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Blow-up Solutions for Mean Field Equations and Toda Systems on Riemann Surfaces

by Zhengni Hu

Supervisor:
Prof. Dr. Thomas Bartsch

Co-Supervisor:
Prof. Dr. Mohameden Ahmedou

Jury:
Prof. Dr. Thomas Bartsch
Prof. Dr. Mohameden Ahmedou
Prof. Dr. Stefan Witzel
Dr. Panagiotis Konstantis

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Dekan: *Stefan Witzel*

Gutachter:	*Thomas Bartsch*	*Mohameden Ahmedou*
	Stefan Witzel	*Panagiotis Konstantis*

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Abstract

This dissertation studies mean field type equations and the $SU(3)$ Toda system, significant topics both in mathematical physics and differential geometry, with wide-ranging applications. Its primary focus lies in the construction of mixed boundary - interior bubbling solutions on Riemann surfaces with boundary, which exhibit blow-up at prescribed numbers of points in the interior and on the boundary as the parameters approach critical values.

For mean field type equations and partial blow-up solutions of Toda systems, variational methods and Lyapunov-Schmidt reduction are employed to construct the solutions. However, for asymmetric blow-up solutions of Toda systems, due to the intricate limit profiles, the Lyapunov-Schmidt reduction cannot be applied directly. Herein, we introduce the “k-symmetric” condition for the surface and utilize singular perturbation methods to construct a family of bubbling solutions that blows up at “k-symmetric centers” of the surface.

The dissertation is organized into two parts: one exploring blow-up solutions for mean field type equations and the other for the $SU(3)$ Toda system. Each part delves into the construction of blow-up solutions under various scenarios.

Keywords: Mean field equations, Toda system, Blow-up solutions, Lyapunov-Schmidt reduction

List of induced articles

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Preface

Mean field equations and Toda systems are important topics in mathematical physics and differential geometry, influencing a broad spectrum of areas. Mean field equations, in particular, are closely related to various problems such as the Kazdan-Warner problem [KW74, DJLW97, NT98], the prescribed Gauss curvature problem [CY87, CD87, CGY93, CY88, CL93, Mos73], the Chern-Simons-Higgs gauge theory [NT99, Tar96, DJLW99b, DJL$^+$01, CY95], statistical mechanics [CLMP92, CLMP95, DJLW99a, CK94, Kie93], and the Keller-Segel system for chemotaxis collapse [SS00, WW02, KS70, Chi84, Bat19]. These connections imply the equation's versatility and profound implications in theoretical and applied mathematics. Toda systems have a strong connection with geometric constructs. They are associated with holomorphic curves on surfaces in the $\mathbb{CP}^N$, flat $SU(N+1)$ connection, complete integrability, and harmonic sequences [Gue97, BJRW88, Dol97, CW87, BW97]. In particular, for the two-dimensional sphere S^2, the solution space of the $SU(3)$ Toda system is identical to the space of holomorphic curves of S^2 in $\mathbb{CP}^2$ [LWY12]. In physics, Toda systems are one of the limiting equations of nonabelian Chern-Simons gauge field theory (refer to [DJPT91, Dun95, NT99, NT00, Yan99, Yan01] and references therein), which implies their significance in understanding complex physical phenomena.

Despite originating from various backgrounds, both mean field equations and Toda systems can be viewed as generalizations of Liouville-type equations. The $SU(3)$ Toda system can be described as a coupled system comprising two mean field equations. The exponential nonlinearity of these equations significantly influences the behavior of the solutions, leading to complex phenomena, including the potential for solutions to exhibit blow-up behavior under specific conditions. Over the last two decades, mean field equations and Toda systems have gained significant attention, stimulating fruitful research on their existence, uniqueness, and blow-up phenomena.

This dissertation delves into the construction of blow-up solutions for mean field equations and the $SU(3)$ Toda system on compact Riemann surfaces with boundary, focusing on problems with Neumann boundary conditions. This setting addresses a gap in current research—predominantly focused on bounded domains in $\mathbb{R}^2$ or compact surfaces without boundaries—and explores blow-up phenomena that may occur on the boundary.

The dissertation is organized as follows:

Chapter 1 serves as the introduction, introducing the research problems, basic settings, and stating the main results of this thesis.

Chapter 2 introduces basic concepts and tools for studying blow-up solutions on Riemann surfaces with boundary. This chapter includes discussions on isothermal coordinates, regularity theory, Green's functions, Kirchhoff-Routh type functions, and Lyapunov-Schmidt reduction.

Subsequently, the dissertation is divided into two parts: Chapter 3 explores blow-up solutions for mean field type equations; Chapter 4 and Chapter 5 delve into blow-up solutions for the $SU(3)$ Toda system.

Chapter 3 specifically aims to construct blow-up solutions on Riemann surfaces with boundaries through the variational method and Lyapunov-Schmidt reduction.

Chapter 4 addresses the construction of blow-up solutions that exhibit partial blow-up phenomena, employing Lyapunov-Schmidt reduction. To deal with the component that does not blow up, we must study the non-degeneracy of singular mean field equations coupled with a balanced condition which is the so-called shadow system.

Chapter 5 studies asymmetric blow-ups of Toda system. The problem introduces additional complexity due to its lack of proper approximation solutions, necessitating alternative techniques or additional assumptions. Under the assumption of a "k-symmetric" property, the construction of blow-up solutions for the surface is realized by singular perturbation methods.

“众里寻他千百度，蓦然回首，那人却在，灯火阑珊处。”

——宋·辛弃疾《青玉案》

"In the crowd I seek him a thousand times; suddenly, turning my head, I find him there where the lights are dim."

-From Qiji Xin, poet of the Song Dynasty, in "Qing Yu An"

Contents

1 Introduction

1.1 Mean field type equations

In the first part, we are interested in the existence of blow-up solutions of the following mean field equations on a Riemann surface (Σ, g) with smooth boundary $\partial\Sigma$:

$$\begin{cases} -\Delta_g u = \lambda \left(\dfrac{Ve^u}{\int_\Sigma Ve^u dv_g} - \dfrac{1}{|\Sigma|_g} \right) & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} u = 0 & \text{on } \partial\Sigma \end{cases}, \tag{1.1.1}$$

where g is the Riemann metric, $\mathring{\Sigma} = \Sigma \setminus \partial\Sigma$ denotes the interior of Σ, ν_g is the unit outward normal to the boundary $\partial\Sigma$, Δ_g is the Laplace-Beltrami operator, dv_g is the volume element in (Σ, g), $|\Sigma|_g = \int_\Sigma dv_g$, $V : \Sigma \to \mathbb{R}_+$ is a smooth positive function, and $\lambda \geq 0$ is a parameter.

The existence of solutions for mean field equations on Riemann surfaces has been widely studied in the past two decades, with most papers dealing with compact surfaces without boundary. In that case, if $\lambda < 8\pi$ the functional

$$J_\lambda(u) := \frac{1}{2} \int_\Sigma |\nabla u|^2 \, dv_g - \lambda \log \left(\int_\Sigma Ve^u dv_g \right)$$

associated to (1.1.1) is bounded below and coercive on a subspace of $H^1(\Sigma)$ with average 0,

$$\overline{\mathrm{H}}^1(\Sigma) := \left\{ u \in H^1(\Sigma) : \int_\Sigma u dv_g = 0 \right\}.$$

Due to the Moser-Trudinger inequality, solutions of (1.1.1) can be obtained by minimizing J_λ. For $\lambda \geq 8\pi$, the functional J_λ is unbounded from below and above, making the problem more complicated. In [ST98], Struwe and Tarantello proved the existence of non-trivial solutions on the flat torus $\Sigma = \mathbb{R}^2/\mathbb{Z}^2$ for $\lambda \in (8\pi, 4\pi^2)$. In [DJLW99a], Ding et al. studied the existence of solutions on general compact Riemann surface with genus ≥ 1 for $\lambda \in (8\pi, 16\pi)$. Lin in [Lin00] considered the case of the 2-dimensional unit sphere S^2 for $\lambda \in (8\pi, 16\pi) \cup (16\pi, 24\pi)$ by computing the Leray-Schauder degree d_λ. In [CL03] Chen and Lin generalized the result of [DJLW99a] by computing d_λ for $\lambda \notin 8\pi\mathbb{N}_+$. If the genus of Σ is at least one, then $d_\lambda \neq 0$, hence (1.1.1) has a solution. Unfortunately, when Σ is a sphere and $\lambda \in (8\pi k, 8\pi(k+1))$ with $k \geq 2$, then $d_\lambda = 0$, so this method does not yield the existence of solutions. The first complete existence result for a general compact surface was given by Djadli [Dja08]. He applied variational methods, and the min-max scheme introduced by Djadli and Malchiodi in [DM08] to obtain the existence of a solution for $\lambda \notin 8\pi\mathbb{N}_+$. In a very recent paper [LSY23], Li, Sun, and Yang considered the mean field equation on compact Riemann surfaces with smooth boundary. They show that the mean field equation with Dirichlet boundary conditions has a solution for $\lambda \notin 8\pi\mathbb{N}_+$ provided the surface is not contractible. They also showed that the mean field equation with Neumann boundary conditions (1.1.1) admits a solution for $\lambda \notin 4\pi\mathbb{N}_+$, even if Σ is contractible.

For (1.1.1), blow-up solutions of mean field equations have been constructed for bounded domains in $\mathbb{R}^2$ with Dirichlet boundary conditions and compact surfaces without boundary. See [BM91, dPKM05, dPW06, EGP05, MW01, NS90, Suz92] for mean field equations on domains, and [BGH$^+$20, EF14, Fig23] for mean field equations on compact closed surfaces, and the references therein.

In Chapter 3, we can consider a more general problem by introducing a linear source term with parameter $\beta \geq 0$ as follows:

$$\begin{cases} -\Delta_g u + \beta u = \lambda \left(\dfrac{Ve^u}{\int_\Sigma Ve^u dv_g} - \dfrac{1}{|\Sigma|_g} \right) & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} u = 0 & \text{on } \partial\Sigma \end{cases}, \tag{1.1.2}$$

where the parameters $\lambda, \beta \in \mathbb{R}$ and V is a non-negative smooth function with a finite set of zeros, denoted by $\{q_1, \cdots, q_\iota\}$ for some $\iota \in \mathbb{N}$. Apparently, when $\beta = 0$ and V is a positive function, (1.1.2) reduces to the classical mean field equation (1.1.1). The formulation in (1.1.2) arises from the steady state of the Keller-Segel system; here, the Neumann boundary condition serves as the natural boundary condition for models of chemotaxis phenomena.

In the one-dimensional case, Schaaf proved the existence of non-trivial solutions using a bifurcation technique in [Sch85]. For the higher-dimensional case with dimension $N \geq 3$, we refer to [AP16, PV15, Bil98] and references therein. We specifically focus on the case where dimension $N = 2$. By Struwe's technique and blow-up analysis, Wang and Wei in [WW02] obtained non-constant solutions for $\beta > \frac{\lambda}{|\Omega|} - \lambda_1$ and $\lambda \in (4\pi, +\infty) \setminus 4\pi\mathbb{N}_+$, where λ_1 is the first eigenvalue of $-\Delta$ with the Neumann boundary condition. Independently, Senba and Suzuki deduced the same result in [SS00]. Battaglia generated their result for $\lambda \in (0, +\infty) \setminus 4\pi\mathbb{N}_+$ and β with any sign in [Bat19]. He proved the existence of nonconstant solutions with some algebraic conditions involved with β, λ and eigenvalues λ_i by the variational method and Morse theory. However, when λ approaches the critical value set $4\pi\mathbb{N}_+$, the blow-up phenomena may occur. Del Pino and Wei in [dPW06] constructed positive value bubbling solutions for the Neumann boundary condition problem on a bounded domain $\Omega \subset \mathbb{R}^2$ for parameter $\beta > 0$

$$\begin{cases} -\Delta u + \beta u = \varepsilon^2 e^u & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases}, \tag{1.1.3}$$

by the Lyapunov-Schmidt reduction as $\varepsilon \to 0$. In particular, the sequence of bubbling solutions blows up at k distinct points $\xi_1, \cdots, \xi_k$ inside the domain Ω and $m-k$ distinct points $\xi_{k+1}, \cdots, \xi_m$ on the boundary of Ω. Moreover, as $\varepsilon \to 0$

$$u_\varepsilon \to \sum_{i=1}^{k} 8\pi\delta_{\xi_i} + \sum_{i=k+1}^{m} 4\pi\delta_{\xi_i},$$

in the sense of measures on Σ, where δ_ξ is Dirac mass concentrated at ξ. To the best of the author's knowledge, it is the first result concerning the construction of blow-up solutions on the boundary for Liouville-type equations with Neumann boundary conditions. The technique so-called "localized energy method" developed to address blow-up points of the boundary in [dPW06] is very important for this project.

Subsequently, Del Pino, Pistoia, and Vaira in [dPPV16] constructed solutions of (1.1.3) that blow up along the whole boundary $\partial\Omega$.

Inspired by [dPW06], we study the blow-up solutions of a generalized mean field equation (1.1.2) and naturally the blow-up solutions of (1.1.1) are constructed. Specifically, given integers $m \geq k \geq 0$, we establish sufficient conditions for blow-up solutions. Moreover, the precise locations of blow-up points are explicitly characterized by the stable critical point set of a reduced function $\mathcal{F}^V_{k,m}$ defined by (2.6.2) in terms of Green's function on Σ and the potential function V. It is noteworthy that we allow V to be 0 at q_i for any $i = 1, \cdots, \iota$ where $\iota \in \mathbb{N}$. So, it is also possible to establish the existence of blow-up solutions for the following singular problem:

$$\begin{cases} -\Delta_g \tilde{u} + \beta\tilde{u} = \lambda \left(\dfrac{\tilde{V} e^{\tilde{u}}}{\int_\Sigma \tilde{V} e^{\tilde{u}} dv_g} - \dfrac{1}{|\Sigma|_g} \right) - \sum_{i=1}^{\iota} n_i \dfrac{\varrho(q_i)}{2} \left(\delta_{q_i} - \dfrac{1}{|\Sigma|_g} \right) & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} \tilde{u} = 0 & \text{on } \partial\Sigma \end{cases}. \tag{1.1.4}$$

Here, $\varrho(x) = 8\pi$ when $x \in \mathring{\Sigma}$ and 4π when $x \in \partial\Sigma$, $\tilde{V}$ is a positive smooth function, and $n_i \in \mathbb{N}$ for $i = 1, \cdots, \iota$. Notably, the problem (1.1.4) is a special example of (1.1.2). We take

$u(x) = \tilde{u}(x) + \sum_{i=1}^{\iota} n_i \frac{\varrho(q_i)}{2} G^g(x, q_i)$ and $V(x) = \tilde{V}(x) e^{-4\pi \sum_{i=1}^{\iota} n_i \frac{\varrho(q_i)}{2} G^g(x,q_i)}$, where $G^g(\cdot, q_i)$ is the Green's function defined in Section 2.4. Then, u satisfies the equations (1.1.2) in which V is a nonnegative smooth function with the zero set $\{q_1, \cdots, q_\iota\}$.

Our main challenge is that the Neumann boundary condition implies the potential occurrence of blow-up points on the boundary. The estimates derived for the interior case cannot be straightforwardly applied to the boundary scenario. To address this difficulty, we employ the approximation solutions from [EF14] for closed Riemann surfaces with some modification on the boundary and then use the Lyapunov-Schmidt reduction and variational methods to establish a sufficient condition for the existence of bubbling solutions for λ approaching the critical value set $4\pi\mathbb{N}_+$.

Throughout this dissertation, we denote $\varrho(x) = 8\pi$ if $x \in \mathring{\Sigma}$ and 4π if $x \in \partial\Sigma$ and $\lambda_{k,m} = 4\pi(m+k)$. We define the configuration set as follows:

$$\Xi_{k,m} = \mathring{\Sigma}^k \times (\partial\Sigma)^{m-k} \setminus \mathbb{F}_{k,m}(\Sigma),$$

where $\mathbb{F}_{k,m}(\Sigma) := \{\xi = (\xi_1, \cdots, \xi_m) : \xi_i = \xi_j \text{ for some } i = j\}$ is called the thick diagonal.

Let $\Sigma' := \{x \in \Sigma : V(x) > 0\}$ and then we define that

$$\Xi'_{k,m} := \Xi_{k,m} \cap (\Sigma')^m.$$

The main theorem in Chapter 3 is as follows:

Theorem 1.1.1. *Given integers* $k \leq m \in \mathbb{N}_+$. *If* $(\emptyset \neq) K \subset\subset \Xi'_{k,m}$ *is a* C^1*-stable critical point set of* $\mathcal{F}^V_{k,m}$ *(see (2.6.2)), then there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *a family of blow-up solutions* u_ε *of (1.1.2) with* $\lambda_\varepsilon \to \lambda_{k,m}$ *can be constructed. Furthermore, solutions* u_ε *blow up precisely at points* $\xi_1, \cdots, \xi_m$ *with* $\xi = (\xi_1, \cdots, \xi_m)$ *in* K, *(up to a subsequence), as* $\varepsilon \to 0$

$$\frac{\lambda_\varepsilon V e^{u_\varepsilon}}{\int_\Sigma V e^{u_\varepsilon} dv_g} \to \sum_{i=1}^{m} \varrho(\xi_i) \delta_{\xi_i},$$

which is convergent as measures on Σ.

We define the set of global minimum points of $\mathcal{F}^V_{k,m}$ as follows:

$$K_{\min} := \left\{ x \in \Xi'_{k,m} : \mathcal{F}^V_{k,m}(\xi) = \inf_{\Xi'_{k,m}} \mathcal{F}^V_{k,m} \right\}. \tag{1.1.5}$$

Corollary 1.1.2. *Given integers* $0 \leq k \leq m \in \mathbb{N}_+$. *Suppose* $K_{min} \neq \emptyset$. *Then, the conclusions in Theorem 1.1.1 hold. Furthermore,* u_ε *has* k *local maximum points* ξ_i^ε *in* $\mathring{\Sigma}$ *for* $i = 1, \cdots, k$

and $m-k$ local maximum points ξ_i^ε restricted to the boundary $\partial\Sigma$ for $i=k+1,\cdots,m$ such that up to a subsequence $(\xi_1^\varepsilon,\cdots,\xi_m^\varepsilon)$ converges to $\xi:=(\xi_1,\cdots,\xi_m)\in K_{min}$ with

$$\lim_{\varepsilon\to 0}\mathcal{F}^V_{k,m}(\xi_1^\varepsilon,\cdots,\xi_m^\varepsilon)=\min_{\Xi'_{k,m}}\mathcal{F}^V_{k,m}=\mathcal{F}^V_{k,m}(\xi).$$

Remark 1.1.3. *If the zero set of V is empty, $\mathcal{F}^V_{k,m}$ is divergent towards $+\infty$ as ξ approaches $\partial\Xi_{k,m}$ (see Lemma 2.6.1). This divergence suggests the presence of at least one global minimum point in the interior of $\Xi_{k,m}$, i.e.*

$$K_{min}\neq\emptyset.$$

Additionally, a local minimum point is stable. Consequently, in this scenario, the blow-up solutions must exist. However, when $V(q)=0$ for some $q\in\Sigma$, a challenge arises. As ξ approaches $\partial\Xi_{k,m}$, there exist cases where the first two terms related to Green's functions go to $+\infty$ while the last term approaches $-\infty$ as ξ approaches to $\partial\Xi'_{k,m}$, leading to complicated asymptotic behaviors of $\mathcal{F}^V_{k,m}$.

1.2 Toda systems

In the second part, we consider the following Toda system on a Riemann surface Σ with Riemann metric g:

$$\begin{cases} -\Delta_g u_1 = 2\rho_1\left(\frac{V_1e^{u_1}}{\int_\Sigma V_1e^{u_1}dv_g}-\frac{1}{|\Sigma|_g}\right)-\rho_2\left(\frac{V_2e^{u_2}}{\int_\Sigma V_2e^{u_2}dv_g}-\frac{1}{|\Sigma|_g}\right) & \text{in } \mathring{\Sigma} \\ -\Delta_g u_2 = 2\rho_2\left(\frac{V_2e^{u_2}}{\int_\Sigma V_2e^{u_2}dv_g}-\frac{1}{|\Sigma|_g}\right)-\rho_1\left(\frac{V_1e^{u_1}}{\int_\Sigma V_1e^{u_1}dv_g}-\frac{1}{|\Sigma|_g}\right) & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g}u_1=\partial_{\nu_g}u_2=0 & \text{on } \partial\Sigma \end{cases}, \tag{1.2.1}$$

where ρ_i is a non-negative parameter and $V_i:\Sigma\to\mathbb{R}_+$ is a smooth positive function for any $i=1,2$. For simplicity, we normalize the area of Σ, i.e. $|\Sigma|_g=1$.

The system (1.2.1) has a variational structure with the corresponding energy functional

$$J_\rho(u_1,u_2)=\int_\Sigma Q(u,u)dv_g+\sum_{i=1}^2\rho_i\left(\frac{1}{|\Sigma|_g}\int_\Sigma u_idv_g-\log\int_\Sigma V_ie^{u_i}dv_g\right),$$

where the parameter $\rho=(\rho_1,\rho_2)$, and the bilinear map

$$Q(v,w)=\frac{1}{3}\langle\nabla v_1,\nabla w_1\rangle_g+\frac{1}{3}\langle\nabla v_2,\nabla w_2\rangle_g+\frac{1}{6}\langle\nabla v_2,\nabla w_1\rangle_g+\frac{1}{6}\langle\nabla v_1,\nabla w_2\rangle_g$$

for $v=(v_1,v_2)$ and $w=(w_1,w_2)$.

The weak solutions of (1.2.1) correspond to the critical points of the functional J_ρ on $H^1(\Sigma)\times$

$H^1(\Sigma)$. Nevertheless, we focus on solutions in the subspace of $H^1(\Sigma)\times H^1(\Sigma)$ where $\int_\Sigma u_i dv_g = 0$ for each $i = 1, 2$. Both (1.2.1) and the energy functional J_ρ exhibit invariance under the addition of a constant; thus, our assumption does not impose a significant limitation. Define the inner product $\langle u, v\rangle = \int_\Sigma \langle \nabla u, \nabla v\rangle_g dv_g$ for any $u, v \in \overline{\mathrm{H}}^1(\Sigma)$. We study the functional J_ρ on the space $\overline{\mathrm{H}}^1(\Sigma) \times \overline{\mathrm{H}}^1(\Sigma)$ instead of $H^1(\Sigma) \times H^1(\Sigma)$.

The problem (1.2.1) is the classical $SU(3)$ Toda system. The study of the Toda system on compact Riemann surfaces without boundary has attracted considerable attention. A primary tool for controlling the exponential terms in the Toda system is the Moser-Trudinger inequality, which allows for the exponential term to be bounded by the H^1-norm. A sharper version of this inequality for the Toda system is given by Jost and Wang in [JW01]:

$$4\pi \log \int_\Sigma e^{u_1} dv_g + 4\pi \log \int_\Sigma e^{u_1} dv_g \le \int_\Sigma Q(u, u) dv_g + C_\Sigma, \quad u = (u_1, u_2) \in \overline{\mathrm{H}}^1(\Sigma) \times \overline{\mathrm{H}}^1(\Sigma). \tag{1.2.2}$$

It implies that if $\rho_i < 4\pi$ for $i = 1, 2$ the functional J_ρ is coercive and bounded from below. Via the standard variational method, J_ρ admits a minimizer. A crucial insight from [JW01] reveals that the functional J_ρ is bounded from below if and only if $\rho_i \le 4\pi$ for any $i = 1, 2$. However, situations where the ρ_i's exceed the threshold of coercivity value 4π become more intricate. With a certain constraint on the Gaussian curvature, Jost, Lin, and Wang [JLW05] prove that J_ρ has a minimizer when $\rho_1 = 4\pi$ and $\rho_2 \in (0, 4\pi)$. Independently, Li [LL05] studies the case $\rho_i = 4\pi$ for both $i = 1, 2$. For surfaces Σ with a positive genus, the Toda system has been shown to possess solutions for $\rho_i \notin 4\pi\mathbb{N}_+$ for $i = 1, 2$ (see [BJMR15]). In general genus settings, Malchiodi and Ndiaye in [MN07] prove the existence of solutions for $\rho_1 \in (4\pi m, 4\pi(m + 1))$ and $\rho_2 \in (0, 4\pi)$, with $m \in \mathbb{N}$. Subsequent research has addressed cases where one of the parameters ρ_i lies in $(4\pi, 8\pi)$, confirming the existence of solutions on surfaces with positive genus (see [MR10]) and without genus restrictions (see [JKM15]). The problem of the existence of general parameters and genus remains an open question.

Suppose $u^n = (u_1^n, u_2^n)$ be a sequence of solutions of (1.2.1) with the parameter $\rho^n = (\rho_1^n, \rho_2^n)$ that converges to $\rho = (\rho_1, \rho_2)$ as $n \to +\infty$. We define the blow-up set for u_i^n as

$$\mathcal{S}_i := \{x \in \Sigma : \exists x_n \to x \text{ such that } u_i^n(x_n) \to +\infty\},$$

and the local limit masses as

$$\sigma_i(x) = \lim_{r\to 0} \lim_{n\to+\infty} \rho_i^n \int_{\mathcal{U}_r(x)} \frac{V_i e^{u_i^n}}{\int_\Sigma V_i e^{u_i^n} dv_g} dv_g \qquad \text{for } i = 1, 2,$$

where $\mathcal{U}_r(x) := \{x' \in \Sigma : d_g(x, x') < r\}$ and $d_g(\cdot, \cdot)$ denotes the distance induced by the geodesic on (Σ, g). Let $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$ which is called the blow-up points set of the Toda system (1.2.1).

For Σ compact Riemann surfaces with smooth boundary, for any $x \in \mathcal{S}$, $(\sigma_1(x), \sigma_2(x))$ can only take the following five possible values (refer to [JLW05] for a domain in $\mathbb{R}^2$). Specifically, we have the following statement:

Proposition 1.2.1. *Let* $(\sigma_1(x), \sigma_2(x))$ *be the local limit masses of Toda system* (1.2.1)*. Then*

$$(\sigma_1(x), \sigma_2(x)) \in \left\{ \left(0, \frac{1}{2}\varrho(x)\right), \left(\frac{1}{2}\varrho(x), 0\right), \left(\frac{1}{2}\varrho(x), \varrho(x)\right), \left(\varrho(x), \frac{1}{2}\varrho(x)\right), (\varrho(x), \varrho(x)) \right\}.$$

Based on the values of $(\sigma_1(x), \sigma_2(x))$ at blow-up point x, we have the following three scenarios:

i). Partial blow-up: $\left(\frac{1}{2}\varrho(x), 0\right), \left(0, \frac{1}{2}\varrho(x)\right)$. In this case, one component is bounded from above around x, and the other blows up at x;

ii). Asymmetric blow-up: $\left(\frac{1}{2}\varrho(x), \varrho(x)\right), \left(\varrho(x), \frac{1}{2}\varrho(x)\right)$. In this case, both components blow up at the same point x but with different rates;

iii). Full blow-up: $(\varrho(x), \varrho(x))$. In this case, both components blow up at the same point x with the same rates.

In the subsequent two chapters, we explore the construction of blow-up solutions, focusing on partial and asymmetric blow-up phenomena, respectively. For the full blow-up scenario, the special case where $u_1 = u_2$ and $\rho_1 = \rho_2$ simplify the system to a mean field equation. For this equation, blow-up solutions have been constructed in both bounded domains, closed Riemann surfaces and compact Riemann surfaces with boundary (see [EGP05, BGH$^+$20, EF14] and Chapter 3, respectively). However, the general case of full blow-ups remains unresolved. Lin, Wei, and Zhao have examined blow-up solutions in the full blow-up scenario, providing five necessary conditions for the existence of such solutions, indicating the increased difficulty of constructing full blow-up solutions in [LWZ12]. The discussion of the full blow-up case is beyond the scope of this thesis due to its complexities.

In Chapter 4, we construct a family of blow-up solutions via the Lyapunov-Schmidt reduction and variational methods, where one component remains uniformly bounded from above, while the other exhibits blow-ups at a prescribed number of points, both in the interior and on the boundary. This construction is based on a non-degeneracy hypothesis of a “shadow system”.

For partial blow-up scenarios, D’Aprile, Pistoia, and Ruiz employ Lyapunov-Schmidt reduction and variational methods to successfully construct partial blow-up solutions, either in simply connected domains or when one of the parameters is sufficiently small in [DPR15]. Independently, [LLWY18] proved the existence of blow-up solutions exhibiting partial blow-up for $\min\{\rho_1, \rho_2\} < 8\pi$ on closed Riemann surfaces by the method of degree counting.

Given integers $m \geq k \geq 0$, consider a sequence $\rho^n := (\rho_1^n, \rho_2^n)$ where $\rho_2^n = \rho_2 \in (0, 2\pi)$ and ρ_1^n approaches $2\pi(m+k)$. We aim to construct a sequence of solutions $u^n = (u_1^n, u_2^n)$ for which u_1^n exhibits blow-up while u_2^n remains bounded from above. We denote

$$\Xi_{k,m}^{\delta} := \{\xi \in \Xi_{k,m} : d_g(\xi_i, \partial\Sigma) \geq \delta \text{ for } i = 1, \cdots, k; d_g(\xi_i, \xi_j) \geq \delta \text{ for } i \neq j\},$$

for any $\delta > 0$, To address the component without blow-ups, we introduce solutions of the following singular mean field equations:

$$\begin{cases} -\Delta_g z(x) = 2\rho_2 \left(\dfrac{\tilde{V}_2(x,\xi) e^{z(x)}}{\int_\Sigma \tilde{V}_2(\cdot,\xi) e^{z} dv_g} - 1 \right) & x \in \Sigma \\ \partial_{\nu_g} z(x) = 0 & x \in \partial\Sigma \end{cases}, \tag{1.2.3}$$

where $\xi \in \Xi_{k,m}$ and

$$\tilde{V}_2(x,\xi) = V_2(x) e^{-\sum_{i=1}^{m} \frac{1}{2}\varrho(\xi_i) G^g(x,\xi_i)},$$

where G^g is the Green's function (for details, refer to Section 2.4). Since $2\rho_2 < 4\pi$, the energy functional

$$I_\xi(z) = \frac{1}{2}\int_\Sigma |\nabla z|_g^2 dv_g - 2\rho_2 \log\left(\int_\Sigma \tilde{V}_2(\cdot,\xi) e^{z} dv_g \right)$$

is coercive by the Moser-Trudinger inequality. Standard variational analysis indicates that (1.2.3) admits a solution in $\overline{\mathrm{H}}^1(\Sigma)$ for $\rho_2 \in (0, 2\pi)$. However, the non-degeneracy of the solutions is not guaranteed. To address this issue, additional conditions are needed.

Define the reduced function as follows for given $V_1 \in C^\infty(\Sigma, \mathbb{R}_+)$:

$$\Lambda_{k,m} : \Xi_{k,m} \to \mathbb{R}, \quad \xi \mapsto \frac{1}{2} I_\xi(z(\cdot,\xi)) - \frac{1}{4}\mathcal{F}_{k,m}^{V_1}(\xi), \tag{1.2.4}$$

where $\mathcal{F}_{k,m}^{V_1}$ is defined by (2.6.2) and $z(\cdot,\xi)$ is a solution of (1.2.3) with respect to ξ.

As in [LLWY18], for fixed $\rho_2 \notin 2\pi\mathbb{N}$ and some $\alpha \in (0,1)$, we consider the following shadow system in the space $C^{2,\alpha}(\Sigma) \times \Xi_{k,m}$ with restriction $\int_\Sigma w dv_g = 0$:

$$\begin{cases} -\Delta_g w = 2\rho_2 \left(\dfrac{V_2 e^{w - \sum_{i=1}^{m} \frac{\varrho(\xi_i)}{2} G^g(x,\xi_i)}}{\int_\Sigma V_2 e^{w - \sum_{i=1}^{m} \frac{\varrho(\xi_i)}{2} G^g(x,\xi_i)} dv_g} - 1 \right) & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} w = 0 & \text{on } \partial\Sigma \\ \nabla f_0(\xi) = 0 & \text{in } \Xi_{k,m} \end{cases}, \tag{1.2.5}$$

where $f_0(\xi_1, \cdots, \xi_m) = \mathcal{F}^{V_1}_{k,m}(\xi) - \sum_{i=1}^m \varrho(\xi_i) w(\xi_i)$.

We consider the following hypothesis:

(H1) *There exists a non-degenerate solution* $(w, \xi) \in C^{2,\alpha}(\Sigma) \times \Xi_{k,m}$ *with* $\int_\Sigma w dv_g = 0$ *of the shadow system* (1.2.5) *for the positive potential function* $(V_1, V_2) \in C^{2,\alpha}(\Sigma, \mathbb{R}_+) \times C^{2,\alpha}(\Sigma, \mathbb{R}_+)$.

Theorem 1.2.2. *Given integers* $m \geq k \geq 0$ *and* $\rho_2 \notin 2\pi\mathbb{N}$. *If the hypothesis (H1) is satisfied for the potential functions* V_1, V_2 *and* (w, ξ), *then there exist an open neighborhood of* ξ *denoted by* $\mathscr{D}(\subset \Xi_{k,m})$ *and a constant* $\varepsilon_0 > 0$ *such that a family of solutions* $\mathbf{u}_\varepsilon = (u_{1,\varepsilon}, u_{2,\varepsilon})$ *of the Toda system* (1.2.1) *corresponding to the parameter* $\rho^\varepsilon = (\rho_1^\varepsilon, \rho_2)$ *can be constructed for* $\varepsilon \in (0, \varepsilon_0)$. *Moreover, as* $\varepsilon \to 0$, $\rho^\varepsilon \to (2\pi(m+k), \rho_2)$, *there exists a sequence* $\xi^\varepsilon = (\xi_1^\varepsilon, \cdots, \xi_m^\varepsilon) \to \xi$ *in* $\Xi_{k,m}$ *such that*

$$2\rho_1^\varepsilon \frac{V_1 e^{u_{1,\varepsilon}}}{\int_\Sigma V_1 e^{u_{1,\varepsilon}} dv_g} \to \sum_{i=1}^k 8\pi\delta_{\xi_i} + \sum_{i=k+1}^m 4\pi\delta_{\xi_i}, \tag{1.2.6}$$

which is convergent as measures on Σ, *and*

$$u_{2,\varepsilon} \quad = \quad w(\cdot, \xi^\varepsilon) - \frac{1}{2}\sum_{i=1}^m \varrho(\xi_i) G^g(\cdot, \xi_i^\varepsilon) + o(1), \tag{1.2.7}$$

which is convergent in $\overline{\mathrm{H}}^1(\Sigma)$.

For the parameter $\rho_2 \in (0, 2\pi)$ sufficiently small, we can deduce the uniqueness and non-degeneracy of the solutions of (1.2.3) by the implicit function theorem (refer to Section 4.1) for any ξ in a compact subset of $\Xi_{k,m}$. On the other hand, $\mathcal{F}^{V_1}_{k,m}(\xi) \to +\infty$ as $\xi \to \partial\Xi_{k,m}$, which implies that the global minimum point of the reduced function $\Lambda_{k,m}$ exists and lies in the interior of $\Xi_{k,m}$. Hence, for $\rho_2 \in (0, 2\pi)$ sufficiently small, the hypothesis (H1) holds true. It leads to the following corollary:

Corollary 1.2.3. *Given integers* $m \geq k \geq 0$. *There exist an open subset* $\mathscr{D} \subset \overline{\mathscr{D}} \subset \Xi_{k,m}$ *and* $\rho_0 \in (0, 2\pi)$ *sufficiently small such that for any fixed* $\rho_2 \in (0, \rho_0)$, *a family of solutions* $\mathbf{u}_\varepsilon = (u_{1,\varepsilon}, u_{2,\varepsilon})$ *of the Toda system* (1.2.1) *corresponding to the parameter* $\rho^\varepsilon = (\rho_1^\varepsilon, \rho_2) \to (2\pi(m+k), \rho_2)$ *can be constructed and they blow up at* k*-points in the interior and* $(m-k)$*-points on the boundary.*

Moreover, there exists a sequence $\xi^\varepsilon \in \Xi_{k,m}$ *converging to some* $\xi \in \overline{\mathscr{D}}$ *such that* (1.2.6) *is valid for* ρ_1^ε *and* $u_{1,\varepsilon}$, (1.2.7) *is valid for* ρ_2 *and* $u_{2,\varepsilon}$, *and* $\inf_{\Xi_{k,m}} \Lambda_{k,m} = \Lambda_{k,m}(\xi)$.

Applying a transversality theorem, the shadow system (1.2.5) is non-degenerate for a generic positive function $(V_1, V_2) \in C^{2,\alpha}(\Sigma, \mathbb{R}_+) \times C^{2,\alpha}(\Sigma, \mathbb{R}_+)$ (refer to Theorem 4.1.8). We select

$(V_1, V_2) \in C^{2,\alpha}(\Sigma, \mathbb{R}_+) \times C^{2,\alpha}(\Sigma, \mathbb{R}_+)$ such that (1.2.5) is nondegenerate for any solution (w, ξ). Using the method of continuity, we deduce the existence of a solution (w, ξ) of the shadow system (1.2.5), leading to our second corollary:

Corollary 1.2.4. *Suppose $m \geq k \geq 0$ are integers, and $\rho_2 \in (0, 2\pi)$. For a generic positive function (V_1, V_2) in $C^{2,\alpha}(\Sigma, \mathbb{R}_+) \times C^{2,\alpha}(\Sigma, \mathbb{R}_+)$, the conclusions of Theorem 1.2.2 hold.*

In chapter 5, we discuss asymmetric blow-up solutions for the $SU(3)$ Toda system. Concerning asymmetric blow-ups, Ao and Wang in [AW14] introduced a family of blow-up solutions for Toda systems with Dirichlet boundary conditions on a unit ball centered at the origin, which exhibits a single blow-up point at the center. D'Aprile, Pistoia and Ruiz extended this result to planar domains that possess a so-called "k-symmetric" property for $\mathrm{k} \geq 2$, as $\rho_1 \to 8\pi$, while the fixed parameter $\rho_2 \in (4\pi, 8\pi)$ in [DPR16]. Furthermore, Musso, Pistoia, and Wei generalized the result of [DPR16] for the $SU(N+1)$ Toda system, applying the same approach, singular perturbation, for $N \geq 2$ in [MPW16].

Let Σ be a Riemann manifold (with or without boundary) with $\dim(\Sigma) = n$. Let H be a closed subgroup of $\mathrm{Iso}(\Sigma)$, where $\mathrm{Iso}(\Sigma)$ is the isometry group on Σ. We denote

$$\Sigma_0 = \{x \in \Sigma : \gamma(x) = x \text{ for any } \gamma \in H\},$$

which is the fixed point set under the action H.

Since isometries preserve the angles and send geodesic to geodesic, as in [IN06, Lemma 2.2], we have the following lemma:

Lemma 1.2.5 (see [IN06])**.** *Let $x \in \Sigma_0$. Then the action of H in a neighborhood U around x is smoothly conjugated to the action of a closed subgroup of the orthogonal group $O(n)$ in the open unit ball $\mathbb{B}_1^n \subset \mathbb{R}^n$ if $x \notin \partial\Sigma$, or $O(n-1) \times \{\mathrm{id}\}$ in the cylinder $\mathbb{B}_1^{n-1} \times [0, 1) \subset \mathbb{R}^{n-1} \times \mathbb{R}_+$ if $x \in \partial\Sigma$.*

In our context, we set (Σ, g) as a compact Riemann surface with smooth boundary. We consider H to be a closed subgroup of $O(3)$, the orthogonal group of degree 3. Let

$$\mathfrak{R}_{\mathrm{k}} = \begin{bmatrix} \cos\frac{\pi}{\mathrm{k}} & \sin\frac{\pi}{\mathrm{k}} & 0 \\ -\sin\frac{\pi}{\mathrm{k}} & \cos\frac{\pi}{\mathrm{k}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in O(3), \tag{1.2.8}$$

for $\mathrm{k} \in \mathbb{N}_+$.

Definition 1.2.6. *Let Σ be a Riemann surface. Given an integer $\mathrm{k} \geq 1$, we say Σ is* k*-symmetric* *if* Σ *can be embedded in* $\mathbb{R}^3$ *and is invariant under* $\mathfrak{R}_{\mathrm{k}}$. *We say* x *is a* k*-symmetric center of* Σ *if* $x \in \Sigma_0$.

Definition 1.2.7. *Let* Σ *be a* k*-symmetric surface. A function* $f : \Sigma \to \mathbb{R}$ *is* $\mathfrak{R}_k$*-invariant if and only if for any* $x \in \Sigma$, $f(x) = f(\mathfrak{R}_{\mathrm{k}} x)$.

By technique reasons, we assume that Σ is a "k-symmetric" surface for $\mathrm{k} \geq 2$. Due to the smoothness of the boundary, we know that $\Sigma_0 \cap \partial\Sigma = \emptyset$.

Theorem 1.2.8. *Given* $\mathrm{k} \geq 2$, *we assume that* Σ *is a* k*-symmetric surface with non-empty fixed point set under the action of the isometric group* $\langle \mathfrak{R}_{\mathrm{k}} \rangle$, *i.e.*

$$\Sigma_0 := \left\{ x \in \Sigma : \mathfrak{R}_{\mathrm{k}}^i(x) = x \text{ for any } i \in \mathbb{N} \right\} \neq \emptyset,$$

and the potential functions V_1, V_2 *are* $\mathfrak{R}_{\mathrm{k}}$*-invariant. For any distinct points* $\xi_1, \cdots, \xi_m \in \Sigma_0$, *there exists a family of solutions* $\mathbf{u}_\varepsilon = (u_{1,\varepsilon}, u_{2,\varepsilon})$ *of* (1.2.1) *which blows up precisely at* $\xi_1, \cdots, \xi_m$ *with* $\rho^\varepsilon := (\rho_1^\varepsilon, \rho_2^\varepsilon) \to (4\pi m, 8\pi m)$ *as* $\varepsilon \to 0$ *and* $(\sigma_1(\xi_j), \sigma_2(\xi_j)) = (4\pi, 8\pi)$ *for* $j = 1, \cdots, m$. *Furthermore, as* $\varepsilon \to 0$

$$\rho_1^\varepsilon \frac{V_1 e^{u_{1,\varepsilon}}}{\int_\Sigma V_1 e^{u_{1,\varepsilon}} dv_g} \to \sum_{j=1}^m 4\pi \delta_{\xi_j} \text{ and } \rho_2^\varepsilon \frac{V_2 e^{u_{2,\varepsilon}}}{\int_\Sigma V_2 e^{u_{2,\varepsilon}} dv_g} \to \sum_{j=1}^m 8\pi \delta_{\xi_j},$$

which are convergent as measures on Σ.

Remark 1.2.9.

- *As in [MPW16], we can extend our result for* $SU(N+1)$ *Toda systems* $(N \geq 1)$ *with minor modification of the approach we applied.*

- *Two-dimensional spheres and half spheres are typical examples of the* k*-symmetric surfaces for any* $\mathrm{k} > 0$ *and their symmetric centers are precisely at the pole points (see Fig. 1.1).*

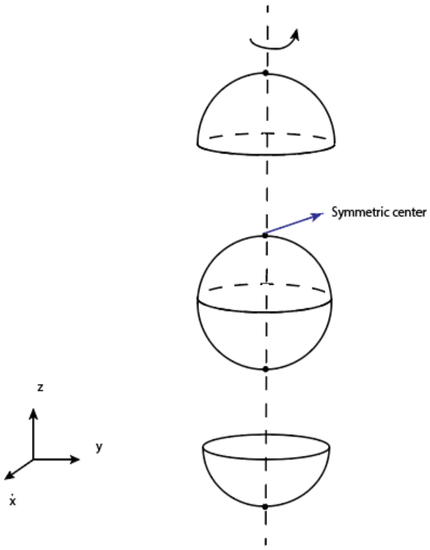

Figure 1.1: Disjoint union of spheres and half spheres

Referring to Lemma 1.2.5, it is observed that for any fixed point $x \in \partial\Sigma$ *under the action of a closed subgroup of the isometric group, the local behavior of the action is equivalent to a closed subgroup of* $O(1)\times\{\mathrm{id}\}$ *acting on the cylinder* $(-1,1)\times[0,1)$. *While* $O(1)=\{1,-1\}$, *a function invariant under reflection fails to demonstrate the invertibility of the linearized operator. This limitation constitutes the primary obstacle preventing the construction of blow-up points on the boundary through this approach.*

2 Preliminaries

2.1 Notations

Throughout this paper, we use the terms “sequence” and “subsequence” interchangeably, as the distinction is not crucial for the context of our analysis. The constant denoted by C in our deduction may assume different values across various equations or even within different lines of equations.

The table below enumerates the mathematical symbols and notations employed throughout this dissertation:

$\mathring{\Sigma}$	The interior of the Riemann surface Σ
$\partial\Sigma$	The boundary of the Riemann surface Σ
Δ_g	The Laplace-Beltrami operator with respect to metric g
dv_g	The area element in (Σ, g)
ds_g	The line element in $\partial\Sigma$ with respect to metric g
$\vert\Sigma\vert_g$	The area of Σ, i.e. $\int_\Sigma dv_g$
ν_g	The unit outward normal of $\partial\Sigma$
$\mathbb{N}$	The set of non-negative integers
$\mathbb{N}_+$	The set of positive integers
$\mathbb{R}_+$	$(0, +\infty)$
$\mathbb{R}^2_+$	$\{y = (y_1, y_2) \in \mathbb{R}^2 : y_2 \geq 0\}$
$\mathbb{B}^n_r(\hat{y})$	$\{y \in \mathbb{R}^n : \vert y - \hat{y}\vert < r\}, \mathbb{B}^2_r(\hat{y})$, particularly $\mathbb{B}_r(\hat{y}) := \mathbb{B}^2_r(\hat{y})$
$\mathbb{B}^n_r$	$\{y \in \mathbb{R}^n : \vert y\vert < r\}, \mathbb{B}^2_r$, particularly $\mathbb{B}_r = \mathbb{B}^2_r$
$\mathbb{B}^+_r$	$\{y = (y_1, y_2) \in \mathbb{R}^2 : \vert y\vert < r, y_2 \geq 0\}$
$\mathbb{D}$	$\{y = (y_1, y_2) \in \mathbb{R}^2 : \vert y\vert \leq 1\}$
$d_g(x, \hat{x})$	The metric distance defined by the geodesics for the Riemann metric g
$d_g(A, B)$	$\inf_{x\in A, \hat{x}\in B} d_g(x, \hat{x})$, for A, B two subsets of Σ

$\mathbb{F}_{k,m}(X,Y)$	$\{\xi := (\xi_1, \cdots, \xi_m) \in X^k \times Y^{m-k} : \xi_i = \xi_j$ for some $i \neq j\}$, for integers $m \geq k \geq 0$
$L^p(\Sigma)$	For $p \in [1, +\infty)$, $L^p(\Sigma) := \{u : \int_\Sigma \|u\|^p dv_g < +\infty\}$ with $\\|u\\|_p := (\int_\Sigma \|u\|^p dv_g)^{1/p}$; for $p = \infty$, $L^\infty(\Sigma) := \{u : \sup_\Sigma \|u\| < +\infty\}$ with $\\|u\\|_\infty := \sup_\Sigma \|u\|$
$W^{s,p}(\Sigma)$	The Sobolev spaces, $W^{s,p}(\Sigma) = \{u \in L^p(\Sigma) : D^\alpha u \in L^p(\Sigma)$ for all α with $\|\alpha\| \leq s\}$, for $s \in \mathbb{N}, p \geq 1$ and D^α denotes the weak derivative for the multi-index α
$W^{s,p}_\partial(\Sigma)$	$\{h\|_{\partial\Sigma} : h \in W^{s,p}(\Sigma)\}$ with $\\|h\\|_{W^{s,p}_\partial(\Sigma)} := \inf\{\\|\psi\\|_{W^{s,p}(\Sigma)} : \psi \in W^{s,p}(\Sigma)$ with $\psi\|_{\partial\Sigma} = h\}$
$W^{s,p}_0(\Sigma)$	$\{u \in W^{s,p}(\Sigma) : \int_\Sigma u dv_g = 0\}$
$H^1(\Sigma)$	$W^{1,2}(\Sigma)$
$\overline{\mathrm{H}}^1(\Sigma)$	$\{u \in H^1(\Sigma) : \int_\Sigma u dv_g = 0\}$ with the inner product $\langle u, v\rangle := \int_\Sigma \langle \nabla u, \nabla v\rangle_g + \beta uv dv_g$ (In Part II, $\beta = 0$)
$C^{s,\alpha}(\Sigma)$	$\Big\{u : \\|u\\|_{C^s} < \infty, [D^s u]_\alpha = \sup_{x,y\in\Sigma, x\neq y} \frac{\|D^s u(x) - D^s u(y)\|}{\|x-y\|^\alpha} < \infty\Big\}$, for a non-negative integer s and a Hölder coefficient $0 < \alpha \leq 1$
$C^\alpha(\Sigma)$	$C^{0,\alpha}(\Sigma)$
$C^{s,\alpha}_0(\Sigma)$	$\{u \in C^{2,\alpha}(\Sigma) : \int_\Sigma u dv_g = 0\}$
$C^\alpha_0(\Sigma)$	$C^{0,\alpha}_0(\Sigma)$
$C^\infty_c(X)$	$C^\infty_c(X) := \{u : u$ is a smooth function on X with a compact support $\}$
$\overline{f}$	$\frac{1}{\|\Sigma\|_g} \int_\Sigma f dv_g$ for any function $f \in L^1(\Sigma)$
$\mathrm{i}(\xi)$	$\mathrm{i}(\xi) = 2$ if $\xi \in \mathring{\Sigma}$ and $\mathrm{i}(\xi) = 1$ if $\xi \in \partial\Sigma$
$\varrho(\xi)$	$\varrho(\xi) = 8\pi$ if $\xi \in \mathring{\Sigma}$ and $\varrho(\xi) = 4\pi$ if $\xi \in \partial\Sigma$
$A \subset\subset B$	A is a compact subset of B
$\mathrm{supp} f$	The support of the function f, i.e. $\overline{\{x : f(x) \neq 0\}}$
δ_ξ	The Dirac measure on Σ concentrated at point ξ
δ_{ij}	The Kronecker symbol, i.e. $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$

2.2 Isothermal coordinates

In our study, we focus on the context of compact Riemann surfaces. While this constraint may appear to be rather strong, it includes a broad range of 2-dimensional manifolds, because any orientable Hausdorff 2-dimensional manifold can be endowed with the structure of a Riemann surface (see [Ber57, Section 2]). Every Riemann surface (Σ, g) is locally conformally flat, and the local coordinates in which g is conformal to the Euclidean metric are called isothermal coordinates (refer to [Che55, Ber57], for instance).

We will introduce a family of isothermal coordinates for the Riemann surface (Σ, g) following the constructions in [EF14, YZ21].

For any $\xi \in \mathring{\Sigma}$, there exists an isothermal coordinate system $(U(\xi), y_\xi)$ such that y_ξ maps an open neighborhood $U(\xi)$ around ξ onto $\mathbb{B}_{2r_\xi}$ denoted by B^ξ in which $g = \sum_{i=1}^{2} e^{\hat{\varphi}_\xi(y_\xi(x))} \mathrm{d}x^i \otimes \mathrm{d}x^i$, where $\frac{1}{2}\hat{\varphi}_\xi$ is the conformal factor. Without loss of generality, we can assume that $y_\xi(\xi) = (0,0)$ and $\overline{U(\xi)} \subset \mathring{\Sigma}$.

When it comes to ξ on the boundary $\partial\Sigma$, we apply a lemma in [YZ21] to map a neighborhood around ξ onto a half ball in $\mathbb{R}^2$ conformally.

Lemma 2.2.1 (Lemma 4 of [YZ21]). *Let (Σ, g) be a compact Riemann surface with smooth boundary $\partial\Sigma$. For any fixed point $\xi \in \partial\Sigma$, there exists a constant $\delta > 0$ and an isothermal coordinate system $(U(\xi), y_\xi; \{y_1, y_2\})$ around ξ such that $y_\xi(\xi) = (0,0)$, $U(\xi) \subset \Sigma$ is a neighborhood of ξ, $y_\xi(U(\xi)) = \mathbb{B}_\delta^+$ and $y_\xi(U(\xi) \cap \partial\Sigma) = \mathbb{B}_\delta^+ \cap \partial\mathbb{R}_+^2$. In this coordinate system, there exists a function $\hat{\varphi}_\xi \in C^\infty(\mathbb{B}_\delta^+)$ such that for any $x \in U(\xi)$, the metric can be written as*

$$g = \sum_{i=1}^{2} e^{\hat{\varphi}_\xi(y_\xi(x))} \mathrm{d}x^i \otimes \mathrm{d}x^i.$$

Let ν_g be a unit outward vector field on the boundary $\partial\Sigma$. For any $x \in U(\xi) \cap \partial\Sigma$, we denote $y = y_\xi(x)$, then

$$(y_\xi)_*(\nu_g)|_x = -e^{-\frac{1}{2}\hat{\varphi}_\xi(y)} \frac{\partial}{\partial y_2}\bigg|_{y = y_\xi(x)}.$$

The proof is originally from [YZ21]. For the reader's convenience, I will briefly show it here.

Proof. Since Σ is compact, there exists another Riemann surface (Σ^*, g^*) with smooth boundary $\partial\Sigma^*$ such that

$$\overline{\Sigma} \subset \Sigma^*, \quad d_{g^*}(\Sigma, \partial\Sigma^*) > 0, \quad g^* = g \text{ on } \Sigma,$$

and ξ is in the interior of Σ^*. By [Che55], there exists a neighborhood U around ξ on Σ such that

$$\tilde{\psi}_1 : U \to \mathbb{B}_r \subset \mathbb{R}^2$$

with $\tilde{\psi}_1(\xi) = (0,0)$ and

$$g = \sum_{i=1}^{2} e^{2f_1(\tilde{\psi}_1(x))} \mathrm{d}x^i \otimes \mathrm{d}x^i,$$

where f_1 is a smooth function with $f_1(0,0) = 0$. Denote $U_1 = U \cap \Sigma$ and $\psi_1 = \tilde{\psi}_1|_{U_1}$. Then,

$$\psi_1 : \overline{U}_1 \to \overline{\Omega}_1$$

is a diffeomorphism with $\psi_1(\xi) = (0,0)$ and $\psi_1(\overline{U}_1 \cap \partial\Sigma) = \Gamma_1 \subset \partial\Omega_1$, where $\Omega_1 \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega_1$ apart from two corner points. Moreover, the metric in $\overline{U}_1$ can be written as

$$g = \sum_{i=1}^{2} e^{2f_1(\psi_1(x))} \mathrm{d}x^i \otimes \mathrm{d}x^i.$$

We denote $\nu_1 = (\psi_1)_*(\nu_g)$. It follows that $\nu_1 = e^{-f_1(x)}\nu_0$, where ν_0 is the unit outward normal at Γ_1 for Ω_1.

To get a smooth domain, we take $\Omega_2 \subset \Omega_1$ to be a smooth domain such that $(0,0)$ is an inner point with respect to its boundary $\partial\Omega_2$ and a smooth curve $\Gamma_2 := \psi_1(\Omega_2 \cap \partial\Sigma) \subset \Gamma_1$.

Applying the Riemann mapping theorem (see [Ahl78, Section 6.1]), there exists a conformal map $\psi_2 : \Omega_2 \to \mathbb{D}$ with $w = \psi_2(z)$ and $\psi_2(0,0) = (0,-1)$, where $z = z_1 + iz_2$. Theorem 3.5 and Theorem 3.6 of [Pom92] imply that ψ_2 can be extended to be a smooth map from $\overline{\Omega}_2$ to $\overline{\mathbb{D}}$. Hence, $\overline{\Omega}_2$ is conformal to a closed disk $\overline{\mathbb{D}}$ in $\mathbb{R}^2$ by ψ_2 and $\psi_2'(z) \neq 0$ for any $z = z_1 + iz_2 \in \overline{\Omega}_2$.

For fixed $q \notin \psi_2(\Gamma_2)$, there exists a Möbius transformation $\zeta = h(w) : \overline{\mathbb{D}} \setminus \{q\} \to \mathbb{R}^2_+$ with $h \circ \psi_2 \circ \psi_1(\xi) = (0,0)$.

We define that $\varphi(z) = f_1(z) - \log|h'(\omega)\psi_2'(z)|$. Let $\tau : \mathbb{R}^2_+ \to \mathbb{R}^2_+, \zeta \mapsto e^{\varphi(0,0)}\zeta$ and $y := \tau(\zeta)$. It follows that $dy = e^{\varphi(0,0)}h'(\omega)\psi_2'(z)dz$. Setting $y_\xi = \tau \circ h \circ \psi_2 \circ \psi_1$, we choose $\delta > 0$ sufficiently small such that $y_\xi^{-1}(\partial\mathbb{B}^+_\delta \cap \partial\mathbb{R}^2_+) \subset \psi_1^{-1}(\Gamma_1)$.

It is easy to check

$$\begin{aligned} g &= e^{2f_1(\psi_1(x))}(dx_1^2 + dx_2^2) = \frac{\exp(2f_1(z))}{|h'(w)\psi_2'(z)|^2}\exp(-2\varphi(0,0))|dz|^2 \\ &= e^{2f_1(z) - 2\varphi(0,0) - 2\log|h'(w)\psi_2'(z)|}(dz_1^2 + dz_2^2) = e^{2\varphi(z) - 2\varphi(0,0)}(dz_1^2 + dz_2^2) \\ &= e^{\hat{\varphi}_\xi(y)}(dy_1^2 + dy_2^2), \end{aligned}$$

where $\hat{\varphi}_\xi(y) = 2\varphi(z) - 2\varphi(0,0) = 2f_1(z) - 2\log|h'(w)\psi_2'(z)| - 2\varphi(0,0)$. By direct calculation, for any $x \in y_\xi^{-1}(\partial\mathbb{B}^+_\delta \cap \partial\mathbb{R}^2_+)$, $(y_\xi)_*(\nu_g)|_x = -e^{-\frac{1}{2}\hat{\varphi}_\xi(y)}\frac{\partial}{\partial y_2}$, where $y = y_\xi(x)$. [Pom92, Theorem

3.6] yields that $\hat{\varphi}_\xi$ is smooth on $\overline{\mathbb{B}}_\delta^+$. $\square$

For $\xi \in \partial\Sigma$ there exists an isothermal coordinate $(U(\xi), y_\xi)$ around ξ such that the image of y_ξ is a half disk $B^\xi := \mathbb{B}_{2r_\xi}^+$ and $y_\xi\left(U(\xi) \cap \partial\Sigma\right) = B^\xi \cap \partial\mathbb{R}_+^2$ with $g = \sum_{i=1}^2 e^{\hat{\varphi}_\xi(y_\xi(x))} \mathrm{d}x^i \otimes \mathrm{d}x^i$, where $\hat{\varphi}_\xi$ is smooth. Without loss of generalization, we can assume that $y_\xi(\xi) = (0,0)$.

Here $\hat{\varphi}_\xi : B^\xi \to \mathbb{R}$ is related to K_g, the Gaussian curvature of Σ, by the equation

$$-\Delta\hat{\varphi}_\xi(y) = 2K_g(y_\xi^{-1}(y))e^{\hat{\varphi}_\xi(y)} \quad \text{for all } y \in B^\xi. \tag{2.2.1}$$

For $\xi \in \Sigma$ and $0 < r \leq 2r_\xi$, we set

$$B_r^\xi := B^\xi \cap \{y \in \mathbb{R}^2 : |y| < r\} \quad \text{and} \quad U_r(\xi) := y_\xi^{-1}(B_r^\xi).$$

Both y_ξ and $\hat{\varphi}_\xi$ are assumed to depend smoothly on ξ as in [EF14]. Additionally, $\hat{\varphi}_\xi$ satisfies $\hat{\varphi}_\xi(0) = 0$ and $\nabla\hat{\varphi}_\xi(0) = 0$. Specifically, as in Lemma 2.2.1, the Neumann boundary conditions are preserved by the isothermal coordinate, i.e. for any $\xi \in \partial\Sigma$ and $x \in y_\xi^{-1}\left(B^\xi \cap \partial\mathbb{R}_+^2\right)$, we have

$$\left(y_\xi\right)_* \left(\nu_g(x)\right) = -e^{-\frac{\hat{\varphi}_\xi(y)}{2}} \frac{\partial}{\partial y_2}\bigg|_{y=y_\xi(x)}. \tag{2.2.2}$$

2.3 Regularity for Neumann boundary problems on surfaces

Lemma 2.3.1. *Let* (Σ) *be a compact Riemann surface with smooth boundary* $\partial\Sigma$. *For any* $\beta \geq 0$, *if* $f \in L^2(\Sigma, g)$ *satisfies*

$$\int_\Sigma f = 0,$$

then there exists a unique weak solution of

$$\begin{cases} -\Delta_g u + \beta u = f & \text{in} \quad \Sigma \\ \partial_{\nu_g} u = 0 & \text{on} \quad \partial\Sigma \\ \int_\Sigma u dv_g = 0 \end{cases}, \tag{2.3.1}$$

i.e. there exists a unique $u \in \overline{H}^1(\Sigma)$ *satisfying*

$$\int_\Sigma \langle\nabla u, \nabla\varphi\rangle_g dv_g + \beta\int_\Sigma u\varphi dv_g = \int_\Sigma f\varphi dv_g + \int_{\partial\Sigma} h\varphi ds_g, \quad \forall\varphi \in H^1(\Sigma).$$

Moreover, for any $p > 1$ if $f \in L^p(\Sigma)$, there exists a $u \in W_0^{2,p}(\Sigma) := W^{2,p}(\Sigma) \cap \{u : \int_\Sigma u dv_g = 0\}$ solving (2.3.1) *with the following $W^{2,p}$-estimate:*

$$\|u\|_{W^{2,p}(\Sigma)} \leq C\|f\|_{L^p(\Sigma)}.$$

For the Poisson equation with homogeneous Neumann boundary condition, the L^p-estimate was proven in [YZ21, Lemma 5]. And we can deduce (2.3.1) by the same approach.

Proof. For the uniqueness, we assume that u_1, u_2 are two weak solutions of (2.3.1) in $\overline{\mathrm{H}}^1(\Sigma)$. It follows that

$$\int_\Sigma \langle \nabla(u_1 - u_2), \nabla\varphi\rangle_g dv_g + \beta \int_\Sigma (u_1 - u_2)\varphi dv_g = 0,$$

for any $\varphi \in H^1(\Sigma)$. Then, $u_1 = u_2$ up to the addition of a constant. Observing that $\int_\Sigma u_1 dv_g = \int_\Sigma u_2 dv_g = 0$, we deduce that $u_1 \equiv u_2$.

We will prove the existence of solutions using variational methods. Consider the energy functional

$$J(u) = \frac{1}{2}\int_\Sigma (|\nabla u|_g^2 + \beta u^2) dv_g - \int_\Sigma f u dv_g.$$

Applying the Hölder inequality and the Poincaré inequality,

$$\left|\int_\Sigma f u dv_g\right| \leq \|f\|_{L^2(\Sigma)}\|u\|_{L^2(\Sigma)} \leq C\|f\|_{L^2(\Sigma)}\|\nabla u\|_{L^2(\Sigma)},$$

which yields that J has a lower bound in $\overline{\mathrm{H}}^1(\Sigma)$. Let u_n be a sequence in $\overline{\mathrm{H}}^1(\Sigma)$ such that J attains the minimum value, i.e.

$$\lim_{n\to+\infty} J(u_n) = \inf_{u\in\overline{\mathrm{H}}^1(\Sigma)} J(u).$$

For any $n \in \mathbb{N}_+$, $J(u_n) \geq \frac{1}{2}\|u_n\|^2 - C\|f\|_{L^2(\Sigma)}\|u_n\|$. Given that $\inf_{u\in\overline{\mathrm{H}}^1(\Sigma)} J(u) \leq J(0) = 0$, u_n is uniformly bounded in $\overline{\mathrm{H}}^1(\Sigma)$. Up to a subsequence, we assume that u_n converges to some $u_0 \in \overline{\mathrm{H}}^1(\Sigma)$ weakly. By the Rellich-Kondrachov theorem, $u_n \to u_0$ strongly in $L^q(\Sigma)$ for any $q > 1$ and almost everywhere. Fatou's lemma implies that

$$J(u_0) \leq \liminf_{n\to+\infty} J(u_n) = \inf_{u\in\overline{\mathrm{H}}^1(\Sigma)} J(u).$$

Thus, u_0 is a minimizer of $J(u)$ on $\overline{\mathrm{H}}^1(\Sigma)$.

Next, we consider the $W^{2,p}$-estimates of the solutions. Employing the isothermal coordinates introduced in Section 2.2 it is sufficient to prove the L^p-regularity locally in an open disk or half-disk in $\mathbb{R}^2$. Specifically, in the case of a half-disk, we can extend the problem by the reflection of the x-axis to a full open disk, considering that $\partial_{\nu_g} u = 0$ on the boundary. This extension allows for the application of the standard local L^p-theory, thereby we can establish the L^p-regularity for the Neumann boundary problem (2.3.1) on a compact Riemann surface Σ. $\square$

Let $W^{s,p}_{\partial}(\Sigma) := \{h|_{\partial\Sigma} : h \in W^{s,p}(\Sigma)\}$ equipped with the norm

$$\|h\|_{W^{s,p}_{\partial}(\Sigma)} := \inf\left\{\|\psi\|_{W^{s,p}(\Sigma)} : \psi \in W^{s,p}(\Sigma) \text{ with } \psi|_{\partial\Sigma} = h\right\},$$

for any $s \in \mathbb{N}$ and $p \in (1, +\infty)$. For the inhomogeneous boudnary condition, we have the following L^p-theory:

Lemma 2.3.2 (Theorem 3.2 of [Weh04])**.** *Suppose that $f \in L^p(\Sigma)$ and $h \in W^{1,p}_{\partial}(\Sigma)$. Let u be a weak solution with $\int_\Sigma u dv_g = 0$ of*

$$\begin{cases} -\Delta_g u + \beta u = f & \textit{in} \quad \Sigma \\ \partial_{\nu_g} u = 0 & \textit{on} \quad \partial\Sigma \\ \int_\Sigma u dv_g = 0 \end{cases}.$$

Then, $u \in W^{2,p}(\Sigma)$ with the estimate

$$\|u\|_{W^{2,p}(\Sigma)} \leq C\left(\|f\|_{L^p(\Sigma)} + \|h\|_{W^{1,p}_{\partial}(\Sigma)}\right).$$

For the case $\beta = 0$, we refer to [ADN59] and [Weh04]. By the same approach, Lemma 2.3.2 can be proven for $\beta > 0$; hence, we omit the details.

Next, we consider the Schauder estimates for the Neumann boundary condition on compact Riemann surfaces.

Lemma 2.3.3. *For any given $\alpha \in (0,1), \beta \geq 0$, let (Σ, g) be a compact Riemann surface with boundary in $C^{2,\alpha}$-class and let $f \in C^{\alpha}(\Sigma), h \in C^{1,\alpha}(\Sigma)$ such that:*

$$\int_\Sigma f dv_g = \int_{\partial\Sigma} h ds_g. \tag{2.3.2}$$

Then, there exists a unique solution to the problem

$$\begin{cases} -\Delta_g u + \beta u = f & \text{in } \Sigma \\ \partial_{\nu_g} u = h & \text{on } \partial\Sigma \end{cases} \tag{2.3.3}$$

in the space $C_0^{2,\alpha}(\Sigma) := C^{2,\alpha}(\Sigma) \cap \{u : \int_\Sigma u dv_g = 0\}$. *Moreover, it has the following Schauder estimate:*

$$\|u\|_{C^{2,\alpha}(\Sigma)} \leq C \left(\|f\|_{C^\alpha(\Sigma)} + \|h\|_{C^{1,\alpha}(\Sigma)} \right),$$

where C *is a constant.*

We refer to the Schauder interior estimates for domains as in [GT01, Corollary 6.3].

Theorem 2.3.4 (Corollary 6.3 of [GT01]). *Let* Ω *be an open subset of* $\mathbb{R}^n$ *and let* $u \in C^{2,\alpha}(\Omega)$ *be a bounded solution in* Ω *of the equation* $Lu = a^{ij} D_{ij} u + b^i D_i u + cu = f$, *where* $f \in C^\alpha(\Omega)$ *and there are positive constants* λ, Λ *such that the coefficients satisfy* $a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$, *for any* $x \in \Omega, \xi \in \mathbb{R}^n$ *and* $\|a^{ij}\|_{C^0(\Omega)} + \|b^i\|_{C^0(\Omega)} + \|c\|_{C^0(\Omega)} \leq \Lambda$. *Then we have the interior estimate: for any* $\Omega' \subset\subset \Omega$,

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C(\|u\|_{C^0(\Omega)} + \|f\|_{C^\alpha(\Omega)}) \tag{2.3.4}$$

where $C = C(n, \Omega', \alpha, \lambda, \Lambda)$ *is a constant.*

The Schauder estimate with oblique derivative boundary conditions is as follows:

Theorem 2.3.5 (Lemma 6.29 of [GT01]). *Let* Ω *be a bounded open set in* $\mathbb{R}^n_+$ *with a boundary portion* T *on* $x_n = 0$. *Suppose that* $u \in C^{2,\alpha}(\Omega \cup T)$ *is a solution in* Ω *of* $Lu = f$ *(as in Theorem 2.3.4) satisfying the boundary condition*

$$N(x')u = \gamma(x')u + \sum_{i=1}^{n} \beta_i(x') D_i u = h(x'), \quad x' \in T, \tag{2.3.5}$$

where $|\beta_n| \geq \kappa > 0$ *for some constant* κ. *Assume that* $f \in C^\alpha(\Omega)$, $h \in C^{1,\alpha}(T)$, $a^{ij}, b^i, c \in C^\alpha(\Omega)$ *and* $\gamma, \beta_i \in C^{1,\alpha}(T)$ *with*

$$\|a^{ij}, b^i, c\|_{C^{0,\alpha}(\Omega)}, \|\gamma, \beta_i\|_{C^{1,\alpha}(T)} \leq \Lambda, \quad i, j = 1, \cdots, n.$$

Then for any $\Omega' \subset\subset \Omega \cup T$,

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C(\|u\|_{C^0(\Omega)} + \|h\|_{C^{1,\alpha}(T)} + \|f\|_{C^\alpha(\Omega)}), \tag{2.3.6}$$

where $C = C(n, \Omega', \alpha, \lambda, \kappa, \Lambda, \text{diam } \Omega)$ *is a constant.*

Proof of Lemma 2.3.3. By combining the isothermal coordinates in Section 2.2 with the results from Theorem 2.3.4 and Theorem 2.3.5, we can infer the lemma.

We consider $u \in C^{2,\alpha}(\Sigma)$ solving (2.3.3). For each point $\zeta \in \Sigma$, there exists an isothermal chart $(U(\zeta), y_\zeta)$ defined in Section 2.2.

Given the compactness of Σ, it can be expressed as a finite union of the form:

$$\Sigma = \bigcup_{i=1}^{l_1+l_2} U_{r_{\zeta_i}}(\zeta_i),$$

where $\zeta_i \in \mathring{\Sigma}$, for $i = 1, \cdots, l_1$ and $\zeta_i \in \partial\Sigma$ for $i = l_1 + 1, \cdots, l_1 + l_2$ and $U_{r_{\zeta_i}} \subset U(\zeta_i)$ is defined in Section 2.2.

Applying Theorem 2.3.4, for each $i = 1, \cdots, l_1$,

$$\|u\|_{C^{2,\alpha}(U_{r_{\zeta_i}}(\zeta_i))} \leq C(\|u\|_{C^0(U(\zeta_i))} + \|f\|_{C^\alpha(U(\zeta_i))}).$$

Then, utilizing the method in [GT01, Theorem 6.31], we estimate $\|u\|_{C^0(U(\zeta_i))}$ in term of $\|f\|_{C^0(\Sigma)}$. Consequently,

$$\|u\|_{C^{2,\alpha}(U_{r_{\zeta_i}}(\zeta_i))} \leq C(\|f\|_{C^\alpha(\Sigma)}).$$

Similarly, Theorem 2.3.5 implies that for $i = l_1 + 1, \cdots, l_1 + l_2$,

$$\|u\|_{C^{2,\alpha}(U_{r_{\zeta_i}}(\zeta_i))} \leq C(\|u\|_{C^0(U(\zeta_i))} + \|h\|_{C^{1,\alpha}(U(\zeta_i)\cap\partial\Sigma)} + \|f\|_{C^\alpha(\Sigma)}).$$

And [GT01, Theorem 6.31] yields that $\|u\|_{C^0(U(\zeta_i))} \leq C\|f\|_{C^0(\Sigma)}$. It follows that

$$\|u\|_{C^{2,\alpha}(U_{r_{\zeta_i}}(\zeta_i))} \leq C(\|f\|_{C^\alpha(\Sigma)}).$$

Summing up the local Schauder estimates for $i = 1, \cdots, l_1 + l_2$, we deduce that

$$\|u\|_{C^{2,\alpha}(\Sigma)} \leq C(\|f\|_{C^\alpha(\Sigma)} + \|h\|_{C^{1,\alpha}(\Sigma)}). \tag{2.3.7}$$

Applying Lemma 2.3.1, when $h \equiv 0$ we have a unique solution $u \in W^{2,2}(\Sigma)$ solving (2.3.3). Then the estimate (2.3.7) implies $u \in C^{2,\alpha}(\Sigma)$. Due to the Fredholm alternative mentioned in [GT01, P. 130], for any inhomogeneous $h \in C^{2,\alpha}(\Sigma)$ satisfying (2.3.2), there exists a unique solution $u \in C_0^{2,\alpha}(\Sigma)$ of (2.3.3). □

2.4 Green's functions

For any fixed $\xi \in \Sigma$, we define the Green's function for Neumann boundary conditions by the following equations:

$$\begin{cases} -\Delta_g G^g(x,\xi) + \beta G^g(x,\xi) = \delta_\xi - \frac{1}{|\Sigma|_g} & x \in \mathring{\Sigma} \\ \partial_{\nu_g} G^g(x,\xi) = 0 & x \in \partial\Sigma \\ \int_\Sigma G^g(x,\xi) dv_g(x) = 0 & \end{cases}, \tag{2.4.1}$$

where $\beta \geq 0, \delta_\xi$ is a Dirac measure concentrated at ξ.

Remark 2.4.1. *We give several important properties of Green's functions.*

a. there exists a unique Green's function $G^g(\cdot,\xi) \in L^1(\Sigma)$ solving (2.4.1) in the distributional sense;

b. for any distinct points $x, \xi \in \Sigma, G^g(x,\xi) = G^g(\xi,x)$;

c. the representation formula holds, for any $h \in C^2(\Sigma)$,

$$h(x) - \frac{1}{|\Sigma|_g}\int_\Sigma h dv_g = \int_\Sigma G^g(\cdot,x)(-\Delta_g + \beta)h dv_g + \int_{\partial\Sigma} G^g(\cdot,x)\partial_{\nu_g} h ds_g, \tag{2.4.2}$$

where dv_g is the area element of (Σ, g) and ds_g is the line element of $\partial\Sigma$;

d. for any distinct points $x, \xi \in \Sigma$, there exists a constant $C > 0$ such that

$$|G^g(x,\xi)| \leq C\left(1 + |\log d_g(x,\xi)|\right), \quad |\nabla_\xi G^g(x,\xi)| \leq C d_g^{-1}(x,\xi),$$

where $d_g(x,\xi)$ denotes the geodesic distance between x and ξ, and ∇_ξ denotes the gradient with respect the variable ξ.

