only one orbit of roots.

The following result about point stabilisers will be needed in our proof of the parametrisation theorem in chapter 4.

1.3.11 Proposition. *Let $\alpha \in \Phi$. Then the stabiliser of α in the Weyl group W is the subgroup of W which is generated by $\{ \sigma_\beta \mid \beta \in \Phi, \beta \cdot \alpha = 0 \}$.*

Proof. This is proven in [Hum90, Theorem 1.12] for reduced root systems, and the general assertion follows easily. □

1.3.B The Presentation of Weyl Groups and the Word Problem

We can find a simple finite presentation of the Weyl group. At first, we need a practical generating set.

1.3.12 Proposition. *Let Δ be a rescaled root base of Φ. Then the Weyl group of Φ is generated by the reflections $\{ \sigma_\delta \mid \delta \in \Delta \}$.*

Proof. This is proven in [Hum90, Theorem 1.5] for reduced root systems and root bases. The general result follows because $\sigma_\alpha = \sigma_{\lambda\alpha}$ for all $\alpha \in \Phi$ and $\lambda \in \mathbb{R} \setminus \{0\}$ and because for any root system Φ, the set Φ^{indiv} is a reduced root system (with the same Weyl group). □

1.3.13 Remark (Presentation of Weyl groups). Let W be the Weyl group of Φ and let Δ be any rescaled root base of Φ. For all $\delta, \delta' \in \Delta$, denote the order of $\sigma_\delta\sigma_{\delta'}$ in W by $m(\delta, \delta')$. Then we have the relation $(\sigma_\delta\sigma_{\delta'})^{m(\delta,\delta')} = 1$ for all $\delta, \delta' \in \Delta$, simply by the definition of $m(\delta, \delta')$. Put $S := \{ \sigma_\delta \mid \delta \in \Delta \}$. It is a non-trivial fact that W is not only generated by S, but that W is in fact isomorphic to the abstract group generated by the set S with respect to the relations $(\sigma_\delta\sigma_{\delta'})^{m(\delta,\delta')} = 1$ for $\delta, \delta' \in \Delta$. A proof of this well-known result can be found in [Hum90, Theorem 1.9]. For $\delta = \delta'$, this relation says that $\sigma_\delta^2 = 1$. Using this fact and Notation 1.1.7, we can rewrite the relation $(\sigma_\delta\sigma_{\delta'})^{m(\delta,\delta')} = 1$ for $\delta \neq \delta'$ as $P_{m(\delta,\delta')}(\sigma_\delta, \sigma_{\delta'}) = P_{m(\delta,\delta')}(\sigma_{\delta'}, \sigma_\delta)$. In this form, these relations are called the *braid relations*.

We now turn to the word problem, which is of course strongly related to the presentation of the Weyl group from Remark 1.3.13. By Remark 1.3.2, it is not surprising that the classical solution of the word problem generalises to the case of rescaled root bases without a problem.

1.3.14 Definition (Homotopy). Let Δ be a rescaled root base and let $\bar{\alpha}, \bar{\beta}$ be two words over $\Delta \cup (-\Delta)$. We say that $\bar{\alpha}, \bar{\beta}$ are *square-homotopic* if there exist words $\bar{\rho}, \bar{\zeta}$ over $\Delta \cup (-\Delta)$ and a root $\delta \in \Delta$ such that $\bar{\alpha} = \bar{\rho}\bar{\zeta}$ and $\bar{\beta} = \bar{\rho}\delta\delta\bar{\zeta}$ or such that $\bar{\beta} = \bar{\rho}\bar{\zeta}$ and $\bar{\alpha} = \bar{\rho}\delta\delta\bar{\zeta}$. We say that $\bar{\alpha}, \bar{\beta}$ are *braid-homotopic* if there exist words $\bar{\rho}, \bar{\zeta}$ and simple roots $\delta, \delta' \in \Delta$ such that $\bar{\alpha} = \bar{\rho}P_m(\delta, \delta')\bar{\zeta}$ and $\bar{\beta} = \bar{\rho}P_m(\delta', \delta)\bar{\zeta}$ where m denotes the order of $\sigma_\delta\sigma_{\delta'}$ in the Weyl group and where P_m is as in Notation 1.1.7. We say that $\bar{\alpha}, \bar{\beta}$ are *elementary homotopic* if they are square-homotopic or braid-homotopic, and we say that they are *homotopic* if they lie in the symmetric reflexive transitive closure of the relation "elementary homotopic".

1.3.15 Note. We define homotopy for words over $\Delta \cup (-\Delta)$ and not merely for words over Δ. In the context of the word problem, this may seem like an unnecessary generalisation because of the relation $\sigma_{-\delta} = \sigma_\delta$. However, we will later use words $\bar{\delta} = (\delta_1, \dots, \delta_k)$ over $\Delta \cup (-\Delta)$ to represent group elements $w_{\bar{\delta}} = w_{\delta_1} \cdots w_{\delta_k}$ where $(w_\delta)_{\delta \in \Delta}$ is a fixed family of so-called Weyl elements in a root graded group and where $w_{-\delta} := w_\delta^{-1}$ for all $\delta \in \Delta$. In this setting, it is not (necessarily) true that $w_{-\delta} = w_\delta$, and thus it makes sense to consider words over $\Delta \cup (-\Delta)$.

1.3.16 Proposition. *Let Δ be a rescaled root base of Φ and let $\bar{\alpha}, \bar{\beta}$ be two words over Δ. Then $\bar{\alpha}, \bar{\beta}$ are homotopic if and only if $\sigma_{\bar{\alpha}} = \sigma_{\bar{\beta}}$.*

Proof. It is clear from Remark 1.3.13 that homotopic words $\bar{\alpha}$, $\bar{\beta}$ automatically satisfy $\sigma_{\bar{\alpha}} = \sigma_{\bar{\beta}}$. The other implication is proven in [Hum90, Theorem 8.1] for reduced root systems and root bases, and the general assertion is an easy consequence. $\square$

1.3.17 Note. Proposition 1.3.16 is only a weaker form of the solution of the word problem in Weyl groups, and not sufficient to actually decide the word problem computationally. However, it will be adequate for our purposes.

1.3.18 Note (continuing 1.3.15). Observe that the words $\bar{\alpha}$ and $\bar{\beta}$ in Proposition 1.3.16 are not allowed to contain letters from $-\Delta$. As a counterexample, the words $\bar{\alpha} := (\delta, -\delta)$ (for any $\delta \in \Delta$) and the empty word represent the same element in the Weyl group, but they are not homotopic. This fact will be relevant in the proof of Lemma 4.5.5.

1.3.C The Length Function

1.3.19 Notation for this section. From now on, we denote by Φ a root system, by Δ a rescaled root base of Φ, by Π the unique positive system in Φ which contains Δ (as in Proposition 1.2.24) and by W the Weyl group of Φ.

The length of any element w in the Weyl group can be characterised as the cardinality of a certain set $N(w)$ of roots. This fact is usually proved by induction on the length of w. In the proof, the set $N(w)$ is "constructed" element by element, which provides a natural ordering of the set $N(w)$. We will show in section 2.4 that in any Φ-graded group, a certain product map with respect to this natural ordering of $N(w)$ is bijective. This will imply that, in the crystallographic setting, the product map with respect to any ordering of the set Π^{indiv} is bijective.

In this subsection, we introduce the aforementioned natural ordering of $N(w)$, using the notion of root sequences. Standard references are [Hum90, 1.6] and [How96], the latter being closer to our presentation. We will also see that there exists a unique element ρ in W of highest length, and that this element satisfies $N(\rho) = \Pi^{\text{indiv}}$. A corollary of our observations is that the length of any $w \in W$ is indeed the cardinality of $N(w)$, but this fact will not be needed later.

In the literature, the results of this section are usually only phrased for reduced root systems. They easily generalise to arbitrary root systems, but we have to make sure to consider Π^{indiv} instead of Π. Further, the main results (Lemma 1.3.25 and Proposition 1.3.28) hold only for proper root bases.

1.3.20 Definition (Length function). Let $w \in W$. An *expression of w* is a word $\bar{\delta} = (\delta_1, \dots, \delta_k)$ over Δ such that $w = \sigma_{\delta_1} \cdots \sigma_{\delta_k}$, and the number k is called the *length of* $\bar{\delta}$. If Δ is not clear from the context, then an expression of w is also called a Δ-*expression of* w. The *length of* w is the minimal length of an expression of w, and it is denoted by $\ell_\Delta(w)$ (or simply by $\ell(w)$). A *reduced expression of* w is an expression of w of length $\ell(w)$, and an arbitrary word $\bar{\delta}$ over Δ is called *reduced* if it is a reduced expression of $\sigma_{\bar{\delta}}$.

1.3.21 Definition. For all $w \in W$, we define

$$N(w) := N_\Pi(w) := \{ \alpha \in \Pi^{\text{indiv}} \mid \alpha^{w^{-1}} \in -\Pi \}.$$

1.3.22 Note. In most references (for example in [Hum90, Section 1.6]), the Weyl group acts on the root system from the left and the root system is reduced. Then $N(w)$ is defined as $\{ \alpha \in \Pi \mid w(\alpha) \in -\Pi \}$.

1.3.23 Lemma. *Let $w \in W$ and let $\alpha \in \Phi^{indiv} \setminus N(w)$. Then there exists a positive system Π' which contains $N(w)$ but not α.*

Proof. Since α is not contained in $N(w)$, we have either $\alpha \in -\Pi$ or $\alpha^{w^{-1}} \in \Pi$. In the first case, we can choose $\Pi' := \Pi$. In the second case, we can choose $\Pi := (-\Pi)^w$. □

1.3.24 Lemma. *For all $\delta \in \Delta$, we have $(\Pi \setminus \mathbb{R}_{>0}\delta)^{\sigma(\delta)} = \Pi \setminus \mathbb{R}_{>0}\delta$. In particular, $N(\sigma_\delta) = \{\delta\}$.*

Proof. It is proven in [Hum90, Proposition 1.4] for reduced Φ that $(\Pi \setminus \{\delta\})^{\sigma(\delta)} = \Pi \setminus \{\delta\}$. The general assertion follows. □

The following result is, essentially, the induction step of Proposition 1.3.28. It provides an explicit description and even a natural ordering of the set $N(w)$. In Proposition 2.3.22, we will show that this ordering has the property of being "extremal", essentially by Lemma 1.3.23.

1.3.25 Lemma. *Assume that Δ is a proper root base. Let $w \in W$ and let $\delta \in \Delta$ such that $\ell(w\sigma_\delta) > \ell(w)$. Then $N(w\sigma_\delta) = N(w)^{\sigma(\delta)} \sqcup \{\delta\}$ (where "$\sqcup$" denotes disjoint union).*

Proof. This statement is the essence of the proof of [Hum90, Lemma 1.6], albeit with different conventions (see Note 1.3.22). □

The natural ordering on $N(w)$ is captured by the notion of root sequences.

1.3.26 Definition (Root sequence). Let $\bar{\delta} = (\delta_1, \dots, \delta_k)$ be a reduced word over Δ. The tuple $(\beta_1, \dots, \beta_k)$ where $\beta_i := \delta_i^{\sigma(\alpha_{i+1} \cdots \alpha_k)}$ for all $i \in [1, k-1]$ and $\beta_k := \delta_k$ is called the *root sequence associated to* $\bar{\delta}$. Further, the tuple $(\beta_k, \dots, \beta_1)$ is called the *inverse root sequence associated to* $\bar{\delta}$.

1.3.27 Remark. Let $\bar{\delta} = (\delta_1, \dots, \delta_k)$ be a reduced word over Δ such that $k > 0$ and put $s := \sigma(\delta_k)$. Denote by $(\alpha_1, \dots, \alpha_{k-1})$ the root sequence of $(\delta_1, \dots, \delta_{k-1})$. Then it is clear from Definition 1.3.26 that the root sequence of $\bar{\delta}$ is $(\alpha_1^s, \dots, \alpha_{k-1}^s, \delta_k)$.

1.3.28 Proposition. *Assume that Δ is a proper root base. Let $w \in W$, let $\bar{\delta} = (\delta_1, \dots, \delta_k)$ be a reduced expression of w and let $\bar{\alpha} = (\alpha_1, \dots, \alpha_k)$ be the associated root sequence. Then $N(w) = \{ \alpha_1, \dots, \alpha_k \}$ and $|N(w)| = k = \ell(w)$.*

Proof. This is clear for $k = 0$ because $N(1_W) = \varnothing$. For $k > 0$, the assertion follows from Lemma 1.3.25 by induction on k. □

We will show in Proposition 2.3.22 that the ordering of $N(w)$ given by an inverse root sequence has the desirable property of being extremal.

1.3.29 Proposition ([Hum90]). *There exists a unique element ρ of W such that $\ell(\rho) \geq \ell(w)$ for all $w \in W$. This element has length $|\Pi^{indiv}|$ and satisfies $\rho^{-1} = \rho$ and $\rho(\Pi) = -\Pi$. In particular, $N(\rho) = \Pi^{indiv}$.*

Proof. This is proven in [Hum90, Theorem 1.8] for reduced root systems. In general, the subset Φ^{indiv} of Φ is a reduced root system which has the same Weyl group as Φ, and the assertion follows easily from this observation. □

1.3.30 Definition (Longest element). The element ρ from Proposition 1.3.29 is called the *longest element of W (with respect to Δ)*.

1.4 Foldings of Root Systems

We have seen in section 1.2 that there are many ways to see root systems as subsystems of other root systems. Foldings, on the other hands, behave more like "quotients of root systems". If a root system Φ' is a folding of Φ, then Φ'-graded groups can be constructed from Φ-graded group in a way which we will describe in section 2.6. We will use this strategy in section 10.4 to construct F_4-graded groups from E_6-graded groups.

In this section, we follow the exposition in [Car72, 13.1, 13.2].

1.4.1 Notation for this section. We denote by Φ a crystallographic root system in the Euclidean space $(V, \cdot)$ such that V is generated by Φ, and we choose a root base Δ of Φ. We denote by W the Weyl group of Φ and by ρ an isomorphism of the Coxeter diagram of Φ with respect to Δ. That is, ρ is a vector space automorphism of V such that $\rho(\Delta) = \Delta$ and such that the group elements $\sigma_\alpha\sigma_\beta$ and $\sigma_{\rho(\alpha)}\sigma_{\rho(\beta)}$ have the same order for all $\alpha, \beta \in \Delta$. We denote by $\tau\colon V \to V$ the unique automorphism of V which maps any $v \in V \setminus \{0\}$ to the unique element $w \in \mathbb{R}_{>0}\rho(v)$ with $w \cdot w = v \cdot v$. Further, we put

$$F := \{\, v \in V \mid \tau(v) = v \,\}$$

and we denote by $\pi\colon V \to F$ the orthogonal projection on F. Finally, we set $\Phi' := \pi(\Phi)$.

1.4.2 Notation. Following the conventions in [Car72, 13.1, 13.2], we let the Weyl group of Φ act on V from the left side in this section.

1.4.3 Note. The map ρ is not necessarily an isometry of V, and the map τ is precisely the unique isometry which maps any $v \in V \setminus \{0\}$ to a positive scalar multiple of $\rho(v)$. If all roots in Φ have the same length, then $\tau = \rho$. In practice, we will only need the case $\Phi = E_6$ in which this assumption is satisfied.

1.4.4 Lemma ([Car72, p. 217]). *Let $v \in V$ and denote by J the orbit of v under τ in V. Then $\pi(v)$ is the average of the vectors in J. That is, we have*

$$\pi(v) = \frac{1}{|J|}\sum_{u \in J} u.$$

Proof. Put $v' := \frac{1}{|J|}\sum_{u \in J} u$. Then $\pi(v') = v'$, so v' is contained in F. Further, since τ is an isometry, we have for all $k \in \mathbb{N}_+$ and $u \in F$ that

$$u \cdot v = \tau^k(u) \cdot \tau^k(v) = u \cdot \tau^k(v).$$

Hence v' is an element of F with $u \cdot v = u \cdot v'$ for all $u \in F$. Since $\pi(v)$ is the unique element with this property, it follows that $\pi(v) = v'$. $\square$

1.4.5 Definition (Folded Weyl group). We put

$$W' := \{ w \in W \mid \tau^{-1} w \tau = w \} = \{ w \in W \mid w\tau = \tau w \}.$$

1.4.6 Lemma. *For all $w \in W'$ and $v \in V$, we have $\pi(w(v)) = w(\pi(v))$.*

Proof. Denote by J and J' the orbits of v and $\pi(v)$ in V, respectively. Since $w\tau^k(v) = \tau^k(w(v))$ for all $k \in \mathbb{N}_+$ by the definition of W', we have $w(J) = J'$. Since $\pi(w(v))$ is the average of J' and $\pi(v)$ is the average of J by Lemma 1.4.4, the assertion follows. $\square$

1.4.7 Lemma ([Car72, 13.1.1]). *For all $w \in W'$, we have $w(F) = F$, and the action of W' on F is faithful.*

Proof. For all $w \in W'$ and all $u \in F = \{ v \in V \mid \tau(v) = v \}$, we have

$$\tau\big(w(v)\big) = w\big(\tau(v)\big) = w(v),$$

so $w(F) \subseteq F$. Since W' consists of automorphism of V, the first assertion follows. The remaining part of the proof can be found in [Car72, 13.1.1]. $\square$

1.4.8 Notation. Let J be any subset of Δ. We denote by W_J the subgroup of W which is generated by $\{ \sigma_\alpha \mid \alpha \in J \}$. This is the Weyl group of the root system $\langle J \rangle_{\mathbb{R}} \cap \Phi$ with respect to the root base J, and we denote its longest element by w_0^J.

1.4.9 Lemma ([Car72, 13.1.2, 13.1.3]). *The following hold:*

(a) For any orbit J of ρ in Δ, the element w_0^J lies in W', and its action on F agrees with the one of $\sigma_{\pi(\alpha)}$ for every $\alpha \in J$.

(b) The set $\{ w_0^J \mid J$ orbit of ρ in $\Delta \}$ generates W'.

The following result is a simple corollary of Lemma 1.4.9, but its specific form will be useful later.

1.4.10 Lemma. *Let $\alpha \in \Phi$, $\beta \in \Delta$ and denote by J the orbit of β under ρ. Then*

$$\pi\big(w_0^J(\alpha)\big) = \sigma_{\pi(\beta)}\big(\pi(\alpha)\big).$$

Proof. By Lemma 1.4.6, $\pi(w_0^J(\alpha)) = w_0^J(\pi(\alpha))$. By Lemma 1.4.9 (a),

$$w_0^J(\pi(\alpha)) = \sigma_{\pi(\beta)}(\pi(\alpha)).$$

The assertion follows. $\square$

1.4.11 Proposition ([Car72, 13.2.2]). *The following hold:*

(a) $\Phi' = \pi(\Phi)$ is a root system in F which spans F.

(b) The set of indivisible elements in $\pi(\Delta)$ is a root base of Φ'.

1.4.12 Proposition. *The following hold:*

(a) Let Π be a positive system in Φ. Then $\pi(\Pi)$ is a positive system in Φ'.

(b) Every root base and every positive system in Φ' is induced by a corresponding object in Φ via π.

(c) Let $\alpha', \beta' \in \Phi'$ be non-proportional. Then there exists a positive system in Φ which contains $\{ \gamma \in \Phi \mid \pi(\gamma) \in \{\alpha', \beta'\} \}$.

Proof. Let Π be a positive system in Φ and denote by $\bar{\Delta}$ the corresponding root base. By Proposition 1.4.11 (b), $\Delta' := \pi(\bar{\Delta})^{\text{indiv}}$ is a root base of Φ'. Now $\pi(\Pi)$ consists of positive linear combinations of Δ' while $\pi(-\Pi)$ consists of negative linear combinations of Δ, and the disjoint union of these two sets is Φ'. Hence $\pi(\Pi)$ is a positive system in Φ'. This proves (a).

Now let Π' be an arbitrary positive system in Φ' and let Π be an arbitrary positive system in Φ. By Proposition 1.3.6, there exists an element w of the Weyl group of Φ' such that $\Pi' = w(\pi(\Pi))$. By (a), we can regard w as an element of W'. Hence by Lemma 1.4.6, $w(\pi(\Pi)) = \pi(w(\Pi))$. Thus Π' is induced by the positive system $w(\Pi)$ in Φ. The assertion for root bases can be proven similarly, so (b) holds.

Now let $\alpha', \beta' \in \Phi'$ be non-proportional. Then there exists a positive system Π' in Φ' which contains α' and β'. Denote by Π a positive system in Φ such that $\pi(\Pi) = \Pi'$. Let $\alpha \in \pi^{-1}(\alpha') \cap \Phi$ be arbitrary, and suppose for a contradiction that $\alpha \notin \Pi$. Then α lies in $-\Pi$, so

$$\alpha' = \pi(\alpha) \in \pi(-\Pi) = -\Pi'.$$

This contradicts the choice of Π'. Hence $\pi^{-1}(\alpha') \cap \Phi$ is contained in Π. In a similar way, we can show that $\pi^{-1}(\beta') \cap \Phi$ is contained in Π. This finishes the proof. $\square$

1.4.13 Definition (Folding). The root system Φ' is called a *folding of* Φ. We will sometimes refer to the map $\pi \colon \Phi \to \Phi'$ as a folding as well.

Chapter 2

Root Graded Groups: Definition and General Observations

In this chapter, we introduce the protagonists of this work: Φ-graded groups where Φ is an arbitrary root system. These are groups G together with a family $(U_\alpha)_{\alpha\in\Phi}$ of subgroups such that, technical details aside, two crucial properties are satisfied: They satisfy some commutator relations and they have Weyl elements. These two conditions are relatively straightforward to define, and we will separately introduce them in sections 2.1 and 2.2, respectively.

For technical reasons, the commutator relations and Weyl elements are not enough to build a satisfactory theory of root graded groups. The reason for this is that we want to decompose certain commutators $[x_\alpha, x_\gamma]$ in G (where α, γ are non-proportional roots and $x_\alpha \in U_\alpha$, $x_\gamma \in U_\gamma$) as products $x_1 \cdots x_m$ where $x_i \in U_{\beta_i}$ and $\beta_1, \dots, \beta_m$ are the roots in the root interval $]\alpha, \gamma[_{\mathrm{Cox}}$. For this decomposition to exist and be unique, we have to assume that the product map (on the set of roots $]\alpha, \gamma[_{\mathrm{Cox}}$) in G is bijective, which need not be the case in general. This should be thought of as a non-degeneracy condition which any root graded group has to satisfy.

Unfortunately, it is not at all clear how exactly this non-degeneracy condition should be formulated. There are several possible candidates which seem like a reasonable choice (and which we compare in Note 2.5.13). In order to understand the situation better, we will first introduce and study the purely combinatorial (that is, root-system-theoretic) notion of closed sets of roots in section 2.3. This concept is indispensable for a proper study of the product maps in root graded groups. In section 2.4, we will investigate several conditions which imply the injectivity or surjectivity of certain product maps. The main result of this section is a simple criterion which guarantees the bijectivity of all product maps on closed sets of roots. This criterion serves as the third axiom for root graded groups, which we can finally introduce in section 2.5. In section 2.6, we describe a construction of Φ'-graded groups from Φ-graded groups for every folding $\pi\colon \Phi \to \Phi'$. In section 2.7, we discuss various special cases of root gradings which have already been considered in the literature, and we summarise the results which have been obtained.

2.1 Groups with Commutator Relations

2.1.1 Notation for this section. We denote by Φ an arbitrary root system.

2.1.A Definition and Comments

We begin by introducing the notion of Φ-pregradings. Without further assumptions, this is not an interesting concept, but it allows us to avoid writing "family of subgroups of G" all the time.

2.1.2 Definition (Pregradings). Let G be a group. A *Φ-pregrading of G* is a family $(U_\alpha)_{\alpha\in\Phi}$ consisting of subgroups of G. These subgroups are called the *root groups of G*. For any subset S of Φ, we denote by U_S the subgroup of G which is generated by $(U_\alpha)_{\alpha\in S}$. For any word $\bar{\alpha} = (\alpha_1, \dots, \alpha_k)$ over Φ, we define $U_{\bar{\alpha}} := U_{\alpha_1} \times \cdots \times U_{\alpha_k}$. If Φ is of rank-2 and $\mathbb{Z} \to \Phi, i \mapsto \alpha_i$ is a fixed rank-2 labeling of Φ (in the sense of Definition 1.2.38), then we put $U_i := U_{\alpha_i}$ for all $i \in \mathbb{Z}$.

2.1.3 Convention. Let G be a group with a Φ-pregrading $(U_\alpha)_{\alpha\in\Phi}$. If X is a property that roots may or may not have, we will also refer to X as a property of the root groups. For example, a root group U_α is called indivisible if α is an indivisible root. Most of the time, we will use this convention when refering to root lengths: A root group U_α is called short or long if α is short or long.

2.1.4 Definition (Group with commutator relations). Let G be a group with a Φ-pregrading $(U_\alpha)_{\alpha\in\Phi}$. We say that *$G$ has Φ-commutator relations with root groups* $(U_\alpha)_{\alpha\in\Phi}$ if the following conditions are satisfied:

(RGG-Com) For all distinct, non-proportional roots α, β, we have

$$[U_\alpha, U_\beta] \subseteq U_{]\alpha,\beta[_{\mathrm{Cox}}}.$$

(RGG-Div) If α is a root such that 2α is also a root, then

$$U_{2\alpha} \subseteq U_\alpha, \qquad [U_\alpha, U_{2\alpha}] = \{1_G\} \quad \text{and} \quad [U_\alpha, U_\alpha] \subseteq U_{2\alpha}.$$

2.1.5 Definition (Group with crystallographic commutator relations). Let G be a group which has Φ-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi}$. We say that *$G$ has crystallographic Φ-commutator relations* if the following stronger version of (RGG-Com) is satisfied:

(RGG-Com-cry) For all distinct, non-proportional roots α, β, we have

$$[U_\alpha, U_\beta] \subseteq U_{]\alpha,\beta[}.$$

This definition merits several comments.

2.1.6 Note (Commutators for opposite and equal roots). We have no commutator relations for $[U_\alpha, U_{-\alpha}]$ and $[U_\alpha, U_\alpha]$ unless 2α is a root, in which case $[U_\alpha, U_\alpha] \subseteq U_{2\alpha}$. However, if the additional axioms of a Φ-graded group (see Definition 2.5.2) are satisfied, the latter problem disappears: We will show (in Propositions 5.4.9, 7.6.2 and 9.5.2) that for all irreducible crystallographic root systems Φ of rank at least 3 as well as for A_2, any Φ-graded group satisfies $[U_\alpha, U_\alpha] = \{1_G\}$ (which means that U_α is abelian) for all roots α for which 2α is not a root. The same holds for root systems of type H because every root in H_3 is contained in a parabolic subsystem of type A_2.

2.1.7 Note (The crystallographic assumption). The notion of crystallographic commutator relations can be defined even when Φ is not crystallographic. In practice, however, we will not consider crystallographic commutator relations if Φ is not crystallographic. Conversely, we will not consider non-crystallographic commutator relations if Φ is crystallographic. The only exception to this rule will be section 7.4, where we study non-crystallographic B_2-gradings. However, this is not due to a genuine interest in non-crystallographic B_n-gradings, but rather motivated by the fact that certain results about crystallographic BC_2-gradings can be more easily formulated in the language of non-crystallographic B_2-gradings.

Recall from Note 1.2.33 that if Φ is crystallographic, then the extra assumption of crystallographic commutator relations says precisely that for some pairs (α, γ) for roots with $]\alpha, \gamma[_{\mathrm{Cox}} = \{\beta\}$ for some root β, we have the trivial commutator relation $[U_\alpha, U_\gamma] = \{1\}$ instead of the commutator relation $[U_\alpha, U_\gamma] \subseteq U_\beta$.

2.1.8 Note (Commutator relations in the sense of Loos-Neher). The specific terminology of "groups with Φ-commutator relations" appears for the first time in [Fau83, p. 183]. It also appears in [LN19] by Loos-Neher, who cite [Fau77] as their inspiration (see [LN19, p. 72]). However, our Definitions 2.1.4 and 2.1.5 differ from the one in [LN19, p. 23] in a few aspects. First of all, while we restrict our attention to root systems Φ, Loos-Neher define arbitrary R-gradings where R is any subset of a free abelian group X such that R contains 0 and spans X. Many of the facts and concepts that we introduce in this chapter are studied in this broad generality in [LN19, Chapter 1]. In particular, R is allowed to be infinite.

However, when we restrict our attention to reduced root systems, we see that our definition is actually less restrictive than the one in [LN19]. For once, Neher-Loos only consider what we call crystallographic commutator relations. Further, the commutator axiom in [LN19] says that $[U_\alpha, U_\beta] \subseteq U_{]\alpha,\beta[}$ for any so-called nilpotent pair (α, β), which includes the case $\alpha = \beta$. In our notation, the interval $]\alpha, \alpha[$ is not defined (see Definition 1.2.30), but in the notation of [LN19], we have $]\alpha, \alpha[= \mathbb{N}_{\geq 2}\alpha \cap \Phi$. For reduced root systems, this set is empty. Thus the commutator axioms in [LN19] require that each root group is abelian if Φ reduced, which (for the root systems we are interested in) is redundant by Note 2.1.6.

2.1.9 Note (Reducedness). The axiom (RGG-Div) can be omitted if Φ is reduced. Recall from Theorem 1.2.53 that the only crystallographic root system which is not reduced is BC_n. In fact, (RGG-Div) is specifically tailored for this root system. For root systems which are neither reduced nor crystallographic, the axioms should be modified.

2.1.10 Note (Commutator formulas). Axioms (RGG-Com) and (RGG-Com-cry) can be seen as "global" conditions: They tell us what happens on the level on root subgroups. However, they provide no "local" information: They do not give a formula for the commutator of two specific root group elements $x_\alpha \in U_\alpha$ and $x_\beta \in U_\beta$. Our main goal for root graded groups is to find such formulas for all pairs (α, β). See Goal 2.5.4 for more details.

2.1.11 Note (The rank-1 case). Assume that $\Phi = A_1$. In other words, Φ consists of exactly two roots α and $-\alpha$. Then there exist no non-proportional roots in Φ, so that every A_1-pregrading has A_1-commutator relations. For this reason, root gradings of type A_1 (or of type $A_1 \times \cdots \times A_1$) are much more difficult to understand than root gradings for other root systems.

As a consequence, we often have to assume that every root in Φ is contained in a root subsystem which is not of type A_1^n for some $n \in \mathbb{N}_+$. Examples in which this assumption is explicit are Lemmas 2.2.5 and 2.2.20. In many other situations, this assumption is not made explicit because it is always satisfied for irreducible root systems of rank at least 2.

Closed root subsystems induce subgroups with commutator relations in a natural way. We will extend this result to root graded groups in Proposition 2.5.11.

2.1.12 Lemma. *Let G be a group which has Φ-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi}$ and let Φ' be a subset of Φ. Denote by H the group which is generated by $(U_\alpha)_{\alpha\in\Phi'}$. If Φ' is a closed root subsystem, then H has Φ'-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi'}$. If Φ' is a crystallographically closed root subsystem and the commutator relations of G are crystallographic, then H has crystallographic Φ'-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi'}$.*

Proof. This follows from the fact that the interval between $\alpha, \beta \in \Phi'$ does not depend on whether we consider α, β as elements of Φ or as elements of Φ'. $\square$

2.1.B Commutator and Product Maps

We will often be faced with situations in which we want to decompose a commutator $[x_1 \cdots x_n, y_1 \cdots y_m]$, where $x_1, \ldots, x_n \in U_\alpha$ and $y_1, \ldots, y_m \in U_\gamma$ for non-proportional roots α, γ, into a product of of more "basic" elements, for example into commutators $[x_i, y_j]$. Such a decomposition is always possible, but its nature depends on the number of roots in $]\alpha,\gamma[_{\mathrm{Cox}}$. If $]\alpha,\gamma[_{\mathrm{Cox}}$ is empty, then $[x_1, \ldots, x_n, y_1, \ldots, y_m] = 1_G$. The following Lemma 2.1.13 describes the situation in which $]\alpha,\gamma[^{\mathrm{indiv}}_{\mathrm{Cox}}$ contains exactly one element. This is always the case in simply-laced root systems, and even in the non-simply-laced case, we can find many pairs of roots (α, γ) which satisfy this condition.

2.1.13 Lemma. *Let G be a group which has Φ-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi}$ and let α, γ be roots such that $]\alpha,\gamma[^{\mathrm{indiv}}_{\mathrm{Cox}}$ has exactly one element. Then we have*

$$[x_\alpha y_\alpha, x_\gamma] = [x_\alpha, x_\gamma][y_\alpha, x_\gamma] \quad \text{and} \quad [x_\alpha, x_\gamma y_\gamma] = [x_\alpha, y_\gamma][x_\alpha, x_\gamma]$$

for all $x_\alpha, y_\alpha \in U_\alpha$ and $x_\gamma, y_\gamma \in U_\gamma$. In particular,

$$[x_\alpha^{-1}, x_\gamma] = [x_\alpha, x_\gamma]^{-1} = [x_\alpha, x_\gamma^{-1}]$$

for all $x_\alpha \in U_\alpha$ and $x_\gamma \in U_\gamma$.

Proof. Denote the unique root in $]\alpha,\gamma[^{\mathrm{indiv}}_{\mathrm{Cox}}$ by β and let $x_\alpha, y_\alpha \in U_\alpha$, $x_\gamma, y_\gamma \in U_\gamma$. Then Relation 1.1.11 (vii) says that

$$[x_\alpha y_\alpha, x_\gamma] = [x_\alpha, x_\gamma]^{y_\alpha}[y_\alpha, x_\gamma] \quad \text{and} \quad [x_\alpha, x_\gamma y_\gamma] = [x_\alpha, y_\gamma][x_\alpha, x_\gamma]^{y_\alpha}.$$

Observe that $[x_\alpha, x_\gamma]$ lies in U_β by (RGG-Com) and that α is adjacent to β. It follows that y_α commutes with $[x_\alpha, x_\gamma]$, so the first assertion follows. In particular, we have

$$1_G = [1_G, x_\gamma] = [x_\alpha x_\alpha^{-1}, x_\gamma] = [x_\alpha, x_\gamma][x_\alpha^{-1}, x_\gamma],$$

so $[x_\alpha, x_\gamma]^{-1} = [x_\alpha^{-1}, x_\gamma]$. Similarly, we can see that $[x_\alpha, x_\gamma]^{-1} = [x_\alpha, x_\gamma^{-1}]$. $\square$

2.1.14 Remark. Lemma 2.1.13 has the minor flaw that the order of the commutators in the formula $[x_\alpha, x_\gamma y_\gamma] = [x_\alpha, y_\gamma][x_\alpha, x_\gamma]$ is reversed. In practice, however, it will always be the case that either U_β or U_γ is abelian, which implies that $[x_\alpha, x_\gamma y_\gamma] = [x_\alpha, x_\gamma][x_\alpha, y_\gamma]$.

If $]\alpha,\gamma[^{\mathrm{indiv}}$ contains more than one root, then the situation is more delicate. Before we can consider this, we have to introduce some notation.

2.1.15 Notation (Product map). Let G be a group with a Φ-pregrading $(U_\alpha)_{\alpha\in\Phi}$, let k be a positive integer and let $\alpha_1, \ldots, \alpha_k$ be pairwise distinct roots. The *product map on $U_{\alpha_1} \times \cdots \times U_{\alpha_k}$* (or simply the *product map on $(\alpha_1, \ldots, \alpha_k)$*) is the map

$$U_{\alpha_1} \times \cdots \times U_{\alpha_k} \to \langle U_{\alpha_1} \cup \cdots \cup U_{\alpha_k} \rangle, (g_1, \ldots, g_k) \mapsto g_1 \cdots g_k.$$

2.1.16 Definition. Let G be a group which has Φ-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi}$. Let α, β be non-proportional, indivisible roots and denote by $(\alpha_1, \dots, \alpha_k)$ the unique interval ordering of $[\alpha,\beta]_{\mathrm{Cox}}^{\mathrm{indiv}}$ such that $\alpha_1 = \alpha$ and $\alpha_k = \beta$ (see Remark 1.2.35). Assume that the product map on $U_{\alpha_2} \times \cdots \times U_{\alpha_{k-1}}$ is bijective. (We will see in Remark 2.4.16 that this assumption is always satisfied if G is rank-2-injective in the sense of Definition 2.1.19.) Then we denote by

$$\big([\,\cdot\,,\,\cdot\,]_{\alpha_i} \colon U_\alpha \times U_\beta \to U_{\alpha_i}, (x_\alpha, x_\beta) \mapsto [x_\alpha, x_\beta]_{\alpha_i}\big)_{i\in[2,k-1]}$$

the unique family of maps with the property that

$$[x_\alpha, x_\beta] = [x_\alpha, x_\beta]_{\alpha_2} \cdots [x_\alpha, x_\beta]_{\alpha_{k-1}}$$

for all $x_\alpha \in U_\alpha$, $x_\beta \in U_\beta$. In other words, for each $i \in [2, k-1]$ we denote by $[x_\alpha, x_\beta]_{\alpha_i}$ the unique element of U_{α_i} with the property that

$$[x_\alpha, x_\beta] \in U_{\alpha_2} \cdots U_{\alpha_{i-1}} [x_\alpha, x_\beta]_{\alpha_i} U_{\alpha_{i+1}} \cdots U_{\alpha_{k-1}}. \tag{2.1}$$

The surjectivity of the product map in Definition 2.1.16 guarantees the existence of an element $[x_\alpha, x_\beta]_{\alpha_i}$ with the property (2.1), and the injectivity of the product map guarantees that it is uniquely determined. Before we can continue, there is a subtlety in Definition 2.1.16 that we have to point out.

2.1.17 Note (on Definition 2.1.16). We emphasise that $[\,\cdot\,,\,\cdot\,]_{\alpha_i}$ is a map on the product $U_\alpha \times U_\beta$ (on the two arguments x_α and x_β), not a map on $U_{]\alpha,\beta[_{\mathrm{Cox}}}$ (on the argument $[x_\alpha, x_\beta]$). The reason for this is that we fix an interval ordering of $[\alpha,\beta]_{\mathrm{Cox}}^{\mathrm{indiv}}$ in Definition 2.1.16, and this is only possible (without ambiguity of choice) if we declare which of the roots α, β should be the first one and which should be the last one. In other words, we cannot (without ambiguity) define a family of maps

$$\big(U_{]\alpha,\beta[_{\mathrm{Cox}}} \to U_{\alpha_i}, g \mapsto g_{\alpha_i}\big)_{i\in[2,k-1]}$$

because it is not clear whether the elements $g_{\alpha_2}, \dots, g_{\alpha_{k-1}}$ are defined by the property $g = g_{\alpha_2} \cdots g_{\alpha_{k-1}}$ or by the property $g = g_{\alpha_{k-1}} \cdots g_{\alpha_2}$.

2.1.18 Remark. The notational subtleties of Note 2.1.17 are insubstantial if $k = 4$ because the root groups U_{α_2} and U_{α_3} commute by (RGG-Com). Indeed, it follows from this that for all $g \in \langle U_{\alpha_2} \cup U_{\alpha_3} \rangle$ there exist unique elements $g_2 \in U_{\alpha_2}$ and $g_3 \in U_{\alpha_3}$ such that $g = g_2 g_3 = g_3 g_2$. Since closed root intervals in crystallographic root systems of rank at least 3 contain at most 4 indivisible roots, this means that we can largely ignore the implications of Note 2.1.17 in this work.

We continue to assume that $k = 4$. Then for all $x_1 \in U_{\alpha_1}$, $x_4 \in U_{\alpha_4}$ and $i \in \{2, 3\}$, it follows from the previous remarks and Relation 1.1.11 (vi) that $[x_1, x_4]_{\alpha_i} = [x_4, x_1]_{\alpha_i}^{-1}$ because

$$[x_1, x_4] = [x_4, x_1]^{-1} = \big([x_4, x_1]_{\alpha_3}[x_4, x_1]_{\alpha_2}\big)^{-1} = [x_4, x_1]_{\alpha_2}^{-1}[x_4, x_1]_{\alpha_3}^{-1}$$

with $[x_4, x_1]_{\alpha_3}^{-1} \in U_{\alpha_3}$ and $[x_4, x_1]_{\alpha_2}^{-1} \in U_{\alpha_2}$. In other words, "$([x_4, x_1]^{-1})_{\alpha_i} = ([x_4, x_1]_{\alpha_i})^{-1}$" (which we put in quotes because the notation $([x_4, x_1]^{-1})_{\alpha_i}$ is not actually defined by Note 2.1.17).

Using Definition 2.1.16, we can prove analogues of Lemma 2.1.13 for the case that $]\alpha,\gamma[_{\mathrm{Cox}}^{\mathrm{indiv}}$ contains more than one element. These formulas involve not only basic commutators $[x_i, y_j]$ but also nested commutators and the maps from Definition 2.1.16. We will derive them for the case that $]\alpha,\gamma[_{\mathrm{Cox}}^{\mathrm{indiv}}$ contains exactly 2 elements in Lemmas 7.4.4, 7.5.3 and 9.4.3. Since these formulas will be used a lot, it is essential that the bijectivity assumption on the product map in Definition 2.1.16 is always met. We will see in Lemma 2.4.15 that we automatically have surjectivity, but injectivity need not hold in general. Thus we introduce the following condition.

2.1.19 Definition (Rank-2-injective). Let G be a group. A Φ-pregrading $(U_\alpha)_{\alpha\in\Phi}$ of G is said to be *rank-2-injective* if for all parabolic root subsystems Φ' of Φ of rank 2, all positive systems Π' in Φ' and all interval orderings $(\alpha_1, \dots, \alpha_k)$ of $(\Pi')^{\mathrm{indiv}}$, the product map on $(\alpha_1, \dots, \alpha_k)$ is injective.

The following result is an easy consequence of rank-2-injectivity.

2.1.20 Lemma. *Let G be a group with a Φ-pregrading $(U_\alpha)_{\alpha\in\Phi}$. If $\alpha_1, \dots, \alpha_k$ are pairwise distinct roots such that the product map on $(\alpha_1, \dots, \alpha_k)$ is injective, then for all distinct $i, j \in [1, k]$, we have $U_{\alpha_i} \cap U_{\alpha_j} = \{1_G\}$. In particular, if G is rank-2-injective, then $U_\alpha \cap U_\beta = \{1_G\}$ for all non-proportional roots α, β.*

Proof. Let $i, j \in [1, k]$ be distinct and let $x \in U_{\alpha_i} \cap U_{\alpha_j}$. Denote by $\varphi_i(x)$ the tuple $(1_G, \dots, 1_G, x, 1_G, \dots, 1_G)$ with x at position i, and define $\varphi_j(x)$ in a similar way. Then both $\varphi_i(x)$ and $\varphi_j(x)$ are mapped to x under the product map on $(\alpha_1, \dots, \alpha_k)$. By the injectivity of this map, it follows that $x = 1_G$, as desired. □

2.1.C Generalised Commutator Relations

Groups with commutator relations satisfy a commutator relation of the form

$$[U_\alpha, U_\beta] \subseteq U_{]\alpha,\beta[_{\mathrm{Cox}}}.$$

The goal of this subsection is to derive a stronger formula of the form

$$[U_A, U_B] \subseteq U_{]A,B[_{\mathrm{Cox}}}$$

where A, B are subsets of Φ satisfying some conditions. This formula is called the *generalised commutator relation*. We closely follow the arguments in [LN19, 3.9 (a)], but we have to be careful because of the differing conventions concerning root intervals (see Note 1.2.31).

2.1.21 Definition (Commutator set). Let A, B be two subsets of Φ. Then the sets

$$]A, B[_{\mathrm{Cox}} := \Phi \cap \left\{ \sum_{i=1}^n \lambda_i \alpha_i + \sum_{j=1}^m \mu_j \beta_j \;\middle|\; \begin{array}{c} n, m \in \mathbb{N}_+, \\ \lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m \in \mathbb{R}_{>0}, \\ \alpha_1, \dots, \alpha_n \in A, \beta_1, \dots, \beta_m \in B \end{array} \right\},$$

$$]A, B[:= \Phi \cap \left\{ \sum_{i=1}^n \lambda_i \alpha_i + \sum_{j=1}^m \mu_j \beta_j \;\middle|\; \begin{array}{c} n, m \in \mathbb{N}_+, \lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m \in \mathbb{N}_+, \\ \alpha_1, \dots, \alpha_n \in A, \beta_1, \dots, \beta_m \in B \end{array} \right\}$$

are called the *commutator set of* (A, B) and the *crystallographic commutator set of* (A, B), respectively.

2.1.22 Remark (Root intervals as commutator sets). Let α, β be non-proportional roots. Then

$$]\alpha, \beta[=]\{\alpha\}, \{\beta\}[\quad \text{and} \quad]\alpha, \beta[_{\mathrm{Cox}} =]\{\alpha\}, \{\beta\}[_{\mathrm{Cox}},$$

so root intervals are a special case of commutator sets. While root intervals are defined only for pairs of non-proportional roots, there is no such restriction for commutator sets. For example, we have

$$]\{\alpha\}, \{\alpha\}[= \Phi \cap \mathbb{N}_{\geq 2}\alpha \quad \text{and} \quad]\{\alpha\}, \{\alpha\}[_{\mathrm{Cox}} = \Phi \cap \mathbb{R}_{>0}\alpha.$$

Note that the set $]\{\alpha\}, \{\alpha\}[$ is precisely the set $]\alpha, \alpha[$ in the notation of [LN19], see Note 1.2.31.

2.1.23 Remark. Let A, A', B, B' be subsets of Φ such that $A \subseteq A'$ and $B \subseteq B'$. Then $]A, B[\subseteq]A', B'[$ and $]A, B[_{\mathrm{Cox}} \subseteq]A', B'[_{\mathrm{Cox}}$. Further, if $C :=]A, B[$, then $]A \cup B, C[\subseteq C$, and similarly for non-crystallographic commutator sets.

We begin with an auxiliary lemma from group theory.

2.1.24 Lemma ([LN19, 3.7]). *Let G be a group, let H be a subgroup and let X_1, X_2 be subsets of G normalising H (meaning that $H^x = H$ for all $x \in X_1 \cup X_2$) such that $[X_1, X_2] \subseteq H$. Denote by G_1 and G_2 the groups generated by X_1 and X_2, respectively. Then G_1, G_2 normalise H and they satisfy $[G_1, G_2] \subseteq H$.*

Proof. It is clear that X_1^{-1} and X_2^{-1} normalise H as well. Hence G_1 and G_2 normalise H. Further, it follows from Relation 1.1.11 (vii) that

$$1_G = [x_1^{-1}x_1, x_2] = [x_1^{-1}, x_2]^{x_1}[x_1, x_2]$$

for all $x_1 \in X_1$, $x_2 \in X_2$, which implies that

$$[x_1^{-1}, x_2] = ([x_1, x_2]^{-1})^{x_1^{-1}} \in H^{x_1^{-1}} = H.$$

Thus we also have $[X_1^{-1}, X_2] \subseteq H$. In a similar way, it can be shown that $[X_1, X_2^{-1}] \subseteq H$ and $[X_1^{-1}, X_2^{-1}] \subseteq H$. Therefore, by replacing X_1 with $X_1 \cup X_1^{-1}$ and X_2 with $X_2 \cup X_2^{-1}$, we can assume that G_1 and G_2 are generated by X_1 and X_2 as monoids, respectively.

Now let $x_1, y_1 \in X_1$ and $x_2 \in X_2$. Then again by Relation 1.1.11 (vii), we have $[x_1y_1, x_2] = [x_1, x_2]^{y_1}[y_1, x_2]$ where $[x_1, x_2]$ and $[y_1, x_2]$ lie in H by assumption. Further, since X_1 normalises H, we also have that $[x_1, x_2]^{y_1}$ lies in H. Thus $[x_1y_1, x_2]$ lies in H. By a straightforward induction using a similar argument, we conclude that $[g_1, g_2]$ lies in H for all $g_1 \in G_1$ and $g_2 \in G_2$, as desired. □

2.1.25 Lemma ([LN19, 3.9]). *Let G be a group with a Φ-pregrading $(U_\alpha)_{\alpha\in\Phi}$ and let A, B, C be subsets of Φ. Assume that for all $\alpha \in A$, $\beta \in B$, we have $[U_\alpha, U_\beta] \subseteq U_C$ and that for all $\alpha \in A$, $\beta \in B$, $\gamma \in C$, we have $[U_\alpha, U_\gamma] \subseteq U_C$ and $[U_\beta, U_\gamma] \subseteq U_C$. Then U_A and U_B normalise U_C and we have $[U_A, U_B] \subseteq U_C$.*

Proof. Put $X_1 := \bigcup_{\alpha\in A} U_\alpha$, $X_2 := \bigcup_{\beta\in B} U_\beta$ and $H := U_C$. The first assumption on commutation relations in G yields that $[X_1, X_2] \subseteq H$ and the second one yields that X_1 and X_2 normalise H. Thus the assertion follows from Lemma 2.1.24. □

2.1.26 Remark (Cones). Let W be a finite-dimensional real vector space and let C be a subset of W. Then C is called a *cone* if it is closed under addition and scalar multiplication with $\mathbb{R}_{>0}$. The *cone of* C, denoted by $\mathrm{Cone}(C)$, is the smallest cone which contains C. Equivalently, it is the convex hull of $\mathbb{R}_{>0}C$, and it is also the additive semigroup generated by $\mathbb{R}_{>0}C$. Note that a cone may, but need not contain 0_W.

2.1.27 Proposition (Generalised commutator relation, [LN19, 3.9]). *Let G be a group with a Φ-pregrading $(U_\zeta)_{\zeta\in\Phi}$ and let A, B be subsets of Φ such that*

$$\mathrm{Cone}(A) \cap \mathrm{Cone}(-B) = \varnothing.$$

Assume that Φ is reduced or crystallographic and that for all roots ζ for which 2ζ is not a root, the root group U_ζ is abelian. Then the following hold:

(a) If G has Φ-commutator relations with root groups $(U_\zeta)_{\zeta\in\Phi}$, then

$$[U_A, U_B] \subseteq U_{]A,B[_{Cox}}.$$

(b) If G has crystallographic Φ-commutator relations with root groups $(U_\zeta)_{\zeta\in\Phi}$, then

$$[U_A, U_B] \subseteq U_{]A,B[}.$$

Proof. We only consider the case that G has Φ-commutator relations with root groups $(U_\zeta)_{\zeta\in\Phi}$. The other assertion can be proven in the same way. Recall from Lemma 1.2.28 that the only positive multiples of a root α which can possibly be a root are $\frac{1}{2}\alpha$, α and 2α. Our goal is to verify the conditions in Lemma 2.1.25 for $C := \,]A, B[_{\mathrm{Cox}}$.

Let $\alpha \in A$ and $\beta \in B$. If $\alpha \neq \beta \neq -\alpha$, then it follows from the commutator relations of G and Remark 2.1.23 that

$$[U_\alpha, U_\beta] \subseteq U_{]\alpha,\beta[_{\mathrm{Cox}}} \subseteq U_C.$$

If $\alpha = \beta$, then

$$[U_\alpha, U_\beta] \subseteq \begin{cases} \{1_G\} & \text{if } 2\alpha \notin \Phi, \\ U_{2\alpha} & \text{if } 2\alpha \in \Phi \end{cases}$$

by assumption and Axiom (RGG-Div), so $[U_\alpha, U_\beta] \subseteq U_C$ in any case. Further, the case $\alpha = -\beta$ cannot occur because $A \cap (-B) \subseteq \mathrm{Cone}(A) \cap \mathrm{Cone}(-B) = \varnothing$. Thus we have $[U_\alpha, U_\beta] \subseteq U_C$ for all $\alpha \in A$ and $\beta \in B$.

Now let $\alpha \in A$ and $\gamma \in C = \,]A, B[_{\mathrm{Cox}}$. Again, if $\alpha \neq \gamma \neq -\alpha$, then

$$[U_\alpha, U_\gamma] \subseteq U_{]\alpha,\gamma[_{\mathrm{Cox}}} \subseteq U_{]A,C[_{\mathrm{Cox}}} \subseteq U_C$$

by Remark 2.1.23. If $\alpha = \gamma$, then

$$[U_\alpha, U_\gamma] \subseteq \begin{cases} \{1_G\} & \text{if } 2\alpha \notin \Phi, \\ U_{2\alpha} & \text{if } 2\alpha \in \Phi \end{cases}$$

by the same argument as in the previous paragraph, so $[U_\alpha, U_\gamma] \subseteq U_C$ in any case. Now suppose for a contradiction that $\alpha = -\gamma$. By the definition of C, there exist $n, m \in \mathbb{N}_+$, $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m \in \mathbb{R}_{>0}$, $\alpha_1, \dots, \alpha_n \in A$ and $\beta_1, \dots, \beta_m \in B$ such that $\gamma = \sum_{i=1}^n \lambda_i \alpha_i + \sum_{j=1}^m \mu_j \beta_j$. Hence

$$\alpha + \sum_{i=1}^n \lambda_i \alpha_i = -\sum_{j=1}^m \mu_j \beta_j \in \mathrm{Cone}(A) \cap \mathrm{Cone}(-B) = \varnothing,$$

which is impossible. We conclude that $[U_\alpha, U_\gamma] \subseteq U_C$ for all $\alpha \in A$ and $\gamma \in C$. By interchanging the roles of A and B, we observe that $[U_\beta, U_\gamma] \subseteq U_C$ for all $\beta \in B$ and $\gamma \in C$ as well. Hence the conditions of Lemma 2.1.25 are satisfied, and it follows that $[U_A, U_B] \subseteq U_{]A,B[_{\mathrm{Cox}}}$. □

2.1.28 Remark. Let Π be a positive system in Φ. Then $\mathrm{Cone}(\Pi) \cap \mathrm{Cone}(-\Pi) = \varnothing$. It follows that for any subsets A, B of Π, the assumption in Proposition 2.1.27 that $\mathrm{Cone}(A) \cap \mathrm{Cone}(-B) = \varnothing$ is satisfied.

Recall from Note 2.1.6 that for Φ-graded groups, the assumption in Proposition 2.1.27 that certain root groups are abelian is automatically satisfied if Φ is of rank at least 3. However, this is a non-trivial fact which we still have to prove.

2.1.29 Proposition. *Let W be a subspace of the Euclidean space V surrounding Φ and denote by $\pi\colon V \to W$ some projection onto W. (That is, φ is a surjective homomorphism* *which fixes W pointwise, but it is not necessarily orthogonal.) Let G be a group which has (crystallographic) Φ-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi}$, and put*

$$\hat{U}_{\alpha'} := \langle U_\alpha \mid \alpha \in \Phi, \pi(\alpha) = \alpha' \rangle$$

for all $\alpha' \in \Phi'$. Assume that $\Phi' := \pi(\Phi)$ is a reduced root system in W and that for all non-proportional $\alpha', \beta' \in \Phi'$, there exists a positive system Π in Φ which contains $\Phi \cap (\pi^{-1}(\alpha') \cup \pi^{-1}(\beta'))$. Assume further that Φ is reduced or crystallographic and that for all roots ζ for which 2ζ is not a root, the root group U_ζ is abelian. Then G also has (crystallographic) Φ'-commutator relations with root groups $(\hat{U}_{\alpha'})_{\alpha'\in\Phi'}$.

Proof. We prove only the crystallographic assertion. The proof for the non-crystallographic assertion is similar. Let α', β' be non-proportional roots in Φ'. Put $A := \pi^{-1}(\alpha') \cap \Phi$ and $B := \pi^{-1}(\beta') \cap \Phi$. By assumption, there exists a positive system containing $A \cup B$, which by Remark 2.1.28 implies that $\mathrm{Cone}(A) \cap \mathrm{Cone}(-B) = \varnothing$. Thus it follows from Proposition 2.1.27 that

$$[\hat{U}_{\alpha'}, \hat{U}_{\beta'}] = [U_A, U_B] \subseteq U_{]A,B[}.$$

For any $\gamma \in \Phi$, we have $U_\gamma \subseteq \hat{U}_{\pi(\gamma)}$, so $U_{]A,B[} \subseteq \hat{U}_{\pi(]A,B[)}$. Further,

$$\begin{aligned}]\alpha', \beta'[=]\pi(\alpha), \pi(\beta)[&= \Phi' \cap \{ n\pi(\alpha) + m\pi(\beta) \mid n, m \in \mathbb{N}_+ \} \\ &= \Phi' \cap \pi(\{ n\alpha + m\beta \mid n, m \in \mathbb{N}_+ \}) \end{aligned}$$

and

$$\begin{aligned} \pi(]\alpha, \beta[) &= \pi\big(\Phi \cap \{ n\alpha + m\beta \mid n, m \in \mathbb{N}_+ \}\big) \\ &= \Phi' \cap \pi\big(\Phi \cap \{ n\alpha + m\beta \mid n, m \in \mathbb{N}_+ \}\big) \\ &\subseteq \Phi' \cap \pi\big(\{ n\alpha + m\beta \mid n, m \in \mathbb{N}_+ \}\big). \end{aligned}$$

Hence $\pi(]\alpha, \beta[)$ is a subset of $]\alpha', \beta'[$. Altogether, we infer that

$$[\hat{U}_{\alpha'}, \hat{U}_{\beta'}] \subseteq U_{]A,B[} \subseteq \hat{U}_{\pi(]A,B[)} \subseteq \hat{U}_{]\alpha',\beta'[},$$

which proves Axiom (RGG-Com-cry). Further, Axiom (RGG-Div) is empty by the assumption that Φ' is reduced. This finishes the proof. □

2.1.30 Note. Let everything be as in Proposition 2.1.29, except that Φ and Φ' are not assumed to be root systems. (The notion of "positive systems in Φ and Φ'" is defined exactly as in 1.2.19.) Then the same proof as above shows that the groups $(\hat{U}_{\alpha'})_{\alpha'\in\Phi'}$ satisfy the desired commutator relations.

2.2 Weyl Elements

2.2.1 Notation for this section. Unless otherwise specified, we denote by Φ an arbitrary root system and by G a group which has Φ-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi}$.

In this section, we study Weyl elements. The second axiom of root graded groups will be that α-Weyl elements exist for all roots α.

2.2.A Definition and Basic Observations

2.2.2 Definition (Weyl elements, Weyl triples). Let G be a group with a Φ-pregrading $(U_\alpha)_{\alpha\in\Phi}$.

(a) Let α be any root. An *α-Weyl element (in G)* is an element $w_\alpha \in U_{-\alpha}U_\alpha U_{-\alpha}$ with the property that $U_\beta^{w_\alpha} = U_{\sigma_\alpha(\beta)}$ for all roots β. The set of all α-Weyl elements is denoted by M_α.

(b) A *Weyl element* is an α-Weyl element for some root α.

(c) Let α be any root. An *α-Weyl triple* is a triple $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ consisting of $a_{-\alpha}, c_{-\alpha} \in U_{-\alpha}$ and $b_\alpha \in U_\alpha$ such that the product $a_{-\alpha}b_\alpha c_{-\alpha}$ is an α-Weyl element, called the *Weyl element corresponding to* $(a_{-\alpha}, b_\alpha, c_{-\alpha})$.

(d) A *Weyl triple* is an α-Weyl triple for some root α.

(e) Let α be any root. An element $b_\alpha \in U_\alpha$ is called *α-invertible* if there exists elements $a_{-\alpha}, c_{-\alpha} \in U_{-\alpha}$ such that $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ is an α-Weyl triple. We denote the set of α-invertible elements of U_α by $U_\alpha^\sharp$. We will also write *Weyl-invertible* in place of α-invertible if the root α is not specified.

(f) If Φ is of rank 2 and $\mathbb{Z} \to \Phi, i \mapsto \alpha_i$ is a fixed rank-2 labeling of Φ (in the sense of Definition 1.2.38), then we put $U_i^\sharp := U_{\alpha_i}^\sharp$ and $M_i := M_{\alpha_i}$ for all $i \in \mathbb{Z}$.

2.2.3 Note. The terminology of "Weyl elements" appears for the first time in [Fau83, p. 184], though of course these elements had already been considered before in the contexts of Chevalley groups, algebraic groups and RGD-systems. Many results about Weyl elements in a more general setting can be found in [LN19, Section 5].

2.2.4 Note. The notion of Weyl elements does not appear to be useful in groups without Φ-commutator relations. However, the way we have phrased Definition 2.2.2 allows us to discuss Weyl elements in groups which have not yet been proven to have commutator relations.

Before we begin a proper investigation of Weyl elements, we observe that under reasonable assumptions, the trivial element 1_G is never α-invertible. These assumptions will always be satisfied if Φ is irreducible of rank at least 2 and the root groups form a root grading in the sense of Definition 2.5.2.

2.2.5 Lemma. *Let G be a group which has Φ-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi}$ and let α be any root. Assume that there exists a root β such that (α, β) is the root base of a closed rank-2 subsystem of Φ which is not of type $A_1 \times A_1$. (In other words, we assume that α is contained in a closed rank-2 root subsystem Φ' of Φ which is not of type $A_1 \times A_1$.) Assume further that $U_\beta \cap U_{\sigma_\alpha(\beta)} = \{1_G\}$ and that $U_\beta \neq \{1_G\}$. Then $1_G \notin U_\alpha^\sharp$.*

Proof. Assume that 1_G is contained in $U_\alpha^\sharp$. Then there exist $a_{-\alpha}, c_{-\alpha} \in U_{-\alpha}$ such that $a_{-\alpha}c_{-\alpha}$ is an α-Weyl element. In other words, there exists an α-Weyl element w_α which is contained in $U_{-\alpha}$. Choose an element $x_\beta \in U_\beta \setminus \{1_G\}$. Then $x_\beta^{w_\alpha} \in U_{\sigma_\alpha(\beta)}$. At the same time, since β is adjacent to $-\alpha$, we have $x_\beta^{w_\alpha} = x_\beta \in U_\beta$. By our assumptions, it follows that $x_\beta = 1_G$, which is a contradiction. □

The following properties of Weyl elements in Proposition 2.2.6 are elementary and will be used all the time. Interestingly, their proof does not rely on the Φ-commutator relations of G at all. We will prove more specific variations of some of these statements in Lemmas 2.2.23 and 2.2.24 for groups which have "unique Weyl extensions".

2.2.6 Proposition. *Let G be a group with a Φ-pregrading $(U_\alpha)_{\alpha\in\Phi}$. Then the following statements hold for all roots α:*

(a) *If $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ is an α-Weyl triple, then $(c_{-\alpha}^{-1}, b_\alpha^{-1}, a_{-\alpha}^{-1})$ is an α-Weyl triple as well (with corresponding Weyl element w_α^{-1}). In particular, the sets M_α and $U_\alpha^\sharp$ are stable under group inversion.*

(b) *Let w_β be a β-Weyl element for some root β. If $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ is an α-Weyl triple with corresponding Weyl element w_α, then $(a_{-\alpha}^{w_\beta}, b_\alpha^{w_\beta}, c_{-\alpha}^{w_\beta})$ is a $\sigma_\beta(\alpha)$-Weyl triple with corresponding Weyl element $w_\alpha^{w_\beta}$. In particular, $M_\alpha^{w_\beta} = M_{\sigma_\beta(\alpha)}$ and $(U_\alpha^\sharp)^{w_\beta} = U_{\sigma_\beta(\alpha)}^\sharp$.*

(c) *If $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ is an α-Weyl triple with corresponding Weyl element $w_\alpha := a_{-\alpha} b_\alpha c_{-\alpha}$, then $(c_{-\alpha}^{w_\alpha^{-1}}, a_{-\alpha}, b_\alpha)$ and $(b_\alpha, c_{-\alpha}, a_{-\alpha}^{w_\alpha})$ are $(-\alpha)$-Weyl triples and all three Weyl triples have the same corresponding Weyl element. In particular, $M_\alpha = M_{-\alpha}$.*

(d) *If $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ is an α-Weyl triple, then $a_{-\alpha}$ and $c_{-\alpha}$ are $(-\alpha)$-invertible.*

(e) *If α is a root such that 2α is also a root, then every 2α-Weyl triple is also an α-Weyl triple.*

Proof. For the whole proof, we fix an arbitrary root α and an α-Weyl triple $(a_{-\alpha}, b_\alpha, c_{-\alpha})$, whose corresponding Weyl element we denote by w_α. For (a), let β be another arbitrary root. Since w_α is an α-Weyl element, we have

$$U_{\sigma_\alpha(\beta)}^{w_\alpha} = U_\beta, \quad \text{so} \quad U_\beta^{w_\alpha^{-1}} = U_{\sigma_\alpha(\beta)}.$$

Further, we have

$$w_\alpha^{-1} = c_{-\alpha}^{-1} b_\alpha^{-1} a_{-\alpha}^{-1} \in U_{-\alpha} U_\alpha U_{-\alpha}.$$

Altogether, we conclude that $(c_{-\alpha}^{-1}, b_\alpha^{-1}, a_{-\alpha}^{-1})$ is an α-Weyl triple with corresponding Weyl element w_α^{-1}. In particular, b_α^{-1} lies in $U_\alpha^\sharp$. This proves (a). Using that $\sigma_\beta \sigma_\alpha \sigma_\beta = \sigma(\alpha^{\sigma(\beta)})$ by Remark 1.2.6, assertion (b) can be proven in a similar way.

We turn to (c). A simple computation, using only the definition of conjugation in groups, shows that

$$b_\alpha c_{-\alpha} a_{-\alpha}^{w_\alpha} = w_\alpha = c_{-\alpha}^{w_\alpha^{-1}} a_{-\alpha} b_\alpha.$$

Further, $a_{-\alpha}^{w_\alpha}$ and $c_{-\alpha}^{w_\alpha^{-1}}$ lie in U_α and $\sigma_{-\alpha} = \sigma_\alpha$, so it follows that $(c_{-\alpha}^{w_\alpha^{-1}}, a_{-\alpha}, b_\alpha)$ and $(b_\alpha, c_{-\alpha}, a_{-\alpha}^{w_\alpha})$ are $(-\alpha)$-Weyl triples, both with corresponding Weyl element w_α. This proves (c), and (d) is a consequence of (c). Finally, (e) holds because $U_{\pm 2\alpha}$ is contained in $U_{\pm\alpha}$ by (RGG-Div). □

2.2.7 Lemma. *Assume that Φ is reduced. Let Δ be a root base of Φ and assume that there exists a δ-Weyl element w_δ for each $\delta \in \Delta$. Then there exists an α-Weyl element for each root α.*

Proof. Let α be an arbitrary root. By Proposition 1.3.4, there exist $\delta, \delta_1, \dots, \delta_n \in \Delta$ such that $\alpha = \delta^{\sigma(\delta_1 \cdots \delta_n)}$. Then it follows from Proposition 2.2.6 (b) that $w_\delta^{w_{\delta_1} \cdots w_{\delta_n}}$ is an α-Weyl element, which finishes the proof. □

2.2.8 Remark. Lemma 2.2.7 and its proof remain valid if Φ is of type BC and Δ is the standard rescaled root base (see Remark 9.1.2). For proper root bases of non-reduced root systems, however, it is not true: The existence of α-Weyl elements does not imply the existence of 2α-Weyl elements.

2.2.9 Lemma. *Assume that G is rank-2-injective. Let α, β, ξ be roots, let $\gamma \in]\alpha, \beta[_{Cox}^{indiv}$ and let w_ξ be a ξ-Weyl element for some root ξ. Then we have*

$$[x_\alpha, x_\beta]_\gamma^{w_\xi} = [x_\alpha^{w_\xi}, x_\beta^{w_\xi}]_{\sigma_\xi(\gamma)}$$

for all $x_\alpha \in U_\alpha$ and all $x_\beta \in U_\beta$. In particular,

$$[x_\alpha, x_\beta]_\gamma^{w_\xi^2} = [x_\alpha^{w_\xi^2}, x_\beta^{w_\xi^2}]_\gamma$$

for all $x_\alpha \in U_\alpha$ and all $x_\beta \in U_\beta$.

Proof. Denote by $(\gamma_1, \dots, \gamma_k)$ be the unique interval ordering of $[\alpha, \beta]_{\mathrm{Cox}}^{\mathrm{indiv}}$ such that $\gamma_1 = \alpha$ and $\gamma_k = \beta$, and let $i \in [1, k]$ such that $\gamma = \gamma_i$. Then $(\gamma_1^{\sigma(\xi)}, \dots, \gamma_k^{\sigma(\xi)})$ is the unique interval ordering of $[\alpha^{\sigma(\xi)}, \beta^{\sigma(\xi)}]_{\mathrm{Cox}}^{\mathrm{indiv}}$ whose first element is $\alpha^{\sigma(\xi)}$ and whose last element in $\beta^{\sigma(\xi)}$. By the definition of $[x_\alpha, x_\beta]_\gamma$, we have

$$[x_\alpha, x_\beta] \in U_{\gamma_1} \cdots U_{\gamma_{i-1}} [x_\alpha, x_\beta]_\gamma U_{\gamma_{i+1}} \cdots U_{\gamma_k}$$

and $[x_\alpha, x_\beta]_\gamma \in U_\gamma$. It follows that

$$\begin{aligned}[x_\alpha^{w_\xi}, x_\beta^{w_\xi}] = [x_\alpha, x_\beta]^{w_\xi} &\in U_{\gamma_1}^{w_\xi} \cdots U_{\gamma_{i-1}}^{w_\xi} [x_\alpha, x_\beta]_\gamma^{w_\xi} U_{\gamma_{i+1}}^{w_\xi} \cdots U_{\gamma_k}^{w_\xi} \\ &= U_{\gamma_1^{\sigma(\xi)}} \cdots U_{\gamma_{i-1}^{\sigma(\xi)}} [x_\alpha, x_\beta]_\gamma^{w_\xi} U_{\gamma_{i+1}^{\sigma(\xi)}} \cdots U_{\gamma_k^{\sigma(\xi)}}\end{aligned}$$

and $[x_\alpha, x_\beta]_\gamma^{w_\xi} \in U_{\sigma_\xi(\gamma)}$. This implies that $[x_\alpha^{w_\xi}, x_\beta^{w_\xi}]_{\sigma_\xi(\gamma)} = [x_\alpha, x_\beta]_\gamma^{w_\xi}$. □

The following result is crucial in [TW02], and it will be an important tool in our computations as well.

2.2.10 Proposition ([TW02, (6.4)]). *Put $n := |\Phi|/2$. Assume that Φ has rank 2 with $n \geq 3$ and that G is rank-2-injective. Further, we fix a rank-2 labeling of Φ and we will use Notation 2.2.2 (f). Let $z \in \mathbb{Z}$, let $b_1 \in U_{z+1}^\sharp$ and let $x_n \in U_{z+n}$. Let $a_{1+n}, c_{1+n} \in U_{z+1+n}^\sharp$ such that (a_{1+n}, b_1, c_{1+n}) is a $(z+1)$-Weyl triple, and denote by w_1 the associated Weyl element. For all $i \in [2, n-1]$, we put $x_i := [b_1, x_n^{-1}]_{z+i} \in U_{z+i}$, so that $[b_1, x_n^{-1}] = x_2 \cdots x_{n-1}$. Then the following hold:*

(a) $x_n^{w_1} = x_2 = [b_1, x_n^{-1}]_{z+2}$. *In particular, x_2 lies in $U_{z+2}^\sharp$ if x_n lies in $U_{z+n}^\sharp$.*

(b) $[x_2, c_{1+n}^{-1}] = x_3 \cdots x_n$.

Proof. Put $w := w_1 = a_{1+n} b_1 c_{1+n}$. Since

$$[a_{1+n}^{-1}, x_n^{-1}] \in [U_{z+n+1}, U_{z+n}] = \{1_G\},$$

it follows from Relation 1.1.11 (viii) that

$$[b_1, x_n^{-1}] = [a_{1+n}^{-1} w c_{1+n}^{-1}, x_n^{-1}] = [w c_{1+n}^{-1}, x_n^{-1}].$$

Therefore,

$$\begin{aligned}x_2 \cdots x_n = [b_1, x_n^{-1}] x_n = [w c_{1+n}^{-1}, x_n^{-1}] x_n &= \big(c_{1+n} w^{-1} x_n w c_{1+n}^{-1} x_n^{-1}\big) x_n \\ = c_{1+n} x_n^w c_{1+n}^{-1} = x_n^w (x_n^w)^{-1} c_{1+n} x_n^w c_{1+n}^{-1} &= x_n^w [x_n^w, c_{1+n}^{-1}].\end{aligned}$$

Note that c_{1+n} lies in U_{z+n+1} and that x_n^w lies in

$$U_{z+n}^w = U_{z+n}^{w_1} = U_{2(z+1)+n-(z+n)} = U_{z+2}$$

by Remark 1.2.37. Thus $[x_n^w, c_{1+n}^{-1}]$ lies in $U_{[z+3, z+n]}$. Therefore, it follows from

$$x_2 x_3 \cdots x_n = x_n^w [x_n^w, c_{1+n}^{-1}]$$

and the rank-2-injectivity of G that $x_2 = x_n^w$ and $x_3 \cdots x_n = [x_n^w, c_{1+n}^{-1}] = [x_2, c_{1+n}^{-1}]$. This finishes the proof. □

2.2.11 Note. Special cases of Proposition 2.2.10 for specific values of n can also be proven in a slightly different way. We will see this in Lemmas 5.4.2 and 7.4.5.

We will also perform a similar computation in Lemma 7.5.12 for a pair of roots which does not form a root base.

2.2.B Balanced Weyl Triples

Any Weyl triple (a, b, c) is associated with four elements of G: a, b, c and $w := abc$. These four elements are related by the equation $w = abc$, so in particular, the knowledge of any three of these group elements determines the missing fourth element. We now investigate under which conditions we can find more relations between these four elements.

2.2.12 Definition (Balanced Weyl triple). A Weyl triple (a, b, c) with corresponding Weyl element $w := abc$ is called *weakly balanced* if $a = c$, and it is called *balanced* if, in addition, it satisfies the equivalent conditions of the following Lemma 2.2.14.

2.2.13 Note. We will also refer to the Weyl element w in Definition 2.2.12 as (weakly) balanced if the corresponding conditions are satisfied, but this is actually abuse of notation: In general, the Weyl triple (a, b, c) is not necessarily uniquely determined by the Weyl element abc. (For examples, see Remarks 7.3.14 and 9.3.12.) However, it will always be clear from the context what we mean.

2.2.14 Lemma ([LN19, 5.15]). *Let α be a root and let $(a_{-\alpha}, b_\alpha, a_{-\alpha})$ be a weakly balanced α-Weyl triple with corresponding Weyl element $w_\alpha := a_{-\alpha} b_\alpha a_{-\alpha}$. Then the following statements are equivalent:*

(a) $b_\alpha^{w_\alpha} = a_{-\alpha}$.

(b) $a_{-\alpha}^{w_\alpha} = b_\alpha$.

(c) $w_\alpha = b_\alpha a_{-\alpha} b_\alpha$.

Proof. We know from Proposition 2.2.6 (c) that the Weyl elements $w'_\alpha := a_{-\alpha}^{w_\alpha^{-1}} a_{-\alpha} b_\alpha$ and $w''_\alpha := b_\alpha a_{-\alpha} a_{-\alpha}^{w_\alpha}$ are both equal to w_α. Clearly, w'_α equals $b_\alpha a_{-\alpha} b_\alpha$ if and only if $a_{-\alpha}^{w_\alpha^{-1}} = b_\alpha$, so (a) is equivalent to (c). Similarly, w''_α equals $b_\alpha a_{-\alpha} b_\alpha$ if and only if $a_{-\alpha}^{w_\alpha} = b_\alpha$, so (b) is equivalent to (c) as well. □

The notion of balanced Weyl triples is due to [LN19, 5.15] while the terminology of weakly balanced Weyl triples is, to our knowledge, new. It is clear that in a weakly balanced Weyl triple, the knowledge of two of the elements a, b and $w := aba$ is enough to reconstruct the third one.

2.2.15 Remark. If $(a_{-\alpha}, b_\alpha, a_{-\alpha})$ is a balanced α-Weyl triple with associated Weyl element w_α, then

$$b_\alpha^{w_\alpha^2} = a_{-\alpha}^{w_{-\alpha}} = b_\alpha.$$

A natural question to ask is whether weakly balanced Weyl triples are automatically balanced. We will see in Lemma 2.2.24 (e) that that this is true if the group has unique Weyl extensions. This extra assumptions will be satisfied in all cases that we are interested in by Proposition 2.2.22. In more general situations, however, the distinction between balanced and weakly balanced Weyl triples is relevant. Still, we make the following observation.

2.2.16 Proposition. *If α is a root such that all α-Weyl triples are weakly balanced, then all α-Weyl triples are balanced. In particular, if all Weyl triples in G are weakly balanced, then all Weyl triples in G are balanced.*

Proof. Let α be a root and assume that all α-Weyl triples are weakly balanced. If there exist no α-Weyl triples, then there is nothing to prove, so we can assume that there exists an α-Weyl triple $(a_{-\alpha}, b_\alpha, a_{-\alpha})$ with corresponding Weyl element $w_\alpha := a_{-\alpha} b_\alpha a_{-\alpha}$. If $(\bar{a}_\alpha, \bar{b}_{-\alpha}, \bar{c}_\alpha)$ is a $(-\alpha)$-Weyl triple, then $(\bar{a}_\alpha^{w_\alpha}, \bar{b}_{-\alpha}^{w_\alpha}, \bar{c}_\alpha^{w_\alpha})$ is an α-Weyl triple by Proposition 2.2.6 (b), so it follows from our assumptions that $\bar{a}_\alpha^{w_\alpha} = \bar{c}_\alpha^{w_\alpha}$. We conclude that all $(-\alpha)$-Weyl triples are weakly balanced as well. Since $(b_\alpha, a_{-\alpha}, a_{-\alpha}^{w_\alpha})$ is a $(-\alpha)$-Weyl triple by Proposition 2.2.6 (c), this implies that $b_\alpha = a_{-\alpha}^{w_\alpha}$, so $(a_{-\alpha}, b_\alpha, a_{-\alpha})$ is balanced. This finishes the proof. □

2.2.17 Note. A second natural question is whether every Weyl triple is weakly balanced. We will show in Proposition 5.4.10 that every Weyl element in A_2-graded groups is weakly balanced and thus, by Proposition 2.2.16 or Lemma 2.2.24 (e), balanced. This implies that the same is true for all roots α in an arbitrary root system which are contained in a (crystallographically) closed subsystem of type A_2. This holds, for example, for the long roots in B_n, the short roots in C_n (where $n \geq 3$) and for all roots in H_3 and H_4. However, it does not hold for the short roots in B_n and the long roots in C_n. We can still prove that all Weyl elements for such roots are balanced (see Lemmas 7.7.3 and 9.6.4), but this proof relies on computations in the coordinatising structures of these groups. The only remaining case are the short roots in BC_n for $n \geq 3$. For these roots, there do in fact exist examples of Weyl elements which are not weakly balanced. We will encounter them in Definition 9.3.10.

2.2.C Unique Weyl Extensions

2.2.18 Definition (Unique Weyl extensions). Let $\alpha \in \Phi$. We say that G *has unique α-Weyl extensions* if for all $b_\alpha \in U_\alpha^\sharp$, there exist *unique* elements $a_{-\alpha}, c_{-\alpha} \in U_{-\alpha}$ such that $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ is an α-Weyl triple. If this is the case, we define $\kappa_\alpha, \lambda_\alpha \colon U_\alpha^\sharp \to U_\alpha$ to be the unique maps for which $\kappa_\alpha(b_\alpha) b_\alpha \lambda_\alpha(b_\alpha)$ is an α-Weyl element for all $b_\alpha \in U_\alpha^\sharp$. Further, we define $\mu_\alpha \colon U_\alpha^\sharp \to M_\alpha, b_\alpha \mapsto \kappa_\alpha(b_\alpha) b_\alpha \lambda_\alpha(b_\alpha)$. We say that G *has unique Weyl extensions* if it has unique α-Weyl extensions for all roots α. We will sometimes leave out the subscripts of the maps κ, λ and μ if they are clear from the context.

The property of having unique Weyl extensions says precisely that a Weyl triple is uniquely determined by its "middle element". Under mild conditions on the root system and the root groups, it is in fact true that a Weyl triple (a, b, c) is uniquely determined by any of the elements a, b, c. This is the statement of the following Lemma 2.2.20, which is a special case of [LN19, 5.20]. It is also proven in a less general context in [TW02, (6.1)]. We begin the proof of Lemma 2.2.20 with an auxiliary lemma.

2.2.19 Lemma. *Let α, β be roots such that $U_\alpha \cap U_{\sigma_\beta(\alpha)} = \{1_G\}$ and assume that there exists a β-Weyl element w_β. Then if x_α is an element of U_α which centralises U_β and $U_{-\beta}$, it follows that $x_\alpha = 1_G$.*

Proof. By assumption, we have $x_\alpha = x_\alpha^{w_\beta}$ where the element on the left-hand side lies in U_α while the element on the right-hand side lies in $U_{\sigma_\beta(\alpha)}$. Since $U_\alpha \cap U_{\sigma_\beta(\alpha)} = \{1_G\}$, it follows that $x_\alpha = 1_G$. □

2.2.20 Lemma ([LN19, 5.20]). *Let α be a root and let*

$$\mathfrak{u} = (a_{-\alpha}, b_\alpha, c_{-\alpha}) \quad \textit{and} \quad \tilde{\mathfrak{u}} = (\tilde{a}_{-\alpha}, \tilde{b}_\alpha, \tilde{c}_{-\alpha})$$

be two α-Weyl triples. Assume that α is contained in a parabolic rank-2 closed subsystem Φ' of Φ which is not of type $A_1 \times A_1$ and that there exists a β-Weyl element for some root $\beta \in \Phi'$ adjacent to α. Assume further that G is rank-2-injective. Then if one of the three statements $a_{-\alpha} = \tilde{a}_{-\alpha}$, $b_\alpha = \tilde{b}_\alpha$, $c_{-\alpha} = \tilde{c}_{-\alpha}$ holds, it follows that $\mathfrak{u} = \tilde{\mathfrak{u}}$.

Proof. Denote by Φ' a parabolic rank-2 subsystem of Φ and by β a root which are chosen as in the assertion. Without loss of generality, we assume that $\Phi' = \Phi$, and we fix the unique rank-2 labeling $i \mapsto \alpha_i$ of Φ with $\alpha_1 = \alpha$ and $\alpha_2 = \beta$. Further, we put $m := |\Phi|/2$. Since Φ is not of type $A_1 \times A_1$, we have $m \geq 3$.

We begin with the case $b_\alpha = \tilde{b}_\alpha$. Put

$$w_\alpha := a_{-\alpha} b_\alpha c_{-\alpha} \quad \text{and} \quad \tilde{w}_\alpha := \tilde{a}_{-\alpha} \tilde{b}_\alpha \tilde{c}_{-\alpha}$$

Then for $\bar{a}_{-\alpha} := \tilde{a}_{-\alpha} a_{-\alpha}^{-1}$, $\bar{c}_{-\alpha} := c_{-\alpha}^{-1} \tilde{c}_{-\alpha}$ and for all $x_m \in U_{\alpha_m}$, we have

$$x_m^{\tilde{w}_\alpha} = x_m^{\bar{a}_{-\alpha} w_\alpha \bar{c}_{-\alpha}} = (x_m^{w_\alpha})^{\bar{c}_{-\alpha}} = x_m^{w_\alpha} [x_m^{w_\alpha}, \bar{c}_{-\alpha}].$$

Note that $x_m^{\tilde{w}_\alpha}$ and $x_m^{w_\alpha}$ lie in $U_{\sigma_\alpha(\alpha_m)} = U_{\alpha_2}$ while $[x_m^{w_\alpha}, \bar{c}_{-\alpha}]$ lies in $U_{]\alpha_2, \alpha_{m+1}[_{\mathrm{Cox}}} = U_{\alpha_3} \cdots U_{\alpha_m}$ for all $x_m \in U_{\alpha_m}$. Since the product map on $\bar{\alpha}$ is injective, we conclude that $x_m^{\tilde{w}_\alpha} = x_m^{w_\alpha}$ and $[x_m^{w_\alpha}, \bar{c}_{-\alpha}] = 1_G$ for all $x_m \in U_{\alpha_m}$. In particular, $\bar{c}_{-\alpha}$ centralises $U_{\alpha_m}^{w_\alpha} = U_{\alpha_2} = U_\beta$. As α is adjacent to β (and thus $-\alpha$ is adjacent to $-\beta$), we also have that $\bar{c}_{-\alpha}$ centralises $U_{-\beta}$. Note further that $U_{-\alpha} \cap U_{\sigma_\alpha(\beta)} = U_{\alpha_{m+1}} \cap U_{\alpha_m} = \{1_G\}$ by Lemma 2.1.20. Thus it follows from Lemma 2.2.19 that $\bar{c}_{-\alpha} = 1_G$. In other words, $c_{-\alpha} = \tilde{c}_{-\alpha}$. Since $(c_{-\alpha}^{-1}, b_\alpha^{-1}, a_{-\alpha}^{-1})$ and $(\tilde{c}_{-\alpha}^{-1}, \tilde{b}_\alpha^{-1}, \tilde{a}_{-\alpha}^{-1})$ are α-Weyl elements as well (by Proposition 2.2.6 (a)), the same chain of arguments yields that $a_{-\alpha} = \tilde{a}_{-\alpha}$, too. This finishes the proof of the case $b_\alpha = b'_\alpha$.

Now assume that $c_{-\alpha} = \tilde{c}_{-\alpha}$. By Proposition 2.2.6 (c), we also have $(-\alpha)$-Weyl triples $(b_\alpha, c_{-\alpha}, a_{-\alpha}^{w_\alpha})$ and $(\tilde{b}_\alpha, \tilde{c}_{-\alpha}, \tilde{a}_{-\alpha}^{\tilde{w}_\alpha})$. Observe that the requirements of the current lemma are also satisfied for $-\alpha$ in place of α because the existence of a β-Weyl element implies the existence of a $(-\beta)$-Weyl element, again by Proposition 2.2.6 (c). Thus the conclusion of the previous paragraph allows us to infer that $b_\alpha = \tilde{b}_\alpha$ and $a_{-\alpha}^{w_\alpha} = \tilde{a}_{-\alpha}^{\tilde{w}_\alpha}$. The second equation says precisely that

$$c_{-\alpha}^{-1} b_\alpha^{-1} a_{-\alpha}^{-1} a_{-\alpha} a_{-\alpha} b_\alpha c_{-\alpha} = \tilde{c}_{-\alpha}^{-1} \tilde{b}_\alpha^{-1} \tilde{a}_{-\alpha}^{-1} \tilde{a}_{-\alpha} \tilde{a}_{-\alpha} \tilde{b}_\alpha \tilde{c}_{-\alpha},$$

which implies that $a_{-\alpha} = \tilde{a}_{-\alpha}$ because we already know that $b_\alpha = \tilde{b}_\alpha$ and $c_{-\alpha} = \tilde{c}_{-\alpha}$. We conclude that $\mathfrak{u} = \tilde{\mathfrak{u}}$ in this case as well. The case $a_{-\alpha} = \tilde{a}_{-\alpha}$ can be proven in the same way by considering the $(-\alpha)$-Weyl triples $(c_{-\alpha}^{w_\alpha^{-1}}, a_{-\alpha}, b_\alpha)$ and $(\tilde{c}_{-\alpha}^{\tilde{w}_\alpha^{-1}}, \tilde{a}_{-\alpha}, \tilde{b}_\alpha)$. ☐

2.2.21 Note. Lemma 2.2.20 does not rule out the possibility of two distinct Weyl triples having the same associated Weyl element. In fact, there exist short Weyl elements in root gradings of types B and BC which have two distinct associated Weyl triples: see Remarks 7.3.14 and 9.3.12. However, in A_2-graded groups, every Weyl element has a unique associated Weyl triple (Proposition 5.4.10 (b)).

For most applications, we can reduce Lemma 2.2.20 to the following statement.

2.2.22 Proposition. *Assume that Φ does not have an irreducible component of type A_1, that $U_\alpha^\sharp \neq \varnothing$ for all roots α and that G is rank-2-injective. Then G has unique Weyl extensions.*

Proof. It follows from the assumption on Φ that every root is contained in a rank-2 subsystem which is not of type $A_1 \times A_1$. Thus the assertion is a consequence of Lemma 2.2.20. ☐

We now investigate some properties of groups with unique Weyl extensions. The following two statements are essentially reformulations of Proposition 2.2.6 using the maps μ, κ and λ. The proofs use the same arguments as in [TW02], but in a more general context.

2.2.23 Lemma ([TW02, (6.2)]). *Let α be a root such that G has unique α-Weyl extensions. Then the following hold:*

(a) $\mu_\alpha(b_\alpha^{-1}) = \mu_\alpha(b_\alpha)^{-1}$, $\kappa_\alpha(b_\alpha^{-1}) = \kappa_\alpha(b_\alpha)^{-1}$ *and* $\lambda_\alpha(b_\alpha^{-1}) = \lambda_\alpha(b_\alpha)^{-1}$ *for all* $b_\alpha \in U_\alpha^\sharp$.

(b) Let β be a root for which there exists a β-Weyl element w_β. Then G has unique $\sigma_\beta(\alpha)$-Weyl extensions. Further, we have

$$\mu_{\sigma_\beta(\alpha)}(b_\alpha^{w_\beta}) = \mu_\alpha(b_\alpha)^{w_\beta},\ \kappa_{\sigma_\beta(\alpha)}(b_\alpha^{w_\beta}) = \kappa_\alpha(b_\alpha)^{w_\beta},\ \lambda_{\sigma_\beta(\alpha)}(b_\alpha^{w_\beta}) = \lambda_\alpha(b_\alpha)^{w_\beta}$$

for all $b_\alpha \in U_\alpha^\sharp$.

Proof. The first assertion follows from Proposition 2.2.6 (a) and the second one from Proposition 2.2.6 (b). ☐

2.2.24 Lemma ([TW02, (6.3)]). *Let α be a root such that G has unique α-Weyl extensions and assume that there exists $b_\alpha \in U_\alpha^\sharp$. Then G has unique $(-\alpha)$-Weyl extensions, the images of κ_α and λ_α are contained in $U_{-\alpha}^\sharp$ and the following hold:*

(a) $\mu_\alpha(b_\alpha) = \mu_{-\alpha}\big(\kappa_\alpha(b_\alpha)\big) = \mu_{-\alpha}\big(\lambda_\alpha(b_\alpha)\big)$.

(b) $\kappa_{-\alpha}\big(\lambda_\alpha(b_\alpha)\big) = b_\alpha = \lambda_{-\alpha}\big(\kappa_\alpha(b_\alpha)\big)$.

(c) $\lambda_{-\alpha}\big(\lambda_\alpha(b_\alpha)\big) = \kappa_\alpha(b_\alpha)^{\mu_\alpha(b_\alpha)}$.

(d) $\kappa_{-\alpha}\big(\kappa_\alpha(b_\alpha)\big) = \lambda_\alpha(b_\alpha)^{\mu_\alpha(b_\alpha)^{-1}}$.

(e) If $\lambda_\alpha(b_\alpha) = \kappa_\alpha(b_\alpha)$, *then*

$$b_\alpha^{\mu_\alpha(b_\alpha)} = \kappa_\alpha(b_\alpha) = \lambda_\alpha(b_\alpha).$$

That is, every weakly balanced α-Weyl triple is balanced.

Proof. It follows from Lemma 2.2.23 (b) that G has unique $(-\alpha)$-Weyl extensions (because $\sigma_\alpha(\alpha) = -\alpha$) and from Proposition 2.2.6 (c) that the images of κ_α and λ_α are contained in $U_{-\alpha}^\sharp$. We know from Proposition 2.2.6 (c) that

$$\big(b_\alpha, \lambda_\alpha(b_\alpha), \kappa_\alpha(b_\alpha)^{\mu_\alpha(b_\alpha)}\big)$$

is a $(-\alpha)$-Weyl triple with associated Weyl element $\mu_\alpha(b_\alpha)$. This implies that

$$\mu_{-\alpha}\big(\lambda_\alpha(b_\alpha)\big) = \mu_\alpha(b_\alpha), \quad \kappa_{-\alpha}\big(\lambda_\alpha(b_\alpha)\big) = b_\alpha, \quad \lambda_{-\alpha}\big(\lambda_\alpha(b_\alpha)\big) = \kappa_\alpha(b_\alpha)^{\mu_\alpha(b_\alpha)}.$$

Similarly, we know from Proposition 2.2.6 (c) that

$$\big(\lambda_\alpha(b_\alpha)^{\mu_\alpha(b_\alpha)^{-1}}, \kappa_\alpha(b_\alpha), b_\alpha\big)$$

is a $(-\alpha)$-Weyl triple with associated Weyl element $\mu_\alpha(b_\alpha)$, which implies that

$$\mu_{-\alpha}\big(\kappa_\alpha(b_\alpha)\big) = \mu_\alpha(b_\alpha), \quad \kappa_{-\alpha}\big(\kappa_\alpha(b_\alpha)\big) = \lambda_\alpha(b_\alpha)^{\mu_\alpha(b_\alpha)^{-1}}, \quad \lambda_{-\alpha}\big(\kappa_\alpha(b_\alpha)\big) = b_\alpha.$$

This finishes the proof of the first four assertions. If $\lambda_\alpha(b_\alpha) = \kappa_\alpha(b_\alpha)$, then it follows from the previous assertions that

$$b_\alpha = \kappa_{-\alpha}\big(\lambda_\alpha(b_\alpha)\big) = \kappa_{-\alpha}\big(\kappa_\alpha(b_\alpha)\big) = \lambda_\alpha(b_\alpha)^{\mu_\alpha(b_\alpha)^{-1}}.$$

Conjugating both sides by $\mu_\alpha(b_\alpha)$, the final assertion follows. ☐

2.2.D The Braid Relations for Weyl Elements

In the proof of the parametrisation theorem in chapter 4, we will require that a fixed family of Weyl elements satisfies the same braid relations as the Weyl group. We will also use these relations in the computation of certain "blueprint rewriting rules" which are needed to apply the blueprint technique. To formulate the braid relations for Weyl elements, we first have to define Δ-systems of Weyl elements.

2.2.25 Notation (System of Weyl elements). Let Δ be a rescaled root base of Φ. A *Δ-system of Weyl elements (in G)* is a family $(w_\delta)_{\delta\in\Delta}$ such that w_δ is a δ-Weyl element for each $\delta \in \Delta$. Given such a Δ-system of Weyl elements, we put $w_{-\delta} := w_\delta^{-1}$ for any $\delta \in \Delta$ and $w_{\bar{\delta}} := w_{\delta_1} \cdots w_{\delta_m}$ for any word $\bar{\delta} = (\delta_1, \dots, \delta_m)$ over $\Delta \cup (-\Delta)$ (with the convention that $w_{\bar{\delta}} = 1_G$ if $\bar{\delta}$ is the empty word).

2.2.26 Definition (Braid relations). Let Δ be a rescaled root base of Φ and let $(w_\delta)_{\delta\in\Phi}$ be a Δ-system of Weyl elements. We say that $(w_\delta)_{\delta\in\Delta}$ *satisfies the braid relations (in G)* if for any distinct $\alpha, \beta \in \Delta$, we have $P_{o(\sigma_\alpha\sigma_\beta)}(w_\alpha, w_\beta) = P_{o(\sigma_\alpha\sigma_\beta)}(w_\beta, w_\alpha)$ where $o(\sigma_\alpha\sigma_\beta)$ is the order of $\sigma_\alpha\sigma_\beta$ in the Weyl group. Similarly, we say that $(w_\delta)_{\delta\in\Delta}$ *satisfies the braid relations modulo $Z(G)$* if the images of $P_{o(\sigma_\alpha\sigma_\beta)}(w_\alpha, w_\beta)$ and $P_{o(\sigma_\alpha\sigma_\beta)}(w_\beta, w_\alpha)$ in $G/Z(G)$ are equal for all distinct $\alpha, \beta \in \Delta$. Further, we say that *G satisfies the braid relations for Weyl elements (modulo $Z(G)$)* if for any root base Δ, each Δ-system of Weyl elements satisfies the braid relations (modulo $Z(G)$).

For the purposes of the parametrisation theorem and the blueprint technique, it would be sufficient to find one Δ-system of Weyl elements which satisfies the braid relations modulo $Z(G)$. In practice, we can prove a stronger statement (Theorem 2.2.34): Any Δ-system of Weyl elements satisfies the braid relations in G, provided that G has unique Weyl extensions and that it is rank-2-injective. This is proven in [TW02, (6.9)] under less general assumptions, but the same arguments remain valid in our context. We present this proof in the remaining part of this subsection.

To avoid notational problems, we briefly rule out the trivial case of the root system $A_1 \times A_1$.

2.2.27 Proposition. *Assume that $\Phi = A_1 \times A_1$. Let $\Delta = \{\alpha, \beta\}$ be a root base of $A_1 \times A_1$ and assume that there exist an α-Weyl element w_α and a β-Weyl element w_β. Then the family $(w_\gamma)_{\gamma\in\Delta}$ satisfies the braid relations.*

Proof. We only have to show that $w_\alpha w_\beta = w_\beta w_\alpha$, which is trivial because the subgroup $\langle U_\alpha, U_{-\alpha}\rangle$ commutes with $\langle U_\beta, U_{-\beta}\rangle$ by the commutator relations. $\square$

2.2.28 Setup. Put $n := |\Phi|/2$. Assume that Φ is reduced of rank 2 with $n \geq 3$ and fix a rank-2 labeling of Φ (in the sense of Definition 1.2.38) as well as elements $u \in U_1^\sharp$, $v \in U_n^\sharp$. Assume further that G is rank-2-injective and that it has unique Weyl extensions for all roots. Define sequences $(e_k)_{k\in\mathbb{N}_+}$ and $(f_k)_{k\in\mathbb{N}_+}$ in G by

$$e_1 := u \in U_1^\sharp, \qquad f_1 := v \in U_n^\sharp = U_{1+(n-1)}^\sharp \qquad \text{and}$$
$$e_k := f_{k-1}^{\mu(e_{k-1})} \in U_k^\sharp, \qquad f_k := \lambda(e_{k-1}) \in U_{k+(n-1)}^\sharp \quad \text{for all } k \in \mathbb{N}_{\geq 2}.$$

(All these elements are Weyl-invertible in the sense of Definition 2.2.2 (e) by Lemma 2.2.23 (b) and Lemma 2.2.24.) For all $k \in \mathbb{N}_+$, we define

$$x_{k,1} := e_k \in U_k^\sharp \quad \text{and} \quad x_{k,n} := f_k \in U_{k+(n-1)}^\sharp.$$

Further, we put
$$x_{k,i} := [e_k, f_k^{-1}]_{k+i-1} \in U_{k+i-1}$$
for all $k \in \mathbb{N}_+$ and all $i \in [2, n-1]$. Thus we have
$$[x_{k,1}, x_{k,n}^{-1}] = x_{k,2} \cdots x_{k,n-1} \tag{2.2}$$
for all $k \in \mathbb{N}_+$. Finally, we put $w_0 := \mu(v)$ and $w_k := \mu(e_k)$ for all $k \in \mathbb{N}_+$. Our goal is to show that (w_0, w_1) satisfies the braid relations, that is, that $P_n(w_0, w_1) = P_n(w_1, w_0)$.

2.2.29 Lemma. *Let everything be as in 2.2.28. Then*
$$f_k = x_{k,n} = x_{k+1,n-1} = x_{k+2,n-2} = \cdots = x_{k+n-1,1} = e_{k+n-1}$$
for all $k \in \mathbb{N}_+$. In particular, $f_k = e_{k+n-1}$ for all $k \in \mathbb{N}_+$.

Proof. Let $k \in \mathbb{N}_+$ be arbitrary. By Proposition 2.2.10 (a) and the definition of e_{k+1}, we have
$$x_{k+1,1} = e_{k+1} = f_k^{\mu(e_k)} = x_{k,n}^{\mu(x_{k,1})} = x_{k,2}.$$
Further,
$$x_{k+1,n} = f_{k+1} = \lambda(e_k) = \lambda(x_{k,1}).$$
Together with (2.2), these equations imply that
$$x_{k+1,2} \cdots x_{k+1,n-1} = [x_{k+1,1}, x_{k+1,n}^{-1}] = [x_{k,2}, \lambda(x_{k,1})^{-1}] = x_{k,3} \cdots x_{k,n}.$$
Since G is rank-2-injective, we infer that $x_{k+1,i} = x_{k,i+1}$ for all $i \in [2, n-1]$. The assertion follows. □

The following statement is a mere corollary of Lemma 2.2.29, but it is worth to be pointed out on its own.

2.2.30 Lemma. *Let G, Φ and the rank-2 labelling be as in 2.2.28. Let $b_1 \in U_1^\sharp$ and $b_n \in U_n^\sharp$. Then for all $i \in [2, n-1]$, the element $[b_1, b_n]_i$ lies in $U_i^\sharp$.*

Proof. Put $u := b_1 \in U_1^\sharp$ and $v := b_n^{-1}$. By Proposition 2.2.6 (a), v is contained in $U_n^\sharp$. Thus we can define sequences $(e_k)_{k\in\mathbb{N}_+}$, $(f_k)_{k\in\mathbb{N}_+}$ and $(x_{k,i})_{k\in\mathbb{N}_+, i\in[2,n-1]}$ as in 2.2.28. Then for all $i \in [2, n-1]$, we have
$$[b_1, b_n]_i = x_{1,i} = x_{i,1} = e_i \in U_i^\sharp,$$
as desired. □

2.2.31 Lemma. *Let everything be as in 2.2.28. Then we have $w_k = w_{k+n}$ for all $k \in \mathbb{N}_0$ and $w_{k-1}^{w_k} = w_{k+1}$ for all $k \in \mathbb{N}_+$.*

Proof. Let $k \in \mathbb{N}_+$. It follows from Lemma 2.2.24 (a), Lemma 2.2.29 and the definition of f_{k+1} that
$$w_k = \mu(e_k) = \mu\big(\lambda(e_k)\big) = \mu(f_{k+1}) = \mu(e_{k+n}) = w_{k+n}.$$
Similarly,
$$w_0 = \mu(v) = \mu(f_1) = \mu(e_n) = w_n.$$
This proves the first claim. Further, we have
$$e_{k+1} = f_k^{\mu(e_k)} = e_{k+n-1}^{w_k}.$$
By Lemma 2.2.23 (b) and the previous claim, this implies that
$$w_{k+1} = \mu(e_{k+1}) = \mu(e_{k+n+1}^{w_k}) = \mu(e_{k+n+1})^{w_k} = w_{k+n-1}^{w_k} = w_{k-1}^{w_k},$$

as desired. ☐

2.2.32 Lemma. *Let everything be as in 2.2.28. Then for all $k \in \mathbb{N}_+$, we have*

$$w_{2k} = w_0^{w_1 (w_0 w_1)^{k-1}} \quad \text{and} \quad w_{2k+1} = w_1^{(w_0 w_1)^k}.$$

Proof. We prove that statement by induction on k. For $k = 1$, it is a direct consequence of Lemma 2.2.31 that

$$w_{2k} = w_2 = w_0^{w_1} = w_0^{w_1 (w_0 w_1)^{k-1}} \qquad \text{and}$$
$$w_{2k+1} = w_3 = w_1^{w_2} = w_1^{w_1^{-1} w_0 w_1} = w_1^{w_0 w_1} = w_1^{(w_0 w_1)^k}.$$

Now assume that the desired statement is true for some $k \in \mathbb{N}_+$. We proceed to show that it is true for $k+1$ as well. Again by Lemma 2.2.31, we have

$$w_{2k+2} = w_{2k}^{w_{2k+1}} = w_{2k+1}^{-1} w_{2k} w_{2k+1}.$$

By the induction hypothesis, it follows that

$$\begin{aligned} w_{2k+2} &= \big(w_1^{(w_0 w_1)^k}\big)^{-1} w_0^{w_1 (w_0 w_1)^{k-1}} w_1^{(w_0 w_1)^k} = \big((w_1^{w_0})^{-1} w_0 (w_1^{w_0})\big)^{w_1 (w_0 w_1)^{k-1}} \\ &= \big(w_0^{-1} w_1^{-1} w_0 w_0 w_0^{-1} w_1 w_0\big)^{w_1 (w_0 w_1)^{k-1}} = \big(w_0^{-1} w_1^{-1} w_0 w_1 w_0\big)^{w_1 (w_0 w_1)^{k-1}} \\ &= (w_0^{w_1 w_0})^{w_1 (w_0 w_1)^{k-1}} = w_0^{w_1 (w_0 w_1)^k}. \end{aligned}$$

In the same way, we have

$$\begin{aligned} w_{2k+3} &= w_{2k+1}^{w_{2k+2}} = w_{2k+2}^{-1} w_{2k+1} w_{2k+2} = \big(w_0^{w_1 (w_0 w_1)^k}\big)^{-1} w_1^{(w_0 w_1)^k} w_0^{w_1 (w_0 w_1)^k} \\ &= \big((w_0^{w_1})^{-1} w_1 w_0^{w_1}\big)^{(w_0 w_1)^k} = \big(w_1^{-1} w_0^{-1} w_1 w_1 w_1^{-1} w_0 w_1\big)^{(w_0 w_1)^k} \\ &= \big(w_1^{-1} w_0^{-1} w_1 w_0 w_1\big)^{(w_0 w_1)^k} = (w_1^{w_0 w_1})^{(w_0 w_1)^k} = w_1^{(w_0 w_1)^{k+1}}. \end{aligned}$$

This finishes the proof. ☐

2.2.33 Remark. Assume that $\bar{\Phi}$ is a root system which is crystallographic and not reduced and let $\bar{G}$ be a group which has $\bar{\Phi}$-commutator relations with root groups $(\bar{U}_\alpha)_{\alpha \in \bar{\Phi}}$. We do not assume that these commutator relations are crystallographic. For any root α, we denote by α' the unique indivisible root in $\mathbb{R}_{>0}\alpha$. Then $\alpha' \in \{\alpha, \alpha/2\}$ for all roots α by Lemma 1.2.28. Thus it follows from axiom (RGG-Div) that $\bar{U}_\alpha \subseteq \bar{U}_{\alpha'}$ for all roots α. Hence every α-Weyl element is also an α'-Weyl element for all roots α and the groups $(\bar{U}_\alpha)_{\alpha \in \bar{\Phi}^{\mathrm{indiv}}}$ satisfy $\bar{\Phi}^{\mathrm{indiv}}$-commutator relations. We conclude that, in order to verify the braid relations for Weyl elements in $(\bar{G}, (\bar{U}_\alpha)_{\alpha \in \bar{\Phi}})$, it suffices to verify the braid relations in $(\bar{G}, (\bar{U}_\alpha)_{\alpha \in \bar{\Phi}^{\mathrm{indiv}}})$.

2.2.34 Theorem ([TW02, (6.9)]). *Assume that Φ is crystallographic or reduced, that G is rank-2-injective and that $U_\alpha^\sharp \neq \varnothing$ for all roots α. Then G satisfies the braid relations for Weyl elements.*

Proof. If Φ is of rank 1, there is nothing to show, so we assume that it has rank at least 2. Let Δ be any root base and let $\alpha, \beta \in \Phi$ be distinct. It suffices to consider the root subsystem spanned by $\{\alpha, \beta\}$, so we can assume that Φ is of rank 2. Further, we can assume by Remark 2.2.33 that Φ is reduced.

Put $n := |\Phi|/2$. If $n = 2$, then the assertion holds by Proposition 2.2.27, so assume that $n \geq 3$. In this situation, we know from Proposition 2.2.22 that G has unique Weyl extensions. Fix the rank-2 labeling of Φ which is induced by (α, β), so that $U_\alpha = U_1$ and $U_\beta = U_n$. Let $u \in U_1^\sharp$, $v \in U_n^\sharp$ and put $w_1 := \mu(u)$,

$w_0 := \mu(v)$. Then we are in the situation of 2.2.28, and we have to show that $P_n(w_1, w_0) = P_n(w_0, w_1)$.

At first, assume that n is even and put $l := n/2 \in \mathbb{N}_+$. Then it follows from Lemma 2.2.32 that

$$w_0 = w_n = w_{2l} = w_0^{w_1(w_0w_1)^{l-1}} = \big(w_1(w_0w_1)^{l-1}\big)^{-1} \cdot w_0 \cdot w_1(w_0w_1)^{l-1}.$$

Multiplying this term from the left side with $w_1(w_0w_1)^{l-1}$, we infer that

$$w_1(w_0w_1)^{l-1}w_0 = w_0w_1(w_0w_1)^{l-1}.$$

In other words, $P_n(w_1, w_0) = P_n(w_0, w_1)$, as desired.

Now assume that n is odd, so that $n = 2l + 1$ for some $l \in \mathbb{N}_{\geq 1}$. Then

$$w_0 = w_n = w_{2l+1} = w_1^{(w_0w_1)^l} = \big((w_0w_1)^l\big)^{-1} \cdot w_1 \cdot (w_0w_1)^l$$

by Lemma 2.2.32. Multiplying from the left side with $(w_0w_1)^l$, we infer that

$$(w_0w_1)^l w_0 = w_1(w_0w_1)^l.$$

Again, this says precisely that $P_n(w_1, w_0) = P_n(w_0, w_1)$, which finishes the proof of the braid relations. □

2.2.35 Note. The main ingenuity in the proof of Theorem 2.2.34 in [TW02, (6.9)] lies in the fact that it manages to cover all values of n at the same time. More explicit variants of the proof which cover only specific root systems will be given in Remarks 5.4.11, 7.6.17 and 9.5.14.

2.3 Closed and Ordered Sets of Roots

2.3.1 Notation for this section. We denote by Φ a root system and we will frequently use Convention 2.3.5.

Our next goal is to study under which conditions the product map on a positive system in Φ is bijective. The notion of closed sets of roots, which we have already introduced in Definition 1.2.39, will be indispensable in this context. The current section is dedicated to a proper investigation of these objects. This can be done on a purely combinatorial, root-system-theoretic level without reference to any Φ-graded group. We could have already covered this subject in chapter 1, but we decided not to because it is less standard than the other topics of chapter 1. Further, the motivation is more obvious after having introduced groups with Φ-commutator relations and product maps.

Apart from some basic properties, our goal is to show that every positive system of Φ has an extremal ordering (except when Φ is not reduced, in which case we have to remove roots until only one root from each ray remains). This is Proposition 2.3.24. In Lemma 2.4.17, we will show that under a suitable condition on a group G with Φ-commutator relations, the product map for any extremal ordering is bijective.

None of the material in this section is new or surprising, but the literature on this topic can be confusing because no single reference provides all the facts that we will list here. Further, there are subtle differences in the used definitions. This concerns, in particular, the case of non-reduced root systems. Some references on this topic are [Bou81, Section VI.1.7] and [Ste67, Lemmas 16–18].

2.3.2 Definition (Ideals). Let Ψ be a subset of Φ. A subset I of Ψ is called an *ideal of* Ψ if $]\alpha,\beta[_{\mathrm{Cox}} \subseteq I$ for all non-proportional $\alpha \in I$ and $\beta \in \Psi$, and it is called a *crystallographic ideal of* Ψ if $]\alpha,\beta[\subseteq I$ for all non-proportional $\alpha \in I$ and $\beta \in \Psi$.

2.3.3 Warning. The terminology "crystallographic ideal" makes it sound like being crystallographic is a property that an ideal may or may not have, but it is actually the other way around: Every ideal is also a crystallographic ideal, but not every crystallographic ideal is an ideal.

2.3.4 Example. For any element w of the Weyl group and any positive system Π in Φ, the set $N(w) := N_\Pi(w) := \{\, \alpha \in \Pi^{\mathrm{indiv}} \mid \alpha^{w^{-1}} \in -\Pi \,\}$ from Definition 1.3.21 is closed.

The same remarks as in Note 2.1.7 apply to (crystallographically) closed set and to (crystallographic) ideals: The crystallographic properties are well-defined even in the non-crystallographic setting, but they are only useful in the crystallographic setting.

2.3.5 Convention. In the following, we will often put the words "crystallographic" and "crystallographically" in brackets. Any assertion of this form should be interpreted as two statements, once with and once without *all* words in brackets. For example, the following Remark 2.3.6 asserts that any ideal is closed and that any crystallographic ideal is crystallographically closed. On the contrary, it does not assert that any crystallographic ideal is closed. By the end of this section, it should be clear that the underlying principle is always the same. In Note 2.3.25, we will indicate how the crystallographic and the non-crystallographic theory could be treated in a more uniform way.

2.3.6 Remark. Let Ψ be a subset of Φ and let I be a subset of Ψ. If I is a (crystallographic) ideal of Ψ, then clearly I is (crystallographically) closed. Conversely, if $I = \Psi \setminus \{\alpha\}$ for some $\alpha \in \Psi$ and Ψ is (crystallographically) closed, then I is automatically a (crystallographic) ideal of Ψ if I is (crystallographically) closed. Further, Ψ is a (crystallographic) ideal of Ψ if and only if Ψ is (crystallographically) closed.

The following two lemmas would fit into section 1.2 as well. However, we will only need them in the proof of Lemma 2.3.9, so we have put them here.

2.3.7 Lemma. *Assume that Φ is crystallographic and let α, β be non-proportional roots. If $\alpha \cdot \beta > 0$, then $\alpha - \beta$ is a root. If $\alpha \cdot \beta < 0$, then $\alpha + \beta$ is a root.*

Proof. This is proven in [Hum72, Lemma 9.4] for reduced root systems, but the same arguments work in the general case. Alternatively, since BC_n is the only crystallographic root system which is not reduced (by Theorem 1.2.53), we could prove the general case by an inspection of this root system. $\square$

2.3.8 Lemma. *Assume that Φ is crystallographic. Let α, β be non-proportional roots and let $i, j \in \mathbb{N}_+$ such that $i\alpha + j\beta$ is a root. Then either $(i-1)\alpha + j\beta$ or $i\alpha + (j-1)\beta$ is a root.*

Proof. Set $\gamma := i\alpha + j\beta$. Since $i \geq 1$ and $j \geq 1$, the sets $\{\, \alpha, \gamma \,\}$ and $\{\, \beta, \gamma \,\}$ are linearly independent. Therefore, it follows from Lemma 2.3.7 that $(i-1)\alpha + j\beta \in \Phi$ if $\gamma \cdot \alpha > 0$ and $i\alpha + (j-1)\beta \in \Phi$ if $\gamma \cdot \beta > 0$. Now suppose that $\gamma \cdot \alpha \leq 0$ and $\gamma \cdot \beta \leq 0$. Then

$$\gamma \cdot \gamma = \gamma \cdot (i\alpha + j\beta) = i(\gamma \cdot \alpha) + j(\gamma \cdot \beta) \leq 0.$$

Since $\cdot$ is positive-definite, it follows that $\gamma \cdot \gamma = 0$ and thus $\gamma = 0$. This contradicts the fact that γ is a root. □

The following result shows that the condition in the definition of crystallographic ideals can be slightly weakened.

2.3.9 Lemma. *Assume that Φ is crystallographic. Let Ψ be a crystallographically closed subset of Φ and let I be a subset of Ψ with the property that for all non-proportional $\alpha \in I$ and $\beta \in \Psi$ for which $\alpha + \beta$ is a root, $\alpha + \beta$ lies in I. Then I is a crystallographic ideal of Ψ.*

Proof. Let $\alpha \in I$, $\beta \in \Psi$ and $\gamma \in]\alpha, \beta[$. Then there exist $i, j \in \mathbb{N}_+$ such that $\gamma = i\alpha + j\beta$. By induction on $i + j$, we show that γ lies in I. If $i + j = 2$, then $\gamma = \alpha + \beta$ and so γ lies in I by the assumption on I. Now assume that $i + j \geq 3$. By Lemma 2.3.8, we have $(i-1)\alpha + j\beta \in \Phi$ or $i\alpha + (j-1)\beta \in \Phi$. The induction hypothesis now implies that $(i-1)\alpha + j\beta \in I$ or $i\alpha + (j-1)\beta \in I$. In the first case, it follows that

$$\gamma = \big((i-1)\alpha + j\beta\big) + \alpha \in I$$

because $\gamma \in \Phi$, $(i-1)\alpha + j\beta \in I$ and $\alpha \in \Psi$. Similarly, in the second case it follows that

$$\gamma = \big(i\alpha + (j-1)\beta\big) + \beta \in I.$$

This finishes the proof. □

We make a brief detour into a slightly different notion of closedness, which will be needed in the proof of Proposition 2.3.12 (and only there). Alternatively, we could prove Proposition 2.3.12 by citing [LN19, 1.14], but this would require a similar amount of notational setup.

2.3.10 Definition (Bourbaki-closed). Assume that Φ is crystallographic. We say that a subset Ψ of Φ is *Bourbaki-closed* if it is closed in the sense of [Bou81, Définition VI.1.4, p. 160], which means that for all $\alpha, \beta \in \Psi$ with $\alpha + \beta \in \Phi$, we have $\alpha + \beta \in \Psi$.

2.3.11 Lemma. *Assume that Φ is crystallographic and let Ψ be a subset of Φ. Then the following hold:*

(a) If Ψ is Bourbaki-closed, then Ψ is crystallographically closed.

(b) Assume that Ψ is crystallographically closed. Then Ψ is Bourbaki-closed if and only if it satisfies the following conditions for all roots α for which 2α is also a root: If $\alpha \in \Psi$ then $2\alpha \in \Psi$; if $2\alpha, -\alpha \in \Psi$ then $\alpha \in \Psi$.

(c) If Φ is reduced, then Ψ is crystallographically closed if and only if it is Bourbaki-closed.

(d) Assume that Ψ is crystallographically closed and that $-\lambda\alpha \notin \Psi$ for all $\alpha \in \Psi$ and $\lambda > 0$. Then Ψ is contained in a Bourbaki-closed set Ψ' with $\Psi' \cap (-\Psi') = \varnothing$.

Proof. By Lemma 2.3.9, any Bourbaki-closed set Ψ is a crystallographic ideal of itself, which by Remark 2.3.6 means precisely that Ψ is crystallographically closed. This proves (a). However, a crystallographically closed set Ψ is not necessarily Bourbaki-closed because the roots α, β in Definition 1.2.39 are required to be non-proportional. Thus a crystallographically closed set Ψ is Bourbaki-closed if and only if for all roots $\alpha \in \Psi$ and for all $\lambda \in \mathbb{R}$ such that $\lambda\alpha \in \Psi$ and $(1+\lambda)\alpha \in \Phi$, we have $(1+\lambda)\alpha \in \Psi$. By Lemma 1.2.28, we can restrict to $\lambda \in \{1/2, 1, 2\}$, which yields precisely the statement of (b). Assertion (c) follows from (b). In (d), we can take $\Psi' := (\Psi \cup 2\Psi) \cap \Phi$. □

2.3.12 Proposition. *Assume that Φ is crystallographic and let Ψ be a crystallographically closed subset of Φ such that for all $\alpha \in \Psi$, we have $-\lambda\alpha \notin \Psi$ for all $\lambda > 0$. (If Φ is reduced, this simply means that $\Psi \cap (-\Psi) = \varnothing$.) Then there exists a positive system Π in Φ which contains Ψ.*

Proof. This is proven in [Bou81, Proposition VI.1.22, p. 163] under the slightly modified assumptions that Ψ is Bourbaki-closed and $\Psi \cap (-\Psi) = \varnothing$. We have proved in Lemma 2.3.11 (d) that Ψ is contained in a set Ψ' with these properties, so the assertion follows. □

2.3.13 Definition (Extremal root). Let Ψ be a subset of Φ and let $\alpha \in \Psi$. Then α is called *extremal in* Ψ if there exists a point p in the Euclidean space $(V, \cdot)$ surrounding Φ which is not orthogonal to any root in Φ such that $\alpha \cdot p > 0$ and $\beta \cdot p < 0$ for all $\beta \in \Psi \setminus \{\alpha\}$.

2.3.14 Remark. Let Ψ be a subset of Φ. Then a root $\alpha \in \Psi$ is extremal in Ψ if and only if there exists a positive system Π in Φ which contains $\Psi \setminus \{\alpha\}$ but not α.

2.3.15 Definition (Root ordering). Let Ψ be a subset of Φ. An *ordering of* Ψ is a tuple $(\alpha_1, \dots, \alpha_m)$ such that $\alpha_1, \dots, \alpha_m$ are pairwise distinct and $\Psi = \{ \alpha_1, \dots, \alpha_m \}$.

2.3.16 Definition (Properties of orderings). Let Ψ be a subset of Φ and let $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$ be an ordering of Ψ.

(a) $\bar{\alpha}$ is called *extremal* if α_i is extremal in $\{ \alpha_i, \dots, \alpha_m \}$ for all $i \in [1, m]$.

(b) $\bar{\alpha}$ is called *(crystallographically) normal* if $\{ \alpha_i, \dots, \alpha_m \}$ is a (crystallographic) ideal of Ψ for all $i \in [1, m]$.

(c) $\bar{\alpha}$ is called *(crystallographically) subnormal* if Ψ is (crystallographically) closed and $\{ \alpha_{i+1}, \dots, \alpha_m \}$ is a (crystallographic) ideal of $\{ \alpha_i, \dots, \alpha_m \}$ for all $i \in [1, m-1]$.

(d) $\bar{\alpha}$ is called a *height ordering* if Ψ is contained in some positive system Π and $\mathrm{ht}(\alpha_1) \le \dots \le \mathrm{ht}(\alpha_m)$ where ht denotes the height function with respect to Π.

We will now investigate various basic properties of the notions defined in Definition 2.3.16.

2.3.17 Remark (Basic properties of extremal orderings). Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$ be an ordering of a subset Ψ of Φ.

(1) If $\bar{\alpha}$ is extremal, then there cannot exist distinct $i, j \in [1, m]$ and $\lambda > 0$ such that $\alpha_i = \lambda\alpha_j$.

(2) Let $i \in [1, m]$ and $\lambda \in \mathbb{R}_{>0}$ such that $\lambda\alpha_i$ is a root. Then $(\alpha_1, \dots, \alpha_m)$ is extremal if and only if $(\alpha_1, \dots, \lambda\alpha_i, \dots, \alpha_m)$ is extremal.

(3) Let ψ be an orthogonal automorphism of the Euclidean space $(V, \cdot)$ surrounding Φ. Then $(\alpha_1, \dots, \alpha_m)$ is an extremal ordering of Ψ if and only if $(\alpha_1^{\psi}, \dots, \alpha_m^{\psi})$ is an extremal ordering of Ψ^{ψ}.

(4) Any tuple which is obtained from $\bar{\alpha}$ by deleting an arbitrary number of entries is also extremal.

2.3.18 Remark (Basic properties of (sub-) normal orderings). Let Ψ be a subset of Φ and let $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$ be an ordering of Ψ.

(1) If $\bar{\alpha}$ is (crystallographically) normal, then it is also (crystallographically) subnormal.

(2) If $\bar{\alpha}$ is (sub-) normal, then it is also crystallographically (sub-) normal.

(3) If $\bar{\alpha}$ is (sub-) normal, then for all for all $i \in [1, m]$, the tuple $(\alpha_i, \ldots, \alpha_m)$ is a (sub-) normal ordering of $\{ \alpha_i, \ldots, \alpha_m \}$. The same assertion holds for the crystallgraphic properties.

(4) If $\bar{\alpha}$ is (crystallographically) normal, then Ψ is a (crystallographic) ideal of itself, and so it is (crystallographically) closed by Remark 2.3.6.

(5) It follows from Remark 2.3.6 that $\bar{\alpha}$ is (crystallographically) subnormal if and only if all sets $\{ \alpha_i, \ldots, \alpha_m \}$ for $i \in [1, m]$ are (crystallographically) closed.

Observe that we had to require in the definition of (crystallographically) subnormal orderings that Ψ is (crystallographically) closed, which was not necessary for (crystallographically) normal orderings.

2.3.19 Example (of a subnormal extremal ordering). Let $S = [\alpha, \beta]_{\mathrm{Cox}}^{\mathrm{indiv}}$ for some indivisible roots α, β and let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ be an interval ordering of S in the sense of Definition 1.2.34. Then it follows from Remark 2.3.18 (5) that $\bar{\alpha}$ is subnormal. Further, it is clear that $\bar{\alpha}$ is extremal.

2.3.20 Lemma. *Assume that Φ is crystallographic, let Ψ be a crystallographically closed subset of Φ and let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m)$ be a height ordering of Ψ. Then $\bar{\alpha}$ is a crystallographically normal ordering of Ψ.*

Proof. Let $i \in [1, m]$. We have to show that $I := \{ \alpha_i, \ldots, \alpha_m \}$ is a crystallographic ideal of Ψ. By the definition of height orderings, there exists a positive subsystem Π of Φ which contains Ψ such that $\mathrm{ht}(\alpha_1) \leq \cdots \leq \mathrm{ht}(\alpha_m)$. By Lemma 2.3.9, it suffices to show that for non-proportional $\alpha \in \Psi$ and $\beta \in I$ for which $\alpha + \beta$ is a root, $\alpha + \beta$ lies in I. Since Ψ is crystallographically closed, $\alpha + \beta$ lies in Ψ, so there exists $j \in [1, m]$ such that $\alpha + \beta = \alpha_j$. Now $\mathrm{ht}(\alpha_j) = \mathrm{ht}(\alpha) + \mathrm{ht}(\beta) > \mathrm{ht}(\beta) \geq \mathrm{ht}(\alpha_i)$, so $j > i$. This implies that α_j lies in I, which finishes the proof. $\square$

2.3.21 Lemma. *Let Ψ be a (crystallographically) closed subset of Φ. Then every extremal ordering $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m)$ of Ψ is (crystallographically) subnormal, and for all $i \in [1, m]$, the set $\{ \alpha_i, \ldots, \alpha_m \}$ is (crystallographically) closed.*

Proof. Since $(\alpha_2, \ldots, \alpha_m)$ is also extremal by Remark 2.3.17 (4), it suffices by induction to show that $\{ \alpha_2, \ldots, \alpha_m \}$ is (crystallographically) closed and a (crystallographic) ideal of Ψ. By Remark 2.3.6, we only have to prove the first statement because it implies the second one. Let $i, j \in [2, m]$ such that α_i, α_j are non-proportional. Write $I :=]\alpha_i, \alpha_j[$ for the proof of the crystallographic assertion and $I :=]\alpha_i, \alpha_j[_{\mathrm{Cox}}$ for the proof of the non-crystallographic assertion. Since Ψ is (crystallographically) closed, we have $I \subseteq \Psi$, so it suffices to show that α_1 is not contained in I. Assume that $\alpha_1 = \lambda\alpha_i + \mu\alpha_j$ for some $\lambda, \mu \in \mathbb{R}_{>0}$ (where $\lambda, \mu \in \mathbb{N}_+$ for the crystallographic assertion). Since α_1 is extremal in Ψ, there exists a point p in the surrounding Euclidean space of Φ such that $\alpha_1 \cdot p > 0$ and $\alpha_i \cdot p, \alpha_j \cdot p < 0$. Then

$$0 < \alpha_1 \cdot p = \lambda(\alpha_i \cdot p) + \mu(\alpha_j \cdot p) < 0,$$

which is a contradiction. $\square$

We now prove some existence results concerning extremal orderings. The following result is essentially a stronger version of Proposition 1.3.28: It provides not only a description of the set $N(w)$, but even an extremal ordering of $N(w)$.

2.3.22 Proposition. *Choose a root base Δ of Φ and denote by Π the corresponding positive system. Let w be an element of the Weyl group of Φ, let $\bar{\delta} = (\delta_1, \ldots, \delta_k)$ be a Δ-expression of w and let $\bar{\alpha} = (\alpha_k, \ldots, \alpha_1)$ be the inverse associated root sequence from Definition 1.3.26. Then $\bar{\alpha}$ is an extremal ordering of the set $N(w) = \{ \beta \in \Pi^{indiv} \mid \beta^{w^{-1}} \in -\Pi \}$ from Definition 1.3.21.*

Proof. If $k = 0$, then $w = 1_W$ and there is nothing to prove. We proceed by induction and assume that $k > 0$. Put $\delta := \delta_k$, $\bar{\delta}' := (\delta_1, \ldots, \delta_{k-1})$ and $w' := \sigma_{\bar{\delta}'}$. Then $\bar{\delta}'$ is a reduced expression of w' and $\ell(w'\sigma_\delta) = \ell(w) > \ell(w')$. Denote the root sequence of $\bar{\delta}'$ by $(\beta_1, \ldots, \beta_{k-1})$. By the induction hypothesis, $(\beta_{k-1}, \ldots, \beta_1)$ is an extremal ordering of $N(w')$. Thus by Remark 2.3.17 (3), $(\beta_{k-1}^{\sigma(\delta)}, \ldots, \beta_1^{\sigma(\delta)})$ is an extremal ordering of $N(w')^{\sigma(\delta)}$. Recall from Remark 1.3.27 that the root sequence of $\bar{\delta}$ is $(\beta_1^{\sigma(\delta)}, \ldots, \beta_{k-1}^{\sigma(\delta)}, \delta)$ and from Lemma 1.3.25 that $N(w) = N(w')^{\sigma(\delta)} \sqcup \{\delta\}$. Hence it remains to show that δ is extremal in $\{\delta\} \cup N(w')^{\sigma(\delta)}$. Equivalently, we have to show that $\delta^{\sigma(\delta)}$ is extremal in $\{\delta^{\sigma(\delta)}\} \cup N(w')$, which holds by Lemma 1.3.23. This finishes the proof. $\square$

2.3.23 Proposition. *Choose a root base Δ of Φ and denote by Π the corresponding positive system. Put $m := |\Pi^{indiv}|$, let $\bar{\delta} = (\delta_1, \ldots, \delta_m)$ be any reduced expression of the longest element ρ of W with respect to Δ and denote by $\bar{\alpha}$ the inverse root sequence associated to $\bar{\delta}$. Let Ψ be any subset of Π^{indiv} and let $\bar{\beta}$ be the tuple which is obtained from $\bar{\alpha}$ by deleting all entries in $\Pi \setminus \Psi$. Then $\bar{\beta}$ is an extremal ordering of Ψ.*

Proof. Since $N(\rho) = \Pi^{\text{indiv}}$ by Proposition 1.3.29, it is a consequence of Proposition 2.3.22 that $\bar{\alpha}$ is an extremal ordering of Π^{indiv}. Thus by Remark 2.3.17 (4), $\bar{\beta}$ is an extremal ordering of Ψ. $\square$

2.3.24 Proposition. *Let Π be a positive system in Φ and let Ψ be a subset of Π consisting of pairwise non-proportional roots. Then there exists an extremal ordering of Ψ.*

Proof. For any root $\alpha \in \Psi$, there exists a unique indivisible root α' (which might be equal to α) such that $\mathbb{R}\alpha = \mathbb{R}\alpha'$. Then $\Psi' := \{ \alpha' \mid \alpha \in \Psi \}$ is a subset of Π^{indiv}. Thus Proposition 2.3.23 yields an extremal ordering $\bar{\alpha}'$ of Ψ. Scaling the roots in $\bar{\alpha}'$ appropriately and applying Remark 2.3.17 (2), we obtain an extremal ordering of Ψ. $\square$

2.3.25 Note. Continuing the remarks in Convention 2.3.5, we briefly describe how the crystallographic and the non-crystallographic considerations in this section (and in the rest of this thesis) could be treated in a more uniform way. Define a *root system with intervals* to be a tuple $\Psi = (\Phi, (\langle\alpha, \beta\rangle)_{\alpha,\beta})$ where Φ is a root system in the regular sense and $(\langle\alpha, \beta\rangle)_{\alpha,\beta}$ is a family of subsets of Φ where α, β runs over all pairs of non-proportional roots in Φ. We require that $\langle\alpha, \beta\rangle \subseteq]\alpha, \beta[_{\text{Cox}}$ for all non-proportional roots α, β. Now a *group with Ψ-commutator relations* is defined as in Definition 2.1.4, except that we must have $[U_\alpha, U_\beta] \subseteq U_{\langle\alpha,\beta\rangle}$. Similarly, the notions of ideals and closed sets of roots are defined with respect to the sets $\langle\alpha, \beta\rangle$.

Now let Φ be a root system. There are only two root systems with intervals corresponding to Φ that are interesting:

$$\tilde{\Phi} := \big(\Phi, (]\alpha, \beta[_{\text{Cox}})_{\alpha,\beta}\big) \quad \text{and} \quad \tilde{\Phi}^{\text{cry}} := \big(\Phi, (]\alpha, \beta[)_{\alpha,\beta}\big).$$

Note that in this setup, the information "whether we use the crystallographic terminology or not" is part of the datum of a root system with intervals. For example, "C_3 regarded as a root system in the Coxeter sense" and "C_3 regarded as a crystallographic root systems" are two distinct objects, namely $\tilde{C}_3$ and $\tilde{C}_3^{\mathrm{cry}}$. Thus C_3-graded groups are precisely $\tilde{C}_3$-graded groups and crystallographic C_3-graded groups are precisely $\tilde{C}_3^{\mathrm{cry}}$-graded groups.

Many (though not all) results in this thesis which concern Φ-graded groups for arbitrary root systems Φ can be stated and proven for Ψ-graded groups where $\Psi = (\Phi, (\langle \alpha, \beta \rangle)_{\alpha,\beta})$ is an arbitrary root system with intervals. In this setup, root gradings, root intervals, ideals and closed sets of roots are understood to be defined with respect to the sets $\langle \alpha, \beta \rangle$. Thus the crystallographic and the non-crystallographic case would not have to be considered separately most of the time. There are only a few results for which this strategy would not work, such as Proposition 2.4.13.

Despite the advantages of this approach, we will not use the language of root systems with intervals in the sequel. The reason for this is that the regular notion of root systems is well-established and that the distinction between the crystallographic and the non-crystallographic case is only a minor nuisance.

2.4 Bijectivity of the Product Map

2.4.1 Notation for this section. We denote by Φ a root system and by G a group with Φ-commutator relations with root groups $(U_\alpha)_{\alpha \in \Phi}$. Further, we will frequently use Convention 2.3.5.

2.4.2 Definition. Let Ψ be a subset of Φ, let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m)$ be an ordering of Ψ and let G be a group with Φ-commutator relations with root groups $(U_\alpha)_{\alpha \in \Phi}$. Then $\bar{\alpha}$ is called *G-injective*, *G-surjective* or *G-bijective* if the product map on $\bar{\alpha}$ is injective, surjective or bijective, respectively.

The goal of this section is to find a criterion which guarantees the existence of a G-bijective ordering on a (crystallographically) closed subset Ψ of Φ. In the crystallographic setting, we will even find that every ordering of Ψ is G-bijective.

2.4.3 Note (The product map in Chevalley groups). Assume that Φ is crystallographic and reduced and that G is a Chevalley group of type Φ (which implies that G has crystallographic Φ-commutator relations). Let Ψ be a crystallographically closed subset of Φ such that $\Psi \cap (-\Psi) = \varnothing$. It is shown in [Ste67, Lemma 17] that every height ordering of Ψ is G-bijective. Further, it is shown in [Ste67, Lemma 18] that the existence of a crystallographically normal G-bijective ordering of Ψ implies that every ordering of Ψ is G-bijective. Since every height ordering is crystallographically normal by Lemma 2.3.20, it follows that every ordering of G is bijective.

Our approach in this section is motivated by the strategy for Chevalley groups outlined above, but some of the arguments do not carry over to our more general setting. We begin this section with an investigation of product maps in arbitrary groups H. This includes Steinberg's result that the existence of a crystallographically normal G-bijective ordering of Ψ implies that every ordering of Ψ is G-bijective (Lemma 2.4.6). Unfortunately, this result is not applicable in our situation, and we state it purely for completeness. In its place, we will use Lemma 2.4.9, which is due to Tits and which says that a similar assertion holds if G has a central series with certain properties. Lemma 2.4.9 came to our attention in the form of [LN19, 3.10].

In the second part of this section, we prove the existence of a G-bijective ordering of Ψ (where Ψ is any subset of a positive system of Φ whose roots are pairwise non-proportional). By the same arguments as in [Ste67, Lemma 17], every subnormal ordering of Ψ is G-surjective (Lemma 2.4.14). However, Steinberg's proof of G-injectivity uses the natural action of a Chevalley group on a certain module, so it cannot be generalised to Φ-graded groups. Instead, we will show in Lemma 2.4.10 that every extremal ordering of Ψ is G-injective. Since extremal orderings exist by Proposition 2.3.24 and are subnormal by Lemma 2.3.21, we infer that there exists a G-bijective ordering of Ψ (Proposition 2.4.18). In general, this existence result is the best we can do because Steinberg's Lemma 2.4.6 does not hold for subnormal orderings and the conditions of Tits' Lemma 2.4.9 are not satisfied for general Φ-gradings. However, Tits' lemma does apply in the crystallographic setting, so we infer that every ordering of Ψ is G-bijective in this case.

2.4.A Purely Group-theoretic Observations

2.4.4 Lemma. *Let H be a group, let $m \in \mathbb{N}_{\geq 1}$ and let $U_1, \dots, U_m$ be subgroups of H. Then the product map $\mu_1 \colon U_1 \times \cdots \times U_m \to H$ is injective, surjective or bijective if and only if the product map $\mu_2 \colon U_m \times \cdots \times U_1 \to H$ is injective, surjective or bijective, respectively.*

Proof. Let $g_i \in U_i$ for all $i \in [1, m]$. Then we have
$$\mu_1(g_1, \dots, g_m) = \mu_2(g_m^{-1}, \dots, g_1^{-})^{-1}.$$
The assertion follows. □

2.4.5 Lemma. *Let H be a group, let $m \in \mathbb{N}_{\geq 1}$ and let $U_1, \dots, U_m$ be subgroups of H such that $U_1 \cup \cdots \cup U_m$ generates H. Put $V_i := \langle U_j \mid j \in [i, m] \rangle$ for all $i \in [1, m]$. If V_{i+1} is normal in V_i for all $i \in [1, m-1]$, then the product map $U_1 \times \cdots \times U_m \to H$ is surjective.*

Proof. For $m = 1$, there is nothing to prove. Now assume that $m = 2$. Since V_2 is normal in V_1, we have that U_1 normalises U_2. Hence by Relation 1.1.11 (iv), $U_1 U_2$ is a subgroup of H. In fact, $U_1 U_2 = H$ because $U_1 \cup U_2$ generates H. Thus the assertion holds for $m = 2$. The assertion for arbitrary m follows by induction. □

For completeness, here is the lemma used in [Ste67] to deduce G-bijectivity of arbitrary orderings.

2.4.6 Lemma ([Ste67, Lemma 18]). *Let H be a group, let $m \in \mathbb{N}_{\geq 1}$ and let $U_1, \dots, U_m$ be subgroups of H such that $U_1 \cup \cdots \cup U_m$ generates H. Put $V_i := \langle U_j \mid j \in [i, m] \rangle$ for all $i \in [1, m]$. Assume that V_i is normal in H for all $i \in [1, m]$ and that the product map $U_1 \times \cdots \times U_m \to H$ is injective. Then for all permutations $\sigma \colon [1, m] \to [1, m]$, the product map $U_{\sigma(1)} \times \cdots \times U_{\sigma(m)} \to H$ is bijective.*

To state Tits' lemma, we briefly recall the notion of central series.

2.4.7 Definition (Central series). Let H be a group. A *central series of H* is a finite subgroup sequence
$$\{1_G\} = Z_n \trianglelefteq Z_{n-1} \trianglelefteq \cdots \trianglelefteq Z_1 = H$$
such that $[H, Z_i] \subseteq Z_{i+1}$ for all $i \in [1, n-1]$. In particular, $Z_1, \dots, Z_n$ are normal subgroups of H.

2.4.8 Lemma. *Let H, K be groups and let $\pi \colon H \to K$ be a surjective homomorphism. If*

$$\{1_H\} = Z_n \trianglelefteq Z_{n-1} \trianglelefteq \cdots \trianglelefteq Z_1 = H$$

is a central series in H, then

$$\{1_K\} = \pi(Z_n) \trianglelefteq \pi(Z_{n-1}) \trianglelefteq \cdots \trianglelefteq \pi(Z_1) = K$$

is a central series in K.

Proof. For all $i \in [1, n-1]$, we have

$$[K, \pi(Z_i)] = [\pi(H), \pi(Z_i)] = \pi([H, Z_i]) \subseteq \pi(Z_{i+1}),$$

as desired. □

2.4.9 Lemma ([Tit87, 4.7, Lemma 2, p. 559]). *Let H be a group and let $H_1, \dots, H_n$ be subgroups such that H is generated by $H_1 \cup \cdots \cup H_n$. Assume that H possesses a central series*

$$\{1\} = Z_{h+1} \trianglelefteq Z_h \trianglelefteq \cdots \trianglelefteq Z_1 = H$$

such that for all $j \in \{1, \dots, h\}$, we have $Z_j \subseteq \langle H_{i(j)}, Z_{j+1} \rangle$ for some $i(j) \in \{1, \dots, n\}$. Then the following hold:

(a) For every permutation σ of the set $\{1, \dots, n\}$, the product map

$$\mu_\sigma \colon H_{\sigma(1)} \times \cdots \times H_{\sigma(n)} \to X$$

is surjective.

(b) If the product map μ_σ is injective for some permutation σ, then it is injective for all permutations σ.

2.4.B Existence of a Bijective Ordering

We now turn away from arbitrary groups and back to the group G with Φ-commutator relations. We begin with our only criterion for G-injectivity.

2.4.10 Lemma. *Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$ be an extremal ordering of Ψ. Assume that for any positive system Π in Φ and any $\beta \in -\Pi$, we have $U_\beta \cap U_\Pi = \{1_G\}$. Then $\bar{\alpha}$ is G-injective.*

Proof. We prove by induction on m that the product map

$$\mu_{\bar{\alpha}} \colon U_{\alpha_1} \times \cdots \times U_{\alpha_m} \to U_\Psi$$

is injective. The case $m = 1$ is trivial, so assume that $m > 1$. Let $(g_1, \dots, g_m)$ and $(h_1, \dots, h_m)$ be two elements of $U_{\alpha_1} \times \cdots \times U_{\alpha_m}$ such that $g_1 \cdots g_m = h_1 \cdots h_m$. Then

$$h_1^{-1} g_1 = h_2 \cdots h_m (g_2 \cdots g_m)^{-1} \in U_{\alpha_1} \cap \langle U_{\alpha_j} \mid j \in [2, m] \rangle. \tag{2.3}$$

Since α_1 is extremal in Ψ, there exists a positive system Π which contains $\{\alpha_2, \dots, \alpha_m\}$ but not α_1, so

$$U_{\alpha_1} \cap \langle U_{\alpha_j} \mid j \in [2, m] \rangle \subseteq U_{\alpha_1} \cap U_\Pi = \{1_G\}. \tag{2.4}$$

Putting (2.3) and (2.4) together, we infer that $g_1 = h_1$ and $g_2 \cdots g_m = h_2 \cdots h_m$. By the induction hypothesis, it follows that $g_j = h_j$ for all $j \in [1, m]$. Thus $\mu_{\bar{\alpha}}$ is injective. □

As a next step, we make the connection between ideals of Φ and normal subgroups of G. This is done in Lemma 2.4.11 for reduced root systems and in Lemma 2.4.12 for non-reduced crystallographic root systems. Both lemmas make essentially the same assertion, and they will be summarised in Proposition 2.4.13.

2.4.11 Lemma. *Assume that Φ is reduced. Let Ψ be a subset of Φ with $\Psi \cap (-\Psi) = \varnothing$ and let I be a subset of Ψ. Assume that I is a (crystallographic) ideal of Ψ (and that G has crystallographic Φ-commutator relations). Then U_I is normal in U_Ψ.*

Proof. Let $\alpha \in I$ and let $\beta \in \Psi$. Let $x_\alpha \in U_\alpha$, $x_\beta \in U_\beta$. We have to show that $[x_\alpha, x_\beta]$ lies in U_I. If β lies in I, then this is clear, so we can assume that β lies in $\Psi \setminus I$. In particular, $\alpha \neq \beta$. Since $\Psi \cap (-\Psi) = \varnothing$, we also have $\alpha \neq -\beta$. As Φ is reduced, this implies that α and β are non-proportional. If G has crystallographic Φ-commutator relations, then $[x_\alpha, x_\beta]$ lies in $U_{]\alpha,\beta[}$, so it lies in U_I if I is a crystallographic ideal of Ψ. In general, $[x_\alpha, x_\beta]$ lies in $U_{]\alpha,\beta[_{\mathrm{Cox}}}$, so it lies in U_I if I is an ideal. This finishes the proof. □

2.4.12 Lemma. *Assume that Φ is crystallographic and not reduced. Let Ψ be a subset of Φ such that $-\lambda\alpha \notin \Psi$ for all $\alpha \in \Psi$ and all $\lambda > 0$. Let I be a subset of Ψ. Assume that I is a (crystallographic) ideal of Ψ (and that G has crystallographic Φ-commutator relations). Then U_I is normal in U_Ψ.*

Proof. Just like in the proof of Lemma 2.4.11, we only have to show that $[x_\alpha, x_\beta]$ lies in U_I where $\alpha \in I$, $\beta \in \Psi \setminus I$, $x_\alpha \in U_\alpha$ and $x_\beta \in U_\beta$. If α and β are non-proportional, we can proceed as in the proof of Lemma 2.4.11. Otherwise we have $\beta = \lambda\alpha$ for some $\lambda \in \mathbb{R}_{>0} \setminus \{1\}$. Since Φ is crystallographic, we must have $\lambda = 2$ or $\lambda = 1/2$ by Lemma 1.2.28. If $\lambda = 2$, then x_α and x_β both lie in U_α by (RGG-Div), and thus $[x_\alpha, x_\beta] \in U_\alpha \subseteq U_I$. If $\lambda = 1/2$, then

$$[x_\alpha, x_\beta] \in [U_{2\beta}, U_\beta] \subseteq [U_\beta, U_\beta] \subseteq U_{2\beta} = U_\alpha \subseteq U_I,$$

again by (RGG-Div). This finishes the proof. □

2.4.13 Proposition. *Assume that Φ is reduced or crystallographic. Let Ψ be a subset of Φ such that $-\lambda\alpha \notin \Psi$ for all $\alpha \in \Psi$ and all $\lambda > 0$. Let I be a subset of Ψ. Assume that I is a (crystallographic) ideal of Ψ (and that G has crystallographic Φ-commutator relations). Then U_I is normal in U_Ψ.*

Proof. This follows from Lemmas 2.4.11 and 2.4.12. □

As a consequence of Proposition 2.4.13, we obtain a criterion for G-surjectivity.

2.4.14 Lemma. *Assume that Φ is reduced or crystallographic. Let Ψ be a subset of Φ such that $-\lambda\alpha \notin \Psi$ for all $\alpha \in \Psi$ and all $\lambda > 0$. Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m)$ be an ordering of Ψ. Assume that $\bar{\alpha}$ is (crystallographically) subnormal (and that G has crystallographic Φ-commutator relations). Then $\bar{\alpha}$ is G-surjective.*

Proof. Put $U_i := U_{\alpha_i}$ for all $i \in [1, m]$ and $V_i := \langle U_j \mid j \in [i, m] \rangle$ for all $i \in [1, m]$. Since $\bar{\alpha}$ is (crystallographically) subnormal, it follows from Proposition 2.4.13 that V_{i+1} is normal in V_i for all $i \in [1, m-1]$. Hence by Lemma 2.4.5, the product map $U_{\alpha_1} \times \cdots \times U_{\alpha_m} \to G$ is surjective, as desired. □

A consequence of Lemma 2.4.14 is that the product map on every closed root interval is surjective. In particular, if G is rank-2-injective (in the sense of Definition 2.1.19), then it is actually “rank-2-bijective”.

2.4.15 Lemma. *Let $S = [\alpha, \beta]_{Cox}^{indiv}$ for non-proportional $\alpha, \beta \in \Phi^{indiv}$ and let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$ be an interval ordering of S in the sense of Definition 1.2.34. Then $\bar{\alpha}$ is G-surjective.*

Proof. We know from Example 2.3.19 that $\bar{\alpha}$ is a subnormal ordering of S. Thus the assertion follows from Lemma 2.4.14. □

2.4.16 Remark. It follows from Lemma 2.4.15 that the maps in Definition 2.1.16 are always well-defined if G is rank-2-injective.

Putting everything together, we obtain a criterion for G-bijectivity.

2.4.17 Lemma. *Let Ψ be a subset of Π. Assume that Ψ is (crystallographically) closed (and that G has crystallographic Φ-commutator relations). Assume further that for any positive system Π' in Φ and any $\gamma \in -\Pi'$, we have $U_\gamma \cap U_{\Pi'} = \{1_G\}$. Then any extremal ordering of Ψ is G-bijective.*

Proof. Any such ordering is G-injective by Lemma 2.4.10 and G-surjective by Lemmas 2.3.21 and 2.4.14. ☐

In particular, G-bijective orderings exist under natural conditions.

2.4.18 Proposition. *Let Π be a positive system in Φ and let Ψ be a subset of Π consisting of pairwise non-proportional roots. Assume that Ψ is (crystallographically) closed (and that G has crystallographic Φ-commutator relations). Assume further that for any positive subsystem Π' of Φ and any $\gamma \in -\Pi$, we have $U_\gamma \cap U_{\Pi'} = \{1_G\}$. Then there exists a G-bijective ordering of Ψ.*

Proof. By Proposition 2.3.24, there exists an extremal ordering of Ψ and this ordering is G-bijective by Lemma 2.4.17. ☐

2.4.19 Note. The assumption in Proposition 2.4.18 that Ψ is (crystallographically) closed is essential for the G-surjectivity of Ψ. Even if Ψ is not (crystallographically) closed, there exists an extremal ordering of Ψ by Proposition 2.3.24, but we cannot apply Lemma 2.4.17. However, such an extremal ordering is still G-injective by Lemma 2.4.10.

2.4.C All Orderings are Bijective for Crystallographic Root Systems

In the crystallographic setting, we can apply Lemma 2.4.9 to infer that all orderings are G-bijective. Here we have to assume that certain root groups are abelian. For Φ-graded groups of rank at least 3, this will be proven to automatically be true (see Note 2.1.6), and the proof of this fact does not rely on Proposition 2.4.20. The first two assertions of the following result are essentially a special case of [LN19, 3.12].

2.4.20 Proposition. *Assume that Φ is crystallographic, that G has crystallographic Φ-commutator relations and that for all roots α for which 2α is not a root, the root group U_α is abelian. Let Ψ be a crystallographically closed subset of Φ such that $-\lambda\alpha \notin \Psi$ for all $\alpha \in \Psi$ and all $\lambda > 0$. Assume further that the roots in Ψ are pairwise non-proportional. Then the following hold:*

(a) *Any ordering of Ψ is G-surjective.*
(b) *If some ordering of Ψ is G-injective, then every ordering of Ψ is G-bijective.*
(c) *If $U_\gamma \cap U_{\Pi'} = \{1_G\}$ for any positive system Π' in Φ and any $\gamma \in -\Pi$, then every ordering of Ψ is G-bijective.*

Proof. By Proposition 2.3.12, there exists a positive system Π in Φ which contains Ψ. Denote by $\mathrm{ht}\colon \Phi \to \mathbb{Z}$ the height function with respect to Π, choose a height ordering $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$ of Ψ and define $Z_i := \langle U_{\alpha_j} \mid j \in [i, m] \rangle$ for all $i \in [1, m]$.

Then by the generalised commutator relation (Proposition 2.1.27), which we can apply by the assumption on certain root groups to be abelian, we have

$$[U_\Psi, Z_i] \subseteq \langle U_\alpha \mid \alpha \in \Phi, \mathrm{ht}(\alpha) > \mathrm{ht}(\alpha_i) \rangle \subseteq Z_{i+1}$$

for all $i \in [1, m-1]$. Hence

$$\{1_G\} \trianglelefteq Z_m \trianglelefteq Z_{m-1} \trianglelefteq \cdots \trianglelefteq Z_1 = U_\Psi$$

is a central series of U_Ψ. Further, it is clear that $Z_i = \langle U_{\alpha_i}, Z_{i+1} \rangle$ for all $i \in [1, m-1]$. Thus the conditions of Lemma 2.4.9 are satisfied. The assertions of (a) and (b) follow. Using Proposition 2.4.18, assertion (c) follows from (b). □

2.5 Root Graded Groups

2.5.1 Notation for this section. We denote by Φ an arbitrary root system.

In Propositions 2.4.18 and 2.4.20, we have found a simple criterion for the existence of G-bijective orderings in positive systems of Φ. This criterion is the final axiom for root graded groups, which should be thought of as a "non-degeneracy condition". In Note 2.5.13, we will discuss several variations of this condition and how they relate to each other.

2.5.2 Definition (Root graded group). Let G be a group. A *Φ-grading of G* is a Φ-pregrading $(U_\alpha)_{\alpha \in \Phi}$ with the following properties:

(i) G is generated by $(U_\alpha)_{\alpha \in \Phi}$ and $U_\alpha \neq \{1_G\}$ for all roots α.

(ii) G has Φ-commutator relations with root groups $(U_\alpha)_{\alpha \in \Phi}$ (in the sense of Definition 2.1.4).

(iii) For all $\alpha \in \Phi$, there exists an α-Weyl element in G (in the sense of Definition 2.2.2).

(iv) For all positive systems Π and all $\alpha \in \Phi \setminus \Pi$, we have $U_\Pi \cap U_\alpha = \{1_G\}$.

If, in addition, G has crystallographic Φ-commutator relations with root groups $(U_\alpha)_{\alpha \in \Phi}$ (in the sense of Definition 2.1.5), then $(U_\alpha)_{\alpha \in \Phi}$ is called a *crystallographic Φ-grading of G*. In this context, the pair $(G, (U_\alpha)_{\alpha \in \Phi})$ is called a (crystallographic) *Φ-graded group*.

2.5.3 Notation. If $(G, (U_\alpha)_{\alpha \in \Phi})$ is a Φ-graded group, then the root system Φ is called the *type of* $(G, (U_\alpha)_{\alpha \in \Phi})$ or simply the *type of* G. We adopt the convention that if X is a property of Φ, then G is also said to have property X. For example, a root graded group is said to be irreducible, simply-laced or of rank r if its type has the corresponding property.

Note that the definition of root graded groups is purely combinatorial: it only involves a root system and its Weyl group. There is no algebraic structure (like a ring or a module) hidden in the axioms, and there is at first glance no reason to believe that such a thing should exist. However, in chapter 3 we will see that Chevalley groups provide a large and well-known class of examples of crystallographic root graded groups which are "defined over a commutative associative ring". The goal of this thesis is to show that every crystallographic root graded group of rank at least 3 is "of algebraic origin" in a similar way. The following remark makes this statement more precise.

2.5.4 Goal (Coordinatisations of root graded groups). Let G be a group with a Φ-grading $(U_\alpha)_{\alpha \in \Phi}$ and denote by $\mathbb{O} := \mathrm{Orb}^{\mathrm{red}}(\Phi) = (O_1, \ldots, O_k)$ the set of indivisible orbits of Φ (see Notation 1.3.9). Let $\mathcal{X} = (X_1, \dots, X_k)$ be a family of algebraic structures (each of which has an underlying group structure) and put $X_\alpha := X_{i(\alpha)}$ for all $\alpha \in \Phi^{\mathrm{indiv}}$ where $i(\alpha)$ is the unique index such that $\alpha \in O_{i(\alpha)}$. A *coordinatisation of G by $\mathcal{X}$* is a family of maps $(\theta_\alpha)_{\alpha \in \Phi}$ with the following properties:

(i) For all $\alpha \in \Phi^{\mathrm{indiv}}$, the map $\theta_\alpha \colon X_\alpha \to U_\alpha$ is an isomorphism between the underlying group of the algebraic structure X_α and the root group U_α.

(ii) Let (α, β) be a pair of non-proportional indivisible roots and denote by $(\gamma_1, \dots, \gamma_p)$ an interval ordering of $]\alpha, \beta[_{\mathrm{Cox}}$. Then there exist maps

$$(f_i \colon X_\alpha \times X_\beta \to X_{\gamma_i})_{i \in [1,p]}$$

such that the following generalisation of the Chevalley commutator formula (see Theorem 3.2.16) holds:

$$[\theta_\alpha(a), \theta_\beta(b)] = \prod_{i=1}^{p} \theta_{\gamma_i}\big(f_i(a,b)\big)$$

for all $a \in X_\alpha$ and $b \in X_\beta$. Further, the maps $f_1, \dots, f_p$ can be described explicitly using only the structural maps of the algebraic structures in $\mathcal{X}$.

The preceding definition should be regarded more as a template than as a precise definition. For each type of Φ in the classification of root systems, we will introduce a more concise definition of coordinatisations of Φ-graded groups: see Definition 5.6.2 for types A, D and E, Definition 7.7.2 for type B, and Definition 9.6.2 for type BC (which includes type C as a special case). For type F_4, technicalities prevent us from using a definition in the exact same spirit, see Definition 10.4.15 and Note 10.4.16.

For all crystallographic root systems Φ of rank at least 3, the goal of this thesis is to find a class $\mathcal{C}_\Phi$ of algebraic structures such that every crystallographic Φ-graded groups is coordinatised by an object $\mathcal{X} \in \mathcal{C}_\Phi$. This is precisely the content of Theorems 5.7.14, 7.11.21, 9.10.26 and 10.7.8.

2.5.5 Note. The axiom that G is generated by $(U_\alpha)_{\alpha \in \Phi}$ is harmless: If G is a group with a Φ-pregrading $(U_\alpha)_{\alpha \in \Phi}$ which satisfies all axioms except for this one, then $(U_\alpha)_{\alpha \in \Phi}$ is a Φ-grading of the group G' which is generated by $(U_\alpha)_{\alpha \in \Phi}$. Since our interest lies in the coordinatisation of the root groups, everything outside of G' is irrelevant for our purposes, so we simply require that $G = G'$.

We will never need the notion of homomorphisms of root graded groups, but there is no harm in stating the obvious definition.

2.5.6 Definition (Homomorphisms of root graded groups). Let $(G, (U_\alpha)_{\alpha \in \Phi})$ and $(G', (U'_\alpha)_{\alpha \in \Phi})$ be Φ-graded groups. A group homomorphism $f \colon G \to G'$ is called a *homomorphism of root graded groups* if $f(U_\alpha) \subseteq U'_\alpha$ for all roots α. A homomorphism of root graded groups is called an *isomorphism of root graded groups* if it is an isomorphism of groups whose inverse is also a homomorphism of root graded groups.

We now record some special cases of previous results from this chapter.

2.5.7 Proposition. *Let G be a group with a Φ-grading $(U_\alpha)_{\alpha \in \Phi}$. Choose a root base Δ of Φ and denote by Π the corresponding positive system. Let w be an element of the Weyl group of Φ, let $\bar{\delta} = (\delta_1, \dots, \delta_k)$ be a reduced Δ-expression of w and let $\bar{\alpha} = (\alpha_1, \dots, \alpha_k)$ be the associated root sequence from Definition 1.3.26. Then the product map on $\bar{\alpha}$ in G is bijective.*

Proof. Denote by $\bar{\alpha}' := (\alpha_k, \dots, \alpha_1)$ the inverse root sequence. Then $N(w) = \{\alpha_1, \dots, \alpha_k\}$ by Proposition 1.3.28, where N is defined as in Definition 1.3.21. Further, $\bar{\alpha}$ is an extremal ordering by Proposition 2.3.22 and $N(w)$ is clearly closed, so the product map on $\bar{\alpha}'$ in G is bijective by Lemma 2.4.17. Using Lemma 2.4.4, we infer that the product map on $\bar{\alpha}$ is bijective as well. □

2.5.8 Proposition. *Assume that Φ is crystallographic and let G be a group with a crystallographic Φ-grading $(U_\alpha)_{\alpha\in\Phi}$. Assume further that for all roots α for which 2α is not a root, the root group U_α is abelian. Let Π be a positive system in Φ and let Ψ be a crystallographically closed subset of Π whose elements are pairwise non-proportional. Then the product map on any ordering of Ψ is bijective.*

Proof. This follows from Proposition 2.4.20 (c). □

By Note 2.1.6, the assumption in Proposition 2.5.8 on certain root groups to be abelian is always satisfied if Φ is of rank at least 3. However, we still have to prove this fact.

2.5.9 Lemma. *Let G be a group with a Φ-grading $(U_\alpha)_{\alpha\in\Phi}$. Then G is rank-2-injective.*

Proof. Let Π' be a positive system in a parabolic rank-2 subsystem of Φ and let $\bar{\alpha} = (\alpha_1, \dots, \alpha_k)$ be an interval ordering of $(\Pi')^{\mathrm{indiv}}$. Then $\Psi := \{\alpha_1, \dots, \alpha_k\}$ is closed and $\bar{\alpha}$ is an extremal ordering of Ψ. Thus it follows from Lemma 2.4.17 that the product map on $\bar{\alpha}$ is bijective, which finishes the proof. □

2.5.10 Theorem (Braid relations). *Let G be a group with a Φ-grading $(U_\alpha)_{\alpha\in\Phi}$ and assume that Φ is crystallographic or reduced. Then G satisfies the braid relations for Weyl elements.*

Proof. By Lemma 2.5.9, G is rank-2-injective. Hence the assertion is a special case of Theorem 2.2.34. □

The following result, the analogue of Lemma 2.1.12 for root gradings, guarantees the existence of many root graded subgroups in a root graded group. It allows us to prove many properties of root graded groups by restricting to the cases of rank 2 or rank 3.

2.5.11 Proposition. *Let G be a group with a Φ-grading $(U_\alpha)_{\alpha\in\Phi}$ and let Φ' be a subset of Φ. Denote by H the group which is generated by $(U_\alpha)_{\alpha\in\Phi'}$. If Φ' is a closed root subsystem, then$(U_\alpha)_{\alpha\in\Phi'}$ is a Φ'-grading of H. If Φ' is a crystallographically closed root subsystem and the commutator relations of G are crystallographic, then $(U_\alpha)_{\alpha\in\Phi'}$ is a crystallographic Φ'-grading of H. For any root α in Φ', every α-Weyl element in G is also an α-Weyl element in H.*

Proof. The assertion about Weyl elements is clear. The subgroup H has the desired (crystallographic) Φ'-commutator relations by Lemma 2.1.12. Further, Axiom 2.5.2 (iv) is satisfied by Remark 1.2.46. □

The existence of Weyl elements implies that many pairs of root groups are isomorphic. However, it does not provide canonical isomorphisms between these root groups. The purpose of the parametrisation theorem, which we will study in chapter 4, is precisely to rectify this problem.

2.5.12 Lemma. *Let G be a group with a Φ-grading $(U_\alpha)_{\alpha\in\Phi}$ and let α, β be roots which lies in the some orbit under the Weyl group. Then U_α is isomorphic to U_β.*

Proof. Choose an element u of the Weyl group such that $\alpha^u = \beta$. By the definition of the Weyl group, there exist roots $\gamma_1, \dots, \gamma_k$ such that $u = \sigma(\gamma_1) \cdots \sigma(\gamma_k)$. For each $i \in [1, k]$, we choose a γ_i-Weyl element w_i, and we put $w := w_1 \cdots w_k$. Then the map $x \mapsto x^w$ is an isomorphism from U_α to U_β. □

We end this section with a discussion of Axiom 2.5.2 (iv).

