



Nirenberg problem on high dimensional spheres: blow up with residual mass phenomenon

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Abstract. In this paper, we extend the analysis of the subcritical approximation of the Nirenberg problem on spheres recently conducted in Malchiodi and Mayer (J Differ Equ 268(5):2089–2124, 2020; Int Math Res Not 18:14123–14203, 2021). Specifically, we delve into the scenario where the sequence of blowing up solutions exhibits a non-zero weak limit, which necessarily constitutes a solution of the Nirenberg problem itself. Our focus lies in providing a comprehensive description of such blowing up solutions, including precise determinations of blow-up points and blow-up rates. Additionally, we compute the topological contribution of these solutions to the difference in topology between the level sets of the associated Euler-Lagrange functional. Such an analysis is intricate due to the potential degeneracy of the solutions involved. We also provide a partial converse, wherein we construct blowing up solutions when the weak limit is non-degenerate.

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1. Introduction and main results

Given a smooth positive function K defined on the standard sphere \mathbb{S}^n , where $n \geq 3$, equipped with its standard metric g_0 , the Nirenberg problem aims to determine a Riemannian metric g conformally equivalent to g_0 such that the scalar curvature $R_g = K$. This can be formulated as solving the following nonlinear problem involving the critical Sobolev exponent:

$$(\mathcal{N}_K) \quad L_{g_0} u = K u^{(n+2)/(n-2)}, \quad u > 0 \quad \text{on } \mathbb{S}^n,$$

where $L_{g_0} := -\Delta_{g_0} + n(n-2)/4$ denotes the conformal Laplacian.

This problem has garnered significant attention over the past half-century, with extensive research contributions. See [3, 5, 6, 10, 13–18, 21, 22, 25–27, 34] and the references therein.

Due to Kazdan-Warner topological obstructions [11, 22], the Nirenberg problem is not solvable for every function K . Thus, the central inquiry revolves around identifying sufficient conditions on K for the problem to be solvable.

Regarding existence results, Euler-Poincaré type criteria have been established by Bahri-Coron [6] and by Chang-Gursky-Yang [14] on S^3 , and by Ben Ayed et al. [10] and Yanyan Li [26] on S^4 . Furthermore, these results have been extended to higher-dimensional spheres under various conditions, including flatness conditions near the critical points of K [25], in the perturbative framework [13], or under pinching conditions [18] and [30].

The Nirenberg problem has a variational structure. Indeed its solutions correspond to the positive critical points of the functional

$$I_K(u) := \frac{1}{2} \|u\|^2 - \frac{1}{p+1} \int_{S^n} K|u|^{p+1}, \quad \text{with } p := \frac{n+2}{n-2}, \quad (1)$$

defined on $H^1(S^n)$ equipped with the scalar product

$$\langle u, w \rangle := \int_{S^n} \nabla u \nabla w + \frac{n(n-2)}{4} \int_{S^n} uw$$

and its associated norm

$$\|u\|^2 = \int_{S^n} |\nabla u|^2 + \frac{n(n-2)}{4} \int_{S^n} u^2.$$

The functional I_K fails to satisfy the Palais-Smale condition and the reason for such a lack of compactness is the existence of almost solutions of the equation (\mathcal{N}_K) . These almost solutions called *bubbles* are defined as follows:

$$\tilde{\delta}_{(a,\lambda)}(x) = c_0 \frac{\lambda^{(n-2)/2}}{(2 + (\lambda^2 - 1)(1 - \cos d_{g_0}(x, a)))^{(n-2)/2}}, \quad \text{with } a \in S^n \text{ and } \lambda > 0, \quad (2)$$

where d_{g_0} is the geodesic distance on (S^n, g_0) and $c_0 = (n(n-2))^{(n-2)/4}$.

After performing a stereographic projection Π with the point $-a$ as pole, the function $\tilde{\delta}_{(a,\lambda)}$ is transformed into

$$\delta_{(0,\lambda)}(y) = c_0 \frac{\lambda^{(n-2)/2}}{(1 + \lambda^2 |y|^2)^{(n-2)/2}}, \quad y \in \mathbb{R}^n, \quad (3)$$

which is a solution of the problem [12]

$$-\Delta u = u^{\frac{n+2}{n-2}}, \quad u > 0 \quad \text{in } \mathbb{R}^n.$$

One way to overcome the lack of compactness of the functional I_K , which goes back to Yamabe [38], is to lower the exponent and first consider the subcritical approximation

$$(\mathcal{N}_{K,\tau}) \quad L_{g_0} u = K u^{((n+2)/(n-2))-\tau}, \quad u > 0 \quad \text{on } S^n,$$

and its associated Euler-Lagrange functional

$$I_{K,\tau}(u) := \frac{1}{2} \|u\|^2 - \frac{1}{p+1-\tau} \int_{S^n} K|u|^{p+1-\tau}, \quad u \in H^1(S^n) \text{ with } p := \frac{n+2}{n-2}.$$

Thanks to elliptic estimates, either the solution u_τ remains uniformly bounded as $\tau \rightarrow 0$ and hence converges strongly to a solution ω of the Nirenberg problem (\mathcal{N}_K) or it blows up. In the latter case, following Schoen [33], Yanyan Li [25, 26], Chen-lin [16, 17] or Druet-Hebey-Robert [20], one performs a refined blow up analysis. It follows from such a blow up analysis that on \mathbb{S}^3 only single blow up can occur, whereas on \mathbb{S}^4 multiple blow up can occur, yet they are isolated simple, see [6, 14, 25, 26]. However, the scenario shifts notably in higher-dimensional spheres. While on $\mathbb{S}^5, \mathbb{S}^6$ blowing up solutions have finite energy [17], Chen-Lin [15] constructed a sequence blowing up solutions of infinite energy. A fact that underscores the profound challenge presented by higher-dimensional spheres. Recently Malchiodi-Mayer [28, 29] have shown, that the blow up points of finite energy blowing up solutions having zero weak limit are all isolated simple. Motivated by the intricate question of the multiplicity of solutions to the Nirenberg problem, we undertake in this paper a systematic analysis of the scenario where a sequence of energy bounded solution of $(\mathcal{N}_{K,\tau})$ has a non zero weak limit. We point out that such a weak limit is necessarily a solution of (\mathcal{N}_K) and it is known that such a situation does not occur if the dimension of the sphere is less than six, see [5, 10, 25, 26]. For spheres of dimension $n \geq 7$ our main result can be stated as follows:

Theorem 1.1. *Let $n \geq 7, 0 < K \in C^3(\mathbb{S}^n)$ having only non-degenerate critical points y_1, \dots, y_q with $\Delta K(y_i) \neq 0$ for $1 \leq i \leq q$. Let (u_τ) be a sequence of energy bounded solutions of $(\mathcal{N}_{K,\tau})$ converging weakly but not strongly $u_\tau \rightharpoonup \omega \neq 0$. Then ω is a solution of (\mathcal{N}_K) . Furthermore there exists an integer $N \leq q$ such that u_τ has to blow up and takes the following form*

$$u_\tau = \omega + \sum_{i=1}^N \frac{1}{K(a_{i,\tau})^{(n-2)/4}} \tilde{\delta}_{a_{i,\tau}, \lambda_{i,\tau}} + v_\tau,$$

where $\tilde{\delta}$ is defined in (2) and $v_\tau \rightarrow 0$ in $H^1(\mathbb{S}^n)$.

Furthermore we have that:

- i) $a_{i,\tau}$ converges to a critical point y_i of K with $\Delta K(y_i) < 0$.
- ii) $\lambda_{i,\tau} \rightarrow \infty, \tau \ln \lambda_{i,\tau} \rightarrow 0$ and $\lambda_{i,\tau} d_{g_0}(a_{i,\tau}, y_i) \rightarrow 0$ as $\tau \rightarrow 0$.
- iii) For $i \neq j, d_{g_0}(a_{i,\tau}, a_{j,\tau}) \geq \frac{1}{2} \min\{d_{g_0}(y_k, y_\ell) : y_k \neq y_\ell \in \mathcal{K}^\infty\}$ where

$$\mathcal{K}^\infty := \{y \in \mathbb{S}^n : \nabla K(y) = 0 \text{ and } \Delta K(y) < 0\}. \tag{4}$$

- iv) There exists a dimensional constant $\kappa_1(n) > 0$ such that

$$\frac{1}{\lambda_{i,\tau}^2} = -\kappa_1(n) \frac{K(y_i)}{\Delta K(y_i)} \tau(1 + o_\tau(1)). \tag{5}$$

In addition, if ω is a non-degenerate critical point of I_K , then u_τ is a non-degenerate critical point of $I_{K,\tau}$ and

$$\begin{aligned} \text{Morse index}(I_{K,\tau}, u_\tau) &= (N + 1) + \text{Morse index}(I_K, \omega) \\ &\quad + \sum_{i=1}^N (n - \text{Morse index}(K, y_i)), \end{aligned}$$

where the morse index of $I_{K,\tau}$ at the critical point u_τ is the number of the negative eigenvalues of the linearized operator

$$L_{g_0} - \left(\frac{n+2}{n-2} - \tau \right) K u_\tau^{\frac{4}{n-2} - \tau}.$$

Remark 1.1.

- a) We point out that Theorem 1.1 remains valid for a sign-changing function K provided the additional condition that 0 is a regular value of K is satisfied. This result stems from the non-existence of positive solutions to the following PDEs:

$$\Delta u = u^{\frac{n+2}{n-2}} \quad \text{and} \quad -\Delta u = x_n u^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n.$$

For further details, see Lin [27].

- b) The existence of blow-up solutions with a non-zero weak limit is excluded if the function K is sufficiently flat near its critical points, with the flatness order satisfying $\beta > \frac{n-2}{2}$. Consequently, under this flatness condition, the phenomena studied in our paper do not occur.
- c) Abbas Bahri [5] constructed a pseudogradient for the Euler-Lagrange functional associated to the limit problem $(\mathcal{N}_{K,0})$ and investigated its non-precompact flow lines. The description of such diverging flow lines corresponds precisely to the analysis of blow-up solutions in the subcritical approximation studied in [29] when the weak limit is zero, and to Theorem 1.1 when the weak limit is non-zero. See [31]. In comparison to Proposition 2' in [5], our analysis is more refined, as it offers a more precise description of the blow-up behavior. However, the primary advantage of our approach is that it allows us to define and compute local degrees of the solutions, leading to the derivation of the existence of arbitrarily many solutions when the function K is close to a constant. See Theorems 1.1 and 1.2 in [1] for further details.

Regarding the proof strategy of Theorem 1.1, it is essential to provide some insights. While conventional blow-up analysis techniques typically rely heavily on precise pointwise C^0 -estimates of the blowing-up solutions u_τ and extensively employ Pohozaev identities [16, 17, 20, 23, 25, 26, 29, 33], our approach diverges from this path. Our strategy hinges on deriving balancing conditions governing the parameters of concentration, utilizing refined asymptotic estimates of the gradient within the so-called “neighborhood at infinity”. These balancing conditions emerge through testing the equation with vector fields representing the leading terms of the gradient concerning the concentration parameters. By analyzing these conditions, we can extract all pertinent information regarding the blow-up phenomenon. We highlight that the potential degeneracy of the weak limit introduces an additional challenging dimension to the problem. Estimating parameters within the nontrivial kernel of the linearized operator becomes particularly arduous in such scenarios. It is worth noting that circumventing the reliance on pointwise estimates and Pohozaev identities can be advantageous, especially in the exploration of non-compact variational problems where non-simple blow-ups may arise, such as in

the singular mean-field equation with quantized singularities [9, 19, 24, 36, 37], and in the Nirenberg problem on half-spheres [2]. The presence of non-simple blow-up points significantly complicates the task of establishing pointwise C^0 -estimates, making our method particularly valuable in such contexts.

In the next theorem we provide, under generic condition, the following converse of Theorem 1.1.

Theorem 1.2. *Let $n \geq 7$, $0 < K \in C^3(\mathbb{S}^n)$. Let y_1, \dots, y_N be distinct non-degenerate critical points of K with $\Delta K(y_i) < 0$ and ω be a non-degenerate solution of (\mathcal{N}_K) . Then, as $\tau \rightarrow 0$, there exists a unique solution $u_{\tau, \omega, y_1, \dots, y_N}$ of $(\mathcal{N}_{K, \tau})$ satisfying: $u_{\tau, \omega, y_1, \dots, y_N} - \omega$ develops exactly one bubble at each point y_i and converges weakly to zero in $H^1(\mathbb{S}^n)$ as $\tau \rightarrow 0$. More precisely there exist $\lambda_{1, \tau}, \dots, \lambda_{N, \tau}$ having the same order as $\tau^{-1/2}$ and points $a_{i, \tau} \rightarrow y_i$ for all i such that*

$$\left\| u_{\tau, \omega, y_1, \dots, y_N} - \omega - \sum_{i=1}^N K(a_i)^{\frac{2-n}{4}} \tilde{\delta}_{a_{i, \tau}, \lambda_{i, \tau}} \right\| \rightarrow 0 \quad \text{and}$$

$$I_\tau(u_{\tau, \omega, y_1, \dots, y_N}) \rightarrow \frac{1}{n} \|\omega\|^2 + \frac{1}{n} S_n \left(\sum_{i=1}^N K(a_i)^{\frac{2-n}{2}} \right),$$

as $\tau \rightarrow 0$, where

$$S_n := c_0^{2n/(n-2)} \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^n} dx = \left(\pi n(n-2) \right)^{n/2} \frac{\Gamma(n/2)}{\Gamma(n)}. \tag{6}$$

In addition, $u_{\tau, \omega, y_1, \dots, y_N}$ is non-degenerate for $I_{K, \tau}$ and

$$\begin{aligned} \text{Morse index}(I_{K, \tau}, u_{\tau, \omega, y_1, \dots, y_N}) &= (N + 1) + \text{Morse index}(I_K, \omega) \\ &\quad + \sum_{i=1}^N (n - \text{Morse index}(K, y_i)). \end{aligned}$$

We emphasize that the precise characterization of blow-up solutions presented in this paper, particularly in Theorems 1.1 and 1.2, serves as a crucial foundation for our subsequent work in [1], which explores the existence of multiple solutions. These results allow us to establish the compactness of solutions and facilitate both the definition and computation of the degree of solutions whose energies lie within specific intervals. In particular, they are pivotal in demonstrating the existence of infinitely many solutions to Nirenberg’s problem under perturbative conditions. For further details, see Theorems 1.1 and 1.2 in [1].

