On Floquet Multipliers of Periodic Solutions of Delay Equations with Monotone Nonlinearities

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ABSTRACT

We exclude Floquet multipliers $\mu \leq -1$ for slowly oscillating periodic solutions of autonomous single-delay equations with strictly decreasing nonlinearities.

Consider a periodic solution $p: \mathbb{R} \to \mathbb{R}$ of equation

$$x'(t) = g(x(t-1)) \tag{1}$$

where g is continuous and satisfies the condition for negative feedback,

$$\xi g(\xi) < 0$$

at least for all $\xi \neq 0$ in $p(\mathbb{R})$.

Suppose that p is slowly oscillating in the sense that zeros are spaced at distances larger than the delay $\tau = 1$, and that all zeros are simple.

It follows that p has zeros, and that they form a sequence of points z_n, $n \in \mathbb{Z}$, which satisfy

$$z_n + 1 < z_{n+1}.$$

We may assume $z_0 = -1$. Then the minimal period π of p is

$$\pi = 1 + z_{2j} \text{ for some } j \in \mathbb{N}.$$

Existence of slowly oscillating periodic solutions is well known. Sufficient conditions are e.g. that the negative feedback condition holds for all $\xi \neq 0$, that g is bounded from above or from below, that $g'(0)$ exists and that the zero solution is linearly unstable[4,5,7].

Observe that in these results, as well as in other, more special ones,

$$j = 1 .$$

The present note is devoted to strictly monotone nonlinearities g. We assume in addition that g is continuously differentiable with

$$g'(\xi) < 0 \text{ for all } \xi \in p(\mathbb{R}) \tag{M}$$

This has strong implications on the Floquet multipliers of p, and thereby on the structure of the semiflow of Eq. (1) in a neighborhood of the orbit formed by the segments p_t, $p_t(\tau) = p(t+\tau)$ for $-1 \le \tau \le 0$, in the phase space

$$C = C([-1,0],\mathbb{R}) .$$

Recall that a Floquet multiplier[3] $\mu \in \mathbb{C}$ is a nonzero point in the spectrum of the continuous linear "period map"

$$U : \varphi \to y_\pi$$

where $y:[-1,\infty) \to \mathbb{C}$ is the solution of the linear variational equation

$$y'(t) = g'(p(t-1))\, y(t-1) \tag{2}$$

with initial value

$$y_0 = \varphi \in C_{\mathbb{C}} = C([-1,0],\mathbb{C}) .$$

The period map U is compact so that each Floquet multiplier μ is an eigenvalue with a generalized eigenspace $G_{\mathbb{C}}(\mu)$ of finite dimension $m(\mu)$.

The Floquet multipliers are real or occur in complex conjugate pairs. The number $1 \in \mathbb{C}$ is a Floquet multiplier with eigenvector

$$(p')_0 = p'|[-1,0] ,$$

as is easily seen after differentiating Eq. (1) for $x = p$.

Incidentally, note that the zeros of p' are given by

$$q_n = z_n + 1 .$$

In particular,

$$\pi = q_{2j} .$$

The solution p' of Eq. (2) is slowly oscillating.

Results from [1] imply

$$\sum_{|\mu| \ge 1} m(\mu) \le 2 , \tag{3}$$

see the remark following the proof of the Proposition below. This leaves us with four possibilities, in view of $m(1) \ge 1$:

1. $m(1) = 1$, $|\mu| < 1$ for every multiplier $\mu \neq 1$.
   (The case of exponential stability of p. For examples, see [1] and [2].)

2. $m(1) = 2$, no multipliers $\mu \neq 1$ with $|\mu| \ge 1$.

3. $m(1) = 1$, and there exists a multiplier $\mu \in (1,\infty)$, $m(\mu) = 1$; all multipliers λ different from 1, μ satisfy $|\lambda| < 1$.

4. $m(1) = 1$, and there exists a multiplier $\mu \in (-\infty,-1]$, $m(\mu) = 1$; all multipliers λ different from 1, μ satisfy $|\lambda| < 1$.

Case 2 corresponds to a local center manifold which determines the behavior of the semiflow close to the orbit of p in C. Examples are suggested by results in [6,2]. Case 3 is hyperbolic, with a two-dimensional local unstable manifold. Examples are contained in [2]. For more, see [8,9].

The purpose of the present note is to rule out case 4.

Theorem. Every multiplier μ with $|\mu| \ge 1$ satisfies $\mu \in [1,\infty)$.

Once again, observe that this is not restricted to the case $\pi = z_2 + 1$ usually considered.

The Theorem excludes local center or unstable manifolds which are topologically Moebius strips, and period doubling bifurcation along branches of periodic solutions for parameterized equations ($g = \alpha f$, f fixed, $\alpha > 0$ a parameter). Also, the occurrence of minimal periods $\pi = 1 + z_{2j}$ with $j > 1$ seems unlikely.

