Delay equations: instability and the trivial fixed point's index

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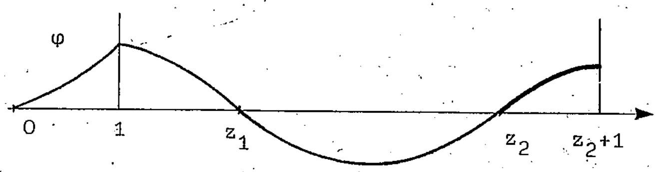
Let us consider equation

$$x(t) = -f(x(t-1)) \qquad \text{(f)}$$

where $f: \mathbb{R} \to \mathbb{R}$ is assumed to be continuous with $f(0) = 0$ and $\xi f(\xi) > 0$ for $\xi \neq 0$. Let K denote the convex closed cone of increasing functions with $\varphi(0) = 0$ in the Banach space C of continuous functions $\varphi: [0,1] \to \mathbb{R}$. Define $T0 := 0$ and $T\varphi := x^{\varphi}(z_2+\cdot)$, with z_2 the second zero in $(1,\infty)$ of the solution $x^{\varphi}: [0,\infty) \to \mathbb{R}$ of equation (f) with initial value $x^{\varphi}|[0,1] = \varphi \neq 0$, $\varphi \in K$. T is a continuous operator mapping K into K, and nonzero fixed points of T define periodic solutions of equation (f). This is just Jones' wellknown idea [2].

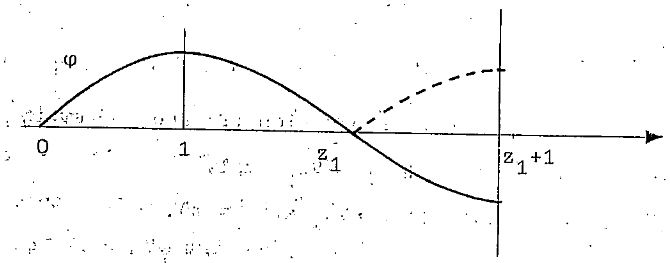


In [5] we showed how to apply Schauder's fixed point theorem to obtain existence of nonzero fixed points, provided f is bounded below or above and $f'(0) > \pi/2$. The object of the present note is to derive

$$\text{ind}\,(0,T,K) = 0$$

from our result on unstable behaviour of slowly oscillating solutions by simply applying the additivity of the index.- The result on index was first proven by R.D. Nussbaum who used it to establish global bifurcation of periodic solutions [3].

A refinement of our argument given below will be employed in a forthcoming paper on secondary bifurcation and loss of symmetry [4] where we compute the index of a nonzero fixed point of the operator $S: \varphi \to -x^{\varphi}(z_1+\cdot)$ with respect to certain subsets of K. If f is odd then nonzero fixed points of S define periodic solutions with period $2z_1$ and the symmetry property $x(t) = -x(z_1+t)$ for all $t \geq 0$.



Throughout the following we assume in addition that f is differentiable at $\xi = 0$ with $f'(0) > \pi/2$, and that $f(\xi) > -B$ for all ξ and some constant $B > 0$. For a solution $x: [0,\infty) \to \mathbb{R}$ of equation (f) and for $t \geq 0$, x_t denotes the element $s \to x(t+s)$ of C. For $\varphi \in C$, x^{φ} denotes the unique solution with initial value $x_0 = \varphi$.

The basic tool used in [5] was the Liapounov functional $V: C \to \mathbb{R}_0^+$ of Hale and Perello [1]. V is continuous and quadratic, i.e. $V(\lambda\varphi) = \lambda^2 V\varphi$ for real λ, and satisfies

$$V\varphi \leq c_2 \|\varphi\|^2 \text{ for all } \varphi \in C \tag{2}$$

with some constant $c_2 > 0$, see e.g. Lemma 1 (i) [5].

We proved the a priori estimate

$$\begin{array}{l} p\|x_t\|^2 \leq Vx_t \text{ for all } t \geq 0, \text{ all solutions } x \text{ of} \\ \text{equation (f) such that } x_0 \in K \text{ and } x_0 \leq B, \end{array} \tag{3}$$

with some $p > 0$, see Theorem 1 [5] and the first part of the proof of Theorem 3 [5].