The subsequent sections of the paper are structured as follows: In Sect. 2, we perform a finite-dimensional reduction while Sect. 3 delves into providing a meticulous expansion of the gradient within the neighborhood at infinity. The refined blow-up analysis of finite energy approximated solutions to problem $(\mathcal{N}_{K, \tau})$, with a non-zero weak limit, is detailed in Sect. 4 while Sect. 5 is dedicated to the proof of Theorem 1.2. Lastly, in the appendix, we gather some indispensable estimates crucial for substantiating various assertions throughout this paper.

2. Finite dimensional reduction

In this section, we perform a Lyapunov-Schmidt-type reduction by demonstrating that any sequence of blow-up solutions can be decomposed into a finite-dimensional component, defined on a space spanned by the bubbles and their derivatives with respect to the parameters of the concentration α_i, λ_i, a_i , along with a small infinite-dimensional remainder, which can be neglected. Although this procedure is now classical, it presents a new aspect in our setting because the solutions involved in the description of blow-up behavior may degenerate. Therefore, the primary objective of this section is to address this potential degeneracy.

Let ω be a solution of (\mathcal{N}_K) and let $N_0(\omega)$ be the kernel of the associated quadratic form defined by:

$$Q_\omega(h) := \|h\|^2 - \frac{n+2}{n-2} \int_{\mathbb{S}^n} K\omega^{4/(n-2)}h^2 \quad \text{for } h \in H^1(\mathbb{S}^n). \quad (7)$$

Let m_2 be the dimension of $N_0(\omega)$ and (e_1, \dots, e_{m_2}) be an orthonormal basis of $N_0(\omega)$. We set

$$H_0(\omega) := \text{span}(\omega) \oplus \text{span}(e_1, \dots, e_{m_2}).$$

Following M. Mayer, see Lemma 3.6 and Proposition 3.7 in [32], we parameterize a neighborhood of ω by

$$\begin{aligned} u_{\alpha,\beta} &:= \alpha\left(\omega + \sum_{i=1}^{m_2} \beta_i e_i + h(\beta)\right) \quad \text{with } h(\beta) \perp H_0(\omega); \\ h(\beta) &= O(\|\beta\|^2) \text{ and } \|h(\beta)\|_{C^2} \rightarrow 0, \end{aligned} \quad (8)$$

where $\beta := (\beta_1, \dots, \beta_{m_2}) \in \mathbb{R}^{m_2}$, α close to 1 and the function $u_{\alpha,\beta}$ satisfies

$$\langle \nabla I_K(u_{\alpha,\beta}), h \rangle := \langle u_{\alpha,\beta}, h \rangle - \int_{\mathbb{S}^n} K u_{\alpha,\beta}^p h = 0 \quad \text{for each } h \in H_0(\omega)^\perp. \quad (9)$$

Next for ω a solution of (\mathcal{N}_K) whose Kernel is of dimension m_2 , $N \in \mathbb{N}_0$ and μ a small positive constant, we define the so called *neighborhood at infinity* $V(\omega, N, \mu)$ as follows:

$$\begin{aligned} V(\omega, N, \mu) &:= \left\{ u \in H^1(\mathbb{S}^n) : \exists \lambda_1, \dots, \lambda_N > \mu^{-1}; \exists a_1, \dots, a_N \in \mathbb{S}^n, \quad \text{with} \right. \\ &\quad \varepsilon_{ij} < \mu; \exists \alpha_0 \in (1 - \mu, 1 + \mu); \exists \beta \in \mathbb{R}^{m_2} \quad \text{with } \|\beta\| \leq c\mu \\ &\quad \left. \text{such that } \|u - \sum_{i=1}^N K(a_i)^{(2-n)/4} \tilde{\delta}_{a_i, \lambda_i} - u_{\alpha_0, \beta}\| < \mu \right\}, \end{aligned} \quad (10)$$

where

$$\varepsilon_{ij} := \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \frac{1}{2} \lambda_i \lambda_j (1 - \cos d(a_i, a_j)) \right)^{(2-n)/2}. \quad (11)$$

Following Bahri-Coron [7] we consider, for $u \in V(\omega, N, \mu)$, the following minimization problem

$$\text{Min} \left\{ \left\| u - \sum_{i=1}^N \alpha_i \tilde{\delta}_{a_i, \lambda_i} - u_{\alpha_0, \beta} \right\| ; \alpha_i > 0 ; \beta \in \mathbb{R}^{m_2} ; \lambda_i > 0, a_i \in \mathbb{S}^n \right\}. \quad (12)$$

We then have the following proposition whose proof is identical, up to minor modification, to the one of Proposition 7 in [7].

Proposition 2.1. *For any $N \in \mathbb{N}_0$ there exists $\mu_0 > 0$ such that if $\mu < \mu_0$ and $u \in V(\omega, N, \mu)$ the minimization problem (12) has, up to permutation of the indices, a unique solution.*

Hence it follows from Proposition 2.1 that every $u \in V(\omega, N, \mu)$ can be written in a unique way as

$$u = \sum_{i=1}^N \alpha_i \tilde{\delta}_{a_i, \lambda_i} + u_{\alpha_0, \beta} + v, \quad \text{where} \quad (13)$$

$$a_i \in \mathbb{S}^n, i \leq N, \quad \alpha_i^{4/(n-2)} K(a_i) = 1 + o(1) \quad \forall i \geq 1, \quad \alpha_0 = 1 + o(1), \quad (14)$$

and $v \in H^1(\mathbb{S}^n)$ satisfying

$$\|v\| < \mu, \quad \langle v, \psi \rangle = 0, \quad \text{for } \psi \in E_{\omega, a, \lambda} \quad \text{where} \quad (15)$$

$$E_{\omega, a, \lambda} := \text{span} \left\{ \tilde{\delta}_i, \frac{\partial \tilde{\delta}_i}{\partial \lambda_i}, \frac{\partial \tilde{\delta}_i}{\partial a_i}, u_{\alpha_0, \beta}, \frac{\partial u_{\alpha_0, \beta}}{\partial \beta_k} ; 1 \leq i \leq N ; ; k \leq m_2 \right\}, \quad (16)$$

where $\tilde{\delta}_i := \tilde{\delta}_{a_i, \lambda_i}$. Furthermore a refined Struwe energy type decomposition ([35]) has been proved in [29]. Namely the following result holds:

Proposition 2.2. ([29], Proposition 3.1) *Let u_τ be an energy bounded solution of $(\mathcal{N}_{K, \tau})$ which blows up. We assume that there exists a positive solution ω of (\mathcal{N}_K) such that $u_\tau \rightarrow \omega$ (but $u_\tau \not\rightarrow \omega$). Then there exists N such that u_τ can be written as*

$$u_\tau := u_{\alpha, \beta} + \sum_{i=1}^N \alpha_i \tilde{\delta}_{a_i, \lambda_i} + v_\tau \in V(\omega, N, \mu) \quad (17)$$

with $v_\tau \in E_{\omega, a, \lambda}^\perp$ and $\tau \ln \lambda_i = o_\tau(1) \quad \forall i$.

In what follows, we assume that u_τ is decomposed as in (17). Now, we are going to estimate the remainder term v_τ in Proposition 2.2. To this aim, we need to give some information about the second variation of the Euler-Lagrange functional with respect to the infinite dimensional variable v . First, for a solution ω of (\mathcal{N}_K) (not necessary a non-degenerate one), we decompose $H^1(\mathbb{S}^n)$ as follows:

$$H^1(\mathbb{S}^n) := N_-(\omega) \oplus H_0(\omega) \oplus N_+(\omega) \quad \text{where } H_0(\omega) := \text{span}\{\omega\} \oplus N_0(\omega) \quad (18)$$

where $N_-(\omega)$, $N_0(\omega)$ and $N_+(\omega)$ are respectively the space of negativity, of nullity and of positivity of the quadratic form Q_ω (defined by (7)) in $\text{span}\{\omega\}^\perp$.

Note that these spaces are orthogonal spaces with respect to $\langle \cdot, \cdot \rangle$ and the bilinear form $B_\omega(\cdot, \cdot)$ ($:= \int_{\mathbb{S}^n} K\omega^{p-1} \cdot \cdot$). Furthermore, the sequence of the eigenvalues (denoted by (σ_i)) corresponding to Q_ω satisfies $\sigma_i \nearrow 1$. Therefore, there exists a constant $c > 0$ such that

$$\begin{aligned} Q_\omega(h) &\leq -c \|h\|^2 \quad \text{for each } h \in N_-(\omega); \\ Q_\omega(h) &\geq c \|h\|^2 \quad \text{for each } h \in N_+(\omega). \end{aligned} \tag{19}$$

Lemma 2.3. *Assume that the ε_{ij} 's are small. Let $v \in E_{\omega,a,\lambda}^\perp$ (defined in (16)). Written v as*

$$v := v_- + v_0 + v_+ \quad \text{with} \quad v_- \in N_-(\omega); \quad v_0 \in H_0(\omega); \quad v_+ \in N_+(\omega) \quad (\text{see (18)}).$$

Let us define

$$Q_{\omega,a,\lambda}(v) := \|v\|^2 - p \sum \int_{\mathbb{S}^n} \tilde{\delta}_i^{p-1} v^2 - p \int_{\mathbb{S}^n} K\omega^{p-1} v^2. \tag{20}$$

Then, $Q_{\omega,a,\lambda}$ is a non-degenerate quadratic form on $E_{\omega,a,\lambda}^\perp$. More precisely, there exists a positive constant \underline{c} such that

$$\|v_0\| = o(\|v\|), \tag{21}$$

$$Q_{\omega,a,\lambda}(v_-) = Q_\omega(v_-) + o(\|v_-\|^2) \leq -\underline{c} \|v_-\|^2, \tag{22}$$

$$Q_{\omega,a,\lambda}(v_+) \geq \underline{c} \|v_+\|^2 + o(\|v\|^2), \tag{23}$$

where Q_ω is defined in (7).

Proof. First, we remark that, (21)–(23) imply that $Q_{\omega,a,\lambda}$ is non-degenerate on $E_{\omega,a,\lambda}^\perp$.

Second, note that the spaces $H_0(\omega)$, $N_-(\omega)$ and $N_+(\omega)$ are orthogonal spaces with respect to $\langle \cdot, \cdot \rangle$ and the associated bilinear form $B_\omega(\cdot, \cdot)$ ($:= \int_{\mathbb{S}^n} K\omega^{p-1} \cdot \cdot$).

We start by proving (21). Since $v_0 \in H_0(\omega)$, it follows that $v_0 = \gamma_0 \omega + \sum \gamma_i e_i$ where (e_1, \dots, e_{m_2}) is an orthonormal basis of $N_0(\omega)$. Using the fact that $v \in E_{\omega,a,\lambda}^\perp$ (which implies that $v \perp u_{\alpha,\beta}$ and $v \perp \partial u_{\alpha,\beta} / \partial \beta_i$ for each i), it follows that

$$\begin{aligned} \gamma_0 &= \langle v_0, \omega \rangle = \langle v, \omega \rangle \\ &= (1/\alpha) \langle v, u_{\alpha,\beta} \rangle - (1/\alpha) \sum \beta_i \langle v, e_i \rangle - \langle v, h(\beta) \rangle = o(\|v\|), \\ \gamma_i &= \langle v_0, e_i \rangle = \langle v, e_i \rangle \\ &= \langle v, e_i + (\partial h(\beta) / \partial \beta_i) \rangle - \langle v, \partial h(\beta) / \partial \beta_i \rangle = o(\|v\|) \quad \forall 1 \leq i \leq m_2 \end{aligned}$$

(by using the smallness of β and $h(\beta)$ in the C^1 sense with respect to β). This ends the proof of (21).

Concerning (22), we have

$$Q_{\omega,a,\lambda}(v_-) = \|v_-\|^2 - p \sum \int \tilde{\delta}_i^{p-1} v_-^2 - p \int K\omega^{p-1} v_-^2 = Q_\omega(v_-) - p \sum \int \tilde{\delta}_i^{p-1} v_-^2.$$

Observe that, since v_- belongs to a fixed finite dimensional space, we derive that $\|v_-\|_\infty \leq c\|v_-\|$ and therefore

$$\int \tilde{\delta}_i^{p-1} v_-^2 \leq \|v_-\|_\infty^2 \int \tilde{\delta}_i^{p-1} = o(\|v_-\|^2) \quad \text{for each } i.$$

Hence the proof of (22) follows by using (19).

It remains to prove (23). Note that, using Proposition 3.1 of [4], there exists a constant $\underline{c}_1 > 0$ such that

$$\|h\|^2 - \frac{n+2}{n-2} \sum_{i=1}^N \int_{\mathbb{S}^n} \tilde{\delta}_{a_i, \lambda_i}^{\frac{4}{n-2}} h^2 \geq \underline{c}_1 \|h\|^2 \quad \text{for each } h \in E_{a, \lambda}^\perp. \quad (24)$$

where $E_{a, \lambda}^\perp$ is introduced in (16) (with $\omega = 0$).