2.5.13 Note (Possible choices of the non-degeneracy condition). Let G be a group which has Φ-commutator relations with root groups $(U_\alpha)_{\alpha \in \Phi}$. We consider the following conditions on G:

(i) For all positive systems Π in Φ, we have $U_\Pi \cap U_{-\Pi} = \{1_G\}$.

(ii) Axiom 2.5.2 (iv): For all positive systems Π and all $\alpha \in \Phi \setminus \Pi$, we have $U_\Pi \cap U_\alpha = \{1_G\}$.

(iii) For any positive system Π, every ordering of Π is G-bijective.

(iv) For any positive system Π, there exists a G-bijective ordering of Π.

(v) G is rank-2-injective.

(vi) For all non-proportional roots α, β, we have $U_\alpha \cap U_\beta = \{1_G\}$.

Then we have implications "(i) $\implies$ (ii) $\implies$ (iv)" and "(iii) $\implies$ (v) $\implies$ (vi)" where the non-trivial implications hold by Proposition 2.4.18 and Lemma 2.1.20. Further, if Φ is crystallographic and G has crystallographic Φ-commutator relations, then we also have "(iii) $\iff$ (iv)" by Proposition 2.4.20 (b). Thus under crystallographic assumptions, we have a clean chain

$$\text{(i)} \implies \text{(ii)} \implies \text{(iii)} \iff \text{(iv)} \implies \text{(v)} \implies \text{(vi)}.$$

Most of our proofs work under the relatively weak assumption (v) on rank-2-injectivity. However, the blueprint technique itself requires the stronger condition (ii), which is why we have chosen it as an axiom for root graded groups. It is interesting to note that the coordinatisation of simply-laced root graded groups in chapter 5 (which does not rely on the blueprint technique) even works under the weaker condition (vi). This suggests that it might be possible to prove the coordinatisation results on non-simply-laced root systems under slightly weaker assumptions than (ii) if one found a way to work without the blueprint technique. However, even Zhang's computations on C_n-graded groups in [Zha14] require condition (iii), so it seems that this is the minimal condition which has to be used in any case.

As a final remark, it is worth mentioning that all specific examples of root graded groups that we construct in this thesis (including all Chevalley groups) even satisfy the strongest condition (i). In these examples, U_Π can be regarded as a group of upper triangular (generalised) matrices while $U_{-\Pi}$ consists of lower triangular matrices.

2.6 Foldings of Root Graded Groups

2.6.1 Notation for this section. We use the same notation as in 1.4.1 for Φ, V, Δ, ρ, τ, $F := \{ v \in V \mid \tau(v) = v \}$, $\pi \colon V \to F$ and $\Phi' := \pi(\Phi)$. In addition, we denote by G a group with a Φ-pregrading $(U_\alpha)_{\alpha \in \Phi}$.

In this section, we describe how the folding $\pi \colon \Phi \to \Phi'$ gives rise to a construction of Φ'-graded groups from Φ-graded groups. This construction is closely related to the construction of the twisted groups from Chevalley groups (see [Car72, 13.4]), but more general. We will use it in section 10.4 to construct examples of F_4-graded groups as foldings of E_6-graded groups.

We use a similar method in [BW] to construct H_3-graded groups as foldings of D_6-graded groups. However, this requires a more general approach because H_3 cannot be realised as a folding of D_6 which is induced by a diagram automorphism, but only by a folding in the more general sense of [Mü92].

By construction, roots in Φ' can be identified with their preimages under π in Φ, that is, with sets of roots in Φ. Thus the following definition of a Φ'-pregrading $(\hat{U}_{\alpha'})_{\alpha' \in \Phi'}$ is not surprising.

2.6.2 Definition (Folded pregrading). For all $\alpha' \in \Phi'$, we put

$$\hat{U}_{\alpha'} := \langle U_\alpha \mid \alpha \in \Phi, \pi(\alpha) = \alpha' \rangle.$$

The Φ'-pregrading $(\hat{U}_{\alpha'})_{\alpha' \in \Phi'}$ is called a *folding of* $(U_\alpha)_{\alpha \in \Phi}$.

The goal of this section is to find conditions on $(G, (U_\alpha)_{\alpha \in \Phi})$ which guarantee that $(G, (\hat{U}_{\alpha'})_{\alpha' \in \Phi})$ is a Φ'-graded group. (Note that group G remains the same in both gradings, only the family of root subgroups changes.) Clearly, we should require that $(G, (U_\alpha)_{\alpha \in \Phi})$ itself is a Φ-graded group, but it turns out that we need a bit more.

2.6.3 Lemma (Commutator relations). *Assume that Φ' is reduced and that for all roots ζ for which 2ζ is not a root, the root group U_ζ is abelian. If G has (crystallographic) Φ-commutator relations with root groups $(U_\alpha)_{\alpha \in \Phi}$, then it also has (crystallographic) Φ'-commutator relations with root groups $(\hat{U}_{\alpha'})_{\alpha' \in \Phi'}$.*

Proof. This follows from Proposition 2.1.29, whose assumptions are satisfied by Proposition 1.4.12 (c). □

2.6.4 Lemma (Non-degeneracy). *If $U_\Pi \cap U_{-\Pi} = \{1_G\}$ for all positive systems Π in Φ, then also $\hat{U}_{\Pi'} \cap \hat{U}_{-\Pi'} = \{1_G\}$ for all positive systems Π' in Φ'.*

Proof. Let Π' be a positive system in Φ'. By Proposition 1.4.12 (b), there exists a positive system Π in Φ such that $\pi(\Pi) = \Pi'$. Then for all $\beta \in \Phi$, we have $\beta \in \Pi$ if and only if $\pi(\beta) \in \Pi'$ because Φ is the disjoint union of Π and $-\Pi$. It follows that

$$\hat{U}_{\Pi'} \cap \hat{U}_{-\Pi'} \subseteq U_\Pi \cap U_{-\Pi} = \{1_G\},$$

as desired. □

2.6.5 Note. We cannot show that the validity of the more general Axiom 2.5.2 (iv) for $(U_\alpha)_{\alpha \in \Phi}$ (that is, $U_\Pi \cap U_\alpha = \{1_G\}$ for all $\alpha \in \Phi \setminus \Pi$) implies the validity of Axiom 2.5.2 (iv) for $(\hat{U}_{\alpha'})_{\alpha' \in \Phi'}$. The reason for this is that root groups in $(\hat{U}_{\alpha'})_{\alpha' \in \Phi'}$ are not necessarily contained in root groups of $(U_\alpha)_{\alpha \in \Phi}$.

2.6.6 Lemma (Weyl elements). *Let J be an orbit of Δ under ρ and let u_α be an α-Weyl element (with respect to $(U_\gamma)_{\gamma \in \Phi}$) for all $\alpha \in J$. Choose $k \in \mathbb{N}_+$ and $\alpha_1, \dots, \alpha_k \in J$ such that $\bar{\alpha} = (\alpha_1, \dots, \alpha_k)$ is an expression of the longest element $w_0 := w_0^J$ of the subgroup $W_J := \langle \sigma_\alpha \mid \alpha \in J \rangle$ of* $\mathrm{Weyl}(\Phi)$*. Put $u := \prod_{i=1}^k u_{\alpha_i}$ and denote by α' the unique element in $\pi(J)$. Then $\hat{U}_{\beta'}^u = \hat{U}_{(\beta')^{\sigma(\alpha')}}$ for all $\beta' \in \Phi'$. If, in addition, the roots in J are pairwise orthogonal, then u is an α'-Weyl element (with respect to $(\hat{U}_{\gamma'})_{\gamma' \in \Phi'}$).*

Proof. Let $\beta \in \Phi$ and put $\beta' := \pi(\beta)$. As a first step, we prove that $U_\beta^u \subseteq \hat{U}_{(\beta')^{\sigma(\alpha')}}$. Since $(u_\gamma)_{\gamma \in J}$ are Weyl elements, U_β^u lies in the root group associated to

$$\beta^{\sigma(\alpha_1)\cdots\sigma(\alpha_k)} = \beta^{w_0} \in \Phi.$$

Now it follows from Lemma 1.4.10 that

$$\pi\big(\beta^{w_0}\big) = \pi(\beta)^{\sigma(\alpha')} = (\beta')^{\sigma(\alpha')}.$$

We conclude that U_β^u is contained in $\hat{U}_{(\beta')^{\sigma(\alpha')}}$.

It follows from the conclusion of the previous paragraph that $\hat{U}_{\beta'}^u \subseteq \hat{U}_{(\beta')^{\sigma(\alpha')}}$ for all $\beta' \in \Phi'$. Note that $(\alpha_k, \dots, \alpha_1)$ is an expression of $(w_0)^{-1} = w_0$, and that u_α^{-1} is also an α-Weyl element for all $\alpha \in J$. Hence we can also apply the conclusion of the previous paragraph to $\bar{u} := u_{\alpha_k}^{-1} \cdots u_{\alpha_1}^{-1} = u^{-1}$, which yields that $\hat{U}_{\beta'}^{\bar{u}} \subseteq \hat{U}_{(\beta')^{\sigma(\alpha')}}$ for all $\beta' \in \Phi'$. Replacing β' by $(\beta')^{\sigma(\alpha')}$ and using that $\bar{u} = u^{-1}$, we infer that $\hat{U}_{(\beta')^{\sigma(\alpha')}} \subseteq \hat{U}_{\beta'}^u$ for all $\beta' \in \Phi'$. We conclude that $\hat{U}_{\beta'}^u = \hat{U}_{(\beta')^{\sigma(\alpha')}}$ for all $\beta' \in \Phi'$, which proves the first part of the claim.

To prove that u is an α'-Weyl element, it remains to show that it is contained in $\hat{U}_{-\alpha'}\hat{U}_{\alpha'}\hat{U}_{-\alpha'}$. By construction, it is contained in $\prod_{i=1}^k U_{-\alpha_i}U_{\alpha_i}U_{-\alpha_i}$. The assumption on J implies that it is a root base of type A_1^k. In particular, $\alpha_1, \dots, \alpha_k$ must be pairwise distinct and $J = \{ \alpha_1, \dots, \alpha_k \}$. We conclude that for all distinct $i, j \in [1, k]$, the groups U_{α_i} and $\langle U_{\alpha_j}, U_{-\alpha_j} \rangle$ commute. Hence the factors of u can be reordered in a way which yields

$$u \in \prod_{i=1}^k U_{-\alpha_i} \prod_{i=1}^k U_{\alpha_i} \prod_{i=1}^k U_{-\alpha_i} \subseteq \hat{U}_{-\alpha'}\hat{U}_{\alpha'}\hat{U}_{-\alpha'}.$$

The assertion follows. □

We can summarise the previous results as follows.

2.6.7 Proposition. *Let $(G, (U_\alpha)_{\alpha \in \Phi})$ be a (crystallographic) Φ-graded group with $U_\Pi \cap U_{-\Pi} = \{1_G\}$ for all positive systems Π in Φ. Assume that Φ' is reduced and that for all roots ζ for which 2ζ is not a root, the root group U_ζ is abelian. Assume further that for each orbit J of Δ under ρ, all roots in J are pairwise orthogonal. Then $(G, (\hat{U}_{\alpha'})_{\alpha' \in \Phi'})$ is a (crystallographic) Φ'-graded group with $\hat{U}_{\Pi'} \cap \hat{U}_{-\Pi'} = \{1_G\}$ for all positive systems Π' in Φ'.*

Proof. This is the culmination of Lemmas 2.6.3, 2.6.4 and 2.6.6. We also use Lemma 2.2.7 to ensure that Weyl elements exist for all roots and not merely for the simple roots. □

Recall from Note 2.1.6 that the assumption on certain root groups to be abelian is automatically satisfied if Φ is irreducible of rank at least 3.

2.7 Root Graded Groups in the Literature

In this section, we briefly review several different notions of root gradings which have been discussed in the literature. All of them are special cases of our definition. We have already given an overview of this topic in the preface of this thesis, so we restrict ourselves to filling in the technical details that are missing. Some references to later parts of this thesis will be necessary at certain points.

We begin with Tits' notion of RGD-systems.

2.7.1 Definition (RGD-system, [Tit92, p. 258]). Let Φ be a root system. An *RGD-system of type* Φ is a Φ-graded group $(G, (U_\alpha)_{\alpha \in \Phi})$ such that $U_\alpha^\sharp = U_\alpha \setminus \{1_G\}$ for all roots α.

As discussed in the preface, predecessors of the RGD-axioms can be found as early as in [Tit62, p. 140], and essentially the same definition is given in [Fau77].

2.7.2 Note. Definition 2.7.1 does not agree precisely with the one given in [Tit92, p. 258]. First of all, the root system Φ is not assumed to be finite in [Tit92], which makes the formulation of the commutator relation axiom slightly more technical. Secondly, an RGD-system in the sense of Tits is not assumed to be generated by $(U_\alpha)_{\alpha \in \Phi}$, but by $(U_\alpha) \cup H$ where H denotes the intersection of the normalisers of all root groups. Since our interest lies in coordinatising the root groups of a root graded group, we are free to replace an RGD-system in Tits' sense by the subgroup generated by all root groups. Thirdly, Axiom 2.5.2 (iv) is replaced by the slightly weaker assumption that for any root base Δ and for all $\delta \in \Delta$, the root group $U_{-\delta}$ is not contained in U_Π where Π denotes the positive system corresponding to Δ. Fourthly, the definition in [Tit92] only requires that $U_\alpha \setminus \{1_G\}$ is a subset of $U_\alpha^\sharp$, but by Lemma 2.2.5, this is equivalent to our requirement that $U_\alpha^\sharp = U_\alpha \setminus \{1_G\}$ (except possibly for root systems with irreducible components of type A_1).

2.7.3 Note (Invertible elements in root gradings). Let G be a root graded group which is coordinatised by some algebraic structure $\mathcal{A}$. We will show in Propositions 5.6.6, 7.7.6 and 9.6.5 and Remark 10.4.17 that for any root α, the elements of $U_\alpha^\sharp$ are in bijective correspondence with the elements of $\mathcal{A}$ which are invertible (in a suitable algebraic sense). Thus RGD-systems are precisely the root graded groups which are coordinatised by "division structures".

We now turn to notions of root gradings which do not have "division assumptions".

2.7.4 Note (Faulkner's A_2-graded groups). In [Fau14, 13.3], Faulkner defines *groups of Steinberg type*. These are the same as A_2-graded groups, except for a few minor differences. Firstly, the commutator axiom (RGG-Com) in Definition 2.1.4 is required to hold for $\alpha = \beta$ as well, which says precisely that all root groups are abelian. (That is, Faulkner uses the same definition of commutator relations as Loos-Neher, see Note 2.1.8.) We will show in Proposition 5.4.9 that this axiom is unnecessary. Secondly, α-Weyl elements in our sense are $(-\alpha)$-Weyl elements in Faulkner's sense, but this is practically irrelevant by Proposition 2.2.6 (c). Thirdly and finally, Axiom 2.5.2 (iv) is not part of the definition of a group of Steinberg type. Instead, the weaker axiom that $U_\alpha \cap U_\beta = \{1_G\}$ for all distinct roots α, β is an additional assumption in Faulkner's coordinatisation theorem. We will see that our proof of the coordinatisation theorem for A_2-gradings (Theorem 5.7.14) works under this more general assumption as well. We conclude that Faulkner's definition is essentially the same as our definition of A_2-graded groups.

We now turn to Shi's definition of root graded groups. It is formulated in [Shi93] only for the simply-laced root systems, but the generalisation to arbitrary (finite) reduced root systems is immediate. Essentially the same definition is used in [Zha14, 3.1.5]. Before we can phrase it, we have to introduce Steinberg groups. Their definition relies on some structure constants in Chevalley groups which will be properly introduced in chapter 3.

2.7.5 Definition (Steinberg group). Let $\Bbbk$ be a commutative associative ring, let Φ be a reduced crystallographic root system and let $c = (c_{\alpha,\beta})_{\alpha,\beta\in\Phi}$ be a family of Chevalley structure constants of type Φ (in the sense of Definition 3.1.18). The *Steinberg group of type Φ over $\Bbbk$ with respect to c* is the abstract group defined by the following presentation: The generators are symbols $\hat{x}_\alpha(r)$ for $\alpha \in \Phi$ and $r \in \Bbbk$ and the following relations hold.

(i) $\hat{x}_\alpha(r+s) = \hat{x}_\alpha(r)\hat{x}_\alpha(s)$ for all $\alpha \in \Phi$ and all $r, s \in \Bbbk$.

(ii) Let α, β be non-proportional roots and define

$$\mathcal{I}(\alpha,\beta) := \{\, (i,j) \in \mathbb{N}_+^2 \mid i\alpha + j\beta \in \Phi \,\}$$

as in Notation 3.2.15. Then for all $r, s \in \Bbbk$, we have a relation

$$[\hat{x}_\alpha(r), \hat{x}_\beta(s)] = \prod_{(i,j)\in\mathcal{I}} \hat{x}_{i\alpha+j\beta}\big(c_{\alpha,\beta,i,j} r^i s^j\big)$$

which has exactly the same form a the Chevalley commutator formula (Theorem 3.2.16). Here the integers $c_{\alpha,\beta,i,j}$ can be computed from c as in Remark 3.2.21.

We denote this group by $\mathrm{St}_{\Phi,c}(\Bbbk)$ or simply by $\mathrm{St}_\Phi(\Bbbk)$. For each root α, the subgroup $\hat{U}_\alpha := \{\, \hat{x}_\alpha(r) \mid r \in \Bbbk \,\}$ is called the *root group of* $\mathrm{St}_{\Phi,c}(\Bbbk)$ *corresponding to α*. For any root α and any invertible element r in $\Bbbk$, we define

$$\hat{w}_\alpha(r) := \hat{x}_{-\alpha}(-r^{-1})\hat{x}_\alpha(r)\hat{x}_{-\alpha}(-r^{-1}).$$

2.7.6 Definition (Shi's notion of root gradings, [Shi93, (2.1)]). Let Φ be a reduced crystallographic root system and let G be a group. A *Φ-grading of G in Shi's sense* is a Φ-pregrading $(U_\alpha)_{\alpha\in\Phi}$ for which there exist a family $c = (c_{\alpha,\beta})_{\alpha,\beta\in\Phi}$ of Chevalley structure constants, a commutative associative non-zero ring $\Bbbk$ and a homomorphism $\varphi\colon \mathrm{St}_\Phi(\Bbbk) \to G$ with the following properties:

(i) There exists a positive system Π in Φ such that the restriction of φ to $\langle \hat{U}_\alpha \mid \alpha \in \Pi \rangle$ is injective.

(ii) $\varphi(\hat{U}_\alpha)$ is contained in U_α for all roots α.

(iii) G is generated by $(U_\alpha)_{\alpha\in\Delta}$.

(iv) G has crystallographic Φ-commutator relations, and all root groups are abelian.

(v) For all distinct roots α, β, we have $U_\alpha \cap U_\beta = \{1_G\}$.

(vi) For all roots α, the element $\varphi(\hat{w}_\alpha(1_\Bbbk))$ is an α-Weyl element in G.

Observe that all root groups in Definition 2.7.6 are not equal to $\{1\}$ because $\Bbbk$ is non-zero.

2.7.7 Note. Let Φ be a reduced crystallographic root system, let α be an arbitrary root and put $w_\alpha := \varphi(\hat{w}_\alpha(1_\mathbb{Z}))$. In [Shi93], it is not assumed that w_α is an α-Weyl element, but only that for all roots β such that (α, β) is a root base of the subsystem that it spans, we have $U_\beta^{w_\alpha} = U_{\sigma_\alpha(\beta)}$. It is then shown in [Shi93, (2.4) (ii)] that the same equation holds for all roots β, which implies that w_α is indeed an α-Weyl element. Thus Definition 2.7.6 is equivalent to Shi's definition in [Shi93, (2.1)] for simply-laced root systems. However, it is not at all clear that [Shi93, (2.1)] also holds for non-simply-laced root systems. Hence Definition 2.7.6 seems to be the correct way to formulate Shi's definition for arbitrary (reduced crystallographic) root systems.

2.7.8 Remark (Comparison). Let Φ be a reduced crystallographic root system. It is immediate that every Φ-graded group in Shi's sense is a crystallographic

Φ-graded group in our sense. Conversely, assume that Φ is simply-laced of rank at least 2 and that $(G, (U_\alpha)_{\alpha\in\Phi})$ is a Φ-graded group in our sense. Our coordinatisation result for such groups (Theorem 5.7.14) says precisely that there exists a ring $\mathcal{R}$ such that G satisfies the same relations as in Definition 2.7.5. In other words, G is a quotient of "$\mathrm{St}_{\Phi,c}(\mathcal{R})$" for an appropriate c, though $\mathcal{R}$ need not be commutative associative. Denote by $\mathcal{k}$ the commutative associative subring of $\mathcal{R}$ which is generated by $1_{\mathcal{R}}$, so that $\mathcal{k}$ is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}_{\geq 2}$. Then there exists a homomorphism $\varphi\colon \mathrm{St}_{\Phi,c}(\mathcal{k}) \to G$ with the properties in Definition 2.7.6. Here the injectivity of φ on $\langle \hat{U}_\alpha \mid \alpha \in \Pi\rangle$ follows from the fact that the commutator relations on $\langle \hat{U}_\alpha \mid \alpha \in \Pi\rangle$ (and thus on $\langle U_\alpha \mid \alpha \in \Pi\rangle$) already determine the group multiplication on this subset. This can be proven as in [TW02, (8.13)], and it is also a consequence of the arguments in section 2.4.

We conclude that Shi's definition of root gradings is in fact equivalent to ours for simply-laced root systems of rank at least 2. For non-simply-laced root systems, the situation is more delicate because the involved coordinatising structures are no longer mere rings. For example, assume that $(G, (U_\alpha)_{\alpha\in B_3})$ is a crystallographic B_3-graded group. Theorem 7.11.21 yields a pointable quadratic module (M, q) over a commutative associative ring $\mathcal{k}$ which coordinatises G via root isomorphisms $(\theta_\alpha)_{\alpha\in B_3}$. Choose $v_0 \in M$ with $q(v_0) = 1_{\mathcal{k}}$. Then we have a homomorphism $\varphi\colon \mathrm{St}_{\Phi,c}(\mathcal{k}) \to G$ (for an appropriate family c) whose image is

$$G' := \langle \{\, \theta_\alpha(r) \mid \alpha \text{ long}, r \in \mathcal{k} \,\} \cup \{\, \theta_\beta(rv_0) \mid \beta \text{ short}, r \in \mathcal{k} \,\}\rangle.$$

However, it is not clear that the restriction of φ to $\langle \hat{U}_\alpha \mid \alpha \in \Pi\rangle$ is injective for some positive system Π. For example, if $\mathcal{k} = \mathbb{Z}/n\mathbb{Z}$, it is not obvious that $mv_0 \neq 0$ for every proper divisor m of n. This illustrates that Definition 2.7.6 is not adequate to capture the more general phenomena which may occur in non-simply-laced root graded groups.

While the involvement of the Steinberg group makes Shi's definition more complicated, it has practical benefits: It simplifies the (sign problem in the) parametrisation of these groups. See Remark 4.1.29 for a few more details.

2.7.9 Coordinatisation results in the literature. Let Φ be a root system. Recall from the preface that RGD-systems of type Φ are essentially the same thing as Moufang buildings of type Φ (see [AB08, Section 7.8] for details). Thus Tits' classification of irreducible Moufang buildings for root systems of rank at least 3 in [Tit74] yields coordinatisation results for irreducible RGD-systems of rank at least 3. It should be noted, however, that the language of [Tit74] is rather different from the one in this thesis. In contrast, the classification of irreducible Moufang buildings of rank 2 (that is, of Moufang polygons) in [TW02] uses a group-theoretic language which is very similar to ours and which provides explicit commutator formulas. In [TW02, (40.22)], the classification of irreducible Moufang buildings of rank at least 3 is deduced from the classification of Moufang polygons, and explicit commutator relations are provided for some (but not all) pairs of roots. By Tits' famous Theorem 4.1.2 (see [Tit74, 4.1.2]), the commutator relations on all pairs of (non-proportional) roots are uniquely determined by the commutator relations in [TW02, (40.22)].

We now turn to Φ-gradings which are not assumed to be RGD-systems. Our coordinatisation result for simply-laced root gradings of rank at least 2 (Theorem 5.7.14) is given in essentially the same way in [Shi93, (2.3)] for root gradings in Shi's sense. For root gradings of types B, BC and F_4, there exist no prior results.

For C_n-graded groups in Shi's sense with $n \geq 3$, a weaker version of our coordinatisation result (Theorem 9.10.26) can be found in 3.4.19 of Zhang's PhD thesis [Zha14]. The result in [Zha14] is obtained under the additional assumption that the root groups are 2-divisible, and it remains a conjecture that the coordinatising ring is alternative. Further, due to a problem in the proof of [Zha14, 3.4.8 (4)], Zhang overlooks the examples of C_n-graded groups which are coordinatised not only by a ring $\mathcal{R}$ with involution σ, but by a tuple $(\mathcal{R}, M)$ where M is an additional module over $\mathcal{R}$. See section 9.3 and, in particular, Remark 9.3.29 for more details on these additional examples. We can solve all these problems in the setting of BC_n-gradings (which properly generalise C_n-gradings) using the blueprint technique.

Chapter 3

Chevalley Groups

The main examples of root graded groups are Chevalley groups, which were introduced by Chevalley in his famous paper [Che55] in 1955. By construction, every Chevalley group is coordinatised by a commutative associative ring $\mathcal{R}$ in the sense that its root groups are isomorphic to the additive group $(\mathcal{R}, +)$. Further, the celebrated Chevalley commutator formula states that commutators of root group elements can be described in terms of the multiplication on $\mathcal{R}$. In this chapter, we outline the construction of Chevalley groups and their basic properties. This will motivate the definition of coordinatisations of root graded groups. Some subtleties, like the sign problem, are already visible in this context. However, this chapter is merely a survey: We will give no proofs and state only those definitions which are either simple or absolutely necessary for this introduction. None of the material is new, but the viewpoint of root graded groups lets us see some results in a different light.

Standard references on this topic are [Hum72] for the basic theory of Lie algebras and [Ste67; Car72] for the material on Chevalley groups. It should be noted that [Hum72, Chapter VII] also contains the construction of Chevalley groups (but not many properties of these groups beyond that) and that [Car72] only covers adjoint Chevalley groups, not general Chevalley groups. Further, all three references have the disadvantage that they only introduce Chevalley groups over fields and not over commutative associative rings. However, all the basic results in this chapter remain valid over commutative associative rings, and only minor modifications of the proofs are necessary. A survey of Chevalley groups in this general setting is [VP96]. It contains all the results which we introduce in this chapter, albeit usually without proofs.

3.1 Semisimple Lie Algebras

Before we can turn to Chevalley groups, we have to introduce semisimple Lie algebras. By "integrating" these algebras, we will obtain Chevalley groups. The standard reference for the material in this section is [Hum72].

3.1.1 Definition (Lie algebra, [Hum72, 1.1]). Let $\Bbbk$ be a commutative associative ring. A *Lie algebra over* $\Bbbk$ is a $\Bbbk$-module L together with a $\Bbbk$-bilinear operation $[\cdot, \cdot] \colon L \times L \to L, (x, y) \mapsto [x, y]$ which satisfies the following properties:

(i) *Antisymmetry*: $[x, x] = 0$ for all $x \in L$.

(ii) *Jacobi identity*: $\big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] = 0$ for all $x, y, z \in L$.

3.1.2 Example. For any associative $\Bbbk$-algebra A, the $\Bbbk$-module A together with the commutator bracket $[\,\cdot\,,\,\cdot\,]\colon A \times A \to A$, $(x, y) \mapsto xy - yx$ is a Lie algebra.

3.1.3 Definition (Ideals and subalgebras, [Hum72, 2.1]). Let $\Bbbk$ be a field, let L be a Lie algebra over $\Bbbk$ and let U be a $\Bbbk$-subspace of L. Then U is called an *ideal of* L if $[x, y] \in U$ for all $x \in L$ and $y \in U$, and it is called a *Lie subalgebra of* L if $[x, y] \in U$ for all $x, y \in U$.

3.1.4 Remark. A Lie algebra is called *semisimple* if it contains no non-trivial solvable ideal ([Hum72, 3.1]). Further, we define a *Cartan subalgebra of* L to be a maximal torus. For any Cartan subalgebra H, we denote by H^* the vector space of $\Bbbk$-linear maps from H to $\Bbbk$.

3.1.5 Notation for this section. From now on, we denote by $\Bbbk$ an algebraically closed field of characteristic 0, by L a semisimple Lie algebra over $\Bbbk$ and by H a Cartan subalgebra of L.

3.1.6 Definition (Root system of a Lie algebra). For any $\Bbbk$-linear map $\alpha\colon H \to \Bbbk$, we call

$$L_\alpha := \{\, x \in L \mid [h, x] = \alpha(h)x \text{ for all } h \in H \,\}$$

the *root subspace of* L *corresponding to* α. We call α a *root of* L *(with respect to* H*)* if $L_\alpha \neq \{0\}$ and $\alpha \neq 0$. We denote the set of all roots of L with respect to H by $\Phi := \Phi(L, H)$ and call it the *root system of* L *(with respect to* H*)*.

3.1.7 Proposition ([Hum72, 8.2, 8.5]). *The Killing form* $\kappa\colon H \times H \to H$ *is non-degenerate, and it induces an inner product* $\kappa_{\mathbb{R}}$ *on the real vector space* $H^* \otimes_{\mathbb{Q}} \mathbb{R}$.

3.1.8 Definition. For each $\alpha \in H^*$, we denote by t_α the unique element of H which satisfies $\kappa(t_\alpha, h) = \alpha(h)$ for all $h \in H$, and we further put $h_\alpha := \frac{2}{\kappa(t_\alpha, t_\alpha)} t_\alpha$.

3.1.9 Remark. For any $\alpha \in H^*$, we have $\alpha(h_\alpha) = 2\frac{\alpha(t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = 2\frac{\kappa(t_\alpha, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = 2$.

3.1.10 Proposition ([Hum72, Theorem 8.5]). *The set* Φ *is a reduced crystallographic root system in the Euclidean space* $(H^* \otimes_{\mathbb{Q}} \mathbb{R}, \kappa_{\mathbb{R}})$.

3.1.11 Theorem (Root space decomposition, [Hum72, 8.1, 8.2, 8.4]). *As a* $\Bbbk$*-vector space, we have the following* root space decomposition of L*:*

$$L = H \oplus \bigoplus_{\alpha \in \Phi(L,H)} L_\alpha.$$

Further, $\dim L_\alpha = 1$ *for all roots* α *and* $[L_\alpha, L_\beta] = L_{\alpha+\beta}$ *for all roots* α, β.

This finishes our outline of the basic theory of semisimple Lie algebras. We now turn to Chevalley bases and their structure constants. The key idea here is that a Chevalley basis equips L with an integral structure, which will later allow us to base change to arbitrary commutative associative rings. This will be crucial for the definition of Chevalley groups over these rings.

3.1.12 Notation for this section. From now on, we denote by $\Phi := \Phi(L, H)$ the root system of L with respect to H and by ℓ the rank of this root system.

3.1.13 Definition (Chevalley basis, [Hum72, Proposition 25.2]). Choose an ordered root base $\Delta = (\alpha_1, \dots, \alpha_\ell)$ of Φ. A *Chevalley basis of* L *of type* Δ *(with respect to* H*)* is a family $\mathcal{C} = (x_\alpha)_{\alpha \in \Phi} \cup (h_i)_{i \in [1,\ell]}$ with the following properties:

(i) $h_i = h_{\alpha_i}$ for all $i \in [1, \ell]$.
(ii) $[x_\alpha, x_{-\alpha}] = h_\alpha$ for all $\alpha \in \Phi$.
(iii) Let α, β be roots such that $\alpha + \beta$ is a root. If the scalars $c_{\alpha,\beta}, c_{-\alpha,-\beta} \in \Bbbk$ are defined by $[x_\alpha, x_\beta] = c_{\alpha,\beta} x_{\alpha+\beta}$ and $[x_{-\alpha}, x_{-\beta}] = c_{-\alpha,-\beta} x_{-\alpha-\beta}$, then $c_{-\alpha,-\beta} = -c_{\alpha,\beta}$.

A *Chevalley basis of L (with respect to H)* is a Chevalley basis of some type.

3.1.14 Note. Let Δ, Δ' be two ordered root bases of Φ and let $\mathcal{C} = (x_\alpha)_{\alpha\in\Phi} \cup (h_i)_{i\in[1,\ell]}$ be a Chevalley basis of type Δ. Then $(x_\alpha)_{\alpha\in\Phi} \cup (h_\delta)_{\delta\in\Delta'}$ is a Chevalley basis of type Δ', and this is the unique Chevalley basis of type Δ' which extends $(x_\alpha)_{\alpha\in\Phi}$. This shows that the crucial objects in a Chevalley basis are the root space elements $(x_\alpha)_{\alpha\in\Phi}$ and not the torus elements $(h_\delta)_{\delta\in\Delta}$. In particular, the type of a Chevalley basis usually plays only a minor role, if at all.

3.1.15 Remark. Let $\mathcal{C} = (x_\alpha)_{\alpha\in\Phi} \cup (h_i)_{i\in[1,\ell]}$ be a Chevalley basis of L. Then $h_i = h_{\alpha_i} = [x_{\alpha_i}, x_{-\alpha_i}]$ for all $i \in [1, \ell]$ where α_i is the corresponding simple root. It follows that $(x_\alpha)_{\alpha\in\Phi}$ generates L as a Lie algebra.

3.1.16 Remark. Let $\mathcal{C} = (x_\alpha)_{\alpha\in\Phi} \cup (h_i)_{i\in[1,\ell]}$ and $\mathcal{C}' = (x'_\alpha)_{\alpha\in\Phi} \cup (h'_i)_{i\in[1,\ell]}$ be two Chevalley bases of L. Since each root space L_α is one-dimensional, there exists a family $(\lambda_\alpha)_{\alpha\in\Phi}$ of non-zero complex coefficients such that $x'_\alpha = \lambda_\alpha x_\alpha$ for all $\alpha \in \Phi$. Since

$$h_\alpha = [x'_\alpha, x'_{-\alpha}] = \lambda_\alpha \lambda_{-\alpha} [x_\alpha, x_{-\alpha}] = \lambda_\alpha \lambda_{-\alpha} h_\alpha,$$

we have $\lambda_{-\alpha} = \lambda_\alpha^{-1}$ for all $\alpha \in \Phi$.

3.1.17 Definition (Chevalley structure constants). Let $\mathcal{C} = (x_\alpha)_{\alpha\in\Phi} \cup (h_i)_{i\in[1,\ell]}$ be a Chevalley basis of L. For any pair of roots α, β, we define $c_{\alpha,\beta} \in \Bbbk$ by the equation $[x_\alpha, x_\beta] = c_{\alpha,\beta} x_{\alpha+\beta}$ if $\alpha + \beta$ is a root, and we define $c_{\alpha,\beta} := 0$ if $\alpha + \beta$ is not a root. The scalars $(c_{\alpha,\beta})_{\alpha,\beta\in\Phi}$ are called the *(Chevalley) structure constants of L (with respect to $\mathcal{C}$)*.

3.1.18 Definition. Let $c = (c_{\alpha,\beta})_{\alpha,\beta\in\Phi}$ be a family of integers. We say that c is a *family of Chevalley structure constants of type* Φ if there exist a semisimple complex Lie algebra L with root system Φ and a Chevalley basis $\mathcal{C}$ of L such that $(c_{\alpha,\beta})_{\alpha,\beta\in\Phi}$ are the Chevalley structure constants of L with respect to $\mathcal{C}$.

Observe that the structure constants of a Chevalley basis $\mathcal{C} = (x_\alpha)_{\alpha\in\Phi} \cup (h_i)_{i\in[1,\ell]}$ depend only on $(x_\alpha)_{\alpha\in\Phi}$ and not on $(h_i)_{i\in[1,\ell]}$.

3.1.19 Example (of a Chevalley basis). Consider the root system $\Phi = A_\ell$ for some $\ell \in \mathbb{N}_+$. Without loss of generality, we can use the standard representation of A_ℓ:

$$A_\ell = \{\, e_i - e_j \mid i \neq j \in [1, \ell+1] \,\}$$

where $(e_1, \ldots, e_{\ell+1})$ is an orthonormal basis of some Euclidean space. (See Remark 5.2.3 for more details.) The corresponding Lie algebra is

$$L := \mathfrak{sl}_{\ell+1}(\Bbbk) := \left\{ A \in M_{\ell+1}(\Bbbk) \;\middle|\; \sum_{i=1}^{\ell+1} A_{ii} = 0 \right\}.$$

For all $i, j \in [1, \ell+1]$, we denote by b_{ij} the matrix in $M_{\ell+1}(\Bbbk)$ with $1_\Bbbk$ at position (i, j) and zero at every other position. For any root $\alpha = e_i - e_j \in A_\ell$, we now put $x_\alpha := x_{ij} := b_{ij}$. Further, we put $h_i := b_{ii} - b_{i+1,i+1}$. Then $H := \langle h_i \mid i \in [1, \ell] \rangle_\Bbbk$, the space of diagonal matrices in L, is a Cartan subalgebra of L, and the spaces $L_\alpha := \langle x_\alpha \rangle$ for all $\alpha \in A_\ell$ are the root spaces of L. Using the formulas

$$b_{ij} \cdot b_{pq} = \begin{cases} b_{iq} & \text{if } j = p, \\ 0 & \text{otherwise,} \end{cases}$$

one can easily compute that $[x_{ij}, x_{jk}] = x_{ik}$ and $[x_{jk}, x_{ij}] = -x_{ik}$ for all pairwise distinct $i, j, k \in [1, \ell+1]$. It follows that $(x_\alpha)_{\alpha \in A_\ell} \cup (h_i)_{i \in [1,\ell]}$ is a Chevalley basis of L and that the numbers $c_{e_i - e_j, e_j - e_k} := 1$ and $c_{e_j - e_k, e_i - e_j} := -1$ for all pairwise distinct $i, j, k \in [1, \ell+1]$ and $c_{\alpha,\beta} := 0$ for all other pairs of roots $\alpha, \beta \in \Phi$ are the structure constants of this Chevalley basis.

3.1.20 Remark. In continuation of Example 3.1.19, we can easily determine all Chevalley bases of type A_1. The only root bases of A_1 are $\Delta = \{ e_1 - e_2 \}$ and $\Delta' = \{ e_2 - e_1 \}$. In Example 3.1.19, we have seen the Chevalley basis

$$(x_{12} = b_{12}, x_{21} = b_{21}, h_1 = b_{11} - b_{22})$$

of type Δ. By Remark 3.1.16, any other Chevalley basis of type Δ must be of the form

$$(x_{12} = \lambda b_{12}, x_{21} = \lambda^{-1} b_{21}, h_1 = b_{11} - b_{22})$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$, and it is easy to see that this is indeed a Chevalley basis for all $\lambda \in \mathbb{C} \setminus \{0\}$. Similarly, the Chevalley bases of type Δ' are exactly the tuples of the form

$$(x_{12} = \lambda b_{12}, x_{21} = \lambda^{-1} b_{21}, h_1 = b_{22} - b_{11})$$

where $\lambda \in \mathbb{C} \setminus \{0\}$.

We now cite some essential properties of Chevalley bases.

3.1.21 Lemma ([Hum72, Proposition 25.2]). *There exists a Chevalley basis of L.*

3.1.22 Lemma ([Hum72, Proposition 25.2]). *Let $\mathcal{C}$ be a Chevalley basis of L. Then $\mathcal{C}$ is a basis of the $\Bbbk$-vector space L, and the $\mathbb{Z}$-span of $\mathcal{C}$ is a Lie subalgebra of L over $\mathbb{Z}$.*

3.1.23 Lemma. *Let $\mathcal{C}$ be a Chevalley basis of L. Then the Chevalley structure constants of L with respect to $\mathcal{C}$ have the following properties:*

(a) For all roots α, β, we have $c_{\alpha,\beta} = -c_{\beta,\alpha}$ and $c_{-\alpha,-\beta} = -c_{\alpha,\beta}$.

(b) If α, β are roots such that $\alpha + \beta$ is a root, then $c_{\alpha,\beta} \in \{ \pm(r+1) \}$ where r is the maximal non-negative integer such that $\beta - r\alpha$ is a root.

(c) All structure constants of L with respect to $\mathcal{C}$ are integers, and their absolute values are uniquely determined by the root system Φ.

(d) If $\mathcal{C}'$ is another Chevalley basis of L with structure constants $(c'_{\alpha,\beta})_{\alpha,\beta \in \Phi}$, then $c'_{\alpha,\beta} \in \{ \pm c_{\alpha,\beta} \}$ for all roots α, β.

(e) If $\mathcal{C}'$ is another Chevalley basis of L with structure constants $(c'_{\alpha,\beta})_{\alpha,\beta \in \Phi}$ such that $c_{\alpha,\beta} = c'_{\alpha,\beta}$ for all roots α, β lying in some positive system Π in Φ, then $c_{\alpha,\beta} = c'_{\alpha,\beta}$ for all roots α, β.

Proof. The property $c_{\alpha,\beta} = -c_{\beta,\alpha}$ follows from the anti-commutativity of the Lie bracket while the second part of (a) is part of the definition of a Chevalley basis. A proof of (b) can be found in [Hum72, Theorem 25.2]. Parts (c) and (d) are consequences of (b). Property (e) holds by [VP96, (14.8)]. □

3.1.24 Remark. It follows from Lemma 3.1.23 (b) that in a simply-laced root system, we have $c_{\alpha,\beta} \in \{\pm 1\}$ for all roots α, β for which $\alpha + \beta$ is a root.

3.1.25 Note (The sign problem for Chevalley bases). We emphasise the statement of Lemma 3.1.23 (d): The structure constants of a Chevalley basis are only unique up to a sign. However, not every choice of signs corresponds to a Chevalley basis. This raises the following natural question: How much freedom do we have in the choice of signs, and can we describe an algorithm which produces an (or any) explicit choice of signs? This problem is well-known in the theory of Chevalley groups, and we will refer to it as the *sign problem (for Chevalley groups)*. In [Car72, Section 4.2], it is shown that the signs can be chosen arbitrarily for so-called extraspecial pairs of roots, and then all other signs are determined by these arbitrary choices. More information can also be found in [VP96, Sections 14, 15].

A direct consequence of the sign problem for Chevalley bases is that the signs in the Chevalley commutator formula are not unique either. Similarly, the signs which appear in certain conjugation formulas for Weyl elements depend on the signs of the Chevalley basis. See Notes 3.2.23 and 3.3.8 for more details. Further, a variation of the sign problem is the main difficulty in the parametrisation of root graded groups. We will elaborate on this in Remark 4.1.14.

The following fact on complex semi-simple Lie algebras will be needed in Lemma 3.3.13. Its proof relies on standard arguments, but it is hard to find an explicit reference in the literature.

3.1.26 Lemma. *Let Δ be a root base of Φ, let $\mathcal{C} = (x_\alpha)_{\alpha\in\Phi} \cup (h_i)_{i\in[1,\ell]}$ be a Chevalley basis of L of type Δ (with respect to H) and let $(c_{\alpha,\beta})_{\alpha,\beta\in\Phi}$ be the corresponding structure constants. Let Δ' be a non-empty subset of Δ and denote by Φ' the root system which is spanned by Δ'. Put $H' := \langle h_\delta \mid \delta \in \Delta' \rangle$ and $L' := \langle (x_\alpha)_{\alpha\in\Phi'} \cup H' \rangle$. Then L' is a semisimple Lie algebra, H' is a Cartan subalgebra of L', the root system of L' with respect to H' can be identified with Φ' in a canonical way and $\mathcal{C}' := (x_\alpha)_{\alpha\in\Phi'} \cup (h_\delta)_{\delta\in\Delta'}$ is a Chevalley basis of L' of type Δ' with respect to H'. Further, the structure constants of $\mathcal{C}'$ are precisely $(c_{\alpha,\beta})_{\alpha,\beta\in\Phi'}$.*

3.2 Construction of the Chevalley Groups

3.2.1 Notation for this section. We denote by $\Bbbk$ an algebraically closed field of characteristic 0, by L a semisimple Lie algebra over $\Bbbk$ with a Cartan subalgebra H and (reduced crystallographic) root system $\Phi := \Phi(L, H)$ and by $\mathcal{C} = (x_\alpha)_{\alpha\in\Phi} \cup (h_i)_{i\in[1,\ell]}$ a Chevalley basis with structure constants $(c_{\alpha,\beta})_{\alpha,\beta\in\Phi}$.

In the previous section, we studied only the Lie algebra L. We now introduce representations of L. For any such representation δ, we will define Chevalley groups of type δ by "integrating δ".

3.2.2 Definition. For any $\Bbbk$-vector space V, we denote by $\mathfrak{gl}(V)$ the Lie algebra consisting of the vector space $\mathrm{End}_\Bbbk(V)$ with the commutator bracket.

3.2.3 Definition (Representations). A *representation of L* is a homomorphism $\delta\colon L \to \mathfrak{gl}(V)$ for some $\Bbbk$-vector space V. It is called *finite-dimensional* if V is finite-dimensional as a $\Bbbk$-vector space and it is called *faithful* if it is injective. In this context, the vector space V is also called an *L-module*. An L-module V is called called *irreducible* if $V \neq \{0\}$ and if there exists no non-trivial submodule of V.

3.2.4 Example. The homomorphism $\mathrm{ad}\colon L \to \mathfrak{gl}(L), y \mapsto \mathrm{ad}_y$ is a representation of L, called the *adjoint representation of L*.

3.2.5 Lemma. *Let $\delta\colon L \to \mathfrak{gl}(V)$ be a finite-dimensional representation of L. Then $\delta(x_\alpha)$ is nilpotent for all $\alpha \in \Phi$.*

Proof. This is a consequence of [Hum72, Lemma 20.1]. □

3.2.6 Strategy. Fix any representation $\delta\colon L \to \mathfrak{gl}(V)$ of L. For all $\alpha \in \Phi$, we can define a map

$$\theta_\alpha\colon (\Bbbk,+) \to \mathrm{Aut}_\Bbbk(V), \lambda \mapsto \exp\big(\lambda\delta(x_\alpha)\big) = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}\delta(x_\alpha)^i.$$

Here the sum in the exponential is actually finite by Lemma 3.2.5, so we need not worry about analytic considerations. It is not too difficult to show that the maps $(\theta_\alpha)_{\alpha\in\Phi}$ are injective homomorphisms (see, however, Remark 3.2.25). We put $U_\alpha := \theta_\alpha(\Bbbk)$ for all $\alpha \in \Phi$. The Chevalley group of type δ over $\Bbbk$ is the group G which is generated by $(U_\alpha)_{\alpha\in\Phi}$, and we will see that $(U_\alpha)_{\alpha\in\Phi}$ is a crystallographic Φ-grading of G.

The construction above was relatively easy because we worked over $\Bbbk$, and it is not at all clear how it should be extended to arbitrary commutative associative rings. The key idea will be to find a suitable $\mathbb{Z}$-structure of V, and these $\mathbb{Z}$-structures will be called admissible lattices.

3.2.7 Definition (Admissible lattices). Let $\delta\colon L \to \mathfrak{gl}(V)$ be a finite-dimensional representation of L. An additive subgroup M of V is called a *lattice* if there exists a $\Bbbk$-basis B of V such that M is the $\mathbb{Z}$-span of B. It is called an *admissible lattice (with respect to $\mathcal{C}$ and δ)* if, in addition, $\frac{\delta(x_\alpha)^n}{n!}(M) \subseteq M$ for all non-negative integers n and all $\alpha \in \Phi$.

It is a non-trivial fact that admissible lattices always exist. This is relatively easy to prove for the adjoint representation, but the general proof uses Kostant's Theorem (see [Hum72, Section 26]).

3.2.8 Proposition ([Hum72, Proposition 25.5]). *The $\mathbb{Z}$-span of $\mathcal{C}$ is an admissible lattice in L with respect to the adjoint representation* $\mathrm{ad}\colon L \to \mathfrak{gl}(L)$.

3.2.9 Proposition ([Hum72, Theorem 27.1]). *Let $\delta\colon L \to \mathfrak{gl}(V)$ be any finite-dimensional representation of L. Then there exists an admissible lattice in V with respect to δ.*

Finally, we have everything we need to construct Chevalley groups.

3.2.10 Remark (Construction of the root homomorphism). Let $\delta\colon L \to \mathfrak{gl}(V)$ be a finite-dimensional faithful representation of L and choose an admissible lattice M in V (which exists by Proposition 3.2.9). Let α be an arbitrary root and let $\mathcal{R}$ be an arbitrary commutative associative ring. For any non-negative integer n, the endomorphism $\frac{\delta(x_\alpha)^n}{n!}$ of V leaves M invariant because M is admissible, and thus $\frac{\delta(x_\alpha)^n}{n!}$ can be regarded as an endomorphism $\theta_{\alpha,n}$ of M. Then we also have an $\mathcal{R}$-endomorphism $\theta_{\alpha,n} \otimes \mathrm{id}_\mathcal{R}$ of $V(\mathcal{R}) := M \otimes_\mathbb{Z} \mathcal{R}$. Now we can define a map

$$\theta_\alpha\colon (\mathcal{R},+) \to \mathrm{End}_\mathcal{R}\big(V(\mathcal{R})\big), r \mapsto \sum_{m=0}^{\infty} r^m(\theta_{\alpha,m} \otimes \mathrm{id}_\mathcal{R})$$

where the sum is actually finite because $\delta(x_\alpha)$ is nilpotent (by Lemma 3.2.5). The map θ_α is actually a homomorphism and thus its image lies in $\mathrm{Aut}_\mathcal{R}(V(\mathcal{R}))$.

3.2.11 Definition (Chevalley groups). Let $\delta\colon L \to \mathfrak{gl}(V)$ be a finite-dimensional faithful representation of L, let M be an admissible lattice in V, let $\mathcal{R}$ be a commutative associative ring and put $V(\mathcal{R}) := M \otimes_{\mathbb{Z}} \mathcal{R}$. For all roots α, we define the corresponding *root homomorphism* to be the homomorphism

$$\theta_\alpha\colon (\mathcal{R}, +) \to \mathrm{Aut}_{\mathcal{R}}(V(\mathcal{R}))$$

from Remark 3.2.10 and we define the corresponding *root subgroup*, denoted by U_α, to be the image of θ_α. The *Chevalley group of type δ over $\mathcal{R}$ (with respect to M and $\mathcal{C}$)* is the subgroup of $\mathrm{Aut}_{\mathcal{R}}(V(\mathcal{R}))$ which is generated by $(U_\alpha)_{\alpha\in\Phi}$. We let it act on $V(\mathcal{R})$ from the left-hand side (which, in particular, determines in which order we compose elements of the Chevalley group). A *Chevalley group of type Φ over $\mathcal{R}$* is a Chevalley group over $\mathcal{R}$ which is defined with respect to some Chevalley basis of L, some finite-dimensional faithful representation δ of L and some admissible lattice M in this representation. We will say that a Chevalley group of type Φ over $\mathcal{R}$ is defined *with respect to c'* if $c' = (c'_{\alpha,\beta})_{\alpha,\beta\in\Phi}$ is the family of Chevalley structure constants of a Chevalley basis with respect to which G is defined.

3.2.12 Note (Adjoint Chevalley groups). The Chevalley groups of type ad are called *adjoint Chevalley groups*. Since the existence of admissible lattices for this representation is relatively easy to verify (see Proposition 3.2.8), the construction of adjoint Chevalley groups is easier than the general construction. In [Car72], adjoint Chevalley groups are actually the only Chevalley groups which are defined. We will see that the Chevalley commutator formula (Theorem 3.2.16) is independent of the chosen representation. Since this formula is the essential property that we are interested in from the viewpoint of root graded groups, we could theoretically restrict ourselves to adjoint Chevalley groups as well. However, the additional effort to define general Chevalley groups is very low as long as we skip the proof of Proposition 3.2.9. Further, while adjoint Chevalley groups are easier to define, we will see in Example 3.2.14 that Chevalley groups for the natural representation of L are easier to compute because this representation is simply the identity map in terms of matrices.

3.2.13 Note (Chevalley groups over rings). In many references, for example in [Hum72; Car72; Ste67], Chevalley groups are only defined over fields, and many results do not extend to rings. For example, adjoint Chevalley groups are nearly always simple (see [Ste67, Chapter 4]), but Chevalley groups over rings containing a proper ideal are not simple. However, the basic construction of Chevalley groups remains the same over rings, and our definition agrees with the one in [VP96, Section 6].

3.2.14 Example (Chevalley groups of type A_ℓ). Consider the Lie algebra $L := \mathfrak{sl}_{\ell+1}(\mathbb{C})$ of type A_ℓ. Denote by $\mathcal{C} = (x_\alpha)_{\alpha\in A_\ell} \cup (h_i)_{i\in[1,\ell]}$ the Chevalley basis of L from Example 3.1.19. Put $V := \mathbb{C}^{\ell+1}$ and denote by $\delta\colon L \to \mathfrak{gl}(V)$ the natural representation of L. Thus for each $x \in L$, the transformation matrix of $\delta(x)$ with respect to the standard basis of V is exactly x. Further, we denote by M the abelian group which is generated by the standard basis in V. Since for all $\alpha \in A_\ell$, we have that $\delta(x_\alpha)$ has integral entries and satisfies $\delta(x_\alpha)^2 = 0$, the lattice M is admissible with respect to $\mathcal{C}$ and δ.

Now let $\mathcal{R}$ be an arbitrary commutative associative ring and let $(\theta_\alpha)_{\alpha\in A_\ell}$ denote the family of root homomorphisms of the Chevalley group G of type δ over $\mathcal{R}$ with respect to M. Then for each $\alpha = e_i - e_j \in A_\ell$, the matrix of $\theta_\alpha(\lambda)$ for $\lambda \in \mathcal{R}$

is exactly $\mathrm{id}_{\ell+1} + \lambda x_\alpha = \mathrm{id}_{n+1} + \lambda b_{ij}$. Matrices of this form are called *elementary matrices*. The Chevalley group G is exactly the group which is generated by these matrices. We call it the *elementary group (over $\mathcal{R}$)*, and we will also denote it by $\mathrm{E}_{n+1}(\mathcal{R})$. Clearly, all elementary matrices have determinant 1, so this group is contained in $\mathrm{SL}_{\ell+1}(\mathcal{R})$. If $\mathcal{R}$ is a field, then we have $\mathrm{SL}_{\ell+1}(\mathcal{R}) = \mathrm{E}_{\ell+1}(\mathcal{R})$.

Observe that elementary matrices and the elementary group can also be defined for an arbitrary associative ring $\mathcal{R}$ which is not assumed to be commutative. This group is no longer a Chevalley group, but it is still an A_ℓ-graded group.

Finally, we can state the Chevalley commutator formula, which is the most important result of this chapter. Not only does it say that a Chevalley group has crystallographic Φ-commutator relations with respect to its root groups, but it also provides explicit formulas for the commutator of two root group elements.

3.2.15 Notation. For any pair of non-proportional roots α, β, we put

$$\mathcal{I}(\alpha,\beta) := \{\, (i,j) \in \mathbb{N}_+^2 \mid i\alpha + j\beta \in \Phi \,\} = \{\, (i,j) \in \mathbb{N}_+^2 \mid i\alpha + j\beta \in \,]\alpha,\beta[\,\,\}.$$

3.2.16 Theorem (Chevalley commutator formula, [Ste67, Lemma 15]). *Let α, β be non-proportional roots and put $\mathcal{I} := \mathcal{I}(\alpha,\beta)$. Endow the set $\mathcal{I}$ with an arbitrary but fixed order. Then there exists a unique family $(c_{\alpha,\beta,i,j})_{(i,j)\in\mathcal{I}}$ of integers (which may depend on the chosen order on $\mathcal{I}$) such that for all commutative associative rings $\mathcal{R}$ and for all Chevalley groups G of type Φ over $\mathcal{R}$, the corresponding root homomorphisms $(\theta_\gamma)_{\gamma\in\Phi}$ satisfy*

$$[\theta_\alpha(r), \theta_\beta(s)] = \prod_{(i,j)\in\mathcal{I}} \theta_{i\alpha+j\beta}\big(c_{\alpha,\beta,i,j} r^i s^j\big)$$

for all $r, s \in \mathcal{R}$. In particular, G has crystallographic Φ-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi}$.

3.2.17 Definition (Chevalley structure constants). The integers $c_{\alpha,\beta,i,j}$ (where α, β are non-proportional roots and $(i,j) \in \mathcal{I}(\alpha,\beta)$) in Theorem 3.2.16 are called the *(Chevalley) structure constants of (the Chevalley group) G*.

We close this section with some complementary remarks.

3.2.18 Note. The structure constants of a Chevalley group G are already determined by the semisimple Lie algebra L from which G is constructed, so it would be more precise to call them "structure constants of L". However, the terminology in Definition 3.2.17 is well-established in the literature.

3.2.19 Remark (Commutator conventions). Recall from Note 1.1.9 that we have defined commutators by $[x,y] := x^{-1}y^{-1}xy$ while some authors prefer the convention $[x,y] := xyx^{-1}y^{-1}$. Of course, the Chevalley structure constants $c_{\alpha\beta ij}$ depend on the convention we use. Passing from one convention to the other one means that we replace r and s by $-r$ and $-s$ on the left-hand side of Theorem 3.2.16, so the structure constant $c_{\alpha\beta ij}$ has to be multiplied by $(-1)^{i+j}$. This has practical relevance for us because [VP96] and [Ste67] use the second convention for commutators while the convention in [Car72] coincides with ours.

3.2.20 Note. In the way Theorem 3.2.16 is stated in [Ste67], it is not actually a statement about the Chevalley group G but rather about the universal enveloping algebra U of L (or, more precisely, about a $\mathbb{Z}$-form of U which is determined by the Chevalley basis $\mathcal{C}$). Even though Steinberg only considers Chevalley groups over fields, it follows from the statement of [Ste67, Lemma 15] that the Chevalley commutator formula holds for Chevalley groups over arbitrary commutative associative rings. A reference which states the Chevalley commutator formula over commutative associative rings is [VP96, p. 94].

3.2.21 Remark (Computation of the structure constants). Since, as explained in Note 3.2.20, the Chevalley commutator cormula is actually a statement about L and its Chevalley basis $\mathcal{C}$, it is not surprising that the Chevalley structure constants $c_{\alpha\beta ij}$ of its Chevalley groups can be computed from the structure constants $c_{\alpha\beta}$ of L. We now explain how. Let α, β be non-proportional roots and define, for each $m \in \mathbb{N}_+$,

$$M_{\alpha,\beta,m} := \frac{1}{m!}\prod_{i=1}^{m} c_{\alpha,\beta+(i-1)\alpha} = \frac{1}{m!}c_{\alpha,\beta}c_{\alpha,\alpha+\beta}\cdots c_{\alpha,(m-1)\alpha+\beta}.$$

Using Lemma 3.1.23 (b), one can show that these numbers are binomial coefficients, so in particular, they are integers (see [Car72, p. 62]). Observe that $M_{\alpha,\beta,1} = c_{\alpha,\beta}$ and $M_{\alpha,\beta,m} = 0$ if $\alpha + (m-1)\beta$ is not a root. Choose any order $\prec$ on $\mathcal{I}(\alpha,\beta)$ with the property that $i+j < i'+j'$ implies $(i,j) \prec (i',j')$. For this order of $\mathcal{I}(\alpha,\beta)$, [Car72, 5.2.2] supplies formulas for the structure constants $c'_{\alpha\beta ij}$ defined by the following slightly different version of the Chevalley commutator formula:

$$\theta_\alpha(u)^{-1}\theta_\beta(t)^{-1}\theta_\alpha(u)\theta_\beta(t) = [\theta_\alpha(u),\theta_\beta(t)] = \prod_{(j,i)}\theta_{i\beta+j\alpha}\big(c'_{i,j,\beta,\alpha}(-t)^i u^j\big).$$

Clearly, these structure constants are related to our structure constants by the formula $c_{\alpha,\beta,i,j} = (-1)^i c_{i,j,\beta,\alpha}$. Thus the formulas in [Car72, 5.2.2] yield:

$$\begin{aligned} c_{\alpha,\beta,j,1} &= -c'_{1,j,\beta,\alpha} = (-1)^{j+1}M_{\alpha,\beta,j},\\ c_{\alpha,\beta,1,i} &= (-1)^i c'_{i,1,\beta,\alpha} = (-1)^i M_{\beta,\alpha,i},\\ c_{\alpha,\beta,3,2} &= c'_{2,3,\beta,\alpha} = -\frac{2}{3}M_{\alpha+\beta,\alpha,2},\\ c_{\alpha,\beta,2,2} &= -c'_{3,2,\beta,\alpha} = -\frac{1}{3}M_{\alpha+\beta,\beta,2}. \end{aligned}$$

Further, all these numbers lie in $\{\pm 1, \pm 2, \pm 3\}$ by [Car72, 5.2.2]. Note that the formula for $c_{\alpha,\beta,i,j}$ differs from the one given in [VP96, p. 95] by the sign $(-1)^{i+j}$ because of the differing convention for commutators (see Remark 3.2.19). Further, note that $c_{\alpha,\beta,1,1} = M_{\alpha,\beta,1} = c_{\alpha,\beta}$.

3.2.22 Example. Assume that the root system Φ is simply-laced. Then we know from Remark 5.2.2 (2) that for all non-proportional roots α, the interval $]\alpha,\beta[$ is either $\{\alpha+\beta\}$ or empty. Hence by Remark 3.2.21, the Chevalley commutator formula has the form

$$[\theta_\alpha(r),\theta_\beta(s)] = \theta_{\alpha+\beta}(c_{\alpha,\beta}rs)$$

for all non-proportional roots α, β for which $\alpha+\beta$ is a root and for all $r,s \in \mathcal{R}$. In the group $\mathrm{E}_{\ell+1}(\mathcal{R})$ from Example 3.2.14, the commutator formula thus has the form

$$[\theta_{e_i-e_j}(r),\theta_{e_j-e_k}(s)] = \theta_{e_i-e_k}(rs) \quad\text{and}\quad [\theta_{e_j-e_k}(r),\theta_{e_i-e_j}(s)] = \theta_{e_i-e_k}(-rs)$$

for all pairwise distinct $i,j,k \in [1,\ell+1]$ and all $r,s \in \mathcal{R}$. Note that each of these two formulas is a simple consequence of the other one by the relation $[x,y]^{-1} = [y,x]$. Further, the first formula remains valid in the group $\mathrm{E}_{\ell+1}(\mathcal{R})$ for any associative ring $\mathcal{R}$ which need not be commutative, but we have to replace rs in the second formula by sr.

3.2.23 Note (The sign problem in the Chevalley commutator formula). We have seen in Remark 3.2.21 that the structure constants in the Chevalley commutator formula can be computed from the structure constants of the Chevalley bases. Since the latter are only unique up to a sign by Note 3.1.25, it follows that the former are only unique up to a sign as well. This is the second occurrence of the sign problem in the theory of Chevalley groups.

3.2.24 Note. We have already remarked in Note 3.2.12 that Carter only covers adjoint Chevalley groups in [Car72]. However, we know from Theorem 3.2.16 that the structure constants do not depend on the choice of a representation, so the formulas in Remark 3.2.21 are still valid for all representations.

3.2.25 Remark (Injectivity of the root homomorphisms). Let α be any root. The homomorphism $\theta_\alpha\colon (\mathcal{R}, +) \to U_\alpha$ is surjective by the definition of U_α, and it is in fact also injective. This is proven in [Ste67, Corollary 1 of Lemma 17] for fields, but the proof does not transfer to rings: it uses that $\lambda v = 0$ implies $\lambda = 0$ where v is a non-zero vector in a module over a field and λ is a scalar. However, it remains true over commutative associative rings (for example, this is stated in [VP96, below (9.1)]), but this is a non-trivial fact.

3.2.26 Lemma. *Let Δ be a root base of Φ such that Δ is the type of $\mathcal{C}$, let Δ' be a subset of Δ and denote by Φ' the root system which is spanned by Δ'. Let $\delta\colon L \to \mathfrak{gl}(V)$ be a finite-dimensional faithful representation of L, let M be an admissible lattice in V with respect to $\mathcal{C}$, let $\mathcal{R}$ be a commutative associative ring and let G be the Chevalley group of type δ over $\mathcal{R}$ with respect to $\mathcal{C}$ and M. Denote the root groups and root homomorphisms of G by $(U_\alpha)_{\alpha\in\Phi}$ and $(\theta_\alpha)_{\alpha\in\Phi}$, respectively. Put $G' := \langle U_\alpha \mid \alpha \in \Phi' \rangle$ and let L' be the Lie subalgebra of L of type Φ' with Chevalley basis $\mathcal{C}'$ from Lemma 3.1.26. Then the restriction δ' of δ to L' is a finite-dimensional faithful representation of L', M is an admissible lattice in V with respect to δ' and $\mathcal{C}'$ and G' is the Chevalley group of type δ' over $\mathcal{R}$ with respect to $\mathcal{C}$ and M. Further, the root groups and root homomorphisms of G' are $(U_\alpha)_{\alpha\in\Phi}$ and $(\theta_\alpha)_{\alpha\in\Phi}$, respectively.*

Proof. The only difficulty lies in proving that $\mathcal{C}'$ is indeed a Chevalley basis, which we already know from Lemma 3.1.26. The remaining assertions are straightforward to check. □

3.3 Weyl Elements in Chevalley Groups

3.3.1 Notation for this section. We denote by $\Bbbk$ an algebraically closed field of characteristic 0, by L a semisimple Lie algebra over $\Bbbk$ with a Cartan subalgebra H and (reduced crystallographic) root system $\Phi := \Phi(L, H)$, by ℓ the rank of Φ and by $\mathcal{C} = (x_\alpha)_{\alpha\in\Phi} \cup (h_i)_{i\in[1,\ell]}$ a Chevalley basis of L (with respect to H) with structure constants $(c_{\alpha,\beta})_{\alpha,\beta\in\Phi}$. We fix a finite-dimensional faithful representation $\delta\colon L \to \mathfrak{gl}(V)$, an admissible lattice M in V and a commutative associative ring $\mathcal{R}$. We denote by G the Chevalley group of type δ over $\mathcal{R}$ (with respect to M), by $(U_\alpha)_{\alpha\in\Phi}$ its root groups and by $(\theta_\alpha)_{\alpha\in\Phi}$ its root homomorphisms.

Since we already know from the Chevalley commutator formula that G has crystallographic Φ-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi}$, it remains to establish the existence of Weyl elements in G. However, we will not settle for a mere existence result. Instead, we will give explicit formulas for the conjugation action of Weyl elements on the root groups, and we will derive properties of the signs which appear in these formulas. In particular, we will encounter the square formula for Weyl elements (see Note 3.3.12) which plays an important role in the theory of root graded groups (see Definitions 4.2.10 (e) and 4.4.4).

3.3.2 Note. Essentially all results in this sections are consequences of [Ste67, Lemma 19 (a), (c)]. It is slightly problematic that Lemma 19 is only formulated for fields. However, the statements of Lemma 19 (a) and (c) remain valid over commutative associative rings: For each equation in Lemma 19 (a) and (c), we can find a certain integral Laurent polynomial

$$f \in \mathbb{Z}[t_1, \ldots, t_n, u_1, \ldots, u_m, u_1^{-1}, \ldots, u_m^{-1}]$$

such that the validity of the desired equation (over $\mathcal{R}$) is equivalent to the assertion that f vanishes at all points in $\mathcal{R}^n \times (\mathcal{R}^\times)^m$ (where $\mathcal{R}^\times$ is the set of invertible elements in $\mathcal{R}$). If the desired results holds over $\mathbb{C}$, then f must be the zero polynomial, and hence it vanishes at all points in any commutative associative ring. Thus if Lemma 19 (a) and (c) hold over $\mathbb{C}$, then they hold over all commutative associative rings. In fact, Steinberg only proves [Ste67, Lemma 19] over $\mathbb{C}$ and then invokes the same argument that we have just described to infer that it holds over arbitrary fields.

Similarly, it can be shown that statement of Lemma 19 (b) in [Ste67] remains true over commutative associative rings. However, the existence assertion in Lemma 19 (b) makes the required argument more subtle. In any case, we will not need this result.

The validity of [Ste67, Lemma 20 (b), (c)] over commutative associative rings can be proven by similar arguments. Alternatively, since these results are direct consequences of Lemma 19 (a) and (c), one can also apply the same proof as in [Ste67], but with references to Lemma 19 replaced by references to the more general assertions for commutative associative rings.

We will show that the elements in the following definition are Weyl elements, and investigate their properties.

3.3.3 Definition. Let α be a root and let r be an invertible element of $\mathcal{R}$. We define the following two elements of G:

$$w_\alpha(r) := \theta_{-\alpha}(-r^{-1})\theta_\alpha(r)\theta_{-\alpha}(-r^{-1}), \qquad w'_\alpha(r) := \theta_\alpha(r)\theta_{-\alpha}(-r^{-1})\theta_\alpha(r).$$

Further, we put $w_\alpha := w_\alpha(1)$ and $w'_\alpha := w'_\alpha(1)$. The elements $(w_\alpha)_{\alpha\in\Phi}$ are called the *standard Weyl elements in G*.

3.3.4 Note. Clearly, we have the following relations:

$$w_\alpha(r)^{-1} = w_\alpha(-r), \qquad w'_\alpha(r)^{-1} = w'_\alpha(-r),$$
$$w_\alpha(r) = w'_{-\alpha}(-r^{-1}), \qquad w_\alpha = w'_{-\alpha}(-1) = (w'_{-\alpha})^{-1}.$$

The reason that we introduce both $w_\alpha(r)$ and $w'_\alpha(r)$ is that [Ste67] uses the elements $w'_\alpha(r)$ while we prefer to work with the elements $w_\alpha(r)$. For this reason, we will first cite the results in [Ste67] involving $w'_\alpha(r)$ and then state the analogous results involving $w_\alpha(r)$ as a corollary.

The essential result for our purposes is the following one.

3.3.5 Lemma. *For any pair of roots α, β, there exists a unique $\eta'_{\mathcal{R}}(\alpha,\beta) \in \{\pm 1_{\mathcal{R}}\}$ (independent of the choice of the representation δ and of the admissible lattice M, but not of the choice of $\mathcal{C}$) such that*

$$w'_\alpha \theta_\beta(\lambda)(w'_\alpha)^{-1} = \theta_{\sigma_\alpha(\beta)}\big(\eta'_{\mathcal{R}}(\alpha,\beta)\lambda\big)$$

for all $\lambda \in \mathcal{R}$. Further, these constants satisfy

$$\eta'_{\mathcal{R}}(\alpha,\beta) = \eta'_{\mathcal{R}}(\alpha,-\beta) \quad \textit{and} \quad \eta'_{\mathcal{R}}(\alpha,\beta)\eta'_{\mathcal{R}}\big(\alpha,\sigma_\alpha(\beta)\big) = (-1)^{\langle\beta|\alpha\rangle}.$$

Proof. The first statement is [Ste67, Lemma 20 (b)] and the relation $\eta'_{\mathcal{R}}(\alpha,\beta) = \eta'_{\mathcal{R}}(\alpha,-\beta)$ is part of [Ste67, Lemma 19 (a)]. For any invertible element r of $\mathcal{R}$, we define $h'_\alpha(r) := w'_\alpha(r)w'_\alpha(-1)$ (as in [Ste67, Lemma 19]). Then [Ste67, Lemma 20 (c)] says that

$$h'_\alpha(r)\theta_\beta(\lambda)h'_\alpha(r)^{-1} = \theta_\beta(r^{\langle\beta|\alpha\rangle}\lambda)$$

for all invertible $r \in \mathcal{R}$ and all $\lambda \in \mathcal{R}$. Applying this statement with $r := -1_{\mathcal{R}}$ (so that $h'_\alpha(r) = w'_\alpha(-1)^2 = (w'_\alpha)^{-2}$), we infer that

$$(w'_\alpha)^{-2}\theta_\beta(\lambda)(w'_\alpha)^2 = \theta_\beta((-1)^{\langle\beta|\alpha\rangle}\lambda)$$

for all $\lambda \in \mathcal{R}$. Thus

$$\begin{aligned}\theta_\beta(\lambda) &= (w'_\alpha)^2\theta_\beta\big((-1)^{\langle\beta|\alpha\rangle}\lambda\big)(w'_\alpha)^{-2} = w'_\alpha\theta_{\sigma_\alpha(\beta)}\big(\eta'_{\mathcal{R}}(\alpha,\beta)(-1)^{\langle\beta|\alpha\rangle}\lambda\big)(w'_\alpha)^{-1} \\ &= \theta_\beta\big(\eta'_{\mathcal{R}}(\alpha,\sigma_\alpha(\beta))\eta'_{\mathcal{R}}(\alpha,\beta)(-1)^{\langle\beta|\alpha\rangle}\lambda\big)\end{aligned}$$

for all $\lambda \in \mathcal{R}$. Since θ_β is injective by Remark 3.2.25, the last assertion follows by putting $\lambda := 1_{\mathcal{R}}$. □

3.3.6 Note. Lemma 19 in [Ste67] is actually formulated for all elements $w'_\alpha(\lambda)$ where $\lambda \in \mathcal{R}$ is invertible, and not just for $\lambda = 1_{\mathcal{R}}$. Lemma 20, on the other hand, is not stated in this generality. An adaption of the proof of Lemma 20 makes it possible to show that

$$w'_\alpha(r)\theta_\beta(\lambda)(w'_\alpha(r))^{-1} = \theta_{\sigma_\alpha(\beta)}\big(\eta'_{\mathcal{R}}(\alpha,\beta)r^{-\langle\beta|\alpha\rangle}\lambda\big)$$

for all invertible $r \in \mathcal{R}$. However, the weaker statement of Lemma 3.3.5 will be sufficient for our purposes.

In order to suit the conventions of this thesis, we will now write down the assertion of Lemma 3.3.5 for the elements w_α in place of w'_α and for $\eta_{\mathcal{R}}(\beta,\alpha)$ in place of $\eta'_{\mathcal{R}}(\alpha,\beta)$.

3.3.7 Lemma. *For any pair of roots α,β, there exists a unique $\eta_{\mathcal{R}}(\beta,\alpha) \in \{\pm 1_{\mathcal{R}}\}$ (independent of the choice of the representation δ and of the admissible lattice M, but not of the choice of $\mathcal{C}$) such that*

$$w_\alpha^{-1}\theta_\beta(\lambda)w_\alpha = \theta_{\sigma_\alpha(\beta)}\big(\eta_{\mathcal{R}}(\beta,\alpha)\lambda\big)$$

for all $\lambda \in \mathcal{R}$. Further, these constants satisfy

$$\eta_{\mathcal{R}}(\beta,\alpha) = \eta_{\mathcal{R}}(-\beta,\alpha) \quad \textit{and} \quad \eta_{\mathcal{R}}(\beta,\alpha)\eta_{\mathcal{R}}(\sigma_\alpha(\beta),\alpha) = (-1)^{\langle\beta|\alpha\rangle}.$$

Proof. Using that $w_\alpha^{-1} = w'_{-\alpha}$, the first assertion follows from Lemma 3.3.5, and we see that $\eta_{\mathcal{R}}(\beta,\alpha) = \eta'_{\mathcal{R}}(-\alpha,\beta)$. Now

$$\eta_{\mathcal{R}}(\beta,\alpha) = \eta'_{\mathcal{R}}(-\alpha,\beta) = \eta'_{\mathcal{R}}(-\alpha,-\beta) = \eta_{\mathcal{R}}(-\beta,\alpha)$$

and

$$\eta_{\mathcal{R}}(\beta,-\alpha)\eta_{\mathcal{R}}\big(\sigma_\alpha(\beta),-\alpha\big) = \eta'_{\mathcal{R}}(\alpha,\beta)\eta'_{\mathcal{R}}\big(\alpha,\sigma_\alpha(\beta)\big) = (-1)^{\langle\beta|\alpha\rangle}.$$

Since $\sigma_\alpha = \sigma_{-\alpha}$, the latter equality implies that

$$\eta_{\mathcal{R}}(\beta,\alpha)\eta_{\mathcal{R}}\big(\sigma_\alpha(\beta),\alpha\big) = (-1)^{\langle\beta|-\alpha\rangle} = (-1)^{-\langle\beta|\alpha\rangle} = (-1)^{\langle\beta|\alpha\rangle},$$

which finishes the proof. □

3.3.8 Note (The sign problem for Weyl elements in Chevalley groups). The numbers $\eta'_{\mathcal{R}}(\alpha, \beta)$ in Lemma 3.3.5 (and, therefore, also the numbers $\eta_{\mathcal{R}}(\beta, \alpha)$ in Lemma 3.3.7) depend only on the root system Φ and the Chevalley basis $\mathcal{C}$ and not on the choice of the representation δ. However, they do depend on the choice of the Chevalley basis $\mathcal{C}$, and thus on the choice of the signs which appear in the structure constants of $\mathcal{C}$. This is the third and final occurrence of the sign problem in the theory of Chevalley groups (see also Notes 3.1.25 and 3.2.23). It is essentially a special case of the sign problem that we will be faced with in the parametrisation of root graded groups. See Remark 4.1.23 for more details.

3.3.9 Remark. Note that the numbers $\eta_{\mathcal{R}}(\beta, \alpha)$ and $\eta'_{\mathcal{R}}(\alpha, \beta)$ in Lemmas 3.3.5 and 3.3.7 lie in $\{\pm 1_{\mathcal{R}}\}$, not in $\{\pm 1_{\mathbb{Z}}\}$. We can of course lift them to numbers $\tilde{\eta}_{\mathcal{R}}(\alpha, \beta) \in \{\pm 1_{\mathbb{Z}}\}$ and $\tilde{\eta}'_{\mathcal{R}}(\beta, \alpha) \in \{\pm 1_{\mathbb{Z}}\}$, but this lift is only uniquely determined if $1_{\mathcal{R}} \neq -1_{\mathcal{R}}$ (that is, if $2_{\mathcal{R}} \neq 0_{\mathcal{R}}$). Nonetheless, the uniqueness assertions in Lemmas 3.3.5 and 3.3.7 remain true if $2_{\mathcal{R}} = 0_{\mathcal{R}}$ because in this case, $\{\pm 1_{\mathcal{R}}\}$ is a singleton and thus, of course, contains a unique element.

In [Ste67, Lemma 19 (a)], it is shown that if $\mathcal{R}$ is a field, then the numbers $\eta'_{\mathcal{R}}(\alpha, \beta)$ "do not depend on $\mathcal{R}$" in the sense that there exists $c(\alpha, \beta) \in \{\pm 1_{\mathbb{Z}}\}$ such that $\eta'_{\mathcal{R}}(\alpha, \beta) = \operatorname{str}_{\mathcal{R}}(c(\alpha, \beta))$ for all fields $\mathcal{R}$ (where $\operatorname{str}_{\mathcal{R}}$ denotes the unique ring homomorphism from $\mathbb{Z}$ to $\mathcal{R}$). The same remains true over commutative associative rings (see Note 3.3.2), and we then of course have $c(\alpha, \beta) = \eta'_{\mathbb{Z}}(\alpha, \beta)$. Similar assertions hold for the elements $\eta_{\mathcal{R}}(\beta, \alpha)$ in Lemma 3.3.7.

We conclude that we have constants $\eta(\beta, \alpha) := \eta_{\mathbb{Z}}(\beta, \alpha)$ which depend only on L (or in other words, on the root system Φ) and the choice of a Chevalley basis (which comes with a choice of Chevalley structure constants). These constants describe (an important part of) the structure of all Chevalley groups of type L which are defined with respect to this fixed Chevalley basis. Equivalently, they describe the structure of all Chevalley groups of type Φ for some fixed family of Chevalley structure constants. We have seen that these constants satisfy the relations

$$\eta(\beta, \alpha) = \eta(-\beta, \alpha) \quad \text{and} \quad \eta(\beta, \alpha)\eta(\sigma_\alpha(\beta), -\alpha) = (-1)^{\langle \beta | \alpha \rangle}.$$

3.3.10 Definition (Chevalley parity map). The map $\eta \colon \Phi \times \Phi \to \{\pm 1_{\mathbb{Z}}\}$ from Remark 3.3.9 is called the *Chevalley parity map for $c = (c_{\alpha,\beta})_{\alpha,\beta \in \Phi}$*.

We will see in Remark 3.3.14 that η can be computed from c, which justifies the name.

3.3.11 Remark. It follows from Lemma 3.3.7 that for all roots α, β, we have

$$\begin{aligned} w_\alpha^{w_\beta} &= \theta_{-\alpha}(-1)^{w_\beta}\theta_\alpha(1)^{w_\beta}\theta_{-\alpha}(-1)^{w_\beta} \\ &= \theta_{-\sigma_\beta(\alpha)}\big(-\eta(-\alpha, \beta)\big)\theta_{\sigma_\beta(\alpha)}\big(\eta(\alpha, \beta)\big)\theta_{-\sigma_\beta(\alpha)}\big(-\eta(-\alpha, \beta)\big) \\ &= \theta_{-\sigma_\beta(\alpha)}\big(-\eta(\alpha, \beta)\big)\theta_{\sigma_\beta(\alpha)}\big(\eta(\alpha, \beta)\big)\theta_{-\sigma_\beta(\alpha)}\big(-\eta(\alpha, \beta)\big) = w_{\sigma_\beta(\alpha)}^{\eta(\alpha,\beta)}. \end{aligned}$$

Recall from Proposition 1.3.5 that every root can be written as $\delta^{\sigma(\bar{\delta})}$ for some $\delta \in \Delta$ and some word $\bar{\delta}$ over Δ. It follows that the whole family $(w_\alpha)_{\alpha \in \Phi}$ of standard Weyl elements can be reconstructed from $(w_\delta)_{\delta \in \Delta}$ using only conjugation and group inversion.

We will later see that for any Δ-system $(w_\delta)_{\delta \in \Delta}$ of Weyl elements in a root graded group H of rank at least 3, we can find a coordinatisation of H such that elements $(w_\delta)_{\delta \in \Delta}$ behave in a similar ways as the standard Weyl elements in Chevalley groups. More precisely, the coordinatisation is constructed in a way which ensures that the elements $(w_\delta)_{\delta\in\Delta}$ satisfy a similar formula as in Lemma 3.3.7.

3.3.12 Note. The formula $\eta(\beta,\alpha)\eta(\sigma_\alpha(\beta),-\alpha) = (-1)^{\langle\beta|\alpha\rangle}$ implies that

$$\theta_\beta(\lambda)^{w_\alpha^2} = \theta_\beta\big((-1)^{\langle\beta|\alpha\rangle}\lambda\big)$$

for all $\lambda \in \mathcal{R}$, so it describes the action of squares of (certain) Weyl elements on the root groups. For this reason, we will also call it the *square formula for Weyl elements*. It will turn out that, except for a few special cases, the action of squares of arbitrary Weyl elements in root graded groups of rank at least 3 adheres to the same formula. See Propositions 5.4.17, 7.6.15 and 9.5.13 In fact, the square formula and the braid relations for Weyl elements are the main results which have to be shown in order to parametrise root graded groups.

Observe that the square formula can be formulated purely in terms of the group G and its root groups $(U_\alpha)_{\alpha\in\Phi}$, without reference to the root isomorphisms: We have

$$x_\beta^{w_\alpha^2} = x_\beta^\varepsilon$$

for all roots α,β and all $x_\beta \in U_\beta$, where $\varepsilon := (-1)^{\langle\beta|\alpha\rangle}$.

3.3.13 Lemma. *For all $\alpha \in \Phi$, we have $\eta(\alpha,\alpha) = \eta(-\alpha,\alpha) = -1$ where $\eta(\alpha,\alpha)$ is the constant from Definition 3.3.10.*

Proof. We know from Remark 3.3.9 that $\eta(\alpha,\alpha) = \eta(-\alpha,\alpha)$, so we only have to show that $\eta(\alpha,\alpha) = -1$. By Lemma 3.2.26, the group G' which is generated by $U_\alpha \cup U_{-\alpha}$ is a Chevalley group of type A_1 over $\mathcal{R}$. Since η can be computed inside G', we can thus assume that $\Phi = A_1$. Since η does not depend on the choice of a representation, it suffices to prove the statement for the natural representation.

It remains to show that for every Chevalley basis $\mathcal{C}$ of L, there exists an admissible lattice in the natural representation of L with respect to $\mathcal{C}$ such that the constants η satisfy $\eta(\alpha,\alpha) = -1$ for all $\alpha \in A_1$. (It then follows that the statement holds for any admissible lattice because these constants do not depend on the choice of a lattice.) This means that we have to show that $\theta_\alpha(1)^{w_\alpha} = \theta_{-\alpha}(-1)$. For the specific Chevalley basis $\mathcal{C} = (x_{12}, x_{21}, h_1)$ and the specific admissible lattice $M = \langle b_1, b_2\rangle_{\mathbb{Z}}$ that are chosen as in Example 3.2.14, we can prove this with a simple computation. Further, we know from Example 3.1.19 that any other Chevalley basis must be of the form $\mathcal{C}' = (\lambda x_{12}, \lambda^{-1}x_{21}, h_1')$ for some $h_1' \in \{h_1, -h_1\}$ and some $\lambda \in \mathbb{C} \setminus \{0\}$. Then $M' := \langle \lambda b_1, b_2\rangle_{\mathbb{Z}}$ is an admissible lattice with respect to $\mathcal{C}'$. Denote the root isomorphisms of the Chevalley group defined with respect to $\mathcal{C}'$ and M' by $(\theta'_\alpha)_{\alpha\in A_1}$. Then we see that for all $\alpha \in A_1$ and all $a \in \mathcal{R}$, the matrix of $\theta'_\alpha(a)$ with respect to $(\lambda b_1, b_2)$ is the same as the matrix of $\theta_\alpha(a)$ with respect to (b_1, b_2). We conclude that the desired statement holds for any Chevalley basis of L, which finishes the proof. □

3.3.14 Remark (Computation of the Chevalley parity map). Let α, β be roots. If $\alpha \in \{\pm\beta\}$, then we know from Lemma 3.3.13 that $\eta(\beta,\alpha) = -1$. If α, β are linearly independent, then we can perform computations similar to the ones in Lemmas 5.4.2 and 7.4.5 to determine $\eta(\beta,\alpha)$ in terms of the structure constants which appear in the Chevalley commutator formula. Since the structure constants in the Chevalley commutator formula can be computed from the structure constants $c = (c_{\gamma,\delta})_{\gamma,\delta\in\Phi}$ of the Chevalley basis by Remark 3.2.21, we conclude that there exists an algorithm to compute $\eta(\beta,\alpha)$ from c. For the simply-laced root systems, the resulting formulas can be found in Lemma 5.5.4.

In order to conclude that G is a Φ-graded group, we have to make a final observation.

3.3.15 Proposition. *There exist an $\mathcal{R}$-basis B of $V(\mathcal{R})$ and a positive system Π in Φ such that the matrices of elements of U_Π are unipotent upper-triangular while matrices of elements of $U_{-\Pi}$ are unipotent lower-triangular. In particular, $U_\Pi \cap U_{-\Pi} = \{1\}$.*

Proof. This is proven in [Ste67, Corollary 3 of Lemma 17] for fields, but the same proof is valid for arbitrary commutative associative rings. □

3.3.16 Theorem. *$(U_\alpha)_{\alpha\in\Phi}$ is a crystallographic Φ-grading of G.*

Proof. By definition, G is generated by $(U_\alpha)_{\alpha\in\Phi}$. The Chevalley commutator formula (Theorem 3.2.16) yields that G has crystallographic Φ-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi}$. By Lemma 3.3.7, $(w_\alpha)_{\alpha\in\Phi}$ is a family of Weyl elements in G. Finally, Axiom Definition 2.5.2 (iv) is satisfied by Proposition 3.3.15. □

3.3.17 Note (see also Note 3.3.2). In its generality, Theorem 3.3.16 relies on the fact that everything we cite from [Ste67] remains true over the arbitrary commutative associative ring $\mathcal{R}$. However, this fact is only needed to ensure the existence of Weyl elements. Alternatively, the existence of Weyl elements can be deduced from the concrete form of the Chevalley commutator formula. We will see a sketch of the necessary computation in Proposition 5.6.6, and we will also use this fact in Propositions 7.7.6 and 9.6.5.

Chapter 4

The Parametrisation Theorem

As explained in the preface and in 2.5.4, the goal of this thesis is to coordinatise root graded groups $(G, (U_\alpha)_{\alpha\in\Phi})$ by algebraic structures. Our general strategy to construct such a coordinatisation can be broken down into two steps. In the first step, we construct a parametrisation of G. That it, we construct a family of groups $(M_\alpha)_{\alpha\in\Phi}$ (where $M_\alpha = M_\beta$ if α, β are conjugate under the Weyl group) and a family $(\theta_\alpha\colon M_\alpha \to U_\alpha)_{\alpha\in\Phi}$ of isomorphisms which satisfies a certain consistency condition. A precise definition of parametrisations will be given in 4.3.4. We emphasise that throughout this thesis, we will carefully distinguish between "parametrisations" in the sense just described and "coordinatisations" in the sense of Goal 2.5.4. From this point on, this distinction will be crucial.

In the second step of our general strategy, we show that the commutation maps which are defined on the parametrising groups satisfy certain identities. This is done using the blueprint technique, which is described in chapter 6. These identities turn out to be the axioms of certain algebraic structures. We conclude that the commutation maps impose an algebraic structure on the parametrising groups, thereby turning the parametrisation into a coordinatisation.

The present chapter is dedicated to the first step in our general strategy, that is, to the construction of a parametrisation. The goal of this chapter is the parametrisation theorem (Theorem 4.5.16), which is a criterion for the existence of a parametrisation. In section 4.1, we give a detailed and slightly informal overview of the technical machinery and language that will be used throughout this chapter. From a purely formal viewpoint, section 4.1 is redundant because all necessary definitions and statements will be presented in full detail in the following sections. However, the reader may find the motivation given in section 4.1 helpful. A detailed overview of the sections in this chapter will be given in Note 4.1.32 after we have introduced the necessary technical language.

4.1 Motivation and Overview of this Chapter

4.1.1 Notation for this section. We denote by Φ a root system of rank at least 3 and by Δ a rescaled root base of Φ. In order to avoid technicalities in this outline, we assume that Φ is reduced. Further, we write $\mathbb{O} := \mathrm{Orb}^{\mathrm{red}}(\Phi) = \mathrm{Orb}(\Phi) = (O_1, \dots, O_k)$ (see Notation 1.3.9). Unless otherwise specified, we denote by $(G, (U_\alpha)_{\alpha\in\Phi})$ a Φ-graded group.

This section serves as an overview of the chapter. All definitions and results that we introduce here will be repeated in the following sections, and we will sometimes leave out technical details. We will address in Remark 4.1.31 why Δ is allowed to be a rescaled root base, and why it is not sufficient for root systems of type BC to only consider proper root bases. Further, we will discuss in Note 4.1.10 the changes which are necessary if Φ is not reduced.

4.1.A Goals

We begin with a summary of the properties which relate the structure of a Chevalley group to the structure of the commutative associative ring over which it is defined.

4.1.2 Summary (Coordinatisations of Chevalley groups). Assume that the root system Φ is crystallographic and let G be a Chevalley group of type Φ with root groups $(U_\alpha)_{\alpha\in\Phi}$. Then the following three properties are satisfied:

(1) There exist an abelian group $(\mathcal{R},+)$ and isomorphisms

$$(\theta_\alpha\colon (\mathcal{R},+) \to U_\alpha)_{\alpha\in\Phi}.$$

(2) There exists a bi-additive map $\mathcal{R}\times\mathcal{R} \to \mathcal{R}$ which turns $\mathcal{R}$ into a commutative associative ring such that the Chevalley commutator formula (Theorem 3.2.16) is satisfied.

(3) We have a "standard" family of Weyl elements $(w_\alpha)_{\alpha\in\Phi}$ which satisfy $\theta_\beta(\lambda)^{w_\alpha} = \theta_{\sigma_\alpha(\beta)}\big(\eta(\beta,\alpha)\lambda\big)$ for all roots α, β and all $\lambda\in\mathcal{R}$, where $\eta(\beta,\alpha)\in\{\pm 1_{\mathbb{Z}}\}$ is a sign which does not depend on λ (and, in the Chevalley group setting, not even on the choice of $\mathcal{R}$ and the representation δ). These Weyl elements are defined with respect to the "standard invertible element" $1_{\mathcal{R}}$ in $\mathcal{R}$.

In the context of root graded groups, we have to adjust our expectations. The following three remarks describe how the three properties in Summary 4.1.2 generalise to root gradings. We have already elaborated on this in great detail in the preface of this thesis, so there will naturally be some overlap. However, we are now in a better position to introduce some more technical details. In particular, the material in Remark 4.1.5 has not been discussed beforehand.

4.1.3 Remark (Parametrising groups). We know from Lemma 2.5.12 that roots α, β which lie in the same orbit under the Weyl group have isomorphic root groups $U_\alpha \cong U_\beta$. However, there is no reason for this to be true for arbitrary pairs (α,β). Thus instead of a single group $(\mathcal{R},+)$ which parametrises all root groups, our goal is to find multiple groups $M_1,\dots,M_k$ and a family $(\theta_\alpha\colon M_i \to U_\alpha)_{\alpha\in O_i}$ of isomorphisms for all $i\in[1,k]$. The groups $M_1,\dots,M_k$ are called the *parametrising groups (of G)*. Given parametrising groups $M_1,\dots,M_k$, we put $M_\alpha := M_{i(\alpha)}$ for all roots α where $i(\alpha)\in[1,k]$ is the unique index such that $\alpha\in O_{i(\alpha)}$.

4.1.4 Remark (Commutation maps). Assume that we have parametrising groups $M_1,\dots,M_k$ and root homomorphisms $(\theta_\alpha)_{\alpha\in\Phi}$ as in Remark 4.1.3. Let α,β be non-proportional roots and let $(\gamma_1,\dots,\gamma_m)$ be the unique interval ordering of $[\alpha,\beta]_{\mathrm{Cox}}$ such that $\gamma_1=\alpha$ and $\gamma_m=\beta$ (as in Remark 1.2.35). Since the root homomorphisms are bijective, the so-called *commutation maps*

$$\psi_{\alpha,\beta,j}\colon M_\alpha\times M_\beta \to M_{\gamma_j}, (a,b)\mapsto \theta_{\gamma_j}^{-1}\big([\theta_\alpha(a),\theta_\beta(b)]_{\gamma_j}\big)$$

are well-defined for all $j\in[2,m-1]$. In other words, we have

$$[\theta_\alpha(a),\theta_\beta(b)] = \prod_{j=2}^{m-1}\theta_{\gamma_j}\big(\psi_{\alpha,\beta,j}(a,b)\big)$$

for all $a \in M_\alpha$, $b \in M_\beta$. This equation already looks like the Chevalley commutator formula. However, it is a purely formal observation. At this point, we know nothing about the maps $\psi_{\alpha,\beta,j}$. Furthermore, we need a large number of different maps $\psi_{\alpha,\beta,j}$ to describe this commutator formula. In contrast, only a single map $f\colon \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is required in Summary 4.1.2 (2), and we know that this map turns $\mathcal{R}$ into a commutative associative ring. In other words, we know that f satisfies the identities

$$\begin{aligned} f(x,y) &= f(y,x), & f\big(f(x,y),z\big) &= f\big(x,f(y,z)\big), \\ f(x+y,z) &= f(x,z)+f(y,z), & f(x,y+z) &= f(x,y)+f(y,z), \\ f(x,1_{\mathcal{R}}) &= x = f(1_{\mathcal{R}},x) \end{aligned} \tag{4.1}$$

for all $x,y,z \in \mathcal{R}$ and for some (necessarily unique) $1_{\mathcal{R}} \in \mathcal{R}$.

Our next two goals from this point on are straightforward: Reduce the number of maps which are needed to describe the commutator relations, and show that the remaining maps satisfy some identities. These identities will be “nice” in the sense that they are precisely the axioms of some algebraic structure, just like the identities in (4.1) are exactly the axioms which turn an abelian group $(\mathcal{R},+)$ into a commutative associative ring $(\mathcal{R},+,f)$ with identity element $1_{\mathcal{R}}$. We will see in Note 4.1.13 that the reduction of the number of necessary commutation maps poses no serious problem. The computation of identities, however, is more complicated and relies on the blueprint technique, which we will introduce in chapter 6.

4.1.5 Remark (The twisting group). Assume again that we have found parametrising groups $M_1,\dots,M_k$ and root homomorphisms $(\theta_\alpha)_{\alpha\in\Phi}$ as in Remark 4.1.3. We will later see several examples of root graded groups which illustrate that we cannot, in general, hope for a formula of the form $\theta_\beta(\lambda)^{w_\alpha} = \theta_{\sigma_\alpha(\beta)}\big(\eta(\beta,\alpha)\lambda\big)$ where $\eta(\beta,\alpha)$ is 1 or -1 (as it is the case for Chevalley groups).

As an example, let $\mathcal{R}$ be any associative ring which is equipped with an involution $\sigma\colon \mathcal{R} \to \mathcal{R}, r \mapsto r^\sigma$. (We will define involutions in Definition 8.3.2, but for the moment, it suffices to think of the complex numbers equipped with the conjugation map.) In section 9.3, we will construct a C_3-graded group in which, for some roots α, β and a “standard” Weyl element w_α, we have $\theta_\beta(\lambda)^{w_\alpha} = \theta_{\sigma_\alpha(\beta)}(\lambda^\sigma)$ for all $\lambda \in \mathcal{R}$. This example shows that, in our more general setup, the formula in Summary 4.1.2 (3) should not only allow signs $\eta(\beta,\alpha) \in \{\pm 1\}$ but more general “twistings” of the parametrising structures.

Further, it turns out to be more practical to only consider a family $(w_\delta)_{\delta\in\Delta}$ of Weyl elements for a rescaled root base Δ, not a “full” family $(w_\alpha)_{\alpha\in\Phi}$ of Weyl elements. The reasoning behind this will be explained in Note 4.1.18.

We conclude that our goal is as follows: We want to find a (finite abelian) group A which acts on the parametrising groups $M_1,\dots,M_k$, a family $(w_\delta)_{\delta\in\Delta}$ of Weyl elements and a map $\eta\colon \Phi\times\Delta \to A$ such that for all $\alpha \in \Phi$ and $\delta \in \Delta$, we have

$$\theta_\alpha(\lambda)^{w_\delta} = \theta_{\sigma_\delta(\alpha)}\big(\eta(\alpha,\delta).\lambda\big) \quad \text{for all } \lambda \in M_\alpha. \tag{4.2}$$

Before we move on, we give a name to the objects that we encountered in Remark 4.1.5. We do not include finiteness in the definition of twisting groups because this is not necessary in the abstract theory. However, in our concrete situation, all twisting groups will be of the form $\{\pm 1\}^p$ for some $p \in [1,3]$.

4.1.6 Definition (Parity map). Let A be an abelian group. A *parity map (with values in A)* is a map $\eta\colon \Phi\times\Delta \to A, (\alpha,\delta) \mapsto \eta_{\alpha,\delta} = \eta(\alpha,\delta)$.

4.1.7 Definition (Parameter system). A *parameter system* is a tuple

$$\mathcal{P} = (A, M_1, \dots, M_k)$$

consisting of groups $M_1, \dots, M_k$ and an abelian group A which acts on each of the sets $M_1, \dots, M_k$. The group A is called the *twisting group of $\mathcal{P}$*. Further, we put $M_\alpha := M_{i(\alpha)}$ for all roots α where $i(\alpha) \in [1, k]$ denotes the unique index such that $\alpha \in O_{i(\alpha)}$.

4.1.8 Definition (Parametrisation). Let $\mathcal{P} = (A, M_1, \dots, M_k)$ be a parameter system, let $(w_\delta)_{\delta\in\Delta}$ be a Δ-system of Weyl elements and let η be a parity map η with values in A. A *parametrisation of G by $\mathcal{P}$ with respect to η and $(w_\delta)_{\delta\in\Delta}$* is a family $(\theta_\alpha\colon M_\alpha \to U_\alpha)_{\alpha\in\Phi}$ of isomorphisms satisfying (4.2). Formula (4.2) is called the *conjugation formula for Weyl elements (with respect to $\mathcal{P}$ and η)*, and $(w_\delta)_{\delta\in\Delta}$ is said to be *compatible with* $(\theta_\alpha)_{\alpha\in\Phi}$ if it satisfies the conjugation formula for Weyl elements. The maps $(\theta_\alpha)_{\alpha\in\Phi}$ are called *root isomorphisms*.

4.1.9 Example. Recall from Remark 4.1.5 that we can construct a C_3-graded group from any associative ring $\mathcal{R}$ with involution σ. In this example, the twisting group is $\{\pm 1\}^2$ and its action on $\mathcal{R}$ is given by $(-1, 1).r = -r$ and $(1, -1).r = r^\sigma$ for all $r \in \mathcal{R}$. (It is worth pointing out that only the short root groups in this example are parametrised by $\mathcal{R}$, but for the motivational purposes of this section, it is sufficient to only consider these root groups.)

4.1.10 Note. We briefly consider what happens if Φ is not reduced. In this case, a parameter systems consists of a parametrising group for each *indivisible* orbit and a parametrisation consists of a root isomorphism for each *indivisible* root. However, this data automatically yields a parametrisation of all remaining, non-indivisble groups: If a root is of the form 2α for some root α, then $U_{2\alpha}$ is a subgroup of U_α, so a root isomorphism $\theta_\alpha\colon M_\alpha \to U_\alpha$ provides a parametrising group $M_{2\alpha} := \theta_\alpha^{-1}(U_{2\alpha})$ for $U_{2\alpha}$ and a corresponding root isomorphism $\theta_{2\alpha} := \theta_\alpha|_{M_{2\alpha}}$. We should also require that any parity map η satisfies $\eta_{\alpha,\delta} = \eta_{\lambda\alpha,\delta}$ for all roots $\alpha \in \Phi$, $\delta \in \Delta$ and all $\lambda > 0$ for which $\lambda\alpha$ is a root. Apart from this technicality, non-reduced root systems are treated in the exact same manner as reduced root systems. For simplicity, we will stick to the assumption that Φ is reduced for the scope of this section. However, we will proceed in full generality in the remaining part of this chapter.

4.1.B The General Strategy

Before we can outline our general strategy to obtain a parametrisation, we have to introduce another definition of twisting groups which is similar to the one in Definition 4.1.7. We will clarify the relation between these two definitions in Remark 4.1.14. Both notions will be needed to formulate the parametrisation theorem.

4.1.11 Definition (Twisting groups for root gradings). A *twisting group for G* is an abelian group A which acts on each root group of G and which satisfies some additional technical conditions which we will specify in Definition 4.3.13.

4.1.12 Strategy. The order of the three points in Summary 4.1.2 reflects the order in which we encountered these three properties in the context of Chevalley groups: The isomorphisms $(\theta_\alpha)_{\alpha\in\Phi}$ appeared from the beginning, then we established the Chevalley commutator formula and then we computed the conjugation formulas for Weyl elements.

Our strategy for root graded groups will follow a different order: Starting from a Δ-system $(w_\delta)_{\delta\in\Delta}$ of Weyl elements, a twisting group A for G and a parity map η, we construct parametrising groups $M_1, \dots, M_k$ on which A acts and root homomorphisms $(\theta_\alpha)_{\alpha\in\Phi}$ such that the conjugation formula (4.2) is satisfied. This construction is precisely the content of the parametrisation theorem. In other words, the parametrisation theorem reduces the problem of constructing $M_1, \dots, M_k$ and $(\theta_\alpha)_{\alpha\in\Phi}$ to the problem of finding an appropriate parity map η_Φ and an appropriate twisting group A_Φ for G such that certain conditions relating G, A_Φ and η_Φ are satisfied. The statement and proof of the parametrisation theorem is independent of the root system.

After $M_1, \dots, M_k$ and $(\theta_\alpha)_{\alpha\in\Phi}$ are constructed, we can define commutation maps as in Remark 4.1.4. It then remains to compute the identities which these commutation maps verify. This is done using the blueprint technique. We will elaborate on this in chapter 6 and focus on the parametrisation theorem for the rest of this chapter.

4.1.13 Note. Recall that we have defined a large number of commutation maps $\psi_{\alpha,\beta,j}$ in Remark 4.1.4. Using the conjugation formula (4.2) and Lemma 2.2.9, we can reduce the number of these maps as follows. Recall that $\psi_{\alpha,\beta,j}$ is defined by the formula

$$[\theta_\alpha(a), \theta_\beta(b)]_{\gamma_j} = \theta_{\gamma_j}\big(\psi_{\alpha,\beta,j}(a,b)\big)$$

for all $a \in M_\alpha$ and $b \in M_\beta$. Here α, β are non-proportional roots and $(\gamma_1, \dots, \gamma_m)$ is an interval ordering of $[\alpha,\beta]_{\mathrm{Cox}}$. For any $\delta \in \Delta$, we can conjugate this equation by w_δ to obtain

$$[\theta_{\sigma_\delta(\alpha)}(\eta_{\alpha,\delta}.a), \theta_{\sigma_\delta(\beta)}(\eta_{\beta,\delta}.b)]_{\sigma_\delta(\gamma_j)} = \theta_{\sigma_\delta(\gamma_j)}\big(\eta_{\gamma_j,\delta}.\psi_{\alpha,\beta,j}(a,b)\big)$$

for all $a \in M_\alpha = M_{\sigma_\delta(\alpha)}$ and $b \in M_\beta = M_{\sigma_\delta(\beta)}$. At the same time, by the definition of $\psi_{\sigma_\delta(\alpha),\sigma_\delta(\beta),j}$, we also have

$$[\theta_{\sigma_\delta(\alpha)}(\eta_{\alpha,\delta}.a), \theta_{\sigma_\delta(\beta)}(\eta_{\beta,\delta}.b)]_{\sigma_\delta(\gamma_j)} = \theta_{\sigma_\delta(\gamma_j)}\big(\psi_{\sigma_\delta(\alpha),\sigma_\delta(\beta),j}(\eta_{\alpha,\delta}.a, \eta_{\beta,\delta}.b)\big)$$

for all $a \in M_{\sigma_\delta(\alpha)}$ and $b \in M_{\sigma_\delta(\beta)}$. Thus we can compute $\psi_{\sigma_\delta(\alpha),\sigma_\delta(\beta),j}$ from $\psi_{\alpha,\beta,j}$ if we know the parity map η and the action of the twisting group. This shows that we only need to compute a low number of commutation maps because all the other ones can be obtained by twisting.

4.1.14 Remark (Twisting groups in the parametrisation theorem). Observe that we have defined two notions of twisting groups: A twisting group for G is a group which acts on each root group while a twisting group of a parameter system $(A, M_1, \dots, M_k)$ is a group A which acts on each of the parametrising groups $M_1, \dots, M_k$. The purpose of the parametrisation theorem can be rephrased as follows: Starting from a twisting group A for G, we want to construct parametrising groups $M_1, \dots, M_k$ on which A acts and root isomorphisms $(\theta_\alpha)_{\alpha\in\Phi}$ such that, among other properties, we have

$$\theta_\alpha(a.x) = a.\theta_\alpha(x) \tag{4.3}$$

for all $a \in A$, all roots α and all $x \in M_\alpha$. This process turns the twisting group A for G into a twisting group of the parameter system $(A, M_1, \dots, M_k)$.

Observe the following subtlety: Let A be a twisting group for G and let α be a root. Then for any isomorphism $\theta_\alpha \colon M_\alpha \to U_\alpha$, we can of course define a unique action of A on M_α which satisfies (4.3). This might make the goal from the previous paragraph seem trivial. However, if we choose another root α' with

$M_\alpha = M_{\alpha'}$, then it is not clear that the aforementioned procedure defines the same action on M_α. Thus we have to carefully choose the isomorphisms $(\theta_\alpha)_{\alpha\in\Phi}$, and we need some conditions on the twisting group A for G which guarantee that for two roots α, α' lying in the same orbit, the actions of A on U_α and $U_{\alpha'}$ are in some way related.

4.1.C The Parametrisation Theorem

The following remarks outline the strategy for our proof of the parametrisation theorem.

4.1.15 Convention. For the sake of a less technical explanation, we will in the following always assume that there is only one orbit of roots under the Weyl group. In the general situation, all constructions that we describe below have to be performed separately for each orbit of roots.

We begin with a first naive construction which will be improved later on.

4.1.16 Construction (of the root isomorphisms). It is not difficult to parametrise a single root group for a fixed root $\alpha_0 \in \Phi$: We simply choose a group M which is isomorphic to U_{α_0} and we choose an isomorphism $\theta_{\alpha_0}\colon M \to U_{\alpha_0}$. Since all root groups are isomorphic by Lemma 2.5.12 and Convention 4.1.15, it is clear that there exist isomorphisms $\theta_\beta\colon M \to U_\beta$ for all $\beta \in \Phi \setminus \{\alpha_0\}$ as well. The proof of Lemma 2.5.12 even tells us how to construct them: simply conjugate by an appropriate sequence of Weyl elements. To make this precise, we fix a rescaled root base Δ of Φ and a Δ-system $(w_\delta)_{\delta\in\Delta}$ of Weyl elements. Then for all $\beta \in \Phi$, we define an isomorphism $\theta_\beta\colon M \to U_\beta$ as follows:

(1) Choose a Weyl group element u with $\alpha_0^u = \beta$.
(2) Choose roots $\delta_1, \dots, \delta_m \in \Delta$ such that $u = \sigma(\delta_1)\cdots\sigma(\delta_m)$. (This is possible by Proposition 1.3.12.)
(3) Define $\theta_\beta(\lambda) := \theta_{\alpha_0}(\lambda)^{w_{\delta_1}\cdots w_{\delta_m}}$ for all $\lambda \in M$.

It is not clear that the map θ_β is independent of the choices that we make in (1) and (2). In fact, it is not. Still, this construction is a good starting point that we will improve upon in the following.

4.1.17 Remark (on Construction 4.1.16). If we could show that the definition of θ_β is independent of the choices made in Construction 4.1.16, then it would follow that

$$\theta_\beta(\lambda)^{w_\delta} = \left(\theta_{\alpha_0}(\lambda)^{w_{\delta_1}\cdots w_{\delta_m}}\right)^{w_\delta} = \theta_{\sigma_\delta(\beta)}(\lambda) \tag{4.4}$$

for all $\lambda \in M$ and all $\delta \in \Delta$, simply by the definition of $\theta_{\sigma_\delta(\beta)}$. This formula is clearly too much to ask for because it does not involve any twisting group. Instead, we will show that θ_β is only *almost* independent of the choices we made above: We can find a twisting group A which acts on M such that, if θ'_β is the root isomorphism defined by a different choice of u' and $\delta'_1, \dots, \delta'_{m'}$, then there exists $a \in A$ such that $\theta'_\beta(\lambda) = \theta_\beta(a.\lambda)$ for all $\lambda \in M$. In other words, θ_β is uniquely determined up to twisting. Even more, we can predict the precise value of a only from $\delta_1, \dots, \delta_m$ and $\delta'_1, \dots, \delta'_{m'}$ using a suitably chosen parity map. A precise first formulation of this statement will be given in Goal 4.1.21, which will be refined in Goal 4.1.26.

4.1.18 Note (Canonicity of the Weyl elements). In a first naive approach, we might have been tempted to fix a "full" system $(w_\gamma)_{\gamma\in\Phi}$ of Weyl elements and not merely a Δ-system $(w_\delta)_{\delta\in\Delta}$ of Weyl elements. However, recall from Summary 4.1.2 (3) that in the setting of Chevalley groups, formula (4.2) holds only for a certain "standard" family $(w_\alpha^{\mathrm{Chev}})_{\alpha\in\Phi}$ of Weyl elements. Further, we have seen in Remark 3.3.11 that $(w_\alpha^{\mathrm{Chev}})_{\alpha\in\Phi}$ is already uniquely determined by $(w_\delta^{\mathrm{Chev}})_{\delta\in\Delta}$. Thus we cannot (or should not) expect that any arbitrary choice of Weyl elements $(w_\gamma)_{\gamma\in\Phi}$ in G can be used to construct a parametrisation which satisfies the conjugation formula. However, it seems plausible that any Δ-system $(w_\delta)_{\delta\in\Delta}$ can be used to construct a parametrisation with respect to which it is "standard" (in the sense that the conjugation formula is satisfied). This is precisely what happens in the parametrisation theorem.

Before we can formulate Goal 4.1.21, we have to introduce some notation concerning parity maps.

4.1.19 Notation. Let $\eta\colon \Phi\times\Delta\to A$ be a parity map (with values in some abelian group A). We define $\eta(\alpha,-\delta) := \eta(\alpha^{\sigma(\delta)},\delta)^{-1}$ for all $\alpha\in\Phi$, $\delta\in\Delta$ and

$$\eta(\alpha,\bar{\delta}) := \prod_{i=1}^{m}\eta(\alpha^{\sigma(\delta_1\cdots\delta_{i-1})},\delta_i) = \eta(\alpha,\delta_1)\cdot\eta(\alpha^{\sigma(\delta_1)},\delta_2)\cdots\eta(\alpha^{\sigma(\delta_1\cdots\delta_{m-1})},\delta_m)$$

for any word $\bar{\delta} = (\delta_1,\dots,\delta_m)$ over $\Delta\cup(-\Delta)$. We use the convention that $\eta_{\alpha,\varnothing} = 1_A$ for the empty word $\varnothing$. We will also sometimes write $\eta_{\alpha,\bar{\delta}}$ in place of $\eta(\alpha,\bar{\delta})$.

4.1.20 Note. Let $(w_\delta)_{\delta\in\Delta}$ be any Δ-system of Weyl elements which is compatible with a family of root isomorphisms $(\theta_\alpha)_{\alpha\in\Phi}$ with respect to a parity map η (in the sense of Definition 4.1.8). Notation 4.1.19 is made precisely to ensure that the following generalisation of (4.2) holds:

$$\theta_\alpha(\lambda)^{w_{\bar{\delta}}} = \theta_{\sigma_{\bar{\delta}}(\alpha)}\big(\eta(\alpha,\bar{\delta}).\lambda\big)$$

for all roots α, all words $\bar{\delta}$ over $\Delta\cup(-\Delta)$ and all $\lambda\in M_\alpha$. Here $w_{\bar{\delta}}$ is defined as in Notation 2.2.25. A detailed proof of this fact can be found in Remark 4.3.5.

4.1.21 Goal. Find a twisting group A for G and a parity map $\eta\colon\Phi\times\Delta\to A$ such that for all words $\bar{\delta}$ and $\bar{\rho}$ over $\Delta\cup(-\Delta)$ with $\alpha_0^{\sigma(\bar{\delta})} = \alpha_0^{\sigma(\bar{\rho})}$, we have

$$x_{\alpha_0}^{w_{\bar{\delta}}} = a.(x_{\alpha_0}^{w_{\bar{\rho}}})$$

where $a := \eta(\alpha_0,\bar{\delta}')^{-1}\eta(\alpha_0,\bar{\delta})$.

We will see in Remark 4.1.25 that due to some technical difficulties, this goal is not well-formulated for the root systems of types B, C, BC and F_4. In Goal 4.1.26, we will propose a more refined and more general version.

We begin our pursuit of Goal 4.1.21 with an investigation of the twisting group. At first, we analyse the effect of the choice which is made in Construction 4.1.16 (2). The choice in Construction 4.1.16 (1) presents more technical difficulties, which we will address in Remark 4.1.25.

4.1.22 Remark (Construction of the twisting group, part 1). The construction of the twisting group A and its action on the root groups depend on the root system Φ, but the general approach is always the same. Recall that we will always have $A = \{\pm 1\}^p$ for some $p\in\{1,2,3\}$, so in each case, we merely have to determine the value of p and the action of each direct factor of A on the root groups.

Let $\bar{\delta}$ and $\bar{\rho}$ be two words over $\Delta\cup(-\Delta)$ such that $\sigma(\bar{\delta})$ and $\sigma(\bar{\rho})$ map α_0 to the same root. For the scope of this remark, we focus on step 4.1.16 (2), so we assume that $\sigma(\bar{\delta}) = \sigma(\bar{\rho})$. To avoid technical difficulties, we also assume that $\bar{\delta}$

and $\bar{\rho}$ do not contain letters from $-\Delta$. Then by the solution of the word problem in Weyl groups (Proposition 1.3.16), we know that $\bar{\rho}$ can be obtained from $\bar{\delta}$ by a sequence of elementary homotopy moves. Thus in order to understand the difference between the actions of $w_{\bar{\delta}}$ and $w_{\bar{\rho}}$, we have to understand the conjugation action of w_ζ^2 for all $\zeta \in \Delta$ and the difference between the conjugation actions of $P_m(w_\zeta, w_\rho)$ and $P_m(w_\rho, w_\zeta)$ for all $\rho, \zeta \in \Delta$ where m denotes the order of $\sigma_\zeta\sigma_\rho$. The latter is easy to understand: We know from Theorem 2.5.10 that Weyl elements satisfy the braid relations, so the actions of $P_m(w_\zeta, w_\rho)$ and $P_m(w_\rho, w_\zeta)$ agree. In fact, it would be sufficient in this situation to know that the braid relations are valid modulo the center.

Thus it remains to study the actions of squares of Weyl elements. These actions are not always trivial, but we will see that in nearly all cases, they are either trivial or the group inversion. Even more, we will show that the square formula for Weyl elements that we have seen in Note 3.3.12 for Chevalley groups is nearly always satisfied for root graded groups. For this reason, the twisting group A will always have one direct factor $\{\pm 1\}$ in which -1 acts on all root groups by group inversion. We adopt the convention that this is always the first factor of A.

We will also encounter some situations in which $p \geq 2$. In these situations, the action of the other direct factors of A on a root group U_α is defined via conjugation by $\bar{w}_\zeta^2$ where ζ is some root (which depends on α) and $\bar{w}_\zeta$ is any ζ-Weyl element. In all cases, we can show that this action is independent of the choice of $\bar{w}_\zeta$ and that we even have some freedom in choosing the root ζ. In particular, the definition of the twisting action is independent of the fixed Δ-system $(w_\delta)_{\delta\in\Delta}$ of Weyl elements.

4.1.23 Remark (Choice of a parity map: The sign problem). It is clear that the parity map η in Goal 4.1.21 cannot be chosen arbitrarily. For example, we will prove that for certain roots $\alpha \in \Phi$ and $\delta \in \Delta$, w_δ^2 acts on U_α by inversion, so $\eta_{\alpha,\delta\delta}$ must be an element of A which acts on U_α by inversion. More generally, $\eta_{\alpha,\delta\delta}$ must act on U_α in the same way as w_δ^2 for all $\alpha \in \Phi$ and $\delta \in \Delta$. In the language of Definition 4.4.3, this says precisely that G must be *square-compatible with respect to η and $(w_\delta)_{\delta\in\Delta}$*. Similarly, the validity of the braid relations shows that we should require $\eta(\alpha, P_m(\delta_1, \delta_2)) = \eta(\alpha, P_m(\delta_2, \delta_1))$ for all $\alpha \in \Phi$ and all $\delta_1, \delta_2 \in \Delta$ where m denotes the order of $\sigma(\delta_1)\sigma(\delta_2)$. This property will be called *braid-invariance (of η)*.

Generally speaking, we will derive several conditions on η which ensure that a "consistent" parametrisation with respect to η is possible. One might expect η to be uniquely determined by these properties, but this is not true. In fact, we will see in Remark 4.3.12 that as soon as we have found one adequate parity map η, we can obtain a large number of similar maps from η by "twisting η". Thus, for lack of an abstract existence result, we will have to explicitly determine a map $\eta\colon \Phi \times \Delta \to A$ satisfying all desired properties. We refer to the problem of finding such a map as the *sign problem (for root graded groups)*. When we specialise to Chevalley groups, it is precisely the sign problem that we have already seen in this context (see Note 3.3.8 and also Notes 3.1.25 and 3.2.23).

The best way to find a suitable parity map η is to "read it off" from an appropriate "model" H, that is, from a Φ-graded group H which is known to be coordinatised by some algebraic structure. For example, in the case of simply-laced root systems, we can take precisely the constants that appear in Chevalley groups (see Lemma 3.3.7). It is easy to verify that a parity map which is defined in this way automatically satisfies all the consistency conditions that we ask for (except for the condition of stabiliser-compatibility, which we will introduce in Remark 4.1.28). For example, it follows from Note 4.1.20 and the braid relations for Weyl elements that any parity map η which comes from a Φ-graded group H satisfies that $\eta(\alpha, P_m(\delta, \delta'))$ and $\eta(\alpha, P_m(\delta', \delta))$ act identically on U_α for all roots α and all $\delta, \delta' \in \Delta$. Under suitable faithfulness conditions, this implies that $\eta(\alpha, P_m(\delta, \delta')) = \eta(\alpha, P_m(\delta', \delta))$. A precise statement of the results of this kind is given in Lemma 4.4.10 and Proposition 4.4.11.

Of course, for the strategy from the previous paragraph to work, we have to require that the parity map η can actually be read off from the group H in a unique way. This essentially amounts to saying that the twisting group A acts faithfully on each root group of H. If this is not the case, then η is not uniquely determined by H. (In practice, we will have to use a slightly weaker condition than true faithfulness, see Note 4.3.9.) For example, if H is a Chevalley group defined over a ring $\mathcal{R}$ with $1_{\mathcal{R}} = -1_{\mathcal{R}}$, then we have

$$\theta_\alpha(\lambda)^{w_\delta} = \theta_{\sigma_\delta(\alpha)}(\lambda) = \theta_{\sigma_\delta(\alpha)}(-\lambda)$$

for all roots $\alpha \in \Phi$, $\delta \in \Delta$ and all $\lambda \in \mathcal{R}$. Thus we cannot read off $\eta(\alpha, \delta)$ from this specific group H. This example shows that the group H has to be *sufficiently generic*, which roughly means that "all twisting phenomena which may arise in general Φ-graded groups can be observed in H". The only type of twisting which occurs in Chevalley groups is additive inversion in the ring, so we can only use Chevalley groups of type Φ as a model if this is the only type of twisting which occurs in all Φ-graded groups. This is true for the simply-laced root systems, but not for the other root systems.

As a more complicated example, consider C_3-graded groups. Recall from Example 4.1.9 that there exist C_3-graded groups with twisting group $A = \{\pm 1\}^2$ where the second factor of A represents an involution σ of some associative ring $\mathcal{R}$. Thus in order to find a suitable parity map η, we need a C_3-graded group H in which all four elements of A act differently on the root groups. Specifically, this means that the maps $\mathrm{id}_{\mathcal{R}}$, $-\mathrm{id}_{\mathcal{R}}$, σ and $-\sigma$ on $\mathcal{R}$ have to be pairwise distinct. In other words, we have to require that $1_{\mathcal{R}} \neq -1_{\mathcal{R}}$ and $\sigma \neq \mathrm{id}_{\mathcal{R}}$ (because $\sigma = -\mathrm{id}_{\mathcal{R}}$ is not possible for a ring involution σ unless $1_{\mathcal{R}} = -1_{\mathcal{R}}$).

4.1.24 Note (Existence problem and coordinatisation problem). Recall from the preface of this thesis that the coordinatisation problem for Φ-graded groups consists of finding a class $\mathcal{C}_\Phi$ of algebraic objects such that each Φ-graded group is coordinatised (in a suitable sense) by an object in this class. The existence problem, on the other hand, consists of finding a Φ-graded group G_X which is coordinatised by X for any object X in $\mathcal{C}$. We have seen in Remark 4.1.23 that our approach to the coordinatisation problem is closely tied to the existence problem: Before we can coordinatise *all* Φ-graded groups, we have to construct *one* sufficiently generic Φ-graded group H.

It is interesting to note that the group H does not have to be as generic as the class $\mathcal{C}_\Phi$ allows, only generic enough to read off the parity map. For example, general C_3-graded groups are coordinatised by alternative rings $\mathcal{R}$ with involution which need not be associative (and by some more data which is irrelevant for this discussion), but the associativity of $\mathcal{R}$ plays no role for the twisting group actions. Thus we can read off the parity map from a group which is defined over an associative ring. This means that our solution of the coordinatisation problem is independent of the general solution of the existence problem. This will be particularly relevant for F_4-graded groups because no complete solution of the existence problem is available for these groups, but we can still construct groups which are generic enough to supply a suitable parity map.

Finally, we turn to the choice in Construction 4.1.16 (1), which causes more technical problems.

4.1.25 Remark (Construction of the twisting group, part 2). Let α_0 be the fixed root in Construction 4.1.16 and let u, v be elements of the Weyl group such that $\alpha_0^u = \alpha_0^v$. Then vu^{-1} stabilises α_0, which by Proposition 1.3.11 means that it is a product $\sigma(\beta_1) \cdots \sigma(\beta_q)$ where $\beta_1, \dots, \beta_q$ are roots orthogonal to α_0. Thus in order to understand the difference between the actions of $w_{\bar{\delta}}$ and $w_{\bar{\delta}'}$, we have to understand the conjugation action of Weyl elements w_β on U_{α_0} for roots β which are orthogonal to α_0.

Our first impulse is to proceed similarly as in Remark 4.1.22: For every root α, choose a root $\beta(\alpha)$ which is orthogonal to α and a $\beta(\alpha)$-Weyl element $\bar{w}_{\beta(\alpha)}$ and declare that (some direct factor of) A acts on U_α through conjugation by $\bar{w}_{\beta(\alpha)}$. However, unlike in Remark 4.1.22, this definition is not independent of the choices of $\beta(\alpha)$ and $\bar{w}_{\beta(\alpha)}$, which renders it unfeasible.

Since the well-definition of the "orthogonal twisting actions" causes problems, we simply take the approach of not defining them at all (for the beginning) and trying to proceed without them. As a consequence, we split the twisting group into two parts: one part A which is a twisting group as before, equipped with the actions from Remark 4.1.23, and one part B which is simply an abstract group without any actions. This leads to Goal 4.1.26.

4.1.26 Goal (Refinement of Goal 4.1.21). Find a twisting group A for G, an abelian group B and parity maps $\eta\colon \Phi \times \Delta \to A$, $\mu\colon \Phi \times \Delta \to B$ such that for all words $\bar{\delta}$ and $\bar{\rho}$ over $\Delta \cup (-\Delta)$ with $\alpha_0^{\sigma(\bar{\delta})} = \alpha_0^{\sigma(\bar{\rho})}$ and $\mu(\alpha_0, \bar{\delta}) = \mu(\alpha_0, \bar{\rho})$, we have

$$x_{\alpha_0}^{w_{\bar{\delta}}} = a.(x_{\alpha_0}^{w_{\bar{\rho}}})$$

where $a := \eta(\alpha_0, \bar{\delta})\eta(\alpha_0, \bar{\rho})^{-1}$.

4.1.27 Remark (Twisting systems). The tuple (A, η, B, μ) in Goal 4.1.26 is called a *partial twisting system for G* because only the first part is equipped with twisting actions on the root groups. See Definition 4.3.18 for a proper definition of partial twisting system which includes all the necessary technical requirements.

For root system of types A, D, E and H, any pair (α, β) of orthogonal roots is automatically adjacent and thus every β-Weyl element acts trivially on U_α. This will allow us to take $B = \{1\}$ for these root systems, thereby simplifying the theory. A partial twisting system (A, η, B, μ) for which B (and, as a consequence, μ) is trivial will be called a *twisting system*. If this is the case, we drop B and μ from the notation and simply refer to (A, η) as a twisting system.

As soon as we have proven the statement in Goal 4.1.26, we can easily define an action of B on each root group, thereby turning $A \times B$ into a twisting group. In other words, the statement in Goal 4.1.26 allows us to turn the partial twisting system (A, η, B, μ) into a twisting system $(A \times B, \eta \times \mu)$.

4.1.28 Remark (on the proof of 4.1.26). The flexibility of not having to define an action of B on the root groups from the beginning comes with a price: the parity map μ has to satisfy some additional properties. In general, these properties are not satisfied for the map η. This illustrates that the conditions in the parametrisation theorem are carefully balanced against each other.

The first necessary condition on μ is *square-invariance*, which means that $\mu(\alpha, \delta\delta) = 1_B$ for all $\alpha \in \Phi$ and $\delta \in \Delta$. Together with braid-invariance (see Remark 4.1.23), this property implies that for all $\alpha \in \Phi$ and all words $\bar{\delta}$ over $\Delta \cup (-\Delta)$, the value of $\mu(\alpha, \bar{\delta})$ depends only on $\sigma(\bar{\delta})$.

Secondly, we have to require a somewhat technical property called *stabiliser compatibility of G with respect to (η, μ) and $(w_\delta)_{\delta \in \Delta}$*. This property roughly says that any $u \in \mathrm{Weyl}(\Phi)$ which stabilises some root α and which satisfies $\mu(\alpha, u) = 1_B$ "behaves tamely". It is specifically tailored to make a proof of Goal 4.1.26 possible, so we have to work some more to verify it for each root system Φ. In fact, it is usually the most difficult property to verify because it is not a consequence of the way the parity map μ is defined. See Remark 4.1.23 and also Note 4.4.2 for more technical details. In every case, Proposition 1.3.11 will play a crucial role.

4.1.29 Remark (4.1.26 for simply-laced root systems). The simply-laced root systems have the property that every pair of orthogonal roots is adjacent. Thus whenever α, β are roots such that $\sigma_\alpha(\beta) = \beta$, the root groups U_α and U_β commute and hence every α-Weyl element acts trivially on U_β. As a consequence, the property of stabiliser compatibility is trivially satisfied in this situation. See Lemma 4.6.4 for a precise formulation of this statement.

Using this simplification and Shi's more restrictive definition of root gradings (2.7.6), Shi proves in [Shi93, (2.11)] that the assertion of Goal 4.1.21 holds for the specific Weyl elements $\varphi(\hat{w}_\alpha(1_k))$ in Axiom 2.7.6 (vi) and for simply-laced root systems. His proof uses in a significant way that these Weyl elements come from the homomorphic image of a Steinberg group, which makes it more concise but less general than our proof of the parametrisation theorem. In particular, the role of the parity map η (and thus of the sign problem) is not as prominent in Shi's work because any Φ-graded group in Shi's sense comes, by definition, with a family $c = (c_{\alpha,\beta})_{\alpha,\beta \in \Phi}$ of Chevalley structure constants which determine the parity map. However, the main ideas in Shi's approach are similar: One has to understand the action of squares of Weyl elements and the actions of Weyl elements on orthogonal root groups (which is trivial in the simply-laced setting).

4.1.30 Note (The rank-2 case). Assume that Φ is irreducible crystallographic of rank 2 and choose a positive system Π in Φ. Recall that irreducible RGD-systems of type Φ were classified by Tits-Weiss in [TW02]. More precisely, they classify *root group sequences*. Here root group sequences are pairs $(U_\Pi, (U_\alpha)_{\alpha \in \Pi})$ consisting of a group U_Π and a family $(U_\alpha)_{\alpha \in \Pi}$ satisfying some axioms. For any RGD-system $(G, (U_\alpha)_{\alpha \in \Phi})$, the pair $(U_\Pi, (U_\alpha)_{\alpha \in \Pi})$ is a root group sequence, and any RGD-system is essentially determined by its root group sequence. For more details, see (8.7), (8.10) and (8.11).

By inspecting the classification of irreducible crystallographic root systems of rank 2, we see that Φ has at most 16 elements, so Π has at most 8 elements. This low number of roots allows Tits-Weiss to parametrise the root groups "by hand", without using a machinery similar to the one in the parametrisation theorem. The same is true for [Fau14, 13.7] in which Faulkner parametrises A_2-graded groups.

4.1.31 Remark (Rescaled root bases and type BC). Recall from Notation 4.1.1 that Δ is not necessarily a proper root base of Φ but only a rescaled root base (in the sense of Definition 1.2.22). For reduced root systems, there is no difference between these notions, so we assume that $\Phi = BC_n$ for some $n \in \mathbb{N}_{\geq 3}$. In the standard representation of BC_n (see Remark 9.1.2), the standard root base of BC_n is

$$\Delta' = \{ e_i - e_{i+1} \mid i \in [1, n-1] \} \cup \{ e_n \}$$

and the only rescaled non-proper root base which can be obtained by rescaling Δ' is

$$\Delta = \{ e_i - e_{i+1} \mid i \in [1, n-1] \} \cup \{ 2e_n \}.$$

The only reason for fixing a rescaled root base is that we want to fix a corresponding system of Weyl elements, so the question is: Should we fix an e_n-Weyl element or a $2e_n$-Weyl element? Since every $2e_n$-Weyl element is also an e_n-Weyl element, every statement which can be proven for a fixed Δ'-system of Weyl elements can also be proven for a Δ-system of Weyl elements. However, a Δ-system of Weyl elements might allow us to prove more. In other words, if we chose a Δ'-system, then we would ignore the information that $U_\delta^\sharp$ is non-empty for all long roots. Therefore, we should prefer Δ over Δ'. This is the reason why many statements concerning arbitrary root systems – the preliminaries in chapter 1, the parametrisation theorem and the blueprint technique – are proven for rescaled root bases not merely for root bases.

4.1.32 Note. We can now give an overview of the sections in this chapter. In section 4.2, we introduce and study parity maps. This will be done on a purely combinatorial level, without reference to any Φ-graded group or any action on root groups. In section 4.3, we define parameter systems, twisting groups and (partial) twisting systems. In section 4.4, we study the interplay between the notions from the previous two sections in the form of compatibility conditions. The parametrisation theorem is proven in section 4.5. In section 4.6, we collect some criteria which will be used in later chapters to prove the compatibility conditions for specific partial twisting systems. In section 4.7, we end with some remarks on how we will apply the parametrisation theorem in the subsequent chapters.

4.2 Parity Maps

4.2.1 Notation for this section. In this section, Φ is a root system and Δ is a rescaled root base of Φ.

In this section, we study the purely combinatorial notion of parity maps, without reference to some Φ-graded group.

4.2.2 Definition (Δ-expressions). For any word $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$ over $\Delta \cup (-\Delta)$, we define the corresponding *inverse word* by $\bar{\alpha}^{-1} := (-\alpha_m, \dots, -\alpha_1)$. For any root α, a *Δ-expression of* α is a word over $\Delta \cup (-\Delta)$ of the form $(\bar{\delta}^{-1}, \delta, \bar{\delta})$ where $\delta \in \Delta$ and $\bar{\delta}$ is a word over $\Delta \cup (-\Delta)$ such that $\delta^{\sigma(\bar{\delta})} = \alpha$.

Note that a Δ-expression of the root α is automatically a Δ-expression of the element $\sigma(\alpha)$ in the sense of Definition 1.3.20.

4.2.3 Remark. Let G be a group with a Φ-pregrading $(U_\alpha)_{\alpha\in\Phi}$, let $(w_\delta)_{\delta\in\Delta}$ be a Δ-system of Weyl elements and let α be a root. Then by Proposition 2.2.6 (b), $w_{\bar{\rho}}$ is an α-Weyl element for any Δ-expression $\bar{\rho}$ of α. Further, $w_{\bar{\alpha}^{-1}} = w_{\bar{\alpha}}^{-1}$ for any word $\bar{\alpha}$ over $\Delta \cup (-\Delta)$. Further, note that we have already seen Δ-expressions in Proposition 1.3.5.

4.2.4 Definition (Parity map). Let A be an abelian group. A *Δ-parity map with values in A* is a map $\eta\colon \Phi \times \Delta \to A, (\alpha, \delta) \mapsto \eta_{\alpha,\delta} = \eta(\alpha, \delta)$ with the property that

$\eta_{\alpha,\delta} = \eta_{\lambda\alpha,\delta}$ whenever λ is a positive real number such that α and $\lambda\alpha$ are roots. Given any such map, we define $\eta(\alpha, -\delta) := \eta(\alpha^{\sigma(\delta)}, \delta)^{-1}$ for all $\alpha \in \Phi$, $\delta \in \Delta$ and

$$\eta_{\alpha,\bar{\delta}} := \prod_{i=1}^{m} \eta(\alpha^{\sigma(\delta_1\cdots\delta_{i-1})}, \delta_i) = \eta(\alpha, \delta_1) \cdot \eta(\alpha^{\sigma(\delta_1)}, \delta_2) \cdots \eta(\alpha^{\sigma(\delta_1\cdots\delta_{m-1})}, \delta_m)$$

for any word $\bar{\delta} = (\delta_1, \ldots, \delta_m)$ over $\Delta \cup (-\Delta)$. We use the convention that $\eta_{\alpha,\varnothing} = 1_A$ for the empty word $\varnothing$. This defines a map

$$\eta\colon \Phi \times \mathcal{L}(\Delta \cup (-\Delta)) \to A$$

which we will sometimes call the *extended* Δ*-parity map*. Here $\mathcal{L}(\Delta \cup (-\Delta))$ denotes the free monoid over the set $\Delta \cup (-\Delta)$, see Definition 1.1.6. When Δ and A are clear from the context, we will simply say that η is a *parity map*.

4.2.5 Example. Assume that Φ is crystallographic and reduced. Then we have a Chevalley parity map $\eta\colon \Phi \times \Phi \to \{\pm 1\}$ from Definition 3.3.10. Its restriction to $\Phi \times \Delta$ is a Δ-parity map with values in $\{\pm 1\}$.

We begin with some easy consequences of Definition 4.2.4.

4.2.6 Remark. Let η be a parity map with values in some abelian group A and let $\alpha \in \Phi$, $\delta \in \Delta$. Then

$$\eta(\alpha^{\sigma(\delta)}, -\delta) = \eta(\alpha^{\sigma(\delta)\sigma(\delta)}, \delta)^{-1} = \eta(\alpha, \delta)^{-1}.$$

This implies that

$$\eta_{\alpha,(\delta,-\delta)} = \eta(\alpha, \delta)\eta(\alpha^{\sigma(\delta)}, -\delta) = \eta(\alpha, \delta)\eta(\alpha^{\sigma(\delta)\sigma(\delta)}, \delta)^{-1} = 1_A$$

and that, similarly, $\eta_{\alpha,(-\delta,\delta)} = 1_A$. Thus for any word $\bar{\delta}$ over $\Delta \cup (-\Delta)$, we can delete any occurence of $(\delta, -\delta)$ for any $\delta \in \Delta \cup (-\Delta)$ without changing the value of $\eta_{\beta,\bar{\delta}}$ for any $\beta \in \Phi$. In other words, the extended parity map $\Phi \times \mathcal{L}(\Delta \cup (-\Delta)) \to A$ defined by η factors through a map $\Phi \times F(\Delta) \to A$ where $F(\Delta)$ denotes the free group on Δ. Further, since $\eta_{\alpha,-(-\delta)} = \eta_{\alpha,\delta} = \eta(\alpha^{\sigma(\delta)}, -\delta)^{-1}$, we see that $\eta(\alpha, -\rho) = \eta(\alpha^{\sigma(\rho)}, \rho)^{-1}$ holds for all $\rho \in \Delta \cup (-\Delta)$ (and not merely for all $\rho \in \Delta$).

4.2.7 Lemma. *Let η be a parity map with values in some abelian group A, let $\alpha \in \Phi$ and let $\bar{\beta} = (\beta_1, \ldots, \beta_m)$ be a word over $\Delta \cup (-\Delta)$ where $m \geq 1$. Then $\eta(\alpha^{\sigma(\bar{\beta})}, \bar{\beta}^{-1}) = \eta(\alpha, \bar{\beta})^{-1}$. In particular, $\eta_{\alpha,\bar{\beta}^{-1}} = \eta_{\alpha,\bar{\beta}}^{-1}$ if $\alpha^{\sigma(\bar{\beta})} = \alpha$.*

Proof. Write $\bar{\beta}^{-1} = (\beta'_1, \ldots, \beta'_m) = (-\beta_m, \ldots, -\beta_1)$. On the one hand,

$$\eta_{\alpha,\bar{\beta}}^{-1} = \left(\prod_{i=1}^{m} \eta(\alpha^{\sigma(\beta_1\cdots\beta_{i-1})}, \beta_i) \right)^{-1} = \prod_{i=1}^{m} \eta(\alpha^{\sigma(\beta_1\cdots\beta_{i-1})}, \beta_i)^{-1}$$

(with the convention that $\sigma(\beta_1 \cdots \beta_{i-1}) = \mathrm{id}_\Phi$ for $i = 1$). On the other hand,

$$\eta(\alpha^{\sigma(\bar{\beta})}, \bar{\beta}^{-1}) = \prod_{i=1}^{m} \eta(\alpha^{\sigma(\bar{\beta})\sigma(\beta'_1\cdots\beta'_{i-1})}, \beta'_i) = \prod_{i=1}^{m} \eta(\alpha^{\sigma(\beta_1\cdots\beta_{m+1-i})}, -\beta_{m+1-i}).$$

Performing the index shift $j := m + 1 - i$, we infer that

$$\eta(\alpha^{\sigma(\bar{\beta})}, \bar{\beta}^{-1}) = \prod_{j=1}^{m} \eta(\alpha^{\sigma(\beta_1\cdots\beta_j)}, -\beta_j).$$

By Remark 4.2.6, we have $\eta(\alpha^{\sigma(\beta_1\cdots\beta_j)}, -\beta_j) = \eta(\alpha^{\sigma(\beta_1\cdots\beta_{j-1})}, \beta_j)^{-1}$ for all $j \in \{1, \ldots, m\}$, so it follows that $\eta(\alpha^{\sigma(\bar{\beta})}, \bar{\beta}^{-1}) = \eta(\alpha, \bar{\beta})^{-1}$. $\square$

4.2.8 Lemma. *Let η be a parity map with values in some abelian group A, let α be a root and let $\bar{\delta}, \bar{\rho}$ be words over $\Delta \cup (-\Delta)$ such that $\alpha^{\sigma(\bar{\delta})} = \alpha$. Then $\eta_{\alpha^{\sigma(\bar{\rho})}, \bar{\rho}^{-1}\bar{\delta}\bar{\rho}} = \eta_{\alpha,\bar{\delta}}$.*

Proof. By induction on the length of $\bar{\rho}$, we can assume that the word $\bar{\rho}$ consists of a single root $\rho \in \Delta \cup (-\Delta)$. Then

$$\eta_{\alpha^{\sigma(\rho)},(-\rho,\bar{\delta},\rho)} = \eta_{\alpha^{\sigma(\rho)},-\rho}\eta_{\alpha,\bar{\delta}}\eta_{\alpha^{\sigma(\bar{\delta})},\rho}.$$

Since $\eta_{\alpha^{\sigma(\rho)},-\rho} = \eta_{\alpha,\rho}^{-1}$ Remark 4.2.6 and since $\alpha^{\sigma(\bar{\delta})} = \alpha$, the assertion follows. □

We now define some properties of parity maps that will be needed throughout this chapter.

4.2.9 Note. Every property X that we define in this chapter will be of the form "α-X holds for all $\alpha \in \Phi$" where α-X is another property which is defined for every root α. However, we will only define the properties α-X for those X for which this terminology will be needed later, and simply define X directly in all other cases.

4.2.10 Definition. Let A be an abelian group and let η be a Δ-parity map with values in A.

(a) We say that η is *trivial* if $\eta_{\alpha,\delta} = 1$ for all $\alpha \in \Phi$ and $\delta \in \Delta$.

(b) For any root γ, we say that η is *γ-braid-invariant* if for all $\alpha, \beta \in \Delta$, we have $\eta(\gamma, P_m(\alpha,\beta)) = \eta(\gamma, P_m(\beta,\alpha))$ where m denotes the order of $\sigma_\alpha\sigma_\beta$ in the Weyl group. We say that η is *braid-invariant* if it is γ-braid-invariant for all $\gamma \in \Phi$.

(c) We say that η is *square-invariant* if for all $\alpha \in \Phi, \delta \in \Delta$, we have $\eta_{\alpha,\delta\delta} = 1_A$.

(d) We say that η is *Weyl-invariant* if it is braid- and square-invariant.

(e) For any root α, we say that η *satisfies the square formula for α* if $A = \{\pm 1\}^p$ for some $p \in \mathbb{N}_+$ and $\eta_{\alpha,\delta\delta} = (-1,1,\dots,1)^{\langle \alpha | \delta \rangle}$ for all $\delta \in \Delta$. Further, we say that η *satisfies the square formula* if it satisfies the square formula for all $\alpha \in \Phi$.

(f) For any root $\alpha \in \Phi$, we say that η is *α-adjacency-trivial* if for any $\beta \in \Phi$ such that α is adjacent to β and $-\beta$ and for any Δ-expression $\bar{\rho}$ of β, we have $\eta_{\alpha,\bar{\rho}} = 1_A$. We say that η is *crystallographically α-adjacency-trivial* if the same property holds with "adjacent" replaced by "crystallographically adjacent". Further, we say that η is *(crystallographically) adjacency-trivial* if it is (crystallographically) α-adjacency-trivial for all roots α.

4.2.11 Remark. Recall from Note 1.2.33 that adjacency implies crystallographic adjacency. It follows that crystallographic adjacency-triviality implies adjacency-triviality.

4.2.12 Note. Braid-invariance and square-invariance are the most essential properties in Definition 4.2.10. The square formula (which we have already seen, in a different form, for Chevalley groups in Note 3.3.12) is merely a useful tool for proving square compatibility for groups which satisfy the square formula for Weyl elements, see Definitions 4.4.3 and 4.4.4. In a similar way, adjacency-triviality is merely a tool for proving stabiliser compatibility in certain cases, see Definition 4.4.5, Proposition 4.6.3, and Lemma 4.6.4. Both the square formula and adjacency-triviality have the useful property that they can be phrased purely in terms of parity maps, without reference to a specific root graded group. We will elaborate on this in Note 4.4.2.

4.2.13 Remark (Weyl-invariant parity maps). Let η be a square-invariant parity map with values in some abelian group A. Then for all $\alpha \in \Phi$ and $\delta \in \Delta$, we have

$$\eta_{\alpha,-\delta} = \eta_{\alpha,-\delta}\eta_{\alpha^{\sigma(-\delta)},\delta\delta} = \eta_{\alpha,(-\delta,\delta,\delta)}.$$

Applying Remark 4.2.6, we infer that $\eta_{\alpha,-\delta} = \eta_{\alpha,\delta}$ for all $\alpha \in \Phi$ and $\delta \in \Delta$. Now assume that η is, in addition, braid-invariant (so that it is Weyl-invariant). By the solution of the word problem in Coxeter groups (Proposition 1.3.16), we then have that $\eta_{\alpha,\bar{\beta}}$ depends only on α and $\sigma(\bar{\beta})$ but not on the representation $\bar{\beta}$ of $\sigma(\bar{\beta})$, which justifies the name "Weyl-invariant". Thus in this case, we can and will regard η as a map defined on $\Phi \times \mathrm{Weyl}(\Phi)$.

The notion of transporter sets has not appeared in the motivational section 4.1 because it is of a rather technical nature. It will become important when we define twisting actions of the group B in a partial twisting system (A, η, B, μ) on the root groups. See Remark 4.1.27 for more details.

4.2.14 Definition (Transporter sets). Let A be an abelian group and let η be a Δ-parity map with values in A. For any roots α, β which lie in the same orbit under the Weyl group, the set

$$A_{\eta,\alpha\to\beta} := \{\, \eta_{\alpha,\bar{\delta}} \mid \bar{\delta} \text{ word over } \Delta \cup (-\Delta) \text{ with } \alpha^{\sigma(\bar{\delta})} = \beta \,\} \subseteq A$$

is called the *transporter set for* (α, β) *with respect to* η. If η is clear from the context, we will usually write $A_{\alpha\to\beta}$ for $A_{\eta,\alpha\to\beta}$.

4.2.15 Remark. Let η be a parity map with values in some abelian group A. Then for all roots α, β, γ which lie in the same orbit as α under the Weyl group, we have $A_{\alpha\to\beta}A_{\beta\to\gamma} \subseteq A_{\alpha\to\gamma}$, $A_{\alpha\to\beta}^{-1} = A_{\beta\to\alpha}$ and $1_A \in A_{\alpha\to\alpha}$. In particular, $A_{\alpha\to\beta}$ is not necessarily a subgroup of A, but $A_{\alpha\to\alpha}$ always is.

In practice, we are only interested in parity maps for which all transporter sets within any orbit are the same (though different orbits may have different transporter sets). The following Lemma 4.2.16 provides some (seemingly weaker) criteria from which this property follows. Here we always denote by $\hat{\alpha}$, $\hat{\beta}$ roots which are prefixed with an existential quantifier and by α, β roots which are prefixed with a universal quantifier.

4.2.16 Lemma. *Let η be a parity map with values in some abelian group A and let O be an orbit of Φ under the Weyl group. Then the following properties are equivalent:*

(i) There exists $\hat{\alpha} \in O$ such that for all $\beta \in O$, we have $1_A \in A_{\hat{\alpha}\to\beta}$.

(ii) There exists $\hat{\alpha} \in O$ such that for all $\beta \in O$, we have $1_A \in A_{\beta\to\hat{\alpha}}$.

(iii) There exist $\hat{\alpha}, \hat{\beta} \in O$ such that $A_{\hat{\alpha}\to\hat{\beta}}$ is a subgroup of A which contains $A_{\hat{\beta}\to\beta}$ for all $\beta \in O$.

(iv) There exist $\hat{\alpha}, \hat{\beta} \in O$ such that $A_{\hat{\alpha}\to\hat{\beta}}$ is a subgroup of A which contains $A_{\alpha\to\beta}$ for all $\alpha, \beta \in O$.

(v) For all $\alpha, \beta, \gamma, \delta \in O$, we have $A_{\alpha\to\beta} = A_{\gamma\to\delta}$.

Proof. The implications "(v) $\implies$ (iv) $\implies$ (iii)" and "(i) $\iff$ (ii)" are clear. Now assume that (iii) holds, and choose $\hat{\alpha}$, $\hat{\beta}$ as in the assertion. We want to prove (i). For all $\beta \in O$, we have

$$A_{\hat{\alpha}\to\hat{\beta}}A_{\hat{\beta}\to\beta} \subseteq A_{\hat{\alpha}\to\beta}$$

by Remark 4.2.15. By assumption, $A_{\hat{\alpha}\to\hat{\beta}}$ contains $A_{\hat{\beta}\to\beta}$. Since it is a subgroup of A, it also contains $A_{\hat{\beta}\to\beta}^{-1}$. Thus the inclusion above implies that 1_A is contained in $A_{\hat{\alpha}\to\beta}$ for all $\beta \in O$, which proves (i).

Now assume that (i) holds. We want to prove (v). Let $\hat{\alpha}$ be as in (i). Note that property (ii) is also satisfied for $\hat{\alpha}$, too, because $A_{\beta\to\hat{\alpha}} = A_{\hat{\alpha}\to\beta}^{-1}$. Further, for all $\beta \in O$, it follows from Remark 4.2.15 that

$$A_{\hat{\alpha}\to\hat{\alpha}}A_{\hat{\alpha}\to\beta} \subseteq A_{\hat{\alpha}\to\beta} \quad \text{and} \quad A_{\hat{\alpha}\to\beta}A_{\beta\to\hat{\alpha}} \subseteq A_{\hat{\alpha}\to\hat{\alpha}}.$$

Since $A_{\hat{\alpha}\to\beta}$ and $A_{\beta\to\hat{\alpha}}$ both contain 1_A, we infer that $A_{\hat{\alpha}\to\hat{\alpha}} \subseteq A_{\hat{\alpha}\to\beta} \subseteq A_{\hat{\alpha}\to\hat{\alpha}}$, so $A_{\hat{\alpha}\to\hat{\alpha}} = A_{\hat{\alpha}\to\beta}$ for all $\beta \in O$. In particular, $A_{\hat{\alpha}\to\beta}$ is a group, so $A_{\beta\to\hat{\alpha}} = A_{\hat{\alpha}\to\beta}^{-1} = A_{\hat{\alpha}\to\beta} = A_{\hat{\alpha}\to\hat{\alpha}}$ for all $\beta \in O$ as well. Now for arbitrary β, γ, we have

$$A_{\beta\to\hat{\alpha}}A_{\hat{\alpha}\to\gamma} \subseteq A_{\beta\to\gamma} \quad \text{and} \quad A_{\hat{\alpha}\to\beta}A_{\beta\to\gamma} \subseteq A_{\hat{\alpha}\to\gamma}$$

where 1_A lies in $A_{\beta\to\hat{\alpha}}$ and in $A_{\hat{\alpha}\to\beta}$. By similar arguments as above, it follows that $A_{\hat{\alpha}\to\gamma} = A_{\beta\to\gamma}$ for all $\beta, \gamma \in O$. Since we already know that $A_{\hat{\alpha}\to\gamma} = A_{\hat{\alpha}\to\hat{\alpha}}$, we conclude that all the groups $(A_{\beta\to\gamma})_{\beta,\gamma\in O}$ are equal to $A_{\hat{\alpha}\to\hat{\alpha}}$. In particular, they are pairwise equal. This finishes the proof of (v). □

4.2.17 Remark. Note that property 4.2.16 (iv) is automatically satisfied if $A_{\hat{\alpha}\to\hat{\beta}} = A$ for some $\hat{\alpha}, \hat{\beta} \in O$. In this case, we have $A_{\alpha\to\beta} = A$ for all $\alpha, \beta \in O$.

4.2.18 Definition (Transporter properties). Let A be an abelian group and let η be a Δ-parity map with values in A.

(a) We say that η is *complete* if $A_{\alpha\to\alpha} = A$ for all roots α.
(b) We say that η is *semi-complete* if for all roots α, the group $A_{\alpha\to\alpha}$ has a group-theoretic complement in A. That is, there exists a subgroup C_α of A such that $A_{\alpha\to\alpha} \cap C_\alpha = \{1\}$ and $A_{\alpha\to\alpha}C_\alpha = A$.
(c) For any orbit O of Φ under the Weyl group, we say that η is *transporter-invariant on O* if it satisfies the equivalent conditions in Lemma 4.2.16 for O. Further, we say that η is *transporter-invariant* if it is transporter-invariant on all orbits of Φ.

4.2.19 Remark. Let η be a transporter-invariant Δ-parity map with values in some abelian group A and let α, β be roots which lie in the same orbit. Since $A_{\alpha\to\beta} = A_{\alpha\to\alpha}$, it follows from Remark 4.2.15 that 1_A is contained in $A_{\alpha\to\beta}$.

4.2.20 Note. Clearly, any complete parity map is also transporter-invariant and semi-complete. In practice, we would like our parity maps to be complete, but there will be situations in which this is not the case. We delay an explanation of this fact until Note 4.3.11. There we will also sketch an alternative approach which would allow us to only consider complete parity maps, but which introduces other difficulties.

The notion of semi-completeness is particularly strange. It will be used in Definition 4.5.9 to construct an action of the group B which appears in a partial twisting system (A, η, B, μ) on each of the root groups. It will be clear how this action should be defined on the transporter sets $B_{\alpha\to\alpha}$. The existence of a complement C_α of $B_{\alpha\to\alpha}$ allows us to extend this action to B by simply declaring that C_α acts trivially on all root groups. In all cases that we consider, the twisting group will be of the form $A = \{\pm 1\}^p$ for some $p \in \{1, 2, 3\}$ and every transporter set in A will simply be the product of some of the p components of A. In this situation, it is clear that we can simply choose the complement to be the product of the remaining components.

We have discussed in Remark 4.1.27 that we want to turn a partial twisting system (A, η, B, μ) into a twisting group $A \times B$ with a parity map $\eta \times \mu$. To this end, we briefly state some properties of products of parity maps for later use.

4.2.21 Notation. Let η, μ be two parity maps with values in some abelian groups A, B, respectively. Then we denote by $\eta \times \mu$ the parity map

$$\Phi \times \Delta \to A \times B, (\alpha, \beta) \mapsto (\eta_{\alpha,\beta}, \mu_{\alpha,\beta}).$$

4.2.22 Remark. If A, B are two abelian groups and $\bar{\eta}$ is a parity map with values in $A \times B$, then there exist unique parity maps η, μ with values in A, B, respectively, such that $\bar{\eta} = \eta \times \mu$.

4.2.23 Definition (Independent parity maps). Let A, B be two abelian groups and let η, μ be parity maps with values in A, B, respectively. The parity maps η and μ are called *independent* if $(A \times B)_{\eta\times\mu,\alpha\to\alpha} = A_{\eta,\alpha\to\alpha} \times B_{\mu,\alpha\to\alpha}$ for all $\alpha \in \Phi$.

4.2.24 Remark. Let η_1, η_2 be two parity maps with values in respective abelian groups A_1 and A_2. Denote by π_1 and π_2 the canonical projection homomorphisms from $A_1 \times A_2$ onto A_1 and A_2, respectively. Then for all roots α, β which lie in the same orbit under the Weyl group and for all $i \in \{1, 2\}$, we have

$$\pi_i\big((A_1 \times A_2)_{\eta_1\times\eta_2,\alpha\to\beta}\big) \subseteq (A_i)_{\eta_i,\alpha\to\beta}.$$

In particular, if $(A \times B)_{\eta\times\mu,\alpha\to\alpha}$ contains $A_{\eta,\alpha\to\alpha}$ and $B_{\mu,\alpha\to\alpha}$ for all roots α, then η and μ are independent.

4.2.25 Lemma. *Let η, μ be two parity maps with values in respective abelian groups A and B, let α be any root and let O be any orbit in Φ.*

(a) For all the following properties, it is true that $\eta \times \mu$ has this property if and only if η and μ both have this property: square-invariance, (α-) braid-invariance, Weyl-invariance, (crystallographic) (α-) adjacency-triviality.

(b) For all the following properties, it is true that η and μ both have this property if $\eta \times \mu$ has this property: completeness, transporter-invariance (on O).

Proof. The assertions of (a) are easy to verify. The assertions of (b) follow from Remark 4.2.24, using criterion 4.2.16 (i) to prove transporter-invariance. □

4.3 Parametrisations and Twisting Structures

4.3.1 Notation for this section. In this section, Φ is a root system an G is a group with a Φ-pregrading $(U_\alpha)_{\alpha\in\Phi}$. Further, we fix a rescaled root base Δ of Φ and we assume that there exists a Δ-system $(w_\delta)_{\delta\in\Delta}$ of Weyl elements in G, which we also fix.

In this section, we study the interplay of parity maps with root graded groups. We begin with the notion of parametrisations.

4.3.2 Definition (Parameter system). Write $\mathrm{Orb}^{\mathrm{red}}(\Phi) = (O_1, \dots, O_k)$ (see Notation 1.3.9). A *parameter system (of type Φ)* is a tuple $\mathcal{P} = (A, M_1, \dots, M_k)$ consisting of an abelian group A and groups $M_1, \dots, M_k$ such that A acts on each of the sets $M_1, \dots, M_k$. The group A is called the *twisting group of $\mathcal{P}$* and the actions of A on $M_1, \dots, M_k$ are called the *twisting actions*. Given a parameter system $\mathcal{P} = (A, M_1, \dots, M_k)$ and a root $\alpha \in \Phi$, we denote by M_α the group $M_{i(\alpha)}$ where $i(\alpha) \in [1, k]$ is the unique index such that $\alpha \in O_{i(\alpha)}$.

4.3.3 Note. The root system Φ appears in the definition of parameter systems of type Φ only insofar that the number k of groups $M_1, \dots, M_k$ is the number of indivisible orbits in Φ. However, the definition of the groups $(M_\alpha)_{\alpha\in\Phi}$ clearly depends on Φ. For this reason, we use the terminology of "parameter systems of type Φ".

Further, observe that we do not require A to act on $M_1, \ldots, M_k$ by group automorphisms. Thus we need not have $a.(xy) = (a.x)(a.y)$ for all $a \in A$ and x, y in one of the parametrising groups. However, this additional condition will often be satisfied.

4.3.4 Definition (Parametrisation). Denote by $\mathrm{Orb}^{\mathrm{red}}(\Phi) = (O_1, \ldots, O_k)$ the tuple of indivisible orbits of Φ, let $\mathcal{P} = (A, M_1, \ldots, M_k)$ be a parameter system of type Φ and let η a Δ-parity map with values in A. A *parametrisation of G by $\mathcal{P}$ with respect to η and $(w_\delta)_{\delta \in \Delta}$* is a family

$$\big(\theta_\alpha \colon M_\alpha \to U_\alpha\big)_{\alpha \in \Phi^{\mathrm{indiv}}}$$

such that for all roots $\alpha \in \Phi^{\mathrm{indiv}}$, $\delta \in \Delta$ and all $x \in M_\alpha$, we have

$$\theta_\alpha(x)^{w_\delta} = \theta_{\sigma_\delta(\alpha)}(\eta_{\alpha,\delta}.x).$$

If β is a root which is not indivisible and β' is the unique indivisible root in $\mathbb{R}_{>0}\beta$, then we define $M_\beta \coloneqq \theta_{\beta'}^{-1}(U_{\beta'})$ and $\theta_\beta \coloneqq \theta_{\beta'}\big|_{M_\beta} \colon M_\beta \to U_\beta$. The family $(\theta_\alpha)_{\alpha \in \Phi}$ will also be called a parametrisation of G, and the maps in this family are called the *root isomorphisms of G*. We will also say that *G is parametrised by $\mathcal{P}$ with respect to η and $(w_\delta)_{\delta \in \Delta}$ and root isomorphisms $(\theta_\alpha)_{\alpha \in \Phi}$*.

See also Note 4.1.10 for a discussion of Definition 4.3.4 in the setting of non-reduced root systems.

4.3.5 Remark. Let everything be as in Definition 4.3.4. Let $\alpha \in \Phi$ and $x \in M_\alpha$. It follows from the equation

$$\theta_\alpha(x)^{w_\delta} = \theta_{\sigma_\delta(\alpha)}(\eta_{\alpha,\delta}.x)$$

that

$$\theta_\alpha(x) = \theta_{\sigma_\delta(\alpha)}(\eta_{\alpha,\delta}.x)^{w_\delta^{-1}}.$$

Replacing α by $\sigma_\delta(\alpha)$ and x by $\eta^{-1}_{\sigma_\delta(\alpha),\delta}.x$, we infer that

$$\theta_{\sigma_\delta(\alpha)}(\eta^{-1}_{\sigma_\delta(\alpha),\delta}.x) = \theta_\alpha(x)^{w_\delta^{-1}}.$$

Thus

$$\theta_\alpha(x)^{w_{-\delta}} = \theta_\alpha(x)^{w_\delta^{-1}} = \theta_{\sigma_\delta(\alpha)}\big(\eta^{-1}_{\sigma_\alpha(\alpha),\delta}.x\big) = \theta_{\sigma_\delta(\alpha)}\big(\eta_{\alpha,-\delta}.x\big).$$

This implies that for all words $\bar{\delta}$ over $\Delta \cup (-\Delta)$, we have

$$\theta_\alpha(x)^{w_{\bar{\delta}}} = \theta_{\alpha^{\sigma(\bar{\delta})}}(\eta_{\alpha,\bar{\delta}}.x).$$

The notion of faithfulness for parameter systems will not be needed during the actual parametrisation process. However, as discussed in Remark 4.1.23, it is necessary to "read off" a parity map from a root graded group with a parametrisation.

