Gated chamber complexes, simplicial arrangements and Coxeter groups

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Contents

Deutschsprachige Einleitung

Man findet für jede irreduzible endliche reelle Spiegelungsgruppe eine Menge von linearen Hyperebenen in einem entsprechenden Vektorraum über $\mathbb{R}$. Diese stellen die Spiegelungshyperebenen der Gruppe dar. Die Zusammenhangskomponenten des Komplements der Vereinigung dieser Hyperebenen besteht aus offenen konvexen Kegeln, welche allgemein als *Kammern* bezeichnet werden.

Im Fall einer diskreten irreduziblen affinen Spiegelungsgruppe, bei welcher die Menge der Spiegelungshyperbenen affine Hyperebenen sind, sind analog die Zusammenhangskomponenten des Komplements offene konvexe Mengen. Man kann diese konvexen Mengen als offene konvexe Kegel auffassen, indem man den affinen Raum $\mathbb{A}^n$ in $\mathbb{R}^{n+1}$ als affine Hyperbene einbettet. Diese Kegel zerlegen jedoch lediglich einen Halbraum des $\mathbb{R}^{n+1}$.

In der allgemeinen Betrachtung von Coxetersystemen (W, S) hat Tits [Tit13] die geometrische Darstellung von Coxetergruppen eingeführt, um eine reelle Darstellung der Gruppe zu erhalten. Bezüglich dieser Darstellung besitzt die Gruppe W eine Wirkung auf einem konvexen Kegel T. Hierbei legen die fundamentalen Spiegelungen in S einen offenen Kegel C innerhalb von T fest, welcher einen Präfundamentalbereich für die Wirkung von W auf T beschreibt. Der Kegel T wird als *Tits-Kegel* bezeichnet. Die Spiegelungshyperebenen liefern einen kanonischen simplizialen Komplex, welcher sich aus Teilmengen des Tits-Kegels zusammensetzt.

Die simplizialen Komplexe, welche man auf diese Weise erhält, haben diverse grundlegende Eigenschaften, welche durch das Axiomensystem für (dünne) Kammerkomplexe zusammengefasst werden. Daher kann man jedem Coxetersystem den entsprechenden Coxeterkomplex zuordnen. Die Kammern dieses Komplexes stehen hierbei in Korrespondenz zu den Gruppenelementen. Insbesondere die Apartments in Tits Gebäudetheorie, welche von der Theorie der algebraischen Gruppen über Körpern motiviert wird, sind Coxeterkomplexe. Ein Gebäude ist nach Definition ein Kammerkomplex, der von Unterkomplexen überdeckt wird, welche wiederum alle isomorph zu einen bestimmten Coxeterkomplex sind. Innerhalb der Gebäudetheorie ist die Existenz von Projektionsabbildungen ein grundlegendes Werkzeug.

Die Kammern eines solchen Komplexes bilden zusammen mit der Metrik des Kammerngraphen einen metrischen Raum. In [DS87] haben Dress und Scharlau eine charakteristische Eigenschaft für Teilmengen dieser metrischen Räume formuliert (im deutschen manchmal als Tor-Eigenschaft bezeichnet, werden wir diese im Folgenden *(DS)-Eigenschaft* nennen). Diese Eigenschaft vereinfacht einige der Argumente von Tits und Scharlau [Sch85] hat hierüber außerdem eine Charakterisierung von Gebäuden angegeben. Der zunächst noch lückenhafte Beweis wurde später von Kasikova und Shult [KS96] vervollständigt.

Ausgehend von den Arbeiten [Sch85] und [DS87] wurden in [Müh94] Kammerkomplexe mit der (DS)-Eigenschaft eingeführt und studiert. Hierbei ist ein Kammerkomplex mit (DS)-Eigenschaft ein Kammerkomplex, in welchem alle Projektionsabbildungen existieren. Ein Hauptresultat in [Müh94] besagt, dass ein solcher Kammerkomplex von endlichem Durchmesser im Wesentlichen schon ein Gebäude ist. Mühlherr führt zudem den Begriff von Fixpunkt-Komplexen ein, welche von einem Kammerkomplex mit (DS)-Eigenschaft und einer Untergruppe seiner Automorphismengruppe abhängen. Er gibt in seiner Thesis ein Kriterium an, wann diese Strukturen wieder ein Kammerkomplex mit (DS)-Eigenschaft sind. Dieses Resultat kann als Grundlage für die Abstiegs-Theorie angesehen werden, welche in [MPW15] entwickelt wurde.

Der erste Teil dieser Arbeit behandelt diese Resultate und verallgemeinert einige der Aussagen in [MPW15] auf beliebige Kammerkomplexe mit (DS)-Eigenschaft. Wir führen den Reduzibilitätstyp für solche Komplexe ein, falls sie solide sind. Außerdem zeigen wir, dass Segmente in Kammerkomplexen konvex sind. Insbesondere verallgemeinern wir eines der Hauptwerkzeuge für die Abstiegstheorie in [MPW15] auf solide Kammerkomplexe mit (DS)-Eigenschaft, welche ein Apartment-System zulassen. Die entsprechende Aussage ist recht technisch und lässt sich in Theorem 1.6.10 finden.

Die geometrische Darstellung einer Coxetergruppe liefert eine Menge von Hyperebenen in einem reellen Vektorraum. Das Konzept von Hyperebenen, die einen Kegel zerlegen, kann auf beliebige (lokal endliche) Mengen von Hyperebenen erweitert werden. Eine solche Verallgemeinerung wird von der Klassifikation der Nichols-Algebren motiviert. Diesen Algebren kann man als Invariante einen *Cartan-Graph* zuordnen (in den Arbeiten von Cuntz und Heckenberger werden Cartan-Graphen noch als *Cartan-Schemata* bezeichnet).

Jeder Cartan-Graph legt ein Weyl-Gruppoid fest, eine kategorielle Verallgemeinerung von Weyl-Gruppen. Im sphärischen Fall wurden diese von Cuntz und Heckenberger in einer Reihe von Arbeiten [CH09a], [CH11], [CH12], [CH15] vollständig klassifiziert. Tatsächlich existiert eine Korrespondenz zwischen Weyl-Gruppoiden eines bestimmten Typs und endlichen kristallographischen simplizialen Arrangements [Cun11].

Das zweite Kapitel soll die Grundlage für eine größere Klasse von simplizialen Arrangements schaffen, welche Hyperebenen-Arrangements umfasst, die nicht endlich, aber lokal endlich in einem offenen konvexen Kegel sind. Letzteren werden wir in Analogie zur geometrischen Darstellung von Coxetergruppen Tits-Kegel nennen.

Ein solches Arrangement liefert einen Kammerkomplex mit (DS)-Eigenschaft und Typ-Funktion. Diese Eigenschaften ermöglichen einen etwas anderen Zugang zu simplizialen Arrangements als jener von Cuntz und Heckenberger. Desweiteren kann man aus jedem zusammenhängenden einfach zusammenhängenden Weyl-Gruppoid eine kanonische Menge von Hyperebenen konstruieren, allerdings a priori keinen konvexen Kegel, welcher von diesen simplizial zerlegt wird. Dieser Kegel kann jedoch aus den kombinatorischen Daten rekonstruiert werden. Insgesamt erhält man eine Korrespondenz zwischen zusammenhängenden einfach zusammenhängenden Weyl-Gruppoiden, welche ein reelles Wurzelsystem zulassen und kristallographischen, simplizialen Arrangements mit reduziertem Wurzelsystem. Dieses Ergebnis wird in Korollar 2.6.24 formuliert.

Da dieses Kapitel auch eine Grundlage für die Theorie der Weyl-Gruppoide bilden soll, wiederholen wir einige Konzepte, die zwar bekannt sind, deren Ursprung in der Literatur allerdings schwer nachzuverfolgen ist. Einige der Tatsachen über Hyperebenen-Arrangements und simpliziale Arrangements, die wir verwenden, sind zwar weit verbreitet, der Vollständigkeit halber geben wir jedoch kurze Beweise an, soweit dies möglich ist.

Einige Kommentare zum Ursprung der Objekte, die wir behandeln:

1. Simpliziale Arrangements wurden zuerst von Melchior [Mel41] und später von Grünbaum [Grü71] eingeführt und untersucht. Kurz darauf wurden diese Arrangements auch durch die Arbeit von Deligne [Del72] bekannt. Da die Spiegelungshyperebenen einer endlichen Coxetergruppe auch ein simpliziales Arrangement darstellen, kann man im Kontext dieser Arrangements auch die $K(\pi, 1)$-Eigenschaft von Komplementen von Spiegelungsarrangements untersuchen. Außerdem eignen sich simpliziale Arrangements gut als Beispiele oder Gegenbeispiele für Überlegungen über allgemeine Hyperebenen-Arrangements.

2. Wir können nicht mit Sicherheit sagen, wo Arrangements von Hyperebenen auf konvexen Kegeln zum ersten Mal erwähnt werden, da dies ein recht natürliches Konzept ist. Dies wird auch in [Par14] ohne weitere Referenz erwähnt. Unsere Definition eines simplizialen Arrangements auf einem offenen konvexen Kegel ist hauptsächlich vom Begriff des Tits-Kegels für Coxetergruppen inspiriert.

3. Die Tatsache, dass Hyperebenen-Arrangements interessante Beispiele für Mengen mit der (DS)-Eigenschaft in metrischen Räumen darstellen, erschien in [BLVS$^+$99] zum ersten Mal. Zumindest im simplizialen Fall wurde dies auch schon früher beobachtet [Tit74].

4. Es ist eine recht natürliche Beobachtung, dass es eine Verbindung zwischen Wurzelsystemen und simplizialen Arrangments gibt. Diese wurde ebenso von Dyer in seinen Arbeiten [Dye11a], [Dye11b] gemacht, taucht vermutlich aber auch schon früher in der Literatur auf.

Bei der Betrachtung von allgemeinen Coxetergruppen ist der zugehörige Kammerkomplex eng mit dem Cayley-Graph der Gruppe verwandt; letzterer ist im Prinzip der

Kammergraph des entsprechenden Kammerkomplexes. Dieser wird beispielsweise auch in der Untersuchung des Isomorphieproblems für Coxetergruppen verwendet.

Da Coxetergruppen durch Erzeugende und Relationen definiert werden können, kann man Dehns Probleme [Deh11] für diese Gruppen formulieren. Das Wortproblem kann mit grundlegender Theorie ([Tit69]) gelöst werden. Das Konjugationsproblem ist lösbar, da Moussong [Mou88] gezeigt hat, dass der Davis-Komplex einer Coxetergruppe die CAT(0)-Eigenschaft hat. Daher wirkt eine Coxetergruppe eigentlich und kokompakt auf einem CAT(0)-Raum, somit ist nach [BH99, Chapter III, 1.11] das Konjugationsproblem für diese Gruppen lösbar. Später hat Daan Krammer [Kra09] einen effizienteren Algorithmus zur Lösung des Konjugationsproblems angegeben.

Das Isomorphieproblem ist weiterhin offen. Es gibt allerdings Arbeiten, die es erlauben, das Problem zunächst auf erzeugende Mengen von Spiegelungen zu reduzieren [HM04], und später auch auf spitzwinklige Mengen von Spiegelungen [MM08]. In diesem Fall wurde von Mühlherr [Müh06] eine Vermutung zur Lösung des Isomorphieproblems aufgestellt. Hiernach erzeugen zwei Coxeter-erzeugende Mengen genau dann isomorphe Gruppen, wenn die erste aus der zweiten durch eine Reihe von Twists hervorgehen kann, wobei ein Twist einen partiellen inneren Automorphismus beschreibt. Diese Vermutung wurde bereits für eine Reihe von Klassen von Coxetergruppen bewiesen.

Die Vermutung wurde von Mühlherr und Weidmann [MW02] für schiefwinklige Coxetersysteme gezeigt, von Caprace und Mühlherr [CM07] für 2-sphärische Coxetersysteme. Ratcliffe und Tschantz [RT08] haben sie für „chordal“ Coxetergruppen bewiesen und ein Resultat von Caprace und Przytycki [CP10] liefert die Vermutung für solche Coxetergruppen, die aufgrund ihres Diagramms keine Twists zulassen. Die Referenzen [MW02] und [RT08] nutzen die Zerlegung einer Coxetergruppe als Graph von Gruppen. Dieser Ansatz ist für diese Spezialfälle sehr effizient, scheint sich jedoch nur schwer auf beliebige Diagramme verallgemeinern zu lassen. Eines der größten Probleme liegt in der Existenz von lokalen Twists, die sich nicht zu globalen Twists erweitern lassen. Die Bedingungen in diesen Arbeiten sind so formuliert, dass lokale Twists kontrolliert werden können.

Wir zeigen die Twist-Vermutung für Diagramme, die keine Rang 3 Unterdiagramme eines bestimmten Typs besitzen, welche wiederum die irreduziblen sphärischen beinhalten. Diese Diagramme werden ausgeschlossen, um Twists von höherem Rang zu vermeiden. Allerdings erlaubt diese Bedingung auch keine Diagramme vom Typ $\tilde{C}_2$ und $\tilde{G}_2$, um technische Details zu vermeiden. Obwohl wir eine größere Klasse von Coxetergruppen abdecken, für die die Twistvermutung bisher noch nicht bewiesen war, benötigen unsere Techniken noch substantielle Verbesserungen, um den allgemeinen Fall behandeln zu können. Trotzdem sind die von uns entwickelten Methoden die ersten, die Rotationstwists auf geometrische Weise behandeln können. Der grundlegende Ansatz hierfür wurde schon in [Müh98] benutzt. Die Coxetersysteme in [MW02] und [RT08] erlauben ebenfalls diese Art von Twists, umgehen sie allerdings durch die Benutzung der Bass-Serre-Theorie.

Das Folgende ist unser Hauptresultat:

Theorem. Sei (W, S) ein irreduzibles, nicht sphärisches Coxetersystem von endlichem Rang größer oder gleich 3, dessen Diagramm keine Unterdiagramme der Typen $\bullet\overset{3}{\text{———}}\bullet\overset{\text{n}}{\text{———}}\bullet$ für $n \geq 3$ oder $\bullet\overset{4}{\text{———}}\bullet\overset{\text{n}}{\text{———}}\bullet$ für $n \geq 4$ enthält. Falls $R \subset S^W$ eine irreduzible spitzwinklige Coxeter-erzeugende Menge für W ist, dann folgt $R \sim_t S$.

Diese Anforderung an das Diagramm werden wir später als Bedingung (E) bezeichnen, was an unser ursprüngliches Vorhaben erinnert, gerade (***e**ven*) Coxetergruppen zu behandeln. Im obigen Theorem bezeichnet $\sim_t$ die oben erwähnte Twist-Äquivalenz.

Introduction

For every irreducible real finite reflection group there exists a set of linear hyperplanes in the respective real vector space, the reflection hyperplanes of the group. The complement of the union of the hyperplanes consists of open convex cones as connected components, and are called *chambers*.

Similarly when considering a discrete irreducible affine reflection group and the set of affine reflection hyperplanes, the connected components of the complement are open convex sets. By embedding the affine space $\mathbb{A}^n$ into the real space $\mathbb{R}^{n+1}$ as an affine hyperplane, one can again consider these connected components as open convex cones. However, in this case these cones decompose a halfspace instead of the whole $\mathbb{R}^{n+1}$.

When considering Coxeter systems (W, S) in general, it was Tits idea [Tit13] to consider the geometric representation of the Coxeter group to obtain a real representation of the group. With respect to this representation, he obtained a group action of W on a convex cone T, and the fundamental reflections in S yield a open cone C inside T, which is a prefundamental domain for the action of W on T. The cone T is the now denoted as the *Tits cone*. The reflection hyperplanes yield a canonical simplicial complex consisting of subsets of the Tits cone.

The simplicial complexes obtained from a Tits cone have several basic properties which are summarized in the axioms of a (thin) chamber complex. As a consequence, one associates to each Coxeter system 'its' Coxeter complex. The chambers of this complex are in one to one correspondence with the group elements. Coxeter complexes are precisely the apartments in Tits' theory of buildings, which is motivated by the theory of algebraic groups over fields. By definition a building is a chamber complex covered by subcomplexes which are all isomorphic to a given Coxeter complex. A basic and crucial fact in the theory of buildings is the existence of projection mappings.

Dress and Scharlau [DS87] later described the gate property for the metric space obtained from the chamber complex by taking as a metric the length of a minimal gallery between two chambers. This property simplifies some of Tits arguments. Also Scharlau gave a characterisation of buildings via the gate property [Sch85]. The proof however was incomplete, Kasikova and Shult [KS96] later gave a complete proof.

Inspired by the papers [Sch85] and [DS87] gated chamber complexes had been introduced and investigated in [Müh94]. A gated chamber complex is by definition a chamber complex, for which all projection mappings exist and it is one of the main results in [Müh94] that a gated chamber complex of finite diameter is essentially a building. Mühlherr also introduces the notion of fixed point complex, associated to a gated chamber complex and a subgroup of its automorphism group. In his thesis he gives a criterion for when this structure is again a gated chamber complex, which can be seen as the foundation for the theory of descent developed in [MPW15].

The first part of this work picks up these results and tries to generalise some of the statements found in [MPW15] for arbitrary gated chamber complexes. We introduce the reducibility type of a firm gated chamber complex and show that segments in gated chamber complexes are convex. We also generalise one of the main tools used in [MPW15] to firm gated chamber complexes which admit an apartment system. For more details refer to the introduction of Chapter 1.

The geometric representation of a Coxeter group yields a set of reflection hyperplanes in a real vector space. One can generalise the concept of reflection hyperplanes in the Tits cone to sets of hyperplanes which are not necessarily reflection hyperplanes of a Coxeter or Weyl group. The motivation for this generalisation comes from the classification of Nichols algebras. These algebras carry as an invariant a *Cartan graph* (which was called *Cartan scheme* in the works by Cuntz and Heckenberger).

Cartan graphs and the associated Weyl groupoids, categorical generalisations of Weyl groups, have been completely classified by Cuntz and Heckenberger in a series of papers [CH09a], [CH11], [CH12], [CH15] in the spherical case. It turns out that Weyl groupoids of a certain type correspond to the crystallographic finite simplicial hyperplane arrangements [Cun11].

The second chapter intends to set a foundation for a more general class of simplicial arrangements, considering hyperplane arrangements which are not finite, but locally finite in an open convex cone, which we call the Tits cone in analogy to the case of Coxeter and Weyl groups.

From such an arrangement we obtain a gated chamber complex with a type function, which allows us some approaches different from those Cuntz and Heckenberger used. In particular we can show that simplicial arrangements correspond to Weyl groupoids permitting a real root system. The introduction of Chapter 2 contains more details on this and points out the relation to Weyl groups.

When considering Coxeter groups in general, the associated chamber complex is closely related to the Cayley graph of the group, which is basically the chamber graph of the respective chamber complex. It is used for example to approach the isomorphism problem for Coxeter groups.

As Coxeter groups can be defined as a class of groups given by a presentation, Dehn's fundamental decision problems [Deh11] can be applied to these groups. The word problem can be solved ([Tit69]) using elementary considerations. The conjugacy problem is solvable due to the work of Moussong [Mou88], who showed that the Davis-Complex of a Coxeter group is CAT(0). Hence Coxeter groups act properly and cocompactly on a CAT(0)-Space. By [BH99, Chapter III, 1.11] therefore the conjugacy problem is solvable. Daan Krammer [Kra09] found an efficient algorithm to solve the conjugacy problem for Coxeter groups.

The isomorphism problem is still an open problem. However, there have been several works to reduce the problem to generating sets consisting of reflections [HM04] and to sharp-angled sets of reflections [MM08]. In this case a complete solution has been conjectured by Mühlherr [Müh06], stating that two Coxeter generating sets generate isomorphic groups, if and only one can be obtained from the other by a series of twists. A twist can be thought of as a partially applied inner automorphism. This conjecture has already been proved in some special cases, for an overview refer to the introduction of Chapter 3. We were able to prove the conjecture for a wider class of Coxeter groups, the statement or our main theorem can also be found in the introduction of Chapter 3.

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1 Gated chamber complexes

1.1 Introduction

Primary examples for chamber complexes are buildings in the sense of Tits (cp. [Tit74]) and, more specific, Coxeter complexes; where the latter are the most important examples for thin chamber complexes. These objects share the property that all residues, seen as sets of chambers, satisfy the gate property. This gives rise to the projection maps for residues. Moreover, in these complexes the image of a residue under a projection is again a residue. We will call chamber complexes satisfying these exact properties *totally gated*.

Except for the canonical examples mentioned above, totally gated chamber complexes do occur naturally when considering fixed-point structures in buildings, which has been noted by Mühlherr in his dissertation (cp. [Müh94]). In [MPW15], Mühlherr, Petersson and Weiss give a sufficient criterion in terms of Tits indices for such a fixed point structure to be again a building.

The intention of this chapter is a generalisation of some of the results of [MPW15] to the more general case of totally gated chamber complexes.

We also answer some open questions which arose from [Müh94], in particular we show that segments in totally gated chamber complexes are always convex. In his dissertation, Mühlherr already provided several results for spherical gated chamber complexes, in particular the existence of apartment systems.

Our final result in Section 6 is basically restricted to firm gated chamber complexes which allow an apartment system, thus it holds in particular for thin complexes. The existence of an apartment system in a firm gated chamber complex remains an open question.

This chapter is organised in the following way:

In Section 2 we recall the basic notions of simplicial complexes and chamber complexes. For the latter we introduce *type functions*, as well as the gate property for subsets, which is one of the main tools used in this chapter.

For the use in the second chapter we also state some connections between certain properties of chamber complexes, such as the existence of a type function and the property that all stars are gated. These statements can all be found in [Müh94].

In Section 3 we show that we can assign to a totally gated firm chamber complex Δ a reducibility type, which is a well known fact for Coxeter complexes and, more generally, buildings. In these special cases, the reducibility type can be derived from the diagram. We assign to every chamber of Δ a Coxeter matrix, and we show that these matrices, even though they can vary for different chambers, have the same reducibility type.

In Section 4 we recall the notion of parallelism for two residues and note some of the immediate consequences. We also note some more specific statements which are useful in the Sections 5 and 6.

In Section 5 we provide a proof that segments, i. e. the union of all chambers on minimal galleries between two given chambers, in totally gated chamber complexes are always convex sets. This result is relatively easy to prove in the case of Coxeter complexes, since segments are intersections of roots. For buildings, the statement can be obtained using the apartment system. The proof provided is independent of these structures.

In Section 6 we prove a more technical statement about irreducible non-spherical parallel residues, which also has an analogue in the case of buildings. For buildings the statement is a critical component in the theory of descent, which is developed in [MPW15]. We were only able to show the result in the thin case. However, there is an immediate generalisation to the case where Δ admits an apartment system.

1.2 Chamber complexes and the gate property

The notation in this chapter is mostly taken from [Dav08, Appendix A] and [Tit74]. Recall the notion of partially ordered sets:

Definition 1.2.1. We will call a pair $(M, \leq)$ consisting of a set M and a partial order $\leq$ on M a *partially ordered set* or shorter a *poset*, and will omit the partial order when it is unambiguous from the context.

For $A \in M$ we write

$$\underline{A} := \{B \in M \mid B \leq A\}.$$

Let $(M, \leq)$, $(N, \subseteq)$ be posets. A *morphism of posets* is a map $\varphi : M \to N$, such that for $A, B \in M$ we have

$$A \leq B \Rightarrow \varphi(A) \subseteq \varphi(B).$$

It is called an *isomorphism* if it is bijective and φ^{-1} is a morphism as well.

Definition 1.2.2. A *simplex* is a poset $(S, \leq)$ isomorphic to $(\mathcal{P}(J), \subseteq)$ for some set J, where $\subseteq$ denotes the set-wise inclusion. A *simplicial complex* is a poset $(\Delta, \leq)$ such that

1) $\underline{A}$ is a simplex for all $A \in \Delta$,

2) $A, B \in \Delta$ have a unique greatest lower bound, denoted by $A \cap B$.

Now let $(\Delta, \leq)$ be a simplicial complex. For $A, B \in \Delta$ we say that A *is a face of* B if $A \leq B$. We will write $A < B$ if $A \leq B$ and $A \neq B$.

Due to the second property, there exists a unique minimal element in Δ which is denoted by $\emptyset$.

A *vertex* of Δ is an element $v \in \Delta$ such that $A \leq v$ and $A \neq v$ imply $A = \emptyset$.

Remark 1.2.3. Another way to define a simplicial complex is to take a set J, and let $\Delta \subset \mathcal{P}(J)$. Then $(\Delta, \subseteq)$ is a poset. It is a simplicial complex if furthermore for $A \in \Delta$ also $\mathcal{P}(A) \subseteq \Delta$. This structure is a simplicial complex by the definition above. Furthermore, if we take as J the set of vertices, every simplicial complex corresponds to such a structure. Definition 1.2.2 makes it easier to describe the star of a simplex, and is therefore preferred.

Definition 1.2.4. Let $A \in \Delta$, then the *rank of* A, $\mathrm{rk}(A)$, is the cardinality of the set of vertices contained in A. We define the *rank of* Δ, $\mathrm{rk}(\Delta) := \sup_{A \in \Delta} \mathrm{rk}(A)$.

For $A \in \Delta$ define the *star of* A as $\mathrm{St}(A) := \{B \in \Delta \mid A \leq B\}$. This is again a simplicial complex with minimal element A.

A *chamber* of Δ is a maximal element in Δ, we will denote the set of chambers as $\mathrm{Cham}(\Delta)$ or $\mathcal{C}$, if Δ is unambiguous.

Let $\alpha : \Delta \to \Delta'$ be a map between simplicial complexes Δ and Δ'. Then α is called a *morphism of simplicial complexes* if it is a morphism of posets and furthermore $\alpha|_A : \underline{A} \to \underline{\alpha(A)}$ is an isomorphism for all $A \in \Delta$.

A *subcomplex* Δ' of Δ is a subset of Δ such that the inclusion $\Delta' \to \Delta$ is a morphism of simplicial complexes.

For $A \leq B$, the *codimension of* A *in* B is the rank of B in $\mathrm{St}(A)$, denoted by $\mathrm{codim}_B(A)$.

We say that A is a *maximal face* of B, if $\mathrm{codim}_B(A) = 1$.

Definition 1.2.5. Let Δ be a simplicial complex. We call Δ a *chamber complex*, if it satisfies:

1) Every $A \in \Delta$ is contained in a chamber.

2) For two chambers $C, D \in \Delta$ there is a sequence $C = C_0, C_1, \ldots, C_k = D$ such that

$$\mathrm{codim}_{C_{i-1}}(C_{i-1} \cap C_i) = \mathrm{codim}_{C_i}(C_{i-1} \cap C_i) \leq 1$$

for $1 \leq i \leq k$.

We call a sequence as in 2) a *gallery (from* C *to* D*)* and k its *length*.

Note that the first property is always satisfied if $\mathrm{rk}(\Delta)$ is finite. In this case it is easy to see (cp. [Müh94, 1.3, p.15]) that every chamber has the same rank. As a consequence of 2) we find that every element in a gallery is again a chamber.

For two chambers $C, D \in \Delta$ and $A \in \Delta$ with $A \leq C$ and $A \leq D$ we have $\operatorname{codim}_C(A) = \operatorname{codim}_D(A)$ (see [Tit74, 1.3]). This allows us to define the *corank of* A as $\operatorname{corank}(A) = \operatorname{codim}_C(A)$ for any chamber C containing A. We call C and D *adjacent*, or $C \sim D$, if $\operatorname{corank}(C \cap D) = 1$.

For $A \in \Delta$ with $\operatorname{corank}(A) = 1$, we call $\operatorname{Cham}(\operatorname{St}(A))$ a *panel* of Δ.

The complex Δ is *meagre* (resp. *thin*, *firm*, *thick*), if every panel contains at most two (exactly two, at least two, at least three) chambers.

A chamber complex Δ is *strongly connected*, if $\operatorname{St}(A)$ is a chamber complex for every $A \in \Delta$.

A *morphism of chamber complexes* is a morphism of simplicial complexes mapping chambers to chambers.

Definition 1.2.6. Let Δ be a chamber complex and I be an index set. A *type function (over I)* of Δ is a morphism of chamber complexes $\tau : \Delta \to \mathcal{P}(I)$. If Δ has a type function τ over I, we say that Δ is *I-numbered*, or *numbered*, when I is not important.

A *weak type function (over I)* of Δ is a family of type functions

$$(\tau_C : \underline{C} \to \mathcal{P}(I))_{C \in \operatorname{Cham}(\Delta)}$$

which is compatible in the sense that $\tau_C|_{\underline{C} \cap \underline{D}} = \tau_D|_{\underline{C} \cap \underline{D}}$ for adjacent chambers C and D.

Given a type function τ we define the *dual type function* $\hat{\tau} : \Delta \to \mathcal{P}(I)$ by $\hat{\tau}(A) = I \setminus \tau(A)$.

Given two adjacent chambers C, D, we say that C and D are *i-adjacent* if $\hat{\tau}(C \cap D) = i$. In this case we write $C \sim_i D$. For a subset $J \subset I$, we write $C \sim_J D$ if there exists a gallery $C = C_0, C_1, \ldots, C_k = D$ from C to D such that for all $0 \leq l \leq k-1$ we find $C_l \sim_i C_{l+1}$ for some $i \in J$.

Let $\gamma = (C_0, C_1, \ldots, C_k)$ be a gallery such that $C_{j-1} \sim_{i_j} C_j$ for $j = 1, \ldots, k$. We call the sequence $(i_1, \ldots, i_k)$ the *type of* γ.

Remark 1.2.7.
1. A type function τ with index set I induces a weak type function $(\tau|_{\underline{C}} : \underline{C} \to \mathcal{P}(I))_{C \in \operatorname{Cham}(\Delta)}$. Conversely, we show in Lemma 1.2.15 that a weak type function $(\tau_C : \underline{C} \to \mathcal{P}(I))_{C \in \operatorname{Cham}(\Delta)}$ on a strongly connected chamber complex gives rise to a type function τ such that $\tau|_{\underline{C}} = \tau_C$. This is a known result and was already mentioned in [Müh94].
2. It follows immediately from the definition that $\sim_J$ is an equivalence relation for all $J \subset I$. This motivates the next definition.

Definition 1.2.8. Let Δ be a numbered chamber complex. We call an equivalence class of $\sim_J$ a *J-residue*. In particular, for a chamber C we denote with $\mathcal{R}_J(C)$ the J-residue containing C, i. e.

$$\mathcal{R}_J(C) := \{D \in \mathcal{C} \mid C \sim_J D\}.$$

A subset $\mathcal{R} \subset \mathcal{C}$ is called a *residue* of Δ, if $\mathcal{R}$ is the J-residue containing C for some $J \subset I$, $C \in \mathcal{C}$. The number $|J|$ is the *rank* of $\mathcal{R}$, J its *type*.

Definition and Remark 1.2.9. Panels are residues of rank 1. If the type of a panel is $\{i\}$, we say it is an i-panel. For $i \in I$, $C \in \mathcal{C}$ we write $\mathcal{R}_i(C)$ instead of $\mathcal{R}_{\{i\}}(C)$.

We will now introduce the main property of interest to us:

Definition 1.2.10. For a metric space (M, d) and $x, y \in M$ we define the *segment between x and y* as

$$\sigma(x, y) = \{z \in M \mid d(x, z) + d(z, y) = d(x, y)\}.$$

Let $x \in M$ and $A \subseteq M$. A point $y \in A$ is called a *gate of x to A* or *the projection of x on A* if $y \in \sigma(x, z)$ for all $z \in A$. The gate of x to A is uniquely determined and we will denote it by $\mathrm{proj}_A(x)$. The set A is called *gated* if every $x \in M$ has a gate to A. We call a subset $A \subset M$ *convex*, if for all $x, y \in A$ also $\sigma(x, y) \subset A$.

The following lemma is immediate from the definitions.

Lemma 1.2.11. *Let Δ be a chamber complex, C, D chambers. Define $d(C, D)$ to be the length of a minimal gallery from C to D. Then* $(\mathrm{Cham}(\Delta), d)$ *is a connected metric space.*

Definition 1.2.12. Let Δ be a chamber complex. We say that Δ is *gated*, if every residue of Δ is gated, and *totally gated*, if furthermore $\mathrm{proj}_{\mathcal{R}}(\mathcal{S})$ is again a residue for all residues $\mathcal{R}$, $\mathcal{S}$ of Δ.

The following lemma is an immediate consequence from the definition [DS87, Proposition 1].

Lemma 1.2.13. *In a metric space all gated subsets are convex. In particular, for a gated chamber complex we find that all residues are convex.*

Given the fundamental notions of chamber complexes, we will end this section by recalling and elaborating some connections between the gate property for stars of Δ, the existence of weak type functions and type functions, and the property of being strongly connected.

We recall the following theorem.

Theorem 1.2.14 (see [Müh94, 1.5.3])**.** *Let Δ be a chamber complex, such that all sets* $\mathrm{Cham}(St(A)) \subset \mathrm{Cham}(\Delta)$ *are gated for $A \in \Delta$ with* $\mathrm{codim}_\Delta(A) \in \{1, 2\}$. *Let C be a chamber and τ be type function of $\underline{C}$. Then there exists a unique weak type function* $(\tau_D)_{D \in \mathrm{Cham}(\Delta)}$ *such that $\tau_C = \tau$.*

We also note a result for type functions of chamber complexes. Mühlherr mentions this in [Müh94] as an easy consequence, but we to elaborate the proof.

Lemma 1.2.15. *Let Δ be a strongly connected chamber complex. If Δ has a weak type function over I, it has a type function over I. In particular, if $(\tau_C)_{C\in\mathrm{Cham}(\Delta)}$ is a weak type function, there exists a type function τ such that $\tau|_{\underline{C}} = \tau_C$ for all $C \in \mathrm{Cham}(\Delta)$.*

Proof. Let $\mathcal{C} = \mathrm{Cham}(\Delta)$, and $(\tau_C)_{C\in\mathcal{C}}$ be a weak type function, so $\tau_C|_{\underline{C}\cap\underline{D}} = \tau_D|_{\underline{C}\cap\underline{D}}$ for all $C, D \in \mathcal{C}$.

Assume $F \in \Delta$, and let C, D be chambers with $F \in \underline{C}, \underline{D}$. Since Δ is strongly connected, $\mathrm{St}(F)$ is connected, and we find $C, D \in St(F)$. Thus we find a gallery $C = C_0, C_1, \ldots, C_{m-1}, C_m = D$ from C to D with all $C_i \in St(F)$, so in particular $F \in \underline{C_i}$ for all $0 \le i \le m$. Also C_{i-1} and C_i are adjacent for $1 \le i \le m$, therefore $\tau_{C_{i-1}}(F) = \tau_{C_i}(F)$ and inductively we obtain $\tau_C(F) = \tau_D(F)$.

This allows us to define $\tau(F) := \tau_C(F)$ for every simplex F and every chamber C containing F. By definition τ coincides with τ_C on all simplices F' contained in C, in particular $\tau|_{\underline{C}}$ is a type function and thus a morphism of chamber complexes.

Finally, τ itself is a morphism, since every simplex F is contained in a chamber C, and $\tau|_{\underline{C}}$ is a morphism. □

1.3 Reducibility of chamber complexes

In this section we introduce the Coxeter matrix of a chamber. We also show that totally gated chamber complexes have a unique reducibility type.

Remark 1.3.1. Let Δ be a numbered chamber complex, and let $A \in \Delta$ such that $\mathrm{codim}_\Delta(A) = 2$ and $\mathrm{Cham}(\mathrm{St}(A))$ is gated.

The main theorem of [Sch85] yields that the chamber graph of $\mathrm{Cham}(\mathrm{St}(A))$ corresponds to the chamber graph of a generalised polygon, if it contains a circuit, and corresponds to the chamber graph of a bipartite tree otherwise.

Definition 1.3.2. Let $I = \{1, \ldots, r\}$ be a finite index set. Let Δ be a strongly connected firm gated chamber complex of rank r with a type function $\tau : \Delta \to \mathcal{P}(I)$ and let $\mathcal{C} = \mathrm{Cham}(\Delta)$.

In this case, for every face $A \in \Delta$ of corank 2, the chamber graph of $\mathrm{St}(A)$ is a generalised n-gon for some $n \ge 2$ or a tree by the previous remark. Under the dual type function $\hat\tau : \Delta \to \mathcal{P}(I), \hat\tau : S \mapsto I \setminus \tau(S)$, $\hat\tau(A)$ is a two-element subset of I.

Fix a chamber $C \in \mathcal{C}$, for a corank 2 face $A \in \underline{C}$ we obtain $\hat\tau(A) = \{i, j\}$ for some $i \ne j \in I$. Set $m_{ij}^{C,\Delta} := n$, if the chamber-graph of $\mathrm{St}(A)$ is a generalised n-gon, and set $m_{ij}^{C,\Delta} = \infty$, if it is a tree.

Define the *Coxeter-matrix at a chamber C in Δ* to be the matrix $M_C^\Delta := (m_{ij}^{C,\Delta})_{i,j\in I}$ such that $m_{ii}^{C,\Delta} = 1$ for all $i \in I$, and $m_{ij}^{C,\Delta}$ is defined as above for $i \ne j$. We omit the data Δ and C when there is ambiguity.

Given the Coxeter-matrix at C, define the *Coxeter diagram of* C to be the labelled simplicial graph (V, E) with $V = I$, and $\{i, j\} \in E$ if and only if $m_{ij}^C \geq 3$. The natural labelling here is the map $\Lambda : E \to \mathbb{N} \cup \{\infty\}$, $\{i, j\} \mapsto m_{i,j}$.

If we want to consider only the simplicial graph (V, E) and forget the labelling Λ, we will talk about the *diagram of* C.

We call the diagram of C *irreducible* if it is connected, and *reducible* otherwise. For a subset $J \subset I$ the partition $\{J_1, \dots, J_k\}$ of J, where J_i is a connected component in $(J, E|_J)$ for $1 \leq i \leq k$, is called the *reducibility type of* C *w. r. t.* J, the sets J_i are the *components of* J *(at* C*)*. We will omit J in the case $J = I$.

We require the following basic observation for residues and stars:

Lemma 1.3.3. *Let* $J \subset I$*,* $C \in \mathcal{C}$*, and* $\mathcal{R} = \mathcal{R}_J(C)$*. Then* $\mathcal{R} = \mathrm{Cham}(\mathrm{St}(A))$*, where* $A \subset C$ *with* $\hat{\tau}(A) = J$*.*

Likewise, for $A \in \Delta$ *with* $\hat{\tau}(A) = J$*,* $\mathrm{Cham}(\mathrm{St}(A))$ *is a* J*-residue.*

Proof. Both claims follow easily by noting that for $A \subset C$ with $\hat{\tau}(A) = J$ and every chamber $D \in \mathcal{C}$ we have

$$(A \subset D \wedge C \sim D) \Leftrightarrow C \sim_i D \text{ for some } i \in J.$$

Then use induction on $d(C, D)$. □

Definition 1.3.4. Given a residue $\mathcal{R}$, the maximal face $A \in \Delta$ with $\mathcal{R} = \mathrm{Cham}(\mathrm{St}(A))$ is called the *core of* $\mathcal{R}$. For brevity, if A is the core of $\mathcal{R}$, write $\underline{\mathcal{R}}$ for the simplicial complex $\mathrm{St}(A)$.

Remark 1.3.5. The maximality of the simplex A is necessary only if Δ is not firm. If $J \subset I$ such that $\mathcal{R}$ is a J-residue, and no chamber in $\mathcal{R}$ has an i-neighbour for $i \in I \setminus J$, then $\mathcal{R}$ is also a $J \cup \{i\}$-residue.

Choosing A maximal means choosing the index set J minimal, this makes $\mathcal{R} = \mathrm{Cham}(\mathrm{St}(A))$ as firm as possible as a chamber complex.

Also note that when describing the residue $\mathcal{R}_J(C)$, the simplex A with $\hat{\tau}(A) = J$ and $\mathrm{Cham}(\mathrm{St}(A)) = \mathcal{R}_J(C)$ is always uniquely determined, the ambiguity above arises only if we forget the index set J.

Residues carry a lot of structure; the following lemma summarizes the properties which are most important for us.

Lemma 1.3.6. *Let* Δ *be a strongly connected firm gated chamber complex of finite rank* $|I|$ *with type function* τ*. Let* $C \in \mathcal{C}$*,* $J \subseteq I$*,* $\mathcal{R} := \mathcal{R}_J(C)$*. Then* $\underline{\mathcal{R}}$ *is a strongly connected gated chamber complex of rank* $|J|$*, with* $\mathrm{Cham}\,\underline{\mathcal{R}} = \mathcal{R}$ *and dual type function* $\hat{\tau}|_{\underline{\mathcal{R}}} : \underline{\mathcal{R}} \to \mathcal{P}(J)$*. In the case that* Δ *is totally gated,* $\underline{\mathcal{R}}$ *is totally gated. If* $M_C^\Delta = (m_{ij})_{i,j \in I}$*, then the matrix at* $C \in \mathcal{R}$ *as a chamber in* $\underline{\mathcal{R}}$ *is* $M_C^{\underline{\mathcal{R}}} = (m_{ij})_{i,j \in J}$*.*

Proof. Assume $\underline{\mathcal{R}} = \mathrm{St}(A)$ for some $A \in \Delta$. Since Δ is strongly connected, $\underline{\mathcal{R}}$ is a chamber complex. Let $B \in \mathrm{St}(A)$, then $\mathrm{St}(B) \subset \mathrm{St}(A)$, and as Δ is strongly connected, so is $\underline{\mathcal{R}}$.

Furthermore, $\mathrm{St}(A)$ is (totally) gated, since Δ is (totally) gated. Also, $\mathrm{St}(A)$ is convex, and stars in $\underline{\mathcal{R}}$ are also stars in Δ. If $\hat{\tau}(A) = J$, by definition of the type function $\hat{\tau}(B) \subset J$ for all $A \subset B$, and $\hat{\tau}|_{\underline{\mathcal{R}}}$ is still a morphism of complexes, so it is a type function.

The facts that the rank of $\underline{\mathcal{R}}$ is $|J|$ and $\mathrm{Cham}(\underline{\mathcal{R}}) = \mathcal{R}$ are immediate from the definition.

It remains to show that $M_C^{\mathcal{R}} = (m_{ij}^{C,\mathcal{R}})_{i,j \in J}$ for all $C \in \mathcal{R}$. So let $C \in \mathcal{R}$, and let $B \subset C$ with $\mathrm{corank}(B) = 2$. Now $B \in \underline{\mathcal{R}}$ if and only if $A \subset B$, therefore $\hat{\tau}(B) = \{i, j\} \subset J$. Since $\mathrm{St}(B) \subset \underline{\mathcal{R}}$, the i, j-th entry of M_C^{Δ} and $M_C^{\mathcal{R}}$ coincide. □

Proposition 1.3.7. *Let Δ be a strongly connected firm totally gated chamber complex of rank 3 with a type function. Then every chamber $C \in \mathcal{C}$ has the same reducibility type.*

In particular, if we find a chamber of Δ with reducibility type $\{\{1\}, \{2, 3\}\}$, every chamber in $\mathcal{C}$ has this type.

Proof. Fix a chamber $C \in \mathcal{C}$. Assume the diagram of C is reducible with $m_{12}^C = 2$, $m_{13}^C = 2$, $m_{23}^C \geq 2$. For $i = 1, 2, 3$ let D_i be a chamber i-adjacent chamber to C and for $i = 2, 3$ let E_i be a chamber i-adjacent to D_1. Note that by our assumptions on the numbers m_{ij}^C these chambers exist. Consequently, $D_1, E_2, E_3 \in \mathcal{A} = \mathcal{R}_{\{2,3\}}(D_1)$. Let $n \in \mathbb{N} \dot{\cup} \{\infty\}$ be the diameter of the chamber graph of $\mathcal{A}$. We obtain

$$\mathrm{proj}_{\mathcal{A}}(C) = D_1, \qquad \mathrm{proj}_{\mathcal{A}}(D_2) = E_2, \qquad \mathrm{proj}_{\mathcal{A}}(D_3) = E_3.$$

Let $\mathcal{B}$ be the $2, 3$-residue containing C, we obtain similarly

$$\mathrm{proj}_{\mathcal{B}}(D_1) = C, \qquad \mathrm{proj}_{\mathcal{B}}(E_2) = D_2, \qquad \mathrm{proj}_{\mathcal{B}}(E_3) = D_3,$$

and since Δ is totally gated, we can conclude $\mathrm{proj}_{\mathcal{B}}(\mathcal{A}) = \mathcal{B}$ and $\mathrm{proj}_{\mathcal{A}}(\mathcal{B}) = \mathcal{A}$. In particular this implies $n = m_{23}$.

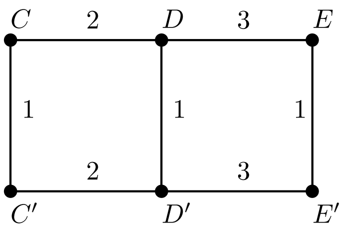
Therefore $\mathrm{proj}_{\mathcal{B}}|_{\mathcal{A}} : \mathcal{A} \mapsto \mathcal{B}$ is a bijection, and since

$$d(C, D_1) = d(D_2, E_2) = d(D_3, E_3) = 1,$$

it follows inductively that $d(E, proj_B(E)) = 1$ for every $E \in \mathcal{A}$. Hence E and $\mathrm{proj}_{\mathcal{B}}(E)$ are 1-adjacent for every $E \in \mathcal{A}$, and in particular the diagrams of D_1, D_2, D_3 coincide with the diagram of C, since Δ is firm.

By symmetry we obtain that the diagram of D_i is irreducible if and only if the diagram of C is irreducible. The statement of the proposition now follows by induction on the distance to C. □

Remark 1.3.8. The notion of Coxeter matrices at chambers can as well be defined for arbitrary strongly connected gated numbered complexes, which are not firm. However, the above statement is wrong in the meagre case, the following complex is a counterexample.



Here dots represent chambers, two chambers connected with an i-labelled edge means that the two chambers are i-adjacent. If there is no i-labelled edge containing a chamber, this chamber has no i-neighbour. In this complex, the chambers D, D' have a reducible Coxeter matrix, since 1 commutes with 2 and 3, but this is not the case for the chambers C, C', E, E'.

To generalise this result to chamber complexes of finite rank, we need the following graph-theoretical result.

Lemma 1.3.9. *Let $\Gamma = (V, E)$ be a finite connected simplicial graph with $|V| = n \geq 3$, $v \in V$. Then either there exist $W_1, W_2 \subset V$ such that*

i) $|W_1| = n - 1 = |W_2|$,

ii) $v \in W_1 \cap W_2$,

iii) $(W_i, E|_{W_i})$ *is connected for* $i = 1, 2$,

iv) $W_1 \cup W_2 = V$.

Or Γ is a string and v an endpoint, i. e. we can assume $V = \{1, \dots, n\}$, $e \in E \Leftrightarrow e = \{i, i+1\}$ for $i \in \{1, \dots, n-1\}$, and $v \in \{1, n\}$.

Proof. Let $m = \max\{d(v, w) \mid w \in V\}$. If $m = n$, we are in the second case, so assume $m < n$. Define $k \in \mathbb{N}$ to be the maximal number such that $|\{w \in V \mid d(v, w) = k\}| \geq 2$, k is well defined with $k \geq 1$ since we are not in the second case. Let $\{w_1, \dots, w_l \in V\} = \{w \in V \mid d(v, w) = k\}$, $l \geq 2$ by choice of k. We distinguish two cases:

Case 1, $k = m$. In this case $W_1 = V \setminus \{w_1\}$, $W_2 = V \setminus \{w_2\}$ satisfy the four above properties.

Case 2, $k < m$. By choice of k there exists in this case an index $1 \leq i \leq l$ and $x_{k+1}, \dots, x_m$ such that $d(x_j, v) = j$ for $k < j \leq m$ and $(w_i, x_{k+1}, \dots, x_m)$ is a path in Γ. The vertices $x_j, k < j \leq m$ are unique by choice of k. Let $w \in \{w_1, \dots, w_l\} \setminus \{w_i\}$. Then $W_1 := V \setminus \{w\}$, $W_2 := V \setminus \{x_m\}$ satisfy the above properties. $\square$

Theorem 1.3.10. *Let Δ be a strongly connected firm totally gated chamber complex of finite rank. Then every chamber $C \in \mathcal{C}$ has the same reducibility type.*

Proof. In rank 1 Δ is just a collection of vertices and the empty set, and all vertices are adjacent. The diagram at every chamber is just the one element in I.

In rank 2 Δ is a generalised n-gon or a tree, and again the statement is immediate.

For rank $r \geq 3$ we prove the statement by induction on r, the basis being $r = 3$, which is shown in 1.3.7. So assume $r > 3$, and let $I = \{1, \ldots, r\}$.

Let C be a chamber, $i \in I$ and assume that the diagram (V, E) of C is irreducible. Assume furthermore that we are in the first case of 1.3.9, then we find $J_1, J_2 \subset I$ with the properties:

i) $|J_1| = r - 1 = |J_2|$,

ii) $i \in J_1 \cap J_2$,

iii) $(J_i, E|_{J_i})$ is connected for $i = 1, 2$,

iv) $J_1 \cup J_2 = I$, or, equivalently, $J_1 \neq J_2$.

Let D_i be i-adjacent to C. Then C, D_i are contained and adjacent in the J_1-residue of C as well as in the J_2-residue of C. Using induction on J_1, we obtain that D_1 has the same reducibility type as C in $\mathcal{R}_{J_1}(C)$ and in $\mathcal{R}_{J_2}(C)$. Then 1.3.6 yields that with respect to D_i the set J_1 is connected as well as J_2. Since both contain i, I is connected and the reducibility type of D_i is $\{I\}$.

Now assume that we do not find a pair J_1, J_2 with the above properties. Then we are in the second case of 1.3.9, hence the diagram of C is a line and i the vertex at one end. W. l. o. g. assume that $i = 1$, and that $i, i + 1$ are adjacent in the diagram of C.

Then consider the two subsets $J_1 = \{1, \ldots, r - 1\}$ and $J_2 = \{1, r - 1, r\}$. We apply induction on the residues $\mathcal{R}_{J_1}(C)$ and $\mathcal{R}_{J_2}(C)$ and can observe that the set J_1 is connected in the diagram at D_1. Furthermore $|J_2| = 3$ and 1.3.7 yields that $\{r - 1, r\}$ is an edge in the diagram of D_1. Again we obtain that the diagram of D_1 is connected.

By induction on the distance to C therefore the diagram of every chamber is irreducible.

Now assume that C has reducibility type $\{J_1, \ldots, J_k\}$, where $k \geq 2$. Clearly $C \in \mathcal{R}_{J_i}(C)$ for all i. In case $k = 2$ assume C and D are i-adjacent chambers for some $i \in J_1$. By our result above and considering the chamber complex $\mathcal{R}_{J_1}(C)$ J_1 is a connected component at D. Therefore the reducibility type of D is a refinement of $\{J_1, J_2\}$.

In case $k > 3$ let D be i-adjacent to C and let $j, l \in I \backslash J_1$, such that $m^C_{ij} = m^C_{il} = m^C_{jl} = 2$, where $M^\Delta_C = (m^C_{ij})_{i,j \in I}$. By 1.3.7 we obtain $m^D_{ij} = m^D_{il} = m^D_{jl} = 2$, by considering $\mathcal{R}_{\{i,j,l\}}(C)$ and using 1.3.6. We also find J_1 to be a connected component at D. We can conclude again, that the reducibility type of D is a refinement of $\{J_1, \ldots, J_k\}$.

By symmetry we obtain that C, D are of the same reducibility type. $\square$

Definition 1.3.11. Let Δ be a strongly connected totally gated chamber complex. With respect to the last statement we say that a partition $\{J_1, \dots, J_k\}$ of I is the *reducibility type of* Δ, if it is the reducibility type of a chamber $C \in \mathcal{C}$.

For $i \in \{1, \dots, k\}$ we call J_i a *component of* I in Δ.