For the case where $\beta = 0$, we refer to [YZ21, Lemma 6]. For $\beta > 0$, these properties can be deduced using the same approach.

Let χ be a radial cut-off function in $C^\infty(\mathbb{R}, [0,1])$ such that

$$\chi(s) = \begin{cases} 1, & \text{if } |s| \leq 1 \\ 0, & \text{if } |s| \geq 2 \end{cases}. \tag{2.4.3}$$

We define the cut-off function $\chi_\xi \in C^\infty(\Sigma, [0,1])$ by

$$\chi_\xi(x) = \begin{cases} \chi(\frac{|y_\xi(x)|}{\bar{\mathrm{r}}_\xi}) & \text{if } x \in U(\xi) \\ 0 & \text{if } x \in \Sigma \setminus U(\xi) \end{cases}, \tag{2.4.4}$$

where $\bar{\mathrm{r}}_\xi \in (0, \frac{1}{2} r_\xi]$ which will be selected later. We define the Robin's function as follows:

$$R^g(\zeta) := \lim_{x \to \zeta} \left(G^g(x,\zeta) + \frac{4}{\varrho(\zeta)} \log d_g(x,\zeta) \right).$$

Observe that for $\zeta \in U(\xi)$, $\lim_{x\to\zeta} \frac{d_g(x,\zeta)}{|y_\xi(x) - y_\xi(\zeta)|} = e^{\frac{1}{2}\varphi_\xi(\zeta)}$. It follows

$$R^g(\zeta) = \lim_{x\to\zeta} \left(G^g(x,\zeta) + \frac{4}{\varrho(\zeta)} \log |y_\xi(x) - y_\xi(\zeta)| \right) + \frac{2}{\varrho(\zeta)} \varphi_\xi(y_\xi(\zeta)). \tag{2.4.5}$$

In particular, using the assumption $\varphi_\xi(y_\xi(\xi)) = \varphi_\xi(0) = 0$, we obtain that

$$R^g(\xi) = \lim_{x\to\xi} \left(G^g(x,\xi) + \frac{4}{\varrho(\xi)} \log |y_\xi(x)| \right).$$

Let

$$\Gamma^g_\xi(x) = \Gamma^g(x,\xi) = \begin{cases} \frac{1}{2\pi} \chi_\xi(x) \log \frac{1}{|y_\xi(x)|} & \text{if } \xi \in \mathring{\Sigma} \\ \frac{1}{\pi} \chi_\xi(x) \log \frac{1}{|y_\xi(x)|} & \text{if } \xi \in \partial\Sigma \end{cases}.$$

Decomposing the Green's function $G^g(x,\xi) = \Gamma^g_\xi(x) + H^g(x,\xi)$, we have the function $H^g_\xi(x) := H^g(x,\xi)$ solving the following equations:

$$\begin{cases} -\Delta_g H^g_\xi + \beta H^g_\xi & = & -\beta \frac{4}{\varrho(\xi)} \chi_\xi \log \frac{1}{|y_\xi|} + \frac{4}{\varrho(\xi)} (\Delta_g \chi_\xi) \log \frac{1}{|y_\xi|} \\ & & + \frac{8}{\varrho(\xi)} \left\langle \nabla \chi_\xi, \nabla \log \frac{1}{|y_\xi|} \right\rangle_g - \frac{1}{|\Sigma|_g}, \qquad \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} H^g_\xi & = & -\frac{4}{\varrho(\xi)} (\partial_{\nu_g} \chi_\xi) \log \frac{1}{|y_\xi|} - \frac{4}{\varrho(\xi)} \chi_\xi \partial_{\nu_g} \log \frac{1}{|y_\xi|}, \quad \text{on } \partial\Sigma \\ \int_\Sigma H^g_\xi dv_g & = & -\frac{4}{\varrho(\xi)} \int_\Sigma \chi_\xi \log \frac{1}{|y_\xi|} dv_g \end{cases}. \tag{2.4.6}$$

Lemma 2.4.2. *For any fixed $\xi \in \Sigma$ and $\alpha \in (0,1)$, H^g_ξ is $C^{1,\alpha}$-smooth. Moreover, H^g_ξ is uniformly bounded in $C^{1,\alpha}(\Sigma)$ for any ξ in any compact subset of $\mathring{\Sigma}$ or on $\partial\Sigma$.*

Proof. We apply the isothermal coordinate $(y_\xi, U(\xi))$ introduced in Section 2.2. By the transformation law for Δ_g under a conformal map, $\Delta_{\tilde{g}} = e^{-\varphi}\Delta_g$ for any $\tilde{g} = e^{\varphi}g$. It follows that $\Delta_g\left(\log\frac{1}{|y_\xi(x)|}\right) = e^{-\varphi_\xi(y)}\,\Delta\log\frac{1}{|y|}\Big|_{y=y_\xi(x)} = -\frac{\varrho(\xi)}{4}\delta_\xi$, where δ_ξ is the Dirac mass concentrated at $\xi\in\Sigma$. For any $x\in U(\xi)\cap\partial\Sigma$,

$$\partial_{\nu_g}\log|y_\xi(x)| \overset{(2.2.2)}{=} -e^{-\frac{1}{2}\varphi_\xi(y)}\frac{\partial}{\partial y_2}\log|y|\Big|_{y=y_\xi(x)} = -e^{-\frac{1}{2}\varphi_\xi(y)}\frac{y_2}{|y|^2}\Big|_{y=y_\xi(x)} \equiv 0.$$

Clearly, $\partial_{\nu_g}\chi(|y_\xi(x)|) = 0$ for $x\in\partial\Sigma\cap U_{\bar{r}_\xi}(\xi)$. It follows that that $\partial_{\nu_g}H^g(\cdot,\xi)$ is smooth on $\partial\Sigma$. $\Delta_g H^g(\cdot,\xi)$ is bounded in $L^p(\Sigma)$, for any $p\geq 1$. Using the L^p-estimate in Lemma 2.3.2, we derive that

$$\|H^g_\xi - \overline{H^g_\xi}\|_{C^{2,\alpha}(\Sigma)} \quad\leq\quad C(\|\partial_{\nu_g}H^g_\xi\|_{W^{1,p}_\partial(\Sigma)} + \|-\Delta_g H^g_\xi\|_{L^p(\Sigma)})$$

for same constant $C>0$ which is independent with ξ. Given $p=\frac{2}{1-\alpha}$ for any $\alpha\in(0,1)$, the Sobolev embedding theorem yields that $H^g_\xi(x)$ in $C^{1,\alpha}(\Sigma)$. Considering that $\|-\Delta_g H^g(\cdot,\xi)\|_{L^p(\Sigma)}$, $\|\partial_{\nu_g}H^g_\xi\|_{C^{1,\alpha}(\partial\Sigma)}$ and $\left|\int_\Sigma H^g_\xi dv_g\right|$ are uniformly bounded for any ξ in any compact subset of $\mathring{\Sigma}$ or on $\partial\Sigma$, we have $H^g_\xi(x)$ is uniformly bounded for any ξ in any compact subset of $\mathring{\Sigma}$ or on $\partial\Sigma$. □

Remark 2.4.3.

- *$H^g_\xi(x)$ is the regular part of $G^g(x,\xi)$. It is easy to check that $H^g(\xi,\xi) = R^g(\xi)$ which is independent of the choice of the cut-off function χ and the local isothermal chart $(y_\xi, U(\xi))$.*
- *For $\beta = 0$, we can obtain higher regularity for the regular part of the Green's functions. Specifically, there exists a unique smooth solution H^g_ξ solving (2.4.6) in $C^\infty(\Sigma)$.*

2.5 C^1-stable critical point set

In this section, we will introduce the definition of "stable" critical point set as in [Li97, dPKM05, EGP05]. This concept serves as an extension of non-degenerate critical points.

Definition 2.5.1. *Let $F: D\to\mathbb{R}$ be a C^1-function and K be a compact subset of critical points of F, i.e*

$$K \subset\subset \{x\in D: \nabla F(x) = 0\}.$$

A critical point set K is C^1-stable if for any closed neighborhood U of K in D, there exists $\varepsilon>0$ such that if $G: D\to\mathbb{R}$ is a C^1-function with $\|F-G\|_{C^1(U)}<\varepsilon$, then G has at least one critical point in U.

Remark 2.5.2. *A compact subset of critical points of F is stable if one of the following conditions is satisfied:*

a). K is a strict local maximum set of F, i.e. for any $x, y \in K$, $F(x) = F(y)$ and for some open neighborhood U of K, $F(x) > F(y)$ for any $x \in K$ and $y \in U \backslash K$;

b). K is a strict local minimum set of F;

c). K is an isolated critical point set with a nontrivial local degree.

2.6 Kirchhoff-Routh type functions

We define the thick diagonal for any X and Y,

$$\mathbb{F}_{k,m}(X,Y) = \{\xi := (\xi_1, \cdots, \xi_m) \in X^k \times Y^{m-k} : \xi_i = \xi_j \text{ for some } i \neq j\}.$$

Denote $\mathbb{F}_{k,m}(\Sigma) = \mathbb{F}_{k,m}(\mathring{\Sigma}, \partial\Sigma)$ for simplicity. Next, we introduce the configuration set $\Xi_{k,m} := \mathring{\Sigma}^k \times (\partial\Sigma)^{m-k} \setminus \mathbb{F}_{k,m}(\Sigma)$. Given $\delta > 0$ and $V \in C^\infty(\Sigma)$ is a non-negative function, we define

$$\begin{aligned} \Xi^\delta_{k,m} := \{\xi \in \Xi_{k,m} : d_g(\xi_i, \partial\Sigma) \geq \delta \text{ for } i = 1, \cdots, k, d_g(\xi_i, \xi_j) \geq \delta \text{ for } i \neq j, \\ |V(\xi_i)| \geq \delta \text{ for } i = 1, \cdots, m\}, \end{aligned} \tag{2.6.1}$$

which is a compact subset of $\mathring{\Sigma}^k \times (\partial\Sigma)^{m-k}$. Particularly, when V is a positive function, we always assume that $\delta > 0$ is sufficiently small such that $\inf_\Sigma V \geq \delta$ and then $\Xi^\delta_{k,m}$ can be simplified as follows:

$$\Xi^\delta_{k,m} := \{\xi \in \Xi_{k,m} : d_g(\xi_i, \partial\Sigma) \geq \delta \text{ for } i = 1, \cdots, k, d_g(\xi_i, \xi_j) \geq \delta \text{ for } i \neq j\}.$$

For any integers $m \geq k \geq 0$, we can define a Kirchhoff-Routh type function on $\Xi_{k,m}$ (see [Lin41, BMP19, ABF23])

$$\mathcal{K}_0(x_1, x_2, \cdots, x_m) = \sum_{j=1}^m \varrho^2(\xi_j) R^g(\xi_j) + \sum_{j' \neq j} \varrho(\xi_j)\varrho(\xi_{j'}) G^g(\xi_j, \xi_{j'}) + \sum_{i=1}^m f_i(\xi_i),$$

where $f_i : \Sigma \to \mathbb{R}$ is smooth for $i = 1, \cdots, m$. In our context, we will show that the existence of blow-up solutions of (1.1.2) is strongly related to the C^1-stable critical point set of a m-vortices Kirchhoff-Routh type function,

$$\mathcal{F}^V_{k,m}(\xi) = \sum_{j=1}^m \varrho^2(\xi_j) R^g(\xi_j) + \sum_{j' \neq j} \varrho(\xi_j)\varrho(\xi_{j'}) G^g(\xi_j, \xi_{j'}) + \sum_{j=1}^m 2\varrho(\xi_j) \log V(\xi_j). \tag{2.6.2}$$

We observe that for any $\alpha \in (0,1)$, $G^g(x,\xi) \in C^\infty(\Sigma \setminus \{\xi\})$ and $H^g_\xi(x)$ is $C^{1,\alpha}(\Sigma)$, too. Thus $\mathcal{F}^V_{k,m}$ is $C^{1,\alpha}(\Xi^\delta_{k,m})$ for any fixed $\delta > 0$.

Lemma 2.6.1. *Suppose that $V > 0$ on Σ. Then, for any $\xi \in \mathring{\Sigma}$, we have:*

$$R^g(\xi,\xi) = H^g(\xi,\xi) \to +\infty \text{ as } \xi \text{ approaches } \partial\Sigma. \tag{2.6.3}$$

Furthermore, for any $\xi = (\xi_1, \dots, \xi_m) \in \Xi_{k,m}$, it holds that

$$\mathcal{F}^V_{k,m}(\xi) \to +\infty,$$

as ξ approaches $\partial\Xi_{k,m}$.

Proof. Since $V(x) > 0$, for any $x \in \Sigma$ the function $\mathcal{F}^V_{k,m}$ is well-defined on

$$\Xi_{k,m} = \mathring{\Sigma}^k \times (\partial\Sigma)^{m-k} \setminus \mathbb{F}_{k,m}(\Sigma).$$

For any $\zeta \in \partial\Sigma$, consider an isothermal chart $(y_\zeta, U(\zeta))$. Set $r_0 = r_\zeta/2$. Then, for any $\xi \in U_{r_\zeta}(\zeta)$, we decompose the Green's function as follows:

$$G^g(x,\xi) = \tilde{H}^g(x,\xi) - \frac{4}{\varrho(\xi)}\chi\left(\frac{|y_\zeta(x) - y_\zeta(\xi)|}{r_0}\right)\log|y_\zeta(x) - y_\zeta(\xi)|,$$

where χ is a cut-off function defined by (2.4.3). Applying the representation formula and divergence theorem, for any $\xi \in U_{r_\zeta}(\zeta)$, we obtain

$$\begin{aligned}
&\tilde{H}^g(\xi,\xi) = \int_\Sigma G^g(x,\xi)(-\Delta_g + \beta)\tilde{H}^g(x,\xi)dv_g(x) + \int_{\partial\Sigma} G^g(x,\xi)\partial_{\nu_g}\tilde{H}^g(x,\xi)ds_g(x) + \mathcal{O}(1)\\
= \;& \int_\Sigma (|\nabla\tilde{H}^g(x,\xi)|_g^2 + \beta|\tilde{H}^g(x,\xi)|^2)dv_g(x) - \frac{1}{4\pi^2}\int_{\partial\Sigma}\partial_{\nu_g}\left(\chi(|y_\zeta(x) - y_\zeta(\xi)|)\log|y_\zeta(x) - y_\zeta(\xi)|\right)\\
&\cdot\chi(|y_\zeta(x) - y_\zeta(\xi)|)\log|y_\zeta(x) - y_\zeta(\xi)|ds_g(x) + \mathcal{O}(1)\\
\geq \;& -\frac{1}{4\pi^2}\int_{\{x:|y_\zeta(x)-y_\zeta(\xi)|<r_0\}\cap\partial\Sigma}\log|y_\zeta(x) - y_\zeta(\xi)|\partial_{\nu_g}\log|y_\zeta(x) - y_\zeta(\xi)|ds_g(x) + \mathcal{O}(1)\\
\geq \;& \frac{1}{4\pi^2}\int_{\{y:|y-y_\zeta(\xi)|<r_0\}\cap\partial\mathbb{R}^2_+}\frac{-y_\zeta(\xi)_2}{|y - y_\zeta(\xi)|^2}\log|y - y_\zeta(\xi)|dy_1 + \mathcal{O}(1)\\
\geq \;& -\frac{1}{4\pi^2}\log(|y_\zeta(\xi)_2|)\int_\mathbb{R}\frac{1}{1+s^2}ds + \mathcal{O}(1) = -\frac{1}{4\pi}\log|y_\zeta(\xi)_2| + \mathcal{O}(1) \to +\infty,
\end{aligned}$$

as $d_g(\xi, \partial\Sigma) \to 0$, where ds_g is the line element of $\partial\Sigma$.

It is straightforward to see that $H^g(\xi,\xi) = \tilde{H}^g(\xi,\xi)$. The first statement is concluded.

Next, we assume that $\xi \in \Xi_{k,m}$.

Claim 2.6.2. There exists a constant c_0 satisfying $G^g(\xi_i,\xi_j) \geq c_0$, for any $\xi_i \neq \xi_j$.

Before proving Claim 2.6.2 we first show how Lemma 2.6.1 follows. We denote that $\mathcal{I}_0 = \{i : 1 \le i \le k, d_g(\xi_i, \partial\Sigma) \to 0 \text{ as } \xi \text{ going to } \partial\Xi_{k,m}\}$. For any $i \in \mathcal{I}_0$, $H^g(\xi_i, \xi_i) \to +\infty$. There exists a compact subset set F of $\mathring{\Sigma}$ such that $\xi_i \in F$ for any $i \in \{1, 2, \cdots, k\} \setminus \mathcal{I}_0$. It follows that any $i \notin \mathcal{I}_0$, $H^g(\xi_i, \xi_i) \ge -\sup_{x\in F} \|H^g_x\|_{C(\Sigma)} > -\infty$.

Case I. $\mathcal{I}_0 \neq \emptyset$. As ξ approaches $\partial\Xi_{k,m}$,

$$\begin{aligned}\mathcal{F}^V_{k,m}(\xi) \ &\ge\ \sum_{i\in\mathcal{I}_0} \varrho(\xi_i)^2 H^g(\xi_i, \xi_i) - \sum_{i\notin\mathcal{I}_0} \sup_{x\in\partial\Sigma\cup F} \varrho(\xi_i)^2 \|H^g(\cdot, x)\|_{C(\Sigma)} \\ &\quad - \sum_{i\neq h} \varrho(\xi_i)\varrho(\xi_h)|c_0| + \sum_{i=1}^{m} 2\varrho(\xi_i) \inf_{x\in\Sigma} \log V(x) \to +\infty.\end{aligned}$$

Case II. $\mathcal{I}_0 = \emptyset$. Then there exists a compact subset F such that $\xi_i \in F$ for any $1 \le i \le k$ and

$$\mathcal{I}_1 := \{(i, j) : i, j = 1, 2, \cdots, m; i \neq j \text{ such that } d_g(\xi_i, \xi_j) \to 0 \text{ as } \xi \to \partial\Xi_{k,m}\}$$

is non-empty. For any $(i, j) \in \mathcal{I}_1$,

$$\begin{aligned}G^g(\xi_i, \xi_j) \ &=\ H^g(\xi_i, \xi_j) + \frac{4}{\varrho(\xi_j)} \chi(|y_{\xi_j}(\xi_i)|/\bar{\mathrm{r}}_{\xi_j}) \log \frac{1}{|y_{\xi_j}(\xi_i)|} \\ &\ge\ -\sup_{x\in F\cup\partial\Sigma} \|H^g(\cdot, x)\|_{C(\Sigma)} + c_1 \frac{4}{\varrho(\xi_j)} \log \frac{1}{|d_g(\xi_i, \xi_j)|},\end{aligned}$$

in which $c_1 > 0$ is a constant. Consequently, as ξ approaches to $\partial\Xi_{k,m}$,

$$\begin{aligned}\mathcal{F}^V_{k,m}(\xi) \ &\ge\ -64\pi^2 m^2 c_0 + 64\pi^2 m \sup_{x\in\partial\Sigma\cup F} \|H^g(\cdot, x)\|_{C(\Sigma)} \\ &\quad + c_1 \sum_{(i,j)\in\mathcal{I}_1} \frac{4}{\varrho(\xi_i)} \log \frac{1}{d_g(\xi_i, \xi_j)} + \sum_{i=1}^{m} 2\varrho(\xi_i) \inf_{x\in\Sigma} \log V(x) \to +\infty.\end{aligned}$$

It remains to establish Claim 2.6.2. We begin by decomposing the Green's function as follows:

$$\begin{aligned}G^g(x, \xi_j) \ &=\ H^g(x, \xi_j) + \frac{4}{\varrho(\xi_j)} \chi(|y_{\xi_j}(x)|/\bar{\mathrm{r}}_{\xi_j}) \log \frac{1}{|y_{\xi_j}(x)|} \\ &\ge\ -\|H^g(\cdot, \xi_j)\|_{C(\Sigma)} + \frac{4}{\varrho(\xi_j)} \chi(|y_{\xi_j}(x)|/\bar{\mathrm{r}}_{\xi_j}) \log \frac{1}{|y_{\xi_j}(x)|}.\end{aligned}$$

If $\xi_j \in \partial\Sigma$, $\|H^g(\cdot, \xi_j)\|_{C(\Sigma)}$ is uniformly bounded. It is clear that $G^g(x, \xi_j) \ge c_0$, for some $c_0 > 0$. Thus, it suffices to focus on the cases where $j = 1, \cdots, k$. We observe that $G^g(x, \xi_j) \in C^{1,\alpha}_{loc}(\Sigma \setminus \{\xi_j\})$ for any $\alpha \in (0, 1)$ and $\lim_{x\to\xi_j} G^g(x, \xi_j) = +\infty$. Let $h(x)$ be the unique solution of the Dirichlet problem:

$$\begin{cases} (-\Delta_g+\beta)h(x) = -\frac{1}{|\Sigma|_g}, & x\in\mathring{\Sigma} \\ h(x)=0 & x\in\partial\Sigma \end{cases}.$$

Define that $\tilde{G}^g(x,\xi_j)=G^g(x,\xi_j)-h(x)$. Then, $-\Delta_g\tilde{G}^g(x,\xi_j)=0$ on $\Sigma\setminus\{\xi_j\}$. Considering that $\lim_{x\to\xi_j}\tilde{G}^g(x,\xi_j)=+\infty$, it follows that

$$\inf_{\Sigma\setminus\{\xi_j\}}\tilde{G}^g(x,\xi_j)=\min_{x\in\partial\Sigma}\tilde{G}^g(x,\xi_j),$$

by the maximum principle. Thus we have for some constants $c_2,c_0>0$

$$\begin{aligned} \inf_{\Sigma\setminus\{\xi_j\}}G(x,\xi_j) &\geq \inf_{\Sigma\setminus\{\xi_j\}}\tilde{G}^g(x,\xi_j)-\|h\|_{C(\Sigma)}\geq\min_{x\in\partial\Sigma}\tilde{G}^g(x,\xi_j)-\|h\|_{C(\Sigma)} \\ &\geq \min_{x\in\partial\Sigma}G^g(\xi_j,x)-2\|h\|_{C(\Sigma)} \\ &\geq -\sup_{x\in\partial\Sigma}\|H^g_x\|_{C(\Sigma)}-2\|h\|_{C(\Sigma)}+c_2\min_{x\in\partial\Sigma}\frac{1}{\pi}\log\frac{1}{d_g(x,\xi_j)}:=c_0. \end{aligned}$$

□

Remark 2.6.3. *In [HB24, Lemma 4.1], for* $\beta=0$ *it shows that*

$$|\nabla\mathcal{F}^V_{k,m}(\xi)|_g\to+\infty$$

as ξ *goes to* $\partial\Xi_{k,m}$.

2.7 Lyapunov-Schmidt reduction

The Lyapunov-Schmidt reduction is a method to study the solutions of nonlinear equations, named after Aleksandr Lyapunov and Erhard Schmidt. Given a nonlinear equation

$$f(x)=0, \tag{2.7.1}$$

where $f:X\to Z$ and X,Z are Banach spaces. If D_Xf is non-degenerate at x_0, we can use the implicit function theorem; otherwise, we can consider using the Lyapunov-Schmidt reduction. We decompose $X=X_1\oplus X_2$ and $Z=Z_1\oplus Z_2$ with $\dim(X_2),\dim(Z_2)<+\infty$. We define projects

$\Pi_i : Z \to Z_i$ for $i = 1, 2$. The equation (2.7.1) is equivalent to the following coupled system

$$\begin{cases} \Pi_1 f(x_1, x_2) = 0 \\ \\ \Pi_2 f(x_1, x_2) = 0 \end{cases},$$

where $x_i \in X_i$, for $i = 1, 2$. Suppose that $D_X \Pi_1 f|_{X_1} : X_1 \to Z_1$ at (x_0, y_0) is an isomorphism, then the first equation satisfies the implicit function theorem. Consequently, there exists a solution $x_1 = x_1(x_2)$ for given x_2. We substitute the solution into the second equation:

$$\Pi_2(f(x_1(x_2), x_2)) = 0,$$

which becomes a finite-dimensional problem.

Part I

Mean field type equations

3 Blow-up solutions for mean field type equations

3.1 Construction of blow-up solutions

3.1.1 Approximation solutions

Let $\bar{f} = \frac{1}{|\Sigma|_g}\int_\Sigma f dv_g$ for any $f \in L^1(\Sigma)$. To study blow-up solutions of (1.1.2), we consider the weak solution of the following problem in the space $\overline{\mathrm{H}}^1(\Sigma)$,

$$\begin{cases} (-\Delta_g + \beta)u = \varepsilon^2 V e^u - \overline{\varepsilon^2 V e^u} & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} u = 0 & \text{on } \partial\Sigma \end{cases}, \tag{3.1.1}$$

such that $\varepsilon^2 V e^u \to \sum_{i=1}^m \varrho(\xi_i)\delta_{\xi_i}$, convergent in a sense of measure on Σ as $\varepsilon \to 0$, for some $\xi = (\xi_1, \cdots, \xi_m) \in \Xi_{k,m}$. If we take $\lambda = \varepsilon^2 \int_\Sigma V e^u dv_g$, the weak solutions of (3.1.1) must be the weak solutions of (1.1.2). So we try to construct a sequence of blow-up solutions of (3.1.1) as $\varepsilon \to 0$ and then pass back to the original problem (1.1.2) as $\lambda \to \lambda_{k,m}$.

It is well-known that $u_{\tau,\eta}(y) = \log\frac{8\tau^2}{(\tau^2\varepsilon^2+|y-\eta|^2)^2}$ for $(\tau,\eta) \in (0,\infty) \times \mathbb{R}^2$, which are all the solutions of the Liouville-type equations on $\mathbb{R}^2$,

$$\begin{cases} -\Delta u = \varepsilon^2 e^u & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^u < \infty \end{cases}.$$

Our goal is to construct approximate solutions of (3.1.1) applying the pull-back of $u_{\tau,\eta}$ to Σ by isothermal coordinates and selecting appropriate values for τ and ξ. Define

$$U_{\tau,\xi}(x) = u_{\tau,0}(y_\xi(x)) = \log\frac{8\tau^2}{(\tau^2\varepsilon^2 + |y_\xi(x)|^2)}, \text{ for all } x \in y_\xi^{-1}(B^\xi).$$

We then introduce a projection operator P, employed to project $U_{\tau,\xi}$ into the space $\overline{\mathrm{H}}^1(\Sigma)$. The projected function $PU_{\tau,\xi}$ is defined as the solution of the following problem:

$$\begin{cases} (-\Delta_g+\beta)PU_{\tau,\xi}(x)=\varepsilon^2\chi_\xi e^{-\varphi_\xi}e^{U_{\tau,\xi}}-\overline{\varepsilon^2\chi_\xi e^{-\varphi_\xi}e^{U_{\tau,\xi}}}, & x\in\mathring{\Sigma}, \\ \partial_{\nu_g}PU_{\tau,\xi}=0, & x\in\partial\Sigma, \\ \int_\Sigma PU_{\tau,\xi}dv_g=0, \end{cases} \tag{3.1.2}$$

For $\beta\neq 0$, the last condition of $\int_\Sigma PU_{\tau,\xi}dv_g=0$ can be inferred from the preceding equations via the divergence theorem. However, the condition is included to address the case where $\beta=0$, ensuring solution criteria for all $\beta\geq 0$. The solution of (3.1.2) is unique in $\overline{\mathrm{H}}^1(\Sigma)$ and $PU_{\tau,\xi}$ in $C^\infty(\Sigma)$ by regularity theory (see Lemma 2.3.3), ensuring that $PU_{\tau,\xi}$ is well-defined. Let

$$\psi^0_{\tau,\eta}(y)=\frac{\partial}{\partial_\tau}u_{\tau,\eta}(x)=\frac{2}{\tau}\frac{|y-\eta|^2-\tau^2\varepsilon^2}{|y-\eta|^2+\tau^2\varepsilon^2},$$

and

$$\psi^j_{\tau,\eta}(y)=\frac{\partial}{\partial\eta_j}u_{\tau,\eta}(x)=4\frac{y_j-\eta_j}{\tau^2\varepsilon^2+|y-\eta|^2},\ \text{for } j=1,2.$$

It is observed that the derivatives above satisfy the equation: $-\Delta\psi=\varepsilon^2e^{u_{\tau,\eta}}\psi$ in $\mathbb{R}^2$, where $\psi=\psi^j_{\tau,\eta}$, for $j=0,1,2$. The function $\Psi^j_{\tau,\xi}$ is defined as the pullback of $\psi^j_{\tau,0}$ under the isothermal coordinate y_ξ, that is, $\Psi^j_{\tau,\xi}(x)=\psi^j_{\tau,0}(y_\xi(x))$, for any $x\in y_\xi^{-1}(B^\xi_{2r_0})$. Let $P\Psi^j_{\tau,\xi}$ be a projection into $\overline{\mathrm{H}}^1(\Sigma)$ of $\Psi^j_{\tau,\xi}$, for $\xi\in\Sigma$ and $j=0,1,\cdots,\mathrm{i}(\xi_i)$, where $\mathrm{i}(x)$ equals 2 if $x\in\mathring{\Sigma}$ and equals 1 if $x\in\partial\Sigma$. $P\Psi^j_{\tau,\xi}$ is defined as the solution of the following equations:

$$\begin{cases} (-\Delta_g+\beta)P\Psi^j_{\tau,\xi}=\varepsilon^2\chi_\xi e^{-\varphi_\xi}e^{U_{\tau,\xi}}\Psi^j_{\tau,\xi}-\overline{\varepsilon^2\chi_\xi e^{-\varphi_\xi}e^{U_{\tau,\xi}}\Psi^j_{\tau,\xi}}, & x\in\mathring{\Sigma}, \\ \partial_{\nu_g}P\Psi^j_{\tau,\xi}=0, & x\in\partial\Sigma, \\ \int_\Sigma P\Psi^j_{\tau,\xi}=0. \end{cases} \tag{3.1.3}$$

By the regularity theory in Lemma 2.3.3, the solution to (3.1.3) is unique and smooth on Σ. Hence, $P\Psi^j_{\tau,\xi}$ is well-defined and in the space $C^\infty(\Sigma)$.

For any $\xi=(\xi_1,\cdots,\xi_m)\in\Xi^\delta_{k,m}$, we can establish an isothermal chart around y_{ξ_i} for each point ξ_i for $i=1,\cdots,m$. Given the compactness of Σ, it is possible to select a uniform radius $r_{\xi_i}>0$ for any $\xi\in\Xi^\delta_{k,m}$, denoted as $4r_0$, and $\bar{\mathrm{r}}_{\xi_i}:=r_0$, where $\bar{\mathrm{r}}_{\xi_i}$ is introduced in Section 2.4. This radius is sufficiently small and depends only on δ and $\partial\Sigma$. Moreover, we ensure that $y_{\xi_i}^{-1}(B^{\xi_i}_{2r_0})\cap\partial\Sigma=\emptyset$ for $i=1,\cdots,k$ and $y_{\xi_i}^{-1}(B^{\xi_i}_{2r_0})\cap y_{\xi_j}^{-1}(B^{\xi_j}_{2r_0})=\emptyset$ for any $i,j=1,2,\cdots,m$

with $i \neq j$. For any $i = 1, \cdots, m$, we define the scaling parameter $\tau_i(\xi)$ as:

$$\tau_i(\xi) = \sqrt{\frac{1}{8} V(\xi_i) e^{\varrho(\xi_i) H^g(\xi_i,\xi_i) + \sum_{j\neq i} \varrho(\xi_j) G^g(\xi_i,\xi_j)}}. \tag{3.1.4}$$

The formulation of the scaling parameter $\tau_i(\xi)$ is chosen for technical considerations. For simplicity, we denote that $\tau_i = \tau_i(\xi)$,

$$U_i = U_{\tau_i(\xi),\xi_i} = \log\left(\frac{8\tau_i^2}{(\tau_i^2\varepsilon^2 + |y_{\xi_i}|^2)^2}\right),$$

$\chi_i = \chi_{\xi_i}$, $\varphi_i = \varphi_{\xi_i}$, $\hat{\varphi}_i = \hat{\varphi}_{\xi_i}$.

We consider the manifold for given integers $m \geq k \geq 0$ and a positive constant $\varepsilon > 0$,

$$\mathcal{M}_\varepsilon^{k,m} := \left\{\sum_{i=1}^m PU_i : \xi_i \in \mathring{\Sigma} \text{ for } i = 1, 2, \cdots, k \text{ and } \xi_i \in \partial\Sigma \text{ for } i = k+1, \cdots, m\right\}.$$

The functions in manifold $\mathcal{M}_\varepsilon^{k,m}$ serve as approximate solutions of the problem (3.1.1). We then denote the projected functions for any $i = 1, 2, \cdots, m$ and $j = 0, 1, \cdots, \mathrm{i}(\xi_i)$

$$P\Psi_i^j := P\Psi_{\tau_i(\xi),\xi_i}^j.$$

These projected functions generates a subspace of $\overline{\mathrm{H}}^1(\Sigma)$,

$$K_\xi := \langle P\Psi_i^j : i = 1, \cdots, m, j = 1, \cdots, \mathrm{i}(\xi_i)\rangle.$$

Furthermore, we introduce an inner product for the space $\overline{\mathrm{H}}^1(\Sigma)$ as follows:

$$\langle \psi, \phi\rangle := \int_\Sigma \langle \nabla\psi, \nabla\phi\rangle_g dv_g + \beta \int_\Sigma \psi\phi dv_g \text{ for any } \psi, \phi \in \overline{\mathrm{H}}^1(\Sigma).$$

The orthogonal complement of K_ξ,

$$K_\xi^\perp = \left\{\phi \in \overline{\mathrm{H}}^1(\Sigma) : \langle\phi, f\rangle = 0 \text{ for all } f \in K_\xi\right\}.$$

We also introduce $\Pi_\xi : \overline{\mathrm{H}}^1(\Sigma) \to K_\xi$ and $\Pi_\xi^\perp : \overline{\mathrm{H}}^1(\Sigma) \to K_\xi^\perp$ as the orthogonal projections onto K_ξ and $K_\xi^\perp$, respectively. The solution u can decompose into two parts: one part lies on the manifold $\mathcal{M}_\varepsilon^{k,m}$; the other part is on $K_\xi^\perp$ near the orthogonal space of the tangent space of the manifold $\mathcal{M}_\varepsilon^{k,m}$, i.e. $u = \sum_{i=1}^m PU_i + \phi_\xi^\varepsilon$, where $\phi_\xi^\varepsilon \in K_\xi^\perp$ is the remainder term.

3.1.2 Lyapunov-Schmidt reduction

Utilizing the Moser-Trudinger type inequality on compact Riemann surfaces, as in [Yan06], we have

$$\sup_{\int_\Sigma |\nabla_g u|^2 dv_g=1, \int_\Sigma u dv_g=0} \int e^{2\pi u^2} dv_g < +\infty.$$

Since $\left(\int_\Sigma |\nabla u|_g^2 dv_g + \beta \int_\Sigma |u|^2 dv_g\right)^{\frac{1}{2}}$ and $\left(\int_\Sigma |\nabla u|_g^2 dv_g\right)^{\frac{1}{2}}$ are equivalent norms in the Hilbert space $\overline{\mathrm{H}}^1(\Sigma)$, it follows that for any $u \in \overline{\mathrm{H}}^1(\Sigma)$

$$\begin{aligned}
\log \int_\Sigma e^u dv_g &\leq \log \int_\Sigma e^{2\pi \frac{u^2}{\|u\|^2} + \frac{1}{8\pi}\|u\|^2} dv_g \quad \text{(by Young's Inequality)} \\
&= \frac{1}{8\pi} \int_\Sigma |\nabla_g u|^2 dv_g + C, \\
&\leq \frac{1}{8\pi C} \langle u, u \rangle + C,
\end{aligned}$$

where $C > 0$ is a constant. Consequently, $\overline{\mathrm{H}}^1(\Sigma) \to L^p(\Sigma), u \mapsto e^u$ is continuous. For any $p > 1$, let $i_p^* : L^p(\Sigma) \to \overline{\mathrm{H}}^1(\Sigma)$ be the adjoint operator corresponding to the immersion $i : \overline{\mathrm{H}}^1(\Sigma) \to L^{\frac{p}{p-1}}$ and $\tilde{i}^* : \cup_{p>1} L^p(\Sigma) \to \overline{\mathrm{H}}^1(\Sigma)$. For any $f \in L^p(\Sigma)$, we define that $i^*(f) := \tilde{i}^*(f - \bar{f})$, i.e. for any $h \in \overline{\mathrm{H}}^1(\Sigma)$, $\langle i^*(f), h \rangle = \int_\Sigma (f - \bar{f}) h dv_g$.

The problem (3.1.1) has the following equivalent form,

$$\begin{cases} u = i^*(\varepsilon^2 V e^u) \\ u \in \overline{\mathrm{H}}^1(\Sigma) \end{cases}.$$

We consider the linear operator

$$L_\xi^\varepsilon : \overline{\mathrm{H}}^1(\Sigma) \to \overline{\mathrm{H}}^1(\Sigma), \phi \mapsto \Pi_\xi^\perp(\phi - i^*(\varepsilon^2 V e^{\sum_{i=1}^m PU_i} \phi)),$$

for any fixed $\xi \in \Xi_{k,m}^\delta$. The following lemma shows that the linear operator is partially invertible for fixed ε, and the norm of the inverse operator is controlled by $|\log \varepsilon|$ as $\varepsilon \to 0$, which is a key lemma to solve the problem (3.1.1).

Lemma 3.1.1. *For any $\delta > 0$, let $\xi = (\xi_1, \cdots, \xi_m) \in \Xi_{k,m}^\delta$. There exists $\varepsilon_0 > 0$ and a constant $c > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\left\| L_\xi^\varepsilon(\phi) \right\| \geqslant \frac{c}{|\log \varepsilon|} \|\phi\|, \quad \forall \phi \in K_\xi^\perp.$$

In particular, the operator $L_\xi^\varepsilon : K_\xi^\perp \to \overline{\mathrm{H}}^1(\Sigma)$ *is invertible and* $\left\|\left(L_\xi^\varepsilon\right)^{-1}\right\| \leqslant |\log\varepsilon|/c.$

By the fundamental work of [EGP05], the proof of Lemma 3.1.1 is rather standard. Following their idea, we will prove it by deducing a contradiction.

Proof of Lemma 3.1.1. Assume the conclusion in Lemma 3.1.1 does not hold. Then there exists $\xi \in \Xi_{k,m}^\delta \subset \Xi_{k,m}$ for some small $\delta > 0$, a sequence $\varepsilon_n \to 0$ and $\phi_n \in K_\xi^\perp$ with $\|\phi_n\| = 1$ and $\|L_\xi^{\varepsilon_n}(\phi)\| = o(\frac{1}{|\log\varepsilon_n|})$. To simplify the notations, we use ε instead of ε_n and ϕ instead of ϕ_n. It follows that

$$\phi - i^*(\varepsilon^2 V e^{\sum_{i=1}^m PU_i}\phi) = \psi + w, \tag{3.1.5}$$

where $\psi \in K_\xi^\perp$ and $w \in K_\xi$. Then $\|\psi\| = o\left(\frac{1}{|\log\varepsilon|}\right) \to 0$, as $\varepsilon \to 0$. (3.1.5) is equivalent that ϕ solves the following problem in the weak sense,

$$\begin{cases} (-\Delta_g + \beta)\phi = \varepsilon^2 V e^{\sum_{i=1}^m PU_i}\phi - \overline{\varepsilon^2 V e^{\sum_{i=1}^m PU_i}\phi} + (-\Delta_g+\beta)(\psi + w), & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g}\phi = 0, & \text{on } \partial\Sigma \end{cases}.$$

Step 1. $\|w\| = o(1)$, as $\varepsilon \to 0$.

Given that $w \in K_\xi$, it can be expressed as $w = \sum_{i=1}^m \sum_{j=1}^{\mathrm{i}(\xi_i)} c_{ij} P\Psi_i^j$. Considering the inner product of (3.1.5) with $P\Psi_{i'}^{j'}$, we have the following equation:

$$\begin{aligned} &\langle \phi, P\Psi_{i'}^{j'}\rangle - \int_\Sigma P\Psi_{i'}^{j'}\left(\varepsilon^2 V e^{\sum_{i=1}^m PU_i}\phi - \frac{1}{|\Sigma|_g}\int_\Sigma \varepsilon^2 V e^{\sum_{i=1}^m PU_i}\phi dv_g\right) dv_g \\ = \quad & \langle \psi, P\Psi_{i'}^{j'}\rangle + \langle w, P\Psi_{i'}^{j'}\rangle. \end{aligned}$$

Since $P\Psi_{i'}^{j'} \in \overline{\mathrm{H}}^1(\Sigma)$ and $\phi \in K_\xi^\perp$, we have $\int_\Sigma P\Psi_{i'}^{j'} dv_g = 0$ and $\langle \psi, P\Psi_{i'}^{j'}\rangle = \langle \phi, P\Psi_{i'}^{j'}\rangle = 0$. It follows

$$-\varepsilon^2 \int_\Sigma V e^{\sum_{i=1}^m PU_i}\phi P\Psi_{i'}^{j'} dv_g = \sum_{i=1}^m \sum_{j=1}^{\mathrm{i}(\xi_i)} c_{ij}\langle P\Psi_i^j, P\Psi_{i'}^{j'}\rangle. \tag{3.1.6}$$

Applying Lemma A.0.5, the right-hand side of (3.1.6) equals

$$\frac{8\varrho(\xi_{i'})D_{j'}}{\pi\tau_{i'}^2\varepsilon^2}c_{i'j'} + \mathcal{O}(\varepsilon^{\alpha_0 - 1}\sum_{i=1}^m\sum_{j=1}^{\mathrm{i}(\xi_i)}|c_{ij}|),$$

where $D_{j'} > 0$ is a constant and $\alpha_0 \in (0,1)$. The left-hand side of (3.1.6) can be expanded as follows:

$$\begin{aligned}\int_\Sigma \varepsilon^2 \left(\textstyle\sum_{i=1}^m \chi_i e^{U_i} - V e^{\sum_{i=1}^m PU_i}\right) P\Psi_{i'}^{j'} \phi dv_g - \textstyle\sum_{i=1}^m \int_\Sigma \varepsilon^2 \chi_i e^{U_i}(P\Psi_{i'}^{j'} - \chi_{i'}\Psi_{i'}^{j'})\phi dv_g \\ -\varepsilon^2 \int_\Sigma \chi_{i'}^2(-e^{-\varphi_{i'}}+1)e^{U_{i'}}\Psi_{i'}^{j'}\phi dv_g - \varepsilon^2 \int_\Sigma \chi_{i'}^2 e^{-\varphi_{i'}} e^{U_{i'}}\Psi_{i'}^{j'}\phi dv_g.\end{aligned}$$

Since $\|\phi\| = 1$ and $\phi \in K_\xi^\perp$, $\int_\Sigma \varepsilon^2 \chi_{i'}^2 e^{-\varphi_{i'}} e^{U_{i'}} \Psi_{i'}^{j'} \phi = \mathcal{O}(\varepsilon^2) + \langle P\Psi_{i'}^{j'}, \phi\rangle = \mathcal{O}(\varepsilon^2)$. By calculation, we have

$$\left|\varepsilon^2 \int_\Sigma (e^{-\varphi_{i'}} - 1)\chi_{i'} e^{U_{i'}} \Psi_{i'}^{j'} dv_g\right| \leq \mathcal{O}\left(\int_{|y|\leq 2r_0} \frac{\tau_i^2 \varepsilon^2 |y|^3 dy}{(\tau_i^2\varepsilon^2 + |y|^2)^3}\right) = \mathcal{O}(\varepsilon).$$

Applying Lemma A.0.5 and Lemma A.0.7,

$$\begin{aligned}\left|\int_\Sigma \varepsilon^2 \left(\sum_{i=1}^m \chi_i e^{U_i} - V e^{\sum_{i=1}^m PU_i}\right) P\Psi_{i'}^{j'} \phi dv_g\right| &\leq C \left\|\varepsilon^2 \left(\sum_{i=1}^m \chi_i e^{U_i} - V e^{\sum_{i=1}^m PU_i}\right)\right\|_p \|\phi\|_q \|P\Psi_{i'}^{j'}\| \\ &\leq \mathcal{O}(\varepsilon^{\frac{2-p}{p}-1}) = \mathcal{O}(\varepsilon^{\frac{2(1-p)}{p}}),\end{aligned}$$

where $p, q \in (1, +\infty)$ with $\frac{1}{p} + \frac{1}{q} < 1$ and $C > 0$ is a constant. Furthermore, Lemma A.0.3 implies $P\Psi_{i'}^{j'} - \chi_{i'}\Psi_{i'}^{j'} = \mathcal{O}(1)$, as $\varepsilon \to 0$. And applying Lemma A.0.2, we deduce for any $i = 1, \cdots, m$

$$\begin{aligned}\left|\varepsilon^2 \int_\Sigma \chi_i e^{U_i} \phi (P\Psi_{i'}^{j'} - \chi_{i'}\Psi_{i'}^{j'}) dv_g\right| &\leq \mathcal{O}(\|\varepsilon^2 \chi_i e^{U_i}\|_p \|\phi\|_q) \\ &\leq \mathcal{O}(\varepsilon^{\frac{2(1-p)}{p}}),\end{aligned}$$

as $\varepsilon \to 0$. Combining these estimates, we conclude

$$\frac{8\varrho(\xi_{i'})D_{j'}}{\pi\tau_{i'}^2\varepsilon^2} c_{i'j'} + \mathcal{O}\left(\varepsilon^{\alpha_0 - 1} \sum_{i=1}^m \sum_{j=1}^{\mathrm{i}(\xi_i)} |c_{ij}|\right) = \mathcal{O}(\varepsilon^{\frac{2(1-p)}{p}}),$$

as $\varepsilon \to 0$. Then $|c_{i'j'}| = \mathcal{O}(\varepsilon^{\frac{2}{p}})$ as $\varepsilon \to 0$, where $p \in (1, 2)$. So

$$\sum_{i=1}^m \sum_{j=1}^{\mathrm{i}(\xi_i)} |c_{ij}| = \mathcal{O}(\varepsilon^{\frac{2}{p}}) \tag{3.1.7}$$

by the arbitrariness of i' and j'. Lemma A.0.5 and (3.1.7) yield that

$$\|w\|^2 = \left\|\sum_{i=1}^m \sum_{j=1}^{\mathrm{i}(\xi_i)} c_{ij} P\Psi_i^j\right\|^2 = \mathcal{O}\left(\sum_{i=1}^m \sum_{j=1}^{\mathrm{i}(\xi_i)} |c_{ij}|^2 \frac{1}{\varepsilon^2} + \mathcal{O}(\varepsilon^{\alpha_0 - 1})\right) \leq \mathcal{O}(\varepsilon^{\frac{4}{p}-2}).$$

Hence, it follows $\|w\| = \mathcal{O}(\varepsilon^{\frac{2-p}{p}})$ for any $p \in (1, 2)$ as $\varepsilon \to 0$.

Step 2. $\langle \phi, P\Psi_i^0 \rangle \to 0$, as $\varepsilon \to 0$.

Following the construction in [EGP05, EF14], we define

$$\omega_i(y) = \frac{4}{3\tau_i} \log(\tau_i^2\varepsilon^2 + |y|^2)\frac{\tau_i^2\varepsilon^2 - |y|^2}{\tau_i^2\varepsilon^2 + |y|^2} + \frac{8}{3\tau_i}\frac{\tau_i^2\varepsilon^2}{\tau_i^2\varepsilon^2 + |y|^2},$$

and

$$t_i(y) = -2\frac{\tau_i^2\varepsilon^2}{\tau_i^2\varepsilon^2 + |y|^2}.$$

We observe that $\omega_i(y)$ and $t_i(y)$ satisfy the following equations, respectively:

$$-\Delta\omega_i - \varepsilon^2 e^{u_{\tau_i,0}}\omega_i = \varepsilon^2 e^{u_{\tau_i,0}}\psi^0_{\tau_i,0} \text{ in } \mathbb{R}^2$$

and

$$-\Delta t_i - \varepsilon^2 e^{u_{\tau_i,0}} t_i = \varepsilon^2 e^{u_{\tau_i,0}} \text{ in } \mathbb{R}^2.$$

By straightforward calculation, we derive that

$$\int_{\mathbb{R}^2} |\nabla\omega_i|^2 = M_i^2(1+o(1))(\log\varepsilon)^2, \quad \int_{\mathbb{R}^2} |\nabla t_i|^2 = O(1), \text{ as } \varepsilon \to 0$$

with $M_i = \frac{32}{3\tau_i}\left(\int_{\mathbb{R}^2}\frac{|y|^2}{(1+|y|^2)^4}\right)^{\frac{1}{2}}$. Let

$$u_i(x) = \chi_i(x)\left(\omega_i(y_{\xi_i}(x)) + \frac{2\varrho(\xi_i)}{3\tau_i}H^g(\xi_i,\xi_i)t_i(y_{\xi_i}(x))\right), \text{ for all } x \in U_{2r_0}(\xi_i).$$

The projection $Pu_i \in \overline{\mathrm{H}}^1(\Sigma)$ of u_i is given by

$$\begin{cases} (-\Delta_g + \beta)Pu_i(x) = -\chi_i\Delta_g u_i(x) + \overline{\chi_i\Delta_g u_i(x)} & x \in \mathring{\Sigma} \\ \partial_{\nu_g} Pu_i(x) = 0 & x \in \partial\Sigma \\ \int_\Sigma Pu_i dv_g = 0 \end{cases}. \tag{3.1.8}$$

Consider $\eta_i = u_i - Pu_i + \frac{2\varrho(\xi_i)}{3\tau_i}H^g(x,\xi_i)$. The integral of η_i on Σ has the following estimate:

$$\int_\Sigma \eta_i dv_g = \mathcal{O}(\varepsilon^2\log^2\varepsilon).$$

If $\xi_i \in \mathring{\Sigma}$, we have $\partial_{\nu_g}\eta_i \equiv 0$ in $\partial\Sigma$. For $\xi_i \in \partial\Sigma$,

$$\partial_{\nu_g}\eta_i = \partial_{\nu_g}u_i + \frac{8}{3\tau_i}\partial_{\nu_g}(\chi_i \log|y_{\xi_i}|) = \mathcal{O}(\varepsilon^2|\log\varepsilon|) \in C^\infty(\partial\Sigma),$$

in view of $\partial_{\nu_g}|y_{\xi_i}(x)| = -e^{-\frac{\hat{\varphi}_i(y)}{2}} \frac{y_2}{|y|}|_{y=y_{\xi_i}(x)} = 0$ for any $x \in U(\xi_i) \cap \partial\Sigma$. Considering that $\int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^3} \log(1+|y|^2)dy = -\frac{\pi}{2}$, and $\int_{\mathbb{R}^2} \frac{2}{(1+|y|^2)^3}dy = \int_{\mathbb{R}^2} \frac{1}{(1+|y|^2)^2}dy = \pi$, we deduce that

$$\left\|(-\Delta_g + \beta)\eta_i\right\|_p = \mathcal{O}(\varepsilon^{\frac{1}{p}}|\log\varepsilon|).$$

Claim 3.1.2. $\|\eta_i - \overline{\eta_i}\|_\infty \leq C\varepsilon^{\frac{1}{p}}|\log\varepsilon|$.

Indeed, we can find a smooth function f with $f|_{\partial\Sigma} = \partial\eta_i|_{\partial\Sigma}$ and $\|f\|_\infty \leq C\varepsilon^2|\log\varepsilon|$ for some constant $C > 0$. By the Schauder estimate in Lemma 2.3.3, there exists a unique solution $u_1 \in C^\infty(\Sigma)$ solving

$$\begin{cases} (-\Delta_g + \beta)u_1 = -\frac{1}{|\Sigma|}\int_\Sigma f ds_g + \beta\frac{1}{|\Sigma|}\int_\Sigma \eta_i dv_g & \text{in } \mathring{\Sigma} \\ \partial_\nu u_1 = f & \text{on } \partial\Sigma \\ \int_\Sigma u_1 dv_g = \int_\Sigma \eta_i dv_g \end{cases}.$$

It follows that $\|u_1 - \overline{u_1}\|_\infty = \mathcal{O}(\varepsilon^2|\log\varepsilon|)$. On the other hand, the regularity theory of Lemma 2.3.1, there exists a unique solution $u_2 \in W^{2,p}$ of the following problem

$$\begin{cases} (-\Delta_g + \beta)u_2 = (-\Delta_g + \beta)\eta_i + \frac{1}{|\Sigma|}\int_\Sigma f ds_g - \beta\frac{1}{|\Sigma|}\int_\Sigma \eta_i dv_g & \text{in } \mathring{\Sigma} \\ \partial_\nu u_2 = 0 & \text{on } \partial\Sigma \\ \int_\Sigma u_2 dv_g = 0 \end{cases},$$

with $\|u_2\|_{W^{2,p}} \leq C\varepsilon^{\frac{1}{p}}|\log\varepsilon|$ for some constant $C > 0$. By the uniqueness, $\eta_i = u_1 + u_2 \in W^{2,p}(\Sigma)$

$$\|\eta_i - \overline{\eta_i}\|_\infty = \mathcal{O}(\varepsilon^{\frac{1}{p}}|\log\varepsilon|).$$

Since $|\bar{\eta}_i| = \mathcal{O}(\varepsilon^2\log^2\varepsilon)$, we deduce that

$$\|\eta_i\|_\infty \leq \mathcal{O}(\varepsilon^{\frac{1}{p}}|\log\varepsilon|), \tag{3.1.9}$$

for $p > 1$. Moreover, for any $x \in \Sigma \setminus \{\xi_i\}$, the following inequality holds:

$$\left|Pu_i(x) - \frac{2\varrho(\xi_i)}{3\tau_i}G^g(x,\xi_i)\right| \leq \mathcal{O}(\varepsilon^{\frac{1}{p}}|\log\varepsilon|). \tag{3.1.10}$$

Additionally, $\|Pu_i\|^2$ is computed directly as

$$\begin{aligned}\|Pu_i\|^2 &= \langle Pu_i, Pu_i\rangle = -\int_\Sigma \chi_i\left(u_i + \frac{2\varrho(\xi_i)}{3\tau_i}H^g_{\xi_i} + \mathcal{O}(\varepsilon^{\frac{1}{p}}|\log\varepsilon|)\right)\Delta_g u_i dv_g\\ &= \mathcal{O}(|\log\varepsilon|^2),\end{aligned}$$

as $\varepsilon \to 0$. Thus

$$\|Pu_i\| = \mathcal{O}(|\log\varepsilon|), \tag{3.1.11}$$

as $\varepsilon \to 0$. Applying Pu_i as a test function for (3.1.5),

$$\langle Pu_i, \phi\rangle - \int_\Sigma \varepsilon^2 V e^{\sum_{h=1}^m PU_h}\phi Pu_i dv_g = \langle Pu_i, w+\psi\rangle.$$

Considering $|\langle Pu_i, w+\psi\rangle| \le \|Pu_i\|(\|w\|+\|h\|) \le \|Pu_i\| o\left(\frac{1}{|\log\varepsilon|}\right) = o(1)$, we deduce that

$$\langle Pu_i, \phi\rangle - \int_\Sigma \varepsilon^2 V e^{\sum_{h=1}^m PU_h}\phi Pu_i dv_g(x) = o(1). \tag{3.1.12}$$

By (3.1.8) and $\|\phi\| = 1$ with the Hölder inequality,

$$\begin{aligned}\langle Pu_i, \phi\rangle &= \int_\Sigma(-\chi_i\Delta_g u_i + \overline{\chi_i\Delta_g u_i})\phi dv_g \qquad (3.1.13)\\ &= \int_\Sigma \varepsilon^2 e^{-\varphi_i}e^{U_i}u_i\phi dv_g + \int_\Sigma \varepsilon^2\chi_i e^{-\varphi_i}e^{U_i}\Psi_i^0\phi dv_g + \int_\Sigma \frac{2\varrho(\xi_i)}{3\tau_i}H^g(\xi_i,\xi_i)\varepsilon^2\chi_i e^{-\varphi_i}e^{U_i}\phi dv_g + \mathcal{O}(\varepsilon^2)\\ &= \int_\Sigma \varepsilon^2 e^{U_i}u_i\phi dv_g + \int_\Sigma \varepsilon^2(e^{-\varphi_i}-1)e^{U_i}u_i\phi dv_g\\ &\quad + \langle P\Psi_i^0, \phi\rangle + \int_\Sigma \frac{2\varrho(\xi_i)}{3\tau_i}H^g(\xi_i,\xi_i)\varepsilon^2\chi_i(e^{-\varphi_i}-1)e^{U_i}\phi dv_g\\ &\quad + \int_\Sigma \frac{2\varrho(\xi_i)}{3\tau_i}(H^g(\xi_i,\xi_i) - H^g(\cdot,\xi_i))\varepsilon^2\chi_i e^{U_i}\phi dv_g + \int_\Sigma \frac{2\varrho(\xi_i)}{3\tau_i}H^g(\cdot,\xi_i)\varepsilon^2\chi_i e^{U_i}\phi dv_g + \mathcal{O}(\varepsilon^2)\\ &= \int_\Sigma \varepsilon^2 e^{U_i}u_i\phi dv_g + \langle P\Psi_i^0, \phi\rangle + \int_\Sigma \frac{2\varrho(\xi_i)}{3\tau_i}H^g(\cdot,\xi_i)\varepsilon^2\chi_i e^{U_i}\phi dv_g + \mathcal{O}(\varepsilon^{\frac{2-q}{q}}),\end{aligned}$$

for any $q \in (1,2)$. On the other hand, Lemma A.0.2, Lemma A.0.7, (3.1.9) and (3.1.11) with the Hölder inequality yield that

$$
\begin{aligned}
&\int_\Sigma \varepsilon^2 V e^{\sum_{h=1}^m PU_h}\phi P u_i dv_g = \sum_{h=1}^m \int_\Sigma \varepsilon^2 \chi_h e^{U_h}\phi P u_i dv_g + \mathcal{O}(\varepsilon^{\frac{2-s}{s}}|\log\varepsilon|) \qquad (3.1.14)\\
= & \sum_{h=1}^m \int_\Sigma \varepsilon^2\chi_h e^{U_h}\cdot\left(u_i + \frac{2\varrho(\xi_i)}{3\tau_i}H^g(\cdot,\xi_i) + \mathcal{O}(\varepsilon^{\frac{1}{p}}|\log\varepsilon|)\right)\phi dv_g + \mathcal{O}(\varepsilon^{\frac{1}{p}}|\log\varepsilon|)\\
= & \int_\Sigma \varepsilon^2 e^{U_i}u_i\phi dv_g + \int_\Sigma \frac{2\varrho(\xi_i)}{3\tau_i}H^g(\cdot,\xi_i)\varepsilon^2\chi_i e^{U_i}\phi dv_g + \sum_{h\neq i}\int_\Sigma \frac{2\varrho(\xi_i)}{3\tau_i}H^g(\xi_h,\xi_i)\varepsilon^2\chi_h e^{U_h}\phi dv_g\\
& + \sum_{h\neq i}\int_\Sigma \frac{2\varrho(\xi_i)}{3\tau_i}(H^g(\cdot,\xi_i) - H^g(\xi_h,\xi_i))\varepsilon^2\chi_h e^{U_h}\phi dv_g + \mathcal{O}(\varepsilon^{\frac{1}{p}+2(\frac{1}{s}-1)}|\log\varepsilon|),\\
= & \int_\Sigma \varepsilon^2 e^{U_i}u_i\phi dv_g + \int_\Sigma \frac{2\varrho(\xi_i)}{3\tau_i}H^g(\cdot,\xi_i)\varepsilon^2\chi_i e^{U_i}\phi dv_g + \sum_{h\neq i}\int_\Sigma \frac{2\varrho(\xi_i)}{3\tau_i}H^g(\xi_h,\xi_i)\varepsilon^2\chi_h e^{U_h}\phi dv_g\\
& + \mathcal{O}(\varepsilon^{\frac{1}{p}+2(\frac{1}{s}-1)}|\log\varepsilon|)
\end{aligned}
$$

for any $s \in (1,2)$. We choose s, p sufficiently close to 1 such that $\frac{1}{p} + 2(\frac{1}{s} - 1) > 0$. Then, (3.1.13) and (3.1.14) imply that

$$
\begin{aligned}
\langle P\Psi_i^0, \phi\rangle &= \sum_{h\neq i}\int_\Sigma \frac{2\varrho(\xi_i)}{3\tau_i}H^g(\xi_h,\xi_i)\varepsilon^2\chi_h e^{U_h}\phi dv_g + o(1)\\
&= \sum_{h\neq i}\int_\Sigma \frac{2\varrho(\xi_i)}{3\tau_i}H^g(\xi_h,\xi_i)\varepsilon^2\chi_h e^{-\varphi_h}e^{U_h}\phi dv_g + o(1)\\
&= \sum_{h\neq i}\frac{2\varrho(\xi_i)}{3\tau_i}H^g(\xi_h,\xi_i)\langle PU_h,\phi\rangle + o(1)
\end{aligned}
$$

as $\varepsilon \to 0$. It is sufficient to show that $\langle PU_h, \phi\rangle = o(1)$ for $h = 1, \cdots, m$. Using $P\Psi_i^0$ as a test function of (3.1.5), we deduce that for $i = 1, \cdots, m$

$$
\begin{aligned}
o(1) &= \langle \phi, P\Psi_i^0\rangle - \int_\Sigma \varepsilon^2 V e^{\sum_{h=1}^m PU_h}\phi P\Psi_i^0 dv_g\\
&= \int_\Sigma \varepsilon^2\chi_i e^{-\varphi_i}e^{U_i}(\Psi_i^0 - P\Psi_i^0)\phi dv_g - \sum_{h\neq i}\int_\Sigma \varepsilon^2\chi_h e^{-\varphi_h}e^{U_h}P\Psi_i^0\phi dv_g + o(1)\\
&= \int_\Sigma \varepsilon^2\chi_i e^{-\varphi_i}e^{U_i}\left(-\frac{2}{\tau_i} + \mathcal{O}(\varepsilon^{1+\alpha_0})\right)\phi dv_g - \sum_{h\neq i}\int_\Sigma \varepsilon^2\chi_h e^{-\varphi_h}e^{U_h}\mathcal{O}(\varepsilon^{1+\alpha_0})\phi dv_g + o(1)\\
&= -\frac{2}{\tau_i}\langle PU_i,\phi\rangle + o(1),
\end{aligned}
$$

where we applied Lemma A.0.2, A.0.3, A.0.5 and A.0.7 along with the facts $\|\phi\| = 1, \|w\| = o(1), \|\psi\| = o(1/|\log\varepsilon|)$. Consequently, $\langle PU_h, \phi\rangle = o(1)$, for $h = 1, \cdots, m$.

Step 3. Construct a contradiction.

We denote $\mathbb{R}_x = \mathbb{R}^2$ if $x \in \mathring{\Sigma}$; $\mathbb{R}_x = \mathbb{R}^2_+ := \{y \in \mathbb{R}^2 : y_2 \geq 0\}$ if $x \in \partial\Sigma$. Let π_N be the stereographic projection through the north pole for the standard unit sphere in $\mathbb{R}^3$. We denote that $S_i = \pi_N^{-1}(\mathbb{R}_{\xi_i})$ for $i = 1, \cdots, m$. Define the following spaces for $\xi = (\xi_1, \cdots, \xi_m) \in \Xi^{\delta}_{k,m}$:

$$L_i := \left\{ \Psi : \mathbb{R}_{\xi_i} \to \mathbb{R} : \left\| \frac{\Psi}{1 + |\cdot|^2} \right\|_{L^2(\mathbb{R}_{\xi_i})} < +\infty \right\},$$

and

$$H_i := \left\{ \Psi : \mathbb{R}_{\xi_i} \to \mathbb{R} : \|\nabla \Psi\|_{L^2(\mathbb{R}_{\xi_i})} + \left\| \frac{\Psi}{1 + |\cdot|^2} \right\|_{L^2(\mathbb{R}_{\xi_i})} < \infty \right\}.$$

The associated norms are defined as the following,

$$\|\Psi\|_{L_i} := \left\| \frac{\Psi}{1 + |\cdot|^2} \right\|_{L^2(\mathbb{R}_{\xi_i})} \text{ and } \|\Psi\|_{H_i} := \|\nabla \Psi\|_{L^2(\mathbb{R}_{\xi_i})} + \left\| \frac{\Psi}{1 + |\cdot|^2} \right\|_{L^2(\mathbb{R}_{\xi_i})}.$$

The maps

$$L_i \to L^2(S_i) : \Psi \mapsto \Psi \circ \pi_N$$

and

$$H_i \to H^1(S_i) : \Psi \mapsto \Psi \circ \pi_N$$

are isometric. Let $\Omega_i^{\varepsilon} := \frac{1}{\tau_i \varepsilon} B_{2r_0}^{\xi_i}$, $\phi_i^{\varepsilon}(y) = \phi(y_{\xi_i}^{-1}(\tau_i \varepsilon y))$ and $\chi_i^{\varepsilon}(y) = \chi(\tau_i \varepsilon |y|)$. Consider

$$\tilde{\phi}_i^{\varepsilon} = \begin{cases} \phi_i^{\varepsilon} \chi_i^{\varepsilon} & y \in \Omega_i^{\varepsilon} \\ 0 & y \in \mathbb{R}_{\xi_i} \setminus \Omega_i^{\varepsilon} \end{cases}.$$

By Lemma A.0.7 with the Hölder inequality, we have

$$\begin{aligned}
\sum_{h=1}^{m} \varepsilon^2 \int_{\Sigma} \chi_h e^{U_h} \phi^2 dv_g &= \varepsilon^2 \int_{\Sigma} V e^{\sum_{h=1}^m PU_h} \phi^2 dv_g + \mathcal{O}\left(\int_{\Sigma} \varepsilon^2 \Big| \sum_{h=1}^{m} \chi_h e^{U_h} - V e^{\sum_{h=1}^m PU_h} \Big| \phi^2 dv_g \right) \\
&= \varepsilon^2 \int_{\Sigma} e^{\sum_{h=1}^m PU_h} \phi^2 + \mathcal{O}\left(\left\| \varepsilon^2 \left(\sum_{h=1}^{m} \chi_h e^{U_h} - V e^{\sum_{h=1}^m PU_h} \right) \right\|_p \|\phi^2\|_q \right) \\
&= \varepsilon^2 \int_{\Sigma} V e^{\sum_{h=1}^m PU_h} \phi^2 dv_g + \mathcal{O}(\varepsilon^{\frac{2-p}{p}}) \\
&= \varepsilon^2 \int_{\Sigma} V e^{\sum_{h=1}^m PU_h} \phi^2 dv_g + o(1),
\end{aligned}$$

as $\varepsilon \to 0$, where $p \in (1, 2)$ and $\frac{1}{p} + \frac{1}{q} = 1$. On the other hand, we use ϕ as a test function of (3.1.5). Since $\|\phi\| = 1$ and $\|\psi\| = o(\frac{1}{\log \varepsilon})$,

$$\varepsilon^2 \int_{\Sigma} V e^{\sum_{h=1}^m PU_h} \phi^2 dv_g = \langle \phi, \phi \rangle - \langle w + \psi, \phi \rangle = 1 + o(1).$$

By direct calculation, we have the following estimates:

$$\begin{aligned}\sum_{i=1}^{m}\varepsilon^2\int_\Sigma \chi_i e^{U_i}\phi^2 dv_g &= \sum_{i=1}^{m}\int_{B^{\xi_i}_{2r_0}} \frac{8\tau_i^2\varepsilon^2\chi^2(|y|/r_0)}{(\tau_i^2\varepsilon^2+|y|^2)^2}(\phi\circ y_{\xi_i}^{-1}(\tau_i\varepsilon y))^2 dy + \mathcal{O}(\varepsilon^2)\\ &= 8\sum_{i=1}^{m}\int_{\mathbb{R}_{\xi_i}} \frac{|\tilde{\phi}_i^\varepsilon(y)|^2}{(1+|y|^2)^2}dy + \mathcal{O}(\varepsilon^2),\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega_i^\varepsilon} |\nabla\tilde{\phi}_i^\varepsilon|^2 dy &= \int_{\frac{1}{\tau_i\varepsilon}B^{\xi_i}_{2r_0}} |\chi_i^\varepsilon\nabla\phi_i^\varepsilon + \phi_i^\varepsilon\nabla\chi_i^\varepsilon|^2 dy\\ &= \mathcal{O}\left(\int_{B^{\xi_i}_{2r_0}} (|\nabla\phi_i(y_{\xi_i}^{-1}(y))|^2 + |\phi(y_{\xi_i}^{-1}(y))|^2)dy\right)\\ &= \mathcal{O}\left(\int_\Sigma |\nabla\phi|_g^2 dv_g + \int_\Sigma e^{-\varphi_i}|\phi(x)|^2 dv_g\right)\\ &= \mathcal{O}\left(\|\phi\|\right) = \mathcal{O}(1).\end{aligned}$$

Hence, $\tilde{\phi}_i^\varepsilon$ is bounded in H_i. We observe that H_i compactly embeds in L_i. Up to a subsequence, as $\varepsilon \to 0$, $\tilde{\phi}_i^\varepsilon \to \tilde{\phi}_i^0$ weakly in H_i and strongly in L_i. It follows that

$$\sum_{i=1}^{m}\|\tilde{\phi}_i^0\|_{L_i}^2 = \frac{1}{8}. \tag{3.1.15}$$

For any $h \in C_c^\infty(\mathbb{R}_i)$, assume that $\operatorname{supp} h \subset \mathbb{B}_{R_0}$. If $\tau_i\varepsilon < \frac{r_0}{R_0}$, then $\operatorname{supp} \nabla\chi\left(\frac{|y|}{r_0}\right) \cap \operatorname{supp} h\left(\frac{1}{\tau_i\varepsilon}y\right) = \emptyset$. For any $\Phi \in \overline{\mathrm{H}}^1(\Sigma)$,

$$0 = \int_{B^{\xi_i}_{2r_0}} \Phi\circ y_{\xi_i}^{-1}\nabla\chi\left(\frac{|\cdot|}{r_0}\right)\cdot\nabla h\left(\frac{\cdot}{\tau_i\varepsilon}\right) = \int_{B^{\xi_i}_{2r_0}} h\left(\frac{\cdot}{\tau_i\varepsilon}\right)\nabla(\Phi\circ y_{\xi_i}^{-1})\cdot\nabla\chi\left(\frac{|\cdot|}{r_0}\right). \tag{3.1.16}$$

In (3.1.16), we take $\Phi = \phi, w$ and ψ, respectively.

Claim 3.1.3. *For any* $\|h\| := (\int_{\mathbb{R}^2}|\nabla h|^2 + |h|^2)^{\frac{1}{2}} \le 1$ *and* $h \in C_c^\infty(\mathbb{R}^2)$, *as* $\varepsilon \to 0$

$$\int_\Sigma \chi_i h^2\left(\frac{1}{\tau_i\varepsilon}y_{\xi_i}\right)dv_g = O(\varepsilon^2) \tag{3.1.17}$$

$$\text{and } \int_\Sigma \chi_i\left|\nabla h\left(\frac{1}{\tau_i\varepsilon}y_{\xi_i}\right)\right|_g^2 dv_g = \mathcal{O}(1). \tag{3.1.18}$$

Indeed,

$$\begin{aligned}
\textstyle\int_\Sigma \chi_i h^2\left(\frac{1}{\tau_i\varepsilon}y_{\xi_i}(x)\right)dv_g(x) &= \textstyle\int_{B^{\xi_i}_{2r_0}} e^{\hat{\varphi}_i(y)}\chi\left(\frac{|y|}{r_0}\right)h^2\left(\frac{1}{\tau_i\varepsilon}y\right)dy \\
&\le \mathcal{O}\left(\varepsilon^2\textstyle\int_{\Omega_i^\varepsilon}|h\left(y\right)|^2dy\right)\le \mathcal{O}(\varepsilon^2)\|h\|^2=\mathcal{O}(\varepsilon^2). \\
\textstyle\int_\Sigma \chi_i\left|\nabla h\left(\frac{1}{\tau_i\varepsilon}y_{\xi_i}(x)\right)\right|_g^2dv_g(x) &= \textstyle\int_{B^{\xi_i}_{2r_0}}\chi\left(\frac{|y|}{r_0}\right)\left|\nabla h\left(\frac{1}{\tau_i\varepsilon}y\right)\right|^2dy\le\mathcal{O}\left(\int_{\Omega_i^\varepsilon}|\nabla h\left(y\right)|^2dy\right)\le\mathcal{O}(1).
\end{aligned}$$

The claim above concluded.

