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On the 21 Card Trick and Verini's Lost Trick

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Abstract. We provide a necessary and sufficient condition for the solvability of generalized variants of the twenty-one card trick (21CT). The 21CT works by dealing the cards into three stacks several times and, after the spectator identifies the stack containing their card each round, arranging the stacks so the performer always knows where the chosen card ends up for the final reveal. Our analysis extends to any number of cards, arranged in multiple stacks, with an arbitrary number of cards per stack and an arbitrary number of iterations of dealing into stacks. We also allow for flexible collection of the stacks induced by face-down or face-up dealing of the deck of cards, ensuring that the unknown card appears at a predetermined position within the deck. Notably, this also allows us to analyze the historically first card trick to appear in print, proposed by Verini in 1542—a lost card trick that was rediscovered only recently: it turns out that the trick *does not* work. But, we also demonstrate how it can be fixed.

MSC Classification:
00A08: Recreational mathematics
01A40: 15th and 16th centuries, Renaissance
05-03: Historical
11A63: Radix representation; digital problems
11B37: Recurrences
68R05: Combinatorics

Additional Key Words and Phrases: card magic, self working trick, 21 card-trick (generalization), Verini's lost card trick, face-down and face-up dealing, solvability, mixed radix, numeral system

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1 Introduction

The twenty-one card trick (21CT) is a classical self working trick with ordinary cards, and not requiring sleight-of-hand. It is described as follows—the description is literally taken from [16] but we have changed the term "heap" to "stack," as indicated by square brackets:

> Deal the cards into three packs, face upwards, and request a spectator to note a card, and remember in which [stack] it is. When you have dealt twenty-one cards, throw the rest aside, these not being employed in the trick. Ask in which [stack] the chosen card is, and place that [stack] between the other two, and deal again as before. Again ask the question, place the [stack] indicated in the middle, and deal again a third time. Note particularly the fourth or middle card of each [stack], as one or other of those three cards will be the card thought of. Ask, for the last time, in which [stack] the chosen card now is, when you may be certain that it was the card which you noted as being the middle card of that [stack].[1]

A sample performance of the 21-card trick is given in Section 2, which also includes some historical remarks and the mathematical notation used. The mathematical principle behind this trick, known as the *redistribution principle*, is the basis of many other classic tricks in

[1] This trick uses a convention we shall call "face-down dealing:" that is, the magician holds the card stack face down in the hand when dealing the cards. We will discuss this aspect at greater depth further below.

magic.[2] In particular, the very first card trick to appear in print, described by Giovanni Battista Verini, is based on the same principle. Verini's "Specchio del mercatate al. S." [21] published in Italy in 1542, not only explains arithmetic principles for merchants but also includes a chapter with several mathematical recreations using playing cards; see Figure 1. One of them is a card guessing trick that uses 52 cards, 4 stacks and 3 iterations; the trick is based on the redistribution principle. Yet it is interesting to note that the (implicit) convention in the description of the trick

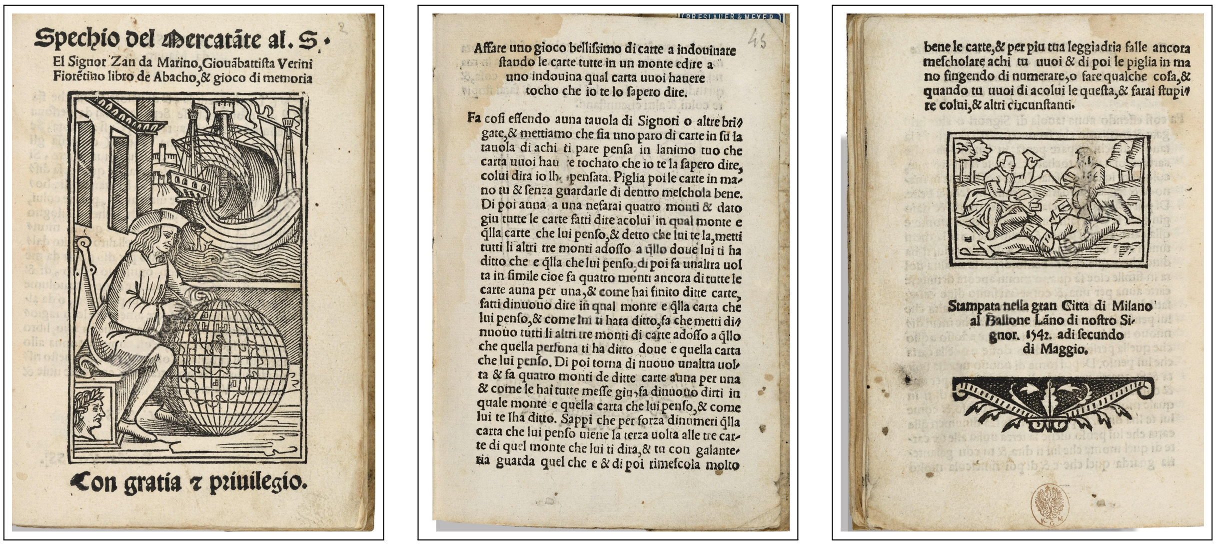


Fig. 1. Verini's description[4] of the 52 card trick in his book "Specchio del mercatante el. S." from 1542, titlepage and folio 45r and 45v (from left to right).

is to turn over the entire deck when dealing out the cards. This is different from the other early sources, and we can speculate that this might hint at several independent initial developments of this card trick. Its inner workings make the analysis a bit more intricate, as we shall see below. Although it is the oldest known card trick, the book was buried in the archives for centuries, and textbooks on magic do not mention the trick. It was rediscovered only in 2022, and the hunt for the book is an interesting story in its own right; see [19]. The present work features the first mathematical analysis of Verini's lost trick. Interestingly, we shall demonstrate the trick is broken, in the sense that it *does not* work in *all cases*. However, there are various ways to amend Verini's trick, with the same or a different number of cards, using the methods developed in this paper.

[2] For more background, see "Conjuring Credits—The Origins of Wonder" at the website https://www.conjuringcredits.com.

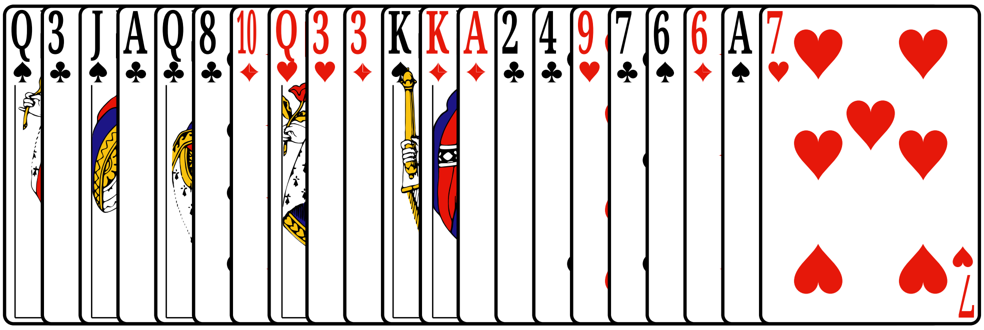
[4] The following English translation is literally taken from [19]: [f. 45r] To make a beautiful game of divining cards, being the cards all in one pile, and saying to someone: Divine which card you want to have chosen that I will be able to tell you. Do thus: Being at a table of lords or other groups, and let's say there is a deck of cards on the table, tell whoever you like: Think in your soul which card you want to have chosen, that I will be able to tell you. He will say: I have thought of it. Then you take the cards in your hand and, without looking inside [the deck], mix them well. Then, one by one, you will make four piles and, once all the cards are dealt, [while left unclear by Verini, the cards must be dealt face up and kept that way throughout for the trick to work] have him tell you in which pile is the card he thought of, and once he has told you, put all the other three piles on top of the one where he told you is the one he thought of. Then do the same again; that is, make four more piles of all the cards, one by one, and as you have finished the said cards, have him tell you in which pile is the card he thought of, and as he will have told you, proceed such that you put all the other

The twenty-one card trick has been studied mathematically for many years; for more information see Section 2. Recently, a particularly detailed analysis was provided by Deb [5], who offered a systematic treatment of the redistribution principle underlying the trick. This work clarifies why the trick succeeds and identifies the conditions under which the chosen card can be determined with certainty. In this paper, we build on this line of work and extend it in several directions. Our first contribution is a general framework that allows the study of arbitrary collection sequences, rather than only the classical "middle stack" procedure. This leads to explicit formulas and simple conditions that characterize when a card trick based on redistribution is solvable. A key advantage of our approach is that it treats face-down and face-up dealing in a unified way, making it possible to directly compare the two settings and to identify essential differences between them. We then apply this framework to Verini's card trick. Using our methods, we show precisely why the original version of the trick fails in certain situations. Finally, we demonstrate how the trick can be repaired by modifying the collection procedure, thereby recovering a working version of Verini's idea. In this way, our results not only clarify the mathematical structure behind classical card tricks, but also shed new light on the earliest known example of such a trick. In order to keep the paper self-contained, it includes also cut-out cards for use in the Appendix.

2 A Few Comments on Card Tricks

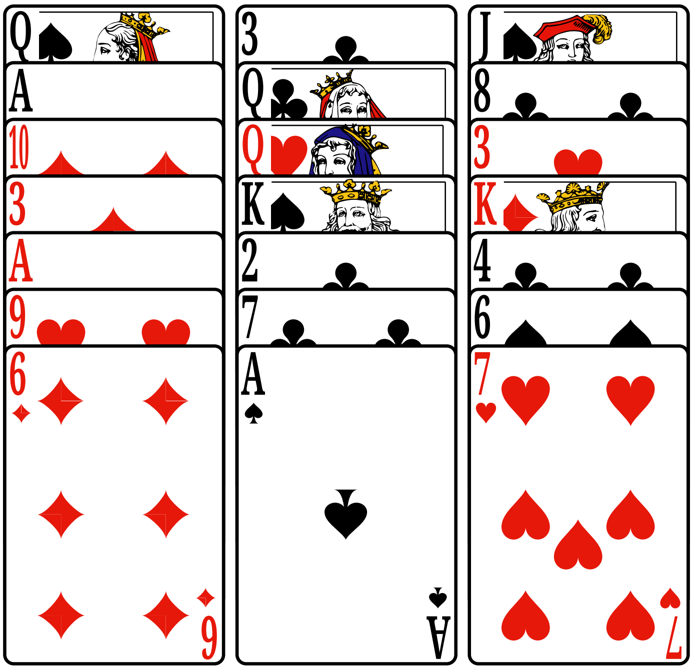
We start with a performance of the original 21 card trick. In the forthcoming we will speak of the magician (henceforth Magi) demonstrating the trick to a spectator (henceforth Audy). Let us show the 21 card trick in more detail in the following example, but let Audy select the cards in order to make the trick more spectacular and we place the stack asked for the last time in the middle of the deck such that card which was noted is being the eleventh card within the whole deck—these changes are implemented in order to make the analysis of the card trick easier to grasp:

Example 1. Audy randomly selects 21 cards, remembers the Jack of Clubs (third from the left), and shuffles the deck, which results in

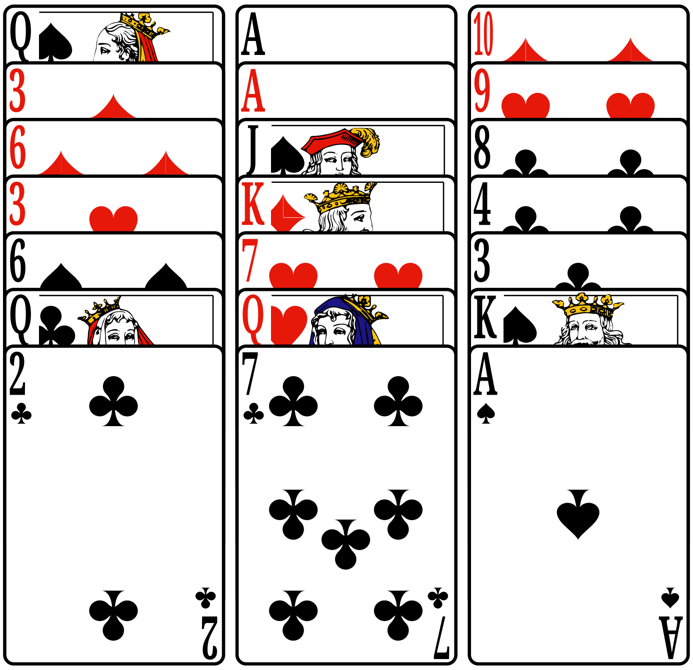


which is handed over to Magi. Then he deals out the cards into three adjacent stacks. On the table these stacks look as follows:

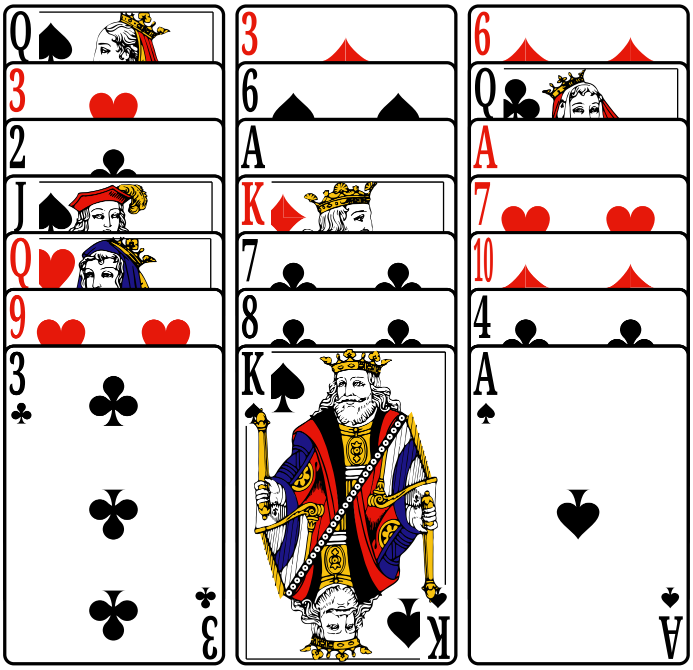
three piles of cards back on top of the one in which that person told you there is the card he thought of. Then, do the same again, one more time, and make four piles of the said cards, one by one, and as you have them all dealt, have him tell you again in which pile is the card he thought of, and when he has told you, be aware that by the power of numbers, the card he thought of comes the third time as the third card of that pile that he will tell you. And you gracefully note what it [the card] is and then shuffle the cards very [f. 45v] well, and for your greater elegance, have them shuffled again by whomever you want, and then take them in your hand, pretending to count them, or to do something, and when you want, say to him: This is it, and you will amaze him and the others around you.



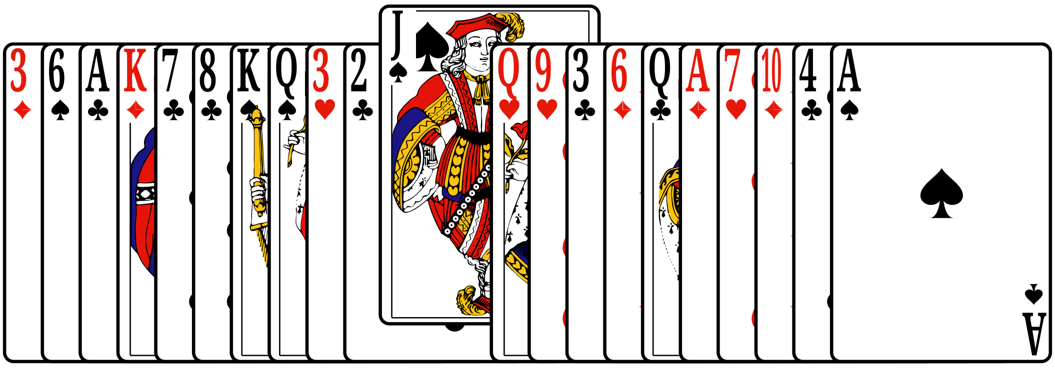
Audy states that her card is in the third stack. Magi collects the stacks into a deck from left to right and places the third stack in the middle of the deck. Then he again deals out the cards into three adjacent stacks, which results in



Before the final round starts, Audy points to the middle stack. Then Magi collects the stacks from left to right such that the stack Audy pointed to is in the middle, and continues by dealing the cards into three stacks again. Magi ends this round with the following card stacks on the table:



Audy's final hint is that the memorized card is in the first stack. Thus, Magi collects the deck by taking the stacks from left to right and placing the first stack in the middle of the pack. Then he ends up with the following order of cards at his hand



where the eleventh card (slightly moved up) is the Jack of Clubs and Audy confirms that this was the memorized card.

In the classic monograph [16] it is also mentioned that the same effect will be produced with any number of cards, so long as such number is odd, and a multiple of three. The process and result will be the same, save that if fifteen cards are used each stack will consist of five cards, and the third card of each will be the middle one; if twenty-seven cards, each stack will consist of nine cards, and the fifth will be the selected one. The latter card trick is also described in [17] in detail. Although it is not explicitly stated in [17], this card trick is closely related to ternary numbers. We will come back to this issue later. It is worth mentioning that up to our knowledge a twenty-four card trick does not appear in the literature on magic tricks, and, as we will see, this is not a coincidence.

The 21 card trick has been extensively featured both in the literature on magic, and on recreational mathematics; and it has a very rich history. A variant of the twenty-one card trick appears already in a book by Horatio Galasso [9], who described a variant of the trick that uses 15 cards in 3 stacks in 1593. For most of the time during past centuries, magic tricks were taught only by oral tradition, so there are not many written descriptions. The variant on 21 cards seems to have been documented first by Bachet in 1612, see [2]. Bachet, not unlike Galasso, merely described the trick and did not include any explanations why it works. Also, he mentioned neither Galasso nor the variant of the trick on 15 cards.

The first proper mathematical analysis for this family of tricks was carried out only much later: standard references for mathematical magic [3, 11] credit Gergonne's 1813 paper [12] for the first rigorous analysis of the trick, this time on 27 cards, and its generalization to n^n cards, for n a positive integer. Gergonne also analyzed the inverse problem, namely, given a deck of n^n cards and a desired final location, in which order to collect the stacks in each round so that the card finally appears at the prescribed location.

Later (1868), Hudson studied variants of the trick with an arbitrary number of stacks and arbitrary number of cards per stack [18], yet the analysis was flawed. The argument was put straight by Dickson (1895), who showed that with n stacks, k iterations and $k+1$ iterations, respectively, are needed for a deck of size n^k, and of $n(n^k+2)$ respectively [6]. Subsequently, Onnen (1909) gave general bounds on the card position in terms of the number of iterations, and gave conditions under which the trick will not yield a unique reveal in the middle position after a given number of iterations [20]. He also analyzed the scenario where the number of stacks varies per iteration, and exemplified that analysis with a new variant, where 48 cards are laid out in 4 stacks in the first iteration, 3 stacks in the second iteration, and again in 4 stacks in the third iteration.

