Existence of a Non-Constant Periodic Solution of a Non-Linear Autonomous Functional Differential Equation Representing the Growth of a Single Species Population

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Summary

We consider an integro-differential equation for the density n of a single species population where the birth rate is constant and the death rate depends on the values of n in an interval of length $\tau-1>0$. We prove the existence of a non-constant periodic solution under the conditions birth rate $b>\pi/2$ and $\tau-1$ small enough. The basic idea of proof (due to R. D. Nussbaum) is to employ a theorem about non-ejective fixed points for a translation operator associated with the solutions of the equation.

1. The growth rate of a single species population is governed by the birth and death rates b and d, which depend on the population density n. Mathematically, we have an equation $\dot{n}=(b-d)\,n$, with b and d depending on the function n. The consideration of these equations with hereditary effects in b and d is motivated by the fact that, without retardation, they would, in general, have no non-negative oscillatory solution (e.g. the logistic equation $\dot{n}(t)=(b-\tilde{d}\,n(t))\,n(t)$, with b and $\tilde{d}$ constant) — in contrast to the results of laboratory growth experiments (see e.g. [3]).

G. Dunkel ([1], 1968) proposed the retarded functional differential equation

$$\dot{n}(t)=\left(b+\int_{\gamma}^{\tau} \Psi\left(n(t-a)\right) dS(a)\right) n(t) \tag{1}$$

as a growth model for a single species population. In (1), the birth rate b is assumed constant. The death rate at time t is given by $-\int_{\gamma}^{\tau} \Psi\left(n(t-a)\right) dS(a)$, with $0\leq\gamma<\tau$, $S: R_0^+\to R$ decreasing, $S=0$ on $[\tau,\infty)$, $S(\gamma)>0$, $\Psi\geq 0$ increasing. Here, S is a fixed survival function ($S(a)$ denotes the fraction of the population surviving to age a). τ is the maximum lifespan and γ is the time required by the death rate to react to changes in the population density. Ψ measures how much an increase in n increases the death rate.

We are interested in the existence of a non-constant periodic solution of eq. (1)*. First, let us write (1) in a mathematically convenient way: If there is exactly one

* A proof of existence was also announced by G. Dunkel in [1].

number $n^* \in R^+$ with $\Psi(n^*)=b/S(\gamma)$ (thus n^* is the saturation level of the population) then we set $x(t):=n(t)/n^*-1$ and obtain from (1)

$$\dot{x}(t)=\int_{\gamma}^{\tau} (\Psi(n^* x(t-a)+n^*)-\Psi(n^*))\, dS(a)\,[x(t)+1].$$

If $\gamma=1$ and $\Psi(n)=\Psi(n^*)\, n/n^*$, we arrive at

$$\dot{x}(t)=-\alpha \int_{1}^{\tau} x(t-a)\, ds(a)\,[x(t)+1], \tag{2}$$

with $s(a):=-S(a)/S(\gamma)$ and $\alpha:=b$.

We shall prove

Theorem 1: *Let $s: R_0^+ \to R$ be an increasing function with $s=-1$ on $[0,1]$, $s<0$ on $(1,\tau)$, $s=0$ on $[\tau,\infty)$. If $\alpha>\pi/2$ there is a constant $c \in R^+$ such that for $\tau \in (1, 1+c)$ eq. (2) has a non-constant differentiable periodic solution x on R with values in $(-1, e^{\alpha\tau}-1]$.*

Corollary: *Let $\gamma=1$, $S=S(1)$ on $[0,1]$, $S>0$ on $(1,\tau)$ and $S=0$ on $[\tau,\infty)$. Let $\Psi(m)=\Psi(n^*)\, m/n^*$, with n^* the only positive solution of $\Psi(m)=b/S(1)$. If $b>\pi/2$ there is a constant $c \in R^+$ such that for $\tau \in (1, 1+c)$, eq. (1) has a positive, non-constant, periodic solution on R.*

Proof: Set $s(a):=-S(a)/S(1)$ and $\alpha:=b$. If x is the solution of (2) guaranteed by theorem 1, then $n:=n^* x+n^*$ satisfies (1).

Proofs of the existence of non-trivial periodic solutions for nonlinear autonomous differential delay equations (*DDEs*) are known since the thesis of G. S. Jones (1962). He considered the equation

$$\dot{y}(t)=-\alpha' y(t-1)[y(t)+1] \tag{3}$$

which is a special case of (2) (set $s:=-1$ on $[0,\tau)$ and $y(t):=x(\tau t)$, $\alpha':=\alpha\tau$). R. B. Grafton (1969) and R. D. Nussbaum ([6], to appear) improved the method of Jones. They treated other equations than (3) also. One common feature of these *DDEs* is the fact that their right sides only involve a finite number of time lags, which are constant or depend on y in an explicit way. Obviously, eq. (2) is of a more general type.

Our proof of theorem 1 follows the idea of Nussbaum [6] using a theorem about non-ejective fixed points for a certain translation operator associated with eq. (2). We employ the following theorem of W. A. Horn [4], which is essentially due to F. E. Browder.

Theorem 2: *Let A be an infinite dimensional closed convex subset of a Banach space X. Let $T: A \to A$ be continuous and compact. Then T has a non-ejective fixed point.*

A point $x_0 \in A$ is called an ejective fixed point of T, if $T(x_0)=x_0$ and if there is a neighborhood U of x_0 with $(\forall\, x \in U-\{x_0\}\ \exists\ n_x \in N: T^{n_x}(x) \notin U)$.
To apply theorem 2, let X denote the space of real-valued continuous functions on $[-\tau, 0]$, with sup-norm. We set $A:=\{f \in X \mid -1 \le f \le 0,\ f$ increasing on $[-(\tau-1), 0],\ f(0)=0\}$. A is closed, convex and of infinite dimension. If $f \in X$,

$f(0)>-1$, there exists an unique function $x_f:[-\tau,\infty)\to R$ satisfying (2) on R^+ and $x_f\,|\,[-\tau,0]=f$.

In section 2, we show that, for every $f\in A-\{0\}$, x_f has a first zero z_f after the first local minimum m_f on R^+ with $x_f(m_f)<0$. We define the "variable translation operator" T by $T(0):=0$, $T(f)(r):=x_f(z_f+r)$ for $r\in[-\tau,0]$, $f\in A-\{0\}$, and, under the hypotheses $\tau-1<(\alpha e^{\alpha\tau})^{-1}$, $\alpha>\pi/2$, s as in theorem 1, we prove

Lemma 3: *The operator $T:A\to X$ is continuous and compact. We have $T(A)\subset A$ and $T(f)>-1$ on A.*

Here, the condition on α could be replaced by $\alpha>1$. "$s<0$ on $(1,\tau)$" is needed to show that the definition of T on $A-\{0\}$ makes sense (see the proof of lemma 1). The smallness condition on $\tau-1$ gives $T(A)\subset A$.

