Unstable Manifolds of Periodic Orbits of a Differential Delay Equation

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ABSTRACT. Unstable sets of periodic solutions to delay equations

$$x'(t) = -\mu x(t) + f(x(t-1))$$

with monotone nonlinearities are shown to be smooth 2-dimensional graphs whose boundaries are formed by further periodic orbits.

1. Introduction

Let a continuously differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ be given, with

$$f(0) = 0 \quad \text{and} \quad \xi f(\xi) < 0 \quad \text{for} \quad \xi \neq 0.$$

Let $\mu \geq 0$. The equation

$$x'(t) = -\mu x(t) + f(x(t-1)) \tag{1.1}$$

models a system governed by delayed negative feedback and decay. Initial data $\phi : [-1, 0] \longrightarrow \mathbb{R}$ in the infinite–dimensional phase space $C = C([-1,0], \mathbb{R})$ define continuous solutions $x^\phi : [-1, \infty) \longrightarrow \mathbb{R}$ which satisfy eq. (1.1) for $t > 0$. The relations

$$F(t, \phi) = x_t^\phi, \quad x_t^\phi = x^\phi(t + \cdot) \in C \quad \text{for} \quad t \geq 0,$$

constitute a semiflow F on C whose dynamics appears in many cases to be structured by periodic orbits. The most accessible periodic solutions are those which are slowly oscillating in the sense that consecutive zeros are spaced at distances

$$z_{j+1} - z_j > 1$$

greater than the delay.

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Results on existence, uniqueness and nonuniqueness, stability and instability, bifurcation and more have been obtained, for example, in [**4, 5, 9, 11, 12, 18–20, 22, 26–31, 33, 35, 38**].

The dominating role of slowly oscillating periodic solutions is to a certain extent explained by a result in [**34**] which guarantees that for $\mu = 0$ and for some f, initial values of eventually slowly oscillating solutions are an open and dense set, and by the Morse decomposition result from [**21**].

The simplest situations occur when f is monotone, and bounded from above or from below. We shall assume in this paper

$$f(0) = 0, \quad f'(\xi) < 0 \quad \text{for all} \quad \xi \in \mathbb{R}, \quad \sup f < \infty$$

and consider slowly oscillating periodic solutions y which are unstable and hyperbolic. Results in [**2, 3, 8, 32**] indicate that such solutions exist for suitable monotone nonlinearities as above. Examples will be given in [**17**].

The unstable set W of a periodic solution y consists of the phase curves $\mathbb{R} \ni t \longrightarrow x_t \in \mathbb{R}$ of solutions which converge to the orbit

$$|\eta| := \{y_s : s \in \mathbb{R}\}$$

as $t \longrightarrow -\infty$.

We are interested in the nature of W. An important intermediate result on the way to the description of W is Theorem 5.3 which asserts that, for y unstable and hyperbolic, there is precisely one Floquet multiplier outside the unit circle, necessarily situated on the ray $(1, \infty)$. This implies that local unstable manifolds $W_{loc} \subset W$ of $|\eta|$ are 2–dimensional and orientable (no Moebius strips).

The main result, Theorem 5.4, states that for y unstable and hyperbolic, W is a smooth, annulus–like graph of dimension two, bordered either by two other periodic orbits, or by one periodic orbit and by the stationary state $0 \in C$.

On the invariant set $\overline{W}$ the semiflow F can be represented by the flow of a planar vectorfield. Phase curves in W spiral away from the orbit of y, towards the closed orbits forming the boundary $\overline{W} \setminus W$.

The closure $\overline{W}$ should be viewed as a part of the global attractor of the semiflow F. Results in [**8, 18, 26, 32**] suggest that there are special cases, with a single unstable periodic orbit and a stable steady state, where $\overline{W}$ is the full attractor.

We briefly mention further aspects of this study. Section 4 deals with the linearization of eq. (1.1) at the zero solution,

$$x'(t) = -\mu x(t) - \alpha x(t-1) \tag{1.2}$$

where $\alpha = -f'(0) > 0$. The semigroup of eq. (1.2) determines an invariant decomposition

$$C = L \oplus Q, \qquad \dim L = 2$$

which serves as a coordinate system for the description of W as a graph in Theorem 5.4.

Let us point out that the set W may be far away from $0 \in C$. The relevant link between the nonlinear semiflow F and the semigroup of eq. (1.2) in our situation is not linearization, but the presence of a set which is invariant for both: This is the set S of nonzero data $\phi \in C$ with at most one change of sign in $[-1, 0]$. S contains segments $x_t \in C$ of slowly oscillating solutions; W and the space L (except 0) belong to S while

$$S \cap Q = \emptyset.$$

Theorem 5.1 and Corollary 5.1 show that all L–projections of orbits of slowly oscillating periodic solutions are simple closed curves which wind around $0 \in L$; they are nested in L.

Several ingredients for the proof of Theorem 5.4 are taken from [**36**] where a somewhat different, partly simpler situation is studied: If the zero solution of eq. (1.1) is linearly unstable then the global unstable set of the stationary point $0 \in C$ contains an invariant smooth 2-dimensional graph bordered by the orbit of a slowly oscillating periodic solution. Other constructions which have no counterpart in [**36**] employ winding numbers and homotopies.

All results in this paper are formulated and proved for eq. (1.1) with $\mu > 0$, i.e. for a system with decay. It is not difficult to extract the version for the case $\mu = 0$, which is technically simpler.

A general reference for most of the basic properties of differential delay equations which are used in the sequel is [**13**]. For calculus in Banach spaces and for properties of the winding number, see [**7**]. For submanifolds and transversality, see [**1**].

2. Notation, Preliminaries

$\mathbb{N}_0$ denotes the set of nonnegative integers. $\mathbb{R}^+$ stands for the interval $[0, \infty)$. Spectra of closed linear operators T in complex Banach spaces are denoted by spec(T), or spec; in case of a real Banach space, spec(T) is the spectrum of the complexification of T.

Consider a subset $X \subset E$ of a real Banach space E, and a point $x \in X$. The set $T_x X$ of tangents to X at x is defined to be the set of all vectors

$$v = Dc(0)1$$

where $c : (-1, 1) \longrightarrow E$ is a differentiable curve with $c(0) = x$ and $c((-1, 1)) \subset X$. Note $0 \in T_x X$. In general, $T_x X$ is *not* a vectorspace. For a differentiable map $g : U \longrightarrow G$, $U \supset X$,

$$Dg(x) T_x X \subset T_{g(x)} g(X).$$

The closure, the interior and the boundary of X are denoted by

$$\overline{X}, \quad X^\circ, \quad \partial X,$$

respectively.

Let Y denote a 2–dimensional normed vectorspace over $\mathbb{R}$. For a map

$$c : [a,b] \longrightarrow Y$$

we denote by

$$| c | := c([a,b])$$

its trace. We say c is (piecewise) smooth if c is (piecewise) of class C^1.

Fix an isomorphism ζ of Y onto the $\mathbb{R}$–vectorspace $\mathbb{C}$. Then the winding number of a closed, piecewise smooth curve c with respect to a point $y \in Y \setminus | c |$ is defined as

$$\text{wind}(y,c) := \text{wind}(\zeta y, \zeta \circ c) \in \mathbb{Z}$$

where on the right hand side we have the familiar winding number of $\mathbb{C}$-valued curves.

If c is a simple closed, piecewise smooth curve, $y \in Y \setminus | c |$, then

$$\text{wind}(y,c) \in \{-1,0,1\}.$$

For such c, the interior and exterior of c are defined by

$$\text{int}(c) := \{y \in Y \setminus | c | : \text{wind}(y,c) \neq 0\}$$

$$\text{ext}(c) := \{y \in Y \setminus | c | : \text{wind}(y,c) = 0\}.$$

Both sets are open and connected; $\text{int}(c)$ is bounded and $\text{ext}(c)$ is unbounded; the boundaries coincide with the trace $| c |$.

The winding number is a homotopy invariant: If

$$\text{hom} : [0,1] \times [a,b] \longrightarrow Y \quad \text{is continuous,} \quad y \in Y \setminus \text{hom}([0,1] \times [a,b]),$$

$$\text{hom}(\cdot, a) = \text{hom}(\cdot, b),$$

and if both $\text{hom}(0,\cdot)$ and $\text{hom}(1,\cdot)$ are piecewise smooth, then

$$\text{wind}(y, \text{hom}(0,\cdot)) = \text{wind}(y, \text{hom}(1,\cdot)).$$

(It is not necessary that the closed curves $\text{hom}(t,\cdot), 0 < t < 1$, are piecewise smooth.)

Let a function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given. A solution of the differential delay equation

$$x'(t) = g(x(t), x(t-1)) \tag{2.1}$$

is either a differentiable function $x : \mathbb{R} \longrightarrow \mathbb{R}$ so that (2.1) is satisfied for all real t, or a continuous function $x : [t_0 - 1, \infty) \longrightarrow \mathbb{R}$, $t_0 \in R$, which is differentiable on (t_0, ∞) and satisfies (2.1) for all $t > t_0$.

Analogously one defines complex-valued solutions in case g is linear, and solutions of nonautonomous equations

$$x'(t) = g(t, x(t-1))$$

for functions $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ or $g : [t_0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}, t_0 \in R$.

The space C, and the complex vectorspace C' of continuous functions $\phi : [-1,0] \longrightarrow \mathbb{C}$, are equipped with the maximum–norm:

$$||\phi|| = \max_{t \in [-1,0]} |\phi(t)|$$

Solutions define phase curves $t \longrightarrow x_t$ with values in C or C' by

$$x_t(s) := x(t+s) \quad \text{for all} \quad s \in [-1,0],$$

provided the interval $[t-1,t]$ belongs to the domain of x.

The term trajectory is used in connection with maps: If P is a mapping, then sequences (ϕ_n) with $\phi_{n+1} = P(\phi_n)$ are called trajectories.

3. Basic Properties of Solutions

Let a continuously differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ be given as in the introduction:

$$f(0) = 0, \quad f'(\xi) < 0 \quad \text{for all} \quad \xi, \quad \sup f < \infty.$$

Let a constant $\mu > 0$ be given.

We collect a series of basic facts on solutions of equation (1.1). For proofs, see e.g. **[36]**.

The phase curves of solutions $x = x^\phi$ of initial value problems

$$x'(t) = -\mu x(t) + f(x(t-1)) \quad \text{for} \quad t > 0, \quad x|[-1,0] = \phi \in C$$

define a continuous semiflow

$$F : \mathbb{R}^+ \times C \ni (t,\phi) \longrightarrow x_t^\phi \in C.$$

We have continuous dependence on initial data also in the following sense:

For $\phi \in C, t \geq 0$, and $\epsilon > 0$ given, there exists $\delta > 0$ such that for all $\psi \in C$ with $|\psi - \phi| \leq \delta$ and for all $s \in [0,t]$,

$$|x^\psi(s) - x^\phi(s)| < \epsilon.$$

Each map $F(t,\cdot)$ is injective. Any two solutions $x : \mathbb{R} \longrightarrow \mathbb{R}$ and $x^* : \mathbb{R} \longrightarrow \mathbb{R}$ with $x_t = x_t^*$ for some $t \in \mathbb{R}$ coincide. Each map $F(t,\cdot), t \geq 1$, is compact.

F is of class C^1 on $(1,\infty) \times C$; each $F(t,\cdot), t \geq 0$, is of class C^1. Each $D_2F(t,\phi)$ is injective, and D_2F is continuous on $\mathbb{R}^+ \times C$.

For $t > 1$ and $\phi \in C$,

$$D_1F(t,\phi)1 = (x')_t \quad \text{where} \quad x = x^\phi,$$

and for $t \geq 0, \phi \in C, \psi \in C$,

$$D_2F(t,\phi)\psi = v_t$$

where $v : [-1,\infty) \longrightarrow \mathbb{R}$ is the solution of the initial value problem for the linear variational equation along $x := x^\phi$,

$$v'(t) = -\mu v(t) + f'(x(t-1))v(t-1), \tag{3.1}$$

$$v_0 = \psi.$$

The maps $V_x(t,0), t \geq 0$, given by $\psi \longrightarrow v_t$ are linear, continuous, and compact for $t \geq 1$. For more on linear variational equations, see e.g. [**6, 13**].

Note that if $x : [-2, \infty) \longrightarrow \mathbb{R}$ is a solution of eq. (1.1), then

$$D_2F(t, x_0)(x')_0 = (x')_t \quad \text{for all} \quad t \geq 0.$$

Among the most elementary properties of single solutions are the following: Every solution $x : [t_0 - 1, \infty) \longrightarrow \mathbb{R}$ is bounded. Its ω–limit set

$$\omega(x) = \{\phi \in C : \text{There is a sequence} \quad (t_n)_0^\infty$$
$$\text{with} \quad t_n \longrightarrow \infty, \quad F(t_n, x_{t_0}) \longrightarrow \phi\}$$

is nonempty, compact, connected. Each $\phi \in \omega(x)$ defines a unique solution $x^* : \mathbb{R} \longrightarrow \mathbb{R}, x_0^* = \phi$, with phase curve in $\omega(x)$. Every positive (negative) solution $x : [t_0 - 1, \infty) \longrightarrow \mathbb{R}$ tends to 0 as $t \longrightarrow \infty$. For every bounded solution $x : \mathbb{R} \longrightarrow \mathbb{R}$,

$$\inf x^{-1}(0) = -\infty.$$

PROOF. Suppose $x(t) \neq 0$ for all $t \leq t_0$. Set $\hat{x}(t) := e^{\mu t}x(t)$. The solution $\hat{x}$ of the equation

$$\hat{x}'(t) = e^{\mu t} f(e^{-\mu(t-1)}\hat{x}(t-1))$$

satisfies $\text{sign}(\hat{x}(t)) = -\text{sign}(\hat{x}'(t)) \neq 0$ for $t \leq t_0$ and $\hat{x}(t) \longrightarrow 0$ as $t \longrightarrow -\infty$, a contradiction. $\square$

An important observation is that initial values in the convex cone

$$K := \{\phi \in C : \phi(-1) = 0, \quad 0 < \phi \quad \text{in} \quad (-1, 0]\}$$

define solutions $x = x^\phi$ which are slowly oscillating functions (with respect to the delay 1) in the sense that any pair of zeros $z' > z$ satisfies

$$z' - z > 1.$$

Moreover, for $\phi \in K$, the zeros of $x := x^\phi$ form a sequence $(z_j)_0^J$, $J \in \mathbb{N}_0$ or $J = \infty$, such that

$$z_j + 1 < z_{j+1} \quad \text{and} \quad x'(z_{j+1}) \neq 0 \quad \text{for} \quad j < J. \tag{3.2}$$

(It can be shown that the case $J < \infty$ occurs for

$$-f'(0) \cdot e^\mu < e^{-1}.$$

See also Section 4: Cases I and II provide solutions of the linearized equation without zeros.)

When convenient we shall also write $z_j(x), z_j(\phi), J(x), J(\phi)$.

Phase curves of solutions starting in K have values in the set S of nonzero functions with at most one change of sign; i.e.

$$\phi \in S \quad \text{if and only if} \quad \phi \neq 0, \quad \text{and there exists}$$
$$z \in [-1, 0] \quad \text{such that} \quad \phi \leq 0 \quad \text{in} \quad [-1, z], \quad 0 \leq \phi \quad \text{in} \quad [z, 0],$$
$$\text{or there is} \quad z \in [-1, 0] \quad \text{with} \quad 0 \leq \phi \quad \text{in} \quad [-1, z], \quad \phi \leq 0 \quad \text{in} \quad [z, 0].$$

S is a cone (if $t > 0$ and $\phi \in S$, then $t\phi \in S$), but not convex. One can show that S is homotopy equivalent to a circle [**10**]. Elementary considerations yield

$$\overline{S} = S \cup \{0\}.$$

The set S is flow-invariant:

$$F(\mathbb{R}^+ \times S) \subset S,$$

and, more generally than above, every $\phi \in S$ defines a solution $x = x^\phi$ such that for t_0 sufficiently large, $x|[t_0, \infty)$ is slowly oscillating.

The monotonicity of f implies an even stronger statement.

PROPOSITION 3.1. 1. *For initial data ϕ, ψ with $\phi - \psi \in S$, there exists $s \in [0, 4]$ such that $F(s, \phi) - F(s, \psi)$ has no zero;*

$$F(t, \phi) - F(t, \psi) \in S \quad \textit{for all} \quad t \geq 0.$$

2. *Let a solution $x : [-1, \infty) \longrightarrow \mathbb{R}$ of eq. (1.1) be given. Every solution $v : [-1, \infty) \longrightarrow \mathbb{R}$ of eq. (3.1), with $v_0 \in S$, satisfies*

$$v_t \in S \quad \textit{for all} \quad t \geq 0,$$

and there exists $s \in [0, 4]$ so that v_s has no zero. On $[4, \infty)$, v is slowly oscillating.

PROOF. 1. Apply [**36, Remark 6.1**] and [**36, Proposition 6.1**] to the solution

$$t \longrightarrow e^{\mu t}(x^\phi(t) - x^\psi(t))$$

of the equation

$$x'(t) = g(t, x(t-1))$$

where

$$g(t, \xi) = e^{\mu t}[f(e^{-\mu(t-1)}(\xi + x^\psi(t-1))) - f(e^{-\mu(t-1)}x^\psi(t-1))]$$
$$= e^{\mu t} \int_{e^{-\mu(t-1)}x^\psi(t-1)}^{e^{-\mu(t-1)}(\xi + x^\psi(t-1))} f'.$$

2. Apply [**36, Remark 6.1, Proposition 6.1**] to the solution $t \longrightarrow e^{\mu t}v(t)$ of the equation

$$d'(t) = e^\mu f'(x(t-1))d(t-1). \quad \square$$

The zeros of any bounded solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ with all values x_t of the phase curve in S form a sequence $(z_j)_{-\infty}^J$, $J \in \mathbb{Z}$ or $J = \infty$, with property (3.2).

PROPOSITION 3.2. 1. *Let $x : \mathbb{R} \longrightarrow \mathbb{R}$ be a solution of eq. (1.1) whose zeros form a sequence $(z_j)_{-\infty}^{J}$, $J \in \mathbb{Z}$ or $J = \infty$, with property (3.2). Then each local extremum of x belongs to some interval $(z_j, z_j + 1]$.*

2. *There exist positive constants b_f, c_f, d_f such that for every solution as in assertion 1,*

$$|x| \le b_f; \quad |x(t) - x(s)| \le c_f \cdot |t - s|$$

$$\textit{and} \quad \|x_t - x_s\| \le c_f \cdot |t - s| \quad \textit{for all real} \quad t, s.$$

Furthermore,

$$\max_{[z_j, z_{j+1}]} |x| \le d_f \max_{[z_{j-1}, z_j]} |x|$$

for all $j \le J - 1$, in case $J \in \mathbb{Z}$, and for all $j \in \mathbb{Z}$, in case $J = \infty$; and

$$\max_{[z_J, \infty)} |x| \le d_f \cdot \max_{[z_{J-1}, z_J]} |x| \quad \textit{in case} \quad J \in \mathbb{Z}.$$

PROOF. 1. For $z_j + 1 \le t < z_{j+1}$, or for $z_j + 1 \le t < \infty$ in case $j = J < \infty$,

$$x'(t) = -\mu x(t) + f(x(t-1)) < f(x(t-1)) \le 0 \quad \text{in case} \quad 0 < x(t),$$

$$x'(t) = -\mu x(t) + f(x(t-1)) > f(x(t-1)) \ge 0 \quad \text{in case} \quad x(t) < 0.$$

Consequently, each local extremum of x lies in an interval $(z_j, z_j + 1)$, and in case $J < \infty$,

$$|x(t)| \le \max_{[z_J, z_J + 1]} |x| \quad \text{for} \quad t \ge z_J + 1.$$

2. Estimate of local maxima:
If $0 < x$ in (z_j, z_{j+1}) and $z_j < t \le z_j + 1$, then

$$x'(t) = -\mu x(t) + f(x(t-1)) \le f(x(t-1)) \le \sup(f),$$

hence

$$0 < x(t) \le \sup(f).$$

Estimate of local minima:
If $x < 0$ in (z_j, z_{j+1}) and $z_j < t \le z_j + 1$, then

$$x'(t) \ge f(x(t-1)) \ge f(\max_{[z_{j-1}, z_j]} x) \ge f(\sup(f));$$

hence

$$x(t) \ge f(\sup(f)).$$

Set $b_f := \max\{\sup(f), -f(\sup(f))\}$. Lipschitz continuity follows from

$$|x'(t)| \le \mu b_f + \max\{f(-b_f), -f(b_f)\}, \quad \text{for} \quad t \in \mathbb{R}.$$