Combining the result in Step 1 and $\|\psi\|=o(\frac{1}{|\log\varepsilon|})$,

$$\|w\|+\|\psi\|=o(1), \tag{3.1.19}$$

as $\varepsilon\to 0$. Assume that $0<\varepsilon<\frac{r_0}{\tau_i R_0}$, as $\varepsilon\to 0$

$$\begin{aligned}
&\int_{\mathbb{R}_{\xi_i}}\nabla\tilde{\phi}_i^\varepsilon\nabla h dy=\int_{B^{\xi_i}_{2r_0}}\nabla\left(\chi\left(\frac{|y|}{r_0}\right)\phi\circ y_{\xi_i}^{-1}(y)\right)\cdot\nabla h\left(\frac{1}{\tau_i\varepsilon}y\right)dy \\
\overset{(3.1.16)}{=}\;&\int_{B^{\xi_i}_{2r_0}}\nabla\phi\circ y_{\xi_i}^{-1}(y)\cdot\nabla\left(\chi\left(\frac{|y|}{r_0}\right)h\left(\frac{1}{\tau_i\varepsilon}y\right)\right)dy=\int_\Sigma\left\langle\nabla\phi,\nabla\left(\chi_i h\left(\frac{1}{\tau_i\varepsilon}y_{\xi_i}\right)\right)\right\rangle_g dv_g \\
\overset{(3.1.5)}{=}\;&-\beta\int_\Sigma\chi_i h\left(\frac{1}{\tau_i\varepsilon}y_{\xi_i}\right)\phi dv_g+\int_\Sigma\varepsilon^2\chi_i Ve^{\sum_{h=1}^m PU_h}\phi h\left(\frac{1}{\tau_i\varepsilon}y_{\xi_i}\right)dv_g \\
&-\frac{1}{|\Sigma|_g}\int_\Sigma\varepsilon^2 Ve^{\sum_{h=1}^m PU_h}\phi dv_g\int_\Sigma\chi_i h\left(\frac{1}{\tau_i\varepsilon}y_{\xi_i}\right)dv_g \\
&+\int_\Sigma\left\langle\nabla(w+\psi),\nabla\left(\chi_i h\left(\frac{1}{\tau_i\varepsilon}y_{\xi_i}\right)\right)\right\rangle_g dv_g \\
=\;&-\tau_i^2\varepsilon^2\beta\int_{\mathbb{R}_{\xi_i}}\tilde{\phi}_i^\varepsilon(y)h(y)dy+\int_\Sigma\varepsilon^2\chi_i Ve^{\sum_{h=1}^m PU_h}\phi h\left(\frac{1}{\tau_i\varepsilon}y_{\xi_i}(x)\right)dv_g(x) \\
&-\tau_i^2\varepsilon^2\int_\Sigma\varepsilon^2 Ve^{\sum_{h=1}^m PU_h}\phi dv_g(x)\int_{\mathbb{R}_{\xi_i}}\chi\left(\frac{\tau_i\varepsilon|y|}{r_0}\right)e^{\varphi_i(\tau_i\varepsilon y)}h(y)dy+o(1),
\end{aligned}$$

for any $h\in C_c^\infty(\mathbb{R}^2)$ with $\|h\|\le 1$. By the Hölder inequality and Lemma A.0.7,

$$\left|\int_{\mathbb{R}_{\xi_i}}\tilde{\phi}_i^\varepsilon(y)h(y)dy\right|\le\|\tilde{\phi}_i^\varepsilon(y)\|_{L_i}\left(\int_{\mathbb{R}_{\xi_i}}(1+|y|^2)^2|h(y)|dy\right)^{\frac{1}{2}}\le C\|\tilde{\phi}_i^\varepsilon(y)\|_{L_i}\|h\|,$$

where $C>0$ is a constant depending on R_0, and

$$\begin{aligned}
&\left|\varepsilon^2\int_\Sigma\varepsilon^2\chi_i Ve^{\sum_{h=1}^m PU_h}\phi\right|\le\varepsilon^2\left\|\varepsilon^2\chi_i Ve^{\sum_{h=1}^m PU_h}\right\|_p\|\phi\| \\
\le\;&\varepsilon^2\left(\left\|\varepsilon^2\chi_i Ve^{\sum_{h=1}^m PU_h}-\varepsilon^2\sum_{i=1}^m\chi_i e^{U_i}\right\|_p+\left\|\varepsilon^2\sum_{i=1}^m\chi_i e^{U_i}\right\|_p\right)\|\phi\| \\
=\;&\mathcal{O}\left(\varepsilon^{\frac{2}{p}}\|\phi\|\right).
\end{aligned}$$

Applying Lemma A.0.1 and (3.1.4),

$$
\begin{aligned}
&\textstyle\int_{\Sigma} \varepsilon^2 \chi_i V e^{\sum_{h=1}^m PU_h} \phi h\left(\frac{1}{\tau_i \varepsilon} y_{\xi_i}(x)\right) dv_g(x) \\
= \;& \textstyle\int_{U_{2r_0}(\xi_i)} \frac{8\tau_i^2\varepsilon^2\chi_i}{\left(\tau_i^2\varepsilon^2+\left|y_{\xi_i}\right|^2\right)^2} \exp\{-\log(8\tau_i^2) + \varrho\left(\xi_i\right) H\left(\xi_i,\xi_i\right) \\
&\textstyle+ \sum_{h\neq i} \varrho\left(\xi_h\right) G\left(\xi_i,\xi_h\right) + \log V\left(\xi_i\right) + \mathcal{O}\left(\left|y_{\xi_i}\right| + \varepsilon^{1+\alpha_0}\right)\} \phi h\left(\frac{1}{\tau_i\varepsilon} y_{\xi_i}\right) dv_g \qquad (3.1.20) \\
= \;& \textstyle\int_{B_{r_0}^{\xi_i}} e^{\hat{\varphi}_i(y)} \frac{8\tau_i^2\varepsilon^2}{\left(\tau_i^2\varepsilon^2+|y|^2\right)^2} \phi \circ y_{\xi_i}^{-1}(y) \chi\left(\frac{|y|}{r_0}\right) h\left(\frac{1}{\tau_i\varepsilon} y\right) dy + o(1) \\
= \;& \textstyle\int_{\mathbb{R}_{\xi_i}} \frac{8}{(1+|y|^2)^2} \tilde{\phi}_i^\varepsilon h(y) dy + o(1).
\end{aligned}
$$

Then, $\tilde{\phi}_i^0$ is a distribution solution for

$$
-\Delta U = \frac{8}{(1+|y|^2)^2} U \text{ in } \mathbb{R}_{\xi_i} \text{ with } \int_{\mathbb{R}_i^2} |\nabla U|^2 dy < \infty. \qquad (3.1.21)
$$

with boundary condition $\partial_{\nu_0} U = 0$ on $\partial\mathbb{R}_{\xi_i}$, where ν_0 is the unit outward normal of $\partial\mathbb{R}_{\xi_i}$. By the regularity theory, $\tilde{\phi}_i^0$ is a smooth solution. It is well-known that any solutions to problem (3.1.21) take the form (see [EGP05, Lemma D.1], for instance) $\tilde{\phi}_i^0(y) = \frac{a_0^i(1-|y|^2)}{1+|y|^2} + \sum_{j=1}^{\mathrm{i}(\xi_i)} \frac{a_j^i y_j}{1+|y|^2}$, where $a_j^i \in \mathbb{R}$ for $i = 1, \cdots, m, j = 0, \cdots, \mathrm{i}(\xi_i)$.

Applying the result from Step 2

$$
\begin{aligned}
&\frac{16}{\tau_i} \int_{\mathbb{R}_{\xi_i}} \frac{|y|^2-1}{(|y|^2+1)^3} \tilde{\phi}_i^0(y) dy = \lim_{\varepsilon\to 0} \frac{16}{\tau_i} \int_{\Omega_i^\varepsilon} \frac{|y|^2-1}{(|y|^2+1)^3} \phi_i^\varepsilon \chi_i^\varepsilon dy \\
= \;& \lim_{\varepsilon\to 0} 16\tau_i\varepsilon^2 \int_{B_{2r_0}^{\xi_i}} \frac{|y|^2-\tau_i^2\varepsilon^2}{(|y|^2+\tau_i^2\varepsilon^2)^3} \phi \circ y_{\xi_i}^{-1}(y) \chi\left(\frac{|y|}{r_0}\right) dy \\
= \;& \lim_{\varepsilon\to 0} \int_{B_{2r_0}^{\xi_i}} \varepsilon^2 e^{u_{\tau_i,0}} \psi_{\tau_i,0}^0 \phi \circ y_{\xi_i}^{-1}(y) \chi(|y|) dy = \lim_{\varepsilon\to 0} \int_{\Sigma} \varepsilon^2 \chi_i e^{-\varphi_i} e^{U_i} \Psi_i^0 \phi dv_g \\
= \;& \lim_{\varepsilon\to 0} \langle P\Psi_i^0, \phi \rangle = 0.
\end{aligned}
$$

For any $i = 1, \cdots, m$ and $j = 1, \cdots, \mathrm{i}(\xi_i)$,

$$
\begin{aligned}
&\frac{32}{\tau_i\varepsilon} \int_{\mathbb{R}_{\xi_i}} \frac{y_j}{(|y|^2+1)^3} \tilde{\phi}_i^0 dy = \lim_{\varepsilon\to 0} \frac{32}{\tau_i\varepsilon} \int_{\Omega_i^\varepsilon} \frac{y_j}{(|y|^2+1)^3} \phi_i^\varepsilon \chi_i^\varepsilon dy \\
= \;& \lim_{\varepsilon\to 0} 32\tau_i^2\varepsilon^2 \int_{B_{2r_0}^{\xi_i}} \frac{y_j}{(|y|^2+\tau_i^2\varepsilon^2)^3} \phi \circ y_{\xi_i}^{-1}(y) \chi\left(\frac{|y|}{r_0}\right) dy \\
= \;& \lim_{\varepsilon\to 0} \int_{B_{2r_0}^{\xi_i}} \varepsilon^2 \chi\left(\frac{|y|}{r_0}\right) e^{u_{\tau_i,0}} \psi_{\tau_i,0}^j \phi \circ y_{\xi_i}^{-1}(y) dy \\
= \;& \lim_{\varepsilon\to 0} \int_{U_{2r_0}(\xi_i)} \varepsilon^2 \chi_i e^{U_i} e^{-\varphi_i} \Psi_i^j \phi(x) dv_g = \lim_{\varepsilon\to 0} \langle P\Psi_i^j, \phi \rangle = 0.
\end{aligned}
$$

Hence, for any $i = 1, \cdots, m, j = 1, \cdots, \mathrm{i}(\xi_i)$, $\int_{\mathbb{R}_{\xi_i}} \frac{|y|^2-1}{(|y|^2+1)^3} \tilde{\phi}_i^0 dy = \int_{\mathbb{R}_{\xi_i}} \frac{y_j}{(|y|^2+1)^3} \tilde{\phi}_i^0 dy = 0$. This implies that $\tilde{\phi}_i^0 \equiv 0$, which contradicts to (3.1.15). $\square$

Fixing $\sum_{i=1}^m PU_i$, we try to obtain the solution of

$$\Pi_\xi^\perp \left(\sum_{i=1}^m PU_i + \phi_\xi^\varepsilon - i^*(\varepsilon^2 e^{\sum_{i=1}^m PU_i + \phi_\xi^\varepsilon}) \right) = 0,$$

for $\phi_\xi^\varepsilon \in K_\xi^\perp$ applying the fixed-point theorem. Then, the problem is reduced to a finite-dimensional one.

Proposition 3.1.4. *For any $\delta > 0$, and $\xi = (\xi_1, \cdots, \xi_m) \in \Xi_{k,m}^\delta$. For any $p \in (1, \frac{6}{5})$ there exist $\varepsilon_0 > 0$ and $R > 0$ (uniformly in ξ) such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists a unique $\phi_\xi^\varepsilon \in K_\xi^\perp$ such that*

$$\Pi_\xi^\perp \left[\sum_{i=1}^m PU_i + \phi_\xi^\varepsilon - i^* \left(\varepsilon^2 e^{\sum_{i=1}^m PU_i + \phi_\xi^\varepsilon} \right) \right] = 0, \tag{3.1.22}$$

with $\|\phi_\xi^\varepsilon\| \le R\varepsilon^{\frac{2-p}{p}} |\log \varepsilon|$.

Proof. Define an operator $T_\xi^\varepsilon : K_\xi^\perp \to \mathbb{R}$ as follows:

$$\begin{aligned} T_\xi^\varepsilon(\phi) &= \left[\left(L_\xi^\varepsilon \right)^{-1} \circ \Pi_\xi^\perp \circ i^* \right] M_\xi^\varepsilon(\phi), \\ M_\xi^\varepsilon(\phi) &= \varepsilon^2 e^{\sum_{i=1}^m PU_i} \left[e^\phi - 1 - \phi \right] + \left[\varepsilon^2 e^{\sum_{i=1}^m PU_i} - \varepsilon^2 \sum_{i=1}^m \chi_i e^{-\varphi_{\xi_i}} e^{U_i} \right], \end{aligned}$$

for $\phi \in K_\xi^\perp$. We observe that $i^*(\varepsilon^2 \sum_{i=1}^m \chi_i e^{-\varphi_{\xi_i}} e^{U_i}) = \sum_{i=1}^m PU_i$. It is obvious that ϕ is a fixed point of T_ξ^ε if and only if ϕ solves (3.1.22) on $K_\xi^\perp$.

Applying Lemma 3.1.1, Lemma A.0.7 and Lemma A.0.8 and the Moser-Trudinger inequality, we obtain

$$\begin{aligned} \|T_\xi^\varepsilon(\phi)\| &\le C|\log \varepsilon| \|i^* \circ M_\xi^\varepsilon(\phi)\| \le C|\log \varepsilon| \left\| M_\xi^\varepsilon(\phi) \right\|_p \\ &\le C|\log \varepsilon| \left(\left\| \varepsilon^2 V e^{\sum_{i=1}^m PU_i} (e^\phi - 1 - \phi) \right\|_p \right. \\ &\quad \left. + \left\| \varepsilon^2 V e^{\sum_{i=1}^m PU_i} - \varepsilon^2 \sum_{i=1}^m \chi_i e^{U_i} \right\|_p \right) \\ &\le C|\log \varepsilon| \left(\|\phi\|^2 e^{c_2 \|\phi\|^2} \varepsilon^{\frac{2-2pr}{pr}} + \varepsilon^{\frac{2-p}{p}} \right), \end{aligned}$$

where $c_2 > 0$ is a constant, $r > 1$ is sufficiently close to 1, and $p \in (1, \frac{6}{5})$. We then fix arbitrary $p \in (1, \frac{6}{5})$ and choose $R > 0$ large enough such that $C(1 + e^{c_2}) \le R$. Next, we select $\varepsilon_1 > 0$ such that $\max \left\{ R\varepsilon^{\frac{2-2pr}{pr} + \frac{2-p}{p}} |\log \varepsilon|, R\varepsilon^{\frac{2-p}{p}} |\log \varepsilon| \right\} \le 1$ for all $\varepsilon \in (0, \varepsilon_1)$. Consequently, for any

$\|\phi\| \leq R\varepsilon^{\frac{2-p}{p}}|\log\varepsilon|$, we have $\|T_\xi^\varepsilon\| \leq R\varepsilon^{\frac{2-p}{p}}|\log\varepsilon|$. And similarly, by Lemma A.0.8 we deduce that

$$\begin{aligned}\|T_\xi^\varepsilon(\phi_1) - T_\xi^\varepsilon(\phi_2)\| &\leq C'|\log\varepsilon| \left\|\varepsilon^2 e^{\sum_{i=1}^m PU_i}(e^{\phi_1} - e^{\phi_2} - (\phi_1 - \phi_2))\right\|_p \\ &\leq C'|\log\varepsilon| e^{c_2(\sum_{j=1}^2 \|\phi_j\|^2)}\varepsilon^{\frac{2-2pr}{pr}}(\sum_{j=1}^2 \|\phi_j\|)\|\phi_1 - \phi_2\| \\ &\leq 2RC'e^{2c_2}\varepsilon^{\frac{2-2pr}{pr}+\frac{1+\alpha_0-p}{p}}\log^2\varepsilon\|\phi_1 - \phi_2\| \leq \frac{1}{2}\|\phi_1 - \phi_2\|,\end{aligned}$$

uniformly for any $\varepsilon \in (0, \varepsilon_2)$ and $\xi \in \Xi_{k,m}^\delta$, where $\varepsilon_2 > 0$ is chosen such that

$$\max\left\{R\varepsilon^{\frac{2-p}{p}}|\log\varepsilon|, 2RC'e^{c_2}\varepsilon^{\frac{2-2pr}{pr}+\frac{2-p}{p}}\log^2\varepsilon\right\} < \frac{1}{2},$$

for any $\varepsilon \in (0, \varepsilon_2)$. We then take $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. Thus, $T_\xi^\varepsilon(\phi)$ is a contraction map on $\left\{\phi \in K_\xi^\perp : \|\phi\| \leq R\varepsilon^{\frac{2-p}{p}}|\log\varepsilon|\right\}$. By the contracting-mapping principle, there exists a unique fixed point of T_ξ^ε on $\left\{\phi \in c : \|\phi\| \leq R\varepsilon^{\frac{2-p}{p}}|\log\varepsilon|\right\}$. □

3.1.3 The reduced functional and its expansion

The associated functional $E_\varepsilon : \overline{\mathrm{H}}^1(\Sigma) \to \mathbb{R}$ of the problem (3.1.1) is defined as following:

$$E_\varepsilon(u) = \frac{1}{2}\int_\Sigma (|\nabla u|_g^2 + \beta|u|^2)dv_g - \varepsilon^2\int_\Sigma Ve^u dv_g, \tag{3.1.23}$$

for any $u \in \overline{\mathrm{H}}^1(\Sigma)$. We expect to find a solution u with the form $\sum_{i=1}^m PU_i + \phi_\xi^\varepsilon$, where ϕ_ξ^ε is obtained by Proposition 3.1.4. Then substituting it in the energy functional E_ε, we have the reduced functional $\tilde{E}_\varepsilon(\xi) := E_\varepsilon(\sum_{i=1}^m PU_i + \phi_\xi^\varepsilon)$ with $\|\phi_\xi^\varepsilon\| \leq R\varepsilon^{\frac{2-p}{p}}|\log\varepsilon|$, i.e.

$$\begin{aligned}\widetilde{E}_\varepsilon(\xi) : &= \frac{1}{2}\int_\Sigma \left(\left|\nabla\left(\sum_{i=1}^m PU_i + \phi_\xi^\varepsilon\right)\right|_g^2 + \beta\left|\sum_{i=1}^m PU_i + \phi_\xi^\varepsilon\right|^2\right)dv_g \\ &\quad -\varepsilon^2\int_\Sigma Ve^{\sum_{i=1}^m PU_i + \phi_\xi^\varepsilon}dv_g.\end{aligned}$$

The reduced functional $\tilde{E}_\varepsilon : \Xi_{k,m} \to \mathbb{R}$ admits a C^1-expansion with respect to ξ in the form of the following asymptotic expansion:

Proposition 3.1.5. *As* $\varepsilon \to 0$,

$$\widetilde{E}_\varepsilon(\xi) = \sum_{i=1}^m \varrho(\xi_i)(3\log 2 - 2\log\varepsilon) - 2\sum_{i=1}^m \varrho(\xi_i) - \frac{1}{2}\mathcal{F}_{k,m}^V(\xi) + o(1),$$

which is C^1-uniformly in any compact sets of $\Xi_{k,m}$.

Proof. Denote $\phi = \phi^{\varepsilon}_{\xi}$ to simplify the notation. Then

$$\begin{aligned}
\widetilde{E}_{\varepsilon}(\xi) \quad = \quad & \frac{1}{2}\left(\sum_{i=1}^{m}\langle PU_i, PU_i\rangle + \sum_{i\neq j}\langle PU_i, PU_j\rangle\right) + \frac{1}{2}\left(\|\phi\|^2 + 2\sum_{i=1}^{m}\langle PU_i, \phi\rangle\right) \\
& - \int_{\Sigma}\varepsilon^2 V e^{\sum_{i=1}^{m} PU_i} dv_g - \int_{\Sigma}\varepsilon^2\left(Ve^{\sum_{i=1}^{m} PU_i+\phi} - Ve^{\sum_{i=1}^{m} PU_i}\right) dv_g.
\end{aligned}$$

We notice that $|e^s - 1| \leq e^{|s|}|s| (\forall s \in \mathbb{R})$. By Lemma A.0.2, we deduce that

$$\begin{aligned}
& \left|\int_{\Sigma}\varepsilon^2\left(Ve^{\sum_{i=1}^{m} PU_i+\phi} - Ve^{\sum_{i=1}^{m} PU_i}\right) dv_g\right| \leq \left|\int_{\Sigma}\varepsilon^2 V e^{\sum_{i=1}^{m} PU_i} e^{|\phi|}|\phi| dv_g\right| \\
\leq \quad & \mathcal{O}\left(\varepsilon^2\left(\int_{\Sigma} e^{r\sum_{i=1}^{m} PU_i} dv_g\right)^{1/r}\left\|e^{|\phi|}\right\|_a\|\phi\|_b\right) \leq \mathcal{O}\left(\left\|\varepsilon^2 V e^{\sum_{i=1}^{m} PU_i}\right\|_r\|\phi\|\right) = o(1),
\end{aligned}$$

as $\varepsilon \to 0$, where $r \in (1,2)$ with $\frac{1}{a} + \frac{1}{r} + \frac{1}{b} = 1$ and $\frac{2(1-r)}{r} + \frac{2-p}{p} > 0$. By Lemma A.0.9 and Lemma A.0.10, as $\varepsilon \to 0$,

$$\widetilde{E}_{\varepsilon}(\xi) = \sum_{i=1}^{m}\varrho(\xi_i)(3\log 2 - 2\log\varepsilon) - 2\sum_{i=1}^{m}\varrho(\xi_i) - \frac{1}{2}\mathcal{F}^{V}_{k,m}(\xi) + o(1). \tag{3.1.24}$$

By (3.1.22), we deduce that

$$\sum_{i=1}^{m} PU_i + \phi - i^*\left(\varepsilon^2 V e^{\sum_{i=1}^{m} PU_i+\phi}\right) = \sum_{s=1}^{m}\sum_{t=1}^{\mathrm{i}(\xi_s)} c^{\varepsilon}_{st} P\Psi^t_s, \tag{3.1.25}$$

where c^{ε}_{st}'s are coefficients. We combine (3.1.25) with Lemma A.0.1, Lemma A.0.15, and Remark A.0.6 to deduce the following estimate: as $\varepsilon \to 0$,

$$\sum_{s=1}^{m}\sum_{t=1}^{\mathrm{i}(\xi_s)}|c^{\varepsilon}_{st}| = \mathcal{O}(\varepsilon^2). \tag{3.1.26}$$

For the C^1-expansion, applying Lemma A.0.3 and Lemma A.0.15, we deduce that

$$\begin{aligned}
& \partial_{(\xi_h)_j} E_{\varepsilon}\left(\sum_{i=1}^{m} PU_i + \phi\right) \\
= \quad & \left\langle\sum_{i=1}^{m} PU_i + \phi - i^*\left(\varepsilon^2 \mathrm{e}^{\sum_{i=1}^{m} PU_i+\phi}\right), \partial_{(\xi_h)_j} PU_h + \sum_{i=1}^{m} P\Psi^0_i \partial_{(\xi_h)_j}\tau_i(\xi) + \partial_{(\xi_h)_j}\phi\right\rangle \\
= \quad & -\frac{1}{2}\frac{\partial\mathcal{F}^V_{k,m}}{\partial(\xi_h)_j}(\xi_1, \cdots, \xi_m) + \left\langle\sum_{s=1}^{m}\sum_{t=1}^{\mathrm{i}(\xi_s)} c^{\varepsilon}_{st} P\Psi^t_s, \sum_{i=1}^{m} P\Psi^0_i \partial_{(\xi_h)_j}\tau_i(\xi) + \partial_{(\xi_h)_j}\phi\right\rangle + o(1) \\
= \quad & -\frac{1}{2}\frac{\partial\mathcal{F}^V_{k,m}}{\partial(\xi_h)_j}(\xi_1, \cdots, \xi_m) + \sum_{s=1}^{m}\sum_{t=1}^{\mathrm{i}(\xi_s)} c^{\varepsilon}_{st}\left\langle P\Psi^t_s, \sum_{i=1}^{m} P\Psi^0_i \partial_{(\xi_h)_j}\tau_i(\xi) + \partial_{(\xi_h)_j}\phi\right\rangle + o(1),
\end{aligned}$$

for any $h = 1, 2, \cdots, m$ and $j = 1, \cdots, \mathrm{i}(\xi_h)$. Utilizing Lemma A.0.5, we have

$$|\langle P\Psi_s^t, P\Psi_i^0\rangle| \le \|P\Psi_s^t\|\|P\Psi_i^0\| = O\left(\frac{1}{\varepsilon}\right).$$

Considering the fact that $\langle P\Psi_s^t, \phi\rangle = 0$ and $|\partial_{(\xi_h)_j} P\Psi_s^t| \le |\partial_{(\xi_h)_j}\Psi_s^t| = \mathcal{O}(\frac{1}{\varepsilon^2})$, we obtain

$$\begin{aligned}\langle P\Psi_s^t, \partial_{(\xi_h)_j}\phi\rangle &= \partial_{(\xi_h)_j}\langle P\Psi_s^t, \phi\rangle - \langle \partial_{(\xi_h)_j} P\Psi_s^t, \phi\rangle \le O\left(\|\phi\| \left\|\partial_{(\xi_h)_j} P\Psi_s^t\right\|\right) \\ &= O\left(\tfrac{\|\phi\|}{\varepsilon^2}\right) = o\left(\tfrac{1}{\varepsilon^2}\right),\end{aligned}$$

through direct calculation. Consequently, we have

$$\left\langle \sum_{s=1}^{m}\sum_{t=1}^{\mathrm{i}(\xi_s)} c_{st}^{\varepsilon} P\Psi_s^t, \sum_{i=1}^{m} \partial_{(\xi_h)_j}\tau_i(\xi) P\Psi_i^0 + \partial_{(\xi_h)_j}\phi \right\rangle = o\left(\frac{1}{\varepsilon^2}\sum_{s=1}^{m}\sum_{t=1}^{\mathrm{i}(\xi_s)} |c_{st}^{\varepsilon}|\right). \tag{3.1.27}$$

It follows that

$$\partial_{(\xi_h)_j}\tilde{E}_{\varepsilon}(\xi) = -\frac{1}{2}\frac{\partial \mathcal{F}_{k,m}^{V}}{\partial\,(\xi_h)_j}(\xi_1, \cdots, \xi_m) + o\left(\frac{1}{\varepsilon^2}\sum_{s=1}^{m}\sum_{t=1}^{\mathrm{i}(\xi_s)} |c_{st}^{\varepsilon}|\right). \tag{3.1.28}$$

Then, (3.1.26) and (3.1.28) imply that for any $h = 1, \cdots, m$ and $j = 1, \cdots, \mathrm{i}(\xi_h)$, as $\varepsilon \to 0$

$$\partial_{(\xi_h)_j} E_{\varepsilon}\left(\sum_{i=1}^{m} PU_i + \phi\right) = -\frac{1}{2}\frac{\partial \mathcal{F}_{k,m}^{V}}{\partial\,(\xi_h)_j}(\xi_1, \cdots, \xi_m) + o(1).$$

□

The following proposition states that $\sum_{i=1}^{m} PU_i + \phi_{\xi}^{\varepsilon}$ being a critical point of E_{ε} in $\overline{\mathrm{H}}^1(\Sigma)$ is equivalent to ξ being a critical point of $\tilde{E}_{\varepsilon}$ in $\Xi_{k,m}$.

Proposition 3.1.6. *The function* $\sum_{i=1}^{m} PU_{\tau_i(\xi),\xi_i} + \phi_{\xi}^{\varepsilon}$ *is a solution of* (3.1.1) *if and only if* ξ *is a critical point of the reduced function*

$$\xi \mapsto \tilde{E}_{\varepsilon}(\xi) = E_{\varepsilon}\left(\sum_{i=1}^{m} PU_{\tau_i(\xi),\xi_i} + \phi_{\xi}^{\varepsilon}\right).$$

Proof. Denote $\phi := \phi_{\xi}^{\varepsilon}$ to simplify the notation. Assume that ξ is a critical point of the reduced function $\tilde{E}_{\varepsilon}$ in $\Xi_{k,m}$. Then ξ satisfies

$$\partial_{(\xi_i)_j}\tilde{E}_{\varepsilon}(\xi) = 0, \tag{3.1.29}$$

for any $i = 1, \cdots, m$ and $j = 1, \cdots, \mathrm{i}(\xi_i)$. According to (3.1.22) in Proposition 3.1.4, the following identity holds:

$$\sum_{h=1}^{m} PU_h + \phi - i^* \left(\varepsilon^2 V e^{\sum_{h=1}^{m} PU_h + \phi}\right) = \sum_{s=1}^{m} \sum_{t=1}^{\mathrm{i}(\xi_s)} c_{st}^{\varepsilon} P\Psi_s^t,$$

where c_{st}^{ε}'s are coefficients. Then, (3.1.29) becomes

$$\left\langle \sum_{s=1}^{m} \sum_{t=1}^{\mathrm{i}(\xi_s)} c_{st}^{\varepsilon} P\Psi_s^t, \partial_{(\xi_i)_j} PU_i + \sum_{h=1}^{m} P\Psi_i^0 \partial_{(\xi_i)_j} \tau_h(\xi) + \partial_{(\xi_i)_j} \phi \right\rangle = 0. \tag{3.1.30}$$

By (3.1.27) in Proposition 3.1.5 and (3.1.30),

$$\sum_{s=1}^{m} \sum_{t=1}^{\mathrm{i}(\xi_s)} c_{st}^{\varepsilon} \left\langle P\Psi_s^t, \partial_{(\xi_i)_j} PU_i \right\rangle = o\left(\frac{1}{\varepsilon^2} \sum_{s=1}^{m} \sum_{t=1}^{\mathrm{i}(\xi_s)} |c_{st}^{\varepsilon}| \right).$$

By Remark A.0.6, we conclude that $c_{ij}^{\varepsilon} = 0$ for any $i = 1, \cdots, m$ and $j = 1, \cdots, \mathrm{i}(\xi_i)$. Hence,

$$\sum_{h=1}^{m} PU_h + \phi - i^* \left(\varepsilon^2 \mathrm{e}^{\sum_{h=1}^{m} PU_h + \phi}\right) = 0. \tag{3.1.31}$$

If $\sum_{h=1}^{m} PU_h + \phi_{\xi}^{\varepsilon}$ is a weak solution of (3.1.1) in $\overline{\mathrm{H}}^1(\Sigma)$, then (3.1.31) is satisfied. Furthermore, (3.1.29) leads to the conclusion that ξ is a critical point of the reduced functional $\tilde{E}_{\varepsilon}$. □

3.2 Proof of main results

Now, we are ready to prove the main results.

Proof of Theorem 1.1.1. Let K be a stable critical point set of $\mathcal{F}_{k,m}^{V}$. As $\varepsilon \to 0$ there exist a sequence of points $\xi^{\varepsilon} = (\xi_1^{\varepsilon}, \cdots, \xi_m^{\varepsilon}) \in \Xi_{k,m}$ such that $\operatorname{dist}(\xi^{\varepsilon}, K) \to 0$ and ξ_{ε} is a critical point of $\tilde{E}_{\varepsilon} : \Xi'_{k,m} \to \mathbb{R}$. Without loss of generality, assume that up to a subsequence

$$\xi^{\varepsilon} = (\xi_1^{\varepsilon}, \cdots, \xi_m^{\varepsilon}) \to \xi = (\xi_1, \cdots, \xi_m) \in K,$$

as $\varepsilon \to 0$. Then define $u_{\varepsilon} = \sum_{i=1}^{m} PU_{\tau_i(\xi^{\varepsilon}), \xi_i^{\varepsilon}} + \phi_{\xi^{\varepsilon}}^{\varepsilon}$. According to Proposition 3.1.6, u_{ε} solves (3.1.1) as $\varepsilon \to 0$, which means that u_{ε} solves problem (1.1.2) in the weak sense for some $\lambda := \lambda_{\varepsilon} = \varepsilon^2 \int_{\Sigma} V e^{u_{\varepsilon}} dv_g$. Applying Lemma A.0.2, LemmaA.0.8 and Lemma A.0.10, $\lambda = \lambda_{k,m} + o(1)$, as $\varepsilon \to 0$.

Claim. For any $\Psi \in C(\Sigma)$, $\varepsilon^2 \int_{\Sigma} V e^{u_{\varepsilon}} \Psi dv_g \to \sum_{i=1}^{m} \varrho(\xi_i) \Psi(\xi_i)$, *as* $\varepsilon \to 0$. In fact, by the inequality $|\mathrm{e}^s - 1| \leqslant \mathrm{e}^{|s|}|s|$ for any $s \in \mathbb{R}$ and Lemma A.0.7, we have

$$\begin{aligned}\varepsilon^2 \int_\Sigma V \mathrm{e}^{u_\varepsilon} \Psi dv_g &= \varepsilon^2 \int_\Sigma V \mathrm{e}^{\sum_{i=1}^m PU_i} \Psi dv_g + o(1) = \sum_{i=1}^m \int_\Sigma \varepsilon^2 \chi_{\xi_i} e^{U_i} \Psi dv_g + o(1) \\ &= \sum_{i=1}^m \varrho(\xi_i) \Psi\left(\xi_i\right) + o(1),\end{aligned}$$

as $\varepsilon \to 0$. Therefore, u_ε is a family of blow-up solutions of (1.1.2) as $\varepsilon \to 0$. The proof is concluded. □

Proof of Corollary 1.1.2. As demonstrated in the proof of Theorem 1.1.1, for any $\varepsilon > 0$ sufficiently small, we can construct $\xi^\varepsilon \in \Xi^\delta_{k,m}$ and λ_ε such that up to a subsequence $\xi^\varepsilon \to \xi \in K_{\min}$, $\lambda_\varepsilon \to \lambda_{k,m} = 4\pi(m+k)$, and $u_\varepsilon = \sum_{i=1}^m PU_{\tau_i(\xi^\varepsilon),\xi_i^\varepsilon} + \phi^\varepsilon_{\xi^\varepsilon}$ solving (1.1.2) for the parameter λ_ε. It follows that

$$\mathcal{F}^V_{k,m}(\xi^\varepsilon) \to \mathcal{F}^V_{k,m}(\xi) = \min_{\xi \in \Xi^\delta_{k,m}} \mathcal{F}^V_{k,m}(\xi), \quad \text{as } \varepsilon \to 0.$$

We recall the following expansion from Proposition 3.1.5,

$$\widetilde{E}_\varepsilon(\xi) = \sum_{i=1}^m \varrho(\xi_i)(3\log 2 - 2\log\varepsilon) - 2\sum_{i=1}^m \varrho(\xi_i) - \frac{1}{2}\mathcal{F}^V_{k,m}(\xi) + o(1)$$

in C^1-sense. As $\varepsilon \to 0$, u_ε is uniformly bounded on $\Sigma \setminus \cup_{i=1}^m U_\epsilon(\xi_i^\varepsilon)$ for sufficiently small $\epsilon > 0$ and $\sup_{U_\epsilon(\xi_i)} u_\varepsilon \to +\infty$, as $\varepsilon \to 0$. Lemma A.0.1 implies that

$$u_\varepsilon = -2\sum_{i=1}^m \chi\left(|y_{\xi_i^\varepsilon}(x)|/r_0\right) \log(\varepsilon^2\tau_i^2(\xi^\varepsilon) + |y_{\xi_i^\varepsilon}(x)|^2) + \mathcal{O}(1), \text{ as } \varepsilon \to 0.$$

Given that $\varepsilon > 0$ is sufficiently small, we have

$$\max_{U_{r_0}(\xi_i^\varepsilon)} u_\varepsilon = \max\left\{u_\varepsilon(x) : |y_{\xi_i^\varepsilon}(x)| < \sqrt{\varepsilon\tau_i(\xi^\varepsilon)}\right\} \text{ for } i = 1, \cdots, m.$$

Then, there exists $\tilde{\xi}_{\varepsilon,i}$ satisfying that $|y_{\xi_{\varepsilon,i}}(\tilde{\xi}_{\varepsilon,i})| < \sqrt{\varepsilon\tau_i(\xi^\varepsilon)}$ attaining the local maximum of u_ε for any $i = 1, \cdots, m$. Moreover, $\tilde{\xi}_\varepsilon := (\tilde{\xi}_{\varepsilon,1}, \cdots, \tilde{\xi}_{\varepsilon,m}) \to \xi$ and

$$\mathcal{F}^V_{k,m}(\tilde{\xi}_\varepsilon) \to \min_{\xi \in \Xi^\delta_{k,m}} \mathcal{F}^V_{k,m}(\xi), \text{ as } \varepsilon \to 0.$$

Applying Theorem 1.1.1, we can conclude the proof. □

Part II

Toda systems

4 Partial blow-up solutions for Toda systems

4.1 Properties of the shadow system

Let $C^{\alpha}_0(\Sigma)$ and $C^{s,\alpha}_0(\Sigma)$ be the spaces of Hölder continuous functions of class C^{α} and $C^{s,\alpha}$ with zero average over Σ for any $s = 1, 2, \alpha \in (0, 1)$, respectively.

Next, we study some important properties of the shadow system (1.2.5). Firstly, we establish a prior estimate for the solutions of (1.2.5).

We define the following shadow system with parameter $t \in [0, 1]$,

$$\begin{cases} -\Delta_g w^t = 2\rho_2 \left(\dfrac{V_2 e^{w^t - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(x,\xi_i)}}{\int_\Sigma V_2 e^{w^t - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(x,\xi_i)} dv_g} - 1 \right) & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} w^t = 0 & \text{on } \partial\Sigma \\ \partial f_t(\xi) = 0 & \text{in } \Xi_{k,m} \end{cases}, \tag{S$_t$}$$

where

$$f_t(\xi_1, \cdots, \xi_m) = \mathcal{F}^{V_1}_{k,m}(\xi) - \textstyle\sum_{i=1}^m (1-t)\varrho(\xi_i) w^t(\xi_i), \text{ for } \xi \in \Xi_{k,m}.$$

Proposition 4.1.1. *Given any $\alpha \in (0, 1)$, there exists a constant $C > 0$ such that for any $t \in [0, 1]$, any ρ_2 in a compact subset of $\mathbb{R}_+ \setminus 2\pi\mathbb{N}_+$ and any solution (w^t, ξ^t) of (S$_t$), we have the following prior estimate,*

$$\|w^t\|_{C^{2,\alpha}(\Sigma)} < C.$$

Proof. Define that

$$\tilde{w}^t = w^t - \log\left(\int_\Sigma V_2 e^{w^t - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(\cdot,\xi_i)} dv_g \right).$$

It follows that $\int_\Sigma V_2 e^{\tilde{w}^t - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(\cdot,\xi_i)} dv_g = 1$. We can rewrite the system (S_t) as follows:

$$\begin{cases} -\Delta_g \tilde{w}^t = 2\rho_2 \left(V_2 e^{\tilde{w}^t - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(x,\xi_i)} - 1 \right) & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} \tilde{w}^t = 0 & \text{on } \partial\Sigma \\ \nabla \left(\mathcal{F}_{k,m}^{V_1}(x) - \sum_{i=1}^m (1-t)\varrho(x_i)\tilde{w}^t(x_i) \right)\Big|_{x=\xi} = 0 & \text{in } \Xi_{k,m} \end{cases}, \tag{4.1.1}$$

For a sequence of $(\tilde{w}_n^t, \xi^n)$ solves (4.1.1) with the parameter $\rho_2^n \to \rho_2$ as $n \to +\infty$. We define that $w_n^t = \tilde{w}_n^t + \log\left(\int_\Sigma V_2 e^{w_t^n - \sum_{i=1}^m \frac{\varrho(\xi_i^n)}{2} G^g(\cdot,\xi_i^n)} dv_g \right)$. We define the blow-up point set as

$$\mathcal{B} = \left\{ x_0 \in \Sigma : \exists x_n \to x_0 \text{ s.t. } \lim_{n\to+\infty} \tilde{w}_n^t(x_n) = +\infty \right\},$$

and the local limit mass

$$\tilde{\sigma}(x) = \lim_{r\to 0} \lim_{n\to+\infty} \int_{U_r(x)} 2\rho_2^n V_2 e^{\tilde{w}_n^t - \sum_{i=1}^m \frac{1}{2}\varrho(\xi_i) G^g(\cdot,\xi_i)} dv_g.$$

Step 1. If $\rho_2 \notin 2\pi\mathbb{N}$, then $\mathcal{B} = \emptyset$. Furthermore, there exists constant $C > 0$ such that $\sup_\Sigma \tilde{w}_n^t \leq C$.

Claim 4.1.2. If $\tilde{\sigma}(x_0) < \frac{1}{2}\varrho(x_0)$, then there exists a small open neighborhood U_{x_0} around x_0 such that $e^{\tilde{w}_n^t} \in L^p(U_{x_0})$ for some $p > 1$ and $\tilde{w}_n^t \in L^\infty(U_{x_0})$. Consequently, $\tilde{\sigma}(x_0) = 0$ and $x_0 \notin \mathcal{B}$.

Proof of Claim 4.1.2. Using the isothermal coordinates, we define $\tilde{v}^n(y) := \tilde{w}_n^t \circ y_{x_0}^{-1}(y)$ and $\tilde{V}_i(y) := V_i \circ y_{x_0}^{-1}(y)$, we can obtain the local version of (4.1.17). Specifically,

$$-\Delta \tilde{v}^n(y) = 2\rho_2^n e^{\hat{\varphi}_{x_0}(y)} \left(\tilde{V}(y) e^{\tilde{v}^n(y) - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(y_{x_0}^{-1}(y),\xi_i)} - 1 \right) \text{ in } B^{x_0}, \tag{4.1.2}$$

and

$$\partial_{y_2} \tilde{v}^n(y) = 0 \text{ on } B^{x_0} \cap \{y \in \mathbb{R}^2 : y_2 = 0\}.$$

When $x_0 \in \partial\Sigma$, we extend $\tilde{v}^n$ to an open ball by reflection of x-axis (refer to [NT91]).

In distribution sense, for sufficiently small $r_0 > 0$, $\tilde{v}^n$ solves

$$-\Delta \tilde{v}^n(y) = \tilde{f}^n \text{ in } \mathbb{B}_{r_0}, \tag{4.1.3}$$

where $\tilde{f}^n = 2\rho_2^n e^{\hat{\varphi}_{x_0}(y)} \left(\tilde{V}(y) e^{\tilde{v}^n(y) - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(y_{x_0}^{-1}(y),\xi_i)} - 1 \right)$ in $\mathbb{B}_{r_0}$. For any real-valued function f, let $f_+ := \max\{0, f\}$. We decompose $(\tilde{v}^n)_+$ into two parts

$$(\tilde{v}^n)_+ = \tilde{v}_1^n + \tilde{v}_2^n, \text{ in } \mathbb{B}_r,$$

for any $r \in (0, r_0)$, where $\tilde{v}_1^n$ is the solution of

$$\begin{cases} -\Delta \tilde{v}_1^n = (\tilde{f}^n)_+ & \text{in } \mathbb{B}_r \\ \tilde{v}_1^n = 0 & \text{on } \partial\mathbb{B}_r \end{cases}$$

and $\tilde{v}_2^n$ is the solution of

$$\begin{cases} -\Delta \tilde{v}_2^n = 0 & \text{in } \mathbb{B}_r \\ \tilde{v}_2^n = (\tilde{v}^n)_+ & \text{on } \partial\mathbb{B}_r \end{cases}.$$

Applying the result given by Brezis and Merle in [BM91, Theorem 1] for $D = \mathbb{B}_r$, we have

$$\int_{\mathbb{B}_r} \exp\left(\frac{(4\pi - \epsilon)(\tilde{v}_1^n)_+}{\left\|(\tilde{f}^n)_+\right\|_{L^1(\mathbb{B}_r)}} \right) \leqslant C\frac{r^2}{\epsilon}$$

for some constant $C > 0$, for any $r \in (0, r_0)$ and $\epsilon \in (0, 4\pi)$.

Due to the assumption, for any $c_0 \in (0, 4\pi)$ there exist constants $r_1 > 0$ and $N \in \mathbb{N}_+$ such that

$$\int_{\mathbb{B}_r} 2\rho_2^n e^{\varphi_{x_0}} \tilde{V} e^{\tilde{v}^n - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(y_{x_0}^{-1}, \xi_i)} \leq 4\pi - c_0, \forall r \in (0, r_1), n \geq N.$$

Observe that there exists a constant $c_1 > 0$ such that $\int_{\mathbb{B}_r} 2\rho_2^n e^{\varphi_{x_0}} \leq c_1 \rho_2^n r^2$. Since $\rho_2^n \to \rho_2$, ρ_2^n is uniformly bounded. Combining the estimates above, we have

$$\int_{\mathbb{B}_r} |(\tilde{f}^n)_+| \leq 4\pi - c_0 + c_1 \sup_n |\rho_2^n| r^2.$$

We take $r^* = \min\left\{ r_0, r_1, \sqrt{\frac{c_0}{2c_1 \sup_{n \in \mathbb{N}} |\rho_2^n|}} \right\}$ and fix a $\epsilon \in (0, 4\pi)$ sufficiently small such that $4\pi - \epsilon > 4\pi - \frac{1}{4}c_0$. For any $n \geq N$ and $r \in (0, r^*)$, $\int_{\mathbb{B}_r} e^{p\tilde{v}^n} \leqslant C$, for some constant C, where $p = \frac{4\pi - \frac{1}{4}c_0}{4\pi - \frac{1}{2}c_0} > 1$. Thus, $(\tilde{f}^n)_+ \in L^p(\mathbb{B}_r)$. By the L^p-theory and Sobolev inequality,

$$\|\tilde{v}_1^n\|_{L^\infty\left(\mathbb{B}_{\frac{1}{2}r}\right)} \leq C,$$

for some constant $C > 0$. On the other hand, the maximum principle implies that $\tilde{v}_2^n > 0$. Given the estimate (4.1), we have $\int_{\mathbb{B}_{\frac{1}{2}r}} \tilde{v}_2^n \leq \int_{\mathbb{B}_{\frac{1}{2}r}} ((\tilde{v}^n)_+ + |\tilde{v}_1^n|) \leq C + \|\tilde{v}^n\|_{L^1(\mathbb{B}_{r^*})}$. Additionally, we observe that $\| - \Delta_g \tilde{w}_n^t \|_{L^1(\Sigma)}$ is uniformly bounded. Using the L^p-theory, we then have $\|\tilde{w}_n^t\|_q \leq C$, for any $q \in (1,2)$ and for some constant $C > 0$. Applying the mean-value theorem of harmonic functions, it follows that

$$\|\tilde{v}_2^n\|_{L^\infty\left(\mathbb{B}_{\frac{1}{4}r}\right)} \leq C \|\tilde{v}_2^n\|_{L^1\left(\mathbb{B}_{\frac{1}{2}r}\right)} \leq C(1 + \|\tilde{v}^n\|_{L^1(\mathbb{B}_{r^*})}) \leq C.$$

To conclude, we proved $\|(v^n)_+\|_{L^\infty(y_{x_0}^{-1}(\mathbb{B}_{\frac{1}{4}r} \cap B^{x_0})} \leq C$ for some constant $C > 0$, i.e. $\tilde{w}_t^n$ is bounded from above in a small neighborhood of x_0. Hence, Claim 4.1.2 is complete.

Given that $\int_\Sigma 2\rho_2^n V_2 e^{\tilde{w}_n^t(y) - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(y_{x_0}^{-1}(y), \xi_i)} \equiv 1$, Claim 4.1.2 indicates that $\mathcal{B}$ must be a finite set. Otherwise, the presence of infinitely many blow-up points in $\mathcal{B} \setminus \{\xi_i\}_{i=1}^m$, each with a local limit mass $\tilde{\sigma}(x) \geq \frac{1}{2}\varrho(x)$ for any $x \in \mathcal{B} \setminus \{\xi_i\}_{i=1}^m$. $2\rho_2 \geq \sum_{x \in \mathcal{B}} \frac{1}{2}\varrho(x)$, which leads to a contradiction.

For

$$-\Delta u(x) = a(x)F(u) + a_0(x), \quad \text{in } U \subset \mathbb{R}^2,$$

we have the following Pohozaev's identity:

$$\begin{aligned} &\textstyle\int_U 2(aF(u) + a_0 u) + \int_U (y \cdot \nabla a)F(u) + (y \cdot \nabla a_0)u \\ &= \textstyle\int_{\partial U} (y \cdot \nabla u)\frac{\partial u}{\partial \nu} - (y \cdot \nu)\frac{|\nabla u|^2}{2} + (y \cdot \nu)(aF(u) + a_0 u), \end{aligned}$$

where U is a smooth domain, $F(u) = \int_0^u f(s)\, ds$ and ν is the unit outward norm of ∂U.

Applying the Pohozaev's identity, we derive that

$$\begin{aligned} &\int_{\partial\mathbb{B}_r} \frac{r}{2}|\nabla \tilde{v}^n|^2 ds_r - r\int_{\partial\mathbb{B}_r} |\partial_\nu \tilde{v}^n|^2 ds_r \qquad (4.1.4) \\ = \quad & r\int_{\partial\mathbb{B}_r} 2\rho_2^n e^{\varphi_{x_0}} e^{\tilde{v}^n - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(y_{x_0}^{-1}(y), \xi_i)} ds_r - 4\int_{\mathbb{B}_r} \rho_2^n e^{\hat{\varphi}_{x_0}} \left(e^{\tilde{v}^n - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(y_{x_0}^{-1}(y), \xi_i)} - \tilde{v}^n \right) dy \\ & - \int_{\mathbb{B}_r} 2\rho_2^n e^{\tilde{v}^n} \langle \nabla (e^{\hat{\varphi}_{x_0}} \tilde{V} e^{-\sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(y_{x_0}^{-1}(y), \xi_i)}), y \rangle \, dy - 2\rho_2^n \int_{\mathbb{B}_r} e^{\hat{\varphi}_{x_0}} y \cdot \nabla \tilde{v}^n \, dy, \end{aligned}$$

where ds_r is the line element on $\partial\mathbb{B}_r$ and ν denotes the unit outward vector on $\partial\mathbb{B}_r$.

Suppose that $\mathcal{B} \neq \emptyset$, we will prove that

$$\int_\Sigma V_2 e^{w_n^t - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(\cdot, \xi_i)} dv_g \to +\infty.$$

Using the Jensen's inequality with $\int_\Sigma w_n^t dv_g = \int_\Sigma G^g(\cdot, \xi_i) dv_g = 0$ for any $i = 1, \cdots, m$, we can deduce that $\int_\Sigma V_2 e^{w_n^t - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(\cdot,\xi_i)} dv_g$ has a positive lower bound. Assume that

$$0 < c^* \leq \int_\Sigma V_2 e^{w_n^t - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(\cdot,\xi_i)} dv_g \leq C^*$$

for some constant $c^*, C^* > 0$.

Up to a subsequence, $V_2 e^{w_n^t - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(\cdot,\xi_i)} dv_g$ is convergent to some Borel measure denoted by μ in sense of measure.

For any $x_0 \in \mathcal{B}$, we choose $\epsilon_0 > 0$ such that $U_{\epsilon_0}(x_0) \cap (\{\xi_i\}_{i=1}^m \cup \mathcal{B} \setminus \{x_0\}) = \emptyset$, where $U_{\epsilon_0}(x_0)$ is definedd by the isothermal coordinates in Section 2.2. Since $\tilde{w}_n^t \in L^\infty_{loc}(\Sigma \setminus \mathcal{B})$, there exists a constant $c' > 0$ such that $\sup_{\partial U_{\epsilon_0}(x_0)} |\tilde{w}_t^n| \leq c'$. For $x_0 \in \partial\Sigma$, we apply the reflection of the x-axis to extend the functions on an open disk. We consider the following equations:

$$\begin{cases} -\Delta \tilde{h}^n = 2\rho_2^n e^{\hat{\varphi} x_0} \left(\tilde{V} e^{\tilde{v}^n - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(y_{x_0}^{-1}(y),\xi_i)} - 1 \right) & \text{in } \mathbb{B}_{\epsilon_0} \\ \tilde{h}^n = -c' & \text{on } \partial\mathbb{B}_{\epsilon_0} \end{cases}$$

By the maximal principle, we deduce that $\tilde{v}^n \geq \tilde{h}^n$ in $\mathbb{B}_{\epsilon_0}$. By the regularity theory, $\tilde{h}^n$ converges to some $\tilde{h}$ weakly in $W^{1,q}(\mathbb{B}_{\epsilon_0})$ for $q \in (1,2)$. It follows that $\tilde{h}$ solves

$$\begin{cases} -\Delta \tilde{h} = \mu & \text{in } \mathbb{B}_{\epsilon_0} \\ \tilde{h} = -c' & \text{on } \partial\mathbb{B}_{\epsilon_0} \end{cases}$$

in the sense of distribution. We consider the Green's function defined by the following equations:

$$\begin{cases} -\Delta G^*(x,x_0) = 4\pi(\delta_0 - 1), & \text{in } \mathbb{B}_{\epsilon_0} \\ G^*(x,x_0) = -c, & \text{on } \partial\mathbb{B}_{\epsilon_0} \end{cases}.$$

By the maximal principal and Claim 4.1.2, we derive that $h \geq G^*(y,x_0)$. Observe that

$$G^*(y,x_0) = 2\log|y| + A_{x_0}(y),$$

where A_{x_0} is a smooth function depending on x_0. Fatou's lemma yields that

$$+\infty = \int_{B_{\varepsilon_0}} G^*(\cdot,x_0) \leq \int_{\mathbb{B}_{\epsilon_0}} e^h \leq \liminf_{n\to+\infty} \int_{\mathbb{B}_{\epsilon_0}} e^{h_n} \leq \liminf_{n\to+\infty} \int_{\mathbb{B}_{\epsilon_0}} e^{\tilde{w}_n^t} \leq CC^*.$$

There is a contradiction. So $\int_\Sigma V_2 e^{w_n^t-\sum_{i=1}^m \frac{\varrho(\xi_i)}{2}G^g(\cdot,\xi_i)}dv_g \to +\infty$, which implies $\tilde{w}_n^t \to -\infty$ uniformly in any compact subset of $\Sigma \setminus \mathcal{B}$.

We recall the definition of $U_{\epsilon_0}(x_0) = y_{x_0}^{-1}(B_{\epsilon_0}^{x_0})$. By (4.1.2), we have

$$-\Delta \tilde{v}^n(y) = 2\rho_2^n \tilde{V}^*(y)|y|^{2\gamma(x_0)}e^{\tilde{v}^n(y)} - 2\rho_2^n, \text{ in } \mathbb{B}_{\epsilon_0}$$

where $V^* \in C^\infty(\mathbb{B}_{\epsilon_0}, \mathbb{R}_+)$ with a lower bound away from 0, and $\gamma(x_0) = \begin{cases} 1 & x_0 \in \{\xi_i\}_{i=1}^m \\ 0 & \text{otherwise} \end{cases}$.

Since $\int_\Sigma \left|V_2 e^{\tilde{w}_n^t-\sum_{i=1}^m \frac{\varrho(\xi_i)}{2}G^g(x,\xi_i)} - 1\right| \leq 2$, by the regularity theory, for any $q \in (1,2)$, we derive that $\|w_n^t\|_{W^{1,q}(\Sigma)} \leq C$, for some constant $C > 0$. For $2\rho_2 = \sum_{x\in\mathcal{B}} \tilde{\sigma}(x)$,

$$w_n^t \to \sum_{x\in\mathcal{B}} \tilde{\sigma}(x)G^g(\cdot, x),$$

weakly in $W^{1,q}(\Sigma)$ and strongly in $C^2_{loc}(\Sigma\backslash\mathcal{B})$, where G^g is the Green's function defined by (2.4.1) with $\beta = 0$. We observe that for any $x \in U_{\epsilon_0}(x_0)$, $-\Delta_g G^g(x,x_0) = \delta_{x_0} - 1$ and $G^g(x,x_0) = \frac{4}{\varrho(x_0)}\log\frac{1}{|y_{x_0}(x)|} + H^g(x,x_0)$, where $H^g(\cdot,x_0) \in C^2(U_{\epsilon_0}^{x_0})$. Passing the limit $n \to +\infty$ first and then $r \to 0$ for (4.1.4), we have

$$\tilde{\sigma}(x_0)^2 = \varrho(x_0)(1+\gamma(x_0))\tilde{\sigma}(x_0).$$

Since $x_0 \in \mathcal{B}$, $\tilde{\sigma}(x_0) > 0$. It follows $\tilde{\sigma}(x_0) = \varrho(x_0)(1+\gamma(x_0)) = \begin{cases} 8\pi(1+\gamma(x_0)) & \text{if } x_0 \in \mathring{\Sigma} \\ 4\pi(1+\gamma(x_0)) & \text{if } x_0 \in \partial\Sigma \end{cases}$.
Since $\gamma(x_0) = 0$ or 1, $2\rho_2 = \sum_{x\in\mathcal{B}}\tilde{\sigma}(x) \in 2\pi\mathbb{N}$. This leads to a contradiction since we assume that $\rho_2 \notin 2\pi\mathbb{N}$.

Step 2. For any $t \in [0,1], \alpha \in (0,1)$, and (w^t, ξ^t) a solution to (S_t),

$$\|w^t\|_{C^{2,\alpha}(\Sigma)} \leq C,$$

for some constant $C > 0$.

Step 1 implies that

$$\sup_\Sigma(w^t - \log(\int_\Sigma V_2 e^{w^t-\sum_{i=1}^m \frac{\varrho(\xi_i^t)}{2}G^g(\cdot,\xi_i^t)}dv_g))$$

is uniformly bounded from above. By the Schauder estimate in Lemma 2.3.3, it follows that $\|w^t\|_{C^{2,\alpha}(\Sigma)} \leq C$, for any $\alpha \in (0,1)$ and some constant $C > 0$. □

The shadow system is a singular mean field equation coupled with a balance condition. An intuitive approach to deducing the non-degenerate solutions is to first analyze the non-degeneracy of the singular mean field equation and then address the balance condition. We introduce the following hypothesis, which was originally introduced by [DPR15] to deal with Dirichlet boundary conditions, and have modified it here to be compatible with Neumann boundary conditions:

(H) *For some* $\xi^0 \in \Xi_{k,m}$*, there exists a non-degenerate solution of the problem* (1.2.3)*, i.e. there exists a solution of* (1.2.3)*, denoted by* $z(\cdot, \xi^0)$*, and the following linear problem:*

$$\begin{cases} -\Delta\psi(x) = 2\rho_2 \left(\dfrac{\tilde{V}_2(x,\xi^0) e^{z(x,\xi^0)} \psi(x)}{\int_\Sigma \tilde{V}_2(\cdot,\xi^0) e^{z(\cdot,\xi^0)} dv_g} - \dfrac{\tilde{V}_2(x,\xi^0) e^{z(x,\xi^0)} \int_\Sigma \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi^0)} \psi dv_g}{\left(\int_\Sigma \tilde{V}_2(\cdot,\xi^0) e^{z(\cdot,\xi^0)} dv_g\right)^2} \right) & \text{in } \mathring{\Sigma}, \\ \partial_{\nu_g} \psi = 0 & \text{on } \partial\Sigma \end{cases} \tag{4.1.5}$$

admits only the trivial solution $\psi = 0$.

Lemma 4.1.3. *Assuming the hypothesis (H) holds for* $(\xi^0, z(\cdot, \xi^0))$*, for any* $\alpha \in (0,1)$*, there exists an open neighborhood* $\mathscr{D}$ *around* ξ^0 *such that the map* $\xi \mapsto z(\cdot, \xi)$ *from* $\mathscr{D}$ *into* $C_0^{2,\alpha}(\Sigma)$ *is* C^1*-differentiable.*

Proof. We observe that for any $\alpha \in (0,1)$,

$$\tilde{V}_2(x,\xi) = V_2(x) e^{-\sum_{i=1}^m \frac{\varrho(\xi_i)}{2} H^g(x,\xi_i)} \Pi_{i=1}^m e^{2\chi_i(x) \log |y_{\xi_i}(x)|} \in C^\alpha(\Sigma),$$

uniformly for $\mathscr{D} \subset\subset \Xi_{k,m}$. Let

$$\Psi : C_0^{2,\alpha}(\Sigma) \times \Xi_{k,m} \to C_0^\alpha(\Sigma) \times \{0\},$$

$$\Psi(z,\xi) = \begin{bmatrix} -\Delta_g z(x) - 2\rho_2 \left(\frac{\tilde{V}_2(x,\xi) e^z}{\int_\Omega \tilde{V}_2(\cdot,\xi) e^z dv_g} - 1 \right) \\ \partial_{\nu_g} z \end{bmatrix}.$$

We observe that z is a solution of (1.2.3) if and only if $\Psi(z,\xi) = 0$. According to the hypothesis (H), for fixed ξ^0, there exists $z = z(\cdot, \xi^0)$ that is the non-degenerate solution of (1.2.3). Then we have $\Psi(z, \xi^0) = 0$ and for any $\psi \in C_0^{2,\alpha}(\Sigma)$

$$D_z \Psi(z, \xi^0)(\psi) = \begin{bmatrix} -\Delta_g \psi - K(\psi) \\ \partial_\nu \psi \end{bmatrix},$$

where $K : C_0^{2,\alpha}(\Sigma) \to C_0^{\alpha}(\Sigma)$,

$$\psi \mapsto 2\rho_2 \left(\frac{\tilde{V}_2(x,\xi^0)e^z\psi(x)}{\int_\Sigma \tilde{V}_2(\cdot,\xi^0)e^z dv_g} - \frac{\tilde{V}_2(x,\xi^0)e^z \int_\Sigma \tilde{V}_2(\cdot,\xi^0)e^z\psi dv_g}{(\int_\Sigma \tilde{V}_2(\cdot,\xi^0)e^z dv_g)^2} \right).$$

By the regularity theory in Section 2.3, the mapping $(-\Delta_g, \partial_{\nu_g}) : C_0^{2,\alpha}(\Sigma) \to C_0^{\alpha}(\Sigma) \times \{0\}$ is an isomorphism. It is noted that the operator K is compact, which preserves the Fredholm index upon addition to $(-\Delta_g, \partial_{\nu_g})$. Hence, $(-\Delta_g - K, \partial_{\nu_g})$ is a Fredholm operator of index zero. Applying the hypothesis (H), the derivative $D_z\Psi(z,\xi^0)(\psi)$ has a trivial kernel, thereby affirming the non-degeneracy of $D_z\Psi(z,\xi^0)$. The implicit function theorem yields the existence of a radius $r > 0$ and a continuously differentiable mapping

$$z : \tilde{B}_r(\xi^0) := \{\xi \in \Xi_{k,m} : d_g(\xi,\xi^0) < r\} \to C_0^{2,\alpha}(\Sigma),$$

$$\xi \mapsto z_\xi,$$

such that $z_{\xi^0} = z(\cdot,\xi^0)$ and z_ξ is the unique solution of $\Psi(z_\xi,\xi) = 0$ for any $\xi \in \tilde{B}_r(\xi^0)$. We take $\mathscr{D} = \tilde{B}_r(\xi^0)$. □

Lemma 4.1.4. *For $\xi \in \Xi_{k,m}$, $i = 1,\cdots,m, j = 1,\cdots,\mathrm{i}(\xi_i)$*

$$\partial_{(\xi_i)_j} I_\xi(z(\cdot,\xi)) = \frac{1}{2}\varrho(\xi_i)\partial_{x_j} z(x,\xi)|_{x=\xi_i}. \tag{4.1.6}$$

Proof. Since $z(\cdot,\xi)$ solves the problem (1.2.3), $I'_\xi(z(\cdot,\xi)) = 0$. By the representation formula (2.4.2), we deduce that

$$\begin{aligned} & \partial_{(\xi_i)_j} I_\xi(z(\cdot,\xi)) \\ = \; & I'_\xi(z(\cdot,\xi))\partial_{(\xi_i)_j} z(\cdot,\xi) - 2\rho_2 \int_\Sigma \frac{\tilde{V}_2(\cdot,\xi)e^{z(\cdot,\xi)}}{\int_\Sigma \tilde{V}_2(\cdot,\xi)e^{z(\cdot,\xi)}dv_g} \left(-\frac{\varrho(\xi_i)}{2}\partial_{(\xi_i)_j} G^g(\cdot,\xi_i) \right) dv_g \\ = \; & \rho_2\varrho(\xi_i) \int_\Sigma \frac{\tilde{V}_2(\cdot,\xi)e^{z(\cdot,\xi)}}{\int_\Sigma \tilde{V}_2(\cdot,\xi)e^{z(\cdot,\xi)}dv_g} \partial_{(\xi_i)_j} G^g(\cdot,\xi_i) dv_g = \frac{1}{2}\varrho(\xi_i)\partial_{x_j} z(x,\xi)|_{x=\xi_i}. \end{aligned}$$

□

It is clear that (H1) implies (H). The hypothesis (H) describes the nondegeneracy of solutions of the singular mean field equations (1.2.3). For $\rho_2 \in (0,2\pi)$, it is generally observed that hypothesis (H) may not hold. Nonetheless, when $\rho_2 > 0$ is chosen to be sufficiently small, this hypothesis is satisfied for any $\mathscr{D}$ that is an open precompact subset of $\Xi_{k,m}$.

Lemma 4.1.5. *Let $\mathscr{D} \subset \overline{\mathscr{D}} \subset \Xi_{k,m}$ be an open subset. There exists $\rho_0 \in (0,2\pi)$ sufficiently* *small such that for any $\rho_2 \in (0, \rho_0)$, there exists a unique solution $z(\cdot, \xi)$ of* (1.2.3) *satisfying (H).*

Proof. To formulate our argument, we select $\mathscr{D}$ as an arbitrary open precompact subset of $\Xi_{k,m}$. We will prove the lemma by contradiction.

Let us suppose that there exists a sequence $\rho_2^n \to 0$, $\xi^n \in \mathscr{D}$, and $z_n^1, z_n^2 \in C_0^{2,\alpha}(\Sigma)$, two distinct solutions of the following problem:

$$\begin{cases} -\Delta_g z(x) = 2\rho_2^n \left(\frac{\tilde{V}_2(x,\xi^n) e^z}{\int_\Sigma \tilde{V}_2(\cdot,\xi^n) e^z dv_g} - 1 \right), & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} z(x) = 0, & \text{on } \partial\Sigma \end{cases}. \tag{4.1.7}$$

Given the compactness of the solutions space for (4.1.7) (refer to [LSY23]), up to a subsequence, for $i = 0, 1$ $\xi^n \to \xi^0 \in \overline{\mathscr{D}}$, $z_n^i \to z^i$ in the Hölder space $C_0^{2,\alpha}(\Sigma)$. As the parameter $\rho_2^n \to 0$ in the context of (4.1.7), it follows $z^i = 0$ for $i = 0, 1$. We introduce a mapping

$$\begin{aligned} \Psi : C_0^{2,\alpha}(\Sigma) \times \Xi_{k,m} \times \mathbb{R} &\to C_0^{\alpha}(\Sigma) \times \{0\}, \\ (z, \xi, \rho) &\mapsto \begin{bmatrix} -\Delta_g z - 2\rho \left(\frac{\tilde{V}_2(x,\xi^n) e^{z_n}}{\int_\Sigma \tilde{V}_2(\cdot,\xi^n) e^{z_n} dv_g} - 1 \right) \\ \partial_{\nu_g} z \end{bmatrix}. \end{aligned}$$

We note that z is a solution of (4.1.7) if and only if $\Psi(z, \xi, \rho) = 0$ and

$$D_z \Psi(z^i, \xi^0, 0)(\psi) = \begin{bmatrix} -\Delta_g \psi \\ \partial_{\nu_g} \psi \end{bmatrix}.$$

Given that $(-\Delta_g, \partial_{\nu_g})$ is an isomorphism from $C_0^{2,\alpha}(\Sigma)$ to $C_0^{\alpha}(\Sigma) \times \{0\}$, the implicit function theorem yields that there exists $r > 0$, $\delta > 0$, U a neighborhood of 0 in $C_0^{2,\alpha}(\Sigma)$ and a C^1-diffeomorphism:

$$z : \{\xi \in \Xi_{k,m} : d_g(\xi, \xi^0) < r\} \times (-\delta, \delta) \to U, \quad (\xi, \rho) \mapsto z_{\xi,\rho},$$

with $\Psi(z_{\xi,\rho}, \xi, \rho) = 0$. Denote $\tilde{B}_r(\xi^0) := \{\xi \in \Xi_{k,m} : d_g(\xi, \xi^0) < r\}$. Additionally, $z = z_{\xi,\rho}$ is the unique solution in U satisfying $\Psi(z, \xi, \rho) = 0$. There exists $N_0 > 0$ sufficiently large such that $z_n^i \in U$ and $\xi^n \in \tilde{B}_r(\xi^0)$ for any $i = 0, 1$ and $n \geq N_0$. Given the uniqueness of $z_{\xi,\rho}$, we deduce that $z_n^i = z_{\xi^n,\rho^n}$ for $i = 0, 1$, leading to a contradiction.

We now will establish the non-degeneracy. Suppose $\rho_2^n \to 0$, $\xi^n \in \mathscr{D}$, z_n a solution of (4.1.7) and ψ_n a nontrivial solution of the following problem:

$$\begin{cases} -\Delta_g \psi = K_n(\psi), & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} \psi = 0 & \text{on } \partial\Sigma \end{cases}, \tag{4.1.8}$$

where $K_n : C_0^{2,\alpha}(\Sigma) \to C_0^{\alpha}(\Sigma)$,

$$K_n(\psi) = 2\rho_2^n \left(\frac{\tilde{V}_2(x,\xi)e^z\psi(x)}{\int_\Sigma \tilde{V}_2(\cdot,\xi)e^z dv_g} - \frac{\tilde{V}_2(x,\xi)e^z \int_\Sigma \tilde{V}_2(\cdot,\xi)e^z\psi dv_g}{(\int_\Sigma \tilde{V}_2(\cdot,\xi)e^z dv_g)^2} \right).$$

Taking the limit of (4.1.8) as $n \to 0$, we have that $\rho_2^n \to 0, z_n \to 0$ in $C^{2,\alpha}(\Sigma)$. Without loss of generality, we assume that $\|\psi_n\|_\infty = 1$. This assumption leads to $\|K_n(\psi_n)\|_\infty \to 0$. By the regularity theory, we obtain that $\psi_n \to 0$ in $C^2(\Sigma)$, which leads to a contradiction. □

On the other hand, we can show that the shadow system (1.2.5) is non-degenerate for generic $(V_1, V_2) \in C^{2,\alpha}(\Sigma, \mathbb{R}_+) \times C^{2,\alpha}(\Sigma, \mathbb{R}_+)$ and for any fixed $\rho_2 \geq 0$, by a well-known transversality theorem which can be found in [Abr63, Qui70], [Hen05] and references therein.

Theorem 4.1.6 (Theorem 5.4 of [Hen05]). *Let $M, \mathcal{V}, N$ be Banach manifolds of class C^r for some $r \in \mathbb{N}_+$, let $\mathcal{D} \subset M \times \mathcal{V}$ be open, let $\mathcal{F} : \mathcal{D} \to N$ be a C^r map, and fix a point $z \in N$. Assume for each $(y, \psi) \in \mathcal{F}^{-1}(z)$ that:*

(1) $D_y\mathcal{F}(y,\psi) : T_yM \to T_zN$ is semi-Fredholm with index $< r$;

(2) $D\mathcal{F}(y,\psi) : T_yM \times T_\psi\mathcal{V} \to T_zN$ is surjective;

(3) $\mathcal{F}^{-1}(z) \to \mathcal{V}$, $(y,\psi) \mapsto \psi$, is σ-proper.

Then for $\mathcal{D}_\psi = \{y \in M : (y,\psi) \in \mathcal{D}\}$,

$$\mathcal{V}_0 := \{\psi \in \mathcal{V} : z \textit{ is not a critical value of } \mathcal{F}(\cdot,\psi) : \mathcal{D}_\psi \to N\}$$

is a residual of $\mathcal{V}$.

For any $p > 1, \alpha \in (0,1)$, we define

$$M := C_0^{2,\alpha}(\Sigma) \times \Sigma^k \times (\partial\Sigma)^{m-k}, \quad \mathcal{V} := C^{2,\alpha}(\Sigma, \mathbb{R}_+) \times C^{2,\alpha}(\Sigma, \mathbb{R}_+), \quad N := C_0^{\alpha}(\Sigma) \times \{0\} \times \mathbb{R}^{m+k},$$

and $\mathcal{D} := C_0^{2,\alpha}(\Sigma) \times \Xi_{k,m} \times \mathcal{V}$ is an open subset of M. We note that for any $\xi \in \mathring{\Sigma}^k \times (\partial\Sigma)^{m-k}$, the tangent space $T_\xi(\mathring{\Sigma}^k \times (\partial\Sigma)^{m-k})$ is isomorphic to $\mathbb{R}^{m+k}$, so we identify the elements in $T_\xi(\mathring{\Sigma}^k \times (\partial\Sigma)^{m-k})$ and $\mathbb{R}^{m+k}$.

Consider the map for $t \in [0,1]$

$$\mathcal{T}_t(w,\xi,V_1,V_2) = \begin{bmatrix} \Delta_g w + 2\rho_2 \left(\dfrac{V_2 e^{w-\sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(\cdot,\xi_i)}}{\int_\Sigma V_2 e^{w-\sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(\cdot,\xi_i)} dv_g} - 1 \right) \\ \partial_{\nu_g} w \\ \nabla_{\xi_1} f_t(\xi) \\ \vdots \\ \nabla_{\xi_m} f_t(\xi) \end{bmatrix}. \tag{4.1.9}$$

Remark 4.1.7.

- *A residual of $\mathcal{V}$ is defined as a countable intersection of open dense subsets of $\mathcal{V}$. Consequently, it is dense in $\mathcal{V}$.*
- *It is clear that $\Xi_{k,m}$ and $C^{2,\alpha}(\Sigma,\mathbb{R}_+)$ are not Banach spaces. However, since Banach manifolds require only local modeling on complete spaces, $\Xi_{k,m}$ and $C^{2,\alpha}(\Sigma,\mathbb{R}_+)$ are Banach manifolds. Consequently, the transversality theorem (4.1.6) is applicable in this context.*

Theorem 4.1.8. *For $t \in [0,1]$, $\mathcal{T}_t$ is C^1-differentiable. Moreover,*

$$\mathcal{V}^t_{reg} := \{(V_1,V_2) \in \mathcal{V} : \text{any solution } (w,\xi) \text{ of } \mathcal{T}_t(\cdot,V_1,V_2) = 0 \text{ is nondegenerate}\}$$

is residual in $\mathcal{V}$.

Proof. It is easy to check the condition (3) in Theorem 4.1.6. Let

$$M_i := \left\{ (w,\xi) \in C_0^{2,\alpha}(\Sigma) \times \Xi_{k,m} : \operatorname{dist}(\xi, \partial\Xi_{k,m}) \geq \frac{1}{2^i} \text{ and } \|w\|_{C^{2,\alpha}(\Sigma)} \leq 2^i \right\},$$

for $i \in \mathbb{N}$. It follows that

$$C_0^{2,\alpha}(\Sigma) \times \Xi_{k,m} = \bigcup_{i=0}^{+\infty} M_i.$$

We consider the map $\mathcal{T}_t^{-1}(0) \cap (M_i \times \mathcal{V}) \to \mathcal{V}, (t,w,\xi,V_1,V_2) \mapsto (V_1,V_2)$. For any $(t^n,w^n,\xi^n,V^n) \in \mathcal{T}_t^{-1}(0) \cap (M_i \times \mathcal{V})$ such that $V^n := (V_1^n,V_2^n) \to V^0 := (V_1^0,V_2^0)$ in $\mathcal{V}$, by the compactness of $\Xi_{k,m}^{\frac{1}{2^i}}$, up to a subsequence, $\xi^n := (\xi_1^n,\cdots,\xi_m^n) \to \xi^0 := (\xi_1^0,\cdots,\xi_m^0) \in \Xi_{k,m}^{\frac{1}{2^i}}$. The Arzelà-Ascoli theorem implies that for any $\alpha' \in (0,\alpha)$, there exists w^0 such that

$$w^n \to w^0 \text{ in } C_0^{2,\alpha'}(\Sigma).$$

Since $(w^n, \xi^n, V^n) \in \mathcal{T}_t^{-1}(0)$, we have (w^0, ξ^0, V^0) satisfying $\mathcal{T}_t(w^0, \xi^0, V^0) = 0$.

$$\begin{cases} -\Delta_g(w^n - w^0) = 2\rho_2 \dfrac{V_2^n e^{w^n - \sum_{i=1}^m \frac{\varrho(\xi_i^n)}{2} G^g(\cdot, \xi_i^n)}}{\int_\Sigma V_2^n e^{w^n - \sum_{i=1}^m \frac{\varrho(\xi_i^n)}{2} G^g(\cdot, \xi_i^n)} dv_g} - 2\rho_2 \dfrac{V_2^0 e^{w^0 - \sum_{i=1}^m \frac{\varrho(\xi_i^0)}{2} G^g(\cdot, \xi_i^0)}}{\int_\Sigma V_2^0 e^{w^0 - \sum_{i=1}^m \frac{\varrho(\xi_i^0)}{2} G^g(\cdot, \xi_i^0)} dv_g} & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g}(w^n - w^0) = 0 & \text{on } \partial\Sigma \end{cases}.$$

We have

$$\| - \Delta_g(w^n - w^0)\|_{C^\alpha(\Sigma)} \le C\|w^n - w^0\|_{C^\alpha(\Sigma)} \to 0,$$

as $n \to +\infty$. Applying the Schauder estimates, we can derive the convergence $w^n \to w^0$ in $C_0^{2,\alpha}(\Sigma)$. Thus, $\mathcal{T}_t^{-1}(0) \to \mathcal{V}, (w, \xi, V) \mapsto V$ is σ-proper.

