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Stationäre Lösungen des N–Wirbel–Problems der Fluiddynamik

Stationary solutions to the N–vortex–problem

Inauguraldissertation
zur Erlangung des Doktorgrades
der Naturwissenschaftlichen Fachbereiche
der Justus–Liebig–Universität Gießen

vorgelegt von

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am
19. Dezember 2013

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Kurzzusammenfassung

Diese Dissertation befasst sich mit Existenz und Eigenschaften stationärer Lösungen für die Bewegung von N Punktwirbeln in einer idealisierten zweidimensionalen Flüssigkeit in einem beschränkten Gebiet Ω, die bestimmt wird durch ein Hamiltonsches System

$$\begin{cases} \Gamma_i \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = \dfrac{\partial H_\Omega}{\partial y_i}(z_1, \ldots, z_N) \\ \Gamma_i \dfrac{\mathrm{d}y_i}{\mathrm{d}t} = -\dfrac{\partial H_\Omega}{\partial x_i}(z_1, \ldots, z_N) \end{cases} \qquad \text{wobei } z_i = (x_i, y_i),\ i = 1, \ldots, N,$$

wobei $H_\Omega(z) := \sum_{j=1}^N \Gamma_j^2 h(z_j) + \sum_{i,j=1, i\neq j}^N \Gamma_i \Gamma_j G(z_i, z_j)$ die sogenannte „Kirchhoff–Routh–path function“ ist, unter verschiedenen Bedingungen an die „Wirbelstärken“ Γ_i, sowie verschiedenen geometrisch–topologischen Annahmen über das Gebiet Ω, wie vor allem Symmetrie und mehrfacher Zusammenhang. Des Weiteren werden mögliche Anwendungen der vorliegenden Resultate auf die Untersuchung der sinh–Poisson–Gleichung sowie der Lane–Emden–Fowler–Gleichung diskutiert.

Abstract

This dissertation is concerned with the study of existence and properties of stationary solutions for the dynamics of N point vortices in an idealised fluid constrained to a two–dimensional domain Ω, which is governed by a Hamiltonian system

$$\begin{cases} \Gamma_i \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = \dfrac{\partial H_\Omega}{\partial y_i}(z_1, \ldots, z_N) \\ \Gamma_i \dfrac{\mathrm{d}y_i}{\mathrm{d}t} = -\dfrac{\partial H_\Omega}{\partial x_i}(z_1, \ldots, z_N) \end{cases} \qquad \text{where } z_i = (x_i, y_i),\ i = 1, \ldots, N,$$

where $H_\Omega(z) := \sum_{j=1}^N \Gamma_j^2 h(z_j) + \sum_{i,j=1, i\neq j}^N \Gamma_i \Gamma_j G(z_i, z_j)$ is the so–called Kirchhoff–Routh–path function under various conditions on the “vorticities” Γ_i and various topological and geometrical assumptions on Ω, notably symmetry and multiple connectivity. Further, possible applications of the results to the study of the sinh–Poisson equation and the Lane–Emden–Fowler equation are discussed.

Contents

1 Introduction

The N–vortex–problem of fluid dynamics is concerned with the dynamics of N point vortices $z_1, \dots, z_N$ in an ideal fluid constrained to a two–dimensional domain Ω with corresponding vortex strengths (so-called vorticities) $\Gamma_1, \dots, \Gamma_N \in \mathbb{R}$, whose absolute values determine the degree to which the surrounding fluid is curled and whose signs determine the direction of revolution for the surrounding fluid. It is governed by a Hamiltonian system

$$\begin{cases} \Gamma_i \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = \dfrac{\partial H_\Omega}{\partial y_i}(z_1, \dots, z_N) \\ \Gamma_i \dfrac{\mathrm{d}y_i}{\mathrm{d}t} = -\dfrac{\partial H_\Omega}{\partial x_i}(z_1, \dots, z_N) \end{cases} \quad \text{where } z_i = (x_i, y_i),\ i = 1, \dots, N, \tag{1.1}$$

which is derived as a limit of the Euler–equations for the motion of the whole fluid. The geometry of the domain comes into play through the hydrodynamic Green's function, a generalisation of the classical Green's function of the first kind for the Laplacian on Ω, which plays a dominant role in the Hamilton function H_Ω.

Since its derivation by Helmholtz, Kirchhoff, Lord Kelvin and Routh in the second half of the 19^{th} century, this model has played a central role in the research on fluid dynamics, motivated by prominent examples of it's applicability in turbulences of the earth's atmosphere and oceans up to the dynamics of an electron plasma, see for example the survey article [1] as well as the monographies [14, 15, 16].

There is plenty of literature in the case that Ω is the whole Euclidean plane and all the vorticities have the same sign. For research about these cases [1, 14, 15, 16] are a very good starting point. In particular, a lot of research has been done considering questions of integrability and ergodicity as well as existence and geometrical form of stationary or periodic solutions.

In these papers it is crucial that if Ω is the whole Euclidean plane, the Hamilton function is explicitly given by

$$H(z_1, \dots, z_N) = -\frac{1}{2\pi} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \Gamma_i \Gamma_j \ln |z_i - z_j|.$$

Much less is known about the behaviour of solutions of (1.1), if Ω is a bounded domain and some of the vorticities Γ_i are positive, others negative, so that the vortices are rotating in different directions. H_Ω is then defined on the so-called configuration space[1]

$$\mathcal{F}_N\Omega := \{(z_1, \dots, z_N) \in \Omega^N : z_i \neq z_j \text{ for } i \neq j\},$$

which is an open subset of Ω^N and therefore of all of $\mathbb{C}^N$.

[1] This space is of great importance in algebraic topology. The term "configuration space" comes from there and is not to be confused with the configuration space of some mechanical system in physics, though both spaces share the same historical origin.

In this case the Hamilton function, which in the literature is commonly called "Kirchhoff–Routh path function" is given by

$$H_\Omega(z_1, \ldots, z_N) = \sum_{j=1}^{N} \Gamma_j^2 h(z_j) + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \Gamma_i \Gamma_j G(z_i, z_j),$$

where $G : \mathcal{F}_2\Omega \to \mathbb{R}$ is the hydrodynamic Green's function with regular part $g(x, y) = G(x, y) + \frac{1}{2\pi} \ln |x - y|$, and $h(x) = g(x, x)$ is the so-called Robin's function.

Basic results concerning G, h and the dynamics of the equation (1.1) in the case $N = 1$ were proven in [10, 11, 13]. For $N \geq 2$ there are plenty of papers of numerical nature by mathematicians, physicists and engineers, especially for special domains, whose Green's function is either explicitly known or can be described by methods of complex analysis.

Contrary to that, there are only a few analytical papers concerned with the case of a general domain Ω, notably [2, 4, 6, 7], where, under some special conditions on Ω and the coefficients Γ_i, critical points of H_Ω and thereby stationary solutions of (1.1) are obtained. In [6] it is assumed that $\Gamma_i = 1$ for all $i \in \{1, \ldots, N\}$ and that Ω is not simply connected. In [7] very special simply connected ("dumbbell shaped") domains are allowed, but again only for the case $\Gamma_i = 1$ for all $i \in \{1, \ldots, N\}$. The paper [2] is concerned with the case $N = 2$, $\Gamma_1 = -1$, $\Gamma_2 = 1$, and Ω an arbitrary bounded domain, this is the first instance where a stationary configuration of counterrotating vortices in an arbitrary domain is found. Lastly, in [4] a stationary solution of N counterrotating vortices lying on the symmetry axis of a axially symmetric domain is found for arbitrary N and $\Gamma_i = (-1)^i$ for $i \in \{1, \ldots, N\}$. Additionally, the much more complicated case of a general bounded domain Ω with $\Gamma_i = (-1)^i$ is settled in [4] successfully for $N = 3$ and $N = 4$.

It shall also be mentioned that, somewhat surprisingly, in the papers [2, 6, 7] the Hamiltonian H_Ω appears as a limit functional for some elliptic boundary value problems in Ω and the existence of critical points of certain perturbations of H_Ω gives rise to solutions of these problems.

The goal of this dissertation is to investigate the existence and properties of critical points of H_Ω (and hence of stationary solutions to equation (1.1)) under various conditions on the vorticities Γ_i as well as some geometrical and topological assumptions (such as symmetry or multiple connectivity) on Ω, but for general $N \in \mathbb{N}$. To some extent we are also able to prove the existence of new nodal solutions to the elliptic boundary value problems considered for example in [2].

Although the problem of finding critical points of H_Ω is finite dimensional, the problem has proven itself to be considerably refractory. The most obvious difficulty is that for an arbitrary domain Ω the Green's function as an essential part of H_Ω is only implicitly given as a solution of a partial differential equation, thus all relevant properties of H_Ω have to be derived through the analysis of the corresponding partial differential equation. More importantly, H_Ω is only defined on the incomplete manifold $\mathcal{F}_N\Omega$ and is for general vorticities Γ_i strongly indefinite. In fact, it may be the case that $H_\Omega(z)$ remains bounded for $\operatorname{dist}(z, \partial\mathcal{F}_N\Omega) \to 0$, which in the model corresponds to collisions of multiple vortices with each other or with $\partial\Omega$. This lack of compactness is crucial, since it prevents us from using standard methods of critical point theory, such as the "mountain–pass"–theorem. Hence a more detailed study of the behaviour of H_Ω is necessary. The usual methods of critical point theory, all of which apply some sort of modified gradient flow of H_Ω are difficult to apply due to the incompleteness of $\mathcal{F}_N\Omega$. Success in applying these methods is therefore intimately connected to a good analytical understanding of collisions, that is of flow lines $z : (t^-(z_0), t^+(z_0)) \to \mathcal{F}_N\Omega$ to the gradient flow of H_Ω satisfying

$$\min\{|z_i(t) - z_j(t)|, \operatorname{dist}(z_i(t), \partial\Omega) : i, j \in \{1, \dots, N\}, i \neq j\} \to 0$$

for $t \to t^+(z_0)$. This, in turn, depends very sensitively on the constellation of the vorticities Γ_i.

The space $\mathcal{F}_N\Omega$ on the other hand exhibits a rich topology even for simply connected Ω, such that, given appropriate compactness properties of H_Ω, finding critical points of H_Ω is a rather easy task.

The bulk of this thesis is therefore concerned with deriving conditions on the vorticities Γ_i and on the domain Ω such that the gradient flow of H_Ω has a compact flow line. The relevant condition on the Γ_i has in part already been conjectured in [4] and is a rather strict one for larger N. In particular, for general Ω, the "model case" $\Gamma_i = (-1)^i$ is not covered by our results, which therefore complement the results given in [4].

If Ω is dihedrally symmetric, we are able to gain some compactness and relax the conditions on the Γ_i. It is here where we are also able to derive new nodal solutions to the elliptic boundary value problems mentioned before.

This thesis is organized as follows. In chapter 2 we give some preliminary results concerning the behaviour of the Green's and Robin–functions. We also give an abstract deformation argument which will be perpetually used throughout the whole thesis. In chapter 3 we consider the (considerably simpler) case of a dihedrally symmetric domain Ω and derive several critical points of different geometrical type for H_Ω. Chapter 4 is concerned with the careful analysis of the behaviour of H_Ω along "collision" flowlines. Chapter 5 then provides linking properties for H_Ω, consequently proving the existence of critical points of H_Ω also in the case of a general domain Ω. Lastly, chapter 6 is dedicated to the discussion of the stability of the previous results as well as possible applications to the elliptic boundary value problems mentioned before.

1.1 Statement of results

In this subsection we give an outline of the theorems proven in this thesis. In order not to get too deep into technicalities already in the introduction, we state the results in a simplified rather than their fully general version in order to give an overview of the topics covered.

Let therefore $\Omega \subset \mathbb{C}$ be a smooth domain and denote the Kirchhoff–Routh–path–function with vorticities $\Gamma \in \mathbb{R}^N$ in Ω by H_Ω^Γ. We will prove more general versions of the following theorems:

Theorem 1.1. *Let* $\Gamma \in \mathbb{R}^N$ *satisfy* $\sum_{\substack{i,j\in J\\ i\neq j}} \Gamma_i\Gamma_j \neq 0$ *and* $\sum_{j\in J}\Gamma_j^2 > \sum_{\substack{i,j\in J\\ i\neq j}} |\Gamma_i\Gamma_j|$ *for all* $J \subset \{1,\dots,N\}$, $|J| \geq 2$, *as well as* $\Gamma_j = (-1)^j|\Gamma_j|$, $j \in \{1,\dots,N\}$, *where* $|\Gamma_j| \leq |\Gamma_{j+1}|$ *for* $j \in \{1,\dots,N-1\}$. *Then* H_Ω^Γ *has a critical point.*

If Ω is not simply–connected, we may exploit the richer topology of the configuration space $\mathcal{F}_N\Omega$ to abolish the necessity for alternating vorticities.

Theorem 1.2. *Assume* Ω *is not simply–connected and let, as before,* $\Gamma \in \mathbb{R}^N$ *satisfy* $\sum_{\substack{i,j\in J\\ i\neq j}} \Gamma_i\Gamma_j \neq 0$ *and* $\sum_{j\in J}\Gamma_j^2 > \sum_{\substack{i,j\in J\\ i\neq j}} |\Gamma_i\Gamma_j|$ *for all* $J \subset \{1,\dots,N\}$, $|J| \geq 2$. *Then* H_Ω^Γ *has a critical point.*

For D_p–symmetric domains Ω, where D_p denotes the symmetry group of a regular p–gon, and where without loss of generality we may take $0 \in \mathbb{C}$ to be the symmetry center of Ω, the conditions on Γ are much less severe.

Theorem 1.3. *Assume* Ω *is* D_p*–symmetric,* $0 \notin \Omega$, *and let* $\Gamma \in \mathbb{R}^{kp}$ *satisfy* $\Gamma_{j+lk} = \Gamma_j = (-1)^j|\Gamma_j|$, *for* $j \in \{1,\dots,k\}$, $l \in \{1,\dots,p-1\}$, *where* $|\Gamma_j| \leq |\Gamma_{j+1}|$ *for* $j \in \{1,\dots,k-1\}$. *Then* H_Ω^Γ *has a critical point, whose components are symmetrically aligned with alternating vortices and increasing or decreasing modulus along the symmetry axes of* Ω.

Further, we have a result stating that under slightly stricter conditions on Γ, we may replace the hole at the symmetry center of Ω by a sufficiently strong vortex.

Theorem 1.4. *Assume* Ω *is* D_p*–symmetric,* $0 \in \Omega$, *and let* $(\Gamma_0,\dots,\Gamma_{kp}) \in \mathbb{R}^{kp+1}$ *satisfy* $\Gamma_{j+lk} = \Gamma_j = (-1)^j|\Gamma_j|$, *for* $j \in \{1,\dots,k\}$, $l \in \{1,\dots,p-1\}$, *where* $|\Gamma_j| < |\Gamma_{j+1}|$ *for* $j \in \{1,\dots,k-1\}$. *There is* $\tilde\Gamma_0 > 0$, *such that for all* $\Gamma_0 > \tilde\Gamma_0$ *the Kirchhoff–Routh–path–function* H_Ω^Γ *has a critical point, whose components are symmetrically aligned with alternating vortices and decreasing modulus along the symmetry axes of* Ω *with the vortex with strength* Γ_0 *placed in the symmetry center of* Ω.

More general versions of the above theorems are found in theorems 5.1, 5.2, concerning the case of an asymmetric domain Ω as well as theorems 3.6 and 3.9, respectively, where symmetric domains are considered.

2 Preliminaries

2.1 Hypotheses and basic notation

Hypothesis 2.1. Let $\Omega \subset \mathbb{C}$ be a bounded domain with C^3–boundary. A fortiori, Ω is finitely connected and satisfies a uniform exterior ball condition, that is there exists a constant $r > 0$ such that for any $x \in \partial\Omega$ there is $x^* \in \mathbb{C}$ such that $B_r(x^*) \subset \mathbb{C} \setminus \Omega$ as well as $x \in \partial B_r(x^*)$. Let $k_0 := \operatorname{rank} \pi_1(\Omega)$, and denote the bounded components (if any) of $\mathbb{C} \setminus \bar{\Omega}$ by Ω_j, $j \in \{1, \ldots, k_0\}$.

For convenience in stating our results and further hypotheses, we start by fixing some useful notation.

Definition 2.2 (Basic notation)**.** The configuration space of N point vortices in Ω is defined as

$$\mathcal{F}_N\Omega := \left\{ z \in \Omega^N : z_i = z_j \Leftrightarrow i = j \right\},$$

which is an open subset of Ω^N and therefore of all of $\mathbb{C}^N$. We denote its boundary in $\mathbb{C}^N$ by $\partial\mathcal{F}_N\Omega$. We set $\Delta_N := \left\{ t \in (0,\infty)^N : t_i < t_{i+1} \ \forall i \in \{1, \ldots, N-1\} \right\}$, and for $a \in \mathbb{C}$ and $v \in S^1$ we define the space of ordered configurations of N vortices along the line $a + \mathbb{R} \cdot v$ through Ω to be the N–dimensional submanifold of $\mathcal{F}_N\Omega$ defined by

$$\mathcal{L}_N(a, v) := (\tilde{a} + \Delta_N \cdot v) \cap \Omega^N = \left\{ (a + t_1 v, \ldots, a + t_n v) \in \Omega^N : (t_1, \ldots, t_n) \in \Delta_N \right\},$$

where $\tilde{a} = (a, \ldots, a) \in \mathbb{C}^N$. The symmetric group Σ_N on N symbols acts freely on $\mathcal{F}_N\Omega$ via

$$\Sigma_N \times \mathcal{F}_N\Omega \ni (\sigma, z) \mapsto \sigma * z := (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(N)}) \in \mathcal{F}_N\Omega,$$

hence it is possible to define

$$\mathcal{L}_N^\sigma(a, v) := \sigma^{-1} * \mathcal{L}_N(a, v)$$

as well as

$$\mathcal{L}_N^\sigma\Omega := \bigcup_{(a,v) \in \Omega \times S^1} \mathcal{L}_N^\sigma(a, v)$$

for $\sigma \in \Sigma_N$. Lastly, for $\emptyset \neq C \subset \{1, \ldots, N\}$ it will be useful to define the orthogonal projection π_C by

$$\pi_C : \mathbb{C}^N \ni z \mapsto (z_j)_{j \in C} \in \mathbb{C}^{|C|}.$$

Definition 2.3 (Reflection at $\partial\Omega$)**.** Since Ω is C^3 there is $\varepsilon > 0$ such that the orthogonal projection

$$p : \Omega_\varepsilon := \{ z \in \Omega : \operatorname{dist}(z, \partial\Omega) < \varepsilon \} \to \partial\Omega$$

is a well–defined C^2–map satisfying $|p(z) - z| = \operatorname{dist}(z, \partial\Omega)$. The reflection at $\partial\Omega$ is

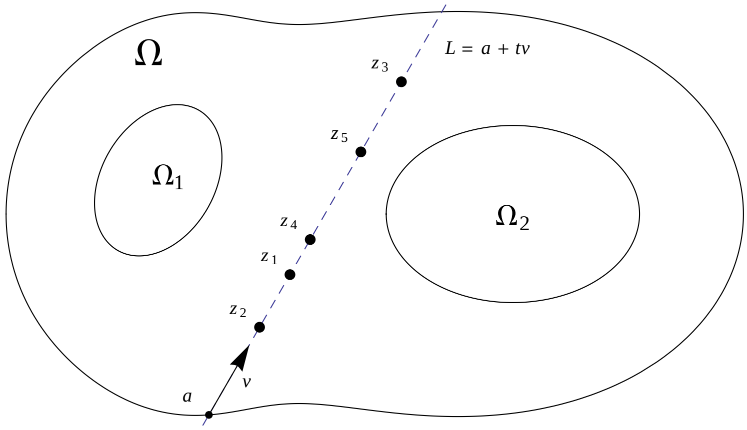

Figure 1: A configuration $z \in \mathcal{L}_5^{(12)(345)}(a, v) \subset \mathcal{F}_5\Omega$.

then defined as the C^2–map[2]

$$\overline{\Omega_\varepsilon} \ni z \mapsto \overline{z} := 2p(z) - z \in \mathbb{C}.$$

Additionally, in what comes we will always abbreviate $d(z) := \mathrm{dist}(z, \partial\Omega)$.

We are now ready to state the general sufficient assumptions on the function G for carrying out our arguments.

Hypothesis 2.4. Let $G : \mathcal{F}_2\Omega \to \mathbb{R}$ satisfy the following hypotheses: G is bounded below by some constant C_0 and has logarithmic singularities on the diagonal in $\Omega \times \Omega$, more precisely, the map $\mathcal{F}_2\Omega \ni (x, y) \mapsto G(x, y) + \frac{1}{2\pi} \ln |x - y| \in \mathbb{R}$ has a continuation $g \in C^1(\Omega^2)$, which is bounded from above by some constant $C_1 > 0$. Thus, we may write

$$G(x, y) = g(x, y) - \frac{1}{2\pi} \ln |x - y|. \tag{2.1}$$

Further, for every $\varepsilon > 0$ there is $C_2 > 0$ depending only on Ω and ε such that

$$|G(x, y)| + |\nabla_x G(x, y)| + |\nabla_y G(x, y)| \leq C_2 \tag{2.2}$$

for every $x, y \in \Omega$ with $|x - y| \geq \varepsilon$. Similarly, there is a constant $C_3 > 0$, also depending only on ε and Ω, such that

$$|\psi(x, y)| + |\nabla_x \psi(x, y)| + |\nabla_y \psi(x, y)| \leq C_3, \tag{2.3}$$

for every $x, y \in \Omega_\varepsilon$, where $\psi(x, y) = g(x, y) - \frac{1}{2\pi} \ln |\bar{x} - y|$ and $x \mapsto \bar{x}$ is reflection at $\partial\Omega$. Further there exists a constant $C_4 > 0$ such that for any line $L = \mathbb{R}v + w \subset \mathbb{C}$ with

[2] Here and in all what follows, when talking about differentiability, we regard $\mathbb{C} = \mathbb{R}^2$, and thus mean differentiability in the real–valued sense. We regard $\Omega \subset \mathbb{C}$ simply because the elegant geometrical properties of complex multiplication allow us to state some things more concisely.

$L \cap \Omega \neq \emptyset$

$$G(w + rv, w + sv) - G(w + rv, w + tv) \geq -C_4. \tag{2.4}$$

for all $r < s < t$, for which the left hand side is defined. Finally, if Ω is invariant under some symmetry group U acting on Ω by linear isometries, the functions g and h respect these symmetries, that is

$$\begin{aligned} g(u \cdot x, v \cdot y) &= g(x, u^{-1}v \cdot y) \\ h(u \cdot x) &= h(x) \end{aligned} \tag{2.5}$$

for any $x, y \in \Omega$, $u, v \in U$.

Definition 2.5. For $\Gamma \in \mathbb{R}^N$ we define the *Kirchhoff–Routh path function* for vortices with vorticities Γ_i, $i \in \{1, \ldots, N\}$ to be the function

$$H_\Omega^\Gamma : \mathcal{F}_N\Omega \ni (z_1, \ldots, z_N) \mapsto \sum_{j=1}^{N} \Gamma_j^2 h(z_j) + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \Gamma_i \Gamma_j G(z_i, z_j) \in \mathbb{R},$$

where the functions G and h satisfy the above hypotheses. If the parameter Γ is understood, we will drop it from the notation, writing H_Ω instead of H_Ω^Γ.

2.2 Preliminary results

Theorem 2.6. *Green's function of the first kind for the Dirichlet Laplacian in Ω satisfies hypothesis 2.4.*

Proof. All of the conditions in 2.4 are either well known properties of the Dirichlet Laplacian, see for example [12] or verified in [4] except for property (2.4), which is a slight sharpening of the result given there.

To see that (2.4) holds assume on the contrary that there is a sequence $L_n = a_n + \mathbb{R}v_n$ of lines with $\Omega \cap L_n \neq \emptyset$ as well as $r_n < s_n < t_n$ such that

$$G(a_n + r_n v_n, a_n + s_n v_n) - G(a_n + r_n v_n, a_n + t_n v_n) \to -\infty \tag{2.6}$$

as $n \to \infty$, where by selecting appropriate subsequences we may take the sequences $(a_n) \subset \overline{\Omega}$ and $(v_n) \subset S^1$ to be convergent to some $a \in \overline{\Omega}$ and $v \in S^1$, respectively.

Now since G is bounded below (2.6) implies

$$G(x_n, z_n) = g(x_n, z_n) - \frac{1}{2\pi} \ln |t_n - r_n| \to \infty,$$

hence $|t_n - r_n| \to 0$, such that $r_n, s_n, t_n \to t$ as $n \to \infty$, since g is bounded from above, and where we abbreviated $x_n := a_n + r_n v_n$, $y_n := a_n + s_n v_n$, $z_n := a_n + t_n v_n$.

If $a + tv \in \Omega$ this leads to a contradiction via

$$G(x_n, y_n) - G(x_n, z_n) \geq C - \frac{1}{2\pi} \ln \frac{s_n - r_n}{t_n - r_n} \geq C - \ln 1 = C,$$

for some constant C, since g is bounded on compact subsets of $\Omega \times \Omega$.

Thus $a + tv \in \partial\Omega$, so if n is large enough, $x_n, y_n, z_n \in \Omega_\varepsilon$, where we may use the approximation

$$G(x_n, y_n) - G(x_n, z_n) = \frac{1}{2\pi} \ln \frac{|\overline{x_n} - y_n|}{|\overline{x_n} - z_n|} \cdot \frac{|x_n - z_n|}{|x_n - y_n|} + O(1)$$

as $n \to \infty$. Considering the differentiable function

$$f : (r_n, \infty) \ni \alpha \mapsto \frac{|\overline{x_n} - a_n - \alpha v_n|^2}{|x_n - a_n - \alpha v_n|^2} \in \mathbb{R}$$

we easily compute

$$\begin{aligned}
f'(\alpha) = 2&\frac{\langle -v_n, \overline{x_n} - a_n - \alpha v_n\rangle |x_n - a_n - \alpha v_n|^2 + \langle v_n, x_n - a_n - \alpha v_n\rangle |\overline{x_n} - a_n - \alpha v_n|^2}{|x_n - a_n - \alpha v_n|^4} \\
&= \frac{(4d_{x_n} \langle v_n, \nu_{x_n}\rangle + \alpha - r_n)(\alpha - r_n)^2 + 2\langle v_n, r_n - \alpha v_n\rangle |\overline{x_n} - a_n - \alpha v_n|^2}{|x_n - a_n - \alpha v_n|^4} \\
&= \frac{2}{|x_n - a_n - \alpha v_n|^4} \Big((2d_{x_n} \langle v_n, \nu_{x_n}\rangle (\alpha - r_n)^2 + (\alpha - r_n)^3 \\
&\quad + (r_n - \alpha)\left(4d_{x_n}^2 + (r_n - \alpha)^2 - 2d_{x_n}(r_n - \alpha)\langle v_n, \nu_{x_n}\rangle\right) \Big) \\
&= \frac{8d_{x_n}^2 (r_n - \alpha)}{|x_n - a_n - \alpha v_n|^4} \leq 0,
\end{aligned}$$

thus f is decreasing, in other words

$$\frac{|\overline{x_n} - y_n|^2}{|x_n - y_n|^2} = f(s_n) \geq f(t_n) = \frac{|\overline{x_n} - z_n|^2}{|x_n - z_n|^2},$$

hence

$$\frac{|\overline{x_n} - y_n|}{|\overline{x_n} - z_n|} \cdot \frac{|x_n - z_n|}{|x_n - y_n|} \geq 1,$$

from which we deduce

$$\begin{aligned}
G(x_n, y_n) - G(x_n, z_n) &= \frac{1}{2\pi} \ln \frac{|\overline{x_n} - y_n|}{|\overline{x_n} - z_n|} \cdot \frac{|x_n - z_n|}{|x_n - y_n|} + O(1) \\
&\geq \ln 1 + O(1) = O(1),
\end{aligned}$$

as $n \to \infty$, which is the desired contradiction. □

Rather than the regular Green's function for the Dirichlet Laplacian, the single most important class of Green's functions G for fluid dynamics is the class of so-called *hydrodynamic Green's functions*, which we will now introduce. An excellent motivation and introduction to the topic of hydrodynamic Green's functions is provided by [11].