There exists $d_f > 0$ such that

$$|f(\xi)| \le d_f \cdot |\xi| \quad \text{for} \quad |\xi| \le b_f.$$

Consider a zero z_j with, say, $x'(z_j) > 0$. Then, for every $t \in (z_j, z_j + 1)$,

$$x'(t) = -\mu x(t) + f(x(t-1)) \le f(x(t-1)) \le d_f \cdot |x(t-1)| \le d_f \max_{[z_{j-1}, z_j]} |x|,$$

and consequently

$$0 < x(t) = \int_{z_j}^{t} x'(s)ds \le d_f \max_{[z_{j-1}, z_j]} |x|;$$

and it becomes obvious how to deduce the estimates involving d_f. □

To investigate slowly oscillating solutions we shall make use of a globally defined return map. Consider the subset

$$K_P := \{\phi \in K \cup (-K) : J(\phi) \ge 2\}$$

of the double cone $K \cup (-K)$. Phase curves $t \longrightarrow x_t^{\phi}$ which start in K_P intersect at $t = z_2(\phi) + 1$ transversally with the hyperplane

$$H \subset C \quad \text{given by} \quad \phi(-1) = 0$$

since the tangent vector $\chi := D_1 F(z_2(\phi) + 1, \phi)1$ satisfies

$$\chi(-1) = x'(z_2) \ne 0.$$

Using the Implicit Function Theorem and continuous dependence on initial data, one constructs an open neighborhood N of K_P and a C^1–map

$$\mathit{stop} : N \longrightarrow (1, \infty)$$

(such that $F(\mathit{stop}(\phi), \phi) \in H$ for $\phi \in N$) with

$$\mathit{stop}(\phi) = z_2(\phi) + 1 \quad \text{for all} \quad \phi \in K_P.$$

The C^1–map

$$P : N \longrightarrow C, \quad P(\phi) = F(\mathit{stop}(\phi), \phi)$$

satisfies

$$P(\phi) = F(z_2(\phi) + 1, \phi) \in K \quad \text{for} \quad \phi \in K \quad \text{and} \quad J(\phi) \ge 2,$$

$$P(\phi) = F(z_2(\phi) + 1, \phi) \in -K \quad \text{for} \quad \phi \in -K \quad \text{and} \quad J(\phi) \ge 2.$$

Let $p_\chi : C \longrightarrow C$ denote the projection onto H, parallel to χ;

$$p_\chi(\psi) = \psi - \frac{\psi(-1)}{\chi(-1)} \cdot \chi.$$

Then one computes, for $\phi \in K_P$ and $\psi \in C$,

$$DP(\phi)\psi = p_\chi D_2 F(z_2(\phi) + 1, \phi)\psi.$$

The same constructions which lead to the return map P yield also a smooth stopping map $P_{1/2}$ on an open neighborhood $N_{1/2}$ of

$$K_{1/2} := \{\phi \in K \cup (-K) : J(\phi) \ge 1\}$$

such that

$$P_{1/2}(\phi) = F(z_1(\phi) + 1, \phi) \quad \text{on} \quad K_{1/2};$$

i.e.,

$$P_{1/2}(\phi) \in -K \quad \text{for} \quad \phi \in K \quad \text{and} \quad J(\phi) \ge 1,$$

$$P_{1/2}(\phi) \in K \quad \text{for} \quad \phi \in -K \quad \text{and} \quad J(\phi) \geq 1.$$

In the sequel we shall consider slowly oscillating periodic solutions

$$y : \mathbb{R} \longrightarrow \mathbb{R}$$

of eq. (1.1). We may, and will, assume

$$z_0(y) = -1 \quad \text{and} \quad y_0 \in K;$$

the minimal period $\tau = \tau(y)$ of y is then of the form

$$\tau = z_{2n}(y) + 1 \quad \text{for some} \quad n \in \mathbb{N}$$

(later we shall prove that $n = 1$), and

$$\eta : [0, \tau] \ni t \longrightarrow y_t \in C$$

is a simple closed smooth curve. When convenient we shall write $\eta(y)$ instead of η.

The local stability properties of y are governed by the Floquet multipliers, i.e. by the spectrum of the linear continuous, compact map $V = V_y(\tau, 0)$ given by the solutions of the linear variational equation

$$v'(t) = -\mu v(t) + f'(y(t-1))v(t-1) \tag{3.3}$$

The number 1 is an eigenvalue of V, and $(y')_0$ is an eigenvector.

The solution y is called hyperbolic if 1 is simple and if there are no other eigenvalues on the unit circle.

If y is unstable, i.e. not stable, and hyperbolic, then necessarily

$$|\lambda| > 1$$

for at least one Floquet multiplier λ.

We end this section with basic facts about the unstable set

$$W := W(y) := \{\phi \in C : \text{There is a solution} \quad x : \mathbb{R} \longrightarrow \mathbb{R} \quad \text{of eq. (1.1)}$$
$$\text{such that} \quad x_0 = \phi \quad \text{and} \quad x_t \longrightarrow |\eta| \quad \text{as} \quad t \longrightarrow -\infty\}.$$

Observe that

$$F(\mathbb{R}^+ \times W) = W.$$

PROPOSITION 3.3. *$W \subset S$.*

PROOF. Let $\phi \in W$. Consider the solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ with $x_0 = \phi$. Use the compactness of $|\eta|$ to find a sequence $t_n \longrightarrow -\infty$ and $t \in \mathbb{R}$ such that $x_{t_n} \longrightarrow y_t$. For some $s > t$, y_s has no zero. By continuity of F, $x_{t_n + s}$ has no zero for $|n|$ sufficiently large; $x_{t_n + s} \in S$. Forward invariance of S yields $\phi = x_0 \in S$. $\square$

Every solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) with phase curve in W is bounded. Using Proposition 3.3, and previous remarks, we infer that the zeros of x form a sequence $(z_j)_{-\infty}^{J}$ where $J = J(x) \in \mathbb{Z}$ or $J = J(x) = \infty$, so that statement (3.2) holds. Proposition 3.2 yields

$$\|x_t\| \leq b_f \quad \text{on} \quad \mathbb{R}.$$

It follows that

$$\overline{W} \subset \overline{F(\{1\} \times \{\phi \in C : \|\phi\| \leq b_f\})}$$

is compact. Also,

$$\overline{W} \subset \overline{S} = S \cup \{0\}.$$

PROPOSITION 3.4. *For every $\phi \in \overline{W} \setminus W$ there is a solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) such that $x_0 = \phi$ and $x_t \in \overline{W} \setminus W$ for all $t \in \mathbb{R}$.*

PROOF. See the proof of **[36, Proposition 8.1]** for the case $\phi \neq 0$. $\square$

We infer

$$F(\mathbb{R}^+ \times (\overline{W} \setminus W)) = \overline{W} \setminus W.$$

The set

$$(\overline{W} \setminus W) \setminus \{0\}$$

is formed by phase curves of bounded, slowly oscillating solutions as above.

4. On the Linear Equation

The solutions $x : [-1, \infty) \longrightarrow \mathbb{R}$ of eq. (1.2), $\alpha := -f'(0)$ as in the introduction, define a C_0–semigroup of operators $T(t) = D_2F(t, 0)$, $t \geq 0$. The spectrum spec of its generator consists of complex conjugate pairs of eigenvalues in the double strips S_k given by

$$2k\pi < |\text{Im}(\lambda)| < 2k\pi + \pi, \quad k \in \mathbb{N},$$

and by at most two eigenvalues in the strip S_0 given by

$$|\text{Im}(\lambda)| < \pi;$$

the total multiplicity of spec in S_0 is 2.

We have

$$\max \text{Re}(\cup_{\mathbb{N}}(\text{spec} \cap S_k)) \quad < \quad \min \text{Re}(\text{spec} \cap S_0).$$

Spectral decomposition and reellification yield a $T(t)$–invariant decomposition

$$C = L \oplus Q \tag{4.1}$$

into closed subspaces. The generator of the induced semigroup on L has spectrum spec $\cap S_0$;

$$\dim L = 2.$$

Three cases are possible for spec $\cap S_0$:

I. $\alpha e^{\mu} < \frac{1}{e}$. spec $\cap S_0$ consists of two simple, real eigenvalues $u_{00} < u_0 < 0$; a basis $\{\beta_1, \beta_2\}$ of L is given by the restrictions of the functions

$$t \longrightarrow e^{u_0 t}, \quad t \longrightarrow e^{u_{00} t}$$

to $[-1,0]$.

II. $\alpha e^{\mu} = \frac{1}{e}$. spec $\cap S_0$ consists of a double eigenvalue

$$u_0 = -1 - \mu;$$

a basis $\{\beta_1, \beta_2\}$ of L is given by

$$t \longrightarrow e^{u_0 t}, \quad t \longrightarrow -te^{u_0 t}.$$

III. $\alpha e^{\mu} > \frac{1}{e}$. spec $\cap S_0$ consists of a complex conjugate pair $\lambda_0 = u_0 + iv_0$, $\overline{\lambda_0}$ of simple eigenvalues; $0 < v_0$. A basis $\{\beta_1, \beta_2\}$ of L is given by

$$t \longrightarrow e^{u_0 t}\sin(v_0 t), \quad t \longrightarrow e^{u_0 t}\cos(v_0 t).$$

REMARK 4.1. 1. Slowly and rapidly oscillating solutions. L consists of the segments x_t of the linear combinations of the functions $\mathbb{R} \longrightarrow \mathbb{R}$ used to define the bases above. All of these linear combinations (except the trivial one) are slowly oscillating solutions of eq. (1.2); we have

$$L \subset S \cup \{0\}.$$

Real–valued solutions associated with eigenvalues $\lambda = u + iv$ outside S_0 have zeros spaced at distances

$$\frac{\pi}{v} < \frac{1}{2};$$

hence they are not slowly oscillating.

2. Stability. Consider

$$v(\mu) \in \left(\frac{\pi}{2}, \pi\right) \quad \text{defined by} \quad v(\mu) = -\mu \tan(v(\mu)).$$

For

$$\alpha < -\frac{\mu}{\cos(v(\mu))} \quad \left(= \frac{v(\mu}{\sin(v(\mu))} > \frac{\pi}{2}\right),$$
$$\mathrm{Re(spec)} < 0;$$

the zero solution of eq. (1.2) is exponentially attractive. It follows that there exists a neighborhood U of 0 and a positive constant c such that

$$||F(t,\phi)|| \le ce^{\frac{u_0 t}{2}} \quad \text{for all} \quad \phi \in U, t \ge 0.$$

At $\alpha = -\frac{\mu}{\cos(v(\mu))}$, $u_0 = 0$, and for $\alpha > -\frac{\mu}{\cos(v(\mu))}$, $u_0 > 0$.

3. For $\alpha e^{\mu} > 1$, every solution $x : [-1,\infty) \longrightarrow \mathbb{R}$ of the nonlinear equation (1.1) has an unbounded zeroset (see [**36, Proposition 6.3, proof, part a**]). We conclude that in case there exists a solution x of eq. (1.1) with $x^{-1}(0)$ bounded from above,

$$\alpha \le 1, \quad \text{and} \quad \mathrm{Re(spec)} < 0.$$

Let $p : C \longrightarrow C$ and $q : C \longrightarrow C$ denote the projections onto L and Q given by (4.1).

LEMMA 4.1. $0 \notin pS$.

PROOF. In case III, see [**36, Lemma 6.3, proof**]. Analogous arguments work in the cases I and II where nontrivial solutions with phase curve in L have at most one zero. □

In other words,

$$S \cap Q = \emptyset,$$

the cone S contains $L \setminus \{0\}$ and stays away from the complementary subspace Q.

Observe that

$$pK \cap (-pK) = \emptyset \tag{4.2}$$

since $p\phi = p\psi$ with $\phi \in K$, $\psi \in -K$ would imply

$$0 = p(\phi - \psi) \in p(K + K) \subset pK \subset pS,$$

a contradiction.

The remainder of this section is devoted to simple closed curves in L which wind around the origin. The subsequent constructions will later be used to define homotopies with values in the set S.

We begin with certain subcones of S. It will become important that these subcones are convex (recall that S is not).

For every integer $k \ge 3$, define

$$K_{k0} := \{\phi \in C : 0 < \phi(t) \quad \text{in} \quad (-1,0)\},$$

$$K_{k1} := \left\{\phi \in C : 0 < \phi(t) \quad \text{in} \quad \left[-1, -\frac{1}{k}\right), \quad \phi'(t) \quad \text{exists} \right.$$
$$\left. \text{and is negative in} \quad \left[-\frac{1}{k}, 0\right]\right\},$$

$$K_{k\kappa} := \left\{\phi \in C : 0 < \phi(t) \quad \text{in} \quad \left[-1, -\frac{\kappa}{k}\right), \quad \phi'(t) \quad \text{exists and is negative} \right.$$
$$\left. \text{in} \quad \left[-\frac{\kappa}{k}, -\frac{\kappa-1}{k}\right], \quad \phi(t) < 0 \quad \text{in} \quad \left(-\frac{\kappa-1}{k}, 0\right]\right\}$$
$$\text{for} \quad \kappa = 2, \dots, k-1,$$

$$K_{kk} := \left\{\phi \in C : \phi'(t) \quad \text{exists and is negative in} \quad \left[-1, -1 + \frac{1}{k}\right], \right.$$
$$\left. \phi(t) < 0 \quad \text{in} \quad \left(-1 + \frac{1}{k}, 0\right]\right\}.$$

Our aim is to find, in each of the cases I, II, III, a simple closed piecewise smooth curve

$$c : [0,b] \longrightarrow C, \quad |c| \subset L,$$

with

$$1 = \mathrm{wind}(0,c)$$

and an integer $k(c) \geq 3$ such that for every integer $k \geq k(c)$, there is a subdivision

$$0 = t_0 < t_1 < \cdots < t_{2k+2} = b$$

so that

$$c(t) \in K_{k\kappa} \quad \text{for} \quad t_\kappa \leq t \leq t_{\kappa+1}, \quad \kappa = 0, \ldots, k;$$
$$c(t) \in -K_{k\kappa} \quad \text{for} \quad t_{k+1+\kappa} \leq t \leq t_{k+1+\kappa+1}, \quad \kappa = 0, \ldots, k.$$

We begin with case III. The slowly oscillating solution

$$x : t \longrightarrow e^{u_0(t+1)} \sin(v_0(t+1))$$

of eq. (1.2) has segments $x_t \in L$; its zeros are given by

$$z_j = \frac{\pi j}{v_0} - 1, \quad j \in \mathbb{Z}.$$

Note

$$z_0 = -1, \quad x_0 \in K, \quad z_2 + 1 = \frac{2\pi}{v_0}.$$

Except for $u_0 = 0$, x is not periodic. For $0 \leq t \leq \frac{2\pi}{v_0}$, we set

$$c(t) := e^{-u_0 t} x_t$$
$$= e^{u_0} \cos(v_0(t+1)) \cdot \beta_1 + e^{u_0} \sin(v_0(t+1)) \cdot \beta_2.$$

This defines a simple closed smooth curve in L. Employing the isomorphism $\zeta : L \longrightarrow \mathbb{C}$ with $\beta_1 \longrightarrow 1, \quad \beta_2 \longrightarrow i$, we have

$$1 = \text{wind}(0, c).$$

Choose an integer $k(c) \geq 3$ so large that

$$x'(t) \neq 0 \quad \text{for} \quad |t - z_j| \leq \frac{1}{k(c)}, \quad j \in \{0, 1, 2\}.$$

One verifies easily that for every integer $k \geq k(c)$,

$$x_t \in K_{k0} \quad \text{for} \quad 0 \leq t \leq z_1,$$

$$x_t \in K_{k\kappa} \quad \text{for} \quad z_1 + \frac{\kappa}{k} \leq t \leq z_1 + \frac{\kappa+1}{k}, \quad \kappa = 0, \ldots, k-1;$$

$$x_t \in -K_{k0} \quad \text{for} \quad z_1 + 1 \leq t \leq z_2;$$

$$x_t \in -K_{k\kappa} \quad \text{for} \quad z_2 + \frac{\kappa}{k} \leq t \leq z_2 + \frac{\kappa+1}{k}, \quad \kappa = 0, \ldots, k-1.$$

The values $c(t)$ have the same property. A subdivision as required is given by

$$0 = t_0 < t_1 = z_1 < t_2 = z_1 + \frac{1}{k} < \cdots < t_{k+1} = z_1 + 1$$

$$< t_{k+2} = z_2 < t_{k+3} = z_2 + \frac{1}{k} < \cdots < t_{2k+2} = z_2 + 1.$$

In cases I and II, the construction of c is different. We shall obtain c as a parameterization of the boundary of the quadrangle

$$\{r \cdot \beta_1 + s \cdot \beta_2 \in L : |r| \leq 1, |s| \leq 1\};$$

in these cases, points on $|c|$ are not multiples of segments of one solution of eq. (1.2).

Case I. Consider the straight line parameterizations

$$c_1 : [-1, 1] \ni s \longrightarrow \beta_1 + s \cdot \beta_2 \in L,$$

$$c_2 : [-1, 1] \ni r \longrightarrow -r \cdot \beta_1 + \beta_2 \in L.$$

It is an elementary exercise to prove the following result.

PROPOSITION 4.1. *Let real numbers r and s be given with $0 < |r| + |s|$. The function*

$$g = g_{rs}, \quad g(t) = re^{u_0 t} + se^{u_{00} t} \quad \textit{for} \quad t \in \mathbb{R},$$

has at most one zero. A zero $z = z(r, s)$ exists if and only if $r \neq 0 \neq s$ and $sign(s) = -sign(r)$. In this case,

$$z = -\frac{1}{u_{00} - u_0} \log\left(-\frac{s}{r}\right), \quad sign(g'(z)) = sign(-s) \neq 0.$$

If g and g' have zeros z and z', respectively, then the distance

$$z - z' = \frac{\log(u_0) - \log(u_{00}}{u_0 - u_{00}} > 0 \tag{4.3}$$

does not depend on r, s.

For c_1, we have

$$c_1(s)(t) = g_{1s}(t) \quad (|s| \leq 1, \quad -1 \leq t \leq 0).$$

Consequently, each $c_1(s)$ has at most one zero; $c_1(s)$ has a zero (in $[-1, 0]$) if and only if

$$-1 \leq s < 0 \quad \text{and} - 1 \leq z(1, s) \leq 0;$$

in this case, the zero of $c_1(s)$ equals

$$z(1, s) = \frac{1}{u_0 - u_{00}} \log(-s).$$

Define $s_1 \in [-1, 0)$ by

$$-1 = \frac{1}{u_0 - u_{00}} \log(-s_1) \quad (= z(1, s_1)).$$

Then

$$c_1(s) \quad \text{has a zero} \quad z(1, s) \quad \text{for} \quad -1 \leq s \leq s_1,$$
$$z(1, -1) = 0,$$

$$\frac{\partial z}{\partial s}(1, s) = \frac{1}{u_0 - u_{00}} \cdot \frac{1}{s} < 0 \quad \text{for} \quad -1 \leq s \leq s_1 \quad (< 0), \tag{4.4}$$

$$z(1, s_1) = -1,$$

and

$$\operatorname{sign}(c_1(s)'(z(1,s)) = \operatorname{sign}(g'_{1s}(z(1,s)) = \operatorname{sign}(-s) > 0 \quad \text{for} \quad -1 \le s \le s_1.$$

Choose $k \in \mathbb{N}, \quad k \geq 3$ so large that

$$\frac{1}{k} < \frac{\log(u_0) - \log(u_{00})}{u_0 - u_{00}}. \tag{4.5}$$

Using (4.4) we find a subdivision

$$-1 = \sigma_0 < \sigma_1 < \cdots < \sigma_k = s_1$$

such that for $\sigma_{\kappa-1} \le s \le \sigma_\kappa, \quad \kappa = 1, \dots, k$, we have

$$-\frac{\kappa}{k} \le z(1,s) \le -\frac{\kappa - 1}{k}.$$

Using (4.3) and (4.5) we infer that for such s and κ,

$$c_1(s)'(t) > 0 \quad \text{for} \quad -\frac{\kappa}{k} \le t \le -\frac{\kappa-1}{k},$$

$$c_1(s)(t) < 0 \quad \text{for} \quad t \in [-1,0] \cap \left(-\infty, -\frac{\kappa}{k}\right],$$

$$0 < c_1(s)(t) \quad \text{for} \quad t \in [-1,0] \cap \left[-\frac{\kappa-1}{k}, \infty\right),$$

or

$$c_1(s) \in -K_{k\kappa}.$$

Next, we see that for $s_1 < s < 0$ and for $0 \le s \le 1$, the function $c_1(s)$ is positive on $(-1, 0)$; i.e.,

$$c_1(s) \in K_{k0}.$$

We turn to c_2. From the definition, $c_2(r)$ is positive on $(-1,0)$ for $|r| \le 1$, hence

$$c_2(r) \in K_{k0} \quad \text{for} \quad -1 \le r \le 1.$$

Set $c_3 := -c_1, \quad c_4 := -c_2$. Now it becomes obvious how to complete the construction.

Case II. Consider c_1 and c_2 defined as in Case I.

PROPOSITION 4.2. *Let real numbers r and s with $0 < |r| + |s|$ be given. The function*

$$g = g_{rs}, \quad g(t) = e^{u_0 t}(r + st) \quad \textit{for} \quad t \in \mathbb{R},$$

has at most one zero. A zero $z = z(r,s)$ exists if and only if $s \neq 0$. In this case,

$$z = -\frac{r}{s}, \quad \textit{sign}(g'(z)) = \textit{sign}(s).$$

If g and g' have zeros z and z', respectively, then

$$z' - z = -\frac{1}{u_0}. \tag{4.6}$$

Note that for $|s| \le 1$ and $-1 \le t \le 0$,

$$c_1(s)(t) = \beta_1(t) + s \cdot \beta_2(t) = e^{u_0 t}(1 + (-s) \cdot t) = g_{1,-s}(t).$$

The function $c_1(s)$ is strictly positive on $(-1,0)$ for $|s| \le 1$, so that $c_1(s) \in K_{k0}$ for all integers $k \geq 3, \quad |s| \le 1$.

