A general first-passage-time model for multivariate credit spreads and a note on barrier option pricing

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Für meine Omas und Opas

Preface

After finishing my diploma in mathematics in November 2002 I was not totally sure about doing a PhD, so I started working with KPMG as a quantitative advisor where my main task was to price any financial product. There I was more and more convinced that I needed deeper theory for my understanding. I went back to the university of Giessen and got a research position within a project that is financed by BMBF (Bundesministerium für Bildung und Forschung) which I gratefully acknowledge. With this dissertation I never went back to pure mathematics, but at least I found my place – somewhere in between deep theory and pure application.

I have to thank many people who helped me to come this far and made my time so enjoyable with workshops, winterschools, coffee breaks and cocktails:
I want to thank my supervisors Prof. Dr. Ludger Overbeck, Prof. Dr. Wolfgang Schmidt, and Prof. Dr. Winfried Stute for many discussions and advices. Prof. Stute, thanks for accompanying me through my whole university life! Three years ago, I had to promise my grandma that she can come to my disputation in order to meet you! I further want to thank all my colleagues and friends at the university of Giessen, the Frankfurt School of Finance & Management, and the university of Ulm; in particular Rolf Klaas, Swantje Becker, Christina Niethammer, Natalie Packham, and last but not least Rüdiger Kiesel. I want to thank my great friends from the good old Coba time: Andreas, Antonis and Tino. It's always a pleasure to catch up with you! Andreas, a very big thank-you goes to you for having spent such an enjoyable time with me here in Frankfurt and such an amount of time with my thesis. I am also very grateful to Nick Bingham for his big effort to insert thousands of commas and hyphens into my writing: I shall remember the nice story of Cinderella! Once more, thank you to all of my friends!

I am most grateful to my best and oldest friends, Meli and Olli, for being there – always when necessary - for almost all my life!
Finally, I want to thank my family: Mama, Papa, Oma und Opa und nochmal Oma und Opa, danke, dass ihr mich so lange unterstützt habt!

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Introduction

A *first-passage time (FPT)* is defined as the first time point a stochastic process crosses some *threshold level*, *default boundary* or *critical barrier* that is usually a constant, but can be a random variable or even a stochastic process itself. When we aim at general results for first-passage times we will call this stochastic process the *first-passage process* or *underlying process*. This model approach – 'a first-passage process falling below some threshold' – is called *threshold model/approach.* A first-passage time will be a *stopping time* with respect to a *filtration* that holds the necessary information to answer the question whether a first passage happened. In this dissertation we will introduce a new class of first-passage processes: Brownian motion time-changed with a continuous stochastic non-decreasing process.

Our main application will be in *credit risk*, where the first-passage process will be interpreted as *firm-value process*, *asset-value process* or *ability-to-pay process* - depending on one's particular interest. The threshold level will be a function of the firm's *liabilities.* The threshold approach thus connects *equity* and *debt* of a firm. In this context the threshold model is also called *structural model*, see Definition 1.8, and the first-passage time is called *default time* because it indicates a company's *default event.* A default event may not imply a total default (liquidation of the firm) but can also indicate a *rating downgrade*[1]. As literature on credit risk and *credit derivatives* we suggest Bluhm & Overbeck (2006), Schönbucher (2003) and Martin, Reitz & Wehn (2006).

There is another application in *barrier option pricing.* Here the underlying process is the option's *underlying.* The threshold level is the *barrier* value and the first-passage time indicates a certain knock-in or knock-out event.

Our focus is on models that yield a *first-passage-time distribution* which

[1] Rating systems measure the creditworthiness of borrowing companies. The borrowers are ranked in ratings. External ratings are assigned by rating agencies - the most famous ones are Standard & Poor's (S&P), Moody's and Fitch - and internal ratings are established by the credit institute itself. An improving creditworthiness turns into a rating upgrade and a worsening creditworthiness into a rating downgrade. The transition probabilities are given by a migration matrix. Compare Bluhm et al. (2003) and Bluhm & Overbeck (2006).

can be represented by an integral and/or series. We will call this an *analytical distribution*. In the mentioned applications, credit risk and barrier option pricing, this allows for a simpler calibration of the model to given market data, either a *default-probability curve* or barrier option prices.

In the following we summarize the advances of the structural model: The *classical structural approach*, introduced by MERTON (1974), considers a geometric Brownian motion as asset-value process and assumes that the firm's debt consist of only one issued *zero-coupon bond* with some *face value*. This classical approach allows for a default only at one date, the *maturity* T of the bond, and default happens when the asset value at maturity is less than the face value K. This is illustrated in Figure 1. In order to

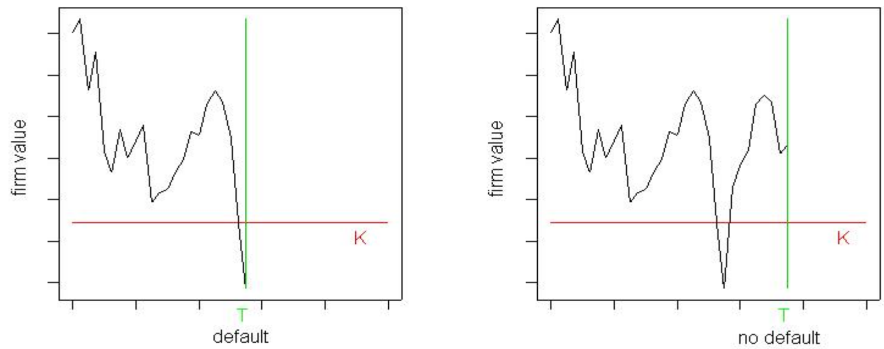

Figure 1: Classical structural approach: default happens at a fixed time T when the firm value at that time, Y_T, lies below the pre-specified default threshold K, that is when $Y_T < K$.

determine the default distribution Merton applied the well-known option pricing result by BLACK & SCHOLES (1973). We name a few extensions of Merton's classical approach: GESKE & JOHNSON (1984) considered coupon-bearing bonds. LELAND (1994) and LELAND & TOFT (1996) applied a generalized geometric Brownian motion and examined an optimal capital structure, i.e. maturity and amount, of corporate debt. Their model is able to predict various *credit-spread term structures*.[2] SHIMKO, TEJIMA & VAN DEVENTER (1993) introduced stochastic interest rates following a *Vasicek process*, see VASICEK (1977). Regarding *Lévy processes*[3] a default-time distribution under the classical approach can be determined when a closed-form probability density is available. This is for example the case for

[2]We introduce credit spreads in Chapter 3 and analyze them in Chapter 4, 5 and 6.

[3]A Lévy process is defined by independent and stationary increments and stochastic continuity, see Def. 1.3 and cf. SATO (1999), KYPRIANOU (2006b) or APPLEBAUM (2005).

the jump-diffusion model, see Zhou (1997a) and Subsection 1.2.6, and for some subordinated Lévy processes, see Subsections 1.2.8 and 1.2.10.

Black & Cox (1976) extended the classical approach to the *first-passage approach*: They considered a geometric Brownian motion as ability-to-pay process continuously in time, an exponentially time-dependent default boundary and allowed for default at any time, whenever the threshold has been hit. Their approach yields an analytical first-passage-time distribution; compare Harrison (1985) and see Section 1.2.1. There are several extensions of this original first-passage approach. A first survey on first-passage times was given by Abrahams (1986). An overview of more recent first-passage models can be found in Bielecki & Rutkowski (2002) and Elizalde (2005) - among these are the following: Kim, Ramaswarny & Sundaresan (1993) introduced stochastic interest rates following a *Cox-Ingersoll-Ross (CIR) process*; see Cox, Ingersoll & Ross (1985). There is a positive probability of default at maturity. Longstaff & Schwartz (1995b) used Vasiceck interest rates. Nielsen, Saà-Requejo & Santa-Clara (1993) applied stochastic interest rates modeled by a *Hull & White process*, and introduced a stochastic default barrier. Both papers, however, show that introducing a stochastic process for the risk-free interest rate has only a small effect on credit spreads. Briys & de Varenne (1997) analyzed correlated asset-value and interest-rate processes. Regarding Lévy processes an analytical FPT formula is not available. But, at least Kyprianou (2006a) and Alili & Kyprianou (2005) were able to determine overshoot and undershoot densities for some specific, time-changed Lévy processes at first-passage, see Subsections 1.2.8 and 1.2.9. For a *jump-diffusion model* with *double-exponentially distributed jumps*, Kou & Wang (2003) derived the Laplace transform of the FPT distribution, see Subsection 1.2.7, in our terms it is not an analytical FPT distribution.

The structural models mentioned above assume that the asset-value process is adapted to the market filtration. In reality this is not true; see Buffett (2002). For this Duffie & Lando (2001) and Giesecke (2004), (2006) analyzed the role of information in structural models and introduced models with *incomplete (accounting) information* about the firm assets, as well as about the liability-dependent threshold barrier. These models belong to the class of *hybrid models* because they combine the advantages of *reduced-form models* and structural models, that is, tractability to market data and economic intuition. For completeness, one more word about the class of reduced-form models. They are also called *intensity-based models* or *hazard-rate models* because default probabilities are modeled through a possibly random *intensity* or *hazard rate (process)*. Default time is an exogenous random variable and the cause of default is not further specified (through a firm value or asset value). These two main model classes, structural and reduced-form models, are compared in Jarrow & Protter (2004). Other hybrid models where default intensity explicitly depends on firm value were considered by MADAN & UNAL (1998) or AMMANN (1999). Ammann herein analyzed *counterparty risk* in a Merton-type framework.

We also want to name some literature of the more recent class of *credit-barrier models*, where the focus lies on an influencing factor other than the firm value. HULL & WHITE (2001) and AVELLANEDA & ZHU (2001) modeled the *distance-to-default* in a structural framework with *time-dependent threshold boundary.* They assumed that the distance-to-default is not observable; instead it is used to build a risk-neutral measure that leads to realistic spread curves. GORDY & HEITFIELD (2001) modeled the distance-to-default through a one-factor model and focus on the changes in distance-to-default over time, in order to model rating transitions. They found that the process of rating transitions is not closely tied to a default-indicating process such as the distance-to-default process. Therefore they suggested to incorporate *through-the-cycle ratings.* The credit-barrier model by ALBANESE ET AL. (2003) and ALBANESE & CHEN (2007) considered the credit rating as driving process which was calibrated to *migration rates*[4] and credit spreads. It captures all important firm information. Models with jumps or stochastic volatility are necessary in order to fit the whole *matrix of migration rates*, that is, not only the probabilities of retaining the same rating level, but also the transition probabilities of ratings changes.

With the just given overview of one-dimensional structural models, including the classical approach, the first-passage approach, hybrid models with incomplete information and credit-barrier models, we have seen that there are structural models, even with jumps, where a classical default-time distribution can be obtained. When regarding first-passage-time models those based on Brownian motion can lead to analytical FPT distributions. But when including jump processes, as far as we know, analytical FPT distributions are not available. But, at least, there are some examples where a numerical approximation for the FPT distribution can be yielded, when a *Laplace transform* or *Fourier transform* of the FPT distribution is available. These will be said to have a *numerical FPT distribution*, in order to differentiate from the *analytical FPT distribution.*

We also want to focus on *multivariate structural models* and desire analytical joint first-passage time distributions that yield *joint default probabilities.* Naturally, we also restrict ourselves to processes based on Brownian motion. For two correlated Brownian motions and an exponentially *time-dependent barrier*, ZHOU (1997b) derived a joint FPT distribution by applying results of COX & MILLER, HARRISON (1985) and REBHOLZ (1994). In this context Rebholz and FISCHER (2003) also considered the two-dimensional Brownian motion with drift. OVERBECK & SCHMIDT (2005) extended the result of Zhou by applying a *deterministic time change*[5] on each Brow-

[4] See footnote 1.

[5] A deterministic time change is a positive non-decreasing function; see Definition 1.1.

nian motion and determined the *joint survival probability* for two correlated processes. In addition, their model perfectly fits the marginal default-probability term structures.

Still, an ability-to-pay process given by a Brownian motion or a deterministically time-changed Brownian motion does not have enough degrees of freedom to adapt marginal default probabilities, joint default probabilities, and *credit-spread dynamics* (the evolution of the credit-spread term structure in time)[6]. Furthermore the dependence between two asset-value processes relies on only one correlation parameter. We analyze the *Merton model* and the *Overbeck & Schmidt model* in the Chapters 4 and 5.

In this dissertation we will consider first-passage-time models that have a continuous stochastic time-change. For this we here give a short history of the time change:
Bochner (1949) first introduced a time-changed Brownian motion. Feller (1966) first presented subordinators as a time change to Markov processes. Clark (1973) introduced Brownian motion with an independent time change as a price process in finance. Monroe (1978) showed that a very general semi-martingale can be embedded in Brownian motion via a time change. Ikeda & Watanabe (1981) studied time-change models for solving SDEs. Øksendal (1990) studied when a stochastic integral can be represented as a time change of a diffusion. Geman, Madan & Yor (2000) and Carr & Wu (2003) introduced subordinated Lévy process. Schoutens (2003), (2004) and Cariboni & Schoutens (2007) used these in derivative pricing.

As the second part of the introduction we here give the structure of this doctoral thesis:

Chapter 1 starts with an introduction to the framework for modeling first-passage times and gives definitions that are used in all the chapters. For the first time we introduce a *general continuous stochastic time-change model* on Brownian motion in a FPT-setting and derive analytical formulas for the FPT distribution in one and several dimensions. The multivariate model introduces a dependence structure via the time change. Our two-dimensional model allows for an additional dependency of correlated Brownian motions, the so-called *Brownian correlation*, and also yields an analytical FPT distribution. We give time-change examples that are close at hand and that yield numerical time-change distributions (in terms of a Laplace transform) and, as a consequence, numerical FPT distributions. Furthermore we give less obvious time-change examples for which we derive an analytical time-change distribution. Whenever a conditional time-change distribution is necessary, which is the case when we determine credit-spread dynamics, numerical approximations are not practical. Our general time-change model can also

[6] See Conclusion 3.5.

be used across assets in order to model dependence structures on the one hand and to model dynamics on the other hand. Up to now, multivariate processes for dependency modeling include only either correlation between the driving processes or a time change. Our bivariate model contains both; see Section 1.3.4.

Chapter 2 analyzes one specific time-change model where the time change is given by the integral over an independent squared Brownian motion. This is the simplest model in our general model class. We refer to it as the *simple time-change model*. We calibrate the model to default-probability curves and yield a good fit for non-investment-grade companies. Using these calibrated model parameters we simulate and plot joint and multivariate default probabilities or survival probabilities, respectively. We analyze the influence of the time change on the joint survival probability and furthermore the relationship between default correlation and event correlation.

Chapter 3 introduces the basic credit product, the credit default swap (CDS), which is a contract between a protection seller and a protection buyer. We determine the CDS spread under annual and under continuous protection payments following Hull & White (2000) and Schmidt (2004b), respectively. Then we introduce the credit-spread dynamics. In order to show that it is reasonable and moreover necessary to consider credit-spread dynamics when modeling credit-spread curves, we empirically study credit-spread volatility of five years markets CDS spreads.

In Chapter 4 we determine the credit spread and the credit-spread dynamics under the FPT approach of the Merton model. The model has no degrees of freedom to influence these credit-spread dynamics. At the end of the chapter we review some extensions of the Merton model. The extensions add randomness to the default barrier (CreditGrades model), the interest rates, or the business clock. Other extensions assume incomplete accounting information or include jumps into the first-passage process. The main advantages of these models are that they can yield a positive instantaneous credit spread and also other credit-spread shapes than the hump-shaped term structure under the Merton model.

Chapter 5 studies the deterministic time-transformation model by Overbeck & Schmidt. The model perfectly fits the default-probability curve and thus the survival-probability term structure. We make an additional assumption of an available default-probability density (derivative of the given default-probability curve) which implies an analytical formula for the credit-spread dynamics. Again credit-spread dynamics are a function of asset value and threshold and cannot be influenced.

In Chapter 6 the general stochastic continuous time-change model is applied to credit-spread modeling. With the time change arbitrarily many degrees of freedom can be included. Furthermore, when choosing a time change with start above zero the model is of incomplete information and yields non-zero instantaneous credit spreads. The time change has several interpretations e.g. as economic time or amount of information flow. The extension of the stochastic time-change model to a multivariate model is straightforward and inserts a dependency via a joint time change. The model is applicable for multivariate products, especially multi-credit products. A time change should not always be chosen all the same for the underlying credits. Instead, the credit-spread dynamics of each underlying have to be studied and the time change should be chosen accordingly. We derive an analytical formula for the continuously-paid credit spread. Assuming an absolutely continuous business clock we can also determine an analytical formula for the credit-spread dynamics. Credit-spread dynamics should in particular be considered when credit contracts have a long time to maturity and when credit products are credit-spread sensitive. We name a few examples: *constant-to-maturity (CMS) swaps*, *credit-spread options*, *credit baskets*, *k-th-to-default swaps*, *collateralized-debt obligations (CDOs)*, *(credit-spread) variance swaps* and *leveraged credit products*; cf. for example Schönbucher (2003) or Hunt & Kennedy (2004). We take the *first-to-default swap* as an example to show that our multivariate model can be used to yield closed formulas for more complex credit products. This is due to the analytical formula for the joint default probability. For all these formulas the conditional time-change density is needed. For this we give two explicit examples, the simple time change of Chapter 2 and a CIR-type time change.

Chapter 7 shows how our stochastic time-change model can be applied to option pricing, especially pricing of barrier options. The time-change model is a stochastic volatility model, and we give the time-change model that is equivalent to the *Heston model* which is well-known for option pricing. Under stochastic volatility models the risk-neutral measure is not unique and we choose to price under the minimal martingale measure. Assuming the general time-change model, allowing for correlation between spot and time change, we derive a closed pricing-formula for the European call. Under no correlation and zero interest rates we show how to derive pricing formulas for barrier options applying our FPT results of Chapter 1. The degrees of freedom of the time change can be used to produce desired volatility features.

Chapter 1

First-passage times under a general continuous stochastic time-change model

This chapter introduces the general stochastic framework which will be the basis throughout this thesis. For definitions and notations from stochastic calculus we follow PROTTER (2004) when considering stochastic processes in general, KARATZAS & SHREVE (1991), STEELE (2001) and KLEBANER (2005) for processes based on Brownian motion and SCHOUTENS (2003), CONT & TANKOV (2004) and KYPRIANOU (2006b) for Lévy processes.

In a structural setting with *underlying process* $(Y_t)_{t\geq 0}$ and a pre-specified *threshold level* K, the *first-passage time (FPT)* is given by the time point the process Y first touches or crosses the threshold. In credit risk the underlying process is the *asset-value process*. The first-passage time will be a *stopping time* with respect to a *filtration* that holds the necessary information – only then we can observe whether a first-passage event happened or not. In general the threshold boundary can be a stochastic process itself, see Figure 1.1, but in that case the asset-value process Y can be adjusted so that the new threshold K is a constant. Note that then the adjusted model has a different interpretation.

We introduce a general continuous time-change model, a process that lives in a transformed time, under another clock, see Definition 1.1. Under this time-change model we derive analytical formulas for first-passage-time distributions. We start with the first-passage model in one dimension, then extend the model to two and higher dimensions. Joint first-passage-time distributions are derived under the following asset dependencies:

- independent Brownian processes but identical time change,
- correlated Brownian processes and identical time change,

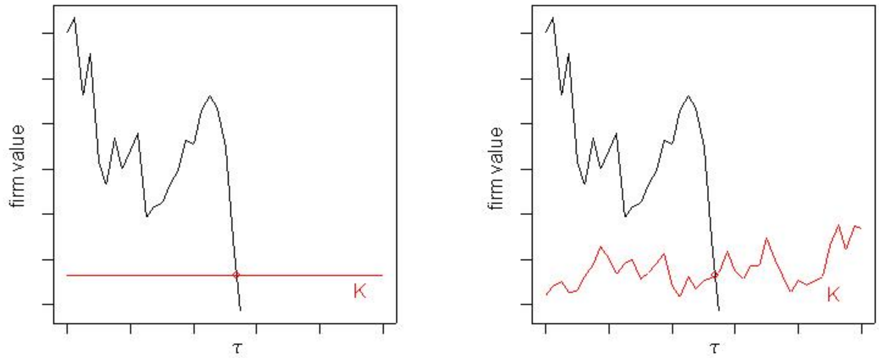

Figure 1.1: Structural FPT approach: default happens at the first-passage time τ, where the firm value process first crosses the default threshold K.

- correlated Brownian processes under separate time changes.

At the end of this chapter we give some time-change examples. Table 1.1 lists Laplace transforms of distributions of time changes close at hand. These lead to numerical FPT distributions. For one thing it is not clear whether there are indeed time-change processes with an analytical distribuition such that an analytical FPT distribution can be obtained. We give examples. These are listed in Table 1.3. In Chapter 6 we analyze the evolution of credit speads under the stochastic time-change model. In order to obtain *analytical conditional survival probabilities*, *analytical first-passage-time distributions* and herewith also *analytical time-change distributions* become necessary.

1.1 First-passage-time framework

We assume a *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω represents the *states of the world*, $\mathcal{F}$ is the *σ-algebra* containing all possible *events* of interest and $\mathbb{P} : \mathcal{F} \to [0,1]$ is the *probability measure*. The probability space is assumed to be equipped with a *filtration* $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. A filtration is a nondecreasing family of sub-σ-algebras of $\mathcal{F}$, that is, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $0 \leq s \leq t$. In this thesis any stochastic process X will live on $\Omega \times [0, \infty)$, sometimes only on $\Omega \times [0, T]$, $T < \infty$, and will be adapted to $\mathbb{F}$, i.e. for all $t \geq 0$, X_t will be $\mathcal{F}_t$-measurable. The underlying process will throughout be denoted by $(Y_t)_{t \geq 0}$ and generates the filtration $\mathbb{F}^Y = (\mathcal{F}_t^Y)$, $\mathcal{F}_0^Y := \{\emptyset, \Omega\}$ and $\mathcal{F}_t^Y := \sigma(Y_s : s \leq t)$, that is the smallest filtration holding all the information about Y, and is called *natural filtration of* Y. Brownian motions will be denoted by W and B and it is assumed that they start in zero.

Definition 1.1 (Time change)
A deterministic time change is a positive, non-decreasing function and a stochastic time change a positive, non-decreasing stochastic process.

The non-decreasing property of the time transformation can be understood as saying that information that has been obtained once, will never be lost. Thus a stochastic time change can contain neither a pure Brownian motion nor negative jumps. A stochastic time change adds stochastic volatility to a process. The original clock will sometimes be called *normal clock* and the new clock will be called *business clock*. The time change may be interpreted as *experienced time*, that runs faster when the information flow is bigger (or speeds up). In other words, experienced time is a measure of the amount of information arrival.[1]

Definition 1.2 (Absolute continuity)
A stochastic process X is called *absolutely continuous* if there exists a process (h_t) such that

$$X_t \quad = \quad h_0 + \int_0^t h_s \,\mathrm{d}s \ .$$

If X is a positive, increasing process we can substitute the positive process (h_t) by a process (g_t) with $g_t^2 = h_t$.

Definition 1.3 (Lévy process)
A real-valued *càdlàg*[2] $\mathbb{F}$-adapted stochastic process L with independent, stationary increments that is stochastic continuous, i.e.

$$\forall t \geq 0, \ \forall \epsilon > 0 \ : \qquad \lim_{s \to t} \mathbb{P}(|L_t - L_s| > \epsilon) = 0 \quad ,$$

is said to be a *Lévy process*.

Every Lévy process $(L_t)_t$ can be associated with an *infinitely divisible*[3] random variable, through L_1. Any infinitely divisible distribution is specified

[1] Geman, Madan & Yor (2000) show that a time change (of a Lévy process) represents a measure of activity in the economy and therefore is a speed of the economy.

[2] Càdlàg is the abbreviation for *continu à droite et limites à gauche* (right continuous paths with left limits).

[3] The distribution of a random variable X is said to be *infinitely divisible* if for all $n \in \mathbb{N}$ there exist iid random variables $X_1^{(n)}, \ldots, X_n^{(n)}$ such that

$$X \stackrel{\mathcal{L}}{=} X_1^{(n)} + \cdots + X_n^{(n)} \quad .$$

by its characteristic triplet (b, A, ν) with $b \in \mathbb{R}$, $A \in \mathbb{R}_+$ and a Borel measure ν satisfying $\nu(\{0\}) = 0$, $\int_{\mathbb{R}}(1 \wedge x^2)\nu(\,\mathrm{d}x) < \infty$, where the so-called *characteristic exponent*

$$\psi_1(u) \quad = \quad iub - \frac{u^2 A}{2} + \int_{\mathbb{R}} (\mathrm{e}^{iux} - 1 - iux\mathbb{I}_{\{|x|<1\}})\nu(\,\mathrm{d}x)$$

yields the characteristic function

$$\mathbb{E}\left[\mathrm{e}^{iuL_1}\right] \quad = \quad \mathrm{e}^{\psi_1(u)} \quad .$$

Thus also L_t is identified through (b, A, ν) (called *Lévy triplet*) and the so-called *Lévy exponent*

$$\psi_t(u) \quad \equiv \quad t\psi_1(u)$$

describes the characteristic function

$$\mathbb{E}\left[\mathrm{e}^{iuL_t}\right] \quad = \quad \mathrm{e}^{\psi_t(u)} \quad .$$

ν is said to be the *Lévy measure*, and its density, when this exists, the *Lévy density*. A linear drift is the simplest Lévy process. Brownian motion (with drift) is the only non-trivial Lévy process with continuous paths. A Lévy process may have small (< 1) *infinite-variation* jumps and large (≥ 1) *finite-variation*[4] jumps. A Lévy process can be split up in the just-mentioned descriptive components:

$$\begin{aligned} L_t \quad &= \quad bt + \sqrt{A}W_t + \int_0^t \int_{\mathbb{R}} x(\mathcal{J}^L - \nu)(\,\mathrm{d}s,\,\mathrm{d}x) \\ &= \quad \underbrace{\Big(b - \int_{|x|\geq 1} x\nu(\,\mathrm{d}x)\Big)}_{\equiv\, b'} t + \underbrace{\sqrt{A}W_t}_{\text{continuous martingale}} \\ &\quad + \underbrace{\int_0^t \int_{|x|\geq 1} x\mathcal{J}^L(\,\mathrm{d}s,\,\mathrm{d}x)}_{\text{Compound poisson process}} + \underbrace{\int_0^t \int_{|x|<1} x\mathcal{J}^L(\,\mathrm{d}s,\,\mathrm{d}x) - t\int_{|x|<1} x\nu(\,\mathrm{d}x)}_{\text{pure jump martingale}} \end{aligned}$$

where $\mathcal{J}^L$ is the *jump measure* (or *Poisson measure*), a random measure counting the jumps of magnitude ≥ 1 resp. < 1. The Lévy measure ν is its *compensator*, in that the integral over the small jumps under the *compensated jump measure* $\mathcal{J}^L - \nu$ is a martingale. This decomposition is called *Lévy-Itô decomposition.*

[4] A process X is said to be of *finite variation* if almost all paths are of finite variation on each compact interval, i.e. for any decreasing partition Π_m of this interval that is tending to zero (for $m \to \infty$, w.r.t. the maximum-norm) it is true that

$$\lim_{m\to\infty} \sum_{\Pi_m} |X_{t_k} - X_{t_{k-1}}| < \infty \quad .$$

Otherwise the process is said to be of *infinite variation.*

As literature on Lévy processes we suggest PAPAPANTOLEON (2005), SATO (1999), KYPRIANOU (2006b), CONT & TANKOV (2004) and APPLEBAUM (2005).

Definition 1.4 (Subordinator)
A *subordinator* is an increasing Lévy process, that is it has a nonnegative drift, no diffusion and only positive jumps that are of finite variation.

Subordinators (without drift) are fully characterized through their jump measure ν. Since subordinators are positive non-decreasing processes they can be used as time-change processes. Examples are the *gamma process* or the *inverse Gaussian process*, whose corresponding Lévy densities are given in Table 1.1.

Definition 1.5 (Stopping time)
On a measurable, filtered space $(\Omega, \mathcal{F}, \mathbb{F})$ a random variable T is called *stopping time* of the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Definition 1.6 (Laplace transform)
The *Laplace transform* of a positive random variable T is given by

$$l_T(u) \quad := \quad \mathbb{E}\left[\mathrm{e}^{-uT}\right] \;=\; \int_0^\infty \mathrm{e}^{-ut}\, \mathbb{P}(T \in \,\mathrm{d}t)\;, \quad u \geq 0\;.$$

Integration by parts yields the following equivalence for the Laplace transform:

$$l_T(u) \quad = \quad u \int_0^\infty \mathrm{e}^{-ut}\, \mathbb{P}(T \leq t)\, \mathrm{d}t\;. \tag{1.1}$$

Definition 1.7 (Default time under the classical structural approach)
Given the underlying process $(Y_t)_{t\geq 0}$, a threshold level K, and a fixed time $T \in [0, \infty)$ the *classical default time* is defined by

$$\tau \quad := \quad \begin{cases} T & \text{if } Y_T < K \\ \infty & \text{otherwise}\;, \end{cases}$$

a discrete random variable on $\mathbb{R} \cup \{\infty\}$ whose distribution is specified by $\mathbb{P}(\tau = T)$.

Definition 1.8 (First passage time (FPT) under the structural approach)
Given the underlying process $(Y_t)_{t\geq 0}$ and a threshold level K, the *first-passage time* (or *default time*) is defined by

$$\tau \quad := \quad \inf\{s \geq 0 : Y_s < K\}\;.$$

The *first-passage-time distribution*, also called *default probability*, is denoted by $\mathbb{P}\,(\tau \leq t)$ and for $u \geq t$ the conditional default probability density given the information $\mathcal{F}_t^Y$ is given by $\mathbb{P}(\tau \in \,\mathrm{d}u \mid \mathcal{F}_t^Y)$.

Since $\{\tau \leq t\} = \{\inf_{0 \leq s \leq t} Y_s < K\} \in \mathcal{F}_t^Y$ for all t, Definition 1.5 tells us that the first-passage time τ is a $\mathbb{F}^Y$-stopping time. Considering a FPT is only interesting, when the first-passage event has not already occurred at time zero; so we will always assume $K \leq Y_0$.

Definition 1.9 (Multivariate first-passage-time distribution)
Given underlying processes $(Y_t^i)_{t \geq 0}$, threshold levels K_i and default times $\tau_i = \inf\{s \geq 0 : Y_s < K\}$, $i = 1, \dots, n$, the *joint first-passage-time distribution* (or *joint default probability (JDP)*) is given by

$$\mathbb{P}(\tau_1 \leq t, \dots, \tau_n \leq t) \quad .$$

Furthermore

$$\mathbb{P}(\tau_1 > t, \dots, \tau_n > t)$$

is called *joint survival probability (JSP)*.

The two-dimensional JDP, JSP and marginal probabilities are in the following relation:

$$\mathbb{P}(\tau_1 \leq t, \tau_2 \leq t) = \mathbb{P}(\tau_1 > t, \tau_2 > t) + \mathbb{P}(\tau_1 \leq t) + \mathbb{P}(\tau_2 \leq t) - 1 \ . \tag{1.2}$$

The following definitions characterize the dependencies in a multivariate model.

Definition 1.10 (Asset correlation)
The correlation between two asset value processes Y^1 and Y^2,

$$\rho_t^A := \operatorname{Corr}\left(Y_t^1, Y_t^2\right) = \frac{\operatorname{Cov}(Y_t^1, Y_t^2)}{\sqrt{\operatorname{Var}(Y_t^1)\operatorname{Var}(Y_t^2)}} \ ,$$

is called *asset correlation*.

Asset correlation does not display the whole dependence between two stochastic processes, as it does not totally specify the joint distribution. We will say that the joint distribution is described by the *asset dependence*. Concerning our processes, asset dependence will be due to correlated driving Wiener processes on the one hand and a joint (or dependent) time change on the other. Therefore the next definition is to distinguish the cause of dependence. Assume the Brownian motions W^1 and W^2 are correlated with constant correlation parameter ρ. Then there exists a Brownian motion $W^\perp$, independent of W^1, such that W^2 can be written in terms of W^1 and $W^\perp$:

$$W_t^2 = \rho W_t^1 + \sqrt{1-\rho^2}\, W_t^\perp \quad \forall t \ .$$

Definition 1.11 (Brownian correlation)
A constant correlation between Brownian motions will be called *Brownian correlation*. If there is zero correlation we say there is *Brownian independence.*

Definition 1.12 (Event correlation / default correlation)
The correlation between the default events $\{\tau_1 \leq t\}$ and $\{\tau_2 \leq t\}$,

$$\begin{aligned}\rho_t^E &:= \operatorname{Corr}\left(\mathrm{I\!I}_{\{\tau_1 \leq t\}}, \mathrm{I\!I}_{\{\tau_2 \leq t\}}\right) = \frac{\operatorname{Cov}\left(\mathrm{I\!I}_{\{\tau_1 \leq t\}}, \mathrm{I\!I}_{\{\tau_2 \leq t\}}\right)}{\sqrt{\operatorname{Var}\left(\mathrm{I\!I}_{\{\tau_1 \leq t\}}\right) \operatorname{Var}\left(\mathrm{I\!I}_{\{\tau_2 \leq t\}}\right)}} \\ &= \frac{\mathbb{P}(\tau_1 \leq t, \tau_2 \leq t) - \mathbb{P}(\tau_1 \leq t)\mathbb{P}(\tau_2 \leq t)}{\sqrt{\mathbb{P}(\tau_1 \leq t)(1 - \mathbb{P}(\tau_1 \leq t))\mathbb{P}(\tau_2 \leq t)(1 - \mathbb{P}(\tau_2 \leq t))}} \,.\end{aligned}$$

is called *event correlation* or *default correlation.*

We define $m_t := \min(\mathbb{P}(\tau_1 \leq t), \mathbb{P}(\tau_2 \leq t))$ and $M_t := \max(\mathbb{P}(\tau_1 \leq t), \mathbb{P}(\tau_2 \leq t))$. Then $0 \leq \mathbb{P}(\tau_1 \leq t, \tau_2 \leq t) \leq m_t$, and some simple algebraic transformations lead to the following *natural bounds* for the event correlation:

$$\frac{-\mathbb{P}(\tau_1 \leq t)\mathbb{P}(\tau_2 \leq t)}{\sqrt{\mathbb{P}(\tau_1 \leq t)(1-\mathbb{P}(\tau_1 \leq t))\mathbb{P}(\tau_2 \leq t)(1-\mathbb{P}(\tau_2 \leq t))}} \leq \rho_t^E \leq \sqrt{\frac{m_t(1-M_t)}{M_t(1-m_t)}} \,. \tag{1.3}$$

The next lemma connects the stochastic integral (w.r.t. Brownian motion) to a time-changed Brownian motion. We will need this relationship in order to determine dynamics under our stochastic time-change model in Section 6.3. The remark following the lemma considers the special case where the time change is deterministic. This will be applied to derive the dynamics under the Overbeck & Schmidt model in Section 5.4. First of all we introduces the quadratic variation.

Definition 1.13 (Quadratic variation)
Let X be a continuous martingale. The *quadratic variation process* $(\langle X \rangle_t)_{t \geq 0}$ is defined by

$$\langle X \rangle_t := X_t^2 - 2\int_0^t X_s \,\mathrm{d}X_s \,.$$

If X is in addition square-integrable, i.e. $\mathbb{E}[X_t^2] < \infty \;\forall t$, the quadratic variation process $\langle X \rangle$ is the unique adapted, increasing process for which $\langle X \rangle_0 = X_0^2 = 0$ and $X^2 - \langle X \rangle$ is a martingale. Then also, if $\Pi_m = \{t_0, \ldots, t_m\}$ are partitions of $[0, t]$ tending to zero (for $m \to \infty$) in terms of the maximum-norm $|\Pi_m| = \max_{1 \leq k \leq m} |t_k - t_{k-1}|$, we have that the so-called *realized* *quadratic variation* or *sample quadratic variation* $\sum_{k=1}^{m}\left(X_{t_k}-X_{t_{k-1}}\right)^2$ tends to the quadratic variation[5], i.e.

$$\sum_{\Pi_m}\left(X_{t_k}-X_{t_{k-1}}\right)^2 \quad \stackrel{|\Pi_m|\to 0}{\longrightarrow} \quad \langle X\rangle_t \quad \text{in probability.}$$

Furthermore, for two stochastic processes X and Y the *realized covariation* is given by

$$\sum_{k=1}^{m}\left(X_{t_k}-X_{t_{k-1}}\right)\left(Y_{t_k}-Y_{t_{k-1}}\right) \ .$$

Lemma 1.14 [6] *(Time-changed Brownian motion)*
Let (g_t) be a stochastic process on $(\Omega,\mathcal{F},\mathbb{P})$ that is adapted to the filtration $\mathbb{F}=(\mathcal{F}_t)$, has càglàd[7] paths and satisfies $\mathbb{E}[\int_0^t g_s^2\,\mathrm{d}s]<\infty$ for every $t\geq 0$. Furthermore $(W_t,\mathcal{F}_t)$ be a Brownian motion. Then the stochastic integral $\int_0^t g_s\,\mathrm{d}W_s$ is a continuous local martingale with quadratic variation

$$\left\langle \int_0^{\cdot} g_s\,\mathrm{d}W_s\right\rangle_t \quad = \quad \int_0^t g_s^2\,\mathrm{d}s \ =: \ G_t \ ,$$

mean zero and variance

$$\mathrm{Var}\left[\int_0^t g_s\,\mathrm{d}W_s\right] \ = \ \mathbb{E}\left[\left(\int_0^t g_s\,\mathrm{d}W_s\right)^2\right] \ = \ \mathbb{E}\left[\left\langle\int_0^{\cdot} g_s\,\mathrm{d}W_s\right\rangle_t\right] \ = \ \mathbb{E}[G_t] \ .$$

Furthermore

$$G_t^{-1} \quad = \quad \inf\left\{u\geq 0 \ : \ \left\langle\int_0^{\cdot} g_s\,\mathrm{d}W_s\right\rangle_u > t\right\}$$

is a $\mathbb{F}$-stopping time. Then there exists a Brownian motion $\left(B_t,\mathcal{F}_{G_t^{-1}}\right)$ such that, a.s.,

$$\int_0^t g_s\,\mathrm{d}W_s \quad = \quad B_{G_t} \ , \quad 0\leq t<\infty \ .$$

Remark 1.15 (Deterministically time-changed Brownian motion)
In particular the last lemma holds for a deterministic time change $G_t = \int_0^t g_s^2\,\mathrm{d}s$. In fact, then $\left(\int_0^t g_s\,\mathrm{d}W_s\right)_t$ and $(W_{G_t})_t$ are both Gauss-processes having the same covariance structure i.e.

$$\int_0^t g_s\,\mathrm{d}W_s \ \stackrel{\mathcal{L}}{=} \ W_{G_t} \ .$$

This will be used in Chapter 5.

[5]Cf. Karatzas & Shreve (1991), Section 1.5.
[6]Cf. Karatzas & Shreve (1991), Section 3.4, B.
[7]Càglàd is the abbreviation for *continu à gauche et limites à droite* (left continuous paths with right limits).

Remark 1.16 (Quadratic variation of time-change processes)
Geman, Madan & Yor (2002) considered the composite process of a Brownian motion W time changed with a right-continuous process A,

$$Y_t \quad = \quad W_{A_t} \ ,$$

and analyzed whether the time change can be observed under the information of the underlying process $\mathbb{F}^Y$. This is important because if the time change is known to all market participants continuously in time, then it is useable for hedging and martingale models are available for pricing. Otherwise pricing under the filtration $\mathbb{F}^Y$ is critical. If the time change is continuous and square integrable, i.e. $\mathbb{E}[A_t^2] < \infty$, it is given by the quadratic variation of the time-changed process (cf. Geman, Madan & Yor (2000)):

$$A_t = \langle Y \rangle_t \ .$$

Geman, Madan & Yor showed that a discontinuous time change cannot be recovered by the observed composite process (i.e. by its realized quadratic variation). For a general discontinuous time change they obtained

$$\mathbb{E}\left[([Y,Y]_t - A_t)^2\right] \quad = \quad \mathbb{E}\left[2[A,A]_t\right] \ > \ 0 \ ,$$

so the time change is not determined by the quadratic variation.[8] They considered a variety of discontinous time changes, analyzed whether the realized quadratic variation is (at least) a sufficient statistic for the time change and found that this is only the case for the Gamma time change.

1.1.1 The aim of this chapter and our application

We aim for a structural first-passage-time model τ that leads to an analytical first-passage-time distribution, in one and more dimensions. We will assume a constant threshold level $K \leq 0$ and for the underlying process Y we search a model class that, in the multivariate model, inherents some dependency structure.

In our application to modeling credit-spread curves, the structural approach has a realistic interpretation of the default event and the available analytical FPT distributions simplify calibrations. Moreover, dependencies in our multivariate model, i.e. business time and asset correlation, can be explained. Last but not least our model yields credit-spread dynamics and allows for input on the credit-spread volatility. This is especially necessary and important for long-term and leveraged credit products.

When it comes to calibrating a specific model (see Chapter 2 and 5) we sometimes assume that the market provides us with a FPT distribution $(F(t))_{0\leq t\leq T}$. Usually, for simplicity, we will assume it is given by the

[8] $[Y,Y]_t$ denotes the quadratic variation for a semimartingale, in general. Our Def. 1.13 is well-defined because in case Y is continuous we have $[Y,Y]_t = \langle Y \rangle_t$.

exponential distribution

$$F(t) \;=\; 1 - \mathrm{e}^{-\lambda t}\,, \quad \lambda > 0. \tag{1.4}$$

Then on a discrete grid $\{t_0 = 0, \ldots, t_m = T\}$ we want to fit the model exactly at least at two points,

$$F(t_i) \;\overset{!}{=}\; \mathbb{P}\left(\tau \le t_i\right)\,,$$

e.g. the liquid credit spread points $t_1 = 5$ years and $t_2 = 7$ years.

1.2 Survey: default-time and first-passage-time models

In this chapter we want to keep the results as general as possible and therefore do not focus on the interpretation in credit risk and the link to equity and debt. This we postpone to the Chapters 3, 4, 5 and 6, where we give an introduction to credit risk, especially to the credit spread, and analyze the *Merton model*, the *Overbeck & Schmidt model* and our *stochastic time change model* in detail.

1.2.1 Brownian motion with drift

Merton (1974) introduced the classical threshold model to finance, where the considered underlying process is given by geometric Brownian motion and the threshold level is supposed to be constant. At a fixed point in time T, one is interested in whether at that time the underlying process crossed the threshold or not. This model is called the *Merton model*. It is equivalent to considering Brownian motion (with volatility and drift parameter), $Y_t = \sigma W_t + \mu t$, and a constant threshold barrier K (given by the natural logarithm of the original threshold value). Please note that throughout this thesis W denotes a Brownian motion starting at zero, μ a constant drift, σ a constant volatility, $K \le 0$ a constant threshold level, Φ the standard normal cumulative distribution function and ϕ its density. The default-time distribution for the classical approach (see Definition 1.7) is specified by

$$\mathbb{P}\left(\tau = T\right) \;=\; \mathbb{P}\left(\sigma W_T + \mu T < K\right) \;=\; \Phi\left(\frac{K - \mu T}{\sigma\sqrt{T}}\right)\,. \tag{1.5}$$

Black & Cox (1976) started to consider the Merton model in a *first-passage time (FPT)* approach. So again geometric Brownian motion was chosen to be the underlying process, and the threshold was assumed to be exponentially time dependent. Equivalently, we consider the first-passage-time problem for Brownian motion with drift and a constant threshold barrier. The FPT distribution can be derived via the reflection principle[9] (since

[9]See Harrison (1985) or Karatzas & Shreve (1991), Section 2.6.

$K \leq W_0$), and yields

$$\begin{aligned} \mathbb{P}(\tau \leq t) &= \mathbb{P}\left(\min_{0\leq s\leq t}[\sigma W_s + \mu s] < K\right) \\ &= \Phi\left(\frac{K-\mu t}{\sigma\sqrt{t}}\right) + \mathrm{e}^{2\frac{\mu K}{\sigma^2}}\,\Phi\left(\frac{K+\mu t}{\sigma\sqrt{t}}\right) . \end{aligned} \tag{1.6}$$

1.2.2 Joint survival probability - Brownian motion

Zhou (2001) determined the JSP of the two-dimensional Brownian motion $(\sigma_1 W^1, \sigma_2 W^2)$, with correlation parameter ρ, not crossing the upper bound (K_1, K_2) by considering the equivalent problem of solving a partial differential equation (PDE). Let

$$\tilde{F}(K_1, K_2, t) = \mathbb{P}\left(\max_{0\leq s\leq t}\sigma_1 W_s^1 < K_1,\ \max_{0\leq s\leq t}\sigma_2 W_s^2 < K_2\right) \tag{1.7}$$

be the joint survival probability and $f(x_1, x_2, t)$ the corresponding transition probability density satisfying

$$\tilde{F}(K_1, K_2, t) = \int_{-\infty}^{K_1}\int_{-\infty}^{K_2} f(x_1, x_2, t)\,\mathrm{d}x_1\,\mathrm{d}x_2 .$$

That is, f is the probability density that a particle being at (x_1, x_2) at time zero and not having yet crossed the boundary (K_1, K_2) will not reach that boundary in the time interval $[0, t]$. The transition probability satisfies the Kolmogorov forward equation[10],

$$\begin{aligned} \frac{\sigma_1^2}{2}\frac{\partial^2 f}{\partial x_1^2} + \rho\sigma_1\sigma_2\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2}\frac{\partial^2 f}{\partial x_2^2} &= \frac{\partial f}{\partial t} \\ x_1 < K_1\ ,\ x_2 < K_2 \quad &, \end{aligned}$$

subject to specific boundary conditions:

$$\begin{aligned} & f(x_1, K_2, t) = f(K_1, x_2, t) = 0\ , \quad t > 0\ , \\ & \int_{-\infty}^{K_1}\int_{-\infty}^{K_2} f(x_1, x_2, t)\,\mathrm{d}x_1\,\mathrm{d}x_2 \leq 1\ , \quad t > 0\ , \\ & f(-\infty, x_2, t) = f(x_1, -\infty, t) = 0 \\ & f(x_1, x_2, 0) = \delta(x_1)\delta(x_2)\ , \end{aligned}$$

where δ is the Dirac delta function. The first three conditions are due to the fact that f is a density and the last one displays the starting value of the process. Zhou derived the solution by various transformations of the PDE,

[10] See for example Cox & Miller (1972), Karatzas & Shreve (1991).

in particular by transforming the (x_1, x_2) coordinates into polar coordinates (r, θ), yielding the PDE

$$\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} \quad = \quad 2\frac{\partial f}{\partial t} \; .$$

As a solution he obtained the transition probability density:

$$f(r,\theta,t) \quad = \frac{2}{\sigma_1\sigma_2\sqrt{1-\rho^2}\alpha t}\,\mathrm{e}^{-\frac{r^2+r_0^2}{2t}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \sin\left(\frac{n\pi\theta}{\alpha}\right) I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{t}\right) \; , \tag{1.8}$$

where the abbreviations r_0, θ_0 and α are given in Theorem 1.17 and I_ν denotes the *modified Bessel function* of the first kind with order ν[11]. The JSP $\tilde{F}$ of equation 1.7 is yielded by integrating f.

We will always consider the contrary problem of not crossing a lower barrier. Thus before applying Zhou's solutions for the JSP $\tilde{F}$ and transition probability density f we have to reflect our first-passage processes at the x-axes. Therefore the next theorem states the JSP result in our terms where surviving means not crossing a lower barrier. We will apply Zhou's results various times.

Theorem 1.17 *(Zhou; JSP of a two-dimensional Brownian motion not crossing a lower barrier)*
Let $Y^1 = \sigma_1 W^1$ and $Y^2 = \sigma_2 W^2$ be correlated Wiener processes with correlation parameter ρ and $K_1 < 0$, $K_2 < 0$ the threshold levels. Then Y^1 and Y^2 have the following joint survival probability:

$$\begin{aligned}
& \mathbb{P}\left(\tau_1 > t, \tau_2 > t\right) \\
\equiv \;& \mathbb{P}\left(\min_{0\le s\le t} \sigma_1 W_s^1 > K_1, \min_{0\le s\le t} \sigma_2 W_s^2 > K_2\right) \\
= \;& \frac{2r_0}{\sqrt{2\pi t}}\,\mathrm{e}^{-\frac{r_0^2}{4t}} \sum_{n=1,3,5,\ldots} \frac{1}{n}\sin\left(\frac{n\pi\theta_0}{\alpha}\right)\left[I_{\frac{1}{2}(\frac{n\pi}{\alpha}+1)}\left(\frac{r_0^2}{4t}\right) + I_{\frac{1}{2}(\frac{n\pi}{\alpha}-1)}\left(\frac{r_0^2}{4t}\right)\right] \; ,
\end{aligned}$$

where

$$\begin{aligned}
\theta_0 \quad &= \quad \begin{cases} \tan^{-1}\left(\frac{\sigma_1 K_2\sqrt{1-\rho^2}}{\sigma_2 K_1 - \rho\sigma_1 K_2}\right) & \textit{if } \frac{\sigma_1 K_2\sqrt{1-\rho^2}}{\sigma_2 K_1 - \rho\sigma_1 K_2} > 0 \\ \pi + \tan^{-1}\left(\frac{\sigma_1 K_2\sqrt{1-\rho^2}}{\sigma_2 K_1 - \rho\sigma_1 K_2}\right) & \textit{otherwise,} \end{cases} \\
r_0 \quad &= \quad \frac{Y_0^2 - K_2}{\sigma_2 \sin(\theta_0)} \; , \\
\alpha \quad &= \quad \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \textit{if } \rho < 0 \\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \textit{otherwise.} \end{cases}
\end{aligned}$$

[11] The Bessel function is given in Def. A.3, Borodin & Salminen (2002) and studied in more detail in Revuz & Yor (2005).

1.2.3 Joint survival probability - Brownian motion with drift

We proceed much as in the previous subsection (for Brownian motion without drift). Denote the JSP of the two-dimensional Brownian motion with drift and correlation parameter ρ by

$$\tilde{F}^{\mu}(K_1, K_2, t) = \mathbb{P}\left(\max_{0\le s\le t}\left(\sigma_1 W_s^1 + \mu_1 s\right) < K_1, \max_{0\le s\le t}\left(\sigma_2 W_s^2 + \mu_2 s\right) < K_2\right) . \tag{1.9}$$

The corresponding transition probability density for a particle being at (x_1, x_2) in time zero and not reaching the boundary (K_1, K_2) in the time interval $[0, t]$ will be denoted by $f^{\mu}(x_1, x_2, t)$. The solution for the transition probability density in polar coordinates can be derived by changing the measure so that the considered Brownian motion with drift becomes a Brownian motion without drift under the new measure, and then using the result for f in equation (1.8). Compare e.g. Fischer (2003). The solution is the following:

$$\begin{aligned} f^{\mu}(r,\theta,t) &= \frac{2}{\sigma_1\sigma_2\sqrt{1-\rho^2}\alpha t}\,\mathrm{e}^{\frac{A_3 t}{2}+A_1 r\cos\theta+A_2 r\sin\theta-\frac{r^2+r_0^2}{2t}} \\ &\quad\cdot\sum_{n=1}^{\infty}\sin\left(\frac{n\pi\theta_0}{\alpha}\right)\sin\left(\frac{n\pi\theta}{\alpha}\right)I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{t}\right) \quad . \end{aligned} \tag{1.10}$$

The JSP $\tilde{F}^{\mu}$ is yielded when integrating the transition probability density in equation (1.10) and is stated in the (2001) paper by ZHOU - an update of ZHOU (1997b). The derivation can be found in FISCHER (2003) resp. REBHOLZ (1994).

Again we give the JSP result in the way that is suitable for us, i.e. for the contrary problem of not crossing a lower barrier. Thus before applying the JSP $\tilde{F}^{\mu}$ and transition probability density f^{μ} we have to reflect our first-passage processes at the x-axes. The result, the JSP for not crossing a lower barrier, is given by the next theorem.