4.3.6 Definition (Faithfulness with respect to parity maps). Let A be any group acting on a set M and let η be a Δ-parity map with values in A. For any root α, we say that the action of A on M is *α-faithful with respect to η*, or simply *$\eta_{\alpha,\cdot}$-faithful* if for all words $\bar{\delta}, \bar{\rho}$ over $\Delta \cup (-\Delta)$ such that $\eta_{\alpha,\bar{\delta}}$ and $\eta_{\alpha,\bar{\rho}}$ act identically on M, we have $\eta_{\alpha,\bar{\delta}} = \eta_{\alpha,\bar{\rho}}$.

4.3.7 Remark. Equivalently, A acts α-faithfully on M with respect to η if and only if the subgroup $\bigcup_{\beta \in O} A_{\eta,\alpha \to \beta}$ acts faithfully on M where O denotes the orbit of α in Φ. If η is transporter-invariant, then $\bigcup_{\beta \in O} A_{\eta,\alpha \to \beta} = A_{\eta,\alpha \to \alpha}$.

4.3.8 Definition (Faithfulness for parameter systems). Let $\mathcal{P} = (A, M_1, \ldots, M_k)$ be a parameter system of type Φ. We say that $\mathcal{P}$ is *faithful* if A acts faithfully on all groups $M_1, \ldots, M_k$. For any Δ-parity map η with values in A, we say that $\mathcal{P}$ is *η-faithful* if for all roots α, the group A acts α-faithfully with respect to η on all groups $M_1, \ldots, M_k$.

In practice, we will only consider transporter-invariant parity maps η and η-faithful parameter systems $(A, M_1, \ldots, M_k)$. In this setting, all groups $A_{\eta,\alpha\to\beta}$ act faithfully on M_i for all α, β in the same orbit and all $i \in [1, k]$.

4.3.9 Note. Let $\mathcal{P} = (A, M_1, \ldots, M_k)$ be a parameter system of type Φ and let η a Δ-parity map with values in A. In all practical situations, the map

$$\eta \colon \Phi \times \mathcal{L}(\Delta \cup (-\Delta)) \to A$$

will always be surjective because otherwise, we would simply shrink the codomain to the image of η. However, this does not imply that the map

$$\eta_{\alpha,\cdot} \colon \mathcal{L}(\Delta \cup (-\Delta)) \to A \tag{4.5}$$

is surjective for all $\alpha \in \Phi$. This is illustrated by Example 4.3.10. As a consequence, faithfulness of all twisting actions of A is a stronger property than η-faithfulness of $\mathcal{P}$. However, η-faithfulness of $\mathcal{P}$ is precisely the property that we need to show that a parity map which is read of from some Φ-graded group automatically satisfies most of the properties that we need (see Lemma 4.4.10).

4.3.10 Example. Let $\mathscr{k}$ be a commutative associative ring and let M be a $\mathscr{k}$-module. Assume that there exist a $\mathscr{k}$-quadratic form $q \colon M \to \mathscr{k}$ and an element $v_0 \in M$ with $q(v_0) = 1_{\mathscr{k}}$, and denote the reflection corresponding to v_0 by $\sigma_{v_0} \colon M \to M$. (We will give precise definitions of quadratic forms and reflections in Definitions 7.1.8 and 7.1.20, but they are actually not relevant for the moment.) Put $A := \{\pm 1\}^2$ and define actions of A on $\mathscr{k}$ and M by

$$(\varepsilon_1, \varepsilon_2).a := \varepsilon_1 a \quad \text{and} \quad (\varepsilon_1, \varepsilon_2).v := \begin{cases} \varepsilon_1 v & \text{if } \varepsilon_2 = 1, \\ \varepsilon_1 \sigma_{v_0}(v) & \text{if } \varepsilon_2 = -1 \end{cases}$$

for all $(\varepsilon_1, \varepsilon_2) \in A$, all $a \in \mathscr{k}$ and all $v \in M$. In other words, the first component of A acts on $\mathscr{k}$ and M by additive inversion while the second component acts trivially on $\mathscr{k}$ and by σ_{v_0} on M. Thus the action of A on M is faithful in the generic case (that is, unless we specifically choose M and v_0 to satisfy, for example, $\sigma_{v_0} = -\mathrm{id}_M$), but the action of A on $\mathscr{k}$ is never faithful. There is no way to avoid this in our setup because we need two twisting actions on M but only one twisting action on $\mathscr{k}$, so there must be a part of A which acts trivially on $\mathscr{k}$.

In section 7.3, we will construct a B_3-graded group H which is parametrised by the parameter system $\mathcal{P} := (A, M, \mathscr{k})$. Since the action of A on $\mathscr{k}$ is not faithful, there is no unique parity map η with respect to which H is parametrised: We can freely choose the second component of $\eta_{\alpha,\delta}$ for all long roots α and all $\delta \in \Delta$. However, we can choose a parity map η by declaring that the second component should always be 1 when it is not uniquely determined. It is not difficult to see that $\mathcal{P}$ is η-faithful for this map η.

4.3.11 Note. The fact that certain components of the twisting group A might only act on some of the parametrising groups but not on all of them, as we have seen in Example 4.3.10, is also responsible for the problems in Note 4.2.20: If all components of A acted on all parametrising groups, then the map (4.5) in Note 4.3.9 would be surjective for all $\alpha \in \Phi$ and thus η would be complete. In particular, any η-faithful parameter system would be faithful. Since this is not the case, we have to work with the weaker conditions of semi-completeness and transporter-invariance that we introduced in Definition 4.2.18.

These problems could be avoided by introducing a separate twisting group A_O for each orbit O of roots (or equivalently, one such twisting group for each parametrising group). In Example 4.3.10, we would have $A_L = \{\pm 1\}$ for the orbit L of long roots and $A_S = \{\pm 1\}^2$ for the orbit S of short roots. The action of A_L on $\Bbbk$ would be given by additive inversion while the action of A_S on M would be given by additive inversion in the first component and by σ_{v_0} in the second component. The advantage of this approach is clear: We have to worry neither about transporter sets and semi-completeness nor about faithfulness with respect to η. However, this not only increases the number of twisting groups that we have to carry along, but also the number of parity maps $\eta_O\colon O \times \Delta \to A_O$. Most root systems have two indivisible orbits S and L of short and long roots, so this would mean that we need twisting groups A_S, A_L, groups B_S, B_L and parity maps η_S, η_L, μ_S, μ_L in order to parametrise the corresponding root graded groups. This would make our notation rather unwieldy, so we have opted to let a single twisting group act on all root groups simultaneously.

4.3.12 Remark (Twisted parametrisations). Let $\mathcal{P} = (A, M_1, \dots, M_k)$ be a parameter system of type Φ, let η be a Δ-parity map with values in A and let $(\theta_\gamma)_{\gamma\in\Phi}$ be a parametrisation of G by $\mathcal{P}$ with respect to η and $(w_\delta)_{\delta\in\Delta}$. Fix an arbitrary root α and an arbitrary element $a \in A$. Assume that the action of a on M_α is a group endomorphism (and thus an isomorphism). We define a twisted root isomorphism for U_α by

$$\theta'_\alpha\colon M_\alpha \to U_\alpha, x \mapsto \theta_\alpha(a.x).$$

The assumption on a assures that θ'_α is indeed a homomorphism. Further, define $\theta'_\beta := \theta_\beta$ for all roots β distinct from α. Then $(\theta'_\gamma)_{\gamma\in\Phi}$ is also a parametrisation of G by $\mathcal{P}$ with respect to $(w_\delta)_{\delta\in\Delta}$, but the corresponding parity map η' is distinct from η. Namely, we have

$$\eta'_{\alpha,\delta} = a\eta_{\alpha,\delta}$$

for all $\delta \in \Delta$ and

$$\eta'_{\beta,\delta} = a^{-1}\eta_{\beta,\delta}$$

for all $\beta \in \Phi$ and $\delta \in \Delta$ such that $\sigma_\delta(\beta) = \alpha$. We say that $(\theta'_\gamma)_{\gamma\in\Phi}$ is *obtained from* $(\theta_\gamma)_{\gamma\in\Phi}$ *by twisting*, and similarly for η' and η. More generally, we say that a parametrisation or a parity map is obtained from another one by twisting if it can be obtained by a finite number of twistings in the previous sense. It is clear that, once we have found one parametrisation and a corresponding parity map, we can construct a large number of different parametrisations and parity maps by twisting. In Remark 4.5.18, we will show that all parity maps which satisfy the necessary compatibility conditions for some root graded group G are the same up to twisting if G is sufficiently generic (which, in more technical terms, translates to some faithfulness assumptions).

We now turn the the structures which are needed as a stepping stone in the construction of a parametrisation: twisting groups and (partial) twisting systems. Since we have already given sufficient motivation in section 4.1, we simply state their definitions.

4.3.13 Definition (Twisting group). A *twisting group for* $(G, (w_\delta)_{\delta\in\Delta})$ *with respect to* Δ is a tuple $(A, (\omega_\alpha)_{\alpha\in\Phi})$ with the following properties:

(i) A is an abelian group.

(ii) For each root α, $\omega_\alpha\colon A \times U_\alpha \to U_\alpha, (a, g) \mapsto a.g$ is a group action of A on the set U_α, called the *twisting action of A on U_α*.

(iii) The twisting action on G commutes with conjugation by the fixed set of Weyl elements and their inverses: For all $\alpha \in \Phi$, $\delta \in \Delta \cup (-\Delta)$ and $a \in A$, we have $(a.x_\alpha)^{w_\delta} = a.(x_\alpha^{w_\delta})$ for all $x_\alpha \in U_\alpha$.

We will usually simply say that A a twisting group for G, leaving the twisting actions, the root base and the Weyl elements implicit.

4.3.14 Note. In every example that we will see, the twisting action is the conjugation by some element of G. Thus it would make sense to write it as a right action instead of as a left action. However, writing it as a left action makes it easier to distinguish between twisting actions and conjugation by elements in $(w_\delta)_{\delta\in\Delta}$, as in Axiom 4.3.13 (iii). Since twisting groups are abelian, this does not make a formal difference in any way.

The following twisting group will play an important role for every root system that we consider.

4.3.15 Example. Consider the multiplicative group $A := \{\pm 1\}$ of order 2 and define that the non-trivial element of A acts on all root groups by group inversion. That is, $a.x_\alpha := x_\alpha^{-1}$ for all $\alpha \in \Phi$, $x_\alpha \in U_\alpha$ and the non-trivial element $a \in A \setminus \{1_A\}$. Then A is a twisting group for G. Note that for all $\alpha \in \Phi$, the twisting action on U_α is compatible with the multiplication on U_α (that is, A acts by group automorphisms) if and only if U_α is abelian.

4.3.16 Note. In every example, property 4.3.13 (iii) follows from the fact that, even though the twisting action can be expressed as conjugation by a certain group element (which is not uniquely determined), it can be defined "naturally" without making any choices (see Remark 4.1.22). For example, the twisting group in Example 4.3.15 acts by inversion, and it is clear that $(x_\alpha^{w_\delta})^{-1} = (x_\alpha^{-1})^{w_\delta}$. In fact, we will always see that property 4.3.13 (iii) is not only satisfied for the fixed Δ-system $(w_\delta)_{\delta\in\Delta}$ of Weyl elements but in fact for all Weyl elements. However, the weaker condition 4.3.13 (iii) will be sufficient to prove the parametrisation theorem.

4.3.17 Definition. Let A be a twisting group for G, let $g \in G$ and let $a \in A$. For any root α, we say that *g acts on U_α by a* if g normalises U_α and $x_\alpha^g = a.x_\alpha$ for all $x_\alpha \in U_\alpha$.

4.3.18 Definition (Partial twisting system). A tuple (A, η, B, μ) is called a *partial twisting system for $(G, (w_\delta)_{\delta\in\Delta})$ with respect to* Δ if it satisfies the following properties:

(i) A is a twisting group for $(G, (w_\delta)_{\delta\in\Delta})$ and η is a parity map with values in A.

(ii) B is an abelian group and μ is a parity map with values in B.

(iii) η is braid-invariant.

(iv) μ is Weyl-invariant and semi-complete.

(v) $\eta \times \mu$ is transporter-invariant and η, μ are independent in the sense of Definitions 4.2.18 and 4.2.23.

We will usually leave the root base Δ implicit.

Recall from Lemma 4.2.25 (b) that, as a consequence of Axiom 4.3.18 (v), the parity maps η and μ in a partial twisting system are transporter-invariant as well.

For root systems of types A, D, E and H, we do not need the generality of partial twisting systems, so we can work with the following definition.

4.3.19 Definition (Twisting system). A *twisting system for* $(G, (w_\delta)_{\delta\in\Delta})$ *with respect to* Δ is a tuple (A, η) where A is a twisting group for $(G, (w_\delta)_{\delta\in\Delta})$ and η is a transporter-invariant braid-invariant parity map with values in A. In other words, it is a tuple (A, η) such that $(A, \eta, \{1\}, (\alpha, \delta) \mapsto 1)$ is a partial twisting system, which is then called the *partial twisting system associated to* (A, η).

4.3.20 Example (Parametrisations of Chevalley groups). We briefly investigate how the notions defined in this section can be used in the context of Chevalley groups. Assume that Φ is crystallographic and reduced. Let $\mathcal{R}$ be a commutative associative ring, let $c = (c_{\alpha,\beta})_{\alpha,\beta\in\Phi}$ a family of Chevalley structure constants of type Φ (in the sense of Definition 3.1.17) and let G be a Chevalley group of type Φ over $\mathcal{R}$ with respect to c. Denote by $(U_\alpha)_{\alpha\in\Phi}$ the root groups of G, by $(\theta_\alpha)_{\alpha\in\Phi}$ the root isomorphisms of G and by $(w_\alpha)_{\alpha\in\Phi}$ the standard Weyl elements from Definition 3.3.3. Then the Chevalley parity map

$$\eta \colon \Phi \times \Phi \to A$$

from Definition 3.3.10 induces a Δ-parity map with values in $A := \{\pm 1_{\mathbb{Z}}\}$. Declaring that A acts on all root groups $(U_\alpha)_{\alpha\in\Phi}$ by inversion, this group is a twisting group for $(G, (w_\delta)_{\delta\in\Delta})$. Even more, (A, η) is a twisting system for $(G, (w_\delta)_{\delta\in\Delta})$. Further, the tuple $\mathcal{P} := (A, \mathcal{R}, \dots, \mathcal{R})$ is a parameter system of type Φ, where $\mathcal{R}$ appears $|\mathrm{Orb}(\Phi)|$ times in the tuple and A acts on all copies of $\mathcal{R}$ by additive inversion. This parameter system is faithful if and only if $1_{\mathcal{R}} \neq -1_{\mathcal{R}}$, or in other words, $2_{\mathcal{R}} \neq 0_{\mathcal{R}}$. Finally, by Lemma 3.3.7, $(\theta_\alpha)_{\alpha\in\Phi}$ is a parametrisation of G by $\mathcal{P}$ with respect to $(w_\delta)_{\delta\in\Delta}$ and η.

4.4 Compatibility Conditions

4.4.1 Notation for this section. We continue to use Notation 4.3.1.

4.4.2 Note. Until now, all conditions that we imposed on parity maps were independent of the twisting actions on G, and vice versa. This clean separation is desirable: We will prove in Lemma 4.4.10 and Proposition 4.4.11 that every parity map η which is read off from an example group H automatically has all the properties that we want, but only with respect to H. For properties which are phrased purely in terms of parity maps, such as braid-invariance, this is no restriction. However, the compatibility conditions that we now introduce relate η only to the specific Φ-graded group H, so we still have to prove that arbitrary Φ-graded groups (equipped with suitably defined twisting actions) are compatible with η. For square compatibility, this is usually easy because in most cases, the square formula provides a clean separation between the combinatoric side and the group-theoretic side. Stabiliser compatibility, on the other hand, will cause more difficulties.

4.4.3 Definition (Square compatibility). Let η be a Δ-parity map with values in some twisting group A for G. For any root α, we say that G is *α-square-compatible with respect to η (and $(w_\delta)_{\delta\in\Delta}$)* if for all $\delta \in \Delta$, the element w_δ^2 acts on U_α by $\eta_{\alpha,\delta\delta}$. Further, we say that G is *square-compatible with respect to η (and $(w_\delta)_{\delta\in\Delta}$)* if it is α-square-compatible with respect to η (and $(w_\delta)_{\delta\in\Delta}$) for all $\alpha \in \Phi$.

4.4.4 Definition (Square formula). Let α be a root. We say that G *satisfies the square formula (for Weyl elements) for* α if for all roots β, we have $x_\alpha^w = x_\alpha^\varepsilon$ for all $x_\alpha \in U_\alpha$ where w is the square of any β-Weyl element and $\varepsilon := (-1)^{\langle \alpha | \beta \rangle}$. Further, we say that G *satisfies the square formula (for Weyl elements)* if it satisfies the square formula for all roots.

4.4.5 Definition (Stabiliser compatibility). Let η, μ be two Δ-parity maps with values in some abelian groups A, B where A is a twisting group for G and μ is Weyl-invariant. For any root $\alpha \in \Phi$, we say that G is *α-stabiliser-compatible with respect to (η, μ) (and $(w_\delta)_{\delta \in \Delta}$)* if for all $u \in \mathrm{Weyl}(\Phi)$ such that $\mu_{\alpha,u} = 1_B$ and $\alpha^u = \alpha$, there exists a word $\bar{\delta}$ over $\Delta \cup (-\Delta)$ such that $\sigma_{\bar{\delta}} = u$ and $w_{\bar{\delta}}$ acts on U_α by $\eta_{\alpha,\bar{\delta}}$. We say that G is *stabiliser-compatible with respect to (η, μ) (and $(w_\delta)_{\delta \in \Delta}$)* if it is α-stabiliser-compatible with respect to (η, μ) (and $(w_\delta)_{\delta \in \Delta}$) for all $\alpha \in \Phi$. If these conditions are satisfied for the trivial parity map μ, we will simply say that G is *(α-) stabiliser-compatible with respect to η (and $(w_\delta)_{\delta \in \Delta}$)*

4.4.6 Remark. If η is a parity map with values in some abelian group A and G is α-stabiliser-compatible with respect to η and $(w_\delta)_{\delta \in \Phi}$, then for any abelian group B and any parity map μ with values in B, the group G is α-stabiliser-compatible with respect to (η, μ) and $(w_\delta)_{\delta \in \Delta}$.

4.4.7 Note. Any element u in Definition 4.4.5 can be written as $u = \sigma(\beta_1) \cdots \sigma(\beta_k)$ for some roots $\beta_1, \ldots, \beta_k$ which are all orthogonal to α by Proposition 1.3.11. Thus one might think that it would be sufficient to require u in Definition 4.4.5 to be of the form σ_β for some root β which is orthogonal to α. However, the condition in Definition 4.4.5 is stronger: We could have $u = \sigma_\beta \sigma_\gamma$ for roots β, γ which are orthogonal to α such that $\mu_{\alpha,\beta} = \mu_{\alpha,\gamma}^{-1}$ (so that $\mu_{\alpha,u} = 1$) but $\mu_{\alpha,\beta} \neq 1 \neq \mu_{\alpha,\gamma}$.

4.4.8 Note (Admissible partial twisting systems). Both square compatibility and stabiliser compatibility are needed in the proof of the parametrisation theorem, so it is tempting to make these conditions part of the definition of partial twisting systems. Instead we take the following approach: For root systems Φ of type B, C, BC or F, we will later (in the corresponding chapters) define the notions of "admissible partial twisting systems", which are partial twisting systems with some additional properties which are specific to the root system in question. For example, the isomorphism types of A and B will be prescribed, and also some specific values of μ. We will then show that any Φ-graded group is automatically square-compatible and stabiliser-compatible with respect to an admissible partial twisting system. Hence it makes sense to not require square and stabiliser compatibility in the definition of partial twisting systems.

4.4.9 Note (Square compatibility with respect to μ). Let (A, η, B, μ) be a partial twisting system for G. Formally, it does not make sense to ask whether G is square-compatible with respect to $\eta \times \mu$ because there is no action of B given on the root groups. However, since μ is square-invariant, we have $\mu_{\alpha,\delta\delta} = 1_B$ for all $\alpha \in \Phi$ and $\delta \in \Delta$ anyway. Thus for any action of B on the root groups, it follows from the assumption that G is square-compatible with respect to η that it is square-compatible with respect to $\eta \times \mu$ as well.

As announced in Remark 4.1.23, we want to define the parity maps η and μ by reading them of from an example group. More precisely, we will define a parity map $\bar{\eta}$ in this way and then use Remark 4.2.22 to split it into two parity maps η, μ such that $\bar{\eta} = \eta \times \mu$. The following result states that a parity map which is defined in this way automatically has many desirable properties. However, the restrictions of Note 4.4.2 apply. For this reason, only Proposition 4.4.11 is truly useful, whereas Lemma 4.4.10 merely serves to illustrate the connection between some of the concepts we have introduced, and to show that they are natural conditions.

4.4.10 Lemma. *Write* $\operatorname{Orb}^{red}(\Phi) = (O_1, \ldots, O_k)$ *for the tuple of indivisible orbits in* Φ. *Let* $\mathcal{P} = (A, M_1, \ldots, M_k)$ *be a parameter system of type* Φ *and let* η *be a* Δ*-parity map with values in* Δ. *Assume that there exists a parametrisation* $(\theta_\alpha)_{\alpha\in\Phi}$ *of* G *by* $\mathcal{P}$ *with respect to* η *and* $(w_\delta)_{\delta\in\Delta}$. *For each root* α, *we define an action* ω_α *of* A *on* U_α *by*

$$a.\theta_\alpha(m) := \theta_\alpha(a.m) \quad \text{for all } m \in M_\alpha.$$

Then the following hold:

(a) $(A, (\omega_\alpha)_{\alpha\in\Phi})$ *is a twisting group for* G.

(b) G *is square-compatible and stabiliser-compatible with respect to* η *and* $(w_\delta)_{\delta\in\Delta}$.

Proof. For (a), the only non-trivial thing to show is that the twisting actions commute with conjugation by the fixed Weyl elements. Let $\alpha \in \Phi$, $\delta \in \Delta \cup (-\Delta)$ and let $x_\alpha \in U_\alpha$. Let $a \in A$ and put $m := \theta_\alpha^{-1}(x_\alpha)$. Then

$$(a.x_\alpha)^{w_\delta} = (a.\theta_\alpha(m))^{w_\delta} = \theta_\alpha(a.m)^{w_\delta} = \theta_{\sigma_\delta(\alpha)}(\eta_{\alpha,\delta}a.m) \qquad \text{and}$$
$$a.(x_\alpha^{w_\alpha}) = a.\big(\theta_\alpha(m)^{w_\delta}\big) = a.\theta_{\sigma_\delta(\alpha)}(\eta_{\alpha,\delta}.m) = \theta_{\sigma_\delta(\alpha)}(a\eta_{\alpha,\delta}.m).$$

Since A is abelian, we conclude that $(a.x_\alpha)^{w_\delta} = a.(x_\alpha^{w_\alpha})$, as desired.

For (b), let $\alpha \in \Phi$, $\delta \in \Delta$ and $x_\alpha \in U_\alpha$. Then we have

$$x_\alpha^{w_\delta^2} = \theta_\alpha(m)^{w_\delta^2} = \theta_{\sigma_\delta(\alpha)}(\eta_{\alpha,\delta}.m)^{w_\delta} = \theta_\alpha(\eta_{\sigma_\delta(\alpha),\delta}\eta_{\alpha,\delta}.m) = \theta_\alpha(\eta_{\alpha,\delta\delta}.m) = \eta_{\alpha,\delta\delta}.x_\alpha$$

where $m := \theta_\alpha^{-1}(x_\alpha)$. Thus G is square-compatible. For stabiliser compatibility, we have to show that for all $\alpha \in \Phi$ and all $u \in \operatorname{Weyl}(\Phi)$ with $\alpha^u = \alpha$, we can find a representation $\bar{\delta}$ of u over Δ such that $w_{\bar{\delta}}$ acts on U_α by $\eta_{\alpha,\bar{\delta}}$. Since G is parametrised with respect to η and $(w_\delta)_{\delta\in\Delta}$, this actually holds for all representations $\bar{\delta}$ of u. □

4.4.11 Proposition. *Let everything be as in Lemma 4.4.10 and let* α *be a root. Assume that the action of* $A_{\eta,\alpha\to\alpha}$ *on* M_α *is* α*-faithful with respect to* η. *Then the following hold:*

(a) η *is* α*-adjacency-trivial. If the commutator relations of* G *are crystallographic, then* η *is crystallographically* α*-adjacency-trivial.*

(b) *If the family* $(w_\delta)_{\delta\in\Delta}$ *satisfies the braid relations modulo* $Z(G)$, *then* η *is* α*-braid-invariant.*

(c) *Assume that* G *satisfies the square formula for Weyl elements and that* $A = \{\pm 1\}^p$ *for some* $p \in \mathbb{N}_+$. *Assume further that* $(-1, 1, \ldots, 1)$ *lies in* $A_{\eta,\alpha\to\alpha}$ *and that it acts on all parametrising groups* $M_1, \ldots, M_k$ *be inversion. Then* η *satisfies the square formula for* α.

Proof. For (a), let β be a root such that α is adjacent to β and $-\beta$ and let $\bar{\rho}$ be a Δ-expression of β. Then $\sigma(\bar{\rho}) = \sigma(\beta)$ and $w := w_{\bar{\rho}}$ is a β-Weyl element. By the assumption on adjacency, it follows that w acts trivially on U_α, so that

$$\theta_\alpha(m) = \theta_\alpha(m)^w = \theta_{\alpha^{\sigma(\bar{\rho})}}(\eta_{\alpha,\bar{\rho}}.m) = \theta_{\alpha^{\sigma(\beta)}}(\eta_{\alpha,\bar{\rho}}.m) = \theta_\alpha(\eta_{\alpha,\bar{\rho}}.m)$$

for all $m \in M_\alpha$. Since the action of A on M_α is α-faithful with respect to η, this implies that $\eta_{\alpha,\bar{\rho}} = 1_A$. Thus η is α-adjacency-trivial. The assertion for the crystallographic case can be proven in the same way.

For (b), let $\beta, \gamma \in \Delta$ be distinct, denote the order of $\sigma_\beta \sigma_\gamma$ by o and put $\bar{\rho} := P_o(\beta, \gamma)$, $\bar{\zeta} := P_o(\gamma, \beta)$. By the braid relation in the Weyl group, we have $\sigma_{\bar{\rho}} = \sigma_{\bar{\zeta}}$ and we denote this element by φ. Now for all $m \in M_\alpha$, we see that $\theta_\alpha(m)^{w_{\bar{\rho}}} = \theta_{\alpha^\varphi}(\eta_{\alpha,\bar{\rho}}.m)$ and $\theta_\alpha(m)^{w_{\bar{\zeta}}} = \theta_{\alpha^\varphi}(\eta_{\alpha,\bar{\zeta}}.m)$. Since $(w_\delta)_{\delta\in\Delta}$ satisfies the braid relations modulo $Z(G)$, these two group elements are equals. We infer that $\eta_{\alpha,\bar{\rho}}$ and $\eta_{\alpha,\bar{\zeta}}$ act identically on M_α. Since the action of A on M_α is $\eta_{\alpha,\cdot}$-faithful by assumption, it follows that $\eta_{\alpha,\bar{\rho}} = \eta_{\alpha,\bar{\zeta}}$, so η is α-braid-invariant.

For (c), let $\delta \in \Delta$ and put $\varepsilon := (-1)^{\langle \alpha | \delta \rangle}$ and $w := w_\delta^2$. Let $m \in M_\alpha$ be arbitrary. Since G satisfies the square formula for Weyl elements, we have

$$\theta_\alpha(\eta_{\alpha,\delta\delta}.m) = \theta_\alpha(m)^w = \theta_\alpha(\varepsilon m).$$

If $\varepsilon = 1$, then it follows from the fact that $A_{\alpha\to\alpha}$ acts faithfully on M_α that $\eta_{\alpha,\delta\delta} = 1_A$. If $\varepsilon = -1$, then we see that $\eta_{\alpha,\delta\delta}$ and $(-1, 1, \dots, 1)$ act identically on M. Since both elements lie in $A_{\alpha\to\alpha}$, it follows that $\eta_{\alpha,\delta\delta} = 1_A$ in this case as well. □

4.5 Proof of the Parametrisation Theorem

4.5.1 Notation for this section. We denote by Φ a root system, by G is a group with a Φ-pregrading $(U_\alpha)_{\alpha\in\Phi}$, by Δ a rescaled root base of Φ, by $(w_\delta)_{\delta\in\Delta}$ a Δ-system of Weyl elements in G and by (A, η, B, μ) a partial twisting system for $(G, (w_\delta)_{\delta\in\Delta})$. We assume that G is square-compatible with respect to η and stabiliser-compatible with respect to (η, μ) and that $(w_\delta)_{\delta\in\Delta}$ satisfies the braid relations modulo $Z(G)$. Further, Notation 4.5.8 holds from the point where it is introduced.

Finally, we have gathered all the necessary tools to prove the parametrisation theorem (Theorem 4.5.16). Our first intermediate goal is Proposition 4.5.6, which is precisely Goal 4.1.26.

4.5.2 Note (Braid relations). If G is a Φ-graded group and Φ is reduced or crystallographic, then it automatically satisfies the braid relations for Weyl elements by Theorem 2.5.10. However, since the proof of the parametrisation theorem works without any reference to Axiom 2.5.2 (iv), we formulate it for more general groups with a Φ-pregrading. Hence we need to make the assumption that $(w_\delta)_{\delta\in\Delta}$ satisfies the braid relations modulo $Z(G)$. Note that the braid relations are only required for this specific system of Weyl elements, not for any system, and that they are only required to hold modulo $Z(G)$.

4.5.3 Lemma. *Let α be a root and let $\bar{\beta} = (\beta_1, \dots, \beta_m)$, $\bar{\gamma} = (\gamma_1, \dots, \gamma_k)$ be two homotopic words over $\Delta \cup (-\Delta)$ such that $\alpha^{\sigma(\bar{\beta})} = \alpha = \alpha^{\sigma(\bar{\gamma})}$. Then the following assertions are equivalent:*

(i) $w_{\bar{\beta}}$ acts on U_α by $\eta_{\alpha,\bar{\beta}}$.

(ii) $w_{\bar{\gamma}}$ acts on U_α by $\eta_{\alpha,\bar{\gamma}}$.

Proof. By induction, we can assume that $\bar{\gamma}$ and $\bar{\beta}$ are either braid-homotopic or square-homotopic. In the case of braid-homotopy, we have $\eta_{\alpha,\bar{\beta}} = \eta_{\alpha,\bar{\gamma}}$ (because η is braid-invariant) and the conjugation actions of $w_{\bar{\beta}}$ and $w_{\bar{\gamma}}$ on G are identical (because $(w_\delta)_{\delta\in\Delta}$ satisfies the braid relations modulo $Z(G)$). Hence the assertion is clear for braid-homotopy. In the case of square-homotopy, we can assume that

$$\bar{\gamma} = (\beta_1, \dots, \beta_p, \delta, \delta, \beta_{p+1}, \dots, \beta_m)$$

for some $p \in [0, m]$ and some $\delta \in \Delta$. Put

$$\bar{\omega} := (\beta_1, \dots, \beta_p), \qquad \bar{\zeta} := (\beta_{p+1}, \dots, \beta_m) \quad \text{and} \quad \rho := \alpha^{\sigma(\bar{\omega})}.$$

Observe that

$$\eta(\alpha, \bar{\gamma}) = \eta(\alpha, \bar{\omega})\eta(\rho, \delta\delta)\eta(\rho, \bar{\zeta}) = \eta(\alpha, \bar{\omega})\eta(\rho, \bar{\zeta})\eta(\rho, \delta\delta) = \eta(\alpha, \bar{\beta})\eta(\rho, \delta\delta)$$

and $w_{\bar{\gamma}} = w_{\bar{\omega}} w_\delta^2 w_{\bar{\zeta}}$. Since G is square-compatible with respect to η and the action of A commutes with conjugation by $w_{\bar{\zeta}}$ (by Axiom 4.3.13 (iii)), we have

$$x_\alpha^{w_{\bar{\gamma}}} = \left(\eta_{\rho,\delta\delta}.x_\alpha^{w_{\bar{\omega}}}\right)^{w_{\bar{\zeta}}} = \eta_{\rho,\delta\delta}.x_\alpha^{w_{\bar{\omega}} w_{\bar{\zeta}}} = \eta_{\rho,\delta\delta}.x_\alpha^{w_{\bar{\beta}}}.$$

Since $\eta_{\alpha,\bar{\gamma}} = \eta_{\alpha,\bar{\beta}}\eta_{\rho,\delta\delta}$, the assertion follows. □

The following statement allows us to restrict our attention to words over Δ (and not over $\Delta \cup (-\Delta)$).

4.5.4 Lemma. *Let α be a root and let $\bar{\beta} = (\beta_1, \dots, \beta_m)$ be a word over $\Delta \cup (-\Delta)$ such that $\alpha^{\sigma(\bar{\beta})} = \alpha$. Choose an arbitrary $i \in [1, m]$ and put*

$$\bar{\beta}' := (\beta_1, \dots, \beta_{i-1}, -\beta_i, \beta_{i+1}, \dots, \beta_m)$$

Then the following assertions are equivalent:

(i) $w_{\bar{\beta}}$ acts on U_α by $\eta_{\alpha,\bar{\beta}}$.

(ii) $w_{\bar{\beta}'}$ acts on U_α by $\eta_{\alpha,\bar{\beta}'}$.

Proof. Note that the word $\bar{\beta}'' := (\beta_1, \dots, \beta_{i-1}, -\beta_i, \beta_i, \beta_i, \beta_{i+1}, \dots, \beta_m)$ is square-homotopic to $\bar{\beta}'$. Thus it follows from Lemma 4.5.3 that (ii) is equivalent to the statement "$w_{\bar{\beta}''}$ acts on U_α by $\eta_{\alpha,\bar{\beta}''}$". However, $w_{\bar{\beta}''} = w_{\bar{\beta}}$ (because $w_{-\beta_i} = w_{\beta_i}^{-1}$) and $\eta_{\alpha,\bar{\beta}''} = \eta_{\alpha,\bar{\beta}}$ (because of Remark 4.2.6). The assertion follows. □

4.5.5 Lemma. *Let $\alpha \in \Phi$ and let $\bar{\delta} = (\delta_1, \dots, \delta_m)$ be a word over $\Delta \cup (-\Delta)$ such that $\alpha^{\sigma(\bar{\delta})} = \alpha$ and $\mu_{\alpha,\bar{\delta}} = 1_B$. Then $w_{\bar{\delta}}$ acts on U_α by $\eta_{\alpha,\bar{\delta}}$.*

Proof. Since G is stabiliser-compatible with respect to (η, μ), there exists a word $\bar{\rho}$ over $\Delta \cup (-\Delta)$ such that $\sigma_{\bar{\rho}} = \sigma_{\bar{\delta}}$ and such that $w_{\bar{\rho}}$ acts on U_α by $\eta_{\alpha,\bar{\rho}}$. Denote by $\bar{\rho}'$ and $\bar{\delta}'$ the words obtained from $\bar{\rho}$ and $\bar{\delta}$ by replacing each letter in $-\Delta$ by its negative. Then $\sigma_{\bar{\rho}'} = \sigma_{\bar{\rho}} = \sigma_{\bar{\delta}} = \sigma_{\bar{\delta}'}$. Hence it follows from Proposition 1.3.16 that $\bar{\rho}'$ and $\bar{\delta}'$ are homotopic. (Here we use that $\bar{\rho}'$ and $\bar{\delta}'$ are words over Δ and not over $\Delta \cup (-\Delta)$. See also Note 1.3.18.) Since $w_{\bar{\rho}}$ acts on U_α by $\eta_{\alpha,\bar{\rho}}$, it follows from Lemma 4.5.4 that $w_{\bar{\rho}'}$ acts on U_α by $\eta_{\alpha,\bar{\rho}'}$. By Lemma 4.5.3, this implies that $w_{\bar{\delta}'}$ acts on U_α by $w_{\bar{\delta}'}$. Again by Lemma 4.5.4, it follows that $w_{\bar{\delta}}$ acts on U_α by $w_{\bar{\delta}}$. This finishes the proof. □

4.5.6 Proposition. *Let $\alpha \in \Phi$ and let $\bar{\beta} = (\beta_1, \dots, \beta_m)$, $\bar{\gamma} = (\gamma_1, \dots, \gamma_k)$ be two words over $\Delta \cup (-\Delta)$ such that $\alpha^{\sigma(\bar{\beta})} = \alpha^{\sigma(\bar{\gamma})}$ and $\mu_{\alpha,\bar{\beta}} = \mu_{\alpha,\bar{\gamma}}$. Put $a := \eta_{\alpha,\bar{\beta}} \cdot \eta_{\alpha,\bar{\gamma}}^{-1}$. Then $x_\alpha^{w_{\bar{\beta}}} = (a.x_\alpha)^{w_{\bar{\gamma}}}$ for all $x_\alpha \in U_\alpha$.*

Proof. Consider $\bar{\gamma}^{-1} = (-\gamma_k, \dots, -\gamma_1)$ and $\bar{\delta} := (\bar{\beta}, \bar{\gamma}^{-1})$. Since $U_\alpha^{\sigma(\bar{\beta})} = U_\alpha^{\sigma(\bar{\gamma})}$, we have $\alpha^{w_{\bar{\delta}}} = \alpha$. Further, the following hold by Lemma 4.2.7:

$$\eta(\alpha, \bar{\delta}) = \eta(\alpha, \bar{\beta})\eta(\alpha^{\sigma(\bar{\beta})}, \bar{\gamma}^{-1}) = \eta(\alpha, \bar{\beta})\eta(\alpha^{\sigma(\bar{\gamma})}, \bar{\gamma}^{-1}) = \eta(\alpha, \bar{\beta})\eta(\alpha, \bar{\gamma})^{-1} = a,$$

$$\mu(\alpha, \bar{\delta}) = \mu(\alpha, \bar{\beta})\mu(\alpha^{\sigma(\bar{\beta})}, \bar{\gamma}^{-1}) = \mu(\alpha, \bar{\beta})\mu(\alpha^{\sigma(\bar{\gamma})}, \bar{\gamma}^{-1}) = \mu(\alpha, \bar{\beta})\mu(\alpha, \bar{\gamma})^{-1} = 1_B.$$

It follows from the second equation and Lemma 4.5.5 that $w_{\bar{\delta}}$ acts on U_α by a, that is, by $\eta(\alpha, \bar{\gamma})^{-1}$. Therefore, we have shown that

$$x_\alpha^{w_{\bar{\beta}} w_{\bar{\gamma}^{-1}}} = x_\alpha^{w_{\bar{\delta}}} = a.x_\alpha$$

for all $x_\alpha \in U_\alpha$. This finishes the proof. $\square$

The following special case of Proposition 4.5.6 will often be useful.

4.5.7 Proposition. *Let $\alpha \in \Phi$ and let $\bar{\beta} = (\beta_1, \dots, \beta_m)$, $\bar{\gamma} = (\gamma_1, \dots, \gamma_k)$ be two words over $\Delta \cup (-\Delta)$ such that $\alpha^{\sigma(\bar{\beta})} = \alpha^{\sigma(\bar{\gamma})}$, $\eta_{\alpha,\bar{\beta}} = \eta_{\alpha,\bar{\gamma}}$ and $\mu_{\alpha,\bar{\beta}} = \mu_{\alpha,\bar{\gamma}}$. Then $x_\alpha^{w_{\bar{\beta}}} = x_\alpha^{w_{\bar{\gamma}}}$ for all $x_\alpha \in U_\alpha$.*

4.5.8 Notation for this section. From now on, we write $\mathrm{Orb}^{\mathrm{red}}(\Phi) = (O_1, \dots, O_k)$ and for each root α, we fix a complement C_α of $B_{\mu,\alpha\to\alpha}$ in B (which exists because μ is semi-complete).

Thanks to Proposition 4.5.7, we can now define an action of B on the root groups. For this, the semi-completeness of μ finally becomes relevant.

4.5.9 Definition. Let $\alpha \in \Phi$. Denote by $\pi_\alpha \colon B \to B_{\alpha\to\alpha}$ the canonical projection with respect to the decomposition $B = B_{\alpha\to\alpha} \times C_\alpha$. We define an action of B on the set U_α by $b.x_\alpha := x_\alpha^{w_{\bar{\delta}}}$ for all $b \in B$ and $x_\alpha \in U_\alpha$ where $\bar{\delta}$ is any word over $\Delta \cup (-\Delta)$ such that $\alpha^{\sigma(\bar{\delta})} = \alpha$, $\eta_{\alpha,\bar{\delta}} = 1_A$ and $\mu_{\alpha,\bar{\delta}} = \pi_\alpha(b)$. (This action is well-defined by Proposition 4.5.7 and because η and μ are independent.) If $\bar{\delta}$ is any word with these properties, we say that *$\bar{\delta}$ induces the action of b on U_α*.

4.5.10 Remark. Equivalently, we could define the action in Definition 4.5.9 by first defining an action of $B_{\alpha\to\alpha}$ on U_α in the same way, but without using π_α, and then declaring that C_α acts trivially on U_α.

4.5.11 Note. Without the assumption that μ is semi-complete, it is not clear that the action of $B_{\alpha\to\alpha}$ in Definition 4.5.9 can be extended to all of B. Since a different choice of the complement C_α would, in general, yield a different action, we have fixed the choice of $(C_\alpha)_{\alpha\in\Phi}$ in Notation 4.5.8. In practice, we will only ever be interested in the action of elements from $B_{\alpha\to\alpha}$, so this choice of complements is harmless in practice.

This observation has two interesting consequences: Firstly, instead of declaring that C_α acts trivially on U_α, we could choose an arbitrary action and the results of this section would remain true (because these results do not make any assertions about this action). Secondly, it implies that suitable modifications of the statements in this section remain true even if μ is not semi-complete if we take care to replace B by $B_{\alpha\to\alpha}$ in appropriate places. However, the assumption of semi-completeness is satisfied in all practical examples and it simplifies notation, which is we why we use it.

As a next goal, we want to show that $A \times B$ is a twisting group for $(G, (w_\delta)_{\delta\in\Delta})$ (Proposition 4.5.14).

4.5.12 Lemma. *For each $\alpha \in \Phi$, the action ω_α of B on U_α in Definition 4.5.9 is a well-defined group action. With these actions, $(B, (\omega_\alpha)_{\alpha\in\Phi})$ is a twisting group for $(G, (w_\delta)_{\delta\in\Delta})$.*

Proof. Let $\alpha \in \Phi$. For every $b \in B$ there exists a word $\bar{\delta}$ which induces the action of b on U_α. To show that this defines a group action, we have to show that

$1_B.x_\alpha = x_\alpha$ and $b_1b_2.x_\alpha = b_1.(b_2.x_\alpha)$ for all $x_\alpha \in U_\alpha$ and all $b_1, b_2 \in B$. The first assertion is clear because the empty word $\varnothing$ satisfies $\mu_{\alpha,\varnothing} = 1_B$, so that

$$1_B.x_\alpha = x_\alpha^{w_\varnothing} = x_\alpha^{1_G} = x_\alpha \quad \text{for all } x_\alpha \in U_\alpha.$$

For the second assertion, let $\bar{\delta}_1, \bar{\delta}_2$ be two words over $\Delta \cup (-\Delta)$ such that

$$\alpha^{\sigma(\bar{\delta}_i)} = \alpha, \qquad \eta_{\alpha,\bar{\delta}_i} = 1_A \quad \text{and} \quad \mu_{\alpha,\bar{\delta}_i} = \pi_\alpha(b_i)$$

for all $i \in \{1, 2\}$. Then the word $\bar{\delta} := (\bar{\delta}_1, \bar{\delta}_2)$ satisfies

$$\alpha^{\sigma(\bar{\delta})} = \alpha, \qquad \eta_{\alpha,\bar{\delta}} = 1_A \quad \text{and} \quad \mu_{\alpha,\bar{\delta}} = \pi_\alpha(b_1b_2),$$

so that

$$b_1b_2.x_\alpha = x_\alpha^{w_{\bar{\delta}}} = \left(x_\alpha^{w_{\bar{\delta}_1}}\right)^{w_{\bar{\delta}_2}} = b_2.(b_1.x_\alpha) \quad \text{for all } x_\alpha \in U_\alpha.$$

Since B is abelian, we conclude that ω_α is indeed an action. (In this context, it is worthwhile to recall Note 4.3.14.)

To see that $(B, (\omega_\alpha)_{\alpha\in\Phi})$ is a twisting group for G, it remains to show that the twisting action is compatible with conjugation by the fixed set of Weyl elements and their inverses. Let $\alpha \in \Phi$, $\delta \in \Delta \cup (-\Delta)$, $x_\alpha \in U_\alpha$ and $b \in B$. Choose words $\bar{\omega}, \bar{\zeta}$ over $\Delta \cup (-\Delta)$ such that $\bar{\omega}$ induces the action of b on U_α and $\bar{\zeta}$ induces the action of b on $U_{\alpha^{\sigma(\delta)}}$. Then

$$(b.x_\alpha)^{w_\delta} = x_\alpha^{w_{\bar{\omega}}w_\delta} \quad \text{and} \quad b.(x_\alpha^{w_\delta}) = x_\alpha^{w_\delta w_{\bar{\zeta}}},$$

and also

$$\eta_{\alpha,\bar{\omega}} = 1_A = \eta_{\alpha^{\sigma(\delta)},\bar{\zeta}} \quad \text{and} \quad \alpha^{\sigma(\bar{\omega})} = \alpha = \alpha^{\sigma(\bar{\zeta})}.$$

We now compute that

$$\eta_{\alpha,\bar{\omega}\delta} = \eta_{\alpha,\bar{\omega}}\eta_{\alpha,\delta} = \eta_{\alpha,\delta}, \qquad \eta_{\alpha,\delta\bar{\zeta}} = \eta_{\alpha,\delta}\eta_{\alpha^{\sigma(\delta)},\bar{\zeta}} = \eta_{\alpha,\delta},$$
$$\mu_{\alpha,\bar{\omega}\delta} = \mu_{\alpha,\bar{\omega}}\mu_{\alpha,\delta} = b\mu_{\alpha,\delta}, \qquad \mu_{\alpha,\delta\bar{\zeta}} = \mu_{\alpha,\delta}\mu_{\alpha^{\sigma(\delta)},\bar{\zeta}} = \mu_{\alpha,\delta}b = b\mu_{\alpha,\delta}.$$

Thus $\eta_{\alpha,\bar{\omega}\delta} = \eta_{\alpha,\delta\bar{\omega}}$ and $\mu_{\alpha,\bar{\omega}\delta} = \mu_{\alpha,\delta\bar{\omega}}$. By an application of Proposition 4.5.7, we conclude that the actions of $w_{\bar{\omega}}w_\delta$ and $w_\delta w_{\bar{\zeta}}$ on U_α are identical, as desired. □

4.5.13 Lemma. *Let $\alpha \in \Phi$. Then the actions of A and B on U_α commute. That is, $a.(b.x_\alpha) = b.(a.x_\alpha)$ for all $a \in A$, $b \in B$ and $x_\alpha \in U_\alpha$.*

Proof. Let $a \in A$, $b \in B$ and $x_\alpha \in U_\alpha$. Choose a word $\bar{\zeta}$ over $\Delta \cup (-\Delta)$ which induces the action of b on U_α. Using Axiom 4.3.13 (iii), we see that

$$a.(b.x_\alpha) = a.x_\alpha^{w_{\bar{\zeta}}} = (a.x_\alpha)^{w_{\bar{\zeta}}} = b.(a.x_\alpha),$$

as desired. □

4.5.14 Proposition. *$A \times B$ is a twisting group for G (with the natural action induced by the actions of A and B) and $(A \times B, \eta \times \mu)$ is a twisting system for G.*

Proof. By Lemma 4.5.13, the actions of A and B induce a natural action of $A \times B$ on each root group. Since the actions of A and B commute with conjugation by Weyl elements, so does the action of $A \times B$. Thus $A \times B$ is indeed a twisting group for G. Further, it is clear that $\eta \times \mu$ is braid-invariant (because both η and μ are, see Lemma 4.2.25 (a)) and transporter-invariant (by Axiom 4.3.18 (v)), so $(A \times B, \eta \times \mu)$ is a twisting system for G. □

4.5.15 Notation. We define $\bar{\eta} := \eta \times \mu$.

4.5.16 Theorem (Parametrisation theorem). *There exist a parameter system $\mathcal{P} = (A \times B, M_1, \dots, M_k)$ of type Φ (in the sense of Definition 4.3.2) whose twisting group is $A \times B$ and a parametrisation $(\theta_\alpha \colon M_\alpha \to U_\alpha)_{\alpha \in \Phi}$ of G by $\mathcal{P}$ with respect to $\bar{\eta}$ and $(w_\delta)_{\delta \in \Delta}$ (in the sense of Definition 4.3.4) such that for all roots α, the twisting action of $A \times B$ on M_α is compatible with the twisting action of $A \times B$ on U_α, meaning that*

$$c.\theta_\alpha(m) = \theta_\alpha(c.m)$$

for all $c \in A \times B$ and all $m \in M_\alpha$.

Proof. For the whole proof, we fix an arbitrary $i \in [1, k]$. Choose an arbitrary root $\alpha_i \in O_i$, a group M_i which is isomorphic to U_{α_i} and an isomorphism $\theta_{\alpha_i} \colon M_i \to U_{\alpha_i}$. For each $\beta \in O_i$, we define an isomorphism

$$\theta_\beta \colon M_i \to U_\beta, x \mapsto \theta_{\alpha_i}(x)^{w_{\bar{\omega}}}$$

where $\bar{\omega}$ is any word over $\Delta \cup (-\Delta)$ such that

$$\alpha_i^{\sigma(\bar{\omega})} = \beta \quad \text{and} \quad \bar{\eta}_{\alpha,\bar{\omega}} = (1_A, 1_B).$$

Since $\bar{\eta}$ is transporter-invariant, such a word $\bar{\omega}$ exists by Remark 4.2.19. Further, we know from Proposition 4.5.7 that θ_β does not depend on the choice of $\bar{\omega}$. Thus θ_β is well-defined.

Now we define an action of $A \times B$ on M_i by $c.x := \theta_{\alpha_i}^{-1}(c.\theta_{\alpha_i}(x))$ for all $c \in A \times B$ and $x \in M_i$, so that $\theta_{\alpha_i}(c.x) = c.\theta_{\alpha_i}(x)$. By definition, this action is compatible with the twisting action of A. Since $A \times B$ is a twisting group for G, it follows for all $\beta \in O_i$ that

$$\theta_\beta(c.x) = \theta_{\alpha_i}(c.x)^{w_{\bar{\delta}}} = \big(c.\theta_{\alpha_i}(x)\big)^{w_{\bar{\delta}}} = c.\theta_{\alpha_i}(x)^{w_{\bar{\delta}}} = c.\theta_\beta(x)$$

for all $x \in M_i$ and $c \in A \times B$, where $\bar{\delta}$ is an arbitrary word over $\Delta \cup (-\Delta)$ such that $\theta_\beta(y) = \theta_{\alpha_i}(y)^{w_{\bar{\delta}}}$ for all $y \in M_i$. Thus the action of $A \times B$ on M_i is compatible with all root isomorphisms $(\theta_\beta)_{\beta \in O_i}$.

Now let $\beta \in O_i$, $\delta \in \Delta$ and $x \in M_i$. We want to show that $\theta_\beta(x)^{w_\delta} = \theta_\gamma(c.x)$ where $\gamma := \beta^{\sigma(\delta)}$ and $c := \bar{\eta}_{\beta,\delta}$. By its definition, c lies in $(A \times B)_{\beta \to \gamma}$, so it also lies in $(A \times B)_{\gamma \to \gamma}$ because $\bar{\eta}$ is transporter-invariant. Thus there exists a word $\bar{\omega}$ over $\Delta \cup (-\Delta)$ such that

$$\bar{\eta}(\gamma, \bar{\omega}) = c \quad \text{and} \quad \gamma^{\sigma(\bar{\omega})} = \gamma.$$

Further, by Remark 4.2.19, there also exists a word $\bar{\zeta}$ over $\Delta \cup (-\Delta)$ such that

$$\bar{\eta}(\alpha_i, \bar{\zeta}) = (1_A, 1_B) \quad \text{and} \quad \alpha_i^{\sigma(\bar{\zeta})} = \beta.$$

By their respective definitions, these words satisfy

$$c.\theta_\gamma(x) = \theta_\gamma(c.x) = \theta_\gamma(x)^{w_{\bar{\omega}}} \quad \text{and} \quad \theta_{\alpha_i}(x)^{w_{\bar{\zeta}}} = \theta_\beta(x). \tag{4.6}$$

Further

$$\begin{aligned}\bar{\eta}(\alpha_i, \bar{\zeta}\delta\bar{\omega}^{-1}) &= \bar{\eta}(\alpha_i, \bar{\zeta})\bar{\eta}(\beta, \delta)\bar{\eta}(\gamma, \bar{\omega}^{-1}) = c\bar{\eta}(\gamma^{\sigma(\bar{\omega})^{-1}}, \bar{\omega})^{-1} = c\bar{\eta}(\gamma, \bar{\omega})^{-1} \\ &= cc^{-1} = (1_A, 1_B).\end{aligned}$$

By the definition of θ_γ, it follows that $w := w_{\bar{\zeta}} w_\delta w_{\bar{\omega}}^{-1}$ satisfies $\theta_{\alpha_i}(x)^w = \theta_\gamma(x)$. Therefore,

$$\theta_\gamma(x)^{w_{\bar{\omega}}} = \theta_{\alpha_i}(x)^{w_{\bar{\zeta}} w_\delta}.$$

Applying both equations in (4.6), we infer that

$$\theta_\gamma(c.x) = \theta_\beta(x)^{w_\delta},$$

which finishes the proof. □

4.5.17 Remark. It is clear from the first lines of the proof that the statement of Theorem 4.5.16 can be slightly strengthened: For all $i \in [1,k]$ choose a root $\alpha_i \in O_i$, a group M_i which is isomorphic to U_{α_i} and an isomorphism $\theta_{\alpha_i} \colon M_i \to U_{\alpha_i}$. Then there exist an action of $A \times B$ on each of the sets $M_1, \dots, M_k$ and a family of isomorphisms $(\theta_\alpha)_{\alpha \in \Phi}$ extending $(\theta_{\alpha_i})_{i \in [1,k]}$ such that the assertions in Theorem 4.5.16 hold. In other words, whenever we can construct multiple parametrisations of a group (for different parity maps, for example), we can always arrange it so that they are parametrised by the same groups $M_1, \dots, M_k$. This technical observation will be useful in the parametrisation of F_4-graded groups, in which we will parametrise certain B_3- and C_3-graded subgroups and then consider their "overlap".

The following remark says that different choices of parity maps yield the same parametrisations up to twisting.

4.5.18 Remark. We briefly consider what changes when we choose another pair η', μ' of Δ-parity maps with values in A and B, respectively. Assume that (A, η', B, μ') satisfies the same assumptions as (A, η, B, μ) in Notation 4.5.1. Specifically, this means that (A, η', B, μ') is a partial twisting system for $(G, (w_\delta)_{\delta \in \Delta})$ and that G is square-compatible with respect to η' and stabiliser-compatible with respect to (η', μ'). Assume further that $B_{\mu, \alpha \to \alpha} = B_{\mu', \alpha \to \alpha}$ for all roots α, and choose the same complement C_α of these groups in B.

The parametrisation theorem yields two parameter systems

$$\mathcal{P} = (A \times B, M_1, \dots, M_k) \quad \text{and} \quad \mathcal{P}' = (A \times B, M_1', \dots, M_k')$$

of type Φ and corresponding parametrisations $(\theta_\alpha)_{\alpha \in \Phi}$ and $(\theta'_\alpha)_{\alpha \in \Phi}$ of G with respect to $(w_\delta)_{\delta \in \Delta}$ and $\eta \times \mu$ or $\eta' \times \mu'$, respectively. For each $i \in [1,k]$, we fix a root $\alpha_i \in O_i$. Using Remark 4.5.17, we can assume that $M_i = M_i'$ and $\theta_{\alpha_i} = \theta'_{\alpha_i}$ for all $i \in [1,k]$. However, observe that the parameter systems $\mathcal{P}$ and $\mathcal{P}'$ are not the same because the actions of $A \times B$ on $M_1, \dots, M_k$ may be different. We solve this notational issue by writing $\omega_i(c, x)$ for the action of $\mathcal{P}$ and $\omega_i'(c, x)$ for the action of $\mathcal{P}'$, where $i \in [1,k]$, $x \in M_\alpha$ and $c \in A \times B$.

We begin by comparing the different twisting actions. Fix $i \in [1,k]$ for the scope of this paragraph. For all $a \in A$, we have

$$\theta'_{\alpha_i}\big(\omega_i'(a, x)\big) = a.\theta_{\alpha_i}(x) = \theta_{\alpha_i}\big(\omega_i(a, x)\big)$$

for all $x \in M_i$ because the twisting action of A on U_α is fixed. It follows that the two actions of A on M_i are identical. Now let $b \in B_{\mu, \alpha_i \to \alpha_i} = B_{\mu', \alpha_i \to \alpha_i}$ and choose a word $\bar{\delta}$ over $\Delta \cup (-\Delta)$ such that $\sigma(\bar{\delta})$ stabilises α_i and such that $(\eta \times \mu)'_{\alpha_i, \bar{\delta}} = b$. Then

$$\begin{aligned}\theta'_{\alpha_i}\big(\omega_i'(b, x)\big) &= \theta'_{\alpha_i}(x)^{w_{\bar{\delta}'}} = \theta_{\alpha_i}(x)^{w_{\bar{\delta}'}} = \theta_{\alpha_i}\big(\omega_i((\eta \times \mu)_{\alpha_i, \bar{\delta}'}, x)\big) \\ &= \theta'_{\alpha_i}\big(\omega_i((\eta \times \mu)_{\alpha_i, \bar{\delta}'}, x)\big)\end{aligned}$$

for all $x \in M_i$. We conclude that, while the actions $\omega_i(b, \cdot\,)$ and $\omega_i'(b, \cdot\,)$ for individual elements b might be distinct, each action $\omega_i(b, \cdot\,)$ can be expressed as $\omega_i'(b', \cdot\,)$ for some $b' \in B$. In fact, it is not difficult to show that the map $b \mapsto b'$ is an automorphism of the group $B_{\mu, \alpha_i \to \alpha_i}$. By choosing the same complement C_α for $B_{\mu, \alpha_i \to \alpha_i}$ and $B_{\mu', \alpha_i \to \alpha_i}$ in Definition 4.5.9, we can extend $b \mapsto b'$ to an automorphism of B which still has the property that $\omega_i(b, \cdot\,) = \omega_i'(b', \cdot\,)$ for all $b \in B$.

We now compare the different root isomorphisms. Again, let $i \in [1,k]$ be arbitrary. Let β be any root in O_i and let $\bar{\delta}$ be a word over $\Delta \cup (-\Delta)$ such that $\sigma(\bar{\delta})$ maps α_i to β. Since $\theta_{\alpha_i} = \theta'_{\alpha_i}$, we then have

$$\theta'_\beta\big(\omega_i((\eta' \times \mu')_{\alpha,\bar{\delta}}, x)\big) = \theta'_{\alpha_i}(x)^{w_{\bar{\delta}}} = \theta_{\alpha_i}(x)^{w_{\bar{\delta}}} = \theta_\beta\big(\omega'_i((\eta \times \mu)_{\alpha,\bar{\delta}}, x)\big)$$

for all $x \in M_i$. Using the conclusion of the previous paragraph, it follows that there exists $c \in A \times B$ such that

$$\theta'_\beta(x) = \theta_\beta\big(\omega_i(c, x)\big)$$

for all $x \in M_i$. We conclude that $(\theta'_\gamma)_{\gamma\in\Phi}$ can be obtained from $(\theta_\gamma)_{\gamma\in\Phi}$ by twisting (in the sense of Remark 4.3.12). Observe that c automatically has the property that $\omega_i(c, \cdot\,)$ is a group automorphism of M_i because both θ_β and θ'_β are homomorphisms.

Finally, assume in addition that $\mathscr{P}$ is $(\eta \times \mu)$-faithful and $\mathscr{P}'$ is $(\eta' \times \mu')$-faithful. (By the previous observations, this is already true if it holds for one of the parameter systems.) Since then $\eta \times \mu$ and $\eta' \times \mu'$ are determined by their corresponding parametrisations $(\theta_\gamma)_{\gamma\in\Phi}$ and $(\theta'_\gamma)_{\gamma\in\Phi}$, it follows that $\eta' \times \mu'$ can be obtained from $\eta \times \mu$ by twisting as well.

4.5.19 Note. In all examples of root graded groups of rank at least 3, the group B is $\{\pm 1\}$ or trivial. It follows that the automorphism $B \to B, b \mapsto b'$ in Remark 4.5.18 must be the identity map. In an imaginary example where B is of the form $\{\pm 1\}^p$, it would however be possible that $b \mapsto b'$ interchanges some components of B.

4.5.20 Note. A consequence of Remark 4.5.18 is that the specific choice of the parity maps η and μ is ultimately not relevant. Thus there is no harm in choosing explicit parity maps η_Φ and μ_Φ during the parametrisation process of Φ-graded groups. The reader who prefers a different choice of parity maps η'_Φ and μ'_Φ does not have to repeat all our proofs: Instead, they can simply take our parametrisation $(\theta_\alpha)_{\alpha\in\Phi}$ (which respects η_Φ and μ_Φ) and twist it appropriately to obtain a parametrisation $(\theta'_\alpha)_{\alpha\in\Phi}$ with respect to η'_Φ and μ'_Φ.

4.6 Criteria for the Compatibility Conditions

4.6.1 Notation for this section. We denote by Φ an arbitrary root system, by Δ a rescaled root base of Φ, by G a group with a Φ-pregrading $(U_\alpha)_{\alpha\in\Phi}$ and by $(w_\delta)_{\delta\in\Delta}$ a Δ-system of Weyl elements in G.

In this chapter, we collect some criteria which will be used to prove the compatibility conditions in the parametrisation theorem. We begin with a criterion for square compatibility. Surprisingly, it will be applicable in many situations, and similar ideas work in all remaining situations.

4.6.2 Lemma. *Let $A := \{\pm 1\}^n$ be a twisting group for $(G, (w_\delta)_{\delta\in\Delta})$ for some $n \in \mathbb{N}_+$ and assume that the first component of A acts on all root groups by inversion. Let α be any root and let η be a Δ-parity map with values in A. Assume that both η and G satisfy the square formula for α. Then G is α-square compatible with respect to η (and any Δ-system of Weyl elements).*

Proof. Let $\delta \in \Delta$, set $w := w_\delta^2$ for some δ-Weyl element w_δ and put $\varepsilon := (-1)^{\langle\alpha|\delta\rangle}$. Then $x_\alpha^w = x_\alpha^\varepsilon$ for all $x_\alpha \in U_\alpha$ because G satisfies the square formula for α and $\eta_{\alpha,\delta\delta} = (\varepsilon, 1, \dots, 1)$ because η satisfies the square formula for α. It follows that w acts on U_α by $\eta_{\alpha,\delta\delta}$, as desired. ☐

The following result is the main criterion for stabiliser compatibility. It is very general and can be applied for all root systems. However, it is unnecessarily complicated in many situations, which is why we state a simpler, less general version in Lemma 4.6.4. We use Convention 2.3.5 to formulate both assertions.

4.6.3 Proposition. *Assume that G has (crystallographic) Φ-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi}$. Let A be a twisting group for G, let η be a (crystallographically) adjacency-trivial parity map with values in A such that G is square-compatible with respect to η and $(w_\delta)_{\delta\in\Delta}$ and let μ be a Weyl-invariant parity map with values in the abelian group $B := \{\pm 1_B\}$. Fix a root α and define the following sets:*

$$\begin{aligned}
\mathcal{O} &:= \{\, \beta\in\Phi \mid \alpha\cdot\beta = 0 \,\},\\
\mathcal{A} &:= \{\, \beta\in\mathcal{O} \mid \alpha \textit{ is (crystallographically) adjacent to } \beta \textit{ and } -\beta \,\},\\
&= \{\, \beta\in\mathcal{O} \mid \alpha \textit{ is (crystallographically) adjacent to } \beta \,\}\\
\bar{\mathcal{A}} &:= \mathcal{O}\setminus\mathcal{A}.
\end{aligned}$$

Assume that we have $\mu_{\alpha,\sigma(\beta)} = 1_B$ for all $\beta\in\mathcal{A}$ and $\mu_{\alpha,\sigma(\beta)} = -1_B$ for all $\beta\in\bar{\mathcal{A}}$. Assume further that for all Δ-positive roots $\beta,\beta'\in\bar{\mathcal{A}}$, there exist words $\bar{\delta},\bar{\delta}'$ over $\Delta\cup(-\Delta)$ such that $\sigma(\bar{\delta}) = \sigma(\beta)$, $\sigma(\bar{\delta}') = \sigma(\beta')$, $\eta_{\alpha,\bar{\delta}} = \eta_{\alpha,\bar{\delta}'}$ and such that $w_{\bar{\delta}}$ and $w_{\bar{\delta}'}$ act identically on U_α. Then G is α-stabiliser-compatible with respect to (η,μ) and $(w_\delta)_{\delta\in\Delta}$.

Proof. Let $u\in\mathrm{Weyl}(\Phi)$ such that $\alpha^u = \alpha$ and $\mu_{\alpha,u} = 1_B$. By Proposition 1.3.11, there exist $\rho_1,\dots,\rho_k\in\mathcal{O}$ such that $u = \sigma_{\rho_1}\cdots\sigma_{\rho_k}$. Since $\sigma_{\rho_i} = \sigma_{-\rho_i}$ for all $i\in[1,n]$, we can choose $\rho_1,\dots,\rho_k$ so that all of them are Δ-positive. At first, we only consider the special case that all roots $\rho_1,\dots,\rho_k$ lie in $\bar{\mathcal{A}}$. We will show that for each $i\in[1,k]$, there exists a word $\bar{\delta}^i$ over $\Delta\cup(-\Delta)$ such that $\sigma(\bar{\delta}^i) = \sigma(\rho_i)$ and such that $w_{\bar{\delta}^1\cdots\bar{\delta}^k}$ acts on U_α by $\eta_{\alpha,\bar{\delta}^1\cdots\bar{\delta}^k}$. By the assumption on the map μ, we have

$$1_B = \mu_{\alpha,u} = \mu_{\alpha,\sigma(\rho_1)\cdots\sigma(\rho_k)} = \prod_{i=1}^{k}\mu_{\alpha,\sigma(\rho_i)} = (-1_B)^k,$$

which implies that k is even. For all $i\in[1,k/2]$, we can now by assumption choose words $\bar{\delta}^{2i-1}$, $\bar{\delta}^{2i}$ over $\Delta\cup(-\Delta)$ such that

$$\sigma(\bar{\delta}^{2i-1}) = \sigma(\rho_{2i-1}),\qquad \sigma(\bar{\delta}^{2i}) = \sigma(\rho_{2i}),\qquad \eta_{\alpha,\bar{\delta}^{2i-1}} = \eta_{\alpha,\bar{\delta}^{2i}}$$

and such that $w_{\bar{\delta}^{2i-1}}$ and $w_{\bar{\delta}^{2i}}$ act identically on U_α. Then $w_{\bar{\delta}^1\cdots\bar{\delta}^k}$ and $\prod_{i=1}^{k/2} w_{\bar{\delta}^{2i}}^2$ act identically on U_α. Since G is square-compatible with respect to η, it follows that $w_{\bar{\delta}^1\cdots\bar{\delta}^k}$ acts on U_α by $\prod_{i=1}^{k/2}\eta_{\alpha,\bar{\delta}^{2i}\bar{\delta}^{2i}}$. Since $\eta_{\alpha,\bar{\delta}^{2i-1}} = \eta_{\alpha,\bar{\delta}^{2i}}$ and $\sigma_{\bar{\delta}^{2i}}(\alpha) = \alpha = \sigma_{\bar{\delta}^{2i-1}}(\alpha)$ for all $i\in[1,k/2]$, we also have that

$$\prod_{i=1}^{k/2}\eta_{\alpha,\bar{\delta}^{2i}\bar{\delta}^{2i}} = \prod_{i=1}^{k/2}\eta_{\alpha,\bar{\delta}^{2i}}\eta_{\alpha,\bar{\delta}^{2i}} = \prod_{i=1}^{k/2}\eta_{\alpha,\bar{\delta}^{2i-1}}\eta_{\alpha,\bar{\delta}^{2i}} = \prod_{j=1}^{k}\eta_{\alpha,\bar{\delta}^j} = \eta_{\alpha,\bar{\delta}^1\cdots\bar{\delta}^k}.$$

It follows that $w_{\bar{\delta}^1\cdots\bar{\delta}^k}$ acts on U_α by $\eta_{\alpha,\bar{\delta}^1\cdots\bar{\delta}^k}$, as desired.

Now we consider the general case. Let $i(1),\dots,i(n),j(1),\dots,j(m)$ be pairwise distinct indices from $[1,k]$ such that

$$\begin{aligned}
\{\, p\in[1,k] \mid \rho_p\in\mathcal{A}\,\} &= \{\, i(1),\dots,i(n)\,\}, &\qquad i(1)<\cdots<i(n),\\
\{\, p\in[1,k] \mid \rho_p\in\bar{\mathcal{A}}\,\} &= \{\, j(1),\dots,j(m)\,\}, &\qquad j(1)<\cdots<j(m).
\end{aligned}$$

Put $u' := \sigma(\rho_{j(1)})\cdots\sigma(\rho_{j(m)})$. By the assumption on μ, we have $\mu_{\alpha,\sigma(\rho_{i(p)})} = 1_B$ for all $p\in[1,n]$, and thus $\mu_{\alpha,u'} = \mu_{\alpha,u} = 1_B$. Hence by the conclusion of the previous paragraph, we can for each $p\in[1,m]$ find a word $\bar{\delta}^p$ over $\Delta\cup(-\Delta)$

such that $\sigma(\bar{\delta}^p) = \sigma(\rho_{j(p)})$ and such that $w_{\bar{\delta}^1\dots\bar{\delta}^m}$ acts on U_α by $\eta_{\alpha,\bar{\delta}^1\dots\bar{\delta}^m}$. For every $p \in [1,n]$, we choose an arbitrary Δ-expression $\bar{\zeta}^p$ of $\rho_{i(p)}$. For each $q \in [1,k]$, we can now define $\bar{\xi}^q := \bar{\delta}^p$ if $q = j(p)$ for some $p \in [1,m]$ and $\bar{\xi}^q := \bar{\zeta}^p$ if $q = i(p)$ for some $p \in [1,n]$. Note that for each $p \in [1,n]$, we have $\eta_{\alpha,\bar{\zeta}^p} = 1_A$ because η is (crystallographically) adjacency-trivial and we also have that $w_{\bar{\zeta}^p}$ acts trivially on U_α because it is a $\rho_{i(p)}$-Weyl element and $\rho_{i(p)}$ lies in $\mathcal{A}$. This implies that $w_{\bar{\xi}^1\dots\bar{\xi}^k}$ and $w_{\bar{\delta}^1\dots\bar{\delta}^m}$ act identically on U_α and that $\eta_{\alpha,\bar{\xi}^1\dots\bar{\xi}^k} = \eta_{\alpha,\bar{\delta}^1\dots\bar{\delta}^m}$. We conclude that $w_{\bar{\xi}^1\dots\bar{\xi}^k}$ acts on U_α by $\eta_{\alpha,\bar{\xi}^1\dots\bar{\xi}^k}$, as desired. □

In the following result, we consider the special case that $\bar{\mathcal{A}}$ is empty. In this situation, we can easily drop the assumptions on the parity map μ. Again, we use Convention 2.3.5.

4.6.4 Lemma. *Assume that G has (crystallographic) Φ-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi}$. Let η be a Δ-parity map with values in some twisting group A for $(G,(w_\delta)_{\delta\in\Delta})$ and let $\alpha \in \Phi$. Put*

$$\Delta^W := \{ \delta^u \mid \delta \in \Delta, u \in \mathrm{Weyl}(\Phi) \}.$$

Assume that η is (crystallographically) α-adjacency-trivial and that any root in Δ^W which is orthogonal to α is also (crystallographically) adjacent to α. Then G is α-stabiliser-compatible with respect to η.

Proof. Let u be an element of the Weyl group of Φ such that $\alpha^u = \alpha$. Then by Proposition 1.3.11, there exist roots $\beta_1,\dots,\beta_m \in \Phi$ which are all orthogonal to α such that $u = \sigma_{\beta_1}\cdots\sigma_{\beta_m}$. For the rest of this paragraph, we fix an arbitrary $i \in [1,m]$. Applying Proposition 1.3.5, we can find a Δ-expression $\bar{\rho}^i$ of $\lambda\beta^i$ for some $\lambda \in \mathbb{R}_{>0}$. By Remark 4.2.3, $w_{\bar{\rho}^i}$ is a $\lambda\beta_i$-Weyl element. Since $\lambda\beta^i$ has a Δ-expression, it lies in Δ^W, and it is orthogonal to α because β^i is. The same assertions hold for $-\lambda\beta^i$. Thus it follows from our assumptions that α is (crystallographically) adjacent to $\lambda\beta_i$ and $-\lambda\beta_i$. Hence $w_{\bar{\rho}^i}$ centralises U_α. Further, $\eta_{\alpha,\bar{\rho}^i} = 1_A$ because η is α-adjacency-trivial. We conclude that $w_{\bar{\rho}^i}$ acts on U_α by $\eta_{\alpha,\bar{\rho}^i}$.

It follows from the conclusion of the previous paragraph that $w_{\bar{\rho}}$ acts on U_α by $\eta_{\alpha,\bar{\rho}}$ where $\bar{\rho} := (\bar{\rho}^1,\dots,\bar{\rho}^m)$. Since $u = \sigma_{\beta_1}\cdots\sigma_{\beta_m} = \sigma_{\bar{\rho}^1}\cdots\sigma_{\bar{\rho}^m} = \sigma_{\bar{\rho}}$, $\bar{\rho}$ is a representation of u. The assertion follows. □

4.6.5 Remark. If Φ is reduced, then the set Δ^W in Lemma 4.6.4 is the whole root system by Proposition 1.3.5. In contrast, assume now that Φ is the non-reduced root system BC_n in standard representation for some $n \geq 2$ and that Δ is the standard rescaled root base of BC_n. (We will formally define these objects in Remark 9.1.2, but we have already seen Δ in Remark 4.1.28.) Then Δ^W consists of all roots which are long or of medium length. Let α be an short root. Then α does indeed have the property that any root β in Δ^W which is orthogonal to α must be crystallographically adjacent to α. This conclusion does not hold if β is short: For example, the roots $\alpha := e_1$ and $\beta := e_2$ are orthogonal but not crystallographically adjacent.

4.7 Remarks on the General Strategy

As we have emphasised before, the parametrisation theorem allows us to approach Φ-graded groups for different root systems Φ in a uniform way. For this reason, chapters 5 and 7 to 10 all follow a common outline. In this section, we describe this outline. We always denote by Φ the specific root system which is studied in the corresponding chapter (or one of these root systems if there are multiple).

4.7.1 The parametrising algebraic structures. Each of the aforementioned chapters begins with an investigation of the algebraic structures that coordinatise Φ-graded groups. Most of the time, these are well-known structures with a highly developed underlying theory. We confine ourselves to covering the basics of these theories, though we will sometimes go beyond what is strictly necessary for our purposes. In the case of root gradings of type $(B)C$, the algebraic structures in question are so involved that they are covered in a separate chapter.

For each algebraic structure, we define a *standard parameter system*, which is a parameter system in the sense of Definition 4.3.2. We will show that every Φ-graded group can be parametrised by a parameter system of this form.

4.7.2 The root system. As a next step, we study the combinatorial properties of Φ. We will give a standard representation of Φ, that is, a specific description of the roots as linear combinations of an orthonormal basis of a Euclidean space. Using this representation, we can easily compute the Cartan integers, which will later allow us to prove the square formula for Weyl elements in Φ-graded groups (in most cases). Further, we will prove some minor lemmas concerning the relationship between arbitrary roots α and β.

4.7.3 Construction of a generic example. Before we turn to the investigation of arbitrary Φ-graded groups, we construct examples of such groups. In these examples, we will see the commutator relations that we ultimately want to establish in every Φ-graded group. Further, we will also see the standard parameter system from 4.7.1 in action.

Ideally, we would construct a Φ-graded group for any object in the class of algebraic structures which were studied in 4.7.1. However, we will restrict ourselves to certain special cases whenever a general construction would be much more complicated. In some cases, it is even true that no general construction is known. Yet, the groups that we construct will be sufficiently generic (in the sense of Remark 4.1.23) to yield adequate parity maps. These parity maps will later be used in the parametrisation of Φ-graded groups.

4.7.4 Computations in Φ-graded groups. As a next step, we study the action of Weyl elements in Φ-graded groups on the root groups, independently of any choice of a parity map. These results form the mathematical core of the study of Φ-graded groups for any specific root system Φ. Our goal is to obtain some fundamental results which will later be used to prove the compatibility conditions which appear in the parametrisation theorem. This means that we have to investigate the actions of squares of Weyl elements on all root groups and the action of β-Weyl elements on any root group which is orthogonal to β. Further, we will give alternative proofs of the braid relations that were established in Theorem 2.5.10.

In order to prove the aforementioned statements, we need two computational tools. At first, we investigate how the commutator maps $[\,\cdot\,,\,\cdot\,]_\alpha$ behave under products in the first and second argument. We will see that these maps are always additive or, in a loose sense, "quadratic" in each argument, which is precisely the behaviour that we have seen in the commutator relations in 4.7.3. Secondly, we need the formulas from Proposition 2.2.10. Using the identities from the previous step, we will give alternative, more explicit proofs of these formulas for each root system.

With these tools in hand, we can approach the problems from the first paragraph. While all computations in the previous paragraph take place in root graded subgroups of rank 2, we will now have to use the rank-3 assumptions in a few places. However, some partial results still hold in the rank-2 situation, and we will prove as much as we can in this generality.

4.7.5 Standard signs in Φ-graded groups. We say that a Φ-pregrading is *coordinatised with standard signs* if there exist an algebraic structure as in 4.7.1 and root isomorphisms which satisfy the precise commutator relations of the example in 4.7.3 (or a suitable generalisation of these relations). The terminology is due to the fact that the signs in these relations are not canonically determined, as we have seen in Remark 4.3.12. Thus we (somewhat arbitrarily) declare that the signs which occur in 4.7.3 are the "standard ones". Using the parity maps from 4.7.3, our goal is to show that every Φ-graded group is coordinatised with standard signs.

Note that the notion of coordinatisations with standard signs is defined for arbitrary pregradings. It is clear that any pregrading which is coordinatised in this way automatically has (crystallographic) Φ-commutator relations, but the existence of Weyl elements is not obvious. However, we can use our specific knowledge of the commutator relations to show that any element of the coordinatising structure which is invertible (in a suitable sense) induces a Weyl element, and that every Weyl element is of this form. Thus any Φ-pregrading which has a coordinatisation with standard signs is automatically a Φ-grading, except that the validity of Axiom 2.5.2 (iv) is not clear.

4.7.6 Note (On the choice of a parity map). After working with parity maps in a purely abstract way for the whole chapter, it might seem disappointing that we ultimately prove our coordinatisation theorems only for a specific choice of standard signs and not in a more abstract way. However, observe that as soon as we have established the existence of one coordinatisation $\theta = (\theta_\alpha)_{\alpha\in\Phi}$ (with standard signs), we can easily twist θ (as in Remark 4.3.12) to obtain coordinatisations of the same group with non-standard signs. By Remark 4.5.18, essentially every coordinatisation can be obtained in this way. Thus it is not a serious restriction to only consider standard signs. See also Note 4.5.20.

4.7.7 Admissible partial twisting systems and the parametrisation of Φ-graded groups. Recall that the parametrisation theorem starts from a partial twisting system (A, η, B, μ) and a Φ-graded group G which satisfies some compatibility conditions. It is clear that these compatibility conditions impose some restrictions on the partial twisting system, so not every imaginable partial twisting system can be used to parametrise Φ-graded groups. A *Φ-admissible partial twisting system* is, by definition, an admissible partial twisting which satisfies some additional conditions which depend on Φ. These conditions are, unlike the compatibility conditions, easy to verify if the parity maps η and μ are explicitly given.

The *standard partial twisting system (of type Φ)* is defined to be the partial twisting system (A, η, B, μ) that we see in 4.7.3, except that we "forget" the actions of B on the root groups. We will show that the standard partial twisting system of type Φ is Φ-admissible. Further, we will prove that every Φ-graded group G satisfies the compatibility conditions with respect to any Φ-admissible partial twisting system. As a consequence, we can apply the parametrisation theorem to G and its standard partial twisting system, which yields a parametrisation of G.

4.7.8 Note (Admissible partial twisting systems). We could in principle work without the abstract notion of admissible partial twisting systems: It would be sufficient to simply show that the standard partial twisting system satisfies all compatibility conditions with respect to any Φ-graded group. However, this would not simplify our proofs in a significant way: We would still, as a first step, show that the standard partial twisting system has some properties (which are exactly the axioms of admissible partial twisting systems) and then, as a second step, use only these properties to prove the compatibility conditions.

4.7.9 Note (Standard signs and twisting systems for simply-laced groups). Our approach to standard signs and standard (partial) twisting systems for the simply-laced root graded groups will be slightly different than described in 4.7.5 and 4.7.7. The reason for this is essentially that we want to cover types A, D and E at the same time, so we cannot use a single explicit standard twisting system or a single explicit set of standard signs. We will elaborate on this in sections 5.5 and 5.6.

4.7.10 Blueprint computations. As a final step, we apply the blueprint technique to compute the commutator relations with respect to the chosen parametrisation. We will elaborate on this in chapter 6. For these computations, it will be crucial that the signs in our parametrisation are explicitly given. We emphasise that there is nothing specific about the standard signs that makes the blueprint computation work: The only relevant fact is that we make *some* explicit choice of signs.

Chapter 5

Root Gradings of Simply-laced Type

In this chapter, we begin our investigation of Φ-graded groups for specific root systems Φ with the case that Φ is simply-laced (that is, of type A, D or E) and of rank at least 2. The main result of this section is Theorem 5.7.14: Every Φ-graded group is coordinatised by a ring which must be associative if Φ is of rank at least 3 and commutative if Φ is of type D or E. It turns out that, after our hard work to establish the parametrisation theorem, there is actually not much left to do to prove this. We can even carry out the complete coordinatisation of these groups without using the blueprint technique.

The coordinatisation result of this section has already been proven by Shi in [Shi93], but for a more restrictive definition of root gradings. (See also section 2.7.) Hence our results are more general than Shi's, though many of the arguments are the same. The main additional difficulty in our generality is the sign problem in the sense of Remark 4.1.23, which we solve using the parametrisation theorem. A corollary of our coordinatisation result is that every simply laced root graded group in our sense is also a root graded group in Shi's sense. Thus the two notions of root gradings are actually equivalent, but this is a non-trivial fact.

This chapter is organised in the way described in section 4.7. We begin with a brief introduction to nonassociative rings. In the following section, we study some purely combinatorial properties of the simply-laced root systems and compute the Cartan integers for these root systems. In section 5.3, we discuss the existence problem for simply-laced root graded groups. The main mathematical work of this chapter happens in section 5.4: We prove that Φ-graded groups satisfy the square formula for Weyl elements. In sections 5.5 and 5.6, we introduce standard twisting systems and standard signs for root graded groups, respectively. Further, we show that every ring which coordinatises a Φ-graded group (with standard signs) must be associative if Φ is of rank at least 3 and commutative if Φ is, in addition, of type D or E. In section 5.7, we carry out the coordinatisation of simply-laced root graded groups.

5.1 Nonassociative Rings

One of our first actions in this thesis was to announce the frightening Convention 1.1.3 that rings are not assumed to be associative. We will sometimes call these objects "nonassociative rings" to emphasise this convention. In this section, we introduce the most basic notions which are relevant in the theory of nonassociative rings. Technically, only the definition of these objects is needed in this chapter, but it seems prudent to go a bit beyond that. Some of the results in this section will only be needed in later chapters.

Of particular interest to us are rings which satisfy the so-called alternative laws, and which are called alternative rings. While we can merely show that every A_2-graded group is coordinatised by a nonassociative ring with no further properties, all known examples of rings which appear as coordinatising rings of A_2-graded groups are in fact alternative. See section 5.3 for a few more details. However, since we do not need the notion of alternative rings until we investigate root graded groups of type $(B)C$ in chapter 9, we delay their introduction until section 8.2.

A standard reference for most of the material in this section is [Sch66].

5.1.1 Definition (Ring). A *ring* is a triple $(\mathcal{R}, +, \cdot)$ consisting of a set $\mathcal{R}$ and two binary operations
$$+, \cdot \colon \mathcal{R} \times \mathcal{R} \to \mathcal{R}$$
such that the following conditions are satisfied:

(i) $(\mathcal{R}, +)$ is an abelian group.

(ii) There exists an element $1_\mathcal{R} \in \mathcal{R}$ such that $1_\mathcal{R} \cdot x = x = x \cdot 1_\mathcal{R}$ for all $x \in \mathcal{R}$.

(iii) The distributive laws are satisfied. That is,
$$(x + y) \cdot z = (x \cdot z) + (y \cdot z) \quad \text{and} \quad x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$
for all $x, y, z \in \mathcal{R}$.

A *subring of* $\mathcal{R}$ is a subset of $\mathcal{R}$ which is closed under addition, additive inversion and multiplication and which contains $1_\mathcal{R}$. An *ideal of* $\mathcal{R}$ is a subgroup I of $(\mathcal{R}, +)$ such that $ax \in I$ and $xa \in I$ for all $a \in \mathcal{R}$ and $x \in I$.

5.1.2 Definition (Homomorphism of rings). Let $\mathcal{R}$, $\mathcal{S}$ be two rings. A map $f \colon \mathcal{R} \to \mathcal{S}$ is called a *homomorphism of rings* if it preserves addition and multiplication and $f(1_\mathcal{R}) = 1_\mathcal{S}$.

5.1.3 Notation for this section. For the rest of this section, $(\mathcal{R}, +, \cdot)$ is a ring.

5.1.4 Notation. The neutral element of the group $(\mathcal{R}, +)$ will always be denoted by $0_\mathcal{R}$ and the identity element from Axiom 5.1.1 (ii) will always be denoted by $1_\mathcal{R}$. Further, we will often leave out the multiplication dot.

5.1.5 Remark. For any ideal I of $\mathcal{R}$, the quotient $\mathcal{R}/I$ has a canonical ring structure, as in the associative setting.

The standard parameter system for a ring can be defined immediately.

5.1.6 Definition (Standard parameter system). Put $A := \{\pm 1\}$ and declare that A acts on $(\mathcal{R}, +)$ by inversion. That is, we put $1_A.r := r$ and $-1_A.r := -r$. Then the pair $(A, \mathcal{R})$ is called the *standard parameter system for* $\mathcal{R}$.

It is a well-established fact that the commutator is a useful tool for studying noncommutative structures. Naturally, we can apply the same idea to nonassociative structures.

5.1.7 Definition (Associator). The *associator map* or simply the *associator* is defined by
$$\mathcal{R} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}, (a, b, c) \mapsto [a, b, c] := (ab)c - a(bc).$$

A ring element of the form $[x, y, z]$ for some $x, y, z \in \mathcal{R}$ will also be called an *associator*. For any subsets A, B, C of $\mathcal{R}$, we denote by $[A, B, C]$ the ideal which is generated by $\{ [a, b, c] \mid a \in A, b \in B, c \in C \}$.

By the distributive laws, the associator map is additive in each component. Many of the additional properties which can be imposed on rings, such as the alternative laws, can be phrased in terms of vanishing of certain associators. We will get back to this in Definition 8.2.2.

5.1.8 Remark. Clearly, $\mathcal{R}/[\mathcal{R}, \mathcal{R}, \mathcal{R}]$ is an associative ring, and any ideal I for which $\mathcal{R}/I$ is associative contains $[\mathcal{R}, \mathcal{R}, \mathcal{R}]$.

It is well-known that any associative ring has a center, which is itself an associative commutative ring. We can define a similar object for nonassociative rings, called the nucleus, but here the definition is a bit more delicate because we have to distinguish between left, middle and right nucleus. We will see in Remark 8.2.8 that this distinction is superfluous for alternative rings.

5.1.9 Definition (Nucleus and center). We define the following subsets of $\mathcal{R}$, called the *left nucleus*, *middle nucleus* and *right nucleus:*

$$\begin{aligned} \mathrm{LNucl}(\mathcal{R}) &:= \{ x \in \mathcal{R} \mid [x, y, z] = 0 \text{ for all } y, z \in \mathcal{R} \}, \\ \mathrm{MNucl}(\mathcal{R}) &:= \{ y \in \mathcal{R} \mid [x, y, z] = 0 \text{ for all } x, z \in \mathcal{R} \}, \\ \mathrm{RNucl}(\mathcal{R}) &:= \{ z \in \mathcal{R} \mid [x, y, z] = 0 \text{ for all } x, y \in \mathcal{R} \}. \end{aligned}$$

Further, we define the *nucleus of* $\mathcal{R}$ by

$$\mathrm{Nucl}(\mathcal{R}) := \mathrm{LNucl}(\mathcal{R}) \cap \mathrm{MNucl}(\mathcal{R}) \cap \mathrm{RNucl}(\mathcal{R}).$$

Finally, the *center of* $\mathcal{R}$ is

$$Z(\mathcal{R}) := \{ x \in \mathrm{Nucl}(\mathcal{R}) \mid xy = yx \text{ for all } x, y \in \mathcal{R} \} \subseteq \mathrm{Nucl}(\mathcal{R}).$$

5.1.10 Remark ([Sch66, p. 13]). By the distributive laws, all the nuclei defined in 5.1.9 are closed under addition and additive inverse. Less obviously, they are also closed under multiplication: Let $x, x', y, z \in \mathcal{R}$. If $x, x' \in \mathrm{LNucl}\,\mathcal{R}$, then

$$\big((xx')y\big)z = \big(x(x'y)\big)z = x\big((x'y)z\big) = x\big(x'(yz)\big) = (xx')(yz),$$

so $xx' \in \mathrm{LNucl}(\mathcal{R})$. If $x, x' \in \mathrm{MNucl}(\mathcal{R})$, then

$$\big(y(xx')\big)z = \big((yx)x'\big)z = (yx)(x'z) = y\big(x(x'z)\big) = y\big((xx')z\big),$$

so $xx' \in \mathrm{MNucl}(\mathcal{R})$. If $x, x' \in \mathrm{RNucl}(\mathcal{R})$, then

$$(yz)(xx') = \big((yz)x\big)x' = \big(y(zx)\big)x' = y\big((zx)x'\big)$$

so $xx' \in \mathrm{RNucl}(\mathcal{R})$. Further, the center of $\mathcal{R}$ has the same properties because

$$(xx')y = x(x'y) = x(yx') = (xy)x' = (yx)x' = y(xx')$$

for all $x, x' \in Z(\mathcal{R})$ and $y \in \mathcal{R}$. Since $1_{\mathcal{R}}$ is clearly contained in the center of $\mathcal{R}$, it follows that all the sets defined in 5.1.9 are subrings of $\mathcal{R}$.

5.1.11 Remark ([McC04, 21.2.1]). Since nuclear elements, in the words of McCrimmon, “slip in and out of parentheses”, we have

$$\begin{aligned} n[x, y, z] &= [nx, y, z], & [xn, y, z] &= [x, ny, z], \\ [x, yn, z] &= [x, y, nz], & [x, y, zn] &= [x, y, z]n \end{aligned}$$

for all $x, y, z \in \mathcal{R}$ and all $n \in \mathrm{Nucl}(\mathcal{R})$. For example,

$$\begin{aligned} n[x,y,z] &= n\big((xy)z\big) - n\big(x(yz)\big) = \big(n(xy)\big)z - (nx)(yz) \\ &= \big((nx)y\big)z - (nx)(yz) = [nx,y,z]. \end{aligned}$$

In Lemma 8.2.12, we will see stronger versions of these formulas for alternative rings .

In an associative ring, an element a is called *invertible* if there exists $b \in \mathcal{R}$ such that $ab = ba = 1_{\mathcal{R}}$. We could make the same definition for nonassociative rings, but the resulting notion of invertibility is not satisfactory. To see why, assume that we have ring elements a, x, y such that $ax = y$. If a is invertible with inverse b and the ring is associative, then we can multiply by b from the left to obtain

$$by = b(ax) = (ba)x = x.$$

Clearly, this computation relies on the associativity law. Thus, if we want that $ax = y$ implies $x = by$, we need to strengthen our assumptions.

5.1.12 Definition (Invertibility, [Fau14, p. 190]). For all $x \in \mathcal{R}$, we define *left and right multiplication maps* $l_x, r_x \colon \mathcal{R} \to \mathcal{R}$ by $l_x(y) := x \cdot y$ and $r_x(y) := y \cdot x$ for all $y \in \mathcal{R}$. For any $x, y \in \mathcal{R}$, we say that y is an *inverse of* x if l_x and r_x are invertible maps such that $l_x^{-1} = l_y$ and $r_x^{-1} = r_y$. Further, we say that x is *invertible* if there exists $z \in \mathcal{R}$ such that z is an inverse of x.

5.1.13 Note. There exists no standard notion of invertibility in nonassociative rings. For example, elements $a, b \in \mathcal{R}$ are called *inverses of each other* in [Sch66, p. 38] if $ab = 1_{\mathcal{R}} = ba$, which by Note 5.1.20 is a weaker notion of invertibility than ours. However, we will see in Lemma 8.2.9 that the two notions are equivalent for alternative rings.

Our notion of invertibility is precisely the one which relates to Weyl-invertible elements in root gradings (in the sense of Definition 2.2.2 (e)). This fact was first observed by Faulkner in [Fau14, Theorem 13.8], and we will prove it in Proposition 5.6.6 (a).

5.1.14 Remark. The product of two invertible elements is not necessarily invertible. This is, of course, a major drawback of our notion of invertibility.

5.1.15 Definition (Division ring). A *division ring* is a ring in which every non-zero element is invertible.

5.1.16 Remark. Let $x, y \in \mathcal{R}$. Then y is an inverse of x if and only if all the maps $l_x \circ l_y$, $l_y \circ l_x$, $r_x \circ r_y$ and $r_y \circ r_x$ are trivial, which is to say that

$$x(yz) = y(xz) = z = (zx)y = (zy)x \quad \text{for all } z \in \mathcal{R}.$$

Thus clearly, if y is an inverse of x, then x is an inverse of y.

5.1.17 Lemma. *For all $x, y \in \mathcal{R}$, the following assertions are equivalent:*

(i) x is an inverse of y.

(ii) $xy = yx = 1_{\mathcal{R}}$ and $[x,y,z] = [y,x,z] = [z,x,y] = [z,y,x] = 0_{\mathcal{R}}$ for all $z \in \mathcal{R}$.

Proof. At first, assume that x is an inverse of y. Putting $z := 1_{\mathcal{R}}$ in Remark 5.1.16 yields that $xy = 1_{\mathcal{R}} = yx$. Further, it follows from Remark 5.1.16 that $x(yz) = z = 1_{\mathcal{R}}z = (xy)z$ for all $z \in \mathcal{R}$, so $[x,y,z] = 0_{\mathcal{R}}$. The remaining statements in (ii) can be proven similarly. In essentially the same way, it follows from (ii) that x is an inverse of y. □

5.1.18 Lemma. *Let $x \in \mathcal{R}$ be invertible. Then the following statements are equivalent for all $y \in \mathcal{R}$:*

(i) y is an inverse of x.

(ii) $xy = 1_{\mathcal{R}}$.

(iii) $yx = 1_{\mathcal{R}}$.

Further, there exists a unique $y \in \mathcal{R}$ satisfying these properties.

Proof. If y is an inverse of x, we have already shown in Lemma 5.1.17 that $xy = 1_{\mathcal{R}} = yx$. Conversely, assume that $y \in \mathcal{R}$ satisfies $xy = 1_{\mathcal{R}}$. Since x is invertible, there exists an element x^{-1} which is an inverse of x. Multiplying the equation $xy = 1_{\mathcal{R}} = xx^{-1}$ by x^{-1} from the left, we obtain $y = x^{-1}$. Thus any $y \in \mathcal{R}$ satisfying $xy = 1_{\mathcal{R}}$ equals x^{-1}. Similarly, we can prove that any $y \in \mathcal{R}$ satisfying $yx = 1_{\mathcal{R}}$ equals x^{-1}. In particular, every inverse of x equals x^{-1}, so all inverses of x are identical. This finishes the proof. □

5.1.19 Notation. If x is an invertible element of $\mathcal{R}$, we denote its unique inverse by x^{-1}.

5.1.20 Note. The assumption that x is invertible in Lemma 5.1.18 is crucial. In general, if x and y are arbitrary elements of $\mathcal{R}$ such that $xy = 1_{\mathcal{R}} = yx$, then by Lemma 5.1.17 it is not clear that x and y are invertible. Further, the product of two invertible ring elements is not necessarily invertible.

For nuclear elements, the warning in Note 5.1.20 does not apply.

5.1.21 Lemma. *Let $x \in \operatorname{Nucl}\mathcal{R}$ and $y \in \mathcal{R}$ such that $xy = 1_{\mathcal{R}} = yx$. Then x is invertible and $x^{-1} = y$.*

Proof. This follows from Lemma 5.1.17. □

We end this section with a brief introduction of modules over nonassociative rings which will be needed in chapter 9. The definition is word by word the same as for associative rings. However, for reasons that we will see in Observation 5.1.23, it is not standard in nonassociative algebra. In fact, we are not aware of a single reference which uses this definition.

5.1.22 Definition (Module). A *right $\mathcal{R}$-module* is an abelian group $(M, +)$ together with a scalar multiplication $\star\colon M \times \mathcal{R} \to M$ satisfying the following conditions for all $r, s \in \mathcal{R}$ and all $v, w \in M$:

(i) $(v + w) \star r = (v \star r) + (w \star r)$.

(ii) $v \star (r + s) = (v \star r) + (v \star s)$.

(iii) $v \star (r \cdot s) = (v \star r) \star s$.

(iv) $v \star 1_{\mathcal{R}} = v$.

It is called *faithful* if for any $r \in \mathcal{R}$ with $v \star r = 0$ for all $v \in M$, we have $r = 0_{\mathcal{R}}$. Further, *left $\mathcal{R}$-modules* are defined similarly.

5.1.23 Observation. Let M be a module over $\mathcal{R}$. Let $a, b, c \in \mathcal{R}$ and let $v \in M$. Using only Axiom 5.1.22 (iii), we see that

$$v \star \big((ab)c\big) = \big(v \star (ab)\big) \star c = \big((v \star a) \star b\big) \star c = (v \star a) \star (bc) = v \star \big(a(bc)\big).$$

Thus M is actually a module over the associative ring $\mathcal{R}/[\mathcal{R}, \mathcal{R}, \mathcal{R}]$. This explains why Definition 5.1.22 is not standard in the theory of nonassociative rings. However, it is of use in the theory of root graded groups: We will see root graded groups of type $(B)C$ in which some root groups are coordinatised by a ring $\mathcal{R}$ while other root groups are coordinatised by an algebraic structure which involves a module over $\mathcal{R}$. In this setting, we do not have the "freedom" to replace $\mathcal{R}$ by $\mathcal{R}/[\mathcal{R},\mathcal{R},\mathcal{R}]$ because the full ring $\mathcal{R}$ is needed as a coordinatising structure for some root groups.

On a final note, recall that a module over an associative ring $\mathcal{R}$ can also be seen as an abelian group $(M,+)$ together with a homomorphism $\mathcal{R} \to \operatorname{End}(M,+)$. Since the ring $\operatorname{End}(M,+)$ is associative, it is natural from this viewpoint that any module structure over $\mathcal{R}$ should factor through $\mathcal{R}/[\mathcal{R},\mathcal{R},\mathcal{R}]$.

5.2 The Simply-laced Root Systems

5.2.1 Notation for this section. Unless otherwise specified, we denote by Φ a simply-laced root system in the sense of Definition 1.2.61.

In this section, we investigate the simply-laced root systems. We begin with some more characterisations of these objects.

5.2.2 Remark. Let Φ be a crystallographic root system. Then the following properties are equivalent:

(1) Φ is simply-laced in the sense of Definition 1.2.61. That is, all irreducible components of Φ are of type A, D or E.

(2) For all non-proportional roots α, β, we have $]\alpha,\beta[_{\mathrm{Cox}} \subseteq \{\alpha+\beta\}$. In particular, $]\alpha,\beta[\, = \,]\alpha,\beta[_{\mathrm{Cox}}$.

(3) If α, β are two non-proportional roots in Φ, then $\Phi \cap \langle \alpha, \beta \rangle_{\mathbb{R}}$ is of type $A_1 \times A_1$ or A_2.

(4) If Δ is a root base of Φ, then for all distinct $\alpha, \beta \in \Delta$, the order of $\sigma_\alpha \sigma_\beta$ is 2 or 3. (This says precisely that all edges in the Coxeter diagram of Φ are simple in the sense of Remark 1.2.55, which motivates the terminology "simply-laced".)

If Φ is irreducible, then these properties are equivalent to the property that all roots have the same length.

We now turn to the standard representations, that is, to the actual definitions of the root systems A and D. The standard representation of D will never be needed, but we give it anyway because it is easy to write down. We skip over the more complicated standard representations of E_6, E_7 and E_8, however. They can be found, like all the other standard representations, in [Hum72, Section 12.1].

5.2.3 Remark (Standard representation of A_n). Let $n \in \mathbb{N}_+$ and let V be a Euclidean space of dimension $n+1$ with orthonormal basis $(e_1, \dots, e_{n+1})$. The *standard representation of A_n* is

$$A_n := \{ e_i - e_j \mid i \neq j \in [1, n+1] \}.$$

Note that a root $e_i - e_j$ is uniquely determined by the tuple (i,j) of indices. For this reason, we will often use the notation ij in place of $e_i - e_j$. For example, we could write U_{ij} for the root group $U_{e_i - e_j}$. The *standard (ordered) root base* is

$$\Delta := \{ e_i - e_{i+1} \mid i \in [1, n] \} \qquad \big(\text{or } \Delta_{\mathrm{ord}} := (e_1 - e_2, \dots, e_n - e_{n+1})\big)$$

and the corresponding positive system is

$$\Pi := \{ e_i - e_j \mid i < j \in [1, n+1] \}.$$

Observe that V is not spanned by A_n.

5.2.4 Remark (Standard representation of D_n). Let V be a Euclidean space of dimension $n \geq 4$ with orthonormal basis $(e_1, \dots, e_n)$. The *standard representation of* D_n is

$$D_n := \{ \varepsilon_1 e_i + \varepsilon_2 e_j \mid i \neq j \in [1, n], \varepsilon_1, \varepsilon_2 \in \{\pm 1\} \}.$$

The *standard root base* is

$$\Delta := \{ e_i - e_{i+1} \mid i \in [1, n-1] \} \cup \{ e_{n-1} + e_n \}$$

and the corresponding positive system is

$$\Pi := \{ e_i - e_j \mid i < j \in [1, n] \} \cup \{ e_i + e_j \mid i \neq j \in [1, n] \}.$$

5.2.5 Remark. It is clear to see that the standard root base of A_{n-1} can be regarded as a subset of the standard root base of D_n. Thus D_n contains A_{n-1} as a root subsystem. This subsystem is, in fact, parabolic.

5.2.6 Definition (A_2-pairs and A_2-triples). Let Φ be any root system. An *A_2-pair (in Φ)* is a tuple (α, γ) of roots such that $\{ \alpha, \gamma \}$ is a root base of the parabolic root subsystem that it spans. An *A_2-triple (in Φ)* is a triple (α, β, γ) of roots such that (α, γ) is an A_2-pair and $\beta = \alpha + \gamma$.

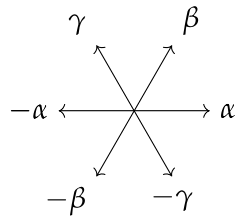


Figure 5.1: An A_2-triple (α, β, γ).

5.2.7 Remark. In an A_2-triple (α, β, γ), the roots α, β, γ are exactly the positive roots of the corresponding A_2-subsystem (with respect to the root base (α, γ)). Further, the reflection σ_α interchanges β with γ, $-\beta$ with $-\gamma$ and α with $-\alpha$. See Figure 5.1 for an illustration.

5.2.8 Note. The terminology of "A_2-pairs" is borrowed from [Shi93, (1.4)]. We will introduce obvious variants of this notion for the root systems B_2 and BC_2 as well (Definitions 7.2.4 and 9.1.7).

We now collect some simple facts about simply-laced root systems.

5.2.9 Remark (The Weyl group of A_n). Let $n \in \mathbb{N}_+$ and let Δ be the standard root base of A_n. An easy computation shows that

$$e_i^{\sigma(e_j - e_k)} = e_{i(j\,k)}$$

for all $i, j, k \in [1, n+1]$ with $j \neq k$ where $(j\ k)$ denotes the transposition which interchanges j and k. It follows that there exists an isomorphism φ from the Weyl group W of A_n to the symmetric group on $[1, n+1]$ which maps $\sigma(e_j - e_k)$ to $(j\ k)$ for all distinct $j, k \in [1, n+1]$. This isomorphism satisfies $e_i^w = e_{i\varphi(w)}$ for all $w \in W$ and $i \in [1, n+1]$. Observe that φ depends on the choice of the ordered orthonormal base $(e_1, \dots, e_{n+1})$.

5.2.10 Remark. If Φ is irreducible and of rank at least 2 (or more generally, if Φ does not contain an irreducible component of type A_1), then every root lies in an A_2-triple.

5.2.11 Lemma. *Let α, β be roots. Then the subsystem which is spanned by α, β is of type A_1, A_2 or $A_1 \times A_1$.*

Proof. This follows from Remark 1.2.56. □

By the following result, we do not have to distinguish between A_2-subsystems, closed A_2-subsystems and parabolic A_2-subsystems of Φ.

5.2.12 Lemma. *Every subsystem of Φ of rank at most 2 is parabolic.*

Proof. Since Φ is reduced, every rank-1 subsystem is parabolic. Now let Φ' be a rank-2 subsystem and denote by Φ'' the parabolic subsystem that Φ' spans. Then Φ'' is of type A_2 or $A_1 \times A_1$ by Lemma 5.2.11. Since none of these root systems contains a proper rank-2 subsystem, it follows that $\Phi' = \Phi''$. Thus Φ' is parabolic. □

5.2.13 Lemma. *If $\alpha, \beta \in \Phi$ are orthogonal, then α is adjacent to β and $-\beta$.*

Proof. Since A_2 has no pair of orthogonal roots, it follows from Lemma 5.2.11 that the subsystem spanned by α, β must be of type $A_1 \times A_1$. The assertion follows. □

We now compute the Cartan integers. They will play an important role in the proof of the square formula for Weyl elements (Proposition 5.4.17).

5.2.14 Proposition (Cartan integers). *Let α, β be roots. The Cartan integer $\langle \beta | \alpha \rangle = 2\frac{\beta \cdot \alpha}{\alpha \cdot \alpha}$ is determined as follows:*

(a) *$\langle \beta | \alpha \rangle = 0$ if and only if α and β are orthogonal (or equivalently, if and only if they lie in no common A_2-subsystem).*

(b) *$\langle \beta | \alpha \rangle = -1$ if and only if (α, β) is an A_2-pair.*

(c) *$\langle \beta | \alpha \rangle = 1$ if and only if $(\alpha, -\beta)$ is an A_2-pair.*

(d) *$\langle \beta | \alpha \rangle = 2$ if and only if $\alpha = \beta$.*

(e) *$\langle \beta | \alpha \rangle = -2$ if and only if $\alpha = -\beta$.*

Further, $\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle$.

Proof. At first, we assume that Φ is irreducible. Since all roots have the same length by Remark 5.2.2, we have $\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle$. The first assertion is clear. If (α, β) is an A_2-pair, then we can easily compute the Cartan numbers

$$\langle \beta | \alpha \rangle = -1, \quad \langle -\beta | \alpha \rangle = 2, \quad \langle \alpha | \alpha \rangle = 2, \quad \langle -\alpha | \alpha \rangle = -2,$$

which proves one implication in the remaining assertions. Conversely, assume that $\langle \beta | \alpha \rangle \neq 0$. Then α and β lie in an A_2-subsystem by the first assertion. Since we have just computed all Cartan integers which appear in a root system of type A_2, the other implications follow.

We now consider the general case. If the subsystem spanned by α and β is of type A_2 or A_1, then they lie in a common irreducible subsystem and the observations from the previous paragraph apply. Otherwise they lie in a subsystem of type $A_1 \times A_1$, so $\langle \alpha | \beta \rangle = 0 = \langle \beta | \alpha \rangle$. In this case, all assertions hold as well. □

We end this section with the notion of positive A_2-pairs. They will be needed to properly define coordinatisations of A_n-graded groups (Definition 5.6.2).

5.2.15 Definition (Positive A_2-pair). Let (α, β), (α', β') be two A_2-pairs in Φ. We say that (α', β') is *(α, β)-positive* if they lie in the same orbit under the Weyl group $\mathrm{Weyl}(\Phi)$, that is, if there exists $u \in \mathrm{Weyl}(\Phi)$ such that $\alpha^u = \alpha'$ and $\beta^u = \beta'$.

The following result is an easy computation. It is the underlying reason why any ring which coordinatises a root graded group of type D or E is commutative while the same is not true for root graded groups of type A.

5.2.16 Lemma ([BM92b, 1.4]). *Let (α_0, β_0) be an A_2-pair in Φ, and assume that Φ is irreducible.*

(a) If Φ is of type A, then the set of A_2-pairs has exactly two orbits under the Weyl group of Φ: The set of (α_0, β_0)-positive A_2-pairs and the set of (β_0, α_0)-positive A_2-pairs.

(b) If Φ is of type D or E, then every A_2-pair is (α_0, β_0)-positive.

The following result on positive A_2-pairs will be needed to prove that the coordinatising ring of an A_3-graded group is associative (Proposition 5.6.9).

5.2.17 Lemma. *Assume that Φ is of rank at least 3 and choose some A_2-pair (α_0, γ_0). Then there exist roots α, β, γ with the following properties:*

(i) $\langle \alpha, \beta, \gamma \rangle_{\mathbb{R}} \cap \Phi$ is a root subsystem of type A_3 with root base $\{ \alpha, \beta, \gamma \}$.

(ii) (α, β) and (β, γ) are (α_0, γ_0)-positive A_2-pairs.

Further, any such choice of roots has the property that $(\alpha + \beta, \gamma)$ and $(\alpha, \beta + \gamma)$ are (α_0, γ_0)-positive A_2-pairs.

Proof. Since Φ is of type A, D or E, it is easy to see that Φ has a parabolic subsystem of type A_3. Thus we can choose roots α, β, γ such that $\{ \alpha, \beta, \gamma \}$ is a root base of a parabolic A_3-subsystem of Φ and such that (α, β), (β, γ) are A_2-pairs. We will show that either both (α, β) and (β, γ) or both (γ, β) and (β, α) are (α_0, γ_0)-positive. Note that (α, β) and (β, γ) lie in the same orbit under the Weyl group because $\sigma_\gamma \sigma_\beta \sigma_\alpha$ maps (α, β) to (β, γ):

$$\alpha \xrightarrow{\sigma(\gamma)} \alpha \xrightarrow{\quad\sigma(\beta)\quad} \alpha + \beta \xrightarrow{\quad\sigma(\alpha)\quad} -\alpha + (\alpha + \beta) = \beta,$$

$$\beta \xrightarrow{\sigma(\gamma)} \beta + \gamma \xrightarrow{\sigma(\beta)} -\beta + (\beta + \gamma) = \gamma \xrightarrow{\quad\sigma(\alpha)\quad} \gamma.$$

By the same argument, (β, α) and (γ, β) lie in the same orbit. Thus for the existence assertion, it only remains to show that (α, β) or (β, α) is (α_0, γ_0)-positive. This holds by Lemma 5.2.16.

Now let α, β, γ be any roots with the desired properties. Then σ_α maps (β, γ) to $(\alpha + \beta, \gamma)$, so $(\alpha + \beta, \gamma)$ is an (α_0, γ_0)-positive A_2-pair (because (β, γ) has the same properties). Similarly, σ_γ maps (α, β) to $(\alpha, \beta + \gamma)$, so $(\alpha, \beta + \gamma)$ is also an (α_0, γ_0)-positive A_2-pair. This finishes the proof. $\square$

5.3 Construction of Simply-laced Root Graded Groups

For simply-laced root graded groups, the existence problem is (nearly) completely solved by the theory of Chevalley groups. In this sections, we collect some remarks on this topic. Recall that the existence problem was defined in Note 4.1.24.

5.3.1 The existence problem for types D and E. For any root system Φ, the Chevalley groups of type Φ constitute examples of Φ-graded groups for any commutative associative ring. Since we will show that every root graded group of type D or E is coordinatised by such a ring, the Chevalley groups thus provide a complete solution of the existence problem for these types. We emphasise that this does not mean that every root graded group of type D or E is isomorphic to a Chevalley group. Instead, it means that every such group G is coordinatised by a commutative associative ring $\mathcal{R}$ which also coordinatises some Chevalley group H such that G and H satisfy the same commutator relations.

5.3.2 The existence problem for type A in rank at least 3. We will show that A_n-graded groups for $n \geq 3$ are coordinatised by associative rings which need not be commutative. Thus the Chevalley groups only provide a partial solution of the existence problem. However, a complete solution of this problem is not far away: Recall from Examples 3.2.14 and 3.2.22 that for any commutative ring $\mathcal{R}$, the group $\mathrm{E}_{n+1}(\mathcal{R})$ is a Chevalley group of type A_n and that the definition of this group makes sense even when $\mathcal{R}$ is not assumed to be commutative. It is not difficult to see that $\mathrm{E}_{n+1}(\mathcal{R})$ is A_n-graded for any associative ring $\mathcal{R}$. Thus this group solves the existence problem for type A_n.

It is noteworthy that, even though Chevalley groups do not solve the existence problem for type A_n, they are still sufficiently generic in the sense of Remark 4.1.23 to read off a parity map for the parametrisation of arbitrary A_n-graded groups.

5.3.3 Note (Existence problem for A_2-graded groups). The rank-2 case (which, we recall, is only of secondary concern in this thesis) is more difficult. For any alternative ring $\mathcal{R}$, Faulkner constructs in [Fau83, Section 3] a Jordan pair $V = V(\mathcal{R})$, a Tits-Kantor-Koecher algebra $T = \mathrm{TKK}(V)$ and a group $G(\mathcal{R}) = G(T)$ such that $G(\mathcal{R})$ is A_2-graded and coordinatised by $\mathcal{R}$. Essentially the same construction, but phrased without the language of Jordan pairs, can also be found in the appendix of [Fau89]. This is the most general known construction of A_2-graded groups. There is no known example of an A_2-graded group which is coordinatised by a ring that is not alternative. However, the problem whether every such ring is alternative remains open.

5.4 Rank-2 Computations

5.4.1 Notation for this section. We denote by Φ any irreducible simply-laced root system of rank at least 2 and by G a group which has Φ-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi}$ (in the sense of Definition 2.1.4). Further, we assume that all for all non-proportional roots α, β, we have $U_\alpha \cap U_\beta = \{1_G\}$.

In this section, we study the action of Weyl elements in simply-laced root graded groups. Our goal is to show that G satisfies the square formula for Weyl elements (Proposition 5.4.17). This is also essentially the content of [Shi93, (2.4)(iii)] and [TW02, (19.3)], but we prove it in a more general setting (see section 2.7). The core arguments, however, are mostly the same.

Most of the time, we will fix some A_2-triple and perform all computations in the corresponding A_2-graded subgroup (see Proposition 2.5.11). Since every root lies in a parabolic subsystem of type A_2 by Remark 5.2.10, this is no restriction of generality.

Observe that we do not assume that $U_\alpha^\sharp$ is non-empty for all roots α. However, we will most of the time make statements about arbitrary Weyl elements, and these statements are of course empty if no such elements exist. All statements which do not involve Weyl elements are independent of their existence, however. Further, note that we do not require Axiom 2.5.2 (iv), simply because all proofs work without difficulty under the weaker assumption that $U_\alpha \cap U_\beta = \{1_G\}$ for all non-proportional roots α, β. See also Note 2.5.13.

We begin with a computational result which will be our main tool in this section. It is essentially the special case of $n = 3$ in Proposition 2.2.10 (which is the equivalent of [TW02, (6.4)]), but we prove it again in a slightly different way. The same arguments can be found in the proof of [Shi93, (2.4)]. The same result for RGD-systems is proven in [TW02, (19.1)].

5.4.2 Lemma. *Let (α, β, γ) be an A_2-triple, let $a_{-\alpha}, c_{-\alpha} \in U_{-\alpha}$, let $b_\alpha \in U_\alpha$ and set $w_\alpha := a_{-\alpha} b_\alpha c_{-\alpha}$. Then the following statements hold:*

(a) *For any $x_\gamma \in U_\gamma$, we have $x_\gamma^{w_\alpha} \in U_\beta$ if and only if $x_\gamma = [[x_\gamma, b_\alpha], c_{-\alpha}]^{-1}$. In this case, $x_\gamma^{w_\alpha} = [x_\gamma, b_\alpha]$.*

(b) *For any $x_\beta \in U_\beta$, we have $x_\beta^{w_\alpha} \in U_\gamma$ if and only if $x_\beta = [[x_\beta, a_{-\alpha}], b_\alpha]^{-1}$. In this case, $x_\beta^{w_\alpha} = [x_\beta, a_{-\alpha}]$.*

Proof. Let $x_\gamma \in U_\gamma$. Using the commutator axiom of root graded groups and Relation 1.1.11 (i), we perform the following computation:

$$x_\gamma^{w_\alpha} = x_\gamma^{b_\alpha c_{-\alpha}} = \big(x_\gamma [x_\gamma, b_\alpha]\big)^{c_{-\alpha}} = x_\gamma^{c_{-\alpha}} [x_\gamma, b_\alpha]^{c_{-\alpha}} = x_\gamma [x_\gamma, b_\alpha] \big[[x_\gamma, b_\alpha], c_{-\alpha}\big].$$

Note that $[[x_\gamma, b_\alpha], c_{-\alpha}]$ lies in U_γ while $[x_\gamma, b_\alpha]$ lies in U_β, so these two terms commute. Thus we can write

$$x_\gamma^{w_\alpha} = x_\gamma \big[[x_\gamma, b_\alpha], c_{-\alpha}\big] [x_\gamma, b_\alpha]$$

with $x_\gamma[[x_\gamma, b_\alpha], c_{-\alpha}] \in U_\gamma$ and $[x_\gamma, b_\alpha] \in U_\beta$. Thus if $x_\gamma = [[x_\gamma, b_\alpha], c_{-\alpha}]^{-1}$, then $x_\gamma^{w_\alpha}$ lies in U_β, and in this case $x_\gamma^{w_\alpha} = [x_\gamma, b_\alpha]$. Conversely, if $x_\gamma^{w_\alpha}$ lies in U_β, then

$$x_\gamma^{w_\alpha} [x_\gamma, b_\alpha]^{-1} = x_\gamma \big[[x_\gamma, b_\alpha], c_{-\alpha}\big]$$

where the left-hand side lies in U_β and the right-hand side lies in U_γ. Since $U_\beta \cap U_\gamma = \{1_G\}$ by assumption, we infer that $x_\gamma[[x_\gamma, b_\alpha], c_{-\alpha}] = 1_G$. In other words, $x_\gamma = [[x_\gamma, b_\alpha], c_{-\alpha}]^{-1}$. This finishes the proof of (a).

Now let $x_\beta \in U_\beta$. We perform the same kind of computation as in (a), but the result is slightly more complicated because x_β and $a_{-\alpha}$ do not commute:

$$\begin{aligned} x_\beta^{w_\alpha} &= \big(x_\beta [x_\beta, a_{-\alpha}]\big)^{b_\alpha c_{-\alpha}} = \big(x_\beta [x_\beta, a_{-\alpha}] \big[[x_\beta, a_{-\alpha}], b_\alpha\big]\big)^{c_{-\alpha}} \\ &= x_\beta [x_\beta, c_{-\alpha}] [x_\beta, a_{-\alpha}] \big[[x_\beta, a_{-\alpha}], b_\alpha\big] \Big[\big[[x_\beta, a_{-\alpha}], b_\alpha\big], c_{-\alpha}\Big]. \end{aligned}$$

Note that x_β and $[[x_\beta, a_{-\alpha}], b_\alpha]$ lie in U_β while all the other factors on the right-hand side lie in U_γ. It follows that if $x_\beta = [[x_\beta, a_{-\alpha}], b_\alpha]^{-1}$, then $x_\beta^{w_\alpha}$ lies in U_γ. Conversely, if $x_\beta^{w_\alpha}$ lies in U_γ, then

$$x_\beta \big[[x_\beta, a_{-\alpha}], b_\alpha\big] = x_\beta^{w_\alpha} \bigg([x_\beta, c_{-\alpha}] [x_\beta, a_{-\alpha}] \Big[\big[[x_\beta, a_{-\alpha}], b_\alpha\big], c_{-\alpha}\Big] \bigg)^{-1}$$

where the left-hand side lies in U_β and the right-hand side lies in U_γ. Since $U_\beta \cap U_\gamma = \{1_G\}$, it follows that $x_\beta = [[x_\beta, a_{-\alpha}], b_\alpha]^{-1}$ and

$$x_\beta^{w_\alpha} = [x_\beta, c_{-\alpha}] [x_\beta, a_{-\alpha}] \Big[\big[[x_\beta, a_{-\alpha}], b_\alpha\big], c_{-\alpha}\Big] = [x_\beta, c_{-\alpha}] [x_\beta, a_{-\alpha}] [x_\beta^{-1}, c_{-\alpha}].$$

Note that $[x_\beta, c_{-\alpha}]$ lies in U_γ, so it commutes with x_β and $a_{-\alpha}$ and thus also with $[x_\beta, a_{-\alpha}]$. Hence we have $x_\beta^{w_\alpha} = [x_\beta, a_{-\alpha}][x_\beta, c_{-\alpha}][x_\beta^{-1}, c_{-\alpha}]$ where $[x_\beta^{-1}, c_{-\alpha}] = [x_\beta, c_{-\alpha}]^{-1}$ by Lemma 2.1.13, so $x_\beta^{w_\alpha} = [x_\beta, a_{-\alpha}]$. This finishes the proof of (b). □

In practice, we will usually apply Lemma 5.4.2 in the form of the following formulas.

5.4.3 Proposition (Corollary of Lemma 5.4.2). *Let (α, β, γ) be an A_2-triple. Assume that $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ is an α-Weyl triple and denote its corresponding Weyl element by w_α. Then the following statements hold:*

(a) For any $x_\gamma \in U_\gamma$, we have $x_\gamma = [[x_\gamma, b_\alpha], c_{-\alpha}]^{-1}$ and $x_\gamma^{w_\alpha} = [x_\gamma, b_\alpha]$.

(b) For any $x_\beta \in U_\beta$, we have $x_\beta = [[x_\beta, a_{-\alpha}], b_\alpha]^{-1}$ and $x_\beta^{w_\alpha} = [x_\beta, a_{-\alpha}]$.

Proof. Since w_α is an α-Weyl element, we have $x_\gamma^{w_\alpha} \in U_\beta$ for all $x_\gamma \in U_\gamma$ and $x_\beta^{w_\alpha} \in U_\gamma$ for all $x_\beta \in U_\beta$, so these statements follow from Lemma 5.4.2. □

5.4.4 Note. In the following, we will often use that for any A_2-triple (α, β, γ), $(-\alpha, \gamma, \beta)$ is an A_2-triple as well. This is easily seen from Figure 5.1. Thus Proposition 5.4.3 (a) not only tells us how α-Weyl elements act on U_γ but also how $(-\alpha)$-Weyl elements act on U_β. Since any α-Weyl element is a $(-\alpha)$-Weyl element as well (by Proposition 2.2.6 (c)), this is a powerful tool. In fact, this provides us with an alternative way to prove Proposition 5.4.3 (b). See Note 5.4.6 for more details.

We will show in Proposition 5.4.10 (a) that every Weyl triple is weakly balanced (and thus balanced by Proposition 2.2.16 or by Lemma 2.2.24 (e)). The following result is a first step into this direction.

5.4.5 Lemma. *Let (α, β, γ) be an A_2-triple and assume that $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ is an α-Weyl triple. Then $[x_\beta, a_{-\alpha}] = [x_\beta, c_{-\alpha}]$ for all $x_\beta \in U_\beta$.*

Proof. Let $x_\beta \in U_\beta$ and denote by $w_\alpha := a_{-\alpha} b_\alpha c_{-\alpha}$ the Weyl element corresponding to $(a_{-\alpha}, b_\alpha, c_{-\alpha})$. We already know from Proposition 5.4.3 (b) that $x_\beta^{w_\alpha} = [x_\beta, a_{-\alpha}]$. Further, we know from Proposition 2.2.6 (c) that $(b_\alpha, c_{-\alpha}, a_{-\alpha}^{w_\alpha})$ is a $(-\alpha)$-Weyl triple with corresponding Weyl element w_α. Applying Proposition 5.4.3 (a) to the A_2-triple $(-\alpha, \gamma, \beta)$ and the $(-\alpha)$-Weyl triple $(b_\alpha, c_{-\alpha}, a_{-\alpha}^{w_\alpha})$, we obtain that $x_\beta^{w_\alpha} = [x_\beta, c_{-\alpha}]$. This finishes the proof. □

5.4.6 Note. Let everything be as in the proof of Lemma 5.4.5. By choosing the Weyl triple $(c_{-\alpha}^{w_\alpha^{-1}}, a_{-\alpha}, b_\alpha)$ instead of $(b_\alpha, c_{-\alpha}, a_{-\alpha}^{w_\alpha})$, we obtain that $x_\beta^{w_\alpha} = [x_\beta, a_{-\alpha}]$. This provides a different proof of (the main result of) Proposition 5.4.3 (b) which does not rely on the computation in the proof of Lemma 5.4.2 (b).

Recall that the axioms of a group with Φ-commutator relations merely require that $[U_\alpha, U_\gamma]$ is a subset of $U_{\alpha+\gamma}$ if (α, γ) is an A_2-triple. By the following result, we actually have equality, provided that there exist enough Weyl elements.

5.4.7 Proposition ([Shi93, (2.4)(i)]). *Let (α, β, γ) be an A_2-triple. Then for all $b_\alpha \in U_\alpha^\sharp$, the map*

$$U_\gamma \to U_\beta, x_\gamma \mapsto [x_\gamma, b_\alpha]$$

is an isomorphism of groups. In particular, $[U_\alpha, U_\gamma] = U_\beta$ if $U_\alpha^\sharp$ is non-empty.

Proof. Choose $a_{-\alpha}, c_{-\alpha} \in U_{-\alpha}$ such that $w_\alpha := a_{-\alpha} b_\alpha c_{-\alpha}$ is an α-Weyl element. By Proposition 5.4.3 (a), the map in the assertion is simply the map

$$U_\gamma \to U_\beta, x_\gamma \mapsto x_\gamma^{w_\alpha}$$

which is clearly a homomorphism. Further, it is bijective because w_α is an α-Weyl element. □

As a consequence of the previous result, the root groups must be abelian if we have sufficiently many Weyl elements.

5.4.8 Lemma. *Let (α, β, γ) be an A_2-triple and assume that $U_\alpha^\sharp$ is non-empty. Then U_β is abelian.*

Proof. We have $U_\beta = [U_\alpha, U_\gamma]$ by Proposition 5.4.7. Since U_β commutes with U_α and U_γ by the commutator axiom, we infer that U_β commutes with itself. In other words, U_β is abelian. □

5.4.9 Proposition. *Assume that $U_\alpha^\sharp$ is non-empty for all roots α. Then all root groups of G are abelian.*

Proof. Every root lies in an A_2-triple by Remark 5.2.10, so the assertion follows from Lemma 5.4.8. □

We are now in a position to prove many desirable properties of Weyl elements in G. While we already know from Proposition 2.2.22 that G has unique Weyl extensions, we will give a new proof of this fact to show how it can be derived from the previous results in this section.

5.4.10 Proposition. *Assume that $U_\beta^\sharp$ is non-empty for all roots β. Then for any root α, the following statements hold:*

(a) *Every α-Weyl triple is balanced.*
(b) *If $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ and $(a'_{-\alpha}, b'_\alpha, c'_{-\alpha})$ are two α-Weyl triples whose corresponding Weyl elements are identical, then $(a_{-\alpha}, b_\alpha, c_{-\alpha}) = (a'_{-\alpha}, b'_\alpha, c'_{-\alpha})$.*
(c) *G has unique α-Weyl extensions.*

Proof. Choose roots β, γ such that (α, β, γ) is an A_2-triple and choose invertible elements $x_\beta \in U_\beta^\sharp$, $x_\gamma \in U_\gamma^\sharp$. We will keep these choices throughout the whole proof.

For (a), let $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ be an α-Weyl triple. By Lemma 5.4.5, we have $[x_\beta, c_{-\alpha}] = [x_\beta, a_{-\alpha}]$, which implies that

$$[c_{-\alpha}, x_\beta] = [x_\beta, c_{-\alpha}]^{-1} = [x_\beta, a_{-\alpha}]^{-1} = [a_{-\alpha}, x_\beta].$$

Since $(-\alpha, \gamma, \beta)$ is an A_2-triple and x_β is invertible, we know from Proposition 5.4.7 that the map $U_{-\alpha} \to U_\gamma, x_{-\alpha} \mapsto [x_{-\alpha}, x_\beta]$ is an isomorphism. Thus it follows that $a_{-\alpha} = c_{-\alpha}$, proving that every α-Weyl triple is weakly balanced. By Proposition 2.2.16, it follows that every α-Weyl triple is balanced, so (a) holds.

Now let $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ and $(a'_{-\alpha}, b'_\alpha, c'_{-\alpha})$ be two α-Weyl triples whose corresponding Weyl elements are identical. Denote their common Weyl element by w_α. Then by Proposition 5.4.3, we have $[x_\beta, a_{-\alpha}] = x_\beta^{w_\alpha} = [x_\beta, a'_{-\alpha}]$ and $[x_\gamma, b_\alpha] = x_\gamma^{w_\alpha} = [x_\gamma, b'_\alpha]$. As in the proof of (a), this implies that $a_{-\alpha} = a'_{-\alpha}$ and $b_\alpha = b'_\alpha$. It follows that $c_{-\alpha} = (a_{-\alpha} b_\alpha)^{-1} w_\alpha = (a'_{-\alpha} b'_\alpha)^{-1} w_\alpha = c'_{-\alpha}$. This finishes the proof of (b).

Now let $b_\alpha \in U_\alpha^\sharp$. By the definition of $U_\alpha^\sharp$ and by (a), there exists $a_{-\alpha} \in U_{-\alpha}$ such that $(a_{-\alpha}, b_\alpha, a_{-\alpha})$ is an α-Weyl triple. To show the uniqueness of $a_{-\alpha}$, we take another element $a'_{-\alpha} \in U_{-\alpha}$ such that $(a'_{-\alpha}, b_\alpha, a'_{-\alpha})$. (Note that $a_{-\alpha}b_\alpha a_{-\alpha}$ and $a'_{-\alpha}b_\alpha a'_{-\alpha}$ are allowed to be distinct.) By Proposition 5.4.3 (b), we have

$$\big[[x_\beta, a_{-\alpha}], b_\alpha\big] = x_\beta^{-1} = \big[[x_\beta, a'_{-\alpha}], b_\alpha\big].$$

Using the same arguments as in (a) and (b) and the fact that b_α is invertible, we infer that $[x_\beta, a_{-\alpha}] = [x_\beta, a'_{-\alpha}]$. Another application of the same argument, using that x_β is invertible, yields $a_{-\alpha} = a'_{-\alpha}$. This finishes the proof of (c). □

5.4.11 Remark (Braid relations). We already know from Theorem 2.5.10 that G satisfies the braid relations for Weyl elements. In the following, we show how this result can be derived from the previous results in this section. The proof is ultimately similar to the one in Theorem 2.5.10, but more explicit because we have fixed the choice of the root system.

Without loss of generality, it suffices to consider the root system A_2, which we consider in its standard representation (from Remark 5.2.3): We write

$$A_2 = \{\, e_i - e_j \mid i, j \in [1,3] \,\}$$

and we choose the standard root base $\Delta = \{\, e_1 - e_2, e_2 - e_3 \,\}$. Let (a_{21}, b_{12}, a_{21}) be an $(e_1 - e_2)$-Weyl triple, let $(a'_{32}, b'_{23}, a'_{32})$ be an $(e_2 - e_3)$-Weyl triple and denote by $w_{12} := a_{21}b_{12}a_{21}$ and $w_{23} := a'_{32}b'_{23}a'_{32}$ the corresponding Weyl elements. We have to show that $w_{12}w_{23}w_{12} = w_{23}w_{12}w_{23}$. Since

$$w_{12}w_{23}w_{12} = w_{23}w_{12}w_{12}^{w_{23}w_{12}}$$

it suffices to show that $w_{12}^{w_{23}w_{12}} = w_{23}$. Recall that $w_{12}^{w_{23}w_{12}}$ is a Weyl element by Proposition 2.2.6 (b). More precisely, its corresponding root is

$$\sigma_{e_1-e_2}(\sigma_{e_2-e_3}(e_1 - e_2)) = \sigma_{e_1-e_2}(e_1 - e_3) = e_2 - e_3$$

and its corresponding Weyl triple is $(a_{21}^{w_{23}w_{12}}, b_{12}^{w_{23}w_{12}}, a_{21}^{w_{23}w_{12}})$. By two applications of Proposition 5.4.3, we have

$$b_{12}^{w_{23}w_{12}} = [b_{12}, b'_{23}]^{w_{12}} = \big[[b_{12}, b'_{23}], a_{21}\big] = \big[[b'_{23}, b_{12}]^{-1}, a_{21}\big].$$

Using Lemma 2.1.13 and Proposition 5.4.3 (a), we infer that

$$b_{12}^{w_{23}w_{12}} = \big[[b'_{23}, b_{12}], a_{21}\big]^{-1} = b'_{23}.$$

Thus we have an $(e_2 - e_3)$-Weyl triple $(a_{21}^{w_{23}w_{12}}, b'_{23}, a_{21}^{w_{23}w_{12}})$ corresponding to $w_{12}^{w_{23}w_{12}}$ and an $(e_2 - e_3)$-Weyl triple $(a'_{32}, b'_{23}, a'_{32})$ corresponding to w_{23}. By Proposition 5.4.10 (c), these two Weyl triples must be identical, so that $w_{12}^{w_{23}w_{12}} = w_{23}$. This finishes the proof of the braid relation.

Our next goal is to prove the square formula for Weyl elements (Proposition 5.4.17). To do this, we consider arbitrary roots α, β and compute for each possible "configuration" of (α, β) how the squares of α-Weyl elements act on U_β. Here by "configurations" we essentially mean the different cases in Proposition 5.2.14.

5.4.12 Reminder (see Definition 1.1.10). Let w be an element of G and let α be a root. We say that *w acts trivially on U_α* if $x_\alpha^w = x_\alpha$ for all $x_\alpha \in U_\alpha$ and we say that *w acts on U_α by inversion* if $x_\alpha^w = x_\alpha^{-1}$ for all $x_\alpha \in U_\alpha$.

The following proof uses the ideas from Note 5.4.4. An alternative proof is given in Remark 5.4.14.

5.4.13 Lemma. *Let (α, β, γ) be an A_2-triple and assume that w_α is an α-Weyl element. Then w_α^2 acts on U_β and on U_γ by inversion.*

Proof. Choose an α-Weyl triple $(a_{-\alpha}, b_\alpha, a_{-\alpha})$ such that $w_\alpha = a_{-\alpha} b_\alpha a_{-\alpha}$ and let $x_\beta \in U_\beta$ be arbitrary. Put $w := w_\alpha^2$. Using the formulas from Proposition 5.4.3 two times, we see that

$$x_\beta^w = [x_\beta, a_{-\alpha}]^{w_\alpha} = [[x_\beta, a_{-\alpha}], b_\alpha].$$

By Proposition 5.4.3 (b), this means that $x_\beta^w = x_\beta^{-1}$, so w acts on U_β by inversion. Since w_α is also a $(-\alpha)$-Weyl element and $(-\alpha, \gamma, \beta)$ is an A_2-triple, it follows that w acts on U_γ by inversion as well. □

5.4.14 Remark (Alternative proof of Lemma 5.4.13). Let $x_\beta \in U_\beta$. In addition to the Weyl triple $(a_{-\alpha}, b_\alpha, a_{-\alpha})$, we consider the Weyl triple $(a_{-\alpha}^{-1}, b_\alpha^{-1}, a_{-\alpha}^{-1})$ whose corresponding Weyl element is w_α^{-1} (see Proposition 2.2.6 (a)). By Proposition 5.4.3 (b), we have

$$x_\beta^{w_\alpha} = [x_\beta, a_{-\alpha}] \quad \text{and} \quad x_\beta^{w_\alpha^{-1}} = [x_\beta, a_{-\alpha}^{-1}].$$

Since $[x_\beta, a_{-\alpha}^{-1}] = [x_\beta, a_{-\alpha}]^{-1}$ by Lemma 2.1.13, we infer that

$$x_\beta^{w_\alpha^{-1}} = (x_\beta^{w_\alpha})^{-1}.$$

This means that w_α^2 acts on x_β by inversion. Using similar arguments, we can deduce from Proposition 5.4.3 (a) that w_α^2 acts on U_γ by inversion.

Lemma 5.4.13 was relatively straightforward to prove because we only had to apply the formulas from Proposition 5.4.3. For the action of w_β on its "own" root group U_β, however, we have no such formula. Instead, we have use the fact that U_β can be written as the commutator of two adjacent root groups, and then apply Lemma 5.4.13.

5.4.15 Lemma. *Assume that $U_\alpha^\sharp$ is non-empty for all roots α and let β be any root. Then for all β-Weyl elements w_β, w_β^2 acts trivially on U_β and $U_{-\beta}$.*

Proof. By Remark 5.2.10, there exist roots α, γ such that (α, β, γ) is an A_2-pair. Let $x_\beta \in U_\beta$ be arbitrary and put $w := w_\beta^2$. By Proposition 5.4.7, there exist $x_\alpha \in U_\alpha$ and $x_\gamma \in U_\gamma$ such that $x_\beta = [x_\alpha, x_\gamma]$. Thus it follows from Lemmas 2.1.13 and 5.4.13 that

$$x_\beta^w = [x_\alpha^w, x_\gamma^w] = [x_\alpha^{-1}, x_\gamma^{-1}] = [x_\alpha, x_\gamma] = x_\gamma.$$

That is, w acts trivially on U_β. Since w_β is also a $(-\beta)$-Weyl element by Proposition 2.2.6 (c), this implies that w acts trivially on $U_{-\beta}$, too. □

5.4.16 Lemma. *Let α, β be orthogonal roots and assume that w_α is an α-Weyl element. Then w_α^2 acts trivially on U_β.*

Proof. It follows from the assumptions and Lemma 5.2.13 that β is adjacent to α and $-\alpha$. This implies that U_β commutes with $\langle U_\alpha, U_{-\alpha} \rangle$ and thus with w_α. The assertion follows. □

The previous results cover all possible "configurations" of pairs of roots (α, β). It is now a straightforward computation to check that the square formula is satisfied in each case.

5.4.17 Proposition. *Assume that $U_\gamma^\sharp$ is non-empty for all roots γ and let α, β be two roots. Let w_α be an α-Weyl element, put $w := w_\alpha^2$ and set $\varepsilon := (-1)^{\langle \beta | \alpha \rangle}$ where $\langle \beta | \alpha \rangle = 2\frac{\beta \cdot \alpha}{\alpha \cdot \alpha}$ is the Cartan integer for (β, α). Then $x_\beta^w = x_\beta^\varepsilon$ for all $x_\beta \in U_\beta$. In other words, G satisfies the square formula for Weyl elements (see Definition 4.4.4).*

Proof. We only have to put Lemmas 5.4.13, 5.4.15 and 5.4.16 together and apply Proposition 5.2.14. □

5.5 Standard Twisting Systems

5.5.1 Notation for this section. We denote by Φ an irreducible simply-laced root system and by $c = (c_{\alpha,\beta})_{\alpha,\beta \in \Phi}$ a family of Chevalley structure constants of type Φ (in the sense of Definition 3.1.18).

In this section, we collect some properties of the Chevalley structure constants of simply-laced type and of the standard twisting systems which they define.

5.5.2 Note. In 4.7.5 and 4.7.7, we declared that the "standard (partial) twisting system" and the "standard signs" for Φ-graded groups are the ones which occur in some kind of "standard example". For the root systems of types B, BC, C and F, we will state explicit formulas for the involved signs and parity maps. In the setting of simply-laced root graded groups, however, this approach is not possibe because we want to cover three types of root systems at the same time. For instance, a parity map η defined on A_n cannot be used to parametrise root graded groups of type D or E.

Instead, we take a more general approach in this chapter: We define not a single standard twisting system, but rather a twisting system for every family c of Chevalley structure constants of simply-laced type (Definition 5.5.6). Similarly, we will define coordinatisations of Φ-graded groups with signs c in Definition 5.6.2. In other words, we can cover all simply-laced root systems at the same time by working with the family c (which is not explicitly given). An additional advantage of this approach is that it makes the connection to Chevalley groups more evident.

It should be noted that the strategy described above is only possible because simply-laced root graded groups have the exact same twisting systems and commutator relations as Chevalley groups. The more complicated root systems will require a different approach.

5.5.3 Note. We will, throughout this section, often consider a Chevalley group H of type Φ over $\mathbb{C}$ with respect to c, and we will denote its root groups, its root isomorphisms and the standard Weyl elements from Definition 3.3.3 by $(U_\alpha)_{\alpha \in \Phi}$, $(\theta_\alpha)_{\alpha \in \Phi}$ and $(w_\alpha)_{\alpha \in \Phi}$, respectively. By choosing the complex numbers as a base ring, we ensure that the set $\{\pm 1_{\mathbb{Z}}\}$ embeds into the base ring. Thus we can write $\theta_\alpha(a)$ for any $a \in \{\pm 1_{\mathbb{Z}}\}$, and $\theta_\alpha(a) = \theta_\alpha(b)$ implies $a = b$ for all $a, b \in \{\pm 1_{\mathbb{Z}}\}$ and all $\alpha \in \Phi$. This is the only property of the base ring that we need, so we could as well replace $\mathbb{C}$ by any commutative associative ring $\mathcal{R}$ with $2_{\mathcal{R}} \neq 0_{\mathcal{R}}$. However, the choice of the complex numbers has the additional benefit that the construction of Chevalley groups over $\mathbb{C}$ is easier than over arbitrary rings by Strategy 3.2.6.

At first, we show how the parity map in simply-laced Chevalley groups can be computed from the family c. Similar computations could be done for all other crystallographic root systems as well.

5.5.4 Lemma. *Let $\eta\colon \Phi \times \Phi \to \{\pm 1_{\mathbb{Z}}\}$ the Chevalley parity map for c from Definition 3.3.10. Then for all roots α, δ, the value of $\eta_{\alpha,\delta}$ is determined by c in the following way:*

(a) $\eta(\alpha, \alpha) = \eta(\alpha, -\alpha) = -1$.
(b) $\eta(\alpha, \delta) = 1$ *if α, δ are orthogonal.*
(c) $\eta(\alpha, \delta) = c_{\alpha,\delta}$ *if (α, δ) is an A_2-pair.*
(d) $\eta(\alpha, \delta) = -c_{\alpha,-\delta}$ *if $(\alpha, -\delta)$ is an A_2-pair.*

Proof. Let H, $(U_\alpha)_{\alpha\in\Phi}$, $(\theta_\alpha)_{\alpha\in\Phi}$ and $(w_\alpha)_{\alpha\in\Phi}$ be as in Note 5.5.3. The first assertion holds by Lemma 3.3.13. If α and δ are orthogonal, then w_δ commutes with U_α and so $\eta(\alpha, \delta) = 1$, which implies the second assertion. Now assume that (α, δ) is an A_2-pair, and put $\beta := \alpha + \delta$. Then it follows from Proposition 5.4.3 (a) that

$$\theta_\beta(\eta_{\alpha,\delta}) = \theta_\alpha(1)^{w_\delta} = [\theta_\alpha(1), \theta_\delta(1)] = \theta_\beta(c_{\alpha,\delta}),$$

so $\eta_{\alpha,\delta} = c_{\alpha,\delta}$. Now assume that $(\alpha, -\delta)$ is an A_2-pair, and put $\gamma := \alpha - \delta$. Then (δ, α, γ) is an A_2-triple, so it follows from Proposition 5.4.3 (b) that

$$\theta_\gamma(\eta_{\alpha,\delta}) = \theta_\alpha(1)^{w_\delta} = [\theta_\alpha(1), \theta_{-\delta}(-1)] = \theta_\gamma(-c_{\alpha,-\delta}).$$

Thus $\eta_{\alpha,\delta} = -c_{\alpha,-\delta}$. This finishes the computation of η. □

5.5.5 Example. We consider the family of Chevalley structure constants for A_n from Example 3.1.19, which is given by

$$c_{e_i-e_j,e_j-e_k} = 1, \qquad c_{e_j-e_k,e_i-e_j} = -1$$

and $c_{\alpha,\beta} = 0$ for all other pairs α, β of roots. Let $i, j, k, l \in [1, n+1]$ such that $i \neq j$ and $k \neq l$. Write $c_{ij,kl}$ for $c_{e_i-e_j,e_k-e_l}$. We want to compute $a := \eta_{e_i-e_j,e_k-e_l}$. By the first two assertions in Lemma 5.5.4, we have $a = -1$ if $|\{i,j\} \cap \{k,l\}| = 2$ and $a = 1$ if $\{i,j\} \cap \{k,l\} = \varnothing$. From now on, we assume that $|\{i,j\} \cap \{k,l\}| = 1$. Then

$$a = \begin{cases} c_{ij,jl} = 1 & \text{if } j = k, \\ -c_{ij,jk} = -1 & \text{if } j = l, \\ -c_{ij,li} = 1 & \text{if } i = k, \\ c_{ij,ki} = -1 & \text{if } i = l. \end{cases}$$

In summary, we have

$$\eta_{e_i-e_j,e_k-e_l} = \begin{cases} -1 & \text{if } l \in \{i,j\}, \\ 1 & \text{otherwise.} \end{cases}$$

5.5.6 Definition (Standard twisting system). Let Δ be a root base of Φ. The *standard twisting system of type Φ defined by c with respect to* Δ is the pair $(A, \eta|_{\Phi\times\Delta})$ where $A := \{\pm 1_{\mathbb{Z}}\}$ and η is the Chevalley parity map for c (which is given by the formulas in Lemma 5.5.4). If G is a group with a Φ-pregrading, then the *standard twisting system for G defined by c with respect to* Δ is the same pair together with the additional information that A acts on all root groups of G by inversion.

5.5.7 Lemma. *Let G be a group with a Φ-pregrading, let Δ be a root base of Φ and let $(w_\delta)_{\delta\in\Delta}$ be a Δ-system of Weyl elements in G. Denote the standard twisting system for G defined by c with respect to Δ by (A, η). Then (A, η) is a twisting system for $(G, (w_\delta)_{\delta\in\Delta})$. Further, η is complete and adjacency-trivial and it satisfies the square formula.*

Proof. By Example 4.3.15, A is a twisting group for $(G, (w_\delta)_{\delta\in\Delta})$. We know from Lemma 5.5.4 (a) that $A_{\alpha\to\alpha} = A$ for all roots α, so η is complete. In particular, it is transporter-invariant by Remark 4.2.17. Using that the map η is read off from the Chevalley group H in Note 5.5.3 and that Weyl elements in this group satisfy the braid relations by Theorem 2.5.10, it follows from Proposition 4.4.11 (b) that η is braid-invariant as well. Similarly, since H satisfies the square formula for Weyl elements by Proposition 5.4.17, it follows from Proposition 4.4.11 (c) that η satisfies the square formula. Finally, η is adjacency-trivial by Proposition 4.4.11 (a). This finishes the proof. □

We end this section with two properties of c and η which, at the moment, are completely unmotivated. Lemma 5.5.8 will be used in the proof of the associativity of the coordinatising ring (Proposition 5.6.9). Lemma 5.5.9 will be applied when we transport the commutator formula on one fixed A_2-pair (α_0, γ_0) to all other A_2-pairs (Lemma 5.7.10).

5.5.8 Lemma. *Assume that Φ is of rank at least 3 and let α, β, γ be roots such that $\{\alpha, \beta, \gamma\}$ is a root base of a parabolic A_3-subsystem of Φ and such that $\alpha \cdot \gamma = 0$. Then $c_{\alpha,\beta}c_{\alpha+\beta,\gamma} = c_{\beta,\gamma}c_{\alpha,\beta+\gamma}$.*

Proof. By the Chevalley commutator formula (Theorem 3.2.16), we have

$$[\theta_\alpha(1), \theta_\gamma(1)] = 1_H \qquad \text{and}$$
$$\big[[\theta_\beta(1), \theta_\alpha(1)^{-1}], [\theta_\beta(1), \theta_\gamma(1)]\big] \in [U_{\alpha+\beta}, U_{\beta+\gamma}] = \{1_H\}$$

Thus it follows from the variant of the Hall-Witt identity given in Lemma 1.1.13 that

$$\big[[\theta_\alpha(1), \theta_\beta(1)], \theta_\gamma(1)\big] = \big[\theta_\alpha(1), [\theta_\beta(1), \theta_\gamma(1)]\big].$$

By the Chevalley commutator formula, this means precisely that

$$\theta_{\alpha+\beta+\gamma}(c_{\alpha,\beta}c_{\alpha+\beta,\gamma}) = \theta_{\alpha+\beta+\gamma}(c_{\beta,\gamma}c_{\alpha,\beta+\gamma}).$$

The assertion follows. □

5.5.9 Lemma. *Let η be the Chevalley parity map for c, let (α, γ) be any A_2-pair and let $\bar{\delta}$ be any word over Φ. Then $\eta_{\alpha,\bar{\delta}}\eta_{\beta,\bar{\delta}}c_{\alpha^{\sigma(\bar{\delta})},\gamma^{\sigma(\bar{\delta})}} = \eta_{\alpha+\gamma,\bar{\delta}}c_{\alpha,\gamma}$.*

Proof. Let H, $(U_\alpha)_{\alpha\in\Phi}$, $(\theta_\alpha)_{\alpha\in\Phi}$ and $(w_\alpha)_{\alpha\in\Phi}$ be as in Note 5.5.3. By the Chevalley commutator formula (Theorem 3.2.16), we have

$$[\theta_\alpha(1), \theta_\gamma(1)] = \theta_{\alpha+\gamma}(c_{\alpha,\gamma})$$

Conjugating this equation by $w_{\bar{\delta}}$, we obtain

$$[\theta_{\alpha^{\sigma(\bar{\delta})}}(\eta_{\alpha,\bar{\delta}}), \theta_{\gamma^{\sigma(\bar{\delta})}}(\eta_{\gamma,\bar{\delta}})] = \theta_{\alpha^{\sigma(\bar{\delta})}+\gamma^{\sigma(\bar{\delta})}}(c_{\alpha,\gamma}\eta_{\alpha+\gamma,\bar{\delta}}).$$

However, we also have

$$[\theta_{\alpha^{\sigma(\bar{\delta})}}(\eta_{\alpha,\bar{\delta}}), \theta_{\gamma^{\sigma(\bar{\delta})}}(\eta_{\gamma,\bar{\delta}})] = \theta_{\alpha^{\sigma(\bar{\delta})}+\gamma^{\sigma(\bar{\delta})}}(c_{\alpha^{\sigma(\bar{\delta})},\gamma^{\sigma(\bar{\delta})}}\eta_{\alpha,\bar{\delta}}\eta_{\gamma,\bar{\delta}}).$$

Since $\theta_{\alpha^{\sigma(\bar{\delta})}+\gamma^{\sigma(\bar{\delta})}}$ is injective, the assertion follows. □

5.6 Standard Signs

5.6.1 Notation for this section. We denote by Φ an irreducible simply-laced root system of rank at least 2, by (α_0, γ_0) an arbitrary A_2-pair in Φ, by $c = (c_{\alpha,\beta})_{\alpha,\beta\in\Phi}$ a family of Chevalley structure constants of type Φ (in the sense of Definition 3.1.18) and by G a group with a Φ-pregrading $(U_\alpha)_{\alpha\in\Phi}$. We assume that there exists a coordinatisation of G by a ring $\mathcal{R}$ based at (α_0, γ_0) with signs c (in the sense of the following Definition 5.6.2), and we fix this coordinatisation.

In this section, we define the notion of coordinatisations of G with signs c, as explained in Note 5.5.2. Further, we will show that any ring which coordinatises G must be associative if $\mathrm{rank}(\Phi) \geq 3$ and, if Φ is of type D or E, even commutative. In particular, these properties are independent of the specific construction the we will undertake in section 5.7.

5.6.2 Definition (Coordinatisations). Choose an A_2-pair (α_0, γ_0) and let $\mathcal{R}$ be a ring. A *coordinatisation of G by $\mathcal{R}$ based at (α_0, γ_0) with signs c* is a family

$$(\theta_\alpha \colon (\mathcal{R}, +) \to U_\alpha)_{\alpha\in\Phi}$$

of isomorphisms such that for all $r, s \in \mathcal{R}$ and all (α_0, γ_0)-positive A_2-pairs (α, γ) (in the sense of Definition 5.2.15), we have the commutator formula

$$[\theta_\alpha(r), \theta_\gamma(s)] = \theta_{\alpha+\gamma}(c_{\alpha,\gamma} rs).$$

5.6.3 Remark. If Φ is of type D or E, then any A_2-pair is (α_0, γ_0)-positive for any other A_2-pair (α_0, γ_0) by Lemma 5.2.16 (b). Hence we can drop the "based at (α_0, γ_0)" part in Definition 5.6.2 for these root systems.

5.6.4 Remark. Let (α, γ) be an A_2-pair which is not (α_0, γ_0)-positive. Then (γ, α) is (α_0, γ_0)-positive by Lemma 5.2.16. Since $c_{\gamma,\alpha} = -c_{\alpha,\gamma}$ by Lemma 3.1.23 (a), it follows that

$$[\theta_\alpha(r), \theta_\gamma(s)] = [\theta_\gamma(s), \theta_\alpha(r)]^{-1} = \theta_{\alpha+\gamma}(-c_{\gamma,\alpha} sr) = \theta_{\alpha+\gamma}(c_{\alpha,\gamma} sr).$$

for all $r, s \in \mathcal{R}$. Thus non-positive A_2-pairs satisfy the same commutator relations as positive A_2-pairs except that rs is replaced by sr. This is the key observation that we will use in Proposition 5.6.11 to show that $\mathcal{R}$ must be commutative if Φ is of type D or E.

5.6.5 Example. Assume that Φ is the root system A_n in standard representation for some $n \geq 2$ and let c be the family of Chevalley structure constants from Example 3.1.19. Further, let $(\alpha_0, \gamma_0) := (e_i - e_j, e_j - e_k)$ for some pairwise distinct $i, j, k \in [1, n+1]$. Then a coordinatisation $(\theta_\alpha)_{\alpha\in A_n}$ by a ring $\mathcal{R}$ based at (α_0, γ_0) with signs c satisfies the same commutator relations as the group $\mathrm{E}_{n+1}(\mathcal{R})$ in Example 3.2.22:

$$[\theta_{e_i-e_j}(r), \theta_{e_j-e_k}(s)] = \theta_{e_i-e_k}(rs) \quad \text{and} \quad [\theta_{e_j-e_k}(r), \theta_{e_i-e_j}(s)] = \theta_{e_i-e_k}(-sr)$$

for all pairwise distinct $i, j, k \in [1, \ell+1]$ and all $r, s \in \mathcal{R}$. In this case, the set of (α_0, γ_0)-positive A_2-pairs (that is, the set of A_2-pairs which have rs in the commutator formula and not sr) coincides with the set of A_2-pairs which have a positive sign in the commutator formula. However, for other choices of (α_0, γ_0) and c, these sets may be distinct.

Note that we have defined coordinatisations for arbitrary Φ-pregradings. However, it is clear from the definition that any pregrading with a coordinatisation has Φ-commutator relations. The following result says that pregradings with standard coordinatisations also have Weyl elements.

5.6.6 Proposition. *For all roots α and all invertible $r \in \mathcal{R}$, define*

$$w_\alpha(r) := \theta_{-\alpha}(-r^{-1})\theta_\alpha(r)\theta_{-\alpha}(-r^{-1}).$$

Then the following hold:

(a) The maps

$$\mathcal{R}^\times \to U_\alpha^\sharp, r \mapsto \theta_\alpha(r) \quad \textit{and} \quad \mathcal{R}^\times \to M_\alpha, r \mapsto w_\alpha(r)$$

are well-defined bijections for all roots α. Here $\mathcal{R}^\times$, $U_\alpha^\sharp$ and M_α denote the sets of invertible elements in $\mathcal{R}$ (in the sense of Definition 5.1.12), the set of α-invertible elements in U_α and the set of α-Weyl elements, respectively.

(b) There exists an α-Weyl element for each root α.

(c) Let Δ be any root base of Φ and denote by (A, η) the standard twisting system for G defined by c with respect to Δ (as in Definition 5.5.6). Then G is parametrised by $(A, \mathcal{R})$ with respect to η and $(w_\delta(1_\mathcal{R}))_{\delta\in\Delta}$.

Proof. Let α be any root and let $w_\alpha = \theta_{-\alpha}(r)\theta_\alpha(s)\theta_{-\alpha}(t)$ be an α-Weyl element for some $r, s, t \in \mathcal{R}$. Choose a root γ such that (α, γ) is a (α_0, γ_0)-positive A_2-pair, and put $\beta := \alpha + \gamma$. Then it follows from Proposition 5.4.3 (a) that for all $a \in \mathcal{R}$, we have

$$\theta_\gamma(a) = \big[[\theta_\gamma(a), \theta_\alpha(s)], \theta_{-\alpha}(t)\big] = [\theta_\beta(c_{\gamma,\alpha}sa), \theta_{-\alpha}(t)] = \theta_\gamma\big(c_{\beta,-\alpha}c_{\gamma,\alpha}\big(t(sa)\big)\big).$$

By performing the same computation in a Chevalley group of type Φ, one can show that $c_{\beta,-\alpha}c_{\gamma,\alpha} = -1$. Thus we infer that s is invertible with inverse $-t$. In a similar way, one can show that the same assertion holds for t replaced by r. Hence every Weyl element in G has the desired form. This says precisely that the second map in (a) is surjective (if it is well-defined).

Now let β be an arbitrary root, let $r \in \mathcal{R}$ be invertible and consider $w_\beta := w_\beta(r)$. Using Lemma 5.4.2, a similar computation as in the previous paragraph shows that $U_\zeta^{w_\beta} = U_{\sigma_\beta(\zeta)}$ for all $\zeta \in \Phi \setminus \{\alpha, -\alpha\}$. It remains to show the same statement for $\zeta \in \{\alpha, -\alpha\}$. Choose an (α_0, γ_0)-positive A_2-pair (α, γ) such that $\beta = \alpha + \gamma$. Then $U_\beta = [U_\alpha, U_\gamma]$ because $\theta_\beta(a) = [\theta_\alpha(c_{\alpha,\beta}^{-1}1_\mathcal{R}), \theta_\gamma(a)]$ for all $a \in \mathcal{R}$. Similarly, $U_{-\beta} = [U_{-\alpha}, U_{-\gamma}]$. It follows that

$$U_{\pm\beta}^{w_\beta} = [U_{\pm\alpha}, U_{\pm\gamma}]^{w_\beta} = [U_{\pm\alpha}^{w_\beta}, U_{\pm\gamma}^{w_\beta}] = [U_{\mp\gamma}, U_{\mp\alpha}] = U_{\mp\beta}.$$

Hence $w_\beta(r)$ is indeed a β-Weyl element. This shows that the second map in (a) is well-defined. Further, it is injective by Proposition 5.4.10 (b). It follows that the first map in (a) is a well-defined bijection as well.

Assertion (b) follows from (a) because $\mathcal{R}^\times$ contains $1_\mathcal{R}$. Finally, assertion (c) can be proven by similar computations as above, using Proposition 5.4.3. ☐

Observe that Proposition 5.6.6 (c) says precisely that the Weyl elements $(w_\alpha(1_\mathcal{R}))_{\alpha\in\Phi}$ satisfy the same conjugation formulas as in Chevalley groups (Lemma 3.3.7).

5.6.7 Remark. It follows from Proposition 5.6.6 that G satisfies $U_\alpha^\sharp = U_\alpha \setminus \{1_G\}$ (the additional condition of being an RGD-system) if and only if $\mathcal{R}$ is a division ring.

5.6.8 Example. Assume that Φ is the root system A_n in standard representation for some $n \geq 2$ and let $(\theta_\alpha)_{\alpha\in\Phi}$ be a coordinatisation of G by a ring $\mathcal{R}$ with signs and base point chosen as in Example 5.6.5. Then by Example 5.5.5, the conjugation formula in Proposition 5.6.6 (c) says that

$$\theta_{kl}(r)^{w_{ij}(1)} = \begin{cases} \theta_{\tau(k),\tau(l)}(r) & \text{if } j \notin \{k, l\}, \\ \theta_{\tau(k),\tau(l)}(-r) & \text{if } j \in \{k, l\} \end{cases}$$

for all $i, j, k, l \in [1, n+1]$ with $i \neq j$ and $k \neq l$ and all $r \in \mathcal{R}$, where τ is the transposition which interchanges i and j.