We turn to c_2. For $|r| \le 1$ and $-1 \le t \le 0$,

$$c_2(r)(t) = -r \cdot \beta_1(t) + \beta_2(t) = e^{u_0 t}(-r + (-1) \cdot t) = g_{-r,-1}(t).$$

It follows that $c_2(r)$ has a zero (in $[-1,0]$) if and only if $0 \le r \le 1$; in this case, the zero equals

$$z(-r,-1) = -r;$$

$$z(0,-1) = 0,$$

$$\frac{\partial}{\partial r}(r \longrightarrow z(-r,-1))(r) = -1 < 0 \quad \text{for} \quad 0 \le t \le 1,$$

$$z(-1,1) = -1,$$

and for $0 \le r \le 1$,

$$\operatorname{sign}(c_2(r)'(z(-r,-1))) = \operatorname{sign}(g'_{-r,-1}(z(-r,-1))) = \operatorname{sign}(-1) = -1. \tag{4.7}$$

For $-1 \le r \le 0$, we see that

$$c_2(r) \quad \text{is strictly positive on} \quad (-1,0).$$

Choose an integer $k \geq 3$ with

$$\frac{1}{k} < -\frac{1}{u_0}. \tag{4.8}$$

For $\frac{\kappa-1}{k} \le r \le \frac{\kappa}{k}, \quad \kappa = 1, \dots, k$, we have

$$-\frac{\kappa}{k} \le z(-r,-1) \le -\frac{\kappa-1}{k}.$$

By (4.8) and (4.6), and by (4.5),

$$c_2(r)'(t) < 0 \quad \text{for} \quad -\frac{\kappa}{k} \le t \le -\frac{\kappa-1}{k};$$

and

$$0 < c_2(r)(t) \quad \text{for} \quad t \in [-1,0] \cap \left(-\infty, -\frac{\kappa}{k}\right],$$

$$c_2(r)(t) < 0 \quad \text{for} \quad t \in [-1,0] \cap \left[-\frac{\kappa-1}{k}, \infty\right);$$

or

$$c_2(r) \in K_{k\kappa}.$$

Now the construction is completed as in case I.

5. Results

THEOREM 5.1. *For every slowly oscillating periodic solution y of eq.* (1.1) *with $y_0 \in K$ and $z_0 = -1$,*

$$\tau = z_2(y) + 1 \quad \text{and} \quad 0 \in int(p \circ \eta).$$

COROLLARY 5.1. *For any two slowly oscillating periodic solutions y^1 and y^2 with initial values in K and $z_0(y^i) = -1$, the simple closed curves*

$$\eta(y^i) : [0, \tau(y^i)] \ni t \longrightarrow y^i_t \in C$$

have nested projections: Either $y^1 = y^2$, or $|p \circ \eta(y^i)| \subset int(p \circ \eta(y^j))$, $\quad i \neq j$.

THEOREM 5.2. *For every slowly oscillating periodic solution y of eq.* (1.1), *y' is also slowly oscillating and has minimal period $\tau(y)$. The zeros of y and y' alternate.*

REMARK 5.1. We shall use the projection p to prove these results. Alternatively, one can work with evaluation maps

$$x_t \longrightarrow (x(t), x'(t)) \quad \text{or} \quad x_t \longrightarrow (x(t), x(t-1))$$

into the plane $\mathbb{R}^2$ (one leaves the phase space of eq. (1.1)), derive the statements about minimal periods and about y', and obtain analogues of the others. See **[23, 24]**.

THEOREM 5.3. *(Floquet multipliers) Let a slowly oscillating periodic solution y of eq.* (1.1) *be given. If y is unstable and hyperbolic, then there exists exactly one Floquet multiplier u with $|u| > 1$. Also, u is simple, and $u \in (1, \infty)$. If y is unstable but not hyperbolic, then the multiplier 1 has multiplicity 2, and $|\lambda| < 1$ for all multipliers $\lambda \neq 1$.*

THEOREM 5.4. *(On the unstable set of y) Suppose y is a slowly oscillating periodic solution of eq.* (1.1) *which is unstable and hyperbolic.*
1. The closure $\overline{W}$ of the unstable set $W = W(y)$ is a graph $\overline{w} : \overline{pW} \longrightarrow Q$. Moreover, $\overline{w}$ is Lipschitz continuous. The set pW is an open subset of L, and $w := \overline{w}|pW$ is C^1.
2. $E := (\overline{W} \setminus W) \cap p^{-1}(ext(p \circ \eta))$ is the orbit of a slowly oscillating periodic solution of eq. (1.1).
3. $I := (\overline{W} \setminus W) \cap p^{-1}(int(p \circ \eta))$ is either the orbit of a slowly oscillating periodic solution of eq. (1.1), *or $I = \{0\}$.*

6. Proofs of Theorem 5.1, Corollary 5.1 and Theorem 5.2

PROPOSITION 6.1. 1. *If y is a slowly oscillating periodic solution of eq.* (1.1), *$y_0 \in K$ and $z_0 = -1$, then*

$$y_t - y_s \in S \quad \text{for} \quad 0 \leq s < t < \tau.$$

2. *For any two slowly oscillating periodic solutions y^i, $\quad i = 1, 2$, with disjoint orbits in C,*

$$y^1_t - y^2_s \in S \quad \text{for all real} \quad t, s.$$

PROOF. 1. Fix $t_0 \in (0, \tau)$ and $t_1 \in (t_0, t_0 + \tau)$ so that $y_{t_0} > 0 > y_{t_1}$. By continuity,

$$M := \{t \in (t_0, t_1) : y_{t_0} - y_s \in S \quad \text{for all} \quad s \in [t, t_1]\} \neq \emptyset,$$

and $t_- := \inf(M) \in [t_0, t_1)$. Suppose $t_0 < t_-$. By continuity, $\quad y_{t_0} - y_{t_-} \in \overline{S} = S \cup \{0\}$. Minimality of τ yields

$$y_{t_0} \neq y_{t_-}, \quad \text{so} \quad y_{t_0} - y_{t_-} \in S.$$

By Proposition 3.1, there exists $t \in [0, 4]$ such that $y_{t_0+t} - y_{t_-+t}$ has zero. By continuity, there exists $\epsilon > 0$ such that for $|s| < \epsilon$

$$y_{t_0+t} - y_{t_-+t+s} \quad \text{has no zero, and is in} \quad S.$$

Fix $j \in \mathbb{N}$ so large that

$$t_0 + t + \epsilon - j\tau < t_-.$$

By periodicity,

$$y_{t_0+t-j\tau} - y_{t_-+t-j\tau+s} \in S \quad \text{for} \quad |s| < \epsilon.$$

Proposition 3.1 gives

$$y_{t_0} - y_{t_-+s} \in S \quad \text{for} \quad |s| < \epsilon,$$

which implies a contradiction to the definition of t_-. We have shown that

$$[y_{t_0} - y_t \in S \quad \text{for all} \quad t \in (t_0, t_1].$$

Analogously one finds

$$y_{t_0} - y_t \in S \quad \text{for all} \quad t \in (t_1, t_0 + \tau).$$

For $0 \leq s < t < \tau$, we have $t_0 + s - t + \tau \in (t_0, t_0 + \tau)$; the relations

$$\begin{aligned} y_t - y_s &= F(t - (t_0 - \tau), y_{t_0 - \tau}) \quad - \quad F(t - (t_0 - \tau), y_{s + (t_0 - \tau) - t}) \\ &= F(\ldots, y_{t_0}) \quad - \quad F(\ldots, y_{t_0 + s - t + \tau}) \end{aligned}$$

and Proposition 3.1, applied to $t - (t_0 - \tau) > 0$, give the assertion. 2. Fix points t_i such that $y^1_{t_1} > 0 > y^2_{t_2}$. Then

$$M := \{t < t_2 : \quad y^1_{t_1} - y^2_s \in S \quad \text{for all} \quad s \in [t, t_2]\} \neq \emptyset.$$

Suppose $t_- := \inf(M) > -\infty$. By continuity,

$$y^1_{t_1} - y^2_{t_-} \in \overline{S} = S \cup \{0\}.$$

As the orbits are disjoint,

$$y^1_{t_1} - y^2_{t_-} \in S.$$

For some $t_0 \in [0,4]$,

$$y^1_{t_1+t_0} - y^2_{t_-+t_0} \quad \text{has no zero}$$

(Proposition 3.1), and there exists $\epsilon > 0$ such that

$$y^1_{t_1+t_0} - y^2_{t_-+t_0+t} \in S \quad \text{for} \quad |t| < 2\epsilon.$$

Let $\tau_i = \tau(y^i)$ for $i = 1, 2$. There exist positive integers $k > \frac{t_0}{\tau_1}$ and l such that

$$|k\tau_1 - l\tau_2| < \epsilon;$$

or $k\tau_1 = l\tau_2 + \hat{t}$ where $|\hat{t}| < \epsilon$. Hence, for $|t| < \epsilon$,

$$y^1_{t_1+t_0-k\tau_1} - y^2_{t_-+t_0-k\tau_1+t} = y^1_{t_1+t_0} - y^2_{t_-+t_0-l\tau_2-\hat{t}+t}$$

$$= y^1_{t_1+t_0} - y^2_{t_-+t_0-\hat{t}+t} \in S.$$

Proposition 3.1 yields

$$y^1_{t_1} - y^2_{t_-+t} \in S \quad \text{for} \quad |t| < \epsilon,$$

which implies a contradiction to the definition of t_-. We have shown that

$$y^1_{t_1} - y^2_t \in S \quad \text{for all} \quad t \leq t_2.$$

Analogously, one obtains

$$y^1_{t_1} - y^2_t \in S \quad \text{for all} \quad t > t_2.$$

Now let real numbers s and t be given. Choose $k \in \mathbb{N}$ so that $s - t_1 + n\tau_1 > 0$. Then

$$y^1_s - y^2_t = F(s - t_1 + n\tau_1, y^1_{t_1 - n\tau_1}) - F(\dots, y^2_{t-(s-t_1+n\tau_1)})$$

$$= F(\dots, y^1_{t_1}) - F(\dots, y^2_{\dots}),$$

and Proposition 3.1 implies the assertion. □

PROOF OF THEOREM 5.1. 1. Recall the closed curves $c : [0,b] \longrightarrow C$, $|c| \subset L$, and the integers $k(c)$ from Section 4.
2. Consider the simple closed curve $\eta : [0,\tau] \longrightarrow C$, $\tau = z_{2n}+1$. As in Section 4, one obtains, for each subinterval

$$[z_{2\nu}+1, z_{2\nu+2}+1], \quad \nu = 0, \dots, n-1$$

an integer $k(\nu)$ so that for each integer $k \geq k(\nu)$ there is a subdivision

$$z_{2\nu}+1 = t^\nu_0 < t^\nu_1 < \dots < t^\nu_{2k+2} = z_{2\nu+2}+1$$

with

$$y_t \in K_{k\kappa} \quad \text{for} \quad t^\nu_\kappa \leq t \leq t^\nu_{\kappa+1}, \quad \kappa = 0, \dots, k,$$

$$y_t \in -K_{k\kappa} \quad \text{for} \quad t^\nu_{k+1+\kappa} \leq t \leq t^\nu_{k+1+\kappa+1}, \quad \kappa = 0, \dots, k.$$

Fix an integer

$$k \geq k(c) + \max_{\nu=0,\dots,n-1} k(\nu).$$

3. Reparameterize η so that one obtains a simple closed piecewise smooth curve $\eta_r : [0,n] \longrightarrow C$ such that for $\nu = 0, \dots, n-1$ and for $\kappa = 0, \dots, 2k+1$,

$$\eta_r\left(\left[\nu + \frac{\kappa}{2k+2}, \nu + \frac{\kappa+1}{2k+2}\right]\right) = \eta([t^\nu_\kappa, t^\nu_{\kappa+1}]),$$

i.e., the interval $[z_{2\nu}+1, z_{2\nu+2}+1]$ corresponds to $[\nu, \nu+1]$ and t^ν_κ corresponds to $\nu + \frac{\kappa}{2k+2}$.
4. Fix a subdivision

$$0 = t_0 < t_1 < \dots < t_{2k+2} = b$$

as in Section 4, i.e.

$$c(t) \in K_{k\kappa} \quad \text{for} \quad t_\kappa \leq t \leq t_{\kappa+1}, \quad \kappa = 0, \dots, k,$$

$$c(t) \in -K_{k\kappa} \quad \text{for} \quad t_{k+1+\kappa} \leq t \leq t_{k+1+\kappa+1}, \quad \kappa = 0, \dots, k.$$

Reparameterize c so that one obtains a simple closed piecewise smooth curve $c_r : [0,1] \longrightarrow C$ with

$$c_r\left(\left[\frac{\kappa}{2k+2}, \frac{\kappa+1}{2k+2}\right]\right) = c([t_\kappa, t_{\kappa+1}]) \quad \text{for} \quad \kappa = 0, \dots, 2k+2.$$

Extend c_r by periodicity to a curve $n \cdot c_r : [0,n] \longrightarrow C$. Note that $|c| \subset L$ gives $p \circ n \cdot c_r = n \cdot c_r$.
5. Define a homotopy of closed curves as follows. For

$$0 \leq \nu \leq n-1, \quad 0 \leq \kappa \leq 2k+1, \quad \nu + \frac{\kappa}{2k+2} \leq t \leq \nu + \frac{\kappa+1}{2k+2}, \quad s \in [0,1],$$

set

$$\hom(s,t) := s \cdot n \cdot c_r(t) + (1-s) \cdot \eta_r(t).$$

Convexity of the subcones $K_{k\kappa}, -K_{k\kappa}$ of S now yields that

$$\hom([0,1] \times [0,n]) \subset S.$$

In particular, $0 \notin p \circ \hom(\dots)$, by Lemma 4.1. It follows that

$$\mathrm{wind}(0, p \circ \eta) = \mathrm{wind}(0, p \circ \eta_r) = \mathrm{wind}(0, p \circ \hom(0,\cdot)) = \mathrm{wind}(0, p \circ \hom(1,\cdot))$$

$$= \mathrm{wind}(0, p \circ n \cdot c_r) = \mathrm{wind}(0, n \cdot c_r) = n \cdot \mathrm{wind}(0, c_r) = n \cdot \mathrm{wind}(0, c) = n;$$

hence $0 \in \mathrm{int}(p \circ \eta)$.
6. Proposition 6.1 and Lemma 4.1 yield

$$0 \neq p(y_s - y_t) = py_s - py_t \quad \text{for} \quad 0 \leq s < t < \tau;$$

p is injective on $|\eta|$. It follows that $p \circ \eta$ is a simple closed curve; therefore $n = 1$, and $\tau = z_2 + 1$. □

PROOF OF COROLLARY 5.1. Theorem 5.1 implies that the minimal period of y^i is $z_2(y^i)+1$. Set $\eta_i := \eta(y^i)$. It follows that in case $y_0^1 \neq y_0^2$ the orbits $|\eta_i|$ are disjoint. Proposition 6.1 and Lemma 4.1 yield

$$|p \circ \eta_1| \cap |p \circ \eta_2| = \emptyset. \tag{6.1}$$

Connect $0 \in \operatorname{int}(p \circ \eta_2) \cap \operatorname{int}(p \circ \eta_1)$ by a straight line to a point $\chi \in |p \circ \eta_1|$ with minimal norm. Then $[0,1)\chi \subset \operatorname{int}(p \circ \eta_1)$. In case $[0,1)\chi \cap |p \circ \eta_2| = \emptyset$, we conclude that $\chi \in \operatorname{int}(p \circ \eta_2)$, and using (6.1), that $|p \circ \eta_1| \subset \operatorname{int}(p \circ \eta_2)$. Otherwise,

$$\emptyset \neq [0,1)\chi \cap |p \circ \eta_2| \subset [0,1)\chi \subset \operatorname{int}(p \circ \eta_1),$$

and (6.1) implies $|p \circ \eta_2| \subset \operatorname{int}(p \circ \eta_1)$. □

PROOF OF THEOREM 5.2. Proposition 6.1 yields

$$(y')_0 = \lim_{t \searrow 0} \frac{1}{t}(y_t - y_0) \in \overline{S} = S \cup \{0\}.$$

Since $y'(-1) > 0$, $(y')_0 \in S$. Proposition 3.1 and periodicity imply that y' is slowly oscillating. As in the proof of Proposition 3.2,

$$y'(t) < 0 \quad \text{on} \quad [0, z_1].$$

There exists a zero of y' in $[-1, z_1]$. As y' is slowly oscillating, there is exactly one zero t_0 of y' in $[-1, 0)$, and $-1 < t_0$. Analogously one sees that there is precisely one zero t_1 of y' in $[z_1, z_2]$, and $z_1 < t_1 < z_1 + 1$. Now it is obvious how to complete the proof. □

7. Proof of Theorem 5.3

We have to generalize results from [5, 37] where eq.(1.1) was studied with $\mu = 0$. Note first that for $\mu > 0$ and for every continuous function $b : \mathbb{R} \longrightarrow \mathbb{R}$, a function $x : \mathbb{R} \longrightarrow \mathbb{R}$ is a solution of the equation

$$x'(t) = -\mu x(t) + b(t) x(t-1) \tag{7.1}$$

if and only if

$$d : t \longrightarrow e^{\mu t} x(t)$$

is a solution of the equation

$$d'(t) = e^{\mu} b(t) d(t-1). \tag{7.2}$$

A differentiable function $x : \mathbb{R} \longrightarrow \mathbb{R}$ is called slowly oscillating *at* t (with respect to the delay 1), if either x has no zero in $[t-1, t]$, or if x has precisely one zero in $[t-1, t]$ and this zero is simple.

COROLLARY 7.1. *Consider solutions x of eq. (7.1) and d of eq. (7.2) as above; x is slowly oscillating at t if and only if d is slowly oscillating at t.*

PROOF. The zeros of x and d coincide. $x(z) = 0 \neq x'(z)$ implies $d(z) = 0$ and $d'(z) = \mu d(z) + e^{\mu z} x'(z) \neq 0$; $d(z) = 0 \neq d'(z)$ implies $x(z) = 0 \neq x'(z)$. □

Assume

$$b(t) < 0 \quad \text{for all} \quad t \in \mathbb{R}$$

from now on. Using **[5, Lemma 1]** and Corollary 7.1 we infer:

LEMMA 7.1. *A solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (7.1) which is slowly oscillating at $t \in \mathbb{R}$, is slowly oscillating at every $s \geq t$.*

Note, also, that every slowly oscillating solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (7.1) is necessarily slowly oscillating at every $t \in \mathbb{R}$; i.e., has only simple zeros. (Proof: The function $d : t \longrightarrow e^{\mu t} x(t)$ is a slowly oscillating solution of eq. (7.2). $d(t) = 0$ and $d'(t) = 0$ would imply $d(t-1) = 0$, a contradiction. So, d has only simple zeros. Apply Corollary 7.1.)

It is easy to see that the set

$$\Sigma \quad \text{of slowly oscillating solutions} \quad x : \mathbb{R} \longrightarrow \mathbb{R} \quad \text{of eq. (7.1)}$$

equals

$$e^{-\mu \cdot} \Sigma_2$$

where Σ_2 denotes the set of slowly oscillating solutions $d : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (7.2).

Let $C(\mathbb{R})$ denote the real vectorspace of continuous functions $x : \mathbb{R} \longrightarrow \mathbb{R}$, equipped with the topology of uniform convergence on compact sets.

LEMMA 7.2. 1. $\overline{\Sigma} \subset \Sigma \cup \{0\}$
2. *For every linear space $Z \subset \Sigma \cup \{0\}$,* $\dim Z \leq 2$.

PROOF. 1. Apply **[5, Lemma 2]** to Σ_2, multiply by the function $\mathbb{R} \ni t \longrightarrow e^{-\mu t} \in \mathbb{R}$, use $\Sigma = e^{-\mu \cdot} \Sigma_2$.
2. $Z \subset \Sigma \cup \{0\} = e^{-\mu \cdot}(\Sigma_2 \cup \{0\})$ implies $e^{\mu \cdot} Z \subset \Sigma_2 \cup \{0\}$. By **[5, Lemma 3]**, $\dim e^{\mu \cdot} Z \leq 2$. Hence $\dim Z \leq 2$. □

The results of **[5, Section 3]** hold for eq. (7.1) without change; the proofs of **[5, Lemmas 4, 5, 6, 7]** and of **[5, Corollary 1]** remain valid.

We consider the linear variational equation (3.3). Recall that with every Floquet multiplier $\lambda \in \operatorname{spec}(V) \setminus \{0\}$, $\operatorname{Im}(\lambda) \geq 0$, is associated a linear space $\mathcal{G}(\lambda) \subset C(\mathbb{R})$ of solutions of eq. (3.3) which are real parts of complex-valued solutions whose segments at $t = 0$ belong to the eigenspace of the spectral set $\{\lambda, \overline{\lambda}\}$. This is as in [5]. We have

$$\dim \mathcal{G}(\lambda) = \mathrm{m}(\lambda)$$

where $\mathrm{m}(\lambda)$ denotes the multiplicity of λ as an eigenvalue of V. The slowly oscillating periodic solution $y' : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (3.3) belongs to the space $\mathcal{G}(1)$.