Theorem 1.18 *(JSP of a two-dimensional Brownian motion with drift not crossing a lower barrier)*
Let $Y^1 = \sigma_1 W^1$ and $Y^2 = \sigma_2 W^2$ be correlated Wiener processes with correlation parameter ρ and $K_1 < 0$, $K_2 < 0$ the threshold levels. Then Y^1 and

Y^2 have the following joint survival probability:

$$\begin{aligned}
& \mathbb{P}\left(\tau_1 > t, \tau_2 > t\right) \\
\equiv\ & \mathbb{P}\left(\min_{0\le s\le t}\left(\sigma_1 W_s^1 + \mu_1 s\right) > K_1, \min_{0\le s\le t}\left(\sigma_2 W_s^2 + \mu_2 s\right) > K_2\right) \\
=\ & \frac{2}{\alpha t}\,\mathrm{e}^{\frac{A_3 t}{2} - \frac{r_0^2}{2t}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \\
& \int_{\theta=0}^{\alpha}\int_{r=0}^{\infty} \sin\left(\frac{n\pi\theta}{\alpha}\right) r\,\mathrm{e}^{A_1 r\cos\theta + A_2 r\sin\theta - \frac{r^2}{2t}}\, I_{\frac{n\pi}{\alpha}}\left(\frac{r r_0}{t}\right)\,\mathrm{d}\theta\,\mathrm{d}r\ ,
\end{aligned}$$

where r_0, θ_0 and α are as in Theorem 1.17 and

$$\begin{aligned}
A_1 &= \frac{\mu_1\sigma_2 - \rho\mu_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2}\ , \\
A_2 &= \frac{\mu_2\sigma_1 - \rho\mu_1\sigma_2}{(1-\rho^2)\sigma_1\sigma_2^2}\ , \\
A_3 &= \frac{\mu_1^2\sigma_2^2 - 2\rho\mu_1\mu_2\sigma_1\sigma_2 + \mu_2^2\sigma_1^2}{(1-\rho^2)\sigma_1^2\sigma_2^2}\ .
\end{aligned}$$

1.2.4 Deterministically time-changed Brownian motion

Let $(\mathcal{T}_t)$ be a continuous deterministic time change with regard to Definition 1.1. Then, by continuity, the first-passage time of the time-changed process $(W_{\mathcal{T}_t})$ has distribution

$$\mathbb{P}\left(\min_{s\le t} W_{\mathcal{T}_s} < K\right) = \mathbb{P}\left(\min_{s\le\mathcal{T}_s} W_s < K\right) = 2\Phi\left(\frac{K}{\sqrt{\mathcal{T}_t}}\right) \quad (1.11)$$

This FPT will be applied in Chapter 5 to calibrate the *Overbeck & Schmidt credit-spread model.*

1.2.5 Joint survival probability - deterministically time-changed Brownian motion

Hull & White (2001) fitted a Merton-type default model on a discrete time grid to the default probability curve of CDS spreads. Therefore the threshold value had to be adapted at each grid point to the given default probability value. In a next step, considering two firms, they calculated the default correlation, as in Def. 1.12, by simulating the instantaneous correlation through proxies such as the company's equity returns and calculating the joint default probabilities.

Overbeck & Schmidt (2005) introduced a continuous-time model by applying a deterministic time change to a Wiener process. Their model yields an analytical first-passage-time distribution and can be perfectly fitted to a given default-probability curve. In Chapter 5 we analyze the so-called *Overbeck & Schmidt model* in more detail. At this point we want to state their result of an analytical joint survival probability for a deterministic time-change model.

Theorem 1.19 *(Overbeck & Schmidt extended*[12]*; Joint survival probability)*
Let W^1 and W^2 be correlated Wiener processes with correlation parameter ρ and $K_1,\ K_2 < 0$ the threshold levels. For deterministic time changes T^1 and T^2 with $T^1_0 = 0 = T^2_0$ and default times τ_i, $i = 1, 2$ defined through the asset-value processes $Y^1_s = W^1_{T^1_s}$ and $Y^2_s = W^2_{T^2_s}$ we have the following expression for the joint survival probability:

$$
\begin{aligned}
& \mathbb{P}\left(\tau_1 > t, \tau_2 > t\right) \\
= \; & \frac{2}{\alpha T}\, \mathrm{e}^{-\frac{r_0^2}{2T}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \\
& \int_{\theta=0}^{\alpha} \int_{r=0}^{\infty} \sin\left(\frac{n\pi\theta}{\alpha}\right) r\, \mathrm{e}^{-\frac{r^2}{2T}}\, I_{\frac{n\pi}{\alpha}}\left(\frac{r r_0}{T}\right) \\
& \left(-1 + 2\Phi\left(r \frac{\mathbb{I}_{T^1_t \le T^2_t} \sin\theta + \mathbb{I}_{T^1_t > T^2_t} [\sqrt{1-\rho^2}\cos\theta + \rho\sin\theta]}{\sqrt{\Delta}}\right)\right) \mathrm{d}\theta\, \mathrm{d}r,
\end{aligned}
$$

where

$$
\begin{aligned}
T &= \min(T^1_t, T^2_t)\ , \\
\Delta &= \max(T^1_t, T^2_t) - \min(T^1_t, T^2_t)\ , \\
\theta_0 &= \begin{cases} \tan^{-1}\left(\frac{K_2\sqrt{1-\rho^2}}{K_1-\rho K_2}\right) & \text{if } \frac{K_2\sqrt{1-\rho^2}}{K_1-\rho K_2} > 0 \\ \pi + \tan^{-1}\left(\frac{K_2\sqrt{1-\rho^2}}{K_1-\rho K_2}\right) & \text{otherwise,} \end{cases} \\
r_0 &= \frac{-K_2}{\sin(\theta_0)}\ , \\
\alpha &= \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{if } \rho < 0 \\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{otherwise.} \end{cases}
\end{aligned}
$$

Note that if $T \equiv T^1_t = T^2_t$, the expression simplifies to the formula that

[12] The survival probability stated in O&S (2005) is correctly given under the assumption $T^2 \geq T^1$ that is applied in their proof. It is not correct for $T^2 < T^1$ since then, following the lines of Zhou (1997b), another coordinate transform has to be done to solve the corresponding PDE. As a consequence, for general T^1 and T^2, the indicator functions had to be included in the integrand of the general O&S result.

directly follows from Zhou's Theorem 1.17:

$$
\begin{aligned}
&\mathbb{P}\left(\tau_1 > t, \tau_2 > t\right) \\
&= \frac{2r_0}{\sqrt{2\pi T}}\,\mathrm{e}^{-\frac{r_0^2}{4T}} \sum_{n=1,3,5,\ldots} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \left[I_{\frac{1}{2}(\frac{n\pi}{\alpha}+1)}\left(\frac{r_0^2}{4T}\right) + I_{\frac{1}{2}(\frac{n\pi}{\alpha}-1)}\left(\frac{r_0^2}{4T}\right)\right] .
\end{aligned}
$$

In the following sections we give a short overview about which FPT results can be reached when including jumps in the underlying process. The jump part makes it difficult to study the distribution of the first-passage time τ because the process does not necessarily hit the boundary K exactly - there can be an *overshoot* $Y_\tau - K > 0$ or *undershoot* $K - Y_\tau > 0$.

1.2.6 Classical jump-diffusion approach

Zhou (1997a) developed a new structural approach by modeling the firm-value process through a *jump-diffusion process*:

$$
\begin{aligned}
Y_t &= \mu t + \sigma W_t + \sum_{i=1}^{N_t} Z_i \ , \\
N_t &\sim Pois(\lambda t) \ , \ Z_i \text{ iid} \sim \mathcal{N}(\mu_Z, \sigma_Z) \ ,
\end{aligned}
\tag{1.12}
$$

where (N_t) determines the number of jumps and Z_i the amplitude of the i-th jump. The advantage of including jumps into a diffusion model is that this allows an instantaneous default of the firm. Furthermore a jump diffusion model can generate various credit-spread shapes including upward-, downward-sloping, flat and hump-shaped. Zhou determined a closed-form default time distribution under the classical approach:

$$
\mathbb{P}\left(\tau = T\right) \ \equiv \ \mathbb{P}(Y_T < K) \ = \ \sum_{n=1}^{\infty} \frac{\exp -\lambda t(\lambda t)^i}{i!} \Phi\left(\frac{K - \mu T - i\mu_Z}{\sqrt{\sigma^2 T + i\sigma_Z^2}}\right) .
$$

1.2.7 First-passage jump-diffusion approach

Kou & Wang (2003) analyzed the FPT of the jump diffusion process in equation (1.12) hitting or crossing an upper[13] barrier K,

$$
\tau \ = \ \inf\{s \geq 0 : Y_s \geq K\} \ ,
$$

where the Z_i are iid double exponentially distributed, i.e.

$$
f_{Z_i}(z) \ = \ p \cdot \eta_1 \,\mathrm{e}^{-\eta_1 z}\, \mathbb{I}_{\{z \geq 0\}} + q \cdot \eta_2 \,\mathrm{e}^{\eta_2 z}\, \mathbb{I}_{\{z<0\}} \ ,
$$

[13]Note that this is the contrary problem since we consider the FPT distribution of crossing a lower barrier.

with $p, q \geq 0$, $p+q=1$ and $\eta_1, \eta_2 > 0$. The FPT distribution has to be split up:

$$\mathbb{P}(\tau \leq t) \quad = \quad \mathbb{P}(\tau \leq t, Y_\tau - K = 0) + \mathbb{P}(\tau \leq t, Y_\tau - K > 0) \ ,$$

that is, the overshoot distribution is necessary, in particular $\mathbb{P}(Y_\tau - K = 0)$ and $\mathbb{P}(Y_\tau - K > x)$, $x > 0$. In the case of double exponentially distributed jumps the overshoot distribution can be derived because it is conditionally memoryless, i.e.

$$\mathbb{P}(Y_\tau - K \geq x \mid Y_\tau - K > 0) \quad = \quad \mathrm{e}^{-\eta_1 x} \ .$$

Furthermore overshoot and FPT are conditionally independent:

$$\begin{aligned} & \mathbb{P}(\ \tau \leq t, Y_\tau - K \geq x \mid \quad Y_\tau - K > 0) \\ = & \ \mathbb{P}(\tau \leq t \mid Y_\tau - K > 0) \quad \mathbb{P}(Y_\tau - K \geq x \mid Y_\tau - K > 0) \ . \end{aligned}$$

When we want to determine the FPT distribution via the reflection principle, the dependence structure between overshoot and terminal value Y_t is needed, but not known:

$$\begin{aligned} \mathbb{P}(\tau \leq t) = & \ \mathbb{P}(\tau \leq t, Y_\tau - K = 0, Y_t \geq K) + \mathbb{P}(\tau \leq t, Y_\tau - K > 0, Y_t \geq K) \\ & + \ \mathbb{P}(\tau \leq t, Y_\tau - K = 0, Y_t < K) + \mathbb{P}(\tau \leq t, Y_\tau - K > 0, Y_t < K) \ . \end{aligned}$$

As a consequence, an explicit FPT distribution is not available. But Kou and Wang were able to compute the Laplace transform[14] of the FPT τ and hence could retrieve the FPT distribution from equation (1.1) by applying the Gaver-Stehfest algorithm, that numerically inverts the Laplace transform on the real line. Further numerical methods for first-passage of jump-diffusion processes can be found in Atiya & Metwally (2005).

1.2.8 Subordinated Lévy processes

In Definition 1.3 and 1.4 we introduced Lévy processes and subordinators, non-decreasing Lévy processes that can be used as a time change. Cariboni & Schoutens (2007) considered a subordinated Wiener process as asset-value process, where the subordinator was given by a pure jump process and interpreted as new business clock. Let us consider a general subordinator J independent of W and the resulting first-passage process

$$Y_t \quad = \quad \sigma W_{J_t} + \mu \ J_t \ .$$

Then Y is again a Lévy process.[15] A subordinator has independent increments, that is, the arriving information is not affected by the amount of

[14] See Definition 1.6.

[15] This even holds when we substitute W by another Levy process independent of J; see Kyprianou (2006b), Lemma 2.15.

information that has already arrived. If the subordinator yields a Laplace transform, then also a Laplace transform is available for the whole process Y. This is for instance the case for the *gamma subordinator* or *inverse Gaussian subordinator* (see Table 1.1), which lead to the *variance gamma model* or to the *normal inverse Gaussian model*. Under the classical approach (Def. 1.7), a closed-form default time distribution is obtained:

$$\begin{aligned} \mathbb{P}\,(\tau = T) &= \mathbb{P}(Y_T < K) \;=\; \mathbb{E}\,[\,\mathbb{P}(Y_T < K \mid J_T)\,] \\ &= \int_0^\infty \mathbb{P}(Y_T < K \mid J_T = x)\;\mathbb{P}(J_T \in \,\mathrm{d}x) \\ &= \int_0^\infty \Phi\left(\frac{K - \mu x}{\sigma\sqrt{x}}\right)\mathbb{P}(J_T \in \,\mathrm{d}x)\;, \end{aligned}$$

where the corresponding result for Brownian motion with drift, equation (1.5), was applied. That is, the default-time distribution is explicit whenever an explicit density of the subordinator $f_{J_t}(x)\,\mathrm{d}x = \mathbb{P}(J_T \in \,\mathrm{d}x)$ is available. Examples are the already mentioned *gamma process* and the *inverse Gaussian process*, see Table 1.1. As mentioned when we defined subordinators,

subordinator J_t	Lévy measure $\nu(x)$	probability density $f_{J_t}(x)$	Laplace transform $\mathbb{E}[\mathrm{e}^{-uJ_t}]$
gamma	$\frac{c\,\mathrm{e}^{-\lambda x}}{x}\mathbb{I}_{x>0}$	$\frac{\lambda^{ct}}{\Gamma(ct)}x^{ct-1}\,\mathrm{e}^{-\lambda x}$	$(1-\frac{u}{\lambda})^{-ct}$
inv. Gaussian	$\frac{c\,\mathrm{e}^{-\lambda x}}{x^{3/2}}\mathbb{I}_{x>0}$	$\frac{ct}{x^{3/2}}\,\mathrm{e}^{2ct\sqrt{\pi\lambda}-\lambda x-\pi\frac{c^2t^2}{x}}$	$\mathrm{e}^{-2ct\sqrt{\pi}(\sqrt{\lambda-u}-\sqrt{\lambda})}$

Table 1.1: Gamma subordinator and inverse-Gaussian subordinator, $c > 0$ accounts for the overall jump intensity (i.e. it determines the time-change speed) and $\lambda > 0$ for the decay rate of big jumps. The denominator $x^{3/2}$ of the inverse Gaussian Lévy measure increases the importance of small jumps, compared to the gamma denominator x.

J is only continuous in the trivial case of a deterministic drift. Interesting subordinators are jump processes and thus lead to a noncontinuous process Y. Then at first-passage, *overshoots* or *undershoots* are possible. As a consequence, an explicit formula for the FPT distribution is not available. Compare Cont & Tankov (2004). Furthermore compare Alili & Kyprianou (2005) and Kyprianou (2006b) for the first-passage of Lévy processes and an analysis of overshoot and undershoot distributions of (independently) exponentially time changed respectively subordinated Lévy processes (especially when the Lévy process is *spectrally one-sided* or a subordinator itself) via the *Wiener-Hopf factorization*.

The multivariate model is given by

$$\begin{aligned} Y_t^i &= \sigma_i W_{J_t}^i + \mu_i J_t \ , \quad i = 1, \ldots, m \ , \\ \rho_{ij} &= \mathrm{Corr}(W_t^i, W_t^j) \quad \forall t \ , \end{aligned} \tag{1.13}$$

where the subordinator inserts a dependency structure - in addition to a possible Brownian correlation through the ρ_{ij}'s. Luciano & Schoutens (2006) introduced the corresponding exponential model, in particular assuming a gamma subordinator (see Table 1.1), for modeling multivariate financial assets:

$$A_t^i = A_0^i \exp\left\{\sigma_i W_{J_t}^i + \mu_i J_t\right\} \ , \quad i = 1, \ldots, m \ ,$$

see also Section 4.6.2 and Chapter 7. One possible interpretation is that the asset-value processes are influenced by the same external, economic information and thus run under the same business clock. This is one of the ideas we adopt to yield dependence, while considering another time-change process.

1.2.9 Stable processes at first passage

A (strictly) stable process X is a special Lévy process. Its distribution is characterized by a constant $\alpha \in (0, 2]$ and the following property:

$$X_t \overset{\mathcal{L}}{=} t^{1/\alpha} X_1 \quad \forall t \geq 0 \ .$$

Kyprianou (2006a) determined the overshoot and undershoot distributions of X at first passage. Furthermore he was able to derive the first-passage-time distribution for the reflected strictly stable process

$$\left\{(z \vee \sup_{0 \leq s \leq t} X_s) - X_t : t \geq 0\right\} \ , \ z \geq 0 \ .$$

This is not possible for the stable process X itself.

1.2.10 Time-changed Lévy processes

Carr & Wu (2003) considered a Lévy process L time changed by an integrated jump process

$$G_t = \int_0^t v(s_-) \, \mathrm{d}s \ ,$$

that is normalized so that $\mathbb{E}[G_t] = t$. The local intensity v is interpreted as *instantaneous (business) activity rate* and can be correlated with innovations in L (e.g. due to leverage effects). Being an integral, G is a continuous process. Carr & Wu determined the characteristic function of

$$Y_t \quad = \quad L_{G_t} \ ,$$

when L and G are independent and when they are dependent. Under independence this is straightforward and yields:

$$\mathbb{E}\left[\mathrm{e}^{iuY_t}\right] \quad = \quad l_{G_t}(\psi_L(u)) \ ,$$

where l_{G_t} is the Laplace transform of the time change and ψ_L is the Lévy exponent; see eq. (1.1). This is useful for option pricing under the corresponding exponential model, and reduces calculation compared to characteristic functions.

Classical default time distribution and FPT distribution are only available when L is a diffusion process. The formulas are explicit only if the distribution of G is explicit. This is exactly the model of our interest; see Section 1.3 and Chapter 6.

1.2.11 Summary and conclusion

In this last review section we have summarized models that are important to understand our further proceeding. We have seen models based on Brownian motion that lead to an analytical classical default-time distribution as well as a first-passage-time distribution. We have learnt that the classical diffusion model with jumps has a closed-form default-time distribution and that, in case the jumps are double exponentially distributed, a Laplace transform of the FPT can be derived and the FPT distribution can be retrieved numerically. There is even a so-called reflected strictly stable process - a very specific Lévy process - where a FPT distribution is available. We aim at an analytical FPT distribution, always having in mind our application of modeling credit spread and credit-spread dynamics. The survey on first-passage processes and first-passage times teaches us that it seems very reasonable to stay with a model based on Brownian motion. Therefore Chapter 4 analyzes *Merton's threshold model.* We will learn that the Merton model fails to produce realistic credit-spread curves and credit-spread dynamics. Chapter 5 then considers the *deterministic time-transformation model.* We will find that, while it perfectly fits a given default-probability curve, it has only one parameter - the correlation parameter - to describe dependencies for two-dimensional default probabilities. Furthermore, the model can not influence multivariate default probabilities of more than two dimensions and it has no degrees of freedom to influence credit-spread dynamics. Therefore, as an extended approach, we apply a stochastic time change on Brownian motion.

1.3 Stochastic time-change model

Consider a Wiener process with drift

$$\sigma W_t + \mu t$$

and substitute the *normal time* t by some *experienced time* G_t as in Definition 1.1. This leads to the first-passage process of our interest:

$$Y_t \quad = \quad \sigma W_{G_t} + \mu \; G_t \; . \tag{1.14}$$

By definition Y is adapted to the filtration $\mathbb{F}^Y$. The corresponding first-passage time is given by

$$\tau \quad = \quad \inf\{s \geq 0 : Y_s < K\} \; = \; \inf\{s \geq 0 : \sigma W_{G_s} + \mu \; G_s < K\}$$

and is a $\mathbb{F}^Y$ stopping time in terms of Definition 1.5. Considering a FPT makes only sense when a first-passage (or *default*) has not already occurred in the past, i.e. when we have not already crossed the barrier K at time zero, that is, we will always assume that

$$K \quad < \quad \sigma W_{G_0} + \mu \; G_0.$$

We interpret the stochastic clock as a business clock that measures the *amount of (stochastic) information arrival*. The extension to a multivariate model is straightforward:

$$\begin{aligned} Y_t^i \quad &= \quad \sigma_i W^i_{G^i_t} + \mu_i G^i_t \; , \quad i = 1, \ldots, m \; , \qquad (1.15) \\ \rho_{ij} \quad &= \quad \mathrm{Corr}(W^i, W^j) \; . \end{aligned}$$

Each underlying process Y^i has an individual driving Wiener process W^i, drift μ_i and uncertainty parameter σ_i. Dependencies are possible between the Brownian motions, between the time changes and between Brownian motions and time changes. We will not allow for the last mentioned dependence between Brownian motion and time change, since then our proofs will not work. In detail, as mentioned already in the introduction of this chapter, we will consider the following dependence structures between the first-passage processes:

- independent Brownian processes but identical time change,
- correlated Brownian processes and identical time change,
- correlated Brownian processes under separate time changes.

When applying the same time change to each process, one can think of underlyings that reach the same (business) information at the same time, with the same speed. If the (external) information is different, independent time changes could be suitable. The correlation parameter ρ_{ij}, of the untransformed Brownian motions, could describe any other dependence structure, some basic joint dependence that does not change with time. In unison with the general time change Definition 1.1 comes the assumption for our stochastic time change model:

Assumption 1.20 *(Continuous time change)*
Let G be a non-decreasing process with continuous paths and starting value $G_0 = g \geq 0$. Furthermore G be independent of the Wiener process W.

Assumption 1.20 accompanies our model (1.14) and will be assumed throughout this thesis. When the starting value g is truely positive the business clock speeds up at the very first moment and thus enables an instant jump of the underlying process. In credit risk this enables an *instantaneous default*. When we slightly change the model by defining the asset-value process and the default barrier

$$\begin{aligned} \tilde{Y}_t &:= W_{G_t - g} \ , \\ \tilde{K} &:= K - W_g \ , \end{aligned} \tag{1.16}$$

then the threshold level $\tilde{K}$ is a random variable (see Figure 4.9) and the *CreditGrades model* by Finger et al. (2002) is a special case of the model. We give a detailed introduction to the CreditGrades model in Section 4.5.1.

1.3.1 What does a continuous stochastic time change look like?

There are two main approaches in order to get a positive, non-decreasing, continuous time change process. The first possibility is to apply an integral to a positive process (g_s):

$$G_t = g + \int_0^t g_s \, \mathrm{d}s \quad g \geq 0 \ . \tag{1.17}$$

Here g_s can be understood as the *increase of information* or *default speed*. A positive process is for instance the square or absolute value of some other process. There are also stochastic processes that are already positive such as the Cox-Ingersoll-Ross process or the generalized Ornstein-Uhlenbeck process with positive jumps, see Section 1.3.6. The integral representation (1.17) makes G an *absolutely continuous process* (Definition 1.2) with starting value g. Conversely, if G is an absolutely continuous process there exists a non-negative process (g_s) and a starting value g such that (1.17) holds.[16]

[16] Cf. Protter (2004).

The second possibility is to define the time change by the supremum of a positive continuous process (which is actually its maximum):

$$G_t \quad = \quad \sup_{0\le s\le t} g_s \quad = \quad \max_{0\le s\le t} g_s \ .$$

Note that our results in the remainder of this chapter are general and thus independent of any particular time-change expression.

1.3.2 First-passage-time distribution

In the following we will abbreviate the time change distribution as follows:

$$\mathbb{P}_{G_t}(\,\mathrm{d}x) \quad \equiv \quad \mathbb{P}(G_t \in \,\mathrm{d}x) \ .$$

Theorem 1.21 *(First-passage time for stochastic continuous time change) Let G be a time change fullfilling Assumption 1.20. Then the default probability for the first-passage process $Y_t = \sigma W_{G_t} + \mu\, G_t$ and threshold barrier $K \le \sigma W_g + \mu\, g$ is given by*

$$\begin{aligned}\mathbb{P}\left(\tau \le t\right) \quad = \quad & \int_g^\infty \left[\Phi\left(\frac{K-Y_0}{\sigma\sqrt{x-g}} - \frac{\mu}{\sigma}\sqrt{x-g}\right)\right. \\ & \left. +\ \mathrm{e}^{2\frac{\mu}{\sigma^2}(K-Y_0)}\,\Phi\left(\frac{K-Y_0}{\sigma\sqrt{x-g}} + \frac{\mu}{\sigma}\sqrt{x-g}\right)\right]\mathbb{P}_{G_t}(\,\mathrm{d}x)\ .\end{aligned}$$

Proof. We use Assumption 1.20, i.e. continuity of the time change and independence of time change and Wiener process. Note that $G_0 = g$ and $Y_0 = \sigma W_g + \mu g$. Then

$$\begin{aligned} & \mathbb{P}\left(\tau \le t\right) \\ = \quad & \mathbb{P}\left(\min_{0\le s\le t}\left[\sigma W_{G_s} + \mu G_s\right] < K\right) \\ = \quad & \mathbb{P}\left(\min_{0\le s\le t}\left[W_{G_s} + \frac{\mu}{\sigma} G_s\right] < \frac{K}{\sigma}\right) \\ \overset{G_t\text{ cont.}}{=} \quad & \mathbb{P}\left(\min_{g\le s\le G_t}\left[W_s + \frac{\mu}{\sigma} s\right] < \frac{K}{\sigma}\right) \\ = \quad & \int_g^\infty \mathbb{P}\left(\min_{g\le s\le G_t}\left[W_s + \frac{\mu}{\sigma} s\right] < \frac{K}{\sigma} \mid G_t = x\right)\mathbb{P}_{G_t}(\,\mathrm{d}x) \\ \overset{G_t\perp(W_s)}{=} \quad & \int_g^\infty \mathbb{P}\left(\min_{g\le s\le x}\left[W_s + \frac{\mu}{\sigma} s\right] < \frac{K}{\sigma}\right)\mathbb{P}_{G_t}(\,\mathrm{d}x) \\ = \quad & \int_g^\infty \mathbb{P}\left(\min_{0\le s\le x-g}\left[W_{g+s} + \frac{\mu}{\sigma}(g+s) - \frac{Y_0}{\sigma}\right] < \frac{K-Y_0}{\sigma}\right)\mathbb{P}_{G_t}(\,\mathrm{d}x)\ . \end{aligned}$$

The value of the integrand is now given by the FPT formula (1.6) for Brownian motion with drift and start at zero. $\square$

In the following we apply this result for the multi-dimensional model.

1.3.3 Multivariate first-passage-time distribution under Brownian independence

In equation (1.15) we introduced the multivariate model. In this section it is assumed that the driving Wiener processes of the the m first-passage processes are uncorrelated (*Brownian independence*, see Def. 1.11). They are only dependent through the same time change, and thus experience identical information at the same time. This is the simple extension to a multi-dimensional default probability and deriving the FPT formula is straightforward: Conditioning on G, we use the independence of the Wiener processes W^i and apply the result for the one-dimensional case (Theorem 1.21).

Corollary 1.22 *(Multi-dimensional default probability under Brownian independence)*
The joint default probability for the multivariate model $Y_t^i := \sigma_i W_{G_t}^i + \mu_i G_t$, $i = 1, \ldots, m$, with threshold levels $K_i \leq Y_0^i$ and uncorrelated Wiener processes is given by

$$
\begin{aligned}
& \mathbb{P}\,(\tau_1 \leq t,\ \ldots,\ \tau_m \leq t) \\
= \ & \int_g^\infty \prod_{i=1}^m \Bigg[\Phi\left(\frac{K_i - Y_0^i}{\sigma_i \sqrt{x-g}} - \frac{\mu_i}{\sigma_i}\sqrt{x-g} \right) \\
& \qquad + \mathrm{e}^{2\frac{\mu_i (K_i - Y_0^i)}{\sigma_i^2}} \Phi\left(\frac{K_i - Y_0^i}{\sigma_i \sqrt{x-g}} + \frac{\mu_i}{\sigma_i}\sqrt{x-g} \right) \Bigg] \mathbb{P}_{G_t}(\,\mathrm{d}x)\ .
\end{aligned}
$$

Proof. Using the conditional independence of the processes Y^i yields

$$
\begin{aligned}
\mathbb{P}\,(\tau_1 \leq t,\ \ldots,\ \tau_m \leq t) &= \int_g^\infty \mathbb{P}\,(\tau_1 \leq t,\ \ldots,\ \tau_m \leq t \mid G_t)\ \mathbb{P}_{G_t} \\
&\overset{\perp}{=} \int_g^\infty \prod_{i=1}^m \mathbb{P}\,(\tau_i \leq t \mid G_t)\ \mathbb{P}_{G_t}
\end{aligned}
$$

Insert the conditional one-dimensional FPT distribution derived in the proof of Theorem 1.21. □

1.3.4 Joint survival probability under Brownian correlation

Applying the same time transformation to each underlying process is one way to insert a dependence structure into a multivariate model. Another possibility is to correlate the driving Wiener processes in addition. For deriving the joint survival probability under Brownian correlation, we restrict ourselves to the two-dimensional case because as a main step of our proofs we will use Zhou's result stated in Theorem 1.17. Note that equation (1.2) then yields the joint default probability.

Theorem 1.23 *(JSP for correlated Brownian motions with drift under a joint time change)*
Let W^1 and W^2 be correlated Wiener processes with correlation parameter ρ. Furthermore let G be the joint time change satisfying Assumption 1.20 and having starting value $G_0 = g$. Then the default times τ_i, $i = 1, 2$ of the processes $Y_t^1 := \sigma_1 W_{G_t}^1 + \mu_1 G_t$ and $Y_t^2 := \sigma_2 W_{G_t}^2 + \mu_2 G_t$ have joint survival probability

$$
\begin{aligned}
&\mathbb{P}(\tau_1 > t, \tau_2 > t) \\
= &\int_{[g,\infty)^2} \frac{2}{\alpha(x-g)} \,\mathrm{e}^{\frac{A_3(x-g)}{2} - \frac{r_0^2}{2(x-g)}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \\
&\int_{\theta=0}^{\alpha} \int_{r=0}^{\infty} \sin\left(\frac{n\pi\theta}{\alpha}\right) r\,\mathrm{e}^{A_1 r\cos\theta + A_2 r\sin\theta - \frac{r^2}{2(x-g)}}\, I_{\frac{n\pi}{\alpha}}\left(\frac{r r_0}{x-g}\right) \mathrm{d}\theta\, \mathrm{d}r \\
&\mathbb{P}_{G_t}(\,\mathrm{d}x)\ ,
\end{aligned}
$$

where I_ν denotes the modified Bessel function[17] *of the first kind with order ν and*

$$
\begin{aligned}
\theta_0 &= \begin{cases} \tan^{-1}\left(\frac{\sigma_1(K_2-Y_0^2)\sqrt{1-\rho^2}}{\sigma_2(K_1-Y_0^1)-\rho\sigma_1(K_2-Y_0^2)}\right) & \textit{if } \frac{\sigma_1(K_2-Y_0^2)\sqrt{1-\rho^2}}{\sigma_2(K_1-Y_0^1)-\rho\sigma_1(K_2-Y_0^2)} > 0 \\ \pi + \tan^{-1}\left(\frac{\sigma_1(K_2-Y_0^2)\sqrt{1-\rho^2}}{\sigma_2(K_1-Y_0^1)-\rho\sigma_1(K_2-Y_0^2)}\right) & \textit{otherwise,} \end{cases} \\
r_0 &= \frac{Y_0^2 - K_2}{\sigma_2 \sin(\theta_0)}\ , \\
\alpha &= \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \textit{if } \rho < 0 \\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \textit{otherwise.} \end{cases} \\
A_1 &= \frac{\mu_1\sigma_2 - \rho\mu_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2}\ , \\
A_2 &= \frac{\mu_2\sigma_1 - \rho\mu_1\sigma_2}{(1-\rho^2)\sigma_1\sigma_2^2}\ , \\
A_3 &= \frac{\mu_1^2\sigma_2^2 - 2\rho\mu_1\mu_2\sigma_1\sigma_2 + \mu_2^2\sigma_1^2}{(1-\rho^2)\sigma_1^2\sigma_2^2}
\end{aligned}
$$

Proof. For each process we follow the first steps of the proof of Theorem 1.21, yielding

$$
\begin{aligned}
&\mathbb{P}(\tau_1 > t, \tau_2 > t) \\
= &\int_g^{\infty} \mathbb{P}\left(\min_{g\le s\le x}\left(\sigma_1 W_s^1 + \mu_1 s\right) > K_1, \min_{g\le s\le x}\left(\sigma_2 W_s^2 + \mu_2 s\right) > K_2\right) \mathbb{P}_{G_t}(\,\mathrm{d}x)\ .
\end{aligned}
$$

[17] See footnote 11.

Then with the integrand we proceed as in the proof of Theorem 1.19 given in the (2005) paper by Overbeck & Schmidt:

$$
\begin{aligned}
&= \int_g^\infty \mathbb{P}\left(\sigma_1 W_s^1 + \mu_1 s > K_1,\ \sigma_2 W_s^2 + \mu_2 s > K_2,\ g \le s \le x\right) \mathbb{P}_{G_t}(\,\mathrm{d}x) \\
&= \int_g^\infty \mathbb{P}\Big(\sigma_1 W_{g+s}^1 + \mu_1(g+s) > K_1,\ \sigma_2 W_{g+s}^2 + \mu_2(g+s) > K_2, \\
&\qquad s \le x-g\Big) \mathbb{P}_{G_t}(\,\mathrm{d}x) \\
&= \int_g^\infty \mathbb{P}\big(-\sigma_1 W_{g+s}^1 - \mu_1(g+s) + Y_0^1 < -K_1 + Y_0^1, \\
&\qquad -\sigma_2 W_{g+s}^2 - \mu_2(g+s) + Y_0^2 < -K_2 + Y_0^2,\ s \le x-g\big) \mathbb{P}_{G_t}(\,\mathrm{d}x) \\
&= \int_g^\infty F^\mu(Y_0^1 - K_1, Y_0^2 - K_2, x-g)\ \mathbb{P}_{G_t}(\,\mathrm{d}x)\ ,
\end{aligned}
$$

where F^μ is the JSP of the two-dimensional Brownian motion with drift $(-\sigma_1 W_{g+s}^1 - \mu_1(g+s) + Y_0^1, -\sigma_2 W_{g+s}^2 - \mu_2(g+s))$, not crossing the upper barrier $(Y_0^1 - K_1, Y_0^2 - K_2)$ by time $x-g$, stated in equation 1.9. The corresponding proper JSP of not crossing the lower barrier is given in Theorem 1.18, in polar coordinates. □

Corollary 1.24 *(JSP for correlated Brownian motions)*
Let W^1 and W^2 be Wiener processes with correlation ρ. The default times τ_i, $i = 1, 2$ of the processes $Y_t^1 := \sigma_1 W_{G_t}^1$ and $Y_t^2 := \sigma_2 W_{G_t}^2$ have joint survival probability

$$
\begin{aligned}
&\mathbb{P}\left(\tau_1 > t, \tau_2 > t\right) \\
&= \frac{2r_0}{\sqrt{2\pi}} \int_g^\infty \frac{1}{\sqrt{x-g}}\, \mathrm{e}^{-\frac{r_0^2}{4(x-g)}} \sum_{n=1,3,5,\ldots} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \\
&\qquad \left[I_{\frac{1}{2}(\frac{n\pi}{\alpha}+1)}\left(\frac{r_0^2}{4(x-g)}\right) + I_{\frac{1}{2}(\frac{n\pi}{\alpha}-1)}\left(\frac{r_0^2}{4(x-g)}\right)\right] \mathbb{P}_{G_t}(\,\mathrm{d}x)\ ,
\end{aligned}
$$

where the θ_0, r_0, α and I_ν are as in Theorem 1.23.

The proof for the corollary follows directly from the proof of Theorem 1.23 when inserting F, the JSP for two-dimensional Brownian motion without drift (given in Zhou (2001)), instead of F^μ, the JSP for two-dimensional Brownian motion with drift.

Remark 1.25 Note the following equivalences for α, θ_0 and r_0:[18]

$$\begin{aligned}
\alpha &= \arccos(-\rho)\\
\theta_0 &= \pi - \arccos\left(\frac{\sigma_2\tilde{K}_1 - \rho\sigma_1\tilde{K}_2}{\sqrt{\sigma_2^2\tilde{K}_1^{\,2} - 2\sigma_1\sigma_2\tilde{K}_1\tilde{K}_2 + \sigma_1^2\tilde{K}_2^{\,2}}}\right)\\
r_0 &= \sqrt{\frac{\sigma_2^2\tilde{K}_1^{\,2} - 2\sigma_1\sigma_2\tilde{K}_1\tilde{K}_2 + \sigma_1^2\tilde{K}_2^{\,2}}{(1-\rho^2)\sigma_1^2\sigma_2^2}}\ ,
\end{aligned}$$

where we abbreviated $\tilde{K}_i \equiv K_i - Y_0^i$.

1.3.5 Joint survival probability under Brownian correlation and separate time changes

In this section we derive the JSP for the two-dimensional model with Brownian correlation, but different time transformations, that might be dependent or independent. When choosing independent time changes, calibration is very much simplified because then at a first step each underlying process Y^i can be calibrated separately (by adapting its volatility σ_i, drift μ_i and time change G^i). A constant dependence between the two underlying process can afterwards be introduced via the Brownian correlation parameter ρ and the relation (1.3).

Theorem 1.26 *(JSP under correlated Brownian motions with drift and separate time changes)*
Let W^1 and W^2 be correlated Wiener processes with correlation parameter ρ, and G^1 and G^2 be time changes satisfying Assumption 1.20 and having joint starting values $G_0^1 = g = G_0^2$. Then the default times τ_i, $i = 1, 2$ of the processes $Y_t^1 := \sigma_1 W^1_{G_t^1} + \mu_1 G_t^1$ and $Y_t^2 := \sigma_2 W^2_{G_t^2} + \mu_2 G_t^2$ have joint survival probability

$$\begin{aligned}
&\mathbb{P}\left(\tau_1 > t, \tau_2 > t\right)\\
= &\int_{[g,\infty)^2} \frac{2}{\alpha(T(x,y)-g)}\, \mathrm{e}^{\frac{A_3(T(x,y)-g)}{2} - \frac{r_0^2}{2(T(x,y)-g)}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right)\\
&\int_{\theta=0}^{\alpha}\int_{r=0}^{\infty} \sin\left(\frac{n\pi\theta}{\alpha}\right) r\, \mathrm{e}^{A_1 r\cos\theta + A_2 r\sin\theta - \frac{r^2}{2(T(x,y)-g)}}\, I_{\frac{n\pi}{\alpha}}\left(\frac{r r_0}{T(x,y)-g}\right)\\
&\left(-1 + 2\Phi\left(r\frac{\mathbb{I}_{x\le y}\sin\theta + \mathbb{I}_{x>y}[\sqrt{1-\rho^2}\cos\theta + \rho\sin\theta]}{\sqrt{\Delta(x,y)}}\right)\right)\mathrm{d}\theta\,\mathrm{d}r\\
&\mathbb{P}_{(G_t^1,G_t^2)}(\,\mathrm{d}x \times\,\mathrm{d}y)\ ,
\end{aligned}$$

[18] These definitions for α, θ_0 and r_0 as well as A_1, A_2 and A_3 as stated in Theorem 1.23 were introduced by Fischer (2003) in order to determine the JSP. Note that we corrected the sign within the arccos of θ_0, added a missing square for K_1 in the denominator of r_0 as well as a factor 2 in the nominator of A_3.

where θ_0, r_0, α, A_1, A_2 and A_3 are as given in Theorem 1.23 and the minimal time T and the time difference Δ are defined by

$$\begin{aligned} T(G_t^1, G_t^2) &= \min(G_t^1, G_t^2) \ , \\ \Delta(G_t^1, G_t^2) &= \max(G_t^1, G_t^2) - \min(G_t^1, G_t^2) \ . \end{aligned}$$

Proof. For the proof we furthermore define the index i, indicating the faster time change,

$$i \equiv i(G_t^1, G_t^2) = \begin{cases} 1 & \text{if } G_t^1 > G_t^2 \\ 2 & \text{otherwise} \end{cases}$$

and follow the first steps as in the proof of Theorem 1.21 for each time change separately. We use the definitions of T, Δ and i and apply the conditional Markov property of Y^i, as in the proof of Overbeck & Schmidt (2005):

$$\begin{aligned}
& \mathbb{P}\left(\tau_1 > t, \tau_2 > t\right) \\
= \quad & \mathbb{P}\left(\min_{0\le s\le t} \sigma_1 W^1_{G^1_s} + \mu_1 G^1_s > K_1,\ \min_{0\le s\le t} \sigma_2 W^2_{G^2_s} + \mu_2 G^2_s > K_2\right) \\
\overset{G_t \text{ cont.}}{=} \quad & \mathbb{P}\left(\min_{g\le s\le G^1_t} \sigma_1 W^1_s + \mu_1 s > K_1,\ \min_{g\le s\le G^2_t} \sigma_2 W^2_s + \mu_2 s > K_2\right) \\
\overset{G_t \perp (W^i_s)}{=} \quad & \int_{[g,\infty)^2} \mathbb{P}\left(\min_{g\le s\le x} \sigma_1 W^1_s + \mu_1 s > K_1,\ \min_{g\le s\le y} \sigma_2 W^2_s + \mu_2 s > K_2\right) \\
& \qquad \mathbb{P}_{(G^1_t, G^2_t)}(\,\mathrm{d}x \times \,\mathrm{d}y) \\
= \quad & \int_{[g,\infty)^2} \mathbb{E}\Big(\mathbb{I}_{\{\sigma_1 W^1_{g+s} > K_1,\ \sigma_2 W^2_{g+s} > K_2,\ s\le T-g\}} \\
& \qquad \mathbb{P}\left(\sigma_i W^i_s + \mu_i s > K_i,\ T \le s \le T+\Delta \mid W^i_s, s\le T\right)\Big) \\
& \qquad \mathbb{P}_{(G^1_t, G^2_t)}(\,\mathrm{d}x \times \,\mathrm{d}y) \\
\overset{\text{Markov}}{=} \quad & \int_{[g,\infty)^2} \mathbb{E}\Big(\mathbb{I}_{\{\sigma_1 W^1_{g+s} + \mu_1(g+s) > K_1,\ \sigma_2 W^2_{g+s} + \mu_2(g+s) > K_2,\ s\le T-g\}} \\
& \qquad \mathbb{P}\left(\sigma_i W^i_s + \mu_i s > K_i, T \le s \le T+\Delta \mid W^i_T\right)\Big) \mathbb{P}_{(G^1_t, G^2_t)}(\,\mathrm{d}x \times \,\mathrm{d}y) \\
= \quad & \int_{[g,\infty)^2} \mathbb{E}\Bigg(\mathbb{I}_{\{-\sigma_1 W^1_{g+s} - \mu_1(g+s) + Y^1_0 < -K_1 + Y^1_0\}} \\
& \qquad \mathbb{I}_{\{-\sigma_2 W^2_{g+s} - \mu_2(g+s) + Y^2_0 < -K_2 + Y^2_0, s\le T-g\}} \\
& \qquad \left(1 - 2\Phi\left(\frac{K_i - \sigma_i W^i_T - \mu_i T}{\sigma_i \sqrt{\Delta}}\right)\right)\Bigg) \mathbb{P}_{(G^1_t, G^2_t)}(\,\mathrm{d}x \times \,\mathrm{d}y) \\
= \quad & \int_{[g,\infty)^2} \int_{-\infty}^{Y^1_0 - K_1} \int_{-\infty}^{Y^2_0 - K_2} f^\mu(x_1, x_2, T-g, \rho) \\
& \qquad \left(1 - 2\Phi\left(\frac{K_i - Y^i_0 + x_i}{\sigma_i \sqrt{\Delta}}\right)\right) \,\mathrm{d}x_1 \,\mathrm{d}x_2\ \mathbb{P}_{(G^1_t, G^2_t)}(\,\mathrm{d}x \times \,\mathrm{d}y) \ ,
\end{aligned}$$

where f^μ is the transition-probability density of $(-\sigma_1 W^1_{g+s} - \mu_1(g+s) + Y^1_0, -\sigma_2 W^2_{g+s} - \mu_2(g+s) + Y^2_0)$ not hitting the barriers $Y^1_0 - K_1$ and $Y^2_0 - K_2$

by time $T-g$. f^μ was given in equation (1.10). □

Remark 1.27 In the last theorem we put the constraint that the time changes have the same starting value. We now show why this is a necessary requirement in order to obtain an analytical formula. Assume $G_0^1 \equiv g^1 \neq g^2 \equiv G_0^2$ and define

$$\begin{aligned}
g &= \max(g^1, g^2) \equiv g^1 \vee g^2 , \\
\wedge &= \begin{cases} 1 & \text{if } g^1 < g^2 \\ 2 & \text{otherwise} \end{cases} \quad \text{i.e.} \quad g^\wedge \equiv g^1 \wedge g^2 \\
\delta &= \max(g^1, g^2) - \min(g^1, g^2) \equiv g^1 \vee g^2 - g^1 \wedge g^2 , \\
i \equiv i(G_t^1, G_t^2) &= \begin{cases} 1 & \text{if } G_t^1 > G_t^2 \\ 2 & \text{otherwise} \end{cases} .
\end{aligned}$$

Then the survival probability can be decomposed as follows

$$\begin{aligned}
& \mathbb{P}\left(\tau_1 > t, \tau_2 > t\right) \\
&= \mathbb{P}\left(\min_{0\le s\le t} \sigma_1 W^1_{G_s^1} > K_1,\ \min_{0\le s\le t} \sigma_2 W^2_{G_s^2} > K_2\right) \\
&= \int_{[g,\infty)^2} \mathbb{E}\Big(\mathbb{I}_{\{\sigma_1 W^1_{g^1+s} > K_1,\ \sigma_2 W^2_{g^2+s} > K_2,\ s \le T-g\}} \mathbb{E}\Big(\mathbb{I}_{\{\sigma_\wedge W^\wedge_{g^\wedge+s} > K_2,\ T-g\le s\le T-g^\wedge\}} \\
&\qquad \mathbb{I}_{\{\sigma_i W^i_{g^i+s} > K_i,\ T-g^i\le s\le T-g^i+\Delta\}} \mid W^1_{g^1+s},\ W^2_{g^2+s}, s \le T-g\Big)\Big) \\
&\qquad \mathbb{P}_{(G_t^1, G_t^2)}(\,\mathrm{d}x \times \,\mathrm{d}y) .
\end{aligned}$$

Now one has to distinguish two cases for the inner conditional expectation (where G_t^1 and G_t^2 are fixed):
In case $\wedge$ and i are not the same, we have $W^\wedge \perp W^i$ and $g^i = g$ which for the inner conditional expectation implies (using the Markov property)

$$\begin{aligned}
& \mathbb{E}\Big(\mathbb{I}_{\{\sigma_\wedge W^\wedge_{g^\wedge+s} > K_\wedge,\ T-g\le s\le T-g^\wedge\}} \mid W^\wedge_{g^\wedge+T-g}\Big) \\
\cdot\ & \mathbb{E}\Big(\mathbb{I}_{\{\sigma_i W^i_{g^i+s} > K_i,\ T-g^i\le s\le T-g^i+\Delta\}} \mid W^i_{g^i+T-g}\Big) \\
=\ & \mathbb{E}\Big(\mathbb{I}_{\{\sigma_\wedge W^\wedge_{g^\wedge+s} > K_\wedge,\ T-g\le s\le T-g^\wedge\}} \mid W^\wedge_{T-\delta}\Big) \\
\cdot\ & \mathbb{E}\Big(\mathbb{I}_{\{\sigma_i W^i_{g+s} > K_i,\ T-g\le s\le T-g+\Delta\}} \mid W^i_T\Big) \\
=\ & \left(1 - 2\Phi\left(\frac{K_\wedge - \sigma_\wedge W^\wedge_{T-\delta}}{\sigma_\wedge\sqrt{\delta}}\right)\right)\left(1 - 2\Phi\left(\frac{K_i - \sigma_i W^i_T}{\sigma_i\sqrt{\Delta}}\right)\right)
\end{aligned}$$

In case $\wedge$ and i coincide, we have $W^\wedge \equiv W^i$ and $g^i = g^\wedge$, which for the inner conditional expectation implies

$$\mathbb{E}\Big(\mathbb{I}_{\{\sigma_\wedge W^\wedge_{g^\wedge+s} > K_\wedge,\ T-g\le s\le T-g^\wedge+\Delta\}} \mid W^\wedge_{g^\wedge+T-g}\Big) = 1 - 2\Phi\left(\frac{K_\wedge - \sigma_\wedge W^\wedge_{T-\delta}}{\sigma_\wedge\sqrt{\Delta-\delta}}\right) .$$

So the inner equation depends on whether $G^1_t \geq G^2_t$ or not. Since this event is random we won't have an analytical formula unless we further require $G^1 \geq G^2$ or $G^1 \leq G^2$ for all t. Then the time changes are not independent and a separate calibration for each asset-value model to its credit-spread curve is not possible.

The last two sections of this chapter give explicit examples for possible time changes. We distinguish between so-called *numerical time-change distributions* and so-called *analytical time-change distributions*:
The next section concentrates on examples where a numerical approximation of the time-change distribution is available through a *Laplace transform*. These time change distributions and the resulting FPT distributions are said to be *numerical*. By contrast, the last section of this chapter gives examples that have an explicit time-change density, in terms of an integral and/or series representation. Then, time-change distribution and FPT distribution are said to be *analytical*.

1.3.6 Numerical time-change densities

The distribution of G_t is uniquely determined by its *characteristic function* $\phi_{G_t}(u) = \mathbb{E}[\mathrm{e}^{iuG_t}]$ and also – since G_t is a positive random variable – by its *Laplace transform* $l_{G_t}(u) = \mathbb{E}[\mathrm{e}^{uG_t}]$[19], which we prefer to consider because it is a transformation on the real line. The time-change distribution can be retrieved numerically, via Laplace inversion[20], when applying equation (1.1) for G_t:

$$l_{G_t}(u) \quad = \quad u \int_0^\infty \mathrm{e}^{-ut}\, \mathbb{P}(G_t \leq t)\, \mathrm{d}t \; .$$

As we mentioned already there are two main approaches to obtaining a positive continuous time change, the integral over a positive process $G_t = g + \int_0^t g_s \,\mathrm{d}s$ or the supremum of a positive continuous process $G_t = \sup_{0 \leq s \leq t} g_s$. Time-change examples close at hand follow the integral approach and are given by integrated positive and mean-reverting processes such as *Cox-Ingersoll-Ross (CIR)* or *generalized Ornstein-Uhlenbeck (OU)*, where the Lévy component is given by a subordinator[21]. The corresponding driving processes $(g_t)_t$ are given by

$$\begin{aligned} \mathrm{d}g_t^{CIR} &= \kappa(\theta - g_t^{CIR})\,\mathrm{d}t + \hat{\sigma}\sqrt{g_t^{CIR}}\,\mathrm{d}B_t \;, \\ \mathrm{d}g_t^{OU} &= -\kappa g_t^{OU}\,\mathrm{d}t + \,\mathrm{d}J_t \;, \end{aligned} \tag{1.18}$$

[19] See Definition 1.6.

[20] The fastest method for retrieving a distribution is by direct integration, which is possible whenever a characteristic function is explicitly known, confer Kilin (2007).