5.6.9 Proposition. *Let $\mathcal{R}$ be a ring which coordinatises G (with signs c and based at some A_2-pair (α_0, γ_0)) and assume that Φ is of rank at least 3. Then the ring $\mathcal{R}$ is associative.*

Proof. Choose roots α, β, γ as in Lemma 5.2.17, and let $r, s, t \in \mathcal{R}$. By the same arguments as in the proof of Lemma 5.5.8, the Hall-Witt identity in the form of Lemma 1.1.13 yields that

$$\big[[\theta_\alpha(r), \theta_\beta(s)], \theta_\gamma(t)\big] = \big[\theta_\alpha(r), [\theta_\beta(s), \theta_\gamma(t)]\big].$$

By the properties of the roots α, β, γ from Lemma 5.2.17, this says precisely that

$$\theta_{\alpha+\beta+\gamma}\big(c_{\alpha,\beta}c_{\alpha+\beta,\gamma}(r \cdot s) \cdot t\big) = \theta_{\alpha+\beta+\gamma}\big(c_{\beta,\gamma}c_{\alpha,\beta+\gamma} r \cdot (s \cdot t)\big).$$

Since $c_{\alpha,\beta}c_{\alpha+\beta,\gamma} = c_{\beta,\gamma}c_{\alpha,\beta+\gamma}$ by Lemma 5.5.8, it follows that $(r \cdot s) \cdot t = r \cdot (s \cdot t)$. Thus $\mathcal{R}$ is associative. □

5.6.10 Remark (Invertible elements are Moufang elements). In [Fau14, Theorem 13.8], it is shown that, in addition to the statement of Proposition 5.6.6 (a), every invertible element $a \in \mathcal{R}$ is a *Moufang element*. This means that for all $y, z \in \mathcal{R}$, the Moufang identities hold:

$$\begin{aligned} a\big(y(az)\big) &= \big(a(ya)\big)z, \\ \big((za)y\big)a &= z\big((ay)a\big), \\ (ay)(za) &= \big(a(yz)\big)a. \end{aligned}$$

By Proposition 5.6.9, this statement is only of interest if Φ is of rank 2, which means that $\Phi = A_2$. We will see in Proposition 8.2.6 that a ring satisfies the Moufang identities for all $a, y, z \in \mathcal{R}$ if and only if it is an alternative ring. That is, $\mathcal{R}$ is alternative if and only if every element is a Moufang element. Since $0_{\mathcal{R}}$ is always a Moufang element, it follows that $\mathcal{R}$ must be alternative if it is a division ring. In other words, any ring which coordinatises an RGD-system of type A_2 must be alternative. For general A_2-gradings, this is not known to be true. See also Note 5.3.3.

5.6.11 Proposition. *Assume that Φ is of type D or E. Then any ring $\mathcal{R}$ which coordinatises G (with signs c) is associative and commutative.*

Proof. Since root systems of type D are E have rank at least 4, the associativity of $\mathcal{R}$ follows from Proposition 5.6.9. For the commutativity, let $r, s \in \mathcal{R}$ and let (α, γ) be any A_2-pair. By Remark 5.6.4, the reversed A_2-pair (γ, α) satisfies the commutator formula

$$\big[\theta_\gamma(r), \theta_\alpha(s)\big] = \theta_{\alpha+\gamma}\big(c_{\gamma,\alpha} sr\big).$$

On the other hand, since (γ, α) is itself positive with respect to any A_2-pair (α_0, γ_0) by Lemma 5.2.16 (b), we also have the usual commutator formula

$$\big[\theta_\gamma(r), \theta_\alpha(s)\big] = \theta_{\alpha+\gamma}\big(c_{\gamma,\alpha} rs\big).$$

It follows that $rs = sr$, as desired. □

5.6.12 Note. Propositions 5.6.9 and 5.6.11 are not specific to standard signs: By Note 4.7.6, every coordinatisation of G with non-standard signs can be obtained from a standard coordinatisation via twisting, and this process does not change the coordinatising ring. Thus Propositions 5.6.9 and 5.6.11 hold for non-standard coordinatisations as well.

5.7 The Coordinatisation

5.7.1 Notation for this section. We denote by Φ any irreducible simply-laced root system of rank at least 2, by Δ a root base of Φ and by G a group which has Φ-commutator relations with root groups $(U_\alpha)_{\alpha\in\Phi}$ and which satisfies $U_\alpha^\sharp \neq \varnothing$ for all roots α. We assume that all for all non-proportional roots α, β, we have $U_\alpha \cap U_\beta = \{1_G\}$. Further, we choose a family $c = (c_{\alpha,\beta})_{\alpha,\beta\in\Phi}$ of Chevalley structure constants of type Φ and we denote by (A, η) the standard twisting system for G defined by c (in the sense of Definition 5.5.6). From Construction 5.7.6 on, we will also fix an arbitrary A_2-pair (α_0, γ_0).

5.7.2 Reminder. We know from Lemma 5.5.7 that η is complete and adjacency-trivial.

In this section, we can finally construct the coordinatising ring for G. We begin by verifying that the assumptions of the parametrisation theorem are satisfied.

5.7.3 Lemma. *G is stabiliser-compatible with respect to η and $(w_\delta)_{\delta\in\Phi}$.*

Proof. By Lemma 5.2.13, any pair of orthogonal roots in a simply-laced root system is adjacent. Further, we know that η is adjacency-trivial. Thus the assertion follows from Lemma 4.6.4. □

5.7.4 Lemma. *G is square-compatible with respect to η and $(w_\delta)_{\delta\in\Phi}$.*

Proof. Since G satisfies the square formula for Weyl elements by Proposition 5.4.17 and η satisfies the square formula, this follows from Lemma 4.6.2. □

5.7.5 Construction (of the coordinatising ring). Having verified that all the conditions in Notation 4.5.1 are satisfied, we can now apply the parametrisation theorem (Theorem 4.5.16). We conclude that there exist a group $(\mathcal{R}, +)$ and a parametrisation $(\theta_\alpha)_{\alpha\in\Delta}$ of G by the parameter system $\mathcal{P} = (A, \mathcal{R})$ with respect to η and $(w_\delta)_{\delta\in\Phi}$. By Proposition 5.4.9, the root groups of G are abelian, so $(\mathcal{R}, +)$ is abelian as well. Further, the action of A on the set $\mathcal{R}$ is given by the equation

$$a.\theta_\alpha(r) = \theta_\alpha(a.r)$$

for all $a \in A$, all roots α and all $r \in \mathcal{R}$. It follows that $1_A.r = r$ and $(-1_A).r = -r$ for all $r \in \mathcal{R}$. Since $(\mathcal{R}, +)$ is abelian, this implies that A acts on $\mathcal{R}$ by group automorphisms, so that $a.(r+s) = (a.r) + (a.s)$ for all $a \in A$ and all $r, s \in \mathcal{R}$.

5.7.6 Construction (of the ring multiplication). For the remainder of this section, we fix an arbitrary A_2-pair (α_0, γ_0). Since c_{α_0,γ_0} lies in $\{\pm 1_\mathbb{Z}\}$ by Remark 3.1.24, we can regard it as an element of A and thus let it act on $\mathcal{R}$. With this convention, we now define a multiplication on $\mathcal{R}$ by

$$\cdot\colon \mathcal{R} \times \mathcal{R} \to \mathcal{R}, (a, b) \mapsto c_{\alpha_0,\gamma_0}.\theta_{\alpha_0+\gamma_0}^{-1}\big([\theta_{\alpha_0}(a), \theta_{\gamma_0}(b)]\big).$$

Put differently, we define $\cdot$ to be the unique multiplication on $\mathcal{R}$ which satisfies

$$[\theta_{\alpha_0}(a), \theta_{\gamma_0}(b)] = \theta_{\alpha_0+\gamma_0}(c_{\alpha_0,\gamma_0}.(a \cdot b))$$

for all $a, b \in \mathcal{R}$. (Here we have used that $c_{\alpha_0,\gamma_0}^{-1} = c_{\alpha_0,\gamma_0}$.) Since $c_{\alpha_0,\gamma_0,1,1} = c_{\alpha_0,\gamma_0}$ by Remark 3.2.21, this is the same formula that is satisfied in Chevalley groups, see Example 3.2.22. We will later show that this multiplication turns $\mathcal{R}$ into a ring.

5.7.7 Note. A priori, the action of A on $\mathcal{R}$ is defined independently of the multiplication on $\mathcal{R}$, and it does not rely on the existence of a "unit element $1_{\mathcal{R}}$". However, we will soon show that $\mathcal{R}$ does indeed have a unit element $1_{\mathcal{R}}$, and then it is clear that $a.r = i(a) \cdot r$ for all $a \in A$ and $r \in \mathcal{R}$ where $i(1_A) := 1_{\mathcal{R}}$ and $i(-1_A) := -1_{\mathcal{R}}$.

It remains to verify two properties: Firstly, that $\mathcal{R}$ is a ring. Secondly, that the commutator formula is satisfied for all (α_0, γ_0)-positive A_2-pairs and not merely for (α_0, γ_0). We begin with the distributive laws. They are proven in essentially the same way in [Shi93, (2.19)] and [TW02, (19.7)].

5.7.8 Lemma (Distributive laws). *The multiplication on $\mathcal{R}$ satisfies*

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{and} \quad (a + b) \cdot c = a \cdot c + b \cdot c$$

for all $a, b, c \in \mathcal{R}$. In particular, we have $(-a) \cdot b = -(a \cdot b) = a \cdot (-b)$ for all $a, b \in \mathcal{R}$.

Proof. Let $a, b, c \in \mathcal{R}$ and denote by $(\alpha, \gamma) := (\alpha_0, \gamma_0)$ the fixed A_2-pair from Construction 5.7.6. By Lemma 2.1.13, we have

$$[\theta_\alpha(a + b), \theta_\gamma(c)] = [\theta_\alpha(a)\theta_\alpha(b), \theta_\gamma(c)] = [\theta_\alpha(a), \theta_\gamma(c)][\theta_\alpha(b), \theta_\gamma(c)].$$

We conclude that

$$\begin{aligned} \theta_{\alpha+\gamma}\big(c_{\alpha,\gamma}.((a + b) \cdot c)\big) &= [\theta_\alpha(a + b), \theta_\gamma(c)] = [\theta_\alpha(a), \theta_\gamma(c)][\theta_\alpha(b), \theta_\gamma(c)] \\ &= \theta_{\alpha+\gamma}(c_{\alpha,\gamma}.(a \cdot c))\theta_{\alpha+\gamma}(c_{\alpha,\gamma}.(b \cdot c)) \\ &= \theta_{\alpha+\gamma}\big(c_{\alpha,\gamma}.(a \cdot c) + c_{\alpha,\gamma}.(b \cdot c)\big). \end{aligned}$$

Thus

$$c_{\alpha,\gamma}.((a + b) \cdot c) = c_{\alpha,\gamma}.(a \cdot c) + c_{\alpha,\gamma}.(b \cdot c).$$

Since the right-hand side equals $c_{\alpha,\gamma}.((a \cdot c) + (b \cdot c))$ by Construction 5.7.5, it follows that $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$. The right distributive law can be proven in a similar way. $\square$

5.7.9 Note. From now on, we will simply write ar in place of $a.r$ for $a \in A$ and $r \in \mathcal{R}$. Since $a.(r \cdot s) = (a.r) \cdot s = r \cdot (a.s)$ for all $r, s \in \mathcal{R}$ and all $a \in A$ by Lemma 5.7.8, this will not cause any confusion.

5.7.10 Lemma. *Let (α, γ) be an A_2-pair which lies in the orbit of (α_0, γ_0) under the Weyl group of Φ (where (α_0, γ_0) is the fixed A_2-pair from Construction 5.7.6). Then we have*

$$[\theta_\alpha(r), \theta_\gamma(s)] = \theta_{\alpha+\gamma}(c_{\alpha,\gamma} r \cdot s) \quad \text{and} \quad [\theta_\gamma(s), \theta_\alpha(r)] = \theta_{\alpha+\gamma}(c_{\gamma,\alpha} r \cdot s)$$

for all $r, s \in \mathcal{R}$. In particular, if (α_0, γ_0) and (α_0', γ_0') are two A_2-pairs which are conjugate under the Weyl group, then replacing (α_0, γ_0) by (α_0', γ_0') in Construction 5.7.6 does not change the multiplication on $\mathcal{R}$.

Proof. By Construction 5.7.6, the first formula is satisfied for $(\alpha, \gamma) = (\alpha_0, \gamma_0)$. Now let u be an arbitrary element of the Weyl group of Φ and put $\alpha := \alpha_0^u$, $\beta := \beta_0^u$. Since the simple reflections generate the Weyl group (Proposition 1.3.12), we can choose a word $\bar{\delta}$ over Δ such that $u = \sigma_{\bar{\delta}}$, and we put $w := w_{\bar{\delta}}$. Let $r, s \in \mathcal{R}$ be arbitrary. Conjugating the equation

$$[\theta_{\alpha_0}(r), \theta_{\gamma_0}(s)] = \theta_{\alpha_0+\gamma_0}(c_{\alpha_0,\gamma_0} r \cdot s)$$

by w, we obtain that

$$[\theta_\alpha(\eta_{\alpha_0,\bar{\delta}}r), \theta_\gamma(\eta_{\gamma_0,\bar{\delta}}s)] = \theta_{\alpha+\gamma}(\eta_{\alpha_0+\gamma_0,\bar{\delta}}c_{\alpha_0,\gamma_0}r \cdot s).$$

By Lemma 5.5.9, this says that

$$[\theta_\alpha(\eta_{\alpha_0,\bar{\delta}}r), \theta_\gamma(\eta_{\gamma_0,\bar{\delta}}s)] = \theta_{\alpha+\gamma}(\eta_{\alpha_0,\bar{\delta}}\eta_{\gamma_0,\bar{\delta}}c_{\alpha,\gamma}r \cdot s).$$

Replacing r by $\eta_{\alpha_0,\bar{\delta}}^{-1}r$ and s by $\eta_{\gamma_0,\bar{\delta}}^{-1}s$, this gives us the desired first formula.

For the second formula, observe that by Relation 1.1.11 (vi), the first formula yields that

$$[\theta_\gamma(s), \theta_\alpha(r)] = [\theta_\alpha(r), \theta_\gamma(s)]^{-1} = \theta_{\alpha+\gamma}(-c_{\alpha,\gamma}.r \cdot s).$$

Since $c_{\gamma,\alpha} = -c_{\alpha,\gamma}$ by Lemma 3.1.23 (a), the second formula follows. □

5.7.11 Construction (of the identity element). Fix an arbitrary simple root δ_0. By Proposition 5.4.10, there exist unique $a_{-\delta_0} \in U_{-\delta_0}$ and $b_{\delta_0} \in U_{\delta_0}$ such that $w_{\delta_0} = a_{-\delta_0}b_{\delta_0}a_{-\delta_0}$. We can thus define an element $1_{\mathcal{R}} := \theta_{\delta_0}^{-1}(b_{\delta_0})$ which, a priori, depends on the choice of δ_0.

5.7.12 Lemma ([TW02, (19.9)]). *We have $r \cdot 1_{\mathcal{R}} = r$ and $1_{\mathcal{R}} \cdot r = r$ for all $r \in \mathcal{R}$. In particular, $1_{\mathcal{R}}$ does not depend on the choice of δ_0 in Construction 5.7.11.*

Proof. Let $r \in \mathcal{R}$ and write $w_{\delta_0} = a_{-\delta_0}b_{\delta_0}a_{-\delta_0}$ as in Construction 5.7.11. Since the Weyl group acts transitively on Φ, there exists a root ρ such that (ρ, δ_0) is an A_2-pair which lies in the orbit of (α_0, γ_0). Recall that $\sigma_{\delta_0}(\rho) = \rho + \delta_0$. Now it follows from Lemmas 5.4.2 and 5.7.10 that

$$\begin{aligned}\theta_{\rho+\delta_0}(c_{\rho,\delta_0}r \cdot 1_{\mathcal{R}}) &= [\theta_\rho(r), \theta_{\delta_0}(1_{\mathcal{R}})] = [\theta_\rho(r), b_{\delta_0}] \\ &= \theta_\rho(r)^{w_{\delta_0}} = \theta_{\rho+\delta_0}(\eta_{\rho,\delta_0}r).\end{aligned}$$

Since (ρ, δ_0) is an A_2-pair, we know from Lemma 5.5.4 that $\eta_{\rho,\delta_0} = c_{\rho,\delta_0}$. Thus $r \cdot 1_{\mathcal{R}} = r$.

The proof of the identity $1_{\mathcal{R}} \cdot r = r$ is similar, but with a twist. This time we choose a root ξ such that (δ_0, ξ) is an A_2-pair in the orbit of (α_0, γ_0). Then again by Lemmas 5.4.2 and 5.7.10, we have

$$\begin{aligned}\theta_{\delta_0+\xi}(c_{\delta_0,\xi}1_{\mathcal{R}} \cdot r) &= [\theta_{\delta_0}(1_{\mathcal{R}}), \theta_\xi(r)] = [\theta_\xi(r), b_{\delta_0}]^{-1} \\ &= (\theta_\xi(r)^{w_{\delta_0}})^{-1} = \theta_{\xi+\delta_0}(-\eta_{\xi,\delta_0}r).\end{aligned}$$

As above, we know from Lemma 5.5.4 that $\eta_{\xi,\delta_0} = c_{\xi,\delta_0}$. Since $c_{\xi,\delta_0} = -c_{\delta_0,\xi}$ by Lemma 3.1.23 (a), we infer that $1_{\mathcal{R}} \cdot r = r$.

Now assume that $1'_{\mathcal{R}}$ is the element which is defined by the choice of another simple root δ'_0. Then it follows from the previous results that $1'_{\mathcal{R}} = 1'_{\mathcal{R}} \cdot 1_{\mathcal{R}} = 1_{\mathcal{R}}$, which finishes the proof. □

5.7.13 Proposition. *$(\mathcal{R}, +, \cdot)$ is a ring.*

Proof. This is a consequence of Lemmas 5.7.8 and 5.7.12. □

We can now state our main result for simply-laced root graded groups. For simplicity, we phrase it for Φ-graded groups (which have to satisfy Axiom 2.5.2 (iv)), but we have seen that it is actually true for any group G as in Notation 5.7.1. Recall further from Remark 5.6.3 that the "base point" of a coordinatisation is only relevant if Φ is of type A and that in this case, it is possible to choose (α_0, γ_0) and c in a way which yields the simple commutator formulas in Example 5.6.5.

5.7.14 Theorem (Coordinatisation theorem for A_n, D_n, E_n). *Let Φ be an irreducible simply-laced root system of rank at least 2 and let $(G, (U_\alpha)_{\alpha \in \Phi})$ be a Φ-graded group. Choose any A_2-pair (α_0, γ_0) and any family $c = (c_{\alpha,\beta})_{\alpha,\beta \in A_n}$ of Chevalley structure constants (in the sense of Definition 3.1.18). Then there exist a ring $\mathcal{R}$ and a coordinatisation of G by $\mathcal{R}$ based at (α_0, γ_0) with signs c. If Φ is of rank at least 3, then $\mathcal{R}$ is necessarily associative. If Φ is of type D or E, then $\mathcal{R}$ is necessarily associative and commutative. Further, if we fix a Δ-system of Weyl elements in G, then we can choose the root isomorphisms $(\theta_\alpha)_{\alpha \in \Phi}$ so that $w_\delta = \theta_{-\delta}(-1_{\mathcal{R}})\theta_\delta(1_{\mathcal{R}})\theta_{-\delta}(-1_{\mathcal{R}})$ for all $\delta \in \Delta$.*

Proof. We have constructed $\mathcal{R}$ in Constructions 5.7.5 and 5.7.6, shown that it is a ring in Proposition 5.7.13 and proven the commutator formulas in Lemma 5.7.10. By Propositions 5.6.9 and 5.6.11, the assertions about the associativity and commutativity of $\mathcal{R}$ hold. By Lemma 5.7.12, the Weyl elements have the desired form. $\square$

Chapter 6

The Blueprint Technique

In [RT87], Ronan-Tits proved the existence of a large class of thick buildings by constructing them from so-called "blueprints". Here a blueprint is, essentially, a local parameter system for a building which encompasses certain commutator relations in the rank-2 residues. A blueprint is said to be realisable if there exists a building which conforms to it, and Ronan-Tits give a simple realisability criterion for blueprints. This criterion involves computations with certain rewriting rules which are induced by a self-homotopy of the longest element in the Coxeter group. For example, they show that a blueprint of type A_3 is realisable if and only if it is associative (in a suitable sense). The corresponding computation is the same one that we will perform in section 6.3.

The blueprint technique is a novel method which consists of performing the same kind of computations as in [RT87] in the context of root graded groups. It makes sense for arbitrary root systems, but it will become apparent in this chapter that it only produces meaningful results if the root system is of rank at least 3. While the computations in the blueprint technique follow the same algorithm as in [RT87], the underlying logic is, in a certain sense, inverted: Ronan-Tits start from explicit commutator relations and then perform computations to derive conditions on these relations. These extra conditions guarantee that a building with the given commutator relations exist. On the other hand, we start with an existing (arbitrary) root graded group G (which one should think of as a "generalised building") and use the blueprint computations to obtain information about the commutation maps in G (in the sense of Remark 4.1.4).

The initial idea to "turn around" the computations of Ronan-Tits in this way was suggested by Bernhard Mühlherr. Originally, the only objective of this approach was to prove that coordinatising rings for C_3-graded groups are alternative. It later turned out that the blueprint technique can be used in a more general way to not only prove identities in the coordinatising algebraic structure (such as the alternative law), but to actually equip the parametrising groups with an algebraic structure. See Note 6.4.4 for more details.

We begin this chapter with a section which describes the general idea of the blueprint technique in a slightly informal way. In the following section, we state and prove the results which make this technique work. In the third section, we show how the blueprint technique can be used to obtain a new proof of the fact that the coordinatising rings for root graded groups of type A in rank at least 3 are associative. We end this chapter with some concluding remarks which, in Strategy 6.4.3, also contain a summary of the blueprint technique. Later in chapters 7 and 9, we will apply the blueprint technique to root graded groups of types B and $(B)C$, respectively. It should be noted that the specific computations for types B and $(B)C$ have not been performed in [RT87].

6.1 The Idea of the Blueprint Technique

6.1.1 Notation for this section. We denote by Φ a root system of rank n, by $\Delta = (\delta_1, \dots, \delta_n)$ a rescaled ordered root base of Φ and by $W := \mathrm{Weyl}(\Phi)$ the Weyl group. Further, we denote by G a Φ-graded group with root groups $(U_\alpha)_{\alpha\in\Phi}$.

In this section, we describe the idea of the blueprint technique. At some points, we will restrict ourselves to the root system $\Phi = A_3$ and the standard root base Δ to simplify notation, but it will always be clear how our arguments should be generalised to arbitrary root systems. Recall from Remark 5.2.9 that the Weyl group of A_3 can be identified with the symmetric group of $[1, 4]$ in a canonical way (as soon as a suitable root basis has been fixed).

6.1.2 Notation. In the context of the blueprint technique, we will often write i for the simple root δ_i. For example, we write U_i for the root group U_{δ_i} or 121 for the word $(\delta_1, \delta_2, \delta_1)$.

At first, we have to introduce homotopy cycles.

6.1.3 Definition (Homotopy cycles). Let w be an arbitrary element of W and let $\bar{\alpha}$ be any expression of w (in the sense of Definition 1.3.20). A *homotopy cycle of* $\bar{\alpha}$ is a sequence of braid homotopy moves (in the sense of Definition 1.3.14) which transforms $\bar{\alpha}$ into itself. A homotopy cycle is said to be *trivial* if every homotopy step is "reversed" by a later step, and *non-trivial* otherwise.

6.1.4 Note. Our definition of triviality for homotopy cycles is admittedly rather vague. However, it will only be used in informal discussions, so this poses no problem.

6.1.5 Example (Homotopy cycles in A_3). Consider the root system $\Phi = A_3$. Here we have the braid moves

$$121 \to 212, \qquad 232 \to 323, \qquad 13 \to 31$$

(and their inverses) which correspond to the braid relations

$$\sigma(\delta_1)\sigma(\delta_2)\sigma(\delta_1) = \sigma(\delta_2)\sigma(\delta_1)\sigma(\delta_2), \qquad \sigma(\delta_2)\sigma(\delta_3)\sigma(\delta_2) = \sigma(\delta_3)\sigma(\delta_2)\sigma(\delta_3),$$
$$\sigma(\delta_1)\sigma(\delta_3) = \sigma(\delta_3)\sigma(\delta_1).$$

Thus the sequences $121 \to 212 \to 121$ and $12113 \to 21213 \to 21231 \to 12131 \to 12113$ are examples of homotopy cycles. Clearly, they are trivial. A reduced expression of the longest element ρ in W (in the sense of Proposition 1.3.29) is given by $\bar{\alpha}_1 = 123121$. A non-trivial homotopy cycle of $\bar{\alpha}_1$ is given in Figure 6.1.

6.1.6 Remark (Homotopy cycles in Cayley graphs). The *Cayley graph of W with respect to the generators $\sigma(\delta_1), \dots, \sigma(\delta_n)$* is, by definition, the graph whose vertex set is W and whose edge set is

$$\{\, \{w, w\sigma(\delta_i)\} \mid w \in W, i \in [1, n] \,\}.$$

Further, any edge of the form $\{w, w\sigma(\delta_i)\}$ is labelled with i. As an example, we consider the Weyl group of A_3. Recall from Remark 5.2.9 that this group can be identified with $\mathrm{Sym}([1, 4])$ such that $\sigma_{e_1-e_2} = (1\,2)$, $\sigma_{e_2-e_3} = (2\,3)$, $\sigma_{e_3-e_4} = (3\,4)$.

(1)	$12\underline{31}21$	(8)	$321\underline{323}$
(2)	$\underline{121}321$	(9)	$3\underline{212}32$
(3)	$21\underline{232}1$	(10)	$\underline{31}2\underline{13}2$
(4)	$2\underline{13}2\underline{31}$	(11)	$1\underline{323}12$
(5)	$23\underline{121}3$	(12)	$123\underline{212}$
(6)	$\underline{232}123$	(13)	123121
(7)	$32\underline{31}23$		

Figure 6.1: A non-trivial homotopy cycle for the longest word in $\mathrm{Weyl}(A_3)$ (taken from [TW02, Fig. 8, p. 472]).

The Cayley graph of $\mathrm{Sym}([1,4])$ with respect to these standard generators is depicted in Figure 6.2 on page 160. Now the (reduced) expressions of any element w correspond to the paths from id to w (of minimal length) and the braid homotopy moves correspond to certain transformation rules for these paths.

The homotopy cycle of the longest element ρ in Figure 6.1 can be interpreted as follows. The Cayley graph in Figure 6.2 can also be drawn on the sphere S^2 with the large outer square (whose top-left corner is id) on the top and the small inner square (whose bottom-right corner is ρ) on the bottom. Then id and ρ lie on opposite points of S^2. Now any minimal path from id to w goes in an approximate semicircle from the top to the bottom, and the homotopy moves in Figure 6.1 "push this semicircle once around (the hole in) S^2". The first and the seventh path of this sequence are depicted in Figures 6.2a and 6.2b, respectively.

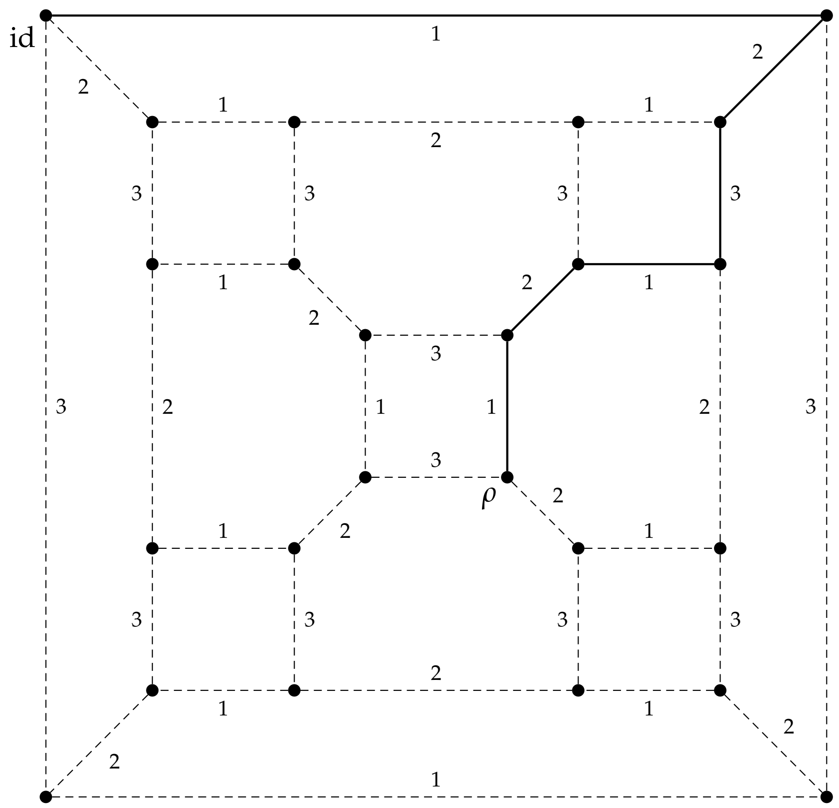
As a corollary of the observations in the previous paragraph, we observe that ρ is the only element of W for which non-trivial homotopy cycles exist: reduced expression of any other word are "too short to be pushed around the sphere". This statement remains true if we replace Φ by any other root system, and it is the reason why we will only consider reduced expressions of the longest element in the blueprint technique.

6.1.7 Note. We have observed in Remark 6.1.6 that every element of W which is not the longest element has only trivial homotopy cycles. If Φ has rank 2, then even the longest element has only trivial homotopy cycles: If $\{ \delta, \delta' \}$ is the root base of such a root system, then the only reduced expressions of the longest element ρ are $P_m(\delta, \delta')$ and $P_m(\delta', \delta)$ for $m := |\Phi|/2$. Hence the only homotopy cycles of ρ are $P_m(\delta, \delta') \to P_m(\delta', \delta) \to P_m(\delta, \delta')$ and $P_m(\delta', \delta) \to P_m(\delta, \delta') \to P_m(\delta, \delta')$ (and iterations of these cycles). Clearly, these cycles are trivial.

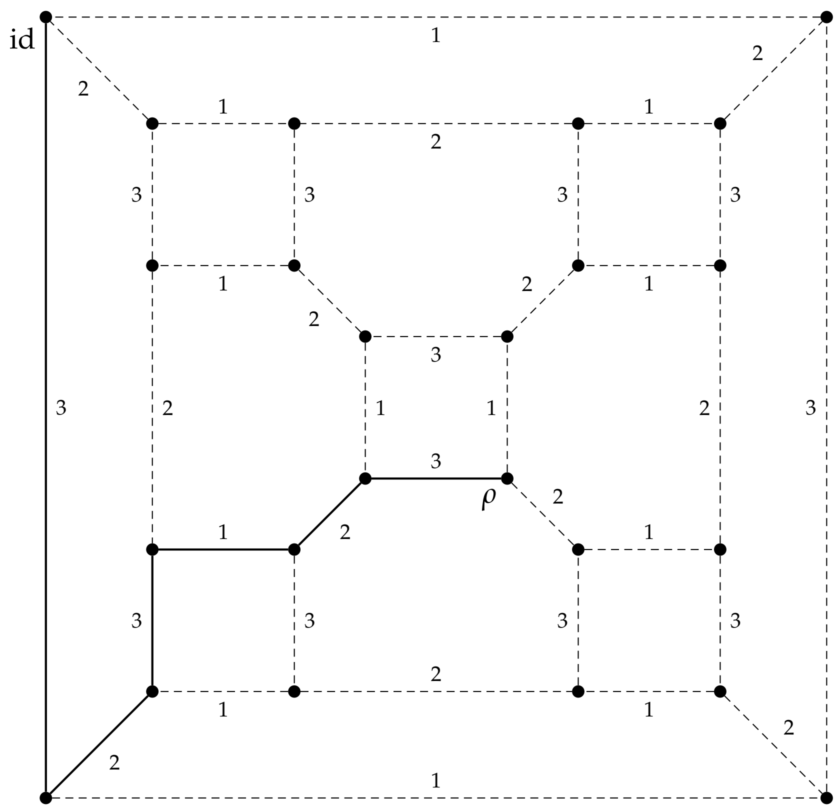
Using the language of homotopy cycles, we can now describe the idea of the blueprint technique.

6.1.8 The blueprint technique. For ease of presentation, we consider only the root system $\Phi = A_3$ in this introduction, but the underlying principle applies to all root systems. We use the standard representation of A_3 from Remark 5.2.3 and we denote by $\Delta = \{ \delta_1, \delta_2, \delta_3 \}$ the standard root base of A_3, so that $\delta_i = e_i - e_{i+1}$ for all $i \in [1,3]$. By Theorem 5.7.14, we have a ring $(\mathcal{R}, +, \cdot)$ and isomorphisms $(\theta_\alpha \colon (\mathcal{R}, +) \to U_\alpha)_{\alpha \in \Phi}$. This ring must be associative by Theorem 5.7.14, but we proceed as if we did not know this, and we try to derive a new proof of this fact.

For each $i \in [1,13]$, we denote by $\bar{\alpha}_i$ the reduced expression of the longest element ρ in the i-th column of Figure 6.1. For example, $\bar{\alpha}_1 = (\delta_1, \delta_2, \delta_3, \delta_1, \delta_2, \delta_1) =$



(a) The path 123121 (rows 1 and 13 in Figure 6.1 on page 159).



(b) The path 323123 (row 7 in Figure 6.1 on page 159).

Figure 6.2: The Cayley graph of $\mathrm{Sym}([1,4])$ for the standard generators s_1, s_2, s_3 and two reduced paths from id to the longest element ρ. See Remark 6.1.6.

$\bar{\alpha}_{13}$ or, in simplified notation, $\bar{\alpha}_1 = 123121 = \bar{\alpha}_{13}$. Further, we choose arbitrary ring elements $a, b, c, d, e, f \in \mathcal{R}$ and consider the tuple

$$y_1 := \big(\theta_1(a), \theta_2(b), \theta_3(c), \theta_1(d), \theta_2(e), \theta_1(f)\big).$$

In other words, y_1 denotes an arbitrary element of

$$U_{\bar{\alpha}_1} = U_1 \times U_2 \times U_3 \times U_1 \times U_2 \times U_1.$$

Recall that Figure 6.1 depicts a non-trivial homotopy cycle of $\bar{\alpha}_1$ which uses the braid moves from Example 6.1.5. Using the theoretical framework of the blueprint technique, we can compute a "sensible" rewriting rule on the tuple y_1 for each of these homotopy moves. For examples, the braid moves $13 \to 31$ and $121 \to 212$ correspond to rewriting rules

$$\begin{aligned} U_1 \times U_3 &\to U_3 \times U_1, \\ \big(\theta_1(x), \theta_3(y)\big) &\mapsto \big(\theta_3(y), \theta_1(x)\big) \qquad \text{and} \\ U_1 \times U_2 \times U_1 &\to U_2 \times U_1 \times U_2, \\ \big(\theta_1(x), \theta_2(y), \theta_1(z)\big) &\mapsto \big(\theta_2(z), \theta_1(-y - zx), \theta_2(x)\big). \end{aligned}$$

Now the main part of the blueprint technique consists of "working down" the column in Figure 6.1 and applying the corresponding rewriting rule to the tuple y_1 in each step. We refer to this procedure as the *blueprint computation*. In the first step, we apply the elementary homotopy $31 \to 13$ to $\bar{\alpha}_1$, which produces $\bar{\alpha}_2 = 121321$. Simultaneously, we apply the corresponding rewriting rule $U_3 \times U_1 \to U_1 \times U_3$ to y_1. This yields a tuple

$$y_2 = \big(\theta_1(a), \theta_2(b), \theta_1(c'), \theta_3(d'), \theta_2(e), \theta_1(f)\big) \in U_{\bar{\alpha}_2}$$

where $c', d' \in \mathcal{R}$ are new ring elements which depend on c and d. In the next step, we apply the elementary homotopy $121 \to 212$ to $\bar{\alpha}_2$ to obtain the word $\bar{\alpha}_3 = 212321$. Again we use the associated rewriting rule to compute a new tuple

$$y_3 = \big(\theta_2(a'), \theta_1(b'), \theta_2(c''), \theta_3(d'), \theta_2(e), \theta_1(f)\big) \in U_{\bar{\alpha}_2}$$

where a', b', c'' depend on a, b and c'. After applying twelve transformations in total, we end up with $\bar{\alpha}_{13} = 123121 = \bar{\alpha}_1$ and a tuple

$$y_{13} = \big(\theta_1(\tilde{a}), \theta_2(\tilde{b}), \theta_3(\tilde{c}), \theta_1(\tilde{d}), \theta_2(\tilde{e}), \theta_1(\tilde{f})\big) \in U_{\bar{\alpha}_{13}} = U_{\bar{\alpha}_1}.$$

For reasons which will be explained later, we know that we must have $y_1 = y_{13}$, so $\tilde{a} = a$, $\tilde{b} = b, \dots, \tilde{f} = f$. Since $\tilde{a}, \dots, \tilde{f}$ can be calculated explicitly (using our rewriting rules), we obtain six relations which hold for arbitrary elements $a, \dots, f \in \mathcal{R}$. When we actually perform these calculations, five of these relations turn out to be trivial, but we will see that the last relation is equivalent to the associativity law. Therefore, the blueprint technique yields that any ring which coordinatises an A_3-graded group must be associative.

6.1.9 Remark (Formal justification). In order to actually perform the calculations in 6.1.8, we need to state the rewriting rules on our tuples. These rules should be chosen in a way which guarantees that, at the end of the computation, we have $y_{13} = y_1$. The idea is as follows: For every reduced expression $\bar{\beta} = (\beta_1, \dots, \beta_m)$ of the longest element, we define a certain map $\gamma_{\bar{\beta}} \colon U_{\beta_1} \times \dots \times U_{\beta_m} \to G$ and we choose our rewriting rules so that $\gamma_{\bar{\alpha}_{i+1}}(y_{i+1}) = \gamma_{\bar{\alpha}_i}(y_i)$ for all i. In other words, we can think of γ as an invariant of the involved tuples, and the rewriting rules are chosen precisely to leave the invariant unchanged. Thus we have

$$\gamma_{\bar{\alpha}_1}(y_1) = \gamma_{\bar{\alpha}_2}(y_2) = \dots = \gamma_{\bar{\alpha}_{k-1}}(y_{k-1}) = \gamma_{\bar{\alpha}_k}(y_k) = \gamma_{\bar{\alpha}_1}(y_k)$$

where k is the length of the homotopy cycle in question. Now it remains to show that $\gamma_{\bar{\alpha}_1}$ is injective, which is done in Proposition 6.2.7.

6.1.10 Remark (Simplification of the blueprint computation). Instead of working down an entire homotopy cycle, we will in practice perform two separate computations: One working "halfway down", which transforms the initial tuple y_1 into some tuple x, and another one working "halfway up", which transforms y_1 into some other tuple x'. By the same arguments as in Remark 6.1.9, we have $\gamma(x') = \gamma(y_1) = \gamma(x'')$, so we infer that $x' = x''$. We will do this because the tuple entries in the blueprint computation become more complicated with each step, so it is more efficient to perform two short computations than to perform one long computation.

6.2 The Formal Framework of the Blueprint Technique

6.2.1 Notation for this section. We denote by Φ a root system of rank n, by $\Delta = (\delta_1, \dots, \delta_n)$ a rescaled ordered root base of Φ, by Π the corresponding positve system in Φ and by $W := \mathrm{Weyl}(\Phi)$ the Weyl group. Further, we denote by G a Φ-graded group with root groups $(U_\alpha)_{\alpha\in\Phi}$ and we fix a Δ-system $(w_\delta)_{\delta\in\Delta}$ of Weyl elements in G.

The first part of this section is devoted to the introduction of the invariant γ that we announced in Remark 6.1.9 and the proof that it is an injective map (Proposition 6.2.7). In the second part, we state some basic results concerning rewriting rules which will be used later on.

6.2.2 Reminder. Recall from Definition 2.1.2 that for any word $\bar{\alpha} = (\alpha_1, \dots, \alpha_k)$ over Φ, we put
$$U_{\bar{\alpha}} := U_{\alpha_1} \times \cdots \times U_{\alpha_k}.$$
This should not be confused with the notation $U_S := \langle U_\alpha \mid \alpha \in S\rangle$ for any subset S of Φ.

6.2.3 Definition (Blueprint invariant). Let $\bar{\delta}$ be a word over Δ. The *blueprint invariant of type $\bar{\delta}$ (with respect to $(w_\delta)_{\delta\in\Delta}$)* is the map
$$\tilde{\gamma}_{\bar{\delta}} \colon U_{\bar{\delta}} \to G, (g_1, \dots, g_m) \mapsto \prod_{i=1}^{m} w_{\delta_i} g_i = w_{\delta_1} g_1 \cdots w_{\delta_m} g_m.$$
Further, the *reduced blueprint invariant of type $\bar{\delta}$ (with respect to $(w_\delta)_{\delta\in\Delta}$)* is the composition of $\tilde{\gamma}_{\bar{\delta}}$ with the canonical projection to $G/Z(G)$:
$$\gamma_{\bar{\delta}} \colon U_{\bar{\alpha}} \to G/Z(G), (g_1, \dots, g_m) \mapsto \tilde{\gamma}_{g_1,\dots,g_m} Z(G).$$

6.2.4 Definition (Blueprint rewriting rules). Let $\bar{\alpha}, \bar{\beta}$ be words over Δ. A *rewriting rule of type $(\bar{\alpha}, \bar{\beta})$ (with respect to $(w_\delta)_{\delta\in\Delta}$)* is a map $U_{\bar{\alpha}} \to U_{\bar{\beta}}$. A rewriting rule φ of type $(\bar{\alpha}, \bar{\beta})$ is called a *blueprint rewriting rule* if it respects the reduced blueprint invariant: $\gamma_{\bar{\beta}} \circ \varphi = \gamma_{\bar{\alpha}}$.

The main result in this section is the fact that the reduced blueprint invariant is an injective map. For the non-reduced blueprint invariant, this is a direct consequence of the injectivity of the product maps (Proposition 2.5.7), but we have to work a bit more to see that it also holds modulo the center.

6.2.5 Lemma. *Let $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$ be a reduced word over Δ (where $m \geq 1$) and let $v := \sigma(\bar{\alpha})$. Then $U_{N(v)} \cap Z(G) = \{1_G\}$ where $Z(G)$ denotes the center of G.*

$$\begin{tikzcd} U_{\bar\alpha} \arrow[rr, "\varphi"] \arrow[dr, "\gamma_{\bar\alpha}"'] & & U_{\bar\beta} \arrow[dl, "\gamma_{\bar\beta}"] \\ & G/Z(G) & \end{tikzcd}$$

Figure 6.3: The condition of being a blueprint rewriting rule.

Proof. It suffices to prove the statement for the root base corresponding to Δ, so we can assume that Δ is a proper root base. This allows us to apply Lemma 1.3.25 and Proposition 1.3.28. The statement is trivial if $m = 1$, so we assume that $m > 1$. Let $\bar\beta = (\beta_1, \dots, \beta_m)$ be the root sequence associated to $\bar\alpha$ as in Definition 1.3.26, so that $N(w) = \{ \beta_1, \cdots, \beta_m \}$ by Proposition 1.3.28. Let $x \in U_{N(w)} \cap Z(G)$. We want to show that $x = 1_G$. By Proposition 2.5.7, the product map on $\bar\beta$ is bijective, so there exist $x_1 \in U_{\beta_1}, \dots, x_m \in U_{\beta_m}$ such that $x = x_1 \cdots x_m$. Choose an arbitrary β_m-Weyl element w_m, and observe that $\beta_m = \alpha_m$ by the definition of root sequences. In particular, β_m is a simple root. Now $x_1 \cdots x_m = x_1^{w_m} \cdots x_m^{w_m}$ because $x_1 \cdots x_m$ is central, which implies that

$$(x_1^{w_m} \cdots x_{m-1}^{w_m})^{-1} x_1 \cdots x_m = x_m^{w_m}.$$

Note that the element on the left-hand side lies in U_Π (because $(\Pi \setminus \mathbb{R}_{>0}\gamma)^{\sigma(\gamma)} = \Pi \setminus \mathbb{R}_{>0}\gamma \subseteq \Pi$ for any $\gamma \in \Delta$ by Lemma 1.3.24) while the element on the right-hand side lies in $U_{-\beta_m}$. Since $U_\Pi \cap U_{-\beta_m} = \{1_G\}$ by Axiom 2.5.2 (iv), it follows that $x_m^{w_m} = 1_G$, so $x_m = 1_G$.

We conclude from the previous paragraph that $x = x_1 \cdots x_{m-1}$. Put $v' := v\sigma(\alpha_m)$. Note that $(\alpha_1, \dots, \alpha_{m-1})$ is a reduced expression of v'. By Lemma 1.3.25,

$$\{ \beta_1, \cdots, \beta_{m-1} \} = N(v) \setminus \{\beta_m\} = N(v) \setminus \{\alpha_m\} = N(w')^{\sigma(\alpha_m)}.$$

Thus $x_1 \cdots x_{m-1}$ lies in $U_{N(v')^{\sigma(\alpha_m)}} = U_{N(v')}^{w_m}$, so $x_1^{w_m} \cdots x_{m-1}^{w_m}$ lies in $U_{N(v')}$. It follows that $x = x_1^{w_m} \cdots x_{m-1}^{w_m} \in U_{N(w')} \cap Z(G)$. Using induction, we infer that $x = 1_G$, as desired. $\square$

6.2.6 Proposition. $U_\Pi \cap Z(G) = \{1_G\}$.

Proof. The longest word ρ satisfies $N(\rho) = \Pi^{\text{indiv}}$ by Proposition 1.3.29. Since $U_{\Pi^{\text{indiv}}} = U_\Pi$, the assertion follows from Lemma 6.2.5. $\square$

6.2.7 Proposition. *Let $\bar\alpha = (\alpha_1, \dots, \alpha_m)$ be a reduced word over Δ. Then the reduced blueprint invariant $\gamma_{\bar\alpha}$ of type $\bar\alpha$ with respect to $(w_\delta)_{\delta\in\Delta}$ is injective.*

Proof. As in Notation 6.1.2, we put $U_i := U_{\delta_i}$ and $w_i := w_{\delta_i}$ for all $i \in [1, n]$. Let $g = (g_1, \dots, g_m)$ and $h = (h_1, \dots, h_m)$ be elements of $U_1 \times \cdots \times U_m$ such that $\gamma_{\bar\alpha}(g) = \gamma_{\bar\alpha}(h)$. Then there exists $z \in Z(G)$ such that

$$w_1 g_1 w_2 g_2 \cdots w_{m-1} g_{m-1} w_m g_m = w_1 h_1 w_2 h_2 \cdots w_{m-1} h_{m-1} w_m h_m z.$$

We want to show that $g = h$. Using that $ab = ba^b$ for all $a, b \in G$, we see that the product on the left side equals

$$w_1 w_2 \cdots w_m g_1^{w_2 \cdots w_m} g_2^{w_3 \cdots w_m} \cdots g_{m-1}^{w_m} g_m.$$

A similar transformation can be done on the right-hand side. We infer that

$$g_1^{w_2 \cdots w_m} g_2^{w_3 \cdots w_m} \cdots g_{m-1}^{w_m} g_m = h_1^{w_2 \cdots w_m} h_2^{w_3 \cdots w_m} \cdots h_{m-1}^{w_m} h_m z. \tag{6.1}$$

Put

$$\bar{g} := (g_1^{w_2\cdots w_m}, g_2^{w_3\cdots w_m}, \ldots, g_{m-1}^{w_m}, g_m), \qquad \bar{h} := (h_1^{w_2\cdots w_m}, h_2^{w_3\cdots w_m}, \ldots, h_{m-1}^{w_m}, h_m).$$

Note that for each $i \in [1,m]$, $g_i^{w_{i+1}\cdots w_m}$ and $h_i^{w_{i+1}\cdots w_m}$ lie in the root group associated to $\alpha_i^{\sigma(\alpha_{i+1}\cdots\alpha_m)}$. It follows that $\bar{g}$ and $\bar{h}$ lie in $U_{\bar{\beta}}$ where $\bar{\beta} = (\beta_1, \ldots, \beta_m)$ denotes the root sequence associated to $\bar{\alpha}$. Thus (6.1) says that $\mu_{\bar{\beta}}(\bar{g}) = \mu_{\bar{\beta}}(\bar{h})z$ where $\mu_{\bar{\beta}}$ denotes the product map on $\bar{\beta}$. Hence

$$z = \mu_{\bar{\beta}}(\bar{h})^{-1}\mu_{\bar{\beta}}(\bar{g}) \in U_\Pi \cap Z(G).$$

By Proposition 6.2.6, it follows that $z = 1_G$, so we actually have $\mu_{\bar{\beta}}(\bar{g}) = \mu_{\bar{\beta}}(\bar{h})$. Using Proposition 2.5.7, we conclude that $\bar{g} = \bar{h}$. Since conjugation by Weyl elements is an automorphism, this yields $g = h$, so $\gamma_{\bar{\alpha}}$ is indeed injective. □

We can now state the essence of the blueprint technique as follows.

6.2.8 Theorem. *Let $\bar{\alpha}, \bar{\beta}$ be two reduced words over Δ such that $\sigma_{\bar{\alpha}} = \sigma_{\bar{\beta}}$, let $x \in U_{\bar{\alpha}}$ and let $\psi_1, \psi_2 \colon U_{\bar{\alpha}} \to U_{\bar{\beta}}$ be two blueprint rewriting rules. Then $\psi_1(x) = \psi_2(x)$.*

Proof. Since ψ_1 and ψ_2 are γ-compatible, we have

$$\gamma_{\bar{\beta}}(\psi_1(x)) = \gamma_{\bar{\alpha}}(x) = \gamma_{\bar{\beta}}(\psi_2(x)).$$

Since $\gamma_{\bar{\beta}}$ is injective by Proposition 6.2.7, the assertion follows. □

6.2.9 Note. In practice, all blueprint rewriting rules in this thesis will be compatible with the non-reduced blueprint invariant, and not merely with the reduced blueprint invariant. Thus it would have been sufficient for our purposes to only prove the injectivity of the non-reduced blueprint invariant, which would have been slightly easier.

The rest of this section is a collection of elementary facts about blueprint rewriting rules.

6.2.10 Remark. Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\bar{\beta} = (\beta_1, \ldots, \beta_m)$ be two words over Δ, let $g_i \in U_{\alpha_i}$ for all $i \in [1,n]$ and let $h_j \in U_{\beta_j}$ for all $j \in [1,m]$. Then

$$\gamma_{\bar{\alpha}\bar{\beta}}(g_1, \ldots, g_n, h_1, \ldots, h_m) = \gamma_{\bar{\alpha}}(g_1, \ldots, g_n)\gamma_{\bar{\beta}}(h_1, \ldots, h_m).$$

6.2.11 Remark. Let $\bar{\alpha}$, $\bar{\beta}$ be words over Φ and let $\varphi \colon U_{\bar{\alpha}} \to U_{\bar{\beta}}$ be a bijective blueprint rewriting rule. Then $\gamma_{\bar{\beta}} \circ \varphi = \gamma_{\bar{\alpha}}$, so

$$\gamma_{\bar{\beta}} = \gamma_{\bar{\beta}} \circ \varphi \circ \varphi^{-1} = \gamma_{\bar{\alpha}} \circ \varphi^{-1}.$$

That is, φ^{-1} is also a blueprint rewriting rule.

6.2.12 Remark. Clearly, the concatenation of two blueprint rewriting rules is again a blueprint rewriting rule.

In the following Lemma 6.2.13 and its proof, we will identify $(\bar{g}, \bar{h})$ (for any tuples $\bar{g} = (g_1, \ldots, g_n)$, $\bar{h} = (h_1, \ldots, h_m)$) with the tuple $(g_1, \ldots, g_n, h_1, \ldots, h_m)$.

6.2.13 Lemma. *Let $\bar{\alpha}$, $\bar{\beta}$, $\bar{\alpha}'$, $\bar{\beta}'$ be words over Φ and let*

$$\varphi \colon U_{\bar{\alpha}} \to U_{\bar{\alpha}'}, \bar{g} \mapsto \varphi(\bar{g}) \quad \text{and} \quad \psi \colon U_{\bar{\beta}} \to U_{\bar{\beta}'}, \bar{h} \mapsto \psi(\bar{h})$$

be blueprint rewriting rules. Further, define

$$\varphi \times \psi \colon U_{\bar{\alpha}\bar{\beta}} = U_{\bar{\alpha}} \times U_{\bar{\beta}} \to U_{\bar{\alpha}'\bar{\beta}'} = U_{\bar{\alpha}'} \times U_{\bar{\beta}'}, (\bar{g}, \bar{h}) \mapsto (\varphi(\bar{g}), \psi(\bar{h})).$$

Then $\varphi \times \psi$ is a blueprint rewriting rule.

Proof. Let $\bar{g} \in U_{\bar{\alpha}}$ and let $\bar{h} \in U_{\bar{\beta}}$. By Remark 6.2.10, we have

$$\gamma_{\bar{\alpha}\bar{\beta}}(\bar{g}, \bar{h}) = \gamma_{\bar{\alpha}}(\bar{g})\gamma_{\bar{\beta}}(\bar{h}).$$

On the other hand,

$$\big(\gamma_{\bar{\alpha}'\bar{\beta}'} \circ (\varphi \times \psi)\big)(\bar{g}, \bar{h}) = \gamma_{\bar{\alpha}'\bar{\beta}'}\big(\varphi(\bar{g}), \psi(\bar{h})\big) = \gamma_{\bar{\alpha}'}\big(\varphi(\bar{g})\big)\gamma_{\bar{\beta}'}\big(\psi(\bar{h})\big).$$

Since φ and ψ are blueprint rewriting rules, the assertion follows. □

The following result says that it is sufficient to compute blueprint rewriting rules for the basic braid moves, for example for $121 \to 212$ in A_3. These basic rewriting rules can then be extended to rewriting rules which are defined on reduced expressions of the longest word.

6.2.14 Proposition. *Let $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_2'$ be words over Δ and let $\varphi\colon U_{\bar{\alpha}_2} \to U_{\bar{\alpha}_2'}$ be a blueprint rewriting rule. Then the map*

$$\mathrm{id} \times \varphi \times \mathrm{id}\colon U_{\bar{\alpha}_1} \times U_{\bar{\alpha}_2} \times U_{\bar{\alpha}_3} \to U_{\bar{\alpha}_1} \times U_{\bar{\alpha}_2'} \times U_{\bar{\alpha}_3}, (\bar{g}_1, \bar{g}_2, \bar{g}_3) \mapsto (\bar{g}_1, \varphi(\bar{g}_2), \bar{g}_3)$$

is a blueprint rewriting rule.

Proof. The identity map is clearly a blueprint rewriting rule, so this is a consequence of Lemma 6.2.13. □

The following blueprint rewriting rule is easy to compute and will be used for all root systems.

6.2.15 Lemma. *Let $\alpha, \beta \in \Phi$ be such that β is adjacent to α and to $-\alpha$. Then the map*

$$\varphi\colon U_\alpha \times U_\beta \to U_\beta \times U_\alpha, (x_\alpha, x_\beta) \mapsto (x_\beta, x_\alpha).$$

is a blueprint rewriting rule.

Proof. Note that $-\beta$ is adjacent to $-\alpha$ because β is adjacent to α. Similarly, $-\beta$ is adjacent to α because β is adjacent to $-\alpha$. Thus $\langle U_\alpha, U_{-\alpha}\rangle$ commutes with $\langle U_\beta, U_{-\beta}\rangle$. Hence

$$\gamma_{\alpha\beta}(x_\alpha, x_\beta) = w_\alpha x_\alpha w_\beta x_\beta = w_\beta x_\beta w_\alpha x_\alpha \qquad \text{and}$$
$$(\gamma_{\beta\alpha} \circ \varphi)(x_\alpha, x_\beta) = \gamma_{\beta\alpha}(x_\beta, x_\alpha) = w_\beta x_\beta w_\alpha x_\alpha$$

for all $x_\alpha \in U_\alpha$ and all $x_\beta \in U_\beta$. In other words, φ is a blueprint rewriting rule. □

6.3 Blueprint Computations for A_3

6.3.1 Notation for this section. We consider the root system A_n for $n \geq 3$ in standard representation and with its standard ordered root base $\Delta = (\delta_1, \ldots, \delta_n)$. Further, we denote by G a group with an A_n-grading $(U_\alpha)_{\alpha \in A_n}$ and by $(\theta_\alpha)_{\alpha \in A_n}$ a coordinatisation of G by some ring $\mathcal{R}$ with signs and base point chosen as in Example 5.6.5. and by $(w_\delta)_{\delta \in \Delta}$ a fixed Δ-system of Weyl elements in G.

6.3.2 Notation. We will usually (and in contrast to Notation 6.1.2) identify any root $e_i - e_j$ with the pair (i, j) and we will always write ij instead of (i, j). In other words, we set $U_{ij} := U_{e_i - e_j}$ and $\theta_{ij} := \theta_{e_i - e_j}$ for all distinct $i, j \in [1, n+1]$. Further, we denote by

$$w_{ij} := w_{ij}(1_{\mathcal{R}}) = \theta_{ji}(-1_{\mathcal{R}})\theta_{ij}(1_{\mathcal{R}})\theta_{ij}(1_{\mathcal{R}})$$

the Weyl element from Proposition 5.6.6.

In this section, we will demonstrate how the blueprint technique can be used to show that the coordinatising ring $\mathcal{R}$ must be associative. In other words, we give a new proof of the A_n-case in Proposition 5.6.9. All computations in this section will take place in the parabolic root subsystem

$$A_3 := \{ e_i - e_j \mid i \neq j \in [1,4] \}$$

of A_n and in the corresponding A_3-graded subgroup of G. No further information can be obtained from the blueprint technique by considering larger subsystems.

At first, we have to compute the blueprint rewriting rules for A_3. We begin by explicitly stating them. In the proof of Lemma 6.3.4, we will see how to arrive at these specific formulas.

6.3.3 Definition (Rewriting rules for A_3). We define the following rewriting rules:

$$\begin{aligned}
&\psi_{123}\colon U_{12} \times U_{23} \times U_{12} \to U_{23} \times U_{12} \times U_{12},\\
&\big(\theta_{12}(a), \theta_{23}(b), \theta_{12}(c)\big) \mapsto \big(\theta_{23}(c), \theta_{12}(-b - ca), \theta_{23}(a)\big);\\
&\psi_{234}\colon U_{23} \times U_{34} \times U_{23} \to U_{34} \times U_{23} \times U_{34},\\
&\big(\theta_{23}(a), \theta_{34}(b), \theta_{23}(c)\big) \mapsto \big(\theta_{34}(c), \theta_{23}(-b - ca), \theta_{34}(a)\big); \qquad \text{and}\\
&\qquad\varphi\colon U_{12} \times U_{34} \to U_{34} \times U_{12},\\
&\qquad\big(\theta_{12}(a), \theta_{34}(b)\big) \mapsto \big(\theta_{34}(b), \theta_{12}(a)\big).
\end{aligned}$$

It is easy to see that these maps are bijections whose inverses are given by

$$\begin{aligned}
&\psi_{123}^{-1}\colon U_{23} \times U_{12} \times U_{12} \to U_{12} \times U_{23} \times U_{12},\\
&\big(\theta_{23}(a), \theta_{12}(b), \theta_{23}(c)\big) \mapsto \big(\theta_{12}(c), \theta_{23}(-b - ac), \theta_{12}(a)\big);\\
&\psi_{234}^{-1}\colon U_{34} \times U_{23} \times U_{34} \to U_{23} \times U_{34} \times U_{23},\\
&\big(\theta_{34}(a), \theta_{23}(b), \theta_{34}(c)\big) \mapsto \big(\theta_{23}(c), \theta_{34}(-b - ac), \theta_{23}(a)\big); \qquad \text{and}\\
&\qquad\varphi^{-1}\colon U_{34} \times U_{12} \to U_{12} \times U_{34},\\
&\qquad\big(\theta_{34}(b), \theta_{12}(a)\big) \mapsto \big(\theta_{12}(a), \theta_{34}(b)\big).
\end{aligned}$$

6.3.4 Lemma. *ψ_{123}, ψ_{123}^{-1}, ψ_{234}, ψ_{234}^{-1}, φ and φ^{-1} are blueprint rewriting rules (with respect to the Weyl elements w_{12}, w_{23}, w_{34}).*

Proof. By Remark 6.2.11, we only need to prove that ψ_{123}, ψ_{234} and φ are blueprint rewriting rules. Further, φ is a blueprint rewriting rule by Lemma 6.2.15. We proceed to prove that ψ_{123} is a blueprint rewriting rule (and the proof for ψ_{234} is identical). Let $a, b, c \in \mathcal{R}$ and set

$$\bar{\alpha} := (e_1 - e_2, e_2 - e_3, e_1 - e_2) \quad \text{and} \quad \bar{\beta} := (e_2 - e_3, e_1 - e_2, e_2 - e_3).$$

On the one hand, we have

$$\begin{aligned}
x &:= (\gamma_{\bar{\beta}} \circ \psi_{123})\big(\theta_{12}(a), \theta_{23}(b), \theta_{12}(c)\big) = \gamma_{\bar{\beta}}\big(\theta_{23}(c), \theta_{12}(-b - ca), \theta_{23}(a)\big)\\
&= w_{23}\theta_{23}(c) w_{12}\theta_{12}(-b - ca) w_{23}\theta_{23}(a)\\
&= w_{23}w_{12}w_{23}\theta_{23}(c)^{w_{12}w_{23}}\theta_{12}(-b - ca)^{w_{23}}\theta_{23}(a).
\end{aligned}$$

Using the conjugation formula from Example 5.6.8, we can compute that

$$\begin{aligned}
\theta_{23}(c)^{w_{12}w_{23}}\theta_{12}(-b - ca)^{w_{23}}\theta_{23}(a) &= \theta_{13}(-c)^{w_{23}}\theta_{13}(-b - ca)\theta_{23}(a)\\
&= \theta_{12}(c)\theta_{13}(-b - ca)\theta_{23}(a)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned} y &:= \gamma_{\bar{\alpha}}\big(\theta_{12}(a), \theta_{23}(b), \theta_{12}(c)\big) = w_{12}\theta_{12}(a)w_{23}\theta_{23}(b)w_{12}\theta_{12}(c) \\ &= w_{12}w_{23}w_{12}\theta_{12}(a)^{w_{23}w_{12}}\theta_{23}(b)^{w_{12}}\theta_{12}(c) \end{aligned}$$

where

$$\theta_{12}(a)^{w_{23}w_{12}}\theta_{23}(b)^{w_{12}}\theta_{12}(c) = \theta_{13}(a)^{w_{12}}\theta_{13}(-b)\theta_{12}(c) = \theta_{23}(a)\theta_{13}(-b)\theta_{12}(c).$$

We have to prove that $x = y$. Since $w_{23}w_{12}w_{23} = w_{12}w_{23}w_{12}$ by Theorem 2.5.10, it only remains to show that

$$\theta_{12}(c)\theta_{13}(-b - ca)\theta_{23}(a) = \theta_{23}(a)\theta_{13}(-b)\theta_{12}(c).$$

In order to prove this, we apply the commutator relations in A_3-graded groups:

$$\begin{aligned} \theta_{23}(a)\theta_{13}(-b)\theta_{12}(c) &= \theta_{23}(a)\theta_{12}(c)\theta_{13}(-b) = \theta_{12}(c)\theta_{23}(a)[\theta_{23}(a), \theta_{12}(c)]\theta_{13}(-b) \\ &= \theta_{12}(c)\theta_{23}(a)[\theta_{12}(c), \theta_{23}(a)]^{-1}\theta_{13}(-b) \\ &= \theta_{12}(c)\theta_{23}(a)\theta_{13}(-ca)\theta_{13}(-b) = \theta_{12}(c)\theta_{13}(-b - ca)\theta_{23}(a). \end{aligned}$$

Thus $x = y$, so ψ_{123} is a blueprint rewriting rule. This finishes the proof. □

Now that we have determined the blueprint rewriting rules for A_3, we can apply the ideas from 6.1.8 and 6.1.10 in practice.

6.3.5 Theorem. *The multiplication $\cdot$ on $\mathcal{R}$ is associative.*

Proof. In this proof, we drop Notation 6.3.2 and go back to Notation 6.1.2 instead. That is, we write θ_i in place of $\theta_{i,i+1}$. Let $a, b, c, d, e, f \in \mathcal{R}$ be arbitrary and set

$$x := \big(\theta_1(a), \theta_2(b), \theta_3(c), \theta_1(d), \theta_2(e), \theta_1(f)\big).$$

A homotopy cycle for the longest word in A_3 is given in Figure 6.4c on page 168. Working down rows 1 to 7 and applying the corresponding blueprint rewriting rules in the process (as in Figure 6.4a), we obtain a tuple $y = (y_1, \dots, y_6)$ where

$$\begin{aligned} &y_1 = \theta_3(f), && y_2 = \theta_2(-e - fd), && y_3 = \theta_3(d), \\ &y_4 = \theta_1\big(c + ea + f(b + da)\big), && y_5 = \theta_2(-b - da), && y_6 = \theta_3(a). \end{aligned}$$

Further, working up rows 13 to 7 yields a tuple $z = (z_1, \dots, z_6)$ where

$$\begin{aligned} &z_1 = \theta_3(f), && z_2 = \theta_2(-e - fd), && z_3 = \theta_3(d), \\ &z_4 = \theta_1\big(c + fb + (e + fd)a\big), && z_5 = \theta_2(-b - da), && z_6 = \theta_3(a). \end{aligned}$$

(For the intermediate steps of these two calculations, see Figures 6.4a and 6.4b.) Since each used rewriting rule is a blueprint rewriting rule, an application of Theorem 6.2.8 yields $y_i = z_i$ for all $i \in [1, 6]$. In particular, $y_4 = z_4$, so

$$c + ea + f(b + da) = c + fb + (e + fd)a \qquad \text{for all } a, b, c, d, e, f \in \mathcal{R}.$$

Setting $b = c = e = 0$, we conclude that $f(da) = (fd)a$ for all $a, f, d \in \mathcal{R}$. That is, $\mathcal{R}$ is associative. □

6.3.6 Note. We could set b, c and e to zero right at the beginning of the computation in Figure 6.4 to obtain an easier (and yet flawless) proof of Theorem 6.3.5. Of course, this is only possible with the knowledge which variables turn out to be immaterial, and we can only obtain this knowledge by performing the full computation.

(1)	$12\underline{31}21$	$\theta_1(a)$	$\theta_2(b)$	$\theta_3(c)$	$\theta_1(d)$	$\theta_2(e)$	$\theta_1(f)$
(2)	$\underline{121}321$	$\theta_1(a)$	$\theta_2(b)$	$\theta_1(d)$	$\theta_3(c)$	$\theta_2(e)$	$\theta_1(f)$
(3)	$21\underline{232}1$	$\theta_2(d)$	$\theta_1(-b-da)$	$\theta_2(a)$	$\theta_3(c)$	$\theta_2(e)$	$\theta_1(f)$
(4)	$2\underline{13}2\underline{31}$	$\theta_2(d)$	$\theta_1(-b-da)$	$\theta_3(e)$	$\theta_2(-c-ea)$	$\theta_3(a)$	$\theta_1(f)$
(5)	$23\underline{121}3$	$\theta_2(d)$	$\theta_3(e)$	$\theta_1(-b-da)$	$\theta_2(-c-ea)$	$\theta_1(f)$	$\theta_3(a)$
(6)	$\underline{232}123$	$\theta_2(d)$	$\theta_3(e)$	$\theta_2(f)$	$\theta_1(c+ea+f(b+da))$	$\theta_2(-b-da)$	$\theta_3(a)$
(7)	323123	$\theta_3(f)$	$\theta_2(-e-fd)$	$\theta_3(d)$	$\theta_1\big(c+ea+f(b+da)\big)$	$\theta_2(-b-da)$	$\theta_3(a)$

(a) The first part of the blueprint computation: rows 1 to 7.

(13)	$123\underline{121}$	$\theta_1(a)$	$\theta_2(b)$	$\theta_3(c)$	$\theta_1(d)$	$\theta_2(e)$	$\theta_1(f)$
(12)	$1\underline{232}12$	$\theta_1(a)$	$\theta_2(b)$	$\theta_3(c)$	$\theta_2(f)$	$\theta_1(-e-fd)$	$\theta_2(d)$
(11)	$\underline{13}2\underline{31}2$	$\theta_1(a)$	$\theta_3(f)$	$\theta_2(-c-fb)$	$\theta_3(b)$	$\theta_1(-e-fd)$	$\theta_2(d)$
(10)	$3\underline{121}32$	$\theta_3(f)$	$\theta_1(a)$	$\theta_2(-c-fb)$	$\theta_1(-e-fd)$	$\theta_3(b)$	$\theta_2(d)$
(9)	$321\underline{232}$	$\theta_3(f)$	$\theta_2(-e-fd)$	$\theta_1\big(c+fb+(e+fd)a\big)$	$\theta_2(a)$	$\theta_3(b)$	$\theta_2(d)$
(8)	$32\underline{13}23$	$\theta_3(f)$	$\theta_2(-e-fd)$	$\theta_1\big(c+fb+(e+fd)a\big)$	$\theta_3(d)$	$\theta_2(-b-da)$	$\theta_3(a)$
(7)	323123	$\theta_3(f)$	$\theta_2(-e-fd)$	$\theta_3(d)$	$\theta_1\big(c+fb+(e+fd)a\big)$	$\theta_2(-b-da)$	$\theta_3(a)$

(b) The first part of the blueprint computation: rows 13 to 7.

(1)	$12\underline{31}21$
(2)	$\underline{121}321$
(3)	$21\underline{232}1$
(4)	$2\underline{13}2\underline{31}$
(5)	$23\underline{121}3$
(6)	$\underline{232}123$
(7)	$32\underline{31}23$
(8)	$321\underline{323}$
(9)	$3\underline{212}32$
(10)	$\underline{31}2\underline{13}2$
(11)	$1\underline{323}12$
(12)	$123\underline{212}$
(13)	123121

(c) A homotopy cycle.

Figure 6.4: The blueprint technique applied to A_3.

6.3.7 Note. Observe that we have never applied the distributive laws in the computation in Figure 6.4. In fact, these laws follow from the blueprint computations: Putting $c = b = 0$ and $f = 1$ in the equation $y_4 = z_4$ in the proof of Theorem 6.3.5, we obtain that

$$ea + da = (e + d)a.$$

Conversely, putting $c = d = 0$ and $d = 1$, we infer that

$$f(b + a) = fb + fa.$$

This proof is independent of the one in Lemma 5.7.8. We will get back to this observation in Remark 6.4.2.

6.3.8 Note. In the same manner as for A_3, we can use the blueprint technique to show that any ring which coordinatises a D_4-graded group is commutative. Since all the root systems E_6, E_7, E_8 and D_n for $n \geq 4$ contain D_4 as a parabolic subsystem, it follows that the same assertion holds for these root systems as well.

6.4 Concluding Remarks

We end this chapter with some remarks.

6.4.1 Remark (Commutation maps). In section 6.3, we have shown that any ring $(\mathcal{R}, +, \cdot)$ which coordinatises an A_3-graded group (with standard signs) is associative. We can also phrase this as follows: We have shown in Lemma 5.7.12, Theorem 6.3.5, and Note 6.3.7 that for any group $(\mathcal{R}, +)$ which coordinatises an A_3-graded group with standard signs, the commutation map $f\colon \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ satisfies the identities (4.1) in Remark 4.1.4 which turn $\mathcal{R}$ into an associative ring. This viewpoint – that the blueprint computations yield identities for the commutation maps which are precisely the axioms of some algebraic structure – will generalise without problems to the more complicated root systems.

6.4.2 Remark (Classes of identities). It will be beneficial to informally distinguish two classes of identities for commutation maps. We say that an identity is a *rank-2 identity* if it follows from computations "within a single rank-2 subgroup", and we say that it is a *higher-rank identity* if it requires computations in a subgroup of rank 3 or higher. For example, the "bi-additivity" of commutators in Lemma 2.1.13 is proven in the rank-2 setting, and so the distributive laws for the ring multiplication in Lemma 5.7.8 are rank-2 identities. We will see other examples of rank-2 identities in Lemmas 7.4.4, 7.5.3 and 9.4.3. On the other hand, the associativity and commutativity laws in Propositions 5.6.9 and 5.6.11 are higher-rank identities.

Both kinds of identities can in principle be derived with the blueprint technique, as we have seen in Theorem 6.3.5 and Note 6.3.7. However, since the rank-2 identities can be proven in an easy and straightforward way, it is usually more efficient to derive them directly and then to use them during the blueprint computation to simplify terms.

Another important type of identities are those which we call *Weyl identities*. These are, by definition, identities which hold for the specific elements that parametrise the fixed Δ-system $(w_\delta)_{\delta \in \Delta}$ of Weyl elements. In the A_3-case, the identity $1_\mathcal{R} \cdot r = r = r \cdot 1_\mathcal{R}$ (where $1_\mathcal{R}$ is defined to be the unique element of $\mathcal{R}$ which satisfies $w_{\delta_0} \in U_{-\delta_0}\theta_{\delta_0}(1_\mathcal{R})U_{-\delta_0}$) is an (and the only) example of a Weyl identity. These identities always follow from formulas as in Proposition 5.4.3.

Technically, they are rank-2 identities, but it is beneficial to consider them separately because they do not follow from the blueprint computations. The reason for this is that the blueprint computations only produce identities which hold for arbitrary parameters whereas the Weyl identities involve the specific elements which parametrise $(w_\delta)_{\delta \in \Delta}$.

6.4.3 Strategy. Let Φ be a root system of rank at least 3 and let G be a Φ-graded group. We can summarise the steps in the blueprint technique for Φ-graded groups as follows:

(1) Compute a reduced expression of the longest element in $\mathrm{Weyl}(\Phi)$ and a non-trivial homotopy cycle of this expression.
(2) Define commutation maps in G, as in Remark 4.1.4 and Note 4.1.13.
(3) Compute the rank-2 identities of the commutation maps, including the Weyl identities.
(4) For each braid homotopy move in $\mathrm{Weyl}(\Phi)$, compute a blueprint rewriting rule. These rewriting rules involve the commutation maps in G.
(5) Perform the blueprint computations, as in Figure 6.4.
(6) Show that the set of identities which result from the blueprint computations is equivalent to a set of "smaller, easier-to-understand" identities. This is done by putting several of the involved variables to zero (or to 1 or to similar "canonical" elements) and then using the resulting identities to simplify the original identities.

6.4.4 Note (The original conception of the blueprint technique). Using the Hall-Witt identity, Zhang shows in [Zha14, 3.4.9 to 3.4.18] that the commutation maps in certain C_3-graded groups can be described explicitly in terms of the algebraic structure of a ring with involution. The original idea of the blueprint technique was to apply it only after these Hall-Witt computations are done, which results in more explicit formulas for the blueprint rewriting rules. This in turn makes the blueprint computation much less cumbersome, essentially because many of the identities which would "normally" result from the blueprint computation have already been verified "by hand". The only new identity which follows from the "original" blueprint computation is the alternative law, which is in fact all that we wanted to prove at this time. It is a crucial observation that the blueprint computations work just as well if the rewriting rules are given only in terms of abstract commutation maps, which significantly streamlines our approach to root graded groups for all root systems. See also Remark 6.4.1.

6.4.5 Remark (Rank higher than 3). The length of the longest word in $\mathrm{Weyl}(\Phi)$ and thus of the blueprint computation increases with the rank of Φ, so there is a high incentive to keep this rank as low as possible. It turns out that, in all types of root systems for which we perform the blueprint computation, there is no need to go higher than rank 3. For example, if we were to perform the blueprint computation for A_4, we would obtain only the associativity of the coordinatising ring but no further identities. For this reason, we will always restrict the blueprint computations to the rank-3 case.

We emphasise that this restriction of the blueprint computation does not lead to a restriction of the generality of our coordinatising results. To see why, observe that every root system of type X_n for $X \in \{ A, B, C, BC \}$ and $n \geq 3$ contains a subsystem of type X_3. Thus every X_n-graded group G contains an X_3-graded subgroup G'. The blueprint computations in rank 3 show that the commutation maps in G' satisfy a certain set of identities. Using the ideas in Note 4.1.13, we can describe all commutation maps in G using only the commutation maps in G'. Thus the blueprint computation yields information about the computation maps in all of G, not only in G'.

Chapter 7

Root Gradings of Type B

In this chapter, we investigate B_n-graded groups for $n \geq 2$, though we will restrict ourselves to the case $n \geq 3$ for the main results. Except for the case of RGD-systems, we are not aware of any prior literature on this topic. The standard reference for RGD-systems of type B_2 is Chapter 23 in [TW02]. Many of the arguments in [TW02] no longer work in the setting of root graded groups. Instead, we have to deploy several "rank-3 arguments".

Unlike the simply-laced root systems, B_n (and every other root systems that we study from this point on) has multiple orbits of roots, so we will for the first time see root graded groups with non-isomorphic pairs of root groups. Since every long root in B_n is contained in an A_2-subsystem (Lemma 7.2.12), we should expect the long root groups to be coordinatised by some ring k. In the end, we will show that this ring must be commutative associative and that the short root groups are coordinatised by a quadratic module M over k.

More precisely, we prove the following two statements. Firstly, there exist an abelian group $(k, +)$ which parametrises the long root groups and another abelian group $(M, +)$ which parametrises the short root groups. Secondly, the commutation maps will be described by a map $k \times k \to k$ which turns k into a commutative associative ring, a map $k \times M \to M$ which turns M into a left k-module, a k-quadratic form $q\colon M \to k$ and a k-bilinear form $f\colon M \times M \to k$ which is the linearisation of q. The commutator formula has the exact same form as the Chevalley commutator formula, except that expressions of the form v^2 and $2vw$ for $v, w \in M$ (which do not make sense for module elements) are replaced by $q(v)$ and $f(v, w)$, respectively.

This chapter is organised in the way described in section 4.7. We begin with a brief introduction to quadratic modules over commutative associative rings. In section 7.2, we study some purely combinatorial properties of the root system B_n and compute its Cartan integers. In section 7.3, we show how a so-called elementary orthogonal group can be constructed from any quadratic module, which provides a complete solution of the existence problem. The rank-2 computations are split into two parts: We begin with the general case of (non-crystallographic) B_2-gradings in section 7.4 and continue with crystallographic B_2-gradings in section 7.5. Results from both sections will be used in chapter 9 as well.

From the second half of the chapter on, we consider B_n-gradings for $n \geq 3$. In section 7.6, we prove that Weyl elements in these groups satisfy the square formula. In sections 7.7 and 7.8, we define the notions of standard partial twisting systems and standard signs for B_n-graded groups, respectively. In section 7.9, we apply the results from the previous sections and the parametrisation theorem to construct parametrising groups $(\Bbbk, +)$ and $(M, +)$. The commutation maps, their rank-2 identities and the blueprint rewriting rules are defined, derived and computed in section 7.10. Finally, we perform the blueprint computations for B_3 in section 7.11, and we state our final result in Theorem 7.11.21.

7.1 Quadratic Maps and Modules

7.1.1 Notation for this section. We denote by $\Bbbk$ an arbitrary commutative associative ring. Unless otherwise specified, all modules are understood to be left modules over $\Bbbk$ (in the standard sense of Definition 5.1.22).

In this section, we introduce the basic terminology of quadratic maps, forms and modules. These objects are often only considered over base fields, but the basic notions (which are all that we need) translate to the case of commutative associative base rings without a change. With this caveat, all the material in this section is standard and can be found, for example, in [EKM08, Section 7]. A reference which treats quadratic modules over arbitrary commutative associative rings is [Knu91], though it only considers quadratic modules which are finitely generated projective.

Some of the results in this section will be generalised in section 8.1 to the more general weakly quadratic maps.

We begin with the notion of quadratic maps between arbitrary $\Bbbk$-modules.

7.1.2 Definition (Polarisation, [EKM08, 7.1]). Let $(M, +), (N, +)$ be two abelian groups and let $f\colon M \to N$ be any map (not necessarily a homomorphism). The *polarisation of f* is the map

$$M \times M \to N, (v, w) \mapsto f(v + w) - f(v) - f(w).$$

More generally, let n be a positive integer, let $(M_1, +), \dots, (M_n, +), (N, +)$ be abelian groups and let $f\colon M_1 \times \dots \times M_n \to N$ be any map. Then for all $i \in [1, n]$, the *polarisation of f at position i* is the map

$$M_1 \times \dots \times M_{i-1} \times M_i \times M_i \times M_{i+1} \times \dots \times M_n \to N,$$
$$(v_1, \dots, v_{i-1}, v_i, w_i, v_{i+1}, \dots, v_n) \mapsto \tilde{f}(v_i + w_i) - \tilde{f}(v_i) - \tilde{f}(w_i)$$

where $\tilde{f}(u_i) := f(v_1, \dots, v_{i-1}, u_i, v_{i+1}, \dots, v_n)$.

7.1.3 Definition (Quadratic map, [EKM08, 7.1], [Knu91, (5.3.5)]). Let M, N be $\Bbbk$-modules. A map $q\colon M \to N$ is called *$\Bbbk$-quadratic* (or simply *quadratic*) if it satisfies the following two conditions:

(i) $q(\lambda v) = \lambda^2 q(v)$ for all $\lambda \in \Bbbk, v \in M$.

(ii) The polarisation $f\colon M \times M \to N, (v, w) \mapsto q(v + w) - q(v) - q(w)$ of f is $\Bbbk$-bilinear. This map is also called the *linearisation of q*.

In practice, we will only be concerned with the case that $N = \Bbbk$, which we study from Definition 7.1.8 on.

7.1.4 Note (Polarisations and linearisations). The terms "polarisation" and "linearisation" are often used interchangeably in the literature. In this thesis, we will use "polarisation" as a general descriptor for the maps defined in Definition 7.1.2 while only the polarisations of quadratic maps will be called "linearisations". In other words, a polarisation will only be called a linearisation if it is actually linear (or rather, bilinear).

7.1.5 Lemma. *Let M, N be $\Bbbk$-modules, let $q\colon M \to N$ be a quadratic map and let f denote its linearisation. Then the following statements hold:*

(a) *$q(v+w) = q(v) + q(w) + f(v,w)$ for all $v, w \in M$.*

(b) *For all $n \in \mathbb{N}_+$ and all $v_1, \ldots, v_n \in M$, we have*

$$q\left(\sum_{i=1}^{n} v_i\right) = \sum_{i=1}^{n} q(v_i) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} f(v_i, v_j).$$

(c) *f is symmetric, that is, $f(v,w) = f(w,v)$ for all $v, w \in M$.*

(d) *$q(0_M) = 0_N$.*

(e) *$f(v,v) = 2_\Bbbk q(v)$ for all $v \in M$.*

Proof. The first assertion is simply a reformulation of the definition of f and the second assertion follows from the first one by induction. The third assertion is clear. The fourth assertion follows from the equation

$$q(0_M) = q(0_M + 0_M) = q(0_M) + q(0_M) + f(0_M, 0_M) = 2_\Bbbk q(0_M).$$

For the last assertion, let $v \in M$. On the one hand,

$$q(2_\Bbbk v) = 2_\Bbbk^2 q(v) = 4_\Bbbk q(v).$$

On the other hand,

$$q(2_\Bbbk v) = q(v+v) = q(v) + q(v) + f(v,v) = 2_\Bbbk q(v) + f(v,v).$$

Thus $4_\Bbbk q(v) = 2_\Bbbk q(v) + f(v,v)$, or in other words, $2_\Bbbk q(v) = f(v,v)$. □

The following Lemma 7.1.6 says that for commutative associative rings in which 2 is invertible, a quadratic map from M to N is essentially the same thing as a symmetric bilinear map from $M \times M$ to N. Thus the study of quadratic maps reduces to the study of symmetric bilinear maps in this case. In the general situation, however, we will see in Example 7.1.10 that a quadratic map "contains more information" than its linearisation. For this reason, quadratic maps are better suited to the study of general commutative associative rings than symmetric bilinear forms.

7.1.6 Lemma. *Assume that $2_\Bbbk$ is invertible and let $f\colon M \times M \to N$ be a symmetric $\Bbbk$-bilinear map between $\Bbbk$-modules M, N. Then*

$$q\colon M \to N, v \mapsto \frac{f(v,v)}{2}$$

is a $\Bbbk$-quadratic map with linearisation f, and it is the unique map with these properties. Thus we have a bijective correspondence between symmetric $\Bbbk$-bilinear maps $M \times M \to N$ and $\Bbbk$-quadratic maps $M \to N$ if $2_\Bbbk$ is invertible.

Proof. It is easy to check that $v \mapsto \frac{f(v,v)}{2}$ defines a quadratic map, and it follows from Lemma 7.1.5 (e) that a quadratic map is uniquely determined by its linearisation. □

7.1.7 Lemma. *Let M, N be $\Bbbk$-modules. Then the set of quadratic maps from M to N is closed under addition and $\Bbbk$-scalar multiplication.*

Proof. This is clear from the definition of quadratic maps. □

From now on, we specialise to the case that the codomain of the quadratic map in question is $\Bbbk$. This is the only case of relevance for our purposes.

7.1.8 Definition (Quadratic form). Let M be a k-module. A *quadratic form on M* is a k-quadratic map $q\colon M \to k$ where k is regarded as a k-module in the natural way. It is called *anisotropic* if $q(v) = 0$ implies $v = 0$ for all $v \in M$.

7.1.9 Definition (Quadratic module). A *quadratic module (over k)* is a pair (M, q) consisting of a (left) k-module M and a quadratic form q on M. It is called *anisotropic* if q is anisotropic.

7.1.10 Example. The map $q\colon k \to k, r \mapsto r^2$ is a quadratic form with linearisation $f\colon k \times k \to k, (r,s) \mapsto 2rs$. In fact, the group that we will construct in section 7.3 for an arbitrary quadratic module (M, q) will be a Chevalley group if we take $M = k$ and $q\colon r \mapsto r^2$.

More generally, the map

$$q\colon k^n \to k, v \mapsto \sum_{i=1}^{n} v_i^2$$

is a quadratic form with linearisation

$$f\colon k^n \times k^n \to k, (v,w) \mapsto 2\sum_{i=1}^{n} v_i w_i.$$

Note that $f = 0$ if $2_k = 0_k$. In particular, if $2_k = 0_k$, then q and the zero map are two quadratic forms which are distinct (unless k is the zero ring) but have the same linearisation. This shows that in general, a quadratic form "contains more information" than its linearisation.

7.1.11 Lemma (Direct sum of quadratic modules). *Let (M, q), (M', q') be two quadratic modules and let f, f' denote the linearisations of q and q', respectively. Then the direct sum $M \oplus M'$ together with the map*

$$q \oplus q'\colon M \oplus M' \to k, (v, v') \mapsto q(v) + q'(v')$$

is a quadratic module, called the direct sum of (M, q), (M', q'). *The linearisation of $q \oplus q'$ is*

$$f \oplus f'\colon (M \oplus M') \times (M \oplus M') \to k, \big((u,u'),(v,v')\big) \mapsto f(u,v) + f(u',v').$$

Proof. This follows from a simple computation. □

7.1.12 Remark (Inner direct sum). Let (M, q) be a quadratic module with subspaces A, B and with linearisation f. We say that M is the *(inner) direct sum of A and B*, and we write $M = A \oplus B$, if M is the (inner) direct sum of A and B as k-modules and $f(a,b) = 0$ for all $a \in A$, $b \in B$. If this is the case, then M is isomorphic to the outer direct sum of A and B in the sense of Lemma 7.1.11. A subspace C of M is called a *direct summand of M* if there exists a subspace D such that $M = C \oplus D$ (as quadratic modules).

7.1.13 Example (Hyperbolic space, [EKM08, p. 40], [Knu91, (5.6.2)]). Let $n \in \mathbb{N}_{\geq 1}$ and let $V := k^n$. Denote by V^* the dual space of V. The *hyperbolic space of dimension $2n$ over k* is the k-module $M := V \oplus V^*$ together with the quadratic form

$$q\colon M \to k, (v, f) \mapsto f(v).$$

If we denote the canonical basis of V by $(e_1, \ldots, e_n)$ and the corresponding dual basis by $(e_{-1}, \ldots, e_{-n})$, then q is given on coordinates by

$$q\colon k^n \oplus k^n \to k, \big((\lambda_1, \ldots, \lambda_n), (\lambda_{-1}, \ldots, \lambda_{-n})\big) \mapsto \sum_{i=1}^{n} \lambda_i \lambda_{-i}.$$

The pairs (e_i, e_{-i}) are called *hyperbolic pairs*. More abstractly, a pair (v, w) of elements in a quadratic module (M', q') with linearisation f' is called a hyperbolic pair if $q'(v) = 0 = q'(w)$, $f'(v, w) = 1$ and $\langle v, w\rangle_{\Bbbk}$ is a direct summand of M' (in the sense of Remark 7.1.12). Such a pair is automatically $\Bbbk$-linearly independent: If $\lambda, \mu \in \Bbbk$ satisfy $\lambda v + \mu w = 0$, then

$$0_{\Bbbk} = f'(v, 0) = f'(v, \lambda v + \mu w) = \lambda f'(v, v) + \mu f'(v, w) = 2\lambda q'(v) + \mu = \mu,$$

and similarly $\lambda = 0_{\Bbbk}$. Thus v, w span a direct summand of M which is isomorphic to the hyperbolic plane $\Bbbk^2$.

7.1.14 Note. The condition in Example 7.1.13 that the module spanned by a hyperbolic pair is a direct summand will not be relevant in our context. It is absent from the definition of hyperbolic pairs in [GPR, 11.17].

7.1.15 Lemma. *Let (M, q) be a quadratic $\Bbbk$-module which contains an element $v_0 \in M$ such that $q(v_0)$ is invertible. Then for all $a \in \Bbbk$ with $av_0 = 0_M$, we have $2a = 0$. In particular, M is a faithful $\Bbbk$-module if $2_{\Bbbk}$ is not a zero divisor.*

Proof. Let $a \in \Bbbk$ such that $av_0 = 0$ and denote the linearisation of q by f. Then $af(v_0, v_0) = f(av_0, v_0) = 0$. At the same time, $af(v_0, v_0) = 2aq(v_0)$ by Lemma 7.1.5 (e). Since $q(v_0)$ is invertible, it follows that $2a = 0$, as desired. In particular, any $a \in \Bbbk$ with $aM = \{0\}$ satisfies $av_0 = 0$ and thus $2a = 0$. Hence M is faithful if $2_{\Bbbk}$ is not a zero divisor. ☐

The notions of orthogonality and orthogonal groups which are known for inner products (see Definitions 1.2.2 and 1.2.4) transfer to the setting of quadratic forms without surprises.

7.1.16 Definition (Orthogonal group). For any quadratic module (M, q), we call

$$\mathrm{O}(q) := \{\, \varphi \in \mathrm{Aut}_{\Bbbk}(M) \mid q(\varphi(v)) = q(v) \text{ for all } v \in M \,\}$$

the *orthogonal group of* q. We use the convention that it acts on M from the left side, so that the composition $\varphi \circ \psi$ of $\varphi, \psi \in \mathrm{O}(q)$ is the map $x \mapsto \varphi(\psi(x))$.

7.1.17 Remark. Let (M, q) be a quadratic module and let $\varphi \in \mathrm{O}(q)$. Then we also have $f(\varphi(u), \varphi(v)) = f(u, v)$ for all $u, v \in M$ where f is the linearisation of q.

7.1.18 Note. Recall from Convention 1.2.3 and Note 1.2.17 that in the context of root systems in a Euclidean space $(V, \cdot)$, we always let automorphisms of V act from the right side. The reason for this is that we often need reflections in the context of Weyl elements, whose conjugation action is also written on the right side. Since this specific reasoning does not apply to quadratic forms, we have the freedom to let $\mathrm{O}(q)$ act on M from the left side, “as usual”.

7.1.19 Definition (Orthogonality). For any quadratic module (M, q), we say that $v, u \in M$ are *orthogonal* if $f(u, v) = 0$ where f is the linearisation of q. For any $\Bbbk$-subspace U of M, we define $U^{\perp} := \{\, v \in M \mid f(u, v) = 0 \text{ for all } u \in U \,\}$ and we put $v^{\perp} := \langle v\rangle_{\Bbbk}^{\perp}$ for all $v \in M$.

Even the notion of reflections, which we defined in Definition 1.2.5 for inner products, can be generalised to the setting of quadratic forms. However, we have to be a bit more careful at this point. Recall from Definition 1.2.5 that the reflection corresponding to a non-zero vector v in a Euclidean space $(V, \cdot)$ is the map

$$\sigma_v\colon V \to V, x \mapsto x^{\sigma_v} := x - 2\frac{x \cdot v}{v \cdot v} v.$$

We can write down the same definition for arbitrary symmetric bilinear forms f as long as we assume that $f(v,v)$ is invertible. However, since $f(v,v) = 2q(v)$ by Lemma 7.1.5 (e), this assumption is never satisfied if $2_\Bbbk$ is not invertible. The solution to this problems consists of removing the coefficient 2 from the definition of σ_v. If $2_\Bbbk$ is invertible, then the resulting definition is equivalent to the naive definition above.

7.1.20 Definition (Reflection, [EKM08, 7.2], [Knu91, (6.1.2)]). Let (M,q) be a quadratic module over $\Bbbk$, denote by f the linearisation of q and let $v \in M$ such that $q(v)$ is invertible in $\Bbbk$. The *reflection corresponding to v* is the map

$$\sigma_v\colon M \to M, u \mapsto u - q(v)^{-1} f(u,v)v.$$

Reflection in quadratic modules satisfy some of the main properties that we are used to from reflections in Euclidean spaces. The most important caveat is that in general, we cannot expect the module M to decompose as $\langle v \rangle_\Bbbk \oplus v^\perp$.

7.1.21 Lemma. *Let (M,q) be a quadratic module over $\Bbbk$, denote by f the linearisation of q and let $v \in M$ such that $q(v)$ is invertible in $\Bbbk$. Then σ_v lies in $\mathrm{O}(q)$ and satisfies $\sigma_v(u) = u$ for all $u \in v^\perp$, $\sigma_v(v) = -v$ and $\sigma_v^2 = \mathrm{id}_M$.*

Proof. It is clear that σ_v is $\Bbbk$-linear and that it is the identity on $v^\perp$. Further, since $f(v,v) = 2q(v)$ by Lemma 7.1.5 (e), we have

$$\sigma_v(v) = v - q(v)^{-1} f(v,v)v = v - 2v = -v.$$

This implies that for all $u \in M$,

$$\begin{aligned}\sigma_v^2(u) &= \sigma_v\big(u - q(v)^{-1} f(u,v)v\big) = \sigma_v(u) - q(v)^{-1} f(u,v)\sigma_v(v) \\ &= u - q(v)^{-1} f(u,v)v + q(v)^{-1} f(u,v)v = u.\end{aligned}$$

Finally, by Lemma 7.1.5 (a),

$$\begin{aligned}q\big(\sigma_v(u)\big) &= q\big(u - q(v)^{-1} f(u,v)v\big) \\ &= q(u) + q\big({-q(v)^{-1}} f(u,v)v\big) + f\big(u, -q(v)^{-1} f(u,v)v\big) \\ &= q(u) + q(v)^{-2} f(u,v)^2 q(v) - q(v)^{-1} f(u,v) f(u,v) = q(u).\end{aligned}$$

This finishes the proof. □

We now turn to pointed quadratic modules.

7.1.22 Definition (Pointed quadratic module). A *pointed quadratic module over $\Bbbk$* is a triple (M,q,e) where (M,q) is a quadratic module over $\Bbbk$ and e is an element of M with $q(e) = 1_\Bbbk$. The element e is called the *base point*. A quadratic module (M,q) is called *pointable* if there exists $e \in M$ such that (M,q,e) is a pointed quadratic module.

7.1.23 Note. Recall from Lemma 5.7.12 that in simply-laced root graded groups, the ring element $1_\Bbbk$ which appears in the decomposition of the fixed Weyl elements turns out to be the unit element in the coordinatising ring. We will observe a similar phenomenon in B_n-graded groups: The module element e which appears in the decomposition of the fixed short Weyl element will ultimately satisfy $q(e) = 1_\Bbbk$. Thus the quadratic module that we construct from any B_n-graded groupg is pointable, and the choice of a Δ-system of Weyl elements determines a "canonical" choice of a base point. However, note that the unit element of a ring is uniquely determined by the ring structure whereas a pointable quadratic module may have many possible base points.

7.1.24 Note (see [GPR, 11.14]). Let (M, q, e) be a pointed quadratic module and denote by f the linearisation of q. One can define a *trace map*

$$\mathrm{tr}\colon M \to \Bbbk, u \mapsto f(e, u) = f(u, e)$$

and a *conjugation map*

$$\overline{\cdot}\colon M \to M, u \mapsto \overline{u} := \mathrm{tr}(u)e - u.$$

Observe that, since $q(e) = 1_\Bbbk$, we have $\overline{\cdot} = -\sigma_e$ where σ_e denotes the reflection corresponding to e.

We will not need the trace and conjugation maps in this generality. However, we will encounter them in Definition 10.1.12 in the special case that (M, q) is a conic algebra. Further, we will see similar concepts in section 8.3.

Finally, we can define the standard parameter system for pointed quadratic modules.

7.1.25 Definition (Standard parameter system). Let (M, q, e) be a pointed quadratic module. Put $A := B := \{\pm 1\}$. Declare that A acts on $\Bbbk$ and M by inversion and that B acts trivially on $\Bbbk$ and by σ_e on M. More precisely, this means that

$$-1_A.a = -a, \quad -1_A.m = -m, \quad -1_B.a = a, \quad -1_B.m = \sigma_e.m$$

for all $a \in \Bbbk$ and all $m \in M$. Then the triple $(A \times B, M, \Bbbk)$ is called the *standard parameter system for* (M, q, e).

7.1.26 Remark. To obtain an induced action of $A \times B$ on M and $\Bbbk$, we have to observe that the actions of A and B commute. This holds because the reflection map σ_{v_0} is linear.

7.2 Root Systems of Type B

7.2.1 Notation for this section. We denote by n an integer at least 2.

In this section, we collect some basic facts about the root system B_n which will be needed later on. Note that we exclude the case $n = 1$ because the resulting root system B_1 would be isomorphic to A_1.

7.2.2 Remark (Standard representation of B_n). Let V be a Euclidean space of dimension n with orthonormal basis $(e_1, \ldots, e_n)$. The *standard representation of* B_n is

$$B_n := \{\, \varepsilon_1 e_i + \varepsilon_2 e_j \mid i \neq j \in [1, n], \varepsilon_1, \varepsilon_2 \in \{\pm 1\} \,\} \cup \{\, \varepsilon e_i \mid i \in [1, n], \varepsilon \in \{\pm 1\} \,\}.$$

The long roots are exactly those which lie in the first set and the short roots are exactly those in the second one. The *standard root base* is

$$\Delta := \{\, e_i - e_{i+1} \mid i \in [1, n-1] \,\} \cup \{e_n\}$$

and the corresponding positive system is

$$\Pi := \{\, e_i - e_j \mid i < j \in [1, n] \,\} \cup \{\, e_i + e_j \mid i \neq j \in [1, n] \,\} \cup \{\, e_i \mid i \in [1, n] \,\}.$$

7.2.3 Remark. It is clear to see that the standard root base of A_{n-1} (from Remark 5.2.3) can be regarded as a subset of the standard root base of B_n. Thus B_n contains A_{n-1} as a parabolic root subsystem. We also see that all roots in the standard representation of D_n (from Remark 5.2.4) are also contained in B_n and that they even form a crystallographically closed subset of B_n. However, this subset is not closed (in the non-crystallographic sense) and, consequently, not parabolic.

7.2.4 Definition (B_2-pairs and B_2-quadruples). Let Φ be any root system. A *B_2-pair (in Φ)* is a pair (α, δ) of roots such that $\langle \alpha, \delta \rangle_{\mathbb{R}} \cap \Phi$ is a root subsystem of Φ of type B_2 with root base (α, δ) with α being the long simple root in this subsystem and δ being the short simple root. A *B_2-quadruple (in Φ)* is a quadruple $(\alpha, \beta, \gamma, \delta)$ of roots in Φ such that (α, δ) is a B_2-pair, $\beta = \alpha + \delta$ and $\gamma = \alpha + 2\delta$.

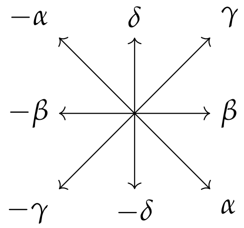


Figure 7.1: A B_2-quadruple $(\alpha, \beta, \gamma, \delta)$.

7.2.5 Remark. In a B_2-quadruple $(\alpha, \beta, \gamma, \delta)$, the roots $\alpha, \beta, \gamma, \delta$ are exactly the positive roots of the corresponding B_2-subsystem (with respect to the root base (α, δ)). All reflection maps on this subsystem can be read off from the diagram in Figure 7.1.

7.2.6 Remark (compare Remark 9.1.11). Let G be a group with a B_2-grading $(U_\alpha)_{\alpha \in B_2}$ and let $(\alpha, \beta, \gamma, \delta)$ be a B_2-quadruple. Then this grading is crystallographic if and only if the commutators $[U_{\varepsilon\alpha}, U_{\sigma\gamma}]$ are trivial for all $\varepsilon, \sigma \in \{\pm 1\}$.

We now state some simple results, mostly on the possible ways in which to roots can "relate" to each other. Since every pair of roots lies in a parabolic rank-2 subsystem, there is only a handful of cases which can occur.

7.2.7 Lemma. *Let α, β be roots. Then the subsystem which is spanned by α, β is of type A_1, A_2, $A_1 \times A_1$ or B_2.*

Proof. This follows from Remark 1.2.56. □

We can imitate the proof of Lemma 5.2.12 to show that B_n contains no non-parabolic subsystems of types A_2 and B_2. Thus we do not have to distinguish between A_2-subsystems, closed A_2-subsystems and parabolic A_2-subsystems in the following, and similarly for B_2-subsystems. However, the same is not true for subsystems of type $A_1 \times A_1$: The short roots in B_2 form a subsystem which is neither parabolic nor crystallographically closed. Similarly, the long roots in B_2 form a subsystem which is crystallographically closed but not parabolic.

7.2.8 Lemma. *Every A_2-subsystem and every B_2-subsystem of B_n is parabolic.*

Proof. Let Φ' be an A_2-subsystem of B_n. Then Φ' is contained in the parabolic root subsystem $\Phi'' := \langle \Phi' \rangle_{\mathbb{R}} \cap \Phi$. Since $A_1 \times A_1$ and B_2 do not contain a subsystem of type A_2, it follows from Lemma 7.2.7 that Φ'' is of type A_2, so $\Phi' = \Phi''$. The proof for B_2-subsystems is similar. □

7.2.9 Remark. The assertion of Lemma 7.2.8 remains true for A_2 and B_2 replaced by A_m and B_m for $m \in [2, n]$.

7.2.10 Lemma. *Let β, δ be two short roots such that $\beta \notin \{\pm\delta\}$. Then there exist long roots α, γ such that $(\alpha, \beta, \gamma, \delta)$ is a B_2-quadruple. In fact, we must have $\alpha = \beta - \delta$ and $\gamma = \beta + \delta$.*

Proof. In the standard representation of B_n, there exist distinct $i, j \in [1, n]$ such that $\beta \in \{\pm e_i\}$ and $\delta \in \{\pm e_j\}$. Now it can be easily seen that the long roots $\alpha := \beta - \delta$ and $\gamma := \beta + \delta$ have the desired property, and that they are the only roots with this property. □

7.2.11 Lemma. *Let ρ, ζ be two orthogonal roots in B_n. Then one of the following two conditions is satisfied:*

(i) ρ is crystallographically adjacent to ζ and $-\zeta$.

(ii) ρ and ζ are both short and there exist long roots α, γ such that $(\alpha, \rho, \gamma, \zeta)$ is a B_2-quadruple.

Proof. Denote by Φ' the root subsystem which is spanned by ρ, ζ. Then Φ' must be of type $A_1 \times A_1$, A_2 or B_2 by Lemma 7.2.7. In the first case, (i) holds. The second case is not possible because there are no orthogonal roots in A_2. Now assume that we are in the third case. Then ρ and ζ must have the same length. If they are both short, then (ii) holds by Lemma 7.2.10. Otherwise (i) holds. This finishes the proof. □

7.2.12 Lemma. *If $n \geq 3$, then every long root in B_n lies in an A_2-subsystem.*

Proof. Roots of the form $e_i - e_j$ span an A_2-subsystem together with $e_j - e_k$ while roots of the form $\pm(e_i + e_j)$ span an A_2-subsystem together with $\pm(e_k - e_j)$, were k is any index from $[1, n] \setminus \{ i, j \}$. □

7.2.13 Lemma. *If $n \geq 3$, then two long roots α, γ in B_n lie in a common A_2-subsystem if and only if they are not orthogonal.*

Proof. If α and γ are orthogonal, then they cannot lie in common A_2-subsystem because A_2 contains no pair of orthogonal roots. Now assume that α and γ are not orthogonal. Note that for any element w of the Weyl group, α and γ lie in a common A_2-subsystem if and only if α^w and β^w lie in a common A_2-subsystem. Since the Weyl group acts transitively on the set of long roots, we can thus assume that $\alpha = e_1 + e_2$.

Since β is long, there exist $i \neq j \in [1, n]$ and $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ such that $\beta = \varepsilon_1 e_i + \varepsilon_2 e_j$. At first, assume that $\{ i, j \} = \{ 1, 2 \}$. Then we must have $\beta \in \{ \pm\alpha \}$ because α and β are not orthogonal. In this case, any A_2-subsystem which contains α must also contain β. Since there exists at least one A_2-subsystem which contains α by Lemma 7.2.12, the assertion follows. Now assume that $\{ i, j \} \neq \{ 1, 2 \}$. Then $\{ 1, 2 \} \cap \{ i, j \}$ contains exactly one element, because otherwise α and γ would be orthogonal. Without loss of generality, assume that $i = 1$. Replacing β by $-\beta$ if necessary, we can further assume that $\varepsilon_1 = 1$, so that $\beta = e_1 + \varepsilon_2 e_j$. Then α and β lie in the common A_2-subsystem $\{ \pm(e_1 + e_2), \pm(e_j - e_2), \pm(e_1 + e_j) \}$. □

7.2.14 Note. If $n = 2$, then long roots α, γ with $\gamma \in \{ \pm\alpha \}$ do not lie in a common A_2-subsystem (because B_2 has no A_2-subsystem) but they are not orthogonal. That is why we have to assume $n \geq 3$ in Lemma 7.2.13.

7.2.15 Lemma. *Let α be a long root and let β be a short root. Then α, β lie in a common B_2-subsystem if and only if they are not orthogonal.*

Proof. Since the statement is invariant under the action of the Weyl group, we can assume without loss of generality that $\beta = e_1$. Let $i \neq j \in [1, n]$ be the unique indices such that $\alpha \in \{ \pm e_i \pm e_j \}$. Then α, β lie in a common B_2-subsystem if and only if $1 \in \{ i, j \}$, which is the case if and only if they are not orthogonal. This finishes the proof. □

In the computation of the Cartan integers, it is practical to consider each orbit of roots separately. By Proposition 1.3.8, the orbits in B_n are precisely the set of long roots and the set of short roots. Thus we end up with four cases to consider. We do not give proofs, but all statements follow from a simple investigation of the standard representation of B_n.

7.2.16 Lemma. *Let α, γ be two long roots. Then the Cartan integer $\langle \alpha | \gamma \rangle = 2\frac{\alpha \cdot \gamma}{\gamma \cdot \gamma}$ is determined as follows:*

(a) $\langle \alpha | \gamma \rangle = 0$ *if and only if α and γ are orthogonal. (If $n \geq 3$, this is equivalent by Lemma 7.2.13 to α and γ lying in no common A_2-subsystem).*

(b) $\langle \alpha | \gamma \rangle = -1$ *if and only if (α, γ) is an A_2-pair.*

(c) $\langle \alpha | \gamma \rangle = 1$ *if and only if $(\alpha, -\gamma)$ is an A_2-pair.*

(d) $\langle \alpha | \gamma \rangle = 2$ *if and only if $\alpha = \gamma$.*

(e) $\langle \alpha | \gamma \rangle = -2$ *if and only if $\alpha = -\gamma$.*

7.2.17 Lemma. *Let β, δ be two short roots. Then the Cartan integer $\langle \beta | \delta \rangle$ is determined as follows:*

(a) $\langle \beta | \delta \rangle = 0$ *if and only if β and δ are orthogonal (or equivalently, if and only if $\beta \notin \{ \pm\delta \}$).*

(b) $\langle \beta | \delta \rangle = 2$ *if and only if $\beta = \delta$.*

(c) $\langle \beta | \delta \rangle = -2$ *if and only if $\beta = -\delta$.*

7.2.18 Lemma. *Let α be a long root and let β be a short root. Then the Cartan integer $\langle \alpha | \beta \rangle$ is determined as follows:*

(a) $\langle \alpha | \beta \rangle = 0$ *if and only if α and β are orthogonal (or, equivalently by Lemma 7.2.15, if and only if they do not lie in a common B_2-subsystem).*

(b) $\langle \alpha | \beta \rangle = 2$ *if and only if (α, β) is a B_2-pair.*

(c) $\langle \alpha | \beta \rangle = -2$ *if and only if $(\alpha, -\beta)$ is a B_2-pair.*

7.2.19 Lemma. *Let α be a long root and let β be a short root. Then the Cartan integer $\langle \beta | \alpha \rangle$ is determined as follows:*

(a) $\langle \beta | \alpha \rangle = 0$ *if and only if α and β are orthogonal (or equivalently, if and only if they do not lie in a common B_2-subsystem).*

(b) $\langle \beta | \alpha \rangle = 1$ *if and only if (α, β) is a B_2-pair.*

(c) $\langle \beta | \alpha \rangle = -1$ *if and only if $(\alpha, -\beta)$ is a B_2-pair.*

Since we will usually only be interested in the number $(-1)^{\langle \alpha | \beta \rangle}$ (or, in other words, the parity of the Cartan integer), the following summary will be very useful.

7.2.20 Proposition. *Let ρ, ζ be two roots in B_n. Then $\langle \rho | \zeta \rangle$ is an even number if and only if one of the following conditions is satisfied:*

(i) *ζ is short.*

(ii) $\rho \in \{ \pm\zeta \}$.

(iii) *ρ and ζ are orthogonal.*

Equivalently, we can replace the last statement by the following one:

(iii′) *ρ is long and and ρ, ζ do not lie in a common subsystem of type A_2 or B_2.*

We end this section with the definition of a certain subset $\hat{B}_n$ of B_n which will prove useful later on.

7.2.21 Remark. As we have discussed in Note 4.1.18, we will not work with a "full" family $(w_\alpha)_{\alpha\in\Phi}$ of Weyl elements during the coordinatisation of B_n-graded groups, but only with a Δ-system $(w_\delta)_{\delta\in\Delta}$ (which is chosen arbitrarily). Since the simple reflections generate the Weyl group by Proposition 1.3.12, this poses no problem on the theoretical side. On the practical side, however, the number of generators which are needed to express an element of the Weyl group with respect to Δ may be rather large, which makes computations more cumbersome. As a compromise, it will in some situations be useful to consider a $\hat{B}_n$-extension of $(w_\delta)_{\delta\in\Delta}$ (in the sense of the following Definition 7.2.23) where $\hat{B}_n$ is the subset

$$\hat{B}_n := \{\, e_i - e_j \mid i \neq j \in [1,n] \,\} \cup \{\, e_i \mid i \in [1,n] \,\}$$

of B_n. In other words, $\hat{B}_n$ is the union of the canonical subsystem of type A_{n-1} and the set of positive short roots. Observe that this set is neither closed nor a root system.

The set $\bar{S} := \{\, \sigma_\alpha \mid \alpha \in \hat{B}_n \,\}$ is a practical set of generators for the Weyl group for the following reason: For all roots $\alpha, \beta \in B_n$ which lie in the same orbit, we need at most four reflections from $\bar{S}$ to go from α to β. In other words, there exist $m \in [0,4]$ and a word $\bar{\delta}$ over $\hat{B}_n$ of length m such that $\alpha^{\sigma(\bar{\delta})} = \beta$. The idea here is to use the subgroup $\mathrm{Weyl}(A_{n-1})$ of $\mathrm{Weyl}(B_n)$ to permute the basis vectors $(e_i)_{i\in[1,n]}$ (which requires at most two reflections) and the reflections $(\sigma_{e_i})_{i\in[1,n]}$ to change the signs of the basis vectors (which also requires at most two reflections).

7.2.22 Definition (Standard Δ-expression). Let Δ denote the standard root base of B_n. For any root $\alpha \in \hat{B}_n$, we define a Δ-expression $\bar{\rho}^\alpha$ of α (in the sense of Definition 4.2.2) as follows:

(a) If $\alpha \in \Delta$, we put $\bar{\rho}^\alpha := (\alpha)$.

(b) If $\alpha = e_i$ for some $i \in [1, n-1]$, we define $\bar{\rho}^{e_i} := (e_i - e_{i+1}, \bar{\rho}^{e_{i+1}}, e_{i+1} - e_i)$.

(c) If $\alpha = e_i - e_j$ for some $i < j \in [1, n-1]$ with $i+1 < j$, we define $\bar{\rho}^{e_i - e_j} := (e_j - e_{j-1}, \bar{\rho}^{e_i - e_{j-1}}, e_{j-1} - e_j)$.

(d) If $\alpha = e_j - e_i$ for some $i < j \in [1, n-1]$ with $i+1 < j$, we define $\bar{\rho}^{e_j - e_i} := (\bar{\rho}^{e_i - e_j})^{-1}$.

The word $\bar{\rho}^\alpha$ will also be called the *standard Δ-expression of α*. We also put $\bar{\rho}^i := \bar{\rho}^{e_i}$ for all $i \in [1,n]$ and $\bar{\rho}^{ij} := \bar{\rho}^{e_i - e_j}$ for all distinct $i, j \in [1,n]$.

7.2.23 Definition ($\hat{B}_n$-extensions). Let Δ denote the standard base of B_n, let G be any group with a B_n-pregrading $(\hat{U}_\alpha)_{\alpha\in B_n}$ and let $(w_\delta)_{\delta\in\Delta}$ be a Δ-system of Weyl elements in G. Then we define a family $(w_\alpha)_{\alpha\in\hat{B}_n}$, called the *standard $\hat{B}_n$-extension of* $(w_\delta)_{\delta\in\Delta}$, by $w_\alpha := w_{\bar{\rho}^\alpha}$ for all $\alpha \in \hat{B}_n$. We will sometimes write w_{ij} for $w_{e_i - e_j}$ and w_i for w_{e_i}.

7.2.24 Note. Since $\sigma_\alpha = \sigma_{-\alpha}$ for all roots α, we could remove all negative roots in A_{n-1} from $\hat{B}_n$ without changing the set $\bar{S}$ of generators in Remark 7.2.21. However, it will be more practical to have both Weyl elements w_{ij} and $w_{ji} = w_{ij}^{-1}$ available. By contrast, we will show in Proposition 7.6.16 that for all $i \in [1,n]$, the short Weyl elements w_i and w_i^{-1} act identically on all root groups. Thus there is no need to include the short negative roots in $\hat{B}_n$.

7.2.25 Note. The main computations during which we use $\hat{B}_n$-extensions are Lemma 7.10.7 and Proposition 7.11.19. The reason why $\hat{B}_n$-extensions are useful in this context are Lemma 7.3.28 and Proposition 7.10.2.

7.3 Construction of B_n-graded Groups

7.3.1 Notation for this section. We denote by $\Bbbk$ a commutative associative ring, by (M, q, v_0) a pointed quadratic module over $\Bbbk$ and by $f\colon M \to \Bbbk$ the linearisation of q. Further, we fix an integer $n \in \mathbb{N}_{\geq 2}$, the root system $\Phi := B_n$ in standard representation (as in Remark 7.2.2) and the standard root base Δ of B_n. We will at some points consider the subset $\hat{B}_n$ of B_n defined in Remark 7.2.21.

The construction of RGD-systems of type B_n from a quadratic module over a field is well-known. For example, the required matrices can be found for $n = 2$ in [VM98, p. 469]. The construction in [VM98] is, in fact, more general than what we need because it starts not from a quadratic module, but from a σ-quadratic module where σ is an anti-involution of the field. Taking σ to be the identity map, we obtain the regular notion of quadratic modules. The construction outlined in this section is essentially an adaption of the one in [VM98] to arbitrary n and with the field replaced by a commutative associative ring. We will change some minor specifics of the construction, like signs and the order of the matrix rows and columns, but in its essence it is the same.

The group $\mathrm{EO}(q)$ that we construct in this section can also be implemented on a computer. All computational results that we state in this section can be verified in this way. See Note 7.3.29 for a few details on the implementation.

7.3.2 Convention. As in Definition 7.1.16, we consider endomorphisms of any module to be acting from the left. Thus the composition $\varphi \circ \psi$ of two such endomorphisms is the map $x \mapsto \varphi(\psi(x))$.

7.3.A Construction

Our example of a B_n-graded group will be a group of orthogonal automorphissm, but not of the quadratic module (M, q). Instead, we have to add n hyperbolic pairs to M (in the sense of Example 7.1.13).

7.3.3 Construction. We put $V_+ := \Bbbk^n$, $V_- := \Bbbk^n$ and $V := M \oplus V_+ \oplus V_-$. We denote by $(b_1, \ldots, b_n)$ the standard basis of V_+, by $(b_{-1}, \ldots, b_{-n})$ the standard basis of V_- and we will always consider V_+ and V_- to be subsets of V without specifying the natural embedding. We will usually denote elements of V_+, V_- by the letters v, w, elements of M by the letters m, u and elements of V by the letters x, y. Further, we define a map

$$V \to \Bbbk, m \oplus (v_1, \ldots, v_n) \oplus (v_{-1}, \ldots, v_{-n}) \mapsto q(m) + \sum_{i=1}^{n} v_i v_{-i}$$

which clearly extends q, and which we denote by q as well.

7.3.4 Lemma. *The map $q\colon V \to \Bbbk$ is a quadratic form on V.*

Proof. In fact, V is simply the direct sum of (M, q) with the hyperbolic space of dimension $2n$ from Example 7.1.13, so this follows from Lemma 7.1.11. $\square$

7.3.5 Remark (Generalised matrices). Since M is not free, we cannot represent automorphisms of V by matrices with coefficients in $\Bbbk$. However, using the direct sum decomposition of V, we can represent them by "generalised matrices" which are elements of

$$\begin{pmatrix} \mathrm{Hom}(M, M) & \mathrm{Hom}(V_+, M) & \mathrm{Hom}(V_-, M) \\ \mathrm{Hom}(M, V_+) & \mathrm{Hom}(V_+, V_-) & \mathrm{Hom}(V_-, V_+) \\ \mathrm{Hom}(M, V_-) & \mathrm{Hom}(V_+, V_-) & \mathrm{Hom}(V_-, V_-) \end{pmatrix},$$

where Hom stands for the group of $\Bbbk$-linear homomorphisms. Further, since V_+ and V_- are free of rank n, we can represent elements of $\mathrm{Hom}(V_\varepsilon, V_\sigma)$ by $(n \times n)$-matrices over $\Bbbk$, elements of $\mathrm{Hom}(V_\varepsilon, M)$ by $(1 \times n)$-matrices with entries in M and elements of $\mathrm{Hom}(M, V_\varepsilon)$ by $(n \times 1)$-matrices with entries in $\mathrm{Hom}(M, \Bbbk)$. Here ε and σ denote arbitrary signs. The elements of $\mathrm{Hom}(M, \Bbbk)$ that we encounter will always be of the form $f(u, \cdot\,)$ for some $u \in M$, which is the map $v \mapsto f(u, v)$.

We can now construct the desired root homomorphisms.

7.3.6 Construction (Long root homomorphisms). Let $i, j \in [1, n]$ be distinct and let $a \in \Bbbk$. We denote by $\theta_{e_i - e_j}(a)$ the unique $\Bbbk$-linear endomorphism of V which is given on the direct summands of V as follows, where m, v^+, v^- denote arbitrary elements of M, V_+, V_-, respectively:

$$\theta_{e_i-e_j}(a)\colon \quad m \mapsto m, \quad v^+ \mapsto v^+ + av_j^+ b_i, \quad v^- \mapsto v^- - av_i^- b_{-j}.$$

Now assume that, in addition, $i < j$. Then we define $\theta_{e_i+e_j}(a)$ and $\theta_{-e_i-e_j}(a)$ by the following formulas:

$$\begin{aligned} \theta_{e_i+e_j}(a)\colon &\quad m \mapsto m, \quad v^+ \mapsto v^+, \quad v^- \mapsto v^- + av_j^- \cdot b_i - av_i^- \cdot b_j, \\ \theta_{-e_i-e_j}(a)\colon &\quad m \mapsto m, \quad v^+ \mapsto v^+ + av_i^+ \cdot b_{-j} - av_j^+ \cdot b_{-i}, \quad v^- \mapsto v^-. \end{aligned}$$

Examples of the corresponding generalised matrices are given in Figure 7.2.

7.3.7 Construction (Short root homomorphisms). Let $i \in [1, n]$ and let $u \in M$. Then we define $\theta_{e_i}(u)$ and $\theta_{-e_i}(u)$ to be the unique $\Bbbk$-linear endomorphisms of V which are given on the direct summands of V as follows, where m, v^+, v^- denote arbitrary elements of M, V_+, V_-, respectively:

$$\begin{aligned} \theta_{e_i}(u)\colon &\ m \mapsto m + f(u,m) \cdot b_i,\ v^+ \mapsto v^+,\ v^- \mapsto v^- - v_i^- \cdot u - v_i^- q(u) \cdot b_i, \\ \theta_{-e_i}(u)\colon &\ m \mapsto m - f(u,m) \cdot b_{-i},\ v^+ \mapsto v^+ + v_i^+ \cdot u - v_i^+ q(v) \cdot b_{-i},\ v^- \mapsto v^-. \end{aligned}$$

Examples of the corresponding generalised matrices are given in Figure 7.3.

7.3.8 Lemma. *For each short root β and for each long root α, the maps $\theta_\alpha\colon (\Bbbk, +) \to \mathrm{End}_\Bbbk(V)$ and $\theta_\beta\colon (M, +) \to \mathrm{End}_\Bbbk(V)$ are injective homomorphisms. In particular, their images lie in $\mathrm{Aut}_\Bbbk(V)$.*

Proof. The injectivity of these maps can be deduced from their matrix representations. The homomorphism property follows from a straightforward matrix computation. □

Now the definition of the elementary orthogonal group, our main example of a B_n-graded group, is not surprising.

7.3.9 Definition (Elementary orthogonal group). For each root α, we denote the image of θ_α in $\mathrm{Aut}_\Bbbk(V)$ by U_α, and we denote by $\mathrm{EO}(q)$ the group which is generated by $(U_\alpha)_{\alpha\in\Phi}$. We call $\mathrm{EO}(q)$ the *elementary orthogonal group of q*.

7.3.10 Note. A straightforward computation shows that each root group is contained in the orthogonal group $\mathrm{O}(q)$ from Definition 7.1.16. (Here q denotes the quadratic form on V and not the one on M.) It follows that the elementary orthogonal group $\mathrm{EO}(q)$ is contained in $\mathrm{O}(q)$, just like the elementary group from Example 3.2.14 is contained in the special linear group. However, we will never formally use this fact, so we do not carry out the tedious computation.

$$\theta_{e_1-e_2}(a) = \left(\begin{array}{c|ccc|ccc} \mathrm{id}_M & & & & & & \\ \hline & 1 & a & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ \hline & & & & 1 & & \\ & & & & -a & 1 & \\ & & & & & & 1 \end{array}\right),$$

$$\theta_{e_1+e_2}(a) = \left(\begin{array}{c|ccc|ccc} \mathrm{id}_M & & & & & & \\ \hline & 1 & & & & a & \\ & & 1 & & -a & & \\ & & & 1 & & & \\ \hline & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{array}\right),$$

$$\theta_{-e_1-e_2}(a) = \left(\begin{array}{c|ccc|ccc} \mathrm{id}_M & & & & & & \\ \hline & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ \hline & & -a & & 1 & & \\ & a & & & & 1 & \\ & & & & & & 1 \end{array}\right).$$

Figure 7.2: The long root homomorphisms for B_3.

7.3.B Weyl Elements

We can define Weyl elements in $\mathrm{EO}(q)$ which look just like the ones in Chevalley groups (Definition 3.3.3), except that the notion of "inverses of elements of M" is not obvious.

7.3.11 Definition (Weyl elements). For all distinct $i, j \in [1, n]$ and all invertible $a \in k$, we define

$$w_{ij}(a) := w_{e_i-e_j}(a) := \theta_{e_j-e_i}(-a^{-1}) \circ \theta_{e_i-e_j}(a) \circ \theta_{e_j-e_i}(-a^{-1}) \in \mathrm{EO}(q)$$

and $w_{ij} := w_{e_i-e_j} := w_{ij}(1_k)$. For all $i \in [1, n]$ and all $u \in M$ for which $q(u)$ is invertible, we define

$$w_i(u) := w_{e_i}(u) := \theta_{-e_i}\big(-q(u)^{-1}u\big) \circ \theta_{e_i}(u) \circ \theta_{-e_i}\big(-q(u)^{-1}u\big)$$

and $w_i := w_{e_i} := w_i(v_0)$.

7.3.12 Note (see also 7.2.21). Observe that we have defined Weyl elements only for the roots in the set $\hat{B}_n$ from Remark 7.2.21. In theory, it would be enough to only consider the Weyl elements $w_{i,i+1}(a)$ for $i \in [1, n-1]$ and $w_n(u)$, that is, the ones which correspond to the root base Δ. However, by the considerations in Remark 7.2.21, it will be more practical to have an explicit α-Weyl element for all $\alpha \in \hat{B}_n$. At the same time, while we could easily define Weyl elements for all remaining roots in B_n as well, there is no additional benefit in doing so.

7.3.13 Definition (Standard system of Weyl elements). The *standard system of Weyl elements for* $\mathrm{EO}(q)$ is the family $(w_\delta)_{\delta \in \Delta}$ given by Definition 7.3.11.

7.3.14 Remark (Short Weyl elements). Let $u \in M$ such that $q(u)$ is invertible. The goal of this remark is to explicitly compute the generalised matrix of $w_1(u)$.

$$\theta_{e_1}(u) = \left(\begin{array}{c|ccc|ccc} \mathrm{id}_M & & & & -u & & \\ \hline f(u,\,\cdot\,) & 1 & & & -q(u) & & \\ & & 1 & & & & \\ & & & 1 & & & \\ \hline & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{array}\right),$$

$$\theta_{-e_1}(u) = \left(\begin{array}{c|ccc|ccc} \mathrm{id}_M & u & & & & & \\ \hline & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ \hline -f(u,\,\cdot\,) & -q(u) & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{array}\right).$$

Figure 7.3: The short root homomorphisms for B_3.

We will leave out the rows and columns which correspond to $(b_i)_{i\in[2,n]}$ and $(b_{-i})_{i\in[2,n]}$ because they are trivial. We begin the computation with

$$\begin{aligned}
&\theta_{e_1}(u)\theta_{-e_1}\big(-q(u)^{-1}u\big) \\
&\quad= \begin{pmatrix} \mathrm{id} & 0 & -u \\ -f(u,\,\cdot\,) & 1 & -q(u) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{id} & -q(u)^{-1}u & 0 \\ 0 & 1 & 0 \\ q(u)^{-1}f(u,\,\cdot\,) & -q(u)^{-1} & 1 \end{pmatrix} \\
&\quad= \begin{pmatrix} \mathrm{id} - q(u)^{-1}f(u,\,\cdot\,)u & -q(u)^{-1}u + q(u)^{-1}u & -u \\ -f(u,\,\cdot\,) + q(u)q(u)^{-1}f(u,\,\cdot\,) & r(u) & -q(u) \\ q(u)^{-1}f(u,\,\cdot\,) & -q(u)^{-1} & 1 \end{pmatrix} \\
&\quad= \begin{pmatrix} \mathrm{id} - q(u)^{-1}f(u,\,\cdot\,)u & 0 & -u \\ 0 & 0 & -q(u) \\ q(u)^{-1}f(u,\,\cdot\,) & -q(u)^{-1} & 1 \end{pmatrix}
\end{aligned}$$

where $r(u) := -q(u)^{-1}f(u,u) + 1 + q(u)q(u)^{-1} = 0$ because $f(u,u) = 2q(u)$ by Lemma 7.1.5 (e). Now

$$\begin{aligned}
w_1(u) &= \theta_{-e_1}\big(-q(u)^{-1}u\big)\theta_{e_1}(u)\theta_{-e_1}\big(-q(u)^{-1}u\big) \\
&= \begin{pmatrix} \mathrm{id} & -q(u)^{-1}u & 0 \\ 0 & 1 & 0 \\ q(u)^{-1}f(u,\,\cdot\,) & -q(u)^{-1} & 1 \end{pmatrix} \\
&\qquad\qquad \cdot \begin{pmatrix} \mathrm{id} - q(u)^{-1}f(u,\,\cdot\,)u & 0 & -u \\ 0 & 0 & -q(u) \\ q(u)^{-1}f(u,\,\cdot\,) & -q(u)^{-1} & 1 \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{id} - q(u)^{-1}f(u,\,\cdot\,)u & 0 & u - q(u)^{-1}q(u)u \\ 0 & 0 & -q(u) \\ \varphi & -q(u)^{-1} & -q(u)^{-1}f(u,u) + q(u)^{-1}q(u) + 1 \end{pmatrix}
\end{aligned}$$

where $\varphi\colon M \to k$ is the map which sends $x \in M$ to

$$\begin{aligned}
&q(u)^{-1}f\big(u, x - q(u)^{-1}f(u,x)u\big) + q(u)^{-1}f(u,x) \\
&\qquad = q(u)^{-1}\big(f(u,x) - q(u)^{-1}f(u,x)f(u,u) + f(u,x)\big) = 0.
\end{aligned}$$

We conclude that

$$w_1(u) = \begin{pmatrix} \mathrm{id} - q(u)^{-1} f(u, \cdot\,)u & 0 & 0 \\ 0 & 0 & -q(u) \\ 0 & -q(u)^{-1} & 0 \end{pmatrix}.$$

Hence $w_1(au) = w_1(u)$ for all $a \in \Bbbk$ with $a^2 = 1_\Bbbk$. In particular, $w_1(u) = w_1(-u)$. We conclude that a short Weyl element may in general be represented by distinct Weyl triples.

The following results show that the elements in Definition 7.3.11 are indeed Weyl elements. Even more, we have specific formulas for their conjugation actions. We can see from these formulas that, in addition to additive inversion on $\Bbbk$ and M, we can expect a second kind of twisting in B_n-graded groups: the reflection σ_{v_0} on M. Note that, unlike the unit element $1_\Bbbk$ in the ring $\Bbbk$, the element v_0 is not uniquely determined by (M, q), but rather an arbitrary (but fixed) element of M with the property that $q(v_0) = 1_\Bbbk$.

7.3.15 Lemma. *Let $i, j \in [1, n]$ be distinct and let $a \in \Bbbk$ be invertible. Then $w := w_{ij}(a)$ is an $(e_i - e_j)$-Weyl element. It satisfies the following formulas for all $b \in \Bbbk$ and $u \in M$:*

(a) $\theta_{e_i-e_j}(b)^w = \theta_{e_j-e_i}(-a^{-2}b)$ *and* $\theta_{e_j-e_i}(b)^w = \theta_{e_i-e_j}(-a^2b)$.

(b) For all $k \in [1, n] \setminus \{i, j\}$, we have

$$\theta_{e_i-e_k}(b)^w = \theta_{e_j-e_k}(a^{-1}b), \qquad \theta_{e_i+e_k}(b)^w = \theta_{e_j+e_k}(\delta^-_{k\in\langle i,j\rangle} a^{-1}b),$$
$$\theta_{e_k-e_i}(b)^w = \theta_{e_k-e_j}(ab), \qquad \theta_{-e_i-e_k}(b)^w = \theta_{-e_j-e_k}(\delta^-_{k\in\langle i,j\rangle} ab).$$

(c) For all $k \in [1, n] \setminus \{i, j\}$, we have

$$\theta_{e_j-e_k}(b)^w = \theta_{e_i-e_k}(-ab), \qquad \theta_{e_j+e_k}(b)^w = \theta_{e_i+e_k}(-\delta^-_{k\in\langle i,j\rangle} ab),$$
$$\theta_{e_k-e_j}(b)^w = \theta_{e_k-e_i}(-a^{-1}b), \qquad \theta_{-e_j-e_k}(b)^w = \theta_{-e_i-e_k}(-\delta^-_{k\in\langle i,j\rangle} a^{-1}b).$$

(d) $\theta_{e_i}(u)^w = \theta_{e_j}(a^{-1}u)$ *and* $\theta_{-e_i}(u)^w = \theta_{-e_j}(au)$.

(e) $\theta_{e_j}(u)^w = \theta_{e_i}(-au)$ *and* $\theta_{-e_j}(u)^w = \theta_{-e_i}(-a^{-1}u)$.

(f) For any root α for which no formula for the conjugation action of w on U_α is given above, this action is trivial.

Proof. This follows from a straightforward computation. See also Note 7.3.29. □

7.3.16 Remark. Let $i, j \in [1, n]$ be distinct and let $a \in \Bbbk$ be invertible. Since $w := w_{ij}(a)$ is a Weyl element by Lemma 7.3.15, it follows from Proposition 2.2.6 (c) that

$$\begin{aligned} w_{ij}(a) &= \theta_{e_i-e_j}(a)\theta_{e_j-e_i}(-a^{-1})\theta_{e_j-e_i}(-a^{-1})^w \\ &= \theta_{e_i-e_j}(a)\theta_{e_j-e_i}(-a^{-1})\theta_{e_i-e_j}(a) = w_{ji}(-a^{-1}). \end{aligned}$$

In particular, $w_{ij} = w_{ji}^{-1}$.

7.3.17 Lemma. *Let $i \in [1, n]$ and let $v \in M$ such that $q(v)$ is invertible. Then $w := w_i(v)$ is an e_i-Weyl element. It satisfies the following formulas for all $b \in \Bbbk$ and $u \in M$:*

(a) For all $k \in [1, n] \setminus \{i\}$, we have

$$\theta_{e_i-e_k}(b)^w = \theta_{-e_i-e_k}\big(\delta^-_{i>k} q(v)^{-1}b\big), \qquad \theta_{e_i+e_k}(b)^w = \theta_{e_k-e_i}\big(\delta^-_{i>k} q(v)^{-1}b\big),$$
$$\theta_{e_k-e_i}(b)^w = \theta_{e_k+e_i}\big(\delta^-_{i>k} q(v)b\big), \qquad \theta_{-e_i-e_k}(b)^w = \theta_{e_i-e_k}\big(\delta^-_{i>k} q(v)b\big).$$

(b) $\theta_{e_i}(u)^w = \theta_{-e_i}\big(q(v)^{-1}\sigma_v(u)\big)$ and $\theta_{-e_i}(u)^w = \theta_{e_i}\big(q(v)\sigma_v(u)\big)$.
(c) For all $k \in [1,n] \setminus \{i\}$, $\theta_{e_k}(u)^w = \theta_{e_k}\big(\sigma_v(u)\big)$ and $\theta_{-e_k}(u)^w = \theta_{-e_k}\big(\sigma_v(u)\big)$.
(d) For any root α for which no formula for the conjugation action of w on U_α is given above, this action is trivial.

Proof. This follows from a straightforward computation. See also Note 7.3.29. □

7.3.C Parity Maps and Twisting Structures

We can read off the correct definition of the twisting groups and parity maps in $\mathrm{EO}(q)$ (with respect to the standard system of Weyl elements) from the above formulas. In other words, the parity maps and twisting groups are chosen precisely to ensure that Lemma 7.3.21 holds.

7.3.18 Notation for this section. From now on, we denote by $(A \times B, M, \hbar)$ the standard parameter system for (M, q, v_0) (as in Definition 7.1.25). Further, we define maps $\eta\colon B_n \times \hat{B}_n \to A$ and $\mu\colon B_n \times \hat{B}_n \to B$ by the formulas in Figure 7.4, where $\hat{B}_n$ is as in Remark 7.2.21. By restricting the second component to Δ, we obtain Δ-parity maps which we also denote by η and μ, and which we call the *standard parity maps of type B_n*.

α	η_{α,e_i-e_j}	μ_{α,e_i-e_j}
$\pm(e_i - e_j)$	-1_A	1_B
$\pm(e_i + e_j)$	1_A	1_B
$\pm(e_i - e_k)$	1_A	1_B
$\pm(e_i + e_k)$	$\delta^-_{k\in\langle i,j\rangle}1_A$	1_B
$\pm(e_j - e_k)$	-1_A	1_B
$\pm(e_j + e_k)$	$-\delta^-_{k\in\langle i,j\rangle}1_A$	1_B
$\pm e_k \pm e_l$	1_A	1_B
$\pm e_i$	1_A	1_B
$\pm e_j$	-1_A	1_B
$\pm e_k$	1_A	1_B

α	η_{α,e_i}	μ_{α,e_i}
$\pm(e_i - e_k)$	$\delta^-_{i>k}1_A$	1_B
$\pm(e_i + e_k)$	$\delta^-_{i>k}1_A$	1_B
$\pm e_k \pm e_l$	1_A	1_B
$\pm e_i$	1_A	-1_B
$\pm e_k$	1_A	-1_B

Figure 7.4: The definition of $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ for all $\alpha \in B_n$ and $\beta \in \hat{B}_n$, see Notation 7.3.18. In the left table, we assume that i, j, k, l are pairwise distinct. In the right table, we assume that i, k, l are pairwise distinct. For small values of n, it is of course not possible to choose three or four pairwise distinct indices, in which case the corresponding rows should be ignored.

7.3.19 Note. The values $\eta_{\alpha,\beta}$ for $\beta \in \hat{B}_n \setminus \Delta$ in Figure 7.4 are not relevant for the definition of the Δ-parity map η because it is defined on the set $\Phi \times \Delta$. However, the assertion of the following Lemma 7.3.21 holds for all $\beta \in \hat{B}_n$, not merely for $\beta \in \Delta$. We will use this observation in Lemma 7.3.28 to express some values of the extended parity map

$$\eta\colon \Phi \times \mathcal{L}(\Delta \cup (-\Delta)) \to A$$

(which we defined in 4.2.4) in terms of the values $(\eta_{\alpha,\beta})_{\alpha\in B_n, \beta\in\hat{B}_n}$. We conclude that, while the values $\eta_{\alpha,\beta}$ for $\beta \in \hat{B}_n \setminus \Delta$ are not needed to define the parity map $\eta\colon \Phi \times \Delta \to A$, they nonetheless express important properties of this map. The fact that we can directly read off these values from Figure 7.4 will allow us to simplify some later computations, for example in Lemma 7.10.7. Similar remarks apply for μ in place of η.

7.3.20 Remark. Consider $A_{n-1} = \{ e_i - e_j \mid i \neq j \in [1, n] \}$ as a subset of B_n in the natural way. Then for all $\alpha, \beta \in A_{n-1}$, the value $\eta_{\alpha,\beta}$ in Figure 7.4 is the same as the one in Example 5.5.5.

7.3.21 Lemma. *Let $\alpha \in B_n$ and $\beta \in \hat{B}_n$. Let $x \in \Bbbk$ if α is long and let $x \in M$ if α is short. Then $\theta_\alpha(x)^{w_\beta} = \theta_{\sigma_\beta(\alpha)}(\eta_{\alpha,\beta}\mu_{\alpha,\beta}.x)$.*

Proof. This follows from Lemmas 7.3.15 and 7.3.17 by putting $a := 1_\Bbbk$ and $v := v_0$, respectively. ☐

7.3.22 Remark. Note that, even though the definition of η and μ is clearly motivated by Lemmas 7.3.15 and 7.3.17, it is completely independent of the choices of $\Bbbk$, M, q and v_0. This is evident from Figure 7.4. As a consequence, we can obtain information about η and μ by performing computations in the group $\mathrm{EO}(q)$ for any fixed choice of the parameters $\Bbbk$, M, q and v_0. In the sequel, we will often choose $\Bbbk := \mathbb{C}$, $M := \mathbb{C}^2$, $q\colon M \to \mathbb{C}, (x, y) \mapsto x^2 + y^2$ and $v_0 := (1, 0)$. This choice of parameters has the following two crucial properties: Firstly, $1_\Bbbk \neq -1_\Bbbk$, so that the inversion map on $(\Bbbk, +)$ is non-trivial; and secondly, that σ_{v_0} is neither the inversion map nor the identity map on $(M, +)$. These properties imply that the standard parameter system $(A \times B, M, \Bbbk)$ from Definition 7.1.25 is $(\eta \times \mu)$-faithful (in the sense of Definition 4.3.8).

7.3.23 Lemma. *The root isomorphisms $(\theta_\alpha)_{\alpha \in B_n}$ from Constructions 7.3.6 and 7.3.7 form a parametrisation of $\mathrm{EO}(q)$ by $(A \times B, M, \Bbbk)$ with respect to $\eta \times \mu$ and $(w_\delta)_{\delta \in \Delta}$.*

Proof. This follows from Lemma 7.3.21. ☐

7.3.24 Lemma. *Let $(w_\alpha)_{\alpha \in \hat{B}_n}$ be as in Definition 7.3.11. Then the following hold:*

(a) *For all pairwise distinct $i, j, k \in [1, n]$, we have*

$$w_{ij}^{w_{jk}} = w_{ik} \quad \text{and} \quad w_{ij}^{w_{kj}} = w_{ki}.$$

For all distinct $i, j \in [1, n]$, we have

$$w_i^{w_{ij}} = w_j \quad \text{and} \quad w_i^{w_{ji}} = w_i^{-1}.$$

(b) *$(w_\alpha)_{\alpha \in \hat{B}_n}$ is the standard $\hat{B}_n$-extension of $(w_\delta)_{\delta \in \Delta}$, the standard system of Weyl elements.*

Proof. The first assertion follows from Lemma 7.3.21 and an inspection of the values in Figure 7.4. The second assertion follows from the first one and from Remark 7.3.16. ☐

7.3.D Commutator Relations

We now proceed to show that $(U_\alpha)_{\alpha \in B_n}$ is a crystallographic B_n-grading of $\mathrm{EO}(q)$.

7.3.25 Proposition. *The group $\mathrm{EO}(q)$ satisfies the following commutator relations. For all distinct $i, j \in [1, n]$, all $a \in \Bbbk$ and all $v \in M$, we have*

$$\begin{aligned}
[\theta_{e_i - e_j}(a), \theta_{e_j}(v)] &= \theta_{e_i}(av)\theta_{e_i + e_j}\big(\delta_{i<j}^{-} a q(v)\big), \\
[\theta_{e_j - e_i}(a), \theta_{-e_j}(v)] &= \theta_{-e_i}(-av)\theta_{-e_i - e_j}\big(\delta_{i<j}^{-} a q(v)\big), \\
[\theta_{e_i + e_j}(a), \theta_{-e_j}(v)] &= \theta_{e_i}\big(\delta_{i<j}^{-} av\big)\theta_{e_i - e_j}\big(\delta_{i<j}^{-} a q(v)\big), \\
[\theta_{-e_i - e_j}(a), \theta_{e_j}(v)] &= \theta_{-e_i}\big(\delta_{i>j}^{-} av\big)\theta_{e_j - e_i}\big(\delta_{i<j}^{-} a q(v)\big).
\end{aligned}$$

For all distinct $i, j \in [1, n]$ and all $v, w \in M$, we have

$$[\theta_{e_i}(v), \theta_{e_j}(w)] = \theta_{e_i+e_j}\big(\delta^-_{i<j} f(v,w)\big),$$
$$[\theta_{e_i}(v), \theta_{-e_j}(w)] = \theta_{e_i-e_j}\big(f(v,w)\big),$$
$$[\theta_{-e_i}(v), \theta_{-e_j}(w)] = \theta_{-e_i-e_j}\big(\delta^-_{i>j} f(v,w)\big).$$

For all pairwise distinct $i, j, k \in [1, n]$ and all $a, b \in \Bbbk$, we have

$$[\theta_{e_i-e_j}(a), \theta_{e_j-e_k}(b)] = \theta_{e_i-e_k}(ab),$$
$$[\theta_{e_i-e_j}(a), \theta_{e_j+e_k}(b)] = \theta_{e_i+e_k}(\delta^-_{k\in\langle i,j\rangle} ab),$$
$$[\theta_{e_i-e_j}(a), \theta_{-e_k-e_i}(b)] = \theta_{-e_j-e_k}(\delta^-_{k\notin\langle i,j\rangle} ab),$$
$$[\theta_{e_i+e_j}(a), \theta_{-e_k-e_i}(b)] = \theta_{e_j-e_k}(\delta^-_{i\in\langle j,k\rangle} ab).$$

Proof. This follows from a straightforward computation. See also Note 7.3.29. □

7.3.26 Proposition. *Let Π denote the standard positive system in B_n. Then the group* $\mathrm{EO}(q)$ *satisfies $U_\Pi \cap U_{-\Pi} = \{1\}$.*

Proof. Recall that the generalised matrices of the root homomorphisms with respect to the decomposition $V = M \oplus V_+ \oplus V_-$ and with respect to the ordered bases $(b_1, \dots, b_n)$ of V_+ and $(b_{-1}, \dots, b_{-n})$ of V_- are given in Figures 7.2 and 7.3. Consider instead the generalised matrices with respect to the decomposition $V = V_+ \oplus M \oplus V_-$ and with respect to the ordered bases $(b_1, \dots, b_n)$ of V_+ and $(b_{-n}, \dots, b_{-1})$ of V_-. For $n = 2$, some of these matrices are displayed in Figure 7.5. With respect to this decomposition, all generalised matrices of positive root isomorphisms are upper triangular while all generalised matrices of negative root isomorphisms are lower triangular. The assertion follows. □

7.3.27 Theorem. *The family $(U_\alpha)_{\alpha\in B_n}$ is a crystallographic B_n-grading of* $\mathrm{EO}(q)$.

Proof. By Lemmas 7.3.15 and 7.3.17, there exist Weyl elements for all simple roots and by Lemma 2.2.7, it follows that there exist Weyl elements for all roots. The remaining conditions are satisfied by Propositions 7.3.25 and 7.3.26. □

7.3.E Concluding Remarks

We state another brief technical observation on the parity maps η and μ which will be useful for later computations. Specifically, it will be used in Proposition 7.10.2. See also Remark 7.2.21.

7.3.28 Lemma. *Let $\alpha \in \hat{B}_n$, let $\bar{\alpha}$ be the standard Δ-expression of α in the sense of Definition 7.2.22 and let ζ be an arbitrary root. Then the element $\eta_{\zeta,\alpha}$ from Figure 7.4 equals the value $\eta_{\zeta,\bar{\alpha}}$ of the extended parity map $\eta\colon \Phi \times \mathcal{L}(\Delta \cup (-\Delta)) \to A$ (from Definition 4.2.4). The same assertion holds for μ in place of η.*

Proof. Observe that the assertion is independent of $\Bbbk$, M, q and v_0, so we have the freedom to choose these parameters as in Remark 7.3.22. Let $x \in \Bbbk = \mathbb{C}$ if ζ is long and let $x \in M = \mathbb{C}^2$ if ζ is short. Recall from Lemma 7.3.24 that the family $(w_\beta)_{\beta\in\hat{B}_n}$ of Weyl elements in $\mathrm{EO}(q)$ from Definition 7.3.11 coincides with the standard $\hat{B}_n$-extension of $(w_\delta)_{\delta\in\Delta}$ by Lemma 7.3.24. This implies that $w_\alpha = w_{\bar{\alpha}}$, so $\theta_\zeta(x)^{w_\alpha} = \theta_\zeta(x)^{w_{\bar{\alpha}}}$. By Lemma 7.3.21, we have

$$\theta_\zeta(x)^{w_\alpha} = \theta_{\zeta^{\sigma(\alpha)}}(\eta_{\zeta,\alpha}\mu_{\zeta,\alpha}.x).$$

$$\theta_{e_1-e_2}(a) = \left(\begin{array}{cc|c|cc} 1 & a & & & \\ & 1 & & & \\ \hline & & \mathrm{id}_M & & \\ \hline & & & 1 & -a \\ & & & & 1 \end{array}\right),$$

$$\theta_{e_1+e_2}(a) = \left(\begin{array}{cc|c|cc} 1 & & & a & \\ & 1 & & & -a \\ \hline & & \mathrm{id}_M & & \\ \hline & & & 1 & \\ & & & & 1 \end{array}\right),$$

$$\theta_{-e_1-e_2}(a) = \left(\begin{array}{cc|c|cc} 1 & & & & \\ & 1 & & & \\ \hline & & \mathrm{id}_M & & \\ \hline a & & & 1 & \\ & -a & & & 1 \end{array}\right),$$

$$\theta_{e_1}(u) = \left(\begin{array}{cc|c|cc} 1 & & f(u,\cdot) & & -u \\ & 1 & & & -q(u) \\ \hline & & \mathrm{id}_M & & \\ \hline & & & 1 & \\ & & & & 1 \end{array}\right),$$

$$\theta_{-e_1}(u) = \left(\begin{array}{cc|c|cc} 1 & & & & \\ & 1 & & & \\ \hline u & & \mathrm{id}_M & & \\ \hline & & & 1 & \\ -q(u) & & -f(u,\cdot) & & 1 \end{array}\right).$$

Figure 7.5: The generalised matrices of some root isomorphisms with respect to the decomposition of V in the proof of Proposition 7.3.26.

On the other hand, since $(\theta_\beta)_{\beta \in B_n}$ is a parametrisation of $\mathrm{EO}(q)$ by $(A \times B, M, \Bbbk)$ with respect to $\eta \times \mu$ and $(w_\delta)_{\delta \in \Delta}$ by Lemma 7.3.23 and since $\bar{\alpha}$ is a Δ-expression of α, we also have

$$\theta_\zeta(x)^{w_{\bar{\alpha}}} = \theta_{\zeta^{\sigma(\bar{\alpha})}}(\eta_{\zeta,\bar{\alpha}} \mu_{\zeta,\bar{\alpha}}.x) = \theta_{\zeta^{\sigma(\alpha)}}(\eta_{\zeta,\bar{\alpha}} \mu_{\zeta,\bar{\alpha}}.x).$$

As the parameter system $(A \times B, M, \Bbbk)$ is $(\eta \times \mu)$-faithful (in the sense of Definition 4.3.8) by Remark 7.3.22, we infer that $\eta_{\zeta,\bar{\alpha}} \mu_{\zeta,\bar{\alpha}} = \eta_{\zeta,\bar{\alpha}} \mu_{\zeta,\bar{\alpha}}$, as desired. ☐

7.3.29 Note (Implementation of the elementary orthogonal group). Every single computation in this section can be performed by hand in principle, but the number of necessary computations is rather large. Hence an implementation of the group $\mathrm{EO}(q)$ in a computer algebra system is desirable, for which we have used the GAP system [Gap]. In the following, we describe the general strategy of our implementation. First of all, observe that it suffices to consider the case that $\Bbbk$ is a (commutative associative) polynomial ring over the integers in sufficiently many indeterminates and that M is a free module of sufficiently high rank over $\Bbbk$. For example, if we prove that

$$[\theta_{e_i-e_j}(t), \theta_{e_j-e_k}(u)] = \theta_{e_i-e_k}(tu)$$

holds for indeterminates t, u in $\Bbbk$, then the same equation holds for all elements t, u in an arbitrary commutative associative ring (and for an arbitrary choice of

(M, q)).

Denote by $(b_1, \ldots, b_m)$ a $\Bbbk$-basis of M. In order to describe the maps q and f, we introduce indeterminates $(u_i)_{i\in[1,m]}$ and $(t_{ij})_{i<j\in[1,m]}$ in $\Bbbk$ which represent the values $(q(b_i))_{i\in[1,m]}$ and $(f(b_i, b_j))_{i<j\in[1,m]}$. Using the formulas in Lemma 7.1.5, we can explicitly define the maps $q\colon M \to \Bbbk$ and $f\colon M \times M \to \Bbbk$ in terms of these indeterminates. In other words, there exists a unique $\Bbbk$-quadratic map $q\colon M \to \Bbbk$ such that $q(b_i) = u_i$ for all $i \in [1, m]$, and similarly for f. Since $(u_i)_{i\in[1,m]}$ and $(t_{ij})_{i<j\in[1,m]}$ are indeterminates, any computation involving $(b_i)_{i\in[1,m]}$ and the values of these module elements under q and f is also valid for any family $(b_i')_{i\in[1,m]}$ of elements in any quadratic module (M', q') with linearisation f' over any commutative associative ring $\Bbbk'$.

Once the ring $\Bbbk$ and the quadratic module (M, q) are represented in GAP, it is a straightforward task to implement the root homomorphisms and to verify that all claimed identities (in particular, the commutator relations and the conjugation formulas for Weyl elements) hold.

7.4 Rank-2 Computations, Non-crystallographic Case

7.4.1 Notation for this section. We denote by G a group which has B_2-commutator relations with root groups $(U_\alpha)_{\alpha\in B_2}$ (in the sense of Definition 2.1.4) and we choose pairwise non-proportional roots $\alpha, \beta, \gamma, \delta$ such that $(\alpha, \beta, \gamma, \delta)$ is an interval ordering of $[\alpha, \delta]_{\mathrm{Cox}}$. Further, we assume that G is rank-2-injective.

Our rank-2 computations are split into two parts. In this section, we derive several formulas in the higher generality of non-crystallographic commutator relations. In the following section, we will specialise to crystallographic commutator relations to simplify our results. This approach has the advantage that the distinction between long and short roots disappears, as we will explain in Note 7.4.3. Thus every computation that can be performed in this generality saves us the trouble of performing two separate computations later on.

7.4.2 Note. Throughout this section, we will often use that root group elements from adjacent root groups commute. However, it is a priori not clear (and, in the general situation of Notation 7.4.1, not even true) that the root groups themselves are abelian. This will only be proven in Proposition 7.6.2 for the higher-rank case.

7.4.3 Note. Since our only assumption on $(\alpha, \beta, \gamma, \delta)$ is that it is an interval ordering of a closed root interval, and since we do not assume that the B_2-commutator relations are crystallographic, everything that we prove in this section will also be true for the root quadruple $(\delta, \gamma, \beta, \alpha)$. In particular, we do not assume that $(\alpha, \beta, \gamma, \delta)$ is a B_2-quadruple, so we do not assume that α, γ are long and β, δ are short.

Before we can perform any serious computations, we have to derive an analogue of Lemma 2.1.13 for open root intervals with two elements.

7.4.4 Lemma. *The following relations hold for all $x_\alpha, x_\alpha' \in U_\alpha$ and all $y_\delta, y_\delta' \in U_\delta$:*

(a) $[x_\alpha, y_\delta y_\delta']_\beta = [x_\alpha, y_\delta']_\beta [x_\alpha, y_\delta]_\beta$. *In particular,* $[x_\alpha, y_\delta^{-1}]_\beta = [x_\alpha, y_\delta]_\beta^{-1}$.

(b) $[y_\delta y_\delta', x_\alpha]_\beta = [y_\delta, x_\alpha]_\beta [y_\delta', x_\alpha]_\beta$. *In particular,* $[y_\delta^{-1}, x_\alpha]_\beta = [y_\delta, x_\alpha]_\beta^{-1}$.

(c) $[x_\alpha, y_\delta y_\delta']_\gamma = [x_\alpha, y_\delta']_\gamma \big[[x_\alpha, y_\delta]_\beta, y_\delta'\big] [x_\alpha, y_\delta]_\gamma$.

(d) $[y_\delta y_\delta', x_\alpha]_\gamma = [y_\delta, x_\alpha]_\gamma \big[[y_\delta, x_\alpha]_\beta, y_\delta'\big] [y_\delta', x_\alpha]_\gamma$.

(e) $[x_\alpha x'_\alpha, y_\delta]_\beta = [x_\alpha, y_\delta]_\beta \big[[x_\alpha, y_\delta]_\gamma, x'_\alpha\big] [x'_\alpha, y_\delta]_\beta$.

(f) $[y_\delta, x_\alpha x'_\alpha]_\beta = [y_\delta, x'_\alpha]_\beta \big[[y_\delta, x_\alpha]_\gamma, x'_\alpha\big] [y_\delta, x_\alpha]_\beta$.

(g) $[x_\alpha x'_\alpha, y_\delta]_\gamma = [x_\alpha, y_\delta]_\gamma [x'_\alpha, y_\delta]_\gamma$. *In particular,* $[x_\alpha^{-1}, y_\delta]_\gamma = [x_\alpha, y_\delta]_\gamma^{-1}$.

(h) $[y_\delta, x_\alpha x'_\alpha]_\gamma = [y_\delta, x'_\alpha]_\gamma [y_\delta, x_\alpha]_\gamma$. *In particular,* $[y_\delta, x_\alpha^{-1}]_\gamma = [y_\delta, x_\alpha]_\gamma^{-1}$.

Proof. Using Relation 1.1.11 (vii), we compute that

$$[x_\alpha, y_\delta y'_\delta] = [x_\alpha, y'_\delta] \cdot [x_\alpha, y_\delta]^{y'_\delta} = [x_\alpha, y'_\delta]_\beta [x_\alpha, y'_\delta]_\gamma [x_\alpha, y_\delta]_\beta^{y'_\delta} [x_\alpha, y_\delta]_\gamma^{y'_\delta}.$$

Applying Relation 1.1.11 (i) and the fact that U_γ commutes with U_δ and with U_β, we infer that

$$\begin{aligned}[x_\alpha, y_\delta y'_\delta] &= [x_\alpha, y'_\delta]_\beta [x_\alpha, y'_\delta]_\gamma [x_\alpha, y_\delta]_\beta \big[[x_\alpha, y_\delta]_\beta, y'_\delta\big] [x_\alpha, y_\delta]_\gamma \\ &= [x_\alpha, y'_\delta]_\beta [x_\alpha, y_\delta]_\beta \cdot [x_\alpha, y'_\delta]_\gamma \big[[x_\alpha, y_\delta]_\beta, y'_\delta\big] [x_\alpha, y_\delta]_\gamma.\end{aligned}$$

Since the product map on (β, γ) is injective by assumption, this implies that (a) and (c) hold. Using Remark 2.1.18 and, in the second computation, Lemma 2.1.13, we infer that

$$[y_\delta y'_\delta, x_\alpha]_\beta = [x_\alpha, y_\delta y'_\delta]_\beta^{-1} = [x_\alpha, y_\delta]_\beta^{-1} [x_\alpha, y'_\delta]_\beta^{-1} = [y_\delta, x_\alpha]_\beta [y'_\delta, x_\alpha]_\beta \qquad \text{and}$$

$$\begin{aligned}[y_\delta y'_\delta, x_\alpha]_\gamma &= [x_\alpha, y_\delta y'_\delta]_\gamma^{-1} = [x_\alpha, y_\delta]_\gamma^{-1} \big[[x_\alpha, y_\delta]_\beta, y'_\delta\big]^{-1} [x_\alpha, y'_\delta]_\gamma^{-1} \\ &= [y_\delta, x_\alpha]_\gamma \big[[y_\delta, x_\alpha]_\beta, y'_\delta\big] [y'_\delta, x_\alpha]_\gamma,\end{aligned}$$

thereby proving (b) and (d).

By Note 7.4.3, the results of the previous paragraph also hold for $(\delta, \gamma, \beta, \alpha)$ in place of $(\alpha, \beta, \gamma, \delta)$. Thus (a) implies (h), (b) implies (g), (c) implies (f) and (d) implies (e). □

Using Lemma 7.4.4, we can give an alternative proof of Proposition 2.2.10, just like we did in Lemma 5.4.2 for simply-laced root graded groups.

7.4.5 Lemma. *Let $x_\delta \in U_\delta$, let $a_{-\alpha}, c_{-\alpha} \in U_{-\alpha}$, let $b_\alpha \in U_\alpha$ and set $w_\alpha := a_{-\alpha} b_\alpha c_{-\alpha}$. Then $x_\delta^{w_\alpha}$ lies in U_β if and only if the following conditions are satisfied:*

(i) $\big[x_\delta, [x_\delta, b_\alpha]_\beta\big] [x_\delta, b_\alpha]_\gamma \big[[x_\delta, b_\alpha]_\beta, c_{-\alpha}\big]_\gamma^{[x_\delta, b_\alpha]_\gamma} = 1_G$.

(ii) $x_\delta \big[[x_\delta, b_\alpha]_\beta, c_{-\alpha}\big]_\delta [[x_\delta, b_\alpha]_\gamma, c_{-\alpha}] = 1_G$.

Further, in this case, $x_\delta^{w_\alpha} = [x_\delta, b_\alpha]_\beta$.

Proof. Using Relation 1.1.11 (i), we compute that

$$x_\delta^{w_\alpha} = x_\delta^{b_\alpha c_{-\alpha}} = \big(x_\delta [x_\delta, b_\alpha]\big)^{c_{-\alpha}} = x_\delta [x_\delta, b_\alpha] \big[[x_\delta, b_\alpha], c_{-\alpha}\big].$$

We want to express $x_\delta^{w_\alpha}$ as a product of root group elements, so we use the maps from Definition 2.1.16 to split each factor into its root group components:

$$x_\delta^{w_\alpha} = x_\delta [x_\delta, b_\alpha]_\beta [x_\delta, b_\alpha]_\gamma \big[[x_\delta, b_\alpha]_\beta [x_\delta, b_\alpha]_\gamma, c_{-\alpha}\big].$$

By Relation 1.1.11 (vii),

$$\begin{aligned}\big[[x_\delta, b_\alpha]_\beta [x_\delta, b_\alpha]_\gamma, c_{-\alpha}\big] &= \big[[x_\delta, b_\alpha]_\beta, c_{-\alpha}\big]^{[x_\delta, b_\alpha]_\gamma} \big[[x_\delta, b_\alpha]_\gamma, c_{-\alpha}\big] \\ &= \big[[x_\delta, b_\alpha]_\beta, c_{-\alpha}\big]_\gamma^{[x_\delta, b_\alpha]_\gamma} \big[[x_\delta, b_\alpha]_\beta, c_{-\alpha}\big]_\delta \big[[x_\delta, b_\alpha]_\gamma, c_{-\alpha}\big].\end{aligned}$$

Further, $x_\delta [x_\delta, b_\alpha]_\beta = [x_\delta, b_\alpha]_\beta x_\delta [x_\delta, [x_\delta, b_\alpha]_\beta]$ by Relation 1.1.11 (v). Thus altogether, using that U_γ commutes with U_β and U_δ, we have

$$x_\delta^{w_\alpha} = [x_\delta, b_\alpha]_\beta \cdot y_\gamma \cdot y_\delta$$

where

$$y_\gamma := \big[x_\delta, [x_\delta, b_\alpha]_\beta\big]\,[x_\delta, b_\alpha]_\gamma\,\big[[x_\delta, b_\alpha]_\beta, c_{-\alpha}\big]_\gamma^{[x_\delta, b_\alpha]_\gamma} \in U_\gamma \qquad \text{and}$$

$$y_\delta := x_\delta\big[[x_\delta, b_\alpha]_\beta, c_{-\alpha}\big]_\delta\big[[x_\delta, b_\alpha]_\gamma, c_{-\alpha}\big] \in U_\delta.$$

Now the assertion follows from the injectivity of the product map on (β, γ, δ). □

We will usually apply Lemma 7.4.5 in the following form. The analogous statement for RGD-systems is proven in [TW02, (21.19)].

