Asymptotic Stability for Some Functional Differential Equations.

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Synopsis

For a non-linear functional differential equation from population biology, a result on asymptotic stability is obtained by investigating the zeros of the characteristic equation of the linearised functional differential equation.

1. It is well known that the zero solution of the autonomous linear differential delay equation

$$x'(t) = -\alpha x(t-1), \quad \alpha>0, \tag{1}$$

is uniformly asymptotically stable if $\alpha<\pi/2$ [see for example, **6** and for definitions **2**].

We are interested in the stability of the zero solution of the equation

$$x'(t) = -\alpha \int_{-r}^{0} x(t+a)ds(a), \tag{2}$$

with $\alpha>0$, $r>0$, s: $[-r, 0]\to R$ increasing and with total variation $V(s)$ not exceeding unity. Obviously, equation (1) is a special case of equation (2).

Our interest in equation (2) comes from its relation to a population growth model proposed by Dunkel [**4**]. He considered the equation

$$n'(t) = [b+\int_{0}^{r} \Psi(n(t-a))dS(a)]n(t) \tag{3}$$

for the density n of a single species population, with b and r positive, ψ: $R_0^+\to R_0^+$ increasing and S: $[0, r]\to R_0^+$ decreasing with $S(0)>0$, $S(r) = 0$.

Now let $\Psi(n) = n$ for $n \geqq 0$. Then we obtain equation (3) from the general rate equation $n' = (b-d)n$ if we assume that the birth rate b of the population is constant, and that the density dependent death rate d at time t is given by $D(n_t)$, where D is a positive continuous *linear* functional on the Banach space C of continuous real-valued functions on the interval $[-r, 0]$, with supremum-norm. n_t is the function defined by $n_t(a)\colon = n(t+a)$ for $-r \leqq a \leqq 0$. The constant solution $\tilde{n}(t) = b/S(0)$ represents the equilibrium state of the population. Setting $x(t)\colon = n(t)S(0)/b-1$, $\alpha\colon = b$, $s(a)\colon = S(-a)/S(0)$ on $[-r, 0]$, we obtain

$$x'(t) = -\alpha \int_{-r}^{0} x(t+a)ds(a)[1+x(t)], \tag{4}$$

with s: $[-r, 0]\to R$ increasing and $V(s) = 1$. The stability properties of the solution $\tilde{n}$ of equation (3) and of the zero solution of equation (4) are the same.

In order to derive a stability result for the zero solution of equation (4), we need the following perturbation theorem.

THEOREM 1. *Let* $L: C_m \to R^m$ *be linear and continuous. Let* $N: C_m \to R^m$ *be continuous with the property* $(\forall \varepsilon > 0 \exists \delta > 0: \| \Phi \| \leqq \delta \Rightarrow | N(\Phi)| \leqq \varepsilon \| \Phi \|)$. *Let the zero solution of the equation* $x'(t) = L(x_t)$ *be uniformly asymptotically stable. Then the zero solution of the equation* $x'(t) = L(x_t) + N(x_t)$ *is uniformly asymptotically stable.*

Here, m is a positive integer, and C_m denotes the Banach space of continuous mappings from the interval $[-r, 0]$ into R^m, with supremum-norm. Theorem 1 is an obvious generalisation of [**2**, Theorem 18.3] in the case of *autonomous* functional differential equations. It can be proved by modifying arguments given in [**2**, ch. 16 and 18]. For details, see [**5**, Lemma 3 and Theorem 2].

Define $L(\Phi) := -\alpha \int_{-r}^{0} \Phi(a) ds(a)$, $N(\Phi) := -\alpha\Phi(0) \int_{-r}^{0} \Phi(a) ds(a)$ for Φ in $C = C_1$.