Now, if $0\in A$ (the only constant function in A) is an ejective fixed point of T for $\alpha>\pi/2$ and $\tau-1<c$ with a positive constant c, theorem 1 follows from theorem 2 by the autonomy of eq. (2) and by the uniqueness of the initial value problem mentioned above.

Section 3 contains the remaining proof of the ejectivity of 0. Here, the assumption $\alpha>\pi/2$ becomes essential. If $\alpha>\pi/2$, there is a complex number λ with positive real part satisfying the characteristic equation $\lambda+\alpha e^{-\lambda}=0$ of the linear *DDE* $\dot z(t)=-\alpha z(t-1)$. E. M. Wright ([7], 1955) used this fact to derive an instability theorem for the nonlinear equation (3). Wright's method would be sufficient for proving ejectivity of $0\in A$ in the case of eq. (3) but is not in the more general case of eq. (2). We only obtain the result of lemma 5. The main tool for deriving the ejectivity of $0\in A$ (stated in lemma 8) from lemma 5 is the extimate $|x(m_j)|\le c_3\,|x(m_{j+1})|$ (lemma 6) concerning successive extrema of a solution x of (2) with initial value $x\,|\,[-\tau,0]$ in $A-\{0\}$. The proof of lemma 6 requires only elementary tools, but is rather lengthy. To obtain lemma 8, we also have to impose new bounds on the size of $\tau-1$.

2. We make the general assumptions $\tau\in R$, $\tau>1$, $\alpha>\pi/2$, $s:R_0^+\to R$ increasing with $s=-1$ on $[0,1]$, $s<0$ on $(1,\tau)$ and $s=0$ on $[\tau,\infty)$. First, we have

Theorem 3: *For all $f\in X$ with $f(0)>-1$ there exists exactly one continuous function $x_f:[-\tau,\infty)\to R$, differentiable on R^+, with $x_f\,|\,[-\tau,0]=f$ and satisfying* (2) *on R^+. We have $x_f>-1$ on R. On compact sets, x_f depends continuously on f:*

$$\forall\,(g\in X:g(0)>-1)\,\forall\,t\in R_0^+\,\forall\,\varepsilon>0\,\exists\,\delta>0\,\forall\,(f\in X:f(0)>-1):$$
$$\|f-g\|<\delta\Rightarrow\sup_{[-\tau,t]}|x_f(r)-x_g(r)|<\varepsilon.$$

Proof: With $f\in X$, $f(0)>-1$, eq. (2) is an inhomogeneous linear ordinary differential equation on $[0,1)$. Hence theorem 3 can be proved by solving ordinary differential equations on $[0,1)$, $[1,2)$, and so on. From $f(0)>-1$ we obtain

$$t_1\ge t_2\ge 0\Rightarrow x_f(t_1)+1=(x_f(t_2)+1)\exp\left(-\alpha\int_{t_2}^{t_1}\int_1^{\tau}x_f(t'-a)\,ds(a)\,dt'\right).\tag{4}$$

(4) implies $x_f>-1$ on R_0^+.

To show that the definition of T given in section 1 makes sense we have to examine the zeros and the local extrema of solutions x_f with $f \in A-\{0\}$. This is done in the next two lemmas.

Lemma 1: *For $f \in A-\{0\}$,*

i) *x_f has a first local maximum M_f in $[1, \tau]$ with $0 < x_f(M_f) \leq e^{\alpha\tau}-1$,*

ii) *x_f decreases on $[M_f, M_f+1]$,*

iii) *if $\tau < 2$, there is a first zero Z_f of x_f in $[M_f, M_f+\tau+1]$, and x_f decreases on $[M_f, Z_f]$.*

iv) *$\tau-1 < (\alpha e^{\alpha\tau})^{-1}$ implies $Z_f > M_f+\tau-1$,*

v) *$Z_f > \tau$ and x_f decreasing on $[M_f, Z_f+1]$.*

Proof: Let $f \in A-\{0\}$. We write x, M, Z instead of x_f, M_f, Z_f.

i) $f \leq 0$ and $x > -1$ on R_0^+ imply by (2) that x increases on $[0, 1]$. $f \neq 0$, $s < 0$ on $[1, \tau)$ and $x(0)=0$ give $x(1) > 0$. If x has no local maximum in $[1, \tau]$, then $x \geq 0$ on $[0, \tau]$, hence, by (2), $\dot{x} \leq 0$ on $[\tau, \tau+1]$. With $\dot{x} \geq 0$ on $(0, 1]$, we obtain a local maximum of x in $[1, \tau]$, which is a contradiction. Formula (4) and $f \geq -1$ imply $x(M)+1 \leq \exp\left(-\alpha \int_0^\tau \int_1^\tau \inf f ds(a)\, dt'\right) \leq e^{\alpha\tau}$.

ii) x is increasing on $[-(\tau-1), M]$. Hence $x(t \dot{-} \cdot) \geq x(M-\cdot)$ on $[1, \tau]$ for $t \in [M, M+1]$. Then, $x > -1$ on R_0^+ and $M \in [1, \tau]$ give for these t:

$$\dot{x}(t) = -\alpha \int_1^\tau x(t-a)\, ds(a) [x(t)+1] \leq -\alpha \int_1^\tau x(M-a)\, ds(a) [x(t)+1] = 0,$$

since $\dot{x}(M)=0$ and $x > -1$ on R_0^+ imply $-\alpha \int_1^\tau x(M-a)\, ds(a) = 0$, by (2).

iii) Let us assume $x > 0$ on $[M, M+\tau+1]$. Then x decreases on $[\tau, M+\tau+1]$ (compare proof of i)). From ii), $2 > \tau$ and $M \geq 1$ we see that x is decreasing on $[M, \tau]$, hence x is decreasing on $[M, M+\tau+1]$. With $\varepsilon := x(M+\tau) > 0$ and $t \in [M+\tau, M+\tau+1]$ we obtain

$$\begin{aligned} \dot{x}(t)/[x(t)+1] &= -\alpha \int_1^\tau x(t-a)\, ds(a) \\ &\leq -\alpha \int_1^\tau \inf x \,|\, [M, M+\tau]\, ds(a) = -\alpha\varepsilon, \end{aligned}$$

since $(t \in [M+\tau, M+\tau+1] \wedge a \in [1, \tau] \Rightarrow t-a \in [M, M+\tau])$.

For these t, $x(t) > 0$ and $1 < a$ imply $\dot{x}(t) < -\varepsilon$. Hence

$$x(M+\tau+1) - x(M+\tau) = \int_{M+\tau}^{M+\tau+1} \dot{x}(r)\, dr < -\varepsilon = -x(M+\tau),$$

$x(M+\tau+1) < 0$, contradiction.