7.4.6 Lemma. *Assume that $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ is an α-Weyl triple and denote by $w_\alpha := a_{-\alpha}b_\alpha c_{-\alpha}$ the corresponding Weyl element. Then the following statements hold for all $x_\delta \in U_\delta$:*

(a) $x_\delta^{w_\alpha} = [x_\delta, b_\alpha]_\beta$.

(b) $\big[x_\delta, [x_\delta, b_\alpha]_\beta\big]\,[x_\delta, b_\alpha]_\gamma\,\big[[x_\delta, b_\alpha]_\beta, c_{-\alpha}\big]_\gamma^{[x_\delta, b_\alpha]_\gamma} = 1_G$.

(c) $x_\delta\big[[x_\delta, b_\alpha]_\beta, c_{-\alpha}\big]_\delta\big[[x_\delta, b_\alpha]_\gamma, c_{-\alpha}\big] = 1_G$.

Proof. This follows from Lemma 7.4.5. □

Similarly to the strategy described in Note 5.4.4, we can apply Lemma 7.4.6 to the root quadruple $(-\alpha, \delta, \gamma, \beta)$, using the $(-\alpha)$-Weyl triples given by Proposition 2.2.6 (c).

7.4.7 Lemma. *Assume that $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ is an α-Weyl triple and denote by $w_\alpha := a_{-\alpha}b_\alpha c_{-\alpha}$ the corresponding Weyl element. Then the following statements hold for all $x_\beta \in U_\beta$:*

(a) $x_\beta^{w_\alpha} = [x_\beta, a_{-\alpha}]_\delta = [x_\beta, c_{-\alpha}]_\delta$.

(b) $\big[x_\beta, [x_\beta, a_{-\alpha}]_\delta\big]\,[x_\beta, a_{-\alpha}]_\gamma\,\big[[x_\beta, a_{-\alpha}]_\delta, b_\alpha\big]_\gamma^{[x_\beta, a_{-\alpha}]_\gamma} = 1_G$.

(c) $x_\beta\big[[x_\beta, a_{-\alpha}]_\delta, b_\alpha\big]_\beta\big[[x_\beta, a_{-\alpha}]_\gamma, b_\alpha\big] = 1_G$.

Proof. By Proposition 2.2.6 (c),

$$(c_{-\alpha}^{w_\alpha^{-1}}, a_{-\alpha}, b_\alpha) \quad \text{and} \quad (b_\alpha, c_{-\alpha}, a_{-\alpha}^{w_\alpha})$$

are $(-\alpha)$-Weyl triples corresponding to the $(-\alpha)$-Weyl element w_α. Applying Lemma 7.4.6 to the second Weyl triple and to the root quadruple $(-\alpha, \delta, \gamma, \beta)$, we see that $x_\beta^{w_\alpha} = [x_\delta, c_{-\alpha}]_\delta$. Applying Lemma 7.4.6 to the first Weyl triple, we obtain the remaining assertions. □

7.4.8 Note. In Lemma 7.4.7, we did not write down all relations which can be obtained by applying Lemma 7.4.6 to the Weyl triple $(b_\alpha, c_{-\alpha}, a_{-\alpha}^{w_\alpha})$. The reason for this is simply that these relations are more cumbersome and that we will not need them in the sequel.

Using Lemma 7.4.6, we obtain a first formula for the action of squares of Weyl elements.

7.4.9 Lemma. *Let $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ be an α-Weyl triple, denote by $w_\alpha := a_{-\alpha}b_\alpha c_{-\alpha}$ the corresponding Weyl element and put $w := w_\alpha^2$. Then $[x_\delta, b_\alpha^{-1}]_\beta^w = [x_\delta, b_\alpha]_\beta$ for all $x_\delta \in U_\delta$.*

Proof. Let $x_\delta \in U_\delta$. By Lemma 7.4.6 (a), we have

$$x_\delta^{w_\alpha} = [x_\delta, b_\alpha]_\beta \quad \text{and} \quad x_\delta^{w_\alpha^{-1}} = [x_\delta, b_\alpha^{-1}]_\beta,$$

applying Proposition 2.2.6 (a) in the second case. Since w_α^2 clearly maps $x_\delta^{w_\alpha^{-1}}$ to $x_\delta^{w_\alpha}$, the assertion follows. □

7.4.10 Note. It is curious that in Lemma 7.4.9, we cannot prove the stronger statement that

$$[x_\delta, x_\alpha]_\beta^w = [x_\delta, x_\alpha^{-1}]_\beta \tag{7.1}$$

for all $x_\alpha \in U_\alpha$ and $x_\delta \in U_\delta$. In the more specialised setting of crystallographic BC_2-graded groups, we will be able to show that every square $v := w_{2\beta}^2$ of a 2β-Weyl element $w_{2\beta}$ has the property that

$$[x_\delta, x_\alpha]_\beta^v = [x_\delta, x_\alpha^{-1}]_\beta$$

for all $x_\alpha \in U_\alpha$ and $x_\delta \in U_\delta$ (see Lemma 9.4.12). This will be enough to conclude that the actions of w and v on U_β are identical, so in particular, w does indeed satisfy (7.1) (see Lemma 9.4.15). However, this proof is only valid for crystallographic BC_2-graded groups, not for the more general case of (non-crystallographic) B_2-graded groups.

7.4.11 Proposition. *For all $b_\alpha \in U_\alpha^\sharp$, the map $U_\delta \to U_\beta, x_\delta \mapsto [x_\delta, b_\alpha]_\beta$ is an isomorphism of root groups.*

Proof. Choose $a_{-\alpha}, c_{-\alpha}$ such that $w_\alpha := a_{-\alpha} b_\alpha c_{-\alpha}$ is an α-Weyl element. By Lemma 7.4.6, the map above is simply the map $x_\delta \mapsto x_\delta^{w_\alpha}$, which is an isomorphism by the definition of $U_\alpha^\sharp$. □

We end this section with some purely notational observations which will make future referencing easier. They follow from the previous results by replacing $(\alpha, \beta, \gamma, \delta)$ with $(\delta, \gamma, \beta, \alpha)$.

7.4.12 Lemma. *Assume that $(a_{-\delta}, b_\delta, c_{-\delta})$ is a δ-Weyl triple and denote by $w_\delta := a_{-\delta} b_\delta c_{-\delta}$ the corresponding Weyl element. Then the following statements hold for all $x_\alpha \in U_\alpha$:*

(a) $x_\alpha^{w_\delta} = [x_\alpha, b_\delta]_\gamma$.

(b) $\big[x_\alpha, [x_\alpha, b_\delta]_\gamma\big]\,[x_\alpha, b_\delta]_\beta \big[[x_\alpha, b_\delta]_\gamma, c_{-\delta}\big]_\beta^{[x_\alpha, b_\delta]_\beta} = 1_G$.

(c) $x_\alpha \big[[x_\alpha, b_\delta]_\gamma, c_{-\delta}\big]_\alpha \big[[x_\alpha, b_\delta]_\beta, c_{-\delta}\big] = 1_G$.

Proof. This is a reformulation of Lemma 7.4.6, using Note 7.4.3. □

7.4.13 Lemma. *Assume that $(a_{-\delta}, b_\delta, c_{-\delta})$ is an δ-Weyl triple and denote by $w_\delta := a_{-\delta} b_\delta c_{-\delta}$ the corresponding Weyl element. Then the following statements hold for all $x_\gamma \in U_\gamma$:*

(a) $x_\gamma^{w_\delta} = [x_\alpha, a_{-\delta}]_\alpha = [x_\alpha, c_{-\delta}]_\alpha$.

(b) $\big[x_\gamma, [x_\gamma, a_{-\delta}]_\alpha\big]\,[x_\gamma, a_{-\delta}]_\beta \big[[x_\gamma, a_{-\delta}]_\alpha, b_\delta\big]_\beta^{[x_\gamma, a_{-\delta}]_\beta} = 1_G$.

(c) $x_\gamma \big[[x_\gamma, a_{-\delta}]_\alpha, b_\delta\big]_\gamma \big[[x_\gamma, a_{-\delta}]_\beta, b_\delta\big] = 1_G$.

Proof. This is a reformulation of Lemma 7.4.7, using Note 7.4.3. □

7.4.14 Proposition. *For all $b_\delta \in U_\delta^\sharp$, the map $U_\alpha \to U_\gamma, x_\alpha \mapsto [x_\alpha, b_\delta]_\gamma$ is an isomorphism of root groups.*

Proof. This is a reformulation of Proposition 7.4.11, using Note 7.4.3. □

7.5 Rank-2 Computations, Crystallographic Case

7.5.1 Notation for this section. We denote by G a group which has crystallographic B_2-commutator relations with root groups $(U_\alpha)_{\alpha\in B_2}$ and we choose a B_2-quadruple $(\alpha,\beta,\gamma,\delta)$. Further, we assume that G is rank-2-injective.

In this section, we mostly simplify the results from the previous section under the additional assumption that the commutator relations are crystallographic. We will also perform a handful of new computations. Due to the crystallographic assumption, the distinction of root lengths matters from now on. In particular, the roles of $(\alpha,\beta,\gamma,\delta)$ and $(\delta,\gamma,\beta,\alpha)$ are no longer interchangeable.

7.5.2 Note. Since the root system B_2 is isomorphic to C_2, all results in this section are also valid for crystallographic C_2-gradings. We will explain how this works in Remark 9.2.8.

7.5.3 Lemma. *The following relations hold for all $x_\alpha, x'_\alpha \in U_\alpha$ and all $y_\delta, y'_\delta \in U_\delta$:*

(a) $[x_\alpha, y_\delta y'_\delta]_\beta = [x_\alpha, y'_\delta]_\beta [x_\alpha, y_\delta]_\beta$. *In particular,* $[x_\alpha, y_\delta^{-1}]_\beta = [x_\alpha, y_\delta]_\beta^{-1}$.

(b) $[y_\delta y'_\delta, x_\alpha]_\beta = [y_\delta, x_\alpha]_\beta [y'_\delta, x_\alpha]_\beta$. *In particular,* $[y_\delta^{-1}, x_\alpha]_\beta = [y_\delta, x_\alpha]_\beta^{-1}$.

(c) $[x_\alpha, y_\delta y'_\delta]_\gamma = [x_\alpha, y'_\delta]_\gamma \big[[x_\alpha, y_\delta]_\beta, y'_\delta\big] [x_\alpha, y_\delta]_\gamma$.

(d) $[y_\delta y'_\delta, x_\alpha]_\gamma = [y_\delta, x_\alpha]_\gamma \big[[y_\delta, x_\alpha]_\beta, y'_\delta\big] [y'_\delta, x_\alpha]_\gamma$.

(e) $[x_\alpha x'_\alpha, y_\delta]_\beta = [x_\alpha, y_\delta]_\beta [x'_\alpha, y_\delta]_\beta$. *In particular,* $[x_\alpha^{-1}, y_\delta]_\beta = [x_\alpha, y_\delta]_\beta^{-1}$.

(f) $[y_\delta, x_\alpha x'_\alpha]_\beta = [y_\delta, x'_\alpha]_\beta [y_\delta, x_\alpha]_\beta$. *In particular,* $[y_\delta, x_\alpha^{-1}]_\beta = [y_\delta, x_\alpha]_\beta^{-1}$.

(g) $[x_\alpha x'_\alpha, y_\delta]_\gamma = [x_\alpha, y_\delta]_\gamma [x'_\alpha, y_\delta]_\gamma$. *In particular,* $[x_\alpha^{-1}, y_\delta]_\gamma = [x_\alpha, y_\delta]_\gamma^{-1}$.

(h) $[y_\delta, x_\alpha x'_\alpha]_\gamma = [y_\delta, x'_\alpha]_\gamma [y_\delta, x_\alpha]_\gamma$. *In particular,* $[y_\delta, x_\alpha^{-1}]_\gamma = [y_\delta, x_\alpha]_\gamma^{-1}$.

Proof. These statements follow from Lemma 7.4.4 with the additional assumption that the commutator relations are crystallographic. More precisely, we only need that U_α and U_γ commute and the only changes are in (e) and (f). □

Since there are two orbits of roots and we want to investigate the actions of (squares of) Weyl elements on all root groups, we have to distinguish four cases. For better clarity, we study each of these cases in a separate subsection. The case of long Weyl elements acting on long root groups will not be covered in this section because we can reduce it to the study of A_2-subsystems in section 7.6.

7.5.A The Action of Long Weyl Elements on Short Root Groups

In this subsection, we show that squares of long Weyl elements act on the short root groups in the same B_2-subsystem by inversion (Lemma 7.5.6).

7.5.4 Lemma. *Assume that $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ is an α-Weyl triple with corresponding Weyl element $w_\alpha := a_{-\alpha} b_\alpha c_{-\alpha}$. Then the following hold:*

(a) $x_\delta^{w_\alpha} = [x_\delta, b_\alpha]_\beta$.

(b) $\big[x_\delta, [x_\delta, b_\alpha]_\beta\big] [x_\delta, b_\alpha]_\gamma \big[[x_\delta, b_\alpha]_\beta, c_{-\alpha}\big]_\gamma^{[x_\delta, b_\alpha]_\gamma} = 1_G$.

(c) $x_\delta = [[x_\delta, b_\alpha]_\beta, c_{-\alpha}]_\delta^{-1}$.

Proof. Using that $U_{-\alpha}$ and U_γ commute in a crystallographic B_2-grading, these statements follow from Lemma 7.4.6. □

7.5.5 Lemma. *Assume that $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ is an α-Weyl triple with corresponding Weyl element $w_\alpha := a_{-\alpha}b_\alpha c_{-\alpha}$. Then the following hold:*

(a) $x_\beta^{w_\alpha} = [x_\beta, a_{-\alpha}]_\delta = [x_\beta, c_{-\alpha}]_\delta$.

(b) $\big[x_\beta, [x_\beta, a_{-\alpha}]_\delta\big]\,[x_\beta, a_{-\alpha}]_\gamma\,\big[[x_\beta, a_{-\alpha}]_\delta, b_\alpha\big]_\gamma^{[x_\beta, a_{-\alpha}]_\gamma} = 1_G$.

(c) $x_\beta = [[x_\beta, a_{-\alpha}]_\delta, b_\alpha]_\beta^{-1}$.

Proof. Using that U_α and U_γ commute in a crystallographic B_2-grading, these statements follow from Lemma 7.4.7. □

7.5.6 Lemma. *Assume that w_α is an α-Weyl element. Then w_α^2 acts on U_β and U_δ by inversion.*

Proof. We apply the same strategy as in the proof of Lemma 5.4.13. Choose an α-Weyl triple $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ such that $w_\alpha = a_{-\alpha}b_\alpha c_{-\alpha}$ and let x_δ be an arbitrary element of U_δ. Put $w := w_\alpha^2$. Using first Lemma 7.5.4 and then Lemma 7.5.5, we have

$$x_\delta^w = [x_\delta, b_\alpha]_\beta^{w_\alpha} = [[x_\delta, b_\alpha]_\beta, c_{-\alpha}]_\delta.$$

Again by Lemma 7.5.4, we infer that $x_\delta^w = x_\delta^{-1}$, so w acts on U_δ by inversion. Since w_α is also a $(-\alpha)$-Weyl element by Proposition 2.2.6 (c) and $(-\alpha, \delta, \gamma, \beta)$ is a B_2-quadruple, it follows that w acts on U_β by inversion as well. □

7.5.7 Remark (Alternative proof of Lemma 7.5.6). Recall that we gave an alternative proof of Lemma 5.4.13 in Remark 5.4.14. Using the same strategy, we can also prove Lemma 7.5.6 in a different way. Let $x_\delta \in U_\delta$. In addition to the Weyl triple $(a_{-\alpha}, b_\alpha, c_{-\alpha})$, we consider the Weyl triple $(c_{-\alpha}^{-1}, b_\alpha^{-1}, a_{-\alpha}^{-1})$ whose corresponding Weyl element is w_α^{-1} (see Proposition 2.2.6 (a)). By Lemma 7.5.4, we have

$$x_\delta^{w_\alpha} = [x_\delta, b_\alpha]_\beta \quad\text{and}\quad x_\delta^{w_\alpha^{-1}} = [x_\delta, b_\alpha^{-1}]_\beta.$$

Using Lemma 7.5.3 (f), we infer that

$$x_\delta^{w_\alpha^{-1}} = (x_\delta^{w_\alpha})^{-1}.$$

This means that w_α^2 acts on U_δ by inversion. By similar arguments, we can deduce from Lemma 7.5.5 that w_α^2 acts on U_β by inversion.

The ideas from the previous paragraph can also be phrased in a slightly different way: Any $x_\beta \in U_\beta$ can, by Proposition 7.4.11, be written as $x_\beta = [x_\delta, b_\alpha^{-1}]_\beta$ for some $x_\delta \in U_\delta$, and it follows from Lemma 7.4.9 that

$$x_\beta^w = [x_\delta, b_\alpha^{-1}]_\beta^w = [x_\delta, b_\alpha]_\beta.$$

By Lemma 7.5.3 (f), it follows that $x_\beta^w = x_\beta^{-1}$.

7.5.8 Proposition. *Assume that $U_\alpha^\sharp$ is non-empty. Then U_β and U_δ are abelian.*

Proof. A group H is abelian if and only if the inversion map $h \mapsto h^{-1}$ is an endomorphism (and thus an automorphism) of H. By Lemma 7.5.6, the inversion maps on U_β and U_δ are the same as conjugation by w_α^2 for any α-Weyl element w_α, and conjugation by any group element is a homomorphism. The assertion follows. □

7.5.9 Note. In the rank-2 setting, we cannot prove that the long root groups are abelian as well, but this poses no problem: In the rank-3 setting, every long root is contained in an A_2-subsystem by Lemma 7.2.12, so every long root group is abelian by Proposition 5.4.9.

7.5.B The Action of Short Weyl Elements on Long Roots

In this subsection, we obtain no conclusive results. We will revisit this case in the rank-3 setting.

7.5.10 Lemma. *Assume that $(a_{-\delta}, b_\delta, c_{-\delta})$ is a δ-Weyl triple and denote by $w_\delta := a_{-\delta}b_\delta c_{-\delta}$ the corresponding Weyl element. Then the following statements hold for all $x_\alpha \in U_\alpha$:*

(a) $x_\alpha^{w_\delta} = [x_\alpha, b_\delta]_\gamma$.

(b) $[x_\alpha, b_\delta]_\beta \big[[x_\alpha, b_\delta]_\gamma, c_{-\delta}\big]_\beta^{[x_\alpha, b_\delta]_\beta} = 1_G$.

(c) $x_\alpha \big[[x_\alpha, b_\delta]_\gamma, c_{-\delta}\big]_\alpha [[x_\alpha, b_\delta]_\beta, c_{-\delta}] = 1_G$.

Proof. This follows from Lemma 7.4.12, using the additional information that U_α and U_γ commute in a crystallographic B_2-grading. □

7.5.11 Lemma. *Assume that $(a_{-\delta}, b_\delta, c_{-\delta})$ is an δ-Weyl triple and denote by $w_\delta := a_{-\delta}b_\delta c_{-\delta}$ the corresponding Weyl element. Then the following statements hold for all $x_\gamma \in U_\gamma$:*

(a) $x_\gamma^{w_\delta} = [x_\gamma, a_{-\delta}]_\alpha = [x_\gamma, c_{-\delta}]_\alpha$.

(b) $[x_\gamma, a_{-\delta}]_\beta \big[[x_\gamma, a_{-\delta}]_\alpha, b_\delta\big]_\beta^{[x_\gamma, a_{-\delta}]_\beta} = 1_G$.

(c) $x_\gamma \big[[x_\gamma, a_{-\delta}]_\alpha, b_\delta\big]_\gamma [[x_\gamma, a_{-\delta}]_\beta, b_\delta] = 1_G$.

Proof. This follows from Lemma 7.4.12, using the additional information that U_α and U_γ commute in a crystallographic B_2-grading. □

7.5.C The Action of Short Weyl Elements on Short Root Groups

Again, there are no conclusive results in this subsection because rank-3 assumptions are necessary.

Note that the following result is not the specialisation of computation in the non-crystallographic setting, but a new one. The reason for this is simply that even under crystallographic assumptions, the computation is tedious, and it would be even more so in the general case.

7.5.12 Lemma. *Let $a_{-\beta}, c_{-\beta} \in U_{-\beta}$, let $b_\beta \in U_\beta$, let $x_\delta \in U_\delta$ and set $w_\beta := a_{-\beta}b_\beta c_{-\beta}$. Then $x_\delta^{w_\beta}$ lies in U_δ if and only if the following conditions are satisfied:*

(i) $$1_G = [x_\delta, c_{-\beta}]\big[[x_\delta, b_\beta], c_{-\beta}\big]_{-\alpha}[x_\delta, a_{-\beta}]\Big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta, c_{-\beta}\Big] \cdot \Big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma, c_{-\beta}\Big]_{-\alpha}.$$

(ii) $\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma = [x_\delta, b_\beta]^{-1}$.

Further, in this case,

$$x_\delta^{w_\beta} = x_\delta\big[[x_\delta, b_\beta], c_{-\beta}\big]_\delta\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta\Big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma, c_{-\beta}\Big]_\delta.$$

Proof. As in Lemmas 5.4.2 and 7.4.5, we use Relation 1.1.11 (i) to perform the following computation:

$$\begin{aligned}
x_\delta^{w_\beta} &= \big(x_\delta[x_\delta, a_{-\beta}]\big)^{b_\beta c_{-\beta}} \\
&= \Big(x_\delta[x_\delta, b_\beta][x_\delta, a_{-\beta}]\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma\Big)^{c_{-\beta}} \\
&= x_\delta[x_\delta, c_{-\beta}][x_\delta, b_\beta]\big[[x_\delta, b_\beta], c_{-\beta}\big][x_\delta, a_{-\beta}]\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta \\
&\quad \cdot \Big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta, c_{-\beta}\Big]\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma\Big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma, c_{-\beta}\Big]
\end{aligned}$$

$$\begin{aligned}&= x_\delta[x_\delta, c_{-\beta}][x_\delta, b_\beta]\big[[x_\delta, b_\beta], c_{-\beta}\big]_{-\alpha}\big[[x_\delta, b_\beta], c_{-\beta}\big]_\delta[x_\delta, a_{-\beta}]\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta \\ &\quad \cdot \big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta, c_{-\beta}\big]\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma\big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma, c_{-\beta}\big]_{-\alpha} \\ &\quad \cdot \big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma, c_{-\beta}\big]_\delta\end{aligned}$$

Note that each of the factors lies in $U_{-\alpha}$, U_δ or U_γ and that each of these root groups commutes with the other two because G has crystallgrophic B_2-commutator relations. (However, we do not have any information on any of the groups $U_{-\alpha}$, U_δ or U_γ being abelian.) Thus we can write $x_\delta^{w_\beta} = y_{-\alpha}y_\delta y_\gamma$ where

$$\begin{aligned}y_{-\alpha} &:= [x_\delta, c_{-\beta}]\big[[x_\delta, b_\beta], c_{-\beta}\big]_{-\alpha}[x_\delta, a_{-\beta}]\big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta, c_{-\beta}\big] \\ &\qquad \cdot \big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma, c_{-\beta}\big]_{-\alpha}, \\ y_\delta &:= x_\delta\big[[x_\delta, b_\beta], c_{-\beta}\big]_\delta\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta\big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma, c_{-\beta}\big]_\delta, \\ y_\gamma &:= [x_\delta, b_\beta]\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma.\end{aligned}$$

Now by the injectivity of the product map on $(-\alpha, \delta, \gamma)$, $x_\delta^{w_\beta}$ lies in U_δ if and only if $y_{-\alpha} = 1_G = y_\gamma$, and in this case, $x_\delta^{w_\beta} = y_\delta$. The assertion follows. □

We know from Note 7.5.9 that all root groups will ultimately be proven to be abelian in the higher-rank setting. With this knowledge, we can simplify the statement of Lemma 7.5.12.

7.5.13 Lemma. *Let everything be as in Lemma 7.5.12, and assume additionally that the root groups $U_{-\alpha}, U_\delta, U_\gamma$ are abelian. Then $x_\delta^{w_\beta}$ lies in U_δ if and only if the following conditions are satisfied:*

(i) $[x_\delta, c_{-\beta}][x_\delta, a_{-\beta}]\big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta, c_{-\beta}\big] = 1_G$.

(ii) $\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma = [x_\delta, b_\beta]^{-1}$.

Further, in this case, $x_\delta^{w_\beta} = x_\delta[[x_\delta, a_{-\beta}], b_\beta]_\delta$.

Proof. Define $y_{-\alpha}, y_\delta, y_\gamma$ as in the proof of Lemma 7.5.12. At first, assume that $x_\delta^{w_\beta}$ lies in U_δ. Then $[[x_\delta, a_{-\beta}], b_\beta]_\gamma = [x_\delta, b_\beta]^{-1}$ by Lemma 7.5.12. Plugging this relation into the definitions of $y_{-\alpha}$ and y_δ, we obtain that

$$\begin{aligned}y_{-\alpha} &= [x_\delta, c_{-\beta}]\big[[x_\delta, b_\beta], c_{-\beta}\big]_{-\alpha}[x_\delta, a_{-\beta}]\big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta, c_{-\beta}\big]\big[[x_\delta, b_\beta]^{-1}, c_{-\beta}\big]_{-\alpha}, \\ y_\delta &= x_\delta\big[[x_\delta, b_\beta], c_{-\beta}\big]_\delta\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta\big[[x_\delta, b_\beta]^{-1}, c_{-\beta}\big]_\delta.\end{aligned}$$

Note that

$$[[x_\delta, b_\beta]^{-1}, c_{-\beta}]_\delta = [[x_\delta, b_\beta], c_{-\beta}]_\delta^{-1} \quad \text{and} \quad [[x_\delta, b_\beta]^{-1}, c_{-\beta}]_{-\alpha} = [[x_\delta, b_\beta], c_{-\beta}]_{-\alpha}^{-1}$$

by Lemma 7.5.3 (e) and Lemma 7.5.3 (g), respectively. Since $U_{-\alpha}$ and U_δ are abelian, it follows that

$$y_{-\alpha} = [x_\delta, c_{-\beta}][x_\delta, a_{-\beta}]\big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta, c_{-\beta}\big] \quad \text{and} \quad y_\delta = x_\delta\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta.$$

Since $y_{-\alpha} = 1_G$ and $y_\delta = x_\delta^{w_\beta}$ by Lemma 7.5.12, this proves the first implication of the assertion.

Conversely, assume that (i) and (ii) are satisfied. Using (ii) and the same computation as in the first paragraph, but reversed, we see that the term in (i) equals $y_{-\alpha}$. Thus $y_{-\alpha} = 1_G$, and so it follows from Lemma 7.5.12 that $x_\delta^{w_\beta}$ lies in U_δ. □

As usual, we give a reformulation of Lemma 7.5.13 under the assumption that w_β is a Weyl element.

7.5.14 Lemma. *Let $(a_{-\beta}, b_\beta, c_{-\beta})$ be a β-Weyl element and denote by $w_\beta := a_{-\beta}b_\beta c_{-\beta}$ the corresponding Weyl element. Assume that the root groups $U_{-\alpha}, U_\delta, U_\gamma$ are abelian. Then the following hold:*

(a) $x_\delta^{w_\beta} = x_\delta[[x_\delta, a_{-\beta}], b_\beta]_\delta$.

(b) $[x_\delta, c_{-\beta}][x_\delta, a_{-\beta}]\big[\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta, c_{-\beta}\big] = 1_G$.

(c) $\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma = [x_\delta, b_\beta]^{-1}$.

Proof. This is a consequence of Lemma 7.5.13. □

7.6 Rank-3 Computations

7.6.1 Notation for this section. We denote by G a group which has crystallographic B_n-commutator relations with root groups $(U_\alpha)_{\alpha\in B_n}$ for some fixed integer $n \geq 3$. We assume that $U_\alpha^\sharp$ is non-empty for all $\alpha \in \Phi$ and that G is rank-2-injective.

In this section, we prove that G satisfies the square formula for Weyl elements (Proposition 7.6.15). Using the square formula, we can give a new proof of the braid relations in G (Remark 7.6.17).

Before we begin, we make a brief observation which will simplify many computations.

7.6.2 Proposition. *All root groups of G are abelian.*

Proof. Every short root lies in a B_2-quadruple, so it follows from Proposition 7.5.8 that all short root groups are abelian. Further, every long roots lies in an A_2-subsystem by Lemma 7.2.12, so all short root groups are abelian by Proposition 5.4.9. □

7.6.A The Action of Long Weyl Elements on Short Root Groups

In this case, everything is essentially done by the rank-2 computations, specifically by Lemma 7.5.6. It remains to put everything together.

7.6.3 Lemma. *Let α be a long root, let β be a short root and let w_α be an α-Weyl element. If α and β are orthogonal, then w_α^2 acts trivially on U_β. Otherwise w_α^2 acts on U_β by inversion.*

Proof. If α and β are orthogonal, then β is adjacent to α and $-\alpha$ by Lemma 7.2.11 because α is long. This implies that w_α^2 acts trivially on U_β. If they are not orthogonal, then they lie in a common B_2-quadruple, so it follows from Lemma 7.5.6 that w_α^2 acts on U_β by inversion. □

7.6.4 Proposition. *Let α be a long root, let β be a short root, let w_α be an α-Weyl element. Put $w := w_\alpha^2$ and set $\varepsilon := (-1)^{\langle\beta|\alpha\rangle}$. Then $x_\beta^w = x_\beta^\varepsilon$ for all $x_\beta \in U_\beta$.*

Proof. We know from Proposition 7.2.20 that $\varepsilon = 1$ if α, β are orthogonal and that $\varepsilon = -1$ if they are not. Thus the assertion follows from Lemma 7.6.3. □

7.6.B The Action of Long Weyl Elements on Long Roots

In this case, we can reduce everything to the case of gradings of type A_2 and $A_1 \times A_1$.

7.6.5 Proposition. *Let α, γ be two long roots and let w_α be an α-Weyl element. Put $w := w_\alpha^2$ and set $\varepsilon := (-1)^{\langle \gamma | \alpha \rangle}$. Then $x_\gamma^w = x_\gamma^\varepsilon$ for all $x_\gamma \in U_\gamma$.*

Proof. If α and γ lie in a common A_2-subsystem, then the assertion holds by Proposition 5.4.17. If they do not lie in a common A_2-subsystem, then they must be orthogonal by Lemma 7.2.13 and thus we have $\varepsilon = 1$. In this case, γ is adjacent to α and $-\alpha$ by Lemma 7.2.11, and thus w_α acts trivially on U_γ. The assertion follows. □

7.6.6 Note. The proof of Proposition 7.6.5 fails if $n = 2$ because Lemma 7.2.13 does not hold in this case. In fact, it is not necessarily true that w_α^2 acts trivially on U_α in a crystallographic B_2-graded group. We will see a counterexample in Note 9.3.30.

7.6.C The Action of Short Weyl Elements on Long Root Groups

In Lemma 7.5.10 (a), we have already derived a simple formula for the action of short Weyl elements on long root groups. In order to arrive at a formula for the action of squares, we need to understand the behaviour of the map $(x_\alpha, x_\delta) \mapsto [x_\alpha, x_\delta]_\gamma$ under inverses in the second component. This is only possible in the rank-3 setting because we have to use Proposition 7.6.5.

7.6.7 Lemma. *Let $(\alpha, \beta, \gamma, \delta)$ be a B_2-quadruple. Then $[x_\alpha, x_\delta^{-1}]_\gamma = [x_\alpha, x_\delta]_\gamma$ for all $x_\alpha \in U_\alpha$ and all $x_\delta \in U_\delta$.*

Proof. Let $x_\alpha \in U_\alpha$ and let $x_\delta \in U_\delta$. Further, choose an arbitrary γ-Weyl element w_γ and set $w := w_\gamma^2$. Since $x_\delta^w = x_\delta^{-1}$ by Lemma 7.6.3 and $x_\alpha^w = x_\alpha$ by Proposition 7.6.5, it follows from Lemma 2.2.9 that

$$[x_\alpha, x_\delta]_\gamma^w = [x_\alpha^w, x_\delta^w]_\gamma = [x_\alpha, x_\delta^{-1}]_\gamma.$$

However, we also know that w acts trivially on U_γ by Proposition 7.6.5, so $[x_\alpha, x_\delta]_\gamma^w = [x_\alpha, x_\delta]_\gamma$. The assertion follows. □

7.6.8 Note. Thinking in terms of the commutator relations that we will ultimately prove for G (see Proposition 7.3.25), Lemma 7.6.7 corresponds to the fact that $q(-v) = q(v)$ for a quadratic form q.

7.6.9 Lemma. *Let $(\alpha, \beta, \gamma, \delta)$ be a B_2-quadruple and let w_δ be a δ-Weyl element. Then w_δ^2 acts trivially on U_α and on U_γ.*

Proof. Choose a δ-Weyl triple $(a_{-\delta}, b_\delta, c_{-\delta})$ whose corresponding Weyl element is w_δ and let $x_\alpha \in U_\alpha$. By Proposition 2.2.6 (a), $(c_{-\delta}^{-1}, b_\delta^{-1}, a_{-\delta}^{-1})$ is a δ-Weyl triple with corresponding Weyl element w_δ^{-1}. Thus it follows from Lemma 7.5.10 (a) that

$$x_\alpha^{w_\delta} = [x_\delta, b_\alpha]_\gamma \quad \text{and} \quad x_\alpha^{w_\delta^{-1}} = [x_\delta, b_\alpha^{-1}]_\gamma.$$

These terms are equal by Lemma 7.6.7. This implies that w_δ^2 acts trivially on x_α and this on all of U_α. Since w_δ is also a $(-\delta)$-Weyl element and $(\gamma, \beta, \alpha, -\delta)$ is a B_2-quadruple, it follows that w_δ^2 acts trivially on U_γ as well. □

7.6.10 Proposition. *Let δ be a short, let α be a long root and let w_δ be a δ-Weyl element. Put $\varepsilon := (-1)^{\langle\alpha|\delta\rangle}$ and $w := w_\delta^2$. Then $x_\alpha^w = x_\alpha^\varepsilon$ for all $x_\alpha \in U_\alpha$. Even more, we always have $\varepsilon = 1$ and w acts trivially on all long root groups.*

Proof. By Lemma 7.2.18, $\langle\alpha|\delta\rangle$ is an even number, so $\varepsilon = 1$. If α and δ lie in a common B_2-subsystem, then w acts trivially on U_α by Lemma 7.6.9. Otherwise they are orthogonal, and thus α is adjacent to δ and $-\delta$ by Lemma 7.2.11. It follows that w acts trivially on U_α in this case, too. □

7.6.D The Action of Short Weyl Elements on Short Root Groups

In this section, the rank-3 assumption is used implicitly when we refer to results from subsection 7.6.C.

We already know that every long Weyl element in G is balanced by the corresponding result for A_2-graded groups (see Proposition 5.4.10 (a)). For short Weyl elements, this is not yet clear and will only follow from a combination of Proposition 7.7.5 and Theorem 7.11.21. However, the following result is a first step into this direction.

7.6.11 Lemma. *Let $(\alpha, \beta, \gamma, \delta)$ be a B_2-quadruple and let $(a_{-\beta}, b_\beta, c_{-\beta})$ be a β-Weyl triple. Then $[x_\delta, a_{-\beta}] = [x_\delta, c_{-\beta}]$ for all $x_\delta \in U_\delta$.*

Proof. By Proposition 2.2.6 (a), $(c_{-\beta}^{-1}, b_\beta^{-1}, a_{-\beta}^{-1})$ is also a β-Weyl triple. Thus it follows from Lemma 7.5.12 (ii) that

$$\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma = [x_\delta, b_\beta]^{-1} = [b_\beta, x_\delta] \quad \text{and} \quad \big[[x_\delta, c_{-\beta}^{-1}], b_\beta^{-1}\big]_\gamma = [b_\beta^{-1}, x_\delta].$$

Note that by Lemma 7.5.3 (g) and Lemma 2.1.13, respectively, we have

$$\big[[x_\delta, c_{-\beta}^{-1}], b_\beta^{-1}\big]_\gamma = \big[[x_\delta, c_{-\beta}], b_\beta^{-1}\big]_\gamma^{-1} \quad \text{and} \quad [b_\beta^{-1}, x_\delta] = [b_\beta, x_\delta]^{-1}.$$

Further, $[[x_\delta, c_{-\beta}], b_\beta^{-1}]_\gamma = [[x_\delta, c_{-\beta}], b_\beta]_\gamma$ by Lemma 7.6.7. Altogether, we conclude that

$$\big[[x_\delta, a_{-\beta}], b_\beta\big]_\gamma = [b_\beta, x_\delta] = [b_\beta^{-1}, x_\delta]^{-1} = \big[[x_\delta, c_{-\beta}^{-1}], b_\beta^{-1}\big]_\gamma^{-1} = [[x_\delta, c_{-\beta}], b_\beta]_\gamma.$$

Since the map $U_{-\alpha} \to U_\gamma, x_{-\alpha} \mapsto [x_{-\alpha}, b_\beta]_\gamma$ is an isomorphism by Proposition 7.4.14, we infer that $[x_\delta, a_{-\beta}] = [x_\delta, c_{-\beta}]$, as desired. □

7.6.12 Lemma. *Let $(\alpha, \beta, \gamma, \delta)$ be a B_2-quadruple and let w_β be a β-Weyl element. Then w_β^2 acts trivially on U_δ.*

Proof. Choose a β-Weyl triple $(a_{-\beta}, b_\beta, c_{-\beta})$ whose corresponding Weyl element is w_β. By Proposition 2.2.6 (a), $(c_{-\beta}^{-1}, b_\beta^{-1}, a_{-\beta}^{-1})$ is a $(-\beta)$-Weyl triple with corresponding Weyl element w_β^{-1}. Applying Lemma 7.5.13 to these two Weyl triples, we see that

$$x_\delta^{w_\beta} = x_\delta\big[[x_\delta, a_{-\beta}], b_\beta\big]_\delta \quad \text{and} \quad x_\delta^{w_\beta^{-1}} = x_\delta\big[[x_\delta, c_{-\beta}^{-1}], b_\beta^{-1}\big]_\delta.$$

Using Lemma 2.1.13 and Lemma 7.5.3 (a), we can simplify the last term as follows:

$$x_\delta^{w_\beta^{-1}} = x_\delta\big[[x_\delta, c_{-\beta}], b_\beta\big]_\delta.$$

Since $[x_\delta, a_{-\beta}] = [x_\delta, c_{-\beta}]$ by Lemma 7.6.11, we conclude that w_β and w_β^{-1} act identically on U_δ. In other words, w_β^2 acts trivially on U_δ, which finishes the proof. □

7.6.13 Lemma. *Let β be a short root and let w_β be a β-Weyl elements. Then w_β^2 acts trivially on U_β and $U_{-\beta}$.*

Proof. Choose roots α, γ, δ such that $(\alpha, \beta, \gamma, \delta)$ is a B_2-quadruple. Further, choose any α-Weyl element w_α. By the definition of Weyl elements, we have $U_\beta = U_\delta^{w_\alpha}$, so U_β is contained in the group generated by $U_\delta \cup U_\alpha \cup U_{-\alpha}$. Since w_β^2 acts trivially on U_δ by Lemma 7.6.12 and trivially on U_α and $U_{-\alpha}$ by Proposition 7.6.10, it follows that w_β^2 acts trivially on U_β as well. Since w_β is also a $(-\beta)$-Weyl element by Proposition 2.2.6 (c), this implies that w_β^2 acts trivially on $U_{-\beta}$ as well. □

7.6.14 Proposition. *Let β, δ be two short roots and let w_β be a β-Weyl element. Put $\varepsilon := (-1)^{\langle \delta | \beta \rangle}$ and $w := w_\beta^2$. Then $x_\delta^w = x_\delta^\varepsilon$ for all $x_\delta \in U_\delta$. Even more, we always have $\varepsilon = 1$ and w acts trivially on all short root groups.*

Proof. By Lemma 7.2.17, $\langle \delta | \beta \rangle$ is always an even number, so $\varepsilon = 1$. It is clear that β and δ lie in a common B_2-subsystem. Thus w acts trivially on U_δ by Lemmas 7.6.12 and 7.6.13, which finishes the proof. □

7.6.E Summary

We can summarise the results of the previous subsections as follows.

7.6.15 Proposition. *Let ξ, ζ be any two roots in B_n and let w_ζ be a ζ-Weyl element. Put $w := w_\zeta^2$ and $\varepsilon := (-1)^{\langle \xi | \zeta \rangle}$. Then $x_\xi^w = x_\xi^\varepsilon$ for all $x_\xi \in U_\xi$. In other words, G satisfies the square formula for Weyl elements (see Definition 4.4.4).*

Proof. This is simply a summary of Propositions 7.6.4, 7.6.5, 7.6.10 and 7.6.14. □

The following result is simply a special case of Proposition 7.6.15, but it deserves to be pointed out.

7.6.16 Proposition. *Let β be a short root and let w_β be a β-Weyl element. Then w_β^2 lies in the center of G.*

Proof. By Propositions 7.6.10 and 7.6.14, w_β^2 acts trivially on all root groups. Since these generate G, it follows that w_β^2 acts trivially on G. In other words, it lies in the center of G. □

7.6.F Final Observations

7.6.17 Remark (Braid relations). We already know from Theorem 2.5.10 that G satisfies the braid relations for Weyl elements, but as in Remark 5.4.11, we can give an alternative proof of this fact using the results in this section. Let Δ be a root base of B_n and denote by (α, δ) the unique B_2-pair which appears in Δ. (Note that since Δ was chosen arbitrarily, this can be any B_2-pair in B_n.) Choose roots β, γ such that $(\alpha, \beta, \gamma, \delta)$ is a B_2-quadruple. We have to show that $w_\alpha w_\delta w_\alpha w_\delta = w_\delta w_\alpha w_\delta w_\alpha$. Since

$$w_\alpha w_\delta w_\alpha w_\delta = w_\delta w_\alpha w_\delta w_\alpha^{w_\delta w_\alpha w_\delta},$$

we only have to show that $w_\alpha = w_\alpha^{w_\delta w_\alpha w_\delta}$. Put $w := w_\delta w_\alpha w_\delta$, and note that $w = w_\delta^2 w_\alpha^{w_\delta}$. Since $w_\alpha^{w_\delta}$ is a γ-Weyl element by Proposition 2.2.6 (b), it lies in $\langle U_\gamma \cup U_{-\gamma} \rangle$ and thus it commutes with w_α. Further, w_δ^2 even lies in the center of G by Proposition 7.6.16. We conclude that $w_\alpha^w = w_\alpha$, which finishes the proof.

7.6.18 Note (on 7.6.17). We emphasise the ever-present assumption in this section that $n \geq 3$. While it is true by Theorem 2.5.10 that any B_2-graded group satisfies the braid relations for Weyl elements (even without crystallographic assumptions), the proof in Remark 7.6.17 only works for crystallographic B_2-graded groups which occur as root graded subgroups of a B_3-graded group. For general B_2-graded groups, it is not necessarily true that w_δ^2 acts trivially on U_α. We will see a counterexample in Note 9.3.30.

Before we end this section, we make a quick detour to prove Proposition 7.6.21. It will be needed in the proof of stabiliser-compatibility for B_n-graded groups (Proposition 7.9.3).

7.6.19 Lemma. *Let $(\alpha, \beta, \gamma, \delta)$ be a B_2-quadruple and choose an α-Weyl element w_α and a δ-Weyl element w_δ. Let ρ be any short root which does not lie in $\{ \pm\beta, \pm\delta \}$. Then the Weyl elements w_δ and $w_\delta^{w_\alpha}$ act identically on U_ρ.*

Proof. The only short roots which lie in a common B_2-subsystem with α are those in $\{ \pm\beta, \pm\delta \}$, so U_ρ commutes with U_α and $U_{-\alpha}$. The assertion follows. $\square$

7.6.20 Lemma. *Let α, β, γ be roots which form a root base for a subsystem of B_n of type B_3 such that (α, β) is an A_2-pair and (β, γ) is a B_2-pair. Choose corresponding Weyl elements w_α, w_β, w_γ. Then the actions of w_γ and $w_\gamma^{w_\beta w_\alpha}$ on U_β are identical.*

Proof. Without loss of generality, we can assume that the B_3-subsystem is in standard form (as in Remark 7.2.2). Thus we assume that there exists an orthonormal basis (e_1, e_2, e_3) of $\langle \alpha, \beta, \gamma \rangle_{\mathbb{R}}$ which satisfies

$$\alpha = e_1 - e_2, \qquad \beta = e_2 - e_3 \quad \text{and} \quad \gamma = e_3.$$

Further, we put

$$w_{12} := w_\alpha, \quad w_{21} := w_{12}^{-1}, \quad w_{23} := w_\beta, \quad w_{32} := w_{23}^{-1}, \quad w_3 := w_\gamma.$$

For an arbitrary $x_2 \in U_{e_2}$, we have to show that

$$x_2^{w_3} = x_2^{w_{21} w_{32} w_3 w_{23} w_{12}}.$$

Since $x_2^{w_{21}}$ lies in U_{e_1}, this element commutes with w_{32}, so

$$x_2^{w_{21} w_{32} w_3 w_{23} w_{12}} = x_2^{w_{21} w_3 w_{23} w_{12}}.$$

Similarly, $x_2^{w_{21} w_3}$ lies in U_{e_1} and thus commutes with w_{23}, so

$$x_2^{w_{21} w_3 w_{23} w_{12}} = x_2^{w_{21} w_3 w_{12}}.$$

Finally, w_3 commutes with w_{12}, so $w_{21} w_3 w_{12} = w_{21} w_{12} w_3 = w_3$. This finishes the proof. $\square$

7.6.21 Proposition. *Write B_n in standard representation and denote by Δ the standard root base of B_n. Let $(w_\delta)_{\delta \in \Delta}$ be a Δ-system of Weyl elements in G and let $(w_\alpha)_{\alpha \in \hat{B}_n}$ be its standard $\hat{B}_n$-extension (as in Definition 7.2.23). Then for all $i, j, k \in [1, n]$ such that $k \notin \{ i, j \}$, the actions of w_{e_i} and w_{e_j} on U_{e_k} and U_{-e_k} are identical.*

Proof. The assertion is trivial if $i = j$, so we can assume that i, j, k are pairwise distinct. For all $i \in [1, n-1]$, we put $w_{i,i+1} := w_{e_i - e_{i+1}}$ and $w_{i+1,i} := w_{i,i+1}^{-1}$. Further, we put $w_i := w_{e_i}$ for all $i \in [1, n]$. We begin by showing that for all $i \in [1, n]$ and all $k \in [1, n-1] \setminus \{i\}$, the actions of w_i and w_n on U_{e_k} and U_{-e_k} are identical. We prove this by induction on i, the case $i = n$ being trivial. Thus assume that $i < n$. If $k \neq i+1$, then e_k and $-e_k$ are not contained in $\{ \pm e_i, \pm e_{i+1} \}$, so it follows from

Lemma 7.6.19 that $w_i = w_{i+1}^{w_{i+1,i}}$ and w_{i+1} act identically on U_{e_k} and U_{-e_k}. By the induction hypothesis, it follows that w_i and w_n act identically on U_{e_k} and U_{-e_k}. Now assume that $k = i+1$. Then $i+2 = k+1 \leq n$, so

$$w_i = w_{i+2}^{w_{i+2,i+1}w_{i+1,i}}.$$

Thus it follows from Lemma 7.6.20 that w_i and w_{i+2} act identically on $U_{e_{i+1}}$ and $U_{-e_{i+1}}$. Again by the induction hypothesis, we infer that w_i and w_n act identically on U_{e_k} and U_{-e_k}.

Now let $i, j, k \in [1, n]$ be pairwise distinct. If $k \neq n$, then it follows from the conclusion of the previous paragraph that w_i and w_j act on U_{e_k} and U_{-e_k} in the same way as w_n, so in particular, w_i and w_j act identically. Now assume that $k = n$. Without loss of generality, assume that $i < j$. Applying Lemma 7.6.19 inductively, we can easily see that all the Weyl elements

$$w_j, \quad w_j^{w_{j,j-1}}, \quad w_j^{w_{j,j-1}w_{j-1,j-2}}, \quad \dots, \quad w_j^{w_{j,j-1}\cdots w_{i+1,i}}$$

act identically on U_{e_n} and U_{-e_n}. Since

$$w_i = w_j^{w_{j,j-1}\cdots w_{i+1,i}},$$

the assertion follows. □

7.7 Standard Signs

7.7.1 Notation for this section. We fix an integer $n \geq 3$ and consider the root system B_n in its standard representation (as in Remark 7.2.2). We denote by G a group with a B_n-pregrading $(U_\alpha)_{\alpha \in B_n}$, by $\Bbbk$ a commutative associative ring and by (M, q) a quadratic module over $\Bbbk$. We assume that there exists a coordinatisation $(\theta_\alpha)_{\alpha \in B_n}$ of G by (M, q) with standard signs (in the sense of the following Definition 7.7.2), and we fix this coordinatisation.

7.7.2 Definition (Coordinatisation with standard signs). Let G be a group with a B_n-pregrading $(U_\alpha)_{\alpha \in B_n}$, let $\Bbbk$ be a commutative associate ring and let (M, q) be a quadratic module over $\Bbbk$. A *coordinatisation of G by (M, q) with standard signs* is a family $(\theta_\alpha)_{\alpha \in B_n}$ with the following properties:

(i) For all roots α, the map θ_α is an isomorphism from $(\Bbbk, +)$ to U_α if α is long and it is an isomorphism from $(M, +)$ to U_α if α is short.

(ii) The same commutator relations as in Proposition 7.3.25 are satisfied. We will refer to them as the *standard commutator relations*.

We will show in Theorem 7.11.21 that every B_n-graded group is coordinatised by some quadratic module with standard signs.

In the remaining part of this section, we investigate some properties of groups with a standard coordinatisation. We begin by showing that all Weyl elements in such groups are balanced, which by Theorem 7.11.21 shows that Weyl elements in all B_n-graded groups are balanced.

7.7.3 Lemma. *Let δ be a short root and let w_δ be a δ-Weyl element in G. If $v', v, v'' \in M$ are such that $w_\delta = \theta_{-\delta}(v')\theta_\delta(v)\theta_{-\delta}(v'')$, then $q(v)$ is invertible and $v' = v'' = -q(v)^{-1}v$. In particular, w_δ is weakly balanced.*

Proof. We prove this for the short root $\delta = e_2$. The assertion for all other short roots can be proven similarly. Let w_2 be a e_2-Weyl element and let $v', v, v'' \in$

M be such that $w_2 = \theta_{-e_2}(v')\theta_{e_2}(v)\theta_{-e_2}(v'')$. For all $a \in k$, we know from Lemma 7.5.10 (a), Lemma 7.5.11 (a) and the standard commutator relations that

$$\theta_{e_1-e_2}(a)^{\hat{w}_2} = [\theta_{e_1-e_2}(a), \theta_{e_2}(v)]_{e_1+e_2} = \theta_{e_1+e_2}\big(-aq(v)\big) \qquad \text{and}$$
$$\theta_{e_1+e_2}(a)^{\hat{w}_2} = [\theta_{e_1+e_2}(a), \theta_{-e_2}(v')]_{e_1-e_2} = \theta_{e_1-e_2}\big(-aq(v')\big).$$

This implies that $\theta_{e_1-e_2}(1_k)^{w_2^2} = (q(v)q(v'))$. On the other hand, we know that $\theta_{e_1-e_2}(1_k)^{w_2^2} = \theta_{e_1-e_2}(1_k)$ by Proposition 7.6.10, so it follows that $q(v)q(v') = 1_k$. In particular, $q(v)$ is invertible. Further, it follows from Lemma 7.5.10 (b) that

$$\begin{aligned} 1_G &= [\theta_{e_1-e_2}(1), \theta_{e_2}(v)]_{e_1}\big[[\theta_{e_1-e_2}(1), \theta_{e_2}(v)]_{e_1+e_2}, \theta_{-e_2}(v')\big]_{e_1} \\ &= \theta_{e_1}(v)\big[\theta_{e_1+e_2}\big(-q(v)\big), \theta_{-e_2}(v')\big]_{e_1} = \theta_{e_1}\big(v + q(v)v'\big). \end{aligned}$$

It follows that $v = -q(v)v'$. Since $q(v)$ is invertible, this implies that $v' = -q(v)^{-1}v$. The same argument can be applied to the Weyl element $w_2^{-1} = \theta_{-e_2}(-v'')\theta_{e_2}(-v)\theta_{-e_2}(-v')$, which then yields that $-v'' = -q(-v)^{-1}(-v)$. Thus we have $v'' = -q(v)^{-1}v = v'$ as well. □

7.7.4 Note. The element $v \in M$ in Lemma 7.7.3 need not be uniquely determined by w_δ: See Remark 7.3.14.

7.7.5 Proposition. *All Weyl elements in G are balanced.*

Proof. For long Weyl elements, we already know this from the corresponding assertion for A_2-graded groups, see Proposition 5.4.10 (a). For short Weyl elements, this follows from Lemma 7.7.3 and Proposition 2.2.16. □

7.7.6 Proposition. *The following hold:*

(a) *Let α be a long root. Define*

$$w_\alpha(r) := \theta_{-\alpha}(-r^{-1})\theta_\alpha(r)\theta_{-\alpha}(-r^{-1})$$

for all invertible $r \in k$. Then the maps

$$k^\times \to U_\alpha^\sharp, r \mapsto \theta_\alpha(r) \quad \text{and} \quad k^\times \to M_\alpha, r \mapsto w_\alpha(r)$$

are well-defined bijections. Here $k^\times$, $U_\alpha^\sharp$ and M_α denote the sets of invertible elements in k (in the sense of Definition 5.1.12), the set of α-invertible elements in U_α and the set of α-Weyl elements, respectively.

(b) *Let δ be a short root and put $M^\times := \{ u \in M \mid q(u) \in k^\times \}$. Define*

$$w_\delta(u) := \theta_{-\delta}\big(-q(u)^{-1}u\big)\theta_\delta(u)\theta_{-\delta}\big(-q(u)^{-1}u\big)$$

for all $u \in M^\times$. Then the maps

$$M^\times \to U_\delta^\sharp, u \mapsto \theta_\delta(u) \quad \text{and} \quad M^\times \to M_\delta, u \mapsto w_\delta(u)$$

are well-defined surjections, and the first one is a bijection.

(c) *The Weyl elements defined above satisfy the same conjugation formulas as in Lemmas 7.3.15 and 7.3.17.*

(d) *Let Δ be any root base of B_n and choose an element $v_0 \in M$ with $q(v_0) = 1_k$. Define A, B, η and μ as in Notation 7.3.18. Put $w_\alpha := w_\alpha(1_k)$ for all long roots in Δ and $w_\beta := w_\beta(v_0)$ for the unique short root in Δ. Then G is parametrised by $(A \times B, M, k)$ with respect to $\eta \times \mu$ and $(w_\delta)_{\delta \in \Delta}$.*

Proof. By Proposition 5.6.6 (a), we already know that the maps in (a) are bijective if they are well-defined. Using computations as in Lemmas 7.5.4 and 7.5.5, one can show that elements of the form $w_\alpha = w_\alpha(r)$ satisfy $U_\beta^{w_\alpha} = U_{\sigma_\alpha(\beta)}$ for all short roots β. This proves (a).

We know from Lemma 7.7.3 that the maps in (b) are surjective if they are well-defined. Further, the first map is injective because θ_δ is injective. Now let $u \in M^\times$ and put $w_\delta := w_\delta(u)$. Using the rank-2 computations and the commutator relations, one can show that $U_\rho^{w_\delta} = U_{\sigma_\delta(\rho)}$ for all $\rho \in B_n \setminus \{\pm\delta\}$. Even more, these computations show that the same conjugation formulas as in Lemmas 7.3.15 and 7.3.17 are satisfied. It remains to prove the assertion for $\rho \in \{\pm\delta\}$. We do this for $\delta = e_2 = \rho$ and the remaining cases can be covered in a similar way. Let $v \in M$ be arbitrary. By the standard commutator relations, we have

$$\begin{aligned}\theta_{e_2}(v) &= [\theta_{e_2-e_1}(1_\Bbbk), \theta_{e_1}(v)]_{e_2} = [\theta_{e_2-e_1}(1_\Bbbk), \theta_{e_1}(v)][\theta_{e_2-e_1}(1_\Bbbk), \theta_{e_1}(v)]_{e_1+e_2}^{-1} \\ &= [\theta_{e_2-e_1}(1_\Bbbk), \theta_{e_1}(v)]\theta_{e_1+e_2}\big(-q(v)\big).\end{aligned}$$

Using that the conjugation formulas in Lemma 7.3.15 are satisfied, we infer that

$$\begin{aligned}\theta_{e_2}(v)^{w_2(u)} &= [\theta_{e_2-e_1}(1_\Bbbk)^{w_2(u)}, \theta_{e_1}(v)^{w_2(u)}]\theta_{e_1+e_2}\big(-q(v)\big)^{w_2(u)} \\ &= \big[\theta_{-e_1-e_2}\big(-q(u)^{-1}\big), \theta_{e_1}\big(\sigma_u(v)\big)\big]\theta_{e_1-e_2}\big(q(u)^{-1}q(v)\big)\end{aligned}$$

where

$$\begin{aligned}\big[\theta_{-e_1-e_2}\big(-q(u)^{-1}\big), \theta_{e_1}\big(\sigma_u(v)\big)\big] &= \theta_{-e_2}\big(q(u)^{-1}\sigma_u(v)\big)\theta_{e_1-e_2}\big(-q(u)^{-1}q\big(\sigma_u(v)\big)\big) \\ &= \theta_{-e_2}\big(q(u)^{-1}\sigma_u(v)\big)\theta_{e_1-e_2}\big(-q(u)^{-1}q(v)\big).\end{aligned}$$

It follows that

$$\theta_{e_2}(v)^{w_2(u)} = \theta_{-e_1}\big(q(u)^{-1}\sigma_u(v)\big),$$

which is precisely the desired conjugation formula in Lemma 7.3.17 (b) for this choice of roots. This finishes the proof of (b).

Assertions (c) and (d) follow from similar computations as above, or have already been proven. □

7.7.7 Note. In the proof of Proposition 7.7.6 (b), it is not valid to argue that

$$\begin{aligned}\theta_{e_2}(v)^{w_2(u)} &= [\theta_{e_2-e_1}(1_\Bbbk), \theta_{e_1}(v)]_{e_2}^{w_2(u)} = [\theta_{e_2-e_1}(1_\Bbbk)^{w_2(u)}, \theta_{e_1}(v)^{w_2(u)}]_{-e_2} \\ &= \big[\theta_{-e_1-e_2}\big(-q(u)^{-1}\big), \theta_{e_1}\big(\sigma_u(v)\big)\big]_{-e_2} = \theta_{-e_2}\big(q(u)^{-1}\sigma_u(v)\big)\end{aligned}$$

because we would have to use Lemma 2.2.9 for the second equality. This would not be correct because $w_2(u)$ is not yet proven to be a Weyl element (and also because G is not assumed to be rank-2-injective). Instead, we have to use the workaround outlined above.

7.7.8 Remark. It follows from Proposition 7.7.6 that G satisfies $U_\alpha^\sharp = U_\alpha \setminus \{1_G\}$ (the additional condition of being an RGD-system) if and only if $\Bbbk$ is a field and (M, q) is anisotropic.

The following result is a special case of Proposition 7.7.6 (b). We prove it explicitly because it will be used in Lemma 7.11.20.

7.7.9 Lemma. *Let β be a short root, let w_β be a β-Weyl element and let $v \in M$ such that $w_\beta = \theta_{-\beta}\big(-q(v)^{-1}v\big)\theta_\beta(v)\theta_{-\beta}\big(-q(v)^{-1}v\big)$. Then for all short roots δ which are orthogonal to β, we have*

$$\theta_\delta(u)^{w_\beta} = \theta_\delta(\sigma_v(u))$$

for all $u \in M$ where $\sigma_v\colon M \to M, u \mapsto u - q(v)^{-1}f(v,u)v$ denotes the reflection associated to v (as in Definition 7.1.20).

Proof. Let $u \in M$. By Lemma 7.2.10, there exist unique long roots α, γ such that $(\alpha, \beta, \gamma, \delta)$ is a B_2-quadruple. Then by Lemma 7.5.14 (a), we have

$$\theta_\delta(u)^{w_\beta} = \theta_\delta(u)\big[\big[\theta_\delta(u), \theta_{-\beta}\big(-q(v)^{-1}v\big)\big], \theta_\beta(v)\big]_\delta.$$

Choose $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ and $i, j \in [1, n]$ such that $\delta = \varepsilon_1 e_i$ and $\beta = \varepsilon_2 e_j$. Using the standard commutator relations, one can show that

$$\big[\big[\theta_\delta(u), \theta_{-\beta}\big(-q(v)^{-1}v\big)\big], \theta_\beta(v)\big]_\delta = \theta_\delta\big(-q(v)^{-1}f(u,v)v\big)$$

in every possible case. See Figure 7.6 for a detailed computation. We conclude that

$$\theta_\delta(u)^{w_\beta} = \theta_\delta\big(u - q(v)^{-1}f(u,v)v\big) = \theta_\delta\big(\sigma_v(u)\big),$$

as desired. □

ε_1	ε_2	$x := \big[\theta_{\varepsilon_1 e_i}(u), \theta_{-\varepsilon_2 e_j}\big(-q(v)^{-1}v\big)\big]$	$\big[x, \theta_{\varepsilon_2 e_j}(v)\big]_{\varepsilon_1 e_i}$
$-$	$+$	$\theta_{-e_i-e_j}\big(-\delta^-_{i>j}q(v)^{-1}f(u,v)\big)$	$\theta_{-e_i}\big(-\delta^-_{i>j}\delta^-_{i>j}q(v)^{-1}f(u,v)v\big)$
$+$	$+$	$\theta_{e_i-e_j}\big(-q(v)^{-1}f(u,v)\big)$	$\theta_{e_i}\big(-q(v)^{-1}f(u,v)v\big)$
$+$	$-$	$\theta_{e_i+e_j}\big(-\delta^-_{i<j}q(v)^{-1}f(u,v)\big)$	$\theta_{e_i}\big(-\delta^-_{i<j}\delta^-_{i<j}q(v)^{-1}f(u,v)v\big)$
$-$	$-$	$\theta_{e_j-e_i}\big(q(v)^{-1}f(v,u)\big)$	$\theta_{-e_i}\big(-q(v)^{-1}f(v,u)v\big)$

Figure 7.6: The computation in Lemma 7.7.9.

7.8 Admissible and Standard Partial Twisting Systems

7.8.1 Notation for this section. We fix an integer $n \geq 3$ and consider the root system B_n in its standard representation (as in Remark 7.2.2) with its standard root base Δ. We choose $\Bbbk := \mathbb{C}$, $M := \mathbb{C}^2$, $q\colon M \to \mathbb{C}$ and $v_0 := (1, 0)$ as in Remark 7.3.22. We define the B_n-graded group $\mathrm{EO}(q)$ with root groups $(U_\alpha)_{\alpha \in B_n}$ as in Definition 7.3.9. We denote the root isomorphisms from Constructions 7.3.6 and 7.3.7 by $(\theta_\alpha)_{\alpha \in B_n}$ and the standard system of Weyl elements from Definition 7.3.13 by $(w_\delta)_{\delta \in \Delta}$. Further, we denote by (A, η, B, μ) the standard partial twisting system of type B_n in the sense of the following Definition 7.8.2.

In this section, we introduce the standard partial twisting system for any B_n-graded group. Its definition is motivated by the twisting structure of the elementary orthogonal group in section 7.3. We will show that it has some desirable properties which serve as the axioms of admissible partial twisting systems. The goal of the following section 7.9 is that any admissible partial twisting system satisfies the conditions of the parametrisation theorem. In the final sections of this chapter, we will then use the standard partial twisting system to construct a parametrisation of any B_n-graded group with standard signs.

7.8.2 Definition (Standard partial twisting system). The *standard partial twisting system of type B_n (with respect to Δ)* is the tuple (A, η, B, μ) where $A := B := \{\pm 1\}$ and where η, μ are the Δ-parity maps from Notation 7.3.18. If G is a group with a B_n-pregrading, then the *standard partial twisting system for G (with respect to Δ)* is the same tuple together with the additional information that A acts on all root groups of G by inversion.

7.8.3 Note. Technically, the standard partial twisting system of type B_n is not a partial twisting system because a partial twisting system has to be defined with respect to a pair $(G, (w_\delta)_{\delta\in\Delta})$ where G is a group with a B_n-pregrading and $(w_\delta)_{\delta\in\Delta}$ is a Δ-system of Weyl elements. In particular, the group A has to be equipped with actions on the root groups of G. For this reason, we distinguish between the "abstract" standard partial twisting system of type B_n and the standard partial twisting system for a specific group G. Note further that it is not obvious that the standard partial twisting system for a B_n-graded group satisfies all the axioms of a partial twisting system in Definition 4.3.18. We will verify this throughout this section.

7.8.4 Remark. We can also regard the groups A and B in the standard partial twisting system (A, η, B, μ) as part of the standard parameter system $(A \times B, M, \hbar)$ from Definition 7.1.25. Note that in the former interpretation, A acts on the root groups and B is equipped with no action while in the latter interpretation, $A \times B$ acts on M and $\hbar$.

7.8.5 Reminder. Recall from Theorem 7.3.27 that $(U_\alpha)_{\alpha\in B_n}$ is a crystallographic B_n-grading of $\mathrm{EO}(q)$. Thus we can apply all rank-2 and rank-3 computations from the previous sections to $\mathrm{EO}(q)$. Further, we know from Lemma 7.3.23 that $\mathrm{EO}(q)$ is parametrised by the standard parameter system $(A \times B, M, \hbar)$ with respect to $\eta \times \mu$ and $(w_\delta)_{\delta\in\Delta}$ and from Remark 7.3.22 that $(A \times B, M, \hbar)$ is $(\eta \times \mu)$-faithful.

We now prove some basic properties of the parity maps η and μ. Since all values of η and μ are given explicitly in Figure 7.4, all these properties could be proven with a straightforward but lengthy computation. However, we will see that they are much easier to derive by performing certain computations in the group $\mathrm{EO}(q)$.

7.8.6 Lemma. *η is braid-invariant and adjacency-trivial and μ is Weyl-invariant and adjacency-trivial.*

Proof. By Theorem 2.5.10, the system $(w_\delta)_{\delta\in\Delta}$ satisfies the braid relations. Thus it follows from Proposition 4.4.11 that $\eta \times \mu$ is braid-invariant and adjacency-trivial, which by Lemma 4.2.25 (a) implies that η and μ have the same properties. Further, an inspection of Figure 7.4 shows that μ is even square-invariant. Hence μ is Weyl-invariant, which finishes the proof. □

7.8.7 Lemma. *μ is semi-complete.*

Proof. Note that the only subgroups of B are $\{1\}$ and B, both of which have a complement. Thus every parity map with values in B is semi-complete, which makes the assertion trivial. □

7.8.8 Lemma. *$\eta \times \mu$ is transporter-invariant and η, μ are independent.*

Proof. At first, we consider the orbit of short roots. Put $\hat{\alpha} := e_1$. Then

$$\begin{aligned}(\eta \times \mu)_{\hat{\alpha},(e_1-e_2,e_1-e_2)} &= (\eta \times \mu)_{e_1,e_1-e_2}(\eta \times \mu)_{e_2,e_1-e_2} = (1_A, 1_A)(-1_A, 1_B)\\ &= (-1_A, 1_B) \qquad \text{and}\\ (\eta \times \mu)_{\hat{\alpha},e_2} &= (1_A, -1_B).\end{aligned}$$

Since these elements generate $A \times B$, it follows that $(A \times B)_{\hat{\alpha}\to\hat{\alpha}} = A \times B$. By Remark 4.2.17, this implies that $\eta \times \mu$ is transporter-invariant on the orbit of short roots.

Now we consider the orbit of long roots. By an inspection of Figure 7.4, it is clear that

$$(A \times B)_{\alpha \to \beta} \subseteq A \times \{1_B\}$$

for all long roots α, β. Further, we have

$$(A \times B)_{e_1 - e_2 \to e_2 - e_1} = A \times \{1_B\}$$

because $(\eta \times \mu)_{e_1 - e_2, e_1 - e_2} = (-1_A, 1_B)$. By criterion 4.2.16 (iv), we infer that $\eta \times \mu$ is transporter-invariant on the orbit of long roots as well.

Finally, the previous computations together with Remark 4.2.24 show that η and μ are independent in the sense of Definition 4.2.23. □

7.8.9 Lemma. *For all roots $\alpha \in \Phi$ and $\delta \in \Delta$, we have $\eta_{\alpha, \delta\delta} = (-1_A)^{\langle \alpha | \delta \rangle}$ where $\langle \alpha | \delta \rangle$ denotes the Cartan integer for (α, δ). In other words, η satisfies the square formula (in the sense of Definition 4.2.10 (e)).*

Proof. It suffices to verify that the conditions of Proposition 4.4.11 (c) are satisfied. We have proven in Proposition 7.6.15 that G satisfies the square formula for Weyl elements. Further, by definition, $A = \{\pm 1\}$ and -1_A acts on both κ and M by inversion. Finally, by the transporter-invariance of η and an inspection of Figure 7.4, we have $A_{\alpha \to \alpha} = A$ for all roots α. This finishes the proof. □

7.8.10 Lemma. *Let $i, j \in [1, n]$ and let $k \in [1, n] \setminus \{i, j\}$. Then the words $\bar{\rho}^i$, $\bar{\rho}^j$ from Definition 7.2.22 satisfy $(\eta \times \mu)_{\varepsilon e_k, \bar{\rho}^i} = (\eta \times \mu)_{\varepsilon e_k, \bar{\rho}^j}$ for all $\varepsilon \in \{\pm 1\}$.*

Proof. We know from Proposition 7.6.21 that $w_{\bar{\rho}^i}$ and $w_{\bar{\rho}^j}$ act identically on $U_{\varepsilon e_k}$. Since the action of $(A \times B, M, \kappa)$ is $(\eta \times \mu)$-faithful by Remark 7.3.22, the assertion follows. □

7.8.11 Lemma. *If ρ, ζ are short roots in B_n, then $\mu_{\rho, \sigma(\zeta)} = -1_B$.*

Proof. For the simple root $\zeta = e_n$, we can immediately read this off from Figure 7.4. Now let $\zeta = e_m$ for some $m \in [1, n-1]$ and put $\delta_i := e_i - e_{i+1}$ for all $i \in [1, n-1]$. Then $\zeta = e_n^{\sigma(\delta_{n-1} \cdots \delta_m)}$, so

$$\begin{aligned} \mu_{\rho, \sigma(\pm\zeta)} &= \mu_{\rho, \delta_m \cdots \delta_{n-1} e_n \delta_{n-1} \cdots \delta_m} = \mu_{\rho, \delta_m \cdots \delta_{n-1}} \mu_{\rho^{\sigma(\delta_m \cdots \delta_{n-1})}, e_n} \mu_{\rho^{\sigma(\delta_m \cdots \delta_{n-1} e_n)}, \delta_{n-1} \cdots \delta_m} \\ &= 1_B (-1_B) 1_B = -1_B. \end{aligned}$$

The assertion follows. □

7.8.12 Note. Lemma 7.8.11 also follows from Lemma 7.3.17 (b) and 7.3.17 (c) because the parameter system is $(\eta \times \mu)$-faithful.

7.8.13 Definition (Admissible partial twisting system). Let G be a group with a B_n-pregrading $(U_\alpha)_{\alpha \in B_n}$ and let $(w'_\delta)_{\delta \in \Delta}$ be a Δ-system of Weyl elements in G. A *B_n-admissible partial twisting system for $(G, (w'_\delta)_{\delta \in \Delta})$* is a partial twisting system (A', η', B', μ') for $(G, (w'_\delta)_{\delta \in \Delta})$ with the following additional properties:

(i) $A' = \{\pm 1\}$ and $-1_{A'}$ acts on all root groups of G by inversion.

(ii) $B' = \{\pm 1\}$.

(iii) η' and μ' are adjacency-trivial.

(iv) η' satisfies the square formula.

(v) For all $i, j \in [1, n]$ and $k \in [1, n] \setminus \{i, j\}$, the words $\bar{\rho}^i$, $\bar{\rho}^j$ from Definition 7.2.22 satisfy $\eta'_{\varepsilon e_k, \bar{\rho}^i} = \eta'_{\varepsilon e_k, \bar{\rho}^j}$ for all $\varepsilon \in \{\pm 1\}$.

(vi) If ρ, ζ are orthogonal short roots in B_n, then $\mu'_{\rho,\sigma(\zeta)} = -1_{B'}$.

We will sometimes refer to such objects as *admissible partial twisting systems* if the root system B_n is clear from the context.

7.8.14 Proposition. *Let G be a group with a B_n-pregrading $(U_\alpha)_{\alpha\in B_n}$. Then for any Δ-system $(w_\delta)_{\delta\in\Delta}$ of Weyl elements in G, the standard partial twisting system (A, η, B, μ) is a B_n-admissible partial twisting system for $(G, (w_\delta)_{\delta\in\Delta})$. In particular, admissible partial twisting systems exist for each group with a B_n-pregrading.*

Proof. We have to check that (A, η, B, μ) satisfies the axioms of a partial twisting system for $(G, (w_\delta)_{\delta\in\Delta})$ (see Definition 4.3.18) and the axioms in Definition 7.8.13. By Example 4.3.15, A is a twisting group for $(G, (w_\delta)_{\delta\in\Delta})$. All the remaining axioms follow from the results in this section. □

7.8.15 Note. We have proven in Lemma 7.8.11 that $\mu_{\rho,\sigma(\zeta)} = -1_B$ holds for any pair of short roots. In Axiom 7.8.13 (vi), however, we only require that this holds for orthogonal pairs (ρ, ζ). In other words, we do not require that $\mu_{\rho,\sigma(\rho)} = -1_B$. In fact, we could for any $i \in [1, n]$ replace the homomorphism θ_{-e_i} in Construction 7.3.7 by $\theta'_{-e_i} \colon v \mapsto \theta_{-e_i}(\sigma_{v_0}(v))$. Then the corresponding Weyl elements would have to be redefined as

$$w_i(u) := w_{e_i}(u) := \theta_{-e_i}\big(-q(u)^{-1}\sigma_{v_0}(u)\big) \circ \theta_{e_i}(u) \circ \theta_{-e_i}\big(-q(u)^{-1}\sigma_{v_0}(u)\big)$$

for all $u \in M$ for which $q(u)$ is invertible. Further, the resulting parity map μ would then satisfy $\mu_{\pm e_i,\sigma(e_i)} = 1_B$, but we would still have $\mu_{\pm e_i,\sigma(e_j)} = -1_B$ for all $j \in [1, n] \setminus \{i\}$. This illustrates that property 7.8.13 (vi) is intrinsic to the group $\mathrm{EO}(q)$ (that is, invariant under a twisting of the parametrisation) while the value of $\mu_{\rho,\sigma(\rho)}$ can be different for distinct parametrisations of $\mathrm{EO}(q)$.

7.9 The Parametrisation

7.9.1 Notation for this section. We fix an integer $n \geq 3$ and consider the root system B_n in its standard representation (as in Remark 7.2.2) with its standard root base Δ. We denote by G a group which has crystallographic B_n-commutator relations with root groups $(U_\alpha)_{\alpha\in B_n}$ such that $U_\alpha^\sharp$ is non-empty for all $\alpha \in \Phi$ and such that G is rank-2-injective. We fix a Δ-system $(w_\delta)_{\delta\in\Delta}$ of Weyl elements and we denote by (A, η, B, μ) a B_n-admissible partial twisting system for $(G, (w_\delta)_{\delta\in\Delta})$.

In this section, we show that G satisfies the conditions in the parametrisation theorem with respect to any B_n-admissible partial twisting system. In particular, this holds for the standard partial twisting system, which will be the only case that we are interested in. After the work in the previous sections, all that is truly left to do is the verification of stabiliser-compatibility. For this we use the criterion from Proposition 4.6.3.

7.9.2 Proposition. *G is square-compatible with respect to η.*

Proof. Since both G and η satisfy the square formula (by Proposition 7.6.15 and Axiom 7.8.13 (iv)), this follows from Lemma 4.6.2. Here we have to use Axiom 9.7.11 (ii). □

7.9.3 Proposition. *G is stabiliser-compatible with respect to (η, μ) and $(w_\delta)_{\delta\in\Delta}$.*

Proof. Let α be any root. We know from Axiom 7.8.13 (iii) that η is α-adjacency-trivial and we want to show that G is α-stabiliser-compatible. If α is long, then it has the property that all roots which are orthogonal to α are adjacent to α, and thus it follows from Lemma 4.6.4 that G is α-stabiliser-compatible with respect to η. Thus it is also α-stabiliser-compatible with respect to (η, μ) by Remark 4.4.6, as desired.

Now assume that α is short. Write $\alpha = \varepsilon e_k$ for some $\varepsilon \in \{\pm 1\}$ and $k \in [1, n]$. As in Proposition 4.6.3, we consider the following sets:

$$\begin{aligned}
\mathcal{O} &:= \{\, \beta \in \Phi \mid \alpha \cdot \beta = 0 \,\} = \{\, \sigma e_i \mid i \in [1,n] \setminus \{k\}, \sigma \in \{\pm 1\} \,\},\\
\mathcal{A} &:= \{\, \beta \in \mathcal{O} \mid \alpha \text{ is crystallographically adjacent to } \beta \,\} = \varnothing,\\
\bar{\mathcal{A}} &:= \mathcal{O} \setminus \mathcal{A} = \mathcal{O}.
\end{aligned}$$

For all $\beta \in \bar{\mathcal{A}}$, we have $\mu_{\alpha,\sigma(\beta)} = -1_B$ by Axiom 7.8.13 (vi). Now let $\beta, \beta' \in \bar{\mathcal{A}}$. Let $i, j \in [1, n]$ and $\varepsilon, \varepsilon' \in \{\pm 1\}$ such that $\beta = \varepsilon e_i$ and $\beta' = \varepsilon' e_j$. Let $\bar{\delta} := \bar{\rho}^i$ and $\bar{\delta}' := \bar{\rho}^j$ be the words from Definition 7.2.22. By definition, they satisfy $\sigma(\bar{\delta}) = \sigma(e_i) = \sigma(\beta)$ and $\sigma(\bar{\delta}') = \sigma(e_j) = \sigma(\beta')$. Further, we have $\eta_{\alpha,\bar{\delta}} = \eta_{\alpha,\bar{\delta}'}$ by Axiom 7.8.13 (v) and the actions of $w_{\bar{\delta}}$ and $w_{\bar{\delta}'}$ are identical by Proposition 7.6.21. Thus all conditions in Proposition 4.6.3 are satisfied, and we conclude that G is α-stabiliser-compatible. This finishes the proof. □

7.9.4 Proposition. *There exist abelian groups $(k, +)$ and $(M, +)$ (each equipped, as a set, with an action of $A \times B$) and a parametrisation $(\theta_\alpha)_{\alpha \in B_n}$ of G by $(A \times B, M, k)$ with respect to $\eta \times \mu$ and $(w_\delta)_{\delta \in \Delta}$ such that the action of A on k and M is given by group inversion.*

Proof. This follows from the parametrisation theorem (Theorem 4.5.16), whose assumptions are satisfied by Propositions 7.9.2 and 7.9.3 and Theorem 2.5.10. □

7.10 Computation of the Blueprint Rewriting Rules

7.10.1 Notation for this section. We fix an integer $n \geq 3$ and consider the root system B_n in its standard representation (as in Remark 7.2.2) with its standard root base Δ. We denote by G a group with a crystallographic B_n-grading $(U_\alpha)_{\alpha \in B_n}$. We fix a Δ-system of Weyl elements $(w_\delta)_{\delta \in \Delta}$ and denote its standard $\hat{B}_n$-extension by $(w_\beta)_{\beta \in \hat{B}_n}$. To simplify notation, we put $w_{ij} := w_{e_i - e_j}$ for all distinct $i, j \in [1, n]$ and we put $w_i := w_{e_i}$ for all $i \in [1, n]$. We denote the standard partial twisting system for G by (A, η, B, μ), by $(k, +)$, $(M, +)$ any groups which satisfy the assertion of Proposition 7.9.4 and by $(\theta_\alpha)_{\alpha \in B_n}$ the corresponding parametrisation of G. The groups k and M are equipped with actions of $A \times B$, and we call the map

$$M \to M, v \mapsto \overline{v} := -1_B.v$$

the *involution on M*. Further, we choose elements $v_{-1}, v_0, v_1 \in M$ such that $w_n = \theta_{-e_n}(v_{-1})\theta_{e_n}(v_0)\theta_{-e_n}(v_1)$.

In this section, we define the commutation maps of the parametrisation $(\theta_\alpha)_{\alpha \in B_n}$ and derive their rank-2 identities. Our goal is to define the blueprint rewriting rules for B_n and to prove that they are indeed blueprint rewriting rules.

We will frequently apply the following result without reference. It says that the action of the $\hat{B}_n$-extension $(w_\beta)_{\beta \in \hat{B}_n}$ of $(w_\delta)_{\delta \in \Delta}$ on the root groups is determined by the values in Figure 7.4.

7.10.2 Proposition. *Let $\alpha \in B_n$ and let $\beta \in \hat{B}_n$. Let $x \in k$ if α is long and let $x \in M$ if α is short. Then $\theta_\alpha(x)^{\hat{w}_\beta} = \theta_{\sigma_\beta(\alpha)}(\eta_{\alpha,\beta}\mu_{\alpha,\beta}.x)$ where $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ are the values in Figure 7.4.*

Proof. Denote by $\bar{\beta}$ the standard Δ-expression of β from Definition 7.2.22. Since G is parametrised by $(A \times B, M, k)$ with respect to $\eta \times \mu$ and $(w_\delta)_{\delta\in\Delta}$, we have

$$\theta_\alpha(x)^{\hat{w}_\beta} = \theta_{\sigma_\beta(\alpha)}(\eta_{\alpha,\bar{\beta}}\mu_{\alpha,\bar{\beta}}.x).$$

We know from Lemma 7.3.28 that $\eta_{\alpha,\bar{\beta}}\mu_{\alpha,\bar{\beta}} = \eta_{\alpha,\beta}\mu_{\alpha,\beta}$, so the assertion follows. □

7.10.3 Lemma. *The involution $M \to M, v \mapsto \overline{v}$ is an automorphism of M with $\overline{\overline{v}} = v$ for all $v \in M$.*

Proof. Since the involution is induced by the action of -1_B and $(-1_B)^2 = 1_B$, we have $\overline{\overline{v}} = v$ for all $v \in M$. Further, since $(\eta \times \mu)_{e_1,e_n} = (1_A, -1_B)$, we have

$$\theta_{e_1}\big(\overline{v+u}\big) = \theta_{e_1}(v+u)^{w_n} = \theta_{e_1}(v)^{w_n}\theta_{e_1}(u)^{w_n} = \theta_{e_1}(\overline{v})\theta_{e_1}(\overline{u}) = \theta_{e_1}(\overline{v}+\overline{u})$$

for all $v, u \in M$, so $\overline{v+u} = \overline{v} + \overline{u}$. □

7.10.4 Lemma. *For all $i \in [1,n]$, we have $w_i = \theta_{-e_i}(v_{-1})\theta_{e_i}(v_0)\theta_{-e_i}(v_1)$ for all $i \in [1,n]$.*

Proof. By the choice of v_{-1}, v_0, v_1, this is clear for $i = n$. Since $w_i = w_n^{w_{ni}}$ and $(\eta \times \mu)_{\pm e_n, e_n - e_i} = (1,1)$, the general assertion follows from Proposition 7.10.2. □

7.10.5 Definition (Commutation maps). We define

$$\begin{aligned}
\cdot : k \times k &\to k, (a,b) \mapsto ab := a \cdot b,\\
g : k \times M &\to M, (a,v) \mapsto g(a,v),\\
q : k \times M &\to k, (a,v) \mapsto q(a,v),\\
f : M \times M &\to k, (u,v) \mapsto f(u,v)
\end{aligned}$$

to be the unique maps which satisfy

$$\begin{aligned}
[\theta_{e_1-e_2}(a), \theta_{e_2-e_3}(b)] &= \theta_{e_1-e_3}(a \cdot b),\\
[\theta_{e_1-e_2}(a), \theta_{e_2}(v)] &= \theta_{e_2}\big(g(a,v)\big)\theta_{e_1+e_2}\big(-q(a,v)\big),\\
[\theta_{e_1}(u), \theta_{e_2}(v)] &= \theta_{e_1+e_2}\big(-f(u,v)\big)
\end{aligned}$$

for all $a, b \in k$ and all $u, v \in M$.

Our goal is to show that $\cdot$ turns k into a commutative associative ring, that g defines a k-modules structure on M, that $M \to k, v \mapsto q(1_k, v)$ is a quadratic form with linearisation f and that $q(r,v) = rq(1_k, v)$ for all $r \in k$ and $v \in M$.

7.10.6 Lemma (Rank-2 identities, part 1). *The maps $\cdot$, g, q and f are additive in all components except for the second component of q, which satisfies*

$$q(a, v+w) = q(a,v) + q(a,w) + f\big(g(a,v), w\big)$$

for all $a \in k$ and all $v, w \in M$.

Proof. The additivity assertions follow from Lemma 2.1.13, as in the proof of Lemma 5.7.8. Now let $a \in k$ and let $v, w \in M$. By Lemma 7.5.3 (c) and the definition of the maps q, g and f, we have

$$\begin{aligned}
&\theta_{e_1+e_2}\big(-q(r, v+w)\big) \\
&\quad = [\theta_{e_1-e_2}(a), \theta_{e_2}(v)\theta_{e_2}(w)]_{e_1+e_2} \\
&\quad = [\theta_{e_1-e_2}(a), \theta_{e_2}(w)]_{e_1+e_2}\big[[\theta_{e_1-e_2}(a), \theta_{e_2}(v)]_{e_1}, \theta_{e_2}(w)\big][\theta_{e_1-e_2}(a), \theta_{e_2}(v)]_{e_1+e_2} \\
&\quad = \theta_{e_1+e_2}\big(-q(a,w)\big)\big[\theta_{e_1}\big(g(a,v)\big), \theta_{e_2}(w)\big]\theta_{e_1+e_2}\big(-q(a,v)\big) \\
&\quad = \theta_{e_1+e_2}\big(-q(a,w) - f\big(g(a,v), w\big) - q(a,v)\big).
\end{aligned}$$

By the bijectivity of $\theta_{e_1+e_2}$, the second assertion follows. □

7.10.7 Lemma. *For all pairwise distinct $i, j, k \in [1, n]$ and all $a, b \in \Bbbk$, the following commutator relations hold:*

$$\begin{aligned}
[\theta_{e_i-e_j}(a), \theta_{e_j-e_k}(b)] &= \theta_{e_i-e_k}(ab), \\
[\theta_{e_i-e_j}(a), \theta_{e_j+e_k}(b)] &= \theta_{e_i+e_k}(\delta^-_{k\in\langle i,j\rangle}ab), \\
[\theta_{e_i-e_j}(a), \theta_{-e_k-e_i}(b)] &= \theta_{-e_j-e_k}(\delta^-_{k\notin\langle i,j\rangle}ab), \\
[\theta_{e_i+e_j}(a), \theta_{-e_k-e_i}(b)] &= \theta_{e_j-e_k}(\delta^-_{i\in\langle j,k\rangle}ab).
\end{aligned}$$

Proof. Let $a, b \in \Bbbk$. We will frequently apply Proposition 7.10.2 without explicitly saying so. By definition, the first assertion holds if $(i, j, k) = (1, 2, 3)$. Thus it suffices to show that if $i, j, k \in [1, n]$ are distinct and σ is a permutation of $[1, n]$, then the validity of the first equation for (i, j, k) implies the validity of the first equation for $(i^\sigma, j^\sigma, k^\sigma)$. Since the group of permutations is generated by transpositions, it suffices to prove this for transpositions σ. Thus let $i, j, k \in [1, n]$ be distinct and such that

$$[\theta_{e_i-e_j}(a), \theta_{e_j-e_k}(b)] = \theta_{e_i-e_k}(a \cdot b) \quad \text{for all } a, b \in \Bbbk \tag{$*$}$$

and let σ be the transposition which interchanges the distinct numbers $p, q \in [1, n]$. If $\{ i, j, k \} \cap \{ p, q \} = \varnothing$, the assertion is clear. If $|\{ i, j, k \} \cap \{ p, q \}| = 1$, conjugating $(*)$ by w_{pq} (if $p \in \{ i, j, k \}$) or by w_{qp} (if $q \in \{ i, j, k \}$) shows that the assertion holds. Now assume that $|\{ i, j, k \} \cap \{ p, q \}| = 2$. If $\{ p, q \} = \{ i, j \}$, then conjugating $(*)$ by w_{ij} and applying Lemma 2.1.13 yields

$$\begin{aligned}
[\theta_{e_i-e_j}(a), \theta_{e_j-e_k}(b)]^{w_{ij}} &= [\theta_{e_i-e_j}(r)^{w_{ij}}, \theta_{e_j-e_k}(s)^{w_{ij}}] = [\theta_{e_j-e_i}(-r), \theta_{e_i-e_k}(-s)] \\
&= [\theta_{e_j-e_i}(r), \theta_{e_i-e_k}(s)] \qquad \text{and} \\
[\theta_{e_i-e_j}(a), \theta_{e_j-e_k}(b)]^{w_{ij}} &= \theta_{e_i-e_k}(r \cdot s)^{w_{ij}} = \theta_{e_j-e_k}(r \cdot s).
\end{aligned}$$

The assertion follows in this case. Similarly, conjugating $(*)$ by w_{jk} or by w_{ik} shows that the assertion holds for the transposition interchanging j and k or i and k, respectively. This finishes the proof of the first equation.

Now conjugating the first equation by w_k yields

$$[\theta_{e_i-e_j}(a), \theta_{e_j+e_k}(\delta^-_{k>j}b)] = \theta_{e_i+e_k}(\delta^-_{k>i}ab) \quad \text{for all } a, b \in \Bbbk,$$

which (by replacing b with $\delta^-_{k>j}b$) implies that

$$[\theta_{e_i-e_j}(a), \theta_{e_j+e_k}(b)] = \theta_{e_i+e_k}(\delta^-_{k>i}\delta^-_{k>j}ab) \quad \text{for all } a, b \in \Bbbk.$$

By going through all possible cases, it is easy to observe that $\delta^-_{k>i}\delta^-_{k>j} = \delta^-_{k\in\langle i,j\rangle}$, which proves the second equation. In a similar manner, we can conjugate the first equation by w_j to obtain

$$[\theta_{e_i+e_j}(\delta^-_{j>i}a), \theta_{-e_j-e_k}(\delta^-_{j>k}b)] = \theta_{e_i-e_k}(ab) \quad \text{for all } a, b \in \Bbbk.$$

This is equivalent to

$$[\theta_{e_i+e_j}(a), \theta_{-e_j-e_k}(b)] = \theta_{e_i-e_k}(\delta^-_{j\in\langle i,k\rangle}) \quad \text{for all } a, b \in \mathcal{k}.$$

Interchanging the roles of i and j yields the fourth equation. Finally, conjugating the fourth equation by w_j, we see that

$$[\theta_{e_i-e_j}(\delta^-_{j>i}a), \theta_{-e_k-e_i}(b)] = \theta_{-e_j-e_k}(\delta^-_{j>k}\delta^-_{i\in\langle j,k\rangle}ab) \quad \text{for all } a, b \in \mathcal{k}.$$

This implies that

$$[\theta_{e_i-e_j}(a), \theta_{-e_k-e_i}(b)] = \theta_{-e_j-e_k}(\delta^-_{j>i}\delta^-_{j>k}\delta^-_{i\in\langle j,k\rangle}ab) \quad \text{for all } a, b \in \mathcal{k}.$$

Consulting the truth table in Figure 7.7, we see that $\delta^-_{j>i}\delta^-_{j>k}\delta^-_{i\in\langle j,k\rangle} = \delta^-_{k\notin\langle i,j\rangle}$, which finishes the proof. $\square$

	$\delta^-_{j>i}$	$\delta^-_{j>k}$	$\delta^-_{i\in\langle j,k\rangle}$	$\delta^-_{j>i}\delta^-_{j>k}\delta^-_{i\in\langle j,k\rangle}$	$\delta^-_{k\notin\langle i,j\rangle}$
$i<j<k$	$-$	$+$	$+$	$-$	$-$
$i<k<j$	$-$	$-$	$+$	$+$	$+$
$j<i<k$	$+$	$+$	$-$	$-$	$-$
$j<k<i$	$+$	$+$	$+$	$+$	$+$
$k<i<j$	$-$	$-$	$-$	$-$	$-$
$k<j<i$	$+$	$-$	$+$	$-$	$-$

Figure 7.7: A truth table which shows that $\delta^-_{j>i}\delta^-_{j>k}\delta^-_{i\in\langle j,k\rangle} = \delta^-_{k\notin\langle i,j\rangle}$ for all pairwise distinct $i, j, k \in [1, n]$.

7.10.8 Remark. We could perform the same computations as in Lemma 7.10.7 for the other commutation maps. However, it will be more efficient to do so at a later point, when we have acquired more information about the maps f, q and g. For the moment, we only note that for all $i < j \in [1, n]$, conjugating the equations in Definition 7.10.5 by $w_{2j}w_{1i}$ (where w_{kk} is interpreted as 1_G) yields that

$$\begin{aligned}
[\theta_{e_i-e_j}(a), \theta_{e_j}(v)] &= \theta_{e_i}\big(g(a,v)\big)\theta_{e_i+e_j}\big(-q(a,v)\big), \\
[\theta_{e_i}(u), \theta_{e_j}(v)] &= \theta_{e_i+e_j}\big(-f(u,v)\big)
\end{aligned}$$

for all $a \in \mathcal{k}$ and all $u, v \in M$. We delay the remaining computations until Proposition 7.11.19.

7.10.9 Lemma (Rank-2 identities, part 2). *$(\mathcal{k}, +, \cdot)$ is a ring. If we denote its identity element by $1_{\mathcal{k}}$, we have $w_{ij} = \theta_{e_j-e_i}(-1_{\mathcal{k}})\theta_{e_i-e_j}(1_{\mathcal{k}})\theta_{e_j-e_i}(-1_{\mathcal{k}})$ for all distinct $i, j \in [1, n]$.*

Proof. We already know from Lemma 7.10.6 that the multiplication satisfies the distributive law, so we only have to verify the existence of an identity element. Let $i, j \in [1, n]$ be distinct and let $x \in \mathcal{k}$ be arbitrary. Since $n \geq 3$, we can choose some $k \in [1, n] \setminus \{i, j\}$. By Proposition 5.4.10, there exist unique $a, b \in \mathcal{k}$ such that $w_{ij} = \theta_{e_j-e_i}(a)\theta_{e_i-e_j}(b)\theta_{e_j-e_i}(a)$ and we have $\theta_{e_j-e_i}(a) = \theta_{e_i-e_j}(b)^{w_{ij}} = \theta_{e_j-e_i}(-b)$, so $a = -b$. Using Proposition 5.4.3 (a), we can now compute that

$$\begin{aligned}
\theta_{e_k-e_j}(x \cdot b) &= [\theta_{e_k-e_i}(x), \theta_{e_i-e_j}(b)] = \theta_{e_k-e_i}(x)^{w_{ij}} = \theta_{e_k-e_j}(x), \\
\theta_{e_i-e_k}(b \cdot x) &= [\theta_{e_i-e_j}(b), \theta_{e_j-e_k}(x)] = [\theta_{e_j-e_k}(x), \theta_{e_i-e_j}(b)]^{-1} \\
&= (\theta_{e_j-e_k}(x)^{w_{ij}})^{-1} = \theta_{e_i-e_k}(x).
\end{aligned}$$

It follows that $\Bbbk$ is a ring with identity element $1_\Bbbk := b$ and that

$$w_{ij} = \theta_{e_j-e_i}(-1_\Bbbk)\theta_{e_i-e_j}(1_\Bbbk)\theta_{e_j-e_i}(-1_\Bbbk).$$

Since the identity element of a ring is uniquely determined by the ring structure, our construction does not depend on our choice of i and j. Thus we also have

$$w_{kl} = \theta_{e_l-e_k}(-1_\Bbbk)\theta_{e_k-e_l}(1_\Bbbk)\theta_{e_l-e_k}(-1_\Bbbk)$$

for any choice of distinct indices $k, l \in [1, n]$. $\square$

7.10.10 Note. The assertion of Lemma 7.10.9 involves only roots which lie in the canonical A_{n-1}-subsystem of B_n. Further, the signs of η on this subsystem are the same as in Example 5.5.5 by Remark 7.3.20 and μ is trivial on this subsystem. Thus Lemma 7.10.9 is also a direct consequence of Lemma 5.7.12.

7.10.11 Lemma (Rank-2 identities, part 3). *We have $g(1_\Bbbk, v) = v$ for all $v \in M$ and $q(a, v_0) = a$ for all $a \in \Bbbk$.*

Proof. Let $v \in M$. By the definition of g, we have

$$\theta_{e_2}(g(1, v)) = [\theta_{e_1-e_2}(1), \theta_{e_2}(v)]_{e_2}.$$

By an application of Remark 2.1.18 and Lemma 7.5.4 (a), we can compute that

$$[\theta_{e_1-e_2}(1), \theta_{e_2}(v)]_{e_2} = [\theta_{e_2}(v), \theta_{e_1-e_2}(1)]_{e_2}^{-1} = \left(\theta_{e_2}(v)^{w_{12}}\right)^{-1} = \theta_{e_2}(v),$$

so $g(1, v) = v$. Now let $a \in \Bbbk$. In a similar way as above, it follows from Proposition 7.10.2 and Lemma 7.5.10 (a) that

$$\theta_{e_1+e_2}\big(-q(a, v_0)\big) = [\theta_{e_1-e_2}(a), \theta_{e_2}(v_0)]_{e_1+e_2} = \theta_{e_1-e_2}(a)^{w_2} = \theta_{e_1+e_2}(-a),$$

so $q(a, v_0) = a$. $\square$

We now compute the blueprint rewriting rules. As explained in Remark 6.4.5, we will only perform the blueprint computation in the rank-3 case: Namely, in the subgroup of G corresponding to the root subsystem spanned by

$$\{ e_1 - e_2, e_2 - e_3, e_3 \}.$$

7.10.12 Definition (Blueprint rewriting rules). We define the following rewriting rules:

$$\begin{aligned}
\psi_{12} \colon U_{e_1-e_2} \times U_{e_2-e_3} \times U_{e_1-e_2} &\to U_{e_2-e_3} \times U_{e_1-e_2} \times U_{e_1-e_2}, \\
\big(\theta_{e_1-e_2}(a), \theta_{e_2-e_3}(b), \theta_{e_1-e_2}(c)\big) &\mapsto \big(\theta_{e_2-e_3}(c), \theta_{e_1-e_2}(-b-ca), \theta_{e_2-e_3}(a)\big), \\
\psi_{12}^{-1} \colon U_{e_2-e_3} \times U_{e_1-e_2} \times U_{e_1-e_2} &\to U_{e_1-e_2} \times U_{e_2-e_3} \times U_{e_1-e_2}, \\
\big(\theta_{e_2-e_3}(a), \theta_{e_1-e_2}(b), \theta_{e_2-e_3}(c)\big) &\mapsto \big(\theta_{e_1-e_2}(c), \theta_{e_2-e_3}(-b-ac), \theta_{e_1-e_2}(a)\big), \\
\varphi_{13} \colon U_{e_1-e_2} \times U_{e_3} &\to U_{e_3} \times U_{e_1-e_2}, \\
\big(\theta_{e_1-e_2}(a), \theta_{e_3}(v)\big) &\mapsto \big(\theta_{e_3}(v), \theta_{e_1-e_2}(a)\big), \\
\varphi_{13}^{-1} \colon U_{e_3} \times U_{e_1-e_2} &\to U_{e_1-e_2} \times U_{e_3}, \\
\big(\theta_{e_3}(v), \theta_{e_1-e_2}(a)\big) &\mapsto \big(\theta_{e_1-e_2}(a), \theta_{e_3}(v)\big)
\end{aligned}$$

and

$$\psi_{23} \colon U_{e_3} \times U_{e_2-e_3} \times U_{e_3} \times U_{e_2-e_3} \to U_{e_2-e_3} \times U_{e_3} \times U_{e_2-e_3} \times U_{e_3}$$

which maps $\big(\theta_{e_3}(v), \theta_{e_2-e_3}(a), \theta_{e_3}(u), \theta_{e_2-e_3}(b)\big)$ to

$$\Big(\theta_{e_2-e_3}(b), \theta_{e_3}\big(\overline{u} - \overline{g(b, \overline{v})}\big), \theta_{e_2-e_3}(x), \theta_{e_3}(-\overline{v})\Big)$$

where $x := a - q(b, -\overline{v}) - f\big(u - g(b, \overline{v}), \overline{v}\big)$.

7.10.13 Lemma. *The maps ψ_{12}, ψ_{12}^{-1}, φ_{13}, φ_{13}^{-1} and ψ_{23} in Definition 7.10.12 are blueprint rewriting rules (with respect to $(w_\delta)_{\delta \in \Delta'}$). Further, ψ_{12} and ψ_{12}^{-1} are inverses of each other, and the same holds for φ_{13} and φ_{13}^{-1}.*

Proof. The second statement about inverses is easy to verify. Further, we know from Lemma 6.2.15 that φ_{13} and φ_{13}^{-1} are blueprint rewriting rules. Observe that the rewriting rule ψ_{12} and its inverse are the same rules that we have used for A_3. Since the restriction of η to $A_2 = \{ e_i - e_{i+1} \mid i \in [1,2] \}$ yields the same parity map that we have used for A_3 (see Remark 7.3.20), the same computation as in Lemma 6.3.4 shows that ψ_{12} and ψ_{12}^{-1} are blueprint rewriting rules. It remains to show that ψ_{23} is a blueprint rewriting rule. For this, let $a, b, c, d \in \Bbbk$, let $v, u, h, k \in M$ and put

$$\bar{\alpha} := (e_3, e_2 - e_3, e_3, e_2 - e_3) \quad \text{and} \quad \bar{\beta} := (e_2 - e_3, e_3, e_2 - e_3, e_3).$$

Further, we set

$$x := \big(\theta_{e_3}(v), \theta_{e_2 - e_3}(a), \theta_{e_3}(u), \theta_{e_2 - e_3}(b)\big).$$

On the one hand, we have

$$\begin{aligned}
\tilde{\gamma}_{\bar{\alpha}}(x) &= w_3 \theta_{e_3}(v) w_{23} \theta_{e_2 - e_3}(a) w_3 \theta_{e_3}(u) w_{23} \theta_{e_2 - e_3}(b) \\
&= w_3 w_{23} w_3 w_{23} \theta_{e_3}(v)^{w_{23} w_3 w_{23}} \theta_{e_2 - e_3}(a)^{w_3 w_{23}} \theta_{e_3}(u)^{w_{23}} \theta_{e_2 - e_3}(b) \\
&= w_3 w_{23} w_3 w_{23} \theta_{e_2}(-v)^{w_3 w_{23}} \theta_{e_2 + e_3}(-a)^{w_{23}} \theta_{e_2}(-u) \theta_{e_2 - e_3}(b) \\
&= w_3 w_{23} w_3 w_{23} \theta_{e_3}(-\overline{v}) \theta_{e_2 + e_3}(-a) \theta_{e_2}(-u) \theta_{e_2 - e_3}(b).
\end{aligned}$$

On the other hand, we put

$$x' := \big(\theta_{e_2 - e_3}(c), \theta_{e_3}(h), \theta_{e_2 - e_3}(d), \theta_{e_3}(k)\big).$$

Then

$$\begin{aligned}
\tilde{\gamma}_{\bar{\beta}}(x') &= w_{23} \theta_{e_2 - e_3}(c) w_3 \theta_{e_3}(h) w_{23} \theta_{e_2 - e_3}(d) w_3 \theta_{e_3}(k) \\
&= w_{23} w_3 w_{23} w_3 \theta_{e_2 - e_3}(c)^{w_3 w_{23} w_3} \theta_{e_3}(h)^{w_{23} w_3} \theta_{e_2 - e_3}(d)^{w_3} \theta_{e_3}(k) \\
&= w_{23} w_3 w_{23} w_3 \theta_{e_2 + e_3}(-c)^{w_{23} w_3} \theta_{e_2}(-h)^{w_3} \theta_{e_2 + e_3}(-d) \theta_{e_3}(k) \\
&= w_{23} w_3 w_{23} w_3 \theta_{e_2 - e_3}(c) \theta_{e_2}(-\overline{h}) \theta_{e_2 + e_3}(-d) \theta_{e_3}(k).
\end{aligned}$$

In order to compare these two terms, we have to change the order of the product in the first one, using the commutator formulas in Remark 7.10.8 as well as Relation 1.1.11 (v):

$$\begin{aligned}
&\theta_{e_3}(-\overline{v}) \theta_{e_2 + e_3}(-a) \theta_{e_2}(-u) \theta_{e_2 - e_3}(b) \\
&\quad = \theta_{e_3}(-\overline{v}) \theta_{e_2 - e_3}(b) \theta_{e_2 + e_3}(-a) \theta_{e_2}(-u) \\
&\quad = \theta_{e_2 - e_3}(b) \theta_{e_3}(-\overline{v}) \big[\theta_{e_3}(-\overline{v}), \theta_{e_2 - e_3}(b)\big] \theta_{e_2 + e_3}(-a) \theta_{e_2}(-u) \\
&\quad = \theta_{e_2 - e_3}(b) \theta_{e_3}(-\overline{v}) \theta_{e_2}\big(-g(b, -\overline{v})\big) \theta_{e_2 + e_3}\big(q(b, -\overline{v})\big) \theta_{e_2 + e_3}(-a) \theta_{e_2}(-u) \\
&\quad = \theta_{e_2 - e_3}(b) \theta_{e_2}\big(g(b, \overline{v}) - u\big) \theta_{e_2 + e_3}\big(q(b, -\overline{v}) - a\big) \theta_{e_3}(-\overline{v}) \\
&\qquad\qquad \cdot \big[\theta_{e_3}(-\overline{v}), \theta_{e_2}\big(g(b, \overline{v}) - u\big)\big] \\
&\quad = \theta_{e_2 - e_3}(b) \theta_{e_2}\big(g(b, \overline{v}) - u\big) \theta_{e_2 + e_3}\big(q(b, -\overline{v}) - a\big) \theta_{e_3}(-\overline{v}) \\
&\qquad\qquad \cdot \theta_{e_2 + e_3}\big(f\big(g(b, \overline{v}) - u, -\overline{v}\big)\big) \\
&\quad = \theta_{e_2 - e_3}(b) \theta_{e_2}\big(g(b, \overline{v}) - u\big) \\
&\qquad\qquad \cdot \theta_{e_2 + e_3}\Big(q(b, -\overline{v}) - a + f\big(u - g(b, \overline{v}), \overline{v}\big)\Big) \theta_{e_3}(-\overline{v}).
\end{aligned}$$

Since $w_3 w_{23} w_3 w_{23} = w_{23} w_3 w_{23} w_3$ by Theorem 2.5.10 and since G is rank-2-injective, we conclude that $\tilde{\gamma}_{\tilde{\alpha}}(x) = \tilde{\gamma}_{\tilde{\beta}}(x')$ if and only if

$$c = b, \qquad -\overline{h} = g(b, \overline{v}) - u,$$
$$-d = q(b, -\overline{v}) - a + f\big(u - g(b, \overline{v}), \overline{v}\big) \quad \text{and} \quad k = -\overline{v}.$$

In other words, $\tilde{\gamma}_{\tilde{\alpha}}(x) = \tilde{\gamma}_{\tilde{\beta}}(x')$ if and only if $x' = \psi_{23}(x)$, which shows that ψ_{23} is a blueprint rewriting rule. This finishes the proof. $\square$

7.11 The Blueprint Computation

7.11.1 Notation for this section. Notation 7.10.1 continues to hold.

Finally, we have everything ready to perform the blueprint computation. This computation takes place within the B_3-subsystem which is spanned by Δ' from Notation 7.10.1, and there are no new identities which could be obtained by performing the computation in larger subsystems.

7.11.2 Remark (Blueprint computation). A homotopy cycle of the longest word in the Weyl group of B_3 is given in Figure 7.8. We begin with the tuple

$$\big(\theta_3(u), \theta_2(a), \theta_1(b), \theta_3(v), \theta_2(c), \theta_3(w), \theta_2(d), \theta_1(r), \theta_2(s)\big)$$

where $\theta_1 := \theta_{e_1 - e_2}$, $\theta_2 := \theta_{e_2 - e_3}$, $\theta_3 := \theta_{e_3}$ and where $u, v, w \in M$ and $a, b, c, d, r, s \in \Bbbk$ are arbitrary. Working down rows 1 to 12 in the homotopy cycle and applying the respective blueprint rewriting rules in the process, we obtain a tuple

$$\big(\theta_2(x_1), \theta_1(x_2), \theta_2(x_3), \theta_3(x_4), \theta_2(x_5), \theta_3(x_6), \theta_1(x_7), \theta_2(x_8), \theta_3(x_9)\big).$$

Conversely, working up from row 23 to row 12, we obtain a tuple

$$\big(\theta_2(y_1), \theta_1(y_2), \theta_2(y_3), \theta_3(y_4), \theta_2(y_5), \theta_3(y_6), \theta_1(y_7), \theta_2(y_8), \theta_3(y_9)\big).$$

Here $x_i, y_i \in M$ for all $i \in \{4, 6, 9\}$ and $x_i, y_i \in \Bbbk$ for all $i \in \{1, 2, 3, 5, 7, 8\}$. Now $x_i = y_i$ for all $i \in [1, 9]$ by Theorem 6.2.8. The precise results of this computation can be found in Figures 7.9 to 7.11. The intermediate steps of the computation have been performed with GAP [Gap].

(1)	$321\underline{3232}12$	(13)	$21323\underline{212}3$
(2)	$321232\underline{31}2$	(14)	$2\underline{13}2\underline{31}213$
(3)	$3\underline{212}32132$	(15)	$23\underline{121}3213$
(4)	$\underline{31}2\underline{13}2132$	(16)	$2321232\underline{13}$
(5)	$1323\underline{121}32$	(17)	$2321\underline{2323}1$
(6)	$1\underline{3232}1232$	(18)	$232\underline{13}2321$
(7)	$1232\underline{31}232$	(19)	$\underline{2323}12321$
(8)	$12321\underline{3232}$	(20)	$323\underline{212}321$
(9)	$123\underline{212}323$	(21)	$32\underline{31}2\underline{13}21$
(10)	$12\underline{31}2\underline{13}23$	(22)	$321323\underline{121}$
(11)	$\underline{121}323123$	(23)	321323212
(12)	$21\underline{2323}123$		

Figure 7.8: A homotopy cycle of the longest word in $\mathrm{Weyl}(B_3)$ (taken from [TW02, Fig. 8, p. 472]).

7.11.3 Note. Throughout this section, we denote by u, v, w arbitrary elements of M and by a, b, c, d, r, s arbitrary elements of $\Bbbk$.

$$(1)\ s = s, \qquad (2)\ -r - sd = -r - ds, \qquad (3)\ d = d,$$
$$(6)\ -v + \overline{g(s,u)} = -v + g(s,\overline{u}), \qquad (9)\ u = u.$$

Figure 7.9: The equations "(i) $x_i = y_i$" for $i \in \{1,2,3,6,9\}$ in Remark 7.11.2.

$$w - g(d,\overline{v}) - \overline{g(r,\overline{u})} = w - g(r,u) - g(ds,u) - \overline{g(d,v)} + \overline{g\big(d,g(s,\overline{u})\big)}$$
$$a - q(s,u) - f(\overline{v},u) + f\big(g(s,u),u\big) = a - q(s,-\overline{u}) - f(v,\overline{u}) + f\big(g(s,\overline{u}),\overline{u}\big)$$

Figure 7.10: The equations $x_4 = y_4$ and $x_8 = y_8$ in Remark 7.11.2, respectively.

Since all identities in Figures 7.9 to 7.11 hold for arbitrary values of the variables, they hold in particular if we replace some variables by 0, 1_k or v_0. This allows us to derive shorter identities. We can then use these shorter identities to simplify the original identities. We continue this process until we have proven that the set of identities in Figures 7.9 to 7.11 is equivalent to the axioms of a quadratic module $(M, v \mapsto q(1_k, v))$ over a commutative ring k with linearisation f and scalar multiplication g. The map $v \mapsto \overline{v}$ will turn out to be the reflection σ_{v_0}.

7.11.4 Lemma (Equations 2, 6). *The ring k is commutative, and we have $g(s,\overline{u}) = \overline{g(s,u)}$ for all $s \in k$, $u \in M$.*

Proof. The first equation follows from equation 2 with $r = 0$, the second one from equation 6 with $v = 0$ (see Figure 7.9). □

Lemma 7.11.4 clearly covers all non-trivial identities which can be derived from Figure 7.9.

7.11.5 Lemma (Equation 4). *We have $g(ds,u) = g\big(d,g(s,u)\big)$ for all $d,s \in k$ and $u \in M$.*

Proof. Using the relation $g(s,\overline{u}) = \overline{g(s,u)}$ from Lemma 7.11.4 and the fact that the involution on M is of order at most 2 by Lemma 7.10.3, we see that both sides of equation 4 in Figure 7.10 contain the term $w - \overline{g(d,v)} - g(r,u)$. Removing these terms from the equation, we are left with $g(ds,u) = \overline{g(d,g(s,\overline{u}))}$, which by the same considerations as before yields the desired relation. □

Again, it is clear that no more identities can be deduced from equation 4.

7.11.6 Lemma (Equation 8). *The following hold:*

(a) $f(\overline{v},u) = f(v,\overline{u})$ and $f(\overline{v},\overline{u}) = f(v,u)$ for all $u,v \in M$.

(b) $q(s,-\overline{u}) = q(s,u)$ for all $s \in k$, $u \in M$.

Proof. Putting $a := s := 0_k$ in equation 8 in Figure 7.10, we see that $f(\overline{v},u) = f(v,\overline{u})$, which implies that $f(\overline{v},\overline{u}) = f(v,\overline{\overline{u}}) = f(v,u)$. Using Lemma 7.11.4, we infer that

$$f\big(g(s,\overline{u}),\overline{u}\big) = f\big(\overline{g(s,u)},\overline{u}\big).$$

Thus we can cancel nearly all terms from equation 8, and what remains is $q(s,-\overline{u}) = q(s,u)$. This finishes the proof. □

Only equations 5 and 7 remain. We begin with equation 7 because it is shorter, except that we quickly derive the following fact from equation 5 to simplify our notation.

$$
\begin{aligned}
x_5 &= c - q(d, -\overline{v}) - f(w, \overline{v}) + f(g(d, \overline{v}), \overline{v}) + ra \\
&\quad - \Big(-b - ad - q(r, -\overline{u}) - f(\overline{w}, \overline{u}) + f\big(\overline{g(d, \overline{v})}, \overline{u}\big) + f\big(g(r, \overline{u}), \overline{u}\big) \Big) s, \\
y_5 &= c + sb - \big(a - q(s, -\overline{u}) - f(v, \overline{u}) + f(g(s, \overline{u}), \overline{u})\big)(-r - ds) \\
&\quad - q\big(d, -v + g(s, \overline{u})\big) - f(\overline{w}, v) + f(\overline{w}, g(s, \overline{u})) + f\big(\overline{g(r, u)}, v\big) \\
&\quad - f\big(\overline{g(r, u)}, g(s, \overline{u})\big) + f\big(\overline{g(ds, u)}, v\big) - f\big(\overline{g(ds, u)}, g(s, \overline{u})\big) \\
&\quad + f\big(g(d, v), v\big) - f(g(d, v), g(s, \overline{u})) - f\big(g(d, g(s, \overline{u})), v\big) \\
&\quad + f\big(g(d, g(s, \overline{u})), g(s, \overline{u})\big), \\
x_7 &= -b - ad - q(r, -\overline{u}) - f(\overline{w}, \overline{u}) + f\big(\overline{g(d, \overline{v})}, \overline{u}\big) + f\big(g(r, \overline{u}), \overline{u}\big), \\
y_7 &= -b - q(r, u) - q(ds, u) - f(w, u) + f\big(g(r, u), u\big) + f\big(g(ds, u), u\big) \\
&\quad - d\big(a - q(s, -\overline{u}) - f(v, \overline{u}) + f\big(g(s, \overline{u}), \overline{u}\big)\big).
\end{aligned}
$$

Figure 7.11: The values of x_5, y_5, x_7, y_7 in Remark 7.11.2.

7.11.7 Lemma. *The ring $\Bbbk$ is associative.*

Proof. Putting all variables except for a, d, s in equation 5 in Figure 7.11 to zero, we obtain that $(ad)s = a(ds)$. □

7.11.8 Remark (Equation 7, simplification). With the knowledge obtained so far, we can cancel many terms in equation 7 in Figure 7.11, which yields the following simplified equation:

$$
\begin{aligned}
f\big(\overline{g(d, \overline{v})}, \overline{u}\big) &= -q(ds, u) + f\big(g(ds, u), u\big) + dq(s, -\overline{u}) \\
&\qquad + df(v, \overline{u}) - df\big(g(s, \overline{u}), \overline{u}\big).
\end{aligned}
$$

By Lemmas 7.11.4 and 7.11.6, this is equivalent to the following equation:

$$
\begin{aligned}
f\big(g(d, v), \overline{u}\big) &= -q(ds, u) + f\big(g(ds, u), u\big) + dq(s, u) \\
&\qquad + df(v, \overline{u}) - df\big(g(s, u), u\big).
\end{aligned}
$$

7.11.9 Lemma (Equation 7). *The following hold:*

(a) $f\big(g(d, v), u\big) = df(v, u)$ *for all $d \in \Bbbk$ and all $u, v \in M$.*

(b) $q(d, u) = dq(1_\Bbbk, u)$ *for all $d \in \Bbbk$ and all $u \in M$.*

Proof. Putting $s := 0$ in equation 7 in Remark 7.11.8, we see that $f\big(g(d, v), \overline{u}\big) = df(v, \overline{u})$. This proves the first assertion because the involution on M is a bijection. Using the first assertion and the associativity of $\Bbbk$, we see that

$$
f\big(g(ds, u), u\big) = (ds)f(u, u) = d\big(sf(u, u)\big) = df\big(g(s, u), u\big).
$$

Thus we can now simplify equation 7 to obtain $q(ds, u) = dq(s, u)$, from which the second assertion follows by putting $s := 1_\Bbbk$. □

7.11.10 Remark. As a variation of Lemma 7.11.9 (a), we also have $f\big(\overline{g(d, v)}, u\big) = df(\overline{v}, u)$ because, by Lemma 7.11.6 (a),

$$
f\big(\overline{g(d, v)}, u\big) = f\big(g(d, v), \overline{u}\big) = df(v, \overline{u}) = df(\overline{v}, u).
$$

7.11.11 Remark. At first glance, the equation $q(ds, u) = dq(s, u)$ that we have proven in Lemma 7.11.9 may seem stronger than the equation $q(d, u) = dq(1_\Bbbk, u)$.

However, the first equation follows from the second one together with the associativity of $\Bbbk$:

$$q(ds,u) = (ds)q(1_{\Bbbk},u) = d\big(sq(1_{\Bbbk},u)\big) = dq(s,u).$$

Thus we do not lose any information by only recording the second equation in Lemma 7.11.9.

7.11.12 Notation. From now on, we put $q(v) := q(1,v)$ for all $v \in M$.

7.11.13 Remark (Equation 5, simplification, part 1). We are now left with only equation 5 in Figure 7.11, the most complicated one. Using the previously derived identities, we can simplify this equation. First of all, note that the right-hand side of this equation contains the terms

$$f\big(\overline{g(ds,u)},v\big) - f\big(g(d,g(s,\overline{u})),v\big) \qquad \text{and}$$
$$-f\big(\overline{g(ds,u)},g(s,\overline{u})\big) + f\big(g(d,g(s,\overline{u})),g(s,\overline{u})\big),$$

both of which are zero. Further, observe that the left-hand side contains

$$c - f(w,\overline{v}) + bs + f\big(g(d,\overline{v}),\overline{v}\big)$$

while the right-hand side contains

$$c - f(\overline{w},v) + sb + f\big(g(d,v),v\big).$$

Since these terms are equal, we can cancel them. What remains are the following expressions for x_5 and y_5:

$$\begin{aligned}
x_5 &= -q(d,-\overline{v}) + ra \\
&\quad - \Big(-ad - q(r,-\overline{u}) - f(\overline{w},\overline{u}) + f\big(\overline{g(d,\overline{v})},\overline{u}\big) + f\big(g(r,\overline{u}),\overline{u}\big)\Big)s, \\
y_5 &= -\big(a - q(s,-\overline{u}) - f(v,\overline{u}) + f(g(s,\overline{u}),\overline{u})\big)(-r-ds) \\
&\quad - q\big(d,-v+g(s,\overline{u})\big) + f\big(\overline{w},g(s,\overline{u})\big) + f\big(\overline{g(r,u)},v\big) \\
&\quad - f\big(\overline{g(r,u)},g(s,\overline{u})\big) - f\big(g(d,v),g(s,\overline{u})\big)
\end{aligned}$$

Applying the distributive law and the previously established identities, we see that

$$\begin{aligned}
x_5 &= -dq(v) + ra + ads + rsq(u) + sf(w,u) - dsf(v,\overline{u}) - rsf(u,u), \\
y_5 &= ar + ads - rsq(u) - ds^2q(u) - rf(v,\overline{u}) - dsf(v,\overline{u}) + rsf(u,u) \\
&\quad + ds^2f(u,u) - dq\big(g(s,\overline{u}) - v\big) + f\big(\overline{w},g(s,\overline{u})\big) + rf(\overline{u},v) - rsf(u,u) \\
&\quad - dsf(v,\overline{u}).
\end{aligned}$$

Clearly, the term

$$ra + ads - dsf(v,\overline{u}) - rsf(u,u)$$

on the left-hand side cancels the term

$$ar + ads - dsf(v,\overline{u}) - rsf(u,u)$$

on the right-hand side, so we are left with

$$\begin{aligned}
x_5 &= -dq(v) + rsq(u) + sf(w,u), \\
y_5 &= -rsq(u) - ds^2q(u) - rf(v,\overline{u}) + rsf(u,u) + ds^2f(u,u) \\
&\quad - dq\big(g(s,\overline{u}) - v\big) + f\big(\overline{w},g(s,\overline{u})\big) + rf(\overline{u},v) - dsf(v,\overline{u}).
\end{aligned}$$

7.11.14 Lemma (Equation 5, part 1). *The following hold:*

(a) *$f(u,v) = f(v,u)$ for all $u,v \in M$.*
(b) *$f\big(v, g(d,u)\big) = df(v,u)$ for all $u,v \in M$, $d \in \Bbbk$.*

Proof. Putting $s := 0$, $d := 0$, $r := 1$ in equation 5 (see Remark 7.11.13), we obtain $0 = -f(v,\overline{u}) + f(\overline{u},v)$. Since the involution on M is a bijection, this implies the first assertion. Now the second assertion together with Lemma 7.11.9 (a) yields that
$$f\big(v, g(d,u)\big) = f\big(g(d,u), v\big) = df(u,v) = df(v,u),$$
as desired. □

7.11.15 Remark (Equation 5, simplification, part 2). Lemma 7.11.14 allows us to simplify equation 5 even further: The term $sf(w,u)$ on the left-hand side cancels the term $f\big(\overline{w}, g(s,\overline{u})\big)$ on the right-hand side. Further, we see that the term $-rf(v,\overline{u}) + rf(\overline{u},v)$ on the right-hand side equals zero. Thus we obtain
$$x_5 = -dq(v) + rsq(u),$$
$$y_5 = -rsq(u) - ds^2q(u) + rsf(u,u) + ds^2f(u,u) - dq\big(g(s,\overline{u}) - v\big) - dsf(v,\overline{u}).$$

7.11.16 Lemma (Equation 5, part 2). *The following hold:*
(a) *$f(u,u) = 2q(u)$ for all $u \in M$.*
(b) *$q(-v) = q(\overline{v}) = q(v)$ for all $v \in M$.*

Proof. Putting $r := 1$, $s := 1$, $d := 0$ and $v := 0$ in equation 5 in Remark 7.11.15, we see that $q(u) = -q(u) + f(u,u)$, which implies the first assertion. Using this identity, we obtain the following simplified version of equation 5:
$$-dq(v) = -ds^2q(u) - dq\big(g(s,\overline{u}) - v\big) - dsf(v,\overline{u}).$$
Putting $u := 0$ and $d := 1$, we see that $q(v) = q(-v)$. Since we know that $q(-\overline{v}) = q(v)$ by Lemma 7.11.6 (b), the second assertion follows. □

7.11.17 Remark. It is not immediately obvious that no further relations can be deduced from equation 5. However, using the rank-2-computation from Lemma 7.10.6, we see that
$$\begin{aligned} dq\big(g(s,\overline{u}) - v\big) &= dq\big(g(s,\overline{u})\big) + dq(-v) + df\big(g(s,\overline{u}), -v\big) \\ &= ds^2q(u) + dq(v) - dsf(\overline{u},v). \end{aligned}$$
Thus the simplified version of equation 5 in the proof of Lemma 7.11.16 reduces to $0 = 0$. It is noteworthy that, up to this point, we have never used the non-linear identity from Lemma 7.10.6 in our computations. This shows that, even if we had not proven this identity in our rank-2-computations, we would have derived it from the blueprint computations.

7.11.18 Summary. The following hold:
(a) The ring $\Bbbk$ is associative and commutative.
(b) The map $g\colon \Bbbk \times M \to M$ defines a $\Bbbk$-scalar multiplication in M which turns it into a $\Bbbk$-module.
(c) The involution on M is a $\Bbbk$-linear automorphism of M with $\overline{\overline{v}} = v$ and $q(\overline{v}) = q(v)$ for all $v \in M$.
(d) The map $f\colon M \times M \to M$ is a symmetric $\Bbbk$-bilinear form on M and the map $q\colon M \to \Bbbk, v \mapsto q(1_{\Bbbk}, v)$ is a $\Bbbk$-quadratic form on M with linearisation f.

7.11.19 Proposition. *G satisfies the same commutator relations as the group $\mathrm{EO}(q)$ in Proposition 7.3.25. In other words, $(\theta_\alpha)_{\alpha \in B_n}$ is a parametrisation of G by (M, q) with standard signs.*

Proof. The third set of relations (concerning commutators of medium-length roots) is satisfied by Lemma 7.10.7. Now let $i < j \in [1, n]$, let $r \in \mathcal{R}$ and let $v \in M$. Throughout this computation, we will frequently apply Lemma 7.11.6 (a) and Lemma 7.11.16 (b). We know from Remark 7.10.8 that

$$[\theta_{e_i - e_j}(r), \theta_{e_j}(v)] = \theta_{e_i}(rv)\theta_{e_i+e_j}\big(-rq(v)\big).$$

Conjugating this equation by w_{ij}, we infer that

$$[\theta_{e_j - e_i}(-r), \theta_{e_i}(-v)] = \theta_{e_j}(rv)\theta_{e_i+e_j}\big(-rq(v)\big).$$

In other words,

$$[\theta_{e_j - e_i}(r), \theta_{e_i}(v)] = \theta_{e_j}(rv)\theta_{e_i+e_j}\big(rq(v)\big).$$

This says precisely that for all distinct $i, j \in [1, n]$, we have

$$[\theta_{e_i - e_j}(r), \theta_{e_j}(v)] = \theta_{e_i}(rv)\theta_{e_i+e_j}\big(\delta^-_{i<j} rq(v)\big), \tag{7.2}$$

which is the first relation in Proposition 7.3.25.

Continue to assume that $r \in \mathcal{R}$, $v \in M$ and that $i, j \in [1, n]$ are distinct (but not necessarily that $i < j$). Conjugating (7.2) by w_j, we see that

$$\big[\theta_{e_i+e_j}(\delta^-_{j>i} r), \theta_{-e_j}(\overline{v})\big] = \theta_{e_i}\big(\overline{rv}\big)\theta_{e_i-e_j}\big(\delta^-_{j>i}\delta^-_{i<j} rq(v)\big).$$

In other words,

$$[\theta_{e_i+e_j}(r), \theta_{-e_j}(v)] = \theta_{e_i}(\delta^-_{j>i} rv)\theta_{e_i-e_j}\big(\delta^-_{j>i} rq(v)\big).$$

This is the third relation in Proposition 7.3.25. Conjugating it by w_i, we obtain

$$\big[\theta_{e_j-e_i}(\delta^-_{i>j} r), \theta_{-e_j}(\overline{v})\big] = \theta_{-e_i}\big(\delta^-_{j>i}\overline{rv}\big)\theta_{-e_i-e_j}\big(\delta^-_{i>j}\delta^-_{j>i} rq(v)\big).$$

This says that

$$\big[\theta_{e_j-e_i}(r), \theta_{-e_j}(v)\big] = \theta_{-e_i}(-rv)\theta_{-e_i-e_j}\big(\delta^-_{j>i} rq(v)\big),$$

which is the second relation in Proposition 7.3.25. Now we conjugate (7.2) by w_i. This yields

$$[\theta_{-e_i-e_j}(\delta^-_{i>j} r), \theta_{e_j}(\overline{v})] = \theta_{-e_i}\big(\overline{rv}\big)\theta_{e_j-e_i}\big(\delta^-_{i>j}\delta^-_{i<j} rq(v)\big).$$

Hence

$$[\theta_{-e_i-e_j}(r), \theta_{e_j}(v)] = \theta_{-e_i}(\delta^-_{i>j} rv)\theta_{e_j-e_i}\big(\delta^-_{i<j} rq(v)\big),$$

which is the fourth relation in Proposition 7.3.25. This finishes the proof of the first set of relations in Proposition 7.3.25.

Now let $i, j \in [1, n]$ be distinct and let $u, v \in M$. We know from Remark 7.10.8 that

$$[\theta_{e_i}(v), \theta_{e_j}(u)] = \theta_{e_i+e_j}\big(-f(v, u)\big)$$

if $i < j$. In this case, conjugating this equation by w_{ij} yields

$$[\theta_{e_j}(v), \theta_{e_i}(-u)] = \theta_{e_i+e_j}\big(-f(v, u)\big).$$

We conclude that

$$[\theta_{e_i}(v), \theta_{e_j}(u)] = \theta_{e_i+e_j}\big(\delta^-_{i<j} f(v, u)\big)$$

holds for all distinct $i, j \in [1, n]$. This is the fifth relation in Proposition 7.3.25. We can conjugate it by w_i to obtain

$$[\theta_{-e_i}(\overline{v}), \theta_{e_j}(\overline{u})] = \theta_{e_j-e_i}\big(\delta^-_{i>j}\delta^-_{i<j}f(v,u)\big),$$

which means that

$$[\theta_{-e_i}(v), \theta_{e_j}(u)] = \theta_{e_j-e_i}\big(-f(v,u)\big). \tag{7.3}$$

Since $[\theta_{-e_i}(v), \theta_{e_j}(u)] = [\theta_{e_j}(u), \theta_{-e_i}(v)]^{-1}$ and f is symmetric, this is the sixth relation in Proposition 7.3.25. Finally, we can conjugate (7.3) by w_j to obtain

$$[\theta_{-e_i}(\overline{v}), \theta_{-e_j}(\overline{u})] = \theta_{-e_j-e_i}\big(-\delta^-_{j>i}f(v,u)\big).$$

In other words,

$$[\theta_{-e_i}(v), \theta_{-e_j}(u)] = \theta_{-e_j-e_i}\big(\delta^-_{j<i}f(v,u)\big),$$

which is the seventh relation in Proposition 7.3.25. This finishes the proof. □

Before we state the main result of this chapter, we make a final observation.

7.11.20 Lemma. *We have $v_{-1} = v_1 = -v_0$ and $q(v_0) = 1$ as well as $\overline{u} = \sigma_{v_0}(u)$ for all $u \in M$. In particular, (M, q, e) is a pointed quadratic module.*

Proof. As in the proof of Lemma 7.7.3, we can easily compute that

$$\theta_{e_1-e_2}(1)^{w_2} = \theta_{e_1+e_2}(-q(v_0)).$$

On the other hand, by Proposition 7.10.2, we know that

$$\theta_{e_1-e_2}(1)^{w_2} = \theta_{e_1+e_2}(-1).$$

It follows that $q(v_0) = 1$. By Lemma 7.7.3, this implies that

$$v_{-1} = v_1 = -q(v_0)^{-1}v_0 = -v_0.$$

Finally, we have shown in Lemma 7.7.9 for every $u \in M$ that $\theta_{e_1}(u)^{w_2} = \theta_{e_1}(\sigma_{v_0}(u))$, but we also know that $\theta_{e_1}(u)^{w_2} = \theta_{e_1}(\overline{u})$. This finishes the proof. □

7.11.21 Theorem (Coordinatisation theorem for B_n). *Let G be a group with a crystallographic B_n-grading $(U_\alpha)_{\alpha\in B_n}$ for some $n \in \mathbb{N}_{\geq 3}$. Then there exist a commutative associative ring $\Bbbk$ and a pointable quadratic module (M, q) over $\Bbbk$ such that G is coordinatised by (M, q) with standard signs (in the sense of Definition 7.7.2). Further, if we fix a Δ-system of Weyl elements in G, then we can choose the root isomorphisms $(\theta_\alpha)_{\alpha\in B_n}$ so that $w_\alpha = \theta_{-\alpha}(-1_\Bbbk)\theta_\alpha(1_\Bbbk)\theta_{-\alpha}(-1_\Bbbk)$ for all long simple roots α and $w_\delta = -\theta_{-\delta}(-v_0)\theta_\delta(v_0)\theta_{-\delta}(-v_0)$ for the short simple root δ where v_0 is some element of M with $q(v_0) = 1$.*

Summary of the proof. Choose a root base Δ of B_n and a Δ-system $(w_\delta)_{\delta\in\Delta}$ of Weyl elements. Denote by (A, η, B, μ) the standard admissible partial twisting system for G, as in Definition 7.8.2. Then by Proposition 7.9.4, there exist abelian groups $(\Bbbk, +)$ and $(M, +)$ on which $A \times B$ acts and a parametrisation $(\theta_\alpha)_{\alpha\in\Phi}$ of G by $(A \times B, M, \Bbbk)$ with respect to $(w_\delta)_{\delta\in\Delta}$ and $\eta \times \mu$. We can define commutation maps as in Definition 7.10.5. By Summary 7.11.18, these maps equip $\Bbbk$ with the structure of a commutative associative ring and M with a $\Bbbk$-module structure and a $\Bbbk$-quadratic form $q\colon M \to \Bbbk$. By Proposition 7.11.19, $(\theta_\alpha)_{\alpha\in\Phi}$ is a coordinatisation of G by (M, q) with standard signs. By Lemmas 7.10.9 and 7.11.20, the Weyl elements $(w_\delta)_{\delta\in\Delta}$ have the desired form. □

Chapter 8

Jordan Modules and Related Structures

Chapters 5 and 7 each started with a section which introduced the coordinatising algebraic structures of relevance in the respective chapter. For root gradings of types C and BC, the corresponding algebraic structures are the so-called Jordan modules. The theory of these objects is rather involved, and we will need several other algebraic structures to define and investigate them properly. Further, Jordan modules are a new algebraic structure which we introduce specifically to describe root gradings of types C and BC. For these reasons, we dedicate an own chapter to the study of these objects.

Section 8.1 begins with an introduction of weakly quadratic maps, which generalise quadratic maps. We will also introduce square-modules $(M, \hat{+})$, which are groups equipped with a weakly quadratic scalar multiplication. They serve as the fundamental language in which we describe the other algebraic structures in this chapter. In sections 8.2 and 8.3, we list some known properties of alternative rings and (nonassociative) rings with involution, respectively. Section 8.4 is dedicated to involutory sets, which are alternative rings $\mathcal{R}$ equipped with an involution and a certain subset of $\mathcal{R}$. In section 8.5, we define pseudo-quadratic modules over involutory sets. Any pseudo-quadratic module M gives rise to a group $T(M)$ which is our main (and in fact, the only known) example of a Jordan module.

In section 8.6, we can finally define Jordan modules. A Jordan module is a pair of groups $(J, \mathcal{R})$ equipped with a family of maps satisfying exactly the identities which result from the blueprint computation for root gradings of type BC. The group $\mathcal{R}$ in a Jordan module is always equipped with the structure of an alternative ring with involution. We will show in section 8.7 that if $2_{\mathcal{R}}$ is invertible, then every Jordan module is of the form $T(M)$ for some pseudo-quadratic module M. Thus Jordan modules should be regarded as a slight generalisation of the class of groups $T(M)$ which allows us to capture certain "characteristic-2 phenomena". It should be noted that we are not aware of examples of Jordan modules which are not of the form $T(M)$, but we cannot exclude their existence.

In section 8.8, we will present a notion of "purely alternative rings" which was introduced by Slater in [Sla67]. We will show that such rings do not admit non-zero modules. This fact will not formally be needed in the sequel. Rather, it illustrates that the non-associativity of the base ring in the definition of pseudo-quadratic modules should be seen as a technicality: Alternative rings admit pseudo-quadratic modules only insofar that they are not purely alternative.

The present chapter is independent from chapter 9, with a few exceptions: The material on pseudo-quadratic modules will be needed in the construction in section 9.3, and some results on alternative rings will be used in section 9.10. Hence the the enthusiastic reader may wish to skip ahead to chapter 9 and return to the current chapter only later.

8.1 Weakly Quadratic Maps

8.1.1 Note. Throughout this chapter, we will often consider groups $(M, \hat{+})$ which are not assumed to be commutative but which are still written additively because we think of them as "modules" and because, in some important cases, they turn out to be abelian. We indicate this non-commutativity by the hat on the operator. The neutral element of such groups will always be denoted by 0 and the inverse of an element $x \in M$ will be denoted by $\hat{-}x$. In particular, we have $\hat{-}(x \hat{+} y) = \hat{-}y \hat{-} x$. We will later see that in root gradings of type C, only the commutative case is relevant.

In this section, we introduce weakly quadratic maps and square-modules. Their theory provides the basic but essential language in which we describe Jordan modules and related algebraic structures. None of the results in this section are difficult, but we are not aware of any literature in which weakly quadratic maps appear.

Weakly quadratic maps generalise quadratic maps (Definitions 7.1.2 and 7.1.3) in two ways: Firstly, the involved groups are allowed to be nonabelian. Secondly, we do not require Axiom 7.1.3 (i).

8.1.2 Definition (Weakly quadratic maps). Let $(M, \hat{+}), (N, \hat{+})$ be groups and let $q\colon M \to N$ be a map. The *polarisation of* q is the map

$$f\colon M \times M \to N, (v, w) \mapsto \hat{-}q(w) \hat{-} q(v) \hat{+} q(v \hat{+} w).$$

We say that q is *weakly quadratic* if its polarisation is bi-additive, in which case the map f is also called the the *linearisation of* q.

8.1.3 Remark. Equivalently, the polarisation of a map $q\colon M \to N$ is the unique map $f\colon M \times M \to N$ which satisfies

$$q(v \hat{+} w) = q(v) \hat{+} q(w) \hat{+} f(v, w)$$

for all $v, w \in M$. In particular, if q is weakly quadratic, we have

$$q(0) = q(0 \hat{+} 0) = q(0) \hat{+} q(0) \hat{+} f(0, 0) = q(0) \hat{+} q(0),$$

so that $q(0) = 0$.

8.1.4 Example. Let M, N be groups. Clearly, any additive or constant map $q\colon M \to N$ is weakly quadratic because its polarisation is the zero map. Further, if M and N are abelian, then they are $\mathbb{Z}$-modules and any $\mathbb{Z}$-quadratic map $q\colon M \to N$ is weakly quadratic. Further, if N is abelian, then the sum of two weakly quadratic maps from M to N is weakly quadratic. Thus in the abelian setting, sums of quadratic, linear and constant maps are examples of weakly quadratic. However, not every weakly quadratic map can be written in this way.

Our interest in weakly quadratic maps stems from the fact that in this chapter, we will frequently see groups which are "modules" over some ring except that the scalar multiplication is only weakly quadratic in the scalar. Such objects will be called square-modules.

8.1.5 Definition (Square-module). Let $\mathcal{R}$ be a ring. A *(right) square-module over $\mathcal{R}$* is a group $(N, \hat{+})$ together with a map $\omega\colon N \times \mathcal{R} \to N$ (called the *square-scalar multiplication*) satisfying the following properties:

(i) ω is additive in the first component.

(ii) For all $v \in N$, the map $\mathcal{R} \to N, r \mapsto \omega(v, r)$ is weakly quadratic.

(iii) $\omega(v, 1_{\mathcal{R}}) = v$ for all $v \in N$.

8.1.6 Definition (Properties of square-modules). Let $(N, \hat{+})$ be a square-module over a ring $\mathcal{R}$ with square-scalar multiplication ω.

(a) We say that N is *properly quadratic* if N is abelian and the map in Axiom 8.1.5 (ii) is $\mathbb{Z}$-quadratic in the sense of Definition 7.1.3.

(b) We say that N is *multiplicative* if $\omega\big(\omega(v, r), s\big) = \omega(v, rs)$ for all $v \in N$ and $r, s \in \mathcal{R}$.

(c) A subgroup U of N is called a *square-submodule* if $\omega(u, r) \in U$ for all $u \in U$ and $r \in \mathcal{R}$.

8.1.7 Remark. Let $\mathcal{R}$ be a ring. A square-module (M, ω) over $\mathcal{R}$ is an $\mathcal{R}$-module (in the regular sense of Definition 5.1.22) if and only if it is abelian, multiplicative and ω is additive in the second component.

The most important examples of square modules in our setting come from alternative rings with nuclear involutions, their involutory sets and pseudo-quadratic modules over these objects: see Example 8.4.2 and Lemmas 8.4.3, 8.4.5 to 8.4.7 and 8.5.34. All the interesting examples of square-modules will be multiplicative. However, these examples often arise as square-submodules of (less interesting) non-multiplicative square-modules, which justifies the higher generality in our definition of square-modules. An example of a square-module which is not known to be multiplicative but which contains interesting multiplicative square-submodules is given in 8.4.2.

8.1.8 Remark. By Remark 8.1.3, the scalar multiplication on a square-module N satisfies $\omega(v, 0_{\mathcal{R}}) = 0_N$ for all $v \in N$.

8.1.9 Definition. Let (N, ω), (N', ω') be two square-modules over a ring $\mathcal{R}$ and let $f\colon N \to N'$ be a map.

(a) We say that f *preserves the square-scalar multiplication* if

$$f\big(\omega(v, r)\big) = \omega'\big(f(v), r\big)$$

for all $v \in N$ and all $r \in \mathcal{R}$.

(b) We say that f is a *homomorphism of square-modules from N to N'* if it is a homomorphism of additive groups which preserves the scalar multiplication. It is called an *isomorphism of square-modules* if it is a bijective homomorphism of square-modules, and it is called an *automorphism of square-modules* if, in addition, $(N, \omega) = (N', \omega')$.

We record the following standard facts from module theory which remain valid in this generality.

8.1.10 Lemma. *Kernels and images of homomorphisms of square-modules are square-submodules. Further, intersections of square-submodules are square-submodules.*

Proof. Let $f\colon (N,\omega) \to (N',\omega')$ be a homomorphism of square-modules over some ring $\mathcal{R}$. If $v \in N$ is such that $f(v) = 0$, then

$$f\big(\omega(v,r)\big) = \omega'\big(f(v),r\big) = \omega'(0,r) = 0$$

for all $r \in \mathcal{R}$, so the kernel of f is a square-submodule of N. Further, for all $v \in N$, we have

$$\omega'\big(f(v),r\big) = f\big(\omega(v,r)\big)$$

for all $r \in \mathcal{R}$, so the image of f is a submodule of N'. Finally, the assertion about intersections is clear. $\square$

8.1.11 Lemma. *Let (M,ω) be a square-module over some ring $\mathcal{R}$ and let U be a square-submodule of M. Denote by $\pi\colon M \to M/U$ the canonical projection, which is a homomorphism of groups. Then there exists a unique map*

$$\tilde{\omega}\colon (M/U) \times \mathcal{R} \to M/U$$

such that

$$\tilde{\omega}\big(\pi(x),r\big) = \pi\big(\omega(x,r)\big)$$

for all $x \in M$ and $r \in \mathcal{R}$. Further, $(M/U,\tilde{\omega})$ is a square-module, π is a homomorphism of square-modules and $(M/U,\tilde{\omega})$ is multiplicative if (M,ω) is.

Proof. This follows from standard arguments. $\square$

On any multiplicative square-module, we have a canonical "involution".

8.1.12 Definition (Involution on square-modules). Let (M,ω) be a multiplicative square-module over some ring $\mathcal{R}$. The map

$$\overline{\cdot}\colon M \to M, x \mapsto \omega(x,-1_{\mathcal{R}})$$

is called the *canonical involution on* (M,ω).

8.1.13 Lemma. *Let (M,ω) be a multiplicative square-module over some ring $\mathcal{R}$. Then the canonical involution on (M,ω) is an automorphism of the square-module (M,ω) which is its own inverse.*

Proof. For all $x \in M$, we have

$$\overline{\overline{x}} = \omega\big(\omega(x,-1_{\mathcal{R}}),-1_{\mathcal{R}}\big) = \omega\big(x,(-1_{\mathcal{R}})(-1_{\mathcal{R}})\big) = \omega(x,1) = x,$$

so $\overline{\cdot}$ is its own inverse. Further, it is a homomorphism of groups because ω is additive in the first component. Finally, we have for all $x \in M$ and $r \in \mathcal{R}$ that

$$\omega(\overline{x},r) = \omega\big(\omega(x,-1_{\mathcal{R}}),r\big) = \omega(x,-r) = \omega\big(\omega(x,r),-1_{\mathcal{R}}\big) = \overline{\omega(x,r)}.$$

Thus $\overline{\cdot}$ is a homomorphism of square-modules, which finishes the proof. $\square$

8.2 Alternative Rings

8.2.1 Notation for this section. We denote by $\mathcal{R}$ an arbitrary ring.

We have already introduced nonassociative rings and the basic related notions in section 5.1. In this section, we record some elementary facts about the special case of alternative rings. We will also see some other related classes of nonassociative rings. A basic reference on this subject is [Sch66, Chapter III], but we will also cite various other sources.

8.2.2 Definition (Associativity properties, [Sch66, Chapter III]). We say that $\mathcal{R}$ is *alternative* if

$$x(xy) = (xx)y \quad \text{and} \quad (xy)y = x(yy)$$

for all $x, y \in \mathcal{R}$. We say that $\mathcal{R}$ is *weakly alternative* if

$$[x,y,z] = -[y,x,z] \quad \text{and} \quad [x,y,z] = -[x,z,y]$$

for all $x, y, z \in \mathcal{R}$. We say that $\mathcal{R}$ is *flexible* if

$$(xy)x = x(yx)$$

for all $x, y \in \mathcal{R}$. Further, for all $a, x, y, z \in \mathcal{R}$, the following relations are called the *Moufang identities*:

$$\big(x(ax)\big)y = x\big(a(xy)\big), \tag{LM}$$

$$y\big(x(ax)\big) = \big((yx)a\big)x, \tag{RM}$$

$$(xy)(ax) = x\big((ya)x\big). \tag{MM}$$

The terminology of "weakly alternative rings" is borrowed from [Fau89, p. 170]. It is not standard.

8.2.3 Example. The standard examples of rings which are alternative but not associative are octonion algebras. These are 8-dimensional composition algebras over an arbitrary field. In fact, if $\mathcal{R}$ is an alternative simple ring, then [Kle53] says that $\mathcal{R}$ is either associative or an octonion algebra over a field. We will encounter a generalisation of composition algebras over commutative associative rings in section 10.2.

8.2.4 Note. We will mostly be concerned with alternative rings in this thesis. The notions of flexible and Moufang rings interest us only insofar that they appear in the characterisation of alternative rings that we give in Proposition 8.2.6. The notion of weakly alternative rings, on the other hand, matters to us for technical reasons: In section 9.10, we will prove that any ring $\mathcal{R}$ which arises from a $(B)C_3$-graded group is always alternative. In the course of the proof, we will first show that $\mathcal{R}$ is weakly alternative (Lemma 9.10.14), which allows us to apply Remark 8.2.8. Only later can we show that $\mathcal{R}$ is actually alternative (Proposition 9.10.16).

8.2.5 Reminder (Alternating maps). Let $n \in \mathbb{N}_+$ and let $f \colon \mathcal{R}^n \to \mathcal{R}$ be a map which is additive in each component. Then f is *alternating* if for all $v_1, \dots, v_n \in \mathcal{R}$ such that at least two of the elements $v_1, \dots, v_n$ are equal, we have $f(v_1, \dots, v_n) = 0_{\mathcal{R}}$. Further, it is called *weakly alternating* or *antisymmetric* if

$$f\big(v_{\sigma(1)}, \dots, v_{\sigma(n)}\big) = (-1)^{\operatorname{sgn}(\sigma)} f(v_1, \dots, v_n)$$

for any permutation σ of $[1, n]$ where $\operatorname{sgn}(\sigma)$ denotes the signum of σ. (It suffices to verify this condition for transpositions σ because they generate the permutation group.) Any alternating map is weakly alternating, and if $2_{\mathcal{R}}$ is not a zero divisor in $\mathcal{R}$, then every weakly alternating map is alternating.

We have the following characterization of alternative rings. The main parts of the proof can be found in [Sch66, III.1]. The fact that alternative rings satisfy property 8.2.6 (iv) is usually referred to as *Artin's Theorem*. We will see variations of this theorem in Lemma 8.2.17 and Proposition 8.3.16.

8.2.6 Proposition (Artin's Theorem). *The following conditions on $\mathcal{R}$ are equivalent:*

(i) $\mathcal{R}$ *is alternative.*

(ii) $\mathcal{R}$ *is alternative and flexible.*

(iii) *The associator map is alternating.*

(iv) *Every subring of* $\mathcal{R}$ *which is generated by at most two elements is associative.*

(v) $\mathcal{R}$ *satisfies the Moufang identities* (LM), (RM) *and* (MM).

(vi) $\mathcal{R}$ *satisfies the left and right Moufang identities* (LM) *and* (RM).

8.2.7 Lemma. *Every alternative ring is weakly alternative. If* $\mathcal{R}$ *is a weakly alternative ring in which* $2_{\mathcal{R}}$ *is not a zero divisor, then* $\mathcal{R}$ *is alternative.*

Proof. This follows from Reminder 8.2.5 and Proposition 8.2.6. □

8.2.8 Remark. Recall the different notions of nuclei from Definition 5.1.9. If $\mathcal{R}$ is weakly alternative, then

$$\mathrm{Nucl}(\mathcal{R}) = \mathrm{LNucl}(\mathcal{R}) = \mathrm{MNucl}(\mathcal{R}) = \mathrm{RNucl}(\mathcal{R}).$$

We will use this fact in Lemma 9.10.14, in a situation in which $\mathcal{R}$ is weakly alternative but not yet proven to be alternative.

Recall from Definition 5.1.12 that the notion of invertibility is slightly surprising (and non-standard) in nonassociative rings. In alternative rings, it turns out that we can use the usual notion of invertibility from associative ring theory.

8.2.9 Lemma. *Assume that* $\mathcal{R}$ *is alternative. For all* $x, y \in \mathcal{R}$*, the following assertions are equivalent:*

(i) x *is an inverse of* y *in the sense of Definition 5.1.12.*

(ii) $xy = 1_{\mathcal{R}} = yx$.

Proof. Assume that $xy = 1_{\mathcal{R}} = yx$. Using the Moufang identities, it is proven in [Sch66, p. 38] that $[x, y, z] = 0_{\mathcal{R}}$ for all $z \in \mathcal{R}$. Since $\mathcal{R}$ is alternative, it follows that $[y, x, z] = [z, x, y] = [z, y, x] = 0_{\mathcal{R}}$ for all $z \in \mathcal{R}$ as well. Thus the assertion follows from Lemma 5.1.17. □

8.2.10 Lemma. *If* $\mathcal{R}$ *is weakly alternative and* x *is an invertible element of the nucleus of* $\mathcal{R}$*, then* x^{-1} *lies in the nucleus as well.*

Proof. Let x be an invertible element of the nucleus and let $y, z \in \mathcal{R}$ be arbitrary. Then we have

$$yz = \big(x(x^{-1}y)\big)z = x\big((x^{-1}y)z\big).$$

Multiplying by x^{-1} from the left side yields $x^{-1}(yz) = (x^{-1}y)z$. That is, we have $[x^{-1}, y, z] = 0_{\mathcal{R}}$ for all $y, z \in \mathcal{R}$. Using Remark 8.2.8, we conclude that x^{-1} lies in the nucleus. □

We will have occasion to use the following identities.

8.2.11 Lemma. *Assume that* $\mathcal{R}$ *is alternative. Then the following assertions hold for all* $x, x', y, z \in \mathcal{R}$*:*

(a) $[x^2, y, z] = x[x, y, z] + [x, y, z]x$.

(b) *The* right bumping formula: $[x, y, z]x = [x, xy, z] = [x, y, xz]$.

(c) *The* left bumping formula: $x[x, y, z] = [x, yx, z] = [x, y, zx]$.

(d) $[x, y, z]x' + [x', y, z]x = [x, x'y, z] + [x', xy, z] = [x, y, x'z] + [x', y, xz]$.

(e) $x[x',y,z] + x'[x,y,z] = [x,yx',z] + [x',yx,z] = [x,y,zx'] + [x',y,zx]$.

Proof. The first three identities are (2.12), (2.13) and (2.14) in [BK51, Lemma 2.2], albeit with a typo in (2.14). The last two identities are linearisations of (b) and (c). By this we mean the following: At first, observe that the formula in (b) is additive in y and z but "quadratic" in x. Replacing x by $x + x'$ in this formula, we obtain

$$\begin{aligned}[x,y,z]x + [x',y,z]x + [x,y,z]x' + [x',y,z]x' \\ = [x,y,xz] + [x',y,xz] + [x,y,x'z] + [x',y,x'z].\end{aligned}$$

All summands in which only one of the variables x and x' appears cancel by (b), so we remain with

$$[x',y,z]x + [x,y,z]x' = [x',y,xz] + [x,y,x'z].$$

This is precisely one of the equations in (d). The remaining assertions can be proven similarly. □

We now state some formulas concerning nuclear elements.

8.2.12 Lemma ("Nuclear Slipping Formula", [McC70, 3.1.6], [McC04, 21.2.1]). *Assume that $\mathcal{R}$ is alternative. Then for all $x, y, z \in \mathcal{R}$ and all $n \in \mathrm{Nucl}(\mathcal{R})$, we have*

$$\begin{aligned}n[x,y,z] = [nx,y,z] = [xn,y,z] = [x,ny,z] = [x,yn,z] \\ = [x,y,nz] = [x,y,zn] = [x,y,z]n.\end{aligned}$$

In particular, nuclear elements commute with associators.

Proof. Let $x, y, z \in \mathcal{R}$ and let $n \in \mathrm{Nucl}(\mathcal{R})$. We know from Remark 5.1.11 that in any nonassociative ring, we have

$$\begin{aligned}n[x,y,z] &= [nx,y,z], & [xn,y,z] &= [x,ny,z], \\ [x,yn,z] &= [x,y,nz], & [x,y,zn] &= [x,y,z]n.\end{aligned}$$

Further, since the associator is alternating, we also have

$$n[x,y,z] = -n[y,x,z] = -[ny,x,z] = [x,ny,z].$$

A similar computations yields that we also have $n[x,y,z] = [x,y,nz]$. Further,

$$[x,y,nz] = [x,yn,z] = -[x,z,yn] = -[x,z,y]n = [x,y,z]n.$$

Combining all statements, the assertion follows. □

8.2.13 Lemma ([McC70, (1.8), (1.9)]). *Assume that $\mathcal{R}$ is alternative. Then the following statements hold, where $[x,y] := xy - yx$ denotes the commutator for $x, y \in \mathcal{R}$:*

(a) $\big[\mathrm{Nucl}(\mathcal{R}), \mathcal{R}\big] \subseteq \mathrm{Nucl}(\mathcal{R})$.

(b) $[x,n][x,y,z] = 0_{\mathcal{R}}$ *for all $x, y, z \in \mathcal{R}$ and all $n \in \mathrm{Nucl}(\mathcal{R})$.*

Proof. Let $x, y, z \in \mathcal{R}$ and let $n \in \mathrm{Nucl}(\mathcal{R})$. Then it follows from Lemma 8.2.12 that

$$\big[[n,x],y,z\big] = [nx,y,z] - [xn,y,z] = n[x,y,z] - n[x,y,z] = 0,$$

which implies the first assertion. The second assertion follows from another application of Lemma 8.2.12 in combination with the left bumping formula (Lemma 8.2.11 (c)): We have

$$\begin{aligned}(xn)[x,y,z] = x\big(n[x,y,z]\big) = x\big([x,y,z]n\big) = \big(x[x,y,z]\big)n \\ = [x,yx,z]n = n[x,yx,z] = n\big(x[x,y,z]\big) = (nx)[x,y,z],\end{aligned}$$

which implies that $[x,n][x,y,z] = 0_{\mathcal{R}}$. □

We end this section with some results on associative subrings of alternative rings.

8.2.14 Definition (Strongly associative set, [GPR, 14.1]). A subset A of $\mathcal{R}$ is called *strongly associative* if $[a, a', x] = 0_{\mathcal{R}}$ for all $a, a' \in A$ and all $x \in \mathcal{R}$.

8.2.15 Lemma ([BK51, Theorem I.2]). *Assume that $\mathcal{R}$ is alternative and let A, B, C be subsets of $\mathcal{R}$ satisfying $[A, A, \mathcal{R}] = [B, B, \mathcal{R}] = [C, C, \mathcal{R}] = [A, B, C] = \{0\}$. (That is, A, B, C are strongly associative sets such that $[A, B, C] = \{0\}$.) Then the subring of $\mathcal{R}$ which is generated by $A \cup B \cup C$ is associative.*

8.2.16 Note. Property 8.2.6 (iv) is a special case of Lemma 8.2.15: Assume that $\mathcal{R}$ is alternative and let $x, y \in \mathcal{R}$. Then the sets $A := \{ x \}$, $B := \{ y \}$ and $C := \{ 0 \}$ satisfy the conditions of Lemma 8.2.15, and so the subring of $\mathcal{R}$ generated by $\{ x, y \}$ is associative. Similarly, it follows from Lemma 8.2.15 that the subring generated by $\{ x, y, z \}$ where $x, y, z \in \mathcal{R}$ satisfy $[x, y, z] = 0$ is alternative.

8.2.17 Lemma ([BK51, Theorem I.3]). *Assume that $\mathcal{R}$ is alternative and let A, B be subsets of $\mathcal{R}$ such that A is an associative subring of $\mathcal{R}$ and $[A, A, B] = [B, B, \mathcal{R}] = \{0\}$. (That is, B is a strongly associative subset of $\mathcal{R}$ and A is an associative subring of $\mathcal{R}$ such that $[A, A, B] = \{0\}$.) Then the subring of $\mathcal{R}$ generated by $A \cup B$ is associative.*

The following corollary of Lemma 8.2.17 is a useful generalisation of Artin's Theorem (Proposition 8.2.6).

8.2.18 Proposition. *Assume that $\mathcal{R}$ is alternative and let $x, y \in \mathcal{R}$. Then the subring generated by $\{ x, y \} \cup \mathrm{Nucl}(\mathcal{R})$ is associative.*

Proof. This follows from Lemma 8.2.17 for $A := \mathrm{Nucl}(\mathcal{R})$ and $B := \{ x, y \}$. $\square$

8.3 Rings with Involution

8.3.1 Notation for this section. Unless otherwise specified, we denote by $\mathcal{R}$ a ring with an involution $\sigma \colon \mathcal{R} \to \mathcal{R}$ in the sense of the following Definition 8.3.2.

In this section, we study involutions on rings. All rings with involution that we are interested in will be alternative. However, we will be faced with situations in which a ring with involution is alternative, but not yet proven to be alternative. For this reason, it is more practical to introduce involutions on arbitrary rings. A valuable, albeit not easily accessible reference on this subject is [McC70, Section 9.5]. Another comprehensive book on the subject in the setting of algebras over a field is [Knu+98].

8.3.A Basic Definitions and Observations

8.3.2 Definition (Involution). An *involution of $\mathcal{R}$* is a map $\sigma \colon \mathcal{R} \to \mathcal{R}, r \mapsto r^\sigma$ satisfying $(r + s)^\sigma = r^\sigma + s^\sigma$, $(r \cdot s)^\sigma = s^\sigma \cdot r^\sigma$ and $(r^\sigma)^\sigma = r$ for all $r, s \in \mathcal{R}$. We will refer to the second property as *anti-compatibility with the ring multiplication*. Elements $r \in \mathcal{R}$ with the property that $r^\sigma = r$ are called *symmetric*, and we denote the set of all symmetric elements by $\mathrm{Fix}(\sigma)$.

8.3.3 Example. The identity map is an involution on $\mathcal{R}$ if and only if $\mathcal{R}$ is commutative. If this is the case, it is called the *trivial involution (on $\mathcal{R}$)*.

8.3.4 Example. The conjugation map $\mathbb{C} \to \mathbb{C}, a + bi \mapsto a - bi$ is an involution on the field of complex numbers. More generally, any conic algebra has a conjugation map which is an involution in certain situations. See Lemmas 10.1.19, 10.1.23 and 10.1.27 for more details.

8.3.5 Remark. The set of symmetric elements is an additive subgroup of $(\mathcal{R}, +)$, and it contains $1_{\mathcal{R}}$ because

$$1_{\mathcal{R}} = (1_{\mathcal{R}}^{\sigma})^{\sigma} = (1_{\mathcal{R}}^{\sigma} \cdot 1_{\mathcal{R}})^{\sigma} = 1_{\mathcal{R}}^{\sigma} \cdot (1_{\mathcal{R}}^{\sigma})^{\sigma} = 1_{\mathcal{R}}^{\sigma} \cdot 1_{\mathcal{R}} = 1_{\mathcal{R}}^{\sigma}.$$

However, it is in general not a subring of $\mathcal{R}$: For all $r, s \in \mathrm{Fix}(\sigma)$, we have $(rs)^{\sigma} = s^{\sigma} r^{\sigma} = sr$ which, in general, is distinct from rs.