In addition, the sequence of the eigenvalues (denoted by (σ_i)) corresponding to Q_ω (defined by (7)) satisfies $\sigma_i \nearrow 1$. Let $N_k(\omega)$ be the eigenspace associated to the eigenvalue σ_k . These spaces are orthogonal with respect to $\langle \cdot, \cdot \rangle$ and the bilinear form B_ω . Let $\sigma_{k_0} := \min\{\sigma_i : \sigma_i > 0\}$. Hence it is easy to see that $N_+(\omega) = \oplus_{k \geq k_0} N_k(\omega)$. Furthermore, for $k \in \mathbb{N}$, it holds

$$Q_\omega(h) \geq \sigma_k \|h\|^2 \quad \text{for each } h \in \oplus_{j \geq k} N_j(\omega). \quad (25)$$

Let k_1 be such that $\sigma_{k_1} \geq 1 - \underline{c}_1/4$. We decompose $N_+(\omega)$ as follows:

$$N_+(\omega) := (\oplus_{k_0 \leq k \leq k_1} N_k(\omega)) \oplus (\oplus_{k > k_1} N_k(\omega)) := N_+^{0,1}(\omega) \oplus N_+^1(\omega).$$

Note that $N_+^{0,1}(\omega)$ is a fixed finite dimensional space. Now, since $v_+ \in N_+(\omega)$, it holds that

$$v_+ := v_0^+ + v_1^+ \quad \text{where } v_0^+ \in N_+^{0,1}(\omega) \text{ and } v_1^+ \in N_+^1(\omega). \quad (26)$$

Hence, using the orthogonality of the spaces $N_+^{0,1}(\omega)$ and $N_+^1(\omega)$, it follows that

$$\begin{aligned} Q_{\omega, a, \lambda}(v_+) &= \|v_0^+\|^2 + \|v_1^+\|^2 - p \sum_{i=1}^N \int_{\mathbb{S}^n} \tilde{\delta}_{a_i, \lambda_i}^{\frac{4}{n-2}} \{(v_0^+)^2 + (v_1^+)^2 + 2v_0^+ v_1^+\} \\ &\quad - p \int_{\mathbb{S}^n} K \omega^{\frac{4}{n-2}} \{(v_0^+)^2 + (v_1^+)^2\}. \end{aligned}$$

Observe that

$$\begin{aligned} \|v_0^+\|^2 - \frac{n+2}{n-2} \int_{\mathbb{S}^n} K \omega^{\frac{4}{n-2}} (v_0^+)^2 &= Q_\omega(v_0^+) \geq \sigma_{k_0} \|v_0^+\|^2 \quad (\text{by using (25)}) \\ \int_{\mathbb{S}^n} \tilde{\delta}_{a_i, \lambda_i}^{\frac{4}{n-2}} (v_0^+)^2 &\leq \|v_0^+\|_\infty^2 \int_{\mathbb{S}^n} \tilde{\delta}_{a_i, \lambda_i}^{\frac{4}{n-2}} = o(\|v_0^+\|^2) \\ \int_{\mathbb{S}^n} \tilde{\delta}_{a_i, \lambda_i}^{\frac{4}{n-2}} |v_0^+| |v_1^+| &\leq \|v_0^+\|_\infty \int_{\mathbb{S}^n} \tilde{\delta}_{a_i, \lambda_i}^{\frac{4}{n-2}} |v_1^+| = o(\|v_0^+\| \|v_1^+\|) = o(\|v_1^+\|^2 + \|v_0^+\|^2) \\ -p \int_{\mathbb{S}^n} K \omega^{\frac{4}{n-2}} (v_1^+)^2 &= Q_\omega(v_1^+) - \|v_1^+\|^2 \geq (\sigma_{k_1} - 1) \|v_1^+\|^2 \geq -(\underline{c}_1/4) \|v_1^+\|^2 \end{aligned}$$

where for the last formula, we have used (25) and the choice of k_1 . Combining these estimates, we get

$$\begin{aligned}
 Q_{\omega,a,\lambda}(v_+) & \geq \{ \|v_1^+\|^2 - p \sum_{i=1}^N \int_{\mathbb{S}^n} \widetilde{\delta}_{a_i,\lambda_i}^{\frac{4}{n-2}}(v_1^+)^2 \} + \sigma_{k_0} \|v_0^+\|^2 - \frac{c_1}{4} \|v_1^+\|^2 + o(\|v_1^+\|^2 + \|v_0^+\|^2).
 \end{aligned}
 \tag{27}$$

Note that the function v_1^+ is not necessarily in $E_{a,\lambda}^\perp$. For this reason we write $v_1^+ := \sum t_i \psi_i + \tilde{v}_1^+$ with $\tilde{v}_1^+ \in E_{a,\lambda}^\perp$ where the ψ_i 's are the functions $\tilde{\delta}_j$'s and their derivatives with respect to λ_j and a_j^k . Then t_i satisfies that

$$\begin{aligned}
 t_i + o\left(\sum |t_k|\right) & = \langle v_1^+, \psi_i \rangle = \langle v, \psi_i \rangle - \langle v_0, \psi_i \rangle - \langle v_-, \psi_i \rangle - \langle v_0^+, \psi_i \rangle \\
 & = O\left(\int \tilde{\delta}_i^p(|v_0| + |v_-| + |v_0^+|)\right) = o(\|v_0\| + \|v_-\| + \|v_0^+\|) = o(\|v\|)
 \end{aligned}$$

by using the fact that these functions are in fixed finite dimensional spaces. Thus we derive that $\|v_1^+\|^2 = \|\tilde{v}_1^+\|^2 + o(\|v\|^2)$ and therefore, using (24), we get

$$\begin{aligned}
 \|v_1^+\|^2 - p \sum_{i=1}^N \int_{\mathbb{S}^n} \widetilde{\delta}_{a_i,\lambda_i}^{\frac{4}{n-2}}(v_1^+)^2 & = \|\tilde{v}_1^+\|^2 - p \sum_{i=1}^N \int_{\mathbb{S}^n} \widetilde{\delta}_{a_i,\lambda_i}^{\frac{4}{n-2}}(\tilde{v}_1^+)^2 + o(\|\tilde{v}_1^+\|^2 + \|v\|^2) \\
 & \geq \frac{1}{2} c_1 \|v_1^+\|^2 + o(\|v\|^2).
 \end{aligned}
 \tag{28}$$

Combining (26), (27) and (28), we get

$$\begin{aligned}
 Q_{\omega,a,\lambda}(v_+) & \geq \frac{1}{4} c_1 \|v_1^+\|^2 + \sigma_{k_0} \|v_0^+\|^2 + o(\|v_1^+\|^2 + \|v_0^+\|^2) + o(\|v\|^2) \\
 & \geq c \|v_+\|^2 + o(\|v\|^2).
 \end{aligned}$$

Thus the result follows. □

Next, we prove the following proposition which gives the estimate of the infinite dimensional part of u_τ .

Proposition 2.4. *Let v_τ be the remainder term defined in Proposition 2.2. Then there holds:*

$$\begin{aligned}
 \|v_\tau\| & \leq cR(\tau, a, \lambda) \quad \text{with} \\
 R(\tau, a, \lambda) & := \tau + \sum_{i=1}^N \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{1 \leq i \neq j \leq N} \varepsilon_{ij}^{\frac{n+2}{2(n-2)}} (\ln \varepsilon_{ij}^{-1})^{\frac{n+2}{2n}}.
 \end{aligned}$$

Proof. Using (18), v_τ can be decomposed as follows

$$v_\tau := v_\tau^- + v_\tau^0 + v_\tau^+ \quad \text{where } v_\tau^0 \in H_0(\omega) ; v_\tau^- \in N_-(\omega) \text{ and } v_\tau^+ \in N_+(\omega).
 \tag{29}$$

Since $v_\tau \in E_{w,a,\lambda}^\perp$, using Lemma 2.3, we get that $v_\tau^0 = o(\|v_\tau\|)$. For the other parts, note that v_τ^+ and v_τ^- are not necessarily in $E_{a,\lambda}^\perp$ but they are in $H_0(\omega)^\perp$.

Now we will focus on estimating v_τ^- . Since u_τ is a solution of $(\mathcal{N}_{K,\tau})$, we get

$$\langle \nabla I_{K,\tau}(u_\tau), v_\tau^- \rangle = 0.$$

Notice that

$$\langle \nabla I_{K,\tau}(u), h \rangle = \langle u, h \rangle - \int_{\mathbb{S}^n} K|u|^{p-1-\tau} u h \quad \text{for each } h \in H^1(\mathbb{S}^n). \quad (30)$$

Now, taking $\bar{u}_\tau := u_\tau - v_\tau$ and using (81), we obtain

$$\begin{aligned} & \sum \alpha_j \langle \tilde{\delta}_j, v_\tau^- \rangle + \langle u_{\alpha,\beta}, v_\tau^- \rangle + \|v_\tau^-\|^2 \\ &= \int K \bar{u}_\tau^{p-\tau} v_\tau^- \\ &= \int K \left(\bar{u}_\tau^{p-\tau} + (p-\tau) \bar{u}_\tau^{p-1-\tau} v_\tau + O(|v_\tau|^{p-\varepsilon}) \right) v_\tau^- \\ &= \int K \bar{u}_\tau^{p-\tau} v_\tau^- + p \int K \bar{u}_\tau^{p-1-\tau} v_\tau v_\tau^- + o(\|v_\tau\| \|v_\tau^-\|) \end{aligned} \quad (31)$$

Using (80), we see that

$$\begin{aligned} \bar{u}_\tau^{p-1-\tau} &= \sum_{i=1}^N (\alpha_i \tilde{\delta}_i)^{p-1-\tau} + u_{\alpha,\beta}^{p-1-\tau} \\ &+ O\left(\sum_{1 \leq j \neq i \leq N} (\tilde{\delta}_i \tilde{\delta}_j)^{(p-1)/2} + \sum_{i=1}^N (\tilde{\delta}_i u_{\alpha,\beta})^{(p-1)/2} \right). \end{aligned}$$

Notice that, estimate (14) implies that

$$u_{\alpha,\beta} = \omega + o(1).$$

Thus, using the fact that v_τ^- is in a finite dimensional space (which implies that $\|\cdot\|_\infty \leq c\|\cdot\|$), we derive that

$$\begin{aligned} & \int K \bar{u}_\tau^{p-1-\tau} v_\tau v_\tau^- \\ &= \sum_{i=1}^N \int K (\alpha_i \tilde{\delta}_i)^{p-1-\tau} v_\tau v_\tau^- + \int K u_{\alpha,\beta}^{p-1-\tau} v_\tau v_\tau^- + o(\|v_\tau\| \|v_\tau^-\|) \\ &= \sum_{i=1}^N O(|v_\tau^-|_\infty \int \tilde{\delta}_i^{p-1} |v_\tau|) + \int K \omega^{p-1} (v_\tau^0 + v_\tau^- + v_\tau^+) v_\tau^- + o(\|v_\tau\| \|v_\tau^-\|) \\ &= \int K \omega^{p-1} (v_\tau^-)^2 + o(\|v_\tau\| \|v_\tau^-\|) \end{aligned} \quad (32)$$

where we have used the orthogonality of v_τ^- , v_τ^0 and v_τ^+ with respect to the bilinear form $(h_1, h_2) \mapsto \int K \omega^{p-1} h_1 h_2$. Thus (31) and (32) imply that

$$-Q_\omega(v_\tau^-) + o(\|v_\tau\| \|v_\tau^-\|) = \sum \alpha_j \langle \tilde{\delta}_j, v_\tau^- \rangle + \langle u_{\alpha,\beta}, v_\tau^- \rangle - \int K \bar{u}_\tau^{p-\tau} v_\tau^- := \ell(v_\tau^-). \quad (33)$$

Observe that, using $\|v_\tau^-\|_\infty \leq c\|v_\tau^-\|$, it holds that

$$|\langle \tilde{\delta}_i, v_\tau^- \rangle| \leq c \int \tilde{\delta}_i^p |v_\tau^-| \leq c \|v_\tau^-\|_\infty \int \tilde{\delta}_i^p \leq c \frac{\|v_\tau^-\|}{\lambda_i^{(n-2)/2}} \quad ; \quad \int \tilde{\delta}_i |v_\tau^-| \leq c \frac{\|v_\tau^-\|}{\lambda_i^{(n-2)/2}}. \tag{34}$$

$$\begin{aligned} \int K \bar{u}_\tau^{p-\tau} v_\tau^- &= \int K u_{\alpha,\beta}^{p-\tau} v_\tau^- + \sum O\left(\int u_{\alpha,\beta}^{p-1} \delta_i |v_\tau^-| + \int \delta_i^{p-\tau} |v_\tau^-|\right) \\ &= \int K u_{\alpha,\beta}^p v_\tau^- + O\left(\|v_\tau^-\| \left(\tau + \sum \frac{1}{\lambda_i^{(n-2)/2}}\right)\right). \end{aligned} \tag{35}$$

Therefore, combining (19), (33)–(35) and (9), we get

$$c \|v_\tau^-\|^2 \leq -Q_\omega(v_\tau^-) \leq c \|v_\tau^-\| \left(\tau + \sum \frac{1}{\lambda_i^{(n-2)/2}} + o(\|v_\tau\|)\right). \tag{36}$$

It remains to estimate the v_τ^+ -part. Recall that $v_\tau^+ \in N_+(\omega)$ and it is not necessarily in $E_{a,\lambda}^\perp$. Note that Eq. (31) holds with v_τ^+ instead of v_τ^- . Observe that, using (80) and (14), it holds

$$\begin{aligned} &\int K \bar{u}_\tau^{p-1-\tau} v_\tau v_\tau^+ \\ &= \sum_{i=1}^N \int K (\alpha_i \tilde{\delta}_i)^{p-1-\tau} v_\tau v_\tau^+ + \int K u_{\alpha,\beta}^{p-1-\tau} v_\tau v_\tau^+ + o(\|v_\tau\| \|v_\tau^+\|) \\ &= \sum_{i=1}^N \int \tilde{\delta}_i^{p-1} v_\tau v_\tau^+ + \int K \omega^{p-1} (v_\tau^0 + v_\tau^- + v_\tau^+) v_\tau^+ + o(\|v_\tau\| \|v_\tau^+\|) \\ &= \sum_{i=1}^N \int \tilde{\delta}_i^{p-1} (v_\tau^+)^2 + \int K \omega^{p-1} (v_\tau^+)^2 + o(\|v_\tau\| \|v_\tau^+\|), \end{aligned} \tag{37}$$

by using the fact that $\|v_\tau^-\|_\infty \leq c\|v_\tau^-\| \leq c\|v_\tau\|$, $\|v_\tau^0\|_\infty \leq c\|v_\tau^0\| \leq c\|v_\tau\|$ and the fact that $\int K \omega^{p-1} v_\tau^- v_\tau^+ = \int K \omega^{p-1} v_\tau^0 v_\tau^+ = 0$. Hence we obtain

$$\begin{aligned} &Q_{\omega,a,\lambda}(v_\tau^+) + o(\|v_\tau\| \|v_\tau^+\|) \\ &= - \sum \alpha_j \langle \tilde{\delta}_j, v_\tau^+ \rangle - \langle u_{\alpha,\beta}, v_\tau^+ \rangle + \int K \bar{u}_\tau^{p-\tau} v_\tau^+ := -\ell(v_\tau^+). \end{aligned} \tag{38}$$

Using Lemma 2.3, we derive that $Q_{\omega,a,\lambda}(v_\tau^+) \geq c\|v_\tau^+\|^2 + o(\|v_\tau\|^2)$, hence it remains to estimate the linear part $\ell(v_\tau^+)$. In fact, using (80), we have

$$\begin{aligned} \int K \bar{u}_\tau^{p-\tau} v_\tau^+ &= \sum_{i=1}^N \int_{\mathbb{S}^n} K (\alpha_i \tilde{\delta}_i)^{p-\tau} v_\tau^+ + \int_{\mathbb{S}^n} K u_{\alpha,\beta}^{p-\tau} v_\tau^+ \\ &\quad + O\left(\sum_{1 \leq i \neq j \leq N} \int (\tilde{\delta}_i \tilde{\delta}_j)^{p/2} |v_\tau^+| + \sum_{i=1}^N \int (\tilde{\delta}_i \omega)^{p/2} |v_\tau^+|\right). \end{aligned} \tag{39}$$