Proof of "x decreasing on $[M, Z]$": If $Z > M+1$ $(>\tau)$, then $x \geq 0$ on $[0, Z]$ implies $\dot{x} \leq 0$ on $[\tau, Z]$, by (2). Now ii) gives the assertion.

iv) From $\tau - 1 < (\alpha e^{\alpha\tau})^{-1}$ we have $\tau < 2$, hence iii) holds. Let us assume $Z - M \leq \tau - 1$. Then $(0-x(M))/(Z-M) \leq -x(M)/(\tau-1)$, hence $\exists\, t \in [M, Z] : \dot{x}(t) \leq -x(M)/(\tau-1)$.

On the other hand $\dot{x}(t) \geq -\alpha \int_1^\tau \sup x \mid [0, Z]\, ds(a)\, [x(t)+1] = -\alpha\, x(M)\, [x(t)+1]$, since part iii) implies $x(M) = \sup x \mid [0, Z] = \sup x \mid [-\tau, Z]$. By i), we arrive at $\dot{x}(t) \geq -\alpha\, x(M)\, e^{\alpha\tau}$. We get $-(\tau-1)^{-1} \geq \dot{x}(t)/x(M) \geq -\alpha\, e^{\alpha\tau}$, contradiction.

v) We have $Z > M + \tau - 1 \geq \tau$. $x > -1$ on R_0^+, $x \geq 0$ on $[0, Z]$ and $Z \geq \tau$ and (2) imply $\dot{x} \leq 0$ on $[Z, Z+1]$.

Lemma 2: *Let $\tau - 1 < (\alpha\, e^{\alpha\tau})^{-1}$. For all $f \in A - \{0\}$,*

i) *there is a first local minimum m_f of x_f in $[Z_f + 1, Z_f + \tau]$. We have $x_f(m_f) < 0$ and*
ii) $\exp(-\alpha\,\tau\,(e^{\alpha\tau} - 1)) - 1 \leq x_f(m_f)$,
iii) *there exists a first zero z_f of x_f on $[m_f, \infty)$. z_f is bounded independent from $f \in A - \{0\}$,*
iv) *x_f increases on $[m_f, z_f]$. We have $z_f > m_f + \tau - 1$.*

Proof: Let $f \in A - \{0\}$. We write M, Z, m, z, x instead of M_f, Z_f, m_f, z_f, x_f.

i) is proved like part i) of lemma 1.

ii) Since $Z > \tau$, we can use formula (4) with $t_2 \in [Z, Z+\tau]$ and $t_1 := Z$ to derive the desired estimate from ($t \in [0, Z + \tau - 1] \Rightarrow x(t) \leq e^{\alpha\tau} - 1$). The implication is a result of lemma 1.

iii) We may assume $x < 0$ on $[m, m + \tau + 1]$.

a) x increases on $[m, m+1]$. Proof: x is decreasing on $[M, m] = [M, Z] \cup [Z, m] \supset [m - \tau, m]$. Hence $x(t-a) \leq x(m-a)$ for $t \in [m, m+1]$ and $a \in [1, \tau]$. For these t, eq. (2) and $x > -1$ on R_0^+ imply

$$\dot{x}(t)/[x(t)+1] = -\alpha \int_1^\tau x(t-a)\, ds(a) \geq -\alpha \int_1^\tau x(m-a)\, ds(a) = 0,$$

hence $\dot{x}(t) \geq 0$.

b) $m \geq Z + 1$ and $2 > \tau$ give $m + 1 - \tau > Z$.

c) x increases on $[m, m + \tau + 1]$. Proof:
$t \in [m+1, m+\tau+1] \wedge a \in [1, \tau] \Rightarrow t - a \in [m + 1 - \tau, m + \tau] \subset [Z, m+\tau] \Rightarrow x(t-a) \leq 0$.
Hence $\dot{x}(t) = -\alpha \int_1^\tau x(t-a)\, ds(a)\, [x(t)+1] \geq 0$. Now a) implies the assertion.

d) $x(m+\tau) < -(1 - 2/\pi)$. Proof: For $t \in [m+\tau, m+\tau+1]$, c) implies

$$\dot{x}(t) = -\alpha \int_1^\tau x(t-a)\, ds(a)\, [x(t)+1] \geq -\alpha\, x(m+\tau)\, [x(m+\tau)+1],$$

since $0 \geq x(m+\tau) \geq x(t-a)$ for $a \in [1, \tau]$ and $0 < x(m+\tau) + 1 \leq x(t) + 1$.

We obtain

$$x(m+\tau+1) - x(m+\tau) = \int_{m+\tau}^{m+\tau+1} \dot{x}(r)\, dr \geq -\alpha\, x(m+\tau)\, [x(m+\tau)+1].$$

Now, $x(m+\tau) \geq -(1 - 2/\pi)$ implies

$$x(m+\tau+1) - x(m+\tau) \geq -\pi/2 \cdot x(m+\tau) \cdot 2/\pi,\; x(m+\tau+1) \geq 0,$$

which is a contradiction.

e) Set $\tilde{m} := \sup\{t \geq m \mid x < -(1-2/\pi)$ on $[m, t]\}$. Then $\infty \geq \tilde{m} > m+\tau$, by d). For $t \in [m+\tau, \tilde{m})$, we set $t_1 := t$, $t_2 := m+\tau$ and use formula (4): Estimating the factor $x(m+\tau)+1$ by $\exp(-\alpha\tau(e^{\alpha\tau}-1))$ and the integrand by $-(1-2/\pi)$, we get $x(t)+1 \geq \exp(-\alpha\tau(e^{\alpha\tau}-1)) \exp((t-m-\tau)\alpha(1-2/\pi))$ and, by $m \leq Z+\tau$, $Z \leq M+\tau+1$, $M \leq \tau$:

$$2/\pi \geq x(t)+1 \geq \exp(-\alpha\tau(e^{\alpha\tau}-1)) \exp((t-4\tau-1)\alpha(1-2/\pi)).$$

We see: $\tilde{m}$ is bounded by a constant depending on α and τ only. Using an argument as in the proof of d) (replace $m+\tau$ by $\tilde{m}$) we obtain a first zero of x in $[\tilde{m}, \tilde{m}+1]$.

iv) Proceeding as in part a) of the proof of iii) we infer $\dot{x} \geq 0$ on $[m, m+1]$. If $z > m+1$, the proofs of b) and c) (with $m+\tau+1$ replaced by z) show $\dot{x} \geq 0$ on $[m+1, z]$. The relation $m+\tau-1 < z$ follows as in the proof of part iv) of lemma 1:

$$m+\tau-1 \geq z \Rightarrow (\exists\, t \in [m, z] : \dot{x}(t) \geq -x(m)/(\tau-1)).$$

On the other hand $\dot{x}(t) \leq -\alpha x(m)[x(t)+1] \leq -\alpha x(m)$, therefore $\alpha \geq (\tau-1)^{-1}$, $\alpha e^{\alpha\tau} \geq (\tau-1)^{-1}$, which is a contradiction.

