Dissertation

Superlinear dynamics of a scalar parabolic equation

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Februar 2007

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und Nils

Zusammenfassung

Die vorliegende Arbeit befasst sich mit dem semilinearen parabolischen Anfangs-Randwertproblem

$$u_t - u_{xx} = f(u), \qquad u(\cdot, 0) = u_0, \quad u(0) = u(1) = 0. \tag{P}$$

Dieses Problem ist das einfachste Modell einer Wärmeleitungs- oder Reaktions-Diffusionsgleichung. Der von uns betrachtete superlineare Fall spielt insbesondere bei der Modellierung von Verbrennungsprozessen eine Rolle, konkrete Anwendungen finden sich in [Hen81, Kapitel 2]. Aus mathematischer Sicht induziert (P) einen Halbfluß φ auf dem Zustandsraum $H_0^1([0,1])$, an dessen dynamischen Eigenschaften wir interessiert sind.

Aus technischer Sicht hat der Halbfluß φ sehr gute Eigenschaften: er ist kompakt, besitzt eine Gradientenstruktur, und ist rückwärtseindeutig (d.h. die Zeit-1-Abbildung ist injektiv). Zudem ist, vereinfacht gesagt, die Zahl der Nullstellen entlang Lösungskurven monoton fallend ([Mat82]). Dieses zweite, diskrete Lyapunov Funktional wurde von vielen Autoren genutzt, um weitere Eigenschaften des Halbflusses zu beweisen. So schneiden sich stabile und instabile Mannigfaltigkeiten transversal ([Hen85]), Nichtdegeneriertheit der Gleichgewichtslösungen ist eine generische Eigenschaft ([BC84]), und im dissipativen Fall ist der globale Attraktor des Flusses ein endlichdimensionaler $\mathcal{C}^1$-Graph. Im dissipativen Fall wurde darüberhinaus mit Hilfe der Transversalitätseigenschaften und Conley Index Methoden von Brunovský und Fiedler ([BF88, BF89]) die Frage der Existenz verbindender Orbits zwischen den Gleichgewichtslösungen vollständig gelöst.

Im superlinearen Fall gibt es unendlich viele Gleichgewichtslösungen, und für diesen Fall scheint es keine vergleichbaren Resultate über verbindende Orbits oder Flussäquivalenz zu geben. Die zahlreichen Arbeiten über dieses und ähnliche superlineare Probleme, z.B. von Marek Fila, Hiroshi Matano, Peter Poláčik, Pavol Quittner und anderen, befassen sich überwiegend mit Blow-Up Lösungen. Über global beschränkte Lösungen scheint wenig bekannt zu sein. In der vorliegenden Arbeit gelingt es uns, für eine sehr große Klasse von superlinearen Nichtlinearitäten genau anzugeben, welche Gleichgewichtslösungen durch heterokline Orbits verbunden werden, und welche nicht. Für eine Teilklasse superlinearer Probleme (diese enthält den Modellfall $f(u) = u|u|^p$ oder auch $f(u) = u|u|^p - \lambda u$) können wir beweisen, daß bestimmte endlichdimensionale invariante Mengen $\mathcal{A}_{n,\infty}$ strukturell stabil sind (d.h. falls f und $\tilde{f}$ „nahe" beieinander liegen, so gibt es einen Homöomorphismus $\mathcal{A}_{n,\infty} \to \tilde{\mathcal{A}}_{n,\infty}$ der Orbits auf Orbits abbildet, und die zeitliche Orientierung der Orbits erhält). Die Mengen $\mathcal{A}_{n,\infty}$ enthalten auch Blow-Up Lösungen (d.h. unbeschränkte Lösungen mit endlicher Existenzzeit), d.h. diese partielle strukturelle Stabilität erstreckt sich auch auf das Blow-Up Verhalten.

Wir erhalten unsere Resultate auf folgende Weise: Eine superlineare Funktion f wird ausserhalb eines kompakten Intervalls so abgeändert, daß ein dissipativer Halbfluss entsteht. Da der Zustandsraum $H_0^1([0,1])$ kompakt nach $\mathcal{C}^0([0,1])$

einbettet, stimmt dieser abgeänderte Halbfluss auf einer Nullumgebung mit dem ursprünglichen Fluß φ überein. Wird nun das Intervall vergrößert, auf dem f unverändert bleibt, so wächst auch diese Nullumgebung entsprechend, und es lassen sich die Ergebnisse über den dissipativen Fall anwenden und auf φ übertragen. Diese Ergebnisse sind jedoch in der Mehrzahl nur für hyperbolische Halbflüsse richtig, was den skizzierten Ansatz technisch erschwert. Beim Abschneiden der superlinearen Funktion entstehen im modifizierten Fluß notwendig zusätzliche Gleichgewichtslösungen. Diese dürfen natürlich nicht ausgeartet sein. Zudem dürfen sie die Struktur der „Originalgleichgewichte“ nicht beeinflussen, um eine Rückübertragung der Ergebnisse auf den superlinearen Fall zu ermöglichen. Für das Resultat über strukturelle Stabilität ist es darüberhinaus notwendig, auch stetige Familien von Funktionen so abzuschneiden, daß wiederum stetige Familien von Funktionen entstehen. Auch die Position, ab der die Funktionen abgeändert werden, muß stetig variiert werden können.

Dies alles sicherzustellen ist der technische Kern dieser Arbeit (Kapitel 3). Konkret werden die Nichtlinearitäten, nach einem kurzen geglätteten Übergang, konstant fortgesetzt — damit erfüllen sie die Wachstumsbedingung für die Existenz eines globalen Attraktors. Dabei geht entscheidend ein, daß die Gleichgewichtslösungen von (P) Lösungen gewöhnlicher Differentialgleichungen zweiter Ordnung sind. Als zweidimensionales System können zur Analyse die Struktur des Phasenraumes und Shooting-Curve Techniken verwendet werden. Mit Hilfe solcher Shooting-Curve Methoden läßt sich insbesondere auch die Nichtdegeneriertheit von Gleichgewichtslösungen charakterisieren. Einige weitere, spezielle Hilfsmittel werden in Abschnitt 3.3.1 entwickelt. Zudem erfüllen die Ableitungen dieser stationären Lösungen nach verschiedenen Größen lineare Differentialgleichungen zweiter Ordnung, auf die Sturmsche Vergleichssätze angewandt werden können.

Mit Hilfe dieser technischen Resultate konstruieren wir in Abschnitt 4.1 eine Folge (φ_n) dissipativer Halbflüsse, die jeweils eine wachsende Zahl von Gleichgewichten der superlinearen Gleichung enthalten. Aus [BF89] folgt leicht, daß unter den angegebenen Bedingungen für hinreichend große n verbindende Orbits bezüglich φ_n existieren. Es ist allerdings nicht klar, daß die verbindenden Orbits komplett im unveränderten Bereich des Halbflusses liegen, wenn dies für die Endpunkte des Orbits gilt. Daher muß noch sichergestellt werden, daß die gefundenen Verbindungen für $n \to \infty$ nicht abreißen. Dies gelingt mit Energieargumenten und dadurch, daß wir die Nullstellenzahl der zusätzlich entstehenden Lösungen in geeigneter Weise kontrollieren können.

Um für „gleichmäßig superlineare“ f genauere Aussagen darüber zu erhalten, in welchem Sinne φ von φ_n approximiert wird, konstruieren wir nun eine stetige Familie $(\varphi_\nu)_\nu$ von dissipativen Halbflüssen, die für ganzzahlige ν mit den vorher Konstruierten übereinstimmen. Somit existiert ein globaler Attraktor von φ_n und es stellt sich die Frage, ob diese invariante Menge für ν nahe bei n in φ_ν erhalten bleibt. Es stellt sich heraus, daß für $\nu \geq n$ lokale Attraktoren von φ_ν existieren, die stetig in ν variieren, und für $n = \nu$ mit dem globalen Attraktor übereinstimmen. Darüberhinaus sind die Flüsse auf diesen lokalen Attraktoren konjugiert, mit anderen Worten: die Struktur bleibt für alle $\nu \geq n$ erhalten, und wird durch die mit wachsendem ν nach und nach hinzukommenden energierreicheren Gleichgewichte nicht beeinflusst.

Mit diesen Resultaten erhält man dann leicht die Konjugation zweier superlinearer Flüsse auf n-dimensionalen invarianten Mengen, die durch Grenzübergang aus den genannten lokalen Attraktoren entstehen. Diese Konstruktion liefert überdies eine Interpretation von Blow-Up als Approximation von ∞ durch eine wachsende Folge von (lokalen/globalen) Attraktoren: Die zusätzlich durch das Abschneiden entstehenden Gleichgewichtslösungen variieren stetig in ν, und konvergieren in der H_0^1-Norm gegen ∞ für $\nu \to \infty$.

Für die hervorragende Betreuung der Arbeit danke ich Herrn Thomas Bartsch, ebenso danke ich Herrn Hans-Otto Walther für die Arbeit als Gutachter. Als Ausdruck meiner Freude über viele Dinge, die während der Arbeit an dieser Dissertation geschehen sind, widme ich diese Arbeit Alexandra, Anna-Katharina und Nils.

Contents

CHAPTER 1

Introduction

We are concerned with solutions of the onedimensional semilinear parabolic problem

$$u_t - u_{xx} = f(u), \qquad u(\cdot, 0) = u_0, \quad u(0) = u(1) = 0 \tag{P}$$

on the unit interval with a smooth nonlinearity f. This problem is the simplest model example of an heat or reaction-diffusion equation. With a superlinear nonlinearity problems of that type appear in the theory of combustion. Several applications are mentioned in [Hen81, Chapter 2]. This problem induces a differentiable semiflow φ on $H_0^1([0,1])$, the dynamics of which we want to investigate. Definitions and details will be given in section 2.

A very important property of this problem is that, roughly speaking, the number of zeros is nonincreasing along solutions of (P). This was showed by Matano ([Mat82]). His result has been used by Angenent ([Ang86]) and Henry ([Hen85]) to show that stable and unstable manifold necessarily intersect transversally. It has also been proved by Brunovsky and Chow ([BC84]) that nondegeneracy of equilibrium solutions is a generic property. The semiflow has several nice features (compactness, backward uniqueness, gradient structure) and finally the equilibrium solutions are solutions of a twodimensional autonomous initial value problem. The twodimensional flow resulting from this IVP can be examined by phase-plane analysis and shooting curve techniques. So from a technical point of view the problem (P) looks rather promising.

There has been a lot of work on the dynamics of φ in the dissipative case. Under the growth condition $\limsup_{|t|\to\infty} t^{-1} f(t) < \pi^2$ the semiflow φ admits a compact global attractor which is the union of the unstable manifolds of all equilibrium (i.e. time-independent) solutions. This means that the essential dynamics of φ can be described on a finite-dimensional invariant set, which happens to be a smooth graph in our case ([Bru90]). This attractor is also the union of the equilibrium solutions and the connecting, or heteroclinic, orbits between equilibria. The question which equilibria are connected by heteroclinic orbits has been solved completely by Brunovský and Fiedler ([BF88],[BF89]) in the (generic) hyperbolic case. They used the nonincrease of the zero-number and Conley Index arguments, extending previous partial results ([CS80], [Hen81, §5.3],[Hen85]). For dissipative Morse-Smale systems Oliva ([Oli02]) obtained a structural stability result, showing that the flows on the attractors of two "close" Morse-Smale semiflows are conjugate. Lu ([Lu94]) transferred a classical result of Palis and Smale ([PS70]) to φ using the existence of an inertial manifold ([CL88]). He showed that two close semiflows are conjugate not only on the attractor, but also on an open neighborhood of the attractor.

The problem (P) (and similar problems) with superlinear nonlinearity is a very active field of research. Several people (e.g. Marek Fila, Hiroshi Matano, Peter Poláčik, Pavol Quittner and others) have proved results about blow-up solutions. About the set of globally bounded solutions however, not much is known. It is known that in this case infinitely many equilibrium solutions exist (see eg. [Str80]), and trajectories either blow up in finite time or converge to one of the equilibria. So we will consider both the set of globally bounded solutions and the union of all unstable manifolds, and finite dimensional approximations or subsets of these. Our first main results is to extend the results of Brunovský and Fiedler to give a complete description of the connecting orbit structure of a very large class of superlinear nonlinearities in Theorem 4.2. Apparently there is no similar result about connecting orbits for superlinear nonlinearities. The second main result is the conjugacy of two close flows on certain finite dimensional invariant sets, including a "stability of blow-up behaviour" result (Theorem 4.5). Although we were not able to prove structural stability w.r.t. one of the infinite-dimensional sets mentioned above, this result also seems to be completely new. This approach could certainly be refined to investigate blow-up phenomena by "dissipative approximation", cf. section 4.3.

Technically we use a straightforward approach. Modifying a given superlinear f outside a compact interval I we get a dissipative semiflow. Due to the compactness of the embedding of the state space $H_0^1([0,1])$ in the space of continuous functions this modified semiflow coincides with the original one on an open ball in X. Increasing I increases this ball, so we get an sequence of dissipative semiflows φ_n. These φ_n necessarily contain equilibrium solutions that do not exist in φ, but we are able to make sure that these additional solutions are nondegenerate. We apply the results on connecting orbits to these φ_n and are able to show, that these connections w.r.t. φ_n persist as $n \to \infty$. For the structural stability results we have to construct a continuous family φ_ν of dissipative flows approximating φ. As $\nu \to \infty$ the number of equilibria increases, so necessarily degenerate stationary solutions appear. By controlling the zero number of these solutions we are able to apply results for global attractors to local attractors of φ_n.

The technical part, cutting of a family of superlinear functions appropriately, will be done in chapter 3 after collecting everything we need to know about the parabolic flow in chapter 2. These results will then be applied in chapter 4.

1.1 Notation

We will work mostly in standard Lebesgue- and Sobolev Spaces. Let $L^2 = L^2([0,1])$ be the space of [equivalence classes of] square integrable functions with the usual norm $\|u\|_2 := (\int u^2)^{\frac{1}{2}}$. The functions in L^2 with (weak) derivative in

L^2 form the space H^1, endowed with the norm $\|u\| = (\|u\|_2^2 + \|u_x\|_2^2)^{\frac{1}{2}}$. Our basic space is $X := H_0^1$, the completion of $\mathcal{C}_c^\infty([0,1])$ ($\mathcal{C}^\infty$ functions with compact support) wrt. $\|\cdot\|$. By Poincaré inequality (cf. section A.2), $\|u\|_2^2 \leq \pi^2\|u_x\|_2^2$ for $u \in X$, implying

$$\|u_x\|_2^2 \leq \|u\|^2 \leq (1+\pi^2)\|u_x\|_2^2.$$

Let H^2 be the space of functions in L^2 with first and second derivative also in L^2, and $L^\infty = L^\infty([0,1])$ the space of essentially bounded functions on $[0,1]$ with $\|u\|_\infty = \operatorname{ess\,sup}\{u(x) : x \in [0,1]\}$.

For $\epsilon > 0$ we denote by $U_\epsilon(u)$ ($B_\epsilon(u)$) the open (closed) ball of radius ϵ around u — if it is clear from the context we do not denote the underlying set or topology, usually it will be $\mathbb{R}$, an interval in $\mathbb{R}$ or X.

For a manifold $A \subset X$ and $x \in A$ T_xA is the tangent space of A in x. Two manifolds A, B intersect transversally, $A \pitchfork B$, if $T_xA + T_xB = X$ for all $x \in A \cap B$. This implies that $A \cap B$ is a submanifold of X, the dimension of which is $\dim A - \operatorname{codim} B$ if both numbers are finite. Two submanifolds A, B of a manifold $C \subset X$ intersect transversally in C, $A \pitchfork_C B$, if $T_xA + T_xB = T_xC$ for every $x \in A \cap B$. For manifolds we use the notation $\partial A := \bar{A} \setminus A$.

For $x \in X$, $B \subset X$ let $\operatorname{dist}(x,B) := \inf_{y \in B}\|x-y\|$. The mapping $x \mapsto \operatorname{dist}(x,B)$ is continuous. For $A, B \subset X$ let $\delta(A,B) := \sup_{x\in A}\operatorname{dist}(x,B)$ (note that $\delta(A,B) \neq \delta(B,A)$ in general). We call a family $(A_t)_t$ of sets upper (lower) semi-continuous at t_0, if $\delta(A_t, A_{t_0}) \xrightarrow{t\to t_0} 0$ $\big(\delta(A_{t_0}, A_t) \xrightarrow{t\to t_0} 0\big)$. We call it continuous at t_0 if it is both upper and lower semi-continuous. Two smooth submanifolds A, B are "ϵ-close in the $\mathcal{C}^1$-topology" if there is a $\mathcal{C}^1$-diffeomorphism $\chi : A \to B$ such that $\|\chi - id\|_{\mathcal{C}^1} < \epsilon$.

CHAPTER 2

The parabolic semiflow

2.1 Existence and basic properties

The equation (P) with a twice differentiable f generates a semiflow on the space X (it is *well-posed* in X, cf. the following proposition). In fact many assertions used in this work hold for f only $\mathcal{C}^1$. We will always assume f to be $\mathcal{C}^2$ though – if weaker assumptions are sufficient we will sometimes remark this explicitly. We rewrite (P) in terms of operators to apply a general theory. Let A be the Dirichlet realization of $u \mapsto u''$ in L^2: A is defined on $H^2 \cap X$, self-adjoint and ≤ 0 ([CH98, Proposition 2.6.1]), so $-A$ is a sectorial operator and generates an analytic semigroup ([Hen81, Example 2 on p. 19; Theorem 1.3.4]). Let $\hat{f} : X \ni u \mapsto f \circ u \in L^2([0,1])$ be the superposition operator induced by f (cf. Lemma A.1). Then the following holds:

Proposition 2.1. *For every $u_0 \in X$ there is a maximal $T^{max}(u_0) \in (0, \infty]$ such that the Cauchy Problem*

$$\begin{cases} \dot{u}(t) + Au(t) = \hat{f}(u) & t > 0 \\ u(0) = u_0 \end{cases}$$

has a solution

$$u \in \mathcal{C}(I^+, X) \cap \mathcal{C}^1(\mathring{I}^+, L^2([0,1])) \cap \mathcal{C}(\mathring{I}^+, H_2^2([0,1])),$$

where $I^+ = I^+(u_0) := [0, T^{max}(u_0))$. These solutions induce a local continuous semiflow φ on X. We set

$$\mathcal{D}^+ = \{(t, u_0) \in \mathbb{R}_0^+ \times X : t \in I^+(u_0)\}$$

and $\mathring{\mathcal{D}}^+ = \mathcal{D}^+ \setminus (\{0\} \times X)$. For any $s \geq 0$ we also set

$$\mathcal{D}_s = \{u_0 \in X : (s, u_0) \in \mathcal{D}^+\}, \quad \mathcal{D}_\infty = \bigcap_{s \geq 0} \mathcal{D}_s.$$

So we can write

$$\varphi^t : \mathcal{D}_t \ni u_0 \mapsto \varphi(t, u_0) \in X$$

and φ has the following additional properties:

(i) *If* $\|\varphi(t,u_0)\|$ *is uniformly bounded on* $I(u_0)$ *then* $T^{max}(u_0) = \infty$.

(ii) $\mathcal{D}$ *is open in* $[0,\infty) \times X$ *and* $\mathcal{D}_s$ *is open in* X *for any* $s \geq 0$.

(iii) φ *is continuous and* $\mathcal{C}^1$ *in its second argument*

(iv) φ *is compact, meaning that for* $T \in (0,\infty]$, $V \subset \mathcal{D}_T$, $0 < \epsilon_1 < \epsilon_2 < T$ *and*

$$M(\epsilon_i) = \bigcup_{t \in [\epsilon_i, T)} \varphi(t,V), \quad i = 1,2$$

boundedness of $M(\epsilon_1)$ *(in* X*) implies boundedness of* $M(\epsilon_2)$ *in* H^2 *and thus (by Sobolev embedding) precompactness of* $M(\epsilon_2)$ *(in* X*). In particular this implies that for* $u_0 \in \mathcal{D}_\infty$ $\varphi([0,\infty),u_0)$ *is precompact if it is bounded.*

(v) u *is a classical (pointwise) solution of* (P) *on* $(0,T^{max}(u_0)) \times \Omega$, *i.e. it is* $\mathcal{C}^1$ *in* (t,x) *and* $\mathcal{C}^2$ *in* x.

(vi) *For any* $t \geq 0$ *both* φ^t *and* $D\varphi^t$ *are injective.*

Proof. All assertions are stated in Theorems A.3 and B.2 in [AB05]. The hypotheses of B.2 demand a polynomial bound on f'. Since in the one-dimensional case $\hat{f}$ is $\mathcal{C}^1$ uniformly on bounded sets due to the continuous embedding $X \hookrightarrow \mathcal{C}([0,1])$ (Lemma A.1), this restriction is not necessary. □

Now that we have a semiflow we can recall the following notions:

Remark 2.2. 1. Time independent solutions of (P), i.e. fixed points of φ, are called *equilibria* (*equilibrium solutions*, *stationary solutions*) of φ. Let E denote the set of equilibria. A solution $v \in E$ is *hyperbolic* or *nondegenerate*, if 0 is no eigenvalue of the linear operator

$$L_v : H^2 \cap X \to L^2 : u \mapsto u_{xx} + f'(v(x))u.$$

L_u is a densely defined self-adjoint operator and bounded from above (which follows from the Kato-Rellich theorem, [RS75, Thm X.12], as $L^2 \ni u \mapsto f'(v) \cdot u \in L^2$ is bounded and symmetric), thus it generates an analytic semigroup $e^{-L_u t}$, see [Hen81, Theorem 1.3.2]. By [Wei76, Satz 8.26] the spectrum of L_u consists only of eigenvalues, and these are simple.

We call φ hyperbolic if all equilibria are hyperbolic. We call a set $M \subset X$ hyperbolic if all $v \in E \cap M$ are hyperbolic. For $v \in E$ let $i(v) = |\{\sigma(L_v) \cap (0,\infty)\}| < \infty$ denote the *Morse index* of v.

2. Let $v \in E$ be hyperbolic, $\lambda_1(v), \lambda_2(v), \dots$ the eigenvalues of the operator L_v defined above with eigenvectors $e_1(v), e_2(v), \dots$. These eigenvectors form a complete orthogonal system in L^2. As $e_i(v) \in X$ we can define $X_1^n(v) = \mathrm{span}\{e_1, \dots, e_n\} \subset X$, $X^n(v) = \mathrm{cl}_X(\mathrm{span}\{e_{n+1}, \dots\})$ and get closed subspaces of X with $X = X_1^n(v) \oplus X^n(v)$. Let $P_1^n(v)$, $P^n(v)$ denote the corresponding projections onto these subspaces.

3. By the backward uniqueness for $u \in X$ there is a minimal $T^{\min} \in [-\infty, 0]$ such that $\varphi(\cdot, u)$ is defined on

$$I(u) := (T^{\min}, T^{\max}).$$

Let $I^-(u) := I(u) \cap (-\infty, 0]$ and $\gamma^\pm(u) := \varphi^{I^\pm}(u)$ denote the *positive/negative halforbit of* u, $\gamma(u) = \gamma^+(u) \cup \gamma^-(u)$ the orbit through u. For $B \subset X$ let $\gamma^{(\pm)}(B) := \bigcup_{u \in B} \gamma^{(\pm)}(u)$.

So we can extend the domain of definition of φ to

$$\mathcal{D} = \{(t, u_0) \in \mathbb{R} \times X : t \in I(u_0)\}$$

and also extend the definition of $\mathcal{D}_t$ to negative times.

4. A set $S \subset X$ is *positively/negatively invariant* if $\gamma^+(S) \subset S$,$\gamma^-(S) \subset S$ respectively, and *invariant* if $\gamma(S) = S$. Clearly S is both positively and negatively invariant if and only if it is invariant. We call S locally (positively/negatively) invariant if for all $x \in S$ there is $t > 0$ such that $\varphi^{(-t,t)}(x) \subset S$ ($\varphi^{[0,t)}(x) \subset S$ / $\varphi^{(-t,0]}(x) \subset S$ respectively).

5. For $B \subset X$ let

$$\omega(B) := \{u \in X : \exists t_n \to \infty \exists u_n \in B \cap \mathcal{D}_{t_n} : \varphi^{t_n}(u_n) \to u\}$$
$$\alpha(B) := \{u \in X : \exists t_n \to \infty \exists u_n \in B \cap \mathcal{D}_{-t_n} : \varphi^{-t_n}(u_n) \to u\}$$

denote the ω- and α-limit sets of B. If $B \subset \mathcal{D}_\infty \cap \mathcal{D}_{-\infty}$ these definitions are equivalent to the usual formulas for semiflows defined for all times:

$$\omega(B \cap D_\infty) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \varphi^t(B \cap D_\infty)}, \qquad \alpha(B \cap D_{-\infty}) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \varphi^{-t}(B \cap D_{-\infty})}$$

If $B = \{u\}$ then $\omega(B) = \emptyset$, $\alpha(B) = \emptyset$ if $T^{\max}(u) < \infty$, $T^{\min}(u) > -\infty$ respectively.

We write $\alpha(u) = \alpha(\{u\})$, $\omega(u) = \omega(\{u\})$. Often $\alpha(u), \omega(u)$ contain only a single element (cf. Proposition 2.3 a)), in these cases we identify these sets with their unique element.

6. Let for $v \in E$

$$W^u(v) := \{y \in \mathcal{D}_{-\infty} : \varphi^{-t}(y) \xrightarrow{t \to \infty} v\}$$
$$W^s(v) := \{y \in \mathcal{D}_\infty : \varphi^t(y) \xrightarrow{t \to \infty} v\}$$

denote the unstable and stable set of v, respectively.

7. The *energy functional*

$$J : X \to \mathbb{R},\ u \mapsto \frac{1}{2}\|u_x\|_2^2 - \int_0^1 F(u(x))\,dx, \qquad F(u) := \int_0^u f(v)\,dv.$$

is $\mathcal{C}^1$ on X and decreases strictly along non-constant solutions: For a solution u

$$\frac{\partial}{\partial t} J(u(t)) = -\int_0^1 u_t^2\,dx \leq 0,$$

so φ is a *gradient-like semiflow.*

8. If f is such that J is bounded from below and $J(u) \to \infty$ as $\|u\| \to \infty$, then we call φ a *dissipative gradient-like semiflow.*

9. We will shortly speak of the restriction $\varphi|_M$ to a set $M \subset X$ instead of the restriction of φ to

$$\tilde{\mathcal{D}} = \left\{ (x,t) \in \mathcal{D} : x \in M, \varphi^{[t,0]\cup[0,t]}(x) \subset M \right\}.$$

Similarly we say that the semiflows φ and ψ coincide on a set M when $\varphi|_M = \psi|_M$ in the sense defined above.