Definition 2.7 (Hydrodynamic Green's function)**.** The *hydrodynamic Green's function with periods* $\gamma_0, \dots, \gamma_{k_0} \in \mathbb{R}$, subjected to the condition $\sum_{j=0}^{k_0} \gamma_j = -1$ is the unique solution $G \in C^2(\mathcal{F}_2\overline{\Omega})$ of the problem

$$\begin{cases} -\Delta G(\cdot, y) = \delta_y & \text{for every } y \in \Omega \\ \langle \nabla_x G(x,y), \tau_x \rangle = 0 & \text{for every } y \in \Omega, x \in \partial\Omega \\ \int_{\partial\Omega_j} \langle \nabla_x G(x,y), \nu_x \rangle \, \mathrm{d}s(x) = \gamma_j & \text{for every } j \in \{0, \dots, k_0\} \\ \int_{\partial\Omega} G(x,y) \langle \nabla_x G(x,z), \nu_x \rangle \, \mathrm{d}s(x) = 0 & \text{for every } y, z \in \Omega, \end{cases}$$

where $\partial\Omega_0 = \partial\Omega \setminus \bigcup_{j=1}^{k_0} \partial\Omega_j$.

Using this definition we have the following

Theorem 2.8. *Any hydrodynamic Green's function satisfies hypothesis 2.4, given the prescribed periods are symmetric if* Ω *is symmetric.*

Proof. Nearly all of this follows from the fact that there is a symmetric positive semidefinite matrix $(g^{kl}) \in \mathbb{R}^{k_0 \times k_0}$, such that

$$G(x,y) = G^0(x,y) + \sum_{k,l=1}^{k_0+1} g^{kl} u_{k-1}(x) u_{l-1}(y),$$

where G^0 is the Green's function of the Dirichlet Laplacian in Ω and the u_k are the unique solutions of

$$\begin{cases} \Delta u_k = 0 & \text{in } \Omega \\ u_k = \delta_{kl} & \text{on } \partial\Omega_l, \end{cases}$$

see [11], proposition 7.By assumption 2.1 on $\partial\Omega$ each of the u_k has bounded gradient and is bounded by the maximum principle. Therefore (2.1), (2.2), (2.4) and (2.5) are immediate and so is (2.3), since

$$\psi(x,y) = g(x,y) - \tfrac{1}{2\pi} \ln|\bar{x} - y| = g_0(x,y) - \sum_{k,l} g^{kl} u_k(x) u_l(y) - \tfrac{1}{2\pi} \ln|\bar{x} - y|,$$

in other words $\psi(x,y) = \psi_0(x,y) - \sum_{k,l} g^{kl} u_k(x) u_l(y)$ and we are done. $\square$

Concerning the analysis of the boundary behaviour of H_Ω, the condition (2.3) is of course crucial. The detailed study of boundary collisions will be postponed until chapter 4, but by then we will need a technical lemma, which may be a simple case of some general theorem known to differential geometers. Its proof though is pretty easy, so that we place it here for further reference.

Lemma 2.9. *There is* $\varepsilon > 0$ *such that*

$$\mathrm{D}p(z)v = \frac{1}{1 - \kappa_z d_z} \langle v, \tau_z \rangle \tau_z$$

$$\mathrm{D}\nu_z v = -\frac{\kappa_z}{1-\kappa_z d_z}\left\langle \tau_z, v\right\rangle \tau_z$$

holds for any $z \in \Omega_\varepsilon$, where τ_z is the unit tangent vector to $\partial\Omega$ at $p(z)$, such that the basis (τ_z, ν_z) is positively oriented and κ_z is the curvature of $\partial\Omega$ at $p(z)$ with respect to the induced orientation of $\partial\Omega$.

Proof. Let $\varepsilon > 0$ so that $p : \Omega_\varepsilon \to \partial\Omega$ is well–defined and $\varepsilon < \frac{1}{\max_{z\in\partial\Omega}|\kappa(z)|}$, where κ is the curvature of $\partial\Omega$. Fix $z \in \Omega_\varepsilon$ and let $\gamma : (-\delta, \delta) \to \partial\Omega$ for some $\delta > 0$ be a local parametrisation of $\partial\Omega$ by arc length such that $\gamma(0) = p(z)$ and $(\dot\gamma(0), \nu_z)$ is positively oriented. Then $\ddot\gamma(t) = \kappa_{\gamma(t)}\nu_{\gamma(t)}$, and further setting

$$F : (-\delta, \delta) \times B_\delta(z) \ni (t, x) \mapsto \langle\dot\gamma(t), x - \gamma(t)\rangle \in \mathbb{R},$$

we observe that F is C^1, and since $z - p(z) \perp T_{p(z)}\partial\Omega$: $F(0, z) = 0$, as well as

$$F_t(0, z) = \langle\ddot\gamma(0), z - \gamma(0)\rangle - \langle\dot\gamma(0), \dot\gamma(0)\rangle = d_z\kappa_z - 1$$

by the chain rule. Since $F_t(0, z) \neq 0$ by construction, the implicit function theorem tells us that there is a C^1–map $x \mapsto \theta(x)$ satisfying $F(\theta(x), x) = 0$ with derivative

$$\nabla\theta(z) = -(F_t(0, z))^{-1}F_x(0, p(z)) = \frac{1}{1-\kappa_z d_z}\tau_z.$$

By the chain rule we infer

$$\mathrm{D}p(z)v = \mathrm{D}(\gamma\circ\theta)(z)v = \dot\gamma(0)\left\langle\nabla\theta(z), v\right\rangle = \frac{1}{1-\kappa_z d_z}\left\langle v, \tau_z\right\rangle\tau_z,$$

as well as

$$\mathrm{D}\nu_z v = J\mathrm{D}(\dot\gamma\circ\theta)(z)v = J\ddot\gamma(0)\left\langle\nabla\theta(z), v\right\rangle = J\kappa_z\nu_z\frac{1}{1-\kappa_z d_z}\left\langle\tau_z, v\right\rangle$$

$$= -\frac{\kappa_z}{1-\kappa_z d_z}\left\langle\tau_z, v\right\rangle\tau_z,$$

which are precisely the claimed formulae and where we used the fact that $\nu_z = J\tau_z$, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. □

Lemma 2.10. *There are $\varepsilon > 0$ and constants $C_6, C_7, C_8 > 0$ depending only on Ω such that the inequalities*

$$\max\{d(x) + d(y), C_6|x - y|\} \le |x - \overline{y}| \le |x - y| + 2d(y) \tag{2.7}$$

$$|x - \overline{y}|^2 \ge C_7|p(x) - p(y)|^2 \tag{2.8}$$

$$\big||x - \overline{y}| - |\overline{x} - y|\big|^2 \le C_8(d(x) + d(y))|p(x) - p(y)|^2 \tag{2.9}$$

hold for any $x, y \in \Omega_\varepsilon$.

PROOF. Concerning the first inequality consider the straight line joining x and $\overline{y}$. This line intersects $\partial\Omega$ at some point $z \in \partial\Omega$, which implies

$$|x - \overline{y}| = |x - z| + |z - \overline{y}| \geq d(x) + d(\overline{y}) = d(x) + d(y).$$

The other direction is immediate from the triangle inequality, since

$$|x - \overline{y}| = |x - y + y - \overline{y}| = |x - y + 2d(y)\nu_y| \leq |x - y| + 2d(y),$$

and the other inequalities are verified as (2.1), (2.4) and (2.5) in [4]. $\square$

We are now ready to define a very important class of parameters.

Definition 2.11 ($\mathcal{L}$–admissible parameters)**.** A parameter $\Gamma \in \mathbb{R}^N$ is called $\mathcal{L}$-admissible if there is $\sigma \in \Sigma_N$ such that $\iota(\sigma * \Gamma) \in \overline{\Delta_N}$ or $-\iota(\sigma * \Gamma) \in \overline{\Delta_N}$, where $\iota : \mathbb{R}^N \to \mathbb{R}^N$ is the involution $(x_j)_{j\in\{1,\ldots,N\}} \mapsto \big((-1)^j x_j\big)_{j\in\{1,\ldots,N\}}$ and the closure is to be taken in $(0,\infty)^N$. Similarly, we call Γ strictly $\mathcal{L}$–admissible, if $\iota(\sigma * \Gamma) \in \Delta_N$ or $-\iota(\sigma * \Gamma) \in \Delta_N$.

With this notation, we have the following theorem, which lies on the very foundation of this thesis. Its proof is similar to the one given for the case of axially symmetric Ω in [4] but works out just as well for general $\mathcal{L}$–admissible parameters Γ without any assumptions on symmetry.

Theorem 2.12. *Let Γ be $\mathcal{L}$–admissible with corresponding permutation $\tilde\sigma \in \Sigma_N$ and let $\sigma \in \{\hat\sigma\tilde\sigma, \tilde\sigma\}$, where $\hat\sigma \in \Sigma_N$ is the order–reversing permutation. Then $H_\Omega\big|_{\mathcal{L}_N^\sigma\Omega}$ is bounded above, and fixing a line $L = a + \mathbb{R}v \subset \mathbb{C}$ with $a \in \mathbb{C} \setminus \Omega$, $v \in S^1$ and $\Omega \cap L \neq \emptyset$, we have that*

$$H_\Omega\big|_{\mathcal{L}_N^\sigma(a,v)}(z) \to -\infty \quad \text{as} \quad z \to \partial\mathcal{L}_N^\sigma(a,v),$$

where the boundary of the N–dimensional submanifold $\mathcal{L}_N^\sigma(a,v)$ of $\mathcal{F}_N\Omega$ is to be taken in L^N.

PROOF. Let Γ be $\mathcal{L}$–admissible. Since the change $\Gamma \mapsto -\Gamma$ leaves H_Ω unaffected we may assume without loss of generality that $\Gamma_j = (-1)^j|\Gamma_j|$ and $|\Gamma_{j+1}| \leq |\Gamma_j|$. Thus H_Ω takes the form

$$\begin{aligned} H_\Omega(x) &= \sum_{j=1}^N \Gamma_j^2 h(x_j) + \sum_{\substack{i,j=1 \\ i\neq j}}^N (-1)^{i+j}|\Gamma_i\Gamma_j| G(x_i, x_j) \\ &= \sum_{j=1}^N \Gamma_j^2 h(x_j) + 2\sum_{i=1}^{N-1} G_i(x), \end{aligned}$$

where for $N - i$ even we have

$$G_i(x) = \sum_{k=1}^{\frac{N-i}{2}} |\Gamma_i| \Big(|\Gamma_{i+2k}| G(x_i, x_{i+2k}) - |\Gamma_{i+2k-1}| G(x_i, x_{i+2k-1}) \Big),$$

whereas for $N-i$ odd

$$G_i(x) = \sum_{k=1}^{\frac{N-i-1}{2}} |\Gamma_i| \Big(|\Gamma_{i+2k}| G(x_i, x_{i+2k}) - |\Gamma_{i+2k-1}| G(x_i, x_{i+2k-1}) \Big) - |\Gamma_i \Gamma_N| G(x_i, x_N)$$

We now are able to infer from hypothesis (2.4) that for any line $L = \{a + tv : t \in \mathbb{R}\}$ with $a \in \partial\Omega$, $v \in S^1$, and such that $\Omega \cap L \neq \emptyset$

$$G(a + rv, a + sv) - G(a + rv, a + tv) \geq -C_4$$

for all $r < s < t$ for which the left hand side is defined, so combining this result with the condition $|\Gamma_{i-1}| \geq |\Gamma_i|$ and the fact that $G \geq C_0$ we get for a $x \in \mathcal{L}_N(a, v)$ and $N - i$ even, that $G_i(x)$ is equal to

$$\sum_{k=1}^{\frac{N-i}{2}} |\Gamma_i| \Bigg[|\Gamma_{i+2k}| \Big(\underbrace{G(x_i, x_{i+2k}) - G(x_i, x_{i+2k-1})}_{\leq C_4} \Big) + \big(\underbrace{|\Gamma_{i+2k}| - |\Gamma_{i+2k-1}|}_{\leq 0} \big) G(x_i, x_{i+2k-1}) \Bigg]$$

$$\leq \sum_{k=1}^{\frac{N-i}{2}} \Big(|\Gamma_i \Gamma_{i+2k}| C_4 + |\Gamma_i| \left(|\Gamma_{i+2k}| - |\Gamma_{i+2k-1}| \right) C_0 \Big),$$

whereas analogously for $N - i$ odd

$$G_i(x) \leq \sum_{k=1}^{\frac{N-i-1}{2}} \Big(|\Gamma_i \Gamma_{i+2k}| C_4 + |\Gamma_i| \left(|\Gamma_{i+2k}| - |\Gamma_{i+2k-1}| \right) C_0 \Big) - \underbrace{|\Gamma_i \Gamma_N| G(x_i, x_N)}_{\geq |\Gamma_i \Gamma_N| C_0}.$$

Since by hypothesis 2.4 h is bounded from above by C_1, this gives the required upper bound.

Now fixing a and v, every $z \in \mathcal{L}_N(a, v)$ has a unique representation $z = \tilde{a} + tv$ with $t \in \Delta_N$, where $\tilde{a} := (a, \ldots, a) \in \mathbb{C}^N$. Setting

$$\mathcal{R} := \left\{ t \in \Delta^N : \tilde{a} + tv \in \mathcal{F}_N\Omega \right\}$$

as well as

$$E : \mathcal{R} \ni t \mapsto H_\Omega(\tilde{a} + tv) \in \mathbb{R},$$

we have to show that $E(t) \to -\infty$ as $\mathrm{dist}(t, \partial\mathcal{R}) \to 0$. Therefore consider a sequence $(t^n) \subset \mathcal{R}$ with the property that $t^n \to \partial\mathcal{R}$ as $n \to \infty$. Let us first consider the case that $d(a + t_k^n v) \to 0$ as $n \to \infty$ for some $k \in \{1, \ldots, N\}$. Since $\sum_{i,j=1, i\neq j}^{N} (-1)^{i+j} |\Gamma_i \Gamma_j| G(a + t_i^n v, a + t_j^n v)$ is bounded from above as $n \to \infty$ and $h(a + t_k^n v) \to -\infty$ for $n \to \infty$ we infer that indeed $E(t^n) \to -\infty$ as claimed.

Hence we may assume that

$$\liminf_{n\to\infty} d(a + t_j^n v) > 0 \tag{2.10}$$

for all $j \in \{1, \dots, N\}$ and that

$$t_{k+1}^n - t_k^n \to 0 \quad \text{as } n \to \infty \tag{2.11}$$

for some $k \in \{1, \dots, N-1\}$. By assumption (2.10) the first two sums in

$$E(t^n) = \sum_{j=1}^N \Gamma_j^2 h(a + t_j^n v) + \sum_{\substack{i,j=1 \\ i \neq j}}^N \Gamma_i \Gamma_j g(a + t_i^n v, a + t_j^n v) - \sum_{\substack{i,j=1 \\ i \neq j}}^N (-1)^{i+j} \frac{|\Gamma_i \Gamma_j|}{2\pi} \ln |t_i^n - t_j^n|$$

remain bounded as $n \to \infty$. We then expand

$$\sum_{\substack{i,j=1 \\ i \neq j}}^N (-1)^{i+j} |\Gamma_i \Gamma_j| \ln |t_i^n - t_j^n| = 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^N (-1)^{i+j} |\Gamma_i \Gamma_j| \ln |t_i^n - t_j^n| = 2 \sum_{i=1}^{N-1} |\Gamma_i| \ln \psi_i(t),$$

where for $N - i$ even

$$\psi_i(t) = \prod_{j=1}^{\frac{N-i}{2}} \frac{|t_{i+2j}^n - t_i^n|^{|\Gamma_{i+2j}|}}{|t_{i+2j-1}^n - t_i^n|^{|\Gamma_{i+2j-1}|}} \geq \prod_{j=1}^{\frac{N-i}{2}} |t_{i+2j-1}^n - t_i^n|^{|\Gamma_{i+2j}| - |\Gamma_{i+2j-1}|} \geq C$$

and for $N - i$ odd

$$\psi_i(t) = \frac{1}{|t_N^n - t_i^n|^{|\Gamma_N|}} \prod_{j=1}^{\frac{N-i-1}{2}} \frac{|t_{i+2j}^n - t_i^n|^{|\Gamma_{i+2j}|}}{|t_{i+2j-1}^n - t_i^n|^{|\Gamma_{i+2j-1}|}} \geq \frac{C}{|t_N^n - t_i^n|^{|\Gamma_N|}}$$

for some constant $C > 0$, since $|\Gamma_{i+2k}| - |\Gamma_{i+2k-1}| \leq 0$. It thus remains to show that $\psi_k(t^n) \to \infty$ for $n \to \infty$ for some $k \in \{1, \dots, N-1\}$. Let k be maximal satisfying (2.11). If $k = N - 1$ we infer $\psi_{N-1}(t^n) \to \infty$ for $n \to \infty$ and the proof is done. Otherwise there is $\delta > 0$ such that

$$\delta \leq |t_{j+1}^n - t_j^n| \leq \frac{1}{\delta} \quad \text{for all } j > k \tag{2.12}$$

and n sufficiently large. Therefore, if $N - k$ is even,

$$\psi_k(t) = \frac{|t_{k+2}^n - t_k^n|^{|\Gamma_{k+2}|}}{|t_{k+1}^n - t_k^n|^{|\Gamma_{k+1}|}} \prod_{j=2}^{\frac{N-k}{2}} \frac{|t_{k+2j}^n - t_k^n|^{|\Gamma_{k+2j}|}}{|t_{k+2j-1}^n - t_k^n|^{|\Gamma_{k+2j-1}|}} \geq \frac{\tilde{C} \cdot \delta^{|\Gamma_{k+2}|}}{|t_{k+1}^n - t_k^n|^{|\Gamma_{k+1}|}} \to \infty$$

as $n \to \infty$ by (2.11) and (2.12), whereas for $N - k$ odd

$$\psi_k(t) = \frac{|t_{k+2}^n - t_k^n|^{|\Gamma_{k+2}|}}{|t_{k+1}^n - t_k^n|^{|\Gamma_{k+1}|}} \cdot \frac{1}{|t_N^n - t_k^n|^{|\Gamma_N|}} \prod_{j=2}^{\frac{N-k}{2}} \frac{|t_{k+2j}^n - t_k^n|^{|\Gamma_{k+2j}|}}{|t_{k+2j-1}^n - t_k^n|^{|\Gamma_{k+2j-1}|}}$$

$$\geq \frac{\tilde{C} \cdot \delta^{|\Gamma_{k+2}|+1}}{|t_{k+1}^n - t_k^n|^{|\Gamma_{k+1}|}} \to \infty \quad \text{as } n \to \infty,$$

again by (2.11) and (2.12) and the proof is done. □

2.3 A general deformation argument

For simplicity in stating our results, we find the following definition useful:

Definition 2.13 (φ-complete deformations)**.** Let X be a topological space and let φ be a flow on $\mathcal{F}_N\Omega$. We call a family $\beta \subset [\alpha] \in [X, \mathcal{F}_N\Omega]$ of homotopic maps φ-complete, if for any $\alpha \in \beta$ and any continuous map $T : \alpha(X) \to [0, \infty)$ such that $T(x) \in [0, t^+(x))$ for all $x \in \alpha(X)$ the map

$$X \ni x \mapsto \varphi(T(\alpha(x)), \alpha(x)) \in \mathcal{F}_N\Omega$$

is in β.

Denoting the gradient flow of H_Ω by

$$\varphi : \bigcup_{z \in \mathcal{F}_N\Omega} (t^-(z), t^+(z)) \times \{z\} \to \mathcal{F}_N\Omega,$$

in the sequel, we will frequently use the following

Lemma 2.14 (general deformation argument)**.** *Suppose there is a subset $\mathcal{L} \subset \mathcal{F}_N\Omega$, such that H_Ω is bounded above on $\mathcal{L}$ by σ, that is*

$$\sup H_\Omega\big|_{\mathcal{L}} = \sigma < \infty. \tag{2.13}$$

Further let X be a topological space, $\beta \subset [\alpha] \in [X, \mathcal{F}_N\Omega]$ be φ-complete, and such that for any representative $\alpha \in \beta$ the intersection

$$\alpha(X) \cap \mathcal{L} \neq \emptyset$$

is nonempty. Then, fixing some representative α_0, there is $x \in \alpha_0(X)$, such that

$$\lim_{t \to t^+(x)} H_\Omega(\varphi(t, x)) < \infty.$$

Proof. Assume not. Then for every $x \in \alpha_0(X)$ there is a minimal $T(x) \in [0, t^+(x))$ such that

$$H_\Omega(\varphi(T(x), x)) = \sigma + 1.$$

Since for each $x \in \alpha_0(X)$ $H_\Omega \circ \psi(\cdot, x)$ is strictly increasing (otherwise we are done), the map

$$T : \alpha_0(X) \ni x \mapsto T(x) \in \mathbb{R}$$

is continuous. It follows, that unambiguously defining

$$\alpha_t : X \ni \xi \mapsto \varphi\left(tT(\alpha_0(\xi)), \alpha_0(\xi)\right) \in \mathcal{F}_N\Omega,$$

where $t \in [0,1]$: $\alpha_0 \simeq \alpha_1$ and $\alpha_1 \in \beta$, since β is φ-complete, hence there is $\xi \in X$ with $\alpha_1(\xi) \in \mathcal{L}$, but $H_\Omega(\alpha_1(\xi)) = \sigma + 1$, in contradiction with (2.13), and we are done. $\square$

3 The symmetric case

3.1 Developing a convenient language

This chapter is devoted to develop the correct notion of symmetry for our problem as well as to devise some useful definitions, which enable us to state our results as concisely as possible, while remaining also in the most general form. The techniques used in the proofs are as basic as is the intuition behind the results, which is quickly explained using pictures rather than words, nonetheless, without devising some appropriate language, the notation will quickly become cluttered.

To get well beyond this, we will supplement the more abstract sounding theorems by more concrete examples later on.

When speaking of stationary solutions to the N–vortex problem, to a physicist, vortices with equal strengths are indistinguishable. To satisfy this intuition, we thus shall work in the quotient space $\mathcal{F}_N\Omega/\Sigma$ of $\mathcal{F}_N\Omega$ under the group action of some appropriate subgroup $\Sigma \leq \Sigma_N$. When some symmetry of Ω, represented by a symmetry group G comes into play, we are thus really interested in the quotient space $(\mathcal{F}_N\Omega/\Sigma)/G$, which is not the nicest space to work with, since in the interesting cases, the action of G on $\mathcal{F}_N\Omega/\Sigma$ is, of course, not free.

Since all calculations take place in (particularly nice) subsets of $\mathcal{F}_N\Omega$ anyway, we thus might as well develop our concept of symmetry for the N–vortex problem solely working on this space.

Definition 3.1 (Symmetric points)**.** As stated in definition 2.2, the symmetric group Σ_N on N symbols acts on $\mathbb{R}^N$ and $\mathcal{F}_N\Omega \subset \mathbb{C}^N$ by permutation of coordinates:

$$\sigma * z = \left(z_{\sigma^{-1}(1)}, \dots, z_{\sigma^{-1}(j)}\right)$$

Further, if Ω is D_p–invariant, the dihedral group D_p also acts diagonally on $\mathcal{F}_N\Omega$ by linear isometries of Ω, in fact, without loss of generality we may take the dihedral group D_p as $\langle g_0, s_0\rangle$, wherein the rotation element g_0 acts on Ω by multiplication with $e^{\frac{2\pi i}{p}}$ and s_0 is complex conjugation.

Let $\Sigma(\Gamma) := (\Sigma_N)_\Gamma$ be the stabilizer of $\Gamma \in \mathbb{R}^N$, and let $U \leq D_p$ be a subgroup. We say that $z \in \mathcal{F}_N\Omega$ is an U–invariant (or U–symmetric) point for Γ, if for any $u \in U$ there is $\sigma \in \Sigma(\Gamma)$ such that

$$u \cdot z = \sigma * z. \tag{3.1}$$

We denote the set of U–symmetric points for Γ by

$$\mathfrak{S}^U_\Gamma\Omega := \{z \in \mathcal{F}_N\Omega :\, \forall u \in U\, \exists\, \sigma \in \Sigma(\Gamma) : u \cdot z = \sigma * z\},$$

and denote by

$$\widehat{\mathfrak{S}}^U_\Gamma\Omega := \left\{z \in \mathfrak{S}^U_\Gamma\Omega :\, \exists\, k \in \{1, \dots, N\} : z_k = 0\right\}.$$

the set of all U–symmetric points for Γ where one vortex is placed in the symmetry center of Ω.

For the discussions to come, we want to restrict H_Ω to certain subsets $\mathfrak{S}^U_\Gamma\Omega$ and $\widehat{\mathfrak{S}}^U_\Gamma\Omega$, respectively, so we want to introduce some useful notation for that.