During the 20th century, the 21 card trick was popularized by textbooks on recreational mathematics and magic [3, 11]. More recent expositions treated the trick as a kind of discrete dynamical system [1, 15] or numeral system [4], to mention a few.

To our knowledge, Verini's 52 card trick has not been analyzed yet from a mathematical perspective. Before analyzing it, we need to clarify an implicit assumption from the example above, specifically how to deal and collect cards when performing the trick.

2.1 How to Deal and Collect Cards—Two Possibilities

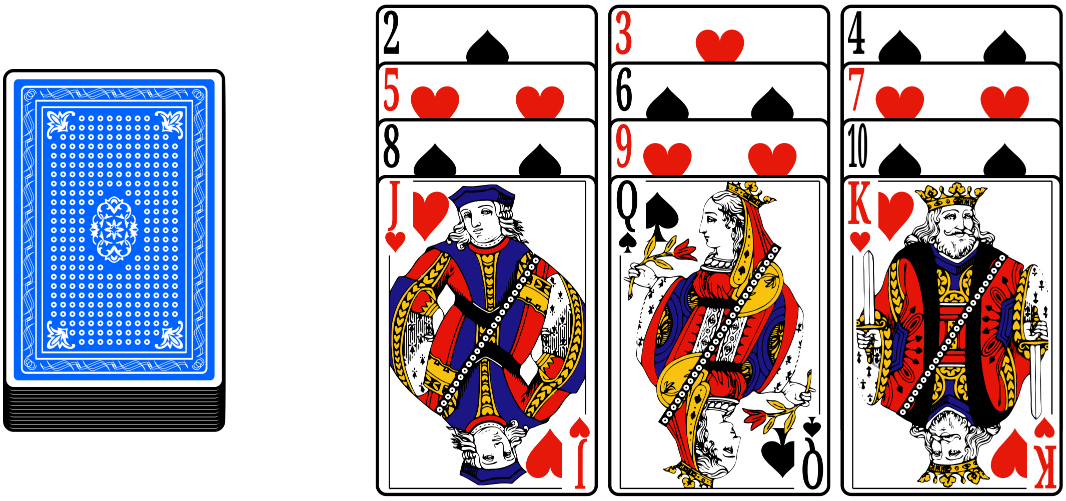
Consider the following cards fanned as follows:

Stacking the cards in the hand and flipping them over gives a deck of cards where we see the backside. We introduce the following two dealing process described below:

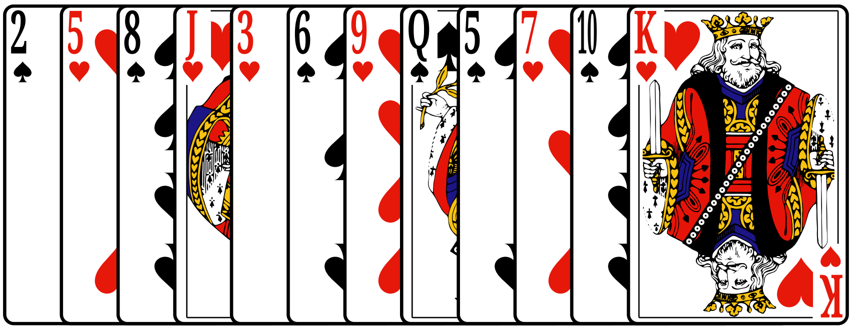
- dealing (and collecting) face-down and
- dealing (and collecting) face-up.

We first describe dealing face-down, which is (in our eyes) the more common dealing and collecting scenario. The classical 21 card trick is performed by dealing face-down, while Verini's trick is carried out in face-up mode. Unless stated otherwise, we assume that cards are dealt face down.

2.1.1 Dealing Face-Down Then start dealing the cards face-down and laying the cards face upward into three stacks results in the deck of cards shown on the left:

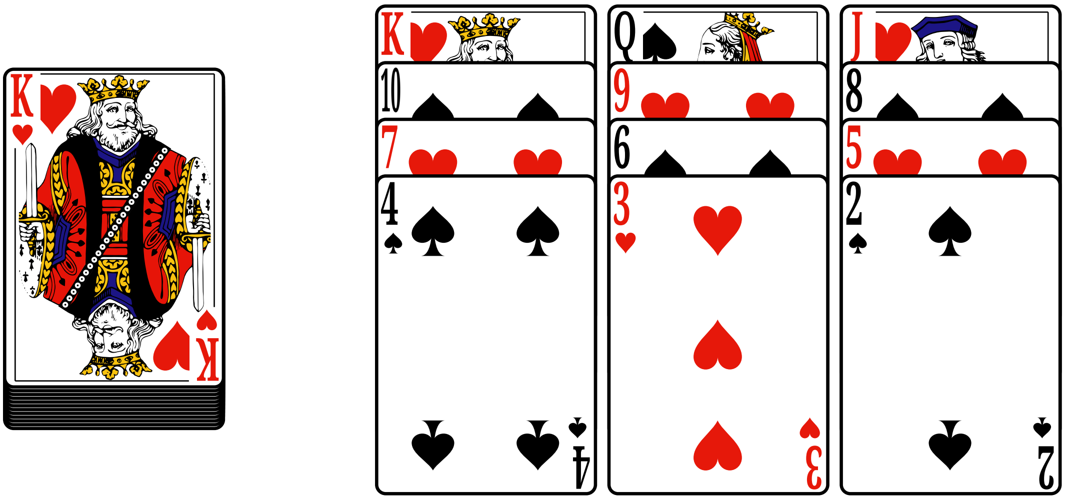


Collecting these stacks by gathering the stacks individually, turning the stack over, and placing the first and middle stack on top of the stack that contains the King of Hearts gives us a deck of cards which looks as follows when fanned out:

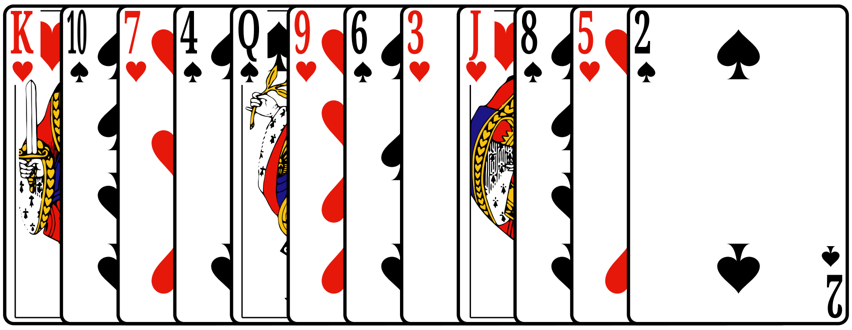


There is a slight variation possible, namely whether the left or middle stack is placed first on top of the stack that contains the King of Hearts. Nevertheless, all cards from the King of Hearts stack are at the right, and in particular, the King of Hearts is the last card in the deck.

2.1.2 Dealing Face-Up But there is also another way to deal and collect the cards. Before the dealing starts, the deck of cards is not flipped over. Hence, during the dealing process one sees the topmost card. Thus, when dealing the cards face-up and laying the cards face upward into three stacks results in the following—deck of cards is shown left again:



Collecting these stacks by placing the middle and the right-most stack on top of the stack that contains the King of Hearts gives us



when fanned out. Again, there maybe a slight variation depending on whether the middle or right-most stack is placed first on top of the stack that contains the King of Hearts. Observe, that dealing this way result in a deck, where the King of Hearts is the first card of the deck.

2.2 Deb's Solution to the Classical 21 Card Trick

We take some notation from [5]: A *card trick* or simply a *trick* is a quadruple (C, n, j, k), where C is the number of *given cards*, n is the number of *stacks to split into*, j with $0 \leq j < n$ is the number of *stacks to put on top of the stack which contains Audy's card*, and k is the number of *iterations to be performed*. Then the card trick (C, n, j, k) is *solvable* if there exists an ℓ such that the magic trick can be performed within k iterations, i.e., Audy's card is revealed in Step 4 as the ℓth card. In this case we simple write $(C, n, j, k) = \ell$. Observe, that in Step 3 always j stacks are put on top of the stack which contains Audy's card. Thus, fixing C, n, and k we are left with n card trick, namely (C, n, j, k) for each j with $0 \leq j < n$.

In order to analyze card tricks we need some more notation: let d_k (s_k, respectively) refer to the deck position (stack position, respectively) of Audy's card after $k \geq 1$ iterations. Thus, d_k (s_k, respectively) denotes the position of the chosen card from the top of the full deck of n distinct cards (within the specific stack that contains it, respectively). With this definition we can describe the dealing and collection process as follows: Assume that $C = mn$. Then define d_0 satisfying $1 \leq d_0 \leq C$ and for $k \geq 1$ let

$$d_k = mj + s_k,$$

where

$$s_k = \left\lceil \frac{d_{k-1}}{n} \right\rceil .$$

More, on floor- and ceiling functions, can be found in the Appendix.

The following was shown in [5, Theorem 3.4]:

Theorem 2. *Let $m = C/n$ and $k \geq 1$. Then*

$$d_k = \begin{cases} mj + \left\lceil \dfrac{mjn\left(\dfrac{n^{k-1}-1}{n-1}\right) + d_0}{n^k} \right\rceil, & \text{if } n > 1, \\ d_0, & \text{if } n = 1, \end{cases}$$

for j with $0 \leq j < n$.

This equation is the key to characterize solvability of card tricks. In a series of theorems in [5, Theorems 4.1–4.5] a complete characterization of solvability was given, where the most complicated case (the last one) uses a divisiblity property, which forces that the border case, where the stack shown by Audy is either always placed by Magi at the top or bottom of the deck of cards, have to be considered separately. The summary of these theorems reads as follows:

Theorem 3. *Let $m = C/n$ and $b = \frac{mj}{n-1}$ for $n > 1$. Then the following holds:*

1. *If $C = 1$, then the trick $(1, 1, 0, k)$ is solvable for any $k \geq 1$ with $(1, 1, 0, k) = 1$.*
2. *If $C > 1$ and $n = 1$, then the trick $(C, 1, 0, k)$ is* not *solvable for any $k \geq 1$.*
3. *In $C > 1$ and $n > 1$ we distinguish several subcases:*
   (a) *The trick $(C, n, 0, k)$ is solvable for any integer $k \geq \log_n(C)$ with $(C, n, 0, k) = 1$.*
   (b) *The trick $(C, n, n-1, k)$ is solvable for any integer $k > \log_n(C-1)$ with the result $(C, n, n-1, k) = C$.*
   (c) *The trick (C, n, j, k) with $0 < j < n-1$ is solvable if and only if $(n-1) \nmid mj$ and $k \geq \log_n t$, where $t = \max\left\{\frac{C-bn}{1-\{b\}}, \frac{bn-1}{\{b\}}\right\}$, with $(C, n, j, k) = mj + \lfloor b \rfloor + 1$.*

It follows a list of 159 card tricks for small C and j with $0 \leq j < n$—see Table 1. Later we show that collection sequence induced by $j = 0$ and $j = n-1$ are always solvable if the collection sequence is long enough. It is worth mentioning that there are certain card tricks with short collections sequences of length two. This is quite appealing for a magic trick.

As the reader may have noticed from inspecting Table 1 the card trick $(15, 3, 1, 3)$ is exactly those of Galasso [9]. It also contains the familiar 21 card trick, but a 24 card variant is notably only listed for both trivial collection sequences. Interestingly, when considering a full 52 card deck, all card tricks with $n = 4$ stacks that arise from uniform collection sequences already occur for $k = 3$.

3 The 21 Card Trick and Verini's Lost Card Trick

This section is two-fold. First we define the notion of a card trick and then explain the solution presented in [5] in more detail. The we generalize the notion of a card trick to cover arbitrary collection sequences of Magi and give a necessary and sufficient condition whenever a collection sequence gives a unique predetermined position.

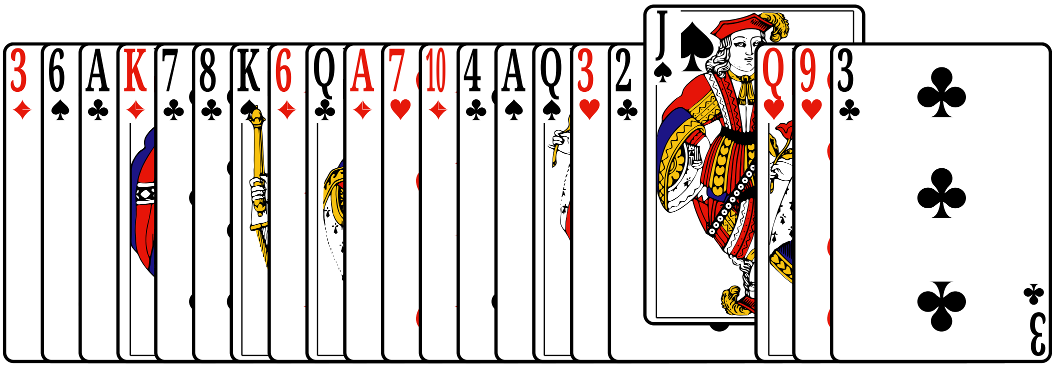
Solvable card trick $(C,n,j,k)=\ell$				
$(15,3,0,3)=1$	$(30,5,3,3)=23$	$(36,6,5,2)=36$	$(45,3,0,4)=1$	$(48,8,5,3)=35$
$(15,3,1,3)=8$	$(30,5,4,3)=30$	$(39,3,0,4)=1$	$(45,3,1,4)=23$	$(48,8,6,3)=42$
$(15,3,2,3)=15$	$(30,6,0,2)=1$	$(39,3,1,4)=20$	$(45,3,2,4)=45$	$(48,8,7,2)=48$
$(18,3,0,3)=1$	$(30,6,5,2)=30$	$(39,3,2,4)=39$	$(45,5,0,3)=1$	$(49,7,0,2)=1$
$(18,3,2,3)=18$	$(32,4,0,3)=1$	$(40,4,0,3)=1$	$(45,5,1,3)=12$	$(49,7,1,2)=9$
$(20,4,0,3)=1$	$(32,4,1,3)=11$	$(40,4,1,3)=14$	$(45,5,2,3)=23$	$(49,7,2,2)=17$
$(20,4,1,3)=7$	$(32,4,2,3)=22$	$(40,4,2,3)=27$	$(45,5,3,3)=34$	$(49,7,3,2)=25$
$(20,4,2,3)=14$	$(32,4,3,3)=32$	$(40,4,3,3)=40$	$(45,5,4,3)=45$	$(49,7,4,2)=33$
$(20,4,3,3)=20$	$(33,3,0,4)=1$	$(40,5,0,3)=1$	$(45,9,0,2)=1$	$(49,7,5,2)=41$
$(21,3,0,3)=1$	$(33,3,1,4)=17$	$(40,5,4,3)=40$	$(45,9,1,3)=6$	$(49,7,6,2)=49$
$(21,3,1,3)=11$	$(33,3,2,4)=33$	$(40,8,0,2)=1$	$(45,9,2,2)=12$	$(50,5,0,3)=1$
$(21,3,2,3)=21$	$(35,5,0,3)=1$	$(40,8,1,3)=6$	$(45,9,3,3)=17$	$(50,5,1,3)=13$
$(24,3,0,3)=1$	$(35,5,1,3)=9$	$(40,8,2,2)=12$	$(45,9,4,2)=23$	$(50,5,3,3)=38$
$(24,3,2,3)=24$	$(35,5,2,3)=18$	$(40,8,3,3)=18$	$(45,9,5,3)=29$	$(50,5,4,3)=50$
$(24,4,0,3)=1$	$(35,5,3,3)=27$	$(40,8,4,3)=23$	$(45,9,6,2)=34$	$(50,10,0,2)=1$
$(24,4,3,3)=24$	$(35,5,4,3)=35$	$(40,8,5,2)=29$	$(45,9,7,3)=40$	$(50,10,1,2)=6$
$(25,5,0,2)=1$	$(35,7,0,2)=1$	$(40,8,6,3)=35$	$(45,9,8,2)=45$	$(50,10,2,2)=12$
$(25,5,1,2)=7$	$(35,7,1,3)=6$	$(40,8,7,2)=40$	$(48,3,0,4)=1$	$(50,10,3,2)=17$
$(25,5,2,2)=13$	$(35,7,2,3)=12$	$(42,3,0,4)=1$	$(48,3,2,4)=48$	$(50,10,4,2)=23$
$(25,5,3,2)=19$	$(35,7,3,2)=18$	$(42,3,2,4)=42$	$(48,4,0,3)=1$	$(50,10,5,2)=28$
$(25,5,4,2)=25$	$(35,7,4,3)=24$	$(42,6,0,3)=1$	$(48,4,3,3)=48$	$(50,10,6,2)=34$
$(27,3,0,3)=1$	$(35,7,5,3)=30$	$(42,6,1,3)=9$	$(48,6,0,3)=1$	$(50,10,7,2)=39$
$(27,3,1,3)=14$	$(35,7,6,2)=35$	$(42,6,2,3)=17$	$(48,6,1,3)=10$	$(50,10,8,2)=45$
$(27,3,2,3)=27$	$(36,3,0,4)=1$	$(42,6,3,3)=26$	$(48,6,2,3)=20$	$(50,10,9,2)=50$
$(28,4,0,3)=1$	$(36,3,2,4)=36$	$(42,6,4,3)=34$	$(48,6,3,3)=29$	$(51,3,0,4)=1$
$(28,4,1,3)=10$	$(36,4,0,3)=1$	$(42,6,5,3)=42$	$(48,6,4,3)=39$	$(51,3,1,4)=26$
$(28,4,2,3)=19$	$(36,4,3,3)=36$	$(42,7,0,2)=1$	$(48,6,5,3)=48$	$(51,3,2,4)=51$
$(28,4,3,3)=28$	$(36,6,0,2)=1$	$(42,7,6,2)=42$	$(48,8,0,2)=1$	$(52,4,0,3)=1$
$(30,3,0,4)=1$	$(36,6,1,2)=8$	$(44,4,0,3)=1$	$(48,8,1,3)=7$	$(52,4,1,3)=18$
$(30,3,2,4)=30$	$(36,6,2,2)=15$	$(44,4,1,4)=15$	$(48,8,2,3)=14$	$(52,4,2,3)=35$
$(30,5,0,3)=1$	$(36,6,3,2)=22$	$(44,4,2,4)=30$	$(48,8,3,2)=21$	$(52,4,3,3)=52$
$(30,5,1,3)=8$	$(36,6,4,2)=29$	$(44,4,3,3)=44$	$(48,8,4,2)=28$	

Table 1. List of 159 solvable card tricks for C with $15 \leq C \leq 52$ on n stacks such that at each stack contains at least five cards, for j with $0 \leq j < n$, and minimal k.