1.4 Parallel residues

Throughout this section, always assume Δ to be a gated chamber complex. We recall the well-known notion of parallelism in chamber complexes and give some immediate consequences. Please note that most of the statements are already known for buildings, and the proofs vary only slightly for gated chamber complexes.

Definition 1.4.1. Let $\mathcal{R}$, $\mathcal{S}$ be two residues in Δ. We say that $\mathcal{R}$ and $\mathcal{S}$ are *parallel in* Δ, if $\mathrm{proj}_{\mathcal{R}}(\mathcal{S}) = \mathcal{R}$ and $\mathrm{proj}_{\mathcal{S}}(\mathcal{R})$.

The complex Δ is called *spherical*, if there exist $C, D \in \mathcal{C}$ such that $\mathrm{proj}_P(D) \neq C$ for all panels P containing C.

A residue is called *spherical* if it is spherical as a chamber complex. A subset $J \subset I$ is called *spherical at* C, if $\mathcal{R}_J(C)$ is spherical. Chambers C and D as in the definition above are *opposite*.

Define the following subsets of I. Let $C, D \in \mathcal{C}$, then

$$J^+(C, D) := \{i \in I \mid \mathrm{proj}_{\mathcal{R}_i(D)}(C) = D\},$$
$$J^-(C, D) := \{i \in I \mid \mathrm{proj}_{\mathcal{R}_i(D)}(C) \neq D\}.$$

Lemma 1.4.2. *Let $C, D \in \mathcal{C}$. We find $i \in J^+(C, D)$ if and only if $d(C, D_i) = d(C, D)+1$ for all chambers D_i which are i-adjacent to D; and $i \in J^-(C, D)$ if and only if there exists a gallery $(C = C_0, C_1, \dots, C_{k-1} = D', C_k = D)$, where $D' = \mathrm{proj}_{\mathcal{R}_i(D)}(C)$.*

Proof. In the first case both statements are equivalent to $D = \mathrm{proj}_{\mathcal{R}_i(D)}(C)$. In the second case, let $i \in J^-(C, D)$ and $D' = \mathrm{proj}_{\mathcal{R}_i(D)}(C)$. The definition of $J^-(C, D)$ yields $D' \neq D$. Therefore $d(C, D') < d(C, D)$, and we can extend a minimal gallery from C to D' to a minimal gallery from C to D. Now assume such a minimal gallery exists, then $d(C, D') > d(C, D)$ holds and in particular $D \neq \mathrm{proj}_{\mathcal{R}_i(D)}(C)$, so $i \in J^-(C, D)$. □

Lemma 1.4.3. *Let $K \subset J \subset I$, $C, D \in \mathcal{C}$. Then*

$$\mathrm{proj}_{\mathcal{R}_K(D)}(C) = \mathrm{proj}_{\mathcal{R}_K(D)}(\mathrm{proj}_{\mathcal{R}_J(D)}(C)).$$

Proof. By definition there exists a minimal gallery from C to $\mathrm{proj}_{\mathcal{R}_K(D)}(C)$ passing through $\mathrm{proj}_{\mathcal{R}_J(D)}(C)$. Hence the distance from $\mathrm{proj}_{\mathcal{R}_J(D)}(C)$ to $\mathrm{proj}_{\mathcal{R}_K(D)}(C)$ is minimal among all chambers in $\mathcal{R}_K(D)$, which proves the statement. □

Corollary 1.4.4. *Let $i \in J \subset I$, $C, D \in \mathcal{C}$. Then*

$$\mathrm{proj}_{\mathcal{R}_i(D)}(C) = \mathrm{proj}_{\mathcal{R}_i(D)}(\mathrm{proj}_{\mathcal{R}_J(D)}(C)).$$

Lemma 1.4.5. *Let $C, D \in \mathcal{C}$, then $J = J^-(C, D)$ is spherical at D. In particular, $D' = \mathrm{proj}_{R_J(D)}(C)$ is opposite to D in $R_J(D)$.*

Proof. Let $J := J^-(C, D)$, $\mathcal{R} := \mathcal{R}_J(D)$, $D' = \mathrm{proj}_{\mathcal{R}}(D)$. By choice of J for every $i \in J$ we find $\mathrm{proj}_{\mathcal{R}_i(D)}(C) \neq D$, we obtain the statement by observing $\mathrm{proj}_{\mathcal{R}_i(D)}(D') = \mathrm{proj}_{\mathcal{R}_i(D)}(C)$ for all $i \in J$ by 1.4.3. □

Lemma 1.4.6. *Let $C, D \in \mathcal{C}$, $(i_1, \ldots, i_k)$ the type of a minimal gallery from C to D. Then the set $J_{C,D} := \{i_1, \ldots, i_k\}$ is independent of the choice of the minimal gallery from C to D.*

Proof. Assume first that C and D are opposite in some residue $\mathcal{R} = \mathcal{R}_J(C)$, and assume $J_{C,D} = J'$. Then C and D are contained and opposite in $\mathcal{R}_{J'}(C)$. Residues are convex and $J = J^-(C, D)$, hence $J \subseteq J'$. Now $J' \subseteq J$ follows again from the convexity of residues, thus we obtain $J_{C,D} = J$.

We prove the general case by induction on $d = d(C, D)$. For $d = 0$ and $d = 1$, the statement is true, since in the first case C, D are opposite in $\mathcal{R}_\emptyset(C)$, in the second they are opposite in $\mathcal{R}_i(C)$, if C, D are i-adjacent.

So let $d \geq 2$, and let $J = J^-(C, D)$. Consider the residue $\mathcal{R} := \mathcal{R}_J(D)$, and let $D' := \mathrm{proj}_{\mathcal{R}}(C)$. In the case where $C \in \mathcal{R}$, the statement is true since C is opposite to D in $\mathcal{R}$. So assume $C \notin \mathcal{R}$. Then $d(C, D) = d(C, D') + d(D', D)$ by definition of D' and $D \neq D'$. Therefore $J_{C,D'}$ is independent of the choice of the minimal gallery from C to D' by induction. Furthermore, D' and D are opposite chambers in $\mathcal{R}$ by 1.4.5, therefore $J_{D',D} = J$, which is independent of the choice of the gallery.

Therefore $J_{C,D} = J \cup J_{C,D'}$ holds and $J_{C,D}$ is hence independent of the choice of the minimal gallery. □

Definition 1.4.7. For $C, D \in \mathcal{C}$ we also call the set $J_{C,D}$ as above the *support of* $\{C, D\}$, written as $\mathrm{supp}\,(C, D)$.

For $X \subset \mathcal{C}$ we write $\mathrm{supp}\,(X) = \bigcup_{C,D \in X} \mathrm{supp}\,(C, D)$.

Remark 1.4.8.

1. It follows immediately that $\mathrm{supp}\,(\mathcal{R}_J(C)) = J$ for all $C \in \mathcal{C}$, $J \subset I$.
2. Also note that $\mathrm{supp}\,(C, D) = \{i\}$ if and only if C, D are i-adjacent.
3. If $C, D, E \in \mathcal{C}$ with $d(C, D) + d(D, E) = d(C, E)$, we find

$$\mathrm{supp}\,(C, E) = \mathrm{supp}\,(C, D) \cup \mathrm{supp}\,(D, E).$$

The following two statements characterise parallel residues and are well-known:

Lemma 1.4.9. *Let $\mathcal{R}$, $\mathcal{S}$ be parallel residues of Δ. Then* $\operatorname{proj}_{\mathcal{R}}|_{\mathcal{S}} : \mathcal{S} \to \mathcal{R}$ *and* $\operatorname{proj}_{\mathcal{S}}|_{\mathcal{R}} : \mathcal{R} \to \mathcal{S}$ *are bijections inverse to each other.*

Lemma 1.4.10. *Let $\mathcal{R}$, $\mathcal{S}$ be residues. Then* $\operatorname{proj}_{\mathcal{S}}(\mathcal{R})$ *and* $\operatorname{proj}_{\mathcal{R}}(\operatorname{proj}_{\mathcal{S}}(\mathcal{R}))$ *are parallel.*

Lemma 1.4.11. *Let $\mathcal{R}$, $\mathcal{S}$ be parallel residues of Δ. Then $d(C, \operatorname{proj}_{\mathcal{S}}(C))$ is independent of the choice of $C \in \mathcal{R}$.*

Proof. Assume $C \sim D \in \mathcal{R}$, let $C' := \operatorname{proj}_{\mathcal{S}}(C)$, $D' := \operatorname{proj}_{\mathcal{S}}(D)$ and let $d = d(C, C')$, $d' = d(D, D')$. By definition of C', D', we find $d(C, D') = d + 1$ and $d(D, C') = d' + 1$.

Assume $d > d'$, then $d(C, C') \geq d' + 1 = d(D, C')$. By 1.4.9 we find $C = \operatorname{proj}_{\mathcal{R}}(C')$, a contradiction to $d(C, C') \geq d(D, C')$, and by symmetry we obtain $d = d'$. The claim holds by induction on $d(C, D)$ for $D \in \mathcal{R}$. □

Definition 1.4.12. Let $\mathcal{R}$, $\mathcal{S}$ be parallel residues of Δ. The previous lemma allows us to define $d(\mathcal{R}, \mathcal{S}) := d(C, \operatorname{proj}_{\mathcal{S}}(C))$ for $C \in \mathcal{R}$. Note that this number also coincides with $\min\{d(C, D) \mid C \in \mathcal{R}, D \in \mathcal{S}\}$.

Corollary 1.4.13. *Let $\mathcal{R}$, $\mathcal{S}$ be parallel residues of Δ. Then either $\mathcal{R} \cap \mathcal{S} = \emptyset$ or $\mathcal{R} = \mathcal{S}$.*

Definition 1.4.14. Let $C \in \mathcal{C}$, $i, j \in I$. We say that i and j *commute at* C, if $m_{ij}^C = 2$.

Likewise, for a subset $X \subset \mathcal{C}$, $J, J' \subset I$ we say that J *and* J' *commute at* X if $m_{ij}^C = 2$ for all $i \in J$, $j \in J'$, $C \in X$, i.e. every element of J commutes with every element of J' at every chamber $C \in X$.

For $C \in \mathcal{C}$, $X \subset \mathcal{C}$ and $J \subset I$ we will write $J_C^{\perp}$ (resp. $J_X^{\perp}$) for all elements in I commuting with J at C (resp. at X).

Lemma 1.4.15. *Let $\mathcal{R}$, $\mathcal{S}$ be parallel residues of Δ with $d(\mathcal{R}, \mathcal{S}) = 1$. Then $\mathcal{R}$ and $\mathcal{S}$ are of the same type J. If C, $\operatorname{proj}_{\mathcal{S}}(C)$ are i-adjacent, then i commutes with J at $\mathcal{R} \cup \mathcal{S}$, and $D \sim_i \operatorname{proj}_{\mathcal{S}}(D)$ for all $D \in \mathcal{R}$.*

Proof. Let $J_{\mathcal{R}}$ be the type of $\mathcal{R}$, $J_{\mathcal{S}}$ be the type of $\mathcal{S}$. Let $C, D \in \mathcal{R}$ be j-adjacent for $j \in J_{\mathcal{R}}$. Further, let $C' := \operatorname{proj}_{\mathcal{S}}(C)$, $D' = \operatorname{proj}_{\mathcal{S}}(D)$ and assume C', D' are l-adjacent for $l \in J_{\mathcal{S}}$. Consequently, (D, D', C') is a minimal gallery from D to C', residues are convex, hence this gallery is contained in the residue $\mathcal{R}_{\{i,j\}}(D)$. We can conclude that D' is i-adjacent to D and D' is j-adjacent to C'.

Induction on $d(C, D)$ yields that every chamber $D \in \mathcal{R}$ is i-adjacent to $\operatorname{proj}_{\mathcal{S}}(D)$. We also showed that $j \in J_{\mathcal{R}}$ implies $j \in J_{\mathcal{S}}$, by symmetry we obtain $J_{\mathcal{R}} = J_{\mathcal{S}}$. Since C, D, D', C' form a circuit of order 4 in the chamber graph, $m_{ij}^C = 2 = m_{ij}^{C'}$ for all $j \in J_{\mathcal{R}}$. By induction on $d(C, D)$ therefore $m_{ij}^D = 2$ for all $j \in J_{\mathcal{R}}$ and all chambers $D \in \mathcal{R}$, and likewise for all $D \in \mathcal{S}$. □

Lemma 1.4.16. *Let $\mathcal{R}$, $\mathcal{S}$ be parallel residues of Δ. Then $J^-(C, \operatorname{proj}_{\mathcal{S}}(C))$ is independent of the choice of $C \in \mathcal{R}$.*

Proof. Fix $C \in \mathcal{R}$ and let D be adjacent to C, let $C' = \mathrm{proj}_{\mathcal{S}}(C)$, $D' = \mathrm{proj}_{\mathcal{S}}(D)$. Let $j \in J^-(C, C')$ and assume C', D' are l-adjacent. Let $\mathcal{R}' := \mathcal{R}_{\{j,l\}}(C')$ and let $E := \mathrm{proj}_{\mathcal{R}'}(D)$. Now both $j, l \in J^-(D, C')$, since $D' = \mathrm{proj}_{\mathcal{R}_l(C')}$ and $\mathrm{proj}_{\mathcal{R}_j(C')}(D) = \mathrm{proj}_{\mathcal{R}_j(C')}(C)$ hold.

Therefore E is opposite to C' in $\mathcal{R}_{\{j,l\}}(C')$, and assume $\mathrm{proj}_{\mathcal{R}_j(D')}(E) = D'$. Then any chamber D'' which is j-adjacent to D' satisfies $d(E, C') = d(E, D'')$, furthermore $D' = \mathrm{proj}_{\mathcal{R}_l(C')}(E)$ since D', C' are l-adjacent and $d(E, D') = d(E, C') - 1$. This is only possible in case $D' = E$, a contradiction. Therefore we obtain by 1.4.3 that $\mathrm{proj}_{\mathcal{R}_j(D')}(D) = \mathrm{proj}_{\mathcal{R}_j(D')}(E) \neq D'$ and $j \in J^-(D, \mathrm{proj}_{\mathcal{S}}(D))$.

The claim follows by induction on $d(C, D)$ for $D \in \mathcal{R}$. □

Definition 1.4.17. Let $\mathcal{R}$, $\mathcal{S}$ be parallel residues. Define $J^-(\mathcal{R}, \mathcal{S}) := J^-(C, \mathrm{proj}_{\mathcal{S}}(C))$ for some $C \in \mathcal{R}$. By 1.4.16 $J^-(\mathcal{R}, \mathcal{S})$ is independent of the choice of C.

Lemma 1.4.18. *Let $\mathcal{R}$ and $\mathcal{S}$ be residues. If $\mathrm{proj}_{\mathcal{S}}|_{\mathcal{R}} : \mathcal{R} \to \mathcal{S}$ is a bijection, then $(\mathrm{proj}_{\mathcal{S}}|_{\mathcal{R}})^{-1} = \mathrm{proj}_{\mathcal{R}}|_{\mathcal{S}}$, and in particular $\mathcal{R}$ and $\mathcal{S}$ are parallel.*

Proof. Let $C \in \mathcal{R}$, $C' = \mathrm{proj}_{\mathcal{S}}(C)$, and let $D = \mathrm{proj}_{\mathcal{R}}(C') \in \mathcal{R}$. Therefore there exists a minimal gallery from C to C' of the form $(C, \dots, D, \dots, C')$, where C' is the only chamber in $\mathcal{S}$. Hence $(D, \dots, C')$ is a minimal gallery from D to $\mathcal{S}$, yielding $\mathrm{proj}_{\mathcal{S}}(D) = C'$. We can conclude $D = C$, which proves our claims. □

The following statement is mentioned for buildings in a slightly different language also in [MPW15].

Lemma 1.4.19. *Let $\mathcal{R}$, $\mathcal{S}$ be parallel residues, where $\mathcal{R}$ is of type J and fix a $C \in \mathcal{R}$. Let $K \subset J$, $K^\circ \subset I \setminus J$, and define $\mathcal{Y} = \mathcal{R}_{K \cup K^\circ}(C)$, $\mathcal{X} = \mathcal{R}_K(C)$, $\mathcal{X}' = \mathrm{proj}_{\mathcal{S}}(\mathcal{X})$, $\mathcal{X}'' = \mathrm{proj}_{\mathcal{Y}}(\mathcal{X}')$. Assume that $\mathcal{X}'$ and $\mathcal{X}''$ are residues. Then $\mathcal{X}$, $\mathcal{X}'$ and $\mathcal{X}''$ are pairwise parallel.*

Proof. The residues $\mathcal{X}$ and $\mathcal{X}'$ are parallel since $\mathrm{proj}_{\mathcal{S}}$ is a bijection. Let $C' = \mathrm{proj}_{\mathcal{S}}(C)$ and $C'' = \mathrm{proj}_{\mathcal{Y}}(C') = \mathrm{proj}_{\mathcal{X}''}(C')$.

By 1.4.3 we find

$$\mathrm{proj}_{\mathcal{X}}|_{\mathcal{X}'} = \mathrm{proj}_{\mathcal{X}}|_{\mathcal{Y}} \circ \mathrm{proj}_{\mathcal{Y}}|_{\mathcal{X}'},$$

and therefore, as the image of $\mathrm{proj}_{\mathcal{Y}}|_{\mathcal{X}'}$ is $\mathcal{X}''$,

$$\mathrm{proj}_{\mathcal{X}}|_{\mathcal{X}'} = \mathrm{proj}_{\mathcal{X}}|_{\mathcal{X}''} \circ \mathrm{proj}_{\mathcal{X}''}|_{\mathcal{X}'}. \tag{1.1}$$

The maps $\mathrm{proj}_{\mathcal{X}}|_{\mathcal{X}'}$ and $\mathrm{proj}_{\mathcal{X}'}|_{\mathcal{X}}$ are inverse bijections, therefore (1.1) yields that $\mathrm{proj}_{\mathcal{X}}|_{\mathcal{X}''}$ is surjective and $\mathrm{proj}_{\mathcal{X}''}|_{\mathcal{X}'}$ is injective. By definition $\mathrm{proj}_{\mathcal{Y}}|_{\mathcal{X}'} = \mathrm{proj}_{\mathcal{X}''}|_{\mathcal{X}'}$ is surjective, hence $\mathrm{proj}_{\mathcal{X}''}|_{\mathcal{X}'}$ is a bijection. The composition of $\mathrm{proj}_{\mathcal{X}''}|_{\mathcal{X}'}$ and $\mathrm{proj}_{\mathcal{X}}|_{\mathcal{X}''}$ is a bijection, so we obtain that $\mathrm{proj}_{\mathcal{X}}|_{\mathcal{X}''}$ and $\mathrm{proj}_{\mathcal{X}''}|_{\mathcal{X}'}$ are bijections. By 1.4.18 we obtain that $\mathcal{X}''$ and $\mathcal{X}$ are parallel residues as well as $\mathcal{X}'$ and $\mathcal{X}''$. □

Lemma 1.4.20. *Let $\mathcal{R}$ and $\mathcal{S}$ be parallel panels. Assume $\mathcal{S}'$ is another panel parallel to $\mathcal{R}$ such that $\mathcal{S} \cap \mathcal{S}' \neq \emptyset$, then $\mathcal{S} = \mathcal{S}'$.*

Proof. If $\mathcal{S} \cap \mathcal{S}'$ contains two chambers C and C', those are i-adjacent for some $i \in I$, thus $\mathcal{S} = \mathcal{R}_i(C) = \mathcal{S}'$. Hence assume $\{C\} = \mathcal{S} \cap \mathcal{S}'$ and $\mathcal{S} = \mathcal{R}_i(C)$, $\mathcal{S}' = \mathcal{R}_j(C)$.

Let $\mathcal{R}' := \operatorname{proj}_{\mathcal{R}_{\{i,j\}}(C)}(\mathcal{R})$. By Lemma 1.4.19 we find $\mathcal{R}'$ to be parallel to both $\mathcal{S}$ and $\mathcal{S}'$. In particular, since $\mathcal{S} \neq \mathcal{S}'$ and $d(\mathcal{S}, \mathcal{S}') = 0$, we find $\mathcal{R}' \neq \mathcal{S}, \mathcal{S}'$. Let $C' := \operatorname{proj}_{\mathcal{R}'}(C)$. Since $\mathcal{R}'$ is parallel to $\mathcal{S}$ and $\mathcal{S}'$, we obtain $\operatorname{proj}_{\mathcal{S}}(C') = C = \operatorname{proj}_{\mathcal{S}'}(C')$, thus $\{i, j\} \subset J^+(C', C)$. This is a contradiction to the fact that $\mathcal{R}_{\{i,j\}}(C)$ is convex. This yields $\mathcal{S} = \mathcal{S}'$. □

The following statement can be found as Lemma 1.4.6 in [Müh94], where it is stated without a proof.

Lemma 1.4.21. *Assume $\mathcal{R}$ and $\mathcal{S}$ are parallel panels, $C \in \mathcal{R}$, $D \in \mathcal{S}$. Let $C' = \operatorname{proj}_{\mathcal{S}}(C)$, $D' = \operatorname{proj}_{\mathcal{R}}(D)$. Then*

$$\sigma(C, D) = \sigma(C, C') \cup \sigma(D, D') = \sigma(C', D').$$

Proof. Induction on $d(\mathcal{R}, \mathcal{S})$. If $\mathcal{R} = \mathcal{S}$, there is nothing to show. If both are contained in a residue of rank 2, $\mathcal{R}$ and $\mathcal{S}$ are opposite panels in an n-gon. In this case $\sigma(C, C')$, $\sigma(D, D')$ are disjoint paths of length n and $\sigma(C, D)$ is a circle.

So let $d(\mathcal{R}, \mathcal{S}) > 0$ and assume they are not contained in a common rank 2 residue. Let $\gamma = (C = C_0, C_1, \ldots, C_{n+1} = D)$ be a minimal gallery from C to D. Since $\operatorname{proj}_{\mathcal{S}}(C) = C'$, $\operatorname{proj}_{\mathcal{S}}(D) = D$, there exists an index m such that $\operatorname{proj}_{\mathcal{S}}(C_m) = C'$, $\operatorname{proj}_{\mathcal{S}}(C_{m+1}) \neq C'$. If $\operatorname{proj}_{\mathcal{S}}(C_{m+1}) \neq D$, we find $d(C_{m+1}, \operatorname{proj}_{\mathcal{S}}(C_{m+1})) < d(C_{m+1}, D) = n - m$ and

$$\begin{aligned} d(C, \operatorname{proj}_{\mathcal{S}}(C_{m+1})) &\leq d(C, C_m) + d(C_m, C_{m+1}) + d(C_{m+1}, \operatorname{proj}_{\mathcal{S}}(C_{m+1}) \\ &= n - m + 1 + m - 1 = n = d(C, C'), \end{aligned}$$

a contradiction to $C' = \operatorname{proj}_{\mathcal{S}}(C)$. Thus $\operatorname{proj}_{\mathcal{S}}(C_{m+1}) = D$.

Assume $C_m \sim_i C_{m+1}$, then the panels $\mathcal{R}_i(C_m)$ and $\mathcal{S}$ are parallel. In the case $m = n$ we have $C_m = C'$, then $\gamma \subset \sigma(C, C') \cup \{D\}$.

In the case $m = 0$ we have $C_m = C$, and we will show that $C_{m+1} = D'$. Let $C \sim_j C_1$ and define $\mathcal{R}' := \mathcal{R}_i(C)$, by construction $\operatorname{proj}_{\mathcal{S}}(\mathcal{R}') = \mathcal{S}$, hence $\mathcal{S}$ and $\mathcal{R}'$ are parallel and $C \in \mathcal{R} \cap \mathcal{R}'$. By Lemma 1.4.20 we find $\mathcal{R} = \mathcal{R}'$.

Observe that $d(D, D') = n$, $d(D, C) = n + 1$ to obtain $C_1 = \operatorname{proj}_{\mathcal{R}}(D) = D'$. Thus $\gamma \subset \sigma(D, D') \cup \{C\}$. Therefore we can exclude these cases and assume $\mathcal{R}_i(C_m) \neq \mathcal{R}, \mathcal{S}$.

For $0 \leq k \leq m$ we find $C_m \in \sigma(C, C')$. Using induction we obtain $\sigma(C_m, D) = \sigma(C_m, C) \cup \sigma(C_{m+1}, D')$. Since we have $C_m \in \sigma(C, C')$ and $C_{m+1} \in \sigma(D, D')$, we obtain further $\sigma(C_m, C') \cup \sigma(C_{m+1}, D') \subset \sigma(C, C') \cup \sigma(D, D')$, as required. □

1.5 Convexity of segments

In this section we show that segments in totally gated chamber complexes are already convex.

Definition 1.5.1. Let Σ be a meagre chamber subcomplex of Δ. For a residue $\mathcal{R}$ of Δ we write

$$\mathcal{R}^{\Sigma} = \mathcal{R} \cap \mathrm{Cham}(\Sigma).$$

Furthermore we call two panels P, Q *adjacent (in Δ)* if they are opposite in a rank 2 residue (of Δ).

Remark 1.5.2. From the definition one does not obtain that $\mathcal{R}^{\Sigma}$ is a residue of $\mathrm{Cham}(\Sigma)$, as it might be the union of residues. If P is a panel of Δ with $P \cap \mathrm{Cham}(\Sigma) \neq \emptyset$, P^{Σ} is again a panel in Σ, and furthermore every panel P' in Σ can be written as P^{Σ} for some panel P of Δ. If P' contains two chambers, the choice for P is unique.

Lemma 1.5.3. *Let Δ be totally gated, and let $C, D \in \mathcal{C}$. Assume $C = C_0, C_1, \ldots, C_m = D$ is a gallery from C to D which is not minimal. Then there exist two indices $0 < k, l < m$ such that $\mathcal{R}_i(C_k)$ and $\mathcal{R}_j(C_l)$ are parallel, where i, j are given by $C_k \sim_i C_{k+1}$, $C_l \sim_j C_{l+1}$.*

Proof. Let l be a minimal index such that $C_0, \ldots, C_l$ is minimal, $C_0, \ldots, C_{l+1}$ is not. Further let k be a maximal index such that $C_{k+1}, \ldots, C_{l+1}$ is minimal, $C_k, \ldots, C_{l+1}$ is not. Then $\mathrm{proj}_{\mathcal{R}_j(C_l)}(C_k) \neq C_l$, $\mathrm{proj}_{\mathcal{R}_j(C_l)}(C_{k+1}) = C_l$. Hence $\mathcal{R}_i(C_k)$ and $\mathcal{R}_j(C_l)$ are parallel. $\square$

Notation 1.5.4. In the following, let $C \neq D \in \mathcal{C}$ be arbitrary chambers. Write $\sigma := \sigma(C, D)$ and $\Sigma := \Sigma(C, D) := \{F \in \Delta \mid F \in \underline{E}, E \in \sigma(C, D)\}$.

Let $D' \in \sigma$ such that $D' \sim_i D$ for an $i \in J^{-}(C, D)$. Let $P := \mathcal{R}_i(D)$, then $P \cap \sigma = \{D, D'\}$.

We can then define

$$X := \{E \in \sigma(C, D) \mid \mathrm{proj}_P(E) = D\},$$
$$X' := \{E \in \sigma(C, D) \mid \mathrm{proj}_P(E) = D'\}.$$

Further let $\mathcal{W}$ consist of all panels of σ contained in a panel of Δ parallel to P, i. e.

$$\mathcal{W} := \{Q^{\sigma} \mid Q \text{ is a panel parallel to } P, |Q^{\sigma}| = 2\}.$$

Recall from [Müh94, Lemma 1.4.5] the following fact:

Lemma 1.5.5. *For any $C, D \in \mathcal{C}$ the complex $\Sigma(C, D)$ is meagre.*

Lemma 1.5.6. *With notation as above we find $\sigma = X \mathbin{\dot\cup} X'$.*

Proof. Assume $E \in \sigma$, $\mathrm{proj}_P(E) \neq D, D'$, then $\mathrm{proj}_P(E) \in \sigma(E,D) \subset \sigma(C,D)$ by [Müh94, Lemma 1.4.5], a contradiction. Hence either $E \in X$ or $E \in X'$ holds. □

Lemma 1.5.7. *Let $Q^\Sigma = \{Q_1, Q_2\}$ be a panel in σ. We find $Q^\Sigma \in \mathcal{W}$ if and only if $Q_1 \in X$, $Q_2 \in X'$ or vice versa.*

Proof. If $Q_1 \in X$ and $Q_2 \in X'$, we find $\mathrm{proj}_P(Q) = P$, and hence P and Q are parallel, and Q^Σ contains two chambers, hence $Q^\Sigma \in \mathcal{W}$.

So let $Q^\Sigma \in \mathcal{W}$, then $\mathrm{proj}_P(Q_1) \neq \mathrm{proj}_P(Q_2)$. Both $\mathrm{proj}_P(Q_1), \mathrm{proj}_P(Q_2)$ are in σ though, thus w. l. o. g. we find $\mathrm{proj}_P(Q_1) = D$, $\mathrm{proj}_P(Q_2) = D'$, as required. □

Definition 1.5.8. In the situation above, define the simplicial graph Γ in the following way. Let $\Gamma = (V, E)$ such that $V = \mathcal{W}$, and $\{R^\Sigma, S^\Sigma\} \in E$ if R, S are adjacent.

Lemma 1.5.9. *Assume $\{R^\Sigma, S^\Sigma\} \in E$, such that R and S are opposite in the rank 2 residue $\mathcal{R}$. Then $\mathcal{R}^\Sigma$ is a rank 2 residue of σ, and R^Σ, S^Σ are opposite in $\mathcal{R}^\Sigma$.*

Furthermore exactly one of the following holds:

i) $d(C,R) > d(C,S)$ and $d(D,R) < d(D,S)$ or

ii) $d(C,R) < d(C,S)$ and $d(D,R) > d(D,S)$.

Proof. Let $R^\Sigma = \{R_1, R_2\}$ and $S^\Sigma = \{S_1, S_2\}$, by 1.5.7 we can assume $\mathrm{proj}_P(R_1) = \mathrm{proj}_P(S_1) = D$, $\mathrm{proj}_P(R_2) = \mathrm{proj}_P(S_2) = D'$.

Let $T = \mathrm{proj}_{\mathcal{R}}(P)$, then by 1.4.19 the panels P, T, R, and P, T, S are pairwise parallel. In particular $T^\Sigma \in \mathcal{W}$, since $\mathrm{proj}_{\mathcal{R}}(D)$ is on a minimal gallery from D to R_1, $\mathrm{proj}_{\mathcal{R}}(D')$ is on a minimal gallery from D' to R_2. So write $T^\Sigma = \{T_1, T_2\}$ with $\mathrm{proj}_P(T_1) = D, \mathrm{proj}_P(T_2) = D'$.

In particular, σ contains minimal galleries from T_i to R_i and S_i for $i = 1, 2$. Since σ is meagre, we obtain that either $T = R$ or $T = S$, and furthermore $\mathcal{R}^\Sigma$ is a rank 2 residue of σ.

If $\mathrm{proj}_{\mathcal{R}}(P) = R$, then we are in case i), if $\mathrm{proj}_{\mathcal{R}}(P) = S$, then we are in case ii). □

Lemma 1.5.10. *The graph Γ is connected. In particular, for $Q^\Sigma \in \mathcal{W}$, there exists a path*

$$Q_0^\Sigma = P^\Sigma, Q_1^\Sigma, \ldots, Q_k^\Sigma = Q^\Sigma$$

in Γ such that $d(C, Q_i^\Sigma) < d(C, Q_{i-1}^\Sigma)$ for $1 \leq i \leq k$.

Proof. Let $Q^\Sigma \in \mathcal{W}$. We proceed by induction on $d(Q^\Sigma, P^\Sigma)$. If $d(Q^\Sigma, P^\Sigma) = 0$, we obtain that $Q = P$, so assume $d(Q^\Sigma, P^\Sigma) > 0$, and let $Q = \{Q_1, Q_2\}$ with $\mathrm{proj}_P(Q_1) = D$, $\mathrm{proj}_P(Q_2) = D'$. Let $Q_1 \sim_i Q_2$.

Let $j \in J^-(Q_1, D)$, $J = \{i, j\}$ and $\mathcal{R} := \mathcal{R}_J(D)$. Then by 1.4.19 we find that Q, $R := \mathrm{proj}_{\mathcal{R}}(P)$, P are parallel. In particular, $R^\Sigma \in \mathcal{W}$, and $d(P,R) < d(C,R)$. Hence

R and Q are adjacent and by induction Γ is connected. Furthermore there exists a path $R_0^\Sigma = P^\Sigma, R_1^\Sigma, \dots, R_m^\Sigma = R$ in Γ, such that $d(C, R_i^\Sigma) < d(C, R_{i-1}^\Sigma)$ for $1 \le i \le m$. By construction of R we find $d(C,Q) < d(C,R)$, R and Q are adjacent, and the second assertion follows. □

Lemma 1.5.11. *Let $\{R^\Sigma, Q^\Sigma\}, \{Q^\Sigma, S^\Sigma\} \in E$, such that $d(P,R) > d(P,Q) < d(P,S)$ and $R \neq S$. Then there exists a panel $T^\Sigma \in \mathcal{W}$ such that $d(P,R) < d(P,T) > d(P,S)$, and a path from Q to T of the form $Q_0 = Q, Q_1, \dots, Q_m = T$ such that $d(P, Q_{i-1}) < d(P, Q_i)$ for $1 \le i \le m$.*

Furthermore, Q, R, S and T are panels in a rank 3 residue $\mathcal{R}$ and $\mathcal{R}^\Sigma$ is a residue of σ.

Proof. Write $Q^\Sigma = \{Q_1, Q_2\}$, $R^\Sigma = \{R_1, R_2\}$, $S^\Sigma = \{S_1, S_2\}$ as before, so $\mathrm{proj}_P(Y_1) = D$, $\mathrm{proj}_P(Y_2) = D'$ for $Y \in \{Q, R, S\}$.

Assume that Q is a panel of type i. Let $j \in J^-(R_1, Q_1)$, $k \in J^-(S_1, Q_1)$. Then i, j, k are pairwise distinct, as R, Q are opposite in an $\{i,j\}$-residue, Q, S are opposite in an $\{i,k\}$-residue, and $S \neq R$. Set $J = \{i,j,k\}$ and $\mathcal{R} := \mathcal{R}_J(Q_1)$.

Then $J \subset J^-(C, Q_1)$, therefore by 1.4.5 we obtain that $T_2 := \mathrm{proj}_{\mathcal{R}}(C)$ is opposite to Q_1 in $\mathcal{R}$, and $T_2 \in \sigma$. In particular $\mathcal{R}^\Sigma$ is a residue in σ. Since T_2 is opposite to Q_1, there exists a chamber T_1 opposite to Q_2 and adjacent to T_2. Since $T_1 \in \sigma(Q_1, T_2)$, we find $T_1 \in \sigma$. Hence $\{T_1, T_2\}$ is a panel in σ. Denote this panel by T^Σ.

Then T is opposite to Q in $\mathcal{R}$, and it remains to show that $T^\Sigma \in \mathcal{W}$. Since $j \in J^-(R_1, Q_1)$, $k \in J^-(S_1, Q_1)$, we find $Q = \mathrm{proj}_{\mathcal{R}}(P)$, therefore we obtain by 1.4.19 that T is parallel to P and that $d(P,R) < d(P,T) > d(P,Q)$.

In particular we find that $\mathcal{R}^\Sigma$ is a spherical residue in σ. Hence all the panels in $\mathcal{R}^\Sigma$ parallel to Q^Σ belong to $\mathcal{W}$, and we find a path in Γ as in the statement. □

Definition 1.5.12. Let $\mathcal{Q} \subset \mathcal{W}$ be the set of all panels Q^Σ, such that there does not exist an edge $\{Q^\Sigma, R^\Sigma\}$ with $d(P,R) > d(P,Q)$.

A path $R_0, \dots, R_k$ in Γ is called *ascending*, if $d(P, R_i) < d(P, R_{i+1})$ for all $0 \le i \le k-1$. Note that a path of length 0 is ascending by definition.

The following lemma is immediate from the definition.

Lemma 1.5.13. *For every $R^\Sigma \in \mathcal{W}$ there exists a $Q^\Sigma \in \mathcal{Q}$ such that R^Σ is on an ascending path from P^Σ to Q^Σ.*

Proposition 1.5.14. *Assume R^Σ, S^Σ, $T^\Sigma \in \mathcal{W}$ such that there exist ascending paths from R^Σ to S^Σ and to T^Σ. Then there exists a panel $U^\Sigma \in \mathcal{W}$ such that there exist ascending paths from S^Σ and from T^Σ to U^Σ.*

Proof. The number $\kappa := d(C,R)$ is bounded by $d(C, D')$ and it attains a minimum at $\kappa_0 = d(C,Q)$ for some $Q^\Sigma \in \mathcal{Q}$. Induction on κ. In case $\kappa = \kappa_0$, $R^\Sigma \in \mathcal{Q}$, and we find $R = S = T$, the statement is true.

So assume $\kappa > \kappa_0$. The statement is trivial, if there exists an ascending path from S^Σ to T^Σ. Assume that such a path does not exist and in particular let R, S, T be distinct. Thus let

i) $R^\Sigma = S_0^\Sigma, S_1^\Sigma, \ldots, S_\mu^\Sigma = S^\Sigma$,

ii) $R^\Sigma = T_0^\Sigma, T_1^\Sigma, \ldots, T_\nu^\Sigma = T^\Sigma$,

the ascending paths with $\mu, \nu \geq 1$. If $S_1 = T_1$, we obtain the statement by induction, using S_1 instead of R, thus let $S_1 \neq T_1$. Then by Lemma 1.5.11 we find $U_0^\Sigma \in \mathcal{W}$ such that there exist ascending galleries from S_1^Σ and T_1^Σ to U_0^Σ.

Hence $d(C, S_1) < d(C, R) > d(C, T_1)$ and we can use induction to obtain the existence of panels S'^Σ, T'^Σ and ascending galleries from U_0^Σ and S^Σ to S'^Σ and from U_0^Σ and T^Σ to T'^Σ. Since $d(C, U_0) < d(C, R)$, we can use induction, substituting R with U_0 to obtain the existence of a panel U^Σ and ascending paths from S'^Σ and T'^Σ to U.

Hence we find ascending paths from S^Σ and T^Σ to U^Σ and are done. □

Corollary 1.5.15. *There exists a unique panel $Q_0^\Sigma \in \mathcal{W}$ such that $d(C, Q_0^\Sigma)$ is minimal.*

Proof. The existence of such a panel is clear, assume Q_1^Σ, Q_2^Σ are two such panels. By 1.5.10 there exist ascending paths from P to Q_1^Σ, Q_2^Σ. Then by 1.5.14 there exist a panel $Q_3^\Sigma \in \mathcal{W}$ and ascending paths from Q_1^Σ and Q_2^Σ to Q_3^Σ. By choice of Q_1, Q_2 we obtain $Q_1 = Q_2 = Q_3$, which proves our claim. □

Corollary 1.5.16. *Let $R^\Sigma \in \mathcal{W}$. Then there exists an ascending path from P^Σ to Q_0^Σ containing R^Σ.*

Proof. There exist ascending paths from P^Σ to Q_0^Σ and R^Σ by Lemma 1.5.10, hence by Proposition 1.5.14 there exist $Q_1 \in \mathcal{W}$ and ascending paths from U_0^Σ and R^Σ to U_1^Σ. By construction of U_0 we obtain $U_0 = U_1$ and get an ascending path from P^Σ to R^Σ and one from R^Σ to U_0^Σ. □

Lemma 1.5.17. *Let $P^\Sigma = R_0^\Sigma, \ldots, R_k^\Sigma$ be an ascending path in Γ, $F_i \in R_i$, such that $\mathrm{proj}_P(F_i) = D$ for all $0 \leq i \leq k$. Then $F_i \in \sigma(D, F_k)$.*

Proof. The statement holds for $i = k$. Now assume the statement is true for F_{j+1}, $j < k$. Then R_j, R_{j+1} are adjacent, assume they are opposite in the rank 2 residue $\mathcal{R}$. Then $R_j = \mathrm{proj}_{\mathcal{R}}(P)$, hence $F_j \in \sigma(D, F_{j+1}) \subset \sigma(D, F_k)$, where the last inclusion holds by induction and [Müh94, Lemma 1.4.5]. □

Lemma 1.5.18. *Let $Q_0^\Sigma = \{E, E'\}$ with $\mathrm{proj}_P(E) = D$, $\mathrm{proj}_P(E') = D'$. Then $X = \sigma(E, D)$, $X' = \sigma(C, D')$.*

Proof. We show $X' = \sigma(C, D')$ first. Let $F \in \sigma(C, D')$, then $\mathrm{proj}_P(F) = D'$, since $D \notin \sigma(C, D')$. Hence $F \in X'$. If $F \in X'$, F is on a minimal gallery from C to D passing through D', thus $F \in \sigma(C, D')$.

Now for the equality $X = \sigma(D, E)$, let $D = D_0, \dots, D_k = E$ be a minimal gallery from D to E. By definition $\mathrm{proj}_P(E) = D$. Assume there exists an index i such that $\mathrm{proj}_P(D_i) = D'$, and assume i is maximal with this property. Then D_{i+1}, D_i are contained in a panel parallel to P and $d(D_{i+1}, D) = d(D_i, D') = d(D_i, D) - 1$, in contradiction to the minimality of the gallery. Thus $\sigma(D, E) \subset X$.

Assume now $F \in X$, so $\mathrm{proj}_P(F) = D$. As F is on a minimal gallery from C to D, say $F_0 = C, F_1, \dots, F_k = F, \dots, F_n = D$, there exists an index $1 \leq j \leq k-1$ such that $\mathrm{proj}_P(F_j) = D'$, $\mathrm{proj}_P(F_{j+1}) = D$, thus $R^\Sigma := \{F_j, F_{j+1}\} \in \mathcal{W}$, and F is on a minimal gallery from F_{j+1} to D.

Consequently, we find by Corollary 1.5.16 an ascending path in Γ of the form $P^\Sigma = P_0^\Sigma, P_1^\Sigma, \dots, R^\Sigma = P_{i_0}^\Sigma, \dots, P_l^\Sigma = Q_0^\Sigma$.

Assume $G_i \in P_i^\Sigma$ such that $\mathrm{proj}_P(G_i) = D$, then we also find a minimal gallery from D to E passing through all the G_i by Lemma 1.5.17. We find that $G_{i_0} = F_{j+1}$ for some i_0, thus $\sigma(F_{j+1}, D) \subseteq \sigma(D, E)$ follows. Since $F \in \sigma(F_{j+1}, D)$, we obtain $F \in \sigma(D, E)$. □

Proposition 1.5.19. *Let $\mathcal{C}$ be a totally gated chamber complex, and let $C, D \in \mathcal{C}$. Then the set $\sigma(C, D) \subset \mathcal{C}$ is convex.*

Proof. We proceed by induction on $d(C, D) = n$. In the case $n = 0, 1$ the statement is immediate, so assume $n \geq 2$. By Lemma 1.5.18 we know that $\sigma = X \,\dot\cup\, X' = \sigma(D, E) \,\dot\cup\, \sigma(C, D')$, by induction X, X' are convex.

We will show in the following that for all $F, F' \in \sigma$ we find $\sigma(F, F') \subset \sigma$. We show this by induction on $d_\sigma(F, F')$, where d_σ denotes the distance in σ. The cases $d_\sigma(F, F') = 0, 1$ are immediate, so assume $d_\sigma(F, F') > 1$.

Since X, X' are convex, the statement is also immediate if $F, F' \in X$, or if $F, F' \in X'$. So assume $F \in X$ and $F' \in X'$.

We show first that there exists a minimal gallery in σ from F to F'. Since σ is connected, we find a gallery from F to F' in σ of minimal length m (which might not be minimal in $\mathcal{C}$), say $F = F_0, F_1, \dots, F_m = F'$. If $R^\Sigma := \{F_0, F_1\} \in \mathcal{W}$, there is nothing to show, because $F_1, \dots, F_m$ is minimal and we find $\mathrm{proj}_R(F') = F_1$, since X' is convex, thus $d(F_0, F') > d(F_1, F')$.

So assume i_0 to be the first index such that $\{F_{i_0}, F_{i_0+1}\} \in \mathcal{W}$. Assume the gallery in σ is not minimal. By 1.5.3 there exists a panel $R_1^\Sigma := \{F_{i_1}, F_{i_1+1}\}$ which is parallel to a panel $R_2^\Sigma := \{F_{i_2}, F_{i_2+1}\}$. Since X and X' are convex, we can assume that $i_1 < i_0$, $i_2 > i_0$. By using induction we can thus assume $i_1 = 0$, $i_2 = m-1$. Further let $F_0 \sim_{j_1} F_1$, $F_1 \sim_{j_2} F_2$.

The induction hypothesis yields $\sigma(F_1, F_m) \subset \sigma$, hence

$$F_1 \neq \mathrm{proj}_{\mathcal{R}_{j_1}(F_1)}(F_m) \in \sigma(F_1, F_m).$$

Since σ is meagre, we find $\mathrm{proj}_{\mathcal{R}_{j_1}(F_1)}(F_m) = F_0$ and $d_\sigma(F_0, F_m) \leq m-2$, a contradiction. Hence the gallery $F_0, \dots, F_m$ is minimal in Δ.

We can now show that any minimal gallery from F to F' is in σ. To do this, we show that for every $j \in J^-(F', F)$ we find $G_j := \mathrm{proj}_{\mathcal{R}_j(F)}(F') \in \sigma(C, D)$. We can then use the identity

$$\sigma(F, F') = \{F\} \cup \bigcup_{j \in J^-(F', F)} \sigma(G_j, F'),$$

which can be found in [Müh94, Lemma 1.4.5]. Let $j_0 \in J^-(F_m, F_0) \setminus \{j_1\}$, and let $J = \{j_0, j_1\}$, $\mathcal{R} := \mathcal{R}_J(F_0)$. In the case that there exists no such j_0, the statement follows immediately. Let $G := \mathrm{proj}_{\mathcal{R}}(F_m)$, then G is on a minimal gallery from F_m to F_1, hence $G \in \sigma$, and by Lemma 1.4.5 G is opposite to F_0. If $G \neq F_m$, we obtain the statement by induction, therefore assume $G = F_m$.

Then either $\mathrm{proj}_{\mathcal{R}}(P)$ or $\mathrm{proj}_{\mathcal{R}}(Q_0)$ is different from the panel in $\mathcal{R}$ containing the set $\{F_{i_0}, F_{i_0+1}\}$, let T be this panel.

We show that there exists a unique panel T such that T is parallel to the panel containing $\{F_{i_0}, F_{i_0+1}\}$ in $\mathcal{R}$.

Consider $P' := \mathrm{proj}_{\mathcal{R}}(P)$ and $Q' := \mathrm{proj}_{\mathcal{R}}(Q_0)$. If $P' = Q'$, we find a minimal gallery from D and E to F passing through $\mathrm{proj}_{\mathcal{R}}(D) = \mathrm{proj}_{\mathcal{R}}(E)$. But $F \in \sigma(D, E)$, hence $d(D, F) + d(E, F) = d(D, E)$. This implies $F = \mathrm{proj}_{\mathcal{R}}(D)$, which is a contradiction to F not being contained in a panel in $\mathcal{W}$. Hence we obtain T to be either in P' or Q', in both cases $\mathcal{R}^\Sigma$ is a rank 2 residue and $\sigma(F, F')$ is contained in σ. $\square$

Corollary 1.5.20. *Assume $\tau : \Delta \to I$ is the type function of Δ. For $C, D \in \mathcal{C}$ the complex $\Sigma := \Sigma(C, D)$ is a meagre gated chamber complex with type function $\tau' = \tau|_\Sigma$. In particular the projection maps for Δ and Σ coincide on* $\mathrm{Cham}(\Sigma)$. *More precisely: If $A \in \Sigma$, $\mathrm{proj}_A^\Sigma : \mathrm{Cham}(\Sigma) \to \mathrm{Cham}(\mathrm{St}_\Sigma(A))$ is the projection map, then*

$$\mathrm{proj}_A^\Sigma = \mathrm{proj}_A |_{\mathrm{Cham}(\Sigma)}.$$

Proof. The complex Σ is a meagre chamber complex by 1.5.19. For any $A \in \Sigma$, we find $\mathrm{St}_\Sigma(A) \subset \mathrm{St}_\Delta(A)$, therefore $\mathrm{Cham}(\mathrm{St}_\Sigma(A)) \subset \mathrm{Cham}(\mathrm{St}_\Delta(A))$. Since $\mathrm{Cham}(\Sigma) = \sigma(C, D)$ is convex, we find $\mathrm{proj}_A^\Delta(E) \in \mathrm{Cham}(\Sigma)$ for all $E \in \mathrm{Cham}(\Sigma)$. Therefore residues in Σ are gated. $\square$

1.6 Irreducible non-spherical parallel residues

In this section we generalise one of the steps in [MPW15] to firm totally gated chamber complexes admitting an apartment system.

Lemma 1.6.1. *If Δ is a meagre gated chamber complex with a type function, parallelism is an equivalence relation on panels.*

Proof. All panels which only contain one chamber are parallel to each other. So let P, Q, R be three panels, each containing 2 chambers, such that P is parallel to Q, Q is parallel to R. Let $P = \{P_1, P_2\}$, $Q = \{Q_1, Q_2\}$, $R = \{R_1, R_2\}$ such that $\mathrm{proj}_Q(P_i) = Q_i$, $\mathrm{proj}_R(Q_i) = R_i$ for $i = 1, 2$.

First consider the case where P and Q are opposite in a rank 2-residue, say $\mathcal{R}$. Since P and Q are opposite, we can write $\mathcal{R} = \sigma(P_1, Q_1) \,\dot{\cup}\, \sigma(P_2, Q_2)$. Observe that since $Q_i = \mathrm{proj}_Q(R_i) = \mathrm{proj}_Q(\mathrm{proj}_{\mathcal{R}}(R_i))$ by 1.4.3, we find $\mathrm{proj}_{\mathcal{R}}(R_i) \in \sigma(P_i, Q_i)$. Hence $\mathrm{proj}_P(R_i) = \mathrm{proj}_P(\mathrm{proj}_{\mathcal{R}}(R_i)) = P_i$, hence P and Q are parallel.

Now let P, Q, R be arbitrary, only assume that $P \neq Q$ and P, Q are not opposite in a rank 2 residue. We prove the statement by induction on $d(P, Q) + d(Q, R)$. Note that $d(P, Q) + d(Q, R) = 0$ implies $P = Q = R$, the statement is immediate.

So assume $P = \mathcal{R}_i(P_1)$, and $j \in J^-(Q_1, P_1)$. Define $\mathcal{R} := \mathcal{R}_{\{i,j\}}(P_1)$, and let $S := \mathrm{proj}_{\mathcal{R}}(Q)$. Then we find $\{i, j\} \subset J^-(Q_1, P_2)$ and by 1.4.5 we find that $\mathrm{proj}_{\mathcal{R}}(Q_2)$ is opposite to P_1. Furthermore by 1.4.16 $\{i, j\} \subset J^-(Q_2, P_1)$ and we also find $\mathrm{proj}_{\mathcal{R}}(Q_1)$ to be opposite to P_2. Hence S is a panel and by 1.4.19 P, Q, S are pairwise parallel. Furthermore $d(Q, S) < d(P, Q)$, hence $d(S, Q) + d(Q, R) < d(P, Q) + d(Q, R)$ and by induction S and R are parallel. By our first consideration we find R and P to be parallel. Being parallel is transitive, and thus an equivalence relation. □

Lemma 1.6.2. *Assume Δ to be a meagre gated chamber complex. Let $C, C' \in \mathcal{C}$ with $d(C, C') = k$. Let $C = C_0, C_1, \dots, C_k = C'$ and $C = D_0, D_1, \dots, D_k = C'$ be two minimal galleries from C to C'. Then for every $0 \leq i \leq k-1$, the panel $\{C_i, C_{i+1}\}$ is parallel to a panel $\{D_j, D_{j+1}\}$ for a unique $0 \leq j \leq k-1$.*

Proof. We can assume w. l. o. g. that the two galleries are disjoint, i. e.

$$\{C_1, \dots, C_{k-1}\} \cap \{D_1, \dots, D_{k-1}\} = \emptyset.$$

We write $P_i := \{C_i, C_{i+1}\}$, $Q_i := \{D_i, D_{i+1}\}$.