10. If ψ is another semiflow on X induced by (P) with nonlinearity g, then α_ψ, J_g, etc. bear the obvious meaning, for a family of semiflows φ_θ we often write α_θ, J_θ etc. when there is no danger of confusion.

11. Let for $i \in \{1,2\}$ $V_i \subset X$ be open sets, $\{0\} \times V_i \subset \mathcal{O}_i \subset \mathbb{R}_0^+ \times V_i$ open such that for all $x \in V_i$ the sets $I_i^+(x) := \{t : (t,x) \in \mathcal{O}_i\}$ are relatively open intervals in $[0,\infty)$, and let $\varphi_i : \mathcal{O}_i \to V_i$ be continuous semiflows. We call these *conjugate* or *equivalent* if there is an homeomorphism $h : V_1 \to V_2$ and a continuous $\tau : \mathcal{O}_1 \to \mathbb{R}_0^+$ with $\tau(x,\cdot) : I_1(x) \to I_2(h(x))$ bijective and increasing for every $x \in V_1$ and

$$h(\varphi_1^t(x)) = \varphi_2^{\tau(x,t)}(h(x)).$$

So h maps orbits of φ_1 onto orbits of φ_2, preserving the sense of direction in time.

Proposition 2.3. ***a)*** *Let $u \in X$. If $\gamma^+(u)$ is bounded in X, then $\omega(u)$ consists of a single equilibrium solution. If $\gamma^-(u)$ is precompact in X, then $\alpha(u)$ is a nonempty, compact, invariant and connected subset of E. If φ is hyperbolic $\alpha(u)$ consists of a single element.*

b) *$u \in E$ is nondegenerate if and only if the spectrum of $D\varphi^t(u)$ is disjoint from the unit circle for $t \geq 0$ (and many authors define nondegeneracy that way).*

c) *For an hyperbolic $u \in E$ the sets $W^u(u)$, $W^s(u)$ are $\mathcal{C}^1$ submanifolds of X with $\dim(W^u(u)) = \operatorname{codim}(W^s(u)) = i(u)$.*

d) *Let f be $\mathcal{C}^2$, $v \in E$ hyperbolic, $n := i(v)$. Then with the notation from Remark 2.2 the following assertions hold:*

$$
\begin{aligned}
&T_vW^u(v) = X_1^n(v)\\
&T_vW^s(v) = X^n(v)\\
&\forall u \in W^u(v)\exists k \in \{1,\dots,n\} : \frac{\varphi^t(u)-v}{\|\varphi^t(u)-v\|_{H^2}} \xrightarrow[X]{t\to-\infty} \frac{\pm e_k}{\|e_k\|_{H^2}}\\
&\forall u \in W^s(v)\exists k \in \{n,\dots\} : \frac{\varphi^t(u)-v}{\|\varphi^t(u)-v\|_{H^2}} \xrightarrow[X]{t\to\infty} \frac{\pm e_k}{\|e_k\|_{H^2}}.
\end{aligned}
$$

Proof. If $\gamma^+(u)$ is bounded it is precompact as φ is compact, so $\omega(u)$ and $\alpha(u)$ are nonempty, closed, invariant, connected and bounded by [Hal88, Lemmas 3.1.1,3.1.2]. Now $J(\gamma^+(u))$ is bounded, so there is a sequence $t_n \to \infty$ such that $\frac{d}{dt}J(\varphi^{t_n}(u)) = \|\frac{d}{dt}\varphi^{t_n}(u)\|_2^2 \to 0$, so $\omega(u)$ (and similarly $\alpha(u)$) contain at least one equilibrium solution. By [Mat78, Theorem A] $\omega(u)$ has at most one element. If $u_1, u_2 \in \alpha(u)$ choose sequences $(s_n)_n, (t_n)_n$ with $t_n \le s_n \le t_{n-1}$ for all n, $s_n \to -\infty$, $t_n \to -\infty$ and $\varphi^{t_n}(u) \to u_1$, $\varphi^{s_n}(u) \to u_2$. As J is strictly decreasing along nonconstant solutions $J(u_1) = J(u_2)$, which implies $\alpha(u) \subset E$. Equilibria are critical points of φ, so they are isolated if φ is hyperbolic. This shows a).

To see b) we note that by [AB05, Theorem A.3] $h(t) = D\varphi^t(v_0)$ is the mild solution of $\dot v(t) - L_u v(t) = 0$, so $D\varphi^t(v_0) = e^{-L_u t}v_0$. Now there are subspaces Y_1, Y_2 of $L^2([0,1])$ such that $L^2([0,1]) = Y_1 \oplus Y_2$ with $\dim(Y_1) < \infty$, the Y_i being invariant w.r.t.. the restrictions L_i of L_u to $Y_i \cap D(L_u)$ and $\sigma(L_1) = \sigma(L_u) \cap (-\infty, 0]$, $\sigma(L_2) = \sigma(L_u) \cap (0,\infty)$ (see section 1.5 of [Hen81] for all this, cf. also Remark 2.2 2). Now by construction $\sigma(L_2)$ is bounded away from 0, so $e^{-L_2 t}$ is strictly contracting for $t > 0$, and $\sigma(L_1) = \{\lambda_1,\dots,\lambda_k\}$, so $\sigma(e^{-L_1 t}) = \{e^{-\lambda_1 t},\dots,e^{-\lambda_k t}\}$ and it follows that $0 \in \sigma(L_u) \iff 1 \in \sigma(D\varphi^t(u))$.

c) This can be found in Henry [Hen85] in chapter 6. Theorem 6.1.9 in [Hen85] though is false as stated, as simple counterexamples show (this Theorem is used to globalize the local (un-)stable manifolds). A correct proof for this can be found in Theorem 2.2 in [AB05] in case of the stable manifold. The same modifications can be done to prove the unstable case.

Alternatively one can use [Bru90], where unstable manifolds are shown to be global $\mathcal{C}^1$-graphs in the case $\limsup_{|s|\to\infty} \frac{f(s)}{s} < \pi^2$. If this condition is not satisfied choose $R > 0$ and $0 < \delta < 1$ and make f constant outside $[-R-\delta, R+\delta]$ with a smoothing on $(-R-\delta,-R) \cup (R, R+\delta)$ to make the modified function $\mathcal{C}^2$. Then $W^u(v) \cap U_R(0)$ is a $\mathcal{C}^1$-submanifold by [Bru90]. By letting $R \to \infty$ the assertion follows because X is compactly embedded in $\mathcal{C}_0$.

d) follows from Theorem 2.1, Lemma 3.1 and Theorem 3.2 in [BF86]. □

Definition 2.4. Let $f \in \mathcal{C}^2$. If φ is hyperbolic, φ^t and $D\varphi^t$ are injective for all $t > 0$, and for $v, w \in E$ the manifolds $W^u(v)$, $W^s(w)$ intersect transversally if they intersect at all, we call φ a *Morse-Smale semiflow.*

For hyperbolic φ by [Hen85, Theorem 7] the stable and unstable manifolds always intersect transversally for $f \in \mathcal{C}^2$, so by Proposition 2.1 (vi) φ is Morse-Smale if φ is hyperbolic.

2.2 Dissipative semiflows

Definition 2.5. a) A set $B \subset X$ *attracts* a set $C \subset X$ under φ if $C \subset \mathcal{D}_\infty$ and $\delta(\varphi^t(C), B) \to 0$ as $t \to \infty$.

b) We call φ *dissipative* if there is a bounded set $B \subset X$ which attracts each bounded set of X under φ. In this case we will also call f dissipative.

c) An invariant set $\mathcal{A}$ is a *global attractor* if it is a *maximal compact invariant set* (i.e. it contains any compact invariant set) which attracts each bounded set $B \subset X$. We will most of the time call global attractors just attractors.

d) A set $\mathcal{A}_\lambda$ is a *local attractor* if it is compact, invariant and there is an open neighborhood U of $\mathcal{A}_\lambda$ such that $\mathcal{A}_\lambda$ attracts U.

So we are ready to define the set of "dissipative functions":

Proposition 2.6. *Let*

$$\tilde{\mathcal{G}}_d := \Big\{ f : \mathbb{R} \to \mathbb{R}\ \mathcal{C}^2, \limsup_{|u| \to \infty} \frac{f(u)}{u} < \pi^2 \Big\}.$$

Then for $f \in \tilde{\mathcal{G}}_d$ the parabolic flow induced by (P) *is dissipative and admits a connected global attractor given by*

$$\mathcal{A} = W^u(E) = \{ y \in \mathcal{D}_{-\infty} : \varphi^{-t}(y) \xrightarrow{t \to \infty} E \}.$$

If $f \in \mathcal{G}_d$ with

$$\mathcal{G}_d := \{ f \in \tilde{\mathcal{G}}_d : \text{ all equilibria of } \varphi \text{ are nondegenerate} \}$$

then E is finite and

$$\mathcal{A} = \bigcup_{u \in E} W^u(u).$$

This condition on f to be dissipative is the standard hypothesis, see for example [Lu94, Bru90, BF89, BF88]. As references for a proof usually [Hen81] or [Hal88] are given, where the condition on f is $\limsup \frac{f(u)}{u} \leq 0$. The estimates in the proof have to be sharpened somewhat in our case, we give a detailed proof in appendix A.2.

***Remark* 2.7.** The bound on f is sharp: For $\frac{f(u)}{u} \to \pi^2$ the assertion may or may not hold, but for $\limsup \frac{f(u)}{u} > \pi^2$ φ is not dissipative. From the proof of Proposition 2.6 in section A.2 it is easy to see that $J(\lambda e_1) \to -\infty$ as $\lambda \to \infty$, so there cannot be a compact attractor.

We note the following corollary:

Corollary 2.8. *For $f \in \mathcal{G}_d$ φ is a (dissipative) Morse-Smale semiflow.*

Lemma 2.9. *Let $f \in \tilde{\mathcal{G}}_d$ and $\mathcal{A}_0$ a local attractor of φ. Then $\mathcal{A}_0$ is stable, i.e.*

$$\forall V \supset \mathcal{A}_0 \text{ open } \exists V \supset W \supset \mathcal{A}_0 \text{ open} : \gamma^+_{v_0}(W) \subset V. \tag{2.1}$$

Proof. $\mathcal{A}_0$ is a local attractor, so we can choose $U \supset \mathcal{A}_0$ open with $\delta(\varphi^t(U), \mathcal{A}_0) \to 0$. In particular there is $t_0 > 0$ such that $\varphi^{[t_0,\infty)}(U) \subset V$. Let $\epsilon := \mathbf{dist}(\partial V, \mathcal{A}_0) > 0$, $u_0 \in \mathcal{A}_0$, $t \in [0, t_0]$. By continuity of φ there is an open $U_{u_0,t} \ni u_0$ such that $\varphi^t(U_{u_0,t}) \subset U_\epsilon(\varphi^t(u_0))$. By compactness of $[0, t_0]$ there is an open $U_{u_0} \ni u_0$ such that $\varphi^{[0,t_0]}(U_{u_0}) \subset U_\epsilon(\mathcal{A}_0)$. By compactness of $\mathcal{A}_0$ there is an open $U \supset W \supset \mathcal{A}_0$ such that $\varphi^{[0,t_0]}(W) \subset U_\epsilon(\mathcal{A}_0) \subset V$. As $\varphi^{[t_0,\infty)}(W) \subset \varphi^{[t_0,\infty)}(U) \subset V$ we have proved (2.1). □

2.3 Non-dissipative semiflows

Increasing $L = \limsup \frac{f(u)}{u}$ the flow φ gets "less and less dissipative": If for example $\pi^2 n^2 < \lim \frac{f(u)}{u} < \pi^2(n+1)^2$ there is a n-dimensional subspace X_1^n of H_1^0 where $J(u) \to -\infty$ as $\|u\| \to \infty$ while it is bounded from below on $(X_1^n)^\perp$, cf. section A.2. On the other hand a proof similar to the one of Proposition 3.10 b) shows that the number of sign changes and the Morse indices of stationary solutions are bounded as long as L is finite (this "zero-number" and its connection to the Morse index will be discussed in detail below).

We are primarily interested in *definite superlinear* flows, i.e. the case $L = \infty$. Now for the non-dissipative semiflow $u \in X$ may have an unbounded positive halforbit existing only for a finite time. Basic tools for this case are a-priori bounds and a "good" behavior of the energy functional, and to ensure these we will have to impose stronger conditions than just $L = \infty$. We define the following set of functions:

$$\tilde{\mathcal{G}} := \Big\{ f : \mathbb{R} \to \mathbb{R}\ \mathcal{C}^2 : \ \exists R > 0, \mu > 2 : \forall |u| \geq R : f(u)u \geq \mu F(u), \\ F(\pm R) > 0 \}, \tag{2.2}$$

where $F(u) = \int_0^u f(x)\,dx$. In particular for $f \in \tilde{G}$ we have $F(u) \geq C \cdot |u|^\mu$ for $|u| \geq R$, and thus $L = \infty$. The reason for considering $f \in \tilde{\mathcal{G}}$ is the following Lemma:

Lemma 2.10 (Quittner)**.** *Let $f \in \tilde{\mathcal{G}}$.*
a) *Let $\delta, C_0 > 0$. Then there exists a constant $C = C(\delta, C_0)$ such that $\|u_0\| \le C_0$ implies*

$$\|\varphi^t(u_0)\| \le C \text{ for } t \in [0, T^{max}(u_0) - \delta), \tag{2.3}$$

where $\infty - \delta = \infty$.

b) *The mapping $X \ni u_0 \mapsto T^{max}(u_0) \in (0, \infty]$ is continuous. If $T^{max}(u_0) < \infty$ then*

$$J(\varphi^t(u_0)) \to -\infty \text{ as } t \uparrow T^{max}(u_0). \tag{2.4}$$

***Remark* 2.11.** This Lemma also opens another way to tackle the connecting orbit problem (cf. Definition 3.18). Together with some standard results it proves that the sets $J^{-1}([a,b])$ are "admissible" in the sense of Rybakowski ([Ryb87]), which allows the application of his homotopy index theory.

Proof. This Lemma is Theorem 6.1 in [Qui03], we have to verify the following four conditions (stated here for an autonomous f): There exist nondecreasing functions $d_2, d_4 : \mathbb{R}^+ \to \mathbb{R}^+$ and constants $d_1, \epsilon > 0$, $\mu > 2$, $a_i \ge 0$ ($i = 1, \ldots, 4$) such that

$$|f(u)| \le d_2(|u|) + a_2 \tag{2.5}$$
$$F(u) \ge d_1 |u|^{2+2\epsilon} - a_1 \tag{2.6}$$
$$f(u)u \ge \mu F(u) - a_3 \tag{2.7}$$
$$|f(u) - f(v)| \le (a_4 + d_4(|u| + |v|))|u - v|. \tag{2.8}$$

For autonomous f (2.5) is always satisfied. As f is locally Lipschitz continuous (2.8) is also clear. Condition (2.7) follows by taking μ, R from the definition of $\tilde{\mathcal{G}}$ and

$$a_3 := \min\{f(u)u - \mu F(u) : |u| \le R\}.$$

Similarly it is enough to show

$$\forall |u| \ge R : F(u) \ge d_1 |u|^{2+2\epsilon}$$

to verify (2.6). Let $\epsilon := \frac{\mu - 2}{2} > 0$, w.l.o.g. $u > 0$. Now

$$\left(\frac{F(u)}{u^{2+2\epsilon}}\right)' = \frac{1}{u^{3+2\epsilon}}\Big(f(u)u - \underbrace{(2+2\epsilon)}_{=\mu} F(u)\Big) \ge 0,$$

i.e. for $u \ge R$

$$\frac{F(u)}{u^{2+2\epsilon}} \ge \underbrace{\frac{F(R)}{R^{2+2\epsilon}}}_{=:d_1 > 0} \iff F(u) \ge d_1 u^{2+2\epsilon}.$$

$\square$

Again we will restrict ourselves to nondegenerate equilibria and most of the time to a subset of "uniformly superlinear" functions:

$$\mathcal{G} := \{f \in \tilde{\mathcal{G}} : \text{ All equilibria of (P) are nondegenerate } \}$$
$$\mathcal{F} := \{f \in \tilde{\mathcal{G}} : \forall u \neq 0 : f'(u)u^2 > f(u)u, f'(0) \neq k^2\pi^2 \text{ for } k \in \mathbb{N}\}$$

We shall prove that $\mathcal{F} \subset \mathcal{G}$ (Corollary 3.12) which is good, as $\mathcal{F}$ is a common class of superlinear functions and contains the model case $u|u|^{p-1}$ $(p > 2)$. One also easily verifies the following

Lemma 2.12. *Let $f \in \mathcal{F}$, then $f(0) = 0$. If $f'(0) \geq 0$ then $f'(x) > 0$ for all $x \neq 0$, if $f'(0) < 0$ then f has precisely one positive and one negative zero.*

The nondegeneracy condition in the definitions of $\mathcal{G}_d$ and $\mathcal{G}$ is not easily verified, except in the special case $\mathcal{F} \subset \mathcal{G}$. But it has been proved that nondegeneracy is a generic condition. To make this precise we fix the topology used:

Definition 2.13. We will use two different topologies on the space $\mathcal{C}^2 = \mathcal{C}^2(\mathbb{R})$ of all twice differentiable functions (cf. [Hir76, Chapter 2.1]).
a) The *weak topology* is the topology induced by the metric

$$d(f,g) := \sum_{n=1}^{\infty} \frac{2^{-n}|f-g|_n}{1+|f-g|_n}$$

where $|f|_n$ is the standard $\mathcal{C}^2$-Norm on $\mathcal{C}^2([-n,n])$. This is the topology of $\mathcal{C}^2$-convergence on compact sets, let $\mathcal{C}^2_w$ denote $\mathcal{C}^2$ endowed with the weak topology. We will also sometimes use the weak topology on $\mathcal{C}^1_w$.

b) Now let $K_i \subset \mathbb{R}$ compact for all $i \in \mathbb{N}$ such that for all $x \in \mathbb{R}$ there is an $K_i \ni x$ and an open neighborhood $U_x \ni x$ which intersects only finitely many K_i. Let further $\{\epsilon_i\}_{i\in\mathbb{N}}$ be a family of positive numbers and $f \in \mathcal{C}^2$. Then the set

$$\{g \in \mathcal{C}^2 : \forall i \in \mathbb{N} \forall k \in \{0,1,2\} \forall x \in K_i : |f^{(k)}(x) - g^{(k)}(x)| < \epsilon_i\}$$

is an open neighborhood of f in the strong topology. The sets of this type form a base for the *strong topology* (or *Whitney topology/fine topology-*) , i.e. strong-open sets are unions of sets of the above type. We write $\mathcal{C}^2_s$ for $\mathcal{C}^2$ endowed with this topology.

***Remark* 2.14.** The strong topology is not metrizable. For $f \in \mathcal{C}^2$

$$U_{\mathcal{C}^2_s,\epsilon}(f) := \{g \in \mathcal{C}^2 : \forall k \in \{0,1,2\} \forall x \in \mathbb{R} : |g^{(k)}(x) - f^{(k)}(x)| < \epsilon\} \subset U_{\mathcal{C}^2_w,3\epsilon}$$

is an open set in $\mathcal{C}^2_s$. It is easily verified that the weak topology is strictly weaker than the strong one, that means that open sets in $\mathcal{C}^2_w$ are also open in $\mathcal{C}^2_s$, but $U_{\mathcal{C}^2_s,\epsilon}(f)$ is not an open set in $\mathcal{C}^2_w$ for any $\epsilon > 0$.

Now we can formulate precisely in which sense hyperbolicity of equilibria is a *generic* property of $f \in \tilde{\mathcal{G}}_d \cup \tilde{\mathcal{G}}$:

Proposition 2.15. *$\tilde{\mathcal{G}}_d$, $\tilde{\mathcal{G}}$ are open in $\mathcal{C}^2_s$, and $\mathcal{G}_d$, $\mathcal{G}$ are residual subsets w.r.t. the strong topology (i.e. countable intersections of strong-open subsets) of $\tilde{\mathcal{G}}_d$, $\tilde{\mathcal{G}}$ respectively. This implies that $\mathcal{G}_d \subset \tilde{\mathcal{G}}_d$, $\mathcal{G} \subset \tilde{\mathcal{G}}$ are dense subsets.*

Proof. We first prove that $\tilde{\mathcal{G}}$ is open. Let $f \in \tilde{\mathcal{G}}$, R,μ as in (2.2) and $g \in U_{\mathcal{C}^2_s,\epsilon}(f)$ for

$$0 < \epsilon < \min\left(\left\{|f(u)| : |u| \geq R\right\} \cup \left\{\frac{F(R)}{R}, \frac{F(-R)}{R}\right\}\right)$$

(this minimum is strictly positive by (2.2). Fix a_1, d_1, ϵ' as in (2.6), pick some $\mu' \in (2, \mu)$ and let $u \geq R$. Then

$$\begin{aligned} g(u)u - \mu' G(u) &\geq (f(u) - \epsilon)u - \mu' \int_0^u f(x) + \epsilon\, dx \\ &\overset{(2.2)}{\geq} (\mu - \mu')F(u) - \epsilon(\mu' + 1)u \\ &\overset{(2.6)}{\geq} u\left[(\mu - \mu')d_1 u^{1+2\epsilon'} - \epsilon(\mu' + 1)\right] - (\mu - \mu')a_1 \end{aligned}$$

so we can choose $R' \geq R$ such that $g(u)u \geq \mu' G(u)$ for $u \geq R'$ and analogously for $u \leq -R'$. By choice of ϵ we also get

$$G(R') \geq F(R) - \epsilon R + \int_R^{R'} f(x) - \epsilon\, dx > 0$$

and similarly $G(-R') > 0$, thus $g \in \tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}$ is open.

By [BC84] the set of hyperbolic $f \in \mathcal{C}^2$ is a residual subset of $\mathcal{C}^2_s$. By [Hir76, Theorem 4.4] residual subsets are dense in $\mathcal{C}^2_s$. But $\tilde{\mathcal{G}}$ is an open subset of $\mathcal{C}^2_s$, so $\mathcal{G}$ is an residual subset of (and thus dense in) $\tilde{\mathcal{G}}$. This implies in particular that for any $f \in \tilde{\mathcal{G}}$ and any $\epsilon > 0$ there exists $g \in U_{\mathcal{C}^2_s,\epsilon}(f) \cap \mathcal{G}$.

Similarly for $f \in \tilde{\mathcal{G}}_d$ one easily checks $U_{\mathcal{C}^2_s,\epsilon}(f) \subset \tilde{\mathcal{G}}_d$ (for arbitrary $\epsilon > 0$ in fact), and as above it follows that $\mathcal{G}_d$ is a residual subset of $\tilde{\mathcal{G}}_d$. $\square$

CHAPTER 3

Technical results

3.1 The zero number

Definition 3.1. For $u \in \mathcal{C}([0,1])$ let $z(u)$ be the number of strict sign changes of u in $(0,1)$, i.e.

$$z(u) := \sup\big(\{0\} \cup \{k \in \mathbb{N} : \exists x_1, \ldots, x_{k+1} \in (0,1), x_1 < x_2 < \cdots < x_k, \\ u(x_i) \cdot u(x_{i+1}) < 0 \text{ for all } 1 \le i \le k\}\big).$$

We call $z(u)$ the *zero number* of u.

The zero number is a "discrete Lyapunov functional" for scalar equations:

Proposition 3.2. ***a)*** *Let $f \in \mathcal{C}^2$ with $f(0) = 0$, $u \in X$, then $z(\varphi^t(u))$ is non-increasing.*

b) *If $z(\varphi^t(u))$ is constant on an interval $I_0 \subset I(u)$, then $(\varphi^t(u))_x(0) \neq 0$ for $t \in I_0$.*

c) *If $z(u) < \infty$ then the set of times $t \in I(u)$ for which $\varphi^t(u)$ has only simple zeros is open dense in $I(u)$.*

d) *If $f \in \mathcal{C}^2$, $v \in E$ and u is a solution of* (P) *defined on an interval I, then $w(t) := u(t) - v$ satisfies the nonautonomous equation $w_t - w_{xx} = g(w,x)$ on I with $g(y,x) := f(y + v(x)) - f(v(x))$. We have $g(0,\cdot) = 0$, $z(w(t))$ is non-increasing and w has only simple zeros on an open dense subset of I.*

e) *If $f \in \mathcal{C}^2$ and $v \in E$ is hyperbolic, then $i(v) \in \{z(v), z(v)+1\}$.*

Proof. a) is proved in Lemma 1.1 of [BF86] for f bounded in $\mathcal{C}^1$. This can easily be transferred to general $f \in \mathcal{C}^2$ by modifying f outside a compact interval, cf. section 3.3. Assertions b), c) are Lemmas 7.4, 7.3 respectively of [BF88], proved there for $\limsup_{|u|\to\infty} f(u)/u < \infty$. As above the assertion follows also for $f \in \mathcal{C}^2$. Statement d) is easily checked by a direct calculation together with the Lemmas in [BF86, BF88] cited above. These can be applied also in this non-autonomous case, because g satisfies the growth condition $\limsup_{|t|\to\infty} \frac{g(t,x)}{t} < \pi^2$ uniformly in x.

e) is Lemma 5.1 in [BF88]. $\square$

Proposition 3.2 together with 2.3 d) immediately yields the following:

Corollary 3.3. *Let $f \in \mathcal{C}^2$, $v \in E$ hyperbolic, then $z(u-v) < i(v)$ for $u \in W^u(v)$ and $z(u-v) \geq i(v)$ for $u \in W^s(v)$.*

Lemma 3.4. *Let $f \in \mathcal{G}_d$, $v, w \in E$, $v \neq w$ and $|v'(0)| \geq |w'(0)|$. Then $z(v-w) = z(v)$ and all zeros of $v-w$ are simple.*

Proof. The assertion $z(v-w) = z(v)$ is Lemma 4.2 in [BF88]. The zeros of $v-w$ are simple by Proposition 3.2 d), c). □

3.2 The T-Map and stationary solutions

To get a first idea of the dynamics of the parabolic semiflow we locate the stationary or equilibrium solutions of (P). Equilibria of (P) are solutions of

$$-u'' = f(u) \qquad u(0) = u(1) = 0, \tag{E}$$

to find these we examine solutions of the initial value problem

$$-u''(t,\eta) = f(u(t)), \qquad u(0) = 0, \qquad u'(0) = \eta \tag{IVP}$$

($' = \frac{d}{dt}$) and try to find η s.t. $u(1,\eta) = 0$. Throughout this section we will use some comparison results about second order ODE. The following is taken from [Har64, Section XI.3]. We consider the equations

$$-u'' = q_1(t) \cdot u \tag{3.1}$$

$$-u'' = q_2(t) \cdot u \tag{3.2}$$

with $q_1, q_2 \in \mathcal{C}([0,1])$. We call (3.1) a *Sturm majorant* of (3.2) if $q_1 \leq q_2$, and a strict Sturm majorant if in addition $q_1(t) < q_2(t)$ for some $t \in (0,1)$.