To compute the derivative of $\mathcal{T}_t$ in the direction of (w, ξ), we proceed as follows. Let $\phi \in C_0^{2,\alpha}(\Sigma), v = (v_1, \cdots, v_m) \in \mathbb{R}^{m+k}$. The derivative of $\mathcal{T}_t$ with respect to (w, ξ) is given by

$$D_{w,\xi}\mathcal{T}_t(w, \xi, V_1, V_2)[\phi, v] = \begin{bmatrix} T_0(w, \xi, V_1, V_2)[\phi, v] \\ \\ T_1(w, \xi, V_1, V_2)[\phi, v] \\ \\ \vdots \\ \\ T_m(w, \xi, V_1, V_2)[\phi, v] \end{bmatrix}, \tag{4.1.10}$$

where

$$\begin{aligned} &T_0(w, \xi, V_1, V_2)[\phi, v] = \\ &\left[\Delta_g \phi + 2\rho_2 \left(\frac{\tilde{V}_2 e^w \phi}{\int_\Sigma \tilde{V}_2 e^w dv_g} - \frac{\tilde{V}_2 e^w \int_\Sigma \tilde{V}_2 e^w \phi dv_g}{\left(\int_\Sigma \tilde{V}_2 e^w dv_g\right)^2} \right) - \rho_2 \sum_{i=1}^m \varrho(\xi_i) \left(\frac{\tilde{V}_2 e^w \nabla_{x_i} G^g(\cdot, x_i)|_{x_i = \xi_i} \cdot v_i}{\int_\Sigma \tilde{V}_2 e^w dv_g} \right) \right. \\ &\left. +\rho_2 \sum_{i=1}^m \left(\frac{\tilde{V}_2 e^w \int_\Sigma \tilde{V}_2 e^w \nabla_{x_i} G^g(\cdot, x_i)|_{x_i = \xi_i} \cdot v_i dv_g}{\left(\int_\Sigma \tilde{V}_2 e^w dv_g\right)^2} \right), \partial_{\nu_g} \phi \right]^\top, \end{aligned}$$

and

$$\begin{aligned} T_i(w, \xi, V_1, V_2)[\phi, v] = {} & \nabla_{x_i}^2 \left(\sum_{i=1}^m \varrho^2(x_i) R^g(\xi_i) + \sum_{\substack{i,i=1 \\ i \neq i}}^m \varrho(x_i)\varrho(x_i) G(x_i, x_i) \right)\Bigg|_{x=\xi} \cdot v_i \\ & + 2\varrho^2(x_i) \nabla_{x_i}^2 \log V_1(x_i)|_{x_i = \xi_i} \cdot v_i - \varrho(\xi_i)(1-t)\nabla_{x_i}^2 w(x_i)|_{x_i = \xi_i} \cdot v_i \\ & - \varrho(\xi_i)(1-t)\nabla_{\xi_i}\phi(\xi_i), \end{aligned}$$

for $i=1,\cdots,m$, where $\tilde{V}_2 = V_2 e^{-\sum_{i=1}^{m}\frac{\varrho(\xi_i)}{2}G^g(\cdot,\xi_i)}$. By the formula of C^1-derivatives, it is easy to know that $\mathcal{T}_t$ is C^1-differentiable for (w,ξ) uniformly for any V_1, V_2 bounded in $C^{2,\alpha}(\Sigma,\mathbb{R}_+)$ and $t\in[0,1]$. We decompose $D_{w,\xi}\mathcal{T}_t$ into the following two linear operators:

$$\mathcal{T}_{11}^t(w,\xi,V_1,V_2)[\phi,v] = \begin{bmatrix} \Delta_g\phi + 2\rho_2\left(\frac{\tilde{V}_2 e^w\phi}{\int_\Sigma \tilde{V}_2 e^w dv_g} - \frac{\tilde{V}_2 e^w \int_\Sigma \tilde{V}_2 e^w \phi dv_g}{\left(\int_\Sigma \tilde{V}_2 e^w dv_g\right)^2}\right) \\ \partial_{\nu_g}\phi \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{4.1.11}$$

and

$$\mathcal{T}_{12}^t(w,\xi,V_1,V_2)[\phi,v] = \begin{bmatrix} T_{10}(w,\xi,V_1,V_2)[\phi,v] \\ T_{11}(w,\xi,V_1,V_2)[\phi,v] \\ \vdots \\ T_{1m}(w,\xi,V_1,V_2)[\phi,v] \end{bmatrix}, \tag{4.1.12}$$

where

$$\begin{aligned} & T_{10}(w,\xi,V_1,V_2)[\phi,v] \\ & = \Bigg[-\rho_2\sum_{i=1}^m \varrho(\xi_i)\left(\frac{\tilde{V}_2 e^w}{\int_\Sigma \tilde{V}_2 e^w dv_g}\nabla_{x_i}G^g(\cdot,x_i)|_{x_i=\xi_i}\cdot v_i\right) \\ & +\rho_2\sum_{i=1}^m\left(\frac{\tilde{V}_2 e^w}{\left(\int_\Sigma \tilde{V}_2 e^w dv_g\right)^2}\int_\Sigma \tilde{V}_2 e^w \nabla_{x_i}G^g(\cdot,x_i)|_{x_i=\xi_i}\cdot v_i dv_g\right), 0\Bigg]^\top, \end{aligned}$$

and $T_{1i}=T_i$ for any $i=1,\cdots,m$. Since the operator $(-\Delta_g,\partial_{\nu_g}) : C_0^{2,\alpha}(\Sigma)\to C_0^{\alpha}(\Sigma)\times\{0\}$ is an isomorphism, and given the compactness of Σ along with the compact embedding theorem for Sobolev spaces, we can infer that the operator

$$\phi\mapsto 2\rho_2\left(\frac{\tilde{V}_2 e^w\phi}{\int_\Sigma \tilde{V}_2 e^w\, dv_g} - \frac{\tilde{V}_2 e^w\int_\Sigma \tilde{V}_2 e^w\phi\, dv_g}{\left(\int_\Sigma \tilde{V}_2 e^w\, dv_g\right)^2}\right)$$

is compact. Consequently, it follows that

$$\dim(\ker(\mathcal{T}_{11}^t(w,\xi,V_1,V_2))) = \operatorname{codim}(\operatorname{Im}(\mathcal{T}_{11}^t(w,\xi,V_1,V_2))) = m+k,$$

indicating that $\mathcal{T}_{11}^t(w,\xi,V_1,V_2)$ is a Fredholm operator with index 0. Similarly, using the compactness of Σ and the compact embedding theorem for Sobolev spaces, $\mathcal{T}_{12}^t(w,\xi,V_1,V_2)$ is also compact. Thus, $D_{w,\xi}\mathcal{T}_t(w,\xi,V_1,V_2)$ is a Fredholm operator with index 0. The condition (1) in Theorem 4.1.6 holds.

Next, we will show that condition (2) holds true. Let $(w,\xi,V_1,V_2) \in \mathcal{T}_t^{-1}(0)$. We have the derivative of $\mathcal{T}_t$ with respect to $V=(V_1,V_2)$ as follows:

$$D_{V_1}\mathcal{T}_t(w,\xi,V_1,V_2)[h_1] = \begin{bmatrix} 0 \\ 0 \\ 2\varrho^2(\xi_1)\left(\frac{\nabla_{\xi_1}h_1(\xi_1)}{V_1(\xi_1)} - \frac{h_1(\xi_1)\nabla_{\xi_1}V_1(\xi_1)}{V_1^2(\xi_1)}\right) \\ \vdots \\ 2\varrho^2(\xi_m)\left(\frac{\nabla_{\xi_m}h_1(\xi_m)}{V_1(\xi_m)} - \frac{h_1(\xi_m)\nabla_{\xi_m}V_1(\xi_m)}{V_1^2(\xi_m)}\right) \end{bmatrix},$$

and

$$D_{V_2}\mathcal{T}_t(w,\xi,V_1,V_2)[h_2] = \begin{bmatrix} 2\rho_2\left(\frac{\tilde{V}_2e^w}{\int_\Sigma \tilde{V}_2e^w dv_g}\frac{h_2}{V_2} - \frac{\tilde{V}_2e^w\int_\Sigma \tilde{V}_2e^w\frac{h_2}{V_2}dv_g}{\left(\int_\Sigma \tilde{V}_2e^w dv_g\right)^2}\right) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

For $v=(v_1,\cdots,v_m)=0\in\mathbb{R}^{m+k}$, and $h_1\in C^{2,\alpha}(\Sigma)$ satisfying that

$$2\varrho^2(\xi_i)\left(\frac{\nabla_{\xi_i}h_1(\xi_i)}{V_1(\xi_i)} - \frac{h_1(\xi_i)\nabla_{\xi_i}V_1(\xi_i)}{V_1^2(\xi_i)}\right) = \varrho(\xi_i)\nabla_{\xi_i}\phi(\xi_i),$$

for $i = 1, \cdots, m$, we have

$$\begin{aligned} & D_{w,\xi}\mathcal{T}_t(w,\xi,V_1,V_2)[\phi, v_1, \dots, v_m] + D_{V_1}\mathcal{T}_t(w,\xi,V_1,V_2)[h_1] \\ = & \begin{bmatrix} \Delta_g \phi + 2\rho_2 \left(\frac{\tilde{V}_2 e^w \phi}{\int_\Sigma \tilde{V}_2 e^w dv_g} - \frac{\tilde{V}_2 e^w \int_\Sigma \tilde{V}_2 e^w \phi dv_g}{\left(\int_\Sigma \tilde{V}_2 e^w dv_g\right)^2} \right) \\ \partial_{\nu_g}\phi \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \end{aligned}$$

Claim 4.1.9. *Let* $L(\phi) = \begin{bmatrix} -\Delta_g \phi - 2\rho_2 \left(\frac{\tilde{V}_2 e^w \phi}{\int_\Sigma \tilde{V}_2 e^w dv_g} - \frac{\tilde{V}_2 e^w \int_\Sigma \tilde{V}_2 e^w \phi dv_g}{\left(\int_\Sigma \tilde{V}_2 e^w dv_g\right)^2} \right) \\ \partial_{\nu_g}\phi \end{bmatrix}$. *If* $\phi \in C_0^{2,\alpha}(\Sigma)$ *such that*

$$\begin{aligned} L(\phi) &= 0, \\ \int_\Sigma \left(\frac{\tilde{V}_2 e^w}{\int_\Sigma \tilde{V}_2 e^w dv_g} \frac{h_2}{V_2} - \frac{\tilde{V}_2 e^w \int_\Sigma \tilde{V}_2 e^w \frac{h_2}{V_2} dv_g}{\left(\int_\Sigma \tilde{V}_2 e^w dv_g\right)^2} \right) \phi dv_g &= 0, \end{aligned}$$

for any $h_2 \in C^{2,\alpha}(\Sigma)$, *then* $\phi = 0$.

To proceed with the proof, we will defer the proof of Claim 4.1.9 for later. Define that

$$H = \operatorname{span}\left\{ \operatorname{Im}(D_{w,\xi}\mathcal{T}_t(w,\xi,V_1,V_2)[\cdot, v] + D_{V_1}\mathcal{T}_t(w,\xi,V_1,V_2)[h_1]) \bigcup \operatorname{Im}(D_{V_2}\mathcal{T}_t(w,\xi,V_1,V_2)) \right\}.$$

We will prove that

$$H = \quad C_0^\alpha(\Sigma) \times \{0\} \times \underbrace{\{0\} \times \cdots \times \{0\}}_{m+k} \quad . \tag{4.1.13}$$

For any $h \in C_0^\alpha(\Sigma)$, since $(-\Delta_g, \partial_{\nu_g}) : C^{2,\alpha}(\Sigma) \to C_0^\alpha(\Sigma) \times \{0\}$ is an isomorphism, there exists $\phi_0 \in C_0^{2,\alpha}(\Sigma)$ with $\partial_{\nu_g}\phi_0 = 0$ such that

$$-\Delta_g \phi_0 = h.$$

We recall that $K(\phi) = 2\rho_2 \left(\frac{\tilde{V}_2 e^w}{\int_\Sigma \tilde{V}_2 e^w dv_g} \phi - \frac{\tilde{V}_2 e^w \int_\Sigma \tilde{V}_2 e^w \phi dv_g}{\left(\int_\Sigma \tilde{V}_2 e^w dv_g\right)^2} \right)$. Then

$$\begin{bmatrix} h \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \Delta_g \phi_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \Delta_g \phi_0 + K(\phi_0) \\ \partial_{\nu_g} \phi_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - D_{V_2} \mathcal{T}_t(w, \xi, V_1, V_2))[V_2 \phi_0] \in H.$$

Hence, (4.1.13) is concluded. Next, we will show that

$$\operatorname{Im}(D_{V_1} \mathcal{T}_t(w, \xi, V_1, V_2)) = \{0\} \times \{0\} \times \mathbb{R}^{m+k}. \tag{4.1.14}$$

We select functions $h_{11}, h_{12} \in C^{2,\alpha}(\Sigma)$, such that

$$h_{1i}(\xi_i) = 0, \quad \text{and} \quad \nabla_{\xi_i} h_{1i}(\xi_i) = 0, \quad 2 \le i \le m, \quad i = 1, 2. \tag{4.1.15}$$

In the case where $\xi_1 \in \mathring{\Sigma}$, it is possible to find functions h_{11}, h_{12} that satisfy (4.1.15) and fulfill the following conditions:

$$\frac{\nabla_{\xi_1} h_{11}(\xi_1)}{V_1(\xi_1)} - \frac{h_{11}(\xi_1) \nabla_{\xi_1} V_1(\xi_1)}{V_1^2(\xi_1)} = (1, 0),$$

and

$$\frac{\nabla_{\xi_1} h_{12}(\xi_1)}{V_1(\xi_1)} - \frac{h_{12}(\xi_1) \nabla_{\xi_1} V_1(\xi_1)}{V_1^2(\xi_1)} = (0, 1).$$

If $\xi_1 \in \partial\Sigma$, we also can find h_{11}, h_{12} that satisfy (4.1.15) such that

$$\frac{\nabla_{\xi_1} h_{11}(\xi_1)}{V_1(\xi_1)} - \frac{h_{11}(\xi_1) \nabla_{\xi_1} V_1(\xi_1)}{V_1^2(\xi_1)} = 1.$$

It follows that

$$[0, 0, c_1, 0, \cdots, 0]^\top \in \operatorname{Im}(D_{V_1} \mathcal{T}_t(w, \xi, V_1, V_2)),$$

for all $c_1 \in \mathbb{R}^2$ when $\xi_1 \in \mathring{\Sigma}$ and $c_1 \in \mathbb{R}$ when $\xi_1 \in \partial\Sigma$. Similarly, appropriate selection of h_{11} and h_{12} allows us to infer that

$$[0, 0, c_1, \cdots, c_m]^\top = \operatorname{Im}(D_{V_1} \mathcal{T}_t(w, \xi, V_1, V_2)),$$

for all $(c_1, \cdots, c_m) \in \mathbb{R}^{m+k}$. Consequently, this leads us to conclude (4.1.14), demonstrating that $D\mathcal{T}_t(w, \xi, V_1, V_2)$ is surjective.

Invoking Theorem 4.1.6, we deduce that

$$\mathcal{V}_0 \quad = \{V \in \mathcal{V} : 0 \text{ is a critical value of } \mathcal{T}_t(\cdot, V) : C_0^{2,\alpha}(\Sigma) \times \Xi_{k,m} \to C_0^{\alpha}(\Sigma) \times \{0\} \times \mathbb{R}^{m+k}\}$$

is a meager subset of $\mathcal{V}$. Consequently, the complement $\mathcal{V}_{reg}^t = \mathcal{V} \setminus \mathcal{V}_0$ is a residual set in $\mathcal{V}$.

Proof of Claim 4.1.9. If $\phi \in \ker(L)$, for any $h \in L^2(\Sigma)$ we have

$$\int_\Sigma (-\Delta_g \phi) h dv_g = 2\rho_2 \left(\frac{\int_\Sigma \tilde{V}_2 e^w \phi h dv_g}{\int_\Sigma \tilde{V}_2 e^w dv_g} - \frac{\int_\Sigma \tilde{V}_2 e^w h dv_g \int_\Sigma \tilde{V}_2 e^w \phi dv_g}{\left(\int_\Sigma \tilde{V}_2 e^w dv_g\right)^2} \right). \tag{4.1.16}$$

Since $C^{2,\alpha}(\Sigma)$ is dense in $L^2(\Sigma)$, (4.1.13) implies that the right hand side of (4.1.16) vanishes, i.e. $\int_\Sigma (-\Delta_g \phi) h dv_g = 0$ for any $h \in L^2(\Sigma)$. Hence, $-\Delta_g \phi = 0$ in $\mathring{\Sigma}$ and $\partial_{\nu_g} \phi = 0$ on $\partial\Sigma$. By the Schauder estimate in Lemma 2.3.3, $\phi = 0$ in $C_0^{2,\alpha}(\Sigma)$. □

Using the method of continuity, we can conclude the following proposition:

Proposition 4.1.10. *There exists a dense subset of $C^{2,\alpha}(\Sigma, \mathbb{R}_+) \times C^{2,\alpha}(\Sigma, \mathbb{R}_+)$, denoted by $\mathcal{V}_{reg}$, such that for any $(V_1, V_2) \in \mathcal{V}_{reg}$ and any $\rho_2 \in (0, 2\pi)$, a non-degenerate solution (w, ξ) of the shadow system* (1.2.5) *exists.*

Proof. We recall the shadow system for (w, ξ) in $C_0^{2,\alpha}(\Sigma) \times \Xi_{k,m}$:

$$\begin{cases} -\Delta_g w = 2\rho_2 \left(\dfrac{V_2 e^{w - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(x, \xi_i)}}{\int_\Sigma V_2 e^{w - \sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(x, \xi_i)} dv_g} - 1 \right) & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} w = 0 & \text{on } \partial\Sigma \\ \nabla f_0(\xi) = 0 & \text{in } \Xi_{k,m} \end{cases},$$

where

$$f_0(x_1, x_2, \cdots, x_m) = \quad \mathcal{F}_{k,m}^{V_1}(x) - \textstyle\sum_{i=1}^m \varrho(x_i) w(x_i).$$

To apply the method of continuity, we introduce a parameter $t \in [0, 1]$ to deform (1.2.5) to a decoupled system. Clearly, when $t = 0$, the shadow system (S_t) equals (1.2.5), and when $t = 1$,

(S_t) is a decoupled system,

$$\begin{cases} -\Delta_g w = 2\rho_2 \left(\dfrac{V_2 e^{w-\sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(x,\xi_i)}}{\int_\Sigma V_2 e^{w-\sum_{i=1}^m \frac{\varrho(\xi_i)}{2} G^g(x,\xi_i)} dv_g} - 1 \right) & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} w = 0 & \text{on } \partial\Sigma \end{cases}, \tag{4.1.17}$$

with $\nabla \mathcal{F}_{k,m}^{V_1}(\xi) = 0$ for ξ in $\Xi_{k,m}$. Observing that $\mathcal{F}_{k,m}^{V_1}(\xi) \to +\infty$ as $\xi \to \partial\Xi_{k,m}$, we can deduce the existence of a global minimum point of $\mathcal{F}_{k,m}^{V_1}$, denoted by $\xi^1 \in \Xi_{k,m}$. For any $\rho_2 \in (0, 2\pi)$ and $\xi \in \Xi_{k,m}$, the singular mean field equation (4.1.17) is solvable via the Moser-Trudinger inequality and standard variational methods. Consequently, a solution (w^1, ξ^1) of (S_t) exists when $t = 1$ and $\rho_2 \in (0, 2\pi)$. Let $\mathbb{Q}_0 := \{t \in [0,1] : t \text{ is a rational number}\}$. Theorem 4.1.8 implies that for any $t \in \mathbb{Q}_0$, $\mathcal{V}_{reg}^t$ is a residual set in $C^{2,\alpha}(\Sigma, \mathbb{R}_+) \times C^{2,\alpha}(\Sigma, \mathbb{R}_+)$. Then, the intersection set $\mathcal{V}_{reg} := \bigcap_{t\in\mathbb{Q}_0} \mathcal{V}_{reg}^t$ remains a residual set in $C^{2,\alpha}(\Sigma, \mathbb{R}_+) \times C^{2,\alpha}(\Sigma, \mathbb{R}_+)$, too. By fixing arbitrary $(V_1, V_2) \in \mathcal{V}_{reg}$, it follows that any solution (w^t, ξ^t) of (S_t) is non-degenerate for any $t \in \mathbb{Q}_0$.

We define that

$$T = \left\{ t \in [0,1] : \exists (w^t, \xi^t) \in C_0^{2,\alpha} \times \Xi_{k,m} \text{ is a non-degenerate solution of } (\mathrm{S}_t) \right\}.$$

Considering that $1 \in \mathbb{Q}_0$, it follows that $1 \in T (\neq \emptyset)$ based on the analysis above. By the method of continuity, it is sufficient to show that T is both close and open in $[0,1]$.

Suppose that $t_n \in T$ such that $t_n \to t_0$ as $n \to +\infty$. Then, there exists $(w^n, \xi^n) := (w^{t_n}, \xi^{t_n})$ solves (S_{t_n}) with ξ^n is a critical point of f_{t_n} in $\Xi_{k,m}$. Proposition 4.1.1 implies that for same $\alpha \in (0,1)$

$$\|w^n\|_{C^{2,\alpha}(\Sigma)} \leq C,$$

for constant $C > 0$. By the Arzelà - Ascoli theorem, for any $\alpha' \in (0, \alpha)$, there exists a $w^0 \in C^{2,\alpha'}(\Sigma)$ such that

$$w^n \to w^0 \text{ as } n \to +\infty,$$

strongly in $C^{2,\alpha'}(\Sigma)$ and $\xi^n \to \xi^0$ with $\xi^0 \in \overline{\Xi}_{k,m}$. [HB24, Lemma 4.1] implies that

$$|\nabla \mathcal{F}_{k,m}^{V_1}(\xi)|_g \to +\infty$$

as $\xi \to \partial\Xi_{k,m}$. Combining with the uniformly bounded of $\|w^n\|_{C^1(\Sigma)}$, there exists $\delta > 0$ such that $\mathrm{dist}(\xi^n, \partial\Xi_{k,m}) \geq \delta$. It follows that $\xi^0 \in \Xi_{k,m}$.

For any $\varphi \in \overline{\mathrm{H}}^1(\Sigma)$, as $n \to +\infty$

$$\int_\Sigma \langle \nabla w^n, \nabla \varphi \rangle_g dv_g \to \int_\Sigma \langle \nabla w^0, \nabla \varphi \rangle_g dv_g,$$

and

$$\begin{aligned} & 2\rho_2 \int_\Sigma \left(\frac{V_2 e^{w^n - \sum_{i=1}^m \frac{\varrho(\xi_i^n)}{2} G^g(x,\xi_i^n)}}{\int_\Sigma V_2 e^{w - \sum_{i=1}^m \frac{\varrho(\xi_i^n)}{2} G^g(x,\xi_i^n)} dv_g} - 1 \right) \varphi dv_g \\ \longrightarrow \quad & 2\rho_2 \int_\Sigma \left(\frac{V_2 e^{w^0 - \sum_{i=1}^m \frac{\varrho(\xi_i^0)}{2} G^g(x,\xi_i^0)}}{\int_\Sigma V_2 e^{w^0 - \sum_{i=1}^m \frac{\varrho(\xi_i^0)}{2} G^g(x,\xi_i^0)} dv_g} - 1 \right) \varphi dv_g. \end{aligned}$$

By Schauder estimates and $\|\Delta_g w^0\|_{C^\alpha} < +\infty$, we have $w^0 \in C_0^{2,\alpha}(\Sigma)$.

Observe that $\nabla \mathcal{F}_{k,m}^{V_1}, \nabla w^0$ is continuous at ξ^0 and for some constant $C > 0$

$$|\nabla f_t(x) - \nabla f_{t_0}(x)| \leq C|t - t_0|.$$

It is easy to verify that ξ^0 is a critical point of f_{t_0}. Indeed,

$$\begin{aligned} \nabla f_{t_0}(\tilde{\xi}^0) &= \nabla(f_{t_0}(\xi^0) - f_{t_0}(\xi^n)) + \nabla(f_{t_0}(\xi^n) - f_{t_n}(\xi^n)) + \nabla f_{t_n}(\xi^n) \\ &= \nabla(f_{t_0}(\xi^0) - f_{t_0}(\xi^n)) + \nabla(f_{t_0}(\xi^n) - f_{t_n}(\xi^n)). \end{aligned}$$

By the arbitrariness of n, we have $\nabla f_{t_0}(\xi^0) = 0$. Hence, (w^0, ξ^0) solves (S_{t_0}). It remains to show the non-degeneracy of the solution (w^0, ξ^0). In the proof of Theorem 4.1.8, it is established that $D_{w,\xi}\mathcal{T}_t$ is a Fredholm operator with index 0. Given $(h, 0, \zeta) \in C_0^\alpha(\Sigma) \times \{0\} \times \mathbb{R}^{m+k}$, the non-degeneracy yields that there exists $(\phi_n, v_n) \in C_0^{2,\alpha}(\Sigma) \times \mathbb{R}^{m+k}$ such that

$$D_{w,\xi}\mathcal{T}_{t_n}(w^n, \xi^n, V_1, V_2)[\phi_n, v_n] = (h, 0, \zeta).$$

Considering that $\|w^n\|_{C^{2,\alpha}(\Sigma)} \leq C$, the operator norm of $D_{w,\xi}\mathcal{T}_{t_n}(w^n, \xi^n, V_1, V_2)$ is uniformly bounded. It follows that ϕ_n is uniformly bounded in $C^{2,\alpha}(\Sigma)$. By the Arzelà-Ascoli theorem, up to a subsequence, $\phi_n \to \phi_0$ weakly in $C^{2,\alpha}(\Sigma)$ and strongly in $C^2(\Sigma)$ for some $\phi_0 \in C_0^{2,\alpha}(\Sigma)$ and $v_n \to v_0$ for some $v_0 \in \mathbb{R}^{m+k}$. Passing the limit $n \to +\infty$,

$$D_{w,\xi}\mathcal{T}_{t_n}(w^n, \xi^n, V_1, V_2)[\phi_n, v_n] \to D_{w,\xi}\mathcal{T}_{t_0}(w^0, \xi^0, V_1, V_2)[\phi_0, v_0],$$

which implies that $D_{w,\xi}\mathcal{T}_{t_0}(w^0, \xi^0, V_1, V_2)[\phi_0, v_0] = (h, 0, \zeta)$. Hence, we prove that (w^0, ξ^0) is a non-degenerate solution of (S_{t_0}).

It is clear that (w, ξ) solves (S_t) if and only if $\mathcal{T}_t(w, \xi, V_1, V_2) = 0$ (see (4.1.9)). Due to the non-degeneracy of $\mathcal{T}_{t_0}$ at (w^0, ξ^0), the implicit function theorem yields in a small open neighborhood of t_0 in $[0,1]$, denoted by I_{t_0},

$$t \mapsto (w^t, \xi^t)$$

is continuous satisfying that (w^t, ξ^t) is the unique solution of $\mathcal{T}_t(\cdot, V_1, V_2) = 0$ and $(w^{t_0}, \xi^{t_0}) = (w^0, \xi^0)$. Since for any $t \in I_{t_0} \cap \mathbb{Q}_0$, (w^t, ξ^t) is a non-degenerate solution of $\mathcal{T}_t(\cdot, V_1, V_2) = 0$, it follows $I_{t_0} \cap \mathbb{Q}_0$. Using the closeness argument we proved before, we can deduce that $I_{t_0} \subset T$.

To sum up, we obtain that the non-empty set T is both close and open in $[0,1]$. Hence, $T = [0,1]$. Consequently, there exists a non-degenerate solution of the shadow system (1.2.5). $\square$

4.2 Construction of partial blow-up solutions

4.2.1 Approximation solutions

To construct blow-up solutions of (1.2.1), it is sufficient to consider the following problem:

$$\begin{cases} -\Delta_g u_1 = (2\varepsilon V_1 e^{u_1} - \overline{2\varepsilon V_1 e^{u_1}}) - \left(\rho_2 \frac{V_2 e^{u_2}}{\int_\Sigma V_2 e^{u_2} dv_g} - \rho_2\right) & \text{in } \mathring{\Sigma} \\ -\Delta_g u_2 = (2\rho_2 \frac{V_2 e^{u_2}}{\int_\Sigma V_2 e^{u_2} dv_g} - 2\rho_2) - \left(\varepsilon V_1 e^{u_1} - \overline{\varepsilon V_1 e^{u_1}}\right) & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} u_1 = \partial_{\nu_g} u_2 = 0 & \text{on } \partial\Sigma \end{cases}, \tag{4.2.1}$$

where $\rho_2 \in (0, 2\pi)$ and ε is a positive parameter. We are going to construct a family blow-up solution $\mathbf{u}_\varepsilon = (u_{1,\varepsilon}, u_{2,\varepsilon})$ of (4.2.1) as $\varepsilon \to 0$ which blows up exactly at $\{\xi_1, \cdots, \xi_m\}$ where $\xi = (\xi_1, \cdots, \xi_m) \in \mathring{\Sigma}^k \times (\partial\Sigma)^{m-k}$ is C^1-stable critical point of a reduced function $\Lambda_{k,m}$ defined by (1.2.4) with the limit mass

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Sigma V_1 e^{u_{1,\varepsilon}} dv_g = \sum_{i=1}^m \frac{1}{2} \varrho(\xi_i) = 2\pi(m+k).$$

Setting $\rho_1 = \varepsilon \int_\Sigma V_1 e^{u_1} dv_g$, then we construct a family blow-up solutions of (1.2.1) as $\rho_1 \to 2\pi(m+k)$. We only have the first component u_1 blows up and its profile is related to the solutions of Liouville-type problems. It is well-known that

$$u_{\tau,\eta}(y) = \log \frac{8\tau^2}{(\tau^2 + |y-\eta|^2)^2}$$

for $(\tau, \eta) \in (0, \infty) \times \mathbb{R}^2$ are all the solutions to the Liouville-type equation,

$$\begin{cases} -\Delta u = e^u \text{ in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} e^u < \infty. \end{cases} \tag{4.2.2}$$

Moreover, we have $\int_{\mathbb{R}^2} e^{u_{\tau,0}(y)} dy = 8\pi$. Applying isothermal coordinate $(y_\xi, U(\xi))$ in Section 2.2, we can pull-back $u_{\tau,0}$ to the Riemann surface around ξ,

$$U_{\tau,\xi} := \log \frac{8\tau^2}{(\tau^2 + |y_\xi(x)|^2)^2} \text{ in } U(\xi).$$

Then we project the local bubbles into the functional space $\overline{\mathrm{H}}^1(\Sigma)$ by following equations:

$$\begin{cases} -\Delta_g PU_{\tau,\xi} = \chi_\xi e^{-\varphi_\xi} e^{U_{\tau,\xi}} - \overline{\chi_\xi e^{-\varphi_\xi} e^{U_{\tau,\xi}}} & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} PU_{\tau,\xi} = 0 & \text{on } \partial\Sigma \\ \int_\Sigma PU_{\tau,\xi} dv_g = 0 & \end{cases} \cdot \tag{4.2.3}$$

For any fixed $\delta > 0$, for any $\xi \in \Xi^\delta_{k,m}$, by the compactness of Σ, we can choose a uniformly $\overline{\mathrm{r}}_{\xi_i} > 0$ denoted by r_0 which only depends on δ such that $r_{\xi_i} \geq 2r_0$ for $i = 1, \cdots, m$. Without loss of generality, we can assume there exists an isothermal chart $(y_{\xi_i}, U_{4r_0}(\xi_i))$ around ξ_i such that $U_{4r_0}(\xi_i) \cap U_{4r_0}(\xi_j) = \emptyset$ for any $i \neq j$ and $U_{4r_0}(\xi_i) \cap \partial\Sigma = \emptyset$ for $i = 1, \cdots, k$. We take the concentration parameter

$$\delta_i = d_i \varepsilon^{\frac{1}{2}}, \tag{4.2.4}$$

for some $d_i > 0$ which will be chosen later. For simplicity of the notations, we denote that $U_i = U_{\delta_i, \xi_i}, PU_i = PU_{\delta_i, \xi_i}, \chi_i = \chi\left(\frac{y_{\xi_i}}{r_0}\right)$ and $\varphi_i = \hat{\varphi}_{\xi_i}(y_{\xi_i})$.

Assuming (H1), the approximation solution $\mathbf{W}_\varepsilon = (W_{1,\varepsilon}, W_{2,\varepsilon})$ is defined by

$$W_{1,\varepsilon} = \sum_{i=1}^m PU_i - \frac{1}{2} z(\cdot, \xi), \text{ and } W_{2,\varepsilon} = z(\cdot, \xi) - \frac{1}{2} \sum_{i=1}^m PU_i,$$

where $z(\cdot, \xi)$ is the solution of (1.2.3) from Lemma 4.1.3. Next, we are going to construct solutions with the form

$$\mathbf{u}_\varepsilon = \mathbf{W}_\varepsilon + \boldsymbol{\phi}_\varepsilon,$$

where $\boldsymbol{\phi}_\varepsilon = (\phi_{1,\varepsilon}, \phi_{2,\varepsilon})$ is the error term. By Lemma B.0.1, we have for $x \in \Sigma$

$$W_{1,\varepsilon}(x) = \sum_{i=1}^{m} \chi_i \log \frac{1}{(\delta_i^2 + |y_{\xi_i}(x)|^2)^2} + \sum_{i=1}^{m} \varrho(\xi_i) H^g(x, \xi_i) - \frac{1}{2} z(x, \xi) + \mathcal{O}(\varepsilon |\log \varepsilon|) \tag{4.2.5}$$

and

$$\begin{aligned} W_{2,\varepsilon}(x) &= z(x,\xi) - \tfrac{1}{2}\textstyle\sum_{i=1}^{m} \chi_i \log \frac{1}{(\delta_i^2 + |y_{\xi_i}(x)|^2)^2} - \tfrac{1}{2}\sum_{i=1}^{m} \varrho(\xi_i) H^g(x, \xi_i) + \mathcal{O}(\varepsilon |\log \varepsilon|) \\ &= \log \tilde{V}_2(x, \xi) - \log V_2(x) + z(x, \xi) + \mathcal{O}(\varepsilon |\log \varepsilon|), \end{aligned} \tag{4.2.6}$$

as $\varepsilon \to 0$. For $i = 1, \cdots, m$, we define

$$\Theta_i(y) = W_{1,\varepsilon} \circ y_{\xi_i}^{-1}(\delta_i y) + \hat{\varphi}_{\xi_i}(\delta_i y) - \chi(\delta_i |y|/r_0) U_i \circ y_{\xi_i}^{-1}(\delta_i y) + \log V_1 \circ y_{\xi_i}^{-1}(\delta_i y) + \log(2\varepsilon), \tag{4.2.7}$$

for $y \in \Omega_i := \frac{1}{\delta_i} B_{2r_0}^{\xi_i}$. Lemma B.0.1 with (4.2.4) imply that

$$\begin{aligned} \Theta_i(y) &= -2\log d_i - 2\log 2 + \varrho(\xi_i) H^g(\xi_i, \xi_i) + \sum_{l \neq i} \varrho(\xi_l) G^g(\xi_i, \xi_l) + \log V_1(\xi_i) \\ &\quad - \frac{1}{2} z(\xi_i, \xi) + \mathcal{O}(\varepsilon |\log \varepsilon| + \varepsilon^{\frac{1}{2}} |y|). \end{aligned} \tag{4.2.8}$$

To ensure that Θ_i is sufficiently small, for $i = 1, \cdots, m$, we choose

$$d_i = \sqrt{\frac{1}{8} e^{\varrho(\xi_i) H^g(\xi_i, \xi_i) + \sum_{l \neq i} \varrho(\xi_l) G^g(\xi_i, \xi_l) + \log V_1(\xi_i) - \frac{1}{2} z(\xi_i, \xi)}}. \tag{4.2.9}$$

4.2.2 Lyapunov-Schmidt reduction

The linearized problem

We recall that $\tilde{i}^* : \cup_{p>1} L^p(\Sigma) \to \overline{\mathrm{H}}^1$ which is defined in Section 3.1.2. For any $f \in L^p(\Sigma)$, we define that $i^*(f) := \tilde{i}^*(f - \bar{f})$, i.e. for any $h \in \overline{\mathrm{H}}^1$, $\langle i^*(f), h \rangle = \int_\Sigma (f - \bar{f}) h \, dv_g$. Let

$$f_1(u_1) = 2\varepsilon V_1 e^{u_1} - \overline{2\varepsilon V_1 e^{u_1}} \text{ and } f_2(u_2) = 2\rho_2 \left(\frac{V_2 e^{u_2}}{\int_\Sigma V_2 e^{u_2} dv_g} - 1 \right) \tag{4.2.10}$$

and

$$F(\mathbf{u}) = \begin{pmatrix} f_1(u_1) - \frac{1}{2} f_2(u_2) \\ \\ f_2(u_2) - \frac{1}{2} f_1(u_1) \end{pmatrix}. \tag{4.2.11}$$

The problem (4.2.1) has the following equivalent form,

$$\begin{cases} \mathbf{u} = (u_1, u_2) = i^*(F(\mathbf{u})), \\ u \in \mathcal{H} \end{cases}. \tag{4.2.12}$$

We consider the following linear operator associated with the problem (4.2.1):

$$\mathcal{L}_{\xi,\varepsilon} : \mathcal{H} \to \mathbb{R}, \quad \boldsymbol{\phi} \mapsto (L^1_{\xi,\varepsilon}(\boldsymbol{\phi}), L^2_{\xi,\varepsilon}(\boldsymbol{\phi})), \tag{4.2.13}$$

where

$$\begin{aligned} L^1_{\xi,\varepsilon}(\boldsymbol{\phi}) : \quad = \quad & -\Delta_g \phi_1 - \sum_{i=1}^m \left(\chi_i e^{-\varphi_i} e^{U_i} \phi_1 - \overline{\chi_i e^{-\varphi_i} e^{U_i} \phi_1} \right) \\ & + \rho_2 \left(\frac{V_2 e^{W_{2,\varepsilon}} \phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} - \frac{V_2 e^{W_{2,\varepsilon}} \int_\Sigma V_2 e^{W_{2,\varepsilon}} \phi_2 dv_g}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g \right)^2} \right) \end{aligned}$$

and

$$\begin{aligned} L^2_{\xi,\varepsilon}(\boldsymbol{\phi}) : \quad = \quad & -\Delta_g \phi_2 - 2\rho_2 \left(\frac{V_2 e^{W_{2,\varepsilon}} \phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} - \frac{V_2 e^{W_{2,\varepsilon}} \int_\Sigma V_2 e^{W_{2,\varepsilon}} \phi_2 dv_g}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g \right)^2} \right) \\ & + \frac{1}{2} \sum_{i=1}^m \left(\chi_i e^{-\varphi_i} e^{U_i} \phi_1 - \overline{\chi_i e^{-\varphi_i} e^{U_i} \phi_1} \right). \end{aligned}$$

Formally, for $i = 1, \cdots, m$, we deduce that the limiting operator of $L^1_{\xi,\varepsilon}$ is given by

$$-\Delta \phi - \frac{8}{(1 + |y|^2)^2} \phi,$$

through appropriate scaling around ξ_i in an isothermal chart (for details, see Lemma 4.2.1). It is well-known that the kernel space is generated by (refers to [DPEM12, BP98], for instance)

$$z^0(y) := \frac{1 - |y|^2}{1 + |y|^2}, z^i(y) := \frac{4y_j}{1 + |y|^2}, j = 1, \cdots, \mathrm{i}(\xi_i).$$

We define that

$$Z_i^0 = \begin{cases} 2\frac{\delta_i^2 - |y_{\xi_i}|^2}{\delta_i^2 + |y_{\xi_i}|^2} & \text{in } U_{4r_0}(\xi_i) \\ 0 & \text{in } \Sigma \setminus U_{4r_0}(\xi_i) \end{cases} \quad \text{and } Z_i^j = \begin{cases} \frac{4(y_{\xi_i})_j}{\delta_i^2 + |y_{\xi_i}|^2} & \text{in } U_{4r_0}(\xi_i) \\ 0 & \text{in } \Sigma \setminus U_{4r_0}(\xi_i) \end{cases},$$

for any $i = 1, \cdots, m$ and $j = 1, \cdots, \mathrm{i}(\xi_i)$ and then projects Z_i^j into the Hilbert space $\overline{\mathrm{H}}^1(\Sigma)$ by following equations:

$$\begin{cases} -\Delta_g PZ_i^j = \chi_i e^{-\varphi_i} e^{U_i} Z_i^j - \overline{\chi_i e^{-\varphi_i} e^{U_i} Z_i^j} & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} PZ_i^j = 0 & \text{on } \partial\Sigma \\ \int_\Sigma PZ_i^j = 0 \end{cases} . \tag{4.2.14}$$

Define the subspace $K_\xi = span\{PZ_i^j : i = 1, \cdots, m, j = 1, \cdots, \mathrm{i}(\xi_i)\} \times \{0\}$. To ensure the invertibility of the linear operator $\mathcal{L}_{\xi,\varepsilon}$, we confine the error term $\boldsymbol{\phi}$ to the orthogonal complement of $\in K_\xi$, denoted by $K_\xi^\perp$, where

$$K_\xi^\perp := \left\{ \boldsymbol{\phi} = (\phi_1, \phi_2) \in \mathcal{H} : \int_\Sigma \langle \nabla\phi_1, \nabla h_1 \rangle_g dv_g = 0 \text{ for any } \mathbf{h} = (h_1, h_2) \in K_\xi \right\}.$$

Furthermore, we introduce the orthogonal projections $\Pi_\xi : \mathcal{H} \to K_\xi$ and $\Pi_\xi^\perp : \mathcal{H} \to K_\xi$.

Lemma 4.2.1. *Let $\mathscr{D}$ be a compact subset of $\Xi_{k,m}$. For any $p > 1$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$,and $\xi \in \mathscr{D}$, any $\mathbf{h} = (h_1, h_2) \in (L^p(\Sigma))^2$ and $\boldsymbol{\phi} = (\phi_1, \phi_2) \in (W^{2,p}(\Sigma))^2 \cap K_\xi^\perp$ is the unique solution of*

$$\begin{cases} \mathcal{L}_{\xi,\varepsilon}(\boldsymbol{\phi}) = \mathbf{h} & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} \boldsymbol{\phi} = 0 & \text{on } \partial\Sigma \\ \int_\Sigma \boldsymbol{\phi} dv_g = 0 \end{cases} , \tag{4.2.15}$$

the following estimate holds

$$\|\boldsymbol{\phi}\| \le C |\log\varepsilon| \|\mathbf{h}\|_p,$$

where $\|\mathbf{h}\|_p = \|h_1\|_p + \|h_2\|_p$.

Proof. We will prove it by contradiction. Suppose Lemma 4.2.1 fails, i.e. there exist $p > 1$, a sequence of $\varepsilon_n \to 0, \xi^n \to \xi^* \in \Xi_{k,m}$, $\mathbf{h}_n := (h_{1,n}, h_{2,n}) \in (L^p(\Sigma))^2$ and $\boldsymbol{\phi}_n := (\phi_{1,n}, \phi_{2,n}) \in (W^{2,p}(\Sigma))^2 \cap K_\xi^\perp$ solves (4.2.15) for $\mathbf{h}_n$ satisfying that

$$\|\boldsymbol{\phi}_n\| = 1 \text{ and } |\log\varepsilon_n| \|\mathbf{h}_n\|_p := |\log\varepsilon_n| \sum_{i=1}^2 \|h_i\|_p \to 0, \tag{4.2.16}$$

as $n \to +\infty$. For simplicity, we still use the notations $\phi_j, h_j, \xi, \varepsilon$ instead of $\phi_{j,n}, h_{j,n}, \xi^n, \varepsilon_n$ for

$j = 1, 2$. We denote $\mathbb{R}_\xi = \mathbb{R}^2$ if $\xi \in \mathring{\Sigma}$ and $\mathbb{R}^2_+$ if $\xi \in \partial\Sigma$. Define that for $j = 1, 2, i = 1, \cdots, m$

$$\tilde{\phi}_{ji}(y) = \begin{cases} \chi(\delta_i|y|/r_0)\phi_j \circ y_{\xi_i}^{-1}(\delta_i y), & y \in \Omega_i := \frac{1}{\delta_i} B_{2r_0}^{\xi_i} \\ 0 & y \in \mathbb{R}_{\xi_i} \setminus \Omega_i \end{cases}.$$

Then we consider the following spaces for $\xi \in \Sigma$

$$\mathrm{L}_\xi := \left\{ u : \left\| \frac{u}{1+|y|^2} \right\|_{L^2(\mathbb{R}_\xi)} < +\infty \right\}$$

and

$$\mathrm{H}_\xi := \left\{ u : \|\nabla u\|_{L^2(\mathbb{R}_\xi)} + \left\| \frac{u}{1+|y|^2} \right\|_{L^2(\mathbb{R}_\xi)} < +\infty \right\}.$$

Step 1. *As* $\varepsilon \to 0$, *for any* $i = 1, \cdots, m$, $\tilde{\phi}_{1i} \to a_{1i}\frac{1-|y|^2}{1+|y|^2}$, *for some* $a_{1i} \in \mathbb{R}$ *weakly in* H_{ξ_i} *and strongly in* L_{ξ_i} *and* $\tilde{\phi}_{2i} \to 0$ *weakly in* $\overline{\mathrm{H}}^1(\Sigma)$ *and strongly in* $L^q(\Sigma)$ *for any* $q \geq 2$.

We first estimate the second component $\tilde{\phi}_{2i}$. Let $\psi_2 \in C_c^\infty(\Sigma \setminus \{\xi_1^*, \cdots, \xi_m^*\})$ with $\int_\Sigma \psi_2 dv_g = 0$. We use ψ_2 as a test function for $L^2_{\xi,\varepsilon}(\boldsymbol{\phi}) = h_2$. Then, it follows that

$$\begin{aligned} &\textstyle\int_\Sigma \langle \nabla \phi_2, \nabla \psi_2 \rangle_g dv_g - 2\rho_2 \int_\Sigma \left(\frac{V_2 e^{W_{2,\varepsilon}} \phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} - \frac{V_2 e^{W_{2,\varepsilon}} \int_\Sigma V_2 e^{W_{2,\varepsilon}} \phi_2 dv_g}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g\right)^2} \right) \psi_2 dv_g \\ &\qquad + \tfrac{1}{2} \textstyle\sum_{i=1}^m \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} \phi_1 \psi_2 dv_g = \int_\Sigma h_2 \psi_2 dv_g. \end{aligned} \tag{4.2.17}$$

The asymptotic expansion for (4.2.6) implies that

$$\begin{aligned} V_2 e^{W_{2,\varepsilon}} &= V_2 e^{z(\cdot,\xi) - \frac{1}{2}\sum_{i=1}^m \chi_i \log \frac{1}{(\delta_i^2 + |y_{\xi_i}|^2)^2} - \frac{1}{2}\sum_{i=1}^m \varrho(\xi_i) H^g(\cdot,\xi_i) + \mathcal{O}(\varepsilon|\log\varepsilon|)} \\ &= \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)} + \mathcal{O}(\varepsilon|\log\varepsilon|), \end{aligned} \tag{4.2.18}$$

in view of $G^g(\cdot,\xi_i) = H^g(\cdot,\xi_i) + \frac{4}{\varrho(\xi_i)}\chi_i \log\frac{1}{|y_{\xi_i}|}$. By the assumption (4.2.16), we have $\phi_1 \rightharpoonup \phi_1^*$ and $\phi_2 \rightharpoonup \phi_2^*$ weakly in $\overline{\mathrm{H}}^1(\Sigma)$ and strongly in $L^q(\Sigma)$ for any $q \geq 2$. The Sobolev's inequality yields that

$$\left| \int_\Sigma h_2 \psi_2 \right| \leq \|\mathbf{h}\|_p \|\psi_2\|_{p'} \leq \|\mathbf{h}\|_p \|\psi_2\| = o(1/|\log\varepsilon|) \to 0,$$

where $p, p' > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Passing the limit of (4.2.17), the estimates (4.2.18) and (4.2.16) yield that

$$\begin{aligned} &\int_\Sigma \langle \nabla \phi_2^*, \nabla \psi_2 \rangle_g dv_g \\ = \;& 2\rho_2 \left(\int_\Sigma \frac{\tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)} \psi_2}{\int_\Sigma \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)} dv_g} dv_g - \frac{\int_\Sigma \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)} \psi_2 dv_g \int_\Sigma \int_\Sigma \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)} \phi_2^* dv_g}{(\int_\Sigma \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)} dv_g)^2} \right) \end{aligned}$$

By assumption, $\|\phi_2^*\| \leq 1$. It follows that $\phi_2^* \in \overline{\mathrm{H}}^1(\Sigma)$ solves the problem (4.1.5) in the weak sense. Through the hypothesis (H1), we obtain that $\phi_2^* = 0$.

Applying ϕ_1 as a test function for $L^1_{\xi,\varepsilon}(\boldsymbol{\phi}) - L^2_{\xi,\varepsilon}(\boldsymbol{\phi}) = h_1 - h_2$, since $\int_\Sigma \phi_1 dv_g = 0$,

$$\begin{aligned}\int_\Sigma (h_1 - h_2)\phi_1 dv_g \quad &= \quad \int_\Sigma \langle \nabla(\phi_1 - \phi_2), \nabla\phi_1\rangle_g dv_g - \frac{3}{2}\sum_{i=1}^m \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} \phi_1^2 dv_g \\ &\quad + 3\rho_2 \int_\Sigma \left(\frac{V_2 e^{W_{2,\varepsilon}} \phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} - \frac{V_2 e^{W_{2,\varepsilon}} \int_\Sigma V_2 e^{W_{2,\varepsilon}} \phi_2 dv_g}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g\right)^2} \right) \phi_1 dv_g. \end{aligned} \tag{4.2.19}$$

The Sobolev's inequality yields that

$$\left| \int_\Sigma (h_1 - h_2)\phi_1 \right| \leq \|\mathbf{h}\|_p \|\phi_1\|_{p'} \leq \|\mathbf{h}\|_p \|\phi_1\| = o(|\log\varepsilon|) \to 0,$$

where $p, p' > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Using the Hölder inequality, we derive that $|\int_\Sigma \langle \nabla\phi_1, \nabla\phi_2\rangle_g dv_g| \leq \|\phi_1\|\|\phi_2\| \leq 1$. Hence,

$$\sum_{i=1}^m \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} \phi_1^2 dv_g = 2\rho_2 \int_\Sigma \left(\frac{V_2 e^{W_{2,\varepsilon}} \phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} - \frac{V_2 e^{W_{2,\varepsilon}} \int_\Sigma V_2 e^{W_{2,\varepsilon}} \phi_2 dv_g}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g\right)^2} \right) \phi_1 dv_g + \mathcal{O}(1). \tag{4.2.20}$$

Applying ϕ_1 as a test function for $L^1_{\xi,\varepsilon}(\boldsymbol{\phi}) = h_1$, we can deduce that

$$\begin{aligned}\int_\Sigma h_1 \phi_1 dv_g \quad &= \quad \int_\Sigma \langle \nabla\phi_1, \nabla\phi_1\rangle_g dv_g - \sum_{i=1}^m \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} \phi_1^2 dv_g \\ &\quad + \rho_2 \int_\Sigma \left(\frac{V_2 e^{W_{2,\varepsilon}} \phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} - \frac{V_2 e^{W_{2,\varepsilon}} \int_\Sigma V_2 e^{W_{2,\varepsilon}} \phi_2 dv_g}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g\right)^2} \right) \phi_1 dv_g, \end{aligned} \tag{4.2.21}$$

in view of $\int_\Sigma \phi_1 = 0$. Similarly, we have

$$\begin{aligned}\sum_{i=1}^m \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} \phi_1^2 dv_g \quad &= \quad \rho_2 \int_\Sigma \left(\frac{V_2 e^{W_{2,\varepsilon}} \phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} - \frac{V_2 e^{W_{2,\varepsilon}} \int_\Sigma V_2 e^{W_{2,\varepsilon}} \phi_2 dv_g}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g\right)^2} \right) \phi_1 dv_g \\ &\quad + \mathcal{O}(1) \\ &\overset{(4.2.20)}{=} \frac{1}{2}\sum_{i=1}^m \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} \phi_1^2 + \mathcal{O}(1). \end{aligned}$$

Consequently,

$$\sum_{i=1}^m \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} \phi_1^2 dv_g = \mathcal{O}(1). \tag{4.2.22}$$

By straightforward calculation, we obtain that

$$\sum_{i=1}^m \int_{\mathbb{R}_{\xi_i}} \frac{1}{(1+|y|^2)^2} (\tilde{\phi}_{1i}(y))^2 dy = \sum_{i=1}^m \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} \phi_1^2 dv_g + \mathcal{O}(1) = \mathcal{O}(1).$$

By the assumption $\|\boldsymbol{\phi}\| = 1$, we immediately have $\int_{\mathbb{R}_{\xi_i}} |\nabla\tilde{\phi}_{1i}|^2 \leq \int_\Sigma |\nabla\phi_1|_g^2 dv_g \leq 1$. It follows that $\tilde{\phi}_{1i}$ is uniformly bounded in H_{ξ_i}. Applying Proposition A.1 in [MPW16], $\mathrm{L}_{\xi_i} \hookrightarrow \mathrm{H}_{\xi_i}$ is compact embedding, then up to a subsequence $\tilde{\phi}_{1i} \to \tilde{\phi}_{1i}^*$, which is weakly convergent in H_{ξ_i} and strongly in L_{ξ_i}. For any $q > 1$, we estimate the L^q-norm of $\chi_i e^{-\varphi_i} e^{U_i}$ by direct calculation as follows:

$$\int_\Sigma \left|\chi_i e^{-\varphi_i} e^{U_i}\right|^q dv_g = \int_{U_{2r_0}(\xi_i)} \frac{\chi^q\left(\frac{|y|}{r_0}\right)\delta_i^{2-2q}}{(1+|y|^2)^{2q}} dy = \mathcal{O}(\delta_i^{2(1-q)}). \tag{4.2.23}$$

For any $\varphi \in C_c^\infty(\mathbb{R}_{\xi_i})$, assume that $\operatorname{supp}\varphi \subset B_{R_0}(0)$. If $\delta_i < \frac{r_0}{R_0}$, then $\operatorname{supp}\nabla\chi\left(\frac{|y|}{r_0}\right) \cap \operatorname{supp}\varphi\left(\frac{1}{\delta_i}y\right) = \emptyset$. For any $\Phi \in \overline{\mathrm{H}}^1(\Sigma)$,

$$\begin{aligned} 0 &= \int_{B_{2r_0}^{\xi_i}} \Phi \circ y_{\xi_i}^{-1}(y)\nabla\chi\left(\frac{|y|}{r_0}\right) \cdot \nabla\varphi\left(\frac{1}{\delta_i}y\right) dy \\ &= \int_{B_{2r_0}^{\xi_i}} h\left(\frac{1}{\delta_i}y\right) \nabla(\Phi \circ y_{\xi_i}^{-1}(y)) \cdot \nabla\chi\left(\frac{|y|}{r_0}\right) dy. \end{aligned} \tag{4.2.24}$$

We observe that for any $q > 1$

$$\int_\Sigma \chi_i \varphi^q \left(\frac{1}{\delta_i} y_{\xi_i}(x)\right) dv_g(x) = O(\delta_i^q) \quad (\delta_i \to 0) \tag{4.2.25}$$

and

$$\int_\Sigma \chi_i \left|\nabla\varphi\left(\frac{1}{\delta_i} y_{\xi_i}(x)\right)\right|_g^2 dv_g(x) = \mathcal{O}(1) \quad (\delta_i \to 0). \tag{4.2.26}$$

Applying (4.2.18), we have

$$\begin{aligned} & 2\rho_2 \left(\frac{V_2 e^{W_{2,\varepsilon}}\phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} - \frac{V_2 e^{W_{2,\varepsilon}} \int_\Sigma V_2 e^{W_{2,\varepsilon}}\phi_2 dv_g}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g\right)^2}\right) \\ = \; & 2\rho_2 \left(\frac{\tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)}\phi_2}{\int_\Sigma \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)} dv_g} - \frac{\tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)} \int_\Sigma \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)}\phi_2 dv_g}{(\int_\Sigma \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)} dv_g)^2}\right) + \mathcal{O}(\varepsilon|\log\varepsilon|). \end{aligned} \tag{4.2.27}$$

It follows that

$$\begin{aligned} & 2\rho_2 \int_\Sigma \left(\frac{V_2 e^{W_{2,\varepsilon}}\phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} - \frac{V_2 e^{W_{2,\varepsilon}} \int_\Sigma V_2 e^{W_{2,\varepsilon}}\phi_2 dv_g}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g\right)^2}\right) \chi_i \varphi\left(\frac{y_{\xi_i}}{\delta_i}\right) dv_g \\ = \; & \mathcal{O}\left(\delta_i^{2(1-\frac{1}{q})}\|\phi_2\|_q + \varepsilon|\log\varepsilon|\right) = o(1), \end{aligned} \tag{4.2.28}$$

in which we applied that $\phi_2 \to 0$ in $L^q(\Sigma)$ for any $q \geq 2$. Assume that $0 < \varepsilon < \frac{r_0^2}{d_i^2 R_0^2}$, as $\varepsilon \to 0$, the estimates (4.2.23)-(4.2.28) with the Hölder inequality imply that

$$
\begin{aligned}
&\int_{\mathbb{R}_{\xi_i}} \nabla\tilde{\phi}_{1i}\nabla\varphi dy = \int_{B^{\xi_i}_{2r_0}} \nabla\left(\chi\left(\frac{|y|}{r_0}\right)\phi_1\circ y_{\xi_i}^{-1}(y)\right)\cdot\nabla\varphi\left(\frac{1}{\delta_i}y\right)dy\\
=\ &\int_{B^{\xi_i}_{2r_0}} \nabla\phi_1\circ y_{\xi_i}^{-1}(y)\cdot\nabla\left(\chi\left(\frac{|y|}{r_0}\right)\varphi\left(\frac{1}{\delta_i}y\right)\right)dy\\
=\ &\int_{\Sigma}\left\langle\nabla\phi_1,\nabla\left(\chi_i\varphi\left(\frac{1}{\delta_i}y_{\xi_i}\right)\right)\right\rangle_g dv_g\\
=\ &\int_{\Sigma}\chi_i e^{-\varphi_i}e^{U_i}\phi_1\varphi\left(\frac{1}{\delta_i}y_{\xi_i}\right)dv_g-\int_{\Sigma}\chi_i e^{-\varphi_i}e^{U_i}\phi_1 dv_g\int_{\Sigma}\chi_i\varphi\left(\frac{1}{\delta_i}y_{\xi_i}\right)dv_g\\
&-\rho_2\int_{\Sigma}\left(\frac{V_2e^{W_{2,\varepsilon}}\phi_2}{\int_{\Sigma}V_2e^{W_{2,\varepsilon}}dv_g}-\frac{V_2e^{W_{2,\varepsilon}}\int_{\Sigma}V_2e^{W_{2,\varepsilon}}\phi_2dv_g}{\left(\int_{\Sigma}V_2e^{W_{2,\varepsilon}}dv_g\right)^2}\right)\chi_i\varphi\left(\frac{1}{\delta_i}y_{\xi_i}\right)dv_g\\
&+\int_{\Sigma}h_1\chi_i\varphi\left(\frac{1}{\delta_i}y_{\xi_i}\right)dv_g\\
=\ &\int_{\mathbb{R}_{\xi_i}}\frac{8}{(1+|y|^2)^2}\tilde{\phi}_{1i}(y)\varphi(y)dy+\mathcal{O}(\delta_i^{1+2(1-q)/q})+o(1)+o(|\log\varepsilon|)\\
=\ &\int_{\mathbb{R}_{\xi_i}}\frac{8}{(1+|y|^2)^2}\tilde{\phi}_{1i}(y)\varphi(y)dy+o(1),
\end{aligned}
$$

for $q>1$ sufficiently close to 1 such that $1+\frac{2(1-q)}{q}>0$. Thus, $\tilde{\phi}_{1i}$ converges to the solution ϕ^*_{1i} of

$$
\begin{cases}
-\Delta\phi=\frac{8}{(1+|y|^2)^2}\phi & \text{in } \mathbb{R}_{\xi_i}\\
\partial_{y_2}\phi=0 & \text{on } \partial\mathbb{R}_{\xi_i}\\
\int_{\mathbb{R}_{\xi_i}}|\nabla\phi(y)|^2dy<+\infty
\end{cases},
\tag{4.2.29}
$$

in the distribution sense. According to the regularity theory, ϕ^*_{1i} is a smooth solution on the space $\mathbb{R}_{\xi_i}$, for any $i=1,\cdots,m$. The result [DPEM12] implies that the solutions space of (4.2.29) is generated by the following functions: for $j=1,\cdots,\mathrm{i}(\xi_i)$

$$
z^0(y):=\frac{1-|y|^2}{1+|y|^2},\quad z^j(y):=\frac{4y_j}{1+|y|^2}.
$$

Since $\boldsymbol{\phi}\in K^{\perp}_{\xi}$, for any $i=1,\cdots,m$ and $j=1,\cdots,\mathrm{i}(\xi_i)$,

$$
\begin{aligned}
&\frac{32}{\delta_i}\int_{\mathbb{R}_{\xi_i}}\frac{y_j}{(|y|^2+1)^3}\tilde{\phi}^*_{1i}dy=\lim_{\varepsilon\to0}\frac{32}{\delta_i}\int_{\Omega_i}\frac{y_j}{(|y|^2+1)^3}\tilde{\phi}_{1i}\chi(\delta_i|y|/r_0)dy\\
=\ &\lim_{\varepsilon\to0}32\int_{B^{\xi_i}_{2r_0}}\frac{y_j}{(|y|^2+\delta_i^2)^3}\phi_i\circ y_{\xi_i}^{-1}(y)\chi\left(\frac{|y|}{r_0}\right)dy\\
=\ &\lim_{\varepsilon\to0}\int_{U_{2r_0}(\xi_i)}\chi_ie^{-\varphi_i}e^{U_i}Z^j_i\phi_idv_g=\lim_{\varepsilon\to0}\langle PZ^j_i,\phi_i\rangle=0.
\end{aligned}
$$

Consequently, $\int_{\mathbb{R}_{\xi_i}}\frac{|y|^2-1}{(|y|^2+1)^3}\tilde{\phi}^*_{1i}dy=\int_{\mathbb{R}_{\xi_i}}\frac{y_j}{(|y|^2+1)^3}\tilde{\phi}^*_{1i}dy=0$. It follows that $\tilde{\phi}^*_{1i}=a_{1i}\frac{1-|y|^2}{1+|y|^2}$.

Step 2. *For $i = 1, \cdots, m$,*

$$\int_{\Omega_i} \frac{8}{(1+|y|^2)^2}\tilde{\phi}_{1i}(y)dy = o(|\log\varepsilon|^{-1}).$$

Applying that PZ_i^0 as a test function of $L^1_{\xi,\varepsilon}(\boldsymbol{\phi}) = h_1$, we have

$$\begin{aligned}
&\int_\Sigma 8\chi_i e^{-\varphi_i}\frac{\delta_i^2}{\left(\delta_i^2+|y_{\xi_i}|^2\right)^2}\phi_1 Z_i^0 dv_g = \int_\Sigma 8\chi_i e^{-\varphi_i}\frac{\delta_i^2}{\left(\delta_i^2+|y_{\xi_i}|^2\right)^2}\phi_i PZ_i^0 dv_g \\
&-\rho_2\int_\Sigma\left(\frac{V_2 e^{W_{2,\varepsilon}}\phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}}dv_g} - \frac{V_2 e^{W_{2,\varepsilon}}\int_\Sigma V_2 e^{W_{2,\varepsilon}}\phi_2 dv_g}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}}dv_g\right)^2}\right)PZ_i^0 dv_g + \int_\Sigma h_1 PZ_i^0 dv_g.
\end{aligned}$$

It follows that

$$\begin{aligned}
0 \;=\; & \int_\Sigma 8\chi_i e^{-\varphi_i}\frac{\delta_i^2}{\left(\delta_i^2+|y_{\xi_i}|^2\right)^2}\phi_i\left(PZ_i^0 - Z_i^0\right)dv_g \\
& -\rho_2\int_\Sigma\left(\frac{V_2 e^{W_{2,\varepsilon}}\phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}}dv_g} - \frac{V_2 e^{W_{2,\varepsilon}}\int_\Sigma V_2 e^{W_{2,\varepsilon}}\phi_2 dv_g}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}}dv_g\right)^2}\right)PZ_i^0 dv_g + \int_\Sigma h_1 PZ_i^0 dv_g.
\end{aligned}$$

By Lemma B.0.2, we have $\|PZ_i^0\|_{p'} = \mathcal{O}(1)$, for $\frac{1}{p}+\frac{1}{p'} = 1$. Then, the assumption $\|\mathbf{h}\|_p = o(|\log\varepsilon|^{-1})$ yields

$$\left|\int_\Sigma h_1 PZ_i^0 dv_g\right| \le \|PZ_i\|_{p'}\|h_1\|_p = o\left(\frac{1}{|\log\varepsilon|}\right).$$

Lemma B.0.2 with (4.2.27) implies that

$$\begin{aligned}
& \rho_2\int_\Sigma\left(\frac{V_2 e^{W_{2,\varepsilon}}\phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}}dv_g} - \frac{V_2 e^{W_{2,\varepsilon}}\int_\Sigma V_2 e^{W_{2,\varepsilon}}\phi_2 dv_g}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}}dv_g\right)^2}\right)PZ_i^0 dv_g \\
=\; & \mathcal{O}\left(\delta_i^{2(1-\frac{1}{q})}|\log\delta_i|\|\phi_2\|_q + \varepsilon|\log\varepsilon|\right) = o(\varepsilon^{\frac{1}{2}}|\log\varepsilon|),
\end{aligned}$$

for $q = 2$. Applying Lemma B.0.2 again, we derive that

$$\begin{aligned}
& (\log\varepsilon)\int_\Sigma 8\chi_i e^{-\varphi_i}\frac{\delta_i^2}{\left(\delta_i^2+|y_{\xi_i}|^2\right)^2}\phi_i\left(PZ_i^0 - Z_i^0\right)dv_g \\
=\; & (\log\varepsilon)\int_\Sigma 8\chi_i e^{-\varphi_i}\frac{\delta_i^2}{\left(\delta_i^2+|y_{\xi_i}|^2\right)^2}\phi_i\left(1+\mathcal{O}(\delta_i^2|\log\delta_i|)\right)dv_g \\
=\; & (1+\mathcal{O}(\delta_i^2|\log\delta_i|))(\log\varepsilon)\int_{\Omega_i}\frac{8}{\left(1+|y|^2\right)^2}\tilde{\phi}_{1i}(y)dy + \mathcal{O}(|\log\varepsilon|\delta_i^2) \\
=\; & (\log\varepsilon)\int_{\Omega_i}\frac{8}{\left(1+|y|^2\right)^2}\tilde{\phi}_{1i}(y)dy + o(1),
\end{aligned}$$

as $\varepsilon\to 0$. Thus, we have $\int_{\Omega_i}\frac{8\log\varepsilon}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_{1i}(y)dy = o(1)$ as $\varepsilon\to 0$.

Step 3. *Construct the contradiction.*

Using PU_i as a test function for (4.2.15), we derive that

$$\begin{aligned}\int_\Sigma 8\chi_i e^{-\varphi_i}\frac{\delta_i^2}{\left(\delta_i^2+|y_{\xi_i}|^2\right)^2}\phi_i dv_g &= \int_\Sigma 8\chi_i e^{-\varphi_i}\frac{\delta_i^2}{\left(\delta_i^2+|y_{\xi_i}|^2\right)^2}\phi_i PU_i dv_g \\ -\rho_2\int_\Sigma\left(\frac{V_2e^{W_{2,\varepsilon}}\phi_2}{\int_\Sigma V_2e^{W_{2,\varepsilon}}dv_g}-\frac{V_2e^{W_{2,\varepsilon}}\int_\Sigma V_2e^{W_{2,\varepsilon}}\phi_2 dv_g}{\left(\int_\Sigma V_2e^{W_{2,\varepsilon}}dv_g\right)^2}\right)&PU_i dv_g+\int_\Sigma h_i PU_i dv_g.\end{aligned}\tag{4.2.30}$$

The left hand side of (4.2.30) equals $\int_\Sigma 8\chi_i e^{-\varphi_i}\frac{\delta_i^2}{\left(\delta_i^2+|y_{\xi_i}|^2\right)^2}\phi_i dv_g = \int_{\Omega_i}\frac{8}{(1+|y|^2)^2}\tilde{\phi}_{1i}(y)dy+\mathcal{O}(\delta_i^2)$ $= o(1)$ by the result of Step 2. Lemma B.0.1 yields that $\|PU_i\|_\infty = \mathcal{O}(|\log\varepsilon|)$. We deduce that

$$\left|\int_\Sigma h_i PU_i dv_g\right| \leq \|h_i\|_p\|PU_i\|_{p'} = o(1),$$

where $\frac{1}{p}+\frac{1}{p'}=1$. By (4.2.27) and Lemma B.0.1, we have

$$\rho_2\int_\Sigma\left(\frac{V_2e^{W_{2,\varepsilon}}\phi_2}{\int_\Sigma V_2e^{W_{2,\varepsilon}}dv_g}-\frac{V_2e^{W_{2,\varepsilon}}\int_\Sigma V_2e^{W_{2,\varepsilon}}\phi_2 dv_g}{\left(\int_\Sigma V_2e^{W_{2,\varepsilon}}dv_g\right)^2}\right)PU_i dv_g = \mathcal{O}\left(\delta_i^{\frac{1}{2}}\|\phi_2\|+\varepsilon|\log\varepsilon|\right) = o(1),$$

for $q=2$. From straightforward calculation, we derive that

$$\begin{aligned}\delta_i\int_{\Omega_i}\frac{8}{(1+|y|^2)^2}|\tilde{\phi}_{1i}(y)||y|dy &\leq 8\delta_i\delta_i^{\frac{2(1-q)}{q}}\|\phi_i\|_{q'}\left(\int_{\mathbb{R}^2}\left(\frac{|y|}{(1+|y|^2)^2}\right)^q dy\right)^{1/q} \\ &= O\left(\delta_i^{\frac{2-q}{q}}\right) = o(1).\end{aligned}$$

where $q\in(1,2)$ such that $\frac{1}{q}+\frac{1}{q'}=1$. Applying Lemma B.0.1, Step 1 and Step 2, we deduce that as $\varepsilon\to 0$

$$\begin{aligned}&\int_\Sigma 8\chi_i e^{-\varphi_i}\frac{\delta_i^2}{\left(\delta_i^2+|y_{\xi_i}|^2\right)^2}\phi_1 PU_i dv_g \\ =\ &\int_\Sigma 8\chi_i e^{-\varphi_i}\frac{\delta_i^2}{\left(\delta_i^2+|y_{\xi_i}|^2\right)^2}\phi_1\left(-2\chi_i\log\left(\delta_i^2+|y_{\xi_i}|^2\right)+\varrho(\xi_i)H^g(\cdot,\xi_i)\ +\mathcal{O}(\delta_i^2|\log\delta_i|)\right)dv_g \\ =\ &\int_{\Omega_i}\frac{8}{(1+|y|^2)^2}\tilde{\phi}_{1i}(y)\left(-4\log\delta_i-2\log\left(1+|y|^2\right)+\varrho(\xi_i)H^g(\xi_i,\xi_i)\right)dy \\ &+O\left(\int_{\Omega_i}\frac{8}{(1+|y|^2)^2}\left|\tilde{\phi}_{1i}(y)\right|\left(\delta_i|y|+\delta_i^2|\log\delta_i|\right)dy\right)+\mathcal{O}(\delta_i^2) \\ \to\ &-2a_{1i}\int_{\mathbb{R}_{\xi_i}}\frac{8}{(1+|y|^2)^2}\frac{1-|y|^2}{1+|y|^2}\log(1+|y|^2)dy = \varrho(\xi_i)a_{1i},\end{aligned}$$

in which the last equality used the fact that $\int_{\mathbb{R}^2}\frac{8}{(1+|y|^2)^2}\frac{1-|y|^2}{1+|y|^2}\log(1+|y|^2)dy = -4\pi$.