Lemma 3.2. *The element σ in* (3.1) *is uniquely determined, in fact, we have a locally constant map*

$$\tau : \mathfrak{S}^U_\Gamma\Omega \ni z \mapsto \tau_z \in \mathrm{Hom}(U, \Sigma(\Gamma)),$$

such that

$$u \cdot z = \tau_z(u)^{-1} * z \tag{3.2}$$

for all $u \in U$, $z \in \mathfrak{S}^U_\Gamma\Omega$.[3]

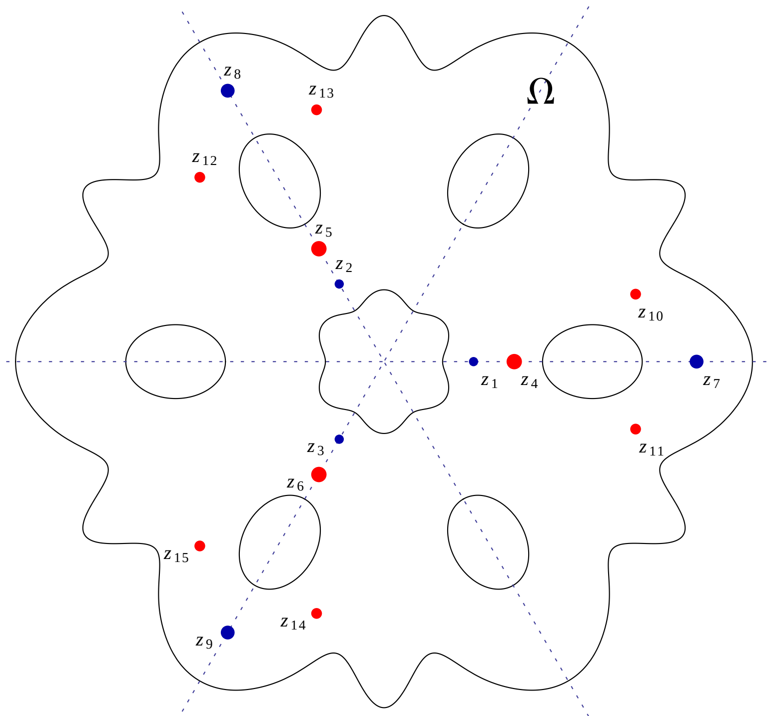

Figure 2: A D_3–symmetric configuration $z \in \mathfrak{S}^{D_3}_\Gamma\Omega \subset \mathcal{F}_{15}\Omega$ for a D_6–symmetric domain Ω satisfying $\tau_z(g_0^2) = (321)(654)(987)(14\,12\,10)(15\,13\,11)$ as well as $\tau_z(s_0) = (10\,11)(23)(56)(89)(13\,14)(12\,15)$

Proof. Let $z \in \mathfrak{S}^U_N\Omega$, $u \in U$, and let $\sigma, \sigma' \in \Sigma(\Gamma)$ satisfy $u \cdot z = \sigma * z = \sigma' * z$, that is $z_{\sigma^{-1}(j)} = z_{(\sigma')^{-1}(j)}$ for every $j \in \{1, \ldots, N\}$. Since $z \in \mathcal{F}_N\Omega$ and the group action of $\Sigma(\Gamma)$ is free on $\mathcal{F}_N\Omega$, this is only possible if $\sigma = \sigma'$, hence there is a map $\tau_z : U \to \Sigma(\Gamma)$

[3]Outside this section the letter τ is reserved for the positively oriented tangent vector field to $\partial\Omega$, which is not needed in this section, so we want this notation to hold only in this section.

satisfying (3.2). This map is clearly a homomorphism, because

$$\tau_z(u\cdot v)^{-1} * z = (u\cdot v)\cdot z = u\cdot(v\cdot z) = u\cdot(\tau_z(v)^{-1} * z),$$

and since U acts diagonally on $\mathcal{F}_N\Omega$, both the U–action and the $\Sigma(\Gamma)$–action commute, therefore this is equal to

$$= \tau_z(v)^{-1} * (u\cdot z) = \tau_z(v)^{-1} * (\tau_z(u)^{-1} * z) = (\tau_z(v)^{-1}\cdot\tau_z(u)^{-1}) * z,$$

hence

$$\tau_z(u\cdot v)^{-1} = \tau_z(v)^{-1}\cdot\tau_z(u)^{-1} = (\tau_z(u)\cdot\tau_z(v))^{-1},$$

which is what we were to show.

Concerning the local constancy of the map τ, we have for $z\in\mathfrak{S}_\Gamma^U\Omega$, $u\in U$:

$$\begin{aligned}|\tau_z(u)^{-1}z-\tau_{z'}(u)^{-1}z| &\le |\tau_z(u)^{-1}z-\tau_{z'}(u)^{-1}z'|+|\tau_{z'}(u)^{-1}z'+\tau_{z'}(u)^{-1}z|\\ &= |u\cdot z-u\cdot z'|+|\tau_{z'}(u)^{-1}(z-z')|\le(1+|\tau_{z'}^{-1}(u)|)|z-z'|,\end{aligned}$$

where we identify $\tau_{z'}(u)$ with the matrix $A\in\mathrm{GL}_N(\mathbb{C})$ representing it and $|\cdot|$ denotes the induced matrix norm. Since $\Sigma(\Gamma)z\subset\mathcal{F}_N\Omega$ is discrete and $\Sigma(\Gamma)$ acts freely on $\mathcal{F}_N\Omega$ this means $\tau_z(u)=\tau_{z'}(u)$ for z' close enough to z and the proof is done. □

Definition 3.3. Throughout this section we select an arbitrary fundamental domain $J=J_z$ for the U–action on $\{1,\dots,N\}$ defined by

$$U\times\{1,\dots,N\}\ni(u,j)\mapsto u\cdot j:=\tau_z(u)(j)\in\{1,\dots,N\},$$

that is a subset $J\subset\{1,\dots,N\}$ such that the map

$$J\ni j\mapsto U\cdot j\in\{1,\dots,N\}/U$$

is bijective.

Proof. This is really a U–action since

$$(uv)\cdot j=\tau_z(uv)(j)=(\tau_z(u)\tau_z(v))(j)=\tau_z(u)\left(\tau_z(v)(j)\right)=u\cdot(v\cdot j).$$

□

Remark (Simplifying notation). The U–action on $\{1,\dots,N\}$ and hence its fundamental domain J of course depend heavily on $z\in\mathfrak{S}_\Gamma^U\Omega$. However, the proofs to come only employ the gradient flow of H_Ω, so hypothesis (2.5) implies that once we chose an initial value $z_0\in\mathfrak{S}_\Gamma^U\Omega$, we may choose the same fundamental domain J for every z^t, $t\in(0,t^+(z_0))$, since $\Sigma(\Gamma)$ acts on $\mathcal{F}_N\Omega$ by isometries, too, and the gradient flow φ of H_Ω is therefore equivariant with respect to those transformations, that is for $z_0\in\mathcal{F}_N\Omega$, $g\in D_p$ and $\sigma\in\Sigma(\Gamma)$ satisfying $g\cdot z_0=\sigma * z_0$ and $t\in(t^-(z),t^+(z))$ we have

$(t^-(z_0), t^+(z_0)) = (t^-(g \cdot z_0), t^+(g \cdot z_0)) = (t^-(\sigma * z_0), t^+(\sigma * z_0))$ and

$$g \cdot \varphi_t(z_0) = \varphi_t(g \cdot z_0) = \varphi_t(\sigma * z_0) = \sigma * \varphi_t(z_0), \tag{3.3}$$

hence the suppression of the z–dependence of J in our notation is justified, thereby considerably increasing readability.

As mentioned before, the different group actions of $\Sigma(\Gamma)$ and U on $\mathcal{F}_N$ may be represented in terms of commuting linear isometries $\mathbb{C}^N \to \mathbb{C}^N$. It is therefore safe to henceforth denote both group actions simply by juxtaposition.

We now continue by developing a formula for H_Ω which takes all its invariance properties into account.

Lemma 3.4. *Let $U \leq G$ be a rotation subgroup, $z \in \mathfrak{S}_\Gamma^U \Omega$ and $J = J_z$ as above. Then*

$$H_\Omega(z) = |U| \left(\sum_{j \in J} \Gamma_j^2 h(z_j) + \sum_{\substack{i,j \in J \\ i \neq j}} \Gamma_i \Gamma_j G(z_i, z_j) + \sum_{u \in U \setminus \{\mathrm{id}\}} \sum_{i,j \in J} \Gamma_i \Gamma_j G(z_i, u z_j) \right), \tag{3.4}$$

if $z_j \neq 0$ for every $j \in J$ as well as

$$\begin{aligned} H_\Omega(z) = \Gamma_n^2 h(0) + |U| \Bigg(&\sum_{j \in J \setminus \{n\}} \Gamma_j^2 h(z_j) + \sum_{\substack{i,j \in J \\ i \neq j}} \sum_{u \in U} \Gamma_i \Gamma_j G(z_i, u z_j) \\ &+ \sum_{j \in J \setminus \{n\}} \sum_{u \in U \setminus \{\mathrm{id}\}} \Gamma_j^2 G(z_j, u z_j) \Bigg), \end{aligned} \tag{3.5}$$

if $z_n = 0$.

Proof.

$$H_\Omega(z) = \sum_{j=1}^N \Gamma_j^2 h(z_j) + \sum_{\substack{i,j=1 \\ i \neq j}}^N \Gamma_i \Gamma_j G(z_i, z_j) = \sum_{j \in J} \sum_{k \in U \cdot j} \Gamma_k^2 h(z_k) + \sum_{\substack{i,j \in J \\ k \in U \cdot i,\, l \in U \cdot j \\ k \neq l}} \Gamma_k \Gamma_l G(z_k, z_l)$$

$$= \sum_{j \in J} \sum_{k \in U \cdot j} \Gamma_k^2 h(z_k) + \sum_{\substack{i,j \in J \\ k \in U \cdot i,\, l \in U \cdot j \\ i \neq j}} \Gamma_k \Gamma_l G(z_k, z_l) + \sum_{\substack{j \in J \\ k,l \in U \cdot j \\ k \neq l}} \Gamma_k \Gamma_l G(z_k, z_l)$$

Now if $k \in U \cdot j$ we have $k = u \cdot j = \tau_z(u)(j)$ for some $u \in U$, so we have $\Gamma_k = \Gamma_j$ by the definition of $\Sigma(\Gamma)$ as well as $z_k = z_{\tau_z(u)(j)} = \tau_z(u)^{-1} z_j = u \cdot z_j$, so $h(z_k) = h(z_j)$ by condition (2.5), and we are left with

$$H_\Omega(z) = \sum_{j\in J} |U\cdot j|\Gamma_j^2 h(z_j) + \sum_{\substack{i,j\in J\\ i\neq j}} \sum_{u,v\in U} \frac{1}{|U_{u\cdot i}||U_{v\cdot j}|}\Gamma_i\Gamma_j G(uz_i, vz_j) + \sum_{\substack{j\in J\\ k,l\in U\cdot j\\ k\neq l}} \Gamma_j^2 G(z_k, z_l)$$

$$= \sum_{j\in J} |U\cdot j|\Gamma_j^2 h(z_j) + \sum_{\substack{i,j\in J\\ i\neq j}} \sum_{u,v\in U} \frac{1}{|U_i||U_j|}\Gamma_i\Gamma_j G(z_i, u^{-1}vz_j) + \sum_{\substack{j\in J\\ k,l\in U\cdot j\\ k\neq l}} \Gamma_j^2 G(z_k, z_l),$$

since the index $k = u\cdot j$ is achieved precisely $|U_k| = |U_{u\cdot j}|$ times as u ranges through U and $|U_{u\cdot j}| = |U_j|$, since $U_{u\cdot j}$ and U_j are conjugate subgroups of U.

Since U is a rotation subgroup of G, we have $U_j = \{\mathrm{id}\}$ if $z_j \neq 0$ and $U_j = U$ otherwise, hence in these cases $|U\cdot j| = |U|$ and $|U\cdot j| = 1$, respectively. Now if $z_j \neq 0$ for every $j\in J$ we therefore obtain

$$H_\Omega(z) = |U|\sum_{j\in J} \Gamma_j^2 h(z_j) + \sum_{\substack{i,j\in J\\ i\neq j}} \sum_{u,v\in U} \Gamma_i\Gamma_j G(z_i, u^{-1}vz_j) + \sum_{\substack{j\in J\\ k,l\in U\cdot j\\ k\neq l}} \Gamma_j^2 G(z_k, z_l)$$

$$= |U|\left(\sum_{j\in J} \Gamma_j^2 h(z_j) + \sum_{\substack{i,j\in J\\ i\neq j}} \Gamma_i\Gamma_j G(z_i, z_j) + \sum_{u\in U\setminus\{\mathrm{id}\}} \sum_{i,j\in J} \Gamma_i\Gamma_j G(z_i, uz_j)\right),$$

since $k = u\cdot j, l = v\cdot j \in U\cdot j$ are equal if and only if $z_k = z_l$, therefore if $u^{-1}v = \mathrm{id}$, which has precisely $|U|$ solutions, and, more generally the equation $w = u^{-1}v$ has precisely $|U|$ solutions for $w\in U$ fixed.

If, on the other hand, $z_n = 0$ for some $n\in J$, we obtain

$$\begin{aligned} H_\Omega(z) = \Gamma_n^2 h(0) + |U|&\left(\sum_{j\in J\setminus\{n\}} \Gamma_j^2 h(z_j) + \sum_{\substack{i,j\in J\setminus\{n\}\\ i\neq j}} \sum_{u\in U} \Gamma_i\Gamma_j G(z_i, uz_j)\right) \\ &+2\sum_{j\in J\setminus\{n\}} \sum_{u,v\in U} \frac{\Gamma_j\Gamma_n}{|U|} G(z_j, 0) + |U| \sum_{j\in J\setminus\{n\}} \sum_{u\in U\setminus\{\mathrm{id}\}} \Gamma_j^2 G(z_j, uz_j) \\ = \Gamma_n^2 h(0) + |U|&\Bigg(\sum_{j\in J\setminus\{n\}} \Gamma_j^2 h(z_j) + \sum_{\substack{i,j\in J\\ i\neq j}} \sum_{u\in U} \Gamma_i\Gamma_j G(z_i, uz_j) \\ &\qquad + \sum_{j\in J\setminus\{n\}} \sum_{u\in U\setminus\{\mathrm{id}\}} \Gamma_j^2 G(z_j, uz_j)\Bigg), \end{aligned}$$

and the proof is done. ☐

3.2 Statement of results

In this section we state our results on critical points of H_Ω for symmetric domains Ω in general form. The next section is concerned with the proofs of the theorems given here, and in section 3.5 we give some important examples and discuss the results further.

The most complicated part of the theorems given here is not the existence of critical points, this is achieved by elementary means by showing that H_Ω restricted to certain symmetric submanifolds of $\mathcal{F}_N\Omega$ assumes a local maximum, but to state a corresponding multiplicity result. In order to do so, we find the following definition useful.

Definition 3.5. The group $\Pi_r := \Sigma_r \times \mathbb{Z}_2^r$ acts on an arbitrary set M of r–dimensional arrays of vectors via

$$\Pi_r \times M \ni ((\sigma, v), (z_1, \dots, z_r)) \mapsto (\sigma, v) \diamond (z_1, \dots, z_r) := \sigma * (v_1 \odot z_1, \dots, v_r \odot z_r) \in M,$$

where $1 \odot z$ is the result of reversing the order of components of z and Σ_r acts on M by permutation of components as usual.

Theorem 3.6. *Suppose Ω has dihedral symmetry of order p and that $0 \notin \Omega$. Then for any rotation subgroup $U \le D_p$ such that there is a $z \in \mathfrak{S}_\Gamma^U\Omega \neq \emptyset$ the Kirchhoff-Routh–path function H_Ω has at least $N_1(\Gamma, \Omega, U)$ distinct U–symmetric critical points, whose components lie on the symmetry axes of the $D_{|U|}$–action on Ω, that is, on*

$$\mathcal{S}_U := \mathbb{R} \cdot \left\{ e^{\frac{2\pi i k}{|U|}} : k \in \{0, \dots, |U| - 1\} \right\} \cap \Omega.$$

Here $N_1(\Gamma, \Omega, U)$ is given by

$$\sum_{r=1}^{l} \binom{l}{r} \cdot \left| \Pi_r \diamond \left\{ (\sigma_C \pi_C \Gamma)_{C \in \mathcal{P}} : \begin{array}{c} \mathcal{P} \text{ partition of } J, |\mathcal{P}| = r, \forall C \in \mathcal{P} : \pi_C\Gamma \\ \text{is } \mathcal{L}\text{–admissible with permutation } \sigma_C \end{array} \right\} \right|,$$

where $J = J_z$ is a fundamental domain for the U–action on $\{1, \dots, N\}$ and l is the number of connected components of $[0, \infty) \cap \Omega$ if $|U|$ is even and of $\mathbb{R} \cap \Omega$ if $|U|$ is odd.

If $|U| = 1$, that is, Ω is only axially symmetric, the condition $0 \notin \Omega$ may be dropped.

Corollary 3.7. *The consequence of the above theorem also holds if one replaces the condition "$0 \notin \Omega$" by "$[0, \infty) \cap \Omega$ is disconnected". If then additionally $0 \in \Omega$, the multiplicity result is more complicated: If $\widehat{\mathfrak{S}}_\Gamma^U\Omega = \emptyset$, one has to replace l in $N_1(\Gamma, \Omega, U)$ by $l-1$, placing all vortices away from the middle. On the other hand, if $\widehat{\mathfrak{S}}_\Gamma^U\Omega \neq \emptyset$, there additionally is the possibility to place a single vortex into the component of $[0, \infty) \cap \Omega$ containing the symmetry center. We are then in the situation of theorem 3.9, and the multiplicity result there holds.*

We are also able to give a positive result if the hole at the symmetry center of Ω is replaced by a sufficiently strong vortex in the symmetry center. In order to do this, we need one additional definition.

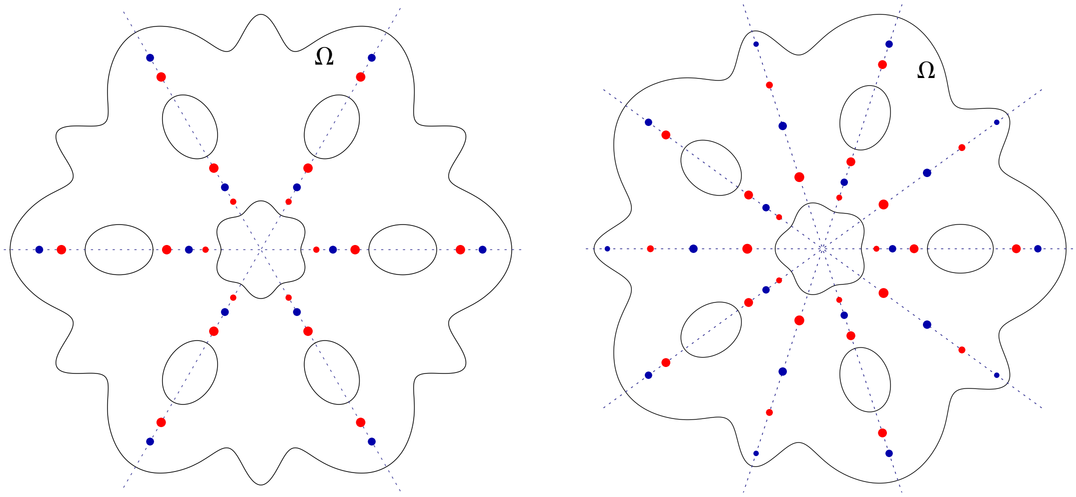

Figure 3: Symmetric critical points of H_Ω according to theorem 3.6 for $|U|$ even and $|U|$ odd, respectively.

Definition 3.8 (Center–admissibility)**.** A strictly $\mathcal{L}$–admissible parameter $\Gamma \in \mathbb{R}^r$ is called center–admissible for a rotation subgroup $U \leq D_p$ with $|U| > 1$ if

$$|\Gamma_{\sigma^{-1}(r)}| \geq (|U|-1) \max_{j\in\{1,\dots,r-2\}} \left\{ \frac{\Gamma^2_{\sigma^{-1}(j+1)} + \Gamma^2_{\sigma^{-1}(j)}}{|\Gamma_{\sigma^{-1}(j+1)}| - |\Gamma_{\sigma^{-1}(j)}|}, |\Gamma_{\sigma^{-1}(j)}| \right\}$$

for $r \geq 3$, where $\sigma \in \Sigma_r$ is such that $-\iota\sigma\Gamma \in \Delta_r$ or $\iota\sigma\Gamma \in \Delta_r$ as in definition 2.11. If $r \in \{1,2\}$ the above condition is empty, in this case we call Γ center–admissible if it is $\mathcal{L}$–admissible.

Theorem 3.9. *Suppose Ω is D_p–symmetric with $0 \in \Omega$. Then for any rotation subgroup $U \leq D_p$ such that there is $z \in \widehat{\mathfrak{S}}^U_\Gamma\Omega \neq \emptyset$, the Kirchhoff–Routh path function H_Ω has at least $N_2(\Gamma, \Omega, U)$ critical points, whose components lie on the symmetry axes $\mathcal{S}_U$ as before and where one of the vortices is located in the symmetry center of Ω. Here, the number $N_2(\Gamma, \Omega, U)$ is given by*

$$\sum_{r=0}^{l-1} \binom{l-1}{r} \left| \bigcup_{\substack{C^*\subset J \\ \pi_{C^*}\Gamma \text{ is c.-a.}}} \{\sigma_{C^*}\pi_{C^*}\Gamma\} \times \left(\Pi_r \diamond \left\{ (\sigma_C\pi_C\Gamma)_{C\in\mathcal{P}} : \begin{array}{c} \mathcal{P} \text{ partition of} \\ J\setminus C^*, |\mathcal{P}| = r, \\ \forall C \in \mathcal{P} : \pi_C\Gamma \text{ is } \mathcal{L}\text{–a.} \end{array} \right\} \right) \right|,$$

where $J = J_z$ is as before, l is the number of connected components of $[0,\infty) \cap \Omega$ and "$\mathcal{L}$–a." and "c.–a." stand for "$\mathcal{L}$–admissible" and "center–admissible", respectively.

Despite their complex statement, proof and concept of these theorems are fairly simple. The proof is constructive in every case: One places certain vortices U–symmetrically onto

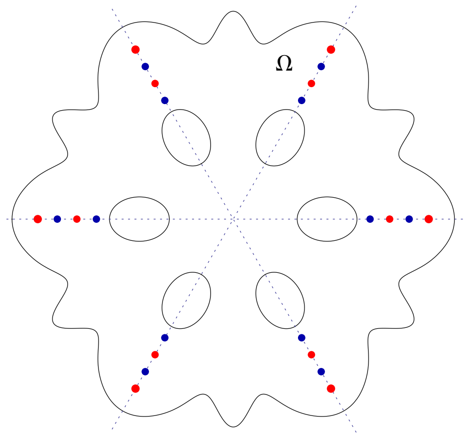

Figure 4: The situation of corollary 3.7.

the symmetry axes of Ω, such that the vorticities are increasing or decreasing along the symmetry axes by modulus and have alternating signs, and this pattern may be broken between different connected components of the symmetry axes. The multiplicity result is then obtained by rigorously counting all the different possibilities one has for doing so.

The proof will be relatively descriptive, and we will discuss simpler cases and examples of this theorem later on.

Of course it is theorem 2.12 concerning the behaviour of the Kirchhoff–Routh path function H_Ω for vortices lying on a line together with a form of the principle of symmetric criticality which lies at the heart of all of this (and of almost all of the results to come).

3.3 Proof of theorem 3.6

This section is devoted to the proofs of theorems 3.6 and its corollaries. We will explicitly construct a starting value $z_0 \in \mathfrak{S}^U_\Gamma\Omega$ which will converge to a critical point $z^* \in \mathfrak{S}^U_\Gamma\Omega$ of H_Ω under the gradient flow φ, while meticulously keeping track of all the choices we could have made, thereby proving the multiplicity result.

We start out noting that in order to prove theorem 3.6 we necessarily have $N = k \cdot |U|$ if there shall be any chance of having an U-symmetric point at all. Further, we may then assume without loss of generality that the fundamental domain $J = J_{z_0}$ is equal to $\{1, \dots, k\}$ and that our vortices are sorted appropriately such that their corresponding vorticities are k-periodic, that is $\Gamma_{j+nk} = \Gamma_j$ for any $j \in J$ and $n \in \{1, \dots, |U|\}$.

The first easy observation we make allows us to generalise theorem 2.12 a little bit.

Lemma 3.10 (Vortex addition lemma)**.** *Suppose we decompose $\{1, \dots, N\}$ as the disjoint union of A and B and further suppose there is $\rho > 0$ such that $|z_i - z_j| > \rho$ for*

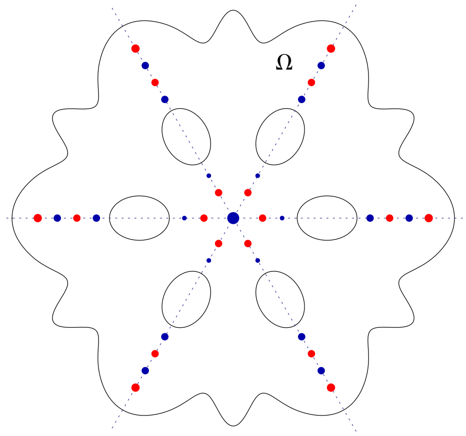

Figure 5: A symmetric critical point according to theorem 3.9.

any $i \in A$, $j \in B$. Then there is a constant $C > 0$ depending only on Ω and ρ such that

$$\left| H_\Omega^\Gamma(z) - H_\Omega^{\pi_A\Gamma}(\pi_A z) - H_\Omega^{\pi_B\Gamma}(\pi_B z) \right| + \left| \nabla H_\Omega^\Gamma(z) - \nabla \widetilde{H_\Omega^{\pi_A\Gamma}}(\pi_A z) - \nabla \widetilde{H_\Omega^{\pi_B\Gamma}}(\pi_B z) \right| \leq C, \tag{3.6}$$

where $\widetilde{H_\Omega^{\pi_A\Gamma}}$ is the obvious continuation of $H_\Omega^{\pi_A\Gamma}$ to $\mathcal{F}_N\Omega$.