Concerning the remainder terms, it follows that

$$\int (\tilde{\delta}_i \tilde{\delta}_j)^{p/2} |v_\tau^+| \leq \|v_\tau^+\| \left(\int (\tilde{\delta}_i \tilde{\delta}_j)^{n/(n-2)} \right)^{(n+2)/(2n)} \leq c \|v_\tau^+\| \varepsilon_{ij}^{\frac{n+2}{2(n-2)}} \ln^{\frac{n+2}{2n}}(\varepsilon_{ij}^{-1}), \tag{40}$$

$$\int_{\mathbb{S}^n} \tilde{\delta}_i^{p/2} |v_\tau^+| \leq \|v_\tau^+\| \left(\int_{\mathbb{S}^n} \tilde{\delta}_i^{n/(n-2)} \right)^{(n+2)/(2n)} \leq c \|v_\tau^+\| \frac{\ln^{(n+2)/(2n)}(\lambda_i)}{\lambda_i^{(n+2)/4}}, \tag{41}$$

where we have used the following estimate (see (E2) of [4])

$$\begin{aligned} \int_{\mathbb{S}^n} (\tilde{\delta}_k \tilde{\delta}_j)^{n/(n-2)} &= \int_{\mathbb{R}^n} (\delta_k \delta_j)^{n/(n-2)} \\ &\leq c \varepsilon_{kj}^{n/(n-2)} \ln(\varepsilon_{kj}^{-1}) \quad \forall k \neq j. \end{aligned} \tag{42}$$

For the other integrals in (39), we recall that $v_\tau^+ \in N_+(\omega)$ and therefore it follows that $v_\tau^+ \in H_0(\omega)^\perp$. Hence using (9), we derive that

$$\begin{aligned} -\langle u_{\alpha,\beta}, v_\tau^+ \rangle + \int_{\mathbb{S}^n} K u_{\alpha,\beta}^{p-\tau} v_\tau^+ &= -\langle u_{\alpha,\beta}, v_\tau^+ \rangle + \int_{\mathbb{S}^n} K u_{\alpha,\beta}^p v_\tau^+ + O(\tau \|v_\tau^+\|) \\ &= O(\tau \|v_\tau^+\|). \end{aligned} \tag{43}$$

Furthermore, using Lemma 6.1, it holds that

$$\begin{aligned} \int_{\mathbb{S}^n} K(\alpha_i \tilde{\delta}_i)^{p-\tau} v_\tau^+ &= \frac{\alpha_i^{p-\tau} c_0^{-\tau} K(a_i)}{\lambda_i^{\tau(n-2)/2}} \int_{\mathbb{S}^n} \tilde{\delta}_i^p v_\tau^+ + O\left(\int_{\mathbb{S}^n} |K(x) - K(a_i)| \tilde{\delta}_i^p |v_\tau^+| \right. \\ &\quad \left. + \tau \int \tilde{\delta}_i^p \ln[2 + (\lambda_i^2 - 1)(1 - \cos d(a_i, x))] |v_\tau^+| \right) \\ &= \frac{\alpha_i^{p-\tau} c_0^{-\tau} K(a_i)}{\lambda_i^{\tau(n-2)/2}} \langle \tilde{\delta}_i, v_\tau^+ \rangle + O\left(\|v_\tau^+\| \left(\tau + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \right) \right). \end{aligned} \tag{44}$$

Therefore we get

$$\begin{aligned} \alpha_i \langle \tilde{\delta}_i, v_\tau^+ \rangle - \int_{\mathbb{S}^n} K(\alpha_i \tilde{\delta}_i)^{p-\tau} v_\tau^+ \\ = \alpha_i \left(1 - \frac{\alpha_i^{p-1-\tau} K(a_i)}{c_0^\tau \lambda_i^{\tau(n-2)/2}} \right) \langle \tilde{\delta}_i, v_\tau^+ \rangle + O\left(\|v_\tau^+\| \left(\tau + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \right) \right). \end{aligned} \tag{45}$$

Finally, using the fact that $v_\tau \in E_{w,a,\lambda}^\perp$, as in the computations of (34), we derive that

$$\langle \tilde{\delta}_i, v_\tau^+ \rangle = \langle \tilde{\delta}_i, v_\tau \rangle - \langle \tilde{\delta}_i, v_\tau^0 \rangle - \langle \tilde{\delta}_i, v_\tau^- \rangle = O\left(\|v_\tau\| / \lambda_i^{(n-2)/2} \right) \quad \forall 1 \leq i \leq N.$$

Thus, using (19), (38) becomes

$$c \|v_\tau^+\|^2 \leq Q_{\omega,a,\lambda}(v_\tau^+) \leq o(\|v_\tau\|^2) + O\left(\|v_\tau\| \left(R(\tau, a, \lambda) \right) \right). \tag{46}$$

Thus using (29), (36) and (46), we derive that

$$\|v_\tau\|^2 = \|v_\tau^+\|^2 + \|v_\tau^-\|^2 + o(\|v_\tau\|^2) \leq c \|v_\tau\| \left(R(\tau, a, \lambda) \right)$$

which completes the proof. □

Next we are going to give some expansions of the gradient of $I_{K,\tau}$ in $V(\omega, N, \mu)$.

3. Expansion of the gradient in the neighborhood at infinity

In this section we perform refined asymptotic expansions of the gradient of the Euler-Lagrange functional $I_{K,\tau}$ in the neighborhood at infinity $V(\omega, N, \mu)$. These asymptotic expansions of the gradient with respect to the variables α_i, λ_i, a_i , which can be viewed as localised Pohozaev identities, lead to *balancing conditions* for these parameters. The key novelty of these expansions lies in the inclusion of terms that involve the solution ω and its interaction with the bubbles.

We first provide the expansions with respect to the *gluing parameter* α_i 's. Namely we prove:

Proposition 3.1. *Let $u := u_{\alpha,\beta} + \sum_{i=1}^N \alpha_i \tilde{\delta}_i + v \in V(\omega, N, \mu)$ with v satisfies (15) and $\tau \ln \lambda_j$ is small for each j . For $1 \leq i \leq N$, it holds:*

$$\langle \nabla I_{K,\tau}(u), \tilde{\delta}_i \rangle = \alpha_i S_n \left(1 - \lambda_i^{-\tau} \frac{n-2}{2} \alpha_i^{\frac{4}{n-2}} K(a_i) \right) + O(R_{\alpha_i} + \|v\|^2)$$

where S_n is defined in Theorem 1.2 and $R_{\alpha_i} := \tau + \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij}$.

Proof. We take $h = \tilde{\delta}_i$ in (30). Observe that, since $v \in E_{w,a,\lambda}^\perp$, we get $\langle v, \tilde{\delta}_i \rangle = 0$. Furthermore, using the fact that $u_{\alpha,\beta}$ is L^∞ bounded, easy computations imply that

$$\langle \tilde{\delta}_j, \tilde{\delta}_i \rangle = \int_{\mathbb{R}^n} \delta_j^p \delta_i = O(\varepsilon_{ij}) \quad \forall j \neq i, \quad \langle u_{\alpha,\beta}, \tilde{\delta}_i \rangle = \int_{\mathbb{S}^n} \tilde{\delta}_i^p(u_{\alpha,\beta}) = O\left(\frac{1}{\lambda_i^{(n-2)/2}}\right), \tag{47}$$

$$\|\tilde{\delta}_i\|^2 = \int_{\mathbb{S}^n} \tilde{\delta}_i^{p+1} = \int_{\mathbb{R}^n} \delta_i^{p+1} = c_0^{p+1} \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^n} = S_n. \tag{48}$$

For the second part, let $\bar{u} := u - v$ then it holds

$$\int_{\mathbb{S}^n} K|u|^{p-1-\tau} u \tilde{\delta}_i = \underbrace{\int_{\mathbb{S}^n} K \bar{u}^{p-\tau} \tilde{\delta}_i}_{(I1)} + (p-\tau) \underbrace{\int_{\mathbb{S}^n} K \bar{u}^{p-1-\tau} v \tilde{\delta}_i}_{(I2)} + O(\|v\|^2). \tag{49}$$

Concerning the integral (I1), it holds

$$\int_{\mathbb{S}^n} K \bar{u}^{p-\tau} \tilde{\delta}_i = \alpha_i^{p-\tau} \int_{\mathbb{S}^n} K \tilde{\delta}_i^{p+1-\tau} + O\left(\sum_{j \neq i} \int_{\mathbb{S}^n} ((\tilde{\delta}_j^p + u_{\alpha,\beta}^p) \tilde{\delta}_i + \tilde{\delta}_i^p (\tilde{\delta}_j + u_{\alpha,\beta}))\right). \tag{50}$$

Note that

$$\int_{\mathbb{S}^n} \tilde{\delta}_j^p \tilde{\delta}_i + \tilde{\delta}_j \tilde{\delta}_i^p = O(\varepsilon_{ij}) \quad ; \quad \int_{\mathbb{S}^n} u_{\alpha,\beta} \tilde{\delta}_i^p \leq \frac{c}{\lambda_i^{(n-2)/2}} \quad ;$$

$$\int_{\mathbb{S}^n} u_{\alpha,\beta}^p \tilde{\delta}_i \leq c \int_{\mathbb{S}^n} \tilde{\delta}_i \leq \frac{c}{\lambda_i^{(n-2)/2}}. \tag{51}$$

Now, since we assumed that $\tau \ln \lambda_i$ is small then expanding K around a_i , we get

$$\begin{aligned} \int_{\mathbb{S}^n} K \tilde{\delta}_i^{p+1-\tau} &= c_0^{-\tau} \lambda_i^{-\tau \frac{n-2}{2}} \int_{\mathbb{S}^n} K \tilde{\delta}_i^{p+1} + O\left(\int_{\mathbb{S}^n} |\tilde{\delta}_i^{-\tau} - c_0^{-\tau} \lambda_i^{-\tau \frac{n-2}{2}}| \tilde{\delta}_i^{p+1}\right) \\ &= c_0^{-\tau} \lambda_i^{-\tau \frac{n-2}{2}} \int_{\mathbb{R}^n} K \delta_i^{p+1} + O(\tau) \\ &= K(a_i) \lambda_i^{-(n-2)/2} S_n + O\left(\tau + \frac{1}{\lambda_i^2}\right) \end{aligned}$$

by using (48), Lemma 6.1 and the fact that $c_0^{-\tau} = 1 + O(\tau)$. Hence the estimate of the first integral is complete.

For the integral (I2), using (84), it follows that

$$\begin{aligned} &\int_{\mathbb{S}^n} K \bar{u}^{p-1-\tau} v \tilde{\delta}_i \\ &= \alpha_i^{p-1-\tau} \int_{\mathbb{S}^n} K \tilde{\delta}_i^{p-\tau} v + O\left(\sum_{j \neq i} \int_{\mathbb{S}^n} |v| (\tilde{\delta}_i \tilde{\delta}_j)^{p/2} + \int_{\mathbb{S}^n} |v| (\tilde{\delta}_i u_{\alpha, \beta})^{p/2}\right) \\ &= O\left(\|v\| \left(\tau + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij}^{\frac{n+2}{2(n-2)}} \ln^{\frac{n+2}{2}}(\varepsilon_{ij}^{-1})\right)\right) \end{aligned}$$

by using (40), (41) and (44) with v instead of v_τ^+ . Thus the proof follows. \square

Next we provide a balancing condition involving the rate of the concentration λ_i and the mutual interaction of bubbles ε_{ij} .

Proposition 3.2. *Let $u := u_{\alpha, \beta} + \sum_{i=1}^N \alpha_i \tilde{\delta}_i + v \in V(\omega, N, \mu)$ with v satisfies (15) and $\tau \ln \lambda_j$ is small for each j . For $1 \leq i \leq N$, it holds:*

$$\begin{aligned} &\langle \nabla I_{K, \tau}(u), \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \rangle \\ &= c_2 \sum_{j \neq i} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \left(1 - \frac{\alpha_j^{p-1} K(a_j)}{\lambda_j^{(n-2)/2}} - \frac{\alpha_i^{p-1} K(a_i)}{\lambda_i^{(n-2)/2}}\right) \\ &\quad + \frac{\alpha_i^p}{\lambda_i^{\tau(n-2)/2}} \left(c_4 \frac{\Delta K(a_i)}{\lambda_i^2} + 2c_5 K(a_i) \tau\right) \\ &\quad + O\left(\sum_{j \neq k} \varepsilon_{kj}^{\frac{n-2}{2}} \ln(\varepsilon_{kj}^{-1}) + \tau^2 + \sum_{k=1}^N \frac{\ln \lambda_k}{\lambda_k^{n/2}} + \frac{|\nabla K(a_i)|^2}{\lambda_i^2} + \frac{1}{\lambda_i^3} + \frac{1}{\lambda_i^{(n-2)/2}} + \|v\|^2\right) \end{aligned}$$

where

$$\begin{aligned} c_2 &:= c_0^{p+1} \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^{(n+2)/2}} dx; \quad , \\ c_4 &:= \frac{n-2}{2n} c_0^{p+1} \int_{\mathbb{R}^n} \frac{|x|^2(|x|^2 - 1)}{(1 + |x|^2)^{n+1}}; \\ c_5 &:= \frac{(n-2)^2}{4} c_0^{p+1} \int_{\mathbb{R}^n} \frac{(|x|^2 - 1) \ln(1 + |x|^2)}{(1 + |x|^2)^{n+1}} dx. \end{aligned}$$

Proof. We take $h = \lambda_i \partial \tilde{\delta}_i / \partial \lambda_i$ in (30) and we will estimate each term. Observe that, since $v \in E_{w,a,\lambda}^\perp$, it holds: $\langle \bar{v}, \lambda_i \partial \tilde{\delta}_i / \partial \lambda_i \rangle = 0$. Furthermore,

$$\begin{aligned} \langle \tilde{\delta}_i, \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \rangle &= \int_{\mathbb{R}^n} \delta_i^p \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = 0, \\ \langle u_{\alpha,\beta}, \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \rangle &= p \int_{\mathbb{S}^n} \tilde{\delta}_i^{p-1} \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} u_{\alpha,\beta} = O\left(\int_{\mathbb{R}^n} \delta_i^p\right) = O\left(\frac{1}{\lambda_i^{(n-2)/2}}\right). \end{aligned} \tag{52}$$

In addition, we have (see Estimate (E1) of [4])

$$\langle \tilde{\delta}_j, \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \rangle = c_2 \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\left(\varepsilon_{ij}^{n/(n-2)} \ln \varepsilon_{ij}^{-1}\right) \quad \forall j \neq i. \tag{53}$$

To proceed further, notice that

$$\begin{aligned} |t + s|^{\gamma-1} (t + s) z &= t^\gamma z + \gamma t^{\gamma-1} s z + O(t^{\gamma-1} s^2 + |s|^{\gamma+1}) \\ &\text{for } \gamma > 0, t > 0, s \in \mathbb{R} \text{ and } |z| \leq ct \end{aligned}$$

for some constant c .