Theorem 3.5. ***a) (Sturm Comparison Theorem)*** *Let (3.1) be a Sturm majorant for (3.2) and u_1 be a solution of (3.1) with exactly $n \geq 1$ zeros $t_1 < t_2 < \cdots < t_n$ in $(0,1]$. Let $u_2 \not\equiv 0$ be a solution of (3.2) satisfying*

$$\frac{u_1'(0)}{u_1(0)} \geq \frac{u_2'(0)}{u_2(0)} \tag{3.3}$$

($\frac{u_i'(0)}{u_i(0)} = \infty$ if $u_i(0) = 0$). Then u_2 has at least n zeros in $(0,1]$.

If either (3.1) is a strict Sturm majorant for (3.2) or (3.3) holds with strict inequality, then u_2 has at least n zeros in $(0,1)$.

b) (Sturm Separation Theorem) *If u_1, u_2 are linearly independent solutions of (3.1), then the zeros of u_1 separate and are separated by those of u_2.*

For superlinear f there are infinitely many stationary solutions.

Theorem 3.6 (Struwe)**.** *For any $f \in \mathcal{C}^2$ with $\frac{f(x)}{x} \to \infty$ as $|x| \to \infty$ there are infinitely many solutions to* (E)*. For any sequence $(u_k)_k$ of distinct solutions with $|u_k'(0)| \to \infty$ we have $z(u_k) \xrightarrow{k\to\infty} \infty$.*

Proof. The first assertion is a special case of Theorem 1 in Struwe ([Str80]), the assertion about $z(u_k)$ is stated in the proof of that Theorem. □

Unless stated otherwise in this section $u(\cdot,\eta,f)$ will denote a solution to (IVP).

Definition 3.7. a) For $f : \mathbb{R} \to \mathbb{R}$ locally Lipschitz, $\eta \in \mathbb{R}$ let $u(\cdot,\eta) = u(\cdot,\eta,f)$ be the solution of the equation (IVP).

b) Let (for given locally Lipschitz continuous f) $D = D_f$ be the set of all $\eta \in \mathbb{R}$, for which $u(x,\eta,f) = 0$ for some $x > 0$ and define

$$T_f = T : D \to \mathbb{R}, \qquad \eta \mapsto \inf\{x > 0 : u(x,\eta,f) = 0\}. \tag{3.4}$$

Remark **3.8.** We can write (IVP) as the twodimensional system

$$\begin{aligned} u' &= v \\ v' &= -f(u) \end{aligned} \tag{SYS}$$

with initial values $u(0) = 0$ and $v(0) = \eta$. The corresponding vectorfield $V(u,v) = (v,-f(u))$ is antisymmetric w.r.t. the u-axis, i.e. $V(u,-v) = (-v,-f(u))$. This simple observation has some important consequences: Orbits of (SYS) are symmetric w.r.t. the u-axis, in particular for $\eta \in D$

$$u'(T(\eta),\eta) = -\eta \tag{3.5}$$

and

$$u'\left(\frac{T(\eta)}{2},\eta\right) = 0.$$

We also see

$$\eta \in D \iff \exists t > 0 : u'(t,\eta) = 0,$$

and that $u(t,\eta)$ is a $(T(\eta) + T(-\eta))$-periodic solution if $\eta,-\eta \in D$. For such a periodic solution we have $u(T(t)+t,\eta) = u(t,-\eta)$. The system (SYS) also has a first integral $E(u,v) = \frac{1}{2}v^2 + F(u)$ with $F(u) := \int_0^u f(t)\,dt$ (which makes it easy to see the v-symmetry of orbits).

By means of the function T equilibrium solutions of (P) and nondegeneracy of these solutions can be characterized as follows:

Proposition 3.9. *Let f $\mathcal{C}^2$, $\eta \neq 0$, $D_s := D \cap -D$.*
a) *If $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^k$, $k \in \mathbb{N}$, then $D \setminus \{0\}$ is open and T is $\mathcal{C}^{k+2}$ in $D \setminus \{0\}$. For $\eta \in D \setminus \{0\}$ we have*

$$T'(\eta) = \frac{u_\eta(T(\eta),\eta)}{\eta}.$$

b) *A solution $u(\cdot,\eta_0)$ of* (IVP) *is a nonnegative (or non-positive) equilibrium solution of* (P) *if and only if $T(\eta_0) = 1$, and nondegenerate if and only if $T'(\eta_0) \neq 0$.*

c) *A solution $u(\cdot,\eta_0)$ of* (IVP) *is an equilibrium solution of* (P) *with $n = 2k+1$ zeroes if and only if the function*

$$T^{(n)} : \eta \mapsto (k+1)T(\eta) + (k+1)T(-\eta)$$

attains the value 1 at η_0, and is nondegenerate if and only if this function has non-vanishing derivative at η_0.

In this case $u(T(\eta_0) + \cdot, \eta_0) = u(\cdot, -\eta_0)$ is also an equilibrium solution with n zeros.

d) *A solution $u(\cdot,\eta_0)$ of* (IVP) *is an equilibrium solution of* (P) *with $n = 2k$ zeroes if and only if the function*

$$T^{(n)} : \eta \mapsto (k+1)T(\eta) + kT(-\eta)$$

attains the value 1 at η_0, and is nondegenerate if and only if this function has non-vanishing derivative at η_0. (Of course b) is just a special case of d) stated explicitly for clarity).

Proof.
a) (cf. Brunovský-Chow [BC84, Thm 2.3]) Fix $\eta_0 \in D \setminus \{0\}$ and $T_0 = T(\eta_0)$. We have $u'(T_0,\eta_0) = -\eta_0 \neq 0$ by (3.5), and solutions to (IVP) are $\mathcal{C}^{k+2}$ if f is $\mathcal{C}^k$. By the Implicit Function Theorem there exists a unique $\mathcal{C}^{k+2}$ function τ defined in a neighborhood of η_0 such that for (ξ,η) close to (T_0,η_0) we have $u(\xi,\eta) = 0$ if and only if $\xi = \tau(\eta)$. By the continuous dependence Theorem T and τ are identical in a small neighborhood of η_0. So $D \setminus \{0\}$ is open and T is $\mathcal{C}^{k+2}$ on $D \setminus \{0\}$. The formula for $T'(\eta)$ is obtained by differentiating the identity $u(T(\eta),\eta) = 0$ and (3.5).

The assertion about $u(\cdot,-\eta_0)$ in c) is clear, cf. Remark 3.8. The other assertions are Theorems 2.5 to 2.7 in [BC84]. □

Proposition 3.10. ***a)*** *Let $f, f_k \in \mathcal{C}^1$, $f_k \to f$ in $\mathcal{C}^1_w$, $\eta \in \mathbb{R}$, $\eta_k \in D_{f_k}$, $\eta_k \to \eta$. If $\eta \in D_f \setminus \{0\}$ then $T_{f_k}(\eta_k) \to T_f(\eta)$ and $T_{f_k'}(\eta_k) \to T_f'(\eta)$. If $\eta \notin D_f$ then $T_{f_k}(\eta_k) \to \infty$.*

b) *If f is $\mathcal{C}^1$, $f(0) = 0$, then*

$$\lim_{\eta \to 0} T(\eta) = \begin{cases} \pi \cdot f'(0)^{-\frac{1}{2}} & f'(0) > 0 \\ \infty & f'(0) \leq 0. \end{cases}$$

c) *If $f \in \mathcal{F}$ then $D = \mathbb{R}$, $T'(\eta) < 0$ for $\eta > 0$, $T'(\eta) > 0$ for $\eta < 0$ and $T(\eta) \to 0$ for $|\eta| \to \infty$.*

d) *If $f \in \tilde{\mathcal{G}}$ then there exists $\bar{\eta} > 0$ such that $\mathbb{R} \setminus (-\bar{\eta}, \bar{\eta}) \subset D$.*

Remark 3.11. In general T is discontinuous at 0, cf. Proposition 3.10 b) where $T(0) = 0$.

Proof.
a) First consider $\eta \in D_f \setminus \{0\}$, w.l.o.g. assume $\eta > 0$. We have

$$u(\cdot, \eta_k, f_k) \xrightarrow{k \to \infty} u(\cdot, \eta, f)$$

uniformly on compact intervals, and the same is true for u', u_η. If $0 < T_f(\eta) < \infty$ then

$$\begin{aligned} &\forall 1 >> \epsilon > 0 \exists k_\epsilon \in \mathbb{N} : k \geq k_\epsilon \\ &\qquad\qquad \Rightarrow u(t, \eta_k, f_k)|_{[\epsilon, T_f(\eta) - \epsilon]} > 0 \wedge u(T_f(\eta) + \epsilon, \eta_k, f_k) < 0, \end{aligned}$$

and because of $u'(0, \eta, f) = \eta \neq 0$ and $u'(\cdot, \eta_k, f_k) \to u'(\cdot, \eta, f)$ uniformly on a neighborhood of 0 we have $T_{f_k}(\eta_k) \in (T_f(\eta) - \epsilon, T_f(\eta) + \epsilon)$. Now $T'_{f_k}(\eta_k) \to T'_f(\eta)$ follows from 3.9 a) and the differentiable dependence theorem.

If $\eta \notin D_f$ then

$$\forall n \in \mathbb{N} \exists k_\epsilon \in \mathbb{N} : k \geq k_\epsilon \Rightarrow \forall x \in \left[\frac{1}{n}, n\right] : u(x, \eta_k, f_k) \neq 0,$$

this implies $T_{f_k}(\eta_k) \to \infty$ as above.

b) Again w.l.o.g. let $\eta > 0$ and

$$q_1(x) = q_1(x, \eta) = \begin{cases} \frac{f(u(x,\eta))}{u(x,\eta)} & u(x, \eta) \neq 0 \\ f'(0) & u(x, \eta) = 0, \end{cases}$$

so $u(\cdot, \eta)$ solves the homogeneous linear equation

$$-v'' = q_1(\cdot, \eta) \cdot v. \tag{3.6}$$

Case 1: $f'(0) > 0$. Let $v_1(x) = \sin\left(\sqrt{f'(0)+\epsilon}\cdot x\right)$, $v_2(x) = \sin\left(\sqrt{f'(0)-\epsilon}\cdot x\right)$ be solutions of the equations

$$-v'' = (f'(0)+\epsilon)v \tag{3.7}$$

and

$$-v'' = (f'(0)-\epsilon)v \tag{3.8}$$

respectively for some $0 < \epsilon < f'(0)$. We have

$$(f'(0)-\epsilon)x \le q_1(x,\eta) \le (f'(0)+\epsilon)x$$

uniformly for $\eta \le \eta_\epsilon$ sufficiently small and $x \le \frac{\pi}{\sqrt{f'(0)-\epsilon}}$, so by the Sturm Comparison Theorem we have

$$\frac{\pi}{\sqrt{f'(0)-\epsilon}} \ge T(\eta) \ge \frac{\pi}{\sqrt{f'(0)+\epsilon}},$$

and the assertion follows.

Case 2: $f'(0) \le 0$. Let $m := \lim_{\eta\downarrow 0} \max u(\cdot,\eta) \ge 0$. If $m = 0$ (which is possible if $f'(0) = 0$) let v_1 be the solution of $-v'' = \epsilon v$; analogous to case 1 we conclude $\frac{\pi}{\sqrt{\epsilon}} < T(\eta)$ uniformly in η small, so $T(\eta) \xrightarrow{\eta\to 0} \infty$.

Now consider the case $m > 0$. As $\max u(\cdot,\eta) = u(\frac{1}{2}T(\eta),\eta)$ we have

$$u\left(\frac{1}{2}T(\eta),\eta\right) \xrightarrow{\eta\to 0} m. \tag{3.9}$$

Now suppose $\liminf_{\eta\to 0} T(\eta) = T_0 < \infty$ then there exists a sequence $(\eta_k)_k$ with $\eta_k \xrightarrow{k\to\infty} 0$ such that $T(\eta_k) \to T_0$ as $k \to \infty$. By the continuous dependence Theorem follows

$$u\left(\frac{1}{2}T(\eta_k),\eta_k\right) \xrightarrow{k\to\infty} 0$$

which contradicts (3.9).

c) We use the interpretation of (IVP) as (SYS) and the first integral $E(u,v) = \frac{1}{2}v^2 + F(u)$, i.e. $E(u(t))$ is constant along solutions. Fix $\eta > 0$. From the definition of $\mathcal{F}$ and Lemma 2.12 we see the existence of $n_{-1} \le 0 \le n_1$ such that $F(n_{-1}) = F(n_1) = 0$, $F(u)u \le 0$ on $[n_{-1},n_1]$, $F(u)u > 0$ on $\mathbb{R}\setminus[n_{-1},n_1]$ and F strictly decreasing on $(-\infty,n_{-1}]$ and strictly increasing on $[n_{-1},\infty)$. So there exist precisely two $u_{-1} < n_{-1} < n_1 < u_1$ with $F(u_{-1}) = F(u_1) = \frac{1}{2}\eta^2$. Thus the only possible intersections with the axes of the (connected component of the) level curves $C(\eta)$ of E through $(0,\eta)$ are $(0,\pm\eta)$, $(u_{\pm 1},0)$. Furthermore $C(\eta)$ is bounded away from 0, and $f \ne 0$ on $\mathbb{R}\setminus[n_{-1},n_1]$, so there are no critical points of E on $C(\eta)$, thus $C(\eta)$ by the implicit function is a smooth curve which coincides with the trajectory through $(0,\eta)$. It has to be compact, because it is bounded, and as it can intersect the v-axis only in $(0,\pm\eta)$ it has to be a closed curve round the origin. This implies $\pm\eta \in D$. $0 \in D$ follows by $f(0) = 0$.

We only prove $T'(\eta) < 0$ for $\eta > 0$ — the assertion for $\eta < 0$ follows analogously (consider $-f(-x)$ instead of $f(x)$). Now let $(u_n)_n$ be a sequence of distinct solutions with $u_n(1) = 0$ and $|u_n'(0)| \to \infty$. By Theorem 3.6 such a sequence exists and we have $z(u_n) \to \infty$, which means by Proposition 3.9 $T^{(z(u_n)+1)}(u_n'(0)) = 1$. This implies $T(|u_n'(0)|) \to 0$, so $T(\eta) \to 0$ as $|\eta| \to \infty$.

By Proposition 3.9 a) it is sufficient to show $u_\eta(T(\eta),\eta) < 0$. Let q_1 be defined as in the proof of b). By the differentiable dependence of the solution on the initial value we have

$$\begin{aligned} -u_\eta''(x,\eta) &= f'(u(x,\eta))u_\eta(x,\eta) \\ u_\eta(0,\eta) &= 0 \\ u_\eta'(0,\eta) &= 1. \end{aligned}$$

The same equation is, with different initial values, also solved by u', so by the Sturm Separation Theorem the zeroes of u_η separate and are separated by those of u'. But clearly $-\frac{1}{2}T(-\eta)$, $\frac{1}{2}T(\eta)$, $T(\eta) + \frac{1}{2}T(-\eta)$ are consecutive zeroes of u' and $u'(0,\eta) = 0$, so the smallest positive zero N of u_η lies in the interval $(\frac{1}{2}T(\eta), T(\eta) + \frac{1}{2}T(-\eta))$ (and is the only zero of u_η in this interval). From $u_\eta'(0,\eta) = 1$ we conclude that u_η is positive on $(0,N)$ and negative on $(N, T(\eta) + \frac{1}{2}T(-\eta))$. It remains to show $N < T(\eta)$.

The derivative $u'(\cdot,\eta)$ solves the linear equation

$$-v'' = q_2 \cdot v \tag{3.10}$$

with

$$q_2(x) = q_2(x,\eta) := f'(u(x,\eta)),$$

and $\frac{f(u)}{u} < f'(u)$ (for $u \neq 0$ and $f \in \mathcal{F}$) is just the condition for (3.6) to be a strict Sturm majorant of (3.10) on the interval $[0, T(\eta)]$, so again by Sturm Comparison u_η has a zero in $(0, T(\eta))$, which means $N < T(\eta)$.

d) By (2.6) there exists $K > 0$ and $n_{-1} < 0 < n_1$ such that $F(n_{-1}) = F(n_1) = K$ and F strictly monotone on $\mathbb{R} \setminus [n_{-1}, n_1]$. Take $\eta \geq \bar{\eta} := \sqrt{2K}$ and proceed as in c).

$\square$

The Propositions 3.10 and 3.9 together with Remark 3.8 immediately yield a complete description of the equilibrium solutions of (P) in the case $f \in \mathcal{F}$. For $f \in \mathcal{G}$ the general structure of equilibria can be described.

Corollary 3.12. ***a)*** *For $f \in \mathcal{G}$ we write the set of nontrivial equilibrium solutions as $\{u_k : k \in \mathbb{Z} \setminus \{0\}\}$ ordered by their initial slope, i.e. $u'_{-(k+1)}(0) < u'_{-k}(0) < 0 < u'_k(0) < u'_{k+1}(0)$. If $z(u_k)$ is odd by Remark 3.8 there exists a $l \in \mathbb{Z}$ such that $z(u_l) = z(u_k)$ and $u'_l(0) = -u'_k(0)$.*

If $[u'_l(0), u'_{l+1}(0)] \subset D_s \setminus \{0\}$, then $z(u_{l+1}) \in \{z(u_l) - 1, z(u_l), z(u_l) + 1\}$. If $z(u_k) = n$, there exists $l \in \mathbb{Z}$ with $z(u_l) = n + 1$.

b) *For every $f \in \mathcal{F}$ and integer $n > i(0)$ the equation* (P) *has precisely two solutions u_n, u_{-n} with $n - 1$ zeros. These are nondegenerate with $i(u_{\pm n}) = n$, $u'_n(0) > 0 > u'_{-n}(0)$, and there are no other nontrivial solutions of* (P)*. In particular $\mathcal{F} \subset \mathcal{G}$.*

Proof. First let $f \in \mathcal{G}$, then all equilibria are nondegenerate. So the set $\{v'(0) : v \in E\}$ has no accumulation point in $\mathbb{R}$, and we can order the equilibria as stated. Let $[u'_l(0), u'_{l+1}(0)] \subset D_s \setminus \{0\}$, $n := z(u_l)$. Then $T^{(n-1)}(u'_l(0)) < T^{(n)}(u'_l(0)) = 1 < T^{(n+1)}(u'_l(0))$. Suppose $m := z(u_{l+1}) > n + 1$, then $T^{(n+1)}(u'_{l+1}(0)) < T^{(m)}(u'_{l+1}(0)) = 1$, so there exists a $\eta \in (u'_l(0), u'_{l+1}(0))$ with $T^{(n+1)}(\eta) = 1$. This means there is an equilibrium solution of (P) "between" u_l and u_{l+1}, which is impossible. For $m < n - 1$ the contradiction follows analogously.

Now let $\eta_0 := \inf\{\eta > 0 : [\eta, \infty) \subset D_s\}$. If $u'_k(0) \in (-\infty, \eta_0) \cup (\eta_0, \infty)$, and w.l.o.g. $k > 0$, there has to be a $l > k$ with $z(u_l) = n + 1$ by the assertion proved above, because $z(u_m) \to \infty$ as $m \to \infty$. If $u'_k(0) \in [-\eta_0, \eta_0]$, then $\eta_0 > 0$ and either $\eta_0 \in \mathbb{R} \setminus D$ or $-\eta_0 \in \mathbb{R} \setminus D$. So $T(\eta) \xrightarrow{\eta \downarrow \eta_0} \infty$ or $T(\eta) \xrightarrow{\eta \uparrow -\eta_0} \infty$, either way for any $m \geq 2$ $T^{(m)}(\eta) \xrightarrow{|\eta| \downarrow \eta_0} \infty$ by Proposition 3.10 a). Theorem 3.6 implies $\liminf_{|\eta| \to \infty} T^{(m)}(\eta) = 0$, so in this case for any $m \geq 2$ there are at least two equilibrium solutions with m zeros. The remaining assertion follows from Remark 3.8.

For $f \in \mathcal{F}$ uniqueness and nondegeneracy of the solutions with n zeros follow from Proposition 3.10 c). If $f'(0) < \pi^2$ then $\lim_{\eta \to 0} T(\eta) > 1$ by Proposition 3.10 b), so there is a positive and a negative equilibrium. If $f'(0) \in (k^2\pi^2, (k+1)^2\pi^2)$, then $\lim_{\eta \to 0} T(\eta) \in (\frac{1}{k+1}, \frac{1}{k})$. and $i(0) = k$. This means

$$\lim_{\eta \to 0} T^{(k)}(\eta) > 1 > \lim_{\eta \to 0} T^{(k+1)}(\eta),$$

so the nontrivial solutions have at least $k + 1 = i(0) + 1$ zeros. □

***Remark* 3.13.** For $f \in \mathcal{G}$ and an equilibrium u_k of course the relation $i(u_k) \in \{z(u_k), z(u_k) + 1\}$ from Proposition 3.2 e) still holds, but in general there is no relation between k and $i(u_k)$ or $z(u_k)$.

3.3 The cutoff function

Given $f \in \mathcal{G}$ we will construct a function $\tilde{f} \in \mathcal{G}_d$ such that $f = \tilde{f}$ on a compact interval $[-M^-, M^+]$. Outside the interval $[-M^- - a_-, M^+ + a_+]$ $\tilde{f}$ will be constant, and in the remaining two gaps we choose a simple construction to make $\tilde{f}$ $\mathcal{C}^2$. While doing all this we have to control the function $\tilde{T}$ associated with $\tilde{f}$.

To be more precise we can find

$$u_n'(0), -u_{-n}'(0) < \eta_1 < u_{n+1}'(0), -u_{-(n+1)}'(0)$$

and get $M^+ := \max u(\cdot, \eta_1, f)$, $M^- := -\min u(\cdot, -\eta_1, f)$. We will find a η_{31} (choosing appropriate a_+, a_-) s.t. $\tilde{T}'$ has no zeroes outside $[-\eta_{31}, \eta_{31}]$. On $(-\eta_{31}, -\eta_1)$ and (η_1, η_{31}) we will be able to control the values of $\tilde{T}$ to make sure $u(1, \eta, \tilde{f}) \neq 0$ for η in these intervals. Note that it is crucial to have these intervals symmetric w.r.t. 0 as we have to sum up multiples of $\tilde{T}$ and $\tilde{T}(-\cdot)$ to make assertions about sign changing solutions. See Figure 3.1 for a schematic picture of the phase-plane in case of the modified function.

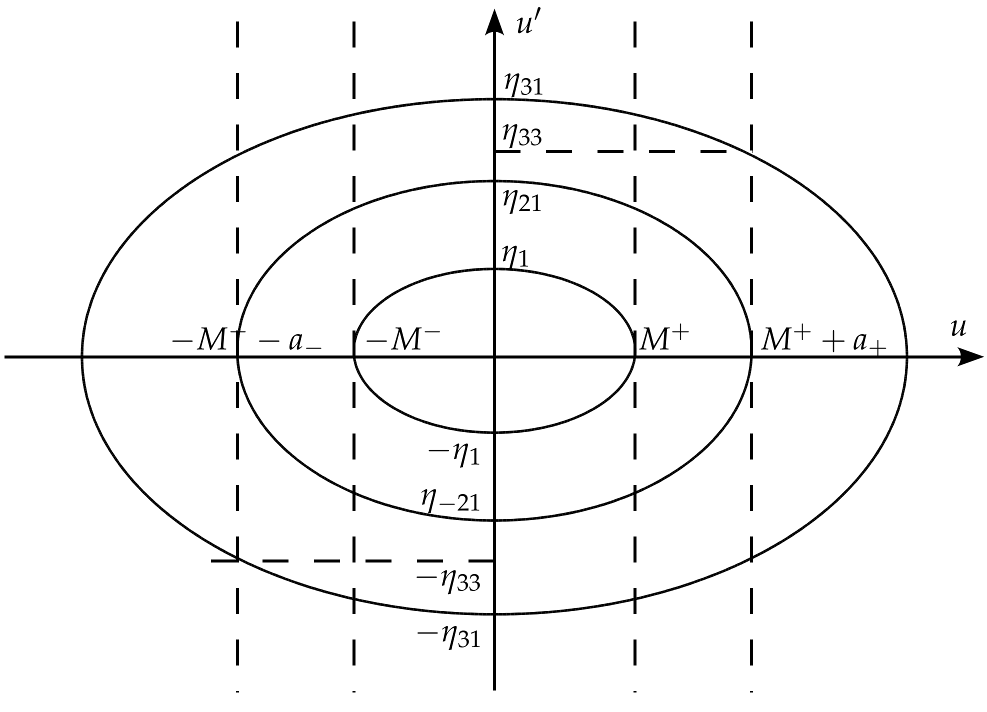

Figure 3.1: phase-space diagram for modified function

As $\tilde{f}$ will be defined piecewise we will first derive properties of equations with constant right hand side and of those with the "bridge function" g (in 3.3.2 defined on $[0, a]$ for simplicity as the problem is autonomous) as right hand side. Then we have to glue together the results for individual parts of trajectories w.r.t. $\tilde{f}$. For these right hand sides we will not only investigate the corresponding functions T_a, T_g, but for the bridge functions we also have to look at trajectories which reach a in finite time, and thus do not become 0 again. Tools for tackling these problems will be derived in the next section. By Remark 3.8 it is sufficient to do most computations for $\eta > 0$ only, the case $\eta < 0$ follows analogously.

The final construction will still be more complicated, as we will have to cut off functions f_θ depending on an additional parameter (with η_1 also depending on θ continuously) in a continuous way (i.e. $\theta \mapsto f_\theta \in \mathcal{G}$ and also $\theta \mapsto \tilde{f}_\theta \in \mathcal{G}_d$ will be continuous). The reason for this will become clear in the applications (cf. Proposition 4.4). So we state our cutoff Proposition, the main technical result of this work. For $v, w \in \tilde{E}_\theta$ (the set of equilibria for the right hand side $\tilde{f}_\theta$) we write $v < w :\iff v'(0) < w'(0)$ and $|v| < |w| :\iff |v'(0)| < |w'(0)|$ to shorten notation.