It follows from **[5, Lemmas 5, 6, 7]** that every nonzero element of the space

$$\oplus_{\lambda \in \operatorname{spec}(V), \operatorname{Im}(\lambda) \geq 0, |\lambda| \geq 1} \mathcal{G}(\lambda)$$

is slowly oscillating. By **[5, Corollary 1]**,

$$\dim \oplus_{\lambda\in \mathrm{spec}(V),\mathrm{Im}(\lambda)\geq 0,|\lambda|\geq 1}\mathcal{G}(\lambda) \leq 2.$$

We infer that

$$\text{either}\quad \mathrm{m}(1)=1,\quad |\lambda|<1 \quad \text{for all} \quad \lambda\in \mathrm{spec}(V)\setminus\{1\},$$

$$\text{or}\quad \mathrm{m}(1)=2,\quad |\lambda|<1 \quad \text{for all} \quad \lambda\in \mathrm{spec}(V)\setminus\{1\},$$

$$\text{or}\quad \mathrm{m}(1)=1,\quad \text{and there exists} \quad u\in \mathrm{spec}(V) \quad \text{in} \quad (-\infty,-1]\cup(1,\infty),$$

$$\mathrm{m}(u)=1,\quad |\lambda|<1 \quad \text{for all} \quad \lambda\in \mathrm{spec}(V)\setminus\{1,u\}.$$

In order to prove Theorem 5.3, it remains to exclude in the last case that

$$u\leq -1.$$

This can be done as in **[37]**. We begin with the following lemma.

LEMMA 7.3. *Let $Z\subset C(\mathbb{R})$ be a linear space of solutions of eq. (7.1) such that every $x\in Z\setminus\{0\}$ is slowly oscillating. Suppose there exists $\underline{x}\in Z\setminus\{0\}$ such that its zeroset is neither bounded from below nor from above.*
1. *Then the same holds true for every $x\neq 0$ in Z, and $x^{-1}(0)$ is given by a strictly increasing sequence of simple zeros*

$$t_{x,n+1}>t_{x,n}+1,\quad n\in\mathbb{Z}.$$

2. *For x,v in $Z\setminus\{0\}$ and $t_{x,n}<t_{v,n}<t_{x,n+1}$,*

$$t_{x,n+1}<t_{v,n+1}.$$

PROOF. 1. The zeros of $\underline{x}$ form a sequence of points $t_n, n\in\mathbb{Z}$, with $t_n+1<t_{n+1}$. Suppose the zeros of some $x\in Z$, $x\neq 0$, are bounded below by some real a. Then $x(t)\neq 0$ for $t<a$, hence

$$\mathrm{sign}(x'(t))=\mathrm{sign}(-\mu x(t)+b(t)x(t-1))=-\mathrm{sign}(x(t))$$

for all $t<a$. Observe that x is not a real multiple of $\underline{x}$. Choose $t_{n+1}<a$ so that

$$\mathrm{sign}(\underline{x})=\mathrm{sign}(x)\quad \text{on} \quad (t_n,t_{n+1}).$$

There exist $c>0$, $t\in(t_n,t_{n+1})$ such that

$$|c\underline{x}|\leq|x| \quad \text{on} \quad (t_n,t_{n+1}) \quad \text{and} \quad c\underline{x}(t)=x(t).$$

Consequently, $c\underline{x}'(t)=x'(t)$; t is a double zero of $c\underline{x}-x\in Z$, so $c\underline{x}-x=0$, a contradiction. If one assumes an upper bound a for the zeros of x then

$$\mathrm{sign}(x'(t))=-\mathrm{sign}(x(t))\quad \text{for} \quad t>a+1,$$

and one can argue as above.
2. Assume

$$t_{x,n}<t_{v,n}<t_{x,n+1} \quad \text{and} \quad t_{v,n+1}\leq t_{x,n+1}.$$

In case $t_{v,n+1}=t_{x,n+1}$, define

$$c:=\frac{v'(t_{x,n+1})}{x'(t_{x,n+1})}.$$

Then $cx-v\in Z$ has a double zero at $t_{x,n+1}$, hence $cx-v=0$, a contradiction to $v(t_{v,n})=0\neq x(t_{v,n})$. In case $t_{v,n+1}<t_{x,n+1}$ there exist $c\in\mathbb{R}$ and $t\in(t_{v,n},t_{x,n+1})$ such that

$$|cv|\leq|x| \quad \text{and} \quad \mathrm{sign}(cv)=\mathrm{sign}(x) \quad \text{in} \quad (t_{v,n},t_{x,n+1}),\quad cv(t)=x(t).$$

It follows that $cv'(t)=x'(t)$; t is a double zero of $cv-x\in Z$, hence $cv-x=0$, a contradiction as before. □

REMARK 7.1. Assertion 2 above expresses a synchronization among solutions in Z.

Now assume that there exists $u\leq -1$ in $\mathrm{spec}(V)$. Set

$$b(t):=f'(y(t-1)),\quad \text{for} \quad t\in\mathbb{R}.$$

The space $Z:=\mathcal{G}(1)\oplus\mathcal{G}(u)$ satisfies the hypotheses in Lemma 7.3. Choose $x\in\mathcal{G}(u)\setminus\{0\}$. $\mathrm{m}(u)=1$ implies

$$x_\tau=Vx_0=u\cdot x_0.$$

There are consecutive zeros

$$-1<t_0<0,\quad t_1,\quad \tau-1<t_2<\tau$$

of y', and

$$\tau=t_2-t_0,$$

by Theorem 5.2 and its proof. It follows that

$$\mathrm{sign}(x(t))=\mathrm{sign}(u\cdot x(t-\tau))=-\mathrm{sign}(x(t-\tau))\quad \text{for} \quad \tau-1\leq t\leq\tau;$$

hence

$$\mathrm{sign}(x(t_2))=-\mathrm{sign}(x(t_0)) \tag{7.3}$$

and

$$\mathrm{sign}(x'(t_2))=-\mathrm{sign}(x'(t_0)). \tag{7.4}$$

We may assume that in the sequence of zeros $t_{x,n}$ of x, given by Lemma 7.1,

$$t_{x,0}\leq t_0<t_{x,1}.$$

The case $t_{x,0} < t_0$. Repeated applications of Lemma 7.3 yield

$$t_{x,1} < t_1 < t_{x,2} < t_2 < t_{x,3}.$$

As zeros are simple, we arrive at a contradiction to (7.3).

The case $t_{x,0} = t_0$. By (7.3), $x(t_2) = 0$. Hence $t_{x,1} \leq t_2$. If $(t_0 <) \quad t_{x,1} < t_1$, or if $t_1 < t_{x,1} < t_2$, then repeated applications of Lemma 7.3 show that t_2 is not a zero of x. This contradicts (7.3). Similarly,

$$t_2 = t_{x,1} \quad (> t_1 > t_0 = t_{x,0})$$

is not possible, due to Lemma 7.1. We have shown

$$t_{x,1} = t_1.$$

Arguments as before yield

$$t_{x,2} = t_2.$$

The simplicity of the zeros now gives

$$\text{sign}(x'(t_0)) = \text{sign}(x'(t_{x,0})) = \text{sign}(x'(t_{x,2})) = \text{sign}(x'(t_2)),$$

a contradiction to (7.4). □

It is convenient to state the following results at this point.

PROPOSITION 7.1. *If y is unstable and hyperbolic, and if u is the Floquet multiplier in $(1,\infty)$, then the set $\mathcal{G}(1) \oplus \mathcal{G}(u) \setminus \{0\}$ consists of slowly oscillating solutions $\mathbb{R} \longrightarrow \mathbb{R}$ of eq. (3.3). We have $\mathcal{G}(1) = \mathbb{R} \cdot y'$, $\quad \mathcal{G}(u) = \mathbb{R} \cdot v$ for some slowly oscillating solution v of eq. (3.3) with $||v_0|| = 1$, $\quad V(y')_0 = (y')_0$ and $Vv_0 = u \cdot v_0$.*

The proof should be obvious from preceding arguments.

8. Graph Representation of W

From now on we assume that the hypothesis of Theorem 5.4 is satisfied: y is a slowly oscillating periodic solution of eq. (1.1) with $y_0 \in K$ and $z_0 = -1$; y is unstable and hyperbolic. Then 1 is a simple Floquet multiplier. According to Theorem 5.3, there is a single Floquet multiplier u outside the unit circle; we have

$$u \in (1,\infty), \quad \text{m}(u) = 1.$$

It is convenient to begin with the return map

$$P_H : N \cap H \ni \phi \longrightarrow P(\phi) \in H.$$

Also, y_0 is a hyperbolic fixed point of P_H. u is a simple eigenvalue of the compact map $DP_H(y_0)$. All other eigenvalues satisfy $|\lambda| < 1$.

This is essentially as in case of a periodic orbit of a vectorfield on a finite-dimensional space [16]. For a proof in our situation, see the forthcoming book [6], or compare [15].

Now, **[14, Theorem 3.1]** (or **[25, Theorem 2.7]**) yields a one-dimensional local unstable manifold W_H of P_H at y_0:

Let G_H and R_H denote the eigenspaces given by the spectral sets $\{u\}$ and $\text{spec}(DP_H(y_0)) \setminus \{u\}$, respectively;

$$H = G_H \oplus R_H; \quad \dim(G_H) = 1.$$

Fix

$$u_H \in (1, u).$$

There exist convex open neighborhoods N_G of 0 in G_H, $\quad N_R$ of 0 in R_H,

$$y_0 + N_G + N_R \subset N \cap H,$$

and a C^1–map

$$w_H : N_G \longrightarrow N_R, \quad w_H(0) = 0 \quad \text{and} \quad Dw_H(0) = 0,$$

such that the shifted graph

$$y_0 + \{\phi + w_H(\phi) : \phi \in N_G\} =: W_H$$

coincides with the set $W(y_0 + N_G + N_R)$ of $\phi \in y_0 + N_G + N_R$ such that there is a trajectory $(\phi_j)_{-\infty}^0$ of P_H with $\phi_0 = \phi$, $\quad \phi_j \in y_0 + N_G + N_R$ for all $j \in -\mathbb{N}_0$; there exists a constant $c_H \geq 0$ such that every trajectory $(\phi_j)_{-\infty}^0$ of P in $y_0 + N_G + N_R$ satisfies

$$||\phi_j - y_0|| \leq c_H \cdot ||\phi_0 - y_0|| \cdot u_H^j \quad \text{for all} \quad j \in -\mathbb{N}_0.$$

REMARK 8.1. $W_H \subset W$.

PROOF. Let $\phi \in W_H$. There is a trajectory $(\phi_j)_{-\infty}^0$ of P with $\phi = \phi_0$ and $\phi_j \longrightarrow y_0$ as $j \longrightarrow -\infty$. This defines a solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) whose zeros are given by a sequence $(z_j)_{-\infty}^J, J \in \mathbb{N} \cup \{\infty\}$, $J \geq 2$, $\quad z_0 = -1$, $\quad$ with property (3.2) such that

$$\phi_j = x_{z_{2j}+1} \quad \text{for} \quad j \leq 0.$$

Continuity of *stop* implies $z_2(\phi_j) + 1 \longrightarrow \tau$ as $j \longrightarrow -\infty$. Now it is easy to conclude that

$$x_t \longrightarrow |\eta| \quad \text{as} \quad t \longrightarrow -\infty,$$

or, $\phi = \phi_0 = x_0 \in W$. □

REMARK 8.2. $P(W_H) \subset W \cap H$.

PROOF. Use $P(W_H) \subset F(\mathbb{R}^+ \times W_H) \subset F(\mathbb{R}^+ \times W) \subset W$. □

We need exponential convergence and an asymptotic phase on the unstable set W. This can be achieved by standard arguments which use the return map P and its local unstable manifold W_H. However, a citeable treatment in the case of semiflows does not seem to be available in the literature, so we include a detailed proof.

Define $\rho > 0$ by

$$e^{\rho\tau} = u_H.$$

PROPOSITION 8.1. *For every solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ of eq.* (1.1) *with phase curve in W, there are $t = t(x) \leq 0$, $\gamma = \gamma(x) \in [0, \tau)$ and a constant $c = c(x) > 0$ such that for all $s \leq t$,*

$$\|x_s - y_{s+\gamma}\| \leq c \cdot e^{\rho s}.$$

1. . Choose a neighborhood N_0 of y_0 and a constant $c > 0$ such that

$$\|Dstop(\phi)\| \leq c \quad \text{on} \quad N_0, \quad \|D_2F(s,\phi)\| \leq c \quad \text{for} \quad 0 \leq s \leq \tau, \quad \phi \in N_0$$

(for the last property, use compactness of $[0, \tau]$ and continuity of D_2F on $[0, \infty) \times C$.)
2. Let a solution $\hat{x} : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) with phase curve in W be given. There exists $\hat{t} \geq 0$ so that for $x := \hat{x}(\cdot - \hat{t})$ we have

$$x_t \in N_0 \quad \text{for all} \quad t \leq 0, \quad J(x) \in \mathbb{N} \quad \text{or} \quad J(x) = \infty,$$

and for the zeros z_j of x

$$z_0 = -1 \quad \text{and} \quad x_{z_{2j}+1} \in K \cap (y_0 + N_G + N_R) \quad \text{for} \quad j \leq 0.$$

Set

$$\phi_j := x_{z_{2j}+1} \quad \text{for} \quad j \leq 0,$$

$$\Delta_j := z_{2j+2} - z_{2j} \quad (= stop(\phi_j)) \quad \text{for} \quad j \leq -1,$$

so that

$$\phi_{j+1} = P(\phi_j) = F(\Delta_j, \phi_j) \quad \text{for} \quad j \leq -1.$$

The estimate

$$|\tau - \Delta_j| = |stop(y_0) - stop(\phi_j)| \leq c \cdot \|y_0 - \phi_j\|$$

$$\leq c \cdot c_H \cdot \|y_0 - \phi_j\| \cdot {u_H}^j \quad \text{for} \quad j \leq 0$$

implies that

$$\gamma := -\sum_1^\infty (\tau - \Delta_{-j})$$

is well–defined. For some $l \in \mathbb{Z}$, $l\tau \leq \gamma < (l+1)\tau$.
3. For $k \in \mathbb{N}$, set $t_k := \gamma + \sum_1^k (\tau - \Delta_{-j})$. Then

$$\|x_{t_k} - k\tau - l\tau - x_{-k\tau} - l\tau\| \leq c_f \cdot |t_k| \qquad \text{(Proposition 3.2)}$$

$$\leq c_f \cdot c \cdot c_H \cdot \|y_0 - \phi_0\| \cdot u_H^{-k} \cdot \frac{1}{u_H - 1}.$$

4. For all $k \in \mathbb{N}$ we obtain

$$\|x_{-k\tau} - l\tau - y_\gamma - l\tau\| \leq$$

$$\|x_{t_k} - k\tau - l\tau - x_{-k\tau} - l\tau\| + \|x_{t_k} - k\tau - l\tau - y_\gamma - l\tau\|$$

$$\leq c_f \cdot c \cdot c_H \cdot \|y_0 - \phi_0\| \cdot \frac{1}{u_H - 1} \cdot u_H^{-k} \quad + \quad \|x_{t_k} - k\tau - l\tau - y_\gamma - l\tau\|$$

$$= \quad \ldots \quad + \|F(\gamma - l\tau, \phi_{-k}) - F(\gamma - l\tau, y_0)\|$$

$$(\text{with} \quad t_k - k\tau = \gamma + \sum_1^k (-\Delta_{-j}) \quad)$$

$$\leq \quad \cdots + c \cdot \|\phi_{-k} - y_0\| \leq \hat{c} \cdot \|\phi_0 - y_0\| \cdot e^{-\gamma(k+l)\tau},$$

$$\text{where} \quad \hat{c} := \left(\frac{c_f}{u_H - 1} + 1\right) \cdot c_H \cdot c \cdot e^{\rho l \tau},$$

or

$$\|x_{-k\tau} - y_{\gamma - l\tau}\| \leq \hat{c} \cdot \|\phi_0 - y_0\| \cdot e^{-\rho k \tau} \quad \text{for integers} \quad k \geq |l| + 1.$$

Let $t \leq -(|l| + 1)\tau$ be given. For some integer $k \geq |l| + 2$,

$$-k\tau < t \leq -k\tau + \tau.$$

Hence,

$$\|x_t - y_{t+(\gamma - l\tau)}\| = \|F(t + k\tau, x_{-k\tau}) - F(t + k\tau, y_{\gamma - l\tau - k\tau})\|$$

$$= \| \quad \ldots \quad - F(t + k\tau, y_{\gamma - l\tau})\| \quad \text{(periodicity)}$$

$$\leq c \cdot \|x_{-k\tau} - y_{\gamma - l\tau}\| \leq c \cdot \hat{c} \cdot \|\phi_0 - y_0\| \cdot e^{-\rho k \tau}$$

$$= c \cdot \hat{c} \cdot \|\phi_0 - y_0\| \cdot e^{-\rho t - \rho k \tau} e^{\rho t} \leq c \cdot \hat{c} \cdot \|\phi_0 - y_0\| \cdot e^{\rho t}.$$

Finally, for $t \leq -\hat{t} - (|l| + 1)\tau$,

$$\|\hat{x}_t - y_{t+\hat{t}+\gamma - l\tau}\| = \|x_{t+\hat{t}} - y_{t+\hat{t}+\gamma - l\tau}\|$$

$$\leq c \cdot \hat{c} \cdot \|\phi_0 - y_0\| \cdot e^{\rho(t+\hat{t})} = c \cdot \hat{c} \cdot \|\phi_0 - y_0\| \cdot e^{\rho \hat{t}} e^{\rho t}.$$

Define the asymptotic phase of $\hat{x}$ by

$$\hat{\gamma} := \hat{t} + \gamma - l\tau - j\tau \in [0, \tau)$$

where $j \in \mathbb{Z}$. $\square$

We turn to the period map $F(\tau, \cdot) : C \longrightarrow C$. As before, there exists a local unstable manifold at y_0. More precisely, there is a smooth map

$$w_\tau : N_\tau \longrightarrow R'_\tau,$$

where N_τ is an open convex neighborhood of 0 in the one–dimensional (reellified) linear unstable eigenspace

$$G_\tau \subset C \quad \text{of} \quad V = D_2F(\tau, y_0),$$

and R'_τ is an open convex neighborhood of 0 in the (reellified) eigenspace R_τ associated with the complementary spectral set

$$\operatorname{spec}(V) \setminus \{u\},$$

such that we have

$$w_\tau(0) = 0, \quad Dw_\tau(0) = 0;$$

and the shifted graph

$$W_\tau := y_0 + \{\chi + w_\tau(\chi) : \chi \in N_\tau\}$$

coincides with the set of all $\phi \in y_0 + N_\tau + R'_\tau$ such that there is a trajectory $(\phi_j)_{-\infty}^0$ of $F(\tau, \cdot)$ with $\phi_0 = \phi$ and

$$(\phi_j - y_0) \cdot u_H^{-j} \in N_\tau + R'_\tau \quad \text{for all} \quad j \in -\mathbb{N}_0.$$

The map $\Phi : W_\tau \longrightarrow W_\tau, \quad \phi \longrightarrow \phi_{-1}$, is a C^1–diffeomorphism onto an open neighborhood W'_τ of y_0 in W_τ. For a proof, we refer to [**14, Theorem 3.1**] on local unstable manifolds of hyperbolic fixed points. Of course, the present situation is non–hyperbolic, but as indicated in [**36, Section 5**], one can easily modify the proof from [**14**] in order to derive the result described above. A very detailed version of the proof of [**14, Theorem 3.1**] may be found in [**25**].

Proposition 7.1 implies that there is a slowly oscillating solution $v : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (3.3), $\|v_0\| = 1$, so that

$$T_{y_0} W_\tau = \quad (G_\tau =) \quad \mathbb{R} \cdot v_0.$$

Now we are in a position to prove a first part of Theorem 5.4, namely that W is given by a map w with domain $pW \subset L$ and range in Q. The assertion is equivalent to the injectivity of the projection p on W. The latter follows from $0 \notin pS$ (Lemma 4.1), provided we can show that nontrivial differences

$$0 \neq \phi - \overline{\phi} \tag{8.1}$$

of points $\phi, \overline{\phi}$ in W belong to S.

So let $\phi \neq \overline{\phi}$ in W be given. There are solutions $x : \mathbb{R} \longrightarrow \mathbb{R}$, $\overline{x} : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) with $x_0 = \phi$, $\overline{x}_0 = \overline{\phi}$, respectively. Let γ and $\overline{\gamma}$ denote the asymptotic phases of x and $\overline{x}$.