Thus, letting $\bar{u} := u - v$, we get

$$\begin{aligned} \int_{\mathbb{S}^n} K |u|^{p-1-\tau} u \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} &= \underbrace{\int_{\mathbb{S}^n} K \bar{u}^{p-\tau} \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i}}_{(I3)} \\ &\quad + (p - \tau) \underbrace{\int_{\mathbb{S}^n} K \bar{u}^{p-1-\tau} v \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i}}_{(I4)} + O(\|v\|^2). \end{aligned} \tag{54}$$

For the integral (I4), let $\Omega_i := \{x : \sum_{j \neq i} \alpha_j \tilde{\delta}_j + u_{\alpha,\beta} \leq \alpha_i \tilde{\delta}_i / 2\}$, it holds

$$\begin{aligned} \int_{\mathbb{S}^n} K \bar{u}^{p-1-\tau} v \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} &= \underbrace{\int_{\mathbb{S}^n} K (\alpha_i \tilde{\delta}_i)^{p-1-\tau} v \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i}}_{(I5)} + O\left(\sum_{j \neq i} \int_{\Omega_i} \tilde{\delta}_i^{p-1} \tilde{\delta}_j |v|\right) \\ &\quad + O\left(\int_{\mathbb{S}^n \setminus \Omega_i} \tilde{\delta}_j^{p-1} \tilde{\delta}_i |v| + \int \tilde{\delta}_i^{p-1} u_{\alpha,\beta} |v| + \int \tilde{\delta}_i u_{\alpha,\beta}^{p-1} |v|\right). \end{aligned} \tag{55}$$

For the last remainder terms, using the Holder's inequalities, we obtain

$$\int \tilde{\delta}_i^{p-1} u_{\alpha,\beta} |v| + \int \tilde{\delta}_i u_{\alpha,\beta}^{p-1} |v| \leq c \|v\| \frac{1}{\lambda_i^2}. \tag{56}$$

Observe that, since $n \geq 7$, it follows that $p - 1 < 1$. Thus, using the Holder's inequalities and (42), we obtain

$$\sum_{j \neq i} \int_{\Omega_i} \tilde{\delta}_i^{p-1} \tilde{\delta}_j |v| + \int_{\mathbb{S}^n \setminus \Omega_i} \tilde{\delta}_j^{p-1} \tilde{\delta}_i |v| \leq c \sum_{j \neq i} \int (\tilde{\delta}_i \tilde{\delta}_j)^{p/2} |v| \leq c \|v\| \varepsilon_{ij}^{p/2} \ln(\varepsilon_{ij}^{-1})^{\frac{n+2}{2n}}.$$

Concerning the integral (I5), following the same computations in (44) and using the fact that $\langle \partial \tilde{\delta}_i / \partial \lambda_i, v \rangle = 0$, we obtain

$$\int_{\mathbb{S}^n} K \delta_i^{p-1-\tau} v \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = O\left(\|v\| \left(\tau + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right)\right) \tag{57}$$

which completes the estimate of (55).

It remains to estimate the integral (I3). Using (82), it holds

$$\begin{aligned} \int_{\mathbb{S}^n} K(\bar{u})^{p-\tau} \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} &= \sum_{k=1}^N \int_{\mathbb{S}^n} K(\alpha_k \tilde{\delta}_k)^{p-\tau} \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} + \int_{\mathbb{S}^n} K(u_{\alpha,\beta})^{p-\tau} \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \\ &\quad + (p-\tau) \int_{\mathbb{S}^n} K(\alpha_i \tilde{\delta}_i)^{p-\tau-1} \left(\sum_{j \neq i} \alpha_j \tilde{\delta}_j + u_{\alpha,\beta} \right) \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \\ &\quad + \sum_{k \neq j} O\left(\int (\tilde{\delta}_k \tilde{\delta}_j)^{\frac{n}{n-2}} + \int (u_{\alpha,\beta} \tilde{\delta}_k)^{\frac{n}{n-2}} \right). \end{aligned} \tag{58}$$

The estimate of the first one of the remainder terms is given in (42). Furthermore, it holds

$$\int_{\mathbb{S}^n} \tilde{\delta}_{a,\lambda}^{\frac{n}{n-2}} u_{\alpha,\beta}^{\frac{n}{n-2}} \leq c \int_{\mathbb{S}^n} \tilde{\delta}_{a,\lambda}^{\frac{n}{n-2}} \leq c \frac{\ln \lambda}{\lambda^{n/2}}. \tag{59}$$

In addition, using (42), (53) and Lemma 6.1, we get, for $k \neq i$,

$$\begin{aligned} &\int_{\mathbb{S}^n} K \tilde{\delta}_k^{p-\tau} \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \\ &= \frac{c_0^{-\tau}}{\lambda_k^{\tau(n-2)/2}} K(a_k) \int_{\mathbb{S}^n} \tilde{\delta}_k^p \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \\ &\quad + O\left(\int_{\mathbb{S}^n} d(x, a_k) \tilde{\delta}_k^p \tilde{\delta}_i + \int_{\mathbb{S}^n} |\tilde{\delta}_k^{-\tau} - \frac{c_0^{-\tau}}{\lambda_k^{\tau(n-2)/2}}| \tilde{\delta}_k^p \tilde{\delta}_i \right) \\ &= \frac{1}{\lambda_k^{\tau(n-2)/2}} K(a_k) c_2 \lambda_i \frac{\partial \varepsilon_{ik}}{\partial \lambda_i} \\ &\quad + O\left(\varepsilon_{ik}^{n/(n-2)} \ln \varepsilon_{ik}^{-1} + \left(\frac{1}{\lambda_k} + \tau\right) \varepsilon_{ik} (\ln \varepsilon_{ik}^{-1})^{(n-2)/n} \right). \end{aligned}$$

In the same way, it holds

$$\begin{aligned} p \int_{\mathbb{S}^n} K \tilde{\delta}_i^{p-\tau-1} \tilde{\delta}_k \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} &= \frac{1}{\lambda_i^{\tau \frac{n-2}{2}}} K(a_i) c_2 \lambda_i \frac{\partial \varepsilon_{ik}}{\partial \lambda_i} \\ &\quad + O\left(\varepsilon_{ik}^{\frac{n}{n-2}} \ln \varepsilon_{ik}^{-1} + \left(\frac{1}{\lambda_i} + \tau\right) \varepsilon_{ik} (\ln \varepsilon_{ik}^{-1})^{\frac{n-2}{n}} \right). \end{aligned}$$

Concerning the first term of (58) for $k = i$, using Lemma 6.1, we get

$$\begin{aligned} & \int_{\mathbb{S}^n} K \tilde{\delta}_i^{p-\tau} \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \\ &= \frac{c_0^{-\tau}}{\lambda_i^{\tau \frac{n-2}{2}}} \left(\int_{\mathbb{S}^n} K \tilde{\delta}_i^p \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \right. \\ & \quad \left. + \tau \frac{n-2}{2} \int_{\mathbb{S}^n} K \tilde{\delta}_i^p \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \ln[2 + (\lambda_i^2 - 1)(1 - \cos d(x, a_i))] \right) + O(\tau^2). \end{aligned} \tag{60}$$

In addition, for r a fixed positive radius, it holds

$$\begin{aligned} \int_{\mathbb{S}^n} K \tilde{\delta}_i^p \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} &= \int_{\mathbb{R}^n} \tilde{K} \delta_i^p \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \\ &= \tilde{K}(\tilde{a}_i) \int_{B(a_i, r)} \delta_i^p \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + \sum_{k=1}^n \frac{\partial \tilde{K}}{\partial x_k}(\tilde{a}_i) \int_{B(a_i, r)} (x - \tilde{a}_i)_k \delta_i^p \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \\ & \quad + \frac{1}{2} \sum_{1 \leq k, j \leq n} \frac{\partial^2 \tilde{K}}{\partial x_k \partial x_j}(\tilde{a}_i) \int_{B(\tilde{a}_i, r)} (x - \tilde{a}_i)_k (x - \tilde{a}_i)_j \delta_i^p \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \\ & \quad + O\left(\int_{B(\tilde{a}_i, r)} |x - \tilde{a}_i|^3 \delta_i^{p+1} + \int_{\mathbb{R}^n \setminus B(\tilde{a}_i, r)} \delta_i^{p+1} \right). \end{aligned}$$

Notice that, by oddness, we have

$$\begin{aligned} \int_{B(a_i, r)} (x - \tilde{a}_i)_k \delta_i^p \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} &= 0 \quad \forall k \quad \text{and} \\ \int_{B(\tilde{a}_i, r)} (x - \tilde{a}_i)_k (x - \tilde{a}_i)_j \delta_i^p \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} &= 0 \quad \text{for } k \neq j. \end{aligned}$$

Thus, using (52), it holds

$$\begin{aligned} \int_{\mathbb{S}^n} K \tilde{\delta}_i^p \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} &= -\tilde{K}(\tilde{a}_i) \int_{\mathbb{R}^n \setminus B(a_i, r)} \delta_i^p \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \\ & \quad + \frac{1}{2} \sum_{1 \leq k \leq n} \frac{\partial^2 \tilde{K}}{\partial x_k^2}(\tilde{a}_i) \int_{\mathbb{R}^n} (x - \tilde{a}_i)_k^2 \delta_i^p \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + O\left(\frac{1}{\lambda_i^3}\right) \\ &= \frac{1}{2} \Delta \tilde{K}(\tilde{a}_i) \frac{n-2}{2} \frac{c_0^{2n/(n-2)}}{\lambda_i^2} \frac{1}{n} \int_{\mathbb{R}^n} \frac{|y|^2(1-|y|^2)}{(1+|y|^2)^{n+1}} dy + O\left(\frac{1}{\lambda_i^3}\right), \end{aligned}$$

In the same way, we have

$$\begin{aligned}
 & \int_{\mathbb{S}^n} K \tilde{\delta}_i^p \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \ln[2 + (\lambda_i^2 - 1)(1 - \cos d(x, a_i))] \\
 &= \int_{\mathbb{R}^n} \tilde{K} \delta_{0, \lambda_i}^p \lambda_i \frac{\partial \delta_{0, \lambda_i}}{\partial \lambda_i} \ln\left[\frac{2 + 2\lambda_i^2 |y|^2}{1 + |y|^2}\right] \\
 &= \int_{\mathbb{R}^n} \tilde{K} \delta_{0, \lambda_i}^p \lambda_i \frac{\partial \delta_{0, \lambda_i}}{\partial \lambda_i} \ln(2 + 2\lambda_i^2 |y|^2) + O\left(\int_{\mathbb{R}^n} \delta_{0, \lambda_i}^{p+1} \ln(1 + |y|^2)\right) \\
 &= K(a_i) c_0^{\frac{2n-2}{n}} \frac{n-2}{2} \int_{\mathbb{R}^n} \frac{(1 - |y|^2) \ln(1 + |y|^2)}{(1 + |y|^2)^{n+1}} dy + O\left(\frac{1}{\lambda_i^2}\right), \tag{61}
 \end{aligned}$$

which achieve the estimate of (60). Note that, for the estimate of the remainder term in (61) we used the fact that $\ln(1 + |t|^2) \leq c|t|^2$ for each $t \in \mathbb{R}$.

It remains to estimate the integrals which appear in (58) and contain $u_{\alpha, \beta}$. Using (51), there hold

$$\begin{aligned}
 & \int_{\mathbb{S}^n} K(\alpha_0 u_{\alpha, \beta})^{p-\tau} \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} = O\left(\int_{\mathbb{S}^n} \tilde{\delta}_i\right) = O\left(\frac{1}{\lambda_i^{(n-2)/2}}\right), \\
 & \int_{\mathbb{S}^n} K \tilde{\delta}_i^{p-\tau-1} u_{\alpha, \beta} \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} = O\left(\int_{\mathbb{S}^n} \tilde{\delta}_i^p\right) = O\left(\frac{1}{\lambda_i^{(n-2)/2}}\right).
 \end{aligned}$$

Combining the previous estimates, the proof follows. □

Finally, we provide the following balancing condition involving the point of concentration a_i .

Proposition 3.3. *Let $u := u_{\alpha, \beta} + \sum_{i=1}^N \alpha_i \tilde{\delta}_i + v \in V(\omega, N, \mu)$ with v satisfies (15) and $\tau \ln \lambda_j$ is small for each j . For $1 \leq i \leq N$, it holds:*

$$\langle \nabla I_{K, \tau}(u), \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_i}{\partial a_i} \rangle = \alpha_i^p (c + o(1)) \frac{\nabla K(a_i)}{\lambda_i} + O\left(\frac{1}{\lambda_i^3} + \frac{1}{\lambda_i^{(n-2)/2}} + \sum_{j \neq i} \varepsilon_{ij} + \|v\|^2\right).$$

Proof. The proof follows as the previous one. Hence we will omit it here. □

In the next section, we are going to prove Theorem 1.1.

4. Asymptotic behavior of blowing up solutions

The focus in this section is a refined blow up analysis of finite energy solutions of the subcritical approximated problem $(\mathcal{N}_{K, \tau})$. Namely, our aim is to prove Theorem 1.1. To do this, we need to fix some important facts.

