Hans-Otto Walther

Hyberbolic periodic solutions, heteroclinic connections and transversal homoclinic points in autonomous differential delay equations

CONTENTS

Abstract: We study autonomous differential delay equations

$$\dot{x}(t) = g(x(t-1))$$

for C^1-nonlinearities $g:\underline{R} \to \underline{R}$ which are periodic in x and sine-like, i.e., with $g(0) = 0$ and with one (simple) zero in the period interval $(0,\omega)$. Such equations model delayed feedback on the circle.

It is proved that within this class there exist nonlinearities with

- hyperbolic periodic solutions close to the equilibria $j\omega$, $j \in \underline{Z}$,

and with

- heteroclinic solutions connecting the periodic orbit close to $j\omega$ to the periodic orbit close to $(j+1)\omega$.

In addition, a transversality property is established.

The corresponding Poincare´map, which has no continuous inverse, constitutes a discrete (semi-) dynamical system in an infinite dimensional phase space with a transversal homoclinic point. A result of Hale and Lin is applied to describe in terms of symbolic dynamics the chaotic motion close to the homoclinic loop of the map, or equivalently, close to the heteroclinic connections of the continuous semiflow.

Key words and phrases: Differential delay equation, autonomous, infinite dimensional system, hyperbolic, periodic orbit, heteroclinic, noninvertible Poincare´map, transversal homoclinic point, symbolic dynamics

INTRODUCTION

Autonomous differential delay equations

$$(g) \qquad \dot{x}(t) = g(x(t-1))$$

with $g:\underline{R} \to \underline{R}$ periodic model delayed feedback for a state variable on the circle.

If γ_z is a zero of g with

$$(\xi - \gamma_z)\cdot g(\xi - \gamma_z) < 0 \quad \text{for} \quad \xi - \gamma_z \neq 0 \quad \text{small}$$

then feedback is negative with respect to the rest point given by γ_z. Continuity of g implies a zero in the interval $(\gamma_z - \omega, \gamma_z)$ where ω denotes the minimal period of g. In the simplest case there is only one zero in $(\gamma_z - \omega, \gamma_z)$; we may assume that g is sine-like in the sense

$$g(0) = 0\,, \quad 0 < g \quad \text{in} \quad (0,\gamma_z)\,, \quad g < 0 \quad \text{in} \quad (\gamma_z,\omega)\,.$$

Such equations occur in a number of applications. We only mention models for phase-locked loops [2,5,30].

Initial values $\phi \in C = C([-1,0],\underline{R})$ determine solutions $x^\phi:[-1,\infty) \to \underline{R}$. Setting

$$X(t,\phi) := x_t^\phi := x^\phi(t+\cdot)\,|\,[-1,0] \quad \text{for} \quad t \geq 0\,, \quad \phi \in C\,,$$

one obtains a continuous semiflow $X:\underline{R}^+ \times C \to C$. X does not extend to a flow; there is no underlying vectorfield, no matter how smooth g is.

The dynamics of these semiflows is very rich. Depending on further properties of g, one finds various kinds of periodic solutions, hetero- and homoclinic solutions, and in parameterized equations, local and nonlocal bifurcations [5,31,33,34,35].

Chaotic motion has been established for special differential delay equations with a somewhat different feedback structure [32,23,10,11,9]. The nonlinearities in [32,10,11,9] are smoothed step functions, especially designed so that periodic and homoclinic solutions are found by explicit computation. Initially different solutions flow together in finite time and enter 2-dimensional subsets of C; one obtains Poincare´maps on one-dimensional domains, and chaos can be derived from results on interval

---

Received by the editors May 23, 1988 and, in revised form October 10, 1988

maps [16,15,24].

The present paper contains a more general approach to chaotic differential delay equations. It does not rely on nonlinearities which are smoothed step functions. Geometric arguments replace explicit calculations. The structure finally implying chaos is more complicated than in the former examples. A reduction to interval maps is no longer at hand.

Before giving results, let us recall the work of Smale from [26,27]: Consider a diffeomorphism f in a finite-dimensional space with hyperbolic fixed point x^*, i.e., there are no eigenvalues of $Df(x^*)$ on the unit circle $\underline{S}$. If the stable and unstable (immersed) manifolds of x^* intersect transversally in a homoclinic trajectory $(x_n)_{-\infty}^{\infty}$, i.e., $f(x_{n-1}) = x_n \underset{\neq}{\to} x^*$ as $|n| \to \infty$, then there is chaos: On an invariant subset, an iterate of f is topologically conjugate to a shift on a space of symbol sequences [6]. See also [17,19,21,22].

Šilnikov, who studied a hyperbolic periodic orbit of a vectorfield with transversal stable and unstable manifolds, constructed a neighborhood so that all trajectories in this neighborhood are represented by symbol sequences [25].

A generalization of Šilnikov's result to infinite dimension, for maps which are not necessarily injective, is due to Hale and Lin (Theorem 5.2 [8]). A new proof of Hale's and Lin's result, now based on hyperbolic structures for arbitrary C^1-maps and on a generalized shadowing lemma [28], has been given in [29].

In the present paper we prove existence of sine-like nonlinearities g with hyperbolic periodic solutions

$$\ldots, x - \omega, x, x + \omega, \ldots$$

and heteroclinic solutions

$$\ldots, h - \omega, h, h + \omega, \ldots$$

of eq. (g);

$$h_t \to \{x_s : s \in \underline{R}\} =: o \text{ as } t \to -\infty, \quad h_t \to o + \omega \text{ as } t \to \infty$$

so that intersections modulo ω of $\underline{R} \ni t \mapsto h_t \in C$ with hyperplanes H and $H + \omega$ define a homoclinic trajectory $(\chi_n)_{-\infty}^{\infty}$ of an associated Poincaré map, with a transversality property (Corollaries 1, 2, 4 in Section IV.4) which permits to apply Theorem 5.2 [8] (with a minor generalization) and to deduce chaotic behavior.

Alternatively, it is possible to apply Theorem 5.1 [29] which is a bit more comfortable and gives the same result. See Section 6 of [29].

The chaotic solutions of eq. (g) are seen close to the periodic solutions $\ldots, x - \omega, x, x + \omega, \ldots$ and their heteroclinic connections $\ldots, h - \omega, h, h + \omega, \ldots$. For example, there are infinitely many distinct orbits in C of periodic solutions of the second kind,

$$y(\cdot + p) = y + n\omega \text{ for some } p > 0 \text{ and some } n \in \underline{N}.$$

Such solutions $y: \underline{R} \to \underline{R}$ describe periodic motion of the state variable on the circle $\underline{R} \bmod \omega$, composed of rotations around and small oscillations near $x(\underline{R}) \bmod \omega$.

We give an outline of the search for sine-like nonlinearities with the desired properties.

Chapter I deals with unstable periodic solutions of equation

$$(ag_0) \qquad \dot{x}(t) = a g_0(x(t-1))$$

with parameter $a > 0$ and $g_0: \underline{R} \to \underline{R}$ C^1-smooth, $g_0(0) = 0$, $g_0'(0) = 1$. Linearization of the semiflow at $0 \in C$ leads to the "characteristic equation"

$$((a)) \qquad z - ae^{-z} = 0$$

for the spectrum of the generator. An inspection of this transcendental equation shows that at

$$a = \frac{3\pi}{2} =: \alpha$$

there is a Hopf bifurcation of periodic solutions with periods close to $\frac{4}{3}$. A positive solution ξ_α of eq. $((\alpha))$ indicates that the bifurcating periodic orbits have at least one Floquet multiplier outside $\underline{S}$ which makes them unstable.

A major difficulty is to control the second Floquet multiplier close to the multiplier 1, which is present due to Hopf bifurcation: Is it equal to 1, inside $\underline{S}$ or outside ? There seems to be no Hopf bifurcation theorem in the literature which leads to conditions on g_0 and its derivatives ensuring that the multiplier close to 1 lies strictly inside $\underline{S}$.

For this reason, theorems on Hopf bifurcation are abandoned. We impose the conditions that for some $\gamma > 0$

g_0 is odd in $[-\gamma,\gamma]$ and

g_0' is strictly decreasing on $[0,\gamma]$.

This permits to obtain a supercritical curve $\alpha < a \to x_{a,0} \in C$ of initial values for bifurcating periodic solutions x_a , by an idea of Kaplan and Yorke [13] and by a result of Nussbaum [20].

Then we can employ the method from [31], originally developped for the less local problem of bifurcation from periodic orbits, and show that for $a - \alpha > 0$ small the second multiplier close to 1 lies strictly inside $\underline{S}$. In other words, the solutions x_a are hyperbolic with one unstable direction, given by the eigenspace of the multiplier outside $\underline{S}$.

Positive feedback

$$0 < \xi g_0(\xi) \quad \text{for} \quad 0 < |\xi| \leq \gamma$$

and monotonicity imply instability of solutions with values in $[-\gamma,\gamma]$, like x_a , also in terms of elementary inequalities for drifting away from each other. Section I.7 contains what is needed for the sequel.

Chapter II is abstract. For a C^1-map f with a hyperbolic fixed point x^* , local stable and unstable manifolds are introduced as in [8] (and [18]); f is not necessarily invertible. The final application of Theorem 5.2 [8] on chaos is prepared, and a detailed proof is given for a technical result on motion towards subsets of the local unstable manifold (Proposition II.1.1). The latter helps to link local behavior near x^* (where f is dominated by $Df(x^*)$) to motion far away.

Section III.1 introduces Poincaré maps

$$P_a : D_a \to H$$

for the periodic solutions x_a . D_a is an open subset of a hyperplane H in C which is transversal to the trajectory $t \mapsto x_{a,t}$ of x_a in C at $t = 0$. The initial values

$$x_{a,0} =: \phi_a^*$$

become hyperbolic fixed points of the maps P_a , with one-dimensional linear unstable space $\underline{R}\lambda_a$.

We know the limiting direction of the linear unstable spaces as $a \downarrow \alpha$. This permits to fix $a > \alpha$ sufficiently close to α so that one can control the position of the solutions starting in the local unstable manifold of ϕ_a^* , in terms of inequalities: We set $g_1 := a g_0$ and drop the index a . Let $U \subset H$ and $S \subset H$ denote the local unstable and local stable manifolds of ϕ^* . Segments y_2 of solutions of eq. (g_1) with initial value in $U \setminus S$ reach an open cone in C with vertex ϕ^* (Proposition III.2.1).

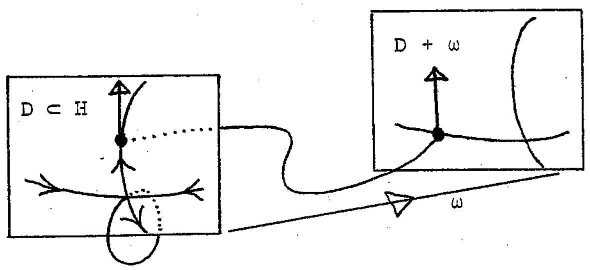
The idea how to find a periodic nonlinearity g with a heteroclinic solution from x to $x + \omega$ is basically as in [32] and in [10,9]. It exploits the delay: Let us assume that g_1 is increasing on all of $\underline{R}^+$ – not yet periodic. The instability results from Section I.7 and Proposition III.2.1 imply that candidates $h : \underline{R} \to \underline{R}$ for heteroclinic solutions, with h_0 in the branch of U above S , increase to values $h(t) > \gamma$, with

$$-\gamma < h < h(t) \quad \text{on} \quad (-\infty,t) \quad \text{and} \quad 0 < \dot{h} \quad \text{on} \quad [t,t+1] .$$

A change of g_1 outside the interval $[-\gamma,h(t)]$ would not affect results for the small amplitude solution x , for P and $h|(-\infty,t+1]$; a modification of g_1 on the interval $(h(t),h(t+1)]$ can be used to steer the solution h of the delay equation to prescribed values on subintervals of $(t+1,t+2]$.

The program is then to change g_1 outside $[-\gamma,\gamma]$

- to a sine-like function g with minimal period ω
- so that a segment h_{t_1} , $t_1 > 0$, of a solution h with $h_0 \in U$ reaches the local stable manifold $S + \omega \subset D + \omega \subset H + \omega$ of the fixed point $\phi^* + \omega$ of the shifted Poincaré map on $D + \omega$
- in such a way that a tangent vector $v \in T_{h_0}U$ is transported to a vector $v_1 \in H$ transversal to $S + \omega$ at h_{t_1} .



As it can not be expected to hit the subset $S + \omega$ of codimension 1 in $H + \omega$ immediately, we look for a one-parameter-family of functions $g = g_b$, $b \geq 4$, with solutions $h = h_b$ such that $h_{4,t}$ reaches $H + \omega$ below $S + \omega$ while for some $b^* > 4$, $h_{b^*,t}$ reaches $H + \omega$ above $S + \omega$ so that continuity implies $h_{b,t_1} \in S + \omega$ for some intermediate parameter b .

Section III.3 contains local preparations for this. It is here that the technical Proposition II.1.1 enters, via Corollary III.2.2 .

Section III.4 characterizes transversality by inequalities for solutions w of the linear variational equation along the heteroclinic solution h ,

$$\dot{w}(t) = g'(h(t-1))w(t-1) \ ,$$

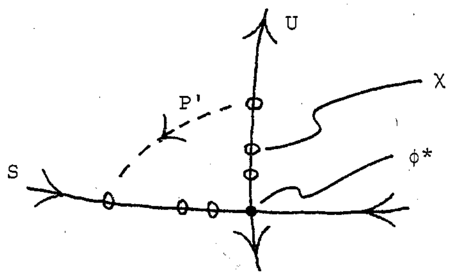
that is, directly in terms of the objects g , h which we are going to construct.

In Chapter IV the program sketched above is carried out. The initial value $\chi = h_0 \in U$ becomes a transversal homoclinic point of a modification P' of the map P : For $\phi \in H$ close to $\chi \neq \phi^*$, P' is given by intersections of trajectories $t \rightarrow X(t,\phi) \approx h_t$ with $H + \omega$ at $t \approx t_1$ and by a shift modulo ω back into H .

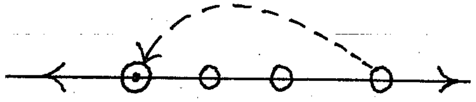
Proposition IV.4.1 exhibits an important difference to the former examples of chaotic differential delay equations [32,23,10,11,9]. The heteroclinic solution does not flow into the limiting periodic orbit in finite time. In particular, the points

$$S \ni (P')^n(\chi) = h_{t_n} - \omega \ , \text{ with } n \in \underline{N} \text{ and } t_n > 0 \ ,$$

are all different from ϕ^* , and the homoclinic sequence is not entirely contained in the one-dimensional submanifold U .



This is in contrast to [32,10,11] where we could reduce the Poincaré map to a map on a one-dimensional submanifold which contained the unstable fixed point and a homoclinic trajectory jumping on the fixed point at some index -



so that chaos followed from Marotto's result on maps in finite dimensional spaces with "snap-back repellers" [16].

Chapter V presents a short application of the work of Hale and Lin [8] on chaos to the map P' .

PRELIMINARIES

The sets of integers, positive integers, real and complex numbers are denoted by $\underline{Z}$, $\underline{N}$, $\underline{R}$ and $\underline{C}$, respectively. $\underline{N}_0 := \underline{N} \cup \{0\}$, $\underline{R}^+ := [0,\infty)$ and $\underline{R}^- := (-\infty,0]$. For $0 \neq x \in \underline{R}$, sign $x = \frac{x}{|x|}$; sign $0 := 0$. $\underline{S}$ is the unit circle in $\underline{C}$.

Let L be a real Banach space with norm $|\ |$. For $r > 0$, $L_r := \{x \in L: |x| < r\}$. L^c denotes the space of continuous linear maps $L \to L$. For $A \in L^c$, the spectrum $\sigma(A)$ is defined as the spectrum of its complexification $\tilde{A} \in \tilde{L}^c$; $\tilde{L}^c$ is the space of continuous $\underline{C}$-linear maps of the complexified space $\tilde{L}$ into $\tilde{L}$.

For a map $f:D \to L$, $D \subset L$, domains of iterates f^n , $n \in \underline{N}$, are denoted by $D_{(n)}$; note $D_{(n+1)} \subset D_{(n)} \subset D_{(1)} = D$ for all $n \in \underline{N}$. A trajectory of f is a sequence $(x_n)_{n\in\underline{Z}} = (x_n)_{-\infty}^{\infty} \in D^{\underline{Z}}$ with $x_{n+1} = f(x_n)$ for all $n \in \underline{Z}$. Similarly for forward and backward trajectories $(x_n)_0^{\infty}$ and $(x_n)_{-\infty}^{0}$.

A fixed point x^* of a C^1-map f with D open is called hyperbolic if $\sigma(Df(x^*)) \cap \underline{S} = \emptyset$. A homoclinic trajectory $(x_n)_{-\infty}^{\infty}$ of x^* is defined by $x_n \to x^*$ as $|n| \to \infty$ and $x_n \neq x^*$ for some n . This includes "snap-back repellers" x^* [16] with an index n such that for $j < n$, $x_j \neq x^*$ while $x_j = x^*$ for $j \geqq n$.

If f is injective then the inverse f^{-1} is defined as a map from $f(D)$ into L .

A tangent to a set $M \subset L$ at a point $x \in M$ is a vector $v \in L$ such that $v = Dc(0)1$ for a C^1-curve $c:(-1,1) \to L$ with $c(0) = x$ and range in M . The set of tangents to M at $x \in M$ is denoted by T_xM .

For C^k-submanifolds of L , we refer to [1].

We collect the basic facts on scalar differential delay equations. Proofs are found in [7], or carried out with the aid of [7].

Let $g:\underline{R} \to \underline{R}$ be given. A solution of equation

(g) $\quad \dot{x}(t) = g(x(t-1))$

is either a differentiable function $x:\underline{R} \to \underline{R}$ which satisfies eq. (g) for all t , or a continuous function $x:[\underline{t}-1,\infty) \to \underline{R}$ which is differentiable and satisfies eq. (g) for all $t > \underline{t}$. If g is linear then complex-valued solutions are defined analogously. It is also clear how to define solutions of nonautonomous equations

$$\dot{x}(t) = A(t)x(t-1)$$

for $A:[\underline{t},\infty) \to \underline{R}$ or $A:\underline{R} \to \underline{R}$.

Let C denote the Banach space of continuous functions $[-1,0] \to \underline{R}$, with the maximum-norm. As complexification of C we consider the space $\tilde{C}$ of continuous functions $[-1,0] \to \underline{C}$, also with the maximum-norm. Constant functions in C or $\tilde{C}$ with value ξ are denoted by ξ , too.

Each solution x of eq. (g) defines a continuous trajectory in C (or in $\tilde{C}$) by $x_t := x(t + \cdot)|[-1,0]$ for all $t \in \underline{R}$ with $[t-1,t]$ in the domain of x .

If the solution carries an index, like x_1 , then we write $x_{1,t}$ for the values of the trajectory.

Let g be continuous, $\underline{t} \in \underline{R}$. Then every $\phi \in C$ determines a unique solution $x:[\underline{t}-1,\infty) \to \underline{R}$ with $x_{\underline{t}} = \phi$, and we have continuous dependence on initial data (restrictions of x to compact intervals depend continuously on ϕ with respect to uniform convergence). This is most easily seen from the formula

$$x(t) - x(\underline{t} + n) = \int_{\underline{t}+n-1}^{t-1} g \circ x$$

for $\underline{t} + n \leqq t \leqq \underline{t} + n + 1$ and $n \in \underline{N}_0$.

Also, if $\tilde{g}:\underline{R} \times \underline{R} \to \underline{R}$ is continuous, if $t \geqq 0$ and $\phi \in C$, and if for $b \in \underline{R}$ $x_b:[-1,\infty) \to \underline{R}$ denotes the solution of

$$\dot{x}(t) = \tilde{g}(b,x(t-1)) , \ x_0 = \phi$$

then the map $\underline{R} \ni b \to x_{b,t} \in C$ is continuous.

Suppose g is C^k, $k \in \underline{N}_0$. Then solutions $x:[\underline{t}-1,\infty) \to \underline{R}$ are C^k on $(\underline{t} + k,\infty)$.

Solutions $x = x^{\phi}$ of initial value problems

$$\dot{x}(t) = g(x(t-1)) \text{ for } t > 0 , \ x_0 = \phi$$

define a continuous semiflow $X = X_g$, $X:\underline{R}^+ \times C \to C$ by $X(t,\phi) = x_t^{\phi}$ for $t \geqq 0$, $\phi \in C$. X is C^k on $(k,\infty) \times C$.

For $t \geqq 1$, the maps $X(t,\cdot)$ are compact (bounded sets are mapped onto relatively compact sets). This follows by Ascoli's theorem since eq. (g) implies that for $x_0 = \phi$ in a bounded set and for $t \geq 0$ given, solutions and their derivatives are bounded on $(0,t]$.

If g is periodic with period ω and if x is a solution of eq. (g) then $x+\omega$, $x-\omega$ are solutions as well, and $X(t,\phi+\omega) = X(t,\phi)+\omega$, $X(t,\phi-\omega) = X(t,\phi)-\omega$ on $\underline{R}^+ \times C$.

Let g be C^1. Then the derivative D_2X exists and is continuous on all of $\underline{R}^+ \times C$; it is given by solutions of the linear variational equation: If $\phi \in C$, $t \geq 0$, $\tilde{\phi} \in C$ then $D_2X(t,\phi)\tilde{\phi} = w_t$ where $w:[-1,\infty) \to \underline{R}$ is the unique solution of the initial value problem

$$\dot{w}(t) = g'(x^{\phi}(t-1))w(t-1) \quad \text{for} \quad t > 0, \ w_0 = \tilde{\phi}.$$

We need an easy consequence for parameterized equations. If $g = ag_0$ with parameter $a > 0$ and $g_0:\underline{R} \to \underline{R}$ C^1, and if continuous maps $a \to \phi_a$ and $a \to \tilde{\phi}_a$ from some interval into C are given then the map

$$a \to D_2X_{ag_0}(2,\phi_a)\tilde{\phi}_a$$

is also continuous.

Let $\xi \in \underline{R}$ with $g(\xi) = 0$ be given. Then $x:\underline{R} \ni t \to \xi \in \underline{R}$ is a constant solution. $\xi \in C$ becomes a stationary point of $X_g = X$. The linear variational equation along x

$$\dot{w}(t) = g'(\xi)w(t-1)$$

is autonomous, and $(t,\phi) \to D_2X(t,\xi)\phi$ defines a C_0-semigroup, called the linearization of X at ξ. For $t \geq 1$, the operators in this semigroup are compact. The spectrum of its generator consists of eigenvalues of finite algebraic multiplicity; it is characterized by the zeros of the transcendental function $z \to z - g'(\xi)e^{-z}$ and their orders (note that the Ansatz $w(t) = e^{zt}$ leads to $z - g'(\xi)e^{-z} = 0$). In case $g'(\xi) \neq 0$ there are infinitely many such zeros, real or complex conjugate pairs. Any set of zeros with $\operatorname{Re} z > u$, $u \in \underline{R}$ given, is finite or empty.

Suppose $x:\underline{R} \to \underline{R}$ is a periodic solution of eq. (g) with g C^1. Let p be the minimal period. The points of $\sigma(D_2X(p,x_0))$ are called Floquet multipliers of x. $z = 1$ is always a Floquet multiplier, namely an eigenvalue with eigenvector $\dot{x}_0$. The periodic solution is said to be hyperbolic if $z = 1$ is isolated and algebraically simple, and if there is no other Floquet multiplier on $\underline{S}$.

In case $p \geq 1$, $D_2X(p,x_0)$ is compact.

A heteroclinic solution of eq. (g) from the periodic solution x to a periodic solution $\tilde{x}$ is a solution $h:\underline{R} \to \underline{R}$ such that h_t converges to the periodic orbit $\{x_s: s \in \underline{R}\}$ as $t \to -\infty$, and $h_t \to \{\tilde{x}_s: s \in \underline{R}\}$ as $t \to \infty$.

References in the sequel are made as in the following example.

Lemma $\left\{\begin{matrix}1\\1.1\\I.1.1\end{matrix}\right\}$ means Lemma 1 in $\left\{\begin{matrix}\text{the same section}\\\text{Section 1 of the same chapter}\\\text{Section 1 of Chapter I}\end{matrix}\right\}$.

I. HYPERBOLIC PERIODIC SOLUTIONS

Hypotheses and summary. We consider solutions of equations

(ag_0) $\quad \dot{x}(t) = ag_0(x(t-1))$

with parameter $a > 0$ and with a C^1-function $g_0 : \underline{R} \to \underline{R}$. We assume that g_0 has the following properties.

(H1) $\quad g_0(0) = 0$, $g_0'(0) = 1$. There exists $\gamma > 0$ such that $0 < g_0'$ on $[-\gamma,\gamma]$ and $g_0(\xi) = -g_0(-\xi)$ for $|\xi| \leqq \gamma$

and

(H2) $\quad g_0'$ is strictly decreasing on $[0,\gamma]$.

It is shown that there is a family of periodic solutions x_a, $a \gtrapprox \frac{3\pi}{2} =: \alpha$, bifurcating from zero such that each x_a is hyperbolic with precisely one Floquet multiplier outside $\underline{S}$.

As a link between local and global behavior of solutions, we provide elementary comparison results for solutions not far from x_a.

---

1. PERIODIC SOLUTIONS

We follow an idea of Kaplan and Yorke [13] and employ a result of Nussbaum [20].

There exists $\eta^* < 0$ such that for $\eta^* \leqq \eta < 0$, the solution (x_η, y_η) of the Hamiltonian system

$$\dot{x} = g_0(y) \ , \ \dot{y} = -g_0(x)$$

with $x_\eta(0) = 0$, $y_\eta(0) = \eta$ is periodic with minimal period $4T_\eta$, and $\eta < y_\eta(t) < 0$ in $(0,T_\eta)$, $y_\eta(T_\eta) = 0$, $|x_\eta(t)| \leqq \gamma$ and $|y_\eta(t)| \leqq \gamma$ for all $t \in \underline{R}$. Moreover, $y_\eta = x_\eta(\cdot + T_\eta)$ since g_0 is odd on $[-\gamma,\gamma]$.

(H2) implies that $\xi \to \frac{g_0(\xi)}{\xi}$ is strictly decreasing on $(0,\gamma]$. Therefore Theorem 1.3 [20] applies (One has to interchange x and y, and must formulate Theorem 1.3 [20] for initial values with η negative). We conclude that the map $\eta \to T_\eta$ is strictly decreasing with $T_\eta \downarrow \frac{\pi}{2}$ as $\eta \uparrow 0$.

Let $a \in (\frac{3\pi}{2}, 3T_{\eta^*}] = (\alpha, 3T_{\eta^*}]$. With $\eta \in [\eta^*,0)$ given by $T_\eta = \frac{a}{3} \in (\frac{\pi}{2}, T_{\eta^*}]$, define $x_a(t) := x_\eta(at)$ for all $t \in \underline{R}$. Then

$$\begin{aligned} \dot{x}_a(t) &= a\dot{x}_\eta(at) = ag_0(y_\eta(at)) = ag_0(x_\eta(at + T_\eta)) \\ &= ag_0(x_\eta(at - 3T_\eta)) = ag_0(x_\eta(a(t-1))) \\ &= ag_0(x_a(t-1)) \end{aligned}$$

for all $t \in \underline{R}$.

Observe $x_a(0) = 0$, $\dot{x}_a(0) < 0$, and $0 < x_a$ in $(-\frac{2}{3},0)$, $x_a < 0$ in $[-1,-\frac{2}{3})$, and the symmetry

((s)) $\quad x_a(t) = -x_a(t - \frac{2}{3})$ for all $t \in \underline{R}$

so that x_a has period $\frac{4}{3}$. $\dot{x}_a$ is strictly decreasing on $(-\frac{1}{3},\frac{1}{3})$.

Set $a_1 := 3T_{\eta^*} > \alpha$, and $\phi_a^* := x_{a,0}$ for $\alpha < a < a_1$. The map $a \to \phi_a^*$ into C is continuous, with limit $0 \in C$ as $a \downarrow \alpha$.

2. LINEARIZATION

For the investigation of Floquet multipliers we collect a few properties of the spectrum of the generator of the linearization at 0 of the semiflow given by eq. (αg_0). These properties are also part of the hypotheses for Hopf bifurcation at $a = \alpha$. One can show that the periodic solutions x_a arise in a Hopf bifurcation, but we do not pursue this. One reason is that there seems to be no Hopf bifurcation theorem in the literature which allows to derive all we need about the solutions x_a. - In this respect it might also be noted that C^1-smoothness as above is a very weak assumption on the nonlinearity g_0.

Linearizing at zero for $a = \alpha$, we obtain the C_0-semigroup of operators $\phi \to y_t$, $t \geq 0$, where $y:[-1,\infty) \to \underline{R}$ is the solution of

$$(\alpha) \qquad \dot{y}(t) = \alpha y(t-1)$$

with $y_0 = \phi$. For the spectrum of its generator G we have

PROPOSITION 1. All zeros of $E:\underline{C} \ni z \to z - \alpha e^{-z} \in \underline{C}$ are simple. There is exactly one zero $\xi_\alpha \in \underline{R}$. ξ_α is positive. $E(\frac{3\pi i}{2}) = 0 = E(-\frac{3\pi i}{2})$. For all $z \in E^{-1}(0) \setminus \{\xi_\alpha, \frac{3\pi i}{2}, -\frac{3\pi i}{2}\}$, $\operatorname{Re} z < 0$.

PROOF. $E(z) = 0 = E'(z) = 1 + \alpha e^{-z}$ would imply $z = -1$, a contradiction to $0 = E'(z)$.

Existence of a unique zero $\xi_\alpha \in \underline{R}$ is obvious, as well as $\xi_\alpha > 0$ and $E(\frac{3\pi i}{2}) = 0 = E(-\frac{3\pi i}{2})$.

We show $z = \frac{3\pi i}{2}$ for any zero $z = u + iv$, $u \geq 0$ and $v \geq 0$, with $z \neq \xi_\alpha$: By uniqueness of ξ_α, $v > 0$. Then $0 < v = \alpha e^{-u}\sin(-v) \leq \alpha = \frac{3\pi}{2}$. In case $0 < v < \frac{3\pi}{2}$, $0 \leq u = \alpha e^{-u}\cos v$ yields $0 < v < \frac{\pi}{2}$, or $v = \alpha e^{-u}\sin(-v) < 0$, a contradiction. Hence $v = \frac{3\pi}{2}$.

By $\frac{3\pi}{2} = \frac{3\pi}{2} e^{-u}\cdot 1$, $u = 0$.

Let W denote the operator $\phi \to y_{\frac{2}{3}}$, with the solution y of the initial value problem $\dot{y}(t) = \alpha y(t-1)$ for $t > 0$, $y_0 = \phi$ as above. This map is not compact but the iterate $V_\alpha := W \circ W$, i.e. the $\frac{4}{3}$-translation along solutions of eq. (α), is. It follows that $\sigma(W)$ has the same properties as the spectrum of a compact operator : Each nonzero point z in $\sigma(W)$ is isolated and a pole of the resolvent

$$\underline{C} \setminus \sigma(W) \ni \zeta \to (\tilde{W} - \zeta)^{-1} \in \tilde{C}^c .$$

The generalized eigenspace (i.e., the image of the eigenprojection associated with the spectral set $\{z\}$) coincides with

$$\ker\ (\tilde{W} - z)^n$$

where $n \in \underline{N}$ is the order of the pole. We have

$$\ker\ (\tilde{W} - z)^{k+1} = \ker\ (\tilde{W} - z)^k \quad \text{for} \quad n \leq k \in \underline{N} ,$$

$$\ker\ (\tilde{W} - z)^{k+1} \underset{\neq}{\supset} \ker\ (\tilde{W} - z)^k \underset{\neq}{\supset} \{0\} \quad \text{for} \quad 1 \leq k < n .$$

In particular, z is an eigenvalue with finite algebraic multiplicity

$$\dim \ker\ (\tilde{W} - z)^n .$$

For proofs, see e.g. Theorem 6 in VII.4.5 [4] and Theorem 18 in VII.3 [4].

Using Lemma 4.1 of Section 7.4 [7] we infer

$$\exp(\tfrac{2}{3}\sigma(G)) \subset \sigma(W) \setminus \{0\} \subset \sigma(W) \subset \exp(\tfrac{2}{3}\sigma(G)) \cup \{0\}$$

and for $0 \neq z \in \sigma(W)$ and $k \in \underline{N}$,

$$\ker\ (\tilde{W} - z)^k = \sum_{z' \in \sigma(G):\ z = \exp(\frac{2}{3}z')} \ker\ (\tilde{G} - z')^k .$$

COROLLARY 1. The eigenvalue -1 of W has algebraic multiplicity 2, the eigenvalue $\exp(\frac{2}{3}\xi_\alpha) > 1$ of W has algebraic multiplicity 1. There exists $\varepsilon \in (0,1)$ with $\sigma(W) \setminus \{-1, \exp(\frac{2}{3}\xi_\alpha)\} \subset \underline{C}_{1-\varepsilon}$.

Observe that $\psi_\alpha:[-1,0] \ni t \to \exp(\frac{4}{3}\xi_\alpha t) \in \underline{R}$ is an eigenfunction for the eigenvalue $\exp(\frac{4}{3}\xi_\alpha)$ of V_α. This is most easily seen directly from eq. (α).

3. TRANSLATION BY VARIATIONAL EQUATIONS ALONG PERIODIC SOLUTIONS

For $a \in (\alpha, a_1)$, let W_a and V_a denote the translations along trajectories given by $\phi \to y_{\frac{2}{3}}$ and $\phi \to y_{\frac{4}{3}}$, respectively, where $y:[-1,\infty) \to \underline{R}$ is the solution of the linear variational equation of x_a

(a) $\dot{y}(t) = a g_0'(x_a(t-1))y(t-1)$

with $y_0 = \phi \in C$. We are interested in the Floquet multipliers, i.e., in $\sigma(V_a)$. $\dot{x}_{a,0} := \dot{x}_a|[-1,0] \in C$ is an eigenvector for the eigenvalue 1 of V_a. The symmetry ((s)) from Section 1 implies that $\dot{x}_{a,0}$ is also an eigenvector for the eigenvalue -1 of W_a, and moreover

(1) $V_a = W_a \circ W_a$.

V_a is compact. As in Section 2 we infer that both sets $\sigma(W_a) \setminus \{0\}$, $\sigma(V_a) \setminus \{0\}$ consist of isolated points which are all eigenvalues with finite algebraic multiplicities. The eigenvalues are real or occur in complex conjugate pairs.