Case 1: $\gamma \neq \overline{\gamma}$. Then

$$x_{-j\tau} \longrightarrow y_\gamma, \quad \overline{x}_{-j\tau} \longrightarrow y_{\overline{\gamma}} \quad \text{as} \quad j \longrightarrow \infty.$$

Now, γ and $\overline{\gamma}$ belong to $[0, \tau)$, and Proposition 6.1 gives

$$y_\gamma - y_{\overline{\gamma}} \in S.$$

According to Proposition 3.1, there exists $s \in [0, 4]$ such that

$$y_{s+\gamma} - y_{s+\overline{\gamma}} \quad \text{has no zero.}$$

The continuity of $F(s, \cdot)$ implies that for all $j \geq \frac{4}{\tau}$ sufficiently large,

$$x_{s-j\tau} - \overline{x}_{s-j\tau} \quad \text{has no zero.}$$

Again by Proposition 3.1,

$$\phi - \overline{\phi} = x_0 - \overline{x}_0 = F(j\tau - s, x_{s-j\tau}) - F(\ldots, \overline{x}_{s-j\tau}) \in S.$$

Case 2: $\gamma = \overline{\gamma}$. Now

$$\|x_{-j\tau+(\tau-\gamma)} - y_0\| \leq c(x) \cdot e^{-j\rho\tau} \quad \text{for all} \quad j \in \mathbb{N},$$

analogously for $\overline{x}$, with positive constants $c(x)$ and $c(\overline{x})$. It follows that there exists $j_0 \in \mathbb{N}$ such that all

$$x_{-j\tau+(\tau-\gamma)}, \quad \overline{x}_{-j\tau+(\tau-\gamma)} \quad \text{with} \quad j \geq j_0$$

belong to the local unstable manifold W_τ; the sequence of normed distances

$$\frac{1}{\|x_{-j\tau+(\tau-\gamma)} - \overline{x}_{-j\tau+(\tau-\gamma)}\|}(x_{-j\tau+(\tau-\gamma)} - \overline{x}_{-j\tau+(\tau-\gamma)}) =: d_j$$

is well–defined (since $\phi \neq \overline{\phi}$) and converges to the unit sphere

$$\{v_0, -v_0\} \subset T_{y_0} W_\tau$$

as $j \longrightarrow \infty$. For some $s \in [0, 4]$, $v_s = V(s, 0)v_0$ has no zero (Proposition 3.1). There exist $\epsilon > 0$ and $\delta > 0$ such that for

$$\operatorname{dist}(\chi, \{v_0, -v_0\}) < \delta,$$
$$|V(s, 0)\chi(t)| > \epsilon \quad \text{on} \quad [-1, 0].$$

Further, there exists $\delta_1 > 0$ such that for $\|\psi\| \leq \delta_1$, $\|\overline{\psi}\| \leq \delta_1$,

$$\begin{aligned} &\|F(s, y_0 + \psi) - F(s, y_0 + \overline{\psi}) - V(s, 0)[\psi - \overline{\psi}]\| \\ &= \left\| \int_0^1 \{D_2F(s, y_0 + \overline{\psi} + \theta(\psi - \overline{\psi})) - D_2F(s, y_0)\}[\psi - \overline{\psi}] d\theta \right\| \\ &\leq \epsilon \cdot \|\psi - \overline{\psi}\|. \end{aligned}$$

Choose an integer $j \geq \frac{4+\tau}{\tau}$ so large that

$$\operatorname{dist}(d_j, \{v_0, -v_0\}) < \delta$$

and

$$\|x_{-j\tau+(\tau-\gamma)} - y_0\| \leq \delta_1, \quad \|\overline{x}_{-j\tau+(\tau-\gamma)} - y_0\| \leq \delta_1.$$

It follows that

$$\left\| \frac{1}{\|x_{-j\tau+(\tau-\gamma)} - \overline{x}_{-j\tau+(\tau-\gamma)}\|} \{x_{s-j\tau+\tau-\gamma} - \overline{x}_{s-j\tau+\tau-\gamma}\} - V(s, 0)d_j \right\| \leq \epsilon,$$

and

$$\frac{1}{\|\ldots\|}\{\ldots\} = d_j$$

has no zero. Consequently, $\{\ldots\} \in S$, and by Proposition 3.1 once again,

$$\phi - \overline{\phi} = x_0 - \overline{x}_0 \in S. \quad \square$$

Corollary 8.1. *For every $\phi \in W$,*

$$T_\phi W \in S \cup \{0\}.$$

Proof. We saw that for any two points $\psi \neq \phi$ in W, $\psi - \phi \in S$. Use $\overline{S} = S \cup \{0\}$, $\mathbb{R} \cdot \overline{S} \subset \overline{S}$. $\square$

9. Smoothness of W

In this section we show that the domain pW of the map w obtained in Section 8 is open, and that w is C^1.

Let p_τ denote the eigenprojection onto G_τ, associated with the spectral set $\{u\}$ of V. The fact that W_τ consists of initial values of backward trajectories of $F(\tau, \cdot)$ which tend to y_0 implies

$$W_\tau \subset W \tag{9.1}$$

PROPOSITION 9.1. *For every neighborhood N_0 of y_0 in C and for every $t_0 \geq 0$,*

$$W = F((t_0, \infty) \times (W_\tau \cap N_0)).$$

PROOF. 1. Positive invariance of W and (9.1) yield $W \supset F(\ldots)$.
2. Let a neighborhood N_0 of y_0 in C be given. Let $\phi \in W$. There exist $\gamma \in [0, \tau)$, $t \leq 0$ and $c > 0$ such that the solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) with $x_0 = \phi$ satisfies

$$\|x_s - y_{s+\gamma}\| \leq c \cdot e^{\rho s} \quad \text{for} \quad s \leq t.$$

In particular,

$$\|x_{-k\tau - j\tau - \gamma} - y_0\| \leq (c \cdot e^{-\rho\gamma} \cdot u_H^{-j}) \cdot u_H^{-k}$$

for all integers $j \geq -\frac{t+\gamma}{\tau}$, and for all $k \in \mathbb{N}_0$. For j sufficiently large we obtain

$$x_{-j\tau-\gamma} \in W_\tau, \quad x_{-j\tau-\gamma} \in N_0, \quad j\tau + \gamma \geq t_0.$$

Hence,

$$\phi = x_0 = F(j\tau + \gamma, x_{-j\tau-\gamma}) \in F((t_0, \infty) \times (W_\tau \cap N_0)). \quad \square$$

COROLLARY 9.1. *W is pathwise connected.*

PROOF. For $\phi \in W = F(\mathbb{R}^+ \times W_\tau)$, $\phi = F(t, \psi)$ with $t \geq 0$, $\psi \in W_\tau$. For $0 \leq s \leq t$, $F(s, \psi) \in W$. The C^1–graph W_τ over the open convex set $N_\tau \subset G_\tau$ connects ψ to y_0 in W. $\square$

PROPOSITION 9.2. *There exist an open neighborhood N_1 of y_0 in C and $\epsilon_1 \in (0, \tau - 1)$ with*

$$F(t, \phi) \notin W_\tau \quad \textit{for} \quad \phi \in N_1 \cap W_\tau, \quad 0 < t < 2\epsilon_1.$$

PROOF. 1. Set $\psi := (y')_0$. Choose $\epsilon \in (0, 1)$, $\epsilon < \|\psi\|$, so small that

$$\psi \notin \left[-\frac{\|\psi\| + \epsilon}{1 - \epsilon}, \frac{\|\psi\| + \epsilon}{1 - \epsilon}\right] \cdot (v_0 + U) \quad + \quad U$$

where $U := \{\phi \in C : \|\phi\| \leq \epsilon\}$. This is possible since ψ and v_0 are linearly independent. We have

$$(0, \infty) \cdot (\psi + U) \cap \mathbb{R} \cdot (v_0 + U) = \emptyset$$

since otherwise,

$$\psi = r \cdot (v_0 + \phi) + \chi \quad \text{with} \quad r \in \mathbb{R}, \phi \in U, \chi \in U;$$

consequently

$$|r|(1 - \epsilon) \leq \|\psi\| + \epsilon$$

which implies a contradiction to the choice of ϵ.
2. Choose an open convex neighborhood N'_τ of 0 in $N_\tau \subset G_\tau$ with

$$Dw_\tau(\chi)v_0 \in U \quad \text{for} \quad \chi \in N'_\tau.$$

For χ, χ' in N'_τ, the integral in

$$\chi_1 + w_\tau(\chi_1) = \chi + w_\tau(\chi) + \int_0^1 [(\chi_1 - \chi) + Dw_\tau(\chi + t(\chi_1 - \chi))(\chi_1 - \chi)]dt$$

has values in $\mathbb{R} \cdot (v_0 + U)$ (use $\chi_1 - \chi = rv_0$ for some $r \in \mathbb{R}$, and convexity). We get

$$y_0 + \chi_1 + w_\tau(\chi_1) \in y_0 + \chi + w_\tau(\chi) + \mathbb{R} \cdot (v_0 + U)$$

for all χ and χ_1 in N'_τ.
3. Recall

$$D_1F(\tau, \Phi(y_0))1 = (y')_0 = \psi.$$

By continuity, there exist $\epsilon_1 \in (0, \tau - 1)$ and a neighborhood N_1 of y_0 in C with the following properties: For all $\phi \in N_1 \cap W_\tau$ and all $t \in [0, 2\epsilon_1]$,

$$D_1F(t + \tau, \Phi(\phi))1 \in \psi + U$$

and

$$p_\tau(F(t, \phi) - y_0) \in N'_\tau.$$

Observe that for phase curves in W with segment $\phi \in W_\tau$ at time $t = 0$, the tangent vectors at times $t \geq 0$ equal

$$D_1F(t + \tau, \Phi(\phi))1.$$

Now assume that for some $\phi \in N_1 \cap W_\tau$ and some $t \in (0, 2\epsilon_1]$,

$$F(t, \phi) \in W_\tau.$$

There are χ, χ_1 in N'_τ with

$$\phi = y_0 + \chi + w_\tau(\chi), \quad F(t, \phi) = y_0 + \chi_1 + w_\tau(\chi_1).$$

Hence,

$$\mathbb{R} \cdot (v_0 + U) \ni F(t, \phi) - \phi$$
$$= t \cdot \int_0^1 D_1F(st + \tau, \Phi(\phi))1ds \in (0, \infty) \cdot (\psi + U),$$

a contradiction to the result of part 1 above. $\square$

Corollary 9.2. *The map*

$$A_1 : (-\epsilon_1, \epsilon_1) \times (W_\tau \cap N_1) \ni (t, \phi) \longrightarrow F(t+\tau, \phi) \in C$$

is injective.

Proof. Assume $F(t+\tau,\phi) = F(s+\tau,\psi)$, $-\epsilon_1 < s \leq t < \epsilon_1$, ϕ and ψ in $W_\tau \cap N_1$. Injectivity of all $F(t', \cdot), t' \geq 0$, yields $\psi = F(t-s, \phi)$. By Proposition 9.1, $t - s = 0$; hence $\psi = \phi$. $\square$

Proposition 9.3. *There exist $\epsilon_2 \in (0, \epsilon_1)$ and an open neighborhood $N_2 \subset N_1$ of y_0 in C, $W_\tau \cap N_2 \subset \Phi(W_\tau)$, such that the C^1–map $p \circ A_2$,*

$$A_2 := A_1 | (-\epsilon_2, \epsilon_2) \times (W_\tau \cap N_2),$$

has injective derivatives.

Proof. 1. At a point $(t, \phi) \in (-\epsilon_1, \epsilon_1) \times (W_\tau \cap N_1)$, a basis of the tangent space to the C^1–manifold $(-\epsilon_1, \epsilon_1) \times (W_\tau \cap N_1)$ is given by the vectors $(1, 0)$ and $(0, v_0 + Dw_\tau(p_\tau(\phi - y_0))v_0)$ in $\mathbb{R} \times C$, and $D(p \circ A_1)(t, \phi)T_{(t,\phi)}(\ldots)$ consists of the linear combinations of the vectors

$$pD_1F(t+\tau, \phi)1 \in C, \quad pD_2F(t+\tau,\phi)[v_0 + Dw_\tau(p_\tau(\phi - y_0))v_0] \in C.$$

2. At $t = 0, \phi = y_0$, these vectors are

$$p(y')_\tau = p(y')_0 \quad \text{and} \quad pVv_0 = u \cdot pv_0, \quad \text{respectively.}$$

We show that $p(y')_0$, pv_0 are linearly independent:
The fact that all nonzero elements of $\mathcal{G}(1) \oplus \mathcal{G}(u) = \mathbb{R} \cdot y' \oplus \mathbb{R} \cdot v$ are slowly oscillating (Proposition 7.1) implies

$$\mathbb{R} \cdot (y')_0 \oplus \mathbb{R} \cdot v_0 \subset S \cup \{0\}.$$

By $0 \notin pS$,

$$0 \neq a_1 \cdot p(y')_0 + a_2 \cdot pv_0 = p(a_1 \cdot (y')_0 + a_2 \cdot v_0)$$

whenever $(a_1, a_2) \neq (0, 0)$.
3. Continuity permits us to find $\epsilon_2 \in (0, \epsilon_1)$ and an open neighborhood $N_2 \subset N_1$ of y_0 in C with $W_\tau \cap N_2 \subset \Phi(W_\tau)$ so that for $|t| < \epsilon_2$ and $\phi \in N_2$,

$$pD_1F(t+\tau,\phi)1 \quad \text{and} \quad pD_2F(t+\tau,\phi)[v_0 + Dw_\tau(p_\tau(\phi - y_0))v_0]$$

are linearly independent. $\square$

Define

$$W_A := A_2((-\epsilon_2, \epsilon_2) \times (W_\tau \cap N_2)) \subset W.$$

Corollary 9.3. *pW_A is an open subset of L, and the map $w|pW_A$ is C^1; W_A is a C^1–submanifold of C.*

Proof. The map $id_{\mathbb{R}} \times \Phi$ defines a diffeomorphism A_0 of the C^1–manifold $(-\epsilon_2, \epsilon_2) \times \Phi^{-1}(W_\tau \cap N_2)$ onto the domain of A_2 (use $W_\tau \cap N_2 \subset \Phi(W_\tau)$). The map

$$B : (-\epsilon_2, \epsilon_2) \times \Phi^{-1}(W_\tau \cap N_2) \longrightarrow L$$

given by $id_{\mathbb{R}} \times \Phi$, A_2, p is a C^1– diffeomorphism onto the open subset pW_A of the 2–dimensional space L (Injectivity follows from Corollary 9.2 and from the injectivity of $p|W$; the injectivity of derivatives follows from Proposition 9.3.). For $\chi = p\phi$ and $\phi \in W_A$,

$$w(\chi) = (id - p)\phi \quad \text{and} \quad \phi = A_2((id_{\mathbb{R}} \times \Phi)(B^{-1}(p\phi)). \quad \square$$

We have

$$W = F(\mathbb{R}^+ \times W_A)$$

since $F(\mathbb{R}^+ \times W_A) \subset F(\mathbb{R}^+ \times W) \subset W$,

$$W = F((\tau, \infty) \times (W_\tau \cap N_2)) \quad \text{(Proposition 9.1)},$$

and for each $t \geq \tau$ and $\phi \in W_\tau \cap N_2$,

$$F(t, \phi) = F(t - \tau, F(\tau, \phi)) = F(t - \tau, A_2(0, \phi)).$$

In order to show that $pW \subset L$ is open, and that w is C^1, it is therefore enough to prove that for every $t \geq 0$,

$$pF(t, \cdot)(W_A) \subset L \quad \text{is open, and} \quad w|pF(t, \cdot)(W_A) \quad \text{is} \quad C^1.$$

So let $t \geq 0$ be given. The C^1–map $p \circ F(t, \cdot)|W_A$ is injective ($p|W$ is injective!); its derivatives are injective since

$$T_\phi W_A \subset T_\phi W \subset S \cup \{0\} \quad \text{(Corollary 8.1)},$$

$$D_2F(t, \phi)S \subset S \quad \text{(Proposition 3.1.2)},$$

$$D_2F(t, \phi) \quad \text{is injective, and} \quad 0 \notin pS.$$

It follows that $p \circ F(t, \cdot)$ defines a C^1–diffeomorphism B_t of W_A onto the open subset $B_t(W_A) = pF(t, \cdot)(W_A) \subset L$. For $\chi \in B_t(W_A)$ and $\chi = p\phi$, $\phi \in F(t, \cdot)(W_A) \subset W$, we obtain

$$w(\chi) = (id - p)\phi \quad \text{and} \quad \phi = F(t, \cdot) \circ B_t^{-1}(p\phi).$$

10. The Lipschitz Condition

We proceed as in [**36, Section 7**]. Lemma 4.1, i.e.

$$0 \notin pS, \tag{10.1}$$

and [**36, proof of Lemma 7.1**] yield the following result.

Lemma 10.1. *Let a subcone $S' \subset S$ be given. The following statements are equivalent.*
1. *There exists $c^* > 0$ with $c^*\|\phi\| \leq \|T(1)\phi\|$ for all $\phi \in S'$.*
2. *There exists $c > 0$ with $c \cdot \|\phi\| \leq \|p\phi\|$ for all $\phi \in S'$.*

In [**36**] we made the hypothesis (H2) that the zero solution of eq. (1.1) is linearly unstable. This is not required in the present paper.

Observe that all prerequisites for [**36, Proposition 7.1**] (with the exception of (10.1) above) and [**36, Proposition 7.1**] itself are derived without using (H2). Therefore we have our final result in this section.

Proposition 10.1. *Let $r > 0$ be given. There exists a constant $c(r) > 0$ with the following property:*
If $x : [t_0 - 1, \infty) \longrightarrow \mathbb{R}$ and $x^* : [t_0 - 1, \infty) \longrightarrow \mathbb{R}$ are solutions of eq. (1.1) with $|x(t)| \leq r$ and $|x^*(t)| \leq r$ for all $t \geq t_0 - 1$, so that

$$d := x - x^* \quad \textit{has no zero in} \quad [t_0 - 1, t_0]$$

then, for all $t \geq t_0 + 2$,

$$c(r)\|d_t\| \leq \|pd_t\|.$$

As $\overline{W}$ is compact there exists $r(W) > 0$ so that every solution of eq. (1.1) with phase curve in $\overline{W}$ is bounded by $r(W)$. Now it is easy to derive

$$\|w(\chi) - w(\chi^*\| \leq \left(1 + \frac{1}{c(r(W))}\right)\|\chi - \chi^*\| \quad \text{for all} \quad \chi \in pW, \chi^* \in pW.$$

For $\chi \neq \chi^*$ in pW, consider the solutions $x : \mathbb{R} \longrightarrow \mathbb{R}$ and $x^* : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) with $x_0 = \chi + w(\chi)$, $x_0^* = \chi^* + w(\chi^*)$. For some $t \leq -6$, $d := x - x^*$ satisfies $d_t \in S$ (see (8.1)). There exists $t_0 \leq -2$ so that

$$d_{t_0} \quad \text{has no zero (Proposition 3.1.2).}$$

Proposition 10.1 yields

$$c(r(W)) \cdot \|\chi + w(\chi) - (\chi^* + w(\chi^*)\| = c(r(W)) \cdot \|d_0\| \leq \|pd_0\| = \|\chi - \chi^*\|.$$

Set

$$l_w := 1 + \frac{1}{c(r(W))}.$$

Finally, we obtain a Lipschitz continuous extension

$$\overline{w} : \overline{pW} \longrightarrow Q$$

of w, with Lipschitz constant l_w, so that

$$\overline{W} = \{\chi + \overline{w}(\chi) : \chi \in \overline{pW}\},$$
$$\overline{W} \setminus W = \{\chi + \overline{w}(\chi) : \chi \in \partial pW\}.$$

11. The Return Map on W

Phase curves on W intersect transversally with the hyperplane H. More precisely, for

$$X := W \cap H$$

we have the following proposition.

Proposition 11.1. *For every solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) with $x_t \in W$ for all t, and $x_0 \in W \cap H$,*

$$(x')_0 = D(t \longrightarrow x_t)(0)1 \in (T_{x_0}W) \setminus H.$$

Proof. (Compare the construction of the map *stop* in Section 3). We have $x(-1) = 0$. Zeros are simple. Hence $0 \neq x'(-1)$, or $(x')_0 \notin H$. ☐

Arguments from the proof of [**36, Proposition 9.1**] yield our next result.

Corollary 11.1. *X is a one-dimensional C^1-submanifold of C.*

Observe that X is the disjoint union of the open subsets

$$X^+ := W \cap K, \quad X^- := W \cap (-K)$$

(since $\phi \in W \cap H$ implies $\phi \in K \cup (-K)$).

The return map P may not be defined on all of X^+ and X^-. So consider

$$O^+ := \{\phi \in X^+ : J(\phi) \geq 2\},$$

$$O^- := \{\phi \in X^- : J(\phi) \geq 2\}.$$

The union of these sets forms the domain of P. They are open in X^+, X^-, respectively (this follows easily from the simplicity of zeros of solutions with phase curve in W, and from continuous dependence on initial data).

Note that the statement

$$J(x) = \infty \quad \text{for all solutions} \quad x : \mathbb{R} \longrightarrow \mathbb{R} \quad \text{of eq. (1.1)} \tag{11.1}$$
$$\text{with phase curve in} \quad W$$

is equivalent to

$$O^+ = X^+ \quad \text{and} \quad O^- = X^-,$$

while in case

$$J(x) < \infty \quad \text{for at least one solution} \quad x : \mathbb{R} \longrightarrow \mathbb{R} \quad \text{of eq. (1.1)} \tag{11.2}$$
$$\text{with phase curve in} \quad W$$

we have

$$O^+ \neq X^+ \quad \text{and} \quad O^- \neq X^-.$$

The arguments from [**36, Proposition 9.3, proof, parts 1, 3, 4**] imply our next result.