Proof. We observe

$$H_\Omega^\Gamma(z) = H_\Omega^{\pi_A\Gamma}(\pi_A z) + H_\Omega^{\pi_B\Gamma}(\pi_B z) + 2 \sum_{i \in A, j \in B} \Gamma_i \Gamma_j G(z_i, z_j).$$

Since $|z_i - z_j| > \rho$ for $i \in A$, $j \in B$, (3.6) is a direct consequence of hypothesis (2.2) together with the triangle inequality. $\square$

Corollary 3.11. *If $L = a + \mathbb{R}\cdot v$ is a line with $a \in \partial\Omega$ and $v \in S^1$ such that $a + (0,\infty)v \cap \Omega$ has l connected components, and $\mathcal{P} = \{C_1, \ldots, C_r\}$ is a partition of $\{1, \ldots, N\}$ with $|\mathcal{P}| = r \leq l$ such that $\pi_{C_j}\Gamma$ is $\mathcal{L}$–admissible with permutation $\tilde{\sigma}_j$ and $\sigma_j \in \{\hat{\sigma}\tilde{\sigma}_j, \tilde{\sigma}_j\}$ for any $j \in \{1, \ldots, r\}$, where, as before, $\hat{\sigma}$ is a permutation which reverses order, then we have*

$$H_\Omega\big|_{\mathcal{E}(a,v)}(z) \to -\infty \quad \textit{for } z \to \partial\mathcal{E}(a,v),$$

where

$$\mathcal{E}_r(a,v) := \left\{ z \in \mathcal{F}_N L : \begin{array}{c} \exists\, L_1, \ldots, L_r \in \Lambda \textit{ pairwise disjoint} \\ \forall j \in \{1, \ldots, r\}, i \in \{1, \ldots, N\} : \\ \pi_{C_j} z \in \mathcal{L}^{\sigma_j}_{|C_j|}(a,v), z_i \in L_j \Leftrightarrow i \in C_j \end{array} \right\},$$

$$\mathcal{E}(a,v) := \bigcup_{r=1}^{l} \mathcal{E}_r(a,v),$$

Λ *is the set of connected components of* $a + (0,\infty)v \cap \Omega$ *and the boundary is to be taken in* L^N*, as in theorem 2.12.*

Proof. Let $z \in \mathcal{E}_r(a,v)$. Since Ω satisfies a uniform exterior ball condition by hypothesis 2.1 we have $\rho := \min\{\mathrm{dist}(L_i, L_j) : i,j \in \{1,\dots,l\}, i \neq j\} > 0$ so that $|z_i - z_j| > \rho$ if $i \in C_m$, $j \in C_n$ for $m \neq n$. Hence we may apply (3.6) inductively to get

$$H_\Omega(z) = \sum_{j=1}^{r} H_\Omega^{\pi_{C_j}\Gamma}(\pi_{C_j} z) + W(z),$$

where $W : \mathcal{E}_r(a,v) \to \mathbb{R}$ is uniformly bounded in the C^1–sense. Since $\pi_{C_j} z \in \mathcal{L}_{|C_j|}^{\sigma_j}(a,v)$ and $\pi_{C_j}\Gamma$ is $\mathcal{L}$–admissible for every $j \in \{1,\dots,r\}$ each of the terms $H_\Omega^{\pi_{C_j}\Gamma}(\pi_{C_j} z)$ is bounded from above by theorem 2.12. Now if $z \to \partial\mathcal{E}(a,v)$ we have $z \to \partial\mathcal{E}_r(a,v)$ for some $r \in \{1,\dots,l\}$ since $\mathrm{dist}(\mathcal{E}_r(a,v), \mathcal{E}_s(a,v)) \geq \rho$ for $r \neq s$ by the uniform exterior ball condition on Ω and therefore $\pi_{C_j} z \to \partial\mathcal{L}_{|C_j|}^{\sigma_j}(a,v)$ for at least one $j \in \{1,\dots,r\}$, hence $H_\Omega(z) \to -\infty$ as claimed. □

In proceeding with the proof of theorem 3.6 will first consider the case that $|U|$ is odd. Define the map

$$s_U : \mathcal{F}_k(\Omega \cap \mathbb{R}) \ni z \mapsto \left(z, e^{\frac{2\pi i}{|U|}} z, \dots, e^{\frac{2\pi(|U|-1)i}{|U|}} z\right) \in \mathcal{F}_N\Omega.$$

s_U is well-defined, linear and injective since $|U|$ is odd, hence, if $\mathcal{E}(a,v)$ is as in Lemma 3.11 with k in place of N, the set $\mathcal{R} := s_U(\mathcal{E}(a,1)) \subset \mathfrak{S}_\Gamma^U\Omega$, where $a = \inf(\Omega \cap \mathbb{R})$, is a differentiable k–dimensional submanifold of $\mathcal{F}_N\Omega$ and s_U is a parametrisation of $\mathcal{R}$.

If, what we now assume, $|U|$ is even, we have to proceed a little differently. Since in this case the counterclockwise rotation by π provides us with an involution of $\mathbb{R} \cap \Omega$, s_U as defined before is not well–defined any more. Instead, we have to define

$$s_U : \mathcal{F}_k(\Omega \cap [0,\infty)) \ni z \mapsto \left(z, e^{\frac{2\pi i}{|U|}} z, \dots, e^{\frac{2\pi(|U|-1)i}{|U|}} z\right) \in \mathcal{F}_N\Omega.$$

s_U is then well-defined, linear and injective as before, and the set $\mathcal{R} := s_U(\mathcal{E}(0,1)) \subset \mathfrak{S}_\Gamma^U\Omega$ is again a differentiable k–dimensional submanifold of $\mathcal{F}_N\Omega$ parametrised by s_U.

Our goal is now to apply an argument similar to the principle of symmetric criticality to $H_\Omega\big|_{\mathcal{R}}$. Using lemma 3.4 we obtain for $z = s_U\tilde{z} \in \mathcal{R}$, $\tilde{z} \in \mathcal{E}(a,1)$ or $\tilde{z} \in \mathcal{E}(0,1)$, respectively:

$$H_\Omega\big|_{\mathcal{R}}(z) = |U| \left(\sum_{j\in J} \Gamma_j^2 h(z_j) + \sum_{\substack{i,j\in J \\ i\neq j}} \Gamma_i\Gamma_j G(z_i,z_j) + \sum_{u\in U\setminus\{\mathrm{id}\}} \sum_{i,j\in J} \Gamma_i\Gamma_j G(z_i, uz_j) \right)$$

$$= |U| \cdot H_\Omega^{\pi_J \Gamma}\big|_{\mathcal{E}(a,1)}(\tilde{z}) + |U| \sum_{u \in U \setminus \{\mathrm{id}\}} \sum_{i,j \in J} \Gamma_i \Gamma_j G(z_i, u z_j), \tag{3.7}$$

independent of the parity of $|U|$.

Since Ω satisfies a uniform exterior ball condition by hypothesis 2.1 there is $\rho > 0$ such that $B_\rho(0) \subset \mathbb{C} \setminus \Omega$, hence

$$|z_i - u z_j| \geq 2\rho \sin\left(\frac{\pi}{|U|}\right) > 0$$

for any $i, j \in J$ and $u \in U \setminus \{\mathrm{id}\}$, which immediately implies that the last term of (3.7) is uniformly bounded on $\mathcal{R}$. Also note that this inequality is trivially fulfilled if $|U| = 1$, which implies that the condition $0 \notin \Omega$ may be dropped in this case.

Corollary 3.11 now implies that $H_\Omega^{\pi_J \Gamma}\big|_{\mathcal{E}(a,v)}$ assumes a local maximum in each connected component of $\mathcal{E}(a, v)$, and so does $H_\Omega\big|_\mathcal{R}$ at some $z^* \in \mathcal{R}$.

It remains to show that the critical point z^* of $H_\Omega|_\mathcal{R}$ is indeed a critical point of H_Ω and to count the connected components of $\mathcal{E}(a, v)$. This is precisely what we do within the next few lemmata.

Lemma 3.12. *$\mathcal{R}$ is invariant under the gradient flow φ of H_Ω.*

Proof. Let $z_0 \in \mathcal{R}$, that is $z_0 = s_U \zeta$ for some $\zeta \in \mathcal{F}_k(\Omega \cap \mathbb{R})$ if $|U|$ is odd and $\zeta \in \mathcal{F}_k(\Omega \cap [0, \infty))$ if $|U|$ is even. Since H_Ω is invariant under the group actions of U and $\Sigma(\Gamma)$, the gradient flow is equivariant with respect to these operations. If $\sigma \in \Sigma(\Gamma)$ is the permutation which shifts indices k times cyclically, we obtain for $t \in (t^-(z_0), t^+(z_0))$ and $j \in \{1, \ldots, |U|\}$

$$\sigma^j \varphi_t(z_0) = \varphi_t(\sigma^j z_0) = \varphi_t(r^j z_0) = r^j \varphi_t(z_0),$$

by construction, where $r = e^{\frac{2\pi i}{|U|}}$, in other words $\varphi_t(z_0) = s_U \pi_J \varphi_t(z_0)$. Now

$$s_0 \pi_J \varphi_t(z_0) = \pi_J s_0 \varphi_t(z_0) = \pi_J \varphi_t(s_0 z_0) = \pi_J \varphi_t(s_0 s_U \zeta) = \pi_J \varphi_t(\tilde{s_U} s_0 \zeta),$$

where $\tilde{s_U} z = (z, r^{-1} z, \ldots, r^{-|U|} z)$, since $s_0 r^j = r^{-j} s_0$. Now $s_0 \zeta = \zeta$, thus $\tilde{s_U} \zeta = \tau z_0$ for some $\tau \in \Sigma(\Gamma)$ whose fixed point set is J. It follows

$$s_0 \pi_J \varphi_t(z_0) = \pi_J \varphi_t(\tilde{s_U} \zeta) = \pi_J \varphi_t(\tau z_0) = \pi_J \tau \varphi_t(z_0) = \pi_J \varphi_t(z_0),$$

in other words we have shown that $\pi_J \varphi_t(z_0) \in \mathcal{F}_k(\mathbb{R} \cap \Omega)$ for all $t \in (t^-(z_0), t^+(z_0))$, which implies that the order of components of $\pi_J z_0 = \zeta$ is preserved under the gradient flow, and of course none of these components can switch into another component of $\Omega \cap \mathbb{R}$ which means $\pi_J \varphi_t(z_0) \in \mathcal{E}(0, 1)$ or $\pi_J \varphi_t(z_0) \in \mathcal{E}(a, 1)$, respectively, hence $\varphi_t(z_0) = s_U \pi_J \varphi_t(z_0) \in \mathcal{R}$ and the proof is done. $\square$

Lemma 3.13. *Any critical point z^* of $H_\Omega\big|_\mathcal{R}$ is also a critical point of H_Ω.*

Proof. Since $\mathcal{R}$ is a k–dimensional differentiable submanifold of $\mathcal{F}_N \Omega$ the fact that z^* is a critical point of $H_\Omega\big|_\mathcal{R}$ means that $\nabla H_\Omega(z^*)$ is orthogonal to $T_{z^*}\mathcal{R}$. On the other hand $\mathcal{R}$ is invariant under the gradient flow φ by Lemma 3.12, thus $\nabla H_\Omega(z^*) \in T_{z^*}\mathcal{R}$, hence $\nabla H_\Omega(z^*) = 0$ and we are done. □

Lemma 3.14. *$\mathcal{R}$ has precisely $N_1(\Gamma, \Omega, U)$ connected components.*

Proof. The connected components of $\mathcal{R}$ are the images of connected components of $\mathcal{E}(a, 1)$, where a is either $\inf(\Omega \cap \mathbb{R})$ if $|U|$ is odd or $a = 0$ if $|U|$ is even. Connected components of these spaces are special connected components of $\mathcal{F}_k L$, and in either case L is the disjoint union of l open intervals. A component of $\mathcal{F}_k L$ is completely specified by placing k vortices into r of the l intervals. Such a component of $\mathcal{F}_k L$ belongs to $\mathcal{E}_r(a, v)$ if and only if the vortices are ordered such that their corresponding vorticities are increasing or decreasing in modulus and have alternating signs within each interval. Summing over r then gives the asserted result. □

This finishes the proof of theorem 3.6, a fortiori we have proved theorem 1.3. If $0 \notin \Omega$ there is also nothing left to show for corollary 3.7. If, on the other hand $0 \in \Omega$ and $[0, \infty) \cap \Omega$ is disconnected, the statement of corollary 3.7 is easily obtained following the lines of the proof for theorem 3.6 with $\rho = \sup L_1$, where L_1 is the first connected component of $[0, \infty) \cap \Omega$ and using $\mathcal{E}(\rho, 1)$ instead of $\mathcal{E}(0, 1)$ and $\mathcal{E}(a, 1)$, which ensures that no vortices are placed into the connected component of $\mathbb{R} \cap \Omega$ containing 0.

3.4 Proof of theorem 3.9

The proof of theorem 3.9 is completely analogous to the proof of theorem 3.6, except that we need a sort of strengthening of theorem 2.12 in order to prevent the vortices from colliding in the symmetry center of Ω, this is what the condition of center–admissibility is made for and the first thing we state here.

Note that throughout this section we leave the rotation subgroup $U \leq D_p$ fixed. If $z_0 \in \widehat{\mathfrak{S}}_\Gamma^U \Omega \neq \emptyset$ we then necessarily have $N = k \cdot |U| + 1$, and without loss of generality we may assume $J = J_{z_0} = \{1, \dots, k+1\}$.

Lemma 3.15. *Let $\Gamma \in \mathbb{R}^N$ be such that $\pi_J \Gamma \in \mathbb{R}^{k+1}$ is center–admissible, $0 \in \Omega$. Without loss of generality we may assume $\Gamma_j = (-1)^j |\Gamma_j|$ for $j \in J$, $|\Gamma_{j+1}| < |\Gamma_j|$ for every $j \in \{1, \dots, k\}$. Defining*

$$\mathcal{R} := \{0\} \times s_U \mathcal{L}_k(0, 1),$$

where s_U is as before, we obtain

$$H_\Omega\big|_{\mathcal{R}}(z) \to -\infty \quad \text{as } z_j \to 0$$

for any $j \subset J$.

Proof. We argue similarly to the proof of theorem 2.12. Equation (3.5) implies

$$
\begin{aligned}
H_\Omega\big|_{\mathcal{R}}(z) = \Gamma_1^2 h(0) + |U|\Bigg(& \sum_{j=2}^{k+1} \Gamma_j^2 h(z_j) + \sum_{\substack{i,j=1\\ i\neq j}}^{k+1} \sum_{u\in U} \Gamma_i\Gamma_j G(z_i, uz_j) \\
& + \sum_{j=2}^{k+1} \sum_{u\in U\setminus\{\mathrm{id}\}} \Gamma_j^2 G(z_j, uz_j)\Bigg),
\end{aligned}
$$

$$
= \Gamma_1^2 h(0) + |U| H_\Omega^{\pi_{J^*}\Gamma}(z) + 2|U|\sum_{j=2}^{k+1}\Gamma_1\Gamma_j G(0,z_j) + |U| \sum_{u\in U\setminus\{\mathrm{id}\}} \sum_{i,j=2}^{k+1} \Gamma_i\Gamma_j G(z_i, uz_j)
$$

where $J^* = J\setminus\{1\}$. Using theorem 2.12 and hypothesis (2.2) together with the fact that $|z_i - uz_j|$ for $u \in U\setminus\{\mathrm{id}\}$ is small only if z_i, z_j are close to $0 \in \Omega$ we obtain that the term $g(z_i, uz_j)$ is bounded by hypothesis 2.4, hence there is a constant $\tilde{C}$ such that

$$
H_\Omega\big|_{\mathcal{R}}(z) \le \tilde{C} - \frac{|U|}{\pi}\sum_{j=2}^{k+1}\Gamma_1\Gamma_j \ln|z_j| - \frac{|U|}{2\pi}\sum_{u\in U\setminus\{\mathrm{id}\}}\sum_{i,j=2}^{k+1}\Gamma_i\Gamma_j\ln|z_i - uz_j|
$$

$$
= \tilde{C} + \frac{|U|}{\pi}\sum_{j=2}^{k+1}(-1)^j|\Gamma_1||\Gamma_j|\ln|z_j| - \frac{|U|}{\pi}\sum_{u\in U\setminus\{\mathrm{id}\}}\sum_{j=2}^{k+1}\sum_{i=j}^{k+1}(-1)^{i+j}|\Gamma_i||\Gamma_j|\ln|z_i - uz_j|
$$

$$
= \tilde{C} + \frac{|U|}{\pi}\sum_{j=2}^{k+1}\left((-1)^j|\Gamma_1||\Gamma_j|\ln|z_j| + \sum_{u\in U\setminus\{\mathrm{id}\}}\sum_{i=j}^{k+1}(-1)^{i+j+1}|\Gamma_i||\Gamma_j|\ln|z_i - uz_j|\right)
$$

$$
= \tilde{C} + \frac{|U|}{\pi}\sum_{j=2}^{k+1}\big((-1)^j|\Gamma_1||\Gamma_j|\ln|z_j| + \psi_j(z)\big)\big)
$$

where

$$
\begin{aligned}
\psi_j(z) = \sum_{u\in U\setminus\{\mathrm{id}\}}\Bigg[& -|\Gamma_j|^2\ln|z_j - uz_j| + \sum_{r=1}^{\frac{k-j}{2}}|\Gamma_j|\ln\frac{|z_{j+2r-1} - uz_j|^{|\Gamma_{j+2r-1}|}}{|z_{j+2r} - uz_j|^{|\Gamma_{j+2r}|}} \\
& + |\Gamma_{k+1}||\Gamma_j|\ln|z_{k+1} - uz_j|\Bigg]
\end{aligned}
$$

if $k-j$ is even and

$$
\psi_j(z) = \sum_{u\in U\setminus\{\mathrm{id}\}}\left(-|\Gamma_j|^2\ln|z_j - uz_j| + \sum_{r=1}^{\frac{k-j-1}{2}}|\Gamma_j|\ln\frac{|z_{j+2r-1} - uz_j|^{|\Gamma_{j+2r-1}|}}{|z_{j+2r} - uz_j|^{|\Gamma_{j+2r}|}}\right)
$$

if $k-j$ is odd. Using the fact that $z \in \mathcal{R}$, we obtain $|z_{j+2r-1} - uz_j| \leq |z_{j+2r} - uz_j|$, which leaves us with

$$\psi_j(z) \leq -(|U|-1)|\Gamma_j|^2 \ln|z_j| + C'$$

for some constant $C' > 0$ in any case. Thus we obtain

$$H_\Omega\big|_{\mathcal{R}}(z) \leq \hat{C} + \frac{|U|}{\pi} \sum_{j=2}^{k+1} \big((-1)^j |\Gamma_1||\Gamma_j| \ln|z_j| - (|U|-1)\Gamma_j^2 \ln|z_j|\big)$$

$$= \hat{C} + \frac{|U|}{\pi} \sum_{j=2}^{k+1} |\Gamma_j| \big((-1)^j |\Gamma_1| - (|U|-1)|\Gamma_j|\big) \ln|z_j| = \hat{C} + \frac{1}{\pi} \ln \widetilde{\psi}(z),$$

where, if k is even,

$$\widetilde{\psi}(z) = \prod_{r=1}^{\frac{k}{2}} \frac{|z_{2r}|^{|\Gamma_{2r}|(|\Gamma_1|-(|U|-1)|\Gamma_{2r}|)}}{|z_{2r+1}|^{|\Gamma_{2r+1}|(|\Gamma_1|+(|U|-1)|\Gamma_{2r+1}|)}} \tag{3.8}$$

$$\leq \prod_{r=1}^{\frac{k}{2}} |z_{2r+1}|^{|\Gamma_1|(|\Gamma_{2r}|-|\Gamma_{2r+1}|)-(|U|-1)\left(|\Gamma_{2r+1}|^2+|\Gamma_{2r}|^2\right)},$$

and if k is odd

$$\widetilde{\psi}(z) = |z_{k+1}|^{|\Gamma_{k+1}|(|\Gamma_1|-(|U|-1)|\Gamma_{k+1}|)} \prod_{r=1}^{\frac{k-1}{2}} \frac{|z_{2r}|^{|\Gamma_{2r}|(|\Gamma_1|-(|U|-1)|\Gamma_{2r}|)}}{|z_{2r+1}|^{|\Gamma_{2r+1}|(|\Gamma_1|+(|U|-1)|\Gamma_{2r+1}|)}} \tag{3.9}$$

$$\leq |z_{k+1}|^{|\Gamma_{k+1}|(|\Gamma_1|-(|U|-1)|\Gamma_{k+1}|)} \prod_{r=1}^{\frac{k-1}{2}} |z_{2r+1}|^{|\Gamma_1|(|\Gamma_{2r}|-|\Gamma_{2r+1}|)-(|U|-1)\left(|\Gamma_{2r+1}|^2+|\Gamma_{2r}|^2\right)}.$$

Since $\pi_J\Gamma$ is center–admissible for U we have that each of the exponents in $\widetilde{\psi}(z)$ is nonnegative, hence $\widetilde{\psi}$ is bounded above since Ω is bounded.

It follows that $H_\Omega\big|_{\mathcal{R}}$ is bounded above, and upon recalling that

$$H_\Omega\big|_{\mathcal{R}}(z) = \Gamma_1^2 h(0) + |U| H_\Omega^{\pi_{J^*}\Gamma}(z) + 2|U| \sum_{j=2}^{k+1} \Gamma_1\Gamma_j G(0,z_j) + |U| \sum_{u\in U\setminus\{\mathrm{id}\}} \sum_{i,j=2}^{k+1} \Gamma_i\Gamma_j G(z_i, uz_j)$$

we infer that $H_\Omega\big|_{\mathcal{R}}(z) \to -\infty$ if $\pi_C z \to 0$ for any $C \subset J$ with $3 \in J$, since then $H_\Omega^{\pi_{J^*}\Gamma}(z)$ does so. If, on the other hand, $z_2 \to 0$ and $z_j > \rho$ for some $\rho > 0$ and $j \in \{3,\dots,k+1\}$ (3.8) and (3.9) imply $\widetilde{\psi}(z) \to 0$, since $|U||\Gamma_1| > (|U|-1)|\Gamma_2|)$ and Ω is bounded, therefore $H_\Omega\big|_{\mathcal{R}}(z) \to -\infty$ and we are done. $\square$

Now the rest of the proof of theorem 3.9 follows easily. Denote the set of connected components of $[0,\infty) \cap \Omega$ by Λ, and let $l = |\Lambda|$. Let $r \in \{1,\dots,l\}$ and suppose

$\mathcal{P} = \{C_1, \dots, C_r\}$ is a partition of J such that $\pi_{C_j}\Gamma$ is $\mathcal{L}$–admissible for $j \in \{1, \dots, r\}$ and that $\pi_{C_1}\Gamma$ is center–admissible. Select connected components $L_1, \dots, L_r \in \Lambda$, where L_1 is the connected component containing 0. Let $\tilde{\mathcal{A}}$ be a connected component of $\mathcal{F}_{k+1}([0,\infty) \cap \Omega)$ such that $z_j \in L_i$ if and only if $j \in C_i$ for $j \in J$, $i \in \{1, \dots, l\}$ and the vortices are ordered with descending moduli inside L_1, placing the strongest vortex in the symmetry center, and that the other vortices are either ordered with ascending or descending moduli inside their respective connected components. Renumbering the vortices we may without loss of generality assume that $|z_i| < |z_{i+1}|$ for all $i \in \{1, \dots, k\}$, a fortiori $z_1 = 0$, and that the vorticities satisfy $\Gamma_{j+ik} = \Gamma_j$ for $i \in \{1, \dots, |U| - 1\}$ and $j \in \{2, \dots, k+1\}$. Defining $\mathcal{A} := \{0\} \times s_U \tilde{\mathcal{A}}$, $I := \{1, j + ki : j \in C_1 \setminus \{1\}, i \in \{0, \dots, |U| - 1\}\}$, $I' := \{1, \dots, N\} \setminus I$ we may write

$$H_\Omega\big|_{\mathcal{A}}(z) = H_\Omega^{\pi_I\Gamma}(\pi_I z) + H_\Omega^{\pi_{I'}\Gamma}(\pi_{I'} z) + W(z),$$

where $W : \mathcal{A} \to \mathbb{R}$ is uniformly bounded by 3.10. Now if $z \to \partial\mathcal{A}$ either $H_\Omega^{\pi_I\Gamma}(\pi_I z) \to -\infty$ by theorem 2.12 and lemma 3.15 or $H_\Omega^{\pi_{I'}\Gamma}(\pi_{I'} z) \to -\infty$ by corrolary 3.7, hence $H_\Omega\big|_{\mathcal{A}}$ assumes a local maximum which is a critical point of H_Ω by arguing as in lemmata 3.12 and 3.13, taking into consideration that $\dot{z}_1 = 0$ for any $z \in \mathcal{A}$ due to the symmetry assumption.

The multiplicity result then follows by rigorously counting all the possibilities for the connected component $\tilde{\mathcal{A}}$. This works out exactly the same way as for lemma 3.14, except we now have to meet the condition that the vorticities of the vortices that are placed into the central connected component of $[0,\infty) \cap \Omega$ need to be center–admissible and descending in modulus away from the symmetry center. This finishes the proof of theorem 3.9 and of course also of theorem 1.4.

3.5 Examples and further discussion

This section is devoted to the discussion of the results obtained in section 3.2 and to some simpler examples. Namely the cases where $\Gamma_i = (-1)^i$ are of significant importance because of their applicability to partial differential equations, which will be discussed in chapter 6. This is most easily done if $p = 1$ or $p > 1$, N even and $0 \notin \Omega$. The case $p = 1$ is the one discussed in [4], and we give a slight sharpening of their theorem here. The other case is a simple application of theorem 3.6, which we state first.

Corollary 3.16. *Let N be even, Ω be D_p–symmetric, $0 \notin \Omega$ and $\Gamma_j = (-1)^j$, $j \in \{1, \dots, N\}$. Then for any common divisor $q > 1$ of p and $\frac{N}{2}$ there are at least*

$$\sum_{r=1}^{l(q)} \sum_{(a,b,c,d) \in \mathcal{V}_{q,r}} \frac{l(q)!}{(l(q) - r)!} \cdot \prod_{j=1}^{\frac{N}{2q}} \frac{1}{a_j! \cdot b_j! \cdot c_j! \cdot d_j!}$$

distinct D_q–symmetric critical points of H_Ω whose components lie on the symmetry axes of the D_q–action on Ω such that the vorticities have alternating signs, where $l(q)$ is the number of connected components of $[0,\infty) \cap \Omega$ if q is even, and of $\mathbb{R} \cap \Omega$, if q is odd and *where*

$$\mathcal{V}_{q,r} := \left\{ (a,b,c,d) \in \mathbb{N}_0^{\frac{2N}{q}} : \begin{array}{l} \sum_{j=1}^{\frac{N}{2q}} (a_j+b_j+c_j+d_j) = r, \sum_{j=1}^{\frac{N}{2q}} a_j = \sum_{j=1}^{\frac{N}{2q}} b_j \\ \sum_{j=1}^{\frac{N}{2q}} [(2j-1)(a_j+b_j) + 2j(c_j+d_j)] = \frac{N}{q} \end{array} \right\}.$$

Corollary 3.17. *If $p = 1$, that is Ω is only axially symmetric, the assumptions "N even" and "$0 \notin \Omega$" are not needed, and the minimum number of critical points of H_Ω is then given by*

$$\sum_{r=1}^{l} \sum_{(a,b,c,d)\in\widetilde{\mathcal{V}}_r} \frac{l!}{(l-r)!} \cdot \prod_{j=1}^{\frac{N}{2}} \frac{1}{a_j! \cdot b_j! \cdot c_j! \cdot d_j!},$$

where l now is the number of connected components of $\mathbb{R} \cap \Omega$ and

$$\widetilde{\mathcal{V}}_r := \left\{ (a,b,c,d) \in \mathbb{N}_0^{4\lfloor\frac{N+1}{2}\rfloor} : \begin{array}{l} \sum_{j=1}^{\lfloor\frac{N+1}{2}\rfloor} (a_j+b_j+c_j+d_j) = r, \sum_{j=1}^{\lfloor\frac{N+1}{2}\rfloor} a_j = \sum_{j=1}^{\lfloor\frac{N+1}{2}\rfloor} b_j - \frac{1+(-1)^N}{2} \\ \sum_{j=1}^{\lfloor\frac{N+1}{2}\rfloor} [(2j-1)(a_j+b_j) + 2j(c_j+d_j)] = N \end{array} \right\}.$$

This is a small sharpening of theorem 3.3 in [4], since now we are allowed to break the pattern of alternating vorticities from one connected component to another.