With the Spectral Mapping Theorem in mind, we first study $\sigma(W_a)$. For $0 \neq z \in \sigma(W_a)$, let $m_a(z)$ denote the algabraic multiplicity. For z in $\underline{C} \setminus \sigma(W_a)$, $m_a(z) := 0$.

Uniform convergence of x_a to 0 on $[-1,\frac{2}{3}]$ as $a \downarrow \alpha$ implies

(2) $W_a \to W$ in C^C as $a \downarrow \alpha$.

Results on continuous dependence of spectra (Chapter IV § 3 no 5 [14]) now yield

COROLLARY 1. There exist $a_2 \in (\alpha, a_1)$ and $\varepsilon \in (0,1)$ with the following properties.

(i) $\text{cl}\, \underline{C}_1 \cap (\exp(\frac{2}{3}\xi_\alpha) + \underline{C}_\varepsilon) = \emptyset = (-1 + \underline{C}_\varepsilon) \cap (\exp(\frac{2}{3}\xi_\alpha) + \underline{C}_\varepsilon)$.

(ii) Let $a \in (\alpha, a_2)$. $\exp(\frac{2}{3}\xi_\alpha) + \underline{C}_\varepsilon$ contains precisely one point $\xi_a \in \sigma(W_a)$. We have $\xi_a > 1$ and $m_a(\xi_a) = 1$. Either $\sigma(W_a) \cap (-1 + \underline{C}_\varepsilon) = \{-1\}$ with $m_a(-1) = 2$, or $m_a(-1) = 1$ and $(-1 + \underline{C}_\varepsilon) \setminus \{-1\}$ contains precisely one point $\xi_a^* \in \sigma(W_a)$. In the last case, $\xi_a^* \in \underline{R}$ and $m_a(\xi_a^*) = 1$. For all z in $\sigma(W_a) \setminus ((-1 + \underline{C}_\varepsilon) \cup (\exp(\frac{2}{3}\xi_\alpha) + \underline{C}_\varepsilon))$, $|z| < 1 - \varepsilon$.

(iii) $\xi_a \to \exp(\frac{2}{3}\xi_\alpha)$ as $a \downarrow \alpha$.

Our aim is to exclude $m_a(-1) = 2$ for each $a > \alpha$ sufficiently close to α, and to show $-1 < \xi_a^* < -1 + \varepsilon$ for the other eigenvalue in $-1 + \underline{C}_\varepsilon$. This will be achieved in Corollary 5.1 below.

4. EIGENVALUES AS ZEROS OF ANALYTIC FUNCTIONS

Let $a \in (\alpha, a_2)$ in this section.

We proceed as in [31] and introduce a function q_a whose zeros coincide with $\sigma(W_a) \setminus \{0\}$. The basic idea for this is as follows. Suppose $z \neq 0$ is an eigenvalue of W_a with eigenvector ϕ. Let w denote the corresponding solution of the variational equation along x_a. Define functions on $[-\frac{1}{3}, 0]$ by

$$\phi_1 := \phi(-\tfrac{2}{3} + \cdot) \; , \; \phi_2 := \phi(-\tfrac{1}{3} + \cdot) \; , \; \phi_3 := \phi(\cdot) \; ,$$

$$w^* := w(\tfrac{1}{3} + \cdot) \; , \; w^{**} := w(\tfrac{2}{3} + \cdot) \; .$$

In terms of these functions, $w_{\frac{2}{3}} = W_a\phi = z\phi$ reads

$$\phi_3 = z\phi_1 \; , \; w^* = z\phi_2 \; , \; w^{**} = z\phi_3 \; .$$

By the variational equation,

$$\begin{aligned} \dot{w}^{**}(\cdot) &= \dot{w}(\tfrac{2}{3} + \cdot) = ag_0'(x_a(\tfrac{2}{3} + \cdot - 1))w(\tfrac{2}{3} + \cdot - 1) \\ &= ag_0'(x_a(-\tfrac{1}{3} + \cdot))\phi_2(\cdot) = \frac{1}{z}\, ag_0'(\ldots)w^*(\cdot) \; , \\ \dot{w}^*(\cdot) &= \dot{w}(\tfrac{1}{3} + \cdot) = ag_0'(x_a(\tfrac{1}{3} + \cdot - 1))w(\tfrac{1}{3} + \cdot - 1) \\ &= ag_0'(x_a(-\tfrac{2}{3} + \cdot)\phi_1(\cdot) = ag_0'(\ldots)\,\frac{1}{z}\,\phi_3(\cdot) = \frac{1}{z^2}\, ag_0'(\ldots)w^{**}(\cdot) \end{aligned}$$

The continuity of w implies boundary conditions

$$w^*(-\tfrac{1}{3}) = \phi_3(0) = \frac{1}{z}\, w^{**}(0) \quad \text{and} \quad w^*(0) = w^{**}(-\tfrac{1}{3}) \; ,$$

or

$$\begin{pmatrix} w^{**} \\ w^* \end{pmatrix}(0) = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} w^{**} \\ w^* \end{pmatrix}(-\tfrac{1}{3})$$

The left hand side is the product of the vector on the right hand side with the fundamental matrix solution of the above system, which equals the identity at $t = -\frac{1}{3}$. So, if $w^{**}(-\frac{1}{3})$ and $w^*(-\frac{1}{3})$ are not both zero a determinant must vanish at z.

It is now clear how to define the function q_a:

For $t \in \underline{R}$, set

$$A_a(t) := ag_0'(x_a(t - \tfrac{1}{3})) \; , \; B_a(t) := ag_0'(x_a(t - \tfrac{2}{3})) = ag_0'(x_a(t)) \; .$$

For $0 \neq z \in \underline{C}$, let $S_{z,a}$ denote the fundamental matrix solution

$$\begin{pmatrix} u_1^{z,a} & u_2^{z,a} \\ v_1^{z,a} & v_2^{z,a} \end{pmatrix} : \underline{R} \to \underline{C}^{2\times 2} \quad \text{of the linear system}$$

$$\dot{u} = \frac{1}{z} A_a(t)v \; , \; \dot{v} = \frac{1}{z^2} B_a(t)u$$

which is equal to the identity matrix at $t = -\frac{1}{3}$. Set

$$Q_{z,a} := S_{z,a}(0) - \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u_1^{z,a}(0) & u_2^{z,a}(0) - z \\ v_1^{z,a}(0) - 1 & v_2^{z,a}(0) \end{pmatrix}$$

and $q_a(z) := \det Q_{z,a}$. The function $q_a : \underline{C} \setminus \{0\} \to \underline{C}$ is analytic (see e.g. Section X.7 [3]). From $\det S_{z,a}(t) = 1$ on $\underline{R}$, we infer

$$q_a(z) = 1 - z + u_2^{z,a}(0) + z \cdot v_1^{z,a}(0) \quad \text{for all} \quad z \in \underline{C} \setminus \{0\} \; .$$

In order to establish relations between the zeros of q_a and eigenvalues of W_a we need a description of the operators $\tilde{W}_a - z$, $0 \neq z \in \underline{C}$, in terms of solutions of inhomogeneous boundary value problems.

It is convenient to introduce the Banach space C' of triples (ϕ_1, ϕ_2, ϕ_3) of continuous functions $[-\frac{1}{3}, 0] \to \underline{C}$ satisfying

$$\phi_1(0) = \phi_2(-\tfrac{1}{3}) \; , \; \phi_2(0) = \phi_3(-\tfrac{1}{3}) \; ,$$

with norm given by $|(\phi_1, \phi_2, \phi_3)| = \max\limits_j \max\limits_t |\phi_j(t)|$.

Let J denote the norm-preserving isomorphism from $\tilde{C}$ onto C' defined by $J\phi = (\phi_1, \phi_2, \phi_3)$, with the restrictions ϕ_1, ϕ_2, ϕ_3 of $\phi(-\frac{2}{3} + \cdot)$, $\phi(-\frac{1}{3} + \cdot)$, ϕ to the interval $[-\frac{1}{3}, 0]$.

Let $0 \neq z \in \underline{C}$, $\chi \in \tilde{C}$, $\phi \in \tilde{C}$. One sees, as at the beginning of the section, that $(\tilde{W}_a - z)\chi = \phi$ implies

$$(1) \qquad \begin{aligned} \dot{u} &= \frac{1}{z} A_a(t)\,[v - \phi_2] \\ \dot{v} &= \frac{1}{z^2} B_a(t)\;[u - \phi_3] - \frac{1}{z} B_a(t)\phi_1 \end{aligned}$$

where $u := (\tilde{W}_a\chi)_3$, $v := (\tilde{W}_a\chi)_2$ and

$$\begin{aligned} v(0) &= u(-\tfrac{1}{3}) \; , \\ u(0) &= \tilde{W}_a\chi(0) = \phi(0) + z\chi(0) = \phi_3(0) + z\chi_3(0) = \phi_3(0) + z \cdot v(-\tfrac{1}{3}) \; . \end{aligned}$$

By variation of constants,

$$\begin{pmatrix} u \\ v \end{pmatrix} = S_{z,a}\left[\begin{pmatrix} u \\ v \end{pmatrix}(-\tfrac{1}{3}) + \int_{-\frac{1}{3}}^{\cdot} S_{z,a}^{-1} \cdot \begin{pmatrix} -\frac{A_a}{z} \cdot \phi_2 \\ \frac{B_a}{z^2} \cdot \phi_3 - \frac{B_a}{z} \cdot \phi_1 \end{pmatrix}\right]$$

Let $I(z, a, \phi, t)$ denote the last integral at $t \in [-\frac{1}{3}, 0]$, with components $I_1(\ldots)$ and $I_2(\ldots)$.

Set $c := \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} := \begin{pmatrix} u \\ v \end{pmatrix}(-\frac{1}{3})$. Then $\begin{pmatrix} \phi_3(0) + z\cdot c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} \phi_3(0) + z\cdot v(-\frac{1}{3}) \\ u(-\frac{1}{3}) \end{pmatrix}$

$= \begin{pmatrix} u \\ v \end{pmatrix}(0) = S_{z,a}(0)c + S_{z,a}(0)\cdot I(z,a,\phi,0)$, or

$$-Q_{z,a}\cdot c = -[S_{z,a}(0) - \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}]\cdot c = L_{z,a}\ \phi$$

with the linear continuous map

$$L_{z,a}:\tilde{C} \ni \phi \to \begin{pmatrix} -\phi_3(0) \\ 0 \end{pmatrix} + S_{z,a}(0)\cdot I(z,a,\phi,0) \in \underline{C}^2 .$$

We have shown the first part of

LEMMA 1. Let $0 \neq z \in \underline{C}$, $\phi \in \tilde{C}$. If $\chi \in \tilde{C}$ and $(\tilde{W}_a - z)\chi = \phi$ then (u,v) defined by $J\tilde{W}_a\chi = (\chi_3,v,u)$ satisfies eq. (1) for all $t \in [-\frac{1}{3},0]$ and $-Q_{z,a}\cdot\begin{pmatrix} u \\ v \end{pmatrix}(-\frac{1}{3}) = L_{z,a}\ \phi$.

Conversely, if $c \in \underline{C}^2$ and

(*) $\quad -Q_{z,a}\cdot c = L_{z,a}\ \phi$,

and if (u,v) is the restriction of the solution of eq. (1) with $\begin{pmatrix} u \\ v \end{pmatrix}(-\frac{1}{3}) = c$ to the interval $[-\frac{1}{3},0]$, then

$$\frac{1}{z}(\frac{1}{z}(u - \phi_3) - \phi_1,\ v - \phi_2,\ u - \phi_3) \in C' ,$$

and $\chi := J^{-1}(\frac{1}{z}(\ldots)) \in \tilde{C}$ satisfies $(\tilde{W}_a - z)\chi = \phi$.

In case $\phi = 0$ and $c \neq 0$, $\chi \neq 0$.

PROOF of the second part. Let $c \neq 0$ with (*) be given.

1. For $t \in [-\frac{1}{3},0]$, set

$$\begin{pmatrix} u \\ v \end{pmatrix}(t) := S_{z,a}(t)[c + I(z,a,\phi,t)]$$

and $\chi_3 := \frac{1}{z}(u - \phi_3)$, $\chi_2 := \frac{1}{z}(v - \phi_2)$, $\chi_1 := \frac{1}{z}(\chi_3 - \phi_1)$. Then

$$\chi_3 - z\chi_1 = \phi_1 ,\quad v - z\chi_2 = \phi_2 ,\quad u - z\chi_3 = \phi_3 .$$

2. We have $(\chi_3,v,u) = J\tilde{\chi}$ with $\tilde{\chi} \in \tilde{C}$. Proof: By definition of u and v , $u(-\frac{1}{3}) = c_1$ and $v(-\frac{1}{3}) = c_2$. By (*) and by definition of u ,

$$zc_2 = c_1u_1^{z,a}(0) + c_2u_2^{z,a}(0) + u_1^{z,a}(0)\cdot I_1(z,a,\phi,0) + u_2^{z,a}(0)\cdot I_2(\ldots)$$

$$-\phi_3(0) = u(0) - \phi_3(0) .$$

Similarly, $c_1 = \ldots = v(0)$. Together, $v(0) = u(-\frac{1}{3})$, and

$$v(-\tfrac{1}{3}) = \tfrac{1}{z}(u(0) - \phi_3(0)) = \chi_3(0) .$$

Recall the definition of J .

3. By linearity, $(\chi_1,\chi_2,\chi_3) = J\chi$ with $\chi \in \tilde{C}$.

4. In order to verify $(\tilde{W}_a - z)\chi = \phi$, or equivalently

$$J(\tilde{W}_a - z)\chi = (\phi_1,\phi_2,\phi_3) ,$$

note first $(\tilde{W}_a\chi)_1 = \chi_3$ so that $((\tilde{W}_a - z)\chi)_1 = \chi_3 - z\chi_1 = \phi_1$.

Next, $(\tilde{W}_a\chi)_2 = w(\frac{1}{3} + \cdot)|[-\frac{1}{3},0]$ where $w:[-1,\infty) \to \underline{C}$ is the solution of

$$\dot{w}(t) = ag_0'(x_a(t-1))w(t-1) ,\quad w_0 = \chi .$$

On $(-\frac{1}{3},0]$ we obtain

$$\dot{w}(\tfrac{1}{3} + \cdot) = B_a\chi_1 = \tfrac{1}{z} B_a z\chi_1 = \tfrac{1}{z} B_a[\chi_3 - \phi_1] = \tfrac{1}{z^2} B_a[u - \phi_3] - \tfrac{1}{z}B_a\phi_1$$
$$= \dot{v} .$$

With $w(\frac{1}{3} - \frac{1}{3}) = w(0) = \chi_3(0) = v(-\frac{1}{3})$, $w(\frac{1}{3} + \cdot) = v = z\chi_2 + \phi_2$ on the interval $[-\frac{1}{3},0]$, or $(\tilde{W}_a\chi)_2 - z\chi_2 = \phi_2$.

Finally, $(\tilde{W}_a\chi)_3 = w(\frac{2}{3} + \cdot)|[-\frac{1}{3},0]$, and on the interval $[-\frac{1}{3},0]$,

$$\dot{w}(\tfrac{2}{3} + \cdot) = A_a\chi_2 = \tfrac{1}{z} A_a[\phi_2 - z\chi_2] - \tfrac{1}{z} A_a\phi_2 = \tfrac{1}{z} A_a v - \tfrac{1}{z} A_a\phi_2 = \dot{u} .$$

With $u(-\frac{1}{3}) = v(0) = w(\frac{1}{3}) = w(\frac{2}{3} - \frac{1}{3})$, we get

$$u = w(\tfrac{2}{3} + \cdot)|[-\tfrac{1}{3},0] ,$$

and therefore $(\tilde{W}_a\chi)_3 = u = z\chi_3 + \phi_3$.

5. The last assertion in the lemma is obvious.

COROLLARY 1. Let $0 \neq z \in \underline{C}$. Then $z \in \sigma(W_a)$ if and only if $q_a(z) = 0$.

PROOF. Each nonzero $z \in \sigma(W_a)$ is an eigenvalue. We saw at the begin of the section that for eigenvalues $z \neq 0$, $q_a(z) = 0$. Conversely, if $q_a(z) = 0$ then there exists $c \in \underline{C}^2 \setminus \{0\}$ with $-Q_{z,a}c = 0$. By Lemma 1, with $\phi = 0$, there is $\chi \in \tilde{C} \setminus \{0\}$ with $(\tilde{W}_a - z)\chi = 0$. Hence z is in $\sigma(W_a)$.

The next objective is to show that algebraic multiplicities coincide with the order $j_a(z)$ of $z \in \underline{C} \setminus \{0\}$, considered as a zero of the analytic function q_a . We follow [31]. Details are a bit more involved.

PROPOSITION 1. The operators $L_{z,a}$, $0 \neq z \in \underline{C}$, are surjective.

PROOF. 1. There is a sequence of points $\phi^{(n)} \in \tilde{C}$ such that $\phi^{(n)} = 1$ for all $n \in \underline{N}$ and $L_{z,a}\phi^{(n)} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as $n \to \infty$. So, vectors close to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are contained in $L_{z,a}\tilde{C}$. We look for linearly independent vectors in $L_{z,a}\tilde{C}$.

2. There exists $\phi \in \tilde{C}$ with $\phi_1 = 0 = \phi_3$ such that the second component of

$$c := L_{z,a}\phi = S_{z,a}(0) \int_{-\frac{1}{3}}^{0} \frac{1}{z} A_a \phi_2 \begin{pmatrix} -v_2^{z,a} \\ v_1^{z,a} \end{pmatrix}$$

does not vanish. Proof: For ψ with $\psi_1 = 0 = \psi_3$,

$$z\,(L_{z,a}\psi)_2 = v_1^{z,a}(0) \int_{-\frac{1}{3}}^{0} A_a \psi_2 (-v_2^{z,a}) + v_2^{z,a}(0) \int_{-\frac{1}{3}}^{0} A_a \psi_2 v_1^{z,a}$$

In case $v_2^{z,a}(0) = 0$, $\det S_{z,a}(0) = 1$ gives $v_1^{z,a}(0) \neq 0$. $A_a > 0$ and $v_2^{z,a}(-\frac{1}{3}) = 1$ allow to find ϕ with $\phi_1 = 0 = \phi_3$ such that $(L_{z,a}\phi)_2 \neq 0$.

In case $v_1^{z,a}(0) = 0$, $v_2^{z,a}(0) \neq 0$. Now $v_1^{z,a}(-\frac{1}{3}) = 0$. But

$$\dot{v}_1^{z,a}(-\tfrac{1}{3}) = \frac{1}{z^2} B_a(-\tfrac{1}{3})\, u_1^{z,a}(-\tfrac{1}{3}) = \frac{1}{z^2} B_a(-\tfrac{1}{3}) \neq 0$$

and $B_a > 0$ allow to find ϕ as desired also in this case.

If $v_1^{z,a}(0) \neq 0 \neq v_2^{z,a}(0)$, choose a sequence of continuous functions $\omega_n : [-\frac{1}{3}, 0] \to \underline{C}$ with $\omega_n(-\frac{1}{3}) = 0 = \omega_n(0)$ for all $n \in \underline{N}$ such that

$$\int_{-\frac{1}{3}}^{0} A_a \omega_n (-v_2^{z,a}) \to -A_a(-\tfrac{1}{3})\, v_2^{z,a}(-\tfrac{1}{3}) = 0 ,$$

$$\int_{-\frac{1}{3}}^{0} A_a \omega_n v_1^{z,a} \to A_a(-\tfrac{1}{3})\, v_1^{z,a}(-\tfrac{1}{3}) = A_a(-\tfrac{1}{3}) > 0$$

as $n \to \infty$. For n sufficiently large and for ϕ with $\phi_2 = \omega_n$ and $\phi_1 = 0 = \phi_3$, $(L_{z,a}\phi)_2 \neq 0$.

3. Consider $\phi \in \tilde{C}$ and c as in part 2. There is a bounded sequence of points $\psi^{(n)} \in \tilde{C}$ with $\psi_1^{(n)} = 0$, $\psi_2^{(n)} = \phi_2$, $\psi_3^{(n)}(0) = c_1$ and $\psi_3^{(n)} | [-\frac{1}{3}, -\frac{1}{n}] = 0$ for all $n \in \underline{N}$. It follows that for $n \to \infty$,

$$L_{z,a}\psi^{(n)} \to \begin{pmatrix} -c_1 \\ 0 \end{pmatrix} + L_{z,a}\phi = \begin{pmatrix} 0 \\ c_2 \end{pmatrix} \neq 0 .$$

REMARK. Lemma 1 and Proposition 1 are the analogue of Lemma 3.1 from [31]. The operator $L_{z,a}$ corresponds to $L(\lambda)$ in [31]. In [31] we did not explicitly show that $L(\lambda)$ is surjective. Let us mention that a proof of this requires that the function $[0,1] \ni t \to g'(x(t)) \in \underline{R}$ (which corresponds to A_a or B_a here) has at most finitely many zeros. This property is satisfied for the functions g in Sections 4-6 of [31] but was forgotten as an extra hypothesis in Section 3 of [31].

For $0 \neq z \in \underline{C}$, set

$$Q^*_{z,a} := \begin{pmatrix} v_2^{z,a}(0) & z - u_2^{z,a}(0) \\ 1 - v_1^{z,a}(0) & u_1^{z,a}(0) \end{pmatrix} .$$

PROPOSITION 2. The analytic map $z \to q_a(z)(\tilde{W}_a - z)^{-1}$ from $\underline{C} \setminus (\sigma(W_a) \cup \{0\})$ into $\tilde{C}^C$ has a continuous extension H_a to $\underline{C} \setminus \{0\}$ with

$$\begin{pmatrix} (H_a(z)\phi)_3 \\ (H_a(z)\phi)_2 \end{pmatrix}(t) = \frac{1}{z} S_{z,a}(t)(-Q^*_{z,a} L_{z,a}\phi)$$

for $t \in [-\frac{1}{3}, 0]$, $0 \neq z \in \sigma(W_a)$, $\phi \in \tilde{C}$.

PROOF. 1. For $\phi \in \tilde{C}$ and $z \in \underline{C}$ with $0 \neq z \notin \sigma(W_a)$, set $\chi_{z,\phi} := (\tilde{W}_a - z)^{-1}\phi$. Then $\chi_{z,\phi} = \frac{1}{z}(\tilde{W}_a \chi_{z,\phi} - \phi)$. It remains to show that the map $H^* : \underline{C} \setminus (\sigma(W_a) \cup \{0\}) \to \tilde{C}^C$, $H^*(z)\phi = q_a(z)\tilde{W}_a \chi_{z,\phi}$, has a continuous extension H' to all of $\underline{C} \setminus \{0\}$.

Consider first the maps H^*_i, $i = 1,2,3$, from $\underline{C} \setminus (\ldots)$ to the complex Banach space of linear continuous maps $\tilde{C} \to C([-\frac{1}{3}, 0], \underline{C})$ which are obtained from the application of J to $H^*(z)\phi$:

$$(H^*_1(z)\phi, H^*_2(z)\phi, H^*_3(z)\phi) = JH^*(z)\phi \in J\tilde{C}$$

for all $z \in \underline{C} \setminus (\ldots)$, $\phi \in \tilde{C}$.

2. Let $z \in \underline{C}$, $0 \neq z \notin \sigma(W_a)$, $\phi \in \tilde{C}$. By Corollary 1, $q_a(z) \neq 0$. We compute $\frac{1}{q_a(z)} H^*_i(z)\phi = (\tilde{W}_a \chi_{z,\phi})_i$ for $i = 1,2,3$:

$$(\tilde{W}_a \chi_{z,\phi})_1 = (\chi_{z,\phi})_3 = \frac{1}{z}((\tilde{W}_a \chi_{z,\phi})_3 - \phi_3) ,$$

and from Lemma 1,

$$\begin{pmatrix} (W_a \chi_{z,\phi})_3 \\ (W_a \chi_{z,\phi})_2 \end{pmatrix} = S_{z,a}[c + I(z,a,\phi,\cdot)]$$

on the interval $[-\frac{1}{3}, 0]$, where $-Q_{z,a}c = L_{z,a}\phi$, or equivalently,

$$c = \frac{-1}{q_a(z)} Q^*_{z,a} L_{z,a}\phi .$$

3. The maps $0 \neq z \to u_i^{z,a}|[-\frac{1}{3},0]$ and $0 \neq z \to v_i^{z,a}|[-\frac{1}{3},0]$, $i = 1,2$, are continuous with respect to uniform convergence on the interval $[-\frac{1}{3},0]$. From the computation above we infer that H^*_2 and H^*_3 , and therefore also the map

$$H^*_1 : z \to H^*_1(z) = \frac{1}{z} H^*_3(z) - \frac{q_a(z)}{z} J_3 ,$$

where $J_3\phi = \phi|[-\frac{1}{3},0]$ for $\phi \in \tilde{C}$,

have continuous extensions H'_i to $\underline{C} \setminus \{0\}$, $i = 1,2,3$.

For example, if $z \neq 0$, if $\phi \in \tilde{C}$ and $t \in [-\frac{1}{3},0]$ then $(H'_3(z)\phi)(t)$ equals the first component of

$$-S_{z,a}(t)Q^*_{z,a}\phi + q_a(z)S_{z,a}(t)I(z,a,\phi,t) .$$

4. For all $z \in \sigma(W_a) \setminus \{0\}$ and all $\phi \in \tilde{C}$,

$$(H'_1(z)\phi, H'_2(z)\phi, H'_3(z)\phi) = \lim_{\zeta \to z} (H^*_1(\zeta)\phi, H^*_2(\zeta)\phi, H^*_3(\zeta)\phi) \in J\tilde{C} .$$

Set $H'(z)\phi := J^{-1}(H'_1(z)\phi, H'_2(z)\phi, H'_3(z)\phi)$, for $0 \neq z \in \underline{C}$ and $\phi \in \tilde{C}$.

5. In case $0 \neq z \in \sigma(W_a)$, $q_a(z) = 0$. This implies the formula for the third and second component of $JH_a(z)\phi$, $\phi \in \tilde{C}$.

COROLLARY 2. Let $0 \neq z \in \sigma(W_a)$, $Q_{z,a} \neq 0$. Then $H_a(z) \neq 0 \in \tilde{C}^{\tilde{C}}$.

PROOF. $Q_{z,a} \neq 0$ yields $Q^*_{z,a} \neq 0$, or $Q^*_{z,a}c \neq 0$ for some $c \in \underline{C}^2$. By Proposition 1, $c = L_{z,a}\phi$ for some $\phi \in \tilde{C}$. Apply the last formula in Proposition 2.

COROLLARY 3. Let $0 \neq z \in \sigma(W_a)$, $Q_{z,a} \neq 0$. Then the geometric eigenspace $\ker(\tilde{W}_a - z)$ has dimension 1, and $m_a(z) = j_a(z)$.

PROOF. 1. Proof of $\dim \ker(\tilde{W}_a - z) = 1$: Recall Lemma 1. Consider the map μ which sends $\chi \in \ker(\tilde{W}_a - z)$ to $\binom{u}{v}(-\frac{1}{3})$ where $J\tilde{W}_a\chi = (\chi_3, v, u)$. We have $\mu(\ker(\ldots)) \subset \{c \in \underline{C}^2 : Q_{z,a}c = 0\}$. μ is injective (Proof: $\mu\chi = 0$ and (1) with $\phi = 0$ give $u = 0 = v$, hence $\chi_3 = \frac{u}{z} = 0$ and $\chi_2 = \frac{v}{z} = 0$, $\chi_1 = \frac{1}{z}\chi_3 = 0$, and $\chi = 0$.).

By $0 \neq z \in \sigma(W_a)$ and $Q_{z,a} \neq 0$,

$$1 \leq \dim \ker(\tilde{W}_a - z) = \dim \mu(\ker(\ldots)) \leq \dim \{\ldots\} = 1 .$$

2. We have $m_a(z) = \dim \ker(\tilde{W}_a - z)^n$ where n is the minimum of integers $k \in \underline{N}$ with $\ker(\tilde{W}_a - z)^k = \ker(\tilde{W}_a - z)^{k+1}$.

By part 1, $n = \Sigma_1^n \dim \ker(\tilde{W}_a - z) \geq \dim \ker(\tilde{W}_a - z)^n = m_a(z)$.

Using also

$$\dim \ker(\tilde{W}_a - z) < \dim \ker(\tilde{W}_a - z)^2 < \ldots < \dim \ker(\tilde{W}_a - z)^n$$
$$= m_a(z)$$

(provided $1 < 2 < \ldots < n$) we infer $m_a(z) = n$.

3. n is equal to the order of the pole of the resolvent of $\tilde{W}_a$ at z . It follows that $n = \min \{k \in \underline{N} : \zeta \to (\zeta - z)^k(\tilde{W}_a - \zeta)^{-1}$ has a continuous extension to $(\underline{C} \setminus \sigma(W_a)) \cup \{z\}\}$. By definition of $j_a(z) =: j$, $q_a(\zeta) = (\zeta - z)^j h(\zeta)$ for ζ in a neighborhood of z , with h analytic and $h(z) \neq 0$. Use Proposition 2 and Corollary 2 to deduce $j = n$, and $j_a(z) = m_a(z)$.

PROPOSITION 3. For $-1 \neq z \neq 0$, $Q_{z,a} \neq 0$.

PROOF. $Q_{z,a} = 0$ implies $u_2^{z,a}(0) = z$ and $v_1^{z,a}(0) = 1$. With $0 = \det Q_{z,a} = q_a(z)$, we find $z = -1$.

5. ZEROS OF q_a

PROPOSITION 1. For all $a \in (\alpha, a_2)$, $\liminf_{\underline{R} \ni z \to -\infty} q_a(z) > 0$.

PROOF. For $\underline{R} \ni z \to -\infty$, the columns of $S_{z,a}(0)$ converge to the values at $t = 0$ of the solutions of $\dot{u} = 0 = \dot{v}$ with initial values $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$ at $t = -\frac{1}{3}$. It follows that $\frac{1}{z} q_a(z) \to -1$.

From Corollaries 3.1 and 4.1, $q_a(-1) = 0$ for $\alpha < a < a_2$. We want to show $m_a(-1) = 1$ and $-1 < \xi_a^*$ for the other eigenvalue of W_a close to -1 (recall the end of Section 3). $m_a(-1) = 1$ will follow from Corollary 4.3 provided we have $Q_{-1,a} \neq 0$ and $q_a'(-1) \neq 0$. The definition of $Q_{-1,a}$ and the subsequent computation show that

$$u_1^{-1,a}(0) > 0 \tag{1}$$

is sufficient for both inequalities.

PROPOSITION 2. Let $a \in (\alpha, a_2)$. Then $v_2^{-1,a}(0) = 0$ and

$$q_a'(-1) = -u_1^{-1,a}(0) \int_{-\frac{1}{3}}^{0} \{A_a[v_2^{-1,a}]^2 + 2B_a[u_2^{-1,a}]^2\} .$$

PROOF. 1. For $0 \neq z \in \underline{C}$,

$$q_a'(z) = -1 + \bar{u}_2^z(0) + v_1^{z,a}(0) + z \cdot \bar{v}_1^z(0) \tag{2}$$

with the solutions $(\bar{u}_i^z, \bar{v}_i^z) : \underline{R} \to \underline{C}^2$, $i = 1,2$, of the systems

$$\begin{aligned} \dot{\bar{u}} &= \frac{1}{z} A_a(t)\bar{v} - \frac{1}{z^2} A_a(t)\, v_i^{z,a}(t) \\ \dot{\bar{v}} &= \frac{1}{z^2} B_a(t)\bar{u} - \frac{2}{z^3} B_a(t)\, u_i^{z,a}(t) \end{aligned} \qquad i = 1,2$$

with initial conditions $0 \in \underline{C}^2$ at $t = -\frac{1}{3}$.

2. Using variations of constants and $\det S_{z,a}(t) = 1$ on $\underline{R}$, we get

$$\begin{pmatrix}\bar{u}_i^z\\ \bar{v}_i^z\end{pmatrix}(0) = S_{z,a}(0) \int_{-\frac{1}{3}}^{0} \begin{pmatrix} v_2^{z,a} & -u_2^{z,a} \\ -v_1^{z,a} & u_1^{z,a} \end{pmatrix} \begin{pmatrix} -\frac{1}{z^2} A_a v_i^{z,a} \\ -\frac{2}{z^3} B_a u_i^{z,a} \end{pmatrix}$$

$$= \int_{-\frac{1}{3}}^{0} \left(\begin{array}{l} u_1^{z,a}(0)\, [-\frac{1}{z^2} A_a v_2^{z,a} v_i^{z,a} + \frac{2}{z^3} B_a u_2^{z,a} u_i^{z,a}] \quad + \\ v_1^{z,a}(0)\, [\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots] \quad + \\ u_2^{z,a}(0)\, [\frac{1}{z^2} A_a v_1^{z,a} v_i^{z,a} - \frac{2}{z^3} B_a u_1^{z,a} u_i^{z,a}] \\ v_2^{z,a}(0)\, [\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots] \end{array} \right)$$

for $i = 1,2$. With (2), we obtain

$$\begin{aligned} q_a'(-1) = -1 + \Big(-u_1^{-1,a}(0) & \int_{-\frac{1}{3}}^{0} \{A_a[v_2^{-1,a}]^2 + 2B_a[u_2^{-1,a}]^2\} \\ & - \int_{-\frac{1}{3}}^{0} \{A_a v_1^{-1,a} v_2^{-1,a} + 2B_a u_1^{-1,a} u_2^{-1,a}\} \Big) + v_1^{-1,a}(0) \\ & + (-1)\Big(-\int_{-\frac{1}{3}}^{0} \{A_a v_1^{-1,a} v_2^{-1,a} + 2B_a u_1^{-1,a} u_2^{-1,a}\} \Big) . \end{aligned}$$

3. Proof of $v_2^{-1,a}(0) = 0$ and $v_1^{-1,a}(0) = 1$: Recall $(W_a + 1)\dot{x}_{a,0} = 0$ and $\dot{x}_a(0) < 0$, $\dot{x}_a(\frac{1}{3}) = 0$. – Lemma 4.1 implies that $\begin{pmatrix}u\\v\end{pmatrix}$ with $J\tilde{W}_a\dot{x}_{a,0} = (\ldots, v, u)$ is a solution of eq. (4.1) with $z = -1$ and $\phi_1 = \phi_2 = \phi_3 = 0$, and we have $u(-\frac{1}{3}) = \dot{x}_a(\frac{1}{3}) = 0$, $v(-\frac{1}{3}) = \dot{x}_a(0) < 0$. Hence

$$\begin{pmatrix}u\\v\end{pmatrix} = \dot{x}_a(0)\begin{pmatrix}u_2^{-1,a}\\v_2^{-1,a}\end{pmatrix} .$$

With $v(0) = \dot{x}_a(-\frac{1}{3})$ and $u(0) = \dot{x}_a(\frac{2}{3}) = -\dot{x}_a(0)$, $S_{-1,a}(0) = \begin{pmatrix} * & -1 \\ * & 0 \end{pmatrix}$ so that $\det S_{-1,a}(0) = 1$ yields $v_1^{-1,a}(0) = 1$.