Corollary 11.2. *P maps* O^+ *diffeomorphically onto* X^+ *and* O^- *diffeomorphically onto* X^-.

We write
$$P^+ : O^+ \longrightarrow X^+, \quad P^- : O^- \longrightarrow X^-$$
for the C^1–diffeomorphisms given by P.

Corollary 11.3. y_0 *and* $y_{z_1(y)+1}$ *are unstable hyperbolic fixed points of* P^+ *and* P^-, *respectively. For both maps, the eigenvalue of the linearization at the fixed point is*
$$u > 1.$$

Proof. 1. Remarks 8.1 and 8.2 yield $W_H \subset X, P(W_H) \subset X$. Hence
$$T_{y_0} W_H \subset T_{y_0} X;$$
in fact,
$$T_{y_0} W_H = T_{y_0} X \quad (= T_{y_0} X^+ = T_{y_0} O^+)$$
since both spaces have dimension 1; and
$$DP(y_0) T_{y_0} W_H \subset T_{P(y_0)} X = T_{y_0} W_H.$$
Recall that $DP(y_0)$ expands the line $T_{y_0} W_H = G_H$ by the factor $u > 1$.

2. Let $z_j = z_j(y)$, for $j \in \mathbb{Z}$. It remains to consider P^- and $y_{z_1+1} \in O^-$. Note first that $P_{1/2}$ maps $O^+ \cup O^-$ into $K_{1/2}$, i.e. into the domain of $P_{1/2}$, and
$$P_{1/2}(O^+) \subset W \cap H = X.$$
Choose a basis vector ϕ^+ of $G_H = T_{y_0} O^+$. Then
$$\phi^- := DP_{1/2}(y_0)\phi^+ \in T_{y_{z_1+1}} X = T_{y_{z_1+1}} O^-.$$
We have $\phi^- \neq 0$ because
$$0 \neq u \cdot \phi^+ = DP(y_0)\phi^+ = DP_{1/2}(y_{z_1+1})(DP_{1/2}(y_0)\phi^+).$$
Finally,
$$\begin{aligned} DP^-(y_{z_1+1})\phi^- &= DP(y_{z_1+1})\phi^- \\ &= (DP_{1/2}(y_{z_2+1}) \circ DP_{1/2}(y_{z_1+1}) \circ DP_{1/2}(y_0))(\phi^+) \\ = DP_{1/2}[DP(y_0)\phi^+] &= DP_{1/2}(y_0)[u \cdot \phi^+] = u \cdot [DP_{1/2}(y_0)\phi^+] = u \cdot \phi^-. \quad \square \end{aligned}$$

It is convenient to introduce local coordinates at the point y_0 of the C^1–submanifold O^+ of C. Choose a C^1–diffeomorphism
$$h : (-1,1) \longrightarrow N^+$$
onto an open neighborhood N^+ of y_0 in O^+, with
$$h(0) = y_0.$$
As y_0 is the only point on $|\eta|$ in O^+,
$$h(a) \notin |\eta| \quad \text{for} \quad 0 < |a| < 1.$$
Proposition 11.1 implies that the tangent vectors
$$(y')_0 \in (T_{y_0} W) \setminus H \quad \text{and} \quad Dh(0)1 \in T_{y_0} O^+ \subset (T_{y_0} W) \cap H$$
are linearly independent. As p defines a diffeomorphism of W onto pW, we obtain the next corollary.

Corollary 11.4. $D(p \circ h)(0)1 \in L$ *and* $p(y')_0 \in L$ *are linearly independent.*

So, for $\epsilon > 0$ sufficiently small, $p \circ h((-\epsilon, 0))$ and $p \circ h((0,\epsilon))$ belong to different components of $L \setminus |p \circ \eta|$. By a modification of h, we achieve
$$p \circ h((-\epsilon, 0)) \subset \operatorname{int}(p \circ \eta) \quad \text{for some} \quad \epsilon \in (0,1).$$
It follows that there exists $\underline{a} \in (0,1)$ such that we have
$$p \circ h((-\underline{a}, 0)) \subset \operatorname{int}(p \circ \eta),$$
$$P^+ \circ h((-\underline{a}, \underline{a})) \subset h((-1,1)),$$
and the C^1–map
$$g : (-\underline{a}, \underline{a}) \longrightarrow (-1,1), \quad g(a) := h^{-1}(P^+(h(a))) \quad \text{for} \quad |a| < \underline{a}$$
satisfies
$$g(0) = 0, \quad g'(0) = u > 1, \quad g'(a) > 1 \quad \text{for} \quad |a| < \underline{a}.$$

Each $a \in (-\underline{a}, \underline{a})$ is the endpoint $a = a_0$ of a backward trajectory $(a_j)_{-\infty}^{0}$ of g;
$$\operatorname{sign}(a_j) = \operatorname{sign}(a) \quad \text{for all} \quad j \quad \text{and} \quad a_j \longrightarrow 0 \quad \text{as} \quad j \longrightarrow -\infty.$$
The solution $x^a : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) given by
$$x_0^a = h(a) \in O^+$$
has segments
$$x_t^a \in W.$$
Its zeros form an increasing sequence of points $z_j = z_j(x^a)$ which has property (3.2). We have
$$x^a_{z_{2j}+1} = h(a_j) \quad \text{for all integers} \quad j \leq 0.$$

12. Simple Closed Curves on W

Let $a \in (-\underline{a}, 0)$ be given. Set $x := x^a$ and consider the sequence of zeros $(z_j)_{-\infty}^{J}$ of x, $\; 2 \leq J \in \mathbb{Z} \cup \{\infty\}$, $\; z_0 = -1$. Now, a is the endpoint $a = a_0$ of a trajectory $(a_j)_{-\infty}^{0}$ of g in $(-\underline{a}, 0)$;
$$a_j \longrightarrow 0 \quad \text{as} \quad j \longrightarrow -\infty$$
and

(12.1) $$h(a_j) = x_{z_{2j}+1} \quad \text{for} \quad j \leq 0.$$

We define piecewise smooth simple closed curves
$$\xi_j : [-1,1] \longrightarrow C, \quad j \in -\mathbb{N}_0$$
with values in W:
$$\text{For} \quad -1 \leq t \leq 0, \quad \xi_j(t) := h(-t \cdot a_j + (1+t) \cdot a_{j-1}).$$

I.e., this piece of ξ_j connects $h(a_j) = x_{z_{2j}+1}$ to $h(a_{j-1}) = x_{z_{2j-2}+1}$ in $O^+ \subset K$.

$$\text{For} \quad 0 \le t \le 1, \quad \xi_j(t) := F(t \cdot (z_{2j} - z_{2j-2}), x_{z_{2j-2}+1}).$$

Here we follow the phase curve from $x_{z_{2j-2}+1}$ to $x_{z_{2j}+1} = \xi_j(-1)$.

In case (11.1) holds, i.e. $J = \infty$ and $O^+ = X^+$, we define ξ_j also for $j \in \mathbb{N}$, by

$$\xi_j(t) := (P^+)^j(\xi_0(t)) \in O^+ \subset K \quad \text{for} \quad -1 \le t \le 0,$$
$$\xi_j(t) := F(t \cdot (z_{2j} - z_{2j-2}), x_{z_{2j-2}+1}) \quad \text{for} \quad 0 \le t \le 1.$$

Whenever ξ_j is defined,

$$\xi_j([-1,0]) \subset W \cap K$$

while $\xi_j|[0,1]$ is a reparameterization of the phase curve between $x_{z_{2j-2}+1}$ and $x_{z_{2j}+1}$.

PROPOSITION 12.1. *Let $k \in \mathbb{N}_0$, $j \in \mathbb{Z}$. Assume $j \le 0$ and $k + j \le 0$, or assume that* (11.1) *holds. Then*

$$\xi_{k+j}([-1,0]) = (P^+)^k(\xi_j([-1,0])).$$

PROOF. In the first case, use $\xi_n([-1,0]) = h([a_n, a_{n-1}])$ for all $n \le 0$, and

$$g^k(a_j) = a_{k+j}, \quad g^k(a_{j-1}) = a_{k+j-1}, \quad \text{and}$$
$$h \circ g^k = (P^+)^k \circ h \quad \text{on} \quad [a_j, a_{j-1}].$$

In the second case, P^+ maps $O^+ = X^+$ one–to–one onto X^+, and it is easy to deduce the assertion. □

PROPOSITION 12.2. *Let $j \in \mathbb{Z}$. Assume $j \le 0$, or assume that* (11.1) *holds. Then*

$$|p \circ \xi_j| \subset int(p \circ \eta).$$

PROOF. The case $j \le 0$: Following ξ_j we see that each point on $|\xi_j|$ connects in $W \setminus |\eta|$ to, say, $h(a_j)$. Recall that

$$ph(a_j) \in \mathrm{int}(p \circ \eta),$$

and use the fact that p maps W one–to–one onto $pW \subset L$. In case (11.1) holds and $j > 0$ we note that

$$\xi_j([-1,0]) = (P^+)^j(\xi_0([-1,0])) \subset K$$

is disjoint with $|\eta|$ since

$$P^+ : X^+ \longrightarrow X^+ \quad \text{is one–to–one}, \quad P^+(y_0) = y_0,$$
$$|\eta| \cap K = \{y_0\}, \quad \xi_0([-1,0]) \cap |\eta| = \emptyset.$$

Also, $x_t \notin |\eta|$ for all $t \in \mathbb{R}$ (otherwise, we obtain a contradiction to (12.1) and $a_j \neq 0$ for all $j \in -\mathbb{N}_0$). Following ξ_j and the phase curve of x we see that each point of $|\xi_j|$ connects in $W \setminus |\eta|$ to $h(a_0)$. Continue as in the first case. □

PROPOSITION 12.3. *Let $j \in \mathbb{Z}$. Assume $j \le 0$, or assume that* (11.1) *holds. Then*

$$|\xi_{j-1}| \cap |\xi_j| = \{x_{z_{2j-2}+1}\},$$

and for integers $k < j$,

$$|\xi_{k-1}| \cap |\xi_j| = \emptyset.$$

PROOF. It is obvious that the intersections of traces contain the sets on the right hand sides. We show that they are also subsets of the right hand side. Let an integer $k \le j$ be given.
1. Proof of

$$\xi_{k-1}((0,1)) \cap |\xi_j| = \emptyset.$$

Otherwise, there exists $\xi_{k-1}(t) = x_s \notin K$ in $|\xi_j|$ where $t \in (0,1)$, $s < z_{2k-2}+1$. Necessarily,

$$x_s = \xi_j(\underline{t}) = x_{\underline{s}} \quad \text{where} \quad \underline{t} \in (0,1) \quad \text{and} \quad z_{2j-2} + 1 < \underline{s}.$$

It follows that x is periodic, and we arrive at a contradiction to (12.1) and $a_j \nearrow 0$ as $j \longrightarrow -\infty$.
2. Proof that

$$\phi \in \xi_{k-1}([-1,0]) \cap |\xi_j| \quad \text{implies} \quad k = j \quad \text{and} \quad \phi = x_{z_{2j-2}+1}.$$

Such ϕ belong to $\xi_j([-1,0]) \subset K$. There exist $t, \underline{t}$ in $[-1,0]$ with

$$\xi_{k-1}(t) = \phi = \xi_j(\underline{t}).$$

Hence,

$$(P^+)^{-j}(\xi_{k-1}(t)) = (P^+)^{-j}(\xi_j(\underline{t})).$$

Proposition 12.1 implies that the left hand side belongs to

$$\xi_{k-j-1}([-1,0]) = h([a_{k-j-1}, a_{k-j-2}]);$$

the right hand side belongs to

$$\xi_0([-1,0]) = h([a_0, a_{-1}]).$$

Strict monotonicity of g yields $k = j$ and

$$(P^+)^{-j}(\phi) = h(a_{-1});$$

$$\phi = (P^+)^j(h(a_{-1})) = (P^+)^j(x_{z_{-2}+1}) = x_{z_{2j-2}+1}. \quad \square$$

Next we consider points on $|\xi_j|$ in $-K$ and their projections in L. Such points have the advantage that close to them ξ_j is smooth (given by the phase curve of x).

PROPOSITION 12.4. *Let $j \in \mathbb{Z}$. Assume $j \leq 1$, or assume that* (11.1) *holds. For every integer $k < j$,*

$$px_{z_{2j-1}+1} \in \mathit{int}(p \circ \xi_k).$$

PROOF. 1. In case $j \leq 1$,

$$x_{z_{2j-1}+1} = P_{1/2}(x_{z_{2j-2}+1}) = P_{1/2}(h(a_{j-1})).$$

Set $a^* := a_{j-1} \in (-\underline{a}, 0)$, and consider the map

$$\Pi : [a^*, 0] \ni \hat{a} \longrightarrow P_{1/2}(h(\hat{a})) \in C.$$

Π maps into $-K \subset H$.

In case that (11.1) holds, and $j > 1$,

$$x_{z_{2j-1}+1} = P_{1/2}(x_{z_{2j-2}+1}) = P_{1/2}((P^+)^{j-1}(x_0))$$
$$= P_{1/2}((P^+)^{j-1}(h(a_0))).$$

Set

$$a^* := a_0 \in (-\underline{a}, 0),$$

and define $\Pi : [a^*, 0] \longrightarrow C$ by

$$\Pi(\hat{a}) := P_{1/2}((P^+)^{j-1}(h(\hat{a}))) \in -K \subset H.$$

2. Π connects $x_{z_{2j-1}+1}$ in $W \cap H$ to $y_{z_1(y)+1} \in |\eta|$. Since all maps involved are given by diffeomorphisms,

$$0 \neq D\Pi(\hat{a})1 \quad \text{for} \quad a^* < \hat{a} < 0.$$

3. Proof of

$$|\xi_k| \cap |\Pi| = \{x_{z_{2k-1}+1}\}.$$

$|\xi_k| \cap |\Pi| \subset \{\dots\}$ follows from $|\Pi| \subset -K$ and from $|\xi_k| \cap (-K) = \{\dots\}$. In case (11.1) holds and $j > 1$ we deduce from $k < j$ that

$$|\xi_k| \ni x_{z_{2k-1}+1} = P_{1/2}(x_{z_{2k-2}+1}) = P_{1/2}((P^+)^{j-1}(x_{z_{2k-2-2(j-1)}+1}))$$
$$= P_{1/2}((P^+)^{j-1}(h(a_{k-j}))) \in |\Pi|,$$

since $a^* = a_0 < a_{k-j} < 0$.

In case $j \leq 1$ we obtain

$$|\xi_k| \ni x_{z_{2k-1}+1} = P_{1/2}(x_{z_{2k-2}+1}) = P_{1/2}(h(a_{k-1})) \in |\Pi|$$

since $a^* = a_{j-1} < a_{k-1} < 0$.

4. The nonzero tangent vectors $D\Pi(a)1 \in H \cap T_{\Pi(a)}W$ where

$$\Pi(a) = x_{z_{2k-1}+1},$$

and $D\xi_k(t)1$ where

$$t \in (0,1), \quad \xi_k(t) = x_{z_{2k-1}+1},$$

are linearly independent since $D\xi_k(t)1$ is a multiple of

$$(x')_{z_{2k-1}+1} \in (T_{\xi_k(t)}W) \setminus H.$$

As p maps W diffeomorphically onto $pW \subset L$ we infer that the projected curves $p \circ \Pi$ and $p \circ \xi_k$ have linearly independent tangent vectors at their intersection point

$$px_{z_{2k-1}+1}.$$

Now, $|p \circ \xi_k| \subset \text{int}(p \circ \eta)$ implies

$$p \circ \Pi(0) = py_{z_1(y)+1} \in |p \circ \eta| \subset \text{ext}(p \circ \xi_k).$$

It follows that the other endpoint of $|p \circ \Pi|$, namely,

$$px_{z_{2j-1}+1} = p \circ \Pi(a^*),$$

belongs to $\text{int}(p \circ \xi_k)$. $\square$

COROLLARY 12.1. *Let integers $k < j$ be given. Assume $j \leq 0$, or assume that* (11.1) *holds. Then* 1. $|p \circ \xi_j| \setminus \{px_{z_{2j-2}+1}\} \subset \mathit{int}(p \circ \xi_{j-1})$,
2. $|p \circ \xi_j| \subset \mathit{int}(p \circ \xi_{k-1})$,
3. $\mathit{int}(p \circ \xi_j) \subset \mathit{int}(p \circ \xi_k)$,
4. $|p \circ \xi_k| \setminus |p \circ \xi_j| \subset \mathit{ext}(p \circ \xi_j)$.

PROOF. 1. Proposition 12.3 implies that each point in

$$|\xi_j| \setminus \{x_{z_{2j-2}+1}\}$$

connects in the complement of $|\xi_{j-1}|$ to $x_{z_{2j-1}+1}$. Apply p and use Proposition 12.4.
2. As before, with the last assertion of Proposition 12.3.
3. Assertions 1,2 and Proposition 12.3 yield

$$|p \circ \xi_j| \subset \text{int}(p \circ \xi_k) \cup |p \circ \xi_k|,$$

and a standard argument completes the proof.
4. This part is a routine consequence of the first assertions. $\square$

COROLLARY 12.2. $\omega(x) \subset \overline{W} \setminus W$.

PROOF. First, $\omega(x) \subset \overline{W}$ is obvious. Next, $px_0 \in \text{int}(p \circ \eta)$ (Proposition 12.2) and

$$x_t \notin |\eta| \quad \text{for all} \quad t$$

imply that $px_t \in \text{int}(p \circ \eta)$ for all $t \in \mathbb{R}$. It follows that

$$p\omega(x) \subset \overline{\text{int}(p \circ \eta)}.$$

Suppose $\omega(x) \cap W \neq \emptyset$. Then there is a solution $\underline{x} : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) with phase curve in $\omega(x) \cap W$. Convergence to $|\eta|$ as $t \longrightarrow -\infty$ implies that there exist $t \in \mathbb{R}$ with

$$p\underline{x}_t \in \text{ext}(p \circ \xi_0)$$

($|p\circ\eta| \subset \mathrm{ext}(p\circ\xi_0)$ since $|p\circ\xi_0| \subset \mathrm{int}(p\circ\eta)$ (Proposition 12.2)). Using $\underline{x}_t \in \omega(x)$ we find $s > z_1 + 1$ such that

$$px_s \in \mathrm{ext}(p\circ\xi_0).$$

Recall $px_{z_1+1} \in \mathrm{int}(p\circ\xi_0)$ (Proposition 12.4). It follows that there exists $t > z_1 + 1$ so that

$$px_t \in |p\circ\xi_0| \quad \text{(in particular,} \quad x_t \in |\xi_0|),$$

$$px_v \in \mathrm{int}(p\circ\xi_0) \quad \text{for} \quad z_1 + 1 \le v < t.$$

In case $x_t \in \xi_0([0,1])$, $x_t = x_v$ for some $v \le z_0 + 1$, and x is periodic which leads to a contradiction.

It remains to consider the case

$$x_t \in \xi_0((-1,0)).$$

Then $x_t = h(a^*)$ where $a_0 < a^* < a_{-1}$. The backward trajectory $(a_j^*)_{-\infty}^0$ of g which ends at $a_0^* = a^*$ satisfies

$$a_j < a_j^* < a_{j-1}.$$

The solution

$$x^* := x(\cdot + t)$$

has zeros z_j^*, $j \in -\mathbb{N}_0$, such that

$$x^*_{z^*_{2j}+1} = h(a_j^*);$$

and

$$x_t^* \notin K \quad \text{for all} \quad t \le 0 \quad \text{not contained in} \quad \{z^*_{2j} + 1 : j \in -\mathbb{N}_0\}.$$

Therefore,

$$K \ni h(a_0) = x_0 = x^*_{-t} = x^*_{z^*_{2j}+1} = h(a_j^*) \quad \text{for some} \quad j \in -\mathbb{N}_0$$

which is a contradiction to $a_0 < a_j^*$. □

Next, we construct homotopies of closed curves in W which connect a reparameterization of η to a reparameterization of a curve ξ_j. We do this for $j = 0$, and in case (11.1) holds, also for every $j \in \mathbb{N}$.

The map

$$m : [a,0) \times [-1,1] \longrightarrow C$$

given by

$$m(s,t) = (P^+)^j \circ h((-t)s + (1+t)g^{-1}(s)) \quad \text{for} \quad t \le 0$$

$$m(s,t) = F(t\cdot(z_2((P^+)^j \circ h \circ g^{-1}(s)) + 1, (P^+)^j \circ h \circ g^{-1}(s)) \quad \text{for} \quad t \ge 0$$

is continuous since both formulae yield the same value $(P^+)^j \circ h \circ g^{-1}(s)$ at points $(s,0)$. Note that

$$m(a,t) = \xi_j(t).$$

The closed curves $m(s,\cdot), a \le s < 0$, are all analogues of ξ_j, composed of a piece

$$(P^+)^j \circ h([s, g^{-1}(s)]) \subset W \cap K,$$

and of a piece of the phase curve through

$$(P^+)^j \circ h(g^{-1}(s)) \quad \text{and} \quad (P^+)^j \circ h(s).$$

The number s serves as the homotopy parameter. The points

$$m(s,1) = F(z_2((P^+)^j \circ h \circ g^{-1}(s)) + 1, (P^+)^j \circ h \circ g^{-1}(s))$$
$$= (P^+)^{j+1} \circ h \circ g^{-1}(s) = (P^+)^j \circ h(s)$$

on the curves $m(s,\cdot)$ fill the homeomorphic image of $h([a,0))$ under the map $(P^+)^j$ (a connected set in the one–dimensional submanifold $X^+ = W \cap K$).