Consequently, $a_{1i}=0$ for any $i=1,\cdots,m$. We ϕ_i as test functions for $L^i_{\xi,\varepsilon}(\boldsymbol{\phi}) = h_i$. For $q\geq 2$, since $\phi_2\to 0$ strongly in $L^q(\Sigma)$ and $\tilde{\phi}_{1i}\to 0$ strongly in L_{ξ_i}, we have the following contradiction: as $\varepsilon \to 0$

$$\begin{aligned}
1 &= \sum_{i=1}^{2} \|\phi_i\|^2 \\
&= \sum_{i=1}^{m} \int_\Sigma 8\chi_i e^{-\varphi_i} \frac{\delta_i^2}{\left(\delta_i^2 + |y_{\xi_i}|^2\right)^2} \phi_1^2 dv_g - \rho_2 \int_\Sigma \left(\frac{V_2 e^{W_{2,\varepsilon}} \phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} - \frac{V_2 e^{W_{2,\varepsilon}} \int_\Sigma V_2 e^{W_{2,\varepsilon}} \phi_2 dv_g}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g\right)^2} \right) \phi_1 dv_g \\
&\quad +2\rho_2 \left(\int_\Sigma \frac{V_2 e^{W_{2,\varepsilon}} \phi_2^2 dv_g}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} - \frac{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} \phi_2 dv_g\right)^2}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g\right)^2} \right) \\
&\quad -\frac{1}{2} \sum_{i=1}^{m} \int_\Sigma 8\chi_i e^{-\varphi_i} \frac{\delta_i^2}{\left(\delta_i^2 + |y_{\xi_i}|^2\right)^2} \phi_1 \phi_2 dv_g + \sum_{i=1}^{2} \int_\Sigma h_i \phi_i dv_g \\
&\le \sum_{i=1}^{m} \|\tilde{\phi}_{1i}\|_{\mathrm{L}_\xi^2}^2 + \mathcal{O}\left(\|\phi_2\|\right) + o\left(\frac{1}{|\log \varepsilon|}\right) \to 0.
\end{aligned}$$

□

Nonlinear problem

The expected solution $\mathbf{W}_\varepsilon + \boldsymbol{\phi}_\varepsilon$ solves (4.2.1) if and only if $\boldsymbol{\phi}_\varepsilon$ solves the following problem:

$$\mathcal{L}_{\xi,\varepsilon}(\boldsymbol{\phi}) = \mathcal{S}_{\xi,\varepsilon}(\boldsymbol{\phi}) + \mathcal{N}_{\xi,\varepsilon}(\phi) + \mathcal{R}_{\xi,\varepsilon}. \tag{4.2.31}$$

Here, the linear operator $\mathcal{L}_{\xi,\varepsilon}$ is defined by (4.2.13), the higher order linear operator $\mathcal{S}_{\xi,\varepsilon}(\boldsymbol{\phi}) := \left(S_{\xi,\varepsilon}^1(\boldsymbol{\phi}), S_{\xi,\varepsilon}^2(\boldsymbol{\phi})\right)$, where for

$$S_{\xi,\varepsilon}^1(\boldsymbol{\phi}) := \left(-\sum_{i=1}^{m} \chi_i e^{-\varphi_i} e^{U_i} + 2\varepsilon V_1 e^{W_{1,\varepsilon}}\right) \phi_1 - \overline{\left(-\sum_{i=1}^{m} \chi_i e^{-\varphi_i} e^{U_i} + 2\varepsilon V_1 e^{W_{1,\varepsilon}}\right) \phi_1} \quad \text{and}$$

$$S_{\xi,\varepsilon}^2(\boldsymbol{\phi}) = -\frac{1}{2} \left(-\sum_{i=1}^{m} \chi_i e^{-\varphi_i} e^{U_i} + 2\varepsilon V_1 e^{W_{1,\varepsilon}}\right) \phi_1 + \frac{1}{2} \overline{\left(-\sum_{i=1}^{m} \chi_i e^{-\varphi_i} e^{U_i} + 2\varepsilon V_1 e^{W_{1,\varepsilon}}\right) \phi_1},$$

the nonlinear term $\mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi}) := \left(N_{\xi,\varepsilon}^1(\boldsymbol{\phi}), N_{\xi,\varepsilon}^2(\boldsymbol{\phi})\right)$, where

$$\begin{aligned}
N_{\xi,\varepsilon}^1(\boldsymbol{\phi}) :&= 2\varepsilon V_1 e^{W_{1,\varepsilon}} \left(e^{\phi_1} - 1 - \phi_1\right) - \overline{2\varepsilon V_1 e^{W_{1,\varepsilon}} \left(e^{\phi_1} - 1 - \phi_1\right)} \\
&\quad - \rho_2 \left(\frac{V_2 e^{W_{2,\varepsilon}+\phi_2}}{\int_\Sigma V_2 e^{W_{2,\varepsilon}+\phi_2} dv_g} - \frac{V_2 e^{W_{2,\varepsilon}} \phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} + \frac{V_2 e^{W_{2,\varepsilon}} \int_\Sigma V_2 e^{W_{2,\varepsilon}} \phi_2 dv_g}{(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g)^2} - \frac{V_2 e^{W_{2,\varepsilon}}}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} \right),
\end{aligned}$$

$$\begin{aligned}
N_{\xi,\varepsilon}^2(\boldsymbol{\phi}) : &= 2\rho_2 \left(\frac{V_2 e^{W_{2,\varepsilon}+\phi_2}}{\int_\Sigma V_2 e^{W_{2,\varepsilon}+\phi_2} dv_g} - \frac{V_2 e^{W_{2,\varepsilon}} \phi_2}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} + \frac{V_2 e^{W_{2,\varepsilon}} \int_\Sigma V_2 e^{W_{2,\varepsilon}} \phi_2 dv_g}{(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g)^2} - \frac{V_2 e^{W_{2,\varepsilon}}}{\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g} \right) \\
&\quad -\varepsilon V_1 e^{W_{1,\varepsilon}} \left(e^{\phi_1} - 1 - \phi_1\right) - \overline{2\varepsilon V_1 e^{W_{1,\varepsilon}} \left(e^{\phi_1} - 1 - \phi_1\right)},
\end{aligned}$$

and the error term $\mathcal{R}_{\xi,\varepsilon} = (R^1_{\xi,\varepsilon}, R^2_{\xi,\varepsilon})$, where

$$R^1_{\xi,\varepsilon} = \Delta_g W_{1,\varepsilon} + 2\varepsilon V_1 e^{W_{1,\varepsilon}} - \overline{2\varepsilon V_1 e^{W_{1,\varepsilon}}} - \rho_2 \left(\frac{V_2 e^{W_{2,\varepsilon}}}{\int_\Sigma V_2 W_{2,\varepsilon} dv_g} - 1 \right),$$
$$\text{and } R^2_{\xi,\varepsilon} = \Delta_g W_{2,\varepsilon} + 2\rho_2 \left(\frac{V_2 e^{W_{2,\varepsilon}}}{\int_\Sigma V_2 W_{2,\varepsilon} dv_g} - 1 \right) - \varepsilon V_1 e^{W_{1,\varepsilon}} + \overline{\varepsilon V_1 e^{W_{1,\varepsilon}}}.$$

Firstly, we will show that the error term goes to zero as $\varepsilon \to 0$.

Lemma 4.2.2. *There exist $p_0 > 1$ and $\varepsilon_0 > 0$ such that for any $p \in (1, p_0)$ we have as $\varepsilon \to 0$*

$$\|\mathcal{R}_{\xi,\varepsilon}\|_p := \sum_{i=1}^2 \|R^i_{\xi,\varepsilon}\|_p = O\left(\varepsilon^{\frac{2-p}{2p}}\right). \tag{4.2.32}$$

Proof. By (4.2.18),

$$\begin{aligned} R^1_{\xi,\varepsilon} &= \sum_{i=1}^m \Delta_g PU_i - \frac{1}{2}\Delta_g z(\cdot,\xi) + 2\varepsilon V_1 e^{W_{1,\varepsilon}} - \overline{2\varepsilon V_1 e^{W_{1,\varepsilon}}} - \rho_2 \left(\frac{\tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)}}{\int_\Sigma \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi) dv_g}} - 1 \right) \\ &\quad + \mathcal{O}(\varepsilon|\log\varepsilon|) \\ &= 2\varepsilon V_1 e^{W_{1,\varepsilon}} - \sum_{i=1}^m \chi_i e^{-\varphi_i} e^{U_i} - \overline{2\varepsilon V_1 e^{W_{1,\varepsilon}} - \sum_{i=1}^m \chi_i e^{-\varphi_i} e^{U_i}} + \mathcal{O}(\varepsilon|\log\varepsilon|). \end{aligned}$$

Similarly, $R^2_{\xi,\varepsilon} = -\varepsilon V_1 e^{W_{1,\varepsilon}} + \frac{1}{2}\sum_{i=1}^m \chi_i e^{-\varphi_i} e^{U_i} + \overline{\varepsilon V_1 e^{W_{1,\varepsilon}} - \frac{1}{2}\sum_{i=1}^m \chi_i e^{-\varphi_i} e^{U_i}} + \mathcal{O}(\varepsilon|\log\varepsilon|)$. For any $p \in (1,2)$, the Hölder inequality yields that

$$\left| \int_\Sigma \sum_{i=1}^m \chi_i e^{-\varphi_i} e^{U_i} - 2\varepsilon V_1 e^{W_{1,\varepsilon}} dv_g \right| \le \left\| \sum_{i=1}^m \chi_i e^{-\varphi_i} e^{U_i} - 2\varepsilon V_1 e^{W_{1,\varepsilon}} \right\|_p |\Sigma|_g^{1-\frac{1}{p}}.$$

Then Lemma B.0.5 implies that for any $p \in (0,1)$, $\|R^i_{\xi,\varepsilon}\|_p = \mathcal{O}(\varepsilon^{\frac{2-p}{2p}}), i = 1, 2$. We take $p_0 = 2$, then the proof is complete. □

The following lemma shows that the higher order linear operator $\mathcal{S}_{\xi,\varepsilon}$ is bounded on $\mathcal{H}$ and the operator norm goes to zero as $\varepsilon \to 0$.

Lemma 4.2.3. *There exists $s_0 > 1$ such that for any $p, r \in (1,2)$ with $pr \in (1, s_0)$, as $\varepsilon \to 0$*

$$\|\mathcal{S}_{\xi,\varepsilon}(\phi)\|_p = O\left(\varepsilon^{\frac{2-pr}{2pr}} \|\phi\|\right), \text{ for } \phi \in \mathcal{H}.$$

Proof. By Lemma B.0.5 and the Hölder inequality, we derive that as $\varepsilon \to 0$

$$\begin{aligned}\|\mathcal{S}_{\xi,\varepsilon}(\boldsymbol{\phi})\|_p = \sum_{i=1}^{2} \|S^i_{\xi,\varepsilon}(\boldsymbol{\phi})\|_p &= O\left(\left\|\left(-\sum_{i=1}^{m} \chi_i e^{-\varphi_i} e^{U_i} + 2\varepsilon V_1 e^{W_{1,\varepsilon}}\right)\phi_1\right\|_p\right)\\ &= O\left(\left(\left\|2\varepsilon V_1 e^{W_{1,\varepsilon}} - \sum_{i=1}^{m} \chi_i e^{-\varphi_i} e^{U_1}\right\|_{pr}\right)\|\phi_1\|_{\frac{pr}{r-1}}\right) = O\left(\varepsilon^{\frac{2-pr}{2pr}}\|\boldsymbol{\phi}\|\right),\end{aligned}$$

for any $p, r \in (1,2)$ with $pr < 2$. □

To study the asymptotic behavior of the non-linear part $\mathcal{N}_{\xi,\varepsilon}$ we have the following lemma:

Lemma 4.2.4. *There exist $c_2, \varepsilon_0 > 0$ and $s_0 > 1$ such that for any $p > 1$, $r > 1$ with $pr \in (1, s_0), \varepsilon \in (0, \varepsilon_0)$,*

$$\|\mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi})\|_p = O\left(\varepsilon^{\frac{1-pr}{pr}} e^{c_2\|\boldsymbol{\phi}\|^2}\|\boldsymbol{\phi}\|^2\right)$$

and

$$\|\mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi}^1) - \mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi}^0)\|_p = \mathcal{O}\left(\varepsilon^{\frac{1-pr}{pr}} e^{c_2\sum_{h=0}^{1}\|\boldsymbol{\phi}^h\|^2}\|\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0\|(\|\boldsymbol{\phi}^1\| + \|\boldsymbol{\phi}^0\|)\right)$$

hold true for any $\boldsymbol{\phi}, \boldsymbol{\phi}^1, \boldsymbol{\phi}^0 \in \{\boldsymbol{\phi} = (\phi_1, \phi_2) \in \mathcal{H} : \|\phi_i\| \le \varepsilon_0, i = 1, 2\}$.

Proof. It is sufficient to prove the second estimate since we can take $\boldsymbol{\phi}^0 = 0$ to deduce the first one. Let f_1, f_2 be defined in (4.2.10). We observe that

$$\begin{aligned} & f_1(W_{1,\varepsilon} + \phi_1^1) - f_1(W_{1,\varepsilon} + \phi_1^0) - f_1'(W_{1,\varepsilon})(\phi_1^1 - \phi_1^0)\\ = \;& 2\varepsilon V_1 e^{W_{1,\varepsilon}+\phi_1^1} - 2\varepsilon V_1 e^{W_{1,\varepsilon}+\phi_1^0} - 2\varepsilon V_1 e^{W_{1,\varepsilon}}(\phi_1^1 - \phi_1^0)\\ & \text{ and } f_2(W_{2,\varepsilon} + \phi_2^1) - f_2(W_{2,\varepsilon} + \phi_2^0) - f_2'(W_{2,\varepsilon})(\phi_2^1 - \phi_2^0)\\ = \;& \frac{\rho_2 V_2 e^{W_{2,\varepsilon}+\phi_2^1}}{\int_\Sigma V_2 e^{W_{2,\varepsilon}+\phi_2^1} dv_g} - \frac{\rho_2 V_2 e^{W_{2,\varepsilon}+\phi_2^0}}{\int_\Sigma V_2 e^{W_{2,\varepsilon}+\phi_2^0} dv_g} - \frac{\rho_2 V_2 W_{2,\varepsilon}(\phi_2^1 - \phi_2^0)}{\int_\Sigma V_2 W_{2,\varepsilon} dv_g}\\ & + \frac{\rho_2 V_2 W_{2,\varepsilon}\int_\Sigma V_2 e^{W_{2,\varepsilon}}(\phi_2^1 - \phi_2^0) dv_g}{\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g\right)^2}.\end{aligned}$$

For $i = 1, 2$, $\theta, \gamma \in (0, 1)$, the mean value theorem yields that for any $p > 1$

$$\begin{aligned} & \|f_i(W_{i,\varepsilon} + \phi_i^1) - f_i(W_{i,\varepsilon} + \phi_i^0) - f_i'(W_{i,\varepsilon})(\phi_i^1 - \phi_i^0)\|_p\\ = \;& \|(f_i'(W_{i,\varepsilon} + \theta\phi_i^1 + (1-\theta)\phi_i^0) - f_i'(W_{i,\varepsilon})(\phi_i^1 - \phi_i^0)\|_p\\ = \;& \|f_i''(W_{i,\varepsilon} + \gamma\theta\phi_i^1 + \gamma(1-\theta)\phi_i^0)(\theta\phi_i^1 + (1-\theta)\phi_i^0)(\phi_i^1 - \phi_i^0)\|_p.\end{aligned}$$

For $i=1$, by the Hölder inequality and Sobolev inequality, we derive that

$$\begin{aligned}
& \|f_1''(W_{1,\varepsilon}+\gamma\theta\phi_1^1+\gamma(1-\theta)\phi_1^0)(\theta\phi_1^1+(1-\theta)\phi_1^0)(\phi_1^1-\phi_1^0)\|_p \\
\leq\ & C\sum_{h=0}^{1}\left(\int_\Sigma \varepsilon^p V_1^p e^{pW_{1,\varepsilon}}(e^{|\phi_1^0|+|\phi_1^0|}|\phi_1^1-\phi_1^0||\phi_1^h|)^p dv_g\right)^{1/p} \\
\leq\ & C\sum_{h=0}^{1}\left(\int_\Sigma \varepsilon^{pr} V_1^{pr} e^{prW_{1,\varepsilon}} dv_g\right)^{\frac{1}{pr}}\left(\int_\Sigma e^{ps(|\phi_1^0|+|\phi_1^1|)} dv_g\right)^{\frac{1}{ps}}\left(\int_\Sigma |\phi_1^1-\phi_1^0|^{pt}|\phi_1^h|^{pt} dv_g\right)^{\frac{1}{pt}} \\
\leq\ & C\sum_{h=0}^{1}\|\varepsilon V_1 e^{W_{1,\varepsilon}}\|_{pr} e^{\frac{ps}{8\pi}(\|\phi_1^1\|^2+\|\phi_1^0\|^2)}\|\phi_1^1-\phi_1^0\|\|\phi_1^h\|,
\end{aligned}$$

where $r,s,t\in(1,+\infty)$, $\frac{1}{r}+\frac{1}{s}+\frac{1}{t}=1$. Applying Lemma B.0.5, we deduce that

$$\begin{aligned}
\|2\varepsilon V_1 e^{W_{1,\varepsilon}}\|_{pr} &\leq \left\|2\varepsilon V_1 e^{W_{1,\varepsilon}}-\sum_{i=1}^{m}\chi_i e^{-\varphi_i}e^{U_i}\right\|_{pr}+\sum_{i=1}^{m}\left\|\chi_i e^{-\varphi_i}e^{U_i}\right\|_{pr)} \\
&\leq \sum_{i=1}^{m}\left\|\chi_i e^{-\varphi_i}e^{U_i}\right\|_{pr}+\mathcal{O}(\varepsilon^{\frac{2-pr}{2}}).
\end{aligned}$$

Lemma B.0.1 implies that

$$\begin{aligned}
\int_\Sigma \chi_i^{pr}e^{-pr\varphi_i}e^{prU_i}dv_g &= \mathcal{O}\left(\delta_i^{2-2pr}\int_{\Omega_i}\left(\frac{1}{(1+|y|^2)^2}\right)^{pr}dy\right)+\mathcal{O}(\delta_i^2) \\
&= \mathcal{O}(\delta_i^{2-2pr})=\mathcal{O}(\varepsilon^{1-pr}).
\end{aligned}$$

Hence,

$$\begin{aligned}
& \left\|2\varepsilon V_1 e^{W_{1,\varepsilon}+\phi_1^1}-2\varepsilon V_1 e^{W_{1,\varepsilon}+\phi_1^0}-2\varepsilon V_1 e^{W_{1,\varepsilon}}(\phi_1^1-\phi_1^0)\right\|_p \qquad (4.2.33) \\
\leq\ & C\sum_{h=0}^{1}\varepsilon^{\frac{1-pr}{pr}}e^{\frac{ps}{8\pi}(\|\phi_1^1\|^2+\|\phi_1^0\|^2)}\|\phi_1^1-\phi_1^0\|\|\phi_1^h\|.
\end{aligned}$$

For $u,v,w\in\overline{\mathrm{H}}^1(\Sigma)$,

$$\begin{aligned}
(2\rho_2)^{-1}f_2''(W_{2,\varepsilon}+u)(v)(w) &= \frac{V_2e^{W_{2,\varepsilon}+u}vw}{\int_\Sigma V_2e^{W_{2,\varepsilon}+u}dv_g}-\frac{V_2e^{W_{2,\varepsilon}+u}v}{(\int_\Sigma V_2e^{W_{2,\varepsilon}+u}dv_g)^2}\int_\Sigma V_2e^{W_{2,\varepsilon}+u}wdv_g \\
&\quad -\frac{V_2e^{W_{2,\varepsilon}+u}}{(\int_\Sigma V_2e^{W_{2,\varepsilon}+u}dv_g)^2}w\int_\Sigma V_2e^{W_{2,\varepsilon}+u}vdv_g \\
&\quad -\frac{V_2e^{W_{2,\varepsilon}+u}}{(\int_\Sigma V_2e^{W_{2,\varepsilon}+u}dv_g)^3}\left(\int_\Sigma V_2e^{W_{2,\varepsilon}+u}vdv_g\right)\left(\int_\Sigma V_2e^{W_{2,\varepsilon}+u}wdv_g\right).
\end{aligned}$$

Given that $\|u\|$ and $\varepsilon > 0$ are sufficiently small,

$$\begin{aligned}\int_\Sigma V_2 e^{W_{2,\varepsilon}+u} dv_g &\geq \int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g - \int_\Sigma V_2 e^{W_{2,\varepsilon}} |e^u - 1| dv_g \\ &\geq \int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g - C \int_\Sigma V_2 e^{W_{2,\varepsilon}} |u| dv_g \\ &\geq \int_\Sigma \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)} dv_g - C\|\tilde{V}_2(\cdot,\xi)e^{z(\cdot,\xi)}\|_p \|u\| + o(1) \\ &\geq C,\end{aligned} \tag{4.2.34}$$

where $C > 0$ is a constant. The Hölder's inequality and (4.2.34) imply that for $a, b, c, d, r, s, t, e, f \in (1, +\infty)$ with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1$, $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1$, $\frac{1}{e} + \frac{1}{f} = 1$,

$$\begin{aligned}&\|f_2''(W_{2,\varepsilon} + \gamma\theta\phi_2^1 + \gamma(1-\theta)\phi_2^0)(\theta\phi_2^1 + (1-\theta)\phi_2^0)(\phi_2^1 - \phi_2^0)\|_p \\ &\leq C \sum_{h=0}^{1} \|V_2 e^{W_{2,\varepsilon}}\|_{pa} \|e^{|\phi_2^1|+|\phi_2^0|}\|_{pb} \|\phi_2^h\|_{pc} \|\phi_2^1 - \phi_2^0\|_{pd} \\ &+C \sum_{h=0}^{1} \|V_2 e^{W_{2,\varepsilon}}\|_{pr}^2 \|e^{|\phi_2^1|+|\phi_2^0|}\|_{ps}^2 \||\phi_2^h|\|_{pt} \|\phi_2^1 - \phi_2^0\|_{pt} \\ &+C \sum_{h=0}^{1} \|V_2 e^{W_{2,\varepsilon}}\|_{pe} \|e^{|\phi_2^1|+|\phi_2^0|}\|_{pf} \|V_2 e^{W_{2,\varepsilon}}\|_{pa} \|e^{|\phi_2^1|+|\phi_2^0|}\|_{pb} \||\phi_2^h|\|_{pc} \|\phi_2^1 - \phi_2^0\|_{pd} \\ &+C \sum_{h=0}^{1} \|V_2 e^{W_{2,\varepsilon}}\|_{pe} \|e^{|\phi_2^1|+|\phi_2^0|}\|_{pf} \|V_2 e^{W_{2,\varepsilon}}\|_{pr}^2 \|e^{|\phi_2^1|+|\phi_2^0|}\|_{ps}^2 \||\phi_2^h|\|_{pt} \|\phi_2^1 - \phi_2^0\|_{pt},\end{aligned}$$

for $\|\boldsymbol{\phi}\|$ sufficiently small. The Moser-Trudinger inequality yields that

$$\|e^{|\phi_2^1|+|\phi_2^0|}\|_q^q \leq C e^{\frac{c}{8\pi}(\|\phi_2^1\|^2 + \|\phi_2^0\|^2)}, \quad q > 1,$$

for some constants $C, c > 0$. Using (4.2.18), we immediately deduce

$$\|V_2 e^{W_{2,\varepsilon}}\|_{pr}^2 \leq C \|\tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)}\|_q \leq C.$$

Combining the estimates above, we deduce that

$$\begin{aligned}&\left\| f_2(W_{2,\varepsilon} + \phi_2^1) - f_2(W_{2,\varepsilon} + \phi_2^0) - f_2'(W_{2,\varepsilon})(\phi_2^1 - \phi_2^0) \right\|_p \\ &\leq C e^{c_2 \sum_{h=0}^{1} \|\phi_2^h\|^2} (\|\phi_2^1\| + \|\phi_2^0\|)(\|\phi_2^1 - \phi_2^0\|),\end{aligned} \tag{4.2.35}$$

for some constants $C, c_2 > 0$. Therefore, we proved that there exist constants $c_2, \varepsilon_0 > 0$ and $s_0 > 1$ for any $p, r > 0$ satisfying $pr \in (1, s_0)$ and $\|\boldsymbol{\phi}^i\| \leq \varepsilon_0$ for $i = 1, 2$

$$\|\mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi}^1) - \mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi}^0)\|_p = \mathcal{O}\left(\varepsilon^{\frac{1-pr}{pr}} e^{c_2 \sum_{h=0}^{1} \|\boldsymbol{\phi}^h\|^2} (\|\boldsymbol{\phi}^1\| + \|\boldsymbol{\phi}^0\|) \|\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0\|\right). \tag{4.2.36}$$

□

Next, for fixed $\xi \in \Xi_{k,m}$, we will find $\boldsymbol{\phi}_\varepsilon$ to solve the problem (4.2.12) in $K_\xi^\perp$, i.e.

$$\boldsymbol{\phi}_\varepsilon = \Pi_\xi^\perp \circ \mathcal{L}_{\xi,\varepsilon}^{-1}(\mathcal{S}_{\xi,\varepsilon}(\boldsymbol{\phi}_\varepsilon) + \mathcal{N}_{\xi,\varepsilon} + \mathcal{R}_{\xi,\varepsilon}) \tag{4.2.37}$$

for $\boldsymbol{\phi}_\varepsilon \in K_\xi^\perp$.

Theorem 4.2.5. *Let $\mathscr{D}$ be a compact subset of $\Xi_{k,m}$, and $\xi = (\xi_1, ..., \xi_m) \in \mathscr{D}$. There exist $p_0 > 1, \varepsilon_0 > 0$ and $R > 0$ (uniformly in ξ) such that for any $p \in (1, p_0)$ and any $\varepsilon \in (0, \varepsilon_0)$ there is a unique $\boldsymbol{\phi}_{\xi,\varepsilon} \in K_\xi^\perp$ solves (4.2.37) satisfying that*

$$\|\boldsymbol{\phi}_{\xi,\varepsilon}\| \le R\varepsilon^{\frac{2-p}{2p}}|\log\varepsilon|.$$

Furthermore, the map $\xi \mapsto \boldsymbol{\phi}_{\xi,\varepsilon}$ is C^1 map with respect to ξ.

Proof. Given that $\xi \in \mathscr{D}$, we define the linear operator

$$\mathcal{T}_{\xi,\varepsilon}(\boldsymbol{\phi}) := \Pi_\xi^\perp \circ \mathcal{L}_{\xi,\varepsilon}^{-1}(\mathcal{S}_{\xi,\varepsilon}(\boldsymbol{\phi}_\varepsilon) + \mathcal{N}_{\xi,\varepsilon} + \mathcal{R}_{\xi,\varepsilon})$$

on $K_\xi^\perp$. For any $\boldsymbol{\phi} \in K_\xi^\perp$, by Lemma 4.2.1-Lemma 4.2.4, there exist constants $s_0 > 1$, $C_0, C, c_2 > 0$ such that

$$\begin{aligned} \|\mathcal{T}_{\xi,\varepsilon}(\boldsymbol{\phi})\| &\le C|\log\varepsilon|\|\mathcal{S}_{\xi,\varepsilon}(\boldsymbol{\phi}) + \mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi}) + \mathcal{R}_{\xi,\varepsilon}\|_p \\ &\le C_0|\log\varepsilon|\left(\varepsilon^{\frac{2-pr}{2pr}}\|\boldsymbol{\phi}\| + \varepsilon^{\frac{1-pr}{pr}}e^{c_2\|\boldsymbol{\phi}\|^2}\|\boldsymbol{\phi}\|^2 + \varepsilon^{\frac{2-p}{2p}}\right), \end{aligned}$$

for any $p, r \in (1,2)$ with $pr \in (1, s_0)$. We take $r = \frac{5}{4}, R = 3C_0$. Then, for arbitrary fixed $p \in (1, \frac{3}{2})$, there exists $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$ we have

$$\max\left\{3C_0(\sqrt{c_2}+1)\varepsilon^{\frac{2-p}{2p}}|\log\varepsilon|, 3C_0e^2\varepsilon^{\frac{1-pr}{pr}+\frac{2-p}{2p}}|\log\varepsilon|\right\} \le 1.$$

Thus, for any $\boldsymbol{\phi} \in \left\{\boldsymbol{\phi} \in K_\xi^\perp : \|\boldsymbol{\phi}\| \le R\varepsilon^{\frac{2-p}{2p}}|\log\varepsilon|\right\}$, $\|\mathcal{T}_{\xi,\varepsilon}(\boldsymbol{\phi})\| \le R\varepsilon^{\frac{2-p}{2p}}|\log\varepsilon|$. For any $\boldsymbol{\phi}^0, \boldsymbol{\phi}^1 \in \left\{\boldsymbol{\phi} \in K_\xi^\perp : \|\boldsymbol{\phi}\| \le R\varepsilon^{\frac{2-p}{2p}}|\log\varepsilon|\right\}$, Lemma 4.2.3 and Lemma 4.2.4 yield that

$$\begin{aligned} \|\mathcal{T}_{\xi,\varepsilon}(\boldsymbol{\phi}^1) - \mathcal{T}_{\xi,\varepsilon}(\boldsymbol{\phi}^0)\| &\le C|\log\varepsilon| \cdot \|\mathcal{S}_{\xi,\varepsilon}(\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0) + \mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi}^1) - \mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi}^0)\|_p \\ &\le C_1|\log\varepsilon|\left(\varepsilon^{\frac{2-pr}{2pr}}\|\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0\| + R\varepsilon^{\frac{1-pr}{pr}+\frac{2-pr}{2pr}}|\log\varepsilon|\|\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0\|\right) \end{aligned}$$

Since $\frac{1-pr}{pr}+\frac{2-pr}{2pr}>0$, there exists $\varepsilon_2<\varepsilon_1$ such that for any $\varepsilon\in(0,\varepsilon_2)$,

$$\max\left\{C_1|\log\varepsilon|\varepsilon^{\frac{2-pr}{2pr}},C_1R\varepsilon^{\frac{1-pr}{pr}+\frac{2-pr}{2pr}}|\log\varepsilon|\right\}\le\frac{1}{4}.$$

We choose $\varepsilon_0=\varepsilon_2$. Consequently, we obtain that $\mathcal{T}_{\xi,\varepsilon}$ is contract mapping on

$$\left\{\boldsymbol{\phi}\in K_\xi^\perp:\|\boldsymbol{\phi}\|\le R\varepsilon^{\frac{2-p}{2p}}|\log\varepsilon|\right\}$$

satisfying that

$$\|\mathcal{T}_{\xi,\varepsilon}(\boldsymbol{\phi}^1)-\mathcal{T}_{\xi,\varepsilon}(\boldsymbol{\phi}^0)\|\le\frac{1}{2}\|\boldsymbol{\phi}^1-\boldsymbol{\phi}^0\|.$$

The Banach fixed-point theorem deduces that there exists a unique $\boldsymbol{\phi}_{\xi,\varepsilon}\in\{\boldsymbol{\phi}\in K_\xi^\perp:\|\boldsymbol{\phi}\|\le R\varepsilon^{\frac{2-p}{2p}}|\log\varepsilon|\}$ solves the problem (4.2.37). Let $F(\mathbf{u})$ be defined by (4.2.11). We define a function $\Phi:\mathscr{D}\times\mathcal{H}\to\mathcal{H}$, $(\xi,\boldsymbol{\phi})\mapsto\boldsymbol{\phi}+\Pi_\xi^\perp\left(W_\varepsilon-i^*\circ F(W_\varepsilon+\Pi_\xi^\perp(\boldsymbol{\phi})\right)$. We observe that $\Phi(\xi,\boldsymbol{\phi}_{\xi,\varepsilon})=0$ and for any $\psi\in\mathcal{H}$

$$\frac{\partial\Phi}{\partial\boldsymbol{\phi}}(\xi,\boldsymbol{\phi}_{\xi,\varepsilon})(\psi)=\psi-\Pi_\xi^\perp\circ i^*(F'(\mathbf{W}_\varepsilon+\boldsymbol{\phi}_{\xi,\varepsilon})(\Pi_\xi^\perp\psi)).$$

Claim 4.2.6. *$\frac{\partial\Phi}{\partial\boldsymbol{\phi}}(\xi,\boldsymbol{\phi}_{\xi,\varepsilon})$ is non-degenerate.*

Indeed,

$$\begin{aligned}\frac{\partial\Phi}{\partial\boldsymbol{\phi}}(\xi,\boldsymbol{\phi}_{\xi,\varepsilon})(\psi)&=\Pi_\xi\psi-\Pi_\xi^\perp\circ i^*\circ(-\mathcal{L}_{\xi,\varepsilon}+\mathcal{S}_{\xi,\varepsilon})(\Pi_\xi^\perp\psi)\\&\quad-\Pi_\xi^\perp\circ i^*\left((F'(\mathbf{W}_\varepsilon+\boldsymbol{\phi}_{\xi,\varepsilon})-F'(\mathbf{W}_\varepsilon)(\Pi_\xi^\perp\psi)\right).\end{aligned}$$

By the mean value theorem, there exists $\theta\in(0,1)$ such that

$$\|F'(\mathbf{W}_\varepsilon+\boldsymbol{\phi}_{\xi,\varepsilon})-F'(\mathbf{W}_\varepsilon)\Pi_\xi^\perp\psi\|_p=\|F'(\mathbf{W}_\varepsilon+\theta\boldsymbol{\phi}_{\xi,\varepsilon})\Pi_\xi^\perp\psi\|_p\le C\varepsilon^{\frac{1-pr}{pr}}\|\boldsymbol{\phi}_{\xi,\varepsilon}\|\|\Pi_\xi^\perp\psi\|.$$

Then, Lemma 4.2.1 and Lemma 4.2.3 imply that for some constant $c>0$

$$\begin{aligned}&\left\|\frac{\partial\Phi}{\partial\boldsymbol{\phi}}(\xi,\boldsymbol{\phi}_{\xi,\varepsilon})(\psi)\right\|\\\ge\;&\|\Pi_\xi^\perp\psi\|+c\|\mathcal{L}_{\xi,\varepsilon}\|\|\Pi_\xi^\perp\psi\|-\|\mathcal{S}_{\xi,\varepsilon}\|\|\Pi_\xi^\perp\psi\|-\|(F'(\mathbf{W}_\varepsilon+\boldsymbol{\phi}_{\xi,\varepsilon})-F'(\mathbf{W}_\varepsilon))\Pi_\xi^\perp\psi\|_p\\\ge\;&\|\Pi_\xi^\perp\psi\|+\frac{c}{|\log\varepsilon|}\|\Pi_\xi^\perp\psi\|-\mathcal{O}(\varepsilon^{\frac{2-pr}{2pr}}\|\Pi_\xi^\perp\psi\|)+\mathcal{O}(\varepsilon^{\frac{1-pr}{pr}}\|\boldsymbol{\phi}_{\xi,\varepsilon}\|\|\Pi_\xi^\perp\psi\|)\\\ge\;&\frac{c}{|\log\varepsilon|}\|\psi\|,\end{aligned}$$

for $p, r > 1$ sufficiently close to 1. Hence, we obtain that $\frac{\partial \Phi}{\partial \boldsymbol{\phi}}(\xi, \boldsymbol{\phi}_{\xi,\varepsilon})$ is invertible with

$$\left\| \left(\frac{\partial \Phi}{\partial \boldsymbol{\phi}}(\xi, \boldsymbol{\phi}_{\xi,\varepsilon}) \right)^{-1} \right\| \leq \frac{1}{c} |\log \varepsilon|.$$

By the implicit function theorem, we have $\xi \mapsto \boldsymbol{\phi}_{\xi,\varepsilon}$ is C^1-differentiable. □

4.2.3 The reduced functional and its expansion

We calculate the energy functional at approximation solutions $\mathbf{W}_\varepsilon$. We define the energy functional corresponding to (4.2.1) as follows

$$E_\varepsilon(\mathbf{u}) = \int_\Sigma Q(u, u) dv_g - \varepsilon \int_\Sigma V_1 e^{u_1} dv_g - \rho_2 \log \left(\int_\Sigma V_2 e^{u_2} dv_g \right), \quad \mathbf{u} = (u_1, u_2) \in \mathcal{H}.$$

Lemma 4.2.7. *Given $m \geq k \geq 0$, we assume that (4.2.4) and (4.2.9) are valid, there exists $\varepsilon_0 > 0$ such that the following expansion holds for $\varepsilon \in (0, \varepsilon_0)$:*

$$E_\varepsilon(\mathbf{W}_\varepsilon) = \Lambda_{k,m}(\xi) - 6\pi(m+k) + 2\pi(m+k)\log 8 - 2\pi(m+k)\log \varepsilon + o(1) \tag{4.2.38}$$

and

$$\partial_\xi E_\varepsilon(\mathbf{W}_\varepsilon) \quad = \quad \partial_\xi \Lambda_{k,m}(\xi) + o(1), \tag{4.2.39}$$

which are convergent in $C(\Xi_{k,m})$ uniformly for any ξ in a compact subset of $\Xi_{k,m}$.

Proof. Let $\mathscr{D}$ be a compact subset of $\Xi_{k,m}$. Then there exists $\varepsilon_0 > 0$ such that $\mathscr{D} \subset \Xi_{k,m}^{\varepsilon_0}$. We consider $\xi \in \Xi_{k,m}^{\varepsilon_0}$.

$$Q(\mathbf{W}_\varepsilon, \mathbf{W}_\varepsilon) = \frac{1}{4} \int_\Sigma \left(|\nabla z(\cdot, \xi)|_g^2 + \left| \nabla \left(\sum_{i=1}^m PU_i \right) \right|_g^2 - \sum_{i=1}^m \langle \nabla PU_i, \nabla z(\cdot, \xi) \rangle_g \right) dv_g$$

The estimate (4.2.18) yields that

$$\begin{aligned}
& \frac{1}{4} \int_\Sigma |\nabla z(\cdot, \xi)|_g^2 - \rho_2 \log \int_\Sigma V_2 e^{W_{\varepsilon,2}} dv_g \\
= \;& \frac{1}{2} \underbrace{\left(\frac{1}{2} \int_\Sigma |\nabla z(\cdot, \xi)|_g^2 - 2\rho_2 \log \int_\Sigma \tilde{V}_2(\cdot, \xi) e^{z(\cdot, \xi)} dv_g \right)}_{= I_\xi(z(\cdot,\xi))} + \mathcal{O}(\varepsilon) \\
= \;& \frac{1}{2} I_\xi(z(\cdot, \xi)) + \mathcal{O}(\varepsilon),
\end{aligned}$$

as $\varepsilon \to 0$. By Lemma B.0.1 and Lemma B.0.5, we deduce that for $p \in (1,2)$

$$\begin{aligned}
&\varepsilon \int_\Sigma V_1 e^{W_{1,\varepsilon}} dv_g \qquad\qquad (4.2.40)\\
= \;& \mathcal{O}\left(\left| \int_\Sigma 2V_1 e^{W_{1,\varepsilon}} - \sum_{i=1}^m \chi_i e^{-\varphi_i} e^{U_i} dv_g \right| \right) + \frac{1}{2} \sum_{i=1}^m \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} dv_g\\
= \;& \mathcal{O}\left(\left\| 2V_1 e^{W_{1,\varepsilon}} - \sum_{i=1}^m \chi_i e^{-\varphi_i} e^{U_i} \right\|_p |\Sigma|^{1-\frac{1}{p}} \right) + \frac{1}{2} \sum_{i=1}^m \varrho(\xi_i) + \mathcal{O}(\varepsilon)\\
= \;& \frac{1}{2} \sum_{i=1}^m \varrho(\xi_i) + \mathcal{O}\left(\varepsilon^{\frac{2-p}{2p}} + \varepsilon \right),
\end{aligned}$$

where we applied that $\int_{|y|<r} \frac{8}{(1+|y|^2)^2} dy = 8\pi - \frac{8\pi}{1+r^2}$. Using $z(\cdot, \xi)$ as a test function for (4.2.3),

$$\begin{aligned}
& \frac{1}{4} \int_\Sigma \langle \nabla PU_i, \nabla z(\cdot, \xi) \rangle_g dv_g = \frac{1}{4} \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} z(\cdot, \xi) dv_g\\
= \;& \frac{1}{4} \int_{\Omega_i} \chi(\delta_i |y|/r_0) \frac{8}{(1+|y|^2)^2} z(y_{\xi_i}^{-1}(\delta_i y), \xi) dy\\
= \;& \frac{1}{4} \varrho(\xi_i) z(\xi_i, \xi) + \mathcal{O}(\varepsilon^{\frac{1}{2}}).
\end{aligned}$$

For any $i, i' = 1, \cdots, m$, using $PU_{i'}$ as a test function for (4.2.3), Lemma B.0.1 with (4.2.9) implies that

$$\begin{aligned}
& \frac{1}{4} \int_\Sigma \langle PU_{i'}, PU_i \rangle_g dv_g = \frac{1}{4} \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} PU_{i'} dv_g\\
= \;& \frac{1}{4} \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} \left(\chi_{i'} (U_{i'} - \log(8\delta_{i'}^2)) + \varrho(\xi_{i'}) H^g(\cdot, \xi_{i'}) + \mathcal{O}(\delta_{i'}^2 |\log \delta_{i'}|) \right)\\
= \;& \begin{cases} -\frac{1}{2} \int_{\Omega_i} \chi(\delta_i |y|/r_0) \frac{8(2\log \delta_i + \log(1+|y|^2))}{(1+|y|^2)^2} dy + \frac{1}{4} \varrho^2(\xi_i) H^g(\xi_i, \xi_i) + \mathcal{O}(\varepsilon^{\frac{1}{2}}) & i' = i\\ \frac{1}{4} \varrho(\xi_i) \varrho(\xi_{i'}) G^g(\xi_i, \xi_{i'}) + \mathcal{O}(\varepsilon^{\frac{1}{2}}) & i' \neq i \end{cases}\\
= \;& \begin{cases} -\frac{1}{2} \varrho(\xi_i) \log \delta_i^2 - \frac{1}{2} \varrho(\xi_i) + \frac{1}{4} \varrho^2(\xi_i) H^g(\xi_i, \xi_i) + \mathcal{O}(\varepsilon^{\frac{1}{2}}) & i' = i\\ \frac{1}{4} \varrho(\xi_i) \varrho(\xi_{i'}) G^g(\xi_i, \xi_{i'}) & i' \neq i \end{cases},
\end{aligned}$$

where we applied that

$$\int_{|y| \le r} \frac{8 \log(1+|y|^2)}{(1+|y|^2)^2} = 8\pi + 8\pi \frac{\log(1+r^2)+1}{1+r^2}$$

and

$$\int_{|y|<r} \frac{8}{(1+|y|^2)^2} dy = 8\pi - \frac{8\pi}{1+r^2}.$$

Combining all the estimates above, we conclude that

$$E_\varepsilon(\mathbf{W}_\varepsilon) = \frac{1}{2} I_\xi(z(\cdot,\xi)) - \frac{1}{4}\mathcal{F}^{V_1}_{k,m}(\xi) - 4\pi(m+k) + 2\pi(m+k)\log 8 - 2\pi(m+k)\log\varepsilon + o(1),$$

as $\varepsilon \to 0$. For any $i = 1, \cdots, m$ and $j = 1, \cdots, \mathrm{i}(\xi_i)$,

$$\begin{aligned}
\partial_{(\xi_i)_j} E_\varepsilon(\mathbf{W}_\varepsilon) &= \int_\Sigma \left(-\frac{2}{3}\Delta_g W_{1,\varepsilon} - \frac{1}{3}\Delta_g W_{2,\varepsilon} - \frac{1}{2} f_1(W_{1,\varepsilon})\right) \partial_{(\xi_i)_j} W_{1,\varepsilon} dv_g \\
&+ \int_\Sigma \left(-\frac{2}{3}\Delta_g W_{2,\varepsilon} - \frac{1}{3}\Delta_g W_{1,\varepsilon} - \frac{1}{2} f_2(W_{2,\varepsilon})\right) \partial_{(\xi_i)_j} W_{2,\varepsilon} dv_g \\
&= \frac{1}{2}\int_\Sigma \left(\sum_{l=1}^m \chi_l e^{-\varphi_l} e^{U_l} - f_1(W_{1,\varepsilon})\right) \partial_{(\xi_i)_j} W_{1,\varepsilon} dv_g \\
&\quad + \frac{1}{2}\int_\Sigma \left(\frac{2\rho_2 \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)}}{\int_\Sigma \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)} dv_g} - f_2(W_{2,\varepsilon})\right) \partial_{(\xi_i)_j} W_{2,\varepsilon} dv_g,
\end{aligned}$$

in view of $\int_\Sigma W_{i,\varepsilon} dv_g = 0$ for $i = 1, 2$. By Lemma B.0.4 and Lemma B.0.3, we have $\|\partial_{(\xi_i)_j} W_{2,\varepsilon}\| = \mathcal{O}(\varepsilon^{-\frac{1}{2}})$. Using Lemma 4.2.18, we deduce that

$$\begin{aligned}
\left|\int_\Sigma \left(\frac{2\rho_2 \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)}}{\int_\Sigma \tilde{V}_2(\cdot,\xi) e^{z(\cdot,\xi)} dv_g} - f_2(W_{2,\varepsilon})\right) \partial_{(\xi_i)_j} W_{2,\varepsilon} dv_g\right| &\leq \mathcal{O}\left(\varepsilon \|\partial_{(\xi_i)_j} W_{2,\varepsilon}\|_{L^1(\Sigma)}\right) \\
&\leq \mathcal{O}\left(\varepsilon^{\frac{1}{2}}\right).
\end{aligned}$$

Lemma B.0.2, Lemma B.0.3 and Lemma B.0.5 yield that for any $j = 1, \cdots, \mathrm{i}(\xi_i)$, and for any $p \in (1, 2)$

$$\begin{aligned}
&\frac{1}{2}\int_\Sigma \left(\sum_{h=1}^m \chi_l e^{-\varphi_h} e^{U_h} - f_1(W_{1,\varepsilon})\right) \partial_{(\xi_i)_j} W_{1,\varepsilon} dv_g \\
= &\frac{1}{2}\sum_{h=1}^m \int_\Sigma \chi_h e^{-\varphi_h} e^{U_h} Z_i^j dv_g - \varepsilon \int_\Sigma V_1 e^{W_{1,\varepsilon}} Z_i^j dv_g \\
&+\mathcal{O}\left(\left\|\sum_{h=1}^m \chi_h e^{-\varphi_l} e^{U_h} - f_1(W_{1,\varepsilon})\right\|_p |\Sigma|^{1-\frac{1}{p}}\right) \\
= &\frac{1}{2}\sum_{h=1}^m \int_\Sigma \chi_l e^{-\varphi_h} e^{U_h} \chi_i Z_i^j dv_g - \varepsilon \int_\Sigma V_1 e^{W_{1,\varepsilon}} \chi_i Z_i^j dv_g + o(1),
\end{aligned}$$

as $\varepsilon \to 0$. By direct calculation, for $j = 1, \cdots, \mathrm{i}(\xi_i)$ we have

$$\int_\Sigma \chi_h e^{-\varphi_h} e^{U_h} \chi_i Z_i^j dv_g = \begin{cases} \frac{1}{\delta_i} \int_{\Omega_i} \chi^2\left(\frac{|y|}{r_0}\right) \frac{32 y_j}{(1+|y|^2)^3} dy & \text{for } h = i \\ 0 & \text{for } h \neq i \end{cases} = 0,$$

where the last equality applied the symmetric property of Ω_i. It is sufficient to calculate the integral $\varepsilon \int_\Sigma V_1 e^{W_{1,\varepsilon}} \chi_i Z_i^j dv_g$. Let $\tau_i(x) = \varrho(\xi_i) H^g(x,\xi_i) + \sum_{h\neq i} \varrho(\xi_h) G^g(x,\xi_h) - \frac{1}{2} z(x,\xi) + \log V_1(x)$.

Recall that $d_i^2 = \frac{1}{4}e^{\tau_i(\xi_i)}$. Using Lemma B.0.1 with (4.2.9), we can derive that $\varepsilon \to 0$

$$\begin{aligned}
& \varepsilon \int_\Sigma V_1 e^{W_{1,\varepsilon}} \chi_i Z_i^j dv_g = 2\varepsilon \int_\Sigma \chi_i e^{\sum_{l=1}^m PU_l - \frac{1}{2}z(\cdot,\xi) + \log V_1} \frac{4(y_{\xi_i})_j}{\delta_i^2 + |y_{\xi_i}|^2} dv_g \\
= \; & \varepsilon \int_\Sigma \chi_i e^{\chi_i(U_i - \log(8\delta_i^2)) + \varrho(\xi_i)H^g(\cdot,\xi_i) + \sum_{l\neq i}\varrho(\xi_l)G^g(\cdot,\xi_l) - \frac{1}{2}z(\cdot,\xi) + \log V_1 + \mathcal{O}(\delta_i^2|\log\delta_i|)} \frac{4(y_{\xi_i})_j}{\delta_i^2 + |y_{\xi_i}|^2} dv_g \\
= \; & \frac{\varepsilon}{\delta_i^3} \int_{\Omega_i} \chi\left(\frac{\delta_i|y|}{r_0}\right) \frac{4y_j}{(1+|y|^2)^3} e^{\tau_i(\xi_i)} \left(1 + \sum_{s=1}^2 \delta_i \partial_{y_s}\tau_i \circ y_{\xi_i}^{-1}(0) y_s + \mathcal{O}(\delta_i^2|y|^2 + \delta_i^2|\log\delta_i|)\right) dy \\
= \; & \frac{e^{\tau_i(\xi_i)}\partial_{y_i}\tau_i \circ y_{\xi_i}^{-1}(0)\varrho(\xi_i)}{8d_i^2} + o(1) = \frac{1}{4}\partial_{(\xi_i)_j}\mathcal{F}_{k,m}^{V_1}(\xi) - \frac{1}{4}\varrho(\xi_i)\partial_{(x_i)_j} z(x,\xi)|_{x=\xi_i} + o(1),
\end{aligned}$$

where we applied the symmetric property of Ω_i and $\int_{\mathbb{R}^2} \frac{4y_j^2}{(1+|y|^2)^3} dy = \pi$. Since $z(\cdot,\xi)$ solves (1.2.3), we have $I'_\xi(z(\cdot,\xi)) = 0$. By representation's formula in Section 2.4, we deduce that

$$\begin{aligned}
& \partial_{(\xi_i)_j} I_\xi(z(\cdot,\xi)) \\
= \; & I'_\xi(z(\cdot,\xi))\partial_{(\xi_i)_j} z(\cdot,\xi) - 2\rho_2 \int_\Sigma \frac{\tilde{V}_2(\cdot,\xi)e^{z(\cdot,\xi)}}{\int_\Sigma \tilde{V}_2(\cdot,\xi)e^{z(\cdot,\xi)} dv_g} \left(-\frac{\varrho(\xi_i)}{2}\partial_{(\xi_i)_j} G^g(\cdot,\xi_i)\right) dv_g \\
= \; & \rho_2 \varrho(\xi_i) \int_\Sigma \frac{\tilde{V}_2(\cdot,\xi)e^{z(\cdot,\xi)}}{\int_\Sigma \tilde{V}_2(\cdot,\xi)e^{z(\cdot,\xi)} dv_g} \partial_{(\xi_i)_j} G^g(\cdot,\xi_i) dv_g \\
= \; & \frac{1}{2}\varrho(\xi_i)\partial_{(x_i)_j} z(x,\xi)|_{x=\xi_i}.
\end{aligned}$$

Hence, we obtain that as $\varepsilon \to 0$

$$\partial_{(\xi_i)_j} E_\varepsilon(\mathbf{W}_\varepsilon) = \frac{1}{2}\partial_{(\xi_i)_j} I_\xi(z(\cdot,\xi)) - \frac{1}{4}\partial_{(\xi_i)_j}\mathcal{F}_{k,m}^{V_1}(\xi) + o(1) = \partial_{(\xi_i)_j}\Lambda_{k,m}(\xi) + o(1).$$

□

Next, we consider the reduced functional $\tilde{E}_\varepsilon : \Xi_{k,m} \to \mathbb{R}, \xi \mapsto E_\varepsilon(\mathbf{W}_\varepsilon + \boldsymbol{\phi})$, where $\boldsymbol{\phi}$ is given by Theorem 4.2.5.

Theorem 4.2.8. *Given integers $m \geq k \geq 0$, we assume that* (4.2.4) *and* (4.2.9) *are valid, there exists $\varepsilon_0 > 0$ such that the expansion holds for $\varepsilon \in (0, \varepsilon_0)$*

$$\tilde{E}_\varepsilon(\xi) = \Lambda_{k,m}(\xi) - 6\pi(m+k) + 2\pi(m+k)\log 8 - 2\pi(m+k)\log\varepsilon + o(1) \tag{4.2.41}$$

and

$$\partial_\xi \tilde{E}_\varepsilon(\xi) \;=\; \partial_\xi \Lambda_{k,m}(\xi) + o(1), \tag{4.2.42}$$

which are convergent in $C(\Xi_{k,m})$ uniformly for any ξ in a compact subset of $\Xi_{k,m}$.

Proof. By direct calculation, we have

$$\begin{aligned}\tilde{E}_\varepsilon(\xi) &= E_\varepsilon(\mathbf{W}_\varepsilon) + \frac{1}{2}\left(\int_\Sigma \left(\sum_{l=1}^m \chi_l e^{-\varphi_l} e^{U_l} - 2\varepsilon V_1 e^{W_{1,\varepsilon}}\right)\phi_1 dv_g - \int_\Sigma \phi_2 \Delta_g z(\cdot,\xi) dv_g\right) \\ &-\varepsilon \int_\Sigma V_1 e^{W_{1,\varepsilon}}\left(e^{\phi_1} - 1 - \phi_1\right) dv_g - \rho_2 \log\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}+\phi_2} dv_g\right) + \rho_2 \log\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g\right).\end{aligned}$$

Recall that $\boldsymbol{\phi} = (\phi_1, \phi_2) \in K_\xi^\perp$ satisfies that for p sufficiently close to 1, $\|\boldsymbol{\phi}\| \le R\varepsilon^{\frac{2-p}{2p}} |\log \varepsilon|$. For $p, r > 1$ sufficiently close to 1, the Hölder inequality implies that

$$\begin{aligned}\left|\int_\Sigma \left(\sum_{l=1}^m \chi_l e^{-\varphi_l} e^{U_l} - 2\varepsilon V_1 e^{W_{1,\varepsilon}}\right)\phi_1 dv_g\right| &\le \left\|\sum_{l=1}^m \chi_l e^{-\varphi_l} e^{U_l} - 2\varepsilon V_1 e^{W_{1,\varepsilon}}\right\|_p \|\phi_1\|_{\frac{p}{p-1}} \\ &\le \mathcal{O}\left(\varepsilon^{\frac{2-p}{2p}}\|\phi_1\|\right) = o(1), \quad \text{(by Lemma B.0.5)} \\ \left|\int_\Sigma \Delta_g z(\cdot,\xi)\phi_2 dv_g\right| &\le \|z(\cdot,\xi)\|\|\phi_2\| = o(1), \\ \left|\varepsilon\int_\Sigma V_1 e^{W_{1,\varepsilon}}\left(e^{\phi_1} - 1 - \phi_1\right) dv_g\right| &\le \mathcal{O}\left(\varepsilon^{\frac{1-pr}{pr}} e^{\frac{ps}{8\pi}\|\phi_1\|^2}\|\phi_1\|^2\right) = o(1). \quad \text{(by (4.2.33))}\end{aligned}$$

By mean value theorem, for some $\theta \in (0,1)$,

$$\begin{aligned}-\rho_2 \log\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}+\phi_2} dv_g\right) + \rho_2 \log\left(\int_\Sigma V_2 e^{W_{2,\varepsilon}} dv_g\right) &= \frac{\int_\Sigma V_2 e^{W_{2,\varepsilon}+\theta\phi_2}\phi_2 dv_g}{\int_\Sigma V_2 e^{W_{2,\varepsilon}+\theta\phi_2} dv_g} \\ &= \mathcal{O}(\|\phi_2\|) = o(1).\end{aligned}$$

Consequently, we obtain that $\tilde{E}_\varepsilon(\xi) = E_\varepsilon(W_\varepsilon) + o(1)$. Immediately, Lemma 4.2.7 yields (4.2.41), which holds in $C(\Sigma)$ and uniformly for any ξ in any compact subset of $\Xi_{k,m}$. Applying Theorem 4.2.5, there exists $\{c_{i,j}^\varepsilon \in \mathbb{R} : i = 1, \cdots, m, j = 1, \cdots, \mathrm{i}(\xi_i)\}$ such that

$$\mathbf{W}_\varepsilon + \boldsymbol{\phi} - i^*(F(\mathbf{W}_\varepsilon + \boldsymbol{\phi})) = \begin{pmatrix} \sum_{i,j} c_{i,j}^\varepsilon P Z_i^j \\ \\ 0 \end{pmatrix}. \tag{4.2.43}$$

And

$$\begin{aligned}\partial_{(\xi_i)_j}\tilde{E}_\varepsilon(\xi) &= \partial_{(\xi_i)_j} E_\varepsilon(\mathbf{W}_\varepsilon) \\ &+ \int_\Sigma \left(-\frac{2}{3}\Delta_g\phi_1 - \frac{1}{3}\Delta_g\phi_2 - \frac{1}{2}(f_1(W_{1,\varepsilon}+\phi_1) - f_1(W_{1,\varepsilon}))\right)\partial_{(\xi_i)_j} W_{1,\varepsilon} dv_g \\ &+ \int_\Sigma \left(-\frac{2}{3}\Delta_g\phi_2 - \frac{1}{3}\Delta_g\phi_1 - \frac{1}{2}(f_2(W_{2,\varepsilon}+\phi_2) - f_2(W_{2,\varepsilon}))\right)\partial_{(\xi_i)_j} W_{2,\varepsilon} dv_g \\ &+ \int_\Sigma \left(-\frac{2}{3}\Delta_g(W_{1,\varepsilon}+\phi_1) - \frac{1}{3}\Delta_g(W_{2,\varepsilon}+\phi_2) - \frac{1}{2}f_1(W_{1,\varepsilon}+\phi_1)\right)\partial_{(\xi_i)_j}\phi_1 dv_g \\ &+ \int_\Sigma \left(-\frac{2}{3}\Delta_g(W_{2,\varepsilon}+\phi_2) - \frac{1}{3}\Delta_g(W_{1,\varepsilon}+\phi_1) - \frac{1}{2}f_2(W_{2,\varepsilon}+\phi_2)\right)\partial_{(\xi_i)_j}\phi_2 dv_g.\end{aligned}$$

Lemma B.0.3 implies that $\partial_{(\xi_i)_j} W_{1,\varepsilon} = PZ_i^j + \mathcal{O}(1)$ and $\partial_{(\xi_i)_j} W_{2,\varepsilon} = -\frac{1}{2} PZ_i^j + \mathcal{O}(1)$. Since $\langle \phi_1, PZ_i^j \rangle = 0$, we have

$$\begin{aligned} \int_\Sigma (-\Delta_g \phi_1) \partial_{(\xi_i)_j} W_{1,\varepsilon} dv_g &= \int_\Sigma \langle \nabla \phi_1, (\nabla PZ_i^j + \mathcal{O}(1)) \rangle_g dv_g \\ &= \langle \phi_1, PZ_i^j \rangle + \mathcal{O}\left(\|\boldsymbol{\phi}\| \right) = o(1). \end{aligned}$$

Using the same reasoning, we can deduce that $\int_\Sigma (-\Delta_g \phi_1) \partial_{(\xi_i)_j} W_{2,\varepsilon} dv_g = o(1)$. It is observed that

$$\begin{aligned} \int_\Sigma (-\Delta_g \phi_2) \left(\partial_{(\xi_i)_j} W_{1,\varepsilon} + 2 \partial_{(\xi_i)_j} W_{2,\varepsilon} \right) dv_g &= \frac{3}{2} \int_\Sigma (-\Delta_g \phi_2) \partial_{(\xi_i)_j} z(\cdot, \xi) dv_g \\ &= \mathcal{O}(\|\phi_2\|) = o(1). \end{aligned}$$

The estimate (4.2.35) yields that for $q > 1$

$$\begin{aligned} &\int_\Sigma (f_2(W_{2,\varepsilon} + \phi_2) - f_2(W_{2,\varepsilon})) \partial_{(\xi_i)_j} W_{2,\varepsilon} dv_g \\ = \;& \mathcal{O}\left((\|\phi_2\|^2 + \|f_2'(W_{2,\varepsilon})\phi_2\|_q \|PZ_i^j\|_{\frac{q}{q-1}} \right) = \mathcal{O}(\|\phi_2\|) = o(1). \end{aligned}$$

The estimate (4.2.33) implies that for $q, r > 1$ sufficiently close to 1

$$\begin{aligned} &\int_\Sigma (f_1(W_{1,\varepsilon} + \phi_1) - f_1(W_{1,\varepsilon})) \partial_{(\xi_i)_j} W_{1,\varepsilon} dv_g \\ = \;& 2\varepsilon \int_\Sigma V_1 e^{W_{1,\varepsilon}} \phi_1 \partial_{(\xi_i)_j} W_{1,\varepsilon} dv_g + \mathcal{O}\left(\int_\Sigma |f_1(W_{1,\varepsilon} + \phi_1) - f_1(W_{1,\varepsilon}) - f_1'(W_{1,\varepsilon})\phi_1 \partial_{(\xi_i)_j} W_{1,\varepsilon}| dv_g \right) \\ = \;& 2\varepsilon \int_\Sigma V_1 e^{W_{1,\varepsilon}} \phi_1 (\chi_i Z_i^j + \mathcal{O}(1)) dv_g + \mathcal{O}\left(\varepsilon^{\frac{1-qr}{qr}} \|\phi_1\|^2 \|PZ_i^j\|_{\frac{q}{q-1}} \right) \\ = \;& 2\varepsilon \int_\Sigma V_1 e^{W_{1,\varepsilon}} \phi_1 \chi_i Z_i^j dv_g + o(1). \end{aligned}$$

By Lemma B.0.5, for any $q \in (1, 2)$

$$\begin{aligned} 2\varepsilon \int_\Sigma V_1 e^{W_{1,\varepsilon}} \phi_1 \chi_i Z_i^j dv_g &= \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} Z_i^j \phi_1 dv_g + \mathcal{O}\left(\left\| 2\varepsilon V_1 e^{W_{1,\varepsilon}} - \sum_{h=1}^m \chi_h e^{-\varphi_h} e^{U_h} \right\|_q \|\phi_1\| \right) \\ &= \langle PZ_i^j, \phi_1 \rangle + \mathcal{O}(\varepsilon^{\frac{2-q}{2q}}) = o(1). \end{aligned}$$

Applying (4.2.43), we derive that for $i=1,\cdots,m, j=1,\cdots,\mathrm{i}(\xi_i)$

$$\begin{aligned}
&\sum_{i',j'} c^{\varepsilon}_{i',j'}\langle PZ_i^j, PZ_{i'}^{j'}\rangle = \left\langle W_{1,\varepsilon}+\phi_1 - i^*(f_1(W_{1,\varepsilon}+\phi_1)-\frac{1}{2}f_2(W_{2,\varepsilon}+\phi_2)), PZ_i^j\right\rangle\\
= &\left\langle W_{1,\varepsilon}-i^*\left(f_1(W_{1,\varepsilon})-\frac{1}{2}f_2(W_{2,\varepsilon})\right), PZ_i^j\right\rangle + \mathcal{O}\left(|\langle \phi_1, PZ_i^j\rangle|\right)\\
&-\int_\Sigma (f_1(W_{1,\varepsilon}+\phi_l)-f_1(W_{1,\varepsilon}))PZ_i^j dv_g + \frac{1}{2}\int_\Sigma (f_2(W_{2,\varepsilon}+\phi_2)-f_2(W_{2,\varepsilon}))PZ_i^j dv_g.
\end{aligned}$$

According to the proof of Lemma 4.2.7,

$$\begin{aligned}
\langle W_{1,\varepsilon}-i^*(f_1(W_{1,\varepsilon})), PZ_i^j\rangle &= \int_\Sigma \left(\sum_{l=1}^m \chi_l e^{-\varphi_l}e^{U_l} - 2\varepsilon V_1 e^{W_{1,\varepsilon}}\right) PZ_i^j dv_g\\
&= 2\partial_{(\xi_i)_j}\Lambda_{k,m}(\xi) + o(1).
\end{aligned}$$

We notice that $\int_\Sigma |PZ_i^j|^2 dv_g = \mathcal{O}(|\log\varepsilon|)$. By (4.2.18), we have

$$\begin{aligned}
-\frac{1}{2}\int_\Sigma f_2(W_{2,\varepsilon})PZ_i^j dv_g &= -\rho_2\int_\Sigma \frac{\tilde{V}_2(\cdot,\xi)e^{z(\cdot,\xi)}}{\int_\Sigma \tilde{V}_2(\cdot,\xi)e^{z(\cdot,\xi)dv_g}} PZ_i^j dv_g + \mathcal{O}\left(\varepsilon\int_\Sigma |PZ_i^j| dv_g\right)\\
&= \mathcal{O}\left((1+\varepsilon)\left\|PZ_i^j\right\|_2\right) = \mathcal{O}\left(|\log\varepsilon|^{\frac{1}{2}}\right).
\end{aligned}$$

Applying (4.2.33) and (4.2.35), we derive that for $q, r>1$ sufficiently close to 1,

$$\begin{aligned}
&\int_\Sigma (f_2(W_{2,\varepsilon}+\phi_2)-f_2(W_{2,\varepsilon}))PZ_i^j dv_g = \mathcal{O}\left((\|\phi_2\|^2 + \|f_2'(W_{2,\varepsilon})\phi_2\|_q)\|PZ_i^j\|_{\frac{q}{q-1}}\right) = o(1),\\
&\text{and } \int_\Sigma (f_1(W_{1,\varepsilon}+\phi_l)-f_1(W_{1,\varepsilon}))PZ_i^j dv_g\\
= \ & 2\varepsilon\int_\Sigma V_1 e^{W_{1,\varepsilon}}\phi_1(\chi_i Z_i^j + \mathcal{O}(1))dv_g + \mathcal{O}\left(\varepsilon^{\frac{1-qr}{qr}}\|\phi_1\|^2\|PZ_i^j\|_{\frac{q}{q-1}}\right)\\
= \ & 2\varepsilon\int_\Sigma V_1 e^{W_{1,\varepsilon}}\phi_1\chi_i Z_i^j dv_g + o(1) = \langle PZ_i^j, \phi_1\rangle + o(1) = o(1).
\end{aligned}$$

Applying Lemma B.0.4, we deduce that $c^{\varepsilon}_{i,j}\frac{8\varrho(\xi_i)D_i}{\delta_i^2\pi} + \mathcal{O}\left(\varepsilon^{-\frac{1}{2}}\sum_{i,j}|c^{\varepsilon}_{i,j}|\right) = \mathcal{O}(|\log\varepsilon|^{\frac{1}{2}})$, as $\varepsilon\to 0$. Considering $\delta_i^2 = \varepsilon d_i^2$, we deduce $\sum_{i,j}|c^{\varepsilon}_{i,j}| = \mathcal{O}(\varepsilon|\log\varepsilon|^{\frac{1}{2}})$. By (4.2.43), we have

$$-\frac{2}{3}\Delta_g(W_{2,\varepsilon}+\phi_2) - \frac{1}{3}\Delta_g(W_{1,\varepsilon}+\phi_1) - \frac{1}{2}f_2(W_{2,\varepsilon}+\phi_2) = -\frac{1}{3}\sum_{i,j}c^{\varepsilon}_{i,j}\Delta_g PZ_i^j$$

and

$$-\frac{2}{3}\Delta_g(W_{1,\varepsilon}+\phi_1) - \frac{1}{3}\Delta_g(W_{2,\varepsilon}+\phi_2) - \frac{1}{2}f_1(W_{1,\varepsilon}+\phi_1) = -\frac{2}{3}\sum_{i,j}c^{\varepsilon}_{i,j}\Delta_g PZ_i^j.$$

Then,

$$\begin{aligned}&\int_\Sigma \left(-\frac{2}{3}\Delta_g(W_{1,\varepsilon}+\phi_1)-\frac{1}{3}\Delta_g(W_{2,\varepsilon}+\phi_2)-\frac{1}{2}f_1(W_{1,\varepsilon}+\phi_1)\right)\partial_{(\xi_i)_j}\phi_1 dv_g\\ =\ &\frac{2}{3}\sum_{i',j'} c^{\varepsilon}_{i',j'}\langle PZ^{i'}_{j'},\partial_{(\xi_i)_j}\phi_1\rangle.\end{aligned}$$

By straightforward calculation, for $q,p>1$ sufficiently close to 1 such that $\frac{2-4q}{2q}+\frac{2-p}{2p}+1>0$

$$\begin{aligned}\langle PZ^{j'}_{i'},\partial_{(\xi_i)_j}\phi_1\rangle &= \partial_{(\xi_i)_j}\langle PZ^{j'}_{i'},\phi_1\rangle-\langle\partial_{(\xi_i)_j}PZ^{j'}_{i'},\phi_1\rangle \qquad (4.2.44)\\ &= \int_\Sigma \partial_{(\xi_i)_j}(-\chi_{i'}e^{-\varphi_{i'}}e^{U_{i'}}Z^{j'}_{i'})\phi_1 dv_g\\ &= \mathcal{O}\left(\|\partial_{(\xi_i)_j}(-\chi_{i'}e^{-\varphi_{i'}}e^{U_{i'}}Z^{j'}_{i'})\|_q\|\phi_1\|\right)\\ &= \mathcal{O}(\varepsilon^{\frac{2-4q}{2q}+\frac{2-p}{2p}}|\log\varepsilon|),\end{aligned}$$

where we applied the condition $\langle PZ^j_i,\phi\rangle=0$. Hence, we obtain that

$$\int_\Sigma \left(-\frac{2}{3}\Delta_g(W_{1,\varepsilon}+\phi_1)-\frac{1}{3}\Delta_g(W_{2,\varepsilon}+\phi_2)-\frac{1}{2}f_1(W_{1,\varepsilon}+\phi_1)\right)\partial_{(\xi_i)_j}\phi_1 dv_g=o(1).$$

Similarly, by the same reasoning, we can deduce that

$$\int_\Sigma \left(-\frac{2}{3}\Delta_g(W_{2,\varepsilon}+\phi_2)-\frac{1}{3}\Delta_g(W_{1,\varepsilon}+\phi_1)-\frac{1}{2}f_2(W_{2,\varepsilon}+\phi_2)\right)\partial_{(\xi_i)_j}\phi_2 dv_g=o(1).$$

Combining all the estimates above, we prove that

$$\partial_{(\xi_i)_j}\tilde{E}_\varepsilon(\xi)=\partial_{(\xi_i)_j}E_\varepsilon(W_\varepsilon)+o(1)=\partial_{(\xi_i)_j}\Lambda_{k,m}(\xi)+o(1).$$

□

4.3 Proof of main results

The next lemma demonstrates that ξ being a critical point of $\tilde{E}_\varepsilon$ is equivalent to $\mathbf{W}_\varepsilon+\boldsymbol{\phi}_{\xi,\varepsilon}$ solving (4.2.1).

Lemma 4.3.1. *For any $\varepsilon\in(0,\varepsilon_0)$ where $\varepsilon_0>0$ sufficiently small, $\xi\in\Xi_{k,m}$ is a critical point of $\xi\mapsto\tilde{E}_\varepsilon(\xi)$ if and only if $\xi\in\Xi_{k,m}$ and $\mathbf{u}=\mathbf{W}_\varepsilon+\boldsymbol{\phi}_{\xi,\varepsilon}$ constructed by Theorem 4.2.5 is a solution of (4.2.1).*

Proof. Suppose that ξ is a critical point of $\tilde{E}_\varepsilon$. Then, for $i=1,\cdots,m, j=1,\cdots,\mathrm{i}(\xi_i)$, Theorem 4.2.5 implies that

$$\begin{aligned}
0 = \partial_{(\xi_i)_j}\tilde{E}_\varepsilon &= \langle \mathbf{u} - i^*(F(\mathbf{u})), \partial_{(\xi_i)_i}(\mathbf{W}_\varepsilon + \boldsymbol{\phi}_{\xi,\varepsilon})\rangle \\
&\overset{(4.2.43)}{=} \sum_{i',j'} c^\varepsilon_{i',j'}\langle PZ^{j'}_{i'}, \partial_{(\xi_i)_i}(W_{1,\varepsilon} + (\boldsymbol{\phi}_{\xi,\varepsilon})_1)\rangle \\
&= \sum_{i',j'} c^\varepsilon_{i',j'}\langle PZ^{j'}_{i'}, PZ^j_i\rangle + \mathcal{O}\left(\sum_{i',j'} c^\varepsilon_{i',i'}(\varepsilon|\log\varepsilon| + |\langle PZ^{j'}_{i'}, \partial_{(\xi_i)_j}(\boldsymbol{\phi}_{\xi,\varepsilon})_1\rangle|\right) \\
&\quad \text{(by Lemma B.0.4 and (4.2.44))} \\
&= c^\varepsilon_{i,j}\frac{8D_i\varrho(\xi_i)}{\pi\delta_i^2} + \mathcal{O}\left(\varepsilon^{-\frac{1}{2}}\sum_{i',j'}|c^\varepsilon_{i',j'}|\right) + o\left(\varepsilon^{-1}\sum_{i',j'}|c^\varepsilon_{i',j'}|\right).
\end{aligned}$$

By the arbitrariness of i, j, it follows as $\varepsilon \to 0$

$$\sum_{i,j} c^\varepsilon_{i,j}\frac{8D_i\varrho(\xi_i)}{\pi\delta_i^2}\left(1 + \mathcal{O}(\varepsilon^{\frac{1}{2}}) + o(1)\right) = 0.$$

There exists $\varepsilon_0 > 0$ sufficiently small such that for $\varepsilon \in (0, \varepsilon_0)$,

$$c^\varepsilon_{i,j} = 0,$$

for $i = 1, \cdots, m, j = 1, \cdots, \mathrm{i}(\xi_i)$. By (4.2.43), $\mathbf{u}$ solves (4.2.1).

Conversely, we assume that for some $\xi \in \Xi_{k,m}$, $\mathbf{u} = \mathbf{W}_\varepsilon + \boldsymbol{\phi}_{\xi,\varepsilon}$ constructed by Theorem 4.2.5 is a solution of (4.2.1). It is easy to see that $\mathbf{u} - i^*(F(\mathbf{u})) = 0$. Consequently, for $i = 1, \cdots, m, j = 1, \cdots, \mathrm{i}(\xi_i)$

$$\partial_{(\xi_i)_j}\tilde{E}_\varepsilon(\xi) = \langle \mathbf{u} - i^*(F(\mathbf{u})), \partial_{(\xi_i)_i}(\mathbf{W}_\varepsilon + \boldsymbol{\phi}_{\xi,\varepsilon})\rangle = 0,$$

which means ξ is a critical point of $\tilde{E}_\varepsilon$ □

Proof of Theorem 1.2.2. Suppose (w, ξ) is a non-degenerate solution of (1.2.5). It is clear that $\mathcal{C} := \{\xi\}$ is a C^1-stable critical point set of $\Lambda_{k,m}$. For $\varepsilon > 0$ sufficiently small, there exists a sequence $\xi^\varepsilon = (\xi^\varepsilon_1, \cdots, \xi^\varepsilon_m) \in \Xi_{k,m}$ such that $d_g(\xi^\varepsilon, \mathcal{C}) < \varepsilon$ and ξ^ε is a critical point of $\tilde{E}_\varepsilon : \Xi_{k,m} \to \mathbb{R}$. Without loss of generality, assume that $\xi^\varepsilon = (\xi^\varepsilon_1, \cdots, \xi^\varepsilon_m) \to \xi = (\xi_1, ..., \xi_m) \in \mathcal{C}$, as $\varepsilon \to 0$. Then define $\mathbf{u}_\varepsilon = \mathbf{W}_\varepsilon + \boldsymbol{\phi}_{\xi^\varepsilon,\varepsilon}$ through Theorem 3.1.4. Moreover, Lemma 4.3.1 yields that $\mathbf{u}_\varepsilon$ solves (4.2.1) as $\varepsilon \to 0$. Let $\rho^\varepsilon_1 = \varepsilon\int_\Sigma V_1 e^{u_{1,\varepsilon}}dv_g$. By (4.2.40) and $|\mathrm{e}^s - 1| \leqslant \mathrm{e}^{|s|}|s|$ for any $s \in \mathbb{R}$, for $q > 1$ sufficiently close to 1

$$\begin{aligned}
\rho^\varepsilon_1 &= \varepsilon\int_\Sigma V_1 e^{W_{1,\varepsilon}}dv_g + \mathcal{O}\left(\varepsilon\int_\Sigma V_1 e^{W_{1,\varepsilon}}|(\boldsymbol{\phi}_{\xi^\varepsilon,\varepsilon})_1|dv_g\right) \\
&= \frac{1}{2}\sum_{i=1}^m \varrho(\xi_i) + \mathcal{O}\left(\|\varepsilon V_1 e^{W_{1,\varepsilon}}\|_q\|\boldsymbol{\phi}_{\xi^\varepsilon,\varepsilon}\|\right) = \frac{1}{2}\sum_{i=1}^m \varrho(\xi_i) + o(1).
\end{aligned}$$

For any $\Psi \in C(\Sigma)$, using Lemma B.0.5, we have

$$\begin{aligned} & 2\rho_1^{\varepsilon} \int_{\Sigma} \frac{V_1 \mathrm{e}^{u_{1,\varepsilon}}}{\int_{\Sigma} V_1 \mathrm{e}^{u_{1,\varepsilon}} dv_g} \Psi dv_g = 2\varepsilon \int_{\Sigma} V_1 \mathrm{e}^{u_{1,\varepsilon}} \Psi dv_g \\ = \ & \sum_{i=1}^{m} \int_{\Sigma} \chi_{\xi_i^{\varepsilon}} e^{-\varphi_{\xi_i^{\varepsilon}}} e^{U_{\delta_i(\xi^{\varepsilon}),\xi_i^{\varepsilon}}} \Psi dv_g + o(1) = \sum_{i=1}^{m} \varrho(\xi_i) \Psi\left(\xi_i\right) + o(1), \end{aligned}$$

as $\varepsilon \to 0$, where $\delta_i(\cdot)$ is defined by (4.2.4). By (4.2.6), we can obtain the estimate (1.2.7). The proof is concluded. □

Proof of Corollary 1.2.3. For any $z \in \overline{\mathrm{H}}^1(\Sigma)$, the Moser-Trudinger inequality implies that

$$I_{\xi}(z) \ = \ \frac{1}{2}\int_{\Sigma} |\nabla z|_g^2 dv_g - 2\rho_2 \log\left(\int_{\Sigma} \tilde{V}_2(\cdot,\xi) e^{z}\right) \geq C,$$

for some constant C (uniformly for $\xi \in \Xi_{k,m}$). For any $\rho_2 \in (0, 2\pi)$, the solution $z(\cdot,\xi)$ of (1.2.3) exists but may be degenerate for $\xi \in \Xi_{k,m}$. We fix an arbitrary $z = z(\cdot,\xi)$ that solves (1.2.3) as a minimizer of the energy functional. Since $z(\cdot,\xi)$ is a minimizer of I_{ξ},

$$\begin{aligned} I_{\xi}(z(\cdot,\xi)) \leq I_{\xi}(0) \ &= \ -2\rho_2 \log\left(\int_{\Sigma} V_2 e^{\sum_{i=1}^{m} \frac{1}{2}\varrho(\xi_i) G^g(\cdot,\xi_i)}\right) \\ &\leq \ \left| 2\rho_2 \left(\sum_{i=1}^{m} \frac{1}{2}\varrho(\xi_i) \int_{\Sigma} G^g(\cdot,\xi_i) dv_g + \sum_{i=1}^{m} \int_{\Sigma} \log V_2 dv_g \right) \right| \\ & \qquad \left(\text{ by the Jensen's inequality and } \int_{\Sigma} G^g(\cdot,\xi_i) dv_g = 0\right) \\ &= \ 2m\rho_2 \max_{\Sigma} |\log V_2|. \end{aligned}$$

Let $\mathfrak{J}_{\xi,\rho_2} := \{z \in \overline{\mathrm{H}}^1(\Sigma) : z \text{ solves (1.2.3) for } \xi \text{ and } \rho_2\}$. As a result, we obtain that for some constant $C_0 > 0$,

$$\sup_{\xi \in \Xi_{k,m}} \ \sup_{\rho_2 \in (0,2\pi)} \ \sup_{z \in \mathfrak{J}_{\xi,\rho_2}} |I_{\xi}(z)| \leq C_0.$$

By Lemma 2.6.1, we know that $\mathcal{F}_{k,m}^{V_1}(\xi) \to +\infty$ as $\xi \to \partial \Xi_{k,m}$. We recall that

$$\Lambda_{k,m}(\xi) = \frac{1}{2} I_{\xi}(z(\cdot,\xi)) - \frac{1}{4}\mathcal{F}_{k,m}^{V_1}(\xi).$$

Fix an arbitrary point $\xi^0 \in \Xi_{k,m}$. There exists $\delta_0 > 0$ sufficiently small such that $\xi^0 \in \Xi_{k,m}^{\delta_0}$ and for any $\xi \in \Xi_{k,m} \setminus \Xi_{k,m}^{\delta_0}$ $\mathcal{F}_{k,m}^{V_1}(\xi) > 4C_0 + \mathcal{F}_{k,m}^{V_1}(\xi^0)$. We take $\mathscr{D}$ to be the interior of $\Xi_{k,m}^{\frac{1}{2}\delta_0}$. Let $\mathcal{C} := \left\{\xi \in \Xi_{k,m} : \mathcal{F}_{k,m}^{V_1}(\xi) = \inf_{\Xi_{k,m}} \Lambda_{k,m}\right\} \subset \Xi_{k,m}^{\delta_0} \subset \mathscr{D}$.