Induction on $d(C, C') = k$. For $d(C, C') = 0, 1$ the statement is empty. If C, C' are contained in a rank 2 residue, either they are chambers in an n-gon and $d(C, C') = n$, in which case C and C' are opposite and the statement holds. Or, C, C' are chambers in an n-gon and $d(C, C') < n$ or are two chambers in a tree, in the last two cases the gallery from C to C' is unique, and we excluded this case.

So assume $k \geq 2$ and that C, C' are not contained in a rank 2 residue. Assume C' and C_{k-1} are j_1-adjacent, C' and D_{k-1} are j_2-adjacent. Then $j_1 \neq j_2$ since the galleries are disjoint. Let $J := \{j_1, j_2\}$, $\mathcal{R} := \mathcal{R}_J(C')$, and $C'' := \mathrm{proj}_{\mathcal{R}}(C)$. Since C, C' are not contained in a rank 2 residue, $C'' \neq C$. The residue $\mathcal{R}$ is finite, and therefore we can assume it is an n-gon. Thus $\mathcal{R} = \sigma(C'', C')$ is a circle of diameter n. Furthermore the minimal galleries from C'' to C' in $\mathcal{R}$ through C_{k-1} and D_{k-1} are disjoint, and contain every panel in $\mathcal{R}$.

Consider the gallery $(C_0, C_1, \dots, C_{k-1})$ and a minimal gallery from C to C_{k-1} passing through C'', which exists by definition of C''. Denote the latter gallery as $(C = E_0, E_1, \dots, E_j = C'', \dots, E_{k-1} = C_{k-1})$. Note that these two galleries may coincide. Define the panel $\{E_i, E_{i+1}\}$ to be P'_i. As $d(C, C_{k-1}) = k-1$, we can apply the induction hypothesis and find for each P_i, $0 \leq i \leq k-2$, a parallel panel $P'_{i'}$.

We find a similar construction for $(D_0, \dots, D_{k-1})$ and a gallery from C to D_{k-1} through C'', which we write as $(C = F_0, \dots F_j = C'', \dots, F_{k-1} = D_{k-1})$. Define further $Q'_i = \{F_i, F_{i+1}\}$. We can assume that for $1 \leq i \leq j$, $F_i = E_i$ holds.

Now if $i' \geq j$, $P'_{i'}$ is a panel in $\sigma(C', C'')$, which has an opposite panel Q in $\sigma(C', C'')$ by induction. If Q is of the form Q_i, there is nothing to show. So assume it is not, then $Q = Q'_l$ for some $l \geq j$. By induction there exists a parallel panel $Q_{l''}$, by Lemma 1.6.1 parallelism is transitive in meagre complexes, so P_i is parallel to $Q_{l''}$.

Assume $i' < j$, then $P'_{i'} = Q'_l$ for some $0 \leq l \leq j-1$, which is parallel to some $Q_{l'}$. Again using Lemma 1.6.1, our claim holds.

The last case we have to check is the panel P_{k-1}. By construction P_{k-1} is a panel in $\mathcal{R}$. This panel is parallel to some panel P in $\mathcal{R}$ by restriction to the rank 2 case, and P is contained in a minimal gallery from C'' to C' passing through D_{k-1}. Note that $P \neq \{D_{k-1}, D_k\}$, since this would imply $D_{k-1} = C_{k-1}$. Therefore P is either of the form $\{D_l, D_{l+1}\}$ or $\{F_l, F_{l+1}\}$, $j \leq l \leq k-1$. In the first case, there is nothing left to show. In the second case, we apply induction and 1.6.1 to obtain a parallel panel in the second gallery.

For the uniqueness, if P_i had two parallel panels Q_l and $Q_{l'}$ we obtain by 1.6.1 that Q_l and $Q_{l'}$ are parallel, which is not possible. □

For totally gated complexes, the above lemma is still valid dropping the assumption of being meagre.

Corollary 1.6.3. *Let Δ be a strongly connected totally gated chamber complex. Let $C, C' \in \mathcal{C}$ with $d(C, C') = k$. Let $C = C_0, C_1, \dots, C_k = C'$ and $C = D_0, D_1, \dots, D_k = C'$ be two minimal galleries from C to C'. Then for every $0 \leq i \leq k-1$, the panel containing $\{C_i, C_{i+1}\}$ is parallel to a panel containing $\{D_j, D_{j+1}\}$ for a unique $0 \leq j \leq k-1$.*

Proof. By using Proposition 1.5.19 we consider the convex meagre chamber complex $\sigma(C, C')$. This is a gated chamber complex, furthermore the projection in $\sigma(C, C')$ coincides with the projection in Δ.

By Lemma 1.6.2 we find in $\sigma(C, C')$ for $P_i = \{C_i, C_{i+1}\}$ a parallel panel $Q_j = \{D_j, D_{j+1}\}$. Let P be the panel containing P_i, Q be the panel containing $\{Q_j, Q_{j+1}\}$. Since $\sigma(C, C')$ is convex, $\{D_j, D_{j+1}\} \subseteq \mathrm{proj}_Q(P)$. As Δ is totally gated, we obtain $\mathrm{proj}_Q(P) = Q$ and by symmetry $\mathrm{proj}_P(Q) = P$. Therefore P and Q are parallel, as desired. □

Lemma 1.6.4. *Assume that Δ is totally gated. Let $\mathcal{R}$ and $\mathcal{S}$ be parallel residues, $j \in J^-(\mathcal{S}, \mathcal{R})$, and let $\mathcal{R}_j(C)$ and $\mathcal{R}_j(D)$ be parallel panels for $C, D \in \mathcal{R}$. Let $J = \operatorname{supp}(D, C)$. Then $j \in J_G^\perp$ for every chamber $G \in \mathcal{R}_J(C)$.*

Proof. Induction on $d(C, D)$. The statement is empty for $d(C, D) = 0$ and follows from 1.4.15 for $d(C, D) = 1$. So let $d(C, D) \geq 2$.

Let $k \in J^-(D, C)$, $Q := \mathcal{R}_j(D)$, $\mathcal{R}' := \mathcal{R}_{\{j,k\}}(C)$. By Lemma 1.4.19 the residue $P := \operatorname{proj}_{\mathcal{R}'}(Q)$ is parallel to $\mathcal{R}_j(C)$ and Q, and since $k \in J^-(D, C)$, $P \neq \mathcal{R}_j(C)$. As $\mathcal{R}$ is convex, $E := \operatorname{proj}_{\mathcal{R}'}(D) \in \mathcal{R}$. Since the type of $\mathcal{R}$ does not contain j, E and C are k-adjacent.

Therefore $d(P, \mathcal{R}_j(C)) = 1$ and from Lemma 1.4.15 we obtain that k and j commute at C. Since $E \in \mathcal{R}_{\{j,k\}}(C)$, k and j also commute at E. Furthermore $d(D, E) = d(D, C) - 1$, applying induction to D and E yields that j and $\operatorname{supp}(D, E)$ commute at E, thus j commutes with $\operatorname{supp}(D, E) \cup \{k\}$ at E. By Remark 1.4.8 we have $\operatorname{supp}(D, E) \cup \{k\} = \operatorname{supp}(C, D) = J$. By Theorem 1.3.10 we obtain $j \in J_G^\perp$ for all $G \in \mathcal{R}_J(E) = \mathcal{R}_J(C)$. □

Lemma 1.6.5. *Let $J = \{j_1, \ldots, j_k\} \subset I$. Then any gallery of type $(j_1, \ldots, j_k)$ is minimal.*

Proof. Induction on $k = |J|$. For $k = 0, 1$ there is nothing to show, so let $k \geq 2$ and let $J' = \{j_1, \ldots, j_{k-1}\}$. Let $C, E \in \mathcal{C}$ such that there exists a gallery of type J, and let $D \in \mathcal{C}$ be the unique chamber which is j_k-adjacent to E, such that there exists a gallery of type $(j_1, \ldots, j_{k-1})$ from C to D, which is minimal by induction. Furthermore C, D are contained in $\mathcal{R} := \mathcal{R}_{J'}(C)$.

Then $\mathcal{R}$ is gated, $E \notin \mathcal{R}$, and $\operatorname{proj}_{\mathcal{R}}(E) = D$. Therefore $(C, \ldots, D, E)$ is a minimal gallery from C to E and has type J. □

Proposition 1.6.6. *Let Δ be totally gated, and let $\mathcal{R}$ be an infinite firm J-residue of Δ. Assume*

$$\mathcal{R} = \bigcup_{i=1}^{k} \mathcal{R}_i,$$

where $\mathcal{R}_i \subsetneq \mathcal{R}$ is a proper sub-residue for $1 \leq i \leq k$. Then J is reducible at $\mathcal{R}$.

Proof. Induction on $|J|$. In case $|J| = 1$, if $\mathcal{R}$ is infinite, it contains infinitely many chambers, which are exactly the $\emptyset$-residues. Thus the statement is empty. In case $|J| = 2$, $\mathcal{R}$ contains infinitely many panels, the statement is empty as well. Therefore assume $|J| \geq 3$ and assume the statement to be true for all residues of type J' with $|J'| < |J|$.

At least one of $\mathcal{R}_i$ contains an infinite number of chambers, suppose this to be $\mathcal{R}_1$. Let $\mathcal{R}' \subset \mathcal{R}_1$ be a sub-residue of $\mathcal{R}_1$ such that the type J' of $\mathcal{R}'$ is a component of J_1 and $\mathcal{R}'$ is infinite.

Let $j \in J$ be such that $j \notin J_1$, and assume that $J' \cup \{j\}$ is irreducible at $\mathcal{R}'$. For every $C \in \mathcal{R}'$ we find some $2 \leq i \leq k$ and $C' \in \mathcal{R}_i$, such that $C' \sim_j C$, since $\mathcal{R}$ is firm. Define for $2 \leq i \leq k$ the set $\mathcal{R}'_i := \mathrm{proj}_{\mathcal{R}'}(\mathcal{R}_i)$, $\mathcal{R}'_i$ is a residue since Δ is totally gated.

Therefore $\mathcal{R}' = \bigcup_{i=2}^{k} \mathcal{R}'_i$, and by induction we obtain that the cover is not proper, hence we can assume $\mathrm{proj}_{\mathcal{R}'}(\mathcal{R}_2) = \mathcal{R}'$. We can furthermore assume that the cover is minimal, thus there exists a chamber $C_0 \in \mathcal{R}'_2 \setminus \bigcup_{i=3}^{k} \mathcal{R}'_i$, which is j-adjacent to C_1 in $\mathcal{R}_2$. Let $\mathcal{R}'' = \mathrm{proj}_{\mathcal{R}_2}(\mathcal{R}')$, then by 1.4.10 $\mathcal{R}'$ and $\mathcal{R}''$ are parallel and $C_1 \in \mathcal{R}''$. Hence $d(\mathcal{R}', \mathcal{R}'') = 1$ and by Lemma 1.4.15 J' commutes with j, in contradiction to the choice of j. We obtain that J' commutes with every $j \notin J_1$, and since J' is a component of J_1, it is a proper component of J, as required. □

Definition 1.6.7. Let Δ be a gated chamber complex of type I. We call a thin convex subcomplex Σ of Δ of type I an *apartment of* Δ. We say that Δ *satisfies condition (AS)*, if any two chambers in $\mathrm{Cham}(\Delta)$ are contained in a common apartment.

Lemma 1.6.8. *If Δ is a totally gated chamber complex, then apartments are totally gated.*

Proof. Let Σ be an apartment, and let $\mathcal{R}^{\Sigma}$, $\mathcal{S}^{\Sigma}$ be residues in Σ. Since Σ is convex, we find residues $\mathcal{R}$, $\mathcal{S}$ of Δ such that $\mathcal{R}^{\Sigma} = \mathcal{R} \cap \Sigma$, $\mathcal{S} \cap \Sigma$. As Δ is totally gated, we obtain that $\mathcal{R}' := \mathrm{proj}_{\mathcal{S}}(\mathcal{R})$ is a residue of Δ.

Since Σ is convex, we find $\mathrm{proj}_{\mathcal{S}}(\mathcal{R}) \cap \Sigma = \mathrm{proj}_{\mathcal{S}^{\Sigma}}(\mathcal{R}^{\Sigma})$, hence $\mathrm{proj}_{\mathcal{S}^{\Sigma}}(\mathcal{R}^{\Sigma})$ is a residue as well. □

Lemma 1.6.9. *Assume Δ is a firm totally gated chamber complex satisfying condition (AS), and let $\mathcal{R}$ and $\mathcal{S}$ be parallel residues. Assume $\mathcal{R}$ is not spherical and of type J, with J irreducible. Let Δ be of type $J \mathbin{\dot\cup} \{j\}$. Then either $\mathcal{R} = \mathcal{S}$, or $d(\mathcal{R}, \mathcal{S}) = 1$ and j commutes with J.*

Proof. Assume $\mathcal{R} \neq \mathcal{S}$. For any chamber $C \in \mathcal{R}$, let $C' = \mathrm{proj}_{\mathcal{S}}(C)$. Furthermore let $C_j = \mathrm{proj}_{\mathcal{R}_j(C)}(C')$. It is immediate that $\{j\} = J^-(C', C)$, hence $C_j \neq C$. First consider the case where Δ is thin.

Consider set $X = \{\mathcal{R}_j(C) \mid C \in \mathcal{R}\}$ and define a relation on X by $\mathcal{R}_1 \approx \mathcal{R}_2$ if and only if $\mathcal{R}_1$ and $\mathcal{R}_2$ are parallel. Since Δ is thin, $\approx$ is an equivalence relation by Lemma 1.6.1.

We will show that the number n of equivalence classes of $\approx$ is exactly $k := d(\mathcal{R}, \mathcal{S})$. So let γ be a minimal gallery from C to C' and let $D \in \mathcal{R}$. By Lemma 1.6.2 $\mathcal{R}_j(D)$ is parallel to a panel on γ, hence $n \leq k$. But all panels on γ are pairwise not parallel, hence $n \geq k$, which proves the claim.

So let $X_1, \ldots, X_k$ denote the equivalence classes of $\approx$. Furthermore let for $1 \leq i \leq k$ be $Y_i = \{C \in \mathcal{R} \mid \mathcal{R}_j(C) \in X_i\}$. Since $\mathcal{R}$ contains an infinite number of chambers, at least one of the sets Y_i must be infinite, say Y_1. Furthermore, the sets Y_i partition the set $\mathcal{R}$.

If $C_i \in \mathcal{R}_i$, by using Lemma 1.6.4 we obtain that Y_i is contained in a residue of the form $\mathcal{R}_i := \mathcal{R}_{J_i}(C_i)$, and J_i commutes with j at every chamber in $\mathcal{R}_{J_i}(C_i)$. In particular, all panels $\mathcal{R}_j(D)$ with $D \in \mathcal{R}_{J_i}(C_i)$ are parallel, hence we can conclude $Y_i = \mathcal{R}_i$ and $\mathcal{R} = \bigcup_{i=1}^{k} \mathcal{R}_k$. Now 1.6.6 yields that this partition cannot be proper, since J is irreducible. Hence find $J_1 = J$ and $k = 1$. Then J and j commute and $d(\mathcal{R}, \mathcal{S}) = 1$.

Now drop the assumption that Δ is thin. Then we find a gallery of type J in $\mathcal{R}$, say $C^0, \ldots, C^k$. Hence $\operatorname{supp}(C^0, C^{k'}) = J \cup \{J\}$, thus by condition (AS) and Lemma 1.6.8 there exists a thin convex totally gated chamber subcomplex Σ containing $\sigma(C^0, C^{k'})$, which is furthermore of type $J \mathbin{\dot\cup} \{j\}$.

We then find $\mathcal{R}^\Sigma$ and $\mathcal{S}^\Sigma$ to be parallel non-spherical J-residues of Σ, thus for $\mathcal{R}^\Sigma$, $\mathcal{S}^\Sigma$, our result holds, in particular $d(C, \operatorname{proj}_{\mathcal{S}^\Sigma}(C)) = 1$ for all $C \in \mathcal{R}^\Sigma$. This implies $d(\mathcal{R}, \mathcal{S}) = 1$, since the distance between parallel residues is well-defined, and the statement then follows from Lemma 1.4.15. ☐

Theorem 1.6.10. *Let Δ be a firm totally gated chamber complex with a type function, which satisfies condition (AS). Assume $\mathcal{R}$ and $\mathcal{S}$ are parallel non-spherical residues, where $\mathcal{R}$ is of type J, J irreducible. Let $C_0 \in \mathcal{R}$, $\operatorname{proj}_{\mathcal{S}}(C_0) = C_k \in \mathbb{S}$, and assume $(C_0, C_1, \ldots, C_k)$ is a minimal gallery from C_0 to C_k. Then for $0 \leq i \leq k$ the residues $\mathcal{R}_J(C_i)$ are all parallel, and in particular $\mathcal{R}_J(C_k) = \mathcal{S}$.*

Proof. We proceed by induction on $k := d(\mathcal{R}, \mathcal{S})$. In case $k = 0$, $\mathcal{R} = \mathcal{S}$ and there is nothing to show. In case $k = 1$, the statement follows from Lemma 1.4.15.

So let $k \geq 2$, $j \in J^-(\mathcal{S}, \mathcal{R})$ and $C \in \mathcal{R}$. Define further $J' := J \cup \{j\}$, $\mathcal{R}' = \mathcal{R}_{J'}(C)$ and $\mathcal{X} := \operatorname{proj}_{\mathcal{R}'}(\mathcal{S})$. By Lemma 1.4.19 we find $\mathcal{R}, \mathcal{X}, \mathcal{S}$ to be pairwise parallel.

The residues $\mathcal{R}$ and $\mathcal{X}$ are contained in $\mathcal{R}'$, which is of type $J \mathbin{\dot\cup} \{j\}$, hence we can apply Lemma 1.6.9 and obtain that $d(\mathcal{R}, \mathcal{X}) = 1$ and the type of $\mathcal{X}$ is J. Since $j \in J^-(\mathcal{S}, \mathcal{R})$, we also find $d(\mathcal{X}, \mathcal{S}) = k - 1$, hence induction can be applied to $\mathcal{X}$ and $\mathcal{S}$. We already obtain from induction at this point that $\mathcal{S} = \mathcal{R}_J(C_k)$, since $\mathcal{X}$ is also of type J. So let $C_0, \ldots, C_k$ be a minimal gallery, where $C_0 \sim_j C_1$ and $C_k = \operatorname{proj}_{\mathcal{S}}(C_0)$, then the residues $\mathcal{R}_i := \mathcal{R}_J(C_i)$ are all parallel for $1 \leq i \leq k$.

By interchanging $\mathcal{R}$ and $\mathcal{S}$ in the argument above, we get that the $\mathcal{R}_i$ are parallel for $0 \leq i \leq k - 1$. By assumption $\mathcal{R} = \mathcal{R}_0$ and $\mathcal{S} = \mathcal{R}_k$ are also parallel, therefore the residues $\mathcal{R}_i$ are pairwise parallel for $0 \leq i \leq k$, as required. ☐

2 Simplicial arrangements and Weyl groupoids

2.1 Introduction

Sets of hyperplanes, which decompose a given cone into simplices, appear naturally when considering the geometric representation of irreducible Coxeter groups [Bou02]. The special cases of Weyl groups occur in the study of finite dimensional Lie Algebras, and the reflection system associated to a Weyl group is an invariant for the semisimple Lie Algebras.

Weyl groups are always equipped with additional combinatorial data, i. e. a root system satisfying the crystallographic property. A root system R in a real vector space V can be abstractly defined as a set $R \subset V^*$ with the properties:

(RS1) $0 \notin R$,

(RS2) $\alpha \in R \implies R \cap \langle \alpha \rangle = \{\pm\alpha\}$,

(RS3) $W.R = R$,

(RS4) $\frac{2(\alpha,\beta)}{(\beta\beta)} \in \mathbb{Z}$ where $(\cdot,\cdot)$ is the standard scalar product.

Here W is the group generated by all reflections along $\ker\alpha$, $\alpha \in R$.

Without the restriction (RS2) one obtains non-reduced root systems, which also have been classified along with the reduced root systems [Bou02].

When omitting the crystallographic property (RS4), one obtains a natural generalisation to Coxeter groups. If in addition the assumption (RS3) is dropped, the remaining structure can be seen as a Coxeter groupoid. As a category, this can be described with the chambers as objects and the linear maps between two chambers as morphisms.

For these groupoids, a different version of the crystallographic property can be formulated, which is equivalent to (RS4) in the case of a group. Groupoids with this property are called Weyl groupoids and appear in the classification of Nichols algebras.

For the classification of the finite dimensional Nichols algebras of diagonal type, it was necessary to decide whether a Cartan graph admits a real root system. Cartan graphs (formerly called *Cartan schemes* by Cuntz and Heckenberger) are a more general notion than generalised Cartan matrices and induce naturally a Weyl groupoid.

The characterisation of arbitrary finite root systems associated to Weyl groupoids has been treated in a series of papers [CH09a], [CH12], [CH11], [CH15] by Cuntz and Heckenberger, meanwhile we have a complete classification of finite Weyl groupoids.

The intention of this chapter is therefore to approach the case of arbitrary Weyl groupoids with root systems. To do this, we consider simplicial arrangements in a more general case than the finite ones.

The most natural step of generalising the finite simplicial arrangements is to consider a set of affine hyperplanes, which decomposes the affine space simplicially. However, such an arrangement can also be viewed - by increasing the dimension by 1 - as a set of linear hyperplanes intersecting an affine (non linear) hyperplane A. The induced simplicial structure on A coincides with the simplicial structure on the half space containing A, which is also a convex open cone. Since the finite simplicial arrangements also yield a simplicial structure on a convex open cone, namely the space itself, it is natural to consider arrangements in arbitrary convex open cones. We will in general denote this cone by T and call it the *Tits cone* of the arrangement.

For a Coxeter group W the Tits cone is usually (cp. [Hum90]) constructed as the union of all translates of the fundamental chamber, i. e.

$$T = \bigcup_{w \in W} w(D),$$

where D is closed. In general, this cone is not an open set, but contains some faces in the boundary of an open cone. For Coxeter groups the concept was first introduced by Tits in [Tit61], a reference can be found in [Tit13]. The object we define as the Tits cone can be thought of as the interior of the classical Tits cone.

Furthermore we will see that any connected simply connected Weyl groupoid with a real root system gives rise to a canonical set of hyperplanes, but not to a convex open cone. However, the cone T can be reconstructed from the combinatorial data, which yields a correspondence between equivalence classes of connected simply connected Cartan graphs permitting a root system and crystallographic arrangements with reduced root systems. This statement can be found in Corollary 2.6.24.

Since this chapter also intends to set the foundation for the theory of Weyl groupoids, we will pick up some concepts which are well known, but whose origin is hard to trace in the literature. Some of the facts about hyperplane arrangements and simplicial arrangements are standard, but for the sake of completeness we give short proofs, where possible.

We make some comments to the origins of our objects of interest.

1. Simplicial arrangements were first introduced and studied by Melchior [Mel41] and subsequently by Grünbaum [Grü71]. Shortly afterwards, simplicial arrangements

attracted attention in the seminal work of Deligne [Del72]: they are a natural context to study the $K(\pi, 1)$ property of complements of reflection arrangements, since the set of reflection hyperplanes of a finite Coxeter group is a simplicial arrangement. They further appeared as examples or counterexamples to conjectures on arrangements.

2. We do not know where arrangements of hyperplanes on convex cones were considered for the first time. The concept seems most natural and they are mentioned in [Par14] without further reference. Of course, our definition of a simplicial arrangement on an open convex cone is inspired by the Tits cone of a Coxeter system.

3. The fact that arrangements of hyperplanes provide interesting examples of gated sets in metric spaces appears in [BLVS+99] for the first time. At least in the simplicial case it was observed much earlier [Tit74].

4. The observation that there is a link between root systems and simplicial arrangements is quite natural. We already mentioned that it was our starting point to investigate the Tits cone of a Weyl groupoid. But it also appears in Dyer's work on rootoids [Dye11a], [Dye11b]. It is conceivable that the observation was made much earlier by other people and is hidden somewhere in the literature.

The structure of this chapter is as follows.

In Section 2 we fix notation and develop the notion of a simplicial cone in a real vector space V. This can be seen either as the cone on a simplex, whose vertices form a basis of V, or as the intersection of half-spaces given by a basis of V^*.

In the first part of Section 3 we define hyperplane arrangements and simplicial arrangements on a convex cone T and introduce root systems for these arrangements. Our main objects of interest are simplicial arrangements, which can be thought of as decompositions of T into simplicial cones, paired with a set of roots for the hyperplanes, which provide additional combinatorial data.

In the second part of Section 3 we introduce the poset $\mathcal{S}$ associated to a simplicial hyperplane arrangement. We show later in Section 4, that $\mathcal{S}$ is actually a gated chamber complex in the sense of Chapter 1. This allows us to define the property of being k-spherical for such hyperplane arrangements. This property gives also rise to a type function of the complex, thus providing an indexing for the root bases introduced in the first part of Section 3.

Section 5 introduces the crystallographic property, which is an analogue to the respective property for root systems of Coxeter groups. The spherical crystallographic simplicial arrangements correspond exactly to the coscorf (**co**nnected **s**imply **co**nnected with **r**eal **f**inite root system) Cartan graphs, so we recall the definitions of Cartan graphs and Weyl groupoids, which are closely related to each other. We also associate in this section to a crystallographic arrangement a Cartan graph in a more or less straightforward way. Furthermore we prove that in rank 3 or higher all 2-spherical crystallographic arrangements satisfy the additive property, which is stronger than the crystallographic property. The additive property arises naturally in the context of Nichols algebras, which motivated Cartan graphs.

In Section 6 we complete the correspondence between crystallographic arrangements and Cartan graphs with real roots. For this purpose we show that the canonical hyperplane arrangement, which can easily be described for any Cartan graph, is indeed a simplicial hyperplane arrangement on a convex open cone.

In the last two sections we describe two basic constructions to obtain lower rank arrangements from a given one. Both are well known in the case of finite arrangements and can be easily generalised.

In Section 7 we describe subarrangements, which correspond to stars in the simplicial complex. We show that the crystallographic property is inherited by subarrangements, and we also give a criterion for when the arrangement is k-spherical.

In Section 8 we introduce restrictions on hyperplanes and focus on those hyperplanes belonging to the arrangement. In this case, the arising objects are again simplicial hyperplane arrangements. The crystallographic property is inherited as well.

Note. This chapter is a joint work with Michael Cuntz and Bernhard Mühlherr.

2.2 Half-spaces and simplicial cones

In this section we will introduce simplicial cones and describe the simplicial structure given by a simplicial cone.

Definition and Remark 2.2.1. Let (V, d) be a connected metric space. For an arbitrary subset $X \subset V$ the *convex hull of* X is the smallest convex set $Y \subset V$, such that $X \subset Y$. For another approach, remember that the segment between $x, y \in V$ is

$$\sigma(x, y) := \{z \in V \mid d(x, z) + d(z, y) = d(x, y)\}.$$

This can be used to set

$$H(X) := \bigcup_{x,y \in X} \sigma(x, y).$$

We can then recursively define

$$\begin{aligned} H^{(0)}(X) &:= X, \\ H^{(n)}(X) &:= H(H^{(n-1)}(X)) \text{ for } 1 \leq n \in \mathbb{N}. \end{aligned}$$

Then the convex hull of X is the set $\bigcup_{n \in \mathbb{N}} H^{(n)}(X)$.

For $V = \mathbb{R}^r$ and a linear independent set $X \subset \mathbb{R}^r$, the convex hull of X can be more easily described as

$$\left\{ \sum_{x \in X} \lambda_x x \mid 0 \leq \lambda_x \leq 1 \text{ for all } x \in X, \sum_{x \in X} \lambda_x = 1 \right\}.$$

In this setting, we will refer to the set

$$\left\{ \sum_{x \in X} \lambda_x x \mid 0 < \lambda_x < 1 \text{ for all } x \in X, \sum_{x \in X} \lambda_x = 1 \right\}$$

as the *open convex hull of* X.

Throughout this chapter, unless otherwise mentioned, all topological properties will refer to the standard topology of $\mathbb{R}^r$. Furthermore, when referring to the metric of $\mathbb{R}^r$, we will use the more common notion of the *interval* between two points x and y:

$$\begin{aligned} [x, y] &:= \sigma(x, y) = \{\lambda x + (1 - \lambda) y \in \mathbb{R}^r \mid \lambda \in [0, 1]\}, \\ (x, y) &:= \sigma(x, y) \setminus \{x, y\} = \{\lambda x + (1 - \lambda) y \in \mathbb{R}^r \mid \lambda \in (0, 1)\}. \end{aligned}$$

Note that the intervals $(x, y]$, $[x, y)$ can be defined analogously.

Definition 2.2.2. Let $V = \mathbb{R}^r$. A subset $K \subset V$ is called a *cone*, if $\lambda v \in K$ for all $v \in K$, $0 < \lambda \in \mathbb{R}$.

For $k \leq r - 1$, the set $S \subset V$ is an *(open) k-simplex in V*, if S is the (open) convex hull of $k + 1$ linearly independent elements in V, called the *vertices of S*, or $V(S)$.

We say $K \subset V$ is *solid in V*, if its interior is non-empty. Equivalently, K is solid in V if there exists a point $x \in T$ and an $\varepsilon > 0$ such that an ε-neighbourhood of x in V is contained in K. In particular, every non-empty open cone is solid.

Let $X \subset V$, then the *cone on* X is defined as $\mathbb{R}_{>0}X = \{\lambda y \in V \mid y \in X, 0 < \lambda \in \mathbb{R}\}$.

Let S be an open $(r-1)$-simplex in V. We call a cone K *open simplicial* if $K = \mathbb{R}_{>0}S$, and *closed simplicial* if $K = \overline{\mathbb{R}_{>0}S}$. Here $^{-}$ denotes the closure in V. We find that $\overline{\mathbb{R}_{>0}S} = \mathbb{R}_{>0}\overline{S} \cup \{0\}$, so a closed simplicial cone can be thought of as a cone on a closed simplex with the origin added.

Furthermore we introduce the following notation. Let $\alpha \in V^*$ be a linear form, then

$$\begin{aligned} \alpha^{\perp} &= \ker \alpha, \\ \alpha^{+} &= \alpha^{-1}(\mathbb{R}_{>0}), \\ \alpha^{-} &= \alpha^{-1}(\mathbb{R}_{<0}). \end{aligned}$$

With this notation we find

$$\begin{aligned} \overline{\alpha^{+}} &= \alpha^{-1}(\mathbb{R}_{\geq 0}) = \alpha^{\perp} \cup \alpha^{+}, \\ \overline{\alpha^{-}} &= \alpha^{-1}(\mathbb{R}_{\leq 0}) = \alpha^{\perp} \cup \alpha^{-}. \end{aligned}$$

Example 2.2.3. The space $V = \mathbb{R}^r$ itself is a convex solid cone. The main application for our notion of cones is the intersection of half-spaces. If $\alpha_1, \dots, \alpha_k \in V^*$ is a set of non-zero linear forms then α_i^+ is a convex open cone. The intersection of cones is again a cone, so

$$\bigcap_{1 \le i \le k} \alpha_i^+$$

is an open convex cone which is either solid or empty. Also note that it is an open set. On the other hand, by taking the half spaces with their boundary, the set

$$\bigcap_{1 \le i \le k} (\alpha_i^+ \cup \alpha_i^\perp)$$

is a closed convex cone, which may be $\{0\}$.

Lemma 2.2.4. *An open or closed simplicial cone is convex and solid.*

Proof. Let K be an open simplicial cone and let S be the open simplex such that $K = \{\lambda s \mid s \in S, 0 < \lambda \in \mathbb{R}\}$. Let $v, w \in K$, then there exist $s, t \in S$ and $\lambda, \mu \in \mathbb{R}_{>0}$ such that $v = \lambda s, w = \mu t$. Let $\alpha, \beta \in \mathbb{R}_{>0}$, and consider the point $\alpha v + \beta w = (\alpha\lambda)s + (\beta\mu)w$. Since S is convex, it contains $\rho s + (1-\rho)t$ for all $\rho \in [0,1]$, so in particular it contains $\frac{\alpha\lambda}{\alpha\lambda+\beta\mu}s + \frac{\beta\mu}{\alpha\lambda+\beta\mu}t$. Therefore we find that

$$\alpha v + \beta w = (\alpha\lambda + \beta\mu)(\frac{\alpha\lambda}{\alpha\lambda + \beta\mu}s + \frac{\beta\mu}{\alpha\lambda + \beta\mu}t) \in T$$

and K is convex. The proof for a closed simplicial cone is basically the same.

If K is an open simplicial cone, it is an open set, and therefore solid. If K is a closed simplicial cone, it contains the closure of an open simplicial cone, and is solid as well. $\square$

Lemma 2.2.5. *If $K = \mathbb{R}_{>0}S$ is an open simplicial cone for an open simplex S with linear independent vertex set $V(S)$, then there exist linear forms $\alpha_1, \dots, \alpha_r \in V^*$ such that $K = \bigcap_{i=1}^r \alpha_i^+$. Up to permutation und positive scalar multiples, $\{\alpha_i \mid 1 \le i \le r\}$ is the dual basis to $V(S)$.*

Proof. Let $K = \mathbb{R}_{>0}S$ for an open simplex S with vertex set $V(S) = \{v_1, \dots, v_r\}$. Let F_i denote the face of $\overline{S}$ containing $V_{\hat{i}} := \{v_j \mid 1 \le i \ne j \le r\}$, so F_i is the convex hull of V_i.

Since $V_{\hat{i}}$ consists of $r-1$ linear independent vectors, it spans a hyperplane in V, which we denote H_i. For a fixed i the set $\mathrm{Ann}(H_i) = \{\alpha \in V^* \mid \alpha(h) = 0 \text{ for all } h \in H_i\}$ is a one dimensional subspace of V^*. Let $\alpha \in \mathrm{Ann}(H_i)$, then by definition $\alpha(v_j) = 0$ for $1 \le i \ne j \le r$. If $\alpha(v_i) = 0$, we can conclude $\alpha = 0$. So $0 \ne \alpha \in \mathrm{Ann}(H_i)$ exists and satisfies $\alpha(v_i) \ne 0$.

So choose $\alpha_i \in \mathrm{Ann}(H_i)$ such that $\alpha_i(v_i) = 1$, α_i is unique as $\dim_{\mathbb{R}}(\mathrm{Ann}(H_i)) = 1$. By construction, the set $\{\alpha_i \mid 1 \leq i \leq r\}$ is the dual basis to $V(S)$.

As S is the open convex hull of its vertices, we obtain

$$S := \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid \sum_{i=1}^{r} \lambda_i = 1, 0 < \lambda_i < 1 \text{ for all } 1 \leq i \leq r \right\}.$$

Thus we find $S \subset \alpha_i^+$ for all $1 \leq i \leq r$. By definition $K = \mathbb{R}_{>0}S$, so $K \subset \alpha_i^+$ and therefore

$$K \subset \bigcap_{i=1}^{r} \alpha_i^+.$$

So let $x \in \bigcap_{i=1}^{r} \alpha_i^+$. By the definition of α_i this implies $x = \sum_{i=1}^{r} \lambda_i v_i$ with $\lambda_i > 0$ for all i. Let $m := \sum_{i=1}^{r} \lambda_i$, then $m > 0$ and $\frac{1}{m} \cdot x \in S$, which proves the equality $K = \bigcap_{i=1}^{r} \alpha_i^+$. □

Remark 2.2.6. The common notation for cones introduces *properness* of a cone T as the property of being closed, convex, solid, and pointed, the latter meaning that $v, -v \in T \implies v = 0$. Thus all closed simplicial cones are proper. In our context being proper is not of interest, the cones we are dealing with are either convex and open or already simplicial.

Proposition 2.2.7. *Let K be an open simplicial cone, such that $\overline{K} = \mathbb{R}_{>0}S \cup \{0\}$ for a closed $r-1$ simplex S. Let B_K be the dual basis to $V(S)$. Define a poset $\mathcal{S}_K$ as $\{\overline{K} \cap \bigcap_{\alpha \in B} \alpha^\perp \mid B \subseteq B_K\}$ with the inclusion as the partial order. Then the faces of S are in one to one correspondence with elements in $\mathcal{S}_K$, more precisely the map $\psi_K : \underline{S} \to \underline{\overline{K}}$, $F \mapsto \mathbb{R}_{>0}F \cup \{0\}$ is an isomorphism of simplicial complexes.*

Proof. Let $V(S) = \{v_1, \ldots, v_r\}$. All simplices contained in S can be identified with the convex hulls of subsets of $V(S)$. As in the proof of Lemma 2.2.5 let F_i be the maximal face of S spanned by $V_{\hat{i}} = \{v_j \mid 1 \leq i \neq j \leq r\}$. Furthermore let H_i be the linear span of F_i, from Lemma 2.2.5 we obtain $H_i = \alpha_i^\perp$ for a unique $\alpha_i \in B$.

Let $J \subseteq \{1, \ldots, r\}$, and F_J be the face of S spanned by $V_J := \{v_i \mid i \in J\}$. We will show that $\mathbb{R}_{>0}F_J \cup \{0\} = \overline{K} \cap \bigcap_{i \notin J} H_i$.

So let $x \in \mathbb{R}_{>0}F_J \cup \{0\}$, so $x = \sum_{j \in J} \lambda_j v_j$, where $\lambda_j \geq 0$. As $F_J \subset S$, we find $x \in \overline{K}$. The α_i are defined by $\alpha_i(v_j) = \delta_{ij}$, hence $\alpha_i(x) = 0$ for $i \notin J$, therefore $x \in \bigcap_{i \notin J} H_i$, which shows one inclusion.

Let on the other hand $x \in \overline{K} \cap \bigcap_{i \notin J} H_i$. Then $\alpha_i(x) = 0$ for all $i \notin J$, thus $x = \sum_{j \in J} \lambda_j v_j$. As x is in $\overline{K} = \bigcap_{i=1}^{r} \overline{\alpha_i^+}$, we find $\lambda_j \geq 0$ for all $j \in J$. Hence $x \in \mathbb{R}_{>0}F_J \cup \{0\}$, and the equality holds.

Every set in $\mathcal{S}_K$ arises as $\overline{K} \cap \bigcap_{i \notin J} H_i$ for some $J \subset \{1, \ldots, r\}$, and every such index set yields a different element in $\mathcal{S}_K$. Hence ψ_K is a well defined and bijective map. By definition it preserves inclusion.

In particular this shows that $(\mathcal{S}_K, \subseteq)$ is a simplicial complex. □

2.3 Simplicial arrangements on a cone

2.3.1 Simplicial arrangements, root systems, and root bases

In the following let $V = \mathbb{R}^r$ and $T \subseteq V$ be an open convex cone. In this section we will establish a notion of hyperplane arrangements on T. The interesting cases the reader may think of are $T = \mathbb{R}^r$ as well as a half-space $T = \alpha^+$ for some $\alpha \in V^*$.

Definition 2.3.1. A *hyperplane arrangement (of rank r)* is a pair $(\mathcal{A}, T)$, where T is a convex open cone in $V = \mathbb{R}^r$, and $\mathcal{A}$ is a (possibly infinite) set of linear hyperplanes such that $H \cap T \neq \emptyset$ for all $H \in \mathcal{A}$. If T is unambiguous from the context, we will also call the set $\mathcal{A}$ a hyperplane arrangement.

Let $X \subset \overline{T}$. Then the *support of* X is defined as $\operatorname{supp}_{\mathcal{A}}(X) = \{H \in \mathcal{A} \mid X \subset H\}$. If $X = \{x\}$ is a singleton, we write $\operatorname{supp}_{\mathcal{A}}(x)$ instead of $\operatorname{supp}_{\mathcal{A}}(\{x\})$, and we will omit the index $\mathcal{A}$, if $\mathcal{A}$ is unambiguous from the context. In this chapter we call the set

$$\sec_{\mathcal{A}}(X) := \bigcup_{x \in X} \operatorname*{supp}_{\mathcal{A}}(x) = \{H \in \mathcal{A} \mid H \cap X \neq \emptyset\}$$

the *section of* X *(in* $\mathcal{A}$*)*. Again, we will omit $\mathcal{A}$ when there is no danger of confusion.

We say that the arrangement $(\mathcal{A}, T)$ is *locally finite*, if for every $x \in T$ there exists a neighbourhood $U_x \subset T$, such that $\sec(U_x)$ is a finite set.

If $(\mathcal{A}, T)$ is locally finite, the connected components of $T \setminus \bigcup_{H \in \mathcal{A}} H$ are open sets and will be called *chambers*, and denoted with $\mathcal{K}(\mathcal{A})$ or just $\mathcal{K}$, if $\mathcal{A}$ is unambiguous from the context.

We associate to an open or closed simplicial cone K the *walls of* K

$$W^K := \{H \text{ hyperplane in } V \mid \langle H \cap \overline{K} \rangle = H, H \cap K^\circ = \emptyset\}.$$

We can now define our main objects of interest:

Definition 2.3.2. Let $T \subseteq \mathbb{R}^r$ be a convex open cone, $\mathcal{A}$ a set of linear hyperplanes. We call a hyperplane arrangement $(\mathcal{A}, T)$ *simplicial*, if

1) $\mathcal{A}$ is locally finite and

2) every $K \in \mathcal{K}(\mathcal{A})$ is an open simplicial cone.

We call a locally finite hyperplane arrangement $(\mathcal{A}, T)$ *thin*, if $W^K \subset \mathcal{A}$ holds for all $K \in \mathcal{K}(\mathcal{A})$.

Remark 2.3.3.

1. The definition of "thin" requires that all possible walls are already in the hyperplane arrangement $\mathcal{A}$. If we do not require this, a bounding hyperplane may arise as a bounding hyperplane of T itself. Consider the case where T itself is an open simplicial cone. Even the empty set would then satisfy that the one chamber, which is T itself, is a simplicial cone. However, it has no walls in $\mathcal{A}$, hence it is not thin.

2. From the definition it follows that in dimensions 0 and 1, there are very few possibilities for simplicial arrangements.

   In dimension 0, the empty set is the only possible hyperplane arrangement, which is also thin. Furthermore the empty set is a simplicial arrangement if T itself is simplicial, as noted above.

   In the case $V = \mathbb{R}$, we find $T = \mathbb{R}$ or $T = \mathbb{R}_{>0}$ or $T = \mathbb{R}_{<0}$. In the first case, $\{0_\mathbb{R}\}$ is a thin simplicial hyperplane arrangement and in the other two cases, $\{0_\mathbb{R}\}$ is a simplicial hyperplane arrangement, but not thin anymore.

3. We will later in this section introduce the notion of k-spherical arrangements, which is a refinement of being thin.

4. In the case where $T = \mathbb{R}^r$ the property of $\mathcal{A}$ being locally finite is equivalent to $\mathcal{A}$ being finite, as 0 is contained in every hyperplane. Furthermore $T = \mathbb{R}^r$ is the only case where $0 \in T$, as the cone over every neighbourhood of 0 is already $\mathbb{R}^r$.

Some of the "classical" cases are the following choices for T.

Definition 2.3.4. We call a locally finite hyperplane arrangement $(\mathcal{A}, T)$ *spherical*, if $T = \mathbb{R}^r$. We call it *affine*, if $T = \gamma^+$ for some $0 \neq \gamma \in V^*$. For an affine arrangement we call γ the *radical* or the *imaginary root* of the arrangement.

Definition and Remark 2.3.5. The cone of a hyperplane arrangement $(\mathcal{A}, T)$ is called T as it resembles the Tits cone for Coxeter groups. It will be called the *Tits cone of the arrangement*. The geometric representation of an irreducible spherical or affine Coxeter group is a prototype of simplicial spherical or affine hyperplane arrangements.

Lemma 2.3.6. *Let $(\mathcal{A}, T)$ be a locally finite hyperplane arrangement. Then for every point $x \in T$ there exists a neighbourhood U_x such that* $\operatorname{supp}(x) = \sec(U_x)$. *Furthermore the set* $\sec(X)$ *is finite for every compact set $X \subset T$.*

Proof. Let $x \in T$, then there exists an open neighbourhood $U \subset T$ of x such that $\sec(U)$ is finite. By taking the smallest open ε-ball contained in U and centred at x, we can assume $U = U_\varepsilon(x)$. Let $H \in \sec(U)$, with $x \notin H$. Let $\delta = d(H, x) > 0$, then $\delta < \varepsilon$, and $U' := U_{\frac{\delta}{2}}(x)$ is an open subset such that $\sec(U') \subseteq \sec(U) \setminus \{H\}$. Since $\sec(U)$ is finite, $\sec(U) \setminus \sec(x)$ is finite. We can therefore repeat this process finitely many times until we find an open ball B such that $\sec(B) = \operatorname{supp}(x)$.

The second assertion is a consequence of the first. Let X be compact, and for $x \in X$ let U_x denote an open subset such that $\sec(U_x) = \operatorname{supp}(x)$. Then

$$\sec(X) \subseteq \bigcup_{x \in X} \sec(U_x)$$

as $X \subseteq \bigcup_{x \in X} U_x$. The U_x are open and X is compact, therefore there exists a finite set $\{x_1, \ldots, x_n\} \subset X$, $n \in \mathbb{N}$, such that

$$X \subseteq \bigcup_{i=1}^{n} U_{x_i}.$$

Consequently, if $H \cap X \neq \emptyset$, then there exists an index i such that $H \cap U_{x_i} \neq \emptyset$ and $H \in \sec(U_{x_i})$. Hence

$$\sec(X) \subseteq \bigcup_{i=1}^{n} \sec(U_{x_i})$$

and $\sec(X)$ is finite. □

Definition 2.3.7. Let $V = \mathbb{R}^r$ and $T \subset V$ be an open convex cone. A *root system (for T)* is a pair (R, T), where $R \subset V^*$ such that

1) $(\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}, T)$ is a thin simplicial hyperplane arrangement,

2) $-\alpha \in R$ for all $\alpha \in R$.

If (R, T) is a root system, $\mathcal{A}$ as above, we call $(\mathcal{A}, T)$ the simplicial hyperplane arrangement *associated to* (R, T).

Let (R, T) be a root system. We call a map $\rho : R \to R$ a *reductor of* R, if for all $\alpha \in R$

1) $\rho(\alpha) = \lambda_\alpha \alpha$ for some $\lambda_\alpha \in \mathbb{R}_{>0}$,

2) $\rho(\langle \alpha \rangle \cap R) = \{\pm \rho(\alpha)\}$.

A root system is *reduced*, if id_R is a reductor. Given (R, T) and a reductor ρ, when no ambiguity can occur, we denote $R^{\mathrm{red}} := \rho(R)$.

We will note some immediate consequences of this definition:

Lemma 2.3.8. *Let (R, T) be a root system, ρ a reductor of R. Then*

i) $\rho(R)$ is a reduced root system in T,

ii) R is reduced if and only if $\langle \alpha \rangle \cap R = \{\pm \alpha\}$ for all $r \in R$,

iii) If R is reduced, id_R is the only reductor of R.

Proof. First as $\rho(\alpha) = \lambda_\alpha \alpha \neq 0$, we find that $\{\alpha^\perp \mid \alpha \in R\} = \{\rho(\alpha)^\perp \mid \alpha \in R\}$, so $\rho(R)$ is a root system. Due to the properties of ρ, we find that $\langle \alpha \rangle \cap \rho(R) = \{\pm\rho(\alpha)\}$, so id_R is a reductor of R.

For the second statement assume $\langle \alpha \rangle \cap R = \{\pm\alpha\}$ for all $\alpha \in R$, then the identity is a reductor. Assume that the identity is a reductor, then for $\alpha, \beta \in R$ with $\beta = \lambda\alpha$, $\lambda \in \mathbb{R}$, we find that $\beta \in \{\pm\alpha\}$.

For the third statement assume that R is reduced, so id_R is a reductor of R. Let $\alpha \in R$, then $\langle \alpha \rangle \cap R = \{\pm\alpha\}$. If ρ is a reductor of R, $\rho(\alpha) = \lambda_\alpha \alpha \in R$, as λ_α is positive, we find $\lambda_\alpha = 1$ and $\rho = \mathrm{id}_R$. □

Remark 2.3.9. In the case where for $\alpha \in R$ the set $\langle \alpha \rangle \cap R$ is finite, the canonical choice for ρ is such that $|\rho(\alpha)|$ is minimal in $\langle r \rangle \cap R$; with respect to an arbitrary scalar product.

The notion of a reductor of R is very general. The intention is to be able to reduce a root system even in the case where for $\alpha \in R$ the set $\langle \alpha \rangle \cap R$ does not have a shortest or longest element. In this most general setting, the existence of a reductor requires the axiom of choice.

In the following we note that root systems always exist for simplicial hyperplane arrangements.

Lemma 2.3.10. *Let $\mathcal{A}$ be a thin simplicial hyperplane arrangement in T, $\mathbb{S}^{r-1}$ the unit sphere in $\mathbb{R}^r$ with respect to the standard metric associated to the standard scalar product $(\cdot,\cdot)$. Let*

$$S = \bigcup_{H \in \mathcal{A}} H^\perp \cap \mathbb{S}^{n-1}.$$

Then for $R = \{(s,\cdot) \in V^* \mid s \in S\}$ the pair (R,T) is a reduced root system associated to $\mathcal{A}$.

Furthermore every reduced root system (R',T) associated to $\mathcal{A}$ is of the form $R' = \{\lambda_\alpha \alpha \mid \lambda_\alpha = \lambda_{-\alpha} \in \mathbb{R}_{>0}, \alpha \in R\}$, and every such set is a reduced root system associated to $\mathcal{A}$.

Proof. As $H \in \mathcal{A}$ is $r-1$-dimensional, $\dim H^\perp = 1$, so $H^\perp \cap \mathbb{S}^{n-1} = \{\pm s\}$ for some vector $s \in V$, with $s^\perp = H$. So R is a root system associated to $\mathcal{A}$. Let $\alpha, \lambda\alpha \in R$ for some $\lambda \in \mathbb{R}$, then $\alpha^\perp = (\lambda\alpha)^\perp$. As the map $s \mapsto (s,\cdot)$ is bijective, we find $\lambda \in \{\pm 1\}$. Therefore R is reduced.

For the second statement note that the hyperplane $H \in A$ determines $s \in H^\perp$ uniquely up to a scalar. So any R' of the given form satisfies $\{\alpha^\perp \mid \alpha \in R'\} = \mathcal{A}$. Furthermore it is reduced, as $\langle \alpha \rangle \cap R' = \{\pm\lambda_\alpha \alpha\}$ for $\alpha \in R$. □

Definition 2.3.11. When fixing a root system (R,T) associated to a thin simplicial hyperplane arrangement $(\mathcal{A},T)$ of rank r, we will from now on call the triple $(\mathcal{A},T,R)$ a *simplicial arrangement (of rank r))*. In view of Lemma 2.3.10 we can always find a root system for a thin simplicial hyperplane arrangement $(\mathcal{A}, T)$, so we find for every such arrangement a simplicial arrangement $(\mathcal{A}, T, R)$.