The transformation

$$\Psi : [a,0) \times [0,1] \longrightarrow [a,0) \times [-1,1]$$

given by

$$\Psi_1(s,t) = s$$

and

$$\Psi_2(s,t) = \frac{t}{s/2a} - 1 \quad \text{for} \quad t \le \frac{s}{2a}$$

$$\Psi_2(s,t) = \frac{t-(s/2a)}{1-(s/2a)} \quad \text{for} \quad \frac{s}{2a} \le t$$

is continuous since both formulae for Ψ_2 yield the same value when

$$t = \frac{s}{2a};$$

$\Psi_2(s,\cdot)$ maps $\left[0, \frac{s}{2a}\right]$ onto $[-1,0]$ and $\left[\frac{s}{2a}, 1\right]$ onto $[0,1]$.

The interval $\left[0, \frac{s}{2a}\right]$ shrinks to $\{0\}$ as $s \nearrow 0$. Now define

$$M(s,t) := m \circ \Psi(s,t) \quad \text{on} \quad [0,a) \times [0,1],$$

$$M(s,t) := \eta(t\cdot\tau) \quad \text{for} \quad s = 0, \quad 0 \le t \le 1.$$

We have that M is continuous on $[a,0) \times [0,1]$; $M(a,\cdot)$ and $M(0,\cdot)$ are reparameterizations of ξ_j and η since

$$M(a,t) = m(a, 2t-1) = \xi_j(2t-1) \quad \text{and} \quad M(0,t) = \eta(t\cdot\tau).$$

Each $M(s,\cdot)$ is a (simple) closed curve on W.

We next show the continuity of M at the points $(0,t)$:
Let a sequence of points (s_n, t_n), $n \in \mathbb{N}$, in the domain of M be given with

$$(s_n, t_n) \longrightarrow (0,t) \quad \text{as} \quad n \longrightarrow \infty.$$

Clearly, as $n \longrightarrow \infty$,

$$M\left(\{s_n\} \times \left[0, \frac{s_n}{2a}\right]\right) = (P^+)^j \circ h([s_n, g^{-1}(s_n)]) \longrightarrow (P^+)^j \circ h(0)$$

$$= y_0 = M(0,0).$$

For a subsequence of points $(s_{\nu(n)}, t_{\nu(n)})$, $\nu : \mathbb{N} \longrightarrow \mathbb{N}$ strictly increasing, such that

$$t_{\nu(n)} \le \frac{s_{\nu(n)}}{2a} \quad \text{for all} \quad n \in \mathbb{N},$$

necessarily $t = 0$, and

$$M(s_{\nu(n)}, t_{\nu(n)}) \longrightarrow M(0,0).$$

For a subsequence of points $(s_{\nu(n)}, t_{\nu(n)})$, $\nu : \mathbb{N} \longrightarrow \mathbb{N}$ strictly increasing, such that

$$t_{\nu(n)} > \frac{s_{\nu(n)}}{2a} \quad (\le 0) \quad \text{for all} \quad n \in \mathbb{N},$$

we obtain, in case $s_{\nu(n)} < 0$,

$$M(s_{\nu(n)}, t_{\nu(n)}) = m(s_{\nu(n)}, \Psi_2(s_{\nu(n)}, t_{\nu(n)}))$$

$$= F(\Psi_2(\ldots) \cdot (z_2((P^+)^j \circ h \circ g^{-1}(s_{\nu(n)})) + 1, (P^+)^j \circ h \circ g^{-1}(s_{\nu(n)})),$$

and in case $s_{\nu(n)} = 0$,

$$M(s_{\nu(n)}, t_{\nu(n)}) = \eta(t_{\nu(n)}\tau).$$

Observe that in case $s_{\nu(n)} < 0$,

$$\Psi_2(s_{\nu(n)}, t_{\nu(n)}) = \frac{t_{\nu(n)} - (s_{\nu(n)}/2a)}{1 - (s_{\nu(n)}/2a)}.$$

For every further subsequence of points $(s_{\nu\circ\kappa(n)}, t_{\nu\circ\kappa(n)})$, $\kappa : \mathbb{N} \longrightarrow \mathbb{N}$ strictly increasing, such that

$$s_{\nu\circ\kappa(n)} < 0 \quad \text{for all} \quad n \in \mathbb{N}$$

we infer

$$\Psi_2(s_{\nu\circ\kappa(n)}, t_{\nu\circ\kappa(n)}) \longrightarrow t \quad \text{as} \quad n \longrightarrow \infty.$$

Continuity of g and h and P^+, and the equations $g(0) = 0$, $h(0) = y_0$, $P^+(y_0) = y_0$ may now be used to derive

$$M(s_{\nu(n)}, t_{\nu(n)}) \longrightarrow F(t \cdot (z_2((P^+)^j \circ h(0)) + 1, (P^+)^j \circ h(0))$$

$$= \eta(t \cdot \tau) = M(0,t),$$

and it is routine to complete the proof.

Proposition 12.5. *$ext(p \circ \xi_0) \cap int(p \circ \eta) \subset pW$. If* (11.1) *holds, then*

$$ext(p \circ \xi_j) \cap int(p \circ \eta) \subset pW \quad \textit{for every} \quad j \in \mathbb{N}.$$

Proof. Set $j := 0$, or assume that (11.1) holds, and let $j \in \mathbb{N}$ in this case. We have

$$|M(1,\cdot)| = |\xi_j|, \quad |M(0,\cdot)| = |\eta|.$$

Hence,

$$\text{ext}(p \circ \xi_j) = \text{ext}(p \circ M(1,\cdot)), \quad \text{ext}(p \circ M(0,\cdot)) = \text{ext}(p \circ \eta).$$

Suppose there exists

$$\chi \in ((\text{ext}(p \circ \xi_j) \cap \text{int}(p \circ \eta)) \setminus pW.$$

Then $\chi \in \text{ext}(p \circ M(1,\cdot))$, and

$$0 = \text{wind}(\chi, p \circ M(1,\cdot)) = \text{wind}(\chi, p \circ M(0,\cdot))$$

since $p \circ M$ is a continuous homotopy of closed curves in $pW, \chi \notin pW$. Therefore

$$\chi \in \text{ext}(p \circ M(0,\cdot)) = \text{ext}(p \circ \eta),$$

a contradiction to $\chi \in \text{int}(p \circ \eta)$. □

Later on we shall sometimes write $\xi_{j,a}$ and $M_{j,a}$ instead of ξ_j and M.

All constructions in this section have counterparts if we begin at the other unstable fixed point of P on $|\eta|$, namely at the fixed point

$$y_{z_1(y)+1} \in O^- \subset -K$$

of the map P^-.

Let $\underline{a}^- < 0$, h^-, g^- denote the analogues of $\underline{a}$, h, g. Every $a^- \in (\underline{a}^-, 0)$ is the endpoint $a^- = a_0^-$ of a trajectory $(a_j^-)_{-\infty}^0$ of g^- in $(\underline{a}^-, 0)$;

$$a_j^- \longrightarrow 0 \quad \text{as} \quad j \longrightarrow -\infty.$$

a^- determines a solution $x^* : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) with phase curve in W, by

$$x_0^* = h^-(a^-) \in -K.$$

The zeros of x^* form a sequence $(z_j^*)_{-\infty}^J$, $3 \le J \in \mathbb{Z}$ or $J = \infty$, with property (3.2), and

$$z_1^* = -1.$$

For applications in the next section, it is convenient to perform a time shift. Set

$$x := x^{a^-} := x^*(\cdot + z_0^* + 1)$$

and

$$z_j := z_j^* - (z_0^* + 1) \quad \text{for all} \quad j.$$

The zeros of x are then given by the sequence $(z_j)_{-\infty}^J$; property (3.2) is satisfied, and

$$z_0 = -1,$$

$$x_0 = x^*_{z_0^*+1} \in K \quad (\text{since} \quad x^*_{z_1^*+1} \in -K),$$

$$x_{z_1+1} = x^*_{z_1^*+1} = x_0^* = h^-(a^-) \in -K.$$

As at the beginning of this section, we construct closed curves

$$\xi_j^- : [-1,1] \longrightarrow C$$

for all $j \in -\mathbb{N}_0$, and in case (11.1) holds, for all integers j, so that

$$\xi_j^-([-1,0]) \subset W \cap (-K),$$

and for $0 \leq t \leq 1$,

$$\xi_j^-(t) = F(t \cdot (z_{2j+1} - z_{2j-1}), x_{z_{2j-1}+1}).$$

13. A Periodic Orbit in $\overline{W} \setminus W$

We assume in this section that (11.1) holds, i.e. $J(x) = \infty$ for all solutions $x : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) with phase curve in W. We look for a periodic orbit in $\overline{W} \setminus W$ which projects into $\mathrm{int}(p \circ \eta) \subset L$.

Consider a solution $x^* = x^{a^*}, \underline{a} < a^* < 0$, as in the preceding section. Then $x_t^* \notin |\eta|$ for all $t \in \mathbb{R}$, and it follows that

$$px_t^* \in \mathrm{int}(p \circ \eta) \quad \text{for all} \quad t \in \mathbb{R} \tag{13.1}$$

since $px_0^* = ph(a^*) \in \mathrm{int}(p \circ \eta)$. The zeros of x^* form a sequence $(z_j^*)_{-\infty}^{\infty}$ with property (3.2), and

$$x^*_{z^*_{2j}+1} \in K, \quad x^*_{z^*_{2j+1}+1} \in -K \quad \text{for all} \quad j \in \mathbb{Z}.$$

The relations

$$x_t \longrightarrow |\eta| \quad \text{as} \quad t \longrightarrow -\infty, \quad |\eta| \cap K = \{y_0\}, \quad |\eta| \cap (-K) = \{y_{z_1(y)+1}\}$$

imply that the trajectories

$$(x^*_{z^*_{2j}+1})_{-\infty}^{\infty} \quad \text{of} \quad P^+ \quad \text{and} \quad (x^*_{z^*_{2j+1}+1})_{-\infty}^{\infty} \quad \text{of} \quad P^-$$

converge to y_0 and $y_{z_1(y)+1}$, respectively, as $j \longrightarrow -\infty$. Using (13.1) we infer that there exists an integer k so that

$$x^*_{z^*_{2k}+1} \in h((\underline{a}, 0)),$$

$$x^*_{z^*_{2k+1}+1} \in h((\underline{a}^-, 0)).$$

Set

$$x := x^*(\cdot + z^*_{2k} + 1) \quad \text{and} \quad z_j := z^*_{j+2k} - (z^*_{2k} + 1) \quad \text{for} \quad j \in \mathbb{Z}.$$

Observe that the sequence $(z_j)_{-\infty}^{\infty}$ has property (3.2);

$$x^{-1}(0) = \{z_j : j \in \mathbb{Z}\}$$

and

$$z_0 = -1,$$

$$x_0 = x^*_{z^*_{2k}+1} = h(a) \quad \text{for some} \quad a \in (\underline{a}, 0),$$

$$x_{z_1+1} = x^*_{z^*_{2k+1}+1} = h(a^-) \quad \text{for some} \quad a^- \in (\underline{a}^-, 0).$$

Therefore

$$x^a = x = x^{a^-}.$$

Let ξ_j and ξ_j^-, $j \in \mathbb{Z}$, denote the closed curves, associated as in Section 12, with x^a and x^{a^-}, respectively. Then

$$\xi_j(1) = x_{z_{2j}+1} \in K, \quad \xi_j^-(1) = x_{z_{2j+1}+1} \in -K \quad \text{for all} \quad j \in \mathbb{Z}.$$

Corollary 13.1. *For integers $k < j - 1$, $\mathit{ext}(p \circ \xi_k^-) \subset \mathit{ext}(p \circ \xi_j)$.*

Proof. The analogue of Corollary 12.1.2 for the curves ξ_κ^- yields

$$|p \circ \xi_j^-| \subset \mathrm{int}(p \circ \xi_k).$$

In particular, the point $px_{z_{2j}+1} \in |p \circ \xi_j|$ belongs to $\mathrm{int}(p \circ \xi_k^-)$. We have

$$|\xi_j| \cap |\xi_k^-| = \emptyset$$

since

$$\xi_j([-1,0]) \cap \xi_k^-([-1,0]) \subset K \cap (-K) = \emptyset,$$
$$\xi_j([0,1]) = \{x_t : z_{2j-2} + 1 \leq t \leq z_{2j} + 1\},$$
$$\xi_k^-([0,1]) = \{x_t : z_{2k-1} + 1 \leq t \leq z_{2k+1} + 1\},$$
$$z_{2k+1} + 1 < z_{2j-2} + 1,$$

and x is not periodic. We conclude that

$$|p \circ \xi_j| \subset \mathrm{int}(p \circ \xi_k^-)$$

from which the assertion follows. ☐

There is a strictly increasing sequence

$$\iota : \mathbb{N} \longrightarrow \mathbb{N}$$

so that the subsequences $(x_{z_{2\iota(j)}})_1^{\infty}$ and $(x_{z_{2\iota(j)+1}})_1^{\infty}$ converge to points ϕ and ϕ^-, respectively, in $\omega(x) \subset \overline{W} \setminus W$ (see Corollary 12.2). These limit points determine solutions b and b^- of eq. (1.1), both defined on $\mathbb{R}$, so that

$$b_{-1} = \phi, \quad b_{-1}^- = \phi^-;$$

the phase curves of b and b^- belong to $\omega(x) \cap (\overline{W} \setminus W)$. On any compact interval $[-2, t], t \geq 0$, we have uniform convergence

$$x(s + z_{2\iota(j)} + 1) \longrightarrow b(s) \quad \text{as} \quad j \longrightarrow \infty, \tag{13.2}$$

$$x(s + z_{2\iota(j)+1} + 1) \longrightarrow b^-(s) \quad \text{as} \quad j \longrightarrow \infty. \tag{13.3}$$

(Working with phase points at $t = z_{2j}$ (and $t = z_{2j+1}$), $j \in \mathbb{Z}$, instead of $t = z_{2j} + 1$ also implies that derivatives $x'(t + z_{2\iota(j)} + 1), t \geq -1$, converge to $b'(t)$ as $j \longrightarrow \infty$. This will be shown and used in the proof of Proposition 13.4 below.) In particular,

$$x_{z_{2\iota(j)}+1} \longrightarrow b_0 \in \overline{K}$$

and

$$x_{z_{2\iota(j)+1}+1} \longrightarrow b_0^- \in \overline{-K}$$

as $j \longrightarrow \infty$. The case $b_0 = 0$ is equivalent to $b = 0$. In case $b_0 \neq 0$, the zeros of b are given by a sequence $(z_j(b))_{-\infty}^{J(b)}$ where $0 \leq J(b) \in \mathbb{Z}$ or $J(b) = \infty$, and $z_0(b) = -1$. Analogously, $b_0^- = 0$ if and only if $b^- = 0$; in case $b^- \neq 0$, the zeros of b^- are given by a sequence $(z_j(b^-))_{-\infty}^{J(b^-)}$ where $1 \leq J(b^-) \in \mathbb{Z}$ or $J(b^-) = \infty$, and $z_1(b^-) = -1$.

Incidentally, note that we can not immediately conclude that b and b^- have the same orbit since there is no estimate of the distances $z_{2\iota(j)+1} - z_{2\iota(j)}$. But we do have the following result.

COROLLARY 13.2. *If $b = 0$ then $b^- = 0$, and*

$$\xi_{\iota(j)+1}([0,1]) \longrightarrow 0, \quad \xi^-_{\iota(j)+1}([0,1]) \longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty.$$

PROOF. 1. Proposition 3.2 implies that for all $j \in \mathbb{N}$,

$$\|x_{z_{2\iota(j)+1}} + 1\| = \max_{[z_{2\iota(j)+1},\, z_{2\iota(j)+2}]} |x| \leq d_f \cdot \max_{[z_{2\iota(j)},\, z_{2\iota(j)+1}]} |x|$$
$$= d_f \cdot \|x_{z_{2\iota(j)}} + 1\|.$$

2. Also by Proposition 3.2, for all $j \in \mathbb{N}$,

$$\max_{[0,1]} \|\xi_{\iota(j)+1}(t)\| = \max_{[z_{2\iota(j)},\, z_{2\iota(j)+2} + 1]} |x| \leq (d_f + 1)^2 \|x_{z_{2\iota(j)}} + 1\|.$$

This yields $\xi_{\iota(j)+1}([0,1]) \longrightarrow 0$. The argument for the curves $\xi^-_{\iota(j)+1}$ is analogous. □

PROPOSITION 13.1. 1. *Let $t \in \mathbb{R}$. We have*

$$pb_t \in int(p \circ \eta) \ni pb_t^-,$$

and for all $j \in \mathbb{Z}$,

$$pb_t \in int(p \circ \xi_j), \quad pb_t^- \in int(p \circ \xi_j^-).$$

In particular,

$$b_t \notin |\eta| \cup |\xi_j| \quad \text{and} \quad b_t^- \notin |\eta| \cup |\xi_j^-|.$$

2. *In case $b \neq 0$, $J(b) = \infty$.*

PROOF. 1. The relations $|\eta| \cup |\xi_j| \subset W$, $b_t \in \overline{W} \setminus W$, imply that some convex neighborhood N_L of pb_t in L is disjoint with $|p \circ \eta| \cup |p \circ \xi_j|$. As $b_t \in \omega(x)$, there exists $s \geq z_{2j+2} + 1$ with $x_s \in p^{-1}(N_L)$. We have

$$x_s \in \xi_k([0,1]) \quad \text{for some} \quad k \geq j+2,$$

and

$$|p \circ \xi_k| \subset \mathrm{int}(p \circ \eta) \cap \mathrm{int}(p \circ \xi_j)$$

(see Proposition 12.2 and Corollary 12.1.2). It follows that

$$b_t \in \mathrm{int}(p \circ \eta) \cap \mathrm{int}(p \circ \xi_j).$$

The proof for b_t^- and ξ_j^- is analogous.

2. Suppose $J(b) < \infty$. Then $b(t) \longrightarrow 0$ as $t \longrightarrow \infty$, and $0 \in C$ is an attractive stationary point of the semiflow F (Section 3, Remark 4.1, parts 2 and 3). For every ψ in some open neighborhood U of 0 in C,

$$F(t, \psi) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty.$$

Choose $t > 0$ with $b_t \in U$. There exists $s > 0$ with $x_s \in U$. Hence $x(t') \longrightarrow 0$ as $t' \longrightarrow \infty$, a contradiction to

$$x_{z_{2\iota(j)}} + 1 \longrightarrow b_0 \neq 0 \quad \text{as} \quad j \longrightarrow \infty. \quad \square$$

Part 2 of Proposition 13.1, statement (13.2) and the simplicity of zeros yield our next result.

COROLLARY 13.3. *In case $b \neq 0$, for every $k \in \mathbb{N}_0$,*

$$z_{2\iota(j)+k} - z_{2\iota(j)} \longrightarrow z_k(b) + 1 \quad \text{as} \quad j \longrightarrow \infty,$$

and if $\epsilon = \epsilon(k) \in (0,1)$ satisfies $z_{k+2}(b) + 1 + \epsilon < z_{k+3}(b)$, then the set $\{x_t : z_{2\iota(j)+k} + 1 \leq t \leq z_{2\iota(j)+k+2} + 1\}$ converges to the set

$$\{b_s : z_k(b) + 1 - \epsilon \leq s \leq z_{k+2}(b) + 1 + \epsilon\}$$

as $j \longrightarrow \infty$.

COROLLARY 13.4. *In case $b \neq 0$,*

$$x_{z_{2\iota(j)+1}} + 1 \longrightarrow b_{z_1(b)} + 1 \quad \text{as} \quad j \longrightarrow \infty,$$

and

$$b^- = b(\cdot + z_1(b) + 1).$$

PROOF. The convergence

$$x_{z_{2\iota(j)}} + 1 \longrightarrow b_0, \quad z_{2\iota(j)+1} - z_{2\iota(j)} \longrightarrow z_1(b) + 1 \quad \text{for} \quad j \longrightarrow \infty$$

and continuity of F imply

$$b_{z_1(b)} + 1 = F(z_1(b) + 1, b_0) = \lim_{j \longrightarrow \infty} F(z_{2\iota(j)+1} - z_{2\iota(j)}, x_{z_{2\iota(j)}} + 1)$$
$$= \lim_{j \longrightarrow \infty} x_{z_{2\iota(j)+1}} + 1 = b_0^-. \quad \square$$

PROPOSITION 13.2. *In case $b \neq 0$, b is periodic with minimal period $z_2(b)+1$.*

PROOF. 1. Suppose $b_0 \neq b_{z_2(b)} + 1$. Then

$$b_{z_1(b)} + 1 \neq b_{z_3(b)} + 1.$$

(Otherwise, b would have period $z_3(b) - z_1(b)$, with $z_2(b)$ being the only zero in the period interval $(z_1(b), z_3(b))$. This would imply

$$z_2(b) + 1 = z_2(b) - z_0(b) = z_3(b) - z_1(b),$$

hence $b_{z_2(b)+1} = b_0$, a contradiction to the assumption above.)