In order to derive (1) we introduce polar coordinates. The columns of $S_{-1,a}$, $\alpha < a < a_2$, i.e. the solutions of

$$\dot{u} = -A_a(t)v \ , \ \dot{v} = B_a(t)u$$

with initial conditions $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ at $t = -\frac{1}{3}$, are given by

$$(u_i^{-1,a}, v_i^{-1,a}) = r_i^a (\cos\theta_i^a, \sin\theta_i^a) \ , \ i = 1,2 \ ,$$

with the solutions $(r_i^a, \theta_i^a) : \underline{R} \to \underline{R}^2$ of the nonlinear, decoupled system

$$\begin{aligned} \dot{r} &= r\,[B_a(t) - A_a(t)] \sin\theta \cdot \cos\theta \\ \dot{\theta} &= A_a(t) \sin^2\theta + B_a(t) \cos^2\theta = A_a(t) + [B_a(t) - A_a(t)] \cos^2\theta \end{aligned} \tag{3}$$

such that $r_1^a(-\frac{1}{3}) = 1 = r_2^a(-\frac{1}{3})$, $\theta_1^a(-\frac{1}{3}) = 0$, $\theta_2^a(-\frac{1}{3}) = \frac{\pi}{2}$. Therefore it is enough to show

PROPOSITION 3. There exists $a_3 \in (\alpha, a_2)$ with

$$0 \leqq \theta_1^a(0) < \frac{\pi}{2}$$

for $\alpha < a < a_3$.

PROOF. 1. There exists $a_3 \in (\alpha, a_2)$ such that $\theta_2^a(0) = \pi$ for a in (α, a_3) . Proof: $v_2^{-1,a}(0) = 0$ for $\alpha < a < a_2$ gives $\theta_2^a(0) \in \underline{Z}\pi$, for such a . Observe that for $a \downarrow \alpha = \frac{3\pi}{2}$, the positive coefficients A_a and B_a in eq. (3) tend to $\frac{3\pi}{2}$ uniformly with respect to $t \in \underline{R}$. The relations $\theta_2^a(-\frac{1}{3}) = \frac{\pi}{2}$ and $0 < \dot{\theta}_2^a$ on $[-\frac{1}{3}, 0]$ exclude $\pi \neq \theta_2^a(0) \in \underline{Z}\pi$ for a sufficiently close to α .

2. Let $a \in (\alpha, a_3)$. Note $0 < \dot{\theta}$ for all solutions of eq. (3) .

3. Proof of $\theta_2^a = \frac{3\pi}{2} - \theta_2^a(-\frac{1}{3} - \cdot)$: Set $\theta^* := \frac{3\pi}{2} - \theta_2^a(-\frac{1}{3} - \cdot)$. Then $\theta^*(-\frac{1}{3}) = \frac{\pi}{2} = \theta_2^a(\frac{1}{3})$. Properties of x_a give

$$g_0'(x_a((-\tfrac{1}{3} - \cdot) - \tfrac{1}{3})) = g_0'(-x_a(- \cdot)) = g_0' \circ x_a \text{ , or}$$

$$A_a(-\tfrac{1}{3} - \cdot) = B_a \quad \text{and} \quad B_a(-\tfrac{1}{3} - \cdot) = A_a \text{ .} \tag{4}$$

Hence

$$\dot{\theta}^* = \dot{\theta}_2^a(-\tfrac{1}{3} - \cdot) = A_a(\ldots) \sin^2 \theta_2^a(\ldots) + B_a(\ldots) \cos^2 \theta_2^a(\ldots)$$
$$= B_a \cos^2 \theta^* + A_a \sin^2 \theta^* \text{ ,}$$

and by uniqueness of solutions, $\theta_2^a = \theta^*$.

4. In particular, $\theta_2^a(-\frac{1}{6}) = \frac{3\pi}{4}$. With $0 < \dot{\theta}_2^a$, we get

$$\theta_2^a((-\tfrac{1}{6}, 0)) \subset (\tfrac{3\pi}{4}, \pi) \quad \text{and} \quad \theta_2^a((-\tfrac{1}{3}, -\tfrac{1}{6})) \subset (\tfrac{\pi}{2}, \tfrac{3\pi}{4}) \text{ .}$$

5. We compare the solution $\theta_0 : \underline{R} \to \underline{R}$ of eq. (3) with $\theta_0(-\frac{1}{6}) = \frac{\pi}{4}$ with the solution θ_2^a .

5.1. On $[-\frac{1}{3}, -\frac{1}{6}))$, $\theta_0 > \theta_2^a - \frac{\pi}{2}$. Proof: Clearly $\theta_0(-\frac{1}{6}) = \theta_2^a(-\frac{1}{6}) - \frac{\pi}{2}$. Let $t \in [-\frac{1}{3}, -\frac{1}{6}]$ with $0 \leqq \theta_0$ in $[t, -\frac{1}{6}]$ be given. On the interval $[t, -\frac{1}{6}]$,

$$\dot{\theta}_0 = [A_a - B_a] \sin^2 \theta_0 + B_a \text{ .}$$

$0 < \dot{\theta}_0$ and $\theta_0(-\frac{1}{6}) = \frac{\pi}{4}$ give $0 \leqq \theta_0 < \frac{\pi}{4}$ on $[t, -\frac{1}{6}]$, so that $\frac{\pi}{2} \leqq \theta_2^a \leqq \frac{3\pi}{4}$ on $[t, -\frac{1}{6}]$, and thereby $\sin^2 \theta_0 < \sin^2 \theta_2^a$ on $[t, -\frac{1}{6}]$.

Hypothesis (H2) implies that $B_a = a g_0' \circ x_a$ is strictly increasing on $[-\frac{1}{3}, 0]$, with $A_a(-\frac{1}{6}) = B_a(-\frac{1}{6})$. Using (4), we infer $A_a - B_a > 0$ on $[-\frac{1}{3}, -\frac{1}{6}]$, and therefore $\dot{\theta}_0 < \dot{\theta}_2^a$ on $[t, -\frac{1}{6}]$. We conclude that $0 \leqq \theta_0$ and $\dot{\theta}_0 < \dot{\theta}_2^a$ on $[-\frac{1}{3}, -\frac{1}{6}]$.

5.2 On $(-\frac{1}{6}, 0]$, $\theta_0 < \theta_2^a - \frac{\pi}{2}$. Proof (as above): We have $\theta_0(-\frac{1}{6}) = \theta_2^a(-\frac{1}{6}) - \frac{\pi}{2}$. Let $t \in (-\frac{1}{6}, 0]$ with $\theta_0 \leqq \frac{\pi}{2}$ on $[-\frac{1}{6}, t]$ be given. On $(-\frac{1}{6}, t]$,

$$\dot{\theta}_0 = [B_a - A_a] \cos^2 \theta_0 + A_a \text{ .}$$

The inequalities $\frac{\pi}{4} \leqq \theta_0 \leqq \frac{\pi}{2}$ and $\frac{3\pi}{4} < \theta_2^a \leqq \pi$ yield $\cos^2 \theta_0 < \cos^2 \theta_2^a$ on $(-\frac{1}{6}, t]$. (H2) and (4) give $B_a - A_a > 0$ on $(-\frac{1}{6}, 0]$. It follows that

$$\dot{\theta}_0 < \dot{\theta}_2^a \quad \text{on} \quad (-\tfrac{1}{6}, t] \text{ ,}$$

and one concludes that $\theta_0 \leqq \frac{\pi}{2}$ and $\dot{\theta}_0 < \dot{\theta}_2^a$ on $(-\frac{1}{6}, 0]$.

6. $\theta_0(-\frac{1}{3}) > \theta_2^a(-\frac{1}{3}) - \frac{\pi}{2} = 0 = \theta_1^a(-\frac{1}{3})$ and $\theta_0(0) < \frac{\pi}{2}$ now imply

$$\theta_1^a(0) < \tfrac{\pi}{2} \text{ .}$$

REMARK. Condition (H2) guaranteed a supercritical family of bifurcating periodic solutions (i.e., for $a > \alpha$), see Section 1. In the proof of Proposition 3 it is used once more, now to the effect that the supercritical solutions x_a have minimal instability: There is only one Floquet multiplier outside $\underline{S}$, namely the multiplier inherited from instability of the zero solution, which can not be avoided because of continuity.

COROLLARY 1. Let $a \in (\alpha, a_3)$. Then $m_a(-1) = 1$ and $-1 < \xi_a^* < -1 + \varepsilon$ for the unique eigenvalue ξ_a^* of W_a in $(-1 + \underline{C}_\varepsilon) \setminus \{-1\}$.

PROOF. 1. $\theta_1^a(0) \in [0, \frac{\pi}{2})$ implies $u_1^{-1,a}(0) > 0$. Therefore $Q_{-1,a} \neq 0$, and Corollary 4.3 gives $m_a(-1) = j_a(-1)$. By Propositions 2 and 3,

$$j_a(-1) = 1 \text{ .}$$

2. Recall Corollary 3.1: We infer

$$1 = \sum_{-1 \neq z < -1 + \varepsilon} m_a(z) = \sum_{-1 \neq z < -1 + \varepsilon} j_a(z)$$

(with Proposition 4.3). q_a is analytic. By Proposition 1 and $q_a'(-1) < 0$ (Proposition 3),

$$\sum_{z < -1} j_a(z) \in 2\underline{Z} \text{ .}$$

It follows that q_a has no zero on $(-\infty, -1)$, and one zero in $(-1, -1 + \varepsilon)$ which must be the unique eigenvalue $\xi_a^* \neq -1$ of W_a in $-1 + \underline{C}_\varepsilon$.

6. HYPERBOLICITY

We return to the operators V_a from Section 3 . Let $a \in (\alpha, a_3)$. For $z \in \underline{C}$ and $n \in \underline{N}$, $(\tilde{V}_a - z^2)^n = (\tilde{W}_a - z)^n \circ (\tilde{W}_a + z)^n$, and the results about $\sigma(W_a)$ imply $1 \in \sigma(V_a)$, $1 < \xi_a^2 \in \sigma(V_a)$, and $|z| < 1$ whenever $\{1, \xi_a^2\} \not\ni z \in \sigma(V_a)$.

Furthermore, if $M_a(z)$ denotes the algebraic multiplicity of z in $\underline{C} \setminus \{0\}$ considered as an eigenvalue of V_a , then $M_a(1) = 1$ and $M_a(\xi_a^2) = 1$.

(We might also have employed the Spectral Mapping Theorem to deduce results for V_a from W_a .)

This means that x_a is a hyperbolic periodic solution of eq. (ag_0) "with one unstable direction".

PROPOSITION 1. There exist $a_4 \in (\alpha, a_3)$ and a continuous map $a \to \psi_a$ from (α, a_4) into C such that each ψ_a is an eigenvector for the eigenvalue ξ_a^2 of V_a with $|\psi_a| = 1$, $0 < \psi_a$, and $\psi_a \to \psi_\alpha$ as $a \downarrow \alpha$.

PROOF. By Corollary 3.1 (iii) , $\xi_a^2 \to \exp(\frac{4}{3}\xi_\alpha)$ as $a \downarrow \alpha$. By (3.2), $V_a \to V_\alpha$ in C^C as $a \downarrow \alpha$. Recall that the function ψ_α , $\psi_\alpha(t) = \exp(\frac{4}{3}\xi_\alpha t)$ for $t \in [-1,0]$, is an eigenvector of V_α for the eigenvalue $\exp(\frac{4}{3}\xi_\alpha)$ (Section 2). Find a disk $\underline{C}_\delta$, $\delta > 0$, and $a' \in (\alpha, a_3)$ with $\{\xi_a^2\} = (\exp(\frac{4}{3}\xi_\alpha) + \mathrm{cl}\, \underline{C}_\delta) \cap \sigma(V_a)$ for $\alpha < a < a'$. For such a , set

$$\psi_a' := \frac{-1}{2\pi i} \int_\Gamma (\tilde{V}_a - z)^{-1} dz \, (\psi_\alpha)$$

with $\Gamma(t) := \exp(\frac{4}{3}\xi_\alpha) + \delta \exp(it)$ for $0 \leqq t \leqq 2\pi$; i.e. , apply the eigenprojection from $\tilde{C}$ onto the generalized eigenspace of ξ_a^2 to the fixed vector ψ_α (compare [4,14]). Then $\psi_a' \to \psi_\alpha$ as $a \downarrow \alpha$. Choose a_4 in (α, a_3) so small that $0 < \psi_a'$ for $\alpha < a < a_4$, and set $\psi_a := \frac{1}{|\psi_a'|} \psi_a'$ for these a .

7. COMPARISON RESULTS

The elementary estimates in this section constitute a link between local and global behavior of solutions of eq. (ag_0) . They rely on two simple facts: Positive initial conditions produce increasing solutions (because of positive feedback, $0 < \xi g_0(\xi)$ for $0 < |\xi| \leqq \gamma$), and a difference of two solutions which is initially positive increases also. The last property is due to monotonicity of g_0 in $[-\gamma, \gamma]$.

PROPOSITION 1. Let $a > 0$. Let solutions $x:[-1,\infty) \to \underline{R}$, $y:[-1,\infty) \to \underline{R}$ of eq. (ag_0) be given. Set $\phi := x_0$, $\psi := y_0$, $d := x - y$.

(i) If $\psi < \phi$ and $|x| \leqq \gamma$, $|y| \leqq \gamma$ on $[-1,t']$ for some $t' \geqq 0$ then $0 < d$ on $(0,t']$ and, for $0 < t \leqq t' + 1$,

$$0 < ag_0'(\gamma) \min_{[-1,0]} d \leqq ag_0'(\gamma)\, d(t-1) \leqq \dot{d}(t) \leqq a\, d(t-1) .$$

(ii) If $\psi < \phi$, $|y| < \frac{\gamma}{4}$ on $[-1,\infty)$ and $|\phi| < \frac{\gamma}{2}$ then there is $t^* > 0$ with $x(t^*) = \gamma$ and $|x| < \gamma$ on $[-1,t^*)$.

If $\phi < \psi$, $|y| < \frac{\gamma}{4}$ on $[-1,\infty)$ and $|\phi| < \frac{\gamma}{2}$ then there is $t^* > 0$ with $x(t^*) = -\gamma$ and $|x| < \gamma$ on $[-1,t^*)$.

PROOF. 1. Note $0 < d$ on $[-1,0]$. Suppose $d(t) \leqq 0$ for some t in $(0,t']$. Then $0 < d$ on $[-1,t^*)$ and $d(t^*) = 0$ for some $t^* \in (0,t']$. Hence

$$0 \geqq \frac{1}{a} \dot{d}(t^*) = g_0(d(t^* - 1) + y(t^* - 1)) - g_0(y(t^* - 1))$$
$$= \int_{y(t^*-1)}^{d(t^*-1) + y(t^*-1)} g_0'(\xi) d\xi .$$

With $0 < d(t^* - 1)$, $|x(t^* - 1)| < \gamma$, $|y(t^* - 1)| < \gamma$:

$$0 \geqq d(t^* - 1) \min_{[-\gamma,\gamma]} g_0' ,$$

a contradiction to $0 < g_0'$ on $[-\gamma,\gamma]$. This proves $0 < d$ on $(0,t']$.

Now let $0 < t \leqq t' + 1$. Using eq. (ag_0) as before, and $g_0'(\gamma) \leqq g_0' \leqq 1$ on $[-\gamma,\gamma]$, we infer

$$a\, d(t-1) \geqq \dot{d}(t) \geqq ag_0'(\gamma)\, d(t-1) > 0 .$$

With $0 < \dot{d}$ on $(0,t'+1]$, $ag_0'(\gamma)\,d(t-1) \geq ag_0'(\gamma)\min_{[-1,0]} d$.

2. We only consider the case $\psi < \phi$. Suppose $|x| \leq \gamma$ on $[-1,\infty)$. By (i), $0 < ag_0'(\gamma)\min_{[-1,0]} d \leq \dot{d}$ on $(0,\infty)$, and $d(t) \to \infty$ as $t \to \infty$, a contradiction to $|y| < \frac{\gamma}{4}$ on $[-1,\infty)$.

It follows that $|x(t^*)| = \gamma$ and $|x| < \gamma$ on $[-1,t^*)$ for some $t^* > 0$. Suppose $x(t^*) = -\gamma$. Then, by (i), $0 < \dot{d}$ on $(0,t^*+1)$, and $d(0) < d(t^*)$, or

$$x(0) - y(0) < -\gamma - y(t^*) \ , \quad x(0) < -\gamma + \frac{\gamma}{2} \ ,$$

a contradiction to $|\phi| < \frac{\gamma}{2}$.

We choose $a_5 \in (\alpha,a_4)$ with $\max_{\underline{R}} |x_a| < \frac{\gamma}{4}$ for $\alpha < a < a_5$, and for each $a \in (\alpha,a_5)$ a number $r_0(a) > 0$ so small that for every solution x of eq. (ag_0) with $x_0 \in \phi_a^* + C_{r_0(a)}$, $|x| < \frac{\gamma}{2}$ on $[-1,2]$.

Proposition 1 yields

COROLLARY 1. Let $a \in (\alpha,a_5)$. For every solution $x:[-1,\infty) \to \underline{R}$ of eq. (ag_0) with $x_0 \in \phi_a^* + C_{r_0(a)}$ and $x_0 > \phi_a^*$ ($x_0 < \phi_a^*$) there exists $t^* = t^*(x_0,a) > 2$ such that

$$|x| < \gamma \text{ on } [-1,t^*) \text{ and } x(t^*) = \gamma \quad (x(t^*) = -\gamma) \ .$$

II. ON HYPERBOLIC FIXED POINTS

Hypotheses and preliminaries. Let $f:D \to L$ be a C^k-map, $k \in \underline{N}$, on an open subset D of a real Banach space L with norm $|\ |$. Suppose f has a hyperbolic fixed point x^* : For $A := Df(x^*)$,

$$\sigma(A) \cap \underline{S} = \emptyset \ .$$

Then there is a decomposition

$$(\oplus) \qquad L = L^u \oplus L^s$$

into A-invariant closed subspaces such that the induced maps $L^u \ni x \to Ax \in L^u$ and $L^s \ni x \to Ax \in L^s$ have spectra outside and inside $\underline{S}$, respectively. L^u is called the linear unstable space; L^s is called the linear stable space. The decomposition $(\oplus)$ defines projections p^u of L onto L^u and p^s of L onto L^s .

Occasionally, we shall write x^u for $p^u x$, x^s for $p^s x$.

The induced map in $L^{u,c}$ is an isomorphism. There exist an equivalent norm $|\ |^*$ on L and a constant $q \in (0,1)$ with

$$(u) \qquad |Av|^* \geq q^{-1}|v|^* \quad \text{on } L^u \ ,$$

$$(s) \qquad |Av|^* \leq q|v|^* \quad \text{on } L^s \ .$$

We say that with respect to $|\ |^*$ A is an "expansion-contraction".

Existence of $|\ |^*$ follows by arguments from the appendix to Chapter 4.5 in [12]; compare also Theorem 4.19 [12] on hyperbolic isomorphisms.

For $M \subset L$ and $r > 0$,

$$M_r^* := \{x \in M : |x|^* < r\} \ .$$

1. LOCAL INVARIANT MANIFOLDS

We restate Theorem 3.1 from [8] on local stable and unstable manifolds for maps which are not necessarily diffeomorphisms:

THEOREM 1. There exist positive reals $r(u)$, $\rho(u)$, $r(s)$, $\rho(s)$ and C^k-maps $u: L^{u*}_{r(u)} \to L^{s*}_{\rho(u)}$, $s: L^{s*}_{r(s)} \to L^{u*}_{\rho(s)}$ with the following properties.

(i) $u(0) = 0$, $s(0) = 0$, $Du(0) = 0$, $Ds(0) = 0$

(ii) $U := x^* + \{x + u(x) : x \in L^{u*}_{r(u)}\} \subset D$

$S := x^* + \{s(x) + x : x \in L^{s*}_{r(s)}\} \subset D$

(iii) $f|S$ is a contraction with respect to $|\ |^*$. $f(S) \subset S$. For each forward trajectory $(x_n)_0^\infty$ of f in $x^* + L^{u*}_{\rho(s)} + L^{s*}_{r(s)}$, $x_0 \in S$.

(iv) There exists an open neighborhood D^u of x^* in L with $f(U \cap D^u) = U$. The restriction $f|U \cap D^u$ is injective with an inverse $f_u^{-1}: U \to L$ which is a contraction with respect to $|\ |^*$. f_u^{-1} defines a C^k-diffeomorphism from the submanifold U onto the submanifold $U \cap D^u$. For every backward trajectory $(x_n)_{-\infty}^0$ of f in $x^* + L^{u*}_{r(u)} + L^{s*}_{\rho(u)}$, $x_0 \in U$.

For a proof, see [8,18]. The "graphs" U and S are called local unstable manifold of x^* and local stable manifold of x^*, respectively. For their tangent spaces, we have

$$T_{x^*}U = L^u \ , \ T_{x^*}S = L^s \ .$$

An obvious consequence of Theorem 1 which does not involve the new norm $|\ |^*$ any more is

COROLLARY 1. (i) Each $x_0 \in S$ continues to a unique forward trajectory of f in S, with $x_n \to x^*$ as $n \to \infty$.

(ii) Each $x_0 \in U$ continues to a unique backward trajectory of f in U, with $x_n \to x^*$ as $n \to -\infty$ and $x_n \in U \cap D^u$ for all $n \in \underline{N}$.

(iii) For $x \in U \cap D^u$, $Df(x)$ induces an isomorphism $T_xU \to T_{f(x)}U$.

In the sequel we prepare the proof of a technical statement on attraction to the local unstable manifold (Proposition 1), and the application of Theorem 5.2 [8] on chaos in Chapter V. Proposition 1 will be used in Chapter III in order to link local to global behavior, in the search for heteroclinic solutions.

As in the case of diffeomorphisms in finite dimensions, one can transform f to a map F with hyperbolic fixed point 0 such that U and S become open subsets of L^u and L^s, respectively.

Choose $r > 0$ with $r < r(u)$, $r < r(s)$,

$$D_I := x^* + L^{u*}_r + L^{s*}_r \subset D \ ,$$

$$|Du(x)|^* < \frac{1}{2} \text{ on } L^{u*}_r \text{ and } |Ds(x)|^* < \frac{1}{2} \text{ on } L^{s*}_r \ .$$

Then

$$U(r) := x^* + \{x + u(x) : x \in L^{u*}_r\} \subset D_I \ ,$$

$$S(r) := x^* + \{s(x) + x : x \in L^{s*}_r\} \subset D_I \ .$$

For $r' > 0$ so small that

$$x^* + L^*_{r'} \subset D^u$$

and

$$[f(x) - x^*]^u \in L^{u*}_r \ , \ [f(x) - x^*]^s \in L^{s*}_r \text{ for all } x \in x^* + L^*_{r'}$$

we obtain

$$f(U(r) \cap [x^* + L^*_{r'}]) \subset U(r) \ , \ f(S(r) \cap [x^* + L^*_{r'}]) \subset S(r) \ .$$

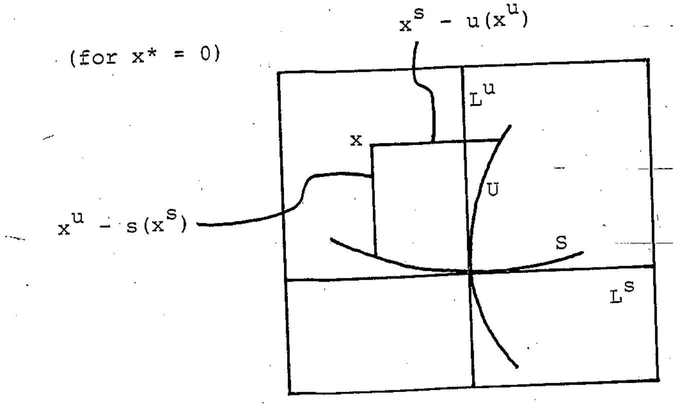
(Proof of the first inclusion: For $x \in U(r) \cap [x^* + L^*_{r'}]$, $x \in U \cap D^u$. Hence $f(x) \in U$, or $f(x) = x^* + y + u(y)$ with $y \in L^{u*}_{r(u)}$. $p^u u(y) = 0$ and $y = y^u$ give $y = [f(x) - x^*]^u \in L^{u*}_{r(u)}$, or $f(x) \in U(r)$.)

Now consider the transformation $T_I : D_I \to L$ defined by

$$T_I(x) = x - x^* - s([x - x^*]^s) - u([x - x^*]^u)$$
$$= [x - x^*]^u - s([x - x^*]^s) + [x - x^*]^s - u([x - x^*]^u)$$

for all $x \in D_I$. Note

$$T_I(x^*) = 0 \text{ and } DT_I(x^*) = id \ .$$



COROLLARY 2. (i) Let $x \in D_I$. Then $x \in U(r)$ is equivalent to

$T_I(x) \in L^u$, and $x \in S(r)$ is equivalent to $T_I(x) \in L^s$.

(ii) $x \in U(r)$ implies $DT_I(x)\ T_xU \subset L^u$. For $x \in S(r)$,

$DT_I(x)\ T_xS \subset L^s$.

The Implicit Function Theorem yields open neighborhoods $D_{II} \subset D_I$ of x^* and V_{II} of $0 \in L$ such that $T_{II} := T_I|D_{II}$ maps D_{II} onto V_{II} and has a C^k-inverse $T_{II}^{-1}:V_{II} \to L$, with T_{II} , T_{II}^{-1} , DT_{II} and DT_{II}^{-1} all bounded and $D_{II} \subset x^* + L^*_{r'}$.

COROLLARY 3. (i) The maps $DT_{II}(x)$, $x \in D_{II}$, are isomorphisms.

(ii) If $x \in D_{II} \cap U$ then $DT_{II}(x)$ induces an isomorphism from T_xU onto L^u .

(iii) If $x \in D_{II} \cap S$ then $DT_{II}(x)$ induces an isomorphism from T_xS onto L^s .

PROOF. 1. Assertion (i) is obvious from the chain rule.

2. Proof of $T_{II}(D_{II} \cap U) \subset V_{II} \cap L^u$: We have

$$D_{II} \cap U \subset D_I \cap U \subset (x^* + L_r^{u*} + L^s) \cap U = U(r) .$$

Hence $T_{II}(D_{II} \cap U) \subset T_I(D_I \cap U(r)) \subset L^u$, by Corollary 2 (i). $T_{II}(D_{II} \cap U) \subset V_{II}$ is clear.

3. Proof of $V_{II} \cap L^u \subset T_{II}(D_{II} \cap U)$: For $y \in V_{II} \cap L^u$, $x := T_{II}^{-1}(y) \in D_{II} \subset D_I$. By Corollary 2 (i) , $x \in U(r) \subset U$.

4. It follows that $T_{II}(D_{II} \cap U) = V_{II} \cap L^u$, and T_{II} induces a C^k-diffeomorphism of the submanifold $D_{II} \cap U$ onto the submanifold $V_{II} \cap L^u$. This implies that all $DT_{II}(x)$, $x \in D_{II} \cap U$, induce isomorphisms of T_xU onto $T_{T_{II}(x)}(V_{II} \cap L^u) = T_{T_{II}(x)}L^u = L^u$.

5. The proof for $x \in D_{II} \cap S$ is analogous.

Note $f \circ T_{II}^{-1}(0) = x^* \in D_{II}$. Continuity implies existence of an open neighborhood $V \subset V_{II}$ of $0 \in L$ such that $f \circ T_{II}^{-1}(V) \subset D_{II}$. $D_V := T_{II}^{-1}(V)$ is an open neighborhood of x^* . The transformed map

$$F:V \ni y \to T_{II}(f(T_{II}^{-1}(y))) \in L$$

is C^k, with $F(0) = 0$ and $DF(0) = A = Df(x^*)$.

COROLLARY 4. (i) For all $x \in D_V$, $f(x) = T_{II}^{-1}(F(T_{II}(x)))$.

(ii) $F(V \cap L^u) \subset L^u$, $F(V \cap L^s) \subset L^s$. The remainder

$R:V \ni y \to F(y) - Ay \in L$

is C^k with $R(0) = 0$, $DR(0) = 0$ and

$R(V \cap L^u) \subset L^u$, $R(V \cap L^s) \subset L^s$.

(iii) For $y \in V \cap L^u$, $p^s\ DR(y)\ p^u = 0$.

For $y \in V \cap L^s$, $p^u\ DR(y)\ p^s = 0$.

PROOF. 1. $x \in D_V$ gives $T_{II}(x) \in V$, and

$$f(x) = T_{II}^{-1}(T_{II}(f(x))) \in f(T_{II}^{-1}(V)) \subset D_{II} = T_{II}^{-1}(V_{II}) .$$

Therefore

$$f(x) = T_{II}^{-1}(T_{II}(f(x))) = T_{II}^{-1}(T_{II}(f(T_{II}^{-1}(T_{II}(x)))))$$

$$= T_{II}^{-1}(F(T_{II}(x)))) .$$

2. For $y \in V \cap L^u$, $T_{II}^{-1}(y) \in D_V \subset D_{II} \subset D_I$ and $y = T_{II}(T_{II}^{-1}(y)) \in L^u$. Corollary 2 (i) yields $T_{II}^{-1}(y) \subset U(r)$. With $T_{II}^{-1}(y) \in D_{II} \subset x^* + L^*_{r'}$, $f(T_{II}^{-1}(y)) \in U(r)$. By $f(T_{II}^{-1}(y)) \in D_{II} \subset D_I$ and by Corollary 2 (i), $F(y) = T_{II}(f(T_{II}^{-1}(y))) \in L^u$. This proves $F(V \cap L^u) \subset$

L^u . The proof of $F(V \cap L^s) \subset L^s$ is analogous. The other parts of assertion (ii) are obvious.

3. Proof of $p^u\, DR(y)\, p^s = 0$ for $y \in V \cap L^s$: $R(V \cap L^s) \subset L^s$ gives $DR(y)\, L^s = DR(y)\, T_y(V \cap L^s) \subset T_{R(y)}L^s = L^s$, or $p^u\, DR(y)\, p^s = 0$.

For the proof of Proposition 1 below we need expansion-contraction properties of the nonlinear map F . Choose $c = c_F > 0$ with

$$q + c < 1 < q^{-1} - c$$

and $r(F) > 0$ so small that

$$V_F := L^{u*}_{r(F)} + L^{s*}_{r(F)} \subset V$$

with

$$c > |p^u\, DR(y)|^* + |p^s\, DR(y)|^* + |DR(y)|^* \text{ on } V_F .$$

COROLLARY 5. For $y \in V_F$,

$$|p^u\, R(y)|^* \le c|y^u|^* ,$$
$$(q^{-1} - c)|y^u|^* \le |p^u\, F(y)|^* \le (|A|^* + c)|y^u|^* ,$$
$$|p^s\, F(y)|^* \le (q + c)|y^s|^* .$$

PROOF. For $y \in V_F$, $y^s \in L^{s*}_{r(F)} = V_F \cap L^s$. By Corollary 4 (ii), $p^u\, R(y) = p^u\, R(y) - p^u\, R(y^s)$. The straight line from y^s to y is contained in V_F so that the Mean Value Theorem implies

$$|p^u\, R(y)|^* \le c|y - y^s|^* = c|y^u|^* .$$

Using (u) and $p^u\, F(y) = Ay^u + p^u\, R(y)$ one obtains the estimate of $p^u\, F(y)$. The estimate of $p^s\, F(y)$ is proved analogously.

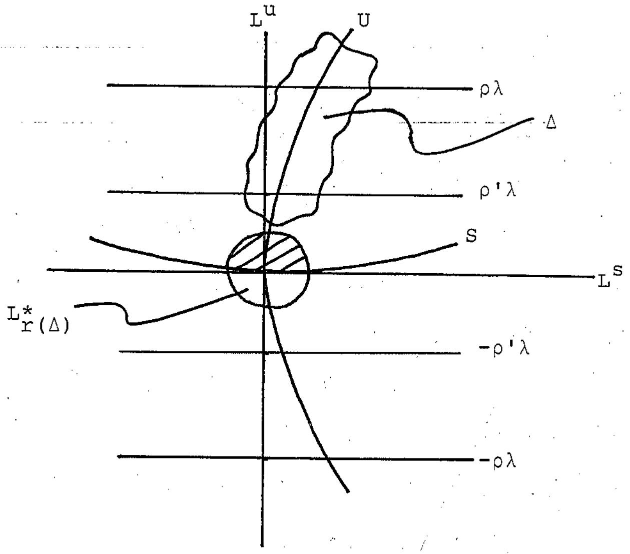
PROPOSITION 1. Assume $\dim L^u = 1$, $L^u = \underline{R}\lambda$ and $A\lambda = \xi\lambda$ with $\xi > 1$, and $[-\rho,\rho]\lambda \subset L^{u*}_{r(u)}$ with $\rho > 0$.

There exists $\rho' \in (0,\rho)$ such that for every open neighborhood Δ of $x^* + \{x + u(x) : x \in [\rho',\rho]\lambda\}$ (of ... $[-\rho,-\rho']\lambda\}$) there is $r(\Delta) > 0$ with the following properties.

(i) $p^s\, L^*_{r(\Delta)} \subset L^{s*}_{r(s)}$

(ii) $x^* + L^*_{r(\Delta)} \subset D$

(iii) For every $x \in x^* + L^*_{r(\Delta)}$ with $[x - x^*]^u - s([x - x^*]^s) \in (0,\infty)\lambda$ (... $\in (-\infty,0)\lambda$) there exist points $x_0 = x$, x_1 , ... , x_j ; $j = j(x) \in \underline{N}$; in D with $x_{k+1} = f(x_k)$ for $k = 0, \dots , j-1$ and $x_j \in \Delta$.



2. PROOF OF PROPOSITION 1.1

We give a proof for the statement not in brackets.