[21] A subordinator is a special time change. It is an increasing Lévy process and therefore needs to have a nonnegative drift, no diffusion and only positive jumps of finite variation. See Definition 1.4.

where $\kappa > 0$ is the *mean-reversion speed* to the *long-term mean* $\theta \geq 0$ (in the CIR case) respectively 0 (in the OU case). $\hat{\sigma} > 0$ is the volatility of the Brownian motion B and J denotes the jumping subordinator. For the gamma and inverse-Gaussian subordinator, the Lévy measure $\nu(x)$ and Laplace transform $l_{J_t}(u) = \mathbb{E}[\mathrm{e}^{-uJ_t}]$ were given in Table 1.1. Table 1.2 lists the Laplace transforms of the time change $G_t = g + \int_0^t g_s \, \mathrm{d}s$, that is $l_{G_t}(u) = \mathbb{E}[\mathrm{e}^{-uG_t}]$, where the driving process is given by a CIR, general-OU and OU-gamma process, respectively. The Laplace transform of the OU-gamma time change is given when the Laplace transform of the gamma subordinator is inserted into the Laplace transform of the general-OU time change (second row of the table).

time change $G_t = g + \int_0^t g_s \, \mathrm{d}s$	Laplace transform $\mathbb{E}[\mathrm{e}^{-uG_t}]$
CIR g^{CIR}	$\frac{\exp\left(\frac{\kappa^2\theta t}{\hat{\sigma}^2}\right)}{\left(\cosh\frac{\gamma t}{2} + \frac{\kappa}{\gamma}\sinh\frac{\gamma t}{2}\right)^{\frac{2\kappa\theta}{\hat{\sigma}^2}}} \exp\left(-\frac{2gu}{\kappa + \gamma\coth\frac{\gamma t}{2}}\right)$ $\gamma = \sqrt{\kappa^2 + 2\hat{\sigma}^2 u}$
general OU g^{OU}	$\exp\left(\frac{gu}{\kappa}(1 - \mathrm{e}^{-\kappa t}) + \int_0^t l_{J_t}\left(\frac{u}{\lambda}(1 - \mathrm{e}^{\kappa(s-t)})\,\mathrm{d}s\right)\right)$
OU-gamma	$\exp\left(\frac{gu}{\kappa}(1 - \mathrm{e}^{-\kappa t}) + \frac{ut\kappa c}{\lambda\kappa - u} + \frac{\lambda\kappa c}{\lambda\kappa - u}\ln\left(1 - \frac{u}{\lambda\kappa}(1 - \mathrm{e}^{-\kappa t})\right)\right)$

Table 1.2: Time-change examples that yield a Laplace transform and thus a numerical time-change distribution.

1.3.7 Explicit time-change densities

Sometimes it is not sufficient to have a numerical approximation of a time-change density, for example when one is interested in an analytical credit-spread formula and wants to derive credit-spread dynamics. In general, whenever a conditional time-change density w.r.t. some filtration $\tilde{\mathbb{F}}$ (that is $\mathbb{P}_{G_t|\tilde{\mathcal{F}}_t}(\mathrm{d}x)$) is needed, it is not enough to have an approximation of $\mathbb{P}_{G_t}(\mathrm{d}x)$. Examples for time changes with analytical density are not obvious!

CIR-type time change

Our first idea is to consider the integrated Cox-Ingersoll-Ross process $G_t^{CIR} = g + \int_0^t g_s^{CIR}\, \mathrm{d}s$. The integrand process (1.18) has explicit solution:

$$\begin{aligned} g_t^{CIR} &= \mathrm{e}^{-\kappa t}\left[g_0^{CIR} + 2\sqrt{g_0^{CIR}}\,\hat{\sigma}\int_0^t \mathrm{e}^{\frac{1}{2}\kappa s}\,\mathrm{d}B_s + \hat{\sigma}^2\Big[\int_0^t \mathrm{e}^{\frac{1}{2}\kappa s}\,\mathrm{d}B_s\Big]^2\right] \\ &= \mathrm{e}^{-\kappa t}\left[\sqrt{g_0^{CIR}} + \hat{\sigma}\int_0^t \mathrm{e}^{\frac{1}{2}\kappa s}\,\mathrm{d}B_s\right]^2 . \end{aligned} \tag{1.19}$$

But the time change G_t^{CIR} has no analytical distribution. Therefore we multiply the integrand g_t^{CIR} with the factor $\mathrm{e}^{\kappa t}$ and define the time change as follows:

Definition 1.28 (CIR-type time transformation) For $g \geq 0$, $g_0^{CIR} \geq 0$, $\hat{\sigma} > 0$, $\kappa > 0$, the *CIR-type time change* is defined by

$$\hat{G}_t = g + \hat{\sigma}^2 \int_0^t \mathrm{e}^{\kappa r}\Big[\frac{\sqrt{g_0^{CIR}}}{\hat{\sigma}} + B_{\int_0^r \mathrm{e}^{\kappa s}\,\mathrm{d}s}\Big]^2 \mathrm{d}r \ .$$

Because of Remark 1.15 $\hat{G}$ is equivalent in distribution to[22]

$$\hat{G}_t \stackrel{\mathcal{L}}{=} g + \int_0^t \mathrm{e}^{2\kappa r}\, g_r^{CIR}\,\mathrm{d}r \ . \tag{1.20}$$

Now we will show that $\hat{G}$ indeed yields a density. We first introduce the abbreviation $G_t \equiv \hat{G}_t - g$ and the notation $\mathbb{P}^x$ for the probability measure where the underlying process starts in x. Thus

$$\mathbb{P}^{\frac{\sqrt{v_0}}{\hat{\sigma}}}(\cdot) \equiv \mathbb{P}\left(\cdot \mid B_0 = \frac{\sqrt{v_0}}{\hat{\sigma}}\right) .$$

Theorem 1.29 *The time change $\hat{G}_t$ has density*

$$\begin{aligned} f_{\hat{G}_t}(x) = {} & \frac{1}{\sqrt{2\pi}\hat{\sigma}^2}\sum_{k=0}^{\infty}\left(\frac{g_0^{CIR}}{2\sigma^2}\right)^k\left(\frac{x-g}{\hat{\sigma}^2}\right)^{-1+\frac{k}{2}}\sum_{j=0}^{\infty}\frac{\Gamma(\frac{1}{2}+k+j)}{\Gamma(\frac{1}{2}+k)j!} \\ & \cdot\exp\left\{-\frac{\left((\frac{1}{2}+2k+2j)\frac{1}{\kappa}(\mathrm{e}^{\kappa t}-1)+\frac{g_0^{CIR}}{2\sigma^2}\right)^2}{2\frac{x-g}{\hat{\sigma}^2}}\right\} \\ & \cdot\sum_{0\leq l\leq\frac{k+1}{2}}(-)^{j+l}2^{\frac{1}{2}+k-l}\frac{(k+1)}{l!(k+1-2l)!}\left(\frac{(\frac{1}{2}+2k+2j)\frac{\mathrm{e}^{\kappa t}-1}{\kappa}+\frac{g_0^{CIR}}{2\sigma^2}}{\frac{\sqrt{x-g}}{\hat{\sigma}}}\right)^{k+1-2l} \end{aligned}$$

---

[22] Cf. Kammer (2002).

where Γ denotes the Gamma function[23]. *In particular the distribution of the time transformation is given by*

$$\mathbb{P}(\hat{G}_t \leq x) \quad = \quad \int_{-\infty}^{\frac{x-g}{\hat{\sigma}^2}} f_{\hat{G}_t}(y)\,\mathrm{d}y \ .$$

Proof. We determine $\mathbb{P}(\hat{G}_t \leq x)$. The density follows by derivation w.r.t. x. For the proof we set $g_0 \equiv g_0^{CIR}$. Then

$$\begin{aligned}
\mathbb{P}(\hat{G}_t \leq x) = \ \mathbb{P}(G_t \leq x - g) \ &= \ \mathbb{P}\left(\int_0^t \mathrm{e}^{\kappa r} \left[\frac{\sqrt{g_0}}{\hat{\sigma}} + B_{\int_0^r \mathrm{e}^{\kappa s}\ \mathrm{d}s} \right]^2 \mathrm{d}r \leq \frac{x-g}{\hat{\sigma}^2} \right) \\
&= \mathbb{P}_{\frac{\sqrt{g_0}}{\hat{\sigma}}} \left(\int_0^t \mathrm{e}^{\kappa r} \left[B_{\int_0^r \mathrm{e}^{\kappa s}\ \mathrm{d}s} \right]^2 \mathrm{d}r \leq \frac{x-g}{\hat{\sigma}^2} \right) \ .
\end{aligned}$$

Substitute $w = \int_0^r \mathrm{e}^{\kappa s}\ \mathrm{d}s \ = \ \frac{1}{\kappa}(\mathrm{e}^{\kappa r} - 1)$, i.e. $\frac{\mathrm{d}w}{\mathrm{d}r} = \mathrm{e}^{\kappa r}$, then

$$\begin{aligned}
\mathbb{P}(G_t \leq x - g) \ &= \ \mathbb{P}_{\frac{\sqrt{g_0}}{\hat{\sigma}}} \left(\int_0^{\frac{1}{\kappa}(\mathrm{e}^{\kappa t} - 1)} B_w^2\,\mathrm{d}w \leq \frac{x-g}{\hat{\sigma}^2} \right) \\
&= \ \int_{-\infty}^{\frac{x-g}{\hat{\sigma}^2}} \mathbb{P}_{\frac{\sqrt{g_0}}{\hat{\sigma}}} \left(\int_0^{\frac{1}{\kappa}(\mathrm{e}^{\kappa t} - 1)} B_w^2\,\mathrm{d}w \in \ \mathrm{d}y \right) \ .
\end{aligned}$$

With Borodin, Salminen (2002) (p. 168, 642) and their notations the integrand can be determined as follows:

$$\begin{aligned}
&\mathbb{P}_{\frac{\sqrt{g_0}}{\hat{\sigma}}} \left(\int_0^{\frac{1}{\kappa}(\mathrm{e}^{\kappa t} - 1)} B_w^2\,\mathrm{d}w \in \ \mathrm{d}y \right) \\
&= \ ec_y(0, \frac{1}{2}, \frac{\mathrm{e}^{\kappa t} - 1}{\kappa}, 0, \frac{g_0}{2\sigma^2})\,\mathrm{d}y \\
&= \ \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{g_0}{2\sigma^2} \right)^k c_y \Big(k, \frac{1}{2} + k, \frac{\mathrm{e}^{\kappa t} - 1}{\kappa}, \frac{g_0}{2\sigma^2} + \frac{k}{\kappa}(\mathrm{e}^{\kappa t} - 1) \Big)\,\mathrm{d}y \\
&= \ \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{g_0}{2\sigma^2} \right)^k \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2} + k + j)}{\Gamma(\frac{1}{2} + k) j!}\ y^{-1+\frac{k}{2}} \\
&\ \cdot \exp\left\{ - \frac{\left((\frac{1}{2} + 2k + 2j)\frac{1}{\kappa}(\mathrm{e}^{\kappa t} - 1) + \frac{g_0}{2\sigma^2} \right)^2}{2y} \right\} \\
&\cdot \sum_{0 \leq l \leq \frac{k+1}{2}} (-)^{j+l} 2^{\frac{1}{2}+k-l} \frac{(k+1)!}{l!(k+1-2l)!} \left(\frac{(\frac{1}{2} + 2k + 2j)\frac{\mathrm{e}^{\kappa t} - 1}{\kappa} + \frac{g_0}{2\sigma^2}}{\sqrt{y}} \right)^{k+1-2l} \mathrm{d}y \\
&\equiv \ f(y)\,\mathrm{d}y \ .
\end{aligned}$$

[23] Cf. e.g. Borodin & Salminen (2002).

Now note that with the *fundamental theorem of calculus*

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(\hat{G}_t \le x) \quad = \quad \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(G_t \le x - g) \; = \; \frac{1}{\hat{\sigma}^2} f\left(\frac{x-g}{\hat{\sigma}^2}\right) \; .$$

□

Remark 1.30 (Drawback of the CIR-type time change)
With the proof of the theorem we have

$$\mathbb{E}G_t \quad \sim \quad \mathbb{E}\left[\int_0^{\frac{1}{\kappa}(\mathrm{e}^{\kappa t}-1)} B_w^2 \,\mathrm{d}w\right] \; = \; \int_0^{\frac{1}{\kappa}(\mathrm{e}^{\kappa t}-1)} w \,\mathrm{d}w \; = \; \frac{1}{2\kappa}(\mathrm{e}^{\kappa t}-1)^2 \; ,$$

that is, on average, for $\kappa > 0$ the business time speeds up as $x \mapsto \mathrm{e}^x$ and does not slow down again.

Remark 1.31 (Another time-change idea)
The drawback can be eliminated by defining another time change:

$$\tilde{G}_t \quad := \quad f(t)G_t \; \stackrel{\mathcal{L}}{=} \; \int_0^t f(t)\,\mathrm{e}^{2\kappa r}\, v(r)\,\mathrm{d}r \; ,$$

where e.g. $f(t) = \mathrm{e}^{-2\kappa t}$, $f(t) = \frac{1}{t}\,\mathrm{e}^{-2\kappa t}$ or anything such that $\mathbb{E}[\tilde{G}_t] \sim t$. The equivalence in distribution follows by (1.20). Note that $\tilde{G}$ is a positive non-decreasing process. Then the first-passage process

$$Y_t \quad = \quad \sigma W_{f(t)G_t} \; ,$$

has default probability

$$\mathbb{P}\left(\tau < t\right) \quad = \quad 2\int_0^\infty \Phi\left(\frac{K - Y_0}{\sigma\sqrt{f(t)x - g}}\right)\mathbb{P}(G_t \in \,\mathrm{d}x) \; .$$

Example 1.32 (CIR-type sample paths)
Figure 1.2 and 1.3 show sample paths for the Brownian motions W and B, the resulting time-change paths G and the first-passage (or asset-value) paths W_G. The parameters were chosen to be $\kappa = 2$, $\hat{\sigma} = 1$ and $\hat{\sigma} = 2$, respectively. The time-change paths have an exponential form. For $\hat{\sigma} = 2$ the business time is much faster than for $\hat{\sigma} = 1$. In Figure 1.2 it reaches 30 and 8, respectively, in a normal unit of time. That is in the same normal time t, G_t with $\hat{\sigma} = 2$ runs faster through the W-path of the first plot (leading to the sixth plot) than with $\hat{\sigma} = 1$ (leading to the fourth plot). Figure 1.3 also illustrates this nicely: In one unit of real time, the slower time change ($\hat{\sigma} = 1$) just reaches the top of the Brownian path (where the time change slows down), whereas for the faster time change ($\hat{\sigma} = 2$) we run over the

Brownian hill which appears like a peak (last plot). The distributions for the two considered time changes, and other values for the speed to mean-reversion κ, are shown in the boxplots of Figure 1.4. Boxplots visualize a distribution by its median, quartiles and extreme values. We see that the parameters of a CIR-time change can yield various distributions for G_t.

Example 1.33 (Multivariate asset-value paths dependent on the same CIR-type time change)
In Figure 1.5 five time-changed Brownian motions are plotted. Thereby, the original Brownian motions are uncorrelated (Brownian independence). All are time-changed with the same CIR-type process having parameters $\kappa = 2$ and $\hat{\sigma} = 1$. Dependence is thus introduced only by the business time. We derived the corresponding multivariate default-probability formula in Subsection 1.3.3, Corollary 1.22. Figure 1.5 shows that as long as the time change is slow (i.e. generates small values), the asset-value paths $W^i_{G_t}$ are not volatile. Vice verca when the time change speeds up we find a lot of variation in the asset-value paths. Note that because of the independence between the W^i's the paths are independent in their up-and-down movements.

Further analytical time-change examples

When inserting $g_0^{CIR} = 0$, $\kappa = 0$ and $\hat{\sigma} = 1$ into the CIR-type time change it reduces to $G_t = \int_0^t B_u^2 \, \mathrm{d}u$. That is said to be the *simple time change*. Table 1.3 states analytical densities for this time change[24], furthermore for the so-called Dufresne time change[25], the time-change process defined by the supremum of Brownian motion[26] and the supremum of the n-dimensional Bessel process[27] $R_s^{(n)} = \sqrt{(B_s^1)^2 + \cdots + (B_s^n)^2}$, $n \in \mathbb{N}$. We derived the time-change densities with the help of the 'Handbook of Brownian Motion' by Borodin & Salminen (2002). We refer to them for the notation and definition of the functions Γ, m, J and j.[28] Note that m is a Laplace inverse having an analytical form. B and $B^1, \ldots, B^n$ denote independent Brownian motions starting at zero. Note that the formulas can be extended for Brownian motions not starting at zero. The parameters $\hat{\sigma}$, κ are constants and $n = 2\nu + 2$ and ν are natural numbers.

[24] The formula for the conditional density is derived in the proof of Theorem 6.7. For the unconditional density insert $B_0 = 0$ instead of B_t leading to the formula in Table 1.3.

[25] Our name comes from the Dufresne identity, cf. Borodin & Salminen (2002).

[26] For differentiating the integral of the time change distribution in Table 1.3 apply the following equivalence (see Appendix A.1):

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-x}^{x} f(u,x) \, \mathrm{d}u \quad = \quad -f(-x,x) + f(x,x) + \int_{-x}^{x} \frac{\mathrm{d}}{\mathrm{d}x} f(u,x) \, \mathrm{d}s$$

[27] For Bessel processes we refer to Revuz & Yor (2005).

[28] The definitions of Γ, m, J and j are given in Borodin & Salminen (2002); see A.2.

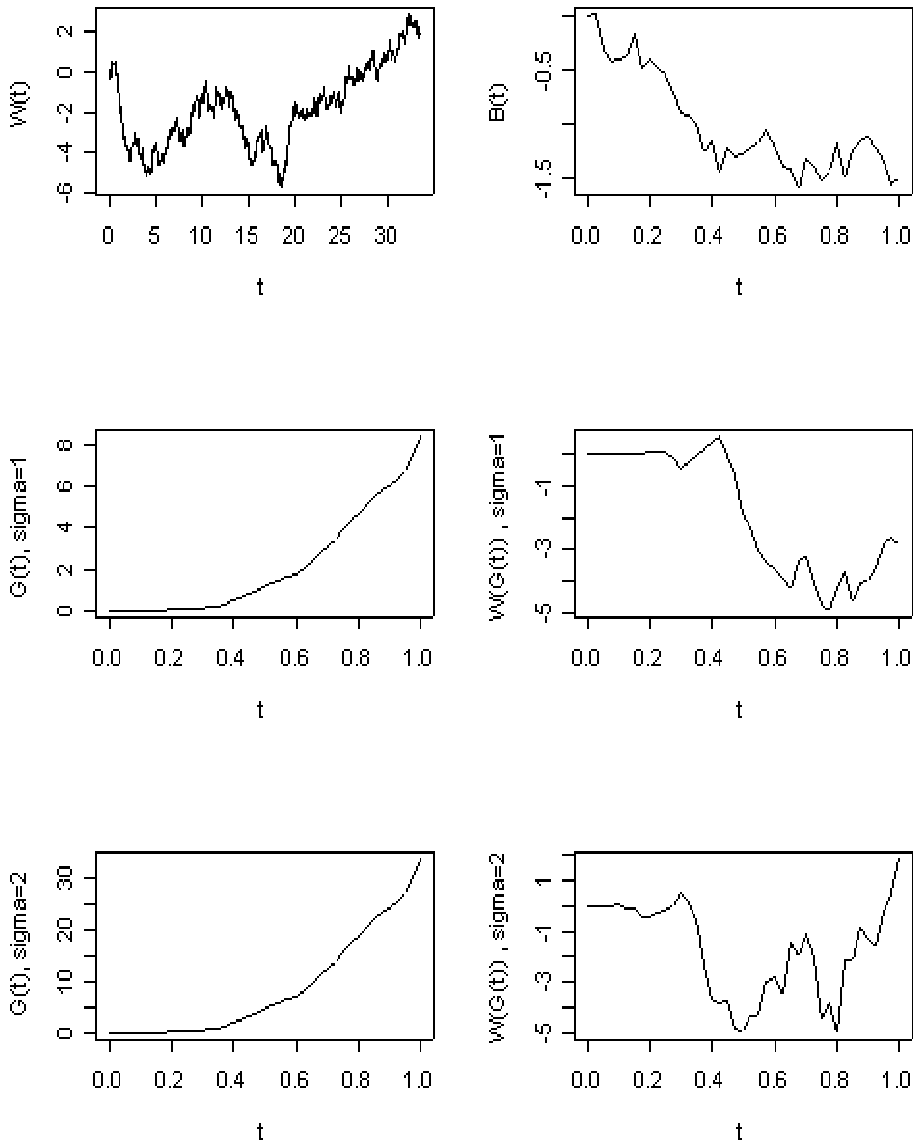

Figure 1.2: Asset-value paths for CIR-type time changes

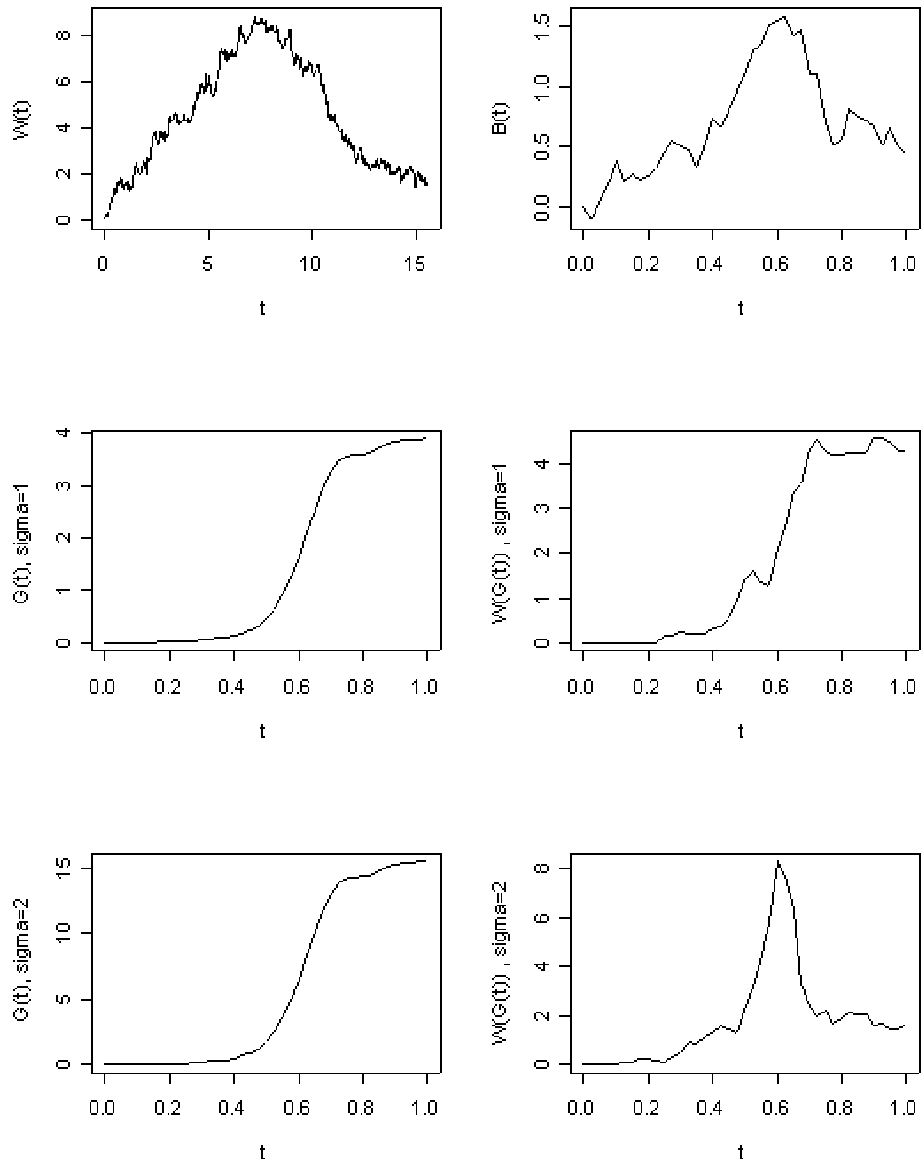

Figure 1.3: Other asset-value paths for CIR-type time changes

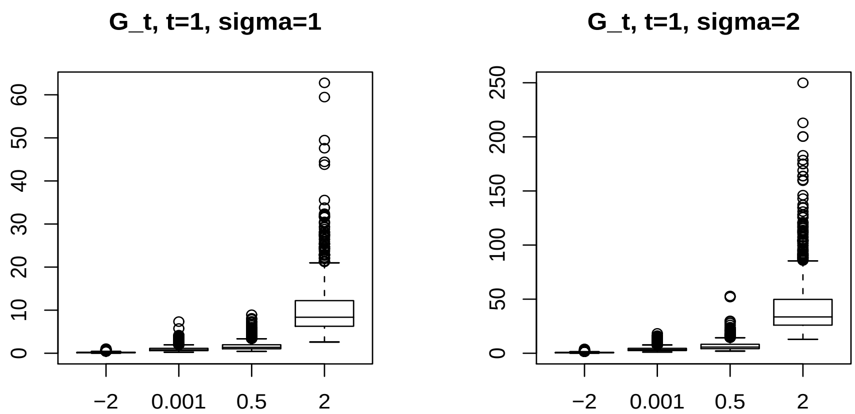

Figure 1.4: CIR-type time-change distributions for different speeds to mean-reversion κ (x-axes)

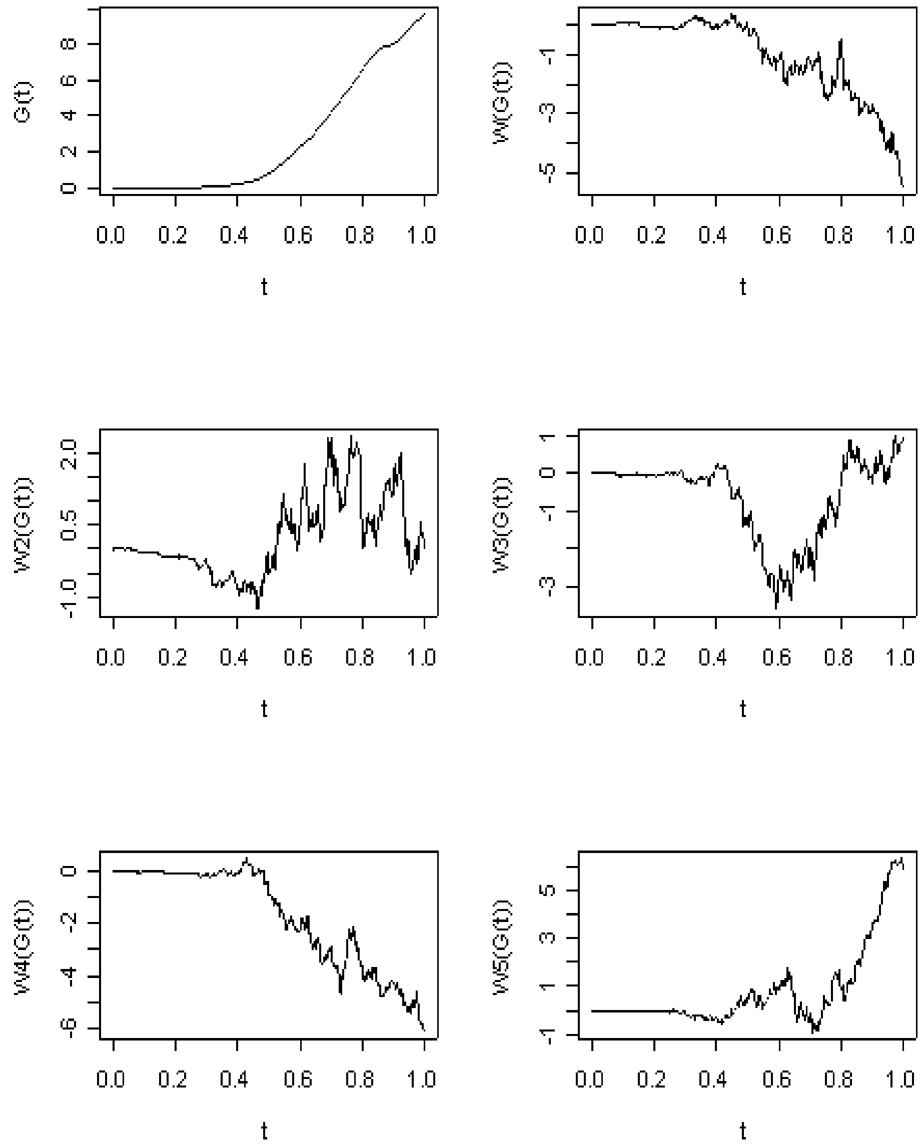

Figure 1.5: Multivariate asset-value process ($\kappa = 2$, $\hat{\sigma} = 1$)

	G_t	$f_{G_t}(x)$
squared BM	$\int_0^t B_s^2 \,\mathrm{d}s$	$\frac{1}{\pi\sqrt{\pi}} \sum_{j=0}^{\infty} (-)^j \frac{\Gamma(\frac{1}{2}+j)}{j!}$ $\cdot \exp^{\frac{((\frac{1}{2}+2j)t)^2}{2y}} \frac{(\frac{1}{2}+2j)t}{x^{\frac{3}{2}}}$
CIR-type	$\hat{G}_t$	$f_{\hat{G}_t}(x)$ (Thm. 1.29)
Dufresne	$\int_0^t \mathrm{e}^{\hat{\sigma} B_s + \hat{\sigma}^2 \kappa s} \,\mathrm{d}s$, $\hat{\sigma} > 0$, $\kappa > -1$	$\hat{\sigma}^{2\kappa+1} x^{\kappa-\frac{1}{2}} 2^{-\kappa-\frac{1}{2}} \,\mathrm{e}^{-\kappa^2 \hat{\sigma}^2 \frac{t}{2} - \frac{1}{\hat{\sigma}^2 x}}$ $\cdot\, m_{\hat{\sigma}^2 t/2} \left(\kappa - \frac{1}{2}, \frac{1}{\hat{\sigma}^2 x}\right)$
sup BM	$\sup_{0\le s\le t} \lvert B_s\rvert$	$\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{\sqrt{2\pi t}} \sum_{k=-\infty}^{\infty} \int_{-x}^{x} (\mathrm{e}^{-(u+4kx)^2/2t}$ $- \mathrm{e}^{-(u+2x+4kx)^2/2t}) \,\mathrm{d}u$
sup Bessel	$\sup_{0\le s\le t} R_s^{(n)}$, $R_s^{(n)} = \sqrt{(B_s^1)^2 + \ldots (B_s^n)^2}$ $n = 2\nu+2$, $R_0^{(n)} = b$	$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=1}^{\infty} \frac{2b^{-\nu} J_\nu(j_{\nu,k} \frac{b}{x})}{j_{\nu,k} x^{-\nu} J_{\nu+1}(j_{\nu,k})} \,\mathrm{e}^{-j_{\nu,k}^2 t/2x^2}$

Table 1.3: Examples for time changes with analytical density

Chapter 2

Analyzing the time-change model with $G_t = \hat{\sigma}^2 \int_0^t B_s^2 \,\mathrm{d}s$

In this chapter we consider the simple time change $G_t = \hat{\sigma}^2 \int_0^t B_s^2 \,\mathrm{d}s$ and analyze its density while varying the parameter $\hat{\sigma}$. We calibrate the model to several default probability curves by fitting two points exactly, which yields a good fit for non-investment-grade ratings. Applying the calibrated parameters we determine the joint default probabilities for two respectively three names, regarding the setting as in Corollary 1.22, that is under zero correlation between the Brownian motions. (Remember our Definition 1.11 of *Brownian independence* and *Brownian correlation*.) We also study joint survival-probability curves under *asset correlation*, regarding Corollary 1.24. *Asset dependence* is due to a common time change and/or Brownian correlation. We analyze the asset dependence produced by the time change and find that it cannot be substituted by adapting the Brownian correlation. Finally we consider the correlation between default events, the so-called *event correlation* or *default correlation*. The relationship to the Brownian correlation is studied and the evolution of default correlation in time is visualized. Note especially that a constant Brownian correlation does not imply a constant default correlation.

Later on in Chapter 6 we analyze this particular simple time change with regard to credit-spread modeling.

2.1 Calibration to a default-probability curve

We calibrate the time-change model with the simple time change $G_t = \hat{\sigma}^2 \int_0^t B_s^2 \,\mathrm{d}s$ to a given default-probability curve. Therefore we discretize the time grid and determine the discretized density

$$\mathbb{P}\left(G_t \in [y, y+\Delta)\right) \quad = \quad \mathbb{P}\left(G_t < y+\Delta\right) - \mathbb{P}\left(G_t < y\right) \ , \qquad (2.1)$$

using the default-probability distribution

$$\begin{aligned}\mathbb{P}\left(G_t < y\right) &= \mathbb{P}\left(\int_0^t B_s^2 \,\mathrm{d}s < \frac{y}{\hat{\sigma}^2}\right) \\ &= \frac{2\sqrt{2}}{\sqrt{\pi}} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(\frac{1}{2}+j)}{j!} \left[1 - \Phi\left(\frac{(4j+1)\hat{\sigma}t}{2\sqrt{y}}\right)\right] . \quad (2.2)\end{aligned}$$

Regarding the default-probability formula of Theorem 1.21, we find that the model has two degrees of freedom for calibration, $\frac{K-Y_0}{\sigma\hat{\sigma}}$ and $\frac{\mu\hat{\sigma}}{\sigma}$. Thus we can fit two points of the curve exactly, which allows us to solve from the analytical FPT formula, Theorem 1.21, for the model parameters easily. Especially in most applications concerning credit spreads (see Chapters 3, 6), it is more important to fit the liquid credit-spread points (usually five and seven years) of the term structure exactly instead of a best fit of the whole curve. The parameter sets of Table 2.1 lead to the same default-probability curve, with fitted points $\mathbb{P}(\tau \leq 0.5) = 0.0064$ and $\mathbb{P}(\tau \leq 5) = 0.0734$. Thereby we fix $g = 0$ and $Y_0 = 0$. The parameter $\hat{\sigma}$ influences the shape of

$\hat{\sigma}$	σ	μ	K
1	1	0.781724	−1.60449
2	0.5	0.195431	−1.60449
5	0.2	0.031269	−1.60449
10	0.1	0.007817	−1.60449
2	1	0.390862	−3.20898
5	1	0.156345	−8.02245

Table 2.1: Parameter sets calibrated to $\mathbb{P}(\tau \leq 0.5) = 0.006$ and $\mathbb{P}(\tau < 5) = 0.073$ ($g = 0$ and $Y_0 = 0$)

the time-change density: Higher values for $\hat{\sigma}$ lead to a flatter density curve with fatter tails. Figure 2.1 shows the densities for G_5 and $\hat{\sigma}$-values $1, 2$ and 5. Note that the scales of the lower plots differ. As we just argued and have seen in Table 2.1, different values of $\hat{\sigma}$ may still lead to the same (marginal) default-probability curve when adapting at the same time the parameter values of K, σ and μ. Thus $\hat{\sigma}$ can be used to influence the joint default probability or credit-spread dynamics, see Chapter 6.

Example 2.1 In Figure 2.2, 2.3 and 2.4 we show default-probability curves coming from the model calibrated to the assumed market default-probability curve that is, for simplicity, given by $F(t) = 1 - \mathrm{e}^{-\lambda t}$ with $\lambda = 1\%$, $\lambda = 7\%$ respectively $\lambda = 10\%$. For this we decided to fix $g = 0$, $Y_0 = 0$, $\sigma = 1$ as well as $\hat{\sigma} = 1$, leading to the simple time change $G_t = \int_0^t B_s^2 \,\mathrm{d}s$. The parameters

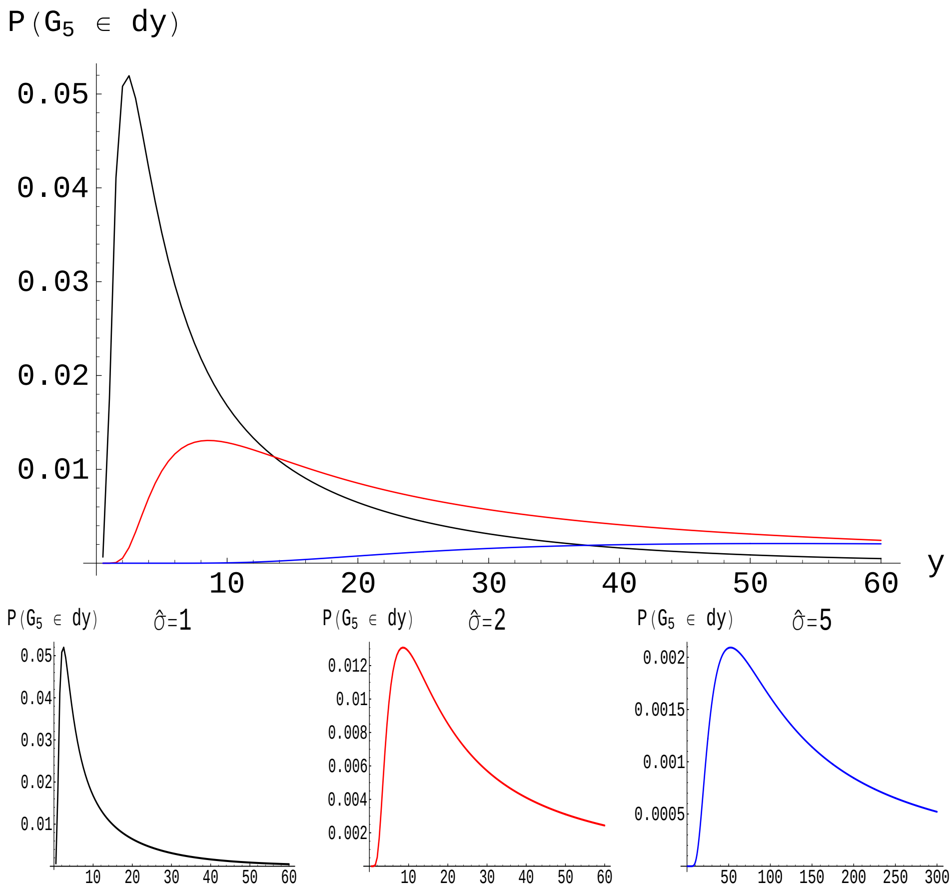

Figure 2.1: Densities of the time change $G_t = \hat{\sigma}^2 \int_0^5 B_s^2 \, \mathrm{d}s$ for $\hat{\sigma} = 1$ (black), $\hat{\sigma} = 2$ (red) and $\hat{\sigma} = 5$ (blue)

μ and K are calibrated to two points, $F(t_1)$ and $F(t_2)$. The parameter sets that lead to the curves in Figure 2.2, 2.3 and 2.4 are given in Table 2.2. The curves calibrated to $F(2), F(5)$ respectively $F(3), F(5)$ respectively $F(5), F(7)$ respectively $F(5), F(10)$ are quite acceptable. Please note that the finally chosen parameter set should emphasize one's personal view on more or less important (liquid) market data points.

Example 2.2 Standard & Poor's publishes average default probabilities (*cumulative default rates*) for different rating classes, see Table 2.3. The average term structures for the years 1981 to 2002 for ratings AAA, BBB, BB, B and CCC are plotted in Figure 2.5 respectively Figure 2.6. Due to the different shapes (concave, linear or convex) of the default-probability curves for different rating classes, we find that our model with the simple time change $G_t = \int_0^t B_s^2 \, \mathrm{d}s$ gives an astonishingly good fit for *speculative grade* default curves (BB, B and CCC), but not such a good fit for *investment-grade* default curves (AAA, AA, A and BBB). Figure 2.6 shows calibrations to the CCC-curve at different time horizons. All these fits are good. Table 2.4 gives the corresponding calibrated parameters.

		$\lambda = 1\%$		$\lambda = 7\%$		$\lambda = 10\%$	
$F(t_1)$	$F(t_2)$	μ	K	μ	K	μ	K
$F(.5)$	$F(5)$	0.9207	−1.5981	0.3802	−1.3489	0.2663	−1.2947
$F(1)$	$F(5)$	0.8753	−1.6732	0.3895	−1.3270	0.2837	−1.2503
$F(2)$	$F(5)$	0.6626	−2.1291	0.3001	−1.5624	0.2164	−1.4346
$F(3)$	$F(5)$	0.5346	−2.5265	0.2290	−1.7989	0.1564	−1.6329
$F(4)$	$F(5)$	0.4520	−2.8601	0.1800	−1.9938	0.1135	−1.7973
$F(5)$	$F(7)$	0.3207	−3.5837	0.0995	−2.3821	0.0422	−2.1162
$F(5)$	$F(10)$	0.2469	−4.1458	0.0342	−2.7687	−0.0107	−2.3910

Table 2.2: Model parameters μ and K calibrated to $F(t) = 1 - \mathrm{e}^{-\lambda t}$ in two points for $\lambda = 1\%$, $\lambda = 7\%$ resp. $\lambda = 10\%$ (for $\sigma = 1$, $\hat{\sigma} = 1$).

2.2 Multivariate default probabilities under Brownian independence

As we mentioned already, $\hat{\sigma}$ influences the dependence in a joint default probability. We assume the multivariate model as of Corollary 1.22. Figure 2.7 shows the one-dimensional default curve and the resulting joint default probabilities for two respectively three names for the parameter set $\hat{\sigma} = 1$, $\sigma_i = 1$, $\mu_i = 0.781724$ and $K_i = -1.60449$ where $i = 1, 2$ resp. $i = 1, 2, 3$.

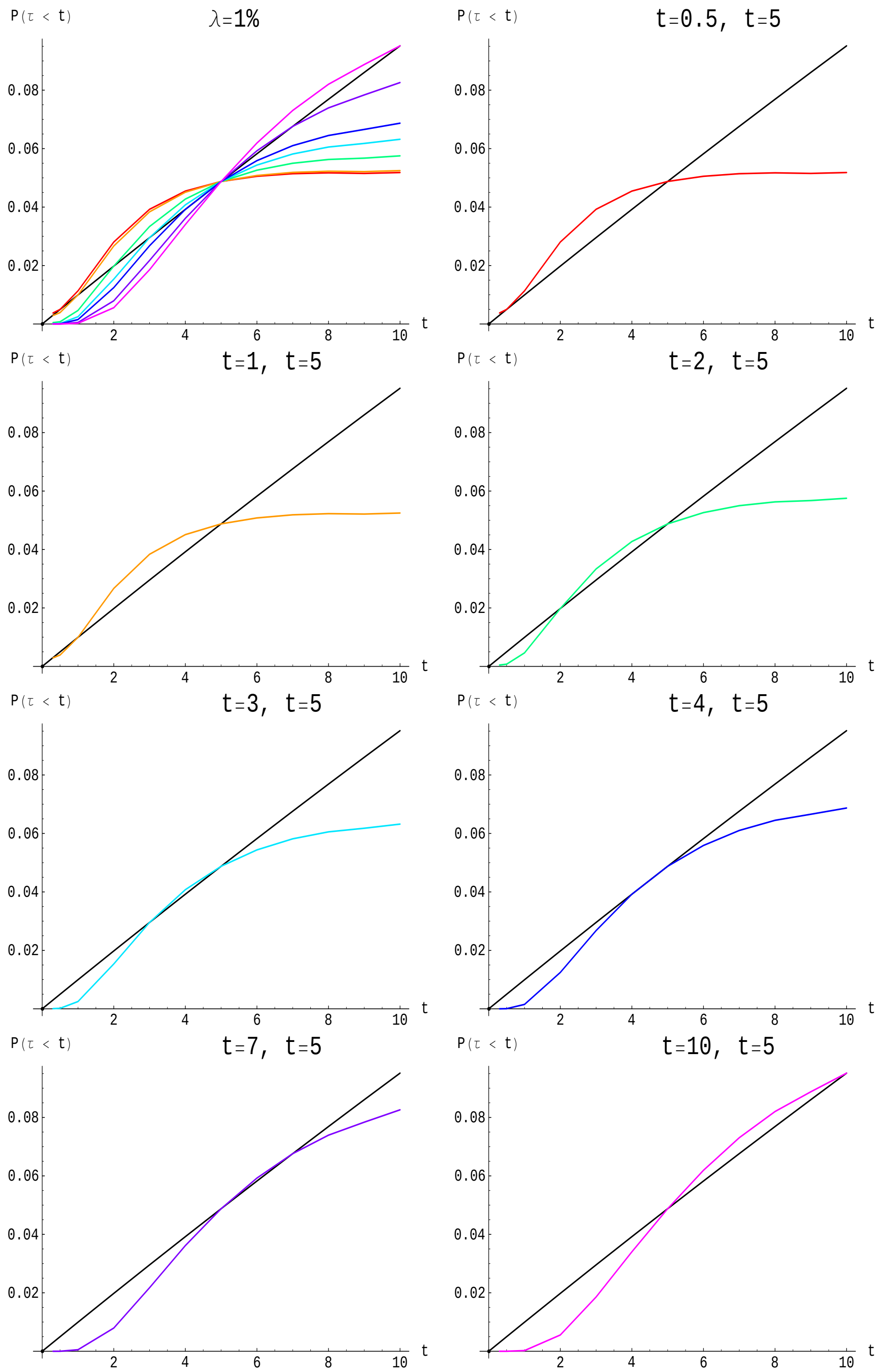

Figure 2.2: Calibration of the simple time change at two points of $F(t) = 1 - \mathrm{e}^{-\lambda t}$ (black curve) with $\lambda = 1\%$

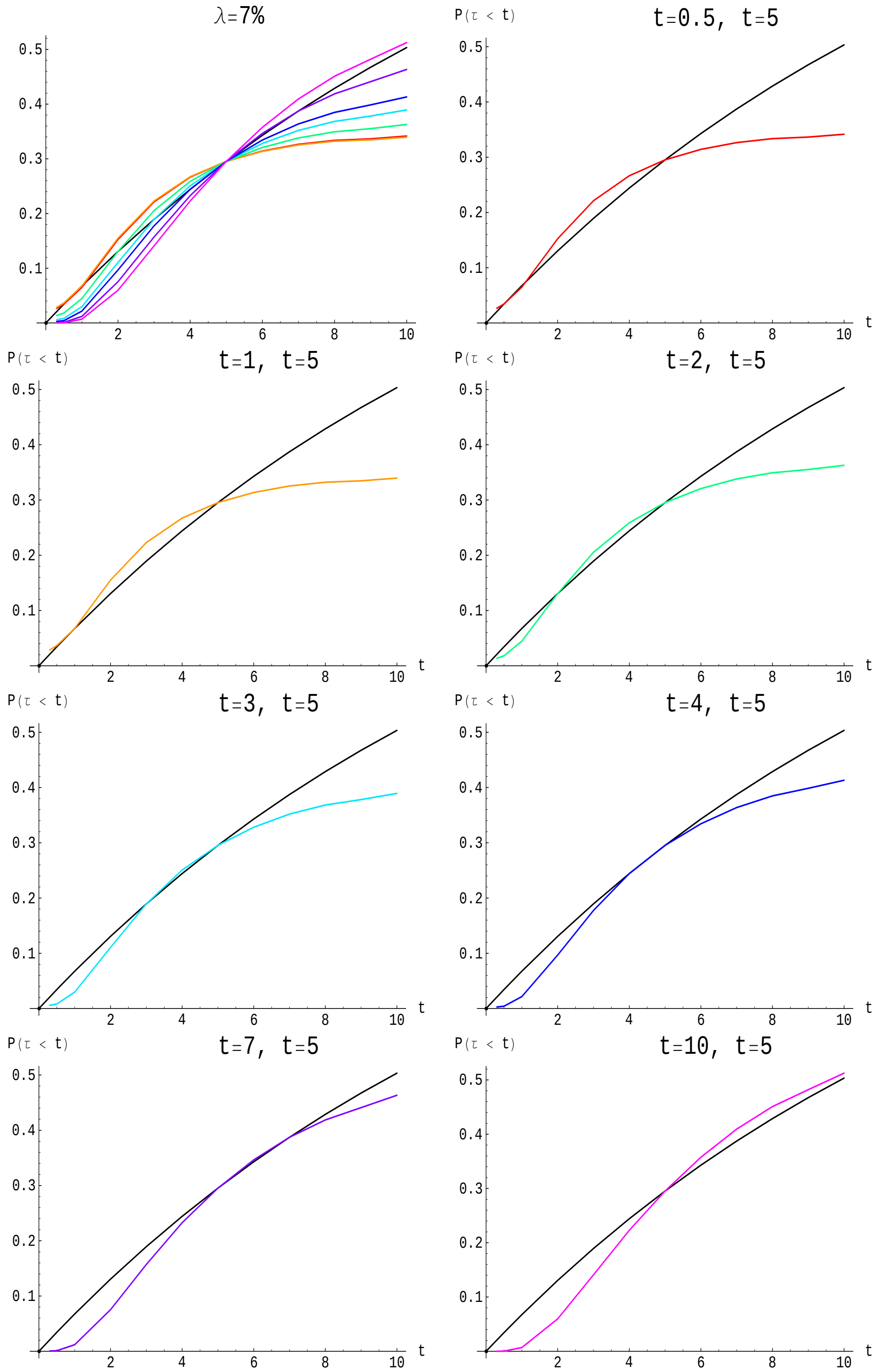

Figure 2.3: Calibration at two points of $F(t) = 1 - \mathrm{e}^{-\lambda t}$ with $\lambda = 7\%$

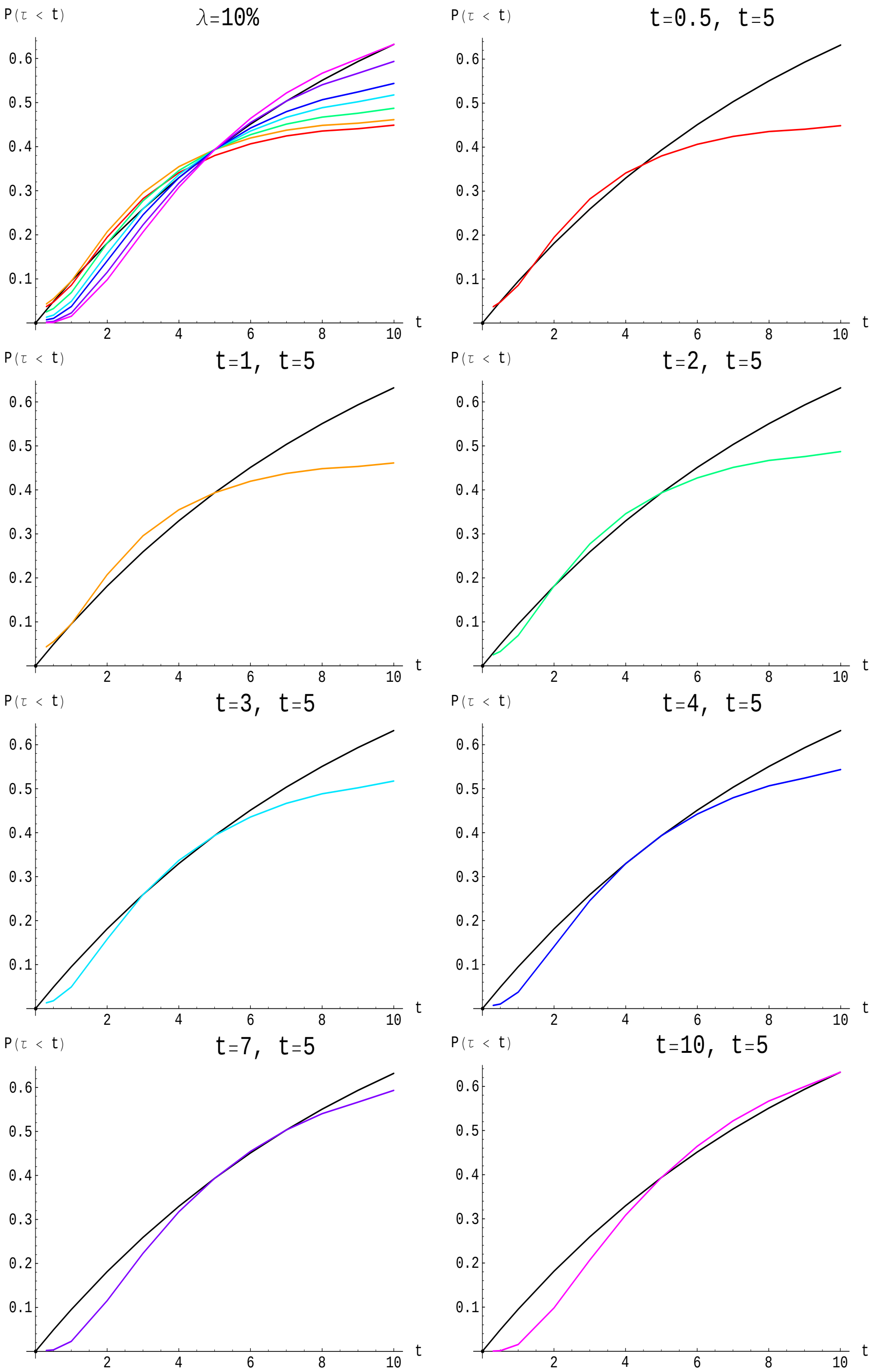

Figure 2.4: Calibration at two points of $F(t) = 1 - \mathrm{e}^{-\lambda t}$ with $\lambda = 10\%$

years	1	2	3	4	5	6	7	8	9	10
rating										
AAA	0.00	0.00	0.03	0.06	0.10	0.17	0.25	0.38	0.43	0.48
AA	0.01	0.03	0.08	0.16	0.27	0.39	0.53	0.65	0.75	0.85
A	0.05	0.15	0.28	0.44	0.62	0.81	1.03	1.25	1.52	1.82
BBB	0.37	0.94	1.52	2.34	3.20	4.02	4.74	5.40	5.99	6.68
BB	1.38	4.07	7.16	9.96	12.34	14.65	16.46	18.02	19.60	20.82
B	6.20	13.27	19.07	23.45	26.59	29.08	31.41	33.27	34.58	35.87
CCC	27.87	36.02	41.79	46.26	50.46	52.17	53.60	54.36	56.16	57.21

Table 2.3: Average S&P default rates [in %]

CCC at t_1	CCC at t_2	μ	K
1	5	0.4389	-0.6965
2	5	0.3829	-0.7689
3	5	0.3069	-0.8882
4	5	0.2331	-1.0336
5	7	0.3330	-0.8442
5	10	0.2413	-0.9947

Table 2.4: Calibration to an average CCC default-rate curve at two points, t_1 and t_2 [in years] (for $\sigma = 1$ and $\hat{\sigma} = 1$)

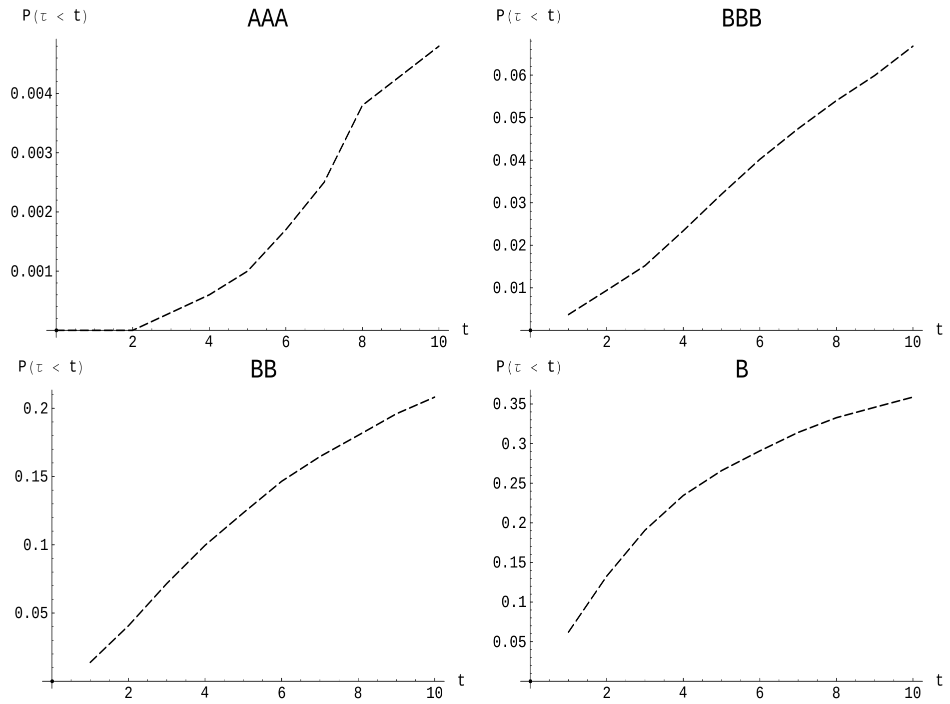

Figure 2.5: Cumulative default-rate curves by S&P

Here unfortunately the influence of $\hat{\sigma}$ on the joint default probabilities is very low. For example, when choosing the parameter set with $\hat{\sigma} = 10$ the dashed curves in Figure 2.8 are obtained and plotted against the curves where $\hat{\sigma} = 1$. Actually, it is rather difficult to compare these default-probability curves at all because for different parameter sets we have to use different grid points for approximation due to the oscillating default-probability integrand. Regarding Corollary 1.22 we conclude that one should choose a time change other than that in Table 1.3, having more degrees of freedom to strengthen the time-change impact.