Let (u_τ) be a sequence of energy bounded solutions of $(\mathcal{N}_{K, \tau})$ converging weakly but not strongly $u_\tau \rightharpoonup \omega \neq 0$. It follows from Propositions 3.1, 3.2 and 3.3 (by taking u_τ instead of u) that the following system holds:

$$\begin{aligned}
 & |1 - \lambda_i^{-\tau} \alpha_i^{\frac{4}{n-2}} K(a_i)| = O\left(R_{\alpha_i} + R(\tau, a, \lambda)^2\right) \quad \forall i \leq N, \\
 (E_i) \quad & -c_2 \sum_{j \neq i} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \alpha_i \left(c_4 \frac{\Delta K(a_i)}{\lambda_i^2 K(a_i)} + 2c_5 \tau \right) = O(R_{2,\lambda}) \quad \forall i \leq N, \\
 (F_i) \quad & \frac{|\nabla K(a_i)|}{\lambda_i} \leq c \left(\frac{1}{\lambda_i^3} + \sum_{j \neq i} \varepsilon_{ij} + R^2(\tau, a, \lambda) + \sum_{1 \leq k \leq N} \frac{\ln \lambda_k}{\lambda_k^{n/2}} \right) \quad \forall i \leq N,
 \end{aligned}$$

where $R(\tau, a, \lambda)$ is defined in Proposition 2.4 and

$$R_{2,\lambda} := \sum_{1 \leq j \neq k \leq N} \varepsilon_{kj}^{\frac{n}{n-2}} \ln(\varepsilon_{kj}^{-1}) + \tau^2 + \sum_{1 \leq k \leq N} \frac{1}{\lambda_k^{5/2}} + \sum_{1 \leq k \leq N} \frac{|\nabla K(a_k)|^2}{\lambda_k^2}. \tag{62}$$

Now, we order the λ_i 's as: $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$ and we start by the following estimate.

Lemma 4.1. *For each solution u_τ in $V(\omega, N, \tau)$, it holds*

$$\sum_{1 \leq k \neq i \leq N} \varepsilon_{ik} + \sum_{1 \leq k \leq N} \frac{|\nabla K(a_k)|}{\lambda_k} + \tau \leq c \frac{1}{\lambda_1^2}.$$

Furthermore, $R_{2,\lambda}$, defined in (62), satisfies: $R_{2,\lambda} = o(1/\lambda_1^{2(n-1)/(n-2)})$.

Proof. We first notice that

$$-\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \geq 0 \quad \text{for each } i \neq j \quad \text{and} \quad -\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \geq c \varepsilon_{ij} \quad \text{if } \lambda_i \geq c' \lambda_j. \tag{63}$$

Hence, summing $2^i \alpha_i (E_i)$, we obtain

$$\sum_{k \neq i} \varepsilon_{ik} + \tau \leq c \left(R_{2,\lambda} + \frac{1}{\lambda_1^2} \right).$$

The result follows by using (F_i) and the definition of $R_{2,\lambda}$ given in (62). \square

Lemma 4.2. *For $i \in \{1, \dots, N\}$, we define*

$$\Gamma_i := \lambda_i^2 \sum_{k \neq i} \varepsilon_{ik} + \frac{|\nabla K(a_i)|/\lambda_i}{\sum_{k \neq i} \varepsilon_{ki} + 1/\lambda_i^2}. \tag{64}$$

Let $D'_1 := \{i : \lim \Gamma_i = \infty\}$ and $D_1 := \{1, \dots, N\} \setminus D'_1$.

(i) *For each $i \in D_1$, there exists a critical point y_i of K such that $\lambda_i |a_i - y_i| \leq C$.*

(ii) *If there exists an index $i \in D'_1$, then it holds that*

$$\sum_{j \geq i} \left(\frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{k \neq j} \varepsilon_{kj} + \tau \right) = o\left(\frac{1}{\lambda_1^{2(n-1)/(n-2)}} \right).$$

(iii) *There exists a critical point y of K such that a_1 converges to y .*

Proof. Let $i \in D_1$, it follows that Γ_i is bounded and therefore it holds that $|\nabla K(a_i)| \leq C/\lambda_i$ which implies the first assertion. Concerning the second one, let $i \in D'_1$, summing $2^j \alpha_j (E_j) + (F_j)/m$ for $j \geq i$ where m is a small constant, it holds that

$$\sum_{j \geq i} \left(\frac{|\nabla K(a_j)|}{\lambda_j} + \sum_{k \neq j} \varepsilon_{kj} + \tau \right) = O\left(\frac{1}{\lambda_i^2} + R_{2,\lambda}\right).$$

Since $i \in D'_1$, it follows that $1/\lambda_i^2$ is small with respect to the left hand side. Thus the proof follows from the estimate of $R_{2,\lambda}$ (see Lemma 4.1).

Concerning the proof of Claim (iii), observe that, if $1 \in D_1$, the result follows from Claim (i), however, if $1 \in D'_1$, from Claim (ii), we deduce that $|\nabla K(a_1)| = o(\lambda_1^{-n/(n-2)})$ which implies the result in this case. Thus the proof of the lemma is complete. \square

Lemma 4.3. *For each $i \neq j$, it holds that: $\varepsilon_{ij} = o(1/\lambda_1^2)$.*

Proof. Observe that, if i or j belongs to D'_1 , then the result follows from Lemma 4.2. In the other case, that is $i, j \in D_1$, using again Lemma 4.2, there exist critical points y_i and y_j such that $\lambda_k d(a_k, y_k) \leq c$ for $k = i, j$. Two cases may occur:

- (a) Either $y_i \neq y_j$, and in this case we get $d(a_i, a_j) \geq c$ and therefore the result follows easily,
- (b) or $y_i = y_j$. Since we have $\lambda_k d(a_k, y_k) \leq c$ and ε_{ij} is small, it follows that $\lambda_i/\lambda_j \rightarrow 0$ or ∞ . Taking $\lambda_i \leq \lambda_j$ and using the fact that Γ_j is bounded, it holds

$$\varepsilon_{ij} \leq \frac{c}{\lambda_j^2} = c \frac{\lambda_i^2}{\lambda_j^2} \frac{1}{\lambda_i^2} = o\left(\frac{1}{\lambda_i^2}\right) = o\left(\frac{1}{\lambda_1^2}\right).$$

Hence the proof is completed. \square

Next we are going to complete the proof of Theorem 1.1. Using the above lemmas, Equations (E_i) can be improved and we get

$$(E'_i) : \quad 2c_5\tau + c_4 \frac{\Delta K(a_i)}{\lambda_i^2 K(a_i)} = o\left(\frac{1}{\lambda_1^2}\right). \tag{65}$$

Now, we claim that

$$(a) \quad \frac{\lambda_N}{\lambda_1} \leq C \quad \text{and} \quad (b) \quad D'_1 = \emptyset. \tag{66}$$

Arguing by contradiction, assume that $\lambda_N/\lambda_1 \rightarrow \infty$. It follows from (E'_N) that $\tau = o(1/\lambda_1^2)$. Putting this information in (E'_1) and using Claim (iii) of Lemma 4.2, we derive a contradiction. Hence all the λ_i 's are of the same order. Concerning assertion (b), it follows from assertion (a) and Lemma 4.1. This completes the proof of the claim.

Combining (66) and assertion (i) of Lemma 4.2, we derive that, for each i , there exists y_{j_i} such that $\lambda_i d(a_i, y_{j_i}) \leq c$. Assume that, for $i \neq k$, we have $y_{j_i} = y_{j_k}$. Thus we obtain that $\lambda_i \lambda_k d(a_i, a_k)^2 \leq c$. This information and the fact that the λ_i 's are of the same order contradict the smallness of

ε_{ik} . Therefore we derive that $d(a_i, a_k) \geq c$ for each $i \neq k$ which implies that $\varepsilon_{ik} = O(1/\lambda_i^{n-2})$ for each $k \neq i$. Now, putting the previous informations in the right hand side of (F'_i) we derive that $\lambda_i d(a_i, y_{j_i}) \rightarrow 0$ as $\tau \rightarrow 0$.

Regarding (5), it follows from (E'_i) and the fact that the λ_i 's are of the same order.

Finally, concerning the morse index of the solution u_τ , the result is done for $\omega = 0$ in [28]. However, for $\omega \neq 0$ and non-degenerate, it follows that $u_{\alpha,\beta} = \alpha\omega$. In this case, we need to add the contribution of α and ω in the morse index formula. Note that, by easy computations, we see that

$$D^2 I_{K,\tau}(u_\tau) \left(\frac{\partial u_\tau}{\partial \alpha}, \frac{\partial u_\tau}{\partial \alpha} \right) = D^2 I_{K,\tau}(u_\tau)(\omega, \omega) + O \left(\left\| \frac{\partial v_\tau}{\partial \alpha} \right\| \right).$$

Note that $D^2 I_{K,\tau}(u_\tau)(\omega, \omega) \leq -c < 0$ and $\left\| \frac{\partial v_\tau}{\partial \alpha} \right\| = o(1)$ (see Lemma 6.3). This implies that the variable α belongs to the negative space. For the contribution of ω , using Lemma 2.3, we see that it is equal to

$$\dim(N_-(\omega)) = \text{morse index}(I_{K,\tau}(\omega)).$$

This completes the proof of the theorem.

5. Construction of blowing up solutions for the approximate problem

This section is devoted to the proof of Theorem 1.2. Namely, we are going to construct blowing up solutions of Problem $(\mathcal{N}_{K,\tau})$. Let $n \geq 7$, $0 < K \in C^3(\mathbb{S}^n)$ and let y_1, \dots, y_N be distinct non-degenerate critical points of K with $\Delta K(y_i) < 0$ and ω be a solution of (\mathcal{N}_K) . In Theorem 1.2 we assume that ω is non-degenerate and therefore the variable β introduced in the function $u_{\alpha,\beta}$ has to be equal to zero, that is, $u_{\alpha,\beta} = \alpha\omega$.

As in [8], the strategy of the proof is the following: we will define a set M_τ whose elements are some points $\mathcal{M} := (\alpha_0, \alpha, \lambda, x, v) \in (\mathbb{R}_+)^{N+1} \times (\mathbb{R}_+)^N \times (\mathbb{S}^n)^N \times H^1(\mathbb{S}^n)$ where v satisfies (15), the other variables satisfy some conditions and N is the number of the bubbles in the desired constructed solution. Precisely, let

$$M_\tau := \{ \mathcal{M} := (\alpha_0, \alpha, \lambda, x, v) \in \mathbb{R}_+ \times (\mathbb{R}_+)^N \times (\mathbb{R}_+)^N \times (\mathbb{S}^n)^N \times H^1(\mathbb{S}^n) : \begin{aligned} &v \text{ satisfies (15); } |\alpha_i^{4/(n-2)} K(x_i) - 1| < \tau \ln^2 \tau \ \forall i \geq 1; \\ &|\alpha_0^{4/(n-2)} - 1| < \tau \ln^2 \tau ; C^{-1} \tau \leq \lambda_i^{-2} \leq C \tau ; d(x_i, y_i) \leq C \tau \ \forall i \geq 1 \}, \end{aligned} \tag{67}$$

where C is a large positive constant.

We remark that, for $\mathcal{M} := (\alpha_0, \alpha, \lambda, x, v) \in M_\tau$, it holds that $u := \alpha_0 \omega + \sum_{i=1}^N \alpha_i \tilde{\delta}_{x_i, \lambda_i} + v \in V(\omega, N, \mu)$ for some μ small.

In addition, we define a function

$$\Psi_\tau : M_\tau \rightarrow \mathbb{R}; \quad \mathcal{M} = (\alpha_0, \alpha, \lambda, x, v) \mapsto I_{K,\tau} \left(\alpha_0 \omega + \sum_{i=1}^N \alpha_i \tilde{\delta}_{x_i, \lambda_i} + v \right) \tag{68}$$

and we need to find a critical point of Ψ_τ .

Recall that the variable v satisfies some orthogonality conditions. Thus using the Lagrange multiplier theorem, it is easy to get the following proposition.

Proposition 5.1. *Let $\mathcal{M} = (\alpha_0, \alpha, \lambda, x, v) \in M_\tau$. \mathcal{M} is a critical point of Ψ_τ if and only if $\alpha_0 \omega + \sum_{i=1}^N \alpha_i \tilde{\delta}_{x_i, \lambda_i} + v$ is a critical point of $I_{K,\tau}$, i.e. if and only if there exists $(A_0, A, B, C) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times (\mathbb{R}^n)^N$ such that the following holds:*

$$\begin{aligned} (E_{\alpha_i}) \quad & \frac{\partial \Psi_\tau}{\partial \alpha_i}(\alpha_0, \alpha, \lambda, x, v) = 0, \quad \forall i = 0, 1, \dots, N, \\ (E_{\lambda_i}) \quad & \frac{\partial \Psi_\tau}{\partial \lambda_i}(\alpha_0, \alpha, \lambda, x, v) = B_i \left\langle \lambda_i \frac{\partial^2 \tilde{\delta}_i}{\partial \lambda_i^2}, v \right\rangle \\ & + \sum_{j=1}^n C_{ij} \left\langle \frac{1}{\lambda_i} \frac{\partial^2 \tilde{\delta}_i}{\partial x_i^j \partial \lambda_i}, v \right\rangle, \quad \forall i = 1, \dots, N, \\ (E_{x_i}) \quad & \frac{\partial \Psi_\tau}{\partial x_i}(\alpha_0, \alpha, \lambda, x, v)|_{T_{x_i}(\mathbb{S}^n)} = B_i \left\langle \lambda_i \frac{\partial^2 \tilde{\delta}_i}{\partial \lambda_i \partial x_i}, v \right\rangle \\ & + \sum_{j=1}^n C_{ij} \left\langle \frac{1}{\lambda_i} \frac{\partial^2 \tilde{\delta}_i}{\partial x_i^j \partial x_i}, v \right\rangle, \quad \forall i = 1, \dots, N, \\ (E_v) \quad & \frac{\partial \Psi_\tau}{\partial v}(\alpha_0, \alpha, \lambda, x, v) = A_0 \omega + \sum_{i=1}^N \left(A_i \tilde{\delta}_i + B_i \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} + \sum_{j=1}^n C_{ij} \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_i}{\partial x_i^j} \right) \end{aligned}$$

where $\tilde{\delta}_i := \tilde{\delta}_{x_i, \lambda_i}$.