1. Consider $\rho > 0$ with $[-\rho,\rho]\lambda \subset L^{u*}_{r(u)}$. Set $D_F := T^{-1}_{II}(V_F)$ and $T_F := T_{II}|D_F$. Choose $\rho'' \in (0,\rho)$ with

$$x^* + \{x + u(x) : x \in [0,\rho'']\lambda\} \subset D_F .$$

Proof that there exists $\rho_1 > 0$ with $[0,\rho_1]\lambda \subset T_F(x^* + \{\ldots\})$:

The estimates of $Du(x)$ and $Ds(x)$ on L^{u*}_r and L^{s*}_r , respectively, in Section 1 and the Mean Value Theorem imply

$$|u(x)|^* \leq \tfrac{1}{2}|x|^* < \tfrac{r}{2} \text{ on } L^{u*}_r , \quad |s(x)|^* \leq \tfrac{1}{2}|x|^* < \tfrac{r}{2} \text{ on } L^{s*}_r .$$

Choose ρ_1 so small that for all $y \in [0,\rho_1]\lambda \subset L^u$,

$$y \in V_F \text{ and } |[T^{-1}_F(y) - x^*]^u|^* \leq \rho''|\lambda|^* .$$

For such y , $z := T^{-1}_F(y)$ lies in $D_F \subset D_{II} \subset D_I \subset D$. Corollary 1.2 (i) gives $z \in U(r)$, or $z = x^* + x + u(x)$ with $x \in L^{u*}_r = (-r,r)\frac{1}{|\lambda|^*}\lambda$. We have $x \in [-\rho'',\rho'']\lambda$ since $|x|^* = |[z - x^*]^u|^* \leq \rho''|\lambda|^*$. It remains to show that $x \in [0,\rho'']\lambda$. $x = \underline{\beta}\lambda$ with $-\rho'' \leq \underline{\beta} \leq \rho''$ and $y = \underline{\alpha}\lambda$ with $0 \leq \underline{\alpha} \leq \rho_1$ imply

$$\underline{\alpha}\lambda = y = T_F(z) = T_F(x^* + \underline{\beta}\lambda + u(\underline{\beta}\lambda)) = \underline{\beta}\lambda - s(u(\underline{\beta}\lambda)) ,$$

see the definition of T_I . We have $s(\ldots) = \underline{\gamma}\lambda$ with $\underline{\alpha}\lambda = (\underline{\beta} - \underline{\gamma})\lambda$. By $\underline{\beta}\lambda = x \in L^{u*}_r$, $|u(\underline{\beta}\lambda)|^* \leq \frac{1}{2}|\underline{\beta}\lambda|^*$. Therefore $u(\underline{\beta}\lambda) \in L^{s*}_r$, and

$$|\underline{\gamma}\lambda|^* = |s(u(\underline{\beta}\lambda))|^* \leq \tfrac{1}{2}|u(\underline{\beta}\lambda)|^* \leq \tfrac{1}{4}|\underline{\beta}\lambda|^* ,$$

or $|\underline{\gamma}| \leq \frac{1}{4}|\underline{\beta}|$. This yields $\underline{\beta} - \underline{\gamma} = 0$ and $\underline{\beta} = 0$ in case $\underline{\alpha} = 0$, and $\underline{\beta} - \underline{\gamma} > 0$ and $\underline{\beta} > 0$ in case $\underline{\alpha} > 0$.

2. The next aim is a variant of Proposition 1.1 for the map F . We show that there exists $\rho'_1 \in (0,\rho_1)$ such that for every open neighborhood of $[\rho'_1,\rho_1]\lambda$ there is $r(\Delta,F) > 0$ with the following properties:

$L^*_{r(\Delta,F)} \subset V_F$,

and for every $y \in L^*_{r(\Delta,F)} \setminus L^s$ with $y^u \in (0,\infty)\lambda$ there exist points $y_0 = y$, y_1 , $\ldots$, y_j ; $j = j(y) \in \underline{N}$; in V_F with

$$y_{k+1} = F(y_k) \text{ for } k = 0 , \ldots , j-1 \text{ and } y_j \in \Delta .$$

2.1. Choose $\rho''_1 \in (0,\rho_1)$ with

(1) $\rho''_1|\lambda|^* < r_F$

and $\rho'_1 \in (0,\rho''_1)$ with

(2) $(|A|^* + c)\rho'_1 < \rho''_1$.

Let an open neighborhood Δ of $[\rho'_1,\rho_1]\lambda$ be given. By compactness, there exists $\varepsilon \in (0,r_F)$ with

(3) $[\rho'_1,\rho_1]\lambda + L^{s*}_{\varepsilon} \subset \Delta$.

Choose $r(\Delta,F) > 0$ so small that for all $y \in L^*_{r(\Delta,F)}$,

(4) $|y^u|^* < \rho'_1|\lambda|^*$,

(5) $|y^s|^* < \varepsilon < r_F$.

It follows that $L^*_{r(\Delta,F)} \subset V_F$.

Let $y \in L^*_{r(\Delta,F)} \setminus L^s$. Then $y^u \neq 0$. Assume $y^u \in (0,\infty)\lambda$.

2.2. There is no forward trajectory $(y_k)^{\infty}_0$ of F in V_F with $y_0 = y$. Proof: Otherwise, Corollary 1.5 yields

$$|y^u_k|^* \geq (q^{-1} - c)^k|y^u|^* > 0 \text{ for all } k \in \underline{N} ,$$

a contradiction to $y_k \in V_F$ for all $k \in \underline{N}$.

2.3. It follows that there exist points $y_0 = y$, y_1 , $\ldots$, y_{k-1} in V_F and $y_k \in L \setminus V_F$ with $y_{\kappa+1} = F(y_\kappa)$ for $\kappa = 0$, $\ldots$, $k-1$. By Corollary 1.5 and by (5),

$$0 < (q^{-1} - c)^{\kappa}|y^u|^* \leq |y^u_\kappa|^* ,$$

(6) $|y^s_\kappa|^* \leq |y^s|^* < \varepsilon < r_F$ for $\kappa = 0$, $\ldots$, k .

We have

$$|y^u_k|^* > \rho'_1|\lambda|^*$$

since otherwise $|y^u_k|^* \leq \rho'_1|\lambda|^* < r_F$ so that $y_k \in V_F$ (with (6)) , a contradiction.

2.4. Let j denote the smallest integer in $\{0 , \ldots , k\}$ with

$$\rho'_1|\lambda|^* < |y^u_j|^* .$$

By (4), $1 \leq j \leq k$. Corollary 1.5 and (2) give

$$|y^u_j|^* = |p^u F(y_{j-1})|^* \leq (|A|^* + c)|y^u_{j-1}|^* \leq (|A|^* + c)\rho'_1|\lambda|^* < \rho''_1|\lambda|^* .$$

We observe $y_j \in V_F$ (from (1) and (6)) and

$$(7) \qquad \rho_1'|\lambda|^* < |y_j^u|^* < \rho_1''|\lambda|^* < \rho_1|\lambda|^* ,$$

$$|y_j^s|^* < \varepsilon \quad \text{(see (6))}.$$

In view of (3) and (7) it remains to show that y_j^u is a positive multiple of λ. The latter is a consequence of $y_0^u = y^u \in (0,\infty)\lambda$, of

$$y_{\kappa+1}^u = Ay_\kappa^u + p^u R(y_\kappa) = \xi y_\kappa^u + p^u R(y_\kappa)$$

and

$$|p^u R(y_\kappa)|^* \leq c|y_\kappa^u|^* \qquad \text{(Corollary 1.5)}$$

for $\kappa = 0, \ldots, j-1$, and of $c < 1 < \varepsilon$.

3. We show that there exists $\rho' \in (0,\rho'')$ with

$$T_F(x^* + \{x + u(x) : x \in [\rho',\rho'']\lambda\}) \supset [\rho_1',\rho_1]\lambda \quad :$$

For all $y \in [\rho_1',\rho_1]\lambda$, $y \neq 0$ and $T_F^{-1}(y) \in U$; in particular, $T_F^{-1}(y) \in U \setminus \{x^*\}$, and thereby $0 \neq [T_F^{-1}(y) - x^*]^u$. Now compactness implies that there exists $\rho' \in (0,\rho'')$ with

$$(8) \qquad |[T_F^{-1}(y) - x^*]^u|^* \geq \rho'|\lambda|^* \quad \text{for all } y \in [\rho_1',\rho_1]\lambda .$$

For such y, $T_F^{-1}(y) = x^* + x + u(x)$ with $x \in [0,\rho'']\lambda$, see part 1. With (8),

$$|x|^* = |[T_F^{-1}(y) - x^*]^u|^* \geq \rho'|\lambda|^* ,$$

or $x \in [\rho',\rho'']\lambda$.

4. Let Δ be an open neighborhood of $x^* + \{x + u(x) : x \in [\rho',\rho]\lambda\} \supset x^* + \{x + u(x) : x \in [\rho',\rho'']\lambda\}$. By the choice of ρ'',

$$x^* + \{x + u(x) : x \in [\rho',\rho'']\lambda\} \subset D_F ,$$

and $\Delta' := T_F(\Delta \cap D_F)$ is an open neighborhood of $[\rho_1',\rho_1]\lambda$, see part 3. Consider $r(\Delta',F)$ according to part 2; $L^*_{r(\Delta',F)} \subset V_F$. Choose $r(\Delta) > 0$ so small that

$$x^* + L^*_{r(\Delta)} \subset D_F , \quad T_F(x^* + L^*_{r(\Delta)}) \subset L^*_{r(\Delta',F)} , \quad p^s L^*_{r(\Delta)} \subset L^{s*}_{r(s)} .$$

Consider $x \in x^* + L^*_{r(\Delta)}$ with $[x - x^*]^u - s([x - x^*]^s) \in (0,\infty)\lambda$. Then $y := T_F(x) \in L^*_{r(\Delta',F)}$. The definition of T_I shows that

$$y^u = [x - x^*]^u - s([x - x^*]^s) \in (0,\infty)\lambda .$$

The result of part 2 guarantees the existence of points $y_0 = y, y_1, \ldots, y_j$; $j \in \underline{N}$; in V_F with $y_{k+1} = F(y_k)$ for $k = 0, \ldots, j-1$ and $y_j \in \Delta'$.

Set $x_k := T_F^{-1}(y_k)$, for $k = 0, \ldots, j$. Then $x_0 = x$, $x_k \in D_F \subset D$ for $k = 0, \ldots, j$ and $x_j \in T_F^{-1}(\Delta') = \Delta \cap D_F \subset \Delta$. Corollary 1.4 (i) gives

$$x_{k+1} = T_F^{-1}(y_{k+1}) = T_{II}^{-1}(F(y_k)) = T_{II}^{-1}(F(T_{II}(x_k))) = f(x_k)$$

for $k = 0, \ldots, j-1$.

III. POINCARE MAPS AND SOLUTIONS CLOSE TO x_a

Summary. The initial values ϕ_a^* of the periodic solutions x_a become hyperbolic fixed points of Poincare´maps P_a on open subsets D_a of a hyperplane $H \subset C$. The linear unstable spaces are one-dimensional, normalized eigenvectors $\lambda_a \in H$ converge as $a \downarrow \alpha$ (Corollary 1.1). This permits to describe the position of segments y_2 of solutions starting in the local unstable manifold of ϕ_a^*, with respect to $x_{a,2}$ in terms of elementary inequalities for $a - \alpha > 0$ sufficiently small. Corollary 1.2 contains preparations for this.

The parameter a is then fixed, and dropped. We continue with

$$g_1 := ag_0 .$$

Section 2 introduces local stable and unstable manifolds S and U for ϕ^*. Proposition 2.1 is the comparison result for solutions y with y_0 in $U \setminus S \subset H$ prepared by Corollary 1.2.

Section 3 contains the strategy for the search of sine-like nonlinearities with heteroclinic solutions h from x (with $h_0 \in U \setminus S$) to another periodic solution, and preparations for this search.

In Section 4, a transversality property is explained and characterized.

1. POINCARE MAPS

Let $\alpha < a < a_5$, and let X_{a0} denote the semiflow $\underline{R}^+ \times C \to C$ given by the solutions $x:[-1,\infty) \to \underline{R}$ of eq. (ag_0). The restriction of X_{a0} to $(1,\infty) \times C$ is C^1. Consider the hyperplane

$$H := \{\phi \in C: \phi(0) = 0\} = \ker ev$$

where $ev: C \ni \phi \to \phi(0) \in \underline{R}$. Note $\phi_a^* = x_{a,0} \in H$.

We have $ev(X_{a0}(\frac{4}{3},\phi_a^*)) = 0$. $\dot{x}_a(\frac{4}{3}) = \dot{x}_a(0) < 0$ implies that the partial derivative of

$$(t,\phi) \to ev(X_{a0}(t,\phi)) = x(t),$$

with the solution $x:[-1,\infty) \to \underline{R}$ of eq. (ag_0) starting at $x_0 = \phi$, with respect to t at $(t,\phi) = (\frac{4}{3},\phi_a^*)$ is different from 0. An application of the Implicit Function Theorem yields a number $r_1(a) \in (0,r_0(a))$ and a C^1-map

$$\tau_a: \phi_a^* + C_{r_1(a)} \to (1,2)$$

with $\tau_a(\phi_a^*) = \frac{4}{3}$, and for all $\phi \in \phi_a^* + C_{r_1(a)}$,

$$ev(X_{a0}(\tau_a(\phi),\phi)) = 0,$$

or equivalently,

$$X_{a0}(\tau_a(\phi),\phi) \in H,$$

and finally

$$\dot{x}(\tau_a(\phi)) < 0$$

with the solution $x:[-1,\infty) \to \underline{R}$ of eq. (ag_0) starting at $x_0 = \phi$. We set

$$N_a := \phi_a^* + C_{r_1(a)} .$$

The map $\underline{P}_a: N_a \ni \phi \to X_{a0}(\tau_a(\phi),\phi) \in H$ is C^1. Let i_H denote the inclusion $H \to C$. Set $D_a := \phi_a^* + H_{r_1(a)} = i_H^{-1}(N_a)$. The Poincare´map

$$P_a: D_a \ni \phi \to \underline{P}_a(i_H(\phi)) \in H$$

is C^1 and has the fixed point ϕ_a^*.

P_a is also compact so that there is no continuous inverse in the infinite-dimensional space H.

Note the formula

$$(1) \qquad P_a^j(\phi) = X_{a0}\left(\sum_{k=0}^{j-1} \tau_a(P_a^k(\phi),\phi)\right)$$

for all ϕ in the domain of the iterate P_a^j, $j \in \underline{N}$.

The inequality $\dot{x}(\tau_a(\phi)) < 0$ for the solution $x:[-1,\infty) \to \underline{R}$ of eq. (ag_0) with $x_0 = \phi \in D_a$ implies $\dot{x}_{\tau_a(\phi)} := \dot{x}(\tau_a(\phi) + \cdot)|[-1,0] \notin H$,

or

$$C = H \oplus \underline{R}\,\dot{x}_{\tau_a(\phi)}$$

so that there is a projection $p_{a,\phi}$ of C onto H, parallel to the tangent vector $D_1X_{a0}(\tau_a(\phi),\phi)1 = \dot{x}_{\tau_a(\phi)}$ of the trajectory

$$\underline{R}^+ \ni t \to X_{a0}(t,\phi) \in C$$

at $t = \tau_a(\phi)$. One computes

$$p_{a,\phi}\psi = \psi - \frac{\psi(0)}{\dot{x}(\tau_a(\phi))}\,\dot{x}_{\tau_a(\phi)} \quad \text{for all } \psi \in C .$$

As in finite dimensions,

$$DP_a(\phi) = p_{a,\phi} \circ D_2X_{a0}(\tau_a(\phi),\phi) \circ i_H \quad \text{for all } \phi \in D_a .$$

More generally, if ϕ is in the domain of P_a^j, $2 \leq j \in \underline{N}$, and if $\phi_0 = \phi$, $\phi_{k+1} = P_a(\phi_k)$, $t_0 = 0$, $t_{k+1} = t_k + \tau_a(\phi_k)$ for $k = 1, \ldots, j-1$ then

$$DP_a^j(\phi) = p_{a,\phi_{j-1}} \circ D_2X_{a0}(t_j,\phi) \circ i_H .$$

In particular,

$$DP_a(\phi_a^*) = p_{a,\phi_a^*} \circ V_a \circ i_H .$$

From the last formula one obtains (compare [31])

$$\sigma(DP_a(\phi_a^*)) = \sigma(V_a) \setminus \{1\}, \quad \sigma(DP(\phi_a^*)) \cap \underline{S} = \emptyset .$$

That is, ϕ_a^* is a hyperbolic fixed point of P_a. The only point in $\sigma(DP_a(\phi_a^*))$ outside $\underline{S}$ is ξ_a^2, an algebraically simple eigenvalue with eigenvector

$$\lambda_a := p_{a,\phi_a^*}\psi_a \in H .$$

Define $\zeta \in H$ by $\zeta(t) := \exp(\frac{4}{3}\xi_\alpha t) - \cos(\frac{3}{2}\pi t)$ for $-1 \leq t \leq 0$.

COROLLARY 1. $\lim_{a \downarrow \alpha} \lambda_a = \zeta$.

PROOF. For $\alpha < a < a_5$ and $t \in \underline{R}$, set $\tilde{u}_a(t) := \frac{1}{\dot{x}_a(0)}\dot{x}_a(t)$ and $\tilde{v}_a(t) := \tilde{u}_a(t-1)$. Then $(\tilde{u}_a,\tilde{v}_a)(0) = (1,0)$, and for all t,

$$\dot{\tilde{u}}_a(t) = ag_0'(x_a(t-1))\,\tilde{u}_a(t-1) = ag_0'(\ldots)\,\tilde{v}_a(t),$$

$$\dot{\tilde{v}}_a(t) = \dot{\tilde{u}}_a(t-1) = ag_0'(x_a(t-2))\,\tilde{u}_a(t-2) =$$

$$= ag_0'(\ldots)\,\tilde{u}_a(t - \tfrac{2}{3}) = -ag_0'(\ldots)\,\tilde{u}_a(t) .$$

Here we used the symmetry ((s)) from Section I.1. For $a \downarrow \alpha$, we have that $(\tilde{u}_a,\tilde{v}_a)$ converges to the solution of the initial value problem

$$\dot{u} = \alpha v, \quad \dot{v} = -\alpha u, \quad (u,v)(0) = (1,0)$$

uniformly on $[-1,0]$. Hence $\frac{1}{\dot{x}_a(0)}\dot{x}_a(t) \to \cos(\frac{3}{2}\pi t)$, uniformly for $t \in [-1,0]$. The definition of p_{a,ϕ_a^*} and Proposition I.6.1 now imply the assertion.

COROLLARY 2. There exist $a \in (\alpha,a_5)$, $\varepsilon > 0$, $\delta > 0$ and $r_2' > 0$ with the following properties.

(i) $\quad 0 < \int_{-1}^t \tilde{\zeta}$ for all $\tilde{\zeta} \in \zeta + C_\varepsilon$, $-1 < t \leq 0$.

(ii) $\quad |g_0'(\xi) - 1| < \frac{\varepsilon}{3|\zeta|}$ for $-\delta < \xi < \delta$

(iii) $\quad |\lambda_a - \zeta| < \frac{\varepsilon}{3}$

(iv) $\quad |\phi_a^*| + 2|r||\lambda_a| < \min\{\delta,\gamma\}$ for $-r_2' < r < r_2'$

(v) $\quad$ The solution $w: [-1,\infty) \to \underline{R}$ of

$$\dot{w}(t) = ag_0'(x_a(t-1))\,w(t-1), \quad w_0 = \lambda_a,$$

satisfies $0 < w_2$; or equivalently,

$$0 < D_2X_{a0}(2,\phi_a^*)\,\lambda_a .$$

PROOF. 1. There exists $t_\zeta \in (-\frac{1}{3},0)$ with $0 < \zeta$ on $[-1,t_\zeta)$, $\zeta < 0$ on $(t_\zeta,0)$. It follows that the map

$$[-1,0] \ni t \to \int_{-1}^t \zeta \in \underline{R}$$

is positive and increasing on $(-1,t_\zeta]$, and decreasing on $(t_\zeta,0]$. One computes $\int_{-1}^0 \zeta > 0$. Hence

$$0 < \int_{-1}^t \zeta \quad \text{for all } t \in (-1,0] .$$

2. Choose $\varepsilon > 0$ such that for all $\tilde{\zeta} \in \zeta + C_\varepsilon$, $\tilde{\zeta} > 0$ on $[-1,-\frac{1}{3}]$ and $\int_{-1}^t \tilde{\zeta} > 0$ on $[-\frac{1}{3},0]$. For such $\tilde{\zeta}$, we obtain part (i) of the assertion.

Choose $\delta > 0$ such that (ii) holds. Corollary 1 implies that there exist $a' \in (\alpha, a_5)$ and $r_2' > 0$ with

$$|\lambda_a - \zeta| < \frac{\varepsilon}{3} \text{ for all } a \in (\alpha, a')$$

and

$$|\phi_a^*| + 2|r||\lambda_a| < \min\{\delta, \gamma\} \text{ for all } a \in (\alpha, a'), \ r \in (-r_2', r_2') .$$

3. For $a_5 > a \downarrow \alpha$, the solution $w_a:[-1,\infty) \to \underline{R}$ of

$$\dot{w}(t) = a g_0'(x_a(t-1))\, w(t-1), \ w_0 = \lambda_a ,$$

converges to the solution $y:[-1,\infty) \to \underline{R}$ of

$$\dot{y}(t) = \alpha y(t-1), \ y_0 = \zeta ,$$

uniformly on $[-1,2]$. The result of part 1 yields $0 < y$ on $(0,1]$. Therefore $0 < \dot{y}$ on $[1,2]$, and $0 < y_2$.

For $a \in (\alpha, a_5)$ as in Corollary 2, we set

$$g_1 := a g_0$$

and drop the index a from now on:

$$\phi^* = \phi_a^*, \ x = x_a, \ r_0 = r_0(a), \ r_1 = r_1(a), \ N = N_a, \ \tau = \tau_a,$$

$$\underline{P} = \underline{P}_a, \ D = D_a, \ P = P_a, \ \lambda = \lambda_a .$$

Instead of

$$\xi_a^2 ,$$

we shall write

$$\mu .$$

We set

$$x_1 := x_{a0} .$$

2. DEPARTURE OF SOLUTIONS FROM THE LOCAL UNSTABLE MANIFOLD OF P

According to Section II.1 there is a decomposition

(1) $\quad H = L^u \oplus L^s$

into $DP(\phi^*)$-invariant closed subspaces. We have

$$L^u = \underline{R}\,\lambda$$

and

$$DP(\phi^*)\,\lambda = \mu\lambda$$

where $\mu > 1$.

Let p^u and p^s denote the projections onto L^u and L^s, given by (1). Occasionally we shall write ϕ^u for $p^u\phi$, ϕ^s for $p^s\phi$.

There is an equivalent norm $|\ |^*$ on H with respect to which $DP(\phi^*)$ induces an expansion on L^u and a contraction on L^s.

For $M \subset H$ and $r > 0$, we set

$$M_r^* := \{\phi \in M: |\phi|^* < r\} .$$

An application of Theorem II.1.1 to P yields radii $r(u)$, $\rho(u)$, $r(s)$, $\rho(s)$ and C^1-maps u, s; a one-dimensional local unstable manifold U of ϕ^*, a local stable manifold S of ϕ^* with codimension 1, and an open neighborhood N^u corresponding to D^u in Theorem II.1.1.

COROLLARY 1. For every solution $y:[-1,\infty) \to \underline{R}$ of eq. (g_1) with $y_0 \in S$,

$$|y| < \frac{\gamma}{2} \text{ on } [-1,\infty) .$$

PROOF. Corollary II.1.1 says that y_0 defines a forward trajectory of P. The definition of P, $\tau(N) \subset (1,2)$, $r_1 < r_0$ and the choice of r_0 (Section I.7) imply the assertion.

COROLLARY 2. Let $\rho > 0$ with $[-\rho,\rho]\lambda \subset L_{r(u)}^{u*}$ be given. There exists $\rho' \in (0,\rho)$ such that for every open subset

$$\Delta \supset \phi^* + \{\phi + u(\phi): \phi \in [\rho',\rho]\lambda\} \text{ of } H$$

$$(\ \ldots \qquad\qquad \in [-\rho,-\rho']\lambda\} \text{ of } H)$$

there is $r(\Delta) \in (0, r_1)$ with the following properties.

(i) $\quad p^s H_{r(\Delta)}^* \subset L_{r(s)}^{s*}$

(ii) For every solution $y:[-1,\infty) \to \underline{R}$ of eq. (g_1) with

$$y_0 \in \phi^* + H^*_{r(\Delta)} \quad \text{and} \quad [y_0 - \phi^*]^u - s([y_0 - \phi^*]^s) \in (0,\infty)\lambda$$
$$(\ldots \quad \text{and} \quad \ldots \quad \in (-\infty,0)\lambda \ )$$

there exists $t(\Delta) = t(\Delta, y_0) > 0$ with

$$|y| < \frac{\gamma}{2} \text{ on } [-1,t(\Delta)] \text{ and } y_{t(\Delta)} \in \Delta .$$

PROOF. Let $\rho > 0$ be given as above. Consider ρ' according to Proposition II.1.1. Let Δ be given as above. Consider $r(\Delta)$ according to Proposition II.1.1. We may assume $r(\Delta) < r_1$. Property (i) is satisfied.

Let $y:[-1,\infty) \to \underline{R}$ be a solution of eq. (g_1) with

$$\phi := y_0 \in \phi^* + H^*_{r(\Delta)} \quad \text{and} \quad [\phi - \phi^*]^u - s([\phi - \phi^*]^s) \in (0,\infty)\lambda$$
$$(\ldots \quad \text{and} \quad \ldots \quad \in (-\infty,0)\lambda \ ) .$$

Then there are points $\phi_0 = \phi$, ϕ_1 , ... , ϕ_j , $j \in \underline{N}$, in D with

$$\phi_{k+1} = P(\phi_k) \quad \text{for } k = 0 , \ldots , j-1$$

and

$$\phi_j \in \Delta .$$

Set $t(\Delta) := \Sigma_{k=0}^{j-1} \tau(\phi_k)$. The definition of P , $\tau(N) \subset (1,2)$, $r_1 < r_0$ and properties of r_0 (Section I.7) imply the remaining part of the assertion.

Corollary 2 will be used in Section 3 below.

The next result describes solutions $y:[-1,\infty) \to \underline{R}$ of eq. (g_1) which start on U close to ϕ^* . Then $y(0) = 0 = x(0)$. It is natural to expect $y_t > x_t$ (or $y_t < x_t$) after a while, for the following reason. The difference $y_0 - \phi^*$ is close to a multiple of the tangent vector λ to U at ϕ^* ; λ is close to ζ (Corollary I.1); ζ is positive on "most" of $[-1,0]$; pairs of initial conditions with positive difference define solutions with increasing difference (Section I.7).

PROPOSITION 1. There exists $r_2 > 0$ with

(i) $(-r_2,r_2)\lambda \subset L^{u*}_{r(u)}$

such that

(ii) for every solution $y:[-1,\infty) \to \underline{R}$ of eq. (g_1) with $y_0 = \phi^* + \phi + u(\phi)$ and $\phi \in (0,r_2)\lambda$ (and $\phi \in (-r_2,0)\lambda$) ,

(ii.1) $|y| < \frac{\gamma}{2}$ on $[-1,2]$

and

(ii.2) $y > x$ on $(0,2]$ ($y < x$ on $(0,2]$) .

PROOF. 1. $u(0) = 0$ and $Du(0) = 0$ permit to find $r_2 \in (0,r_2') \cap (0,1)$ with (i) and

$$|u(r\lambda)| \leq |r\lambda| \min \{1, \frac{\varepsilon}{3|\lambda|}\} \quad \text{for } -r_2 < r \leq r_2 .$$

2. Let $r \in (-r_2,r_2)$, $\phi := r\lambda$. Consider the solution $y:[-1,\infty) \to \underline{R}$ with $y_0 = \phi^* + \phi + u(\phi) \in U \subset \phi^* + C_{r_0}$. The choice of r_0 (Section I.7) implies $|y| < \frac{\gamma}{2}$ on $[-1,2]$. This proves assertion (ii.1).

3. Eq. (g_1) , $g_1 = ag_0$ (see Section 1) and $y(0) = 0 = x(0)$ yield

$$y(t) - x(t) = a \int_{-1}^{t-1} \{g_0 \circ [\phi^* + r\lambda + u(r\lambda)] - g_0 \circ \phi^*\}$$

for $0 \leq t \leq 1$.

The Mean Value Theorem implies that for every $v \in [-1,0]$ there exists $\theta_v \in [0,1]$ with

$$g_0([\ldots](v)) - g_0(\phi^*(v)) - r\zeta(v) =$$
$$g_0'(\phi^*(v) + \theta_v [r\lambda + u(r\lambda)](v)) [\ldots](v) - r\zeta(v) =$$
$$g_0'(\ldots)[r\lambda(v) - r\zeta(v)] + (g_0'(\ldots) - 1) r\zeta(v) + g_0'(\ldots) u(r\lambda)(v) .$$

The argument of g_0' is bounded by $\min \{\delta,\gamma\}$ (see part 1 and Corollary 1.2 (iv)) so that Corollary 1.2 (ii) yields

$$|g_0'(\ldots)| \leq 1 \quad \text{and} \quad |g_0'(\ldots) - 1| < \frac{\varepsilon}{3|\zeta|} .$$

With Corollary 1.2 (iii), we obtain

$$|g_0([\ldots](v)) - g_0(\phi^*(v)) - r\zeta(v)| < |r|\frac{\varepsilon}{3} + \frac{\varepsilon}{3|\zeta|}|r||\zeta| + |r\lambda|\frac{\varepsilon}{3|\lambda|}$$
$$= |r|\varepsilon .$$

In case $0 < r < r_2$, division by r yields

$$|\frac{1}{r}(g_1([\ldots](v)) - g_1(\phi^*(v))) - \zeta(v)| < \varepsilon \quad \text{on } [-1,0] .$$

With Corollary 1.2 (i), we infer

$$0 < \frac{1}{r}\int_{-1}^{t-1} \{\ldots\} \quad \text{on } (0,1]$$

which implies

$$y > x \quad \text{on } (0,1] .$$

For some $t \in (1,2)$, $y_t > x_t$. The choice of a_5 in Section I.7, $g_1 = ag_0$ with $\alpha < a < a_5$ and assertion (ii.1) permit to apply Proposition I.7.1 (i); we obtain $y_2 > x_2$.

In case $-r_2 < r < 0$, we find

$$0 < \frac{1}{r}\int_{-1}^{t-1} \{\dots\} \quad \text{for} \quad 0 < t \leq 1 \;;$$

hence

$$a\int_{-1}^{t-1} \{\dots\} < 0 \;,$$

and

$$y < x \quad \text{on} \quad (0,1] \;.$$

As above, one concludes that $y < x$ on all of $(0,2]$.

3. INTERSECTING H ABOVE AND BELOW THE LOCAL STABLE MANIFOLD

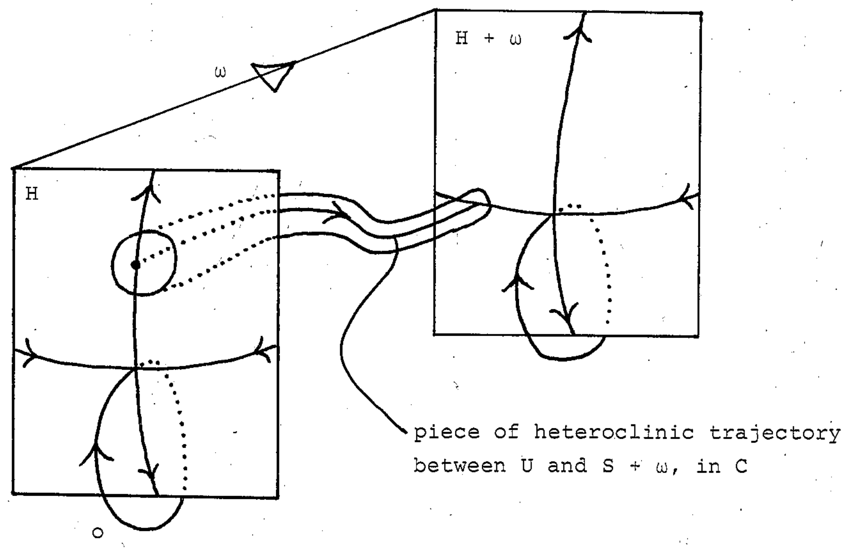
Set $o := \{x_t : t \in \underline{R}\}$. Recall $|x| < \frac{\gamma}{4}$.

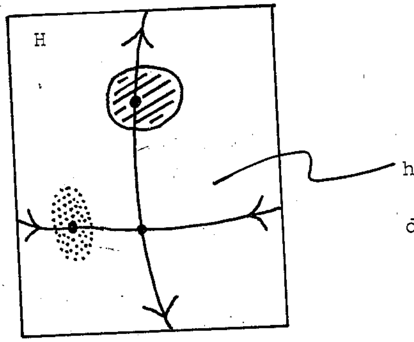
REMARK 1. Our aim is to change g_1 outside the interval $[-\gamma,\gamma]$ to a periodic function g which is sine-like in the sense that there is only one zero in the period interval $(0,\omega)$, in such a way that there is a heteroclinic solution $h:\underline{R} \to \underline{R}$ of eq. (g):

$h_0 \in U$ (and $h_t \to o$ as $t \to -\infty$),

and for some $t > 0$, $h_t \in S + \omega$ (so that $h_{t'}$ converges to the periodic orbit $o + \omega$ as $t' \to \infty$).

h_0 will then become a homoclinic point of a modification P' of P which on a suitably small neighborhood of h_0 in H is given by following the semiflow X of eq. (g) until it reaches $H + \omega$, and by a translation modulo ω back to H.





Let us defer the discussion of transversality of the homoclinic point h_0 to Section 4, and concentrate on the search for g with a heteroclinic solution. The target set $S + \omega$ will be a submanifold of codimension 1 in the hyperplane $H + \omega$ which divides a neighborhood of $\phi^* + \omega$ into upper and lower halves, both open in $H + \omega$. It seems not likely that one can construct g so that a solution h reaches the "thin" set $S + \omega$; it will certainly be easier to find g so that a solution reaches the open halves above and below $S + \omega$. This is what we shall pursue, with an application of the Intermediate Value Theorem in mind, in order to find a nonlinearity with a heteroclinic solution.