2.3 Joint survival probabilities under Brownian correlation

We will instead analyze the joint survival probability (JSP) under asset correlation and joint time change $G_t = \hat{\sigma}^2 \int_0^t B_s^2 \, \mathrm{d}s$, i.e. under the conditions of Theorem 1.24. We fix the parameters $g = 0$, $Y_0 = 0$, $\sigma = 1$ and vary $\hat{\sigma} = 1$ respectively 5, $K = -1.4$ respectively -0.5 and $\rho = 0.9, 0.5, 0.1, -0.5$. The resulting JSP curves are shown in Figure 2.9 where the curves with different correlation parameters are plotted against each other, and Figure 2.10, where the curves with different time-change densities, i.e. parameter

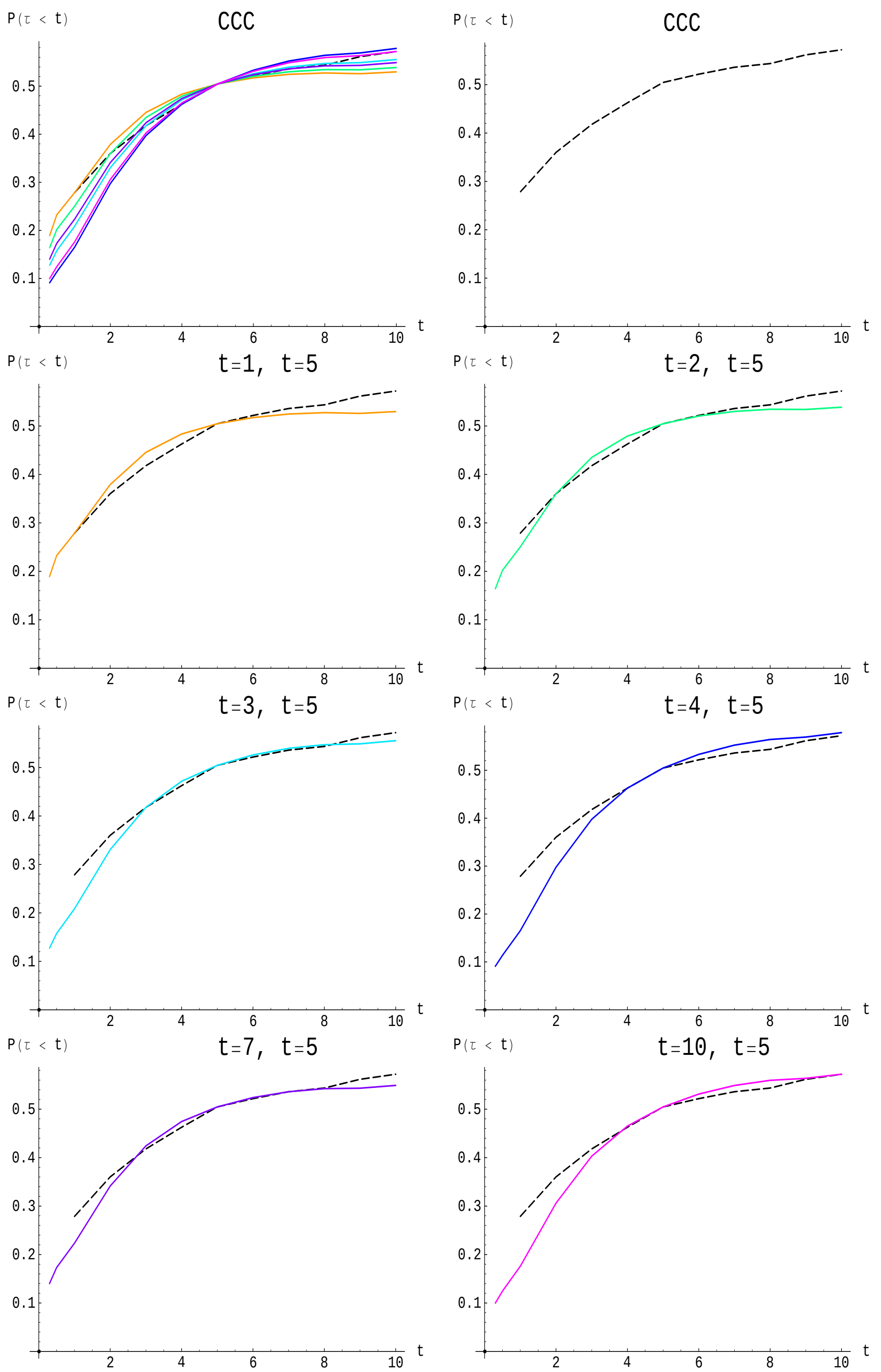

Figure 2.6: Calibration at two points of the CCC cumulative default-rate curve of S&P

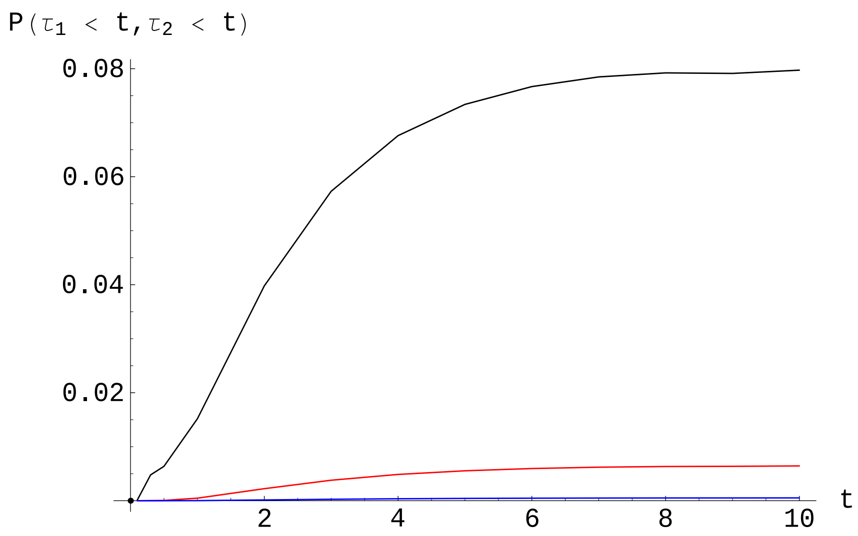

Figure 2.7: Default-probability curve (black) and joint default-probability curves for two (red) resp. three (blue) uncorrelated assets (for $\hat{\sigma} = 1$, $\sigma_i = 1$, $\mu_i = 0.781724$ and $K_i = -1.60449$)

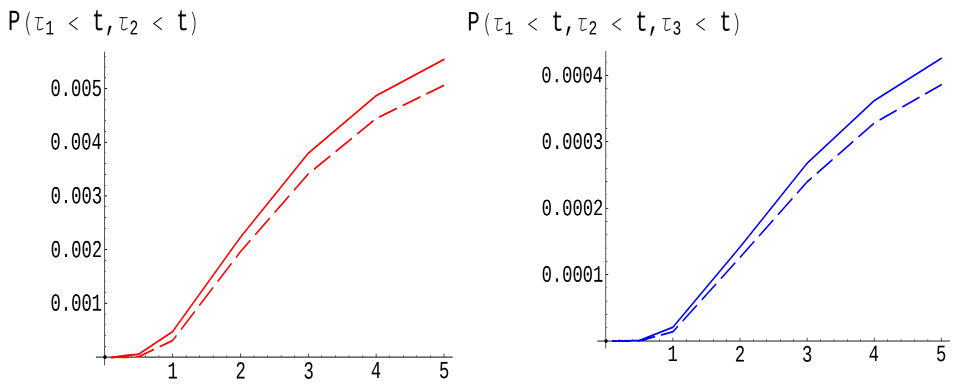

Figure 2.8: Joint default-probability curves for two (red) resp. three (blue) assets when $\hat{\sigma} = 1$ (solid) resp. $\hat{\sigma} = 10$ (dashed)

$\hat{\sigma}$, are plotted against each other. We see that the higher the correlation the higher the JSP. Furthermore with a higher time-change volatility (influenced by $\hat{\sigma}$), the JSP curves become steeper (that is, the JSP decreases more rapidly). Also a threshold level of $K = -1.4$ leads to instantaneous survival probabilities of one, whereas a threshold level of $K = -0.5$ only leads to 0.6 (for $\hat{\sigma} = 1$) respectively 0.4 (for $\hat{\sigma} = 5$). Figures 2.9 and 2.10 have shown that both ρ and $\hat{\sigma}$ influence shape and, especially, steepness of the JSP curves. We want to analyze whether the influence is identical, that is, whether the time change is redundant. Therefore we compare five years JSPs under the following two assumed models:
1) two asset-value processes W^1_G and $W^\perp_G$ having same time change but uncorrelated Brownian motions, i.e. $\mathrm{Corr}(W^1, W^\perp) = 0$ (as in Corollary 1.22),
2) two asset-value processes without time change, W^1 and W^2, but correlated, $\mathrm{Corr}(W^1, W^2) = \rho$ (as in Zhou's Theorem 1.17).
We choose different parameters $\hat{\sigma}$ and solve the following equation for ρ:

$$\mathbb{P}\left(\min_{0\le s\le t} \sigma W^1_{G_s} > K_1,\ \min_{0\le s\le t} \sigma W^\perp_{G_s} > K_2\right) \tag{2.3}$$

$$= \mathbb{P}\left(\min_{0\le s\le t} \sigma W^1_s > K_1,\ \min_{0\le s\le t} \sigma W^2_s > K_2\right), \tag{2.4}$$

$$G_s = \hat{\sigma}^2 \int_0^s B_u^2 \,\mathrm{d}u\ , \quad W^1 \perp W^\perp\ , \quad \mathrm{Corr}(W^1, W^2) = \rho\ .$$

Thereby we keep the marginals in (2.3) constant at $\mathbb{P}(\tau_i \le 5) = 0.378$ (by fixing $K_i = -2.39$ and adapting $\sigma = \frac{1}{\hat{\sigma}}$). Table 2.5 gives the calculated JSPs and corresponding Brownian correlation parameters and marginal default probabilities of (2.4) that are given by $\mathbb{P}(\tau_i \le 5) = 2\Phi\left(\frac{K_i}{\sigma\sqrt{5}}\right)$. With increasing time-change factor $\hat{\sigma}$ the JSP increases and the Brownian correlation decreases, which is due to the decreasing marginals in (2.4). We conclude that the dependence structure from the time change and the Brownian correlation parameter differs. The parameter ρ should be used to include a basic (constant) dependence and the time change should be introduced for stochastic dependence in addition.

2.4 Event correlation versus Brownian correlation

The correlation between the asset-value processes Y_1 and Y_2 is called *asset correlation*, or rather *asset dependence*, because it may not only be due to the correlation between the Brownian motions but also due to a joint (or at least dependent) time change, see Definition 1.10. In order to differentiate the cause of dependence we also introduced the term *Brownian correlation* in Definition 1.11. *Event correlation*, also called *default correlation*, is the correlation between the default events $\{\tau_1 < t\}$ and $\{\tau_2 < t\}$, see Definition 1.12. Recall that the formulas in Theorem 1.23 and Theorem 1.26 for the

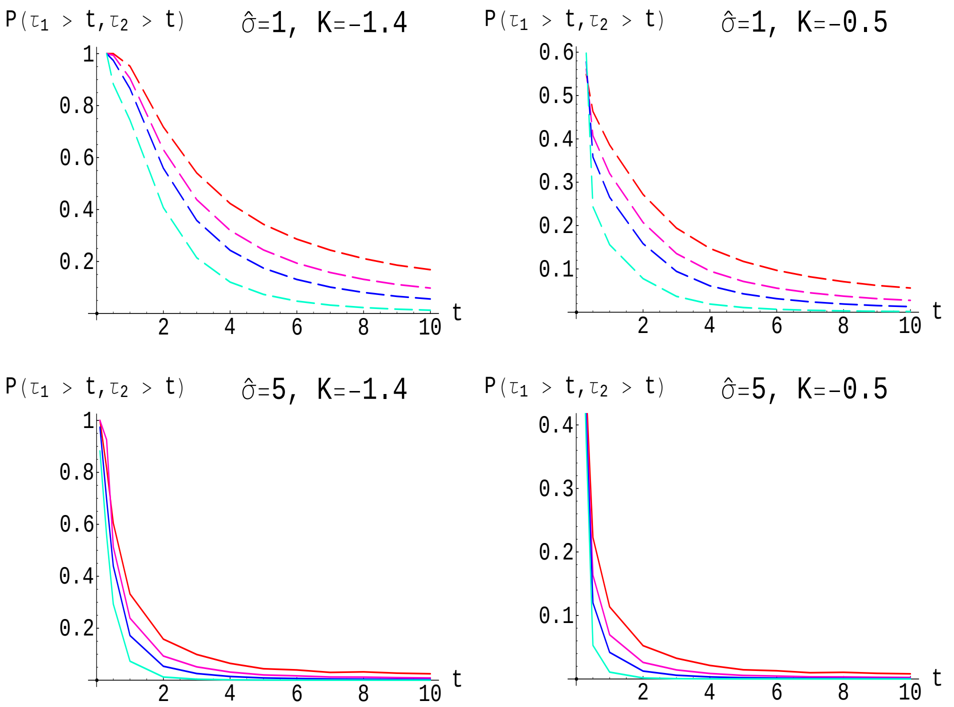

Figure 2.9: Joint survival-probability curves for Brownian correlation parameter $\rho = 0.9$ (red), 0.5 (pink), 0.1 (blue) and -0.5 (turquoise)

$\hat{\sigma}$	JSP	ρ	$\mathbb{P}(\tau_i \leq 5)$ of (2.4)
0.9	0.4186	0.3853	0.3359
1.2	0.4258	-0.0796	0.1994
1.5	0.4292	-0.3802	0.1087
1.8	0.4311	-0.5669	0.0543

Table 2.5: Varying time-change parameter $\hat{\sigma}$, resulting JSPs and corresponding Brownian correlation parameters for constant marginals in (2.3) and marginals in (2.4) as stated.

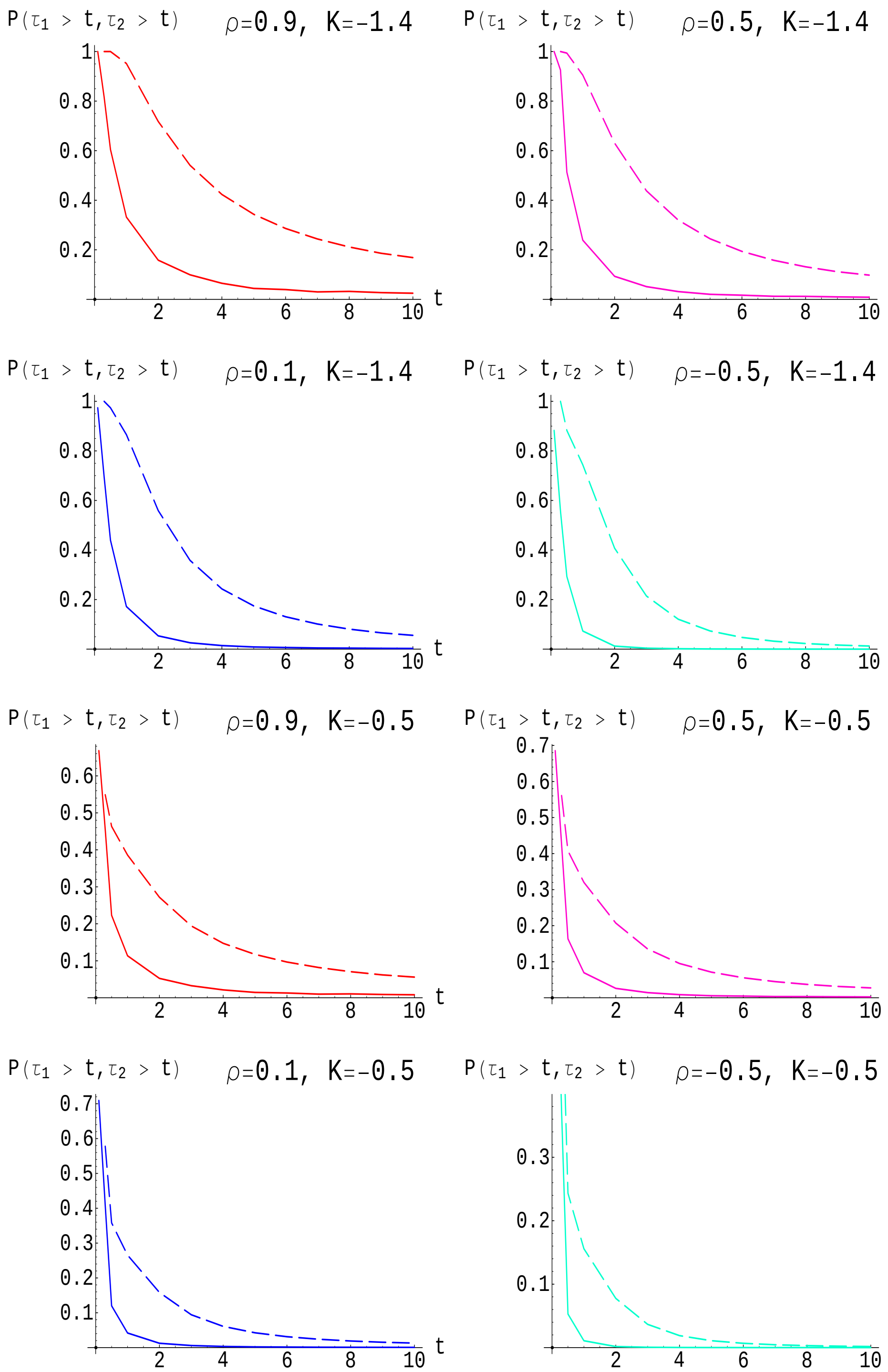

Figure 2.10: Joint survival-probability curves for several Brownian correlations ρ and $\hat{\sigma} = 1$ (dashed) resp. $\hat{\sigma} = 5$ (solid)

joint default probability $\mathbb{P}(\tau_1 \le t, \tau_2 \le t)$ depend on the Brownian correlation ρ. Thus event correlation and Brownian correlation are related through the joint default probability. For this relationship we want to show some numbers. In Example 2.1 we calibrated the model with the simple time change $G_t = \hat{\sigma}^2 \int_0^t B_s^2 \, \mathrm{d}s$ to a given default-probability curve and thereby fitted two points of the term structure exactly. Now let $F_1(t)$ and $F_2(t)$ be given default-probability curves and t a point of exact fit for both curves. Then the event correlation can be written as follows:

$$\rho^E = \frac{\mathbb{P}(\tau_1 \le t, \tau_2 \le t) - F_1(t)F_2(t)}{\sqrt{F_1(t)(1 - F_1(t))F_2(t)(1 - F_2(t))}} \,. \tag{2.5}$$

Table 2.6 gives Brownian correlations corresponding to event correlation values of $0.1, 0.3, 0.5, 0.7$ and 0.9 for default curves $F_i(t) = 1 - \mathrm{e}^{-\lambda_i t}$ with default rate $\lambda_i = 0.07$ respectively $\lambda_i = 0.1$ $(i = 1, 2)$ being exactly fitted at the points $t = 5$ and $t = 10$. We chose default curves for non-investment-grade rating classes because we saw that they yielded the best fit. The empty cells in the second tabular for 0.9 are due to the upper bound (omitting the argument)

$$\rho^E \le \sqrt{\frac{\min(F_1, F_2)(1 - \max(F_1, F_2))}{\max(F_1, F_2)(1 - \min(F_1, F_2))}} \,,$$

coming from condition (1.3). We summarize the numbers as follows: When fixing time, Brownian correlation increases with default correlation. When fixing event correlation, Brownian correlation increases with time.

2.4.1 Event correlation against time

When we consider a constant Brownian correlation parameter over time, this does not necessarily mean that the event correlation is also constant over time. Figure 2.11 shows event-correlation term structures for time-change factors $\hat{\sigma} = 1$ (dashed curves) respectively $\hat{\sigma} = 5$ (solid curves), threshold levels $K = -0.5$ respectively $K = -1$, and Brownian correlation levels $\rho = 0.1$ (blue curves), 0.3 (red curves) and 0.5 (pink curves). For $\rho = 0.1$ and $\rho = 0.5$ corresponding survival-probability term structures were shown in Figure 2.9. We find that the default-correlation term structure is hump-shaped. The curve level increases with increasing threshold level (i.e. for worsening names) and its steepness grows with the time-change factor $\hat{\sigma}$. Note that there are different time scales for $\hat{\sigma} = 1$ and $\hat{\sigma} = 5$.

$\lambda_1 = .07$ $\lambda_2 = .07$	$t = 5$		$t = 10$	
ρ^E	ρ	JDP	ρ	JDP
0.1	0.4467	0.1080	0.7097	0.2784
0.3	0.5900	0.1496	0.8598	0.3284
0.5	0.7190	0.1913	0.9552	0.3784
0.7	0.8277	0.2329	0.9971	0.4284
0.9	~ 1	0.2745	~ 1	0.4784

$\lambda_1 = .07$ $\lambda_2 = .1$	$t = 5$		$t = 10$	
ρ^E	ρ	JDP	ρ	JDP
0.1	0.3544	0.1384	0.5319	0.3423
0.3	0.5214	0.1830	0.7342	0.1906
0.5	0.6743	0.2276	0.8817	0.4388
0.7	0.8044	0.2722	0.9711	0.4870
0.9				

$\lambda_1 = .1$ $\lambda_2 = .1$	$t = 5$		$t = 10$	
ρ^E	ρ	JDP	ρ	JDP
0.1	0.2633	0.1787	0.3627	0.4228
0.3	0.4520	0.2264	0.6022	0.4693
0.5	0.62678	0.2744	0.7881	0.5158
0.7	0.7765	0.3219	0.9158	0.5624
0.9	0.8917	0.3696	0.9844	0.6089

Table 2.6: Event correlation and corresponding Brownian correlation for five resp. ten years JDPs under marginal default-rates of 0.07 and 0.1

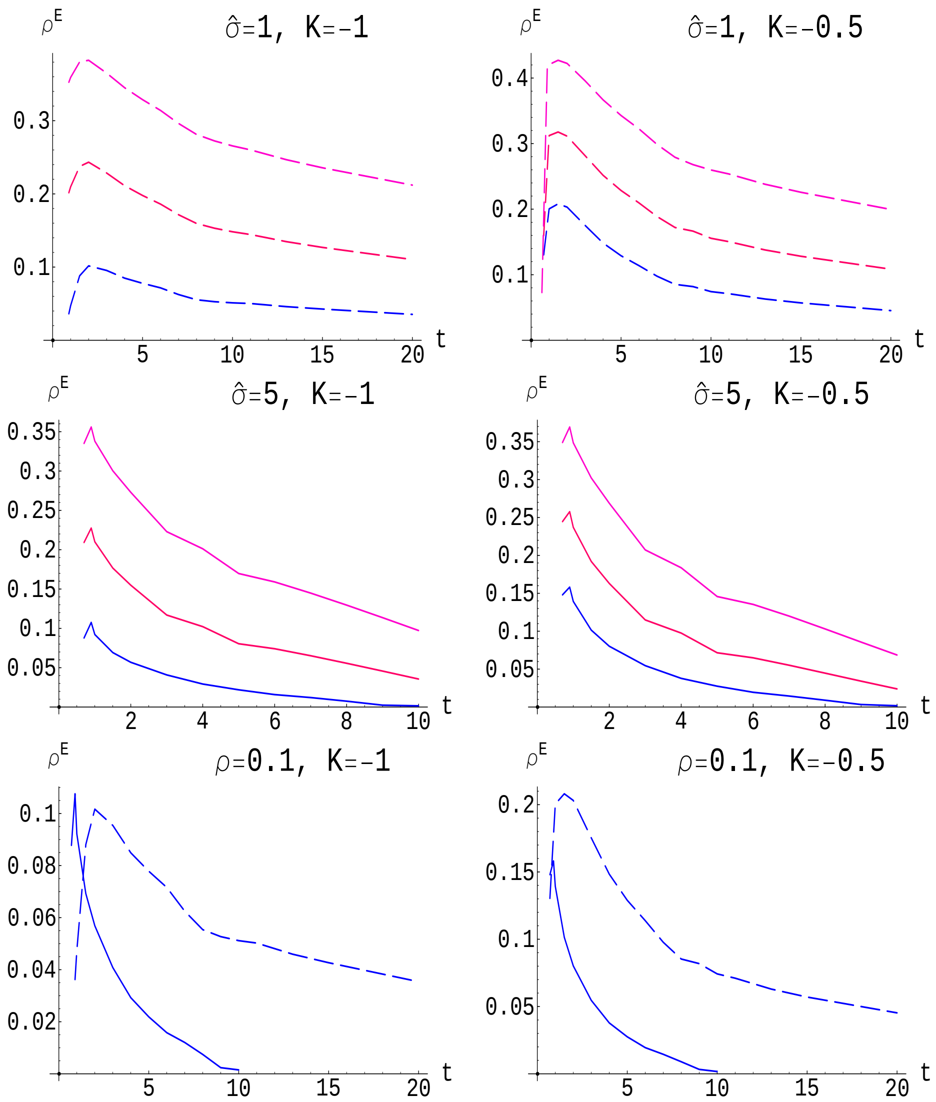

Figure 2.11: Event correlation versus time, for fixed Brownian correlation $\rho = 10\%$ (blue), 30% (red) resp. 50% (pink)

Chapter 3

Credit spread: structural modeling approach and empirics

This chapter introduces in general the *structural* or *threshold approach* for modeling credit-spread curves and its dynamics in time. The dynamics will be described via a *stochastic differential equation (SDE)*. Here we first define the *default event* of an entity, the corresponding *survival probability* and the *credit spread*. Then we explain what we mean by *credit-spread dynamics* and *credit-spread volatility*. The second part of this chapter empirically analyzes the credit-spread volatility with market data from IBM and General Motors.

Under a structural approach, the default event is triggered by an *asset-value* or *firm-value process* falling below some pre-specified *default boundary* (depending on the firm's debt). The default time equals the first-passage time – this is the link to the previous chapter – and we will apply the previous general results on modeling credit spreads in Chapter 6. Asset-value process and default barrier of a threshold model link *equity* (stocks) and *debt* (bonds) of a firm's *capital structure* and provide equity-based default probabilities and credit spreads. The difficulty of the structural model is its calibration to a market-given term structure of default probabilities. But, extensions for multiply correlated credits are straightforward, as we have seen in the preceding chapter when we introduced our multivariate model.

In the subsequent Chapters 4, 5 and 6, we will analyze the dynamics of the credit-spread term structure under the so-called *Merton model*, *Overbeck & Schmidt model* and our *stochastic time-change model* (introduced in the previous chapter). We will not focus on these models here.

3.1 Credit-risk framework

In *credit risk* a first-passage process Y is interpreted as *ability-to-pay process* or *asset-value process* and we assume it is observed at the market.[1] Thus the filtration $\mathbb{F}^Y = (\mathcal{F}_t^Y)$ with $\mathcal{F}_0^Y = \{\emptyset, \Omega\}$ and $\mathcal{F}_t^Y = \sigma(Y_s : s \leq t)$ holds the available market information. In Definition 1.8 we introduced the *first-passage time (FPT)*,

$$\tau \quad = \quad \inf\{s \geq 0 : Y_s < K\} \ .$$

When Y falls below the threshold K, τ indicates a *default event* or *credit event* of an entity and is therefore called *default time*. Note that for us it does not matter whether a credit event indicates a *credit-rating downgrade* (e.g. due to the company's failure to pay some outstanding credit amount in time) or a total default (liquidation of the firm). Furthermore τ is a $\mathbb{F}^Y$-stopping time. The *first-passage-time distribution* $\mathbb{P}(\tau \leq t)$ is called *default probability*. We assume that the market provides us with a whole default probability curve

$$F(t) \quad = \quad \mathbb{P}\left(\tau \leq t\right) \quad , \ t \geq 0 \ ,$$

which was introduced in equation (1.4). The entity's actual *survival probability* will be denoted by

$$Q(0,t) \quad = \quad \mathbb{P}(\tau > t) \quad , \ t \geq 0.$$

Definition 3.1 (Conditional or future survival probability)
For $0 \leq t \leq T$ the *conditional* or *future survival probability given the information* $\mathcal{F}_t^Y$, or for short just *survival probability*, is defined as

$$Q(t,T) \quad := \quad \mathbb{P}\left(\tau > T \mid \mathcal{F}_t^Y\right) \ .$$

The *term structure of survival probabilities* is obtained when considering all $T \geq t$. Note that $Q(0,T)$ is a number, whereas $Q(t,T)$ is a random variable. $Q(t,T)$ is well-defined in the sense that inserting $t = 0$ into Definition 3.1 leads to

$$Q(0,T) = \mathbb{P}(\tau > T \mid \mathcal{F}_0^Y) = \mathbb{P}(\tau > T) \ ,$$

because $\mathcal{F}_0^Y$ is the trivial σ-algebra.
In Definition 1.8 we introduced the conditional default-probability density given the information $\mathcal{F}_t^Y$ - there is the following relation to the survival probability:

$$Q(t,T) \quad = \quad 1 - \int_t^T \mathbb{P}(\tau \in \,\mathrm{d}u \mid \mathcal{F}_t)\,\mathrm{d}u \ . \tag{3.1}$$

[1] For estimating a company's asset value from market data compare e.g. Jones, Mason and Rosenfeld (2004) and Hull, Nelken and White (1984) - both papers suggest implementations for the Merton model. See also Chapter 4.

Remark 3.2 (Survival probability)
The indicator function $\mathbb{I}_{\{\tau>t\}}$ is $\mathcal{F}_t^Y$-measurable, therefore we can write

$$Q(t,T) \;=\; \mathbb{E}\Big(\mathbb{I}_{\{\tau>T\}} \mid \mathcal{F}_t^Y\Big) \;=\; \mathbb{I}_{\{\tau>t\}}\, \mathbb{E}\Big(\mathbb{I}_{\{\tau>T\}} \mid \mathcal{F}_t^Y\Big) \;.$$

Furthermore we determine the expectation of $Q(t,T)$:

$$\mathbb{E}\big(Q(t,T)\big) \;=\; \mathbb{E}\Big(\mathbb{E}\Big(\mathbb{I}_{\{\tau>T\}} \mid \mathcal{F}_t\Big)\Big) \;=\; \mathbb{E}\Big(\mathbb{I}_{\{\tau>T\}}\Big) \;=\; Q(0,T) \quad .$$

The expected survival probability of the time interval $[t,T]$ is given by the actual survival probability of the time interval $[0,T]$. Especially the expectation does not depend on t.

Definition 3.3 (Discount factor)
The present value (at time t) of one unit of currency received at $u \geq t$ is called *discount factor* and denoted $D(t,u)$. Under constant interest rates r and *continuous discounting* it is given by

$$D(t,u) \;=\; \mathrm{e}^{-r(u-t)} \;.$$

Remark 3.4 (Defaultable zero bond)
A zero bond B pays one unit of currency at maturity T, i.e. $B(T,T)=1$, in case of no-default of the reference entity. Let τ denote the default time (classic approach). Assuming that the bond is a liquid market instrument, then it is a martingale under the real measure $\mathbb{P}$, and under no-arbitrage and constant interest rates we have

$$\begin{aligned} B(t,T) \;&=\; \mathbb{E}\Big(\mathrm{e}^{-r(T-t)}\, B(T,T)|\mathcal{F}_t\Big) \;=\; \mathrm{e}^{-r(T-t)}\, \mathbb{E}\Big(\mathbb{I}_{\{\tau>T\}}|\mathcal{F}_t\Big) \\ &=\; \mathrm{e}^{-r(T-t)}\, Q(t,T) \;. \end{aligned}$$

That is the zero bond price is given by the survival probability of the reference entity.

3.2 Credit-default-swap spread

With the definitions of the last section we now can define the *credit-default-swap spread* or simply just *credit spread*. First of all we explain the *credit default swap (CDS)*:

A CDS is a contract between a *protection seller* and a *protection buyer*. The protection seller offers protection against default of a *reference entity* during a certain time period, say between t and *maturity* T. Therefore the protection buyer regularly pays an insurance fee, the *credit spread* $s(t,T)$, but only as long as the entity is not defaulted. In case of a default before maturity of the CDS the protection seller pays a claim amount as agreed in the contract to the protection buyer, which depends on the *recovery rate*[2] R of the entity. We call $M = T - t$ the contract's *time to maturity*.

In order to determine a formula for the credit spread, one can make several assumptions about payment times and amounts in case of default (of the reference entity as well as of each contract holder) and also in case of no-default. We summarize the *Hull & White approach* (2000) for annually spread payments and the *Schmidt approach* (2004b) for continuous spread payments. Both assume no counterparty default risk. The Schmidt approach is the basis for our studies.

3.2.1 Annual credit-spread payments

Hull & White (2000) determined the fair credit spread in case of annual insurance payments. Therefore they introduced the value $U(t,T)$ of payments at the rate of one unit currency per year, starting at contract start t and ending at contract maturity T. They assumed that in case of default at τ the recovery value of the entity is determined by the face value plus accrued interest $A(t,\tau)$ and is given by $(1+A(t,\tau))R$. This is realistic (with regard to bankruptcy laws) for the claim of a defaultable bond, according to J.P. Morgan (1999) and Jarrow & Turnbull (1995). Under no default up to t, the credit spread $s(t,T)$ is fair when the present value of the credit-spread payments equals the present value of the recovery payment in case of default between t and T. So on the one hand, there is the claim amount of the protection buyer, $1-(1+A(t))R$, that is assumed to be paid directly at default and has present value

$$\int_t^T \big(1 - R - A(t,u)R\big)\mathbb{P}(\tau \in \mathrm{d}u \mid \mathcal{F}_t) D(t,u)\,\mathrm{d}u \ .$$

And on the other hand, there are the spread payments by the protection buyer, in case of no default. The corresponding present value is determined by the survival probability $Q(t,T)$ and the rate of annual payments $U(t,T)$, and is given by

$$s(t,T)Q(t,T)U(t,T) \ .$$

In case of default at time τ, let $t^\star$ be the last payment date before default, then the protection buyer makes the annual spread payments till $t^\star$, $U(\tau) \equiv U(t^\star)$. And in addition he pays the accrual of the spread payment for the

[2]The *recovery rate* is that part of the underlying face value that is regained after default. The conditional probability distribution of the recovery rate (meaning conditional on default) describes the *recovery risk*, that is the uncertainty about the severity of a loss upon default. Cf. Schönbucher (2003).

outstanding period $\tau - t^\star$. The accrual we denote by $E(\tau)$. The present value of the spread payments in the default event is

$$s(t,T)\int_t^T \mathbb{P}(\tau \in \mathrm{d}u \mid \mathcal{F}_t)\big(U(u)+E(u)\big)\,\mathrm{d}u\ .$$

This yields

$$s(t,T) \quad = \quad \frac{\int_t^T \big(1-R-A(t,u)R\big)D(t,u)\mathbb{P}(\tau \in \mathrm{d}u \mid \mathcal{F}_t)\,\mathrm{d}u}{\int_t^T \mathbb{P}(\tau \in \mathrm{d}u \mid \mathcal{F}_t)\big(U(t,u)+E(u)\big)\,\mathrm{d}u + Q(t,T)U(t,T)}\ .$$

With equation (3.1) this equals

$$s(t,T) \quad = \quad \frac{\int_t^T \big(1-R-A(t,u)R\big)D(t,u)\mathbb{P}(\tau \in \mathrm{d}u \mid \mathcal{F}_t)\,\mathrm{d}u}{U(t,T)+\int_t^T \mathbb{P}(\tau \in \mathrm{d}u \mid \mathcal{F}_t)\big(U(t,u)-U(t,T)+E(u)\big)\,\mathrm{d}u}\ .$$

3.2.2 Continuous credit-spread payments

Schmidt (2004b) assumed continuous credit-spread payments. In case of default before maturity the protection seller pays $1-R$ at default, that is his payment leg has present value

$$(1-R)\int_t^T D(t,u)\mathbb{P}(\tau \in \mathrm{d}u \mid \mathcal{F}_t)\,\mathrm{d}u\ .$$

Therefore the protection buyer continuously pays the credit spread as long as no credit event occurs, but only till contract maturity, that is until $\tau \wedge T$. His payment leg has present value

$$-s(t,T)\int_t^{\tau\wedge T} D(t,u)Q(t,u)\,\mathrm{d}u\ .$$

Under no default until t, the fair credit spread is obtained when the present value of the two payment legs are equal:

$$-s(t,T)\int_t^{\tau\wedge T} D(t,u)Q(t,u)\,\mathrm{d}u \quad = \quad (1-R)\int_t^T D(t,u)\mathbb{P}(\tau \in \mathrm{d}u \mid \mathcal{F}_t)\,\mathrm{d}u\ .$$

Note that $\int_t^{\tau\wedge T} Q(t,u)\,\mathrm{d}u = \int_t^T Q(t,u)\,\mathrm{d}u$, since $Q(t,u) = 0$ for $u \geq \tau$. Then, conditional on $\tau > t$, the continuously paid credit spread is given by

$$s(t,T) \quad = \quad \frac{(1-R)\int_t^T D(t,u)\mathbb{P}(\tau \in \mathrm{d}u \mid \mathcal{F}_t)\,\mathrm{d}u}{\int_t^T D(t,u)Q(t,u)\,\mathrm{d}u}\ . \tag{3.2}$$

3.2.3 Continuous credit-spread payments in discrete time

Consider the discrete time grid $t_0 = t, t_1, \ldots, t_n = T$. In case of no credit event, the protection buyer pays the credit spread at each time step, leading to the present value:

$$s(t,T)(t_i - t_{i-1}) \sum_{i=1}^{n} D(t,t_i) Q(t,t_i) \ .$$

In case of default, say between t_{i-1} and t_i, we assume that the credit spread is paid until the middle of the time interval $(t_{i-1}, t_i]$, that is $\frac{1}{2}(t_{i-1} + t_i)$. The probability of default in that time interval is given by

$$\mathbb{P}(\tau \in (t_{i-1}, t_i] \mid \mathcal{F}_t) \quad = \quad Q(t, t_{i-1}) - Q(t, t_i) \ .$$

Herewith the payment leg of the protection buyer has present value

$$\begin{aligned} & s(t,T) \sum_{i=1}^{n} (t_i - t_{i-1}) D(t,t_i) Q(t,t_i) \\ + \quad & s(t,T) \sum_{i=1}^{n} (t_i - t_{i-1}) D\left(t, \frac{t_{i-1} + t_i}{2}\right) [Q(t,t_{i-1}) - Q(t,t_i)] \ , \end{aligned}$$

which leads to the following discretization of the credit-spread formula (3.2):

$$\begin{aligned} & s(t,T) \ = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.3) \\ & \frac{(1-R) \ \sum_{i=1}^{n} (t_i - t_{i-1}) D(t,t_i) \ [Q(t,t_{i-1}) - Q(t,t_i)]}{\sum_{i=1}^{n} (t_i - t_{i-1}) \ \left[D(t,t_i) Q(t,t_i) + D(t, \frac{1}{2}(t_{i-1} + t_i))[Q(t,t_{i-1}) - Q(t,t_i)]\right]} \ . \end{aligned}$$

When the time grid is equidistant the spread formula becomes

$$s(t,T) \quad = \quad \frac{(1-R) \ \sum_{i=1}^{n} D(t,t_i) \ [Q(t,t_{i-1}) - Q(t,t_i)]}{\sum_{i=1}^{n} \left[D(t,t_i) Q(t,t_i) + D(t, \frac{1}{2}(t_{i-1} + t_i))[Q(t,t_{i-1}) - Q(t,t_i)]\right]} \ .$$

3.2.4 Credit-spread formula

We assume that the payoff is contingent on default by the reference entity only, that is there is no counterparty default risk. For simplicity we do not consider discount factors and make the following assumption:

Assumption 3.5 *(CDS credit spread)*

- *The credit spread is paid continuously in time.*
- *There is no counterparty default risk*
- *Riskless interest rates are zero.*

Under no default until t, the fair continuously-paid credit spread is given by equation (3.2). That is, under Assumption 3.5 we have

$$s(t,T) \quad = \quad \frac{(1-R)(1-Q(t,T))}{\int_t^T Q(t,u)\,\mathrm{d}u}\ . \tag{3.4}$$

Usually a CDS spread is considered in terms of *time to maturity* M, i.e. $T = t + M$ and

$$s(t,t+M) \quad = \quad \frac{(1-R)(1-Q(t,t+M))}{\int_t^{t+M} Q(t,u)\,\mathrm{d}u}\ . \tag{3.5}$$

Note that when the CDS starts at a future time point t, the spread $s(t,T)$ is called *forward credit spread*.

3.3 Credit-spread curve and its dynamics

A *credit-spread term structure* (*credit-spread curve*) at time t contains various time to maturities M: $\quad (s(t,t+M))_M$,
for example one, three and six months as well as one up to ten years. The credit-spread curve varies in shape and level when moving on in time t. There is a so-called *credit-spread dynamic*. We are going to model credit-spread dynamics via a stochastic differential equation (SDE). For this we have to specify drift factor and volatility term, the so-called *credit-spread volatility*.

The remainder of this chapter analyzes market histories of CDS spreads in order to estimate credit-spread volatility. Furthermore the relation credit-spread versus credit-spread volatility is studied.

3.4 Estimating credit-spread volatility

We use histories of five years CDS spreads from IBM and General Motors (GM), respectively, in order to determine estimates for the *credit-spread volatility* (short: *vol*). The credit-spread histories are shown in Figure 3.1. Both data sets have a time period where credit-spread volatility is high, which leads to a 'peak'. The estimation of the credit-spread volatility is done in the same way as stock volatility is estimated. Hereto we refer to Hull (2003). Say we have $n+1$ credit-spread observations s_i, $i = 0, 1, \dots n$, and let Δt be the time in years between two observations. At the market, for liquid data, there are about 260 mid-day quotes per year, so we set $\Delta t = \frac{1}{260}$ and further choose $n = 130$, that is, data of half a year, for estimating credit-spread volatility.

3.4.1 Volatility estimate of a log-normal spread process

Assuming a geometric Brownian motion for the spread process s,

$$\mathrm{d}s = s\sigma^{LN}\,\mathrm{d}W_i\ , \tag{3.6}$$

the increments of geometric Brownian motion, i.e. the *spread yields*, at $t_i,\ i = 1, \ldots, n$ are given by

$$u_i = \ln\left(\frac{s_i}{s_{i-1}}\right)\ ,$$

and yield the standard deviation

$$S = \sqrt{\tfrac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2} = \sqrt{\tfrac{1}{n-1}\sum_{i=1}^{n}u_i^2 - \tfrac{1}{n(n-1)}\left(\sum_{i=1}^{n}u_i\right)^2}\ , \tag{3.7}$$

where $\bar{u} \equiv \frac{1}{n}\sum_{i=1}^{n}u_i$ abbreviates the mean. Geometric Brownian motion has standard deviation $\sigma\sqrt{\Delta t}$; thus the volatility estimate is given by

$$\hat{\sigma}^{LN} = \frac{S}{\sqrt{\Delta t}}\ .$$

The volatility estimates of IBM and GM 5-years credit spreads, for the time period 2002 to 2005, are shown in Figure 3.2. The first two plots show the credit-spread volatility estimates against time. Using independent data sets for estimation leads to the blue step functions. Each step level indicates the credit-spread vol estimate that was computed with the 130 credit spread data corresponding to the underlying trading days. The change in the credit-spread volatility can be better seen when an estimate at each trading day is computed. For an estimate at each time point we take the credit-spread data of the corresponding last 130 trading days, i.e. when we use the data overlapping. The daily estimates are plotted in pink. In our examples, under the log-normal model, credit-spread vol ranges between 20% and 90%. The volatility of the credit-spread vol (*vol of vol*) is quite high. For comparison, stock volatility lies between 10% and 90% and FX (foreign exchange) volatility between 2% and 20%. The last two plots of Figure 3.2 illustrate that there is a relation between credit spread and credit-spread vol. Increasing IBM credit spreads yield increasing spread volatility. The relationship for the GM data is not so clear, but this might be due to a rating change within this time period.

3.4.2 Volatility estimate of a normal spread process

Assuming a normally distributed spread process,

$$\mathrm{d}s_i = \sigma^{N}\,\mathrm{d}W_i\ , \tag{3.8}$$

the yield spread equals the difference of the spreads

$$u_i = s_i - s_{i-1} .$$

Computing the standard deviation of the u_i's as in equation (3.7) and dividing it by the square of the time increment leads to the volatility estimate $\hat{\sigma}^N$ of the normal spread model. The corresponding computed credit-spread volatilities of the IBM and GM data are shown in Figure 3.3. The first two plots show spread volatility against time and the last two against spread. Again the blue step levels (in the first two plots) are estimates coming from independent data sets and the pink paths show the credit-spread vol evolution when the data was used to generate moving averages (always taking credit-spread quotes of the previous 130 trading days) for estimation. As under the log-normal model, the plots the last two plots of Figure 3.3 show a relationship between credit spread and credit-spread vol, now on another scale.

Remark 3.6 (Comparing log-normal and normal vol)
Comparing equation (3.6) and (3.8) leads to the following relationship between log normal and normal vol:

$$s \cdot \sigma^{LN} = \sigma^N .$$

This is why the credit-spread volatility estimates are on a different scale.

3.5 Conclusion

At the beginning of this chapter we introduced our aim, to model credit-spread curves and their dynamics in time. We want to do this via a SDE. In the previous section we estimated and analyzed the historical volatilty of 5-years credit spreads. We found that, clearly, credit-spread volatility is a process itself and furthermore it depends on the spread level, although the mode of dependence seems to be different for different data sets. The IBM data shows a leverage effect, that is, the credit-spread vol increases with the credit-spread level. We find that a local-volatility model seems to be appropriate in order to model the credit-spread process and especially its volatility. We aim for a geometric Brownian motion SDE in order to describe the spread dynamics:

$$\mathrm{d}s_t = s(t)\,[\mu(t, s_t)\,\mathrm{d}t + \sigma(t, s_t)\,\mathrm{d}W_t] .$$

Goals of the next Chapters 4, 5 and 6 are to derive the credit-spread SDEs under the *Merton model*, the *Overbeck & Schmidt model* and our *stochastic time-change model*. For this the simple credit-spread formula (3.4) will be assumed and our tool will be Itô's Lemma.