The results of Theorem 1.2 will be obtained through a careful analysis of the previous equations on M_τ . Note that

$$\begin{aligned} \frac{\partial \Psi_\tau}{\partial v}(\alpha_0, \alpha, \lambda, x, v) &= \nabla I_{K,\tau}(u); \\ \frac{\partial \Psi_\tau}{\partial \alpha_0}(\alpha_0, \alpha, \lambda, x, v) &= \langle \nabla I_{K,\tau}(u), \omega \rangle \end{aligned}$$

and for each $i \geq 1$, we have

$$\begin{aligned} \frac{\partial \Psi_\tau}{\partial \alpha_i}(\alpha_0, \alpha, \lambda, x, v) &= \langle \nabla I_{K,\tau}(u), \tilde{\delta}_i \rangle; \\ \frac{\partial \Psi_\tau}{\partial x_i}(\alpha_0, \alpha, \lambda, x, v) \Big|_{T_{x_i}(\mathbb{S}^n)} &= \left\langle \nabla I_{K,\tau}(u), \alpha_i \frac{\partial \tilde{\delta}_i}{\partial x_i} \right\rangle \Big|_{T_{x_i}(\mathbb{S}^n)} \quad ; \\ \frac{\partial \Psi_\tau}{\partial \lambda_i}(\alpha_0, \alpha, \lambda, x, v) &= \left\langle \nabla I_{K,\tau}(u), \alpha_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \right\rangle \end{aligned}$$

where $u := \alpha_0 \omega + \sum_{i=1}^N \alpha_i \tilde{\delta}_{x_i, \lambda_i} + v$.

We remark that the last three equations are estimated in Propositions 3.1, 3.2 and 3.3.

Proposition 5.2. *Let $u := \alpha_0 \omega + \sum_{i=1}^N \alpha_i \tilde{\delta}_{x_i, \lambda_i} \in V(\omega, N, \mu)$. Then there exists a unique $\bar{v} \in E_{\omega, x, \lambda}^\perp$ satisfying*

$$\langle \nabla I_{K,\tau}(u + \bar{v}), h \rangle = 0 \quad \forall h \in E_{\omega, x, \lambda}^\perp. \tag{69}$$

This function \bar{v} satisfies

$$\|\bar{v}\| \leq cR(\tau, a, \lambda)$$

where $R(\tau, a, \lambda)$ is defined in Proposition 2.4.

Proof. To prove (69), we need to expand $I_{K,\tau}(u+v)$ with respect to $v \in E_{\omega, a, \lambda}^\perp$. Recall that

$$I_{K,\tau}(u + v) := \frac{1}{2} \|u + v\|^2 - \frac{1}{p + 1 - \tau} \int_{\mathbb{S}^n} K |u + v|^{p+1-\tau}.$$

Note that, for each $x, y \in \mathbb{R}$ and $\gamma \in (2, 3)$, we have

$$|x + y|^\gamma = |x|^\gamma + \gamma |x|^{\gamma-2} xy + \frac{1}{2} \gamma(\gamma - 1) |x|^{\gamma-2} y^2 + O(|y|^\gamma).$$

For $n \geq 7$ and τ small, it holds that $p + 1 - \tau \in (2, 3)$ and therefore we get

$$\begin{aligned} \int_{\mathbb{S}^n} K |u + v|^{p+1-\tau} &= \int_{\mathbb{S}^n} K |u|^{p+1-\tau} + (p + 1 - \tau) \int_{\mathbb{S}^n} K |u|^{p-1-\tau} uv \\ &\quad + \frac{1}{2} (p + 1 - \tau)(p - \tau) \int_{\mathbb{S}^n} K |u|^{p-1-\tau} v^2 + O\left(\int_{\mathbb{S}^n} K |v|^{p+1-\tau}\right). \end{aligned}$$

Since $u \in V(\omega, N, \mu)$ with $\tau \ln \lambda_i$ is small for each i , it follows that

$$\int_{\mathbb{S}^n} K |u|^{p-1-\tau} v^2 = \sum_{i=1}^N \int_{\mathbb{S}^n} \tilde{\delta}_i^{p-1} v^2 + \int_{\mathbb{S}^n} K \omega^{p-1} v^2 + o(\|v\|^2).$$

Hence we get

$$I_{K,\tau}(u + v) = I_{K,\tau}(u) - f(v) + \frac{1}{2} Q_{\omega, a, \lambda}(v) + o(\|v\|^2) \tag{70}$$

where

$$f(v) := \int_{\mathbb{S}^n} K|u|^{p-1-\tau}uv \quad \text{and}$$

$$Q_{\omega,a,\lambda}(v) := \|v\|^2 - p \sum_{i=1}^N \int_{\mathbb{S}^n} \tilde{\delta}_i^{p-1}v^2 - p \int_{\mathbb{S}^n} K\omega^{p-1}v^2.$$

Thus (69) follows from Proposition 2.3 (which tells us that $Q_{\omega,a,\lambda}$ is non-degenerate on the space $E_{\omega,a,\lambda}^\perp$).

Now it remains to estimate $\|\bar{v}\|$. Note that, using the Lagrange multiplier theorem, (69) implies the existence of some constants $(A_0, A, B, C) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times (\mathbb{R}^n)^N$ such that

$$\nabla I_{K,\tau}(u + \bar{v}) = A_0\omega + \sum_{i=1}^N \left(A_i\tilde{\delta}_i + B_i\lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} + C_i \cdot \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_i}{\partial x_i} \right) \tag{71}$$

where $\tilde{\delta}_i := \tilde{\delta}_{x_i,\lambda_i}$. Since ω is assumed to be non-degenerate, we derive that

$$H^1(\mathbb{S}^n) = N_-(\omega) \oplus \text{span}\{\omega\} \oplus N_+(\omega), \tag{72}$$

that is $H_0(\omega)$ (defined in (18)) becomes $H_0(\omega) = \text{span}\{\omega\}$.

As in the proof of Proposition 2.4, we divide \bar{v} into

$$\bar{v} = \bar{v}_- + \bar{v}_0 + \bar{v}_+$$

and we have $\|\bar{v}_0\| = o(\|\bar{v}\|)$ (by using Proposition 2.3).

Multiplying (71) by \bar{v}_- and using the fact that $\langle \bar{v}_-, \omega \rangle = 0$, we get

$$\langle \nabla I_{K,\tau}(u + \bar{v}), \bar{v}_- \rangle = \sum O \left(\int_{\mathbb{S}^n} \tilde{\delta}_i^p |\bar{v}_-| \right) = \sum O \left(\frac{\|\bar{v}_-\|}{\lambda_i^{(n-2)/2}} \right)$$

(since \bar{v}_- belongs to a fixed finite dimensional space which implies that $\|\bar{v}_-\|_\infty \leq c\|\bar{v}_-\|$).

The left hand side is computed in (31) with v_τ instead of \bar{v} . Following the computations done in (31)-(36), we derive that

$$\|\bar{v}_-\|^2 \leq c\|\bar{v}_-\| \left(\tau + \sum \frac{1}{\lambda_i^{(n-2)/2}} + o(\|\bar{v}\|) \right).$$

Now multiplying (71) by \bar{v}_+ and using the fact that $\langle \bar{v}_+, \omega \rangle = 0$, we get

$$\langle \nabla I_{K,\tau}(u + \bar{v}), \bar{v}_+ \rangle = \sum \left(A_i \langle \tilde{\delta}_i, \bar{v}_+ \rangle + B_i \langle \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i}, \bar{v}_+ \rangle + \langle C_i \cdot \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_i}{\partial a_i}, \bar{v}_+ \rangle \right).$$

The left hand side is computed in (31) with v_τ instead of \bar{v} . Recall that \bar{v}_+ does not belong to $E_{\omega,a,\lambda}^\perp$ but $\bar{v} \in E_{\omega,a,\lambda}^\perp$. Thus

$$\langle \tilde{\delta}_i, \bar{v}_+ \rangle = \langle \tilde{\delta}_i, \bar{v} \rangle - \langle \tilde{\delta}_i, \bar{v}_- \rangle - \langle \tilde{\delta}_i, \bar{v}_0 \rangle = O \left(\int \tilde{\delta}_i^p (|\bar{v}_-| + |\bar{v}_0|) \right) = O \left(\frac{\|\bar{v}\|}{\lambda_i^{(n-2)/2}} \right).$$

The same holds for the other scalar products.

Following the computations done in the proof of Proposition 2.4, we get

$$\|\bar{v}_+\|^2 \leq o(\|\bar{v}\|^2) + c\|\bar{v}\|R(\tau, a, \lambda),$$

where $R(\tau, a, \lambda)$ is introduced in Proposition 2.4.

Finally, since $\|\bar{v}\|^2 = \|\bar{v}_+\|^2 + \|\bar{v}_-\|^2 + \|\bar{v}_0\|^2$ we derive the desired estimate.

The proof of Proposition 5.2 is thereby completed. \square

Proposition 5.3. *Let $u := \alpha_0\omega + \sum_{i=1}^N \alpha_i \tilde{\delta}_{x_i, \lambda_i} + \bar{v}$. For each $1 \leq i \leq N$, It holds:*

$$\langle \nabla I_{K, \tau}(u), \omega \rangle = \alpha_0 \|\omega\|^2 (1 - \alpha_0^{p-1}) + O\left(\tau + \|\bar{v}\|^2 + \sum \frac{1}{\lambda_i^{(n-2)/2}} + \frac{1}{\lambda_i^4}\right).$$

Proof. Since \bar{v} satisfies (15), using (30) and (47), it holds

$$\begin{aligned} \langle \nabla I_{K, \tau}(u), \omega \rangle &= \langle u, \omega \rangle - \int K|u|^{p-\tau-1}u\omega \\ &= \alpha_0 \|\omega\|^2 + \sum O\left(\frac{1}{\lambda_i^{(n-2)/2}}\right) - \int K|u|^{p-\tau-1}u\omega. \end{aligned}$$

Furthermore, let $\bar{u} := \alpha_0\omega + \sum_{i=1}^N \alpha_i \tilde{\delta}_{x_i, \lambda_i}$, we have

$$\int_{\mathbb{S}^n} K|u|^{p-1-\tau}u\omega = \int_{\mathbb{S}^n} K\bar{u}^{p-\tau}\omega + (p - \tau) \int_{\mathbb{S}^n} K\bar{u}^{p-1-\tau}\bar{v}\omega + O(\|\bar{v}\|^2).$$

Using (56) and the fact that $\langle w, \bar{v} \rangle = 0$, it follows that

$$\begin{aligned} \int_{\mathbb{S}^n} K\bar{u}^{p-1-\tau}\bar{v}\omega &= \alpha_0^{p-1-\tau} \int_{\mathbb{S}^n} K\omega^{p-\tau}\bar{v} + \sum_{j \geq 1} O\left(\int_{\mathbb{S}^n} \omega^{p-1}\tilde{\delta}_j|\bar{v}| + \int_{\mathbb{S}^n} \omega\tilde{\delta}_j^{p-1}|\bar{v}|\right) \\ &= O\left(\|\bar{v}\|\left(\tau + \sum_{j \geq 1} \frac{1}{\lambda_j^2}\right)\right). \end{aligned}$$

In addition, using (51), it holds

$$\begin{aligned} \int_{\mathbb{S}^n} K\bar{u}^{p-\tau}\omega &= \int_{\mathbb{S}^n} K(\alpha_0\omega)^{p-\tau}\omega + \sum_{j \geq 1} O\left(\int_{\mathbb{S}^n} \omega^p\tilde{\delta}_j + \int_{\mathbb{S}^n} \omega\tilde{\delta}_j^p\right) \\ &= \alpha_0^p \|\omega\|^2 + O\left(\tau + \sum_{j \geq 1} \frac{1}{\lambda_j^{(n-2)/2}}\right). \end{aligned}$$

Hence the proof follows. \square

Proof of Theorem 1.2. Let $(\alpha_0, \alpha, \lambda, x, 0) \in M_\tau$. The proof goes along the ideas introduced in [8]. We need to solve the system $((E_{\alpha_0}), (E_\alpha), (E_\lambda), (E_x), (E_v))$ introduced in Proposition 5.1.

Proposition 5.2 implies that \bar{v} is a critical point of $I_{K, \tau}$ in the space $E_{\omega, x, \lambda}^\perp$. Thus we get

$$\nabla I_{K, \tau}\left(\alpha_0\omega + \sum_{i=1}^N \alpha_i \tilde{\delta}_{x_i, \lambda_i} + \bar{v}\right)_{|E_{\omega, x, \lambda}^\perp} = 0$$

and therefore (by the Lagrange multiplier theorem) there exist some constants $(A_0, A, B, C) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times (\mathbb{R}^n)^N$ such that

$$\begin{aligned} & \nabla I_{K,\tau} \left(\alpha_0 \omega + \sum_{i=1}^N \alpha_i \tilde{\delta}_{x_i, \lambda_i} + \bar{v} \right) \\ &= A_0 \omega + \sum_{i=1}^N \left(A_i \tilde{\delta}_{x_i, \lambda_i} + B_i \lambda_i \frac{\partial \tilde{\delta}_{x_i, \lambda_i}}{\partial \lambda_i} + C_i \cdot \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_{x_i, \lambda_i}}{\partial x_i} \right) \end{aligned}$$

which implies that the equation (E_v) in Proposition 5.1 is satisfied.