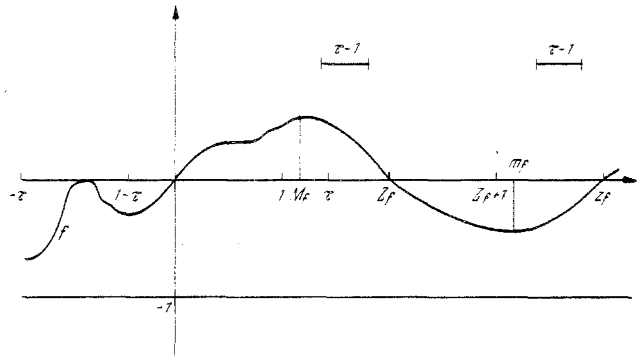

Fig. 1

Proof of lemma 3: Lemma 2 shows that the definition of the operator T on A makes sense. $T(A) \subset A$ follows from $Z_f+1 \leq m_f$, $m_f+\tau-1 < z_f$, x_f increasing on $[m_f, z_f]$ (for $f \in A - \{0\}$), and from $T(f) > -1$ on A (part ii) of lemma 2).

T is compact: On $[z_f-\tau, z_f]$, x_f and $\dot{x}_f$ (by eq. (2)) are bounded independent from $f \in A - \{0\}$ (lemma 2 and lemma 1). Ascoli's theorem implies the assertion.

T is continuous in $0 \in A$ because on $A - \{0\}$ z_f is bounded and because theorem 1 holds.

We prove the continuity of T in $f \in A - \{0\}$. If $g \in A - \{0\}$ we have

$$\| T(f) - T(g) \| = \sup_{[-\tau, 0]} | x_f(z_f+r) - x_g(z_g+r) |$$

$$\leq \sup_{[-\tau, 0]} | x_f(z_f+r) - x_g(z_f+r) | + \sup_{[-\tau, 0]} | x_g(z_f+r) - x_g(z_g+r) |.$$

By the boundedness of $\dot{x}_g$, the second term is less than

$$\sup_{[-\tau,0]}\left|\int_{z_g+r}^{z_f+r}\dot{x}_g(t)\,dt\right|\leq c_0\,|z_f-z_g|.$$

Hence our assertion follows from theorem 1 and from the continuity of the mapping $N: A-\{0\}\ni f\mapsto z_f\in R^+$. Proof of the continuity of N:

a) $f\in A-\{0\}\Rightarrow \dot{x}_f(Z_f)<0\wedge \dot{x}_f(z_f)>0$. Proof: We set $x:=x_f$, $M:=M_f$, $Z:=Z_f$, $m:=m_f$, $z:=z_f$. First, let us assume $\dot{x}(Z)=0$. Then $\int_1^\tau x(Z-a)\,ds(a)=0$. Therefore, $Z-\tau<M$ (by $x\,|\,[M,Z)>0$) and $x=0$ on $[0,Z-\tau]$ (by $0<Z-\tau$, $x\geq 0$ on $[0,Z]$, $\dot{x}\geq 0$ on $(0,M]$).

Setting $t^*:=\sup\{t\in R_0^+\mid x=0 \text{ on } [0,t]\}$, $t^{**}:=\sup\{t\in R^+\mid \dot{x}\geq 0 \text{ on } (0,t]\}$, we have $t^*+1\leq t^{**}$, $M\leq t^{**}$, $Z-\tau\leq t^*$. Hence $Z-1\leq t^*+\tau-1\leq t^*+1\leq t^{**}$, $[Z-\tau,Z-1]\subset[0,t^{**}]$.

On the other hand $\exists\,\tilde{t}\in(\tau,Z):\dot{x}(\tilde{t})<0$. Hence $0<\tilde{t}-\tau<Z-\tau$, $\tilde{t}-1<Z-1$, $\int_1^\tau x(\tilde{t}-a)\,ds(a)>0$.

x increasing on $[0,t^{**}]$ and $[Z-\tau,Z-1]\subset[0,t^{**}]$ give

$$\int_1^\tau x(Z-a)\,ds(a)\geq\int_1^\tau x(\tilde{t}-a)\,ds(a)>0,\ \dot{x}(Z)<0,$$

which is a contradiction. Now suppose $\dot{x}(z)=0$. As before we obtain $\int_1^\tau x(z-a)\,ds(a)=0$, $z-\tau<m$, $x=0$ on $[Z,z-\tau]$. We have $Z<z-\tau$ (lemma 2). Hence $\dot{x}(Z)=0$, contradiction.

b) Now let $f\in A-\{0\}$, $0<\varepsilon<\tau/2$. For every $\varepsilon'>0$ let $C(\varepsilon')$ denote the connected component of $\{t\in R^+\mid |x_f(t)|<\varepsilon'\}$ containing Z_f, $c(\varepsilon')$ the connected component containing z_f. If $\varepsilon'<\min\{x_f(M_f),\,-x_f(m_f),\,x_f(z_f+1)\}$, the set

$$\{t\in R^+\mid |x_f(t)|<\varepsilon'\}\cap[M_f,z_f+1]$$

decomposes into the two disjoint open sets $C(\varepsilon')$ and $c(\varepsilon')$. Part a) implies:

$$\exists\,\varepsilon'\in(0,\min\{x_f(M_f),\,-x_f(m_f),\,x_f(z_f+1)\}):$$
$$C(\varepsilon')\subset\{t\in R\mid |t-Z_f|<\varepsilon\}\wedge c(\varepsilon')\subset\{t\in R\mid |t-z_f|<\varepsilon\}.$$

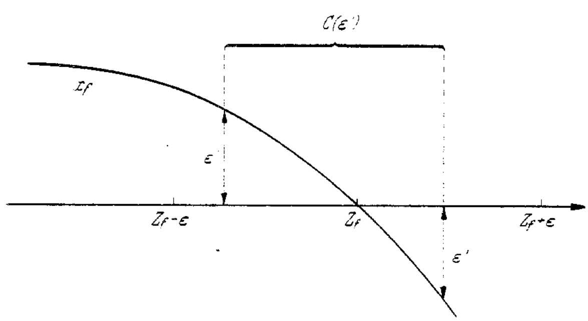

Fig. 2

By theorem 1 there is a $\delta>0$ (depending on f and ε') with

$$g\in A-\{0\}\wedge\|f-g\|<\delta\Rightarrow\sup_{[-\tau,z_f+1]}|x_f(t)-x_g(t)|<\varepsilon'. \tag{5}$$

c) The continuity of N will follow if we can prove that $z_g\in c(\varepsilon')$ for $g\in A-\{0\}$, $\|f-g\|<\delta$.

d) For such g, we have:
$x_g(t)=0\wedge t\in[M_f,z_f+1]\Rightarrow|x_f(t)|<\varepsilon'\wedge t\in[M_f,z_f+1]\Rightarrow t\in C(\varepsilon')\cup c(\varepsilon')$.