8.3.6 Lemma. *Let $r \in \mathcal{R}$ be invertible. Then r^{σ} is invertible and $(r^{\sigma})^{-1} = (r^{-1})^{\sigma}$.*

Proof. For all $s \in \mathcal{R}$, we have

$$s^{\sigma} = \big(r^{-1}(rs)\big)^{\sigma} = (s^{\sigma} r^{\sigma})(r^{-1})^{\sigma}.$$

It follows that $(sr^{\sigma})(r^{-1})^{\sigma} = s$ for all $s \in \mathcal{R}$. Similarly, one can show that the other properties in Definition 5.1.12 are satisfied. Thus r^{σ} is invertible and its inverse is $(r^{-1})^{\sigma}$. □

8.3.7 Notation. For any invertible $r \in \mathcal{R}$, we denote the element $(r^{\sigma})^{-1} = (r^{-1})^{\sigma}$ by $r^{-\sigma}$.

We can define norm and trace maps on any ring with involution. The latter in particular will be crucial in the theory of Jordan modules.

8.3.8 Definition (Traces and norms). The map $\mathrm{Tr} = \mathrm{Tr}_{\sigma} \colon \mathcal{R} \to \mathcal{R}, r \mapsto r + r^{\sigma}$ is called the *trace (on $\mathcal{R}$)* and the map $N = N_{\sigma} \colon \mathcal{R} \to \mathcal{R}, r \mapsto r^{\sigma} r$ is called the *norm (on $\mathcal{R}$)*. We will also refer to ring elements which lie in the image of these maps as *traces* and *norms* and denote the sets of such elements by $\mathrm{Tr}(\mathcal{R})$ and $N(\mathcal{R})$, respectively. Further, we define a bi-additive map

$$\mathcal{R} \times \mathcal{R} \to \mathrm{Tr}(\mathcal{R}), (a, b) \mapsto \mathrm{Tr}(a, b) := \mathrm{Tr}(ab^{\sigma})$$

which we also denote by Tr or Tr_{σ}.

8.3.9 Note. In the definition of the norm, we make the somewhat arbitrary choice that $N(r) = r^{\sigma} r$ and not $N(r) = rr^{\sigma}$. Observe that the set $N(\mathcal{R})$ is independent of this choice because the norm of r in the first sense equals the norm of r^{σ} in the second sense. In particular, the notions of nuclear and central involutions that we will introduce in Definition 8.3.12 do not depend on this choice. Further, we will show in Lemma 8.3.20 that the two definitions coincide if the involution σ is central.

8.3.10 Lemma. *The following hold:*

(a) Every trace and every norm in $\mathcal{R}$ is symmetric, and the trace is an additive map.

(b) If $2_{\mathcal{R}}$ is invertible, then $N(\mathcal{R}) \subseteq \mathrm{Tr}(\mathcal{R}) = \mathrm{Fix}(\sigma)$.

Proof. The first assertion is clear. Now assume that $2_{\mathcal{R}}$ is invertible and let $r \in \mathcal{R}$ be symmetric. Then

$$r = \frac{r + r}{2} = \frac{r}{2} + \frac{r^{\sigma}}{2} = \mathrm{Tr}(r/2),$$

which shows that $\mathrm{Fix}(\sigma)$ is contained in $\mathrm{Tr}(\mathcal{R})$. Thus $N(\mathcal{R}) \subseteq \mathrm{Fix}(\sigma) = \mathrm{Tr}(\sigma)$, as desired. □

The following result shows that the mere existence of an involution on a ring makes it easier to prove that a ring is (weakly) alternative. We will use these facts in the blueprint computations for BC, see Lemma 9.10.14 and Proposition 9.10.16.

8.3.11 Lemma. *The following hold:*

(a) We have $[x,y,z]^\sigma = -[z^\sigma, y^\sigma, x^\sigma]$ *for all* $x,y,z \in \mathcal{R}$.

(b) Let $\mathcal{R}'$ *be any ring which admits an involution* σ *and which satisfies* $[x,y,z] = -[x,z,y]$ *for all* $x,y,z \in \mathcal{R}'$. *Then* $\mathcal{R}'$ *is weakly alternative.*

(c) Let $\mathcal{R}'$ *be any ring which admits an involution* σ *and which satisfies* $[x,y,y] = 0$ *for all* $x,y \in \mathcal{R}$. *Then* $\mathcal{R}'$ *is alternative.*

Proof. The first assertion is a simple computation:

$$[x,y,z]^\sigma = \big((xy)z - x(yz)\big)^\sigma = z^\sigma(y^\sigma x^\sigma) - (z^\sigma y^\sigma)x^\sigma = -[z^\sigma, y^\sigma, x^\sigma].$$

The second assertion follows from the first one because

$$[x,y,z] = -[z^\sigma, y^\sigma, x^\sigma]^\sigma = [z^\sigma, x^\sigma, y^\sigma]^\sigma = -[y,x,z]$$

for all $x,y,z \in \mathcal{R}'$, which together with the property $[x,y,z] = -[x,z,y]$ means that $\mathcal{R}'$ is weakly alternative. The third assertion can be proven in a similar way:

$$[x,x,y] = -[y^\sigma, x^\sigma, x^\sigma]^\sigma = -0^\sigma = 0$$

for all $x,y \in \mathcal{R}'$, and thus $\mathcal{R}'$ is alternative. □

If $\mathcal{R}$ is not associative, we usually need additional assumptions on the involution σ to develop a satisfactory theory. In the context of Jordan modules, nuclear involutions are particularly important. The stronger notion of central involutions is not relevant for Jordan modules, but we will see it again in section 10.1.

8.3.12 Definition (Nuclear and central involutions). The involution σ is called *nuclear* if every norm lies in the nucleus of $\mathcal{R}$. It is called *central* if every norm lies in the center of $\mathcal{R}$.

If the ring $\mathcal{R}$ is equipped with the structure of an algebra over some commutative associative ring $\Bbbk$, then we can also define the notion of scalar involutions in a similar way. See Definition 10.1.7.

8.3.13 Remark. Assume that σ is nuclear. Then for all $r \in \mathcal{R}$, the element

$$(1+r)^\sigma(1+r) = 1 + r^\sigma + r + r^\sigma r$$

lies in the nucleus. As the nucleus is closed under addition, this implies that the trace $r + r^\sigma$ lies in the nucleus. Thus if σ is nuclear, then all traces lie in the nucleus as well. Similarly, if σ is central, then all traces lie in the center. If $2_{\mathcal{R}}$ is invertible, then it is clear that the converses of these statements are also true because then every norm is a trace by Lemma 8.3.10 (b). In Propositions 8.3.17 and 8.3.21, we will see that the same remains true for alternative rings even without assumptions on the invertibility of $2_{\mathcal{R}}$.

8.3.B Nuclear Involutions

We now study nuclear involutions. A recurring theme here is that many associativity results involving a ring element r remain true if we replace some (but not necessarily all) occurrences of r by r^σ. The main example of this behaviour is Artin's Theorem for nuclear involutions, see Proposition 8.3.16. Another main result in this subsection is Proposition 8.3.17.

8.3.14 Lemma. *Assume that every trace is nuclear. Then for all $r, s, t \in \mathcal{R}$, we have $[r^\sigma, s, t] = -[r, s, t]$, $[r, s^\sigma, t] = -[r, s, t]$ and $[r, s, t^\sigma] = -[r, s, t]$. In particular, if $\mathcal{R}$ is alternative, then $[r, r^\sigma, s]$, $[r, s, r^\sigma]$ and $[s, r, r^\sigma]$ are zero for all $r, s \in \mathcal{R}$.*

Proof. Let $r, s, t \in \mathcal{R}$. Since σ is nuclear, we have $[r + r^\sigma, s, t] = 0$. This implies that $[r^\sigma, s, t] = -[r, s, t]$. The other assertions can be proven similarly. □

8.3.15 Lemma (Bumping formulas for nuclear involutions). *Assume that $\mathcal{R}$ is alternative and that every trace is nuclear. Then for all $x, y, z \in \mathcal{R}$, the following variations of the bumping formulas from Lemma 8.2.11 are satisfied:*

$$[x^\sigma, y, z]x = [x^\sigma, xy, z] = [x^\sigma, y, xz] \quad \text{and} \quad x[x^\sigma, y, z] = [x^\sigma, yx, z] = [x^\sigma, y, zx]$$

Proof. This is a consequence of Lemma 8.3.14 and the original bumping formulas. For example,

$$[x^\sigma, y, z]x = -[x, y, z]x = -[x, xy, z] = [x^\sigma, xy, z].$$

The other formulas can be derived in the same fashion. □

8.3.16 Proposition (Artin's Theorem for nuclear involutions). *Assume that $\mathcal{R}$ is alternative and that the involution on $\mathcal{R}$ is nuclear. Then for all $r, s \in \mathcal{R}$, the subring of $\mathcal{R}$ generated by $\{ r, s, r^\sigma, s^\sigma \} \cup \mathrm{Nucl}(\mathcal{R})$ is associative.*

Proof. Set $A := \{ r, r^\sigma \}$, $B := \{ s, s^\sigma \}$ and $C := \{ 0 \}$. Then by Lemma 8.3.14, we have

$$[A, A, \mathcal{R}] = [B, B, \mathcal{R}] = [C, C, \mathcal{R}] = [A, B, C] = \{0\}.$$

Thus it follows from Lemma 8.2.15 that the subring D of $\mathcal{R}$ which is generated by $\{ r, r^\sigma, s, s^\sigma \}$ is associative. Applying Lemma 8.2.17, we conclude that the subring generated by $D \cup \mathrm{Nucl}\,\mathcal{R}$ is associative. □

8.3.17 Proposition ([McC70, Section 9.5]). *Assume that $\mathcal{R}$ is alternative. Then the following assertions on the involution σ are equivalent.*

(i) Every norm lies in the nucleus of $\mathcal{R}$. That is, σ is nuclear.

(ii) Every trace lies in the nucleus of $\mathcal{R}$.

Proof. We already know from Remark 8.3.13 that every trace lies in the nucleus if σ is nuclear. Conversely, assume that every trace is nuclear. Let $x, y, z \in \mathcal{R}$. By the linearised right bumping formula (Lemma 8.2.11 (d)), we have

$$[x, x^\sigma, y]z + [z, x^\sigma, y]x = [x, zx^\sigma, y] + [z, xx^\sigma, y].$$

Since the associator is alternating, we can shift all associators on the right-hand side of the equation one step to the left, which yields that

$$[xx^\sigma, y, z] = [x, x^\sigma, y]z + [z, x^\sigma, y]x - [zx^\sigma, y, x].$$

Using Lemma 8.3.14, we obtain that

$$[xx^\sigma, y, z] = -[x, x, y]z + [z^\sigma, x, y]x + [xz^\sigma, y, x] = [x, y, z^\sigma]x - [x, y, xz^\sigma]$$

By the right bumping formula (Lemma 8.2.11 (b)), we conclude that $[xx^\sigma, y, z] = 0$. This shows that every norm is nuclear as well. □

8.3.18 Lemma. *If σ is nuclear and $2_{\mathcal{R}}$ is invertible, then*

$$N(\mathcal{R}) \subseteq \mathrm{Tr}(\mathcal{R}) = \mathrm{Fix}(\sigma) \subseteq \mathrm{Nucl}(\mathcal{R}).$$

Proof. We have seen in Lemma 8.3.10 (b) that $N(\mathcal{R}) \subseteq \mathrm{Tr}(\mathcal{R}) = \mathrm{Fix}(\sigma)$. Since σ is nuclear, we have that $\mathrm{Tr}(\mathcal{R}) \subseteq \mathrm{Nucl}(\mathcal{R})$. □

8.3.19 Lemma. *Assume that $\mathcal{R}$ is alternative and that the involution σ is nuclear. Then the map* $\mathrm{Tr}\colon \mathcal{R} \to \mathcal{R}, a \mapsto a + a^\sigma$ *is associative in the sense that* $\mathrm{Tr}\big((ab)c\big) = \mathrm{Tr}\big(a(bc)\big)$ *for all $a, b, c \in \mathcal{R}$.*

Proof. Let $a, b, c \in \mathcal{R}$. Since

$$\mathrm{Tr}\big((ab)c\big) = (ab)c + c^\sigma(b^\sigma a^\sigma) \quad \text{and} \quad \mathrm{Tr}\big(a(bc)\big) = a(bc) + (c^\sigma b^\sigma)a^\sigma,$$

we have to show that $[a, b, c] = [c^\sigma, b^\sigma, a^\sigma]$. This follows from Lemma 8.3.14 and the fact that the associator is alternating. □

8.3.C Central Involutions

As already mentioned, central involutions are not relevant for the theory of Jordan modules. Still, it is worth recording that an analogue of Proposition 8.3.17 remains true for this type of involution.

8.3.20 Lemma. *Assume that every trace in σ is central. Then $aa^\sigma = a^\sigma a$ and* $\mathrm{Tr}(ab^\sigma) = \mathrm{Tr}(a^\sigma b)$ *for all $a, b \in \mathcal{R}$.*

Proof. Let $a, b \in \mathcal{R}$. Then $a(a + a^\sigma) = (a + a^\sigma)a$, which implies that $aa^\sigma = a^\sigma a$. Linearising this identity, we obtain that

$$ab^\sigma + ba^\sigma = a^\sigma b + b^\sigma a.$$

In other words, $\mathrm{Tr}(ab^\sigma) = \mathrm{Tr}(a^\sigma b)$, as desired. □

8.3.21 Proposition ([McC70, Section 9.5]). *Assume that $\mathcal{R}$ is alternative. Then the following assertions on the involution σ are equivalent.*

(i) Every norm lies in the center of $\mathcal{R}$. That is, σ is central.

(ii) Every trace lies in the center of $\mathcal{R}$.

Proof. We already know from Remark 8.3.13 that every trace lies in the center if σ is central. Now assume that every trace lies in the center. In particular, every trace is nuclear. Thus we already know from Proposition 8.3.17 that every norm is nuclear, which is to say that σ is nuclear. Using Lemmas 8.3.14 and 8.3.20, we now compute that

$$\begin{aligned}(aa^\sigma)b &= a(a^\sigma b) = a\big(\mathrm{Tr}(a^\sigma b) - b^\sigma a\big) = a\,\mathrm{Tr}(a^\sigma b) - a(b^\sigma a) = \mathrm{Tr}(a^\sigma b)a - a(b^\sigma a) \\ &= (ab^\sigma)a + (ba^\sigma)a - a(b^\sigma a) = (ba^\sigma)a = b(a^\sigma a).\end{aligned}$$

for all $a, b \in \mathcal{R}$. In other words, the norm aa^σ lies in the center, as desired. □

8.4 Involutory Sets

8.4.1 Notation for this section. Unless otherwise specified, we denote by $\mathcal{R}$ an alternative ring with nuclear involution σ.

In this section, we study involutory sets, which are a slight refinement of alternative rings with a nuclear involution. The terminology of "involutory sets" was introduced in [TW02, (11.1)], but more general notions had been studied beforehand. We will discuss this in Note 8.4.10.

The following map plays a major role in the context of involutory sets. We will always interpret it in the language of square-scalar multiplications from section 8.1.

8.4.2 Example. Let $\mathcal{R}$ be an alternative ring with nuclear involution. We define a square-scalar multiplication on $\mathcal{R}$ by

$$\omega\colon \mathcal{R}\times\mathcal{R}\to\mathcal{R}, (r,s)\mapsto s^\sigma rs = (s^\sigma r)s = s^\sigma(rs),$$

which is well-defined by Proposition 8.3.16. It is called the *canonical square-scalar multiplication on* $\mathcal{R}$. With this structure, $\mathcal{R}$ is a properly quadratic square-module over itself. It is also multiplicative if $\mathcal{R}$ is associative, but it is not known to us whether the same is true for alternative rings.

In the following, we will frequently apply Proposition 8.3.16 without comment to avoid excessive use of brackets. Further, we will always use the square-module structure on $\mathcal{R}$ given by Example 8.4.2. We begin by identifying several interesting square-submodules of $\mathcal{R}$.

8.4.3 Lemma. $\mathrm{Fix}(\sigma)$ *is a square-submodule of* $\mathcal{R}$.

Proof. It is clear that $\mathrm{Fix}(\sigma)$ is an additive subgroup of $\mathcal{R}$, and we further have $(r^\sigma sr)^\sigma = r^\sigma s^\sigma r = r^\sigma sr$ for all $r\in\mathcal{R}$, $s\in\mathrm{Fix}(\sigma)$. ☐

8.4.4 Lemma. $N(\mathcal{R})$ *is stable under the square-scalar multiplication of* $\mathcal{R}$.

Proof. Let $r,s\in\mathcal{R}$. Then $r^\sigma N(s)r = r^\sigma(s^\sigma s)r = (r^\sigma s^\sigma)(sr) = N(sr)$ by Proposition 8.3.16, so the assertion follows. ☐

8.4.5 Lemma ([McC70, Section 9.5]). *The nucleus of* $\mathcal{R}$ *is a multiplicative square-submodule of* $\mathcal{R}$.

Proof. Let $x,y,z\in\mathcal{R}$ and let $n\in\mathrm{Nucl}(\mathcal{R})$. We want to show that $x^\sigma nx$ lies in the nucleus. Observe that $x^\sigma nx = [x^\sigma,n]x + nx^\sigma x$ where $nx^\sigma x$ lies in the nucleus because the involution is nuclear. Thus

$$[x^\sigma nx,y,z] = \big[[x^\sigma,n]x,y,z\big] + [nx^\sigma x,y,z] = \big[[x^\sigma,n]x,y,z\big].$$

Since $[x^\sigma,n]$ lies in the nucleus by Lemma 8.2.13 (a), it follows from Remark 5.1.11 and Lemma 8.3.14 yields that

$$[x^\sigma nx,y,z] = \big[[x^\sigma,n]x,y,z\big] = [x^\sigma,n][x,y,z] = -[x^\sigma,n][x^\sigma,y,z].$$

Using Lemma 8.2.13 (b), we infer that $[x^\sigma nx,y,z] = 0_{\mathcal{R}}$. By Remark 8.2.8, this shows that $x^\sigma nx$ lies in the nucleus. Thus the nucleus is indeed a square-submodule of $\mathcal{R}$, and it is multiplicative by Artin's Theorem for nuclear involutions (Proposition 8.3.16). ☐

8.4.6 Lemma. *The map* $\mathrm{Tr}\colon \mathcal{R}\to\mathrm{Nucl}(\mathcal{R})\cap\mathrm{Fix}(\sigma), r\mapsto r+r^\sigma$ *is a homomorphism of square-modules. In particular,* $\mathrm{Tr}(\mathcal{R})$ *is a multiplicative square-submodule of* $\mathrm{Nucl}(\mathcal{R})\cap\mathrm{Fix}(\sigma)$.

Proof. For all $r,s\in\mathcal{R}$, we have

$$\mathrm{Tr}(r^\sigma sr) = r^\sigma sr + (r^\sigma sr)^\sigma = r^\sigma sr + r^\sigma s^\sigma r = r^\sigma(s+s^\sigma)r = r^\sigma\,\mathrm{Tr}(s)r$$

which proves the first statement. It follows that $\mathrm{Tr}(\mathcal{R})$ is the image of a homomorphism of square-modules, so it is a square-submodule of $\mathrm{Nucl}(\mathcal{R})\cap\mathrm{Fix}(\sigma)$ by Lemma 8.1.10. Since the square-module $\mathrm{Nucl}(\mathcal{R})$ is multiplicative, so is every square-submodule. ☐

8.4.7 Lemma. *The additive subgroup generated by* $N(\mathcal{R})\cup\mathrm{Tr}(\mathcal{R})$ *is a multiplicative square-submodule of* $\mathrm{Fix}(\sigma)\cap\mathrm{Nucl}(\mathcal{R})$.

Proof. It is a square-submodule of $\mathcal{R}$ by Lemmas 8.4.4 and 8.4.6, it is clearly contained in $\mathrm{Fix}(\sigma)$ and it is contained in $\mathrm{Nucl}(\mathcal{R})$ by Proposition 8.3.17. □

With these elementary observations in mind, we can now define involutory sets.

8.4.8 Definition (Involutory set). A tuple $(\mathcal{R}, \mathcal{R}_0, \sigma)$ is called a *pre-involutory set (in $(\mathcal{R}, \sigma)$)* if $\mathcal{R}$ is an alternative ring, σ is a nuclear involution on $\mathcal{R}$ and $\mathcal{R}_0$ is a square-submodule of $\mathcal{R}$ such that

$$\mathrm{Tr}(\mathcal{R}) \subseteq \mathcal{R}_0 \subseteq \mathrm{Fix}(\sigma) \cap \mathrm{Nucl}(\mathcal{R}).$$

It is called an *involutory set* if, in addition, $\mathcal{R}_0$ contains $1_{\mathcal{R}}$. It is called *associative* if the ring $\mathcal{R}$ is associative and it is called an *involutory division set* if $\mathcal{R}$ is a division ring (in the sense of Definition 5.1.15).

8.4.9 Remark (on Definition 8.4.8). Let $(\mathcal{R}, \mathcal{R}_0, \sigma)$ be a pre-involutory set. If it is an involutory set, then for all $r \in \mathcal{R}$, we have

$$N(r) = r^\sigma 1_{\mathcal{R}} r \in r^\sigma \mathcal{R}_0 r \subseteq \mathcal{R}_0.$$

Thus $\mathcal{R}_0$ contains $N(\mathcal{R})$. Conversely, if $\mathcal{R}_0$ contains $N(\mathcal{R})$, then it contains $N(1_{\mathcal{R}}) = 1_{\mathcal{R}}$.

8.4.10 Note (Involutory sets in the literature). The terminology of "involutory sets" was introduced in [TW02, (11.1)], but in a less general context: Involutory sets in the sense of [TW02] are exactly the associative involutory division sets in our sense. Since [TW02] considers only RGD-systems, the division assumption on involutory sets is not surprising. Further, since every simple alternative ring is either associative or a Cayley-Dickson algebra by [Kle53], an involutory division set is either associative or comes from a Cayley-Dickson algebra. Involutory division sets of the latter form are called *honorary involutory sets* in [TW02, (38.11)]. Hence the non-associative case is not actually absent from [TW02], but simply given a different name.

It should be noted that involutory sets in our sense are the same thing ample subspaces (of $\mathcal{R}$) in [McC70, Section 9.5]. Further, they are a special case of form parameters in the sense of [HO89, 5.1.C] and [Knu91, p. 23], except that all rings in the latter references are associative.

The following observation illustrates that the notion of involutory sets is only a slight refinement of alternative rings with a nuclear involution and that its main purpose is to capture certain "characteristic-2 phenomena".

8.4.11 Remark. Assume that $2_{\mathcal{R}}$ is invertible in $\mathcal{R}$. Then

$$\mathrm{Tr}(\mathcal{R}) \cup N(\mathcal{R}) = \mathrm{Tr}(\mathcal{R}) = \mathrm{Fix}(\sigma) = \mathrm{Fix}(\sigma) \cap \mathrm{Nucl}(\mathcal{R})$$

by Lemma 8.3.18. It follows that $(\mathcal{R}, \mathrm{Fix}(\sigma), \sigma)$ is the only pre-involutory set in $(\mathcal{R}, \sigma)$, and it is in fact an involutory set. Thus if we restrict our attention to rings in which $2_{\mathcal{R}}$ is invertible, then an involutory set carries exactly the same data as an alternative ring with nuclear involution.

In the general situation, denote by T_1 the additive group that is generated by $\mathrm{Tr}(\mathcal{R}) \cup N(\mathcal{R})$. Then

$$\big(\mathcal{R}, \mathrm{Tr}(\mathcal{R}), \sigma\big) \quad \text{and} \quad (\mathcal{R}, T_1, \sigma)$$

are the minimal pre-involutory set and the minimal involutory set in $(\mathcal{R}, \sigma)$, respectively. Further,

$$(\mathcal{R}, \mathrm{Nucl}(\mathcal{R}) \cap \mathrm{Fix}(\sigma), \sigma)$$

is the maximal pre-involutory set in $(\mathcal{R}, \sigma)$, and it is also an involutory set.

We close this section with some basic facts about involutory sets.

8.4.12 Remark. Let $(\mathcal{R}, \mathcal{R}_0, \sigma)$ be a pre-involutory set and assume that $\mathcal{R} = \mathcal{R}_0$. Then $\mathcal{R} = \mathrm{Fix}(\sigma) \cap \mathrm{Nucl}(\mathcal{R})$, so σ must be the identity map and $\mathcal{R}$ must be associative. By Example 8.3.3, the first fact implies that $\mathcal{R}$ must be commutative as well. We conclude that the case $\mathcal{R} = \mathcal{R}_0$ can only arise under very specific circumstances.

8.4.13 Remark. Let $(\mathcal{R}, \mathcal{R}_0, \sigma)$ and $(\mathcal{R}', \mathcal{R}_0', \sigma')$ be two pre-involutory sets. Then $(\mathcal{R} \oplus \mathcal{R}', \mathcal{R}_0 \oplus \mathcal{R}_0', \sigma \oplus \sigma')$ is also an involutory set where $\sigma \oplus \sigma'$ is the nuclear involution which maps (r, s) to $(r^\sigma, s^{\sigma'})$. Note that this involutory set is never an involutory division set unless $\mathcal{R}$ or $\mathcal{R}'$ is zero. Further, it is an involutory set if and only if both $(\mathcal{R}, \mathcal{R}_0, \sigma)$ and $(\mathcal{R}', \mathcal{R}_0', \sigma')$ are involutory sets.

8.4.14 Lemma. *Let $(\mathcal{R}, \mathcal{R}_0, \sigma)$ be a pre-involutory set, let $h \in \mathcal{R}_0$ and let $r, s \in \mathcal{R}$. Then $r^\sigma hs + s^\sigma hr$ lies in $\mathcal{R}_0$.*

Proof. Note that, since h is nuclear, the expression $r^\sigma hs + s^\sigma hr$ is well-defined. Further,

$$r^\sigma hs + s^\sigma hr = (r+s)^\sigma h(r+s) - r^\sigma hr - s^\sigma hs$$

where the element on the right-hand side lies in $\mathcal{R}_0$ because each summand does. This finishes the proof. □

8.4.15 Remark. Even when h is an arbitrary element of $\mathcal{R}$, the expression on the right-hand side of the equation in the proof of Lemma 8.4.14 is still well-defined by Proposition 8.3.16. This observation yields that

$$(r^\sigma h)s + (s^\sigma h)r = r^\sigma(hs) + s^\sigma(hr)$$

for all $r, s, h \in \mathcal{R}$, but this element does not necessarily lie in $\mathcal{R}_0$.

The following argument is used several times in [TW02, Chapter 11], where all involutory sets are assumed to have division. It has some important consequences for pseudo-quadratic modules over these involutory sets, which we collect in Lemmas 8.5.20 and 8.5.21. In our more general settings, these arguments do not apply. We merely state them to illustrate the additional difficulties that arise in the non-division setting.

8.4.16 Lemma. *Assume that $\mathcal{R}$ is a division ring and that $\mathcal{R}_0 \neq \mathcal{R}$. Let $s \in \mathcal{R}$ such that $sr \in \mathcal{R}_0$ for all $r \in \mathcal{R}$. Then $s = 0_{\mathcal{R}}$.*

Proof. Assume that $s \neq 0$ and choose $t \in \mathcal{R} \setminus \mathcal{R}_0$. Then by putting $r := s^{-1}t$, we infer that $t = sr \in \mathcal{R}_0$, which is a contradiction. □

8.5 Pseudo-quadratic Modules

In this section, we study pseudo-quadratic forms on modules. These objects were introduced over associative division rings by Tits in [Tit74, Section 8.2] during the classification of thick spherical buildings of rank at least 3. Unsurprisingly, they also play in important role in the classification of thick spherical Moufang buildings of rank 2 in [TW02]. Most arguments in this section stem from the latter reference, though in some cases, they only prove weaker results in our more general setting.

Any pseudo-quadratic module is defined with respect to some involutory set. As usual, we cannot restrict our attention to division rings in the context of root graded groups. In contrast, it is a surprising observation that we even have to allow alternative rings in this setting. (Recall from Definition 8.4.8 that the ring $\mathcal{R}$ in an involutory set is assumed to be alternative, and also that the involution σ is nuclear.) However, it will turn out that this is more of a technical nuisance and that all "interesting" phenomena occur within the associative world. We will investigate this more deeply in section 8.8.

It is noteworthy that pseudo-quadratic modules over associative rings are a special case of quadratic modules over rings with a form parameter in the sense of [HO89, Section 5.1D] (see also Note 8.4.10). However, we are not aware of a reference which specifically covers the special case of pseudo-quadratic forms in the generality of non-division base rings. Further, we do not know a previous reference which treats this subject in the nonassociative setting.

8.5.A Definition and Examples

Before we can define pseudo-quadratic forms, we have to introduce skew-hermitian forms. These objects relate to pseudo-quadratic maps in a similar way as symmetric bilinear forms relate to quadratic maps. In this section, we will only encounter skew-hermitian forms on $\mathcal{R}$-modules. However, we will also need skew-hermitian forms on square-modules over $\mathcal{R}$ to define Jordan modules in section 8.6, which is why we define them in this generality.

8.5.1 Definition (Sesquilinear form). Let $\mathcal{R}$ be an alternative ring with involution σ and let (M,ω) be a right square-module over $\mathcal{R}$. A *sesquilinear form on M with respect to σ* is a bi-additive map $f\colon M\times M\to \operatorname{Nucl}(\mathcal{R})$ which satisfies

$$f\big(\omega(u,r),\omega(v,s)\big) = r^\sigma f(u,v)s$$

for all $r,s\in\mathcal{R}$ and $u,v\in M$. It is called *hermitian with respect to σ* if, in addition, it satisfies $f(u,v)^\sigma = f(v,u)$ for all $u,v\in M$. It is called *skew-hermitian with respect to σ* if it satisfies $f(u,v)^\sigma = -f(v,u)$. The zero map $M\times M\to\operatorname{Nucl}(\mathcal{R})$ is called the *trivial sesquilinear form*.

8.5.2 Note. The requirement in Definition 8.5.1 that the image of f lies in the nucleus of $\mathcal{R}$ is necessary because otherwise, the formula $f(ur,vs) = r^\sigma f(u,v)s$ is not well-defined. Alternatively, we could define a sesquilinear form on M to be a bi-additive map $f\colon M\times M\to\mathcal{R}$ which satisfies the two separate formulas $f(ur,v) = r^\sigma f(u,v)$ and $f(u,vs) = f(u,v)s$ for all $u,v\in M$ and all $r,s\in\mathcal{R}$. If f is a sesquilinear form in this sense, then

$$\big(r^\sigma f(u,v)\big)s = f(ur,v)s = f(ur,vs) = r^\sigma f(u,vs) = r^\sigma\big(f(u,v)s\big)$$

for all $u,v\in M$ and $r,s\in\mathcal{R}$. In other words, the image of f necessarily lies in the middle nucleus of $\mathcal{R}$. For alternative rings, we know from Remark 8.2.8 that the middle nucleus coincides with the nucleus, so the two definitions of sesquilinear forms are equivalent.

8.5.3 Definition (Pseudo-quadratic module). Let $(\mathcal{R},\mathcal{R}_0,\sigma)$ be a pre-involutory set and let M be a right module over $\mathcal{R}$. A *pseudo-quadratic form on M with respect to $(\mathcal{R},\mathcal{R}_0,\sigma)$* is a map $q\colon M\to\operatorname{Nucl}(\mathcal{R})$ for which there exists a skew-hermitian form f on M such that the following properties are satisfied:

(i) $q(u+v) \equiv q(u) + q(v) + f(u,v) \pmod{\mathcal{R}_0}$ for all $u, v \in M$.

(ii) $q(ur) \equiv r^\sigma q(u) r \pmod{\mathcal{R}_0}$ for all $u \in M$, $r \in \mathcal{R}$.

A map f with these properties is called a *skew-hermitian form associated to q*. The form q is called *anisotropic* if, in addition, the following axiom is satisfied:

(iii) $q(u) \equiv 0 \pmod{\mathcal{R}_0}$ implies $u = 0_M$ for all $u \in M$.

Two pseudo-quadratic forms q, q' are called *equivalent* if $q(v) \equiv q(v') \pmod{\mathcal{R}_0}$ for all $v \in M$. A pseudo-quadratic form is called *trivial* if it is equivalent to the zero map. A *pseudo-quadratic module (over $(\mathcal{R}, \mathcal{R}_0, \sigma)$)* is a triple (M, q, f) consisting of right module M over $\mathcal{R}$, a pseudo-quadratic form q on M and a skew-hermitian form f associated to q. If $\mathcal{R}$ is an associative division ring, these objects will also be called *pseudo-quadratic spaces*. A pseudo-quadratic module (M, q, f) is called *anisotropic* if q is anisotropic.

8.5.4 Remark. We have already seen in Remark 8.4.12 that a pre-involutory set can satisfy $\mathcal{R} = \mathcal{R}_0$ only under very specific assumptions. Assume that this is the case. Then all axioms in Definition 8.5.3 are trivially satisfied, so every map $q\colon M \to \mathcal{R}$ is a pseudo-quadratic form. Further, the only anisotropic pseudo-quadratic module in this situation is the zero module.

In [TW02], all pseudo-quadratic modules of interest are anisotropic. It follows from previous remarks that if (M, q, f) is an anisotropic pseudo-quadratic module over $(\mathcal{R}, \mathcal{R}_0, \sigma)$, then either $M = \{0\}$ or $\mathcal{R} \neq \mathcal{R}_0$.

8.5.5 Note. In [TW02, (11.19)], it is shown that in any pseudo-quadratic module (M, q, f) over an associative involutory division set $(\mathcal{R}, \mathcal{R}_0, \sigma)$ with $\mathcal{R} \neq \mathcal{R}_0$, the map f is uniquely determined by q. (We will repeat the proof of this fact in Lemma 8.5.21.) Using Remark 8.5.4, it follows that the same is true for every anisotropic pseudo-quadratic module. As a consequence, the map f is not part of the structure of a pseudo-quadratic space in the definition in [TW02, (11.17)].

8.5.6 Example. For any $\mathcal{R}$-module M, the zero map from M to $\mathcal{R}$ is a pseudo-quadratic form. Thus $(M, 0, 0)$ is a pseudo-quadratic module.

8.5.7 Example. Let $(\mathcal{R}, \mathcal{R}_0, \sigma)$ be an involutory set and let $n \in \mathbb{N}_+$. Assume that there exists $h \in \mathrm{Nucl}(\mathcal{R})$ such that $h^\sigma = -h$, and choose such an element. We define the following maps:

$$f\colon \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}, (u,v) \mapsto 2\sum_{i=1}^n u_i^\sigma h v_i,$$

$$q\colon \mathcal{R}^n \to \mathcal{R}, u \mapsto \sum_{i=1}^n u_i^\sigma h u_i \ \Big(= \frac{1}{2} f(u,u) \text{ if } 2_{\mathcal{R}} \text{ is invertible}\Big).$$

Note that if $2_{\mathcal{R}} = 0_{\mathcal{R}}$, then f is the zero map, but q is not necessarily trivial. It is clear that f is skew-hermitian with respect to σ. We want to show that q is pseudo-quadratic with associated pseudo-quadratic form f. Let $u, v \in \mathcal{R}^n$ and let $r \in \mathcal{R}$. It is clear that

$$q(ur) = \sum_{i=1}^n (u_i r)^\sigma h u_i r = r^\sigma \left(\sum_{i=1}^n u_i^\sigma h u_i \right) r = r^\sigma q(u) r.$$

(Note that this equality holds in $\mathcal{R}$, not merely modulo $\mathcal{R}_0$.) Further,

$$\begin{aligned} q(u+v) &= \sum_{i=1}^{n} \big(u_i^\sigma h u_i + v_i^\sigma h u_i + u_i^\sigma h v_i + v_i^\sigma h v_i \big) \\ &= q(u) + q(v) + \sum_{i=1}^{n} \big(v_i^\sigma h u_i + u_i^\sigma h v_i \big). \end{aligned}$$

By the choice of h, we have $u_i^\sigma h v_i = -(v_i^\sigma h u_i)^\sigma$ for all $i \in [1,n]$. Put $r_i := v_i^\sigma h u_i$ for all $i \in [1,n]$. Then we see that

$$\begin{aligned} q(u+v) &= q(u) + q(v) + \sum_{i=1}^{n} (r_i - r_i^\sigma) = q(u) + q(v) + 2\sum_{i=1}^{n} r_i - \sum_{i=1}^{n} (r_i + r_i^\sigma) \\ &= q(u) + q(v) + f(u,v) - \sum_{i=1}^{n} (r_i + r_i^\sigma). \end{aligned}$$

Since $\mathcal{R}_0$ contains all traces, it follows that $q(u+v) \equiv q(u) + q(v) + f(u,v) \pmod{\mathcal{R}_0}$.

8.5.8 Example. We can take $(\mathcal{R}, \mathcal{R}_0, \sigma) := (\mathbb{C}, \mathbb{R}, \sigma)$ (where σ denotes complex conjugation), $h := i$ and $n := 1$ in Example 8.5.7. Then for all $a, b, c, d \in \mathbb{R}$, we have

$$\begin{aligned} f(a+bi, c+di) &= 2(a-bi)i(c+di) = 2(b+ai)(c+di) \\ &= 2(bc-ad) + 2(bd+ac)i, \\ q(a+bi) &= \frac{1}{2} f(a+bi, a+bi) = (a^2+b^2)i. \end{aligned}$$

That is, q is precisely the usual norm function on $\mathbb{C}$, but multiplied with i.

The fact that the images of q and f need to lie in the nucleus of $\mathcal{R}$ illustrates that pseudo-quadratic modules do not really belong to the nonassociative world. However, we can still obtain examples of pseudo-quadratic modules over alternative rings which are not associative using direct sum constructions.

8.5.9 Remark (Direct sums of pseudo-quadratic modules). Let $X = (\mathcal{R}, \mathcal{R}_0, \sigma)$ and $X' = (\mathcal{R}', \mathcal{R}_0', \sigma')$ be two involutory sets and let (M, q, f) and (M', q', f') be pseudo-quadratic modules over X and X', respectively. We have seen in Remark 8.4.13 that $X \oplus X' := (\mathcal{R} \oplus \mathcal{R}', \mathcal{R}_0 \oplus \mathcal{R}_0', \sigma \oplus \sigma')$ is also an involutory set. We can equip the group $M \oplus M'$ with an $(\mathcal{R} \oplus \mathcal{R}')$-module structure by putting

$$(u, u') \cdot (r, r') := (ur, u'r')$$

for all $u \in M$, $u' \in M'$, $r \in \mathcal{R}$ and $r' \in \mathcal{R}'$. It is now easy to verify that $(M \oplus M', q \oplus q', f \oplus f')$ is a pseudo-quadratic module over $X \oplus X'$. This pseudo-quadratic module is anisotropic if and only if both (M, q, f) and (M', q', f') are anisotropic.

8.5.10 Note (Pseudo-quadratic modules over alternative rings). Let everything be as in Remark 8.5.9, and assume in addition that $(M', q', f') = (\{0\}, 0, 0)$. Then $M \oplus M'$ can be identified with M, so Remark 8.5.9 yields that the pseudo-quadratic module (M, q, f) over $(\mathcal{R}, \mathcal{R}_0, \sigma)$ can also be regarded as a pseudo-quadratic module over $(\mathcal{R}, \mathcal{R}_0, \sigma) \oplus (\mathcal{R}', \mathcal{R}_0', \sigma')$. Here $(\mathcal{R}', \mathcal{R}_0', \sigma')$ can be an arbitrary involutory set. Taking $(\mathcal{R}', \mathcal{R}_0', \sigma')$ to not be associative, we conclude that every pseudo-quadratic module can be regarded as a pseudo-quadratic module over an involutory set which is not associative.

While the previous example shows that pseudo-quadratic modules over rings which are not associative exist, it should be regarded as a technicality: Only the associative ideal $\mathcal{R} \oplus \{0\}$ of the alternative ring $\mathcal{R} \oplus \mathcal{R}'$ contributes to the (pseudo-quadratic) module structure of M. We would like to say that, while $\mathcal{R} \oplus \mathcal{R}'$ is alternative but not associative, it is not "purely alternative" either. In this example, the ring splits into an alternative part and an associative, but we cannot expect a similar decomposition to exist for arbitrary alternative rings. Thus it is not at all clear what a reasonable definition of "purely alternative rings" should be.

Following [Sla67], we will present a reasonable abstract notion of pure alternativity for rings in section 8.8. We will show that purely alternative rings do not admit any non-zero modules, so in particular, they do not admit non-zero pseudo-quadratic modules. This fact substantiates our previous claim that "pseudo-quadratic modules do not belong to the nonassociative world". However, we emphasise that we are not aware of a theorem which allows us to decompose an arbitrary alternative ring into an associative part and a purely alternative part. Thus there is still some interest in allowing the underlying involutory set $(\mathcal{R}, \mathcal{R}_0, \sigma)$ to not be associative.

8.5.B Basic Properties

8.5.11 Notation for this section. From now on, we denote by $(\mathcal{R}, \mathcal{R}_0, \sigma)$ a pre-involutory set, by M a right module over $\mathcal{R}$ and by q a pseudo-quadratic form with respect to $(\mathcal{R}, \mathcal{R}_0, \sigma)$. Further, unless otherwise specified, the symbol "$\equiv$" will always be used to denote congruence modulo $\mathcal{R}_0$.

8.5.12 Remark. Let $h\colon M \to \mathcal{R}_0$ be any map. Then $q + h$ is also a pseudo-quadratic form on M because all axioms are defined modulo $\mathcal{R}_0$. Further, a skew-hermitian form $f\colon M \times M \to \mathcal{R}$ is associated to $q + h$ if and only if it is associated to q. Thus the equivalence class of q can be identified with the induced map $\tilde{q}\colon M \to \operatorname{Nucl}(\mathcal{R})/\mathcal{R}_0$. (In fact, this is exactly the approach that is taken in [HO89, 5.1D].) However, it will be important in some contexts that we have chosen a fixed coset representative $q(v)$ in $\mathcal{R}$ for every $v \in M$. See, for example, Note 8.5.27.

8.5.13 Remark. It follows from Axiom 8.5.3 (ii) that

$$q(0_M) = q(0_M 0_{\mathcal{R}}) \equiv 0_{\mathcal{R}}^{\sigma} q(0_M) 0_{\mathcal{R}} = 0_{\mathcal{R}}.$$

8.5.14 Remark. Let $r, s \in \mathcal{R}$ such that $r \equiv s$ and let $t \in \mathcal{R}$. Since $t^{\sigma} \mathcal{R}_0 t \subseteq \mathcal{R}_0$, it follows that $t^{\sigma} r t \equiv t^{\sigma} s t$. If $\mathcal{R}_0 t \subseteq \mathcal{R}_0$, then we also have $rt \equiv st$, but this is not true in general. Similarly, all $t \in \mathcal{R}$ such that $t\mathcal{R}_0 \subseteq \mathcal{R}_0$ have the property that $r \equiv s$ implies $ts \equiv ts$. For example, these properties hold for all t in the image of $\mathbb{Z}$ in $\mathcal{R}$.

The following result is the analogue of Lemmas 7.1.5 and 7.1.6 for pseudo-quadratic modules.

8.5.15 Lemma. *Let f be a skew-hermitian form associated to q. Then the following hold:*

- *(a)* $q(0_M) \equiv 0_{\mathcal{R}}$.
- *(b)* $f(v,v) \equiv 2q(v)$ *for all* $v \in M$.
- *(c)* $r^{\sigma} \equiv -r$ *for all* $r \in \mathcal{R}$.
- *(d)* $f(v,u) \equiv f(u,v)$ *for all* $u, v \in M$.

(e) *If $2_{\mathcal{R}}$ is invertible, then the map $q' \colon M \to \mathcal{R}, v \mapsto \frac{f(v,v)}{2}$ is a pseudo-quadratic form which is equivalent to q and which satisfies $q'(vr) = r^\sigma q(v) r$ (in $\mathcal{R}$, not just modulo $\mathcal{R}_0$) for all $v \in M$ and all $r \in \mathcal{R}$.*

(f) *If $2_{\mathcal{R}}$ is invertible and f is trivial, then q is trivial as well.*

Proof. First of all, note that

$$q(0_M) = q(0_M + 0_M) \equiv q(0_M) + q(0_M) + f(0_M, 0_M) = 2q(0_M).$$

This implies that $q(0_M) \equiv 0_{\mathcal{R}}$. In a similar way, we have for all $v \in M$ that

$$q(2v) \equiv 2^\sigma q(v) 2 = 4q(v) \quad \text{and} \quad q(2v) = q(v+v) \equiv 2q(v) + f(v,v).$$

It follows that $2q(v) \equiv f(v,v)$. Since $\mathcal{R}_0$ contains all traces, we have $r + r^\sigma \equiv 0$ for all $r \in \mathcal{R}$. In other words, $r^\sigma \equiv -r$ for all $r \in \mathcal{R}$. This implies that $f(u,v) \equiv f(v,u)$ for all $u, v \in M$ because f is skew-hermitian.

Now assume that $2_{\mathcal{R}}$ is invertible and define $q' \colon M \to \mathcal{R}, v \mapsto \frac{f(v,v)}{2}$. Then by (d) and the definition of q',

$$\begin{aligned} q'(v+u) &= \frac{f(v+u, v+u)}{2} = \frac{f(v,v)}{2} + \frac{f(u,u)}{2} + \frac{f(v,u)}{2} + \frac{f(u,v)}{2} \\ &= q'(v) + q'(u) + \frac{f(v,u)}{2} + \frac{f(u,v)}{2} \equiv q'(v) + q'(u) + f(v,u) \end{aligned}$$

for all $u, v \in M$. Further,

$$q'(vr) = \frac{f(vr, vr)}{2} = \frac{r^\sigma f(v,v) r}{2} = r^\sigma q(v) r$$

for all $v \in M$ and $r \in \mathcal{R}$. Thus (e) holds. Now assume in addition that f is trivial. Then q' is trivial by definition. Since q' is equivalent to q, it follows that q is trivial as well, which finishes the proof. □

In analogy to the orthogonal group of a quadratic module (Definition 7.1.16), we can define the unitary group of a pseudo-quadratic module.

8.5.16 Definition (Unitary group, [HO89, 5.2A]). The group

$$\mathrm{U}(M) := \{ \varphi \in \mathrm{Aut}_{\mathfrak{K}}(M) \mid q(\varphi(v)) \equiv q(v) \text{ and } f(\varphi(u), \varphi(v)) \text{ for all } u, v \in M \}$$

is called *unitary group of* $M = (M, q, f)$. We use the convention that it acts on M from the left side, so that the composition $\varphi \circ \psi$ of $\varphi, \psi \in \mathrm{O}(q)$ is the map $x \mapsto \varphi(\psi(x))$.

8.5.17 Remark. In the following, we will often consider elements $h \in \mathcal{R}$ with the property that $h \equiv q(v)$ for some $v \in M$. Since $\mathcal{R}_0$ and the image of q are, both by definition, contained in the nucleus of $\mathcal{R}$, all such elements are nuclear.

An important result about pseudo-quadratic spaces over associative involutory set is [TW02, (11.19)]. It is not true in the present generality. The following Lemmas 8.5.18 and 8.5.19 are the parts which remain valid in our general setting. Lemmas 8.5.20 and 8.5.21 are exactly the assertion of [TW02, (11.19)], except that the assumption on $\mathcal{R}$ to be associative turns out to be unnecessary. We will not have occasion to apply any of these results, but we state them for the record.

8.5.18 Lemma. *Let $v \in M$ and let $r, s, h \in \mathcal{R}$ such that $h \equiv q(v)$. Then $f(vs, vr) \equiv s^\sigma h r + r^\sigma h s$.*

Proof. On the one hand,

$$q(v(s+r)) \equiv q(vs) + q(vr) + f(vs, vr) \equiv s^\sigma q(v) s + r^\sigma q(v) r + f(vs, vr).$$

On the other hand,

$$q\big(v(s+r)\big) \equiv (s+r)^\sigma q(v)(s+r) \equiv s^\sigma q(v)s + r^\sigma q(v)r + s^\sigma q(v)r + r^\sigma q(v)s.$$

This implies that the assertion holds for $h = q(v)$. In general, we have $h = q(v) + t$ for some $t \in \mathcal{R}_0$, and it remains to show that $s^\sigma tr + r^\sigma ts \equiv 0$. This is exactly the statement of Lemma 8.4.14, so the proof is finished. □

8.5.19 Lemma. *Let $v \in M$ and let $h \in \mathcal{R}$ such that $h \equiv q(v)$. Then*

$$\big(f(v,v) - h + h^\sigma\big)r \equiv 0 \quad \text{for all } r \in \mathcal{R}.$$

In particular, $f(v,v) \equiv h - h^\sigma \equiv q(v) - q(v)^\sigma$.

Proof. Putting $s := 1_{\mathcal{R}}$ in Lemma 8.5.18 yields $f(v,v)r - hr - r^\sigma h \equiv 0$. Since $r^\sigma h \equiv -(r^\sigma h)^\sigma = -h^\sigma r$, the assertion follows. □

Observe that the assertion of Lemma 8.5.19 is stronger than the special case $f(v,v) \equiv h - h^\sigma$ by Remark 8.5.14. However, for division rings, we have the following result.

8.5.20 Lemma. *Assume that $\mathcal{R}$ is a division ring and that $\mathcal{R}_0 \neq \mathcal{R}$. Then for all $v \in M$ and all $h \in \mathcal{R}$ with $h \equiv q(v)$, we have $f(v,v) = h - h^\sigma$ (in $\mathcal{R}$, not just modulo $\mathcal{R}_0$). In particular, $f(v,v) = q(v) - q(v)^\sigma$ for all $v \in M$.*

Proof. For any $v \in M$ and any $h \in q(v) + \mathcal{R}_0$, it follows from Lemmas 8.4.16 and 8.5.19 that $f(v,v) - h + h^\sigma$ is zero. Since $q(v)$ lies in $q(v) + \mathcal{R}_0$, the second assertion follows. □

8.5.21 Lemma. *Assume that $\mathcal{R}$ is a division ring and that $\mathcal{R}_0 \neq \mathcal{R}$. Then there exists exactly one skew-hermitian form which is associated to q.*

Proof. Assume that f, f' are two such forms. Then $g := f - f'$ is a sesquilinear form whose image is contained in $\mathcal{R}_0$. For all $u, v \in M$ and $r \in \mathcal{R}$, it follows that $g(u,v)r = g(u,vr) \in \mathcal{R}_0$. By Lemma 8.4.16, this implies that $g(u,v) = 0_{\mathcal{R}}$ for all $u, v \in \mathcal{R}$, so $f = f'$. □

Motivated by Lemma 8.5.20, we introduce the following definition.

8.5.22 Definition (Standard pseudo-quadratic module). A pseudo-quadratic module (M, q, f) over $(\mathcal{R}, \mathcal{R}_0, \sigma)$ is called *standard* if $f(v,v) = q(v) - q(v)^\sigma$ for all $v \in M$.

8.5.23 Note. Being standard is a rather weak condition which is automatically satisfied in many situations:

(1) We have seen in Lemma 8.5.20 that every pseudo-quadratic module over a division ring with $\mathcal{R} \neq \mathcal{R}_0$ is standard. Further, the zero module is clearly standard as well. By Remark 8.5.4, this means that every pseudo-quadratic module that appears in [TW02] is standard.
(2) Assume that $2_{\mathcal{R}}$ is invertible and let (M, q, f) be a pseudo-quadratic module. It follows from Lemma 8.5.15 (e) that q is equivalent to a pseudo-quadratic form q' such that (M, q', f) is a standard pseudo-quadratic module.

We will see that the standard condition is necessary to construct a Jordan module from a pseudo-quadratic module (Example 8.6.7). Further, every pseudo-quadratic module which arises from a Jordan module (that is, from a root grading of type C or BC) is standard (Lemma 8.7.14). We conclude that standardness is the right condition to extend the theory of pseudo-quadratic modules to our general setting.

8.5.C The Group $T(M)$

8.5.24 Notation for this section. In addition to Notation 8.5.11, we will from now on fix a skew-hermitian form f such that (M, q, f) is a pseudo-quadratic module over $(\mathcal{R}, \mathcal{R}_0, \sigma)$.

For any pseudo-quadratic module M, we can define a group $T(M)$. If M is standard, this group can be equipped with the structure of a Jordan module (Example 8.6.7). In section 8.7, we will show that every Jordan module arises in this way if $2_{\mathcal{R}}$ is invertible.

8.5.25 Definition. We define

$$T(M) := T(M, q, f) := \{\, (u, h) \in M \times \mathcal{R} \mid q(u) \equiv h \,\}$$

and $(u, h) \cdot (v, k) := \big(u + v, h + k + f(u, v)\big)$ for all $(u, h), (v, k) \in T(q)$.

Observe that the multiplication on $T(M)$ depends on the choice of the skew-hermitian form f. Further, this multiplication is in general not abelian because f is, in general, not symmetric.

8.5.26 Remark. We have a canonical bijection

$$M \times \mathcal{R}_0 \to T(M), (u, h) \mapsto \big(u, q(u) + h\big)$$

of sets which, in general, is not a group homomorphism with respect to the canonical group structure on $M \times \mathcal{R}_0$.

8.5.27 Note. If q' is another pseudo-quadratic form which is equivalent to q, then $T(M, q, f)$ and $T(M, q', f)$ are not only isomorphic, but in fact the same set with the same group multiplication. However, the bijection in Remark 8.5.26 depends on the choice of q.

8.5.28 Remark. If q is trivial, then $T(M) = M \times \mathcal{R}_0$ as sets and the bijection in Remark 8.5.26 is the identity map. However, the multiplication on $T(M)$ is not necessarily the component-wise addition on $M \times \mathcal{R}_0$. This is only the case if f is trivial as well. Note that if $2_{\mathcal{R}}$ is invertible, then the triviality of f implies the triviality of q by Lemma 8.5.15 (f).

8.5.29 Remark. The map $\mathcal{R}_0 \to T(M), r_0 \mapsto (0_M, r_0)$ is an injective homomorphism of square-modules. It is well-defined because $q(0_M) \equiv 0_{\mathcal{R}}$ by Remark 8.5.13.

8.5.30 Remark. Let $(u, h), (v, k) \in T(M)$. Then

$$(u, h) \cdot (v, k) = (v, k) \cdot (u, h) \cdot \big(0, f(u, v) - f(v, u)\big)$$

where $f(u, v) - f(v, u) = f(u, v) + f(u, v)^{\sigma} = \mathrm{Tr}_{\sigma}(f(u, v))$.

8.5.31 Lemma. *The multiplication on $T(M)$ defines a group structure with identity element $(0_M, 0_{\mathcal{R}})$ and inverses given by*

$$(u, h)^{-1} = \big(-u, f(u, u) - h\big)$$

for all $(u, v) \in T(q)$.

Proof. For all $(u, h), (v, k) \in T(M)$, it follows from the axioms of a pseudo-quadratic form that

$$q(u + v) \equiv q(u) + q(v) + f(u, v) \equiv h + k + f(u, v),$$

so $(u+v, h+k+f(u,v))$ lies in $T(q)$. We infer that the multiplication on $T(M)$ is well-defined. Further, it is associative because for any $(u_1,h_1),(u_2,h_2),(u_3,h_3) \in T(M)$, both $\big((u_1,h_1)(u_2,h_2)\big)(u_3,h_3)$ and $(u_1,h_1)\big((u_2,h_2)(u_3,h_3)\big)$ equal

$$\big(u_1+u_2+u_3, h_1+h_2+h_3+f(u_1,u_2)+f(u_1,u_3)+f(u_2,u_3)\big).$$

It is clear that $(0,0)$ is an identity element in $T(q)$. Finally, for all $(u,h) \in T(q)$, we have

$$\begin{aligned}(u,h)\big(-u, f(u,u)-h\big) &= \big(0, h-h+f(u,u)-f(u,u)\big) = (0,0)\\ &= \big(-u, f(u,u)-h\big)(u,h),\end{aligned}$$

so (u,h) is invertible with inverse $\big(-u, f(u,u)-h\big)$. This finishes the proof. □

8.5.32 Remark. Note that, if (M,q,f) is standard, then the inverse of $(u,h) \in T(q)$ is given by $(-u, -h^\sigma)$.

The following observation will be used to define (non-balanced) Weyl elements in root gradings of type BC. See Definition 9.3.10.

8.5.33 Lemma. *Let (M,q,f) be a pseudo-quadratic module over $(\mathcal{R},\mathcal{R}_0,\sigma)$ and let $(u,h) \in T(M)$ such that h is invertible. Then $(uh^{-1}, h^{-\sigma})$ and $(-uh^{-\sigma}, h^{-\sigma})$ lie in $T(M)$.*

Proof. Since (u,h) lies in $T(M)$, there exists $r_0 \in \mathcal{R}_0$ such that $q(u) = h + r_0$. Thus

$$\begin{aligned} q(uh^{-1}) &\equiv h^{-\sigma}q(u)h^{-1} = h^{-\sigma} + h^{-\sigma}r_0h^{-1} \equiv h^{-\sigma} \qquad \text{and}\\ q(-uh^{-\sigma}) &\equiv h^{-1}q(u)h^{-\sigma} = h^{-\sigma} + h^{-1}r_0h^{-\sigma} \equiv h^{-\sigma}.\end{aligned}$$

This proves the claim. □

We now define several maps on $T(M)$ which will later be seen to turn $T(M)$ into a Jordan module over $(\mathcal{R},\sigma)$. As a first step, we define a square-scalar multiplication on $T(M)$.

8.5.34 Lemma. *Define*

$$\varphi\colon T(M)\times\mathcal{R} \to T(M), \big((v,h),r\big) \mapsto (vr, r^\sigma hr).$$

Then $(T(M),\varphi)$ is a well-defined multiplicative square-module.

Proof. For all $(v,h) \in T(M)$ and all $r \in \mathcal{R}$, it follows from Axiom 8.5.3 (ii) and Remark 8.5.14 that

$$q(vr) \equiv r^\sigma q(v)r \equiv r^\sigma hr,$$

so that $(vr, r^\sigma hr)$ lies in $T(M)$. Thus φ is well-defined. It is clear that $\varphi(t, 1_{\mathcal{R}}) = t$ for all $t \in T(M)$. For all $r,s \in \mathcal{R}$ and all $(v,h) \in T(M)$, we have $(vr)s = v(rs)$ by Axiom 5.1.22 (iii) and $s^\sigma(r^\sigma hr)s = (s^\sigma r^\sigma)h(rs)$ by Proposition 8.3.16 and Remark 8.5.17. Hence φ is multiplicative.

To see that φ is weakly quadratic in the second component, let $(v,h) \in T(M)$ and let $r,s \in \mathcal{R}$. By the formula for inverses in $T(M)$ from Lemma 8.5.31, we have

$$\begin{aligned}&\varphi\big((v,h),s\big)^{-1}\varphi\big((v,h),r\big)^{-1}\\ &\qquad= (vs, s^\sigma hs)^{-1}(vr, r^\sigma hr)^{-1}\\ &\qquad= \big(-vs, f(vs,vs)-s^\sigma hs\big)\big(-vr, f(vr,vr)-r^\sigma hr\big)\\ &\qquad= \big(-vs-vr, f(vs,vs)-s^\sigma hs+f(vr,vr)-r^\sigma hr+f(vs,vr)\big).\end{aligned}$$

Further,

$$\varphi\big((v,h),r+s\big) = (vr+vs, r^\sigma hr + r^\sigma hs + s^\sigma hr + s^\sigma hs).$$

Thus the polarisation of φ in the second component is

$$\begin{aligned}
&\varphi\big((v,h),s\big)^{-1}\varphi\big((v,h),r\big)^{-1}\varphi\big((v,h),r+s\big)\\
&\quad= \big(0_M, f(vs,vs) - s^\sigma hs + f(vr,vr) - r^\sigma hr + f(vs,vr)\\
&\qquad\qquad + r^\sigma hr + r^\sigma hs + s^\sigma hr + s^\sigma hs - f(vs+vr, vs+vr)\big)\\
&\quad= \big(0_M, r^\sigma hs + s^\sigma hr - f(vr,vs)\big).
\end{aligned}$$

This expression is bi-additive in (r,s), so φ is weakly quadratic in the second component. Finally, we have for all $(v,h),(v',h') \in T(M)$ and $r \in \mathcal{R}$ that

$$\begin{aligned}
\varphi\big((v,h)(v',h'),r\big) &= \varphi\big((v+v', h+h'+f(v,v')),r\big)\\
&= \big(vr+v'r, r^\sigma hr + r^\sigma h'r + r^\sigma f(v,v')r\big)\\
&= \big(vr+v'r, r^\sigma hr + r^\sigma h'r + f(vr,v'r)\big)\\
&= (vr, r^\sigma hr)(v'r, r^\sigma h'r) = \varphi\big((v,h),r\big)\varphi\big((v',h'),r\big).
\end{aligned}$$

Hence φ is additive in the first component. This finishes the proof. □

8.5.35 Remark. If (M,q,f) is standard, then the polarisation of φ in Lemma 8.5.34 in the second component is

$$\big(0_M, r^\sigma hs + s^\sigma hr - f(vr,vs)\big) = \big(0_M, s^\sigma hr + s^\sigma h^\sigma s\big) = \big(0_M, \mathrm{Tr}_\sigma(s^\sigma hr)\big).$$

Here the "polarisation in the second component" is defined as in 7.1.2.

8.5.36 Observation. By Definition 8.1.12, we have a canonical involution on the multiplicative square-module $(T(M),\varphi)$. It is precisely the map

$$\overline{\cdot}\colon T(M) \to T(M), (u,h) \mapsto (-u,h).$$

As a next step, we define maps π_1 and T_1.

8.5.37 Lemma. *The projection map*

$$\pi_1\colon T(M) \to \mathrm{Nucl}(\mathcal{R}), (v,h) \mapsto h$$

is compatible with the square-scalar multiplication φ from Lemma 8.5.34.

Proof. We denote the canonical square-scalar multiplication on $\mathrm{Nucl}(\mathcal{R})$ by ω (see Example 8.4.2 and Lemma 8.4.5). At first, observe that the image of π_1 is indeed contained in the nucleus of $\mathcal{R}$ by Remark 8.5.17. Now for all $t = (v,h) \in T(M)$ and for all $r \in \mathcal{R}$, we have

$$\pi_1\big(\varphi(t,r)\big) = \pi_1\big((vr, r^\sigma hr)\big) = r^\sigma hr = \omega(h,r) = \omega\big(\pi_1(t),r\big).$$

This finishes the proof. □

8.5.38 Lemma. *The map*

$$\mathrm{T}_1\colon \mathcal{R} \to T(M), r \mapsto \big(0_M, \mathrm{Tr}_\sigma(r)\big) = (0_M, r+r^\sigma)$$

is a homomorphism of square-modules.

Proof. This map is the composition of the map $\mathrm{Tr}_\sigma\colon \mathcal{R} \to \mathcal{R}_0$ from Lemma 8.4.6 with the embedding $\mathcal{R}_0 \to T(M)$ from Remark 8.5.29. Since both maps are homomorphisms of square-modules, the assertion follows. □

We have now defined all maps that are needed to equip M with the structure of a Jordan module. Before we turn to the definition of Jordan modules, we make a final observation.

8.5.39 Remark (Direct products and $T(M)$). Write $X := (\mathcal{R}, \mathcal{R}_0, \sigma)$, let $X' = (\mathcal{R}', \mathcal{R}'_0, \sigma')$ be another involutory sets and let (M', q', f') be a pseudo-quadratic module over X'. By Remark 8.5.9, we can equip $M \oplus M'$ with the structure of a pseudo-quadratic module over $X \oplus X'$. We then have

$$T(M \oplus M') = \left\{ \big((u,u'),(h,h')\big) \;\middle|\; \begin{array}{c} u \in M, u' \in M', h \in \mathcal{R}, h' \in \mathcal{R}', \\ (q \oplus q')(u,u') \equiv (h,h') \pmod{\mathcal{R}_0 \oplus \mathcal{R}'_0} \end{array} \right\},$$

which is clearly isomorphic to

$$T(M) \oplus T(M') = \left\{ \big((u,h),(u',h')\big) \;\middle|\; \begin{array}{c} u \in M, u' \in M', h \in \mathcal{R}, h' \in \mathcal{R}', \\ q(u) \equiv h \pmod{\mathcal{R}_0}, \\ q'(u') \equiv h' \pmod{\mathcal{R}'_0} \end{array} \right\}.$$

Under this identification, the maps φ, π_1 and T_1 defined on $M \oplus M'$ are precisely the respective sums of the maps φ, π_1, T_1 defined on M and M'. In the (not yet introduced) terminology of Jordan modules, this says precisely that the identification between $T(M \oplus M')$ and $T(M) \oplus T(M')$ is an isomorphism of Jordan modules.

8.6 Jordan Modules

Finally, we can define to the protagonists of this chapter.

8.6.A Definition and Examples

8.6.1 Definition (Jordan module). Let $\mathcal{R}$ be an alternative ring with a nuclear involution σ and denote the square-scalar multiplication on $\mathcal{R}$ from Example 8.4.2 by ω. A *Jordan module over* $(\mathcal{R}, \sigma)$ is a tuple $\mathcal{J} = (J, \varphi, \pi_1, \mathrm{T}_1, \psi)$ consisting of a multiplicative right square-module (J, φ) over $\mathcal{R}$, a map

$$\pi_1 \colon (J, \varphi) \to (\mathrm{Nucl}(\mathcal{R}), \omega)$$

which preserves the square-scalar multiplication, a homomorphism

$$\mathrm{T}_1 \colon (\mathcal{R}, \omega) \to (J, \varphi)$$

of square-modules and a skew-hermitian form

$$\psi \colon J \times J \to \mathrm{Nucl}(\mathcal{R})$$

with respect to σ such that the following conditions are satisfied:

(i) $\mathrm{T}_1(a^\sigma) = \mathrm{T}_1(a)$ and $\mathrm{T}_1\big((ab)c\big) = \mathrm{T}_1\big(a(bc)\big)$ for all $a, b, c \in \mathcal{R}$. Thus we can omit brackets in the argument of T_1 without ambiguity.

(ii) $\pi_1(u)^\sigma = \pi_1(u) - \psi(u,u)$ for all $u \in J$.

(iii) $\psi\big(\mathrm{T}_1(a), u\big) = 0 = \psi\big(u, \mathrm{T}_1(a)\big)$ for all $a \in \mathcal{R}$, $u \in J$.

(iv) The following "linearisation properties" hold for all $u, v \in J$ and all $a, b \in \mathcal{R}$:

$$\pi_1(u \hat{+} v) = \pi_1(u) + \pi_1(v) + \psi(u,v), \tag{8.1}$$

$$\varphi(u, a+b) = \varphi(u,a) \hat{+} \varphi(u,b) \hat{+} \mathrm{T}_1\big(b^\sigma \pi_1(u) a\big), \tag{8.2}$$

$$u \hat{+} v = v \hat{+} u \hat{+} \mathrm{T}_1\big(\psi(u,v)\big). \tag{8.3}$$

(v) For all $r \in \mathcal{R}$, we have $\pi_1(\mathrm{T}_1(r)) = r + r^\sigma$.

(vi) There exists $v_0 \in J$ such that $\pi_1(v_0) = 1_{\mathcal{R}}$ and $\psi(v_0, u) = 0_{\mathcal{R}} = \psi(u, v_0)$ for all $u \in J$.

Further, the Jordan module $\mathcal{J}$ is called *abelian* if $(J, \hat{+})$ is an abelian group, it is called *of type C* if ψ is the zero map and it is called *anisotropic* if $\pi_1(u) = 0_{\mathcal{R}}$ implies $u = 0$ for all $u \in J$. If the Jordan module $\mathcal{J}$ is fixed, we will refer to π_1 as the *Jordan module projection*, to T_1 as the *Jordan module trace* and to ψ as the *Jordan module skew-hermitian form*. We will sometimes refer to J as a Jordan module if the remaining objects in $\mathcal{J}$ are clear from the context.

8.6.2 Remark. The requirements on the maps $\pi_1, \mathrm{T}_1, \psi$ which are given in the first paragraph of Definition 8.6.1 can be expressed by the following formulas:

$$\begin{aligned}
\pi_1\big(\varphi(u,a)\big) &= a^\sigma \pi_1(u) a, \\
\mathrm{T}_1(a+b) &= \mathrm{T}_1(a) \hat{+} \mathrm{T}_1(b), \\
\varphi\big(\mathrm{T}_1(a), b\big) &= \mathrm{T}_1(b^\sigma a b), \\
\psi(u \hat{+} u', v \hat{+} v') &= \psi(u,v) + \psi(u,v') + \psi(u',v) + \psi(u',v'), \\
\psi(u,v)^\sigma &= -\psi(v,u), \\
\psi\big(\varphi(u,a), \varphi(v,b)\big) &= a^\sigma \psi(u,v) b
\end{aligned}$$

for all $u, v, u', v' \in J$ and all $a, b \in \mathcal{R}$.

8.6.3 Note. We will show in chapter 9 that Jordan modules of type C are precisely those Jordan modules which coordinatise C_n-graded groups for $n \geq 3$. General Jordan modules appear in the coordinatisation of BC_n-graded groups.

8.6.4 Note. The name “Jordan module” is motivated by the fact that every Jordan module of type C can be equipped with the structure of a quadratic unital Jordan algebra, except that these Jordan algebras are only “weakly quadratic”. We will make this observation more precise in subsection 8.6.C.

We now consider a sequence of three examples of Jordan modules, each being a generalisation of the previous one. Hence it suffices to verify that the last example satisfies the axioms of a Jordan module, which we do in Lemma 8.6.8. We will show in section 8.7 that if $2_{\mathcal{R}}$ is invertible, then every Jordan module is of the form in Example 8.6.7. We do not know an example of a Jordan module which is not of this form.

8.6.5 Example. Let $(\mathcal{R}, \mathcal{R}_0, \sigma)$ be an involutory set. Denote by (J, φ) the square-module $\mathcal{R}_0$ with its canonical square-scalar multiplication and by $\pi_1 \colon J \to \mathrm{Nucl}(\mathcal{R})$ the canonical embedding of J into $\mathcal{R}$. Further, we put

$$\mathrm{T}_1 := \mathrm{Tr}_\sigma \colon \mathcal{R} \to J, r \mapsto r + r^\sigma$$

and we denote by $\psi \colon J \times J \to \mathcal{R}$ the zero map. Then $\mathcal{J} := (J, \varphi, \pi_1, \mathrm{T}_1, \psi)$ is an abelian Jordan module over $(\mathcal{R}, \mathcal{R}_0, \sigma)$ of type C. Since π_1 is an injective homomorphism of groups, $\mathcal{J}$ is anisotropic.

8.6.6 Example. Let $(\mathcal{R}, \mathcal{R}_0, \sigma)$ be an involutory set and let M be a right $\mathcal{R}$-module. Put $J := M \times \mathcal{R}_0$, which is an additive abelian group, and

$$\varphi \colon J \times \mathcal{R} \to J, \big((v,h), r\big) \mapsto (vr, r^\sigma h r).$$

Further, we denote by $\pi_1 \colon J \to \mathcal{R}$ the canonical projection on the second component and by $\psi \colon J \times J \to \mathcal{R}$ the zero map. Finally, set

$$\mathrm{T}_1 \colon \mathcal{R} \to J, r \mapsto (0_M, r + r^\sigma).$$

Then $\mathcal{J} := (J, \varphi, \pi_1, \mathrm{T}_1, \psi)$ is an abelian Jordan module over $(\mathcal{R}, \mathcal{R}_0, \sigma)$ of type C. The special case $M = \{0\}$ is precisely Example 8.6.5. Further, $\mathcal{J}$ is anisotropic if and only if $M = \{0\}$.

8.6.7 Example. Let $(\mathcal{R}, \mathcal{R}_0, \sigma)$ be an involutory set and let (M, q, f) be a standard pseudo-quadratic module over $(\mathcal{R}, \mathcal{R}_0, \sigma)$. Denote by $J := T(M) \subseteq M \times \mathcal{R}$ the group from Definition 8.5.25 and by $\varphi\colon J \times \mathcal{R} \to J$ the square-scalar multiplication from Lemma 8.5.34. Further, denote by $\pi_1\colon J \to \mathrm{Nucl}(\mathcal{R})$ the projection map from Lemma 8.5.37. Finally, we put

$$\mathrm{T}_1\colon \mathcal{R} \to J, r \mapsto (0_M, r + r^\sigma) \quad \text{and} \quad \psi\colon J \times J \to \mathcal{R}, \big((v,h),(v',h')\big) \mapsto f(v,v').$$

Then $\mathcal{J} := (J, \varphi, \pi_1, \mathrm{T}_1, \psi)$ is a Jordan module over $(\mathcal{R}, \mathcal{R}_0, \sigma)$. A Jordan module which is isomorphic to one of this form is called *of pseudo-quadratic type*.

Note that if q and f are trivial (so that the pseudo-quadratic module (M, q, f) is essentially an $\mathcal{R}$-module with no additional structure), then it follows from Remark 8.5.28 that this Jordan module is precisely the one from Example 8.6.6. Note further that f is trivial if and only if $T(M)$ is of type C, and that the triviality of f implies the triviality of q if $2_{\mathcal{R}}$ is invertible (Lemma 8.5.15 (f)).

8.6.8 Lemma. *Let $(\mathcal{R}, \mathcal{R}_0, \sigma)$ be an involutory set and let (M, q, f) be a standard pseudo-quadratic module over $(\mathcal{R}, \mathcal{R}_0, \sigma)$. Then the tuple $\mathcal{J} := (J, \varphi, \pi_1, \mathrm{T}_1, \psi)$ from Example 8.6.7 is indeed a Jordan module.*

Proof. By Lemma 8.5.34, (J, φ) is a multiplicative square-module. The map π_1 is compatible with the square-scalar multiplication by Lemma 8.5.37. By Lemma 8.5.38, the map T_1 is a homomorphism of square-modules. Since f is a skew-hermitian form, the same holds for ψ. Axiom 8.6.1 (i) holds by Lemma 8.3.19. Axiom 8.6.1 (ii) holds because (J, q, f) is standard. By the definition of ψ, T_1 and π_1, Axioms 8.6.1 (iii) and 8.6.1 (v) are satisfied. Equation (8.1) holds by the definition of the multiplication on $T(M)$. Equation (8.2) holds by Remark 8.5.35. Equation (8.3) holds by Remark 8.5.30. In Axiom 8.6.1 (vi), we can take $v_0 := (0, 1_{\mathcal{R}})$. This finishes the proof. □

8.6.9 Lemma. *Let $(\mathcal{R}, \mathcal{R}_0, \sigma)$ be an involutory set, let (M, q, f) be a standard pseudo-quadratic module over $(\mathcal{R}, \mathcal{R}_0, \sigma)$ and let $\mathcal{J} := (J, \varphi, \pi_1, \mathrm{T}_1, \psi)$ be the Jordan module from Example 8.6.7. Then (M, q, f) is anisotropic if and only if $\mathcal{J}$ is anisotropic.*

Proof. At first, assume that (M, q, f) is anisotropic. Let $(u, h) \in J$ such that $\pi_1(u, h) = 0$, that is, such that $h = 0_{\mathcal{R}}$. Then $q(u) \equiv h = 0$ modulo $\mathcal{R}_0$. Since q is anisotropic, it follows that $u = 0$. Thus $(u, h) = 0_J$, so $\mathcal{J}$ is anisotropic.

Now assume that $\mathcal{J}$ is anisotropic and let $u \in M$ such that $q(u) \equiv 0$ modulo $\mathcal{R}_0$. Then $(u, 0)$ lies in J and we have $\pi_1(u, 0) = 0_{\mathcal{R}}$. Since $\mathcal{J}$ is anisotropic, it follows that $(u, 0) = 0_J$, so $u = 0_M$. Hence (M, q, f) is anisotropic. □

8.6.B Basic Properties

8.6.10 Notation for this section. From now on, we denote by $\mathcal{R}$ an alternative ring with a nuclear involution σ and by $\mathcal{J} = (J, \varphi, \pi_1, \mathrm{T}_1, \psi)$ a Jordan module over $(\mathcal{R}, \sigma)$.

8.6.11 Definition (Homomorphism of Jordan modules). Let

$$\mathcal{J}' = (J', \varphi', \pi_1', \mathrm{T}_1', \psi')$$

be another Jordan module over $(\mathcal{R}, \sigma)$. A *homomorphism from $\mathcal{J}$ to $\mathcal{J}'$* is a homomorphism $f\colon (J, \varphi) \to (J', \varphi')$ of square-modules such that $\pi_1 = \pi_1' \circ f$, $\mathrm{T}_1' = f \circ \mathrm{T}_1$ and $\psi = \psi' \circ (f \times f)$. An *isomorphism of Jordan modules* is a bijective homomorphism of Jordan modules and an *automorphism of Jordan modules* is an isomorphism from a Jordan module to itself.

8.6.12 Definition (Jordan submodule). A *Jordan submodule of $\mathcal{J}$* is a square-submodule U of (J, φ) which contains the image of T_1 and which satisfies Axiom 8.6.1 (vi).

Clearly, every Jordan submodule of $\mathcal{J}$ is a Jordan submodule itself with respect to the restrictions of the maps φ, π_1, T_1 and ψ.

Any Jordan module has a canonical involution. It will appear as one of the twisting actions in the standard parameter system of type BC (Definition 8.6.20).

8.6.13 Definition (Jordan module involution). The canonical involution on the multiplicative square-module (J, φ) in the sense of Definition 8.1.12 is called the *Jordan module involution (on $\mathcal{J}$)*. It is the map

$$\overline{\cdot}\colon J \to J, u \mapsto \overline{u} := \varphi(u, -1).$$

8.6.14 Lemma. *The Jordan module involution is an automorphism of J which fixes the image of T_1 element-wise.*

Proof. Let $u, v \in J$ and $r \in \mathcal{R}$. Then we have

$$\begin{aligned}
\pi_1(\overline{u}) &= \pi_1\big(\varphi(u, -1_{\mathcal{R}})\big) = (-1_{\mathcal{R}})^\sigma \pi_1(u)(-1_{\mathcal{R}}) = \pi_1(u), \\
\psi(\overline{u}, \overline{v}) &= \psi\big(\varphi(u, -1_{\mathcal{R}}), \varphi(v, -1_{\mathcal{R}})\big) = (-1_{\mathcal{R}})^\sigma \psi(u, v)(-1_{\mathcal{R}}) = \psi(u, v), \\
\overline{\mathrm{T}_1(r)} &= \varphi\big(\mathrm{T}_1(r), -1_{\mathcal{R}}\big) = \mathrm{T}_1\big((-1_{\mathcal{R}})^\sigma r(-1_{\mathcal{R}})\big) = \mathrm{T}_1(r).
\end{aligned}$$

Since $\overline{\cdot}$ is a homomorphism of square-modules by Lemma 8.1.13, the assertion follows. $\square$

8.6.15 Remark. For all $u \in J$, we have

$$\begin{aligned}
0_J = \varphi(u, 0_{\mathcal{R}}) &= \varphi\big(u, 1_{\mathcal{R}} + (-1_{\mathcal{R}})\big) \\
&= \varphi(u, 1_{\mathcal{R}}) \mathbin{\hat{+}} \varphi(u, -1_{\mathcal{R}}) \mathbin{\hat{+}} \mathrm{T}_1\big((-1)^\sigma \pi_1(u) 1_{\mathcal{R}}\big) \\
&= u \mathbin{\hat{+}} \overline{u} \mathbin{\hat{-}} \mathrm{T}_1\big(\pi_1(u)\big).
\end{aligned}$$

This shows that $\mathrm{T}_1(\pi_1(u)) = u \mathbin{\hat{+}} \overline{u}$. Together with Axiom 8.6.1 (v), we conclude that the diagram in Figure 8.1 commutes.

$$\begin{tikzcd}
\mathcal{R} \arrow[r, "a \mapsto a + a^\sigma"] \arrow[d, "\mathrm{T}_1"'] & \mathcal{R} \arrow[d, "\mathrm{T}_1"] \\
J \arrow[r, "u \mapsto u \hat{+} \overline{u}"'] \arrow[ur, "\pi_1"] & J.
\end{tikzcd}$$

Figure 8.1: The commutative square in Remark 8.6.15.

The properties of being abelian and of being of type C are closely related, but not equivalent.

8.6.16 Lemma. *The following statements hold:*

(a) *$\mathcal{J}$ is abelian if and only if $\mathrm{T}_1 \circ \psi = 0$.*

(b) *$\mathcal{J}$ is of type C if and only if $\pi_1\colon J \to \mathcal{R}$ is a homomorphism of square-modules.*

(c) *If $\mathcal{J}$ is of type C, then it is abelian.*

(d) *If $\mathcal{J}$ is abelian, then ψ is symmetric.*

(e) *ψ is symmetric if and only if its image lies in the set $\{\, r \in \mathcal{R} \mid r^\sigma = -r \,\}$ of skew-symmetric elements.*

Proof. Assertions (a) and (b) follow from Axiom 8.6.1 (iv). Further, (a) implies (c). Now assume that $\mathcal{J}$ is abelian. Then for all $u, v \in J$, it follows from Axiom 8.6.1 (iv) that

$$\pi_1(v) + \pi_1(u) + \psi(v,u) = \pi_1(v \hat{+} u) = \pi_1(u \hat{+} v) = \pi_1(u) + \pi_1(v) + \psi(u,v).$$

Since $(\mathcal{R}, +)$ is abelian, we infer that ψ is symmetric. This proves (d).

Now let $u, v \in J$. Since ψ is skew-hermitian, we have $\psi(u,v)^\sigma = -\psi(v,u)$. Thus the following assertions are equivalent:

$$\psi(u,v) = \psi(v,u) \iff \psi(u,v)^\sigma = \psi(v,u)^\sigma \iff \psi(u,v)^\sigma = -\psi(u,v).$$

It follows that (e) holds. $\square$

8.6.17 Remark (The radical). Denote by

$$R := \mathrm{Rad}(\psi) = \{\, u \in R \mid \psi(x,u) = 0 = \psi(u,x) \text{ for all } x \in J \,\}$$

the radical of the skew-hermitian form ψ. This is a square-submodule of J. Further, Axiom 8.6.1 (iii) says precisely that the image of T_1 is contained in R. Similarly, Axiom 8.6.1 (vi) says that there exists $v_0 \in \mathrm{Rad}(\psi)$ such that $\pi_1(v_0) = 1_{\mathcal{R}}$. Thus

$$\mathcal{J}' := \big(R, \varphi|_{R\times\mathcal{R}}, \pi_1|_R, \mathrm{T}_1, \psi|_{R\times R}\big)$$

is also a Jordan module over $(\mathcal{R}, \sigma)$. In fact, $\psi|_{R\times R}$ is the zero map, so $\mathcal{J}'$ is of type C.

8.6.18 Remark. Assume that $2_{\mathcal{R}}$ is invertible and put $v_0' := \mathrm{T}_1(2_{\mathcal{R}}^{-1}) \in J$. Then $\pi_1(v_0') = 2_{\mathcal{R}}^{-1} + (2_{\mathcal{R}}^{-1})^\sigma = 1_{\mathcal{R}}$ by Axiom 8.6.1 (v) and $v_0' \in \mathrm{Rad}(\psi)$ by Axiom 8.6.1 (iii). We conclude that Axiom 8.6.1 (vi) is redundant if $2_{\mathcal{R}}$ is invertible.

8.6.19 Remark. We can define the following maps:

$$\pi\colon J \times \mathcal{R} \to \mathcal{R}, (u,a) \mapsto a\pi_1(u) \quad \text{and} \quad \mathrm{T}\colon \mathcal{R} \times \mathcal{R} \to J, (a,b) \mapsto \mathrm{T}_1(ab^\sigma).$$

It is straightforward to verify that the following properties, which we will need later, follow from the axioms of a Jordan module:

$$\begin{aligned}
\overline{\varphi(u,a)} &= \varphi\big(\varphi(u,a), -1\big) = \varphi(u, -a) = \varphi\big(\varphi(u,-1), a\big) = \varphi(\overline{u}, a),\\
\overline{\mathrm{T}_1(a)} &= \varphi\big(\mathrm{T}_1(a), -1\big) = \mathrm{T}_1\big((-1)^\sigma a(-1)\big) = \mathrm{T}_1(a),\\
\psi(\overline{u}, v) &= (-1)^\sigma \psi(u,v) = \psi(u,v)(-1) = \psi(u, \overline{v}),\\
\pi(\overline{u}, a) &= a\pi_1\big(\varphi(u,-1)\big) = a(-1)^\sigma \pi_1(u)(-1) = \pi(u,a),\\
\pi\big(\mathrm{T}(a,b), c\big) &= c\pi_1\big(\mathrm{T}_1(ab^\sigma)\big) = c(ab^\sigma) + c(b^\sigma a)\\
&= (ca)b^\sigma + (cb^\sigma)a - [c,a,b^\sigma] - [c,b^\sigma,a] = (ca)b^\sigma + (cb^\sigma)a,\\
\pi\big(\varphi(u,a), b\big) &= b\pi_1\big(\varphi(u,a)\big) = ba^\sigma \pi_1(u)a,\\
\mathrm{T}(a,b) &= \mathrm{T}_1(ab^\sigma) = \mathrm{T}_1\big((ab^\sigma)^\sigma\big) = \mathrm{T}_1(ba^\sigma) = \mathrm{T}(b,a),\\
\mathrm{T}(ab, c) &= \mathrm{T}_1(abc^\sigma) = \mathrm{T}_1(cb^\sigma a^\sigma) = \mathrm{T}(cb^\sigma, a) = \mathrm{T}(a, cb^\sigma),\\
\varphi\big(\mathrm{T}(a,b), c\big) &= \varphi\big(\mathrm{T}_1(ab^\sigma), c\big) = \mathrm{T}_1(c^\sigma ab^\sigma c) = \mathrm{T}(c^\sigma a, c^\sigma b) = \mathrm{T}(c^\sigma b, c^\sigma a)
\end{aligned}$$

for all $a, b, c \in \mathcal{R}$ and all $u \in J$. (Here we have used Proposition 8.3.16 to write the term $ba^\sigma \pi_1(u)a$ without brackets.)

The maps π and T are the ones which describe the commutator relations in root gradings of type BC. Thus in some sense, they are more fundamental for our group-theoretic interests than the maps π_1 and T_1. However, the latter allow for a more concise treatment of Jordan modules in the algebraic setting. In particular, the fact that π_1 and T_1 are homomorphisms of square-modules (except for the minor nuisance that π_1 is not necessarily additive) and that we have the commutative square in Remark 8.6.15 cannot be expressed as succinctly in terms of the maps π and T.

Jordan modules arise from root gradings of type $(B)C$ in the form of the following parameter system.

8.6.20 Definition (Standard parameter system). Define $A := \{\pm 1\}^2$ and $B := \{\pm 1\}$. Denote the elements of A by $(\pm 1_A, \pm 1_A)$ and the elements of B by $\pm 1_B$. We declare that the first component of A acts on $(\mathcal{R}, +)$ and $(J, \hat{+})$ by inversion and that the second component of A acts trivially on $\mathcal{R}$ and by the Jordan module involution on J. Further, we declare that B acts by σ on $\mathcal{R}$ and trivially on T. More precisely, these declarations mean that

$$(-1_A, 1_A).r = -r, \quad (-1_A, 1_A).u = \hat{-}u, \quad (1_A, -1_A).r = r, \quad (1_A, -1_A).u = \overline{u},$$
$$-1_B.r = r^\sigma, \qquad -1_B.u = u$$

for all $r \in \mathcal{R}$ and all $u \in J$. Then the triple $(A \times B, J, \mathcal{R})$ is called the *standard parameter system for $\mathcal{J}$*.

8.6.21 Remark. The action of $A \times B$ on T and $\mathcal{R}$ in Definition 8.6.20 is induced by the respective actions of A and B. Such an induced action exists because the actions of A and B commute.

8.6.22 Reminder. If the Jordan module $\mathcal{J}$ is of the form $T(M, q, f)$ for a standard pseudo-quadratic module (M, q, f) (as in Example 8.6.7), then for all $(u, h) \in T(M, q, f)$, the inverse $\hat{-}(u, h)$ equals $(-u, -h^\sigma)$ (see Lemma 8.5.31 and Remark 8.5.32). In this situation, the Jordan module involution is given by $\overline{(u, h)} = (-u, h)$ (see Observation 8.5.36).

8.6.C Jordan Modules and Jordan Algebras

We now investigate the relation of Jordan modules to Jordan algebras. Notation 8.6.10 continues to hold.

8.6.23 Definition (Weak Jordan algebra). A *weak Jordan algebra* is a tuple $(J, U, 1_J)$ consisting of an abelian group $J = (J, +)$, an element $1_J \in J$ and a weakly quadratic map $U\colon J \to \mathrm{Hom}(J, J), x \mapsto U_x$ such that the following properties are satisfied for all $x, y, z, v \in J$, where

$$\{\cdot, \cdot, \cdot\}\colon J \times J \times J \to J, (x, y, z) \mapsto U_{x+z}y - U_x y - U_z y$$

denotes the tri-linearisation of U:

$$\begin{aligned} U_{U_x y} v &= U_x U_y U_x v, \\ U_x(\{y, x, z\}) &= \{U_x y, z, x\}, \\ U_1 v &= v, \\ \{U_x y, v, \{x, y, z\}\} &= U_x U_y(\{x, v, z\}) + \{x, U_y U_x v, z\}, \end{aligned}$$

$$\{U_x y, v, U_z y\} + U_{\{x,y,z\}} v = U_x U_y U_z v + U_z U_y U_x v + \{x, U_y(\{x,v,z\}), z\},$$
$$U_x(\{y,v,z\}) + \{x, \{y,x,z\}, v\} = \{U_x y, z, v\} + \{\{x,y,v\}, z, x\}.$$

Further, $(J, U, 1_J)$ is called a *Jordan algebra* if, in addition, the map U is (properly) $\mathbb{Z}$-quadratic in the sense of Definition 7.1.3.

8.6.24 Note. What we call "Jordan algebra" could, in more detail, be called "unital quadratic Jordan algebra over $\mathbb{Z}$". Our definition agrees with the one in [Jac81, 1.3.4], but it is phrased in a slightly different way. Observe that the last three identities in Definition 8.6.23 are precisely the polarisations of the first two identities. (See, for example, [McC66, p. 1072].) It follows from this fact that $(J, U, 1_J)$ is a Jordan algebra if and only if for all commutative associative rings $\mathscr{k}$, the scalar extension $(J_{\mathscr{k}}, U_{\mathscr{k}}, (1_J)_{\mathscr{k}})$ satisfies the first three identities in Definition 8.6.23. This is precisely the definition that is given in [Jac81].

The observation in the previous paragraph relies on the fact that a quadratic map U has a unique and well-defined extension $U_{\mathscr{k}}$ to any scalar extension. (For a proof, see for example [GPR, 11.5].) This is not clear for weakly quadratic maps. Thus there is no way to phrase the definition of weak Jordan algebras in the language of scalar extensions. In particular, scalar extension of weak Jordan algebras do not necessarily have the structure of a weak Jordan algebra, and need not even exist.

8.6.25 Remark (Jordan algebras from Jordan modules). Assume that $\mathcal{J}$ is of type C (that is, that $\psi = 0$). We define a map $U \colon J \to \mathrm{Hom}_{\mathbb{Z}}(J, J), x \mapsto U_x$ by

$$U_x \colon J \to J, v \mapsto \varphi\big(v, \pi(x)\big)$$

for all $x \in J$. Further, we define a tri-additive map

$$\{\cdot, \cdot, \cdot\} \colon J \times J \times J \to J,$$
$$(x,y,z) \mapsto U_{x+z} y - U_x y - U_z y = \mathrm{T}_1\big(\pi(x)^\sigma \pi(y) \pi(z)\big).$$

Since the map $J \times J \to \mathrm{Hom}(J, J), (x,z) \mapsto \{x, \cdot, z\}$ is the polarisation of U, it follows that U is weakly quadratic. A straightforward computation shows that (J, U, v_0) is a weak Jordan algebra where v_0 is any element as in Axiom 8.6.1 (vi).

8.7 A Near-classification of Jordan Modules

8.7.1 Notation for this section. Unless otherwise stated, we denote by $\mathcal{R}$ an alternative ring with a nuclear involution σ and by $\mathcal{J} = (J, \varphi, \pi_1, \mathrm{T}_1, \psi)$ a Jordan module over $(\mathcal{R}, \sigma)$.

In this section, we show that under some weak assumptions, every Jordan module $\mathcal{J}$ is of pseudo-quadratic type in the sense of Example 8.6.7. These assumptions will always be satisfied if $2_{\mathcal{R}}$ is invertible.

Note that, a priori, the Jordan module $\mathcal{J}$ is defined over the ring $\mathcal{R}$ with a nuclear involution σ whereas a pseudo-quadratic module M is defined over an involutory set $(\mathcal{R}, \mathcal{R}_0, \sigma)$. Thus as a first step, we should locate an appropriate subset $\mathcal{R}_0$ of $\mathcal{R}$. We know from Remark 8.4.11 that this set is uniquely determined if $2_{\mathcal{R}}$ is invertible, but in general it is not. It turns out that our choice of $\mathcal{R}_0$ will depend on the choice of a so-called involutory submodule of J.

8.7.2 Definition (Involutory submodule). An *involutory submodule of $\mathcal{J}$* is a square-submodule D of J which satisfies

$$\mathrm{T}_1(\mathcal{R}) \subseteq D \subseteq \mathrm{Rad}(\psi)$$

where $\mathrm{Rad}(\psi)$ denotes the radical of ψ (see Remark 8.6.17). It is called *unital* if there exists $d \in D$ such that $\pi_1(d) = 1_{\mathcal{R}}$. It is called an *embedding involutory submodule* if the restriction $\pi_1|_D : D \to \mathcal{R}$ is injective.

8.7.3 Remark. Let D be an involutory submodule of $\mathcal{J}$. Since D is contained in $\mathrm{Rad}(\psi)$, it follows from Remark 8.6.17 and Lemma 8.6.16 that ψ is zero on $D \times D$, that D is abelian and that the restriction of π_1 to D is a homomorphism of square-modules.

The following result justifies the terminology of "involutory submodules".

8.7.4 Lemma. *Let D be an involutory submodule of $\mathcal{J}$. Then the triple $(\mathcal{R}, \pi_1(D), \sigma)$ is a pre-involutory set. It is an involutory set if and only if D is unital.*

Proof. It follows from Lemma 8.1.10 and Remark 8.7.3 that $\pi_1(D)$ is a square-submodule of $\mathrm{Nucl}(\mathcal{R})$. Further, since $\psi|_{D\times D}$ is the zero map, it follows from Axiom 8.6.1 (ii) that $\pi_1(D)$ is contained in $\mathrm{Fix}(\sigma)$. Finally, as D contains $\mathrm{T}_1(\mathcal{R})$, Axiom 8.6.1 (v) implies that $\pi_1(D)$ contains $\mathrm{Tr}(\mathcal{R})$. We conclude that $(\mathcal{R}, \pi_1(D), \sigma)$ is a pre-involutory set. The second assertion is trivial. □

8.7.5 Example (of an involutory submodule). Assume that J is the Jordan module $T(M)$ from Example 8.6.7 for some pseudo-quadratic module (M, q, f) over a pre-involutory set $(\mathcal{R}, \mathcal{R}_0, \sigma)$. Then $D := \{0_J\} \times \mathcal{R}_0 \subseteq T(M)$ is an embedding involutory submodule of $T(M)$ (which is unital if and only if $(\mathcal{R}, \mathcal{R}_0, \sigma)$ is an involutory set).

We have seen in Example 8.7.5 that the existence of a embedding involutory submodule of J is a necessary condition for J to be of pseudo-quadratic type. The main result of this section is precisely that this is also a sufficient condition.

Before we turn to the proof of our main result, we investigate the existence of embedding involutory submodules. It is clear that both $\mathrm{T}_1(\mathcal{R})$ and $\mathrm{Rad}(\psi)$ are involutory submodules of $\mathcal{J}$. Further, since the element v_0 from Axiom 8.6.1 (vi) lies in $\mathrm{Rad}(\psi)$, the square-submodule generated by $\mathrm{T}_1(\mathcal{R}) \cup \{v_0\}$ is a unital involutory submodule of $\mathcal{J}$. Thus the main difficulty lies finding an involutory submodule which is embedding. Under suitable assumptions on 2-torsion and related phenomena, we will show that $\mathrm{T}_1(\mathcal{R})$ satisfies this condition. In the general situation, however, we have no result which guarantees the existence of an embedding involutory submodule in J.

Recall from Remark 8.7.3 that the restriction of π_1 to $\mathrm{T}_1(\mathcal{R})$ is a homomorphism of groups. Hence to show that $\mathrm{T}_1(\mathcal{R})$ is embedding, it suffices to verify that the preimage of $0_{\mathcal{R}}$ in $\mathrm{T}_1(\mathcal{R})$ under π_1 is trivial.

8.7.6 Lemma. *Assume that there exists no 2-torsion on $\mathrm{T}_1(\mathcal{R})$. (That is, if $d \in \mathrm{T}_1(\mathcal{R})$ satisfies $d \dotplus d = 0_J$, then $d = 0_J$.) Then $\mathrm{T}_1(\mathcal{R})$ is a embedding involutory submodule of $\mathcal{J}$.*

Proof. Let $d \in \mathrm{T}_1(\mathcal{R})$ such that $\pi_1(d) = 0_{\mathcal{R}}$. We want to show that $d = 0_J$. Applying Remark 8.6.15, we see that

$$0_J = \mathrm{T}_1(0_{\mathcal{R}}) = \mathrm{T}_1\big(\pi_1(d)\big) = d \dotplus \overline{d}.$$

Since d lies in $\mathrm{T}_1(\mathcal{R})$, we have $\overline{d} = d$ by Lemma 8.6.14. Thus $d \dotplus d = 0_J$, which implies that $d = 0_J$. We infer that the restriction of π_1 to $\mathrm{T}_1(\mathcal{R})$ is injective. □

8.7.7 Lemma. *Assume that $2_{\mathcal{R}}$ is invertible. Then $\mathrm{T}_1(\mathcal{R})$ is a unital embedding involutory submodule of $\mathcal{J}$.*

Proof. By Remark 8.6.18, we have $\pi_1(v_0') = 1_{\mathcal{R}}$ for $v_0' := \mathrm{T}_1(2_{\mathcal{R}}^{-1})$, so $\mathrm{T}_1(\mathcal{R})$ is unital. Now let $d \in \mathrm{T}_1(\mathcal{R})$ such that $\pi_1(d) = 0_{\mathcal{R}}$. We want to show that $d = 0_J$. Choose $r \in \mathcal{R}$ such that $d = \mathrm{T}_1(r)$. It follows from Axiom 8.6.1 (v) that

$$r + r^\sigma = \pi_1\big(\mathrm{T}_1(r)\big) = \pi_1(d) = 0_{\mathcal{R}}.$$

Hence $d' := \mathrm{T}_1(2_{\mathcal{R}}^{-1} r)$ also satisfies

$$\pi_1(d') = \pi_1\big(\mathrm{T}_1(2_{\mathcal{R}}^{-1} r)\big) = \frac{r}{2} + \left(\frac{r}{2}\right)^\sigma = \frac{1}{2}(r + r^\sigma) = 0_{\mathcal{R}}.$$

Again by Axiom 8.6.1 (v) and Lemma 8.6.14, we infer that

$$\begin{aligned} 0_J = \mathrm{T}_1(0_{\mathcal{R}}) &= \mathrm{T}_1\big(\pi_1(d')\big) = d' \hat{+} \overline{d'} = d' \hat{+} d' \\ &= \mathrm{T}_1(2_{\mathcal{R}}^{-1} r) \hat{+} \mathrm{T}_1(2_{\mathcal{R}}^{-1} r) = \mathrm{T}_1(2_{\mathcal{R}}^{-1} r + 2_{\mathcal{R}}^{-1} r) = \mathrm{T}_1(r) = d. \end{aligned}$$

Thus the restriction of π_1 to $\mathrm{T}_1(\mathcal{R})$ is injective, as desired. □

8.7.8 Notation for this section. From now on, we fix an involutory submodule D of $\mathcal{J}$. We put $M := J/D$ and we denote by $\tilde{\varphi}\colon M \times \mathcal{R} \to M$ the multiplicative square-scalar multiplication on M from Lemma 8.1.11. That is, we have

$$\rho\big(\varphi(u,r)\big) = \tilde{\varphi}\big(\rho(u), r\big)$$

for all $u \in M$, $r \in \mathcal{R}$ where $\rho\colon J \to M$ denotes the canonical homomorphism of square-modules.

For the first few observations, we do not have to assume that D is embedding. This will only be necessary in Lemma 8.7.19.

We begin by equipping M with the structure of a pseudo-quadratic module. First of all, we have to verify that $(M, \tilde{\varphi})$ is an $\mathcal{R}$-module and not merely a square-module.

8.7.9 Lemma. *The group $(M, \hat{+})$ (whose group structure is induced by the one on J) is abelian.*

Proof. Let $u, v \in J$. Then

$$u \hat{+} v = v \hat{+} u \hat{+} \mathrm{T}_1\big(\psi(u,v)\big)$$

by Axiom 8.6.1 (iv). Applying ρ, and using that D contains the image of T_1, we infer that $\rho(u) \hat{+} \rho(v) = \rho(v) \hat{+} \rho(u)$. Since ρ is surjective, it follows that M is abelian. □

8.7.10 Lemma. *$(M, \tilde{\varphi})$ is an $\mathcal{R}$-module (in the regular sense of Definition 5.1.22).*

Proof. By construction, $(M, \tilde{\varphi})$ is a square-module. Thus it only remains to show that $\tilde{\varphi}$ is additive in the second component. Let $u \in J$ and let $r, s \in \mathcal{R}$. We know from Axiom 8.6.1 (iv) that

$$\tilde{\varphi}(u, r+s) = \tilde{\varphi}(u,r) \hat{+} \tilde{\varphi}(u,s) \hat{+} \mathrm{T}_1\big(s^\sigma \pi_1(u) r\big).$$

Applying ρ and using that D contains the image of T_1, we infer that

$$\tilde{\varphi}\big(\rho(u), r+s\big) = \tilde{\varphi}\big(\rho(u), r\big) \hat{+} \tilde{\varphi}\big(\rho(u), s\big).$$

Since ρ is surjective, the assertion follows. □

We now construct the pseudo-quadratic structure on M.

8.7.11 Lemma. *There exists a unique map $f\colon M \times M \to \mathrm{Nucl}(\mathcal{R})$ such that*

$$f\big(\rho(u), \rho(v)\big) = \psi(u,v)$$

for all $u, v \in J$. This map is a skew-hermitian form with respect to σ.

Proof. For all $u, v \in M$ and all $d, d' \in D$, we have $\psi(u+d, v+d') = \psi(u,v)$ because D is contained in the radical of ψ. Thus such a map f exists. Since ρ is surjective, f is uniquely determined. Further, it is skew-hermitian because ψ is. □

Recall from Remark 8.5.12 that the equivalence class of a pseudo-quadratic form $q\colon M \to \mathrm{Nucl}(\mathcal{R})$ can be identified in a canonical way with the induced map $\tilde{q}\colon M \to \mathrm{Nucl}(\mathcal{R})/\mathcal{R}_0$. Since the group $T(N)$ depends only on the equivalence class of a pseudo-quadratic module N, it thus makes sense to begin by constructing $\tilde{q}$ and then lifting it (in an arbitrary way) to a map $q\colon M \to \mathrm{Nucl}(\mathcal{R})$.

8.7.12 Lemma. *There exists a unique map $\tilde{q}\colon M \to \mathrm{Nucl}(\mathcal{R})/\pi_1(D)$ such that $\tilde{q}(\rho(u)) = \pi_1(u) + \pi_1(D)$ for all $u \in J$.*

Proof. For any subgroup A of J, the map $\pi_1\colon J \to \mathrm{Nucl}(\mathcal{R})$ induces a map $J/A \to \mathrm{Nucl}(\mathcal{R})/\pi_1(A)$. The assertion follows by taking $A := D$. □

8.7.13 Notation. We choose a map $q\colon M \to \mathrm{Nucl}(\mathcal{R})$ such that $q(u) \in \tilde{q}(u)$ for all $u \in M$.

8.7.14 Lemma. *The tuple (M, q, f) is a standard pseudo-quadratic module with respect to the pre-involutory set $(\mathcal{R}, \pi_1(D), \sigma)$.*

Proof. For all $u, v \in J$, it follows from Axiom 8.6.1 (iv) that

$$\begin{aligned}\tilde{q}\big(\rho(u)+\rho(v)\big) &= \tilde{q}\big(\rho(u+v)\big) = \pi_1(u+v) + \pi_1(D)\\ &= \pi_1(u) + \pi_1(v) + \psi(u,v) + \pi_1(D)\\ &= \tilde{q}\big(\rho(u)\big) + \tilde{q}\big(\rho(v)\big) + f\big(\rho(u),\rho(v)\big) + \pi_1(D).\end{aligned}$$

Further, since π_1 is compatible with the square-scalar multiplication, we have

$$\begin{aligned}\tilde{q}\big(\tilde{\varphi}(\rho(u),r)\big) &= \tilde{q}\big(\rho(\varphi(u,r))\big) = \pi_1\big(\varphi(u,r)\big) + \pi_1(D)\\ &= r^\sigma \pi_1(u) r + \pi_1(D) \supseteq r^\sigma\big(\pi_1(u) + \pi_1(D)\big)r = r^\sigma \tilde{q}\big(\rho(u)\big)r.\end{aligned}$$

for all $u \in J$ and $r \in \mathcal{R}$. It follows that

$$q(x+y) \equiv q(x) + q(y) + f(x,y) \quad \text{and} \quad q\big(\tilde{\varphi}(x,r)\big) \equiv r^\sigma q(x) r$$

modulo $\pi_1(D)$ for all $x, y \in M$ and all $r \in \mathcal{R}$. We conclude that (M, q, f) is a pseudo-quadratic module.

To see that (M, q, f) is standard, let $\tilde{u} \in M$ be arbitrary and choose $u \in J$ such that $\tilde{u} = \rho(u)$. Then it follows from Axiom 8.6.1 (ii) and the definition of f that

$$\pi_1(u) - \pi_1(u)^\sigma = \pi_1(u) - \pi_1(u) + \psi(u,u) = f(\tilde{u}, \tilde{u}).$$

Since $q(\tilde{u})$ lies in $\tilde{q}(\tilde{u}) = \pi_1(u) + \pi_1(D)$, there exists $d \in D$ such that $q(\tilde{u}) = \pi_1(u) + \pi_1(d)$. Then

$$q(\tilde{u}) - q(\tilde{u})^\sigma = \pi_1(u) - \pi_1(u)^\sigma$$

because $\pi_1(d)$ lies in $\mathrm{Fix}(\sigma)$ by Lemma 8.7.4. It follows that $q(\tilde{u}) - q(\tilde{u})^\sigma = f(\tilde{u}, \tilde{u})$, so we conclude that (M, q, f) is standard. □

Having constructed a pseudo-quadratic structure on M, we can now consider the group $T(M) = T(M, q, f)$. It remains to find an isomorphism between this group and J.

8.7.15 Definition. We define a map

$$\alpha\colon J \to T(M), u \mapsto \big(\rho(u), \pi_1(u)\big).$$

By the definition of q, the image of α is indeed contained in $T(M)$, so α is well-defined.

8.7.16 Lemma. *The map α is a homomorphism of groups.*

Proof. Let $u, v \in J$ and put $\tilde{u} := \rho(u)$, $\tilde{v} := \rho(v)$. On the one hand,

$$\alpha(u \hat{+} v) = \big(\rho(u \hat{+} v), \pi_1(u \hat{+} v)\big) = \big(\tilde{u} \hat{+} \tilde{v}, \pi_1(u) + \pi_1(v) + \psi(u,v)\big)$$

by Axiom 8.6.1 (iv). On the other hand,

$$\alpha(u)\alpha(v) = \big(\tilde{u}, \pi_1(u)\big)\big(\tilde{v}, \pi_1(v)\big) = \big(\tilde{u} \hat{+} \tilde{v}, \pi_1(u) + \pi_1(v) + f(\tilde{u}, \tilde{v})\big).$$

We conclude that α is a homomorphism. $\square$

8.7.17 Lemma. *The map α is a homomorphism of Jordan modules (with respect to the Jordan module structure on $T(M)$ from Example 8.6.7 and in the sense of Definition 8.6.11).*

Proof. Let $u, v \in M$, let $r \in \mathcal{R}$ and put $\tilde{u} := \rho(u)$, $\tilde{v} := \rho(v)$. Denote the structure maps of the Jordan module $T(M)$ by φ', π_1', T_1' and ψ'. Then we have

$$\begin{aligned}
\alpha\big(\varphi(u,r)\big) &= \Big(\rho\big(\varphi(u,r)\big), \pi_1\big(\varphi(u,r)\big)\Big) = \big(\tilde{\varphi}(\tilde{u},r), r^\sigma \pi_1(u) r\big) \\
&= \varphi'\big((\tilde{u}, \pi_1(u)), r\big) = \varphi'\big(\alpha(u), r\big), \\
\pi_1'\big(\alpha(u)\big) &= \pi_1'\big(\tilde{u}, \pi_1(u)\big) = \pi_1(u), \\
\alpha\big(\mathrm{T}_1(r)\big) &= \Big(\rho\big(\mathrm{T}_1(r)\big), \pi_1\big(\mathrm{T}_1(r)\big)\Big) = (0_M, r + r^\sigma) = \mathrm{T}_1'(r), \\
\psi'\big(\alpha(u), \alpha(v)\big) &= \psi'\Big(\big(\tilde{u}, \pi_1(u)\big), \big(\tilde{v}, \pi_1(v)\big)\Big) = f(\tilde{u}, \tilde{v}) = \psi(u,v).
\end{aligned}$$

This shows that α is a homomorphism of Jordan modules. $\square$

8.7.18 Lemma. *The map α is surjective.*

Proof. Let $(\tilde{u}, h) \in T(M)$ be arbitrary and choose $u \in J$ such that $\tilde{u} = \rho(u)$. By the definition of $T(M)$, we have $h \equiv q(\tilde{u})$ modulo $\pi_1(D)$, which says precisely that $h \in \tilde{q}(\tilde{u}) = \pi_1(u) + \pi_1(D)$. Thus there exists $d \in D$ such that $h = \pi_1(u) + \pi_1(d)$. Now

$$\alpha(u \hat{+} d) = \big(\tilde{u} + \rho(d), \pi_1(u+d)\big) = \big(\tilde{u}, \pi_1(u) + \pi_1(d) + \psi(u,d)\big) = (\tilde{u}, h).$$

This shows that α is surjective. $\square$

8.7.19 Lemma. *The kernel of α is the kernel of $\pi_1|_D$. In particular, α is injective if and only if D is embedding.*

Proof. Let $u \in J$. Then $\alpha(u) = (\rho(u), \pi_1(u))$. Thus α lies in the kernel of α if and only if it lies in the kernels of π_1 and ρ. Since the kernel of ρ is D, the assertion follows. $\square$

We can now put everything together to obtain the following main result.

8.7.20 Theorem. *Let $\mathcal{R}$ be an alternative ring, let σ be a nuclear involution on $\mathcal{R}$ and let $\mathcal{J} = (J, \varphi, \pi_1, \mathrm{T}_1, \psi)$ be a Jordan module over $(\mathcal{R}, \sigma)$. Choose any involutory submodule D of $\mathcal{J}$. Put $M := J/D$. Then π_1 induces a pseudo-quadratic module structure on M with respect to $(\mathcal{R}, \pi_1(D), \sigma)$ and there exists a surjective homomorphism $\alpha\colon J \to T(M)$ of Jordan modules whose kernel is precisely the kernel of $\pi_1|_D$.*

Using the criterion from Lemma 8.7.7, we can simplify the main result as follows.

8.7.21 Theorem. *Let $\mathcal{R}$ be an alternative ring such that $2_{\mathcal{R}}$ is invertible, let σ be a nuclear involution on $\mathcal{R}$ and let $\mathcal{J}$ be a Jordan module over $(\mathcal{R}, \sigma)$. Then $\mathcal{J}$ is of pseudo-quadratic type. That is, there exists a pseudo-quadratic module M over $(\mathcal{R}, \mathrm{Tr}_\sigma(\mathcal{R}), \sigma)$ such that $\mathcal{J}$ is isomorphic to the Jordan module $T(M)$ from Example 8.6.7.*

We can also give a precise classification of Jordan modules of type C if $2_{\mathcal{R}}$ is invertible.

8.7.22 Theorem. *Let $\mathcal{R}$ be an alternative ring such that $2_{\mathcal{R}}$ is invertible, let σ be a nuclear involution on $\mathcal{R}$ and let $\mathcal{J}$ be a Jordan module over $(\mathcal{R}, \sigma)$ of type C. Put $\mathcal{R}_0 := \mathrm{Tr}_\sigma(\mathcal{R}) = \mathrm{Fix}(\sigma)$. Then there exist an $\mathcal{R}$-module M over $\mathcal{R}$ such that $\mathcal{J}$ is isomorphic to the Jordan module $\mathcal{R}_0 \times M$ which was constructed in Example 8.6.6.*

Proof. This follows from Theorem 8.7.21 and the concluding remarks in Example 8.6.7. □

8.7.23 Remark. Let $(\mathcal{R}, \mathcal{R}_0, \sigma)$ be an involutory set and let N be a pseudo-quadratic module over $(\mathcal{R}, \mathcal{R}_0, \sigma)$. Recall from Example 8.7.5 that $D := \{0_J\} \times \mathcal{R}_0$ is a unital embedding involutory submodule of $T(N)$, independent of any assumption on the invertibility of $2_{\mathcal{R}}$. For this choice of D, the isomorphism α is (up to canonical identifications) the identity map.

Using Remark 8.7.23, we can prove the following corollary of Theorem 8.7.20.

8.7.24 Proposition. *Let $(\mathcal{R}, \mathcal{R}_0, \sigma)$ be an involutory set, let N be a pseudo-quadratic module over $(\mathcal{R}, \mathcal{R}_0, \sigma)$ and assume that $\mathcal{J}$ is the Jordan module $T(N)$. Let $\mathcal{R}_0'$ be a subset of $\mathcal{R}_0$ such that $(\mathcal{R}, \mathcal{R}_0', \sigma)$ is an involutory set. Then there exists a pseudo-quadratic module N' over $(\mathcal{R}, \mathcal{R}_0', \sigma)$ such that $T(N)$ is isomorphic to $T(N')$ as a Jordan module.*

Proof. By Example 8.7.5, $D' := \{0_N\} \times \mathcal{R}_0' \subseteq T(N)$ is an embedding involutory submodule of $T(M)$. Further, we have $\pi_1(D') = \mathcal{R}_0'$. Thus it follows from Theorem 8.7.20 that $T(N)$ is isomorphic as a Jordan module to $T(N')$ for some pseudo-quadratic module N' over $(\mathcal{R}, \mathcal{R}_0', \sigma)$. □

8.8 Purely Alternative Rings and Jordan Modules

Recall from Note 8.5.10 that we can easily construct non-trivial pseudo-quadratic modules M over rings of the form $\mathcal{R} \oplus \mathcal{R}'$ where $\mathcal{R}$ is associative and $\mathcal{R}'$ is alternative. However, we observed that only the associative ideal $\mathcal{R} \oplus \{0\}$ of $\mathcal{R} \oplus \mathcal{R}'$ is relevant for the pseudo-quadratic module structure on M. Thus we attributed the existence of a pseudo-quadratic module over $\mathcal{R} \oplus \mathcal{R}'$ to the fact that $\mathcal{R} \oplus \mathcal{R}'$ is not "purely alternative", though we gave no precise definition of this property.

In this section, we present two closely related definitions of pure alternativity which were proposed by Slater in [Sla67] for not necessarily unital rings. We will show that purely alternative rings in the stronger sense do not admit any non-zero modules and that purely alternative rings in the weaker sense do not admit pseudo-quadratic modules with non-trivial skew-hermitian form. Some more information on purely alternative rings can be found in [McC70, Section 9.4].

The material in this section is largely independent from the remaining part of this thesis and will not be referenced in any formal argument. Instead, its main purpose is to convince the reader that pseudo-quadratic modules are objects which do not properly belong to the world of alternative rings.

To avoid confusion, we explicitly point out that Jordan modules, unlike pseudo-quadratic modules, have their place in the alternative setting: We have already seen an important class of Jordan modules over arbitrary alternative rings with nuclear involution in Example 8.6.5. This class consists precisely of the Jordan modules $T(M)$ where $M = \{0\}$.

8.8.1 Definition (Nuclear ideal). Let $\mathcal{R}$ be a ring. An ideal of $\mathcal{R}$ is called *nuclear* if it is contained in the nucleus of $\mathcal{R}$.

8.8.2 Remark. The nucleus of a ring $\mathcal{R}$ is not (necessarily) a nuclear ideal because it is not (necessarily) an ideal. However, there exists a unique maximal nuclear ideal of $\mathcal{R}$: the sum of all nuclear ideals.

8.8.3 Lemma. *Let $\mathcal{R}$ be an alternative ring and let N be a nuclear ideal. Then the left annihilator*

$$L := \{ a \in \mathcal{R} \mid aN = \{0\} \}$$

is an ideal of $\mathcal{R}$.

Proof. It is clear that L is closed under addition. Now let $a \in L$ and $x \in \mathcal{R}$. We want to show that ax and xa lie in L, so let $n \in N$ be arbitrary. We clearly have $(xa)n = x(an) = x0_{\mathcal{R}} = 0_{\mathcal{R}}$ because n lies in the nucleus. Hence xa lies in L. Further, since N is an ideal, we have $xn \in N$. Thus $(ax)n = a(xn) = 0_{\mathcal{R}}$ because $a \in L$. The assertion follows. □

8.8.4 Definition (Purely alternative ring, [Sla67, 4.1, 4.2]). Let $\mathcal{R}$ be an alternative ring. We say that $\mathcal{R}$ is *purely alternative in the weak sense* if it has no non-zero nuclear ideal. We say that $\mathcal{R}$ is *purely alternative in the strong sense* if $[\mathcal{R}, \mathcal{R}, \mathcal{R}] = \mathcal{R}$. Here $[\mathcal{R}, \mathcal{R}, \mathcal{R}]$ denotes the ideal generated by all associators.

8.8.5 Note. Purely alternative rings in the weak sense are simply called purely alternative rings in [Sla67, 4.1]. Our notion of purely alternative rings in the strong sense is suggested in [Sla67, 4.2] as another “reasonable” definition. Further, it is then shown that any purely alternative ring in the strong sense is also purely alternative in the weak sense if a mild extra condition is satisfied. This extra condition is always satisfied in our setting because rings are assumed to be unital. We will see the precise formulation of this argument in Lemma 8.8.10.

8.8.6 Example (Direct sums). Let $\mathcal{R}$ be an associative ring and let $\mathcal{R}'$ be an alternative ring. Assume that both rings are not zero. Then $\mathcal{R} \oplus \{0\}$ is a nuclear ideal in $\mathcal{R} \oplus \mathcal{R}'$, so $\mathcal{R} \oplus \mathcal{R}'$ is not purely alternative in the weak sense. Further, the associator ideal of $\mathcal{R} \oplus \mathcal{R}'$ is contained in $\{0\} \oplus \mathcal{R}'$, so $\mathcal{R} \oplus \mathcal{R}'$ is not purely alternative in the strong sense either.

8.8.7 Example. Let $\mathcal{R}$ be a ring which is simple in the sense that it contains no non-trivial ideal. Then $\mathcal{R}$ is purely alternative (in either sense) if and only if it is alternative and not associative.

(Note that in the famous theorem from [Kle53] that every simple alternative ring is either associative or a Cayley-Dickson algebra over a field, a ring is called simple if it has no non-trivial ideal *and* contains at least one element which is not nilpotent.)

8.8.8 Remark. An alternative ring is purely alternative in the strong sense if and only if it has no non-zero homomorphic image which is associative.

8.8.9 Lemma ([Sla67, Section 3]). *Let $\mathcal{R}$ be an alternative ring and let N be a nuclear ideal of $\mathcal{R}$. Then $[\mathcal{R},\mathcal{R},\mathcal{R}]N = \{0\}$.*

Proof. Let $x, y, z \in \mathcal{R}$ and let $n \in N$. It follows from the nuclear slipping formula (Lemma 8.2.12) that $[x, y, z]n = [x, y, zn]$. Since N is an ideal, zn lies in N and thus in the nucleus. Hence $[x, y, zn] = 0_{\mathcal{R}}$. This shows that $[x, y, z]$ lies in the left annihilator of N. By Lemma 8.8.3, it follows that the whole ideal $[\mathcal{R},\mathcal{R},\mathcal{R}]$ which is generated by all associators is trivial. □

8.8.10 Lemma. *Let $\mathcal{R}$ be a purely alternative ring in the strong sense. Then $\mathcal{R}$ is also a purely alternative ring in the weak sense.*

Proof. It follows from Lemma 8.8.9 that $\mathcal{R}N = \{0\}$ for any nuclear ideal N. Since $\mathcal{R}$ contains a unit element $1_{\mathcal{R}}$, this implies that every nuclear ideal is zero. This says precisely that $\mathcal{R}$ is purely alternative in the weak sense. □

We can now prove the results that we announced in the introduction of this section.

8.8.11 Proposition. *Let $\mathcal{R}$ be a purely alternative ring in the strong sense and let M be an $\mathcal{R}$-module. Then $M = \{0\}$.*

Proof. We have seen in Observation 5.1.23 that modules over $\mathcal{R}$ are actually modules over $\mathcal{R}/[\mathcal{R},\mathcal{R},\mathcal{R}]$. In other words, we have $vr = 0_M$ for all $v \in M$ and all $r \in [\mathcal{R},\mathcal{R},\mathcal{R}]$. Since $\mathcal{R}$ is purely alternative in the strong sense, this means that $M\mathcal{R} = \{0_M\}$. At the same time, we have $M1_{\mathcal{R}} = M$. We infer that $M = \{0\}$. □

8.8.12 Proposition. *Let $(\mathcal{R}, \mathcal{R}_0, \sigma)$ be an involutory set in which $\mathcal{R}$ is purely alternative in the weak sense and let (M, q, f) be a pseudo-quadratic module over $(\mathcal{R}, \mathcal{R}_0, \sigma)$. Then $f = 0$. In particular, q is trivial if $2_{\mathcal{R}}$ is invertible.*

Proof. Denote by I the additive subgroup of $(\mathcal{R}, +)$ which is generated by the image of f. It follows from the sesquilinearity of f that I is an ideal of $\mathcal{R}$. Further, I is nuclear by Definition 8.5.1. Since $\mathcal{R}$ is purely alternative in the weak sense, we infer that $I = \{0\}$. Hence $f = 0$. If $2_{\mathcal{R}}$ is invertible, then the triviality of f implies the triviality of q by Lemma 8.5.15 (f). □

Chapter 9

Root Gradings of Types C and BC

In this chapter, we study C_n-graded groups and BC_n-graded groups for $n \geq 2$, though we will assume that $n \geq 3$ for the main results. We will see that C_n-graded groups can be regarded as special cases of BC_n-graded groups. This allows us to restrict our attention to BC_n-gradings most of the time. Similarly to the situation in root gradings of type B, every medium-length root in BC_n is contained in an A_2-subsystem (Lemma 9.1.13), so it is clear that there exists a ring $\mathcal{R}$ which coordinatises the medium-length root groups. We will show that $\mathcal{R}$ is alternative and that it has a nuclear involution σ. Further, we will parametrise the short root groups by a (not necessarily abelian) group $(J, \hat{+})$. The commutator relations yield maps φ, π_1, T_1, ψ which equip J with the structure of a Jordan module over $(\mathcal{R}, \sigma)$.

Some partial results on C_n-gradings for $n \geq 3$ have already been obtained by Zhang in his PhD thesis [Zha14]. See 2.7.9 and also Note 6.4.4 for more details. For RGD-systems of type BC_2, the standard reference are Chapters 25 and 26 in [TW02].

As usual, this chapter follows the outline described in section 4.7, except that we have already covered all the relevant algebraic structures in chapter 8. Hence we can begin with the investigation of the root systems C_n and BC_n in section 9.1. In section 9.2, we will uncover several connections between root gradings of types B, C and BC. In particular, we will show how root gradings of type C can be regarded as root gradings of type BC with additional properties. We will also see that the rank-2 results from chapter 7 can also be applied, in a modified way, to root gradings of type BC. Section 9.3 presents the construction of a BC_n-graded group from a standard pseudo-quadratic module. This does not provide a complete solution of the construction problem for BC_n-graded groups, and we will summarise what precisely remains to be done.

After this point, the coordinatisation of BC_n-graded groups begins. In sections 9.4 and 9.5, we will perform the necessary rank-2 and rank-3 computations. We introduce the notions of standard signs and standard partial twisting systems for BC_n-graded groups in sections 9.6 and 9.7, respectively. In section 9.8, we construct the parametrising groups $(\mathcal{R}, +)$ and $(J, \hat{+})$. The commutation maps, their rank-2 identities and the blueprint rewriting rules are defined, derived and computed in section 9.9. Finally, we perform the blueprint computations for BC_n in section 9.10, and we state our final result in Theorem 9.10.26.

9.1 Root Systems of Types C and BC

9.1.1 Notation for this section. We denote by n an integer at least 1.

In this section, we collect some basic facts about root systems of types C and BC which will be needed later on. We will see in Remark 9.2.6 that root gradings of type C_n are special cases of BC_n-graded groups. Hence we can restrict ourselves to root systems of type BC most of the time.

9.1.2 Remark (Standard representation of BC_n). Let V be a Euclidean space of dimension n with orthonormal basis $(e_1, \dots, e_n)$. The *standard representation of* BC_n is

$$\begin{aligned} BC_n &:= \{\, \varepsilon_1 e_i + \varepsilon_2 e_j \mid i \neq j \in [1,n], \varepsilon_1, \varepsilon_2 \in \{\pm 1\} \,\} \cup \{\, \varepsilon e_i \mid i \in [1,n], \varepsilon \in \{\pm 1\} \,\} \\ &\qquad \cup \{\, 2\varepsilon e_i \mid i \in [1,n], \varepsilon \in \{\pm 1\} \,\} \end{aligned}$$

The long roots are exactly those which lie in the third set, the medium-length roots are those in the first set and the short roots are those in the second one. The *standard root base* is

$$\Delta := \{\, e_i - e_{i+1} \mid i \in [1, n-1] \,\} \cup \{e_n\},$$

the *standard rescaled root base* is

$$\Delta := \{\, e_i - e_{i+1} \mid i \in [1, n-1] \,\} \cup \{2e_n\}$$

and the corresponding positive system is

$$\begin{aligned} \Pi &:= \{\, e_i - e_j \mid i < j \in [1,n] \,\} \cup \{\, e_i + e_j \mid i \neq j \in [1,n] \,\} \\ &\qquad \cup \{\, \lambda e_i \mid i \in [1,n], \lambda \in \{1,2\} \,\}. \end{aligned}$$

Recall Remark 4.1.31 for a discussion why we usually prefer the standard rescaled root base over the standard root base.

9.1.3 Remark (Standard representation of C_n). Let V be a Euclidean space of dimension n with orthonormal basis $(e_1, \dots, e_n)$. Assume that $n \geq 2$. (As for B_n, we exclude the case $n = 1$ because the resulting root system C_1 would be isomorphic to A_1.) The *standard representation of* C_n is

$$C_n := \{\, \varepsilon_1 e_i + \varepsilon_2 e_j \mid i \neq j \in [1,n], \varepsilon_1, \varepsilon_2 \in \{\pm 1\} \,\} \cup \{\, 2\varepsilon e_i \mid i \in [1,n], \varepsilon \in \{\pm 1\} \,\}.$$

We can see it as a crystallographically closed subset of (the standard representation of) BC_n in a natural way. The long roots are exactly those which lie in the second set and the short roots are those in the first one. The *standard root base* is

$$\Delta := \{\, e_i - e_{i+1} \mid i \in [1, n-1] \,\} \cup \{2e_n\}$$

and the corresponding positive system is

$$\Pi := \{\, e_i - e_j \mid i < j \in [1,n] \,\} \cup \{\, e_i + e_j \mid i \neq j \in [1,n] \,\} \cup \{\, 2e_i \mid i \in [1,n] \,\}.$$

9.1.4 Warning. Roots of the form $\pm e_i \pm e_j$ are short roots when considered as roots in C_n but of medium length when considered roots in BC_n. However, it will always be clear from the context in which root system we are working.

9.1.5 Remark. As was the case for B_n (see Remark 7.2.3), C_n contains A_{n-1} as a parabolic root subsystem and D_n as a root subsystem. However, in this case, the root subsystem D_n of C_n is not crystallographically closed because $2e_1 = (e_1 - e_2) + (e_1 + e_2)$. Further, BC_n contains both B_n and C_n as root subsystems, but only C_n is crystallographically closed and none of them are parabolic.

9.1.6 Observation. Since $\sigma_{\lambda\alpha} = \sigma_\alpha$ for all $\lambda \in \mathbb{R} \setminus \{0\}$, it is clear that the root systems B_n, C_n and BC_n have the same Weyl group.

9.1.7 Definition (BC_2-pairs and BC_2-quadruples). Let Φ be any root system. A *BC_2-pair (in Φ)* is a pair (α, δ) of roots such that $\langle \alpha, \delta \rangle_{\mathbb{R}} \cap \Phi$ is a root subsystem of Φ of type BC_2 with root base (α, δ) with α being the medium-length simple root in this subsystem and δ being the short simple root. A *BC_2-quadruple* is a quadruple $(\alpha, \beta, \gamma, \delta)$ of roots in Φ such that (α, δ) is a BC_2-pair, $\beta = \alpha + \delta$ and $\gamma = \alpha + 2\delta$.

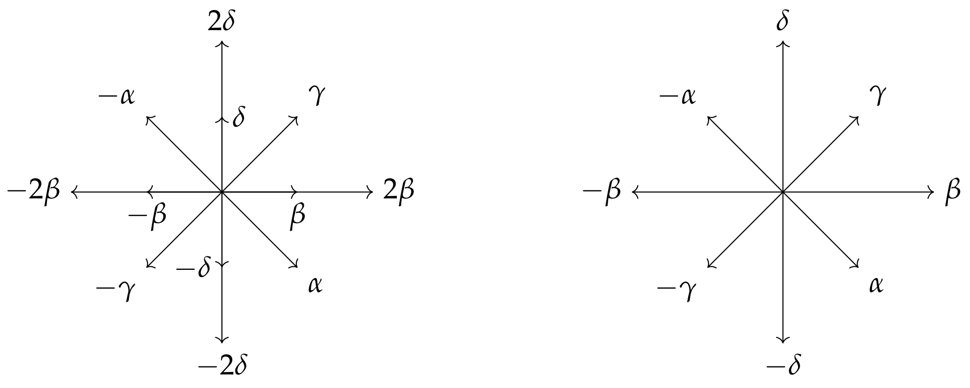
We can define C_2-pairs and C_2-quadruples in an analogous way, although they will rarely be needed.

9.1.8 Definition (C_2-pairs and C_2-quadruples). Let Φ be any root system. A *C_2-pair (in Φ)* is a pair (α, δ) of roots such that $\langle \alpha, \delta \rangle_{\mathbb{Z}} \cap \Phi$ is a root subsystem of Φ of type C_2 with root base (α, δ) such that δ is longer than α. A *C_2-quadruple* is a quadruple $(\alpha, \beta, \gamma, \delta)$ of roots in Φ such that (α, δ) is a C_2-pair, $\beta = 2\alpha + \delta$ and $\gamma = \alpha + \delta$.

9.1.9 Warning. Note that we consider $\langle \alpha, \delta \rangle_{\mathbb{Z}} \cap \Phi$ in Definition 9.1.8: The parabolic root subsystem $\langle \alpha, \delta \rangle_{\mathbb{R}} \cap \Phi$ is allowed to be of type BC_2.

9.1.10 Remark. Assume that $(\alpha, \beta, \gamma, \delta)$ is a BC_2-quadruple. Then the scaled tuple $(\alpha, 2\beta, \gamma, 2\delta)$ is a C_2-quadruple.

9.1.11 Remark (compare Remark 7.2.6). Let G be a group with a C_2-grading $(U_\alpha)_{\alpha \in B_2}$ and let $(\alpha, \beta, \gamma, \delta)$ be a C_2-quadruple. Then this grading is crystallographic if and only if the commutators $[U_{\varepsilon\beta}, U_{\sigma\delta}]$ are trivial for all $\varepsilon, \sigma \in \{\pm 1\}$.



(a) A BC_2-quadruple $(\alpha, \beta, \gamma, \delta)$.

(b) A C_2-quadruple $(\alpha, \beta, \gamma, \delta)$.

Figure 9.1: C_2- and BC_2-quadruples.

We now prove similar properties as in section 7.2. Unless otherwise specified, the word "root" always refers to roots in BC_n.

9.1.12 Lemma. *Let α, β be roots in BC_n. Then the subsystem which is spanned by α, β is of type A_1, BC_1, A_2, $A_1 \times A_1$ or BC_2.*

Proof. This follows from Remark 1.2.56. □

9.1.13 Lemma. *Assume that $n \geq 3$ and let α be a medium-length root in BC_n. Then there exists an A_2-subsystem of BC_n which contains α.*

Proof. We can use the same A_2-subsystems as in Lemma 7.2.12 for B_n. □

9.1.14 Lemma. *Every subsystem of BC_n of type A_2 or BC_2 is parabolic.*

Proof. This can be proven in the same way as Lemma 7.2.8. □

9.1.15 Remark. In the standard representation, the subsystems of BC_n of type BC_2 are exactly the sets of the form $\langle e_i, e_j \rangle_{\mathbb{R}} \cap \Phi$ for distinct $i, j \in [1, n]$.

9.1.16 Lemma. *The Weyl group of BC_n acts transitively on the set of BC_2-pairs and on the set of BC_2-quadruples.*

Proof. Let (α, δ), (α', δ') be two BC_2-pairs. By Lemma 1.2.41 (b), there exist root bases Δ and Δ' containing (α, δ) and (α', δ'), respectively. By Proposition 1.3.6, there exists an element u of the Weyl group which maps Δ to Δ'. Note that (α, δ) is the unique pair of elements in Δ which forms a BC_2-pair, and similarly for (α', δ') and Δ'. It follows from this characterisation that u maps (α, δ) to (α', δ'). Thus the Weyl group acts transitively on the set of BC_2-pairs. This implies that it acts transitively on the set of BC_2-quadruples as well. □

9.1.17 Lemma. *Let $(\alpha, \beta, \gamma, \delta)$ be a BC_2-quadruple in BC_n and assume that $n \geq 3$. Then there exist medium-length roots γ_1, γ_2 such that the following conditions are satisfied:*

- *(i) $(\gamma_1, \gamma, \gamma_2)$ is an A_2-triple. In particular, $\gamma = \gamma_1 + \gamma_2$.*
- *(ii) (α, γ_1) is an A_2-pair.*
- *(iii) $(\alpha, -\gamma_2)$ is an A_2-pair.*

Proof. Without loss of generality, we can use the standard representation of BC_n and assume that $\alpha = e_1 - e_2$ and $\gamma = e_1 + e_2$. Then the roots $\gamma_1 := e_2 - e_3$ and $\gamma_2 := e_1 + e_3$ have the desired properties. □

9.1.18 Lemma. *Assume that $n \geq 2$. Let $\alpha \in BC_n$ be a medium-length root and let $\delta \in BC_n$ be short. Then α and δ lie in a common BC_2-subsystem if and only if $\alpha \cdot \delta \neq 0$.*

Proof. This follows from Remark 9.1.15. □

Let v, w be any non-zero vectors in the Euclidean space surrounding BC_n. By the definition of the Cartan numbers, we have $\langle \lambda v | w \rangle = \lambda \langle v | w \rangle$ and $\langle v | \lambda w \rangle = \lambda^{-1} \langle v | w \rangle$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. Thus as soon as we have computed the Cartan integers for all short and medium-length roots, we can easily derive the Cartan integers for all roots in BC_n. Since the Cartan integer of short and medium-length roots stay the same when we consider them as roots in B_n, we can deduce the following characterisation from Proposition 7.2.20.

9.1.19 Proposition. *Let ρ, ζ be two roots in BC_n. Then $\langle \rho | \zeta \rangle$ is an even number if and only if one of the following conditions is satisfied:*

- *(a) ρ is long.*
- *(b) ζ is short.*
- *(c) $\rho \in \{ \pm\zeta \}$.*
- *(d) ρ and ζ are orthogonal.*

The following lemma will be used in Proposition 9.8.3 to prove stabiliser-compatibility for BC_n-graded groups. Observe that the set $\bar{\mathcal{A}}$ is defined exactly as in Proposition 4.6.3.

9.1.20 Lemma. *Assume that $n \geq 2$. Let α be a medium-length root, choose roots β, γ, δ such that $(\alpha, \beta, \gamma, \delta)$ is a BC_2-quadruple and put*

$$\bar{\mathcal{A}} := \alpha^{\perp} \cap \{\, \rho \in BC_n \mid \rho \text{ is not crystallographically adjacent to } \alpha \,\}.$$

Then $\bar{\mathcal{A}} = \{\, \gamma, -\gamma \,\}$. In particular, for any root base Δ of BC_n, there exists exactly one Δ-positive root in $\bar{\mathcal{A}}$, and this root has medium length.

Proof. We use the standard representation of BC_n. It is clear that γ and $-\gamma$ lie in $\bar{\mathcal{A}}$, so it remains to prove the reverse inclusion. By the transitivity of the Weyl groups on the set of medium-length roots (Proposition 1.3.8), we can assume that $\alpha = e_1 - e_2$. If ρ is a long or short root which is orthogonal to α, then ρ and $-\rho$ are adjacent to α, so ρ does not lie in $\bar{\mathcal{A}}$. Now assume that ρ is a medium-length root in $\bar{\mathcal{A}}$. Since ρ is orthogonal to α, we must have either $\rho \in \{\, \gamma, -\gamma \,\}$ or $\rho = \varepsilon_1 e_i + \varepsilon_2 e_j$ for some $\varepsilon_1, \varepsilon_2 \in \{\, \pm 1 \,\}$ and $i, j \in [3, n]$. In the latter case, ρ is crystallographically adjacent to α, which is not possible if $\rho \in \bar{\mathcal{A}}$. We conclude that $\rho \in \{\, \gamma, -\gamma \,\}$. □

9.1.21 Lemma. *Assume that $n \geq 2$. Let α, α' be two orthogonal medium-length roots which are crystallographically adjacent. Then α and $-\alpha'$ are crystallographically adjacent as well.*

Proof. Since $n \geq 2$, we can choose roots β, γ, δ such that $(\alpha, \beta, \gamma, \delta)$ is a BC_2-quadruple. Then it follows from Lemma 9.1.20 that the set

$$\bar{\mathcal{A}} := \alpha^{\perp} \cap \{\, \rho \in BC_n \mid \rho \text{ or } -\rho \text{ is not crystallographically adjacent to } \alpha \,\}$$

equals $\{\, \gamma, -\gamma \,\}$. Now suppose for a contradiction that α and $-\alpha'$ are not crystallographically adjacent. Then $-\alpha'$ lies in $\bar{\mathcal{A}}$. Thus $\alpha' \in \{\, \gamma, -\gamma \,\}$. However, both γ and $-\gamma$ are not crystallographically adjacent to α whereas α' is by assumption. Thus we obtain a contradiction, which finishes the proof. □

9.1.22 Lemma. *Assume that $n \geq 2$. Then the Weyl group of BC_n acts transitively on the set*

$$S := \left\{ (\alpha, \gamma) \;\middle|\; \begin{array}{l} \alpha, \gamma \text{ are of medium length, orthogonal} \\ \text{and not crystallographically adjacent} \end{array} \right\}.$$

Proof. The set S consists precisely of the pairs (α, γ) for which there exist roots β, δ such that $(\alpha, \beta, \gamma, \delta)$ is a BC_2-quadruple. Thus the assertion follows from Lemma 9.1.16. □

Recall that we have defined a certain subset $\hat{B}_n$ of B_n in Remark 7.2.21. Since C_n and BC_n have the same Weyl group as B_n, the same reasoning leads us to the definition of the set $\hat{C}_n \subseteq C_n \subseteq BC_n$. We will mainly use this set in Lemma 9.9.8 and Proposition 9.10.24. In this context, it is worth recalling Remark 4.1.31.

9.1.23 Definition. For all $n \in \mathbb{N}_{\geq 2}$, we define the following subset of C_n and BC_n:

$$\hat{C}_n := \{\, e_i - e_j \mid i \neq j \in [1, n] \,\} \cup \{\, 2e_i \mid i \in [1, n] \,\} \subseteq C_n.$$

9.1.24 Definition (Standard Δ-expression). Let Δ denote the standard root base of C_n, considered as a subset of BC_n. For any root $\alpha \in \hat{C}_n$, we define a Δ-expression $\bar{\rho}^{\alpha}$ of α (in the sense of Definition 4.2.2) as follows:

(a) If $\alpha \in \Delta$, we put $\bar{\rho}^{\alpha} := (\alpha)$.

(b) If $\alpha = 2e_i$ for some $i \in [1, n-1]$, we define $\bar{\rho}^{e_i} := (e_i - e_{i+1}, \bar{\rho}^{e_{i+1}}, e_{i+1} - e_i)$.

(c) If $\alpha = e_i - e_j$ for some $i < j \in [1, n-1]$ with $i+1 < j$, we define $\bar{\rho}^{e_i - e_j} := (e_j - e_{j-1}, \bar{\rho}^{e_i - e_{j-1}}, e_{j-1} - e_j)$.
(d) If $\alpha = e_j - e_i$ for some $i < j \in [1, n-1]$ with $i+1 < j$, we define $\bar{\rho}^{e_j - e_i} := (\bar{\rho}^{e_i - e_j})^{-1}$.

The word $\bar{\rho}^\alpha$ will also be called the *standard Δ-expression of α*. We also put $\bar{\rho}^i := \bar{\rho}^{e_i}$ for all $i \in [1, n]$ and $\bar{\rho}^{ij} := \bar{\rho}^{e_i - e_j}$ for all distinct $i, j \in [1, n]$.

9.1.25 Note. Let $n \in \mathbb{N}_{\geq 2}$. Denote by Δ_B the standard root base of B_n and by Δ_C the standard root base of C_n. Let $\alpha \in A_{n-1}$. Then α can be regarded as a root in B_n and as a root in C_n, so we have a standard Δ_B-expression $\bar{\rho}^\alpha_B$ and a standard Δ_C-expression $\bar{\rho}^\alpha_C$. In fact, these expression only contain roots from the canonical A_{n-1}-subsystems of B_n and C_n, and we have $\bar{\rho}^\alpha_B = \bar{\rho}^\alpha_C$.

Now let $i \in [1, n]$. Denote by $\bar{\rho}^i_B$ the standard Δ_B-expression of e_i and by $\bar{\rho}^i_C$ the standard Δ_C-expression of $2e_i$. Then $\bar{\rho}^i_C$ is the word obtained from $\bar{\rho}^i_B$ by replacing each occurrence of e_n by $2e_n$.

9.1.26 Definition ($\hat{C}_n$-extensions). Denote by Δ the standard rescaled root base of BC_n. Let G be any group with a BC_n-pregrading $(\hat{U}_\alpha)_{\alpha \in BC_n}$ and let $(w_\delta)_{\delta \in \Delta}$ be a Δ-system of Weyl elements in G. Then we define a family $(w_\alpha)_{\alpha \in \hat{C}_n}$, called the *standard $\hat{C}_n$-extension of* $(w_\delta)_{\delta \in \Delta}$, by $w_\alpha := w_{\bar{\rho}^\alpha}$ for all $\alpha \in \hat{C}_n$. We will sometimes write w_{ij} for $w_{e_i - e_j}$ and w_i for w_{2e_i}.

9.2 The Relationship between Root Gradings of Types *B*, *C* and *BC*

9.2.1 Notation for this section. We denote by n and integer at least 2 and by G an arbitrary group.

In this section, we investigate how (crystallographic) root gradings of types B, C and BC can be constructed from each other. All assertions in this sections are immediate consequences of the definition of root gradings. The most important results are Remark 9.2.8, which says that crystallographic B_2-gradings are the same as crystallographic C_2-gradings, and Remark 9.2.5, which says that crystallographic BC_n-graded groups have crystallographic C_n-graded subgroups. Both statements together yield that a crystallographic BC_n-graded group has many crystallographic B_2-graded subgroups to which we can apply the results of section 7.5. Further, Remark 9.2.6 yields that we do not have to consider crystallographic C_n-gradings separately because they can be seen as a special case of crystallographic BC_n-gradings.

We begin with the situation of non-crystallographic gradings. These observations are not intrinsically relevant for us because we are ultimately interested in crystallographic root gradings. Rather, they illustrate that the crystallographic condition is the key property which differentiates root gradings of types B, C and BC from each other.

9.2.2 Remark (The non-crystallographic case for B and C). If we consider root gradings which are not assumed to be crystallographic, the distinction between B_n and C_n disappears. To make this precise, consider B_n and C_n as root systems in the same Euclidean space and denote by $f\colon B_n \to C_n$ the map which sends $\pm e_i$ to $\pm 2e_i$ and which fixes all other roots. Let $(U_\alpha)_{\alpha \in B_n}$ be a B_n-pregrading of G. Put $U'_\beta := U_{f^{-1}(\beta)}$ for all $\beta \in C_n$. Then $(U_\alpha)_{\alpha \in B_n}$ is a (not necessarily crystallographic)

B_n-grading of G if and only if $(U'_\beta)_{\beta \in C_n}$ is a (not necessarily crystallographic) C_n-grading of G.

9.2.3 Remark (The non-crystallographic case for *BC*). Let $(U_\alpha)_{\alpha \in BC_n}$ be a (not necessarily crystallographic) BC_n-grading of G. Then the group $U_{\pm 2e_i}$ is a subgroup of $U_{\pm e_i}$ for all $i \in [1, n]$. Further, $(U_\alpha)_{\alpha \in B_n}$ is a B_n-gradins of G (and thus also a C_n-grading of G).

Conversely, let $(U_\alpha)_{\alpha \in C_n}$ be a C_n-grading of G. To this grading we can associate a BC_n-grading $(U_\alpha)_{\alpha \in BC_n}$ by setting $U_{\pm e_i} := U_{\pm 2e_i}$. By the observations in the previous paragraph, we can construct BC_n-gradings from B_n-gradings in a similar way.

We conclude that a BC_n-grading $(U_\alpha)_{\alpha \in BC_n}$ is the same thing as a B_n-grading $(U_\alpha)_{\alpha \in B_n}$ (or a C_n-grading) together with an additional family $(U_\beta)_{\beta \in BC_n \setminus B_n}$ of subgroups of the short root groups such that the axioms of a root grading are satisfied for $(U_\alpha)_{\alpha \in BC_n}$. This additional requirement says precisely that the following statements are satisfied:

(i) For each $\alpha \in B_n$, there exists an α-Weyl element w_α with respect to $(U_\beta)_{\beta \in B_n}$ such that $U_{\pm 2e_i}^{w_\alpha} = U_{\pm \sigma_\alpha(2e_i)}$ for all $i \in [1, n]$.

(ii) For all $i \in [1, n]$, there exists a $\pm 2e_i$-Weyl element. That is, there exists a $(\pm e_i)$-Weyl element which lies in $U_{\mp 2e_i} U_{\pm 2e_i} U_{\mp 2e_i}$.

We now turn to the case of crystallographic root gradings.

9.2.4 Remark (The crystallographic case for *B* and *C*). It is clear from Remarks 7.2.6 and 9.1.11 that the map $f\colon B_n \to C_n$ from Remark 9.2.2 no longer induces an equivalence between crystallographic B_n-gradings and crystallographic C_n-gradings. If $n = 2$, then we can replace f by a different map to obtain an equivalence between crystallographic B_2-gradings and crystallographic C_2-gradings. We will see this in Remark 9.2.8. If $n > 2$, however, crystallographic B_n-gradings and crystallographic C_n-gradings are not related.

Some of the constructions in Remark 9.2.2 remain valid in the crystallographic setting.

9.2.5 Remark (C_n-graded subgroups in BC_n-graded groups). Let $(U_\alpha)_{\alpha \in BC_n}$ be a crystallographic BC_n-grading of G. Since C_n is a closed root subsystem of BC_n, it follows from Proposition 2.5.11 that $(U_\alpha)_{\alpha \in C_n}$ is a crystallographic C_n-grading, but of $\langle U_\alpha \mid \alpha \in C_n \rangle$ and not of G. However, $(U_\alpha)_{\alpha \in B_n}$ is not necessarily a crystallographic B_n-grading of G: For example, it is not necessarily true that $U_{e_1 - e_2}$ commutes with $U_{e_1 + e_2}$.

9.2.6 Remark (C_n-graded groups as BC_n-graded groups). Now let $(U_\alpha)_{\alpha \in C_n}$ be a crystallographic C_n-grading of G. Putting $U_{\pm e_i} := U_{\pm 2e_i}$ for all $i \in [1, n]$, we obtain a crystallographic BC_n-grading $(U_\alpha)_{\alpha \in BC_n}$ of G. Thus we can regard crystallographic C_n-gradings as special cases of crystallographic BC_n-gradings. For this reason, we will not specifically consider C_n-gradings most of the time in this chapter.

9.2.7 Remark (BC_n-graded groups as C_n-graded groups). Let $(U_\alpha)_{\alpha \in BC_n}$ be a crystallographic BC_n-grading of G. Put $U'_{2\alpha} := U_\alpha$ for all short roots α in BC_n and $U'_\beta := U_\beta$ for all medium-length roots β in BC_n. Then $(U'_\alpha)_{\alpha \in C_n}$ is a crystallographic C_n-grading of G if and only if for all distinct $i, j \in [1, n]$ and for all signs

$\varepsilon, \sigma \in \{\pm 1\}$, we have that $U_{\varepsilon e_i}$ commutes with $U_{\sigma e_j}$. In other words, a crystallographic BC_n-grading can be regarded as a crystallographic C_n-grading if and only if pairs of orthogonal short root groups commute. A crystallographic BC_n-grading is called *proper* if it cannot be regarded as a crystallographic C_n-gradings in this way.

9.2.8 Remark (B_2-graded groups are C_2-graded groups). Let (α, δ) be a B_2-pair in B_2 and let (α', δ') be a C_2-pair in C_2. Then the unique vector space isomorphism $g\colon \langle \alpha, \delta \rangle_{\mathbb{R}} \to \langle \alpha', \delta' \rangle_{\mathbb{R}}$ which maps α to δ' and δ to α' induces an isomorphism from B_2 to C_2 (in the sense of Definition 1.2.49). Now let G be a group, let $(U_\alpha)_{\alpha \in B_2}$ be a B_2-pregrading of G and put $U'_\beta := U_{g^{-1}(\beta)}$ for all $\beta \in C_2$. Then $(U_\alpha)_{\alpha \in B_2}$ is a crystallographic B_2-grading if and only if $(U'_\beta)_{\beta \in C_2}$ is a crystallographic C_2-grading. Thus crystallographic B_2-gradings and crystallographic C_2-gradings are essentially the same thing.

We will usually apply the previous observation in the following form. Let G be a group with a crystallographic C_2-grading $(U_\zeta)_{\zeta \in C_2}$ and let $(\alpha, \beta, \gamma, \delta)$ be a C_2-quadruple. Then we can regard $(U_\zeta)_{\zeta \in C_2}$ as a crystallographic B_2-grading of G for which $(\delta, \gamma, \beta, \alpha)$ is a B_2-quadruple. To this grading, we can apply the results from section 7.5. In other words, we can use the results from section 7.5 by reversing the roles of α and δ (and consequently, of β and γ).

Recall from Remark 9.2.5 that any BC_n-graded group has a canonical crystallographic C_n-graded subgroup and hence many crystallographic C_2-graded subgroups. We can apply the observations of the previous paragraph to these subgroups as well.

9.3 Construction of BC_n-graded Groups

9.3.1 Notation for this section. We denote by $(\mathcal{R}, \mathcal{R}_0, \sigma)$ and associative involutory set, by (M, q, f) a standard pseudo-quadratic module over $(\mathcal{R}, \mathcal{R}_0, \sigma)$ and by $T := T(M)$ the group from Definition 8.5.25. Further, we fix an integer $n \in \mathbb{N}_{\geq 2}$, the root system $\Phi := BC_n$ in standard representation (as in Remark 9.1.2) and we denote by Δ the standard rescaled root base of BC_n.

The goal of this section is the construction of a BC_n-graded group $\mathrm{EU}(q)$ from the data in Notation 9.3.1. This construction is similar to the one of the elementary orthogonal group in section 7.3. As a consequence, both sections follow essentially the same outline, and some remarks remain true almost verbatim. In Note 9.3.32, we will explain to which extent the construction in this section solves the existence problem for crystallographic BC_n-graded groups. Observe that both f and q are allowed to be zero. In this case, (M, q, f) is simply an $\mathcal{R}$-module with no additional structure, and the group T equals $\mathcal{R}_0 \times M$ by Remark 8.5.28.

Just like the elementary orthogonal orthogonal group, we have implemented the group $\mathrm{EU}(q)$ in GAP [Gap] to perform all the computations in this section. See Note 9.3.33 for a few details on the implementation.

9.3.2 Convention. In this section, we will consider modules over the ring $\mathcal{R}$. We use the convention that $\mathcal{R}$-scalars act on such modules from the right while endomorphisms act from the left. In particular, the composition $\varphi \circ \psi$ of two such endomorphisms φ, ψ is the map $x \mapsto \varphi(\psi(x))$.

9.3.A Construction

9.3.3 Construction. We put $V_+ := \mathcal{R}^n$, $V_- := \mathcal{R}^n$ and $V := M \oplus V_+ \oplus V_-$. We denote by $(b_1, \dots, b_n)$ the standard basis of V_+, by $(b_{-1}, \dots, b_{-n})$ the standard basis of V_- and we will always consider V_+ and V_- to be subsets of V without specifying the natural embedding. We will usually denote elements of V_+, V_- by the letters v, w, elements of M by the letters m, u and elements of V by the letters x, y. We can represent automorphisms of V using generalised matrices in the same way as in Remark 7.3.5.

9.3.4 Construction (Medium-length root homomorphisms). Let $i, j \in [1, n]$ be distinct and let $a \in \Bbbk$. We denote by $\theta_{e_i - e_j}(a)$ the unique $\mathcal{R}$-linear endomorphism of V which is given on the direct summands of V as follows, where m, v^+, v^- denote arbitrary elements of M, V_+, V_-, respectively:

$$\theta_{e_i - e_j}(a)\colon \quad m \mapsto m, \quad v^+ \mapsto v^+ + b_i \cdot a v_j^+, \quad v^- \mapsto v^- - b_j \cdot a^\sigma v_i^-.$$

Now assume that, in addition, $i < j$. Then we define $\theta_{e_i + e_j}$ and $\theta_{-e_i - e_j}$ by the following formulas:

$$\begin{aligned} \theta_{e_i + e_j}(a)\colon &\quad m \mapsto m, \quad v^+ \mapsto v^+, \quad v^- \mapsto v^- + b_i \cdot a v_j^- + b_j \cdot a^\sigma v_i^-, \\ \theta_{-e_i - e_j}(a)\colon &\quad m \mapsto m, \quad v^+ \mapsto v^+ + b_{-j} \cdot a v_i^+ + b_{-i} \cdot a^\sigma v_j^+, \quad v^- \mapsto v^-. \end{aligned}$$

Examples of the corresponding generalised matrices are given in Figure 9.2.

$$\theta_{e_1 - e_2}(a) = \left(\begin{array}{c|ccc|ccc} \mathrm{id}_M & & & & & & \\ \hline & 1 & a & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ \hline & & & & 1 & & \\ & & & & -a^\sigma & 1 & \\ & & & & & & 1 \end{array}\right),$$

$$\theta_{e_1 + e_2}(a) = \left(\begin{array}{c|ccc|ccc} \mathrm{id}_M & & & & & & \\ \hline & 1 & & & & a & \\ & & 1 & & a^\sigma & & \\ & & & 1 & & & \\ \hline & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{array}\right),$$

$$\theta_{-e_1 - e_2}(a) = \left(\begin{array}{c|ccc|ccc} \mathrm{id}_M & & & & & & \\ \hline & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ \hline & & a^\sigma & & 1 & & \\ & a & & & & 1 & \\ & & & & & & 1 \end{array}\right)$$

Figure 9.2: The medium-length root homomorphisms for BC_3.

9.3.5 Construction (Short root homomorphisms). Let $i \in [1, n]$ and let $(u, h) \in T$. Then we define $\theta_{2e_i}(u, h)$ and $\theta_{-2e_i}(u, h)$ to be the unique $\mathcal{R}$-linear endomorphisms of V which are given on the direct summands of V as follows, where m,

v^+, v^- denote arbitrary elements of M, V_+, V_-, respectively:

$$\theta_{2e_i}(u,h)\colon\ m \mapsto m - b_i \cdot f(u,m),\ v^+ \mapsto v^+,\ v^- \mapsto v^- - u \cdot v_i^- + b_i \cdot hv_i^-,$$
$$\theta_{-2e_i}(u,h)\colon\ m \mapsto m - b_{-i} \cdot f(u,m),\ v^+ \mapsto v^+ + u \cdot v_i^+ - b_{-i} \cdot hv_i^+,\ v^- \mapsto v^-.$$

Examples of the corresponding generalised matrices are given in Figure 9.3.

$$\theta_{e_1}(u,h) = \left(\begin{array}{c|ccc|ccc}
\mathrm{id}_M & & & & -u & & \\
\hline
-f(u,\,\cdot\,) & 1 & & & h & & \\
 & & 1 & & & & \\
 & & & 1 & & & \\
\hline
 & & & & 1 & & \\
 & & & & & 1 & \\
 & & & & & & 1
\end{array}\right),$$

$$\theta_{-e_1}(u,h) = \left(\begin{array}{c|ccc|ccc}
\mathrm{id}_M & u & & & & & \\
\hline
 & 1 & & & & & \\
 & & 1 & & & & \\
 & & & 1 & & & \\
\hline
-f(u,\,\cdot\,) & -h & & & 1 & & \\
 & & & & & 1 & \\
 & & & & & & 1
\end{array}\right)$$

Figure 9.3: The short root homomorphisms for BC_3.

9.3.6 Lemma. *For each medium-length root α and for each short root β, the maps $\theta_\alpha\colon (\mathcal{R},+) \to \mathrm{End}_{\mathcal{R}}(V)$ and $\theta_\beta\colon T \to \mathrm{End}_{\mathcal{R}}(V)$ are injective homomorphisms. In particular, their images lie in* $\mathrm{Aut}_{\mathcal{R}}(V)$.

Proof. The injectivity of these maps can be deduced from their matrix representations. The homomorphism property follows from a straightforward matrix computation. □

We can now define our main example of a BC_n-graded group.

9.3.7 Definition (Elementary unitary group). For each root α, we denote the image of θ_α by U_α, and we denote by $\mathrm{EU}(q)$ the group which is generated by $(U_\alpha)_{\alpha\in\Phi}$. We call $\mathrm{EU}(q)$ the *elementary unitary group of q*.

9.3.8 Warning. The group $\mathrm{EU}(q)$ depends not only on q but also on the choice of f, which we suppress in our notation.

9.3.9 Note. The same remarks as in Note 7.3.10 hold for the elementary unitary group: A straightforward computation shows that this group is contained in the unitary group $\mathrm{U}(q)$ from Definition 8.5.16, but we will have no need to formally use this fact.

9.3.B Weyl Elements

Weyl elements for medium-length roots are defined in the same way as usual. However, the formula for short Weyl elements, which is nearly the same as in [TW02, (32.9)], is more intricate. In particular, observe that short Weyl elements are not necessarily balanced (in the sense of Definition 2.2.12).

9.3.10 Definition (Weyl elements). For all distinct $i, j \in [1, n]$ and all invertible $a \in \mathcal{R}$, we define

$$w_{ij}(a) := w_{e_i - e_j}(a) := \theta_{e_j - e_i}(-a^{-1}) \circ \theta_{e_i - e_j}(a) \circ \theta_{e_j - e_i}(-a^{-1})$$

and $w_{ij} := w_{e_i - e_j} := w_{ij}(1_{\mathcal{R}})$. For all $i \in [1, n]$ and all $(u, h) \in T$ for which h is invertible, we define

$$w_i(u, h) := w_{2e_i}(u, h) := \theta_{-2e_i}\big(uh^{-1}, h^{-\sigma}\big) \circ \theta_{2e_i}(u, h) \circ \theta_{-2e_i}\big(-uh^{-\sigma}, h^{-\sigma}\big)$$

and $w_i := w_{2e_i} := w_i(0_M, 1_{\mathcal{R}})$. Note that $w_i(u, h)$ is well-defined by Lemma 8.5.33 and that $(0_M, 1_{\mathcal{R}})$ lies in T because $(\mathcal{R}, \mathcal{R}_0, \sigma)$ is an involutory set (and not merely a pre-involutory set).

9.3.11 Definition (Standard system of Weyl elements). The *standard system of Weyl elements for* $\mathrm{EU}(q)$ is the family $(w_\delta)_{\delta \in \Delta}$ given by Definition 9.3.10.

Similar remarks as in Note 7.3.12 apply to Definition 9.3.10: We have only defined Weyl elements for the subset $\hat{C}_n$ of BC_n, but this will pose no problem in the sequel.