3.1 Face-Down Dealing: Generalization to Arbitrary Collection Sequences

At first we have to generalize the notion of a card trick. A *generalised card trick* or simply a *generalized trick* is a quadruple $(C, n, \vec{\jmath}, k)$, where C, n, and k is defined as in a card trick, and $\vec{\jmath} = (j_k, j_{k-1}, \ldots, j_1)$ is a vector of length k where j_i, for $1 \leq i \leq k$ describes the number of stacks to put on top of the stack which contains Audy's card in the ith iteration. Thus, a card trick is nothing other than a generalized card trick with the *collection sequence* $\underbrace{(j, j, \ldots, j)}_{k \text{ times}}$. In general there are n^k different collection sequences. Observe, that the ordering of the collection sequence reflects a digit interpretation, allowing $\vec{\jmath}$ to be treated analogously to a mixed-radix number with j_1 the least significant component. Obviously, the concept of *solvability* of card tricks carries over to generalized card tricks. We give a small example.

Example 4. Consider the generalized card trick $(21, 3, (2, 1, 1), 3)$, i.e., $C = 21$, $n = 3$, and $k = 3$ as in the example from the previous section. Now the collection sequence is $(2, 1, 1)$ and *not* $(1, 1, 1)$. Thus, except for the last step, the card trick is performed as before. Only in the last step, Magi places two stacks on top of the stack identified by Audy. This results in Magi's card hand



where he finally reveals the eighteenth card from the deck.

Let d_0 satisfy $1 \leq d_0 \leq C$ as before. For the recurrences of d_k and s_k for the generalized card trick we obviously get

$$d_k = mj_k + s_k,$$

where

$$s_k = \left\lceil \frac{d_{k-1}}{n} \right\rceil$$

for $k \geq 1$. Now we are ready to generalize Theorem 2 using these recurrences.

Theorem 5. *Let $m = C/n$ and $k \geq 1$. Then*

$$d_k = \begin{cases} mj_k + \left\lceil \dfrac{mn\left(\sum_{i=1}^{k-1} j_i \cdot n^{i-1}\right) + d_0}{n^k} \right\rceil, & \text{if } n > 1, \\ d_0, & \text{if } n = 1, \end{cases}$$

for $\vec{j} = (j_k, j_{k-1}, \ldots, j_1)$ with $0 \leq j_i < n$ for $1 \leq i \leq k$.

Proof. The result for $n = 1$ is immediate. Thus, in the furthcoming assume $n > 1$. We prove the statement by induction on k. Let $k = 1$, then $d_1 = mj_1 + \left\lceil \frac{d_0}{n} \right\rceil$ and the formula is equal to

$$d_1 = mj_1 + \left\lceil \frac{mn\left(\sum_{i=1}^{1-1} j_i \cdot n^{i-1}\right) + d_0}{n^1} \right\rceil = mj_1 + \left\lceil \frac{mn \cdot 0 + d_0}{n} \right\rceil = mj_1 + \left\lceil \frac{d_0}{n} \right\rceil.$$

Next we continue with the induction step from k to $k+1$ with the appropriate induction hypothesis on k. Let

$$d_{k+1} = mj_{k+1} + s_k = mj_{k+1} + \left\lceil \frac{d_k}{n} \right\rceil = mj_{k+1} + \left\lceil \frac{mj_k + \left\lceil \frac{mn\left(\sum_{i=1}^{k-1} j_i \cdot n^{i-1}\right) + d_0}{n^k} \right\rceil}{n} \right\rceil ,$$

using the induction hypothesis in the last step.The latter term simplifies by moving mj_k into the inner ceiling by multiplying it with n^k. Then factoring mn and moving the remaining $j_k n^k$ into the sum as the last term gives us

$$\left\lceil \frac{\left\lceil \frac{mn\left(\sum_{i=1}^{k} j_i \cdot n^{i-1}\right) + d_0}{n^k} \right\rceil}{n} \right\rceil = \left\lceil \frac{mn\left(\sum_{i=1}^{k} j_i \cdot n^{i-1}\right) + d_0}{n^{k+1}} \right\rceil ,$$

by further simplification using the calculation rules of floors and ceilings. Combining this together leads us to

$$d_{k+1} = mj_{k+1} + \left\lceil \frac{mn\left(\sum_{i=1}^{k} j_i \cdot n^{i-1}\right) + d_0}{n^{k+1}} \right\rceil$$

which is the desired result.

Next, we use the above formula to calculate the final positions for the $C = 21$ possible start positions d_0 manipulated in the three stages ($k = 3$) on three stacks ($n = 3$). By testing all 27 possible collection sequences, we can see how the order affects the final result. This reveals patterns, symmetries, and which different sequences produce the same outcome. All 27 results are listed in Table 2 for reference. Thus, 9 out of the 27 possible collection sequences yield a unique final position. Hence, the question arises how to identify those collection sequences which obey a unique final position d_k.

3.1.1 When a Collection Sequence Yields a Unique Reveal

The case $n = 1$ is already solved by Theorem 3 also in the generalized case, namely $(C, 1, \underbrace{(0, 0, \dots, 0)}_{k \text{ times}})$ is solvable if $C = 1$ with the unique final position 1, while in all other cases, i.e., $C > 1$, the generalized card trick is *not* solvable. Thus in the forthcoming we assume $n > 1$ without further noticing. For a

	d_0																					
Coll. seq.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	d_k-set
$(0,0,0)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	$\{1\}$
$(0,0,1)$	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	$\{1,2\}$
$(0,0,2)$	2	2	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	3	$\{2,3\}$
$(0,1,0)$	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	4	4	4	$\{3,4\}$
$(0,1,1)$	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	$\{4\}$
$(0,1,2)$	4	4	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	$\{4,5\}$
$(0,2,0)$	5	5	5	5	5	5	5	5	5	6	6	6	6	6	6	6	6	6	6	6	6	$\{5,6\}$
$(0,2,1)$	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	7	7	7	7	7	7	$\{6,7\}$
$(0,2,2)$	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	$\{7\}$
$(1,0,0)$	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	$\{8\}$
$(1,0,1)$	8	8	8	8	8	8	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	$\{8,9\}$
$(1,0,2)$	9	9	9	9	9	9	9	9	9	9	9	9	10	10	10	10	10	10	10	10	10	$\{9,10\}$
$(1,1,0)$	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	11	11	11	$\{10,11\}$
$(1,1,1)$	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	$\{11\}$
$(1,1,2)$	11	11	11	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	$\{11,12\}$
$(1,2,0)$	12	12	12	12	12	12	12	12	12	13	13	13	13	13	13	13	13	13	13	13	13	$\{12,13\}$
$(1,2,1)$	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	14	14	14	14	14	14	$\{13,14\}$
$(1,2,2)$	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	$\{14\}$
$(2,0,0)$	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	$\{15\}$
$(2,0,1)$	15	15	15	15	15	15	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	$\{16,15\}$
$(2,0,2)$	16	16	16	16	16	16	16	16	16	16	16	16	17	17	17	17	17	17	17	17	17	$\{16,17\}$
$(2,1,0)$	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	18	18	18	$\{17,18\}$
$(2,1,1)$	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	$\{18\}$
$(2,1,2)$	18	18	18	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	$\{18,19\}$
$(2,2,0)$	19	19	19	19	19	19	19	19	19	20	20	20	20	20	20	20	20	20	20	20	20	$\{19,20\}$
$(2,2,1)$	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	21	21	21	21	21	21	$\{20,21\}$
$(2,2,2)$	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	$\{21\}$

Table 2. Final positions d_3 for $C = 21$, $n = 3$, and $k = 3$ by evaluating the formula shown in Theorem 5 on each collection sequence $\vec{\jmath} = (j_1, j_2, j_3)$ with $j_i \in \{0, 1, 2\}$ for $1 \leq i < n$ subject to every start position d_0 with $1 \leq d_0 \leq 21$. The gray shaded rows describe collection sequences where the final position d_k is either unique or not.

characterization of the solvable cases in the generalized setting we need some further notation. To this end let $C = mn$ with $n > 1$ and $k \geq 1$. For a collection sequence $\vec{\jmath} = (j_k, j_{k-1}, \ldots, j_1)$ with $0 \leq j_i < n$ for $1 \leq i \leq k$, for convenience define the integer

$$S(\vec{\jmath}) := \sum_{i=1}^{k-1} j_i n^{i-1}. \tag{1}$$

It is worth noting that the upper bound in the sum is $k-1$ and *not* k. Thus, we obviously have $0 \leq S(\vec{\jmath}) \leq n^{k-1} - 1$ and there are n^k different collection sequences $\vec{\jmath}$. Recall from Theorem 5 (specialized notation) that

$$d_k = mj_k + \left\lceil \frac{mn \cdot S(\vec{\jmath}) + d_0}{n^k} \right\rceil = mj_k + \left\lceil \frac{C \cdot S(\vec{\jmath}) + d_0}{n^k} \right\rceil,$$

where $d_0 \in \{1, 2, \ldots, C\}$ is the initial deck position. The following criterion decides whether the map $d_0 \mapsto d_k$ is constant.

Theorem 6. *With the notation as above, the final position d_k is independent of the initial position $d_0 \in \{1, 2, \dots, C\}$, i.e., the sequence $\vec{\jmath}$ yields a unique reveal, if and only if*

$$\big(C \cdot S(\vec{\jmath})\big) \bmod n^k + C \leq n^k. \tag{2}$$

Equivalently, letting $A = C \cdot S(\vec{\jmath})$, the integers $A+1, A+2, \dots, A+C$ must all lie strictly between two consecutive multiples of n^k; in that case the common value is

$$d_k = mj_k + \left\lceil \frac{A+1}{n^k} \right\rceil = mj_k + \left\lceil \frac{A+C}{n^k} \right\rceil.$$

Proof. Write $A := C \cdot S(\vec{\jmath})$ and for $d_0 \in \{1, 2, \dots, C\}$ put $q(d_0) := \left\lceil \frac{A+d_0}{n^k} \right\rceil$. The final position is $d_k = mj_k + q(d_0)$, so the value of d_k is independent of d_0 if and only if $q(d_0)$ is constant on the elements from the set $\{1, 2, \dots, C\}$. Since the function $x \mapsto \lceil x/n^k \rceil$ is nondecreasing and integer-valued on integers, $q(d_0)$ attains its minimum at $d_0 = 1$ and its maximum at $d_0 = C$. Hence $q(d_0)$ is constant on $\{1, 2, \dots, C\}$ if and only if

$$\left\lceil \frac{A+1}{n^k} \right\rceil = \left\lceil \frac{A+C}{n^k} \right\rceil.$$

The last equality holds exactly when there exists an integer $t \geq 0$ with

$$t \cdot n^k < A + 1 \leq A + C \leq (t+1) \cdot n^k, \tag{3}$$

i.e., when all integers $A+1, A+2, \dots, A+C$ lie in the single half-open interval $(t \cdot n^k, (t+1) \cdot n^k]$. We need to choose t such that $A = t \cdot n^k + r$ with $0 \leq r \leq n^k - 1$. Thus, $r \equiv A \bmod n^k$, and since $0 \leq A \leq n^k - 1$ always holds, condition (3) is equivalent to $A \mod n^k + C \leq n^k$, which is condition (2) from the statement of the theorem. This proves the theorem.

Let us apply the above stated theorem to our running example:

Example 7. For the classical 21CT we have $C = 21$, $n = 3$, $k = 3$, and $m = 7$. We consider two sequences:

- For the standard sequence $\vec{\jmath} = (1, 1, 1)$ one computes $S(\vec{\jmath}) = 1 \cdot n^0 + 1 \cdot n^1 = 1 + 3 = 4$, $C \cdot S = 21 \cdot 4 = 84$, and $84 \bmod 27 = 3$, and indeed $3 + C = 24 \leq 27$, so the criterion is satisfied and d_k is constant (the constant is 11 in this case). Checking the numbers directly with the formula reproduces the expected result.
- In contrast, for the sequence $\vec{\jmath} = (0, 0, 1)$ we have $S(\vec{\jmath}) = 0 \cdot n^1 + 1 \cdot n^0 = 0 + 1 = 1$, $C \cdot S = 21 \cdot 1 = 21$, and $21 \bmod 27 = 21$, and $21 + C = 42 \not\leq 27$, thus, the criterion is *not* satisfied and d_k is *not* constant. Again, checking the numbers directly with the formula confirms the result.

See the gray shaded rows in Table 2.

The condition given in Theorem 6 allows us to easily compute the solvability for various generalized card tricks. We summarize the status of solvability for

$$C \in \{3, 6, 9, 12, 15, 18, 21, 24, 27\} \cup \{\, C \mid C \geq 30 \text{ and } C \bmod 3 = 0 \,\},$$

with $n = 3$, and $k = 2$ (left) or $k = 3$ (right) in Table 3—a number indicates solvability and the unique reveal, while a dash indicates that the corresponding card trick with the stated collection

Coll. seq.	Deck size C			
	3	6	9	≥ 12
$(0,0)$	1	1	1	–
$(0,1)$	1	–	2	–
$(0,2)$	1	2	3	–
$(1,0)$	2	3	4	–
$(1,1)$	2	–	5	–
$(1,2)$	2	4	6	–
$(2,0)$	3	5	7	–
$(2,1)$	3	–	8	–
$(2,2)$	3	6	9	–

Coll. seq.	Deck size C									
	3	6	9	12	15	18	21	24	27	≥ 30
$(0,0,0)$	1	1	1	1	1	1	1	1	1	–
$(0,0,1)$	1	1	1	1	–	–	–	–	2	–
$(0,0,2)$	1	1	1	–	2	2	–	–	3	–
$(0,1,0)$	1	1	2	2	–	3	–	–	4	–
$(0,1,1)$	1	–	2	–	3	–	4	–	5	–
$(0,1,2)$	1	2	2	3	–	4	–	–	6	–
$(0,2,0)$	1	2	3	–	4	5	–	–	7	–
$(0,2,1)$	1	2	3	4	–	–	–	–	8	–
$(0,2,2)$	1	2	3	4	5	6	7	8	9	–
$(1,0,0)$	2	3	4	5	6	7	8	9	10	–
$(1,0,1)$	2	3	4	5	–	–	–	–	11	–
$(1,0,2)$	2	3	4	–	7	8	–	–	12	–
$(1,1,0)$	2	3	5	6	–	9	–	–	13	–
$(1,1,1)$	2	–	5	–	8	–	11	–	14	–
$(1,1,2)$	2	4	5	7	–	10	–	–	15	–
$(1,2,0)$	2	4	6	–	9	11	–	–	16	–
$(1,2,1)$	2	4	6	8	–	–	–	–	17	–
$(1,2,2)$	2	4	6	8	10	12	14	16	18	–
$(2,0,0)$	3	5	7	9	11	13	15	17	19	–
$(2,0,1)$	3	5	7	9	–	–	–	–	20	–
$(2,0,2)$	3	5	7	–	12	14	–	–	21	–
$(2,1,0)$	3	5	8	10	–	15	–	–	22	–
$(2,1,1)$	3	–	8	–	13	–	18	–	23	–
$(2,1,2)$	3	6	8	11	–	16	–	–	24	–
$(2,2,0)$	3	6	9	–	14	17	–	–	25	–
$(2,2,1)$	3	6	9	12	–	–	–	–	26	–
$(2,2,2)$	3	6	9	12	15	18	21	24	27	–

Table 3. Solvability for card tricks with $n = 3$ stacks and $k = 2$ (left) or $k = 3$ (right) *face-down dealing* phases with different values of C depending on the collection sequence.

sequence is *not* solvable; see also the table in Appendix B for $k = 4$. From the table one can read of in the case of $C = 24$, $n = 3$, $m = 8$, and the collection sequence $\vec{\jmath} = (1, 1, 1)$ the card trick is *not* solvable. This is due to the fact that $S(\vec{\jmath}) = 1 \cdot n^1 + 1 \cdot n^0 = 3 + 1 = 4$, $C \cdot S = 24 \cdot 4 = 96$, and $96 \bmod 3^3 = 15$ and the criterion $15 + C = 39 \leq 27$ is *not* satisfied. Hence, d_k is *not* constant. This explains why a similar card trick with 24 cards did not appear anywhere in the of (magic) literature. It is worth mentioning that both other uniform sequences $(0, 0, 0)$ and $(2, 2, 2)$ lead to a solvable generalised card trick, i.e., the collection sequence yields a unique reveal, namely $d_k = 1$ and $d_k = C$, respectively.

Moreover, note that the card trick with 27 cards, $n = 3$, and $k = 3$ mirrors the relationship between the collection sequence—interpreted as a ternary number—and the final position of the selected card [17].

Now we are interested in how many unique solutions exists for a given generalized card trick. By Theorem 6 this boils down to the question on how many collection sequences $\vec{\jmath}$ exist such that

$$\big(C \cdot S(\vec{\jmath})\big) \bmod n^k + C \leq n^k?$$

The following theorem answers this question; for the proof, we assume the reader is familiar with basic group theory, namely subgroups of finite groups and Lagrange's Theorem, as covered by [10, Ch. 7].

Theorem 8. *Let $m = C/n$ and and $k \geq 1$ be natural numbers. Consider the collection sequences $\vec{\jmath} = (j_k, j_{k-1}, \ldots, j_1)$ with $0 \leq j_i < n$ for $1 \leq i \leq k$. The number of such $\vec{\jmath}$, where*

$$\big(C \cdot S(\vec{\jmath})\big) \bmod n^k + C \leq n^k,$$

is given by

$$N(C,n,k) = \begin{cases} n^k - C + \gcd(C, n^k), & \text{if } C \leq n^k, \\ 0, & \text{if } C > n^k. \end{cases}$$

Proof. First observe, that the mapping $S(\cdot)$ sends the n^k different collection sequences surjectively to the set of integers $\{0, 1, \ldots, n^{k-1} - 1\}$ and each element from the image has exactly n preimages. Thus, it suffices to prove the following statement: Let j be an integer with $0 \leq j \leq n^{k-1} - 1$. The number of j's with $0 \leq j \leq n^{k-1} - 1$ satisfying $(Cj \bmod n^k) + C \leq n^k$ is $n^{k-1} - m + \gcd(m, n^{k-1})$ if $m \leq n^{k-1}$, and 0, otherwise. Hence, multiplying by n gives the desired value of $N(C,n,k)$ as stated above.