We fix $\xi^0 \in \mathcal{C}$ and then we have a non-degenerate solution of (4.1.8) with $\rho_2 \in (0, \rho_0)$, denoted by w^0, where $\rho_0 > 0$ is sufficiently small according to Lemma 4.1.5. It follows (H1) holds for same (w^0, ξ^0) Applying Theorem 1.2.2, we complete the proof. $\square$

Proof of Corollary 1.2.4. As in the proof of Theorem 4.1.8, we have

$$\mathcal{V}_{reg} = \{(V_1, V_2) \in C^{2,\alpha}(\Sigma, \mathbb{R}_+) \times C^{2,\alpha}(\Sigma, \mathbb{R}_+) : (\mathrm{S}_t) \text{ is non-degenerate for } t \in [0,1] \cap \mathbb{Q}\}$$

is a residual set in $C^{2,\alpha}(\Sigma, \mathbb{R}_+) \times C^{2,\alpha}(\Sigma, \mathbb{R}_+)$. We fix an arbitrary $(V_1, V_2) \in \mathcal{V}_{reg}$. Using Proposition 4.1.10, there exists a non-degenerate solution (w, ξ) of (1.2.5). The hypothesis (H1) holds for $\rho_2 \in (0, 2\pi)$. Employing Theorem 1.2.2, we can conclude Corollary 1.2.4. $\square$

5 Asymmetric blow-up solutions for Toda systems

5.1 The construction of asymmetric blow-up solutions

5.1.1 Approximation solutions

We consider Σ is a "k-symmetric" surface with non-empty fixed point set Σ_0 under the action of orthogonal subgroup $\langle \mathfrak{R}_{\mathrm{k}} \rangle$, for $\mathrm{k} \geq 2$, where $\mathfrak{R}_{\mathrm{k}}$ is defined by (1.2.8). Given $\{\xi_1, \cdots, \xi_m\} \subset \Sigma_0 \cap \mathring{\Sigma}$, we will construct a family of solutions blowing up at $\{\xi_1, \cdots, \xi_m\}$. To construct blow-up solutions of (1.2.1), it is sufficient to consider the following problem

$$\begin{cases} -\Delta_g u_1 = 2\varepsilon(V_1 e^{u_1} - \overline{V_1 e^{u_1}}) - \varepsilon(V_2 e^{u_2} - \overline{V_2 e^{u_2}}) & \text{in } \mathring{\Sigma} \\ -\Delta_g u_2 = 2\varepsilon(V_2 e^{u_2} - \overline{V_2 e^{u_2}}) - \varepsilon(V_1 e^{u_1} - \overline{V_1 e^{u_1}}) & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} u_1 = \partial_{\nu_g} u_2 = 0 & \text{on } \partial\Sigma \end{cases}, \tag{5.1.1}$$

where ε is a positive parameter. In fact, for fixed $\xi = (\xi_1, \cdots, \xi_m) \in \Xi_{m,m} := \mathring{\Sigma}^m \setminus \mathbb{F}_{m,m}(\Sigma)$, we are going to construct a family of blow-up solutions $u_\varepsilon = (u_{1,\varepsilon}, u_{2,\varepsilon})$ of (5.1.1) as $\varepsilon \to 0$ with the local limit mass

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Sigma V_i e^{u_{i,\varepsilon}} dv_g = 2^{i+1}\pi m, i = 1, 2.$$

Setting $\rho_i = \varepsilon \int_\Sigma V_i e^{u_i} dv_g$ for $i = 1, 2$, then we can construct a family blow-up solutions of (1.2.1) as $\rho = (\rho_1, \rho_2) \to (4\pi m, 8\pi m)$. For the asymmetric blow-up case, we have no uniform limit profile like in fully blow-up cases. But after a proper scaling around ξ for the system (5.1.1), for any

$i = 1, 2$ we have the limit profile

$$\begin{cases} -\Delta w = |y|^{\alpha_i - 2} e^w & \text{in } \mathbb{R}_\xi \\ \partial_{y_2} w = 0 & \text{on } \partial\mathbb{R}_\xi \\ \int_{\mathbb{R}_\xi} |y|^{\alpha_i - 2} e^w < \infty \end{cases}, \tag{5.1.2}$$

where $\alpha_i = 2^i, \mathbb{R}_\xi = \mathbb{R}^2$ or $\mathbb{R}^2_+$ depending on the scaling center ξ in the interior or on the boundary.

For any $\alpha \geq 2$, we introduce the radially symmetric solutions of the singular Liouville problem (5.1.2),

$$w^\alpha_\tau(y) := \log \frac{2\alpha^2 \tau^\alpha}{(\tau^\alpha + |y|^\alpha)^2} \quad y \in \mathbb{R}^2, \tau > 0.$$

Moreover, we have $\int_{\mathbb{R}^2} |y|^{\alpha - 2} e^{w^\alpha_\tau(y)} dy = 4\pi\alpha$. For any $\xi \in \{\xi_1, \cdots, \xi_m\}$, applying isothermal coordinate $(y_\xi, U(\xi))$, we can pull-back w^α_τ to the Riemann surface around ξ,

$$U^\alpha_{\tau,\xi} := \log \frac{2\alpha^2 \tau^\alpha}{(\tau^\alpha + |y_\xi(x)|^\alpha)^2} \text{ in } U(\xi).$$

Then we project the local bubbles into the functional space $\overline{\mathrm{H}}^1(\Sigma)$ by following equations:

$$\begin{cases} -\Delta_g PU^\alpha_{\tau,\xi} = \chi_\xi e^{-\varphi_\xi} |y_\xi|^{\alpha - 2} e^{U^\alpha_{\tau,\xi}} - \overline{\chi_\xi e^{-\varphi_\xi} |y_\xi|^{\alpha - 2} e^{U^\alpha_{\tau,\xi}}} & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} PU^\alpha_{\tau,\xi} = 0 & \text{on } \partial\Sigma \\ \int_\Sigma PU^\alpha_{\tau,\xi} dv_g = 0 \end{cases}.$$

We take

$$\alpha_i = 2^i \tag{5.1.3}$$

and the concentration parameter

$$\delta_{i,j} = d_{i,j} \varepsilon^{2^{2-2i}}, \tag{5.1.4}$$

where $d_{i,j} > 0$ solves the following identities for $i = 1, 2$

$$\begin{aligned} &\alpha_i \log d_{i,j} - \sum_{i' > i} \alpha_{i'} \log d_{i',j} \\ &= -2 \log \alpha_i + \frac{\varrho(\xi_j)}{2} \left(\alpha_i - \frac{1}{2} \sum_{i' \neq i} \alpha_{i'} \right) \cdot \left(H^g(\xi_j, \xi_j) + \sum_{j' \neq j} G^g(\xi_{j'}, \xi_j) \right) + \log V_i(\xi_j). \end{aligned} \tag{5.1.5}$$

Let

$$\mathcal{A}_{1j} = \left\{ x \in \Sigma : |y_{\xi_j}(x)| \leq \sqrt{\delta_{1,j}\delta_{2,j}} \right\} \text{ and } \mathcal{A}_{2j} = \Sigma \setminus \mathcal{A}_{1j} \tag{5.1.6}$$

for any $j = 1, \cdots, m$. For simplicity of the notations, we denote that $U_j^i = U^{\alpha_i}_{\xi_j,\delta_{i,j}}, PU_j^i = PU^{\alpha_i}_{\xi_j,\delta_{i,j}}, \chi_j = \chi(|y_{\xi_j}|/\bar{\mathrm{r}}_{\xi_j})$ and $\varphi_j = \hat{\varphi}_{\xi_j}(y_{\xi_j})$. The approximation solution $\mathbf{W}_\varepsilon = (W_{1,\varepsilon}, W_{2,\varepsilon})$ is defined by

$$W_{i,\varepsilon} = \sum_{j=1}^{m} PU_j^i - \frac{1}{2} \sum_{\substack{i' = 1 \\ i' \neq i}}^{2} \sum_{j=1}^{m} PU_j^{i'}, \qquad \text{for } i = 1, 2.$$

We define that

$$\mathcal{H}_{\mathrm{k}} := \left\{ u = (u_1, u_2) \in \mathcal{H} : u_i \text{ is } \mathfrak{R}_{\mathrm{k}}\text{-invariant for } i = 1, 2 \right\}.$$

Next, we are going to construct solutions with the form $\mathbf{u}_\varepsilon = \mathbf{W}_\varepsilon + \boldsymbol{\phi}_\varepsilon$, where $\boldsymbol{\phi}_\varepsilon = (\phi_{1,\varepsilon}, \phi_{2,\varepsilon}) \in \mathcal{H}_{\mathrm{k}}$ is the error term.

5.1.2 The linearized operator

We consider the following linear operator associated with the problem (5.1.1):

$$\mathcal{L}_{\xi,\varepsilon}(\boldsymbol{\phi}) := (L^1_{\xi,\varepsilon}(\boldsymbol{\phi}), L^2_{\xi,\varepsilon}(\boldsymbol{\phi})), \tag{5.1.7}$$

where for any $i = 1, 2$

$$\begin{aligned} L^i_{\xi,\varepsilon}(\boldsymbol{\phi}) : \quad = \quad & -\Delta_g \phi_i - \sum_{j=1}^{m} \left(\chi_j e^{-\varphi_j} |y_{\xi_j}|^{\alpha_i - 2} e^{U_j^i} \phi_i - \overline{\chi_j e^{-\varphi_j} |y_{\xi_j}|^{\alpha_i - 2} e^{U_j^i} \phi_i} \right) \\ & + \frac{1}{2} \sum_{\substack{i' = 1 \\ i' \neq i}}^{2} \sum_{j=1}^{m} \left(\chi_j e^{-\varphi_j} |y_{\xi_j}|^{\alpha_{i'} - 2} e^{U_j^{i'}} \phi_{i'} - \overline{\chi_j e^{-\varphi_j} |y_{\xi_j}|^{\alpha_{i'} - 2} e^{U_j^{i'}} \phi_{i'}} \right). \end{aligned}$$

The key lemma is the non-degeneracy of the linear operator $\mathcal{L}_{\xi,\varepsilon}$. Formally, for $i = 1, 2, j = 1, \cdots, m$, we can derive the local limit operator of $L^i_{\xi,\varepsilon}$ is $-\Delta\phi - 2\alpha_i^2 \frac{|y|^{\alpha_i - 2}}{(1+|y|^{\alpha_i})^2}\phi$ by a proper scaling around ξ_j on an isothermal chart (the detail refers to Lemma 5.1.1). The result in [DPEM12] implies that the kernel space is generated by (in polar coordinate (r, θ))

$$\phi^0(r,\theta) := \frac{1 - |r|^{\alpha_i}}{1 + |r|^{\alpha_i}}, \quad \phi^1(r,\theta) := \frac{|r|^{\frac{\alpha_i}{2}}}{1 + |r|^{\alpha_i}} \cos\frac{\alpha_i}{2}\theta, \quad \phi^2(r,\theta) := \frac{|r|^{\frac{\alpha_i}{2}}}{1 + |r|^{\alpha_i}} \sin\frac{\alpha_i}{2}\theta.$$

The k-symmetric condition of $\mathcal{H}_{\mathrm{k}}$ excludes ϕ^1 and ϕ^2. To obtain the invertibility of the linearized operator $\mathcal{L}_{\xi,\varepsilon}$, we need to lay out ϕ^0 by introducing a family of test functions in $\overline{\mathrm{H}}^1(\Sigma)$. Let

$$z_i(y) = \frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}}.$$

It is easy to see that $-\Delta z_i(y) = 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2} z_i(y), y \in \mathbb{R}^2$. For any $j = 1, \cdots, m$, we define

$$Z_{ij}(x) = \begin{cases} z_i\left(\frac{|y_{\xi_j}(x)|}{\delta_{i,j}}\right), & x \in U(\xi_j) \\ 0, & x \in \Sigma \setminus U(\xi_j) \end{cases}.$$

Then, we project Z_{ij} into the space $\overline{\mathrm{H}}^1(\Sigma)$ by following equations:

$$\begin{cases} -\Delta_g PZ_{ij} = \chi_j e^{-\varphi_j} |y_{\xi_j}|^{\alpha_i-2} e^{U_j^i} Z_{ij} - \overline{\chi_j e^{-\varphi_j} |y_{\xi_j}|^{\alpha_i-2} e^{U_j^i} Z_{ij}} & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} PZ_{ij} = 0 & \text{on } \partial\Sigma \\ \int_\Sigma PZ_{ij} dv_g = 0 \end{cases}. \tag{5.1.8}$$

We define that $\mathfrak{L}_{\mathrm{k}}^0 := \{h = (h_1, h_2) : \int_\Sigma h_i dv_g = 0 \text{ and } h_i \text{ is } \mathfrak{R}_{\mathrm{k}} \text{ invariant for } i = 1, 2\}$.

Lemma 5.1.1. *For any $p > 1$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\mathbf{h} = (h_1, h_2) \in (L^p(\Sigma))^2 \cap \mathfrak{L}_{\mathrm{k}}^0$ and $\boldsymbol{\phi} = (\phi_1, \phi_2) \in (W^{2,p}(\Sigma))^2 \cap \mathcal{H}_{\mathrm{k}}$ is the unique solution of*

$$\begin{cases} \mathcal{L}_{\xi,\varepsilon}(\boldsymbol{\phi}) = \mathbf{h} & \text{in } \mathring{\Sigma} \\ \partial_{\nu_g} \boldsymbol{\phi} = 0 & \text{on } \partial\Sigma \\ \int_\Sigma \boldsymbol{\phi} dv_g = 0 \end{cases}, \tag{5.1.9}$$

with $\|\boldsymbol{\phi}\| \le C|\log\varepsilon| \|\mathbf{h}\|_p$, where $\|\mathbf{h}\|_p = \|h_1\|_{L^p(\Sigma)} + \|h_2\|_{L^p(\Sigma)}$.

Proof. We prove it by contradiction. Suppose Lemma 5.1.1 fails, i.e., there exist $p > 1$ and a sequence of $\varepsilon_n \to 0$ and $\mathbf{h}_n := (h_{1,n}, h_{2,n}) \in (L^p(\Sigma))^2 \cap \mathfrak{L}_{\mathrm{k}}^0$ and $\boldsymbol{\phi}_n := (\phi_{1,n}, \phi_{2,n}) \in (W^{2,p}(\Sigma))^2 \cap \mathcal{H}_{\mathrm{k}}$ solves (5.1.9) for $\mathbf{h}_n$ satisfying

$$\|\boldsymbol{\phi}_n\| = 1 \text{ and } |\log\varepsilon_n| \|\mathbf{h}_n\|_p := |\log\varepsilon_n| \sum_{i=1}^2 \|h_{i,n}\|_{L^p(\Sigma)} \to 0,$$

as $n \to +\infty$. For simplicity, we still use the notations ϕ_i, h_i, ε instead of $\phi_{i,n}, h_{i,n}, \varepsilon_n$ for $i = 1, 2$.

Let $\mathbb{R}_\xi = \begin{cases} \mathbb{R}^2 & \text{if } \xi \in \mathring{\Sigma} \\ \mathbb{R}^2_+ & \text{if } \xi \in \partial\Sigma \end{cases}$. We define that for $i = 1, 2, j = 1, \cdots, m$

$$\tilde{\phi}_{ij}(y) = \begin{cases} \chi\left(\frac{\delta_{i,j}|y|}{\bar{r}_{\xi_j}}\right)\phi_i \circ y_{\xi_j}^{-1}(\delta_{i,j}y), & y \in \Omega_{ij} := \frac{1}{\delta_{i,j}}B^{\xi_j} \\ 0 & y \in \mathbb{R}_{\xi_j} \setminus \Omega_{ij} \end{cases}.$$

Then we consider the following spaces for $\alpha \geq 2$ and $\xi \in \Sigma$, $\mathrm{L}^\alpha_\xi := \left\{u : \left\|\frac{|y|^{\frac{\alpha-2}{2}}}{1+|y|^\alpha}u\right\|_{L^2(\mathbb{R}_\xi)} < +\infty\right\}$ and $\mathrm{H}^\alpha_\xi := \left\{u : \|\nabla u\|_{L^2(\mathbb{R}_\xi)} + \left\|\frac{|y|^{\frac{\alpha-2}{2}}}{1+|y|^\alpha}u\right\|_{L^2(\mathbb{R}_\xi)} < +\infty\right\}$.

Step 1. *$\tilde{\phi}_{ij} \to a_{ij}\frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}}$ as $\varepsilon \to 0$ for some $a_{ij} \in \mathbb{R}$, which is weakly in $\mathrm{H}^{\alpha_i}_{\xi_j}$ and strongly in $\mathrm{L}^{\alpha_i}_{\xi_j}$.*

By (5.1.9), it follows that

$$\begin{aligned} & h_1 - h_2 \\ = \; & -\Delta_g(\phi_1 - \phi_2) - \frac{3}{2}\sum_{j=1}^m \left(\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_1-2}e^{U_j^1}\phi_1 - \overline{\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_1-2}e^{U_j^1}\phi_1}\right) \\ & + \frac{3}{2}\sum_{j=1}^m \left(\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_2-2}e^{U_j^2}\phi_2 - \overline{\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_2-2}e^{U_j^2}\phi_2}\right). \end{aligned} \tag{5.1.10}$$

Using ϕ_i as a test function of (5.1.10), we can deduce that

$$\begin{aligned} & \int_\Sigma (h_1 - h_2)\phi_i dv_g \\ = \; & \int_\Sigma \langle \nabla(\phi_1 - \phi_2), \nabla\phi_i\rangle_g dv_g - \frac{3}{2}\sum_{j=1}^m \int_\Sigma \chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_1-2}e^{U_j^1}\phi_1\phi_i dv_g \\ & + \frac{3}{2}\sum_{j=1}^m \int_\Sigma \chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_2-2}e^{U_j^2}\phi_2\phi_i dv_g. \end{aligned} \tag{5.1.11}$$

in view of $\int_\Sigma \phi_i dv_g = 0$. The Sobolev's inequality and Hölder inequality yield that

$$\left|\int_\Sigma (h_1 - h_2)\phi_i\right| \leq \|\mathbf{h}\|_p\|\phi_i\|_{p'} \leq \|\mathbf{h}\|_p\|\phi_i\| = o(1/|\log\varepsilon|) \to 0,$$

where $p, p' > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Hence, we derive that for $i, i' = 1, 2$ with $i' \neq i$

$$\sum_{j=1}^m \int_\Sigma \chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2}e^{U_j^i}\phi_i^2 dv_g = \sum_{j=1}^m \int_\Sigma \chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_{i'}-2}e^{U_j^{i'}}\phi_{i'}\phi_i dv_g + \mathcal{O}(1). \tag{5.1.12}$$

On the other hand, using ϕ_i as a test function of $L^i_{\xi,\varepsilon}(\boldsymbol{\phi}) = h_i$, similarly, we have for $i' \neq i$

$$\begin{aligned}\sum_{j=1}^{m}\int_{\Sigma}\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2}e^{U_j^i}\phi_i^2 dv_g &= \frac{1}{2}\sum_{j=1}^{m}\int_{\Sigma}\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_{i'}-2}e^{U_j^{i'}}\phi_{i'}\phi_i dv_g + \mathcal{O}(1)\\ &\overset{(5.1.12)}{=} \frac{1}{2}\sum_{j=1}^{m}\int_{\Sigma}\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2}e^{U_j^i}\phi_i^2 + \mathcal{O}(1).\end{aligned}$$

Consequently, for $i = 1, 2$

$$\sum_{j=1}^{m}\int_{\Sigma}\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2}e^{U_j^i}\phi_i^2 dv_g = \mathcal{O}(1). \tag{5.1.13}$$

By a straightforward calculation,

$$\sum_{j=1}^{m}\int_{\mathbb{R}_{\xi_j}}\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}(\tilde{\phi}_{ij}(y))^2 dy = \sum_{j=1}^{m}\int_{\Sigma}\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2}e^{U_j^i}\phi_i^2 dv_g = \mathcal{O}(1).$$

Additionally, by the assumption $\|\boldsymbol{\phi}\| = 1$, we immediately have

$$\int_{\mathbb{R}_{\xi_j}}|\nabla\tilde{\phi}_{ij}|^2 \leq \int_{\Sigma}|\nabla\phi_i|_g^2 dv_g \leq 1.$$

It follows that $\tilde{\phi}_{ij}$ is uniformly bounded in $\mathrm{H}^{\alpha_i}_{\xi_j}$. Applying [MPW16, Proposition A.1], $\mathrm{L}^{\alpha_i}_{\xi_j} \hookrightarrow \mathrm{H}^{\alpha_i}_{\xi_j}$ is a compact embedding. Then up to a subsequence, we have

$$\tilde{\phi}_{ij} \to \tilde{\phi}^0_{ij}$$

weakly convergent in $\mathrm{H}^{\alpha_i}_{\xi_j}$ and strongly in $\mathrm{L}^{\alpha_i}_{\xi}$. For any $q > 1$,

$$\int_{\Sigma}\left|\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2}e^{U_j^i}\right|^q dv_g = \int_{U_{2r_0}(\xi_j)}\chi^q(|y|/r_0)\delta_{i,j}^{2-2q}\frac{|y|^{q(\alpha_i-2)}}{(1+|y|^{\alpha_i})^2}dy = \mathcal{O}(\delta_{i,j}^{2(1-q)}). \tag{5.1.14}$$

By changing variables of (5.1.9), (5.1.14) yields that

$$\begin{cases} -\Delta\tilde{\phi}_{ij}(y) = 2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_{ij} + \psi_{ij} & \text{in } \Omega_{ij}\\ \partial_{y_2}\tilde{\phi}_{ij} = 0 & \text{on } \Omega_{ij}\cap\{y_2 = 0\}\end{cases}, \tag{5.1.15}$$

where $\psi_{ij} = -\frac{1}{2}\sum_{i'\neq i}2\alpha_{i'}^2\frac{\delta_{i,j}^{\alpha_{i'}}\delta_{i',j}^{\alpha_{i'}}|y|^{\alpha_{i'}-2}}{(\delta_{i',j}^{\alpha_{i'}}+\delta_{i,j}^{\alpha_{i'}}|y|^{\alpha_{i'}})^2} + \delta_{i,j}^2 h_i\circ y_{\xi_j}^{-1}(\delta_{i,j}y) + \mathcal{O}(\delta_{i,j}^2\sum_{i'=1}^2\delta_{i',j}^{\frac{2}{q}-2})$, for $q \in (1, \frac{4}{3})$.

Next, we are going to show that $\tilde{\phi}_{ij}$ converges to the solution of

$$\begin{cases} -\Delta\phi = 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\phi & \text{in } \mathbb{R}_{\xi_j} \\ \partial_{y_2}\phi = 0 & \text{on } \partial\mathbb{R}_{\xi_j} \\ \int_{\mathbb{R}^2} |\nabla\phi(y)|^2 dy < +\infty \end{cases}, \tag{5.1.16}$$

in the distribution sense. We take an arbitrary $\varphi \in C_c^\infty(\mathbb{R}_{\xi_j})$ with

$$\operatorname{supp}(\varphi) \subset \left\{ y \in \Omega_{ij} : \sqrt{\delta_{i-1,j}/\delta_{i,j}} \le |y| < \sqrt{\delta_{i+1,j}/\delta_{i,j}} \right\},$$

for $\varepsilon > 0$ sufficiently small, where we denote $\delta_{0,j} = 0$ and $\delta_{3,j} = +\infty$. Applying φ as a test function for (5.1.15), we deduce that

$$\begin{aligned} &\int_{\operatorname{supp}(\varphi)} \nabla\tilde{\phi}_{ij} \cdot \nabla\varphi - 2\alpha_i^2 \int_{\operatorname{supp}(\varphi)} \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_{ij}\varphi = \int_{\operatorname{supp}(\varphi)} \psi_{ij}\varphi \\ = \quad & \mathcal{O}\left(\frac{1}{2}\int_{\mathcal{A}_{ij}} \sum_{i'\neq i} 2\alpha_{i'}^2 e^{-\varphi_j} \frac{\delta_{i',j}^{\alpha_{i'}}|y_{\xi_j}|^{\alpha_{i'}-2}}{(\delta_{i',j}^{\alpha_{i'}} + |y_{\xi_j}|^{\alpha_{i'}})^2}\phi_{ij}dv_g\right) + \int_{\delta_{i,j}\operatorname{supp}(\varphi)} h_i \circ y_{\xi_j}^{-1}(y)\varphi(\delta_{i,j}^{-1}y)dy + o(1) \\ = \quad & \mathcal{O}\left(\left(\int_{\sqrt{\delta_{i-1,j}\delta_{i,j}} \le |y| < \sqrt{\delta_{i,j}\delta_{i+1,j}}} \left(\sum_{i'\neq i} \frac{2\alpha_{i'}^2\delta_{i',j}^{\alpha_{i'}}|y|^{\alpha_{i'}-2}}{(\delta_{i',j}^{\alpha_{i'}} + |y|^{\alpha_{i'}})^2}\right)^q dy\right)^{\frac{1}{q}} \|\phi_i\|\right) \\ & +\mathcal{O}\left(\left(\int_{\delta_{i,j}\operatorname{supp}(\varphi)} |\varphi(\delta_{i,j}^{-1}y)|^{p'}dy\right)^{\frac{1}{p'}} \|h_i\|_p\right) + o(1) = o(1), \end{aligned}$$

where $\mathcal{A}_{ij}$ is defined by (5.1.6), $\frac{1}{p} + \frac{1}{p'} = 1, q \in (1, \frac{4}{3})$. Therefore, we obtain that ϕ_{1j}^0 solves (5.1.16) with $\alpha_i = 2$ in sense of distribution on $\mathbb{R}_{\xi_j}$ and ϕ_{2j}^0 solves (5.1.16) with $\alpha_i = 4$ in sense of distribution on $\mathbb{R}_{\xi_j} \setminus \{0\}$. According to the regularity theory, ϕ_{1j}^0 is a smooth solution on the whole space $\mathbb{R}_{\xi_j}$, as well as ϕ_{2j}^0 for any $j = 1, \cdots, m$. The result in [DPEM12] implies that the solution space of (5.1.16) is generated by the following functions (in polar coordinate (r,θ))

$$\phi^0(r,\theta) := \frac{1-|r|^{\alpha_i}}{1+|r|^{\alpha_i}}, \quad \phi^1(r,\theta) := \frac{|r|^{\frac{\alpha_i}{2}}}{1+|r|^{\alpha_i}} \cos\frac{\alpha_i}{2}\theta, \quad \phi^2(r,\theta) := \frac{|r|^{\frac{\alpha_i}{2}}}{1+|r|^{\alpha_i}} \sin\frac{\alpha_i}{2}\theta.$$

Since $\boldsymbol{\phi} \in \mathcal{H}_{\mathrm{k}}$, it follows that for some $a_{ij} \in \mathbb{R}$,

$$\tilde{\phi}_{ij}^0 = a_{ij}\frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}}.$$

Step 2. *For* $i = 1, 2, j = 1, \cdots, m$, $\int_{\Omega_{ij}} 2\alpha_i^2 \frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_{ij}(y)dy = o(|\log\varepsilon|^{-1})$.

Applying that PZ_{ij} as a test function of (5.1.9), we have

$$\int_\Sigma 2\alpha_i^2 \chi_j e^{-\varphi_j} \frac{\delta_{i,j}^{\alpha_i}|y_{\xi_j}|^{\alpha_i-2}}{\left(\delta_{i,j}^{\alpha_i}+|y_{\xi_j}|^{\alpha_i}\right)^2}\phi_i Z_{ij} dv_g = \int_\Sigma 2\alpha_i^2 \chi_j e^{-\varphi_j} \frac{\delta_{i,j}^{\alpha_i}|y_{\xi_j}|^{\alpha_i-2}}{\left(\delta_{i,j}^{\alpha_i}+|y_{\xi_j}|^{\alpha_i}\right)^2}\phi_i PZ_{ij} dv_g$$
$$-\frac{1}{2}\sum_{\substack{i'=1\\ i'\neq i}}^{2}\int_\Sigma 2\alpha_{i'}^2 \chi_j e^{-\varphi_j} \frac{\delta_{i',j}^{\alpha_{i'}}|y_{\xi_j}|^{\alpha_{i'}-2}}{\left(\delta_{i',j}^{\alpha_{i'}}+|y_{\xi_{i'}}|^{\alpha_{i'}}\right)^2}\phi_{i'} PZ_{ij} dv_g + \int_\Sigma h_i PZ_{ij} dv_g.$$

It follows that

$$\begin{aligned} 0 \;=\;& \int_\Sigma 2\alpha_i^2 \chi_j e^{-\varphi_j} \frac{\delta_{i,j}^{\alpha_i}|y_{\xi_j}|^{\alpha_i-2}}{\left(\delta_{i,j}^{\alpha_i}+|y_{\xi_j}|^{\alpha_i}\right)^2}\phi_i \left(PZ_{ij}-Z_{ij}\right) dv_g \\ & -\frac{1}{2}\sum_{\substack{i'=1\\ i'\neq i}}^{2}\int_\Sigma 2\alpha_{i'}^2 \chi_j e^{-\varphi_j} \frac{\delta_{i',j}^{\alpha_{i'}}|y_{\xi_j}|^{\alpha_{i'}-2}}{\left(\delta_{i',j}^{\alpha_{i'}}+|y_{\xi_{i'}}|^{\alpha_{i'}}\right)^2}\phi_{i'} PZ_{ij} dv_g + \int_\Sigma h_i PZ_{ij} dv_g. \end{aligned}$$

By Lemma C.0.2, we have $\|PZ_{ij}\|_{L^{p'}(\Sigma)} = \mathcal{O}(1)$, for $\frac{1}{p}+\frac{1}{p'}=1$. Then, by the assumption $\|\mathbf{h}\|_p = o(|\log\varepsilon|^{-1})$ and the Hölder inequality, we have

$$\left|\int_\Sigma h_i PZ_{ij} dv_g\right| \le \|PZ_{ij}\|_{p'}\|h_i\|_p = o(|\log\varepsilon|^{-1}).$$

Applying Lemma C.0.2 again, we derive that

$$\begin{aligned} & (\log\varepsilon)\int_\Sigma 2\alpha_i^2 \chi_j e^{-\varphi_j} \frac{\delta_{i,j}^{\alpha_i}|y_{\xi_j}|^{\alpha_i-2}}{\left(\delta_{i,j}^{\alpha_i}+|y_{\xi_j}|^{\alpha_i}\right)^2}\phi_i \left(PZ_{ij}-Z_{ij}\right) dv_g \\ = \;& (\log\varepsilon)\int_\Sigma 2\alpha_i^2 \chi_j e^{-\varphi_j} \frac{\delta_{i,j}^{\alpha_i}|y_{\xi_j}|^{\alpha_i-2}}{\left(\delta_{i,j}^{\alpha_i}+|y_{\xi_j}|^{\alpha_i}\right)^2}\phi_i \left(1+\mathcal{O}(\delta_{i,j}^{\alpha_i}|\log\delta_{i,j}|)\right) dv_g \\ = \;& (1+\mathcal{O}(\delta_{i,j}^{\alpha_i}|\log\delta_{i,j}|))(\log\varepsilon)\int_{\Omega_{ij}} 2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_{ij}(y)dy + \mathcal{O}(|\log\varepsilon|\delta_{i,j}^{\alpha_i}) \\ = \;& (\log\varepsilon)\int_{\Omega_{ij}} 2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_{ij}(y)dy + o(1), \end{aligned}$$

as $\varepsilon\to 0$. For $i\neq i'$, (C.0.1) indicates that

$$PZ_{ij}\circ y_{\xi_j}^{-1}(\delta_{i',j}y) = \begin{cases} \mathcal{O}\left(\left(\frac{\delta_{i,j}}{\delta_{i',j}}\right)^{\alpha_i}\frac{1}{|y|^{\alpha_i}}+\delta_{i,j}^{\alpha_i}\right) & \text{if } i<i' \\ 2+\mathcal{O}\left(\left(\frac{\delta_{i',j}}{\delta_{i,j}}\right)^{\alpha_i}|y|^{\alpha_i}+\delta_{i,j}^{\alpha_i}\right) & \text{if } i'<i \end{cases}. \tag{5.1.17}$$

It follows that

$$
\begin{aligned}
&\frac{1}{2}(\log\varepsilon)\int_{\Sigma}2\alpha_{i'}^2\chi_j e^{-\varphi_j}\frac{\delta_{i',j}^{\alpha_{i'}}|y_{\xi_j}|^{\alpha_{i'}-2}}{\left(\delta_{i',j}^{\alpha_{i'}}+|y_{\xi_j}|^{\alpha_{i'}}\right)^2}\phi_{i'}PZ_{ij}dv_g\\
=\quad&\frac{1}{2}(\log\varepsilon)\int_{\Omega_{i'j}}2\alpha_{i'}^2\frac{|y|^{\alpha_{i'}-2}}{(1+|y|^{\alpha_{i'}})^2}\tilde{\phi}_{i'j}(y)PZ_{ij}\circ y_{\xi_j}^{-1}(\delta_{i',j}y)dy+\mathcal{O}(|\log\varepsilon|\delta_{i',j}^{\alpha_{i'}})\\
\overset{(5.1.17)}{=}\quad&\begin{cases}\mathcal{O}\left(|\log\varepsilon|\int_{\Omega_{i'j}}2\alpha_{i'}^2\frac{|y|^{\alpha_{i'}-2}}{(1+|y|^{\alpha_{i'}})^2}|\tilde{\phi}_{i'j}(y)|\left(\left(\frac{\delta_{i,j}}{\delta_{i',j}}\right)^{\alpha_i}\frac{1}{|y|^{\alpha_i}}+\delta_{i,j}^{\alpha_i}\right)dy\right) & \text{if } i<i'\\
2(\log\varepsilon)\int_{\Omega_{i'j}}2\alpha_{i'}^2\frac{|y|^{\alpha_{i'}-2}}{(1+|y|^{\alpha_{i'}})^2}\tilde{\phi}_{i'j}(y)dy & \\
+\mathcal{O}\left(|\log\varepsilon|\int_{\Omega_{i'j}}2\alpha_{i'}^2\frac{|y|^{\alpha_{i'}-2}}{(1+|y|^{\alpha_{i'}})^2}|\tilde{\phi}_{i'j}(y)|\left(\left(\frac{\delta_{i',j}}{\delta_{i,j}}\right)^{\alpha_i}|y|^{\alpha_i}+\delta_{i,j}^{\alpha_i}\right)dy\right) & \text{if } i'<i\end{cases}\\
=\quad&\begin{cases}o(1) & \text{if } i<i'\\
2(\log\varepsilon)\int_{\Omega_{i'j}}2\alpha_{i'}^2\frac{|y|^{\alpha_{i'}-2}}{(1+|y|^{\alpha_{i'}})^2}\tilde{\phi}_{i'j}(y)dy+o(1) & \text{if } i'<i\end{cases}.
\end{aligned}
$$

Thus, we have for $i=1,2$, $(\log\varepsilon)\int_{\Omega_{ij}}2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_{ij}(y)dy=o(1)$ as $\varepsilon\to 0$.

Step 3. *Construct the contradiction.*

Using PU_j^i as a test function for (5.1.9), we derive that

$$
\begin{aligned}
&\int_{\Sigma}2\alpha_i^2\chi_j e^{-\varphi_j}\frac{\delta_{i,j}^{\alpha_i}|y_{\xi_j}|^{\alpha_i-2}}{\left(\delta_{i,j}^{\alpha_i}+|y_{\xi_j}|^{\alpha_i}\right)^2}\phi_i dv_g \qquad (5.1.18)\\
=\ &\int_{\Sigma}2\alpha_i^2\chi_j e^{-\varphi_j}\frac{\delta_{i,j}^{\alpha_i}|y_{\xi_j}|^{\alpha_i-2}}{\left(\delta_{i,j}^{\alpha_i}+|y_{\xi_j}|^{\alpha_i}\right)^2}\phi_i PU_j^i dv_g-\frac{1}{2}\sum_{\substack{i'=1\\ i'\neq i}}^{2}\int_{\Sigma}2\alpha_{i'}^2\chi_j e^{-\varphi_j}\frac{\delta_{i',j}^{\alpha_{i'}}|y_{\xi_j}|^{\alpha_{i'}-2}}{\left(\delta_{i',j}^{\alpha_{i'}}+|y_{\xi_{i'}}|^{\alpha_{i'}}\right)^2}\phi_{i'}PU_j^i dv_g\\
&+\int_{\Sigma}h_i PU_j^i dv_g.
\end{aligned}
$$

The left hand side of (5.1.18) implies that

$$
\begin{aligned}
\int_{\Sigma}2\alpha_i^2\chi_j e^{-\varphi_j}\frac{\delta_{i,j}^{\alpha_i}|y_{\xi_j}|^{\alpha_i-2}}{\left(\delta_{i,j}^{\alpha_i}+|y_{\xi_j}|^{\alpha_i}\right)^2}\phi_i dv_g &= \int_{\Omega_{ij}}2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_{ij}(y)dy+\mathcal{O}(\delta_{i,j}^{\alpha_i})\\
&= o(1),
\end{aligned}
$$

by the result of Step 2. Lemma C.0.1 yields that $\|PU_j^i\|_{L^\infty(\Sigma)}=\mathcal{O}(|\log\varepsilon|)$. We drive that

$$
\left|\int_{\Sigma}h_i PU_j^i dv_g\right|\leq\|h_i\|_p\|PU_j^i\|_{p'}=o(1),
$$

by the Hölder inequality, where $\frac{1}{p}+\frac{1}{p'}=1$. By straightforward calculation, we have

$$\begin{aligned}
\delta_{i,j}\int_{\Omega_{ij}} 2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}|\tilde{\phi}_{ij}(y)||y|dy &\leq 2\alpha_i^2\delta_{i,j}\delta_{i,j}^{\frac{2(1-q)}{q}}\|\phi_i\|_{q'}\left(\int_{\mathbb{R}^2}\left(\frac{|y|^{\alpha_j-1}}{(1+|y|^{\alpha_j})^2}\right)^q dy\right)^{1/q}\\
&= O\left(\delta_{i,j}^{\frac{2-q}{q}}\right)=o(1).
\end{aligned}$$

where $q\in(1,2)$ such that $\frac{1}{q}+\frac{1}{q'}=1$. For $i=1,2$, applying Lemma C.0.1, Step 1 and Step 2, we deduce that

$$\begin{aligned}
&\int_\Sigma 2\alpha_i^2\chi_j e^{-\varphi_j}\frac{\delta_{i,j}^{\alpha_i}|y_{\xi_j}|^{\alpha_i-2}}{\left(\delta_{i,j}^{\alpha_i}+|y_{\xi_j}|^{\alpha_i}\right)^2}\phi_i PU_j^i dv_g\\
=\;&\int_\Sigma 2\alpha_i^2\chi_j e^{-\varphi_j}\frac{\delta_{i,j}^{\alpha_i}|y_{\xi_j}|^{\alpha_i-2}}{\left(\delta_{i,j}^{\alpha_i}+|y_{\xi_j}|^{\alpha_i}\right)^2}\phi_i\left(-2\chi_j\log\left(\delta_{i,j}^{\alpha_i}+|y_{\xi_j}|^{\alpha_i}\right)+\frac{\alpha_i\varrho(\xi_j)}{2}H^g(\cdot,\xi_j)\right.\\
&\left.+\mathcal{O}(\delta_{i,j}^{\alpha_i}|\log\delta_{i,j}|)\right)dv_g\\
=\;&\int_{\Omega_{ij}}2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\tilde{\phi}_{ij}(y)\left(-2\alpha_i\log\delta_{i,j}-2\log\left(1+|y|^{\alpha_i}\right)+\frac{\alpha_i\varrho(\xi_j)}{2}H^g(\xi_j,\xi_j)\right)dy\\
&+O\left(\int_{\Omega_{ij}}2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\left|\tilde{\phi}_{ij}(y)\right|\left(\delta_{i,j}|y|+\delta_{i,j}^{\alpha_i}|\log\delta_{i,j}|\right)dy\right)+\mathcal{O}(\delta_{i,j}^{\alpha_i})\\
\to\;&-2a_{ij}\int_{\mathbb{R}_{\xi_j}}2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}}\log(1+|y|^{\alpha_i})dy=\frac{\varrho(\xi_j)\alpha_i}{2}a_{ij},
\end{aligned}$$

as $\varepsilon\to 0$, in which the last equality used the fact that $\int_{\mathbb{R}^2}2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}}\log(1+|y|^{\alpha_i})dy=-2\pi\alpha_i$. For any $i'\neq i$, by Lemma C.0.1, Step 1 and Step 2 again, it follows that as $\varepsilon\to 0$,

$$\begin{aligned}
&\int_\Sigma 2\alpha_{i'}^2\chi_j e^{-\varphi_j}\frac{\delta_{i',j}^{\alpha_{i'}}|y_{\xi_j}|^{\alpha_{i'}-2}}{\left(\delta_{i',j}^{\alpha_{i'}}+|y_{\xi_j}|^{\alpha_{i'}}\right)^2}\phi_{i'}PU_j^i dv_g\\
=\;&\int_\Sigma 2\alpha_{i'}^2\chi_j e^{-\varphi_j}\frac{\delta_{i',j}^{\alpha_{i'}}|y_{\xi_j}|^{\alpha_{i'}-2}}{\left(\delta_{i',j}^{\alpha_{i'}}+|y_{\xi_j}|^{\alpha_{i'}}\right)^2}\phi_{i'}\left(-2\chi_j\log\left(\delta_{i,j}^{\alpha_i}+|y_{\xi_j}|^{\alpha_i}\right)+\frac{\alpha_i\varrho(\xi_j)}{2}H^g(\cdot,\xi_j)\right.\\
&\left.+\mathcal{O}(\delta_{i,j}^{\alpha_i}|\log\delta_{i,j}|)\right)dv_g\\
&=\int_{\Omega_{i'j}}2\alpha_{i'}^2\frac{|y|^{\alpha_{i'}-2}}{(1+|y|^{\alpha_{i'}})^2}\tilde{\phi}_{i'j}(y)\left(-2\log\left(\delta_{i,j}^{\alpha_i}+\delta_{i',j}^{\alpha_i}|y|^{\alpha_i}\right)+\frac{\alpha_i\varrho(\xi_j)}{2}H^g(\xi_j,\xi_j)\right)dy+\\
&+O\left(\int_{\Omega_{i'j}}2\alpha_{i'}^2\frac{|y|^{\alpha_{i'}-2}}{(1+|y|^{\alpha_{i'}})^2}\left|\tilde{\phi}_{i'j}(y)\right|\left(\delta_{i',j}|y|+\delta_{i,j}^{\alpha_i}|\log\delta_{i,j}|\right)dy\right)+\mathcal{O}(\delta_{i',j}^{\alpha_{i'}})\\
\to\;&\begin{cases}-2\alpha_i a_{i'j}\int_{\mathbb{R}_{\xi_j}}2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}}\log|y|dy & \text{if } i<i'\\ 0 & \text{if } i>i'\end{cases}=\begin{cases}\varrho(\xi_j)a_{i'j} & \text{if } i<i'\\ 0 & \text{if } i>i'\end{cases},
\end{aligned}$$

in which we used $\int_{\mathbb{R}^2}2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}}dy=0$ and $\int_{\mathbb{R}^2}2\alpha_i^2\frac{|y|^{\alpha_i-2}}{(1+|y|^{\alpha_i})^2}\frac{1-|y|^{\alpha_i}}{1+|y|^{\alpha_i}}\log|y|dy=-4\pi$. The right hand side of (5.1.18) has the following estimate:

$$\int_\Sigma \frac{2\alpha_i^2\chi_j e^{-\varphi_j}\delta_{i,j}^{\alpha_i}|y_{\xi_j}|^{\alpha_i-2}}{\left(\delta_{i,j}^{\alpha_i}+|y_{\xi_j}|^{\alpha_i}\right)^2}\phi_i PU_j^i dv_g - \frac{1}{2}\sum_{\substack{i'=1\\ i'\neq i}}^{2}\int_\Sigma \frac{2\alpha_{i'}^2\chi_j e^{-\varphi_j}\delta_{i',j}^{\alpha_{i'}}|y_{\xi_j}|^{\alpha_{i'}-2}}{\left(\delta_{i',j}^{\alpha_{i'}}+|y_{\xi_{i'}}|^{\alpha_{i'}}\right)^2}\phi_{i'} PU_j^i dv_g + o(1)$$

$$= \begin{cases} \frac{\varrho(\xi_j)\alpha_i}{2}\left(a_{ij}-\sum_{i'>i}a_{i'j}\right)+o(1) & \text{if } i=1 \\ \frac{\varrho(\xi_j)\alpha_i}{2}a_{ij}+o(1) & \text{if } i=2 \end{cases}.$$

Consequently, $a_{ij}=0$ for any $i=1,2$, $j=1,\cdots,m$. We will use ϕ_i as test functions for (5.1.9). For $i=1$, we deduce that

$$\begin{aligned} \|\phi_1\|^2 &= \sum_{j=1}^m\int_\Sigma 2\alpha_1^2\chi_j e^{-\varphi_j}\frac{\delta_{1,j}^{\alpha_1}|y_{\xi_j}|^{\alpha_1-2}}{\left(\delta_{1,j}^{\alpha_1}+|y_{\xi_j}|^{\alpha_1}\right)^2}\phi_1^2 dv_g \\ &-\frac{1}{2}\sum_{j=1}^m\int_\Sigma 2\alpha_2^2\chi_j e^{-\varphi_j}\frac{\delta_2^{\alpha_2}|y_{\xi_j}|^{\alpha_2-2}}{\left(\delta_2^{\alpha_2}+|y_{\xi_j}|^{\alpha_2}\right)^2}\phi_2\phi_1 dv_g+\int_\Sigma h_1\phi_i dv_g \\ &\leq \sum_{j=1}^m\|\tilde\phi_{1j}\|^2_{\mathrm{L}_{\xi_j}^{\alpha_1}}+\frac{1}{2}\sum_{j=1}^m\|\tilde\phi_{1j}\|_{\mathrm{L}_{\xi_j}^{\alpha_1}}\|\tilde\phi_{2j}\|_{\mathrm{L}_{\xi_j}^{\alpha_2}}+o(|\log\varepsilon|)\to 0, \end{aligned}$$

as $\varepsilon\to 0$. For $i=2$, we obtain that

$$\begin{aligned} \|\phi_2\|^2 &= \sum_{j=1}^m\int_\Sigma 2\alpha_2^2\chi_j e^{-\varphi_j}\frac{\delta_{2,j}^{\alpha_2}|y_{\xi_j}|^{\alpha_2-2}}{\left(\delta_{2,j}^{\alpha_2}+|y_{\xi_j}|^{\alpha_2}\right)^2}\phi_2^2 dv_g \\ &-\frac{1}{2}\sum_{j=1}^m\int_\Sigma 2\alpha_1^2\chi_j e^{-\varphi_j}\frac{\delta_1^{\alpha_1}|y_{\xi_j}|^{\alpha_1-2}}{\left(\delta_1^{\alpha_1}+|y_{\xi_j}|^{\alpha_1}\right)^2}\phi_2\phi_1 dv_g+\int_\Sigma h_2\phi_i dv_g \\ &= \sum_{j=1}^m\|\tilde\phi_{2j}\|^2_{\mathrm{L}_{\xi_j}^{\alpha_2}}-\frac{1}{2}\sum_{j=1}^m\int_\Sigma 2\alpha_1^2\chi_j e^{-\varphi_j}\frac{\delta_1^{\alpha_1}|y_{\xi_j}|^{\alpha_1-2}}{\left(\delta_1^{\alpha_1}+|y_{\xi_j}|^{\alpha_1}\right)^2}\phi_1\phi_2 dv_g+o(|\log\varepsilon|), \end{aligned}$$

as $\varepsilon\to 0$. Hence,

$$\lim_{\varepsilon\to 0}\sum_{j=1}^m\int_\Sigma 2\alpha_1^2\chi_j e^{-\varphi_j}\frac{\delta_1^{\alpha_1}|y_{\xi_j}|^{\alpha_1-2}}{\left(\delta_1^{\alpha_1}+|y_{\xi_j}|^{\alpha_1}\right)^2}\phi_1\phi_2 dv_g=-2. \tag{5.1.19}$$

For (5.1.11), we take $i=1$ and it follows

$$\begin{aligned} o(|\varepsilon|) &= \|\phi_1\|^2-\int_\Sigma\langle\nabla\phi_1,\nabla\phi_2\rangle_g dv_g-\frac{3}{2}\sum_{j=1}^m\int_\Sigma 2\alpha_1^2\chi_j e^{-\varphi_j}\frac{\delta_{1,j}^{\alpha_1}|y_{\xi_j}|^{\alpha_1-2}}{\left(\delta_{1,j}^{\alpha_1}+|y_{\xi_j}|^{\alpha_1}\right)^2}\phi_1^2 dv_g \\ &+\frac{3}{2}\sum_{j=1}^m\int_\Sigma 2\alpha_2^2\chi_j e^{-\varphi_j}\frac{\delta_2^{\alpha_2}|y_{\xi_j}|^{\alpha_2-2}}{\left(\delta_2^{\alpha_2}+|y_{\xi_j}|^{\alpha_2}\right)^2}\phi_2\phi_1 dv_g \\ &= -\int_\Sigma\langle\nabla\phi_1,\nabla\phi_2\rangle_g dv_g, \end{aligned}$$

where we applied $\|\phi_1\| \to 0$ and $\tilde{\phi}_{ij} \to 0$ strongly in $\mathrm{L}^{\alpha_i}_{\xi_j}$. If we take $i=2$ for (5.1.11), it follows

$$\begin{aligned} o(|\varepsilon|) &= -\|\phi_2\|^2 + \int_\Sigma \langle \nabla\phi_1, \nabla\phi_2\rangle_g dv_g + \frac{3}{2}\sum_{j=1}^m \int_\Sigma 2\alpha_2^2 \chi_j e^{-\varphi_j} \frac{\delta_{2,j}^{\alpha_2}|y_{\xi_j}|^{\alpha_2-2}}{\left(\delta_{2,j}^{\alpha_1} + |y_{\xi_j}|^{\alpha_2}\right)^2}\phi_2^2 dv_g \\ &- \frac{3}{2}\sum_{j=1}^m \int_\Sigma 2\alpha_1^2 \chi_j e^{-\varphi_j} \frac{\delta_1^{\alpha_1}|y_{\xi_j}|^{\alpha_1-2}}{\left(\delta_1^{\alpha_1} + |y_{\xi_j}|^{\alpha_1}\right)^2}\phi_1\phi_2 dv_g \\ &= -1 - \frac{3}{2}\sum_{j=1}^m \int_\Sigma 2\alpha_1^2 \chi_j e^{-\varphi_j} \frac{\delta_1^{\alpha_1}|y_{\xi_j}|^{\alpha_1-2}}{\left(\delta_1^{\alpha_1} + |y_{\xi_j}|^{\alpha_1}\right)^2}\phi_1\phi_2 dv_g, \end{aligned}$$

where we applied $\|\phi_2\| \to 1$, $\int_\Sigma \langle \nabla\phi_1, \nabla\phi_2\rangle_g dv_g \to 0$ and $\tilde{\phi}_{ij} \to 0$ strongly in $\mathrm{L}^{\alpha_i}_{\xi_j}$. So far, we have derived that

$$\sum_{j=1}^m \int_\Sigma 2\alpha_1^2 \chi_j e^{-\varphi_j} \frac{\delta_1^{\alpha_1}|y_{\xi_j}|^{\alpha_1-2}}{\left(\delta_1^{\alpha_1} + |y_{\xi_j}|^{\alpha_1}\right)^2}\phi_1\phi_2 dv_g = -\frac{2}{3},$$

which contradicts with (5.1.19). Lemma 5.1.1 is concluded. □

We recall the definition of i^* and the detail refers to Section 3.1.2. For any $p > 1$, let $i_p^* : L^p(\Sigma) \to \overline{\mathrm{H}}^1$ be the adjoint operator corresponding to the immersion $i : \overline{\mathrm{H}}^1 \to L^{\frac{p}{p-1}}$ and $\tilde{i}^* : \cup_{p>1} L^p(\Sigma) \to \overline{\mathrm{H}}^1$. For any $f \in L^p(\Sigma)$, we define that $i^*(f) := \tilde{i}^*(f - \bar{f})$, i.e. for any $h \in \overline{\mathrm{H}}^1$, $\langle i^*(f), h\rangle = \int_\Sigma (f - \bar{f}) h dv_g$. Let

$$f_1(u_1) = 2\varepsilon V_1 e^{u_1} - \overline{2\varepsilon V_1 e^{u_1}} \text{ and } f_2(u_2) = 2\varepsilon V_2 e^{u_2} - \overline{2\varepsilon V_2 e^{u_2}} \tag{5.1.20}$$

and

$$F(\mathbf{u}) = \begin{pmatrix} f_1(u_1) - \frac{1}{2} f_2(u_2) \\ \\ f_2(u_2) - \frac{1}{2} f_1(u_1) \end{pmatrix}. \tag{5.1.21}$$

The problem (5.1.1) has the following equivalent form,

$$\begin{cases} \mathbf{u} = (u_1, u_2) = i^*(F(\mathbf{u})), \\ \\ u \in \mathcal{H} \end{cases}. \tag{5.1.22}$$

We define that $\mathcal{L}(\boldsymbol{\phi}) = i^*\left(\mathcal{L}_{\xi,\varepsilon}(\boldsymbol{\phi})\right)$, for $\boldsymbol{\phi} = (\phi_1, \phi_2) \in \mathcal{H}_{\mathrm{k}}$.

Proposition 5.1.2. *There exist $\varepsilon_0 > 0$ and $C > 0$ such that for any $\mathbf{h} \in \mathcal{H}_{\mathrm{k}}$, there exist a constant $C > 0$ and a unique solution $\boldsymbol{\phi} \in \mathcal{H}_{\mathrm{k}}$ solving*

$$\mathcal{L}(\boldsymbol{\phi}) = \mathbf{h} \tag{5.1.23}$$

with $\|\mathcal{L}^{-1}\| \leq C|\log\varepsilon|$, *for any* $\varepsilon \in (0, \varepsilon_0)$.

Proof. We observe that $\boldsymbol{\phi} \mapsto i^*\left(\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2}e^{U_j^i}\phi_i - \overline{\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2}e^{U_j^i}\phi_i}\right)$ is a compact operator in $\mathcal{H}_{\mathrm{k}}$ for any $j = 1, \cdots, m, i = 1, 2$. Consequently, we infer that $\mathcal{L}$ is a Fredholm operator in $\mathcal{H}_{\mathrm{k}}$. Suppose that $\boldsymbol{\phi}$ solves $\mathcal{L}(\boldsymbol{\phi}) = 0$. By Lemma 5.1.1, we deduce that $\boldsymbol{\phi} = 0$. From this deduction and the application of Fredholm's alternative, we can establish the uniqueness and existence of solutions for (5.1.23) in $\mathcal{H}_{\mathrm{k}}$.

It remains to get the estimate of the inverse operator of $\mathcal{L}$. Given any $\mathbf{h} \in \mathcal{H}_{\mathrm{k}}$, we already proved that there exists a unique $\boldsymbol{\phi} = (\phi_1, \phi_2) \in \mathcal{H}_{\mathrm{k}}$ such that $i^* \circ \mathcal{L}_{\xi,\varepsilon}(\boldsymbol{\phi}) = \mathbf{h}$. Lemma 5.1.1 implies that

$$\|\mathcal{L}^{-1}(\boldsymbol{\phi})\| = \|\boldsymbol{\phi}\| \leq C|\log\varepsilon|\|\mathcal{L}_{\xi,\varepsilon}(\boldsymbol{\phi})\|_p \leq C|\log\varepsilon|\|\mathcal{L}(\boldsymbol{\phi})\| \leq C|\log\varepsilon|\|\mathbf{h}\|,$$

where $p > 1$. □

5.1.3 Nonlinear problem

The expected solution $\mathbf{W}_\varepsilon + \boldsymbol{\phi}_\varepsilon$ solves (5.1.1) if and only if $\boldsymbol{\phi}_\varepsilon$ solves the following problem in $\mathcal{H}_{\mathrm{k}}$:

$$\mathcal{L}_{\xi,\varepsilon}(\boldsymbol{\phi}) = \mathcal{S}_{\xi,\varepsilon}(\boldsymbol{\phi}) + \mathcal{N}_{\xi,\varepsilon}(\phi) + \mathcal{R}_{\xi,\varepsilon}. \tag{5.1.24}$$

Here, the linear operator $\mathcal{L}_{\xi,\varepsilon}$ is defined by (5.1.7), the higher order linear operator $\mathcal{S}_{\xi,\varepsilon}(\boldsymbol{\phi}) := \left(S^1_{\xi,\varepsilon}(\boldsymbol{\phi}), S^2_{\xi,\varepsilon}(\boldsymbol{\phi})\right)$, where for $i = 1, 2$,

$$\begin{aligned} S^i_{\xi,\varepsilon}(\boldsymbol{\phi}) := &\left(-\sum_{j=1}^m \chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2}e^{U_j^i} + 2\varepsilon V_i e^{W_{i,\varepsilon}}\right)\phi_i - \overline{\left(-\sum_{j=1}^m \chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2}e^{U_j^i} + 2\varepsilon V_i e^{W_{i,\varepsilon}}\right)\phi_i} \\ &+ \sum_{\substack{i'=1 \\ i'\neq i}}^2 \left(\left(\sum_{j=1}^m \frac{1}{2}\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_{i'}-2}e^{U_j^{i'}} - \varepsilon V_{i'} e^{W_{i',\varepsilon}}\right)\phi_{i'} - \overline{\left(\sum_{j=1}^m \frac{1}{2}\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_{i'}-2}e^{U_j^{i'}} - \varepsilon V_{i'} e^{W_{i',\varepsilon}}\right)\phi_{i'}}\right), \end{aligned}$$

the nonlinear term $\mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi}) := \left(N^1_{\xi,\varepsilon}(\boldsymbol{\phi}), N^2_{\xi,\varepsilon}(\boldsymbol{\phi})\right)$, where for $i = 1, 2$,

$$\begin{aligned} N^i_{\xi,\varepsilon}(\boldsymbol{\phi}) : &= 2\varepsilon V_i e^{W_{i,\varepsilon}}\left(e^{\phi_i} - 1 - \phi_i\right) - \overline{2\varepsilon V_i e^{W_{i,\varepsilon}}\left(e^{\phi_i} - 1 - \phi_i\right)} \\ &\quad - \sum_{\substack{i'=1 \\ i'\neq i}}^2 \left(\varepsilon V_{i'} e^{W_{i',\varepsilon}}\left(e^{\phi_{i'}} - 1 - \phi_{i'}\right) - \overline{\varepsilon V_{i'} e^{W_{i',\varepsilon}}\left(e^{\phi_{i'}} - 1 - \phi_{i'}\right)}\right), \end{aligned}$$

and the error term $\mathcal{R}_{\xi,\varepsilon} := (R^1_{\xi,\varepsilon}, R^2_{\xi,\varepsilon})$, where for $i = 1, 2$,

$$R^i_{\xi,\varepsilon} = \Delta_g W_{i,\varepsilon} + 2\varepsilon V_i e^{W_{i,\varepsilon}} - \sum_{\substack{i'=1\\ i'\neq i}}^{2} \varepsilon V_{i'} e^{W_{i',\varepsilon}} - \overline{2\varepsilon V_i e^{W_{i,\varepsilon}} - \sum_{\substack{i'=1\\ i'\neq i}}^{2} \varepsilon V_{i'} e^{W_{i',\varepsilon}}}.$$

Firstly, we will show that the error term goes to zero along $\varepsilon \to 0$. To address this, we introduce a family of functions to deal with the interactions among projected bubbles. For $i = 1, 2, j = 1, \cdots, m$, and $y \in \Omega_{ij}$

$$\begin{aligned}\Theta_{ij}(y) := \exp\Big\{&\varphi_j \circ y_{\xi_j}^{-1}(\delta_{i,j} y) + \Big(PU^i_j - U^i_j - \tfrac{1}{2}\textstyle\sum_{i'\neq i} PU^{i'}_j\Big) \circ y_{\xi_j}^{-1}(\delta_{i,j} y)\\ &+ \textstyle\sum_{j'\neq j}\Big(PU^i_{j'} - \tfrac{1}{2}\sum_{i'\neq i} PU^{i'}_{j'}\Big) \circ y_{\xi_j}^{-1}(\delta_{i,j} y) + \log V_i \circ y_{\xi_j}^{-1}(\delta_{i,j} y) + \log(2\varepsilon) - (\alpha_i - 2)\log|y|\Big\}.\end{aligned} \tag{5.1.25}$$

We take $d_{i,j}$ with the value in (5.1.5) to ensure that Θ_{ij} is sufficiently small for $i = 1, 2, j = 1, \cdots, m$ as detailed in Lemma C.0.3.

Lemma 5.1.3. *There exist $p_0 > 1$ and $\varepsilon_0 > 0$ such that for any $p \in (1, p_0)$ and $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\|\mathcal{R}_{\xi,\varepsilon}\|_p := \sum_{i=1}^{2} \|R^i_{\xi,\varepsilon}\|_p = O\left(\varepsilon^{\frac{2-p}{4p}}\right). \tag{5.1.26}$$

Proof.

$$\begin{aligned} R^i_{\xi,\varepsilon} &= \Delta_g W_{i,\varepsilon} + 2\varepsilon V_i e^{W_{i,\varepsilon}} - \sum_{\substack{i'=1\\ i'\neq i}}^{2} \varepsilon V_{i'} e^{W_{i',\varepsilon}} - \overline{2\varepsilon V_i e^{W_{i,\varepsilon}} - \sum_{\substack{i'=1\\ i'\neq i}}^{2} \varepsilon V_{i'} e^{W_{i',\varepsilon}}}\\ &= -\sum_{j=1}^{m}\left(\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2} e^{U^i_j} - \frac{1}{2}\sum_{\substack{i'=1\\ i'\neq i}}^{2} \chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_{i'}-2} e^{U^{i'}_j}\right) + 2\varepsilon V_i e^{W_{i,\varepsilon}} - \sum_{\substack{i'=1\\ i'\neq i}}^{2} \varepsilon V_{i'} e^{W_{i',\varepsilon}}\\ &\quad \overline{+\sum_{j=1}^{m}\left(\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2} e^{U^i_j} - \frac{1}{2}\sum_{\substack{i'=1\\ i'\neq i}}^{2} \chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_{i'}-2} e^{U^{i'}_j}\right) - 2\varepsilon V_i e^{W_{i,\varepsilon}} + \sum_{\substack{i'=1\\ i'\neq i}}^{2} \varepsilon V_{i'} e^{W_{i',\varepsilon}}}. \end{aligned}$$

For any $p > 1$, the Hölder inequality yields that

$$\left|\int_\Sigma \sum_{j=1}^{m} \chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2} e^{U^i_j} - 2\varepsilon V_i e^{W_{i,\varepsilon}} dv_g\right| \leq \left\|\sum_{j=1}^{m} \chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2} e^{U^i_j} - \varepsilon V_i e^{W_{i,\varepsilon}}\right\|_p,$$

in view of $|\Sigma|_g := \int_\Sigma dv_g = 1$. It is sufficient to calculate $\|\sum_{j=1}^{m} \chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2} e^{U^i_j} - \varepsilon V_i e^{W_{i,\varepsilon}}\|_p$ for $i = 1, 2$. Immediately, by Lemma C.0.4, we deduce (5.1.26). □

The following lemma shows that the higher order linear operator $\mathcal{S}_{\xi,\varepsilon}$ is bounded on $\mathcal{H}$ and the operator norm vanishes as $\varepsilon \to 0$.

Lemma 5.1.4. *There exist* $p_0 > 1$ *and* $\varepsilon_0 > 0$ *such that for any* $p \in (1, p_0)$ *and* $\varepsilon \in (0, \varepsilon_0)$*,*

$$\|\mathcal{S}_{\xi,\varepsilon}(\boldsymbol{\phi})\|_p = O\left(\varepsilon^{\frac{2-p}{4p}}\|\boldsymbol{\phi}\|\right), \text{ for } \boldsymbol{\phi} \in \mathcal{H}.$$

Proof. Using Lemma C.0.4 with the Hölder inequality and Sobolev's inequality, we derive that

$$\begin{aligned}
\|\mathcal{S}_{\xi,\varepsilon}(\boldsymbol{\phi})\|_p = \sum_{i=1}^{2}\|S^i_{\xi,\varepsilon}(\boldsymbol{\phi})\|_p &= O\left(\sum_{i=1}^{2}\left\|\left(-\sum_{j=1}^{m}\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2}e^{U^i_j} + 2\varepsilon V_i e^{W_{i,\varepsilon}}\right)\phi_i\right\|_{L^p(\Sigma)}\right)\\
&+ O\left(\sum_{i=1}^{2}\left|\int_\Sigma\left(-\sum_{j=1}^{m}\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2}e^{U^i_j} + 2\varepsilon V_i e^{W_{i,\varepsilon}}\right)\phi_i dv_g\right|\right)\\
&= O\left(\sum_{i=1}^{2}\left(\left\|2\varepsilon V_i e^{W_{i,\varepsilon}} - \sum_{j=1}^{m}\chi_j e^{-\varphi_j}|y_{\xi_j}|^{\alpha_i-2}e^{U^i_j}\right\|_p\right)\|\phi_i\|_{\frac{p}{p-1}}\right)\\
&= O\left(\varepsilon^{\frac{2-p}{2p}}\sum_{i=1}^{2}\|\phi\|\right) = O\left(\varepsilon^{\frac{2-p}{2p}}\|\boldsymbol{\phi}\|\right) \quad (\varepsilon \to 0),
\end{aligned}$$

for any $p > 1$ sufficiently close to 1. □

To study the asymptotic behavior of the non-linear part $\mathcal{N}_{\xi,\varepsilon}$, we have the following lemma:

Lemma 5.1.5. *There exist* $s_0 > 1$ *and* $\varepsilon_0 > 0$ *such that for any* $p > 1$*,* $r > 1$ *with* $pr \in (1, s_0)$ *and* $\varepsilon \in (0, \varepsilon_0)$*,*

$$\|\mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi})\|_p = O\left(\varepsilon^{\frac{2(1-pr)}{pr}}\|\boldsymbol{\phi}\|\right)$$

and

$$\|\mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi}^0) - \mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi}^1)\|_p = O\left(\varepsilon^{\frac{2(1-pr)}{pr}}\|\boldsymbol{\phi}^0 - \boldsymbol{\phi}^1\|(\|\boldsymbol{\phi}^1\| + \|\boldsymbol{\phi}^0\|)\right)$$

hold true for any $\boldsymbol{\phi}, \boldsymbol{\phi}^0, \boldsymbol{\phi}^1 \in \{\boldsymbol{\phi} = (\phi_1, \phi_2) \in \mathcal{H} : \|\phi_i\| \leq 1, i = 1, 2\}$*.*

Proof. Analogous to the proof of Lemma 4.2.4, we can obtain the result. The detail is omitted here. □

Next, we will use the fixed point theorem to construct solutions for the non-linear problem (5.1.24).

Proposition 5.1.6. *Given* $p_0 > 1$*,* $\varepsilon_0 > 0$ *and* $R_0 > 0$ *such that for any* $p \in (1, p_0)$*,* $\varepsilon \in (0, \varepsilon_0)$ *and* $R \geq R_0$*, there exists a unique* $\boldsymbol{\phi}_\varepsilon = (\phi^1_\varepsilon, \phi^2_\varepsilon) \in \mathcal{H}_{\mathrm{k}}$ *for* $\varepsilon \in (0, \varepsilon_0)$*, such that* $\mathbf{W}_\varepsilon + \boldsymbol{\phi}_\varepsilon$ *solves* *the Toda system* (5.1.1) *with the parameter* ε*, satisfying*

$$\|\boldsymbol{\phi}_\varepsilon\| \le R\varepsilon^{\frac{2-p}{4p}}|\log\varepsilon|.$$

Proof. We define the operator

$$\mathcal{T}_{\xi,\varepsilon}(\boldsymbol{\phi}) := \mathcal{L}_{\xi,\varepsilon}^{-1}(\mathcal{S}_{\xi,\varepsilon}(\boldsymbol{\phi}) + \mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi}) + \mathcal{R}_{\xi,\varepsilon})$$

on $\mathcal{H}_{\mathrm{k}}$. To find $\boldsymbol{\phi}_\varepsilon \in \mathcal{H}_{\mathrm{k}}$ such that $\mathbf{W}_\varepsilon + \boldsymbol{\phi}_\varepsilon$ solves (5.1.1), it is sufficient to find a fixed point of $\mathcal{T}_{\xi,\varepsilon} : \mathcal{H}_{\mathrm{k}} \to \mathcal{H}_{\mathrm{k}}$. We choose $p_0, r > 1$ sufficiently close to 1 such that

$$\frac{2-p_0}{4p_0} + \frac{2(1-p_0r)}{p_0r} > 0.$$

By Lemma 5.1.1 and Lemma 5.1.3-5.1.5, we deduce that there exist $R_0 > 1$ sufficiently large and $\varepsilon_1 > 0$ sufficiently small such that for any $\varepsilon \in (0,\varepsilon_1), R \ge R_0, p \in (1,p_0)$

$$\begin{aligned}
\|\mathcal{T}_{\xi,\varepsilon}(\boldsymbol{\phi})\| &\le C|\log\varepsilon|\left(\|\mathcal{S}_{\xi,\varepsilon}(\boldsymbol{\phi})\|_p + \|\mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi})\|_p + \|\mathcal{R}_{\xi,\varepsilon}\|_p\right)\\
&\le C|\log\varepsilon|\left(\varepsilon^{\frac{2-p}{4p}}\|\boldsymbol{\phi}\| + \varepsilon^{\frac{2-p}{4p}} + \varepsilon^{\frac{2(1-pr)}{pr}}\|\boldsymbol{\phi}\|\right)\\
&\le R\varepsilon^{\frac{2-p}{4p}}|\log\varepsilon|,
\end{aligned}$$

for any $\boldsymbol{\phi} \in \{\boldsymbol{\phi} \in \mathcal{H} : \|\boldsymbol{\phi}\| \le R\varepsilon^{\frac{2-p}{4p}}|\log\varepsilon|\}$. Next, we will prove that $\mathcal{T}_{\xi,\varepsilon}$ is a contract mapping. Given $\boldsymbol{\phi}^0, \boldsymbol{\phi}^1 \in \{\boldsymbol{\phi} \in \mathcal{H} : \|\boldsymbol{\phi}\| \le R\varepsilon^{\frac{2-p}{4p}}|\log\varepsilon|\}$, Lemma 5.1.1 and Lemma 5.1.3-5.1.5 imply that there exists $\varepsilon_0 \in (0,\varepsilon_1)$ such that for any $\varepsilon \in (0,\varepsilon_0)$

$$\begin{aligned}
&\|\mathcal{T}_{\xi,\varepsilon}(\boldsymbol{\phi}^1) - \mathcal{T}_{\xi,\varepsilon}(\boldsymbol{\phi}^0)\| \le C|\log\varepsilon|\left(\|S_{\xi,\varepsilon}(\boldsymbol{\phi}^0 - \boldsymbol{\phi}^1)\|_p + \|\mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi}^0) - \mathcal{N}_{\xi,\varepsilon}(\boldsymbol{\phi}^1)\|_p\right)\\
\le\ & C|\log\varepsilon|\left(\varepsilon^{\frac{2-p}{4p}}\|\boldsymbol{\phi}^0 - \boldsymbol{\phi}^1\| + \varepsilon^{\frac{2(1-pr)}{pr}}(\|\boldsymbol{\phi}^0\| + \|\boldsymbol{\phi}^1\|)\|\boldsymbol{\phi}^0 - \boldsymbol{\phi}^1\|\right)\\
\le\ & C\left(\varepsilon^{\frac{2-p}{4p}}|\log\varepsilon| + \varepsilon^{\frac{2-p}{4p}+\frac{2-p}{4p}}|\log\varepsilon|^2\right)\|\boldsymbol{\phi}^0 - \boldsymbol{\phi}^1\| \le \frac{1}{2}\|\boldsymbol{\phi}^0 - \boldsymbol{\phi}^1\|.
\end{aligned}$$

Utilizing the contraction mapping principle, we establish that for any $\varepsilon \in (0,\varepsilon_0)$ there exists a fixed point $\boldsymbol{\phi}_\varepsilon \in \left\{\boldsymbol{\phi} \in \mathcal{H}_{\mathrm{k}} : \|\boldsymbol{\phi}\| \le R\varepsilon^{\frac{2-p}{4p}}|\log\varepsilon|\right\}$ for the operator $\mathcal{T}_{\xi,\varepsilon}$. □

5.2 Proof of main results

Proof of Theorem 1.2.8. Based on the result in Proposition 5.1.6, it is sufficient to prove that $\rho^\varepsilon = (\rho_1^\varepsilon, \rho_2^\varepsilon) \to (4\pi m, 8\pi m)$ as $\varepsilon \to 0$. Applying Lemma C.0.3, we can deduce that

$$\begin{aligned}
\rho_i^\varepsilon &:= \int_\Sigma \varepsilon V_i e^{u_{i,\varepsilon}} dv_g \\
&= \sum_{j=1}^m \int_{U(\xi_j)} \frac{\alpha_i^2 |y_{\xi_j}|^{\alpha_i - 2}}{(\delta_{ij}^{\alpha_i} + |y_{\xi_j}(x)|^{\alpha_i})^2} \exp\left\{ \left(PU_j^i - U_j^i - \frac{1}{2} \sum_{i' \neq i} PU_j^{i'} \right) \right. \\
&\quad \left. + \sum_{j' \neq j} \left(PU_{j'}^i - \frac{1}{2} \sum_{i' \neq i} PU_{j'}^{i'} \right) + \log V_i + \log(2\varepsilon) - (\alpha_i - 2) \log |y_{\xi_j}| + \phi_{1,\varepsilon} \right\} + \mathcal{O}(\varepsilon) \\
&= \sum_{j=1}^m \int_{\Omega_{ij}} \frac{\alpha_i^2 |y|^{\alpha_i - 2}}{(1 + |y|^{\alpha_i})^2} e^{\Theta_{ij}} dy + o(1) = \sum_{j=1}^m \int_{\Omega_{ij}} \frac{\alpha_i^2 |y|^{\alpha_i - 2}}{(1 + |y|^{\alpha_i})^2} (1 + \delta_{i,j} |y| + \varepsilon^{\frac{3}{4}}) dy + o(1) \\
&= \sum_{j=1}^m \frac{\alpha_i \varrho(\xi_j)}{4} + o(1) = \sum_{j=1}^m 2\pi \alpha_i = \begin{cases} 4\pi m & \text{for } i = 1 \\ \\ 8\pi m & \text{for } i = 2 \end{cases},
\end{aligned}$$

where we applied the integrals $\int_{\mathbb{R}^2} \frac{1}{(1+|y|^2)^2} dy = \pi$ and $\int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^4)^2} dy = \frac{\pi}{2}$. Theorem 1.2.8 is concluded. □

A Estimates for Chapter 3

This section provides detailed proofs of crucial estimates for $PU_{\tau,\xi}$ for $\tau \in (0,\infty)$ and $\xi \in \Sigma$. For any ξ in a compact subset of $\mathring{\Sigma}$ or on $\partial\Sigma$, we assume that r_ξ constructed in Section 2.2 to be $2r_0$ and $\bar{\mathrm{r}}_\xi = r_0$, where r_0 is a positive constant.