Then Lemma 1 (ii) [5] yields constants $c_3 > 0$ and $\delta = \delta_p > 0$ such that

$$\begin{array}{l} \text{for every solution } x \text{ of equation (f) and for every} \\ t \geq 0,\ \|x_t\| \leq \delta \text{ and } p^2\|x_t\|^2 \leq Vx_t \text{ imply} \\ c_3 Vx_t \leq (\text{derivative of } s \to Vx_s \text{ at } s = t). \end{array} \tag{4}$$

Now a slight modification of the proof of Theorem 1 [5] results in

Corollary 1. There are positive constants $A < B$ and a such that for all $\varphi \in K$ with $0 < \|\varphi\| \leq B$,

(i) there exists $t \geq 0$ with $A \leq \|x_s^\varphi\|$ for all $s \geq t$,

(ii) $a \leq V\varphi$ implies $a \leq Vx_t^\varphi$ for all $t \geq 0$,

(iii) $V\varphi \leq a$ implies $\|\varphi\| < A$.

Proof. For $\varphi \in K$ with $0 < \|\varphi\| \leq B$, set $x := x^\varphi$.

(i) There is $u \geq 0$ with $\|x_u\| > \delta$. Proof: Assume $\|x_u\| \leq \delta$ for all $u \geq 0$.

By (3) and $\varphi \neq 0$, $0 < Vx_0$. (4) and (3) give $0 < Vx_0 \exp(c_3 u) \leq Vx_u$ for all $u \geq 0$, contradiction.

We have $\delta \leq \|x_v\|$ for all $v \geq u$, or $p^2\delta^2 c_2^{-1} \leq \|x_v\|^2$ for all $v \geq u$. Proof: Let $v > u$ be given with $\|x_v\| < \delta$. There is $w \in (u,v)$ such that $\|x_w\| = \delta$ and $\|x_t\| < \delta$ for $w < t \leq v$, since $t \to x_t$ is continuous. (3) and (4) imply Vx_t increasing on (w,v). Hence $p^2\delta^2 = p^2\|x_w\|^2 \leq Vx_w \leq Vx_v \leq c_2\|x_v\|^2$. Set $A := \min\{B/2, \delta, p\sqrt{c_2}\}$.

(ii) Choose $\delta' > 0$ so small that $\delta' < A \leq \delta$. Let $p^2\delta'^2 \leq V\varphi$ and $t > 0$. $\delta' \leq \|x_t\|$ implies $p^2\delta'^2 \leq p^2\|x_t\|^2 \leq Vx_t$. If $\|x_t\| < \delta'$ and $\|x_s\| \leq \delta'$ for all $s \in [0,t)$, then (3) and (4) yield $Vx_0 \leq Vx_t$, by $\delta' < \delta$. Hence $p^2\delta'^2 \leq V\varphi = Vx_0 \leq Vx_t$. If $\|x_t\| < \delta'$ and $\delta' < \|x_s\|$ for some $s \in [0,t)$ then there exists $u \in (s,t)$ with $\|x_u\| = \delta'$ and $\|x_v\| < \delta'$ for all $v \in (u,t]$. (3), (4) and $\delta' < \delta$ give $Vx_t \geq Vx_u \geq p^2\|x_u\|^2 = p^2\delta'^2$. Set $a := p^2\delta'^2$.

(iii) $V\varphi \leq a = p^2\delta'^2$ yields $\|\varphi\|^2 \leq \delta'^2 < A^2$, by (3).

From Corollary 1 and from $f > -B$ we might conclude that the set $D := \{\varphi \in K: a \leq V\varphi \text{ and } \varphi \leq B\}$ -which is bounded away from the trivial fixed point $\varphi = 0$ - is invariant with respect to T, and that for every nonzero $\varphi \in K$, all higher iterates $T^n\varphi = x^\varphi_{z_{2n}}$ lie in the subset $\{\varphi \in K: A \leq \|\varphi\| \leq B\}$ of D.