Definition 2.3.12. Let $(\mathcal{A}, T, R)$ be a simplicial arrangement, and fix a reductor ρ of R. Let K be a chamber. The *root basis of* K is the set

$$B^K := \{\alpha \in R^{\mathrm{red}} \mid \alpha^\perp \in W^K, \alpha(x) > 0 \text{ for all } x \in K\}.$$

Lemma 2.3.13. *Let K be an open simplicial cone and $B_K \subset V^*$ such that $K = \bigcap_{\alpha \in B_K} \alpha^+$ and $|B_K| = r$. Then B_K is a basis of V^*.*

Proof. If S is a simplex such that $K = \mathbb{R}_{>0} S$ and $V(S)$ is a basis, by Lemma 2.2.5 B_K is a dual basis to $V(S)$. □

Lemma 2.3.14. *With notation as in 2.3.13, if $\alpha \in B_K$, then $\alpha^\perp \in W^K$.*

Proof. Let $\alpha \in B_K$ and $K = \mathbb{R}_{>0} S$ for a simplex S with vertex set $V(S) = \{v_1, \dots, v_r\}$. In particular by Lemma 2.2.5 we can assume $\alpha(v_1) = 1$, $\alpha(v_i) = 0$ for $i = 2, \dots, r$. As $v_2, \dots, v_r \in \alpha^\perp \cap \overline{S}$, we find that $\overline{K} \cap \alpha^\perp$ contains $v_2, \dots, v_r$. Since $v_2, \dots, v_r$ is a basis for $\alpha^\perp$, we find that $\langle \overline{K} \cap \alpha^\perp \rangle = \alpha^\perp$.

As S can be written as

$$S = \left\{ \sum_{i=1}^{r} \lambda_i v_i \mid \sum_{i=1}^{r} \lambda_i = 1, 0 < \lambda_i < 1 \text{ for all } 1 \le i \le r \right\},$$

we find $S \cap \alpha^\perp = \emptyset$. Since $\alpha^\perp$ is a cone and S is convex, $K \cap \alpha^\perp = \emptyset$ holds. Therefore $\alpha^\perp \in W^K$. □

Lemma 2.3.15. *Let S be an $r-1$ simplex with linearly independent vertex set $V(S)$. If H is a wall of $\mathbb{R}_{>0} S$, then H contains $r-1$ elements of $V(S)$.*

Proof. Let H be a wall of $K := \mathbb{R}_{>0} S$, and $V_H = V(S) \cap H$. The set S is convex and, due to the properties of walls, contained in a unique half space of H. In particular, there exists a half space H^+ such that every vertex of S is either in H^+ or in H. The intersection $\overline{S} \cap H$ is therefore just the convex hull of V_H. Now note that

$$\langle \overline{K} \cap H \rangle = \langle \overline{S} \cap H \rangle = \langle V_H \rangle,$$

this implies our claim. □

From the previous statements we can justify that root bases are actually bases of the dual space:

Corollary 2.3.16. *Let $(\mathcal{A}, T, R)$ be a simplicial arrangement, $K \in \mathcal{K}$. Then B^K is a basis of V^*.*

Proof. By definition of B^K we know $K \subseteq \bigcap_{\alpha \in B^K} \alpha^+$.

By Lemma 2.2.5 $K = \{x \in T \mid \alpha_i(x) > 0 \text{ for } 1 \leq i \leq r\}$ for some $\alpha_1, \dots, \alpha_r \in V^*$, so by Lemma 2.3.14 we find $\{\alpha_i^\perp \mid i = 1, \dots, r\} \subseteq W^K$ and equality follows from Lemma 2.3.15. Up to a scalar we can assume $B^K = \{\alpha_1, \dots \alpha_r\}$. Since the α_i are different by definition, we obtain $|B^K| = r$. Thus we can use Lemma 2.3.13 to obtain the statement. □

The following lemma is crucial for the theory and motivates the notion of root bases.

Lemma 2.3.17. *Let $(\mathcal{A}, T, R)$ be a simplicial arrangement, K a chamber. Then $R \subset \pm \sum_{\alpha \in B^K} \mathbb{R}_{\geq 0} \alpha$. In other words, every root is a non-negative or non-positive linear combination of B^K.*

Proof. The proof works exactly as in the spherical case, see [Cun11, Lemma 2.2]. □

It will be useful to identify simplicial arrangements which basically only differ by the choice of basis, therefore we make the following definition.

Definition 2.3.18. Let $(\mathcal{A}, T, R)$ and $(\mathcal{A}', T', R')$ be simplicial arrangements. Then $(\mathcal{A}, T, R)$ and $(\mathcal{A}', T', R')$ are called *combinatorially equivalent*, if there exists $g \in \mathrm{GL}(V)$ such that $g\mathcal{A} = \mathcal{A}'$, $g * R = R'$, $g(T) = T'$. Here $*$ denotes the dual action of $GL(V)$ on V^*, defined by $g * \alpha = \alpha \circ g^{-1}$.

2.3.2 The simplicial complex associated to a simplicial arrangement

Definition and Remark 2.3.19. Let $(\mathcal{A}, T)$ be a simplicial hyperplane arrangement, which is not necessarily thin. The set of chambers $\mathcal{K}$ gives rise to a poset $\mathcal{S} := \mathcal{S}(\mathcal{A}, T)$ in the following way.

$$\mathcal{S} := \left\{ \overline{K} \cap \bigcap_{H \in \mathcal{A}'} H \mid K \in \mathcal{K}, \mathcal{A}' \subseteq W^K \right\} = \bigcup_{K \in \mathcal{K}} \mathcal{S}_K,$$

with inclusion giving a poset-structure. Note that we do not require any of these intersections to be in T. By construction they are contained in the closure of T, as K is an open subset in T.

We will at this point just note that $\mathcal{S}$ is a simplicial complex. The proof is intuitive, but gets quite lengthy. The steps toward it and the proof itself can be found in Chapter 2.4.1.

Proposition 2.3.20. *For a simplicial hyperplane arrangement $(\mathcal{A}, T)$, the poset $\mathcal{S}(\mathcal{A}, T)$ is a simplicial complex.*

Definition and Remark 2.3.21. The complex $\mathcal{S}$ is furthermore a chamber complex, which justifies the notion of chambers. As for simplicial complexes, we call two chambers $K, K' \in \mathcal{K}$ *adjacent* if $\mathrm{codim}_K(\overline{K} \cap \overline{K'}) = 1$.

We have a slight ambiguity of notation at this point, as a chamber in the simplicial complex $\mathcal{S}$ is the closure of a chamber in $\mathcal{K}$. For the readers convenience, a chamber from now on will always be an element $K \in \mathcal{K}$, while we will refer to a chamber in $\mathcal{S}$ as a *closed chamber*, written as $\overline{K}$ and implying the existence of an element $K \in \mathcal{K}$. We will show in the next section that the closed chambers are indeed chambers in a classical sense.

Remark 2.3.22.

1. Depending on T, the above mentioned simplicial complex can also be seen as canonically isomorphic to the simplicial decomposition of certain objects, arising by intersection with the respective simplicial cones. In case $T = \mathbb{R}^r$, $\mathcal{S}$ corresponds to a simplicial decomposition of the sphere $\mathbb{S}^{n-1}$. If $T = \alpha^+$ for $\alpha \in V^*$, we find $\mathcal{S}$ to be a decomposition of the affine space $\mathbb{A}^{n-1}$, which we identify with the set $\alpha^{-1}(1)$. If T is the light cone and $\mathcal{A}$ is CH-like as defined in Definition 2.3.23 below, we can find a corresponding decomposition of $\mathbb{H}^{n-1}$.

2. In the literature (see [Bou02, Chapter V, §1]) the simplicial complex associated to a finite or affine simplicial hyperplane arrangement is defined in a slightly different manner. Let V be a Euclidean space, $\mathcal{A}$ a locally finite set of (possibly affine) hyperplanes. Define for $A \in \mathcal{A}$, $v \in V \setminus A$ the set $D_H(v)$ to be the halfspace with respect to H containing v. Set

$$v \sim w :\Leftrightarrow w \in \bigcap_{v \in H \in \mathcal{A}} H \cap \bigcap_{v \notin H \in \mathcal{A}} D_H(v).$$

   Then $\sim$ is an equivalence relation, its classes are called *facets*. Facets correspond to simplices, and form a poset with respect to the inclusion $F \leq F' :\Leftrightarrow F \subseteq \overline{F'}$.

   One obtains immediately that every point in V is contained in a unique facet. However, when the space is not the entire euclidean space but a proper convex open cone, it is desirable to consider some of the points in the boundary, as they contribute to the simplicial structure of $\mathcal{S}$. Therefore, we prefer our approach before the classical one.

Using the simplicial complex $\mathcal{S}$, it is now possible to refine the notion of a thin arrangement.

Definition 2.3.23. We call a simplicial hyperplane arrangement $(\mathcal{A}, T)$ *k-spherical* for $k \in \mathbb{N}_0$ if every simplex S of $\mathcal{S}$, such that $\mathrm{codim}(S) = k$, meets T. We say $(\mathcal{A}, T)$ is *CH-like* if it is $r-1$-spherical, and $(\mathcal{A}, T)$ is called *spherical* if it is r-spherical.

Remark 2.3.24. Immediate from the definition are the following observations:

1. Every simplicial hyperplane arrangement of rank r is 0-spherical, as $K \in \mathcal{K}$ is constructed as an open subset of T.
2. $(\mathcal{A}, T)$ is thin if and only if it is 1-spherical.
3. As $\mathcal{S}$ is a simplicial complex w. r. t. set-wise inclusion, being k-spherical implies being $(k-1)$-spherical for $1 \leq k \leq r$.
4. An arrangement is r-spherical if and only if $0 \in T$, which is equivalent to $T = V$. In this case, $\mathcal{A}$ is finite, and $(\mathcal{A}, T)$ is spherical as in Definition 2.3.4.
5. Examples of CH-like arrangements are all arrangements belonging to affine Weyl groups, where the affine $r-1$-plane is embedded into a real vector space of dimension r, as well as all arrangements belonging to compact hyperbolic Coxeter groups, where T is the light cone.
6. As a generalisation of 5., take a Coxeter system (W, S) of finite rank. Then W is said to be k-spherical if every rank k subset of S generates a finite Coxeter group. The geometric representation of W then yields a hyperplane arrangement which is k-spherical in the way defined above. Furthermore, it will not be k'-spherical for any $k' > k$. Therefore, being k-spherical can be seen as a generalisation of the respective property of Coxeter groups.
7. An example of an arrangement which is 1-spherical but not CH-like is the real hyperplane arrangement of the universal Coxeter group on three generators in the light cone. Here the fundamental domain is a hyperbolic triangle with vertices in the boundary of hyperbolic 2-space.
8. An equivalent condition for $\mathcal{A}$ to be k-spherical, which we will use often, is that every $(r-k-1)$-simplex meets T. This uses the fact that simplices of codimension k are exactly $(r-k-1)$-simplices.

An important observation is the fact that the cone T can be reconstructed from the chambers.

Lemma 2.3.25. *For a simplicial hyperplane arrangement* $(\mathcal{A}, T)$ *we find*

$$\overline{T} = \overline{\bigcup_{K \in \mathcal{K}} K}.$$

Furthermore, $(\mathcal{A}, T)$ *is CH-like if and only if*

$$T = \bigcup_{K \in \mathcal{K}} \overline{K}$$

holds.

Proof. As $\bigcup_{K\in\mathcal{K}} K \subset T$, the inclusion $\overline{\bigcup_{K\in\mathcal{K}} K} \subset \overline{T}$ holds. If $x \in T$, either $x \in K$ for some $K \in \mathcal{K}$, or x is contained in a finite number of hyperplanes, thus in a simplex in $\mathcal{S}$ and also in the closure of a chamber, and therefore $x \in \overline{\bigcup_{K\in\mathcal{K}} K}$.

So let $x \in \overline{T} \setminus T$. Assume further that x is not contained in any simplex in $\mathcal{S}$, else the statement follows immediately as above. Therefore $\operatorname{supp}(x) = \emptyset$, and as T is convex, this means x is in the boundary of T. So let $U := U_\delta(x)$ be the open δ-ball with centre x, then $U \cap T \neq \emptyset$. As U is open, $U \cap T$ is open again. The set $\sec(U \cap T)$ can not be empty, else $U_x \cap T$ is not contained in a chamber, in a contradiction to the construction of chambers. So let $\mathcal{K}_0$ be the set of chambers K with $K \cap U \neq \emptyset$. If x is not contained in the closure of $\bigcup_{K\in\mathcal{K}_0} K$, we find a $\delta > \varepsilon > 0$ such that the open ball $U_\varepsilon(x)$ does not intersect any $K \in \mathcal{K}_0$. But $U_\varepsilon(x) \cap T$ must again meet some chambers K, which are then also in $\mathcal{K}_0$, a contradiction. So $x \in \overline{\bigcup_{K\in\mathcal{K}} K}$ and equality holds.

For the second statement, if $(\mathcal{A}, T)$ is CH-like, note that every $x \in T$ is contained in some simplex $F \in \mathcal{S}$, so $T \subset \bigcup_{K\in\mathcal{K}} \overline{K}$ holds. Now if $x \in \overline{K}$ for some $K \in \mathcal{K}$, x is contained in some simplex $F \subset \overline{K}$. Now $\mathcal{A}$ is CH-like, so F meets T. Boundaries of simplices are simplices, hence the intersection $F \cap (\overline{T} \setminus T)$ is again a simplex in $\mathcal{S}$, being CH-like yields that this intersection is empty, which proves the other inclusion.

The other direction of the second statement is immediate from the definition of being $r-1$-spherical. □

Remark 2.3.26. For a simplicial arrangement $(\mathcal{A}, T, R)$ it will be a consequence of Section 2.6 that we can also describe T as the convex closure of $T_0 := \bigcup_{K\in\mathcal{K}} K$, or alternatively as

$$T = \bigcup_{x,y\in T_0} [x, y].$$

We will require the existence of a type function of $\mathcal{S}$ (for the definition, see 1.2.6), which is given by the following proposition and proven in Chapter 2.4.2.

Proposition 2.3.27. *Let $(\mathcal{A}, T)$ be a simplicial hyperplane arrangement. The complex $\mathcal{S} := \mathcal{S}(\mathcal{A}, T)$ is a chamber complex of rank r with*

$$\operatorname{Cham}(\mathcal{S}) = \{\overline{K} \mid K \in \mathcal{K}\}.$$

The complex $\mathcal{S}$ is totally gated and strongly connected. Furthermore there exists a type function $\tau : \mathcal{S} \to I$ of $\mathcal{S}$, where $I = \{1, \ldots, r\}$. The complex $\mathcal{S}$ is thin if and only if $(\mathcal{A}, T)$ is thin, and $\mathcal{S}$ is spherical if and only if $(\mathcal{A}, T)$ is spherical.

2.4 The chamber complex $\mathcal{S}$

2.4.1 The simplicial structure of $\mathcal{S}$

We show that the poset $\mathcal{S}$ associated to the simplicial hyperplane arrangement $(\mathcal{A}, T)$ is actually a simplicial complex and furthermore a chamber complex with its set of chambers being $\mathcal{K}$. We already showed in Lemma 2.2.5 that the simplicial structure on a closed chamber $\overline{K}$ is induced from the simplicial structure of $\overline{S}$, where S is an open simplex such that $K = \mathbb{R}_{>0}S$.

Lemma 2.4.1. *Let S, S' be $r-1$ simplices in $\mathbb{R}^r$ such that $V(S) = \{v_1, \dots, v_r\}$, $V(S') = \{v'_1, \dots, v'_r\}$ are linearly independent and $\mathbb{R}_{>0}S = \mathbb{R}_{>0}S'$. Then, up to permutation, $v_i = \lambda_i v'_i$ for some $\lambda_i \in \mathbb{R}_{>0}$ for all $1 \leq i \leq r$.*

Proof. One can choose positive scalars $\mu_i \in \mathbb{R}_{>0}$ such that $w_i = \mu_i v'_i \in S$. Then $w_i = \sum_{k=1}^r \kappa_{ik} v_i$ with $\sum_{k=1}^r \kappa_{ik} = 1$, $0 \leq \kappa_{ik} \leq 1$. Furthermore the w_i also span a simplex S'' generating $\mathbb{R}_{>0}S$ as a cone, so we find $v_i = \sum_{j=1}^r \nu_{ij} w_i$ with $\sum_{j=1}^r \nu_{ij} = 1$ and $0 \leq \nu_{ij} \leq 1$.

Therefore the matrix $M_{V(S)}^{V(S'')}(\mathrm{id})$ describing the base change is non-negative, and the same holds for its inverse $M_{V(S'')}^{M(S)}(\mathrm{id})$. It is well known (see Theorem 4.6 in Chapter 3 of [BP79] for example) that the inverse of a non-negative matrix is non-negative if and only if the matrix is monomial. Adding the fact that the sum of every column adds up to 1, we get that $M_{V(S)}^{V(S'')}(\mathrm{id})$ and $M_{V(S'')}^{V(S)}(\mathrm{id})$ are already permutation matrices. $\square$

Corollary 2.4.2. *Let $K = \mathbb{R}_{>0}S$ be the cone on an open $(r-1)$-simplex S, with linearly independent vertices $V(S)$, and let $\beta \in V^*$. Then $\beta(v) \geq 0$ for all $v \in V(S)$ if and only if $\beta \in \sum_{\alpha \in B_K} \mathbb{R}_{\geq 0}\alpha$. Likewise $\beta(v) \leq 0$ for all $v \in V(S)$ if and only if $\beta \in -\sum_{\alpha \in B_K} \mathbb{R}_{\geq 0}\alpha$.*

Proof. Let $\beta = \sum_{\alpha \in B_K} \lambda_\alpha \alpha$. By Lemma 2.2.5 we find that B_K is dual to $V(S)$ up to positive scalar multiples. Denote with $\alpha_v \in B_K$ the dual to $v \in V(S)$. So $\beta = \sum_{v \in V(S)} \lambda_v \alpha_v$. Applying this to $V(S)$ yields $\beta(v) = \lambda_v$ for all $v \in V(S)$. This immediately yields both equivalences. $\square$

Definition 2.4.3. Note that every $H \in \mathcal{A}$ separates V into half-spaces, so it does with T, as T is convex and $H \cap T \neq \emptyset$. One way to describe them is the following way: Choose an arbitrary linear form $\alpha \in V^*$ such that $\alpha^\perp = H$. Then α^+ and α^- ($\overline{\alpha^+}$ and $\overline{\alpha^-}$) are the two *open (closed) half-spaces bounded by H.*

For an arbitrary subset $X \subset T$, $X \not\subset H$, if X is contained in one open (resp. closed) half space bounded by H, we denote this particular open (resp. closed) half-space by $D_H(X)$ (resp. $\overline{D_H(X)}$). In this case we write $-D_H(X)$ (resp. $\overline{-D_H(X)}$) for the unique open (resp. closed) half-space not containing X. By definition every chamber K is contained in a unique open half space of $H \in \mathcal{A}$, therefore the sets $D_H(K)$ exist for all $H \in \mathcal{A}$, $K \in \mathcal{K}$. Let K, L be chambers, we say that $H \in \mathcal{A}$ *separates* K and L, if $D_H(K) = -D_H(L)$. We also say that two closed chambers $\overline{K}, \overline{L} \in \mathrm{Cham}(\mathcal{S})$ are *separated* by $H \in \mathcal{A}$, if H separates K and L.

Lemma 2.4.4. *Let $F \in \mathcal{S}$. For every $H \in \mathcal{A}$, either $F \subset H$ or F is contained in a unique closed half space of H, thus in the second case $\overline{D_H(F)}$ is well defined. Furthermore $F \cap H \in \mathcal{S}$, and if $F \in \overline{K}$ for some $K \in \mathcal{K}$, then $F \cap H \in \overline{K}$.*

Proof. In case $F \subset H$, there is nothing to show, so assume $F \not\subset H$. Then the first statement is an immediate consequence of Corollary 2.4.2 and the fact that the elements K are defined as connected components of $V \setminus \bigcup_{H \in \mathcal{A}} H$. Let $\alpha \in V^*$ such that $H = \alpha^\perp$ and $D_H(F) = \alpha^+$.

Let F be the convex hull of the vertices $\mathbb{R}_{>0}v_1, \dots, \mathbb{R}_{>0}v_k$, where $k \geq 1$ as we assume $F \not\subset H$. For the same reason we can assume $v_i, \dots, v_k \notin H$ for some $i < k$, and without loss of generality we can assume $v_1, \dots, v_{j-1} \in H$. Then $F \cap H$ is the convex hull of $\mathbb{R}_{>0}v_1, \dots, \mathbb{R}_{>0}v_{i-1}$. By Proposition 2.2.7 $\underline{K}$ is a simplicial complex, hence $F \cap H$ is a simplex in $\mathcal{S}$ contained in $\underline{K}$. $\square$

In the case where $(\mathcal{A}, T)$ is not thin, we need to be able to handle hyperplanes which are not in $\mathcal{A}$ but occur as a wall of a chamber.

Lemma 2.4.5. *Assume $K \in \mathcal{K}$, $H \in W^K$ and $H \notin \mathcal{A}$. Then $T \cap H = \emptyset$. In particular for all $K_1, K_2 \in \mathcal{K}$ the set $D_H(K_1)$ is well defined and $\overline{D_H(K_1)} = \overline{D_H(K_2)}$ holds. More generally, if $F_1, F_2 \in \mathcal{S}$ with $F_1, F_2 \not\subset H$, we find $\overline{D_H(F_1)} = \overline{D_H(F_2)}$.*

Proof. Assume $T \cap H \neq \emptyset$, then H separates T into two half spaces. As H is a wall of K and $H \notin \mathcal{A}$, we find that K is not a connected component of $T \setminus \bigcup_{H \in \mathcal{A}} H$. Therefore $T \cap H = \emptyset$, which proves our claim. $\square$

Proof of 2.3.20. Let $F \in \mathcal{S}$, then $F = \overline{K} \cap \bigcap_{H \in \mathcal{A}_1} H$ for some $K \in \mathcal{K}$, $\mathcal{A}_1 \subset W^K$. In particular $F \in \mathcal{S}_K$, by Proposition 2.2.7 F is a simplex, as $\mathcal{S}_K$ is a simplicial complex.

So let $F' \in \mathcal{S}$, we have to show $F \cap F' \in \mathcal{S}$. The set-wise intersection $F \cap F'$ is not empty, as it contains 0_V.

Assume $F' = \overline{K'} \cap \bigcap_{H \in \mathcal{A}_2} H$, for $K' \in \mathcal{K}$, $\mathcal{A}_2 \subset W^{K'}$. In the case that $\mathcal{A}_2 = \emptyset$, we find $F' = \overline{K'}$.

Let S, S' be the simplices such that $K = \mathbb{R}_{>0}S$, $K' = \mathbb{R}_{>0}S'$. If $K = K'$, there is nothing to show, as $\underline{K}$ is a simplicial complex by Proposition 2.2.7. So from now on let $K \neq K'$.

Assume first $\mathcal{A}_2 = \emptyset$, so $F' = \overline{K'}$. The set $\overline{K'}$ can be written as $\overline{K'} = \bigcap_{H \in W^{K'}} \overline{D_H(K')}$ by Lemma 2.2.5. By Lemma 2.4.4 the intersection $F \cap \overline{D_H(K')}$ for $H \in W^{K'}$ is either F, in case that $D_H(K')$ contains F, or equals $F \cap H$. In both cases, it is again a simplex in $\underline{K}$. We can conclude that $F \cap \overline{K'} \in \mathcal{S}$ for every $F \in \mathcal{S}$.

Now let $\mathcal{A}_2 \neq \emptyset$, then $F \cap \bigcap_{H \in \mathcal{A}_2} H$ is a simplex in $\underline{K}$ by Lemma 2.4.4.

But $F \cap F'$ can be written as $F \cap \bigcap_{H \in \mathcal{A}_2} H \cap \overline{K'}$, using the previous part of the proof shows our claim. $\square$

2.4.2 The gate property for $\mathcal{S}$

In the following, assume again that $V = \mathbb{R}^r$ and $(\mathcal{A}, T)$ is a simplicial hyperplane arrangement of rank r, with the respective simplicial complex $\mathcal{S} = \mathcal{S}(\mathcal{A}, T)$.

The set T itself is a metric space as a convex open subset of $\mathbb{R}^r$. We will denote this metric as d_T.

Remember that $\mathcal{A}$ is locally finite in T, which implies by Lemma 2.3.6 that if we take a compact subset $X \subseteq T$, the set $\sec(X)$ is finite.

At this point we can justify the notion of the chambers $\mathcal{K}$.

Lemma 2.4.6. $\mathrm{Cham}(\mathcal{S}) = \{\overline{K} \mid K \in \mathcal{K}\}$.

Proof. By definition every maximal element in $\mathcal{S}$ is of the form $\overline{K}$ for some $K \in \mathcal{K}$. Assume $\overline{K} \subset \overline{K'}$ for $K, K' \in \mathcal{K}$, then $\overline{K}$ is contained in some $H \in \mathcal{A}$ by Proposition 2.2.7, which contradicts the definition of $\mathcal{K}$. This proves $\mathrm{Cham}(\mathcal{S}) = \{\overline{K} \mid K \in \mathcal{K}\}$. $\square$

To prove that $\mathcal{S}$ is already a chamber complex, we need a bit more information about the distance between two chambers, depending on the number of hyperplanes separating them.

Definition 2.4.7. We denote by $S(K, L) := \{H \in \mathcal{A} \mid D_H(K) \neq D_H(L)\}$ the set of hyperplanes separating K, L.

We introduce another way to describe chambers in $\mathcal{K}$.

Lemma 2.4.8. *Let* $\overline{K} \in \mathrm{Cham}(\mathcal{S})$*, then* $\overline{K} = \bigcap_{H \in W^K} D_H(K)$.

Proof. This is a direct consequence of the fact that $\overline{K} = \{x \in T \mid \alpha(x) \geq 0 \text{ for all } \alpha \in B_K\}$ and $W^K = \{\alpha^\perp \mid \alpha \in B_K\}$ for some basis $B_K \subset V^*$. For $\alpha \in B_K$ with $\alpha^\perp = H \in W^K$ we find $D_H(K) = \{v \in T \mid \alpha(v) \geq 0\}$. $\square$

Definition and Remark 2.4.9. Remember that two closed chambers $\overline{K}, \overline{L}$ are adjacent, if $\mathrm{codim}_{\overline{K}}(\overline{K} \cap \overline{L}) = 1 = \mathrm{codim}_{\overline{L}}(\overline{K} \cap \overline{L})$. In particular, if K, L are adjacent, (K, L) is a unique minimal gallery from K to L, as defined for chamber complexes. For a hyperplane H we will say that K, L are *adjacent by* H if $\langle \overline{K} \cap \overline{L} \rangle = H$. It is immediate from the definition, that $\overline{K}$, $\overline{L}$ are adjacent if and only if there exists a hyperplane $H \in \mathcal{A}$, such that K and L are adjacent by H.

We can formulate a slightly stronger version of this remark:

Lemma 2.4.10. *Assume* K, L *are adjacent by* H*, then* H *is the unique hyperplane separating* K *and* L*.*

Proof. If K, L are adjacent by H, then as $H \in W^K$ the set $\overline{K} \cap \overline{L}$ is a unique maximal face of $\overline{K}$, spanning a unique hyperplane, which is H, so H separates K and L. Assume $H \neq H' \in \mathcal{A}$. Then $\overline{K}$ and $\overline{L}$ are contained in unique closed subspaces of H', therefore we find $D_{H'}(K) = D_{H'}(\overline{K} \cap H) = D_{H'}(\overline{L} \cap H) = D_{H'}(L)$. So H' does not separate K and L. $\square$

Lemma 2.4.11. *Let* $K, L \in \mathcal{K}$*. Then* $S(K, L)$ *is finite.*

Proof. Choose two points $v \in K$, $w \in L$. As T is convex, the line $[v,w]$ is contained in T, and is compact. Therefore, as $\mathcal{A}$ is locally finite, by Lemma 2.3.6 $\sec([v,w])$ is finite.

Let $H \in \mathcal{A}$ and assume $H \cap [v,w]$ is empty, then $D_H([v,w])$ is well-defined and $D_H(K) = D_H([v,w]) = D_H(L)$, so H does not separate K and L. We can conclude $S(K,L) \subseteq \sec([v,w])$. $\square$

Lemma 2.4.12. *Let $K, L \in \mathcal{K}$. Then there exists a minimal gallery of length $|S(K,L)|$ connecting K and L.*

Proof. By Lemma 2.4.11 $S(K,L)$ is finite, so let $n = |S(K,L)|$. For $n = 0$, we have $D_H(K) = D_H(L)$ for every $H \in A$ and all $H \in W^K \cup W^L$ by Lemma 2.4.5. By Lemma 2.4.8 this implies $K = L$. Let $n = 1$, in particular $K \neq L$ and say $S(K,L) = \{H\}$. Now $K \neq L$ implies $W^K \neq W^L$ by Lemma 2.4.8. Assume $L \subset D_{H'}(K)$ for all $H' \in W^K$, this is a contradiction to Lemma 2.4.8, so we have $H \in W^K$, as H is the unique wall separating K and L by Lemma 2.4.10. A similar argument yields $H \in W^L$, so $\operatorname{codim}_{\overline{K}}(\overline{K} \cap H) = 1 = \operatorname{codim}_{\overline{L}}(\overline{L} \cap H)$.

Assume that $\overline{K} \cap H \neq \overline{L} \cap H$. Both $\overline{K} \cap H$ and $\overline{L} \cap H$ are maximal faces of $\overline{K}$ (resp. $\overline{L}$), there exists a hyperplane $H \neq H'$ with the property $D_{H'}(\overline{K} \cap H) \neq D_{H'}(\overline{L} \cap H)$. But this implies $D_{H'}(K) \neq D_{H'}(\overline{L})$, a contradiction to $S(K,L) = \{H\}$. We get $1 = \operatorname{codim}_{\overline{K}}(\overline{K} \cap \overline{L}) = \operatorname{codim}_{\overline{K}}(\overline{K} \cap \overline{L}) = \operatorname{codim}_{\overline{L}}(\overline{L} \cap H) = 1$, and (K,L) is gallery of length one connecting K, L.

We now use induction on n, let $S(K,L) = \{H_1, \ldots, H_n\}$. Using Lemma 2.4.8 as in the case $n = 1$ above we can sort $S(K,L)$ in a way such that $H_1 \in W^K$. Then there exists a unique chamber K' adjacent to K such that $S(K,K') = \{H_1\}$. We will show $S(K',L) = \{H_2, \ldots, H_n\}$, then the statement of the Lemma follows by induction. Since we have $S(K,K') = \{H_1\}$, we have $D_{H_i}(K) = D_{H_i}(K')$ for $i = 2, \ldots, n$. Assume there exists an additional $H' \in \mathcal{A}$, $H' \neq H_i$ for $i = 1, \ldots, n$ with the property that H' separates K' and L, then H' also separates K and L, a contradiction. By induction this yields a gallery of length n from K to L.

To see that this gallery is minimal we use Lemma 2.4.10. So let $\overline{K} = \overline{K}_0, \overline{K}_1, \ldots, \overline{K}_m = \overline{L}$ be a gallery connecting K, L and let for $i = 1, \ldots, m$ the hyperplane $H_i \in \mathcal{A}$ such that $\overline{K}_{i-1} \cap \overline{K}_i \subset H_i$. Note that H_i is unique with that property by Lemma 2.4.10. Assume $H \in S(K,L)$ then there exists an index $0 \leq i < m$ such that H separates K_i and L but not K_{i+1} and L. So $D_H(K_i) = -D_H(L)$ and $D_H(K_{i+1}) = D_H(L)$, H separates K_i and K_{i+1}. We obtain $H = H_i$, as K_i, K_{i+1} are adjacent and H_i is unique with that property by Lemma 2.4.10.

Thus $S(K,L) \subset \{H_1, \ldots, H_m\}$ and this yields $|S(K,L)| \leq m$. Above we constructed a gallery of length $|S(K,L)|$, this shows that the gallery is already minimal. $\square$

Definition 2.4.13. For a simplex $F \in \mathcal{S}$ we set

$$\mathcal{K}_F := \{K \in \mathcal{K} \mid F \subset \overline{K}\}$$

and

$$\mathcal{A}_F := \{H \in \mathcal{A} \mid F \subset H\}.$$

With this notation, we have $\mathrm{Cham}(St(F)) = \{\overline{K} \mid K \in \mathcal{K}_F\}$.

Lemma 2.4.14. *Let $K, L \in \mathcal{K}_F$, then $S(K, L) \subset \mathcal{A}_F$.*

Proof. The statement is true for $F = \emptyset$ due to Lemma 2.4.5. So let $F \neq \emptyset$. Assume $H \notin \mathcal{A}_F$, then the half-space $D_H(F)$ is well defined and unique by Lemma 2.4.4. As a consequence we have $D_{H'}(K) = D_{H'}(F) = D_{H'}(L)$, and H' does not separate K and L. □

Proposition 2.4.15. *The simplicial complex $\mathcal{S}$ is a strongly connected chamber complex.*

Proof. The complex $\mathcal{S}$ is a chamber complex, since every simplex is contained in a chamber and two chambers $K, L \in \mathcal{K}$ are connected by a gallery of length $|S(K, L)|$ by Lemma 2.4.12 and $|S(K, L)|$ is finite by Lemma 2.4.11. For two elements $K, L \in \mathcal{K}$ we can therefore define the distance $d_{\mathcal{S}}(K, L)$ as the length of a minimal gallery connecting K, L.

Let F be a simplex in $\mathcal{S}$, and consider the simplicial complex $\mathrm{St}(F)$ with chambers $\mathcal{K}_F$. Let $K, L \in \mathcal{K}_F$ and assume $d_{\mathcal{S}}(K, L) = n = |S(K, L)| \geq 1$, so $K \neq L$. The fact that $K, L \in \mathcal{K}_F$ implies $F \in \overline{K} \cap \overline{L}$.

We need to show that there exists a gallery in $\mathrm{St}(F)$ from K to L, which we do by induction on $d_{\mathcal{S}}(K, L)$. For $d_{\mathcal{S}}(K, L) = 1$ we have that K, L are adjacent. So let $d_{\mathcal{S}}(K, L) = n$ and assume $K' \in \mathcal{K}$ with the properties that K, K' are adjacent, $K \cap K' \subset H_1$ and $S(K, L) = \{H_1, \ldots, H_n\}$. Then $H_1 \in \mathcal{A}_F$ by Lemma 2.4.14, and $F \in \overline{K} \cap H_1$ implies $F \subset \overline{K} \cap \overline{K'}$. In particular we get $\overline{K'} \in \mathcal{K}_F$ and by induction there exists a gallery from K' to L in $\mathcal{K}_F$, so we are done. □

Definition 2.4.16. As in the proof of the last proposition, we will denote the length of a minimal gallery between two chambers K, L with $d_{\mathcal{S}}(K, L)$. When there is not chance of confusion with the metric on V, we will sometimes omit the index $\mathcal{S}$.

Lemma 2.4.17. *Let $F \in \mathcal{S}$, $K \in \mathcal{K}$. Then there exists a unique chamber $G \in \mathcal{K}_F$, such that $D_H(K) = D_H(G)$ for all $H \in \mathcal{A}_F$.*

Proof. We prove existence first. Let $K \in \mathcal{K}$ and consider the intersection

$$S := \bigcap_{H \in \mathcal{A}_F} \overline{D_H(K)}.$$

Since $F \subset H$ for every $H \in \mathcal{A}_F$, we have $F \subset S$. Now $\overline{D_H(K)}$ is a union of closed chambers for arbitrary $K \in \mathcal{K}, H \in \mathcal{A}$, therefore S is a union of closed chambers. Let $x \in F$, U a neighbourhood of x such that $\sec(U \cap T) \subset \mathcal{A}_F$. Let $y \in K$ and consider the segment $[x,y]$. Then $[x,y] \cap (U \setminus \{x\}) \neq \emptyset$ and $[x,y]$ is not contained in any hyperplane, as $y \in K$. Hence $[x,y] \cap (U \setminus \{x\})$ is contained in chamber G, which satisfies $G \subset S$ and $F \subset \overline{G}$.

So let G, G' be two chambers in $\mathcal{K}_F$ such that $G, G' \subset S$. Then there exists an element $H \in \mathcal{A}$ such that H separates G and G'. Then Lemma 2.4.14 implies $H \in \mathcal{A}_F$, but then by definition of S we have $D_H(G) = D_H(K) = D_H(G')$, a contradiction. □

Proposition 2.4.18. *The chamber complex $\mathcal{S}$ is gated.*

Proof. Let $F \in \mathcal{S}$ be a simplex, $K \in \mathcal{K}$ and $S := \bigcap_{H \in \mathcal{A}_F} D_H(K)$. Then by Lemma 2.4.17 there exists a unique chamber $G_K \in \mathcal{K}_F$ with $G_K \subset S$. We will prove that G_K is a gate of K on $\mathrm{Cham}(St(F))$. So let $L \in \mathcal{K}_F$. If $H \in \mathcal{A}$ separates G_K and L, we obtain from Lemma 2.4.14 that $H \in \mathcal{A}_F$. On the other hand we get that if $H' \in \mathcal{A}$ separates G_K and K, by construction of G_K we find $H' \notin \mathcal{A}_F$. Now assume $H \in S(K,L) \cap \mathcal{A}_F$, we find that since $H \in \mathcal{A}_F$, H does not separate G_K and K and therefore must separate G_K and L. Assume on the other hand that $H \in S(K,L) \cap (\mathcal{A} \setminus \mathcal{A}_F)$, then it cannot separate L and G_K, and therefore must separate G_K and K. Summarized this yields

$$\begin{aligned} S(K,L) &= (S(K,L) \cap \mathcal{A}_F) \,\dot\cup\, (S(K,L) \cap (\mathcal{A} \setminus \mathcal{A}_F)) \\ &= S(K,G_K) \,\dot\cup\, S(G_K,L). \end{aligned}$$

We thus find for all $L \in \mathcal{K}_F$ that $d_{\mathcal{S}}(K,L) = d_{\mathcal{S}}(K,G_K) + d_{\mathcal{S}}(G_K,L)$. So G_K is indeed a gate for K on $\mathrm{St}(F)$. □

Remark 2.4.19. The above proposition is actually true for (not necessarily simplicial) locally finite hyperplane arrangements, in a slightly different language. Since an arbitrary locally finite hyperplane arrangement $(\mathcal{A},T)$ does not yield a simplicial complex $\mathcal{S}$, take the chamber graph Γ instead, whose vertices are $\mathcal{K}$, and two vertices K, L are contained in an edge if K and L are adjacent. This is a metric space with respect to the graph distance. One can show that for $x \in \overline{T}$ the subsets $\mathcal{K}_x = \{K \in \mathcal{K} \mid x \in \overline{K}\}$ are connected and gated.

We now make use of Lemma 1.2.15 in Chapter 1. Combined with Theorem 1.2.14, we obtain the following theorem.

Theorem 2.4.20. *The complex $\mathcal{S}$ has a type function. In particular, if we have a type function τ_K of a closed chamber $\overline{K}$, this extends uniquely to a type function of $\mathcal{S}$.*

Remark 2.4.21. The construction of the weak type function is actually quite simple. Begin with a chamber K and consider a type function τ of $\overline{K}$. Let L be adjacent to K such that $F = \overline{K} \cap \overline{L}$. Then set $\tau_L|_{\underline{F}} = \tau|_{\underline{F}}$. Let $i \in I$ be the unique index such that $i \notin \tau(F)$, then τ maps the vertex not contained in F to i, so τ_L must map the vertex v in $\overline{L}$ not contained in F to i as well. So as every simplex $S \neq \emptyset$ is either contained in F or contains v, if S is contained in F then $\tau_L(S)$ is already defined, if it contains v set

$\tau_L(S) = \tau(S \cap F) \cup \{i\}$. One can check that τ_L is a morphism of chamber complexes, and furthermore τ_L is the only possible type function of $\underline{\overline{L}}$ satisfying $\tau_L|_{\underline{F}} = \tau|_{\underline{F}}$.

In this way we can inductively construct type functions for all chambers with arbitrary distance to K. This construction works always, however being well defined arises as a problem: Given a chamber L with $d_{\mathcal{S}}(K, L) = n \geq 2$, there may be two chambers K_1, K_2 with $d_{\mathcal{S}}(K, K_1) = d_{\mathcal{S}}(K, K_2) = n - 1$ and K_1, K_2 adjacent to L. Then L has induced type functions from K_1 as well as from K_2. Now Theorem 2.4.20 yields that these two induced type functions coincide, and thus the method gives us a weak type function of $\mathcal{S}$.

To complete the proof of Proposition 2.3.27, we also need the following observation.

Lemma 2.4.22. *The simplicial complex $\mathcal{S}$ is thin (resp. spherical) if and only if the simplicial hyperplane arrangement $(\mathcal{A}, T)$ is thin (resp. spherical).*

Proof. Thin: Since $V \setminus H$ has two connected components, the complex $\mathcal{S}$ is meager. It is thin if and only if for every chamber C and every wall $H \in W^C$ there exists a chamber C_H which is H-adjacent to C. In this case $\overline{C} \cap \overline{C_H} \subset T$, since T is convex. Then $\overline{C} \cap \overline{C_H} \subset H$, therefore H meets T and is contained in $\mathcal{A}$.

Spherical: Assume $(\mathcal{A}, T)$ is spherical, then $T = V$ by definition and $\mathcal{A}$ is finite. Hence also $\mathcal{K}$ is finite and $\mathcal{S}$ is thin, and therefore spherical.

Let $\mathcal{S}$ be spherical. By definition we find two opposite chambers C and C'. As $\mathcal{S}$ is meagre by construction, we also know $\mathcal{K} = \sigma(C, C')$. In particular $S(C, C') = \mathcal{A}$, and $\mathcal{K}$ is finite as well as $\mathcal{A}$.

Assume C has no i-adjacent chamber for an $i \in I$, then $\mathrm{proj}_{\mathcal{R}_i(C)}(D) = C$ for all $D \in \mathcal{K}$, a contradction. Thus $\mathcal{S}$ is also thin. By our previous argument therefore $(\mathcal{A}, T)$ is thin.

Let $x \in \partial T$. Since T can not be written as a finite union of hyperplanes, we can assume that $x \notin H$ for all $H \in \mathcal{A}$. Thus take a neighbourhood U of x and consider the chambers intersecting U. By taking a smaller U we can also assume that only a single chamber D intersects U. Thus there exists a wall of D not meeting T, a contradiction to $\mathcal{A}$ being thin. Hence ∂T is empty and $T = V$ holds. $\square$

We end this section by showing that simplicial arrangements are also examples of totally gated chamber complexes, even though we will not use this property.

Lemma 2.4.23. *Let $F \in \mathcal{S}$, $H \in \mathcal{A}$. Then either $F \subset H$, or $D_H(K) = D_H(K')$ for all $K, K' \in \mathcal{K}_F$.*

Proof. This is a consequence of Lemma 2.4.14, which yields that if $D_H(K) = D_H(K')$ for some $K, K' \in \mathcal{K}_F$, we find $F \subset H$.

Now assume both statements hold, so let $F \subset H$ and $D_H(K) = D_H(K')$ for some $K, K' \in \mathcal{K}_F$. Since there exists a chamber $K \in \mathcal{K}_F$, such that $H \in W^K$, we obtain from the construction of the gate of the set $\mathrm{Cham}(\mathrm{St}(F))$ in Lemma 2.4.17 that $D_H(K) = D_H(K')$ for all $K' \in \mathcal{K}$. But then $-D_H(K) \cap T =$, a contradiction. $\square$

Theorem 2.4.24. *The complex $\mathcal{S}$ is totally gated.*

Proof. Let $\mathcal{R}$, $\mathcal{R}'$ be residues of $\mathcal{S}$, and F, F' be simplices such that $\mathcal{R} = \mathrm{Cham}(\mathrm{St}(F))$, $\mathcal{R}' = \mathrm{Cham}(\mathrm{St}(F'))$. Then define $A_1 := \{H \in \mathcal{A}_{F'} \mid D_H(\mathcal{R}) \text{ is well defined }\}$ and $A_2 := \mathcal{A}_{F'} \setminus A_1$. By 2.4.23 we obtain that for $H \in \mathcal{A}_{F'}$ we have $F \in H$ if and only if $H \in A_2$. Let $\overline{C} \in \mathcal{R}$, $\overline{K}' = \mathrm{proj}_{\mathcal{R}'}(\overline{K})$, then by construction $\mathrm{proj}_{\mathcal{R}'}(\mathcal{R}) \subset D_H(K')$ for all $H \in A_1$.

Define $F'' := \overline{K'} \cap \bigcap_{H \in A_2} H$. By construction $F'' \in \mathcal{S}$ and in particular $F'' \in \mathrm{St}(F')$. Let $\mathcal{R}'' := \mathrm{Cham}(\mathrm{St}(F''))$, then we obtain $\mathcal{R}'' \subset \mathcal{R}'$. Since $F'' \subset \overline{K'}$, we find $F'' \subset \overline{D_H(\mathcal{R})}$ for all $H \in A_1$. Since $\mathrm{proj}_{\mathcal{R}'}(\mathcal{R}) \subset D_H(K')$, we find $\mathrm{proj}_{\mathcal{R}'}(\mathcal{R}) \subset \bigcap_{H \in A_1} D_H(K')$. Furthermore $K \in \mathcal{R}'$ and $K \subset D_H(K')$ for all $H \in A_1$ implies $F'' \subset \overline{K}$ and hence $K \in \mathcal{R}''$, we obtain $\mathrm{proj}_{\mathcal{R}'}(\mathcal{R}) \subset \mathcal{R}''$.

So let $L \in \mathcal{R}''$, and define $A_1^L := W^L \cap \mathcal{A}_{F'} \cap A_1$, $A_2^L := W^L \cap \mathcal{A}_{F'} \cap A_2$. Then $L \in D_H(\mathcal{R})$ for all $H \in A_1^L$. Furthermore $\bigcap_{H \in A_2^L} H$ contains F, and $\bigcap_{H \in A_2^L} D_H(L)$ contains a chamber in $\mathcal{R}$. Thus

$$\bigcap_{H \in W^L \cap \mathcal{A}_{F'}} D_H(L)$$

contains a chamber in $\mathcal{R}$, and $L \in \mathrm{proj}_{\mathcal{R}'}(\mathcal{R})$, as required. $\square$

2.5 The crystallographic property

2.5.1 Crystallographic arrangements

With respect to Lemma 2.3.17 we can make the following definition:

Definition 2.5.1. We call a simplicial arrangement $(\mathcal{A}, T, R)$ a *crystallographic arrangement*, if it satisfies

$$R \subset \pm \sum_{\alpha \in B^K} \mathbb{N}_0 \alpha$$

for all $K \in \mathcal{K}$.

Note that when talking about B^K we implicitly also fix a reductor ρ. However, in the crystallographic case there is not much choice for ρ, as the following statement shows:

Lemma 2.5.2. *If $(\mathcal{A}, T, R)$ is a crystallographic arrangement, the map $\rho : R \to R$ which maps α to the shortest element in $\langle \alpha \rangle \cap R$ is a well defined reductor, and R^{red} is reduced with respect to ρ.*

Proof. As the elements in B^K are reduced, assume $\alpha \in B^K$ and $\lambda\alpha \in R$ for $0 < \lambda < 1$. Since B^K is a basis, $\lambda\alpha \notin \mathbb{N}\alpha$. Therefore for $\alpha \in R$ a minimal element in $\langle\alpha\rangle \cap R$ must exist, and $\rho(\alpha)$ must be this element. □

From now on, let $(\mathcal{A}, T, R)$ be a crystallographic arrangement.

We will now take a closer look at the relations between the bases of adjacent chambers. Again the proof follows [Cun11, Lemma 2.8] closely.

Lemma 2.5.3. *Let $K, L \in \mathcal{K}$ be adjacent chambers. Assume $\overline{K} \cap \overline{L} \subset \alpha_1^\perp$ for $\alpha_1 \in B^K$. If $\beta \in B^L$, then either*

i) $\beta = -\alpha_1$ or

ii) $\beta \in \sum_{\alpha \in B^K} \mathbb{N}_0\alpha$.

holds.

Proof. Since $(\mathcal{A}, T, R)$ is crystallographic, by Lemma 2.3.17 we can assume β is either as in case ii) or $-\beta = \sum_{\alpha \in B^c} \lambda_\alpha \alpha$ with $\lambda_\alpha \in \mathbb{N}_0$. Using that for arbitrary $\varphi, \psi \in V^*$ we have $\overline{\varphi^+} \cap \overline{\psi^+} \subseteq \overline{(\varphi+\psi)^+}$, we get $\overline{K} = \bigcap_{\alpha \in B^K} \overline{\alpha^+} \subset \overline{(-\beta)^+}$. We also observe that $\overline{L} \subset \overline{\beta^+}$, therefore we get $\overline{K} \cap \overline{L} \subset \overline{\beta^+} \cap \overline{(-\beta)^+} = \beta^\perp$.

By choice of α_1 we find $\alpha_1^\perp = \beta^\perp$ and $\beta = -\alpha_1$. □

Definition and Remark 2.5.4. Assume for $K \in \mathcal{K}$ that B^K is indexed in some way, i.e. $B^K = \{\alpha_1, \dots, \alpha_r\}$. For any set I, define the map $\kappa_I : \mathcal{P}(I) \to \mathcal{P}(I)$ by $\kappa_I(J) = I \setminus J$. Set $\kappa := \kappa_{\{1,\dots,r\}}$.

For every simplex $F \subset \overline{K}$ there exists a description of the form $F = \overline{K} \cap \bigcap_{\alpha \in B_F} \alpha^\perp$ for some $B_F \subset B^K$ by Proposition 2.2.7, which gives an index set $J_F = \{i \mid \alpha_i \in B_F\}$.

Finally this gives rise to a type function of $\overline{K}$ in $\mathcal{S}$, by taking the map $\tau_{\overline{K}} : F \mapsto \kappa(J_F)$. By Theorem 2.4.20 the map $\tau_{\overline{F}}$ yields a unique type function τ of the whole simplicial complex $\mathcal{S}$. So let $L \in \mathcal{K}$ be another chamber, then the restriction $\tau|_{\overline{L}}$ is a type function of $\overline{L}$ as well. Assume $B^L = \{\beta_1, \dots, \beta_r\}$, this yields a second type function of $\overline{L}$ in the same way we acquired a type function of $\overline{K}$ before,

$$\tau_{\overline{L}} : F \mapsto \kappa(\{i \mid F \subset \beta_i^\perp\}).$$

We now call the indexing of B^L *compatible with* B^K, if $\tau_{\overline{L}} = \tau|_{\overline{L}}$.

Note that since the type function τ is unique, there is a unique indexing of B^L compatible with B^K.

The following proposition 2.5.6 can also be found in [Cun11], however this can be modified by the following Lemma, which introduces compatibility in the above sense into the argument.

Lemma 2.5.5. *Let $(\mathcal{A}, T, R)$ be a simplicial arrangement. Let $K, L \in \mathcal{K}$ be adjacent chambers and choose an indexing $B^K = \{\alpha_1, \dots, \alpha_r\}$. Let the indexing of $B^L = \{\beta_1, \dots, \beta_r\}$ be compatible with B^K. Assume $\overline{K} \cap \overline{L} \subset \alpha_k^\perp$ for some $1 \le k \le r$. Then $\beta_i \in \langle \alpha_i, \alpha_k \rangle$.*

Proof. If $i = k$, this is immediate from Lemma 2.5.3. So let $i \neq k$ and assume w.l.o.g. $k = 1$.

Consider the type function $\tau_{\overline{K}}$ of $\overline{K}$ and $\tau_{\overline{L}}$ of $\overline{L}$ as restrictions of the unique type function τ. Being compatible yields that $\tau(\overline{K} \cap \overline{L}) = \tau(\overline{K} \cap \alpha_1^\perp) = \{2, \dots, r\}$.

Then $\tau(\overline{K} \cap \overline{L} \cap \alpha_i^\perp) = \{2, \dots, r\} \setminus \{i\} = \tau(\overline{K} \cap \overline{L} \cap \beta_i^\perp)$ holds and we get $\tau(\overline{K} \cap \alpha_i^\perp \cap \alpha_1^\perp) = \{2, \dots, n\} \setminus \{i\} = \tau(\overline{d} \cap \beta_i^\perp \cap \beta_1^\perp)$. We conclude that $\beta_i^\perp \cap \beta_1^\perp = \beta_i^\perp \cap \beta_1^\perp$, and as $\alpha_1 = -\beta_1$ we find $\beta_i^\perp \cap \alpha_1^\perp = \alpha_i^\perp \cap \alpha_1^\perp$.