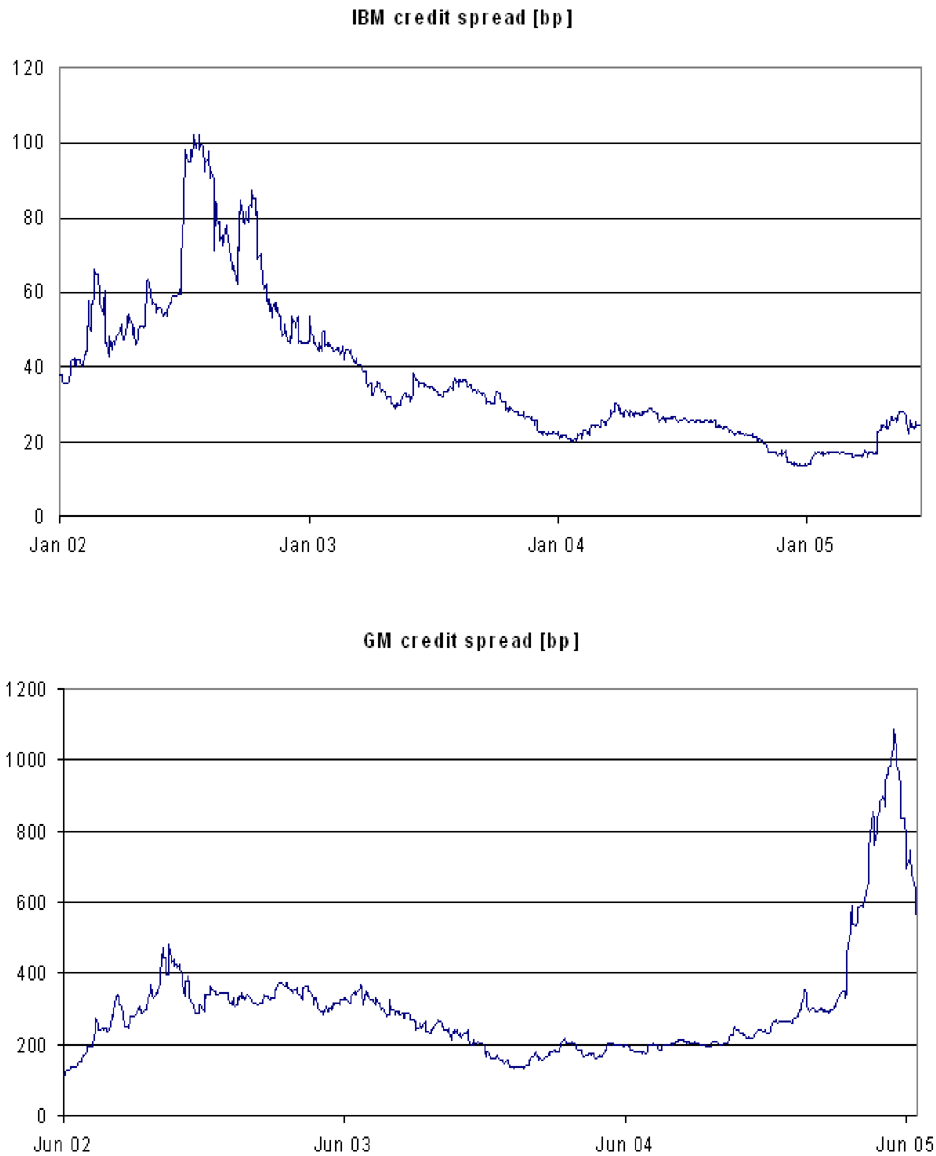

Figure 3.1: 5-years credit-spread histories of IBM and GM

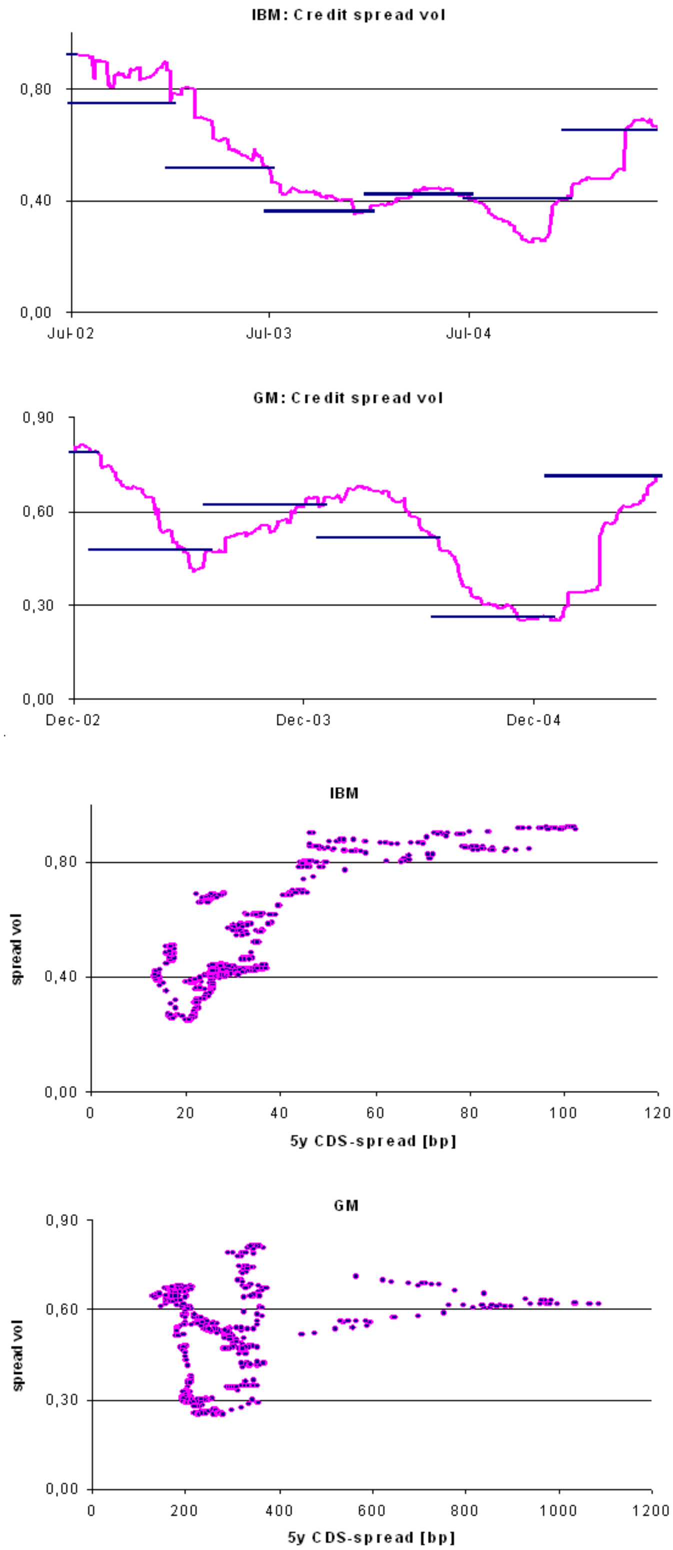

Figure 3.2: Credit-spread vol estimation under the log-normal model

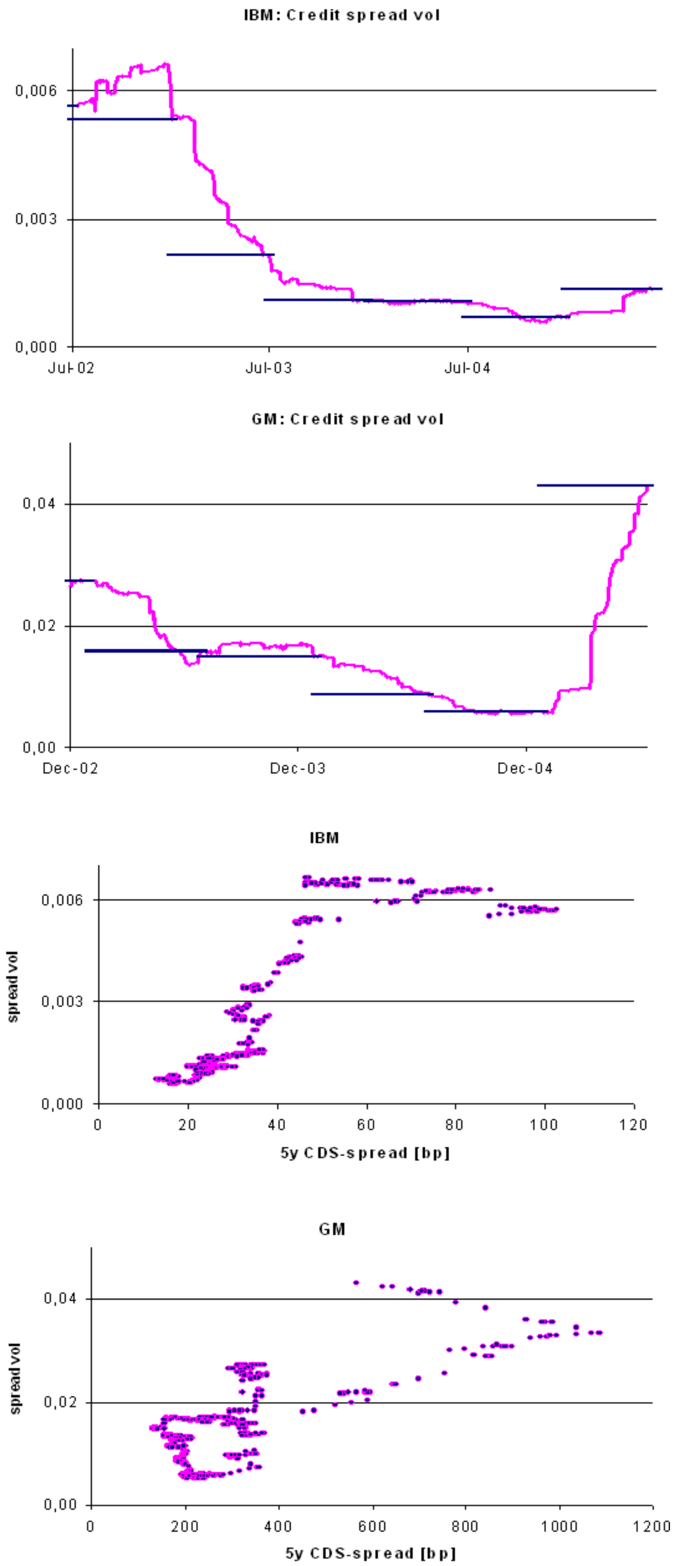

Figure 3.3: Credit-spread vol estimation under the normal model

Chapter 4

Credit spread under the Merton model

Merton (1974) proposed a model for assessing a company's credit risk by characterizing its *equity* as a call on its assets. The firm value or asset value is modeled by geometric Brownian motion. The firm's *outstanding debt* specifies a constant *default level* or *threshold level*. Then under the classical approach a *credit event* occurs at one fixed point in time T, when the value of the asset value lies below the threshold, i.e. $Y_T < K$. The model thus links credit risk to equity and debt of a firm's capital structure. We will call this classical approach the *classical Merton model*. The extension of the classical default makes a default possible continuously in time and happens the first time the asset value (i.e. geometric Brownian motion) falls below the pre-specified barrier. A model that allows for a default at any time is called *first-passage-time model* or just *first-passage model*. In this chapter we analyze credit spread and credit-spread dynamics under the so-called *Merton first-passage model* or simply *Merton model*. It is well-known that the model implies hump-shaped credit-spread curves with zero spread at zero maturity. That is, in contrast to reality, no instantaneous default is possible. We determine the model-implied credit-spread dynamics of the assumed credit-spread formula (3.4) via the Itô rule and find that it is given in terms of the asset value, i.e. Brownian motion, that is assumed to be given by the market. Especially, the Merton model has no degree of freedom to influence the spread dynamics.

Implementations for the Merton model were suggested by Jones, Mason and Rosenfeld (1984), who estimated the company's assets and asset volatilities from the market value of equity and instantaneous equity volatility, and more recently by Hull, Nelken and White (2004), who estimated the parameters based on the implied volatility of options on the company's stock.

4.1 Model framework

We just described the first-passage approach under the Merton model where default happens when geometric Brownian motion falls below the debt level D:

$$\tau \quad = \quad \inf\{s \geq 0 : A_0 \, \mathrm{e}^{W_s} < D\} \ .$$

Equivalently and instead, we consider a Wiener process W as the asset value process that indicates a default whenever it falls below the constant threshold K (where $D \equiv A_0 \, \mathrm{e}^K$). That is, we are interested in the default time

$$\tau \quad = \quad \inf\{s \geq 0 : W_s < K\} \ . \tag{4.1}$$

Under the Merton model the filtrations $\mathbb{F}^Y$ is the natural filtration generated by the Wiener process W_t,

$$\mathcal{F}_t^Y \quad = \quad \sigma\left(W_s : s \leq t\right) \ .$$

We determine the survival probability defined in 3.1:

Theorem 4.1 *(Survival probability under the Merton model)*
The survival probability is given by

$$Q(t,T) \quad = \quad \mathbb{I}_{\{\tau > t\}} \left[1 - 2\ \Phi\left(\frac{K - W_t}{\sqrt{T-t}}\right)\right] \ ,$$

and for fixed time to maturity M,

$$Q(t,t+M) \quad = \quad \mathbb{I}_{\{\tau > t\}} \left[1 - 2\ \Phi\left(\frac{K - W_t}{\sqrt{M}}\right)\right] \ .$$

Note that $Q(t,T)$ is a $\mathcal{F}_t^Y$-measurable random variable!

Proof. The filtration was chosen such that $\mathbb{I}_{\{\tau > t\}}$ is $\mathcal{F}_t^Y$ measurable. With Remark 3.2) the survival probability of the entity is determined by

$$Q(t,T) \quad = \quad \mathbb{I}_{\{\tau > t\}} \mathbb{E}\Big(\mathbb{I}_{\{\inf_{t \leq s \leq T} W_s > K\}} |\ \mathcal{F}_t^Y\Big) \ .$$

In the following we consider the case where $\tau > t$. Then the Markov property of $(W, \mathbb{F})$ yields

$$\begin{aligned} Q(t,T) \quad &= \quad \mathbb{P}\Big(\inf_{t \leq s \leq T} W_s > K \mid W_t\Big) \\ &= \quad \mathbb{P}\Big(\inf_{t \leq s \leq T} (W_s - W_t) > K - W_t \mid W_t\Big) \ . \end{aligned}$$

For $s \geq t$ set $\tilde{W}_{s-t} = W_s - W_t$. Then $(\tilde{W}_u)_{u>0}$ is a Brownian motion starting in zero. Furthemore set $\tilde{K}(W_t) = K - W_t$; then

$$Q(t,T) \;=\; \mathbb{P}\Big(\inf_{0\leq u\leq T-t} \tilde{W}_u > \tilde{K}(W_t) \mid W_t\Big) \;=\; 1 - 2\,\Phi\left(\frac{\tilde{K}(W_t)}{\sqrt{T-t}}\right) ,$$

where for the last equality we used that $\tilde{K}(W_t) < 0$ (since $\tau > t$) and applied the well-known result for the exit time of Brownian motion, derived in Harrison (1985), see (1.6). The general result for Brownian motion starting at any value W_t is stated in Borodin, Salminen (2002), page 251.
□

4.1.1 Speed-of-default probability

For fixed K and M and under the assumption that $\tau > t$ (which especially means that we must have $W_0 > K$), we determine the derivative of the distribution of $\inf_{t\leq s\leq T} W_s$ in W_t, that is of

$$\mathbb{P}\Big(\inf_{t\leq s\leq t+M} W_s \leq K \mid W_t\Big) \;=\; 2\,\Phi\left(\frac{K - W_t}{\sqrt{M}}\right) ,$$

yielding

$$f_K^M(W_t) \;=\; -\frac{2}{\sqrt{M}}\,\phi\left(\frac{K-W_t}{\sqrt{M}}\right) .$$

The derivative f_K^M can be interpreted as *default speed*, depending on the actual asset value level W_t. It describes how fast the default probability increases or decreases. Figure 4.1 shows the default speed for $K = -8.1$, which corresponds to a starting survival probability of $Q(0,10) = 99\%$. The default speed is fast when the starting value W_t is close to the threshold K. This impact is bigger when maturity is closer.

4.2 Calibration - threshold level

For calibrating the Merton model to market data there is only one degree of freedom and that is in the threshold level K. How to choose the K?
One can do this by fixing today's survival probability for one time horizon T, that is $Q(0,T)$, because then, under $W_0 > K$, Theorem 4.1 implies the following threshold level:

$$K \;=\; \Phi^{-1}\Big(\frac{1}{2}(1 - Q(0,T))\Big)\cdot\sqrt{T} + W_0 ,$$

which in turn fixes the whole term structure of survival probabilities $\big(Q(0,t)\big)_{t\geq 0}$,

$$Q(0,t) \;=\; 1 - 2\,\Phi\left(\frac{K - W_0}{\sqrt{t}}\right) .$$

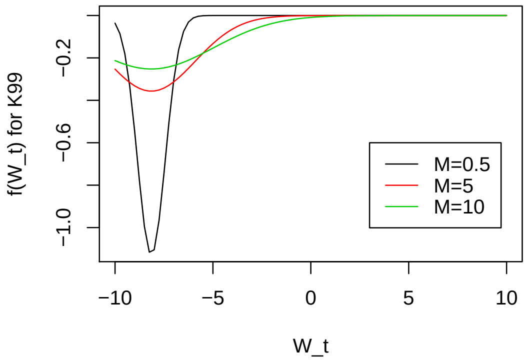

Figure 4.1: Default speed for $K = -8.1$

Example 4.2 Figure 4.2 shows survival-probability term structures for $T = 10$ and the models with $K = -1, \ldots, K = -5$. The instantaneous survival probabilities (in each model) equal one, so the instantaneous credit spreads are zero. The second plot of Figure 4.4 shows the corresponding credit-spread curves at time $t = 0$.

4.3 Credit spread

Corollary 4.3 *(Credit spread under the Merton model) Conditional on no-default up to t, the spread is given by*

$$s(t,T) \quad = \quad \frac{2\ (1-R)\ \Phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)}{T-t-2\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\ du} \ ,$$

and for fixed time to maturity M,

$$s(t,t+M) \quad = \quad \frac{2\ (1-R)\ \Phi\left(\frac{K-W_t}{\sqrt{M}}\right)}{M-2\int_t^{t+M} \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\ du} \ .$$

We will sometimes abbreviate $s_t^M \equiv s(t,t+M)$.
Proof. Insert the Merton survival probability into the spread formula (3.4) and (3.5), respectively. □

4.3.1 Instantaneous spread

The instantaneous credit spread is the short-end maturity spread where $T = t$, that is $M = 0$. We determine the instantaneous credit spread by taking the limit $M \to 0$. We assume no default before t, so especially $W_t > K$. Applying the *rule of de L'Hospital* and the *fundamental theorem of calculus* leads indeed to an instantaneous credit spread of zero:

$$\begin{aligned}
\lim_{M\downarrow 0} s(t,t+M) &= 2\,(1-R)\frac{\lim_{M\downarrow 0}\ \Phi\left(\frac{K-W_t}{\sqrt{M}}\right)}{\lim_{M\downarrow 0}\left(M - 2\int_t^{t+M}\Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,du\right)} \\
&= 2\,(1-R)\frac{\lim_{M\downarrow 0}\ \phi\left(\frac{K-W_t}{\sqrt{M}}\right)(K-W_t)\cdot\left(-\frac{1}{2}\right)\frac{1}{M^{3/2}}}{\lim_{M\downarrow 0}\left(1-2\Phi\left(\frac{K-W_t}{\sqrt{M}}\right)\right)} \\
&= 2\,(1-R)(K-W_t)\cdot\left(-\frac{1}{2}\right)\frac{\lim_{M\downarrow 0}\ \phi\left(\frac{K-W_t}{\sqrt{M}}\right)\frac{1}{M^{3/2}}}{1-0} \\
&= 0\ ,
\end{aligned}$$

since $e^{-\frac{1}{2}\frac{1}{x}}$ converges faster for $x \downarrow 0$ than $\frac{1}{x^{3/2}}$. So the instantaneous spread vanishes in the Merton model. Note that the limit for $W_t \downarrow K$ exists and is finite for $M > 0$:

$$\lim_{W_t\downarrow K} s(t,t+M) = 2\,(1-R)\frac{\Phi(0)}{M-0} = \frac{1-R}{M}\ .$$

4.3.2 Simulations

We divide the time interval $[0,T]$ into $n = 200$ equidistant intervals of length $\frac{T}{n}$ and set $t_0 = 0, t_1 = \frac{T}{n}, \ldots, t_i, \ldots, t_n = T$. Simulating

$$\Delta W_{t_j} \sim \Phi\left(0,\ \tfrac{T}{n}\right) \tag{4.2}$$

yields a discrete Brownian path $(W_{t_i})_i$ with

$$W_{t_i} = W_0 + \sum_{j=1}^{i}\Delta W_{t_j}\ .$$

Then with Theorem 4.1 the survival probability at t_i is determined by

$$Q(t_i,T) = \mathbb{I}_{\{\inf_{0\le j\le i} W_{t_j} > K\}}\left[1 - 2\ \Phi\left(\frac{K-W_{t_i}}{\sqrt{T-t_i}}\right)\right]$$

and for fixed time to maturity $T = t_i + M$ by

$$Q(t_i,t_i+M) = \mathbb{I}_{\{\inf_{0\le j\le i} W_{t_j} > K\}}\left[1 - 2\ \Phi\left(\frac{K-W_{t_i}}{\sqrt{M}}\right)\right]\ .$$

The discrete formula for the continuous credit spread was given in equation (3.3). For zero interest rates it simplifies to

$$s(t_i, T) = \frac{(1-R)(1-Q(t_i,T))}{\frac{T}{n}\sum_{k=i+1}^{n} Q(t_i,t_k)} , \tag{4.3}$$

where

$$Q(t_i, t_k) = \mathbb{I}_{\{\inf_{0\le j\le i} W_{t_j} > K\}} \left[1 - 2\ \Phi\left(\frac{K - W_{t_i}}{\sqrt{t_k - t_i}}\right)\right] .$$

For a constant time to maturity M, we fix the number of grid points that lie in the interval $[t_i, t_i + M]$, say m. Then the equidistant grid intervals are of length $\frac{M}{m}$, which yields the following credit-spread formula

$$s(t_i, t_i + M) = \frac{(1-R)(1-Q(t_i, t_i+M))}{\frac{M}{m}\sum_{k=i+1}^{i+m} Q(t_i,t_k)} .$$

Figure 4.3 shows two possible survival probability paths (left-hand side) and the corresponding credit-spread paths (right-hand side). The Brownian motion underlying the upper plots survives the time interval $[0, 1]$, and the Brownian motion underlying the lower plots defaults since it crosses the threshold shortly before $t = 1$. That is why the corresponding credit-spread path is not plotted till $t = 1$.

Dynamics of credit-spread curves in time

We are interested in the evolution of the credit-spread curves in time. Figure 4.4 illustrates this evolution under the Merton model by considering the term structure $(s(t, t + M))_M$ for several threshold levels K (i.e. several models) at five consecutive points in time, $t = 0, 2.5, 5, 7.5$ and 10. The underlying simulated Brownian path is drawn in the first plot. The other plots show the credit-spread term structures resulting from the actual state of the Brownian motion. A credit-spread curve of a model (specified by K) is only plotted as long as no default happens. Thus in the first term structure plot ($t = 0$) all curves for all K are shown. At $t = 2.5$ the Brownian motion has already crossed the border $K = -1$, that is the asset value process under the model $K = -1$ (black curve) is defaulted and therefore not drawn. And so on.

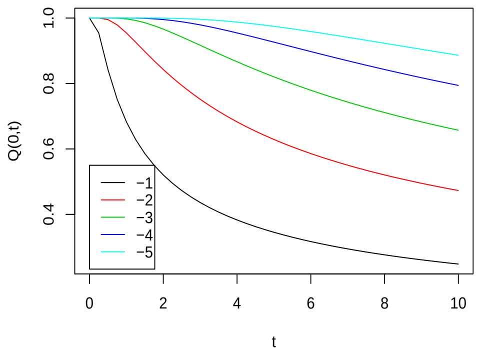

Figure 4.2: Survival-probability term structures for $K = -1, -2, -3, -4, -5$ and $T = 10$.

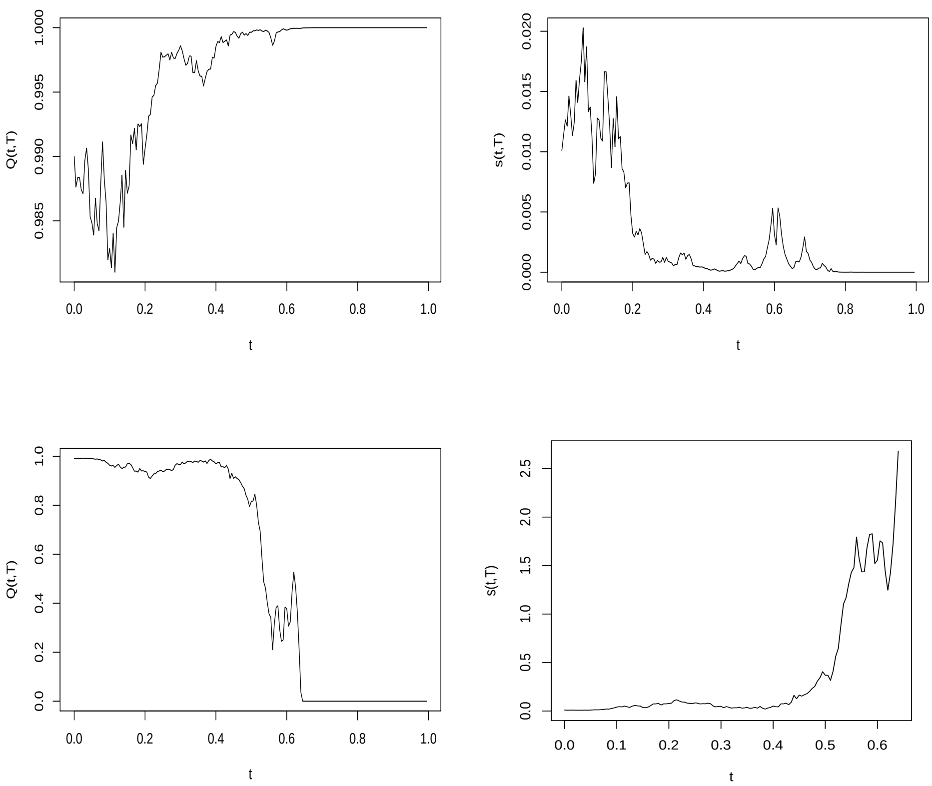

Figure 4.3: Survival-probability paths (left-hand side) and corresponding credit-spread paths (right-hand side) from a surviving (upper plots) and a defaulting (lower plots) Brownian motion.

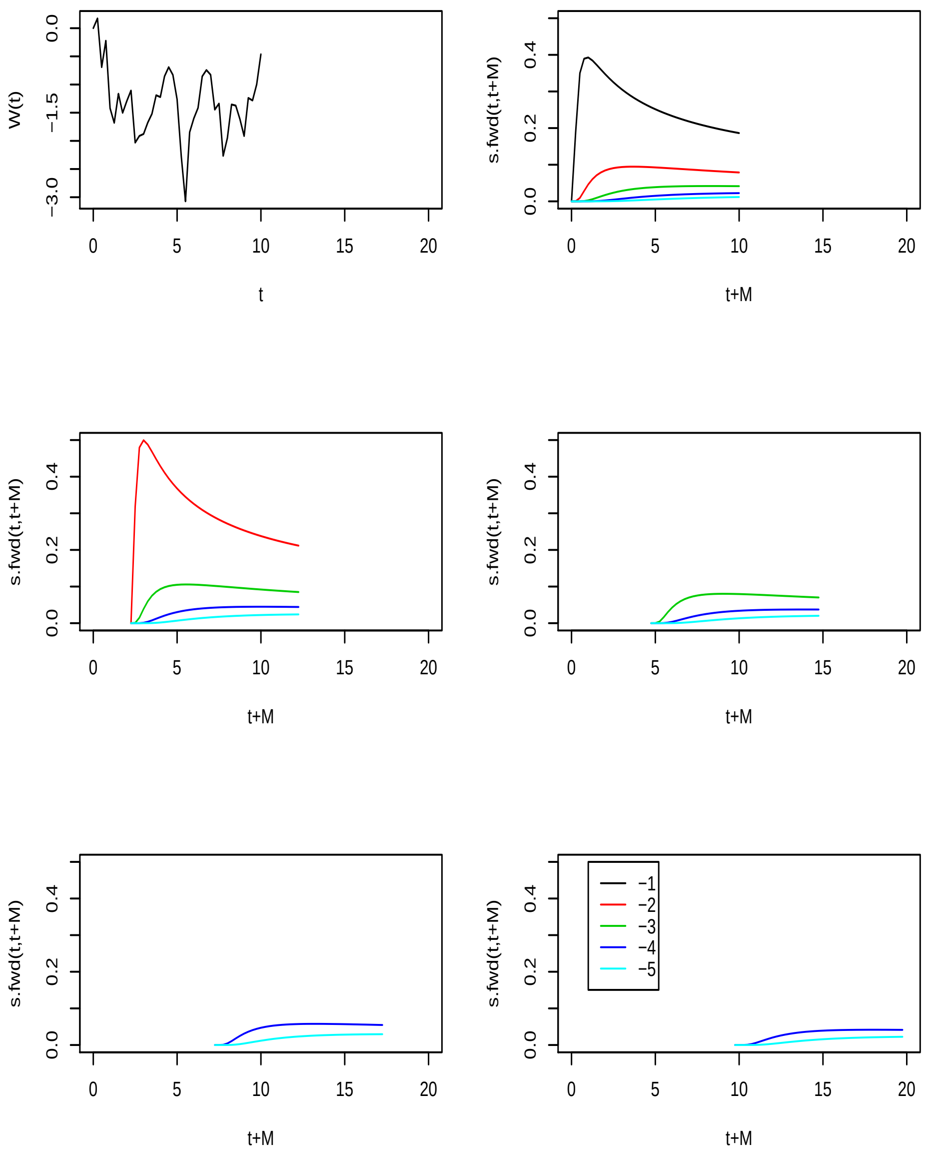

Figure 4.4: Brownian path and corresponding credit-spread term structures for $K = -1, -2, -3, -4, -5$ at $t = 0,\ 2.5,\ 5,\ 7.5,\ 10$

4.4 Credit-spread dynamics

In Corollary 4.3 we derived the credit-spread formula under the Merton model,

$$s(t,T) \quad = \quad \frac{2\,(1-R)\;\Phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)}{T-t-2\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u}\;.$$

When we fix T, as in the next subsection, we simply write $s(t)$. Now we want to determine the spread dynamics in terms of a SDE. We do this by applying *Itô's Lemma* on the credit-spread formula.

4.4.1 Spread dynamics for fixed maturity T

The spread formula is a function f of time t and the value of Brownian motion $x = W_t$, that is $f(t,x) = s(t)$. We introduce the following abbreviation for the partial derivatives:

$$f_t \equiv \frac{\partial f}{\partial t}\;,\quad f_x \equiv \frac{\partial f}{\partial x}\;,\quad f_{xx} \equiv \frac{\partial^2 f}{\partial x^2}\;.$$

By Itô's Lemma we determine the *credit-spread dynamics* (under $\tau > t$):

$$\begin{aligned}
\mathrm{d}s_t \quad &= \quad \left(f_t + \frac{1}{2} f_{xx}\right)\,\mathrm{d}t + f_x\,\mathrm{d}W_t \\
&= \quad s(t)\left[\frac{\left(f_t + \frac{1}{2}f_{xx}\right)}{s(t)}\,\mathrm{d}t + \frac{f_x}{s(t)}\,\mathrm{d}W_t\right] \\
&= \quad s(t)\left[\mu\,\mathrm{d}t - \sigma\,\mathrm{d}W_t\right]\;, \qquad (4.4)
\end{aligned}$$

with $\mu \;:=\; \frac{\left(f_t+\frac{1}{2}f_{xx}\right)}{s}$ and $\sigma \;:=\; -\frac{f_x}{s}$.
Then μ and σ can be interpreted as drift and volatility (vol) of credit spread. The volatility is defined with a negative sign because –as we will see– the derivative f_x is always negative. In our next step we determine the partial derivatives f_t, f_x and f_{xx}:

Proposition 4.4 *(Partial derivatives under the Merton model, fixed T)*
The credit-spread dynamics under the Merton model are given through the following partial derivatives:

$$\begin{aligned}
\frac{f_t(t,W_t)}{1-R} = &\frac{\frac{K-W_t}{(T-t)^{3/2}}\;\phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)}{T-t-2\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u} - 2\frac{\Phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)}{\left[T-t-2\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\right]^2} \\
&+\;2\;\frac{(K-W_t)\Phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)\int_t^T \frac{\phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)}{(u-t)^{3/2}}\,\mathrm{d}u}{\left[T-t-2\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\right]^2}\;,
\end{aligned}$$

$$\frac{f_x(t,W_t)}{-2(1-R)} = \frac{\frac{\phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)}{\sqrt{T-t}}}{T-t-2\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u} + 2\frac{\Phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)\int_t^T \frac{\phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)}{\sqrt{u-t}}\,\mathrm{d}u}{\left[T-t-2\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\right]^2}$$

$$\frac{f_{xx}(t,W_t)}{-2(1-R)} = \frac{\frac{K-W_t}{(T-t)^{3/2}}\,\phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)}{T-t-2\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u} - 8\frac{\Phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)\left[\int_t^T \frac{\phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)}{\sqrt{u-t}}\,\mathrm{d}u\right]^2}{\left[T-t-2\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\right]^3},$$
$$+\ \frac{-4\,\frac{\phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)}{\sqrt{T-t}}\int_t^T \frac{\phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)}{\sqrt{u-t}}\,\mathrm{d}u + 2(K-W_t)\Phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)\int_t^T \frac{\phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)}{(u-t)^{3/2}}\,\mathrm{d}u}{\left[T-t-2\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\right]^2}.$$

Note that f_x is always negative!

Proof. In order to determine the partial derivatives of the credit-spread formula given in Corollary 4.3, we use the following derivatives of standard normal density and standard normal distribution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\,\phi(x) &= -x\cdot\phi(x)\\ \frac{\mathrm{d}}{\mathrm{d}t}\Phi\left(\frac{K-W_t}{\sqrt{T-t}}\right) &= \phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)\cdot\frac{1}{2}\frac{K-W_t}{(T-t)^{3/2}}\ .\end{aligned}$$

Furthermore we need to differentiate a time-dependent integral and apply (A.1) (Appendix)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_t^T f(s,t)\,\mathrm{d}s = -f(t,t)+\int_t^T \frac{\mathrm{d}}{\mathrm{d}t}f(s,t)\,\mathrm{d}s\ .$$

We obtain the following main components for the partial derivatives:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u &= -\lim_{u\to t}\Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\\ &\quad+\int_t^T \phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\cdot\frac{1}{2}\frac{K-W_t}{(u-t)^{3/2}}\,\mathrm{d}u\\ &= -1+\frac{1}{2}\int_t^T \phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\cdot\frac{K-W_t}{(u-t)^{3/2}}\,\mathrm{d}u\\ \frac{\mathrm{d}}{\mathrm{d}W_t}\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u &= \int_t^T \frac{\mathrm{d}}{\mathrm{d}W_t}\Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\\ &= -\int_t^T \phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\cdot\frac{1}{\sqrt{u-t}}\,\mathrm{d}u\ ,\end{aligned}$$

which lead to the stated expressions of the proposition. $\square$

Corollary 4.5 *(Spread dynamics under the Merton model, fixed T)*
Under the Merton model the credit spread has the following dynamics:

$$
\begin{aligned}
\frac{\mathrm{d}s_t}{2(1-R)} \;=\; & \left(\frac{-\Phi\left(\frac{K-W_t}{\sqrt{T-t}}\right) + 2\frac{\phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)}{\sqrt{T-t}} \int_t^T \frac{\phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)}{\sqrt{u-t}}\,\mathrm{d}u}{\left[T-t-2\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\right]^2} \right. \\
& \left. +\; 4\,\frac{\Phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)\left[\int_t^T \frac{\phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)}{\sqrt{u-t}}\,\mathrm{d}u\right]^2}{\left[T-t-2\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\right]^3} \right) \mathrm{d}t \\
& - \left(\frac{\frac{\phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)}{\sqrt{T-t}}}{T-t-2\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u} + 2\frac{\Phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)\int_t^T \frac{\phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)}{\sqrt{u-t}}\,\mathrm{d}u}{\left[T-t-2\int_t^T \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\right]^2} \right) \mathrm{d}W_t\ .
\end{aligned}
$$

Proof. Insert the partial derivatives of Proposition 4.4 into equation (4.4). The first and third terms of f_t cancel out with terms of f_{xx}. □

With Corollary 4.5 the spread $s_t = s_0 + \int_0^t \mathrm{d}s_u$ can be computed.

Remark 4.6 (Local volatility of the credit spread)
Credit spread and credit-spread vol are functions of time and Brownian motion: $s_t = f(t, W_t)$ and $\sigma(t, W_t)$. For fixed t, under the general credit-spread condition of no pre-default, in particular $W_t \geq K$, the derivative of the default probability (see equation 4.2 and Figure 4.1) shows that the default probability is monotone in W_t (and has a maximum at $W_t = K$). Then, this also holds for the credit spread. By the *theorem of the inverse function*, for each t, $f(t,\cdot)$ yields an inverse $g(t,\cdot)$ in W_t. Then the spread volatility $\sigma(t,\cdot)$ can be written as a function of spread:

$$
\sigma(t, W_t) \;=\; -\frac{f_x(t, W_t)}{f(t, W_t)} \;=\; -\frac{f_x(t, g(t, s(t)))}{s(t)} \;=:\; \bar{\sigma}(t, s(t))\ .
$$

In the literature $\bar{\sigma}(t,\cdot)$ is referred to as *local volatility* or *instantaneous volatility*. See for example Rebonato (1999).

4.4.2 Spread dynamics for fixed time to maturity M

As in the last section, we here derive the spread dynamics when time to maturity M is fixed (instead of $T = t + M$) while we move in time t. The credit-spread formula was determined in Corollary 4.3:

$$
s(t, t+M) \;=\; \frac{2\,(1-R)\,\Phi\left(\frac{K-W_t}{\sqrt{M}}\right)}{M - 2\int_t^{t+M} \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u}\ .
$$

Again the spread is a function in t and W_t, that is, there is a function f^M such that $f^M(t, W_t) \equiv s_t^M \equiv s(t, t+M)$. We abbreviate the partial derivatives of f^M as for those of f. Via Itô's rule we derive the spread dynamics for fixed M:

$$\mathrm{d}s_t^M = s(t)\left[\mu^M(t, W_t)\,\mathrm{d}t - \sigma^M(t, W_t)\,\mathrm{d}W_t\right] , \tag{4.5}$$

with $\mu^M := \frac{(f_t^M + \frac{1}{2} f_{xx}^M)}{s^M}$ and $\sigma^M := -\frac{f_x^M}{s^M}$.

Proposition 4.7 *(Partial derivatives under the Merton model, fixed M) Under the Merton model the partial derivatives for fixed time to maturity M are given by*

$$\begin{aligned} f_x^M &= f_x \,|_{T=t+M} \,, \\ f_{xx}^M &= f_{xx} \,|_{T=t+M} \,, \\ \frac{f_t^M(t, W_t)}{1-R} &= 4\,\frac{\Phi\left(\frac{K-W_t}{\sqrt{M}}\right)^2 - \Phi\left(\frac{K-W_t}{\sqrt{M}}\right)}{\left[M - 2\int_t^{t+M} \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\right]^2} \\ &\quad + 2\frac{\Phi\left(\frac{K-W_t}{\sqrt{M}}\right)(K-W_t)\int_t^{t+M} \frac{\phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)}{(u-t)^{3/2}}\,\mathrm{d}u}{\left[M - 2\int_t^{t+M} \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\right]^2} . \end{aligned}$$

Proof. The derivatives with respect to x coincide with those in Proposition 4.4; we just insert $t+M$ instead of T. That is, $f_x^M = f_x$ and $f_{xx}^M = f_{xx}$. For the derivative with respect to t we need the derivative of the time-dependent integral (see (A.2) of the Appendix)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{t+M} f(s,t)\,\mathrm{d}s = -f(t,t) + f(t+M, t) + \int_t^T \frac{\mathrm{d}}{\mathrm{d}t} f(s,t)\,\mathrm{d}s .$$

This leads to the main component of f_t^M:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{t+M} \Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u \\ &= \Phi\left(\frac{K-W_t}{\sqrt{M}}\right) - \lim_{u\to t}\Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right) + \int_t^{t+M} \phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\frac{1}{2}\frac{K-W_t}{(u-t)^{3/2}}\,\mathrm{d}u \\ &= \Phi\left(\frac{K-W_t}{\sqrt{M}}\right) - 1 + \frac{1}{2}\int_t^{t+M} \phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\cdot\frac{K-W_t}{(u-t)^{3/2}}\,\mathrm{d}u . \end{aligned}$$

The partial derivative is then given by

$$\frac{f_t^M(t, W_t)}{1-R} = 4\,\Phi\left(\frac{K-W_t}{\sqrt{M}}\right)\cdot\frac{\Phi\left(\frac{K-W_t}{\sqrt{M}}\right) - 1 + \frac{1}{2}\int_t^{t+M}\phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\cdot\frac{K-W_t}{(u-t)^{3/2}}\,\mathrm{d}u}{\left[M - 2\int_t^{t+M}\Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\right]^2} ,$$

which is the stated expression. $\square$

Corollary 4.8 *(Spread dynamics under the Merton model, fixed M)* *Under the Merton model the spread dynamics for fixed time to maturity M are described by*

$$
\begin{aligned}
\frac{\mathrm{d}s(t)}{2(1-R)} \;=\;& \\
&\left(\frac{1}{2}\,\frac{\frac{K-W_t}{M^{3/2}}\,\phi\left(\frac{K-W_t}{\sqrt{M}}\right)}{M-2\int_t^{t+M}\Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u}+4\,\frac{\Phi\left(\frac{K-W_t}{\sqrt{M}}\right)\left[\int_t^{t+M}\frac{\phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)}{\sqrt{u-t}}\,\mathrm{d}u\right]^2}{\left[M-2\int_t^{t+M}\Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\right]^3}\right.\\
&\left.+\;2\,\frac{\Phi\left(\frac{K-W_t}{\sqrt{M}}\right)^2-\Phi\left(\frac{K-W_t}{\sqrt{M}}\right)+\frac{\phi\left(\frac{K-W_t}{\sqrt{M}}\right)}{\sqrt{M}}\int_t^{t+M}\frac{\phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)}{\sqrt{u-t}}\,\mathrm{d}u}{\left[M-2\int_t^{t+M}\Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\right]^2}\right)\mathrm{d}t\\
-&\left(\frac{\frac{\phi\left(\frac{K-W_t}{\sqrt{M}}\right)}{\sqrt{M}}}{M-2\int_t^{t+M}\Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u}+2\,\frac{\Phi\left(\frac{K-W_t}{\sqrt{M}}\right)\int_t^{t+M}\frac{\phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)}{\sqrt{u-t}}\,\mathrm{d}u}{\left[M-2\int_t^{t+M}\Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\,\mathrm{d}u\right]^2}\right)\mathrm{d}W_t\,.
\end{aligned}
$$

Proof. We insert these partial derivatives of Proposition 4.7 into the Itô dynamics of equation (4.5). The second summand of f_t^M cancels out with a term in f_{xx}^M. □

The spread vol (omitting the arguments) is then given by the term in front of the Brownian increment, divided by the spread:

$$
\sigma \;=\; \frac{2(1-R)}{s_t}\left[\frac{\frac{\phi\left(\frac{K-W_t}{\sqrt{M}}\right)}{\sqrt{M}}}{M-2\int_t^{t+M}\Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\mathrm{d}u}+2\frac{\Phi\left(\frac{K-W_t}{\sqrt{M}}\right)\int_t^{t+M}\frac{\phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)}{\sqrt{u-t}}\mathrm{d}u}{\left[M-2\int_t^{t+M}\Phi\left(\frac{K-W_t}{\sqrt{u-t}}\right)\mathrm{d}u\right]^2}\right].\qquad(4.6)
$$

4.4.3 Simulation: spread dynamics and spread paths

For $R = 0$, $T = 10$ and $Q(0,T) = 99\%$ (i.e. $K = -8.145$ and $s(0,T) = 0.001$), a Brownian path is simulated at $n = 40$ grid points. Figure 4.5 shows the Brownian path, the resulting survival probability path (applying Theorem 4.1), the spread path (Corollary 4.3), the spread increments (Theorem 4.8) and the resulting spread path. Note that, indeed, the spread paths determined by the two approaches coincide (up to numerical differences). Numerical reasons lead to small errors in the integral approximation which can imply slightly negative spreads when using the Itô approach. These errors can be reduced by using other integral approximations such as the Euler approximation. Refining the grid by considering $n = 400$ grid points yields Figure 4.6. Again slightly negative spreads are obtained by the Itô approach!

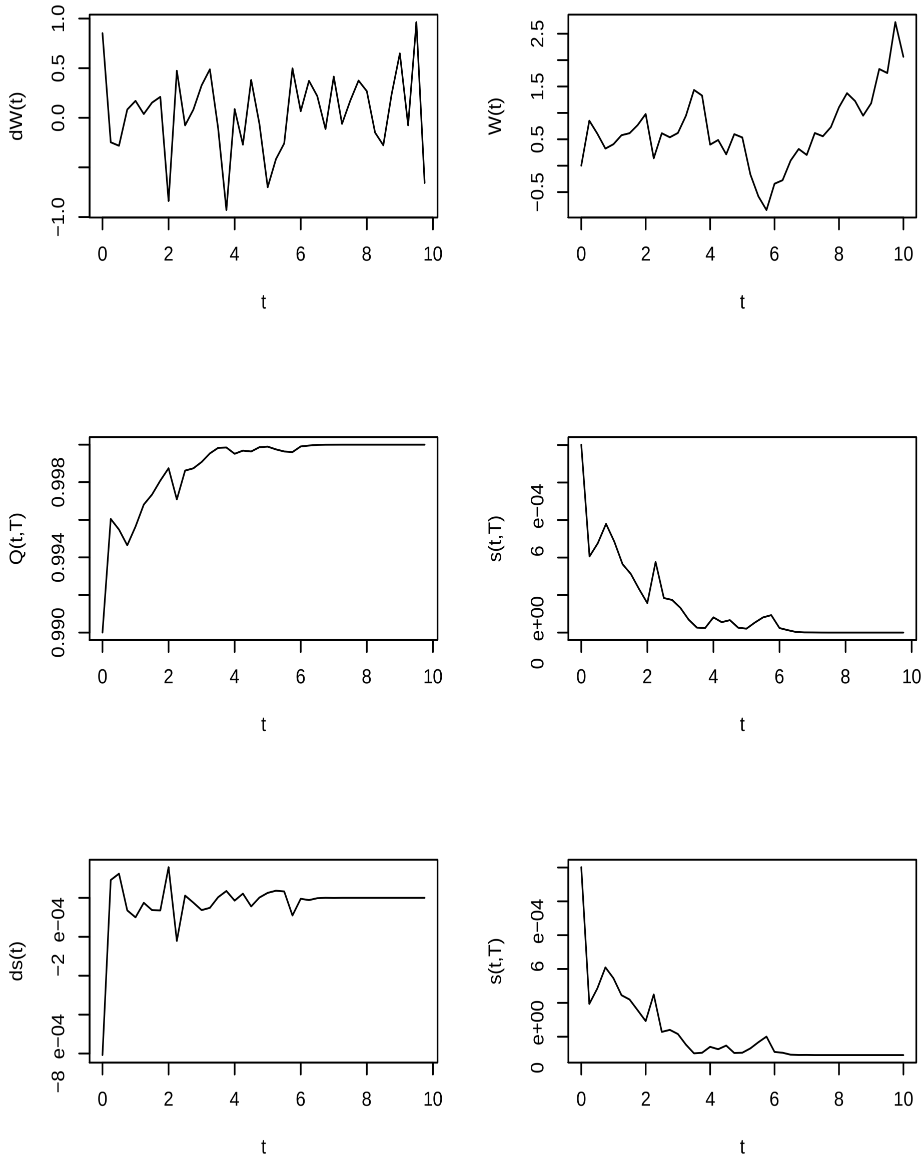

Figure 4.5: Brownian path and resulting spread dynamics ($n = 40$)

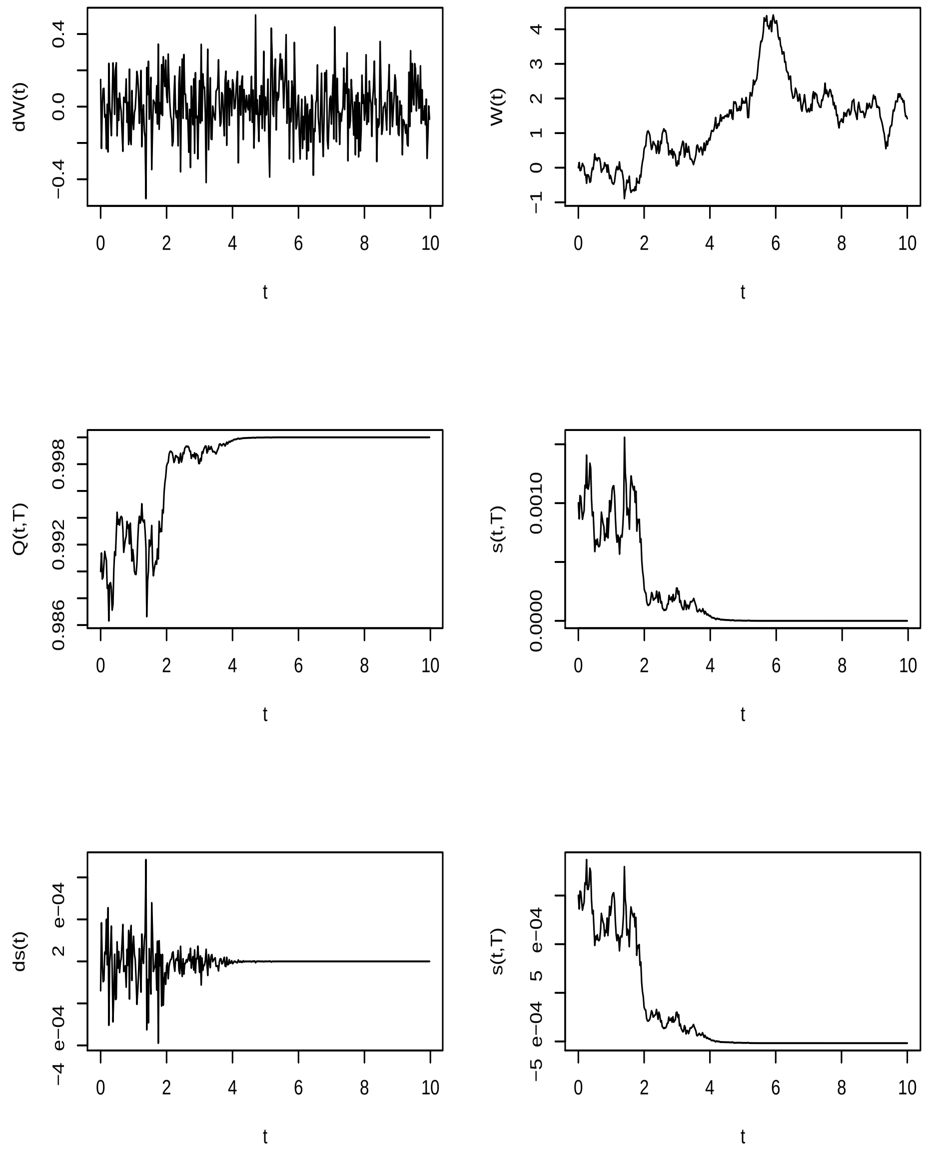

Figure 4.6: Brownian path and resulting spread dynamics ($n = 400$)

4.4.4 Simulation: spread volatility

We have already argued in Remark 4.6 that under the Merton model, spread volatility is not only a function of the actual value W_t, but (for fixed t) can be seen as a function of the actual spread value s_t. Empirically we studied the relation between credit spread and spread volatility in Section 3.4. We found that historical data displays a dependence between the two (at least for liquid data). Now we want to visualize the dependence implied by the Merton model:
We consider the future time point $t = 1$ and choose the threshold level K such that $Q(0, t+M) \equiv \mathbb{P}\Big(\inf_{0\le s\le t+M} W_s > K \mid W_0 = 0\Big) = 99\%$. Furthermore we assume $\tau > t$ and consider future values $W_t = a$ that lie within the 3σ-distance of $W_0 = 0$ (i.e. $a \in [-3\sqrt{t}, 3\sqrt{t}]$). For time to maturities $M = 1$ and $M = 5$, respectively, Figure 4.7 and 4.8 plot the derivative $f_x(W_t)$, the survival probability $Q(t, t+M) = \mathbb{P}\Big(\inf_{t\le s\le t+M} W_s > K \mid W_t = a\Big)$, the corresponding spread $s(t, W_t)$, the spread vol $\sigma_t(W_t)$, and the *local spread vol* $\bar{\sigma}^t(s_t)$. For increasing time to maturity M both spread level and spread vol decrease. The fourth plot of Figure 4.7 shows that spread vol strongly rises when W_t gets close to the default level K. The last plots of each Figure show that the Merton model implies a positive dependence between spread and spread vol (except for very small spreads), but in a way that does for example not fit to the IBM and GM data of Section 3.4.

The next section gives an overview of extensions of the Merton model by adding randomness to the default barrier, interest rates, business clock, or including jumps.