Let us recall that for every nonzero $\varphi \in K$, the zeros in $(1,\infty)$ of the solution x^φ form a sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n+1 < z_{n+1}$ such that in $(z_n+1, z_{n+1}+1)$, $\dot{x} > 0$ for n odd and $0 > \dot{x}$ for n even.

This implies $T^n\varphi = x^{\varphi}_{z_{2n}}$ for all $n \in \mathbb{N}$.

We have in particular

Corollary 2. There are no nonzero fixed points of T in the neighborhood $\{\varphi \in K: V\varphi \leq a\}$ of $\varphi = 0$.

Proof. Let $0 \neq \varphi \in K$, $V\varphi \leq a$. Hence $\varphi \leq B$, by $A < B$ and (iii). By (i), $A \leq \|x^{\varphi}_{z_{2n}}\|$ for all but a finite number of indices. Therefore by (iii), $V\varphi \leq a < VT^n\varphi$ for some $n \in \mathbb{N}$, and consequently $\varphi \neq T\varphi$.

Corollary 3. $\operatorname{ind}(0,T,K) = 0$.

Proof. $f > -B$ yields $T\varphi < B$ for all $\varphi \in K$. Moreover T is continuous and compact from K into the retract $K_B := \{\varphi \in K: \varphi \leq B\}$ of K, see e.g. [5]. So it is enough to show $\operatorname{ind}(0,T,K_B) = 0$. By $\operatorname{ind}(K_B,T,K_B) = 1$ and by Corollary 2, this will follow from the additivity of the index, provided we can show $\operatorname{ind}(\{\varphi \in K_B: a < V\varphi\},T,K_B) = 1$. To see this, we may employ the homotopies $H: (\varphi,t) \to (1-t)T\varphi + (tB/\|T\varphi\|)T\varphi$ and $G: (\varphi,t) \to (1-t)(B/\|T\varphi\|)T\varphi + t\psi$, with $\psi \in K_B$ and $\|\psi\| = \psi(0) = B$ fixed, both defined on $\{\varphi \in K_B: a \leq V\varphi\} \times [0,1]$. H and G are continuous and compact; $a \leq V\varphi$ implies $a \leq Vx^{\varphi}_{z_2} = VT\varphi$ by Corollary 1,(ii), hence $T\varphi \neq 0$.

H and G map into K_B. For H, this follows from $\|H(\varphi,t)\| \leq (1-t)B + tB$ and from $H(\varphi,t) = \lambda T\varphi$ with $\lambda = (1-t)+(tB/\|T\varphi\|) \geq 1$ which gives $a \leq VT\varphi \leq \lambda^2 VT\varphi = V(\lambda T\varphi) = V(H(\varphi,t))$. – Since $\|\varphi\| = \varphi(0)$ for $\varphi \in K$, we have $\|G(\varphi,t)\| = G(\varphi,t)(0) = B$, or $G(\varphi,t) \in K_B$.

$H(\varphi,t) \neq \varphi$ and $G(\varphi,t) \neq \varphi$ for all $\varphi \in K_B$ with $a = V\varphi$ and all $t \in [0,1]$: For G this is obvious from $\|\varphi\| < A < B$ if $a = V\varphi$ (Corollary 1) and from $\|G(\varphi,t)\| = B$. – Assume $H(\varphi,t) = \varphi$ and $a = V\varphi$. Then for $\lambda = (1-t)+(tB/\|T\varphi\|) \geq 1$, $a = V\varphi = \lambda^2 VT\varphi \geq \lambda^2 a$, since $VT\varphi \geq a$ by Corollary 1,(ii). We infer $\lambda = 1$. By $T\varphi < B$, this implies $t = 0$. But then $a = V\varphi$ and $\varphi = H(\varphi,0) = T\varphi$, contradiction to Corollary 2.

Now $H(\cdot,0) = T$, $H(\cdot,1) = G(\cdot,0)$ and $G(\cdot,1) = \psi$, therefore $\operatorname{ind}(\{\varphi \in K_B: a < V\varphi\},T,K_B) = 1$.

Of course, the last proof shows existence of nonzero fixed points once more.

References

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