The key to (3) and to the Theorem is that slowly oscillating solutions are not only important in the dynamics of Eq. (1), but also for Eq. (2). This is a consequence of condition (M) which makes the coefficient $g'(p(t-1))$ negative. (A first indication that slowly oscillating solutions are important in existence. All continuous functions $x:[-1,\infty) \to \mathbb{R}$ which satisfy Eq.(2) for $t > 0$ and have at most one zero in $[-1,0]$ are "slowly oscillating for $t > 1$ ".)

It is convenient to recall a few facts from [1]. Let a Floquet multiplier μ be given. Write $G_{\mathbb{C}} := G_{\mathbb{C}}(\mu)$, $m := m(\mu)$. There exist a $m \times m$-matrix B, with spectrum $\{\mu\}$, and a π-periodic map $w:\mathbb{R} \to \mathbb{C}^m$ such that each function

$$x^c:\mathbb{R} \to \mathbb{C}, \quad x^c(t) = w(t)^{tr} e^{tB} c ,$$

where

$$c \in \mathbb{C}^m ,$$

is a solution to Eq. (2) so that

$$x_0^c \in G_{\mathbb{C}} .$$

These solutions form a subspace $\mathcal{G}_{\mathbb{C}} = \mathcal{G}_{\mathbb{C}}(\mu)$ of dimension m in the complex vectorspace of all continuous functions $y:\mathbb{R} \to \mathbb{C}$. Reellification yields subspaces $\mathcal{G} = \mathcal{G}(\mu)$ of the real vectorspace X of continuous functions $x:\mathbb{R} \to \mathbb{R}$:

$$\mathcal{G}(\mu) := \begin{cases} \mathcal{G}_{\mathbb{C}}(\mu) \cap X & \text{if } \mu \in \mathbb{R} \quad ; \ \dim \mathcal{G}(\mu) = m(\mu) \\ (\mathcal{G}_{\mathbb{C}}(\mu) \oplus \mathcal{G}_{\mathbb{C}}(\bar{\mu})) \cap X & \text{if } \operatorname{Im}\mu > 0 \ ; \ \dim \mathcal{G}(\mu) = 2\,m(\mu) \end{cases}$$

Proposition. Every $x \neq 0$ in $\bigoplus_{|\mu| \geq 1,\, \operatorname{Im}\mu \geq 0} \mathcal{G}(\mu)$ is slowly oscillating.

Proof. Recall that $p' \in \mathcal{G}(1)$ is slowly oscillating, and apply Lemmas 5,6,7 of [1]. **QED.**

In order to derive (3), use in addition Corollary 1 of [1].

The proof of the Theorem relies on two lemmas which hold for slightly more general equations than Eq. (2).

Let a continuous, bounded, negative function $b:\mathbb{R} \to \mathbb{R}$ be given. Let $L \subset X$ denote a linear space of solutions of the equation

$$x'(t) = b(t)\,x(t-1) .$$

Suppose every $x \neq 0$ in L is slowly oscillating.

Lemma 1. If there exists $x \neq 0$ in L such that its zeroset has neither a lower nor an upper bound, then the same holds true for all $y \neq 0$ in L.

Proof. The hypothesis implies that the zeros of x form a sequence of points t_n, $n \in \mathbb{Z}$, with $t_n + 1 < t_{n+1}$. Let $y \in L$, $y \neq 0$, be given. Suppose the zeros of y are bounded below by some real a. Then

$$\operatorname{sign} y'(t) = -\operatorname{sign} y(t) \text{ for } t < a .$$

Note $y \notin \mathbb{R}x$. Choose $t_{n+1} < a$ such that

$$\operatorname{sign} x = \operatorname{sign} y \text{ on } (t_n, t_{n+1}) =: I .$$

There exist $c > 0$ and $t \in I$ with

$$|cx| \leq |y| \text{ on } I \text{ and } cx(t) = y(t) .$$

It follows that $cx'(t) = y'(t)$, and t is a double zero of $cx - y \in L$. Hence $cx - y = 0$, a contradiction.

If one assumes an upper bound a for the zeros of y then

$$\operatorname{sign} y'(t) = -\operatorname{sign} y(t) \quad \text{for } t > a + 1 ,$$

and one can argue as before. **QED.**

Lemma 2. Suppose there exists $x \in L$, $x \neq 0$, so that its zeroset has neither a lower nor an upper bound.