4.5 Survey: extensions of the Merton model

4.5.1 CreditGrades model

The *CreditGrades* (2002) paper by Deutsche Bank, Goldman Sachs, JP Morgan and RiskMetrics Group suggests a generalization of the Merton model by choosing a *random default barrier* (a random variable not a process), which depends on the amount of assets remaining for the debt holders in case of default. The model is illustrated in Figure 4.9; default happens if the random barrier is crossed. By adding randomness to the default barrier the model becomes incomplete: a default cannot be predicted the moment before. This leads to a non-zero instantaneous credit spread. Let L denote the *random average recovery on debt*, having mean $\bar{L}$ and an extreme variance, because the recovery value strongly depends on whether default is triggered by operational or financial difficulties, i.e. whether the company will be restructured or liquidated. Let D be the firm's *debt per share*. As in

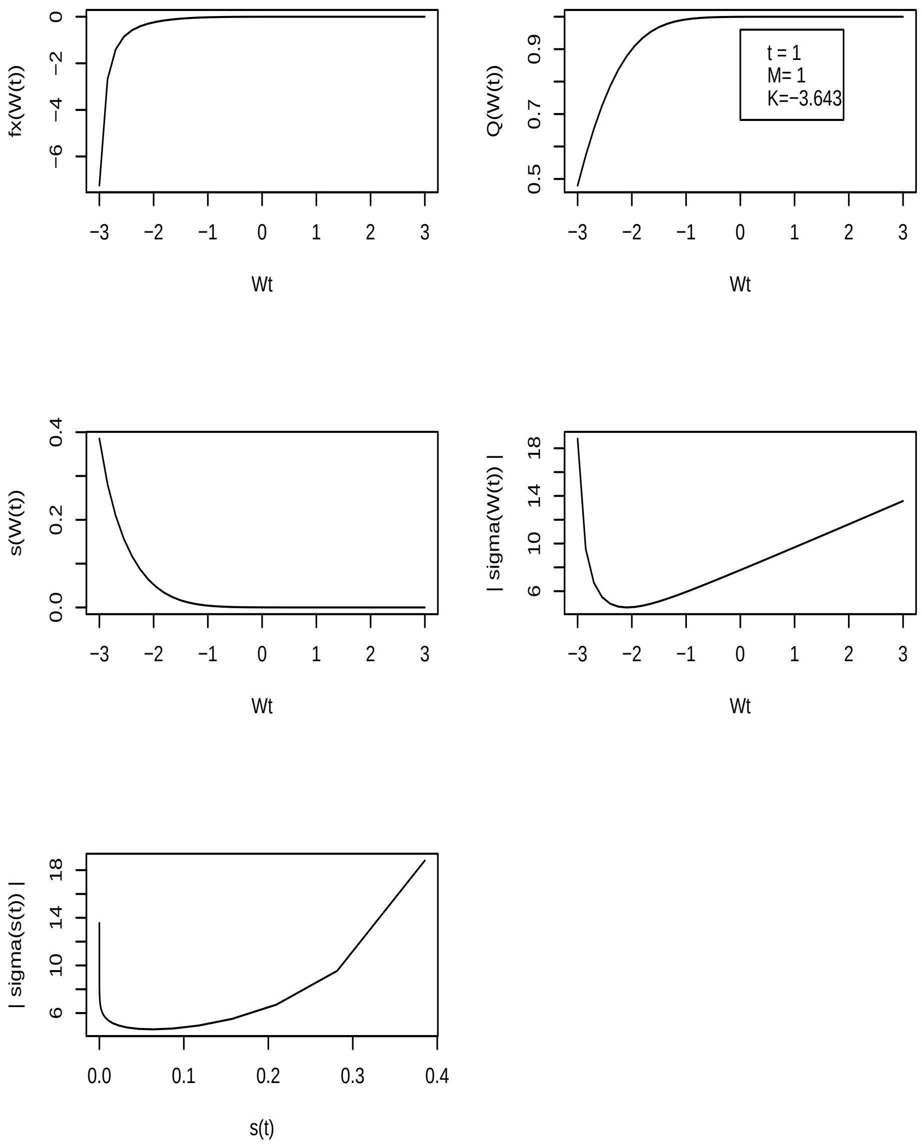

Figure 4.7: Spread vol for $t = 1$, $M = 1$ and $Q(0, t + M) = 99\%$

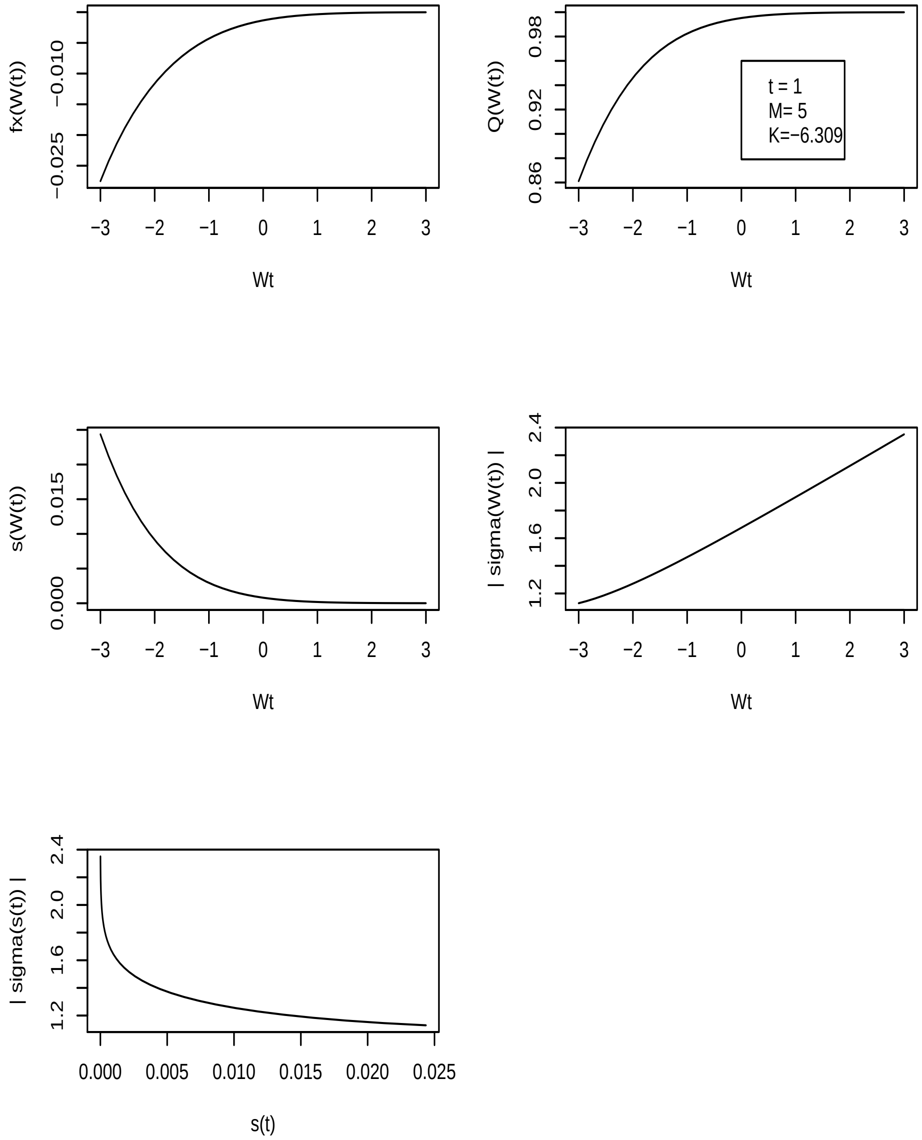

Figure 4.8: Spread vol for $t = 1$, $M = 5$ and $Q(0, t + M) = 99\%$

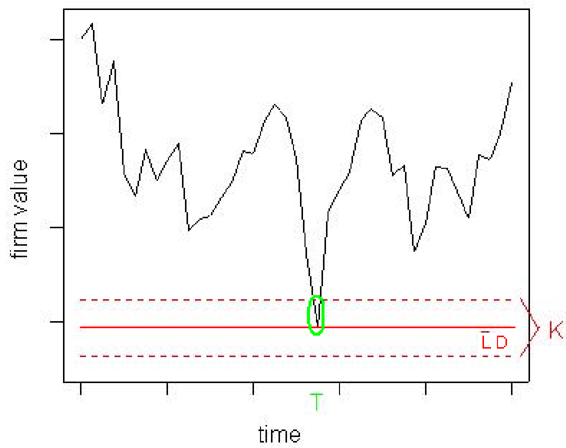

Figure 4.9: CreditGrades model: default happens at T if the random barrier $\tilde{K} \equiv \bar{L}D\,\mathrm{e}^{\lambda Z - \frac{\lambda^2}{2}}$ lies within the green area, otherwise the firm survives.

the Merton model, the asset value process is given by geometric Brownian motion; furthermore the default barrier is assumed to be log-normal:

$$\begin{aligned} \frac{\mathrm{d}V_t}{V_t} &= \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \ , \\ \tilde{K} \equiv LD &= \bar{L}D\,\mathrm{e}^{\lambda Z - \frac{\lambda^2}{2}} \ , \quad Z \sim \mathcal{N}(0,1) \ , \ Z \perp W \ . \end{aligned}$$

Default does not occur as long as the asset value process does not fall below the threshold, that is as long as

$$\begin{aligned} V_0\,\mathrm{e}^{\sigma W_t + \mu t - \sigma^2 \frac{t}{2}} &\geq \bar{L}D\,\mathrm{e}^{\lambda Z - \frac{\lambda^2}{2}} \\ \Leftrightarrow \quad \underbrace{\sigma W_t + \mu t - \lambda Z - \frac{\sigma^2 t}{2}}_{\equiv Y_t} &\geq \underbrace{\ln\left(\frac{\bar{L}D}{V_0}\right) - \frac{\lambda^2}{2}}_{\equiv K} \ . \end{aligned}$$

The first-passage process Y and the constant threshold K yield the first-passage time τ with regard to Definition 1.8. Expectation and variance of Y are given by

$$\mathbb{E}Y_t = \mu t - \tfrac{1}{2}\sigma^2 t \quad , \quad \operatorname{Var} Y_t = \sigma^2 t + \lambda^2 \ .$$

The CreditGrades paper approximates ability-to-pay process and random barrier by introducing a new process starting in the past. This approximation makes a default in the past possible! For an assumed zero drift

($\mu = 0$) the survival probability of the approximated process is given by the one-dimensional standard normal distributions as follows:

$$Q(0,t) \approx \Phi\Big(-\frac{A_t}{2} + \frac{\ln(d)}{A_t}\Big) - d \cdot \Phi\Big(-\frac{A_t}{2} - \frac{\ln(d)}{A_t}\Big) \ ,$$

where d and A_t are as in the next theorem. VERAART (2005) found that the CreditGrades default probability approximation might not be good enough for a highly leveraged debt value D. We will not explain the CreditGrades approximation; instead we determine the exact survival probability formula for the original process Y:

Theorem 4.9 *(Survival probability under the CreditGrades model) Assume $\mu = 0$. Under $Y_0 \geq K$ (no default at time zero), the survival probability is given by*

$$\begin{aligned} \mathbb{P}(\tau > t \mid Y_0 \geq K) \;=\; & \frac{1}{\Phi(\frac{\ln(d)}{\lambda} - \frac{\lambda}{2})}\Bigg[\Phi_2\left(-\frac{\lambda}{2} + \frac{\ln(d)}{\lambda}, -\frac{A_t}{2} + \frac{\ln(d)}{A_t}, \frac{\lambda}{A_t}\right) \\ & - d \cdot \Phi_2\left(\frac{\lambda}{2} + \frac{\ln(d)}{\lambda}, -\frac{A_t}{2} - \frac{\ln(d)}{A_t}, -\frac{\lambda}{A_t}\right)\Bigg] \ , \end{aligned}$$

where Φ_2 is the two-dimensional standard normal distribution and

$$d \;:=\; \tfrac{V_0}{LD}\, \mathrm{e}^{\lambda^2} \ , \qquad A_t^2 \;:=\; \sigma^2 t + \lambda^2 \ .$$

Proof. In order to determine the survival probability conditional on survival until $t = 0$, abbreviate $z_0 = -\frac{K}{\lambda}$, note that $\mathbb{P}(Z \leq z_0) > 0$, condition on Z and use the independence of W and Z and then apply the FPT result for Brownian motion with drift, equation (1.6):

$$\begin{aligned} & \mathbb{P}\Big(\min_{0\leq s\leq t} Y_s \geq K \mid Y_0 \geq K\Big) \\ = \; & \mathbb{P}\left(\min_{0\leq s\leq t}\left[\sigma W_s - \frac{\sigma^2}{2}s\right] \geq K + \lambda Z \mid Z \leq -\frac{K}{\lambda}\right) \\ = \; & \frac{1}{\mathbb{P}(Z \leq z_0)}\mathbb{P}\left(\min_{0\leq s\leq t}\left[\sigma W_s - \frac{\sigma^2}{2}s\right] \geq K + \lambda Z \ , \ Z \leq z_0\right) \\ = \; & \frac{1}{\Phi(z_0)}\mathbb{E}\left[\mathbb{E}\left[\mathbb{I}_{\{\min_{0\leq s\leq t}[\sigma W_s - \frac{\sigma^2}{2}s] \geq K + \lambda Z\}}\mathbb{I}_{\{Z \leq z_0\}} \mid Z\right]\right] \\ = \; & \frac{1}{\Phi(z_0)}\mathbb{E}\left[\mathbb{I}_{\{Z < z_0\}}\mathbb{E}\left[\mathbb{I}_{\{\min_{0\leq s\leq t}[\sigma W_s - \frac{\sigma^2}{2}s] \geq K + \lambda Z\}} \mid Z\right]\right] \end{aligned}$$

$$
\begin{aligned}
&\overset{Z\perp W}{=} \frac{1}{\Phi(z_0)} \int_{-\infty}^{z_0} \mathbb{P}\left(\min_{0\le s\le t} \left[\sigma W_s - \frac{\sigma^2}{2} s\right] \ge K + \lambda z \right) \phi(z)\,\mathrm{d}z \\
&= \frac{1}{\Phi(z_0)} \int_{-\infty}^{z_0} \left[\Phi\Big(\frac{-K-\lambda z - \frac{\sigma^2}{2}t}{\sigma\sqrt{t}}\Big) - \mathrm{e}^{-K-\lambda z}\,\Phi\Big(\frac{K+\lambda z - \frac{\sigma^2}{2}t}{\sigma\sqrt{t}}\Big) \right] \phi(z)\,\mathrm{d}z \\
&= \frac{1}{\Phi(z_0)} \Bigg[\Phi_2\left(z_0, \frac{-K-\frac{\sigma^2}{2}t}{\sqrt{\sigma^2 t + \lambda^2}}, \frac{\lambda}{\sqrt{\sigma^2 t + \lambda^2}} \right) \\
&\qquad - \mathrm{e}^{-K+\frac{\lambda^2}{2}}\,\Phi_2\left(z_0 + \lambda, \frac{-\lambda^2 + K - \frac{\sigma^2}{2}t}{\sqrt{\sigma^2 t + \lambda^2}}, -\frac{\lambda}{\sqrt{\sigma^2 t + \lambda^2}} \right) \Bigg] ,
\end{aligned}
$$

where in the last step we applied the following equality

$$
\int_{-\infty}^{z_0} \mathrm{e}^{vz}\,\phi(z)\Phi(az+b)\,\mathrm{d}z \quad = \quad \mathrm{e}^{\frac{v^2}{2}}\,\Phi_2\Big(z_0 - v, \frac{av+b}{\sqrt{1+a^2}}, -\frac{a}{\sqrt{1+a^2}}\Big) ,
$$

confer for example Ikeda et Al. (1996). Inserting $K = -\ln(d) + \frac{\lambda^2}{2}$ yields the stated result. □

4.5.2 Longstaff and Schwartz: CIR interest rates

Longstaff & Schwartz (1995b) generalized the Merton model by introducing stochastic interest rates r that follow a mean-reverting CIR-process and are correlated with the asset-value process Y (through the Brownian motions W and B). The risk-neutral dynamics are given by

$$
\begin{aligned}
\mathrm{d}Y_t &= (r_t - \delta - \frac{\sigma^2}{2})\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \\
\mathrm{d}r_t &= \kappa(\theta - r_t)\,\mathrm{d}t + \eta\,\mathrm{d}B_t ,
\end{aligned}
$$

where δ is the constant dividend rate and σ respective η are constant volatility rates.

4.5.3 Collin-Dufresne and Goldstein: mean-reverting interest rates and barrier

The extension of the Longstaff & Schwartz model by Collin-Dufresne & Goldstein (2001) lies in choosing a random mean-reverting barrier process that depends on the asset value Y. The interpretation: Firms adjust outstanding debt levels in response to changes in firm value. The log-threshold level follows the process

$$
\mathrm{d}K_t = \lambda(Y_t - \nu - \phi(r_t - \theta) - K_t)\,\mathrm{d}t ,
$$

for some constants λ, ν and ϕ. So if indeed K_t is less than $Y_t - \nu - \phi(r_t - \theta)$ the firm acts to increase K_t, and vice versa.

Both models, Longstaff & Schwartz and Collin-Dufresne & Goldstein, predict negligible credit spreads for very short maturities, which contradicts reality. Comparing the models:
For investment-grade[1] firms the extended model leads to higher spreads for longer maturities, and thus performs better. For speculative-grade firms the model extension generates credit spreads that are larger and less sensitive to changes in firm value, which is more consistent with empirical studies.

4.5.4 Jacobs and Li: two-factor model

Jacobs & Li (2004) modeled the dynamics of credit spreads on corporate bonds with stochastic volatility. They used a two-factor affine model for the credit spread λ_t. The first factor, s_t, can be interpreted as the *level* of the spread and the second, v_t, as *volatility* of the spread. The stochastic volatility model consists of two mean-reverting processes

$$\begin{aligned} \lambda_t &= c + s_t + \delta_1(f_{1t} - \bar{f_{1t}}) + \delta_2(f_{2t} - \bar{f_{2t}}) , \\ \mathrm{d}s_t &= \alpha(\bar{s} - s_t)\,\mathrm{d}t + \sqrt{v_t}\,\mathrm{d}W_t , \\ \mathrm{d}v_t &= \kappa(\bar{v} - v_t)\,\mathrm{d}t + \sigma\sqrt{v_t}\,\mathrm{d}B_t . \end{aligned}$$

Here f_{1t} and f_{2t} are the factors from the two-factor affine model for the riskfree interest rate, each following an independent square-root diffusion process, and $\bar{f_{1t}}$, $\bar{f_{2t}}$ are the resulting means. So δ_1 and δ_2 determine the dependence on the riskless interest rate!
The dynamics of credit spreads are determined in two steps: First the parameters of the riskless interest rate are estimated using U.S. Treasury bond prices and running an extended Kalman filter (EKF). In the second step these estimates for the interest rate are assumed to be true and an EKF is run using corporate bond prices of one firm to estimate the parameters of the corresponding credit-spread process. The EKF estimates an unobservable state vector ξ_t using some observable vector y_t that is a function of the unobservable vector, i.e.

$$\begin{aligned} \xi_{t+1} &= F\xi_t + \epsilon_{t+1} , \\ y_t &= H(\xi_t) + \omega_t \end{aligned}$$

where ϵ_t, ω_t are white noise. The advantage of the EKF approach is that it uses cross-sectional and time-series information.

4.5.5 Duffie and Lando: incomplete accounting information

Duffie & Lando (2001) studied credit-spread term structures on corporate bonds under perfect and imperfect *accounting information*, see Figure 4.10.

[1] See Example 2.2.

Incomplete information models yield credit-spread curves with positive instantaneous spread. Classical structural models with complete (investors) information yield a predictable default time, leading to a zero credit spread at maturity zero. There is no short-term credit risk. In reality investors only receive imperfect information, in the Duffie and Lando case periodically through accounting reports. That is they do not have a precise view on default risk, particularly for short time horizons. Just before a company's default, the reported asset values in a balance-sheet can differ enormously from the actual asset values. We say that investors cannot *anticipate* the default and the default time is *totally inaccessible*[2]. Duffie and Lando derive a distribution of the company's assets conditional on imperfect accounting information. This approach yields the existence of a default-arrival intensity process and thus *combines the structural and the reduced-form approach.*

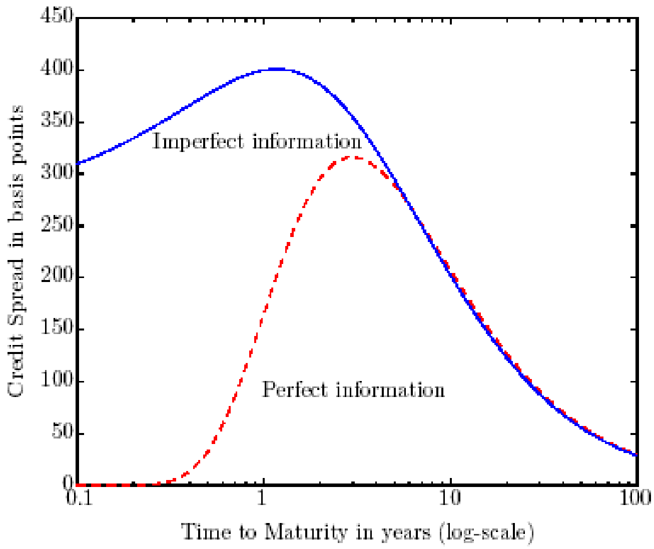

Figure 4.10: Credit-spread term structure under complete (dashed curve) and incomplete (solid curve) information (from DUFFIE & LANDO (2001))

4.5.6 Giesecke and Goldberg: incomplete default-barrier information

GIESECKE & GOLDBERG (2004) introduced the first-passage model in which investors cannot observe the time-invariant random default barrier (that is a random variable not a process) before default (as in the CreditGrades model). But investors observe the asset value. Therefore the running minimum of the asset value process is a natural upper bound for the random

[2] *Totally inaccessibilty* is for example defined in MEYER (1966), D42.

barrier. Concerning the instantaneous credit spread, it is zero when the asset process is above its minimum and infinity when it is below its minimum. In contrast to the model by Duffie & Lando, this model does not yield an intensity process.
GIESECKE (2006) introduced the so-called *trend of a default model* in order to analyze incomplete-information models (incomplete information about asset-value process, barrier level or both) and thus was able to put them into a *generalized reduced-form approach*. Whenever this generalization coincides with the classical intensity model, the trend can be interpreted as *cumulative intensity*.

4.5.7 Zhou: jump-diffusion model

ZHOU (1997a) added a compound Poisson process to Merton's diffusion model. We introduced his so-called *jump-diffusion model* in Subsection 1.2.6, equation (1.12):

$$\begin{aligned} Y_t &= \mu t + \sigma W_t + \sum_{i=1}^{N_t} Z_i \ , \\ N_t &\sim Pois(\lambda t) \ , \ Z_i \text{ iid} \sim \mathcal{N}(\mu_Z, \sigma_Z) \ , \end{aligned}$$

Zhou analyzed size of credit spreads and shapes of credit-spread curves. In contrast to a usual diffusion approach (such as the Merton model), his model is able to match a positive instantaneous default probability and thus an positive instantaneous credit spread. Furthermore it can reproduce various shapes of credit-spread curves including upward-, downward-sloping, flat and hump-shaped.

4.5.8 Comparing structural and reduced-form models

DUFFIE & SINGLETON (2003) compared model-implied spreads from a reduced-form model, a structural model and a model mixture. There are different shapes of spread curves at short and long maturities. A Merton style structural model leads to a *hump-shaped* spread curve starting at zero. The *reduced-form model* is based on an exogeneously specified square-root default-intensity process. The resulting spread curve starts above zero and is increasing to a certain level, where it becomes flat at the end. The *Collin-Dufresne & Goldstein mixture model* combines a target leverage ratio for the *distance-to-default process* and a first-passage model. The resulting spread curve starts again at zero but is only slightly hump-shaped.

4.6 Survey: multivariate extensions of the Merton credit-spread model

Often multivariate dependencies within multi-credit products are modeled via a copula[3], especially under the reduced-form approach. A copula has a static dependency structure and does not show any dynamics. That is, asset dependence does not have any volatility risk and neiter do joint credit spreads.

Our multivariate model, equation (1.15), does show *dependency risk*: The common time change is a stochastic process. Also the time-change volatility and the time change derivative (the so-called *default speed*) are stochastic processes themselves. A time change (distribution) can be constructed to fit the spread dynamics according to one's imagination.

4.6.1 Brownian correlation in bivariate models

In Subsection 1.2.5 we introduced the two-dimensional models with Brownian correlation: Hull & White (2001) (in discrete time) and Overbeck & Schmidt (2005) (in continuous time). Both approaches fit the market's marginal default-probability curves at each grid/time point. The corresponding correlation parameter is matched to the (market) joint default probability at a fixed time horizon.

Our two-dimensional model of Theorem 1.23 adds to the Brownian correlation the dependence inserted by a joint time change. Asset dependencies by the Brownian correlation lead to a default correlation which is not necessary very strong. The time change can be used to strengthen the default correlation and also to influence the dependence structure at any other time horizon.

4.6.2 Cariboni and Schoutens: variance-gamma model

Luciano & Schoutens (2006) considered the *multivariate variance-gamma model* (equation 1.14) as a multi-firm default model. So far, they calibrated the one-dimensional model to CDS spreads by applying a numerical method that solves partial integro-differential equations and is used in pricing digital barrier options. Positive instantaneous credit spreads are yielded. Under the variance-gamma model and other (non-trivial) subordinated Lévy processes explicit credit-spread dynamics cannot be derived via the Itô approach (because the credit-spread formula has no analytical form, see the introduction to Subsection 1.3.6). The dependence intensity varies stochastically (through the gamma time change), but is the same between every two firms.

[3] An introduction to copulas is given by Nelsen (1999).

4.7 Conclusion: Merton-type models

This chapter analyzed the Merton model and Merton-alike models with regard to modeling the credit-spread curve and its dynamics. These models yield a credit-spread volatility that is a function of the actual spread and asset value (which is assumed to be observed). They leave no free-space for giving input on the behavior of the credit-spread curve. In Chapter 6.3 we will therefore introduce our stochastic time-change model.

Chapter 5

Credit spread under the Overbeck & Schmidt model

The Overbeck & Schmidt (2005) approach is an extension of the Merton model. The extension consists of a *deterministic time change* of the Wiener process. We defined the time change in general in Definition 1.1. We interpret it as an indication of the available amount of information. The ability-to-pay process lives in an *experienced time* instead of *normal time*. The so-called *Overbeck & Schmidt model* or *deterministic time-change model* yields an analytical FPT distribution, so the time change can be used to perfectly fit a market given default-probability curve. We determine survival probability, credit spread and credit-spread dynamics, especially credit-spread volatility. The credit-spread dynamics are a deterministic function of the actual values of underlying process and credit spread. Credit-spread volatility can not be influenced.

5.1 Model framework

Overbeck & Schmidt (2005) assumed that a continuous default probability function $(F(t))_{0\leq t<\infty}$ with $F(0) = 0$ is given by the market. They aimed at a default time τ that satisfies (1.4),

$$F(t) \quad = \quad \mathbb{P}\left(\tau \leq t\right) \qquad \forall t \geq 0 \ .$$

Therefore they set

$$\begin{aligned} \hat{\tau} &= \inf\{s \geq 0 : \hat{W}_{\mathcal{T}_s} < K\} \ , \qquad (5.1) \\ \mathcal{T}_t &= \left[\frac{K}{\Phi^{-1}\left(\frac{F(t)}{2}\right)}\right]^2 \ . \end{aligned}$$

Then indeed, for $t \geq 0$ the FPT formula (1.11) for the deterministic time change yields

$$\mathbb{P}(\hat{\tau} \leq t) = 2\Phi\left(K/\sqrt{\mathcal{T}_t}\right) = 2\Phi\left(K \Big/ \frac{K}{\Phi^{-1}\left(\frac{F(t)}{2}\right)}\right) = F(t)\ . \tag{5.2}$$

In the following, in order to determine credit-spread dynamics, we start with another Brownian motion W, assume that the distribution function F admits a density $f = F'$, and define the time of default as follows:

$$\tau = \inf\{t \geq 0 : \int_0^t \sigma_s \, \mathrm{d}W_s < K\} \tag{5.3}$$

$$\sigma_s = \sqrt{-\left[\frac{K}{\Phi^{-1}\left(\frac{F(s)}{2}\right)}\right]^3 \frac{f(s)}{K\varphi\left(\Phi^{-1}\left(\frac{F(s)}{2}\right)\right)}}\ ,$$

where again $K \leq 0$ (no pre-default). σ_s can be interpreted as *default speed* and specifies the speed with which one runs through the Brownian path W. Since F admits a density it is especially continuous. Since, furthermore, $F(0) = 0$ the zero maturity default speed is infinite:

$$\lim_{s\downarrow 0} \sigma_s^2 = -\underbrace{\left[\frac{K}{\Phi^{-1}\left(\frac{F(s)}{2}\right)}\right]^3}_{\to 0} \frac{1}{K \cdot \varphi\Big(\underbrace{\Phi^{-1}\left(\frac{F(s)}{2}\right)}_{\to -\infty}\Big)} = \infty\ , \tag{5.4}$$

since the exponential function in $\varphi(x)$ converges faster (in $x = \Phi^{-1}(F(s)/2)$) to zero than x^3 to infinity. This enables a default at the very next time step.

By Remark 1.15 the stochastic integral underlying the default time (5.3) yields a Brownian motion $\hat{W}$ such that

$$\int_0^t \sigma_s \, \mathrm{d}W_s = \hat{W}_{\mathcal{T}_t}\ ,$$

$$\mathcal{T}_t = \int_0^t (\sigma_s)^2 \, \mathrm{d}s = \left[\frac{K}{\Phi^{-1}\left(\frac{F(t)}{2}\right)}\right]^2 . \tag{5.5}$$

That is, the asset-value process $Y_t = \int_0^t \sigma_s \mathrm{d}W_s$ is equal to a Brownian motion with an absolutely continuous time change. Then also (5.3) and (5.1) are the same default times.

Example 5.1 (Default speed and deterministic time change)
Assume $F(t) = 1 - \mathrm{e}^{-\lambda t}$, with density $f(t) = \lambda\, \mathrm{e}^{-\lambda t}$. Furthermore let $K = -2.578$, which yields the time change $\mathcal{T}_1 = 1$. We want to visualize the default speed σ_s and the time transformation $\mathcal{T}_t$. Note that the time-change integral in (5.5) should not be approximated simply by the sum $\frac{T}{n} \sum_{i=1}^{n} (\sigma_i)^2$ because of property (5.4). Applying the analytical formula in (5.5) leads to Figure 5.1.

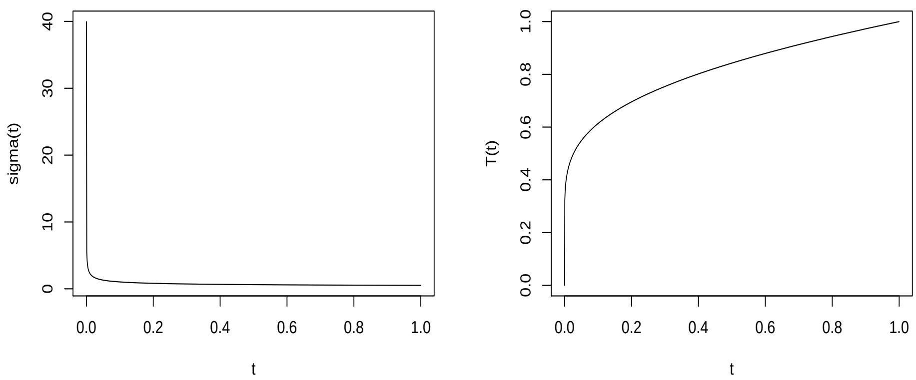

Figure 5.1: Default speed $(\sigma_t)_t$ and analytical time change $(\mathcal{T}_t)_t$

In Subsection 1.2.5 we stated the joint survival probability under the Overbeck & Schmidt model (at time zero). Now we want to determine the conditional survival probability (Definition 3.1). Therfore we have to specify a filtration. As in the general situation of Chapter 3, we assume the information flow is given by the underlying process $Y_t = \hat{W}_{\mathcal{T}_t}$:

$$\mathcal{F}_t^Y \;=\; \sigma\left(\hat{W}_{\mathcal{T}_s} : s \le t\right) . \tag{5.6}$$

Then for every t, $\{\tau > t\}$ is $\mathcal{F}_t^Y$-measurable. That is, τ is a $\mathbb{F}$-stopping time.

Theorem 5.2 *(Survival probability under the Overbeck & Schmidt model) Under $\tau > t$, the survival probability is given by*

$$Q(t,T) = \; 1 - 2\,\Phi\left(\frac{K - \hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T - \mathcal{T}_t}}\right) \;=\; 1 - 2\,\Phi\left(\frac{K - \int_0^t \sigma_s \,\mathrm{d}W_s}{\sqrt{\int_t^T (\sigma_s)^2 \,\mathrm{d}s}}\right) .$$

For fixed time to maturity M (under $\tau > t$) this leads to

$$Q(t, t+M) \quad = \quad 1 - 2\,\Phi\left(\frac{K - \hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_{t+M} - \mathcal{T}_t}}\right) .$$

Proof. We follow the same steps as in the Merton case (Theorem 4.1) and use the Markov property of $(\hat{W}_{\mathcal{T}_t}, \mathcal{F}_t^Y)$. Under $\tau > t$, Definition 3.1 of the survival probability yields

$$\begin{aligned} Q(t,T) \quad &= \mathbb{E}\Big(\mathbb{I}_{\{\inf_{t\le s\le T} \hat{W}_{\mathcal{T}_s} > K\}} \mid \mathcal{F}_t^Y\Big) \quad = \mathbb{P}\Big(\inf_{t\le s\le T} \hat{W}_{\mathcal{T}_s} > K \mid \hat{W}_{\mathcal{T}_t}\Big) \\ &= \mathbb{P}\Big(\inf_{\mathcal{T}_t\le s\le \mathcal{T}_T} \hat{W}_s > K \mid \hat{W}_{\mathcal{T}_t}\Big) \quad = 1 - 2\,\Phi\left(\frac{K - \hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T - \mathcal{T}_t}}\right) , \end{aligned}$$

where for the last equality we used that $K < \hat{W}_{\mathcal{T}_t}$ (since $\tau > t$) and the FPT result for Brownian motion, equation (1.6). □

5.2 Calibration - threshold level

We have seen in equation (5.2) that the deterministic time-change model fits any default-probability curve F (which yields a density). The model yields one more degree of freedom, in the threshold K. It can be used to require $\mathcal{T}_T = T$, which (with (5.5)) leads to

$$T \quad = \quad \int_0^T (\sigma_s)^2 \,\mathrm{d}s \quad = \quad \left[\frac{K}{\Phi^{-1}\left(\frac{F(T)}{2}\right)}\right]^2 ,$$

that is, a strike level of

$$K \quad = \quad \Phi^{-1}\Big(\frac{F(T)}{2}\Big) \cdot \sqrt{T} \,. \tag{5.7}$$

Example 5.3 (Survival probability term structures)
For simplicity we set $F(t) = 1 - \mathrm{e}^{-\lambda t}$ and fix the threshold as in (5.7). Figure 5.2 shows survival probability curves for $\lambda = 1\%, \ldots, 5\%$ and $T = 10$, i.e. for $K = -5.3, -4.2, -3.6, -3.1, -2.7$.

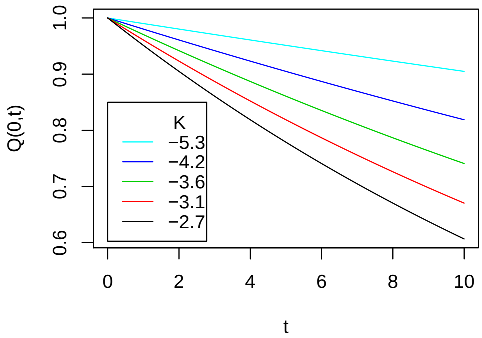

Figure 5.2: Survival-probability term structure for $K = -5.3$, -4.2, -3.6, -3.1, -2.7 and $T = 10$.

5.3 Credit Spread

Corollary 5.4 *(Credit spread under the Overbeck & Schmidt model) Conditional on no-default up to t, the spread is given by*

$$s(t,T) = 2\,(1-R)\frac{\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)}{T-t-2\int_t^T \Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\,\mathrm{d}u},$$

and for fixed time to maturity

$$s(t,t+M) = 2\,(1-R)\frac{\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_{t+M}-\mathcal{T}_t}}\right)}{M-2\int_t^{t+M} \Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\,\mathrm{d}u}.$$

For $t > 0$ this is the forward credit spread.

Proof. Insert the survival probability from Theorem 5.2 into the spread formula (3.4) and (3.5), respectively. □

5.3.1 Instantaneous spread

We determine the zero maturity spread $\lim_{M\downarrow 0} s(t,t+M)$ for an assumed default-probability curve F with density f under no default until t (especially $W_{\mathcal{T}_t} > K$). First of all we determine the derivative of the time change

(5.5):

$$\frac{\mathrm{d}}{\mathrm{d}M}\mathcal{T}_{t+M} = 2\,\frac{K}{\Phi^{-1}\left(\frac{F(t+M)}{2}\right)}\cdot\frac{-K}{\Phi^{-1}\left(\frac{F(t+M)}{2}\right)^2}\cdot\frac{\mathrm{d}}{\mathrm{d}M}\Phi^{-1}\left(\frac{F(t+M)}{2}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}M}\Phi^{-1}\left(\frac{F(t+M)}{2}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}M}\frac{F(t+M)}{2}}{\Phi'\left(\Phi^{-1}\left(\frac{F(t+M)}{2}\right)\right)} = \frac{\frac{1}{2}\,f(t+M)}{\varphi\left(\Phi^{-1}\left(\frac{F(t+M)}{2}\right)\right)}\,,$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}M}\mathcal{T}_{t+M} = -\frac{K^2}{\Phi^{-1}\left(\frac{F(t+M)}{2}\right)^3}\cdot\frac{f(t+M)}{\varphi\left(\Phi^{-1}\left(\frac{F(t+M)}{2}\right)\right)}\,. \tag{5.8}$$

Applying the *rule of de L'Hospital* and the *fundamental theorem of calculus* yields a zero instantaneous credit spread:

$$\begin{aligned}
&\lim_{M\downarrow 0}\frac{s(t,t+M)}{1-R}\\
&= 2\,\frac{\lim_{M\downarrow 0}\;\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_{t+M}-\mathcal{T}_t}}\right)}{\lim_{M\downarrow 0} M - 2\int_t^{t+M}\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\,\mathrm{d}u}\\
&= 2\,\frac{\lim_{M\downarrow 0}\;\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_{t+M}-\mathcal{T}_t}}\right)(K-\hat{W}_{\mathcal{T}_t})\cdot\left(-\frac{1}{2}\right)\frac{\frac{\mathrm{d}}{\mathrm{d}M}\mathcal{T}_{t+M}}{(\mathcal{T}_{t+M}-\mathcal{T}_t)^{3/2}}}{\lim_{M\downarrow 0}\left(1-2\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_{t+M}-\mathcal{T}_t}}\right)\right)}\\
&= -\,(K-\hat{W}_{\mathcal{T}_t})\cdot\frac{\lim_{M\downarrow 0}\;\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_{t+M}-\mathcal{T}_t}}\right)\frac{\frac{\mathrm{d}}{\mathrm{d}M}\mathcal{T}_{t+M}}{(\mathcal{T}_{t+M}-\mathcal{T}_t)^{3/2}}}{1-0}\\
&= (K-\hat{W}_{\mathcal{T}_t})\frac{K^2}{\Phi^{-1}\left(\frac{F(t)}{2}\right)^3}\frac{f(t)}{\varphi\left(\Phi^{-1}\left(\frac{F(t)}{2}\right)\right)}\cdot\lim_{M\downarrow 0}\frac{\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_{t+M}-\mathcal{T}_t}}\right)}{(\mathcal{T}_{t+M}-\mathcal{T}_t)^{3/2}}\\
&= 0\,,
\end{aligned}$$

since $\mathrm{e}^{-\frac{1}{2}\frac{1}{x}}$ converges faster to zero for $x\downarrow 0$ than $\frac{1}{x^{3/2}}$.

5.3.2 Simulations

We discretize the time grid $t_0 = 0, t_1 = \frac{T}{n}, \ldots, t_{n+1} = t+M$ and simulate the survival probability of Theorem 5.2 as follows:

$$Q(t_i, t_i+M) = \mathbb{I}_{\{\inf_{0\le j\le i}\hat{W}_{\mathcal{T}_{t_j}} > K\}}\left[1 - 2\,\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_{t_i}}}{\sqrt{\mathcal{T}_{t_i+M}-\mathcal{T}_{t_i}}}\right)\right]\,,$$

and analogously the survival probability $Q(t_i, t_k)$ needed for the credit spread simulation (see (4.4)). We simulate Brownian increments that depend on the time change (and indirectly on the threshold K),

$$\Delta \hat{W}_{\mathcal{T}_{t_j}} \quad \sim \quad \Phi\left(0, \mathcal{T}_{t_j} - \mathcal{T}_{t_{j-1}}\right) \ ,$$

in order to get a path of the time transformed Brownian motion

$$\hat{W}_{t_i} \quad = \quad \hat{W}_0 + \sum_{j=1}^{i} \Delta \hat{W}_{t_j} \ .$$

Dynamics of credit-spread curves in time

In Subsection 4.3.2 we have shown possible credit-spread dynamics under the Merton model. Here we show two possible credit-spread evolutions under the deterministic time-change model, assuming the simplified default probability curve $F(t) = 1 - \mathrm{e}^{-\lambda t}$ with $\lambda = 1\%$ and 5%, respectively. We let $T = 10$, $M = 10$ which implies a barrier level of $K = -5.28$ (and a survival probability of $Q(0,T) = 90.48\%$), and $K = -2.69$ ($Q(0,T) = 60.65\%$), respectively. The simulated Brownian path in Figure 5.3 crosses the barrier at time $t = 2.75$ and thus implies a default. The default point is shown as a red dot in the Brownian path of the first plot. credit-spread term structures are only shown up to the default time. The Brownian motion simulated with $\lambda = 5\%$ is shown in Figure 5.4. It does not cross the barrier, that is, it does not default. In the first plot the red dots on the Brownian path mark the Brownian levels for which credit-spread curves are plotted.

5.4 Credit-spread dynamics

For determining the dynamics we keep the CDS maturity T fixed an write $s(t)$ instead of $s(t,T)$. Furthermore we consider the spread as a function f of time t and state variable $x = \hat{W}_{\mathcal{T}_t}$, that is $f(t,x) = s(t)$. We use the same abbreviations f_t, f_x, f_{xx} for the partial derivatives as in the Merton Subsection 4.4.1. Conditional on $\tau > t$ we determine the spread dynamics with Itô's formula:

$$\mathrm{d}s_t \quad = \quad s(t)\left[\hat{\mu}(t, \hat{W}_{\mathcal{T}_t})\,\mathrm{d}t - \hat{\sigma}(t, \hat{W}_{\mathcal{T}_t})\,\mathrm{d}\hat{W}_{\mathcal{T}_t}\right] \ , \tag{5.9}$$

where $\hat{\mu} := \frac{\left(f_t + \frac{1}{2} f_{xx}\right)}{s}$ and $\hat{\sigma} := -\frac{f_x}{s}$. In terms of the original Brownian motion W, this is equivalent to

$$\mathrm{d}s_t \quad = \quad s(t)\left[\hat{\mu}\left(t, \int_0^t \sigma_s\,\mathrm{d}W_s\right)\,\mathrm{d}t - \hat{\sigma}\left(t, \int_0^t \sigma_s\,\mathrm{d}W_s\right)\sigma_t\,\mathrm{d}W_t\right] \ .$$

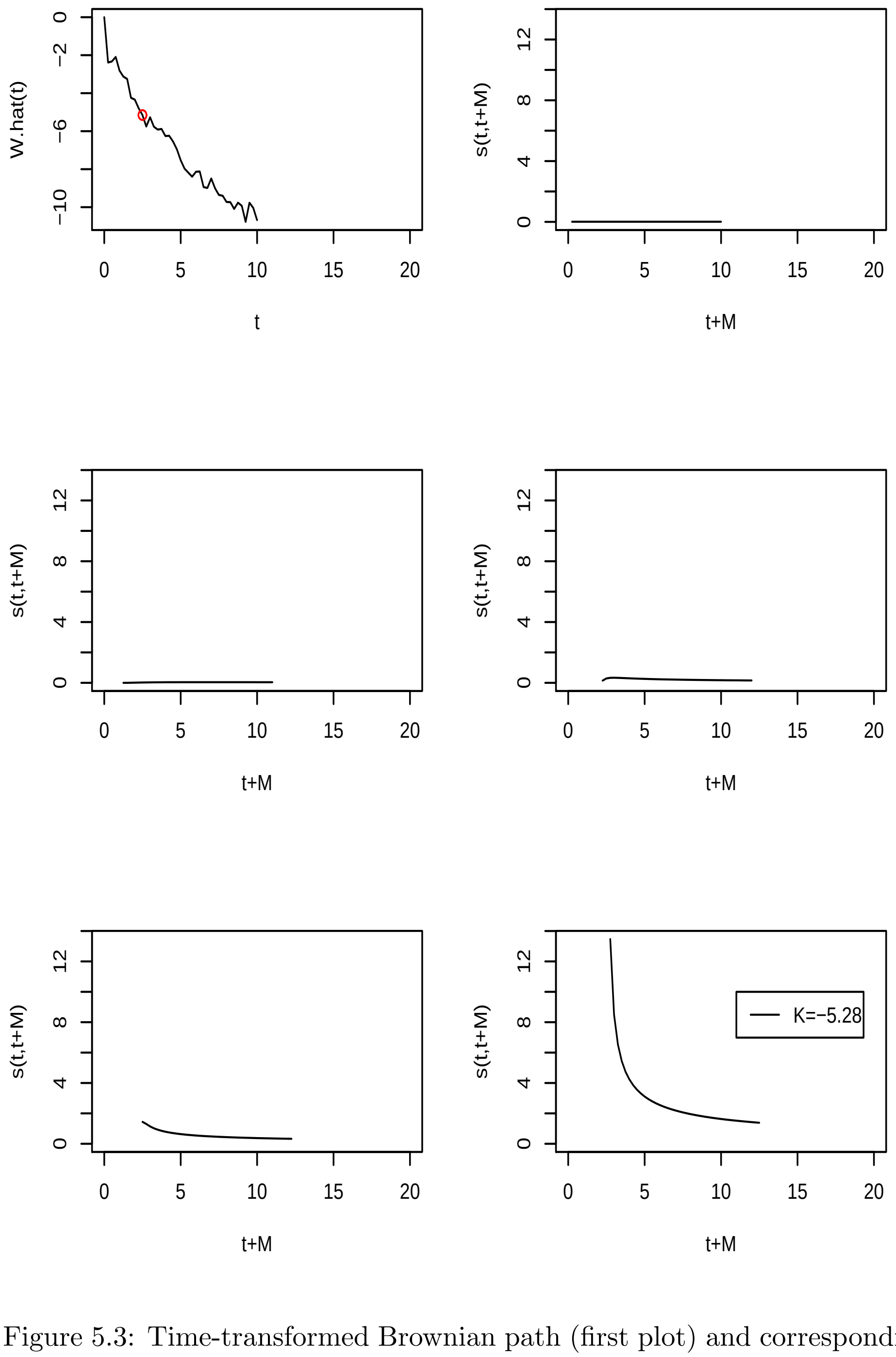

Figure 5.3: Time-transformed Brownian path (first plot) and corresponding credit-spread term structures till default for the threshold $K = -5.28$

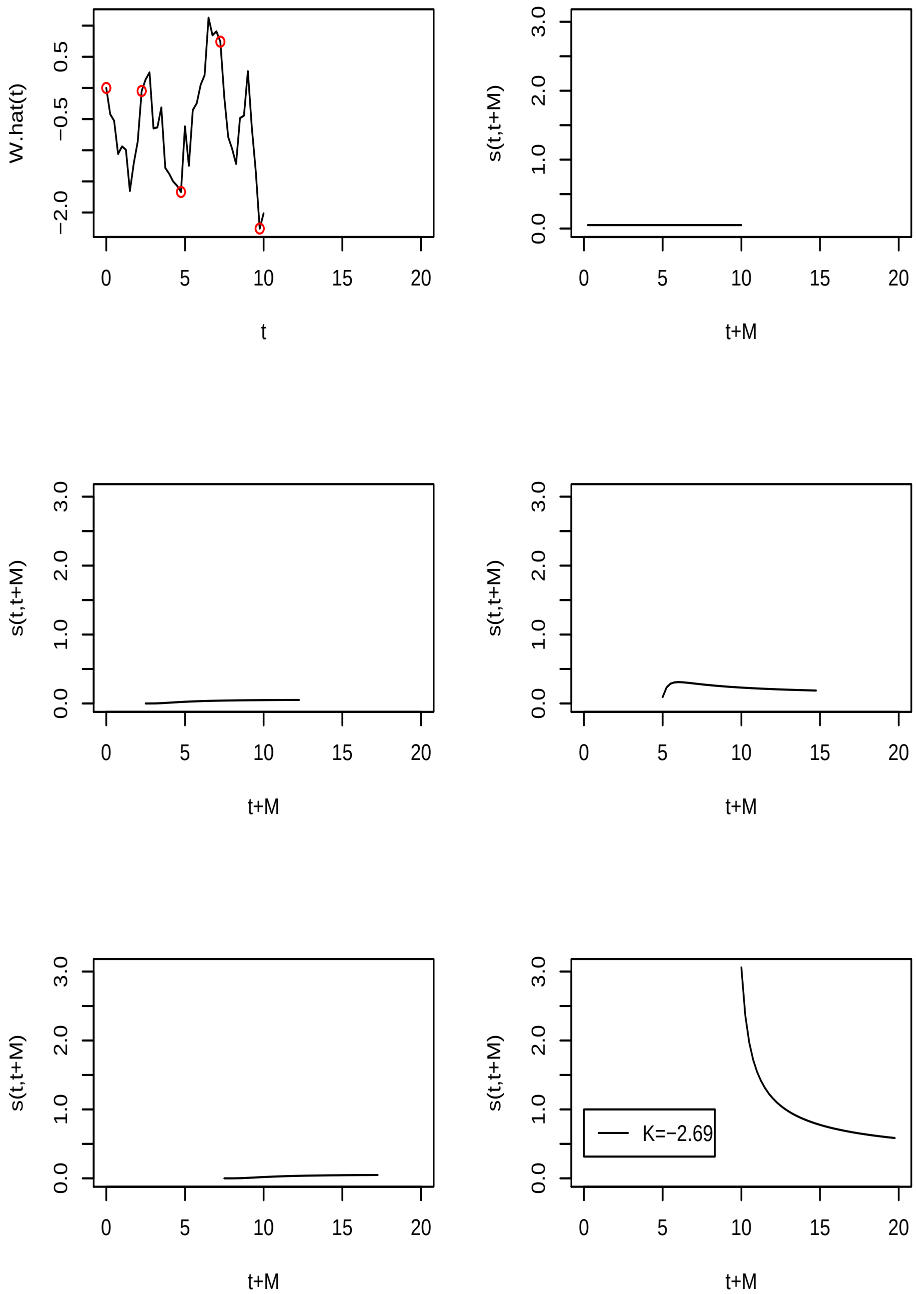

Figure 5.4: Time-transformed Brownian path and corresponding credit-spread term structures for the threshold $K = -2.69$

$\hat{\mu}$ can be interpreted as drift and $\sigma_t \cdot \hat{\sigma}$ as local volatility of credit spread. The default speed σ_t is the main difference to the credit-spread vol of the Merton model. The negative sign in the definition of $\hat{\sigma}$ makes sense because we will find that f_x is a negative function. In the next proposition we determine the partial derivatives f_t, f_x and f_{xx}.