The zero solution of the equation $x'(t) = L(x_t) = -\alpha \int_{-r}^{0} x(t+a) ds(a)$ is uniformly asymptotically stable, if and only if every complex number $\lambda = \mu + i\nu$ with

$$\lambda + \alpha \int_{-r}^{0} \exp[\lambda a] ds(a) = 0 \tag{5}$$

has negative real part μ [**1**]. We shall prove

THEOREM 2. *Let* $\alpha > 0$ *and* $r > 0$. *If* $s: [-r, 0] \to R$ *is a non-constant increasing function with* $V(s) \leqq 1$ *and if* $\alpha r < \pi/2$, *then every solution of equation* (5) *has negative real part.*

Since the equation $y'(t) = -y(t - \pi/2)$ has a non-trivial periodic solution (cos) the bound on αr in Theorem 2 cannot be increased. (Furthermore, equation (1) is unstable, if $\alpha > \pi/2$ [see **6**].)

COROLLARY 1. *If* $\alpha r < \pi/2$, *then the zero solution of equation* (4) *is uniformly asymptotically stable.*

Proof. With L and N defined as above, we have $| N(\Phi)| \leqq \alpha \| \Phi \|^2$. By Theorem 2, $\alpha r < \pi/2$ guarantees the uniform asymptotic stability of the zero solution of the equation $x'(t) = L(x_t)$. Hence Corollary 1 follows from Theorem 1.

Another consequence of Theorem 2 is connected with work of Myškis and Yorke [**7**]. Yorke treated the equation

$$x'(t) = -f(t, x_t) \tag{6}$$

with $f: R_0^+ \times C \to R$ continuous and satisfying the conditions

(i) $\exists \alpha \geqq 0 \forall \Phi \in C \forall t \geqq 0: -\alpha \| \min(\Phi, 0) \| \leqq f(t, \Phi) \leqq \alpha \| \max(\Phi, 0) \|$;

(ii) for all sequences $t_n \to \infty$, Φ_n in C converging to a constant non-zero function in C, $f(t_n, \Phi_n)$ does not converge to zero.

Yorke's main result [**7**, Theorem 1.1] implies that the zero solution of equation (6) is uniformly asymptotically stable if $\alpha r < 3/2$. For linear equations, a very similar result had been proved earlier by Myškis [**3**]. Moreover, Myškis constructed a linear non-autonomous equation satisfying conditions (i and ii) with $\alpha r = 3/2$ and with the zero solution not asymptotically stable. Therefore the result of Yorke is optimal in the general case of linear equations. But it is not if we only admit *autonomous* linear equations. We have

COROLLARY 2. *Let* $L: C \to R$ *be linear and continuous,* $L \neq 0$. *If* L *satisfies condition* (i) *with* $\alpha r < \pi/2$, *then the zero solution of the equation* $x'(t) = -L(x_t)$ *is uniformly asymptotically stable.*

Proof. The estimate on L is equivalent to $L(\Phi) = \alpha \int_{-r}^{0} \Phi(a) ds(a)$ on C, with $s: [-r, 0] \to R$ increasing and $0 < V(s) \leqq 1$.

2. The proof of Theorem 2 is simple: Assume the hypotheses of Theorem 2 fulfilled for α, r and s. Let

$$\lambda + \alpha \int_{-r}^{0} \exp[\lambda a] ds(a) = 0 \tag{5}$$

for a complex number $\lambda = \mu + i\nu$, with μ and ν real. With $\mu \geqq 0$, equation (5) implies

$$| \nu | = \alpha \left| \int_{-r}^{0} \exp[\mu a] \sin[\nu a] ds(a) \right| \leqq \alpha V(s) \sup_{[-r, 0]} | \exp[\mu a] \sin[\nu a] | \leqq \alpha V(s) < \pi/2r$$

and

$$0 \geqq (-\alpha)^{-1} \mu = \int_{-r}^{0} \exp[\mu a] \cos[\nu a] ds(a) = V(s) \exp[\mu a^*] \cos[\nu a^*]$$

with $a^* \in [-r, 0]$. By $V(s) > 0$ and $| \nu a^* | < \pi/2$, we obtain a contradiction.

Notation. R denotes the set of real numbers. R_0^+. is the set of non-negative real numbers.

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