The following lemma is the asymptotic expansion of $PU_{\tau,\xi}$ as $\varepsilon \to 0$.

Lemma A.0.1. *The function $PU_{\delta,\xi}$ satisfies*

$$PU_{\tau,\xi} = \chi_\xi \left(U_{\tau,\xi} - \log\left(8\tau^2\right)\right) + \varrho(\xi)H^g(x,\xi) + \mathcal{O}(\varepsilon^{1+\alpha_0}) \text{ as } \varepsilon \to 0,$$

for any $\alpha_0 \in (0,1)$ and the convergent is locally uniform for ξ in $\mathring{\Sigma}$ and $\partial\Sigma$ and also locally uniform for τ in $(0,+\infty)$. In particular,

$$PU_{\tau,\xi} = \varrho(\xi)G^g(x,\xi) + \mathcal{O}(\varepsilon^{1+\alpha_0}) \text{ as } \varepsilon \to 0,$$

locally uniformly in $\Sigma\backslash\{\xi\}$.

Proof. Let $\eta_{\tau,\xi}(x) = PU_{\tau,\xi} - \chi_\xi(U_{\tau,\xi} - \log 8\tau^2) - \varrho(\xi)H^g(x,\xi)$. If $\xi \in \mathring{\Sigma}$,

$$\partial_{\nu_g}\eta_{\tau,\xi} = 2\partial_{\nu_g}\chi_\xi \log\left(1 + \frac{\tau^2\varepsilon^2}{|y_\xi(x)|^2}\right) - 2\chi_\xi\partial_{\nu_g}\log\left(1 + \frac{\tau^2\varepsilon^2}{|y_\xi(x)|^2}\right) \equiv 0$$

on $\partial\Sigma$. We observe that for any $x \in \partial\Sigma \cap U(\xi)$

$$\partial_{\nu_g}|y_\xi(x)|^2 = -e^{-\frac{1}{2}\hat{\varphi}_\xi(y)}\frac{\partial}{\partial y_2}|y|^2\Big|_{y=y_\xi(x)} = 0.$$

If $\xi \in \partial\Sigma$, for any $x \in \partial\Sigma$, as $\varepsilon \to 0$.

$$\partial_{\nu_g}\eta_{\tau,\xi}(x) \quad = \quad 2(\partial_{\nu_g}\chi_\xi)\frac{\tau^2\varepsilon^2}{|y_\xi(x)|^2} - 2\chi_\xi\partial_{\nu_g}\log\left(1 + \frac{\tau^2\varepsilon^2}{|y_\xi(x)|^2}\right) + \mathcal{O}(\varepsilon^4) = \mathcal{O}(\varepsilon^2).$$

Then for any $\xi \in \Sigma$ we have $\partial_{\nu_g} \eta_{\tau,\xi} = \mathcal{O}(\varepsilon^2)$. For any $A \subset \mathbb{R}^2$, denote $aA := \{ay : y \in A\}$.

$$\begin{aligned}
&\int_\Sigma \eta_{\tau,\xi} dv_g = -\int_\Sigma \chi_\xi (U_{\tau,\xi} - \log(8\tau^2)) + \varrho(x)\Gamma_\xi(x) dv_g(x) \\
= \;& -\int_\Sigma \chi_\xi \log \frac{|y_\xi(x)|^4}{(\tau^2\varepsilon^2 + |y_\xi(x)|^2)^2} dv_g(x) \\
= \;& 2\int_{B^\xi_{r_0}} \log \frac{\tau^2\varepsilon^2 + |y|^2}{|y|^2} e^{-\hat{\varphi}_\xi(y)} dy + 2\int_{B^\xi_{2r_0} \setminus B_{r_0}(0)} \chi(|y|/r_0) \left(\frac{\tau^2\varepsilon^2}{|y|^2} + \mathcal{O}(\varepsilon^4) \right) e^{\hat{\varphi}_\xi(y)} dy \\
= \;& 2\tau^2\varepsilon^2 \int_{\frac{1}{\tau\varepsilon}(B^\xi_{r_0} \cap B_{r_0}(0))} \log\left(1 + \frac{1}{|y|^2}\right) e^{-\hat{\varphi}(\tau\varepsilon y)} dy + \mathcal{O}(\varepsilon^2) \\
= \;& 2\tau^2\varepsilon^2 (1 + \mathcal{O}(\varepsilon)) \int_{B_{r_0/(\tau\varepsilon)}(0)} \log\left(1 + \frac{1}{|y|^2}\right) dy + \mathcal{O}(\varepsilon^2) \\
= \;& \mathcal{O}(\varepsilon^2 |\log\varepsilon|),
\end{aligned}$$

where we applied

$$\begin{aligned}
&\int_{|y| < \frac{r_0}{\tau\varepsilon}} \log\left(1 + \frac{1}{|y|^2}\right) dy = 2\pi \int_0^{r_0/(\tau\varepsilon)} \log\left(1 + \frac{1}{r^2}\right) r dr = \pi \int_0^{r_0^2/(\tau\varepsilon)^2} \log\left(1 + \frac{1}{t}\right) dt \\
= \;& \pi \frac{r_0^2}{\tau^2\varepsilon^2} \log\left(1 + \frac{\tau^2\varepsilon^2}{r_0^2}\right) - \pi \int_0^{r_0^2/(\tau\varepsilon)^2} \left(1 - \frac{1}{1+t}\right) dt \\
= \;& \pi \frac{r_0^2}{\tau^2\varepsilon^2} \left(1 + \frac{\tau^2\varepsilon^2}{r_0^2} + \mathcal{O}(\varepsilon^4)\right) - \pi \frac{r_0^2}{\tau^2\varepsilon^2} + \pi \log\left(1 + \frac{r_0^2}{\tau^2\varepsilon^2}\right) = \mathcal{O}(|\log\varepsilon|).
\end{aligned}$$

For any $x \in U_{2r_0}(\xi)$, $-\Delta_g U_{\tau,\xi} = e^{-\hat{\varphi}_\xi(y)} \Delta u_{\tau,0}|_{y = y_\xi(x)} = e^{-\varphi_\xi} e^{U_{\tau,\xi}}$. It follows that

$$\begin{aligned}
(-\Delta_g + \beta)\eta_{\tau,\xi} &= (-\Delta_g + \beta)\left(PU_{\tau,\xi} - \chi_\xi(U_{\tau,\xi} - \log 8\tau^2) - \varrho(\xi) H^g_\xi\right) \\
&= (\Delta_g \chi_\xi) \log \frac{|y_\xi|^4}{(\tau^2\varepsilon^2 + |y_\xi|^2)^2} + 2\left\langle \nabla\chi_\xi, \nabla \log \frac{|y_\xi|^4}{(\tau^2\varepsilon^2 + |y_\xi|^2)^2} \right\rangle_g \\
&\quad + \frac{1}{|\Sigma|_g}\left(\varrho(\xi) - \int_\Sigma \varepsilon^2 \chi_\xi e^{-\varphi_\xi} e^{U_{\tau,\xi}} dv_g\right) + 2\beta \log\left(1 + \frac{\tau^2\varepsilon^2}{|y_\xi|^2}\right).
\end{aligned}$$

We observe that $\Delta_g \chi_\xi \equiv 0$ and $\nabla\chi_\xi \equiv 0$ in $U_{2r_0}(\xi) \setminus U_{r_0}(\xi)$. For any $x \in U_{2r_0}(\xi) \setminus U_{r_0}(\xi)$, we have

$$U_{\tau,\xi} - \log(8\tau^2) + 4\log|y_\xi(x)| = -2\log\left(1 + \frac{\tau^2\varepsilon^2}{|y_\xi(x)|^2}\right) = -2\tau^2\varepsilon^2 |y_\xi(x)|^{-2} + \mathcal{O}(\varepsilon^4)$$

and

$$\nabla\left(U_{\tau,\xi} - \log(8\tau^2) + 4\log|y_\xi(x)|\right) = -2\tau^2\varepsilon^2 \nabla |y_\xi(x)|^{-2} + \mathcal{O}(\varepsilon^4).$$

$$\begin{aligned}\int_\Sigma \varepsilon^2\chi_\xi e^{-\varphi_\xi}e^{U_{\tau,\xi}}dv_g &= \int_{B^\xi_{2r_0}}\varepsilon^2\chi(|y|/r_0)\frac{8\tau^2}{(\tau^2\varepsilon^2+|y|^2)^2}dy\\ &= \int_{B^\xi_{r_0}}\varepsilon^2\chi(|y|/r_0)\frac{8\tau^2}{(\tau^2\varepsilon^2+|y|^2)^2}dy+\mathcal{O}(\varepsilon^2)\\ &= \varrho(\xi)+\mathcal{O}(\varepsilon^2),\end{aligned}$$

where we applied the fact that $\int_{|y|<r}\frac{\tau^2\varepsilon^2}{(\tau^2\varepsilon^2+|y|^2)^2}dy=\pi-\frac{\pi\tau^2\varepsilon^2}{r^2}+\frac{\pi\tau^4\varepsilon^4}{(r^2+\tau^2\varepsilon^2)r^2}$ for any $r\geq 0$. Hence, for any $p>1$, $\|(-\Delta_g+\beta)\eta_{\tau,\xi}\|_p=\mathcal{O}(\varepsilon^2+\beta\varepsilon^{\frac{2}{p}})$. By the L^p-theory in Lemma 2.3.2, we have

$$\|\eta_{\tau,\xi}-\overline{\eta_{\tau,\xi}}\|_{W^{2,p}(\Sigma)}\leq C\left(\|\partial_{\nu_g}\eta_{\tau,\xi}\|_{W^{1,p}_\partial(\Sigma)}+\|(-\Delta_g+\beta)\eta_{\tau,\xi}\|_{L^p(\Sigma)}\right)\leq C(\varepsilon^2+\beta\varepsilon^{\frac{2}{p}}),$$

for $p>1$. Using the Sobolev's inequality, $\|\eta_{\tau,\xi}-\overline{\eta_{\tau,\xi}}\|_\infty\leq C(\varepsilon^2+\beta\varepsilon^{\frac{2}{p}})$. We take $p\in(1,2)$ such that $\alpha_0=\frac{2}{p}-1>0$, then as $\varepsilon\to 0$

$$\eta_{\tau,\xi}=\mathcal{O}(\varepsilon^{1+\alpha_0}),$$

uniformly in $C(\Sigma)$. □

Lemma A.0.2. *If* $p\geq 1$ *then* $\|\varepsilon^2\chi_\xi e^{U_{\tau,\xi}}\|_p=\mathcal{O}(\varepsilon^{\frac{2(1-p)}{p}})$*, which is uniform for* ξ *in* Σ *and locally uniform for* τ *in* $(0,+\infty)$*.*

Proof.

$$\begin{aligned}\int_\Sigma(\varepsilon^2\chi_\xi e^{U_{\tau,\xi}})^p dv_g &= \int_{B^\xi_{2r_0}}e^{\hat{\varphi}_\xi(y)}\frac{(8\tau^2\varepsilon^2)^p}{(\tau^2\varepsilon^2+|y|^2)^p}dy\\ &= \int_{B^\xi_{2r_0}}\frac{(8\tau^2\varepsilon^2)^p}{(\tau^2\varepsilon^2+|y|^2)^p}dy+\int_{B^\xi_{2r_0}}(e^{\hat{\varphi}_\xi(y)}-1)\frac{(8\tau^2\varepsilon^2)^p}{(\tau^2\varepsilon^2+|y|^2)^p}dy\\ &= (\tau^2\varepsilon^2)^{1-p}\int_{\frac{1}{\tau\varepsilon}B^\xi_{2r_0}}(1+\mathcal{O}(\tau\varepsilon|y|))\frac{8}{(1+|y|^2)^2}dy\\ &= \mathcal{O}(\varepsilon^{2(1-p)}).\end{aligned}$$

Thus $\|\varepsilon^2\chi_\xi e^{-\varphi_\xi}e^{U_{\tau,\xi}}\|_p=\mathcal{O}(\varepsilon^{\frac{2(1-p)}{p}})$ uniformly in $\xi\in\Sigma$ and τ is bounded away from zero. □

Next, we give the asymptotic expansion of $P\Psi^j_{\tau,\xi}$ as $\varepsilon\to 0$, analogue to $PU_{\tau,\xi}$.

Lemma A.0.3. *For any* $\alpha_0\in(0,1)$*,*

$$P\Psi^0_{\tau,\xi}(x)=\chi_\xi\left(\Psi^0_{\tau,\xi}(x)-\frac{2}{\tau}\right)+\mathcal{O}(\varepsilon^{1+\alpha_0})=-4\chi_\xi(x)\frac{\tau\varepsilon^2}{\tau^2\varepsilon^2+|y_\xi(x)|^2}+\mathcal{O}(\varepsilon^{1+\alpha_0}),$$

in $C(\Sigma)$ *as* $\varepsilon\to 0$*. And*

$$P\Psi^0_{\tau,\xi}(x)=\mathcal{O}(\varepsilon^{1+\alpha_0})\ \textit{as}\ \varepsilon\to 0,$$

in $C_{loc}(\Sigma\setminus\{\xi\})$ uniformly for ξ in any compact subset of $\mathring{\Sigma}$ or on $\partial\Sigma$ and τ is bounded away from zero.

For $j=1,\cdots,\mathrm{i}(\xi)$, $P\Psi^j_{\tau,\xi}(x)=\chi_\xi(x)\Psi^j_{\tau,\xi}(x)+\varrho(\xi)H^j(x,\xi)+\mathcal{O}(\varepsilon^{\alpha_0})$, in $C(\Sigma)$ as $\varepsilon\to 0$, where $H^j(x,\xi)$ is the unique solution of the following problem

$$\begin{cases}(-\Delta_g+\beta)H^j(x,\xi) & =-\beta\frac{4}{\varrho(\xi)}\chi_\xi\frac{y_\xi(x)_j}{|y_\xi(x)|^2}+\frac{4}{\varrho(\xi)}(\Delta_g\chi_\xi)\frac{y_\xi(x)_j}{|y_\xi(x)|^2}\\ & \quad+\frac{8}{\varrho(\xi)}\left\langle\nabla\chi_\xi,\nabla\left(\frac{y_\xi(x)_j}{|y_\xi(x)|^2}\right)\right\rangle_g, \qquad x\in\mathring{\Sigma}\\ \partial_{\nu_g}H^j(x,\xi) & =-\frac{4}{\varrho(\xi)}\partial_{\nu_g}\left(\frac{y_\xi(x)_j}{|y_\xi(x)|^2}\right)\chi_\xi-\frac{4}{\varrho(\xi)}\frac{y_\xi(x)_j}{|y_\xi|^2}\partial_{\nu_g}\chi_\xi, \quad x\in\partial\Sigma\\ \int_\Sigma H^j(x,\xi)dv_g & =-\frac{4}{\varrho(\xi)}\int_\Sigma\frac{y_\xi(x)_j}{|y_\xi(x)|^2}\chi_\xi(x)dv_g\end{cases}. \tag{A.0.1}$$

In addition, the convergences above are uniform for ξ in any compact subset of $\mathring{\Sigma}$ or on $\partial\Sigma$ and τ bounded away from zero.

Proof. We recall that $\Psi^0_{\tau,\xi}(x)=\frac{2}{\tau}\frac{|y_\xi(x)|^2-\tau^2\varepsilon^2}{|y_\xi(x)|^2+\tau^2\varepsilon^2}=\frac{2}{\tau}\left(1-\frac{2\tau^2\varepsilon^2}{|y_\xi(x)|^2+\tau^2\varepsilon^2}\right), x\in U_{2r_0}(\xi)$. Let $\eta_{\tau,\xi}=P\Psi^0_{\tau,\xi}-\chi_\xi\left(\Psi^0_{\tau,\xi}-\frac{2}{\tau}\right)$. For $x\in\partial\Sigma$,

$$\partial_{\nu_g}\left(\chi_\xi\left(\Psi^0_{\tau,\xi}(x)-\frac{2}{\tau}\right)\right) \quad = \quad -\partial_{\nu_g}\chi_\xi\frac{4\tau\varepsilon^2}{|y_\xi(x)|^2+\tau^2\varepsilon^2}+\chi_\xi\frac{8\tau\varepsilon^2|y_\xi(x)|^2}{(|y_\xi(x)|^2+\tau^2\varepsilon^2)^2}\partial_{\nu_g}\log|y_\xi(x)|.$$

If $\xi\in\mathring{\Sigma}$, $\partial_{\nu_g}\eta_{\tau,\xi}\equiv 0$ in $\partial\Sigma$; if $\xi\in\partial\Sigma$, $\partial_{\nu_g}\eta_{\tau,\xi}=\mathcal{O}(\varepsilon^2)$ on $\partial\Sigma$. By direct calculation, we have

$$\begin{aligned}\int_\Sigma\chi_\xi\left(\Psi^0_{\tau,\xi}-\frac{2}{\tau}\right)dv_g &= 2\int_\Sigma\chi_\xi\frac{\tau\varepsilon^2}{|y_\xi(x)|^2+\tau^2\varepsilon^2}dv_g(x)=2\tau\varepsilon^2\int_{B^\xi_{2r_0}}\frac{1}{|y|^2+\tau^2\varepsilon^2}e^{\hat{\varphi}_\xi(y)}dy\\ &= 2\tau\varepsilon^2\int_{B^\xi_{2r_0}}\frac{1}{|y|^2+\tau^2\varepsilon^2}dy+2\tau\varepsilon^2\int_{B^\xi_{2r_0}}\frac{1}{|y|^2+\tau^2\varepsilon^2}(e^{\hat{\varphi}_\xi(y)}-1)dy\\ &= \mathcal{O}(\varepsilon^2\log\varepsilon),\end{aligned}$$

and

$$\begin{aligned}&(-\Delta_g+\beta)\eta_{\tau,\xi}(x)=(-\Delta_g+\beta)\left(P\Psi^0_{\tau,\xi}-\chi_\xi\left(\Psi^0_{\tau,\xi}-\frac{2}{\tau}\right)\right)\\ =\quad&(\Delta_g\chi_\xi)\left(\Psi^0_{\tau,\xi}-\frac{2}{\tau}\right)+2\langle\nabla\chi_\xi,\nabla\Psi^0_{\tau,\xi}\rangle_g-\overline{\varepsilon^2\chi_\xi e^{-\varphi_\xi}e^{U_{\tau,\xi}}\Psi^0_{\tau,\xi}}+\beta\chi_\xi\frac{4\tau\varepsilon^2}{|y_\xi|^2+\tau^2\varepsilon^2}\\ =\quad&\beta\chi_\xi\frac{4\tau\varepsilon^2}{|y_\xi|^2+\tau^2\varepsilon^2}+\mathcal{O}(\varepsilon^2),\end{aligned}$$

where we applied the fact for any fixed $r>0$, $\int_{|y|<r}\frac{\tau^2\varepsilon^2-|y|^2}{(\tau^2\varepsilon^2+|y|^2)^3}=\mathcal{O}(\varepsilon^2)$ as $\varepsilon\to 0$. Via the

L^p-theory in Lemma 2.3.2, for any $p > 1$, there exists a constant $C > 0$ such that

$$\begin{aligned} \|\eta_{\tau,\xi} - \overline{\eta_{\tau,\xi}}\|_{W^{2,p}(\Sigma)} &\leq C\left(\|\partial_{\nu_g}\eta_{\tau,\xi}\|_{W^{1,p}_{\partial}(\Sigma)} + \|(-\Delta_g + \beta)\eta_{\tau,\xi}\|_{L^p(\Sigma)}\right) \\ &\leq C(\varepsilon^2 + \beta\varepsilon^{\frac{2}{p}}|\log\varepsilon|^{\frac{1}{p}}). \end{aligned} \tag{A.0.2}$$

Using the Sobolev's inequality, $\|\eta_{\tau,\xi} - \overline{\eta_{\tau,\xi}}\|_{L^\infty(\Sigma)} \leq C(\varepsilon^2 + \beta\varepsilon^{\frac{2}{p}}|\log\varepsilon|^{\frac{1}{p}})$. We choose $p \in (1,2)$ such that $\alpha_0 < \frac{2}{p} - 1$, then $\eta_{\tau,\xi} = \mathcal{O}(\varepsilon^{1+\alpha_0})$, uniformly in $C(\Sigma)$. We recall that $\Psi^j_{\tau,\xi}(x) = \frac{4y_\xi(x)_j}{\tau^2\varepsilon^2 + |y_\xi(x)|^2}$, for any $x \in U_{2r_0}(\xi)$. If $\xi \in \mathring{\Sigma}$, $\partial_{\nu_g}H^j(x,\xi) = 0$ for any $x \in \partial\Sigma$. If $\xi \in \partial\Sigma$, for any $x \in \partial\Sigma$ by direct calculation,

$$\chi_\xi(x)\partial_{\nu_g}\left(\frac{y_\xi(x)_1}{|y_\xi(x)|^2}\right) = 0.$$

Denote $\partial_{\nu_g}H^j(\xi,\xi) := 0$, then $\partial_{\nu_g}H^j(\cdot,\xi) \in C^\infty(\partial\Sigma)$. By Lemma 2.3.3, there is a unique solution to (A.0.1) in $C^{1,\alpha}(\partial\Sigma)$ for any $\alpha \in (0,1)$. Let $\zeta_{\tau,\xi}(x) = P\Psi^j_{\tau,\xi}(x) - \chi_\xi(x)\Psi^j_{\tau,\xi}(x) - \varrho(\xi)H^j(x,\xi)$. Since $\int_B \frac{\varepsilon^3 y_j}{(\varepsilon^2+|y|^2)^3}dy = 0$ for $j = 1,2$ and $B = \mathbb{B}_r$ or $j = 1$ and $B = \mathbb{B}_r \cap \{y_2 \geq 0\}$, we have the following estimates:

$$\overline{\varepsilon^2\chi_\xi e^{-\varphi_\xi}e^{U_{\tau,\xi}}\Psi^j_{\tau,\xi}} = \int_{B^\xi_{2r_0}} \frac{8\tau^2\varepsilon^2\chi(|y|/r_0)y_j}{(\tau^2\varepsilon^2 + |y|^2)^3}dy = \int_B \frac{8\tau^2\varepsilon^2 y_j}{(\tau^2\varepsilon^2 + |y|^2)^3}dy + \mathcal{O}(\varepsilon^2) = \mathcal{O}(\varepsilon^2),$$

$$\begin{aligned} (-\Delta_g + \beta)\zeta_{\tau,\xi} &= -\frac{4\tau^2\varepsilon^2(y_\xi)_j}{(\tau^2\varepsilon^2 + |y_\xi|^2)|y_\xi|^2}\Delta_g\chi_\xi - 8\tau^2\varepsilon^2\left\langle \nabla\chi_\xi, \nabla\left(\frac{(y_\xi)_j}{(\tau^2\varepsilon^2 + |y_\xi|^2)|y_\xi|^2}\right)\right\rangle_g \\ &\quad -\overline{\varepsilon^2\chi_\xi e^{-\varphi_\xi}e^{U_{\tau,\xi}}\Psi^j_{\tau,\xi}} + 4\beta\chi_\xi\frac{\tau^2\varepsilon^2(y_\xi)_j}{(\tau^2\varepsilon^2 + |y_\xi|^2)|y_\xi|^2} \\ &= 4\beta\chi_\xi\frac{\tau^2\varepsilon^2(y_\xi)_j}{(\tau^2\varepsilon^2 + |y_\xi|^2)|y_\xi|^2} + \mathcal{O}(\varepsilon^2), \end{aligned}$$

and

$$\int_\Sigma \zeta_{\tau,\xi}dv_g = 4\int_\Sigma \chi_\xi(x)\frac{\tau^2\varepsilon^2 y_\xi(x)_j}{|y_\xi(x)|^2(\tau^2\varepsilon^2 + |y_\xi(x)|^2)}dv_g(x) = 4\int_{B^\xi_{2r_0}} \chi\left(\frac{|y|}{r_0}\right)e^{\hat{\varphi}_\xi(y)}\frac{\tau^2\varepsilon^2 y_j}{|y|^2(\tau^2\varepsilon^2 + |y|^2)}.$$

If $\xi \in \mathring{\Sigma}$ and $j = 1,2$

$$\begin{aligned} &\left|\int_{\mathbb{B}_{2r_0}} \chi(|y|/r_0)\frac{\tau^2\varepsilon^2(e^{\hat{\varphi}_\xi(y)} - 1)y_j}{|y|^2(\tau^2\varepsilon^2 + |y|^2)}dy\right| \leq C\int_{|y|<2r_0} \frac{\tau^2\varepsilon^2|D^2\hat{\varphi}_\xi(0)y^2 + o(|y|^3)|}{|y|(\tau^2\varepsilon^2 + |y|^2)}dy \\ = \ & C\int_{|y|<2r_0/(\tau\varepsilon)} \frac{\tau^3\varepsilon^3|y|(1 + \tau\varepsilon o(|y|))}{(1 + |y|^2)}dy \leq C2\pi\int_0^{\frac{2r_0}{\tau\varepsilon}} \frac{\tau^3\varepsilon^3 r^2}{(1 + r^2)}dr = \mathcal{O}(\varepsilon^2). \end{aligned}$$

In this case, we have

$$\begin{aligned}
&\int_\Sigma \zeta_{\tau,\xi} dv_g = 4\int_{B_{2r_0}(0)} \chi\left(\frac{|y|}{r_0}\right) e^{\hat{\varphi}_\xi(y)} \frac{\tau^2\varepsilon^2 y_j}{|y|^2(\tau^2\varepsilon^2+|y|^2)}dy \\
= \quad & -4\int_{\mathbb{B}_{2r_0}} \chi\left(\frac{|y|}{r_0}\right) \frac{\tau^2\varepsilon^2 y_j}{|y|^2(\tau^2\varepsilon^2+|y|^2)}dy + \int_{\mathbb{B}_{2r_0}} \chi\left(\frac{|y|}{r_0}\right)(e^{\hat{\varphi}_\xi(y)}-1)\frac{\tau^2\varepsilon^2 y_j}{|y|^2(\tau^2\varepsilon^2+|y|^2)}dy = \mathcal{O}(\varepsilon^2).
\end{aligned}$$

For $\xi \in \partial\Sigma$ and $j=1$,

$$|\textstyle\int_\Sigma \zeta_{\tau,\xi} dv_g| \quad = \quad 4\left|\int_{B^\xi_{2r_0}} \chi\left(\frac{|y|}{r_0}\right) e^{\hat{\varphi}_\xi(y)} \frac{\tau^2\varepsilon^2 y_j}{|y|^2(\tau^2\varepsilon^2+|y|^2)}dy\right| = \mathcal{O}(\varepsilon^2),$$

then, it follows for $\xi \in \partial\Sigma$, $\int_\Sigma \zeta_{\tau,\xi} dv_g = \mathcal{O}(\varepsilon^2)$. If $\xi \in \mathring{\Sigma}$, $\partial_{\nu_g}\zeta_{\tau,\xi}(x) \equiv 0$ for any $x \in \partial\Sigma$. If $x \in \partial\Sigma$, by calculation, we deduce that

$$\begin{aligned}
\partial_{\nu_g}\zeta_{\tau,\xi}(x) &= -\partial_{\nu_g}\chi_\xi(\Psi^j_{\tau,\xi} + \varrho(\xi)H^j(x,\xi)) - \chi_\xi\partial_{\nu_g}\left(\Psi^j_{\tau,\xi} + \varrho(\xi)H^j(x,\xi)\right) \\
&= (\partial_{\nu_g}\chi_\xi)\frac{4\tau^2\varepsilon^2 y_\xi(x)_j}{(\tau^2\varepsilon^2+|y_\xi(x)|^2)|y_\xi(x)|^2} + \chi_\xi\partial_{\nu_g}\frac{4\tau^2\varepsilon^2 y_\xi(x)_j}{(\tau^2\varepsilon^2+|y_\xi(x)|^2)|y_\xi(x)|^2} \\
&= \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Applying the regularity theory in Lemma 2.3.1 and 2.3.3, for any $p \in (1,2)$, we deduce that

$$\|\zeta_{\tau,\xi} - \overline{\zeta_{\tau,\xi}}\|_\infty \le C\left(\|\partial_{\nu_g}\zeta_{\tau,\xi}\|_{L^\infty(\partial\Sigma)} + \|(-\Delta_g+\beta)\zeta_{\tau,\xi}\|_{L^p(\Sigma)}\right) \le C(\varepsilon^2 + \beta\varepsilon^{\frac{1}{p}}).$$

We take $p \in (0,1)$ such that $\alpha_0 = \frac{1}{p}$. Then as $\varepsilon \to 0$, we have

$$\eta_{\tau,\xi} = \mathcal{O}(\varepsilon^{\alpha_0})$$

uniformly in $C(\Sigma)$. □

Remark A.0.4. *$\partial_\tau PU_{\tau,\xi} = P\Psi^0_{\tau,\xi}$ by the uniqueness of the solution to the problem (3.1.3). However, $\partial_{\xi_j}PU_{\tau,\xi} \neq P\Psi^j_{\tau,\xi}$. Analogous to the proof of Lemma A.0.3, we can obtain the following expansion, for any $\alpha_0 \in (0,1)$,*

$$\partial_{\xi_j}PU_{\tau,\xi} = \partial_{\xi_j}(\chi_\xi U_{\tau,\xi}) + \varrho(\xi)\partial_{\xi_j}H^g_\xi + \mathcal{O}(\varepsilon^{\alpha_0}) \text{ as } \varepsilon \to 0, \tag{A.0.3}$$

in $C(\Sigma)$, which is uniformly convergent for ξ in any compact subset of $\mathring{\Sigma}$ or on $\partial\Sigma$ and τ in any compact subset of $(0,\infty)$.

Indeed, we notice that for any $y \in U_{2r_0}(\xi)$ *as* $y \to 0$

$$\partial_{\xi_j} |y_\xi(x)|^2\Big|_{x=y_\xi^{-1}(y)} = -2y_j + \mathcal{O}(|y|^3).$$

Let $\zeta^*_{\tau,\xi} = \partial_{\xi_j} PU_{\tau,\xi} - \partial_{\xi_j}(\chi_\xi U_{\tau,\xi}) - \varrho(\xi)\partial_{\xi_j} H^g(x,\xi)$. *It is easy to obtain*

$$(-\Delta_g + \beta)\zeta^*_{\tau,\xi} = -\beta\partial_{\xi_j}\left(\chi_\xi U_{\tau,\xi} + 4\chi_\xi \log|y_\xi|\right) + \mathcal{O}(\varepsilon^2|\log\varepsilon|), \qquad in\ \mathring{\Sigma}$$

$$\int_\Sigma \zeta^*_{\tau,\xi} dv_g = \mathcal{O}(\varepsilon^2|\log\varepsilon|),$$

and $\partial_{\nu_g}\zeta^*_{\tau,\xi} = \mathcal{O}(\varepsilon^2)$ *on* $\partial\Sigma$. *Applying the regularity theory in Lemma 2.3.1 and Lemma 2.3.3, we have*

$$\zeta^*_{\tau,\xi} = \mathcal{O}(\varepsilon^2|\log\varepsilon| + \beta\varepsilon^{\frac{1}{p}}),$$

in $C(\Sigma)$, *for any* $p \in (1,2)$. *We take* $p \in (1,2)$ *such that* $\alpha_0 = \frac{1}{p}$, *then we deduce* (A.0.3).

The asymptotic "orthogonality" properties of $P\Psi^j_i$.

Lemma A.0.5. *For any* $\alpha_0 \in (0,1)$, *we have as* $\varepsilon \to 0$ *for* $j, i = 0, \cdots, \mathrm{i}(\xi)$,

$$\left\langle P\Psi^i_{\tau,\xi}, P\Psi^j_{\tau,\xi}\right\rangle = \begin{cases} \frac{8\varrho(\xi)D_i}{\pi\tau^2}\delta_{ij} + \mathcal{O}(\varepsilon^{\alpha_0}) & when\ i\ or\ j = 0 \\ \frac{8\varrho(\xi)D_i}{\pi\tau^2\varepsilon^2}\delta_{ij} + \mathcal{O}(\varepsilon^{\alpha_0-1}) & otherwise \end{cases},$$

and

$$\left\langle P\Psi^i_{\tau^0,\xi_0}, P\Psi^j_{\tau^1,\xi_1}\right\rangle = \begin{cases} \mathcal{O}(\varepsilon^{\alpha_0}) & when\ i\ or\ j = 0 \\ \mathcal{O}(\varepsilon^{\alpha_0-1}) & otherwise \end{cases},$$

where three different points $\xi, \xi_0, \xi_1 \in \Sigma$ *and* $\tau, \tau^0, \tau^1 > 0$ *and the* δ_{ij} *is the Kronecker symbol, and* $D_0 = \int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^4}dy$, $D_1 = D_2 = \int_{\mathbb{R}^2}\frac{|y|^2}{(1+|y|^2)^4}dy$.

Proof.

$$\begin{aligned}\langle P\Psi^i_{\tau,\xi}, P\Psi^j_{\tau,\xi}\rangle &= \int_\Sigma \varepsilon^2\chi_\xi e^{-\varphi_\xi} e^{U_{\tau,\xi}}\Psi^i_{\tau,\xi} P\Psi^j_{\tau,\xi} dv_g \\ &= \int_{\Sigma\cap U_{2r_0}(\xi)} + \int_{\Sigma\setminus U_{2r_0}(\xi)} \varepsilon^2\chi_\xi e^{-\varphi_\xi} e^{U_{\tau,\xi}}\Psi^i_{\tau,\xi} P\Psi^j_{\tau,\xi} dv_g.\end{aligned}$$

For $i=j=0$, by Lemma A.0.3,

$$\begin{aligned}
&\int_{\Sigma\cap U_{2r_0}(\xi)}\varepsilon^2\chi_\xi e^{-\varphi_\xi}e^{U_{\tau,\xi}}\Psi^0_{\tau,\xi}P\Psi^0_{\tau,\xi}dv_g\\
=\;&16\tau\varepsilon^2\int_{B^\xi_{2r_0}}\chi\left(\frac{|y|}{r_0}\right)\frac{|y|^2-\tau^2\varepsilon^2}{(\tau^2\varepsilon^2+|y|^2)^3}\left(-\frac{4\tau\varepsilon^2\chi\left(\frac{|y|}{r_0}\right)}{\tau^2\varepsilon^2+|y|^2}+\mathcal{O}(\varepsilon^{1+\alpha_0})\right)dy\\
=\;&\frac{64}{\tau^2}\int_{\frac{1}{\tau\varepsilon}B^\xi_{r_0}}\frac{1-|y|^2}{(1+|y|^2)^4}+\mathcal{O}(\varepsilon^{1+\alpha_0}).
\end{aligned}$$

Considering that $\frac{64}{\tau^2}\int_{\frac{1}{\tau\varepsilon}B^\xi_{r_0}}\frac{1-|y|^2}{(1+|y|^2)^4}=\frac{8\varrho(\xi)}{\tau^2\pi}\int_{\mathbb{R}^2}\frac{1-|y|^2}{(1+|y|^2)^4}dy+\mathcal{O}(\varepsilon^2)$, as $\varepsilon\to 0$, $\langle P\Psi^0_{\tau,\xi},P\Psi^0_{\tau,\xi}\rangle=\int_\Sigma\varepsilon^2\chi_\xi e^{-\varphi_\xi}e^{U_{\tau,\xi}}\Psi^0_{\tau,\xi}P\Psi^0_{\tau,\xi}dv_g=\frac{8\varrho(\xi)D_0}{\pi\tau^2}+\mathcal{O}(\varepsilon^{1+\alpha_0})$, where $D_0=\int_{\mathbb{R}^2}\frac{1-|y|^2}{(1+|y|^2)^4}dy$.

Similarly, for $j=0$ and $i=1,\cdots,\mathrm{i}(\xi)$, we have

$$\begin{aligned}
\langle P\Psi^i_{\tau,\xi},P\Psi^0_{\tau,\xi}\rangle\;&=\;\varepsilon^2\int_{\Sigma\cap U_{2r_0}(\xi)}\chi_\xi e^{-\varphi_\xi}e^{U_{\tau,\xi}}\Psi^i_{\tau,\xi}P\Psi^0_{\tau,\xi}dv_g\\
&=\;32\tau^2\varepsilon^2\int_{B^\xi_{2r_0}}\chi\left(\frac{|y|}{r_0}\right)\frac{y_i}{(\tau^2\varepsilon^2+|y|^2)^3}\left(-\frac{4\tau\varepsilon^2\chi\left(\frac{|y|}{r_0}\right)}{\tau^2\varepsilon^2+|y|^2}+\mathcal{O}(\varepsilon^{1+\alpha_0})\right)dy=\mathcal{O}(\varepsilon^{\alpha_0}).
\end{aligned}$$

Applying Lemma A.0.3, for $\xi\in\mathring{\Sigma}$ we have

$$\begin{aligned}
&\varepsilon^2\int_{\Sigma\cap U_{2r_0}(\xi)}\chi_\xi e^{-\varphi_\xi}e^{U_{\tau,\xi}}\Psi^i_{\tau,\xi}P\Psi^j_{\tau,\xi}dv_g\\
=\;&32\tau^2\varepsilon^2\int_{B^\xi_{2r_0}}\chi\left(\frac{|y|}{r_0}\right)\frac{y_i}{(\tau^2\varepsilon^2+|y|^2)^3}\left(\chi\left(\frac{|y|}{r_0}\right)\frac{4y_j}{\tau^2\varepsilon^2+|y|^2}+\varrho(\xi)H^j(y_\xi^{-1}(y),\xi)+\mathcal{O}(\varepsilon^{\alpha_0})\right)dy\\
=\;&\frac{128}{\tau^2\varepsilon^2}\int_{\frac{1}{\tau\varepsilon}B^\xi_{r_0}}\frac{y_iy_j}{(1+|y|^2)^4}dy+\int_{B^\xi_{r_0}}\frac{32\tau^2\varepsilon^2\varrho(\xi)y_i}{(\tau^2\varepsilon^2+|y|^2)^3}(H^j(y_\xi^{-1}(y),\xi)-H^j(\xi,\xi))dy\\
&+32\tau^2\varepsilon^2\varrho(\xi)H^j(\xi,\xi)\int_{B^\xi_{r_0}}\frac{y_i}{(\tau^2\varepsilon^2+|y|^2)^3}dy+\mathcal{O}(\varepsilon^{\alpha_0-1})\\
=\;&\frac{128}{\tau^2\varepsilon^2}\int_{\frac{1}{\tau\varepsilon}B^\xi_{r_0}}\frac{y_iy_j}{(1+|y|^2)^4}dy+\mathcal{O}\left(\int_{B^\xi_{r_0}}\frac{32\tau^2\varepsilon^2|y|^2}{(\tau^2\varepsilon^2+|y|^2)^3}dy\right)+\mathcal{O}(\varepsilon^{\alpha_0-1})\\
=\;&\frac{8\varrho(\xi)D_i}{\pi\tau^2\varepsilon^2}\delta_{ij}+\mathcal{O}(\varepsilon^{\alpha_0-1}),
\end{aligned}$$

as $\varepsilon\to 0$, where $D_i=\int_{\mathbb{R}^2}\frac{|y|^2}{(1+|y|^2)^4}dy$. For $\xi\in\partial\Sigma$, applying Lemma A.0.3 again, we have

$$\begin{aligned}
&\varepsilon^2\int_{\Sigma\cap U_{2r_0}(\xi)}e^{U_{\tau,\xi}}\Psi^1_{\tau,\xi}P\Psi^1_{\tau,\xi}dv_g\\
=\;&\int_{B^\xi_{2r_0}}\chi\left(\frac{|y|}{r_0}\right)\frac{32\tau^2\varepsilon^2y_1}{(\tau^2\varepsilon^2+|y|^2)^3}\left(\frac{4\chi\left(\frac{|y|}{r_0}\right)y_1}{\tau^2\varepsilon^2+|y|^2}+\varrho(\xi)H^1(y_\xi^{-1}(y),\xi)+\mathcal{O}(\varepsilon^{\alpha_0})\right)\\
=\;&\frac{128}{\tau^2\varepsilon^2}\int_{\frac{1}{\tau\varepsilon}B^\xi_{r_0}}\frac{y_1^2}{(1+|y|^2)^4}+\mathcal{O}(\varepsilon^{\alpha_0-1}).
\end{aligned}$$

We observe that as $\varepsilon \to 0$

$$\begin{aligned} \left| \tfrac{128}{\tau^2\varepsilon^2} \textstyle\int_{\frac{1}{\tau\varepsilon} B^{\xi}_{r_0}} \tfrac{y_1^2}{(1+|y|^2)^4} - \tfrac{128}{\tau^2\varepsilon^2} \textstyle\int_{\mathbb{R}^2_+} \tfrac{y_1^2}{(1+|y|^2)^4} \right| &\leq \tfrac{128}{\tau^2\varepsilon^2} \textstyle\int_{\mathbb{R}^2_+ \setminus \frac{1}{\tau\varepsilon} B^{\xi}_{r_0}} \tfrac{1}{(1+|y|^2)^3} dy \\ &\leq \mathcal{O}(\varepsilon^2), \end{aligned}$$

and

$$\varepsilon^2 \int_{\Sigma \setminus U_{2r_0}(\xi)} \chi_\xi(x) e^{-\varphi_\xi(x)} e^{U_{\tau,\xi}} \Psi^i_{\tau,\xi} P\Psi^j_{\tau,\xi} dv_g = \mathcal{O}(\varepsilon^2 \|P\Psi^j_{\tau,\xi}\|) = \mathcal{O}(\varepsilon),$$

for $i, j \in \{1,2\}$, if $\xi \in \mathring{\Sigma}$, i,j=1,2, and if $\xi \in \partial\Sigma$, $i, j = 1$. Thus we have $\langle P\Psi^i_{\tau,\xi}, P\Psi^j_{\tau,\xi} \rangle = \frac{8\varrho(\xi)D_i}{\pi\tau^2\varepsilon^2}\delta_{ij} + \mathcal{O}(\varepsilon^{\alpha_0-1})$. By assumption, $r_0 > 0$ small enough such that $U_{2r_0}(\xi_0) \cap U_{2r_0}(\xi_1) = \emptyset$, and for $l = 0, 1$, if $\xi_l \in \mathring{\Sigma}$, $U_{2r_0}(\xi_l) \subset\subset \Sigma$.

$$\langle P\Psi^i_{\tau^0,\xi_0}, P\Psi^j_{\tau^1,\xi_1} \rangle = \int_{\Sigma \setminus U_{2r_0}(\xi_0)} + \int_{\Sigma \cap U_{2r_0}(\xi_0)} \varepsilon^2 \chi_{\xi_0} e^{-\varphi_\xi} e^{U_{\tau^0,\xi_0}} \Psi^i_{\tau^0,\xi_0} P\Psi^j_{\tau^1,\xi_1} dv_g.$$

As $\varepsilon \to 0$, we have

$$\int_{\Sigma \setminus U_{2r_0}(\xi_0)} \varepsilon^2 \chi_{\xi_0}(x) e^{-\varphi_\xi(x)} e^{U_{\tau^0,\xi_0}} \Psi^i_{\tau^0,\xi_0} P\Psi^j_{\tau^1,\xi_1} dv_g = \mathcal{O}(\varepsilon^2 \|P\Psi^j_{\tau^1,\xi_1}\|) = \mathcal{O}(\varepsilon).$$

By Lemma A.0.3, for $j \neq 0$

$$\begin{aligned} & \textstyle\int_{U_{2r_0}(\xi_0) \cap \Sigma} \varepsilon^2 \chi_{\xi_0} e^{-\varphi_{\xi_0}} e^{U_{\tau^0,\xi_0}} \Psi^i_{\tau^0,\xi_0} P\Psi^j_{\tau^1,\xi_1} dv_g \\ = & \textstyle\int_{U_{2r_0}(\xi_0)} \varepsilon^2 \chi_{\xi_0} e^{-\varphi_{\xi_0}} e^{U_{\tau^0,\xi_0}} \Psi^i_{\tau^0,\xi_0} \left(\chi_{\xi_1} \frac{4y_{\xi_1}(x)_j}{\tau^2\varepsilon^2 + |y_{\xi_1}(x)|^2} + \varrho(\xi_1) H^j(x,\xi_1) + \mathcal{O}(\varepsilon^{\alpha_0}) \right) \\ = & \varrho(\xi_i) H^j(\xi_0,\xi_1) \textstyle\int_{U_{2r_0}(\xi_0)} \varepsilon^2 \chi_{\xi_0} e^{-\varphi_{\xi_0}} e^{U_{\tau^0,\xi_0}} \Psi^i_{\tau^0,\xi_0} dv_g \\ & + \mathcal{O}\left(\textstyle\int_{U_{2r_0}(\xi_0)} \varepsilon^2 \chi_{\xi_0} e^{-\varphi_{\xi_0}} e^{U_{\tau^0,\xi_0}} \Psi^i_{\tau^0,\xi_0} (|y_{\xi_0}| + \varepsilon^{\alpha_0}) dv_g \right) \\ = & \mathcal{O}(\varepsilon^{\alpha_0 - 1}); \end{aligned}$$

for $j = 0$,

$$\begin{aligned} & \textstyle\int_{U_{2r_0}(\xi_0) \cap \Sigma} \varepsilon^2 \chi_{\xi_0} e^{-\varphi_{\xi_0}} e^{U_{\tau^0,\xi_0}} \Psi^i_{\tau^0,\xi_0} P\Psi^j_{\tau^1,\xi_1} dv_g \\ = & \textstyle\int_{U_{2r_0}(\xi_0)} \varepsilon^2 \chi_{\xi_0} e^{-\varphi_{\xi_0}} e^{U_{\tau^0,\xi_0}} \Psi^i_{\tau^0,\xi_0} \left(-\chi_{\xi_1} \frac{4\tau\varepsilon^2}{\tau^2\varepsilon^2 + |y_{\xi_1}|^2} + \mathcal{O}(\varepsilon^{\alpha_0+1}) \right) = \mathcal{O}(\varepsilon^{\alpha_0}). \end{aligned}$$

Therefore, for any $\xi_1 \neq \xi_0$,

$$\langle P\Psi^i_{\tau^0,\xi_0}, P\Psi^j_{\tau^1,\xi_1}\rangle = \begin{cases} \mathcal{O}(\varepsilon^{\alpha_0}) & \text{when } i \text{ or } j = 0 \\ \mathcal{O}(\varepsilon^{\alpha_0-1}) & \text{otherwise} \end{cases}.$$

□

Remark A.0.6. *Analogue to the proof in Lemma A.0.7, for any $\alpha_0 \in (0,1)$, we have as $\varepsilon \to 0$ for $i = 0, \cdots, \mathrm{i}(\xi)$ and $j = 1, \cdots, \mathrm{i}(\xi)$,*

$$\langle P\Psi^i_{\tau,\xi}, \partial_{\xi_j} PU_{\tau,\xi}\rangle = \frac{8\varrho(\xi)D_i}{\pi\tau^2\varepsilon^2}\delta_{ij} + \mathcal{O}(\varepsilon^{\alpha_0-1}),$$

and for any $i = 0, \cdots, \mathrm{i}(\xi_0)$ and $j = 1, \cdots, \mathrm{i}(\xi_1)$

$$\langle P\Psi^i_{\tau^0,\xi_0}, \partial_{\xi_j} PU_{\tau^1,\xi_1}\rangle = \begin{cases} \mathcal{O}(\varepsilon^{\alpha_0}) & \textit{when } i \textit{ or } j = 0 \\ \mathcal{O}(\varepsilon^{\alpha_0-1}) & \textit{otherwise} \end{cases},$$

where three different points $\xi, \xi_0, \xi_1 \in \Sigma$ and uniformly in $\tau, \tau^0, \tau^1 > 0$ and the δ_{ij} is the Kronecker symbol, and $D_0 = \int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^4}dy$, $D_1 = D_2 = \int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4}dy$.

In the next part, we consider $\xi = (\xi_1, \cdots, \xi_m) \in \Xi^\delta_{k,m}$ for given $\delta > 0$. It gives some technique lemmas to prove Proposition 3.1.4 which reduces the problem into a finite-dimensional one.

Lemma A.0.7. *Let $\xi = (\xi_1, \cdots, \xi_m) \in \Xi^\delta_{k,m}$ (see (2.6.1)). For any $p \in [1, 2)$, there is a positive constant $c := c(p)$ such that for any $\varepsilon > 0$,*

$$\left\| \varepsilon^2 V e^{\sum_{i=1}^m PU_i} - \varepsilon^2 \sum_{i=1}^m \chi_i e^{U_i} \right\|_p \leq c\varepsilon^{\frac{2-p}{p}}.$$

Proof. Let $\mathcal{D} \subset \Xi_{k,m}$ be a compact subset. Then there exists $\delta > 0$ such that $\mathcal{D} \subset \Xi^\delta_{k,m}$. There is a uniform $r_0 > 0$ for any $\xi \in \Xi^\delta_{k,m}$. By calculation, we deduce that

$$\begin{aligned}\int_\Sigma \left| \varepsilon^2 V e^{\sum_{i=1}^m PU_i} - \varepsilon^2 \sum_{i=1}^m \chi_i e^{U_i} \right|^p dv_g &= \sum_{i=1}^m \int_{\Sigma \cap U_{2r_0}(\xi)} \left| \varepsilon^2 V e^{\sum_{i=1}^m PU_i} - \varepsilon^2 \sum_{h=1}^m \chi_h e^{U_h} \right|^p dv_g \\ &\quad + \int_{\Sigma \setminus \cup_{i=1}^m U_{2r_0}(\xi)} \left| \varepsilon^2 V e^{\sum_{i=1}^m PU_i} - \varepsilon^2 \sum_{h=1}^m \chi_h e^{U_h} \right|^p dv_g,\end{aligned}$$

and as $\varepsilon \to 0$, $\int_{\Sigma \setminus \cup_{i=1}^m U_{2r_0}(\xi)} |\varepsilon^2 V e^{\sum_{i=1}^m PU_i} - \varepsilon^2 \sum_{h=1}^m \chi_h e^{U_h}|^p dv_g = \mathcal{O}(\varepsilon^{2p})$. By Lemma A.0.1, for any $x \in U_{2r_0}(\xi_h)$

$$\begin{aligned}\sum_{i=1}^{m} PU_i - \chi_h U_h &= \left(\sum_{i \neq h} \varrho(\xi_i) G^g(\xi_h, \xi_i) + \varrho(\xi_h) H^g(\xi_h, \xi_h) - \log(8\tau_h^2) \right) + \mathcal{O}(|y_{\xi_h}| + \varepsilon^{1+\alpha_0}) \\ &= -\log V(\xi_h) + \mathcal{O}(\varepsilon^{1+\alpha_0} + |y_{\xi_h}|).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{U_{2r_0}(\xi_h)\cap\Sigma} |\varepsilon^2 V e^{\sum_{i=1}^m PU_i} - \varepsilon^2 \chi_h e^{U_h}|^p dv_g \\ &= \int_{U_{r_0}(\xi_h)\cap\Sigma} \left| \varepsilon^2 e^{U_h} (e^{\sum_{i=1}^m PU_i - \chi_h U_h + \log V(x)} - 1) \right|^p dv_g(x) + \mathcal{O}(\varepsilon^{2p}) \\ &= \mathcal{O}\left(\int_{U_{r_0}(\xi_h)\cap\Sigma} \varepsilon^{2p} e^{pU_h} (|y_{\xi_h}(x)| + \varepsilon^{1+\alpha_0})^p dv_g(x) \right) + \mathcal{O}(\varepsilon^{2p}) \\ &= \mathcal{O}\left(\int_{B_{r_0}^{\xi_h}} \left(\frac{8\tau_h^2 \varepsilon^2 (|y| + \varepsilon^{1+\alpha_0})}{(\tau_h^2 \varepsilon^2 + |y|^2)^2} \right)^p dy + \varepsilon^{2p} \right) = \mathcal{O}(\varepsilon^{2-p}),\end{aligned}$$

where $p \in [1, 2)$. $\square$

Lemma A.0.8. *For any $p \geq 1$ and $r > 1$, there are positive constants c_1, c_2 such that for any $\varepsilon > 0$, the following estimates hold for any $\phi_1, \phi_2 \in \overline{\mathrm{H}}^1$.*

$$\|\varepsilon^2 V e^{\sum_{i=1}^m PU_i} (e^{\phi_1} - 1 - \phi_1)\|_p \leq c_1 e^{c_2 \|\phi_1\|^2} \varepsilon^{\frac{(2-2pr)}{pr}} \|\phi_1\|^2, \tag{A.0.4}$$

and

$$\|\varepsilon^2 V e^{\sum_{i=1}^m PU_i} (e^{\phi_1} - e^{\phi_2} - (\phi_1 - \phi_2))\|_p \leq c_1 e^{c_2 (\|\phi_1\|^2 + \|\phi_2\|^2)} \varepsilon^{\frac{(2-2pr)}{pr}} (\|\phi_1\| + \|\phi_2\|) \|\phi_1 - \phi_2\|.$$

Proof. By the mean value theorem, for some $s \in (0, 1)$

$$|(e^{\phi_1} - e^{\phi_2} - (\phi_1 - \phi_2)| \leq \left| e^{s\phi_1 + (1-s)\phi_2} - 1 \right| |\phi_1 - \phi_2| \leq e^{|\phi_1| + |\phi_2|} |\phi_1 - \phi_2| (|\phi_1| + |\phi_2|).$$

By applying the Hölder Inequality, Sobolev Inequality, and Moser-Trudinger Inequality, we derive the following estimate:

$$
\begin{aligned}
&\left(\int_\Sigma V^p e^{p\sum_{i=1}^m PU_i}|e^{\phi_1}-e^{\phi_2}-(\phi_1-\phi_2)|^p dv_g\right)^{1/p}\\
\leq\ & C\sum_{h=1}^2\left(\int_\Sigma V^p e^{p\sum_{i=1}^m PU_i}(e^{|\phi_1|+|\phi_2|}|\phi_1-\phi_2||\phi_h|)^p dv_g\right)^{1/p}\\
\leq\ & C\sum_{h=1}^2\left(\int_\Sigma V^{pr}e^{pr\sum_{i=1}^m PU_i}dv_g\right)^{\frac{1}{pr}}\left(\int_\Sigma e^{ps(|\phi_1|+|\phi_2|)}dv_g\right)^{\frac{1}{ps}}\left(\int_\Sigma|\phi_1-\phi_2|^{pt}|\phi_h|^{pt}dv_g\right)^{\frac{1}{pt}}\\
\leq\ & C\sum_{h=1}^2\left(\int_\Sigma V^{pr}e^{pr\sum_{i=1}^m PU_i}dv_g(x)\right)^{\frac{1}{pr}}e^{\frac{ps}{8\pi}(\|\phi_1\|^2+\|\phi_2\|^2)}\|\phi_1-\phi_2\|\|\phi_h\|,
\end{aligned}
$$

where $r,s,t\in(1,+\infty)$, $\frac{1}{r}+\frac{1}{s}+\frac{1}{t}=1$. By Lemma A.0.1, it follows that

$$
\begin{aligned}
&\int_{\cup_{i=1}^m U_{2r_0}(\xi_i)}V^{pr}e^{pr\sum_{i=1}^m PU_i}dv_g\\
=\ & \sum_{i=1}^m\int_{U_{2r_0}(\xi_i)}\exp\left\{pr\chi_iU_i+pr\left(\sum_{h\neq i}G^g(\xi_i,\xi_h)+\varrho(\xi_i)H^g(\xi_i,\xi_i)\right.\right.\\
&\left.\left.+\log V(\xi_i)-\log(8\tau_i^2)\right)+\mathcal{O}(\varepsilon^{1+\alpha_0}+|y_{\xi_i}|)\right\}dv_g\\
\leq\ & C\left(\sum_{i=1}^m\int_{U_{2r_0}(\xi_i)}e^{pr\chi_iU_i}(1+\mathcal{O}(\varepsilon^{1+\alpha_0}+|y_{\xi_i}(x)|))dv_g(x)\right)\\
\leq\ & C\left(\sum_{i=1}^m\int_{B_{2r_0}^{\xi_i}}e^{\hat{\varphi}_{\xi_i}(y)}\left(\frac{8\tau_i^2}{(\tau_i^2\varepsilon^2+|y|^2)^2}\right)^{pr}(1+\mathcal{O}(\varepsilon^{1+\alpha_0}+|y|))dy\right)\\
\leq\ & C\varepsilon^{2-4pr}.
\end{aligned}
$$

By the definition of PU_i, $PU_i=\mathcal{O}(1)$ in $\Sigma\setminus U_{2r_0}(\xi_i)$. It follows that

$$
\sum_{\Sigma\setminus\cup_{i=1}^m U_{2r_0}(\xi_i)}e^{pr\sum_{i=1}^m PU_i}=\mathcal{O}(1).
$$

Therefore, the estimate (A.0.5) holds and if we take $\phi_2\equiv0$, we obtain the estimate (A.0.4). □

The last part of Appendix A gives some technique lemmas to obtain the C^1-expansion of the reduced functional $\tilde{E}_\varepsilon$.

Lemma A.0.9. *As $\varepsilon\to0$, the following asymptotic expansions hold*

$$
\begin{aligned}
\langle PU_i,PU_i\rangle\ =\ & \varrho(\xi_i)(6\log2-4\log\varepsilon-2\log(8\tau_i^2)+\varrho(\xi_i)H^g(\xi_i,\xi_i)-2)\\
&+\mathcal{O}(\varepsilon|\log\varepsilon|),
\end{aligned}
$$

and for any $i\neq j$, $\langle PU_i,\nabla PU_j\rangle=\varrho(\xi_i)\varrho(\xi_j)G^g(\xi_i,\xi_j)+\mathcal{O}(\varepsilon)$.

Proof. Applying Lemma A.0.1 with (3.1.4), we drive that as $\varepsilon \to 0$

$$
\begin{aligned}
&\langle PU_i, PU_i\rangle = \int_\Sigma |\nabla PU_i|_g^2 + \beta |PU_i|^2 dv_g = \varepsilon^2 \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} PU_i dv_g \\
= \quad & \int_{U_{r_0}(\xi_i)} \frac{8\tau_i^2\varepsilon^2}{(\tau_i^2\varepsilon^2 + |y_{\xi_i}|^2)^2} e^{-\varphi_i} \left(\log \frac{1}{(\tau_i^2\varepsilon^2 + |y_{\xi_i}|^2)^2} + \varrho(\xi_i) H^g(\xi_i, \xi_i) \right. \\
& \left. + \mathcal{O}(|y_{\xi_i}| + \varepsilon^{1+\alpha_0}) \right) dv_g + \mathcal{O}(\varepsilon^2) \\
= \quad & \int_{B_{r_0}^{\xi_i}} \frac{8\tau_i^2\varepsilon^2}{(\tau_i^2\varepsilon^2 + |y|^2)^2} \left(\log \frac{\tau_i^4\varepsilon^4}{(\tau_i^2\varepsilon^2 + |y|^2)^2} - 2\log(\tau_i^2\varepsilon^2) + \varrho(\xi_i) H^g(\xi_i, \xi_i) \right. \\
& \left. + \mathcal{O}(|y| + \varepsilon^{1+\alpha_0}) \right) dy + \mathcal{O}(\varepsilon^2) \\
= \quad & \varrho(\xi_i)(6\log 2 - 4\log\varepsilon - 2\log(8\tau_i^2) + \varrho(\xi_i) H^g(\xi_i, \xi_i) - 2) + \mathcal{O}(\varepsilon|\log\varepsilon|),
\end{aligned}
$$

where we applied the fact that for any $r > 0$, as $\varepsilon \to 0$, $\int_{|y|<r} \frac{\varepsilon^2}{(\varepsilon^2+|y|^2)^2} dy = \pi - \frac{\pi\varepsilon^2}{r^2} + \frac{\pi\varepsilon^4}{(r^2+\varepsilon^2)r^2}$ and $\int_{|y|<r} \frac{\varepsilon^2 \log(\frac{\varepsilon^2+|y|^2}{\varepsilon^2})}{(\varepsilon^2+|y|^2)^2} dy = \pi + \frac{\pi\varepsilon^2\log(\varepsilon^2)}{r^2} + \mathcal{O}(\varepsilon^2)$. For any $i \neq j$, by Lemma A.0.1, as $\varepsilon \to 0$

$$
\begin{aligned}
&\langle PU_i, PU_j\rangle = \varepsilon^2 \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} PU_j dv_g \\
= \quad & \int_{U_{2r_0}(\xi_i)} \frac{8\tau_i^2\varepsilon^2}{(\tau_i^2\varepsilon^2 + |y_{\xi_i}(x)|^2)^2} e^{-\varphi_i(x)} (\varrho(\xi_j) G^g(\xi_i, \xi_j) + \mathcal{O}(|y_{\xi_i}(x)| + \varepsilon^{1+\alpha_0})) + \mathcal{O}(\varepsilon^2 \|PU_j\|) \\
= \quad & 8\varrho(\xi_j) G^g(\xi_i, \xi_j) \int_{B_{2r_0}^{\xi_i}} \frac{\tau_i^2\varepsilon^2}{(\tau_i^2\varepsilon^2 + |y|^2)^2} dy + \mathcal{O}(\varepsilon) = \varrho(\xi_i)\varrho(\xi_j) G^g(\xi_i, \xi_j) + \mathcal{O}(\varepsilon).
\end{aligned}
$$

□

Lemma A.0.10. $\varepsilon^2 \int_\Sigma V e^{\sum_{i=1}^m PU_i} = \sum_{i=1}^m \varrho(\xi_i) + o(1) = 4\pi(m+k) + o(1)$ *as* $\varepsilon \to 0$.

Proof. Applying Lemma A.0.1 and (3.1.4), as $\varepsilon \to 0$

$$
\begin{aligned}
&\varepsilon^2 \int_\Sigma V e^{\sum_{i=1}^m PU_i} dv_g \\
= \quad & \sum_{i=1}^m \varepsilon^2 \int_{U_{2r_0}(\xi_i)} e^{\chi_i U_i + \varrho(\xi_i) H^g(\cdot, \xi_i) - \log 8\tau_i^2 + \sum_{j\neq i} \varrho(\xi_j) G^g(\cdot, \xi_j) + \mathcal{O}(\varepsilon^{1+\alpha_0})} dv_g + \mathcal{O}(\varepsilon^2) \\
= \quad & \sum_{i=1}^m \int_{U_{r_0}(\xi_i)} \frac{8\tau_i^2\varepsilon^2 e^{\varrho(\xi_i) H^g(\xi_i, \xi_i) - \log(8\tau_i^2) + \log V(\xi_i) + \sum_{j\neq i} \varrho(\xi_j) G^g(\xi_i, \xi_j)}}{(\tau_i^2\varepsilon^2 + |y_\xi(x)|^2)^2} \\
& (1 + \mathcal{O}(|y_\xi| + \varepsilon^{1+\alpha_0})) dv_g + \mathcal{O}(\varepsilon^2) \\
= \quad & \sum_{i=1}^m \int_{B_{r_0}^{\xi_i}} \frac{8\tau_i^2\varepsilon^2 e^{\hat{\varphi}_i(y)}}{(\tau_i^2\varepsilon^2 + |y|^2)^2} (1 + \mathcal{O}(|y| + \varepsilon^{1+\alpha_0})) dy + \mathcal{O}(\varepsilon^2) \\
= \quad & \sum_{i=1}^m \int_{\frac{1}{\tau_i\varepsilon} B_{r_0}^{\xi_i}} (1 + \mathcal{O}(\varepsilon|y|))(1 + \mathcal{O}(\varepsilon|y| + \varepsilon^{1+\alpha_0})) \frac{8}{(1+|y|^2)^2} dy + \mathcal{O}(\varepsilon^2) \\
= \quad & \sum_{i=1}^m \varrho(\xi_i) + \mathcal{O}(\varepsilon).
\end{aligned}
$$

□

Lemma A.0.11. *Let $i, h = 1, 2, \cdots, m$ and $j = 1, \cdots, \mathrm{i}(\xi_i)$. Then, as $\varepsilon \to 0$,*

$$\varepsilon^2 \int_\Sigma \chi_h e^{U_h} \partial_{(\xi_i)_j} PU_i dv_g = \frac{\delta_{ih}}{2} \varrho(\xi_i)^2 \partial_{(\xi_i)_j} H^g(\xi_i, \xi_i) + (1 - \delta_{ih}) \varrho(\xi_i) \varrho(\xi_h) \partial_{(\xi_i)_j} G^g(\xi_h, \xi_i) + \mathcal{O}(\varepsilon).$$

Proof. We decompose the integral into the following two parts:

$$\varepsilon^2 \int_\Sigma \chi_h e^{U_h} \partial_{(\xi_i)_j} PU_i = \varepsilon^2 \left(\int_{\Sigma \cap U_{2r_0}(\xi_h)} + \int_{\Sigma \setminus U_{2r_0}(\xi_h)} \right) \chi_h e^{U_h} \partial_{(\xi_i)_j} PU_i.$$

It is clear that $\int_{\Sigma \setminus U_{2r_0}(\xi_h)} \varepsilon^2 \chi_h e^{U_h} \partial_{(\xi_i)_j} PU_i = 0$. For $h \neq i$, $U_{2r_0}(\xi_h) \cap U_{2r_0}(\xi_i) = \emptyset$ by the choice of r_0. Notice that

$$\begin{aligned} \partial_{(\xi_i)_j} |y_\xi(x)|^2 |_{x = y_{\xi_i}^{-1}(y)} &= -2 \langle (y_{\xi_i})_* \partial_{(\xi_i)_j} y_\xi^{-1}(y), y \rangle = -2 \langle (y_{\xi_i})_* \partial_{(\xi_i)_j} (\xi_i + y + \mathcal{O}(|y|^2)), y \rangle \\ &= -2 y_j + \mathcal{O}(|y|^3)(|y| \to 0). \end{aligned}$$

In this case, by Remark A.0.4

$$\begin{aligned} \int_{U_{2r_0}(\xi_h)} \varepsilon^2 \chi_h e^{U_h} \partial_{(\xi_i)_j} PU_i dv_g &= \int_{U_{2r_0}(\xi_h)} \varepsilon^2 \chi_h e^{U_h} (\partial_{(\xi_i)_j} (\chi_i U_i) + \varrho(\xi_i) \partial_{(\xi_i)_j} H^g_{\xi_i} + \mathcal{O}(\varepsilon^{\alpha_0})) \\ &= \varrho(\xi_h) \varrho(\xi_i) \partial_{(\xi_i)_j} H^g(\xi_h, \xi_i) + o(1). \end{aligned}$$

Claim A.0.12. *as $\varepsilon \to 0$,*

$$\begin{aligned} \int_{U_{2r_0}(\xi_h)} \varepsilon^2 \chi_i e^{U_i} \frac{2 \partial_{(\xi_i)_j} |y_{\xi_i}|^2}{\tau_i^2 \varepsilon^2 + |y_{\xi_i}|^2} dv_g &= \mathcal{O}(\varepsilon^2) + \int_{U_{r_0}(\xi_h)} \varepsilon^2 e^{U_i} \frac{4(-(y_{\xi_i})_j + \mathcal{O}(|y_{\xi_i}|^3))}{\tau_i^2 \varepsilon^2 + |y_{\xi_i}|^2} dv_g \\ &= o(1). \end{aligned}$$

Indeed, if $\xi_i \in \mathring{\Sigma}$, we have $|e^{\hat{\varphi}_i(y)} - 1| = |\hat{\varphi}_{\xi_i}(y)| + \mathcal{O}(|\hat{\varphi}_{\xi_i}(y)|^2) = \mathcal{O}(|y|^2)$ as $y \to 0$.

$$\begin{aligned} \int_{U_{r_0}(\xi_i) \cap \Sigma} \varepsilon^2 \chi_i e^{U_i} \frac{2 \partial_{(\xi_i)_j} |y_{\xi_i}(x)|^2}{\tau_i^2 \varepsilon^2 + |y_{\xi_i}(x)|^2} dv_g(x) &= \int_{B^{\xi_i}_{r_0}} \varepsilon^2 e^{\hat{\varphi}_{\xi_i}(y)} \frac{32 \tau_i^2 \varepsilon^2 (-y_j + \mathcal{O}(|y|^2))}{(\tau_i^2 \varepsilon^2 + |y|^2)^3} dy + \mathcal{O}(\varepsilon^2) \\ &= \int_{B^{\xi_i}_{r_0}} \varepsilon^2 (1 + \mathcal{O}(|y|^2)) \frac{-32 \tau_i^2 y_j + \mathcal{O}(|y|^3)}{(\tau_i^2 \varepsilon^2 + |y|^2)^3} dy = \mathcal{O}(\varepsilon). \end{aligned}$$

Claim A.0.12 is concluded. For $i, h = 1, 2, \cdots, m$ and $j = 1, \cdots, \mathrm{i}(\xi_i)$. By Remark A.0.4,

$$\begin{aligned}
&\int_\Sigma \varepsilon^2\chi_i e^{U_i}\partial_{(\xi_i)_j}PU_i dv_g\\
=\;&\int_\Sigma \frac{8\tau_i^2\varepsilon^2\chi_i}{(\tau_i^2\varepsilon^2+|y_{\xi_i}|^2)^2}\left(\chi_i\frac{2\partial_{(\xi_i)_j}|y_\xi|^2}{\tau_i^2\varepsilon^2+|y_{\xi_i}|^2}+\varrho(\xi_i)\partial_{(\xi_i)_j}H^g_{\xi_i}+\mathcal{O}(\varepsilon^{\alpha_0})\right)dv_g\\
=\;&\int_{U_{r_0}(\xi_i)}\varepsilon^2\chi_i(x)e^{U_i(x)}\frac{2\partial_{(\xi_i)_j}|y_\xi(x)|^2}{\tau_i^2\varepsilon^2+|y_{\xi_i}(x)|^2}dv_g(x)\\
&+\frac{1}{2}\varrho(\xi_i)\partial_{(\xi_i)_j}H^g(\xi_i,\xi_i)\int_{U_{r_0}(\xi_i)}\frac{8\tau_i^2\varepsilon^2}{(\tau_i^2\varepsilon^2+|y_{\xi_i}(x)|^2)^2}dv_g(x)+\mathcal{O}(\varepsilon^{\alpha_0})\\
=\;&\frac{1}{2}\varrho(\xi_i)^2\partial_{(\xi_i)_j}H^g(\xi_i,\xi_i)+o(1).
\end{aligned}$$

For $i \neq h$, via Lemma A.0.2, we drive that

$$\begin{aligned}
&\int_{U_{2r_0}(\xi_h)\cap\Sigma} \varepsilon^2\chi_h e^{U_h}\partial_{(\xi_i)_j}PU_i dv_g\\
=\;&\int_{U_{2r_0}(\xi_h)\cap\Sigma} \frac{8\tau_h^2\varepsilon^2\chi_h}{(\tau_h^2\varepsilon^2+|y_{\xi_h}|^2)^2}\left(\chi_i\frac{2\partial_{(\xi_i)_j}|y_{\xi_i}|^2}{\tau_i^2\varepsilon^2+|y_{\xi_i}|^2}+\varrho(\xi_i)\partial_{(\xi_i)_j}H^g_{\xi_i}+\mathcal{O}(\varepsilon^{\alpha_0})\right)dv_g\\
=\;&\int_{U_{2r_0}(\xi_h)\cap\Sigma}\chi_h(x)\frac{8\tau_h^2\varepsilon^2}{(\tau_h^2\varepsilon^2+|y_{\xi_h}|^2)^2}\left(\varrho(\xi_i)\partial_{(\xi_i)_j}G^g(\cdot,\xi_i)+\mathcal{O}(\varepsilon^{\alpha_0})\right)dv_g\\
=\;&\varrho(\xi_i)\varrho(\xi_h)\partial_{(\xi_i)_j}G^g(\xi_h,\xi_i)+\mathcal{O}(\varepsilon^{\alpha_0}).
\end{aligned}$$

Combining all the estimates above, Lemma A.0.11 is concluded. □

Lemma A.0.13. *Let $i = 1, 2, \cdots, m$ and $j = 1, \cdots, \mathrm{i}(\xi_i)$. As $\varepsilon \to 0$,*

$$\varepsilon^2\int_\Sigma Ve^{\sum_{h=1}^m PU_h}\partial_{(\xi_i)_j}PU_i dv_g=\frac{1}{2}\partial_{(\xi_i)_j}\mathcal{F}^V_{k,m}(\xi)+o(1).$$

Proof. First, we divide the integral into three parts to calculate:

$$\begin{aligned}
&\varepsilon^2\int_\Sigma Ve^{\sum_{h=1}^m PU_h}\partial_{(\xi_i)_j}PU_i dv_g\\
=\;&\varepsilon^2\left(\int_{\Sigma\setminus\cup_{h=1}^m U_{2r_0}(\xi_h)}+\int_{U_{2r_0}(\xi_i)\cap\Sigma}+\int_{\cup_{l\neq i}U_{2r_0}(\xi_l)}\right)Ve^{\sum_{h=1}^m PU_h}\partial_{(\xi_i)_j}PU_i dv_g.
\end{aligned}$$

The first term can be easily estimated by Remark A.0.4. As $\varepsilon \to 0$, we have

$$\begin{aligned}
&\int_{\Sigma\setminus\cup_{h=1}^m U_{2r_0}(\xi_h)}\varepsilon^2 Ve^{\sum_{h=1}^m PU_h}\partial_{(\xi_i)_j}PU_i dv_g\\
=\;&\mathcal{O}\left(\varepsilon^2\int_{\Sigma\setminus\cup_{h=1}^m U_{2r_0}(\xi_h)}\left|\partial_{\xi_j}(\chi_\xi U_{\tau,\xi})+\varrho(\xi)\partial_{\xi_j}H^g_\xi+\mathcal{O}(\varepsilon^{\alpha_0})\right|dv_g\right)\\
=\;&\mathcal{O}(\varepsilon^2).
\end{aligned}$$

We observe that for any $i = 1, 2, \cdots, m$ and $j = 1, \cdots, \mathrm{i}(\xi_i)$, as $|y| \to 0$, $\partial_{(\xi_i)_j} H^g(\xi_i, \xi_i) = 2\partial_{x_j} H^g(x, \xi_i)|_{x=\xi_i}$, $e^{\hat{\varphi}_i(y)} = 1 + \mathcal{O}(|y|^2)$, and $\partial_{(\xi_i)_j} |y_\xi(x)|^2|_{x=y_{\xi_i}^{-1}(y)} = -2y_j + \mathcal{O}(|y|^3)$. Applying Lemma A.0.1 and Remark A.0.4 with (3.1.4), we derive that

$$
\begin{aligned}
& \int_{U_{2r_0}(\xi_i)} \varepsilon^2 V e^{\sum_{h=1}^m PU_h} \partial_{(\xi_i)_j} PU_i dv_g \\
= & \int_{U_{2r_0}(\xi_i)} \left(\frac{\varepsilon^2 V e^{\varrho(\xi_i) H^g_{\xi_i} + \sum_{l \neq i} \varrho(\xi_l) G^g(\cdot, \xi_l) + \mathcal{O}(\varepsilon^{1+\alpha_0})}}{(\tau_i^2 \varepsilon^2 + |y_{\xi_i}|^2)^2} \right) \\
& \left(-\frac{2\chi_i \partial_{(\xi_i)_j} |y_{\xi_i}|^2}{(\tau_i^2 \varepsilon^2 + |y_{\xi_i}|^2)} + \varrho(\xi_i) \partial_{(\xi_i)_j} H^g_{\xi_i} + \mathcal{O}(\varepsilon^{\alpha_0}) \right) dv_g \\
= & \int_{B^{\xi_i}_{r_0}} \frac{8\tau_i^2 \varepsilon^2 e^{\hat{\varphi}_{\xi_i}(y)}}{(\tau_i^2 \varepsilon^2 + |y|^2)^2} \exp \Bigg\{ \varrho(\xi_i) H^g(y_{\xi_i}^{-1}(y), \xi_i) + \sum_{h \neq i} \varrho(\xi_h) G^g(y_{\xi_i}^{-1}(y), \xi_h) \\
& + \log V(y_{\xi_i}^{-1}(y)) - \log(8\tau_i^2) + \mathcal{O}(\varepsilon^{1+\alpha_0}) \Bigg\} \left(\left. \frac{-2\partial_{(\xi_i)_j} |y_{\xi_i}(x)|^2}{(\tau_i^2 \varepsilon^2 + |y_{\xi_i}(x)|^2)} \right|_{x=y_{\xi_i}^{-1}(y)} \right. \\
& \left. + \frac{1}{2} \varrho(\xi_i) \partial_{(\xi_i)_j} H^g(\xi_i, \xi_i) + \mathcal{O}(|y| + \varepsilon^{\alpha_0}) \right) dy + \mathcal{O}(\varepsilon^2) \\
= & \int_{\frac{1}{\tau_i \varepsilon} B^{\xi_i}_{r_0}} \frac{8}{(1 + |y|^2)^2} (1 + \mathcal{O}(\varepsilon^2 |y|^2)) \left(1 + \frac{1}{2} \tau_i \varepsilon \sum_{s=1}^2 \varrho(\xi_i) \partial_{(\xi_i)_s} H^g(\xi_i, \xi_i) y_s \right. \\
& \left. + \tau_i \varepsilon \sum_{h \neq i} \varrho(\xi_h) \sum_{s=1}^2 \partial_{(\xi_i)_s} G^g(\xi_i, \xi_h) y_s + \tau_i \varepsilon \sum_{s=1}^2 \partial_{(\xi_i)_s} \log V(\xi_i) y_s + \mathcal{O}(\tau_i^2 \varepsilon^2 |y|^2 + \varepsilon^{1+\alpha_0}) \right) \\
& \cdot \left(\frac{1}{\tau_i \varepsilon} \frac{4y_j}{1 + |y|^2} + \frac{\varrho(\xi_i)}{2} \partial_{(\xi_i)_j} H^g(\xi_i, \xi_i) + \mathcal{O}(\varepsilon |y| + \varepsilon^{\alpha_0}) \right) dy + \mathcal{O}(\varepsilon^2) \\
= & \frac{1}{2} \varrho(\xi_i)^2 \partial_{(\xi_i)_j} H^g(\xi_i, \xi_i) + \frac{1}{2} \varrho(\xi_i)^2 \partial_{(\xi_i)_j} H^g(\xi_i, \xi_i) \\
& + \sum_{h \neq i} \varrho(\xi_i) \varrho(\xi_h) \partial_{(\xi_i)_j} G^g(\xi_i, \xi_h) + \varrho(\xi_i) \partial_{(\xi_i)_j} \log V(\xi_i) + o(1) \\
= & \varrho(\xi_i)^2 \partial_{(\xi_i)_j} H^g(\xi_i, \xi_i) + \sum_{h \neq i} \varrho(\xi_i) \varrho(\xi_h) \partial_{(\xi_i)_j} G^g(\xi_i, \xi_h) + \varrho(\xi_i) \partial_{(\xi_i)_j} \log V(\xi_i) + o(1),
\end{aligned}
$$

where we applied $\int_{\mathbb{R}^2} \frac{1}{(1+|y|^2)^2} dy = \pi = 2 \int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^3} dy$.