Proposition 3.14. *Let $f_\theta \in \mathcal{G}$ for $\theta \in [0, 1]$ such that*

$$[0, 1] \ni \theta \mapsto f_\theta \in \mathcal{G} \subset \mathcal{C}^2_w$$

is continuous, $n \in \mathbb{N}$, $m \in \{n, n + 1\}$. Define

$$\begin{aligned} \mathcal{D}_\theta &:= \{\eta \in \mathbb{R} : \exists t > 0 : u(t, \eta, f_\theta) = 0\} \\ T_\theta &: \mathcal{D}_\theta \ni \eta \mapsto \inf\{t > 0 : u(t, \eta, f_\theta) = 0\} \in \mathbb{R} \end{aligned}$$

and let $u_{k,\theta}$ be the k-th nontrivial solution of

$$-u'' = f_\theta(u), \qquad u(0) = u(1) = 0$$

(cf. Corollary 3.12). Suppose $z(u_{m+1}) > z(u_n)$ and there exists a continuous $\eta_1 : [0, 1] \to \mathbb{R}$ such that $\eta_1(\theta) \in \mathcal{D}_\theta$ for all $\theta \in [0, 1]$ and

$$\begin{gathered} \max\{u'_{n,\theta}(0), -u'_{-n,\theta}(0)\} < \eta_1(\theta) < \min\{u'_{m+1,\theta}(0), -u'_{-(m+1),\theta}(0)\}, \\ T^{(n)}_\theta(\pm\eta_1(\theta)) < 1 < T^{(m+1)}_\theta(\pm\eta_1(\theta)). \end{gathered} \tag{3.11}$$

Then for each $\theta \in [0, 1]$ there exists $\tilde{f}_\theta \in \mathcal{G}_d$, uniquely determined by f_θ and $\eta_1(\theta)$, such that the mapping

$$[0, 1] \ni \theta \mapsto \tilde{f}_\theta \in \mathcal{G}_d \subset \mathcal{C}^2_s$$

is continuous and $f_\theta \equiv \tilde{f}_\theta$ on $[-M(\theta), M(\theta)]$ with $[0, 1] \ni \theta \mapsto M(\theta) \in (0, \infty)$ continuous and $M(\theta)$ uniquely determined by f_θ and $\eta_1(\theta)$. Consequently φ_θ and $\tilde{\varphi}_\theta$ coincide on $\{u \in X : \|u\|_\infty \leq M(\theta)$. If $f_\theta \equiv f$ for all θ then $M(\theta)$ is increasing if $\eta_1(\theta)$ is increasing, and if $n \to \infty$ both $\eta_1(\theta) \to \infty$ and $M(\theta) \to \infty$ uniformly in θ.

For the set $\tilde{E}_\theta$ of stationary solutions of (P) *with r.h.s. $\tilde{f}_\theta$ we have*

$$\begin{aligned}&\tilde{E}_\theta = \{u_{k,\theta} : |k| \le n\} \cup \{\tilde{u}_{k,\theta} : 1 \le |k| \le n\} \cup R_\theta\\ &\tilde{u}_{k,\theta} \text{ nondegenerate}, \quad i(\tilde{u}_{k,\theta}) = z(\tilde{u}_{k,\theta}) = |k| - 1, \quad \tilde{u}'_{-|k|}(0) < 0 < \tilde{u}'_{|k|}(0)\\ &v \in R_\theta \Rightarrow n + 1 \le z(v) \le m.\end{aligned}$$

The mappings $\theta \mapsto u_{k,\theta}$ and $\theta \mapsto \tilde{u}_{k,\theta}$ are continuous for any k and

$$\tilde{u}_{-1} < \cdots < \tilde{u}_{-n} < u_{-n} < \cdots < u_{-1} < 0 < u_1 < \cdots < u_n < \tilde{u}_n < \cdots < \tilde{u}_1, \tag{3.12}$$

$$0 < |u_{\pm 1}| < \cdots < |u_{\pm n}| < |\tilde{u}_{\pm n}| < \cdots < |\tilde{u}_{\pm 1}|, \tag{3.13}$$

The proof will fill the rest of section 3.3. We will first introduce some phase-plane analysis tools. Next we will define the "bridge function" g and compute several properties of the flow induced by g. Finally in 3.3.3 we will put everything together to define $\tilde{f}_\theta$ and prove Proposition 3.14.

3.3.1 Phase-plane analysis tools

Lemma 3.15. *Let $a > 0$, $f : \mathbb{R} \to \mathbb{R}$ $\mathcal{C}^1$. Let $u(\cdot, \eta)$ denote the solution of the initial value problem* (IVP) *and $D := \{\eta \ge 0 : \exists t \ge 0 : u(t, \eta) = a\}$. For $\eta \in D$ let*

$$\begin{aligned}\tau(\eta) &:= \inf\{t > 0 : u(t, \eta) = a\} < \infty\\ \psi(\eta) &:= u'(\tau(\eta), \eta).\end{aligned}$$

Then τ, ψ are continuous on D, τ is $\mathcal{C}^3$ and ψ is $\mathcal{C}^2$ on $\mathring{D}$, and the following assertions hold:

(i) D is an unbounded subinterval of $(0, \infty)$, let $\eta_0 := \inf D$. If $f|_{[0,a]} > 0$ then D is closed and $\psi(D) = [0, \infty)$.

(ii) Let $F(x) := \int_0^x f(t)\,dt$, then

$$\forall \eta \in D : \psi(\eta) = \sqrt{\eta^2 - 2F(a)},$$

in particular if $\psi(\eta_0) = 0$ then

$$\forall \eta \in D : \psi(\eta) = \sqrt{\eta^2 - \eta_0^2}.$$

If $f|_{[0,a]} > 0$ then $\psi'(\eta) = \frac{\eta}{\psi(\eta)} \ge 1$.

(iii) We have $\tau' < 0$ and $\tau(\eta) \to 0$, $\tau'(\eta) \to 0$ as $\eta \to \infty$. The derivative of τ is

$$\tau'(\eta) = -\psi'(\eta) \frac{u_\eta(\tau(\eta), \eta)}{\eta} = \frac{-u_\eta(\tau(\eta), \eta)}{\psi(\eta)}.$$

(iv) *The derivative u_η of u w.r.t. the initial value η satisfies*

$$\forall t \in (0, \tau(\eta)] : \; u_\eta(t, \eta) > 0$$

and

$$0 = -f(a) u_\eta(\tau(\eta), \eta) - u'_\eta(\tau(\eta), \eta)\psi(\eta) + \eta.$$

(v) *Let $f_k \in \mathcal{C}^1$, $f_k \to f$ in $\mathcal{C}^1_w$, τ_k, ψ_k, D_k be defined for f_k as τ, ψ, D for f. If $\eta_k \to \eta \in \mathring{D}$, then $\eta_k \in \mathring{D}_k$ for k large, and*

$$\tau_k(\eta_k) \to \tau(\eta), \qquad \tau'_k(\eta_k) \to \tau'_k(\eta).$$

If $\eta_k \in D_k$, $\eta_k \to \eta_0 \in D$, then

$$\tau_k(\eta_k) \to \tau(\eta_0).$$

Proof. Setting $M > \max\{|f(\alpha)| : 0 \leq \alpha \leq a\}$, we first show that $[\sqrt{2aM}, \infty) \subset D$. Let $V(\alpha, \beta) = (\beta, -f(\alpha))$ the vectorfield of (SYS), and define for $\eta \geq \sqrt{2aM}$ the functions

$$\zeta : [0, a] \to [0, \infty), \qquad \alpha \mapsto \eta + \alpha \frac{M}{\eta}$$
$$\xi : [0, a] \to [0, \infty), \qquad \alpha \mapsto \sqrt{\eta^2 - 2\alpha M}.$$

We will show that trajectories of (SYS) can leave the set

$$C(\eta) = \{(\alpha, \beta) : 0 \leq \alpha \leq a, \xi(\alpha) \leq \beta \leq \zeta(\alpha)\}$$

through $\{a\} \times (\xi(a), \zeta(a))$ only. This implies that the trajectory through $(0, \eta)$ reaches the set $\{a\} \times \mathbb{R}$, which happens in finite time as $(a, 0)$ is the only possible zero of V in $C(\eta)$. Thus $\eta \in D$. We calculate

$$\zeta'(\alpha) = \frac{M}{\eta}, \qquad \xi'(\alpha) = \frac{-M}{\sqrt{\eta^2 - 2\alpha M}}$$
$$V(\alpha, \zeta(\alpha)) = \left(\eta + \alpha \tfrac{M}{\eta}\right) \cdot \left(1, \frac{-f(\alpha)}{\eta + \alpha \frac{M}{\eta}}\right)$$
$$V(\alpha, \xi(\alpha)) = \sqrt{\eta^2 - 2\alpha M} \cdot \left(1, \frac{-f(\alpha)}{\sqrt{\eta^2 - 2\alpha M}}\right).$$

Now

$$\frac{-f(\alpha)}{\eta + \alpha \frac{M}{\eta}} < \frac{M}{\eta} = \zeta'(\alpha),$$

so a trajectory being in $A(\eta := \operatorname{graph}\zeta$ at some time will be in $\mathring{C}(\eta)$ immediately afterwards. Similarly

$$\frac{-f(\alpha)}{\sqrt{\eta^2 - 2aM}} > \frac{-M}{\sqrt{\eta^2 - 2aM}},$$

so a trajectories being in $\operatorname{graph}\xi$ at some time will also be in $\mathring{C}(\eta)$ immediately afterwards. This proves $\eta \in D$.

If $D \ni \tilde{\eta} < \eta$ then the trajectory trough $(0, \eta)$ stays between the trajectory through $(0, \tilde{\eta})$ and $A(\eta)$, so $\eta \in D$ and D is an interval.

Let $\eta \in \mathring{D}$, so $u'(\tau(\eta), \eta) > 0$ and consider

$$U : (t, \eta) \mapsto u(t, \eta) - a.$$

Then U is $\mathcal{C}^3$, $U(\tau(\eta), \eta) = 0$ and $U_t(\tau(\eta), \eta) = u'(\tau(\eta), \eta) > 0$. So by the implicit function theorem there is a $\delta > 0$ such that $\tau|_{(\eta-\delta,\eta+\delta)}$ is $\mathcal{C}^3$. Thus τ is $\mathcal{C}^3$ and ψ is $\mathcal{C}^2$ because u' is $\mathcal{C}^2$. We will prove the continuity of τ on D below after we have shown $\tau' < 0$.

Let $\eta \in \mathring{D}$, $t \in [0, \tau(\eta)]$. The functions u', u_η solve the initial value problems

$$\begin{aligned} u'''(t,\eta) &= -f'(u(t,\eta)) \cdot u'(t,\eta) \qquad & u'(0,\eta) &= \eta \qquad & u''(0,\eta) &= -f(0) \\ u''_\eta(t,\eta) &= -f'(u(t,\eta)) \cdot u_\eta(t,\eta) \qquad & u_\eta(0,\eta) &= 0 \qquad & u'_\eta(0,\eta) &= 1 \end{aligned}$$

respectively. Multiplying the first equation by u_η, the second by u' and subtracting the results we obtain

$$u'''(t,\eta)u_\eta(t,\eta) - u''_\eta(t,\eta)u'(t,\eta) = 0.$$

Integrating this from 0 to $t \leq \tau(\eta)$ yields

$$0 = u''(t,\eta)u_\eta(t,\eta) - u'_\eta(t,\eta)u'(t,\eta) + \eta, \tag{3.14}$$

i.e. for $t = \tau(\eta)$

$$0 = -f(a)u_\eta(\tau(\eta),\eta) - u'_\eta(\tau(\eta),\eta)\psi(\eta) + \eta. \tag{3.15}$$

One easily checks

$$\frac{d}{dt}\left(\frac{1}{2}(u'(t,\eta))^2 + F(u(t,\eta))\right) = 0$$

so

$$\forall t : \ \frac{1}{2}\left(u'(t,\eta)\right)^2 + F(u(t,\eta)) \equiv \frac{1}{2}\eta^2 \tag{3.16}$$

which yields $\psi(\eta)^2 + 2F(a) = \eta^2$, so $\psi(\eta) = \sqrt{\eta^2 - 2F(a)}$. If $\psi(\eta_0) = 0$ then $\eta_0^2 = 2F(a)$ and $\psi(\eta) = \sqrt{\eta^2 - \eta_0^2}$

Differentiating $a = u(\tau(\eta), \eta)$ w.r.t. η yields

$$0 = \underbrace{u'(\tau(\eta), \eta)}_{=\psi(\eta)} \cdot \tau'(\eta) + u_\eta(\tau(\eta), \eta). \tag{3.17}$$

The function $u'(\cdot, \eta)$ has no zero in $[0, \tau(\eta))$, so by the Sturm Separation Theorem $u_\eta(\cdot, \eta)$ cannot have a zero in $(0, \tau(\eta)]$. By (3.17) we have

$$\tau'(\eta) = -\frac{u_\eta(\tau(\eta), \eta)}{\psi(\eta)}, \tag{3.18}$$

and by (3.14) for $0 \leq t \leq \tau(\eta)$

$$u_\eta(t, \eta) = 0 \Rightarrow u'_\eta(t, \eta) = \frac{\eta}{u'(t, \eta)} > 0,$$

i.e. $u_\eta(t, \eta) > 0$ on $(0, \tau(\eta)]$ by the initial values. So $\tau'(\eta) < 0$ by (3.18).

Now we prove the continuity of τ on D. Suppose $\eta_0 \in D$ and let $\tau_0 := \lim_{\eta \downarrow \eta_0} \tau(\eta) \in (0, \infty]$. First suppose $\tau_0 = \infty$. In this case $u(\cdot, \eta) \to u(\cdot, \eta_0)$ uniformly on $[0, n]$ as $\eta \downarrow \eta_0$ for any $n \in \mathbb{N}$. Now $n < \tau(\eta)$ for η sufficiently close to η_0, consequently $u(\cdot, \eta_0)|_{[0,n]} < a$. This implies $\tau(\eta_0) = \infty$, which contradicts $\eta_0 \in D$. Next suppose $\tau_0 < \infty$, again fix $n \in \mathbb{N}$. Then $u(\cdot, \eta) \to u(\cdot, \eta_0)$ uniformly on $[0, \tau_0 - \frac{1}{n}]$, so $u(t, \eta_0) < a$ for $0 \leq t \leq \tau_0 - \frac{1}{n}]$, i.e. $\tau(\eta_0) \geq \tau_0$. But $u(\tau(\eta), \eta) \to u(\tau_0, \eta_0) = a$ by continuous dependence. This implies $\tau(\eta_0) \leq \tau_0$, so τ is continuous in η_0.

Let $\eta > \eta_0$ and $t_0 \in [0, \tau(\eta)]$ such that $u'(t_0, \eta) = \min u'(\cdot, \eta)|_{[0,\tau(\eta)]} \geq 0$. By (3.16) this means $F(u(t_0, \eta)) = \max F|_{[0,a]} =: C$, so

$$\min u'(\cdot, \eta)|_{[0,\tau(\eta)]} = \sqrt{\eta^2 - 2C}$$

for any $\eta \in D$. So we can estimate

$$a = \int_0^{\tau(\eta)} u'(t, \eta)\, dt \geq \tau(\eta)\sqrt{\eta^2 - 2C}$$

thus

$$\tau(\eta) \leq \frac{a}{\sqrt{\eta^2 - 2C}} \xrightarrow{\eta \to \infty} 0.$$

Now for $t \le \tau(\eta)$ and $M > \max\{|f'(x)| : x \in [0,a]\}$ we compute

$$\begin{aligned}
0 \le u_\eta(t,\eta) &= \int_0^t \left(1 - \int_0^s f'(u(r,\eta))u_\eta(r,\eta)\,dr\right) ds \\
&\le t + \int_0^t \int_0^s M \underbrace{u_\eta(r,\eta)}_{\ge 0}\,dr\,ds \\
&\le \tau(\eta) + M\tau(\eta) \int_0^t u_\eta(r,\eta)\,dr.
\end{aligned}$$

By Gronwall's inequality

$$0 \le u_\eta(t,\eta) \le \tau(\eta) \cdot \exp\left(M\tau(\eta)\int_0^t 1\,ds\right) \xrightarrow{\eta\to\infty} 0,$$

so $\tau'(\eta) \to 0$ as $\eta \to \infty$ by (3.18) (note that $\psi(\eta) \to \infty$ as $\eta \to \infty$).

Under the assumptions of (v) we get $\tau_k(\eta_k) \to \tau(\eta)$ in both cases, analogously to the proof of Proposition 3.10 a). Let $\eta_k \to \eta \in \mathring{D}$, then there is a $\delta > 0$ such that $I := [\eta - \delta, \eta + \delta] \subset D$. By continuous dependence and compactness of I

$$\exists \epsilon > 0 \forall \bar{\eta} \in I : u(\tau(\bar{\eta}) + \epsilon, \bar{\eta}) > a.$$

Also by continuous dependence for k large enough $u(\tau(\bar{\eta}) + \epsilon, \eta_k) > a$ for all $\bar{\eta} \in I$, which means $I \subset D_k$. But then $\eta_k \in \mathring{I} \subset \mathring{D}_k$ for all k large enough. Now $\tau_k'(\eta_k) \to \tau'(\eta)$ is a consequence of (iii) and the continuous dependence theorem, so we have proved (v).

Finally consider the special case $f|_{[0,a]} > 0$. For any $\eta \in D$ we have

$$\begin{aligned}
\psi(\eta) = \eta + \int_0^{\tau(\eta)} -f(u(t),\eta)\,dt &\le \eta - \min f|_{[0,a]} \cdot \tau(\eta) \\
\Rightarrow \tau(\eta) &\le \frac{\eta - \psi(\eta)}{\min f|_{[0,a]}} \le \frac{\eta}{\min f|_{[0,a]}}.
\end{aligned}$$

(In particular $\psi(\eta) \le \eta \Rightarrow \psi'(\eta) = \frac{\eta}{\psi(\eta)} \ge 1$.) That means τ is bounded on $(\eta_0, \eta_0 + 1]$, thus $\eta_0 \in D$ by the continuous dependence Theorem. But this implies $\psi(\eta_0) = 0$, otherwise τ would be defined on a neighborhood of η_0 by the implicit function theorem. From $0 = \psi(\eta_0) = \sqrt{\eta_0^2 - 2F(a)}$ we get $\eta_0^2 = 2F(a)$, which yields the formula for ψ^{-1}. □

Lemma 3.16. *In addition to the hypotheses of Lemma 3.15 let* $0 < m \le f|_{[0,a]} \le M$. *Then* $\sqrt{2am} \le \eta_0 \le \sqrt{2aM}$ *and* $T(\eta) < \infty$ *for* $0 \le \eta \le \eta_0$. *We further get the following estimates:*

$$\frac{\eta^2}{2m} \ge \max u(\cdot,\eta) \ge \frac{\eta^2}{2M} \qquad 0 \le \eta \le \eta_0 \tag{3.19}$$

$$\tau(\eta) \le \frac{\eta - \sqrt{\eta^2 - 2aM}}{M} \qquad \eta \ge \sqrt{2aM} \tag{3.20}$$

$$\tau(\eta) \ge \frac{\eta - \sqrt{\eta^2 - 2am}}{m} \qquad \eta \in D \tag{3.21}$$

$$\psi(\eta) \ge \sqrt{\eta^2 - 2aM} \qquad \eta \ge \sqrt{2aM} \tag{3.22}$$

$$\psi(\eta) \le \sqrt{\eta^2 - 2am} \qquad \eta \in D \tag{3.23}$$

$$\frac{2\eta}{m} \ge T(\eta) \ge \frac{2\eta}{M} \qquad 0 \le \eta \le \eta_0. \tag{3.24}$$

Proof. The functions $v(t,\eta) := -\frac{m}{2}t^2 + \eta t$, $w(t,\eta) := -\frac{M}{2}t^2 + \eta t$ satisfy the i.v.p. (IVP) with right hand sides m, M respectively. We easily compute

$$\max v(\cdot,\eta) = v\left(\frac{\eta}{m},\eta\right) = \frac{\eta^2}{2m}, \quad \max w(\cdot,\eta) = w\left(\frac{\eta}{M},\eta\right) = \frac{\eta^2}{2M}.$$

As in the proof of Lemma 3.15 we see $\eta \in D$ for $\eta \ge \sqrt{2aM}$, that is $\sqrt{2aM} \ge \eta_0$. For $\eta \ge \eta_0$ and $t \in [0, \tau(\eta_0)]$ or for $\eta \le \eta_0$ and $t \in [0, T(\eta)]$ we get

$$u'(t,\eta) = \eta - \int_0^t f(u(s,\eta))\,ds \le \eta - tm = v'(t,\eta),$$

i.e. $a = u(\tau(\eta_0),\eta_0) \le v(\tau(\eta_0),\eta_0)$. As $\max v(\cdot,\sqrt{2am}) = a$ and $\max v(\cdot,\eta)$ increases in η this implies $\eta_0 \ge \sqrt{2am}$.

Similarly for $\eta \le \eta_0$ we get $v(T(\eta),\eta) \ge 0 \ge w(T(\eta),\eta)$ and $v(t,\eta) > 0$ for $t \le T(\eta)$, thus follows (3.24). Also

$$\frac{\eta^2}{2m} = \max v(\cdot,\eta) \ge v\left(\frac{1}{2}T(\eta),\eta\right) \ge u\left(\frac{1}{2}T(\eta),\eta\right) = \max u(\cdot,\eta)$$

and as $\frac{\eta}{M} \le \frac{1}{2}T(\eta)$ by (3.24)

$$\frac{\eta^2}{2M} = w\left(\frac{\eta}{M},\eta\right) \le u\left(\frac{\eta}{M},\eta\right) \le \max u(\cdot,\eta),$$

thus follows (3.19).

(3.22) and (3.23) are direct consequences of Lemma 3.15 (ii).

To see (3.20) let $\eta \ge \sqrt{2aM}$, $0 \le t \le \tau(\eta)$, then $a = u(\tau(\eta),\eta) \ge w(\tau(\eta),\eta)$ and $u(t,\eta) \ge w(t,\eta)$ for all $t \le \tau(\eta)$ which implies $\tau(\eta) \le \frac{\eta-\sqrt{\eta^2-2aM}}{M} = \min\{t > 0 : w(t,\eta) = a\}$ by an direct calculation. (3.21) follows analogously. $\square$

3.3.2 The bridge function

Let $c_1, c_2 \in \mathbb{R}$, $0 < d_2 < d_1$, $a > 0$ and define the auxiliary functions

$$\sigma : \mathbb{R} \to (0,\infty), \qquad \sigma(x) := \begin{cases} \exp(\frac{x^4}{x^2-1}) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

$$p : \mathbb{R} \to \mathbb{R}, \qquad p(x) := \frac{c_2}{2}x^2 + c_1 x + (d_1 - d_2).$$

By a straightforward calculation σ is $\mathcal{C}^\infty$ and

$$\sigma'(x) = \sigma(x) \cdot \frac{2x^5 - 4x^3}{(x^2-1)^2}$$
$$\sigma''(x) = \sigma(x) \cdot x \cdot \frac{x(2x^4 - 4x^2)^2 + (10x^3 - 12x)(x^2-1)^2 - 4(2x^5 - 4x^3)(x^2-1)}{(x^2-1)^4},$$

so

$$\sigma(0) = 1, \quad \sigma'(0) = \sigma''(0) = 0, \quad \sigma(1) = \sigma'(1) = \sigma''(1) = 0$$
$$\|\sigma\|_\infty = 1, \quad \sigma'|_{(0,1)} < 0.$$

Define

$$g(x) = g(x,a) = g(x; a, c_1, c_2, d_1, d_2) := p(x)\sigma\Big(\frac{x}{a}\Big) + d_2.$$

Then g is $\mathcal{C}^2$ in (x,a) and the mapping

$$(a, c_1, c_2, d_1, d_2) \mapsto g(\cdot; a, c_1, c_2, d_1, d_2) \in \mathcal{C}^2_s$$

is continuous ($g(x) = d_2$ for $|x| \geq a$). We calculate

$$g'(x,a) = (c_2 x + c_1)\sigma\Big(\frac{x}{a}\Big) + p(x)\sigma'\Big(\frac{x}{a}\Big)\frac{1}{a}$$
$$g''(x,a) = c_2\sigma\Big(\frac{x}{a}\Big) + \frac{2}{a}(c_2 x + c_1)\sigma'\Big(\frac{x}{a}\Big) + \frac{p(x)}{a^2}\sigma''\Big(\frac{x}{a}\Big)$$
$$g(0,a) = d_1, \quad g'(0,a) = c_1, \quad g''(0,a) = c_2$$
$$g(a,a) = d_2, \quad g'(a,a) = 0, \quad g''(a,a) = 0$$
$$g_a(x,a) := \frac{d}{da}g(x,a) = -p(x)\sigma'\Big(\frac{x}{a}\Big)\frac{x}{a^2}.$$

Let

$$a_0 = a_0(c_1, c_2, d_1, d_2) := \min\left\{\sqrt{\frac{d_1 - d_2}{2|c_2|}}, \frac{d_1 - d_2}{4|c_1|}, \sqrt{\frac{d_1\|\sigma'\|_\infty}{|c_2|}}, \frac{d_1\|\sigma'\|_\infty}{|c_1|}, 1\right\}, \tag{3.25}$$

so a_0 is a continuous function in all variables. Let $0 \le x \le a \le a_0$, then from (3.25) we get the estimates

$$\begin{aligned} |p(x) - (d_1 - d_2)| &\le \frac{|c_2|}{2}x^2 + |c_1|x \le \frac{d_1 - d_2}{2} \\ |p'(x) \cdot a| &\le |c_2|a^2 + |c_1|a \le 2d_1\|\sigma'\|_\infty, \end{aligned}$$

which yield

$$\begin{aligned} 0 < \frac{d_1 - d_2}{2} &\le p(x) \le \tfrac{3}{2}(d_1 - d_2) \\ d_2 \le g(x,a) &= p(x)\sigma\big(\tfrac{x}{a}\big) + d_2 \le \tfrac{3}{2}(d_1 - d_2) + d_2 \le 2d_1 \\ |g'(x,a)| &\le \frac{|p'(x) \cdot a|}{a} + |p(x)|\frac{\|\sigma'\|_\infty}{a} \\ &\le \big(2d_1 + \tfrac{3}{2}(d_1 - d_2)\big)\frac{\|\sigma'\|_\infty}{a} \le \frac{4d_1\|\sigma'\|_\infty}{a}. \end{aligned}$$