e) x_g has zeros in $C(\varepsilon')$ and in $c(\varepsilon')$. Proof:

$$M_f\le t\le\inf C(\varepsilon')\vee\sup c(\varepsilon')\le t\le z_f+1\Rightarrow x_f(t)\ge\varepsilon'\Rightarrow x_g(t)>0,\text{ by (5).}$$

$$\sup C(\varepsilon')\le t\le\inf c(\varepsilon')\Rightarrow x_f(t)\le-\varepsilon'\Rightarrow x_g(t)<0,\text{ by (5).}$$

f) $Z_g\in C(\varepsilon')$. Proof: Z_g is the first zero of x_g on $[M_g,\infty)$. If $M_f\le M_g$, Z_g is the first zero of x_g after M_f, since x_g increases on $[0,M_g]$ and $x_g(M_f)>0$ (compare e)). In the other case we have by $M_g<M_f\le\tau<Z_g$ that Z_g is the first zero on $[M_f,\infty)$. d) and e) imply $Z_g\in C(\varepsilon')$.

g) $z_g\in c(\varepsilon')$. Proof: By a) we have $x<0$ on $(Z_g,Z_g+\tau)$. Then z_g is the first zero of x_g on (Z_g,∞). $c(\varepsilon')$ contains a zero z of x_g with $z>Z_g$ (by e) and f)). Hence $z_g\le z$ and $z_g\in C(\varepsilon')\cup c(\varepsilon')$ (by d)). $z_g>Z_g+\tau$ and $(\textit{diameter } C(\varepsilon'))<2\varepsilon<\tau$ imply $z_g\notin C(\varepsilon')$.

3. In the following, we assume that the hypotheses of lemma 3 are fulfilled.

For $0\in A$ to be an ejective fixed point of T, it is sufficient to show:

$$\exists\varepsilon>0\ \forall f\in A-\{0\}\ \exists n(f)\in N:\|T^{n(f)}(f)\|\ge\varepsilon.$$

We shall derive this result in a finite sequence of lemmas under a smallness condition on $\tau-1$.

Let $f\in A-\{0\}$. Proceeding as in lemma 1 and lemma 2, we obtain a sequence of zeros z_j and a sequence of local extrema m_j of $x:=x_f$ with the properties

$$z_0:=0,\ z_1:=Z_f,\ z_2:=z_f,\ m_1=M_f,\ m_2:=m_f, \tag{6}$$

$$j\in N\text{ odd}\Rightarrow(x\text{ decreasing on }[m_j,m_{j+1}])\wedge(x\text{ increasing on }[m_{j+1},m_{j+2}]), \tag{7}$$

$$j\in N\text{ odd}\Rightarrow 0<x(m_j)\le e^{\alpha\tau}-1,$$
$$j\in N\text{ even}\Rightarrow\exp\left(-\alpha\tau(e^{\alpha\tau}-1)\right)-1\le x(m_j)<0, \tag{8}$$

$$j\in N\text{ odd}\Rightarrow z_j\le m_j+\tau+1,$$
$$j\in N\text{ even}\Rightarrow z_j<m_j+c(\alpha,\tau),\text{ with }c(\alpha,\tau)\in R^+\text{ independent from }f\in A-\{0\}\text{ and }j\in N, \tag{9}$$

$$\forall j\in N\cup\{0\}:m_{j+1}\in[z_j+1,z_j+\tau],$$
$$\forall j\in N:m_j+\tau-1<z_j. \tag{10}$$

We start with

Lemma 4: $\alpha>\pi/2\Rightarrow(\exists\lambda\in C:\operatorname{Re}\lambda>0\wedge 0<\operatorname{Im}\lambda<\pi\wedge\lambda+\alpha e^{-\lambda}=0)$.

A proof can be found in [7].

Lemma 5: *If λ is a solution of $\lambda+\alpha e^{-\lambda}=0$ with* $\operatorname{Re}\lambda>0$ *and* $0<\operatorname{Im}\lambda<\pi$, *there are constants $c_1\in R^+$, $c_2\in R^+$ such that for all integers $j\geq 3$ the following estimate holds*:

$$c_1|x(m_j)|\leq(\sup_{t\geq m_j}|x(t)|)^2+c_2(\tau-1)\sup_{t\geq m_j-2}|x(t)|, \tag{11}$$

where c_1 and c_2 depend on α, $\operatorname{Re}\lambda$, $\operatorname{Im}\lambda$ *only.*

Proof: We follow Nussbaum's proof of lemma 4.4 in [6] (which is a variant of Wright's proof of theorem 3 in [7]). Let λ as in lemma 4. Consider a local extremum $m:=m_j$ (with $j\geq 3$) of x. An integration by parts yields

$$\int_m^\infty \dot{x}\,e^{-\lambda t}\,dt=\lambda\int_m^\infty x\,e^{-\lambda t}\,dt-x(m)\,e^{-\lambda m}. \tag{12}$$

With $\Delta(t):=x(t-1)-\int_1^\tau x(t-a)\,ds(a)$, eq. (2) gives

$$\int_m^\infty \dot{x}\,e^{-\lambda t}\,dt=-\alpha\int_m^\infty[x(t-1)-\Delta(t)]\,[x(t)+1]\,e^{-\lambda t}\,dt$$
$$=-\alpha\int_m^\infty x(t-1)e^{-\lambda t}\,dt-\alpha\int_m^\infty x(t-1)\,x(t)\,e^{-\lambda t}\,dt+\alpha\int_m^\infty \Delta(t)\,[x(t)+1]\,e^{-\lambda t}\,dt.$$

Setting

$$R(m):=-\alpha\int_m^\infty x(t-1)\,x(t)\,e^{-\lambda t}\,dt+\alpha\int_m^\infty \Delta(t)\,[x(t)+1]\,e^{-\lambda t}\,dt$$

we obtain

$$\int_m^\infty \dot{x}\,e^{-\lambda t}\,dt=R(m)-\alpha\int_m^\infty x(t-1)\,e^{-\lambda t}\,dt=R(m)-\alpha\,e^{-\lambda}\int_{m-1}^\infty x\,e^{-\lambda t}\,dt.$$

Equation (12) gives

$$R(m)-\alpha\,e^{-\lambda}\int_{m-1}^\infty x\,e^{-\lambda t}\,dt=\lambda\int_m^\infty x\,e^{-\lambda t}\,dt-x(m)\,e^{-\lambda m}.$$

With $\lambda+\alpha\,e^{-\lambda}=0$:

$$R(m)=-\lambda\int_{m-1}^m x\,e^{-\lambda t}\,dt-x(m)\,e^{-\lambda m}. \tag{13}$$

We have

$$-\lambda\int_{m-1}^m x\,e^{-\lambda t}\,dt=x(m)\,e^{-\lambda m}-x(m-1)\,e^{-\lambda(m-1)}-\int_{m-1}^m \dot{x}\,e^{-\lambda t}\,dt.$$

Substitution in (13) and multiplication of the resulting equation by $\exp(\lambda(m-1/2))$ yields

$$-x(m-1)\,e^{\lambda/2}-\int_{m-1}^m \dot{x}(t)\,\exp(-\lambda(t+1/2-m))\,dt=R(m)\,\exp(\lambda(m-1/2)). \tag{14}$$