DEFINITION. Let $\phi \in H$ with $[\phi - \phi^*]^s \in L^{s*}_{r(s)}$ be given. ϕ is said to lie above (below) S if and only if

$$[\phi - \phi^*]^u - s([\phi - \phi^*]^s) \in (0,\infty)\lambda$$

$$(\ldots \qquad \ldots \in (-\infty,0)\lambda \ ) .$$

Clearly $\phi \in S$ is equivalent to

$$[\phi - \phi^*]^u - s([\phi - \phi^*]^s) = 0 .$$

The next result provides inequalities for initial values close to ϕ^* which guarantee that solutions reach H above or below S. This will facilitate the construction of g considerably. The proof uses Corollary 2.2, i.e., a consequence of the technical Proposition II.1.1 on attraction to subsets of local unstable manifolds, together with Proposition 2.1 and parts of Section I.7 which are specific results for eq. (g_1).

PROPOSITION 1. There exists an open neighborhood $N_+ \subset N$ of ϕ^* in C with the following properties.

For every solution $y:[-1,\infty) \to \underline{R}$ of eq. (g_1) with $y_0 = \phi \in N_+$,

$$|y| < \frac{\gamma}{2} \text{ on } [-1,2] \text{ and } [y_{\tau(\phi)} - \phi^*]^s \in L^{s*}_{r(s)} ,$$

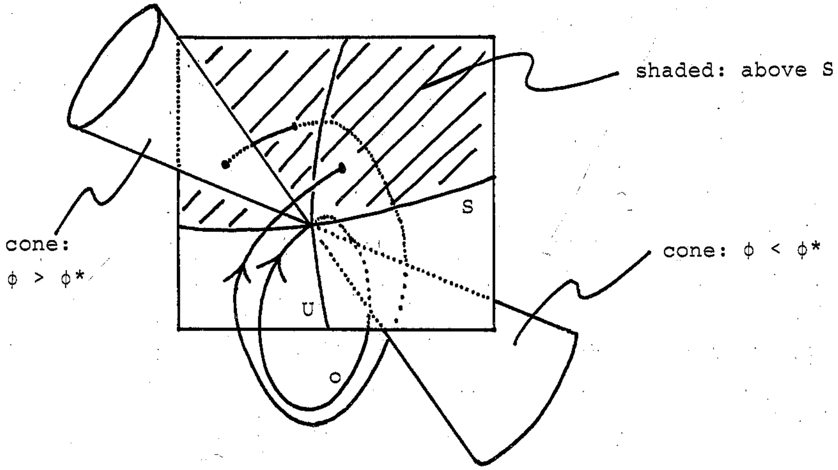
and $y_{\tau(\phi)}$ is above, on or below S.

In case $\phi^* < \phi$ ($\phi < \phi^*$), $y_{\tau(\phi)}$ is above (below) S, and there exists $\tilde{t} = \tilde{t}(\phi) > 0$ with

$$|y| < \gamma \text{ on } [-1,\tilde{t}) \text{ and } y(\tilde{t}) = \gamma \quad (\ldots \text{ and } y(\tilde{t}) = -\gamma) .$$

In case $y_{\tau(\phi)} \in S$,

$$|y| < \frac{\gamma}{2} \text{ on } [-1,\infty) .$$



A trajectory $0 \leq t \to x_t \in C$ starting in N_+ with $x_0 > \phi^*$ reaches H after a time of flight $\tau(x_0) \approx \frac{4}{3}$ above S.

REMARK 2. Observe that the intersection of the open cone with vertex ϕ^* defined by $\phi < \phi^*$ or $\phi > \phi^*$ with H is empty.

PROOF of Proposition 1. 1. Choose $\rho \in (0,r_2)$ with $[-\rho,\rho]\lambda \subset L^{u*}_{r(u)}$. Consider $\rho^+ \in (0,\rho)$ such that the assertion of Corollary 2.2 holds true, with $\rho' = \rho^+$. We show that there exists an open neighborhood Δ^+ of $\phi^* + \{\phi + u(\phi) : \phi \in [\rho^+,\rho]\lambda\}$ in C such that for every solution $y:[-1,\infty) \to \underline{R}$ of eq. (g_1) with $y_0 \in \Delta^+$,

$$|y| < \frac{\gamma}{2} \text{ on } [-1,2] \text{ and } x < y \text{ on } [1,2] \quad :$$

Because of $\rho < r_2$, Proposition 2.1 implies that for every $\tilde{\phi}$ in $\phi^* + \{\phi + u(\phi) : \phi \in [\rho^+,\rho]\lambda\}$ there is an open neighborhood $\tilde{\Delta}$ in C with

$$|y| < \frac{\gamma}{2} \text{ and } x < y \text{ on } [1,2] \text{ for every solution } y:[-1,\infty) \to \underline{R}$$
of eq. (g_1) with $y_0 \in \tilde{\Delta}$.

2. Similarly, there is $\rho^- \in (0,\rho)$ so that the assertion in brackets of Corollary 2.2 holds true, with $\rho' = \rho^-$, and there is an open neighborhhod Δ^- of $\phi^* + \{\phi + u(\phi) : \phi \in [-\rho,-\rho^-]\lambda\}$ with

$$|y| < \frac{\gamma}{2} \text{ on } [-1,2] \text{ and } x > y \text{ on } [1,2] \text{ for all solutions}$$
$y:[-1,\infty) \to \underline{R}$ of eq. (g_1) with $y_0 \in \Delta^-$.

3. Corollary 2.2, applied to $\Delta^+ \cap H$ and to $\Delta^- \cap H$, yields existence of $r^+ = r(\Delta^+ \cap H)$ and $r^- = r(\Delta^- \cap H)$ in $(0,r_1)$ with

$$(*) \qquad [\phi - \phi^*]^s \in L^{s*}_{r(s)} \text{ for all } \phi \in \phi^* + H^*_{r^*}$$
where $r^* := \min\{r^+,r^-\}$

such that for every solution $y:[-1,\infty) \to \underline{R}$ of eq. (g_1) with

$$y_0 \in \phi^* + H^*_{r^*} \text{ and } [y_0 - \phi^*]^u - s([y_0 - \phi^*]^s) \in (0,\infty)\lambda$$
$$(\ldots \text{ and } \ldots \in (-\infty,0)\lambda\ )$$

there exists $t(y) = t(\Delta^+ \cap H, y_0) > 0$ $(\ldots t(y) = t(\Delta^- \cap H, y_0) > 0\ )$ with

$$|y| < \frac{\gamma}{2} \text{ on } [-1,t(y)) \text{ and } y_{t(y)} \in \Delta^+ \cap H \quad (\ldots \in \Delta^- \cap H\ ).$$

4. Corollary 2.1 yields $|y| < \frac{\gamma}{2}$ on $[-1,\infty)$ for solutions $y:[-1,\infty) \to \underline{R}$ of eq. (g_1) with $y_0 \in \phi^* + H^*_{r^*}$ and $y_0 \in S$; i.e.,

$$[y_0 - \phi^*]^u - s([y_0 - \phi^*]^s) = 0 .$$

5. Parts 3,1,2 above imply that for solutions $y:[-1,\infty) \to \underline{R}$ of eq. (g_1) with $y_0 \in \phi^* + H^*_{r^*}$ and $[y_0 - \phi^*]^u - s([y_0 - \phi^*]^s) \in (0,\infty)\lambda$ $(\ldots \in (-\infty,0)\lambda\ )$,

$$|y| < \frac{\gamma}{2} \text{ in } [-1,t(y)+2] \text{ and } x < y \text{ in } [t(y)+1, t(y)+2]$$
$$(\ldots \text{ and } y < x \text{ in } \ldots\ ).$$

Recall $|x| < \frac{\gamma}{4}$. Proposition I.7.1 (ii) yields existence of $t^* = t^*(y)$ in $(t(y)+2,\infty)$ with $|y| < \gamma$ in $[-1,t^*)$ and $y(t^*) = \gamma$ $(\ldots = -\gamma\ )$.

6. Choose an open neighborhood $N_+ \subset N$ of ϕ^* in C so small that $N_+ \subset C_{\frac{\gamma}{2}}$ (recall $|x| < \frac{\gamma}{4}$), and for every solution $y:[-1,\infty) \to \underline{R}$ of eq. (g_1) with $y_0 \in N_+$,

$$|y| < \frac{\gamma}{2} \text{ on } [-1,2] \text{ and } y_{\tau(y_0)} \in \phi^* + H^*_{r^*}$$

(recall $1 < \tau(y_0) < 2$).

For such solutions, $[y_{\tau(y_0)} - \phi^*]^s \in L^{s*}_{r(s)}$, see $(*)$, and $y_{\tau(y_0)}$ is above, on or below S.

The case $x_0 < y_0 \in N_+$: Proposition I.7.1 (ii) implies that for some $\tilde{t} = \tilde{t}(y_0) > 0$, $|y| < \gamma$ in $[-1,\tilde{t})$ and $y(\tilde{t}) = \gamma$. By the choice of N_+ $\tilde{t} > 2 > \tau(y_0)$. $y_{\tau(y_0)} \in \phi^* + H^*_{r^*}$ is above, on or below S.
Assume "below". Then the solution $\tilde{y}:[-1,\infty) \to \underline{R}$ of eq. (g_1) with $\tilde{y}_0 = y_{\tau(y_0)}$ satisfies $|\tilde{y}| < \gamma$ in $[-1,t^*(\tilde{y}))$ and $\tilde{y}(t^*(\tilde{y})) = -\gamma$, see part 5. With $\tilde{y} = y(\cdot + \tau(y_0))|[-1,\infty)$, we obtain a contradiction to the properties of $\tilde{t}(y_0)$.
Assume "on". Then $|\tilde{y}| < \frac{\gamma}{2}$ in $[-1,\infty)$, see part 4. This contradicts $\tau(y_0) < \tilde{t}(y_0)$ and $y(\tilde{t}(y_0)) = \gamma$.

It follows that $y_{\tau(y_0)}$ is above S.

The proof for the case $N_+ \ni y_0 < x_0$ proceeds analogously.

The assertion for $y_{\tau(y_0)} \in S$ follows from $1 < \tau(y_0) < 2$, $|y| < \frac{\gamma}{2}$ in $[-1,2]$, and from part 4.

REMARK 3. Having Proposition 1, we shall look for periodic extensions g of $g_1|[-\gamma,\gamma]$ so that there are solutions h of eq. (g) which start in U and have segments $h_{t_+} > \phi^* + \omega$ $(h_{t_+} < \phi^* + \omega\ )$, $t_+ > 0$, in $N_+ + \omega$ where ω is the minimal period of g. The next result prepares targets modulo ω for this, i.e., functions $\phi^{**} + r > \phi^*$ $(\phi^{**} - r < \phi^*\ )$, $r > 0$, in N_+ for which it will be possible to achieve

$$h_{t_+}(t) = \phi^{**}(t) + r + \omega \quad (\ldots = \phi^{**}(t) - r + \omega\ ),$$

not immediately on all of $[-1,0]$ but first only on some subinterval

$$I \supset [-1,-\tfrac{1}{3}] .$$

We do not simply set $\phi^{**} := \phi^*$, for the following reason. A situation with $h_{t_+} = \phi^* + r + \omega$ $(\ldots = \phi^* - r + \omega)$ forces g to have zeros at $h(t_+ - \frac{4}{3})$ and at $h(t_+ - 2)$, because of eq. (g_1) and $\dot{x}(-1) = 0$, $\dot{x}(-\frac{1}{3}) = 0$. This would at least present an additional difficulty in the search for sine-like functions g with only one zero in the period interval $(0,\omega)$.

Instead, we modify ϕ^* in order to avoid the zero of $\dot{x}$ at -1.

We choose $r_3 \in (0,\frac{\gamma}{4})$ such that

(1) $\phi^* + C_{2r_3} \subset N_+$

and

(2) $\phi^*(-1) + 2r_3 < 0$.

PROPOSITION 2. There exists $t_* \in (-1,-\frac{2}{3})$ such that for $\phi^{**} \in C$ with $\phi^{**}(t) = \phi^*(t)$ in $[t_*,0]$ and $\dot{\phi}^{**}(t) = \dot{\phi}^*(t_*)$ in $[-1,t_*)$,

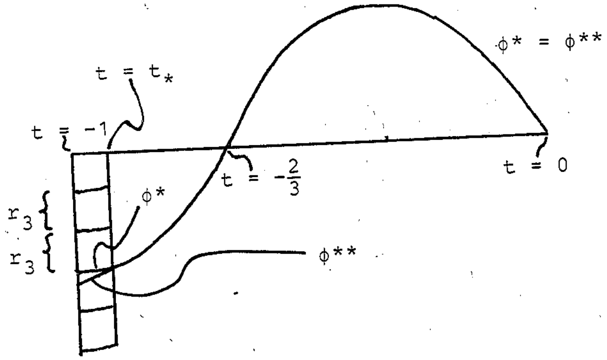
(i) $\phi^{**} - r_3 < \phi^* < \phi^{**} + r_3$.

In particular,

(ii) $\phi^{**} \in \phi^* + C_{r_3}$

and

(iii) $\phi^* - 2r_3 < \phi^{**} - r_3 < \phi^* < \phi^{**} + r_3 < \phi^* + 2r_3$.



PROOF of Proposition 2. Recall $\dot{x}(-1) = 0$. Choose $t_* \in (-1,-\frac{2}{3})$ with

$$0 < \dot{x}(t_*) < r_3 \quad\text{and}\quad x(t_*) < x(-1) + r_3 .$$

Consider ϕ^{**} as above. We have $\phi^{**}(t) = \phi^*(t)$ in $[t_*,0]$, and for $-1 \leqq t < t_*$,

$$\phi^*(t) < \phi^{**}(t_*) = x(t_*) < x(-1) + r_3 \leqq x(t) + r_3$$

(with $0 < \dot{x}$ in $(-1,-\frac{1}{3})$), and

$$\phi^{**}(t) \geqq \phi^{**}(-1) = x(t_*) + (-1 - t_*)\dot{x}(t_*)$$
$$> x(t_*) + (-1 - t_*)r_3 \qquad \text{(by the choice of } t_* \text{)}$$
$$> x(t) - r_3 \qquad \text{(with } 0 < \dot{x} \text{ in } (-1,-\tfrac{1}{3}) \text{ and } -1 - t_* > -1 \text{).}$$

In order to deduce from $h_{t_+}(t) = \phi^{**}(t) + r_3 + \omega$ $(.. = \phi^{**}(t) - r_3 + \omega)$ on some subinterval

$$I \supset [-1,-\tfrac{1}{3}]$$

of $[-1,0]$ that $h_{t_+} \in \phi^* + C_{2r_3} + \omega$ and $h_{t_+}(t) > \phi^*(t) + \omega$ $(h_{t_+}(t) < \ldots)$ on all of $[-1,0]$, we need a further elementary consideration.

We choose $r_4 \in (0,r_3)$ so small that

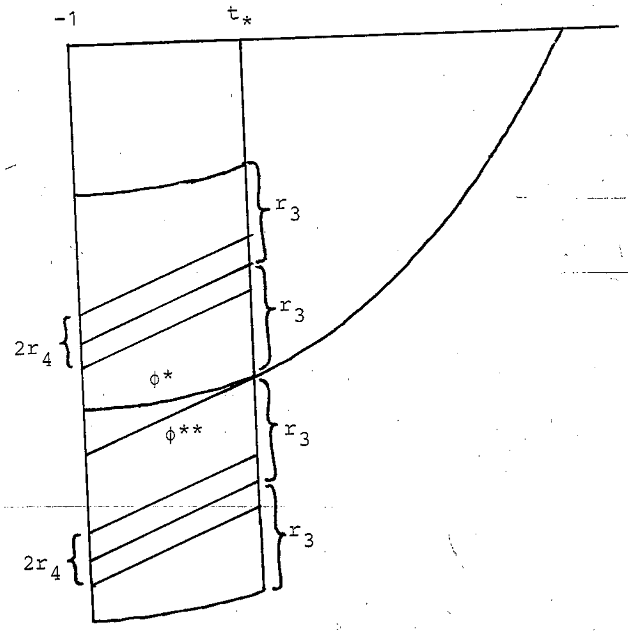
(3) $\phi^* - 2r_3 < \phi^{**} - r_3 - r_4 < \phi^{**} - r_3 + r_4 < \phi^*$
$< \phi^{**} + r_3 - r_4 < \phi^{**} + r_3 + r_4 < \phi^* + 2r_3$

and

(4) $g_1(\frac{\gamma}{4}) + r_4 < g_1(\gamma)$.

In addition, we fix a constant

$$c_1 < g_1(-\gamma) < 0 .$$



PROPOSITION 3. There exists $\theta \in (-\frac{1}{3},0)$ such that for every differentiable $\phi \in C$ with $c_1 < \dot{\phi} \leqq 0$ on $[-\frac{1}{3},0]$ the following holds.

(i) If $x(\theta) - r_3 - r_4 \leqq \phi(\theta) \leqq x(\theta) + r_3 + r_4$ then $x(t) - 2r_3 < \phi(t) < x(t) + 2r_3$ for all $t \in [\theta,0]$.

(ii) If $x(\theta) + r_3 - r_4 \leqq \phi(\theta)$ then $x(t) < \phi(t)$ for $\theta \leqq t \leqq 0$.

(iii) If $\phi(\theta) \leqq x(\theta) - r_3 + r_4$ then $\phi(t) < x(t)$ for $\theta \leqq t \leqq 0$.

PROOF. $r_4 < r_3$ and $x(0) = 0$ permit to choose $\theta \in (-\frac{1}{3},0)$ with

$$0 < \theta c_1 < r_3 - r_4 \quad \text{and} \quad x(\theta) < r_3 - r_4 .$$

Consider ϕ as above. For $\theta \leqq t \leqq 0$,

$$\phi(\theta) \geqq \phi(t) = \phi(\theta) + \int_\theta^t \dot{\phi} \geqq \phi(\theta) + (t - \theta)c_1 .$$

If in addition $\phi(\theta) \leqq x(\theta) + r_3 + r_4$ $\{\phi(\theta) \leqq x(\theta) - r_3 + r_4\}$

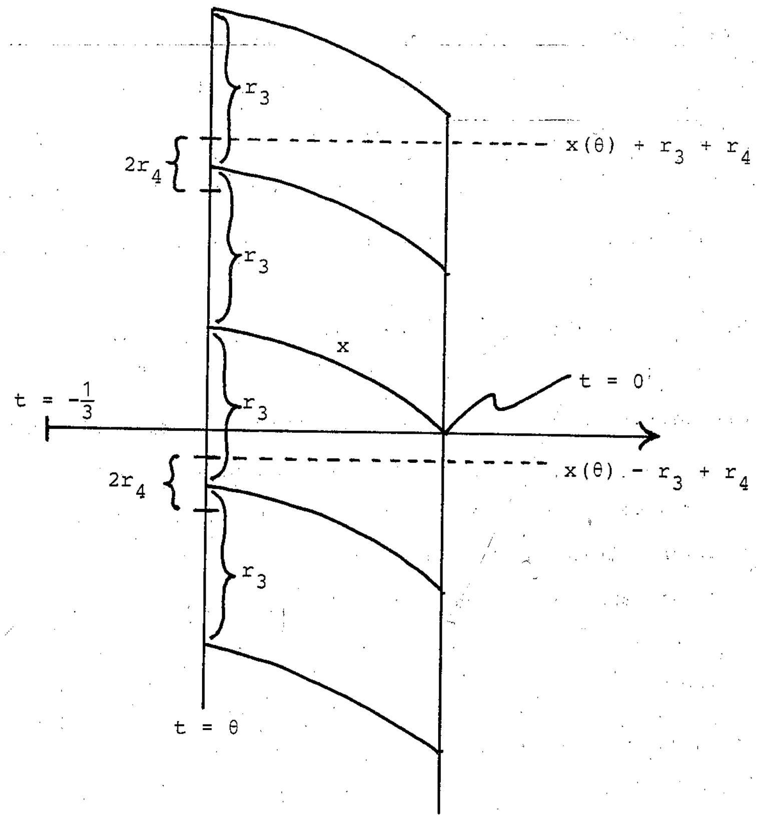
then we obtain

$$\phi(t) \leqq \phi(\theta) \leqq x(\theta) + r_3 + r_4 < 2r_3 \qquad \text{(by the choice of } \theta \text{)}$$
$$\leqq x(t) + 2r_3 \qquad \text{(by } 0 \leqq x \text{ in } [-\tfrac{2}{3},0] \text{)}$$

{then we obtain

$$\phi(t) \leqq \phi(\theta) \leqq x(\theta) - r_3 + r_4 < 0 \qquad \text{(by the choice of } \theta \text{)}$$
$$\leqq x(t) \qquad \text{(by } 0 \leqq x \text{ in } [-\tfrac{2}{3},0] \text{)}\}.$$

The choice of θ in the proof of Proposition 3:



Proof of the lower estimate: If in addition

$$x(\theta) - r_3 - r_4 \leqq \phi(\theta) \qquad \{x(\theta) + r_3 - r_4 \leqq \phi(\theta)\}$$

then

$$\phi(t) \geqq x(\theta) - r_3 - r_4 - \theta c_1 > x(t) - 2r_3$$

(with the choice of θ, and with $\dot{x} < 0$ on $(-\frac{1}{3},0]$)

{then

$$\phi(t) \geqq x(\theta) + r_3 - r_4 - \theta c_1 > x(t)$$

(with the choice of θ, and $\dot{x} < 0$ on $(-\frac{1}{3},0]$)}.

4. A CONDITION FOR TRANSVERSALITY

Recall that we want to find a periodic extension g of $g_1|[-\gamma,\gamma]$, with period ω, such that there is a heteroclinic solution h, with $\chi := h_0 \in U$ and $h_{t_+} \in N_+ + \omega$ for some $t_+ > 0$, so that after an additional time of flight $\tau(h_{t_+} - \omega)$ the segment $h_t \in C$ reaches $S + \omega$ in $H + \omega$. In terms of the modified Poincaré map P' from Remark 3.1,

$$P'(\chi) \in S .$$

The transversality property we have in mind includes

$$H = T_{P'(\chi)}S \oplus DP'(\chi)\, T_\chi U$$

or equivalently, since $T_{P'(\chi)}S$ has codimension 1 while $\dim T_\chi U = 1$,

$$DP'(\chi)v \notin T_{P'(\chi)}S \quad \text{for some nonzero } v \in T_\chi U .$$

In the present section we look for sufficient (in fact, equivalent) conditions for this which do not involve P, P' or τ and which can be expressed in terms of the nonlinearity g alone.

For t_+ as above, there is a decomposition

$$DP'(\chi) = D\underline{P}(h_{t_+} - \omega) \circ D_2X(t_+,\chi) ,$$

because of periodicity; recall $\underline{P} = \underline{P}_a$ and $\underline{P}_a(\phi) = X_{a0}(\tau_a(\phi),\phi)$ on $N_a = N \supset N_+$. Let $v \in T_\chi U \setminus \{0\}$ be given. Since $h_{t_+} - \omega \in N_+$ in the last formula and $D_2X(t_+,\chi)v \in C$ are obtained from h and from the linear variational equation along h, i.e., both in terms of g and g', we only look for a suitable condition on points $y_0 \in N_+$ with $\underline{P}(y_0) \in S$ and on vectors $w_0 \in C$ which ensures

$$D\underline{P}(y_0)w_0 \notin T_{\underline{P}(y_0)}S .$$

PROPOSITION 1. Let $y:[-1,\infty) \to \underline{R}$ be a solution of eq. (g_1) with $y_0 \in N_+$ and $\underline{P}(y_0) \in S$. Let $w:[-1,\infty) \to \underline{R}$ be a solution of

$$\dot{w}(t) = g_1'(y(t-1)w(t-1) .$$

If there exists $t \geq 0$ with $w(t') \neq 0$ for all $t' \in [t-1,t]$ then

$$D\underline{P}(y_0)w_0 \notin T_{\underline{P}(y_0)}S .$$

Moreover, the forward iterates of $\chi_1 := \underline{P}(y_0)$ and $v_1 := D\underline{P}(y_0)w_0$ given by $\chi_{n+1} = P(\chi_n) \in S$ and $v_{n+1} = DP(\chi_n)v_n$ for $n \in \underline{N}$ satisfy

$$v_n \notin T_{\chi_n}S \quad \text{for all } n \in \underline{N} .$$

PROOF. 1. Assume $0 < w$ in $[t-1,t]$ for some $t \geq 0$. (In case $w_t < 0$, consider $-w$ instead of w.) By Proposition 3.1, $|y| < \frac{\gamma}{2}$ on $[-1,\infty)$, and the coefficient in the linear variational equation along y is bounded from below by $g_1'(\frac{\gamma}{2}) > 0$. With $0 < w_t$, we infer $w(t') \to \infty$ as $t' \to \infty$.

2. Define $t_1 := \tau(y_0)$ and $t_{n+1} := t_1 + \Sigma_{k}^{n}{}_{1}\, \tau(\chi_k)$ for $n \in \underline{N}$. Then

$$\chi_n = X_1(t_n,y_0) = y_{t_n} \quad \text{for all } n \in \underline{N} ,$$

$$v_n = D(P^{n-1} \circ \underline{P})(y_0)w_0 = p_{(n)} \circ D_2X_1(t_n,y_0)w_0 \quad \text{for all } n \in \underline{N} ,$$

where $p_{(n)}\phi = \phi - \frac{\phi(0)}{\dot{y}(t_n)}\dot{y}_{t_n}$ for all $\phi \in C$ (compare Section 1).

3. Let $n \in \underline{N}$. Consider the solutions $\underline{y}:[-1,\infty) \to \underline{R}$ of eq. (g_1) with $\underline{y}_0 = \chi_n = y_{t_n}$, and $\underline{w}:[-1,\infty) \to \underline{R}$ of

$$\dot{w}(t') = g_1'(\underline{y}(t' - 1))w(t' - 1) , \; w_0 = v_n .$$

We show that there exists $\underline{t} > 0$ with $\underline{w} > 0$ on $[\underline{t}-1,\underline{t}]$:
From $\underline{y} = y(\cdot + t_n)|[-1,\infty)$ we see that $\underline{w}$, $w(\cdot + t_n)$ and $\dot{y}(\cdot + t_n)$ satisfy the same linear variational equation on $\underline{R}^+$. Part 2 implies

$$\underline{w}_0 = v_n = p_{(n)}\, w_{t_n} = w_{t_n} - \frac{w(t_n)}{\dot{y}(t_n)}\dot{y}_{t_n}$$

for the initial values. Hence

$$\underline{w} = w(\cdot + t_n) - \frac{w(t_n)}{\dot{y}(t_n)}\dot{y}(\cdot + t_n) \quad \text{on } [-1,\infty) .$$

Note $\dot{y}(t' + t_n) \in g_1([-\frac{\gamma}{2},\frac{\gamma}{2}])$ for $-1 \leq t'$. Recall part 1.

4. Suppose $v_n \in T_{\chi_n}S$. There is a C^1-curve $\rho:(-1,1) \to H$ with $\rho((-1,1)) \subset S$, $\rho(0) = \chi_n$, $D\rho(0)1 = v_n$. $0 < \underline{w}_{\underline{t}}$ gives $0 < D_2X_1(t,\rho(0))\underline{w}_0$. Continuity implies that there are a constant function $\phi > 0$ in C and $\varepsilon > 0$ such that

$$D_2X_1(\underline{t},\rho(\cdot))\, D\rho(\cdot)\, 1 \geq \phi \quad \text{on } [-\varepsilon,\varepsilon] .$$

Let $\tilde{y}:[-1,\infty) \to \underline{R}$ denote the solution of eq. (g_1) with $\tilde{y}_0 = \rho(\varepsilon) \in S$. By Corollary 2.1, $|\tilde{y}| < \frac{\gamma}{2}$ on $[-1,\infty)$. $|y| < \frac{\gamma}{2}$ on $[-1,\infty)$ and $\underline{y}(t) = y(t + t_n)$ for $t \geq -1$ (see part 3) give $|\underline{y}| < \frac{\gamma}{2}$ on $[-1,\infty)$, too. We obtain

$$\tilde{y}_{\underline{t}} - \underline{y}_{\underline{t}} = X_1(\underline{t},\rho(\varepsilon)) - X_1(\underline{t},\rho(0))$$

$$= \int_0^{\varepsilon} D(\varepsilon'' \to X_1(\underline{t},\rho(\varepsilon'')))(\varepsilon')1\, d\varepsilon' = \int_0^{\varepsilon} D_2X_1(\underline{t},\rho(\varepsilon'))D\rho(\varepsilon')1\, d\varepsilon'$$

$$\geqq \varepsilon\phi > 0$$

(with the monotonicity of the C-valued Riemann integral).

An application of Proposition I.7.1 (i) to the difference

$$d:[-1,\infty) \ni t \to \tilde{y}(t + \underline{t}) - \underline{y}(t + \underline{t}) \in \underline{R}$$

yields

$$0 < g_1'(\gamma)\ \varepsilon\phi(0) \leqq \dot{d} \quad \text{on} \quad [-1,\infty) ,$$

a contradiction to boundedness of $\tilde{y}$ and $\underline{y}$.

REMARK. The condition in Proposition 1 is also necessary for

$$v_n \notin T_{\chi_n} S \quad \text{for all} \quad n \in \underline{N} .$$

Sketch of proof. Let y , w be given as in Proposition 1. The points χ_n in S tend to ϕ^* as $n \to \infty$. By an inclination lemma for maps which are not necessarily invertible (see [8,36,18]),

$$\frac{1}{|v_n|} v_n \to \pm \frac{1}{|\lambda|} \lambda \quad \text{as} \quad n \to \infty$$

(Here one uses $v_n \notin T_{\chi_n} S$.). Corollary 1.2 (v) and continuity now imply that for n sufficiently large, $D_2X_1(2,\chi_n)v_n$ has no zero in $[-1,0]$. Consider $t_n := \tau(y_0) + \Sigma_{k}^{n-1}{}_1\ \tau(\chi_k)$ as in the proof of Proposition 1. The solution $\tilde{w}:[-1,\infty) \to \underline{R}$ of the linear variational equation along $y(\cdot + t_n)$ with $\tilde{w}_0 = v_n$ satisfies $|\tilde{w}(t')| \to \infty$ as $t' \to \infty$ because $\tilde{w}_2 = D_2X_1(2,\chi_n)v_n$ has no zero and $g_1'(y(\cdot + t_n - 1)) \geqq g_1'(\frac{\gamma}{2}) > 0$. $w(\cdot + t_n)$ and $\dot{y}(\cdot + t_n)$ satisfy the same linear variational equation as $\tilde{w}$, and for the initial values we have

$$v_n = D(P^{n-1} \circ \underline{P})(y_0)w_0 = P_{(n)} \circ D_2X_1(t_n,y_0)w_0$$

$$= w_{t_n} - \frac{w(t_n)}{\dot{y}(t_n)} \dot{y}_{t_n} .$$

Therefore $\tilde{w} = w(\cdot + t_n) - \frac{w(t_n)}{\dot{y}(t_n)} \dot{y}(\cdot + t_n)$ on $[-1,\infty)$, and boundedness of $\dot{y}$ yields $|w(t')| \to \infty$ as $t' \to \infty$.

For the application of Proposition 1 we need

PROPOSITION 2. Let $y:\underline{R} \to \underline{R}$ be a solution of eq. (g_1) with $y_0 \in N_+$ and $\underline{P}(y_0) \in S$. Let $w:[-1,\infty) \to \underline{R}$ be a solution of

$$\dot{w}(t) = g_1'(y(t-1))w(t-1) ,$$

and let $c \in \underline{R}$ be given such that $w_0 - c\dot{y}_0$ has no zero.

Then there exists $t \geqq 0$ such that w_t has no zero.

PROOF. $y_0 \in N_+$ and $\underline{P}(y_0) \in S$ imply $|y| < \frac{\gamma}{2}$ on $[-1,\infty)$ so that the coefficient of the linear variational equation along y is bounded from below by $g_1'(\frac{\gamma}{2}) > 0$. It follows that the solution $w - c\dot{y}$ tends to ∞ , or to $-\infty$, as $t \to \infty$. Note that $\dot{y}(t) \in g_1([-\frac{\gamma}{2},\frac{\gamma}{2}])$ on $\underline{R}^+$.

IV. HETEROCLINIC CONNECTIONS BETWEEN PERIODIC ORBITS, HOMOCLINIC POINTS OF POINCARE MAPS, TRANSVERSALITY

Summary. The program sketched in Remark III.3.1 and in Section III.4.1 is carried out. We choose an initial value χ on the upper branch of U and a tangent vector $v \in T_\chi U$. The nonlinearity g_1 is deformed outside the interval $[-\gamma,\gamma]$ to a one-parameter family of periodic nonlinearities g_b, $4 \leqq b \leqq b^*$, all with a fixed period ω, such that at a time $t_+ > 0$ the solutions h_b of eq. (g_b) through χ satisfy

$$h_{b,t_+} \in N_+ + \omega \text{ , with } h_{4,t_+} < \phi^* + \omega < h_{b^*,t_+} .$$

Continuity and Proposition III.3.1 then yield $b \in (4,b^*)$ with

$$(*) \qquad \underline{P}(h_{b,t_+} - \omega) \in S .$$

In the construction of the functions g_b, care is taken that the solutions w_b of the linear variational equation along h_b, with $w_{b,0} = v$, share a property which in case $(*)$ implies that

$$"y_0" = h_{b,t_+} - \omega \quad \text{and} \quad "w_0" = w_{b,t_+}$$

satisfy the condition of Proposition III.4.1 for transversality.

For $g := g_b$ with $(*)$ the modified return map P' is defined. χ becomes a transversal homoclinic point of P', with respect to the hyperbolic fixed point ϕ^*.

1. ESCAPE FROM x IN DOMAINS OF MONOTONICITY

It is convenient to begin with the additional assumption

(A) $\quad 0 < g_1'$ on (γ,∞) .

We choose an initial value $\chi \in U$ above S and a tangent vector $v \in T_\chi U \subset H$ according to

PROPOSITION 1. There exists $r \in (0,r_2)$ so that

$$\chi := \phi^* + r\lambda + u(r\lambda) \quad \text{and} \quad v := Du(r\lambda)\lambda + \lambda \in T_\chi U$$

have the following properties.

(i) There are a trajectory $(\chi_n)_{-\infty}^{0}$ of P in U with $\chi_0 = \chi$ and $\chi_n \to \phi^*$ as $n \to -\infty$, and a solution $h_1 : \underline{R} \to \underline{R}$ of eq. (g_1) with $h_{1,t_n} = \chi_n$ for all $n \in -\underline{N}_0$

where $(t_n)_{-\infty}^{0}$ in $\underline{R}^-$ is given by $t_0 = 0$ and $t_{n-1} + \tau(\chi_{n-1}) = t_n$ for all $n \in -\underline{N}_0$. We have

(i.1) $\dot{h}_1(0) < 0$.