Proof. We prove corollaries 3.16 and 3.17 at once. This is clearly an application of theorem 3.6, we just have to evaluate the number $N_1(\Gamma, \Omega, U)$ for $U = \mathbb{Z}_q$. To this end fix a common divisor $q > 1$ of p and $\frac{N}{2}$ and put $k := \frac{N}{q}$. In the case $\Gamma_i = (-1)^i$ $\mathcal{L}$–admissibility simply means we have to alternate the signs of vorticities for adjacent vortices along a line, possibly breaking this pattern between two connected components of $\Omega \cap [0,\infty)$ and $\Omega \cap \mathbb{R}$, respectively, depending on the parity of Ω and rotate this configuration $q-1$ times to get the other vortex locations. To do this, necessarily k has to be even, and it in fact is by assumption. We are thus left to count the number of alternating configurations of k vortices put into r of the l connected components. To count these effectively, we form four different groups of configurations of vortices ("chains") which can occur in one single connected component. We possibly have a_j chains of vortices of length $2j-1$ which start with $+1$, b_j chains of length $2j-1$ starting with -1, c_j chains of even length $2j$ starting with $+1$ and finally d_j chains of length $2j$ starting with -1. In order for this to be a proper description of the situation, j has to range from 1 to $\frac{N}{2q}$, and there have to be precisely r chains, that is

$$\sum_{j=1}^{\frac{N}{2q}} (a_j+b_j+c_j+d_j) = r.$$

Further, we have to meet the condition that the sum of all vorticities vanishes if $q > 1$. This is the case if and only if

$$\sum_{j=1}^{\frac{N}{2q}} a_j = \sum_{j=1}^{\frac{N}{2q}} b_j.$$

Finally, there are precisely k vortices to be distributed, in other words

$$\sum_{j=1}^{\frac{N}{2q}} [(2j-1)(a_j+b_j)+2j(c_j+d_j)] = k.$$

$\mathcal{V}_{q,r}$ is precisely the set of all tuples (a,b,c,d) satisfying these constraints. Each of the a_j chains of length $2j-1$ are indistinguishable from each other and similarly for the other groups of chains. The number of orderings of such things is given by the multinomial coefficient

$$r! \cdot \prod_{j=1}^{\frac{N}{2q}} \frac{1}{a_j! \cdot b_j! \cdot c_j! \cdot d_j!}.$$

Summing over $\mathcal{V}_{k,r}$ in combination with the fact that there are $\binom{l}{r}$ ways to choose r out of l connected components gives the expected result. Finally, if $q = 1$ the condition that N is even may be dropped, hence it may be possible for the total vorticity to be .1 instead of 0. The definition of $\widetilde{\mathcal{V}}_r$ precisely takes care of that. $\square$

Example. If, for example $N = 24$, $\Gamma_i = (-1)^i$ and Ω has the symmetry group of a regular hexagon such that $\Omega \cap \mathbb{R}$ has four connected components this gives 20260 axially symmetric, 780 D_3–symmetric and 12 D_6–symmetric critical points for H_Ω. There are also 36 D_2–symmetric critical points, but these are already included in the axially symmetric ones.

One may ask whether the geometry of symmetric critical points of H_Ω described within theorem 3.6, theorem 3.9 and their corollaries is the most general one or even the only one possible. In both cases the answer is negative. This is most easily seen when Ω is U–symmetric and $|U|$ is even as well as $0 \notin \Omega$. In this case there exists a second set of symmetry axes, namely the rotations of the axis $\mathbb{R}e^{\frac{\pi i}{p}}$. Applying theorem 3.6 for a fixed rotation subgroup U to Ω with vorticities Γ_1 and to $\tilde{\Omega} := e^{\frac{\pi i}{|U|}}\Omega$ with vorticities Γ_2 such that $\mathfrak{S}_\Gamma^U \Omega \neq \emptyset$ for $\Gamma \in \{\Gamma_1, \Gamma_2\}$ we are left with two U–symmetric critical points z_1^* and $\tilde{z}_2^*$ of $H_\Omega^{\Gamma_1}$ and $H_{\tilde{\Omega}}^{\Gamma_2}$. It follows easily that $z_2^* := e^{-\frac{\pi i}{p}}\tilde{z}_2^*$ is a critical point of $H_\Omega^{\Gamma_2}$ and, applying the vortex addition lemma 3.10 and lemma 3.12, $\varphi((z_1^*, z_2^*), t) \to z^*$ for $t \to t^+((z_1^*, z_2^*))$, where φ denotes the gradient flow of H_Ω^Γ for $\Gamma = (\Gamma_1, \Gamma_2)$ and z^* is a critical point of H_Ω^Γ with vortices distributed on *all* of the U–symmetry axes of Ω.

The question if it is necessary, that all of the components of critical points have to lie on the symmetry axes of Ω is harder to answer, but the answer is also negative.

Theorem 3.18. *Let Ω be symmetric with respect to the reflection $s_0 : z \mapsto \bar{z}$ on $\mathbb{R}$,*

$N = 3$, $\Gamma_1 = \Gamma_3 = -\frac{1}{4}$, $\Gamma_2 = 1$ *and consider symmetric constellations of vortices*

$$\tilde{\mathcal{F}}_2\Omega := \{(z_1, z_2, z_3) \in \mathcal{F}_3\Omega : z_2 \in \Omega \cap \mathbb{R}, z_3 = \overline{z_1}\}.$$

Then H_Ω has a critical point $z^* \in \tilde{\mathcal{F}}_2\Omega$.

PROOF. Since $\tilde{\mathcal{F}}_2\Omega$ is s_0–invariant, it follows similarly to lemma 3.13 that a critical point of $H_\Omega|_{\tilde{\mathcal{F}}_2\Omega}$ is also a critical point of H_Ω.

We begin by choosing appropriate coordinates on $\tilde{\mathcal{F}}_2\Omega$: $x_2 := x \in \Omega \cap \mathbb{R}$, $x_1 := x + re^{i\varphi}$, $x_3 := \overline{x_1}$, where $r > 0$ and $\varphi \in (0, \pi)$ are chosen such that $(x_1, x_2, x_3) \in \mathcal{F}_3\Omega$. With respect to these coordinates $H_\Omega|_{\tilde{\mathcal{F}}_2\Omega}$ has the form

$$H_\Omega(x, x + re^{i\varphi}, x + re^{-i\varphi}) = h(x) + \frac{1}{8}h(x_1) + 2\left[\frac{1}{16}G(x_1, \overline{x_1}) - \frac{1}{2}G(x, x_1)\right]$$

$$= h(x) + \frac{1}{8}h(x_1) + 2\left[\frac{1}{16}g(x_1, \overline{x_1}) - \frac{1}{32\pi}\ln|2r\sin\varphi| - \frac{1}{2}g(x, x_1) + \frac{1}{4\pi}\ln r\right]$$

$$= h(x) + \frac{1}{8}h(x + re^{i\varphi}) + \frac{1}{8}g(x + re^{i\varphi}, x + re^{-i\varphi}) - g(x, x + re^{i\varphi}) - \frac{1}{16\pi}\ln\frac{\sin\varphi}{r^7} - \frac{1}{16\pi}\ln 2$$

and we infer, using that the g– and h–terms are bounded, that

$$H_\Omega(x_1, x_2, \overline{x_1}) \to \infty \tag{3.10}$$

as $\varphi \to 0$ or $\varphi \to \pi$, as we keep x and r fixed such that $B_r(x) \subset \Omega$.

On the other hand we have the following

Lemma 3.19. *H_Ω is bounded above on $\tilde{\mathcal{L}}_3\Omega := \mathcal{L}_3\Omega \cap \tilde{\mathcal{F}}_2\Omega$.*

PROOF. We have for $z = (z_1, z_2, \overline{z_1}) \in \tilde{\mathcal{L}}_3\Omega$:

$$H_\Omega(z) = h(z_2) + \frac{1}{8}h(z_1) + \frac{1}{8}G(z_1, \overline{z_1}) - G(z_2, z_1)$$

$$\leq \frac{9}{8}C_1 + \frac{1}{8}\left(G(z_1, \overline{z_1}) - G(z_2, z_1)\right) - \frac{7}{8}G(z_2, z_1) \leq \frac{9}{8}C_1 + \frac{1}{8}C_4 - \frac{7}{8}C_0$$

by hypothesis 2.4 and we are done. □

We now are ready to apply a mountain-pass-type argument to $H_\Omega|_{\tilde{\mathcal{F}}_2\Omega}$. To this end, fix $x_0 \in \Omega \cap \mathbb{R}$ and $r_0 > 0$ such that $B_{2r_0}(x_0) \subset \Omega$. Denote the supremum of H_Ω on $\tilde{\mathcal{L}}_3\Omega$ by σ. Next we define a path

$$\tilde{\gamma}_0 : (0, \pi) \ni t \mapsto (x_0 + r_0e^{it}, x_0, x_0 + r_0e^{-it}) \in \tilde{\mathcal{F}}_2\Omega.$$

From (3.10) we infer that there is $\tilde{\delta} > 0$ such that $H_\Omega(\tilde{\gamma}_0(\delta)) \geq \sigma + 1$, $H_\Omega(\tilde{\gamma}_0(\pi - \delta)) \geq \sigma + 1$ for every $\delta \in (0, \tilde{\delta})$. Further observe that

$$\mathrm{Re}(\tilde{\gamma}_{0,1}(t) - \tilde{\gamma}_{0,2}(t)) \to r_0 \quad (t \to 0),$$

therefore there is $\delta \in (0, \tilde{\delta})$ such that

$$\operatorname{Re}(\tilde{\gamma}_{0,1}(\delta) - \tilde{\gamma}_{0,2}(\delta)) = \rho_0 > 0.$$

Set $\gamma_0 := \tilde{\gamma}_0|_{[\delta, \pi-\delta]}$ and define the following class of paths in $\tilde{\mathcal{F}}_2\Omega$:

$$\beta := \left\{ \gamma \in C^0([\delta, \pi - \delta], \tilde{\mathcal{F}}_2\Omega) : \gamma \overset{h}{\simeq} \gamma_0, H_\Omega(h([0,1] \times \{\delta, \pi - \delta\})) \geq \sigma + 1 \right\}.$$

Denoting the gradient flow of H_Ω by φ we have the following

Lemma 3.20. *β is φ-complete and for each $\gamma \in \beta$ we have*

$$\gamma([\delta, \pi - \delta]) \cap \tilde{\mathcal{L}}_3\Omega \neq \emptyset. \tag{3.11}$$

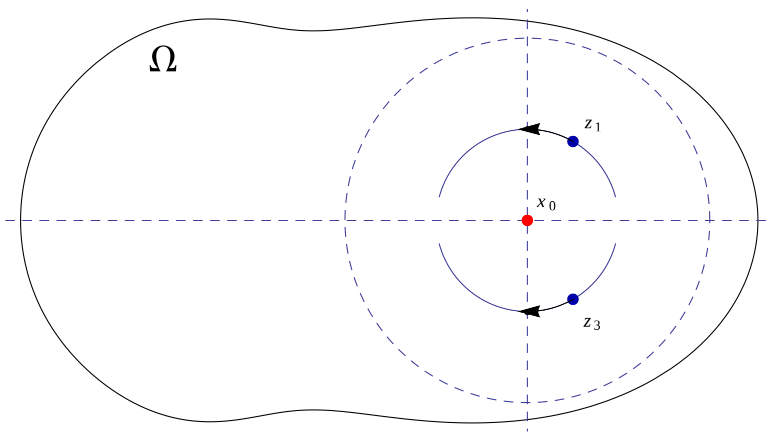

Figure 6: The map γ_0.

Proof. Clearly β is φ-complete, since $H_\Omega(\varphi(x, \cdot))$ is non-decreasing. For the second part let $\gamma \in \beta$ and let Φ be a homotopy connecting γ_0 and $\gamma_1 := \gamma$ as in the definition of β, that is is

$$H_\Omega\left(\Phi([0,1] \times \{\delta, \pi - \delta\})\right) \geq \sigma + 1. \tag{3.12}$$

Define $H : [0,1] \times [\delta, \pi - \delta] \to \mathbb{R}$, $H(s,t) := \operatorname{Re}\left[\Phi_1(s,t) - \Phi_2(s,t)\right]$. We claim that $H(s, \delta) > 0$ for all $s \in [0,1]$. To see this, observe that $H(0, \delta) = \rho_0 > 0$ and assume there is $s_0 \in [0,1]$ such that $H(s_0, \delta) \leq 0$. Then, by the intermediate value theorem, there is $s_1 \in [0,1]$, such that $H(s_1, \delta) = 0$, which means that $\Phi(s_1, \delta) \in \tilde{\mathcal{L}}_3\Omega$, hence $H_\Omega(h(s_1, \delta)) \leq \sigma$, in contradiction with (3.12). Hence $H(s, \delta) > 0$ for all $s \in [0,1]$, and a similar argument shows $H(s, \pi - \delta) < 0$ for all $s \in [0,1]$. In particular, again by the intermediate value theorem, there is $t_0 \in (\delta, \pi - \delta)$ such that $H(1, t_0) = 0$, that is $\gamma(t_0) \in \tilde{\mathcal{L}}_3\Omega$. □

We see by applying Lemma 2.14 to β, that there is some $z_0 \in \tilde{\mathcal{F}}_2\Omega$ satisfying

$$\lim_{t \to t^+(z_0)} H_\Omega(\varphi(z_0, t)) < \infty.$$

In slight anticipation of chapter 4 we may note that $\Gamma = \left(-\frac{1}{4}, 1, -\frac{1}{4}\right)$ is Δ–admissible and ∂–admissible, hence proposition 4.3 implies that H_Ω has a critical point $z^* \in \tilde{\mathcal{F}}_2\Omega$. $\square$

4 Singularities of H_Ω

We will now leave the case of symmetric Ω for the rest of this thesis and work with a general bounded domain Ω which still satisfies hypothesis 2.1. During the last chapter we have taken massive advantage of the fact that the vortices were constrained to a line, hence their order would not change under the gradient flow, which induced some compactness into the problem of finding critical points. This is not to be expected if we work on a general domain Ω, hence we are led to investigate possible conditions on Γ and, from time to time, also on Ω which enable us to give some positive results.

As work on this thesis started out, the paper [4] was the main starting point for the research conducted here. As of this writing, it still is the only reference known to the author dealing with both general Ω *and* alternating signs of vorticities at once. By the time the problem of collisions of vortices under the gradient flow of H_Ω in the interior of the domain seemed to be the largely unsolved one, except in the cases $\Gamma_i = (-1)^i$ and $N \leq 4$, whereas the problem of vortices colliding with $\partial\Omega$ was dealt within [4] in great generality, but as it turned out, the result concerning the latter case given there is erroneous. More precisely, it is the proof of lemma 4.2 in [4] that contains an error. Although the statement of lemma 4.2 appears to be wrong in general, the authors [3] were able to recover their results for $N \leq 4$ and general Γ_i using different methods, but for larger values of N the problem turned out to be the more difficult one, leading to more severe conditions on the vorticity vector Γ than are needed for smaller values of N. In fact, it turns out ([3]) that for $N \leq 4$ the condition $\sum_{\substack{i,j \in J \\ i \neq j}} \Gamma_i \Gamma_j < 0$ for any $J \subset \{1, \dots, N\}$, $|J| \geq 3$ is sufficient for recovering the results of [4]. This is only slightly more restrictive than our condition of Δ–admissibility below, which has been previously conjectured to be sufficient in [4]. The author believes, partly based on numerical simulations of the problem, that the severe restriction of ∂–admissibility is in fact unnecessary and may be completely abolished or at least be weakened, as it is done by Bartsch et al. in the case $N \leq 4$.

This section is devoted to the study of H_Ω near collisions with the boundary $\partial\Omega$ or with each other away from the boundary and to give conditions on Γ and Ω which prevent these. We start out by simply stating the relevant conditions on Γ.

Definition 4.1 (Δ–admissibility)**.** We call a parameter $\Gamma \in \mathbb{R}^N$ Δ–admissible, if for every $C \subset \{1, \dots, N\}$, $|C| \geq 2$:

$$\sum_{\substack{i,j \in C \\ i \neq j}} \Gamma_i \Gamma_j \neq 0.$$

Definition 4.2 (∂–admissibility)**.** We call a parameter $\Gamma \in \mathbb{R}^N$ ∂–admissible, if for every $C \subset \{1, \dots, N\}$, $|C| \geq 2$:

$$\sum_{i \in C} \Gamma_i^2 > \sum_{\substack{i,j \in C \\ i \neq j}} |\Gamma_i \Gamma_j|.$$

If Ω is strictly convex, this condition may be replaced by

$$\sum_{i\in C}\Gamma_i^2 > \sum_{\substack{i,j\in C\\ \Gamma_i\Gamma_j<0}} |\Gamma_i\Gamma_j|.$$

The intuition behind these definitions is simple: the idea is that Δ–admissibility prevents collisions of vortices inside the "diagonal" $\Delta = \{z\in\Omega^N : z_i = z_j \text{ for some } i \neq j\}$ from happening while the energy H_Ω of the system remains finite. Since $\Gamma_i\Gamma_j G(z_i, z_j)$ becomes large if z_i and z_j collide inside Ω, we may regard the quantity $\sum_{\substack{i,j\in C\\ i\neq j}}\Gamma_i\Gamma_j$ as a kind of "collision weight" associated to the vortices with indices in C. Since $\Gamma_j^2 h(z_j)\to -\infty$ if $z_j \to \partial\Omega$, we may, by the same intuitive reasoning, regard the quantity $\sum_{i\in C}\Gamma_i^2$ as a kind of weight for the boundary interaction of the vortices z_i, $i\in C$, and the condition of ∂–admissibility then simply states that the boundary interaction outweighs the collision weight.

The goal of this section is to prove the following

Proposition 4.3. *Let $\Gamma\in\mathbb{R}^N$ be ∂–admissible and Δ–admissible. Then there is $\delta > 0$ such that $|\nabla H_\Omega(z)| > 1$ for every z in $\mathcal{M}_\delta$, where*

$$\mathcal{M}_\delta := \{z\in\mathcal{F}_N\Omega : |z_i - z_j| \le \delta \text{ or } d(z_j)\le\delta \text{ for some } i,j\in\{1,\dots,N\}, i\neq j\}\,.$$

In particular, H_Ω satisfies the Palais–Smale–condition. Further, if there is $z\in\mathcal{F}_N\Omega$ such that

$$\lim_{t\to t^+(z)} H_\Omega(\varphi(z,t)) < \infty,$$

we have $t^+(z) = \infty$ and there is a sequence $s_n\to\infty$ such that defining $z^{s_n} := \varphi(z, s_n)$, we have $z^{s_n}\to z^*\in\mathcal{F}_N\Omega$ where $\nabla H_\Omega(z^*) = 0$, hence H_Ω has a critical point.

The proof of proposition 4.3 of course involves a detailed study of the behaviour of H_Ω near its singularities. The functional H_Ω has singularities at the boundary $\partial\mathcal{F}_N\Omega$ of $\mathcal{F}_N\Omega$ in $\mathbb{C}^N$. This boundary consists of points $z\in\overline{\Omega}^N$ with $z_j\in\partial\Omega$ or $z_i = z_j$ for some indices $i,j\in\{1,\dots,N\}$, $i\neq j$, corresponding to collisions of vortices with the boundary or with each other in $\overline{\Omega}$, respectively.

In order to deal with the problem of collisions effectively, we first introduce some convenient notation for dealing with different types of collisions of vortices within $\overline{\Omega}$, corresponding to the respective parts of the boundary $\partial\mathcal{F}_N\Omega$ of $\mathcal{F}_N\Omega$ in $\mathbb{C}^N$.

First note that collisions of vortices correspond to partitions of the set $\{1,\dots,N\}$ as follows: Given a point $z\in\overline{\Omega}^N$, we define

$$\mathcal{P}_z := \{C\subset\{1,\dots,N\} : z_i = z_j \Leftrightarrow i,j\in C\}\,,$$

which is clearly a partition of $\{1,\dots,N\}$. We call an element $C\in\mathcal{P}$ a cluster, if it has more than one element itself. Denote the subset of clusters of $\mathcal{P}_z$ by $\mathcal{C}(\mathcal{P}_z)$. Now for $C\in\mathcal{P}_z$ define z_C to be the unique element of $\{z_j : j\in C\}$. With this notation, the proof splits essentially into two major cases of types of collisions which have to be excluded. The one which can be settled most easily is the case of interior collisions, that is, given an initial value $z_0 \in \mathcal{F}_N\Omega$ there exists a point $z \in \partial\mathcal{F}_N\Omega$, such that the partition $\mathcal{P}_z$ has a cluster $C \in \mathcal{C}(\mathcal{P})$ satisfying $z_C \in \Omega$, such that $\varphi(z_0,t)_j \to z_C$ as $t \to t^+(z_0)$ if and only if $j \in C$. We denote the set of interior collision points as

$$\partial_{\text{int}}\mathcal{F}_N\Omega = \{z \in \partial\mathcal{F}_N\Omega : \ \exists\ C \in \mathcal{C}(\mathcal{P}_z) \text{ such that } z_C \in \Omega\}.$$

Note that this does include the case of vortices colliding with the boundary, as long as there are some other vortices, which collide inside Ω at the same time.

The second case, which in the following is termed "boundary collisions" is more complicated to settle. In this case the collision point $z \in \partial\mathcal{F}_N\Omega$ satisfies $z_C \in \partial\Omega$ for each cluster $C \in \mathcal{C}(\mathcal{P})$, and it holds that $\varphi(z_0,t)_j \to z_C$ as $t \to t^+(z_0)$ for all $j \in C$, $C \in \mathcal{C}(\mathcal{P}_z)$. We denote the set of boundary collisions with

$$\partial_{\text{bdry}}\mathcal{F}_N\Omega = \{z \in \partial\mathcal{F}_N\Omega : \ \forall\ C \in \mathcal{C}(\mathcal{P}_z) : z_C \in \partial\Omega\}.$$

Clearly $\partial\mathcal{F}_N\Omega$ is the disjoint union of these two sets.

Before we turn to the proof of proposition 4.3, we state some essential lemmata which help us settle the above two cases.

4.1 Interior collisions

Lemma 4.4. *Let Γ be Δ–admissible and for any partition $\mathcal{P}$ of $\{1,\dots,N\}$, $C \in \mathcal{C}(\mathcal{P})$ define*

$$J_C : \mathcal{F}_N\mathbb{C} \ni z \mapsto \sum_{\substack{i,j\in C\\ i\neq j}} \Gamma_i\Gamma_j \ln|z_i - z_j| \in \mathbb{R},$$

$$J_{\mathcal{P}} : \mathcal{F}_N\mathbb{C} \ni z \mapsto \sum_{C\in\mathcal{C}(\mathcal{P})} J_C(z) \in \mathbb{R}.$$

Further define the constant C_Γ by

$$C_\Gamma := \min_{\substack{\mathcal{P}\,\text{partition}\\ \text{of}\,\{1,\dots,N\}\\ \mathcal{C}(\mathcal{P})\neq\emptyset}} \ \min_{C\in\mathcal{C}(\mathcal{P})} \left| \sum_{\substack{i,j\in C\\ i\neq j}} \Gamma_i\Gamma_j \right|,$$

which is positive since Γ is Δ–admissible. With this notation the inequality

$$|\nabla J_{\mathcal{P}_z}(w)| \geq C_\Gamma \max_{C\in\mathcal{C}(\mathcal{P}_z)} \left(\sum_{i\in C} |w_i - z_C|^2 \right)^{-\frac{1}{2}}.$$

holds for every $z \in \partial\mathcal{F}_N\mathbb{C}$, $w \in \mathcal{F}_N\mathbb{C}$.

Proof. Fix points $z \in \partial\mathcal{F}_N\mathbb{C}$, $w \in \mathcal{F}_N\mathbb{C}$ and some cluster $C \in \mathcal{C}(\mathcal{P}_z)$, put $\widetilde{z_C} :=$

$(z_C, \dots, z_C) \in \mathbb{C}^N$ and define

$$j_C : (0, \infty) \ni r \mapsto J_C(\widetilde{z_C} + r(w - \widetilde{z_C})) \in \mathbb{R}.$$

Then

$$j_C'(1) = \sum_{\substack{i,j \in C \\ i \neq j}} \Gamma_i \Gamma_j \neq 0$$

and letting $c_C := |j_C'(1)|$ we infer

$$0 < c_C = |j'(1)| = |\langle \nabla J_C(w), z - \widetilde{z_C} \rangle| \leq |\nabla J_C(w)| \cdot \left(\sum_{i \in C} |w_i - z_C|^2 \right)^{\frac{1}{2}}$$

for any $w \in \mathcal{F}_N\mathbb{C}$. Together with $\min\limits_{C \in \mathcal{C}(\mathcal{P}_z)} c_C \geq C_\Gamma$ and

$$|\nabla J_{\mathcal{P}_z}(w)| = \left(\sum_{C \in \mathcal{C}(\mathcal{P}_z)} |\nabla J_C(w)|^2 \right)^{\frac{1}{2}} \geq \max_{C \in \mathcal{C}(\mathcal{P}_z)} |\nabla J_C(w)|$$

the claim follows. $\square$

Lemma 4.5. *Let Γ be Δ–admissible and $\bar{z} \in \partial_{\text{int}}\mathcal{F}_N\Omega$ with corresponding partition $\mathcal{P}_{\bar{z}}$, and let $C \in \mathcal{C}(\mathcal{P}_{\bar{z}})$ be an interior collision cluster, that is $\bar{z}_C \in \Omega$. There exists $\delta > 0$, such that for each $z \in U_\delta(\bar{z}) \cap \mathcal{F}_N\Omega$:*

$$|\nabla H_\Omega(z)| \geq \frac{C_\Gamma}{4\pi} \left(\sum_{i \in C} |z_i - \bar{z}_C|^2 \right)^{-\frac{1}{2}}.$$

Proof. We decompose H_Ω as

$$H_\Omega(z) = -\frac{1}{2\pi} J_{\mathcal{P}_{\bar{z}}}(z) + K(z),$$

where

$$K(z) = \sum_{j=1}^N \Gamma_j^2 h(z_j) + \sum_{I \in \mathcal{C}(\mathcal{P}_{\bar{z}})} \sum_{\substack{i,j \in I \\ i \neq j}} \Gamma_i \Gamma_j g(z_i, z_j) + \sum_{\substack{i,j \in \{1,\dots,N\} \\ \exists\, I,J \in \mathcal{P}_{\bar{z}}, i \in I, j \in J \\ I \cap J = \emptyset}} \Gamma_i \Gamma_j G(z_i, z_j).$$

Fixing some interior collision cluster C, we have

$$|\nabla H_\Omega(z)| \geq |\nabla H_\Omega(z)|_C = \left| -\frac{1}{2\pi} \nabla J_{\mathcal{P}_{\bar{z}}}(z) + \nabla K(z) \right|_C$$

$$\geq \frac{1}{2\pi}|\nabla J_{\mathcal{P}_{\bar z}}(z)|_C - |\nabla K(z)|_C = \frac{1}{2\pi}|\nabla J_C(z)| - |\nabla K(z)|_C$$

$$\begin{aligned} \geq \frac{1}{2\pi}|\nabla J_C(z)| - \sum_{j\in C}\Gamma_j^2|\nabla h(z_j)| - \sum_{\substack{i,j\in C\\ i\neq j}} |\Gamma_i\Gamma_j|\left(|\nabla_{z_i} g(z_i,z_j)| + |\nabla_{z_j} g(z_i,z_j)|\right) \\ - \sum_{i\in C, j\notin C} |\Gamma_i\Gamma_j||\nabla_{z_i} G(z_i,z_j)|, \end{aligned} \tag{4.1}$$

where for $\zeta \in \mathbb{C}^N$: $|\zeta|_C := |\pi_C\zeta|$, and $\pi_C : \mathbb{C}^N \to \{z \in \mathbb{C}^N : z_j = 0 \text{ if } j \notin C\}$ is the orthogonal projection. For $z \in \mathbb{C}^N$ define

$$r_C(z) := \min\left\{\min_{i\in C, j\notin C} |z_i - z_j|, \min_{j\in C} \operatorname{dist}(z_j, \partial\Omega)\right\}.$$

Since $\delta_0 := r_C(\bar z) > 0$ and r_C is clearly continuous, there is $\tilde\delta > 0$, such that $r_C(z) \geq \frac{\delta_0}{2}$ for every $z \in U_{\tilde\delta}(\bar z) \cap \mathcal{F}_N\Omega$, which by means of hyothesis 2.4 implies that on $U_{\tilde\delta}(\bar z) \cap \mathcal{F}_N\Omega$ the last terms of (4.1) are bounded by some constant $\tilde C \geq 0$.