Let $G(a) := \int_0^a g(t,a)\,dt$. On $[0,a]$ by the above estimates we have $p > 0$ so $g_a \ge 0$ and

$$\frac{d}{da}G(a) = \underbrace{g(a,a)}_{=d_2>0} + \int_0^a g_a(x,a)\,dx > 0$$

and $G(a) \le 2ad_1 \to 0$ as $a \to 0$. Define analogously to section 3.3.1 functions $\tau_{22}(\cdot,a)$, $\psi_{22}(\cdot,a)$ (with f replaced by $g(\cdot,a)$ — the reason for the strange indices will become clear in section 3.3.3) defined on $[\eta_{22}(a),\infty)$ and a function $T_{2,a}$ as in section 3.2. By Lemma 3.15 we have

$$\psi_{22}(\eta,a) = \sqrt{\eta^2 - 2G(a)},$$

so

$$\begin{aligned} \eta_{22}(a) &= \sqrt{2G(a)} \to 0 \ (a \to 0) \\ \eta_{22}'(a) &> 0. \end{aligned} \tag{3.26}$$

By Lemma 3.16 and $g(x) \ge d_2$ we have for $0 \le \eta \le \eta_{22}(a)$

$$T_{2,a}(\eta) \le \frac{2\eta}{d_2} \le \frac{2\eta_{22}(a)}{d_2} \le \frac{2}{d_2}\sqrt{2ad_1} \to 0 \text{ as } a \to 0.$$

For $\eta \ge \eta_{22}(a)$ we derive

$$\tau_{22}(\eta,a) \le \tau_{22}(\eta_{22}(a),a) = \frac{1}{2}T_{2,a}(\eta_{22}(a)) \le \frac{1}{d_2}\sqrt{2ad_1} \to 0 \ (a \to 0). \tag{3.27}$$

We derive another estimate on τ_{22}. Let $\eta_{33} > 0$, $\eta \geq \eta_{32}(a) := (\psi_{22}(\cdot, a))^{-1}(\eta_{33}) = \sqrt{\eta_{33}^2 + 2G(a)}$ and $t \in [0, \tau_{22}(\eta, a)]$. We have $u'(t, \eta, a) \geq \psi_{22}(\eta, a) \geq \eta_{33}$ because $g > 0$, thus

$$a \geq \tau_{22}(\eta, a) \cdot \eta_{33} \Rightarrow \tau_{22}(\eta, a) \leq \frac{a}{\eta_{33}}. \tag{3.28}$$

With this we estimate $\tau_{22}'(\eta, a) = \frac{d}{d\eta}\tau_{22}(\eta, a)$ for $\eta \geq \eta_{32}(a)$. By Lemma 3.15

$$\tau_{22}'(\eta, a) = \frac{-u_\eta(\tau_{22}(\eta, a), \eta)}{\psi_{22}(\eta, a)} \geq \frac{-u_\eta(\tau_{22}(\eta, a), \eta)}{\eta_{33}}.$$

So next we estimate u_η. As u_η, u' satisfy the same linear equation $u_\eta(\cdot, \eta, a)$ cannot have a zero in $(0, \tau_{22}(\eta, a)]$ by Sturm comparison. Let $t \in (0, \tau_{22}(\eta, a)]$:

$$\begin{aligned} u_\eta(t, \eta, a) &= t - \int_0^t \int_0^s \underbrace{g'(u(r, \eta), a)}_{\geq -\frac{4d_1\|\sigma'\|_\infty}{a}} \underbrace{u_\eta(r, \eta, a)}_{\geq 0} \, dr \, ds \\ &\leq t + \frac{4d_1\|\sigma'\|_\infty}{a} \cdot \underbrace{\tau_{22}(\eta, a)}_{\leq \frac{a}{\eta_{33}}} \cdot \int_0^t u_\eta(r, \eta, a) \, dr. \end{aligned}$$

By Gronwall's inequality:

$$\begin{aligned} u_\eta(t, \eta, a) &\leq t + \frac{4d_1\|\sigma'\|_\infty}{\eta_{33}} \int_0^t s \cdot \exp\left[\int_s^t \frac{4d_1\|\sigma'\|_\infty}{\eta_{33}} \, dr\right] ds \\ &\leq t + \frac{4d_1\|\sigma'\|_\infty}{\eta_{33}} \exp\left[\frac{4d_1\|\sigma'\|_\infty t}{\eta_{33}}\right] \frac{t^2}{2} \\ &\overset{t \leq \tau_{22} \leq \frac{a}{\eta_{33}}}{\leq} \frac{a}{\eta_{33}} + \frac{2d_1 a^2\|\sigma'\|_\infty}{\eta_{33}^3} \exp\left[\frac{4ad_1\|\sigma'\|_\infty}{\eta_{33}^2}\right], \end{aligned}$$

so

$$\tau_{22}'(\eta, a) \geq -\frac{a}{\eta_{33}^2} - \frac{2d_1 a^2\|\sigma'\|_\infty}{\eta_{33}^4} \exp\left[\frac{4ad_1\|\sigma'\|_\infty}{\eta_{33}^2}\right] \xrightarrow{a \to 0} 0. \tag{3.29}$$

3.3.3 Proof of the cutoff-proposition

Figure 3.2 shows (part of) the values defined below in the phase-plane.

We write $n = k_+ + k_-$, $m + 1 = l_+ + l_-$ with $(k_+ - k_-), (l_+ - l_-) \in \{0, 1\}$. Then we have $T_\theta^{(n)}(\eta) = k_+ T_\theta(\eta) + k_- T_\theta(-\eta)$, $T_\theta^{(m+1)}(\eta) = l_+ T_\theta(\eta) + l_- T_\theta(-\eta)$ (cf. Proposition 3.9), and similar decompositions exist for all T-maps.

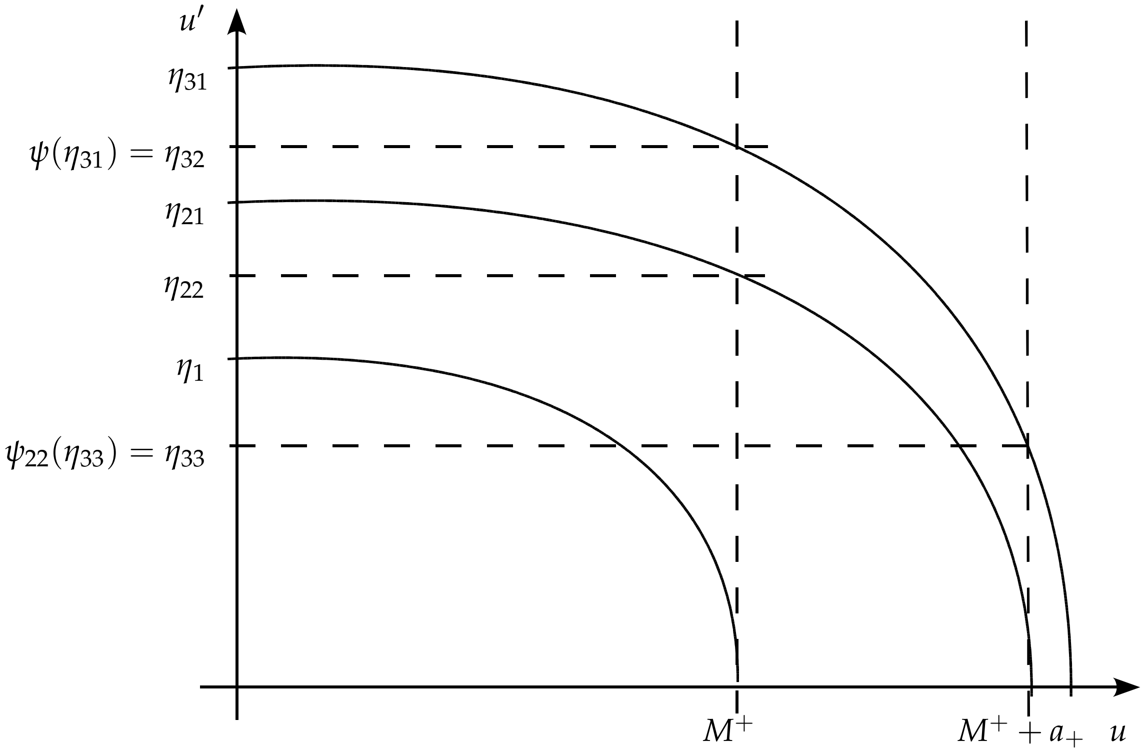

Figure 3.2: more detailed phase-space diagram for cut-off function, θ's omitted

By the hypothesis $(\theta, u) \mapsto f_\theta(u)$ and $(\theta, u) \mapsto f'_\theta(u)$ are continuous. By the continuous dependence Theorem

$$(t, \eta, \theta) \mapsto \begin{pmatrix} u(t, \eta, f_\theta) \\ u'(t, \eta, f_\theta) \end{pmatrix}$$

is continuous, and as u_η satisfies the equation $-u''_\eta = f'(u)u_\eta$ also $(t, \eta, \theta) \mapsto u_\eta(t, \eta, f_\theta)$ is continuous. Finally by Proposition 3.10 a) the functions $(\theta, \eta) \mapsto T_\theta(\eta)$, $(\theta, \eta) \mapsto T'_\theta(\eta)$ are continuous.

Define for $\theta \in [0, 1]$

$$\begin{aligned} M^+(\theta) &:= \max u(\cdot, \eta_1(\theta), f_\theta) \\ M^-(\theta) &:= -\min u(\cdot, \eta_1(\theta), f_\theta) \\ M(\theta) &:= \min\{M^+(\theta), M^-(\theta)\}, \end{aligned}$$

which by a look on the phase plane implies

$$f_\theta(M^+(\theta)) > 0 > f_\theta(-M^-(\theta)).$$

Let for all $k \in \mathbb{Z}$, $\sigma(k,\theta) := u'_{k,\theta}(0)$ be the initial slope of the k-th stationary solution w.r.t. f_θ. Let for $\eta \in [\eta_1(\theta), \infty)$

$$\begin{aligned} \tau_{+1,\theta}(\eta) &:= \inf\{t > 0 : u(t,\eta,\theta) \geq M^+(\theta)\} \\ \psi_{+1}(\eta,\theta) &:= u(\tau_{+1,\theta},\eta,\theta) \\ \tau_{-1,\theta}(\eta) &:= \inf\{t > 0 : -u(t,-\eta,\theta) \geq M^-(\theta)\} \\ \psi_{-1}(\eta,\theta) &:= -u(\tau_{-1,\theta},-\eta,\theta). \end{aligned}$$

Then by Lemma 3.15 (v) the functions $(\theta,\eta) \mapsto \tau_{\pm 1,\theta}(\eta)$ and $(\theta,\eta) \mapsto \tau'_{\pm 1,\theta}(\eta)$ are continuous on $\{(\theta,\eta) : \theta \in [0,1], |\eta| > \eta_1(\theta)\}$.

We construct continuous functions $\epsilon_1, \epsilon_2 : [0,1] \to (0,\infty)$ such that for all $\theta \in [0,1]$ the following estimates hold:

$$\begin{aligned} T_\theta^{(n)}(\eta_1(\theta)) + n\epsilon_1(\theta) &< 1 \\ T_\theta^{(n)}(-\eta_1(\theta)) + n\epsilon_1(\theta) &< 1, \end{aligned} \tag{3.30}$$

$$\begin{aligned} &\forall\theta \in [0,1]\forall\eta \in [\eta_1(\theta), \eta_1(\theta)+\epsilon_2(\theta)] : \\ &\qquad l_+2\tau_{+1,\theta}(\eta) + l_-2\tau_{-1,\theta}(\eta) > 1 \end{aligned} \tag{3.31}$$

$$\begin{aligned} 0 < u_\eta(\tau_{+1,\theta}(\eta),\eta,f_\theta) &\leq \frac{2\eta_1(\theta)}{f_\theta(M^+(\theta))} \\ 0 < -u_\eta(\tau_{-1,\theta}(\eta),-\eta,f_\theta) &\leq \frac{2\eta_1(\theta)}{-f_\theta(-M^-(\theta))}. \end{aligned} \tag{3.32}$$

We have to justify (3.30)-(3.32). By (3.11) we can take

$$\epsilon_1(\theta) := \frac{1}{2n}\min\left\{1 - T_\theta^{(n)}(\eta_1(\theta)), 1 - T_\theta^{(n)}(-\eta_1(\theta))\right\}$$

to satisfy (3.30). By (3.11) we have

$$T_\theta^{(n+1)}(\eta_1(\theta)) = 2l_+\tau_{+1,\theta}(\eta_1(\theta)) + 2l_-\tau_{-1,\theta}(\eta_1(\theta)) > 1.$$

By Lemma 3.15 (iv) we have $0 < u_\eta(\tau_{+1,\theta}(\eta),\eta,f_\theta)$ for all $\eta > \eta_1(\theta)$, and also

$$\begin{aligned} &f_\theta(M^+(\theta)) \cdot u_\eta(\tau_{+1,\theta}(\eta),\eta_1(\theta),f_\theta) = \eta_1(\theta) \\ &\Rightarrow u_\eta(\tau_{+1,\theta}(\eta_1(\theta)),\eta_1(\theta),f_\theta) = \frac{\eta_1(\theta)}{f_\theta(M^+(\theta))} < \frac{2\eta_1(\theta)}{f_\theta(M^+(\theta))}, \end{aligned} \tag{3.33}$$

and similarly

$$-u_\eta(\tau_{-1,\theta}(\eta_1(\theta)),-\eta_1(\theta),f_\theta) < \frac{2\eta_1(\theta)}{-f_\theta(-M^-(\theta))}.$$

To get ϵ_2 note that by (3.11) we have

$$T_\theta^{(n+1)}(\eta_1(\theta)) = 2l_+\tau_{+1,\theta}(\eta_1(\theta)) + 2l_-\tau_{-1,\theta}(\eta_1(\theta)) > 1,$$

and for θ fixed $\eta \mapsto 2l_+\tau_{+1,\theta}(\eta) + 2l_-\tau_{-1,\theta}(\eta)$ is strictly decreasing by Lemma (iii). So for $\theta \in [0,1]$ there is a unique $\tilde{\epsilon}_2(\theta)$ such that

$$2l_+\tau_{+1,\theta}(\eta_1(\theta) + \tilde{\epsilon}_2(\theta)) + 2l_-\tau_{-1,\theta}(\eta_1(\theta) + \tilde{\epsilon}_2(\theta)) = \frac{T_\theta^{(n+1)}(\eta_1(\theta)) + 1}{2},$$

and $\theta \mapsto \tilde{\epsilon}_2(\theta)$ is continuous.

Now define

$$\begin{aligned} C(\theta) &:= \max\left\{ |u'_\eta(\tau_{+1,\theta}(\eta), \eta, f_\theta)|, |u'_\eta(\tau_{-1,\theta}(-\eta), \eta, f_\theta)| : \eta \in \left[\eta_1(\theta), \tfrac{3}{2}\eta_1(\theta)\right] \right\} \\ &> 0 \\ \tilde{\tilde{\epsilon}}_2(\theta) &:= \max\left\{ 0 < \epsilon \le \frac{1}{2}\eta_1(\theta) : \psi_{+1}(\eta_1(\theta) + \epsilon, \theta) \le \frac{\eta_1(\theta)}{2C(\theta)}, \right. \\ &\qquad \left. \psi_{-1}(\eta_1(\theta) + \epsilon, \theta) \le \frac{\eta_1(\theta)}{2C(\theta)} \right\}. \end{aligned}$$

Then $\tilde{\tilde{\epsilon}}_2$ is continuous because $\theta \mapsto \frac{\eta_1(\theta)}{2C(\theta)}$ is continuous, and $\frac{d}{d\eta}\psi_{\pm 1}(\cdot, \theta) > 0$. By Lemma 3.15 (iv) we have $0 < u_\eta(\tau_{1,\theta}(\eta_1(\theta)), \eta, f_\theta)$ for all $\eta \ge \eta_1(\theta)$, and for $\eta_1(\theta) \le \eta \le \eta_1(\theta) + \tilde{\tilde{\epsilon}}_2(\theta)$

$$\begin{aligned} u_\eta(\tau_{1,\theta}(\eta), \eta, f_\theta) &= \frac{\eta - u'_\eta(\tau_{1,\theta}(\eta), \eta, f_\theta) \cdot \psi_{+1}(\eta, \theta)}{f_\theta(M^+(\theta))} \\ &\le \frac{\frac{3}{2}\eta_1(\theta) + C(\theta) + \frac{\eta_1(\theta)}{2C(\theta)}}{f_\theta(M^+(\theta))} = \frac{2\eta_1(\theta)}{f_\theta(M^+(\theta))}. \end{aligned}$$

Similarly for $\eta_1(\theta) \le \eta \le \eta_1(\theta) + \tilde{\tilde{\epsilon}}_2(\theta)$

$$-u_\eta(\tau_{-1,\theta}(\eta_1(\theta), -\eta_1(\theta), f_\theta) \le \frac{2\eta_1(\theta)}{-f_\theta(-M^-(\theta))},$$

so taking $\epsilon_2 := \min\{\tilde{\epsilon}_2, \tilde{\tilde{\epsilon}}_2\}$ (3.31),(3.32) hold.

Further define

$$\begin{aligned} d_{+2}(\theta) &:= \frac{f_\theta(M^+(\theta))}{3} > 0, \qquad d_{-2}(\theta) := \frac{-f_\theta(-M^-(\theta))}{3} > 0 \\ \eta_{33}(\theta) &:= \min\left\{ \frac{\epsilon_1(\theta) \cdot f_\theta(M^+(\theta))}{16}, \epsilon_2(\theta), \frac{-\epsilon_1(\theta) f_\theta(-M^-(\theta))}{16} \right\} > 0. \end{aligned} \tag{3.34}$$

Now define bridge functions $g_{+\theta}(\cdot, a_+)$, $g_{-\theta}(\cdot, a_-)$ as in section 3.3.2 with parameters $d_{+2}(\theta), d_{-2}(\theta)$ as above, $d_{+1}(\theta) := f_\theta(M^+(\theta))$, $d_{-1}(\theta) := -f_\theta(-M^-(\theta))$, $c_{+1}(\theta) := f'_\theta(M^+(\theta))$, $c_{+2}(\theta) := f''_\theta(M^+(\theta))$, $c_{-1}(\theta) := -f'_\theta(-M^-(\theta))$, $c_{-2}(\theta) := -f''_\theta(-M^-(\theta))$. Also define $\eta_{\pm 22}(a_\pm), \tau_{\pm 22}(\cdot, a_\pm), \psi_{\pm 22}(\cdot, a_\pm)$ as in section 3.3.2. With this we can define

$$\tilde{f}_\theta(x) := \begin{cases} f_\theta(x) & x \in [-M^-(\theta), M^+(\theta)] \\ g_{+\theta}(x - M^+(\theta)) & x \in [M^+(\theta), M^+(\theta) + a_+] \\ -g_{-\theta}(-x - M^-(\theta)) & x \in [-M^-(\theta) - a_-, -M^-(\theta)] \\ d_{+2}(\theta) & x \geq M^+(\theta) + a_+ \\ -d_{-2}(\theta) & x \leq -M^-(\theta) - a_-, \end{cases}$$

where $a_+ \leq a_+^0(\theta)$, $a_- \leq a_-^0(\theta)$ and $a_\pm^0(\theta)$ are defined as in (3.25). By construction of $g_{\pm\theta}$ the mapping $\theta \mapsto \tilde{f}_\theta \in \mathcal{C}_s^2$ is continuous if we choose $a_\pm$ depending continuously on θ — we will do this in (3.35), (3.36).

Defining $G_{\pm\theta}(a) := \int_0^a g_{\pm\theta}(t, a_\pm)\, dt$ we get from (3.26)

$$\begin{aligned} \eta_{+22}(\theta, a_+) &= \sqrt{2G_{+\theta}(a_+)} \\ \eta_{-22}(\theta, a_-) &= \sqrt{2G_{-\theta}(a_-)} \\ \frac{d}{da}\eta_{+22}(\theta, a_+) > 0, &\qquad \frac{d}{da}\eta_{-22}(\theta, a_-) > 0 \\ a_+ &= \max u(\cdot, \eta_{+22}(\theta, a_+), g_{+\theta}) \\ a_- &= \max u(\cdot, \eta_{-22}(\theta, a_-), g_{-\theta}). \end{aligned}$$

We calculate (Lemma 3.15 (ii))

$$\begin{aligned} \eta_{\pm 21}(\theta, a_\pm) &:= (\psi_{+1}(\cdot, \theta)^{-1})(\eta_{\pm 22}(\theta, a_\pm)) \\ &= \sqrt{\eta_1(\theta)^2 + \eta_{\pm 22}(\theta, a_\pm)^2}, \\ \eta_{\pm 32}(\theta, a_\pm) &:= (\psi_{\pm 22}(\cdot, \theta, a_\pm)^{-1})(\eta_{33}(\theta)) \\ &= \sqrt{\eta_{33}(\theta)^2 + \eta_{\pm 22}(\theta, a_\pm)^2}, \\ \eta_{\pm 31}(\theta, a_\pm) &:= (\psi_{+1}(\cdot, \theta)^{-1})(\eta_{\pm 32}(\theta, a_\pm)) \\ &= \sqrt{\eta_1(\theta)^2 + \eta_{33}(\theta)^2 + \eta_{\pm 22}^2(\theta, a_\pm)}. \end{aligned}$$

With this notation we fix our final $a_\pm(\theta)$ in two steps. First define

$$\begin{aligned} a^1_+(\theta) := \max\Bigg\{ & 0 < a \le a_+(\theta) : \frac{2\eta_{+22}(\theta,a)}{d_{+2}(\theta)} \le \frac{\epsilon_1(\theta)}{4}, \\ & \frac{1}{f_\theta(M^+(\theta))} \ge \frac{a}{\eta_{33}(\theta)^2} + \frac{2d_{+1}(\theta)a^2\|\sigma'\|_\infty}{\eta_{33}^4} \exp\left(\frac{4ad_{+1}(\theta)\|\sigma'\|_\infty}{\eta_{33}^2}\right), \\ & \eta_{+31}(\theta,a) \le \eta_1(\theta) + \epsilon_2(\theta)\Bigg\}, \end{aligned} \tag{3.35}$$

and $a^1_-(\theta)$ analogously. By (3.26) $a^1_\pm$ are continuous. Now

$$\begin{aligned} a_+(\theta) &:= \max\{0 < a \le a^1_+(\theta) : \eta_{+31}(\theta,a) \le \eta_{-31}(\theta, a^1_-(\theta))\} \\ a_-(\theta) &:= \max\{0 < a \le a^1_-(\theta) : \eta_{-31}(\theta,a) \le \eta_{+31}(\theta, a^1_+(\theta))\}. \end{aligned} \tag{3.36}$$

With these values fixed we suppress the dependence on $a_\pm$ in the remainder of the proof. Again by (3.26) $\theta \mapsto a_\pm(\theta)$ are continuous. By (3.36) we have

$$\eta_{31}(\theta) := \eta_{+31}(\theta) = -\eta_{-31}(\theta) \text{ for all } \theta \in [0,1]$$

and

$$\eta_{\pm 21}(\theta) \le \eta_{31}(\theta) \le \eta_1(\theta) + \epsilon_2(\theta).$$

By Proposition 3.9 it remains to show the following

Claim 1: For all $\theta \in [0,1]$ $\tilde{T}_\theta$ satisfies

a) $\tilde{T}_\theta(\eta) = T_\theta(\eta)$ for $\eta \in [-\eta_1(\theta), \eta_1(\theta)]$.

b) $\tilde{T}^{(n)}_\theta(\eta) < 1 < \tilde{T}^{(m+1)}_\theta(\eta)$ for $\eta \in [\eta_1(\theta), \eta_{31}(\theta)] \cup [-\eta_{31}(\theta), -\eta_1(\theta)]$.

c) $\tilde{T}'_\theta(\eta) > 0 > \tilde{T}'_\theta(-\eta)$ for $\eta > \eta_{31}(\theta)$.

Proof a) is clear. We prove b), c) w.l.o.g. only in the case $\eta > 0$. First let $\eta \in [\eta_1(\theta), \eta_{+21}(\theta)]$. Let $T_{\pm a,\theta}$ be the T-map induced by the bridge function $g_{\pm\theta}$.