1. If $y \in L$, $y \neq 0$, then the zeros of y form a sequence of points $t_{n,y}$, $n \in \mathbb{Z}$, with $t_{n,y} + 1 < t_{n+1,y}$.
2. For y, v in L, $y \neq 0 \neq v$, and for $n \in \mathbb{Z}$ with

$$t_{n,y} < t_{n,v} < t_{n+1,y} , \tag{4}$$

$$t_{n+1,y} < t_{n+1,v} .$$

Proof. The first assertion is a consequence of Lemma 1. To prove part 2, assume (4), and $t_{n+1,v} \leq t_{n+1,y}$.

In case $t_{n+1,v} = t_{n+1,y}$, set $c := v'(t_{n+1,y}) / y'(t_{n+1,y})$. Then $cy - v$ is in L and has a double zero at $t_{n+1,y}$. Hence $cy - v = 0$, which contradicts $v(t_{n,v}) = 0 \neq y(t_{n,v})$.

In case $t_{n+1,v} < t_{n+1,y}$, there exist $c \in \mathbb{R}$ and $t \in (t_{n,v}, t_{n+1,v}) =: I$ such that

$$|cv| \le |y| \text{ in } I \text{ and } cv(t) = y(t) \text{ and } \operatorname{sign} cv = \operatorname{sign} y \text{ in } I .$$

It follows that $cv'(t) = y'(t)$, and $cv - y \in L$ has a double zero at t. Hence $cv - y = 0$, a contradiction as before. **QED.**

Lemmas 1 and 2 express a synchronization property of nontrivial solutions in L.

Proof of the Theorem. Suppose there exists a multiplier $\mu \in (-\infty, -1]$. Then $m(\mu) = 1$. Choose $y \neq 0$ in $\mathcal{G}(\mu)$. y is slowly oscillating, due to the Proposition.

An application of Lemma 2 for $b(t) := g'(p(t-1))$, $L := \mathcal{G}(\mu) \oplus \mathcal{G}(1)$ and $x := p' \in \mathcal{G}(1)$ shows that the zeros of y form a sequence of points t_n, $n \in \mathbb{Z}$, with $t_n + 1 < t_{n+1}$.

We have $y_\pi = U y_0 = \mu y_0$, hence

$$y(\pi) = \mu y(0) \text{ and } y'(\pi) = \mu y'(0) .$$

We may assume that t_0 is the largest non-positive zero of y. Recall that the zeros q_n of p' satisfy $q_0 = 0$ and $q_{2j} = \pi$.

In case $t_0 < 0$, $y(0) \neq 0$ and $t_0 < q_0 < t_1$. Repeated application of part 2 of Lemma 2 yields

$$q_{n-1} < t_n < q_n \text{ for all integers } n \ge 1 .$$

In particular,

$$\pi = q_{2j} \in (t_{2j}, t_{2j+1}) \tag{5}$$

Hence

$$\begin{aligned} 0 \neq \operatorname{sign} y(0) &= \operatorname{sign} y'(t_0) = \operatorname{sign} y'(t_{2j}) && (\text{ due to simplicity of zeros }) \\ &= \operatorname{sign} y(q_{2j}) && (\text{ by (5)}) \\ &= \operatorname{sign} y(\pi) , \end{aligned}$$

a contradiction to $y(\pi)/y(0) = \mu < 0$.

In case $t_0 = 0$, $y'(0) \neq 0$. We obtain a common zero of p' and y at $t = \pi$ since $y(\pi) = \mu y(0) = 0 = p'(0) = p'(\pi)$. It follows that

$$y(t) \neq 0 \text{ for all integers } n \le 2j-1 \text{ and all } t \in (q_n, q_{n+1})$$

– otherwise, repeated application of Lemma 2 would exclude the common zero of p' and y at $\pi = q_{2j}$.

Therefore

$$y^{-1}(0) \cap (-\infty, \pi] \subset \{ q_n : n \le 2j \} = (p')^{-1}(0) \cap (-\infty, \pi] .$$

Analogously,

$$(p')^{-1}(0) \cap (-\infty, \pi] \subset y^{-1}(0) \cap (-\infty, \pi] .$$

Hence

$$y^{-1}(0) \cap [0, \pi] = \{ q_n : n = 0, \dots, 2j \} ,$$

and

$$\begin{aligned} 0 \neq \operatorname{sign} y'(\pi) &= \operatorname{sign} y'(q_{2j}) = \operatorname{sign} y'(q_0) && (\text{ by simplicity }) \\ &= \operatorname{sign} y'(0) , \end{aligned}$$

a contradiction to $y'(\pi)/y'(0) = \mu < 0$. **QED.**

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Addendum. It is in fact true that, under the hypotheses on the nonlinearity g made in this paper, every slowly oscillating periodic solution p with a zero $z_0 = -1$ has minimal period $z_2 + 1$ (personal communication by J. Mallet-Paret). A proof based on techniques from [8] will be given in [9].