Now applying lemma 4.4 yields

$$|\nabla H_\Omega(z)| \geq \frac{C_\Gamma}{2\pi}\left(\sum_{i\in C}|z_i - \bar z_C|^2\right)^{-\frac{1}{2}} - \tilde C.$$

Since $\left(\sum_{i\in C}|z_i - \bar z_C|^2\right)^{-\frac{1}{2}} \to \infty$ for $z \to \bar z$, we may choose some $\delta \in (0, \tilde\delta)$, such that for every $z \in U_\delta(\bar z) \cap \mathcal{F}_N\Omega$:

$$\left(\sum_{i\in C}|z_i - \bar z_C|^2\right)^{-\frac{1}{2}} \geq \frac{4\pi\tilde C}{C_\Gamma},$$

so that

$$|\nabla H_\Omega(z)| \geq \frac{C_\Gamma}{4\pi}\left(\sum_{i\in C}|z_i - \bar z_C|^2\right)^{-\frac{1}{2}},$$

which is what we were to show. □

4.2 Boundary collisions

Now we study the behaviour of H_Ω near $\partial_{\mathrm{bdry}}\mathcal{F}_N\Omega$. Let therefore be $\bar z \in \partial_{\mathrm{bdry}}\mathcal{F}_N\Omega$, that is $\mathcal{P}_{\bar z}$ is a partition of $\{1, \dots, N\}$, such that we have distinct points $\bar z_C \in \partial\Omega$ for every cluster $C \in \mathcal{C}(\mathcal{P}_{\bar z})$. It may as well be that $\mathcal{C}(\mathcal{P}_{\bar z}) = \emptyset$. In this case we have $\bar z \in \mathcal{F}_N\partial\Omega$.

Settling the case of interior collisions is relatively easy since, away from $\partial\Omega$, the logarithmic singularity of G dominates the interaction between vortices. If two vortices x, y are near to the boundary and to each other, this is no longer true, since then the term $g(x, y)$ cannot be neglected. The next lemma is the key to the understanding of the interaction taking place between vortices near the boundary.

Lemma 4.6. *Setting*

$$A(x,y) := 2\pi \left(\langle \nabla_x G(x,y), d_x \nu_x \rangle + \langle \nabla_y G(x,y), d_y \nu_y \rangle \right)$$

for $x, y \in \Omega_\varepsilon$, we have

$$A(x,y) = \frac{\langle x - \bar{y}, \nu_y \rangle^2}{|x - \bar{y}|^2} - \frac{\langle x - y, \nu_y \rangle^2}{|x - y|^2} + o(1),$$

as well as

$$|A(x,y)| \leq 1 + o(1)$$

as $x, y \to x_0 \in \partial\Omega$. Moreover, if Ω is strictly convex, we have

$$o(1) \leq A(x,y) \leq 1 + o(1)$$

Proof. By hypothesis 2.4 we may write

$$G(x,y) = \frac{1}{2\pi} \ln \frac{|\bar{x} - y|}{|x - y|} + O(1),$$

as $x, y \to x_0 \in \partial\Omega$, and the approximation holds in the C^1–sense, therefore (since $\bar{x} = 2p(x) - x$)

$$2\pi \nabla_x G(x,y) = (2\mathrm{D}p(x) - \mathrm{id}) \frac{\bar{x} - y}{|\bar{x} - y|^2} - \frac{x - y}{|x - y|^2} + O(1)$$

$$= \frac{y - \bar{x}}{|\bar{x} - y|^2} - \frac{x - y}{|x - y|^2} + \frac{2}{1 - \kappa_x d_x} \left\langle \frac{\bar{x} - y}{|\bar{x} - y|^2}, \tau_x \right\rangle \tau_x + O(1),$$

which leads to

$$A(x,y) = \left\langle \frac{y - \bar{x}}{|\bar{x} - y|^2} - \frac{x - y}{|x - y|^2}, d_x \nu_x \right\rangle + \left\langle \frac{x - \bar{y}}{|\bar{y} - x|^2} - \frac{y - x}{|x - y|^2}, d_y \nu_y \right\rangle + o(1)$$

$$= \left\langle \frac{y - x + 2d_x \nu_x}{|\bar{x} - y|^2}, d_x \nu_x \right\rangle + \left\langle \frac{x - y + 2d_y \nu_y}{|\bar{y} - x|^2}, d_y \nu_y \right\rangle - \left\langle \frac{x - y}{|x - y|^2}, d_x \nu_x - d_y \nu_y \right\rangle + o(1)$$

$$= 2 \left(\frac{d_x^2}{|\bar{x} - y|^2} + \frac{d_y^2}{|\bar{y} - x|^2} \right) + \left\langle y - x, \frac{d_x \nu_x}{|\bar{x} - y|^2} \right\rangle + \left\langle x - y, \frac{d_y \nu_y}{|\bar{y} - x|^2} \right\rangle - \left\langle \frac{x - y}{|x - y|^2}, d_x \nu_x - d_y \nu_y \right\rangle + o(1)$$

$$= 2 \left(\frac{d_x^2}{|\bar{x} - y|^2} + \frac{d_y^2}{|\bar{y} - x|^2} \right) + \left\langle x - y, \frac{d_y \nu_y}{|\bar{y} - x|^2} - \frac{d_x \nu_x}{|\bar{x} - y|^2} \right\rangle - 1 + \left\langle \frac{x - y}{|x - y|^2}, p(x) - p(y) \right\rangle + o(1)$$

Since $p(x) - p(y) = \mathrm{D}p(y)(x-y) + o(|x-y|)$ we have, using lemma 2.9

$$A(x,y) = 2\left(\frac{d_x^2}{|\bar{x}-y|^2} + \frac{d_y^2}{|\bar{y}-x|^2}\right) + \left\langle x-y, \frac{d_y\nu_y}{|\bar{y}-x|^2} - \frac{d_x\nu_x}{|\bar{x}-y|^2}\right\rangle - 1$$
$$+ \frac{\langle x-y,\tau_y\rangle^2}{(1-\kappa_y d_y)|x-y|^2} + o(1)$$

$$= 2\left(\frac{d_x^2}{|\bar{x}-y|^2} + \frac{d_y^2}{|\bar{y}-x|^2}\right) - \left\langle x-y, \frac{d_x\nu_x - d_y\nu_y}{|x-\bar{y}|^2}\right\rangle + \frac{\langle x-y,\tau_y\rangle^2}{(1-\kappa_y d_y)|x-y|^2} - 1$$
$$- \langle x-y, d_x\nu_x\rangle\left(\frac{1}{|\bar{x}-y|^2} - \frac{1}{|x-\bar{y}|^2}\right) + o(1). \tag{4.2}$$

We shall now see, that the whole last line of (4.2) is in fact $o(1)$:

$$\left|\langle x-y, d_x\nu_x\rangle\left(\frac{1}{|\bar{x}-y|^2} - \frac{1}{|x-\bar{y}|^2}\right)\right| \le d_x|x-y| \cdot \frac{1}{|\bar{x}-y|^2|x-\bar{y}|^2} \cdot \left||\bar{x}-y|^2 - |x-\bar{y}|^2\right|$$

$$\le \frac{d_x|x-y|}{|\bar{x}-y|^2|x-\bar{y}|^2} \cdot c(d_x+d_y)|p(x)-p(y)|^2 \le \tilde{c}\frac{d_x(d_x+d_y)|\bar{x}-y|^2}{|\bar{x}-y|^2|x-\bar{y}|^2} \cdot |x-y|$$
$$\le \tilde{c}|x-y| = o(1)$$

for some constants $c, \tilde{c} > 0$, where we used lemma 2.10 repeatedly. For the sake of a more readable presentation, we continue by estimating the second term of (4.2) separately.

$$\left\langle x-y, \frac{d_x\nu_x - d_y\nu_y}{|x-\bar{y}|^2}\right\rangle = \frac{|x-y|^2}{|x-\bar{y}|^2} - \left\langle \frac{x-y}{|x-\bar{y}|^2}, p(x)-p(y)\right\rangle$$

$$= \frac{|x-y|^2}{|x-\bar{y}|^2} - \left\langle \frac{x-y}{|x-\bar{y}|^2}, \frac{1}{1-\kappa_y d_y}\langle x-y,\tau_y\rangle\,\tau_y + o(|x-y|)\right\rangle$$

$$= \frac{|x-y|^2}{|x-\bar{y}|^2} - \frac{\langle x-y,\tau_y\rangle^2}{(1-\kappa_y d_y)|x-\bar{y}|^2} + o(1)$$

Therefore

$$A(x,y) = 2\left(\frac{d_x^2}{|\bar{x}-y|^2} + \frac{d_y^2}{|\bar{y}-x|^2}\right) - \frac{|x-y|^2}{|x-\bar{y}|^2} + \frac{\langle x-y,\tau_y\rangle^2}{1-\kappa_y d_y}\left(\frac{1}{|x-\bar{y}|^2} + \frac{1}{|x-y|^2}\right) - 1 + o(1)$$

$$= 2\left(\frac{d_x^2}{|\bar{x}-y|^2} + \frac{d_y^2}{|\bar{y}-x|^2}\right) - \frac{|x-y|^2}{|x-\bar{y}|^2} + \langle x-y,\tau_y\rangle^2\left(\frac{1}{|x-\bar{y}|^2} + \frac{1}{|x-y|^2}\right) - 1$$
$$+ \langle x-y,\tau_y\rangle^2\left(\frac{1}{|x-\bar{y}|^2} + \frac{1}{|x-y|^2}\right)\left(\frac{1}{1-\kappa_y d_y} - 1\right) + o(1).$$

Since $|\bar{y}-x| \ge \hat{c}|x-y|$ for some $\hat{c} > 0$ and $d_y = o(1)$, whereas κ_y is bounded, the last line of the preceding formula is again $o(1)$. Since $|x-y|^2 = \langle x-y,\tau_y\rangle^2 + \langle x-y,\nu_y\rangle^2$ we may then rewrite $A(x,y)$ as

$$A(x,y) = 2\left(\frac{d_x^2}{|\bar{x}-y|^2} + \frac{d_y^2}{|\bar{y}-x|^2}\right) - \frac{|x-y|^2}{|x-\bar{y}|^2} + \langle x-y, \tau_y\rangle^2 \left(\frac{1}{|x-\bar{y}|^2} + \frac{1}{|x-y|^2}\right) - 1 + o(1)$$

$$= 2\left(\frac{d_x^2}{|\bar{x}-y|^2} + \frac{d_y^2}{|\bar{y}-x|^2}\right) - \langle x-y, \nu_y\rangle^2 \left(\frac{1}{|x-\bar{y}|^2} + \frac{1}{|x-y|^2}\right) + o(1)$$

$$= 2\frac{d_x^2 + d_y^2}{|x-\bar{y}|^2} - \langle x-y, \nu_y\rangle^2 \left(\frac{1}{|x-\bar{y}|^2} + \frac{1}{|x-y|^2}\right)$$

$$-2d_x^2\left(\frac{1}{|x-\bar{y}|^2} - \frac{1}{|\bar{x}-y|^2}\right) + o(1).$$

Again, the last line is $o(1)$, since

$$\left| d_x^2 \left(\frac{1}{|x-\bar{y}|^2} - \frac{1}{|\bar{x}-y|^2}\right)\right| = \frac{d_x^2 \left| |\bar{x}-y|^2 - |x-\bar{y}|^2 \right|}{|\bar{x}-y|^2 |x-\bar{y}|^2} \le c\frac{d_x^2 |p(x)-p(y)|^2}{|\bar{x}-y|^2|x-\bar{y}|^2} \cdot (d_x + d_y) = o(1),$$

which is also obtained using lemma 2.10. In the following we abbreviate $\alpha = \langle x-y, \nu_y\rangle$, $\beta = \langle x-y, \tau_y\rangle$, hence

$$x - y = \alpha\nu_y + \beta\tau_y,$$

$$x - \bar{y} = x - y + 2d_y\nu_y = (\alpha + 2d_y)\nu_y + \beta\tau_y,$$

$$d_x^2 = d_y^2 + \mathrm{D}d^2(y)(x-y) + \tfrac{1}{2}\mathrm{D}^2 d^2(y)[x-y, x-y] + o(|x-y|^2)$$

$$= d_y^2 + 2d_y\langle x-y, \nu_y\rangle + \langle x-y, (\mathrm{id} - \mathrm{D}p(y))(x-y)\rangle + o(|x-y|^2)$$

$$= d_y^2 + 2\alpha d_y + \alpha^2 + \beta^2 - \frac{1}{1-\kappa_y d_y}\beta^2 + o(|x-y|^2)$$

$$= d_y^2 + 2\alpha d_y + \alpha^2 + \beta^2\left(1 - \frac{1}{1-\kappa_y d_y}\right) + o(|x-y|^2)$$

$$= d_y^2 + 2\alpha d_y + \alpha^2 + o(|x-y|^2),$$

which implies

$$A(x,y) = \frac{4d_y^2 + 4\alpha d_y + 2\alpha^2 - \alpha^2}{(\alpha+2d_y)^2 + \beta^2} - \frac{\alpha^2}{\alpha^2+\beta^2} + o(1) = \frac{(\alpha+2d_y)^2}{(\alpha+2d_y)^2+\beta^2} - \frac{\alpha^2}{\alpha^2+\beta^2} + o(1),$$

which is precisely our first claim. The second claim also follows easily, since $\frac{\xi^2}{\xi^2+\beta^2} \in [0,1]$ for every $\xi \in \mathbb{R}$.

We will now show that the above is in fact nonnegative up to an error of $o(1)$ if Ω is strictly convex.

First observe that $A(x,y) = 0$ for $\beta = 0$. On the other hand, setting $f(t) := \frac{t^2}{t^2+\beta^2}$ for

$\beta \neq 0$ and $t \in \mathbb{R}$ we may apply the mean value theorem to get

$$A(x,y) = f(\alpha + 2d_y) - f(\alpha) + o(1) = 2d_y f'(\xi) + o(1) = \frac{4d_y \beta^2 \xi}{(\xi^2 + \beta^2)^2} + o(1)$$

for some $\xi \in [\alpha, \alpha + 2d_y]$.

Now define $\alpha = \langle x - y, \nu_y \rangle$ and $\alpha' := \langle y - x, \nu_x \rangle$. In the sequel, we will show that for any $x, y \in \Omega_\varepsilon$ one of both scalar products is nonnegative if $|x - y|$ is small enough.

Assume on the contrary that there are sequences $(x^n), (y^n) \subset \Omega_\varepsilon$ with $|x^n - y^n| \to 0$ as $n \to \infty$ such that $\alpha_n = \langle x^n - y^n, \nu_{y^n} \rangle < 0$ and $\alpha'_n = \langle y^n - x^n, \nu_{x^n} \rangle < 0$ for every $n \in \mathbb{N}$. Then we have

$$\begin{aligned} 0 > \alpha_n + \alpha'_n &= \langle x^n - y^n, \nu_{y^n} - \nu_{x^n} \rangle = \langle x^n - y^n, -\mathrm{D}\nu_{y^n}(x^n - y^n) + o(|x^n - y^n|) \rangle \\ &= \left\langle x^n - y^n, \frac{\kappa_{y^n}}{1 - \kappa_{y^n} d_{y^n}} \langle x^n - y^n, \tau_{y^n} \rangle \tau_{y^n} + o(|x^n - y^n|) \right\rangle \\ &= \frac{\kappa_{y^n}}{1 - \kappa_{y^n} d_{y^n}} \beta_n^2 + o(|x^n - y^n|^2), \end{aligned}$$

hence $\beta_n^2 := \langle x^n - y^n, \tau_{y^n} \rangle^2 = o(|x^n - y^n|^2)$, since $\kappa_y \geq \tilde{\varepsilon}$ for all $y \in \Omega_\varepsilon$ and some $\tilde{\varepsilon} > 0$ if Ω is strictly convex. Since $\alpha_n^2 + \beta_n^2 = |x^n - y^n|^2$ we infer

$$\frac{\alpha_n^2}{|x^n - y^n|^2} = 1 - \frac{\beta_n^2}{|x^n - y^n|^2} \to 1$$

as $n \to \infty$, which implies

$$\frac{\alpha_n}{|x^n - y^n|} \to -1$$

as $n \to \infty$, since we have assumed $\alpha_n < 0$. It follows that

$$-1 \leftarrow \frac{\alpha_n}{|x^n - y^n|} > \frac{\alpha_n + \alpha'_n}{|x^n - y^n|} = \frac{\kappa_{y^n}}{1 - \kappa_{y^n} d_{y^n}} \frac{\beta_n^2}{|x^n - y^n|} + o(|x^n - y^n|) \to 0$$

as $n \to \infty$, which is the desired contradiction.

Hence for every $x, y \in \Omega_\varepsilon$ sufficiently close to each other, one of the two scalar products α, α' is nonnegative, and since $A(x,y)$ is symmetric in x, y by definition, we might interchange the roles of x and y to assume $\alpha \geq 0$, which in turn implies $\xi \geq 0$ and we are done. $\square$

Lemma 4.7. *Let Γ be ∂–admissible, $\bar{z} \in \partial_{bdry}\mathcal{F}_N\Omega$ and let $C \in \mathcal{P}_{\bar{z}}$ satisfy $\bar{z}_C \in \partial\Omega$. There is $\delta > 0$ such that for every $z \in U_\delta(\bar{z}) \cap \mathcal{F}_N\Omega$*

$$|\nabla H_\Omega(z)| \geq \frac{\varepsilon_C}{2\pi} \left(\sum_{j \in C} d(z_j)^2 \right)^{-\frac{1}{2}},$$

where the constant $\varepsilon_C > 0$ is given by

$$\varepsilon_C := \frac{1}{2}\left(\sum_{i\in C}\Gamma_i^2 - \sum_{\substack{i,j\in C\\ i\neq j}}|\Gamma_i\Gamma_j|\right),$$

if Ω is not strictly convex, and by

$$\varepsilon_C := \frac{1}{2}\left(\sum_{i\in C}\Gamma_i^2 - \sum_{\substack{i,j\in C\\ \Gamma_i\Gamma_j<0}}|\Gamma_i\Gamma_j|\right)$$

instead, if it is strictly convex.

Proof. Note that in each case $\varepsilon_C > 0$ by the condition of ∂–admissibility. There is $\tilde{\delta} > 0$, such that $z_j \in \Omega_\varepsilon$ for any $j \in C$, $z \in U_{\tilde{\delta}}(\bar{z}) \cap \mathcal{F}_N\Omega$. We thus may consider the function

$$\Phi_C : U_{\tilde{\delta}}(\bar{z}) \cap \mathcal{F}_N\Omega \ni z \mapsto \pi\sum_{j\in C} d(z_j)^2 \in [0,\infty)$$

and simply compute

$$\langle\nabla H_\Omega(z), \nabla\Phi_C(z)\rangle = 2\pi\sum_{j\in C}\left\langle\nabla_{z_j}H_\Omega(z), d(z_j)\nu(z_j)\right\rangle$$

$$= 2\pi\sum_{j\in C}\left\langle\Gamma_j^2\nabla h(z_j) + 2\sum_{i\in C\setminus\{j\}}\Gamma_i\Gamma_j\nabla_{z_j}G(z_j,z_i) + O(1), d(z_j)\nu(z_j)\right\rangle$$

$$= 2\pi\sum_{j\in C}\left\langle\Gamma_j^2\nabla h(z_j), d(z_j)\nu(z_j)\right\rangle + 4\pi\sum_{\substack{i,j\in C\\ i\neq j}}\Gamma_i\Gamma_j\left\langle\nabla_{z_j}G(z_j,z_i), d(z_j)\nu(z_j)\right\rangle + o(1)$$

$$= \sum_{j\in C}\Gamma_j^2\left\langle\frac{\nu(z_j)}{d(z_j)}, d(z_j)\nu(z_j)\right\rangle + 2\sum_{\substack{i,j\in C\\ i<j}}\Gamma_i\Gamma_j A(z_i,z_j) + o(1)$$

as $z \to \bar{z}$. Here we have used the fact that $\nabla_{z_j}G(z_j,z_i) = O(1)$ for $j \in C$, $i \notin C$ as $z \to \bar{z}$ by hypothesis 2.4.

In case Ω is not strictly convex, we may estimate this by

$$\geq \sum_{j\in C}\Gamma_j^2 - 2\sum_{\substack{i,j\in C\\ i<j}}|\Gamma_i\Gamma_j| + o(1) = \sum_{j\in C}\Gamma_j^2 - \sum_{\substack{i,j\in C\\ i\neq j}}|\Gamma_i\Gamma_j| + o(1) = 2\varepsilon_C + o(1)$$

as $z \to \bar{z}$ by use of lemma 4.6.

If, on the other hand, Ω is strictly convex, we may again use lemma 4.6 and similarly conclude

$$\begin{aligned}\langle \nabla H_\Omega(z), \nabla \Phi_C(z)\rangle &\geq \sum_{j\in C}\Gamma_j^2 - 2\sum_{\substack{i,j\in C\\ i<j,\,\Gamma_i\Gamma_j<0}}|\Gamma_i\Gamma_j| + o(1)\\ &= \sum_{j\in C}\Gamma_j^2 - \sum_{\substack{i,j\in C\\ \Gamma_i\Gamma_j<0}}|\Gamma_i\Gamma_j| + o(1) = 2\varepsilon_C + o(1)\end{aligned}$$

as $z \to \bar{z}$. In any case we obtain that there is $\delta \in (0, \tilde{\delta})$ such that

$$\langle \nabla H_\Omega(z), \nabla \Phi_C(z)\rangle \geq \varepsilon_C$$

for every $z \in U_\delta(\bar{z}) \cap \mathcal{F}_N\Omega$.

On the other hand we have

$$\langle \nabla H_\Omega(z), \nabla \Phi_C(z)\rangle \leq |\nabla H_\Omega(z)| \cdot |\nabla \Phi_C(z)| = 2\pi \cdot |\nabla H_\Omega(z)| \cdot \left(\sum_{j\in C} d(z_j)^2\right)^{\frac{1}{2}}$$

by simply applying the Cauchy–Schwarz–inequality, hence we obtain

$$|\nabla H_\Omega(z)| \geq \frac{\varepsilon_C}{2\pi}\left(\sum_{j\in C} d(z_j)^2\right)^{-\frac{1}{2}}$$

for every $z \in U_\delta(\bar{z}) \cap \mathcal{F}_N\Omega$ and we are done. $\square$

4.3 Proof of proposition 4.3

Equipped with these estimates we now turn to the proof of proposition 4.3, which is comprised of the next few lemmata.

Lemma 4.8. *There is $\delta > 0$ such that $|\nabla H_\Omega(z)| > 1$ for every $z \in \mathcal{M}_\delta$.*

Proof. Assume on the contrary that there are sequences $\delta_n \to 0$, $\delta_n > 0$ and $z^n \in \mathcal{M}_{\delta_n}$ such that $|\nabla H_\Omega(z^n)| \leq 1$ for all $n \in \mathbb{N}$. Then, upon choosing a convergent subsequence, we may assume $z^n \in U_{\delta_n}(\bar{z})$ for some $\bar{z} \in \partial\mathcal{F}_N\Omega$ and every $n \in \mathbb{N}$. Now if $\bar{z} \in \partial_{\text{int}}\mathcal{F}_N\Omega$ there are $n_0 \in \mathbb{N}$ and $C \in \mathcal{C}(\mathcal{P}_{\bar{z}})$ satisfying $\bar{z}_C \in \Omega$ such that

$$|\nabla H_\Omega(z^n)| \geq \frac{C_\Gamma}{4\pi}\left(\sum_{i\in C}|z_i^n - \bar{z}_C|^2\right)^{-\frac{1}{2}} \geq \frac{C_\Gamma}{4\pi\sqrt{|C|}\delta_n}$$

for every $n \geq n_0$ by lemma 4.5. Similarly, if $\bar{z} \in \partial_{\text{bdry}}\mathcal{F}_N\Omega$ there are $n_0 \in \mathbb{N}$ and $C \in \mathcal{P}_{\bar{z}}$

satisfying $\bar z_C \in \partial\Omega$ such that

$$|\nabla H_\Omega(z^n)| \geq \frac{\varepsilon_C}{2\pi}\left(\sum_{j\in C} d(z_j^n)^2\right)^{-\frac{1}{2}} \geq \frac{\varepsilon_C}{2\pi\sqrt{|C|}\delta_n}$$

for every $n \geq n_0$ by lemma 4.7. This, however, contradicts the fact that $|\nabla H_\Omega(z^n)| \leq 1$ for all $n \in \mathbb{N}$ and the proof is done. $\square$

Since $\overline{\mathcal{M}_\delta}$ is a neighbourhood of $\partial\mathcal{F}_N\Omega$ in $\overline{\Omega}^N$, this in particular shows that H_Ω satisfies the Palais–Smale–condition.