We have

$$\begin{aligned}
\tilde{T}_\theta^{(n)}(\eta) =& 2k_+ \underbrace{\tau_{+1,\theta}(\eta)}_{\le \tau_{+1,\theta}(\eta_1(\theta))} + k_- \underbrace{2\tau_{-1,\theta}(\eta)}_{\le T_\theta(-\eta_1(\theta))} \\
&+ k_+ \underbrace{T_{a+,\theta}(\underbrace{\psi_{+1}(\eta,\theta)}_{\le \eta_{+22}(\theta)})}_{\overset{\text{Lemma 3.15}}{\le} \frac{2\eta_{+22}(\theta)}{d_{+2}(\theta)} \overset{(3.35)}{\le} \frac{\epsilon_1(\theta)}{4}} + k_- \underbrace{T_{a_-,\theta}(\psi_{-1}(\eta,\theta))}_{\le \frac{\epsilon_1(\theta)}{4}} \\
\le& k_+ T_\theta(\eta_1(\theta)) + k_- T(-\eta_1(0)) + \frac{n}{4}\epsilon_1(\theta) \\
=& T_\theta^{(n)}(\eta_1(\theta)) + \frac{n}{4}\epsilon_1(\theta) \overset{(3.30)}{<} 1.
\end{aligned}$$

and as $\eta_{+21}(\theta) \le \eta_{31}(\theta) \overset{(3.35)}{\le} \eta_1(\theta) + \epsilon_2(\theta)$ by (3.31) we have

$$\tilde{T}_\theta^{(m+1)}(\eta) \ge 2l_+\tau_{+1,\theta}(\eta) + 2l_-\tau_{-1,\theta}(\eta) > 1$$

Now let $\eta \in [\eta_{+21}(\theta), \eta_{31}(\theta)]$ and $T^c_{d_{\pm 2}(\theta)}$ be the T-map related to the constant right-hand side $d_{\pm 2}(\theta)$ – by Lemma 3.16 $T^c_{d_{\pm 2}(\theta)}(\eta) = \frac{2|\eta|}{d_{\pm 2}(\theta)}$. We calculate

$$\begin{aligned}
\tilde{T}_\theta^{(n)}(\eta) =& \underbrace{2k_+\tau_{+1,\theta}(\eta) + 2k_-\tau_{-1,\theta}(\eta)}_{\le T_\theta^{(n)}(\eta_1(\theta))} \\
&+ k_+ 2\underbrace{\tau_{+22,\theta}(\psi_{+1,\theta}(\eta))}_{\le \tau_{+22,\theta}(\eta_{+22}(\theta))} + k_- \underbrace{2\tau_{-22,\theta}(\psi_{-1,\theta}(\eta))}_{\le T_{a_-,\theta}(\eta_{-22}(\theta)) \le \frac{\epsilon_1(\theta)}{4}} \\
&+ k_+ T^c_{d_{+2}(\theta)}(\underbrace{\psi_{+21,\theta}(\eta)}_{\le \eta_{33}(\theta)}) + k_- \underbrace{T^c_{d_{-2}(\theta)}(\psi_{-21,\theta}(\eta))}_{\le \frac{2\eta_{33}(\theta)}{d_{-2}(\theta)}} \\
\le& T_\theta^{(n)}(\eta_1(\theta)) + \frac{n}{4}\epsilon_1(\theta) + k_+\frac{2\eta_{33}(\theta)}{d_{+2}(\theta)} + k_-\frac{2\eta_{33}(\theta)}{d_{-2}(\theta)} \\
\overset{(3.34)}{\le}& T_\theta^{(n)}(\eta_1(\theta)) + \frac{n}{4}\epsilon_1(\theta) + \frac{3n}{8}\epsilon_1(\theta) \overset{(3.30)}{<} 1.
\end{aligned}$$

As above we also have $\tilde{T}_\theta(\eta)^{(m+1)} > 1$, so we have proved b). Finally let $\eta >$

$\eta_{31}(\theta)$. We get

$$
\begin{aligned}
\tilde{T}'_\theta(\eta) &= 2\tau'_{+1,\theta}(\eta) + 2\tau'_{+22}(\psi_{+1,\theta}(\eta))\psi'_{+1,\theta}(\eta)\\
&\quad + \underbrace{{T^c_{d_{+2}(\theta)}}'(\psi_{+21,\theta}(\eta))}_{=\frac{2}{d_{+2}(\theta)}} \cdot \underbrace{\psi'_{+21,\theta}(\eta)}_{=\psi'_{+22,\theta}(\psi_{+1,\theta}(\eta))\psi'_{+1,\theta}(\eta)}\\
&\overset{3.15\text{ (iii)}}{=} \psi'_{+1,\theta}(\eta)\left[\underbrace{\frac{-2u_\eta(\tau_{+1,\theta}(\eta),\eta,f_\theta)}{\eta}}_{\overset{(3.32)}{>} -\frac{4\eta_1(\theta)}{f_\theta(M^+(\theta))\eta_1(\theta)}} + \underbrace{2\tau'_{+22,\theta}(\psi_{+1,\theta}(\eta))}_{\overset{(3.29),\ (3.35)}{>} \frac{-2}{f_\theta(M^+(\theta))}}\right.\\
&\qquad\qquad \left. + \frac{2}{d_{+2}(\theta)}\underbrace{\psi'_{+22,\theta}(\psi_{1,\theta}(\eta))}_{\overset{\text{Lem 3.15 (ii)}}{\geq} 1}\right]\\
&> 0.
\end{aligned}
$$

This proves the claim and concludes the proof of the Proposition.

3.4 Local attractors and linearization

First we note that we can linearize φ locally at nondegenerate equilibria. This is the Hartman-Grobman Theorem, proved for scalar parabolic PDE in [Lu91]:

Theorem 3.17 (Lu)**.** *If $v \in E$ is hyperbolic, then there exist neighborhoods V of v and U of 0 in X, and a homeomorphism $\Phi : V \to U$ such that if $u(t,x)$ is a solution of* (P) *and $u(t,\cdot) \in V$, then $\Phi(u(t,x))$ is a solution of the linear equation*

$$w_t = L_v w \qquad w(0) = w(1) = 0. \tag{3.37}$$

If $w(t,x)$ is a solution of (3.37) *and $w(t,\cdot) \in U$, then $\Phi^{-1}(w(t,x))$ is a solution of* (P)*.*

Definition 3.18. a) Let $f \in \mathcal{C}^2$, $v, w \in E$. If there is $u \in X$ such that $\alpha(u) = v$ and $\omega(u) = w$ then $\gamma(u)$ is called a *connecting orbit* from v to w and we say that v connects to w w.r.t. φ, $v \searrow^\varphi w$ (and we often omit φ). The union of all connecting orbits from v to w is the *connecting set* $\mathcal{C}_\varphi(v,w)$ (which of course may be empty if $v \not\searrow^\varphi w$). In particular $v \searrow^\varphi v$ and $C(v,v) = \{v\}$.

b) If $v_0 \searrow^\varphi v_1 \searrow^\varphi \dots \searrow^\varphi v_k$ then we call $(v_0,\dots,v_k)$ a *connecting chain* from v_0 to v_k.

In working with connecting orbits the following Lemma is fundamental:

Lemma 3.19 (Global λ-Lemma)**.** *Let $f \in \mathcal{C}^2$, $v \in E$ hyperbolic. Let N be an φ-invariant submanifold of X having a point q of transversal intersection with $W^s(v)$. Let B^u be an embedded open disc in $W^u(v)$ centered at v. Then, given $\epsilon > 0$, there exists a submanifold of N ϵ-close to B^u in the $\mathcal{C}^1$-sense.*

Proof. This is Proposition 6.2.10 in [HMO02]. It is formulated there in the general context of dissipative flows, but only the smoothness of the time-one-map is used in the proof to apply a local-λ-Lemma for smooth maps (Proposition 6.1.6 in [HMO02]). □

Now we define a useful short-hand term:

Definition 3.20. Let $v \in E$ be hyperbolic, U a neighborhood of v. A set $S := \partial U(v) \cap W^u(v)$ is a *fundamental domain* for v if all orbits in $W^u(v)$ have precisely one point of intersection with S, and the Hartman-Grobman Theorem can be applied on $U(v)$. Clearly there always exists a fundamental domain.

Proposition 3.21. *Let $f \in \mathcal{C}^2$, $v, w, \bar{v} \in E$ be hyperbolic. Then*
a) *$C(v,w)$ is an $(i(v) - i(w))$-dimensional $\mathcal{C}^1$-submanifold of X.*

b) *Let $v \searrow^{\varphi} \bar{v} \searrow^{\varphi} w$. Then*

$$\forall u_1 \in C(v,\bar{v}) \forall u_2 \in C(\bar{v},w) \forall \epsilon > 0 \exists u_0 \in C(v,w) : \gamma(u_0) \subset U_\epsilon(\gamma(u_1) \cup \gamma(u_2)).$$

In particular $v \searrow^{\varphi} w$.

c) *The sets $\partial W^u(v)$, $\overline{W^u(v)}$ are φ-invariant. If $\overline{C(v,w)}$ is hyperbolic then*

$$\partial C(v,w) = \bigcup_{\substack{i=1,2,\, v_i \in E \\ v \searrow^{\varphi} v_i \searrow^{\varphi} w}} C(v_1, v_2).$$

Proof. This is proved for $f \in \mathcal{G}_d$ in Lemma 3.4 in [BF89]. We will give a detailed proof for our case, as φ may have degenerate equilibria and orbits maybe unbounded.

a) By definition $C(v,w) = W^u(v) \cap W^s(w)$, this intersection is transversal by [Hen85, Theorem 7]. By Proposition 2.3 c) $W^u(v)$ is an $i(v)$-dimensional submanifold of X and $W^s(w)$ is an $i(w)$-codimensional submanifold of X, so the assertion follows.

b) Let $u_1 \in C(v,\bar{v})$, $u_2 \in C(\bar{v},w)$, $n := i(\bar{v})$, $\epsilon > 0$. Choose open neighborhoods $U(\bar{v}) \subset U_\epsilon(\bar{v})$, $U(v) \subset U_\epsilon(v)$, $U(w) \subset U_\epsilon(w)$ such that φ can be linearized on these sets as in Theorem 3.17 and that trajectories leaving one of these neighborhoods never return to it. Let B be an open disc in $W^u(\bar{v}) \cap U(v)$, we can assume without loss that $u_1 \in U(\bar{v})$ and $u_2 \in B$. Let $N_0 \subset W^u(v) \cap U(\bar{v})$ be an n-dimensional open disc centered at u_1, $\overline{N_0} \subset W^u(v) \cap U(\bar{v})$, nowhere tangent to φ and $W^s(\bar{v}) \pitchfork N_0$, $W^s(\bar{v}) \cap N_0 = \{u_1\}$. Then $N := \gamma(N_0)$ is a

$(n+1)$-dimensional submanifold of $W^u(v)$, and by construction it is φ-invariant and transversal to $W^s(\bar{v})$. Note that $W^s(w) \pitchfork W^u(\bar{v})$ (so $W^s(w) \pitchfork B$).

Now the global λ-Lemma assures the existence of a sequence of n-dimensional submanifolds B_k of N converging to B in the $\mathcal{C}^1$-topology, i.e. there are $\psi_k : B \to B_k \subset N$ $\mathcal{C}^1$ with $\|\psi_k - \mathrm{id}\|_{\mathcal{C}^1} \to 0$. Now we apply Lemma A.2 with $M_1 = B \cap U_{\delta_k}(u_2)$, $\delta_k \to 0$, $M_2 = W^s(w)$, $\psi = \psi_k|_{M_1}$ and see that for k large enough $B_k \cap W^s(w) \neq \emptyset$, so there is a sequence of points $w_k \in B_k \cap W^s(w) \subset C(v,w)$ with $w_k \to u_2$. In particular $C(v,w) \neq \emptyset$.

Let $\tilde{w}_k := \gamma(w_k) \cap N_0$, without loss $\tilde{w}_k \to \tilde{w}_0 \in \overline{N_0} \subset W^u(v)$. Clearly $\mathbf{dist}(\bar{v}, \gamma(\tilde{w}_k)) = \mathbf{dist}(\bar{v}, \gamma(w_k)) \to 0$, so there are $t_k \to \infty$ such that $\varphi^{t_k}(\tilde{w}_k) \to \bar{v}$. Suppose $\tilde{w}_0 \notin W^s(\bar{v})$. Then by the linearization on $U(\bar{v})$ there exists a $t_0 > 0$ such that $J(\varphi^{t_0}(\tilde{w}_0)) < J(\bar{v})$, so for k large $J(\varphi^{t_0}(\tilde{w}_k)) < J(\bar{v})$, which contradicts $\varphi^{t_k}(w_k) \to \bar{v}$. By construction of N we have $\tilde{w}_0 \in \gamma(u_1)$, so $\tilde{w}_0 = u_1$.

Choose $t^- < 0 < t^+$ such that $\varphi^{t^-}(u_1) \in U(v)$, $\varphi^{t^+}(u_2) \in U(w)$. Now $\varphi(\cdot, \tilde{w}_k) \to \varphi(\cdot, u_1)$, $\varphi(\cdot, w_k) \to \varphi(\cdot, u_2)$ uniformly on $[t^-, 0]$, $[0, t^+]$ respectively. So for k large enough $\varphi^{t^-}(\tilde{w}_k) \in U(v)$, $\varphi^{t^+}(w_k) \in U(w)$ and $\varphi^{[t^-,0]}(\tilde{w}_k) \subset U_\epsilon(\gamma(u_1))$, $\varphi^{[0,t^+]}(w_k) \subset U_\epsilon(\gamma(u_2))$. As $U(v) \cup U(\bar{v}) \cup U(w) \subset U_\epsilon(\gamma(u_1) \cup \gamma(u_2))$ the assertion follows.

c) Now let $\overline{C(v,w)}$ be hyperbolic. "$\supset$": Let $v_i \in E$, $v \searrow^\varphi v_i \searrow^\varphi w$, $i = 1,2$ and $u \in C(v_1, v_2)$. Then $v_i \in \overline{W^u(v)}$, so they are hyperbolic. By b) $C(v, v_i) \cup \{v_i\} \cup C(v_i, w) \subset \partial C(v,w)$. Another application of b) yields $C(v_1, v_2) \subset \partial C(v_1, w) \subset \partial C(v,w)$, in particular $u \in C(v,w)$.
"$\subset$": Let $u \in \partial C(v,w)$. First suppose $u \in E$, then it is hyperbolic by the hypothesis. Choose $\epsilon > 0$ such that $U_\epsilon(u) \cap E = \{u\}$, let $u_k \in C(v,w) \cap U_\epsilon(u)$ with $u_k \to u$. Let $t^- := \inf\{t > 0 : \|\varphi^{-t}(u_k) - u\| = \epsilon\}$, $t_k^+ := \inf\{t > 0 : \|\varphi^t(u_k) - u\| = \epsilon\}$, $u_k^- = \varphi^{-t_k^-}(u_k)$, $u_k^+ := \varphi^{t_k^+}(u_k)$. The set $\{\tilde{u} \in \overline{C(v,w)}, \|\tilde{u} - v\| = \epsilon\}$ is compact, so w.l.o.g. $u_k^\pm \to u^\pm \in \overline{C(v,w)}$, $\|u^\pm - u\| = \epsilon$. By the φ-invariance of $\overline{C(v,w)}$ and the hyperbolicity assumption we have $\alpha(u^\pm), \omega(u^\pm) \in \overline{W^u(v)}$, and by construction $\alpha(u^+) = \omega(u^-) = u$. Proceeding inductively we get a connecting chain from v to w containing u. By b) $v \searrow^\varphi u \searrow^\varphi w$.

Now let $u \notin E$, then $v_1 := \alpha(u), v_2 := \omega(u) \in \partial C(v,w)$ are hyperbolic and $v \searrow^\varphi v_i \searrow^\varphi w$. □

An important technical property of local attractors needed below is their upper semicontinuity w.r.t. to a parameter:

Lemma 3.22. *Let $I_0 \subset [0,\infty)$, $\nu_0 \in I_0$. For each $\nu \in I_0$ let $f_\nu \in \tilde{\mathcal{G}}_d$, $\nu \mapsto f_\nu \in \mathcal{C}^2_w$ continuous. Suppose $\mathcal{A}_0$ is a local attractor w.r.t. φ_{ν_0}, then there is a $h > 0$ and $\mathcal{A}_0 \subset N_1 \subset X$ open such that for $|\nu - \nu_0| \leq h$ the semiflow φ_ν has a compact local attractor $\mathcal{A}_\nu$ which attracts N_1. Moreover $\delta(\mathcal{A}_\nu, \mathcal{A}_{\nu_0}) \to 0$ as $\nu \to \nu_0$.*

The proof is rather technical and stated in section A.3.

Theorem 3.23. *Let $f \in \tilde{G}_d$, $\mathcal{A}_0$ be a hyperbolic local attractor, $v \in E_0 := E \cap \mathcal{A}_0$, $n := \max\{i(w) : w \in E_0\}$. Then there is an open neighborhood U of $P_1^n(v)\mathcal{A}_0$ in $X_1^n(v)$ and a $\mathcal{C}^1$-function $h : U \to X^n(v)$ such that $\mathcal{A}_0 \subset \operatorname{graph}(h) =: \mathcal{N}$, and $\mathcal{N}$ is positively and locally negatively invariant.*

Proof. The assertion has been proved by Brunovský ([Bru90]) for the hyperbolic global attractor of a dissipative flow. This proof can be modified in a straightforward way to prove Theorem 3.23. We will not repeat the whole proof here but list the main steps and how the arguments carry over to our situation.

Step 1 First ([Bru90, §3 up to Proposition 3.3]) $\mathcal{A}_0$ is shown to be the graph of a function h by showing that $z(v_1 - v_2) < n$ for all $v_1, v_2 \in \mathcal{A}_0$, using the general fact from Sturm-Liouville theory that $z(v - w) \geq n$ if $v - w \in X^n \setminus \{0\}$. In [Bru90, Lemma 3.1] it is shown that $z(v_1 - v_2) < \max\{i(v) : v \in E\}$ for $f \in \mathcal{G}_d$. We can prove in an identical way that $z(v_1 - v_2) < n$ for $v_1, v_2 \in E_0$, but only for $f \in \mathcal{G}_d$. For $f \in \tilde{G}_d$ we can choose $f_k \in \mathcal{G}_d$ with $f_k \to f$ in $\mathcal{C}_w^2$ by Proposition 2.15. By Lemma 3.22 for k large enough φ_k admits a local attractor $\mathcal{A}_k$ such that $\delta(\mathcal{A}_k, \mathcal{A}_0) \to 0$. Let $E_k := \{u \in \mathcal{A}_k : u \text{ is an equilibrium of } \varphi_k\}$, then $\max\{i_k(v) : v \in E_k\} \leq n$ for almost all k. If not then w.l.o.g.

$$\forall k \in \mathbb{N}\, \exists v_k \in E_k : i_k(v_k) > n.$$

We can assume $v_k \to v_0 \in \mathcal{A}_0$, and for $t \in \mathbb{R}$ we have $\varphi^t(v_0) = \lim_{k\to\infty} \varphi_k^t(v_k) = v_0$, so $v_0 \in E_0$. Now $i(v_0) \leq n$, and by the phase-plane analysis (Propositions 3.10 a) and 3.12) we get $i_k(v_k) \leq n$. So choosing $v_1, v_2 \in E_0$ and $v_1^k, v_2^k \in E_k$ with $v_i^k \to v_i$ in $\mathcal{C}^1$ (by the implicit function theorem) analogously to [Bru90, Lemma 3.1] we get $z(v_1^k - v_2^k) < n$. By Proposition 3.2 d), b) $v_1 - v_2$ has only simple zeros, so $z(v_1 - v_2) = \lim_{k\to\infty} z(v_1^k - v_2^k) < n$.

The rest of this step ([Bru90, Lemma 3.2, Proposition 3.3]) can be proved exactly as in [Bru90].

Step 2 Now the map h is shown to be Lipschitz continuous ([Bru90, Lemma 3.5, Corollary 3.6]). To get this the existence of a $\mathcal{C}^1$ inertial manifold is used. This existence is proved by Chow and Lu ([CL88]) for $f \in \tilde{\mathcal{G}}_d$, i.e. without any hyperbolicity assumption. The following indirect proof of existence of a Lipschitz constant then can be repeated without any changes in our case.

Step 3 Finally (pp 308-312) the map h is extended to be $\mathcal{C}^1$ on a neighborhood of $P_1^n(v)\mathcal{A}$. This extension procedure is done in a finite-dimensional setting by the inertial manifold. It is done inductively, starting with the sources and working downwards to the sinks. As mentioned the finite dimensional reduction works without hyperbolicity assumption. The induction can be started at "sources w.r.t. $\mathcal{A}_0$", and as $E \setminus E_0$ has to be bounded away from $\mathcal{A}_0$ (otherwise they would accumulate in the compact global attractor of φ, and thus there was a degenerate equilibrium in $\mathcal{A}_0$) it can proceed to the sinks within E_0 without change of the proof.

□

Corollary 3.24. *Let $\mathcal{A}_0$ be a hyperbolic local attractor, $\mathcal{N}$ as stated in Theorem 3.23, $v \in E \cap \mathcal{A}_0$, $n = \dim(\mathcal{A}_0)$. Then $\varphi|_{\mathcal{N}}$ is locally conjugate at v to the linear flow induced by*

$$u_t = L_1^n(v), \quad u(0) = u(1) = 0$$

on $X_1^n(v)$ with $L_1^n(v) \in \mathcal{L}(X_1^n(v))$ being the restriction of L_v.

Proof. By Theorem 3.23 $\varphi|_{\mathcal{N}}$ is conjugate to it's projection onto $X_1^n(v)$. But this finite-dimensional flow can be linearized at hyperbolic equilibria as stated by the Hartman-Grobman Theorem ([Pug69]). □

CHAPTER 4

Applications of cut-off

4.1 Connecting orbits

Brunovský and Fiedler ([BF88], [BF89]) gave a complete description of the connecting orbit structure in the dissipative case. The following assertion is Theorem 1.3 in [BF89]:

Theorem 4.1 (Brunovský-Fiedler)**.** *Let $f \in \mathcal{G}_d$, then a given $v \in E$ connects precisely to those $w \in E$ for which $i(w) < i(v)$ and for which there is no $\bar{w}$ with $\bar{w}'(0)$ between $v'(0)$ and $w'(0)$ satisfying $z(v - \bar{w}) \leq z(w - \bar{w})$.*

We will be able to show the following result:

Theorem 4.2. *The conclusions of Theorem 4.1 are true for $f \in \mathcal{G}$. For $f \in \mathcal{F}$ and $v, w \in E$ this means*

$$v \searrow^{\varphi} w \iff i(v) > i(w).$$

Proof. We start with the case $f \in \mathcal{F}$. Let $v, w \in E$, $N := \max\{i(v), i(w)\}$. We want to apply Proposition 3.14 with $f_\theta \equiv f$, $n = m > i(0)$ and a constant $\eta_1 = \eta_1(\theta)$. Now either $z(u_n)$ or $z(u_{n+1})$ is odd, so by Remark 3.8 we have either $u_n'(0) = -u_{-n}'(0)$ or $u_{n+1}'(0) = -u_{-(n+1)}'(0)$, consequently we can choose

$$\eta_1 \in \Big(\max\{u_n'(0), -u_{-n}'(0)\},\ \min\{u_{n+1}'(0), -u_{-(n+1)}'(0)\}\Big).$$

By Proposition 3.10 c) $T'(\eta) < 0 < T'(-\eta)$ for all $\eta > 0$, so $T^{(n)\prime}(\eta), T^{(n+1)\prime}(\eta) < 0 < T^{(n)\prime}(-\eta), T^{(n+1)\prime}(-\eta)$, so the conditions (3.11) are satisfied. This gives us a sequence of $f_n \in \mathcal{G}_d$ and associated hyperbolic parabolic semiflows φ_n on X. The set of equilibria of φ_n is

$$E_n = \{u_k \in E : |k| \leq n\} \cup \{\tilde{u}_{k,n} : 1 \leq |k| \leq n\}$$

where $\tilde{u}_{k,n}$ is nondegenerate, $i(\tilde{u}_{k,n}) = z(\tilde{u}_{k,n}) = |k|$. By Proposition 3.14 we have $v, w \in E_n$ for all $n \geq N$. We also fix for each $n \in \mathbb{N}$ a maximal $M(n)$ such that $\varphi_n \equiv \varphi$ on $\{u \in X : \|u\|_\infty < M(n)\}$, clearly $M(n) \xrightarrow{n \to \infty} \infty$.

Define $\alpha_n, \omega_n, W^u_{\mathrm{loc},n}, J_n, T_n$ for φ_n / f_n as for φ / f. Let $\mathcal{A}_n$ be the global attractor of φ_n, then $\mathcal{A}_n$ is a n-dimensional graph (cf. [Bru90]), so $\varphi_n|_{\mathcal{A}_n}$ is a finite dimensional flow. For $B \subset \mathcal{A}_n$ let $\mathbf{int}_n\, B$ and $\partial_n B$ denote the interior and the boundary of B relative to $\mathcal{A}_n$ respectively.

By Lemma 2.10 b) $T^{\max}(u_0) = \infty$ if $J(\gamma^+(u_0))$ is bounded from below, and by Lemma 2.10 a) this implies that in this case $\|\varphi^t(u_0)\|$ is uniformly bounded for all $t \geq 0$.

Suppose first $i(v) > i(w)$, then we have

$$\forall n \geq N : v \searrow^{\varphi_n} w.$$

This follows from Theorem 4.1 and will be proved separately in Corollary 4.3. We will show that this already implies $v \searrow^{\varphi} w$.

Now choose $N' \geq N$ such that $M(N') > \|v\|_\infty$, then there exists $\epsilon > 0$ such that on $B := B_\epsilon(v) \cap \mathcal{A}_{N'}$ all φ_n coincide for $n \geq N'$ by the embedding $X \hookrightarrow C^0$.

Thus exist $u_n \in \partial_{N'} B$ with $\alpha_n(u_n) = v$, $\omega_n(u_n) = w$, and $\|\varphi_n^t(u_n)\|_\infty \leq M(N')$ for $t < 0$. $\partial_{N'} B$ is compact, so w.l.o.g. there exists $u_0 \in \partial_N B$ such that $u_n \to u_0$. We have $\varphi^t(u_n) \to \varphi^t(u_0)$ uniformly on $[-R, 0]$ for any $R > 0$, so $\varphi^t(u_0) \to v$ as $t \to -\infty$.

Claim 1: $J(\varphi^t(u_0)) \geq J(w)$ for all $t \in (-\infty, T^{\max}(u_0))$.

Proof Suppose $\exists\, t_0 \in (0, T^{\max}(u_0)) : J(\varphi^{t_0}(u_0)) < J(w)$. Then $\exists K > 0$ with $\|\varphi^t(u_0)\|_\infty < K$ for all $0 \leq t \leq t_0$. Choose $N_K \in \mathbb{N}$ with $M(N_K) \geq K$, so that

$$\forall n \geq N_K \forall 0 \leq t \leq t_0 : \varphi_n^t(u_0) = \varphi^t(u_0).$$

Define $t_{0,n} := \inf\{t > 0 : \varphi_n^t(u_n) \neq \varphi^t(u_n)\} \geq 0$ for $n \geq N_K$.

Now there exists $N_K' \geq N_K$ such that for all $n \geq N_K' : t_{0,n} > t_0$. To see this assume on the contrary that $\forall n \in \mathbb{N} : t_{0,n} \leq t_0$, w.l.o.g. $t_{0,n} \to t_{0,\infty} \in [0, t_0]$. By the continuous dependence Theorem we get $\varphi_n^{t_{0,n}}(u_n) = \varphi^{t_{0,n}}(u_n) \to \varphi^{t_{0,\infty}}(u_0)$, but $\|\varphi_n^{t_{0,n}}(u_n)\|_\infty \geq M(n) \to \infty$ by construction, so $\|\varphi^{t_{0,n}}(u_n)\| \to \infty$ which contradicts $t_{0,\infty} \leq t_0 < T^{\max}$.

So for $n \geq N_K'$ we have $\varphi_n^t(u_n) = \varphi^t(u_n)$ for all $0 \leq t \leq t_0$, again by continuous dependence $\varphi_n^{t_0}(u_n) \to \varphi^{t_0}(u_0)$, i.e. $J_n(\varphi_n^{t_0}(u_n)) = J(\varphi^{t_0}(u_n)) < J(w) = J_n(w)$ for n large enough. This contradicts $w = \omega_n(u_n)$, so $J(\varphi^t(u_0)) \geq J(w)$ for all $t \in [0, T^{\max}(u_0))$ and the claim is proved.

This implies $T^{\max}(u_0) = \infty$ and we get $\omega(u_0) = v_1 \in E$ with $J(v) > J(v_1) \geq J(w)$. If $v_1 = w$ the proof is complete, so assume $v_1 \neq w$.