(ii) h_1 is C^2 , and

(ii.1) $-\gamma < h_1$ on $\underline{R}$.

(ii.2) $|h_1| < \frac{\gamma}{2}$ on $(-\infty,2]$.

(ii.3) $x < h_1$ on $[1,2]$.

There exists $t_1^* > 2$ with $t_1^* \notin \underline{N}$ such that

(ii.4) $h_1 \leqq h_1(t_1^*)$ on $(-\infty,t_1^*]$,

(ii.5) $\gamma + g_1(\gamma) \leqq h_1(t_1^*)$,

(ii.6) $0 < \dot{h}_1$ on $[t_1^*,\infty)$.

(iii) The solution $w_1 : [-1,\infty) \to \underline{R}$ of

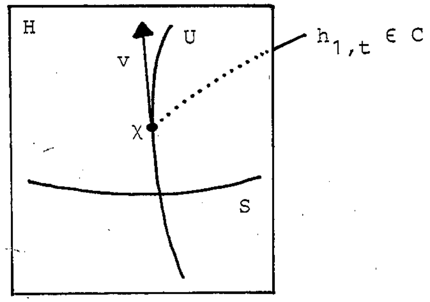
$$\dot{w}(t) = g_1'(h_1(t-1))w(t-1) \text{ , } w_0 = v$$

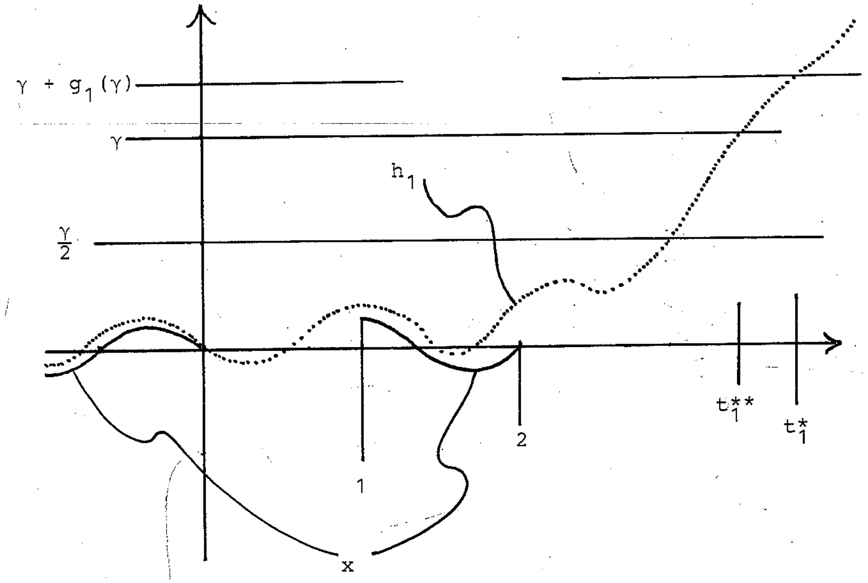
satisfies

(iii.1) $0 < w_1$ on $[1,\infty)$,

(iii.2) $0 < \dot{w}_1$ on $[2,\infty)$.

COMMENT. " $t_1^* \notin \underline{N}$ " in (ii) is technically convenient for Proposition 2 below.





PROOF of Proposition 1. 1. For $0 < r < r_2$, $r\lambda \in L^{u*}_{r(u)}$ (see Proposition III.2.1 (i)) so that $u(r\lambda)$ is defined. We have

$$\chi_r^* := \phi^* + r\lambda + u(r\lambda) \to \phi^* \quad \text{as} \quad r \downarrow 0 ,$$

hence $g_1(\chi_r^*(-1)) \to g_1(x(-1)) < 0$.

Differentiation of the curve $(-r, r_2 - r) \ni t \to \chi_{r+t}^* \in H$ at $t = 0$ yields $v_r := Du(r\lambda)\lambda + \lambda \in T_{\chi_r^*}U$. By $Du(0) = 0$, $\lim_{r \downarrow 0} v_r = \lambda \neq 0$.

In view of Corollary III.1.2 (v) we choose $r \in (0, r_2)$ with

$$g_1(\chi_r^*(-1)) < 0 \quad \text{and} \quad D_2X_1(2, \chi_r^*)v_r > 0 ,$$

and set $\chi := \chi_r^*$, $v := v_r$.

2. $\chi_0 := \chi \in U$ extends to a (discrete) backward trajectory $(\chi_n)_{-\infty}^0$ of P (Corollary II.1.1 (ii)). Using the solutions of eq. (g_1) which start at χ_n one can construct h_1 so that assertion (i) holds true (observe $\dot{h}_1(0) = g_1(\chi_r^*(-1)) < 0$). We have

$$(1) \qquad t_{n-1} + 1 < t_n < t_{n-1} + 2 \quad \text{for all} \quad n \in -\underline{N}_0 .$$

3. As a solution on $\underline{R}$, h_1 is C^1. Eq. (g_1) with g_1 C^1 implies that $\dot{h}_1$ is also C^1 so that h_1 is C^2.

4. $h_{1,t_n} = \chi_n \in U \subset D \subset \phi^* + C_{r_0}$, the choice of r_0 in Section I.7 and (1) yield

$$|h_1| < \frac{\gamma}{2} \quad \text{on} \quad (-\infty, 2] .$$

In addition, $x_2 < h_{1,2}$ because of $h_{1,0} = \chi_0 = \chi_r^*$, $0 < r < r_2$ (see Proposition III.2.1 (ii.2)). Using Proposition I.7.1 (ii) we infer that there exists $t_1^{**} > 2$ with

$$h_1(t_1^{**}) = \gamma \quad \text{and} \quad |h_1| < \gamma \quad \text{on} \quad [-1, t_1^{**}) .$$

5. We prove $\frac{\gamma}{2} < h_1$ on $[t_1^{**}, t_1^{**} + 1]$: Set $t' := t_1^{**} - 2 > 0$. Consider restrictions of $h_1(2 + \cdot)$ and $x(2 + \cdot)$ to $[-1, \infty)$. Note

$$(2) \qquad h_1(2 + t') - x(2 + t') = \gamma - x(2 + t') > \gamma - \frac{\gamma}{4} = \frac{3\gamma}{4} .$$

Proposition I.7.1 (i) is applicable, because of $x(2 + \cdot) < h_1(2 + \cdot)$ on $[-1, 0]$, $|h_1| \leq \gamma$ on $(-\infty, t_1^{**}]$, $|x| < \frac{\gamma}{4}$. We get

$$0 < \dot{h}_1(2 + \cdot) - \dot{x}(2 + \cdot) \quad \text{on} \quad [t', t' + 1] .$$

With (2),

$$h_1(2 + \cdot) - x(2 + \cdot) > \frac{3\gamma}{4} \quad \text{on} \quad [t', t' + 1] ,$$

i.e. for $t_1^{**} \leq t \leq t_1^{**} + 1$, $t-2 \in [t', t' + 1]$ and

$$h_1(t) = h_1(2 + t - 2) > \frac{3\gamma}{4} + x(2 + t - 2) > \frac{3\gamma}{4} - \frac{\gamma}{4} = \frac{\gamma}{2} .$$

6. By (A), $0 < g_1(\gamma) \leq g_1$ on $[\gamma, \infty)$. Using part 5 and eq. (g_1), we obtain

$$0 < h_1 \quad \text{on} \quad [t_1^{**}, \infty) \quad \text{and} \quad 0 < g_1(\tfrac{\gamma}{2}) \leq \dot{h}_1 \quad \text{on} \quad [t_1^{**} + 1, \infty) ,$$

hence $h_1(t) \to \infty$ as $t \to \infty$. Now it is easy to find t_1^* as desired.

7. Proof of assertion (iii) : By the choice of $v = v_r$ in part 1,

$$w_{1,2} = D_2X_1(2,\chi)v > 0$$

so that the differential equation for w_1, (ii.1) and $0 < g_1'$ on $[-\gamma,\infty)$ yield $0 < \dot{w}_1$ on $[2,\infty)$.

The first modification of g_1 outside the interval $[-\gamma,\gamma]$ is a deformation to a constant in some interval to the right of

$$h_1(t_1^* + 1) \qquad > h_1(t_1^*) > \gamma + g_1(\gamma) .$$

This step is already important for transversality; it permits to obtain basic relations (Proposition 2 below) for the verification of the condition

$$" \; |w_0 - c\dot{y}_0| > 0 \; "$$

in Proposition III.4.2. (The application of Proposition 2 below follows in the proof of Proposition 3.2 below. For more explanations, see Remark 3.1.)

Define $\gamma_1 := h_1(t_1^* + 1)$. We have

$$\gamma < \gamma + g_1(\gamma) < \gamma_1 < h_1([t_1^* + 1]) < h_1(t_1^* + 2)$$

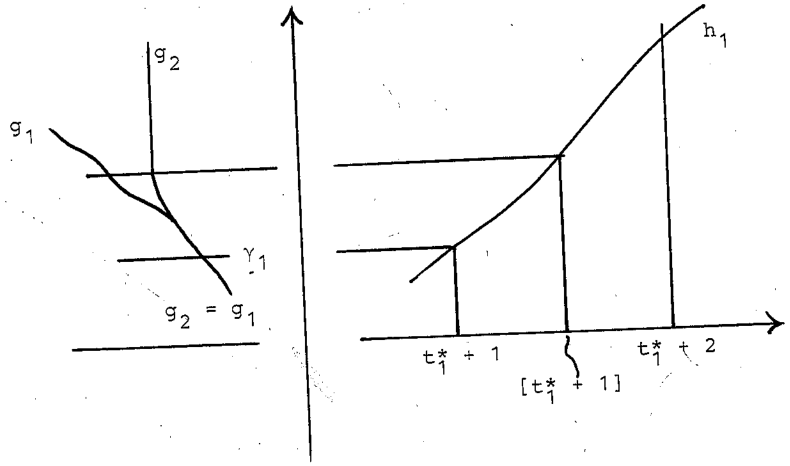
where $[t_1^* + 1]$ is the integer in $(t_1^* + 1, t_1^* + 2)$.

Choose a C^1-function $g_2:\underline{R} \to \underline{R}$ with $g_2 = g_1$ on $(-\infty,\gamma_1]$,

$$0 < g_2'(\xi) \quad \text{for} \quad \gamma_1 < \xi < h_1([t_1^* + 1]) ,$$
$$0 = g_2'(\xi) \quad \text{for} \quad h_1([t_1^* + 1]) \leq \xi .$$

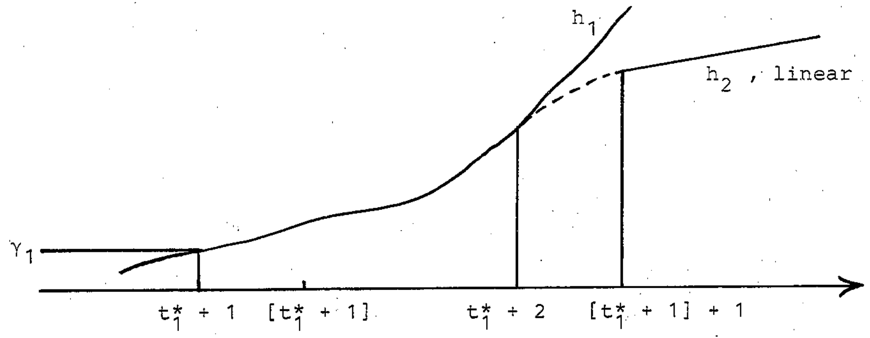
Clearly

$$0 < g_2 \text{ on } (0,\infty) \text{ and } 0 < g_2'(\xi) \text{ for } -\gamma \leq \xi < h_1([t_1^* + 1]) .$$



The restriction of h_1 to $(-\infty,t_1^* + 2]$ satisfies eq. (g_2) since $g_2 \circ h_1 = g_1 \circ h_1$ on $(-\infty,t_1^* + 1]$. It follows that there is a solution $h_2:\underline{R} \to \underline{R}$ of eq. (g_2) which is C^2, with

$$h_2 = h_1 \text{ on } (-\infty,t_1^* + 2] \text{ and } 0 < \dot{h}_2 \text{ on } [t_1^*,\infty) ,$$
$$\dot{h}_2(t) = g_2(h_2([t_1^* + 1])) \text{ for all } t \geq [t_1^* + 1] + 1 .$$



For the solution $w_2:[-1,\infty) \to \underline{R}$ of

$$\dot{w}(t) = g_2'(h_2(t-1))w(t-1) , \quad w_0 = v$$

we get

$$w_2 = w_1 \quad \text{on} \quad [-1,t_1^* + 2] ,$$
$$0 < \dot{w}_2(t) \quad \text{for} \quad t_1^* + 2 \leq t < [t_1^* + 1] + 1 ,$$
$$0 = \dot{w}_2(t) \quad \text{for} \quad [t_1^* + 1] + 1 \leq t ,$$
$$0 < w_2([t_1^* + 1] + 1) .$$

PROPOSITION 2. With $c^* := \dfrac{w_2([t_1^* + 1] + 1)}{\dot{h}_2([t_1^* + 1] + 1)} > 0$,

(i) $w_2(t) = c^* \dot{h}_2(t)$ for all $t \geq [t_1^* + 1] + 1$,

and

(ii) there exists $\varepsilon^* > 0$ with

$$w_2(t) - c^* \dot{h}_2(t) \neq 0 \quad \text{for} \quad [t_1^* + 1] + 1 - \varepsilon^* \leq t < [t_1^* + 1] + 1 .$$

PROOF. 1. Assertion (i) is obvious from the remarks above.

2. We show $0 < g_2'(h_2(t-1))$ for all $t < [t_1^* + 1] + 1$:

For $t \leq t_1^* + 2$, $h_2(t-1) = h_1(t-1) \leq h_1(t_1^* + 1)$, hence

$$h_2(t-1) \in [-\gamma, h_1(t_1^* + 1)]$$

so that

$$g_2'(h_2(t-1)) > 0 \, .$$

If $t_1^* + 2 < t < [t_1^* + 1] + 1$ then

$$-\gamma < h_1(t_1^* + 1) = h_2(t_1^* + 1) < h_2(t-1) < h_2([t_1^* + 1])$$
$$= h_1([t_1^* + 1]) \, ,$$

and the choice of g_2 yields $0 < g_2'(h_2(t-1))$.

3. By $w_{2,0} = v \in H$, $w_2(0) = 0$. Hence

$$\dot{w}_2(1) = g_2'(\ldots)w_2(0) = 0 > c^* \, g_2'(h_2(0))\dot{h}_1(0) \qquad \text{(part 2, Proposition 1 (i.1))}$$

$$= c^* \, g_2'(h_2(0))\dot{h}_2(0) = c^* \, \ddot{h}_2(1) \, ,$$

and for some $\varepsilon' > 0$,

$$w_2(t) - c^* \, \dot{h}_2(t) \neq 0 \quad \text{for all} \quad t \in [1 - \varepsilon', 1) \, .$$

4. Suppose assertion (ii) is false. Then there is a smallest n in $\{1 , \ldots , [t_1^* + 1] + 1\}$ such that for every $\varepsilon > 0$ there exists t_ε in $[n - \varepsilon, n)$ with $w_2(t_\varepsilon) - c^* \, \dot{h}_2(t_\varepsilon) = 0$. By part 3, $2 \leq n$. Consider $n - 1$. For some $\varepsilon'' > 0$,

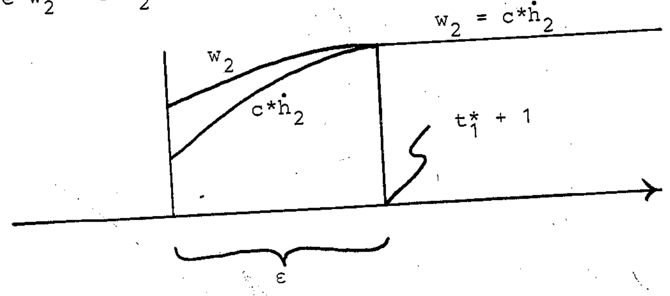
$$w_2(t) - c^* \, \dot{h}_2(t) \neq 0 \quad \text{for all} \quad t \in [n - 1 - \varepsilon'', n - 1) \, .$$

Using part 2 we infer

$$\dot{w}_2(t) - c^* \, \ddot{h}_2(t) = g_2'(h_2(t-1))\{w_2(t-1) - c^* \, \dot{h}_2(t-1)\} \neq 0$$

for $n - \varepsilon'' \leq t < n$, which implies a contradiction to the properties of n .

The case $w_2 > c^*\dot{h}_2$ in Proposition 2(ii) :



The next step is a deformation of g_2 to an <u>increasing</u> function on an interval with left endpoint <u>inside</u> the interval

$$(h_2([t_1^* + 1]), h_2([t_1^* + 1] + 1)) \, .$$

This is necessary for transversality:

Suppose g is a (periodic) continuation of the restriction of g_2 to the interval $[-\gamma, h_2([t_1^* + 1] + 1)]$. For a solution h of eq. (g) with $h_0 = \chi$ and for the solution w of the linear variational equation along h with $w_0 = v$, we infer

$$w(t) = w_2(t) = c^* \, \dot{h}_2(t) = c^* \, \dot{h}(t)$$
$$\text{for} \quad [t_1^* + 1] + 1 \leq t \leq [t_1^* + 1] + 2 \, ,$$

since $g_2'(\xi) = 0$ for $h_2([t_1^* + 1]) \leq \xi \leq h_2([t_1^* + 1] + 1)$. Thereby

$$w_t = c^* \, \dot{h}_t \quad \text{for all} \quad t \geq [t_1^* + 1] + 2 \, .$$

For the modified Poincare´map P' (see Remark III.3.1 and Section III.4), $DP'(\chi)v$ will be given by the projection of some $w_t \in C$, $t \geq [t_1^* + 1] + 2$, onto H , parallel to $\dot{h}_t$ – the result is

$$0 \in T_{P'(\chi)}S \, .$$

Set $t_3^* := [t_1^* + 1] + 1$ so that

$$t_1^* + 2 < t_3^* \, .$$

Choose $t_2^* < t_3^*$ so close to t_3^* that

$$t_3^* + \theta < t_2^* \quad \text{and} \quad t_3^* - \varepsilon^* < t_2^* \, .$$

Clearly

$$t_1^* < t_3^* - 2 < t_3^* - 1 < t_3^* - \frac{1}{3} < t_3^* + \theta \leq \max\{t_3^* + \theta, t_3^* - \varepsilon^*\}$$
$$< t_2^* < t_3^* \, .$$

Set $\gamma_2 := h_2(t_2^*) > \gamma_1$ and choose a C^1-function $g_3 : \underline{R} \to R_-$ with

$$g_3 = g_2 \quad \text{on} \quad (-\infty, \gamma_2] \, ,$$
$$0 < g_3' \quad \text{on} \quad (\gamma_2, \infty) \, .$$

The properties of h_2 , in particular

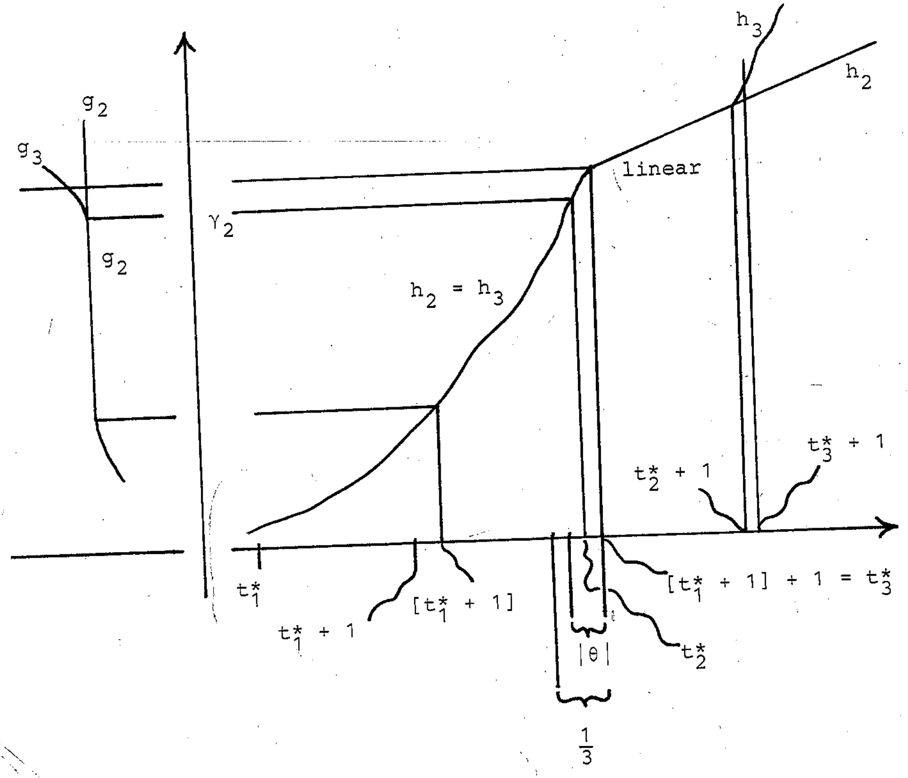
$$-\gamma < h_2(t-1) \leq h_2(t_2^*) = \gamma_2 \quad \text{for} \quad t \leq t_2^* + 1$$

and

$$\dot{h}_2(t) = g_2(h_2(t-1)) = g_3(h_2(t-1)) \quad \text{for} \quad t \leq t_2^* + 1 \, ,$$

imply that there is a C^2-solution $h_3 : \underline{R} \to \underline{R}$ of eq. (g_3) with

$$
(3)\quad \begin{cases} h_3 = h_2 & \text{on } (-\infty, t_2^* + 1] \,, \\ -\gamma < h_3 & \text{on } \underline{R} \,, \\ 0 < \dot{h}_3 & \text{on } [t_1^*, \infty) \,, \\ \gamma + g_1(\gamma) < h_3 & \text{on } (t_1^*, \infty) \,, \\ h_3(t) = \max\limits_{(-\infty,t]} h_3 & \text{for } t \geq t_1^* \,. \end{cases}
$$



2. PERIODIC NONLINEARITIES

Set $\gamma_3 := h_3(t_3^*) > \gamma_2$.

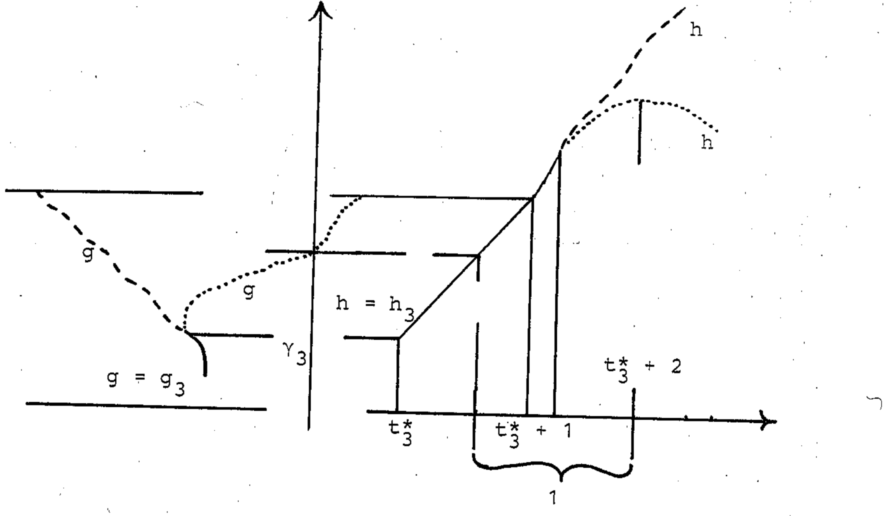
REMARK 1 - the basic observation for the search of a periodic continuation g of the restriction of g_3 to the interval $[-\gamma,\gamma_3]$ such that there is a heteroclinic solution h of eq, (g) from x to another periodic solution:

The delay in the differential equation implies that for any continuation g of $g_3|[-\gamma,\gamma_3]$, the solution $h:[-1,\infty) \to \underline{R}$ with $h_0 = \chi$ coincides with h_3 not only on $[-1,t_3^*]$ but also on the longer interval $[-1, t_3^* + 1]$. This property allows to choose g on intervals

$$h_3(I) \subset (\gamma_3,h_3(t_3^* + 1) \quad \text{with} \quad I \subset (t_3^*,t_3^* + 1]$$

so that eq. (g) steers $\dot{h}(t)$, $t \in I + 1$, close to prescribed values.

Two choices of g outside $[-\gamma,\gamma_3]$:



We shall steer $\dot{h}$ close to $\dot{\phi}^{**}$ ($\phi^{**} \in N_+$ was not far from ϕ^*, see Section III.3), and we shall find $t_+ \approx t_2^* + 2$ with

$$h_{t_+} \in N_+ + \omega$$

where ω is the minimal period of g.

The subtleties in the construction of nonlinearities until now and furtheron are mainly caused by the objective of "sine-like" functions, with not too many zeros per period, and by the aim of transversality.

We fix some $t_4^* > t_3^*$ so close to t_3^* that

$$t_3^* + \theta < t_4^* + \theta < t_2^* < t_3^* ,$$

and define a strictly increasing finite sequence of points $\gamma_n > \gamma_3$:

$$\gamma_4 := h_3(t_4^*) ,$$

$$\gamma_z := h_3(t_z) \quad \text{where} \quad t_z := t_4^* + \frac{2}{3} .$$

γ_z will be the unique zero strictly between 0 and the minimal period of the nonlinearities we are going to construct.

$$\gamma_5 := h_3(t_5^*) \quad \text{where} \quad t_5^* := t_4^* + 1 + \theta > t_z ,$$

$$\gamma_6 := h_3(t_6^*) \quad \text{where} \quad t_6^* := t_2^* + 1 > t_5^* ,$$

$$\gamma_7 := h_3(t_7^*) \quad \text{where} \quad t_7^* := t_3^* + 1 > t_6^* ,$$

$$\gamma_8 := h_3(t_8^*) \quad \text{where} \quad t_8^* := t_4^* + 1 > t_7^* .$$

$\omega := \gamma_8 + \gamma$ will be the minimal period of the nonlinearities to be constructed.

Set $t_+ := t_4^* + 2 = t_8^* + 1$. We shall have

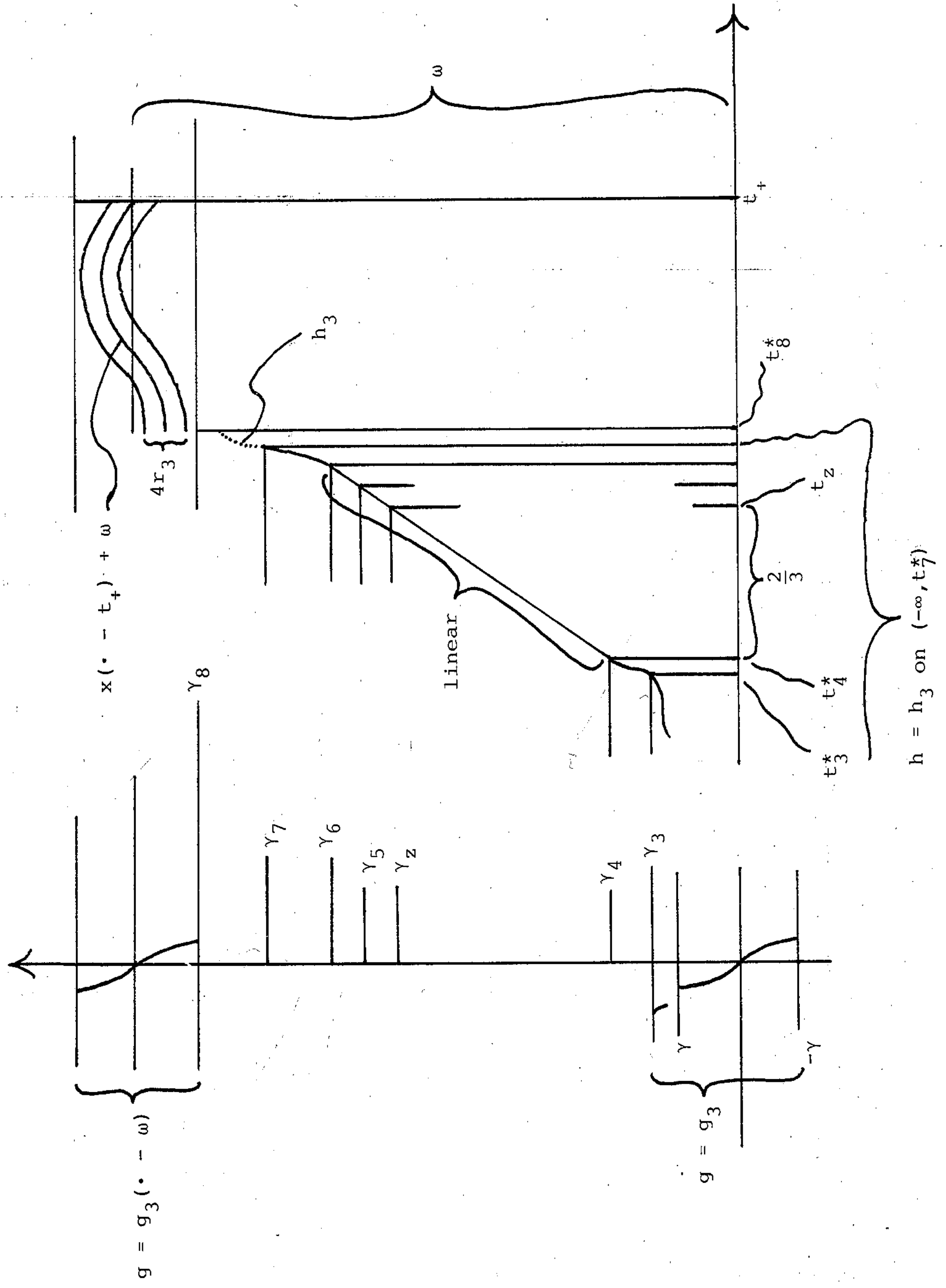
$$h_{t_+} \in \phi^* + C_{2r_3} + \omega \subset N_+ + \omega$$

for solutions h with $h_0 = \chi \in U$.

Define

$$g_4(\xi) := g_3(\xi) \quad \text{for} \quad -\gamma \leq \xi \leq \gamma_3 ,$$

$$g_4(\xi) := g_3(\xi - \omega) \quad \text{for} \quad \omega - \gamma = \gamma_8 \leq \xi \leq \gamma_8 + 2\gamma = \omega + \gamma .$$



The properties of ϕ^{**} (Proposition III.3.2) imply

$$0 < \dot{\phi}^{**}(t - (t_4^* + 1)) \quad \text{for} \quad t_4^* \leqq t < t_4^* + \frac{2}{3} = t_z ,$$
$$\dot{\phi}^{**}(t - (t_4^* + 1)) < 0 \quad \text{for} \quad t_z < t \leqq t_4^* + 1 = t_8^* .$$

Recall that h_3 maps the interval $[t_4^*, t_4^* + 1 + \theta]$ strictly increasing onto the interval $[\gamma_4, \gamma_5]$.

It follows that we can define $g_4(\xi)$ for $\gamma_4 \leqq \xi \leqq \gamma_5$ by a C^1-function such that

$$(1) \qquad |g_4(h_3(t)) - \dot{\phi}^{**}(t - (t_4^* + 1))| < r_4$$
$$\text{for all} \quad t \in [t_4^*, t_4^* + 1 + \theta] \;=\; [t_4^*, t_5^*] ,$$
$$0 < g_4 \quad \text{on} \quad [\gamma_4, \gamma_z) \quad \text{and} \quad g_4 < 0 \quad \text{on} \quad (\gamma_z, \gamma_5] .$$

(Note that we can not simply set

$$g_4(\xi) := \dot{\phi}^{**}((h_3|[\ldots])^{-1}(\xi) - (t_4^* + 1))$$

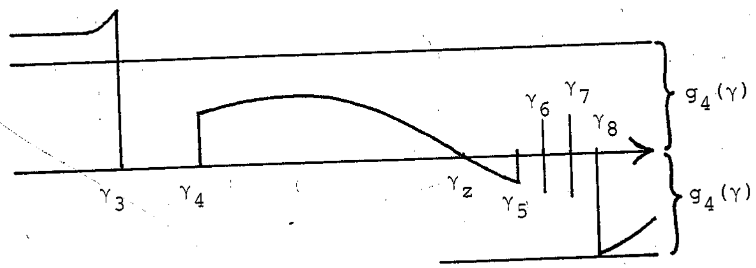
since $\dot{\phi}^{**}$ is not differentiable at $t_* \in (-1, -\frac{2}{3})$:

$$\lim_{t \uparrow t_*} \ddot{\phi}^{**}(t) = 0 \neq g_1'(x(t_* - 1))\dot{x}(t_* - 1) = \ddot{x}(t_*) = \lim_{t \downarrow t_*} \ddot{\phi}^{**}(t) .)$$

REMARK 2. The purpose of (1) is to obtain $\dot{h}_4(t_+ + \cdot) \approx \dot{\phi}^{**}$ on $[-1, \theta]$ for the solution h_4 of eq. (g_4) with $h_4 = h_3$ on $(-\infty, t_3^* + 1]$ - compare Remark 1.

COROLLARY 1. (i) $g_4(\gamma_8) < g_4(\xi)$ for $\gamma_z \leqq \xi \leqq \gamma_5$.

(ii) For $\gamma_4 \leqq \xi \leqq \gamma_z$ and $\gamma \leqq \xi' \leqq \gamma_3$, $g_4(\xi) < g_4(\gamma) \leqq g_4(\xi')$.



PROOF of Corollary 1. For $\gamma_z \leqq \xi \leqq \gamma_5$,

$$g_4(\xi) > \min \dot{\phi}^{**} - r_4 \geqq \min \dot{x} - r_4 \geqq g_1(-\tfrac{\gamma}{4}) - r_4 > g_1(-\gamma)$$
$$= g_4(\gamma_8)$$

(by (III.3.4)), and for $\gamma_4 \leqq \gamma \leqq \gamma_z$,

$$g_4(\xi) < \max \dot{\phi}^{**} + r_4 \leqq \max \dot{x} + r_4 \leqq g_1(\tfrac{\gamma}{4}) + r_4 < g_1(\gamma) = g_4(\gamma) .$$

Use $0 \leqq g_3' = g_4'$ on $[\gamma, \gamma_3]$.