Proposition 5.5 *(Partial derivatives under the Overbeck&Schmidt model) The spread dynamics under the Overbeck&Schmidt model $Y_t = \hat{W}_{\mathcal{T}_t}$ are determined by the following partial derivatives:*

$$\frac{f_t(t,\hat{W}_{\mathcal{T}_t})}{1-R} = 2\,\frac{(K-\hat{W}_{\mathcal{T}_t})\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)\frac{\mathrm{d}\mathcal{T}_t}{\mathrm{d}t}\cdot\int_t^T \frac{\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)}{(\mathcal{T}_u-\mathcal{T}_t)^{3/2}}\,\mathrm{d}u}{\left[T-t-2\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\,\mathrm{d}u\right]^2}$$
$$+\frac{\frac{\mathrm{d}\mathcal{T}_t}{\mathrm{d}t}\cdot\frac{K-\hat{W}_{\mathcal{T}_t}}{(\mathcal{T}_T-\mathcal{T}_t)^{3/2}}\,\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)}{T-t-2\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\,\mathrm{d}u} - 2\,\frac{\Phi\left(\frac{K-W_t}{\sqrt{T-t}}\right)}{\left[T-t-2\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\,\mathrm{d}u\right]^2}$$

$$\frac{f_x(t,\hat{W}_{\mathcal{T}_t})}{-2(1-R)} = \frac{\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)/\sqrt{\mathcal{T}_T-\mathcal{T}_t}}{T-t-2\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\,\mathrm{d}u} + 2\frac{\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)\int_t^T\frac{\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\,\mathrm{d}u}{\left[T-t-2\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\,\mathrm{d}u\right]^2}$$

$$\frac{f_{xx}(t,\hat{W}_{\mathcal{T}_t})}{-2(1-R)} = \frac{\frac{K-\hat{W}_{\mathcal{T}_t}}{(\mathcal{T}_T-\mathcal{T}_t)^{3/2}}\,\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)}{T-t-2\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\,\mathrm{d}u} - 8\frac{\Phi\left(\frac{K-W_t}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)\left[\int_t^T\frac{\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\,\mathrm{d}u\right]^2}{\left[T-t-2\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\,\mathrm{d}u\right]^3}$$
$$+\frac{-4\,\frac{\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\int_t^T\frac{\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\,\mathrm{d}u + 2(K-W_t)\Phi\left(\frac{K-W_t}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)\int_t^T\frac{\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)}{(\mathcal{T}_u-\mathcal{T}_t)^{3/2}}\,\mathrm{d}u}{\left[T-t-2\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\,\mathrm{d}u\right]^2},$$

where $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_t$ in f_t is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_t = -\frac{K^2}{\Phi^{-1}\left(\frac{F(t)}{2}\right)^3}\cdot\frac{\frac{\mathrm{d}}{\mathrm{d}t}F(t)}{\varphi\left(\Phi^{-1}\left(\frac{F(t)}{2}\right)\right)}.$$

Proof. We only give the basic derivatives needed to derive the partial derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right) \quad = \quad \varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)\cdot\frac{1}{2}\frac{K-\hat{W}_{\mathcal{T}_t}}{(\mathcal{T}_T-\mathcal{T}_t)^{3/2}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_t\ ,$$

where $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_t$ is given by (5.8) for $M=0$. Applying (A.1) (Appendix) yields

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\textstyle\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\mathrm{d}u &= -\lim_{u\to t}\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)+\textstyle\int_t^T\frac{\mathrm{d}}{\mathrm{d}t}\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\mathrm{d}u\\ &= -1+\tfrac{1}{2}\tfrac{\mathrm{d}\mathcal{T}_t}{\mathrm{d}t}\textstyle\int_t^T\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\cdot\frac{K-\hat{W}_{\mathcal{T}_t}}{(\mathcal{T}_u-\mathcal{T}_t)^{3/2}}\,\mathrm{d}u \quad ,\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}\hat{W}_{\mathcal{T}_t}}\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\mathrm{d}u \quad = \quad -\textstyle\int_t^T\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\cdot\frac{1}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\,\mathrm{d}u\ .$$

The stated partial derivatives follow directly. □

Theorem 5.6 *(Spread dynamics under the Overbeck & Schmidt model) Under the Overbeck & Schmidt model $Y_t=\hat{W}_{\mathcal{T}_t}$, the credit spread has the following dynamics:*

$$\begin{aligned}\frac{\mathrm{d}s_t}{2(1-R)} \quad = \quad &\left(-\left(\frac{K^2}{\Phi^{-1}\left(\frac{F(t)}{2}\right)^3}\cdot\frac{\frac{\mathrm{d}}{\mathrm{d}t}F(t)}{\varphi\left(\Phi^{-1}\left(\frac{F(t)}{2}\right)\right)}+1\right)\right.\\
&\cdot\left(\frac{1}{2}\frac{\frac{K-\hat{W}_{\mathcal{T}_t}}{(\mathcal{T}_T-\mathcal{T}_t)^{3/2}}\,\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)}{T-t-2\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\mathrm{d}u}+\frac{\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)\int_t^T\frac{\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\mathrm{d}u}{\left[T-t-2\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\mathrm{d}u\right]^2}\right)\\
&+\frac{-\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)+2\frac{\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\int_t^T\frac{\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\mathrm{d}u}{\left[T-t-2\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\mathrm{d}u\right]^2}\\
&\left.+\ 4\,\frac{\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)\left(\int_t^T\frac{\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\mathrm{d}u\right)^2}{\left[T-t-2\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\mathrm{d}u\right]^3}\right)\mathrm{d}t\\
&-\left(\frac{\frac{\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}}{T-t-2\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\mathrm{d}u}+2\frac{\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)\int_t^T\frac{\varphi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\mathrm{d}u}{\left[T-t-2\int_t^T\Phi\left(\frac{K-\hat{W}_{\mathcal{T}_t}}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)\mathrm{d}u\right]^2}\right)\mathrm{d}\hat{W}_{\mathcal{T}_t}\ .\end{aligned}$$

Proof. We insert the partial derivatives and $\frac{\mathrm{d}\mathcal{T}_t}{\mathrm{d}t}$ of Proposition 5.5 into equation (5.9). The first and second term of f_t coincide with terms of f_{xx} apart from the term $\frac{\mathrm{d}\mathcal{T}_t}{\mathrm{d}t}$. The stated expression follows. □

Remark 5.7 (credit-spread volatility)
The spread volatility under the Overbeck & Schmidt model is given by

$$\hat{\sigma} \cdot \sigma_t \;=\; 2(1-R)\frac{\sigma_t}{s_t} \cdot \left[\frac{\frac{\varphi\left(\frac{K-\int_0^t \sigma_s \, \mathrm{d}W_s}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right)}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}}{T-t-2\int_t^T \Phi\left(\frac{K-\int_0^t \sigma_s \, \mathrm{d}W_s}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right) \mathrm{d}u} + 2\frac{\Phi\left(\frac{K-\int_0^t \sigma_s \, \mathrm{d}W_s}{\sqrt{\mathcal{T}_T-\mathcal{T}_t}}\right) \int_t^T \frac{\varphi\left(\frac{K-\int_0^t \sigma_s \, \mathrm{d}W_s}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right)}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}} \mathrm{d}u}{\left[T-t-2\int_t^T \Phi\left(\frac{K-\int_0^t \sigma_s \, \mathrm{d}W_s}{\sqrt{\mathcal{T}_u-\mathcal{T}_t}}\right) \mathrm{d}u\right]^2} \right] .$$

As in the Merton model (Remark 4.6), for fixed t (under no pre-default) there exists an inverse $\hat{g}$ such that the spread volatility can be transformed into a local volatility $\bar{\hat{\sigma}}$ (a function in s_t):

$$\sigma_t \cdot \hat{\sigma}(t, \hat{W}_{\mathcal{T}_t}) \;=\; -\sigma_t \frac{f_x(t, \hat{W}_{\mathcal{T}_t})}{f(t, \hat{W}_{\mathcal{T}_t})} \;=\; -\sigma_t \frac{f_x(t, \hat{g}(t, s(t)))}{s(t)} \;=:\; \bar{\hat{\sigma}}(s_t) \; .$$

5.4.1 Simulation: spread dynamics and spread path

We assume $F(t) = 1 - \mathrm{e}^{-\lambda t}$ with $\lambda = 1\%$, $R = 0$ and $T = 10$ ($Q(0,T) = 90.48\%$ and $K = -5.277$), and divide the time interval into $n = 400$ grid points. Figure 5.5 shows a defaulting time-transformed Wiener process, corresponding survival probability and credit-spread path (Corollary 5.4). The two last plots show the spread increments calculated with Theorem 5.6 (where some slightly negative spreads were obtained), and the resulting spread path. Just before the time of default, the survival probability jumps to zero and the spread increases enormously (and is not plotted after default). Figure 5.6 shows another simulation where $Q(0,T) = 99\%$ and $K = -8.145$ and no default happens. Both Figures show that the spread paths determined by the two simulation approaches basically coincide.

5.4.2 Simulation: spread volatility

We choose the threshold level K (and thus the default intensity λ) such that

$$Q(0, t+M) \;=\; \mathbb{P}\Big(\inf_{0 \le s \le \mathcal{T}_{t+M}} \hat{W}_s > K \mid \hat{W}_0 = 0 \Big) \;=\; 99\%$$

for $M = 5$. Assuming $\tau > t$ and $\hat{W}_{\mathcal{T}_t} = a > K$, where a lies within the 3σ-distance of $\hat{W}_0 = 0$, we determine the derivative $f_x(\hat{W}_{\mathcal{T}_t})$, the survival probability $Q(t, t+M) = \mathbb{P}\Big(\inf_{t \le s \le t+M} \hat{W}_s > K \mid \hat{W}_{\mathcal{T}_t} = a \Big)$, corresponding spread and spread vol. Graphs of spread and spread vol are plotted in Figure 5.7 and 5.8 for $t = 1$ and $t = 5$, respectively. We received simulation errors when the time-transformed Wiener process was close to the threshold level and also when it was so big that the spread was basically zero. In the fourth plot of Figure 5.8 the error is plotted. In the other plots we therefore restricted the x-axis.

5.5 Conclusion

This chapter analyzed the deterministic time-change model for modeling credit-spread curves and spread dynamics. The deterministic time change can be chosen such that the actual credit-spread curve is perfectly fitted. Credit-spread volatility is more than twice as big as under the Merton model. The main difference in the spread volatility originates from the default speed σ_t. Still credit-spread volatility is a deterministic function in the actual spread and actual asset value. Credit-spread volatility and the here inherent risk can not be influenced by the model. Therefore we will consider the stochastic time-change model in the next chapter.

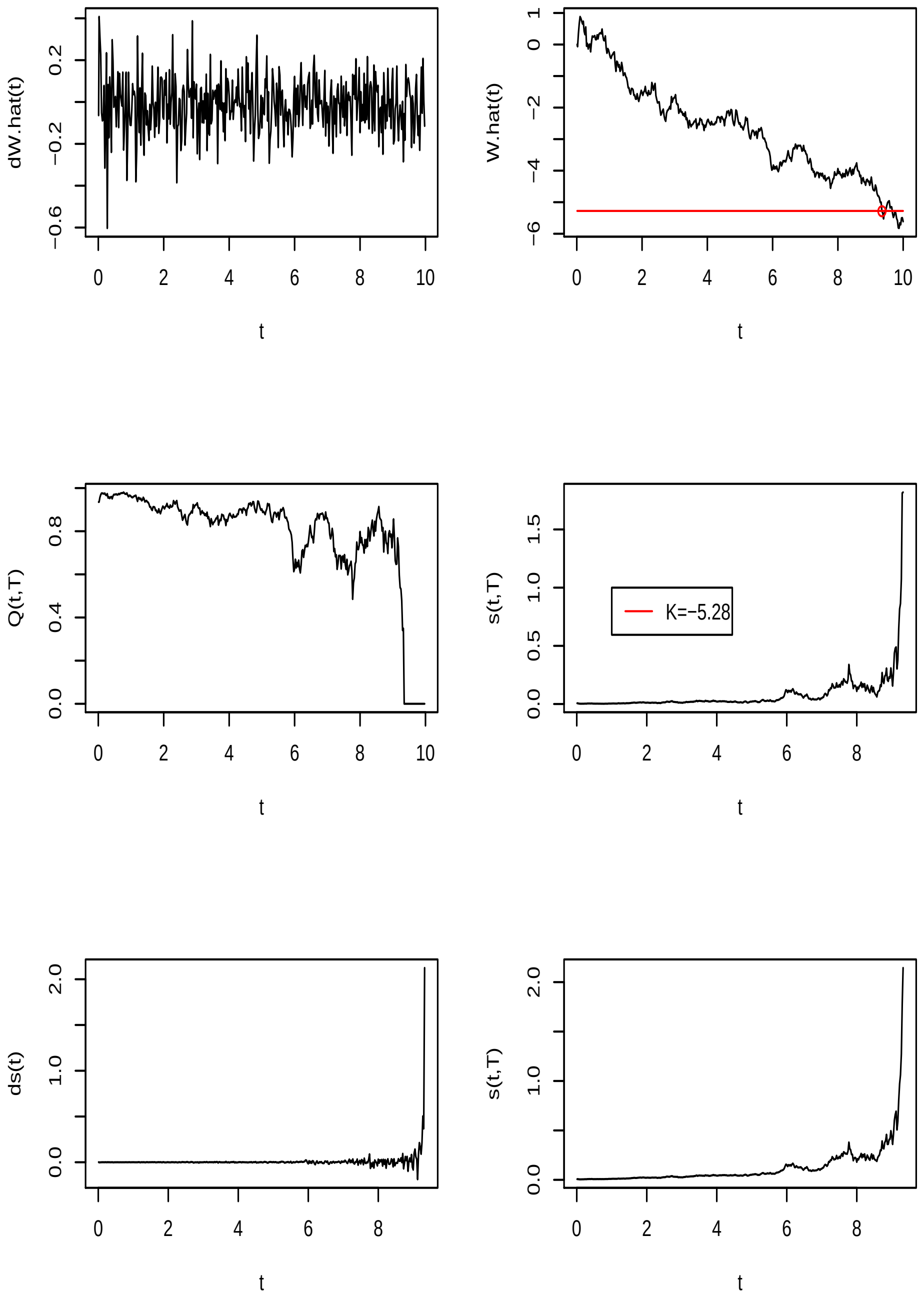

Figure 5.5: Defaulting time-transformed Brownian motion, survival probability, spread dynamics and resulting credit-spread paths

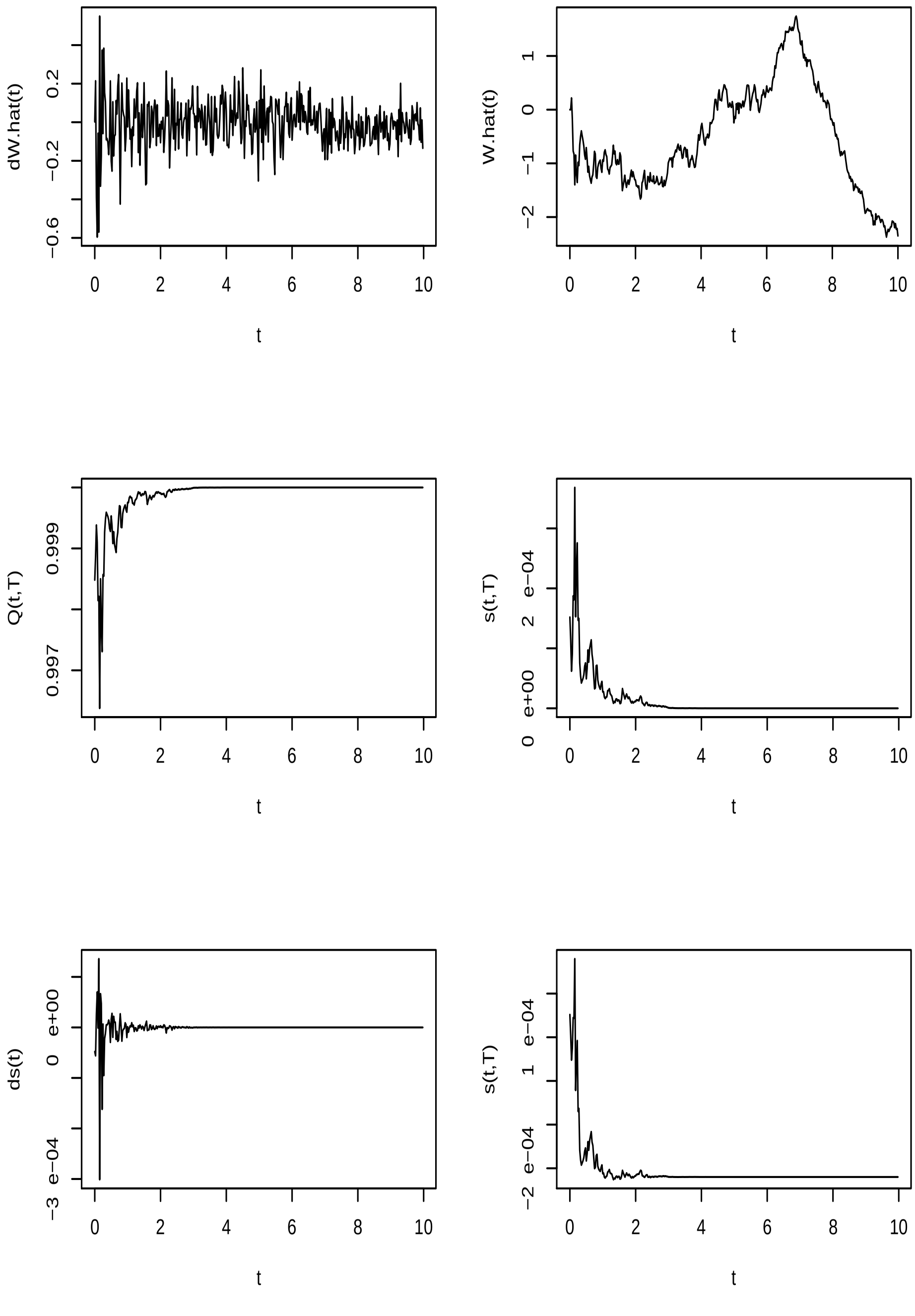

Figure 5.6: Time-transformed Brownian motion, survival probability, spread dynamics and resulting credit-spread paths

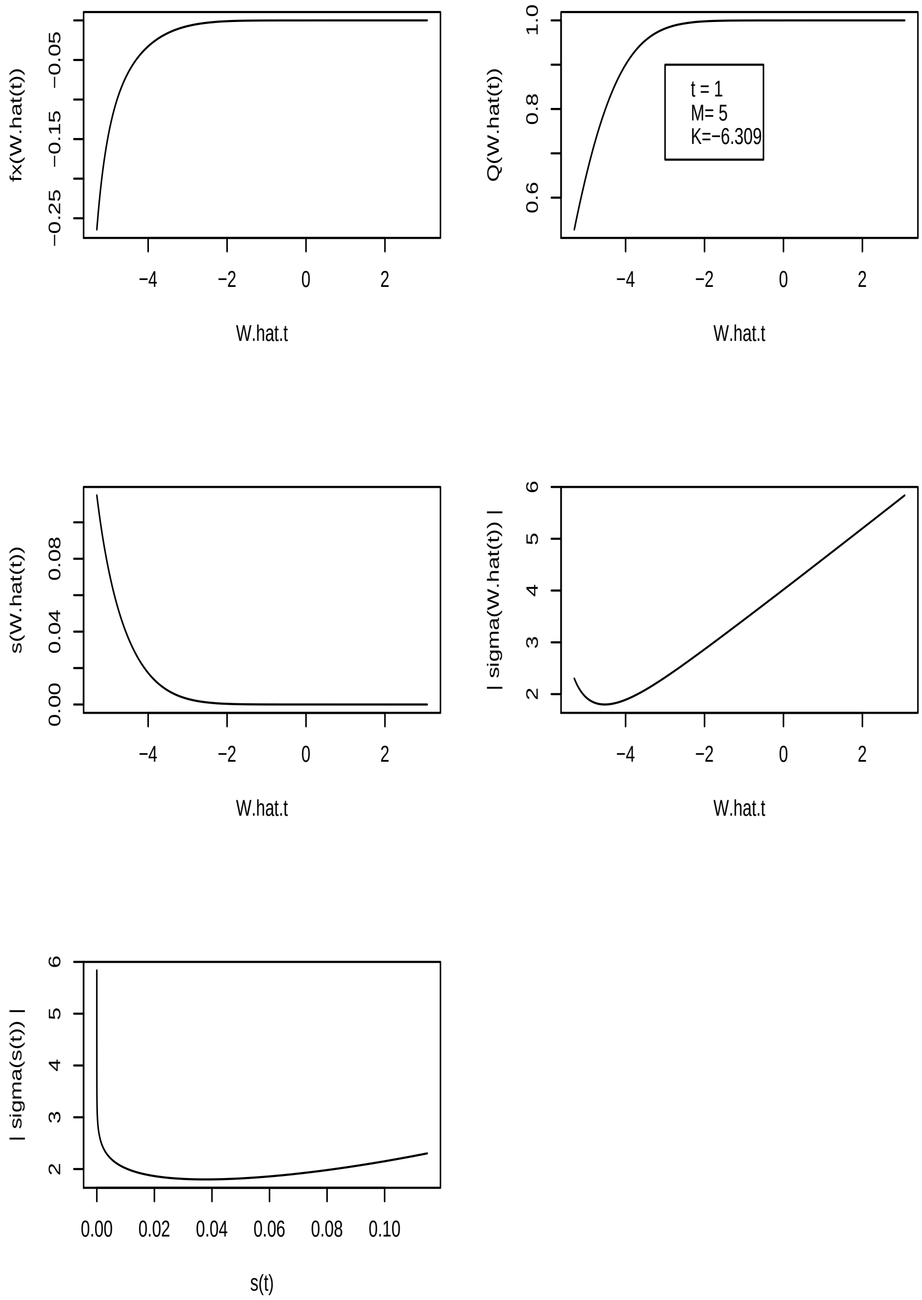

Figure 5.7: Spread vol for $t = 1$, $M = 5$

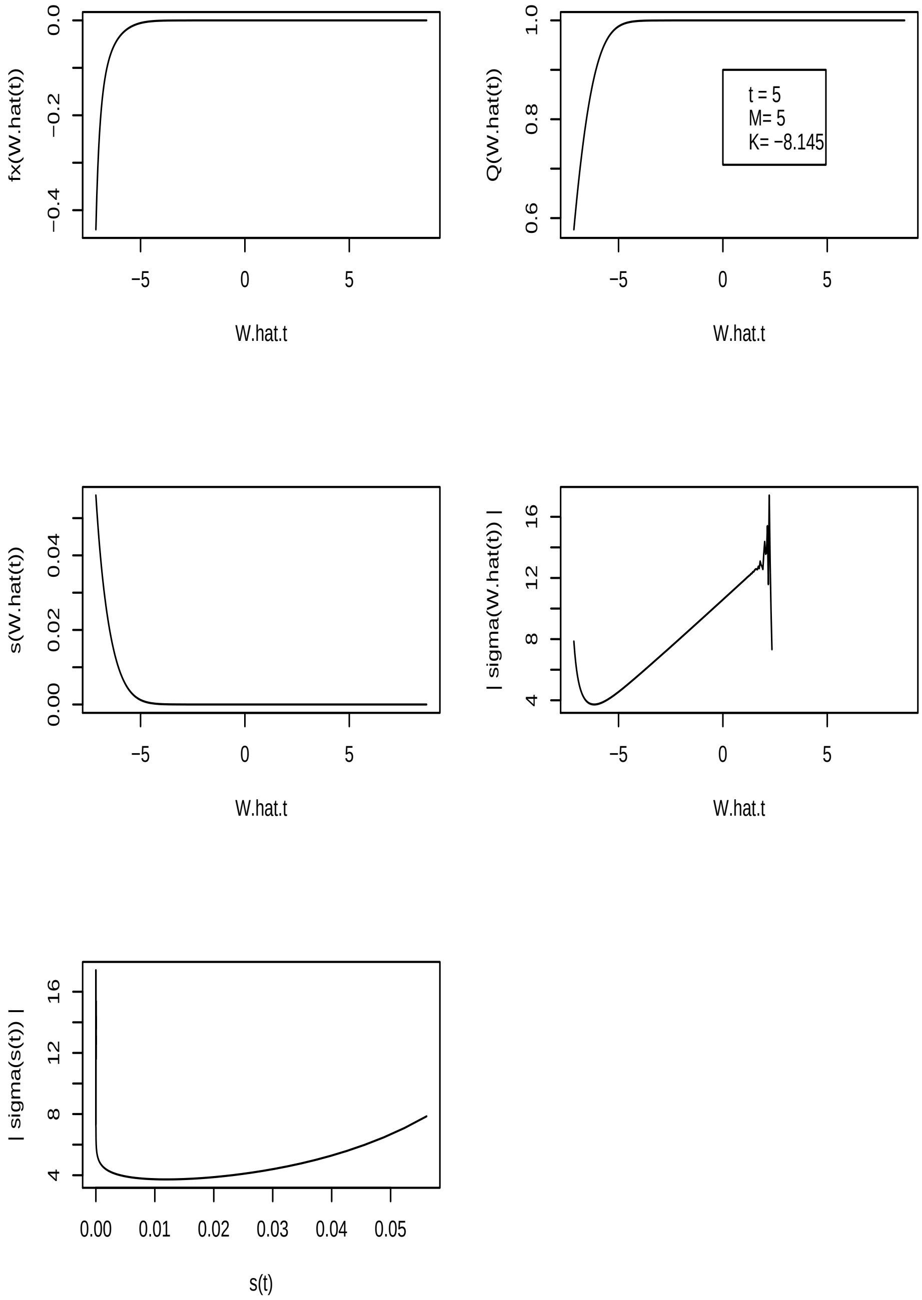

Figure 5.8: Spread vol for $t = 5$, $M = 5$

Chapter 6

Credit spread under the stochastic continuous time-change model

We apply the general continuous stochastic time-change model, introduced in Chapter 1, to credit-spread modeling. When using a positive starting value for the time change, $G_0 > 0$, the model includes incomplete information, as in the CreditGrades model. We derive an analytical formula for the survival probability and herewith for the credit spread. Under the additional assumption that the time change is absolutely continuous (i.e., has an integral representation) we can also derive credit-spread dynamics. Our tool is again Itô's rule.

6.1 Model framework

In Section 1.3 we introduced our stochastic time-change model: the first-passage process (1.14)

$$Y_t \;=\; \sigma W_{G_t} + \mu\, G_t \;,$$

where the time change G satisfies Assumption 1.20, and the default time

$$\tau \;=\; \inf\{s \geq 0 : Y_s < K\} \;.$$

We derive the survival probability under the information given by the asset value process Y (Definition 3.1): $\mathcal{F}_t^Y \;=\; \sigma\,(Y_s : s \leq t)$.
So $\{\tau > t\}$ is $\mathcal{F}_t^Y$-measurable and we know whether a default happens or not.

The next theorem states that the probability of surviving the time interval $[t, T]$ is given by the integral over the conditional time-change distribution. The conditional normality of Y yields that the integrand is given by the FPT expressions for Brownian motion with drift.

Theorem 6.1 *(Survival probability under continuous stochastic time change) Under survival up to t (i.e. $\tau > t$), the probability of surviving T is given by*

$$
\begin{aligned}
Q(t,T) &= 1 - \int_0^\infty \left[\Phi\left(\frac{K - Y_t}{\sigma\sqrt{z}} - \frac{\mu}{\sigma}\sqrt{z} \right) \right. \\
&\quad \left. + \, \mathrm{e}^{2\frac{\mu}{\sigma^2}(K-Y_t)} \, \Phi\left(\frac{K - Y_t}{\sigma\sqrt{z}} + \frac{\mu}{\sigma}\sqrt{z} \right) \right] \mathbb{P}(G_T - G_t \in \, \mathrm{d}z | \, \mathcal{F}_t^Y) \; .
\end{aligned}
$$

Proof. Under $\tau > t$, by Remark 3.2 and continuity of G, the survival probability is determined by

$$
\begin{aligned}
Q(t,T) &= \mathbb{E}\Big(\mathbb{I}_{\{\inf_{t \le s \le T}[\sigma W_{G_s} + \mu G_s] > K\}} \mid \mathcal{F}_t^Y \Big) \\
&\overset{G_t \text{ cont.}}{=} \mathbb{E}\Big(\mathbb{I}_{\{\inf_{G_t \le s \le G_T}\left[W_s + \frac{\mu}{\sigma}s\right] > \frac{K}{\sigma}\}} \mid \mathcal{F}_t^Y \Big) \\
&= \mathbb{E}\Big(\mathbb{I}_{\{\inf_{0 \le s \le G_T - G_t}\left[W_{G_t+s} + \frac{\mu}{\sigma}(G_t+s)\right] > \frac{K}{\sigma}\}} \mid \mathcal{F}_t^Y \Big) \; .
\end{aligned}
$$

We consider the Brownian motion with drift $\frac{\mu}{\sigma}$ and start at $\frac{Y_t}{\sigma}$

$$
\tilde{W}_s \; := \; \frac{Y_t}{\sigma} + W_s + \frac{\mu}{\sigma}s \; = \; W_{G_t} + W_s + \frac{\mu}{\sigma}(G_t + s) \; .
$$

Conditional on G_t, we have equivalence in distribution to

$$
\tilde{W}_s \; \overset{\mathcal{L}|G_t}{=} \; W_{G_t+s} + \frac{\mu}{\sigma}(G_t + s) \; .
$$

So we condition on the larger filtration $\mathcal{F}_t^Y \vee G_t \vee G_T$ and use that then $\tilde{W}$ only depends on the actual value Y_t and not on the whole information $\mathcal{F}_t^Y$ ('conditional Markov property'):

$$
\begin{aligned}
Q(t,T) &= \mathbb{E}\Big(\mathbb{P}_{\frac{Y_t}{\sigma}}\Big(\inf_{0 \le s \le G_T - G_t} \tilde{W}_s > \frac{K}{\sigma} \mid \mathcal{F}_t^Y \vee G_t \vee G_T \mid \mathcal{F}_t^Y \Big) \\
&= \mathbb{E}\Big(\mathbb{P}_{\frac{Y_t}{\sigma}}\Big(\inf_{0 \le s \le G_T - G_t} \tilde{W}_s > \frac{K}{\sigma} \mid Y_t \vee (G_T - G_t) \Big) \mid \mathcal{F}_t^Y \Big) \\
&= \mathbb{E}\left(1 - \left[\Phi\left(\frac{\frac{K}{\sigma} - \frac{Y_t}{\sigma}}{\sqrt{G_T - G_t}} - \frac{\mu}{\sigma}\sqrt{G_T - G_t} \right) \right. \right. \\
&\quad \left. \left. + \, \mathrm{e}^{2\frac{\mu}{\sigma}(\frac{K}{\sigma} - \frac{Y_t}{\sigma})} \Phi\left(\frac{\frac{K}{\sigma} - \frac{Y_t}{\sigma}}{\sqrt{G_T - G_t}} + \frac{\mu}{\sigma}\sqrt{G_T - G_t} \right) \right] \mid \mathcal{F}_t^Y \right) \; .
\end{aligned}
$$

In the last step we inserted the FPT result for Brownian motion with drift (formula (1.6)) which holds because $Y_t \ge K$. The expression claimed follows.

□

Remark 6.2 (Survival probability under $\mathcal{F}_t^Y$-measurable G_t)
By Remark 1.16, the continuous time change G_t is $\mathcal{F}_t^Y$-measurable since it is given through the quadratic variation of the underlying process, $\langle Y \rangle_t = \sigma^2 G_t$. Thus the survival probability can be determined by

$$Q(t,T) = 1 - \int_0^\infty \left[\Phi\left(\frac{K - Y_t}{\sigma\sqrt{z - G_t}} - \frac{\mu}{\sigma}\sqrt{z - G_t} \right) + e^{2\frac{\mu}{\sigma^2}(K - Y_t)} \Phi\left(\frac{K - Y_t}{\sigma\sqrt{z - G_t}} + \frac{\mu}{\sigma}\sqrt{z - G_t} \right) \right] \mathbb{P}(G_T \in \, \mathrm{d}z | \, \mathcal{F}_t^Y) \; .$$

6.2 Credit spread

We give the credit-spread formula for an underlying process with zero drift. The general formula is attained when inserting $Q(t,T)$ of Theorem 6.1 into the general spread formula (3.4).

Corollary 6.3 *(Credit spread under continuous stochastic time change)*
The credit spread for the underlying process $Y_t = W_{G_t}$ conditional on no default up to t is given by

$$s(t,T) = 2(1-R) \, \frac{\int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_T - G_t \in \, \mathrm{d}z \mid \mathcal{F}_t^Y)}{T - t - 2\int_t^T \int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_u - G_t \in \, \mathrm{d}z \mid \mathcal{F}_t^Y) \, \mathrm{d}u} \, ,$$

and for fixed time to maturity M when inserting $T = t + M$.

Proof. Insert the survival probability formula of Theorem 6.1 with $\mu = 0$ and $\sigma = 1$ into the spread formula (3.4). □

6.2.1 Simulations

We discretize the credit-spread formula for $Y_t = W_{G_t}$ on the equidistant time grid $t_0 = t, t_1 = t + \frac{T-t}{n}, \ldots, t_n = T$. Conditional on no default up to t, the discretization of the simplified spread formula was given in (4.3):

$$s(t_i, T) = \frac{(1-R)(1 - Q(t_i,T))}{\frac{T}{n}\sum_{k=i+1}^{n+1} Q(t_i, t_k)} \, .$$

Under the stochastic time-change model we have

$$Q(t_i, t_k) = 1 - 2 \sum_{y=0,\Delta,2\Delta,\ldots} \Phi\left(\frac{K - Y_{t_i}}{\sigma\sqrt{z}}\right) \mathbb{P}(G_{t_k} - G_{t_i} \in [y, y+\Delta) \mid \mathcal{F}_{t_i}^Y) \, .$$

Furthermore we apply the simple time change $G_t = \int_0^t B_s^2 \,\mathrm{d}s$ with discretization

$$\begin{aligned} & \mathbb{P}(G_{t_k} - G_{t_i} \in [y, y+\Delta) \mid \mathcal{F}_{t_i}^Y) \\ = \; & \mathbb{P}(G_{t_k} - G_{t_i} < y+\Delta \mid \mathcal{F}_{t_i}^Y) - \mathbb{P}(G_{t_k} - G_{t_i} < y \mid \mathcal{F}_{t_i}^Y) \end{aligned}$$

and default probability distribution (2.2). We apply the model parameters calibrated in Chapter 2 to $F(t) = 1 - \mathrm{e}^{-\lambda t}$ and to the average CCC-curve (speculative-grade), respectively, at $t = 5$ and $t = 10$. These parameters were stated in the last rows of Table 2.2 and Table 2.4. The simulations yield the default probability curves and credit spread curves in Figure 6.1. In Figure 6.2 we show the influence of a starting value $G_0 = g > 0$ when applying the time change $G_t = g + \int_0^t B_s^2 \,\mathrm{d}s$ and using the same calibration parameters as before. Of course, in applications each model must be calibrated, but here we just want to analyze the influence of g. Simulated default-probability curves and credit-spread curves for various levels $g = 0, 1, 10, 20, 50$ are shown in Figure 6.2. $g > 0$ leads to non-zero instantaneous credit spreads on the one hand, and to steeper curves on the other.

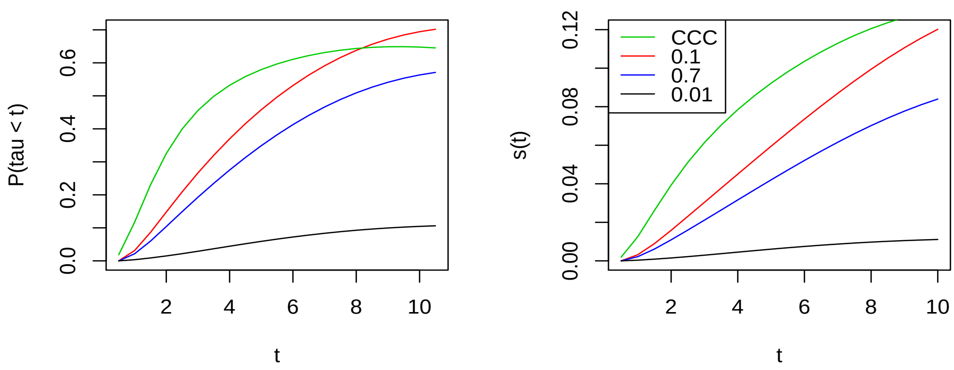

Figure 6.1: Default-probability curves and credit-spread curves under $G_t = \int_0^t B_s^2 \,\mathrm{d}s$ for model parameters calibrated to $F(t) = 1 - \mathrm{e}^{-\lambda t}$ at $F(t_5)$ and $F(t_{10})$, with $\lambda = 1\%$ (black), $\lambda = 7\%$ (blue), $\lambda = 10\%$ (red) and to the average CCC-curve (green).

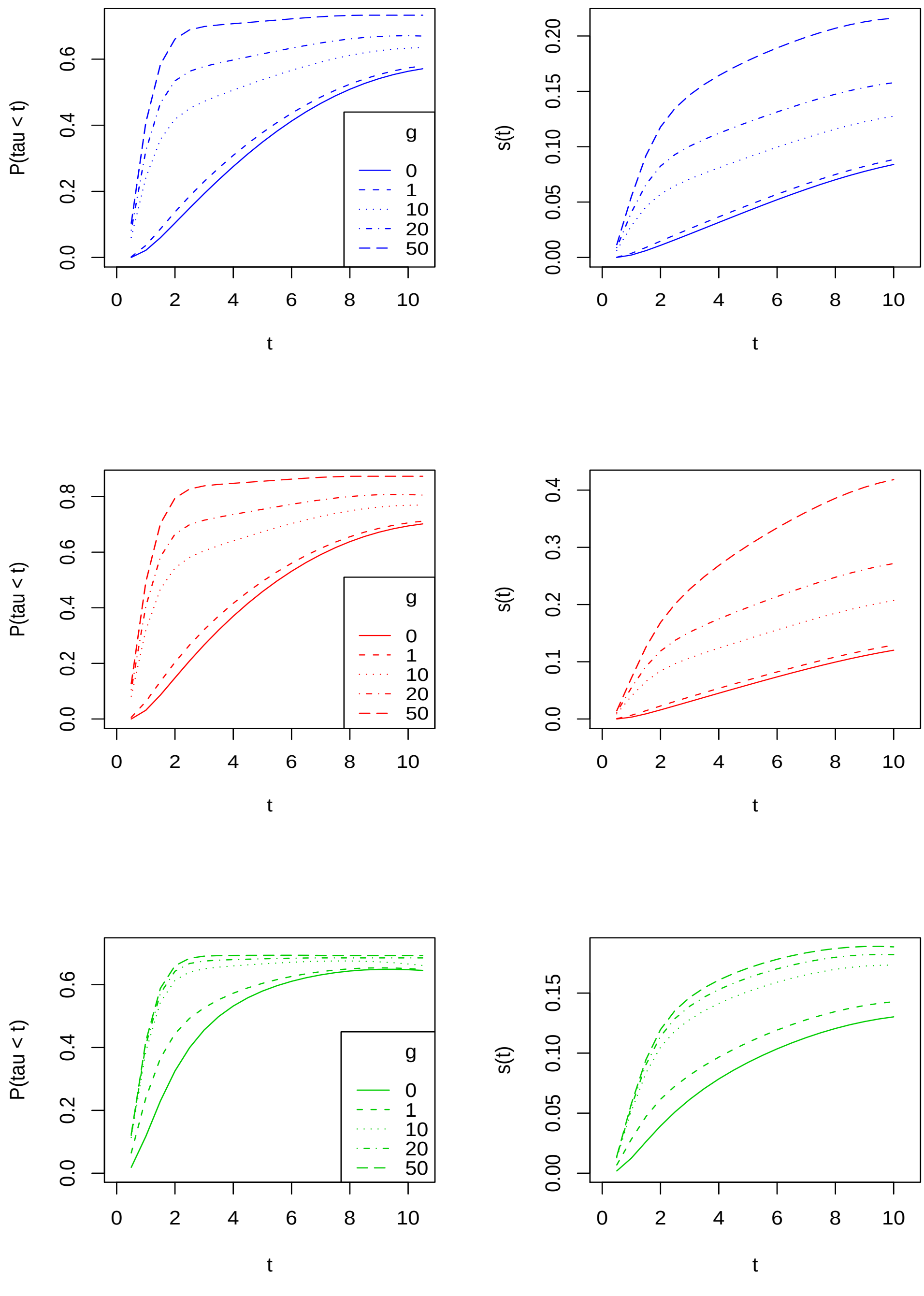

Figure 6.2: Default-probability curves and credit-spread curves under $G_t = g + \int_0^t B_s^2 \, \mathrm{d}s$ for various starting values g. The model parameters are calibrated for $g = 0$ to $F(t) = 1 - \mathrm{e}^{-\lambda t}$ at $F(t_5)$ and $F(t_{10})$, with $\lambda = 7\%$ (blue), $\lambda = 10\%$ (red) and to the average CCC-curve (green).

6.3 Credit-spread dynamics

The goal of this section is to determine credit-spread dynamics under the continuous stochastic time-change model $Y_t = W_{G_t}$, where W is specified below. As under the previous models we will therefore apply Itô's rule on the credit-spread formula of Corollary 6.3 where $T = t + M$ and M is fixed. We therefore abbreviate $s^M(t) = s(t, t+M)$. As in the previous chapters we introduce the spread function f that is a function in t and the state variable $y = Y_t$, i.e. $f(t,y) = s^M(t)$, and its partial derivatives f_t, f_y, f_{yy}. Then Itô's rule yields the following dynamics:

$$\mathrm{d}s_t^M \quad = \quad s_t^M \left[\mu^s(t, Y_t)\,\mathrm{d}t - \sigma^s(t, Y_t)\,\mathrm{d}Y_t\right] \ ,$$

with drift $\mu^s := \frac{f_t + \frac{1}{2} f_{yy}}{s^M}$ and $\sigma^s := -\frac{f_y}{s^M}$. σ^s is not the term that can be interpreted as credit-spread volatility. Therefore we desire dynamics in terms of some Browninan motion $\tilde{W}$:

$$\mathrm{d}s_t^M \quad = \quad s_t^M \left[\mu^s(t, Y_t)\,\mathrm{d}t - \sigma^{sv}(t, Y_t)\,\mathrm{d}\tilde{W}_t\right] \ ,$$

and then σ^{sv} can be interpreted as credit spread volatility.

In contrast to the Overbeck & Schmidt model the time change is now stochastic. As a consequence, Y_t does not in general have a representation in terms of a stochastic integral w.r.t. Brownian motion. Therefore we make the following assumption:

Assumption 6.4 *(Absolute continuity)*
We assume that there is a stochastic process (g_t) with $\mathbb{E}[\int_0^t g_s^2\,\mathrm{d}s] < \infty$ for all $t \geq 0$ such that the time change (G_t) is given by

$$G_t \quad = \quad g_0^2 + \int_0^t g_s^2\,\mathrm{d}s \ ,$$

that is by Definition 1.2, G is absolutely continuous.

By Lemma 1.14 the stochastic integral $\int_0^t g_s\,\mathrm{d}\tilde{W}_s$ is a continuous local martingale with quadratic variation

$$\left\langle \int_0^{\cdot} g_s\,\mathrm{d}\tilde{W}_s \right\rangle_t \quad = \quad \int_0^t g_s^2\,\mathrm{d}s \ = \ G_t - g_0^2 \ .$$

Furthermore there is a Brownian motion W, for which we define $Y_t = W_{G_t}$, such that

$$\mathrm{d}Y_t \ = \ g_t\,\mathrm{d}\tilde{W}_t \qquad \Longleftrightarrow \qquad Y_t \ = \ Y_0 + \int_0^t g_s\,\mathrm{d}\tilde{W}_s \ . \tag{6.1}$$

Then of course, Y is a continuous local martingale and has quadratic variation

$$\langle Y\rangle_t \quad = \quad G_t - g_0^2 \; .$$

Thus Assumption 6.4 and in particular (6.1) lead to the desired credit-spread dynamics:

$$\mathrm{d}s_t^M \quad = \quad s_t^M \,\left[\mu^s(t,Y_t)\,\mathrm{d}t - \sigma^s(t,Y_t) g_t\,\mathrm{d}\tilde{W}_t\right] \; ,$$

and in particular to the spread volatility $\sigma^{sv} = \sigma^s g$. In out next step we determine the partial derivatives for the spread dynamics:

Proposition 6.5 *(Partial derivatives under continuous stochastic time change) We abbreviate*

$$\begin{aligned}
\alpha_t(Y_t) &= M - 2\int_t^{t+M}\int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_u - G_t \in \,\mathrm{d}z \mid \mathcal{F}_t^Y) \\
\beta_t(Y_t) &= \int_t^{t+M}\int_0^\infty \frac{1}{\sqrt{z}}\varphi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_u - G_t \in \,\mathrm{d}z \mid \mathcal{F}_t^Y)\,\mathrm{d}u \; .
\end{aligned}$$

Then, under the stochastic time-change model $Y_t = W_{G_t}$, the partial derivatives are given by

$$\begin{aligned}
\frac{f_t^M}{2(1-R)} &= \frac{\int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(G_{t+M} - G_t \in \,\mathrm{d}z \mid \mathcal{F}_t^Y)}{\alpha_t(Y_t)} \\
&\quad - 2\frac{\int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_{t+M} - G_t \in \,\mathrm{d}z \mid \mathcal{F}_t^Y)}{\alpha_t(Y_t)^2} \\
&\quad + 2\frac{\left[\int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_{t+M} - G_t \in \,\mathrm{d}z \mid \mathcal{F}_t^Y)\right]^2}{\alpha_t(Y_t)^2} \\
&\quad + 2\frac{\int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_{t+M} - G_t \in \,\mathrm{d}z \mid \mathcal{F}_t^Y)}{\alpha_t(Y_t)^2} \\
&\quad \cdot \int_t^{t+M}\int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(G_u - G_t \in \,\mathrm{d}z \mid \mathcal{F}_t^Y)\,\mathrm{d}u \; , \\
\frac{f_y^M}{2(1-R)} &= -\frac{\int_0^\infty \frac{1}{\sqrt{z}}\varphi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_{t+M} - G_t \in \,\mathrm{d}z \mid \mathcal{F}_t^Y)}{\alpha_t(Y_t)} \\
&\quad - 2\frac{\int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_{t+M} - G_t \in \,\mathrm{d}z \mid \mathcal{F}_t^Y)}{\alpha_t(Y_t)^2}\,\beta_t(Y_t) \; ,
\end{aligned}$$

$$
\begin{aligned}
\frac{f^M_{yy}}{2(1-R)} \quad = \quad & - \frac{\int_0^\infty \frac{K-Y_t}{z^{3/2}} \varphi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_{t+M} - G_t \in \mathrm{d}z \mid \mathcal{F}^Y_t)}{\alpha_t(Y_t)} \\
& - 2 \frac{\int_0^\infty \frac{1}{\sqrt{z}} \varphi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_{t+M} - G_t \in \mathrm{d}z \mid \mathcal{F}^Y_t)}{\alpha_t(Y_t)^2} \beta_t(Y_t) \\
& + 2 \frac{\int_0^\infty \frac{1}{\sqrt{z}} \varphi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_{t+M} - G_t \in \mathrm{d}z \mid \mathcal{F}^Y_t)}{\alpha_t(Y_t)^2} \beta_t(Y_t) \\
& + 2 \frac{\int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_{t+M} - G_t \in \mathrm{d}z \mid \mathcal{F}^Y_t)}{\alpha_t(Y_t)^2} \\
& \cdot \int_t^{t+M} \int_0^\infty \frac{K-Y_t}{z^{3/2}} \varphi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_u - G_t \in \mathrm{d}z \mid \mathcal{F}^Y_t) \,\mathrm{d}u \\
& - 4 \frac{\int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_{t+M} - G_t \in \mathrm{d}z \mid \mathcal{F}^Y_t)}{\alpha_t(Y_t)^3} \beta_t(Y_t)^2 \ ,
\end{aligned}
$$

Proof. The partial derivative f^M_t: Apply (A.2) from the Appendix,

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_t^{t+M} f(s,t) \,\mathrm{d}s = -f(t,t) + f(t+M,t) + \int_t^{t+M} \frac{\mathrm{d}}{\mathrm{d}t} f(s,t) \,\mathrm{d}s \ ,
$$

in order to determine

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t} \int_t^{t+M} \int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_u - G_t \in \mathrm{d}z \mid \mathcal{F}^Y_t) \,\mathrm{d}u \\
= \quad & - \int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_t - G_t \in \mathrm{d}z \mid \mathcal{F}^Y_t) \\
& + \int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_{t+M} - G_t \in \mathrm{d}z \mid \mathcal{F}^Y_t) \\
& + \int_t^{t+M} \int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{P}(G_u - G_t \in \mathrm{d}z \mid \mathcal{F}^Y_t) \,\mathrm{d}u \\
= \quad & -1 + \int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_{t+M} - G_t \in \mathrm{d}z \mid \mathcal{F}^Y_t) \\
& + \int_t^{t+M} \int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{P}(G_u - G_t \in \mathrm{d}z \mid \mathcal{F}^Y_t) \,\mathrm{d}u \ .
\end{aligned}
$$

The partial derivative f^M_y: Note that

$$
\frac{\mathrm{d}}{\mathrm{d}Y_t} \mathbb{P}(G_u - G_t \in \mathrm{d}z \mid \mathcal{F}^Y_t) \quad = \quad 0 \quad \forall\, t \le u \le t+M \ .
$$

The partial derivative f^M_{yy}: Use that

$$
\frac{\mathrm{d}}{\mathrm{d}x} \, \varphi(x) \quad = \quad -x \cdot \varphi(x) \ .
$$

Furthermore note that $\beta_t(Y_t) = -\frac{\mathrm{d}}{\mathrm{d}Y_t}\alpha_t(Y_t)$. □

Theorem 6.6 *(Spread volatility under continuous stochastic time change) Under Assumption 6.4 the credit-spread volatility for the time change process $Y_t = W_{G_t}$ is given by*

$$\begin{aligned}\sigma^s \cdot g_t \quad = \quad & \frac{2(1-R)}{s_t^M}\, g_t \cdot \\ & \left\{ \frac{\int_0^\infty \frac{1}{\sqrt{z}}\varphi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_{t+M} - G_t \in \, \mathrm{d}z \mid \mathcal{F}_t^Y)}{\alpha_t(Y_t)} \right. \\ & \left. + \, 2\frac{\int_0^\infty \Phi\left(\frac{K-Y_t}{\sqrt{z}}\right) \mathbb{P}(G_{t+M} - G_t \in \, \mathrm{d}z \mid \mathcal{F}_t^Y)}{\alpha_t(Y_t)^2}\, \beta_t(Y_t) \right\} ,\end{aligned}$$

with $\alpha_t(Y_t)$ and $\beta_t(Y_t)$ as stated in Proposition 6.5.

Proof. Insert $\mathrm{d}Y_t = \sigma^s g_t \, \mathrm{d}\tilde{W}_t$ into the dynamics (6.1) with the partial derivatives from Proposition 6.5. □

6.4 First-to-default swap

We consider a basket of n credits and are interested in the first default of a credit within this basket. The first-to-default (FTD) time is given by

$$\tau^{[1]} \quad = \quad \min(\tau_1, \ldots, \tau_n)$$

and the FTD probability by

$$\mathbb{P}(\tau^{1st} \leq t) \quad = \quad 1 - \mathbb{P}(\tau_1 > t, \ldots, \tau_n > t) \; . \tag{6.2}$$

A first-to-default swap is a contract that offers protection against that first credit event. Therefore the protection buyer pays the FTD spread $s^{[1]}(t,T)$ to the protection seller until contract maturity T, but only as long as the first default has not happened. In case of default before maturity, the protection seller pays an amount as agreed to the protection buyer. We assume he pays $1 - R_{[1]}$, where $R_{[1]}$ is the recovery rate of the credit that is first defaulted.

6.4.1 First-to-default spread on two credits

In the case when the basket contains only two credits, we can derive an analytical formula for the FTD spread under our two-dimensional stochastic time-change model. The FTD time is given by $\tau^{1st} = \tau_1 \wedge \tau_2$ and the FTD probability by

$$\mathbb{P}(\tau^{1st} \leq t) \quad = \quad 1 - \mathbb{P}(\tau_1 > t, \tau_2 > t) \; .$$

When using the same time change for both credits the joint survival probability is given by Theorem 1.23, and when using different time changes it is given by Theorem 1.26. The conditional JSP corresponding to Theorem 1.23 is obtained when integrating over the conditional time-change density $\mathbb{P}(G_T - G_t \in \,\mathrm{d}z\,|\mathcal{F}_t^Y)$ instead of $\mathbb{P}_{G_t}(\,\mathrm{d}x)$ and substituting $x-g$ by z. For the recovery rates we assume $R^1 = R^2 \equiv R$. Then conditional on $\tau^{1st} > t$, the formula for the FTD spread is determined similar to the spread formula (3.2) and is given by

$$s^{1st}(t,T) \quad = \quad \frac{(1-R)\int_t^T D(t,u)\mathbb{P}(\tau^{1st} \in \,\mathrm{d}u \mid \mathcal{F}_t^Y)}{\int_t^T D(t,u)Q^{1st}(t,u)\,\mathrm{d}u}\,, \tag{6.3}$$

where $D(t,u)$ is the discount factor and

$$Q^{1st}(t,T) \quad = \quad \mathbb{P}\left(\tau^{1st} > T \mid \mathcal{F}_t^Y\right) \quad = \quad \mathbb{P}(\tau_1 > t, \tau_2 > t \mid \mathcal{F}_t^Y)\,,$$

the conditional survival probability.