Next, we estimate the numbers A_0, A, B, C by taking the scalar product of (E_v) with $\omega, \tilde{\delta}_i, \lambda_i \partial \tilde{\delta}_i / \partial \lambda_i$ and $\lambda_i^{-1} \partial \tilde{\delta}_i / \partial x_i$ respectively. Thus we derive a quasi-diagonal system in the variables A_0, A, B and C_i 's. The right hand side is given by (using Propositions 3.1, 3.2, 3.3 and 5.3, the fact that $(\alpha_0, \alpha, \lambda, x, 0) \in M_\tau$ and $\partial \Psi_\tau / \partial v = \nabla I_\tau(u)$ with $u := \alpha_0 \omega + \sum_{i=1}^N \alpha_i \tilde{\delta}_i + \bar{v}$)

$$\begin{aligned} \left\langle \frac{\partial \Psi_\tau}{\partial v}, \omega \right\rangle &= O(\tau \ln^2 \tau) \quad ; \quad \left\langle \frac{\partial \Psi_\tau}{\partial v}, \tilde{\delta}_i \right\rangle = O(\tau \ln^2 \tau); \\ \left\langle \frac{\partial \Psi_\tau}{\partial v}, \lambda_i \frac{\partial \tilde{\delta}_i}{\partial \lambda_i} \right\rangle &= O(\tau) \quad ; \quad \left\langle \frac{\partial \Psi_\tau}{\partial v}, \frac{1}{\lambda_i} \frac{\partial \tilde{\delta}_i}{\partial x_i} \right\rangle = O(\tau \ln^2 \tau). \end{aligned}$$

Hence we deduce that

$$\begin{aligned} A_0 &= O(\tau \ln^2 \tau) \quad ; \quad A_i = O(\tau \ln^2 \tau) \quad ; \quad B_i = O(\tau); \\ C_i &= O(\tau \ln^2 \tau) \quad \text{for } i = 1, \dots, N. \end{aligned} \tag{73}$$

Furthermore, since $(\alpha_0, \alpha, \lambda, x, 0) \in M_\tau$ then Proposition 5.2 implies that $\|\bar{v}\| = O(\tau)$ and therefore the system $((E_{\alpha_0}), (E_{\alpha_i}), (E_{\lambda_i}), (E_{x_i}))$ (introduced in Proposition 5.1) is equivalent to

$$(S) : \quad \begin{cases} \langle \nabla I_{K,\tau}(u), \omega \rangle = 0 \\ \langle \nabla I_{K,\tau}(u), \tilde{\delta}_i \rangle = 0 \quad \text{for } i = 1, \dots, N; \\ \langle \nabla I_{K,\tau}(u), \lambda_i \partial \tilde{\delta}_i / \partial \lambda_i \rangle = O(\tau^2 \ln^2 \tau) \quad \text{for } i = 1, \dots, N; \\ \langle \nabla I_{K,\tau}(u), \lambda_i^{-1} \partial \tilde{\delta}_i / \partial x_i \rangle = O(\tau^2 \ln^2 \tau) \quad \text{for } i = 1, \dots, N. \end{cases}$$

Now we introduce the following change of variables:

$$\beta_0 := 1 - \alpha_0^{4/(n-2)}, \tag{74}$$

$$\beta_i := 1 - \alpha_i^{4/(n-2)} K(x_i), \quad i = 1, \dots, N; \tag{75}$$

$$\frac{1}{\lambda_i^2} := - \frac{c_4 K(y_i)}{c_3 (\Delta K)(y_i)} \tau (1 + \Lambda_i), \quad i = 1, \dots, N; \tag{76}$$

$$x_i := \frac{y_i + \xi_i}{|y_i + \xi_i|} \quad \text{with } \xi_i \in T_{y_i}(\mathbb{S}^n), \quad i = 1, \dots, N. \tag{77}$$

Using this change of variables and Propositions 3.1, 3.2, 3.3 and 5.3, the previous system (S) becomes

$$(S') \begin{cases} \beta_i = O(\tau \ln \tau), & \text{for } i = 0, 1, \dots, N, \\ \Lambda_i = O(\tau^{1/5}), & \text{for } i = 1, \dots, N, \\ D^2K(y_i)(\xi_i, \cdot) = O(|\xi^2| + \tau^{2/n} \ln^2 \tau + \tau^{1/5}), & \text{for } i = 1, \dots, N. \end{cases} \quad (78)$$

Since the critical points y_i 's are assumed to be non-degenerate, then, using the fixed point theorem, we derive the existence of $(\beta_0^\tau, \beta^\tau, \xi^\tau, \Lambda^\tau)$ such that the system (S') is satisfied. Hence the existence of a critical point of $I_{K,\tau}$ (by Proposition 5.1).

Note that, for $\omega = 0$, the unicity of such a solution is proved in [28]. The same argument holds for $\omega \neq 0$ non-degenerate.

Lastly, for the Morse index, the same argument used in the proof of Theorem 1.1 holds which completes the proof of Theorem 1.2. \square

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6. Appendix

In this appendix we collect various estimates needed through the paper.

Lemma 6.1. [2] *Let $a \in \mathbb{S}^n$ and $\lambda > 0$ be large.*

(i) Assume that $\tau \ln \lambda$ is small enough, then it holds

$$\delta_{a,\lambda}^{-\tau}(x) = c_0^{-\tau} \lambda^{-\tau(n-2)/2} \left(1 + \frac{n-2}{2} \tau \ln(2 + (\lambda^2 - 1)(1 - \cos d(a, x))) \right) + O\left(\tau^2 \ln(2 + (\lambda^2 - 1)(1 - \cos d(a, x)))\right) \quad \text{for each } y \in \mathbb{S}^n. \quad (79)$$

(ii) For each $\gamma > 0$ and each $\beta \in [0, n/(n-2))$, it holds

$$0 < \int_{\mathbb{S}^n} \delta_{a,\lambda}^{p+1-\beta}(x) \ln^\gamma \left(2 + (\lambda^2 - 1)(1 - \cos d(a, x)) \right) dx = O\left(\frac{1}{\lambda^{\beta(n-2)/2}}\right).$$

Next, we state some elementary estimates needed in the paper. Their proofs follow from Taylor’s expansion.

Lemma 6.2.

i) Let $t_i > 0$ for $i \in \{1, \dots, \ell\}$ and $a, b \in \mathbb{R}$, there hold

$$\left| \left(\sum_{1 \leq i \leq \ell} t_i \right)^\gamma - \sum_{1 \leq i \leq \ell} t_i^\gamma \right| \leq c \begin{cases} \sum_{1 \leq i \neq j \leq \ell} (t_i t_j)^{\gamma/2} & \text{if } 0 < \gamma \leq 2 \\ \sum_{1 \leq i \neq j \leq \ell} t_i^{\gamma-1} t_j & \text{if } \gamma > 2 \end{cases}. \quad (80)$$

$$\left| |a+b|^\gamma - |a|^\gamma - \gamma |a|^{\gamma-2} ab \right| \leq c \begin{cases} |b|^\gamma + |a|^{\gamma-2} b^2 & \text{if } \gamma > 2 \\ |b|^\gamma & \text{if } 1 < \gamma \leq 2 \end{cases}. \quad (81)$$

ii) For $1 < \gamma \leq 3$ and $t_1, \dots, t_\ell > 0$, it holds

$$\left| \left(\sum_{1 \leq i \leq \ell} t_i \right)^\gamma - \sum_{1 \leq i \leq \ell} t_i^\gamma - \gamma t_1^{\gamma-1} \left(\sum_{2 \leq j \leq \ell} t_j \right) \right| t_1 \leq c \sum_{1 \leq k \neq j \leq \ell} (t_k t_j)^{(\gamma+1)/2}. \quad (82)$$

iii) For $\gamma > 1$, and $x, y > 0$, it holds

$$(x+y)^\gamma = x^\gamma + O(x^{\gamma-1}y + y^\gamma). \quad (83)$$

iv) For $\gamma < 1$, and $x, y > 0$, it holds

$$(x+y)^\gamma x = x^{\gamma+1} + O((xy)^{(\gamma+1)/2}). \quad (84)$$

Finally, we prove a crucial estimate of the derivative of the infinite dimensional variable with respect to the parameter α .

Lemma 6.3. The function v_τ , defined in Proposition 2.2, satisfies

$$\left\| \frac{\partial v_\tau}{\partial \alpha} \right\| = o(1).$$

Proof. Recall that we are in the case where $u_\tau = \sum_{i=1}^N \alpha_i \tilde{\delta}_{a_i, \lambda_i} + \alpha \omega + v_\tau$. First, we claim that $\partial v_\tau / \partial \alpha \in E_{\omega, a, \lambda}^\perp$.

Indeed, since $v_\tau \in E_{\omega, a, \lambda}^\perp$, it follows that $\langle \tilde{\delta}_{a_i, \lambda_i}, v_\tau \rangle = 0$ for each α . Therefore, taking the derivative with respect to α , we deduce that $\langle \tilde{\delta}_{a_i, \lambda_i}, \partial v_\tau / \partial \alpha \rangle = 0$. The other orthogonality constraints can be proved in the same way. Hence our claim follows.

Second, note that u_τ satisfies the problem $(\mathcal{N}_{K,\tau})$ and therefore the function v_τ satisfies the following PDE

$$L_{g_0} v_\tau := \left(-\Delta + \frac{n(n-2)}{4} \right) v_\tau = K u_\tau^{p-\tau} - \sum_{i=1}^N \alpha_i \tilde{\delta}_{a_i, \lambda_i}^p - \alpha K \omega^p$$

which implies that

$$\left(-\Delta + \frac{n(n-2)}{4} \right) \frac{\partial v_\tau}{\partial \alpha} = (p-\tau) K u_\tau^{p-\tau-1} \left(\omega + \frac{\partial v_\tau}{\partial \alpha} \right) - K \omega^p. \tag{85}$$

Following the proof of Lemma 5.2 and using (72), we write

$$\frac{\partial v_\tau}{\partial \alpha} = \left(\frac{\partial v_\tau}{\partial \alpha} \right)_- + \left(\frac{\partial v_\tau}{\partial \alpha} \right)_0 + \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+. \tag{86}$$

Since $\partial v_\tau / \partial \alpha \in E_{\omega, a, \lambda}^\perp$, from (21), we deduce that

$$\left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_0 \right\| = o \left(\left\| \frac{\partial v_\tau}{\partial \alpha} \right\| \right). \tag{87}$$

Concerning the other components, for example $(\partial v_\tau / \partial \alpha)_+$, multiplying (85) by $(\partial v_\tau / \partial \alpha)_+$ and integrating over \mathbb{S}^n , we get

$$\left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \right\|^2 = (p-\tau) \int_{\mathbb{S}^n} K u_\tau^{p-\tau-1} \left(\omega + \frac{\partial v_\tau}{\partial \alpha} \right) \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ - \int_{\mathbb{S}^n} K \omega^p \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+. \tag{88}$$

Note that, since $(\partial v_\tau / \partial \alpha)_+ \in N_+(\omega)$, it follows that the last integral in (88) is equal to zero. For the other integral, we have

$$\begin{aligned} & \int_{\mathbb{S}^n} K u_\tau^{p-\tau-1} \omega \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \\ &= \alpha^{p-\tau-1} \int_{\mathbb{S}^n} K \omega^{p-\tau} \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ + \sum_{i=1}^N O \left(\int_{\mathbb{S}^n} \left\{ \omega^{p-1} \tilde{\delta}_{a_i, \lambda_i} + \tilde{\delta}_{a_i, \lambda_i}^{p-1} \omega \right\} \left| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \right| \right) \\ &= o \left(\left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \right\| \right) \end{aligned} \tag{89}$$

and

$$\begin{aligned} & (p-\tau) \int_{\mathbb{S}^n} K u_\tau^{p-\tau-1} \frac{\partial v_\tau}{\partial \alpha} \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \\ &= p \sum_{i=1}^N \int_{\mathbb{S}^n} \tilde{\delta}_{a_i, \lambda_i}^{p-1} \frac{\partial v_\tau}{\partial \alpha} \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ + p \int_{\mathbb{S}^n} K \omega^{p-1} \frac{\partial v_\tau}{\partial \alpha} \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \\ &+ o \left(\left\| \frac{\partial v_\tau}{\partial \alpha} \right\| \left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \right\| \right). \end{aligned} \tag{90}$$

Since $H_0(\omega)$, $N_-(\omega)$ and $N_+(\omega)$ are orthogonal spaces for $\langle \cdot, \cdot \rangle$ and the bilinear form $\int K \omega^{p-1} \cdot$, we derive that

$$\int_{\mathbb{S}^n} K \omega^{p-1} \frac{\partial v_\tau}{\partial \alpha} \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ = \int_{\mathbb{S}^n} K \omega^{p-1} \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+^2. \tag{91}$$

Now, since $H_0(\omega)$ and $N_-(\omega)$ are finite dimensional spaces, we get

$$\begin{aligned} & \int_{\mathbb{S}^n} \tilde{\delta}_{a_i, \lambda_i}^{p-1} \frac{\partial v_\tau}{\partial \alpha} \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \\ &= \int_{\mathbb{S}^n} \tilde{\delta}_{a_i, \lambda_i}^{p-1} \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+^2 + o\left(\left(\left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_0 \right\|_\infty + \left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_- \right\|_\infty \right) \int_{\mathbb{S}^n} \tilde{\delta}_{a_i, \lambda_i}^{p-1} \left| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \right| \right) \\ &= \int_{\mathbb{S}^n} \tilde{\delta}_{a_i, \lambda_i}^{p-1} \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+^2 + o\left(\left\| \frac{\partial v_\tau}{\partial \alpha} \right\| \left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \right\| \right). \end{aligned} \tag{92}$$

However, since ω is bounded and $\langle \partial v_\tau / \partial \alpha, \omega \rangle = 0$, the first integral in (88) satisfies

$$\begin{aligned} & \int_{\mathbb{S}^n} K \omega^{p-\varepsilon-1} \omega \frac{\partial v_\tau}{\partial \alpha} \\ &= \alpha^{p-\varepsilon-1} \int_{\mathbb{S}^n} K \omega^{p-\varepsilon} \frac{\partial v_\tau}{\partial \alpha} + o\left(\sum_{i=1}^N \int_{\mathbb{S}^n} \left\{ \omega^{p-\varepsilon-1} \tilde{\delta}_{a_i, \lambda_i} + \tilde{\delta}_{a_i, \lambda_i}^{p-\varepsilon-1} \omega \right\} \left| \frac{\partial v_\tau}{\partial \alpha} \right| \right) \\ &= o\left(\left\| \frac{\partial v_\tau}{\partial \alpha} \right\| \right). \end{aligned}$$

Thus, using (89)-(92), the equation (88) becomes

$$\begin{aligned} \left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \right\|^2 &= p \int_{\mathbb{S}^n} K \omega^{p-1} \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+^2 + p \sum_{i=1}^N \int_{\mathbb{S}^n} \tilde{\delta}_{a_i, \lambda_i}^{p-1} \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+^2 \\ &\quad + o\left(\left\| \frac{\partial v_\tau}{\partial \alpha} \right\| \left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \right\| + \left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \right\| \right) \end{aligned}$$

which implies, by using Lemma 2.3, that

$$c \left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \right\|^2 \leq Q_{\omega, a, \lambda} \left(\left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \right) = o\left(\left\| \frac{\partial v_\tau}{\partial \alpha} \right\| \left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \right\| + \left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \right\| \right)$$

and therefore

$$\left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_+ \right\| = o(1) + o\left(\left\| \frac{\partial v_\tau}{\partial \alpha} \right\| \right). \tag{93}$$

In the same way, multiplying (85) by $(\partial v_\tau / \partial \alpha)_-$ and integrating over \mathbb{S}^n , we get

$$\left\| \left(\frac{\partial v_\tau}{\partial \alpha} \right)_- \right\| = o(1) + o\left(\left\| \frac{\partial v_\tau}{\partial \alpha} \right\| \right). \tag{94}$$

Combining (87), (93) and (94), we get that

$$\left\| \frac{\partial v_\tau}{\partial \alpha} \right\| = o(1)$$

which completes the proof of the lemma. □

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