REMARK 3. The gap between γ_3 and γ_4 will be closed in such a way that h_4 (see Remark 2) reaches the value $\phi^{**}(-1) + \omega - r_3$ at $t = t_8^*$. Observe that h_4 will then satisfy

$$h_4(t_8^*) = h_4(t_7^*) + \int_{t_7^*}^{t_8^*} \dot{h}_4 = \gamma_7 + \int_{t_3^*}^{t_4^*} g_4 \circ h_4 = \gamma_7 + \int_{t_3^*}^{t_4^*} g_4 \circ h_3 .$$

h_3 maps the interval $[t_3^*, t_4^*]$ strictly increasing onto the interval $[\gamma_3, \gamma_4]$, and

$$\gamma_7 < \gamma_8 < -\frac{\gamma}{4} + \gamma_8 + \gamma - 2r_3 < \phi^{**}(-1) + \omega - r_3 ,$$

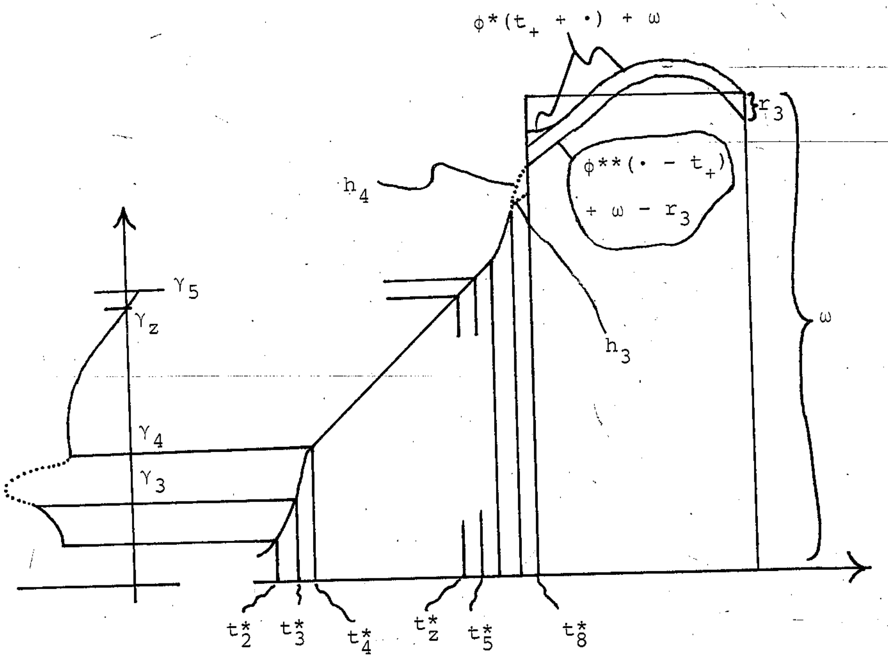
by $|x| < \frac{\gamma}{4}$, by the choice of r_3 in Section III.3 and by Proposition III.3.2 (i).

It follows that we can define g_4 on (γ_3, γ_4) such that $g_4|[-\gamma, \gamma_5]$ is C^1, with

$$g_4(\xi) > 0 \quad \text{for} \quad \gamma_3 < \gamma < \gamma_4$$

and

$$(2) \qquad \gamma_7 + \int_{t_3^*}^{t_4^*} g_4 \circ h_3 = \phi^{**}(-1) + \omega - r_3 .$$



REMARK 4. (1) and (2) together will result in

$$h_4(t_+ + \cdot) \approx \phi^{**} + \omega - r_3 \quad \text{on} \quad [-1,\theta] ,$$

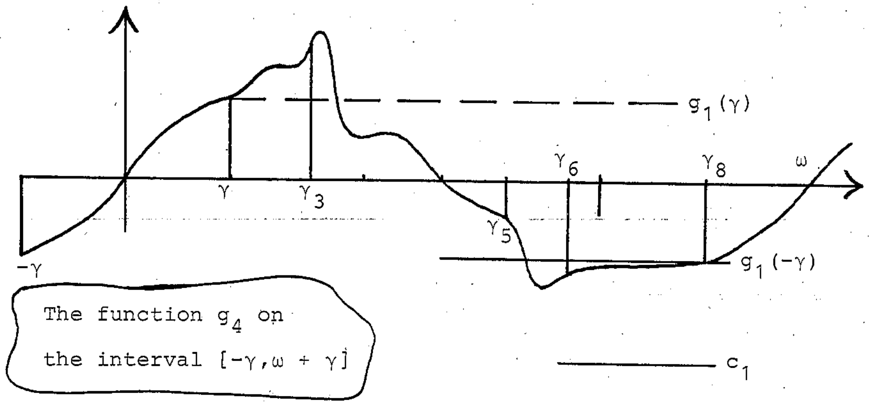
compare Remarks 2 and 3.

Next, the gap between γ_5 and γ_8 is closed in such a way that $g_4|[-\gamma,\omega+\gamma]$ is C^1, and

(3) $\quad c_1 < g_4(\xi) < 0$ for $\gamma_z < \xi \leqq \gamma_8$,

which is possible because of $c_1 < g_1(-\gamma)$, see Section III.3, and Corollary 1 (i), and

(4) $\quad 0 < g_4'(\xi)$ for $\gamma_6 \leqq \xi < \gamma_8$.



The function g_4 on the interval $[-\gamma,\omega+\gamma]$

REMARK 5. Condition (3) will imply $c_1 < \dot{h}_4(t_+ + \cdot)$ on $[\theta,0]$ so that one can use Proposition III.3.3 and extend

$$h_4(t_+ + \cdot) \approx \phi^{**} + \omega - r_3$$

from the interval $[-1,\theta]$ to all of $[-1,0]$.

Condition (4) is important for transversality.

Finally, the ω-periodic continuation g_4 of the function constructed so far is embedded into a family of ω-periodic C^1-functions $g_b:\underline{R} \to \underline{R}$, $b \geqq 4$, such that solutions h_b of eq. (g_b) with $h_b = h_3$ on $(-\infty,t_7^*)$ will have values $h_b(t_8^*)$ which increase from

$$h_4(t_8^*) = \phi^{**}(-1) + \omega - r_3$$

to ∞ as $b \to \infty$:

We choose an ω-periodic C^1-function $g^*:\underline{R} \to \underline{R}$ with

$$g^* = 0 \quad \text{on} \quad [0,\gamma_3] \cup [\gamma_4,\omega] \quad \text{and} \quad 0 < g^* \quad \text{on} \quad (\gamma_3,\gamma_4) ,$$

and set

$$g_b := g_4 + (b-4)g^* \quad \text{for all} \quad b \geqq 4 .$$

COROLLARY 2. (i) For all $b \geqq 4$,

$$g_b = g_3 \quad \text{on} \quad [-\gamma,\gamma_3] \quad \text{and} \quad g_b = g_4 \quad \text{on} \quad [\gamma_4,\omega+\gamma] ,$$

$$0 < g_b \quad \text{on} \quad (0,\gamma_z) \quad \text{and} \quad g_b < 0 \quad \text{on} \quad (\gamma_z,\omega) .$$

(ii) There exists $b^* > 4$ with

$$\phi^{**}(-1) + \omega - r_3 \leq \gamma_7 + \int_{t_3^*}^{t_4^*} g_b \circ h_3 \leq \phi^{**}(-1) + \omega + r_3$$

for all $b \in [4,b^*]$, and

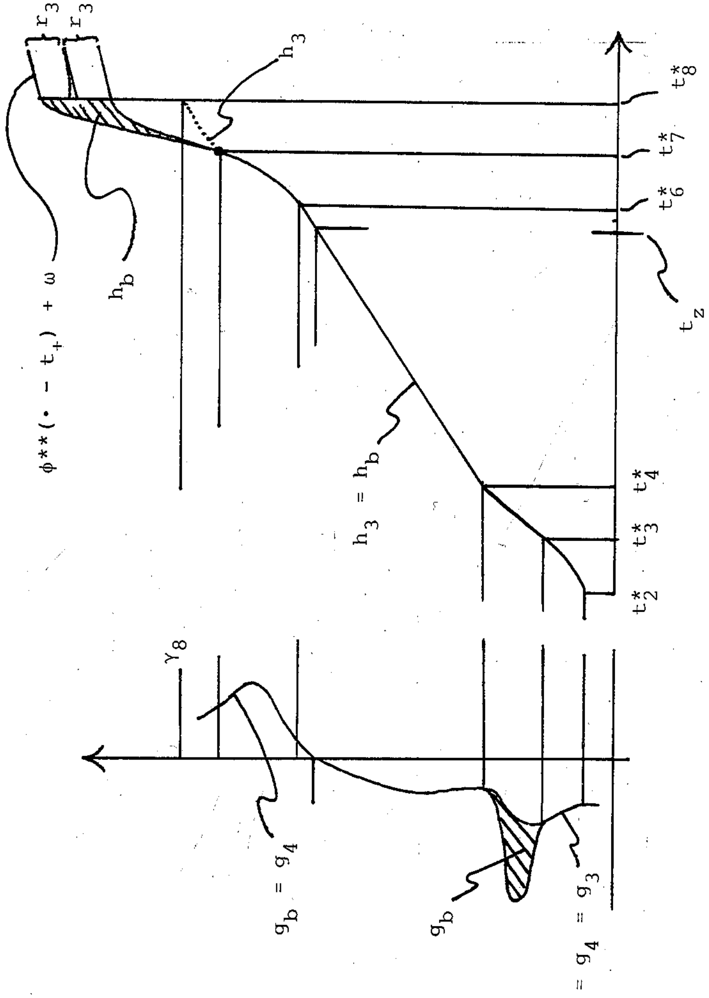
$$\gamma_7 + \int_{t_3^*}^{t_4^*} g_{b^*} \circ h_3 = \phi^{**}(-1) + \omega + r_3 \ .$$

PROOF of assertion (ii). On the interval $[t_3^*,t_4^*]$,

$$g_b \circ h_3 = g_4 \circ h_3 + (b - 4)g^* \circ h_3 \ .$$

Recall (2) and use

$$\int_{t_3^*}^{t_4^*} g^* \circ h_3 \geq \frac{\int_{t_3^*}^{t_4^*} g^* \circ h_3 \dot{h}_3}{\max\limits_{[t_3^*,t_4^*]} \dot{h}_3} = \frac{\int_{\gamma_3}^{\gamma_4} g^*}{\max\limits_{[t_3^*,t_4^*]} \dot{h}_3} > 0 \ .$$



3. TRANSPORT OF $\chi \in U$ AND $v \in T_\chi U$ FOR $0 \leqq t \leqq t_+$

PROPOSITION 1. (i) Let $b \geqq 4$. There is a solution $h_b : \underline{R} \to \underline{R}$ of eq. (g_b) with $h_b = h_3$ on $(-\infty, t_7^*]$ (and $h_{b,0} = \chi \in U$), and with $0 < \dot{h}_b$ and $\gamma_z < h_b$ on $(t_z, t_8^*]$.

(ii) For $4 \leqq b \leqq b^*$, $h_b < \omega$ on $[t_z, t_8^*]$ and

$$h_{b,t_+} - \omega \in \phi^* + C_{2r_3} \subset N_+ .$$

(iii) $h_{4,t_+} - \omega < \phi^* < h_{b^*,t_+} - \omega$.

PROOF. 1. Let $b \geqq 4$. For $t \leqq t_7^* = t_3^* + 1$,

$$-\gamma < h_3(t-1) \leqq h_3(t_3^*) = \gamma_3 ,$$

see (1.3). With $g_b = g_3$ on $[-\gamma, \gamma_3]$,

$$\dot{h}_3(t) = g_3(h_3(t-1)) = g_b(h_3(t-1)) .$$

This yields existence of h_b with $h_b = h_3$ on $(-\infty, t_7^*]$. We have

$$h_b(t_z) = h_3(t_z) = \gamma_z ,$$

$$\dot{h}_b(t) = \dot{h}_3(t) > 0 \quad \text{for } t_z < t \leqq t_7^* ,$$

see (1.3). Let $t_7^* < t \leqq t_8^*$. Then $t_3^* < t - 1 \leqq t_4^*$, and

$$h_b(t-1) = h_3(t-1) \in (\gamma_3, \gamma_4] \subset (0, \omega) ,$$

$$\dot{h}_b(t) = g_b(h_b(t-1)) > 0 .$$

Consequently, $\gamma_z < h_b$ on $(t_z, t_8^*]$.

2. Let $4 \leqq b \leqq b^*$. For $t_z \leqq t \leqq t_8^*$,

$$h_b(t) \leqq h_b(t_8^*) = h_b(t_7^*) + \int_{t_7^*}^{t_8^*} \dot{h}_b = h_3(t_7^*) + \int_{t_3^*}^{t_4^*} g_b \circ h_b$$

$$= \gamma_7 + \int_{t_3^*}^{t_4^*} g_b \circ h_3 \leqq \phi^{**}(-1) + \omega + r_3 \qquad \text{(by Corollary 2.2 (ii))}$$

$$\leqq \phi^*(-1) + \omega + r_3 < \omega \qquad \text{(by Proposition III.3.2 (iii) and by (III.3.2))}.$$

3. Let $-1 \leqq t \leqq \theta$. With eq. (g_b) and Corollary 2.2 (i), we get

$$h_b(t_+ + t) = h_b(t_7^*) + \int_{t_3^*}^{t_4^* + 1 + t} g_b \circ h_b$$

$$= \gamma_7 + \int_{t_3^*}^{t_4^*} g_b \circ h_3 + \int_{t_4^*}^{t_4^* + 1 + t} g_4 \circ h_3 .$$

(On $(-\infty, t_7^*]$, $h_b = h_3$. h_3 maps $[t_4^*, t_7^*]$ onto $[\gamma_4, \gamma_7]$ where $g_b = g_4$. We have

$$t_4^* + 1 + t = t_8^* + t \leqq t_8^* + \theta = t_5^* < t_7^* \text{ .)}$$

Using Corollary 2.2 (ii) and (2.1) we infer

$$\phi^{**}(-1) + \omega - r_3 + [\phi^{**}(t) - \phi^{**}(-1) - r_4] \leqq h_b(t_+ + t)$$

$$\leqq \phi^{**}(-1) + \omega + r_3 + [\phi^{**}(t) - \phi^{**}(-1) + r_4] ,$$

or

(1) $\phi^{**}(t) - r_3 - r_4 \leqq h_b(t_+ + t) - \omega \leqq \phi^{**}(t) + r_3 + r_4$.

For $b = 4$, with (2.2) instead of Corollary 2.2 :

(2) $h_4(t_+ + t) - \omega \leqq \phi^{**}(t) - r_3 + r_4$.

For $b = b^*$, with the last assertion of Corollary 2.2 :

(3) $\phi^{**}(t) + r_3 - r_4 \leqq h_{b^*}(t_+ + t) - \omega$.

4. Combining (1) with (III.3.3) we obtain

(4) $\phi^*(t) - 2r_3 < h_b(t_+ + t) - \omega < \phi^*(t) + 2r_3$
for $-1 \leqq t \leqq \theta$ and $4 \leqq b \leqq b^*$.

Moreover, $\phi^*(\theta) = \phi^{**}(\theta)$ gives

(5) $\phi^*(\theta) - r_3 - r_4 \leqq h_b(t_+ + \theta) - \omega \leqq \phi^*(\theta) + r_3 + r_4$.

For $b = 4$, (2) and $\phi^{**} \leqq \phi^*$ and $r_4 < r_3$ yield

(6) $h_4(t_+ + t) - \omega \leqq \phi^*(t) - r_3 + r_4 < \phi^*(t)$ for $-1 \leqq t \leqq \theta$.

For $b = b^*$, (3) and (III.3.3) give

(7) $\phi^*(t) < h_{b^*}(t_+ + t) - \omega$ for $-1 \leqq t \leqq \theta$

while

(8) $\phi^*(\theta) + r_3 - r_4 \leqq h_{b^*}(t_+ + \theta) - \omega$.

5. Estimate of $\dot{h}_b(t_+ + \cdot)$ for $-\frac{1}{3} \leqq t \leqq 0$ and $4 \leqq b \leqq b^*$:

$$\dot{h}_b(t_+ + t) = g_b(h_b(t_8^* + t)) \in g_b \circ h_b([t_z^*, t_8^*])$$

$$\subset g_b([\gamma_z, \omega]) \qquad \text{(with } \gamma_z < h_b < \omega \text{ on } (t_z^*, t_8^*] \text{)}$$

$$\subset (c_1, 0) \qquad \text{(with (2.3) and Corollary 2.2 (i))}.$$

6. Application of Proposition III.3.3 : Let $4 \leqq b \leqq b^*$. With part 5 and (5), we get

$$\phi^*(t) - 2r_3 < h_b(t_+ + t) - \omega < \phi^*(t) + 2r_3 \quad \text{for } \theta \leqq t \leqq 0 .$$

Together with (4),

$$h_{b,t_+} - \omega \in \phi^* + C_{2r_3} \subset N_+ \qquad \text{(see (III.3.1))},$$

and assertion (ii) is proved.

In case $b = 4$, (6) for $t = \theta$ and part 5 permit to use Proposition III.3.3 (iii). It follows that

$$h_4(t_+ + t) - \omega < \phi^*(t) \quad \text{for} \quad \theta \leqq t \leqq 0 .$$

With (6), we get

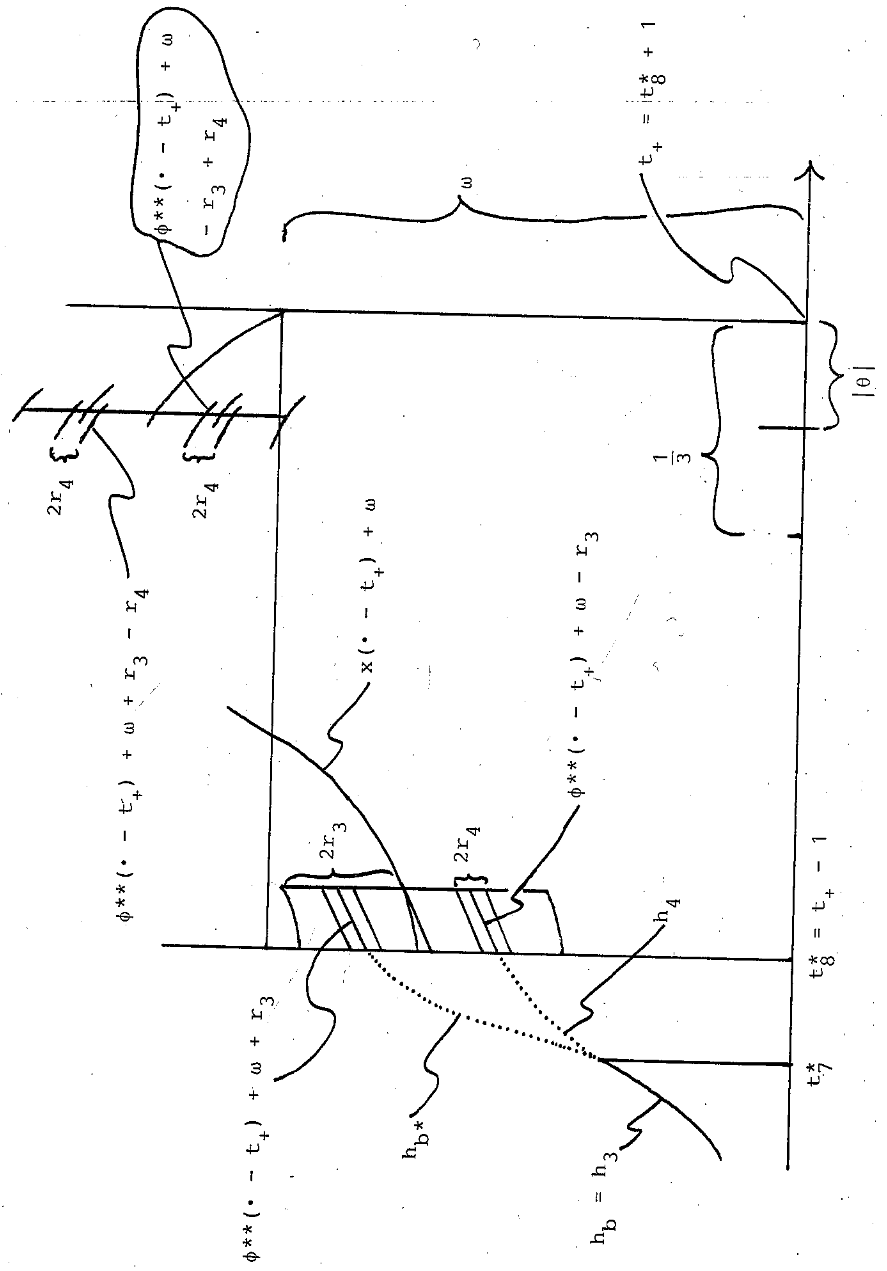
$$h_{4,t_+} - \omega < \phi^* .$$

In case $b = b^*$, (8) and part 5 permit to use Proposition III.3.3 (ii). It follows that

$$\phi^*(t) < h_{b^*}(t_+ + t) - \omega \quad \text{for} \quad \theta \leqq t \leqq 0 .$$

With (7), we get

$$\phi^* < h_{b^*,t_+} - \omega .$$



For transversality, we verify a no-zero condition as in Proposition III.4.2 :

PROPOSITION 2. Let $4 \leqq b \leqq b^*$. The solution $w_b : [-1,\infty) \to \underline{R}$ of

$$\dot{w}(t) = g_b'(h_b(t-1))w(t-1) \quad , \quad w_0 = v \in T_X U$$

satisfies

$$w_b(t) - c^* \dot{h}_b(t) \neq 0 \quad \text{for} \quad t_+ - 1 \leqq t \leqq t_+ \ .$$

Before giving the proof, let us describe the concept which led to Proposition 2.

REMARK 1. The foundations for Proposition 2 were laid in Section 1, and by (2.4). Proposition 1.2 (ii) can now be restated as

$$(*) \qquad w_b < c^* \dot{h}_b \ \text{on} \ [t_2^*,t_3^*) \ , \ \text{or} \ \ w_b > c^* \dot{h}_b \ \text{on} \ [t_2^*,t_3^*) \ .$$

An inequality like

$$w_b < c^* \dot{h}_b \quad \text{on} \quad [t_+ - 1,t_+] = [t_8^*,t_8^* + 1]$$

relies on behavior of w_b and $\dot{h}_b$ on the interval $[t_2^*,t_2^* + 1] = [t_2^*,t_6^*]$ and on sign conditions for the coefficient $g_b'(h_b(\cdot-1))$ in the linear differential equation for w_b and $c^* \dot{h}_b$.

g_b was designed to steer h_{b,t_+} close to ϕ^* . This forces the sign of $g_b'(h_b(\cdot-1))$ to change several times on the interval $[t_2^*,t_8^* + 1]$, and it seems difficult if not hopeless to control the sign of the solution $w_b - c^* \dot{h}_b$ on the interval $[t_6^*,t_8^* + 1]$.

The construction of g_b with $g_2'(\xi) = 0$ for all $\xi \geqq h_1([t_1^* + 1])$ implies that w_b and $\dot{h}_b$ become constant on $[t_3^*,t_6^*]$. By the choice of c^* , $w_b = c^* \dot{h}_b$ on this interval - so that the linear variational equation shows that w_b and $c^* \dot{h}_b$ differ by a constant on the interval

$$[t_3^* + 1,t_6^* + 1] = [t_7^*,t_6^* + 1] \ ,$$

no matter how oscillatory the coefficient $g_b'(h_b(\cdot-1))$ is on the interval $[t_7^*,t_6^* + 1]$.

In order to determine the sign of this constant, i.e.,

$$\text{sign}\ [w_b(t_7^*) - c^* \dot{h}_b(t_7^*)] \ ,$$

we have

$$w_b(t_6^*) - c^* \dot{h}_b(t_6^*) = 0$$

and (*), so that integration of the linear variational equation will give a result provided there is a sign condition for the coefficient $g_b'(h_b(\cdot-1))$ on the interval $(t_6^*,t_7^*]$.

Such a sign condition is

$$0 < g_3' \quad \text{on} \quad (\gamma_2,\infty) \supset (\gamma_2,\gamma_3]$$

from Section 1.

For the final comparison of w_b and $c^* \dot{h}_b$ on the remaining "small" interval $[t_6^* + 1,t_8^* + 1]$ we need

$$0 < g_4' \quad \text{on} \quad [\gamma_6,\gamma_8) \ ,$$

i.e., the condition (2.4).

PROOF of Proposition 2. 1. We show $h_b = h_2$ on $(-\infty,t_6^*]$ and $w_b = w_2$ on $[-1,t_6^*]$: For $t \leqq t_6^* = t_2^* + 1$,

$$h_2(t) = h_3(t) = h_b(t) \qquad \text{(see (1.3) and Proposition 1).}$$

Let $0 \leqq t \leqq t_6^*$. Then $-1 \leqq t - 1 \leqq t_2^*$, and therefore

$$-\gamma < h_3(t-1) \leqq h_3(t_3^*) = \gamma_3 \qquad \text{(see (1.3).}$$

With $g_b = g_3$ on $[-\gamma,\gamma_3]$,

$$g_b'(h_b(t-1)) = g_3'(h_2(t-1)) = g_2'(h_2(t-1))$$

(recall

$$-\gamma < h_2(t-1) \leqq h_2(t_2^*) = \gamma_2$$

and $g_3 = g_2$ on $[-\gamma,\gamma_2]$). It follows that w_b and w_2 satisfy the same linear differential delay equation for $0 \leqq t \leqq t_6^*$, and

$$w_{b,0} = v = w_{2,0} \ .$$

2. Part 1, the choice of c^* in Proposition 1.2 and

$$[t_1^* + 1] + 1 = t_3^* < t_6^*$$

yield

$$(9) \qquad w_b(t_3^*) = c^* \dot{h}_b(t_3^*) \ .$$

Part 1, Proposition 1.2 (ii) and

$$t_3^* - \varepsilon^* \leqq t_2^* < t_3^*$$

give

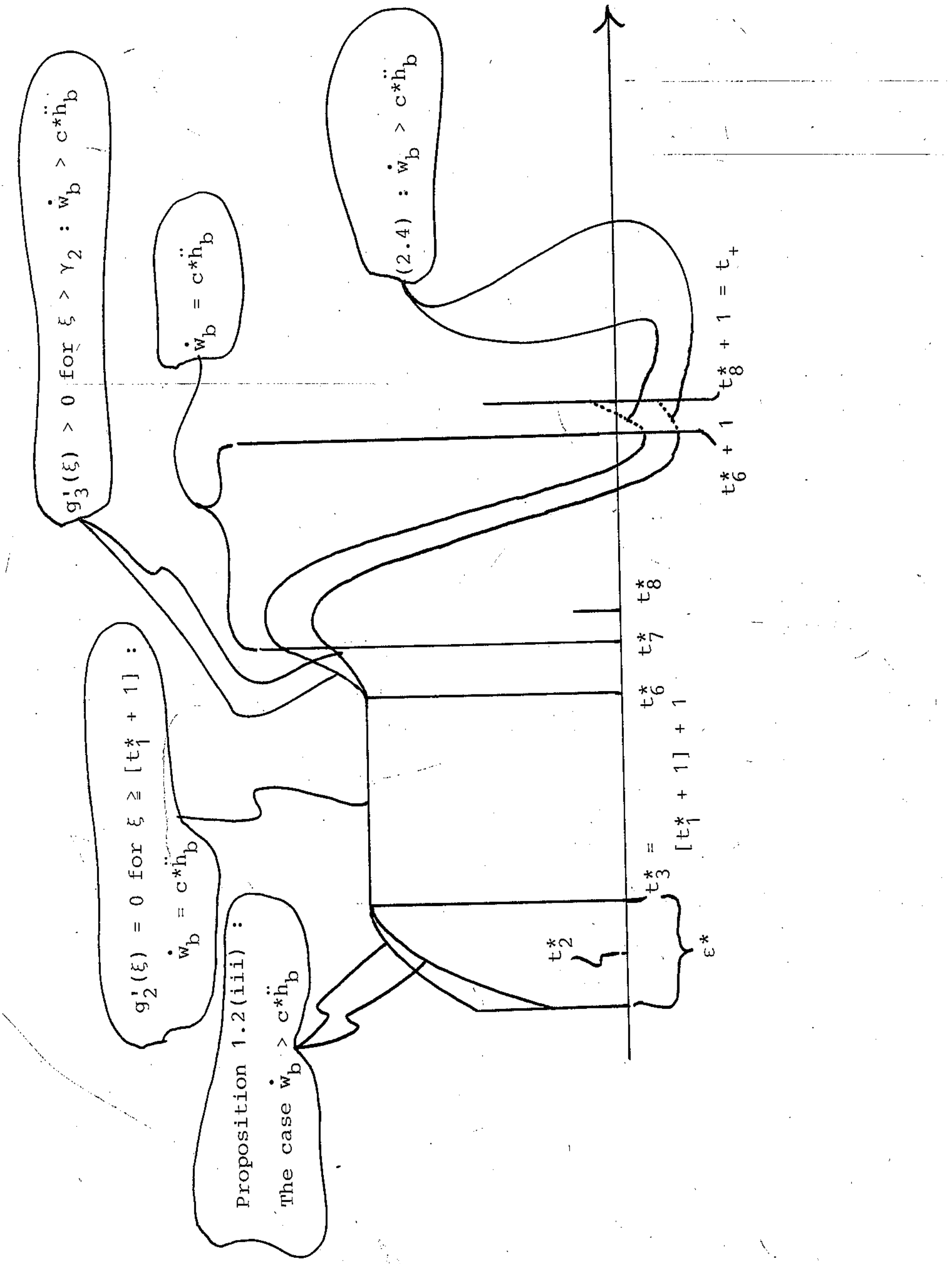
$$(10) \qquad w_b(t) \neq c^* \dot{h}_b(t) \quad \text{for} \quad t_2^* \leqq t < t_3^* \ .$$

From part 1 and from

$$\dot{w}_2(t) = 0 = \ddot{h}_2(t) \quad \text{for} \quad t \geqq [t_1^* + 1] + 1 = t_3^*$$

we obtain

$$\dot{w}_b = 0 = c^* \ddot{h}_b \quad \text{on} \quad [t_3^*,t_6^*]$$



so that (9) implies

(11) $\quad w_b = c^*\,\dot{h}_b \quad \text{on} \quad [t_3^*, t_6^*]$.

3. Suppose

(12) $\quad w_b > c^*\,\dot{h}_b \quad \text{on} \quad [t_2^*, t_3^*)$.

We show $w_b > c^*\,\dot{h}_b$ on $(t_6^*, t_6^* + 1]$:

3.1. Let $t_6^* < t < t_7^*$. Then $t_2^* < t - 1 < t_3^*$, and

$$g_b'(h_b(t-1)) = g_3'(h_3(t-1)) > 0$$

(with

$$\gamma_2 = h_3(t_2^*) < h_3(t-1) \qquad \text{(see (1.3))}$$

and $0 < g_3'$ on (γ_2, ∞))

so that (12) implies

$$\dot{w}_b(t) = g_b'(h_b(t-1))w_b(t-1) > g_b'(h_b(t-1))\; c^*\,\dot{h}_b(t-1) = c^*\,\ddot{h}_b(t) .$$

3.2. For $t_7^* \leq t \leq t_6^* + 1$, $t_3^* \leq t - 1 \leq t_6^*$, and (11) gives

$$\dot{w}_b(t) = g_b'(h_b(t-1))w_b(t-1) = g_b'(h_b(t-1))\; c^*\,\dot{h}_b(t-1) = c^*\,\ddot{h}_b(t) .$$

3.3. Using (11) for $t = t_6^*$ and the results of parts 3.1 and 3.2 we obtain the assertion.

4. Proof of $g_b'(h_b(t-1)) > 0$ for $t_6^* + 1 \leq t \leq t_8^* + 1$:

Let $t_6^* \leq t - 1 \leq t_8^*$. From Proposition 1,

$$h_b(t_6^*) = h_3(t_6^*) = \gamma_6 , \quad 0 < \dot{h}_b \quad \text{in} \quad (t_z, t_8^*] \supset [t_6^*, t_8^*] ,$$

$$h_b(t_8^*) < \omega .$$

It follows that $h_b(t-1) \in [\gamma_6, \omega]$. Recall

$$g_b' = g_4' \quad \text{on} \quad [\gamma_4, \omega + \gamma] \supset [\gamma_6, \omega] \qquad \text{(Corollary 2.2 (i))},$$

$$g_4' > 0 \quad \text{on} \quad [\gamma_6, \gamma_8] \qquad \text{(see (2.4))},$$

$$g_4' = g_1'(\cdot - \omega) > 0 \quad \text{on} \quad [\gamma_8, \omega + \gamma] \supset [\gamma_8, \omega] .$$

5. In case (12), $w_b > c^*\,\dot{h}_b$ on $[t_+ - 1, t_+] = [t_8^*, t_8^* + 1]$. Proof: By part 3,

$$w_b(t_6^* + 1) > c^*\,\dot{h}_b(t_6^* + 1) ,$$

and for $t_6^* + 1 < t < t_8^* + 1$,

$$\dot{w}_b(t) = g_b'(h_b(t-1))w_b(t-1) > g_b'(h_b(t-1))\; c^*\,\dot{h}_b(t-1) = c^*\,\ddot{h}_b(t)$$

(with parts 4 and 3; $t_6^* < t - 1 < t_8^* < t_6^* + 1$), so that

$$w_b > c^*\,\dot{h}_b \quad \text{on} \quad [t_6^* + 1, t_8^* + 1] .$$

Recall part 3, and $t_6^* < t_8^* < t_6^* + 1$.

6. The proof for the case $w_b < c^*\,\dot{h}_b$ on $[t_2^*, t_3^*)$ is analogous.

4. HETEROCLINIC SOLUTIONS AND TRANSVERSAL HOMOCLINIC POINTS

The construction of the functions g_b yields

COROLLARY 1. (Heteroclinic solutions) There exists $b \in (4,b^*)$ with

$$h_{b,t} \to o + \omega \quad \text{as} \quad t \to \infty , \quad h_{b,t} \to o \quad \text{as} \quad t \to -\infty .$$

We have

$$P(h_{b,t_+} - \omega) \in S \subset H ,$$

$$h_{b,t_n} = \chi_n \quad \text{for} \quad n \in -\underline{N}_0 ,$$

$$|h_b(t) - \omega| < \frac{\gamma}{2} \quad \text{for} \quad t \geq t_+ - 1 ,$$

$$|h_b(t)| < \frac{\gamma}{2} \quad \text{for} \quad t \leq 0 .$$

PROOF. 1. The map $[4,b^*] \ni b \to h_{b,t_+} \in C$ is continuous, with

$$\chi_b^+ := h_{b,t_+} - \omega \in N_+ \quad \text{for} \quad 4 \leq b \leq b^* ,$$

$$\chi_4^+ < \phi^* < \chi_{b^*}^+ \qquad \text{(see Proposition 3.1).}$$

Proposition III.3.1 implies that each $P(\chi_b^+) \in H$ is below, on or above S, with $P(\chi_4^+)$ below S and $P(\chi_{b^*}^+)$ above S. By continuity, we obtain $b \in (4,b^*)$ with $P(\chi_b^+) \in S$.