Since $C = mn$, we have $Cj = mnj$. Observe that for any integer j, the remainder of mnj modulo n^k equals n times the remainder of mj modulo n^{k-1}. Indeed, writing $mj = qn^{k-1} + r$ with $0 \leq r < n^{k-1}$ gives $Cj = n(mj) = qn^k + nr$, and since $0 \leq nr < n^k$, it follows that $Cj \bmod n^k = nr = n(mj \bmod n^{k-1})$. Consequently, the inequality $(Cj \bmod n^k) + C \leq n^k$ becomes

$$n(mj \bmod n^{k-1}) + mn \leq n^k.$$

Dividing by the positive integer n yields the equivalent condition

$$mj \bmod n^{k-1} \leq n^{k-1} - m. \tag{4}$$

Let $d = \gcd(m, n^{k-1})$, and write $m = dm'$ and $n^{k-1} = dN$, so that $\gcd(m', N) = 1$. Consider the additive group $\mathbb{Z}/n^{k-1}\mathbb{Z}$ that corresponds to addition modulo n^{k-1}. Define the function $\phi : G \to G$ by $\phi(j) = mj \bmod n^{k-1}$. Then the image of ϕ, that is, the set $\mathrm{im}(\phi) = \{\, \phi(j) \mid j \in G\}$, consists of all multiples of d modulo n^{k-1}:

$$\mathrm{im}(\phi) = \{0, d, 2d, \ldots, (N-1)d\}.$$

The kernel of ϕ, that is, the set

$$\ker(\phi) = \{\, j \in G \mid \phi(j) = 0 \,\} = \{\, j \in G \mid mj \equiv 0 \pmod{n^{k-1}} \,\}.$$

A direct calculation using $m = dm'$ and $n^{k-1} = dN$ with $\gcd(m', N) = 1$ shows that $\ker(\phi) = \{\, 0, N, 2N, \ldots, (d-1)N \,\}$, and thus $|\ker(\phi)| = d$. For $r \in im(\phi)$, choose $j_0 \in G$ with $\phi(j_0) = r$. Then the set of all $j \in G$ with $\phi(j) = r$ is the coset $j_0 + \ker(\phi)$. Translation by j_0 yields a bijection between the kernel and the coset $j_0 + \ker(\phi)$, hence all cosets are of the same size. Therefore, as j runs through the complete residue system $\{0, 1, \ldots, n^{k-1} - 1\}$ modulo n^{k-1}, each element of $\mathrm{im}(\phi)$ occurs exactly $|\ker(\phi)| = d$ times.

Now condition (4) requires that the residue $r = mj \bmod n^{k-1}$ satisfies $r \leq n^{k-1} - m = dN - dm' = d(N - m')$. Since r is always a multiple of d, write $r = id$ with $0 \leq i \leq N - 1$. Then (4) is equivalent to $id \leq d(N - m')$, i.e.,

$$i \leq N - m'. \tag{5}$$

If $m > n^{k-1}$, then $m' > N$ because $m = dm'$ and $n^{k-1} = dN$. Thus, $N - m' < 0$ and no non-negative integer i satisfies the inequality (5); hence in this case there are no solutions j.

Assume now that $m \leq n^{k-1}$, i.e., $m' \leq N$. Then the admissible values of i are $i = 0, 1, \ldots, N - m'$, giving $N - m' + 1$ distinct residues r. Since each such residue is attained by exactly d values of j, the total number of j satisfying (4) equals

$$d \cdot (N - m' + 1) = dN - dm' + d = n^{k-1} - m + d.$$

Recalling that $d = \gcd(m, n^{k-1})$ completes the proof

We now characterize those generalized card tricks where *all* possible collection sequences $\vec{\jmath} \in \{0, \ldots, n-1\}^k$ produce a unique final position, i.e., when every word of length k gives a constant map $d_0 \mapsto d_k$.

Theorem 9. *Fix C, n, k with $n > 1$ and let C be a multiple of n. Every collection sequence $\vec{\jmath} \in \{0, \ldots, n-1\}^k$ yields a unique reveal (that is, the final position d_k is independent of d_0) if and only if C divides n^k, i.e., $C \mid n^k$.*

Proof. By the definition of the greatest common divisor, $\gcd(C, n^k) \mid C$, and therefore $\gcd(C, n^k) \leq C$. Then, with Theorem 8, we find that

$$N(C, n, k) = n^k - C + \gcd(C, n^k) \leq n^k - C + C = n^k.$$

This shows that n^k is an upper bound for $N(C, n, k)$ over all integers $C \geq 1$. Now assume that $C \mid n^k$. Then $\gcd(C, n^k) = C$, and thus

$$N(C, n, k) = n^k - C + C = n^k.$$

Hence the upper bound is attained for every positive divisor C of n^k.

Conversely, suppose that $N(C, n, k) = n^k$, for some $C \geq 1$. Then $n^k - C + \gcd(C, n^k) = n^k$, which implies $\gcd(C, n^k) = C$. By the defining property of the greatest common divisor, this equality holds if and only if $C \mid n^k$. Therefore, the function $N(C, n, k)$ attains its maximum value n^k exactly whenever C divides n^k.

Observe, that by the above theorem the number of such C (with $C = m \cdot n$ and given k) that have a unique reveal for all collection sequences is exactly equal to the number of positive divisors of n^{k-1}. If n has the prime factorization $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, then $n^{k-1} = \prod_{i=1}^{r} p_i^{\alpha_i(k-1)}$, and the number of divisors is

$$\prod_{i=1}^{r} (\alpha_i(k-1) + 1),$$

which is the number we are interested in.

Example 10. For $n = 3$ and $k = 2$ we obtain two decks that have a unique reveal for all collection sequences, because by our previous argumentation we have $n = 3^1$ and thus, $\alpha = 1$, and in turn $\alpha(k-1) + 1 = 1(2-1) + 1 = 2$. If we consider the case, where $k = 2$, then $\alpha(k-1) = 1(3-1) + 1 = 1 \cdot 2 + 1 = 3$. Therefore, in this case we obtain three decks that obey a unique reveal for all collection sequences. Compare these results with Table 3.

3.1.2 On Trivial Uniform Collection Sequences

Consider the collection sequences

$$\vec{0} = (\underbrace{0, 0, \ldots, 0}_{k \text{ times}}) \quad \text{and} \quad n \vec{-} 1 = (\underbrace{n-1, n-1, \ldots, n-1}_{k \text{ times}}),$$

which we both call *trivial*. What concerns a card trick w.r.t. these trivial collection sequence and the solvability of the trick? This is easily answered, namely in both cases the trick is in fact always solvable if k is chosen large enough. This is seen as follows: First, recall that $S(\vec{0}) = 0$ and $S(\vec{n-1}) = n^{k-1} - 1$ Thus, $C \cdot S(\vec{0}) = 0$ and $C \cdot S(\vec{n-1}) = m(n^k - n)$, since $C = mn$. Hence, the solvability property $(C \cdot S(\vec{\jmath})) \bmod n^k + C \leq n^k$ given in Thoerem 6 results in $C \leq n^k$, for $\vec{\jmath} = \vec{0}$, and $(-mn \bmod n^k) + C \leq n^k = C - mn \leq n^k = 0 \leq n^k$ provided $mn \leq n^k$, for $\vec{\jmath} = \vec{n-1}$. Thus, since $C \leq n^k$ is required in both cases we find that $k \geq \log_n C$, where $\log_n(\cdot)$ refers to the base n logarithm.

What can be said about d_k in both cases? We have $d_k = mj_k + \left\lceil \frac{C \cdot S(\vec{\jmath})+1}{n^k} \right\rceil$. Thus, for $\vec{\jmath} = \vec{0}$ we get $d_k = 1$ and for $\vec{n-1}$ we obtain $d_k = m(n-1) + \left\lceil \frac{m(n^k-n)+1}{n^k} \right\rceil = mn$. This is the reason why in Table 1 only non-trivial collection sequences are listed.

3.1.3 When a Collection Sequence is Universal

What about a *universal* collection sequences where all deck sizes are solvable? A closer look at Table 3 suggests that universal sequences may exist, but that they are rare. These two observations hold in general, as will be shown in the following theorem.

Theorem 11. *Let $n, k \geq 2$. There is* no *collection sequence $\vec{\jmath}$, except for $(j_k, \underbrace{0, \ldots, 0}_{k-1 \text{ times}})$ and $(j_k, \underbrace{n-1, \ldots, n-1}_{k-1 \text{ times}})$, for any $0 \leq j_k < n$, such that*

$$\big(C \cdot S(\vec{\jmath})\big) \bmod n^k + C \leq n^k,$$

for every C with $C = mn$, for $1 \leq m \leq n^{k-1}$.

Proof. First, we argue as in the proof of Theorem 8. Recall that the mapping $S(\cdot)$ maps the n^k different collection sequences surjectively to the set of integers $\{0, 1, \ldots, n^{k-1} - 1\}$ and each element from the image has exactly n preimages. It is easy to see that the excluded collection sequences are either mapped by $S(\cdot)$ to 0 or $n^{k-1} - 1$. Thus, we can restate the property that we have to prove as: there is no j with $1 \leq j < n^{k-1} - 1$ such that

$$(C \cdot j) \bmod n^k + C \leq n^k, \tag{6}$$

for every C with $C = mn$, for $1 \leq m \leq n^{k-1}$, while for $j = 0$ or $j = n^{k-1} - 1$ the inequality (6) holds true.

Let $C = mn$ with $1 \leq m \leq n^{k-1}$. Substituting into inequality (6), we obtain

$$(mnj \bmod n^k) + mn \leq n^k.$$

Since $n^k = n \cdot n^{k-1}$ and mnj are both divisible by n, we factor by n resulting in: $mnj \bmod n^k = n\big(mj \bmod n^{k-1}\big)$, so the inequality becomes $n(mj \bmod n^{k-1}) + mn \leq n^k$. Dividing both sides by n yields the equivalent inequality

$$(mj \bmod n^{k-1}) + m \leq n^{k-1}. \tag{7}$$

Assume, for contradiction, that there exists a j with $1 \leq j < n^{k-1} - 1$ such that inequality (7) holds for all $1 \leq m < n^{k-1}$. Consider $m = n^{k-1} - 1$. Then

$$mj = (n^{k-1} - 1)j \equiv -j \pmod{n^{k-1}}$$

implies $mj \bmod n^{k-1} = n^{k-1} - j$. Substituting into inequality (7) gives

$$(n^{k-1} - j) + (n^{k-1} - 1) \leq n^{k-1},$$

which implies $2n^{k-1} - j - 1 \leq n^{k-1}$ and in turn $j \geq n^{k-1} - 1$, contradicting the assumption $j < n^{k-1} - 1$. Hence, no such j exists.

It remains to prove the statement for the boundary cases, namely $j = 0$ and $j = n^{k-1} - 1$, induced by the collection sequences $(j_k, \underbrace{0, \ldots, 0}_{k-1 \text{ times}})$ and $(j_k, \underbrace{n-1, \ldots, n-1}_{k-1 \text{ times}})$, for any j_k with $0 \leq j_k < n$:

- Case $j = 0$: Then $(Cj \bmod n^k) + C = 0 + C = C \leq n^k$, so inequality (6) holds.
- Case $j = n^{k-1} - 1$: Then $Cj = mn(n^{k-1} - 1) = m(n^k - n) \equiv -mn \pmod{n^k}$ implies that $Cj \bmod n^k = n^k - mn$. Thus, $(Cj \bmod n^k) + C = (n^k - mn) + mn = n^k$, and inequality (6) is satisfied, too.

This completes the proof.

3.2 Face-Up Dealing: Verini's Lost Trick and its Generalization

When collecting the card deck face-up, the recursion for the positions changes, since the stack where the named card resides is reversed in its order. Thus, the recursion reads as

$$d_k = mj_k + (m+1) - s_k,$$

where

$$s_k = \left\lceil \frac{d_{k-1}}{n} \right\rceil.$$

This can be simplified to

$$d_k = mj_k + (m+1) - \left\lceil \frac{d_{k-1}}{n} \right\rceil = mj_k + (m+1) + \left\lfloor \frac{-d_{k-1}}{n} \right\rfloor = mj_k + \left\lfloor \frac{(m+1)n - d_{k-1}}{n} \right\rfloor.$$

Again, we are interested in a closed formula for d_k. To this end we transform the recurrence under consideration such that it becomes easily solvable. Also, instead of d_k, it makes sense to consider the quantity $x_k = C + 1 - d_k$, which is the index of the card after k iterations when counting backwards. We will make heavy use of the following identities on floors and ceilings:

$$\left\lfloor \frac{m}{n} \right\rfloor = \left\lceil \frac{m-n+1}{n} \right\rceil \quad \text{and} \quad \left\lceil \frac{m}{n} \right\rceil = \left\lfloor \frac{m+n-1}{n} \right\rfloor$$

for integers m and n with $n > 0$.

In the recurrence

$$d_k = mj_k + \left\lfloor \frac{(m+1)n - d_{k-1}}{n} \right\rfloor$$

we substitute d_{k-1} by $d_{k-1} = C + 1 - x_{k-1}$, where $C = mn$, and get

$$\begin{aligned} d_k &= mj_k + \left\lfloor \frac{mn + n - C - 1 + x_{k-1}}{n} \right\rfloor \\ &= mj_k + \left\lfloor \frac{n - 1 + x_{k-1}}{n} \right\rfloor \\ &= mj_k + \left\lceil \frac{x_{k-1}}{n} \right\rceil. \end{aligned}$$

Observe, when d_k runs form 1 up to C, then x_k runs down from C to 1. Hence, they are "mirrors" of each other w.r.t. the position within the deck.

Next we substitute for x_k on the left hand-side of the recurrence, which leads us to

$$C + 1 - x_k = mj_k + \left\lceil \frac{x_{k-1}}{n} \right\rceil$$

and solving for x_k gives $x_k = C+1-mj_k - \lceil \frac{x_{k-1}}{n} \rceil$. Rearranging the term using $C = mn$ results in

$$x_k = m(n-j_k) + 1 + \left\lfloor \frac{-x_{k-1}}{n} \right\rfloor = m(n-j_k) + \left\lfloor 1 + \frac{-x_{k-1}}{n} \right\rfloor = m(n-j_k) + \left\lfloor \frac{n - x_{k-1}}{n} \right\rfloor,$$

which in turn results further in

$$x_k = m(n-j_k) + \left\lceil \frac{n - x_{k-1} - n + 1}{n} \right\rceil = m(n-j_k) + \left\lceil \frac{1 - x_{k-1}}{n} \right\rceil.$$

Finally, let $\ell_i = n - j_i$, for $1 \leq i \leq k$, and factoring a minus one out of the ceiling gives a recurrence that reads as

$$x_k = m\ell_k - \left\lfloor \frac{x_{k-1} - 1}{n} \right\rfloor.$$

This completes our transformation of the original recurrence. Now we are interested in solving for x_k. We find the following situation—for the convenience of the reader we define:

$$\tilde{S}(\vec{\ell}) := \sum_{i=1}^{k-1} (-1)^{k-i-1} \ell_i n^{i-1},$$

for $\vec{\ell} = (\ell_k, \ell_{k-1}, \ldots, \ell_1)$ with $1 \leq \ell_i < n$, for $1 \leq i \leq k$. Then the theorem on reads as follows:

Theorem 12. *Let $m = C/n$ and $k \geq 1$. Then*

$$x_k = \begin{cases} m\ell_k - \left\lfloor \dfrac{C \cdot \tilde{S}(\vec{\ell}) + x_0 - 1}{n^k} \right\rfloor, & \text{if } n > 1 \text{ and } k \text{ is odd,} \\ m\ell_k - \left\lfloor \dfrac{C \cdot \tilde{S}(\vec{\ell}) - x_0}{n^k} \right\rfloor, & \text{if } n > 1 \text{ and } k \text{ is even,} \\ m\ell_k, & \text{if } n = 1, \end{cases}$$

for $\vec{\ell} = (\ell_k, \ell_{k-1}, \ldots, \ell_1)$ with $1 \leq \ell_i < n$, for $1 \leq i \leq k$.

Proof. The result for $n = 1$ is immediate. Thus, in the forthcoming assume $n > 1$. We prove the statement by induction on k. Let $k = 1$, then $x_1 = m\ell_1 - \lfloor \frac{x_0 - 1}{n} \rfloor$ and the formula is equal to

$$x_1 = m\ell_1 - \left\lfloor \frac{mn \left(\sum_{i=1}^{1-1} (-1)^{k-i-1} \ell_i \cdot n^{i-1} \right) + (-1)^{[1 \bmod 2]+1} x_0 - [1 \bmod 2]}{n^1} \right\rfloor$$

$$= m\ell_1 - \left\lfloor \frac{mn \cdot 0 + x_0 - 1}{n} \right\rfloor = m\ell_1 - \left\lfloor \frac{x_0 - 1}{n} \right\rfloor$$

as desired.

Next we continue with the induction step from k to $k+1$ with the appropriate induction hypothesis on k. Let

$$x_{k+1} = m\ell_{k+1} - \left\lfloor \frac{x_k - 1}{n} \right\rfloor$$

$$= m\ell_{k+1} - \left\lfloor \frac{\left(m\ell_k - \left\lfloor \frac{mn\left(\sum_{i=1}^{k-1}(-1)^{k-i-1}\ell_i \cdot n^{i-1}\right) + (-1)^{[k \bmod 2]+1}x_0 - [k \bmod 2]}{n^k} \right\rfloor \right) - 1}{n} \right\rfloor,$$

using the induction hypothesis in the last step.

We first concentrate on the numerator of the right-most term, and in particular on the bracketed expression. Here the term $m\ell_k$ can be moved into the floor-expression, by first shifting the minus inward and flipping to a ceiling-expression, and the adding $m\ell_k \cdot n^k$, which results in the numerator

$$\left(\left\lceil \frac{m\ell_k n^k - mn\left(\sum_{i=1}^{k-1}(-1)^{k-i-1}\ell_i \cdot n^{i-1}\right) - (-1)^{[k \bmod 2]+1}x_0 + [k \bmod 2]}{n^k} \right\rceil \right) - 1$$

$$= \left\lceil \frac{mn\ell_k n^{k-1} - mn\left(\sum_{i=1}^{k-1}(-1)^{k-i-1}\ell_i \cdot n^{i-1}\right) - (-1)^{[k \bmod 2]+1}x_0 + [k \bmod 2] - n^k}{n^k} \right\rceil$$

Next we apply the ceiling-floor identity which gives us

$$\left\lfloor \frac{mn\ell_k n^{k-1} - mn\left(\sum_{i=1}^{k-1}(-1)^{k-i-1}\ell_i \cdot n^{i-1}\right) - (-1)^{[k \bmod 2]+1}x_0 + [k \bmod 2] - n^k + n^k - 1}{n^k} \right\rfloor.$$

With $[k \bmod 2] - 1 = -[(k+1) \bmod 2]$, collecting the appropriate terms together yields

$$\left\lfloor \frac{mn\left(\sum_{i=1}^{k}(-1)^{(k+1)-i-1}\ell_i \cdot n^{i-1}\right) + (-1)^{[(k+1) \bmod 2]+1}x_0 - [(k+1) \bmod 2]}{n^k} \right\rfloor.$$

Putting the simplified term back into the equation for x_{k+1} results in

$$x_{k+1} = m\ell_{k+1} - \left\lfloor \frac{\left\lfloor \frac{mn\left(\sum_{i=1}^{k}(-1)^{(k+1)-i-1}\ell_i \cdot n^{i-1}\right) + (-1)^{[(k+1) \bmod 2]+1}x_0 + [(k+1) \bmod 2]}{n^k} \right\rfloor}{n} \right\rfloor$$

$$= m\ell_{k+1} - \left\lfloor \frac{mn\left(\sum_{i=1}^{k}(-1)^{(k+1)-i-1}\ell_i \cdot n^{i-1}\right) + (-1)^{[(k+1) \bmod 2]+1}x_0 - [(k+1) \bmod 2]}{n^{k+1}} \right\rfloor .$$

This proves the stated claim.