Then we find $\langle \beta_i, \alpha_1 \rangle = \langle \alpha_i, \alpha_1 \rangle$, so β_i is a linear combination of α_1 and α_i, which proves our claim. $\square$

Proposition 2.5.6. *Let $(\mathcal{A}, T, R)$ be a crystallographic arrangement, $K, L \in \mathcal{K}$ be adjacent chambers. Choose an indexing $B^K = \{\alpha_1, \dots, \alpha_r\}$ and let the indexing of $B^L = \{\beta_1, \dots, \beta_r\}$ be compatible with B^K. Assume $\overline{K} \cap \overline{L} \subset \alpha_k^\perp$ for some $1 \le k \le r$.*

Then there exist $c_i \in \mathbb{Z}$ for $i = 1, \dots, r$ such that $\beta_i = c_i \alpha_k + \alpha_i$. Furthermore, $c_k = -2$ and $c_i \in \mathbb{N}_0$ for $i \neq k$.

Proof. Without loss of generality we can assume $k = 1$, then $\beta_1 = -\alpha_1 = \alpha_1 - 2\alpha_1$. Consider the linear transformation σ, mapping r_i to s_i. This is an element in $\mathrm{GL}_r(\mathbb{R})$ with entries in $\mathbb{Z}$, since the arrangement is crystallographic and B^K, B^L are bases. By symmetry the inverse has also entries in $\mathbb{Z}$, so $\sigma \in \mathrm{GL}_r(\mathbb{Z})$ holds. Hence the matrix of σ with respect to bases B^K and B^L is

$$\begin{pmatrix} -1 & c_2 & \dots & c_r \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix},$$

where by Lemma 2.5.3 the c_i are in $\mathbb{N}$ and $A \in GL_{r-1}(\mathbb{Z})$ with nonnegative entries. Since the matrix of σ^{-1} is of the same form, $A^{-1} \in \mathrm{GL}_{r-1}(\mathbb{Z})$ with nonnegative entries. It is well known (see Theorem 4.6 in Chapter 3 in [BP79] for example) that this implies that A is monomial, and since its entries are in $\mathbb{Z}$, A is a permutation matrix. We know therefore that $\beta_i = \sigma(\alpha_i) = c_i \alpha_1 + \alpha_{\pi(i)}$ for some permutation π. It remains to show that A is in fact the identity matrix. This is a consequence of Lemma 2.5.5, as $\beta_k = \lambda_1 \alpha_1 + \lambda_k \alpha k$ for $\lambda_1, \lambda_k \in \mathbb{R}$. Using that B^K, B^L are bases we find $\pi = \mathrm{id}_{\{1,\dots,r\}}$. $\square$

2.5.2 Cartan graphs and Weyl groupoids

We recall the notion of a Weyl groupoid which was introduced by Heckenberger and Yamane [HY08] and reformulated in [CH09b].

Definition 2.5.7. Let $I := \{1, \dots, r\}$ and $\{\alpha_i \,|\, i \in I\}$ the standard basis of $\mathbb{Z}^I$. A *generalised Cartan matrix* $C = (c_{ij})_{i,j \in I}$ is a matrix in $\mathbb{Z}^{I \times I}$ such that

(M1) $c_{ii} = 2$ and $c_{jk} \leq 0$ for all $i, j, k \in I$ with $j \neq k$,

(M2) if $i, j \in I$ and $c_{ij} = 0$, then $c_{ji} = 0$.

Definition 2.5.8. Let A be a non-empty set, $\rho_i : A \to A$ a map for all $i \in I$, and $C^a = (c^a_{ij})_{i,j \in I}$ a generalised Cartan matrix in $\mathbb{Z}^{I \times I}$ for all $a \in A$. The quadruple

$$\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$$

is called a *Cartan graph* if

(C1) $\rho_i^2 = \mathrm{id}$ for all $i \in I$,

(C2) $c^a_{ij} = c^{\rho_i(a)}_{ij}$ for all $a \in A$ and $i, j \in I$.

Definition 2.5.9. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$ be a Cartan graph. For all $i \in I$ and $a \in A$ define $\sigma^a_i \in \mathrm{Aut}(\mathbb{Z}^I)$ by

$$\sigma^a_i(\alpha_j) = \alpha_j - c^a_{ij}\alpha_i \qquad \text{for all } j \in I. \tag{2.1}$$

The *Weyl groupoid of* $\mathcal{C}$ is the category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are compositions of maps σ^a_i with $i \in I$ and $a \in A$, where σ^a_i is considered as an element in $\mathrm{Hom}(a, \rho_i(a))$. The cardinality of I is the *rank of* $\mathcal{W}(\mathcal{C})$.

Definition 2.5.10. A Cartan graph is called *standard*, if $C^a = C^b$ for all $a, b \in A$. The Cartan graph is called *connected* if its Weyl groupoid is connected, that is, if for all $a, b \in A$ there exists $w \in \mathrm{Hom}(a, b)$. The Cartan graph is called *simply connected*, if $\mathrm{Hom}(a, a) = \{\mathrm{id}^a\}$ for all $a \in A$. There is a straight forward notion of *equivalence* of Cartan graphs which we skip here.

Let $\mathcal{C}$ be a Cartan graph. For all $a \in A$ let

$$(R^{\mathrm{re}})^a = \{\mathrm{id}^a \sigma_{i_1} \cdots \sigma_{i_k}(\alpha_j) \,|\, k \in \mathbb{N}_0,\, i_1, \dots, i_k, j \in I\} \subseteq \mathbb{Z}^I.$$

The elements of the set $(R^{\mathrm{re}})^a$ are called *real roots* (at a). The pair $(\mathcal{C}, ((R^{\mathrm{re}})^a)_{a \in A})$ is denoted by $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$. A real root $\alpha \in (R^{\mathrm{re}})^a$, where $a \in A$, is called positive (resp. negative) if $\alpha \in \mathbb{N}_0^I$ (resp. $\alpha \in -\mathbb{N}_0^I$).

Definition 2.5.11. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan graph. For all $a \in A$ let $R^a \subseteq \mathbb{Z}^I$, and define $m^a_{i,j} = |R^a \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$ for all $i, j \in I$ and $a \in A$. We say that

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$$

is a *root system of type* $\mathcal{C}$, if it satisfies the following axioms.

(R1) $R^a = R^a_+ \cup -R^a_+$, where $R^a_+ = R^a \cap \mathbb{N}_0^I$, for all $a \in A$.

(R2) $R^a \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i \in I$, $a \in A$.

(R3) $\sigma^a_i(R^a) = R^{\rho_i(a)}$ for all $i \in I$, $a \in A$.

(R4) If $i, j \in I$ and $a \in A$ such that $i \neq j$ and $m^a_{i,j}$ is finite, then $(\rho_i\rho_j)^{m^a_{i,j}}(a) = a$.

The root system $\mathcal{R}$ is called *finite* if for all $a \in A$ the set R^a is finite. By [CH09b, Prop. 2.12], if $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then $\mathcal{R} = \mathcal{R}^{\mathrm{re}}$, and hence $\mathcal{R}^{\mathrm{re}}$ is a root system of type $\mathcal{C}$ in that case. Roots which are not real roots are called *imaginary roots.*

Remark 2.5.12. If $\mathcal{C}$ is a Cartan graph and there exists a root system of type $\mathcal{C}$, then $\mathcal{C}$ satisfies

(C3) If $a, b \in A$ and $\mathrm{id} \in \mathrm{Hom}(a, b)$, then $a = b$.

We recall the notion of parabolic subgroupoids and introduce the definition of residues, which is more suited for our purposes.

Definition 2.5.13. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan graph with Weyl groupoid $\mathcal{W}(\mathcal{C})$. For $J \subseteq I$, the *parabolic subgroupoid* $\mathcal{W}_J(\mathcal{C})$ is the category with the same objects as $\mathcal{W}(\mathcal{C})$ and with morphisms generated by σ^a_j, where $a \in A$ and $j \in J$.

For $a \in A$, let C^a_J be the restriction of C^a to the indices J, i.e. $C^a_J := (c^a_{ij})_{i,j\in J}$.

The J*-residue (containing* a*)* is a connected component of $\mathcal{W}_J(\mathcal{C})$ (containing a), and a subgroupoid $\mathcal{S}$ of $\mathcal{W}(\mathcal{C})$ is called a residue, if it is a J-residue for some $J \subset I$.

Remark 2.5.14. With notation as above, we immediately find the following:

1. $\mathcal{W}(\mathcal{C}) = \mathcal{W}_I(\mathcal{C})$,
2. $\mathcal{W}(\mathcal{C})$ is connected if and only if $\mathcal{W}(\mathcal{C})$ is an I-residue,
3. $\mathcal{W}_J(\mathcal{C})$ is the disjoint union of the J-residues of $\mathcal{W}(\mathcal{C})$.

We find the following as an almost immediate consequence of the definition and has been noted by Heckenberger and Welker before, for a proof see [HW11, Proposition 2.11]:

Proposition 2.5.15. *Let* $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ *be a Cartan graph,* $J \subset I$*. Define*

$$\mathcal{C}_J := (J, A, (\rho_j)_{j\in J}, (C^a_J)_{a\in A}).$$

Then $\mathcal{C}_J$ *is a Cartan graph and* $\mathcal{W}(\mathcal{C}_J)$ *is canonically isomorphic to* $\mathcal{W}_J(\mathcal{C})$*.*

Corollary 2.5.16. *With notation as above, let $b \in A$, $\Pi_J = \langle \rho_j \mid j \in J \rangle \leq \mathrm{Sym}(A)$. Define*

$$\mathcal{C}_J^b := (J, \Pi_J(b), (\rho_j)_{j\in J}, (C_J^a)_{a\in \Pi_J(b)}).$$

Then $\mathcal{C}_J^b$ is a connected Cartan graph and $\mathcal{W}(\mathcal{C}_J^b)$ is canonically isomorphic to the J-residue of $\mathcal{W}(\mathcal{C})$ containing b.

Furthermore, if $\mathcal{C}$ is simply connected, then $\mathcal{C}_J^b$ is simply connected.

2.5.3 Cartan graphs and crystallographic arrangements

In this section we will point out one part of the correspondence between reduced root systems of crystallographic arrangements and real root systems of a Cartan graph.

Proposition 2.5.6 allows us to make the following definition:

Definition 2.5.17. Let $(\mathcal{A}, T, R)$ be a crystallographic arrangement. Let K be a chamber and $B^K = \{\alpha_1, \dots, \alpha_r\}$. Let K^i be the chamber i-adjacent to K, i.e. $\overline{K} \cap \overline{K^i} \subset \alpha_i^\perp$. Let $B^{K^j} = \{\beta_1^j, \dots, \beta_r^j\}$ be indexed compatibly, then by Proposition 2.5.6 we find that $\beta_i^j = c_i^j \alpha_j + \alpha_i$ with $c_i^j \in \mathbb{N}$ for $i \neq j$ and $c_i^i = -2$. We will call the matrix $C^K = (-c_i^j)_{1\leq i,j\leq r}$ the *Cartan matrix at K*.

Furthermore, in the above setting we denote by $\varphi_{K^i,K} \in \mathrm{End}_{\mathbb{R}}(V^*)$ the linear extension of the map $\alpha_j \mapsto \beta_j^i$ for all $j = 1, \dots, r$. We will also think of $\mathbb{Z}^r$ as a subset of $\mathbb{R}^r$ and hence interpret the maps $\sigma_i^K : \mathbb{Z}^r \to \mathbb{Z}^r$ arising in the definition of the Weyl groupoid as maps in $\mathrm{End}(\mathbb{R}^r)$.

The following lemma shows that the notion of a Cartan matrix at a chamber K is justified, as it is indeed a generalised Cartan matrix. For the sake of brevity, we omit the fact that it should be called the "generalised Cartan matrix at a chamber". In this work we won't define what a (non-generalised) Cartan matrix is.

Lemma 2.5.18. *Let $K \in \mathcal{K}$, C^K the Cartan matrix at K. Then C^K is a generalised Cartan matrix.*

Proof. The matrix C^K satisfies (M1) from the definition by Proposition 2.5.6. So assume $c_i^j = 0$. This implies $\beta_i^j = \alpha_i$. By construction $\beta_j^i = c_j^i \alpha_i + \alpha_j$. Assume $c_j^i > 0$ and let $v = -c_j^i \alpha_j^\vee + \alpha_i^\vee \in V$, where $\{\alpha_i^\vee \mid i \in I\}$ is the dual basis to B. Then $\beta_i^j(v) = \alpha_i(v) = 1$, $\beta_j^j(v) = -\alpha_j(v) = c_j^i$ and therefore $\beta_j^i(v) = 0$. The last equality means $v \in (\beta_j^i)^\perp$, which contradicts the simplicial structure of $\mathcal{S}$. So (M2) holds. □

Proposition 2.5.19. *Let $(\mathcal{A}, T, R)$ be a crystallographic arrangement and assume the B^K are indexed compatibly for all $K \in \mathcal{K}$. Set $I := \{1, \dots, r\}$, $A := \mathcal{K}$, C^K the generalised Cartan matrix at K, and for $i \in I$ let $\rho_i : A \to A$, $K \mapsto K^i$, where K^i is the chamber i-adjacent to K.*

Then $\mathcal{C} := \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ is a connected Cartan graph.

Proof. For $K \in \mathcal{K}$ by Lemma 2.5.18 the Cartan matrix C^K at K is a generalised Cartan matrix.

The maps ρ_i are well defined, as for $K \in \mathcal{K}$, there exists by Proposition 2.3.27 a unique chamber K^i which is i-adjacent to K, since $\mathcal{S}(\mathcal{A}, T)$ is thin. Since K is then also i-adjacent to K^i, as the indexing of the root basis is compatible, ρ_i is an involution. Thus $\mathcal{C}$ satisfies (C1).

It remains to check (C2). So let K, L be i-adjacent with $B^K = \{\alpha_1, \dots, \alpha_r\}$, $B^L = \{\beta_1, \dots, \beta_r\}$. So we find $\rho_i(K) = L$. If $i = j$, $c_{ii}^K = 2 = c_{jj}^L$, so assume $i \neq j$.

We find the i,j-th entry of C^K to be the number $-c$ such that $\beta_j = c\alpha_i + \alpha_j$. The i,j-th entry of C^{K^i} is the number $-d$ defined by $\alpha_j = d\beta_i + \beta_j$. We obtain $\alpha_j = -d\alpha_i + c\alpha_i + \alpha_j$, and therefore $c = d$ by using the linear independence of α_i, α_j. Therefore $\mathcal{C}$ satisfies (C2) and is a Cartan graph.

The Cartan graph $\mathcal{C}$ is connected: Since $\mathcal{S}$ is a chamber complex, we can find a gallery between two chambers K and K'. Let $(K = K_0, \dots, K_m = K')$ be such a gallery. Assume that K_{j-1} and K_j are i_j-adjacent. Then the map $\sigma_{i_j}^{K_{m-1}} \cdots \sigma_{i_1}^{K_0}$ is in $\mathrm{Hom}(K_0, K_m) = \mathrm{Hom}(K, K')$. □

Definition 2.5.20. Given a crystallographic arrangement $(\mathcal{A}, T, R)$, we will denote the Cartan graph defined in Proposition 2.5.19 by $\mathcal{C}(\mathcal{A}, T, R)$.

Let $K \in \mathcal{K}$, then $\phi_K : V^* \to \mathbb{R}^r$ denotes the coordinate map of V^* with respect to the basis B^K. As R is crystallographic, we find $\phi_K(R) \subset \mathbb{Z}^r \subset \mathbb{R}^r$.

Furthermore let $R^K := \phi_K(R)$ for $K \in \mathcal{K}$.

Remark 2.5.21. From Lemma 2.3.25 it follows that instead of knowing T, it is in fact enough to know a single chamber $K_0 \in \mathcal{K}$ together with its root basis, as every chamber in $\mathcal{K}$ can be obtained from K_0 by repeatedly looking at adjacent chambers.

Lemma 2.5.22. *Let K, K' be i-adjacent chambers. Then the identities*

$$\phi_{K'} \circ \varphi_{K',K} = \phi_K \qquad \textit{and}$$
$$\phi_{K'} = \sigma_i^K \circ \phi_K$$

hold.

Proof. This is a straight forward calculation using the definition of ϕ_k and σ_i^K. □

Using induction on the above expressions immediately yields the following statement.

Corollary 2.5.23. *Let K_0, K_m be arbitrary chambers, $(K_0, \dots, K_m)$ be a gallery such that K_{j-1}, K_j are i_j adjacent. Then*

$$\phi_{K_m} \circ (\varphi_{K_m, K_{m-1}} \cdots \varphi_{K_1, K_0}) = \phi_{K_0} \qquad \textit{and}$$
$$\phi_{K_m} = (\sigma_{i_m}^{K_{m-1}} \cdots \sigma_{i_1}^{K_0}) \circ \phi_{K_0}$$

holds.

Proposition 2.5.24. *The Cartan graph $\mathcal{C}(\mathcal{A},T,R)$ is simply connected.*

Proof. Let $K \in \mathcal{K}$ and $w \in \mathrm{Hom}(K,K)$ with $w = \mathrm{id}^K \sigma_{i_m^{K_{m-1}}} \cdots \sigma_{i_1^K}$. In particular $K = K_0, K_1, \dots, K_m = K$ is a gallery from K to K.

By Corollary 2.5.23 w is the identity on $\mathbb{Z}^r$. □

The following proposition uses results for arrangements at points from Section 2.7.

Proposition 2.5.25. *Let $(\mathcal{A},T,R)$ be a reduced crystallographic arrangement in $T \subset \mathbb{R}^r$, and $\mathcal{C} = \mathcal{C}(\mathcal{A},T,R)$. Then the sets R^K are exactly the real roots of $\mathcal{C}$ at K, and $\mathcal{R} = \mathcal{R}(\mathcal{C},(R^K)_{K\in\mathcal{K}})$ is a root system of type $\mathcal{C}$.*

Proof. We show that $\mathcal{R}$ is a root system of type $\mathcal{C}$. By Corollary 2.4.2 and the crystallographic property (R1) holds, and (R2) holds since R is reduced.

To show (R3), assume K, K' are i-adjacent. In particular this means $\rho_i(K) = K'$. Now

$$\sigma_i^K(R^K) = \sigma_i^K \phi_K(R) = \phi_{K'}(R) = R^{K'} = R^{\rho_i(K)}$$

by Corollary 2.5.23, so (R3) holds.

So let $i \neq j \in I$, and $K \in \mathcal{K}$, such that $m_{ij} := m_{ij}^K = |R^K \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$ is finite.

Assume $B^K = \{\beta_1,\dots,\beta_r\}$, then this term is equivalent to $|R \cap \langle\beta_i,\beta_j\rangle|$, as ϕ_K maps β_k to α_k for $k \in I$. Take the simplex $F \in \mathcal{S}$, $F \subset \overline{K}$, such that the type function of F is $\{i,j\}$, in particular F is a 2-simplex and $F = \alpha_i^\perp \cap \alpha_j^\perp \cap \overline{K}$. Take $x \in F$ such that $\mathrm{supp}\,(x) \cap W^K = \{\alpha_i^\perp, \alpha_j^\perp\}$, and consider the arrangement $(\mathcal{A}_x, R^x)$. Then $\mathcal{A}_x$, R^x has exactly m_{ij} elements, by Corollary 2.7.8 $(\mathcal{A}_x^\pi, R^x)$ is a spherical arrangement, and $\mathcal{K}^x$ consists of $2m_{ij}$ chambers. The induced simplicial complex has a unique induced type function by $\{i,j\}$. Therefore $(\rho_i\rho_j)^{m_{ij}}$ corresponds to a unique gallery $(K = K_0, K_1, \dots, K_n)$ of length $2m_{ij}$. Thus we obtain $K_n = K$ and $(\rho_i\rho_j)^{m_{ij}}(K) = K$.

It remains to show that R^K are actually the real roots at K. Since $\varphi_{K',K}$ maps roots to roots, by Corollary 2.5.23 $(R^{\mathrm{re}})^K \subset R^K$, so we need to check the other inclusion.

Let $\beta \in R$, and set $\beta_K = \phi_K(\beta)$. Let $K' \in \mathcal{K}_0$, such that $\beta \in B^{K'}$, and let $(K = K_0, K_1, \dots, K_m = K')$ be a gallery from K to K' with K_{i-1}, K_i being j_i-adjacent.

Let $\alpha \in B^K$ such that

$$\varphi_{K,K_1} \circ \cdots, \circ \varphi_{K_{m_1},K'}(\beta) = \alpha.$$

Then by Corollary 2.5.23

$$\beta_K = \phi_K(\beta) = \sigma_{j_1}^{K_1} \circ \cdots \circ \sigma_{j_m}^{K_m} \phi_{K'}(\beta),$$

where we used the fact that due to (C2) $\sigma_i^{K^i}\sigma_i^K = \mathrm{id}_{\mathbb{Z}^r}$, if K^i is i-adjacent to K. Now $\beta \in K'$ yields that $\phi_{K'}(\beta)$ is in the standard basis, which proves $\beta_K \in (R^{\mathrm{re}})^K$. Hence $R^K = (R^{\mathrm{re}})^K$, which proves our assumption. □

Remark 2.5.26. It is easy to see that combinatorially equivalent crystallographic arrangements $(\mathcal{A}, T, R)$ and $(\mathcal{A}', T', R')$ yield equivalent Cartan graphs $\mathcal{C}(\mathcal{A}, T, R)$ and $\mathcal{C}(\mathcal{A}', T', R')$. Choosing a different type function of the simplicial complex $\mathcal{S}$ also gives rise to equivalent Cartan graphs, which in fact only differ by a permutation of I.

2.5.4 The additive property

In this section we will discuss the additive property of root systems. Again we use some results for subarrangements, which we will prove later in Section 2.7.

Definition 2.5.27. Let $(\mathcal{A}, T, R)$ be a simplicial arrangement, and fix $K \in \mathcal{K}$. Set

$$R^+ := R \cap \sum_{\alpha \in B^K} \mathbb{R}_{\geq 0}\alpha,$$
$$R^- := R \cap \sum_{\alpha \in B^K} \mathbb{R}_{\leq 0}\alpha,$$

and call R^+ the *positive roots (w. r. t. K)* and R^- the *negative roots (w. r. t. K)*.

Lemma 2.5.28. *If $(\mathcal{A}, T, R)$ is a simplicial arrangement, then $R = R^+ \mathbin{\dot\cup} R^-$ for every $K \in \mathcal{K}$.*

Proof. Let $\alpha \in R$, then $\alpha \in R^+$ or $\alpha \in R^-$ by Lemma 2.3.17.

The sets R^+, R^- are disjoint since $\sum_{\alpha \in B^K} \mathbb{R}_{\geq 0}\alpha \cap \sum_{\alpha \in B^K} \mathbb{R}_{\leq 0}\alpha = \{0\}$, and $0 \notin R$. □

Definition 2.5.29. Let $(\mathcal{A}, T, R)$ be a simplicial arrangement in T, and let $K \in \mathcal{K}$. We say that B^K *satisfies the additive property*, or shorter that B^K *is additive*, if for $\alpha \in R^+$ either we find $\alpha \in B^K$ or $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1, \alpha_2 \in R^+$.

If B^K is additive for all $K \in \mathcal{K}$, then $(\mathcal{A}, T, R)$ is said to be *additive*.

Remark 2.5.30.

1. If $(\mathcal{A}, T, R)$ is additive, then $(\mathcal{A}, T, R)$ is also crystallographic. This is just a consequence from the definition.

2. In [CH11, Corollary 3.8] Cuntz and Heckenberger showed that spherical crystallographic arrangements in dimension 2 are additive. This statement is used in [CH12, Theorem 2.10] to show that every crystallographic spherical arrangement is additive, thus for spherical arrangements the additive property and the crystallographic property are equivalent. Note that both formulations above actually refer to Weyl groupoids.

3. An example of an affine crystallographic arrangement which is not additive in the above sense is the root system of $\tilde{A}_1$, which is

$$R(\tilde{A}_1) = \{\alpha_1 + k\gamma, \alpha_2 + k\gamma\},$$

where $\{\alpha_1, \alpha_2\}$ is a Basis of $(\mathbb{R}^2)^*$ and $\gamma = \alpha_1 + \alpha_2$. We find a chamber K such that $B^K = \{\alpha_1, \alpha_2\}$, but $2\alpha_1 + \alpha_2$ is neither in B^K nor a sum of two positive roots.

We will continue to give a criterion for a crystallographic arrangement to be additive. It can be seen in the above remark, that this can not hold in the general case. The idea for the proof of the following statement is based on [CH12, Theorem 2.10], but adapted to our notation.

Proposition 2.5.31. *Assume that $(\mathcal{A}, T, R)$ is a crystallographic arrangement in $T \subset \mathbb{R}^r$ with $r \geq 3$. If $(\mathcal{A}, T, R)$ is 2-spherical and R is reduced, then it is additive.*

Proof. Let $K_0 \in \mathcal{K}$ and $\beta \in R^+$ with respect to K_0. This yields that $K_0 \subset \beta^+$. Let $K \in \mathcal{K}$, such that $\beta \in B^K$ and assume $d(K_0, K) = m$. Fix a minimal gallery $\gamma = (K_0, K_1, \ldots, K_m = K)$. Let $B^K = \{\beta_1, \ldots, \beta_r\}$, where $\beta = \beta_1$. If $m = 0$, β is already in B^{K_0} and we are done.

So let $m \geq 1$ and assume $\beta \notin B^{K_0}$.

Assume further that K and K_{m-1} are i-adjacent. Since $D_{\beta^\perp}(K_0) = D_{\beta^\perp}(K)$ and a minimal gallery between K_0 and K can not cross $\beta^\perp$, we find $i \neq 1$. So let $F := \beta_1^\perp \cap \beta_i^\perp \cap \overline{K}$, then F is an $n-3$-simplex by construction.

As $(\mathcal{A}, T, R)$ is 2-spherical, $F \cap T$ is not empty. Let $x \in F \cap T$ such that $\operatorname{supp}(x) \cap W^K = \{\beta_1^\perp, \beta_i^\perp\}$.

For the following notation see Section 2.7. The set R_x is contained in $\langle \beta_1, \beta_i \rangle$, $\mathcal{K}_x$ corresponds to the chambers of $\operatorname{St}(F)$. Now $\operatorname{St}(F)$ is a gated set by Proposition 2.4.18, so let $G \in \mathcal{K}_x$ be the unique gate from K_0 to $\mathcal{K}_x$. Then $B^G \cap R_x = \{\alpha_1, \alpha_2\}$ for some $\alpha_1, \alpha_2 \in \langle \beta_1, \beta_i \rangle$. Let $H_i = \alpha_i^\perp$ for $i = 1, 2$. By construction of G we find $D_{H_i}(G) = D_{H_i}(K)$ for $i = 1, 2$ and therefore $K \subset \alpha_1^+ \cap \alpha_2^+$, by Corollary 2.4.2 the roots α_1, α_2 are positive with respect to K.

By Corollary 2.7.12 R_x itself is a crystallographic root system in dimension 2, and $\{\alpha_1, \alpha_2\}$ is a root basis. By construction $G \subset \beta^+$, as $\beta_1 \in R^x$, and again by Corollary 2.4.2 we obtain that β is a positive linear combination of α_1, α_2. From [CH11, Corollary 3.8] it follows that R_x is additive, so β is either in $\{\alpha_1, \alpha_2\}$ or sum of two positive roots α_1', α_2' in $R^+ \cap \langle \alpha_1, \alpha_2 \rangle$. In the latter case we are done, as α_1', α_2' are also positive with respect to K, since they are positive linear combinations of α_1, α_2, which are positive w. r. t. K.

So it remains to check that $\beta \neq \alpha_1, \alpha_2$. As G is the gate from K to $\mathcal{K}_x$, there exists an index $0 \leq j \leq m$ such that $K_j = G$ in the above gallery. Assume $\beta = \alpha_1$, then $\beta \in B^G$ and the minimality of the gallery yields $j = m$. But we assumed K_{m-1} and G are i-adjacent, which means that $K_{m-1} \in \mathcal{K}_x$, a contradiction to the gate property.

So $\beta \neq \alpha_1, \alpha_2$ and we are done. □

Remark 2.5.32. The conditions in Proposition 2.5.31 are necessary, as we pointed out in Remark 2.5.30. The statement however does not take into consideration imaginary roots, and might be true for a wider class of crystallographic arrangements, when taking into account the imaginary roots.

2.6 The geometric realisation of a connected simply connected Weyl groupoid

In the previous section, we constructed a connected simply connected Cartan graph from a given crystallographic simplicial arrangement. The aim of this section is to give a canonical crystallographic simplicial arrangement associated to a given connected simply connected Cartan graph with real root system.

For this chapter, assume $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ to be a connected simply connected Cartan graph of rank r with real roots $\mathcal{R}^{\mathrm{re}} = \mathcal{R}(\mathcal{C}, ((R^{\mathrm{re}})^a)_{a\in A})$, and fix some $a \in A$. Furthermore, assume that $\mathcal{R}^{\mathrm{re}}$ is a root system of type $\mathcal{C}$. When considering [CH09b, Proposition 2.9], this becomes equivalent to require the existence of a root system of type $\mathcal{C}$.

Definition 2.6.1. Let $V = \mathbb{R}^r$ and $B := \{\alpha_i \mid i \in I\}$ be the standard basis of $\mathbb{Z}^r$. Assume $\{\beta_i \mid i \in I\}$ is a basis of V^*. Let $\psi : \mathbb{Z}^r \to V^*$ be the unique $\mathbb{Z}$-linear map given by $\alpha_i \mapsto \beta_i$.

Define for some $a \in A$ the set $R := \psi((R^{\mathrm{re}})^a)$ and $\mathcal{A} := \{r^\perp \mid r \in R\}$. For $b \in A$ with $\mathrm{Hom}(a, b) = \{w\}$, define the map $\psi_b : (R^{\mathrm{re}})^b \to R$ as $\psi_b = \psi w^{-1}$. In particular we find $\psi_a = \psi|_{(R^{\mathrm{re}})^a}$

Further let

$$B^b := \psi_b(B)$$

for all $b \in A$.

Given B^b, set

$$K^b := \bigcap_{\beta\in B^b} \beta^+,$$

and let $\mathcal{K} = \{K^b \mid b \in A\}$. Note that B^b, K^b are defined (and well defined) for all $b \in A$, since $\mathcal{C}$ is connected (and simply connected).

There is a natural notion of the *walls of* K^b, which can be written as

$$W^b := \{\alpha^\perp \mid \alpha \in K^b\}.$$

Let $w \in \mathrm{Hom}(a, b)$ and $i \in I$. We call $K^b \neq K^{b'}$ *i-adjacent* if

$$\langle \overline{K^b} \cap \overline{K^{b'}} \rangle = \psi_b(\alpha_i)^\perp.$$

We say K^b and $K^{b'}$ are *adjacent* if they are i-adjacent for some $i \in I$.

Corollary 2.6.2. *With the above definitions, we find for arbitrary $b, b' \in A$ and $w \in \mathrm{Hom}(b, b')$ that*

$$\psi_b = \psi_{b'} w.$$

Lemma 2.6.3. *For $b \in A$, K^b is a simplicial cone, and H does not meet K^b for all $H \in \mathcal{A}$.*

Proof. The set K^b is a simplicial cone by Lemma 2.2.5, as the sets B^b are bases by construction. Let $w \in \mathrm{Hom}(a,b)$, then $w((R^{\mathrm{re}})^a) = (R^{\mathrm{re}})^b$, and we have $\psi_b(B) = B^b$ as well as $\psi_b((R^{\mathrm{re}})^b) = R$. Therefore $R \subset \pm \sum_{\alpha \in B^b} \mathbb{N}_0 \alpha$. By Corollary 2.4.2 we obtain for every $H \in \mathcal{A}$ that every vertex of an open simplex S such that $K^b = \mathbb{R}_{>0} S$, which is not contained in H, is on the same side of H. □

Lemma 2.6.4. *Let $b \in A$, $H \in W^b$, then $\overline{K^b} \cap H$ spans H.*

Proof. This follows from Lemma 2.2.5, as $\overline{K^b} \cap H$ contains $r-1$ vertices of the complex $\overline{K^b}$, which are linearly independent. □

We introduce notation for half spaces as in Section 2.4. The reason we introduce this notation again, is that it is not obvious that the notation as used before is well defined.

Definition and Remark 2.6.5. For $H \in \mathcal{A}$, Lemma 2.6.3 yields that every K^b is contained in a unique half space associated to H. We denote this halfspace by $D_H(K^b)$. For the half space not containing K^b we write $-D_H(K^b)$.

We say that H *separates* K^b and $K^{b'}$ for $b, b' \in A$ if $D_H(K^b) = -D_H(K^{b'})$, and set

$$S(K^b, K^{b'}) = \{H \in \mathcal{A} \mid D_H(K^b) = -D_H(K^{b'})\}.$$

Furthermore let T be the convex hull of all K^b, $b \in A$.

We will need the following characterisation of walls.

Lemma 2.6.6. *Assume $b \in A$ and let $H \subset V$ be a hyperplane. Then $H \in W^b$ if and only if $H \cap K^b = \emptyset$ and $\langle H \cap \overline{K^b} \rangle = H$.*

Proof. Assume $H \in W^b$ and let $\alpha \in B^b$ such that $\alpha^\perp = H$. Since $K^b \subset \alpha^+$, $K^b \cap H = \emptyset$. By definition of K^b the set $\overline{K^b} \cap H$ is not empty. Let S be a closed simplex such that $\overline{K^b} = \mathbb{R}_{>0} S \cup \{0\}$, by Lemma 2.2.5 it follows that there exists a maximal face F of S contained in H. But F has an $n-2$-dimensional affine space as its affine span, therefore its linear span is a hyperplane. Furthermore $F \subset H \cap \overline{K^b}$, hence we find $\langle H \cap \overline{K^b} \rangle = H$.

Now assume $H \cap K^b = \emptyset$ and $\langle H \cap \overline{K^b} \rangle = H$ both hold. The set K^b is a simplicial cone, from Lemma 2.2.5 we obtain that there exist elements $\beta_1, \dots, \beta_r \in V^*$ such that

$$\overline{K^b} = \bigcap_{i=1}^{r} \beta_i^+.$$

By using Lemma 2.4.1 we can assume $B^b = \{\beta_1, \dots, \beta_r\}$. Let S be as above, then we find a maximal face F of S such that $F \subset H$, but every face of S is contained in a unique hyperplane $\beta_i^\perp$, which proves our claim. □

Lemma 2.6.7. *The map $A \to \mathcal{K}$, $b \mapsto K^b$, is a bijection.*

Proof. As $\mathcal{R}^{\mathrm{re}}$ is a root system of type $\mathcal{C}$, it follows from [HY08, Lemma 8, (iii)], that $\mathcal{C}$ satisfies (C3), which implies the statement. □

Proposition 2.6.8. *Let $b, b' \in A$, $i \in I$, and let $S := \bigcap_{i \neq j \in I} (\psi_b(\alpha_j)^+ \cap \psi_{b'}(\alpha_j)^+)$. Then the following are equivalent:*

i) K^b *and* $K^{b'}$ *are* i*-adjacent,*

ii) $\rho_i(b) = b'$,

iii) $\forall K \in \mathcal{K} : (K \subset S \Leftrightarrow K \in \{K^b, K^{b'}\})$,

iv) $S(K^b, K^{b'}) = \{\psi_b(\alpha_i)^\perp\}$.

Proof. ii) $\implies$ iii),iv): Assume $\rho_i(b) = b'$, then we find $B^{b'} = \{\psi_b(\alpha_j - c^b_{ij}\alpha_i) \mid j \in I\}$, by definition we have $\psi_{b'}(\alpha_j) = \psi_b(\alpha_j - c^b_{ij}\alpha_i)$. Therefore we find $K^b \subset \psi_b(\alpha_j)^+$ for all $j \in I$, and hence also $K^b \subset \psi_{b'}(\alpha_j)^+$ for all $i \neq j \in I$. The analogue statement holds for $K^{b'}$, so we find $K^b, K^{b'} \subset S$. Now assume $K \in \mathcal{K}$, $K \subset S$, and consider the case $K \subset \psi_b(\alpha_i)^+$. Then we obtain $K \subset \bigcap_{i \in I} \psi_b(\alpha_i)^+$ and thus $K \subset K^b$. By Lemma 2.6.3 this already implies $K = K^b$. In the case $K \subset \psi_b(\alpha_i)^-$ we obtain in the same way $K = K^{b'}$. Thus iii) holds. This also implies $\psi_b(\alpha_i)^\perp \in S(K^b, K^{b'})$. From the above we obtain

$$\begin{aligned} S &= (S \cap \psi_b(\alpha_i)^+) \ \dot\cup\ (S \cap \psi_b(\alpha_i)^-) \ \dot\cup\ (S \cap \psi_b(\alpha_i)^\perp) \\ &= K^b \ \dot\cup\ K^{b'} \ \dot\cup\ (S \cap \psi_b(\alpha_i)^\perp, \end{aligned}$$

so assume $H \in S(K^b, K^{b'})$. Then H must meet S, but cannot meet K^b or $K^{b'}$ by Lemma 2.6.3. The intersection $H \cap S$ is open in $\psi_b(\alpha_i)^\perp$. Hence the two hyperplanes must coincide. This shows iv).

iv) $\implies$ iii), i): Let $S(K^b, K^{b'}) = \{\psi_b(\alpha_i)^\perp\}$. By definition $\psi_b(\alpha_i)^\perp$ is a wall of K^b. Assume $\psi_b(\alpha_i)^\perp$ is not a wall of $K^{b'}$, then $K^b \subset \bigcap_{i \in I} \psi_{b'}(\alpha_i)^+$, since the $\psi_{b'}(\alpha_i)$ are exactly the walls of $K^{b'}$. Then we find $K^b = K^{b'}$ by Lemma 2.6.3, but then $S(K^b, K^{b'}) = \emptyset$, a contradiction, hence $\psi_b(\alpha_i)^\perp \in W^{b'}$ holds. For $j \neq i$ we find $D_{\psi_b(\alpha_j)^\perp}(K^b) = D_{\psi_b(\alpha_j)^\perp}(K^{b'})$, and the same holds for $\psi_{b'}(\alpha_j)^\perp$. Therefore S contains both K^b and $K^{b'}$. Assume $K \subset S$, then K is on either side of $\psi_b(\alpha_i)^\perp$. In case $D_{\psi_b(\alpha_i)^\perp}(K) = D_{\psi_b(\alpha_i)^\perp}(K^b)$, $K = K^b$ holds, so assume $D_{\psi_b(\alpha_i)^\perp}(K) = D_{\psi_b(\alpha_i)^\perp}(K^{b'})$. As $\psi_b(\alpha_i)^\perp$ is a wall of $K^{b'}$ different from $\psi_{b'}(\alpha_j)^\perp$ for $j \neq i$, we already find $\psi_b(\alpha_i)^\perp = \psi_{b'}(\alpha_i)^\perp$. We obtain $K \subset \bigcap_{i \in I} \psi_{b'}(\alpha_i)^+$, and therefore $K = K^{b'}$, which shows iii). Furthermore we see that

$S \cap \psi_b(\alpha_i)^\perp$ is not empty, as S is a convex set containing points in $\psi_b(\alpha_i)^+$ and in $\psi_b(\alpha_i)^-$. In particular we showed

$$\begin{aligned} S &= (S \cap \psi_b(\alpha_i)^+) \,\dot\cup\, (S \cap \psi_b(\alpha_i)^-) \,\dot\cup\, (S \cap \psi_b(\alpha_i)^\perp) \\ &= K^b \,\dot\cup\, K^{b'} \,\dot\cup\, (S \cap \psi_b(\alpha_i)^\perp, \end{aligned}$$

and $(S \cap \psi_b(\alpha_i)^\perp \subset \overline{K^b} \cap \overline{K^{b'}}$. Consider open balls $U \subset K^b$, $U' \subset K^{b'}$. The convex hull of U and U' is again open, and therefore intersects $\psi_b(\alpha_i)^\perp$ in a subset U'', which is open in $\psi_b(\alpha_i)^\perp$. Hence U'' spans $\psi_b(\alpha_i)^\perp$. Now U'' is contained in $\overline{K^b}$ as well as $\overline{K^{b'}}$, so we find that K^b and $K^{b'}$ are i-adjacent, which shows i).

i) $\implies$ iv): Let K^b and $K^{b'}$ be i-adjacent, so $\langle \overline{K^b} \cap \overline{K^{b'}} \rangle = \psi_b(\alpha_i)^\perp$. Let $H = \psi_b(\alpha_i)^\perp$. Assume $H' \in \mathcal{A}$ separates K^b and $K^{b'}$. Then $(\overline{K^b} \cap H) \cap (\overline{K^{b'}} \cap H)$ will be contained in H'. So for this intersection to span a hyperplane, we require $H = H'$. Therefore iv) holds.

iii) $\implies$ ii): We have the equality $S \cap \psi_b(\alpha_i)^+ = K^b$ by definition. The intersection $S \cap \psi_b(\alpha_i)^-$ must contain $K^{b'}$. Furthermore we find that $\psi_b(\alpha_i)^\perp$ and $\psi_{b'}(\alpha_i)^\perp$ separate K^b and $K^{b'}$, else we would again obtain $K^b \subset \underline{K^{b'}}$. Since S is convex, we actually find an open subset $U' \subset S$ which is in $\psi_b(\alpha_i)^\perp \cap \overline{K^b}$. Since S is open, this is contained in an open subset $U \subset S$, such that $U \cap \psi_b(\alpha_i)^\perp = U'$.

Assume $\rho_i(b) = b''$, then ii) $\implies$ iv) $\implies$ i) yields that

$$S' = \bigcap_{i \neq j \in I} (\psi_b(\alpha_j)^+ \cap \psi_{b''}(\alpha_j)^+$$

contains exactly the chambers K^b and $K^{b''}$ and that K^b, $K^{b''}$ are i-adjacent.Furthermore we obtain

$$S' = K^b \,\dot\cup\, K^{b''} \,\dot\cup\, (S' \cap \psi_b(\alpha)^\perp.$$

By construction S' contains U', therefore it also contains an open set U'' such that $U'' \cap \psi_b(\alpha_i)^\perp = U'$. But then $U \cap U''$ will be open, is contained in S and meets $K^{b''}$. So we find $K^{b'} = K^{b''}$ and hence $b' = b''$ by Lemma 2.6.7. This shows ii) and finishes the proof. □

Lemma 2.6.9. *For every $H \in \mathcal{A}$, there exists $b \in A$ such that $H \in W^b$.*

Proof. As $H \in \mathcal{A}$, we find $\alpha \in R$ such that $H = \alpha^\perp$, thus there exists $b \in A$ and $i \in I$ such that $\alpha = \phi_b(\alpha_i)$, and consequently $H \in W^b$. □

Lemma 2.6.10. *We find $|S(K^b, K^{b'})| = 0$ if and only if $b = b'$, and $|S(K^b, K^{b'})| = 1$ if and only if K^b and $K^{b'}$ are adjacent.*

Proof. Assume $|S(K^b, K^{b'})| = 0$ holds, and let $B^b := \{\beta_1, \dots, \beta_r\}$. We then find $K^{b'} \subseteq \bigcap_{i=1}^r \beta_i^+$. Assuming that the inclusion is proper provides a contradiction to the fact that no hyperplane in $W^{b'}$ meets K^b in Lemma 2.6.3.

In the case $b = b'$, the statement in Lemma 2.6.7 yields $|S(K^b, K^{b'})| = 0$, which shows the first equivalence.

For the second statement assume $S(K^b, K^{b'}) = \{H\}$, and assume $H \notin W^b$. Then $K^{b'} \subset D_{H'}(K^b)$ for all $H' \in W^b$, and therefore $K^{b'} \subset K^b$ holds. We obtain $b = b'$ and $|S(K^b, K^{b'})| = 0$ in contradiction to our assumptions.

Therefore H is a wall of K^b as well as $K^{b'}$, hence $H = \psi_b(\alpha_i)^\perp$ for some $i \in I$. The statement follows from Proposition 2.6.8, iv) $\implies$ i).

If K^b, $K^{b'}$ are i-adjacent, Proposition 2.6.8, i) $\implies$ iv), yields

$$|S(K^b, K^{b'})| = 1.$$

□

Remark 2.6.11. The distance function d known from chamber complexes, defined by taking a minimal gallery $K^{b_0}, \dots, K^{b_m}$ between chambers K^{b_0} and K^{b_m}, where K^{b_i} and $K^{b_{i+1}}$ are adjacent, and setting $d(K^{b_0}, K^{b_m}) = m$, is a well defined metric on the set $\mathcal{K}$, and turns $\mathcal{K}$ into a metric space $(\mathcal{K}, d)$. The fact that d is a metric can be checked in the same way as if $\mathcal{K}$ was constructed from a simplicial arrangement.

In particular it should be noted that the distance between arbitrary chambers is always finite, as $\mathcal{C}$ is connected.

Proposition 2.6.12. *Assume $K^b \in \mathcal{K}$ and $x \in \overline{K^b}$ such that $\mathcal{A}_x := \operatorname{supp}(x) = \{H \in \mathcal{A} \mid x \in H\}$ is finite. Let $R_x = \{\alpha \in R \mid \alpha^\perp \in \mathcal{A}_x\}$, $W = \langle R_x \rangle$, $V_x = V/W^\perp$. Construct $I_x = \{i \in I \mid \psi_b(\alpha_i) \in R_x\}$, and set $\Pi_x = \langle \rho_i \mid i \in I_x \rangle$. Further define $A_x := \Pi_x(b)$ and $C_x^{b'} = (c_{i,j}^{b'})_{i,j \in I_x}$. Set*

$$\mathcal{C}_x := \mathcal{C}(I_x, A_x, (\rho_i)_{i \in I_x}, (C_x^{b'})_{b' \in A_x}),$$

then $\mathcal{C}_x$ is a connected simply connected Cartan graph with real roots at c being the set $\psi_c^{-1}(R_x)$ for $c \in A_x$. Here R_x is isomorphic to a subset of $(V_x)^*$ via $\alpha(v + W^\perp) := \alpha(v)$.

Proof. First notice that $\mathcal{C}_x$ is indeed a connected and simply connected Cartan graph by Corollary 2.5.16 as it is an I_x-residue of $\mathcal{C}$.

Denote by $(R_x^{\mathrm{re}})^c$ for $c \in A_x$ the real roots at c given by the Cartan graph $\mathcal{C}_x$. By taking the standard basis $\{\alpha_1, \dots, \alpha_r\}$, we can consider $\mathbb{Z}^{I_x}$ as the $\mathbb{Z}$-span of $\{\alpha_i \mid i \in I_x\}$ in $\mathbb{Z}^I$.

Consider the constructions of

$$(R_x^{\mathrm{re}})^c = \{\mathrm{id}^c \sigma_{i_1} \cdots \sigma_{i_k}(\alpha_j) \mid k \in \mathbb{N}_0, i_1, \dots, i_k, j \in I_x\} \quad \text{and}$$
$$(R^{\mathrm{re}})^c = \{\mathrm{id}^c \sigma_{i_1} \cdots \sigma_{i_k}(\alpha_j) \mid k \in \mathbb{N}_0, i_1, \dots, i_k, j \in I\},$$

it follows immediately that $(R_x^{\mathrm{re}})^c \subseteq \psi_c^{-1}(R_x)$ for $c \in A_x$. In particular $(R_x^{\mathrm{re}})^c$ is finite and $\mathcal{C}_x$ is connected simply connected with real finite root system.

In particular, for $c \in A_x$ the real roots $(R_x^{\mathrm{re}})^c$ at c form a spherical simplicial arrangement of rank $|I_x|$ via the map ψ_c in the space V_x. Call this arrangement $\mathcal{A}'$, the corresponding root system R' and the chambers $\mathcal{K}'$. Also denote the canonical projections with $\pi_x : V \to V/W^\perp$ and $\pi_x^\perp : V^* \to (V_x)^*$, $\pi_x^\perp(\alpha)(y + W^\perp) = \alpha(y)$.

Let $c \in A$, $d \in A_x$, and let $(K')^c$ be the chamber associated to c in $\mathcal{C}_x$, and $(B')^c$ the respective root basis in $(V_x)^*$.

We will show that for all $c \in A_x$ we find $\pi_x(K^c) = (K')^c$ and $\pi_x^\perp(B^c \cap R_x) = (B')^c$. For the simplicial arrangement associated to $\mathcal{C}_x$ we need some notation as in Definition 2.6.1. Let $\psi' : \mathbb{Z}^{I^x} \to (V_x)^*$, mapping α_i to β_i. So we find $\psi' = \psi_b|_{\mathbb{Z}^{I_x}}$. The associated root system is then given by $R' = \psi'((R_x^{\mathrm{re}})^b)$. For $c \in A_x$ and $\mathrm{Hom}(c, b) = \{w\}$ we can then define $\psi'_c = \psi' w$ and find $\psi_c((R_x^{\mathrm{re}})^c) = R'$.

We find by definition of I_x that $B^c \cap R_x = \psi_c(\{\alpha_i \mid i \in I_x\})$ and $(B')^c = \psi'_c(\{\alpha_i \mid i \in I_x\})$ via the embedding of α_i, $i \in I_x$ into $\mathbb{Z}^I$. By definition we can write $\psi'_c = \psi' w$, $\psi_c = \psi_b w$ for $\mathrm{Hom}(c, b) = \{w\}$. As noted above we find $\psi' = \psi_b|_{\mathbb{Z}^{I_x}}$, so $\psi'_c = \psi_b|_{\mathbb{Z}^{I_x}} w$, which yields the equality $\pi_x^\perp(B^c \cap R_x) = (B')^c$. Given this, we immediately obtain $\pi_x(K^c) = (K')^c$ by considering how K^c, $(K')^c$ are defined.

Now assume $\alpha \in R_x$. If it is not in $\psi_b(R_x^{\mathrm{re}})$, it meets a chamber in $\mathcal{K}'$, since the elements in $\mathcal{K}'$ are the connected components of $V_x \setminus \bigcup_{H \in \mathcal{A}_x} H$, which is not possible by Lemma 2.6.3.

We can conclude $R_x = R'$, as required. □

Remark 2.6.13. It is an easy observation with proof similar to the proof of Lemma 2.4.14, that the separating hyperplanes for two chambers $K^b, K^{b'}$ with $x \in \overline{K^b} \cap \overline{K^{b'}}$ are all contained in A_x. In combination with the next lemma, this yields that the Cartan graph $\mathcal{C}_x$ is independent of the choice of K^b. In other words, in this case we find $b' \in \Pi_x(b)$.

The next lemma yields a characterisation of the distance d, which we already established for simplicial arrangements.

Lemma 2.6.14. *For $b, b' \in A$ we find $d(K^b, K^{b'}) = |S(K^b, K^{b'})|$.*

Proof. Let $d := d(K^b, K^{b'})$, and set $m = |S(K^b, K^{b'})|$ in case this is finite. For $d = 0, 1$ the statement follows from Lemma 2.6.10. We prove $|S(K^b, K^{b'})| \leq d$ by induction on d, so assume $d \geq 2$. For $c, c' \in A$ with $d(K^c, K^{c'}) < d$ we know $d(K^c, K^{c'}) = |S(K^c, K^{c'})|$. Let $K^b = K^{b_0}, K^{b_1}, \ldots, K^{b_d} = K^{b'}$ be a minimal gallery from K^b to $K^{b'}$. We find $K^{b_1}, \ldots, K^{b_d}$ to be a minimal gallery from K^{b_1} to $K^{b'}$, and hence $d(K^{b_1}, K^{b'}) = d - 1 = |S(K^{b_1}, K^{b'})|$ by induction, and $d(K^b, K^{b_1}) = 1 = |S(K^b, K^{b_1})|$.