Lemma 4.9. *Let $z \in \mathcal{F}_N\Omega$ satisfy*

$$\lim_{t\to t^+(z)} H_\Omega(\varphi(z,t)) = c_0 < \infty.$$

Then $t^+(z) = \infty$.

Proof. In general we have for $s,t \in [0,t^+(z))$, $s<t$ and $z^t := \varphi(z,t)$

$$\begin{aligned}|z^t - z^s| &\leq \int_s^t |\nabla H_\Omega(z^\tau)|\mathrm{d}\tau \leq \sqrt{t-s}\sqrt{\int_s^t |\nabla H_\Omega(z^\tau)|^2 \mathrm{d}\tau} = \sqrt{t-s}\sqrt{H_\Omega(z^t)-H_\Omega(z^s)} \\ &\leq \sqrt{|t-s|}\sqrt{c_0 - H_\Omega(z^s)} \end{aligned} \tag{4.3}$$

Now if $t^+(z) < \infty$ we may take the limit $t \to t^+(z)$ on the right hand side of (4.3) to obtain that for every $\varepsilon > 0$ there is $t_0 \in [0,t^+(z))$ and any $s,t \in [t_0,t^+(z))$, $s<t$: $|z^t - z^s| < \varepsilon$, hence $z^t \to \bar z$ as $t \to t^+(z)$ for some $\bar z \in \overline{\mathcal{F}_N\Omega} = \overline{\Omega}^N$, since $\overline{\Omega}^N$ is compact.

Let $\delta > 0$ be such that the consequences of lemmata 4.5 and 4.7 hold.

If $\bar z \in \partial_{\mathrm{bdry}}\mathcal{F}_N\Omega$ we find $C \in \mathcal{P}_{\bar z}$ and $t_0 \in [0,t^+(z))$ such that for every $t \in [t_0,t^+(z))$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_C(z^t) = \left\langle \nabla H_\Omega(z^t), \nabla\Phi_C(z^t)\right\rangle \geq \varepsilon_C > 0,$$

by application of lemma 4.7 which is a contradiction.

If, on the other hand, $\bar z \in \partial_{\mathrm{int}}\mathcal{F}_N\Omega$, we have $C \in \mathcal{C}(\mathcal{P}_{\bar z})$ as well as $t_0 \in [0,t^+(z))$, such that for all $t \in [t_0,t^+(z))$

$$|\nabla H_\Omega(z^t)| \geq \frac{C_\Gamma}{4\pi}\left(\sum_{i\in C}|z_i^t - \bar z_C|^2\right)^{-\frac{1}{2}}$$

by lemma 4.5. We thus may compute for $s \in [t_0,t^+(z))$, $t \in (s,t^+(z))$

$$H_\Omega(z^t) - H_\Omega(z^s) \geq \int_s^t \frac{C_\Gamma}{4\pi}|\nabla H_\Omega(z^\tau)|\left(\sum_{i\in C}|z_i^\tau - \bar z_C|^2\right)^{-\frac{1}{2}}\mathrm{d}\tau$$

$$= \frac{C_\Gamma}{4\pi} \int_s^t |\dot z^\tau| \left(\sum_{i \in C} |z_i^\tau - \bar z_C|^2 \right)^{-\frac{1}{2}} \mathrm{d}\tau \geq \frac{C_\Gamma}{4\pi} \int_s^t |\pi_C \dot z^\tau| \left(\sum_{i \in C} |z_i^\tau - \bar z_C|^2 \right)^{-\frac{1}{2}} \mathrm{d}\tau$$

$$\geq \frac{C_\Gamma}{4\pi} \int_s^t \left| \left\langle \pi_C \dot z^\tau, \frac{\pi_C (z^\tau - \bar z)}{|\pi_C (z^\tau - \bar z)|} \right\rangle \right| \left(\sum_{i \in C} |z_i^\tau - \bar z_C|^2 \right)^{-\frac{1}{2}} \mathrm{d}\tau$$

$$= \frac{C_\Gamma}{4\pi} \int_s^t \frac{\left| \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\sum_{i \in C} |z_i^\tau - \bar z_C|^2 \right)^{\frac{1}{2}} \right|}{\left(\sum_{i \in C} |z_i^\tau - \bar z_C|^2 \right)^{\frac{1}{2}}} \mathrm{d}\tau \geq -\frac{C_\Gamma}{4\pi} \int_s^t \frac{\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\sum_{i \in C} |z_i^\tau - \bar z_C|^2 \right)^{\frac{1}{2}}}{\left(\sum_{i \in C} |z_i^\tau - \bar z_C|^2 \right)^{\frac{1}{2}}} \mathrm{d}\tau$$

$$= \frac{C_\Gamma}{8\pi} \ln \frac{\sum_{i \in C} |z_i^s - \bar z_C|^2}{\sum_{i \in C} |z_i^t - \bar z_C|^2} \to \infty \quad (t \to t^+(z)),$$

contrary to our assumption. It follows that $t^+(z) = \infty$, which is what we were to show. □

The next lemma finishes the proof of proposition 4.3.

Lemma 4.10. *If there is $z \in \mathcal{F}_N \Omega$ satisfying*

$$\lim_{t \to t^+(z)} H_\Omega(\varphi(z,t)) = c_0 < \infty,$$

there is a sequence $(z^n) \subset \mathcal{F}_N \Omega$ and a point $z^* \in \mathcal{F}_N \Omega$ such that $z^n \to z^*$ and $\nabla H_\Omega(z^n) \to 0$ as $n \to \infty$.

Proof. Since $t^+(z) = \infty$ by lemma 4.9, consider a sequence $(t_n) \subset [0, \infty)$, $t_n \to \infty$, such that $z^{t_n} \to z^*$ as $n \to \infty$.

Let us assume at first that there are $n_0 \in \mathbb{N}$, $\delta > 0$, such that $\delta \leq t_{n+1} - t_n$ and $|\nabla H_\Omega(z^t)| \geq \frac{1}{\sqrt{n}}$ for all $n \geq n_0$ and for all $t \in [t_n, t_n + \delta]$. Then we have

$$H_\Omega(z^{t_n}) \geq H_\Omega(z^{t_{n_0}}) + \sum_{j=n_0}^{n} \int_{t_j}^{t_j+\delta} |\nabla H_\Omega(z^s)|^2 \mathrm{d}s = H_\Omega(z^{t_{n_0}}) + \sum_{j=n_0}^{n} \frac{\delta}{j} \to \infty \quad (n \to \infty),$$

contrary to our main assumption.

Thus there exists a sequence $\delta_n \to 0$, $\delta_n > 0$ for all $n \in \mathbb{N}$, such that with $s_n := t_n + \delta_n$: $|\nabla H_\Omega(z^{s_n})| < \frac{1}{\sqrt{n}}$. It follows that $z^{s_n} \to z^*$ as $n \to \infty$ because of (4.3), since $|s_n - t_n| = \delta_n \to 0$. Abbreviating z^{s_n} by z^n we have $\nabla H_\Omega(z^n) \to 0$ as well as $z^n \to z^*$ as $n \to \infty$. Lemma 4.8 now implies $z^* \in \mathcal{F}_N \Omega$ and the proof is finished. □

This also finishes the proof of proposition 4.3. All that is left to do for proving our main results concerning asymmetric domains is to provide a sort of linking argument for H_Ω, that is finding a point $z \in \mathcal{F}_N \Omega$ such that $H_\Omega(z^t)$ has a finite limit for $t \to \infty$. This is done by applying lemma 2.14 within the next section.

5 Linking phenomena for H_Ω

The goal of this section is to prove the following two theorems.

Theorem 5.1. *For any $N \in \mathbb{N}$ and any $\mathcal{L}$–admissible, ∂–admissible and Δ–admissible parameter $\Gamma \in \mathbb{R}^N$, the Kirchhoff–Routh path function H_Ω has a critical point in $\mathcal{F}_N\Omega$.*

Theorem 5.2. *Suppose that $\pi_1(\Omega) \neq 0$. Then for any $N \in \mathbb{N}$ and for any ∂–admissible and Δ–admissible parameter $\Gamma \in \mathbb{R}^N$, the Kirchhoff–Routh path function H_Ω has a critical point in $\mathcal{F}_N\Omega$.*

5.1 The simply connected case

This subsection is concerned with the proof of theorem 5.1.

Let Γ be ∂– and Δ–admissible, and let Γ be $\mathcal{L}$–admissible with corresponding permutation σ. Reordering the vortices, we may without loss of generality assume that $\sigma = \mathrm{id}$ and hence abbreviate $\mathcal{L}_N\Omega := \mathcal{L}_N^{\mathrm{id}}\Omega$.

The theorem is trivial for $N \leq 2$, for then $H_\Omega(z) \to -\infty$ for $z \to \partial\Omega$ and consequently H_Ω assumes a local maximum in $\mathcal{F}_N\Omega$, since $h(z_j) \to -\infty$ for $z_j \to \partial\Omega$, $j \in \{1, 2\}$ as well as $\Gamma_1\Gamma_2 G(z_1, z_2) \to -\infty$ if $|z_1 - z_2| \to 0$, since $\Gamma_1\Gamma_2 < 0$.

In what comes we thus consider the case $N \geq 3$ and begin to construct an explicit linking for H_Ω.

Without loss of generality we may assume $0 \in \Omega$. Choose $\rho > 0$ such that $\overline{B_{N\rho}(0)} \subset \Omega$. Define

$$\gamma_0 : \mathbb{T}^{N-2} := (S^1)^{N-2} \to \mathcal{F}_N\Omega$$

$$\gamma_{0,1}(\zeta_1, \ldots, \zeta_{N-2}) := 0, \qquad \gamma_{0,N}(\zeta_1, \ldots, \zeta_{N-2}) := (N-1)\rho,$$

$$\gamma_{0,j}(\zeta_1, \ldots, \zeta_{N-2}) = (j-1)\rho\zeta_{j-1},$$

for $j \in \{2, \ldots, N-1\}$, where $\gamma_{0,j}$ denotes the j-th component of γ_0. Setting

$$\Gamma_0 := \left\{\gamma \in C^0(\mathbb{T}^{N-2}, \mathcal{F}_N\Omega) : \gamma \simeq \gamma_0\right\},$$

we have the following

Lemma 5.3. *For every $\gamma \in \Gamma_0$: $\gamma(\mathbb{T}^{N-2}) \cap \mathcal{L}_N\Omega \neq \emptyset$.*

Proof. Let $\tilde{H} : \mathbb{T}^{N-2} \times [0,1] \to \mathcal{F}_N\Omega$ be a deformation from γ_0 to γ. For $t \in [0,1]$ define

$$h_t : \mathbb{T}^{N-2} \times [0,1]^{N-2} \to \mathbb{C}^{N-2}$$

by setting

$$h_{t,j}(\zeta, s) := s_j\left(\tilde{H}_j(\zeta, t) - \tilde{H}_{j+1}(\zeta, t)\right) + (1 - s_j)\left(\tilde{H}_{j+2}(\zeta, t) - \tilde{H}_{j+1}(\zeta, t)\right)$$

for $j \in \{1, \ldots, N-2\}$. Obviously $\gamma(\zeta) \in \mathcal{L}_N\Omega$ if and only if $h_1(\zeta, s) = 0$ for some $s \in [0,1]^{N-2}$. Furthermore $h_t(\zeta, s) \neq 0$ for all $s \in \partial\left([0,1]^{N-2}\right)$, $t \in [0,1]$, since $\tilde{H}(\zeta, t) \in$

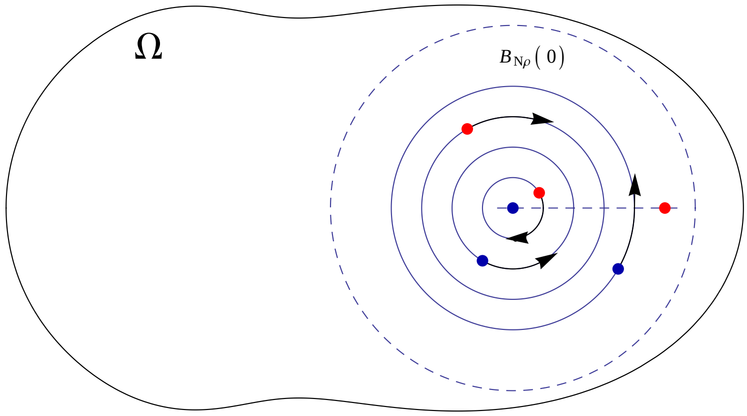

Figure 7: The map γ_0.

$\mathcal{F}_N\Omega$, so the map

$$\begin{gathered}\tilde{g} : \left(\mathbb{T}^{N-2}\times [0,1]^{N-2}\times [0,1], \mathbb{T}^{N-2}\times \partial\left([0,1]^{N-2}\right) \times [0,1]\right) \to \left(\mathbb{C}^{N-2}, \mathbb{C}^{N-2}\setminus\{0\}\right) \\ \tilde{g}(\zeta, s, t) := h_t(\zeta, s)\end{gathered} \tag{5.1}$$

is well–defined and continuous.

Using the Künneth–formula for the pair

$$(X, A) := \left(\mathbb{T}^{N-2} \times [0,1]^{N-2}, \mathbb{T}^{N-2} \times \partial\left([0,1]^{N-2}\right)\right) \cong \left(\mathbb{T}^{N-2} \times D^{N-2}, \mathbb{T}^{N-2} \times S^{N-3}\right)$$

we easily get that

$$H_{2N-4}\left(\mathbb{T}^{N-2} \times [0,1]^{N-2}, \mathbb{T}^{N-2} \times \partial\left([0,1]^{N-2}\right)\right) \cong \mathbb{Z},$$

where we are using singular homology with coefficients in $\mathbb{Z}$. Since $\tilde{g}$ is a homotopy of pairs by (5.1), h_t induces a homomorphism in homology

$$h_* : H_{2N-4}\left(\mathbb{T}^{N-2} \times [0,1]^{N-2}, \mathbb{T}^{N-2} \times \partial\left([0,1]^{N-2}\right)\right) \to H_{2N-4}\left(\mathbb{C}^{N-2}, \mathbb{C}^{N-2} \setminus \{0\}\right),$$

which is independent of $t \in [0,1]$. We claim that the degree $h_*(1) \in \mathbb{Z}$ of h_0 is nonzero.

Observe that h_0 has a unique zero at $p := (\zeta_0, s_0) := \left(1, \ldots, 1, \frac{1}{2}, \ldots, \frac{1}{2}\right)$. Abbreviating $(Y, B) := \left(\mathbb{C}^{N-2}, \mathbb{C}^{N-2} \setminus \{0\}\right)$ we thus have the following commutative diagram:

$$\begin{tikzcd}
(X, A) \arrow[rr, "h_0"] \arrow[d, hook, "i"] & & (Y, B) \\
(X, X \setminus \{p\}) \arrow[urr, "u"] & & (U_\varepsilon(p), U_\varepsilon(p) \setminus \{p\}), \arrow[ll, hook', "j"'] \arrow[u, "w = h_0|_{(U_\varepsilon(p), U_\varepsilon(p)\setminus\{p\})}"']
\end{tikzcd}$$

where u in the middle is given by $u : (X, X \setminus \{p\}) \to (Y, B), x \mapsto h_0(x)$. Taking the $(2N-4)$-th homology, we notice that the restriction homomorphism i_* is an isomorphism since A is the boundary of the ∂–manifold X, where $X \setminus A$ is orientable and i_* maps a generator $\{X\}$ of $H_{2N-4}(X, A)$, which is a fundamental class corresponding to a global orientation of X to a local orientation of X, i.e. a generator of the local homology group $H_{2N-4}(X, X \setminus \{p\})$ of X. See, for example [18], chapter V, theorem 13.1 for further details and rigorous proofs. Since j_* is an excision isomorphism, we have

$$h_* = u_* i_* = w_* j_*^{-1} i_*,$$

so we are done if the map w_* to the right is an isomorphism. But this is surely the case, as $w_*(1) \in \mathbb{Z}$ for small $\varepsilon > 0$ is the local degree of the differentiable map h_0 at p and is nonzero, which can be easily computed as follows: We have (regarding $\mathbb{C}^{N-2}$ as $\mathbb{R}^{2N-4}$)

$$\frac{\partial}{\partial \zeta_{j-1}} h_{0,j}(p) = \frac{1}{2}(j-1)\rho \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \frac{\partial}{\partial \zeta_j} h_{0,j}(p) = -j\rho \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \frac{\partial}{\partial \zeta_{j+1}} h_{0,j}(p) = \frac{1}{2}(j+1)\rho \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\frac{\partial}{\partial s_j} h_{0,j}(p) = -2\rho \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

whereas all the other partial derivatives vanish. Reordering the Jacobian of h_0 at p, such that the first $N-2$ rows correspond to the imaginary parts of the $h_{0,j}$, we get that

$$\mathrm{D}h_0(p) = \begin{pmatrix} A_N & 0 \\ 0 & B_N \end{pmatrix},$$

where $B_N = \mathrm{diag}(-2\rho) \in \mathbb{R}^{(N-2)\times(N-2)}$. Developing the last $N-2$ columns of $\det \mathrm{D}h_0(p)$, and using multilinearity to get rid of the ρ-factors, we get

$$|\det \mathrm{D}h_0(p)| = 2^{N-2}\rho^{2N-4} \,\mathrm{abs} \begin{vmatrix} -1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{1}{2} & -2 & \frac{3}{2} & 0 & & & \vdots \\ 0 & 1 & -3 & 2 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & \frac{N-5}{2} & -(N-4) & \frac{N-3}{2} & 0 \\ \vdots & & & 0 & \frac{N-4}{2} & -(N-3) & \frac{N-2}{2} \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{N-3}{2} & -(N-2) \end{vmatrix}$$

By looking at the first $N-3$ rows of the above matrix, we see that if $v \in \mathbb{R}^{N-2}$ is in the kernel, v must have all components equal to some $v_0 \in \mathbb{R}$. The last row then implies that $v_0 = 0$, hence $\det \mathrm{D}h_0(p) \neq 0$ and we are done. $\square$

By applying lemma 2.14 to the homotopy class Γ_0, we get that there is $\zeta \in \mathbb{T}^{N-2}$ such that H_Ω remains bounded along the trajectory of $\gamma_0(\zeta)$. Since H_Ω satisfies the Palais–Smale–condition, the proof of theorem 5.1 is finished.

5.2 The case $\pi_1(\Omega) \neq 0$

This subsection is devoted to the proof of theorem 5.2. Hence let the paramter Γ be ∂– and Δ–admissible.

Let $\pi_1(\Omega) \neq 0$, that is $k_0 = \operatorname{rank} \pi_1(\Omega) \geq 1$ and select a bounded component Ω_1 of $\mathbb{C} \setminus \bar{\Omega}$. Without loss of generality we may assume $0 \in \Omega_1$. Defining

$$\mathcal{S} := \left\{ z \in \mathcal{F}_N\Omega : \ \forall j \in \{1, \dots, N\} : \frac{z_j}{|z_j|} = e^{\frac{2\pi i j}{N}} \right\},$$

we have the following

Lemma 5.4. *$H_\Omega\big|_{\mathcal{S}}$ is bounded from above.*

Proof. Hypothesis 2.1 implies that Ω satisfies an exterior ball condition, hence there is $\rho > 0$ such that $|z_i - z_j| > \rho$ for $i, j \in \{1, \dots, N\}$, $i \neq j$. Using hypothesis 2.4, the assertion is immediate. $\square$

Since $\bar{\Omega}$ is a ∂–manifold with $\partial\Omega_1$ a component of $\partial\Omega$ there exists a collar of $\partial\Omega_1 \cong S^1$, that is, an open neighborhood U of $\partial\Omega_1$ in $\bar{\Omega}$ and a homeomorphism

$$\tilde{h} : S^1 \times [0, 1) \to U$$

satisfying $h(S^1 \times \{0\}) = \partial\Omega_1$. Setting

$$h_j := \tilde{h}|_{S^1 \times \left\{\frac{j}{N+1}\right\}} : S^1 \to \Omega$$

for $j \in \{1, \dots, N\}$, the h_j are Jordan curves with disjoint images enclosing Ω_1. Thus

$$\gamma_0 := h_1 \times \cdots \times h_N : \mathbb{T}^N \to \mathcal{F}_N\Omega$$

is well defined, and setting

$$\Gamma_0 := \{\gamma : \mathbb{T}^N \to \mathcal{F}_N\Omega : \gamma \simeq \gamma_0\},$$

we have the following

Lemma 5.5. *For all $\gamma \in \Gamma_0$*

$$\gamma\left(\mathbb{T}^N\right) \cap \mathcal{S} \neq \emptyset.$$

Proof. Let $\gamma \in \Gamma_0$, and let $\tilde{H} : \mathbb{T}^N \times [0, 1] \to \mathcal{F}_N\Omega$ be a homotopy connecting γ and γ_0. Setting

$$r : \Omega \ni z \mapsto \frac{z}{|z|} \in S^1,$$

$$\Psi := r \times \cdots \times r : \mathcal{F}_N\Omega \to \mathbb{T}^N,$$

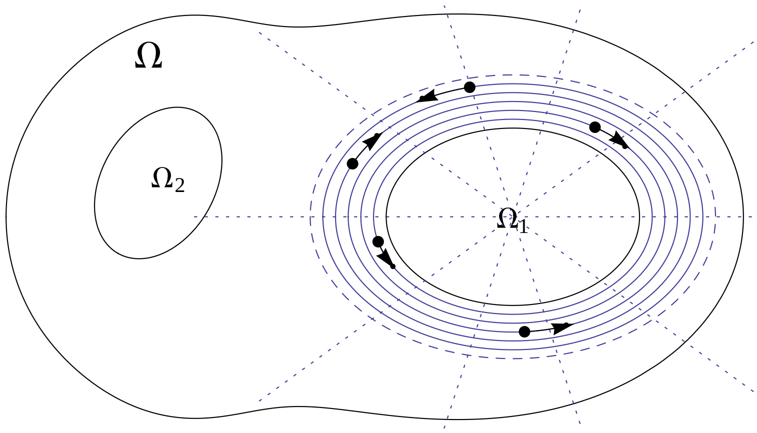

Figure 8: The map γ_0.

Ψ is well–defined and continuous, and the assertion is equivalent to $\bar{e} \in \Psi(\gamma(\mathbb{T}^N))$, where $\bar{e} = \left(e^{\frac{2\pi ij}{N}}\right)_{j\in\{1,\ldots,N\}} \in \mathbb{T}^N$. Now for every $t \in [0,1]$, the map $f_t := \Psi \circ \tilde{H}(\cdot, t)$ induces a homomorphism

$$f_* : \mathbb{Z} \cong H_N(\mathbb{T}^N) \to H_N(\mathbb{T}^N) \cong \mathbb{Z},$$

in singular homology which is independent of t. Since h_j is a homeomorphism onto its image and $h_i \simeq h_j$ for $i, j \in \{1, \ldots, N\}$, the map $r \circ h_j : S^1 \to S^1$ has winding number ± 1, hence induces an isomorphism $(r \circ h_j)_* = (r \circ h_1)_* : h_1(S^1) \to h_1(S^1)$. Now if $\{S^1\} \in H_1(S^1) \cong \mathbb{Z}$ is a generator, $\{\mathbb{T}^N\} = \{S^1\} \times \cdots \times \{S^1\}$ is a generator of $H_N(\mathbb{T}^N)$ and we compute

$$f_*(\{\mathbb{T}^N\}) = (\Psi \circ \gamma_0)_*(\{S^1\} \times \cdots \times \{S^1\}) = ((r \circ h_1) \times \cdots \times (r \circ h_N))_* (\{S^1\} \times \cdots \times \{S^1\})$$

$$= (r \circ h_1)_*(\{S^1\}) \times \cdots \times (r \circ h_1)_*(\{S^1\}) = \pm\{S^1\} \times \cdots \times \{S^1\} = \pm\{\mathbb{T}^N\},$$

hence f_* is an isomorphism. Now if $\bar{e} \notin \Psi(\gamma(\mathbb{T}^N))$, the isomorphism f_* factorizes over $H_N(\mathbb{T}^N \setminus \{\bar{e}\})$, that is we have a commutative diagram

$$\begin{tikzcd}
\mathbb{Z} \cong H_N(\mathbb{T}^N) \arrow[rr, "f_*", "\cong"'] \arrow[dr] & & H_N(\mathbb{T}^N) \cong \mathbb{Z} \\
& H_N(\mathbb{T}^N \setminus \{\bar{e}\}) \arrow[ur, "j_*"'] &
\end{tikzcd}$$

where $j : \mathbb{T}^N \setminus \{\bar{e}\} \hookrightarrow \mathbb{T}^N$. Further we have the exact sequence

$$H_N(\mathbb{T}^N \setminus \{\bar{e}\}) \xrightarrow{j_*} H_N(\mathbb{T}^N) \xrightarrow{\cong} H_N(\mathbb{T}^N, \mathbb{T}^N \setminus \{\bar{e}\}).$$

The restriction homomorphism to the right is an isomorphism since $\mathbb{T}^N$ is a compact orientable connected N–dimensional manifold, see for example [18], chapter V, theorem 12.1. Since the sequence is exact, we conclude that the homomorphism j_* is trivial, which is a contradiction and the proof of 5.5 is complete. $\square$

6 Outlook and application to partial differential equations

The purpose of this section is to discuss the stability of the results given in the previous sections as well as to give applications of our theorems to the nonlinear elliptic problems mentioned in the introduction, namely the sinh–Poisson–equation and the Lane–Emden–Fowler equation.

It is intuitively clear that the critical points of H_Ω constructed in the previous chapters are saddle points of H_Ω, hence are unstable as stationary points for the gradient flow of H_Ω. On the other hand, being derived largely by topological means and given the axiomatic setting of section 2, which in particular allows us to disturb the Green's–function G as a main ingredient of the functional H_Ω by an arbitrary function in $C^1(\overline{\Omega}^2)$, it is to be expected that our results are stable in the sense that one can disturb H_Ω quite a lot at least in compact subsets of $\mathcal{F}_N\Omega$ and still derive the existence of a critical point.

This is precisely what we do in the first subsection. We then go on to apply these results to find sign–changing solutions to the sinh–Poisson– and the Lane–Emden–Fowler–equation. Unfortunately, the case of $|\Gamma_j| \neq 1$ is of no use for these questions, so the only new solutions arising to these problems are those where the case $\Gamma_j = (-1)^j$ is allowed, namely the situation of theorem 3.6.