Choose $N_1 \geq N'$ such that $\varphi_{N_1}^t(u_0) = \varphi^t(u_0)$ for all $t \in \mathbb{R}$, $\epsilon_1 > 0$ such that $\varphi_{N_1} \equiv \varphi$ on $\tilde{B}_1 = B_{\epsilon_1}(v_1) \cap \mathcal{A}_{N_1}$, and that by Corollary 3.24 $\varphi_{N_1}|_{\tilde{B}_1}$ is orbit equivalent to its linearization at v_1. Then by this linearization and the gradient structure of φ_{N_1} we can find a closed $B_1 \subset \tilde{B}_1$ such that any trajectory leaving B_1 never reenters B_1 and that $v_1 \in \mathbf{int}_{N_1}\, B_1$.

Now choose $t_0 > 0$ such that $\varphi^{t_0}(u_0) \in \mathrm{int}_{N_1} B_1$. As above by the continuous dependence $\varphi_n^t(u_n) \to \varphi^t(u_0)$ uniformly on $[0, t_0]$, more precisely for n large $\|\varphi_n^t(u_n)\|_\infty \le M(N_1)$ on $[0, t_0]$, so $\varphi_n^t(u_n) = \varphi^t(u_n)$ for all $t \in (-\infty, t_0]$ and $\varphi^{t_0}(u_n) \in \mathrm{int}_{N_1} B_1$. For such n we can define

$$t_n^{(1)} := \sup\{t > 0 : \varphi_n^t(u_n) \in B_1\} < \infty$$

and

$$u_n^{(1)} := \varphi^{t_n^{(1)}}(u_n) \in \partial_{N_1} B_1$$

(Clearly $\|\varphi^t(u_n^{(1)})\|_\infty \le M(N_1)$ for all $t < 0$ and $\varphi^t(u_n^{(1)}) \to v$ as $t \to -\infty$). W.l.o.g. $u_n^{(1)} \to u_0^{(1)} \in \partial_{N_1} B_1$.

Claim 2: $\varphi^t(u_0^{(1)}) \xrightarrow{t \to -\infty} v_1$

Proof By similar arguments as above we see that there exist $\tilde{t}_n > 0$ such that $\varphi^{\tilde{t}_n}(u_n) \to v_1$. For n large enough $\varphi^{\tilde{t}_n}(u_n) \in \mathrm{int}_{N_1} B_1$, so $s_n := \tilde{t}_n - t_n^{(1)} < 0$. If s_n was bounded then w.l.o.g. $s_n \to s_0 \le 0$, then $\varphi^{s_0}(u_0^{(1)}) = \lim_{n\to\infty} \varphi^{s_n}(u_n^{(1)}) = v_1$ which contradicts $v_1 \in E$, so s_n is unbounded.

We can assume $s_{n+1} \le s_n$, so by the definition of B_1 and by $\varphi^{s_n}(u_n^{(1)}) \in B_1$ for all n we have $\varphi^{s_n}(u_{n+k}^{(1)}) \in B_1$ for all n, k, thus $\varphi^{s_n}(u_0^{(1)}) = \lim_{k\to\infty} \varphi^{s_n}(u_k^{(1)}) \in B_1$ which implies $\varphi^t(u_0^{(1)}) \in B_1$ for all $t \le 0$, this proves the claim.

We can now inductively repeat the same arguments with $u_n^{(1)}, u_0^{(1)}$ instead of u_n, u_0 and find a sequence $v_1, v_2, \dots \in E$ with $J(v) > J(v_1) > J(v_2) > \dots$. As $\{\tilde{v} \in E : J(v) > J(\tilde{v}) \ge J(w)\}$ is finite there is a $k \in \mathbb{N}$ such that $v_k = w$. Similar as above $\varphi_n^t(u_n^{(k-1)})$ can be bounded uniformly on $[0, \infty)$ for large n. So we find that for n large enough $\|\varphi_n^t(u_n)\|_\infty \le M(N_k)$ for all $t \in \mathbb{R}$. In other words: For n large enough we have $\varphi^t(u_n) = \varphi_n^t(u_n)$ for any $t \in \mathbb{R}$.

Now assume $v \searrow^{\varphi} w$ and fix $u_0 \in X$ with $\alpha(u_0) = v$, $\omega(u_0) = w$. Let $C \ge \|\gamma(u_0)\|_\infty$ and $n \in \mathbb{N}$ such that $M(n) \ge C$. Then $\alpha_n(u_0) = v$, $\omega(u_0) = w$, that is $v \searrow^{\varphi_n} w$. But by Theorem 4.1 this is only possible if $i(v) > i(w)$. This completes the proof in the case $f \in \mathcal{F}$.

In the case $f \in \mathcal{G}$ the proof proceeds exactly as the proof of the case $f \in \mathcal{F}$, except that we do not cut off f above *every* pair of solutions.

All equilibria are nondegenerate, so we can write

$$E = \begin{cases} \{u_k : k \in \mathbb{Z} \setminus \{0\}\} & 0 \notin E \\ \{u_k : k \in \mathbb{Z}\} & 0 \in E \end{cases}$$

with $u'_{-(k+1)}(0) < u'_{-k}(0) < 0 < u'_k(0) < u'_{k+1}(0)$ for all $k \in \mathbb{N}$.

By the relation $T^{(n)} < T^{(n+1)}$ for all $n \in \mathbb{N}$ we have

$$\forall k \in \mathbb{Z} \setminus \{0,1\} : |z(u_k) - z(u_{k-1})| \in \{0,1\} \tag{4.1}$$

(cf. Corollary 3.12). By Theorem 3.6 we also have

$$z(u_k) \to \infty \text{ as } |k| \to \infty. \tag{4.2}$$

Define

$$\begin{aligned} n_1 &:= \min\{n \in \mathbb{N} : n \text{ odd }, \exists k : z(\tilde{u}_k) = n\} \\ k_n &:= \max\{k \in \mathbb{N} : z(u_k) = n_1 + 2(n-1)\} \text{ for } n \in \mathbb{N} \\ k_{-n} &:= -\max\{k \in \mathbb{N} : z(u_{-k}) = n_1 + 2(n-1)\} \text{ for } n \in \mathbb{N}. \end{aligned}$$

Then for $n \in \mathbb{N}$ $k_{n+1} \geq k_n + 2$, $z(u_{k_n}) = z(u_{k_{-n}})$ odd, and $u'_{k_n}(0) = -u'_{k_{-n}}(0)$. By construction $T^{(n_1+2(n-1))}(\eta) < 1$ for $\eta > u'_{k_n}(0)$, so choosing $\eta_1 > u'_{k_n}(0)$ close enough to $u'_{k_n}(0)$ this η_1 satisfies (3.11). Now apply Proposition 3.14 with $\tilde{n} = \tilde{m} = k_n$, η_1 as above and $f_\theta \equiv f$.

The rest of the proof can be done in the same way as for $f \in \mathcal{F}$: If $i(w) < i(v)$ and no $\bar{w} \in E$ with $\bar{w}'(0)$ between $v'(0)$ and $w'(0)$ satisfies $z(v - \bar{w}) \leq z(w - \bar{w})$ then $v \searrow^{\varphi_{k_n}} w$ for all n large enough. As above this implies $v \searrow^{\varphi} w$. If $v \searrow^{\varphi} w$ then $v \searrow^{\varphi_{k_n}} w$ for some n, so the given conditions are satisfied by Theorem 4.1. $\square$

In the case $f \in \mathcal{F}$ the connecting orbits w.r.t. φ_n can easily be described in an explicit form:

Corollary 4.3. *Let $f \in \mathcal{F}$. For $v \in E_n$ let $\Omega_n(v) := \{w \in E_n : v \searrow^{\varphi_n} w\} \setminus \{v\}$. Let $k_0 := i(0) + 1$, then we have the following result:*

$$\begin{aligned} \Omega_n(0) &= \{\tilde{u}_k : |k| < k_0\} \\ \Omega_n(u_k) &= \{u_l, \tilde{u}_l : |l| < |k|\} \cup \{\tilde{u}_k\} \\ &\qquad \textit{for all } k \in \{\pm k_0, \dots, \pm n\} \\ \Omega_n(\tilde{u}_k) &= \{\tilde{u}_l : |l| < |k|\} \\ &\qquad \textit{for all } k \in \{\pm 1, \dots, \pm n\}. \end{aligned}$$

Proof. For $v, w \in E_n$ we write $v < w :\iff v'(0) < w'(0)$ and $|v| < |w| :\iff |v'(0)| < |w'(0)|$ to shorten notation. With this notation we have

$$\tilde{u}_{-1} < \cdots < \tilde{u}_{-n} < u_{-n} < \cdots < u_{-k_0} < 0 < u_{k_0} < \cdots < u_n < \tilde{u}_n < \cdots < \tilde{u}_1,$$

$$0 < |u_{\pm k_0}| < \cdots < |u_{\pm n}| < |\tilde{u}_{\pm n}| < \cdots < |\tilde{u}_{\pm 1}|,$$

$i(\tilde{u}_k) = |k| - 1$, $i(u_k) = |k|$ by Corollary 3.12 and Proposition 3.14.

Let $v, w \in E_n$, $v \neq w$, first consider $v < w$. We want to find out in which cases we have $v \searrow^{\varphi_n} w$. We need to consider only w with $i(v) > i(w)$, since this condition is necessary.

Case 1: $v = \tilde{u}_{-k}$ for $k \in \{1, \dots, n\}$, then

$$k - 1 = i(v) > i(w) \iff w \in \{u_{-(k-2)}, \dots, u_{k-2}\} \cup \{\tilde{u}_{k-1}, \dots, \tilde{u}_1\}.$$

If $w \in \{u_{-(k-2)}, \dots, u_{k-2}\}$ then

$$z(v - u_{-k}) \overset{\text{Lemma 3.4}}{=} z(v) = k - 1 = z(u_{-k}) = z(u_{-k} - w),$$

so u_{-k} blocks the connection and we have $v \not\searrow^{\varphi_n} w$. If $w \in \{\tilde{u}_{k-1}, \dots, \tilde{u}_1\}$ and $\bar{w} \in E_n$ with $v < \bar{w} < w$, then $|w| > |v| > |\bar{w}|$. Thus

$$z(v - \bar{w}) = z(v) = k - 1 > z(w) = z(w - \bar{w})$$

and $v \searrow^{\varphi_n} w$.

Case 2: $v = u_{-k}$, $k \in \{k_0, \dots, n\}$, then

$$\begin{aligned} k = i(v) > i(w) &\iff \\ &w \in \{u_{-(k-1)}, \dots, u_{-k_0}, 0, u_{k_0}, \dots, u_{k-1}\} \cup \{\tilde{u}_k\} \cup \{\tilde{u}_{k-1}, \dots, \tilde{u}_1\}. \end{aligned}$$

If $w \in \{u_{-(k-1)}, \dots, u_{-k_0}, 0, u_{k_0}, \dots, u_{k-1}\}$, then $|v| > |\bar{w}|$ for $v < \bar{w} < w$, so $z(v - \bar{w}) = z(v) = k - 1$, and as both $z(w), z(\bar{w}) \leq k - 2$ we get $v \searrow^{\varphi_n} w$. If $w = \tilde{u}_k$ then u_k blocks the connection: $|z(v - u_k)| = k - 1 = z(w) = z(w - u_k)$, so $v \not\searrow^{\varphi_n} w$.

If finally $w \in \{\tilde{u}_{k-1}, \dots, \tilde{u}_1\}$ then for $v < \bar{w} < w$ we have $|w| > |\bar{w}|$, so $z(w - \bar{w}) = z(w) \leq k - 2$. Now either $|\bar{w}| \geq |v|$ or $|\bar{w}| < |v|$. In the first case $z(v - \bar{w}) = z(\bar{w}) \geq z(w) = z(w - \bar{w})$, the latter case yields $z(v - \bar{w}) = z(v) = k - 1$; either way we get $v \searrow^{\varphi_n} w$.

Case 3: $v \geq 0$, then $i(v) > i(w) \iff w \in \{\tilde{u}_k : k - 1 < i(v)\}$. For all $v < \bar{w} < w$ we have $z(v - \bar{w}) = z(\bar{w}) > z(w) = z(w - \bar{w})$, so $v \searrow^{\varphi_n} w$.

For $w > v$ we analogously get symmetric results, so the assertion follows. □

4.2 Continuous approximation of f

With the help of Proposition 3.14 and the connecting orbit structure we can approximate f with a continuum of dissipative functions f_ν and obtain a more detailed picture of the dynamics of φ, including blow-up behavior.

Proposition 4.4. *Let $f \in \mathcal{F}$, $i(0) =: k_0$, $E = \{0, u_{\pm(k_0+1)}, \dots\}$. Then there exists a continuous mapping $[k_0, \infty) \ni \nu \mapsto f_\nu \in \mathcal{G}_d \subset \mathcal{C}^2_w$ such that the following assertions hold:*

(i) For the set E_ν of stationary solutions of the dissipative flows φ_ν the following holds: Let $n \in \mathbb{N}$, $n \geq k_0$, $\nu \in [n, n+1)$, then

$$E_\nu = \{v \in E : i(v) \leq n\} \dot\cup \{\tilde{u}_{\pm 1,\nu}, \dots, \tilde{u}_{\pm n,\nu}\} \dot\cup R_\nu,$$

all $u \in E_\nu \setminus R_\nu$ are hyperbolic, the mapping $[|k|, \infty) \ni \nu \mapsto \tilde{u}_{k,\nu} \in X$ is continuous for all $k \in \mathbb{Z} \setminus \{0\}$, $R_n = \emptyset$ and $z(v) = n+1$ for $v \in R_\nu$, $\nu \in (n, n+1)$. Define the following invariant sets:

$$\begin{aligned}
\mathcal{A} &:= \bigcup_{v \in E} W^u(v) & \mathcal{B} &:= \{u \in \mathcal{A} : T^{max}(u) = \infty\} \\
\mathcal{A}_\nu &:= W^u(E_\nu) & \mathcal{B}_\nu &:= \{u \in \mathcal{A}_\nu : \alpha_\nu(u), \omega_\nu(u) \in E\} \\
\mathcal{A}_{n,\nu} &:= \{u \in \mathcal{A}_\nu : z(\alpha(u)) < n\} & \mathcal{B}_{n,\nu} &:= \mathcal{A}_{n,\nu} \cap \mathcal{B}_\nu \\
\mathcal{A}_{n,\infty} &:= \{u \in \mathcal{A} : z(\alpha(u)) < n\} & \mathcal{B}_{n,\infty} &:= \mathcal{A}_{n,\infty} \cap \mathcal{B}
\end{aligned}$$

(ii) $\mathcal{A}_{n,\nu}$ is a n-dimensional local attractor of φ_ν.

(iii) For $n \geq k_0$ and $\nu \geq n$ the semiflows $\varphi_n|_{\mathcal{A}_n}$ and $\varphi_\nu|_{\mathcal{A}_{n,\nu}}$ are conjugate, i.e. there is a homeomorphism $h_{n,\nu} : \mathcal{A}_n \to \mathcal{A}_{n,\nu}$ mapping orbits onto orbits and preserving the sense of direction in time. The mapping $\nu \mapsto h_{n,\nu} \in \mathcal{C}^0(\mathcal{A}_n, X)$ is continuous.

(iv) $\forall n \in \mathbb{N} \exists N \in \mathbb{N} \forall \nu \geq N : \mathcal{B}_{n,\nu} = \mathcal{B}_{n,\infty}$ and φ, φ_ν coincide on $\mathcal{B}_{n,\nu}$.

(v) Let $M_\nu := \max\{M \in (0,\infty) : f_\nu \equiv f$ on $(-M, M)\}$, then $\mathcal{E}_{n,\nu} = \{u \in \mathcal{A}_{n,\nu} : \|\gamma_\nu^-(u)\|_\infty < M_\nu\} \subset \mathcal{A}_{n,\infty}$ is ascending in ν with $\bigcup_{\nu \geq n} \mathcal{E}_{n,\nu} = \mathcal{A}_{n,\infty}$.

(vi) Let $v, w \in E$, $\nu \geq n := i(v)$. Then the connecting manifold $C_\nu(v,w)$ is connected.

(vii) $J_\nu(\tilde{u}_{k,\nu}) \xrightarrow{\nu \to \infty} -\infty$ for all $k \in \mathbb{Z} \setminus \{0\}$.

(viii) $\forall n \geq k_0 \exists U \subset X^n_1(0)$ open $\exists g : U \to X^n(0)\ \mathcal{C}^1 : \mathcal{B}_{n,\infty} \subset \mathcal{A}_{n,\infty} \subset \operatorname{graph} g$.

Proof. **(i)** is a direct application of Proposition 3.14 together with Proposition 3.10 c). As shown in the proof of 4.2 (case $f \in \mathcal{F}$) for $k > k_0$ there are $\max\{|u'_{\pm k}(0)|\} < a_k < \min\{|u'_{\pm(k+1)}(0)|\}$ — we choose $a_k := \frac{1}{2}(\max\{|u'_{\pm k}(0)|\} + \min\{|u'_{\pm(k+1)}(0)|\})$ for uniqueness and continuous dependence on f. Let $\eta_1(\theta) := (1-\theta)a_k + \theta a_{k+1}$ and apply Proposition 3.14 with this η and constant f. We obtain a continuum $\tilde{f}_\theta$ and set $f_\nu := \tilde{f}_{\theta+k}$.

(ii): We use that for any $t > 0$ φ^t_ν is a continuous mapping from X to $\mathcal{C}^1 = \mathcal{C}^1([0,1])$. To see this choose a sequence $(w_k)_k$ in X with $w_k \to w \in X$. $\{w_k : k \in \mathbb{N}\}$ is bounded in X, so $\{\varphi^t_\nu(w_k) : k \in \mathbb{N}\}$ is bounded in H^2 by Proposition 2.1 (iv). By Sobolev embedding $\{\varphi^t_\nu(w_k) : k \in \mathbb{N}\}$ is precompact in

$\mathcal{C}^1$. Thus any subsequence of $(\varphi_\nu^t(w_k))$ has a subsequence $(\varphi_\nu^t(w_{k_l}))$ such that $\|\varphi_\nu^t(w_{k_l}) - \tilde{w}\|_{\mathcal{C}^1} \to 0$. This implies $\varphi_\nu^t(w_k) \to \tilde{w}$ in X, so $\tilde{w} = \varphi_\nu^t(w)$ and it follows that $\|\varphi_\nu^t(w_k) - \varphi_\nu^t(w)\|_{\mathcal{C}^1} \to 0$.

Define

$$Z_\nu^n := \{u \in X : \exists t \in (0,1) : z(\varphi_\nu^t(u)) < n\}.$$

Then Z_ν^n is open: Let $w \in Z_\nu^n$, $t \in (0,1)$ such that $z(\varphi_\nu^t(w)) < n$ and all zeros of $\varphi_\nu^t(w)$ are simple (Proposition 3.2 c)). There is a neighborhood U of $\varphi_\nu^t(w)$ in $\mathcal{C}^1$ such that $z|_U < n$. By the continuity of $\varphi_\nu^t : X \to \mathcal{C}^1$ the set $(\varphi^t)^{-1}(U) \subset Z_\nu^n$ is an open neighborhood of w in X, so Z_ν^n is open. Next choose $c \in \mathbb{R} \setminus J_\nu^{-1}(E_\nu)$ with $c > \max J_\nu(\mathcal{A}_{n,\nu})$ and let $U := Z_\nu^n \cap J_\nu^{-1}((-\infty, c))$. Then U is positively invariant, open and bounded because φ_ν is a dissipative gradient-like semiflow and J_ν is continuous. Given $v \in E_\nu$ choose $w_k \in U$ with $w_k \to v$. There exist t_n such that $z(\varphi_\nu^{t_k}) < n$, we can assume $t_n \to 1$ by Proposition 3.2 c). Then $\varphi_\nu^{t_n}(w_k) \xrightarrow{\mathcal{C}^1} \varphi_\nu^1(v) = v$, so $z(v) < n$ and we have proved $\overline{U} \cap E_\nu \subset \mathcal{A}_{n,\nu} \cap E_\nu$. The other inclusion is trivial.

Suppose $\mathcal{A}_{n,\nu}$ does not attract U. Then

$$\exists \epsilon > 0, t_m \to \infty, w_m \in U : \mathrm{dist}(\varphi_\nu^{t_m}(w_m), \mathcal{A}_{n,\nu}) \geq \epsilon.$$

But $\mathcal{A}_\nu$ does attract U, so w.l.o.g. $\varphi_\nu^{t_m}(w_m) \to w_0 \in (\mathcal{A}_\nu \setminus \mathcal{A}_{n,\nu}) \cap \overline{U}$. Choosing $s_k \to -\infty$ such that $\|\varphi_\nu^{s_k}(w_0) - \alpha(w_0)\| < \frac{1}{k}$ and an appropriate subsequence of $(t_m)_m$ we get

$$\varphi_\nu^{t_{m_k}+s_k}(w_{m_k}) \to \alpha(w_0) \in (\mathcal{A}_\nu \setminus \mathcal{A}_{n,\nu}) \cap \overline{U} \cap E_\nu = \emptyset,$$

which is impossible as $w_0 \in \mathcal{A}_\nu$.

(iii) In [Oli02] it is proved that for functions $f, g \in \mathcal{G}_d$ being close in the weak topology there is a homeomorphism $\mathcal{A}_f \to \mathcal{A}_g$ mapping orbits onto orbits and preserving sense of direction in time. The proof can be repeated unchanged for local attractors, which yields the stated result. The only properties of the attractor used in the proof are it's compactness and invariance and upper semicontinuity (i.e. $\delta(\mathcal{A}_g, \mathcal{A}_f) \to 0$ as $g \to f$), maximality of the compact invariant set is not used. The properties of the flow, in particular the existence of a partial order structure on the set of equilibria induced by $v \searrow^\varphi w$, are all satisfied and stated in section 2. The upper semicontinuity of local attractors has been proved in Lemma 3.22. The continuity statement follows from the construction of the homeomorphisms in [Oli02] by invariant foliations varying smoothly in $f \in \mathcal{G}_d$.

(iv): $\mathcal{B}_{n,\infty} \cap E$ is finite, so by Lemma 2.10 $\mathcal{B}_{n,\infty}$ is bounded in X, thus it is bounded in $\mathcal{C}^0$. Defining M_ν as in (v) clearly $M_\nu \geq \|\mathcal{B}_{n,\infty}\|_\infty$ for ν large enough, which implies $\mathcal{B}_{n,\infty} \subset \mathcal{B}_{n,\nu}$ for ν large enough. The other inclusion follows from the proof of Theorem 4.2 (case $f \in \mathcal{F}$).

(v) is clear because M_ν is nondecreasing by construction of f_ν.

(vi): We can assume without loss that $\dim(C_\nu(v,w)) > 0$. By (iii) $C_\nu(v,w)$ is homeomorphic to $C_n(v,w)$, so it is enough to show that $C_n(v,w)$ is connected.

Recall that by Theorem 3.23 for $k \in \mathbb{N}$ there are open $U_k \subset X_1^k(u_k)$ and continuously differentiable $g_k : U_k \to X^k(u_k)$ such that $\mathcal{A}_k \subset \mathcal{N}_k = \mathrm{graph}(g_k)$ and $\mathcal{N}_k$ is positively and locally negatively invariant w.r.t. φ_k.

In the case $i(w) = n-1$ the restricted flow $\varphi_\nu|_{\mathcal{N}_n}$ is locally at w conjugate to a linear flow with a one-dimensional stable manifold (cf. Corollary 3.24). So by this linearization we see that there are precisely two orbits converging to w in $\mathcal{N}_n$. But in this case $u_n \searrow w$ and $u_{-n} \searrow w$, so the assertion follows.

Now consider the case $n - i(w) = k > 1$. Again linearize $\varphi_n|_{\mathcal{N}_n}$ in a neighborhood $U \subset \mathcal{N}_n$ of w. By Proposition 3.21

$$U = (\mathcal{A}_{n-1,n} \cup C_n(u_n, w) \cup C_n(u_{-n}, w)) \cap U. \tag{4.3}$$

By Theorem 3.23 $\mathcal{A}_{n-1,n} \cap U = \mathcal{N}_{n-1} \cap U$ is a $\mathcal{C}^1$-hypersurface in U (decreasing U if necessary), more precisely a graph over $P_1^{n-1}(w)U$. By Proposition 2.3 d) $W_n^s(w)$ is locally a graph over $P^{i(w)}(w)U$, so $W_n^s(w) \pitchfork_U \mathcal{A}_{n-1,n}$, and $(U \cap W_n^s(w)) \setminus \mathcal{A}_{n-1,n}$ has exactly two connected components. By (4.3) $(U \cap W^s(w)) \setminus \mathcal{A}_{n-1,n} = (C_n(u_n,w) \cup C_n(u_{-n},w)) \cap U$, so $C_n(u_n,w) \cap U$, $C_n(u_{-n},w) \cap U$ are connected. This proves (vi).

(vii): Let $k \in \mathbb{Z} \setminus \{0\}$, w.l.o.g. $k > 0$, S a fundamental domain for $u_k \in E$ w.r.t. φ_ν for all $\nu \geq k$. Choose a sequence $\nu_n \geq k$, $\nu_n \to \infty$, then

$$\forall n \in \mathbb{N} \exists_1 v_n \in S : \omega_{\nu_n}(v_n) = \tilde{u}_{k,\nu_n}.$$

W.l.o.g. $v_n \to v \in S$.

Claim 1: $T^{\max}(v) < \infty$

Proof Suppose not, then $\omega(v) = w \in E$ with $z(w) = i(w) - 1 < i(u_k) - 1 = z(u_k)$. Choose $N \in \mathbb{N}$ with $M_{\nu_N} > \|C(u_k, w)\|_\infty$, so for all $n \geq N$ we have $\gamma(v) = \gamma_{\nu_n}(v)$. Further there is an open neighborhood U of w in $\mathcal{A}_{N,\nu_N}$ such that $\|U\|_\infty < M_{\nu_N}$ and $z|_U = z(w) < z(v)$. This is because $\mathcal{A}_{N,\nu_N}$ is (part of) a smooth graph over the finite-dimensional $X_1^n(v)$, where the $X-$ and $\mathcal{C}^1-$ norms are equivalent. There is also a finite $T > 0$ such that $\varphi^t(v) \in U$ for all $t \geq T$. So we can conclude for $n \geq N$ large enough

$$\varphi_{\nu_n}^{[0,T]}(v_n) = \varphi^{[0,T]}(v_n) \subset \mathcal{A}_{N,\nu_n}, \quad z(\varphi_{\nu_n}^T(v_n)) = z(w) < z(u_k) = z(\tilde{u}_{k,\nu_n}),$$

which contradicts $\varphi_{\nu_n}^t(v_n) \to \tilde{u}_{k,\nu_n}$ as $t \to \infty$ and the claim is proved.