In analogy to the $S(\cdot)$-function in (1) we define an alternating sum variant as follows:

$$\tilde{S}(\vec{\ell}) := \sum_{i=1}^{k-1}(-1)^{k-i-1}\ell_i n^{i-1}, \tag{8}$$

for the sequence $\vec{\ell} = (\ell_k, \ell_{k-1}, \ldots, \ell_1)$ with $1 \leq \ell_i \leq n$ for $1 \leq i \leq k$. Observe, that $\vec{\ell} = (n - j_k, n - j_{k-1}, \ldots, n - j_1)$ corresponds to the collection sequence $\vec{\jmath}$. As for the function $S(\cdot)$ also for $\tilde{S}(\cdot)$ the upper bound in the sum is $k-1$ and *not* k. In the following subsubsection we show how the $S(\cdot)$- and $\tilde{S}(\cdot)$-function are related to each other.

3.2.1 On Collection Sequences and Certain Sums and Alternating Sums

First we show how the $S(\cdot)$- and the $\tilde{S}(\cdot)$-function related to each other. The following theorem shows that the function $\tilde{S}$ can be seen as a mixed radix numeral system with alternating signs; see [7, 14] for more background.

Theorem 13. *Let $\vec{\ell} \in \{1, 2, \ldots, n\}^k$. There there exist integers j_i with $0 \leq j_i < n$ such that*

$$\tilde{S}(\vec{\ell}) = S(\vec{\jmath}) + \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even.} \end{cases}$$

Proof. First, factor the alternating sign $(-1)^{k-i-1} = (-1)^{k-1}(-1)^i$ such that

$$\tilde{S}(\vec{\ell}) = \sum_{i=1}^{k-1}(-1)^{k-i-1}\ell_i n^{i-1} = (-1)^{k-1}\sum_{i=1}^{k-1}(-1)^i \ell_i n^{i-1}.$$

Then we distinguish two case, namely whether k is odd or even:

1. Let k be odd. Then define

$$j_i := \begin{cases} n - \ell_i, & \text{if } i \text{ is odd,} \\ \ell_i - 1, & \text{if } i \text{ is even.} \end{cases}$$

Observe, that $0 \leq j_i = n - \ell_i < n$, for odd i, and $0 \leq j_i = \ell_i - 1 < n$, for even i. Thus, let $\vec{\jmath} = (j_k, j_{k-1}, \ldots, j_1)$. Now, $(-1)^i \ell_i n^{i-1} = -\ell_i n^{i-1} = j_i n^{i-1} - n^i$, for odd i, and $(-1)^i \ell_i n^{i-1} = \ell_i n^{i-1} = j_i n^{i-1} + n^{i-1}$, for even i. Summing over all i with $1 \leq i \leq k-1$ and using the fact that $(-1)^{k-1} = 1$, since k is odd, results in

$$\begin{aligned}\tilde{S}(\vec{\ell}) &= (-1)^{k-1} \sum_{i=1}^{k-1} (-1)^i \ell_i n^{i-1} \\ &= \sum_{\substack{i=1 \\ k \text{ odd}}}^{k-1} (j_i n^{i-1} - n^i) + \sum_{\substack{i=1 \\ k \text{ even}}}^{k-1} (j_i n^{i-1} + n^{i-1}) = \sum_{i=1}^{k-1} j_i n^{i-1} = S(\vec{\jmath}).\end{aligned}$$

2. Let k be even. Similarly as in the previous case we now define

$$j_i := \begin{cases} \ell_i - 1, & \text{if } i \text{ is odd,} \\ n - \ell_i, & \text{if } i \text{ is even.} \end{cases}$$

Then we argue in similar vein as above. First, $0 \leq j_i = \ell_i - 1 < n$, for odd i, and $0 \leq j_i = n - \ell_i < n$, for even i. Again, let $\vec{\jmath} = (j_k, j_{k-1}, \ldots, j_1)$. Now, $(-1)^{i+1} \ell_i n^{i-1} = \ell_i n^{i-1} = j_i n^{i-1} + n^{i-1}$, for odd i, and $(-1)^{i+1} \ell_i n^{i-1} = -\ell_i n^{i-1} = j_i n^{i-1} - n^i$, for even i. Summing over all i with $1 \leq i \leq k-1$ and using the fact that $(-1)^{k-1} = -1$, since k is even, gives us

$$\begin{aligned}\tilde{S}(\vec{\ell}) &= (-1)^{k-1} \sum_{i=1}^{k-1} (-1)^i \ell_i n^{i-1} = \sum_{i=1}^{k-1} (-1)^{i+1} \ell_i n^{i-1} \\ &= \sum_{\substack{i=1 \\ k \text{ odd}}}^{k-1} (j_i n^{i-1} + n^{i-1}) + \sum_{\substack{i=1 \\ k \text{ even}}}^{k-1} (j_i n^{i-1} - n^{i-1}) = 1 + \sum_{i=1}^{k-1} j_i n^{i-1} = 1 + S(\vec{\jmath}).\end{aligned}$$

This completes the proof.

As an immediate consequence of the correspondence we obtain the following statement, which we state without proof:

Corollary 14. *For every sequence $\vec{\jmath} \in \{0, 1, \ldots, n-1\}^k$ and $\vec{\ell} \in \{1, 2, \ldots, n\}^k$ it holds that*

1. $0 \leq S(\vec{\jmath}) \leq n^{k-1} - 1$ *and*
2. $0 \leq \tilde{S}(\vec{\ell}) \leq n^{k-1} - 1$, *if k is odd, and $1 \leq \tilde{S}(\vec{\ell}) \leq n^{k-1}$, if k is even.*

Moreover, every integer in the respective interval can be obtained as a function value by a suitable choice of $\vec{\jmath}$ respectively $\vec{\ell}$.

Although dealing from the bottom of the deck is allowed in magic, which is not fair in normal card play, the Magi is probably interested in having a closed formula for the final position also in case the card trick is performed by face-up dealing. Next we present this closed formula:

Corollary 15. *Let $m = C/n$ and $k \geq 1$. Fix a collection sequence $\vec{\jmath} = (j_k, j_{k-1}, \ldots, j_1)$ with $0 \leq j_i < n$ for $1 \leq i \leq k$. Then*

$$d_k = \begin{cases} mt_k + \left\lceil \dfrac{C \cdot S(\vec{t}) + (C+1) - d_0}{n^k} \right\rceil & \text{if } n > 1 \text{ and } k \text{ is odd,} \\ mt_k + \left\lceil \dfrac{C \cdot S(\vec{t}) + d_0}{n^k} \right\rceil & \text{if } n > 1 \text{ and } k \text{ is even,} \\ mt_k + 1, & \text{if } n = 1, \end{cases}$$

where $\vec{t}$ is defined by

$$t_i = \begin{cases} j_i & \textit{if } k \textit{ and } i \textit{ are both odd or both are even,} \\ (n - j_i) - 1 & \textit{otherwise,} \end{cases}$$

for $1 \le i \le k$.

Proof. By Theorem 12 we have

$$x_k = \begin{cases} m\ell_k - \left\lfloor \dfrac{C \cdot \tilde{S}(\vec{\ell}) + x_0 - 1}{n^k} \right\rfloor, & \text{if } n > 1 \text{ and } k \text{ is odd,} \\ m\ell_k - \left\lfloor \dfrac{C \cdot \tilde{S}(\vec{\ell}) - x_0}{n^k} \right\rfloor, & \text{if } n > 1 \text{ and } k \text{ is even,} \\ m\ell_k, & \text{if } n = 1. \end{cases}$$

Recall that $\ell_i = n - j_i$, for $1 \le i \le k$, by definition. In order to overcome the alternating sum we use the transformation to the vector $\vec{t}$ described in the proof of Theorem 13. Then the alternatig sum can be replaced by a non-alternatig sum on $\vec{t}$; do not forget the correction term 1 in case k is even. Finally, replacing x_k by $(C+1) - d_k$ or $(mn+1) - d_k$, respectively, we obtain

$$\begin{aligned} d_k &= (C+1) - m\ell_k + \left\lfloor \frac{C \cdot S(\vec{t}) + (C+1) - d_0 - 1}{n^k} \right\rfloor \\ &= m(n - \ell_k) + 1 + \left\lceil \frac{C \cdot S(\vec{t}) + C - d_0 - n^k + 1}{n^k} \right\rceil \\ &= mt_k + \left\lceil \frac{n^k + C \cdot S(\vec{t}) + C - d_0 - n^k + 1}{n^k} \right\rceil \\ &= mt_k + \left\lceil \frac{C \cdot S(\vec{t}) + (C+1) - d_0}{n^k} \right\rceil, \end{aligned}$$

for odd k and by similar calculations

$$d_k = mt_k + \left\lceil \frac{C \cdot S(\vec{t}) + d_0}{n^k} \right\rceil,$$

for even k. The case $n = 1$ will be $d_k = mt_k$ as desired. This proves the stated claim.

Now let us use the formula from Corollary 15 to calculate the final positions for the $C = 21$ possible start positions d_0 manipulated in the three stages ($k = 3$) on three stacks ($n = 3$) but now dealing face-up. By testing all 27 possible collection sequences, we can see how the order affects the final result. All 27 results are listed in Table 4; compare this with Table 2 for face-down dealing. Thus, 9 out of the 27 possible collection sequences yield a unique final position. Like in the case of face-down dealing, also here the question arises how to identify those collection sequences which yield a unique final position d_k.

3.2.2 When a Collection Sequence Yields a Unique Reveal Because of the intricate nature of dealing the cards face-up, it seems difficult to derive an exact characterization of face-up solvability. Recall, for Corollary 14 that $0 \le \tilde{S}(\vec{\ell}) \le n^{k-1} - 1$, if k is odd, and $1 \le \tilde{S}(\vec{\ell}) \le n^{k-1}$, if k is even. The following criterion decides whether the map $x_0 \mapsto x_k$ is constant.

Coll. seq.	d_0																					d_k-set
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	
$(0,0,0)$	6	6	6	6	6	6	6	6	6	6	6	6	5	5	5	5	5	5	5	5	5	$\{5,6\}$
$(0,0,1)$	7	7	7	7	7	7	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	$\{6,7\}$
$(0,0,2)$	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	$\{7\}$
$(0,1,0)$	4	4	4	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	$\{3,4\}$
$(0,1,1)$	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	$\{4\}$
$(0,1,2)$	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	4	4	4	$\{4,5\}$
$(0,2,0)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	$\{1\}$
$(0,2,1)$	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	$\{1,2\}$
$(0,2,2)$	3	3	3	3	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	$\{2,3\}$
$(1,0,0)$	13	13	13	13	13	13	13	13	13	13	13	13	12	12	12	12	12	12	12	12	12	$\{12,13\}$
$(1,0,1)$	14	14	14	14	14	14	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	$\{13,14\}$
$(1,0,2)$	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	$\{14\}$
$(1,1,0)$	11	11	11	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	$\{10,11\}$
$(1,1,1)$	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	$\{11\}$
$(1,1,2)$	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	11	11	11	$\{11,12\}$
$(1,2,0)$	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	$\{8\}$
$(1,2,1)$	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	8	8	8	8	8	8	$\{8,9\}$
$(1,2,2)$	10	10	10	10	10	10	10	10	10	9	9	9	9	9	9	9	9	9	9	9	9	$\{9,10\}$
$(2,0,0)$	20	20	20	20	20	20	20	20	20	20	20	20	19	19	19	19	19	19	19	19	19	$\{19,20\}$
$(2,0,1)$	21	21	21	21	21	21	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	$\{20,21\}$
$(2,0,2)$	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	$\{21\}$
$(2,1,0)$	18	18	18	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	$\{17,18\}$
$(2,1,1)$	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	$\{18\}$
$(2,1,2)$	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	18	18	18	$\{18,19\}$
$(2,2,0)$	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	$\{15\}$
$(2,2,1)$	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	15	15	15	15	15	15	$\{16,15\}$
$(2,2,2)$	17	17	17	17	17	17	17	17	17	16	16	16	16	16	16	16	16	16	16	16	16	$\{16,17\}$

Table 4. Final positions d_3 for $C = 21$, $n = 3$, and $k = 3$ by evaluating the formula from Corollary 15 on each collection sequence $\vec{\jmath} = (j_1, j_2, j_3)$ with $j_i \in \{0,1,2\}$ for $1 \leq i < n$ subject to every start position d_0 with $1 \leq d_0 \leq 21$. The gray shaded rows describe collection sequences where the final position d_k is either unique or not.

Theorem 16. *With the usual notation, the final position x_k is independent of the initial position $x_0 \in \{1, 2, \ldots, C\}$, i.e., the sequence $\vec{\ell}$ yields a unique reveal, if and only if*

$$\begin{cases} \big(C \cdot \tilde{S}(\vec{\ell})\big) \bmod n^k + C \leq n^k, & \text{if } k \text{ is odd,} \\ \big(C \cdot \tilde{S}(\vec{\ell}) - C\big) \bmod n^k + C \leq n^k, & \text{if } k \text{ even.} \end{cases} \tag{9}$$

Proof. We partially proceed as in the proof of Theorem 6. Set $A := C \cdot \tilde{S}(\vec{\ell})$ and for $x_0 \in \{1, 2, \ldots, C\}$ put $q(x_0) := \lfloor \frac{A - x_0}{n^k} \rfloor$ if k is even and $q(x_0) := \lfloor \frac{A + x_0 - 1}{n^k} \rfloor$ if k is odd. The final position is $x_k = mj_k + q(x_0)$, so the value of x_k is independent of x_0 if and only if $q(x_0)$ is constant on the elements from the set $\{1, 2, \ldots, C\}$. Now we distinguish two cases:

1. If k is odd, then we have

$$q(x_0) = \left\lfloor \frac{A + x_0 - 1}{n^k} \right\rfloor = \left\lceil \frac{A + x_0 - 1 - n^k + 1}{n^k} \right\rceil = \left\lceil \frac{(A - n^k) + x_0}{n^k} \right\rceil.$$

   Then we apply the same argumentation as in the proof of Theorem 6 and obtain the condition $(A - n^k) \bmod n^k + C \leq n^k$, which in turn is equivalent to $A \bmod n^k + C \leq n^k$ as stated.

2. If k is even, then we have to consider the term

$$q(x_0) = \left\lfloor \frac{A - x_0}{n^k} \right\rfloor = \left\lceil \frac{A - x_0 - n^k + 1}{n^k} \right\rceil = \left\lceil \frac{(A - n^k + 1) - x_0}{n^k} \right\rceil$$

in detail. Substituting $x_0 = C + 1 - d_0$ leads to

$$q(x_0) = \left\lceil \frac{(A - n^k + 1) - C - 1 + d_0}{n^k} \right\rceil = \left\lceil \frac{(A - n^k - C) + d_0}{n^k} \right\rceil .$$

As in the previous case we apply the argumentation as in the proof of Theorem 6 and obtain the condition $(A - n^k - C) \bmod n^k + C \leq n^k$, which in turn is equivalent to $(A - C) \bmod n^k + C \leq n^k$, and this equals the stated property.

This completes the analysis and proves the stated claim.

We are now in position to analyze Verini's trick. Let us apply the above theorem on that example.

Example 17 (Verini's lost trick). Thus, $C = 52$, $n = 4$, $k = 3$, and $m = 13$. The collection sequence $\vec{\jmath} = (3, 3, 3)$, which gives rise to $\vec{\ell} = (1, 1, 1)$. Then $\tilde{S}(\vec{\ell}) = 1 \cdot 4^2 - 1 \cdot 4 + 1 \cdot 4^0 = 16 - 4 + 1 = 13$ and since k is odd we have to check for $(C \cdot \tilde{S}(\vec{\ell})) \bmod n^k + C \leq n^k$. But this condition is not satisfied, because $(52 \cdot 13) \bmod 4^3 = 676 \bmod 64 = 36$ and $36 + 52 = 88 \not\leq 64 = 4^3$. Therefore Verini's card trick, where the dealing is performed in face-up manner is *not* uniquely solvable, in contrast to the claim of Verini in his book; see [19].

The solvability status of generalized card tricks when dealing face-up for

$$C \in \{3, 6, 9, 12, 15, 18, 21, 24, 27\} \cup \{\, C \mid C \geq 30 \text{ and } C \bmod 3 = 0 \,\},$$

with $n = 3$, and $k = 2$ (left) or $k = 3$ (right) is shown in Table 5; for $k = 4$ see Appendix C. Thus there are many opportunities to modify Verini's trick, and obtain a card trick that actually works.

Next, we are counting the number of unique solutions for a card trick when performed by face-up dealing.

Theorem 18. *Let $m = C/n$ and and $k \geq 1$ be natural numbers. Consider the collection sequences $\vec{\ell} = (\ell_k, \ell_{k-1}, \ldots, \ell_1)$ with $1 \leq \ell_i \leq n$ for $1 \leq i \leq k$. The number of such $\vec{\ell}$, where*

$$\begin{cases} (C \cdot \tilde{S}(\vec{\ell})) \bmod n^k + C \leq n^k, & \text{if } k \text{ is odd,} \\ \left(C \cdot \tilde{S}(\vec{\ell}) - C\right) \bmod n^k + C \leq n^k, & \text{if } k \text{ even} \end{cases}$$

is given by

$$N(C, n, k) = \begin{cases} n^k - C + \gcd(C, n^k), & \text{if } C \leq n^k, \\ 0, & \text{if } C > n^k. \end{cases}$$

Proof. Observe, that the value of $\tilde{S}(\vec{\ell})$ runs through all residues $\{0, 1, \ldots, n^{k-1} - 1\}$ of n^k when k is odd; this is also true, when k is even, for $\tilde{S}(\vec{\ell}) - 1$, but each residue is shifted. Thus, in both cases we can argue as in the proof of Theorem 8. Therefore, we obtain that $N(C, n, k) = n^k - C + \gcd(C, n^k)$, if $C \leq n^k$, regardless of whether k is odd or even, and $N(C, n, k) = 0$, otherwise.