So we can assume $S(K^b, K^{b_1}) = \{H_1\}$, $S(K^{b_1}, K^{b'}) = \{H_2, \ldots, H_d\}$.

Assume H separates K^b and $K^{b'}$, and let $1 \leq j \leq d$ be the first index, such that $D_H(K^{b_{j-1}}) = D_H(K^b)$, $D_H(K^{b_j}) = D_H(K^{b'})$. This index exists, otherwise we find

$D_H(K^b) = D_H(K^{b'})$. If $j = 1$, we find $H = H_1$, else we find $H \in S(K^{b_1}, K^{b'})$, and we can conclude

$$S(K^b, K^{b'}) \subset S(K^b, K^{b_1}) \cup S(K^{b_1}, K^{b'})$$

and thus $m \leq d$.

To show equality we show that there exists a gallery of length m connecting K^b and $K^{b'}$. As we now know that $S(K^b, K^{b'})$ is actually finite, let $S(K^b, K^{b'}) = \{H'_1, \dots, H'_m\}$. There exists a hyperplane in $S(K^b, K^{b'})$ which is a wall of K^b, otherwise we find for every wall of K^b, that $K^{b'}$ is on the same side, which yields $K^b = K^{b'}$ and $b = b'$, in contradiction to $d(K^b, K^{b'}) \geq 2$.

So assume H'_1 is a wall of K^b, then we find $H'_1 = \psi_b(\alpha_i)^\perp$ for some $i \in I$. So let $b_1 \in A$ such that $\rho_i(b) = b_1$, then K^{b_1} is i-adjacent to K^b by Proposition 2.6.8. We obtain $S(K^b, K^{b_1}) = \{H'_1\}$, and as H'_1 is the only hyperplane separating K^b and K^{b_1} by Lemma 2.6.10, we obtain that $D_{H'_i}(K^b) = D_{H'_i}(K^{b_1}) = -D_{H'_i}(K^{b'})$ for $i = 2, \dots, m$.

Furthermore note that every H'_i for $2 \leq i \leq m$ separates K^b and $K^{b'}$, and therefore separates K^{b_1} and $K^{b'}$ as well. On the other hand if H separates K^{b_1} and $K^{b'}$, it also separates K^b and $K^{b'}$. We obtain $S(K^{b_1}, K^{b'}) = \{H'_2, \dots, H'_m\}$, and by induction we find a gallery of length m connecting K^b and $K^{b'}$. Hence $d \leq m$, which proves our claim. □

Corollary 2.6.15. *Assume $b, b' \in A$ and $K^b = K^{b_0}, \dots, K^{b_m} = K^{b'}$ is a minimal gallery from K^b to $K^{b'}$. Then this gallery crosses no hyperplane in $\mathcal{A}$ more than once.*

Another consequence of the last Lemma is the following statement.

Lemma 2.6.16. *Let $b, b' \in A$, $x \in K^b$, $y \in K^{b'}$. For every point $z \in [x, y]$ there exists a neighbourhood U_z such that $\sec(U_z) = \{H \in \mathcal{A} \mid H \cap U_z \neq \emptyset\}$ is finite.*

Proof. Take an open neighbourhood U_x of x in K^b, and an open neighbourhood U_y of y in $K^{b'}$. Take the union $U := \bigcup_{x' \in U_x, y' \in U_y} [x', y']$. Then U contains $[x, y]$ and every hyperplane that meets U separates K^b and $K^{b'}$. Hence the set $\sec(U)$ is finite by Lemma 2.6.14. We find $\varepsilon, \delta \in \mathbb{R}_{>0}$ such that the open balls $B_\varepsilon(x)$, $B_\delta(y)$ satisfy $B_\varepsilon(x) \subset U_x$, $B_\delta(y) \subset U_y$. Assume $\varepsilon \geq \delta$, and let $z' \in B_\delta(z)$. Write $z' = z + v$ for $v \in V$. Then $x + v \in B_\delta(x)$, $y + v \in B_\delta(y)$ and

$$z' = z + v \in [x, y] + v = [x', y'] \subset U.$$

Choosing U_z as $B_\delta(z)$ therefore satisfies $|\sec(U_z)| < \infty$. □

The cone T is defined as the convex hull of all K^b, $b \in A$, we give an alternative description below.

Lemma 2.6.17. *Let $b, b' \in A$ with $d(K^b, K^{b'}) = m$, and let $\Gamma(b, b')$ the set of minimal galleries from K^b to $K^{b'}$. Let $x \in K^b$, $y \in K^{b'}$. Then*

$$[x, y] \subset \bigcup_{K^c \in \gamma \in \Gamma(b,b')} \overline{K^c}.$$

Proof. The interval $[x, y]$ only meets the finite set of hyperplanes $S(K^b, K^{b'})$ by Lemma 2.6.16. As x, y are not contained in any hyperplane in $\mathcal{A}$, the set $[x, y]$ is not contained in a hyperplane as well.

Let $x_1, \ldots, x_k \in [x, y]$ be the points such that $\sec((x_i, x_{i+1})) = \emptyset$, $\operatorname{supp}(x_j) \neq \emptyset$ for all $1 \le i \le k-1$, $1 \le j \le k$, and $\sec([x, y]) = \bigcup_{i=1}^{k} \operatorname{supp}(x_i)$. These points exist by Lemma 2.6.14.

We show the statement by induction on k. Assume $k = 0$, then $b = b'$ and $K^b = K^{b'}$ by Lemma 2.6.7. As $x, y \in K^b$ and K^b is convex, $\sigma(x, y) \subset K^b$, as required.

If $k = 1$, then $(x, x_1) \subset K^b$, $(x_1, y) \subset K^{b'}$. Hence $[x, x_1] \cup [x_1, y] \subset \overline{K^b} \cup \overline{K^{b'}}$. Thus our claim holds for $k = 1$.

So let $k \ge 2$. We show that every open interval (x_j, x_{j+1}) is contained in some chamber K^{c_j} for $1 \le j < k$. It is enough to show that (x_1, x_2) is contained in a chamber, then the statement follows inductively by substituting x with a point on (x_1, x_2). As $x_1 \in \overline{K^b}$, let $J \subset I$ such that $j \in J$ if and only if $x_1 \in \psi_b(\alpha_j)^\perp$.

Define $R_{x_1} := \{\alpha \in R \mid x_1 \in \alpha^\perp\}$, $W = \langle R_{x_1} \rangle$, let $\pi_{x_1}^* : V^* \to (V/W^\perp)^*$,$\pi_{x_1}^*(\alpha)(v + W^\perp) = \alpha(v)$, $\pi_{x_1} : V \mapsto V/W^\perp, v \mapsto v + W^\perp$.

As $\operatorname{supp}(x_1)$ is finite, we can apply Proposition 2.6.12 to find a spherical Cartan graph $\mathcal{C}_{x_1}$, together with a set of chambers in one to one correspondence to the objects A_{x_1}. In particular, as $\mathcal{C}_{x_1}$ is spherical, there exists an object $c_1 \in A_{b,J}$, such that the chamber K^{c_1} is opposite to K^b in the spherical arrangement associated to $\mathcal{C}_{x_1}$. Let $B^b \cap R_{x_1} = \{\beta_1, \ldots, \beta_l\}$, then $W^{c_1} = \{-\beta_1, \ldots, -\beta_l\}$, hence $(x_1, x_2) \subset K^{c_1}$ follows.

We can conclude that (x_j, x_{j+1}) is contained in K^{c_j} for $1 \le j < k$.

Let $z \in (x_j, x_{j+1})$ for some $1 \le j < k$. By counting separating hyperplanes we obtain $d(K^b, K^{c_j}) + d(K^{c_j}, K^{b'}) = d(K^b, K^{b'})$, hence there is a minimal gallery $K^b = K^{b_0}, \ldots, K^{b_\lambda} = K^{c_j}, \ldots, K^{b'} = K^{b_m}$.

By induction we obtain

$$\begin{aligned}
[x, y] = [x, z] \cup [z, y] &\subset \bigcup_{K^{c'} \in \gamma \in \Gamma(b,c)} \overline{K^{c'}} \cup \bigcup_{K^{c'} \in \gamma \in \Gamma(c,b')} \overline{K^{c'}} \\
&\subset \bigcup_{K^{c'} \in \gamma \in \Gamma(b,b')} \overline{K^{c'}},
\end{aligned}$$

since every minimal gallery containing K^c yields minimal galleries from K^b to K^c as well as from K^c to $K^{b'}$. □

Lemma 2.6.18. *For $b, b' \in A$ the set $\{K^c \mid K^c \in \gamma \in \Gamma(b, b')\}$ is finite.*

Proof. There exist only finitely many chambers adjacent to K^b, inductively there exist only finitely many chambers K^c such that $d(K^b, K^c) \leq d(K^b, K^{b'})$, and $\{K^c \mid K^c \in \gamma \in \Gamma(b, b')\}$ is contained in this set. □

Corollary 2.6.19. *Let $x \in \overline{K^b}$, $y \in \overline{K^{b'}}$ for $b, b' \in A$. Then*

$$[x, y] \subset \bigcup_{K^c \in \gamma \in \Gamma(b,b')} \overline{K^c}.$$

Proof. Let $p : [0, 1] \mapsto [x, y]$ be a continuous parametrisation with $p(0) = x$,$p(1) = y$.

Let $x' \in K^b$, $y' \in K^{b'}$ and parametrise $[x, x']$, $[y, y']$ continuously with $p_x : [0, 1] \to [x, x']$, $p_y : [0, 1] \to [y, y']$. Then by Lemma 2.6.17 we have

$$[p_x(\varepsilon), p_y(\varepsilon)] \subset \bigcup_{K^c \in \gamma \in \Gamma(b,b')} \overline{K^c}$$

for all $0 < \varepsilon \leq 1$. As $\bigcup_{K^c \in \gamma \in \Gamma(b,b')} \overline{K^c}$ is by Lemma 2.6.18 a finite union of closed chambers, it is closed again. Since p_x, p_y are continuous, we also find

$$[x, y] = [p_x(0), p_y(0)] \subset \bigcup_{K^c \in \gamma \in \Gamma(b,b')} \overline{K^c}.$$

□

Lemma 2.6.20. *Let $T_0 := \bigcup_{b \in A} K^b$. The set T satisfies*

$$T = \bigcup_{x,y \in T_0} [x, y].$$

Proof. Let $T' := \bigcup_{x,y \in T_0} [x, y]$, then the inclusion $T' \subset T$ is clear from the definition. We show that T' is convex.

Let $z, z' \in T'$, then we find $b, b', c, c' \in A$ with $x \in K^b$, $x' \in K^{b'}$, $y \in K^c$, $y' \in K^{c'}$, such that $z \in [x, y]$, $z' \in [x', y']$. It follows from Lemma 2.6.17 that there exist chambers K^d, $K^{d'}$ with $z \in \overline{K^d}$, $z' \in \overline{K^{d'}}$. By Corollary 2.6.19 we obtain

$$[z, z'] \subset \bigcup_{K^c \in \gamma \in \Gamma(d,d')} \overline{K^c}.$$

Let $z'' \in [z, z']$, then $\operatorname{supp}(z'')$ is finite, and there exists a chamber $K^{d''}$ such that $z'' \in \overline{K^{d''}}$. We obtain by Proposition 2.6.12 that there exists an object d^* opposite to d'' in the spherical Weyl groupoid induced at z''. Therefore K^{d^*} is opposite to $K^{d''}$ in the respective spherical arrangement, and z'' is on a segment between a point in $K^{d''}$ and K^{d^*}. □

Proposition 2.6.21. *The triple $(\mathcal{A}, T, R)$ is a crystallographic arrangement. Furthermore, R is reduced.*

Proof. By Lemma 2.6.20 every point $z \in T$ is on a segment $\sigma(x,y)$, $x \in K^b$, $y \in K^{b'}$, and by Lemma 2.6.16 we find a neighbourhood U_z such that $\sec(U_z)$ is finite. Therefore $\mathcal{A}$ is a locally finite hyperplane arrangement in T, and T is an open convex cone by definition.

Let K be a connected component of $T \setminus (\bigcup_{H \in \mathcal{A}} H)$, and let $x \in K$. As a direct result of Lemma 2.6.17 x is contained in $\overline{K^b}$ for some $b \in A$. By definition of K it follows that $K \subset K^b$, and by Lemma 2.6.3 we get equality.

Furthermore $\mathcal{A}$ is thin, as by definition every wall of K^b, $b \in A$, is in $\mathcal{A}$. From Lemma 2.6.9 it follows that every $H \in \mathcal{A}$ is a wall of some chamber, so it meets T.

It follows also by definition that $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$, so R is a root system for $\mathcal{A}$. Furthermore R is crystallographic since $R = \psi_b(R^b)$, so every root is a positive or negative integral linear combination of B^b for all $b \in A$.

Finally R is reduced since the roots $(R^{\mathrm{re}})^a$ satisfy property (R2). □

Remark 2.6.22. Choosing a different object $a' \in A$ to begin with yields a combinatorially equivalent crystallographic arrangement, as those differ exactly by an element in $\mathrm{GL}_r(\mathbb{Z})$.

Since equivalent Cartan graphs have the same sets of real roots, they also yield combinatorially equivalent crystallographic arrangements.

Definition 2.6.23. Given a connected simply connected Cartan graph $\mathcal{C}$ permitting a root system of type $\mathcal{C}$, we call the crystallographic arrangement $(\mathcal{A}, T, R)$ as constructed above *the geometric realisation of* $\mathcal{C}$.

Corollary 2.6.24. *Let $\mathfrak{A}$ be the set of all crystallographic arrangements with reduced root systems, $\mathfrak{C}$ the set of all connected, simply connected Cartan graphs which permit a root system. Let $\cong$ denote combinatorial equivalence on $\mathfrak{A}$ as well as equivalence on $\mathfrak{C}$. Then there is a canonical bijection*

$$\Lambda : \mathfrak{A}/_{\cong} \to \mathfrak{C}/_{\cong}, \quad \overline{(\mathcal{A}, T, R)} \mapsto \overline{\mathcal{C}(\mathcal{A}, T, R)}.$$

2.7 Subarrangements at a point

Let $(\mathcal{A}, R, T)$ be a simplicial arrangement. Consider a subset $X \subset \overline{T}$. Remember the notion of the section of X, i.e.

$$\sec(X) := \{H \in \mathcal{A} \mid H \cap X \neq \emptyset\}.$$

For a locally finite arrangement and compact $X \subset T$, $\sec(X)$ is finite by Lemma 2.3.6. For $x \in \overline{T}$ we write $\mathrm{supp}\,(x)$ instead of $\sec(\{x\})$, which is also finite for $x \in T$ in a locally finite arrangement.

We want to say more about hyperplane arrangements arising as $\mathrm{supp}\,(x)$ for points $x \in \overline{T}$. So let $x \in \overline{T}$ and set $R_x := \{\alpha \in R \mid \alpha^\perp \in \mathrm{supp}\,(x)\}$, $\mathcal{A}_x := \{\alpha^\perp \mid \alpha \in R^x\} = \mathrm{supp}\,(x)$, we call $\mathcal{A}_x$ the *induced arrangement at* x or the *subarrangement at* x. We define the *chambers at* x, $\mathcal{K}_x := \{K \in \mathcal{K} \mid x \in \overline{K}\}$, as well as a respective set of roots $B_x^K := B^K \cap R_x$.

The set $\mathcal{A}_x$ is clearly finite if $x \in T$.

Lemma 2.7.1. *With notation as above, let* $K \in \mathcal{K}_x$*, then* $\langle B_x^K \rangle = \langle R_x \rangle$*. In particular,* B_x^K *is a basis of* $\langle R_x \rangle$*.*

Proof. The inclusion $\langle B_x^K \rangle \subset \langle R_x \rangle$ holds due to $B_x^K \subset R_x$.

Now the space $(B_x^K)^\perp = \{v \in V \mid \alpha(v) = 0 \ \forall\ \alpha \in B_x^K\}$ is a subspace of V and $(B_x^K)^\perp \cap \overline{K}$ is a face of $\overline{K}$ containing x. Assume F is a face of $(B_x^K)^\perp \cap \overline{K}$ containing x. Then F is also a face of $\overline{K}$ and has the structure $F = \overline{K} \cap \bigcap_{i=1}^m \alpha_i^\perp$ for some $\alpha_1, \ldots, \alpha_m \in B^K$. Then by definition $x \in \alpha_i^\perp$ for $i = 1, \ldots, m$, and $(B_x^K)^\perp \cap \overline{K} \subset F$, so $(B_x^K)^\perp \cap \overline{K}$ is a minimal face of $\overline{K}$ containing x.

The inclusion $\langle B_x^K \rangle \subset \langle R_x \rangle$ implies $(R_x)^\perp \subset (B_x^K)^\perp$, therefore the set $(R_x)^\perp \cap \overline{K}$ is also a face of $\overline{K}$, furthermore it contains x by definition. We can conclude $(B_x^K)^\perp \subset (R_x)^\perp$, which yields the equality.

The second claim follows since $B_x^K := B^K \cap R_x$ and the elements in B^K are linearly independent. □

Definition and Remark 2.7.2. For $0 \neq x \in \overline{T}$, R_x generates a subspace of V of dimension at most $r-1$. Note that the space V will therefore have a dimension which is too high to describe the set $\mathcal{A}_x$ as a simplicial arrangement as before, as the connected components of $V \setminus \bigcup_{H \in \mathcal{A}_x} H$ are not simplicial. For the roots R_x, its sufficient to look at the space $W := \langle R_x \rangle$. Then $W^\perp = \{v \in V \mid \alpha(v) = 0 \text{ for all } \alpha \in W\}$. The vector space we will need to consider will be $V_x := V/W^\perp$, let $\pi : V \to V/W^\perp$ be the projection. The space W is canonically isomorphic to $(V_x)^*$, via

$$\pi^* : W \to (V/W^\perp)^*, \qquad \alpha \mapsto (v + W^\perp \mapsto \alpha(v)).$$

Note that this is well defined by the definition of W. So from now on we can think of R_x as a subset of $(V_x)^*$, so set for $\alpha \in R_x$ and $v \in V$:

$$\alpha(\pi(v)) := \pi^*(\alpha)(\pi(v)) = \alpha(v).$$

Denote $\mathcal{A}_x^\pi := \{\alpha^\perp \leq V_x \mid \alpha \in R_x\}$, we will consider this as an hyperplane arrangement in the cone $T_x = \pi(T)$.

To describe $\mathcal{A}_x$ as an arrangement in the way mentioned above requires more formalism, which we omit in the general case. Most of the time we are interested only in R_x and its combinatorial properties, therefore it does not matter whether we consider these in W or in V^*. But formally the transition to T_x is necessary to obtain a root system in the strict sense.

We will consider V_x as a topological space with respect to the topology in W.

Lemma 2.7.3. *The set T_x is an open convex cone in V_x.*

Proof. The set $\pi(T)$ is open as the image of an open set and π is open. Furthermore, if $[y, z]$ is an intervall in $\pi(T)$, then there exist $y', z' \in T$ with $y = y' + W^\perp$, $z = z' + W^\perp$. The intervall $[y', z']$ is contained in T as T is convex, and we find $\pi([y', z']) = [y, z]$.

Finally, T_x is a cone, since for $y \in T_x$ we find $y' \in T$ with $y = y' + W^\perp$. Since T is a cone, for every $\lambda > 0$ we have $\lambda y' \in T$, so $\lambda y \in T_x$. □

Now we can begin to gather properties of $\mathcal{A}_x$.

Lemma 2.7.4. *The hyperplane arrangement $(\mathcal{A}_x^\pi, T_x)$ is locally finite.*

Proof. Let $y \in \pi(T)$, then there exists an $y' \in T$ such that $y = y' + W$. Since $\mathcal{A}$ is locally finite, there exists a neighbourhood $U \subset T$ containing y' such that $\{H \in \mathcal{A}_x \mid H \cap U \neq \emptyset\}$ is finite. As π maps open sets to open sets and furthermore $\pi(v) \in \alpha^\perp$ if and only if $v \in \alpha^\perp$ holds, we find $\pi(U) \subset T_x$, and $\{H \in \mathcal{A}_x^\pi \mid H \cap \pi(U)\}$ is finite. □

Proposition 2.7.5. *The hyperplane arrangement $(\mathcal{A}_x^\pi, T_x)$ is simplicial. The chambers of this arrangement correspond to $\mathcal{K}_x$.*

Moreover, if $x \in T$, then $(\mathcal{A}_x^\pi, T_x)$ is spherical. If $x \notin T$, $\dim V_x = m$, and $(\mathcal{A}, T)$ is k-spherical for some $k \in \mathbb{N}$, then $(\mathcal{A}_x^\pi, T_x)$ is $\min(m, k)$-spherical.

In particular, $(\mathcal{A}_x^\pi, T_x)$ is thin with root system R_x.

Proof. Assume $\dim V_x = m$, so $\dim\langle R_x\rangle = m$ and $\dim W^\perp = r - m$.

Let $K \in \mathcal{K}_x$, $\pi : V \to V_x$ denote the standard epimorphism. Let $\mathcal{K}'$ be the connected components of $T_x \setminus \bigcup_{\alpha \in R_x} \alpha^\perp$. The set $\pi(B_x^K)$ is a basis for V_x by Lemma 2.7.1 and we find $\pi(K) \subset T_x \cap \bigcap_{\alpha \in B_x^K} \pi^*(\alpha)^+$ by definition. Denote this intersection by K'.

First we show that $K' \in \mathcal{K}'$. Assume there exists $\beta \in R_x$ such that there exist $y', z' \in K'$ with $\beta(y') > 0$, $\beta(z') < 0$. Then we find $y, z \in T$ with the properties: $\beta(y) > 0$, $\beta(z) < 0$, $\alpha(y) > 0, \alpha(z) > 0$ for all $\alpha \in B_x^K$. For any $0 < \lambda < 1$ and $\alpha \in B_x^K$ we find $\alpha(y - \lambda(y - x)) = \alpha(y)(1 - \lambda) > 0$ and $\alpha(z - \lambda(z - x)) < 0$ as $\alpha(x) = 0$. So for $0 < \lambda_y, \lambda_z < 1$ the points $y - \lambda_y(y - x)$, $z - \lambda_z(z - x)$ still satisfy the above inequalities.

Now let $\alpha \in B^K \setminus B_x^K$, then $\alpha(x) > 0$. So choosing $0 < \lambda_y, \lambda_z < 1$ large enough we find that the points $y_1 := y - \lambda_y(y - x)$, $z_1 := z - \lambda_z$ are close enough to x to satisfy $\beta(y_1) > 0$, $\beta(z_1) < 0$ and $\alpha(y_1) > 0 < \alpha(z_1)$ for all $\alpha \in B^K$. So $y_1, z_1 \in K$, in contradiction to the simplicial structure of $\mathcal{S}$.

It remains to prove that K' is again a simplicial cone. Since K is a simplicial cone, there exists an closed simplex S with the property $\overline{K} = \mathbb{R}_{>0}S \cup \{0\}$. Let F_x denote the minimal face of S such that $x \in \mathbb{R}_{>0}F_x$, and let $V(F_x)$ be the vertex set of F_x. Then the vertices $V(S) \setminus V(F_x)$ span a face of S, denote this face by F'. The vertices $\pi(V(F'))$ are linearly independent: Assume $\sum_{v\in V(F')} \lambda_v \pi(v) = 0$, then $\sum_{v\in V(F')} \lambda_v v \in W^\perp$. Furthermore note that $W^\perp$ is spanned by $V(F_x)$. So we get a linear combination of the form $\sum_{v\in V(F')} \lambda_v v = \sum_{v\in V(F_x)} \lambda_v v$. Linear independence of $V(S)$ yields $\lambda_v = 0$ for all $v \in V(S)$.

This also gives us $|V(F')| \leq m$. Assuming inequality, we find more than $r-m$ vertices in $W^\perp$, a contradiction to $\dim W^\perp = r-m$. So $\pi(F')$ spans indeed an $m-1$-simplex in V_x.

We show that $\overline{K'} = \mathbb{R}_{>0}\pi(F') \cup \{0\}$. In general for every $v \in V(\overline{K})$ there exists a unique $\alpha \in B^K$ such that $\alpha(v) > 0$ by Lemma 2.2.5, all other $\alpha \neq \beta \in B^c$ satisfy $\beta(v) = 0$. Since for $v \in V(F')$ the vector v is not contained in any proper face of $\overline{K}$ containing x, we find a unique $\alpha \in B_x^K$ such that $\alpha(v) > 0$.

Let $y \in \mathbb{R}_{>0}\pi(F')$, then $y = \sum_{v\in V(F')} \lambda_v \pi(v)$ with $\lambda_v \geq 0$ for all $v \in V(F')$. Then $\alpha(v) \geq 0$ for all $v \in F'$, $\alpha \in B_x^K$ yields $\alpha(\pi(v)) \geq 0$. Therefore $y \in \overline{K'}$.

So assume $0 \neq y \notin \mathbb{R}_{>0}\pi(F')$, then $y = \sum_{v\in V(F')} \lambda_v \pi(v)$ and there exists a $w \in \pi(V(F'))$ such that $\lambda_w < 0$. Now there exists a unique $\alpha \in B_x^K$ such that $\alpha(w) > 0$, so we find $\alpha(\sum_{v\in V(F')} \lambda_v v) < 0$. Therefore $\alpha(y) < 0$ holds, and we obtain $y \notin \overline{K'}$.

The arrangement $(\mathcal{A}_x^\pi, T_x)$ therefore is simplicial.

For the second part of the statement, assume $x \in T$, then $\pi(T) = V_x$ and $\mathcal{A}_x^\pi$ is spherical.

Now assume $x \notin T$ and $\mathcal{A}$ is k-spherical. Let F' be as above, then F' is isomorphic as a simplicial complex to the closed chamber $\mathbb{R}_{>0}\pi(F')) \cup \{0\}$, and a face of F' meets T if and only if a face of $\mathbb{R}_{>0}\pi(F') \cup \{0\}$ meets T_x. Now F' is an $m-1$-simplex, as it is spanned by m vertices, likewise F_x is an $r-m-1$-simplex.

Let $F_1 \subset F'$ be a face of F', and assume F_1 is an l-simplex. Then $V(F_1) \cup V(F_x)$ generate an $r-m+l$-simplex F_2 of $\mathcal{S}$. As $\mathcal{A}$ is k-spherical, F_2 therefore meets T if $r-k-1 \leq r-m+l$ is satisfied, or equivalently $m-k-1 \leq l$. Under this condition also $\pi_x(F_2) = \pi_x(F_1)$ meets T_x, we can conclude that $\mathcal{A}_x^\pi$ is k-spherical. Since $\mathcal{A}_x^\pi$ will not be more than m-spherical, since it is an arrangement of rank m, we find that $\mathcal{A}_x^\pi$ is $\min(k,m)$-spherical.

So as $(\mathcal{A}, T)$ is thin, $(\mathcal{A}_x^\pi, T_x)$ is a thin simplicial hyperplane arrangement with root system R_x. Since chambers $K \in \mathcal{K}_x$ and in $K' \in \mathcal{K}'$ are uniquely determined by the sets B_x^K and $B^{K'}$ we find that π induces a bijection between K_x and K'. □

Remark 2.7.6. 1. The simplicial arrangement $(\mathcal{A}_x^\pi, T_x, R_x)$ does not strictly depend on the point x, but on the subspace spanned by x, as one gets the same arrangement for every λx, $0 < \lambda \in \mathbb{R}$.

2. We showed in the prove above implicitly that the simplicial complex $\mathcal{S}_x$ associated to the arrangement $(\mathcal{A}_x, T_x)$ is isomorphic to the star of F_x in $\mathcal{S}$, where $\mathcal{S}$ is the complex associated to $(\mathcal{A}, T)$, and $F_x \in \mathcal{S}$ is the smallest simplex containing x.

3. When $x \notin T$, $(\mathcal{A}_x^\pi, T_x)$ may become k'-spherical for some $k' > \min(k, m)$. The reason for this is that $\mathcal{A}$ not being k'-spherical does not imply that every $r - k' - 1$-simplex contained in a chamber in $\mathcal{K}_x$ does not meet T.

Note that the above statements make sense if $R_x = \emptyset$, this occurs if and only if either $x \in T$ is in the interior of a chamber, or $x \notin T$ does not meet any hyperplane $H \in \mathcal{A}$. However, in this case we have $\langle B_x^K \rangle = \{0\}$ and the induced arrangement is the empty arrangement. This is not a problem, since in this case $W^\perp = V$ and $V_x = \{0\}$, but this case is somewhat trivial. Another trivial case occurring can be $x = \{0\}$, in which case $R_x = R$, $\mathcal{A}_x = \mathcal{A}$ and $T_x = T$.

Therefore the requirement $R \neq R_x \neq \emptyset$ is quite natural to make, in particular the assumption $R_x \neq \emptyset$ is helpful at times.

Lemma 2.7.7. *The set $\mathcal{K}_x$ is connected.*

Proof. The set $\mathcal{A}_x^\pi$ is a simplicial arrangement on T_x and therefore $\mathcal{K}'$ is connected by Proposition 2.4.15. Thus $\mathcal{K}_x$ is connected as well, as π preserves adjacency. $\square$

We can also give an exact criterion to when $\mathcal{A}_x^\pi$ is a spherical arrangement:

Corollary 2.7.8. *Assume $(\mathcal{A}, T, R)$ is a simplicial arrangement with rank $r \geq 2$. Let $x \in \overline{T}$. Then $\mathcal{A}_x$ and R_x are finite if and only if $x \in T$.*

In particular, a simplicial arrangement is finite if and only if it is spherical.

Proof. If $x \in T$, $\mathcal{A}_x$, R_x are finite by Proposition 2.7.5. So assume $\mathcal{A}_x$, R_x are finite and let $x \in \overline{T}$. Then also $\mathcal{K}_x$ is a finite set, so let $K \in \mathcal{K}_x$ and by Lemma 2.7.7 we find $K' \in \mathcal{K}_x$ such that $d(K, K')$ is maximal. Hence the indusced simplicial complex $\mathcal{S}_x$ associated to $(\mathcal{A}_x, T_x)$ is spherical, by Proposition 2.3.27 we find that $(\mathcal{A}_x, T_x)$ itself is spherical.

The last statement is obtained by taking $x = 0_V$. $\square$

Lemma 2.7.9. *The root system R is reduced if and only if R_x is reduced for every $x \in T$.*

Proof. This follows immediately since R_x is constructed as a subset of R and $R = \bigcup_{x \in T} R_x$. $\square$

Lemma 2.7.10. *Let $(\mathcal{A}, T, R)$ be a simplicial arrangement. Let $x \in \overline{T}$ with $R_x \neq \emptyset$. Let $K, L \in \mathcal{K}^x$ be adjacent by α_1, and $B^K = \{\alpha_1, \ldots, \alpha_r\}$, $B^L = \{\beta_1, \ldots, \beta_r\}$ indexed compatibly with B^K. Then $B_x^K \to B_x^L, \alpha_i \mapsto \beta_i$ is a bijection .*

Proof. We know by Lemma 2.7.1 that $\langle B_x^K \rangle = \langle B_x^L \rangle$. Since B_x^K, B_x^L consist of linear independent vectors, we get $|B_x^K| = |B_x^L|$.

It remains to show that the map is well defined. For $\alpha_i \in B_x^K$ we find $x \in \beta_i^\perp$. We have $x \in \alpha_1^\perp$ and $x \in \alpha_i^\perp$. Now $\beta_i = \lambda_1 \alpha_1 + \lambda_i \alpha_i$ for some $\lambda_1, \lambda_i \in \mathbb{Z}$ by Lemma 2.5.5, so $\beta_i(x) = \lambda_1 \alpha_1(x) + \lambda_i \alpha_i(x) = 0$ and we are done. $\square$

We obtain the following:

Corollary 2.7.11. *Let $(\mathcal{A}, T, R)$ be a crystallographic arrangement with notation as in 2.7.10. Then $\sigma_{K,L}|_{B_x^K}$ is a bijection from B_x^K to B_x^L mapping α_i to β_i.*

Proposition 2.7.12. *Let $x \in \overline{T}$ with $R_x \neq \emptyset$ and let $(\mathcal{A}, T, R)$ be a crystallographic arrangement in T. Then the simplicial arrangement $(\mathcal{A}_x^\pi, T_x, R_x)$ at x satisfies $R_x \subset \sum_{\alpha \in B_x^K} \mathbb{Z}\alpha$ for all $K \in \mathcal{K}$. In particular, $(\mathcal{A}_x^\pi, T_x, R_x)$ is crystallographic.*

Proof. Let $K, L \in \mathcal{K}_x$ be adjacent, assume $B^K = \{\alpha_1, \dots, \alpha_r\}$, and let $B^L = \{\beta_1, \dots, \beta_r\}$ be indexed compatibly with B^K. Assume w.l.o.g. $\overline{K} \cap \overline{L} \subset \alpha_1^\perp \cap \overline{K}$ and $B_x^K = \{\alpha_1, \dots, \alpha_m\}$, $B_x^L = \{\beta_1, \dots, \beta_m\}$ for some $1 \leq m \leq r$. We know by the previous Corollary 2.7.11 that the mapping $\sigma_{K,L}|_{B_x^K} : B_x^K \to B_x^L, \alpha_i \mapsto c_i \alpha_1 + \alpha_i$ is a bijection. In particular, we get $B_x^L \subset \sum_{\alpha \in B_x^K} \mathbb{Z}\alpha$.

Now let $\alpha \in R_x$. Since $x \in \alpha^\perp$, x is contained in some simplex, and therefore also in some maximal simplex. Thus there exists a chamber $L \in \mathcal{K}_x$, such that $\alpha \in B_x^L$.

Since $\mathcal{K}_x$ is connected by Lemma 2.7.7, there exists a chain $K_0, K_1, \dots, K_{k-1}, K_k$ such that $K_0 = K$, $K_k = L$, and K_{i-1}, K_i are adjacent. It follows by induction, where we use the fact that σ_{K_{i-1},K_i} maps roots to integral linear combinations of $B_x^{K_{i-1}}$, yields $B_x^{K_i} \subset \sum_{j=1}^m \mathbb{Z}\alpha_i$ for all $0 \leq i \leq k$. Hence the statement holds. $\square$

Proposition 2.7.13. *Assume $r \neq 2$ and let $(\mathcal{A}, T, R)$ be a simplicial arrangement. If for all $0 \neq x \in \overline{T}$ the arrangements $(\mathcal{A}_x^\pi, T_x, R_x)$ are crystallographic, then $(\mathcal{A}, T, R)$ is crystallographic.*

If $(\mathcal{A}, T, R)$ is CH-like, then $(\mathcal{A}, T, R)$ is crystallographic if for all $0 \neq x \in T$ the arrangements $(\mathcal{A}_x^\pi, T_x, R_x)$ are crystallographic.

Proof. If $r = 1$ we can conclude that $T = \mathbb{R}$ or $T = \mathbb{R}_{>0}$. In the first case the root system R is 1-dimensional and crystallographic. In the second case there does not exist a thin hyperplane arrangement, as every hyperplane is just $\{0\}$ and this does not intersect $\mathbb{R}_{>0}$. So let $r \geq 3$.

We know that $\mathcal{K}$ is connected by Proposition 2.4.15, so it is enough to show that for two adjacent chambers $K, L \in \mathcal{K}$ we find $B^L \subset \sum_{\alpha \in B^K} \mathbb{Z}\alpha$. The proposition follows then by induction on the length of a minimal gallery between arbitrary chambers $K', L' \in \mathcal{K}$.

Assume further that $B^K = \{\alpha_1, \dots, \alpha_n\}$, $B^L = \{\beta_1, \dots, \beta_n\}$ is indexed compatibly with B^K and K, L are adjacent in α_1. Take two distinct vertices v_1, v_2 of $\overline{K}$ such that

$v_1, v_2 \in \alpha_1^\perp$. Note that if $(\mathcal{A}, T, R)$ is CH-like, these vertices are contained in T. As $\overline{L}$ is a simplicial cone, there exists a unique maximal face not containing v_1, which must contain v_2. Therefore by Proposition 2.7.12

$$B^L \subset B^L_{v_1} \cup B^L_{v_2} \subset \sum_{\alpha \in B^K_{v_1}} \mathbb{Z}\alpha \cup \sum_{\alpha \in B^K_{v_2}} \mathbb{Z}\alpha \subset \sum_{\alpha \in B^K} \mathbb{Z}\alpha.$$

So we are done. □

Example 2.7.14. The requirement $r \neq 2$ in Proposition 2.7.13 is actually necessary. Take the root system of type $\tilde{A}_1$, $R = \{e_i^\vee + k\gamma \mid i = 1, 2, k \in \mathbb{Z}\}$, where the e_i are the standard base vectors and $\gamma = e_1^\vee + e_2^\vee$. Denote with $v_\lambda = \lambda e_1 + (1-\lambda)e_2$ and take as a vertex set the set $\{v_\lambda \mid \lambda \in \mathbb{Z}\}$, which is a lattice in the affine space $W = \{v \in \mathbb{R}^2 \mid \gamma(v) = 1\}$. An open simplex s_λ is just the open convex hull of v_λ and $v_{\lambda+1}$, and the chambers K_λ are the respective cones $\mathbb{R}_{>0}s_\lambda$ in $T = \gamma^+$.

Given the chambers as above, we find the bases $B^\lambda := B^{K_\lambda}$ as $B^0 = \{e_1^\vee, e_2^\vee\}$ and

$$\begin{aligned} B^n &= \{e_2^\vee + n\gamma, -(e_2^\vee + (n-1)\gamma\}, \\ B^{-n} &= \{e_1^\vee + n\gamma, -(e_1^\vee + (n-1)\gamma\} \end{aligned}$$

where $n \in \mathbb{N}$. Note that the order in which the elements of the sets are written down does not give a compatible indexing with each other. Now for $n \in \mathbb{N}, m \in \mathbb{N}_0$ the arrangements at the vertices are given by the roots

$$\begin{aligned} R_{v_n} &= \{\pm(e_2^\vee + (n-1)\gamma)\}, \\ R_{-m} &= \{\pm(e_1^\vee + m\gamma\}. \end{aligned}$$

The root system R itself is crystallographic as well as the arrangements at the points v_λ. Also note that $R = \bigcup_{\lambda \in \mathbb{Z}} B^\lambda$.

One can modify R by defining the root system

$$\tilde{R} = \{(k+1)(e_i^\vee + k\gamma) \mid i = 1, 2, k \in \mathbb{Z}\}.$$

Since $\tilde{R}$ does only contain multiples of the elements in R, we find that the arrangement induced by R and by $\tilde{R}$ is actually the same, so we find the same set of chambers K_λ, which are induced by the same simplices on the same vertex set. Then the root system at the vertices are

$$\begin{aligned} \tilde{R}_{v_n} &= \{\pm n(e_2^\vee + (n-1)\gamma)\}, \\ \tilde{R}_{v_{-m}} &= \{\pm(m+1)(e_1^\vee + m\gamma\}. \end{aligned}$$

This implies that for every point in T the induced arrangement is crystallographic. Note that every point in T different from the v_λ is either a multiple of some v_λ and therefore induces the same arrangement or is in the interior of a simplicial cone and induces the empty arrangement.

Now consider the root bases with respect to $\tilde{R}$, which we shall call $\tilde{B}$ to distinguish them from the sets B^λ. These are of the form.

$$\begin{aligned}\tilde{B}^n &= \{(n+1)(e_2^\vee + n\gamma), -n(e_2 \vee +(n-1)\gamma\},\\ \tilde{B}^{-n} &= \{e_1^\vee + n\gamma, -(e_1^\vee + (n-1)\gamma\}.\end{aligned}$$

So $\tilde{B}^0 = \{e_1^\vee, e_2^\vee\}$, $\tilde{B}^1 = \{2e_1^\vee + 4e_2^\vee, -e_2^\vee\}$. Now $e_1^\vee = \frac{1}{2}(2e_1^\vee + 4e_2^\vee) + 2(-e_2^\vee)$, so $\tilde{R}$ does not satisfy the crystallographic property.

Remark 2.7.15. 1. The restriction $0 \neq x$ is required as $\mathcal{A}_0 = \mathcal{A}$, therefore the statement in Proposition 2.7.13 becomes a tautology if we do not omit the case $x \neq 0$.

2. For the proof of Proposition 2.7.13 it is actually sufficient to assume the crystallographic property for all induced arrangements $(\mathcal{A}_x, T_x, R_x)$ where $0 \neq x$ is contained in some vertex v in the simplicial complex $\mathcal{S}$. It is also not hard to see that being crystallographic in such a point implies R_y being crystallographic for all y such that the minimal simplex F_y containing y is contained in $\mathrm{St}(v)$. Thus R_y is crystallographic for all $y \in \overline{T}$.

3. While in Example 2.7.14 we considered an affine example, one can also think about a spherical example in dimension 2:

   Start with the arrangement of A_2,

   $$R = \{\pm e_1^\vee, \pm e_2^\vee, \pm(e_1^\vee + e_2^\vee)\}.$$

   This is a simplicial crystallographic spherical arrangement. Now consider the set of roots

   $$\tilde{R} = \{\pm e_1^\vee, \pm e_2^\vee, \pm 2(e_1^\vee + e_2^\vee)\},$$

   which induces the same set of hyperplanes, the arrangement at every point $x \neq 0$ is either empty or 1-dimensional and therefore crystallographic. But $\tilde{R}$ itself is not crystallographic.

2.8 Restrictions of hyperplane arrangements

In this section, we will discuss how a thin simplicial arrangement in T induces a simplicial arrangement on certain hyperplanes. In the classical theory of hyperplane arrangements this is also called the restriction of an arrangement [cp. [OT92]].

Definition 2.8.1. Let $(\mathcal{A}, T, R)$ be a simplicial arrangement. Let $H \leq V$ be a hyperplane in V. Set

$$\mathcal{A}^H := \{H' \cap H \leq H \mid H' \in \mathcal{A} \setminus \{H\}, H' \cap H \cap T \neq \emptyset\},$$

this is a set of hyperplanes in H which have non-empty intersection with $T \cap H$, if $T \cap H$ is not empty itself. Define

$$\pi_H^* : V^* \to H^*, \alpha \mapsto \alpha|_H,$$

and set

$$R^H := \pi_H^*(R) \setminus (\{0\} \cup \{\alpha \in \pi_H^*(R) \mid \alpha^\perp \cap H \cap T = \emptyset\}).$$

We can also define the connected components of $H \setminus \bigcup_{H' \in \mathcal{A}^H} H'$ as $\mathcal{K}^H$.

Remark 2.8.2. Note that in the case $r = 0$ there exists no hyperplane which is not in the arrangement. In the case $r = 1$ the set $\mathcal{A}^H$ is just the point $\{0\}$ or empty. Our statements will remain true in these cases, but most of the time they will be empty.

In particular we will examine the case where $H \in \mathcal{A}$, since otherwise we will not necessarily see an induced simplicial complex.

An interesting special case occurs for affine arrangements with radical γ and $H = \gamma^\perp$, since this might yield as R^H a root system associated to a spherical simplicial arrangement (cmp. Definition of an affine simplicial arrangement).

Lemma 2.8.3. *With notation as above, we find $\mathcal{A}^H = \{\alpha^\perp \cap H \mid \pi_H^*(\alpha) \in R^H\}$.*

Proof. This follows immediately from the definition. □

Lemma 2.8.4. *Let $H \in \mathcal{A}$ and $K' \in \mathcal{K}^H$. Then there exists a chamber $K \in \mathcal{K}$, such that $H \in W^K$ and $\overline{K'} = H \cap \overline{K}$.*

Proof. Let $x \in K'$. Since $\mathcal{S}$ is a simplicial complex, there exists a chamber $K \in \mathcal{K}$ with $H \in W^K$ and $x \in \overline{K}$. It remains to show that $\overline{K} \cap H = \overline{K'}$.

Assume there exists a $y \in K'$ with $y \notin \overline{K}$. This implies w. l. o. g. that there exists an $\alpha \in R$ with $\alpha(x) \geq 0$ and $\alpha(y) < 0$. So since K' is convex there exists a $z \in K'$ with $\alpha(z) = 0$ and $z \in \alpha^\perp \cap H$. In particular, $\alpha^\perp \cap H \in \mathcal{A}^H$. This is a contradiction to x and y being in the same connected component K'. So $\overline{K'} \subset \overline{K} \cap H$.

Now let $y \in \overline{K} \cap H$, and assume it is not in K'. Then there exists a hyperplane $H' \in \mathcal{A}^H$ and $\alpha \in R^H$ such that $H' = \alpha^\perp$ and w. l. o. g. $\alpha(x) > 0$, $\alpha(y) \leq 0$. Again we get a contradiction to x, y being in the same simplex $\overline{K}$, we obtain $\overline{K'} \supset \overline{K} \cap H$ and thus the desired equality. □

Corollary 2.8.5. *The elements $K' \in \mathcal{K}^H$ are simplicial cones in $H \cap T$.*

Proof. If $r = 0$, the statement is empty, so let $r \geq 1$. The last Lemma yields that for $K' \in \mathcal{K}_H$ the set $\overline{K'}$ is actually an maximal face of some chamber $K \in \mathcal{K}$. This is a simplicial cone by Proposition 2.2.7. □

The following observation is immediate from a geometric point of view, but necessary to point out:

Lemma 2.8.6. *Let $H \leq V$ be an arbitrary hyperplane, and let $\alpha \in R$. Then we find $\pi_H^*(\alpha)^\perp = \alpha^\perp \cap H$, $\pi_H^*(\alpha)^+ = \alpha^+ \cap H$ and $\pi_H^*(\alpha)^- = \alpha^- \cap H$.*

Proof. First note that $\pi_H^*(\alpha)(x) = \alpha(x)$ by definition of π_H^*. The equalities follow immediately by considering the cases $\alpha(x) = 0$ or $\alpha(x) > 0$. □

The above lemma immediately yields:

Corollary 2.8.7. $\mathcal{A}^H = \{\alpha^\perp < H \mid \alpha \in R^H\}$.

Lemma 2.8.8. *Let $(\mathcal{A}, T, R)$ be a simplicial arrangement, $K \in \mathcal{K}$, $H \in W^K$. Let $B := \pi_H^*(B^K) \setminus \{0\}$. Then*

i) $H \cap \overline{K} = \overline{K'}$ for a unique $K' \in \mathcal{K}^H$,

ii) $\langle \overline{K'} \cap \alpha^\perp \rangle = \alpha^\perp$ and $\alpha^\perp \cap K' = \emptyset$ for $\alpha \in B$.

iii) $K' = \{x \in H \mid \alpha(x) > 0$ for all $\alpha \in B\}$.

Proof. Part i) is clear by the definition of $\mathcal{K}$ and $\mathcal{K}^H$, since $H \cap \overline{K}$ is a unique maximal face of $\overline{K}$.

For the second statement assume $H = \beta^\perp$ for $\beta \in B^K$. We use that $\overline{K'}$ is a maximal face of $\overline{K}$. The maximal faces of $\overline{K'}$ are exactly the sets of the form $\overline{K} \cap H \cap \alpha^\perp = \overline{K'} \cap \alpha^\perp$ for $\alpha \in B^K \setminus \{\beta\}$, by Lemma 2.8.6 we obtain that the faces can also be written as $\overline{K'} \cap \alpha^\perp$ for $\alpha \in B$. As the maximal face $\overline{K'} \cap \alpha^\perp$ spans a hyperplane in H contained in $\alpha^\perp$, we conclude $\langle \overline{K'} \cap \alpha^\perp \rangle = \alpha^\perp$ for $\alpha \in B$ and ii) holds.

Assertion iii) is a direct consequence of Lemma 2.8.6. □

Proposition 2.8.9. *Let $(\mathcal{A}, T, R)$ be a k-spherical simplicial arrangement in T, $k \geq 1$ and $H \in \mathcal{A}$. Then $(\mathcal{A}^H, T \cap H)$ is a $k-1$-spherical simplicial hyperplane arrangement. If $(\mathcal{A}^H, T \cap H)$ is thin, $(R^H, T \cap H)$ is a root system for $\mathcal{A}^H$.*

Proof. Note that R^H does not contain 0 by definition, and if $\alpha \in R^H$, we find $\alpha' \in R$ with $\alpha = \pi_H^*(\alpha')$, so $-\alpha' \in R$ and $-\alpha \in R^H$.

By Lemma 2.8.6 we know that $\mathcal{A}^H = \{\alpha^\perp \leq H \mid \alpha \in R^H\}$ and by definition we have $\alpha^\perp \cap H \cap T \neq \emptyset$. Furthermore we know that the connected components in $\mathcal{K}^H$ are simplicial cones by Corollary 2.8.5.

Let $K' \in \mathcal{K}^H$ and $K \in \mathcal{K}$ such that K' is a face of K. Let $\beta \in B^K$ such that $\beta^\perp = H$. By ii) in Lemma 2.8.8 we find $W^{K'} = \{\pi_H^*(\alpha)^\perp \mid \alpha \in B^K \setminus \{\beta\}\}$, and together with iii) in Lemma 2.8.8 we find reduced roots $B^{K'} = \{\lambda_\alpha \alpha \in (R^H)^{\mathrm{red}} \mid \alpha \in \pi_H^*(B^K) \setminus \{0\}, \lambda_\alpha \in \mathbb{R}_{>0}\}$ with the property $K' = \{x \in H \mid \alpha(x) > 0$ for all $\alpha \in B^{K'}\}$. Hence $\mathcal{A}^H$ is a simplicial arrangement in $T \cap H$, and $\mathcal{A}^H = \{\alpha^\perp \mid \alpha \in R^H\}$ holds. So if $\mathcal{A}^H$ is thin, R^H is a root system for $\mathcal{A}^H$.

Now assume F is an m-simplex in the simplicial complex $\mathcal{S}^H$ associated to the simplicial hyperplane arrangement $(\mathcal{A}^H, T \cap H)$. Since this is a subset of $\mathcal{S}$, F meets $T \cap H$ if and only if F meets T, and therefore $(\mathcal{A}^H, T \cap H)$ is $k-1$-spherical if $(\mathcal{A}, T)$ is k-spherical. □

Corollary 2.8.10. *Assume that $(\mathcal{A}^H, T \cap H, R^H)$ is a thin simplicial arrangement. Let $K \in \mathcal{K}$ such that $H \in W^H$. Then the set $\pi_H^*(B^K) \setminus \{0\}$ is a basis of H.*

Proof. By Proposition 2.8.9 the set $B^{K'}$ is a basis of H, by Lemma 2.8.8 iii) we find that the elements in $\pi_H^*(B^K) \setminus \{0\}$ are non zero scalar multiples of $B^{K'}$. □

Definition 2.8.11. We will call $(\mathcal{A}^H, T \cap H)$ for $H \in \mathcal{A}$ as above the *induced simplicial hyperplane arrangement (by $(\mathcal{A}, T)$) on H* or the *restriction of $(\mathcal{A}, T)$ to H.*

Remark 2.8.12. Given a simplicial arrangement $(\mathcal{A}, T, R)$, in general it seems that properties of R are hard to transfer to restrictions. This can be noticed when looking at the root system of F_4 or $\tilde{F_4}$ in the example below. Here we start with the strongest properties we have, i.e. a crystallographic reduced root system associated to a Weyl group. However, inducing on reflection hyperplanes yields in some cases only a root system associated to a non-standard Cartan graph, which is not reduced anymore. However, the crystallographic property is inherited. We will dedicate the rest of this section to show that this is always the case for restrictions.

From now on, assume that $(\mathcal{A}, T, R)$ is a crystallographic simplicial arrangement. In this case, we can state a stronger version of Lemma 2.8.8.

Proposition 2.8.13. *Let $(\mathcal{A}, T, R)$ be an 2-spherical crystallographic simplicial arrangement in T and $H \in \mathcal{A}$. Let $K \in \mathcal{K}$ such that $H \in W^K$ and $K' \in \mathcal{K}^H$ such that $\overline{K'} = \overline{K} \cap H$. Then*

i) $W^{K'} = \{\alpha^\perp \mid \alpha \in \pi_H^*(B^K) \setminus \{0\}\}$,

ii) For $\alpha \in \pi_H^*(B^K) \setminus \{0\}$ *we have* $\beta \in R^H \cap \langle \alpha \rangle \implies \beta = \lambda\alpha,\ \lambda \in \mathbb{Z}$,

iii) $\pi_H^*(B^K) \setminus \{0\} \subset (R^H)^{\mathrm{red}}$,

iv) $\pi_H^*(B^K) \setminus \{0\} = B^{K'}$.