9.3.12 Remark (Short Weyl elements, see also 7.3.14). Let $(u, h) \in T$ be such that h is invertible. The goal of this remark is to explicitly compute the generalised matrix of $w_1(u, h)$. We will leave out the rows and columns which correspond to $(b_i)_{i \in [2,n]}$ and $(b_{-i})_{i \in [2,n]}$ because they are trivial. We begin the computation with

$$\begin{aligned}
&\theta_{e_1}(u, h)\theta_{-e_1}(-uh^{-\sigma}, h^{-\sigma}) \\
&= \begin{pmatrix} \mathrm{id} & 0 & -u \\ -f(u, \cdot\,) & 1 & h \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{id} & -uh^{-\sigma} & 0 \\ 0 & 1 & 0 \\ h^{-\sigma} f(u, \cdot\,) & -h^{-\sigma} & 1 \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{id} - uh^{-1} f(u, \cdot\,) & -uh^{-\sigma} + uh^{-\sigma} & -u \\ -f(u, \cdot\,) + hh^{-1} f(u, \cdot\,) & f(u, u)h^{-\sigma} + 1 - hh^{-\sigma} & h \\ h^{-1} f(u, \cdot\,) & -h^{-\sigma} & 1 \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{id} - uh^{-1} f(u, \cdot\,) & 0 & -u \\ 0 & 0 & h \\ h^{-1} f(u, \cdot\,) & -h^{-\sigma} & 1 \end{pmatrix}.
\end{aligned}$$

In the last step, we have used that $f(u, u) = h - h^\sigma$ because $h \in q(u) + \mathcal{R}_0$ and (M, q, f) is standard. Now

$$\begin{aligned}
w_1(u, h) &= \theta_{-e_1}(uh^{-1}, h^{-\sigma})\theta_{e_1}(u, h)\theta_{-e_1}(-uh^{-\sigma}, h^{-\sigma}) \\
&= \begin{pmatrix} \mathrm{id} & uh^{-1} & 0 \\ 0 & 1 & 0 \\ -h^{-\sigma} f(u, \cdot\,) & -h^{-\sigma} & 1 \end{pmatrix} \begin{pmatrix} \mathrm{id} - uh^{-1} f(u, \cdot\,) & 0 & -u \\ 0 & 0 & h \\ h^{-1} f(u, \cdot\,) & -h^{-\sigma} & 1 \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{id} - uh^{-1} f(u, \cdot\,) & 0 & -u + uh^{-1}h \\ 0 & 0 & h \\ \varphi & -h^{-\sigma} & -h^{-\sigma} f(u, u) - h^{-\sigma} h + 1 \end{pmatrix}
\end{aligned}$$

where $\varphi\colon M \to \mathcal{R}$ is the map which sends $x \in M$ to

$$\begin{aligned}
&-h^{-\sigma} f\big(u, x - uh^{-1} f(u, x)\big) + h^{-1} f(u, x) \\
&\qquad = -h^{-\sigma} f(u, x) + h^{-\sigma} f(u, u)h^{-1} f(u, x) + h^{-1} f(u, x) = 0.
\end{aligned}$$

We conclude that

$$w_1(u, h) = \begin{pmatrix} \mathrm{id} - uh^{-1} f(u, \cdot\,) & 0 & 0 \\ 0 & 0 & h \\ 0 & -h^{-\sigma} & 0 \end{pmatrix}.$$

Hence $w_1(ua, h) = w_1(u, h)$ where a is any element of the center of $\mathcal{R}$ with $a^\sigma a = 1_{\mathcal{R}}$ and $aa^\sigma = 1_{\mathcal{R}}$. (The first equation implies that (ua, h) lies in T and the second one implies that $w_1(ua, h) = w_1(u, h)$.) In particular, $w_1(u, v) = w_1(-u, v)$ where $(-u, v)$ is the element obtained from (u, v) by applying the Jordan module involution. Hence there exist short Weyl elements which can be represented by distinct Weyl triples. We will see in Note 9.5.15 that the same behaviour can be observed in all BC_3-graded groups.

We now prove that the elements in Definition 9.3.10 are indeed Weyl elements. Since the resulting formulas are rather intricate, we explicitly state the formulas for the standard Weyl elements separately. Only these simplified formulas will be needed to see the twisting actions on the root groups.

9.3.13 Lemma. *Let $i, j \in [1, n]$ be distinct and let $a \in \mathcal{R}$ be invertible. Then $w := w_{ij}(a)$ is an $(e_i - e_j)$-Weyl element. It satisfies the following formulas for all $b \in \mathcal{R}$ and all $(u, h) \in T$:*

(a) *We have*

$$\theta_{e_i-e_j}(b)^w = \theta_{e_j-e_i}(-a^{-1}ba^{-1}), \quad \theta_{e_i+e_j}(b)^w = \theta_{e_i+e_j}\big(-\delta^\sigma_{i>j}\big(a\delta^\sigma_{i<j}(b)a^{-\sigma}\big)\big),$$
$$\theta_{e_j-e_i}(b)^w = \theta_{e_i-e_j}(-aba), \quad \theta_{-e_i-e_j}(a)^w = \theta_{-e_i-e_j}\big(-\delta^\sigma_{i>j}\big(a^\sigma\delta^\sigma_{i<j}(b)a^{-1}\big)\big).$$

(b) *For all $k \in [1, n] \setminus \{ i, j \}$, we have*

$$\theta_{e_i-e_k}(b)^w = \theta_{e_j-e_k}(a^{-1}b), \qquad \theta_{e_i+e_k}(b)^w = \theta_{e_j+e_k}\big(\delta^\sigma_{k<j}\big(a^{-1}\delta^\sigma_{k<i}(b)\big)\big),$$
$$\theta_{e_k-e_i}(b)^w = \theta_{e_k-e_j}(ba), \qquad \theta_{-e_i-e_k}(b)^w = \theta_{-e_j-e_k}\big(\delta^\sigma_{j<k}\big(a^\sigma\delta^\sigma_{i<k}(b)\big)\big).$$

(c) *For all $k \in [1, n] \setminus \{ i, j \}$, we have*

$$\theta_{e_j-e_k}(b)^w = \theta_{e_i-e_k}(-ab), \qquad \theta_{e_j+e_k}(b)^w = \theta_{e_i+e_k}\big(-\delta^\sigma_{k<i}\big(a\delta^\sigma_{k<j}(b)\big)\big),$$
$$\theta_{e_k-e_j}(b)^w = \theta_{e_k-e_i}(-ba^{-1}), \quad \theta_{-e_j-e_k}(b)^w = \theta_{-e_i-e_k}\big(-\delta^\sigma_{i<k}\big(a^{-\sigma}\delta^\sigma_{j<k}(b)\big)\big).$$

(d) $\theta_{e_i}(u, h)^w = \theta_{e_j}\big(ua^{-\sigma}, a^{-1}ha^{-\sigma}\big)$ *and* $\theta_{-e_i}(u, h)^w = \theta_{-e_j}\big(ua, a^\sigma ha\big)$.

(e) $\theta_{e_j}(u, h)^w = \theta_{e_i}\big(-ua^\sigma, aha^\sigma\big)$ *and* $\theta_{-e_j}(u, h)^w = \theta_{-e_i}\big(-ua^{-1}, a^{-\sigma}ha^{-1}\big)$.

(f) *For any root α for which no formula for the conjugation action of w on U_α is given above, this action is trivial.*

Proof. This follows from a straightforward computation. See also Note 9.3.33. □

9.3.14 Lemma (Corollary of 9.3.13). *Let $i, j \in [1, n]$ be distinct. Then $w := w_{ij}$ is an $(e_i - e_j)$-Weyl element. It satisfies the following formulas for all $b \in \mathcal{R}$ and all $(u, h) \in T$:*

(a) *We have*

$$\theta_{e_i-e_j}(b)^w = \theta_{e_j-e_i}(-b), \qquad \theta_{e_i+e_j}(b)^w = \theta_{e_i+e_j}(-b^\sigma),$$
$$\theta_{e_j-e_i}(b)^w = \theta_{e_i-e_j}(-b), \qquad \theta_{-e_i-e_j}(a)^w = \theta_{-e_i-e_j}(-b^\sigma).$$

(b) *For all $k \in [1, n] \setminus \{ i, j \}$, we have*

$$\theta_{e_i-e_k}(b)^w = \theta_{e_j-e_k}(b), \qquad \theta_{e_i+e_k}(b)^w = \theta_{e_j+e_k}\big(\delta^\sigma_{k\in\langle i,j\rangle}(b)\big),$$
$$\theta_{e_k-e_i}(b)^w = \theta_{e_k-e_j}(b), \qquad \theta_{-e_i-e_k}(b)^w = \theta_{-e_j-e_k}\big(\delta^\sigma_{k\in\langle i,j\rangle}(b)\big).$$

(c) *For all $k \in [1, n] \setminus \{ i, j \}$, we have*

$$\theta_{e_j-e_k}(b)^w = \theta_{e_i-e_k}(-b), \qquad \theta_{e_j+e_k}(b)^w = \theta_{e_i+e_k}\big(-\delta^\sigma_{k\in\langle i,j\rangle}(b)\big),$$
$$\theta_{e_k-e_j}(b)^w = \theta_{e_k-e_i}(-b), \qquad \theta_{-e_j-e_k}(b)^w = \theta_{-e_i-e_k}\big(-\delta^\sigma_{k\in\langle i,j\rangle}(b)\big).$$

(d) $\theta_{e_i}(u,h)^w = \theta_{e_j}(u,h)$ *and* $\theta_{-e_i}(u,h)^w = \theta_{-e_j}(u,h)$.

(e) $\theta_{e_j}(u,h)^w = \theta_{e_i}(-u,h)$ *and* $\theta_{-e_j}(u,h)^w = \theta_{-e_i}(-u,h)$.

(f) *For any root α for which no formula for the conjugation action of w on U_α is given above, this action is trivial.*

Proof. This follows from Lemma 9.3.13 by putting $a := 1_{\mathcal{R}}$. □

9.3.15 Lemma. *Let $i \in [1,n]$ and let $(v,t) \in T$ be such that t is invertible. Then $w := w_i(v,t)$ is an e_i-Weyl element. It satisfies the following formulas for all $b \in \mathcal{R}$ and all $(u,h) \in T$:*

(a) *For all $k \in [1,n] \setminus \{i\}$, we have*

$$\theta_{e_i-e_k}(b)^w = \theta_{-e_i-e_k}\big(\delta^\sigma_{i>k}(b^\sigma t^{-\sigma})\big), \qquad \theta_{e_i+e_k}(b)^w = \theta_{e_k-e_i}\big(-\delta^\sigma_{i<k}(b)t^{-\sigma}\big),$$
$$\theta_{e_k-e_i}(b)^w = \theta_{e_k+e_i}\big(\delta^\sigma_{i<k}(bt)\big), \qquad \theta_{-e_i-e_k}(b)^w = \theta_{e_i-e_k}\big(-t^\sigma\delta^\sigma_{i<k}(b)\big).$$

(b) $\theta_{e_i}(u,h)^w = \theta_{-e_i}\big(ut^{-\sigma} + vt^{-\sigma}f(v,u)t^{-\sigma}, t^{-1}ht^{-\sigma}\big)$ *and* $\theta_{-e_i}(u,h)^w = \theta_{e_i}\big(-ut - vt^{-\sigma}f(v,u)t, t^\sigma ht\big)$.

(c) $\theta_{e_j}(u,h)^w = \theta_{e_j}\big(u + vt^{-\sigma}f(v,u),h\big)$ *and* $\theta_{-e_j}(u,h)^w = \theta_{-e_j}\big(u + vt^{-\sigma}f(v,u),h\big)$.

(d) *For any root α for which no formula for the conjugation action of w on U_α is given above, this action is trivial.*

Proof. This follows from a straightforward computation. See also Note 9.3.33. □

9.3.16 Lemma (Corollary of 9.3.15). *Let $i \in [1,n]$. Then $w := w_i$ is an e_i-Weyl element. It satisfies the following formulas for all $b \in \mathcal{R}$ and all $(u,h) \in T$:*

(a) *For all $k \in [1,n] \setminus \{i\}$, we have*

$$\theta_{e_i-e_k}(b)^w = \theta_{-e_i-e_k}\big(\delta^\sigma_{i<k}(b)\big), \qquad \theta_{e_i+e_k}(b)^w = \theta_{e_k-e_i}\big(-\delta^\sigma_{i<k}(b)\big),$$
$$\theta_{e_k-e_i}(b)^w = \theta_{e_k+e_i}\big(\delta^\sigma_{i<k}(b)\big), \qquad \theta_{-e_i-e_k}(b)^w = \theta_{e_i-e_k}\big(-\delta^\sigma_{i<k}(b)\big).$$

(b) $\theta_{e_i}(u,h)^w = \theta_{-e_i}(u,h)$ *and* $\theta_{-e_i}(u,h)^w = \theta_{e_i}(-u,h)$.

(c) *For any root α for which no formula for the conjugation action of w on U_α is given above, this action is trivial.*

Proof. This follows from Lemma 9.3.15 by putting $(v,t) := (0_M, 1_{\mathcal{R}})$. □

9.3.17 Remark (compare 7.3.16). Let $i,j \in [1,n]$ be distinct and let $a \in \mathcal{R}$ be invertible. Since $w := w_{ij}(a)$ is a Weyl element by Lemma 9.3.13, it follows from Proposition 2.2.6 (c) that

$$\begin{aligned} w_{ij}(a) &= \theta_{e_i-e_j}(a)\theta_{e_j-e_i}(-a^{-1})\theta_{e_j-e_i}(-a^{-1})^w \\ &= \theta_{e_i-e_j}(a)\theta_{e_j-e_i}(-a^{-1})\theta_{e_i-e_j}(a) = w_{ji}(-a^{-1}). \end{aligned}$$

In particular, $w_{ij} = w_{ji}^{-1}$.

9.3.18 Remark. Let $i,j \in [1,n]$ be distinct and put $w := w_i^2$. It follows from Lemma 9.3.16 (b) that $\theta_{e_i}(u,h)^w = \theta_{e_i}(-u,h)$ for all $(u,h) \in T$. That is, w acts on U_{e_i} by the Jordan module involution (see Observation 8.5.36). In particular, the action of w on U_{e_i} is, in general, neither trivial nor by inversion, so it does not satisfy the square formula for Weyl elements. By Lemma 9.3.14 (d) and 9.3.14 (e), w_{ij}^2 acts on U_{e_i} by the Jordan module involution as well.

The observations in this remark also say that the Jordan module involution coincides with the short involution that we will define in 9.5.4 on the short root groups of any BC_n-graded group.

9.3.C Parity Maps and Twisting Structures

As in Notation 7.3.18, the formulas above tell us how to define the twisting groups and parity maps for $\mathrm{EU}(q)$. They are chosen precisely to make sure that Lemma 9.3.22 holds.

9.3.19 Notation for this section. From now on, we denote by $(A \times B, T, \mathcal{R})$ the standard parameter system (in the sense of Definition 8.6.20) for the Jordan module T which is constructed from the pseudo-quadratic module (M, q, f) as in Example 8.6.7. Further, we define maps

$$\eta\colon BC_n \times \hat{C}_n \to A \quad \text{and} \quad \mu\colon B_n \times \hat{C}_n \to B$$

by the formulas in Figure 7.4, where $\hat{C}_n$ is as in Definition 9.1.23. By restricting the second components of η and μ to Δ, we obtain Δ-parity maps which we also denote by η and μ, and which we call the *standard parity maps of type BC_n*.

α	η_{α,e_i-e_j}	μ_{α,e_i-e_j}
$\pm(e_i - e_j)$	$(-1_A, 1_A)$	1_B
$\pm(e_i + e_j)$	$(-1_A, 1_A)$	-1_B
$\pm(e_i - e_k)$	$(1_A, 1_A)$	1_B
$\pm(e_i + e_k)$	$(1_A, 1_A)$	$\delta^{-}_{k\in\langle i,j\rangle} 1_B$
$\pm(e_j - e_k)$	$(-1_A, 1_A)$	1_B
$\pm(e_j + e_k)$	$(-1_A, 1_A)$	$\delta^{-}_{k\in\langle i,j\rangle} 1_B$
$\pm e_k \pm e_l$	$(1_A, 1_A)$	1_B
$\pm(2)e_i$	$(1_A, 1_A)$	1_B
$\pm(2)e_j$	$(1_A, -1_A)$	1_B
$\pm(2)e_k$	$(1_A, 1_A)$	1_B

α	η_{α,e_i}	$\mu_{\alpha,2e_i}$
$e_i - e_k$	$(1_A, 1_A)$	$\delta^{-}_{i<k} 1_B$
$e_k - e_i$	$(1_A, 1_A)$	$\delta^{-}_{i<k} 1_B$
$\pm(e_i + e_k)$	$(-1_A, 1_A)$	$\delta^{-}_{i<k} 1_B$
$\pm e_k \pm e_l$	$(1_A, 1_A)$	1_B
$(2)e_i$	$(1_A, 1_A)$	1_B
$-(2)e_i$	$(1_A, -1_A)$	1_B
$\pm(2)e_k$	$(1_A, 1_A)$	1_B

Figure 9.4: The definition of $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ for all $\alpha \in BC_n$ and $\beta \in \hat{C}_n$, see Notation 9.3.19. In the left table, we assume that i, j, k, l are pairwise distinct. In the right table, we assume that i, k, l are pairwise distinct. For small values of n, it is of course not possible to choose three or four pairwise distinct indices, in which case the corresponding rows should be ignored.

9.3.20 Note. Similar remarks as in Note 7.3.19 apply in this situation: The values $\eta_{\alpha,\beta}$ for $\alpha \in BC_n$ and $\beta \in \hat{C}_n \setminus \Delta$ are not need in the definition of the parity map $\eta\colon BC_n \times \Delta \to A$, but they will appear in Lemma 9.3.31.

9.3.21 Remark (compare 7.3.20). Consider the subset

$$A_{n-1} = \{\, e_i - e_j \mid i \neq j \in [1, n] \,\}$$

of B_n. Then for all $\alpha, \beta \in A_{n-1}$, the first component of $\eta_{\alpha,\beta}$ in Figure 7.4 is the same as the value in Example 5.5.5 while the second component is trivial.

9.3.22 Lemma. *Let $\alpha \in BC_n$ and $\beta \in \hat{C}_n$. Let $x \in \mathcal{R}$ if α is medium-length and let $x \in T$ if α is short. Then $\theta_\alpha(x)^{w_\beta} = \theta_{\sigma_\beta(\alpha)}(\eta_{\alpha,\beta}\mu_{\alpha,\beta}.x)$.*

Proof. This follows from Lemmas 9.3.14 and 9.3.16. □

9.3.23 Remark. As in Remark 7.3.22, we can perform computations in the group $\mathrm{EU}(q)$ for some fixed choice of the parameters in Notation 9.3.1 to obtain information about the parity maps η and μ. In the sequel, we will often choose

$(\mathcal{R}, \mathcal{R}_0, \sigma) := (\mathbb{C}, \mathbb{R}, \sigma)$ where σ is complex conjugation and $M := \mathbb{C}$ with q and f defined as in Example 8.5.8. Then the inversion map on $\mathcal{R}$ and the ring involution are non-trivial and distinct. Further, the inversion map on T and the Jordan module involution on T are non-trivial and distinct. These observations imply that the parameter system $(A \times B, T, \mathcal{R})$ is $(\eta \times \mu)$-faithful (in the sense of Definition 4.3.8).

9.3.24 Lemma. *The root isomorphisms $(\theta_\alpha)_{\alpha \in BC_n}$ from Constructions 7.3.6 and 7.3.7 form a parametrisation of $\mathrm{EU}(q)$ by $(A \times B, T, \mathcal{R})$ with respect to $\eta \times \mu$ and $(w_\delta)_{\delta \in \Delta}$.*

Proof. This follows from Lemma 9.3.22. □

9.3.25 Lemma. *Let $(w_\alpha)_{\alpha \in \hat{C}_n}$ be as in Definition 7.3.11. Then the following hold:*

(a) $w_{ij}^{w_{jk}} = w_{ik}$ and $w_{ij}^{w_{kj}} = w_{ki}$ for all pairwise distinct $i, j, k \in [1, n]$, and $w_i^{w_{ij}} = w_j$ and $w_i^{w_{ji}} = w_i^{-1}$ for all distinct $i, j \in [1, n]$.

(b) $(w_\alpha)_{\alpha \in \hat{C}_n}$ is the standard $\hat{C}_n$-extension of $(w_\delta)_{\delta \in \Delta}$.

Proof. The first assertion follows from Lemma 9.3.22 and an inspection of Figure 9.4. The second assertion follows from the first one and Remark 9.3.17. □

9.3.D Commutator Relations

We now proceed to show that $(U_\alpha)_{\alpha \in BC_n}$ is a crystallographic BC_n-grading of $\mathrm{EU}(q)$.

9.3.26 Proposition. *The group $\mathrm{EU}(q)$ satisfies the following commutator relations. For all pairwise distinct $i, j, k \in [1, n]$ and all $a, b \in \mathcal{R}$, we have*

$$\begin{aligned}
[\theta_{e_i - e_j}(a), \theta_{e_j - e_k}(b)] &= \theta_{e_i - e_k}(ab), \\
[\theta_{e_i - e_j}(a), \theta_{e_j + e_k}(b)] &= \theta_{e_i + e_k}\big(\delta^\sigma_{k<i}(a \delta^\sigma_{k<j}(b))\big), \\
[\theta_{e_i - e_j}(a), \theta_{e_i + e_j}(b)] &= \theta_{e_i}\big(0_M, \mathrm{Tr}(a, \delta^\sigma_{i>j}(b))\big), \\
[\theta_{e_i - e_j}(a), \theta_{-e_i - e_j}(b)] &= \theta_{-e_j}\big(0_M, \mathrm{Tr}(a^\sigma, \delta^\sigma_{j<i}(b))\big), \\
[\theta_{e_i - e_j}(a), \theta_{-e_i - e_k}(b)] &= \theta_{-e_j - e_k}\big(-\delta^\sigma_{j<k}(a^\sigma \delta^\sigma_{i<k}(b))\big), \\
[\theta_{e_i + e_j}(a), \theta_{-e_j - e_k}(b)] &= \theta_{e_i - e_k}\big(\delta^\sigma_{j<i}(a) \delta^\sigma_{j<k}(b)\big)
\end{aligned}$$

where $\mathrm{Tr}(a, b) := ab^\sigma + ba^\sigma$. For all distinct $i, j \in [1, n]$ and all $(u, h), (v, k) \in T(q)$, we have

$$\begin{aligned}
[\theta_{e_i}(u, h), \theta_{e_j}(v, k)] &= \begin{cases} \theta_{e_i + e_j}\big(f(u, v)\big) & \text{if } i < j, \\ \theta_{e_i + e_j}\big(-f(v, u)\big) & \text{if } i > j, \end{cases} \\
[\theta_{e_i}(u, h), \theta_{-e_j}(v, k)] &= \theta_{e_i - e_j}\big(-f(u, v)\big), \\
[\theta_{-e_i}(u, h), \theta_{-e_j}(v, k)] &= \begin{cases} \theta_{-e_i - e_j}\big(f(v, u)\big) & \text{if } i < j, \\ \theta_{-e_i - e_j}\big(-f(u, v)\big) & \text{if } i > j. \end{cases}
\end{aligned}$$

For all distinct $i, j \in [1, n]$, all $a \in \mathcal{R}$ and all $(u, h) \in T(q)$, we have

$$\begin{aligned}
[\theta_{e_j}(u, h), \theta_{e_i - e_j}(a)] &= \theta_{e_i}\big(-ua^\sigma, aha^\sigma\big)\theta_{e_i + e_j}\big(-\delta^\sigma_{i>j}(ah)\big), \\
[\theta_{-e_i}(u, h), \theta_{e_i - e_j}(a)] &= \theta_{-e_j}\big(ua, a^\sigma ha\big)\theta_{-e_i - e_j}\big(\delta^\sigma_{i>j}(a^\sigma h)\big), \\
[\theta_{-e_i}(u, h), \theta_{e_i + e_j}(a)] &= \theta_{e_j}\big(-u\delta^\sigma_{i>j}(a), \delta^\sigma_{i<j}(a) h \delta^\sigma_{i>j}(a)\big)\theta_{e_j - e_i}\big(\delta^\sigma_{i<j}(a)h\big), \\
[\theta_{e_i}(u, h), \theta_{-e_i - e_j}(a)] &= \theta_{-e_j}\big(-u\delta^\sigma_{i<j}(a), \delta^\sigma_{i>j}(a) h \delta^\sigma_{i<j}(a)\big)\theta_{e_i - e_j}\big(h^\sigma \delta^\sigma_{i<j}(a)\big).
\end{aligned}$$

Proof. This follows from a straightforward but lengthy computation. See also Note 9.3.33. □

9.3.27 Proposition. *Let* Π *denote the standard positive system in* BC_n. *Then the group* $\mathrm{EU}(q)$ *satisfies* $U_\Pi \cap U_{-\Pi} = \{1\}$.

Proof. We consider the same rearrangement of the decomposition of V as in Proposition 7.3.26. With respect to this rearrangement, the group U_Π consists of upper triangular matrices while $U_{-\Pi}$ consists of lower triangular matrices. The assertion follows. □

9.3.28 Theorem. *The family* $(U_\alpha)_{\alpha \in BC_n}$ *is a crystallographic* BC_n*-grading of* $\mathrm{EU}(q)$.

Proof. By Lemmas 9.3.13 and 9.3.15, there exist Weyl elements for all roots in the standard rescaled root base of BC_n. By Remark 2.2.8, it follows that there exist Weyl elements for all roots. The remaining conditions are satisfied by Propositions 9.3.26 and 9.3.27. □

9.3.E Concluding Remarks

9.3.29 Remark. It follows from the commutator relations in Proposition 9.3.26 that $\mathrm{EU}(q)$ can be regarded as a crystallographic C_n-graded group (in the sense of Remark 9.2.7) if and only if $f = 0$. If $2_{\mathcal{R}}$ is invertible, this property implies that q is trivial by Lemma 8.5.15 (f). In the case that both f and q are zero, our construction produces a crystallographic C_n-graded group whose long root groups are parametrised by $\mathcal{R}_0 \times M$ where M is an $\mathcal{R}$-module with no additional structure.

9.3.30 Note. Assume that $M \neq \{0\}$, $q = 0$, $f = 0$ and $n = 2$, so that $\mathrm{EU}(q)$ is a crystallographic C_2-graded group by Remark 9.3.29. Then $T = M \times \mathcal{R}_0$ by Remark 8.5.28. We have seen in Remark 9.3.18 that $w := w_1^2$ and $v := w_{21}^2$ act on U_{e_1} by the Jordan module involution on T. Since $M \neq \{0\}$, this involution is neither the identity nor the inversion map.

Now put $\alpha := e_1$ and $\delta := e_2 - e_1$ and regard $\mathrm{EU}(q)$ as a crystallographic B_2-graded group, as in Remark 9.2.8. Then (α, δ) is a B_2-pair in the B_2-grading of $\mathrm{EU}(q)$. The observation in the previous paragraph implies that $\mathrm{EU}(q)$ is a crystallographic B_2-graded group in which the squares of α-Weyl elements and δ-Weyl do not act trivially on U_α. This is the counterexample that was promised in Notes 7.6.6 and 7.6.18.

The following result is the analogue of Lemma 7.3.28 for the root system BC. It will be used in the proof of Proposition 9.9.2

9.3.31 Lemma. *Let* $\alpha \in \hat{C}_n$, *let* $\bar{\alpha}$ *be the standard* Δ*-expression of* α *in the sense of Definition 9.1.23 and let* ζ *be an arbitrary root. Then the element* $\eta_{\zeta,\alpha}$ *from Figure 9.4 equals the value* $\eta_{\zeta,\bar{\alpha}}$ *of the extended parity map* $\eta\colon \Phi \times \mathcal{L}(\Delta \cup (-\Delta)) \to A$ *(from Definition 4.2.4). The same assertion holds for* μ *in place of* η.

Proof. This can be proven in exactly the same way as Lemma 7.3.28, using the choice of parameters in Remark 9.3.23. □

9.3.32 Note (The existence problem). For a complete solution of the existence problem for BC_n-graded groups, we would have to construct a BC_n-graded group from an arbitrary Jordan module T over an alternative ring $\mathcal{R}$ with nuclear involution σ. This section covers exactly the special case that $\mathcal{R}$ is associative and that T is of pseudo-quadratic type in the sense of Example 8.6.7.

The second restriction is relatively minor: We know from Theorem 8.7.21 that every Jordan module is of pseudo-quadratic type if 2 is invertible in the base ring. Further, we are not aware of an example of a Jordan module which is not of pseudo-quadratic type.

The restriction on $\mathcal{R}$ to be associative is more intricate. Recall from Proposition 8.8.12 that a purely alternative ring $\mathcal{R}$ does not admit a pseudo-quadratic module (M, q, f) with $f \neq 0$. Thus it seems reasonable to exclude alternative rings which are not associative from the discussion if $f \neq 0$. By Remark 9.3.29, the assumption that $f \neq 0$ says precisely that $(U_\alpha)_{\alpha \in BC_n}$ is a *proper* crystallographic BC_n-grading in the sense of Remark 9.2.7. Equivalently, it says that the Jordan module $T(M)$ is not of type C.

We conclude that the construction in this section is close to a complete solution of the existence problem for proper crystallographic BC_n-gradings. However, the assumption that $\mathcal{R}$ is associative is an important restriction in the setting of crystallographic C_n-gradings. In [Zha14, Section 4.2], Zhang gives a construction of crystallographic C_n-graded groups for alternative rings $\mathcal{R}$ with nuclear involution in which $2_{\mathcal{R}}$ is invertible. This construction builds on [LN19, 21.12] and the standard Jordan matrix algebras in [McC66]. The Jordan module which coordinatises the long root groups in this example is always $\mathrm{Tr}_\sigma(\mathcal{R}) = \mathrm{Fix}(\sigma)$. In other words, this construction covers precisely the Jordan modules in Example 8.6.5 for which $2_{\mathcal{R}}$ is invertible. We are confident that a generalisation of Zhang's strategy can be used to solve the existence problem for arbitrary Jordan modules of type C. We plan to address this problem in future work.

9.3.33 Note (Implementation of the elementary unitary group). The elementary unitary group can be implemented in GAP [Gap] in a similar way as the elementary orthogonal group (see Note 7.3.29), but there are a few additional complications. At first, we consider the implementation of $\mathcal{R}$. Again it suffices to consider the case that $\mathcal{R}$ is a polynomial ring over the integers, but this time the indeterminates in $\mathcal{R}$ are not assumed to commute. Further, the involution σ on $\mathcal{R}$ is implemented by introducing for each indeterminate t_i in $\mathcal{R}$ an indeterminate u_i in $\mathcal{R}$ and by declaring that σ interchanges t_i and u_i.

The module M can, as for the elementary orthogonal group, be assumed to be free, and the map f can be implemented in terms of a family of indeterminates. The map q causes more problems. In our implementation, it is in fact not represented by a GAP function at all. Consequently, the short root homomorphisms are defined on all of $M \times \mathcal{R}$ and not only on T (because we cannot represent T internally without representing q). Of course, most formulas in this section are not valid in this generality. In order to take advantage of the defining identities of T, we use the following strategy: Let $(u, h) \in T$ where u is a basis vector of M and h is an indeterminate in $\mathcal{R}$. Since the pseudo-quadratic module (M, q, f) is standard, we have the identity $f(u, u) = h - h^\sigma$. Hence we can replace each occurrence of the indeterminate representing $f(u, u)$ by $h - h^\sigma$. Since, in fact, an element $(v, t) \in M \times \mathcal{R}$ lies in T if and only if $f(v, v) = t - t^\sigma$, this "rewriting rule" is sufficient to reduce all claimed identities in this section to the equation $0 = 0$. As a corollary, this approach shows that the property of (M, q, f) to be standard is not only sufficient but also necessary for the group $\mathrm{EU}(q)$ to satisfy the desired commutator relations.

9.4 Rank-2 Computations

9.4.1 Notation for this section. We denote by G a group which has crystallographic BC_2-commutator relations with root groups $(U_\alpha)_{\alpha \in BC_2}$ (in the sense of Definition 2.1.4) and we choose a BC_2-quadruple $(\alpha, \beta, \gamma, \delta)$. Further, we assume that G is rank-2-injective.

This section is structured similarly to section 7.5. Our goal is to understand the action of squares of Weyl elements on all root groups. In most cases, we will see that this action obeys the square formula. However, we have already seen in the example of the elementary unitary group that this is not always the case (see Remark 9.3.18).

Recall from Remark 9.2.3 that $(U_\alpha)_{\alpha \in B_2}$ is a (non-crystallographic) B_2-grading of G. Thus we can freely apply the results from section 7.4. By Reminder 9.4.2, we can also apply all results from section 7.5 if we reverse the roles of α and 2δ (and, consequently, of 2β and γ).

9.4.2 Reminder. Recall from Remark 9.2.5 that $(U_\alpha)_{\alpha \in C_2}$ is a crystallographic C_2-grading of the subgroup of G that it generates. Thus it is also a crystallographic B_2-grading by Remark 9.2.8. In this B_2-grading, $(2\delta, \gamma, 2\beta, \alpha)$ is a B_2-quadruple.

In addition to Lemma 7.4.4, we will frequently use the following computation rule.

9.4.3 Lemma. *The following relations hold for all $x_\alpha \in U_\alpha$ and $y_{2\delta}, y'_{2\delta} \in U_{2\delta}$:*

(a) $[x_\alpha, y_{2\delta} y'_{2\delta}]_\gamma = [x_\alpha, y'_{2\delta}]_\gamma [x_\alpha, y_{2\delta}]_\gamma$.

(b) $[y_{2\delta} y'_{2\delta}, x_\alpha]_\gamma = [y_{2\delta}, x_\alpha]_\gamma [y'_{2\delta}, x_\alpha]_\gamma$.

Proof. By Lemma 7.4.4, we have the following relations:

$$[x_\alpha, y_{2\delta} y'_{2\delta}]_\gamma = [x_\alpha, y'_{2\delta}]_\gamma \big[[x_\alpha, y_{2\delta}]_\beta, y'_{2\delta}\big] [x_\alpha, y_{2\delta}]_\gamma,$$
$$[y_{2\delta} y'_{2\delta}, x_\alpha]_\gamma = [y_{2\delta}, x_\alpha]_\gamma \big[[y_{2\delta}, x_\alpha]_\beta, y'_{2\delta}\big] [y'_{2\delta}, x_\alpha]_\gamma.$$

Since $y'_{2\delta}$ commutes with U_β, the assertions follow. $\square$

9.4.4 Note. We will only investigate the actions of squares of ζ-Weyl elements on root groups U_ρ when ρ is not long and ζ is not short. We do not have to consider long roots ρ because the short root groups are contained in the long root groups. Thus if we know the action of w_ζ^2 on all short and medium-length root groups, then we know its action on all root groups. Further, we do not consider short roots ζ by the observations in Remark 4.1.31.

9.4.A The Action of Long Weyl Elements on Medium-length Root Groups

All results in this subsection follow from the corresponding ones in subsection 7.5.A using Reminder 9.4.2. Thus we obtain that squares of long Weyl elements act on the medium-length root groups in the same BC_2-subsystem by inversion.

9.4.5 Lemma. *Assume that $(a_{-2\delta}, b_{2\delta}, c_{-2\delta})$ is a 2δ-Weyl triple and denote by $w_{2\delta} := a_{-2\delta} b_{2\delta} c_{-2\delta}$ the corresponding Weyl element. Then the following statements hold for all $x_\alpha \in U_\alpha$:*

(a) $x_\alpha^{w_{2\delta}} = [x_\alpha, b_{2\delta}]_\gamma$.

(b) $\left[x_\alpha, [x_\alpha, b_{2\delta}]_\gamma\right] [x_\alpha, b_{2\delta}]_\beta \left[[x_\alpha, b_{2\delta}]_\gamma, c_{-2\delta}\right]_\beta^{[x_\alpha, b_{2\delta}]_\beta} = 1_G$.

(c) $x_\alpha = \left[[x_\alpha, b_{2\delta}]_\gamma, c_{-2\delta}\right]_\alpha^{-1}$.

Proof. Since U_β and $U_{2\delta}$ commute in a crystallographic B_2-grading, these statements follow from Lemma 7.4.12. Alternatively, this follows from Lemma 7.5.4 by the considerations in Reminder 9.4.2. □

9.4.6 Lemma. *Assume that $(a_{-2\delta}, b_{2\delta}, c_{-2\delta})$ is a 2δ-Weyl triple and denote by $w_{2\delta} := a_{-2\delta} b_{2\delta} c_{-2\delta}$ the corresponding Weyl element. Then the following statements hold for all $x_\gamma \in U_\gamma$:*

(a) $x_\gamma^{w_{2\delta}} = [x_\gamma, a_{-2\delta}]_\alpha = [x_\gamma, c_{-2\delta}]_\alpha$.

(b) $\left[x_\gamma, [x_\gamma, a_{-2\delta}]_\alpha\right] [x_\gamma, a_{-2\delta}]_\beta \left[[x_\gamma, a_{-2\delta}]_\alpha, b_{2\delta}\right]_\beta^{[x_\gamma, a_{-2\delta}]_\beta} = 1_G$.

(c) $x_\gamma = \left[[x_\gamma, a_{-2\delta}]_\alpha, b_{2\delta}\right]_\gamma^{-1}$.

Proof. Since U_β and $U_{2\delta}$ commute in a crystallographic B_2-grading, these statements follow from Lemma 7.4.13. Alternatively, this follows from Lemma 7.5.5 by the considerations in Reminder 9.4.2. □

9.4.7 Lemma. *Let $w_{2\delta}$ be a 2δ-Weyl element. Then $w_{2\delta}^2$ acts on U_α and U_γ by inversion.*

Proof. Using Reminder 9.4.2, this follows from Lemma 7.5.6. Alternatively, we can mimic the proof of Lemma 7.5.6 (or the alternative proof described in Remark 7.5.7) with references to Lemmas 7.5.4 and 7.5.5 replaced by references to Lemmas 9.4.5 and 9.4.6. □

9.4.8 Note. Recall that we gave an alternative proof of Lemma 7.5.6 in Remark 7.5.7. In the same way, Lemmas 9.4.3 and 9.4.5 can be used to obtain another proof of Lemma 9.4.7.

9.4.9 Proposition. *Assume that $U_{2\delta}^\sharp$ is non-empty. Then U_α and U_γ are abelian.*

Proof. This follows from Proposition 7.5.8 using Reminder 9.4.2, or alternatively by mimicing the proof of Proposition 7.5.8 with references to Lemma 7.5.6 replaced by references to Lemma 9.4.7. □

9.4.10 Note. Since we are ultimately interested in the case of rank at least 3, Proposition 9.4.9 is of no practical use for us: Since every medium-length root of BC_n for $n \geq 3$ lies in an A_2-subsystem, we already know from Proposition 5.4.7 that all medium-length root groups are abelian (if there exist invertible elements, which is always the case in a root graded group).

9.4.B The Action of Long Weyl Elements on Short Root Groups

9.4.11 Lemma. *Let $w_{2\beta}$ be a 2β-Weyl element. Then $w_{2\beta}^2$ acts trivially on U_δ.*

Proof. This is trivial because δ is crystallographically adjacent to 2β and -2β. □

We have seen in Remark 9.3.18 that in the elementary unitary group, the square of a certain "standard" 2β-Weyl element acts on U_β by the Jordan module involution. The following formula provides a description of this involution in the general situation. Note that it holds not only for some specific Weyl element, but in fact for all 2β-Weyl elements.

9.4.12 Lemma. *Let $w_{2\beta}$ be a 2β-Weyl element and put $w := w_{2\beta}^2$. Then $[x_\delta, x_\alpha]_\beta^w = [x_\delta, x_\alpha^{-1}]_\beta$ for all $x_\alpha \in U_\alpha$ and $x_\delta \in U_\delta$. In particular, the action of w on U_β does not depend on the choice of $w_{2\beta}$ if $U_\alpha^\sharp$ is non-empty.*

Proof. We know that w acts trivially on U_δ (by Lemma 9.4.11) and by inversion on U_α (by Lemma 9.4.7). Thus by an application of Lemma 2.2.9, we see that the action of w on U_β satisfies the first assertion.

Now assume, in addition, that $U_\alpha^\sharp$ is non-empty. Then every $x_\beta \in U_\beta$ can be written as $x_\beta = [x_\delta, x_\alpha]_\beta$ for some $x_\delta \in U_\delta$ and $x_\alpha \in U_\alpha^\sharp$ by Proposition 7.4.11. It follows that the action of w on U_β is completely determined by the formula in the first assertion. In particular, it does not depend on the choice of $w_{2\beta}$. □

9.4.C The Action of Medium-length Weyl Elements on Short Root Groups

We begin with a repetition of some of the formulas in subsection 7.5.B.

9.4.13 Lemma. *Assume that $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ is an α-Weyl triple and denote by $w_\alpha := a_{-\alpha} b_\alpha c_{-\alpha}$ the corresponding Weyl element. Then the following statements hold for all $x_\delta \in U_\delta$:*

(a) $x_\delta^{w_\alpha} = [x_\delta, b_\alpha]_\beta$.

(b) $[x_\delta, b_\alpha]_\gamma^{-1} = \big[[x_\delta, b_\alpha]_\beta, c_{-\alpha}\big]_\gamma^{[x_\delta, b_\alpha]_\gamma}$.

(c) $x_\delta \big[[x_\delta, b_\alpha]_\beta, c_{-\alpha}\big]_\delta [[x_\delta, b_\alpha]_\gamma, c_{-\alpha}] = 1_G$.

Proof. This follows from Lemma 7.4.6 with the additional information that U_β and U_δ commute. Alternatively, this follows from Lemma 7.5.10 using the considerations in Reminder 9.4.2. □

9.4.14 Lemma. *Assume that $(a_{-\alpha}, b_\alpha, c_{-\alpha})$ is an α-Weyl triple and denote by $w_\alpha := a_{-\alpha} b_\alpha c_{-\alpha}$ the corresponding Weyl element. Then the following statements hold for all $x_\beta \in U_\beta$:*

(a) $x_\beta^{w_\alpha} = [x_\beta, a_{-\alpha}]_\delta = [x_\beta, c_{-\alpha}]_\delta$.

(b) $\big[x_\beta, [x_\beta, a_{-\alpha}]_\delta\big] [x_\beta, a_{-\alpha}]_\gamma \big[[x_\beta, a_{-\alpha}]_\delta, b_\alpha\big]_\gamma^{[x_\beta, a_{-\alpha}]_\gamma} = 1_G$.

(c) $x_\beta \big[[x_\beta, a_{-\alpha}]_\delta, b_\alpha\big]_\beta [[x_\beta, a_{-\alpha}]_\gamma, b_\alpha] = 1_G$.

Proof. This follows from Lemma 7.4.7 with the additional information that U_β and U_δ commute. Alternatively, this follows from Lemma 7.5.11 using the considerations in Reminder 9.4.2. □

We can prove the same formula as in Lemma 9.4.12 for the actions of w_α^2 on U_β, but with a caveat: A priori, it is not clear that the element $x_\alpha \in U_\alpha$ in this formula can be chosen arbitrarily. This will only follow from an application of Lemma 9.4.12. Recall that this curiosity has already been announced in Note 7.4.10.

9.4.15 Lemma. *Let w_α be an α-Weyl element and let $w_{2\beta}$ be a 2β-Weyl element. Then the actions of w_α^2 and $w_{2\beta}^2$ on U_β are identical. More precisely, we have $[x_\delta, x_\alpha]_\beta^w = [x_\delta, x_\alpha^{-1}]_\beta$ for all $x_\alpha \in U_\alpha$, $x_\delta \in U_\delta$ and $w \in \{ w_\alpha^2, w_{2\beta}^2 \}$.*

Proof. Choose $a_{-\alpha}, c_{-\alpha} \in U_{-\alpha}$ and $b_\alpha \in U_\alpha$ such that $w_\alpha = a_{-\alpha} b_\alpha c_{-\alpha}$. Then by Lemmas 7.4.9 and 9.4.12, both w_α^2 and $w_{2\beta}^2$ map $[x_\delta, b_\alpha^{-1}]_\beta$ to $[x_\delta, b_\alpha]_\beta$ for all $x_\delta \in U_\delta$. Since b_α^{-1} lies in $U_\alpha^\sharp$ by Proposition 2.2.6 (a), we know from Proposition 7.4.11 that any $x_\beta \in U_\beta$ can be written as $x_\beta = [x_\delta, b_\alpha^{-1}]_\beta$ for some $x_\delta \in U_\delta$. Thus the actions of w_α^2 and $w_{2\beta}^2$ are completely determined by this property. In particular, these actions are the same, so it follows from Lemma 9.4.12 that they map $[x_\delta, x_\alpha]_{x_\beta}$ to $[x_\delta, x_\alpha^{-1}]_\beta$ for all $x_\alpha \in U_\alpha$ and all $x_\delta \in U_\delta$. This finishes the proof. □

9.4.16 Lemma. *Let w_α be an α-Weyl element and let w_β be a β-Weyl element. Then w_α^4 and w_β^4 act trivially on U_β.*

Proof. This is a consequence of Lemma 9.4.15. □

Using the results of this subsection, we can show that all long root groups are abelian if sufficiently many medium-length Weyl elements exist.

9.4.17 Lemma. *Let $b_\alpha \in U_\alpha^\sharp$. Then $U_{2\beta}$ is generated by $U_\gamma \cup U_{2\delta} \cup \{b_\alpha\}$.*

Proof. Let $x_{2\beta} \in U_{2\beta}$ and let $a_{-\alpha}, c_{-\alpha} \in U_{-\alpha}$ such that $w_\alpha := a_{-\alpha} b_\alpha c_{-\alpha}$ is an α-Weyl element. Put $x_{2\delta} := x_{2\beta}^{w_\alpha} \in U_{2\delta}$. Then by Lemma 9.4.13 (a),

$$x_{2\beta} = x_{2\delta}^{w_\alpha^{-1}} = [x_{2\delta}, b_\alpha^{-1}]_\beta.$$

Thus $[x_{2\delta}, b_\alpha^{-1}] = x_{2\beta}[x_{2\delta}, b_\alpha^{-1}]_\gamma$. It follows that

$$x_{2\beta} = [x_{2\delta}, b_\alpha^{-1}][x_{2\delta}, b_\alpha^{-1}]_\gamma^{-1} \in \langle b_\alpha, U_\gamma, U_{2\delta} \rangle.$$

This finishes the proof. □

9.4.18 Lemma. *If $U_\alpha^\sharp$ is not empty, then $U_{2\beta}$ is abelian.*

Proof. Choose an arbitrary element $b_\alpha \in U_\alpha^\sharp$. By the crystallographic commutator relations, $U_{2\beta}$ commutes with $U_\gamma \cup U_{2\delta} \cup \{b_\alpha\}$. Hence it follows from Lemma 9.4.17 that $U_{2\beta}$ commutes with itself. □

9.5 Rank-3 Computations

9.5.1 Notation for this section. We denote by G a group which has crystallographic BC_n-commutator relations with root groups $(U_\alpha)_{\alpha \in BC_n}$ for some fixed integer $n \geq 3$. We assume that $U_\alpha^\sharp$ is non-empty for all roots α and that G is rank-2-injective.

The main result of this section is Proposition 9.5.13, which provides a formula for the action of squares of Weyl elements. In contrast to root gradings of types A, B, D and E, this formula does not obey the square formula for Weyl elements in all cases. Instead, we will see an additional involution on the short root groups, which we call the *short involution*. The short involution will later turn out to be precisely the Jordan module involution on the Jordan module which coordinatises the short root groups.

The main part of the proof of Proposition 9.5.13 is already done by the rank-2 computations in the previous section. We will need the rank-3 assumption only to determine the action of medium-length Weyl elements on medium-length root groups.

As usual, we can give an alternative proof of the braid relations for Weyl elements using the main result of this action (Remark 9.5.14). This time, however, there is a caveat: We can only prove the braid relations modulo the center in this way, and only for the standard rescaled root basis of BC_n.

Before we begin, we make a quick observation.

9.5.2 Proposition. *All long and medium-length root groups are abelian.*

Proof. For long root groups, this follows from Lemma 9.4.18. Every medium-length root is contained in a subsystem of type A_2, so the second assertion holds by Proposition 5.4.9. $\square$

9.5.A The Short Involution

9.5.3 Note. In the following, we will often say "let δ be a short root" and then make statements about the long root 2δ. Since every long root can be written this way, this simply means that we study an arbitrary long root and call it 2δ.

9.5.4 Definition (Short involution). Let β be a any short root. The *(short) involution on U_β* is the map $U_\beta \to U_\beta, x_\beta \mapsto x_\beta^* := x_\beta^w$ where $w := w_{2\beta}^2$ for an arbitrary 2β-Weyl element $w_{2\beta}$. If the root β is not specified or if it is clear from the context, we will sometimes call this map the *short involution*.

9.5.5 Note. It is of course misleading to talk about *the* short involution since there is not only one such map but one for each short root group. Whenever we say "the short involution has some property", we mean "for every short root β, the involution on U_β has this property".

9.5.6 Note. Recall from Remark 9.2.6 that to any group G' with a C_n-grading $(U'_\alpha)_{\alpha \in C_n}$ we can associate a canonical BC_n-grading $(U'_\alpha)_{\alpha \in BC_n}$. This allows us to define the short involution on C_n-graded groups as well. That is, for every long root β of C_n we have an endomorphism of $U_\beta = U_{\beta/2}$ which is induced by the conjugation with the square of an arbitrary β-Weyl element, and we call this endomorphism the short involution on U_β. Thus confusingly, the short involution on C_n-graded groups is defined on the long root groups. We accept this because, over the course of this chapter, we will nearly always work with the more general case of BC_n-graded groups and rarely refer to the special case of C_n-graded groups.

9.5.7 Lemma. *Let β be a short root. Then the short involution (on U_β) has the following properties:*

(a) It is well-defined. That is, it does not depend on the choice of $w_{2\beta}$.

(b) It is a group automorphism of U_β of order 1 or 2. In particular, $(x_\beta^{-1})^* = (x_\beta^*)^{-1}$ for all $x_\beta \in U_\beta$.

(c) If α, γ, δ are roots such that $(\alpha, \beta, \gamma, \delta)$ is a BC_2-quadruple, then $[x_\delta, x_\alpha]_\beta^* = [x_\delta, x_\alpha^{-1}]_\beta$ for all $x_\alpha \in U_\alpha$ and $x_\delta \in U_\delta$.

(d) Let ρ be any root and let w_ρ be a ρ-Weyl element. Then $(x_\beta^*)^{w_\rho} = (x_\beta^{w_\rho})^*$ for all $x_\beta \in U_\beta$.

Proof. The first and the third assertion were proven in Lemma 9.4.12. Since the short involution is simply conjugation by some group element, it is a group automorphism. Further, by the formula in (c) and by Proposition 7.4.11, it is clear that $(x_\beta^*)^* = x_\beta$ for all $x_\beta \in U_\beta$, so the short involution is of order 1 or 2.

The last assertion is, again, essentially a consequence of the formula in (c) and Proposition 7.4.11. Let $x_\beta \in U_\beta$ be arbitrary, choose roots α, γ, δ such that

$(\alpha, \beta, \gamma, \delta)$ is a BC_2-quadruple and choose $x_\alpha \in U_\alpha$, $x_\delta \in U_\delta$ such that $x_\beta = [x_\delta, x_\alpha]$. Then by Lemma 2.2.9, we have

$$\begin{aligned}(x_\beta^*)^{w_\rho} &= ([x_\delta, x_\alpha]_\beta^*)^{w_\rho} = [x_\delta, x_\alpha^{-1}]_\beta^{w_\rho} = [x_\delta^{w_\rho}, (x_\alpha^{-1})^{w_\rho}]_{\sigma_\rho(\beta)} \\ &= [x_\delta^{w_\rho}, (x_\alpha^{w_\rho})^{-1}]_{\sigma_\rho(\beta)} = [x_\delta^{w_\rho}, x_\alpha^{w_\rho}]_{\sigma_\rho(\beta)}^* = ([x_\delta, x_\alpha]_\beta^{w_\rho})^* = (x_\beta^{w_\rho})^*.\end{aligned}$$

This finishes the proof. □

9.5.B The Action of Long Weyl Elements on Medium-length Root Groups

9.5.8 Lemma. *Let δ be a short root, let α be a medium-length root and let $w_{2\delta}$ be a 2δ-Weyl element. If α and δ lie in a common BC_2-subsystem, then $w_{2\delta}^2$ acts on U_α by inversion. Otherwise $w_{2\delta}^2$ acts trivially on U_α.*

Proof. The first assertion follows from Lemma 9.4.7. In the second case, α is adjacent to 2δ and -2δ, so $w_{2\delta}^2$ acts trivially on U_α. □

9.5.9 Proposition. *Let δ be a short root, let α be a medium-length root and let $w_{2\delta}$ be a 2δ-Weyl element. Put $w := w_{2\delta}^2$ and $\varepsilon := (-1)^{\langle \alpha | 2\delta \rangle}$. Then $x_\alpha^w = x_\alpha^\varepsilon$ for all $x_\alpha \in U_\alpha$.*

Proof. It follows from Proposition 9.1.19 that $\varepsilon = 1$ if and only if $\alpha \cdot \delta = 0$. By Lemma 9.1.18, this means that $\varepsilon = -1$ if and only if α and δ lie in a common BC_2-subsystem. Thus the assertion is simply a reformulation of Lemma 9.5.8. □

9.5.C The Action of Long Weyl Elements on Short Root Groups

9.5.10 Proposition. *Let β, δ be two short roots, let $w_{2\delta}$ be a 2δ-Weyl element and put $w := w_{2\delta}^2$. Then the following hold:*

(a) *If $\beta \in \{\pm\delta\}$, then w acts on U_β through the short involution. That is, we have $x_\beta^w = x_\beta^*$ for all $x_\beta \in U_\beta$.*
(b) *If $\beta \notin \{\pm\delta\}$, then w acts trivially on U_β. In particular, we have $x_\beta^w = x_\beta^\varepsilon$ for all $x_\beta \in U_\beta$ where $\varepsilon := (-1)^{\langle \beta | \delta \rangle} = (-1)^0 = 1$.*

Proof. The first statement is in fact the definition of the short involution, see Definition 9.5.4. The second statement is trivial because β is crystallographically adjacent to $-\delta$ and δ. □

9.5.D The Action of Medium-length Weyl Elements on the Short Root Groups

9.5.11 Proposition. *Let α be a medium-length root, let δ be a short root, let w_α be an α-Weyl element and put $w := w_\alpha^2$. Then the following hold:*

(a) *If α and δ lie in a common BC_2-subsystem, then w acts on U_δ through the short involution. That is, we have $x_\delta^w = x_\delta^*$ for all $x_\delta \in U_\delta$.*
(b) *If α and δ do not lie in a common BC_2-subsystem, then w acts trivially on U_δ. In particular, we have $x_\delta^w = x_\delta^\varepsilon$ for all $x_\delta \in U_\delta$ where $\varepsilon := (-1)^{\langle \delta | \alpha \rangle} = 1$.*

Proof. At first, assume that α and δ lie in a common BC_2-subsystem. Then there exist roots β, γ such that either $(\alpha, \delta, \gamma, \beta)$ or $(\alpha, \beta, \gamma, \delta)$ is a BC_2-quadruple. In the first case, assertion (a) follows directly from Lemma 9.4.15. In the second case, it is also a consequence of Lemma 9.4.15, but we have to use in addition that

w_α is also a $(-\alpha)$-Weyl element (which holds by Proposition 2.2.6 (c)) and that $(-\alpha, \delta, \gamma, \beta)$ is a BC_2-quadruple. This finishes the proof of (a).

Now assume that α and δ do not lie in a common BC_2-subsystem. Then they must be orthogonal by Lemma 9.1.18, so $\langle \delta | \alpha \rangle = 0$ and δ is adjacent to α and $-\alpha$. This proves the second assertion. □

9.5.E The Action of Medium-length Weyl Elements on Medium-length Root Groups

In contrast to the previous subsections, we now have to use that $n \geq 3$.

9.5.12 Proposition. *Let α, γ be two medium-length roots and let w_α be an α-Weyl element. Put $w := w_\alpha^2$ and $\varepsilon := (-1)^{\langle \gamma | \alpha \rangle}$. Then $x_\gamma^w = x_\gamma^\varepsilon$ for all $x_\gamma \in U_\gamma$.*

Proof. Denote by Φ' the root subsystem of BC_n which is spanned by α and γ. By Lemma 9.1.12 and because α, γ are both of medium length, Φ' is of type A_1, $A_1 \times A_1$, A_2 or BC_2. If it is of type $A_1 \times A_1$, then α, γ are orthogonal (so $\langle \gamma | \alpha \rangle = 0$ and $\varepsilon = 1$) and w acts trivially on U_γ. If Φ' is of type A_1 or A_2, then it lies in a subsystem of type A_2 and the assertion holds by Proposition 5.4.17.

Now assume that Φ' is of type BC_2. Then there exist short roots β, δ such that $(\alpha, \beta, \gamma, \delta)$ is a BC_2-quadruple. Choose roots γ_1 and γ_2 as in Lemma 9.1.17 and let $x_\gamma \in U_\gamma$ be arbitrary. Since $(\gamma_1, \gamma, \gamma_2)$ is an A_2-triple, it follows from Proposition 5.4.7 that there exist $x_1 \in U_{\gamma_1}$ and $x_2 \in U_{\gamma_2}$ such that $x_\gamma = [x_1, x_2]$. Note that by the choice of γ_1, γ_2 and by Lemma 5.4.13, w acts on U_{γ_1} and U_{γ_2} by inversion. Thus $x_\gamma^w = [x_1^w, x_2^w] = [x_1^{-1}, x_2^{-1}]$, which equals $[x_1, x_2]$ by Lemma 2.1.13. In other words, w acts trivially on U_γ. Since α and γ are orthogonal, we further have $\varepsilon = 1$, so this finishes the proof. □

9.5.F Summary

We can summarise the results of this section as follows. By Note 9.4.4, all interesting cases are covered.

9.5.13 Proposition. *Let ρ, ζ be any two roots in BC_n such that ρ is either short or medium-length and such that ζ is either medium-length or long. Let w_ζ be a ζ-Weyl element and put $w := w_\zeta^2$ and $\varepsilon := (-1)^{\langle \rho | \zeta \rangle}$. Then $x_\rho^w = x_\rho^\varepsilon$ for all $x_\rho \in U_\rho$ unless one of the following conditions is satisfied:*

(i) ρ is short, ζ is long and $\zeta \in \{\pm 2\rho\}$.

(ii) ρ is short, ζ is medium-length and $\rho \cdot \zeta \neq 0$.

In each of these exceptional cases, we have $\varepsilon = 1$ and w acts on U_ρ through the short involution from Definition 9.5.4.

Proof. By the combined statements of Propositions 9.5.9 to 9.5.12, it only remains to prove the final assertion on ε. In the first case, we have

$$\langle \rho | \zeta \rangle = 2 \frac{\rho \cdot \zeta}{\zeta \cdot \zeta} = 2 \cdot \frac{\pm 2}{4} = \pm 1$$

and in the second case,

$$\langle \rho | \zeta \rangle = 2 \frac{\pm 1}{2} = \pm 1.$$

Thus $\varepsilon = 1$ in both cases. □

Using Proposition 9.5.13, we can verify the braid relations modulo the center.

9.5.14 Remark (Braid relations). Recall from Theorem 2.5.10 that G satisfies the braid relations. For the root systems A_2 and B_3, we gave alternative proofs of the braid relations using the specific computations in root gradings of these types (see Remarks 5.4.11 and 7.6.17). We can do the same for BC_3, but with some restrictions: Firstly, we can only verify the braid relations modulo $Z(G)$ in this way. Secondly, the proof only works for the standard rescaled root base Δ of BC_n (see Remark 9.1.2), but not for a proper root base.

Let w_{12} be a $(e_1 - e_2)$-Weyl element and let w_2 be a $2e_2$-Weyl element. In the following, we denote congruence modulo $Z(G)$ by the symbol $\equiv$. We have to show that

$$w_{12}w_2w_{12}w_2 \equiv w_2w_{12}w_2w_{12}.$$

Since

$$w_2w_{12}w_2w_{12} = w_{12}w_2w_{12}w_2^{w_{12}w_2w_{12}},$$

it suffices to show that $w_2^{w_{12}w_2w_{12}} \equiv w_2$. Observe that $w_{12}w_2w_{12} = w_2^{w_{12}^{-1}} w_{12}^2$ and that $w_2^{w_{12}^{-1}}$ commutes with w_2 (because it is contained in $\langle U_{2e_1}, U_{-2e_1}\rangle$). Hence $w_2^{w_{12}w_2w_{12}} = w_2^{w_{12}^2}$. Thus it remains to show that $w_2w_{12}^2 \equiv w_{12}^2w_2$.

Let α be an arbitrary root. We have to show that the conjugation actions of $w_2w_{12}^2$ and $w_{12}^2w_2$ on U_α are identical. At first, assume that α is of the form $\alpha = \pm e_i \pm e_j$ for some distinct $i, j \in [1, n]$, and put $\gamma := \alpha^{\sigma(2e_2)}$. Then by Proposition 9.5.13, w_{12}^2 acts on U_α by $(-1)^{\langle \alpha | e_1 - e_2\rangle}$. Further, since both α and γ are of medium length, it follows from Proposition 9.1.19 that $(-1)^{\langle \alpha | e_1 - e_2\rangle}$ and $(-1)^{\langle \gamma | e_1 - e_2\rangle}$ have the same parity. Thus w_{12}^2 acts on U_α in the same way that it acts on U_γ. Since both the identity map and group inversion commute with conjugation by w_2, we conclude that the actions of $w_2w_{12}^2$ and $w_{12}^2w_2$ on U_α are identical.

Now assume that α is of the form $\alpha = \pm e_i$ for some $i \in [1, n]$. If $i \neq 2$, then w_2 centralises U_α and thus $w_2w_{12}^2$ and $w_{12}^2w_2$ act identically on U_α. Now assume that $\alpha = \pm e_2$. Note that $U_{\pm e_2} = U_{\pm e_1}^{w_{12}}$. By what we have already shown, the actions of $w_2w_{12}^2$ and $w_{12}^2w_2$ on $U_{\pm e_1}$ and on w_{12} are identical, so it follows that their actions on $U_{\pm e_2}$ are identical as well. Finally, assume that $\alpha = \pm 2e_i$ for some $i \in [1, n]$. Then U_α is contained in the root group $U_{\pm e_i}$ and the assertion follows from what we have already shown. This finishes the proof.

9.5.15 Note. Since we know from Theorem 2.5.10 that the braid relations are satisfied in G (and not merely modulo the center), it follows from the computations in Remark 9.5.14 that we actually have $w_2w_{12}^2 = w_{12}^2w_2$. In other words, the Weyl elements w_2 and w_2^w are identical where $w := w_{12}^2$. Recall from Proposition 9.5.13 that w acts on U_{2e_2} and U_{-2e_2} by the short involution. Thus for any $2e_2$-Weyl triple (a_{-2}, b_2, c_{-2}), the Weyl triple $(a_{-2}^*, b_2^*, c_{-2}^*)$ has the same associated Weyl element. This is precisely the behaviour that we have observed in Remark 9.3.12.

9.6 Standard Signs

9.6.1 Notation for this section. We fix an integer $n \geq 3$ and consider the root system BC_n in its standard representation (as in Remark 9.1.2). We denote by G a group with a BC_n-pregrading $(U_\alpha)_{\alpha \in B_n}$, by $\mathcal{R}$ an alternative ring with a nuclear involution σ and by $\mathcal{J} = (J, \varphi, \pi_1, \mathrm{T}_1, \psi)$ a Jordan module over $(\mathcal{R}, \sigma)$. We assume that there exists a coordinatisation $(\theta_\alpha)_{\alpha \in BC_n}$ of G by $\mathcal{J}$ with standard signs (in the sense of the following Definition 9.6.2), and we fix this coordinatisation.

9.6.2 Definition (Standard signs). Let $n \in \mathbb{N}_{\geq 2}$ and let G be a group with a BC_n-pregrading $(U_\alpha)_{\alpha \in BC_n}$. Let $\mathcal{R}$ be an alternative ring with nuclear involution σ and let $\mathcal{J} = (J, \varphi, \pi_1, \mathrm{T}_1, \psi)$ be a Jordan module over $(\mathcal{R}, \sigma)$. A *coordinatisation of G by $\mathcal{J}$ with standard signs* is a family $(\theta_\alpha)_{\alpha \in BC_n}$ with the following properties:

(i) For all roots α, the map θ_α is an isomorphism from $(\mathcal{R}, +)$ to U_α if α is medium-length and it is an isomorphism from $(J, \hat{+})$ to U_α if α is short.

(ii) The following commutator relations, called the *standard commutator relations*, are satisfied. For all pairwise distinct $i, j, k \in [1, n]$ and all $a, b \in \mathcal{R}$, we have

$$\begin{aligned}
[\theta_{e_i-e_j}(a), \theta_{e_j-e_k}(b)] &= \theta_{e_i-e_k}(ab), \\
[\theta_{e_i-e_j}(a), \theta_{e_j+e_k}(b)] &= \theta_{e_i+e_k}\big(\delta^\sigma_{k<i}(a\delta^\sigma_{k<j}(b))\big), \\
[\theta_{e_i-e_j}(a), \theta_{-e_k-e_i}(b)] &= \theta_{-e_j-e_k}\big(-\delta^\sigma_{j<k}(a^\sigma\delta^\sigma_{i<k}(b))\big), \\
[\theta_{e_i+e_j}(a), \theta_{-e_k-e_i}(b)] &= \theta_{e_j-e_k}\big(\delta^\sigma_{i<j}(a)\delta^\sigma_{i<k}(b)\big).
\end{aligned}$$

For all distinct $i, j \in [1, n]$ and all $a, b \in \mathcal{R}$, we have

$$\begin{aligned}
[\theta_{e_i-e_j}(a), \theta_{e_i+e_j}(b)] &= \theta_{e_i}\big(\mathrm{T}_1(a\delta^\sigma_{i<j}(b))\big), \\
[\theta_{e_i-e_j}(a), \theta_{-e_i-e_j}(b)] &= \theta_{-e_j}\big(\mathrm{T}_1\big(a^\sigma\delta^\sigma_{i<j}(b)\big)\big).
\end{aligned}$$

For all distinct $i, j \in [1, n]$, all $v \in M$ and all $a \in \mathcal{R}$, we have

$$\begin{aligned}
[\theta_{e_j}(v), \theta_{e_i-e_j}(a)] &= \theta_{e_i}\big(\varphi(v, -a^\sigma)\big)\theta_{e_i+e_j}\big(-\delta^\sigma_{i>j}\big(a\pi_1(v)\big)\big), \\
[\theta_{e_j}(v), \theta_{-e_i-e_j}(a)] &= \theta_{-e_i}\big(\varphi(v, -\delta^\sigma_{i>j}(a))\big)\theta_{e_j-e_i}\big(\pi_1(v)^\sigma\delta^\sigma_{i>j}(a)\big), \\
[\theta_{-e_i}(v), \theta_{e_i-e_j}(a)] &= \theta_{-e_j}\big(\varphi(v, a)\big)\theta_{-e_i-e_j}\big(-\delta^\sigma_{i>j}\big(a^\sigma\pi_1(v)\big)\big), \\
[\theta_{-e_i}(v), \theta_{e_i+e_j}(a)] &= \theta_{e_j}\big(\varphi(v, -\delta^\sigma_{i>j}(a))\big)\theta_{e_j-e_i}\big(\delta^\sigma_{i<j}(a)\pi_1(u)\big).
\end{aligned}$$

For all distinct $i, j \in [1, n]$ and all $u, v \in M$, we have

$$\begin{aligned}
[\theta_{e_i}(u), \theta_{e_j}(v)] &= \begin{cases} \theta_{e_i+e_j}\big(\psi(u, v)\big) & \text{if } i < j, \\ \theta_{e_i+e_j}\big(-\psi(v, u)\big) & \text{if } i > j, \end{cases} \\
[\theta_{e_i}(u), \theta_{-e_j}(v)] &= \theta_{e_i-e_j}\big(-\psi(u, v)\big) \\
[\theta_{-e_i}(u), \theta_{-e_j}(v)] &= \begin{cases} \theta_{-e_i-e_j}\big(\psi(v, u)\big) & \text{if } i < j, \\ \theta_{-e_i-e_j}\big(-\psi(u, v)\big) & \text{if } i > j. \end{cases}
\end{aligned}$$

9.6.3 Note. The commutator relations in Definition 9.6.2 are exactly the same that we have seen in Proposition 9.3.26, but expressed in the language of Jordan modules. In other words, if we take $\mathcal{J}$ to be a Jordan module of pseudo-quadratic type as in Example 8.6.6, then the commutator relations in Definition 9.6.2 and Proposition 9.3.26 are the same. In a similar way, the conjugation formulas in root gradings of type BC are the same as the ones in in Lemmas 9.3.13 and 9.3.15, but expressed in the language of Jordan modules. For example, the formula $\theta_{e_i}(u, h)^w = \theta_{e_j}\big(ua^{-\sigma}, a^{-1}ha^{-\sigma}\big)$ for all $(u, h) \in T$ in Lemma 9.3.13 (d) should be interpreted as $\theta_{e_i}(u)^w = \theta_{e_j}\big(\varphi(u, a^{-\sigma})\big)$ for all $u \in J$.

We will show in Theorem 9.10.26 that every BC_n-graded group for $n \geq 3$ is coordinatised by a Jordan module with standard signs.

As in Lemma 7.7.3, we can show that all long Weyl elements in BC_n-graded groups with a standard coordinatisation are balanced.

9.6.4 Lemma. *Let δ be a short root and let $w_{2\delta}$ be a 2δ-Weyl element in G. Choose $v', v, v'' \in J$ such that $w_{2\delta} = \theta_{-\delta}(v')\theta_\delta(v)\theta_{-\delta}(v'')$. Then $\pi_1(v)$ is invertible with inverse $\pi_1(v')$ and $v' = v'' = \varphi(\overline{v}, \pi_1(v)^{-\sigma})$. In particular, $w_{2\delta}$ is weakly balanced.*

Proof. We prove this only for $\delta = e_2$, the remaining cases being similar. Let w_2 be a $2e_2$-Weyl element and let $v', v, v'' \in J$ be such that $w_2 = \theta_{-e_2}(v')\theta_{e_2}(v)\theta_{-e_2}(v'')$. For all $a \in \mathcal{R}$, we know from Lemmas 9.4.5 (a) and 9.4.6 (a) and the standard commutator relations that

$$\begin{aligned}\theta_{e_1-e_2}(a)^{w_2} &= [\theta_{e_1-e_2}(a), \theta_{e_2}(v)]_{e_1+e_2} = \theta_{e_1+e_2}\big(a\pi_1(v)\big),\\ \theta_{e_1+e_2}(a)^{w_2} &= [\theta_{e_1+e_2}(a), \theta_{-e_2}(v')]_{e_1-e_2} = \theta_{e_1-e_2}\big(-a\pi_1(v')\big).\end{aligned}$$

Together with Proposition 9.5.13, this implies that

$$\begin{aligned}\theta_{e_1-e_2}(-1_{\mathcal{R}}) &= \theta_{e_1-e_2}(1_{\mathcal{R}})^{w_2^2} = \theta_{e_1-e_2}\big(-\pi_1(v)\pi_1(v')\big),\\ \theta_{e_1+e_2}(-1_{\mathcal{R}}) &= \theta_{e_1+e_2}(1_{\mathcal{R}})^{w_2^2} = \theta_{e_1+e_2}\big(-\pi_1(v')\pi_1(v)\big).\end{aligned}$$

We infer that $\pi_1(v)$ is invertible with inverse $\pi_1(v')$.

Further, we know from Lemma 9.4.5 (b) that

$$\begin{aligned}1_G = \big[&\theta_{e_1-e_2}(a), [\theta_{e_1-e_2}(a), \theta_{e_2}(v)]_{e_1+e_2}\big][\theta_{e_1-e_2}(a), \theta_{e_2}(v)]_{e_1}\\ &\cdot \big[[\theta_{e_1-e_2}(a), \theta_{e_2}(v)]_{e_1+e_2}, \theta_{-e_2}(v'')\big]_{e_1}^{[\theta_{e_1-e_2}(a), \theta_{e_2}(v)]_{e_1}}\end{aligned}$$

for all $a \in \mathcal{R}$ where

$$\begin{aligned}\big[\theta_{e_1-e_2}(a), [\theta_{e_1-e_2}(a), \theta_{e_2}(v)]_{e_1+e_2}\big] &= \big[\theta_{e_1-e_2}(a), \theta_{e_1+e_2}\big(a\pi_1(v)\big)\big]\\ &= \theta_{e_1}\big(\mathrm{T}_1(a\pi_1(v)^{\sigma}a^{\sigma})\big),\\ [\theta_{e_1-e_2}(a), \theta_{e_2}(v)]_{e_1} &= \theta_{e_1}\big(\hat{-}\varphi(v, -a^{\sigma})\big),\\ \big[[\theta_{e_1-e_2}(a), \theta_{e_2}(v)]_{e_1+e_2}, \theta_{-e_2}(v'')\big]_{e_1} &= \big[\theta_{e_1+e_2}\big(a\pi_1(v)\big), \theta_{-e_2}(v'')\big]_{e_1}\\ &= \theta_{e_1}\big(\hat{-}\varphi\big(v'', -\pi_1(v)^{\sigma}a^{\sigma}\big)\big).\end{aligned}$$

Observe that

$$\begin{aligned}\theta_{e_1}\big(\hat{-}\varphi\big(v'', -\pi_1(v)^{\sigma}a^{\sigma}\big)\big) &= \theta_{e_1}\big(\varphi\big(v'', -\big(a\pi_1(v)\big)^{\sigma}\big)\big)^{-1}\\ &= [\theta_{e_2}(v''), \theta_{e_1-e_2}\big(a\pi_1(v)\big)]_{e_1},\end{aligned}$$

so this element lies in U_{2e_1} because $\theta_{e_2}(v'')$ lies in U_{2e_2}. In particular, it commutes with $[\theta_{e_1-e_2}(a), \theta_{e_2}(v)]_{e_1}$. Thus it follows from the previous computations that

$$0_J = \mathrm{T}_1\big(a\pi_1(v)^{\sigma}a^{\sigma}\big) \mathbin{\hat{-}} \varphi(v, -a^{\sigma}) \mathbin{\hat{-}} \varphi\big(v'', -\pi_1(v)^{\sigma}a^{\sigma}\big)$$

for all $a \in \mathcal{R}$. Putting $a := 1_{\mathcal{R}}$ in this equation, we infer that

$$0_J = v \mathbin{\hat{+}} \overline{v} \mathbin{\hat{-}} \overline{v} \mathbin{\hat{-}} \overline{\varphi\big(v'', \pi_1(v)^{\sigma}\big)} = v \mathbin{\hat{-}} \overline{\varphi\big(v'', \pi_1(v)^{\sigma}\big)}.$$

In other words,

$$v = \varphi\big(\overline{v''}, \pi_1(v)^{\sigma}\big).$$

Applying the map $\varphi(\overline{\cdot}, \pi_1(v)^{-\sigma})$ to both sides of the equation, we obtain

$$\varphi\big(v, \pi_1(v)^{-\sigma}\big) = v''.$$

Since the $2e_2$-Weyl element $w_2 = \theta_{-e_2}(v')\theta_{e_2}(v)\theta_{-e_2}(v'')$ is arbitrary, the same conclusion holds for the Weyl element $w_2 = \theta_{-e_2}(\hat{-}v'')\theta_{e_2}(\hat{-}v)\theta_{-e_2}(\hat{-}v')$, which yields

$$\varphi\big(\hat{-}v, \pi_1(v)^{-\sigma}\big) = \hat{-}v'.$$

This implies that

$$v' = \varphi\big(v, \pi_1(v)^{-\sigma}\big) = v''$$

because φ is additive in the first component. This finishes the proof for the root

$2\delta = 2e_2$. The remaining cases can be covered in a similar way. □

We also have the usual identification between Weyl-invertible group elements and "invertible" elements of the coordinatising algebraic structure.

9.6.5 Proposition. *The following hold:*

(a) *Let α be a medium-length root. Define*

$$w_\alpha(r) := \theta_{-\alpha}(-r^{-1})\theta_\alpha(r)\theta_{-\alpha}(-r^{-1})$$

for all invertible $r \in \mathcal{R}$. Then the maps

$$\mathcal{R}^\times \to U_\alpha^\sharp, r \mapsto \theta_\alpha(r) \quad \text{and} \quad \mathcal{R}^\times \to M_\alpha, r \mapsto w_\alpha(r)$$

are well-defined bijections. Here $\mathcal{R}^\times$, $U_\alpha^\sharp$ and M_α denote the sets of invertible elements in $\mathcal{R}$ (in the sense of Definition 5.1.12), the set of α-invertible elements in U_α and the set of α-Weyl elements, respectively.

(b) *Let δ be a short root and put $J^\times := \{ u \in J \mid \pi_1(u) \in \mathcal{R}^\times \}$. Define*

$$w_\delta(u) := \theta_{-\delta}\big(\varphi(\overline{u}, \pi_1(u)^{-\sigma})\big)\theta_\delta(u)\theta_{-\delta}\big(\varphi(\overline{u}, \pi_1(u)^{-\sigma})\big)$$

for all $u \in J^\times$. Then the maps

$$J^\times \to U_\delta^\sharp, u \mapsto \theta_\delta(u) \quad \text{and} \quad J^\times \to M_\delta, u \mapsto w_\delta(u)$$

are well-defined surjections, and the first one is a bijection.

(c) *The Weyl elements defined above satisfy the same conjugation formulas as in Lemmas 9.3.13 and 9.3.15, but expressed in the language of Jordan modules. (See Note 9.6.3.)*

(d) *Let Δ be the standard rescaled root base of BC_n and choose an element $v_0 \in J$ with $\pi_1(v_0) = 1_{\mathcal{R}}$. Define A, B, η and μ as in Notation 9.3.19. Put $w_\alpha := w_\alpha(1_{\mathcal{R}})$ for all medium-length roots in Δ and $w_\beta := w_\beta(v_0)$ for the unique long root in Δ. Then G is parametrised by $(A \times B, J, \mathcal{R})$ with respect to $\eta \times \mu$ and $(w_\delta)_{\delta\in\Delta}$.*

Proof. This can be proven in a similar way as Proposition 7.7.6. For the proof of (b), we need Lemma 9.6.4. □

9.6.6 Remark. It follows from Proposition 9.6.5 that G satisfies $U_\alpha^\sharp = U_\alpha \setminus \{1_G\}$ (the additional condition of being an RGD-system) if and only if $\mathcal{R}$ is a division ring and $\mathcal{J}$ is anisotropic.

9.7 Admissible and Standard Partial Twisting Systems

9.7.1 Notation for this section. We fix an integer $n \geq 3$ and consider the root system BC_n in its standard representation with its standard rescaled root base Δ (as in Remark 9.1.2). We choose $(\mathcal{R}, \mathcal{R}_0, \sigma) := (\mathbb{C}, \mathbb{R}, \sigma)$, $M := \mathbb{C}$, $q\colon M \to \mathbb{C}$ and $f\colon M \times M \to \mathcal{R}$ as in Remark 9.3.23. We define the BC_n-graded group $\mathrm{EU}(q)$ with root groups $(U_\alpha)_{\alpha\in BC_n}$ as in Definition 9.3.7. We denote the root isomorphisms from Constructions 9.3.4 and 9.3.5 by $(\theta_\alpha)_{\alpha\in B_n}$ and the standard system of Weyl elements from Definition 7.3.13 by $(w_\delta)_{\delta\in\Delta}$. Further, we denote by (A, η, B, μ) the standard partial twisting system of type BC_n in the sense of the following Definition 9.7.2.

This section is structured similarly to section 7.8: Motivated by the elementary unitary group from section 9.3, we introduce the standard partial twisting system of type BC. We will show that it is admissible in the sense that it satisfies a number of desirable properties. In section 9.8, we will see that every admissible partial twisting system satisfies the condition of the parametrisation theorem.

9.7.2 Definition (Standard partial twisting system). The *standard partial twisting system of type BC_n (with respect to Δ)* is the tuple (A, η, B, μ) where $A := \{\pm 1\}^2$, $B := \{\pm 1\}$ and where η, μ are the Δ-parity maps from Notation 9.3.19. If G is a group with a crystallographic BC_n-grading, then the *standard partial twisting system for G (with respect to Δ)* is the same tuple together with the additional information that A acts on all root groups of G as follows: The first component of A acts on all root groups by inversion while the second component acts trivially on all medium-length root groups and by the short involution (from Definition 9.5.4) on the short root groups.

Similar comments as in Note 7.8.3 and Remark 7.8.4 apply to Definition 9.7.2.

9.7.3 Remark. Let G be a group with a crystallographic BC_n-grading $(\hat{U}_\alpha)_{\alpha \in BC_n}$ and let (A, η, B, μ) be the standard partial twisting system for G. We have to verify that the action of A on the root groups of G is well-defined. For the action on the medium-length root groups, this is evident. For the action on the short root groups, we have to verify that the short involution is of order at most 2 and that it commutes with the group inversion. Both assertions hold by Lemma 9.5.7 (b). Further, the twisting action of A commutes with conjugation by Weyl elements by Lemma 9.5.7 (d). We conclude that A satisfies all the axioms of a twisting group for $(G, (w_\delta)_{\delta \in \Delta})$ (see Definition 4.3.13).

9.7.4 Note. The assumption in Definition 9.7.2 that G has a crystallographic BC_n-grading is necessary to ensure that the short involution exists. In the B_n-situation in Definition 7.8.2, no such assumption was necessary.

9.7.5 Reminder (compare 7.8.5). Recall from Theorem 9.3.28 that $(U_\alpha)_{\alpha \in BC_n}$ is a crystallographic BC_n-grading of $\mathrm{EU}(q)$. Thus we can apply all rank-2 and rank-3 computations from the previous sections to $\mathrm{EU}(q)$. Further, we know from Lemma 9.3.24 that $\mathrm{EU}(q)$ is parametrised by the standard parameter system $(A \times B, T(M), \mathcal{R})$ (from Definition 8.6.20) with respect to $\eta \times \mu$ and $(w_\delta)_{\delta \in \Delta}$ and from Remark 7.3.22 that $(A \times B, T(M), \mathcal{R})$ is $(\eta \times \mu)$-faithful.

We now verify some basic properties of the standard partial twisting system of type BC by performing computations in the group $\mathrm{EU}(q)$.

9.7.6 Lemma. *η is braid-invariant and adjacency-trivial and μ is Weyl-invariant and adjacency-trivial.*

Proof. Using that $(w_\delta)_{\delta \in \Delta}$ satisfies the braid relations by Theorem 2.5.10, this can be proven in the same way as Lemma 7.8.6. □

9.7.7 Lemma. *μ is semi-complete.*

Proof. This follows from the fact that the only subgroups of B are $\{1\}$ and B, just like in Lemma 7.8.7. □

9.7.8 Lemma. *$\eta \times \mu$ is transporter-invariant and η, μ are independent.*

Proof. At first, we consider the orbit of short roots. Put $\hat{\alpha} := e_1$. We can read off from Figure 9.4 that $(A \times B)_{\hat{\alpha}\to\beta}$ is contained in $\{1_A\} \times \{\pm 1_A\} \times \{1_B\}$ for all short roots β. Further, we have $(A \times B)_{\hat{\alpha}\to\hat{\alpha}} = \{1_A\} \times \{\pm 1_A\} \times \{1_B\}$ because

$$\begin{aligned}(\eta \times \mu)_{\hat{\alpha},(e_1-e_2,e_1-e_2)} &= (\eta \times \mu)_{e_1,e_1-e_2}(\eta \times \mu)_{e_2,e_1-e_2} = (1_A, 1_A, 1_B)(1_A, -1_A, 1_B) \\ &= (1_A, -1_A, 1_B).\end{aligned}$$

Thus it follows from criterion 4.2.16 (iii) (with $\hat{\beta} := \hat{\alpha}$) that $\eta \times \mu$ is transporter-invariant on the orbit of short roots. The same computations apply for $\hat{\alpha} := 2e_1$, so it is transporter-invariant on the orbit of long roots as well.

Now we consider the orbit of medium-length roots. Put $\hat{\alpha} := e_1 + e_2$. Again, we can read off from Figure 9.4 that $(A \times B)_{\hat{\alpha}\to\beta}$ is contained in $\{\pm 1_A\} \times \{1_A\} \times \{\pm 1_B\}$ for all medium-length roots β. Further,

$$\begin{aligned}(\eta \times \mu)_{\hat{\alpha},e_1-e_2} &= (-1_A, 1_A, -1_B) \qquad \text{and} \\ (\eta \times \mu)_{\hat{\alpha},(e_2-e_3,e_2-e_3)} &= (\eta \times \mu)_{e_1+e_2,e_2-e_3}(\eta \times \mu)_{e_1+e_3,e_2-e_3} \\ &= (1_A, 1_A, 1_B)(-1_A, 1_A, 1_B) = (-1_A, 1_A, 1_B).\end{aligned}$$

Since these two elements generate $\{\pm 1_A\} \times \{1_A\} \times \{\pm 1_B\}$, it follows that

$$(A \times B)_{\hat{\alpha}\to\hat{\alpha}} = \{\pm 1_A\} \times \{1_A\} \times \{\pm 1_B\}.$$

Hence again by criterion 4.2.16 (iii) (with $\hat{\beta} := \hat{\alpha}$), we infer that $\eta \times \mu$ is transporter-invariant on the orbit of medium-length roots.

Finally, the previous computations together with Remark 4.2.24 show that η and μ are independent in the sense of Definition 4.2.23. □

9.7.9 Lemma. *Let ρ, ζ be any two roots in BC_n such that ρ is either short or medium-length and such that ζ is either medium-length or long. Then the following hold:*

(a) If ρ is short, ζ is long and $\zeta \in \{\pm 2\rho\}$, then $\eta_{\rho,\zeta\zeta} = (1, -1)$.

(b) If ρ is short, ζ is medium-length and $\rho \cdot \zeta \neq 0$, then $\eta_{\rho,\zeta\zeta} = (1, -1)$.

(c) In all cases not covered by the previous assertions, we have $\eta_{\rho,\zeta\zeta} = \big((-1)^{\langle\rho|\zeta\rangle}, 1\big)$.

Proof. This follows from Proposition 9.5.13 because $(A \times B, T(M), \mathcal{R})$ is $(\eta \times \mu)$-faithful. In the first two parts, we also use that the Jordan module involution on $T(M)$ (through which the second component of A acts on $T(M)$) coincides with the short involution on $\mathrm{EU}(q)$ by Remark 9.3.18. □

9.7.10 Lemma. *Let α, γ be two medium-length roots which are orthogonal. Then $\mu_{\alpha,\sigma(\gamma)} = 1_B$ if α and γ are crystallographically adjacent and $\mu_{\alpha,\sigma(\gamma)} = -1_B$ if they are not.*

Proof. At first, assume that α and γ are crystallographically adjacent. By Proposition 1.3.5, there exist $\lambda \in \mathbb{R}_{>0}$ such that $\lambda\gamma$ is a root and a Δ-expression $\bar{\delta}$ of $\lambda\gamma$. Since γ is of medium length, we have $\lambda = 1$. By Lemma 9.1.21, α is crystallographically adjacent to $-\gamma$ as well, so the γ-Weyl element $w_{\bar{\delta}}$ in $\mathrm{EU}(q)$ centralises U_α. On the other hand, we have

$$\theta_\alpha(r)^{w_{\bar{\delta}}} = \theta_\alpha(\eta_{\alpha,\bar{\delta}}\mu_{\alpha,\bar{\delta}}.r)$$

for all $r \in \mathcal{R}$ by Lemma 9.3.24. Since the parameter system $(A \times B, T(M), \mathcal{R})$ is $(\eta \times \mu)$-faithful by Reminder 9.7.5, it follows that $\eta_{\alpha,\bar{\delta}} = (1_A, 1_A)$ and $\mu_{\alpha,\bar{\delta}} = 1_B$. This implies that $\mu_{\alpha,\sigma(\gamma)} = 1_B$ because $\bar{\delta}$ is a Δ-expression of γ.

Now consider the case that α and γ are not crystallographically adjacent. By Lemma 9.1.22, there exists an element u of $\mathrm{Weyl}(BC_n)$ such that $(\alpha, \gamma)^u =$

$(e_1 + e_2, e_1 - e_2)$. Let $\bar{\rho}$ be any word over Δ such that $u = \sigma(\bar{\rho})$. Then it follows from Lemma 4.2.8 that

$$\mu_{\alpha,\sigma(\gamma)} = \mu_{\alpha^{\sigma(\bar{\rho})},\sigma(\bar{\rho})^{-1}\sigma(\gamma)\sigma(\bar{\rho})} = \mu_{\alpha^u,\sigma(\gamma^u)} = \mu_{e_1+e_2,\sigma(e_1-e_2)} = \mu_{e_1+e_2,e_1-e_2}.$$

We can read off from Figure 9.4 that $\mu_{e_1+e_2,e_1-e_2} = -1_B$. Thus the claimed assertion follows. □

9.7.11 Definition (Admissible partial twisting system). Let G be a group with a BC_n-grading $(U_\alpha)_{\alpha\in B_n}$ and let $(w'_\delta)_{\delta\in\Delta}$ be a Δ-system of Weyl elements in G. A *BC_n-admissible partial twisting system for* $(G, (w'_\delta)_{\delta\in\Delta})$ is a partial twisting system (A', η', B', μ') for $(G, (w'_\delta)_{\delta\in\Delta})$ with the following additional properties:

(i) $A' = \{\pm 1\}^2$ and $B' = \{\pm 1\}$.

(ii) The element $(-1_{A'}, 1_{A'})$ acts on all root groups of G by inversion whereas $(1_{A'}, -1_{A'})$ acts trivially on the medium-length root groups and by the short involution on the short and long root groups.

(iii) η' and μ' are adjacency-trivial.

(iv) η' satisfies the formulas in Lemma 9.7.9.

(v) If α, γ are orthogonal medium-length roots, then $\mu_{\alpha,\gamma}$ is given by the formula in Lemma 9.7.10.

We will sometimes refer to such objects as *admissible partial twisting systems* if the root system BC_n is clear from the context.

9.7.12 Proposition. *Let G be a group with a BC_n-grading $(U_\alpha)_{\alpha\in B_n}$. Then for any Δ-system $(w_\delta)_{\delta\in\Delta}$ of Weyl elements in G, the standard partial twisting system (A, η, B, μ) is a BC_n-admissible partial twisting system for $(G, (w_\delta)_{\delta\in\Delta})$. In particular, admissible partial twisting systems exist for each group with a BC_n-grading.*

Proof. We have to check that (A, η, B, μ) satisfies the axioms of a partial twisting system for $(G, (w_\delta)_{\delta\in\Delta})$ (see Definition 4.3.18) and the axioms in Definition 7.8.13. These properties follow from the results in this section. □

9.8 The Parametrisation

9.8.1 Notation for this section. We fix an integer $n \geq 3$ and consider the root system BC_n in its standard representation with its standard rescaled root base Δ (as in Remark 9.1.2). We denote by G a group which has crystallographic BC_n-commutator relations with root groups $(U_\alpha)_{\alpha\in B_n}$ such that $U_\alpha^\sharp$ is non-empty for all $\alpha \in \Phi$ and such that G is rank-2-injective. We fix a Δ-system $(w_\delta)_{\delta\in\Delta}$ of Weyl elements and we denote by (A, η, B, μ) a BC_n-admissible partial twisting system for $(G, (w_\delta)_{\delta\in\Delta})$.

The goal of this section is to show that G satisfies the conditions in the parametrisation theorem with respect to the admissible partial twisting system (A, η, B, μ). As in section 7.9, the main effort lies in the verification of stabiliser compatibility, for which we will use the criterion from Proposition 4.6.3.

9.8.2 Proposition. *G is square-compatible with respect to η.*

Proof. This follows from Proposition 9.5.13 and Lemma 9.7.9. Here we use Axiom 9.7.11 (ii). □

9.8.3 Proposition. *G is stabiliser-compatible with respect to (η, μ).*

Proof. Let α be any root. We know from Axiom 9.7.11 (iii) that η is α-adjacency-trivial and we want to show that G is α-stabiliser-compatible. At first, assume that α is short. Then it follows from Lemma 4.6.4 and Remark 4.6.5 that G is α-stabiliser compatible. Since any long root group is contained in a short root group, it follows that G is α-stabiliser-compatible for any long root as well.

Now assume that α is medium-length. Then there exist distinct $i, j \in [1, n]$ and signs $\varepsilon_i, \varepsilon_j \in \{\pm 1\}$ such that $\alpha = \varepsilon_i e_i + \varepsilon_j e_j$. As in Proposition 4.6.3, we consider the following sets:

$$\begin{aligned}
&\mathcal{O} := \{\, \beta \in \Phi \mid \alpha \cdot \beta = 0 \,\}, \\
&\mathcal{A} := \{\, \beta \in \mathcal{O} \mid \alpha \text{ is crystallographically adjacent to } \beta \text{ and } -\beta \,\} \\
&\bar{\mathcal{A}} := \mathcal{O} \setminus \mathcal{A}.
\end{aligned}$$

By Lemma 9.1.20, we have

$$\begin{aligned}
\bar{\mathcal{A}} &= \{\, \varepsilon_i e_i - \varepsilon_j e_j, \varepsilon_j e_j - \varepsilon_i e_i \,\} \qquad \text{and} \\
\mathcal{A} &= \Phi \cap \langle e_k \mid k \in [1, n] \setminus \{i, j\} \rangle.
\end{aligned}$$

Recall that $\mu_{\alpha,\sigma(\beta)} = 1_B$ for all $\beta \in \mathcal{A}$ and $\mu_{\alpha,\sigma(\beta)} = -1_B$ for all $\beta \in \bar{\mathcal{A}}$ by Axiom 9.7.11 (v). Further, observe that $\bar{\mathcal{A}}$ contains exactly one Δ-positive root. This implies that the condition in Proposition 4.6.3 for all Δ-positive roots $\beta, \beta' \in \bar{\mathcal{A}}$ is trivially satisfied (because $\beta = \beta'$). Hence all conditions in Proposition 4.6.3 hold. We conclude that G is α-stabiliser-compatible with respect to (η, μ) and $(w_\delta)_{\delta \in \Delta}$. □

9.8.4 Proposition. *There exist an abelian group $(\mathcal{R}, +)$, a group $(J, \hat{+})$ (both equipped, as sets, with an action of $A \times B$) and a family $(\theta_\alpha)_{\alpha \in BC_n}$ of maps such that the following conditions are satisfied:*

(i) $(\theta_\alpha)_{\alpha \in BC_n}$ is a parametrisation of G by $(A \times B, J, \mathcal{R})$ with respect to $\eta \times \mu$ and $(w_\delta)_{\delta \in \Delta}$.

(ii) The action of $(-1_A, 1_A)$ on $\mathcal{R}$ and on J is given by group inversion, the action of $(1_A, -1_A)$ on $\mathcal{R}$ is trivial and the action of $(1_A, -1_A)$ on J is given by the formula

$$\theta_\beta\big((1_A, -1_A).v\big) = \theta_\beta(v)^*$$

for all $v \in J$ and all short roots β where $\theta_\beta(v)^*$ denotes the short involution (from Definition 9.5.4).

Proof. This follows from the parametrisation theorem (Theorem 4.5.16), whose assumptions are satisfied by Propositions 9.8.2 and 9.8.3. □

9.9 Computation of the Blueprint Rewriting Rules

9.9.1 Notation for this section. We fix an integer $n \geq 3$ and consider the root system BC_n in its standard representation with its standard rescaled root base Δ (as in Remark 9.1.2). We denote by G a group with a crystallographic BC_n-grading $(U_\alpha)_{\alpha \in B_n}$. We fix a Δ-system of Weyl elements $(w_\delta)_{\delta \in \Delta}$ and denote its standard $\hat{C}_n$-extension by $(w_\beta)_{\beta \in \hat{B}_n}$. To simplify notation, we put $w_{ij} := w_{e_i - e_j}$ for all distinct $i, j \in [1, n]$ and we put $w_i := w_{2e_i}$ for all $i \in [1, n]$. We denote the standard partial twisting system for G (from Definition 9.7.2) by (A, η, B, μ), by $(\mathcal{R}, +)$, $(J, \hat{+})$ any groups which satisfy the assertion of Proposition 9.8.4 and by $(\theta_\alpha)_{\alpha \in BC_n}$ the corresponding parametrisation of G. The groups $\mathcal{R}$ and J are equipped with actions of $A \times B$, and we call the maps

$$\sigma\colon \mathcal{R} \to \mathcal{R}, r \mapsto r^\sigma := -1_B.r \quad \text{and} \quad \overline{\cdot}\colon J \to J, v \mapsto \overline{v} := (1_A, -1_A).v$$

the *involutions on $\mathcal{R}$ and J*, respectively. Further, we choose elements $v_{-1}, v_0, v_1 \in J$ such that $w_n = \theta_{-e_n}(v_{-1})\theta_{e_n}(v_0)\theta_{-e_n}(v_1)$.

We will frequently use the following result without reference. It says that the action of the $\hat{C}_n$-extension $(w_\beta)_{\beta \in \hat{C}_n}$ of $(w_\delta)_{\delta \in \Delta}$ on the root groups is determined by the values in Figure 9.4.

9.9.2 Proposition. *Let $\alpha \in BC_n$ and let $\beta \in \hat{C}_n$. Let $x \in \mathcal{R}$ if α is of medium length and let $x \in J$ if α is short. Then $\theta_\alpha(x)^{\hat{w}_\beta} = \theta_{\sigma_\beta(\alpha)}(\eta_{\alpha,\beta}\mu_{\alpha,\beta}.x)$ where $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$ are the values in Figure 9.4.*

Proof. Denote by $\bar{\beta}$ the standard Δ-expression of β from Definition 9.1.24. Since G is parametrised by $(A \times B, J, \mathcal{R})$ with respect to $\eta \times \mu$ and $(w_\delta)_{\delta \in \Delta}$, we have

$$\theta_\alpha(x)^{\hat{w}_\beta} = \theta_{\sigma_\beta(\alpha)}(\eta_{\alpha,\bar{\beta}}\mu_{\alpha,\bar{\beta}}.x).$$

We know from Lemma 9.3.31 that $\eta_{\alpha,\bar{\beta}}\mu_{\alpha,\bar{\beta}} = \eta_{\alpha,\beta}\mu_{\alpha,\beta}$, so the assertion follows. □

9.9.3 Lemma. *The involutions on $\mathcal{R}$ and J are group automorphisms with $(r^\sigma)^\sigma = r$ for all $r \in \mathcal{R}$ and $\overline{\overline{v}} = v$ for all $v \in J$.*

Proof. Since the involutions are induced by the actions of -1_B and $(1_A, -1_A)$, respectively, and since both these elements have order 2, we have $(r^\sigma)^\sigma = r$ for all $r \in \mathcal{R}$ and $\overline{\overline{v}} = v$ for all $v \in J$. Further, since $(\eta \times \mu)_{e_2, e_1 - e_2} = (1, -1, 1)$, we have

$$\theta_{e_1}(\overline{u \mathbin{\hat{+}} v}) = \theta_{e_2}(u \mathbin{\hat{+}} v)^{w_{12}} = \theta_{e_2}(u)^{w_{12}}\theta_{e_2}(v)^{w_{12}} = \theta_{e_1}(\overline{u})\theta_{e_1}(\overline{v}) = \theta_{e_1}(\overline{u} \mathbin{\hat{+}} \overline{v})$$

for all $u, v \in J$, which proves that the involution on J is an automorphism. Similarly,

$$\begin{aligned}\theta_{e_2+e_1}\big((r+s)^\sigma\big) &= \theta_{e_2-e_1}(r+s)^{w_1} = \theta_{e_2-e_1}(r)^{w_1}\theta_{e_2-e_1}(s)^{w_1} \\ &= \theta_{e_2+e_1}(r^\sigma)\theta_{e_2+e_1}(s^\sigma) = \theta_{e_2+e_1}(r^\sigma + s^\sigma)\end{aligned}$$

for all $r, s \in \mathcal{R}$. □

9.9.4 Lemma. *For all $i \in [1, n]$, we have $w_i = \theta_{-e_i}(v_{-1})\theta_{e_i}(v_0)\theta_{-e_i}(v_1)$.*

Proof. By the choice of v_{-1}, v_0, v_1, this is clear for $i = n$. Since

$$w_i = w_n^{w_{ni}} \quad \text{and} \quad (\eta \times \mu)_{\pm e_n, e_n - e_i} = (1, 1, 1)$$

the general assertion follows from Proposition 9.9.2. □

9.9.5 Definition (Commutator maps). We define

$$\begin{aligned}\cdot\colon \mathcal{R} \times \mathcal{R} \to \mathcal{R}, (a, b) &\mapsto ab := a \cdot b, \\ \pi\colon J \times \mathcal{R} \to \mathcal{R}, (v, a) &\mapsto \pi(v, a), \\ \varphi\colon J \times \mathcal{R} \to J, (v, a) &\mapsto \varphi(v, a), \\ \mathrm{T}\colon \mathcal{R} \times \mathcal{R} \to J, (a, b) &\mapsto \mathrm{T}(a, b), \\ \psi\colon J \times J \to \mathcal{R}, (u, v) &\mapsto \psi(u, v)\end{aligned}$$

to be the unique maps which satisfy the commutator relations

$$\begin{aligned}
[\theta_{e_1-e_2}(a), \theta_{e_2-e_3}(b)] &= \theta_{e_1-e_3}(ab), \\
[\theta_{e_2}(v), \theta_{e_1-e_2}(a)] &= \theta_{e_1+e_2}\big(-\pi(v,a)\big)\theta_{e_1}\big(\varphi(v,-a^\sigma)\big), \\
[\theta_{e_1-e_2}(a), \theta_{e_1+e_2}(b)] &= \theta_{e_1}\big(\mathrm{T}(a,b)\big), \\
[\theta_{e_1}(u), \theta_{e_2}(v)] &= \theta_{e_1+e_2}\big(\psi(u,v)\big)
\end{aligned}$$

for all $a, b \in \mathcal{R}$ and all $u, v \in J$.

9.9.6 Goal. In Definitions 9.10.8 and 9.10.19, we will use π and T to define maps π_1 and T_1 by specialising one of the components to the element $1_{\mathcal{R}}$ (which is not yet defined). See also Remark 8.6.19. Our goal is to show that $\cdot$ equips $\mathcal{R}$ with an alternative ring structure with respect to which σ is a nuclear involution and that the maps π_1, φ, T_1, ψ equip J with a Jordan module structure over $(\mathcal{R}, \sigma)$. The involution $\overline{\cdot}$ will turn out to be the Jordan module involution.

Parts of the following lemma are analogous to [Zha14, 3.4.8].

9.9.7 Lemma (Rank-2 computations, part 1). *The following statements hold:*

(a) The maps $\cdot$, π, φ, T *and* ψ *are additive in all components except (possibly) for the first component of* π *and the second component of* φ.

(b) For all u *in the image of* T, *we have* $\psi(u,v) = 0 = \psi(v,u)$ *and* $u \hat{+} v = v \hat{+} u$ *for all* $v \in J$.

(c) $\pi(u \hat{+} v, a) = \pi(u,a) + \pi(v,a) + \psi\big(\varphi(u,a^\sigma), v\big)$ *for all* $u, v \in J$, $a \in \mathcal{R}$.

(d) $\varphi(u, a+b) = \varphi(u,a) \hat{+} \varphi(u,b) \hat{+} \mathrm{T}\big(a^\sigma, \pi(u, b^\sigma)\big)$ *for all* $u \in J$, $a, b \in \mathcal{R}$.

Proof. The maps $\cdot$, T and ψ are bi-additive by Lemma 2.1.13 and Remark 2.1.14. Further, π is additive in the second component by Lemma 7.4.4 (h) and φ is additive in the first component by Lemma 7.4.4 (b). This proves (a).

Now let u be in the image of T. Then $\theta_{e_1}(u)$ lies in $[U_{e_1-e_2}, U_{e_1+e_2}]$ and thus in U_{2e_1}. Since U_{2e_1} commutes with U_{e_1} and with U_{e_2}, this implies that $u \hat{+} v = v \hat{+} u$ and $\psi(u,v) = 0$ for all $v \in J$. Further, since

$$\theta_{e_2}(u) = \theta_{e_1}(u)^{w_{12}} \in U_{2e_1}^{w_{12}} = U_{2e_2}$$

we also have $\psi(v,u) = 0$ for all $v \in J$.

Now let $u, v \in J$ and let $a \in \mathcal{R}$. Then by Lemma 7.4.4 (d), we have

$$\begin{aligned}
\theta_{e_1+e_2}\big(\pi(u \hat{+} v, a)\big) &= [\theta_{e_2}(u \hat{+} v), \theta_{e_1-e_2}(-a)]_{e_1+e_2} \\
&= [\theta_{e_2}(u), \theta_{e_1-e_2}(-a)]_{e_1+e_2}\big[[\theta_{e_2}(u), \theta_{e_1-e_2}(-a)]_{e_1}, \theta_{e_2}(v)\big] \\
&\quad \cdot [\theta_{e_2}(v), \theta_{e_1-e_2}(-a)]_{e_1+e_2} \\
&= \theta_{e_1+e_2}\big(\pi(u,a)\big)\big[\theta_{e_1}\big(\varphi(u,a^\sigma)\big), \theta_{e_2}(v)\big]\theta_{e_1+e_2}\big(\pi(v,a)\big) \\
&= \theta_{e_1+e_2}\big(\pi(u,a) + \pi(v,a) + \psi\big(\varphi(u,a^\sigma), v\big)\big).
\end{aligned}$$

It follows that (c) holds. Finally, let $u \in J$ and let $a, b \in \mathcal{R}$. Then by Lemma 7.4.4 (f),

$$\begin{aligned}
\theta_{e_1}\big(\varphi(u, a+b)\big) &= \big[\theta_{e_2}(u), \theta_{e_1-e_2}\big(-(a+b)^\sigma\big)\big]_{e_1} \\
&= \big[\theta_{e_2}(u), \theta_{e_1-e_2}(-b^\sigma)\theta_{e_1-e_2}(-a^\sigma)\big]_{e_1} \\
&= \big[\theta_{e_2}(u), \theta_{e_1-e_2}(-a^\sigma)\big]_{e_1}\big[[\theta_{e_2}(u), \theta_{e_1-e_2}(-b^\sigma)]_{e_1+e_2}, \theta_{e_1-e_2}(-a^\sigma)\big] \\
&\quad \cdot [\theta_{e_2}(u), \theta_{e_1-e_2}(-b^\sigma)]_{e_1}
\end{aligned}$$

Here

$$\begin{aligned}
\left[\theta_{e_2}(u), \theta_{e_1-e_2}(-a^\sigma)\right]_{e_1} &= \theta_{e_1}\big(\varphi(u,a)\big), \\
\big[[\theta_{e_2}(u), \theta_{e_1-e_2}(-b^\sigma)]_{e_1+e_2}, \theta_{e_1-e_2}(-a^\sigma)\big] &= \big[\theta_{e_1+e_2}\big(\pi(u,b^\sigma)\big), \theta_{e_1-e_2}(-a^\sigma)\big] \\
&= \big[\theta_{e_1-e_2}(-a^\sigma), \theta_{e_1+e_2}\big(\pi(u,b^\sigma)\big)\big]^{-1} \\
&= \theta_{e_1}\big(\mathrm{T}\big(a^\sigma, \pi(u,b^\sigma)\big)\big), \\
[\theta_{e_2}(u), \theta_{e_1-e_2}(-b^\sigma)]_{e_1} &= \theta_{e_1}\big(\varphi(u,b)\big).
\end{aligned}$$

Since the image of T centralises J, we conclude that

$$\theta_{e_1}\big(\varphi(u,a+b)\big) = \theta_{e_1}\big(\varphi(u,a) \mathbin{\hat{+}} \varphi(u,b) \mathbin{\hat{+}} \mathrm{T}\big(a^\sigma, \pi(u,b^\sigma)\big)\big).$$

This finishes the proof. □

The following computation will not be needed until the end of the following section, but there is no reason to delay it either.

9.9.8 Lemma. *For all pairwise distinct $i,j,k \in [1,n]$ and all $a,b \in \mathcal{R}$, the following commutator relations hold:*

$$\begin{aligned}
[\theta_{e_i-e_j}(a), \theta_{e_j-e_k}(b)] &= \theta_{e_i-e_k}(ab), \\
[\theta_{e_i-e_j}(a), \theta_{e_j+e_k}(b)] &= \theta_{e_i+e_k}\big(\delta^\sigma_{k<i}(a\delta^\sigma_{k<j}(b))\big), \\
[\theta_{e_i-e_j}(a), \theta_{-e_k-e_i}(b)] &= \theta_{-e_j-e_k}\big(-\delta^\sigma_{j<k}(a^\sigma\delta^\sigma_{i<k}(b))\big), \\
[\theta_{e_i+e_j}(a), \theta_{-e_k-e_i}(b)] &= \theta_{e_j-e_k}\big(\delta^\sigma_{i<j}(a)\delta^\sigma_{i<k}(b)\big).
\end{aligned}$$

Proof. We can conjugate the equations in Definition 9.9.5 by the same Weyl elements as in Lemma 7.10.7, but the result is slightly different because the values of the involved parity maps are different. □

9.9.9 Remark. We could perform the same computations as in Lemma 9.9.8 for the other commutator maps. However, it will be more efficient to do so at a later point, when we have acquired more information about the maps π, φ, T and ψ. For the moment, we only note that for all $i<j \in [1,n]$, conjugating the equations in Definition 9.9.5 by $w_{2j}w_{1i}$ (where w_{kk} is interpreted as 1_G) yields that

$$\begin{aligned}
[\theta_{e_j}(v), \theta_{e_i-e_j}(a)] &= \theta_{e_i+e_j}\big(-\pi(v,a)\big)\theta_{e_i}\big(\varphi(v,-a^\sigma)\big), \\
[\theta_{e_i-e_j}(a), \theta_{e_i+e_j}(b)] &= \theta_{e_i}\big(\mathrm{T}(a,b)\big), \\
[\theta_{e_i}(u), \theta_{e_j}(v)] &= \theta_{e_i+e_j}\big(\psi(u,v)\big)
\end{aligned}$$

for all $a \in \mathcal{R}$ and all $u,v \in J$.

9.9.10 Lemma (Rank-2 computations, part 2). *$(\mathcal{R},+,\cdot)$ is a ring. If we denote its identity element by $1_\mathcal{R}$, we have $w_{ij} = \theta_{e_j-e_i}(-1_\mathcal{R})\theta_{e_i-e_j}(1_\mathcal{R})\theta_{e_j-e_i}(-1_\mathcal{R})$ for all distinct $i,j \in [1,n]$.*

Proof. We have already proven in Lemma 9.9.7 (a) that the multiplication satisfies the distributive law. It remains to verify the existence of an identity element. This can be done in the same way as for B_n in the proof of Lemma 7.10.9. □

9.9.11 Lemma (Rank-2 computations, part 3). *We have*

$$\varphi(v,1^\sigma_\mathcal{R}) = v \quad \textit{and} \quad \varphi(v,-1^\sigma_\mathcal{R}) = \overline{v}$$

for all $v \in J$ and $\pi(v_0,a) = a$ for all $a \in \mathcal{R}$.

Proof. Let $v \in J$. Since $w_{12} = \theta_{e_2-e_1}(-1_{\mathcal{R}})\theta_{e_1-e_2}(1_{\mathcal{R}})\theta_{e_2-e_1}(-1_{\mathcal{R}})$ by Lemma 9.9.10, it follows from Lemma 9.4.13 (a) and Lemma 9.4.14 (a) that

$$\begin{aligned}\theta_{e_1}(\overline{v}) &= \theta_{e_2}(v)^{w_{12}} = [\theta_{e_2}(v), \theta_{e_1-e_2}(1_{\mathcal{R}})]_{e_1} = \theta_{e_1}\big(\varphi(v, -1^{\sigma}_{\mathcal{R}})\big),\\ \theta_{e_1}(v) &= \theta_{e_2}(v)^{w_{21}} = [\theta_{e_2}(v), \theta_{e_1-e_2}(-1_{\mathcal{R}})]_{e_1} = \theta_{e_1}\big(\varphi(v, 1^{\sigma}_{\mathcal{R}})\big).\end{aligned}$$

This proves the first assertion. Now let $a \in \mathcal{R}$ and recall from Lemma 9.9.4 that $w_2 = \theta_{-e_2}(v_{-1})\theta_{e_i}(v_0)\theta_{-e_2}(v_1)$. Thus by Lemma 9.4.5 (a),

$$\begin{aligned}\theta_{e_1+e_2}(a) &= \theta_{e_1-e_2}(a)^{w_2} = [\theta_{e_1-e_2}(a), \theta_{e_2}(v_0)]_{e_1+e_2} = [\theta_{e_2}(v_0), \theta_{e_1-e_2}(a)]^{-1}_{e_1+e_2}\\ &= \theta_{e_1+e_2}\big(\pi(v_0, a)\big).\end{aligned}$$

This proves the second assertion. □

We will show in Lemma 9.10.4 that $1^{\sigma}_{\mathcal{R}} = 1_{\mathcal{R}}$, thereby simplifying the formulas in Lemma 9.9.11.

We can now compute the rewriting rules. As was the case for B_n, we will only consider the BC_3-subsystem spanned by $\{ e_1 - e_2, e_2 - e_2, e_3 \}$. See, however, Remark 9.10.3.

9.9.12 Definition (Blueprint rewriting rules). We define the following rewriting rules:

$$\begin{aligned}&\psi_{12}\colon U_{e_1-e_2} \times U_{e_2-e_3} \times U_{e_1-e_2} \to U_{e_2-e_3} \times U_{e_1-e_2} \times U_{e_1-e_2},\\ &\big(\theta_{e_1-e_2}(a), \theta_{e_2-e_3}(b), \theta_{e_1-e_2}(c)\big) \mapsto \big(\theta_{e_2-e_3}(c), \theta_{e_1-e_2}(-b-ca), \theta_{e_2-e_3}(a)\big),\\ &\psi_{12}^{-1}\colon U_{e_2-e_3} \times U_{e_1-e_2} \times U_{e_1-e_2} \to U_{e_1-e_2} \times U_{e_2-e_3} \times U_{e_1-e_2},\\ &\big(\theta_{e_2-e_3}(a), \theta_{e_1-e_2}(b), \theta_{e_2-e_3}(c)\big) \mapsto \big(\theta_{e_1-e_2}(c), \theta_{e_2-e_3}(-b-ac), \theta_{e_1-e_2}(a)\big),\end{aligned}$$

$$\begin{aligned}&\varphi_{13}\colon U_{e_1-e_2} \times U_{2e_3} \to U_{2e_3} \times U_{e_1-e_2},\\ &\big(\theta_{e_1-e_2}(a), \theta_{2e_3}(v)\big) \mapsto \big(\theta_{2e_3}(v), \theta_{e_1-e_2}(a)\big),\\ &\varphi_{13}^{-1}\colon U_{2e_3} \times U_{e_1-e_2} \to U_{e_1-e_2} \times U_{2e_3},\\ &\big(\theta_{2e_3}(v), \theta_{e_1-e_2}(a)\big) \mapsto \big(\theta_{e_1-e_2}(a), \theta_{2e_3}(v)\big)\end{aligned}$$

and

$$\psi_{23}\colon U_{2e_3} \times U_{e_2-e_3} \times U_{2e_3} \times U_{e_2-e_3} \to U_{e_2-e_3} \times U_{2e_3} \times U_{e_2-e_3} \times U_{2e_3}$$

which maps $\big(\theta_{e_3}(v), \theta_{e_2-e_3}(a), \theta_{e_3}(u), \theta_{e_2-e_3}(b)\big)$ to

$$\left(b^{\sigma}, \overline{\varphi(\overline{v}, -b^{\sigma})} \mathbin{\hat{+}} u \mathbin{\hat{+}} \overline{\mathrm{T}(b, a^{\sigma})}, -\pi(\overline{v}, b) - a^{\sigma} - \psi\big(\varphi(\overline{v}, -b^{\sigma}) \mathbin{\hat{+}} \overline{u}, \overline{v}\big), \overline{v}\right).$$

9.9.13 Lemma. *The maps ψ_{12}, ψ_{12}^{-1}, φ_{13}, φ_{13}^{-1} and ψ_{23} in Definition 9.9.12 are blueprint rewriting rules (with respect to $(w_\delta)_{\delta\in\Delta'}$). Further, ψ_{12} and ψ_{12}^{-1} are inverses of each other, and the same holds for φ_{13} and φ_{13}^{-1}.*

Proof. The second statement about inverses is straightforward to verify. Further, we know from Lemma 6.2.15 that φ_{13} and φ_{13}^{-1} are blueprint rewriting rules. Observe that the rewriting rule ψ_{12} and its inverse are the same rules that we have used for A_3. Since the restriction of η to $A_2 = \{ e_i - e_{i+1} \mid i \in [1,2] \}$ yields the same parity map that we have used for A_3 (see Remark 9.3.21), the same computation as in Lemma 6.3.4 shows that ψ_{12} and ψ_{12}^{-1} are blueprint rewriting rules. It remains to show that ψ_{23} is a blueprint rewriting rule. For this, let $a, b, c, d \in \mathcal{R}$, let $v, u, h, k \in M$ and put

$$\bar{\alpha} := (e_3, e_2 - e_3, e_3, e_2 - e_3) \quad \text{and} \quad \bar{\beta} := (e_2 - e_3, e_3, e_2 - e_3, e_3).$$

Further, we set $x := \big(\theta_{e_3}(v), \theta_{e_2-e_3}(a), \theta_{e_3}(u), \theta_{e_2-e_3}(b)\big)$. On the one hand, we have

$$\begin{aligned}
\tilde{\gamma}_{\bar{\alpha}}(x) &= w_3\theta_{e_3}(v)w_{23}\theta_{e_2-e_3}(a)w_3\theta_{e_3}(u)w_{23}\theta_{e_2-e_3}(b) \\
&= w_3w_{23}w_3w_{23}\theta_{e_3}(v)^{w_{23}w_3w_{23}}\theta_{e_2-e_3}(a)^{w_3w_{23}}\theta_{e_3}(u)^{w_{23}}\theta_{e_2-e_3}(b) \\
&= w_3w_{23}w_3w_{23}\theta_{e_2}(\overline{v})^{w_3w_{23}}\theta_{e_2+e_3}(a)^{w_{23}}\theta_{e_2}(\overline{u})\theta_{e_2-e_3}(b) \\
&= w_3w_{23}w_3w_{23}\theta_{e_3}(\overline{v})\theta_{e_2+e_3}(-a^\sigma)\theta_{e_2}(\overline{u})\theta_{e_2-e_3}(b).
\end{aligned}$$

On the other hand, we put $x' := \big(\theta_{e_2-e_3}(c), \theta_{e_3}(h), \theta_{e_2-e_3}(d), \theta_{e_3}(k)\big)$. Then

$$\begin{aligned}
\tilde{\gamma}_{\bar{\beta}}(x') &= w_{23}\theta_{e_2-e_3}(c)w_3\theta_{e_3}(h)w_{23}\theta_{e_2-e_3}(d)w_3\theta_{e_3}(k) \\
&= w_{23}w_3w_{23}w_3\theta_{e_2-e_3}(c)^{w_3w_{23}w_3}\theta_{e_3}(h)^{w_{23}w_3}\theta_{e_2-e_3}(d)^{w_3}\theta_{e_3}(k) \\
&= w_{23}w_3w_{23}w_3\theta_{e_2+e_3}(c)^{w_{23}w_3}\theta_{e_2}(\overline{h})^{w_3}\theta_{e_2+e_3}(d)\theta_{e_3}(k) \\
&= w_{23}w_3w_{23}w_3\theta_{e_2-e_3}(c^\sigma)\theta_{e_2}(\overline{h})\theta_{e_2+e_3}(d)\theta_{e_3}(k).
\end{aligned}$$

In order to compare these two terms, we have to change the order of the product in the first one, using the commutator formulas in Remark 9.9.9 as well as Relation 1.1.11 (v):

$$\begin{aligned}
&\theta_{e_3}(\overline{v})\theta_{e_2+e_3}(-a^\sigma)\theta_{e_2}(\overline{u})\theta_{e_2-e_3}(b) \\
&\quad= \theta_{e_3}(\overline{v})\theta_{e_2+e_3}(-a^\sigma)\theta_{e_2-e_3}(b)\theta_{e_2}(\overline{u}) \\
&\quad= \theta_{e_3}(\overline{v})\theta_{e_2-e_3}(b)\theta_{e_2+e_3}(-a^\sigma)[\theta_{e_2+e_3}(-a^\sigma), \theta_{e_2-e_3}(b)]\theta_{e_2}(\overline{u}) \\
&\quad= \theta_{e_3}(\overline{v})\theta_{e_2-e_3}(b)\theta_{e_2+e_3}(-a^\sigma)[\theta_{e_2-e_3}(b), \theta_{e_2+e_3}(a^\sigma)]\theta_{e_2}(\overline{u}) \\
&\quad= \theta_{e_3}(\overline{v})\theta_{e_2-e_3}(b)\theta_{e_2+e_3}(-a^\sigma)\theta_{e_2}\big(\overline{u} \hat{+} \mathrm{T}(b, a^\sigma)\big) \\
&\quad= \theta_{e_2-e_3}(b)\theta_{e_3}(\overline{v})[\theta_{e_3}(\overline{v}), \theta_{e_2-e_3}(b)]\theta_{e_2}\big(\overline{u} \hat{+} \mathrm{T}(b, a^\sigma)\big)\theta_{e_2+e_3}(-a^\sigma) \\
&\quad= \theta_{e_2-e_3}(b)\theta_{e_3}(\overline{v})\theta_{e_2}\big(\varphi(\overline{v}, b)\big)\theta_{e_2+e_3}\big(-\pi(\overline{v}, b)\big)\theta_{e_2}\big(\overline{u} \hat{+} \mathrm{T}(b, a^\sigma)\big)\theta_{e_2+e_3}(-a^\sigma) \\
&\quad= \theta_{e_2-e_3}(b)\theta_{e_2}\big(\varphi(\overline{v}, b) \hat{+} \overline{u} \hat{+} \mathrm{T}(b, a^\sigma)\big)\theta_{e_2+e_3}(-\pi(\overline{v}, b) - a^\sigma) \\
&\qquad\quad \cdot \big[\theta_{e_3}(\overline{v}), \theta_{e_2}\big(\varphi(\overline{v}, b) \hat{+} \overline{u} \hat{+} \mathrm{T}(b, a^\sigma)\big)\big]\theta_{e_3}(\overline{v}) \\
&\quad= \theta_{e_2-e_3}(b)\theta_{e_2}\big(\varphi(\overline{v}, b) \hat{+} \overline{u} \hat{+} \mathrm{T}(b, a^\sigma)\big) \\
&\qquad\quad \cdot \theta_{e_2+e_3}\big(-\pi(\overline{v}, b) - a^\sigma - \psi\big(\varphi(\overline{v}, b) \hat{+} \overline{u} \hat{+} \mathrm{T}(b, a^\sigma), \overline{v}\big)\big)\theta_{e_3}(\overline{v})
\end{aligned}$$

Note that

$$\psi\big(\varphi(\overline{v}, b) \hat{+} \overline{u} \hat{+} \mathrm{T}(b, a^\sigma), \overline{v}\big) = \psi\big(\varphi(\overline{v}, b) \hat{+} \overline{u}, \overline{v}\big)$$

by Lemma 9.9.7 (b). Recall that $w_3w_{23}w_3w_{23} = w_{23}w_3w_{23}w_3$ by Theorem 2.5.10 and that G is rank-2-injective. We conclude that $\tilde{\gamma}_{\bar{\alpha}}(x) = \tilde{\gamma}_{\bar{\beta}}(x')$ if the following conditions are satisfied:

$$\begin{aligned}
c &= b^\sigma, & h &= \overline{\varphi(\overline{v}, b)} \hat{+} u \hat{+} \overline{\mathrm{T}(b, a^\sigma)}, \\
d &= \overline{v}, & k &= -\pi(\overline{v}, b) - a^\sigma - \psi\big(\varphi(\overline{v}, b) \hat{+} \overline{u}, \overline{v}\big).
\end{aligned}$$

These conditions are equivalent to the equation $x' = \psi_{23}(x)$, which shows that ψ_{23} is a blueprint rewriting rule. This finishes the proof. □

9.10 Blueprint Computations

Notation 9.9.1 continues to hold.

9.10.1 Remark (Blueprint computation). We now have everything ready to perform the blueprint computation for BC_3. Since B_3 and BC_3 have the same Weyl group, we can use the same homotopy cycle as in Figure 7.8 on page 219.

We begin with the tuple

$$\big(\theta_3(u), \theta_2(a), \theta_1(b), \theta_3(v), \theta_2(c), \theta_3(w), \theta_2(d), \theta_1(r), \theta_2(s)\big)$$

where $\theta_1 := \theta_{e_1-e_2}$, $\theta_2 := \theta_{e_2-e_3}$, $\theta_3 := \theta_{e_3}$ and where $u, v, w \in J$ and $a, b, c, d, r, s \in \mathcal{R}$ are arbitrary. Working down rows 1 to 12 in the homotopy cycle and applying the respective blueprint rewriting rules in the process, we obtain a tuple

$$\big(\theta_2(x_1), \theta_1(x_2), \theta_2(x_3), \theta_3(x_4), \theta_2(x_5), \theta_3(x_6), \theta_1(x_7), \theta_2(x_8), \theta_3(x_9)\big).$$

Conversely, working up from row 23 to row 12, we obtain a tuple

$$\big(\theta_2(y_1), \theta_1(y_2), \theta_2(y_3), \theta_3(y_4), \theta_2(y_5), \theta_3(y_6), \theta_1(y_7), \theta_2(y_8), \theta_3(y_9)\big)$$

where $x_i, y_i \in J$ for all $i \in \{4, 6, 9\}$ and $x_i, y_i \in \mathcal{R}$ for all $i \in \{1, 2, 3, 5, 7, 8\}$. Now $x_i = y_i$ for all $i \in [1, 9]$ by Theorem 6.2.8. The results of this computation can be found in Figures 9.5 to 9.7. The intermediate steps of the above computation have been performed with GAP [Gap].

$$(1)\ s^\sigma = s^\sigma, \quad (2)\ -r^\sigma - s^\sigma d^\sigma = -r^\sigma - (ds)^\sigma, \quad (3)\ d^\sigma = d^\sigma, \quad (9)\ u = u$$

Figure 9.5: The equations "(i) $x_i = y_i$" for $i \in \{1, 2, 3, 9\}$ in Remark 9.10.1.

$$\overline{\varphi(u, -s^\sigma)} \mathbin{\hat{+}} \overline{v} \mathbin{\hat{+}} \overline{\mathrm{T}(s, a^\sigma)} = \varphi(\overline{u}, -s^\sigma) \mathbin{\hat{+}} \overline{v} \mathbin{\hat{+}} \mathrm{T}(s, a^\sigma),$$
$$-\pi(u, s) - a^\sigma - \psi\big(\varphi(u, -s^\sigma) \mathbin{\hat{+}} v, u\big) = -\pi(\overline{u}, s) - a^\sigma - \psi\big(\varphi(\overline{u}, -s^\sigma) \mathbin{\hat{+}} \overline{v}, \overline{u}\big)$$

Figure 9.6: The equations $x_6 = y_6$ and $x_8 = y_8$ in Remark 9.10.1, respectively.

$$\begin{aligned}
x_4 &= \overline{\varphi(\overline{u}, -r^\sigma)} \mathbin{\hat{+}} \overline{\varphi(\overline{v}, -d^\sigma)} \mathbin{\hat{+}} w \mathbin{\hat{+}} \overline{\mathrm{T}(d, c^\sigma)} \mathbin{\hat{+}} \overline{\mathrm{T}(r, -b^\sigma - (ad^\sigma)^\sigma)} \\
y_4 &= \overline{\varphi\big(\varphi(\overline{u}, -s^\sigma) \mathbin{\hat{+}} \overline{v} \mathbin{\hat{+}} \mathrm{T}(s, a^\sigma), -d^\sigma\big)} \mathbin{\hat{+}} \overline{\varphi\big(u, r^\sigma + (ds)^\sigma\big)} \mathbin{\hat{+}} w \\
&\quad \mathbin{\hat{+}} \overline{\mathrm{T}(-r - ds, b^\sigma)} \mathbin{\hat{+}} \overline{\mathrm{T}\big(d, c^\sigma + (sb)^\sigma - [A(-r^\sigma - (ds)^\sigma)]^\sigma\big)} \\
&\quad \text{where } A := -\pi(\overline{u}, s) - a^\sigma - \psi\big(\varphi(\overline{u}, -s^\sigma) \mathbin{\hat{+}} \overline{v}, \overline{u}\big), \\
x_5 &= -\pi(\overline{v}, d) - c^\sigma - \psi\big(\varphi(\overline{v}, -d^\sigma) \mathbin{\hat{+}} \overline{w}, \overline{v}\big) + ra \\
&\quad - \big(-\pi(\overline{u}, r) + b^\sigma + (ad^\sigma)^\sigma - \psi(B, \overline{u})\big)s^\sigma \\
&\quad \text{where } B := \varphi(\overline{u}, -r^\sigma) \mathbin{\hat{+}} \varphi(\overline{v}, -d^\sigma) \mathbin{\hat{+}} \overline{w} \mathbin{\hat{+}} \mathrm{T}(d, c^\sigma), \\
y_5 &= -\pi\big(\varphi(\overline{u}, -s^\sigma) \mathbin{\hat{+}} \overline{v} \mathbin{\hat{+}} \mathrm{T}(s, a^\sigma), d\big) - c^\sigma - (sb)^\sigma \\
&\quad + \big[\big(-\pi(\overline{u}, s) - a^\sigma - \psi\big(\varphi(\overline{u}, -s^\sigma) \mathbin{\hat{+}} \overline{v}, \overline{u}\big)\big)\big(-r^\sigma - (ds)^\sigma\big)\big]^\sigma \\
&\quad - \psi\big(C \mathbin{\hat{+}} \overline{w} \mathbin{\hat{+}} \mathrm{T}(-r - ds, b^\sigma), \varphi(\overline{u}, -s^\sigma) \mathbin{\hat{+}} \overline{v} \mathbin{\hat{+}} \mathrm{T}(s, a^\sigma)\big), \\
&\quad \text{where } C := \varphi\big(\varphi(\overline{u}, -s^\sigma) \mathbin{\hat{+}} \overline{v} \mathbin{\hat{+}} \mathrm{T}(s, a^\sigma), -d^\sigma\big) \mathbin{\hat{+}} \varphi\big(u, r^\sigma + (ds)^\sigma\big), \\
x_7 &= -\pi(\overline{u}, r) + b^\sigma + (ad^\sigma)^\sigma - \psi\big(\varphi(\overline{u}, -r^\sigma) \mathbin{\hat{+}} \varphi(\overline{v}, -d^\sigma) \mathbin{\hat{+}} \overline{w} \mathbin{\hat{+}} \mathrm{T}(d, c^\sigma), \overline{u}\big), \\
y_7 &= \pi(u, -r - ds) + b^\sigma + \psi\big(\varphi(u, r^\sigma + (ds)^\sigma) \mathbin{\hat{+}} \overline{w}, u\big) \\
&\quad - d\big(-\pi(\overline{u}, s) - a^\sigma - \psi\big(\varphi(\overline{u}, -s^\sigma) \mathbin{\hat{+}} \overline{v}, \overline{u}\big)\big)
\end{aligned}$$

Figure 9.7: The values of $x_4, y_4, x_5, y_5, x_7, y_7$ in Remark 9.10.1.

9.10.2 Note. In the following computations, we have to be careful to remember that the group J is not abelian. For example, it is not possible to "subtract $\overline{v}$ from equation 6" to obtain

$$\overline{\varphi(u, -s^\sigma)} \mathbin{\hat{+}} \overline{\mathrm{T}(s, a^\sigma)} = \varphi(\overline{u}, -s^\sigma) \mathbin{\hat{+}} \mathrm{T}(s, a^\sigma)$$

because $\overline{v}$ lies "in the middle" of equation 6. (However, it is still true that the identity above is a consequence of equation 6 because we can simply put $v := 0_J$.) Similarly, we will avoid refering to inverses in the group $(J, \hat{+})$ whenever possible.

As in section 7.11, the goal of this section is to show that equations 1 to 9 are equivalent to the axioms of a Jordan module. Clearly, equations 1, 3 and 9 are trivial.

9.10.3 Remark (Associativity for $n \geq 4$). If $n \geq 4$, then BC_n contains a parabolic subsystem of type A_3 whose root groups are parametrised by the ring $\mathcal{R}$. In this case, the blueprint computations for A_3 yield that $\mathcal{R}$ must be associative. However, this is the only identity that can be obtained from the assumption that $n \geq 4$. Note that there is no need to perform a (long) blueprint computation in a rank-4 subsystem: It suffices to perform two separate (short) blueprint computations in the subsystems A_3 and BC_3. Of course, the blueprint computation for A_3 has already been done in section 6.3.

9.10.4 Lemma. *The map σ is an involution of the ring $\mathcal{R}$. In particular, $1_{\mathcal{R}}^{\sigma} = 1_{\mathcal{R}}$.*

Proof. The first assertion follows from equation 2 in Figure 9.5. The second assertion is a consequence of the first one by Remark 8.3.5. ☐

Lemma 9.10.4 clearly covers all non-trivial identities which can be derived from Figure 9.5.

9.10.5 Lemma (Equation 6). *We have $\overline{\varphi(u,s)} = \varphi(\overline{u}, s)$ and $\overline{\mathrm{T}(a,b)} = \mathrm{T}(a,b)$ for all $a, b, s \in \mathcal{R}$ and all $u \in J$.*

Proof. The first assertion follows from equation 6 in Figure 9.6 by putting $v := 0$ and $a := 0$. The second assertion follows from the same equation by putting $u := 0$ and $v := 0$. ☐

9.10.6 Lemma (Equation 8). *We have $\psi(\overline{u}, \overline{v}) = \psi(u, v)$ and $\pi(\overline{u}, s) = \pi(u, s)$ for all $u, v \in J$ and all $s \in \mathcal{R}$.*

Proof. The first assertion follows from equation 8 in Figure 9.6 by putting $s := 0$. Using Lemma 9.10.5, we can now simplify equation 8 to obtain

$$-\pi(u,s) - a^{\sigma} - \psi\big(\varphi(u, -s^{\sigma}) \hat{+} v, u\big) = -\pi(\overline{u}, s) - a^{\sigma} - \psi\big(\varphi(u, -s^{\sigma}) \hat{+} v, u\big).$$

This is equivalent to the second assertion. ☐

Again, it is clear that Lemmas 9.10.5 and 9.10.6 cover everything that can be deduced from Figure 9.6.

9.10.7 Lemma (Equation 7). *The following hold for all $u, v \in J$ and all $a, b \in \mathcal{R}$:*

(a) $\psi(\overline{u}, v) = \psi(u, \overline{v}) = -\psi(u, v)$.

(b) $\psi\big(\varphi(u,a), v\big) = a^{\sigma}\psi(u, v)$.

(c) $\pi(u, a) = a\pi(u, 1_{\mathcal{R}})$.

(d) $(ab)\pi(u, 1_{\mathcal{R}}) + (ab)\psi(u,u) = a\big(b\pi(u, 1_{\mathcal{R}})\big) + a\big(b\psi(u,u)\big)$.

Proof. Putting all variables except for u and w in equation 7 to zero, we see that $-\psi(\overline{w}, \overline{u}) = \psi(\overline{w}, u)$. Together with Lemma 9.10.6 and the fact that $\overline{\cdot}$ is of order at most 2, we infer that (a) holds. Putting all variables except for u, v, and d to zero, we obtain

$$-\psi\big(\varphi(\overline{v}, -d^\sigma), \overline{u}\big) = d\psi(\overline{v}, \overline{u}),$$

which proves (b).

Observe that the term $b^\sigma + da^\sigma$ appears on both sides of equation 7, that $\psi(\overline{w}, \overline{u}) = -\psi(\overline{w}, u)$ by (a), that $\psi\big(\mathrm{T}(d, c^\sigma), \overline{u}\big) = 0_{\mathcal{R}}$ by Lemma 9.9.7 (b) and that $\pi(\overline{u}, r) = \pi(u, r)$ and $\pi(\overline{u}, s) = \pi(u, s)$ by Lemma 9.10.6. It follows that equation 7 is equivalent to $x_7' = y_7'$ where

$$\begin{aligned} x_7' &= -\pi(u, r) - \psi\big(\varphi(\overline{u}, -r^\sigma) \hat{+} \varphi(\overline{v}, -d^\sigma), \overline{u}\big), \\ y_7' &= \pi(u, -r - ds) + \psi\big(\varphi(u, r^\sigma + (ds)^\sigma), u\big) \\ &\qquad - d\big(-\pi(u, s) - \psi(\varphi(\overline{u}, -s^\sigma) \hat{+} \overline{v}, \overline{u})\big). \end{aligned}$$

Using (a), the bi-additivity of ψ and the additivity of π in the second component as well as Lemma 9.10.5, we see that

$$\begin{aligned} x_7' &= -\pi(u, r) - \psi\big(\varphi(\overline{u}, -r^\sigma), \overline{u}\big) - \psi\big(\varphi(\overline{v}, -d^\sigma), \overline{u}\big) \\ &= -\pi(u, r) + r\psi(u, u) + d\psi(v, u) \qquad \text{and} \\ y_7' &= -\pi(u, r) - \pi(u, ds) + (r + ds)\psi(u, u) + d\pi(u, s) - d\big(s\psi(u, u)\big) + d\psi(v, u). \end{aligned}$$

Cancelling the term $-\pi(u, r) + r\psi(u, u) + d\psi(v, u)$ on both sides, it follows that equation 7 is equivalent to

$$0_{\mathcal{R}} = -\pi(u, ds) + (ds)\psi(u, u) + d\pi(u, s) - d\big(s\psi(u, u)\big).$$

Putting $s := 1_{\mathcal{R}}$, we infer that (c) is true. With this knowledge, equation 7 is equivalent to

$$d\big(s\pi(u, 1_{\mathcal{R}})\big) + (ds)\psi(u, u) = (ds)\pi(u, 1_{\mathcal{R}}) + d\big(s\psi(u, u)\big),$$

which is exactly (d). □

We have seen in Lemma 9.10.7 (c) that the map π is uniquely determined by the map $\pi(\,\cdot\,, 1_{\mathcal{R}})$. This motivates the following definition.

9.10.8 Definition. We define a map $\pi_1 \colon J \to \mathcal{R}, u \mapsto \pi(u, 1_{\mathcal{R}})$.

9.10.9 Note. We will later see that a stronger version of Lemma 9.10.7 (d) holds: In fact, we have $(ab)\pi(u, 1) = a\big(b\pi(u, 1)\big)$ and $(ab)\psi(u, u) = a\big(b\psi(u, u)\big)$. We will also see that $\mathcal{R}$ is alternative, so by Remark 8.2.8, these equations imply that $\pi(u, 1)$ and $\psi(u, u)$ lie in the nucleus of $\mathcal{R}$. Even more, we will see that $\psi(u, v)$ lies in the nucleus for all $u, v \in J$.

Once more, the proof of Lemma 9.10.7 shows that no further identities can be obtained from equation 7. Now the only remaining equations are 4 and 5. Even though equation 4 is shorter, we will continue with equation 5 since it is an equation in $\mathcal{R}$ and not in J. This makes it easier to evaluate because $(\mathcal{R}, +)$ is abelian.

9.10.10 Lemma (Equation 5, part 1). *The following identities hold for all $u, v \in J$ and all $a, s, d \in \mathcal{R}$:*

(a) $\psi(u, v)^\sigma = -\psi(v, u)$.

(b) $\psi\big(u, \varphi(v, a)\big) = \psi(u, v)a$.

(c) $\pi\big(\mathrm{T}(s, a), d\big) = (ds)a^\sigma + (da)s^\sigma$.

Proof. We consider equation 5 in Figure 9.7. Putting all variables except for u, v and r to zero, we obtain

$$0 = \big(\psi(\overline{v},\overline{u})r^\sigma\big)^\sigma - \psi\big(\varphi(u,r^\sigma),\overline{v}\big).$$

With $r := 1_{\mathcal{R}}$, Lemma 9.9.11 and Lemma 9.10.7 (a), this yields (a). Now (b) is a consequence of (a) and the previous result 9.10.7 (b) because

$$\psi\big(u,\varphi(v,a)\big) = -\psi\big(\varphi(v,a),u\big)^\sigma = -\big(a^\sigma\psi(v,u)\big)^\sigma = \psi(u,v)a.$$

Replacing all variables in equation 5 except for a, d and s by zero, we see that

$$-(ad^\sigma)^\sigma s^\sigma = -\pi\big(\mathrm{T}(s,a^\sigma),d\big) + \big(a^\sigma(ds)^\sigma\big)^\sigma - \psi\big(\varphi\big(\mathrm{T}(s,a^\sigma),-d^\sigma\big),\mathrm{T}(s,a^\sigma)\big).$$

Recall that the last summand on the right-hand side is trivial by Lemma 9.9.7 (b). Thus we obtain

$$\pi\big(\mathrm{T}(s,a^\sigma),d\big) = (ds)a + (da^\sigma)s^\sigma.$$

This proves (c). □

9.10.11 Remark. Let $u,v \in J$ and $a,b \in \mathcal{R}$. By the same computation as in Note 8.5.2, it follows from Lemma 9.10.7 (b) and Lemma 9.10.10 (b) that

$$\big(a\psi(u,v)\big)b = a\big(\psi(u,v)b\big).$$

In other words, the image of ψ lies in the middle nucleus of $\mathcal{R}$.

9.10.12 Lemma (Equation 5, part 2). *The following hold for all $u \in J$:*

(a) $\pi_1(u)^\sigma = \pi(u) - \psi(u,u)$.

(b) $\pi_1(u)$ *lies in the middle nucleus of* $\mathcal{R}$.

Proof. We consider equation 5 in Figure 9.7 and put all variables except for u, r and s to zero. This yields

$$-\big(-\pi(\overline{u},r) - \psi\big(\varphi(\overline{u},-r^\sigma),\overline{u}\big)\big)s^\sigma = \Big(\big(-\pi(\overline{u},s) - \psi\big(\varphi(\overline{u},-s^\sigma),\overline{u}\big)\big)(-r^\sigma)\Big)^\sigma \\ - \psi\big(\varphi(u,r^\sigma),\varphi(\overline{u},-s^\sigma)\big).$$

We can simplify this equation to obtain

$$\pi(u,r)s^\sigma - r\psi(u,u)s^\sigma = r\pi(u,s)^\sigma + r\psi(u,u)s^\sigma - r\psi(u,u)s^\sigma,$$

or in other words,

$$\pi(u,r)s^\sigma - r\psi(u,u)s^\sigma = r\pi(u,s)^\sigma.$$

For $r := s := 1_{\mathcal{R}}$, this is exactly assertion (a). Using (a) and Lemma 9.10.7 (c), the equation that we have just established simplifies as follows:

$$\big(r\pi_1(u)\big)s^\sigma - r\psi(u,u)s^\sigma = r\big(s\pi_1(u)\big)^\sigma = r\big(\pi_1(u)s^\sigma\big) - r\psi(u,u)s^\sigma.$$

This implies that $\big(r\pi_1(u)\big)s^\sigma = r\big(\pi_1(u)s^\sigma\big)$. In other words, $\pi_1(u)$ lies in the middle nucleus, which finishes the proof of (b). □

9.10.13 Remark. Lemma 9.10.7 (c) together with Lemma 9.10.12 (a) yields that

$$\pi(u,a)^\sigma = \big(a\pi_1(u)\big)^\sigma = \pi_1(u)a^\sigma - \psi(u,u)a^\sigma$$

for all $u \in J$ and $a \in \mathcal{R}$.

Recall the definition of nuclei from Definition 5.1.9.

9.10.14 Lemma. *The ring $\mathcal{R}$ is weakly alternative and satisfies*

$$\mathrm{Nucl}(\mathcal{R}) = \mathrm{LNucl}(\mathcal{R}) = \mathrm{MNucl}(\mathcal{R}) = \mathrm{RNucl}(\mathcal{R}).$$

Further, the images of π_1 and ψ lie in $\mathrm{Nucl}(\mathcal{R})$ *and the involution σ is nuclear.*

Proof. For any $a \in \mathcal{R}$, we have $\pi_1(\mathrm{T}(1_{\mathcal{R}}, a)) = a + a^\sigma$ by Lemma 9.10.10 (c). In particular, $a + a^\sigma$ lies in the image of π_1, so it lies in the middle nucleus by Lemma 9.10.12 (b). By the same computation as in Lemma 8.3.14, this implies that

$$[d, a^\sigma, s] = -[d, a, s] \quad \text{for all } a, d, s \in \mathcal{R}. \tag{9.1}$$

Now let $a, d, s \in \mathcal{R}$ be arbitrary. On the one hand, Lemma 9.10.10 (c) yields that

$$\pi\big(\mathrm{T}(s, a), d\big) = (ds)a^\sigma (da)s^\sigma.$$

On the other hand, it follows from Lemma 9.10.7 (c) that

$$\pi\big(\mathrm{T}(s, a), d\big) = d\pi\big(\mathrm{T}(s, a), 1_{\mathcal{R}}\big) = d\big(sa^\sigma + as^\sigma\big).$$

Together, these two identities imply that $[d, s, a^\sigma] = -[d, a, s^\sigma]$. Invoking (9.1), we infer that $-[d, s^\sigma, a^\sigma] = [d, a^\sigma, s^\sigma]$. This says precisely that the conditions of Lemma 8.3.11 (b) are satisfied, so $\mathcal{R}$ is weakly alternative.

By Remark 8.2.8, it follows from the weak alternativity of $\mathcal{R}$ that nucleus, left nucleus, middle nucleus and right nucleus coincide. Since we have already seen in Remark 9.10.11, Lemma 9.10.12 (b) and (9.1) that the images of π_1 and ψ and all traces of $\mathcal{R}$ lie in the middle nucleus, the remaining assertions follow. □

9.10.15 Lemma (Equation 5, part 3). *We have $\pi\big(\varphi(u, s), d\big) = (ds^\sigma)\pi_1(u)s$ for all $u \in J$ and $d, s \in \mathcal{R}$.*

Proof. We put all variables in equation 5 except for u, d and s to zero. This yields

$$\begin{aligned} 0 = -\pi\big(\varphi(\overline{u}, -s^\sigma), d\big) &+ \Big(\big(-\pi(\overline{u}, s) - \psi(\varphi(\overline{u}, -s^\sigma), \overline{u})\big)\big(-(ds)^\sigma\big)\Big)^\sigma \\ &- \psi\Big(\varphi\big(\varphi(\overline{u}, -s^\sigma), -d^\sigma\big) \hat{+} \varphi\big(u, (ds)^\sigma\big), \varphi(\overline{u}, -s^\sigma)\Big). \end{aligned}$$

Using Lemmas 9.10.5 to 9.10.7 and also Lemma 9.10.10 (a), we observe that

$$\begin{aligned} &\Big(\big(-\pi(\overline{u}, s) - \psi(\varphi(\overline{u}, -s^\sigma), \overline{u})\big)\big(-(ds)^\sigma\big)\Big)^\sigma \\ &\qquad = -(ds)\Big(-\pi(u, s)^\sigma - \psi\big(\varphi(u, -s^\sigma), u\big)^\sigma\Big) \\ &\qquad = (ds)\Big(\pi(u, s)^\sigma - \psi\big(u, \varphi(u, -s^\sigma)\big)\Big) \\ &\qquad = (ds)\big(\pi(u, s)^\sigma + \psi\big(u, u\big)s^\sigma\big) = (ds)\pi(u, s)^\sigma + (ds)\psi(u, u)s^\sigma \end{aligned}$$

and

$$\begin{aligned} &\psi\Big(\varphi\big(\varphi(\overline{u}, -s^\sigma), -d^\sigma\big) \hat{+} \varphi\big(u, (ds)^\sigma\big), \varphi(\overline{u}, -s^\sigma)\Big) \\ &\qquad = \psi\Big(\varphi\big(\varphi(\overline{u}, -s^\sigma), -d^\sigma\big), \varphi(\overline{u}, -s^\sigma)\Big) + \psi\Big(\varphi\big(u, (ds)^\sigma\big), \varphi(\overline{u}, -s^\sigma)\Big) \\ &\qquad = d\psi\big(\varphi(\overline{u}, -s^\sigma), \overline{u}\big)s^\sigma - (ds)\psi(u, \overline{u})s^\sigma \\ &\qquad = -\big(ds\psi(u, u)\big)s^\sigma + (ds)\psi(u, u)s^\sigma. \end{aligned}$$

Therefore,

$$\begin{aligned}0 &= -\pi\big(\varphi(u,-s^\sigma),d\big) + (ds)\pi(u,s)^\sigma + (ds)\psi(u,u)s^\sigma \\ &\qquad + \big(ds\psi(u,u)\big)s^\sigma - (ds)\psi(u,u)s^\sigma \\ &= -\pi\big(\varphi(u,-s^\sigma),d\big) + (ds)\pi(u,s)^\sigma + \big(ds\psi(u,u)\big)s^\sigma.\end{aligned}$$

Since

$$\begin{aligned}(ds)\pi(u,s)^\sigma &= (ds)\pi_1(u)s^\sigma - (ds)\psi(u,u)s^\sigma \qquad \text{and} \\ (ds)\psi(u,u)s^\sigma &= \big(ds\psi(u,u)\big)s^\sigma\end{aligned}$$

by Remark 9.10.13 and Lemma 9.10.14, respectively, we infer that

$$\pi\big(\varphi(u,-s^\sigma),d\big) = (ds)\pi_1(u)s^\sigma.$$

Replacing s by $-s^\sigma$, the assertion follows. □

9.10.16 Proposition. *The ring $\mathcal{R}$ is alternative.*

Proof. Let $s,d \in \mathcal{R}$ and recall that the fixed element $v_0 \in J$ from Notation 9.9.1 satisfies $\pi_1(v_0) = 1_{\mathcal{R}}$ by Lemma 9.9.11. Thus on the one hand, Lemma 9.10.15 yields that

$$\pi\big(\varphi(v_0,s),d\big) = (ds^\sigma)\pi_1(v_0)s = (ds^\sigma)s.$$

On the other hand, and together with Lemma 9.10.7 (c), Lemma 9.10.15 implies that

$$\pi\big(\varphi(v_0,s),d\big) = d\big(\pi\big(\varphi(v_0,s),1_{\mathcal{R}}\big)\big) = d\big(s^\sigma\pi_1(v_0)s\big) = d(s^\sigma s).$$

Thus $[d,s^\sigma,s] = 0$. Since the involution σ is nuclear by Lemma 9.10.14, it follows that $[d,s,s] = 0$ for all $s,d \in \mathcal{R}$. By an application of Lemma 8.3.11 (c), we conclude that $\mathcal{R}$ is alternative. □

9.10.17 Remark. It is not clear that no further identities can be deduced from equation 5 in Figure 9.7. We verify this by simplifying this equation with our recently obtained knowledge. For the simplification of y_5, we begin with the first summand. Repeatedly applying Lemma 9.9.7 (c) as well as the identities from this section and Lemma 9.9.7 (b), we see that

$$\begin{aligned}&\pi\big(\varphi(\overline{u},-s^\sigma) \hat{+} \overline{v} \hat{+} \mathrm{T}(s,a^\sigma),d\big) \\ &\quad = \pi\big(\varphi(\overline{u},-s^\sigma) \hat{+} \overline{v},d\big) + \pi\big(\mathrm{T}(s,a^\sigma),d\big) + \psi\big(\varphi(\varphi(\overline{u},-s^\sigma) \hat{+} \overline{v},d^\sigma),\mathrm{T}(s,a^\sigma)\big) \\ &\quad = \pi\big(\varphi(\overline{u},-s^\sigma) \hat{+} \overline{v},d\big) + \pi\big(\mathrm{T}(s,a^\sigma),d\big) \\ &\quad = \pi\big(\varphi(\overline{u},-s^\sigma),d\big) + \pi\big(\overline{v},d\big) + \psi\big(\varphi(\varphi(\overline{u},-s^\sigma),d^\sigma),\overline{v}\big) + \pi\big(\mathrm{T}(s,a^\sigma),d\big) \\ &\quad = (ds)\pi_1(u)s^\sigma + d\pi_1(v) - ds\psi(u,v) + (ds)a + (da^\sigma)s^\sigma.\end{aligned}$$

Further,

$$\begin{aligned}&\big[\big(-\pi(\overline{u},s) - a^\sigma - \psi\big(\varphi(\overline{u},-s^\sigma) \hat{+} \overline{v},\overline{u}\big)\big)\big(-r^\sigma - (ds)^\sigma\big)\big]^\sigma \\ &\qquad = (r+ds)\big(\pi(u,s)^\sigma + a + \psi\big(\varphi(u,-s^\sigma) \hat{+} v,u\big)^\sigma\big) \\ &\qquad = (r+ds)\big(\pi_1(u)s^\sigma - \psi(u,u)s^\sigma + a - \psi\big(u,\varphi(u,-s^\sigma) \hat{+} v\big)\big) \\ &\qquad = (r+ds)\big(\pi_1(u)s^\sigma - \psi(u,u)s^\sigma + a + \psi(u,u)s^\sigma - \psi(u,v)\big) \\ &\qquad = (r+ds)\big(\pi_1(u)s^\sigma + a - \psi(u,v)\big).\end{aligned}$$

Finally,

$$\begin{aligned}
-\psi\big(C \hat{+} \overline{w} \hat{+} \mathrm{T}(-r-ds, b^\sigma), \varphi(\overline{u}, -s^\sigma) \hat{+} \overline{v} \hat{+} \mathrm{T}(s, a^\sigma)\big) \\
&= -\psi\big(C \hat{+} \overline{w}, \varphi(\overline{u}, -s^\sigma) \hat{+} \overline{v}\big) \\
&= -\psi(C,u)s^\sigma + \psi(C,v) + \psi(w,u)s^\sigma - \psi(w,v)
\end{aligned}$$

where, for any $x \in J$,

$$\begin{aligned}
\psi(C,x) &= \psi\Big(\varphi\big(\varphi(\overline{u}, -s^\sigma) \hat{+} \overline{v} \hat{+} \mathrm{T}(s, a^\sigma), -d^\sigma\big) \hat{+} \varphi\big(u, r^\sigma + (ds)^\sigma\big), x\Big) \\
&= -d\psi\big(\varphi(\overline{u}, -s^\sigma) \hat{+} \overline{v} \hat{+} \mathrm{T}(s, a^\sigma), x\big) + (r+ds)\psi(u,x) \\
&= -ds\psi(u,x) + d\psi(v,x) + (r+ds)\psi(u,x).
\end{aligned}$$

Altogether, we conclude that

$$\begin{aligned}
y_5 = &-(ds)\pi_1(u)s^\sigma - d\pi_1(v) + ds\psi(u,v) - (ds)a - (da^\sigma)s^\sigma - c^\sigma - b^\sigma s^\sigma \\
&+ r\big(\pi_1(u)s^\sigma + a - \psi(u,v)\big) + (ds)\big(\pi_1(u)s^\sigma + a - \psi(u,v)\big) \\
&+ (ds)\psi(u,u)s^\sigma - d\psi(v,u)s^\sigma - (r+ds)\psi(u,u)s^\sigma \\
&- ds\psi(u,v) + d\psi(v,v) + (r+ds)\psi(u,v) + \psi(w,u)s^\sigma - \psi(w,v).
\end{aligned}$$

Note that all the summands $(ds)\pi_1(u)s^\sigma$, $(ds)a$, $r\psi(u,v)$ and $(ds)\psi(u,u)s^\sigma$ appear once with a positive and once with a negative sign in the expression of y_5, and $(ds)\psi(u,v)$ even appears twice with a positive and twice with a negative sign. Thus we have

$$\begin{aligned}
y_5 = &-d\pi_1(v) - (da^\sigma)s^\sigma - c^\sigma - b^\sigma s^\sigma + r\big(\pi_1(u)s^\sigma + a\big) \\
&- d\psi(v,u)s^\sigma - r\psi(u,u)s^\sigma + d\psi(v,v) + \psi(w,u)s^\sigma - \psi(w,v).
\end{aligned}$$

For the simplification of x_5, we first observe that

$$\begin{aligned}
\psi(B, \overline{u}) &= \psi\big(\varphi(\overline{u}, -r^\sigma) \hat{+} \varphi(\overline{v}, -d^\sigma) \hat{+} \overline{w} \hat{+} \mathrm{T}(d, c^\sigma), \overline{u}\big) \\
&= \psi\big(\varphi(u, -r^\sigma) \hat{+} \varphi(v, -d^\sigma) \hat{+} w, u\big) \\
&= -r\psi(u,u) - d\psi(v,u) + \psi(w,u).
\end{aligned}$$

Therefore,

$$\begin{aligned}
x_5 = &-d\pi_1(v) - c^\sigma + d\psi(v,v) - \psi(w,v) + ra + r\pi_1(u)s^\sigma - b^\sigma s^\sigma - (da^\sigma)s^\sigma \\
&- r\psi(u,u)s^\sigma - d\psi(v,u)s^\sigma + \psi(w,u)s^\sigma.
\end{aligned}$$

It is now easy to verify that the equation $x_5 = y_5$ is equivalent to $0 = 0$. Thus there exist no other identities which can be deduced from equation 5.

9.10.18 Lemma (Equation 4, part 1). *The following hold for all $u \in J$ and $a, b, c \in \mathcal{R}$:*

(a) $\mathrm{T}(ab, c) = \mathrm{T}(a, cb^\sigma)$.

(b) $\mathrm{T}(a, b) = \mathrm{T}(b, a)$.

(c) $\varphi(u, -a) = \overline{\varphi(u, a)}$.

(d) $\varphi\big(\mathrm{T}(a,b), c\big) = \mathrm{T}(c^\sigma b, c^\sigma a)$.

(e) $u \hat{+} v = v \hat{+} u \hat{+} \mathrm{T}\big(1_{\mathcal{R}}, \psi(u,v)\big)$.

Proof. We consider equation 4 in Figure 9.7. Putting all variables except for b, d and s to zero, we obtain

$$0_J = \overline{\mathrm{T}(-ds, b^\sigma)} \hat{+} \overline{\mathrm{T}(d, (sb)^\sigma)}.$$

In other words, $\mathrm{T}(ds, b^\sigma) = \mathrm{T}(d, b^\sigma s^\sigma)$. This proves (a).

Replacing all variables in equation 4 except for a, d and r by zero, we see that

$$\overline{\mathrm{T}(r, -(ad^\sigma)^\sigma)} = \overline{\mathrm{T}(d, -((-a^\sigma)(-r^\sigma))^\sigma)}.$$

This is equivalent to the identity $\mathrm{T}(r, da^\sigma) = \mathrm{T}(d, ra)$, which for $a := 1_{\mathcal{R}}$ implies that (b) holds.

Now we substitute each variable in equation 4 except for u and r by zero. In this way, we obtain

$$\overline{\varphi(\overline{u}, -r^\sigma)} = \overline{\varphi(u, r^\sigma)}.$$

Since

$$\overline{\varphi(\overline{u}, -r^\sigma)} = \overline{\overline{\varphi(u, -r^\sigma)}}$$

by Lemma 9.10.5, assertion (c) follows.

Putting all variables in equation 4 except for a, d and s to zero, we obtain

$$0_J = \overline{\varphi\big(\mathrm{T}(s, a^\sigma), -d^\sigma\big)} \mathbin{\hat{+}} \overline{\mathrm{T}\big(d, -\big((-a^\sigma)(-(ds)^\sigma)\big)^\sigma\big)}.$$

Note that, by (c) and Lemma 9.10.5, we have

$$\varphi\big(\mathrm{T}(s, a^\sigma), -d^\sigma\big) = \overline{\varphi\big(\mathrm{T}(s, a^\sigma), d^\sigma\big)} = \varphi\big(\overline{\mathrm{T}(s, a^\sigma)}, d^\sigma\big) = \varphi\big(\mathrm{T}(s, a^\sigma), d^\sigma\big).$$

Hence

$$\varphi\big(\mathrm{T}(s, a^\sigma), d^\sigma\big) = \mathrm{T}\big(d, (ds)a\big),$$

which together with (a) shows that (d) holds.

Next we put all variables in equation 4 except for u, v, d and r to zero. This yields

$$\overline{\varphi(\overline{u}, -r^\sigma)} \mathbin{\hat{+}} \overline{\varphi(\overline{v}, -d^\sigma)} = \overline{\varphi(\overline{v}, -d^\sigma)} \mathbin{\hat{+}} \overline{\varphi(u, r^\sigma)} \mathbin{\hat{+}} \overline{\mathrm{T}\big(d, -\big(\big(-\psi(\overline{v}, \overline{u})\big)(-r^\sigma)\big)^\sigma\big)}.$$

Using Lemmas 9.10.5 and 9.10.6, we can remove most occurrences of $\overline{\cdot}$ in this equation to obtain

$$\varphi(u, -r^\sigma) \mathbin{\hat{+}} \varphi(v, -d^\sigma) = \varphi(v, -d^\sigma) \mathbin{\hat{+}} \overline{\varphi(u, r^\sigma)} \mathbin{\hat{+}} \mathrm{T}(d, -r\psi(v, u)^\sigma).$$

Putting $d := r := -1_{\mathcal{R}}$ and using (c) and Lemma 9.9.11, we infer that

$$u \mathbin{\hat{+}} v = v \mathbin{\hat{+}} u \mathbin{\hat{+}} \mathrm{T}\big(1_{\mathcal{R}}, \psi(u, v)\big),$$

which is exactly the assertion of (e). □

Motivated by Lemma 9.10.18 (a) and 9.10.18 (b), we make the following definition, which is similar to Definition 9.10.8.

9.10.19 Definition. We define a map $\mathrm{T}_1 \colon \mathcal{R} \to J, a \mapsto \mathrm{T}(a, 1_{\mathcal{R}}) = \mathrm{T}(1_{\mathcal{R}}, a)$.

9.10.20 Remark. Let $a, b, c \in \mathcal{R}$. Then Lemma 9.10.18 yields the following properties of the map T_1:

$$\begin{aligned}
\varphi\big(\mathrm{T}_1(a), c\big) &= \mathrm{T}(c^\sigma, c^\sigma a) = \mathrm{T}(1_{\mathcal{R}}, c^\sigma ac) = \mathrm{T}_1(c^\sigma ac),\\
\mathrm{T}_1(a) &= \mathrm{T}(1_{\mathcal{R}}a, 1_{\mathcal{R}}) = \mathrm{T}(1_{\mathcal{R}}, 1_{\mathcal{R}}a^\sigma) = \mathrm{T}_1(a^\sigma),\\
\mathrm{T}_1\big((ab)c\big) &= \mathrm{T}\big((ab)c, 1_{\mathcal{R}}\big) = \mathrm{T}(ab, c^\sigma) = \mathrm{T}(a, c^\sigma b^\sigma) = \mathrm{T}\big(1_{\mathcal{R}}, (c^\sigma b^\sigma)a^\sigma\big)\\
&= \mathrm{T}_1\big((c^\sigma b^\sigma)a^\sigma\big) = \mathrm{T}_1\big(a(bc)\big).
\end{aligned}$$

Further, Lemma 9.10.10 (c) implies that

$$\pi_1\big(\mathrm{T}_1(a)\big) = \pi\big(\mathrm{T}(a, 1_{\mathcal{R}}), 1_{\mathcal{R}}\big) = a + a^\sigma.$$

This last property is in some sense dual to the following Lemma 9.10.21 (a).

9.10.21 Lemma (Equation 4, part 2). *The following identities hold for all $u \in J$ and all $s, d \in \mathcal{R}$:*

(a) $\mathrm{T}_1\big(\pi_1(u)\big) = u + \overline{u} = \overline{u} + u$.

(b) $\varphi\big(\varphi(u,s), d\big) = \varphi(u, sd)$.