Now let $t_n := T^{\max}(v) - \frac{1}{n} > 0$ for n large enough. Then as above $\varphi_{\nu_l}^{t_n}(v_l) \xrightarrow{l\to\infty} \varphi^{t_n}(v)$, in particular $J_{\nu_l}(\varphi_{\nu_l}^{t_n}(v_l)) \xrightarrow{l\to\infty} J(\varphi^{t_n}(v)) \xrightarrow{n\to\infty} -\infty$. But for all t we have $J_{\nu_l}(\tilde{u}_{k,\nu_l}) \leq J_{\nu_l}(\varphi_{\nu_l}^t(v_l))$, so the assertion is proved.

(viii): This is a direct corollary of Theorem 3.23 and (v).

□

4.3 Orbit equivalence and blow-up

For $f \in \mathcal{F}$ we are now able to transfer global stability results from the dissipative case. We can prove the stability of n-dimensional subsets of the flow, namely the sets $\mathcal{A}_{n,\infty}$ in the following theorem. Note that these sets include blow-up trajectories, so we can show some kind of stability of blow-up phenomena. One more step in this direction is Corollary 4.6, where we show the existence of blow-up trajectories with certain properties.

Theorem 4.5. *Let $f \in \mathcal{F}$, $n > i(0)$. We use the notation and the dissipative flows introduced in Proposition 4.4. Choose $\sigma \in \mathbb{R}$ from the same connected component of $\mathbb{R} \setminus \{k^2\pi^2 : k \in \mathbb{N}\}$ that $f'(0)$ lies in, and define $\tilde{f}(x) := x^3 + \sigma x$. Then $\tilde{f} \in \mathcal{F}$ and the flows $\varphi_\nu|_{\mathcal{A}_{n,\nu}}$ and $\tilde{\varphi}_\nu|_{\tilde{\mathcal{A}}_{n,\nu}}$ are conjugate for any $\nu \in [n, \infty)$, as well as the flows $\varphi|_{\mathcal{A}_{n,\infty}}$ and $\tilde{\varphi}|_{\tilde{\mathcal{A}}_{n,\infty}}$.*

Proof. We will first prove the assertion for $\nu \in [n, \infty)$. By Proposition 4.4 (iii) it is enough to show it for $\nu = n$. For $\theta \in [0, 1]$, $x \in \mathbb{R}$ define

$$\tilde{h}_\theta(x) = \tilde{h}(\theta, x) := (1-\theta) f(x) + \theta \tilde{f}(x).$$

Then

$$\begin{aligned} \tilde{h}'_\theta(x) \cdot x^2 &= (1-\theta) f'(x) x^2 + \theta \tilde{f}'(x) x^2 > \tilde{h}_\theta(x) x, \\ \tilde{h}'_\theta(0) &= (1-\theta) f'(0) + \theta\sigma \notin \{k^2\pi^2 : k \in \mathbb{N}\}. \end{aligned}$$

Similarly we check $h_\theta \in \tilde{\mathcal{G}}$, so $\tilde{h}_\theta \in \mathcal{F}$ for all $\theta \in [0, 1]$ (in particular $\tilde{f} \in \mathcal{F}$) — let φ_θ denote the associated parabolic flow. By construction $i_\theta(0)$ is independent of θ. We write the set of equilibria of φ_θ as $\{0, u_{\theta,\pm(i(0)+1)}, \dots\}$. By the continuous dependence Theorem $\theta \mapsto u'_{\theta,l}(0)$ is a continuous function for all $|l| \geq i(0) + 1$. So we can define a continuous function by $\eta_1 : [0, 1] \to \mathbb{R}$, $\eta_1(\theta) := \frac{1}{2}(\max\{u'_{\theta,\pm n}(0)\} + \min\{u'_{\theta,\pm(n+1)}(0)\})$ as in the hypotheses of Proposition 3.14. Then by construction $\varphi_{0,n} = \varphi_n$ and $\varphi_{1,n} = \tilde{\varphi}_n$.

Applying Proposition 3.14 (again with $m = n$) we get a continuum of functions $h_\theta \in \mathcal{G}_d$ with $h_0 = f_n$, $h_1 = \tilde{f}_n$. Again $\varphi_{\theta,n}$ is a dissipative Morse-Smale semiflow for any $\theta \in [0, 1]$, so by Oliva [Oli02] $\varphi_n|_{\mathcal{A}_{n,n}} \sim \tilde{\varphi}_n|_{\tilde{\mathcal{A}}_{n,n}}$.

Now choose $\mathbb{N} \ni \nu \geq n$ so large that $M_\nu > \|\mathcal{B}_{n,\infty}\|_\infty$ and $J_\nu(v) < -1$ for any $v \in E_{n,\nu} \setminus E$ by Proposition 4.4 (vii) and $c \in (-1, 0)$ such that $c \notin J_\nu(E_\nu)$. Let $A = A(\nu) := \{u \in \mathcal{A}_{n,\nu} : J_\nu(u) > c\}$, by the connecting orbit structure and the choice of ν and c we have $A \cap E_\nu = \{0, u_{\pm i(0)+1}, \dots, u_{\pm n}\}$, A is locally positive and globally negatively invariant w.r.t. φ, and

$$\bigcup_{k=i(0)+1}^{n} W^u_\nu(u_{\pm k}) \cup W^u_\nu(0) \supset A \supset \mathcal{B}_{n,\infty}.$$

So there is a $T > 0$ such that for $B := \varphi_\nu^{-T}(A)$ we have $\|B\|_\infty < M_\nu$, thus $\mathcal{A}_{n,\infty} \supset B \supset \mathcal{B}_{n,\infty}$. By construction of B for $u \in \partial B$ we have $\gamma(u) \cap \partial B = \{u\}$, and by Lemma 2.10 the mapping $X \ni u \mapsto T_\varphi^{\max}(u)$ is continuous. This means that the mappings

$$\begin{aligned} \tau &: \mathcal{A}_{n,\infty} \setminus \overline{A} \to [0,\infty), & \tau(u) &:= \inf\{t > 0 : \varphi^{-t}(u) \in \partial A\} \\ g &: \mathcal{A}_{n,\infty} \setminus \overline{A} \to \partial A, & g(u) &:= \varphi^{-\tau(u)}(u) \end{aligned}$$

are continuous. So we can finally define $h : \mathcal{A}_{n,\infty} \to \tilde{\mathcal{A}}_{n,\infty}$ by

$$h(u) := \begin{cases} h_0(u) & u \in \overline{A} \\ \tilde{\varphi}\Big(T_{\tilde{\varphi}}^{\max}(h_0(g(u))) \cdot \frac{\tau(u)}{T_\varphi^{\max}(u)}, h_0(g(u))\Big) & u \in \mathcal{A}_{n,\infty} \setminus \overline{A}. \end{cases}$$

This mapping is a homeomorphism $\mathcal{A}_{n,\infty} \to \tilde{\mathcal{A}}_{n,\infty}$ mapping orbits onto orbits and preserving sense of direction in time, as is easily verified. □

Corollary 4.6. *Let $f \in \mathcal{F}$, $n > i(0)$, $l \in \mathbb{Z}$, $|l| \in \{i(0)+1, \ldots n\}$. Then (with the notation from Proposition 4.4) there is a $v_0 \in \mathcal{A}_{n,\infty}$ with $\varphi^t(v_0) \to \infty$ as $t \to T^{max}(v_0)$ and $z(u) = |l| - 1$, $\operatorname{sign}(u'(0)) = \operatorname{sign}(l)$ for all $u \in \gamma(v_0)$.*

Proof. We will consider the case $l > 0$, the second case follows analogously. Choose $\epsilon > 0$ and $N \geq n$ such that

$$\forall k \geq N: \ \varphi_k\big|_{U_\epsilon(u_l) \cap \mathcal{A}_{n,\infty}} = \varphi\big|_{U_\epsilon(u_l) \cap \mathcal{A}_{n,\infty}}, \quad \partial U_\epsilon(u_l) \cap W^u(u_l) \text{ is closed.}$$

Then

$$\forall k \geq N \, \exists v_k \in \partial U_\epsilon(u_l) \cap W^u(u_l) : \varphi_k^t(v_k) \xrightarrow{t \to \infty} \tilde{u}_{l,k}$$

by Proposition 4.4 and Corollary 4.3. W.l.o.g. $v_k \to v_0 \in \partial B_\epsilon(u_n) \cap W^u(u_l)$. Let $t_0 \in \tilde{I}(v_0) = \{t \in I(v_0) : \varphi^t(v_0) \text{ has only simple zeros}\}$, then

$$z(\varphi^{t_0}(v_0)) = \lim_{k \to \infty} z\left(\varphi_k^{t_0}(v_k)\right) = l - 1$$

($z(\varphi_k^t(v_k)) = l - 1$ for all t because $\gamma_k(v_k)$ connects two equilibria with $l-1$ zeros). By Proposition 3.2 c) $\tilde{I}(v_0)$ is dense in $I(v_0)$, so $z(\varphi^t(v_0)) = l - 1$ for all $t \in I(v_0)$. But $\{v \in E : z(v) = l - 1\} = \{u_l, u_{-l}\}$ by Corollary 3.12, $\alpha(v_0) = u_l$ by construction, and $u_l \not\searrow^\varphi u_{-l}$ by Theorem 4.2. This means $\omega(v_0) = \emptyset$, so by Proposition 2.3 a) $\|\gamma^+(v_0)\|$ is unbounded. For $u \in \gamma(v_0)$ we have $u'(0) > 0$ by Proposition 3.2 b) because $u_l'(0) > 0$. □

***Remark* 4.7.** A more detailed analysis of the properties of $\mathcal{A}_{n,\infty}$ could lead to a better understanding of blow-up phenomena. It could be a first goal to prove that $P_1^n(v)\mathcal{A}_{n,\infty} = X$. We were only able to prove $P_1^1(v)\tilde{u}_{1,\nu} \to \infty$, which implies $P_1^n(v)\tilde{u}_{n,\nu} \to \infty$ if f is odd.

APPENDIX A

Some technical proofs

A.1 The superposition operator $\hat{f}$

Lemma A.1. *For f $\mathcal{C}^1$ the operator $\hat{f} : X \to L^2([0,1])$ is continuously differentiable uniformly on bounded sets.*

Proof. We have to prove that $X \ni v \mapsto f'(u)v \in L^2$ is the Gateaux - derivative of $\hat{f}$ in u, and that the mapping

$$X \ni u \mapsto \Big(X \ni v \mapsto f'(u)v \in L^2 \Big) \in \mathcal{L}(X, L^2)$$

is uniformly continuous on bounded sets. First let $u, v \in X$, $t \in \mathbb{R}$ and $x \in [0,1]$. Then if $v(x) \neq 0$

$$\begin{aligned}
&\left| \frac{f(u(x) + tv(x)) - f(u(x))}{t} - f'(u(x))v(x) \right| \\
&= \left| v(x) \left(\frac{f(u(x) + tv(x)) - f(u(x))}{tv(x)} - f'(u(x)) \right) \right| \\
&\leq \|v\|_\infty \cdot \max\left\{ |f'(\zeta) - f'(\xi)| : |\zeta| \leq \|u\|_\infty, |\xi - \zeta| \leq t\|v\|_\infty \right\} \\
&\xrightarrow{t \to 0} 0.
\end{aligned}$$

The inequality above obviously holds for $v(x) = 0$ as well, so we have shown

$$\frac{f(u(\cdot) + tv(\cdot)) - f(u(\cdot)}{t} \to f'(u(\cdot)) \cdot v(\cdot)$$

in $\mathcal{C}^0([0,1])$. This implies L^2-convergence.

By the Sobolev embedding Theorem ([Ada75]) there is a constant $C > 0$ such that $\|u\|_\infty \leq C\|u\|$ for all $u \in X$. Let $u, w, v \in X$, $\|v\| = 1$, $w \in U_1(u)$ and M' the Lipschitz constant of f' on $[-\|u\|_\infty - C, \|u\|_\infty + C]$:

$$\begin{aligned}
\|f'(u)v - f'(w)v\|_2^2 &= \int_0^1 \big((f'(u(x)) - f'(w(x)))v(x) \big)^2 \, dx \\
&\leq \int_0^1 M'^2 |u(x) - w(x)|^2 v(x)^2 \, dx \leq M'^2 C^2 \|u - w\|_2^2 \leq M'^2 C^2 \|u - w\|,
\end{aligned}$$

so this mapping is (Lipschitz) continuous uniformly on bounded sets. $\square$

A.2 A class of dissipative functions

Proof of Proposition 2.6. The assertion follows by Theorem 3.8.5 in [Hal88], if φ is a dissipative gradient-like semiflow, E is bounded, and φ is *asymptotically smooth*, which means that for any nonempty, closed, bounded and positive invariant set $B \subset X$ there is a compact set K which attracts B.

By [Hal88, Corollary 3.2.2] asymptotical smoothness follows from the compactness of the flow (Proposition 2.1 (iv)), and as φ is gradient-like it remains to show boundedness of E, boundedness of J from below, and $J(u) \to \infty$ as $\|u\| \to \infty$.

X has an orthogonal base of eigenfunctions $e_1, e_2, \dots$ of $-\frac{\partial^2}{\partial x^2}$ to the eigenvalues $\pi^2, 2^2\pi^2, \dots$ such that $u = \sum_{k=1}^{\infty} \langle e_k, u \rangle e_k$ for $u \in X$. So

$$\begin{aligned} \|u_x\|_2^2 &= \int_0^1 (-u_{xx}) u \, dx = \int_0^1 \left(\sum_{k=1}^{\infty} \langle e_k, u \rangle k^2 \pi^2 e_k \right) \cdot u \, dx \\ &\geq \pi^2 \int_0^1 u^2 \, dx = \pi^2 \|u\|_2^2, \end{aligned}$$

which is of course just Poincarés inequality implying $\|u\|^2 \leq (1 + \pi^{-2}) \|u_x\|_2^2$. Now by the condition on f there is a $\delta > 0$ such that $\limsup_{|s| \to \infty} \frac{f(s)}{s} < \pi^2 - \delta$. For $s > 0$ we obtain

$$F(s) - \frac{1}{2}(\pi^2 - \delta)s^2 = \int_0^s \left[\frac{f(t)}{t} - (\pi^2 - \delta) \right] t \, dt \leq C_\delta.$$

A similar estimate holds for $s < 0$, so

$$\exists C_\delta > 0 \forall s \in \mathbb{R} : F(s) \leq \frac{1}{2}(\pi^2 - \delta)s^2 + C_\delta,$$

and we get the estimate

$$\begin{aligned} J(u) &= \frac{1}{2}\|u_x\|_2^2 - \int_0^1 F(u) \, dx \geq \frac{1}{2}\|u_x\|_2^2 - \frac{1}{2}(\pi^2 - \delta)\|u\|_2^2 - C_\delta \\ &\geq \frac{\delta}{2\pi^2}\|u_x\|_2^2 - C_\delta \geq \frac{\delta}{2\pi^2(1 + \pi^{-2})}\|u\|^2 - C_\delta. \end{aligned}$$

So J is bounded from below and $J(u) \xrightarrow{\|u\| \to \infty} \infty$.

To see that E is bounded let $u \in E$, so $-u_{xx} = f(u)$ which is equivalent to

$$\forall v \in X : \int_0^1 u_x v_x \, dx = \int_0^1 f(u) v \, dx.$$

By the conditions on f there are constants $M, \delta > 0$ such that $\frac{f(s)}{s} < \pi^2 - \delta$ for all $|s| \geq M$, and setting $v = u$ in the equation above we get

$$\|u_x\|_2^2 = \int_0^1 u_x^2\,dx = \int_0^1 f(u)u\,dx = \int_{\{|u|<M\}} f(u)u\,dx + \int_{\{|u|\geq M\}} f(u)u\,dx$$
$$\leq \underbrace{M \max f|_{[-M,M]}}_{=:K} + \int_0^1 (\pi^2 - \delta)u^2\,dx \leq K + \frac{\pi^2 - \delta}{\pi^2}\|u_x\|_2^2,$$

so $\frac{\|u\|^2}{1+\pi^{-2}} \leq \|u'\|_2^2 \leq \frac{K\pi^2}{\delta}$ and E is bounded. □

A.3 Upper semicontinuity of local attractors

Proof of Lemma 3.22. We have to verify the assumptions of Theorem 2.4 in [HLR88] where all assertions of Lemma 3.22 are proved. First choose any open bounded neighborhood $\mathcal{A}_0 \subset U \subset X$, $\nu \in I_0$ and a compact interval $I = [t^-, t^+] \subset (0, \infty)$. Choose $\delta_0 > 0$ such that $U_{2\delta_0}(\mathcal{A}_0) \subset U$, open sets $N_1^k \supset \mathcal{A}_0$ such that $\gamma_{\nu_0}^+(N_1^k) \subset U_{\frac{\delta_0}{4k}}(\mathcal{A}_0)$, $N_2 := U_{\frac{\delta_0}{2}}(\mathcal{A}_{n,\nu_0})$, $N_3 := U_{2\delta_0}(\mathcal{A}_{n,\nu_0})$ (note that $\mathcal{A}_0$ ist stable by Lemma 2.9). Then clearly $N_1^k \subset N_2 \subset U_{\delta_0}(N_2) \subset N_3 \subset U$, and the following three conditions are easily verified:

1. $\forall k \in \mathbb{N} \forall t \geq 0 : \varphi_{\nu_0}^t(N_1^k) \subset N_2$.
2. $\exists k \in \mathbb{N}, t_0 > 0\, h_0 > 0, \forall 0 \leq t \leq t_0\, \forall |\nu - \nu_0| \leq h_0 : \varphi_\nu^t(N_1^k) \subset N_2$
   (If not, then
   $$\forall k \in \mathbb{N} \exists 0 \leq t_k \leq \frac{1}{k} \exists \nu_k \in B_{\frac{1}{k}}(\nu_0) \exists u_k \in N_1^k : \mathrm{dist}(\varphi_{\nu_k}^{t_k}(u_k), \mathcal{A}_0) \geq \frac{\delta_0}{2}.$$
   But $u_k \in U_{\frac{\delta_0}{4k}}(\mathcal{A}_0)$, so w.l.o.g. $u_k \to u \in \mathcal{A}_0$ thus $\|\varphi_{\nu_k}^{t_k}(u_k) - u\| \to 0 < \frac{\delta_0}{2}$, a contradiction.)
3. $\forall |\nu - \nu_0| \leq 1\, \forall u \in U_{\delta_0}(N_2) \exists t = t(u, \nu) > 0 : \varphi_\nu^{[0,t]}(u) \subset N_3$.

Fix $k \in \mathbb{N}$ as given by 2 and let $N_1 := N_1^k$.

Next we have to verify a certain continuity of φ_ν w.r.t. ν. We use the variation-of-constants formula

$$\varphi_\nu^t(u) = e^{-At}u + \int_0^t e^{-A(t-s)} f_\nu(\varphi_\nu^s(u))\,ds \tag{A.1}$$

([Hen81, Lemma 3.3.2]), where A is the Dirichlet realization of $-\Delta$ in L^2 and e^{-At} is the analytic semigroup induced by A (cf. section 2.1). We also use the fractional powers A^α of A for $0 \le \alpha \le 1$ with domains $X^\alpha = D(A^\alpha)$ with the graph norms $\|u\|_\alpha = \|A^\alpha u\|_{L^2}$. We have $X^0 = L^2$, $X^1 = H^2 \cap X$ and $X^{\frac{1}{2}} = X$ with $\|\cdot\|$ and $\|\cdot\|_{\frac{1}{2}}$ being equivalent norms (see [Hen81], section 1.4.). In this proof we will write $\|\cdot\|_{L^2}$ instead of $\|\cdot\|_2$ to distinguish it clearly from the graph norms.

By the assumptions on f_ν there is a constant $C(U)$ such that

$$\begin{aligned}\sup\left\{|f_{\nu_0}(t) - f_\nu(t)| : \nu \in B_h(\nu_0), |t| \le \sup\{\|u\|_\infty : u \in U\}\right\}&\\ \le C(U)\cdot o(1) \text{ as } h \to 0.&\end{aligned}$$

This means that for any $0 \le t \le t^+$ and any $u_0 \in U$ with $\varphi_\nu^{[0,t]}, \varphi_{\nu_0}^{[0,t]} \subset U$ we have

$$\begin{aligned}
\|\varphi_\nu^t(u_0) - \varphi_{\nu_0}^t(u_0)\| &\le C \cdot \int_0^t \|A^{\frac{1}{2}} e^{-A(t-s)}(f_\nu(\varphi_\nu^s(u_0)) - f_{\nu_0}(\varphi_{\nu_0}^s(u_0))\|_{L^2}\, ds \\
&\le C \int_0^t (t-s)^{-\frac{1}{2}} \big(\|f_{\nu_0}(\varphi_\nu^s(u_0)) - f_{\nu_0}(\varphi_{\nu_0}^s(u_0))\|_{L_2} + \\
&\qquad \|f_\nu(\varphi_\nu^s(u_0)) - f_{\nu_0}(\varphi_\nu^s(u_0))\|_{L_2}\big)\, ds \\
&\le \int_0^t C \cdot (t-s)^{-\frac{1}{2}} \cdot \big(C(U) \cdot \|\varphi_\nu^s(u_0) - \varphi_{\nu_0}^s(u_0)\| + C(U) \cdot o(1)\big) \\
&\le C(h, I, U) \cdot o(1) + \int_0^t C(U)(t-s)^{-\frac{1}{2}} \|\varphi_\nu^s(u_0) - \varphi_{\nu_0}^s(u_0)\|\, ds \text{ as } h \downarrow 0.
\end{aligned}$$

By Gronwall's inequality

$$\|\varphi_\nu^t(u_0) - \varphi_{\nu_0}^t(u_0)\| \le C(h, I, U) \cdot o(1) \text{ as } h \downarrow 0,$$

which in the notation of [HLR88] means that "φ_ν conditionally approximates φ_{ν_0} on U on compact sets of $[0, \infty)$".

Also, as proved in section A.2, all φ_ν are asymptotically smooth, so the assumptions of Theorem 2.4 in [HLR88] are satisfied.

□

A.4 Persistence of transversal intersections

Lemma A.2. *Let X be a Banach space, M_1, M_2 C^1-submanifolds of X, $\dim M_1 < \infty$, $\operatorname{codim} M_2 < \infty$, $M_1 \pitchfork M_2$, $u_0 \in C := M_1 \cap M_2$, $\psi : U := U_\epsilon(u_0) \cap M_1 \to X$ continuous.*

Then there is an $\epsilon' > 0$ *such that* $\psi(U) \cap M_2 \neq \emptyset$ *if* $\|\psi - \mathrm{id}\|_{\mathcal{C}^0} < \epsilon'$.

Proof. **Case 1:** $\dim M_1 = \operatorname{codim} M_2$. Then we can assume without loss $C = \{u_0\}$, and also $u_0 = 0$. Let $X_i = T_0 M_i$, $i = 1, 2$, then $X = X_1 \oplus X_2$, let P_i be the projections onto X_i along X_{3-i}. For $\delta > 0$ small enough there are $\mathcal{C}^1$-functions

$$g_i : U_\delta(0, X_i) \to X_{3-i}, \qquad i = 1, 2$$

and an open neighborhood $U_\delta \subset U_\epsilon$ of 0 such that

$$M_i \cap U_\delta = \operatorname{graph} g_i.$$

Define

$$\begin{aligned} h_1 &: U_\delta(0, X_1) \to X, & x_1 &\mapsto (x_1, g_1(x_1)) \\ h_2 &: U_\delta \to X, & (x_1, x_2) &\mapsto (x_1 - g_2(x_2), x_2). \end{aligned}$$

Now choose $U'_\delta \subset U_\delta$ open such that $\psi(U'_\delta) \subset U_\delta$ for all ψ with $\|\psi - \mathrm{id}\|_{\mathcal{C}^0} \leq \frac{\delta}{2}$, and let $U_1 := h_1^{-1}(U'_\delta) \subset X_1$. Define

$$f_\psi : U_1 \to X_1, \quad f_\psi = P_1 \circ h_2 \circ \psi \circ h_1,$$

then

$$\begin{aligned} f_\psi(x) = 0 &\iff h_2 \circ \psi \circ h_1(x) \in X_2 \\ &\iff P_1(\psi \circ h_1(x)) = g_2(P_2(\psi \circ h_1(x)) \\ &\iff \psi(\underbrace{h_1(x)}_{\in M_1}) \in M_2 \cap U_\delta \Rightarrow \psi(M_1) \cap M_2 \neq \emptyset, \end{aligned}$$

so it is sufficient to prove that f_ψ has a zero in U_1 if $\|\psi - \mathrm{id}\|_{\mathcal{C}^0}$ is small enough. We want to prove this using the Brouwer degree. Let

$$f_{\mathrm{id}} : U_1 \to X_1, \quad f_{\mathrm{id}} = P_1 \circ h_2 \circ \mathrm{id} \circ h_1 = \mathrm{id} - g_2 \circ g_1,$$

then

$$\begin{aligned} f_{\mathrm{id}}(x) = 0 &\iff h_2 \circ h_2(x) \in X_2 \iff P_1(h_1(x)) = g_2(P_2(h_1(x))) \\ &\iff \underbrace{h_1(x)}_{\in M_1} \in M_2 \iff h_1(x) = 0 \\ &\iff x = 0. \end{aligned}$$

We have

$$Df_{\mathrm{id}}(0) = \mathrm{id} - Dg_2(g_1(0)) \circ \underbrace{Dg_1(0)}_{=0} = \mathrm{id},$$

so $\deg(f_{\mathrm{id}}, U_1, 0) = 1$.

Now there is an $0 < \epsilon' < \frac{\delta}{2}$ such that $0 \notin f_\psi(\partial U_1)$ if $\|\psi - \mathrm{id}\|_{C^0} < \epsilon'$, so for such ψ

$$\deg(f_\psi, U_1, 0) = \deg(f_{\mathrm{id}}, U_1, 0) = 1.$$

This implies $0 \in f_\psi(U_1)$.

Case 2: $\dim M_1 > \operatorname{codim} M_2$, again let w.l.o.g. $u_0 = 0$. Replace M_1 by $\tilde{M}_1 := (T_0 C)^\perp \cap M_1$ (and decrease ϵ if necessary to make this intersection transversal). Then $\dim \tilde{M}_1 = \operatorname{codim}_{M_1} C = \operatorname{codim} M_2$ because $C \subset M_1$ and $\dim C = \dim M_1 - \operatorname{codim} M_2$.

Now apply case 1, and the assertion follows. □

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