6.4.2 First-to-default spread on n credits

When having a basket with n credits our multivariate model with Brownian independence (1.15) can be applied. The JSP in (6.2) and the conditional JSP needed for the FTD spread (6.3) can be derived analogously to the joint default probability of Corollary 1.22. The conditional JSP is given by

$$\begin{aligned} Q^{1st}(t,T) \quad &= \quad \mathbb{P}\,(\tau_1 > t,\ \ldots,\ \tau_m > t \mid \mathcal{F}_t^Y) \\ &= \quad \int_0^\infty \prod_{i=1}^m \Bigg[1 - \Phi\left(\frac{K_i - Y_0^i}{\sigma_i\sqrt{z}} - \frac{\mu_i}{\sigma_i}\sqrt{z}\right) \\ &\qquad - \mathrm{e}^{2\frac{\mu_i(K_i - Y_0^i)}{\sigma_i^2}}\,\Phi\left(\frac{K_i - Y_0^i}{\sigma_i\sqrt{z}} + \frac{\mu_i}{\sigma_i}\sqrt{z}\right)\Bigg]\mathbb{P}(G_T - G_t \in \,\mathrm{d}z|\,\mathcal{F}_t^Y)\,. \end{aligned}$$

Assuming $R^1 = \cdots = R^n \equiv R$ and $\tau^{1st} > t$, formula (6.3) then yields the FTD spread for the n-credits basket.

The next section gives examples for analytical conditional time-change densities that yield an explicit FTD spread. This makes calibration to market prices easy.

6.5 Explicit conditional time-change densities

In this section we want to give two examples for explicit conditional time-change densities $\mathbb{P}(G_T - G_t \in \,\mathrm{d}z \mid \mathcal{F}_t^Y)$, the *simple time change* and the *CIR-type time change*.

6.5.1 The simple time change

The simple time change was introduced at the end of Subsection 1.3.7:

$$G_t \quad = \quad \int_0^t B_u^2 \, \mathrm{d}u \; .$$

We determine the conditional time-change density $\mathbb{P}(G_T - G_t \in \mathrm{d}z | \; B_t)$ and show that it is equal to $\mathbb{P}(G_T - G_t \in \mathrm{d}z | \; \mathcal{F}_t^Y)$.

Theorem 6.7 *(Conditional density of the simple time change)*
The density of the simple time change increment conditional on the actual information $\mathcal{F}_t^Y$ is given by

$$\begin{aligned}
&\mathbb{P}(G_T - G_t \in \mathrm{d}y \mid \mathcal{F}_t^Y) \\
&= \frac{\mathrm{d}y}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{B_t^2}{2}\right)^k \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k+j)}{\Gamma(\frac{1}{2}+k)j!} \; y^{-1+\frac{k}{2}} \\
&\cdot \exp\left\{ -\frac{\left((\frac{1}{2}+2k+2j)(T-t) + \frac{B_t^2}{2} \right)^2}{2y} \right\} \\
&\cdot \sum_{0 \le l \le \frac{k+1}{2}} (-)^{j+l} 2^{\frac{1}{2}+k-l} \frac{(k+1)!}{l!(k+1-2l)!} \left(\frac{(\frac{1}{2}+2k+2j)(T-t) + \frac{B_t^2}{2}}{\sqrt{y}} \right)^{k+1-2l} .
\end{aligned}$$

Proof. First we determine $\mathbb{P}(G_T - G_t \in \mathrm{d}z | \; B_t)$:
For this we insert the time change, modify the integral bounds and substitute the original Brownian motion B by the Brownian motion $\tilde{B}_s := B_{t+s}$ starting in $\tilde{B}_0 = B_t$:

$$\mathbb{P}\left(\int_t^T B_s^2 \, \mathrm{d}s \in \mathrm{d}y \mid B_t \right) \quad = \quad \mathbb{P}\left(\int_0^{T-t} \tilde{B}_s^2 \, \mathrm{d}s \in \mathrm{d}y \mid \tilde{B}_0 \right) \; . \qquad (6.4)$$

The last probability is given in BORODIN, SALMINEN (2002) (pages 168, 642). With their notation it is equal to

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\tilde{B}_0^2}{2} \right)^k c_y\Big(k, \frac{1}{2}+k, T-t, \frac{\tilde{B}_0^2}{2} + k(T-t)\Big) \, \mathrm{d}y \; ,$$

which is equal to the stated expression. Now we show that $\mathbb{P}(G_T - G_t \in \mathrm{d}z | \; \mathcal{F}_t^Y) = \mathbb{P}(G_T - G_t \in \mathrm{d}z | \; B_t)$:
For this we note that the obtained expression only depends on B_t^2, that is

$$\mathbb{P}(G_T - G_t \in \mathrm{d}y \mid B_t) \quad = \quad \mathbb{P}(G_T - G_t \in \mathrm{d}y \mid B_t^2) \; .$$

By Remark 1.16 the continuous time change is equal to the quadratic variation of the asset-value process: $G_t = \langle Y \rangle_t$. Thus G_t is $\mathcal{F}_t^Y$-measurable and this is also true for $B_t^2 = \frac{\mathrm{d}}{\mathrm{d}t} G_t$. Hence

$$\begin{aligned}
\mathbb{P}\left(G_T - G_t \in \mathrm{d}y \mid \mathcal{F}_t^Y\right) &= \mathbb{P}\left(\int_t^T B_u^2 \, \mathrm{d}u \in \mathrm{d}y \mid \sigma\left(Y_s,\ G_s,\ B_s^2 : s \le t\right)\right) \\
&= \mathbb{P}\left(\int_t^T B_u^2 \, \mathrm{d}u \in \mathrm{d}y \mid B_t^2\right) ,
\end{aligned}$$

since the integral is independent of the Brownian path W and the time change G_s for $s \le t$, and a Markov process w.r.t. $\sigma\left(\ B_s^2 : s \le t\right)$. $\square$

6.5.2 The CIR-type time change

Now we consider the CIR-type time change from Def. 1.28 with $g = 0$:

$$\hat{G}_t = \hat{\sigma}^2 \int_0^t \mathrm{e}^{\kappa r} \left[\frac{\sqrt{g_0}}{\hat{\sigma}} + B_{\int_0^r \mathrm{e}^{\kappa s}\, \mathrm{d}s}\right]^2 \mathrm{d}r \stackrel{\mathcal{L}}{=} \int_0^t \mathrm{e}^{2\kappa r}\, g_r^{CIR}\, \mathrm{d}r .$$

$\left(g_r^{CIR}\right)$ is a positive process and the solution of the Cox-Ingersoll-Ross SDE, given in (1.18). The next theorem determines the conditional density.

Theorem 6.8 *(Conditional density of the CIR-type time change)*
Let $b = \frac{1}{\hat{\sigma}}\, \mathrm{e}^{\frac{1}{2}\kappa t} \sqrt{g_t^{CIR}}$. The conditional density of the time-change increment is given by

$$\begin{aligned}
&\mathbb{P}(\hat{G}_T - \hat{G}_t \in \mathrm{d}y \mid \mathcal{F}_t^Y) \\
&= \frac{\mathrm{d}y}{\hat{\sigma}^2\sqrt{2\pi}} \sum_{k=0}^{\infty} \left(\frac{b^2}{2}\right)^k \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k+j)}{\Gamma(\frac{1}{2}+k) j!}\, y^{-(1+\frac{k}{2})} \\
&\cdot \exp\left\{ -\frac{\left((\frac{1}{2}+2k+2j)\frac{1}{\kappa}(\mathrm{e}^{\kappa T}-\mathrm{e}^{\kappa t}) + \frac{b^2}{2}\right)^2}{2y} \right\} \\
&\sum_{0 \le l \le \frac{k+1}{2}} (-)^{j+l} 2^{\frac{1}{2}+k-l} \frac{(k+1)}{l!(k+1-2l)!} \left(\frac{(\frac{1}{2}+2k+2j)\frac{1}{\kappa}(\mathrm{e}^{\kappa T}-\mathrm{e}^{\kappa t}) + \frac{b^2}{2}}{\sqrt{y}} \right)^{k+1-2l} .
\end{aligned}$$

Proof. We determine the density conditional on g_t^{CIR} and find that it only depends on $(g_t^{CIR})^2$. By the same arguments as for the simple time change this is equal to the density conditional on $\mathcal{F}_t^Y$:

$$\begin{aligned}
\mathbb{P}(\hat{G}_T - \hat{G}_t \in \mathrm{d}y \mid \mathcal{F}_t^Y) &= \mathbb{P}(\hat{G}_T - \hat{G}_t \in \mathrm{d}y \mid (g_t^{CIR})^2) \\
&= \mathbb{P}(\hat{G}_T - \hat{G}_t \in \mathrm{d}y \mid g_t^{CIR}) .
\end{aligned}$$

Now we determine $\mathbb{P}(\hat{G}_T - \hat{G}_t \in \mathrm{d}y \mid g_t^{CIR})$:
It is derived analogously to the proof for the unconditional density of Theorem 1.29. We set

$$X_t \;=\; \hat{\sigma}\,\mathrm{e}^{\frac{1}{2}\kappa t}\left[\frac{\sqrt{g_0}}{\hat{\sigma}} + B_{\int_0^t \mathrm{e}^{\kappa s}\,\mathrm{d}s}\right] \;\overset{\mathcal{L}}{=}\; \mathrm{e}^{\kappa t}\sqrt{g_t^{CIR}}\;.$$

Then the time-change increment is given by

$$\hat{G}_T - \hat{G}_t \;=\; \int_t^T X_r^2\,\mathrm{d}r\;.$$

For a given value g_t^{CIR} we have $X_t = \mathrm{e}^{\kappa t}\sqrt{g_t^{CIR}}$ and herewith

$$b \;=\; \frac{X_t}{\hat{\sigma}}\,\mathrm{e}^{-\frac{1}{2}\kappa t} \;=\; \frac{\sqrt{g_0}}{\hat{\sigma}} + B_{\frac{1}{\kappa}(\mathrm{e}^{\kappa t}-1)}\;.$$

Then the conditional density of the time-change increment is given by

$$\begin{aligned}
&\mathbb{P}(\hat{G}_T - \hat{G}_t \in \mathrm{d}y \mid g_t^{CIR})\\
=\;&\mathbb{P}\left(\hat{\sigma}^2\int_t^T \mathrm{e}^{\kappa r}\left[\frac{\sqrt{g_0}}{\hat{\sigma}} + B_{\int_0^r \mathrm{e}^{\kappa s}\,\mathrm{d}s}\right]^2 \mathrm{d}r \in \mathrm{d}y \mid X_t\right)\;.
\end{aligned}$$

Here we substitute $w = \int_0^r \mathrm{e}^{\kappa s}\,\mathrm{d}s = \frac{1}{\kappa}(\mathrm{e}^{\kappa r}-1)$, change the integral bounds, and introduce the Brownian motion $\tilde{B}_w = \frac{\sqrt{g_0}}{\hat{\sigma}} + B_{w+\frac{1}{\kappa}(\mathrm{e}^{\kappa t}-1)}$ with start in $\tilde{B}_0 = b$, which leads to

$$\begin{aligned}
=\;&\mathbb{P}\left(\hat{\sigma}^2\int_{\frac{1}{\kappa}(\mathrm{e}^{\kappa t}-1)}^{\frac{1}{\kappa}(\mathrm{e}^{\kappa T}-1)}\left[\frac{\sqrt{g_0}}{\hat{\sigma}} + B_w\right]^2 \mathrm{d}w \in \mathrm{d}y \mid X_t\right)\\
=\;&\mathbb{P}\left(\hat{\sigma}^2\int_0^{\frac{1}{\kappa}(\mathrm{e}^{\kappa T}-\mathrm{e}^{\kappa t})}\left[\frac{\sqrt{g_0}}{\hat{\sigma}} + B_{w+\frac{1}{\kappa}(\mathrm{e}^{\kappa t}-1)}\right]^2 \mathrm{d}w \in \mathrm{d}y \mid X_t\right)\\
=\;&\mathbb{P}\left(\hat{\sigma}^2\int_0^{\frac{1}{\kappa}(\mathrm{e}^{\kappa T}-\mathrm{e}^{\kappa t})}\tilde{B}_w^2\,\mathrm{d}w \in \mathrm{d}y \mid \tilde{B}_0 = b\right)\;.
\end{aligned}$$

The last probability can be computed analogously to the one in the proof for the simple time change, equation (6.4). With the notation of Borodin, Salminen (2002) (p. 168, 642) it is equal to

$$= \;\frac{1}{\hat{\sigma}^2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{b^2}{2}\right)^k c_y\Big(k,\frac{1}{2}+k,\frac{1}{\kappa}(\mathrm{e}^{\kappa T}-\mathrm{e}^{\kappa t}),\frac{b^2}{2}+\frac{k}{\kappa}(\mathrm{e}^{\kappa T}-\mathrm{e}^{\kappa t})\Big)\,\mathrm{d}y\;,$$

which leads directly to the stated expression. $\square$

6.6 Conclusion

Under our stochastic time-change model with an absolutely continuous business clock we derived an analytical formula for credit spread and credit-spread dynamics. For the simple time change (that was deeply studied in Chapter 2) and the CIR-type time change we explicitly derived the conditional time-change density which is needed for the credit-spread modeling. There are various applications for our (still very general) credit-spread model: For any credit product the underlying credit-spread dynamics should be modeled, in addition to adapting the actual credit-spread curves. In particular the credit-spread volatility is important because it indicates the risk of changes in the credit-spread curve. This is especially necessary for contracts with long time to maturities and credit-spread sensitive products such as the *credit-spread option*, the *variance swap* (on credit-spread) and credit products with *leverage effects*. Our model has three advantages: It yields an analytical credit-spread formula that can be easily calibrated to a given credit-spread curve. A time change with arbitrarily many degrees of freedom can be chosen and fitted to the desired dynamics, especially the credit-spread volatility. Last but not least the time change, and in the two-dimensional case in addition a Brownian correlation parameter, can be used to insert a dependency structure into a multi-credit product. That dependency can be explained as business time or amount of information flow or simply as asset dependence. The multi-dimensional models also yield analytical default probabilities and thus 'joint credit spreads' such as the FTD spread.

One question remains that cannot be answered in general: How do the credit-spread dynamics look like?
In our empirical analysis, in the second part of Chapter 3, we have seen that this has to be studied separately for each credit underlying.

Chapter 7

Outlook: applications to option pricing

In option pricing it is important that the underlying model displays historical and implied volatility features: Historical data show price jumps, *leverage effects* (i.e. past returns are negatively correlated with future volatility), and *volatility clustering* (high/small absolute returns are followed by high/small absolute returns). The model should be able to produce a *volatility surface*, a *smile* (symmetric) or *scew* (asymmetric) along the *moneyness* axis, and a term structure along the time axis. *Stochastic volatility models* with (and without) jump components come into question. The *Heston model*, a generalized geometric Brownian motion with a variance driven by a *Cox-Ingersoll-Ross process*, is well-known – especially in *foreign-exchange option pricing*. It does not include a jump part. Correlating the Brownian components of spot and variance makes possible a volatility skew, smile and leverage effect. The Heston model is of value because Heston (1993) derived a semi-analytical pricing formula for the plain-vanilla call. So far no closed formulas are available for *exotic options*, especially not for *barrier options*.

Examples for stochastic volatility models that include jumps are the *Bates model* (generalizing the Heston model by including a compound Poisson component into the spot process) and the *Barndorff-Nielsen&Shephard model* (having an Ornstein-Uhlenbeck variance process and correlated jumps in spot and variance). Both models are able to show all important volatility features. But – to the best of our knowledge – no closed pricing formulas are available for any kind of option.

As literature for option pricing we refer to Hunt & Kennedy (2004) in general and to Cont & Tankov (2004) and Schoutens (2003) in particular with regard to Lévy processes.

In this chapter we are going to apply our stochastic time-change model to option pricing. The time-change model is a stochastic volatility model that inserts stochastic volatility through the time-change process.[1] For exemplification, we determine the stochastic time-change model that is equivalent (in distribution) to the Heston model. Stochastic volatility models lead to an incomplete market because volatility is not (directly) tradable. For that reason the pricing measure is not unique and we decide in favor of the minimal martingale measure. Under our general model, allowing for correlation between spot and time change, we derive a closed formula for the price of a European call. Under no correlation and zero interest rates we furthermore derive pricing formulas for several barrier options, using our first-passage time (FPT) results of Chapter 1. The time change can be used and adapted to produce the desired volatility features. Certainly, because of continuity, price jumps are not possible under our model.

7.1 Heston model

Heston (1993) introduced a stochastic volatility model where the spot process is given by a generalized geometric Brownian motion and the variance by a Cox-Ingersoll-Ross (CIR) process (see (1.18)):

$$\begin{aligned} \mathrm{d}S_t &= S_t[\mu\,\mathrm{d}t + \sqrt{g_t^{CIR}}\,\mathrm{d}W_t] \\ \mathrm{d}g_t^{CIR} &= \kappa(\theta - g_t^{CIR})\,\mathrm{d}t + \hat{\sigma}\sqrt{g_t^{CIR}}\,\mathrm{d}B_t\ . \end{aligned}$$

The Brownian motions W and B are correlated with correlation parameter ρ. Furthermore, throughout this chapter r denotes a constant interest rate.

7.1.1 Revisited: original Heston call price

Heston derived a semi-analytical solution for the call price by solving a partial differential equation. For this he assumed that the price of volatility risk, that is necessary for the pricing measure, is linear in v, that is, $\lambda(S,v,t) = \lambda v$. Defining the constants, $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, $a = \kappa\theta$, $b_1 = \kappa + \lambda - \rho\sigma$, $b_2 = \kappa + \lambda$, he obtained the following call-price formula:

$$C(S,v,t) \quad = \quad SP_1(\ln(S),v,t) - K\,\mathrm{e}^{-r(T-t)}\,P_2(\ln(S),v,t)\ ,$$

[1] Geman, Madan & Yor (2000), Schoutens (2003), and Carr & Wu (2003) introduced time-changed Lévy process to option pricing. The first two considered subordinators as time changes and Carr & Wu assumed that the time change is locally deterministic, i.e. has an integral representation and thus belongs to our general class of continuous time changes.

where for $j = 1, 2$,

$$
\begin{aligned}
P_j(x,v,T) &= \frac{1}{2} + \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Re}\left[\frac{e^{-i\phi\ln(K)}\varphi_j(x,v,T-t;\phi)}{i\phi}\right] d\phi \ , \\
\varphi_j(x,v,\tau;\phi) &= \mathrm{e}^{C_j(\tau;\phi)+D_j(\tau;\phi)v+i\phi x} \ , \\
C_j(\tau;\phi) &= r\phi i\tau + \frac{a}{\hat{\sigma}^2}\Big\{(b_j - \rho\sigma\phi i + d_j)\tau - 2\ln\Big[\frac{1-g_j\,\mathrm{e}^{d_j\tau}}{1-g_j}\Big]\Big\} \ , \\
D_j(\tau;\phi) &= \frac{b_j - \rho\hat{\sigma}a\phi i + d_j}{\hat{\sigma}^2}\Big[\frac{1-\mathrm{e}^{d_j\tau}}{1-g_j\,\mathrm{e}^{d\tau}}\Big] \ , \\
g_j &= \frac{b_j - \rho\hat{\sigma}\phi i + d_j}{b_j - \rho\hat{\sigma}\phi i - d_j}\Big[\frac{1-\mathrm{e}^{d_j\tau}}{1-g_j\,\mathrm{e}^{d\tau}}\Big] \ , \\
d_j &= \sqrt{(\rho\hat{\sigma}\phi i - b_j)^2 - \hat{\sigma}^2(2u_j\phi i - \phi^2)} \ .
\end{aligned}
$$

7.1.2 Revisited: analytical Heston call price

The filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ is endowed with the true probability measure $\mathbb{P}$ and a filtration $\mathbb{F}$ holding the information about the Brownian motions W and B. We determine the *minimal martingale measure* $\mathbb{Q}$ w.r.t. $\mathbb{F}$ in order to derive *plain-vanilla* prices, following the lines of Bingham & Kiesel (1998), first edition, Chapter 7:

Let $S_t = S_0 + M_t + A_t$, $t \in [0,T]$ and $T < \infty$, be a continuous $\mathbb{P}$-semimartingale[2] with square-integrable $\mathbb{P}$-martingale M, finite variation process $A_t = \int_0^t \alpha_u \,\mathrm{d}\langle M\rangle_u$, and predictable process α. S will be the discounted price process that shall become a martingale under the new measure $\mathbb{Q}$.

Definition 7.1 (Minimal martingale measure)
A martingale measure $\mathbb{Q}$ is called *minimal* if any square-integrable $\mathbb{P}$-martingale which is orthogonal to M remains a martingale under $\mathbb{Q}$.

Theorem 7.2 *(Bingham and Kiesel)*
The minimal martingale measure $\mathbb{Q}$ is unique. It exists if and only if the likelihood process

$$
L_t \;=\; \exp\left\{ -\int_0^t \alpha_u \,\mathrm{d}M_u - \frac{1}{2}\int_0^t \alpha_u^2 \,\mathrm{d}\langle M\rangle_u \right\}
$$

is a square-integrable martingale under $\mathbb{P}$. In that case $\mathbb{Q}$ is given by the Girsanov density $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = L_T$.

[2] An adapted process $(X, \mathbb{F})$ is said to be a local semimartingale if it has the decomposition $X_t = M_t + A_t$, where M is a local martingale and A is càdlàg, with path of finite variation on compacts.

Then in particular

$$\mathbb{E}_{\mathbb{Q}}(S_T|\mathcal{F}_t) \quad = \quad \frac{\mathbb{E}_{\mathbb{P}}(S_T L_T|\mathcal{F}_t)}{L_t} \ .$$

Back to the Heston model: In KAMMER (2002) we derived an analytical pricing formula for the European call under independent Brownian motions W and B, and the information given by

$$\mathcal{F}_t = \sigma((B_u)_{u\geq 0}, W_s : s \leq t) \ ,$$

containing all the information about the Brownian path B. The proof uses the fact that the spot process has a closed-form solution (due to the closed-form solution of the CIR variance; see (1.19)):

$$S_t \quad = \quad S_0 \exp\left\{\mu t - \frac{1}{2}\int_0^t g_s^{CIR}\,\mathrm{d}s + \int_0^t \sqrt{g_s^{CIR}}\,\mathrm{d}W_s\right\} \ . \tag{7.1}$$

The call price is determined under the minimal martingale measure and the following likelihood process is derived for changing measures:

$$L_t \quad = \quad \exp\left\{ - \int_0^t \frac{\mu - r}{\sqrt{g_u^{CIR}}}\,\mathrm{d}W_u - \frac{1}{2}\int_0^t \frac{(\mu-r)^2}{g_u^{CIR}}\,\mathrm{d}u \right\} \ . \tag{7.2}$$

Remember that $g_t^{CIR} > 0$. The price for the European call with payoff $C_T = (S_t - K)_+$ under the information $\mathcal{F}_t$ is then given by

$$C_t = S_t\Phi(d_+) - K\,\mathrm{e}^{-r(T-t)}\,\Phi(d_-) \ , \tag{7.3}$$

where

$$\begin{aligned}
d_+ &= \sqrt{(T-t)}\sqrt{\bar{v} - 2(\mu - r) + (\mu - r)^2\bar{V}} + d \ , \\
d_- &= \sqrt{(T-t)}(\mu - r)\sqrt{\bar{V}} + d \ , \\
d &= \frac{\ln(\frac{S_t}{K}) + (\mu - \frac{1}{2}\bar{v})(T-t)}{\sqrt{(T-t)\bar{v}}} \ , \\
\bar{v} &= \frac{1}{T-t}\int_t^T g_s^{CIR}\,\mathrm{d}s \ , \\
\bar{V} &= \frac{1}{T-t}\int_t^T \frac{1}{g_s^{CIR}}\,\mathrm{d}s \ .
\end{aligned}$$

The risk-neutral spot dynamics and thus the call price depend on μ. Choosing μ gives us one degree of freedom, and of course μ can be set equal to r.

Now we show that the Heston model can be seen as a specific time-change model. Consider the Heston spot process (7.1) and the time change given by the integrated CIR-process

$$G_t^{CIR} \;=\; \int_0^t g_s^{CIR}\,\mathrm{d}s\ .$$

Since $\int_0^t \sqrt{g_s^{CIR}}\,\mathrm{d}W_s$ is an Itô integral and as such a continuous local martingale the spot process is equivalent in distribution to (cf. Lemma 1.14)

$$S_t \;\overset{\mathcal{L}}{=}\; S_0 \exp\left\{\mu t - \frac{1}{2}G_t^{CIR} + W_{G_t^{CIR}}\right\}$$

– a special time-change model.

7.2 Stochastic time-change model

We consider the general *exponential time-change model*:

$$S_t \;=\; S_0 \exp\left\{\mu t - \frac{1}{2}G_t + W_{G_t}\right\}\ , \tag{7.4}$$

and the *exponent process*

$$Y_t \;=\; \ln(\tfrac{S_t}{S_0}) \;=\; \mu t - \frac{1}{2}G_t + W_{G_t}\ , \tag{7.5}$$

where the time change has an integral representation

$$G_t \;=\; \int_0^t g_s\,\mathrm{d}s\ , \quad g_s > 0 \;\; \forall s \;\; \text{a.s.}\ ,$$

that is dependent on a Brownian motion B and satisfies Assumption 1.20. But for European option prices we allow for correlation between W and B. Note that $G_0 > 0$ enables a price jump at the very beginning (but only there). We derive prices under the filtration given by the underlying process:

$$\mathcal{F}_t^Y \;=\; \sigma(Y_s : s \le t) \;=\; \sigma(S_s : s \le t)\ ,$$

and hereto also consider the larger filtration containing the whole path B:

$$\mathcal{F}_t^Y \vee B \;=\; \sigma\left((B_u)_{u\ge 0}, Y_s : s \le t\right) \;=\; \sigma\left((g_u)_{u\ge 0}, Y_s : s \le t\right)\ .$$

7.2.1 European call

We determine the likelihood process to change measures analogously to (7.2) (cf. Kammer (2002)) by using Theorem 7.2. There is a Brownian motion

$\hat{B}$ independent of B (see (1.3), such that $W_t = \rho B_t + \sqrt{1-\rho^2}\hat{B}_t$ for all t. Then with (7.4) the discounted spot process has the integral representation

$$\begin{aligned}\tilde{S}_t = & \ \mathrm{e}^{-rt} S_t \\ = & \ S_0 + (\mu - r)\int_0^t S_u \,\mathrm{d}u + \rho \int_0^t S_u \sqrt{g_u}\,\mathrm{d}B_u + \sqrt{1-\rho^2}\int_0^t S_u\sqrt{g_u}\,\mathrm{d}\hat{B}_u\end{aligned}$$

and martingale part $M_t = \rho \int_0^t S_u\sqrt{g_u}\,\mathrm{d}B_u + \sqrt{1-\rho^2}\int_0^t S_u\sqrt{g_u}\,\mathrm{d}\hat{B}_u$ (under $\mathbb{F}^Y$). Setting $\alpha_t = \frac{\mu - r}{S_t g_t}$ and applying Theorem 7.2 yields the likelihood process:

Corollary 7.3 *(Likelihood process – minimal martingale measure)*
The likelihood process for the general time-change model, for a change from $\mathbb{P}$ to the minimal martingale measure $\mathbb{Q}$ under the filtration $\mathcal{F}_t^Y$, is given by

$$L_t = \exp\left\{ -\int_0^t \frac{\mu - r}{\sqrt{g_u}}\,\mathrm{d}W_u - \frac{1}{2}\int_0^t \frac{(\mu-r)^2}{g_u}\,\mathrm{d}u \right\}.$$

With the likelihood process pricing formulas for plain-vanilla options can be obtained. We give the formula for the European call, and to shorten the expression we assume $\mu = r$.

Theorem 7.4 *(European call under the stochastic time-change model)*
Let W and B be correlated Brownian motions with correlation parameter ρ and assume $\mu = r$. The price for the European call with strike K and maturity T under the information $\mathcal{F}_t^Y$, by changing measures with the likelihood process L_t of Corollary 7.3, is given by

$$C_t = \int_\Omega \left[S_t\,\mathrm{e}^{-\rho^2 \frac{1}{2(T-t)}(G_T - G_t) + \rho\int_t^T \sqrt{g_u}\,\mathrm{d}B_u}\,\Phi(d_+) - K\,\mathrm{e}^{-r(T-t)}\,\Phi(d_-)\right]\mathrm{d}\mathbb{P}_{\mathcal{F}_t^Y}$$

with

$$\begin{aligned} d_+ &= \sqrt{(1-\rho^2)(G_T - G_t)} + d_- \ , \\ d_- &= \frac{\ln(\frac{S_t}{K}) + r(T-t) - \frac{1}{2}(G_T - G_t) + \rho\int_t^T \sqrt{g_u}\,\mathrm{d}B_u}{\sqrt{(1-\rho^2)(G_T - G_t)}} \ . \end{aligned}$$

Proof. We apply the likelihood process of Corollary 7.3 to change measures. Using the *tower property* we consider the expectation w.r.t. the larger filtration $\mathcal{F}_t^Y \vee B$

$$\begin{aligned}\mathrm{e}^{r(T-t)}\,C_t &= \mathbb{E}_{\mathbb{Q}}\left[(S_T - K)^+ | \mathcal{F}_t^Y\right] = \mathbb{E}_{\mathbb{P}}\left[\frac{L_T}{L_t}(S_T-K)^+|\mathcal{F}_t^Y\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[\frac{L_T}{L_t}(S_T-K)^+|\mathcal{F}_t^Y \vee B\right]|\mathcal{F}_t^Y\right] ,\end{aligned}$$

that is determined by following the lines of the proof of (7.3) and given in A.3. □

For zero correlation this leads to an analytical call-price formula whenever an analytical conditional time-change density is available:

$$\begin{aligned} C_t &= \int_0^\infty \left[S_t\Phi(d_+) - K\,\mathrm{e}^{-r(T-t)}\,\Phi(d_-)\right] \mathbb{P}_{G_T - G_t|\mathcal{F}_t^Y}(\,\mathrm{d}z) \\ d_\pm &= \frac{\ln(\frac{S_t}{K}) + r(T-t) \pm \frac{1}{2}z}{\sqrt{z}} \,. \end{aligned}$$

7.2.2 Barrier options

We aim at pricing formulas for the following barrier options:
one-touch and *no-touch options*, as well as *knock-in* and *knock-out calls* and *puts*.

We assume $\mu = r$. Then the spot price

$$S_t = S_0 \exp\left\{rt - \frac{1}{2}G_t + W_{G_t}\right\} \tag{7.6}$$

is risk-neutral because the discounted spot process

$$\tilde{S}_t = \mathrm{e}^{-rt}\,S_t = S_0 \exp\left\{-\frac{1}{2}G_t + W_{G_t}\right\}$$

is already a martingale. We do not need to change measure for pricing. We assume that the Brownian motions W and B are uncorrelated (as in Assumption 1.20) and interest rates are zero. Then $\tilde{S} = S$. Note that indeed interest rates might be zero (e.g. Japan). Furthermore this applies when in a foreign-exchange market domestic and foreign interest rates coincide ($r_d = r_f$), or when in the equity market the interest rate equals the continuous dividend yield ($r = d$).

We apply our first-passage time results of Chapter 1 in order to determine prices for barrier options. The first-passage process is given by (7.5) where $\mu = 0$

$$Y_t = -\frac{1}{2}G_t + W_{G_t} \,,$$

and the first-passage time by

$$\tau = \inf\{t \geq 0 : Y_t < \tilde{K}\} \,, \quad \tilde{K} = \ln\left(\frac{K}{S_0}\right) \,.$$

Touch options

A *one-touch option* OT (*digital-knock-in*) respectively *no-touch option* NT (*digital-knock-out*) has payoff

$$\begin{aligned} OT_T &= \mathbb{I}_{\{\inf_{t\le T} S_t < K\}} \\ NT_T &= \mathbb{I}_{\{\inf_{t\le T} S_t > K\}} \quad = \; 1 - OT_T \; . \end{aligned}$$

Theorem 7.5 *(One-touch option)*
The price for a one-touch and no-touch option is given by the (conditional) first-passage time distribution:

$$\begin{aligned} OT_t &= \begin{cases} 1 & \text{if } \tau < t \\ \mathbb{P}\left(\tau < T \mid \mathcal{F}_t^Y\right) & \text{if } \tau \ge t \end{cases} \\ NT_t &= \begin{cases} 0 & \text{if } \tau < t \\ 1 - \mathbb{P}\left(\tau < T \mid \mathcal{F}_t^Y\right) & \text{if } \tau \ge t \end{cases} \; , \end{aligned}$$

where

$$\begin{aligned} \mathbb{P}\left(\tau < t \mid \mathcal{F}_t^Y\right) \; = & \int_0^\infty \Bigg[\Phi\left(\frac{K}{\sigma\sqrt{x}} + \frac{1}{2\sigma}\sqrt{x}\right) \\ & + \mathrm{e}^{-\frac{1}{\sigma^2}(K-Y_0)} \, \Phi\left(\frac{K}{\sigma\sqrt{x}} - \frac{1}{2\sigma}\sqrt{x}\right) \Bigg] \mathbb{P}(G_T - G_t \in \, \mathrm{d}x |\, \mathcal{F}_t^Y) \; . \end{aligned}$$

Proof. Interest rates are zero, so we do not have to consider discount factors. We consider the digital-knock-in:
The payoff can be written in terms of the first-passage time

$$OT_T \;=\; \mathbb{I}_{\{\inf_{t\le T} Y_t < \tilde{K}\}} \;=\; \mathbb{I}_{\{\tau < T\}} \; .$$

Then

$$OT_t \;=\; \mathbb{E}[\mathbb{I}_{\{\tau<T\}} \mid \mathcal{F}_t^Y] \; ,$$

which is the conditional default probability given in Section 3.1.
The digital-knock-out follows directly because of $NT_T = 1 - OT_T$. □

Touch options on two assets

We here consider two assets S^1 and S^2 and two strike levels K_1 and K_2 on which the digital payoff depends. A one-touch OT2 and a no-touch NT2 on two assets have the following payoffs

$$\begin{aligned} OT2_T &= \mathbb{I}_{\{\inf_{t\le T}(S_t^1, S_t^2) < (K_1, K_2)\}} \\ NT2_T &= \mathbb{I}_{\{\inf_{t\le T}(S_t^1, S_t^2) > (K_1, K_2)\}} \; . \end{aligned}$$

As in the one-asset case, we introduce the first-passage processes

$$Y_t^1 \;=\; -\tfrac{1}{2}G_t^1 + W_{G_t^1}^1 \;, \qquad Y_t^2 \;=\; -\frac{1}{2}G_t^2 + W_{G_t^2}^2$$

and their first-passage times

$$\tau_1 \;=\; \inf\{t \geq 0 : Y_t^1 < \tilde{K}_1\} \;, \qquad \tau_2 \;=\; \inf\{t \geq 0 : Y_t^2 < \tilde{K}_2\} \;,$$

where the threshold levels are given by

$$\tilde{K}_1 \;=\; \ln\left(\tfrac{K_1}{S_0^1}\right) \;, \qquad \tilde{K}_2 \;=\; \ln\left(\frac{K_2}{S_0^2}\right) \;.$$

Both τ_i are stopping times w.r.t.

$$\mathcal{F}_t \;:=\; \sigma(Y_s^1, Y_s^2 : s \leq t) \;=\; \sigma(S_s^1, S_s^2 : s \leq t) \;.$$

Theorem 7.6 *(One-touch option on two assets)*
Let W^1 and W^2 be Wiener processes with correlation ρ. Furthermore let G^1 and G^2 be time changes that satisfy Assumption 1.20 with $G_t^1 = g = G_t^2$ and that are independent of W^1 and W^2. The price for a one-touch respective no-touch dependent on two assets is then given by the joint default respective joint survival probability in that

$$OT_t \;=\; \begin{cases} 1 & \text{if } \tau_1 < t \text{ and } \tau_2 < t \\ \mathbb{P}\left(\tau_1 \leq T, \tau_2 \leq T \mid \mathcal{F}_t\right) & \text{if } \tau_1 < t \text{ or } \tau_2 < t \end{cases}$$

$$NT_t \;=\; \begin{cases} 0 & \text{if } \tau_1 < t \text{ or } \tau_2 < t \\ \mathbb{P}\left(\tau_1 > T, \tau_2 > T \mid \mathcal{F}_t\right) & \text{if } \tau_1 < t \text{ and } \tau_2 < t \end{cases}$$

where $\mathbb{P}(\tau_1 > T, \tau_2 > T \mid \mathcal{F}_t)$ is given by Theorem 1.26 when substituting the unconditional time-change density $\mathbb{P}_{(G_t^1, G_t^2)}(\,\mathrm{d}x \times \mathrm{d}y)$ by the conditional density $\mathbb{P}_{(G_t^1, G_t^2)|\tilde{\mathcal{F}}_t}(\,\mathrm{d}x \times\, \mathrm{d}y)$ and

$$\begin{aligned} & \mathbb{P}(\tau_1 \leq T, \tau_2 \leq T \mid \mathcal{F}_t) \\ = \;& \mathbb{P}(\tau_1 > T, \tau_2 > T \mid \mathcal{F}_t) + \mathbb{P}(\tau_1 \leq T \mid \mathcal{F}_t) + \mathbb{P}(\tau_2 \leq T \mid \mathcal{F}_t) - 1 \;. \end{aligned}$$

with $\mathbb{P}(\tau_i \leq T \mid \mathcal{F}_t)$, $i = 1, 2$, as in Theorem 7.5.

Barrier options

With the model (7.6) we can also derive a closed pricing form for barrier options. A *down-and-in call (DIC)* respectively *put (DIP)* knocks in when a lower boundary has been hit, an *up-and-in call (UIC)* respectively *put*

(UIP) knocks in when an upper boundary has been hit. The payoffs are as follows:

$$\begin{aligned} DIC_T &= (\tilde{S}_T - K)_+ \mathbb{I}_{\{\inf_{t\le T} \tilde{S}_t < H\}} \\ DIP_T &= (K - \tilde{S}_T)_+ \mathbb{I}_{\{\inf_{t\le T} \tilde{S}_t < H\}} \\ UIC_T &= (\tilde{S}_t - K)_+ \mathbb{I}_{\{\inf_{t\le T} \tilde{S}_t > H\}} \\ UIP_T &= (K - \tilde{S}_T)_+ \mathbb{I}_{\{\inf_{t\le T} \tilde{S}_t > H\}} \,. \end{aligned}$$

We give the price for the down-and-in call, the other pricing formulas can be determined analogously.

Theorem 7.7 *(Down-and-in call)*
Let H be the knock-in barrier, K the strike and T the maturity of the down-and-in call. Furthermore set $\tilde{H} = \ln\left(\frac{H}{\tilde{S}_0}\right)$. Under $\tau > t$, i.e., no knock-in yet, the price of a down-and-in call is given by

$$\begin{aligned} DIC_t &= \int_\Omega \Bigg[\tfrac{\tilde{S}_t}{\sqrt{2\pi(G_T - G_t)}} \int_{\tilde{H}}^\infty \mathrm{e}^{\frac{1}{2}z + \frac{1}{8}(G_T-G_t) - \frac{(|z-\tilde{H}|-\tilde{H})^2}{2(G_T-G_t)}} \,\mathrm{d}z \\ &- \tfrac{K}{\sqrt{2\pi(G_T - G_t)}} \int_{\tilde{H}}^\infty \mathrm{e}^{-\frac{1}{2}z + \frac{1}{8}(G_T-G_t) - \frac{(|z-\tilde{H}|-\tilde{H})^2}{2(G_T-G_t)}} \,\mathrm{d}z \Bigg] \quad \mathrm{d}\mathbb{P}_{G_T - G_t | \mathcal{F}_t^Y} \,. \end{aligned}$$

Proof.

$$\begin{aligned} DIC_t &= \mathbb{E}\left[(S_T - K)_+ \mathbb{I}_{\{\inf_{t\le T} S_t < H\}} \mid \mathcal{F}_t^Y \right] \\ &= \mathbb{E}\left[(\exp Y_T - K)_+ \mathbb{I}_{\{\inf_{t\le T} Y_t < \tilde{H}\}} \right] \end{aligned}$$

We use that G and W are independent and apply the known density $\mathbb{P}(\inf_{t\le T} W_t \le \tilde{H},\ W_T \in \mathrm{d}z)$; cf. Borodin & Salminen (2002). □

7.2.3 The Dufresne time change

As an example we consider the so-called *Dufresne time change*

$$G_t = \int_0^t g_s \,\mathrm{d}s = \int_0^t \mathrm{e}^{\hat{\sigma}B_s - \hat{\sigma}^2 \kappa s} \,\mathrm{d}s \,,$$

where $g_t = \mathrm{e}^{\hat{\sigma}B_t - \hat{\sigma}^2\kappa t}$ and $\hat{\sigma} > 0,\ 0 < \kappa \le \frac{1}{2}$. It has expectation (applying Fubini)

$$\begin{aligned} \mathbb{E}[G_t] &= \int_0^t \mathbb{E}\left[\mathrm{e}^{\hat{\sigma}B_s} \mathrm{e}^{-\hat{\sigma}^2\kappa s} \right] \mathrm{d}s = \int_0^t \mathbb{E}\left[\mathrm{e}^{\hat{\sigma}B_s} \right] \mathrm{e}^{-\hat{\sigma}^2\kappa s} \,\mathrm{d}s \\ &= \int_0^t \mathrm{e}^{\frac{1}{2}\hat{\sigma}^2 s - \hat{\sigma}^2\kappa s} \,\mathrm{d}s = \begin{cases} t & \text{if } \kappa = \frac{1}{2} \\ \frac{1}{\hat{\sigma}^2(\frac{1}{2}-\kappa)} \left[\mathrm{e}^{\hat{\sigma}^2(\frac{1}{2}-\kappa)t} - 1 \right] & \text{otherwise.} \end{cases} \end{aligned}$$

Since $0 < \kappa \le \frac{1}{2}$, we have $\mathbb{E}[G_t] \to \infty$ for $t \to \infty$. The option prices are determined by inserting the density for the Dufresne time change (see Table 1.3):

$$\mathbb{P}(G_t \in \mathrm{d}x) = \hat{\sigma}^{-2\kappa+1} x^{-\kappa-\frac{1}{2}} 2^{\kappa-\frac{1}{2}} \mathrm{e}^{-\kappa^2 \hat{\sigma}^2 \frac{t}{2} - \frac{1}{\hat{\sigma}^2 x}} \cdot m_{\hat{\sigma}^2 t/2} \left(-\kappa - \frac{1}{2}, \frac{1}{\hat{\sigma}^2 x} \right) \mathrm{d}x \,.$$

7.3 Idea: adding a leverage effect

We expand the model by a random variable Z that is independent of the processes (W_t) and (g_t):

$$\begin{aligned} S_t &= S_0 \exp\{W_{G_t} + \rho Z\} \\ G_t &= \int_0^t g_s^2 \,\mathrm{d}s + Z^2 \,. \end{aligned}$$

Z influences the leverage effect and also enables a volatility skew. Furthermore Z^2 leads to an a.s. positive starting value G_0 and enforces the variance to speed up and enables an instantaneous price jump.

Using the independence of Z the first-passage time distribution of the exponent process $Y_t = W_{G_t} + \rho Z$ can be determined by conditioning (on Z and G_t) and thus also barrier option prices.

Appendix A

Technical details

A.1 General derivative for a time-dependent integral

We determine the derivative of

$$F(t) \quad = \quad \int_{g(t)}^{h(t)} f(s,t)\,\mathrm{d}s \ .$$

For this let

$$\begin{aligned} G(t) &= (g(t), h(t), t) \ , \\ H(u,v,w) &= \int_u^v f(s,w)\,\mathrm{d}s \ , \\ F(t) &= H \circ G(t) \ , \end{aligned}$$

then

$$\begin{aligned} F'(t) &= < \nabla H(G(t)), G'(t) > \ , \\ D_u H(u,v,w) &= -f(u,w) \ , \\ D_v H(u,v,w) &= f(v,w) \ , \\ D_w H(u,v,w) &= \int_u^v D_w f(s,w)\,\mathrm{d}s \ , \end{aligned}$$

where ∇ denotes the gradient, and herewith

$$\begin{aligned} F'(t) &= < \nabla H(g(t), h(t), t), (g'(t), h'(t), t) > \\ &= -f(g(t),t) \cdot g'(t) + f(h(t),t) \cdot h'(t) + \int_{g(t)}^{h(t)} \frac{\mathrm{d}}{\mathrm{d}t} f(s,t)\,\mathrm{d}s \ . \end{aligned}$$

We are interested in the following special cases:

Example 1

Let T be a constant, set $h : t \mapsto T$ and $g \equiv id$. Then $h'(t) = 0$ and $g'(t) = 1$, and thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_t^T f(s,t)\,\mathrm{d}s \quad = \quad -f(t,t) + \int_t^T \frac{\mathrm{d}}{\mathrm{d}t} f(s,t)\,\mathrm{d}s \tag{A.1}$$

Example 2

Let M be a constant, set $h : t \mapsto t + M$ and $g \equiv id$, then $h'(t) = 1$ and $g'(t) = 1$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_t^{t+M} f(s,t)\,\mathrm{d}s = -f(t,t) + f(t+M,t) + \int_t^{t+M} \frac{\mathrm{d}}{\mathrm{d}t} f(s,t)\,\mathrm{d}s \tag{A.2}$$

A.2 Gamma, Bessel and modified Bessel function

We cite definitions of Borodin, Salminen (2002), page 637 f.

Definition A.1 (Gamma function)
The Gamma function is defined as

$$\Gamma(x) \quad := \quad \int_0^\infty u^{x-1}\,\mathrm{e}^{-u}\,\mathrm{d}u, \qquad \mathrm{Re}\,(x) > 0\ .$$

Definition A.2 (Bessel function)
The Bessel function of order $\nu \geq 0$ is defined by

$$J_\nu(x) \quad := \quad \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{\nu+2k}}{k!\ \Gamma(\nu+k+1)}\ .$$

$0 < j_{\nu,k} < j_{\nu,k} < \dots$ denote the positive zeros of $J_\nu(x)$.

Definition A.3 (Modified Bessel function)
The modified Bessel function of order ν is given by

$$I_\nu(x) \quad := \quad \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{\nu+2k}}{k!\ \Gamma(\nu+k+1)}\ .$$

A.3 Proof: European call of Theorem 7.4

Proof. We give the main steps. The true measure conditional on $\mathcal{F}_t^Y$ is denoted by $\mathbb{P}_{\mathcal{F}_t^Y}$. The general likelihood process L_t was given in Corollary 7.3, but we only consider the case $\mu = r$, that is, $L_t = 1$. Define

$$c \quad := \quad \frac{\ln(\frac{K}{S_t}) - r(T-t) + \frac{1}{2}(G_T - G_t) - \rho \int_t^T \sqrt{g_s}\,\mathrm{d}B_s}{\sqrt{1-\rho^2}}$$

and note that c is $\mathcal{F}_t^Y \vee B$-measurable. Furthermore $(B_s)_{s\geq 0}$ is $\mathcal{F}_t^Y \vee B$-measurable and $(\hat{B}_s)_{s>t}$ is independent of $\mathcal{F}_t^Y \vee B$. Then

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left[\frac{L_T}{L_t}(S_T-K)^+ |\ \mathcal{F}_t^Y \vee B\right] \\
&= \ \mathbb{E}_{\mathbb{P}}\left[S_T\frac{L_T}{L_t}\mathbb{I}_{\{S_T>K\}}|\ \mathcal{F}_t^Y \vee B\right] - K\mathbb{E}_{\mathbb{P}}\left[\frac{L_T}{L_t}\mathbb{I}_{\{S_T>K\}}|\ \mathcal{F}_t^Y \vee B\right] \\
&= \ \int_{\{S_T>K\}} S_t\,\mathrm{e}^{\mu(T-t)-\frac{1}{2}\int_t^T g_s\,\mathrm{d}s+\rho\int_t^T\sqrt{g_s}\,\mathrm{d}B_s+\sqrt{1-\rho^2}\int_t^T\sqrt{g_s}\,\mathrm{d}\hat{B}_s}\ \mathrm{d}\mathbb{P}_{\mathcal{F}_t^Y} \\
&\quad - K\int_{\{S_T>K\}} 1\,\mathrm{d}\mathbb{P}_{\mathcal{F}_t^Y} \\
&= \ S_t\,\mathrm{e}^{\mu(T-t)-\frac{1}{2}\int_t^T g_s\,\mathrm{d}s+\rho\int_t^T\sqrt{g_s}\,\mathrm{d}B_s}\int_{\left\{\int_t^T\sqrt{g_s}\,\mathrm{d}\hat{B}_s>c\right\}} \mathrm{e}^{\sqrt{1-\rho^2}\int_t^T\sqrt{g_s}\,\mathrm{d}\hat{B}_s}\ \mathrm{d}\mathbb{P}_{\mathcal{F}_t^Y} \\
&\quad - K\int_{\left\{\int_t^T\sqrt{g_s}\,\mathrm{d}\hat{B}_s>c\right\}} 1\,\mathrm{d}\mathbb{P}_{\mathcal{F}_t^Y} \\
&= \ S_t\,\mathrm{e}^{(\mu-\frac{1}{2}\bar{g})(T-t)+\rho\int_t^T\sqrt{g_s}\,\mathrm{d}B_s}\,\mathrm{e}^{\frac{1}{2}(1-\rho^2)(G_T-G_t)} \\
&\quad \cdot\Phi\left(\sqrt{1-\rho^2}\sqrt{G_T-G_t}-\frac{c}{\sqrt{G_T-G_t}}\right) \\
&\quad - K\cdot\Phi\left(-\frac{c}{\sqrt{G_T-G_t}}\right)
\end{aligned}
$$

Now insert c, d_+, and d_- as defined. □

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List of Abbreviations

càdlàg	process with right continuous paths and left limits	3
CDS	credit default swap	xviii
CIR	Cox-Ingersoll-Ross, name of a process	30
FPT	first-passage time	xiii
FTD	first-to-default (swap)	123
JDP	joint default probability	6
JSP	joint survival probability	6
PDE	partial differential equation	11
SDE	stochastic differential equation	59
S&P	Standard & Poor's, rating agency	xiii
OU	Ornstein-Uhlenbeck, name of a process	30

List of Symbols

B	Brownian motion; (B_t)	2
$\mathbb{F}^Y$	natural filtration of the process Y; $(\mathcal{F}_t^Y)$	2
F	default-probability curve of the market	9
G_t	continuous stochastic time change	21
$\hat{G}_t$	Cox-Ingersoll-Ross-type time change	32
G_t^{CIR}	Cox-Ingersoll-Ross time change	32
g	starting value of the stochastic time change; G_0	22
g_t	time-change integrand; increase of information / default speed	22
g_t^{CIR}	Cox-Ingersoll-Ross process	30
g_t^{OU}	Ornstein-Uhlenbeck process	30
Γ	Gamma function	33
I_ν	modified Bessel function of the first kind with order ν	12
J_t	subordinator	17
K	face value of a bond or threshold level of a structural model	xiv
$l_T(u)$	Laplace transform of a positive random variable T; $\mathbb{E}\left[e^{-uT}\right]$	5
M	time to maturity; T-t	62
μ	drift parameter	10
$\mathbb{P}$	true probability measure	2
$\mathbb{P}^x$	probability measure where the underlying process starts at x	32
Φ	cumulative standard Normal distribution	10
φ	standard Normal probability density	98
$Q(t,T)$	survival probability	60
R	recovery rate	62
r	interest rate	61
ρ	Brownian correlation	7
ρ_t^A	asset correlation	6
ρ_t^E	event / default correlation	7
$s(t,T)$	credit spread	61
σ	volatility parameter	10
σ_t	deterministic time-change integrand; default speed	98
T	maturity of a bond or credit-default swap	xiv
T_t	continuous deterministic time change	14
τ	default time, classical or first-passage	5
W	Brownian motion; (W_t)	2

Index

Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Literatur verfasst habe.

Frankfurt, den 12. September 2007 Stefanie Kammer