2. The solution $y:[-1,\infty) \to \underline{R}$ of eq. (g_1) with $y_0 = \chi_b^+ = h_{b,t_+} - \omega$ satisfies

$$|y(t)| < \frac{\gamma}{2} \quad \text{for} \quad t \in [-1,2] \supset [-1,\tau(\chi_b^+)]$$

and $y_{\tau_+} = P(\chi_b^+) \in S$ where $\tau_+ := \tau(\chi_b^+)$. With Corollaries II.1.1 (i) and III.2.1, we obtain

$$|y(t)| < \frac{\gamma}{2} \quad \text{for all} \quad t \geq -1 ,$$

$$P^n(y_{\tau_+}) \to \phi^* \quad \text{as} \quad n \to \infty .$$

$\tau(N) \subset (1,2)$ and continuous dependence on initial data yield $y_t \to o$ as $t \to \infty$. $g_b = g_b(\cdot + \omega)$ implies that $h_b(\cdot + t_+) - \omega$ is a solution of eq. (g_b). With $g_b = g_1$ on $[-\gamma,\gamma]$ and $|y| < \frac{\gamma}{2}$, we infer

$$h_b(t + t_+) - \omega = y(t) \quad \text{for} \quad t \geq -1 .$$

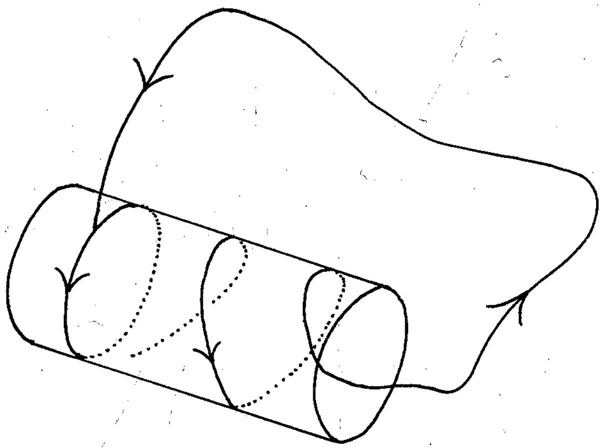
3. The assertions for $t \leq 0$ and $n \in -\underline{N}_0$ follow from Proposition 1.1 since $h_b = h_3 = h_1$ on $(-\infty,0]$.

From now on we consider $g := g_b$ and $h := h_b$ given by Corollary 1. The index b is dropped whenever convenient.

The next result exhibits a difference to the examples of chaotic differential delay equations from [32,23,10,11,9] :

PROPOSITION 1. $h_t \notin o + \omega$ for all $t \geq t_+$.

COMMENT. In [32,10,11,9], nonlinearities were carefully designed smoothed step functions so that solutions of the differential delay equations could explicitly be computed. The semiflow entered a 2-dimensional subset of C containing a periodic orbit, and there was a homoclinic trajectory which merged into the periodic orbit in finite time.



Both facts together made it possible to obtain a Poincaré map on a one-dimensional subset, with a "snap-back repeller" [16] (see Preliminaries) - so that chaotic motion in [32,10,11] was a consequence of results for interval maps [16,15,24].

Proposition 1 shows that the present situation is not as simple.

PROOF of Proposition 1. 1. Proof of $h_{t_+} - \omega \neq x_t$ for all $t \in \underline{R}$:

$$h(t_8^*) = h(t_+ - 1) \geqq x(-1) + \omega - 2r_3 \qquad \text{(Proposition 3.1 (ii))}$$
$$> -\frac{\gamma}{4} + \omega - \frac{\gamma}{2} \qquad \text{(choice of } r_3 \text{ in Section III.3)}$$
$$> \omega - \gamma = \gamma_8$$

and

$$h(t_8^* + \theta) = h(t_8^*) \leqq h(t_7^*) \qquad \text{(Proposition 3.1 (i))}$$
$$= h_3(t_7^*) \qquad \text{(Proposition 3.1 (i))}$$
$$= \gamma_7 < \gamma_8$$

imply that for some $t \in [\theta,0]$, $h(t_8^* + t) = \gamma_8$. Therefore

$$\dot{h}(t_+ + t) = g(h(t_+ + t - 1)) = g(\gamma_8) = g(-\gamma) = g_1(-\gamma) ,$$

while

$$\dot{x}(t') \in g_1([-\tfrac{\gamma}{4},\tfrac{\gamma}{4}]) \text{ for all } t' \in \underline{R} .$$

2. Let $t \in \underline{R}$. The solutions $h(\cdot + t_+) - \omega$ and $x(\cdot + t)$ of eq. (g) are bounded on $[-1,\infty)$ by $\frac{\gamma}{2}$ (Corollary 1). The restriction of g to the interval $[-\gamma,\gamma]$ is injective since $0 < g_1' = g'$ on $[-\gamma,\gamma]$. Using part 1 and eq. (g) one concludes by induction that

$$h_{t_+ + n} - \omega \neq x_{t+n} \quad \text{for all } n \in \underline{N}_0 \text{ and all } t \in \underline{R} .$$

Suppose now that

$$h_t - \omega = x_{\underline{t}} \quad \text{for some } t \geqq t_+ \text{ and some } \underline{t} \in \underline{R} .$$

It follows that the solutions $h - \omega$ and $x(\cdot + \underline{t} - t)$ of eq. (g) coincide on the interval $[t - 1,\infty)$. Hence

$$h_{t_+ + n} - \omega = x_{t_+ + n + \underline{t} - t} \quad \text{for some } n \in \underline{N}_0 ,$$

a contradiction.

We complete the points χ_n , $n \in -\underline{N}_0$, to a sequence in $D^{\underline{Z}}$ which tends to ϕ^* as $n \to \infty$: Set $\chi^+ := h_{t_+} - \omega$ (as in the proof of Corollary 1), and $\chi_1 := \underline{P}(\chi^+) \in S$, $t_1 := t_+ + \tau(\chi^+)$. It follows that there are a sequence of points $\chi_n = P(\chi_{n-1}) \in S$, $2 \leqq n \in \underline{N}$, with

$$\chi_n \to \phi^* \text{ as } n \to \infty ,$$

and a sequence of reals t_n , $2 \leqq n \in \underline{N}$, with

$$t_n = t_{n-1} + \tau(\chi_{n-1}) \in t_{n-1} + (1,2) \quad \text{for all integers } n \geqq 2 ;$$
$$t_n \to \infty \text{ as } n \to \infty .$$

COROLLARY 2. (i) $\chi_n = h_{t_n} - \omega$ for all $n \in \underline{N}$.

(ii) $\chi_n \neq \phi^*$ for all $n \in \underline{Z}$.

(iii) There exists $r_5 > 0$ with the following properties.

(iii.1) $\phi^* \notin \chi_0 + \text{cl } H_{r_5} \subset D$.

(iii.2) $\chi_n \notin \chi_0 + \text{cl } H_{r_5}$ for all $n \in \underline{Z} \setminus \{0\}$.

(iii.3) For every solution $y:[-1,\infty) \to \underline{R}$ of eq. (g) with $y_0 \in \chi_0 + H_{r_5}$,

$$y_{t_+} - \omega \in N_+ .$$

PROOF. 1. ω-periodicity of g implies that $h - \omega$ is a solution of eq. (g). $g = g_1$ on $[-\gamma,\gamma]$ and Corollary 1 show that the restriction of $h - \omega$ to the interval $[t_+ - 1,\infty)$ is a solution of eq. (g_1). Now assertion (i) is obvious from the definitions of χ^+ , $(\chi_n)_1^\infty$, $\underline{P}$, P .

2. Proof of assertion (ii) : Assertion (i), Proposition 1 and $t_n > t_+$ exclude $\chi_n = \phi^*$ for $n \in \underline{N}$. Suppose $\chi_n = \phi^*$ for some $n \in -\underline{N}_0$. Then $h_{t_n} = \phi^*$ (with Corollary 1), and consequently

$$h_t = x_{t - t_n} \in o \quad \text{for } t \geqq t_n ,$$
$$|h(t)| < \frac{\gamma}{4} < \omega \quad \text{for } t \geqq t_n - 1 ,$$

a contradiction to

$$t_n - 1 < 0 < t_+ \text{ and } |h(t_+)| + \frac{\gamma}{2} > \omega \qquad \text{(Corollary 1).}$$

3. Suppose $\chi_n = \chi_0$ with $0 \neq n \in \underline{Z}$. In case $n > 0$, $h_0 = h_{t_n} - \omega$ so that the solutions h and $h(\cdot + t_n) - \omega$ of eq. (g) coincide on $[-1,\infty)$. With $0 < t_n$,

$$h(jt_n) = h(0) + j\omega \to \infty \text{ as } \underline{N} \ni j \to \infty ,$$

a contradiction to Corollary 1.

In case $n < 0$, $h_{t_n} = h_0$, and $h(\cdot + t_n) = h$ on $[-1,\infty)$. For j in $\underline{N}$ with $t_+ + jt_n \in (t_n,0] \subset \underline{R}^-$,

$$h(t_+) = h(t_+ + jt_n) \in h(\underline{R}^-) \subset (-\tfrac{\gamma}{2},\tfrac{\gamma}{2}) ,$$

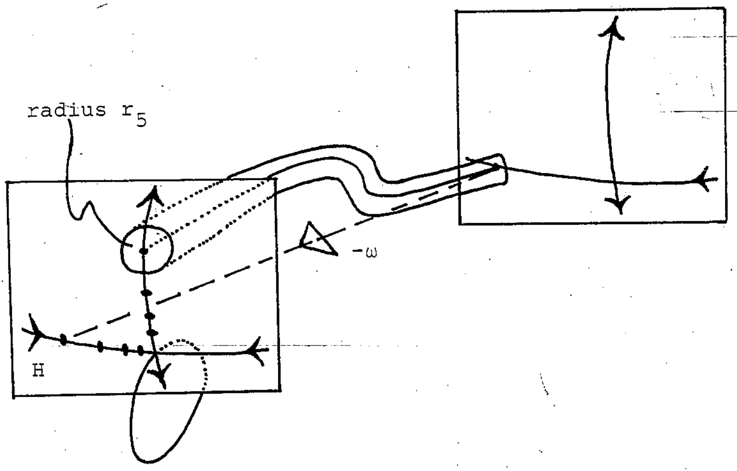
a contradiction to Corollary 1.

4. In view of assertion (ii), part 3 and $\chi_n \to \phi^*$ as $|n| \to \infty$, there exists $r > 0$ with

$$\phi^* \notin \chi_0 + \text{cl } H_r \not\ni \chi_n \quad \text{for } 0 \neq n \in \underline{Z} .$$

$h_0 = \chi_0$ and $h_{t_+} - \omega \in N_+$ permit to find $r_5 \in (0,r)$ so small that

$\chi_0 + \text{cl}\, H_{r_5} \subset D$, and for all solutions $y:[-1,\infty) \to \underline{R}$ of eq. (g) with $y_0 \in \chi_0 + H_{r_5}$, $y_{t_+} - \omega \in N_+$.



We modify the Poincaré map P so that $(\chi_n)_{-\infty}^{\infty}$ becomes a homoclinic trajectory. With r_5 according to Corollary 2, we set

$D' := \{\phi \in D: |\phi - \chi_0| \neq r_5\}$,

$P'(\phi) := P(\phi)$ if $\phi \in D$ and $|\phi - \chi_0| > r_5$,

$P'(\phi) := \underline{P}(y_{t_+} - \omega)$ for $\phi \in \chi_0 + H_{r_5}$ where $y:[-1,\infty) \to \underline{R}$ is the solution of eq. (g) with $y_0 = \phi$.

The map P' is C^1. On the open subset $D \setminus (\chi_0 + \text{cl}\, H_{r_5})$ of D , P' is a shift along solutions y until a return of their segments y_t to H . On the open subset $\chi_0 + H_{r_5}$ of H , P' describes solutions whose segments pass through $N_+ + \omega$ and hit the hyperplane $H + \omega$ afterwards.

COROLLARY 3. ϕ^* is a hyperbolic fixed point of P' , and $(\chi_n)_{-\infty}^{\infty}$ is a homoclinic trajectory of P' .

PROOF. We show $P'(\chi_n) = \chi_{n+1}$ for $n \in \underline{Z}$: By definition of χ_1 and P' ,

$$\chi_1 = \underline{P}(h_{t_+} - \omega) = P'(h_0) = P'(\chi_0) .$$

For $n \neq 0$, $\chi_n \in D$ and $|\chi_n - \chi_0| > r_5$ (Corollary 2) so that $P'(\chi_n) = P(\chi_n)$. If $n < 0$, $P(\chi_n) = \chi_{n+1}$ by Proposition 1.1. If $n > 0$, $P(\chi_n) = \chi_{n+1}$ by the construction before Corollary 2.

Transversality: Consider the tangent vector

$v_0 := v \in T_{\chi_0} U$ (Proposition 1.1)

and its forward iterates defined by

$v_{n+1} = DP'(\chi_n) v_n$ for $n \in \underline{N}_0$.

Recall $\chi_n \in S$ for $n \in \underline{N}$.

COROLLARY 4. $v_n \notin T_{\chi_n} S$ for all $n \in \underline{N}$.

PROOF. 1. Let X denote the semiflow given by the solutions $y:[-1,\infty) \to \underline{R}$ of eq. (g). We have

$$v_1 = DP'(\chi_0) v_0 = [D\underline{P}(X(t_+,\chi_0) - \omega) \circ D_2 X(t_+,\chi_0)]\, v_0$$
$$= D\underline{P}(h_{t_+} - \omega) w_{t_+}$$

with the solution $w:[-1,\infty) \to \underline{R}$ of

$\dot w(t) = g'(h(t-1)) w(t-1)$, $w_0 = v_0$.

2. Application of Proposition III.4.2 to $y := h(\cdot + t_+) - \omega$ and $\underline{w} := w(\cdot + t_+)$: Corollary 1 and $g = g_1$ on the interval $[-\gamma,\gamma]$ imply that for all $t > 0$,

$$\dot y(t) = \dot h(t + t_+) = g(h(t + t_+ - 1)) = g(h(t + t_+ - 1) - \omega)$$
$$= g_1(y(t-1)) .$$

Note $y_0 \in N_+$ and $\underline{P}(y_0) = \chi_1 \in S$. For all $t > 0$,

$$\dot{\underline{w}}(t) = \dot w(t + t_+) = g'(h(t + t_+ - 1)) w(t + t_+ - 1)$$
$$= g_1'(y(t.1)) \underline{w}(t-1) .$$

Proposition 3.2 guarantees that $\underline{w}_0 - c^* \dot y_0 = w_{t_+} - c^* \dot h_{t_+}$ has no zero.

It follows that for some $t \geq 0$, $\underline{w}_t = w_{t + t_+}$ has no zero.

3. Application of Proposition III.4.1 : For all $n \in \underline{N}$,

$$\chi_{n+1} = P'(\chi_n) = P(\chi_n) \quad \text{and} \quad v_{n+1} = DP'(\chi_n) v_n = DP(\chi_n) v_n$$

so that we obtain

$$v_1 = D\underline{P}(h_{t_+} - \omega) w_{t_+} = D\underline{P}(y_0) \underline{w}_0 \notin T_{\underline{P}(y_0)} S = T_{\chi_1} S ,$$

and for $2 \leq n \in \underline{N}$,

$$v_n \notin T_{\chi_n} S .$$

V. ON CHAOTIC BEHAVIOR

A prototype of chaotic behavior is given by the shift in two symbols

$$\sigma: (b_n)_{n \in \underline{Z}} \to (b_{n+1})_{n \in \underline{Z}}$$

on the space $\{0,1\}^{\underline{Z}}$. With the discrete topology on the symbol space $\{0,1\}$ and the product topology on the sequence space, the latter is compact, and the shift is a homeomorphism with complicated trajectories. See [6] for details.

Smale [26,27] proved that diffeomorphisms with transversal homoclinic points have invariant sets on which iterates are topologically conjugate to a symbol shift. See also [17,19,21,22] and Šilnikov's work [25].

These results can be generalized to smooth maps which are not necessarily injective. See [8], or the approach in [29] where hyperbolic structures for arbitrary C^1-maps and shadowing are used.

Of course, one can not expect that restrictions of such maps are equivalent to a shift, which is one-to-one. Nontrivial equivariance seems possible, or better, equivalence of a symbol shift with the shift induced by the map considered on a space of trajectories.

The latter is what Theorem 5.2 [8] asserts. To describe this more precisely, let $F:V \to L$ be a map on an open subset V of a real Banach space L. (This is a bit more general than in [8] where $V = L$.) The trajectories $(y_n)_{-\infty}^{\infty}$ of F form a closed subset Y of $V^{\underline{Z}}$ (with the discrete topology on V). On Y, F acts as the shift

$$\sigma_F: (y_n)_{n \in \underline{Z}} \to (y_{n+1})_{n \in \underline{Z}} = (F(y_n))_{n \in \underline{Z}} .$$

For a finite collection of pairwise disjoint open subsets $\Sigma_0, \dots, \Sigma_m$ of V, let I_Σ denote the map from the set Y_Σ of trajectories in $\Sigma_0 \cup \dots \cup \Sigma_m$ into the space $\Sigma := \{\Sigma_0, \dots, \Sigma_m\}^{\underline{Z}}$ defined by

$$I_\Sigma(y)_n = \Sigma_\mu \text{ if } y_n \in \Sigma_\mu .$$

With the discrete topology on the symbol space $\{\Sigma_0, \dots, \Sigma_m\}$, I_Σ becomes continuous.

Suppose F is C^1, $0 \in V$ is a hyperbolic fixed point, and there is a homoclinic trajectory $(y_n)_{-\infty}^{\infty}$. The result of Hale and Lin guarantees (under additional hypotheses including transversality) that there exist an integer $k > 0$ and sets $\Sigma_0, \dots, \Sigma_m$; $m \geq 2$, as above with

$$0 \in \Sigma_0 \quad \text{and} \quad y_n \in \Sigma_0 \cup \dots \cup \Sigma_m \quad \text{for all} \quad n \in \underline{Z}$$

such that I_Σ defines a homeomorphism onto the subset $\Sigma_I \subset \Sigma$ given by

$$\begin{aligned} b_{i+1} &= \Sigma_{j+1} \quad \text{for } j = 1, \dots, m && \text{in case } b_i = \Sigma_j , \\ b_{i+j} &= \Sigma_0 \quad \text{for } j = 1, \dots, k && \text{in case } b_i = \Sigma_m , \\ b_{i+1} &= \Sigma_1 && \text{in case } b_i = \Sigma_0 \neq b_{i+1} \end{aligned}$$

which makes the diagram

$$\begin{tikzcd} Y_\Sigma \arrow[r, "\sigma_F"] \arrow[d, "I_\Sigma"'] & Y_\Sigma \arrow[d, "I_\Sigma"] \\ \Sigma_I \arrow[r, "\sigma"'] & \Sigma_I \end{tikzcd}$$

commutative.

The additional hypotheses mentioned above require an equivalent norm $|\ |^*$ on L, constants $q \in (0,1)$, $r = r_F > 0$, $c = c_F > 0$ with

$$q + c < 1$$

and an integer $j = j_F > 0$ so that the following holds, with $A = DF(0)$.

(1) $|Av|^* \geq q^{-1}|v|^*$ on the linear unstable space L^u,
$|Av|^* \leq q|v|^*$ on the linear stable space L^s.

(2) $|p^u[DF(y) - A]|^* < c$ and $|p^s[DF(y) - A]|^* < c$ for all y in $\mathrm{cl}\, L_r^* = \{y \in L: |y|^* \leq r\}$, with the projections p^u of L onto L^u and p^s of L onto L^s given by $L = L^u \oplus L^s$.

(3) $\{0\} \neq \tilde{U}(r) \subset L^u$ and $\tilde{S}(r) \subset L^s$ for the local invariant manifolds $\tilde{U}(r)$ and $\tilde{S}(r)$ obtained from the application of Theorem II.1.1 to the restriction $F|L_r^*$.

(4) DF is uniformly continuous on some neighborhood of $0 \in L$.

(5) $y_{-j} \in \tilde{U}(r)$, $y_j \in \tilde{S}(r)$, and F^{2j} maps an open neighborhood Δ_u of y_{-j} in $\tilde{U}(r)$ diffeomorphically onto its image $F^{2j}(\Delta_u)$ which is a C^1-submanifold of L with

$$T_{y_j} F^{2j}(\Delta_u) \oplus T_{y_j} \tilde{S}(r) = L .$$

The notation in (1) - (5) is as in Chapter II.

We indicate how to verify these hypotheses for a restricted iterate of the map P' in suitable coordinates, provided the underlying nonlinearity g is C^2. It is clear from the constructions in Chapters I, III, IV that C^2-smoothness can be achieved.

Then the semiflow X of eq. (g) is C^2 on $(2,\infty) \times C$, and one can use the Implicit Function Theorem to show that all iterates P^n and $(P')^n$, $2 \leq n \in \underline{N}$, are C^2.

1. In order to satisfy (4), one introduces neighborhoods of 0 in L^u and in L^s so that the restrictions of the maps u and s from Section III.2 are C^2. (P^2 is C^2, and close to ϕ^* local stable and unstable manifolds for P^2 coincide with the local stable and unstable manifolds S and U of P from Section III.2.)

As in Section II.1, there is a C^2-diffeomorphism $T:D_T \to H$ from an open neighborhood D_T of ϕ^* onto an open neighborhood V_T of $0 \in H$, with

$$T(\phi^*) = 0 \ , \quad DT(\phi^*) = id \ ,$$

$$T(U \cap D_T) = L^u \cap V_T \ , \quad T(S \cap D_T) = L^s \cap V_T \ .$$

2. Choose $\nu \in \underline{N}$ with

$$\chi_{n\nu} \in U \cap D_T \quad \text{for} \quad n \in -\underline{N} \ , \quad \chi_{n\nu} \in S \cap D_T \quad \text{for} \quad n \in \underline{N} \ .$$

The domain $D'_{(2\nu)}$ of the iterate $(P')^{2\nu}$ is an open neighborhood of the fixed point ϕ^* which contains all χ_n, $n \in \underline{Z}$. It follows that the points $\ldots \chi_{-3\nu}, \chi_{-\nu}, \chi_{\nu}, \chi_{3\nu}, \ldots$, or better

$$\chi_n^* := \chi_{(2n-1)\nu} \quad \text{for} \quad n \in \underline{Z}$$

form a homoclinic trajectory of the restriction

$$(P')^{2\nu}|D'_{(2\nu)} \cap D_T \ .$$

We have $\chi_n^* \neq \phi^*$ for all $n \in \underline{Z}$.

In order to get a map with range in D_T, set

$$D^* := [D'_{(2\nu)} \cap D_T] \cap ((P')^{2\nu})^{-1}(D_T)$$

and

$$P^* := (P')^{2\nu}|D^* \ .$$

Clearly

$$P^*(D^*) \subset D_T \ , \quad \phi^* \in D^* \ , \quad \chi_n^* \in D^* \quad \text{for all} \quad n \in \underline{Z} \ .$$

P^* is C^2 since it is given by an iterate of $(P')^2$.

Corollary IV.4.4, Theorem II.1.1 (iv) and the definition of P' yield

(6) $$D(P^*)^{l-n}(\chi_n^*)v' \notin T_{\chi_l^*}S \quad \text{for all integers} \quad n \leq 0 \ , \quad l > 0$$

$$\text{if} \quad 0 \neq v' \in T_{\chi_n^*}U \ .$$

3. Set $V := T(D^*)$. The transformed map $F:V \to H$, $F(\phi) = T(P^*(T^{-1}(\phi)))$ for $\phi \in V$, is C^2 so that (4) is satisfied, and we have $F(0) = 0$, $F(V) \subset V_T$.

The points $\phi_n := T(\chi_n^*)$, $n \in \underline{Z}$, form a homoclinic sequence of F, $\phi_n \to 0$ as $|n| \to \infty$ and $\phi_n \neq 0$ for all $n \in \underline{Z}$.

Set $A := DF(0) = DP(\phi^*)^{2\nu}$. We see that 0 is a hyperbolic fixed point of F, and we obtain (1) with L^u, L^s, $|\ |^*$ from Section III.2, and with some $q \in (0,1)$. Choose $c = c_F > 0$ with

$$q + c < 1 < q^{-1} - c$$

and $r' > 0$ so small that

(2') $$\text{for all} \quad \phi \in L^{u*}_{r'} + L^{s*}_{r'} \subset V \ ,$$

$$|p^u[DF(\phi) - A]|^* + |p^s[DF(\phi) - A]|^* + |DF(\phi) - A|^* < c \ .$$

4. In order to obtain (3) one shows first that there exists $r = r_F$ in $(0,r')$ with

(7) $$cl\ H_r^* \subset L^{u*}_{r'} + L^{s*}_{r'} \quad \text{and} \quad F(L^{u*}_r) \subset L^u \ , \quad F(L^{s*}_r) \subset L^s \ :$$

Theorem II.1.1 implies that there is a neighborhood $\tilde{D} \subset D_T$ of ϕ^* in H such that for $\phi \in U \cap \tilde{D}$, the iterates $P(\phi), \ldots, P^{2\nu}(\phi)$ are defined with

$$U \ni P(\phi) = P'(\phi) \ , \ \ldots \ , \ U \ni P^{2\nu}(\phi) = (P')^{2\nu}(\phi) \ ,$$

while for $\phi \in S \cap \tilde{D}$,

$$S \ni P(\phi) = P'(\phi) \ , \ \ldots \ , \ S \ni P^{2\nu}(\phi) = (P')^{2\nu}(\phi) \ .$$

Choose $r = r_F \in (0,r')$ so small that

$$H_r^* \subset V \ , \quad cl\ H_r^* \subset L^{u*}_{r'} + L^{s*}_{r'} \ , \quad T^{-1}(H_r^*) \subset \tilde{D} \ .$$

For $\phi \in L^{u*}_r$, use

$$P^*(T^{-1}(\phi)) = P^{2\nu}(T^{-1}(\phi)) \in U \ ,$$

$$F(\phi) = T(P^*(T^{-1}(\phi))) \in T(U \cap D_T) \subset L^u \ ;$$

similarly for $\phi \in L^{s*}_r$.

5. The first inclusion in (7) and (2') yield (2).

6. Verification of (3). Consider the local stable and unstable manifolds $\tilde{S}(r)$ and $\tilde{U}(r)$ obtained from the application of Theorem II.1.1 to $F|H_r^*$.

Proof of $\tilde{U}(r) \subset L^u$: Let $\psi \in \tilde{U}(r)$. There is a trajectory $(\psi_n)_{-\infty}^0$ of F in H_r^* with $\psi = \psi_0$. Recall Corollary II.1.5 and its proof - the properties (7) and (2') imply

$$|p^s\psi|^* = |p^s F(\psi_{-1})|^* \leqq (q + c)|p^s \psi_{-1}|^* \leqq \ldots \leqq (q + c)^n r'$$

for all $n \in \underline{N}$. Hence $p^s\psi = 0$, $\psi \in L^u$.

The proof of $\tilde{S}(r) \subset L^s$ is analogous, with $1 < q^{-1} - c$.

$\{0\} \underset{\neq}{\subset} \tilde{U}(r)$ is clear since $\tilde{U}(r)$ and $\tilde{S}(r)$ are open subsets of L^u and L^s , respectively.

7. Verification of the transversality condition (5). Choose $j = j_F$ in $\underline{N}$ with

$$\phi_{-j} \in \tilde{U}(r) \subset L^u , \quad \phi_j \in \tilde{S}(r) \subset L^s .$$

Observe that for all ϕ in the domain $V_{(2j)} \subset V$ of F^{2j} ,

$$F^{2j}(\phi) = T((P^*)^{2j}(T^{-1}(\phi))) .$$

We have

$$\phi_{-j} \in V_{(2j)} , \quad \phi_j = F^{2j}(\phi_{-j}) .$$

Property (6) implies that $DF^{2j}(\phi_{-j})$ maps the one-dimensional tangent space $T_{\phi_{-j}}\tilde{U}(r) = L^u$ onto a one-dimensional space which is a complement of the closed subspace $T_{\phi_j}\tilde{S}(r) = L^s$ in H . It follows that F^{2j} maps an open neighborhood Δ_u of ϕ_j in $\tilde{U}(r)$, i.e., an open subset of L^u , diffeomorphically onto its image which is a one-dimensional C^2-submanifold of H , transversal to $\tilde{S}(r) \subset L^s$ at ϕ_j .

Mathematisches Institut
Universität München
Theresienstr. 39
D 8000 München 2
W. Germany

REFERENCES

[1] R. Abraham, J. Robbin: Transversal Mappings and Flows. Benjamin, New York 1967
[2] C.I. Cowan, Z.J. Jelonek: Synchronized Systems with Time Delay in the Loop. Proc. Inst. Radio Engrs. 41, 388 - 397 (1957)
[3] J. Dieudonne´: Foundations of Modern Analysis. Academic Press, New York 1960
[4] N. Dunford, J.T. Schwartz: Linear Operators I. Interscience, New York 1967
[5] T. Furumochi: Existence of Periodic Solutions of One-Dimensional Differential-Delay Equations. Tôhoku Math. J. 30, 13 - 35 (1978)
[6] W.H. Gottschalk, G.A. Hedlund: Topological Dynamics. Am. Math. Soc., Providence 1955
[7] J.K. Hale: Theory of Functional Differential Equations. Springer, New York 1977
[8] J.K. Hale, X.B. Lin: Symbolic Dynamics and Nonlinear Semiflows. Ann. Mat. Pura Appl. 144, 229 - 259 (1986)
[9] J.K. Hale, X.B. Lin: Examples of Transverse Homoclinic Orbits in Delay Equations. Nonlinear Analysis 10, 693 - 709 (1986)
[10] U. an der Heiden, H.O. Walther: Existence of Chaos in Control Systems with Delayed Feedback. J. Diff. Equ. 47, 273 - 295 (1983)
[11] U. an der Heiden, H.O. Walther: Chaos in Differential Delay Equations. In: The 9th Int. Conf. on Nonlinear Oscillations. Vol.2, 88 - 91. Kiev 1984
[12] M.C. Irwin: Smooth Dynamical Systems. Academic Press, London 1980
[13] J.L. Kaplan, J.A. Yorke: Ordinary Differential Equations which yield Periodic Solutions of Differential-Delay Equations. J. Math. Analysis Appl. 48, 317 - 324 (1974)
[14] T. Kato: Perturbation Theory for Linear Operators. 2nd Ed., Springer, Berlin 1976
[15] T.Y. Li, J.A. Yorke: Period Three implies Chaos. Am. Math. Monthly 82, 985 - 992 (1975)
[16] F.R. Marotto: Snap-Back Repellers imply Chaos in $\underline{R}^n$. J. Math. Analysis Appl. 63, 199 - 223 (1978)
[17] J.K. Moser: Stable and Random Motions in Dynamical Systems. Princeton Univ. Press, Princeton 1973
[18] A. Neugebauer: Invariante Mannigfaltigkeiten und Neigungslemmata für Abbildungen in Banachräumen. Diploma thesis, Math. Inst. Univ. München, 1988

[19] S.E. Newhouse: Lectures on Dynamical Systems. In: Dynamical Systems. CIME Lectures Bressanone 1980. Birkhäuser, Boston 1980

[20] R.D. Nussbaum: Uniqueness and Nonuniqueness for Periodic Solutions of $x'(t) = -g(x(t-1))$. J. Diff. Equ. 34, 25 - 54 (1979)

[21] K.J. Palmer: Exponential Dichotomies and Transversal Homoclinic Points. J. Diff. Equ. 55, 225 - 256 (1984)

[22] K.J. Palmer: Exponential Dichotomies, the Shadowing Lemma and Transversal Homoclinic Points. In: Dynamics Reported. Vol. 1, 265 - 306. Teubner/Wiley, Stuttgart 1988

[23] H. Peters: Globales Lösungsverhalten zeitverzögerter Differentialgleichungen am Beispiel von Modellfunktionen. Ph.D. thesis, Bremen 1980

[24] A.N. Šarkovskiĭ: Coexistence of Cycles of a Continuous Map of a Line into itself. Ukrainian Mat. Z. 16, 61 - 71 (1964)

[25] L.P. Šilnikov: On a Poincaré-Birkhoff Problem. Math. U.S.S.R. Sbornik 3, 353 - 371 (1967)

[26] S. Smale: Diffeomorphisms with Many Periodic Points. In: Differential and Combinatorial Topology, 63 - 80. Princeton Univ. Press, Princeton 1965

[27] S. Smale: Differentiable Dynamical Systems. Bull. Am. Math. Soc. 73, 747 - 817 (1967)

[28] H. Steinlein, H.O. Walther: Hyperbolic Sets and Shadowing for Noninvertible Maps. In: Proc. Advanced Topics in the Theory of Dynamical Systems (Trento 1987). Academic Press, to appear

[29] H. Steinlein, H.O. Walther: Hyperbolic Sets, Transversal Homoclinic Trajectories and Symbolic Dynamics for C^1-Maps in Banach Spaces. Preprint 1988, submitted

[30] Y. Ueda: Self-Excited Oscillations and their Bifurcations in Nonlinear Differential-Difference Equations. In: 24th Midwest Symp. on Circuits and Systems. Univ. of New Mexico, Albuquerque 1981

[31] H.O. Walther: Bifurcation from Periodic Solutions in Functional Differential Equations. Math. Z. 182, 269 - 289 (1983)

[32] H.O. Walther: Homoclinic Solution and Chaos in $\dot{x}(t) = f(x(t-1))$. Nonlinear Analysis 5, 775 - 788 (1981)

[33] H.O. Walther: Bifurcation from a Heteroclinic Solution in Differential Delay Equations. Trans. Am. Math. Soc. 290, 213 - 233 (1985)

[34] H.O. Walther: Bifurcation from a Saddle Connection in Functional Differential Equations: An Approach with Inclination Lemmas. DISSERTATIONES MATH., to appear

[35] H.O. Walther: Homoclinic and Periodic Solutions of Scalar Differential Delay Equations. In: Proc. of the Semester on Dynamical Systems and Ergodic Theory. Banach Center Publications, Warsaw, to appear

[36] H.O. Walther: Inclination Lemmas with Dominated Convergence. J. Appl. Math. Phys. 32, 327 - 337 (1987)