A Remark on Quantum Brownian Motion[0)]

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Abstract
An implicit exact solution is found for the semiclassical limit of Roumen Tsekov's quantum Smoluchowski equation for the harmonic oscillator.

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Last year Roumen Tsekov [1] derived a solution of his quantum Smoluchowski equation for the quantum harmonic oscillator, which is a Gaussian distribution with dispersion given by Equ. (3) of [1] (see also [2]):

$$\frac{\partial}{\partial t}\sigma^2 = 2D\left(1+\sigma^2\int_0^\beta d\beta \frac{\hbar^2}{4m\sigma^4} - \beta m\omega_0^2\sigma^2\right) , \tag{1}$$

where β >0 is the inverse temperature of the bath, ω_0 the frequency of the oscillator and D>0 is the Einstein diffusion constant. For small displacement $\lambda^2 = \frac{\hbar^2\beta}{4m}$ we approximate this equation by

$$\frac{\partial}{\partial t}\sigma^2 = 2D(1+\frac{\lambda^2}{\sigma^2} - \beta m\omega_0^2\sigma^2) . \tag{2}$$

At first we found a stationary solution, i.e. if $\frac{\partial}{\partial t}\sigma^2 = 0$. This gives an algebraic quadratic equation with only one positive solution σ_∞ :

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$$\sigma_\infty^2 = \frac{1}{2\chi^2}\left(1+\sqrt{4\lambda^2\chi^2+1}\right) \tag{3}$$

For the time-dependent solution we integrate

$$\int \frac{\sigma^2 d\sigma^2}{\left(\sigma^2+\lambda^2-\chi^2\sigma^4\right)} = 2D\int dt = 2Dt + C$$

where $\chi^2 = \beta m \omega_0^2$, and thus

$$\int \frac{zdz}{\left(z+\lambda^2-\chi^2 z^2\right)} = \int \frac{zdz}{\left[(\lambda^2+\frac{1}{4\chi^2}) - \left(\chi z - \frac{1}{2\chi}\right)^2\right]}$$

This integral consists of two summands. The first is, up to a constant factor given by

$$\int \frac{d(\chi z - \frac{1}{2\chi})}{(\lambda^2+\frac{1}{4\chi^2})-(\chi z-\frac{1}{2\chi})^2} = \int \frac{dy}{\psi - y^2} = -\frac{1}{2\sqrt{\psi}}\ln C\frac{y-\sqrt{\psi}}{y+\sqrt{\psi}}$$

The second integral summand is up to a constant factor

$$\int \frac{ydy}{(\psi - y^2)}; y = \chi z - \frac{1}{2\chi}$$

with $\psi = \lambda^2 + \frac{1}{4\chi^2}$.

However we know that

$$\int \frac{ydy}{(\psi - y^2)} = -\frac{1}{2}\ln C(y^2-\psi)$$

Therefore

$$2Dt + C = \int \frac{1}{\chi^2} \frac{(\chi\sigma^2 - \frac{1}{2\chi})d(\chi\sigma^2 - \frac{1}{2\chi})}{\left[\psi - (\chi\sigma^2 - \frac{1}{2\chi})\right]} + \int \frac{1}{2\chi^3} \frac{d(\chi\sigma^2 - \frac{1}{2\chi})}{\left[\psi - (\chi\sigma^2 - \frac{1}{2\chi})\right]}$$

The implicit solution is

$$Const. \bullet e^{2\chi^2 Dt} = \frac{\left[\overline{y} + \sqrt{\psi}\right]^{a-1}}{\left[\overline{y} - \sqrt{\psi}\right]^{a}} \tag{4}$$

with $a = \frac{1}{2}(1 + \frac{1}{\sqrt{4\lambda^2\chi^2 + 1}})$

and $\overline{y} = \chi\sigma^2 - \frac{1}{2\chi}$. One sees easily that the solution,
independent of the initial values tend for $t \to \infty$ exactly to σ_∞^2, because
$\overline{y} \to \sqrt{\psi}$ necessarily.
If at time t=0 the inertial width is σ_0 we calculate the integration constant $Const.$ to be in every case equal to

$$Const. = \frac{(\sigma_0^2 + \sigma_\infty^2 - \frac{1}{\chi^2})^{a-1}}{\chi(\sigma_0^2 - \sigma_\infty^2)^a} \ . \tag{5}$$

Thus the final implicit solution is

$$e^{2\chi^2 Dt} = \frac{(\sigma_0^2 - \sigma_\infty^2)^a (\sigma^2 + \sigma_\infty^2 - \frac{1}{\chi^2})^{a-1}}{(\sigma_0^2 + \sigma_\infty^2 - \frac{1}{\chi^2})^{a-1} (\sigma^2 - \sigma_\infty^2)^a} \ . \tag{6}$$

Most recently, this solution, which already having been presented in [3], has been quoted in [4], where also the starting differential equation (2) appears in a similar context.

References:

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[3] Joachim A. Messer: ***A Remark on Nuclear Friction***, http://www.ptjm.org (5th March 2008)

[4] Roumen Tsekov: ***Thermo-Quantum Diffusion,*** arXiv:0803.4409 (31st March 2008)

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