Nevertheless, every solution of a Poisson–equation on Ω gives rise to a stationary solution for the Euler–equations of an incompressible fluid, and so do the solutions to the sinh–Poisson– and Lane–Emden–Fowler equations. Only recently [5] have constructed a more complicated equation whose solutions concentrate around a critical point of H_Ω, but without giving any existence results. They are able to obtain their result by admitting general vorticities Γ, and our results complement theirs by giving corresponding existence results, consequently leading to new stationary vorticity solutions for the Euler equations.

6.1 Stability of previous results

The first results we state here are some fairly standard arguments which specify, how robust our results on symmetrical domains are with respect to symmetrical perturbations.

Theorem 6.1. *Let $\Omega \subset \mathbb{C}$ be a D_p–symmetric domain satisfying hypothesis 2.1 and let $V \in C^1(\overline{\Omega}^N)$, $n \in \mathbb{N}$ be a D_p–and $\Sigma(\Gamma)$–invariant functional. Then for every critical point z^* of H_Ω according to theorems 3.6, 3.9 and corollaries thereof the functional $H_\Omega + V$ has a corresponding critical point.*

Proof. Theorems 3.6 and 3.9 deliver us with local maxima of some reduced functional $E : W \to \mathbb{R}$ for some open and connected set $W \subset \mathbb{R}^k$. Since $\overline{\Omega}^N$ is compact and $E(z) \to -\infty$ as $z \to \partial W$ by theorem 2.12 and lemma 3.15, respectively, for instance in the situation of theorem 3.6, the functional $E + V \circ s_U$ also has a local maximum, which is also a critical point of the whole functional by the invariance assumption. □

Theorem 6.2. *Let $\Omega \subset \mathbb{C}$ be a D_p–symmetric domain satisfying hypothesis 2.1, let $\Gamma \in \mathbb{R}^N$ be $\mathcal{L}$–admissible and let $V_n \in C^1(\mathcal{F}_N\Omega)$, $n \in \mathbb{N}$ be D_p– and $\Sigma(\Gamma)$–invariant* *functionals, and let further $V_n \to 0$ uniformly on compact subsets of $\mathcal{F}_N\Omega$. There is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and for any U–symmetric critical point z^* of H_Ω according to theorems 3.6, 3.9 and corollaries thereof the functional $H_\Omega + V_n$ has a U–symmetric critical point z^n.*

Proof. As noted before, theorems 3.6 and 3.9 deliver us with a local maximum z^* of some reduced functional $E : W \to \mathbb{R}$ for some open connected set $W \subset \mathbb{R}^k$. Now for small $\varepsilon > 0$ the set $K_\varepsilon := \{z \in W : E(z) \geq E(z^*) - \varepsilon\}$ is compact by theorem 2.12 and lemma 3.15, respectively. Since $V_n \to 0$ uniformly on compact subsets of $\mathcal{F}_N\Omega$ there is $n_0 \in \mathbb{N}$ such that, for instance in the situation of the proof of theorem 3.6 for any $n \geq n_0$: $\sup_{z\in K_\varepsilon} |V_n(s_U z)| < \frac{\varepsilon}{2}$. It follows that for every $z \in \partial K_\varepsilon$:

$$E(z) + V_n(s_U(z)) = E(z^*) - \varepsilon + V_n(s_U(z)) \leq E(z^*) + V_n(s_U(z^*)),$$

hence $E + V_n \circ s_U$ achieves a local maximum z_n in K_ε. By the invariance assumption z_n is a critical point of $H_\Omega + V_n$. □

For the nonsymmetric case of course similar stability properties hold.

Theorem 6.3. *Let $V \in C^1(\overline{\Omega}^N)$ and $\Gamma \in \mathbb{R}^N$ be ∂– and Δ–admissible. Then if either $\pi_1(\Omega) \neq 0$ or Γ is $\mathcal{L}$–admissible, $H_\Omega + V$ has a critical point.*

Sketch of proof. This is proven by verifying that all the estimates on H_Ω and ∇H_Ω remain untouched when switching from H_Ω to $H_\Omega + V$. If, for example, H_Ω is bounded above on $\mathcal{L} \subset \mathcal{F}_N\Omega$, so is $H_\Omega + V$, since $\overline{\Omega}^N$ is compact. Similarly there is a constant $C > 0$ such that $|\nabla V(z)| \leq C$ for all $z \in \mathcal{F}_N\Omega$, so by the triangle inequality and the fact that $|\nabla H_\Omega(z)| \to \infty$ as $z \to \partial\mathcal{F}_N\Omega$ under the above assumptions it follows, that all of the arguments hold for $H_\Omega + V$ as well. □

Theorem 6.4. *Let Γ be Δ– and ∂–admissible. Let $\delta > 0$ such that $|\nabla H_\Omega(z)| > 1$ for all $z \in \mathcal{M}_\delta$. If $V \in C^1(\mathcal{F}_N\Omega)$ satisfies $|V(z) - H_\Omega(z)| < \frac{\delta}{8}$ and $|\nabla V(z) - \nabla H_\Omega(z)| < \frac{1}{4}$ for all $z \in \overline{\mathcal{F}_N\Omega \setminus \mathcal{M}_{\delta/4}}$. Then if either Γ is $\mathcal{L}$–admissible or $\pi_1(\Omega) \neq 0$, the functional V has a critical point.*

Sketch of proof. This is done precisely as in [4]: Since one does not know much about the behaviour of ∇V near the boundary $\partial\mathcal{F}_N\Omega$, in particular one does not know whether V satisfies the Palais–Smale–condition, one constructs a suitable pseudo–gradient vector field v which "blends" ∇V into ∇H_Ω near $\partial\mathcal{F}_N\Omega$. One then proceeds to find some point $z \in \mathcal{F}_N\Omega$ whose energy remains finite along the gradient flow of H_Ω via the linking arguments in chapter 5 and using an analogue of lemma 2.14 one subsequently shows that this z in fact leads also to a critical point of V under the flow corresponding to the vector field v. For further details we refer the reader to section 5 and 6 of [4]. □

6.2 The sinh–Poisson equation

In this section we apply theorem 6.2 to the sinh–Poisson equation. We cite the relevant result from [4] without proof. For further details we refer the reader to the original references [2, 6].

Theorem 6.5. *Every critical point of the functional*

$$\widetilde{I}_\rho(z) : \mathcal{F}_N\Omega \ni z \mapsto -8N\pi \ln \rho + 24N\pi \ln 2 - 16N\pi - 32\pi^2 H_\Omega(z) + r_\rho(z) \in \mathbb{R},$$

where r_ρ is $\Sigma(\Gamma)$– and D_p–invariant if Ω is D_p–symmetric and $r_\rho \to 0$ in $C^1_{\mathrm{loc}}(\mathcal{F}_N\Omega)$ as $\rho \to 0$, gives rise to a solution u_ρ of

$$\begin{cases} -\Delta\psi = \rho \sinh \psi & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Applying theorems 6.5, 3.16 and 6.2 together we immediately get

Theorem 6.6. *Let N be even, $p > 1$ and Ω be D_p–symmetric, $0 \notin \Omega$ and $\Gamma_j = (-1)^j$, $j \in \{1, \dots, N\}$. Then there is $\rho_0 > 0$ such that for any $\rho \in (0, \rho_0)$ and any common divisor $q > 1$ of p and $\frac{N}{2}$ there are at least*

$$\sum_{r=1}^{l(q)} \sum_{(a,b,c,d)\in\mathcal{V}_{q,r}} \frac{l(q)!}{(l(q)-r)!} \cdot \prod_{j=1}^{\frac{N}{2q}} \frac{1}{a_j! \cdot b_j! \cdot c_j! \cdot d_j!}$$

distinct symmetric solutions u_ρ of (6.1). These solutions have the property that for any sequence $\rho_n \to 0$ there exists $z^* \in \mathfrak{S}_\Gamma^{D_q}\Omega$ such that the vorticity field satisfies

$$\rho_n \sinh u_{\rho_n} \rightharpoonup 8\pi \sum_{j=1}^{N} (-1)^j \delta_{z_j^*}$$

weakly in the sense of measures in $\overline{\Omega}$ along a subsequence, where z^* is a critical point of H_Ω whose components lie on the symmetry axes of the D_q–action on Ω such that the vorticities have alternating signs and we adopted the notation of corollary 3.16.

Proof. Define for $\rho > 0$:

$$V_\rho : \mathcal{F}_N\Omega \ni z \mapsto H_\Omega(z) - \frac{1}{32\pi^2} r_\rho(z) \in \mathbb{R}.$$

Then V_ρ is $\Sigma(\Gamma)$– and D_p–invariant and since $r_\rho \to 0$ in $C^1_{\mathrm{loc}}(\mathcal{F}_N\Omega)$ the assertion follows from theorem 6.2 together with the observation that any critical point of V_ρ is also a critical point of $\widetilde{I}_\rho$. $\square$

6.3 The Lane–Emden–Fowler equation

The procedure is entirely analogous to the one used before. The main references in this subsection are [8, 9] and, of course [4], from where we adopt the notation.

Theorem 6.7. *Every critical point of the functional*

$$\widetilde{J}_p(z) : \mathcal{F}_N\Omega \ni z \mapsto N\frac{4\pi e}{p} - N\frac{8\pi e\ln p - c}{p^2} - \frac{32\pi^2 e}{p^2}H_\Omega(z) + \frac{1}{p^2}r_p(z) \in \mathbb{R},$$

where r_p is $\Sigma(\Gamma)$– and $D_{\tilde{p}}$–invariant if Ω is $D_{\tilde{p}}$–symmetric and $r_p \to 0$ in $C^1_{\mathrm{loc}}(\mathcal{F}_N\Omega)$ as $p \to \infty$, gives rise to a solution u_p of

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

Here, c is a fixed constant, see [4] for further detail.

Applying precisely the same procedures we get

Theorem 6.8. *Let N be even, $\tilde{p} > 1$ and Ω be $D_{\tilde{p}}$–symmetric, $0 \notin \Omega$ and $\Gamma_j = (-1)^j$, $j \in \{1, \dots, N\}$. Then there is $p_0 > 0$ such that for any $p > p_0$ and any common divisor $q > 1$ of $\tilde{p}$ and $\frac{N}{2}$ there are at least*

$$\sum_{r=1}^{l(q)} \sum_{(a,b,c,d)\in\mathcal{V}_{q,r}} \frac{l(q)!}{(l(q)-r)!} \cdot \prod_{j=1}^{\frac{N}{2q}} \frac{1}{a_j! \cdot b_j! \cdot c_j! \cdot d_j!}$$

distinct symmetric solutions u_p of (6.2). These solutions have the property that for any sequence $p_n \to \infty$ there exists $z^* \in \mathfrak{S}_\Gamma^{D_q}\Omega$ such that

$$p_n u_{p_n}|u_{p_n}|^{p_n-1} \rightharpoonup 8\pi e\sum_{j=1}^{N}(-1)^j\delta_{z_j^*}$$

weakly in the sense of measures in $\overline{\Omega}$ along a subsequence, where z^* is a critical point of H_Ω whose components lie on the symmetry axes of the D_q–action on Ω such that the vorticities have alternating signs and we adopted the notation of corollary 3.16 and replaced p by $\tilde{p}$.

Proof. Define for $p > 0$:

$$V_p : \mathcal{F}_N\Omega \ni z \mapsto H_\Omega(z) - \frac{1}{32\pi^2 e}r_p(z) \in \mathbb{R}.$$

Then V_p is $\Sigma(\Gamma)$– and D_p–invariant and since $r_p \to 0$ in $C^1_{\mathrm{loc}}(\mathcal{F}_N\Omega)$ as $p \to \infty$ the assertion follows from theorem 6.2 together with the observation that any critical point of V_p is also a critical point of $\widetilde{J}_p$. □

Remark. It is also possible to obtain symmetric solutions of (6.1) and (6.2) in the case that N is odd and $0 \in \Omega$, provided that $\mathbb{R} \cap \Omega$ has more than one connected component: One places the first vortex in the symmetry center and at most one other (differently oriented) vortex into the connected component of $[0,\infty)\cap\Omega$. One then places the additional vortices alternatingly into the other connected components of $[0,\infty)\cap\Omega$. Since the condition of center–admissibility is fulfilled for $\Gamma=(-1,1)$ or $\Gamma=-1$, the vortex addition lemma 3.10 together with lemma 3.15 and corollary 3.7 ensures that H_Ω has a symmetric critical point, which is of course stable with respect to symmetric C^1–perturbations and hence produces solutions for (6.1) and (6.2).

6.4 Connection to the Euler equations

The study of Poisson problems is intimately connected to stationary solutions of the Euler equations for incompressible two–dimensional fluids by means of the following method, which can be found, for example, in [17].

Considering a solution of $\Delta\psi=f(\psi)$ in Ω for an arbitrary $f\in C^1(\mathbb{R})$, the functions

$$v := J\cdot\nabla\psi,$$

$$p := F(\psi)-\frac{1}{2}\left|\nabla\psi\right|^2,$$

where $J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ and $F(s)=\int_0^s f(t)\,\mathrm{d}t$ form a stationary solution of the Euler equations

$$\begin{cases}\partial_t v+(v\cdot\nabla)v+\nabla p=0\\ \nabla\cdot v=0\end{cases}\tag{6.3}$$

with velocity v and pressure p, since

$$\nabla\cdot v=\nabla\cdot(J\cdot\nabla\psi)=-\partial_1\partial_2\psi+\partial_2\partial_1\psi=0,$$

as well as

$$\begin{aligned}\partial_t v+(v\cdot\nabla)v+\nabla p&=\left((J\cdot\nabla\psi)\cdot\nabla\right)(J\cdot\nabla\psi)+\nabla F(\psi)-\frac{1}{2}\nabla\left|\nabla\psi\right|^2\\
&=\begin{pmatrix}-\partial_2\psi\cdot\partial_1+\partial_1\psi\cdot\partial_2\\ -\partial_2\psi\cdot\partial_1+\partial_1\psi\cdot\partial_2\end{pmatrix}\begin{pmatrix}-\partial_2\psi\\ \partial_1\psi\end{pmatrix}+f(\psi)\nabla\psi-\frac{1}{2}\begin{pmatrix}\partial_1\left((\partial_1\psi)^2+(\partial_2\psi)^2\right)\\ \partial_2\left((\partial_1\psi)^2+(\partial_2\psi)^2\right)\end{pmatrix}\\
&=\begin{pmatrix}\partial_2\psi\cdot\partial_1\partial_2\psi-\partial_1\psi\cdot\partial_2^2\psi-\partial_1\psi\cdot\partial_1^2\psi-\partial_2\psi\cdot\partial_1\partial_2\psi\\ -\partial_2\psi\cdot\partial_1^2\psi+\partial_1\psi\cdot\partial_2\partial_1\psi-\partial_1\psi\cdot\partial_1\partial_2\psi-\partial_2\psi\cdot\partial_2^2\psi\end{pmatrix}-(\Delta\psi)\nabla\psi\\
&=(\Delta\psi)\nabla\psi-(\Delta\psi)\nabla\psi=0,\end{aligned}$$

where we adopted the usual physics jargon. The quantity $\omega=\nabla\times v=\partial_1 v_2-\partial_2 v_1$ is called the vorticity of the solution v. Observe that in our situation $\omega=\Delta\psi=f(\psi)$, thus the obtained solutions to the equations considered before give stationary solutions of the Euler equations with prescribed vorticity functions.

On the other hand, by the Biot–Savart law

$$v = -\omega * \frac{Jx}{|x|^2},$$

where $*$ denotes convolution, the velocity v can be recovered from ω, and admitting singular solutions of the form $\omega = \sum_{j=1}^N \Gamma_j \delta_{z_j(t)}$, corresponding to

$$v(x) = -\frac{1}{2\pi}\sum_{j=1}^N \Gamma_j \frac{J(x - z_j(t))}{|x - z_j(t)|^2}$$

via the Biot–Savart law we obtain that the vortex centres $(z_j(t))_{j\in\{1,\dots,N\}}$ obey the Hamiltonian system (1.1) studied here, see [11] for rigorous arguments on this.

From this view, theorems 6.6 and 6.8 deliver us with stationary solutions to the Euler equations where the singular vorticity has been regularized to smooth "vorticity blobs".

For the sinh–Poisson equation and the Lane–Emden–Fowler equation this process is limited to the case that $|\Gamma_j| = 1$ for all $j \in \{1,\dots,N\}$. But recently, by studying the semilinear elliptic problem

$$\begin{cases} -\varepsilon^2\Delta u = \sum_{\substack{j\in\{1,\dots,N\}\\ \Gamma_j>0}} \chi\Omega_j \left(u - q - \frac{\Gamma_j}{2\pi}\ln\frac{1}{\varepsilon}\right)_+^p & \\ \qquad - \sum_{\substack{j\in\{1,\dots,N\}\\ \Gamma_j<0}} \chi\Omega_j \left(-u + q + \frac{\Gamma_j}{2\pi}\ln\frac{1}{\varepsilon}\right)_+^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.4}$$

where Ω_j are mutually disjoint subdomains of Ω such that $z_j^* \in \Omega_j$ for a critical point z^* of the Kirchhoff–Routh path function and χ_{Ω_j} is the characteristic function of Ω_j, $j \in \{1,\dots,N\}$, and applying their results to the Euler equations as sketched above, the authors of [5] succeeded in proving the following

Theorem 6.9. *Suppose $\Omega \subset \mathbb{C}$ is a bounded simply–connected smooth domain. Taking the Kirchhoff–Routh path function as*

$$\widehat{H}_\Omega(z) = H_\Omega(z) + 2\sum_{j=1}^N \Gamma_j \psi_0(z_j),$$

where ψ_0 is uniquely defined up to a constant as the solution of

$$\begin{cases} -\Delta\psi_0 = 0 & \text{in } \Omega \\ -\langle \tau, \nabla\psi_0\rangle = v_n & \text{on } \partial\Omega \end{cases}$$

and $v_n \in L^s(\partial\Omega)$ for some $s > 1$ satisfying $\int_{\partial\Omega} v_n = 0$, we have that for any C^1–stable critical point z^* of $\widehat{H}_\Omega$ there is $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$ the Euler equations

(6.3) *have a solution* v_ε *with outward boundary flux given by* v_n *and vorticity* ω_ε *satisfying*

$$\operatorname{supp}\omega_\varepsilon \subset \bigcup_{j=1}^{N} B_{C\varepsilon}(z_j),$$

where $C > 0$ *is a constant independent of* ε *and* $z_j \in \Omega_j$ *for* $j \in \{1, \dots, N\}$. *Moreover, as* $\varepsilon \to 0$,

$$(z_1, \dots, z_N) \to (z_1^*, \dots, z_N^*)$$

$$\int_{B_{C\varepsilon}(z_j)} \omega_\varepsilon \to \Gamma_j \qquad \forall\, j \in \{1, \dots, N\}, \text{ as well as}$$

$$\int_\Omega \omega_\varepsilon \to \sum_{j=1}^{N} \Gamma_j.$$

This is theorem 1.1 of [5] translated into our notation. Combined with the results in this thesis we may now supplement this theorem by the existence result 5.1, giving us the following theorem, which marks a suitable endpoint for this thesis.

Theorem 6.10. *Let* Ω *be a smooth simply–connected bounded domain, let* $v_n \in C^1(\partial\Omega)$ *and let* Γ *be* $\mathcal{L}$–, Δ– *and* ∂–*admissible. Then there is* $\varepsilon_0 > 0$ *such that for every* $\varepsilon \in (0, \varepsilon_0)$ *the Euler–equations* (6.3) *have a solution* v_ε *satisfying the properties stated in theorem 6.9.*

Sketch of proof. In the case that $\psi_0 = 0$ the theorem follows easily from theorem 6.4, which basically states that the critical points of $H_\Omega = \widehat{H}_\Omega$ are C^1–stable. If $\psi_0 \neq 0$ we define

$$V : \overline{\Omega}^N \ni z \mapsto 2\sum_{j=1}^{N} \Gamma_j \psi_0(z_j) \in \mathbb{R},$$

so that $\widehat{H}_\Omega = H_\Omega + V$ and, since $\psi_0 \in C^1(\overline{\Omega})$ by standard elliptic theory, more precisely, since by integrating the boundary condition we get that ψ_0 satisfies the Dirichlet problem

$$\begin{cases} -\Delta\psi_0 = 0 & \text{in } \Omega \\ \quad -\psi_0 = V_n & \text{on } \partial\Omega \end{cases}$$

for some $V_n \in C^2(\partial\Omega)$ and V_n has a continuation $\tilde{V_n} \in C^2(\overline{\Omega})$, for $\partial\Omega$ is of class C^3, the assertion that $\psi_0 \in C^1(\overline{\Omega})$ and consequently $V \in C^1(\overline{\Omega}^N)$ follows by theorem 6.14 of [12]. The assertion of the theorem follows now from theorem 6.3 and 6.4 applied to $\widehat{H}_\Omega$. This is possible since the proof of 6.4 in [4] only uses the properties of H_Ω which can also be verified for $\widehat{H}_\Omega$, just as in the proof of theorem 6.3, more precisely, the only ingredient needed is that there is $\delta > 0$ such that $|\nabla H_\Omega| > 1$ on $\mathcal{M}_\delta$, which, of course, can also be verified for $\widehat{H_\Omega}$ instead of H_Ω. $\square$

List of Symbols

$*$	group action of Σ_N on $\mathcal{F}_N\Omega$ and $\mathbb{R}^N$ by permutation of coordinates, page 7
$\diamond$	group action of Π_r on arbitrary sets of r–dimensional arrays of vectors via component–wise order–reversal and permutation of coordinates, page 23
$\odot$	group action of $\mathbb{Z}_2$ on $\mathbb{R}^k$ defined by $1 \odot v = \hat{\sigma} * v$, where $\hat{\sigma} \in \Sigma_k$ reverses the order of components, page 23
$\mathcal{L}$–admissible	The components of Γ can be ordered to have alternating sign and ascending modulus, page 13
Δ–admissible	for every subset $C \subset \{1, \ldots, N\}$, $\lvert C\rvert \geq 2$: $\sum_{\substack{i,j \in C \\ i \neq j}} \Gamma_i \Gamma_j \neq 0$, page 37
∂–admissible	for every subset $C \subset \{1, \ldots, N\}$, $\lvert C\rvert \geq 2$: $\sum_{i \in C} \Gamma_i^2 > \sum_{\substack{i,j \in C \\ i \neq j}} \lvert\Gamma_i \Gamma_j\rvert$. If Ω is strictly convex: $\sum_{i \in C} \Gamma_i^2 > \sum_{\substack{i,j \in C \\ \Gamma_i\Gamma_j < 0}} \lvert\Gamma_i \Gamma_j\rvert$, page 38
$\mathcal{C}(\mathcal{P}_z)$	set of clusters of $\mathcal{P}_z$, $\{C \in \mathcal{P}_z : \lvert C\rvert \geq 2\}$, page 38
$\partial_{\mathrm{bdry}}\mathcal{F}_N\Omega$	boundary collisions, $\{z \in \partial\mathcal{F}_N\Omega :\ \forall\ C \in \mathcal{C}(\mathcal{P}_z) : z_C \in \partial\Omega\}$, page 39
$\partial_{\mathrm{int}}\mathcal{F}_N\Omega$	interior collisions, $\{z \in \partial\mathcal{F}_N\Omega :\ \exists\ C \in \mathcal{C}(\mathcal{P}_z)$ such that $z_C \in \Omega\}$, page 39
$d(z)$	$\mathrm{dist}(z, \partial\Omega)$, page 8
$\mathcal{F}_N\Omega$	ordered configuration space, equal to $\left\{z \in \Omega^N : z_i = z_j \Leftrightarrow i = j\right\}$, page 7
G	called Green's function, if not otherwise stated: a function satisfying hypothesis 2.4, page 8
g	Regular part of the Green's function G, $g(x,y) = G(x,y) + \frac{1}{2\pi}\ln\lvert x-y\rvert$, page 8
h	Robin function of the domain Ω, $h(x) = g(x,x)$, page 8
H_Ω^Γ	Kirchhoff–Routh path function with vorticity Γ, $H_\Omega^\Gamma(z) = \sum_{j=1}^N \Gamma_j^2 h(z_j) + \sum_{\substack{i,j=1 \\ i \neq j}}^N \Gamma_i\Gamma_j G(z_i, z_j)$, page 9
$\mathcal{L}_N(a,v)$	space of ordered configurations of N vortices along the line $a + \mathbb{R}v$, page 7
$\mathcal{L}_N^\sigma(a,v)$	$\sigma^{-1} * \mathcal{L}_N(a,v)$ for some $\sigma \in \Sigma_N$, page 7
$\mathcal{L}_N^\sigma\Omega$	$\bigcup_{(a,v)\in\Omega\times S^1} \mathcal{L}_N^\sigma(a,v)$, page 7
$\mathcal{M}_\delta$	$\{z \in \mathcal{F}_N\Omega : \exists\,(i,j) \in \mathcal{F}_2\{1,\ldots,N\} : \lvert z_i - z_j\rvert \leq \delta \vee d(z_j) \leq \delta\}$, page 38
Ω_ε	$\{z \in \Omega : \mathrm{dist}(z, \partial\Omega) < \varepsilon\}$, page 7

$\mathcal{P}_z$ — partition of $\{1,\dots,N\}$ defined by $\{C \subset \{1,\dots,N\} : z_i = z_j \Leftrightarrow i,j \in C\}$, page 38

π_C — projection onto the components corresponding to the indices in C, page 7

Π_r — the group $\Sigma_r \times \mathbb{Z}_2^r$, page 23

p — $p : \Omega_\varepsilon \to \partial\Omega$, orthogonal projection onto $\partial\Omega$, page 7

$\Sigma(\Gamma)$ — stabilizer of $\Gamma \in \mathbb{R}^N$ under the obvious Σ_N–action on $\mathbb{R}^N$, page 18

$\mathfrak{S}_\Gamma^U\Omega$ — U–symmetric points for Γ, $\{z \in \mathcal{F}_N\Omega :\ \forall u \in U\ \exists \sigma \in \Sigma(\Gamma) : u \cdot z = \sigma * z\}$, page 19

$\widehat{\mathfrak{S}}_\Gamma^U\Omega$ — U–symmetric points for Γ with one vertex placed in the symmetry center of Ω, $\big\{z \in \mathfrak{S}_\Gamma^U\Omega :\ \exists k \in \{1,\dots,N\} : z_k = 0\big\}$, page 19

$\overline{z}$ — for $z \in \Omega_\varepsilon$. Reflection of z at ∂_Ω, page 8

z_C — for $z \in \partial\mathcal{F}_N\Omega$ and $C \in \mathcal{P}_z$, the unique element of $\{z_j : j \in C\}$, page 38

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Danksagung

An dieser Stelle möchte ich meiner Familie für ihre stete Unterstützung danken. Ebenso gilt mein Dank Christian Rehm für seine Hilfe beim Korrekturlesen dieser Arbeit.

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Heckholzhausen, den 12. Mai 2014

Christian Kuhl