	Deck size C			
Coll. seq.	3	6	9	≥12
$(0,0)$	1	2	3	–
$(0,1)$	1	–	2	–
$(0,2)$	1	1	1	–
$(1,0)$	2	4	6	–
$(1,1)$	2	–	5	–
$(1,2)$	2	3	4	–
$(2,0)$	3	6	9	–
$(2,1)$	3	–	8	–
$(2,2)$	3	5	7	–

	Deck size C									
Coll. seq.	3	6	9	12	15	18	21	24	27	≥30
$(0,0,0)$	1	2	3	–	4	5	–	–	7	–
$(0,0,1)$	1	2	3	4	–	–	–	–	8	–
$(0,0,2)$	1	2	3	4	5	6	7	8	9	–
$(0,1,0)$	1	1	2	2	–	3	–	–	4	–
$(0,1,1)$	1	–	2	–	3	–	4	–	5	–
$(0,1,2)$	1	2	2	3	–	4	–	–	6	–
$(0,2,0)$	1	1	1	1	1	1	1	1	1	–
$(0,2,1)$	1	1	1	1	–	–	–	–	2	–
$(0,2,2)$	1	1	1	–	2	2	–	–	3	–
$(1,0,0)$	2	4	6	–	9	11	–	–	16	–
$(1,0,1)$	2	4	6	8	–	–	–	–	17	–
$(1,0,2)$	2	4	6	8	10	12	14	16	18	–
$(1,1,0)$	2	3	5	6	–	9	–	–	13	–
$(1,1,1)$	2	–	5	–	8	–	11	–	14	–
$(1,1,2)$	2	4	5	7	–	10	–	–	15	–
$(1,2,0)$	2	3	4	5	6	7	8	9	10	–
$(1,2,1)$	2	3	4	5	–	–	–	–	11	–
$(1,2,2)$	2	3	4	–	7	8	–	–	12	–
$(2,0,0)$	3	6	9	–	14	17	–	–	25	–
$(2,0,1)$	3	6	9	12	–	–	–	–	26	–
$(2,0,2)$	3	6	9	12	15	18	21	24	27	–
$(2,1,0)$	3	5	8	10	–	15	–	–	22	–
$(2,1,1)$	3	–	8	–	13	–	18	–	23	–
$(2,1,2)$	3	6	8	11	–	16	–	–	24	–
$(2,2,0)$	3	5	7	9	11	13	15	17	19	–
$(2,2,1)$	3	5	7	9	–	–	–	–	20	–
$(2,2,2)$	3	5	7	–	12	14	–	–	21	–

Table 5. Solvability for card tricks with $n = 3$ stacks and $k = 2$ (left) or $k = 3$ (right) *face-up dealing* phases with different values of C depending on the collection sequence.

As in the case of face-down dealing we now can also characterize *all* possible sequences $\vec{\ell} \in \{1, 2, \ldots, n\}^k$ that produce a unique final position, i.e., when every word of length k gives a constant map $x_0 \mapsto x_k$. Since the proof is almost identical to that of Theorem 9 we omit it.

Theorem 19. *Fix C, n, k with $n > 1$ and let C be a multiple of n. Every collection sequence $\vec{\ell} \in \{1, 2, \ldots, n\}^k$ yields a unique reveal (that is, the final position x_k is independent of x_0) if and only if C divides n^k, i.e., $C \mid n^k$.* □

3.2.3 On Trivial Uniform and Universal Collection Sequences The results in Table 5 already show that face-up dealing behaves differently from face-down dealing, as illustrated by the comparison with the corresponding face-down results in Table 3; see also the tables in Appendix B and C. For example, the trivial collection sequences $\vec{0}$ and $n \vec{-} 1$ always produce solvable tricks with face-down dealing, but this is no longer true for face-up dealing, which can be a bit inconvenient for an artist when performing a card trick in face-up dealing mode. Moreover, these tables also show that universal sequences exist. We do not consider this issue in the forthcoming. We find it more important to list the solvability of face-up dealt card tricks for uniform collection sequences as it was done in Table 1 for face-down dealing. Table 6 summarizes our findings. From the table we see that in order make Verini's original card trick on $C = 52$ card and $n = 4$ stacks one can either increase the number of phases from three to four which

Solvable card trick $(C,n,j,k)=\ell$ for face-up dealing				
$(15,3,0,3)=14$	$(30,5,2,3)=19$	$(36,6,2,2)=25$	$(42,7,4,3)=20$	$(48,8,0,2)=47$
$(15,3,1,4)=11$	$(30,5,3,3)=14$	$(36,6,3,2)=20$	$(42,7,5,3)=15$	$(48,8,1,2)=42$
$(15,3,2,4)=7$	$(30,5,4,3)=9$	$(36,6,4,2)=15$	$(42,7,6,3)=10$	$(48,8,2,3)=37$
$(18,3,0,3)=17$	$(30,6,0,2)=29$	$(36,6,5,2)=10$	$(44,4,0,3)=43$	$(48,8,3,3)=31$
$(18,3,1,4)=13$	$(30,6,1,2)=25$	$(39,3,0,4)=39$	$(44,4,1,4)=35$	$(48,8,4,2)=26$
$(18,3,2,3)=8$	$(30,6,2,3)=21$	$(39,3,1,4)=29$	$(44,4,2,4)=26$	$(48,8,5,2)=21$
$(20,4,0,3)=19$	$(30,6,3,4)=17$	$(39,3,2,5)=19$	$(44,4,3,4)=17$	$(48,8,6,3)=15$
$(20,4,1,3)=15$	$(30,6,4,3)=12$	$(40,4,0,3)=39$	$(45,3,0,4)=45$	$(48,8,7,3)=10$
$(20,4,2,3)=11$	$(30,6,5,3)=8$	$(40,4,1,3)=31$	$(45,3,1,5)=33$	$(49,7,0,2)=48$
$(20,4,3,3)=7$	$(32,4,0,3)=31$	$(40,4,2,3)=23$	$(45,3,2,5)=22$	$(49,7,1,2)=42$
$(21,3,0,3)=20$	$(32,4,1,3)=25$	$(40,4,3,3)=15$	$(45,5,0,3)=44$	$(49,7,2,2)=36$
$(21,3,1,5)=15$	$(32,4,2,3)=18$	$(40,5,0,3)=39$	$(45,5,1,3)=37$	$(49,7,3,2)=30$
$(21,3,2,4)=10$	$(32,4,3,3)=12$	$(40,5,1,4)=33$	$(45,5,2,3)=29$	$(49,7,4,2)=24$
$(24,3,0,3)=23$	$(33,3,0,4)=33$	$(40,5,2,3)=26$	$(45,5,3,4)=22$	$(49,7,5,2)=18$
$(24,3,1,4)=18$	$(33,3,1,5)=24$	$(40,5,3,3)=19$	$(45,5,4,3)=14$	$(49,7,6,2)=12$
$(24,3,2,4)=12$	$(33,3,2,5)=16$	$(40,5,4,4)=13$	$(45,9,0,2)=44$	$(50,5,0,3)=49$
$(24,4,0,3)=23$	$(35,5,0,3)=34$	$(40,8,0,2)=39$	$(45,9,1,2)=40$	$(50,5,1,3)=41$
$(24,4,1,4)=19$	$(35,5,1,4)=29$	$(40,8,1,2)=35$	$(45,9,2,3)=35$	$(50,5,2,4)=33$
$(24,4,2,4)=14$	$(35,5,2,4)=23$	$(40,8,2,3)=31$	$(45,9,3,2)=31$	$(50,5,3,3)=24$
$(24,4,3,3)=9$	$(35,5,3,3)=17$	$(40,8,3,2)=26$	$(45,9,4,3)=26$	$(50,5,4,3)=16$
$(25,5,0,2)=24$	$(35,5,4,3)=11$	$(40,8,4,3)=22$	$(45,9,5,2)=22$	$(50,10,0,2)=49$
$(25,5,1,2)=20$	$(35,7,0,2)=34$	$(40,8,5,3)=17$	$(45,9,6,3)=17$	$(50,10,1,2)=45$
$(25,5,2,2)=16$	$(35,7,1,2)=30$	$(40,8,6,2)=13$	$(45,9,7,2)=13$	$(50,10,2,2)=40$
$(25,5,3,2)=12$	$(35,7,2,3)=26$	$(40,8,7,3)=8$	$(45,9,8,3)=8$	$(50,10,3,2)=36$
$(25,5,4,2)=8$	$(35,7,3,3)=21$	$(42,3,0,4)=42$	$(48,3,0,4)=48$	$(50,10,4,2)=31$
$(27,3,0,3)=26$	$(35,7,4,2)=17$	$(42,3,1,5)=31$	$(48,3,1,4)=36$	$(50,10,5,2)=27$
$(27,3,1,3)=19$	$(35,7,5,4)=13$	$(42,3,2,4)=21$	$(48,3,2,4)=24$	$(50,10,6,2)=22$
$(27,3,2,3)=12$	$(35,7,6,3)=8$	$(42,6,0,3)=41$	$(48,4,0,3)=47$	$(50,10,7,2)=18$
$(28,4,0,3)=27$	$(36,3,0,4)=36$	$(42,6,1,3)=35$	$(48,4,1,4)=38$	$(50,10,8,2)=13$
$(28,4,1,4)=22$	$(36,3,1,4)=27$	$(42,6,2,3)=29$	$(48,4,2,4)=28$	$(50,10,9,2)=9$
$(28,4,2,3)=16$	$(36,3,2,4)=18$	$(42,6,3,3)=23$	$(48,4,3,3)=18$	$(51,3,0,4)=51$
$(28,4,3,3)=10$	$(36,4,0,3)=35$	$(42,6,4,3)=17$	$(48,6,0,3)=47$	$(51,3,2,5)=25$
$(30,3,0,4)=30$	$(36,4,1,3)=28$	$(42,6,5,3)=11$	$(48,6,1,4)=41$	$(52,4,0,3)=51$
$(30,3,1,4)=22$	$(36,4,2,4)=21$	$(42,7,0,2)=41$	$(48,6,2,3)=34$	$(52,4,1,4)=41$
$(30,3,2,4)=15$	$(36,4,3,4)=14$	$(42,7,1,2)=36$	$(48,6,3,3)=27$	$(52,4,2,3)=30$
$(30,5,0,3)=29$	$(36,6,0,2)=35$	$(42,7,2,3)=31$	$(48,6,4,3)=20$	$(52,4,3,4)=20$
$(30,5,1,3)=24$	$(36,6,1,2)=30$	$(42,7,3,3)=26$	$(48,6,5,3)=13$	

Table 6. List of 184 solvable card tricks for C with $15 \le C \le 52$ on n stacks such that at each stack contains at least five cards, for j with $0 \le j < n$ for face-up dealing, and minimal k.

makes the unknown card to appear in position 20; compare with Table 1 from face-down dealing, where the 52 card trick can be performed with three dealing phases on each uniform collection sequence. Yet another possibility is to stay with three face-up dealing phases and four stacks, but then one has to change the collection pattern either to $(0,0,0)$ or $(2,2,2)$. The latter seems more appealing. In the former case the final position of the card is 51 while in the latter case it is 30. Whether Verini was aware that his trick is flawed is not documented in the literature as far as we know.

Finally, we close with a variant of Verini's card trick that resembles that for face-down dealing [8]; the main difference is to remember another table, which is shown below, the rest of the trick works in the same manner. At the start of the trick, a final position d_4 for the chosen card is preselected, and Audy mentally selects a card from a full 52-card deck. The deck is then distributed into four stacks *via* face-up dealing multiple times, with Audy indicating the stack containing their card each time, and by picking up the stacks in a specific collection sequence order after each deal, for redeals, Audy's chosen card ultimately appears at the preselected position. To this end Magi memorizes the collection sequence table:

1. $(0,3,0)$			
2. $(0,2,1)$	5. $(1,2,1)$	8. $(2,2,1)$	11. $(3,2,1)$
3. $(0,1,2)$	6. $(1,1,2)$	9. $(2,1,2)$	12. $(3,1,2)$
4. $(0,0,3)$	7. $(1,0,3)$	10. $(2,0,3)$	13. $(3,0,3)$

The table indicates the collection sequences for the first three face-up deals (for numbers 1–13). For the fourth deal Magi proceeds as follows:

- if d_4 is 13 or less, do not place any stack on top of the named one,
- if d_4 is between 14–26, then put one stack on top of that Audy pointed to,
- if d_4 lies in 27–39, then two stacks are placed on top of by Audy named one, and finally,
- if d_4 is greater than 39, then place all other stacks on top of Audy's stack.

In each case, the first three deals are carried out in the face-up way exactly as the collection sequence in table specifies.

For instance, if the final position is 42, the $42 - 39 = 3$ and Magi first deals face-up the collection sequence $(0,1,2)$ and finally in the fourth round places $3 = 42 \div 13$ on top of the named stack, since $42 = 3 \cdot 13 + 3$. Then the card is found at the 42nd position. For $d_4 = 18$ use the 5th collection sequence $(1,2,1)$, since $18 - 3 = 5$ and in the fourth face-up deal, place one stack on top of Audy's named stack, since 13 is subtracted once from 18 to become a number in 1–13. Then the card will appear on the 18th position. A rigorous proof that this procedure always works is left to the interested reader.

4 Conclusion

In this work, we have analyzed two ancient card tricks, namely the classic 21 card trick and Verini's lost trick. Like in the previous literature, we analyzed a generalization which allows for an arbitrary number of cards in the deck, number of stacks and number of cards per stack, respectively. We also introduced the concept of a collection sequence, and gave a precise characterization of those generalized card tricks that are solvable. Compared to the conditions on solvability from the previous literature across the centuries, e.g., [5, 20] our characterization is not procedural, but consists of a single formula.

With Verini's card trick, the convention of face-up dealing renders the analysis more challenging: the reason is the permanent reversal of card stacks. We developed a similar machinery for face-up dealing as we did for face-down dealing. While the formulas are a bit more intricate, there are structural similarities, which allow us to reconstruct the theory that we have developed above for the 21 card trick. We thus arrived at the first rigorous mathematical analysis of Verini's lost card trick—recall that, although it is almost 500 years old, it was rediscovered only few years ago. The most surprising result of the present work is of course the outcome of the analysis: Verini's trick is flawed, in the sense that the magician might guess the wrong card, depending on the initial position chosen by the person from the audience.

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A Appendix—Useful Floor and Ceiling Identities

The *floor function* is the function that takes as input a real number x , and gives as output the greatest integer less than or equal to x, denoted $\lfloor x \rfloor$. Similarly, the *ceiling function* maps x to the least integer greater than or equal to x, denoted $\lceil x \rceil$. For example,

- for floor: $\lfloor 2.4 \rfloor = 2$, $\lfloor -2.4 \rfloor = -3$, and
- for ceiling: $\lceil 2.4 \rceil = 3$ and, $\lceil -2.4 \rceil = -2$.

For formally,

$$\lfloor x \rfloor = \max\{\, n \in \mathbb{Z} \mid n \leq x \,\}$$

and

$$\lceil x \rceil = \min\{\, n \in \mathbb{Z} \mid n \geq x \,\},$$

where $\mathbb{Z}$ refers to the set of all integers. Thus, for any integer n we $\lfloor n \rfloor = \lceil n \rceil = n$. What follows are the most important identities on floor- and ceiling-expression.

Let n be an integer and x and arbitrary real. Then

$$n + \lceil x \rceil = \lceil n + x \rceil \quad \text{and} \quad \lceil \lceil x \rceil \rceil = \lceil x \rceil;$$

similar equations hold for the floor function. Moreover,

$$-\lceil x \rceil = \lfloor -x \rfloor \quad \text{and} \quad -\lfloor x \rfloor = \lceil -x \rceil.$$

Also, $\lceil x \rceil + n = \lceil x + n \rceil$; similarly for the floor function. The most important identities are the following two ones that allows us to change from a expression with ceiling to an expression with floors and *vice versa*—sometimes these are called the *floor-ceiling-identties.* It holds

$$\left\lfloor \frac{m}{n} \right\rfloor = \left\lceil \frac{m-n+1}{n} \right\rceil \quad \text{and} \quad \left\lceil \frac{m}{n} \right\rceil = \left\lfloor \frac{m+n-1}{n} \right\rfloor$$

for integers m and n with $n > 0$. Finally we make use of the identities

$$\left\lceil \frac{\lceil x \rceil}{n} \right\rceil = \left\lceil \frac{x}{n} \right\rceil \quad \text{and} \quad \left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor,$$

for integer n and arbitrary real x. For further readings on the floor- and ceiling-functions we refer to, e.g., [13].