Proof. Assertion i) is an immediate consequence of Lemma 2.8.8 ii).

To check ii), let $\alpha_1 \in B^K$ such that $\alpha_1^\perp = H$, let $\alpha \in R$ and $R_\alpha := \{\beta \in R \mid \beta^\perp \cap H = \alpha^\perp \cap H\} = \{\beta \in R \mid \pi_H^*(\beta) \in \langle \pi_H^*(\alpha) \rangle \setminus \{0\}\}$ as in the proof of Proposition 2.8.9. Let $x \in \alpha^\perp \cap H \cap T$, and consider the arrangement $(\mathcal{A}_x, T_x R_x)$. This is a crystallographic arrangement by Corollary 2.7.12, as $(\mathcal{A}, T, R)$ is crystallographic. With respect to this arrangement consider the linearly independent set B_x^K, then $\alpha_1, \alpha \in B_x^K$. Let $B_x^K = \{\alpha_1, \alpha, \tau_3, \dots, \tau_m\}$ for some $m \in \mathbb{N}$. Now for an arbitrary $\beta \in R_\alpha$, we know $\beta \in R_x$ since $x \in \alpha^\perp \cap H = \beta^\perp \cap H$. Therefore $\beta = \lambda_1\alpha_1 + \lambda_2\alpha + \sum_{i=3}^m \lambda_i\tau_i$ with $\lambda_i \in \mathbb{Z}$ for $i = 1, 2, \dots, m$ either all positive or negative.

By Corollary 2.8.10 we have that $\pi_H^*(B^K) \setminus \{0\}$ is a linearly independent set. As a consequence the application of π_H^* yields

$$\pi_H^*(\beta) = \lambda_2\pi_H^*(\alpha) + \sum_{i=3}^m \lambda_i\pi_H^*(\tau_i),$$

and thus by the choice of β we find $\pi_H^*(\beta) = \lambda_2\pi_H^*(\alpha)$ with $\lambda_2 \in \mathbb{Z}$, as desired. This shows ii).

Assertion iii) is a consequence of ii), with respect to the standard reductor, which also exists due to ii). Finally, iv) is immediate from iii) and Lemma 2.8.8. □

Remark 2.8.14. The above proposition also yields that to acquire $(R^H)^{\mathrm{red}}$ it is sufficient to consider the $\pi_H^*(B^K)$ for all chambers K with $H \in W^K$. In other words,

$$(R^H)^{\mathrm{red}} = \bigcup_{H \in W^K} \pi_H^*(B^K) \setminus \{0\}.$$

Proposition 2.8.15. *Let $(\mathcal{A}, T, R)$ be an 2-spherical crystallographic simplicial arrangement in T and $H \in \mathcal{A}$. Then $(\mathcal{A}^H, T \cap H, R^H)$ is a crystallographic arrangement.*

Proof. Let $H = \alpha_1^\perp$ for some $\alpha_1 \in R^{\mathrm{red}}$. Let $K' \in \mathcal{K}_H$ and let $K \in \mathcal{K}$ such that $\overline{K'} \subset \overline{K}$, by Proposition 2.8.13 iv) we know $B^K = \{\alpha_1, \dots, \alpha_r\}$ with $B^{K'} = \{pi_H^*(\alpha_i) \mid i = 2, \dots r\}$. Take $\beta_H \in R^H$, $\beta \in R$ such that $\pi_H^*(\beta) = \beta_H$. Since (A, T, R) is crystallographic, $\beta = \sum_{i=1}^n \lambda_i\alpha_i$ with $\lambda_i \in \mathbb{Z}$. Therefore we get

$$\pi_H^*(\beta) = \sum_{i=1}^r \lambda_i\pi_H^*(\alpha_i) = \sum_{i=2}^n \lambda_i\pi_H^*(\alpha_i),$$

so R^H is indeed crystallographic. □

Remark 2.8.16. In the case where $(\mathcal{A}, T, R)$ is not 2-spherical, but still thin, the restriction $(\mathcal{A}^H, T \cap H)$ might be thin nonetheless, and in this case $(A^H, T \cap H, R^H)$ is again a crystallographic arrangement.

Example 2.8.17. The property of being reduced is not inherited by R^H. Also, if R is a root system, R^H constructed in the way above does not need to be a root system as well, as the following example shows. Take the root system of $\tilde{F_4}$ (which certainly is reduced). Let $\iota : \mathbb{R}^4 \to (\mathbb{R}^4)^*, v \mapsto (v, \cdot)$ be the standard isomorphism, where $(\cdot, \cdot)$ is the standard scalar product. Note that ι takes the standard basis $\{e_1, e_2, e_3, e_4\}$ to its dual $\{e_1^\vee, e_2^\vee, e_3^\vee, e_4^\vee\}$. We will denote elements of $(\mathbb{R}^4)^*$ as vectors with respect to this basis.

A set of simple roots for F_4 in $\mathbb{R}^4$ is for example(cp. [Bou02]):

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right\}.$$

We denote these linear forms by $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ in the order as above. Then the whole root system $R(F_4)$ can be described as ι of

i) vectors with two components 1 or -1, 0 otherwise,

ii) vectors with one component 1 or -1, 0 otherwise,

iii) vectors with all four components $\frac{1}{2}$ or $-\frac{1}{2}$.

So there are 24 roots of type i), 8 of type ii) and 16 of type iii). We will compute orthogonal projections of $R(F_4)$ on two simple roots. For $1 \leq i \neq j \leq 4$ let $\pi_{ij} : (\mathbb{R}^4)^* \to (\varphi_i^\perp \cap \varphi_j^\perp)^*$ denote the respective restriction. The respective projections are:

$$\pi_{12}(R(F_4)) = \left\{ \pm \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \pm \begin{pmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \pm \begin{pmatrix} -1 \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \pm \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \right.$$
$$\left. \pm \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \pm \begin{pmatrix} 0 \\ \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \pm \begin{pmatrix} \frac{1}{2} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix}, \pm \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix}, 0 \right\},$$

$$\pi_{13}(R(F_4)) = \left\{ \pm \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \pm \begin{pmatrix} -1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \right.$$
$$\left. \pm \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \pm \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \pm \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, 0 \right\},$$

$$\pi_{14}(R(F_4)) = \left\{ \pm \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}, \pm \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix}, \pm \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \\ \frac{3}{4} \end{pmatrix}, \pm \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \pm \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}, \right.$$

$$\left. \pm \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \pm \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix}, 0 \right\},$$

$$\pi_{23}(R(F_4)) = \left\{ \pm \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \right.$$

$$\left. \pm \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, 0 \right\},$$

$$\pi_{24}(R(F_4)) = \left\{ \pm \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \pm \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \right.$$

$$\left. \pm \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \pm \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, 0 \right\},$$

$$\pi_{34}(R(F_4)) = \left\{ \pm \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{pmatrix}, \pm \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \\ 0 \end{pmatrix}, \pm \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix}, \right.$$

$$\left. \pm \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, 0 \right\}.$$

We denote with R_{ij} the set $\pi_{ij}(R(F_4))\backslash\{0\}$. These are non-reduced crystallographic rank two root systems. After reducing, consider the images of the two remaining elements of the simple roots. Writing the roots as linear combinations of the two yields:

- R_{23} and R_{24} are combinatorially equivalent to B_2.
- R_{12} and R_{34} are combinatorially equivalent to G_2.

- R_{13} and R_{14} are combinatorially equivalent to $R(1,2,2,2,1,4)$.

Here $R(1,2,2,2,1,4)$ denotes the rank two root system associated to the sequence $(1,2,2,2,1,4)$ according to the classification of spherical rank two Weyl groupoids by Cuntz and Heckenberger in [CH09a].

3 Coxeter groups without small triangle subgroups

3.1 Introduction

The twist conjecture (see [Müh06] for details) is motivated by the isomorphism problem for Coxeter groups, and in fact a proof of this conjecture would yield a complete solution. The conjecture has been proved for skew-angled Coxeter systems [MW02], for 2-spherical Coxeter systems [CM07], for chordal Coxeter systems [RT08] and for twist rigid Coxeter systems [CP10]. The references [MW02] and [RT08] use the decomposition of the Coxeter system as a graph of groups. Although this approach turns out to be very efficient for the special cases considered, it seems to be very difficult to generalise it to arbitrary diagrams. The main difficulty arises when there are local twists which do not extend to global twists. The conditions required in those papers are designed to have control over the local twists.

This chapter follows a strategy which had been used for the right angled case in [Müh98]. Although the twist conjecture hasn't been formulated when that paper was written, its validity for the right-angled case is proved there. The strategy is to introduce a distance matrix for a Coxeter generating set consisting of reflections and to show that one can reduce it by elementary twists. This works very well in the right-angled case, but it becomes considerably more complicated if there are edges with finite labels in the diagram.

In this chapter we prove the conjecture for diagrams which do not have certain rank 3 subdiagrams, including the irreducible spherical ones. The exclusion of those diagrams is essential to avoid higher rank twists, yet our condition does not allow some other types of diagrams including $\tilde{C}_2$ and $\tilde{G}_2$, which is designed to avoid technical details which become quite involved. However, although we cover a large class of Coxeter systems here for which the twist conjecture is not proved yet, our technique certainly needs substantial improvements in order to treat the general case. Yet the methods we develop are the first to directly handle rotation twists in a geometric way, using an approach which is derived from [Müh98]. While the skew-angled and chordal Coxeter systems allow this type of twists as well, the works in [MW02] and [RT08] avoid this by using Bass-Serre theory.

Here is the main result of this chapter:

Main Theorem. Suppose that (W, S) is an irreducible non-spherical Coxeter system of finite rank greater or equal 3, such that its diagram contains no subdiagrams of the type $\bullet\overset{3}{\text{———}}\bullet\overset{n}{\text{———}}\bullet$ for $n \geq 3$ or $\bullet\overset{4}{\text{———}}\bullet\overset{n}{\text{———}}\bullet$ for $n \geq 4$. If $R \subset S^W$ is an irreducible sharp-angled Coxeter generating set for W, then $R \sim_t S$.

We will later in 3.2.1 denote this condition on the diagram as condition (E), referring to the original intention to handle even Coxeter groups. The definition of sharp-angled can be reviewed in Section 3.2.2, the definition of twist-equivalence $\sim_t$ can be found in Section 3.2.5.

It is worthwhile to mention that our result covers the skew-angled Coxeter systems for which the twist conjecture was proved in [MW02].

In Section 2 we fix notation and recall definitions concerning the Cayley graph of a Coxeter system and its roots and walls. Most of the properties stated are taken from [MW02]. We also introduce longest reflections and their properties (3.2.3) as well as two notions of separation (3.2.4) and recall the definition of twists (3.2.5).

In Section 3 we give a characterisation for a Coxeter generating set satisfying our conditions to be geometric. This will act as a base of induction for our main theorem. In the main part of this section we show that whenever neither a reflection in R nor a longest reflection separates two other reflections in R, the set R is already geometric.

In Section 4 we prove our main theorem, distinguishing three different settings of the positions of the walls in the Cayley graph. We show that in each case we find a twist or a series of twists such that the resulting Coxeter generating set has a reduced distance matrix. To do this we first prove some properties of the distances in the Cayley graph and introduce interior separation, a stronger notion of separation taking into consideration walls of longest reflections.

Note. This chapter has been published in Innovations in Incidence Geometry, see [Wei11].

3.2 Preliminaries

3.2.1 Coxeter matrices, systems, diagrams

Let I be a finite set. A *Coxeter matrix over I* is a symmetric matrix $M = (m_{ij})_{i,j\in I}$ with entries in $\mathbb{N} \cup \{\infty\}$ such that $m_{ii} = 1$ for all $i \in I$, $m_{ij} \geq 2$ for all $i \neq j \in I$.

Given a Coxeter matrix M, (W, S) is a *Coxeter system of type M* if W is a group, $S = \{s_i \mid i \in I\} \subset W$ and $\langle S \mid (s_i s_j)^{m_{ij}}, i, j \in I\rangle$ is a presentation for W. For a Coxeter matrix M the *Coxeter diagram* is the undirected graph $\Gamma = (V, E)$ with $V = I$, $E = \{\{i, j\} \mid 2 < m_{ij}\}$ and the labelling $\tau : E \to \mathbb{N}$, $\{i, j\} \mapsto m_{ij}$. The *rank* of the diagram, of the Coxeter matrix, of the Coxeter system is $|I| = |S|$. A group W is called a *Coxeter group* if there exists a subset $S \subset W$ such that (W, S) is a Coxeter system.

If W is a Coxeter group, $R \subset W$ is *universal* if $(\langle R\rangle, R)$ is a Coxeter system. A subset R is a *Coxeter generating set* if R is universal and $\langle R\rangle = W$, i.e. if (W, R) is a Coxeter system. A universal set R is *irreducible*, if there is no non-trivial partition $R = R_1 \dot{\cup} R_2$ such that $o(r_1 r_2) = 2$ holds for all $r_i \in R_i$, $i = 1, 2$. If R is Coxeter generating, it gives rise to a unique Coxeter matrix, justifying our notion of the *diagram of* R. For the subsets $S' \subset S$, $S' = \{s_i \mid i \in J\}$ the *special subgroups* are $W_J := \langle S'\rangle$. In this case, S' is a Coxeter generating set for W_J.

A diagram or subset $J \subset I$ is *spherical* if the generated Coxeter group W_J is finite. We say a diagram satisfies condition (E), if it does not contain subdiagrams of type $\bullet\overset{3}{\text{——}}\bullet\overset{n}{\text{——}}\bullet$ or $\bullet\overset{4}{\text{——}}\bullet\overset{n}{\text{——}}\bullet$ for $n \geq 3$.

Note that the diagram $\bullet\overset{3}{\text{——}}\bullet\overset{n}{\text{——}}\bullet$ is spherical for $2 \leq n \leq 5$.

Let R be a Coxeter generating set, for $J \subset R$ we set $J^\perp = \{r \in R\backslash J \mid rj = jr \;\forall\, j \in J\}$.

3.2.2 The Cayley graph, roots, walls, residues

Consider a Coxeter system (W, S) of type M over I. Then (C, P) with $C = W$ and $P = \{\{w, ws\} \mid w \in W, s \in S\}$ is an undirected graph. Let $\tau : P \to S, \{w, ws\} \mapsto s$ be a labelling. If for all $w \in W$, $P(w) = \{e \in P \mid w \in e\}$ the restriction $\tau|_{P(w)}$ is a bijection, then $\mathcal{C} = (C, P, \tau)$ is the *Cayley graph of* (W, S). The set C is the set of *chambers*, P the set of *panels*. We denote with $\delta : C \times C \to \mathbb{N}$ the distance function on the Cayley graph. For subsets $A, B \subset C$, define $\delta(A, B) = \min\{\delta(a, b) \mid a \in A, b \in B\}$. A *gallery of length* m, $\gamma = (c_0, \ldots, c_m)$, in $\mathcal{C}$ is a path of length m in (C, P), it is *minimal* if $\delta(c_0, c_m) = m$. We will sometimes identify a gallery with its set of chambers $\bigcup_{0\leq i\leq m}\{c_i\}$.

The group W acts on the chambers of $\mathcal{C}$, denoted by $w.c = wc \in C$ for $w \in W$. Regarding this action we have $(w.p)^\tau = p^\tau$ for $p \in P$, so τ is W-invariant.

The elements in S^W are called *reflections of* W *(with respect to* S*)*. Let $r \in S^W$, $P_r = \{p \in P \mid r.p = p\}$. The graph $(C,\ P \setminus P_r)$ has two connected components (see [Ron09], Proposition 2.6), called the *roots associated to* r. The set $C(r) = \bigcup_{p\in P_r} p$ is the *wall of* r. For any chamber $c \in C$, $H(r, c)$ is the unique root associated to r containing c. For $A \subset C$, if A is contained in one root, $H(r, A)$ is the well-defined root associated to r containing A. If H is a root associated to r, $-H$ is the unique root associated to r not equal to H. Therefore, if $c \in C(r)$, then $-H(r, c) = H(r, r.c)$. For $r, s \in S^W$ we define $\delta(r, s) := \delta(C(r), C(s))$.

Now let $c \in C$, $J \subset I$. The set $R_J(c) := cW_J$ is called a *J-residue*. A subset $A \subset C$ is called *residue* if it is a J-residue for some $J \subset I$. A residue A is *spherical* if it is a J-residue and J is spherical. Let $s, t \in S^W$, then we will denote with $A_{s,t}$ an arbitrary maximal spherical residue of the form $R_{\{s,t\}}(c)$, i.e. a residue stabilized by $\langle s, t\rangle$. In particular, the existence of $A_{s,t}$ implies that the product st has finite order.

We will need some basic properties of roots, walls and residues. Geometric versions of these statements can be found in [MW02], we will recall the results we need. The following Lemma is a well known fact, for more details see [Bou02].

Lemma 3.2.1. *([MW02], Lemma 2.3) A subgroup $U \leq W$ is finite if and only if it stabilizes a spherical residue.*

Lemma 3.2.2. *([MW02], Lemma 2.6) Let $U \leq W$ be finite, $\langle U, \{s\}\rangle$ be infinite for an $s \in S^W$. Then every spherical residue stabilized by U is contained in the same unique root associated to s.*

In the situation of the previous lemma, the notation of $H(s, U)$ for the root containing all spherical residues stabilized by U is justified whenever U is finite, $\langle s, U\rangle$ is infinite. In particular, we will write $H(s,t) := H(s, \langle t\rangle)$ if $o(st) = \infty$. Note that since $C(t)$ consists of chambers included in $\{t\}$-residues, we have $H(s,t) = H(s, C(t))$.

Remark 3.2.3. For convenience with our notation, we write $x^w = wxw^{-1}$ for $x \in W$ for the action of W on W by conjugation.

Lemma 3.2.4. *([MW02], Lemma 3.1)*

a) W acts on the set of walls and on the set of roots associated to $r \in S^W$. Let $w \in W$, then $w.C(r) = C(r^w)$. If H_r is a root associated to r, then $w.H_r$ is a root associated to r^w.

b) A root H associated to an element $r \in S^W$ is convex.

Let $U \leq W$. A subset $F \subset C$ is a *fundamental domain for U* if $C = \dot{\bigcup}_{u \in U} u.F$. Let $s, t \in S^W$ and let H_s, H_t be roots associated to s, t. The set $\{H_s, H_t\}$ is a *geometric pair* if $H_s \cap H_t$ is a fundamental domain for $\langle s, t\rangle$. Consider a set Φ of roots, it is *2-geometric* if each pair of roots in Φ is geometric, and *geometric* if it is 2-geometric and $\bigcap_{H \in \Phi} H \neq \emptyset$. A pair $\{H_s, H_t\}$ is *weakly geometric* if $\{H_s, H_t\}$ or $\{-H_s, -H_t\}$ is a geometric pair. A set Φ of roots is *weakly 2-geometric* if each pair of roots is weakly geometric. The set $R \subset S^W$ is geometric (2-geometric, weakly 2-geometric) if there exists a set $\Phi(R)$ of roots associated to the elements in R, such that $\Phi(R)$ is geometric (2-geometric, weakly 2-geometric). The set $R \subset S^W$ is *sharp-angled* if all $\{s, t\} \subset R$ are geometric. We note that if R is geometric with geometric set of roots $\Phi(R)$, then $F := \bigcap_{H \in \Phi(R)} H$ is a fundamental domain for $\langle R\rangle$ and $C(r) \cap F \neq \emptyset$ for all $r \in R$.

The following is a summary of Lemma 4.3, 4.4, 4.5 in [MW02], we will make constant use of these statements.

Lemma 3.2.5. *Let $R \subset S^W$ be a sharp-angled Coxeter generating set, $s, t \in R$. Then:*

a) If $o(st) = 2$, then $\{H_s, H_t\}$ is a geometric pair for all roots H_s, H_t associated to s, t.

b) If $2 < o(st) < \infty$ and H_s is a root associated to s, there is a unique root H_t associated to t such that $\{H_s, H_t\}$ is a geometric pair. Then $\{-H_s, -H_t\}$ is a geometric pair as well, $\{\pm H_s, \mp H_t\}$ is not geometric.

c) If st has infinite order, there exist unique roots H_s, H_t associated to s, t such that $\{H_s, H_t\}$ is a geometric pair. Then $-H_s \subset H_t$, $-H_t \subset H_s$ and $-H_s \cap -H_t = \emptyset$.

We will denote the intersections of the geometric pairs in part b) and c) of the previous lemma as the *standard fundamental domains.* Note that if st has infinite order, the standard fundamental domain is uniquely determined, if $2 < o(st) < \infty$, there are two standard fundamental domains $F := H_s \cap H_t$ and $-F := -H_s \cap -H_t$ for a geometric pair $\{H_s, H_t\}$ and $w_{\{s,t\}}.F = -F$ holds for the longest element $w_{\{s,t\}}$ in $\langle s, t\rangle$ (see Section 3.2.3 for details).

A generalisation of part c) of the previous lemma is the following:

Lemma 3.2.6. *If R is universal, irreducible and non-spherical such that R is geometric, then the geometric set of roots $\Phi(R)$ is unique.*

Proof. This follows directly from 3.2.5 if two elements in R have infinite order. If R is 2-spherical, we can make use of Proposition 7.2 in [CM07]. This yields that if $R \setminus \{r\}$ is spherical for an $r \in R$, such a geometric set is unique. Now we can consider the smallest irreducible non-spherical set $\bar{R} \subset R$ such that $\bar{R} \setminus \{r\}$ is spherical for some $r \in \bar{R}$. For $\bar{R}$ we already have a unique geometric set of roots, therefore the geometric set of roots for R is unique. □

For the readers convenience, we will also repeat a useful property in skew-angled Coxeter systems:

Lemma 3.2.7. *([MW02], Lemma 6.3) Let R be universal, $r, s, t \in R$ pairwise non-commuting elements. Then the product $rsrt$ has infinite order.*

3.2.3 Longest reflections and their basic properties

Consider a sharp-angled Coxeter generating set $R \subset S^W$ and a subset $J = \{s, t\} \subset R$ with $2 < o(st) < \infty$. We have a length function on W with respect to the generating set R and denote with w_J the longest element in $\langle J\rangle$. Define the *longest reflections* s_t, t_s in $\langle J\rangle$ as the elements of $\langle J\rangle \cap S^W$ of maximal length. If $o(st)$ is even, we define s_t to be the longest reflection commuting with s, t_s to be the longest reflection commuting with t. In this case we have $s_t = w_J s$, $t_s = w_J t$. If $o(st)$ is odd, we simply have $s_t = t_s = w_J$.

Remark 3.2.8. Since the reflections s_t, t_s are associated to the highest roots, the notion of a *highest reflection* for s_t and t_s is suggesting itself. We decided to denote them longest reflections, referring to the length function in W with respect to the Coxeter generating set R.

Also note that, given two Coxeter generating sets R, R' both containing J, the length functions on $\langle J\rangle$ with respect to R and with respect to R' are equal.

We need the following properties of s_t, t_s:

Lemma 3.2.9. *Let $R \subset S^W$ be a sharp-angled universal set satisfying (E). Let $J = \{s,t\} \subset R$, $2 < o(st) < \infty$. For all $u \in R \setminus J$ such that $J \cup \{u\}$ is irreducible $o(us_t) = o(ut_s) = \infty$ holds.*

Proof. This is a conclusion from Corollary 9.5 and Lemma 9.8 in [CM07]. If the diagram of $\{s,t,u\}$ is a tree, the statement follows from Corollary 9.5. If it is not a tree, this is Lemma 9.8. □

Remark 3.2.10. Note that (E) is critical for Lemma 3.2.9 to hold. In fact, (E) is the weakest assumption one can make on a sharp-angled universal set R, such that Lemma 3.2.9 still holds.

Proposition 3.2.11. *Consider R as in Lemma 3.2.9. Let $J = \{s,t\}$ with $2 < o(st) < \infty$, $u \in R \setminus J$. The sets $\{u, s_t\}$, $\{u, t_s\}$ are sharp-angled.*

Proof. If $J \cup \{u\}$ is irreducible, then $o(us_t) = \infty = o(ut_s)$ holds by 3.2.9. Thus we can consider the sets of roots $\{H(u,s_t), H(s_t,u)\}$ and $\{H(u,t_s), H(t_s,u)\}$ associated to the sets $\{u,s_t\}$, $\{u,t_s\}$. Define $F = H(u,s_t) \cap H(s_t,u) \neq \emptyset$. Let $x \in \langle u, s_t\rangle$, then $F \cap x.F = \emptyset$ for $x \neq 1_W$ and $\bigcup_{x \in \langle u,s_t\rangle} x.F = C$ hold. The set $\{u,s_t\}$ is geometric, the same holds for $\{u,t_s\}$.

If $J \cup \{u\}$ is reducible, this implies $o(us) = o(ut) = 2$. But then $o(ut_s) = o(us_t) = 2$ holds, let H_{s_t}, H_u arbitrary roots associated to s_t, u. For $F = H_{s_t} \cap H_u$ we have $u.F = H_{s_t} \cap -H_u$, $s_t.F = -H_{s_t} \cap H_u$, $us_t.F = -H_{s_t} \cap -H_u$. So $\bigcup_{x \in \langle u,s_t\rangle} x.F = C$ and $x.F \cap F = \emptyset$ for all $x \in \langle u, s_t\rangle$, $x \neq 1_W$. The set $\{u,s_t\}$ is geometric, the same holds for $\{u,t_s\}$. □

Lemma 3.2.12. *Consider R as in Lemma 3.2.9. Let $J = \{s,t\} \subset R$, $2 < o(st) < \infty$. Consider $u = u_0, \ldots, u_k = v \in R \setminus (J \cup J^\perp)$, $u_i u_{i+1}$ having finite order for $i = 0, \ldots k-1$. The roots $H(s_t,u), H(s_t,v)$ are well-defined and equal.*

Proof. Because of 3.2.9 $o(us_t) = o(vs_t) = \infty$ holds and $H(s_t,u), H(s_t,v)$ are well-defined. Furthermore $s_t u_i$ has infinite order for $i = 0, \ldots, k$. Assume $H(s_t,u) \neq H(s_t,v)$, then by using [MW02] Lemma 4.6 we obtain a reflection u_j such that the product $s_t u_j$ has finite order, a contradiction. □

In the beginning of Section 4 we will state further properties on the order of products of longest reflections.

3.2.4 Separating reflections and interiors

We extend the notion of separation used in [Müh98] for right-angled Coxeter systems to arbitrary sharp-angled Coxeter generating sets $R \subset S^W$. Consider a sharp-angled subset $\{s,u,v\} \subset S^W$. We define $s \in [u,v]$ and say *s separates u and v* if $o(uv) = \infty, o(su), o(sv) > 2$ and all roots H_s associated to s satisfy the following condition: Let

$\{H_u, H_v\}$ be the unique geometric set of roots associated to u, v, if $\{H_u, H_s\}$ is geometric, then $\{H_v, -H_s\}$ is geometric. In other words: uv has infinite order and the set $\{s, u, v\}$ is not geometric.

We will also need a slightly sharper notion of separation. We say that *s separates u and v reducibly*, $s \in_r [u, v]$, if $s \in [u, v]$ and $\delta(u^s, v) < \delta(u, v)$.

Remark 3.2.13. We will show later in 3.4.8, that the property $\delta(v^u, w) < \delta(v, w)$ is sufficient for $u \in_r [v, w]$.

We define for a sharp-angled Coxeter generating set $R \subset S^W$ the *interior of R* to be the set $R^\circ := \{r \in R \mid \exists\ s, t \in R : r \in_r [s, t]\}$. Define $R_2 := \{s_t \in R^W \mid s, t \in R, 2 < o(st) < \infty\}$ to be the set of longest reflections. Due to 3.2.11 the sets $\{s_t, u, v\}$ are sharp-angled for all $u, v \in R \setminus \{s, t\}$, thus we can define the *interior of R_2* to be the set $R_2^\circ := \{s_t \in R_2 \mid \exists\ u, v \in R \setminus \{s, t\} :\ s_t \in [u, v]\}$.

3.2.5 Twists

For a Coxeter generating set R and $J, K, L \subset R$ satisfying

1. J is irreducible spherical,
2. $o(kl) = \infty$ for all $k \in K, l \in L$,
3. $R = J \mathbin{\dot\cup} J^\perp \mathbin{\dot\cup} K \mathbin{\dot\cup} L$,

we say the pair (J, L) is *R-admissible*. For an R-admissible pair (J, L) define $T_{(J,L)}(R) := J \mathbin{\dot\cup} J^\perp \mathbin{\dot\cup} K \mathbin{\dot\cup} L^{w_J}$, called *the twist of R by J*.

Remark 3.2.14. If R is Coxeter generating, $T_{(J,L)}(R)$ is a Coxeter generating set as well. See [BMMN02] for basic properties of twists as well as for a proof that $T_{(J,L)}(R)$ is indeed Coxeter generating. Our condition (E) implies for admissible pairs (J, L) that J either consists of one element or generates a finite dihedral group. We will use the fact that in the case of $|J| = 1$ the diagram of $T_{(J,L)}(R)$ coincides with the diagram of R, the same holds in the case $J = \{s, t\}$, $o(st)$ even. It is easy to see, that if R is sharp-angled, $T_{(J,L)}(R)$ is sharp-angled as well.

Two Coxeter generating sets R, $\bar{R}$ are *twist-equivalent*, $R \sim_t \bar{R}$, if there exists a series of Coxeter generating sets $R = R_0, \ldots, R_m = \bar{R}$, such that R_{i+1} is a twist of R_i by some $J \subset R_i$ for $i = 0, \ldots, m-1$. The relation $\sim_t$ is an equivalence relation on the set of sharp-angled Coxeter generating sets (cf. [BMMN02], Chapter 4).

As we can interpret twists as operations on the diagram of a Coxeter group, we will need that condition (E) is preserved by twists:

Lemma 3.2.15. *Suppose R, R' are Coxeter generating sets for W and (J, L) is an R-admissible pair such that $R' = T_{(J,L)}(R)$. Then R satisfies (E) if and only if R' satisfies (E).*

Proof. Assume R satisfies (E), consider an admissible pair (J, L). Following the above remark, the diagram of R' is the same as the diagram of R if $|J| = 1$ or $J = \{s, t\}$ with $o(st)$ even. The only remaining case is $J = \{s, t\}$ and $o(st)$ odd. Let $R = J \cup J^{\perp} \cup L \cup K$, and assume the diagram of R' contains one of the rank 3 diagrams in question, say $U = \{r_1, r_2, r_3\}$. The set U can not contain elements from both K and L, since their product has infinite order. The diagram of $J \cup J^{\perp} \cup L^{w_J} \subset R'$ is the same as the one of $J \cup J^{\perp} \cup L_J \subset R$, $J \cup J^{\perp}$ being w_J invariant. Therefore R' satisfies (E), since R satisfies (E). By symmetry R' satisfying (E) implies that R satisfies (E), which completes our proof. □

3.3 A characterisation of geometric sets

In this section we will characterise geometric sets using the distances between reflections. For this purpose we will introduce the distance matrix of a Coxeter generating set, as already used in [Müh98]. In particular we will show that R is already conjugate to S if no element in R or no longest reflection in any rank 2 group separates any two fundamental reflections.

Definition 3.3.1. Say we have a Coxeter generating set $R = \{r_i \mid i \in I\}$ for a finite I. Define the distance matrix $D_1(R) = (\delta(r_i, r_j))_{i,j \in I}$. For two Coxeter generating sets $R = \{r_i \mid i \in I\}$, $S = \{s_i \mid i \in I\}$ of same rank $|I|$ we say $D_1(R) < D_1(S)$ if there is a permutation $\sigma : (i, j) \mapsto (i', j')$ in $\mathrm{Sym}(I \times I)$ such that $\delta(r_{i'}, r_{j'}) \leq \delta(s_i, s_j)$ for all $i, j \in I$, and $\delta(r_{i'}, r_{j'}) < \delta(s_i, s_j)$ for at least one pair (i, j).

We can use the distance matrix to characterise if a Coxeter generating set is conjugate to S by adapting Lemma 2.8 from [Müh98]:

Theorem 3.3.1. *Suppose $R \subset S^W$ is a Coxeter generating set which is sharp-angled, irreducible and non-spherical of finite rank at least 3. If R satisfies (E), the following are equivalent:*

a) R is geometric.

b) $R^{\circ} = \emptyset$ and $R_2^{\circ} = \emptyset$.

c) R is conjugate to S.

d) $D_1(R) = 0$.

For the definition of R°, R_2°, see Section 3.2.5. Almost all of the arguments to prove this can be copied from [Müh98], but the implication b) $\Rightarrow$ a) does not follow immediately. As a main step in this deduction, we will prove the following proposition:

Proposition 3.3.2. *Let (W,S) be a Coxeter system, $R \subset S^W$ a Coxeter generating set for W such that R is irreducible, non-spherical, sharp-angled and the diagram for (W,R) satisfies condition (E). If $\{r \in R \mid \exists\, u,v \in R: \ r \in [u,v]\} = \emptyset$ and $R_2^\circ = \emptyset$, then R is conjugate to S.*

To show this, it suffices to show, under our conditions on R, the existence of a weakly 2-geometric set of roots associated to R. We can then make use of Theorem 4.2 in [CM07]:

Theorem 3.3.2. ***(Caprace, Mühlherr, 2007)*** *Any finite, universal and weakly 2-geometric set of reflections is geometric.*

Note that the above mentioned result can also be deduced from [HRT97], as the authors also pointed out in [CM07]. Yet the version cited is more applicable due to the geometric language it uses.

We will prove that trees and chord-free circuits of arbitrary length in the diagram yield geometric sets of roots if R satisfies $\{r \in R \mid \exists\, u,v \in R: \ r \in [u,v]\} = \emptyset$ and $R_2^\circ = \emptyset$. Furthermore, for the rest of this section assume that (W,S) is a Coxeter system and $R \subset S^W$ is a sharp-angled and universal set which satisfies $\{r \in R \mid \exists\, u,v \in R: r \in [u,v]\} = \emptyset$ and $R_2^\circ = \emptyset$.

Lemma 3.3.3. *Let R be irreducible and non-spherical. Assume the diagram of R is a tree. Let $r \in R$, and choose an arbitrary root H_r associated to r. Then there exists a unique weakly 2-geometric set of roots Φ associated to R such that $H_r \in \Phi$. In particular, R is geometric.*

Proof. Choose an arbitrary root H_r associated to r. Consider the distance d in the diagram. We prove the lemma by induction on $\max\{d(r,r') \mid r' \in R\}$. If the maximal distance to r is 0, we are done, since $R = \{r\}$. Assume we have proved the lemma for all R' and $r \in R'$ satisfying $\max\{d(r,r') \mid r' \in R'\} = m$ and we have a set R and an element $r \in R$ satisfying $\max\{d(r,r') \mid r' \in R\} = m+1$. So we can find a weakly 2-geometric set of roots $\bar{\Phi}$ associated to $\bar{R} = \{\bar{r} \in R \mid d(r,\bar{r}) \leq m\}$ containing H_r, using that the diagram of $\bar{R}$ is also a tree. Consider $t \in R \setminus \bar{R}$. Since the diagram of R is a tree and $\bar{R}$ is connected, there is exactly one $\bar{t} \in \bar{R}$ such that $o(t\bar{t}) > 2$ and $o(tt') = 2$ for all other $t' \in R \setminus \{t\}$. In $\bar{\Phi}$ a root $H_{\bar{t}}$ associated to $\bar{t}$ is contained, and there exists a unique root H_t associated to t satisfying that $\{H_t, H_{\bar{t}}\}$ is a weakly geometric pair. Since $o(tr') = 2$ for all $r' \in R \setminus \{\bar{t}\}$, $\bar{\Phi} \cup \{H_t \mid t \in R \setminus \bar{R}\}$ is a weakly 2-geometric set for R containing H_r. By 3.3.2, R is geometric. □

Definition 3.3.4. A diagram of a universal set R is a *chord-free circuit of length $m+1$*, if there is an indexing $R = \{r_0, \ldots r_m\}$ such that $o(r_i r_{i+1}) > 2$ for $i = 0, \ldots, m-1$, $o(r_0 r_m) > 2$, and $o(r_i r_j) = 2$ for all other $i \neq j$.

Lemma 3.3.5. *Let R be irreducible and non-spherical. Assume the diagram of R is a chord-free circuit of rank 5 or greater and does not contain irreducible spherical rank 3 subdiagrams. Then R is geometric.*

Proof. Denote $R = \{r_0, \dots, r_m\}$ such that $o(r_i r_{i+1}) > 2$ for $i = 0, \dots, m-1$ and $o(r_m r_0) > 2$, $o(r_i r_j) = 2$ else.

The diagram of $\{r_0, \dots, r_{m-1}\}$ is a tree, yielding with 3.2.6 a unique geometric set of roots $\{H_0, \dots, H_{m-1}\}$, H_i associated to r_i. The set $\{r_{m-3}, r_{m-2}, r_{m-1}\}$ is irreducible non-spherical, thus by 3.2.6 the set $\{H_{m-3}, H_{m-2}, H_{m-1}\}$ of associated roots is the unique geometric set of roots and gives rise to a unique root H_m associated to r_m such that $\{H_{m-3}, H_{m-2}, H_{m-1}, H_m\}$ is geometric. We have to show that H_0, H_m is a geometric pair, then the set $\{H_0, \dots, H_m\}$ is geometric.

Using the fact that $\{H_{m-3}, H_{m-2}, H_{m-1}, H_m\}$ is unique geometric, by considering irreducible non-spherical sets whose diagrams are trees we get the following unique geometric sets of roots associated to the corresponding elements in R:

$$\{H_{m-2}, H_{m-1}, H_m\}, \{H_{m-2}, H_{m-1}, H_m, H_0'\}, \{H_{m-1}, H_m, H_0'\},$$
$$\{H_{m-1}, H_m, H_0', H_1'\}, \{H_m, H_0', H_1'\}, \{H_m, H_0', H_1', H_2'\}.$$

The last set is associated to $\{r_m, r_0, r_1, r_2\}$. Now $\{r_0, r_1, r_2\}$ is geometric with unique geometric set $\{H_0, H_1, H_2\}$, this shows $H_i = H_i'$ for $i = 0, 1, 2$ and in particular $\{H_0, H_m\}$ is geometric. □

Lemma 3.3.6. *Assume $|R| = 3$. Then R is geometric.*

Proof. R is geometric by 3.2.5 if it is reducible. So let R be irreducible. If $R = \{s, t, u\}$ is 2-spherical, then it is geometric by [CM07]. Assume $o(su) = \infty$, then there is a unique geometric pair of roots $\{H_s, H_u\}$ associated to s, u. Assume further all roots H_t associated to t satisfy that $\{H_s, H_t\}$ is geometric, $\{H_t, H_u\}$ is not geometric. Then we already have $t \in [s, u]$, contrary to our assumption on R. Thus, there must exist a root H_t such that $\{H_s, H_t, H_u\}$ is 2-geometric and thus geometric. □

For the next lemma we omit the properties $\{r \in R \mid \exists\, u, v \in R : \; r \in [u, v]\} = \emptyset$, $R_2^\circ = \emptyset$ on R.

Lemma 3.3.7. *Let $s, t, u, v \in R$. Let $2 < o(uv) < \infty$, $o(st) = \infty$, $s, t \notin \{u, v\}^\perp$. Let H_s, H_t, H_u, H_v be roots associated to s, t, u, v, such that the sets $\{H_s, H_u, H_v\}$ and $\{H_t, -H_u, -H_v\}$ are geometric. Let $F = H_u \cap H_v$, $F' = -H_u \cap -H_v$. Then*

a) $C(s) \cap F \neq \emptyset$,

b) $H(u_v, F) = -H(u_v, F')$,

c) $u_v, v_u \in [s, t]$.

Proof. For a) we have that $\{H_s, H_u, H_v\}$ is geometric, therefore $H_s \cap H_u \cap H_v \neq \emptyset$ and is a fundamental domain for $\langle s, u, v\rangle$. This fundamental domain contains chambers of $C(s)$, proving our first statement. For b) we note that $H(u_v, F)$ is well-defined. Furthermore we have $F' = w_{\{u,v\}}.F$, which proves b). Using $s, t \notin \{u, v\}^{\perp}$, we gain $o(u_v s) = \infty = o(u_v t)$, therefore b) yields $H(u_v, s) = -H(u_v, t)$ and c) holds. □

Lemma 3.3.8. *Let R be irreducible and non-spherical of rank 4 satisfying condition (E). Assume the diagram of R is a chord-free circuit. Then R is geometric.*

Proof. The result is clear if $R = \{s, t, u, v\}$ is 2-spherical by [CM07]. Say $o(su) = o(tv) = 2$, since the diagram of R is a chord-free circuit and assume further $o(st) = \infty$. If $o(uv) = \infty$ and $o(tu), o(vs)$ are finite, the diagram is a chord-free circuit in the sense of [MW02] and therefore geometric.

If $o(tu), o(vs)$ are infinite as well, the diagram is right-angled. In this setting, every pair $\{H_s, H_u\}$ of roots associated to s, u is geometric, and we can make the choice $H_s := H(s, t) = H(s, v)$ and $H_u := H(u, t) = H(u, v)$. The equality $H(s, t) = H(s, v)$ holds since $o(tv) = 2 < \infty$ and $H(s, t) = H(s, A_{t,v}) = H(s, v)$ for a spherical residue $A_{t,v}$ stabilized by $\langle t, v\rangle$. In the same way we can choose $H_t := H(t, s) = H(t, u)$ and $H_v := H(v, s) = H(v, u)$. The set $\{H_s, H_t, H_u, H_v\}$ is 2-geometric by construction. The intersection $H_s \cap H_t \cap H_u \cap H_v \cap A_{t,v}$ is not empty, since $A_{t,v} \subset H_s \cap H_u$, furthermore $A_{t,v} \cap H_t \cap H_v \neq \emptyset$, and $\{H_s, H_t, H_u, H_v\}$ is geometric, thus R is geometric.

If $o(tu)$ is finite, $o(vs)$ infinite, consider v, s instead of s, t and t, u instead of u, v. So we can assume $o(st) = \infty > o(uv)$.

In this case we have a unique geometric pair of roots $\{H_s, H_t\}$ associated to s, t. We denote H_u, H_v the unique roots associated to u, v such that $\{H_s, H_t, H_u\}$ is geometric and $\{H_s, H_t, H_v\}$ is geometric. Assume $\{H_u, H_v\}$ is not a geometric pair. Then $\{H_t, H_u, -H_v\}$ is 2-geometric as well as $\{H_s, -H_u, H_v\}$. If both sets are geometric, we can use 3.3.7 and have $u_v, v_u \in [s, t]$, in contradiction to $R_2^\circ = \emptyset$. If both sets are 2-geometric but not geometric, the sets $\{-H_s, -H_v, H_u\}$, $\{-H_t, H_v, -H_u\}$ are each geometric, and the same argument holds.

So assume $\{H_s, H_v, -H_u\}$ and $\{-H_t, H_v, -H_u\}$ are geometric. This is a contradiction if $o(tu) = \infty$, since then $-H_u$ can not be part of a geometric pair with any root associated to t, $\{H_t, H_u\}$ is the only geometric pair associated to these roots. So we assume $o(tu) < \infty$. Spherical rank 2 residues stabilized by $\langle s, u\rangle$ (say $A_{s,u}$), $\langle u, v\rangle$ (say $A_{u,v}$) are contained in $H(t_u, A_{u,v}) = H(t_u, v) = H(t_u, s) = H(t_u, A_{s,u})$, else $t_u \in [s, v]$. Now $A_{s,u}, A_{u,v}$ have non-empty intersection with both roots associated to u, therefore both residues have non-empty intersection with one of the standard fundamental domains for the $\langle t, u\rangle$ action. The reflection t_u separates the two fundamental domains for this action. If $A_{s,u}, A_{u,v}$ are separated by t, such that $H(t, A_{s,u}) = -H(t, A_{u,v})$, they have non-empty intersection with different fundamental domains and $t_u \in [s, v]$ holds. So we have $H_t = H(t, A_{s,u}) = H(t, A_{u,v})$. This contradicts the fact that $\{-H_t, H_v, -H_u\}$ is geometric, because this requires $A_{u,v} \subset -H_t$.

Therefore $\Phi := \{H_s, H_t, H_u, H_v\}$ is a geometric set. $\square$

Proof of 3.3.2.
Choose an arbitrary reflection $r \in R$ and a root H_r associated to r. Consider an arbitrary reflection $s \in R \setminus \{r\}$ and a path connecting them in the diagram, say $\gamma_s = (r = r_0, \ldots, r_m = s)$. We define roots H_{r_i} associated to r_i, $i > 0$, inductively such that $\{H_{r_i}, H_{r_{i-1}}\}$ is a weakly geometric pair. The root $H_{r_m} = H_s$ associated to s then does not depend on the choice of the path γ_s, since all trees and chord-free circuits are geometric.

Since R is irreducible, we find for every $s \in R$ a path γ_s connecting r and s yielding a root H_s associated to s. The set $\Phi := \{H_s \mid s \in R\}$ is well-defined and weakly 2-geometric, therefore R is geometric due to 3.3.2. $\square$

The proven statement will allow us to characterise a geometric set by considering the sets $\{r \in R \mid \exists\, u, v \in R : \ r \in [u, v]\}$ and R_2°.

To complete the proof of the implication b) $\Rightarrow$ a) in 3.3.1 we will prove the useful property that $\{r \in R \mid \exists\, u, v \in R : \ r \in [u, v]\} = \emptyset$ if and only if $R^\circ = \emptyset$.

Definition 3.3.9. Let $\gamma = (c_0, \ldots, c_m)$ be a gallery in (C, P). We say γ *crosses* $r \in S^W$, if there is an index $0 \leq i < m$ such that $H(r, c_i) = -H(r, c_{i+1})$. In this situation, $\{c_i, c_{i+1}\}$ is a panel in P_r. It is easy to see that a minimal gallery crossing r crosses r only once.

Lemma 3.3.10. *Let $r, s, t \in S^W$. If a minimal gallery connecting $C(s)$ to $C(t)$ crosses r, then $\delta(s, t) > \delta(s, t^r)$.*

Proof. Let $\gamma = (c_0, \ldots, c_m)$ be this minimal gallery with $c_0 \in C(s), c_m \in C(t)$, i the index such that $H(r, c_i) = -H(r, c_{i+1})$. Then $c_i = r.c_{i+1}$ and $\gamma = (c_0, \ldots, c_i = r.c_{i+1}, \ldots, r.c_m)$ is a gallery of shorter length connecting $C(s)$ to $C(t^r)$. $\square$

Lemma 3.3.11. *Let $R' = \{r_0, \ldots, r_m\} \subset S^W$ and let $\{H_0, \ldots, H_m\}$ be a set of roots associated to the elements in R'. Assume $D = \bigcap_{i=0}^m H_i \neq \emptyset$. If $\gamma = (c_0, \ldots, c_m)$ is a gallery satisfying $c_0 \notin D, c_m \in D$, then γ crosses one element in R'.*

Proof. Assume not, then $H(r_i, c_0) = H(r_i, c_m)$ for $i = 0, \ldots, r_m$. Since $H_i = H(r_i, c_m)$ for $i = 0, \ldots, m$, this yields $c_0 \in \bigcap_{i=0}^m H_i = D$, contradicting our assumptions. $\square$

Lemma 3.3.12. *Suppose we have a universal, sharp-angled set $R \subset S^W$, $\{r, s, t\} \subset R$ and $r \in [s, t]$. Then:*

a) If $o(rs) = \infty = o(rt)$, then $\delta(s, t) > \delta(s, t^r)$.

b) If $o(rs) < \infty > o(rt)$, then $\delta(s, t) > \delta(s, t^r)$.

c) If $o(rs) < \infty = o(rt)$, then $\delta(s, t) > \delta(s, t^r)$ or $\delta(r, t) > \delta(r, t^s)$.

Proof. Assertion a) is obvious, since a minimal gallery connecting s, t crosses r.
For b), if a minimal gallery emanating from s to t crosses r, we are done. Else we can say that the minimal gallery $\gamma = (c_0, \ldots, c_m)$ with $c_0 \in C(s)$, $c_m \in C(t)$ is included in a root H_r associated to r. We set $\{H_r, H_s\}$ the geometric pair associated to r, s such that $\gamma \subset H_s$ and H_t the root associated to t such that $\{H_s, H_t\}$, $\{-H_r, H_t\}$ are geometric pairs. W.l.o.g. we can assume $\gamma \subset H_r \cap H_s$, else exchange s and t.

We have $c_m \in H_t$. Denote $t' = t^r$ and let $H_{t'} = H(t', c_m)$. Then the pair $\{r, t'\}$ is geometric. Let $H'_{t'}$ denote the root associated to t' such that $H'_{t'} = H(t', s)$. This is well-defined since $o(st') = \infty$, using 3.2.7. Since $H'_{t'} \cap H_r$ is a fundamental domain for $\langle r, t' \rangle = \langle r, t \rangle$ and $C(s) \subset H'_{t'}$, we can show $H'_{t'} = -H_{t'}$. Assume $H'_{t'} = H_{t'}$, then $c_m \in C(t) \cap H'_{t'} \cap H_r = \emptyset$, a contradiction. So we find an index $i < m$ satisfying $c_i \in H'_{t'}$, $c_{i+1} \in H_{t'}$ and $\delta(s, t^r) < \delta(s, t)$.

For c), let $\{H_r, H_s\}$, $\{-H_r, -H_s\}$ be the geometric pairs of roots associated to r, s. Consider a minimal gallery $\gamma_s = (c_0, \ldots, c_m)$ from $C(t) \ni c_0$ to $C(s) \ni c_m$. If it crosses r, then $\delta(s, t) > \delta(s, t^r)$. So assume γ_s does not cross r. Furthermore we can assume that a minimal gallery $\gamma_r = (c'_0, \ldots, c'_k)$ from $C(t)$ to $C(r)$ does not cross s, else $\delta(r, t) > \delta(r, t^s)$, as required. So we have $\delta(s, t) = m$, $\delta(r, t) = k$ and we can assume $k \leq m$. Then the gallery $r.\gamma_r = (r.c'_0, \ldots, r.c'_k)$ connects $C(t^r)$ to $C(r)$. If $r.\gamma_r$ crosses s, we are done since we find $\delta(t^r, s) < k \leq m = \delta(s, t)$. Assume it does not cross s, then $C(t^r) \subset F$ for a fundamental domain $F = H_r \cap H_s$ and a geometric pair $\{H_r, H_s\}$ associated to r, s, since $r.c'_0 \in F$. But then $\{t^r, r, s\}$ are geometric, and $\delta(t, s) = \delta((t^r)^r, s) > \delta(t^r, s)$ holds. $\square$

Corollary 3.3.13. *Suppose we have $R, \{r, s, t\} \subset R$ as in 3.3.12, $r \in [s, t]$. If $o(rs) < \infty > o(rt)$ or $o(rs) = \infty = o(rt)$, then $r \in_r [s, t]$. If $o(rs) < \infty = o(rt)$, $r \in_r [s, t]$ or $s \in_r [r, t]$.*

Proof. This is immediate from 3.3.12 and the definition of reducible separation. $\square$

Proof of 3.3.1.
Assertion a) implies the existence of a fundamental domain $\{c\}$ for the W-action on $\mathcal{C}$, $c \in C(r)$ for all $r \in R$. This shows a) $\Rightarrow$ d) and a) $\Rightarrow$ c), the latter since c corresponds to an element $w \in W$ and $R = S^w = wSw^{-1}$. The implication b) $\Rightarrow$ a) follows from 3.3.2, since $R^\circ = \emptyset \Leftrightarrow \{r \in R \mid \exists\, s, t \in R : r \in [s, t]\} = \emptyset$ by 3.3.13. If c) holds, R is geometric since S is geometric, so c) $\Rightarrow$ a).