The real part of the left side of (14) is

$$l:=-x(m-1)\,e^{\operatorname{Re}\lambda/2}\cos(\operatorname{Im}\lambda/2)$$
$$-\int_{m-1}^m \dot{x}(t)\,\exp(-\operatorname{Re}\lambda\,(t-m+1/2))\cos(\operatorname{Im}\lambda\,(t-m+1/2))\,dt.$$

We look for a lower estimate of l:

a) If $t\in[m-1,m]$, $\dot{x}(t)$ and $x(m-1)$ have the same sign. Proof: If $x(m)=x(m_j)$ is a local minimum we have $m_j-1\geq z_{j-1}$ and $\dot{x}\leq 0$ on $[z_{j-1},m_j]$. Same proof, if $x(m)$ is a local maximum.

b) $t \in [m-1, m]$, $\operatorname{Re}\lambda > 0$ and $0 < \operatorname{Im}\lambda < \pi$ imply $\cos(\operatorname{Im}\lambda(t-m+1/2)) \geq \cos(\operatorname{Im}\lambda/2) > 0$ and $\exp(-\operatorname{Re}\lambda(t-m+1/2)) \geq \exp(-\operatorname{Re}\lambda/2)$.

From a) and b) we infer

$$\begin{aligned} |l| &\geq |x(m-1)\, e^{\operatorname{Re}\lambda/2} \cos(\operatorname{Im}\lambda/2) + e^{-\operatorname{Re}\lambda/2} \cos(\operatorname{Im}\lambda/2)\,(x(m)-x(m-1))| \\ &= |x(m-1)\cos(\operatorname{Im}\lambda/2)\,(e^{\operatorname{Re}\lambda/2} - e^{-\operatorname{Re}\lambda/2}) + x(m)\, e^{-\operatorname{Re}\lambda/2}\cos(\operatorname{Im}\lambda/2)|. \end{aligned}$$

$x(m)x(m-1) \geq 0$ and $e^{\operatorname{Re}\lambda/2} > e^{-\operatorname{Re}\lambda/2}$, therefore

$$|l| \geq |x(m)|\, e^{-\operatorname{Re}\lambda/2} \cos(\operatorname{Im}\lambda/2). \tag{15}$$

Estimate of the real part of the right side of (14):

$$\begin{aligned} &|\operatorname{Re}(\exp(\lambda(m-1/2))\, R(m))| \\ &\leq \left|-\alpha \int_m^\infty x(t-1)\, x(t)\, \operatorname{Re}\exp(-\lambda(t-m+1/2))\, dt\right| \\ &+ \left|\alpha \int_m^\infty \left[x(t-1) - \int_1^\tau x(t-a)\, ds(a)\right] [x(t)+1]\, \operatorname{Re}\exp(-\lambda(t-m+1/2))\, dt\right|. \end{aligned}$$

Since $m = m_j \geq z_{j-1}+1$, $|x|$ increasing on $[z_{j-1}, m_j]$, the first term is majorized by

$$\alpha\left(\sup_{t \geq m} |x(t)|\right)^2 \int_m^\infty |\operatorname{Re}\exp(-\lambda(t-m+1/2))|\, dt. \tag{16}$$

Second term: The mean value theorem for Lebesgue-Stieltjes-integrals and the properties of s imply

$$x(t-1) - \int_1^\tau x(t-a)\, ds(a) = x(t-1) - x(t'), \text{ with } t' \in [t-\tau, t-1],$$

hence by the usual mean value theorem and by $t \geq m = m_j$, $j \geq 3$:

$$\ldots = \dot{x}(t'')(t-1-t'), \text{ with } t'' \in [t', t-1].$$

By (2) and by $0 < x+1 \leq e^{\alpha\tau}$:

$$|\dot{x}(t'')| \leq \alpha\, e^{\alpha\tau} \sup_{r \in [t''-\tau, t''-1]} |x(r)|.$$

Using $0 \leq x+1 \leq e^{\alpha\tau}$ and $0 \leq t-1-t' \leqq \tau-1$, we can now estimate the second term by

$$(\tau-1) \sup_{t \geq m-2\tau} |x(t)|\, \alpha^2 e^{2\alpha\tau} \int_m^\infty |\operatorname{Re}\exp(-\lambda(t-m+1/2))|\, dt. \tag{17}$$

The remaining integral is less than $(\operatorname{Re}\lambda)^{-1} e^{-\operatorname{Re}\lambda/2}$.

From (14)—(17) we infer:

$$\begin{aligned} e^{-\operatorname{Re}\lambda/2} \cos(\operatorname{Im}\lambda/2)\, |x(m)| &\leq \alpha(\operatorname{Re}\lambda)^{-1} e^{-\operatorname{Re}\lambda/2} \left(\sup_{t \geq m} |x(t)|\right)^2 + \\ &(\tau-1)\,\alpha^2 e^{2\alpha\tau} (\operatorname{Re}\lambda)^{-1} e^{-\operatorname{Re}\lambda/2} \sup_{t \geq m-2\tau} |x(t)|. \end{aligned}$$

Now the assertion of lemma 5 follows from

$$m - 2\tau = m_j - 2\tau \geq m_{j-2} \text{ and } \tau < 2.$$

Lemma 6: *If $\tau - 1 < (4\alpha e^{\alpha\tau})^{-1}$, there is a constant $c_3 \in R^+$, depending only on α, such that for all $j \in N$*

$$|x(m_j)| \leq c_3 \, |x(m_{j+1})|. \tag{18}$$

Proof: Let $d := (\alpha \, e^{\alpha\tau})^{-1}$.

a) We have $|x(t)| \geq 1/2 \cdot |x(m_j)|$ on $[m_j, m_j + d/2]$. Proof: If m_j is a local minimum and if $z_{j-1} \in [t-\tau, t-1]$, $t \in [m_j, m_j+1]$, we get from (2)

$$0 \leq \dot{x}(t) = -\alpha\,[x(t)+1] \left[\int_1^{t-z_{j-1}} x(t-a)\, ds(a) + \int_{t-z_{j-1}}^{\tau} x(t-a)\, ds(a) \right].$$

The second integral is non-negative since $a \in [t-z_{j-1}, \tau]$ and $t \geq m_j$ imply $t-a \in [z_{j-1}-\tau, z_{j-1}] \subset [z_{j-2}, z_{j-1}]$, hence $x(t-a) \geq 0$.

Therefore,

$$0 \leq \dot{x}(t) \leq -\alpha\,[x(t)+1] \int_1^{t-z_{j-1}} x(t-a)\, ds(a) \leq -\alpha\, e^{\alpha\tau} \inf x \,|\, [z_{j-1}, m_j] = -d^{-1} x(m_j).$$

If $z_{j-1} \leq t-\tau$ we easily get the same result.