B Appendix—CT Solvability Using Face-Down Dealing ($n = 3$)

	Deck size C																											
Coll. seq.	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	≥ 84
(0, 0, 0, 0)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	–
(0, 0, 0, 1)	1	1	1	1	1	1	1	1	1	1	1	1	1	–	–	–	–	–	–	–	–	–	–	–	–	–	2	–
(0, 0, 0, 2)	1	1	1	1	1	1	1	1	1	–	–	–	–	2	2	2	2	2	–	–	–	–	–	–	–	–	3	–
(0, 0, 1, 0)	1	1	1	1	1	1	–	–	2	2	2	2	2	–	–	–	–	3	3	3	–	–	–	–	–	–	4	–
(0, 0, 1, 1)	1	1	1	1	1	–	2	2	2	2	–	–	–	3	3	3	–	–	–	–	4	–	–	–	–	–	5	–
(0, 0, 1, 2)	1	1	1	1	–	2	2	2	2	–	3	3	3	–	–	–	4	4	–	–	–	5	–	–	–	–	6	–
(0, 0, 2, 0)	1	1	1	–	2	2	2	–	3	3	3	–	–	4	4	–	–	5	5	–	–	–	6	–	–	–	7	–
(0, 0, 2, 1)	1	1	1	2	2	2	–	3	3	3	–	4	4	–	–	5	–	–	–	6	–	–	–	–	–	–	8	–
(0, 0, 2, 2)	1	1	1	2	2	2	3	3	3	–	4	4	–	5	5	–	6	6	–	–	7	–	–	8	–	–	9	–
(0, 1, 0, 0)	1	1	2	2	2	3	3	3	4	4	–	5	5	–	6	6	–	7	–	–	8	–	–	9	–	–	10	–
(0, 1, 0, 1)	1	1	2	2	–	3	3	–	4	–	5	5	–	6	–	–	7	–	8	–	–	9	–	–	–	–	11	–
(0, 1, 0, 2)	1	1	2	2	3	3	–	4	4	5	5	–	6	–	7	–	–	8	–	9	–	–	–	–	–	–	12	–
(0, 1, 1, 0)	1	1	2	2	3	3	4	4	5	5	–	6	–	7	–	8	–	9	–	–	–	–	–	–	–	–	13	–
(0, 1, 1, 1)	1	–	2	–	3	–	4	–	5	–	6	–	7	–	8	–	9	–	10	–	11	–	12	–	13	–	14	–
(0, 1, 1, 2)	1	2	2	3	3	4	4	5	5	6	–	7	–	8	–	9	–	10	–	–	–	–	–	–	–	–	15	–
(0, 1, 2, 0)	1	2	2	3	3	4	–	5	6	6	7	–	8	–	9	–	–	11	–	12	–	–	–	–	–	–	16	–
(0, 1, 2, 1)	1	2	2	3	–	4	5	–	6	–	7	8	–	9	–	–	11	–	12	–	–	14	–	–	–	–	17	–
(0, 1, 2, 2)	1	2	2	3	4	4	5	6	6	7	–	8	9	–	10	11	–	12	–	–	14	–	–	16	–	–	18	–
(0, 2, 0, 0)	1	2	3	3	4	5	5	6	7	–	8	9	–	10	11	–	12	13	–	–	15	–	–	17	–	–	19	–
(0, 2, 0, 1)	1	2	3	3	4	5	–	6	7	8	–	9	10	–	–	12	–	–	–	15	–	–	–	–	–	–	20	–
(0, 2, 0, 2)	1	2	3	–	4	5	6	–	7	8	9	–	–	11	12	–	–	14	15	–	–	–	18	–	–	–	21	–
(0, 2, 1, 0)	1	2	3	4	–	5	6	7	8	–	9	10	11	–	–	–	14	15	–	–	–	18	–	–	–	–	22	–
(0, 2, 1, 1)	1	2	3	4	5	–	6	7	8	9	–	–	–	12	13	14	–	–	–	–	18	–	–	–	–	–	23	–
(0, 2, 1, 2)	1	2	3	4	5	6	–	–	8	9	10	11	12	–	–	–	–	16	17	18	–	–	–	–	–	–	24	–
(0, 2, 2, 0)	1	2	3	4	5	6	7	8	9	–	–	–	–	13	14	15	16	17	–	–	–	–	–	–	–	–	25	–
(0, 2, 2, 1)	1	2	3	4	5	6	7	8	9	10	11	12	13	–	–	–	–	–	–	–	–	–	–	–	–	–	26	–
(0, 2, 2, 2)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	–
(1, 0, 0, 0)	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	–
(1, 0, 0, 1)	2	3	4	5	6	7	8	9	10	11	12	13	14	–	–	–	–	–	–	–	–	–	–	–	–	–	29	–
(1, 0, 0, 2)	2	3	4	5	6	7	8	9	10	–	–	–	–	16	17	18	19	20	–	–	–	–	–	–	–	–	30	–
(1, 0, 1, 0)	2	3	4	5	6	7	–	–	11	12	13	14	15	–	–	–	–	21	22	23	–	–	–	–	–	–	31	–
(1, 0, 1, 1)	2	3	4	5	6	–	9	10	11	12	–	–	–	17	18	19	–	–	–	–	25	–	–	–	–	–	32	–
(1, 0, 1, 2)	2	3	4	5	–	8	9	10	11	–	14	15	16	–	–	–	21	22	–	–	–	27	–	–	–	–	33	–
(1, 0, 2, 0)	2	3	4	–	7	8	9	–	12	13	14	–	–	18	19	–	–	23	24	–	–	–	29	–	–	–	34	–
(1, 0, 2, 1)	2	3	4	6	7	8	–	11	12	13	–	16	17	–	–	21	–	–	–	26	–	–	–	–	–	–	35	–
(1, 0, 2, 2)	2	3	4	6	7	8	10	11	12	–	15	16	–	19	20	–	23	24	–	–	28	–	–	32	–	–	36	–
(1, 1, 0, 0)	2	3	5	6	7	9	10	11	13	14	–	17	18	–	21	22	–	25	–	–	29	–	–	33	–	–	37	–
(1, 1, 0, 1)	2	3	5	6	–	9	10	–	13	–	16	17	–	20	–	–	24	–	27	–	–	31	–	–	–	–	38	–
(1, 1, 0, 2)	2	3	5	6	8	9	–	12	13	15	16	–	19	–	22	–	–	26	–	29	–	–	–	–	–	–	39	–
(1, 1, 1, 0)	2	3	5	6	8	9	11	12	14	15	–	18	–	21	–	24	–	27	–	–	–	–	–	–	–	–	40	–
(1, 1, 1, 1)	2	–	5	–	8	–	11	–	14	–	17	–	20	–	23	–	26	–	29	–	32	–	35	–	38	–	41	–
(1, 1, 1, 2)	2	4	5	7	8	10	11	13	14	16	–	19	–	22	–	25	–	28	–	–	–	–	–	–	–	–	42	–
(1, 1, 2, 0)	2	4	5	7	8	10	–	13	15	16	18	–	21	–	24	–	–	29	–	32	–	–	–	–	–	–	43	–
(1, 1, 2, 1)	2	4	5	7	–	10	12	–	15	–	18	20	–	23	–	–	28	–	31	–	–	36	–	–	–	–	44	–
(1, 1, 2, 2)	2	4	5	7	9	10	12	14	15	17	–	20	22	–	25	27	–	30	–	–	35	–	–	40	–	–	45	–
(1, 2, 0, 0)	2	4	6	7	9	11	12	14	16	–	19	21	–	24	26	–	29	31	–	–	36	–	–	41	–	–	46	–
(1, 2, 0, 1)	2	4	6	7	9	11	–	14	16	18	–	21	23	–	–	28	–	–	–	35	–	–	–	–	–	–	47	–
(1, 2, 0, 2)	2	4	6	–	9	11	13	–	16	18	20	–	–	25	27	–	–	32	34	–	–	–	41	–	–	–	48	–
(1, 2, 1, 0)	2	4	6	8	–	11	13	15	17	–	20	22	24	–	–	–	31	33	–	–	–	40	–	–	–	–	49	–
(1, 2, 1, 1)	2	4	6	8	10	–	13	15	17	19	–	–	–	26	28	30	–	–	–	–	39	–	–	–	–	–	50	–
(1, 2, 1, 2)	2	4	6	8	10	12	–	–	17	19	21	23	25	–	–	–	–	34	36	38	–	–	–	–	–	–	51	–
(1, 2, 2, 0)	2	4	6	8	10	12	14	16	18	–	–	–	–	27	29	31	33	35	–	–	–	–	–	–	–	–	52	–
(1, 2, 2, 1)	2	4	6	8	10	12	14	16	18	20	22	24	26	–	–	–	–	–	–	–	–	–	–	–	–	–	53	–
(1, 2, 2, 2)	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	–
(2, 0, 0, 0)	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	–
(2, 0, 0, 1)	3	5	7	9	11	13	15	17	19	21	23	25	27	–	–	–	–	–	–	–	–	–	–	–	–	–	56	–
(2, 0, 0, 2)	3	5	7	9	11	13	15	17	19	–	–	–	–	30	32	34	36	38	–	–	–	–	–	–	–	–	57	–
(2, 0, 1, 0)	3	5	7	9	11	13	–	–	20	22	24	26	28	–	–	–	–	39	41	43	–	–	–	–	–	–	58	–
(2, 0, 1, 1)	3	5	7	9	11	–	16	18	20	22	–	–	–	31	33	35	–	–	–	–	46	–	–	–	–	–	59	–
(2, 0, 1, 2)	3	5	7	9	–	14	16	18	20	–	25	27	29	–	–	–	38	40	–	–	–	49	–	–	–	–	60	–
(2, 0, 2, 0)	3	5	7	–	12	14	16	–	21	23	25	–	–	32	34	–	–	41	43	–	–	–	52	–	–	–	61	–
(2, 0, 2, 1)	3	5	7	10	12	14	–	19	21	23	–	28	30	–	–	37	–	–	–	46	–	–	–	–	–	–	62	–
(2, 0, 2, 2)	3	5	7	10	12	14	17	19	21	–	26	28	–	33	35	–	40	42	–	–	49	–	–	56	–	–	63	–
(2, 1, 0, 0)	3	5	8	10	12	15	17	19	22	24	–	29	31	–	36	38	–	43	–	–	50	–	–	57	–	–	64	–
(2, 1, 0, 1)	3	5	8	10	–	15	17	–	22	–	27	29	–	34	–	–	41	–	46	–	–	53	–	–	–	–	65	–
(2, 1, 0, 2)	3	5	8	10	13	15	–	20	22	25	27	–	32	–	37	–	–	44	–	49	–	–	–	–	–	–	66	–
(2, 1, 1, 0)	3	5	8	10	13	15	18	20	23	25	–	30	–	35	–	40	–	45	–	–	–	–	–	–	–	–	67	–
(2, 1, 1, 1)	3	–	8	–	13	–	18	–	23	–	28	–	33	–	38	–	43	–	48	–	53	–	58	–	63	–	68	–
(2, 1, 1, 2)	3	6	8	11	13	16	18	21	23	26	–	31	–	36	–	41	–	46	–	–	–	–	–	–	–	–	69	–
(2, 1, 2, 0)	3	6	8	11	13	16	–	21	24	26	29	–	34	–	39	–	–	47	–	52	–	–	–	–	–	–	70	–
(2, 1, 2, 1)	3	6	8	11	–	16	19	–	24	–	29	32	–	37	–	–	45	–	50	–	–	58	–	–	–	–	71	–
(2, 1, 2, 2)	3	6	8	11	14	16	19	22	24	27	–	32	35	–	40	43	–	48	–	–	56	–	–	64	–	–	72	–
(2, 2, 0, 0)	3	6	9	11	14	17	19	22	25	–	30	33	–	38	41	–	46	49	–	–	57	–	–	65	–	–	73	–
(2, 2, 0, 1)	3	6	9	11	14	17	–	22	25	28	–	33	36	–	–	44	–	–	–	55	–	–	–	–	–	–	74	–
(2, 2, 0, 2)	3	6	9	–	14	17	20	–	25	28	31	–	–	39	42	–	–	50	53	–	–	–	64	–	–	–	75	–
(2, 2, 1, 0)	3	6	9	12	–	17	20	23	26	–	31	34	37	–	–	–	48	51	–	–	–	62	–	–	–	–	76	–
(2, 2, 1, 1)	3	6	9	12	15	–	20	23	26	29	–	–	–	40	43	46	–	–	–	–	60	–	–	–	–	–	77	–
(2, 2, 1, 2)	3	6	9	12	15	18	–	–	26	29	32	35	38	–	–	–	–	52	55	58	–	–	–	–	–	–	78	–
(2, 2, 2, 0)	3	6	9	12	15	18	21	24	27	–	–	–	–	41	44	47	50	53	–	–	–	–	–	–	–	–	79	–
(2, 2, 2, 1)	3	6	9	12	15	18	21	24	27	30	33	36	39	–	–	–	–	–	–	–	–	–	–	–	–	–	80	–
(2, 2, 2, 2)	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	–

C Appendix—CT Solvability Using Face-Up Dealing ($n = 3$)

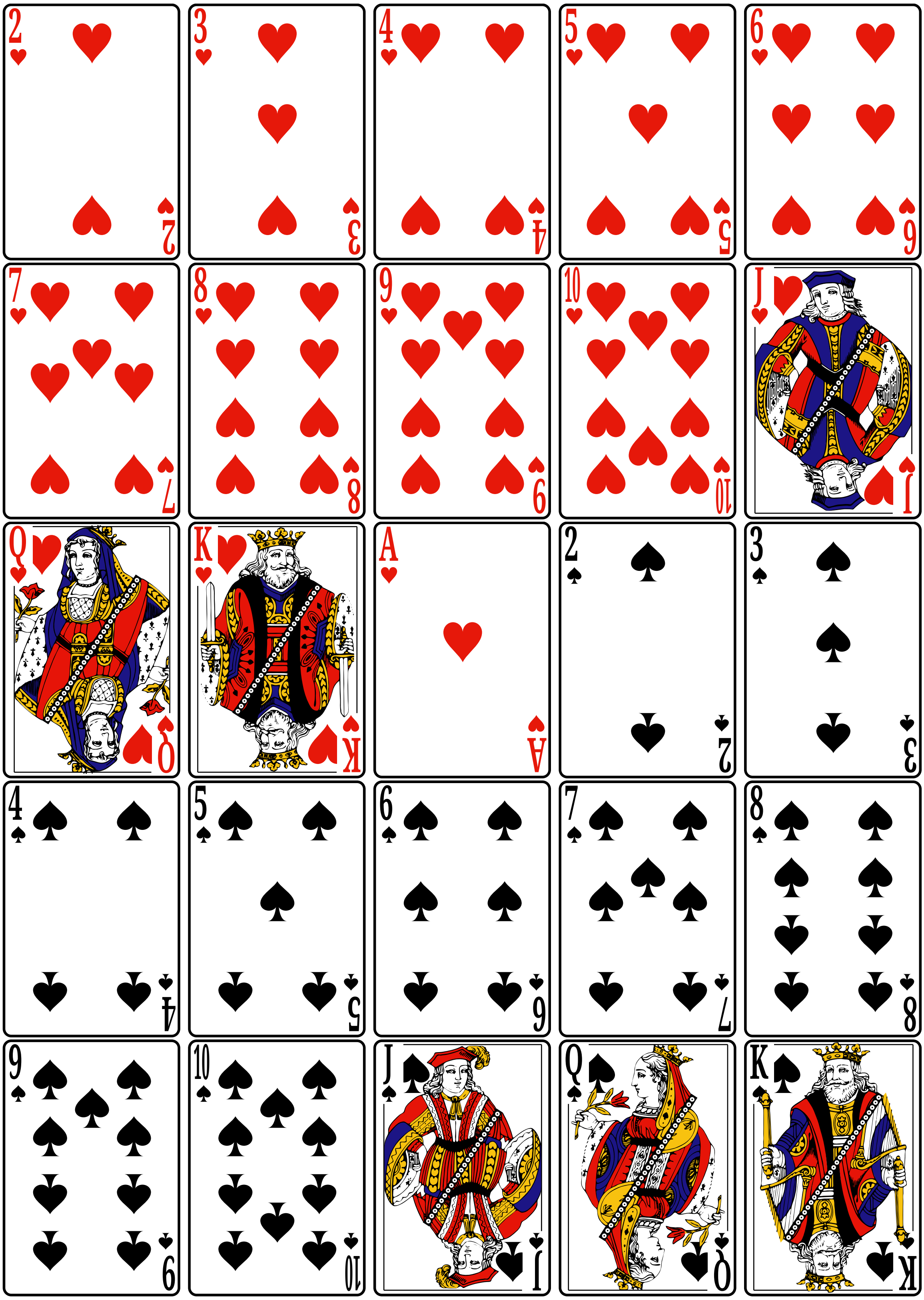
	Deck size C																											
Coll. seq.	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	≥ 84
(0, 0, 0, 0)	1	2	3	–	4	5	6	–	7	8	9	–	–	11	12	–	–	14	15	–	–	–	18	–	–	–	21	–
(0, 0, 0, 1)	1	2	3	3	4	5	–	6	7	8	–	9	10	–	–	12	–	–	–	15	–	–	–	–	–	–	20	–
(0, 0, 0, 2)	1	2	3	3	4	5	5	6	7	–	8	9	–	10	11	–	12	13	–	–	15	–	–	17	–	–	19	–
(0, 0, 1, 0)	1	2	3	4	5	6	–	–	8	9	10	11	12	–	–	–	–	16	17	18	–	–	–	–	–	–	24	–
(0, 0, 1, 1)	1	2	3	4	5	–	6	7	8	9	–	–	–	12	13	14	–	–	–	–	18	–	–	–	–	–	23	–
(0, 0, 1, 2)	1	2	3	4	–	5	6	7	8	–	9	10	11	–	–	–	14	15	–	–	–	18	–	–	–	–	22	–
(0, 0, 2, 0)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	–
(0, 0, 2, 1)	1	2	3	4	5	6	7	8	9	10	11	12	13	–	–	–	–	–	–	–	–	–	–	–	–	–	26	–
(0, 0, 2, 2)	1	2	3	4	5	6	7	8	9	–	–	–	–	13	14	15	16	17	–	–	–	–	–	–	–	–	25	–
(0, 1, 0, 0)	1	1	2	2	3	3	–	4	4	5	5	–	6	–	7	–	–	8	–	9	–	–	–	–	–	–	12	–
(0, 1, 0, 1)	1	1	2	2	–	3	3	–	4	–	5	5	–	6	–	–	7	–	8	–	–	9	–	–	–	–	11	–
(0, 1, 0, 2)	1	1	2	2	2	3	3	3	4	4	–	5	5	–	6	6	–	7	–	–	8	–	–	9	–	–	10	–
(0, 1, 1, 0)	1	2	2	3	3	4	4	5	5	6	–	7	–	8	–	9	–	10	–	–	–	–	–	–	–	–	15	–
(0, 1, 1, 1)	1	–	2	–	3	–	4	–	5	–	6	–	7	–	8	–	9	–	10	–	11	–	12	–	13	–	14	–
(0, 1, 1, 2)	1	1	2	2	3	3	4	4	5	5	–	6	–	7	–	8	–	9	–	–	–	–	–	–	–	–	13	–
(0, 1, 2, 0)	1	2	2	3	4	4	5	6	6	7	–	8	9	–	10	11	–	12	–	–	14	–	–	16	–	–	18	–
(0, 1, 2, 1)	1	2	2	3	–	4	5	–	6	–	7	8	–	9	–	–	11	–	12	–	–	14	–	–	–	–	17	–
(0, 1, 2, 2)	1	2	2	3	3	4	–	5	6	6	7	–	8	–	9	–	–	11	–	12	–	–	–	–	–	–	16	–
(0, 2, 0, 0)	1	1	1	1	1	1	1	1	1	–	–	–	–	2	2	2	2	2	–	–	–	–	–	–	–	–	3	–
(0, 2, 0, 1)	1	1	1	1	1	1	1	1	1	1	1	1	1	–	–	–	–	–	–	–	–	–	–	–	–	–	2	–
(0, 2, 0, 2)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	–
(0, 2, 1, 0)	1	1	1	1	–	2	2	2	2	–	3	3	3	–	–	–	4	4	–	–	–	5	–	–	–	–	6	–
(0, 2, 1, 1)	1	1	1	1	1	–	2	2	2	2	–	–	–	3	3	3	–	–	–	–	4	–	–	–	–	–	5	–
(0, 2, 1, 2)	1	1	1	1	1	1	–	–	2	2	2	2	2	–	–	–	–	3	3	3	–	–	–	–	–	–	4	–
(0, 2, 2, 0)	1	1	1	2	2	2	3	3	3	–	4	4	–	5	5	–	6	6	–	–	7	–	–	8	–	–	9	–
(0, 2, 2, 1)	1	1	1	2	2	2	–	3	3	3	–	4	4	–	–	5	–	–	–	6	–	–	–	–	–	–	8	–
(0, 2, 2, 2)	1	1	1	–	2	2	2	–	3	3	3	–	–	4	4	–	–	5	5	–	–	–	6	–	–	–	7	–
(1, 0, 0, 0)	2	4	6	–	9	11	13	–	16	18	20	–	–	25	27	–	–	32	34	–	–	–	41	–	–	–	48	–
(1, 0, 0, 1)	2	4	6	7	9	11	–	14	16	18	–	21	23	–	–	28	–	–	–	35	–	–	–	–	–	–	47	–
(1, 0, 0, 2)	2	4	6	7	9	11	12	14	16	–	19	21	–	24	26	–	29	31	–	–	36	–	–	41	–	–	46	–
(1, 0, 1, 0)	2	4	6	8	10	12	–	–	17	19	21	23	25	–	–	–	–	34	36	38	–	–	–	–	–	–	51	–
(1, 0, 1, 1)	2	4	6	8	10	–	13	15	17	19	–	–	–	26	28	30	–	–	–	–	39	–	–	–	–	–	50	–
(1, 0, 1, 2)	2	4	6	8	–	11	13	15	17	–	20	22	24	–	–	–	31	33	–	–	–	40	–	–	–	–	49	–
(1, 0, 2, 0)	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	–
(1, 0, 2, 1)	2	4	6	8	10	12	14	16	18	20	22	24	26	–	–	–	–	–	–	–	–	–	–	–	–	–	53	–
(1, 0, 2, 2)	2	4	6	8	10	12	14	16	18	–	–	–	–	27	29	31	33	35	–	–	–	–	–	–	–	–	52	–
(1, 1, 0, 0)	2	3	5	6	8	9	–	12	13	15	16	–	19	–	22	–	–	26	–	29	–	–	–	–	–	–	39	–
(1, 1, 0, 1)	2	3	5	6	–	9	10	–	13	–	16	17	–	20	–	–	24	–	27	–	–	31	–	–	–	–	38	–
(1, 1, 0, 2)	2	3	5	6	7	9	10	11	13	14	–	17	18	–	21	22	–	25	–	–	29	–	–	33	–	–	37	–
(1, 1, 1, 0)	2	4	5	7	8	10	11	13	14	16	–	19	–	22	–	25	–	28	–	–	–	–	–	–	–	–	42	–
(1, 1, 1, 1)	2	–	5	–	8	–	11	–	14	–	17	–	20	–	23	–	26	–	29	–	32	–	35	–	38	–	41	–
(1, 1, 1, 2)	2	3	5	6	8	9	11	12	14	15	–	18	–	21	–	24	–	27	–	–	–	–	–	–	–	–	40	–
(1, 1, 2, 0)	2	4	5	7	9	10	12	14	15	17	–	20	22	–	25	27	–	30	–	–	35	–	–	40	–	–	45	–
(1, 1, 2, 1)	2	4	5	7	–	10	12	–	15	–	18	20	–	23	–	–	28	–	31	–	–	36	–	–	–	–	44	–
(1, 1, 2, 2)	2	4	5	7	8	10	–	13	15	16	18	–	21	–	24	–	–	29	–	32	–	–	–	–	–	–	43	–
(1, 2, 0, 0)	2	3	4	5	6	7	8	9	10	–	–	–	–	16	17	18	19	20	–	–	–	–	–	–	–	–	30	–
(1, 2, 0, 1)	2	3	4	5	6	7	8	9	10	11	12	13	14	–	–	–	–	–	–	–	–	–	–	–	–	–	29	–
(1, 2, 0, 2)	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	–
(1, 2, 1, 0)	2	3	4	5	–	8	9	10	11	–	14	15	16	–	–	–	21	22	–	–	–	27	–	–	–	–	33	–
(1, 2, 1, 1)	2	3	4	5	6	–	9	10	11	12	–	–	–	17	18	19	–	–	–	–	25	–	–	–	–	–	32	–
(1, 2, 1, 2)	2	3	4	5	6	7	–	–	11	12	13	14	15	–	–	–	–	21	22	23	–	–	–	–	–	–	31	–
(1, 2, 2, 0)	2	3	4	6	7	8	10	11	12	–	15	16	–	19	20	–	23	24	–	–	28	–	–	32	–	–	36	–
(1, 2, 2, 1)	2	3	4	6	7	8	–	11	12	13	–	16	17	–	–	21	–	–	–	26	–	–	–	–	–	–	35	–
(1, 2, 2, 2)	2	3	4	–	7	8	9	–	12	13	14	–	–	18	19	–	–	23	24	–	–	–	29	–	–	–	34	–
(2, 0, 0, 0)	3	6	9	–	14	17	20	–	25	28	31	–	–	39	42	–	–	50	53	–	–	–	64	–	–	–	75	–
(2, 0, 0, 1)	3	6	9	11	14	17	–	22	25	28	–	33	36	–	–	44	–	–	–	55	–	–	–	–	–	–	74	–
(2, 0, 0, 2)	3	6	9	11	14	17	19	22	25	–	30	33	–	38	41	–	46	49	–	–	57	–	–	65	–	–	73	–
(2, 0, 1, 0)	3	6	9	12	15	18	–	–	26	29	32	35	38	–	–	–	–	52	55	58	–	–	–	–	–	–	78	–
(2, 0, 1, 1)	3	6	9	12	15	–	20	23	26	29	–	–	–	40	43	46	–	–	–	–	60	–	–	–	–	–	77	–
(2, 0, 1, 2)	3	6	9	12	–	17	20	23	26	–	31	34	37	–	–	–	48	51	–	–	–	62	–	–	–	–	76	–
(2, 0, 2, 0)	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75	78	81	–
(2, 0, 2, 1)	3	6	9	12	15	18	21	24	27	30	33	36	39	–	–	–	–	–	–	–	–	–	–	–	–	–	80	–
(2, 0, 2, 2)	3	6	9	12	15	18	21	24	27	–	–	–	–	41	44	47	50	53	–	–	–	–	–	–	–	–	79	–
(2, 1, 0, 0)	3	5	8	10	13	15	–	20	22	25	27	–	32	–	37	–	–	44	–	49	–	–	–	–	–	–	66	–
(2, 1, 0, 1)	3	5	8	10	–	15	17	–	22	–	27	29	–	34	–	–	41	–	46	–	–	53	–	–	–	–	65	–
(2, 1, 0, 2)	3	5	8	10	12	15	17	19	22	24	–	29	31	–	36	38	–	43	–	–	50	–	–	57	–	–	64	–
(2, 1, 1, 0)	3	6	8	11	13	16	18	21	23	26	–	31	–	36	–	41	–	46	–	–	–	–	–	–	–	–	69	–
(2, 1, 1, 1)	3	–	8	–	13	–	18	–	23	–	28	–	33	–	38	–	43	–	48	–	53	–	58	–	63	–	68	–
(2, 1, 1, 2)	3	5	8	10	13	15	18	20	23	25	–	30	–	35	–	40	–	45	–	–	–	–	–	–	–	–	67	–
(2, 1, 2, 0)	3	6	8	11	14	16	19	22	24	27	–	32	35	–	40	43	–	48	–	–	56	–	–	64	–	–	72	–
(2, 1, 2, 1)	3	6	8	11	–	16	19	–	24	–	29	32	–	37	–	–	45	–	50	–	–	58	–	–	–	–	71	–
(2, 1, 2, 2)	3	6	8	11	13	16	–	21	24	26	29	–	34	–	39	–	–	47	–	52	–	–	–	–	–	–	70	–
(2, 2, 0, 0)	3	5	7	9	11	13	15	17	19	–	–	–	–	30	32	34	36	38	–	–	–	–	–	–	–	–	57	–
(2, 2, 0, 1)	3	5	7	9	11	13	15	17	19	21	23	25	27	–	–	–	–	–	–	–	–	–	–	–	–	–	56	–
(2, 2, 0, 2)	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	–
(2, 2, 1, 0)	3	5	7	9	–	14	16	18	20	–	25	27	29	–	–	–	38	40	–	–	–	49	–	–	–	–	60	–
(2, 2, 1, 1)	3	5	7	9	11	–	16	18	20	22	–	–	–	31	33	35	–	–	–	–	46	–	–	–	–	–	59	–
(2, 2, 1, 2)	3	5	7	9	11	13	–	–	20	22	24	26	28	–	–	–	–	39	41	43	–	–	–	–	–	–	58	–
(2, 2, 2, 0)	3	5	7	10	12	14	17	19	21	–	26	28	–	33	35	–	40	42	–	–	49	–	–	56	–	–	63	–
(2, 2, 2, 1)	3	5	7	10	12	14	–	19	21	23	–	28	30	–	–	37	–	–	–	46	–	–	–	–	–	–	62	–
(2, 2, 2, 2)	3	5	7	–	12	14	16	–	21	23	25	–	–	32	34	–	–	41	43	–	–	–	52	–	–	–	61	–