Proof. In this proof, we will only consider equation 4 in Figure 9.7 with all variables except for u, d and s replaced by zero. This produces the following identity:

$$\begin{aligned} 0 = {} & \overline{\varphi\big(\varphi(\overline{u}, -s^\sigma), -d^\sigma\big)} \mathbin{\hat{+}} \overline{\varphi\big(u, (ds)^\sigma\big)} \\ & \mathbin{\hat{+}} \overline{\mathrm{T}\Big(d, -\big[\big(-\pi(\overline{u}, s) - \psi\big(\varphi(\overline{u}, -s^\sigma), \overline{u}\big)\big)\big(-(ds)^\sigma\big)\big]^\sigma\Big)} \end{aligned}$$

Using the previously computed identities, we can simplify this equation to obtain

$$0 = \varphi\big(\varphi(u, -s^\sigma), -d^\sigma\big) \mathbin{\hat{+}} \overline{\varphi\big(u, s^\sigma d^\sigma\big)} \mathbin{\hat{+}} \mathrm{T}\Big(d, (ds)\big(-\pi(u,s)^\sigma + \big(s\psi(u,u)\big)^\sigma\big)\Big)$$

where

$$\begin{aligned} & \mathrm{T}\Big(d, (ds)\big(-\pi(u,s)^\sigma + \big(s\psi(u,u)\big)^\sigma\big)\Big) \\ & \qquad = \mathrm{T}\Big(d, (ds)\big(-\pi_1(u)s^\sigma + \psi(u,u)s^\sigma - \psi(u,u)s^\sigma\big)\Big) \\ & \qquad = \mathrm{T}\big(1_{\mathcal{R}}, -ds\pi_1(u)s^\sigma d^\sigma\big) = \mathrm{T}_1\big(-ds\pi_1(u)s^\sigma d^\sigma\big). \end{aligned}$$

Replacing s and d by $-s^\sigma$ and $-d^\sigma$, respectively, we conclude that

$$0 = \varphi\big(\varphi(u,s), d\big) \mathbin{\hat{+}} \overline{\varphi(u, sd)} \mathbin{\hat{+}} \mathrm{T}_1\big(-d^\sigma s^\sigma \pi_1(u) sd\big). \tag{9.2}$$

Putting $s := d := 1_{\mathcal{R}}$, we obtain $\mathrm{T}_1(\pi_1(u)) = u \mathbin{\hat{+}} \overline{u}$. Since $\pi_1(u) = \pi_1(\overline{u})$ by Lemma 9.10.6, it follows that $\mathrm{T}_1(\pi_1(u)) = \overline{u} \mathbin{\hat{+}} u$ holds as well. This finishes the proof of (a).

Using (a) and Lemma 9.10.15, we see that

$$\begin{aligned} \mathrm{T}_1\big(-d^\sigma s^\sigma \pi_1(u) sd\big) &= \hat{-}\, \mathrm{T}_1\big(\pi_1\big(\varphi(u,sd)\big)\big) = \hat{-}\big(\varphi(u,sd) \mathbin{\hat{+}} \overline{\varphi(u,sd)}\big) \\ &= \big(\hat{-}\overline{\varphi(u,sd)}\big) \mathbin{\hat{+}} \big(\hat{-}\varphi(u,sd)\big). \end{aligned}$$

With this information, (9.2) becomes

$$0 = \varphi\big(\varphi(u,s), d\big) \mathbin{\hat{+}} \big(\hat{-}\varphi(u,sd)\big),$$

which proves (b). □

9.10.22 Remark. Similarly as in Remark 9.10.17, we will now show that no further identities can be deduced from equation 4 in Figure 9.7. The right-hand side of this equation can be written as

$$\begin{aligned} y_4 = {} & \varphi(u, s^\sigma d^\sigma) \mathbin{\hat{+}} \varphi(v, -d^\sigma) \mathbin{\hat{+}} \mathrm{T}_1(dsad^\sigma) \mathbin{\hat{+}} \overline{\varphi(u, r^\sigma + s^\sigma d^\sigma)} \\ & \mathbin{\hat{+}} w \mathbin{\hat{+}} \mathrm{T}_1(-rb - dsb) \mathbin{\hat{+}} \mathrm{T}_1(dc) \mathbin{\hat{+}} \mathrm{T}_1(dsb) \\ & \mathbin{\hat{+}} \mathrm{T}_1\Big(d, (r + ds)\big[-\pi(u,s)^\sigma - a - \psi\big(\varphi(u, -s^\sigma) \mathbin{\hat{+}} v, u\big)^\sigma\big]\Big). \end{aligned}$$

By Lemma 9.9.7 (d),

$$\overline{\varphi(u, r^\sigma + s^\sigma d^\sigma)} = \overline{\varphi(u, r^\sigma)} \mathbin{\hat{+}} \overline{\varphi(u, s^\sigma d^\sigma)} \mathbin{\hat{+}} \mathrm{T}_1\big(ds\pi_1(u) r^\sigma\big).$$

Further,

$$\begin{aligned}
&\mathrm{T}_1\Big(d, (r+ds)\big[-\pi(u,s)^\sigma - a - \psi\big(\varphi(u,-s^\sigma) \dotplus v, u\big)^\sigma\big]\Big)\\
&\quad= \mathrm{T}_1\Big(d, (r+ds)\big[-\pi_1(u)^\sigma s^\sigma - a + \big(s\psi(u,u)\big)^\sigma - \psi(v,u)^\sigma\big]\Big)\\
&\quad= \mathrm{T}_1\Big(d, (r+ds)\big[-\pi_1(u)^\sigma s^\sigma - a - \psi(u,u)s^\sigma + \psi(u,v)\big]\Big)\\
&\quad= \mathrm{T}_1\big(-r\pi_1(u)^\sigma s^\sigma d^\sigma - rad^\sigma - r\psi(u,u)s^\sigma d^\sigma + r\psi(u,v)d^\sigma\big)\\
&\qquad\quad \dotplus \mathrm{T}_1\big(-ds\pi_1(u)^\sigma s^\sigma d^\sigma - dsad^\sigma - ds\psi(u,u)s^\sigma d^\sigma + ds\psi(u,v)d^\sigma\big).
\end{aligned}$$

We conclude that

$$\begin{aligned}
y_4 = {}& \varphi(u, s^\sigma d^\sigma) \dotplus \varphi(v, -d^\sigma) \dotplus \mathrm{T}_1(dsad^\sigma) \dotplus \overline{\varphi(u, r^\sigma)} \dotplus \overline{\varphi(u, s^\sigma d^\sigma)}\\
&\dotplus \mathrm{T}_1\big(ds\pi_1(u)r^\sigma\big) \dotplus w \dotplus \mathrm{T}_1(-rb - dsb) \dotplus \mathrm{T}_1(dc) \dotplus \mathrm{T}_1(dsb)\\
&\mathrm{T}_1\big(-r\pi_1(u)^\sigma s^\sigma d^\sigma - rad^\sigma - r\psi(u,u)s^\sigma d^\sigma + r\psi(u,v)d^\sigma\big)\\
&\dotplus \mathrm{T}_1\big(-ds\pi_1(u)^\sigma s^\sigma d^\sigma - dsad^\sigma - ds\psi(u,u)s^\sigma d^\sigma + ds\psi(u,v)d^\sigma\big).
\end{aligned}$$

Using that the image of T_1 lies in the center of J by Lemma 9.9.7 (b), that the summands $\mathrm{T}_1(dsad^\sigma)$ and $\mathrm{T}_1(dsb)$ appear once with a positive and once with a negative sign, and that

$$\mathrm{T}_1\big(-r\pi_1(u)^\sigma s^\sigma d^\sigma\big) = \mathrm{T}_1\big(-\big(r\pi_1(u)^\sigma s^\sigma d^\sigma\big)^\sigma\big) = -\,\mathrm{T}_1\big(ds\pi_1(u)r^\sigma\big),$$

we can simplify this expression of y_4 as follows:

$$\begin{aligned}
y_4 = {}& \varphi(u, s^\sigma d^\sigma) \dotplus \varphi(v, -d^\sigma) \dotplus \overline{\varphi(u, r^\sigma)} \dotplus \overline{\varphi(u, s^\sigma d^\sigma)} \dotplus w \dotplus \mathrm{T}_1(-rb)\\
&\dotplus \mathrm{T}_1(dc) \dotplus \mathrm{T}_1(-rad^\sigma) \dotplus \mathrm{T}_1\big(-r\psi(u,u)s^\sigma d^\sigma\big) \dotplus \mathrm{T}_1\big(r\psi(u,v)d^\sigma\big)\\
&\dotplus \mathrm{T}_1\big(-ds\pi_1(u)s^\sigma d^\sigma\big) \dotplus \mathrm{T}_1\big(-ds\psi(u,u)s^\sigma d^\sigma\big) \dotplus \mathrm{T}_1\big(ds\psi(u,v)d^\sigma\big).
\end{aligned}$$

The left-hand side of equation 4 is easier to simplify:

$$x_4 = \varphi(u, -r^\sigma) \dotplus \varphi(v, -d^\sigma) \dotplus w \dotplus \mathrm{T}_1(dc) \dotplus \mathrm{T}_1(-rb) \dotplus \mathrm{T}_1(-rad^\sigma).$$

Subtracting $w \dotplus \mathrm{T}_1(dc) \dotplus \mathrm{T}_1(-rb) \dotplus \mathrm{T}_1(-rad^\sigma)$ from the right side from both x_4 and y_4, we see that equation 4 is equivalent to $x_4' = y_4'$ where

$$\begin{aligned}
x_4' &= \varphi(u, -r^\sigma) \dotplus \varphi(v, -d^\sigma),\\
y_4' &= \varphi(u, s^\sigma d^\sigma) \dotplus \varphi(v, -d^\sigma) \dotplus \overline{\varphi(u, r^\sigma)} \dotplus \overline{\varphi(u, s^\sigma d^\sigma)} \dotplus \mathrm{T}_1\big(-r\psi(u,u)s^\sigma d^\sigma\big)\\
&\quad \dotplus \mathrm{T}_1\big(r\psi(u,v)d^\sigma\big) \dotplus \mathrm{T}_1\big(-ds\pi_1(u)s^\sigma d^\sigma\big)\\
&\quad \dotplus \mathrm{T}_1\big(-ds\psi(u,u)s^\sigma d^\sigma\big) \dotplus \mathrm{T}_1\big(ds\psi(u,v)d^\sigma\big).
\end{aligned}$$

Observe that by Lemma 9.10.15 and Lemma 9.10.21 (a),

$$\mathrm{T}_1\big(-ds\pi_1(u)s^\sigma d^\sigma\big) = \mathrm{T}_1\big(-\pi_1\big(\varphi(u, s^\sigma d^\sigma)\big)\big) = \big(\mathbin{\hat{-}}\overline{\varphi(u, s^\sigma d^\sigma)}\big) \dotplus \big(\mathbin{\hat{-}}\varphi(u, s^\sigma d^\sigma)\big).$$

Thus

$$\begin{aligned}
y_4' = {}& \varphi(u, s^\sigma d^\sigma) \dotplus \varphi(v, -d^\sigma) \dotplus \overline{\varphi(u, r^\sigma)} \dotplus \varphi(\mathbin{\hat{-}}u, s^\sigma d^\sigma) \dotplus \mathrm{T}_1\big(-r\psi(u,u)s^\sigma d^\sigma\big)\\
&\dotplus \mathrm{T}_1\big(r\psi(u,v)d^\sigma\big) \dotplus \mathrm{T}_1\big(-ds\psi(u,u)s^\sigma d^\sigma\big) \dotplus \mathrm{T}_1\big(ds\psi(u,v)d^\sigma\big).
\end{aligned}$$

Further, by Lemma 9.10.18 (e),

$$\begin{aligned}
\varphi(u, -r^\sigma) \dotplus \varphi(v, -d^\sigma) &= \varphi(v, -d^\sigma) \dotplus \varphi(u, -r^\sigma) \dotplus \mathrm{T}_1\big(\psi\big(\varphi(u, -r^\sigma), \varphi(v, -d^\sigma)\big)\big)\\
&= \varphi(v, -d^\sigma) \dotplus \varphi(u, -r^\sigma) \dotplus \mathrm{T}_1\big(r\psi(u,v)d^\sigma\big),\\
\varphi(u, s^\sigma d^\sigma) \dotplus \varphi(v, -d^\sigma) &= \varphi(v, -d^\sigma) \dotplus \varphi(u, s^\sigma d^\sigma) \dotplus \mathrm{T}_1\big(-ds\psi(u,v)d^\sigma\big).
\end{aligned}$$

Therefore, subtracting $\varphi(v, -d^\sigma) \dotplus \mathrm{T}_1(r\psi(u,v)d^\sigma)$ from the left side of the equation $x_4' = y_4'$, we infer that equation 4 is equivalent to $x_4'' = y_4''$ where $x_4'' = \varphi(u, -r^\sigma)$ and

$$y_4' = \varphi(u, s^\sigma d^\sigma) \dotplus \overline{\varphi(u, r^\sigma)} \dotplus \varphi(\hat{-}u, s^\sigma d^\sigma) \dotplus \mathrm{T}_1\big(-r\psi(u,u)s^\sigma d^\sigma\big) \\ \dotplus \mathrm{T}_1\big(-ds\psi(u,u)s^\sigma d^\sigma\big).$$

Again by Lemma 9.10.18 (e), we have

$$\varphi(u, s^\sigma d^\sigma) \dotplus \overline{\varphi(u, r^\sigma)} = \overline{\varphi(u, r^\sigma)} \dotplus \varphi(u, s^\sigma d^\sigma) \dotplus \mathrm{T}_1\big(ds\psi(u, \overline{u})r^\sigma\big)$$

where

$$\mathrm{T}_1\big(ds\psi(u,\overline{u})r^\sigma\big) = \mathrm{T}_1\big(-ds\psi(u,u)r^\sigma\big) = \mathrm{T}_1\big(-r\psi(u,u)^\sigma s^\sigma d^\sigma\big) \\ = \mathrm{T}_1\big(r\psi(u,u)s^\sigma d^\sigma\big).$$

Therefore, $y_4'' = \overline{\varphi(u, r^\sigma)}$. Since $\overline{\varphi(u, r^\sigma)} = \varphi(u, -r^\sigma)$ by Lemma 9.10.18 (c), it follows that equation 4 is equivalent to $0_J = 0_J$. Thus our computations are finished.

9.10.23 Summary. It follows from the collection of identities in this section and the previous one that $\mathcal{R}$ is an alternative ring, σ is a nuclear involution on $\mathcal{R}$, $(M, \varphi, \pi_1, \mathrm{T}_1, \psi)$ is a Jordan module over $(\mathcal{R}, \mathcal{R}_0, \sigma)$ and $\overline{\cdot}$ is the Jordan module involution on M. If $n \geq 4$, then we know in addition that $\mathcal{R}$ is associative (see Remark 9.10.3). Further, the computations in Remark 8.6.2 show that every Jordan module satisfies all the identities that we have collected in this section and the previous one. In other words, Jordan modules are exactly the algebraic structures which are characterised by the results of the blueprint computations. It remains to compute the commutator relations which are not covered by Lemma 9.9.8 and Remark 9.9.9, which is a straightforward task.

9.10.24 Proposition. $(\theta_\alpha)_{\alpha \in BC_n}$ *is a coordinatisation of* G *by* $(M, \varphi, \pi_1, \mathrm{T}_1, \psi)$ *with standard signs.*

Proof. We already know from Lemma 9.9.8 and Remark 9.9.9 that the first set of the standard commutator relations in Definition 9.6.2 holds and that the first relation in each of the other sets holds if $i < j$. We begin with the second set of relations. We already know that

$$[\theta_{e_i - e_j}(a), \theta_{e_i + e_j}(b)] = \theta_{e_i}\big(\mathrm{T}_1(ab^\sigma)\big)$$

holds for $i < j$. Conjugating this identity by w_{ij}, we obtain that

$$[\theta_{e_j - e_i}(-a), \theta_{e_i + e_j}(-b^\sigma)] = \theta_{e_j}\big(\mathrm{T}_1(ab^\sigma)\big).$$

It follows that the first relation in the second set is true. Conjugating this identity by w_i, we infer that

$$[\theta_{-e_i - e_j}(\delta^\sigma_{i<j}(a)), \theta_{e_j - e_i}(-\delta^\sigma_{i<j}(b))] = \theta_{-e_i}\big(\mathrm{T}_1(a\delta^\sigma_{i<j}(b))\big),$$

which proves the second relation.

Now we turn to the third set of equations. We already know that

$$[\theta_{e_j}(v), \theta_{e_i - e_j}(a)] = \theta_{e_i}\big(\varphi(v, -a^\sigma)\big)\theta_{e_i + e_j}\big(-\pi(v, a)\big).$$

It follows from this equation by conjugation with w_{ji} that

$$[\theta_{e_i}(v), \theta_{e_j - e_i}(-a)] = \theta_{e_j}\big(\varphi(v, a^\sigma)\big)\theta_{e_i + e_j}\big(\big(a\pi_1(v)\big)^\sigma\big).$$

We infer that the first equation holds. Conjugating it by w_i, we see that

$$[\theta_{e_j}(v), \theta_{-e_i - e_j}\big(\delta^\sigma_{i<j}(a)\big)] = \theta_{-e_i}\big(\varphi(v, -a^\sigma)\big)\theta_{e_j - e_i}\big(\pi_1(v)^\sigma a^\sigma\big).$$

The second equation follows. Conjugating the first equation by w_j, we compute that

$$\big[\theta_{-e_j}(v), \theta_{e_i+e_j}\big(\delta^{\sigma}_{i>j}(a)\big)\big] = \theta_{e_i}\big(\varphi(v, -a^{\sigma})\big)\theta_{e_i-e_j}\big(a\pi_1(v)\big).$$

The fourth equation follows from this one by interchanging the roles of i and j. Conjugating the fourth equation by w_j, we obtain

$$\big[\theta_{-e_i}(v), \theta_{e_i-e_j}\big(-\delta^{\sigma}_{i>j}(a)\big)\big] = \theta_{-e_j}\big(\varphi(v, -\delta^{\sigma}_{i>j}(a))\big)\theta_{-e_i-e_j}\big(\delta^{\sigma}_{i>j}\big(\delta^{\sigma}_{i<j}(a)\pi_1(u)\big)\big).$$

This proves the third equation.

Finally, we turn to the last set of equations. Once again, we already know that

$$[\theta_{e_i}(u), \theta_{e_j}(v)] = \theta_{e_i+e_j}\big(\psi(u, v)\big).$$

Conjugating this identity by w_{ij}, we obtain that

$$[\theta_{e_i}(u), \theta_{e_j}(\overline{v})] = \theta_{e_i+e_j}\big(-\psi(u, v)^{\sigma}\big) = \theta_{e_i+e_j}\big(\psi(v, u)\big).$$

Thus the first relation holds. Conjugating this relation by w_j, it follows that

$$[\theta_{e_i}(u), \theta_{-e_j}(v)] = \begin{cases} \theta_{e_i-e_j}\big(-\psi(u, v)\big) & \text{if } i < j \\ \theta_{e_i-e_j}\big(\psi(v, u)^{\sigma}\big) = \theta_{e_i-e_j}\big(-\psi(u, v)\big) & \text{if } i > j, \end{cases}$$

proving the second identity. Finally, the last identity follows from the second one by conjugation with w_i. □

9.10.25 Lemma. *We have $v_{-1} = v_1 = \overline{v_0}$.*

Proof. By Proposition 9.10.24, we can apply Lemma 9.6.4. This yields that

$$v_{-1} = v_1 = \varphi\big(\overline{v_0}, \pi_1(v_0)^{-\sigma}\big).$$

Since $\pi_1(v_0) = 1_{\mathcal{R}}$ by Lemma 9.9.11, the assertion follows. □

We can now state the main result of this chapter. Recall the near-classification results for Jordan modules in Theorems 8.7.20 to 8.7.22.

9.10.26 Theorem (Coordinatisation theorem for BC_n). *Let $n \in \mathbb{N}_{\geq 3}$ and let G be a group with a crystallographic BC_n-grading $(U_\alpha)_{\alpha \in BC_n}$. Then there exist an alternative ring $\mathcal{R}$ with nuclear involution σ and a Jordan module $\mathcal{J} = (J, \varphi, \pi_1, \mathrm{T}_1, \psi)$ over $(\mathcal{R}, \sigma)$ such that G is coordinatised by $\mathcal{J}$ with standard signs. The ring $\mathcal{R}$ must be associative if $n \geq 4$ and the Jordan module $\mathcal{J}$ must be of type C if $(U_\alpha)_{\alpha \in BC_n}$ arises from a crystallographic C_n-grading as in Remark 9.2.6. Further, if we fix a Δ-system of Weyl elements in G, then we can choose the root isomorphisms $(\theta_\alpha)_{\alpha \in BC_n}$ so that $w_\delta = \theta_{-\delta}(-1_{\mathcal{R}})\theta_\delta(1_{\mathcal{R}})\theta_{-\delta}(-1_{\mathcal{R}})$ for all long simple roots δ and $w_\delta = \theta_{-\delta}(\overline{v_0})\theta_\delta(v_0)\theta_{-\delta}(\overline{v_0})$ for the short simple root δ where v_0 is some element of J with $\pi_1(v_0) = 1$ and $\overline{\cdot}$ denotes the Jordan module involution.*

Summary of the proof. Choose the standard rescaled root base Δ of BC_n and a Δ-system $(w_\delta)_{\delta \in \Delta}$ of Weyl elements. Denote by (A, η, B, μ) the standard admissible partial twisting system for G, as in Definition 9.7.2. Then by Proposition 9.8.4, there exist groups $(\mathcal{R}, +)$ and $(J, \hat{+})$ on which $A \times B$ acts and a parametrisation $(\theta_\alpha)_{\alpha \in BC_n}$ of G by $(A \times B, J, \mathcal{R})$ with respect to $(w_\delta)_{\delta \in \Delta}$ and $\eta \times \mu$. We can define commutation maps as in Definition 9.9.5. By Summary 9.10.23, these maps equip $\mathcal{R}$ with the structure of an alternative ring with nuclear involution (which must be associative if $n \geq 4$) and the group J with the structure of a Jordan module $\mathcal{J}$. By Proposition 9.10.24, $(\theta_\alpha)_{\alpha \in BC_n}$ is a coordinatisation of G by $\mathcal{J}$ with standard signs. By Lemmas 9.9.10 and 9.10.25, the Weyl elements $(w_\delta)_{\delta \in \Delta}$ have the desired form.

If $(U_\alpha)_{\alpha \in BC_n}$ arises from a crystallographic C_n-grading, then pairs of orthogonal short root groups commute (see Remark 9.2.7). This says precisely that $\psi = 0$, or in other words, that $\mathcal{G}$ is of type C. $\square$

Chapter 10

Root Gradings of Type F

In this final chapter, we investigate F_4-graded groups. Our goal is to show that each such group is coordinatised by a multiplicative conic alternative algebra $\mathcal{A}$ over a commutative associative ring $\Bbbk$. These objects should be thought of as suitable generalisations of composition algebras. If the base ring $\Bbbk$ is a field of characteristic not 2 and the norm function on $\mathcal{A}$ is anisotropic, then $\mathcal{A}$ is a composition division algebra in the classical sense (Lemma 10.2.17). These assumptions are always satisfied for multiplicative conic alternative algebras which coordinatise RGD-systems of type F_4.

While an explicit standard representation for the root system F_4 is available, it is much less practical to work with than the standard representations for root systems of types A, B, C, BC and D. This makes it much more difficult to concisely write down all commutator relations in F_4-graded groups. For this reason, we will only specify the commutator relations which can be seen in some (standard) root base. All other pairs of non-proportional roots adhere to commutator relations of the same form, but with twisted signs.

Recall that the main work in the previous chapters consisted of rank-2 and rank-3 computations. Since every proper root subsystem of F_4 is of type A, B or C (or a product of such root systems), most of the necessary computations have already been performed in the previous chapters. In fact, we can directly apply the coordinatisation theorems for B_3-graded groups and C_3-graded groups to certain subgroups G_B and G_C of an arbitrary F_4-graded group G. All that remains to do is to investigate the "overlap" of G_B and G_C, by which we mean the root groups which are contained in both groups. By comparing the commutator relations of G_B and G_C on these root groups, we obtain additional identities which turn the coordinatising structure of the short root groups into a multiplicative conic alternative algebra.

The outline of this chapter is as follows. We introduce conic algebras and related algebraic structures in sections 10.1 and 10.2 and the root system F_4 in section 10.3. In section 10.4, we construct an F_4-graded group for each multiplicative conic algebra which is associative, commutative and faithful. We will also define the notion of coordinatisations with standard signs and the standard parity maps in this section. In section 10.5, we perform a few computations which are specific to F_4-gradings. In section 10.6, we introduce the standard partial twisting system for F_4-graded groups and show that every F_4-graded group has a parametrisation by a pair of abelian groups $(\mathcal{A}, \Bbbk)$. In section 10.7, we use the coordinatisation theorems for B_3 and C_3 to equip $\Bbbk$ with the structure of an associative commutative ring and $\mathcal{A}$ with the structure of a multiplicative conic alternative algebra over $\Bbbk$. We state this main result in Theorem 10.7.8.

10.1 Conic Algebras and Related Structures

10.1.1 Notation for this section. We denote by $\Bbbk$ an arbitrary commutative associative ring. Unless otherwise specified, all modules and algebras are defined over $\Bbbk$.

10.1.A Algebras and Scalar Involutions

Before we can define conic algebras, we should clarify what an algebra is. We will also introduce the concept of algebras with scalar involutions, which are closely related to conic algebras. (For a precise formulation of this statement, see Remark 10.1.20.) While the notion of scalar involutions will not, strictly speaking, be necessary to study conic algebras, it fits nicely into the framework of nuclear and central involutions that we have introduced in section 8.3.

There are two equivalent ways of describing algebras, both of which will be useful in the sequel: As rings equipped with a structural homomorphism, and as modules equipped with a bilinear multiplication.

10.1.2 Definition (Algebra). A *$\Bbbk$-algebra* is a pair consisting of a ring $\mathcal{A}$ and a homomorphism $\Bbbk \to \mathrm{Z}(\mathcal{A})$, called the *structural homomorphism of $\mathcal{A}$* and usually denoted by $\mathrm{str}_{\mathcal{A}}$. We will often simply call $\mathcal{A}$ a $\Bbbk$-algebra, leaving the structural homomorphism implicit. The elements of $\mathcal{A}$ which lie in the image of $\mathrm{str}_{\mathcal{A}}$ are called *scalars*. Further, an algebra is called *faithful* if its structural homomorphism is injective.

Recall from Definition 5.1.9 that elements in the center of a ring are not only assumed to commute with all elements of the ring, but also to lie in the nucleus.

10.1.3 Remark (Algebras as modules). Given a $\Bbbk$-algebra $(\mathcal{A}, \mathrm{str}_{\mathcal{A}})$, we can define a right $\Bbbk$-module structure on $\mathcal{A}$ by

$$\star\colon \mathcal{A} \times \Bbbk \to \mathcal{A}, (a, \lambda) \mapsto \mathrm{str}_{\mathcal{A}}(\lambda)a = a\,\mathrm{str}_{\mathcal{A}}(\lambda).$$

Since $\Bbbk$ is commutative, we can also consider $\mathcal{A}$ as a left $\Bbbk$-module with the same scalar multiplication, and we well use both notations depending on the circumstances. We will always use the symbol $\star$ to denote the scalar multiplication on a $\Bbbk$-algebra.

In fact, we could define a $\Bbbk$-algebra to be a ring $\mathcal{A}$ together with a $\Bbbk$-module structure $\star$ on $\mathcal{A}$ such that the ring multiplication of $\mathcal{A}$ is $\Bbbk$-bilinear. Then the map $\mathrm{str}_{\mathcal{A}}\colon \Bbbk \to \mathcal{A}, \lambda \mapsto 1_{\mathcal{A}} \star \lambda$ is a ring homomorphism, and its image lies in the center of $\mathcal{A}$ because the ring multiplication is $\Bbbk$-bilinear.

10.1.4 Notation. Let $\mathcal{A}$ be a $\Bbbk$-algebra, let $\lambda \in \Bbbk$ and let $a \in \mathcal{A}$. We will often write λa for the element $\lambda \star a = \mathrm{str}_{\mathcal{A}}(\lambda)a$ to simplify notation. In particular, we will often write $\lambda 1_{\mathcal{A}}$ for $\mathrm{str}_{\mathcal{A}}(\lambda)$. However, we will never write λ for $\mathrm{str}_{\mathcal{A}}(\lambda)$ because the map $\mathrm{str}_{\mathcal{A}}$ is not necessarily injective.

The notion of faithfulness for algebras agrees with the one for modules.

10.1.5 Lemma. *Let $\mathcal{A}$ be a $\Bbbk$-algebra. Then $\mathcal{A}$ is faithful as a $\Bbbk$-algebra (that is, $\mathrm{str}_{\mathcal{A}}$ is injective) if and only if it is faithful as a $\Bbbk$-module (that is, if $\lambda \in \Bbbk$ satisfies $\lambda\mathcal{A} = \{0\}$, then $\lambda = 0$).*

Proof. Assume that $\mathcal{A}$ is faithful as a module and let $\lambda \in \ker(\mathrm{str}_{\mathcal{A}})$. Then we have $\lambda \star a = \mathrm{str}_{\mathcal{A}}(\lambda)a = 0_{\mathcal{A}} \cdot a = 0_{\mathcal{A}}$ for all $a \in \mathcal{A}$. Thus $\lambda = 0$ because $\mathcal{A}$ is faithful as a module. This shows that $\mathcal{A}$ is faithful as an algebra.

Now assume that $\mathcal{A}$ is faithful as an algebra and let $\lambda \in \Bbbk$ such that $\lambda \star a = 0_{\mathcal{A}}$ for all $a \in \mathcal{A}$. In particular, we then have $\mathrm{str}_{\mathcal{A}}(\lambda) = \mathrm{str}_{\mathcal{A}}(\lambda) \cdot 1_{\mathcal{A}} = \lambda \star 1_{\mathcal{A}} = 0_{\mathcal{A}}$. Since $\mathcal{A}$ is faithful as an algebra, it follows that $\lambda = 0$, so $\mathcal{A}$ is faithful as a module as well. $\square$

The property of a ring involution (in the sense of Definition 8.3.2) to be compatible with the algebra structure can be phrased in two equivalent ways.

10.1.6 Lemma. *Let $\mathcal{A}$ be a $\Bbbk$-algebra and let σ be an involution of the ring $\mathcal{A}$ (in the sense of Definition 8.3.2). Then the following assertions are equivalent:*

(i) The map σ is linear with respect to the $\Bbbk$-module structure of $\mathcal{A}$.

(ii) The image of the structural homomorphism $\mathrm{str}_{\mathcal{A}} \colon \Bbbk \to \mathcal{A}$ is contained in $\mathrm{Fix}(\sigma)$.

Proof. The involution σ is $\Bbbk$-linear if and only if $(a \star \lambda)^\sigma = a^\sigma \star \lambda$ for all $\lambda \in \Bbbk$ and $a \in \mathcal{A}$. Observe that

$$\begin{aligned} (a \star \lambda)^\sigma &= \big(a\,\mathrm{str}_{\mathcal{A}}(\lambda)\big)^\sigma = \mathrm{str}_{\mathcal{A}}(\lambda)^\sigma a^\sigma \qquad \text{and} \\ a^\sigma \star \lambda &= a^\sigma\,\mathrm{str}_{\mathcal{A}}(\lambda) = \mathrm{str}_{\mathcal{A}}(\lambda) a^\sigma \end{aligned}$$

for all $\lambda \in \Bbbk$ and $a \in \mathcal{A}$. Thus σ is clearly linear if the image of $\mathrm{str}_{\mathcal{A}}$ is contained in $\mathrm{Fix}(\sigma)$. Conversely, by putting $a := 1_{\mathcal{A}}$ in the equations above, we see that $\mathrm{str}_{\mathcal{A}}(\lambda)^\sigma = \mathrm{str}_{\mathcal{A}}(\lambda)$ for all $\lambda \in \Bbbk$ if σ is linear. $\square$

With the notion of algebra involutions at hand, we can now define scalar involutions.

10.1.7 Definition (Scalar involution). Let $\mathcal{A}$ be a $\Bbbk$-algebra. An *(algebra) involution of $\mathcal{A}$* is an involution σ of the ring $\mathcal{A}$ which satisfies the equivalent properties of Lemma 10.1.6. It is called *scalar involution* if all norms (in the sense of Definition 8.3.8) are scalars. That is, $a^\sigma a$ lies in $\mathrm{str}_{\mathcal{A}}(\Bbbk)$ for all $a \in \mathcal{A}$.

Since all scalars in a $\Bbbk$-algebra $\mathcal{A}$ lie in the center, every scalar involution of $\mathcal{A}$ is central (and thus nuclear) in the sense of Definition 8.3.12. Conversely, every central involution can be regarded as a scalar involution, though we may have to change the ground ring.

10.1.8 Lemma ([McC70, Section 9.5]). *Let $\mathcal{R}$ be a ring with a central involution σ and put $Z^\sigma(\mathcal{R}) := Z(\mathcal{R}) \cap \mathrm{Fix}(\sigma)$. Then $\mathcal{R}$ is a faithful $Z^\sigma(\mathcal{R})$-algebra with scalar involution σ.*

Proof. Since $Z^\sigma(\mathcal{R})$ is a subring of $\mathcal{R}$, the natural inclusion $i \colon Z^\sigma(\mathcal{R}) \to \mathcal{R}$ provides a faithful $Z^\sigma(\mathcal{R})$-algebra structure on $\mathcal{R}$. For all $\lambda \in Z^\sigma(\mathcal{R})$ and all $a \in \mathcal{R}$, we have

$$(a\lambda)^\sigma = \lambda^\sigma a^\sigma = \lambda a^\sigma = a^\sigma \lambda,$$

so σ is an involution with respect to this algebra structure on $\mathcal{R}$. Further, all norms of σ lie in the center of $\mathcal{R}$ because σ is central and they satisfy $(a^\sigma a)^\sigma = a^\sigma a$ for all $a \in \mathcal{R}$. We conclude that all norms lie in $Z^\sigma(\mathcal{R})$, which shows that σ is a scalar involution $\square$

10.1.9 Remark. Let $\mathcal{A}$ be a $\Bbbk$-algebra with an algebra involution σ. Recall from Remark 8.3.13 and Propositions 8.3.17 and 8.3.21 that all traces in $\mathcal{A}$ are nuclear (or central) if σ is nuclear (or central), and that the converse is true if $\mathcal{A}$ is alternative or if $2_{\mathcal{A}}$ is invertible. By the same argument as in Remark 8.3.13, all traces in $\mathcal{A}$ are scalar if σ is scalar and the converse is true if $2_{\mathcal{A}}$ is invertible. If we know that $\mathcal{A}$ is alternative, the assumption that $2_{\mathcal{A}}$ is invertible can be relaxed slightly: It suffices that some trace in $\mathcal{A}$ is invertible, see [McC70, Section 9.5].

10.1.B Conic Algebras

10.1.10 Convention. In this section, we will denote the linearisation of any $\Bbbk$-quadratic form $q\colon M \to \Bbbk$ by q as well, so that $q(x,y) := q(x+y) - q(x) - q(y)$ for all $x, y \in M$.

We can now turn to the protagonists of this section. In this subsection, we will closely follow the exposition in [GPR], which in turn follows [McC85].

10.1.11 Definition (Conic algebra, [GPR, 16.1]). A *conic algebra over $\Bbbk$* is a pair $(\mathcal{A}, \mathrm{n})$ consisting of a $\Bbbk$-algebra $\mathcal{A}$ and a $\Bbbk$-quadratic form $\mathrm{n}\colon \mathcal{A} \to \Bbbk$ (in the sense of Definition 7.1.8) such that $\mathrm{n}(1_{\mathcal{A}}) = 1_{\Bbbk}$ and such that the following *Cayley-Hamilton equation* is satisfied for all $a \in \mathcal{A}$:

$$a^2 - \mathrm{n}(a, 1_{\mathcal{A}})a + \mathrm{n}(a)1_{\mathcal{A}} = 0_{\mathcal{A}}.$$

The map n is called the *norm of $\mathcal{A}$*.

If $(\mathcal{A}, \mathrm{n})$ is a conic algebra, then $(\mathcal{A}, \mathrm{n}, 1_{\mathcal{A}})$ is a pointed quadratic module. Thus by Note 7.1.24, we have corresponding conjugation and trace maps.

10.1.12 Definition (Conjugation and trace, [GPR, 16.1]). Let $(\mathcal{A}, \mathrm{n})$ be a conic algebra over $\Bbbk$. The map

$$\mathrm{tr}\colon \mathcal{A} \to \Bbbk, a \mapsto \mathrm{n}(a, 1_{\mathcal{A}})$$

is called the *trace of* $(\mathcal{A}, \mathrm{n})$ and the map

$$\overline{\cdot} = -\sigma_{1_{\mathcal{A}}}\colon \mathcal{A} \to \mathcal{A}, a \mapsto \mathrm{tr}(a)1_{\mathcal{A}} - x$$

is called the *conjugation of* $(\mathcal{A}, \mathrm{n})$.

The fact that conic algebras are pointable quadratic modules also has the following consequence.

10.1.13 Lemma. *Assume that $2_{\Bbbk}$ is not a zero divisor and let $(\mathcal{A}, \mathrm{n})$ be a conic algebra over $\Bbbk$. Then $\mathcal{A}$ is a faithful $\Bbbk$-algebra.*

Proof. This follows from Lemma 7.1.15. □

10.1.14 Note. In [McC70, Sections 2.1, 2.2] and [McC85, p. 86], an algebra $\mathcal{A}$ is called *of degree 2* if there exist a quadratic form $\mathrm{n}\colon \mathcal{A} \to \Bbbk$ and a linear form $\mathrm{tr}\colon \mathcal{A} \to \Bbbk$ such that $\mathrm{n}(1_{\mathcal{A}}) = 1_{\Bbbk}$, $\mathrm{tr}(1_{\mathcal{A}}) = 2_{\Bbbk}$ and

$$a^2 - \mathrm{tr}(a)a + \mathrm{n}(a)1_{\mathcal{A}} = 0$$

for all $a \in \mathcal{A}$. Thus conic algebras are precisely those algebras of degree 2 in which $\mathrm{tr}(a) = \mathrm{n}(a, 1_{\mathcal{A}})$ for all $a \in \mathcal{A}$. If $\mathcal{A}$ is an algebra of degree 2 with norm n and trace tr, then $\mathrm{n}(a, 1_{\mathcal{A}})1_{\mathcal{A}} = \mathrm{tr}(a)1_{\mathcal{A}}$ for all $a \in \mathcal{A}$ by [McC85, (0.8)]. Hence for faithful algebras, the two notions coincide. In particular, they coincide if $2_{\Bbbk}$ is not a zero divisor because Lemma 10.1.13 is true for algebras of degree 2 as well. Even in the non-faithful setting, many results in this section remain true for algebras of degree 2, but we will have no need to use this fact.

Here are two examples of conic algebras. The first one motivates the name of the Cayley-Hamilton equation and the second one will be useful in the construction of F_4-graded groups in section 10.4.

10.1.15 Example. Let $\mathcal{A}$ be the $\Bbbk$-algebra of (2×2)-matrices over $\Bbbk$ and put $\mathrm{n}(a) := \det(a)$ for all $a \in \mathcal{A}$. Then $(\mathcal{A}, \mathrm{n})$ is a conic algebra over $\Bbbk$. Its trace is the usual trace map

$$\mathrm{tr}\colon \mathcal{A} \to \Bbbk, a \mapsto a_{11} + a_{22}$$

and its conjugation is

$$\overline{\cdot}\colon \mathcal{A} \to \mathcal{A}, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

The *Cayley-Hamilton theorem* says that any $a \in \mathcal{A}$ is a root of its characteristic polynomial, and this is precisely the Cayley-Hamilton equation in this situation.

10.1.16 Example ([GPR, 16.2 (d)]). Put $\mathcal{A} := \Bbbk \times \Bbbk$ and $\mathrm{n}(a, b) := ab$ for all $(a, b) \in \mathcal{A}$. Then $(\mathcal{A}, \mathrm{n})$ is a conic algebra over $\Bbbk$ with trace $\mathrm{tr}(a, b) = a + b$ and conjugation $\overline{(a, b)} = (b, a)$ for all $(a, b) \in \mathcal{A}$. Further, the linearisation of n is

$$\mathcal{A} \times \mathcal{A} \to \Bbbk, \big((a, b), (c, d)\big) \mapsto ad + cb.$$

Observe that, forgetting the multiplication on $\mathcal{A}$, the quadratic module $(\mathcal{A}, \mathrm{n})$ is the hyperbolic plane (that is, the hyperbolic space of dimension 2) from Example 7.1.13.

Conic algebras arise from F_4-graded groups in the form of the following parameter system.

10.1.17 Definition (Standard parameter system). Define $A := B := \{\pm 1\}$. We declare that A acts on $\mathcal{A}$ and $\Bbbk$ by inversion and that B acts trivially on $\Bbbk$ and by conjugation on $\mathcal{A}$ (that is, by the switching involution). More precisely, this means that

$$-1_A.\lambda = -\lambda, \quad -1_A.a = -a, \quad -1_B.\lambda = \lambda, \quad -1_B.a = \overline{a}$$

for all $\lambda \in \Bbbk$ and all $a \in \mathcal{A}$. Then the triple $(A \times B, \mathcal{A}, \Bbbk)$ is called the *standard parameter system for* $(\mathcal{A}, \mathrm{n})$.

We now investigate some basic properties of conic algebras. The following identities are straightforward to verify.

10.1.18 Lemma ([GPR, 16.5]). *Let* $(\mathcal{A}, \mathrm{n})$ *be a conic algebra over* $\Bbbk$. *Then the following hold:*

(a) $\mathrm{n}(1_{\mathcal{A}}) = 1_{\Bbbk}$ *and* $\mathrm{tr}(1_{\mathcal{A}}) = 2_{\Bbbk}$.

(b) $\overline{1_{\mathcal{A}}} = 1_{\mathcal{A}}$ *and* $\overline{\overline{a}} = a$ *for all* $a \in \mathcal{A}$.

(c) $\overline{a}a = \mathrm{n}(a)1_{\mathcal{A}} = a\overline{a}$ *and* $a + \overline{a} = \mathrm{tr}(a)1_{\mathcal{A}}$ *for all* $a \in \mathcal{A}$.

(d) $a\overline{b} + b\overline{a} = \mathrm{n}(a, b)1_{\mathcal{A}}$ *for all* $a, b \in \mathcal{A}$.

(e) $\mathrm{n}(\overline{a}) = \mathrm{n}(a)$, $\mathrm{n}(\overline{a}, \overline{b}) = \mathrm{n}(a, b)$ *and* $\mathrm{tr}(\overline{a}) = \mathrm{tr}(a)$ *for all* $a, b \in \mathcal{A}$.

(f) *The conjugation and trace maps are* $\Bbbk$*-linear.*

In order to show that the conjugation on a conic algebra is an (algebra) involution, it only remains to show that it is anti-compatible with the multiplication. This is, however, not true in general. Instead, we have the following characterisation.

10.1.19 Lemma ([GPR, 16.10]). *Let $(\mathcal{A}, \mathrm{n})$ be a conic algebra over $\Bbbk$. Then the conjugation of $\mathcal{A}$ is an algebra involution if and only if* $\mathrm{tr}(ab)1_{\mathcal{A}} = \mathrm{n}(a, \overline{b})1_{\mathcal{A}}$ *(that is,* $\mathrm{n}(ab, 1_{\mathcal{A}})1_{\mathcal{A}} = \mathrm{n}(a, \overline{b})$*) for all $a, b \in \mathcal{A}$.*

10.1.20 Remark (Scalar involutions and conic algebras, [McC85, 1.1]). Let $\mathcal{A}$ be a $\Bbbk$-algebra. For the scope of this remark, a *linear involution of $\Bbbk$* is defined to be the same thing as an algebra involution except that it is not assumed to be anti-compatible with the ring multiplication. The notion of scalar linear involutions is defined in the same way as for algebra involutions. It remains true in this generality that all traces with respect to a scalar involution are scalars and that, by the same proof as in Lemma 8.3.20, we have $aa^\sigma = a^\sigma a$ for all $a \in \mathcal{A}$ if σ is a scalar linear involution.

Now assume that $(\mathcal{A}, \mathrm{n})$ is a conic algebra over $\Bbbk$. Then Lemma 10.1.18 (c) says precisely that the linear involution $\overline{\cdot}$ is scalar, though of course it need not be a ring involution. In particular, it is also central and nuclear.

Conversely, let $\mathcal{A}$ be a $\Bbbk$-algebra with a scalar linear involution σ. Denote the image of $\Bbbk$ in $\mathcal{A}$ under the structural homomorphism by $\tilde{\Bbbk}$. (If $\mathcal{A}$ is a faithful $\Bbbk$-algebra, then we can identify $\Bbbk$ with $\tilde{\Bbbk}$.) We define

$$\mathrm{tr}\colon \mathcal{A} \to \tilde{\Bbbk}, a \mapsto \mathrm{Tr}_\sigma(a) = a + a^\sigma \quad \text{and} \quad \mathrm{n}\colon \mathcal{A} \to \tilde{\Bbbk}, a \mapsto N_\sigma(a) = aa^\sigma = a^\sigma a.$$

Then for all $a \in \mathcal{A}$, we have

$$a^2 - \mathrm{tr}(a)a + \mathrm{n}(a)1_{\mathcal{A}} = a^2 - (a + a^\sigma)a + a^\sigma a = 0_{\mathcal{A}} \qquad \text{and}$$
$$\mathrm{n}(a, 1_{\mathcal{A}}) = (a + 1_{\mathcal{A}})(a + 1_{\mathcal{A}})^\sigma - aa^\sigma - 1_{\mathcal{A}} = a + a^\sigma.$$

Hence $(\mathcal{A}, \mathrm{n})$ is a faithful conic $\tilde{\Bbbk}$-algebra with trace tr and conjugation σ. In particular, every faithful $\Bbbk$-algebra with a scalar linear involution is also a faithful conic algebra over $\Bbbk$. Without faithfulness assumptions, however, it is not at all clear that n and tr can be lifted from $\tilde{\Bbbk}$ to $\Bbbk$.

We conclude from the previous two paragraphs that faithful $\Bbbk$-algebras with a scalar linear involution are the same thing as faithful conic algebras.

The following identities, which are well-known in the theory of composition algebras over a field, turn out to be equivalent.

10.1.21 Lemma ([GPR, 16.12]). *Let $(\mathcal{A}, \mathrm{n})$ be a conic algebra over $\Bbbk$. Then the following assertions are equivalent:*

(a) $\mathrm{n}(a, ba) = \mathrm{n}(ba, a) = \mathrm{tr}(b)\,\mathrm{n}(a)$ *for all $a, b \in \mathcal{A}$.*
(b) $\mathrm{n}(a, ab) = \mathrm{n}(ab, a) = \mathrm{tr}(b)\,\mathrm{n}(a)$ *for all $a, b \in \mathcal{A}$.*
(c) $\mathrm{n}(ab, c) = \mathrm{n}(a, c\overline{b})$ *for all $a, b, c \in \mathcal{A}$.*
(d) $\mathrm{n}(ab, c) = \mathrm{n}(b, \overline{a}c)$ *for all $a, b, c \in \mathcal{A}$.*
(e) $\mathrm{tr}(ab) = \mathrm{n}(a, \overline{b})$ *and* $\mathrm{tr}\big((ab)c\big) = \mathrm{tr}\big(a(bc)\big)$ *for all $a, b, c \in \mathcal{A}$.*

10.1.22 Definition (Norm-associativity, [GPR, 16.12]). A conic algebra is called *norm-associative* if it satisfies the equivalent conditions in Lemma 10.1.21.

By Lemma 10.1.19 and Lemma 10.1.21 (e), the conjugation in norm-associative conic algebras is an algebra involution. It turns out that these algebras are automatically flexible (in the sense of Definition 8.2.2) as well.

10.1.23 Lemma ([GPR, 16.14]). *Let $(\mathcal{A}, \mathrm{n})$ be a norm-associative conic algebra over $\Bbbk$. Then $\mathcal{A}$ is flexible and the conjugation of $\mathcal{A}$ is an algebra involution.*

10.1.24 Note. The converse of Lemma 10.1.23 is nearly true: If $(\mathcal{A}, \mathrm{n})$ is a flexible conic algebra whose conjugation is an involution, and if in addition $\mathcal{A}$ is projective as a $\Bbbk$-module, then $(\mathcal{A}, \mathrm{n})$ is norm-associative ([GPR, 16.14]). Further, the projectiveness assumption can be dropped if the algebra structure on $\mathcal{A}$ is faithful ([GPR, 16.15]).

In general, the norm of a conic algebra $(\mathcal{A}, \mathrm{n})$ is not uniquely determined by the algebra $\mathcal{A}$, as is illustrated by the family of examples in [GPR, Exercise 17.8]. However, we have the following result.

10.1.25 Lemma ([GPR, 16.16]). *Let $\mathcal{A}$ be a $\Bbbk$-algebra and let $\mathrm{n}, \mathrm{n}' \colon \mathcal{A} \to \Bbbk$ be two quadratic forms which turn $\mathcal{A}$ into a conic algebra. If $\mathcal{A}$ is projective as a $\Bbbk$-module, then* $\mathrm{n} = \mathrm{n}'$.

The following property is eponymous in the theory of composition algebras.

10.1.26 Definition (Multiplicative conic algebra). A conic algebra $(\mathcal{A}, \mathrm{n})$ over $\Bbbk$ is called *multiplicative* if its norm permits composition in the sense that $\mathrm{n}(ab) = \mathrm{n}(a)\,\mathrm{n}(b)$ for all $a, b \in \mathcal{A}$.

10.1.27 Lemma ([GPR, 17.2]). *Multiplicative conic algebras are norm-associative. In particular, they are flexible and their conjugation is a scalar algebra involution.*

Proof. This holds by [GPR, 17.2], except for the assertion that the conjugation is scalar, which holds by Remark 10.1.20. □

In the following, we list some properties of conic algebras which are, in addition, alternative.

10.1.28 Lemma (Kirmse's identities, [GPR, 17.4]). *Let $(\mathcal{A}, \mathrm{n})$ be a conic alternative algebra. Then we have*

$$a(\overline{a}b) = \mathrm{n}(a)b = (b\overline{a})a$$

for all $a, b \in \mathcal{A}$.

10.1.29 Note ([GPR, 17.4]). Let $(\mathcal{A}, \mathrm{n})$ be a conic alternative algebra. One can show that

$$\mathrm{n}(aba) = \mathrm{n}(a)^2\,\mathrm{n}(b)$$

for all $a, b \in \mathcal{A}$. In this sense, conic alternative algebras are "close to being multiplicative". Yet, [GPR, Exercise 17.8] provides examples of conic alternative algebras which are not multiplicative. If $\mathcal{A}$ is projective as a $\Bbbk$-module, however, $(\mathcal{A}, \mathrm{n})$ is indeed multiplicative by [GPR, 17.6].

10.1.30 Lemma ([GPR, 17.5]). *Let $(\mathcal{A}, \mathrm{n})$ be a conic alternative algebra over $\Bbbk$ and let $a \in \mathcal{A}$. Then a is invertible (in $\mathcal{A}$) if and only if $\mathrm{n}(a)$ is invertible (in $\Bbbk$). In this case, $a^{-1} = \mathrm{n}(a)^{-1}\overline{a}$ and $\mathrm{n}(a^{-1}) = \mathrm{n}(a)^{-1}$.*

10.1.C F_4-data

We now define an algebraic structure that we call "F_4-datum". In section 10.7, we will see that F_4-data are precisely the kind of structures that coordinatise F_4-graded groups. The goal of this section is to show, on a purely algebraic level, that an F_4-datum contains the same information as a multiplicative conic alternative algebra.

10.1.31 Definition (F_4-datum). An F_4-*datum* is a tuple

$$\big(\Bbbk, \mathcal{A}, \star, \mathrm{n}, \langle \cdot \mid \cdot \rangle, \overline{\cdot}, \mathrm{str}_{\mathcal{A}}, \varphi\big)$$

with the following properties:

(i) $\Bbbk$ is a commutative assocative ring.

(ii) $\mathcal{A}$ is an alternative ring and $\star$ is a right $\Bbbk$-module structure on $\mathcal{A}$ (which, a priori, is not assumed to be compatible with the ring structure).

(iii) $\mathrm{n}\colon \mathcal{A} \to \Bbbk$ is a quadratic form with $\mathrm{n}(1_{\mathcal{A}}) = 1_{\Bbbk}$, and $\langle \cdot \mid \cdot \rangle\colon \mathcal{A} \times \mathcal{A} \to \Bbbk$ is the linearisation of n.

(iv) $\overline{\cdot}\colon \mathcal{A} \to \mathcal{A}$ is a nuclear involution of the ring $\mathcal{A}$, and we have $\overline{\cdot} = -\sigma_{1_{\mathcal{A}}}$ where $\sigma_{1_{\mathcal{A}}}$ denotes the reflection corresponding to $1_{\mathcal{A}}$ in the quadratic module $(\mathcal{A}, \mathrm{n})$ (in the sense of Definition 7.1.20).

(v) $\mathrm{str}_{\mathcal{A}}\colon \Bbbk \to \mathrm{Nucl}(\mathcal{A})$ is a map which satisfies $a \star \lambda = a * \mathrm{str}_{\mathcal{A}}(\lambda)$ for all $a \in \mathcal{A}$ and $\lambda \in \Bbbk$ (where $*$ denotes the ring multiplication on $\mathcal{A}$).

(vi) φ is the map $\Bbbk \times \mathcal{A} \to \Bbbk, (\lambda, a) \mapsto \lambda\, \mathrm{n}(a)$.

(vii) $(\Bbbk, \varphi, \mathrm{str}_{\mathcal{A}}, \langle \cdot | 1_{\mathcal{A}} \rangle, 0)$ is a Jordan module of type C over $(\mathcal{A}, \overline{\cdot})$.

10.1.32 Note. The definition of F_4-data contains a lot of redundancy. For example, the maps $\langle \cdot \mid \cdot \rangle$, $\overline{\cdot}$ and φ are uniquely determined by the remaining part of an F_4-datum. For our application of F_4-data in section 10.7, however, it will be practical to keep this redundancy as part of the definition.

10.1.33 Proposition. *Let $(\mathcal{A}, \mathrm{n})$ be a multiplicative conic alternative algebra over $\Bbbk$ and define*

$$\varphi\colon \Bbbk \times \mathcal{A} \to \Bbbk, (\lambda, a) \mapsto \lambda\, \mathrm{n}(a).$$

Then $(\Bbbk, \mathcal{A}, \star, \mathrm{n}, \langle \cdot \mid \cdot \rangle, \overline{\cdot}, \mathrm{str}_{\mathcal{A}}, \varphi)$ is an F_4-datum where $\star$ denotes the $\Bbbk$-module structure on $\mathcal{A}$ (from Remark 10.1.3), $\langle \cdot \mid \cdot \rangle$ denotes the linearisation of n, $\overline{\cdot}$ *is the conjugation on $\mathcal{A}$, $\mathrm{str}_{\mathcal{A}}$ is the structural homomorphism of the $\Bbbk$-algebra $\mathcal{A}$ and the $\Bbbk$-module structure on $\mathcal{A}$ is taken to be the one induced by the $\Bbbk$-algebra structure.*

Proof. Axioms 10.1.31 (i), 10.1.31 (ii), 10.1.31 (iii) are trivial. The conjugation map $\overline{\cdot}$ is a nuclear involution by Lemma 10.1.27 and it equals $-\sigma_{1_{\mathcal{A}}}$ by its definition. Hence Axiom 10.1.31 (iv) is satisfied. Further, Axioms 10.1.31 (v) and 10.1.31 (vi) hold by the definitions of $\star$ and of φ, respectively. Denote by $\mathrm{tr} := \langle \cdot | 1_{\mathcal{A}} \rangle$ the trace of the conic algebra $(\mathcal{A}, \mathrm{n})$. It remains to show that $(\Bbbk, \varphi, \mathrm{str}_{\mathcal{A}}, \mathrm{tr}, 0)$ satisfies the axioms of a Jordan module of type C over $(\mathcal{A}, \overline{\cdot})$ from Definition 8.6.1. That is, $\pi_1 := \mathrm{str}_{\mathcal{A}}$ plays the role of the Jordan module projection and $\mathrm{T}_1 := \mathrm{tr}$ plays the role of the Jordan module trace.

At first, we verify that the maps φ, π_1 and T_1 have the desired properties in the first paragraph of Definition 8.6.1. We will always denote the square-scalar multiplication on $\mathcal{R}$ from Example 8.4.2 by ω. It is clear that φ is additive in the first component and weakly quadratic in the second component, so $(\Bbbk, \varphi)$ is a square-module. Since $\Bbbk$ is associative and n admits composition, it is a multiplicative square-module.

Since $\mathcal{A}$ is a $\Bbbk$-algebra, the image of $\mathrm{str}_{\mathcal{A}} = \pi_1$ lies in the center and thus in the nucleus of $\mathcal{A}$. Further, $\mathrm{str}_{\mathcal{A}}$ preserves the Jordan scalar multiplication: For all $a \in \mathcal{A}$ and $\lambda \in \Bbbk$, we have

$$\begin{aligned} \mathrm{str}_{\mathcal{A}}\big(\varphi(\lambda, a)\big) &= \mathrm{str}_{\mathcal{A}}\big(\lambda\, \mathrm{n}(a)\big) = \mathrm{str}_{\mathcal{A}}(\lambda)\, \mathrm{str}_{\mathcal{A}}\big(\mathrm{n}(a)\big) = \mathrm{str}_{\mathcal{A}}(\lambda)\bar{a}a \\ &= \bar{a}\, \mathrm{str}_{\mathcal{A}}(\lambda)a = \bar{a}\pi_1(\lambda)a = \omega\big(\pi_1(\lambda), a\big) \end{aligned}$$

by Lemma 10.1.18 (c) and because $\Bbbk$ is commutative associative.

It is clear that T_1 is additive. Now let $a, b \in \mathcal{A}$. Then

$$\mathrm{T}_1\big(\omega(a,b)\big) = \mathrm{tr}\big(\bar{b}ab\big) = \mathrm{n}(\bar{b}ab, 1_{\mathcal{A}}),$$

which by Lemma 10.1.21 (d) equals $\mathrm{n}(ab, b)$. By Lemma 10.1.21 (a), this term equals $\mathrm{tr}(a)\,\mathrm{n}(b)$. It follows that

$$\mathrm{T}_1\big(\omega(a,b)\big) = \mathrm{tr}(a)\,\mathrm{n}(b) = \varphi\big(\mathrm{T}_1(a), b\big),$$

so T_1 is a homomorphism of square-modules. We conclude that the maps φ, π_1 and T_1 have all the desired properties in the first paragraph of Definition 8.6.1.

We now turn to the remaining axioms in Definition 8.6.1. Since $\bar{\cdot}$ is an algebra involution, the image of $\mathrm{str}_{\mathcal{A}}$ lies in $\mathrm{Fix}(\bar{\cdot})$ (by Lemma 10.1.6), which implies that Axiom 8.6.1 (ii) holds. Axiom 8.6.1 (i) holds by Lemma 10.1.18 (e) and Lemma 10.1.21 (e), the latter being applicable by Lemma 10.1.27. Axiom 8.6.1 (iii) is trivial because the Jordan module skew-hermitian form is zero.

Now consider Axiom 8.6.1 (iv). Equation (8.1) holds because $\pi_1 = \mathrm{str}_{\mathcal{A}}$ is additive and equation (8.3) holds because $(\Bbbk, +)$ is abelian. Further, for all $\lambda \in \Bbbk$ and $a, b \in \mathcal{A}$, we have

$$\varphi(\lambda, a+b) = \lambda\,\mathrm{n}(a+b) = \lambda\,\mathrm{n}(a) + \lambda\,\mathrm{n}(b) + \lambda\,\mathrm{n}(a,b)$$

where

$$\lambda\,\mathrm{n}(a,b) = \lambda\,\mathrm{n}(b,a) = \lambda\,\mathrm{n}(\bar{b},\bar{a}) = \mathrm{n}\big(\bar{b}\,\mathrm{str}_{\mathcal{A}}(\lambda)a, 1_{\mathcal{A}}\big) = \mathrm{T}_1\big(\bar{b}\pi_1(\lambda)a\big)$$

by Lemma 10.1.18 (e). This shows that equation (8.2) holds as well, and so Axiom 8.6.1 (iv) is satisfied.

By Lemma 10.1.18 (c), we have

$$\pi_1\big(\mathrm{T}_1(a)\big) = \mathrm{str}_{\mathcal{A}}\big(\mathrm{tr}(a)\big) = \mathrm{tr}(a)1_{\mathcal{A}} = a + a^{\sigma}$$

for all $a \in \mathcal{A}$, so Axiom 8.6.1 (v) holds. Finally, Axiom 8.6.1 (vi) is satisfied for $v_0 := 1_{\Bbbk}$ because $\pi_1(1_{\Bbbk}) = \mathrm{str}_{\mathcal{A}}(1_{\Bbbk}) = 1_{\mathcal{A}}$. ☐

10.1.34 Proposition. *Let $(\Bbbk, \mathcal{A}, \star, \mathrm{n}, \langle \cdot \mid \cdot \rangle, \bar{\cdot}, \mathrm{str}_{\mathcal{A}}, \varphi)$ be an F_4-datum. Then the following hold:*

- *(a) The ring $\mathcal{A}$ together with the $\Bbbk$-module structure $\star$ is a $\Bbbk$-algebra (in the sense of Remark 10.1.3), and its structural homomorphism is exactly $\mathrm{str}_{\mathcal{A}}$.*
- *(b) $(\mathcal{A}, \mathrm{n})$ is a multiplicative conic alternative algebra over $\Bbbk$ with conjugation $\bar{\cdot}$.*

Proof. Putting $a := 1_{\mathcal{A}}$ in Axiom 10.1.31 (v), we see that

$$\mathrm{str}_{\mathcal{A}}(\lambda) = 1_{\mathcal{A}} * \mathrm{str}_{\mathcal{A}}(\lambda) = 1_{\mathcal{A}} \star \lambda \tag{10.1}$$

for all $\lambda \in \Bbbk$. In particular, $\mathrm{str}_{\mathcal{A}}(1_{\Bbbk}) = 1_{\mathcal{A}}$ and $\mathrm{str}_{\mathcal{A}}$ is additive. Further, it follows from Axiom 10.1.31 (v) that

$$\mathrm{str}_{\mathcal{A}}(\lambda\mu) = 1_{\mathcal{A}} * \mathrm{str}_{\mathcal{A}}(\lambda\mu) = 1_{\mathcal{A}} \star \lambda\mu = (1_{\mathcal{A}} \star \lambda) \star \mu = \mathrm{str}_{\mathcal{A}}(\lambda) \star \mu$$

for all $\lambda, \mu \in \Bbbk$. By another application of Axiom 10.1.31 (v), we infer that

$$\mathrm{str}_{\mathcal{A}}(\lambda\mu) = \mathrm{str}_{\mathcal{A}}(\lambda) * \mathrm{str}_{\mathcal{A}}(\mu)$$

for all $\lambda, \mu \in \Bbbk$, so $\mathrm{str}_{\mathcal{A}}$ is a homomorphism of rings. Further, since $\mathrm{str}_{\mathcal{A}}$ is the Jordan module projection in an abelian Jordan module (by Axiom 10.1.31 (vii)), we know from Axiom 8.6.1 (ii) that $\overline{\mathrm{str}_{\mathcal{A}}(\lambda)} = \mathrm{str}_{\mathcal{A}}(\lambda)$ for all $\lambda \in \Bbbk$. Moreover, $\bar{\cdot}$ is $\Bbbk$-linear because it equals $-\sigma_{1_{\mathcal{A}}}$ by Axiom 10.1.31 (iv). In other words,

$\overline{a \star \lambda} = \overline{a} \star \lambda$ for all $a \in \mathcal{A}$ and $\lambda \in \Bbbk$. We infer that

$$\mathrm{str}_{\mathcal{A}}(\lambda) * \overline{a} = \overline{\mathrm{str}_{\mathcal{A}}(\lambda)} * \overline{a} = \overline{a * \mathrm{str}_{\mathcal{A}}(\lambda)} = \overline{a \star \lambda} = \overline{a} \star \lambda = \overline{a} * \mathrm{str}_{\mathcal{A}}(\lambda)$$

for all $a \in \mathcal{A}$ and $\lambda \in \Bbbk$. This says precisely that the image of $\mathrm{str}_{\mathcal{A}}$ is contained in the center of $\mathcal{A}$. Thus $(\mathcal{A}, \mathrm{str}_{\mathcal{A}})$ is a $\Bbbk$-algebra, and by (10.1), its $\Bbbk$-scalar multiplication is exactly $\star$. This proves (a).

We now prove that $(\mathcal{A}, \mathrm{n})$ is a conic algebra. We already know from Axiom 10.1.31 (iii) that n is a quadratic form with $\mathrm{n}(1_{\mathcal{A}}) = 1_{\Bbbk}$, so it only remains to verify the Cayley-Hamilton equation. As usual, we denote the square-scalar multiplication on $\mathcal{A}$ by ω. Using that the Jordan module projection $\mathrm{str}_{\mathcal{A}}$ preserves the square-scalar multiplications by Axiom 10.1.31 (vii), we find that

$$\mathrm{str}_{\mathcal{A}}\big(\mathrm{n}(a)\big) = \mathrm{str}_{\mathcal{A}}\big(\varphi(1_{\Bbbk}, a)\big) = \omega\big(\mathrm{str}_{\mathcal{A}}(1_{\Bbbk}), a\big) = \overline{a}\,\mathrm{str}_{\mathcal{A}}(1_{\Bbbk})a = \overline{a}a$$

for all $a \in \mathcal{A}$. Together with Axioms 8.6.1 (v) and 10.1.31 (v) and the fact that the image of $\mathrm{str}_{\mathcal{A}}$ is central, this implies that

$$\begin{aligned} a \star \langle a | 1_{\mathcal{A}} \rangle &= a\,\mathrm{str}_{\mathcal{A}}(\langle a | 1_{\mathcal{A}} \rangle) = \mathrm{str}_{\mathcal{A}}(\langle a | 1_{\mathcal{A}} \rangle)a = (a + \overline{a})a \\ &= a^2 + \overline{a}a = a^2 + \mathrm{str}_{\mathcal{A}}\big(\mathrm{n}(a)\big) = a^2 + 1_{\mathcal{A}} \star \mathrm{n}(a) \end{aligned}$$

for all $a \in \mathcal{A}$. This says precisely that the Cayley-Hamilton equation is satisfied, so $(\mathcal{A}, \mathrm{n})$ is a conic algebra. By Axiom 10.1.31 (ii), it is also alternative.

It remains to show that $(\mathcal{A}, \mathrm{n})$ is multiplicative. By the definition of φ (see Axiom 10.1.31 (vi)) and the fact that it is the scalar multiplication of the multiplicative square-module $\Bbbk$ by Axiom 10.1.31 (vii), we have

$$\mathrm{n}(ab) = \varphi(1_{\Bbbk}, ab) = \varphi\big(\varphi(1_{\Bbbk}, a), b\big) = \varphi\big(\mathrm{n}(a), b\big) = \mathrm{n}(a)\,\mathrm{n}(b)$$

for all $a, b \in \mathcal{A}$. Thus n permits composition, which finishes the proof. □

10.2 Composition Algebras and Pre-composition Algebras

In the previous section, we have seen that multiplicative conic alternative algebras – that is, the objects that will turn out to coordinatise F_4-gradings – have many desirable properties. In this section, we address the question which additional properties a conic algebra should satisfy in order to deserve the name "composition algebra". There exists no standard terminology for these objects over commutative associative rings, but we will follow the conventions of [GPR, Chapter IV]. Composition algebras play no particular role in our treatment of F_4-graded groups (except for the fact that they are special cases of multiplicative conic alternative algebras), so the content of this subsection will not be needed in the subsequent parts of this chapter.

10.2.A Regularity Conditions on Quadratic Forms

Before can state the definition of composition algebras, we have to introduce the notions of regular and non-singular quadratic forms. These properties replace the usual requirement of non-degeneracy in the classical theory, which is inadequate over rings because it is not stable under base change. Again, there is no standard terminology. In the following, we present the conventions of [GPR, 11.9, 11.11].

10.2.1 Definition (Radical). Let M be a $\Bbbk$-module and let $f\colon M \times M \to \Bbbk$ be a symmetric $\Bbbk$-bilinear form. The *radical of f* is

$$\mathrm{Rad}(f) := \{\, v \in M \mid f(u, v) = 0 \text{ for all } u \in M \,\}.$$

If $q\colon M \to \Bbbk$ is a $\Bbbk$-quadratic form with linearisation f, then the *radical of q* is

$$\operatorname{Rad}(q) := \{\, v \in \operatorname{Rad}(f) \mid q(v) = 0 \,\}.$$

10.2.2 Definition (Dual module). Let M be a $\Bbbk$-module. The *dual of M* is $M^* := \operatorname{Hom}_{\Bbbk}(M, \Bbbk)$. If $f\colon M \times M \to \Bbbk$ be a symmetric $\Bbbk$-bilinear form, we put

$$\tilde{f}\colon M \to M^*, v \mapsto \big(u \mapsto f(v,u)\big).$$

10.2.3 Definition (Regularity conditions on quadratic forms). Let M be a $\Bbbk$-module, let $q\colon M \to \Bbbk$ be a $\Bbbk$-quadratic form and denote by $f\colon M \times M \to \Bbbk$ its linearisation.

(a) q is called *non-degenerate* if $\operatorname{Rad}(q)$ is zero.

(b) q is called *weakly regular* if $\operatorname{Rad}(f)$ is zero.

(c) q is called *non-singular* if M is projective and for all $\Bbbk$-algebras K which are fields, the scalar extension q_K is non-degenerate.

(d) q is called *regular* if M is finitely generated projective and the map $\tilde{f}$ is an isomorphism.

10.2.4 Remark (Base change). Let M be a $\Bbbk$-module and let $q\colon M \to \Bbbk$ be a $\Bbbk$-quadratic form. For any $\Bbbk$-algebra R, the base change $(M^*)_R$ of the dual module can be identified with the dual $(M_R)^*$ of the base change in a canonical way if M is finitely generated projective (see [GPR, 9.15]). From this it follows that the notion of regularity is invariant under base change. The same holds for non-singularity. However, non-degeneracy is not invariant under base change, and this defect is precisely the motivation to introduce non-singularity.

Some important relations between the regularity conditions can be found in Figure 10.1. They are either straightforward to prove or can be found in [GPR, 11.9, 11.11].

$$\begin{tikzcd}
\text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{weakly regular} \arrow[l, Rightarrow, bend right, "{\Bbbk\text{ field},\ \dim M<\infty}"'] \arrow[d, Rightarrow, bend right] \\
\text{non-singular} \arrow[r, Rightarrow, "\Bbbk\text{ field}"'] & \text{non-degenerate} \arrow[ul, Rightarrow, "{\Bbbk\text{ field}, \operatorname{char}(\Bbbk)\neq 2, \dim M<\infty}"] \arrow[u, Rightarrow, bend right, "{\Bbbk\text{ field}, \operatorname{char}(\Bbbk)\neq 2}"']
\end{tikzcd}$$

Figure 10.1: The relationship between the regularity conditions for a quadratic form $q\colon M \to \Bbbk$.

10.2.B Composition Algebras

With the notions of non-singularity and regularity at hand, we can now define composition algebras over commutative associative rings.

10.2.5 Definition (Composition algebra, [GPR, 19.5]). A *composition algebra over $\Bbbk$* is a $\Bbbk$-algebra $\mathcal{A}$ which satisfies the following conditions.

(a) $\mathcal{A}$ is projective as a $\Bbbk$-module.

(b) The rank function $\operatorname{spec}(\Bbbk) \to \mathbb{N}_0 \cup \{\infty\}, \mathfrak{p} \mapsto \operatorname{rank}(\mathcal{A}_\mathfrak{p})$ is locally constant with respect to the Zariski topology on $\operatorname{spec}(\Bbbk)$. Here $\operatorname{spec}(\Bbbk)$ denotes the set of prime ideals of $\Bbbk$ and $\mathcal{A}_\mathfrak{p}$ denotes the localisation of $\mathcal{A}$ at $\mathfrak{p}$.

(c) There exists a non-singular quadratic form $\mathrm{n}\colon \mathcal{A} \to \Bbbk$ that permits composition in the sense that $\mathrm{n}(1_\mathcal{A}) = 1_\Bbbk$ and $\mathrm{n}(ab) = \mathrm{n}(a)\,\mathrm{n}(b)$ for all $a, b \in \mathcal{A}$.

Further, it is called a *regular composition algebra* if n can be chosen to be a regular quadratic form.

10.2.6 Remark (Composition algebras over fields). Assume that $\Bbbk$ is a field. Then Axiom 10.2.5 (b) is trivially satisfied because $\Bbbk$ has only one prime ideal, the zero ideal. Further, every vector space over $\Bbbk$ is projective. Thus 10.2.5 (c) is the only non-trivial axiom over fields. However, even over fields, there is no agreement in the literature on the right definition of composition algebras because the regularity condition on the norm may be chosen differently. See, however, Note 10.2.16.

10.2.7 Note. Some authors do not require the existence of an identity element in the definition of composition algebras. In the context of root graded groups, however, this convention is not practical.

Composition algebras fit into the framework of conic algebras as follows.

10.2.8 Proposition ([GPR, 19.12]). *Let $\mathcal{A}$ be a $\Bbbk$-algebra. Then the following assertions are equivalent:*

(i) $\mathcal{A}$ is a composition algebra (respectively, a regular composition algebra).

(ii) $\mathcal{A}$ is finitely generated projective as a $\Bbbk$-module and there exists a non-singular (respectively, regular) quadratic form $\mathrm{n}\colon \mathcal{A} \to \Bbbk$ such that $(\mathcal{A}, \mathrm{n})$ is a conic alternative algebra.

In this situation, the norm n is uniquely determined by the $\Bbbk$-algebra $\mathcal{A}$. Further, the conic algebra $(\mathcal{A}, \mathrm{n})$ is norm-associative and its conjugation is a scalar algebra involution.

Proof. The equivalence of the two assertions is proven in [GPR, 19.12]. The uniqueness of n holds by Lemma 10.1.25. It follows from Lemma 10.1.27 that the conic algebra is norm-associative and that its conjugation is a scalar algebra involution. □

Observe that the norm in condition 10.2.8 (ii) is necessarily multiplicative by Note 10.1.29.

10.2.9 Lemma ([GPR, 19.11]). *Assume that $\Bbbk$ is a field of characteristic not 2. Then every composition algebra over $\Bbbk$ is regular.*

10.2.10 Note. Axiom 10.2.5 (b) is invoked in [GPR, 19.11] to show that a composition algebra is finitely generated as a $\Bbbk$-module. Conversely, any finitely generated projective $\Bbbk$-module satisfies Axiom 10.2.5 (b). Thus we could replace Axiom 10.2.5 (b) by "$\mathcal{A}$ is finitely generated as a $\Bbbk$-module" without changing the meaning of the word "composition algebra". However, recall that in the classical setting over fields, the possibility of non-finitely generated composition algebras is not ruled out from the beginning, and it is instead a theorem that they must be finitely generated as modules. Axiom 10.2.5 (b) makes it possible to carry this result over to the more general situation.

10.2.C Pre-composition Algebras

In order to discuss composition algebras in the classical situation (that is, over fields), the notion of pre-composition algebras will be useful.

10.2.11 Definition (Pre-composition algebra, [GPR, 19.1]). A *pre-composition algebra over $\Bbbk$* is a $\Bbbk$-algebra $\mathcal{A}$ which satisfies the following conditions.

(a) $\mathcal{A}$ is projective as a $\Bbbk$-module.

(b) There exists a non-degenerate quadratic form $\mathrm{n}\colon \mathcal{A} \to \Bbbk$ that permits composition in the sense that $\mathrm{n}(1_{\mathcal{A}}) = 1_{\Bbbk}$ and $\mathrm{n}(ab) = \mathrm{n}(a)\,\mathrm{n}(b)$ for all $a, b \in \mathrm{n}$.

We have the following analogue of Proposition 10.2.8 for pre-composition algebras.

10.2.12 Proposition ([GPR, 17.6, 19.3]). *Let $\mathcal{A}$ be a $\Bbbk$-algebra. Then the following assertions are equivalent:*

(i) $\mathcal{A}$ is a pre-composition algebra.

(ii) $\mathcal{A}$ is projective as a $\Bbbk$-module and there exists a non-degenerate quadratic form $\mathrm{n}\colon \mathcal{A} \to \Bbbk$ such that $(\mathcal{A}, \mathrm{n})$ is a conic alternative algebra.

In this situation, the norm n is uniquely determined by the $\Bbbk$-algebra $\mathcal{A}$. Further, the conic algebra $(\mathcal{A}, \mathrm{n})$ is norm-associative and its conjugation is a scalar algebra involution.

Again, the norm in condition 10.2.12 (ii) is necessarily multiplicative by Note 10.1.29.

10.2.13 Remark. A composition algebra over $\Bbbk$ is a pre-composition algebra if and only if its norm is non-degenerate. By Figure 10.1, this is always the case if the composition algebra is regular of if $\Bbbk$ is a field. However, as [GPR, 19.6] illustrates, it is not true in general.

Unlike composition algebras, pre-composition algebras are not stable under base change, which makes them significantly less interesting.

10.2.14 Example ([GPR, 19.4]). Let K be a purely inseparable field extension of a field $\Bbbk$ of characteristic 2 such that $K^2 \subseteq \Bbbk$. Then K is a pre-composition algebra with anisotropic norm $x \mapsto x^2$. This norm is non-degenerate, but not weakly regular because its linearisation is the zero map. Further, there exists a scalar extension of $\Bbbk$ over which K is no longer a pre-composition algebra. In particular, K is not a composition algebra.

10.2.15 Lemma. *Assume that $\Bbbk$ is a field of characteristic not 2. Then every pre-composition algebra over $\Bbbk$ is a regular composition algebra.*

Proof. At first, assume only that $\Bbbk$ is a field, but make no assumption on its characteristic. In [SV00, 1.2.1], Springer-Veldkamp define a composition algebra over $\Bbbk$ to be a pre-composition algebra over $\Bbbk$ whose norm is weakly regular.[1] If $\Bbbk$ is not of characteristic 2, then this definition is equivalent to the notion of pre-composition algebras.

[1] More precisely, they define a composition algebra to be an algebra which satisfies Axiom 10.2.5 (c) for a "nondegenerate" norm, where by "nondegenerate" they mean what we call "weakly regular" (see [SV00, p. 2]).

Now assume, in addition, that $\Bbbk$ is not of characteristic 2. Then by the conclusion of the previous paragraph, it follows from [SV00, 1.6.2] that any pre-composition algebra $\mathcal{A}$ over $\Bbbk$ is finite-dimensional. By Figure 10.1, this implies that $\mathcal{A}$ is a regular composition algebra. $\square$

10.2.16 Note. Putting Lemmas 10.2.9 and 10.2.15 together, we see that the notions of composition algebras, regular composition algebra and pre-composition algebras coincide over fields of characteristic not 2. As Example 10.2.14 illustrates, this ceases to be true over general fields. See also [SV00, 1.2.2] for more remarks on this topic.

The following result will be useful in the discussion of conic algebras which coordinatise RGD-systems of type F_4.

10.2.17 Lemma. *Assume that $\Bbbk$ is a field and let $(\mathcal{A}, \mathrm{n})$ be a conic alternative algebra which is anisotropic as a quadratic space in the sense of Definition 7.1.9 (but which is not assumed to be multiplicative). Then $\mathcal{A}$ is a pre-composition algebra in which every non-zero element is invertible. If $\Bbbk$ is not of characteristic 2, then $\mathcal{A}$ is a regular composition algebra in which every non-zero element is invertible.*

Proof. Since n is anisotropic, its radical is zero, which says that n is non-degenerate. Hence $\mathcal{A}$ is a pre-composition algebra by Proposition 10.2.12. If $\Bbbk$ is not of characteristic 2, then it follows from Lemma 10.2.15 that $\mathcal{A}$ is a regular composition algebra. Further, $\mathcal{A}$ is a division algebra by Lemma 10.1.30. $\square$

10.3 The Root System F_4

As usual, we begin our discussion of the root system F_4 with a standard representation, even though it is too cumbersome to be of practical use.

10.3.1 Remark (Standard representation of F_4). Let V be a Euclidean space of dimension n with orthonormal basis $(b_1, \dots, b_4)$. The *standard representation of F_4* is

$$\begin{aligned} F_4 := \{\, \varepsilon_1 b_i + \varepsilon_2 b_j \mid i \neq j \in [1, n], \varepsilon_1, \varepsilon_2 \in \{\pm 1\} \,\} \cup \{\, \varepsilon b_i \mid i \in [1, n], \varepsilon \in \{\pm 1\} \,\} \\ \cup \left\{ \frac{1}{2}(\varepsilon_1 b_1 + \varepsilon_2 b_2 + \varepsilon_3 b_3 + \varepsilon_4 b_4) \,\middle|\, \varepsilon_1, \dots, \varepsilon_4 \in \{\pm 1\} \right\}. \end{aligned}$$

Its *standard root base* consists of

$$f_1 := b_2 - b_3, \quad f_2 := b_3 - b_4, \quad f_3 := b_4, \quad f_4 := \frac{1}{2}(b_1 - b_2 - b_3 - b_4),$$

so that f_1, f_2 are long while f_3, f_4 are short. In total, F_4 has 48 roots.

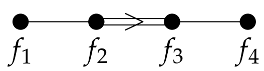


Figure 10.2: The Dynkin Diagram of F_4.

We will always consider root bases of F_4 to be in the following order, which we call standard.

10.3.2 Definition (Standard order). Let $\Delta = (f_1, f_2, f_3, f_4)$ be an ordered root base of F_4. We say that Δ is in *standard order* if $f_1, \dots, f_4$ are as in Figure 10.2. That is, (f_1, f_2) and (f_3, f_4) are A_2-pairs and (f_2, f_3) is a B_2-pair (which, in particular, means that f_2 is longer than f_3).

10.3.3 Definition (Canonical root subsystems). Let $\Delta = (f_1, f_2, f_3, f_4)$ be a root base of F_4 in standard order. The *canonical subsystem of type B_3 (with respect to Δ)* is the root subsystem generated by $\{ f_1, f_2, f_3 \}$. The *canonical subsystem of type C_3 (with respect to Δ)* is the root subsystem generated by $\{ f_2, f_3, f_4 \}$.

10.3.4 Lemma. *Every root in F_4 is contained in a parabolic subsystem of type B_3, in a parabolic subsystem of type C_3 and in a parabolic subsystem of type A_2.*

Proof. By an inspection of Figure 10.2, we see that the root f_2 has the desired properties. Since the Weyl group acts transitively on the set of long roots by Proposition 1.3.8, it follows that the assertion holds for all long roots. By the same argument, it holds for all short roots because it holds for f_3. □

In section 10.4, we will construct examples of F_4-graded groups as foldings of E_6-graded groups. In this setting, we will describe the root system F_4 as follows.

10.3.5 Construction (F_4 as a folding of E_6). Let $\Delta^E = (e_1, \dots, e_6)$ be an ordered root base of E_6 which is indexed[2] as in Figure 10.3, and denote by ρ the unique non-trivial automorphism of this diagram (also as in Figure 10.3). Denote by $(V, \cdot)$ the six-dimensional Euclidean space surrounding E_6. Since all roots of E_6 have the same length, ρ is an isometry, so $\rho = \tau$ in the notation of 1.4.1. Denote by F the fixed space of τ. Then

$$F = \langle e_2, e_4, e_3 + e_5, e_1 + e_6 \rangle_{\mathbb{R}} \quad \text{and} \quad F^{\perp} = \langle e_3 - e_5, e_1 - e_6 \rangle_{\mathbb{R}}.$$

Denote by $\pi\colon V \to F$ the orthogonal projection on F (whose kernel is $F^{\perp}$). Then it follows from Proposition 1.4.11 that $\Phi' := \pi(E_6)$ is a root system of rank 4 in F and that $\Delta' := \pi(\Delta^E)$ is a root base of Φ'. An inspection shows that Φ' is of type F_4 and that the vectors

$$f_1 := e_2, \quad f_2 := e_4, \quad f_3 := \pi(e_3) = \pi(e_5) = \frac{1}{2}(e_3 + e_5),$$
$$f_4 := \pi(e_1) = \pi(e_6) = \frac{1}{2}(e_1 + e_6)$$

are in standard order. See, for example, [Car72, 13.3.3].

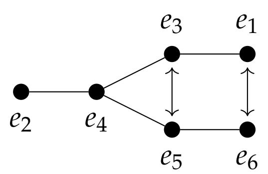


Figure 10.3: The non-trivial diagram automorphism ρ of E_6.

10.3.6 Remark. The long roots in F_4 are precisely those whose preimage in E_6 under π contains exactly one root (namely, the root itself) while the short roots in F_4 are those whose preimage contains exactly two roots (whose common image in F_4 is their average by Lemma 1.4.4). In particular, every long root in F_4 is also a root in E_6. Further, the two roots in the preimage of a short root are orthogonal (and thus adjacent). This is clear from Figure 10.3 for simple roots and it follows for arbitrary roots by the transitivity properties of the Weyl group.

[2]This seemingly unnatural ordering of Δ^E agrees with the one in [Hum72]. We use it because it is the one used by the GAP computer algebra system.

10.4 Construction of F_4-graded Groups via Foldings

10.4.1 Notation for this section. We consider the root system F_4 as a folding of E_6 as in Construction 10.3.5. In particular, we denote by $(V, \cdot)$ the Euclidean space surrounding E_6, by $\rho = \tau$ the non-trivial diagram automorphism of E_6, by F the fixed space of ρ, by $\pi\colon V \to F$ the orthogonal projection onto F and by $\Delta^E = (e_1, \dots, e_6)$ and $\Delta^F = (f_1, \dots, f_4) = \pi(\Delta^E)$ the corresponding root bases. Further, we fix a family $c = (c_{\alpha,\beta})_{\alpha,\beta \in E_6}$ of Chevalley structure constants of type E_6 and by $(G, (U_\alpha)_{\alpha \in E_6})$ a Chevalley group of type E_6 over the commutative associative ring $\mathcal{A}$ with respect to c. We denote the corresponding root isomorphisms by $(\theta_\alpha^E)_{\alpha \in E_6}$.

10.4.2 Reminder (see Example 10.1.16). The commutative associative ring $\mathcal{A} \times \mathcal{A}$ is a multiplicative conic algebra over $\mathcal{A}$ with norm $\mathrm{n}(a,b) := ab$, trace $\mathrm{tr}(a,b) = a + b$ and conjugation $\overline{(a,b)} = (b,a)$ for all $(a,b) \in \mathcal{A}$.

We already know from Proposition 2.6.7 that the E_6-graded group G has a crystallographic F_4-grading $(\hat{U}_{\alpha'})_{\alpha' \in F_4}$ given by

$$\hat{U}_{\alpha'} := \langle U_\alpha \mid \alpha \in E_6, \pi(\alpha) = \alpha' \rangle$$

for all $\alpha' \in F_4$. In this section, we want to show that this F_4-grading is coordinatised by the multiplicative conic algebra $\mathcal{A} \times \mathcal{A}$ over $\mathcal{A}$. In the first subsection, we describe the naive approach to this question. Unfortunately, this strategy turns out to be unsuitable because certain signs do not fit together. In the second subsection, we construct a refined coordinatisation of $(\hat{U}_\alpha)_{\alpha \in F_4}$ by twisting certain root isomorphism of the naive coordinatisation. Since the choice of root isomorphisms which have to be twisted depends on the family c, we have to fix c from this point on. For computational practicality, we choose c to be the family of structure constants that is used in the computer algebra system GAP [Gap].

In the final subsection, we assume that $\mathcal{A}$ is equipped with the additional structure of a faithful multiplicative conic algebra over a commutative associative ring $\Bbbk$. We then construct a crystallographic F_4-graded subgroup of G which is coordinatised by the conic algebra $\mathcal{A}$. This solves the existence problem for F_4-graded groups for multiplicative conic alternative algebras which are, in addition, commutative, associative and faithful.

10.4.A The Naive Coordinatisation

At first, we need to introduce a total order on F_4. There is no abstract reason to prefer any one order over another, but we need to fix some ordering in the definition of the (short) root homomorphisms (see Definitions 10.4.6 and 10.4.10).

10.4.3 Reminder (Lexicographic order). Let $n \in \mathbb{N}_+$. The *strict lexicographic order on* $\mathbb{R}^n$, denoted by $<_{\mathrm{lex}}$, is the strict total order defined as follows: For all $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \mathbb{R}^n$, we have $a <_{\mathrm{lex}} b$ if and only if there exists $i \in [1, n]$ such that $a_j = b_j$ for all $j \in [1, i-1]$ and $a_i < b_i$. Further, the *lexicographic order on* $\mathbb{R}^n$ is the total order $\leq_{\mathrm{lex}}$ which is defined as follows: $a \leq_{\mathrm{lex}} b$ holds if and only if $a = b$ or $a <_{\mathrm{lex}} b$.

10.4.4 Definition (Total order on F_4). We define a total order $\prec$ on the positive system Π_F of F_4 corresponding to Δ^F by

$$\begin{aligned} 1000 &\prec 0100 \prec 0010 \prec 0001 \prec 1100 \prec 0110 \prec 0011 \prec 1110 \prec 0120 \\ &\prec 0111 \prec 1120 \prec 1111 \prec 0121 \prec 1220 \prec 1121 \prec 0122 \prec 1221 \\ &\prec 1122 \prec 1231 \prec 1222 \prec 1232 \prec 1242 \prec 1342 \prec 2342 \end{aligned}$$

where a sequence $a_1a_2a_3a_4$ represents $\sum_{i=1}^4 a_i f_i$. We extend $\prec$ to a total order on F_4 as follows:

(i) For all $\alpha \in \Pi_F$, $\beta \in -\Pi_F$, we declare that $\alpha \prec \beta$.

(ii) For all $\alpha, \beta \in -\Pi_F$, we declare that $\alpha \prec \beta$ if and only if $-\alpha \prec -\beta$.

10.4.5 Note. The ordering of the positive roots in Definition 10.4.4 is a height ordering in the sense of Definition 2.3.16, and it is also the one used in [VP96, p. 96].

10.4.6 Definition (Naive coordinatisation). The *naive coordinatisation of* $(\hat{U}_{\alpha'})_{\alpha' \in F_4}$ consists of the isomorphisms

$$\theta^{\mathrm{n}}_{\alpha'} \colon (\mathcal{k}, +) \to \hat{U}_{\alpha'} = U_{\alpha'}, a \mapsto \theta_{\alpha'}(a)$$

for all long roots $\alpha' \in F_4$ and

$$\theta^{\mathrm{n}}_{\beta'} \colon (\mathcal{k} \times \mathcal{k}, +) \to \hat{U}_{\beta'} = U_{\beta_1} U_{\beta_2}, (a, b) \mapsto \theta_{\beta_1}(a)\theta_{\beta_2}(b)$$

for all short roots $\beta' \in F_4$ with corresponding preimage $\pi^{-1}(\beta') \cap E_6 = \{ \beta_1, \beta_2 \}$ in E_6 where $\beta_1 \prec \beta_2$.

By Remark 10.3.6, the maps in Definition 10.4.6 are well-defined and isomorphisms. Note that we need some total order on F_4 to decide without ambiguity whether (a, b) should be mapped to $\theta_{\beta_1}(a)\theta_{\beta_2}(b)$ or $\theta_{\beta_2}(a)\theta_{\beta_1}(b)$.

10.4.7 Definition (Naive Weyl elements). For all roots $\alpha \in F_4$, we define the *naive standard α-Weyl element* to be

$$w^{\mathrm{n}}_\alpha := \theta^{\mathrm{n}}_{-\alpha}(-1_\alpha)\theta^{\mathrm{n}}_\alpha(1_\alpha)\theta^{\mathrm{n}}_{-\alpha}(-1_\alpha)$$

where $1_\alpha = 1_{\mathcal{A}}$ if α is long and $1_\alpha = (1_{\mathcal{A}}, 1_{\mathcal{A}})$ if α is short.

10.4.8 Remark. Let $\alpha, \beta, \delta \in F_4$ such that α is long and β is short and let $a, b \in \mathcal{A}$. Put $w := w^{\mathrm{n}}_\delta$. A straightforward computation shows that in all possible cases, we have

$$\theta^{\mathrm{n}}_\alpha(a)^w \in \{ \theta^{\mathrm{n}}_{\sigma_\delta(\alpha)}(\varepsilon a) \mid \varepsilon \in \{\pm 1\} \} \qquad \text{and}$$
$$\theta^{\mathrm{n}}_\beta(a, b)^w \in \{ \theta^{\mathrm{n}}_{\sigma_\delta(\beta)}(\varepsilon a, \sigma b), \theta^{\mathrm{n}}_{\sigma_\delta(\beta)}(\varepsilon b, \sigma a) \mid \varepsilon, \sigma \in \{\pm 1\} \}.$$

This suggests that the twisting group for the coordinatisation of $(\hat{U}_{\alpha'})_{\alpha' \in F_4}$ should be $\mathbb{Z}_2^3$ where the actions of the three components on $\mathcal{A} \times \mathcal{A}$ should be given by $(a, b) \mapsto (-a, -b)$, $(a, b) \mapsto (b, a)$ and $(a, b) \mapsto (-a, b)$ (or $(a, b) \mapsto (a, -b)$). However, this is a fallacy: By twisting the coordinatisation $(\theta^{\mathrm{n}}_{\alpha'})_{\alpha' \in F_4}$ appropriately, we can arrange it so that only the inversion action $(a, b) \mapsto (-a, -b)$ and the involution $(a, b) \mapsto (b, a)$ are required. Note that the latter is precisely the conjugation of the conic algebra $\mathcal{A} \times \mathcal{A}$.

10.4.B A Refined Approach

10.4.9 Notation for this section. From now on, we assume that c is the family of structure constants of the simple Lie algebra over $\mathbb{Q}$ of type E_6 which is constructed in GAP [Gap] with the command `SimpleLieAlgebra(“E”, 6, Rationals)`.

10.4.10 Definition (Refined coordinatisation). Put

$$\mathcal{T}' := \left\{ \begin{matrix} f_2+f_3, f_3+f_4, f_1+f_2+f_3, f_2+2f_3+f_4, f_1+f_2+2f_3+f_4, \\ f_1+2f_2+2f_3+f_4, f_1+2f_2+3f_3+2f_4 \end{matrix} \right\} \subseteq F_4,$$
$$\mathcal{T} := \mathcal{T}' \cup (-\mathcal{T}').$$

The *refined coordinatisation of* $(G, (\hat{U}_{\alpha'})_{\alpha' \in F_4})$ consists of the isomorphisms

$$\theta_\alpha := \theta^{\mathrm{n}}_{\alpha'} : (\Bbbk, +) \to \hat{U}_{\alpha'} = U_{\alpha'}, a \mapsto \theta_{\alpha'}(a)$$

for all long roots $\alpha' \in F_4$ and

$$\theta_{\beta'} : (\Bbbk \times \Bbbk, +) \to \hat{U}_{\beta'} = U_{\beta_1} U_{\beta_2}, (a, b) \mapsto \begin{cases} \theta_{\beta_1}(a)\theta_{\beta_2}(b) & \text{if } \beta' \notin \mathcal{T}, \\ \theta_{\beta_1}(a)\theta_{\beta_2}(-b) & \text{if } \beta' \in \mathcal{T} \end{cases}$$

for all short roots $\beta' \in F_4$ with corresponding preimage $\pi^{-1}(\beta') \cap E_6 = \{ \beta_1, \beta_2 \}$ in E_6 where $\beta_1 \prec \beta_2$.

10.4.11 Note. The set $\mathcal{T}'$ in Definition 10.4.10 is chosen precisely so that the following Proposition 10.4.13 holds. The specific form of $\mathcal{T}'$ has been determined by trial and error.

10.4.12 Definition (Refined Weyl elements). For all roots $\alpha \in F_4$, we define the *refined standard α-Weyl element* to be

$$w_\alpha := \theta_{-\alpha}(-1_\alpha)\theta_\alpha(1_\alpha)\theta_{-\alpha}(-1_\alpha)$$

where $1_\alpha = 1_{\mathcal{A}}$ if α is long and $1_\alpha = (1_{\mathcal{A}}, 1_{\mathcal{A}})$ if α is short.

10.4.13 Proposition. *Denote by $(A \times B, \mathcal{A} \times \mathcal{A}, \mathcal{A})$ the standard parameter system for the conic algebra $\mathcal{A} \times \mathcal{A}$ over $\mathcal{A}$ in the sense of Definition 10.1.17. Then there exist unique Δ^F-parity maps $\eta : F_4 \times \Delta^F \to A$ and $\mu : F_4 \times \Delta^F \to B$ such that $(\theta_\alpha)_{\alpha \in F_4}$ is a parametrisation of $(G, (\hat{U}_\alpha)_{\alpha \in F_4})$ by $(A \times B, \mathcal{A} \times \mathcal{A}, \mathcal{A})$ with respect to $(w_\delta)_{\delta \in \Delta^F}$ and $\eta \times \mu$ and such that $\mu_{\alpha,\delta} = 1_B$ for all long roots $\alpha \in F_4$ and all $\delta \in \Delta^F$. Further, these maps satisfy $\eta(\alpha, \delta) = \eta(-\alpha, \delta)$ and $\mu(\alpha, \delta) = \mu(-\alpha, \delta)$ for all $\alpha \in F_4$ and $\delta \in \Delta^F$.*

Proof. For the existence, we have to verify that

$$\theta_\alpha(a)^{w_\delta} \in \{ \theta_{\sigma_\delta(\alpha)}(\varepsilon a) \mid \varepsilon \in \{\pm 1\} \} \qquad \text{and}$$
$$\theta_\beta(a, b)^{w_\delta} \in \{ \theta_{\sigma_\delta(\beta)}(\varepsilon a, \varepsilon b), \theta_{\sigma_\delta(\beta)}(\varepsilon b, \varepsilon a) \mid \varepsilon \in \{\pm 1\} \}.$$

for all short roots $\beta \in F_4$, for all long roots $\alpha \in F_4$, for all $\delta \in \Delta^F$ and all $a, b \in \mathcal{A}$. This is a straightforward computation, which we have performed in GAP [Gap]. The uniqueness follows from the fact that A and B act faithfully on $\mathcal{A} \times \mathcal{A}$, that A acts faithfully on $\mathcal{A}$ and from the additional requirement that $\mu_{\alpha,\delta} = 1_B$ for all long roots $\alpha \in F_4$ and all $\delta \in \Delta^F$. The property that $\eta(\alpha, \delta) = \eta(-\alpha, \delta)$ and $\mu(\alpha, \delta) = \mu(-\alpha, \delta)$ for all $\alpha \in F_4$ and $\delta \in \Delta^F$ is also a mere computation. □

The precise values of η and μ can be found in Figure 10.4.

10.4.14 Definition (Standard parity maps). Let Φ' be any root system of type F_4 and let $\Delta' = (f'_1, f'_2, f'_3, f'_4)$ be any root base of Φ' in standard order. Denote by $\varphi : \Phi' \to F_4$ the unique isomorphism of root systems with $\varphi(f'_i) = f_i$ for all $i \in [1, 4]$. Then the *standard Δ'-parity maps for Φ'* are the Δ'-parity maps $\eta', \mu' : \Phi' \times \Delta' \to \mathbb{Z}_2$ defined by

$$\eta'(\alpha, \delta) := \eta\big(\varphi(\alpha), \varphi(\delta)\big) \quad \text{and} \quad \mu'(\alpha, \delta) := \mu\big(\varphi(\alpha), \varphi(\delta)\big)$$

for all $\alpha \in \Phi'$ and $\delta \in \Delta'$ where η, μ are as in Proposition 10.4.13.

α	$\bar{\eta}(\alpha, f_1)$	$\bar{\eta}(\alpha, f_2)$	$\bar{\eta}(\alpha, f_3)$	$\bar{\eta}(\alpha, f_4)$
1000	$(-1,1)$	$(-1,1)$	$(1,1)$	$(1,1)$
0100	$(1,1)$	$(-1,1)$	$(-1,1)$	$(1,1)$
0010	$(1,1)$	$(1,1)$	$(-1,1)$	$(-1,1)$
0001	$(1,1)$	$(1,1)$	$(1,1)$	$(-1,1)$
1100	$(-1,1)$	$(1,1)$	$(-1,1)$	$(1,1)$
0110	$(1,1)$	$(-1,1)$	$(-1,-1)$	$(-1,1)$
0011	$(1,1)$	$(1,1)$	$(-1,1)$	$(1,1)$
1110	$(-1,1)$	$(1,1)$	$(-1,-1)$	$(-1,1)$
0120	$(1,1)$	$(1,1)$	$(-1,1)$	$(-1,1)$
0111	$(1,1)$	$(-1,1)$	$(1,1)$	$(1,1)$
1120	$(-1,1)$	$(-1,1)$	$(-1,1)$	$(-1,1)$
1111	$(-1,1)$	$(1,1)$	$(1,1)$	$(1,1)$
0121	$(1,1)$	$(1,1)$	$(-1,1)$	$(-1,-1)$
1220	$(1,1)$	$(1,1)$	$(1,1)$	$(-1,1)$
1121	$(-1,1)$	$(-1,1)$	$(-1,1)$	$(-1,-1)$
0122	$(1,1)$	$(1,1)$	$(1,1)$	$(-1,1)$
1221	$(1,1)$	$(1,1)$	$(-1,1)$	$(-1,-1)$
1122	$(-1,1)$	$(-1,1)$	$(1,1)$	$(-1,1)$
1231	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$
1222	$(1,1)$	$(1,1)$	$(-1,1)$	$(-1,1)$
1232	$(1,1)$	$(1,1)$	$(-1,-1)$	$(-1,1)$
1242	$(1,1)$	$(-1,1)$	$(-1,1)$	$(1,1)$
1342	$(-1,1)$	$(1,1)$	$(1,1)$	$(1,1)$
2342	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$

Figure 10.4: The values of η and μ in Proposition 10.4.13, where $\bar{\eta} := \eta \times \mu$. In the first column, a sequence $a_1a_2a_3a_4$ represents $\sum_{i=1}^{4} a_i f_i$. Further, $\bar{\eta}(\alpha, \delta) = \bar{\eta}(-\alpha, \delta)$ for all $\alpha \in F_4$ and $\delta \in \Delta^F$.

Alternatively, the standard parity maps with respect to any root base in standard order can explicitly be defined to be the values given by Figure 10.4.

We can now compute the commutator relations in G with respect to $(\theta_\alpha)_{\alpha \in F_4}$.

10.4.15 Definition (Coordinatisation with standard signs). Denote by $\Delta' = (f_1', f_2', f_3', f_4')$ any root base of F_4 in standard order, let H be a group with an F_4-pregrading $(U_\alpha')_{\alpha \in F_4}$ and let $(w_\delta)_{\delta \in \Delta}$ be a Δ-system of Weyl elements. Let $(\mathcal{B}, \mathrm{n})$ be a multiplicative conic alternative algebra with conjugation $\bar{\cdot}$ and let $(A \times B, \mathcal{B}, \hbar)$ denote the standard parameter system for $(\mathcal{B}, \mathrm{n})$. A *coordinatisation of H by $(\mathcal{B}, \mathrm{n})$ with standard signs with respect to $(w_\delta)_{\delta \in \Delta}$* is a family $(\theta_\alpha')_{\alpha \in F_4}$ of maps such that the following conditions are satisfied:

(i) $(\theta_\alpha')_{\alpha \in F_4}$ is a parametrisation of $(H, (U_\alpha')_{\alpha \in F_4})$ by $(A \times B, \mathcal{B}, \hbar)$ with respect to $(w_\delta)_{\delta \in \Delta^F}$ and $\eta \times \mu$. Here η and μ denote the standard Δ'-parity maps of F_4 (from Definition 10.4.14).

(ii) The following commutator relations hold for all $\lambda, \mu \in k$ and all $c, d \in \mathcal{B}$:

$$\begin{aligned}
[\theta_{f_1}(\lambda), \theta_{f_2}(\mu)] &= \theta_{f_1+f_2}(-\lambda\mu), \\
[\theta_{f_2}(\lambda), \theta_{f_3}(c)] &= \theta_{f_2+f_3}(-c\,\mathrm{str}_{\mathcal{A}}(\lambda))\theta_{f_2+2f_3}(-\lambda\,\mathrm{n}(c)), \\
[\theta_{f_2+f_3}(c), \theta_{f_3}(d)] &= \theta_{f_2+2f_3}(\mathrm{n}(c,d)), \\
[\theta_{f_4}(c), \theta_{f_3}(d)] &= \theta_{f_3+f_4}(cd).
\end{aligned}$$

In the third relation, $\mathrm{n}\colon \mathcal{B} \times \mathcal{B} \to k$ denotes the linearisation of n. These relations are called the *standard commutator relations*.

10.4.16 Note. For all root systems Φ except for F_4, we defined standard coordinatisations by prescribing formulas for the commutator $[U_\alpha, U_\beta]$ for all pairs (α, β) of non-proportional roots. The number of such pairs is $|\Phi| \cdot (|\Phi| - 2)$ (if Φ is reduced), which is a large number even if Φ is "small". Until now, we could always bypass this problem by partitioning the set of root pairs into a manageable number of subsets and giving one commutator formula which holds for all pairs in a given subset. For example, we needed 13 formulas to describe the commutator relations in B_n (Proposition 7.3.25), and only one formula to describe simply-laced root gradings (Definition 5.6.2). In these examples, we did not have to require that a fixed family of Weyl elements acts on the root groups in a certain way because this is a consequence of the commutator formulas (see Propositions 5.6.6 (c), 7.7.6 (d) and 9.6.5 (d)).

For F_4, we are not aware of a way to partition the set of root pairs in a similar manner. Instead, we only describe the commutator relations on a small number of pairs and require in addition that the coordinatisation is compatible with a certain Δ'-system of Weyl elements. For any pair (α, β) of non-proportional roots in F_4, there exists an element u of the Weyl group such that (α^u, β^u) is pair with a prescribed (standard) commutator relation. Conjugating the standard commutator relation by an appropriate sequence of Weyl elements, we can thus compute the commutator relation for an arbitrary pair (α, β).

10.4.17 Remark (Weyl elements in standard coordinatisations). Let $(\mathcal{B}, \mathrm{n})$ be a multiplicative conic alternative algebra over an associative commutative ring k. Let $(H, (V_\gamma)_{\gamma \in F_4})$ be an F_4-graded group and let $(\theta_\gamma^H)_{\gamma \in F_4}$ be a coordinatisation of H by $(\mathcal{B}, \mathrm{n})$ with standard signs (and with respect to some family of Weyl elements). Recall from Lemma 10.1.30 that the set $\mathcal{B}^\times$ of invertible elements in $\mathcal{B}$ is precisely $\{ b \in \mathcal{B} \mid \mathrm{n}(b) \in k^\times \}$. By a similar computation as in Propositions 7.7.6 and 9.6.5, the maps

$$\theta_\alpha\colon k^\times \to V_\alpha^\sharp \quad \text{and} \quad \theta_\beta\colon \mathcal{B}^\times \to V_\beta^\sharp$$

are well-defined bijections for all long roots α and all short roots β. It follows that if H is an RGD-system, then k is a field and n is anisotropic. By Lemma 10.2.17, this implies that $(\mathcal{A}, \mathrm{n})$ is a pre-composition division algebra, and it is even a regular composition algebra if k is not of characteristic 2.

10.4.18 Proposition. *The F_4-graded group $(G, (\hat{U}_\alpha)_{\alpha \in F_4})$ is coordinatised by the conic algebra $\mathcal{A} \times \mathcal{A}$ over $\mathcal{A}$ with standard signs and with respect to $(w_\delta)_{\delta \in \Delta}$.*

Proof. We already know from Proposition 10.4.13 that Axiom 10.4.15 (i) is satisfied. It is a straightforward computation, which we have performed in GAP [Gap], to show that the desired commutator relations hold. $\square$

10.4.19 Remark. Let η and μ denote the standard Δ-parity maps for F_4. Only the following specific values of these parity maps will be needed in sections 10.6 and 10.7:

$$\begin{aligned}
(\eta \times \mu)_{f_2,f_1} &= (1,1), & (\eta \times \mu)_{f_2,f_1f_1} &= (-1,1),\\
(\eta \times \mu)_{f_2,f_1f_2} &= (1,1), & (\eta \times \mu)_{f_2,f_3} &= (-1,1),\\
(\eta \times \mu)_{f_3,f_4} &= (-1,1), & (\eta \times \mu)_{f_3,f_4f_4} &= (-1,1)\\
(\eta \times \mu)_{f_3,f_2} &= (1,1), & (\eta \times \mu)_{f_3,f_4f_3} &= (1,1),\\
(\eta \times \mu)_{f_3,(-f_2,f_3,f_2)} &= (-1,-1).
\end{aligned}$$

They can be read off from Figure 10.4, or follow from a short computation.

The following result will be generalised in Lemma 10.6.7 and then used to apply Proposition 4.6.3.

10.4.20 Lemma. *Let $\alpha \in F_4$ and let $\delta \in \Delta$ such that $\alpha \cdot \delta = 0$. If α is crystallographically adjacent to δ, then $\mu_{\alpha,\delta} = 1_B$. Otherwise, $\mu_{\alpha,\delta} = -1_B$.*

Proof. If α is crystallographically adjacent to δ, then it is also crystallographically adjacent to $-\delta$, so w_δ acts trivially on U_α. Hence $\mu_{\alpha,\delta} = 1_B$ in this case. The second assertion follows from an inspection of Figure 10.4. □

10.4.C A More General Construction

10.4.21 Notation for this section. From now on, we assume that $\Bbbk$ is a commutative associative ring and that $\mathcal{A}$ is equipped with the structure of a multiplicative conic algebra over $\Bbbk$ which is faithful, commutative and associative. We denote the norm on $\mathcal{A}$ by n and its conjugation by $\overline{\cdot}$. Further, we denote by $\tilde{\Bbbk}$ the image of $\Bbbk$ in $\mathcal{A}$ and by $\tilde{\mathcal{A}}$ the subalgebra $\{ (a, \overline{a}) \mid a \in \mathcal{A} \}$ of $\mathcal{A} \times \mathcal{A}$.

In Proposition 10.4.18, we have solved the existence problem for the specific kind of conic algebra from Example 10.1.16. The goal of this subsection is to construct a subgroup of G which is coordinatised by the conic algebra $\tilde{\mathcal{A}}$ over $\tilde{\Bbbk}$. Since $\tilde{\mathcal{A}}$ can be identified with $\mathcal{A}$ and $\tilde{\Bbbk}$ can be identified with $\Bbbk$, this solves the existence problem for multiplicative conic algebras which are faithful, commutative and associative. Observe that the faithfulness assumption is always satisfied by Lemma 10.1.13 if $2_\Bbbk$ is not a zero divisor in $\Bbbk$.

10.4.22 Definition. For all long roots α in F_4, we denote by $\bar{U}_\alpha$ the image of $\tilde{\Bbbk}$ under θ_α in $\hat{U}_\alpha$. For all short roots β, we denote by $\bar{U}_\beta$ the image of $\tilde{\mathcal{A}}$ under θ_β in $\hat{U}_\beta$. Further, we denote by $\bar{\theta}_\alpha$ and $\bar{\theta}_\beta$ the induced isomorphisms

$$\bar{\theta}_\alpha := \theta_\alpha \circ \mathrm{str}_{\mathcal{A}} \colon \Bbbk \to \bar{U}_\alpha \quad \text{and} \quad \bar{\theta}_\beta \colon \mathcal{A} \to \bar{U}_\beta, a \mapsto \theta_\beta(a, \overline{a})$$

and by $\bar{G}$ the subgroup of G which is generated by $(\bar{U}_\gamma)_{\gamma \in F_4}$.

10.4.23 Proposition. *The pair $(\bar{G}, (\bar{U}_\gamma)_{\gamma \in F_4})$ is an F_4-graded group which is coordinatised by the conic algebra $(\mathcal{A}, \mathrm{n})$ over $\Bbbk$ with standard signs and with respect to $(w_\delta)_{\delta \in \Delta}$.*

Proof. Observe that the Weyl elements $(w_\delta)_{\delta \in \Delta}$ do indeed lie in $\bar{G}$. Further, $\tilde{\Bbbk}$ and $\tilde{\mathcal{A}}$ are invariant under the action of the twisting group $A \times B$ in the standard parameter system $(A \times B, \mathcal{A} \times \mathcal{A}, \mathcal{A})$ for $\mathcal{A} \times \mathcal{A}$ and $\mathcal{A}$. It follows that

$$\bar{U}_\gamma^{w_\delta} = \bar{U}_{\sigma_\delta(\gamma)} \tag{10.2}$$

for all $\gamma \in F_4$ and all $\delta \in \Delta$. Further, $\bar{G}$ is generated by $(\bar{U}_\gamma)_{\gamma \in F_4}$ by definition, and $\bar{G}$ satisfies Axiom 2.5.2 (iv) because each root group of $\bar{G}$ is contained in a root group of G.

We now verify the commutator relations. For all $a, b, c \in \mathcal{A}$, we know from Proposition 10.4.18 that the commutator relation

$$[\theta_{f_2}(a), \theta_{f_3}(b,c)] = \theta_{f_2+f_3}\big(-(b,c)\cdot(a,a)\big)\theta_{f_2+2f_3}(-abc)$$

holds. This implies that

$$\begin{aligned}[\bar{\theta}_{f_2}(\lambda), \bar{\theta}_{f_3}(b)] &= [\theta_{f_2}(\lambda 1_{\mathcal{A}}), \theta_{f_3}(b, \bar{b})] = \theta_{f_2+f_3}(-\lambda b, -\lambda\bar{b})\theta_{f_2+2f_3}(-\lambda b\bar{b}) \\ &= \bar{\theta}_{f_2+f_3}\big(-b \operatorname{str}_{\mathcal{A}}(\lambda)\big)\bar{\theta}_{f_2+2f_3}\big(-\lambda \operatorname{n}(b)\big)\end{aligned}$$

for all $\lambda \in \Bbbk$ and $b \in \mathcal{A}$. Here we used that $b\bar{b} = \operatorname{n}(b)1_{\mathcal{A}}$ by Lemma 10.1.18 (c). Thus the desired commutator relation for $[\bar{U}_{f_2}, \bar{U}_{f_3}]$ is satisfied. Further, we know that

$$[\theta_{f_2+f_3}(a,b), \theta_{f_3}(c,d)] = \theta_{f_2+2f_3}\big(\big\langle (a,b)\,\big|\,(c,d)\big\rangle\big)$$

for all $a, b, c, d \in \mathcal{A}$ where $\langle\,\cdot \mid \cdot\,\rangle$ denotes the linearisation of the norm on $\mathcal{A} \times \mathcal{A}$. By Example 10.1.16, $\big\langle (a,b)\,\big|\,(c,d)\big\rangle = ad + bc$. We infer that

$$\begin{aligned}\big[\bar{\theta}_{f_2+f_3}(a), \bar{\theta}_{f_3}(c)\big] &= [\theta_{f_2+f_3}(a,\bar{a}), \theta_{f_3}(c,\bar{c})] = \theta_{f_2+2f_3}(a\bar{c} + \bar{a}c) \\ &= \theta_{f_2+2f_3}\big(\operatorname{n}(a,c)1_{\mathcal{A}}\big) = \bar{\theta}_{f_2+2f_3}\big(\operatorname{n}(a,c)\big)\end{aligned}$$

for all $a, b \in \mathcal{A}$. Here we used that $a\bar{c} + \bar{a}c = \operatorname{n}(a,c)1_{\mathcal{A}}$ by Lemma 10.1.18 (d) Hence the desired commutator relation for $[\bar{U}_{f_2+f_3}, \bar{U}_{f_3}]$ is satisfied as well. The commutator relations for $[\bar{U}_{f_1}, \bar{U}_{f_2}]$ and $[\bar{U}_{f_4}, \bar{U}_{f_3}]$ can be verified in a similar, straightforward manner. Together with (10.2), it follows that $\bar{G}$ has crystallographic F_4-commutator relations with root groups $(\bar{U}_\gamma)_{\gamma \in F_4}$.

Finally, we have to show that the coordinatisation $(\bar{\theta}_\gamma)_{\gamma \in F_4}$ is compatible with the Weyl elements $(w_\delta)_{\delta \in \Delta}$ and the standard Δ-parity maps η, μ. That is, we have to show that

$$\bar{\theta}_\alpha(\lambda)^{w_\delta} = \bar{\theta}_{\sigma_\delta(\alpha)}\big(\eta_{\alpha,\delta}\mu_{\alpha,\delta}.\lambda\big) \quad \text{and} \quad \bar{\theta}_\beta(a)^{w_\delta} = \bar{\theta}_{\sigma_\beta(\alpha)}\big(\eta_{\alpha,\delta}\mu_{\alpha,\delta}.a\big)$$

for all long roots α, all short roots β, all $\delta \in \Delta$, all $\lambda \in \Bbbk$ and all $a \in \mathcal{A}$. Since a similar statement is true for the coordinatisation $(\theta_\gamma)_{\gamma \in F_4}$, we only have to verify that the restriction of the standard parameter system $(A \times B, \mathcal{A} \times \mathcal{A}, \mathcal{A})$ to $(\tilde{\mathcal{A}}, \tilde{\Bbbk})$ is the standard parameter system for $\tilde{\mathcal{A}}$. Evidently, the inversion action on $\mathcal{A} \times \mathcal{A}$ and $\mathcal{A}$ restricts to the inversion action on $\tilde{\mathcal{A}}$ and $\tilde{\Bbbk}$. Further, the switching involution $(a,b) \mapsto (b,a)$ restricts to the conjugation $(a,\bar{a}) \mapsto (\bar{a}, \bar{\bar{a}}) = (\bar{a}, a)$ on $\tilde{\mathcal{A}}$. This finishes the proof. □

10.5 Computations in F_4-graded Groups

10.5.1 Notation for this section. We fix a root base Δ of F_4.

Most rank-2 and rank-3 computations that are necessary to parametrise F_4-graded groups have already been performed in the chapters for B_3 and C_3. In this section, we collect a small number of additional results that are required to apply the parametrisation theorem for F_4-graded groups.

We begin with a verification of one of the conditions in Proposition 4.6.3. We have already shown that it holds for $\Phi \in \{B_3, C_3\}$. By a reduction to rank-3 subsystems, we can lift the result to F_4.

10.5.2 Lemma. *Let $\Phi \in \{ B_3, C_3, F_4 \}$, let Δ be a root base of Φ and let $\alpha \in \Phi$. As in Proposition 4.6.3, we define sets*

$$\begin{aligned} \mathcal{O} &:= \{ \beta \in \Phi \mid \alpha \cdot \beta = 0 \}, \\ \mathcal{A} &:= \{ \beta \in \mathcal{O} \mid \alpha \text{ is (crystallographically) adjacent to } \beta \text{ and } -\beta \}, \\ \bar{\mathcal{A}} &:= \mathcal{O} \setminus \mathcal{A}. \end{aligned}$$

Then for all Δ-positive roots $\beta_1, \beta_2 \in \bar{\mathcal{O}}$, there exist words ${}_1\bar{\delta}$, ${}_2\bar{\delta}$ over $\Delta \cup (-\Delta)$ such that $\sigma({}_1\bar{\delta}) = \sigma(\beta_1)$, $\sigma({}_2\bar{\delta}) = \sigma(\beta_2)$ and such that for any group G which has crystallographic Φ-commutator relations with root groups $(U_\rho)_{\rho \in F_4}$ and for any Δ-system $(w_\delta)_{\delta \in \Delta}$ of Weyl elements in G, the actions of $w_{{}_1\bar{\delta}}$ and $w_{{}_2\bar{\delta}}$ on U_α are identical.

Proof. If α is long, then $\bar{\mathcal{A}}$ is empty, so we can assume that α is short. If $\Phi = C_3$, then $\bar{\mathcal{A}}$ contains exactly one element Δ-positive root (see Lemma 9.1.20), so the desired statement is trivially satisfied (because we can choose ${}_1\bar{\delta} = {}_2\bar{\delta}$). For $\Phi = B_3$, the desired statement holds by Proposition 7.6.21.

We proceed to show that the claim holds for $\Phi = F_4$ as well. Throughout the proof, we denote by G a group which has crystallographic F_4-commutator relations with root groups $(U_\rho)_{\rho \in F_4}$ and by $(w_\delta)_{\delta \in \Delta}$ a Δ-system of Weyl elements in G. We will make sure that all constructions in this proof are independent of the choices of G, $(U_\rho)_{\rho \in F_4}$ and $(w_\delta)_{\delta \in \Delta}$. Denote by Φ' a parabolic subsystem of F_4 of rank 3 which contains α, β_1, β_2. Note that Φ' must be of type $A_2 \times A_1$, B_3 or C_3. In the first case, we can simply take ${}_1\bar{\delta}$ and ${}_2\bar{\delta}$ to be Δ-expressions of β_1 and β_2, respectively, and then both $w_{{}_1\bar{\delta}}$ and $w_{{}_2\bar{\delta}}$ act trivially on U_α because orthogonal roots in $A_2 \times A_1$ are adjacent. Thus from now on, we can assume that Φ' is of type B_3 or C_3.

Choose an arbitrary root base Δ' of Φ'. By Lemma 1.3.7, there exists $u \in \mathrm{Weyl}(\Phi)$ such that Δ' is a subset of Δ^u. By Proposition 1.3.12, we can find a word $\bar{\zeta} = (\zeta_1, \dots, \zeta_k)$ over $\Delta \cup (-\Delta)$ such that $u = \sigma(\bar{\zeta})$. Choose roots $\delta_1, \delta_2, \delta_3 \in \Delta$ such that $\Delta' = \{ \delta'_1, \delta'_2, \delta'_3 \}$ where $\delta'_i := \delta_i^u$ for all $i \in \{ 1, 2, 3 \}$. Further, define $w'_{\delta'_i} := w_{\delta_i}^{w_{\bar{\zeta}}}$ for all $i \in \{ 1, 2, 3 \}$, so that $(w'_{\delta'})_{\delta' \in \Delta'}$ is a Δ'-system of Weyl elements in the group generated by $(U_\rho)_{\rho \in \Phi'}$. Since the assertion is known to be true for $\Phi \in \{ B_3, C_3 \}$, there exist words ${}_1\bar{\delta}', {}_2\bar{\delta}'$ over $\Delta' \cup (-\Delta')$ such that $\sigma({}_1\bar{\delta}') = \sigma(\beta_1)$, $\sigma({}_2\bar{\delta}') = \sigma(\beta_2)$ and such that the actions of $w'_{{}_1\bar{\delta}'}$ and $w'_{{}_2\bar{\delta}'}$ on U_α are identical. For each $i \in \{ 1, 2 \}$, we can write ${}_i\bar{\delta}'$ as

$$\big(\varepsilon_{i,1}\delta'_{n(i,1)}, \dots, \varepsilon_{i,m_i}\delta'_{n(i,m_i)}\big)$$

for integers $m_i \in \mathbb{N}_0$, $n(i,1), \dots, n(i,m_i) \in \{ 1, 2, 3 \}$ and $\varepsilon_{i,1}, \dots, \varepsilon_{i,m_i} \in \{ \pm 1 \}$ because ${}_i\bar{\delta}'$ is a word over $\Delta' \cup (-\Delta')$. Now we can define

$${}_i\bar{\delta} := \big(\bar{\zeta}^{-1}, \varepsilon_{i,1}\delta_{n(i,1)}, \dots, \varepsilon_{i,m_i}\delta_{n(i,m_i)}, \bar{\zeta}\big) \quad \text{for all } i \in \{ 1, 2 \}.$$

In the following, for any word $\bar{\rho}$ over $\Delta \cup (-\Delta)$ we will write $w(\bar{\rho})$ instead of $w_{\bar{\rho}}$ for better legibility, and similarly for $w'(\bar{\rho}')$. Then for all $i \in \{ 1, 2 \}$, we have

$$\begin{aligned} w({}_i\bar{\delta}) &= \left(\prod_{j=1}^{m} w(\delta_{n(i,j)})^{\varepsilon_{i,j}} \right)^{w(\bar{\zeta})} = \prod_{j=1}^{m} w(\delta_{n(i,j)})^{\varepsilon_{i,j} w(\bar{\zeta})} = \prod_{j=1}^{m} w'(\delta'_{n(i,j)})^{\varepsilon_{i,j}} \\ &= w'({}_i\bar{\delta}') \end{aligned}$$

and, similarly,

$$\sigma({}_i\bar{\delta}) = \left(\prod_{j=1}^{m_i} \sigma(\delta_{n(i,j)})\right)^{\sigma(\bar{\zeta})} = \prod_{j=1}^{m_i} \sigma(\delta_{n(i,j)})^{\sigma(\bar{\zeta})} = \prod_{j=1}^{m_i} \sigma\big(\delta_{n(i,j)}^{\sigma(\bar{\zeta})}\big) = \prod_{j=1}^{m_i} \sigma(\delta'_{n(i,j)})$$
$$= \sigma({}_i\bar{\delta}') = \sigma(\beta_i).$$

We conclude that the words ${}_1\bar{\delta}$ and ${}_2\bar{\delta}$ have the desired properties. □

10.5.3 Remark. Denote by η and μ the standard Δ-parity maps for F_4. By the uniqueness of these maps, the words ${}_1\bar{\delta}$ and ${}_2\bar{\delta}$ in Lemma 10.5.2 automatically satisfy $\eta(\alpha, {}_1\bar{\delta}) = \eta(\alpha, {}_2\bar{\delta})$ and $\mu(\alpha, {}_1\bar{\delta}) = \mu(\alpha, {}_2\bar{\delta})$.

10.5.4 Lemma. *Let G be a group with a crystallographic F_4-grading $(U_\alpha)_{\alpha\in\Phi}$ and let Φ_C be a parabolic subsystem of F_4 of type C_3. Let H be the corresponding Φ_C-graded subgroup of G. Then the short involution on H (which, by Note 9.5.6, is defined on the long root groups of H) is trivial.*

Proof. Let α be a long root in Φ and let w_α be any α-Weyl element. By an application of Lemma 10.3.4, we find a parabolic A_2-subsystem Φ_A of F_4 which contains α. Hence it follows from Lemma 5.4.15 that w_α^2 acts trivially on U_α. This action is precisely the short involution on U_α (by Definition 9.5.4), which finishes the proof. □

10.5.5 Lemma. *Let G be a group with a crystallographic F_4-grading $(U_\alpha)_{\alpha\in\Phi}$. Then G satisfies the square formula for Weyl elements.*

Proof. Let α, β be roots and let Φ be a parabolic rank-3 subsystem of F_4 which contains α and β. Then Φ is of type B_3, C_3 or $A_2 \times A_1$. In each case, the square formula is satisfied by Propositions 5.4.17, 7.6.15 and 9.5.13. In the case where Φ is of type C_3, we have to use in addition that the short involution in G is trivial, which holds by Lemma 10.5.4. □

10.6 The Parametrisation

In this section, we define the standard partial twisting system for F_4, show that it satisfies all desired compatibility conditions, and use it to parametrise arbitrary F_4-graded groups.

10.6.1 Notation for this section. We choose a root base Δ of F_4 in standard order and we denote by (A, η, B, μ) the standard partial twisting system of type F_4 with respect to Δ in the sense of the following Definition 10.6.2.

10.6.2 Definition (Standard partial twisting system). The *standard partial twisting system of type F_4 (with respect to Δ)* is the tuple (A, η, B, μ) where $A := B := \{\pm 1\}$ and where η, μ are the standard Δ-parity maps for F_4. If H is a group with an F_4-pregrading, then the *standard partial twisting system for G* is the same tuple together with the additional information that A acts on all root groups of H by inversion.

We begin with the purely combinatorial properties of the parity maps.

10.6.3 Lemma. *η is braid-invariant, μ is Weyl-invariant and both are adjacency-trivial. Further, η satisfies the square formula.*

Proof. Braid-invariance and adjacency-triviality follow from Proposition 4.4.11. The square formula follows from Proposition 4.4.11 together with Lemma 10.5.5. The square-invariance of μ can be verified computationally. □

10.6.4 Lemma. *$\eta \times \mu$ is transporter-invariant and η, μ are independent.*

Proof. We begin with the orbit of long roots. Put $\hat{\alpha} := f_2$. For all long roots β, we know from Proposition 10.4.13 that $(A \times B)_{\hat{\alpha} \to \beta}$ is contained in $A \times \{1_B\}$. Further,

$$(\eta \times \mu)_{f_2, f_1 f_1} = (-1, 1)$$

by Remark 10.4.19. Hence $(A \times B)_{\hat{\alpha} \to \hat{\alpha}} = A \times \{1_B\}$. Thus it follows from criterion 4.2.16 (iii) (with $\hat{\beta} := \hat{\alpha}$) that $\eta \times \mu$ is transporter-invariant on the orbit of long roots.

For the orbit of short roots, we consider $\hat{\beta} := f_3$. Again by Remark 10.4.19, we have

$$(\eta \times \mu)_{f_3, f_4 f_4} = (-1, 1) \quad \text{and} \quad (\eta \times \mu)_{f_3, (-f_2, f_3, f_2)} = (-1, -1).$$

Since $\sigma(f_4 f_4)$ and $\sigma(-f_2, f_3, f_2)$ stabilise f_3, it follows that $(A \times B)_{\hat{\alpha} \to \hat{\alpha}} = A \times B$. Hence again by criterion 4.2.16 (iii) (with $\hat{\beta} := \hat{\alpha}$), we infer that $\eta \times \mu$ is transporter-invariant on the orbit of short roots as well.

Finally, the previous computations together with Remark 4.2.24 show that η and μ are independent. □

10.6.5 Lemma. *μ is semi-complete.*

Proof. This follows from the fact that the only subgroups of B are $\{1\}$ and B, just like in Lemma 7.8.7. □

We conclude that the standard partial twisting system is indeed a partial twisting system.

10.6.6 Lemma. *Let $(G, (U_\alpha)_{\alpha \in F_4})$ be a crystallographic F_4-graded group and choose any Δ-system $(w_\delta)_{\delta \in \Delta}$ of Weyl elements in G. Then the standard partial twisting system (A, η, B, μ) for G (in the sense of Definition 10.6.2) is a partial twisting system for $(G, (w_\delta)_{\delta \in \Delta})$ (in the sense of Definition 4.3.18).*

Proof. By Example 4.3.15, A is a twisting group for $(G, (U_\alpha)_{\alpha \in F_4})$. The remaining properties are satisfied by Lemmas 10.6.3 to 10.6.5. □

It remains to verify the compatibility conditions. We begin with a generalisation of Lemma 10.4.20.

10.6.7 Lemma. *Let $\alpha, \beta \in F_4$ be orthogonal. Then $\mu_{\alpha, \sigma(\beta)} = 1_B$ if α and β are crystallographically adjacent and $\mu_{\alpha, \sigma(\beta)} = -1_B$ otherwise.*

Proof. By Lemma 10.4.20, the assertion is true if β lies in Δ. In general, we know from Proposition 1.3.4 that there exists $u \in \mathrm{Weyl}(F_4)$ such that $\delta := \beta^u$ lies in Δ. Choose a word $\bar{\rho}$ over Δ such that $\sigma(\bar{\rho}) = u$. Then it follows from Lemma 4.2.8 that

$$\eta(\alpha, \sigma_\beta) = \eta(\alpha^{\sigma(\bar{\rho})}, \sigma_{\bar{\rho}}^{-1} \sigma_\beta \sigma_{\bar{\rho}}).$$

Here $\sigma_{\bar{\rho}}^{-1} \sigma_\beta \sigma_{\bar{\rho}} = \sigma(\beta^{\sigma(\bar{\rho})}) = \sigma(\delta)$. Note that α and β are orthogonal if and only if α^u and δ are orthogonal, and the same holds for crystallographic adjacency. Hence the general assertion follows from the special case in Lemma 10.4.20. □

10.6.8 Proposition. *Let $(G, (U_\alpha)_{\alpha \in F_4})$ be a crystallographic F_4-graded group and let $(w_\delta)_{\delta \in \Delta}$ be a Δ-system of Weyl elements in G. Then G is square-compatible with respect to η and $(w_\delta)_{\delta \in \Delta}$.*

Proof. We know from Lemma 10.5.5 that G satisfies the square formula for Weyl elements and from Lemma 10.6.3 that η satisfies the square formula. Hence the assertion follows from Lemma 4.6.2. □

10.6.9 Proposition. *Let $(G, (U_\alpha)_{\alpha \in F_4})$ be a crystallographic F_4-graded group and let $(w_\delta)_{\delta \in \Delta}$ be a Δ-system of Weyl elements in G. Then G is stabiliser-compatible with respect to (η, μ) and $(w_\delta)_{\delta \in \Delta}$.*

Proof. By Lemmas 10.5.2 and 10.6.7 and Remark 10.5.3, the conditions of Proposition 4.6.3 are satisfied. The assertion follows. □

Finally, we can apply the parametrisation theorem.

10.6.10 Proposition. *Let $(G, (U_\alpha)_{\alpha \in F_4})$ be a crystallographic F_4-graded group, let $(w_\delta)_{\delta \in \Delta}$ be a Δ-system of Weyl elements in G and let (A, η, B, μ) be the standard partial twisting system for G. Then there exist abelian groups $(\mathscr{k}, +)$ and $(\mathcal{A}, +)$ (both equipped, as sets, with an action of $A \times B$) such that G is parametrised by $(A \times B, \mathcal{A}, \mathscr{k})$ with respect to $\eta \times \mu$ and $(w_\delta)_{\delta \in \Delta}$ and such that the action of A on $\mathscr{k}$ and $\mathcal{A}$ is given by group inversion.*

Proof. This follows from the parametrisation theorem (Theorem 4.5.16), whose conditions are satisfied by Propositions 10.6.8 and 10.6.9. □

10.7 The Coordinatisation

10.7.1 Notation for this section. We denote by $\Delta = (f_1, f_2, f_3, f_4)$ a root base of F_4 in standard order and by $(V, \cdot)$ the Euclidean space which is generated by Δ. We consider a group G group with a crystallographic F_4-grading $(U_\alpha)_{\alpha \in F_4}$ and a Δ-system of Weyl elements $(w_\delta)_{\delta \in \Delta}$. We denote by (A, η, B, μ) the standard partial twisting system for G (from Definition 10.6.2), by $(\mathscr{k}, +)$, $(\mathcal{A}, +)$ abelian groups as in Proposition 10.6.10 (which are equipped, as sets, with actions of $A \times B$) and by $(\theta_\alpha)_{\alpha \in F_4}$ the corresponding parametrisation of G. The action of -1_B on $\mathcal{A}$ is denoted by

$$\overline{\cdot} \colon \mathcal{A} \to \mathcal{A}, a \mapsto \overline{a} := -1_B.a.$$

We choose elements $1_{\mathscr{k}} \in \mathscr{k}$ and $1_{\mathcal{A}} \in \mathcal{A}$ such that

$$w_{f_2} \in U_{-f_2}\theta_{f_2}(1_{\mathscr{k}})U_{-f_2} \quad \text{and} \quad w_{f_3} \in U_{-f_3}\theta_{f_3}(1_{\mathcal{A}})U_{-f_3}.$$

Put $A' := \{\pm 1\}^2$. We identify A with the subgroup $\{ (\pm 1, 1) \}$ of A' and we declare that the second component of A' acts trivially on all root groups. Further, we denote by G_B and G_C the root graded subgroups of G which correspond to the canonical root subsystems of types B_3 and C_3, respectively (as in Definition 10.3.3).

It follows from Proposition 5.4.10 (b) that $1_{\mathscr{k}}$ and $1_{\mathcal{A}}$ are uniquely determined by w_{f_2} and w_{f_3}.

10.7.2 Remark (on the proofs of Lemmas 10.7.3 and 10.7.5). In the proof of Lemma 10.7.3, we will apply the parametrisation theorem for B_3-graded groups (Proposition 7.9.4) to the subgroup G_B of G. A priori, this yields a new group $\mathscr{k}'$

which parametrises the long root groups and a new group $\mathcal{A}'$ which parametrises the short root groups. Further, the groups $\Bbbk'$ and $\mathcal{A}'$ are, as sets, equipped with an action of $A \times B$. Afterwards, Theorem 7.11.21 provides the commutator relations in G_B with respect to $\Bbbk'$ and $\mathcal{A}'$.

By Remark 4.5.17, we can choose $\Bbbk'$ to be equal (and not just isomorphic) to $\Bbbk$ and $\mathcal{A}'$ to be equal to $\mathcal{A}'$. However, it is a priori not clear that the actions of $A \times B$ on $\Bbbk'$ and $\mathcal{A}'$ agree with the respective actions on $\Bbbk$ and $\mathcal{A}$ after this identification. In other words, the notation $(a, b).x$ is (possibly) ambiguous for $a \in A$, $b \in B$ and $x \in \Bbbk \cup \mathcal{A}$. For the action of A, no ambiguity arises because we know that -1_A acts by group inversion in both cases. Further, the action of B on $\Bbbk$ will never be relevant (see Note 4.5.11). However, it will turn out that the action of B on $\mathcal{A}$ does not, in general, agree with the action of B on $\mathcal{A}'$. This inconvenience is not relevant in the proof of Lemma 10.7.3, so we delay a more detailed investigation of this phenomenon until Remark 10.7.4.

A similar remark holds for the proof of Lemma 10.7.5, where we apply the parametrisation theorem for C_3-graded groups (Proposition 9.8.4). One notable difference is that the parametrisation theorem for C_3-graded groups provides actions of the large group $A' \times B$ on the parametrising structure, not of the group $A \times B$. However, we can quickly focus our interest on the action of this subgroup. Further, we will have to investigate the relationship between the two actions of B on $\mathcal{A}$ during the proof of Lemma 10.7.5 (and not, as in Lemma 10.7.3, as an afterthought). It will turn out that the two actions of B on $\mathcal{A}$ are, in fact, identical.

10.7.3 Lemma. *There exist a commutative associative ring structure on $\Bbbk$ whose identity element is $1_\Bbbk$, a (right) $\Bbbk$-module structure $\star$ on $\mathcal{A}$ and a $\Bbbk$-quadratic form $\mathrm{n}\colon \mathcal{A} \to \Bbbk$ with $\mathrm{n}(1_{\mathcal{A}}) = 1_\Bbbk$ such that the following commutator relations hold for all $\lambda, \mu \in \Bbbk$ and all $c, d \in \mathcal{A}$, where $\langle \cdot \mid \cdot \rangle$ denotes the linearisation of n:*

$$\begin{aligned}
[\theta_{f_1}(\lambda), \theta_{f_2}(\mu)] &= \theta_{f_1+f_2}(-\lambda\mu), \\
[\theta_{f_2}(\lambda), \theta_{f_3}(c)] &= \theta_{f_2+f_3}(-c \star \lambda)\theta_{f_2+2f_3}(-\lambda\, \mathrm{n}(c)), \\
[\theta_{f_2+f_3}(c), \theta_{f_3}(d)] &= \theta_{f_2+2f_3}(\langle c|d\rangle).
\end{aligned}$$

Proof. Denote by $\Delta^B := (b_1, b_2, b_3) := (f_1, f_2, f_3)$ the ordered root base of the canonical B_3-subsystem of F_4 and by $\eta^B\colon B_3 \times \Delta^B \to A$ and $\mu^B\colon B_3 \times \Delta^B \to B$ the Δ^B-parity maps which we defined in 7.3.18. This means that, if we choose an orthonormal basis (e_1, e_2, e_3, e_4) of V such that $b_1 = e_1 - e_2$, $b_2 = e_2 - e_3$ and $b_3 = e_3$, then the maps η^B and μ^B are given by the exact same formulas as in Figure 7.4 on page 189. Observe that (A, η^B, B, μ^B) is the standard partial twisting system for G_B with respect to Δ^B (from Definition 7.8.2).

We know from Proposition 7.9.4 that there exist groups $(\Bbbk', +)$ and $(\mathcal{A}', +)$ (each equipped with an action of $A \times B$), an isomorphism $\theta^B_\alpha\colon (\Bbbk', +) \to U_\alpha$ for each long root α in B_3 and an isomorphism $\theta^B_\beta\colon (\mathcal{A}', +) \to U_\beta$ for each short root β in B_3 such that G_B is parametrised by $(A \times B, \mathcal{A}', \Bbbk')$ with respect to $\eta^B \times \mu^B$ and $(w_\delta)_{\delta \in \Delta^B}$ and such that the action of A on $\mathcal{A}'$ and $\Bbbk'$ is given by group inversion. Denote the corresponding root isomorphisms by $(\theta^B_\alpha)_{\alpha \in B_3}$. By Remark 4.5.17, we can achieve that $\Bbbk' = \Bbbk$, $\mathcal{A}' = \mathcal{A}$, $\theta^B_{f_2} = \theta_{f_2}$ and $\theta^B_{f_3} = \theta_{f_3}$. (See, however, the warning in Remark 10.7.2.)

We now turn to the relationship between the two families of root isomorphisms. We know from Remark 10.4.19 that

$$(\eta \times \mu)_{f_2,f_1} = (1,1), \qquad (\eta \times \mu)_{f_2,f_1f_2} = (1,1), \qquad (\eta \times \mu)_{f_2,f_3} = (-1,1) \quad \text{and} \quad (\eta \times \mu)_{f_3,f_2} = (1,1).$$

Using the table in Figure 7.4, we can also compute that

$$\begin{aligned}
(\eta^B \times \mu^B)_{f_2,f_1} &= (\eta^B \times \mu^B)_{e_2-e_3,e_1-e_2} = (-1,1),\\
(\eta^B \times \mu^B)_{f_2,f_1f_2} &= (\eta^B \times \mu^B)_{e_2-e_3,(e_1-e_2,e_2-e_3)}\\
&= (\eta^B \times \mu^B)_{e_2-e_3,e_1-e_2}(\eta^B \times \mu^B)_{e_1-e_3,e_2-e_3}\\
&= (-1,1)(-1,1) = (1,1),\\
(\eta^B \times \mu^B)_{f_2,f_3} &= (\eta^B \times \mu^B)_{e_2-e_3,e_3} = (-1,1),\\
(\eta^B \times \mu^B)_{f_3,f_2} &= (\eta^B \times \mu^B)_{e_3,e_2-e_3} = (-1,1).
\end{aligned}$$

Since

$$f_2^{\sigma(f_1)} = f_1 + f_2, \quad f_2^{\sigma(f_1f_2)} = f_1, \quad f_2^{\sigma(f_3)} = f_2 + 2f_3 \text{ and } f_3^{\sigma(f_2)} = f_2 + f_3,$$

it follows that

$$\begin{aligned}
\theta_{f_1+f_2}(a) &= \theta_{f_2}(a)^{w_{f_1}} = \theta^B_{f_2}(a)^{w_{f_1}} = \theta^B_{f_1+f_2}(-a),\\
\theta_{f_1}(a) &= \theta_{f_2}(a)^{w_{f_1}w_{f_2}} = \theta^B_{f_2}(a)^{w_{f_1}w_{f_2}} = \theta^B_{f_1}(a),\\
\theta_{f_2+2f_3}(u) &= \theta_{f_2}(-u)^{w_{f_3}} = \theta^B_{f_2}(-u)^{w_{f_3}} = \theta^B_{f_2+2f_3}(u),\\
\theta_{f_2+f_3}(u) &= \theta_{f_3}(u)^{w_{f_2}} = \theta^B_{f_3}(u)^{w_{f_2}} = \theta^B_{f_2+f_3}(-u).
\end{aligned}$$

We can now investigate the commutator relations. From Theorem 7.11.21, we know that there exist a commutative associative ring structure on $\mathscr{k}$ whose identity element is $1_{\mathscr{k}}$, a (left) $\mathscr{k}$-module structure $\star'$ on $\mathcal{A}$ and a $\mathscr{k}$-quadratic form $\mathrm{n}\colon \mathcal{A} \to \mathscr{k}$ with $\mathrm{n}(1_{\mathcal{A}}) = 1_{\mathscr{k}}$ such that G_B is coordinatised by $(\mathcal{A}, \mathscr{k})$ with standard signs. In particular, this means that the following commutator relations hold for all $\lambda, \mu \in \mathscr{k}$ and all $u, v \in \mathcal{A}$, where $\langle \cdot \mid \cdot \rangle$ denotes the linearisation of n:

$$\begin{aligned}
[\theta^B_{f_1}(\lambda), \theta^B_{f_2}(\mu)] &= \theta^B_{f_1+f_2}(\lambda\mu),\\
[\theta^B_{f_2}(\lambda), \theta^B_{f_3}(c)] &= \theta^B_{f_2+f_3}(\lambda \star' c)\theta^B_{f_2+2f_3}\big(-\lambda\, \mathrm{n}(c)\big),\\
[\theta^B_{f_2+f_3}(c), \theta^B_{f_3}(d)] &= \theta^B_{f_2+2f_3}\big(-\langle c|d\rangle\big).
\end{aligned}$$

Using the formulas above for the root isomorphisms, and replacing the left module structure $\star'$ by the corresponding right module structure defined by $c \star \lambda := \lambda \star' c$, we conclude that the desired commutator relations hold. This finishes the proof. $\square$

10.7.4 Remark. Recall from Remark 10.7.2 that we have two actions of B on the module $\mathcal{A}$ from Lemma 10.7.3. The goal of this remark is two compare these two actions. First of all, we have to make our notation precise: For all $c \in \mathcal{A}$, we define $\bar{c} := -1_B.c$ where $-1_B.u$ denotes the action from Proposition 10.6.10 (as in Notation 10.7.1) and we define $\bar{c}^B := -1_B.c$ where $-1_B.c$ denotes the action from Proposition 7.9.4.

Recall from Remark 10.4.19 that $(\eta \times \mu)_{f_3,(-f_2,f_3,f_2)} = (-1_A, -1_B)$. Further, we can compute that

$$\begin{aligned}
(\eta^B \times \mu^B)_{f_3,(-f_2,f_3,f_2)} &= (\eta^B \times \mu^B)_{e_3,(e_3-e_2,e_3,e_2-e_3)}\\
&= (\eta^B \times \mu^B)_{e_3,e_3-e_2}(\eta^B \times \mu^B)_{e_2,e_3}(\eta^B \times \mu^B)_{e_2,e_2-e_3}\\
&= (1,1)(1,-1)(1,1) = (1,-1).
\end{aligned}$$

For all $c \in \mathcal{A}$, it follows that

$$\theta_{f_3}(-\bar{c}) = \theta_{f_3}(c)^{w_{f_2}^{-1} w_{f_3}} = \theta_{f_3}^B(c)^{w_{f_2}^{-1} w_{f_3}} = \theta_{f_3}^B(\bar{c}^B) = \theta_{f_3}(\bar{c}^B),$$

so $\bar{c}^B = -\bar{c}$.

Since $\bar{\cdot}^B$ is precisely the action of B on $\mathcal{A}$ that we have studied in section 7.11, it follows from Lemma 7.11.20 that it is the reflection corresponding to $1_{\mathcal{A}}$ in the sense of Definition 7.1.20. That it, $\bar{\cdot}^B = \sigma_{1_{\mathcal{A}}}$. Thus it follows from the conclusion of the previous paragraph that $\bar{\cdot} = -\sigma_{1_{\mathcal{A}}}$.

10.7.5 Lemma. *There exist an alternative ring structure on $\mathcal{A}$ whose identity element is $1_{\mathcal{A}}$ and maps φ, $\operatorname{str}_{\mathcal{A}}$, T_1 such that $(\Bbbk, \varphi, \operatorname{str}_{\mathcal{A}}, \mathrm{T}_1, 0)$ is a Jordan module of type C over $\mathcal{A}$, the map $\bar{\cdot}$ from Notation 10.7.1 is a nuclear involution of $\mathcal{A}$, $\operatorname{str}_{\mathcal{A}}(1_\Bbbk) = 1_{\mathcal{A}}$ and the following commutator relations hold for all $\lambda \in \Bbbk$ and all $c, d \in \mathcal{A}$:*

$$\begin{aligned}
[\theta_{f_4}(c), \theta_{f_3}(d)] &= \theta_{f_3+f_4}(cd), \\
[\theta_{f_2}(\lambda), \theta_{f_3}(c)] &= \theta_{f_2+2f_3}\big(-\varphi(\lambda, -\bar{c})\big)\theta_{f_2+f_3}\big(-c\operatorname{str}_{\mathcal{A}}(\lambda)\big), \\
[\theta_{f_2+f_3}(c), \theta_{f_3}(c)] &= \theta_{f_2+2f_3}\big(\mathrm{T}_1(c\bar{d})\big).
\end{aligned}$$

Proof. Denote by $\Delta^C := (c_1, c_2, c_3) := (f_4, f_3, f_2)$ the ordered root base of the canonical C_3-subsystem of F_4 and by $\eta^C \colon C_3 \times \Delta^C \to A'$ and $\mu^C \colon C_3 \times \Delta^C \to B$ the Δ^C-parity maps which we defined in 9.3.19. This means that, if we choose an orthonormal basis (e_1, e_2, e_3, e_4) of V such that $c_1 = e_1 - e_2$, $c_2 = e_2 - e_3$ and $c_3 = 2e_3$, then the maps η^B and μ^B are given by the exact same formulas as in Figure 9.4 on page 278. Further, we denote by $\tilde{\eta}^C \colon C_3 \times \Delta^C \to A \leq A'$ the projection of η^C to the first component. Observe that (A', η^C, B, μ^C) is the standard partial twisting system for G_C with respect to Δ^C (from Definition 9.7.2). (Here we use that the short involution on G_C is trivial by Lemma 10.5.4, so that the action of A' on the root groups has the desired form.)

We know from Proposition 9.8.4 that there exist an abelian group $(\Bbbk', +)$ and a group $(\mathcal{A}', +)$ (each group being equipped with an action of $A' \times B$), an isomorphism $\theta_\alpha^C \colon (\Bbbk', +) \to U_\alpha$ for each long root α in C_3 and an isomorphism $\theta_\beta^C \colon (\mathcal{A}', +) \to U_\beta$ for each short root β in C_3 such that G_C is parametrised by $(A' \times B, \mathcal{A}', \Bbbk')$ with respect to $\tilde{\eta}^C \times \mu^C$ and $(w_\delta)_{\delta \in \Delta^C}$ and such that the action of $(-1_A, 1_A)$ on $\mathcal{A}'$ and $\Bbbk'$ is given by group inversion, the action of $(1_A, -1_A)$ on $\mathcal{A}'$ is trivial and the action of $(1_A, -1_A)$ is given by the short involution. Since the short involution on G_C is trivial, we see that $(1_A, -1_A)$ acts trivially on both groups $\Bbbk'$ and $\mathcal{A}'$. Thus the second component of A' is irrelevant. In a more technical language, this means that G_C is parametrised by $(\mathcal{A}', \Bbbk')$ with respect to $\eta^C \times \mu^C$ (and not $\tilde{\eta}^C \times \mu^C$) and $(w_\delta)_{\delta \in \Delta^C}$. By Remark 4.5.17, we can achieve that $\Bbbk' = \Bbbk$, $\mathcal{A}' = \mathcal{A}$, $\theta_{f_2}^B = \theta_{f_2}$ and $\theta_{f_3}^B = \theta_{f_3}$ (which shows that, in particular, $\mathcal{A}'$ must be abelian).

As in the proof of Lemma 10.7.3, Remark 10.7.2 applies in our situation. We begin by proving that the two actions of B on $\mathcal{A}$ are, in fact, equal. By the same arguments as in Remark 10.7.4, it suffices to show that

$$(\eta \times \mu)_{f_3, (-f_2, f_3, f_2)} = (\eta^C \times \mu^C)_{f_3, (-f_2, f_3, f_2)}.$$

We already know from Remark 10.4.19 that

$$(\eta \times \mu)_{f_3, (-f_2, f_3, f_2)} = (-1_A, -1_B).$$

Further, we can compute that

$$\begin{aligned}(\eta^C \times \mu^C)_{f_3,(-f_2,f_3,f_2)} &= (\eta^C \times \mu^C)_{e_2-e_3,(-2e_3,e_2-e_3,2e_3)} \\ &= (\eta^C \times \mu^C)_{e_2-e_3,-2e_3}(\eta^C \times \mu^C)_{e_2+e_3,e_2-e_3}(\eta^C \times \mu^C)_{e_2+e_3,2e_3} \\ &= (\eta^C \times \mu^C)^{-1}_{e_2+e_3,2e_3}(\eta^C \times \mu^C)_{e_2+e_3,e_2-e_3}(\eta^C \times \mu^C)_{e_2+e_3,2e_3} \\ &= (-1,1)(-1,-1)(-1,1) = (-1_A,-1_B).\end{aligned}$$

This shows that the two actions of B on $\mathcal{A}$. In the following, we will denote them by $\overline{\cdot}$, as in Notation 10.7.1.

We can now turn to the relationship between the two families of root isomorphisms. We know from Remark 10.4.19 that

$$\begin{aligned}(\eta \times \mu)_{f_2,f_3} &= (-1,1), & (\eta \times \mu)_{f_3,f_2} &= (1,1), \\ (\eta \times \mu)_{f_3,f_4} &= (-1,1), & (\eta \times \mu)_{f_3,f_4f_3} &= (1,1).\end{aligned}$$

Further, we can compute that

$$\begin{aligned}(\eta^C \times \mu^C)_{f_2,f_3} &= (\eta^C \times \mu^C)_{2e_3,e_2-e_3} = (1,1), \\ (\eta^C \times \mu^C)_{f_3,f_2} &= (\eta^C \times \mu^C)_{e_2-e_3,2e_3} = (1,1), \\ (\eta^C \times \mu^C)_{f_3,f_4} &= (\eta^C \times \mu^C)_{e_2-e_3,e_1-e_2} = (-1,1), \\ (\eta^C \times \mu^C)_{f_3,f_4f_3} &= (\eta^C \times \mu^C)_{e_2-e_3,(e_1-e_2,e_2-e_3)} \\ &= (\eta^C \times \mu^C)_{e_2-e_3,e_1-e_2}(\eta^C \times \mu^C)_{e_1-e_3,e_2-e_3} \\ &= (-1,1)(-1,1) = (1,1)\end{aligned}$$

and

$$f_2^{\sigma(f_3)} = f_2 + 2f_3, \quad f_3^{\sigma(f_2)} = f_2 + f_3, \quad f_3^{\sigma(f_4)} = f_3 + f_4 \text{ and } f_3^{\sigma(f_4f_3)} = f_4.$$

As in the proof of Lemma 10.7.3, it follows that

$$\begin{aligned}\theta_{f_2+2f_3}(\lambda) &= \theta^C_{f_2+2f_3}(-\lambda), & \theta_{f_2+f_3}(c) &= \theta^C_{f_2+f_3}(c) \\ \theta_{f_3+f_4}(c) &= \theta^C_{f_3+f_4}(c), & \theta_{f_4}(u) &= \theta^C_{f_4}(c)\end{aligned}$$

for all $\lambda \in \Bbbk$ and all $c \in \mathcal{A}$.

Finally, we can investigate the commutator relations. By Theorem 9.10.26, there exist an alternative ring structure on $\mathcal{A}$ whose identity element is $1_\mathcal{A}$, a nuclear involution on $\mathcal{A}$ (which coincides with the map $\overline{\cdot}$ defined in Notation 10.7.1) and maps φ, $\operatorname{str}_\mathcal{A}$, T_1, ψ such that $\mathcal{J} = (\Bbbk, \varphi, \operatorname{str}_\mathcal{A}, \mathrm{T}_1, \psi)$ is a Jordan module over $\mathcal{A}$ (where $\psi = 0$ because G_C is C_3-graded), $\operatorname{str}_\mathcal{A}(1_\Bbbk) = 1_\mathcal{A}$ and G_C is coordinatised (as a BC_3-graded group) by $\mathcal{J}$ with standard signs. In particular, this means that the following standard commutator relations from Definition 9.6.2 hold for all $\lambda \in \Bbbk$ and all $c, d \in \mathcal{A}$:

$$\begin{aligned}[\theta^C_{f_4}(c), \theta^C_{f_3}(d)] &= \theta^C_{f_3+f_4}(cd), \\ [\theta^C_{f_2}(\lambda), \theta^C_{f_3}(c)] &= \theta^C_{f_2+2f_3}\big(\varphi(\lambda, -\bar{c})\big)\theta^C_{f_2+f_3}\big(-c \operatorname{str}_\mathcal{A}(\lambda)\big), \\ [\theta^C_{f_3}(c), \theta^C_{f_2+f_3}(d)] &= \theta^C_{f_2+2f_3}\big(\mathrm{T}_1(c\bar{d})\big).\end{aligned}$$

Using the formulas above for the root isomorphisms, the assertion follows. □

We have the following identities which relate the maps from Lemma 10.7.3 to the maps from Lemma 10.7.3.

10.7.6 Lemma. *The following hold:*

(a) $\bar{c} = -\sigma_{1_{\mathcal{A}}}(c)$ *for all* $c \in \mathcal{A}$ *where* $\sigma_{1_{\mathcal{A}}}$ *denotes the reflection in the quadratic module* $(\mathcal{A}, \mathrm{n})$ *corresponding to* $1_{\mathcal{A}}$ *(in the sense of Definition 7.1.20).*

(b) $c \star \lambda = c \operatorname{str}_{\mathcal{A}}(\lambda)$ *for all* $c \in \mathcal{A}$ *and all* $\lambda \in \Bbbk$.

(c) $\varphi(\lambda, c) = \lambda \, \mathrm{n}(c)$ *for all* $\lambda \in \Bbbk$, $c \in \mathcal{A}$.

(d) $\langle c | d \rangle = \mathrm{T}_1(c\bar{d})$ *for all* $c, d \in \mathcal{A}$. *In particular,* $\langle c | 1_{\mathcal{A}} \rangle = \mathrm{T}_1(c)$ *for all* $c \in \mathcal{A}$.

Proof. Assertion (a) holds by Remark 10.7.4. For the remaining assertions, let $\lambda \in \Bbbk$ and $c, d \in \mathcal{A}$ be arbitrary. The rank-2 root subsystem generated by $\{ f_2, f_3 \}$ is contained in both the canonical B_3-subsystem and the canonical C_3-subsystem, so it follows from Lemmas 10.7.3 and 10.7.5 that we have the following "overlapping" commutator relations:

$$\begin{aligned} \theta_{f_2+f_3}(-c \star \lambda)\theta_{f_2+2f_3}(-\lambda \, \mathrm{n}(c)) &= [\theta_{f_2}(\lambda), \theta_{f_3}(c)] \\ &= \theta_{f_2+2f_3}\big(-\varphi(\lambda, -\bar{c})\big)\theta_{f_2+f_3}\big(-c \operatorname{str}_{\mathcal{A}}(\lambda)\big), \\ \theta_{f_2+2f_3}\big(\langle c | d \rangle\big) = [\theta_{f_2+f_3}(c), \theta_{f_3}(d)] &= \theta_{f_2+2f_3}\big(\mathrm{T}_1(c\bar{d})\big). \end{aligned}$$

We conclude that

$$c \star \lambda = c \operatorname{str}_{\mathcal{A}}(\lambda), \quad \lambda \, \mathrm{n}(c) = \varphi(\lambda, -\bar{c}), \quad \langle c | d \rangle = \mathrm{T}_1(c\bar{d}).$$

This proves (b) and (d). For (c), observe that $\mathrm{n}(c) = \mathrm{n}(\sigma_{1_{\mathcal{A}}}(c))$ because $\sigma_{1_{\mathcal{A}}}$ lies in the orthogonal group $\mathrm{O}(\mathrm{n})$ by Lemma 7.1.21. Since $\bar{c} = -\sigma_{1_{\mathcal{A}}}(c)$ by assertion (a), we infer that

$$\varphi(\lambda, -\bar{c}) = \lambda \, \mathrm{n}(c) = \lambda \, \mathrm{n}\big(\sigma_{1_{\mathcal{A}}}(c)\big) = \lambda \, \mathrm{n}(-\bar{c}).$$

As the map $c \mapsto -\bar{c}$ is bijective, it follows that (c) holds. □

10.7.7 Summary. Altogether, we have assembled the following algebraic structures which coordinatise G and describe its commutator relations:

(1) An associative commutative ring $\Bbbk$ with identity element $1_{\Bbbk}$.

(2) An alternative ring $\mathcal{A}$ with identity element $1_{\mathcal{A}}$ and with a $\Bbbk$-module structure $\star$.

(3) A $\Bbbk$-quadratic form $\mathrm{n}\colon (\mathcal{A}, \star) \to \Bbbk$ with $\mathrm{n}(1_{\mathcal{A}}) = 1_{\Bbbk}$ and its linearisation $\langle \cdot | \cdot \rangle \colon \mathcal{A} \times \mathcal{A} \to \Bbbk$.

(4) A nuclear involution $\bar{\cdot}$ on $\mathcal{A}$.

(5) Maps φ, $\operatorname{str}_{\mathcal{A}}$ and T_1 such that $(\Bbbk, \varphi, \operatorname{str}_{\mathcal{A}}, \mathrm{T}_1, 0)$ is a Jordan module of type C.

Further, we know that the identities in Lemma 10.7.6 hold. We conclude that

$$\mathcal{F} := \big(\Bbbk, \mathcal{A}, \star, \mathrm{n}, \langle \cdot | \cdot \rangle, \operatorname{str}_{\mathcal{A}}, \varphi\big)$$

is an F_4-datum. By Proposition 10.1.34, it follows that the ring $\mathcal{A}$ is a $\Bbbk$-algebra with structural homomorphism $\operatorname{str}_{\mathcal{A}}$ and associated scalar multiplication $\star$ and that $(\mathcal{A}, \mathrm{n})$ is a multiplicative conic alternative algebra over $\Bbbk$ with conjugation $\bar{\cdot}$. By Proposition 10.1.33, the multiplicative conic alternative algebra structure on $\mathcal{A}$ determines the remaining objects in $\mathcal{F}$.

10.7.8 Theorem (Coordinatisation theorem for F_4). *Let G be a group with a crystallographic F_4-grading $(U_\alpha)_{\alpha \in F_4}$ and let $(w_\delta)_{\delta \in \Delta}$ be a Δ-system of Weyl elements in G. Then there exist a commutative associative ring $\Bbbk$ and a multiplicative conic alternative algebra $(\mathcal{A}, \mathrm{n})$ over $\Bbbk$ such that $(G, (U_\alpha)_{\alpha \in F_4})$ is coordinatised $(\mathcal{A}, \mathrm{n})$ with standard signs and with respect to $(w_\delta)_{\delta \in \Delta}$ (in the sense of Definition 10.4.15).*

Proof. This is a consequence of Summary 10.7.7. □

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