Then, for $t \in [m_j, m_j + d/2]$,

$$x(t) - x(m_j) = \int_{m_j}^{t} \dot{x}(r)\, dr \leq (t-m_j)\,(-1/d)\, x(m_j) \leq -x(m_j)/2,$$

$$x(t) \leq x(m_j)/2 \leq 0, \; |x(m_j)|/2 \leq |x(t)|.$$

If m_j is a local maximum the same method yields $0 \geq \dot{x} \geq -x(m_j)/d$ on $[m_j, m_j+1]$, which in turn implies a).

b) On $[m_j+\tau, m_j+\tau+d/4]$, we have $|\dot{x}(t)| \geq c_4 |x(m_j)|$, with c_4 depending only on α. Proof: $t \in [m_j+\tau, m_j+\tau+d/4] \Rightarrow [t-\tau, t-1] \subset [m_j, m_j+\tau-1+d/4] \subset [m_j, m_j+d/2]$, since $\tau - 1 < d/4$. a) implies $[m_j, m_j+d/2] \subset [m_j, z_j)$. On $[m_j, z_j)$, x does not change its sign. Hence, since $|x|$ decreases on $[m_j, z_j]$, $|\dot{x}(t)| = \alpha\,[x(t)+1] \left| \int_1^{\tau} x(t-a)\, ds(a) \right|$

$= \alpha[x(t)+1] \int_1^{\tau} |x(t-a)|\, ds(\alpha) \geq \alpha[x(t)+1]\,|x(t-1)| \geq \alpha \exp(-\alpha\tau(e^{\alpha\tau}-1))\,|x(m_j)|/2.$

(Since $t-1 \leq m_j + d/2$, we can apply a).) Set $c_4 := \alpha \exp(-2\alpha(e^{2\alpha}-1))/2$.

c) Now we are able to estimate $|x(m_j)|$ by $|x(m_{j+1})|$. Remark: $m_{j+1} > m_j + \tau + d/4$, because $m_{j+1} \geq m_j + \tau$, and $m_{j+1} \in [m_j+\tau, m_j+\tau+d/4]$ would imply $|\dot{x}(m_{j+1})| > 0$ (with b)), contradiction.

d) Suppose $z_j < m_j + \tau + d/8$. Then

$$|x(m_{j+1})| = \left| \int_{z_j}^{m_j+1} \dot{x}(r)\, dr \right| = \int_{z_j}^{m_j+1} |\dot{x}(r)|\, dr \geq \int_{m_j+\tau+d/8}^{m_j+\tau+d/4} |\dot{x}(r)|\, dr \geq d\,c_4\,|x(m_j)|/8, \tag{19}$$

by b).

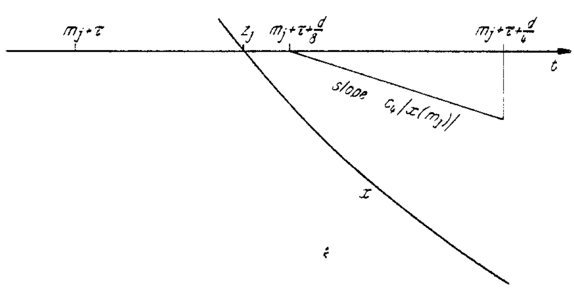

Fig. 3

e) Suppose $z_j \geq m_j + \tau + d/8$. We have $[m_j, m_j + \tau + d/16] \subset [m_j, z_j)$, $|x|$ decreases on $[m_j, z_j]$. For $t \in [m_j, m_j + \tau + d/16]$ we obtain using b)

$$
\begin{aligned}
|x(t)| \geq |x(m_j + \tau + d/16)| &= \left| \int_{m_j+\tau+d/16}^{z_j} \dot{x}(r)\, dr \right| = \int_{m_j+\tau+d/16}^{z_j} |\dot{x}(r)|\, dr \\
&\geq \int_{m_j+\tau+d/16}^{m_j+\tau+d/8} |\dot{x}(r)|\, dr \geq d\, c_4\, |x(m_j)|/16.
\end{aligned} \tag{20}
$$

Case I: $z_j \leq m_j + \tau + 1$. We estimate $|\dot{x}|$ on $[z_j, z_j + d/16]$: For these t, $m_j \leq t - \tau < t - 1 \leq m_j + \tau + d/16 < z_j$. (20) implies $|\dot{x}(t)| = \alpha\,[x(t)+1]\, \left| \int_1^\tau x(t-a)\, ds(a) \right| = \alpha\,[x(t)+1] \int_1^\tau |x(t-a)|\, ds(a) \geq \alpha \exp(-\alpha\,\tau\,(e^{\alpha\tau}-1))\, d\, c_4\, |x(m_j)|/16 \geq d\, c_4^2\, |x(m_j)|/8$.

Hence

$$
\begin{aligned}
|x(m_{j+1})| &= \left| \int_{z_j}^{m_{j+1}} \dot{x}(r)\, dr \right| = \int_{z_j}^{m_{j+1}} |\dot{x}(r)|\, dr \\
&\geq \int_{z_j}^{z_j+d/16} |\dot{x}(r)|\, dr \geq d^2\, c_4^2\, |x(m_j)|/128.
\end{aligned} \tag{21}
$$

Case II: $z_j > m_j + \tau + 1$. This case is only possible if m_j is a local minimum. We set $\tilde{m}_j := \sup\{t \geq m_j \mid x \leq -(1-2/\pi) \text{ on } [m_j, t]\}$. The proof of lemma 2 gives $\tilde{m}_j > m_j + \tau$ and $z_j \in [\tilde{m}_j, \tilde{m}_j + 1]$. We argue as before: x increases on $[m_j, \tilde{m}_j]$, hence for $t \in [\tilde{m}_j, \tilde{m}_j + 1]$

$$
\begin{aligned}
\dot{x}(t) &\geq -\alpha\,[x(t)+1]\, x(t-1) \geq \alpha \exp(-\alpha\,\tau\,(e^{\alpha\tau}-1))\,(1-2/\pi) \\
&\geq 2\, c_4\, (1-2/\pi).
\end{aligned} \tag{22}
$$

If $z_j \leq \tilde{m}_j + 1/2$, we get

$$
\begin{aligned}
x(m_{j+1}) &= \int_{z_j}^{m_{j+1}} \dot{x}(r)\, dr \geq \int_{z_j}^{z_j+1/2} \dot{x}(r)\, dr \geq c_4\,(1-2/\pi) \\
&> c_4\,(1-2/\pi)\, |x(m_j)|, \ \text{since } |x(m_j)| < 1.
\end{aligned} \tag{23}
$$

Now suppose $z_j > \tilde{m}_j + 1/2$. Then

$$
\begin{aligned}
t \in [m_j, \tilde{m}_j + 1/4] &\Rightarrow -x(t) \geq -x(\tilde{m}_j + 1/4) = \int_{\tilde{m}_j+1/4}^{z_j} \dot{x}(r)\, dr \\
&\geq \int_{\tilde{m}_j+1/4}^{\tilde{m}_j+1/2} \dot{x}(r)\, dr \geq c_4\,(1-2/\pi)/2, \ \text{by (22)}.
\end{aligned} \tag{24}
$$

For $t \in [z_j, z_j + 1/4]$ we have $[t-\tau, t-1] \subset [m_j, \tilde{m}_j + 1/4]$, hence $\dot{x}(t) \geq -\alpha\,[x(t)+1] \cdot x(t-1)$ (since x increases on $[m_j, z_j]$) and $\dot{x}(t) \geq \alpha \exp(-\alpha\,\tau\,(e^{\alpha\tau}-1))\, c_4\,(1-2/\pi)/2$, by (24).