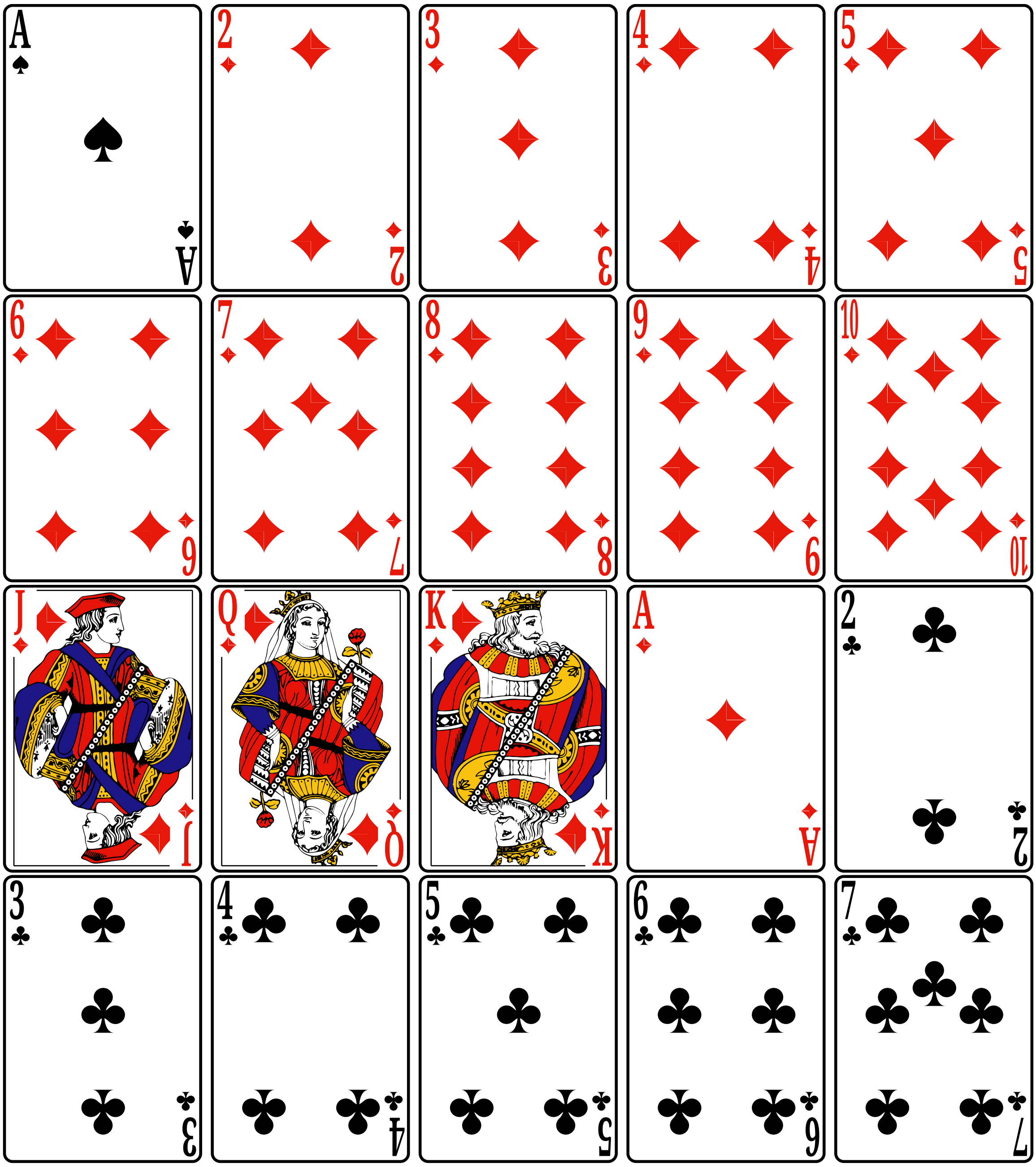
D Appendix—Verini's 52 Card Trick

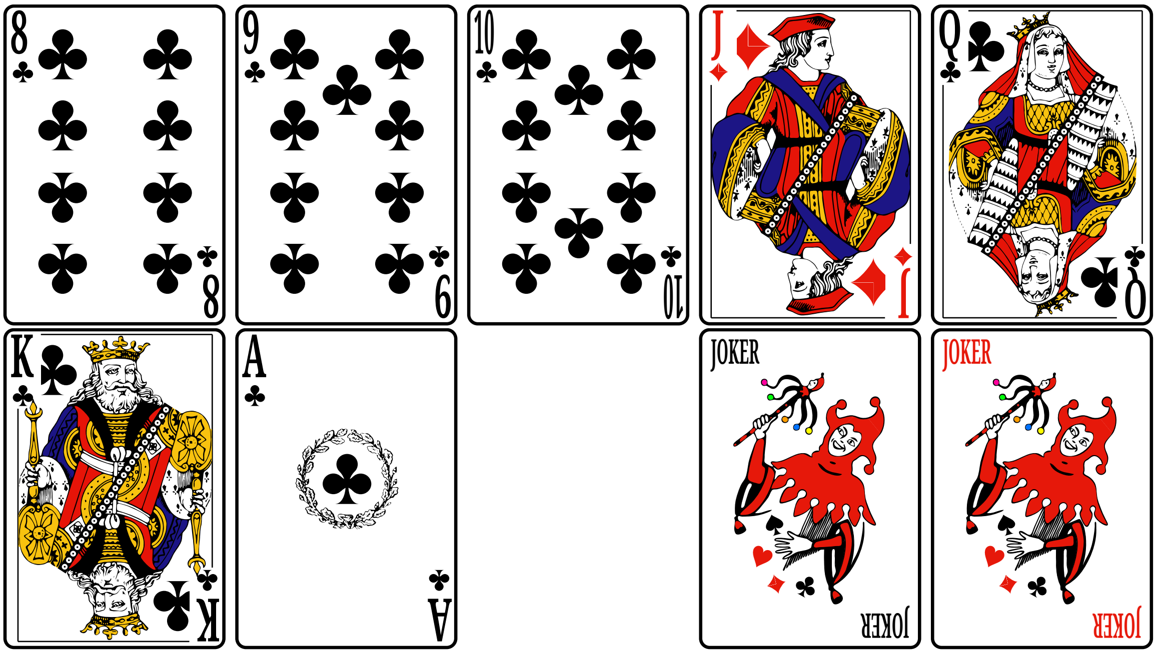
Verini's card trick with 52 cards is performed by dealing and collecting the cards *face-up* into $n = 4$ stacks. Each time, the three stacks that are left aside must be placed on top, which corresponds to the collection sequence $(3, 3, 3)$ in the table—recall, that the collection sequence must be applied from *right-to-left*. Below is a complete list of all possible *final positions* d_k when starting from a given position d_0.

	d_0																																																			
Coll. seq.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	d_k
(0, 0, 0)	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	{10, 11}
(0, 0, 1)	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	{11, 12}
(0, 0, 2)	13	13	13	13	13	13	13	13	13	13	13	13	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	{12, 13}
(0, 0, 3)	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	13	{13}
(0, 1, 0)	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	{8, 7}
(0, 1, 1)	9	9	9	9	9	9	9	9	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	{8, 9}
(0, 1, 2)	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	{9}
(0, 1, 3)	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	9	9	9	9	{9, 10}
(0, 2, 0)	5	5	5	5	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	{4, 5}
(0, 2, 1)	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	{5}
(0, 2, 2)	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	5	5	5	5	5	5	5	5	{5, 6}
(0, 2, 3)	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	{6, 7}
(0, 3, 0)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	{1}
(0, 3, 1)	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	{1, 2}
(0, 3, 2)	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	{2, 3}
(0, 3, 3)	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	{3, 4}
(1, 0, 0)	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	{24, 23}
(1, 0, 1)	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	{24, 25}
(1, 0, 2)	26	26	26	26	26	26	26	26	26	26	26	26	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	{25, 26}
(1, 0, 3)	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26	{26}
(1, 1, 0)	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	{20, 21}
(1, 1, 1)	22	22	22	22	22	22	22	22	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	{21, 22}
(1, 1, 2)	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	22	{22}
(1, 1, 3)	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	22	22	22	22	{22, 23}
(1, 2, 0)	18	18	18	18	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	{17, 18}
(1, 2, 1)	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	18	{18}
(1, 2, 2)	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	18	18	18	18	18	18	18	18	{18, 19}
(1, 2, 3)	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19	{19, 20}
(1, 3, 0)	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	{14}
(1, 3, 1)	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	14	14	14	14	14	14	14	14	14	14	14	14	{14, 15}
(1, 3, 2)	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	{16, 15}
(1, 3, 3)	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	{16, 17}
(2, 0, 0)	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	{36, 37}
(2, 0, 1)	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	37	{37, 38}
(2, 0, 2)	39	39	39	39	39	39	39	39	39	39	39	39	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	38	{38, 39}
(2, 0, 3)	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	39	{39}
(2, 1, 0)	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	{33, 34}
(2, 1, 1)	35	35	35	35	35	35	35	35	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	34	{34, 35}
(2, 1, 2)	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	35	{35}
(2, 1, 3)	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	36	35	35	35	35	{35, 36}
(2, 2, 0)	31	31	31	31	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	{30, 31}
(2, 2, 1)	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	31	{31}
(2, 2, 2)	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	31	31	31	31	31	31	31	31	{32, 31}
(2, 2, 3)	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	33	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	32	{32, 33}
(2, 3, 0)	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	27	{27}
(2, 3, 1)	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	27	27	27	27	27	27	27	27	27	27	27	27	{27, 28}
(2, 3, 2)	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	28	{28, 29}
(2, 3, 3)	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	{29, 30}
(3, 0, 0)	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	{49, 50}
(3, 0, 1)	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	{50, 51}
(3, 0, 2)	52	52	52	52	52	52	52	52	52	52	52	52	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	51	{51, 52}
(3, 0, 3)	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	52	{52}
(3, 1, 0)	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	{46, 47}
(3, 1, 1)	48	48	48	48	48	48	48	48	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	47	{48, 47}
(3, 1, 2)	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	48	{48}
(3, 1, 3)	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	49	48	48	48	48	{48, 49}
(3, 2, 0)	44	44	44	44	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	{43, 44}
(3, 2, 1)	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	{44}
(3, 2, 2)	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	44	44	44	44	44	44	44	44	{44, 45}
(3, 2, 3)	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	{45, 46}
(3, 3, 0)	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	{40}
(3, 3, 1)	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	40	40	40	40	40	40	40	40	40	40	40	40	{40, 41}
(3, 3, 2)	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	41	{41, 42}
(3, 3, 3)	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	43	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	{42, 43}

E Appendix—Card Deck to Cut Out



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