We show d) $\Rightarrow$ b): Assume we have $r \in R^\circ$, then there exist $s, t \in R$ such that $r \in [s, t]$, $\delta(s^r, t) < \delta(s, t) = 0$, a contradiction. The same argument holds if $r' \in [s, t]$ for an $r' \in R_2^\circ$.

We proved a) $\Leftrightarrow$ c); a) $\Rightarrow$ d) $\Rightarrow$ b) $\Rightarrow$ a), thus the proposition holds. $\square$

3.4 J-reductions

Throughout this section we will prove our main theorem using a reduction of the distance matrix of R. The proof consists of the distinction of three cases, dependent on the sets R_2° and R°. These cases are described below. Recall from section 3.2.4 the definitions of $R_2 = \{s_t \in R^W \mid s,t \in R, 2 < o(st) < \infty\}$, $R_2^\circ = \{s_t \in R_2 \mid \exists\, u,v \in R \setminus \{s,t\} : \; s_t \in [u,v]\}$.

In this section we will always assume that $R \subset S^W$ is sharp-angled, universal, irreducible and non-spherical of finite rank at least 3. We will also assume that the diagram of R satisfies condition (E).

3.4.1 Proof of the main theorem

Proof of the main result.
Let (W,S) be a Coxeter system, S irreducible non-spherical satisfying (E), and let R be an irreducible, sharp-angled Coxeter generating set. By considering the Cayley graph $\mathcal{C}$ of (W,R), we can switch the roles of R and S. In Corollary A.4 in [CP10] it is proved that if R is sharp-angled in the Cayley graph of S, S is also sharp-angled in the Cayley graph of R. Thus we can assume we have an arbitrary Coxeter system (W,S) and an irreducible sharp-angled Coxeter generating set R satisfying (E).

We prove the theorem by induction on the entries in $D_1(R)$. If $D_1(R) = 0$, R is conjugate to S by 3.3.1.

So assume $D_1(R) > 0$. Then by 3.3.1 $R^\circ \neq \emptyset$ or $R_2^\circ \neq \emptyset$.

Case 1: If $R_2^\circ \neq \emptyset$, we will construct a sharp-angled Coxeter generating set $\bar{R}$ in Section 3.4.4, resulting from R by a series of twists. We will show in 3.4.17 or 3.4.21 that $\bar{R}$ satisfies $D_1(\bar{R}) < D_1(R)$.

Case 2: Assume $R_2^\circ = \emptyset$ and there exist $s,t \in R$ such that $o(st)$ is even, $H_s \cap H_t = F$ is a standard fundamental domain for $\langle s,t\rangle$ and $C(r) \subset H(s_t,F) \cap -H(t_s,F)$ for all $r \in R \setminus (\{s,t\} \cup \{s,t\}^\perp)$. In Section 3.4.6 we will construct a sharp-angled Coxeter generating set $\bar{R}$, and we will show in 3.4.25 that $D_1(\bar{R}) < D_1(R)$ holds.

Case 3: Assume $R_2^\circ = \emptyset$ and there do not exist $s,t \in R$ as in Case 2. Then we construct a sharp-angled Coxeter generating set $\bar{R}$ in Section 3.4.5, which again satisfies $D_1(\bar{R}) < D_1(R)$, this will be shown in 3.4.24.

In every case we can find a Coxeter generating set $\bar{R}$, twist-equivalent to R and satisfying $D_1(\bar{R}) < D_1(R)$. Furthermore $\bar{R}$ satisfies condition (E) by 3.2.15. Using the induction hypothesis now implies that $\bar{R}$ is already twist equivalent to S, this proves our theorem. □

Before we can continue to prove the three mentioned cases in the proof in the sections 3.4.4 to 3.4.6, we will need some more properties of longest reflections. We will also need a more precise understanding of the Case c) in Lemma 3.3.12, dependent on the order of the product rs. These will be stated in 3.4.2.

We will then in 3.4.3 introduce the notion of interior separation, a notion stronger than reducible separation. This concept is useful for handling the cases occurring in the proof of the main theorem.

Remark 3.4.1. A note on the figures, which will occasionally be used in this section to illustrate some of the geometric ideas behind the technical proofs. We will depict the Cayley graph as a circle in the style of the Poincaré disc model for the hyperbolic plane, even though in general the Cayley graph does not result from a tessellation of a hyperbolic space.

We will always depict reflections of the Coxeter generating set we are currently considering as solid lines, attached to the boundary of the circle. If the context requires a certain root to be chosen, we will emphasize the corresponding half space with short solid lines emanating from the reflection line. Two lines intersecting means the product of the corresponding reflections having finite order, and infinite order otherwise. Conjugates of reflections will be represented by dashed lines, we will use this in particular for reflections resulting from the application of a twist. We will mark the transition caused by a twist as a dotted arrow.

Please note that the figures' sole purpose is to give a geometric intuition to the methods we use, they are not part of our proofs.

3.4.2 Longest reflections and reducible separation

In the following part we will prove further properties of longest reflections and their products. We need this in particular for Lemma 3.4.4, which is necessary to handle rotation twists. Furthermore, we will give criteria for when separation implies reducible separation, based on the results in Lemma 3.3.12.

Lemma 3.4.2. *Assume $s_t, u_v \in R_2$. If $s_t u_v$ has finite order greater 2, then $v = t$ or $u_v = t_s$. In particular $2 < o(s_t u_t) < \infty$ implies $o(st), o(tu) \in \{3, 4\}$.*

Proof. Assume $\{s,t\}$ and $\{u,v\}$ are disjoint, $\{s,t\} \not\subset \{u,v\}^{\perp}$. Using Table 1, p.529 in [CM07], computing the product of the longest roots shows the following facts.

First, we have $o(s_t u_v) = \infty$ if two of $o(su), o(tu), o(sv), o(tv)$ are greater or equal to 3, the diagram is not a tree.

Second, if only one of the above mentioned orders is ≥ 3, assume $o(tu) \geq 3$, and the diagram is a tree, then due to condition (E) our Lemma holds as well.

So the sets $\{s,t\}$, $\{u,v\}$ are not disjoint, assume we have the set $\{s,t,u\}$. We will calculate the orders of the longest reflections. If two of the orders of st, tu, su are ≥ 5, then all longest reflections in different rank 2 sets have infinite order, in particular we can assume that the diagram is not a tree. Furthermore, if $o(st), o(tu)$ are both odd, there is nothing to show, since $s_t = t_s$. So assume one of the orders, say $o(st)$, is even. A calculation shows $t_s t_u$, $t_s u_t$ have infinite order, showing our lemma.

The last assertion follows from condition (E), if one of the orders, say $o(st)$, is less than 5, (E) implies $o(su) \geq 3$. Then a calculation yields $o(s_t u_t) = \infty$ whenever $o(ut) \geq 5$. $\square$

Lemma 3.4.3. *Let $\{s,t\} \neq \{u,v\}$ and $s_t, u_v \in R_2$. Then $\{s_t, u_v\}$ is geometric, with geometric pair $\{H(s_t, A_{u,v}), H(u_v, A_{s,t})\}$.*

Proof. If $o(s_t u_v) = \infty$ or s_t,u_v commute, the statement clearly holds. So let $2 < o(s_t u_v) < \infty$, by 3.4.2 we know $t = v$. By 3.4.2 the order is infinite in case $o(st) \geq 5$ or $o(tu) \geq 5$, so we can assume $o(st), o(tu)$ being 3 or 4.

Then $s_t = tst$ and $u_t = tut$. Now $\{s,u\}$ is geometric with geometric pair of roots $\{H(s, A_{t,u}), H(u, A_{s,t})\}$. Thus the pair $\{H(tst, t.A_{t,u}), H(tut, t.A_{s,t})\}$ is geometric as well. Using that $A_{t,u}, A_{s,t}$ are stabilized by t proves the lemma. $\square$

Lemma 3.4.4. *Assume $R_2^\circ \neq \emptyset$. Then we can find $s,t \in R$ with $s_t \in R_2^\circ$ such that either st has odd order or such that st has even order and there exists a root H associated to s_t satisfying:*

Whenever $x_y \in R_2^\circ$ with $|\{x,y\} \cap \{s,t\}| = 1$, then $A_{x,y} \subset H$.

Proof. Assume for $s_t \in R_2^\circ$ the setup of $o(st)$ even and H a root associated to s_t such that $A_{x,y} \subset -H$, $A_{x',y'} \subset H$ for $x_y, x'_{y'} \in R_2^\circ$ and $|\{x,y\} \cap \{s,t\}| = 1 = |\{x',y'\} \cap \{s,t\}|$. If s_t does not satisfy one of these criteria, the lemma already holds.

Construct a maximal sequence $(r_0, \ldots, r_m)$ with $r_i \in R_2$ (not necessarily in R_2°) such that:

1. If $r_i = u_v$, then $o(uv)$ is even and $u_v \in R_2^\circ$ or $v_u \in R_2^\circ$ hold, for $i = 0, \ldots, m$.
2. If $r_i = u_v, r_{i+1} = u'_{v'}$, then $|\{u,v\} \cap \{u',v'\}| = 1$ for $i = 0, \ldots, m-1$.
3. If $r_{i+1} = u_v$, then $r_i u_v, r_i v_u$ have infinite order for $i = 0, \ldots, m-1$.
4. For $0 \leq i < m$, $H(r_i, A_{i-1}) = -H(r_i, A_{i+1})$.

Here A_i is a spherical residue stabilized by $\langle u,v \rangle$ for $r_i = u_v$ and $A_{-1} := A_{x',y'}$. We build the sequence such that $r_0 = s_t$ or $r_0 = t_s$, thus the sequence is non-empty. The conditions 3. and 4. imply $H(r_i, r_j)$ are defined and equal for all $j < i$. Therefore, $C(r_{i+1}) \subset \bigcap_{j=1}^{i} H(r_j, r_{j+1})$ and in particular this sequence is finite, since R_2 is finite.

So assume $r_m = u_v$, and w.l.o.g. we can assume $u_v \in R_2^\circ$. Set $H = H(u_v, r_j)$ for $j < m$. Now assume we have $a_b \in R_2^\circ$ with $|\{a,b\} \cap \{u,v\}| = 1$ and $A_{a,b} \subset -H$.

If $o(ab)$ is odd, we are done, so let $o(ab)$ be even. In case $o(u_v a_b) = \infty = o(u_v b_a)$ the sequence $(r_0, \ldots, r_m, r_{m+1} = a_b)$ satisfies 1. through 4., contradicting the maximality of the sequence. In case one of $o(u_v a_b)$ or $o(u_v b_a)$ is finite, by using 3.4.2 the products $v_u a_b$ and $v_u b_a$ have infinite order. For the sequence $(r_0, \ldots, r_{m-1}, r'_m = v_u, a_b)$ the statements 1., 2. and 3. hold by definition, for 4. we already have $H(u_v, A_j) = -H(u_v, A_{a,b})$ for all $j < m$. The reflections u_v, v_u both separate the fundamental domains $F := H_u \cap H_v$ and

$F' := -H_u \cap -H_v$ for one choice of a geometric pair $\{H_u, H_v\}$ associated to u, v. Since a_b intersects either u or v by 2. and the same holds for r_{m-1}, a_b and r_{m-1} each have non-empty intersection with one of the fundamental domains F or F'. Since u_v separates A_j and $A_{a,b}$ these fundamental domains are different, therefore $H(v_u, A_j) = -H(v_u, A_{a,b})$ holds as well. This contradicts the maximality of the sequence $(r_0, \dots, r_{m-1}, u_v)$, proving the existence of a longest reflection $u_v \in R_2^\circ$ and an associated root H such that $A_{a,b} \subset H$ whenever we have $a_b \in R_2^\circ$ with $|\{a, b\} \cap \{u, v\}| = 1$. □

Lemma 3.4.5. *Let $r, s, t \in R$, $r \in [s, t]$, $o(rs)$ even and finite, $o(rt) = \infty$. If $C(t)$ is not contained in a fundamental domain $H_1 \cap H_2$ for roots associated to r_s, s_r, every minimal gallery connecting $C(t)$ and $C(s)$ or $C(t)$ and $C(r)$ is contained in $H(r_s, t) \cap H(s_r, t)$.*

Proof. Assume $\gamma = (c_0, \dots, c_m)$ is a minimal gallery connecting $C(t) \ni c_0$ to $C(r) \ni c_m$ crossing r_s. Then $\gamma' = (c_0, \dots, c_i = r_s.c_{i+1}, \dots, r_s.c_m)$ for some index i is a gallery of length less than m connecting $C(t)$ and $C(r)$. So assume γ crosses s_r. The root $H(r, t)$ contains a fundamental domain $H_1 \cap H_2$ for roots associated to r_s, s_r not containing $C(t)$. So there is an index i such that $c_i \notin H_1 \cap H_2$, $c_{i+1} \in H_1 \cap H_2$. Since $C(r)$ has no chambers in this fundamental domain, there is an index $j > i$ satisfying $c_j \in H_1 \cap H_2, c_{j+1} \notin H_1 \cap H_2$. But γ cannot cross s_r twice, so by 3.3.11 it crosses r_s, and we are done. □

Lemma 3.4.6. *Assume Case c) of 3.3.12 and rs having even order. Denote with H_1, H_2 roots associated to r_s, s_r such that $H_1 \cap H_2 =: F$ is a fundamental domain for $\langle s, r\rangle$ and such that $H(s, F) = H(s, t)$, $H(r, F) = H(r, t)$. Then:*

a) If $C(t) \subset F$, then $\delta(r, t^s) < \delta(r, t)$ and $\delta(s, t^r) < \delta(s, t)$.

b) If $C(t) \subset -H_1 \cap H_2$, then $\delta(r, t^s) < \delta(r, t)$.

c) If $C(t) \subset H_1 \cap -H_2$, then $\delta(s, t^r) < \delta(s, t)$.

Proof. For a), consider a minimal gallery $\gamma_r = (c_0, \dots, c_m)$ emanating from t to r. We have $C(s) \cap F = \emptyset = C(r) \cap F$. Thus, γ must cross r_s or s_r by 3.3.11 and can not cross r_s, since $r_s r = r r_s$ holds. Since $C(r) \cap F = \emptyset$, there is an index i satisfying $c_i \in H_2, c_{i+1} \in -H_2$. Consider the gallery $s.\gamma_r = (s.c_0, \dots, s.c_i, s.c_{i+1}, \dots, s.c_m)$, which is a minimal gallery connecting t^s and r^s. So $r^s = r^{s_r}$ holds, yielding a gallery $\gamma' = (s.c_0, \dots, s.c_i = s_r s.c_{i+1}, \dots, s_r s.c_m)$ of length less than m connecting sts to r. The same holds for a gallery emanating from t to s.
In the case of b) a minimal gallery connecting t and r crosses s, since $C(r) \subset -H(s, t) \cup H_1$ holds and a minimal gallery can not cross r_s. The same holds in the case of c). □

Lemma 3.4.7. *Assume Case c) of Lemma 3.3.12 and rs having odd order. If $\delta(t, r) \leq \delta(t, s)$, then $\delta(s, t^r) < \delta(s, t)$ and $r \in_r [s, t]$ holds.*

Proof. Let γ be a minimal gallery connecting $C(s)$ to $C(t)$. The result is immediate, if γ crosses r. So assume γ does not cross r. Then $r.\gamma$ is a gallery connecting $C(t^r)$ to $C(s^r)$. Now we can have the situation of $C(t^r)$ being in a standard fundamental domain $H_r \cap H_s$ for $\{r, s\}$ and thus $r.\gamma$ crosses s by 3.3.11. Otherwise $C(t^r)$ is not contained in such a fundamental domain. A minimal gallery γ' connecting $C(r)$ and $C(t)$ can not cross s due to our assumption $\delta(t, r) \le \delta(t, s)$. We conclude that $r.\gamma'$ emanates from $C(r)$ to $C(t^r)$, crosses s and the fact $\delta(t, r) \le \delta(t, s)$ gives rise to a gallery of length less than $\delta(t, r)$ connecting $C(s)$ and $C(t^r)$, as required. □

Lemma 3.4.8. *Let $r, s, t \in R$. Then $\delta(s, t^r) < \delta(s, t)$ holds if and only if $r \in_r [s, t]$.*

Proof. If $r \in_r [s, t]$, $\delta(s, t^r) < \delta(s, t)$ holds by definition. So assume $\delta(s, t^r) < \delta(s, t)$. Since $\delta(s, t) > \delta(s, t^r) \ge 0$, $o(st) = \infty$. Furthermore we can suppose $o(rs), o(rt) > 2$, else $\delta(s, t) = \delta(s, t^r)$. Consider the geometric pair of roots $\{H_s, H_t\}$ associated to s, t. Assume there is a root H_r such that $\{H_r, H_s\}$, $\{H_r, H_t\}$ is geometric, then the triple $\{H_r, H_s, H_t\}$ is already geometric. Let $t' = t^r$, then $o(t's) = \infty$ and $\{H(t', s), -H_r\}$ is a geometric pair. Now $H(s, t) = H(s, t')$ holds, and therefore we have $r \in [s, t']$. Now we have $\delta(s, t') < \delta(s, t'^r)$, so by 3.3.13 we are in the situation $o(rs) < \infty$, $o(rt) = o(rt') = \infty$, else $\delta(s, t') > \delta(s, t'^r)$ holds. Let $F = H_s \cap H_t$ be the fundamental domain of $\langle r, s\rangle$ containing $C(t)$, then $C(t') \subset r.F$. Now $F \cup r.F \subset H(r_s, F) \cap H(s_r, F)$ and $\delta(r, t') < \delta(s, t')$ holds. By 3.4.6, 3.4.7 we have $\delta(s, t') > \delta(s, t'^r) = \delta(s, t)$, a contradiction. □

3.4.3 Interior separation

Definition 3.4.9. Let $r, s, t \in R$, $T \subset R_2$. For $c \in C$, set $D(T, c) := \bigcap_{t\in T} H(t, c)$. We have $D(T, c) = C$ for $T = \emptyset$ and arbitrary $c \in C$. For $D := D(T, c)$ define $C_D(u) := C(u) \cap D$ for $u \in S^W$.

We say $r \in_D [s, t]$ if $H(r, C_D(s)), H(r, C_D(t))$ are well-defined (i.e. $C_D(s)$, $C_D(t)$ are not empty and contained in a unique root associated to r) and they satisfy $H(r, C_D(s)) = -H(r, C_D(t))$.

Note that since D is convex, it contains any gallery from $C_D(s)$ to $C_D(t)$. Since those are on two different sides of r if $r \in_D [s, t]$, such a gallery crosses r, and $C(r) \cap D \ne \emptyset$. If on the other hand a minimal gallery from $C_D(s)$ to $C_D(t)$ in D crosses r and those roots are well-defined, $r \in_D [s, t]$ holds.

Example. If $r \in [s, t]$, with $o(rs) = \infty = o(rt)$ we have $r \in_C [s, t]$. Furthermore, if for $x \in R_2$ we have a $c \in C$ such that $H(x, c)$ has non-empty intersection with $C(r), C(s), C(t)$, in particular if x commutes with r, s, t, for $D = H(x, c)$ we have $r \in_D [s, t]$.

Now let $r \in [s, t]$ such that $o(r, s) < \infty = o(rt)$. Define $D = H(r_s, t)$. Then either $r \in_D [s, t]$ or $s \in_D [r, t]$ holds.

We will give a criterion on D for the roots $H(r, C_D(s))$ to be well-defined.

Lemma 3.4.10. *Let $r, s \in R$, $T \subset R_2, c \in C(r), D = D(T, c)$ such that $C(s) \cap D \neq \emptyset$. If rs has infinite order, then $H(r, C_D(s))$ exists. If $2 < o(rs) < \infty$, $H(r, C_D(s))$ exists if $r_s \in T$ or $s_r \in T$.*

Proof. If rs has infinite order, $H(r, C(s))$ exists and coincides with $H(r, C_D(s))$ since $C_D(s) \subset C(s)$.

Now let $2 < o(rs) < \infty$. If $r_s \in T$ or $s_r \in T$, then assume there are chambers c, d in $C_D(s) \subset C(s)$ with the property $H(r, c) = -H(r, d)$. We can assume that c, d are contained in opposite fundamental domains for the action of $\langle r, s\rangle$, eventually considering $s.c$ or $s.d$ instead of c or d. So $H(r_s, c) = -H(r_s, c')$, a contradiction. Thus, $C_D(s)$ is contained in a unique root associated to r. □

Lemma 3.4.11. *Let $r, s, t \in R$, $D = D(T, c)$ such that $r \in_D [s, t]$. If $o(rs) < \infty$, then $r_s \in T$ or $s_r \in T$.*

Proof. Since $C_D(s)$ is well-defined, $C_D(s) \subset H(t, d)$ for some $d \in C$ and all $t \in T$, furthermore $C_D(s) \cap -H(r, C_D(s)) = \emptyset$ holds. Then there exists a $u \in T$ with the property $H(u, C_D(s)) = -H(u, c_s)$ for all $c_s \in C(s) \cap -H(r, C_D(s))$. The product su therefore has finite order. The product ru has finite order as well, assume not, then $C_D(s) \subset H(u, r)$, since $C(r) \cap D \neq \emptyset$, but $H(u, r)$ does not satisfy $H(u, C_D(s)) = -H(u, c_s)$ for all $c_s \in C(s) \cap -H(r, C_D(s))$. If $ru = ur$, $su = us$ both hold, both roots associated to u contain chambers in $C(s) \cap -H(r, C_D(s))$. So at least one of the orders must be greater than 2.

If $o(ru) > 2$, $u \in \{r_x, x_r\}$ for some $x \in R$ and since $2 < o(rs) < \infty$ we get $o(su) = \infty$ except for the case $u \in \{r_s, s_r\}$. If $o(su) > 2$, the same argument holds, yielding $r_s \in T$ or $s_r \in T$. □

Lemma 3.4.12. *Let $r, s \in R$, $D = D(T, c)$ such that $H(r, C_D(s)), H(s, C_D(r))$ exist. Then the pair $\{H(r, C_D(s)), H(s, C_D(r))\}$ is geometric.*

Proof. The lemma is true if $o(rs)$ is infinite or 2. Otherwise let x be a reflection in $\{r_s, s_r\}$ in T, which exists by 3.4.11. The root $H(x, C_D(s)) = H(x, C_D(r))$ contains a unique fundamental domain for the $\langle r, s\rangle$-action on C of the form $H_r \cap H_s$ for some choice of geometric pair $\{H_r, H_s\}$ associated to r, s, which contains chambers from $C_D(s)$ and from $C_D(r)$. Therefore $H_r = H(r, C_D(s))$ and $H_s = H(s, C_D(r))$, proving the lemma. □

Lemma 3.4.13. *Let $r, s, t \in R$, $D = D(T, c)$ such that $H(r, C_{D'}(s))$, $H(r, C_{D'}(t))$ are defined. Let further $T' \subset T$, $D' = D(T', c)$, such that the roots $H(r, C_{D'}(s))$, $H(r, C_{D'}(t))$ are defined. Then $r \in_D [s, t] \Leftrightarrow r \in_{D'} [s, t]$.*

Proof. This results directly from $D \subset D'$ and $C_D(s) \subset C_{D'}(s)$. □

The previous lemma allows us in particular for $r \in_D [s, t]$ to retreat to the case $D = D(T, c)$ with T consisting of one element in r_s, s_r if the order $o(rs)$ is finite and one element from r_t, t_r, if $o(rt)$ is finite.

Corollary 3.4.14. *Let $r, s, t \in R$, $D = D(T, c)$ such that $r \in_D [s, t]$. Then $r \in [s, t]$.*

Proof. By 3.4.12 we get two geometric pairs of roots $\{H(r, C_D(s)), H(s, C_D(r))\}$ and $\{H(r, C_D(t)), H(t, C_D(r))\}$. It suffices to show $o(st) = \infty$, then we know $H(t, C_D(r)) = H(t, s)$ and $H(s, C_D(r)) = H(s, t)$ since a minimal gallery connecting $C_D(s)$ to $C_D(t)$ crosses r.

If both orders rs, rt are infinite, there is nothing to show. If rs has finite order, rt has infinite order, $C(s) \subset H(r, C_D(s)) \cup -H(r_s, C_D(s))$ and $C(t) \subset H(r_s, C_D(s)) \cap -H(r, C_D(s))$ hold. Thus there can be no spherical residue stabilized by $\langle s, t\rangle$. Assume both orders rs, rt are finite. Let $u \in \{r_s, s_r\} \cap T$, $u' \in \{r_t, t_r\} \cap T$, then

$$C(s) \subset (H(r, C_D(s)) \cap H(u, t)) \cup (-H(r, C_D(s)) \cap -H(u, t)),$$
$$C(t) \subset (H(r, C_D(t)) \cap H(u', s)) \cup (-H(r, C_D(t)) \cap -H(u', s))$$

hold, a spherical residue $A_{s,t}$ stabilized by $\langle s, t\rangle$ is in the intersection of the two, which is

$$(H(r, C_D(s)) \cap H(u, t) \cap -H(u', s)) \cup (H(r, C_D(t)) \cap -H(u, t) \cap H(u', s)),$$

the union being disjoint. But $(H(r, C_D(s)) \cap H(u, t) \cap -H(u', s))$ contains no panels stabilized by s, $(H(r, C_D(t) \cap -H(u, t) \cap H(u', s))$ contains no panels stabilized by t, thus such a spherical residue can not exist, proving $o(st) = \infty$. □

Lemma 3.4.15. *Let $D = D(T, c)$, $r, s, t \in R$. Then $r \in_D [s, t]$ implies $r \in_r [s, t]$.*

Proof. Let $r \in_D [s, t]$. This implies $C_D(s), C_D(t)$ are non-empty and in different unique roots associated to r.

We know $r \in [s, t]$ by 3.4.14. We have to show $\delta(s, t) > \delta(s, t^r)$ by 3.4.8. This results directly from 3.3.12 for $o(rs), o(rt)$ both infinite or both finite.

In the case $o(sr)$ even, $o(rt)$ infinite we have $r_s \in T$ or $s_r \in T$. The lemma holds since minimal galleries between r, s, t never cross both longest reflections in the even case by using 3.4.5. We have yet to deal with the following case: $o(sr) < \infty$ odd, $o(rt) = \infty$. If a minimal gallery γ between $C(s), C(t)$ crosses r, meaning $\gamma \subset H(r_s, t)$, we are done, so assume $\gamma = (c_0, \ldots, c_m)$, $c_0 \in C(s), c_m \in C(t)$ does not cross r. Then it crosses r_s. In particular, $\delta(s, t) > \delta(r, t)$ holds, else we find a gallery of length less than $\delta(r, t)$ connecting r, t. So we can use 3.4.7 and have $\delta(s, t^r) < \delta(s, t)$, $r \in_r [s, t]$ holds. □

3.4.4 $\{s, t\}$-reductions

Assume that $R_2^\circ \neq \emptyset$. This implies the rank of R being at least 4.

We consider the set $J = \{s, t\}$ with $s_t \in R_2^\circ$. Recall the statement from Lemma 3.4.4, that we can find $s_t \in R_2^\circ$ with $o(st)$ odd or $o(st)$ even and a root H associated to s_t such that $A_{x,y} \subset H$ whenever $x_y \in R_2^\circ$ exists with $|\{x, y\} \cap J| = 1$.

First assume st having odd order. Since $s_t \in R_2^\circ$, we find $u, v \in R$ with $s_t \in [u, v]$. We define sets L_v, K_v the following way: For $r \in R \backslash (J \cup J^\perp)$ set $r \in L_v$ if $H(s_t, r) = H(s_t, u)$, and set $r \in K_v$ if $H(s_t, r) = H(s_t, v)$. Since $o(s_t r) = \infty$ for all $r \in R \setminus (J \cup J^\perp)$, this construction yields $R = J \dot\cup J^\perp \dot\cup K_v \dot\cup L_v$. In addition:

Lemma 3.4.16. *The pair (J, L_v) defined as above is an R-admissible pair.*

Proof. Since $H(s_t, l) = -H(s_t, k)$ whenever $l \in L_v$, $k \in K_v$, we have $s_t \in [l, k]$ and $o(lk) = \infty$ holds for all such l, k. □

Proposition 3.4.17. *Set $\bar{R} := T_{(J,L_v)}(R)$. Then $D_1(\bar{R}) < D_1(R)$.*

Proof. For $l \in L_v, k \in K_v$ a minimal gallery emanating from $C(l)$ to $C(k)$ crosses s_t, yielding a shorter gallery emanating from $C(l^{s_t})$ to $C(k)$. Thus, $\delta(l, k) > \delta(l^{s_t}, k)$ holds at least for the pair $l = u, k = v$. The relations in W yield $s^{s_t} = t$, $t^{s_t} = s$. So we have for all $l \in L_v$: $\delta(l, s) = \delta(l^{s_t}, t)$, $\delta(l, t) = \delta(l^{s_t}, s)$. Then, using a permutation mapping (l, s) to (l, t) and vice versa, we gain $D_1(\bar{R}) < D_1(R)$. □

Now assume $o(st)$ is even, there exists a root H associated to s_t such that $A_{x,y} \subset H$ whenever x_y exists with $|\{x, y\} \cap J| = 1$ by 3.4.4. Let H_s, H_t be roots associated to s, t such that $H_s \cap H_t =: F$ is a fundamental domain and $H = H(s_t, F)$. In case $R_2^\circ \setminus (\{s_t, t_s\} \cup \{r \in R_2^\circ \mid rs_t = s_t r\})$ is empty, choose an arbitrary geometric pair $\{H_s, H_t\}$.

We note that due to our assumptions on s_t, whenever we take $u_v \in R_2$, $|\{u, v\} \cap J| = 1$, with $A_{u,v} \subset -H$, this yields $C(r) \subset H(u_v, F) = H(u_v, A_{s,t})$ for all $r \in R \setminus (\{u, v\} \cup \{u, v\}^\perp)$, else $u_v \in [r, t]$ if $s \in \{u, v\}$ or $u_v \in [r, s]$ if $t \in \{u, v\}$ in contradiction to our assumptions on s_t.

Define $T_s = \{s_t\} \cup \{u_v \in R_2 \mid A_{u,v} \subset -H, |\{u, v\} \cap J| = 1\}$. Let $c \in w_J.F \cap C(s) \cap C(t)$ and define $D_s = D(T_s, c)$. Then for all $r \in R$ satisfying $C(r) \subset -H(s_t, F)$ the intersection $C(r) \cap D_s$ is non-empty and the roots $H(s, C_{D_s}(r))$, $H(t, C_{D_s}(r))$ are defined by 3.4.10.

Now we define two sets L_s, K_s satisfying $R = \{s\} \dot\cup s^\perp \dot\cup K_s \dot\cup L_s$. For $r \in R \setminus (\{s\} \cup s^\perp)$, set $r \in L_s$ if $C_{D_s}(r) \neq \emptyset$ and $s \in_{D_s} [r, t]$. Else $r \in K_s$.

Lemma 3.4.18. *The pair $(\{s\}, L_s)$ is an R-admissible pair.*

Proof. Consider $l \in L_s$, $k \in K_s$. If $C_{D_s}(k) \neq \emptyset$ we have $s \in_{D_s} [l, k]$. This implies $s \in_r [l, k]$ by 3.4.15, $o(lk) = \infty$ holds and we are done. If $C_{D_s}(k) = \emptyset$, this implies $C(k) \subset H(s_t, F)$, and $s_t \in_r [l, k]$ holds. Therefore $o(lk) = \infty$ holds in all cases and $(\{s\}, L_s)$ is an R-admissible pair. □

Set $R' := T_{(\{s\},L_s)}(R)$. Note that since $o(lk) = \infty$ for all $l \in L_s, k \in K_s$, $o(l^s k) = \infty$, this results from 3.2.7, or from the fact that the diagram is not changed by a rank 1 twist. Furthermore for $l \in L_s$ we have $o(lt) = \infty$ since $s \in_r [t, l]$, so $o(l^s t) = \infty$ as well. In consequence, the pair $J \subset R'$ and the root $H(t_s, F)$ still satisfy the property $A_{x,y} \subset H(t_s, F)$ whenever x_y exists with $|\{x, y\} \cap J| = 1$.

We apply the same for t. To be exact, we define $T_t = \{t_s\} \cup \{u_v \in R'_2 \mid A_{u,v} \subset -H, |\{u, v\} \cap J| = 1\}$ and set $D_t = D(T_t, c)$. Again for all $r \in R$ satisfying $C(r) \subset -H(t_s, F)$ the intersection $C(r) \cap D_t$ is non-empty and $H(s, C_{D_t}(r))$, $H(t, C_{D_t}(r))$ are defined.

Define L_t, K_t in the same manner. For $r \in R' \setminus (\{t\} \cup \{t\}^\perp)$ set $r \in L_t$ if $C_{D_t}(r) \neq \emptyset$, $r \notin L_s^s$ and $t \in_{D'} [r, s]$. Else $r \in K_t$.

Lemma 3.4.19. *The pair $(\{t\}, L_t)$ is an R'-admissible pair.*

Proof. The proof copies from the proof of 3.4.18, except for $l \in L_t$, $k \in L_s^s$. This case results from $L_t \subset K_s$, and as mentioned above $o(lk) = \infty$ follows from 3.2.7 or the fact that rank 1 twists preserve the diagram. □

Set $R'' := T_{(\{t\},L_t)}(R')$. Now define L_J, K_J. For $r \in R'' \setminus (J \cup J^\perp)$ set $r \in L_J$ if $C(r) \subset -H(s_t, F) \cup -H(t_s, F)$, else $r \in K_J$. Clearly (J, L_J) is an R''-admissible pair and $L_s^s \cup L_t^t \subset L_J$ holds. We then define $\bar{R} := T_{(J,L_J)}(R'')$.

Remark 3.4.20. For the set K_J we have $K_J = \{r \in R \mid C(r) \subset H(s_t, F) \cap H(t_s, F)\} \subset K_s \cap K_t$. Define $L_0 := \{r \in R \mid C_{D_s}(r) \neq \emptyset, s \notin_D [t, r]\} \cap \{r \in R' \mid C_{D_t}(r) \neq \emptyset, t \notin_{D_t} [s, r]\}$. For $r \in s^\perp$ one of $r \in L_0$, $r \in J^\perp$, $r \in K_t$ or $r \in L_t$ holds. If $r \in K_t \setminus K_J$, meaning $C(r) \subset -H(t_s, F)$ and $t \notin_{D_t} [s, r]$, either $r \in K_s$, and thus $r \in L_0$ holds, or $r \in L_s$. Therefore we have:

$$\begin{aligned}
R &= J \,\dot\cup\, J^\perp \,\dot\cup\, L_s \,\dot\cup\, L_t \,\dot\cup\, L_0 \,\dot\cup\, K_J, \\
R' &= J \,\dot\cup\, J^\perp \,\dot\cup\, L_s^s \,\dot\cup\, L_t \,\dot\cup\, L_0 \,\dot\cup\, K_J, \\
R'' &= J \,\dot\cup\, J^\perp \,\dot\cup\, L_s^s \,\dot\cup\, L_t^t \,\dot\cup\, L_0 \,\dot\cup\, K_J, \\
\bar{R} &= J \,\dot\cup\, J^\perp \,\dot\cup\, L_s^{sw_J} \,\dot\cup\, L_t^{tw_J} \,\dot\cup\, L_0^{w_J} \,\dot\cup\, K_J \\
&= J \,\dot\cup\, J^\perp \,\dot\cup\, L_s^{s_t} \,\dot\cup\, L_t^{t_s} \,\dot\cup\, L_0^{w_J} \,\dot\cup\, K_J.
\end{aligned}$$

The transition of R to $\bar{R}$ in the case of st having order 4 is shown schematically in Figure 3.1.

Proposition 3.4.21. *The Coxeter generating set $\bar{R}$ satisfies $D_1(\bar{R}) < D_1(R)$.*

Proof. Distances to elements in $J^\perp$ are preserved. The same holds for the distances from L_s to s, from L_t to t, from L_0 and K_J to J. Since $s \in_{D_s} [l, t]$ for all $l \in L_s$ and $t \in_{D_t} [l', s]$ for all $l' \in L_t$, distances from L_s to t and from L_t to s are reduced.

The sets L_s, K_J are separated by s_t, in the sense that each pair of elements is separated by s_t, so $\delta(l, k) > \delta(l^{s_t}, k)$ holds for $l \in L_s$, $k \in K_J$ by 3.3.10. The same argument holds for L_t, K_J, which are separated by t_s.

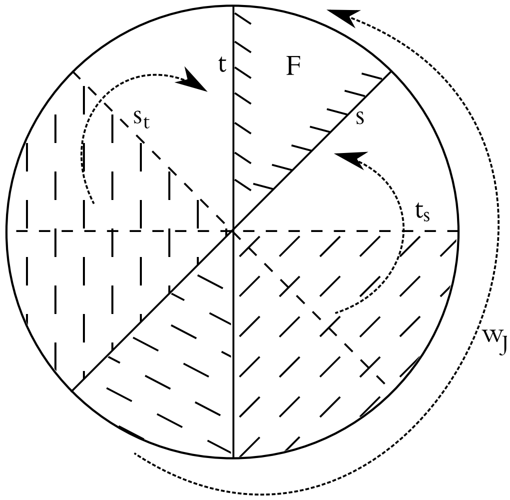


Figure 3.1: {s,t}-reductions in the even case

Assume we have $l \in L_s$, $l' \in L_0$. Then $s \in_{D_s} [l,l']$ holds and $\delta(l,l') > \delta(l^s,l') = \delta(l^{s_t},l'^{w_J})$. The same holds for $l \in L_t$, $l' \in L_0$.

Let $l \in L_s$, $l' \in L_t$. Then consider a minimal gallery $\gamma = (c_0,\ldots,c_m)$ of length m, with $c_0 \in C(l), c_m \in C(l')$. Assume there are indices i,j such that $c_i \in H_s, c_{i+1} \in -H_s, c_j \in -H_t, c_{j+1} \in H_t$, so we assume γ crosses s and t. W.l.o.g. $i < j$. Then $\gamma' = (s.c_0,\ldots s.c_i = c_{i+1},\ldots,c_j = t.c_{j+1},\ldots,t.c_m)$ is a gallery of length $m-2$ connecting $C(l^s)$ to $C(l'^t)$. We find $\gamma'' := w_J.\gamma' = (s_t.c_0,\ldots,s_t.c_i = w_J.c_{i+1},\ldots,w_J.c_j = t_s.c_{j+1},\ldots,t_s.c_m)$ is a gallery of length $m-2$ connecting $C(l^{s_t})$ to $C(l'^{t_s})$, as required. Assume γ does not cross s. This implies $o(sl) < \infty$, l_s or $s_l \in T_s$. Denote this reflection x and set $D_x = D(\{x,s_t\},c)$. We have $s \in_{D_s} [l,t]$ by construction of L_s. Consequently $s \in_{D_x} [l,t]$ and $s \in_{D_x} [l,l']$ since $H(s,C_{D_x}) = H(s,l')$, hereby we gain $\delta(l^s,l') < \delta(l,l')$. Due to $t \in_{D_t} [s,l']$, $t \in_{D_t} [l^s,l']$ holds as well, since we have $\delta(l,s) = 0 = \delta(l^s,s)$. This yields $H(t,C_{D_t}(s)) = H(t,C_{D_t}(l)) = H(t,C_{D_t}(l^s))$ and $\delta(l^s,l'^t) < \delta(l^s,l')$ holds, as required.

Finally consider $l \in L_0$, $k \in K_J$ and a minimal gallery $\gamma = (c_0,\ldots,c_m)$, $c_0 \in C(l)$, $c_m \in C(k)$. Clearly γ crosses s_t and t_s. If it crosses s or t as well or k commutes with s or t, $w_J = ss_t = tt_s$ yields a shorter gallery emanating from l^{w_J} to k.

Now assume γ does not cross s and t and w.l.o.g. assume $\gamma \subset H_s \cap -H_t$. The other case, $\gamma \subset -H_s \cap H_t$, follows in the same manner substituting s and t. The fact $s \notin_D [l,t]$ implies $o(ls) < \infty$, else γ crosses s. If l,s commute, we are done, since γ crosses s_t, so assume $o(ls) > 2$.

In the case $o(kt) = \infty$, either $C(k) \subset H_t$, a contradiction to $\gamma \subset -H_t$, or $C(k) \subset -H_t$. In the last case $C(k) \subset H(s_t,F) \cap H(t_s,F)$ implies $C(k) \subset H_s = -H(s,C_{D_s}(l))$. For $D_x = D(\{l_s\},c)$ we get $s \in_{D_x} [l,k]$, and $\delta(l^s,k) < \delta(l,k)$ holds.

Now let $2 < o(kt) < \infty$. Furthermore we can assume $2 < o(ks) < \infty$ since $o(ks) = \infty$ implies again $s \in_{D_x} [k, l]$. Therefore the set $\{k, s, t\}$ is geometric with geometric set of roots $\{H_k, H_s, H_t\}$, since $C(k) \subset H(s_t, F) \cap H(t_s, F)$. The root H_k associated to k satisfies $H_k = H(k, A_{s,t}) = H(k, l)$. The pair $\{H(s, C_{D_s}(l)), H(l, C_{D_s}(s))\}$ is geometric by 3.4.12. For $D := D(\{l_s\}, c)$ the same holds. Since we can assume $D_s \subset D$, we have $H(l, C_D(s)) = H(l, A_{s,t}) = H(l, k)$ and $H(s, C_D(l)) = H(s, C_D(t)) = -H_s$ yields $s \in [l, k]$. Now we can use 3.3.12 and have $\delta(l^s, k) < \delta(l, k)$, as required.

As a final step we need to show that there are at least two reflections whose distance is reduced in $\bar{R}$, using $s_t \in R_2^\circ$. If L_s or L_t is non-empty, the distance to t or s is reduced. So assume they are empty, then L_0 and K_J must be non-empty and as shown above for $l \in L_0$, $k \in K_J$ the inequality $\delta(l^{w_J}, k) < \delta(l, k)$ holds. □

3.4.5 r-reductions

We now assume that R satisfies $R_2^\circ = \emptyset$ and $R^\circ \neq \emptyset$. Throughout this section we will also assume the following condition $(*)$ on R:

Consider an arbitrary pair $s, t \in R$, $2 < o(st) < \infty$ even and $u \notin \{s, t\} \cup \{s, t\}^\perp$. Denote with $F := H_s \cap H_t, -F := -H_s \cap -H_t$ the standard fundamental domains for the action of $\langle s, t\rangle$. Then either $H(s_t, u) = H(s_t, F)$ and $H(t_s, u) = H(t_s, F)$ hold for all $u \in R \setminus (\{s, t\} \cup \{s, t\}^\perp)$ or $H(s_t, u) = H(s_t, -F)$ and $H(t_s, u) = H(t_s, -F)$ hold for all $u \in R \setminus (\{s, t\} \cup \{s, t\}^\perp)$. In other words, $C(u)$ is not contained in the fundamental domain generated by the geometric pair of roots associated to $\{s_t, t_s\}$.

Since we further require R_2° to be empty, then all $u \notin \{s, t\} \cup \{s, t\}^\perp$ are on the same side of s_t, t_s. Also we have $\{s, t\} \cup \{s, t\}^\perp \neq R$, since R is irreducible. So we see that such a u always exists.

Define $T := R_2$. The intersection

$$D = \bigcap_{\substack{2<o(st)<\infty,\\ u\notin\{s,t\}\cup\{s,t\}^\perp}} H(s_t, u)$$

is non-empty, and for a $c \in D$ we have $D = D(T, c)$. Furthermore, $C_D(r) \neq \emptyset$ for all $r \in R$ due to $(*)$ and $H(r', C_D(r))$ is defined for all $r, r' \in R$ with $rr' \neq r'r$.

Lemma 3.4.22. *If $R^\circ \neq \emptyset$, there exist $r, s, t \in R$ such that $r \in_D [s, t]$.*

Proof. The assumption $R^\circ \neq \emptyset$ yields r, s, t such that $r \in_r [s, t]$. The roots $H(r, C_D(s))$, $H(r, C_D(t))$ are well-defined, and $r \in_D [s, t]$ holds if $o(rs), o(st)$ are both infinite. If they are both finite and a minimal gallery between $C(s)$, $C(t)$ does not cross r, it is easy to see that it crosses r_s or r_t. We conclude that every gallery not crossing r_s or r_t crosses r, proving $r \in_D [s, t]$.

Consider the case $o(rs) < \infty$, $o(rt) = \infty$. Let $D' = D(\{r_s, s_r\}, c)$. If the minimal gallery γ connecting $C_{D'}(s), C_{D'}(t)$ crosses r or the minimal gallery γ' connecting $C_{D'}(r), C_{D'}(t)$ crosses s, this yields $r \in_{D'} [s, t]$ or $s \in_{D'} [r, t]$. If neither γ nor

γ' cross r, s, the first chambers in γ, γ' are not contained in the fundamental domain $F = H(r, C_{D'}(s)) \cap H(s, C_{D'}(r)) \subset H(r_s, c) \cap H(s_r, c)$ by 3.3.11. Therefore, $\gamma \subset -H(r, C_{D'}(s))$, $\gamma' \subset -H(s, C_{D'}(r))$ and $C(t) \subset -H(r, C_{D'}(s)) \cap -H(s, C_{D'}(r)) = w_{\{r,s\}}.F$. But longest reflections separate the two standard fundamental domains, thus $w_{\{r,s\}}.F \cap H(r_s, c) = \emptyset$, a contradiction. □

Now let $r \in R^{\circ}$, $s, t \in R$ such that $r \in_D [s, t]$, these exist by 3.4.22. We find an R-admissible pair $(\{r\}, L_r)$ by defining L_r, K_r the following way. For $r' \in R \setminus (\{r\} \cup r^{\perp})$ we define $r' \in L_r \Leftrightarrow C_D(r') \subset H(r, C_D(s))$ and $r' \in K_r \Leftrightarrow C_D(r') \subset H(r, C_D(t))$. This yields a partition $R = \{r\} \,\dot{\cup}\, r^{\perp} \,\dot{\cup}\, L_r \,\dot{\cup}\, K_r$.

Lemma 3.4.23. *$(\{r\}, L_r)$ is an R-admissible pair.*

Proof. Let $l \in L_r, k \in K_r$, then by construction $r \in_D [l, k]$ and $r \in_r [l, k]$ by 3.4.15, thus $o(lk) = \infty$. Thus the pair $(\{r\}, L_r)$ is admissible. □

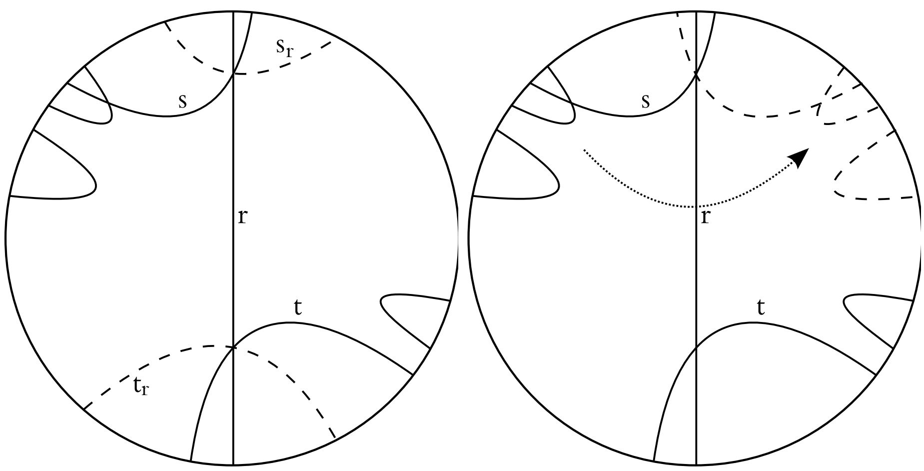


Figure 3.2: r-reductions using interior separation

Define for an $r \in R^{\circ}$ the set $\bar{R} = T_{(\{r\}, L_r)}(R)$. See Figure 3.2 for an example of the above construction, with the property $o(rs) < \infty > o(rt)$. The longest reflections here give rise to a convex set D, which can be seen in the first depiction as the space between the longest reflections s_r, t_r.

Proposition 3.4.24. *The Coxeter generating set $\bar{R}$ satisfies $D_1(\bar{R}) < D_1(R)$.*

Proof. Let $l \in L_t, k \in K_t$, both sets are not empty since $s \in L_t, t \in K_t$. Then $r \in_D [l, k]$ and $r \in_r [l, k]$ by 3.4.15. Thus, $\delta(l^r, k) < \delta(l, k)$. Distances to $r, r^{\perp}$ are preserved. □

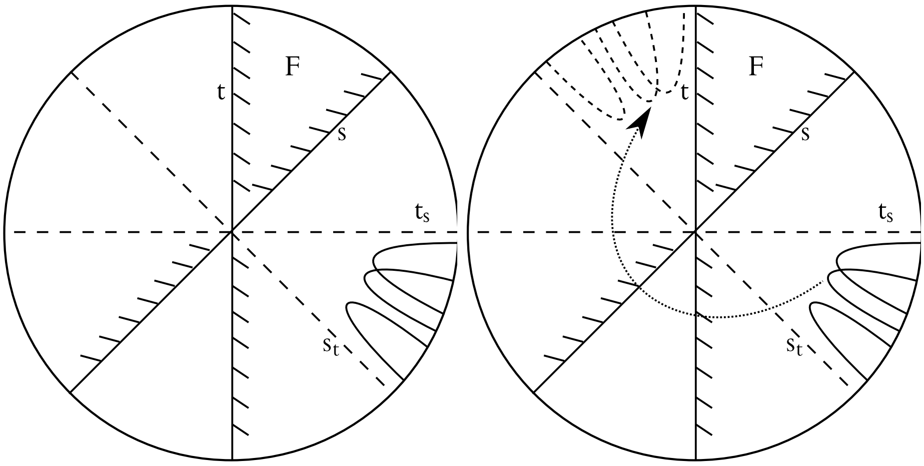


Figure 3.3: r-reductions in an exceptional case

3.4.6 r-reductions in an exceptional case

In order to reduce distances in every case, we have yet to deal with one case.

Assume we have $R_2^\circ = \emptyset$ and $R^\circ \neq \emptyset$. If we can not apply a reduction as constructed in 3.4.5, we can find $J = \{s, t\}$, $2 < o(st) < \infty$ even, together with a standard fundamental domain $F = H_s \cap H_t$, such that we can find an $r \in R \setminus (J \cup J^\perp)$ satisfying $C(r) \subset H(s_t, F) \cap -H(t_s, F)$. Since $R_2^\circ = \emptyset$ and $o(r's_t) = o(r't_s) = \infty$ for all $r' \in R \setminus (J \cup J^\perp)$, we have $C(r') \subset H(s_t, F) \cap -H(t_s, F)$ for all $r' \in R \setminus (J \cup J^\perp)$. In particular, if r commutes with t, it commutes with s as well.

Define $L_s = R \setminus (J \cup s^\perp)$, $K_s = \{t\}$, then $(\{s\}, L_s)$ is clearly an R-admissible pair. Let $\bar{R} = T_{(\{s\}, L_s)}$. An example of the sets R and $\bar{R}$ for a sample of reflections in L_s can be found in Figure 3.3.

Proposition 3.4.25. *The Coxeter generating set $\bar{R}$ satisfies $D_1(\bar{R}) < D_1(R)$.*

Proof. For $l \in L_s$ $\delta(l^s, t) < \delta(l, t)$ holds by 3.4.6. $\square$

Index

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Ich erkläre: Ich habe die vorgelegte Dissertation selbständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt, die ich in der Dissertation angegeben habe. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben, die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht. Bei den von mir durchgeführten und in der Dissertation erwähnten Untersuchungen habe ich die Grundsätze guter wissenschaftlicher Praxis, wie sie in der „Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis“ niedergelegt sind, eingehalten.

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