For any $h \neq i$, analogue to the proof for $h = i$, we can obtain

$$
\int_{U_{2r_0}(\xi_h)} \varepsilon^2 V e^{\sum_{l=1}^m PU_l} \partial_{(\xi_i)_j} PU_i dv_g = \varrho(\xi_i) \varrho(\xi_h) \partial_{(\xi_i)_j} G^g(\xi_h, \xi_i) + o(1).
$$

Combining the estimates above,

$$
\begin{aligned}
& \varepsilon^2 \int_\Sigma V e^{\sum_{h=1}^m PU_h} \partial_{(\xi_i)_j} PU_i dv_g \\
= & \partial_{(\xi_i)_j} \left(\sum_{h=1}^m \varrho(\xi_h)^2 H^g(\xi_h, \xi_h) + \sum_{l \neq h} \varrho(\xi_h) \varrho(\xi_l) G^g(\xi_h, \xi_l) + \sum_{h=1}^m \varrho(\xi_h) \log V(\xi_h) \right) + o(1).
\end{aligned}
$$

□

Lemma A.0.14. *Let $i,h=1,\ldots,m..$ Then as $\varepsilon\to 0$,*

$$\left\|\varepsilon^2\chi_h e^{U_h}\left(\partial_{(\xi_i)_j}PU_i-\chi_i\partial_{(\xi_i)_j}U_i\right)\right\|_p\le O\left(\varepsilon^{\frac{2(1-p)}{p}}\right).$$

Proof. By Remark A.0.4, $\partial_{(\xi_i)_j}PU_i-\chi_i\partial_{(\xi_i)_j}U_i=\mathcal{O}(1)$. Then, applying Lemma A.0.2,

$$\left\|\varepsilon^2\chi_h e^{U_h}\left(\partial_{(\xi_i)_j}PU_i-\chi_i\partial_{(\xi_i)_j}U_i\right)\right\|_p\le O\left(\left\|\varepsilon^2\chi_h e^{U_h}\right\|_p\right)=O\left(\varepsilon^{\frac{2(1-p)}{p}}\right).$$

□

Lemma A.0.15. *Given $\delta>0$, let $\xi=(\xi_1,\cdots,\xi_m)\in\Xi^\delta_{k,m}$. Let $\phi\in K^\perp_\xi$ and $\|\phi\|\le\mathcal{O}(\varepsilon^{\frac{2-p}{p}}|\log\varepsilon|)$, where $p\in(1,\frac{6}{5})$. Then for $i=1,2,\cdots,m$ and $j=1,\cdots,\mathrm{i}(\xi_i)$, as $\varepsilon\to 0$,*

$$\left\langle\sum_{h=1}^m PU_h+\phi-i^*(\varepsilon^2Ve^{\sum_{h=1}^m PU_h+\phi}),\partial_{(\xi_i)_j}PU_i\right\rangle=-\frac{1}{2}\frac{\partial\mathcal{F}^V_{k,m}}{\partial(\xi_i)_j}(\xi)+o(1),\tag{A.0.5}$$

which is uniformly convergent for ξ in $\Xi^\delta_{k,m}$.

Proof. For $y=y_{\xi_i}(x)$, $\partial_{(\xi_i)_j}|y_{\xi_i}(x)|^2=-2y_j+\mathcal{O}(|y|^3)$. Since $\|\phi\|=o(1)$ and $\langle P\Psi^i_j,\phi\rangle=0$, we have

$$\begin{aligned}\left\langle\phi,\partial_{(\xi_i)_j}PU_i\right\rangle&=\int_\Sigma\varepsilon^2e^{-\varphi_i}e^{U_i}\phi\partial_{(\xi_i)_j}\chi_i dv_g+\int_\Sigma\varepsilon^2e^{-\varphi_i}\chi_ie^{U_i}\phi\partial_{(\xi_i)_j}U_idv_g \qquad (A.0.6)\\&+\int_\Sigma\varepsilon^2\chi_ie^{U_i}\phi\partial_{(\xi_i)_j}e^{-\varphi_i}dv_g\\&=\int_\Sigma\varepsilon^2\chi_ie^{-\varphi_i}e^{U_i}\phi\Psi^j_idv_g+\mathcal{O}\left(\int_{B^{\xi_i}_{2r_0}}\frac{\tau_i^2\varepsilon^2\chi\left(\frac{|y|}{r_0}\right)(\tau_i^2\varepsilon^2|y|^2+|y|^4+|y|^3)}{(\tau_i^2\varepsilon^2+|y|^2)^3}|\phi|dy\right)\\&=\langle\phi,P\Psi^j_i\rangle+o(1)=o(1),\end{aligned}$$

for any $i=1,\cdots,m$ and $j=1,\cdots,\mathrm{i}(\xi_i)$. Considering that $\int_\Sigma\partial_{(\xi_i)_j}PU_idv_g=0$ and $\chi_i\cdot\chi_h\equiv 0$

for any $i \neq h$, we have

$$\begin{aligned}
&\left\langle \sum_{h=1}^{m} PU_h + \phi - i^*(\varepsilon^2 V e^{\sum_{h=1}^m PU_h + \phi}), \partial_{(\xi_i)_j} PU_i \right\rangle \\
= \quad & \sum_{h=1}^{m} \langle PU_h, \partial_{(\xi_i)_j} PU_i \rangle + \langle \phi, \partial_{(\xi_i)_j} PU_i \rangle - \varepsilon^2 \int_\Sigma V e^{\sum_{h=1}^m PU_h + \phi} \partial_{(\xi_i)_j} PU_i dv_g \\
\overset{(A.0.6)}{=} \quad & \sum_{h=1}^{m} \int_\Sigma \varepsilon^2 \chi_h e^{-\varphi_h} e^{U_h} \partial_{(\xi_i)_j} PU_i dv_g - \varepsilon^2 \int_\Sigma V e^{\sum_{h=1}^m PU_h} (e^\phi - \phi - 1) \partial_{(\xi_i)_j} PU_i dv_g \\
& - \varepsilon^2 \int_\Sigma \left(V e^{\sum_{h=1}^m PU_h} - \sum_{h=1}^{m} \chi_h e^{U_h} \right) \phi \partial_{(\xi_i)_j} PU_i dv_g + \sum_{h \neq i} \varepsilon^2 \int_\Sigma \chi_h e^{U_h} \phi \chi_i (\partial_{(\xi_i)_j} U_i - \chi_i \partial_{(\xi_i)_j} U_i) dv_g \\
& - \varepsilon^2 \int_\Sigma V e^{\sum_{h=1}^m PU_h} \partial_{(\xi_i)_j} PU_i dv_g + o(1).
\end{aligned}$$

By Lemma A.0.5 and Lemma A.0.8,

$$\begin{aligned}
& \left| \varepsilon^2 \int_\Sigma V e^{\sum_{h=1}^m PU_h} (e^\phi - \phi - 1) \partial_{(\xi_i)_j} PU_i dv_g \right| \leq \| \varepsilon^2 h e^{\sum_{h=1}^m PU_h} (e^\phi - \phi - 1) \|_p \| \partial_{(\xi_i)_j} PU_i \|_q \\
\leq \quad & c \|\phi\|^2 \varepsilon^{\frac{2-2pr}{pr}} | \partial_{(\xi_i)_j} PU_i |_{L^q(\Sigma)} \leq c \|\phi\|^2 \varepsilon^{\frac{2-3pr}{pr}},
\end{aligned}$$

where $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and for any $r > 1$. By Lemma A.0.14,

$$\begin{aligned}
\left| \varepsilon^2 \int_\Sigma \sum_{h=1}^{m} \chi_h e^{U_h} \phi (\chi_i \partial_{(\xi_i)_j} U_i - \partial_{(\xi_i)_j} PU_i) dv_g \right| \quad & \leq \quad c \sum_{h=1}^{m} \|\phi\| \| \varepsilon^2 \chi_h e^{U_h} (\chi_i \partial_{(\xi_i)_j} U_i - \partial_{(\xi_i)_j} PU_i) \|_p \\
& \quad\ \leq c \|\phi\| \varepsilon^{\frac{2(1-p)}{p}}.
\end{aligned}$$

By Lemma A.0.7,

$$\begin{aligned}
\left| \varepsilon^2 \int_\Sigma (\sum_{h=1}^{m} \chi_h e^{U_h} - V e^{\sum_{h=1}^m PU_h}) \phi \partial_{(\xi_i)_j} PU_i \right| \quad & \leq \quad c \varepsilon^2 \|\phi\| \left\| \sum_{h=1}^{m} \chi_h e^{U_h} - V e^{\sum_{h=1}^m PU_h} \right\|_p \| \partial_{(\xi_i)_j} PU_i \| \\
& \leq \quad c \|\phi\| \varepsilon^{\frac{2-p}{p} - 1} = c \|\phi\| \varepsilon^{\frac{2(1-p)}{p}}.
\end{aligned}$$

In view of $\partial_{(\xi_i)_j} |y_{\xi_i}(x)|^2 = -2 y_{\xi_i}(x)_j + \mathcal{O}(|y_{\xi_i}(x)|^3)$ as $x \to \xi_i$, as $\varepsilon \to 0$

$$\begin{aligned}
& \varepsilon^2 \int_\Sigma \chi_i e^{U_i} \phi \chi_i \partial_{(\xi_i)_j} U_i dv_g \\
= \quad & \varepsilon^2 \int_\Sigma e^{U_i} \phi \chi_i e^{-\varphi_i} (1 + \mathcal{O}(|y_{\xi_i}|^2)) \left(P\Psi_i^j + \mathcal{O}\left(\frac{|y_{\xi_i}|^3}{\tau_i^2 \varepsilon^2 + |y_{\xi_i}|^2} \right) \right) dv_g + \mathcal{O}(\varepsilon^2) \\
= \quad & \langle \phi, P\Psi_j^i \rangle + \mathcal{O}(\varepsilon) = o(1).
\end{aligned}$$

On the other hand, applying Lemma A.0.11 and Lemma A.0.13, we deduce that

$$\begin{aligned}
&\sum_{h=1}^{m} \varepsilon^2 \int_\Sigma \chi_h e^{-\varphi_h} e^{U_h} \partial_{(\xi_i)_j} PU_i - \varepsilon^2 \int_\Sigma V e^{\sum_{h=1}^m PU_h} \partial_{(\xi_i)_j} PU_i \\
= \ &\sum_{h=1}^{m} \varepsilon^2 \int_\Sigma \chi_h e^{U_h} \partial_{(\xi_i)_j} PU_i - \varepsilon^2 \int_\Sigma V e^{\sum_{h=1}^m PU_h} \partial_{(\xi_i)_j} PU_i + o(1) = -\frac{1}{2} \partial_{(\xi_i)_j} \mathcal{F}_{k,m}^V(\xi) + o(1).
\end{aligned}$$

For any $p \in (1, \frac{6}{5})$, take $r > 1$ close to 1 enough such that $\frac{4-2p}{p} + \frac{2-3pr}{pr} > 0$. Hence, we have as $\varepsilon \to 0$

$$\left\langle \sum_{h=1}^{m} PU_h + \phi - i^*(\varepsilon^2 V e^{\sum_{h=1}^m PU_h + \phi}), \partial_{(\xi_i)_j} PU_i \right\rangle = -\frac{1}{2} \partial_{(\xi_i)_j} \mathcal{F}_{k,m}^V(\xi) + o(1).$$

□

B Estimates for Chapter 4

The following lemma is the asymptotic expansion of PU_i.

Lemma B.0.1. *As* $\delta_i \to 0$, $PU_i = \chi_i(U_i - \log(8\delta_i^2)) + \varrho(\xi_i)H^g(\cdot,\xi_i) + \mathcal{O}(\delta_i^2|\log\delta_i|)$ *as* $\delta_i \to 0$. *For any* $x \in \Sigma \setminus \{\xi_i\}$,

$$PU_i = \varrho(\xi_i)G^g(\cdot,\xi_i) + \mathcal{O}(\delta_i^2|\log\delta_i|),$$

which is convergent in $C(\Sigma)$.

Proof. Let $\eta_i = PU_i - \chi_i \cdot (U_i - \log(8\delta_i^2)) - \varrho(\xi_i)H^g(\cdot,\xi_i)$.

If $\xi_i \in \mathring{\Sigma}$, $\partial_{\nu_g}\eta_i \equiv 0$ on $\partial\Sigma$. We observe that for any $x \in \partial\Sigma \cap U(\xi_i)$

$$\partial_{\nu_g}|y_{\xi_i}(x)|^2 = -e^{-\frac{1}{2}\hat{\varphi}_{\xi_i}(y)}\left.\frac{\partial}{\partial y_2}|y|^2\right|_{y=y_{\xi_i}(x)} = 0.$$

If $\xi_i \in \partial\Sigma$, for any $x \in \partial\Sigma$, we have as $\delta_i \to 0$

$$\begin{aligned}\partial_{\nu_g}\eta_i(x) &= 2\partial_{\nu_g}\left(\chi_i\log\left(1+\frac{\delta_i^2}{|y_{\xi_i}(x)|^2}\right)\right)\\ &= 2(\partial_{\nu_g}\chi_i)\frac{\delta_i^2}{|y_{\xi_i}(x)|^2} - 2\chi_i\partial_{\nu_g}\log\left(1+\frac{\delta_i^2}{|y_{\xi_i}(x)|^2}\right) + \mathcal{O}(\delta_i^4)\\ &= \mathcal{O}(\delta_i^2).\end{aligned}$$

Thus, for any $i = 1,\cdots,m$, $\partial_{\nu_g}\eta_i = \mathcal{O}(\delta_i^2)$ as $\delta_i \to 0$.

$$\begin{aligned}&\int_\Sigma \eta_i\,dv_g = \int_\Sigma 2\chi_i\log\left(1+\frac{\delta_i^2}{|y_{\xi_i}(x)|^2}\right)\,dv_g(x)\\ =\;& 2\int_{B_{r_0}^{\xi_i}} e^{\hat{\varphi}_{\xi_i}(y)}\log\left(1+\frac{\delta_i^2}{|y|^2}\right)dy + 2\int_{B_{2r_0}^{\xi_i}\setminus B_{r_0}(0)}\chi(|y|/r_0)e^{\hat{\varphi}_{\xi_i}(y)}\left(\frac{\delta_i^2}{|y|^2}+\mathcal{O}(\delta_i^4)\right)dy\\ =\;& 2\delta_i^2\int_{\frac{1}{\delta_i}(B_{r_0}^{\xi_i}\cap B_{r_0}(0))}\log\left(1+\frac{1}{|y|^2}\right)e^{-\hat{\varphi}(\delta_i y)}dy + \mathcal{O}(\delta_i^2)\\ =\;& 2\delta_i^2(1+\mathcal{O}(\delta_i))\int_{B_{r_0/\delta_i}(0)}\log\left(1+\frac{1}{|y|^2}\right)dy + \mathcal{O}(\delta_i^2)\\ =\;& \mathcal{O}(\delta_i^2|\log\delta_i|),\end{aligned}$$

where we applied the fact that

$$
\begin{aligned}
0 &\leq \int_{|y|<\frac{r_0}{\delta_i}} \log\left(1+\frac{1}{|y|^2}\right) dy = 2\pi \int_0^{r_0/\delta_i} \log\left(1+\frac{1}{r^2}\right) rdr \\
&\leq \pi \int_0^{r_0^2/(\tau\rho)^2} \log\left(1+\frac{1}{t}\right) dt \leq 2\pi \int_1^{r_0/\delta_i} r^{-1} dr + \mathcal{O}(1) \leq \mathcal{O}(|\log \delta_i|).
\end{aligned}
$$

For any $x \in U_{2r_0}(\xi)$, $-\Delta_g U_i = e^{-\varphi_i} e^{U_i}$. It follows that

$$
\begin{aligned}
-\Delta_g \eta_i &= 2(\Delta_g \chi_i) \log \frac{|y_{\xi_i}|^2}{\delta_i^2+|y_{\xi_i}|^2} + 4\left\langle \nabla\chi_i, \nabla \log \frac{|y_{\xi_i}|^2}{\delta_i^2+|y_{\xi_i}|^2} \right\rangle_g \\
&\quad + \frac{1}{|\Sigma|_g}\left(\varrho(\xi_i) - \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} dv_g\right).
\end{aligned}
$$

We observe that $\Delta_g \chi_i \equiv 0$ and $\nabla \chi_i \equiv 0$ in $U_{2r_0}(\xi_i) \setminus U_{r_0}(\xi_i)$. For any $x \in U_{2r_0}(\xi_i) \setminus U_{r_0}(\xi_i)$, we have as $\delta_i \to 0$

$$
-2\log\left(1+\frac{\delta_i^2}{|y_{\xi_i}(x)|^2}\right) = -2\delta_i^2 |y_{\xi_i}(x)|^{-2} + \mathcal{O}(\delta_i^4)
$$

and

$$
-2\nabla \log\left(1+\frac{\delta_i^2}{|y_{\xi_i}(x)|^2}\right) = -2\delta_i^2 \nabla |y_{\xi_i}(x)|^{-\alpha_i} + \mathcal{O}(\delta_i^4).
$$

Moreover, a straightforward calculation of the integral implies that

$$
\begin{aligned}
\int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} dv_g &= \int_{B^\xi_{2r_0}} 8\chi(|y|/r_0) \frac{\delta_i^2}{(\delta_i^2+|y|^2)^2} dy \\
&= \int_{B^\xi_{r_0}} 8\frac{\delta_i^2}{(\delta_i^2+|y|^2)^2} dy + \mathcal{O}(\delta_i^2) = \varrho(\xi_i) + \mathcal{O}(\delta_i^2),
\end{aligned}
$$

where we applied the fact that $\int_{|y|<r} 8\frac{\delta_i^2}{(\delta_i^2+|y|^2)^2}\, dy = 8\pi\left(1-\frac{\delta_i^2}{\delta_i^2+r^2}\right)$ for any $r \geq 0$. Hence, as $\delta_i \to 0$, $-\Delta_g \eta_i = \mathcal{O}(\delta_i^2)$. By Lemma 2.3.3, we derive that as $\delta_i \to 0$, $\eta_i = \mathcal{O}(\delta_i^2 |\log \delta_i|)$, which is convergent in $C(\Sigma)$.

$\square$

Lemma B.0.2. *For $i = 1, \cdots, m$ and $j = 1, \cdots, \mathrm{i}(\xi_i)$*

$$
PZ_i^0(x) = \chi_i(x)\left(Z_i^0(x)+1\right) + \mathcal{O}(\delta_i^2|\log\delta_i|) = 4\chi_i(x)\frac{\delta_i^2}{\delta_i^2+|y_{\xi_i}(x)|^2} + \mathcal{O}(\delta_i^2|\log\delta_i|),
$$

in $C^1(\Sigma)$ as $\delta_i \to 0$. And $PZ_i^0(x) = (\delta_i^2|\log\delta_i|)$, in $C^1_{loc}(\Sigma\setminus\{\xi_i\})$ as $\rho \to 0$.

$PZ_i^j(x) = \chi_i(x) Z_i^j(x) + \mathcal{O}(1)$, in $C^1(\Sigma)$ as $\delta_i \to 0$, and $PZ_i^j(x) = \mathcal{O}(1)$, in $C^1_{loc}(\Sigma\setminus\{\xi_i\})$ as $\delta_i \to 0$. In addition, the convergences above are uniform for ξ_i in any compact subset of $\mathring{\Sigma}$ or $\xi_i \in \partial\Sigma$.

Proof. As the proof follows the same reasoning as in Lemma B.0.1, we forego detailing it here for brevity. ☐

Lemma B.0.3. *As $\varepsilon \to 0$, for $i, i' = 1, \cdots, m$ and $i = 1, \cdots, \mathrm{i}(\xi_i)$,*

$$\partial_{(\xi_i)_j} PU_{i'} = \delta_{i'i} PZ_i^j - PZ_{i'}^0 \partial_{(\xi_i)_j} \log d_{i'} + \mathcal{O}(1),$$

in $C(\Sigma)$ uniformly for ξ in any compact subset of $\Xi_{k,m}$, where δ_{ij} is the Kronecker symbol, i.e.,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ \\ 0 & \text{otherwise} \end{cases}.$$

Proof. We observe that $\partial_{(\xi_i)_j} PU_i = \delta_{i'i} \partial_{(\xi_i)_j} PU_{\rho,\xi_i}|_{\rho=\delta_i} + \partial_\rho PU_{\rho,\xi_{i'}}|_{\rho=\delta_{i'}} \varepsilon^{\frac{1}{2}} \partial_{(\xi_i)_j} d_{i'}$. Since

$$\partial_{(\xi_i)_j} |y_{\xi_i}(x)|^2 = -2(y_{\xi_i})_j + \mathcal{O}(|y_{\xi_i}|^3),$$

for any $x \in U(\xi_i)$, we have

$$\partial_{(\xi_i)_j} U_{\rho,\xi_i}|_{\rho=\delta_i} = \frac{4(y_{\xi_i})_j}{\delta_i^2 + |y_{\xi_i}|^2} + \mathcal{O}\left(\frac{|y_{\xi_i}|^2}{\delta_i^2 + |y_{\xi_i}|^3}\right) = Z_i^j + \mathcal{O}\left(\frac{|y_{\xi_i}|^2}{\delta_i^2 + |y_{\xi_i}|^3}\right).$$

By (4.2.3), we derive that

$$\begin{aligned} -\Delta_g(\partial_{(\xi_i)_j} PU_{\rho,\xi_i}|_{\rho=\delta_i} - PZ_i^j) &= \chi_i e^{-\varphi_i} e^{U_{\delta_i,\xi_i}} (\partial_{(\xi_i)_j} U_i - Z_i^j) - \overline{\chi_i e^{-\varphi_i} e^{U_{\delta_i,\xi_i}} (\partial_{(\xi_i)_j} U_i - Z_i^j)} \\ &\quad + \mathcal{O}(1) \\ &= \mathcal{O}\left(\frac{\delta_i^2 |y_{\xi_i}(x)|^3}{(\delta_i^2 + |y_{\xi_i}|^2)^3}\right) + \mathcal{O}(1). \end{aligned}$$

For any $p \in (1,2)$, $\|-\Delta_g(\partial_{(\xi_i)_j} PU_{\rho,\xi_i}|_{\rho=\delta_i} - PZ_i^j)\|_p = \mathcal{O}(1)$, $\int_\Sigma (\partial_{(\xi_i)_j} PU_{\rho,\xi_i}|_{\rho=\delta_i} - PZ_i^j)\, dv_g = 0$, and for any $x \in \partial\Sigma$, $\partial_{\nu_g}(\partial_{(\xi_i)_j} PU_{\rho,\xi_i}|_{\rho=\delta_i} - PZ_i^j) = 0$. Hence, applying the Schauder estimates in Lemma 2.3.3,

$$\|\partial_{(\xi_i)_j} PU_{\rho,\xi_i}|_{\rho=\delta_i} - PZ_i^j\|_{C^{2,\alpha}(\Sigma)} = \mathcal{O}(1).$$

Considering that $\delta_i \partial_\rho U_{\rho,\xi_i}|_{\rho=\delta_i} = \frac{2(|y_{\xi_i}|^2 - \delta_i^2)}{|y_{\xi_i}|^2 + \delta_i^2} = -Z_i^0$, it follows that $\delta_i \partial_\rho U_{\rho,\xi_i}|_{\rho=\delta_i} = PZ_i^0$. Therefore,

$$\partial_{(\xi_i)_j} PU_i = PZ_i^j - PZ_i^0 \partial_{(\xi_i)_j} \log d_i + \mathcal{O}(1)$$

and for $i' \neq i$

$$\partial_{(\xi_i)_j} PU_{i'} = -PZ_{i'}^0 \partial_{(\xi_i)_j} \log d_{i'} + \mathcal{O}(1).$$

$\square$

The asymptotic "orthogonality" properties of PZ_i^j as $\rho \to 0$.

Lemma B.0.4. *As $\rho \to 0$ for $i, i' = 1, \cdots, m$, $j = 0, \cdots, \mathrm{i}(\xi_i)$ and $j' = 0, \cdots, \mathrm{i}(\xi_{i'})$*

$$\langle PZ_i^j, PZ_{i'}^{j'} \rangle = \begin{cases} \delta_{i'i}\delta_{j'j}\frac{8\varrho(\xi_i)D_i}{\pi} + \mathcal{O}(\varepsilon \log|\varepsilon|) & \text{when } i \text{ or } i' = 0 \\ \delta_{i'i}\delta_{j'j}\frac{8\varrho(\xi_i)D_i}{\pi\delta_i^2} + \mathcal{O}(\varepsilon^{-\frac{1}{2}}) & \text{otherwise} \end{cases},$$

where δ_{ij} is the Kronecker symbol, and $D_0 = \int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^4}dy$, $D_1 = D_2 = \int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4}dy$.

Proof.

$$\begin{aligned} \langle PZ_i^j, PZ_{j'}^{i'} \rangle &= \int_\Sigma \chi_i e^{-\varphi_i} e^{U_i} Z_i^j PZ_{i'}^{j'} \, dv_g \\ &= \int_{\Sigma \cap U_{2r_0}(\xi_i)} + \int_{\Sigma \setminus U_{2r_0}(\xi_i)} \chi_i e^{-\varphi_i} e^{U_i} Z_i^j PZ_{i'}^{j'} \, dv_g. \end{aligned}$$

For $i = i' = 0$, by Lemma B.0.2,

$$\begin{aligned} & \int_{\Sigma \cap U_{2r_0}(\xi_i)} \chi_i e^{-\varphi_i} e^{U_i} Z_i^j PZ_{i'}^{j'} \, dv_g \\ = \; & 16 \int_{B_{2r_0}^{\xi_i}} \chi\left(\frac{|y|}{r_0}\right) \frac{|y|^2 - \delta_i^2}{(\delta_i^2 + |y|^2)^3} \left(\frac{4\delta_{i'i}\delta_i^2 \chi\left(\frac{|y|}{r_0}\right)}{\delta_i^2 + |y|^2} + \mathcal{O}(\varepsilon|\log\varepsilon|) \right) dy \\ = \; & 64\delta_{j'i} \int_{\Omega_i} \frac{1 - |y|^2}{(1 + |y|^2)^4} \, dy + \mathcal{O}(\varepsilon|\log\varepsilon|). \end{aligned}$$

As $\varepsilon \to 0$, $\langle PZ_i^0, PZ_{i'}^0 \rangle = \delta_{i'i}\frac{8\varrho(\xi_i)D_0}{\pi} + \mathcal{O}(\varepsilon|\log\varepsilon|)$, where $D_0 = \int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^4} \, dy$. Similarly, for $i' = 0$ and $i = 1, 2$ for $\xi_i \in \mathring{\Sigma}$ and $i = 1$ for $\xi_i \in \partial\Sigma$, we have

$$\begin{aligned} \langle PZ_i^j, PZ_{i'}^{j'} \rangle &= \int_{\Sigma \cap U_{2r_0}(\xi_i)} \chi_i e^{-\varphi_i} e^{U_i} Z_i^j PZ_{i'}^{j'} \, dv_g \\ &= 32 \int_{B_{2r_0}^{\xi_i}} \chi\left(\frac{|y|}{r_0}\right) \frac{y_j}{(\delta_i^2 + |y|^2)^3} \left(-\frac{4\delta_{i'i}\delta_i^2 \chi\left(\frac{|y|}{r_0}\right)}{\delta_i^2 + |y|^2} + \mathcal{O}(\varepsilon|\log\varepsilon|) \right) dy \\ &= \mathcal{O}(\varepsilon|\log\varepsilon|), \end{aligned}$$

where we applied the symmetric property $\int_{B_{2r_0}^{\xi_i}} \chi\left(\frac{|y|}{r_0}\right) \frac{y_j}{(\delta_i^2+|y|^2)^4} \, dy = 0$. Applying Lemma B.0.2, for $j = 1, \cdots, \mathrm{i}(\xi_i)$, $j' = 1, \cdots, \mathrm{i}(\xi_{i'})$,

$$\begin{aligned}
&\int_{\Sigma \cap U_{2r_0}(\xi)} \chi_i e^{-\varphi_i} e^{U_i} Z_i^j P Z_{i'}^{j'} \, dv_g(x) \\
= \;& 32 \int_{B_{2r_0}^{\xi_i}} \chi\left(\frac{|y|}{r_0}\right) \frac{\delta_i^2 y_j}{(\delta_i^2 + |y|^2)^3} \left(\delta_{i'i} \chi\left(\frac{|y|}{r_0}\right) \frac{4 y_{j'}}{\delta_i^2 + |y|^2} + \mathcal{O}(1) \right) dy \\
= \;& \frac{128}{\delta_i^2} \int_{\frac{1}{\delta_i} B_{r_0}^{\xi}} \frac{\delta_{i'i} y_j y_{j'}}{(1 + |y|^2)^4} dy + \mathcal{O}(\varepsilon^{-\frac{1}{2}}) = \frac{8 \delta_{i'i} \delta_{j'j} \varrho(\xi_i) D_i}{\pi \delta_i^2} + \mathcal{O}(\varepsilon^{-\frac{1}{2}}) \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

where $D_i = \int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4} dy$. □

Lemma B.0.5. *There exists $p_0 > 1$ such that for any $p \in (1, p_0)$*

$$\left\| 2\varepsilon V_1 e^{W_{1,\varepsilon}} - \sum_{i=1}^m \chi_i e^{-\varphi_i} e^{U_i} \right\|_p = \mathcal{O}\left(\varepsilon^{\frac{2-p}{2p}} \right),$$

as $\varepsilon \to 0$.

Proof. Applying Lemma B.0.1, we have $x \in \Sigma \setminus \bigcup_{i=1}^m U_{r_0}(\xi_i)$,

$$W_{1,\varepsilon}(x) = \sum_{i=1}^m PU_i - \frac{1}{2} z(\cdot, \xi) = \sum_{i=1}^m \varrho(\xi_i) G^g(x, \xi_i) - \frac{1}{2} z(x, \xi) + \mathcal{O}\left(\delta_i^2 |\log \delta_i| \right) = \mathcal{O}(1).$$

Let Θ_i be defined by (4.2.7). The estimate (4.2.8) leads to

$$\begin{aligned}
&\int_\Sigma \left| 2\varepsilon V_1 e^{W_{1,\varepsilon}} - \sum_{i=1}^m \chi_i e^{-\varphi_i} e^{U_i} \right|^p dv_g = \sum_{i=1}^m \int_{U_{r_0}(\xi_i)} \left| 2\varepsilon V_1 e^{W_{1,\varepsilon}} - e^{-\varphi_i} e^{U_i} \right|^p dv_g \\
&+ \mathcal{O}\left(\varepsilon^p + \delta_i^{2p} \right) \\
= \;& \sum_{i=1}^m \delta_i^2 \int_{\frac{1}{\delta_i} B_{r_0}^{\xi_i}(\xi_i)} e^{(1-p)\varphi_{\xi_i} \circ y_{\xi_i}^{-1}(\delta_i y)} \left| 2\varepsilon V_1 \circ y_{\xi_i}^{-1}(\delta_i y) e^{\varphi_{\xi_i} \circ y_{\xi_i}^{-1}(\delta_i y)} e^{W_{1,\varepsilon} \circ y_{\xi_i}^{-1}(\delta_i y)} - e^{U_i \circ y_{\xi_i}^{-1}(\delta_i y)} \right|^p dy \\
&+ \mathcal{O}\left(\varepsilon^p + \delta_i^{2p} \right) \\
= \;& 8^p \sum_{i=1}^m \int_{\Omega_i} \frac{|\delta_i|^{2-2p}}{(1 + |y|^2)^{2p}} \left| e^{\Theta_i(y)} - 1 \right|^p dy + \mathcal{O}\left(\varepsilon^p + \delta_i^{2p} \right) \\
= \;& 8^p \sum_{i=1}^m \int_{\Omega_i} \frac{|\delta_i|^{2-p} |y|^p}{(1 + |y|^2)^{2p}} \, dy + \mathcal{O}\left(\varepsilon^p |\log \varepsilon|^p \right) = \mathcal{O}(\varepsilon^{\frac{2-p}{2}}).
\end{aligned}$$

□

C Estimates for Chapter 5

The following lemma shows the asymptotic behavior of PU^i_j.

Lemma C.0.1. *For* $i = 1, \cdots, m$ *and* $j = 1, \cdots, \mathrm{i}(\xi_i)$,

$$PU^i_j = \chi_j \cdot (U^i_j - \log(2\alpha_i^2 \delta_{i,j}^{\alpha_i})) + \frac{\alpha_i \varrho(\xi_j)}{2} H^g(\cdot, \xi_j) + \mathcal{O}(\delta_{i,j}^{\alpha_i} |\log \delta_{i,j}|)$$

as $\delta_{i,j} \to 0$. *For any* $x \in \Sigma \setminus \{\xi_j\}$, $PU^i_j = \frac{\alpha_i \varrho(\xi_j)}{2} G^g(\cdot, \xi_j) + \mathcal{O}(\delta_{i,j}^{\alpha_i} |\log \delta_{i,j}|)$, *as* $\delta_{i,j} \to 0$.

Proof. Let $\eta_{ij} = PU^i_j - \chi_j \cdot (U^i_j - \log(2\alpha_i^2 \delta_{i,j}^{\alpha_i})) - \frac{\alpha_i \varrho(\xi_j)}{2} H^g(\cdot, \xi_j)$. If $\xi_j \in \mathring{\Sigma}$, $\partial_{\nu_g} \eta_{ij} \equiv 0$ on $\partial\Sigma$. We observe that for any $x \in \partial\Sigma \cap U(\xi_j)$

$$\partial_{\nu_g} |y_{\xi_j}(x)|^2 = -e^{-\frac{1}{2}\hat{\varphi}_{\xi_j}(y)} \frac{\partial}{\partial y_2} |y|^2 \Big|_{y = y_{\xi_j}(x)} = 0.$$

If $\xi_j \in \partial\Sigma$, for any $x \in \partial\Sigma$, we have as $\delta_{i,j} \to 0$

$$\begin{aligned}
\partial_{\nu_g} \eta_{ij}(x) &= 2\partial_{\nu_g} \left(\chi_j \log \left(1 + \frac{\delta_{i,j}^{\alpha_i}}{|y_{\xi_j}(x)|^{\alpha_i}} \right) \right) \\
&= 2(\partial_{\nu_g} \chi_j) \frac{\delta_{i,j}^{\alpha_i}}{|y_{\xi_j}(x)|^{\alpha_i}} - 2\chi_j \partial_{\nu_g} \log \left(1 + \frac{\delta_{i,j}^{\alpha_i}}{|y_{\xi_j}(x)|^{\alpha_i}} \right) + \mathcal{O}(\delta_{i,j}^{2\alpha_i}) \\
&= \mathcal{O}(\delta_{i,j}^{\alpha_i}).
\end{aligned}$$

Thus, for any $i = 1, 2, j = 1, \ldots, m$, $\partial_{\nu_g} \eta_{ij} = \mathcal{O}(\delta_{i,j}^{\alpha_i})$ as $\delta_{i,j} \to 0$.

$$\begin{aligned}
&\int_\Sigma \eta_{ij} dv_g = \int_\Sigma 2\chi_j \log \left(1 + \frac{\delta_{i,j}^{\alpha_i}}{|y_{\xi_j}(x)|^{\alpha_i}} \right) dv_g(x) \\
=\ & 2 \int_{B_{r_0}^{\xi_j}} e^{\hat{\varphi}_{\xi_j}(y)} \log \left(1 + \frac{\delta_{i,j}^{\alpha_i}}{|y|^{\alpha_i}} \right) dy + 2 \int_{B_{2r_0}^{\xi_j} \setminus \mathbb{B}_{r_0}} \chi(|y|/r_0) e^{\hat{\varphi}_{\xi_j}(y)} \left(\frac{\delta_{i,j}^{\alpha_i}}{|y|^{\alpha_i}} + \mathcal{O}(\delta_{i,j}^{2\alpha_i}) \right) dy \\
=\ & 2\delta_{i,j}^2 \int_{\frac{1}{\delta_{i,j}} B_{r_0}^{\xi_j}} \log \left(1 + \frac{1}{|y|^{\alpha_i}} \right) e^{\hat{\varphi}(\delta_{i,j} y)} dy + \mathcal{O}(\delta_{i,j}^{\alpha_i}) \\
=\ & 2\delta_{i,j}^2 (1 + \mathcal{O}(\delta_{i,j})) \int_{\frac{1}{\delta_{i,j}} B_{r_0}^{\xi_j}} \log \left(1 + \frac{1}{|y|^{\alpha_i}} \right) dy + \mathcal{O}(\delta_{i,j}^{\alpha_i}) \\
=\ & \mathcal{O}(\delta_{i,j}^2 |\log \delta_{i,j}|),
\end{aligned}$$

where we applied the fact that

$$
\begin{aligned}
0 &\leq \int_{|y|<\frac{r_0}{\delta_{i,j}}} \log\left(1+\frac{1}{|y|^{\alpha_i}}\right) dy = 2\pi \int_0^{r_0/\delta_{i,j}} \log\left(1+\frac{1}{r^{\alpha_i}}\right) rdr \\
&\leq \pi \int_0^{r_0^2/(\tau\rho)^2} \log\left(1+\frac{1}{t}\right) dt \leq 2\pi \int_1^{r_0/\delta_{i,j}} r^{1-\alpha_i} dr + \mathcal{O}(1) \leq \mathcal{O}(|\log \delta_{i,j}|).
\end{aligned}
$$

For any $x \in U_{2r_0}(\xi)$, $-\Delta_g U_j^i = e^{-\varphi_j} e^{U_j^i}$. It follows that

$$
\begin{aligned}
-\Delta_g \eta_{ij} &= 2(\Delta_g \chi_j) \log \frac{|y_{\xi_j}|^{\alpha_i}}{\delta_{i,j}^{\alpha_i} + |y_{\xi_j}|^{\alpha_i}} + 4 \left\langle \nabla \chi_j, \nabla \log \frac{|y_{\xi_j}|^{\alpha_i}}{\delta_{i,j}^{\alpha_i} + |y_{\xi_j}|^{\alpha_i}} \right\rangle_g \\
&\quad + \frac{1}{|\Sigma|_g} \left(\frac{1}{2} \alpha_i \varrho(\xi_j) - \int_\Sigma \chi_j e^{-\varphi_j} |y_{\xi_j}|^{\alpha_i - 2} e^{U_j^i} dv_g \right).
\end{aligned}
$$

We observe that $\Delta_g \chi_j \equiv 0$ and $\nabla \chi_j \equiv 0$ in $U_{2r_0}(\xi_j) \setminus U_{r_0}(\xi_j)$. For any $x \in U_{2r_0}(\xi_j) \setminus U_{r_0}(\xi_j)$, we have as $\delta_{i,j} \to 0$

$$
-2 \log \left(1 + \frac{\delta_{i,j}^{\alpha_i}}{|y_{\xi_j}(x)|^{\alpha_i}} \right) = -2 \delta_{i,j}^{\alpha_i} |y_\xi(x)|^{-\alpha_i} + \mathcal{O}(\delta_{i,j}^{2\alpha_i})
$$

and

$$
-2 \nabla \log \left(1 + \frac{\delta_{i,j}^{\alpha_i}}{|y_{\xi_j}(x)|^{\alpha_i}} \right) = -2 \delta_{i,j}^{\alpha_i} \nabla |y_{\xi_j}(x)|^{-\alpha_i} + \mathcal{O}(\delta_{i,j}^{2\alpha_i}).
$$

Moreover, a straightforward calculation of the integral implies that

$$
\begin{aligned}
\int_\Sigma \chi_j e^{-\varphi_j} |y_{\xi_j}|^{\alpha_i - 2} e^{U_j^i} dv_g &= \int_{B^\xi_{2r_0}} 2\alpha_i^2 \chi(|y|/r_0) \frac{\delta_{i,j}^{\alpha_i} |y|^{\alpha_i - 2}}{(\delta_{i,j}^{\alpha_i} + |y|^{\alpha_i})^2} dy \\
&= \int_{B^\xi_{r_0}} 2\alpha_i^2 \frac{\delta_{i,j}^{\alpha_i} |y|^{\alpha_i - 2}}{(\delta_{i,j}^{\alpha_i} + |y|^{\alpha_i})^2} dy + \mathcal{O}(\delta_{i,j}^{\alpha_i}) \\
&= \frac{\alpha_i \varrho(\xi_j)}{2} + \mathcal{O}(\delta_{i,j}^{\alpha_i}),
\end{aligned}
$$

where we applied the fact that $\int_{|y|<r} 2\alpha_i^2 \frac{\delta_{i,j}^{\alpha_i} |y|^{\alpha_i-2}}{(\delta_{i,j}^{\alpha_i} + |y|^{\alpha_i})^2} dy = 4\pi\alpha_i \left(1 - \frac{\delta_{i,j}^{\alpha_i}}{\delta_{i,j}^{\alpha_i} + r^{\alpha_i}}\right)$ for any $r \geq 0$. Hence, as $\delta_{i,j} \to 0$, $-\Delta_g \eta_{ij} = \mathcal{O}(\delta_{i,j}^{\alpha_i})$. By Lemma 2.3.3, we derive that as $\delta_{i,j} \to 0$, $\eta_{ij} = \mathcal{O}(\delta_{i,j}^{\alpha_i} |\log \delta_{i,j}|)$ in $C(\Sigma)$. □

Using the same approach of Lemma C.0.1, we can deduce the asymptotic expansion of PZ_{ij}.

Lemma C.0.2. *For any $i = 1, 2$, $j = 1, \ldots, m$, as $\delta_{i,j} \to 0$*

$$
PZ_{ij} = Z_{ij} + 1 + \mathcal{O}(\delta_{i,j}^{\alpha_i} |\log \delta_{i,j}|) = \frac{2\delta_{i,j}^{\alpha_i}}{\delta_{i,j}^{\alpha_i} + |y_{\xi_j}|^{\alpha_i}} + \mathcal{O}(\delta_{i,j}^{\alpha_i} |\log \delta_{i,j}|). \tag{C.0.1}
$$

Lemma C.0.3. *Let Θ_{ij} be defined by (5.1.25). For $i = 1, 2, j = 1, \dots, m$*

$$|\Theta_{ij}(y)| = \mathcal{O}\left(\delta_{i,j}|y| + \varepsilon^{\frac{3}{4}}\right), \quad y \in \frac{1}{\delta_{i,j}}\mathcal{A}_{ij},$$

and particularly, $\sup_{\frac{1}{\delta_{i,j}}\mathcal{A}_{ij}} |\Theta_{ij}(y)| = \mathcal{O}(1)$ *as* $\varepsilon \to 0$.

Proof. Lemma C.0.1 implies that

$$\begin{aligned}
&\Theta_{ij}(y) = -\log(2\alpha_i^2) - \alpha_i \log\delta_{i,j} + \frac{\alpha_i \varrho(\xi_j)}{2}\left(H^g(\xi_j,\xi_j) + \sum_{j'\neq j} G^g(\xi_{j'},\xi_j)\right)\\
&-\frac{1}{2}\sum_{i'<i}\left(-2\alpha_{i'}\log(\delta_{i,j}|y|) + \frac{\alpha_{i'}\varrho(\xi_j)}{2}\left(H^g(\xi_j,\xi_j) + \sum_{j'\neq j} G^g(\xi_{j'},\xi_j)\right) + \mathcal{O}\left(\frac{\delta_{i',j}^{\alpha_{i'}}}{\delta_{i,j}^{\alpha_{i'}}|y|^{\alpha_{i'}}}\right)\right)\\
&-\frac{1}{2}\sum_{i'>i}\left(-2\alpha_{i'}\log\delta_{i',j} + \frac{\alpha_{i'}\varrho(\xi_j)}{2}\left(H^g(\xi_j,\xi_j) + \sum_{j'\neq j} G^g(\xi_{j'},\xi_j)\right) + \mathcal{O}\left(\frac{\delta_{i,j}^{\alpha_{i'}}|y|^{\alpha_{i'}}}{\delta_{i',j}^{\alpha_{i'}}}\right)\right)\\
&+\log V_i(\xi_j) + \log(2\varepsilon) - (\alpha_i - 2)\log(\delta_{i,j}|y|) + \mathcal{O}\left(\sum_{l=1}^{2}\delta_{l,j}^{\alpha_l}|\log\delta_l| + \delta_{i,j}|y|\right)\\
= &\left(-\alpha_i\log\delta_{i,j} + \sum_{i'>i}\alpha_{i'}\log\delta_{i',j} - \log(2\alpha_i^2) + \frac{\varrho(\xi_j)}{2}\left(\alpha_i - \frac{1}{2}\sum_{i'\neq i}\alpha_{i'}\right)\right.\\
&\left.\cdot\left(H^g(\xi_j,\xi_j) + \sum_{j'\neq j} G^g(\xi_{j'},\xi_j)\right) + \log V_i(\xi_j) + \log(2\varepsilon)\right) + \left(\sum_{i'<i}\alpha_{i'} - (\alpha_i - 2)\right)\log(\delta_{i,j}|y|)\\
&+\sum_{i'<i}\mathcal{O}\left(\frac{\delta_{i',j}^{\alpha_{i'}}}{\delta_{i,j}^{\alpha_{i'}}|y|^{\alpha_{i'}}}\right) + \sum_{i'>i}\mathcal{O}\left(\frac{\delta_{i,j}^{\alpha_{i'}}|y|^{\alpha_{i'}}}{\delta_{i',j}^{\alpha_{i'}}}\right) + \mathcal{O}\left(\sum_{l=1}^{2}\delta_{l,j}^{\alpha_l}|\log\delta_l| + \delta_{i,j}|y|\right).
\end{aligned}$$

Recall $\alpha_i, \delta_{i,j}$ and $d_{i,j}$ are defined by (5.1.3), (5.1.4) and (5.1.5), respectively. Immediately, the last and second terms vanish on the equation above given the fact that

$$\begin{aligned}
&\alpha_i\log\delta_{i,j} - \sum_{i'>i}\alpha_{i'}\log\delta_{i',j} + \log(2\alpha_i^2) - \frac{\varrho(\xi_j)}{2}\left(\alpha_i - \frac{1}{2}\sum_{i'\neq i}\alpha_{i'}\right)\left(H^g(\xi_j,\xi_j) + \sum_{j'\neq j} G^g(\xi_{j'},\xi_j)\right)\\
&-\log V_i(\xi_j) - \log(2\varepsilon) = 0,
\end{aligned}$$

and $\sum_{i'<i}\alpha_{i'} - (\alpha_i - 2) = 0$, for any $i = 1, 2$ and $j = 1, \dots, m$. Hence, we derive that

$$\Theta_{ij}(y) = \sum_{i'<i}\mathcal{O}\left(\frac{\delta_{i',j}^{\alpha_{i'}}}{\delta_{i,j}^{\alpha_{i'}}|y|^{\alpha_{i'}}}\right) + \sum_{i'>i}\mathcal{O}\left(\frac{\delta_{i,j}^{\alpha_{i'}}|y|^{\alpha_{i'}}}{\delta_{i',j}^{\alpha_{i'}}}\right) + \mathcal{O}\left(\sum_{l=1}^{2}\delta_{l,j}^{\alpha_l}|\log\delta_l| + \delta_{i,j}|y|\right).$$

The estimates (5.1.3) and (4.2.4) imply that $\mathcal{O}(\delta_{i,j}^{\alpha_i}|\log\delta_{i,j}|) = \mathcal{O}(\varepsilon\log|\varepsilon|)$, for $i = 1, 2$ and

$j = 1, \dots, m$. For $y \in \frac{1}{\delta_{i,j}} \mathcal{A}_{ij}$, we have $\sqrt{\delta_{i-1,j}/\delta_{i,j}} \le |y| < \sqrt{\delta_{i+1,j}/\delta_{i,j}}$. If $i' < i$,

$$\mathcal{O}\left(\frac{\delta_{i',j}^{\alpha_{i'}}}{\delta_{i,j}^{\alpha_{i'}} |y|^{\alpha_{i'}}}\right) = \mathcal{O}\left(\left(\frac{\delta_{i',j}^2}{\delta_{i,j}\delta_{i-1,j}}\right)^{\alpha_{i'}/2}\right) = \mathcal{O}\left(\left(\frac{\delta_{i-1,j}}{\delta_{i,j}}\right)^{\alpha_{i'}/2}\right) = \mathcal{O}\left(\varepsilon^{\frac{3}{2}\cdot 2^{2-2i+i'}}\right) = \mathcal{O}(\varepsilon^{\frac{3}{4}});$$

if $i' > i$,

$$\sum_{i'>i} \mathcal{O}\left(\frac{\delta_{i,j}^{\alpha_{i'}} |y|^{\alpha_{i'}}}{\delta_{i',j}^{\alpha_{i'}}}\right) = \mathcal{O}\left(\left(\frac{\delta_{i,j}\delta_{i+1,j}}{\delta_{i',j}^2}\right)^{\alpha_{i'}/2}\right) = \mathcal{O}\left(\left(\frac{\delta_{i,j}}{\delta_{i+1,j}}\right)^{\alpha_{i'}/2}\right) = \mathcal{O}\left(\varepsilon^{\frac{3}{8}\cdot 2^{2-2i+i'}}\right) + \mathcal{O}(\varepsilon^{\frac{3}{2}}).$$

Moreover, $\mathcal{O}(\delta_{i,j}|y|) = \mathcal{O}(1)$. Lemma C.0.3 is complete. □

Lemma C.0.4. *For $i = 1, 2$, there exists $p_0 > 1$ such that for any $p \in (1, p_0)$*

$$\left\| 2\varepsilon V_i e^{W_{i,\varepsilon}} - \sum_{j=1}^m \chi_j e^{-\varphi_j} |y_{\xi_j}|^{\alpha_i - 2} e^{U_j^i} \right\|_p = \mathcal{O}\left(\varepsilon^{\frac{2-p}{4p}}\right),$$

as $\varepsilon \to 0$.

Proof. Applying Lemma C.0.1, we have for any $i = 1, 2$ and $x \in \Sigma \setminus \bigcup_{j=1}^m U_{r_0}(\xi_j)$,

$$\begin{aligned} W_{i,\varepsilon}(x) &= \sum_{j=1}^m PU_j^i - \frac{1}{2}\sum_{\substack{i'=1\\ i'\ne i}}^2 \sum_{j=1}^m PU_j^{i'} = \sum_{j=1}^m \frac{\varrho(\xi_j)}{2}\left(\alpha_i - \frac{1}{2}\sum_{\substack{i'=1\\ i'\ne i}}^2 \alpha_{i'}\right) G^g(x,\xi_j) + \mathcal{O}\left(\sum_{i=1}^2 \delta_{i,j}^{\alpha_i} |\log \delta_{i,j}|\right) \\ &= \mathcal{O}(1). \end{aligned}$$

By straightforward calculation, we deduce that

$$\begin{aligned} &\int_\Sigma \left| 2\varepsilon V_i e^{W_{i,\varepsilon}} - \sum_{j=1}^m \chi_j e^{-\varphi_j} |y_{\xi_j}|^{\alpha_i-2} e^{U_j^i} \right|^p dv_g = \sum_{j=1}^m \int_{U_{r_0}(\xi_j)} \left| 2\varepsilon V_i e^{W_{i,\varepsilon}} - e^{-\varphi_j} |y_{\xi_j}|^{\alpha_i-2} e^{U_j^i} \right|^p dv_g \\ &+\mathcal{O}\left(\varepsilon^p + \delta_{i,j}^{\alpha_i p}\right) \\ = \ &\sum_{j=1}^m \int_{\frac{1}{\delta_{i,j}} B_{r_0}^{\xi_j}(\xi_j)} \left| 2\varepsilon V_i \circ y_{\xi_j}^{-1}(\delta_{i,j} y) e^{\varphi_{\xi_j} \circ y_{\xi_j}^{-1}(\delta_{i,j} y)} e^{W_{i,\varepsilon} \circ y_{\xi_j}^{-1}(\delta_{i,j} y)} - |\delta_{i,j} y|^{\alpha_i - 2} e^{U_j^i \circ y_{\xi_j}^{-1}(\delta_{i,j} y)} \right|^p dv_g \\ &+\mathcal{O}\left(\varepsilon^p + \delta_{i,j}^{\alpha_i p}\right) \\ = \ &2\alpha_i^2 \sum_{j=1}^m \int_{\Omega_{ij}} \frac{|\delta_{i,j}|^{2-2p} |y|^{(\alpha_i-2)p}}{(1+|y|^{\alpha_i})^{2p}} \left| e^{\Theta_{ij}(y)} - 1 \right|^p dy + \mathcal{O}\left(\varepsilon^p + \delta_{i,j}^{\alpha_i p}\right). \end{aligned}$$

Observing that $\Omega_{ij} \subset \bigcup_{l=1}^2 \frac{1}{\delta_{i,j}} \mathcal{A}_{lj}$, we intend to compute the integral $\int_{\Omega_{ij}} \frac{|\delta_{i,j}|^{2-2p}|y|^{(\alpha_i-2)p}}{(1+|y|^{\alpha_i})^{2p}} \left|e^{\Theta_{ij}(y)} - 1\right|^p dy$

by dividing it into two regions: $\Omega_{ij} \cap \frac{1}{\delta_{i,j}}\mathcal{A}_{1j}$ and $\Omega_{ij} \cap \frac{1}{\delta_{i,j}}\mathcal{A}_{2j}$. Lemma C.0.3 yields that

$$\begin{aligned}
&\int_{\Omega_{ij}\cap\frac{1}{\delta_{i,j}}\mathcal{A}_{ij}} \frac{|\delta_{i,j}|^{2-2p}|y|^{(\alpha_i-2)p}}{(1+|y|^{\alpha_i})^{2p}} \left|e^{\Theta_{ij}(y)}-1\right|^p dy \\
= \;& \mathcal{O}\left(\int_{\Omega_{ij}\cap\frac{1}{\delta_{i,j}}\mathcal{A}_{ij}} \frac{|\delta_{i,j}|^{2-2p}|y|^{(\alpha_i-2)p}}{(1+|y|^{\alpha_i})^{2p}} \left|\Theta_{ij}(y)\right|^p dy\right) \\
= \;& \mathcal{O}\left(\int_{\Omega_{ij}\cap\frac{1}{\delta_{i,j}}\mathcal{A}_{ij}} |\delta_{i,j}|^{2-2p}\frac{|y|^{(\alpha_i-2)p}}{(1+|y|^{\alpha_i})^{2p}} \left|\delta_{i,j}|y|+\varepsilon^{\frac{3}{4}}\right|^p dy\right) \\
= \;& \mathcal{O}\left(\delta_{i,j}^{2-p}+\delta_{i,j}^{2-2p}\varepsilon^{\frac{3}{4}p}\right) \overset{(4.2.4)}{=} \mathcal{O}(\varepsilon^{(1-p)2^{3-2i}+\frac{3}{4}p}+\varepsilon^{(2-p)2^{2-2i}}) = \mathcal{O}(\varepsilon^{\frac{2-p}{4}}).
\end{aligned}$$

For $i' \neq i$,

$$\begin{aligned}
&\int_{\Omega_{ij}\cap\frac{1}{\delta_{i,j}}\mathcal{A}_{i'j}} \frac{|\delta_{i,j}|^{2-2p}|y|^{(\alpha_i-2)p}}{(1+|y|^{\alpha_i})^{2p}} \left|e^{\Theta_{ij}(y)}-1\right|^p dy \qquad\qquad (\text{C.0.2}) \\
\leq \;& 2^p\int_{\Omega_{ij}\cap\frac{1}{\delta_{i,j}}\mathcal{A}_{i'j}} \frac{|\delta_{i,j}|^{2-2p}|y|^{(\alpha_i-2)p}}{(1+|y|^{\alpha_i})^{2p}} dy + 2^p\int_{\Omega_{ij}\cap\frac{1}{\delta_{i,j}}\mathcal{A}_{i'j}} \frac{|\delta_{i,j}|^{2-2p}|y|^{(\alpha_i-2)p}}{(1+|y|^{\alpha_i})^{2p}} \left|e^{\Theta_{ij}(y)}\right|^p dy.
\end{aligned}$$

For the first term of (C.0.2), applying (4.2.4) we have

$$\begin{aligned}
&\int_{\Omega_{ij}\cap\frac{1}{\delta_{i,j}}\mathcal{A}_{i'j}} \frac{|\delta_{i,j}|^{2-2p}|y|^{(\alpha_i-2)p}}{(1+|y|^{\alpha_i})^{2p}} dy = \mathcal{O}\left(\int_{\frac{\sqrt{\delta_{i'-1,j}\delta_{i',j}}}{\delta_{ij}}\leq|y|\leq\frac{\sqrt{\delta_{i',j}\delta_{i'+1,j}}}{\delta_{ij}}} \frac{|\delta_{i,j}|^{2-2p}|y|^{(\alpha_i-2)p}}{(1+|y|^{\alpha_i})^{2p}} dy\right) \\
= &\begin{cases} O\left(\delta_{i,j}^{2-2p}\left(\frac{\sqrt{\delta_{i',j}\delta_{i'+1,j}}}{\delta_{i,j}}\right)^{(\alpha_i-2)p+2}\right) = O\left(\delta_{i,j}^{2-2p}\left(\frac{\delta_{i',j}}{\delta_{i'+1,j}}\right)^{\frac{(\alpha_i-2)p+2}{2}}\right) & \text{if } i > i' \\ O\left(\delta_{i,j}^{2-2p}\left(\frac{\delta_{i,j}}{\sqrt{\delta_{i'-1,j}\delta_{i',j}}}\right)^{(\alpha_j+2)p-2}\right) = O\left(\delta_{i,j}^{2-2p}\left(\frac{\delta_{i'-1,j}}{\delta_{i',j}}\right)^{\frac{(\alpha_i+2)p-2}{2}}\right) & \text{if } i < i' \end{cases}, \\
= &\begin{cases} O\left(\varepsilon^{-(p-1)2^{3-2i}+\frac{3}{4}(2^{i-1}p-p+1)}\right) & \text{if } i > i' \\ O\left(\varepsilon^{-(p-1)2^{3-2i}+\frac{3}{4}(2^{i-1}p+p-1)}\right) & \text{if } i < i' \end{cases} = \mathcal{O}(\varepsilon^{\frac{2-p}{4}}),
\end{aligned}$$

for $p > 1$ sufficiently close to 1. To estimate the second term of (C.0.2), via Lemma 5.1.18 we derive that

$$\begin{aligned}
&\int_{\Omega_{ij}\cap\frac{1}{\delta_{i,j}}\mathcal{A}_{i'j}} \frac{|\delta_{i,j}|^{2-2p}|y|^{(\alpha_i-2)p}}{(1+|y|^{\alpha_i})^{2p}} \left|e^{\Theta_{ij}(y)}\right|^p dy \\
= \;& \mathcal{O}\left(\varepsilon^p\delta_{i,j}^{2-2p}\int_{\Omega_{ij}\cap\frac{1}{\delta_{i,j}}\mathcal{A}_{i'j}} \left(\frac{\Pi_{i'\neq i}(\delta_{i',j}^{\alpha_{i'}}+\delta_{i,j}^{\alpha_{i'}}|y|^{\alpha_{i'}})}{(1+|y|^{\alpha_i})^2}\right)^p dy\right).
\end{aligned}$$

Recall that $\delta_{i,j} = \mathcal{O}(\varepsilon)$ and $\delta_{2,j} = \mathcal{O}(\varepsilon^{\frac{1}{4}})$. We have for $i = 1$

$$\begin{aligned}
& \varepsilon^p \delta_{1,j}^{2-2p} \int_{\Omega_{1j} \cap \frac{1}{\delta_{1,j}} \mathcal{A}_{2j}} \left(\frac{\delta_{2,j}^{\alpha_2} + \delta_{1,j}^{\alpha_2} |y|^{\alpha_2})}{(1+|y|^{\alpha_1})^2} \right)^p dy \\
= \;& \varepsilon^p \delta_{1,j}^{2-2p} \delta_{2,j}^{\alpha_2 p} \mathcal{O}\left(\int_{|y| \geq \sqrt{\frac{\delta_{2,j}}{\delta_{1,j}}}} \frac{1}{|y|^{2\alpha_1 p}} dy \right) + \varepsilon^p \delta_{1,j}^{2-2p+\alpha_2 p} \mathcal{O}\left(\int_{\Omega_{ij} \cap \left\{ |y| \geq \sqrt{\frac{\delta_{2,j}}{\delta_{1,j}}} \right\}} 1 dy \right) \\
= \;& \mathcal{O}\left(\varepsilon^p \delta_{1,j}^{2-2p+1+\alpha_1 p} \delta_{2,j}^{\alpha_2 p - \alpha_1 p - 1} + \varepsilon^p \delta_{1,j}^{\alpha_2 p - 2p} \right) = \mathcal{O}(\varepsilon^p);
\end{aligned}$$

for $i = 1$.

$$\begin{aligned}
& \varepsilon^p \delta_{2,j}^{2-2p} \int_{\Omega_{2j} \cap \frac{1}{\delta_{2,j}} \mathcal{A}_{1j}} \left(\frac{\delta_{1,j}^{\alpha_1} + \delta_{2,j}^{\alpha_1} |y|^{\alpha_1})}{(1+|y|^{\alpha_2})^2} \right)^p dy \\
= \;& \varepsilon^p \delta_{2,j}^{2-2p} \delta_{1,j}^{\alpha_1 p} \mathcal{O}\left(\int_{|y| \leq \sqrt{\frac{\delta_{1,j}}{\delta_{2,j}}}} 1 dy \right) + \varepsilon^p \delta_{2,j}^{2-2p+\alpha_1 p} \mathcal{O}\left(\int_{|y| \leq \sqrt{\frac{\delta_{1,j}}{\delta_{2,j}}}} |y|^{\alpha_1 p} dy \right) \\
= \;& \mathcal{O}\left(\varepsilon^p \delta_{1,j}^{\alpha_1 p + 1} \delta_{2,j}^{1-2p} + \varepsilon^p \delta_{1,j}^{\frac{1}{2}\alpha_1 p + 1} \delta_{2,j}^{1-2p+\frac{1}{2}\alpha_1 p} \right) = \mathcal{O}(\varepsilon^p).
\end{aligned}$$

Hence, we obtain that

$$\int_{\Omega_{ij} \cap \frac{1}{\delta_{i,j}} \mathcal{A}_{i'j}} \frac{|\delta_{i,j}|^{2-2p} |y|^{(\alpha_i - 2)p}}{(1+|y|^{\alpha_i})^{2p}} \left| e^{\Theta_{ij}(y)} \right|^p dy = \mathcal{O}(\varepsilon^{\frac{2-p}{4}}).$$

$\square$

Lemma C.0.5. *For any $p \geq 1$ and $r > 1$, there are positive constants c_1, c_2 such that for any $\varepsilon > 0$, the following estimates hold for any $\phi_1, \phi_2 \in \overline{\mathrm{H}}^1$ and any $i = 1, 2$:*

$$\|\varepsilon V_i e^{W_{i,\varepsilon}} (e^{\phi_1} - 1 - \phi_1)\|_p \leq c_1 e^{c_2 \|\phi_1\|^2} \rho^{\frac{2-2pr}{4pr}} \|\phi_1\|^2, \tag{C.0.3}$$

and

$$\begin{aligned}
& \|\varepsilon V_i e^{W_{i,\varepsilon}} (e^{\phi_1} - e^{\phi_2} - (\phi_1 - \phi_2))\|_p \qquad \text{(C.0.4)} \\
\leq \;& c_1 e^{c_2(\|\phi_1\|^2 + \|\phi_2\|^2)} \rho^{\frac{2-2pr}{4pr}} (\|\phi_1\| + \|\phi_2\|) \|\phi_1 - \phi_2\|.
\end{aligned}$$

Proof. By the mean value theorem, for some $s \in (0,1)$

$$|(e^{\phi_1} - e^{\phi_2} - (\phi_1 - \phi_2)| \leq \left| e^{s\phi_1 + (1-s)\phi_2} - 1 \right| |\phi_1 - \phi_2| \leq e^{|\phi_1| + |\phi_2|} |\phi_1 - \phi_2| (|\phi_1| + |\phi_2|).$$

The Hölder inequality, Sobolev inequality and Moser-Trudinger inequality yield that

$$\begin{aligned}
&\left(\int_\Sigma V_i^p e^{p\sum_{j=1}^m (PU_j^i - \frac{1}{2}\sum_{i'\neq i} PU_j^{i'})} |e^{\phi_1} - e^{\phi_2} - (\phi_1 - \phi_2)|^p dv_g\right)^{1/p} \\
\leq\ & C\sum_{h=1}^2 \left(\int_\Sigma V_i^p e^{p\sum_{j=1}^m (PU_j^i - \frac{1}{2}\sum_{i'\neq i} PU_j^{i'})} (e^{|\phi_1|+|\phi_2|}|\phi_1 - \phi_2||\phi_h|)^p dv_g\right)^{1/p} \\
\leq\ & C\sum_{h=1}^2 \left(\int_\Sigma V_i^{pr} e^{pr\sum_{j=1}^m (PU_j^i - \frac{1}{2}\sum_{i'\neq i} PU_j^{i'})} dv_g\right)^{\frac{1}{pr}} \left(\int_\Sigma e^{ps(|\phi_1|+|\phi_2|)} dv_g\right)^{\frac{1}{ps}} \\
& \left(\int_\Sigma |\phi_1 - \phi_2|^{pt} |\phi_h|^{pt} dv_g\right)^{\frac{1}{pt}} \\
\leq\ & C\sum_{h=1}^2 \left(\int_\Sigma V_i^{pr} e^{pr\sum_{j=1}^m (PU_j^i - \frac{1}{2}\sum_{i'\neq i} PU_j^{i'})} dv_g(x)\right)^{\frac{1}{pr}} e^{\frac{ps}{8\pi}(\|\phi_1\|^2 + \|\phi_2\|^2)} \|\phi_1 - \phi_2\| \|\phi_h\|,
\end{aligned}$$

where $r, s, t \in (1, +\infty)$, $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1$. Applying Lemma C.0.4, we deduce that

$$\begin{aligned}
\|2\varepsilon V_i e^{W_{i,\varepsilon}}\|_{pr} &\leq \left\| 2\varepsilon V_i e^{W_{i,\varepsilon}} - \sum_{j=1}^m \chi_j e^{-\varphi_j} |y_{\xi_j}|^{\alpha_i - 2} e^{U_j^i} \right\|_{pr} + \sum_{j=1}^m \left\| \chi_j e^{-\varphi_j} |y_{\xi_j}|^{\alpha_i - 2} e^{U_j^i} \right\|_{pr} \\
&\leq \sum_{j=1}^m \left\| \chi_j e^{-\varphi_j} |y_{\xi_j}|^{\alpha_i - 2} e^{U_j^i} \right\|_{pr} + \mathcal{O}(\varepsilon^{\frac{2-pr}{4}}).
\end{aligned}$$

Lemma C.0.1 implies that

$$\begin{aligned}
\int_\Sigma \chi_j^{pr} e^{-pr\varphi_j} |y_{\xi_j}|^{pr(\alpha_i - 2)} e^{prU_j^i} dv_g(x) &= \mathcal{O}\left(\delta_{i,j}^{2-2pr} \int_{\Omega_{ij}} \left(\frac{|y|^{\alpha_i - 2}}{(1 + |y|^{\alpha_i})^2}\right)^{pr} dy\right) + \mathcal{O}(\delta_{i,j}^{\alpha_i}) \\
&= \mathcal{O}(\delta_{i,j}^{2-2pr}) = \mathcal{O}(\varepsilon^{\frac{2-2pr}{4}}).
\end{aligned}$$

Let $\phi_2 \equiv 0$, and then (C.0.3) follows. □

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Statement

I declare that I have completed this dissertation single-handedly without the unauthorized help of a second party and only with the assistance acknowledged therein. I have appropriately acknowledged and cited all text passages that are derived verbatim from or are based on the content of published work of others, and all information relating to verbal communications. I consent to the use of an anti-plagiarism software to check my thesis. I have abided by the principles of good scientific conduct laid down in the charter of the Justus Liebig University Gießen “Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis” in carrying out the investigations described in the dissertation.

Gießen, 10.10.2024 Zhengni Hu