Finally, we arrive at

$$
\begin{aligned}
x(m_{j+1}) = \int_{z_j}^{m_{j+1}} \dot{x}(r)\,dr \geq \int_{z_j}^{z_j+1/4} \dot{x}(r)\,dr \geq c_4^2(1-2/\pi)/4 \\
\geq c_4^2(1-2/\pi)\,|\,x(m_j)|/4.
\end{aligned} \tag{25}
$$

f) The estimate (18) follows from (19), (21), (23), (25) if we replace d by $(\alpha\, e^{2\alpha})^{-1}$ in these estimates.

We want to transform the inequalities (11) and (18) into an estimate concerning an iterate of the operator T. To do this we need the following lemma.

Lemma 7: *For the integers $j \geq 3$, we have*
i) j *odd* $\Rightarrow x(m_j) \leq \alpha\,\tau\, e^{\alpha\tau}\,\|\,T^{(j-1)/2}(f)\,\|$,
ii) j *even* $\Rightarrow |\,x(m_j)| \leq (\alpha\,\tau\, e^{\alpha\tau})^2\,\|\,T^{(j-2)/2}(f)\,\|$.

Proof: i) $x(m_j)$ is a local maximum and equals $\int_{z_{j-1}}^{m_j}[1+x(t)](-\alpha)\int_1^\tau x(t-a)\,ds(a)\,dt$.
$t \in [z_{j-1}, m_j]$ and $x \geq 0$ on $[z_{j-1}, m_j]$ imply

$$
\begin{aligned}
-\int_1^\tau x(t-a)\,ds(a) \leq -\int_1^\tau \inf x\,|\,[z_{j-1}-\tau, z_{j-1}]\,ds(a) \\
= -\inf x\,|\,[z_{j-1}-\tau, z_{j-1}] = \|\,T^{(j-1)/2}(f)\,\|,
\end{aligned}
$$

since the iterates of T are given by the restrictions of x to $[z_j-\tau, z_j]$, with j even. Therefore

$$
x(m_j) \leq \int_{z_{j-1}}^{m_j} \alpha\, e^{\alpha\tau}\,\|\,T^{(j-1)/2}(f)\,\|\,dt \leq \alpha\,\tau\, e^{\alpha\tau}\,\|\,T^{(j-1)/2}(f)\,\|.
$$

ii) Now $x(m_j)$ is a local minimum. Proceeding as in the proof of i), we obtain for $t \in [z_{j-1}, m_j]$

$$
-\int_1^\tau x(t-a)\,ds(a) \geq -\sup x\,|\,[z_{j-1}-\tau, z_{j-1}] \geq -x(m_{j-1}),
$$

hence

$$
x(m_j) = \int_{z_{j-1}}^{m_j}[x(t)+1](-\alpha)\int_1^\tau x(t-a)ds(a)dt \geq -\alpha\tau e^{\alpha\tau}x(m_{j-1}) \geq -(\alpha\tau e^{\alpha\tau})^2\,\|\,T^{(j-2)/2}(f)\|,
$$

by i).

Lemma 8: *There is a constant $c>0$ such that for $\tau-1<c$, $0 \in A$ is an ejective fixed point of the operator T.*

Proof:

a) Consider $j \in N$. Let $\tau-1<(4\,\alpha\, e^{\alpha\tau})^{-1}$. From lemma 6 we infer $\sup_{t \geq m_j}|\,x(t)|$ $\leq c_3^2 \sup_{t \geq m_{j+2}} |\,x(t)|$. Proof: $t \geq m_j \Rightarrow t \in [m_j, m_{j+2}] \vee t \geq m_{j+2}$, $t \in [m_j, m_{j+2}] \Rightarrow |\,x(t)|$ $\leq \max\{|\,x(m_j)|, |\,x(m_{j+1})|, |\,x(m_{j+2})|\}$. By (18), $|\,x(t)| \leq c_3^2\,|\,x(m_{j+2})| \leq c_3^2 \sup_{t \geq m_{j+2}}|\,x(t)|$. (Obviously c_3 can be chosen ≥ 1.)

b) Now consider a zero z_k of x with $k \geq 3$. Set $\delta := \sup_{t \geq z_k} |\,x(t)|$. We have $\delta > 0$ (see (8)). There is a $m_j \geq z_k$ with $j \geq 3$ and $|\,x(m_j)| \geq \delta/2$. (j depends on x, hence on f.)

By lemma 5 we have, using a),

$$c_1\,\delta/2 \le \delta^2 + c_2\,(\tau-1) \sup_{t \ge m_j - 2} |x(t)| \le \delta^2 + c_3^2\, c_2\,(\tau-1)\,\delta.$$

If in addition $\tau - 1 < c_1/2\, c_2\, c_3^2$, we obtain

$$0 < (c_1/2 - (\tau-1)\, c_2\, c_3^2) \le \delta \le 2\,|x(m_j)| \le 2\,\alpha\,\tau\, e^{\alpha\tau}\, \|T^{(j-1)/2}(f)\|,$$

if j is odd, or $\le 2\,(\alpha\,\tau\, e^{\alpha\tau})^2\, \|T^{(j-2)/2}(f)\|$, if j is even. The constants c_1, c_2, c_3 depend only on α.

Setting $c := \min\{c_1/2\, c_2\, c_3^2,\ 1/4\,\alpha\, e^{2\alpha}\}$ and $\varepsilon := (c_1/2 - (\tau-1)\, c_2\, c_3^2)/2\,(\alpha\,\tau\, e^{\alpha\tau})^2$, we obtain for $\tau - 1 < c$

$$\exists\, \varepsilon > 0\ \forall f \in A - \{0\}\ \exists\, n(f) \in N : \|T^{n(f)}(f)\| \ge \varepsilon.$$

Notations: By R, R^+, R_0^+ we mean the real, positive real and non-negative real numbers respectively. N, N_0 denote the positive or non-negative integers. A dot — like in $\dot{x}(t)$ — indicates differentiation. If x is a function from the interval $[-\tau, \infty)$ into R and if $[a, b] \subset [-\tau, \infty)$, then $x\,|\,[a, b]$ is the function $[a, b] \ni t \mapsto x(t) \in R$.

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