Choose $\epsilon \in (0,1)$ with $z_4(b)+1+\epsilon < z_5(b)$. For $z_2(b)+1-\epsilon \leq t \leq z_4(b)+1+\epsilon$,

$$b_t \in -K \quad \text{if and only if} \quad t = z_3(b)+1.$$

It follows that for every t as above,

$$b_t \neq b_{z_1(b)+1} \quad (\in -K).$$

Note $\overline{K} = \{\phi \in C : 0 \leq \phi\}$, and $\overline{K} \cap (-K) = \emptyset$. We conclude that

$$d := \operatorname{dist}(pb_{z_1(b)+1}, p(\overline{W \cap K}) \cup \{pb_t : z_2(b)+1-\epsilon \leq t \leq z_4(b)+1+\epsilon\})$$

is strictly positive.

2. Corollary 13.4 and Corollary 13.3 permit us to find $j \in \mathbb{N}$ with

$$\|px_{z_{2\iota(j)+1}+1} - pb_{z_1(b)+1}\| < \frac{d}{2}$$

and

$$\operatorname{dist}(px_t, \{pb_s : z_2(b)+1-\epsilon \leq s \leq z_4(b)+1+\epsilon\}) < \frac{d}{2}$$

for $z_{2\iota(j)+2}+1 \leq t \leq z_{2\iota(j)+4}+1$.

It follows that for $0 \leq t \leq 1$, the points

$$\begin{aligned} p \circ \xi_{\iota(j)+2}(t) &= pF(t \cdot \{z_{2[\iota(j)+2]} - z_{2[\iota(j)+2]-2}, x_{z_{2[\dots]-2}+1}) \\ &= px_{t \cdot (z_{2\iota(j)+4} - z_{2\iota(j)+2}) + z_{2\iota(j)+2}+1} \end{aligned}$$

belong to the $\frac{d}{2}$–neighborhood of

$$\{pb_s : z_2(b)+1-\epsilon \leq s \leq z_4(b)+1+\epsilon\},$$

so

$$\frac{d}{2} < \operatorname{dist}(p \circ \xi_{\iota(j)+2}([0,1]), pb_{z_1(b)+1}).$$

Since $\xi_{\iota(j)+2}([-1,0]) \subset W \cap K$,

$$p \circ \xi_{\iota(j)+2}([-1,0]) \subset p(\overline{W \cap K});$$

therefore,

$$\frac{d}{2} < \operatorname{dist}(p \circ \xi_{\iota(j)+2}([-1,0]), pb_{z_1(b)+1}).$$

Combining these we have

$$\frac{d}{2} < \operatorname{dist}(|p \circ \xi_{\iota(j)+2}|, pb_{z_1(b)+1}) \tag{13.4}$$

3. Corollary 12.1.4 says that the point

$$px_{z_{2\iota(j)+1}+1} \in \left\{\chi \in L : \|\chi - pb_{z_1(b)+1}\| < \frac{d}{2}\right\}$$

on $|p \circ \xi_{\iota(j)+1}|$ belongs to $\operatorname{ext}(p \circ \xi_{\iota(j)+2})$. Using (13.4) we infer that

$$pb_{z_1(b)+1} \in \operatorname{ext}(p \circ \xi_{\iota(j)+2}),$$

as well, and this is a contradiction to part 1 of Proposition 13.1. □

In case $b \neq 0$ we set

$$\tau_b := z_2(b)+1$$

and define a simple closed smooth curve $\beta : [0, \tau_b] \longrightarrow C$ by

$$\beta(t) := b_t.$$

In this case, $b^- = b(\cdot + z_1(b)+1)$ has the same period τ_b, and for the simple closed curve $\beta^- : [0, \tau_b] \ni t \longrightarrow b_t^- \in C$,

$$|\beta^-| = |\beta|.$$

At the end of this section we shall prove assertion 3 of Theorem 5.4, i.e.,

$$(\overline{W} \setminus W) \cap p^{-1}(\operatorname{int}(p \circ \eta)) = |\beta| \quad \text{if} \quad b \neq 0,$$

$$(\overline{W} \setminus W) \cap p^{-1}(\operatorname{int}(p \circ \eta)) = \{0\} \quad \text{if} \quad b = 0,$$

in case that (11.1) holds.

The first, simple step towards this is the following.

PROPOSITION 13.3. 1. *In case* $b = 0$, $\quad pW \subset L \setminus \{0\}$.
2. *In case* $b \neq 0$, $\quad pW \subset ext(p \circ \beta)$.

PROOF. 1. The first part follows from $pW \subset S$, $\quad 0 \notin pS$.
2. Recall $|p \circ \beta| \subset \operatorname{int}(p \circ \eta)$ (part 1 of Proposition 13.1). Hence

$$|p \circ \eta| \subset \operatorname{ext}(p \circ \beta).$$

For each $\psi \in W$, a phase curve through ψ connects ψ in $W = W \setminus |\beta|$ to a point in the open neighborhood $p^{-1}(\operatorname{ext}(p \circ \beta))$ of $|\eta|$. Apply p and deduce $p\psi \in \operatorname{ext}(p \circ \beta)$. □

PROPOSITION 13.4. 1. *In case* $b \neq 0$,

$$ext(p \circ \beta) \cap int(p \circ \eta) \subset \cup_{\mathbb{N}}\, ext(p \circ \xi_j).$$

2. *In case* $b = 0$,

$$int(p \circ \eta) \setminus \{0\} \subset \cup_{\mathbb{N}}\, ext(p \circ \xi_j).$$

PROOF. 1. The case $b \neq 0$. Let $\chi \in \operatorname{ext}(p \circ \beta) \cap \operatorname{int}(p \circ \eta)$ be given. Recall (4.2) which implies

$$L = (L \setminus pK) \cup (L \setminus p(-K)).$$

1.1. The case $\chi \in L \setminus pK$.
1.1.1. $L \setminus \{\chi\}$ is an open neighborhood of $|p \circ \beta|$. Corollary 13.3 implies that there exists $j_0 \in \mathbb{N}$ such that for all integers $j \geq j_0$,

$$p \circ \xi_{\iota(j)+1}([0,1]) \subset L \setminus \{\chi\}.$$

We introduce homotopies

$$\hom_j : [0,1] \times [-1,1] \longrightarrow L, \quad j \in \mathbb{N} \quad \text{and} \quad j \geq j_0$$

which deform $p \circ \xi_{\iota(j)+1}([-1,0])$ in the convex set pK into the line segment from $p\xi_{\iota(j)+1}(-1)$ to $p\xi_{\iota(j)+1}(0)$:

$$\mathrm{hom}_j(s,t) := s \cdot (-t \cdot p\xi_{\iota(j)+1}(-1) + (1+t) \cdot p\xi_{\iota(j)+1}(0)) + (1-s) \cdot p\xi_{\iota(j)+1}(t)$$

$$\text{for} \quad t \leq 0,$$

$$\mathrm{hom}_j(s,t) := p\xi_{\iota(j)+1}(t) \quad \text{for} \quad 0 \leq t.$$

Observe that hom_j is continuous, and that each $\mathrm{hom}_j(s,\cdot)$ is a closed curve which is piecewise smooth. We have

$$\chi \notin \mathrm{hom}_j([0,1] \times [-1,1])$$

since

$$\mathrm{hom}_j([0,1] \times [-1,0]) \subset pK \subset L \setminus \{0\}, \quad \mathrm{hom}_j([0,1] \times [0,1]) \subset L \setminus \{\chi\}.$$

It follows that the winding numbers of $p \circ \xi_{\iota(j)+1} = \mathrm{hom}_j(0,\cdot)$ and $\mathrm{hom}_j(1,\cdot)$ with respect to χ coincide.

1.1.2. The advantage of $\mathrm{hom}_j(1,\cdot)$ over $p \circ \xi_{\iota(j)+1}$ is that the complex-valued integrands of the Riemann integrals over $[-1,1]$ which define

$$\mathrm{wind}(\chi, \mathrm{hom}_j(1,\cdot))$$

converge uniformly to the integrand of the integral which defines the winding number

$$\mathrm{wind}(\chi, \overline{p \circ \beta})$$

where the curve

$$\overline{p \circ \beta} : [-1,1] \longrightarrow L$$

is given by

$$\overline{p \circ \beta}(t) = pb_0 \quad \text{for} \quad t \leq 0, \quad \overline{p \circ \beta}(t) = p\beta(t \cdot \tau_b) \quad \text{for} \quad 0 \leq t.$$

This uniform convergence is a consequence of the following simple facts. First,

$$\xi_{\iota(j)+1}(0) = x_{z_{2\iota(j)}} + 1 \quad \text{and} \quad \xi_{\iota(j)+1}(-1) = \xi_{\iota(j)+1}(1) = x_{z_{2\iota(j)+2}} + 1$$

both converge to $b_0 = b_{z_2(b)} + 1$ as $j \longrightarrow \infty$. This implies that uniformly for $t \in [-1,0]$,

$$\mathrm{hom}_j(1,t) = -t \cdot p\xi_{\iota(j)+1}(-1) + (1+t) \cdot p\xi_{\iota(j)+1}(0) \longrightarrow pb_0 = \overline{p \circ \beta}(t)$$

as $j \longrightarrow \infty$, and

$$D_2\mathrm{hom}_j(1,t) = p\xi_{\iota(j)+1}(0) - p\xi_{\iota(j)+1}(-1) \longrightarrow 0 = D\overline{p \circ \beta}(t)1$$

as $j \longrightarrow \infty$.

Secondly,

$$z_{2\iota(j)+2} - z_{2\iota(j)} \longrightarrow z_2(b) + 1 \quad \text{and} \quad x_{z_{2\iota(j)}} + 1 \longrightarrow b_0$$

and uniform continuity of $p \circ F$ on the compact set

$$[0, z_2(b) + 2] \times (\{b_0\} \cup \{x_{z_{2\iota(j)}} + 1 : j \in \mathbb{N}\})$$

imply that uniformly for $t \in [0,1]$,

$$\mathrm{hom}_j(1,t) = p\xi_{\iota(j)+1}(t) = pF(t \cdot (z_{2\iota(j)+2} - z_{2\iota(j)}), x_{z_{2\iota(j)}} + 1)$$

$$\longrightarrow pF(t \cdot (z_2(b) + 1), b_0) = \overline{p \circ \beta}(t) \quad \text{as} \quad j \longrightarrow \infty.$$

Finally, observe that for $0 < t < 1$,

$$D_2\mathrm{hom}_j(1,t) = pD\xi_{\iota(j)+1}(t)1$$

$$= p((z_{2\iota(j)+2} - z_{2\iota(j)}) \cdot (x')_{z_{2\iota(j)}} + 1 + t \cdot (z_{2\iota(j)+2} - z_{2\iota(j)})$$

and

$$D_2\overline{p \circ \beta}(t)1 = pD\beta(t)1 = p((z_2(b) + 1) \cdot (b')_{t \cdot (z_2(b)+1)});$$

eq. (1.1) yields

$$(x')_{\ldots} = \mu x_{\ldots} + f \circ x_{\ldots - 1}$$

$$= \mu \cdot F(t \cdot (z_{2\iota(j)+2} - z_{2\iota(j)}), x_{z_{2\iota(j)}} + 1) + f \circ F(t \cdot (z_{2\iota(j)+2} - z_{2\iota(j)}), x_{z_{2\iota(j)}})$$

and analogously

$$(b')_{t \cdot (z_2(b)+1)} = \mu \cdot F(t \cdot (z_2(b) + 1), b_0) + f \circ F(t \cdot (z_2(b) + 1), b_{-1}).$$

Using uniform continuity as above, and in addition

$$x_{z_{2\iota(j)}} \longrightarrow b_{-1} \quad \text{as} \quad j \longrightarrow \infty,$$

we conclude that uniformly for $t \in (0,1)$,

$$D_2\mathrm{hom}_j(1,t)1 \longrightarrow D\overline{p \circ \beta}(t)1 \quad \text{as} \quad j \longrightarrow \infty.$$

1.1.3. The convergence of integrands obtained in part 1.1.2, and the fact that winding numbers are integers, yield finally that for j sufficiently large

$$0 = \mathrm{wind}(\chi, p \circ \beta) \qquad \text{(by hypothesis)}$$

$$= \mathrm{wind}(\chi, \overline{p \circ \beta}) = \mathrm{wind}(\chi, \mathrm{hom}_j(1,\cdot)) = \mathrm{wind}(\chi, p \circ \xi_{\iota(j)+1}),$$

i.e.,

$$\chi \in \mathrm{ext}(p \circ \xi_{\iota(j)+1}).$$

1.2. The case $\chi \in L \setminus p(-K)$. We have

$$\chi \in \mathrm{ext}(p \circ \beta) = \mathrm{ext}(p \circ \beta^-)$$

since $|\beta| = |\beta^-|$. We work with the curves $\xi^-_{\iota(j)+1}$, instead of $\xi_{\iota(j)+1}$, argue as in case 1.1 and find, for some $j \in \mathbb{N}$,

$$\chi \in \mathrm{ext}(p \circ \xi^-_{\iota(j)+1}).$$

Corollary 13.1 yields

$$\chi \in \mathrm{ext}(p \circ \xi_k) \quad \text{for integers} \quad k > \iota(j) + 2.$$

2. The case $b=0$. Let $\chi \in \operatorname{int}(p\circ\eta)\setminus\{0\}$ be given. Choose a convex neighborhood N_L of 0 in L such that $\chi \notin N_L$.
2.1. The case $\chi \in L\setminus pK$. Corollary 13.2 implies that for $j \in \mathbb{N}$ sufficiently large,

$$p\xi_{\iota(j)+1}([0,1]) \subset N_L.$$

Also,

$$p\xi_{\iota(j)+1}([-1,0]) \subset pK \subset L\setminus\{\chi\}.$$

We infer

$$\chi \notin |p\circ\xi_{\iota(j)+1}|, \quad |p\circ\xi_{\iota(j)+1}| \subset N_L \cup pK.$$

Now, pK is a convex cone with $0 \in \overline{pK}$. Observe that $\chi \in L\setminus(N_L\cup pK)$ can be radially connected in $L\setminus(N_L\cup pK)$, i.e. without crossing $|p\circ\xi_{\iota(j)+1}|$, to points with arbitrarily large modulus. This implies $\chi \in \operatorname{ext}(p\circ\xi_{\iota(j)+1})$.
2.2. The case $\chi \in L\setminus p(-K)$. We consider the curves $\xi^-_{\iota(j)+1}$, argue as before and find

$$\chi \in \operatorname{ext}(p\circ\xi^-_{\iota(j)+1}) \quad \text{for some} \quad j \in \mathbb{N}.$$

Corollary 13.1 yields

$$\chi \in \operatorname{ext}(p\circ\xi_k) \quad \text{for integers} \quad k > \iota(j)+2. \quad \square$$

PROOF OF ASSERTION 3 OF THEOREM 5.4 IN CASE THAT (11.1) HOLDS. 1. Using Proposition 13.4 and Proposition 12.5 we infer that in case $b \neq 0$,

$$\operatorname{ext}(p\circ\beta)\cap\operatorname{int}(p\circ\eta) \subset pW\cap\operatorname{int}(p\circ\eta)$$

while for $b=0$,

$$\operatorname{int}(p\circ\eta)\setminus\{0\} \subset pW\cap\operatorname{int}(p\circ\eta).$$

Proposition 13.3 now implies

$$pW\cap\operatorname{int}(p\circ\eta) = \operatorname{ext}(p\circ\beta)\cap\operatorname{int}(p\circ\eta) \quad \text{if} \quad b\neq 0,$$

$$pW\cap\operatorname{int}(p\circ\eta) = \operatorname{int}(p\circ\eta)\setminus\{0\} \quad \text{if} \quad b=0.$$

2. Part 1 of Proposition 13.1 gives

$$|\beta| \subset (\overline{W}\setminus W)\cap p^{-1}(\operatorname{int}(p\circ\eta)) \quad \text{if} \quad b\neq 0,$$

$$0 \in (\overline{W}\setminus W)\cap p^{-1}(\operatorname{int}(p\circ\eta)) \quad \text{if} \quad b=0.$$

It remains to show

$$(\overline{W}\setminus W)\cap p^{-1}(\operatorname{int}(p\circ\eta)) \subset |\beta| \quad \text{if} \quad b\neq 0,$$

$$(\overline{W}\setminus W)\cap p^{-1}(\operatorname{int}(p\circ\eta)) \subset \{0\} \quad \text{if} \quad b=0.$$

3. The case $b\neq 0$. By Proposition 13.3, $pW \subset \operatorname{ext}(p\circ\beta)$. Hence

$$p\overline{W} \subset \overline{pW} \subset \overline{\operatorname{ext}(p\circ\beta)} = |p\circ\beta|\cup\operatorname{ext}(p\circ\beta),$$

and therefore

$$\begin{aligned} p(\overline{W}\setminus W)\cap\operatorname{int}(p\circ\eta) &= (p\overline{W}\setminus pW)\cap\operatorname{int}(p\circ\eta) \\ &\subset ((|p\circ\beta|\cup\operatorname{ext}(p\circ\beta))\setminus pW)\cap\operatorname{int}(p\circ\eta) \\ &\subset (|p\circ\beta|\cup[\operatorname{ext}(p\circ\beta)\cap\operatorname{int}(p\circ\eta)])\setminus pW \\ &\subset |p\circ\beta| \qquad \text{(see part 1 above).} \end{aligned}$$

It follows that

$$(\overline{W}\setminus W)\cap p^{-1}(\operatorname{int}(p\circ\eta)) \subset |\beta|.$$

4. The case $b=0$. Then

$$\begin{aligned} p(\overline{W}\setminus W)\cap\operatorname{int}(p\circ\eta) &= (p\overline{W}\setminus pW)\cap\operatorname{int}(p\circ\eta) \\ &\subset \operatorname{int}(p\circ\eta)\setminus pW \subset (\{0\}\cup pW)\setminus pW \qquad \text{(see part 1)} \\ &= \{0\}; \end{aligned}$$

it follows that

$$(\overline{W}\setminus W)\cap p^{-1}(\operatorname{int}(p\circ\eta)) \subset \{0\}. \quad \square$$

14. The Case of Eventually Monotone Solutions

In this section we assume that (11.2) holds, i.e. there exists a solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) with phase curve in W and

$$J(x) < \infty. \tag{14.1}$$

We shall prove that in this case

$$\{0\} = (\overline{W}\setminus W)\cap p^{-1}(\operatorname{int}(p\circ\eta)). \tag{14.2}$$

Together with the result of the preceding section this will complete the proof of part 3 of Theorem 5.4.

The first observations are the following. The stationary point $0 \in C$ of the semiflow F is stable and attractive (Remark 4.1, parts 3 and 2). We have

$$x_t \notin |\eta| \quad \text{for all} \quad t \in \mathbb{R} \tag{14.3}$$

since otherwise x would be periodic and we would have a contradiction to existence and boundedness of zeros.

A remark in Section 3 yields

$$x(t) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty.$$

Hence

$$\{0\} = \omega(x).$$

By Theorem 5.1, $0 \in \operatorname{int}(p\circ\eta)$; the open neighborhood $\operatorname{int}(p\circ\eta)$ of 0 contains points px_t. Using (14.3) we infer

$$px_t \in \operatorname{int}(p\circ\eta) \quad \text{for all} \quad t \in \mathbb{R}.$$

The latter implies that for some integer $j \leq J(x)$,

$$x_{z_j}+1 \in h((-\underline{a},0)).$$

Relabeling the zeros $z_k = z_k(x)$ of x if necessary we achieve

$$2 \leq J(x), \quad z_0 = -1, \quad x_0 = h(a) \quad \text{for some} \quad a \in (-\underline{a}, 0).$$

Now we are in the situation of Corollary 12.1, and Corollary 12.2 yields

$$\omega(x) \subset \overline{W} \setminus W.$$

Recall from Section 12 the simple closed curve $\xi_0 = \xi_{0,a}$ associated with a, and the homotopy $M = M_{0,a}$ of closed curves which deforms a reparameterization $M(a, \cdot)$ of ξ_0 in W into the reparameterization $M(0, \cdot)$ of η:

$$M(a,t) = \xi_0(2t-1) \quad \text{and} \quad M(0,t) = \eta(t \cdot \tau) \quad \text{for} \quad 0 \leq t \leq 1.$$

The curves

$$p \circ F(j\tau, \cdot) \circ \xi_0, \quad j \in \mathbb{N},$$

are simple, closed and piecewise smooth.

PROPOSITION 14.1. *For every* $j \in \mathbb{N}$,

$$p(\overline{W} \setminus W) \cap int(p \circ \eta) \subset int(p \circ F(j\tau, \cdot) \circ \xi_0).$$

PROOF. Let $\chi \in p(\overline{W} \setminus W) \cap \mathrm{int}(p \circ \eta)$. In particular, $\chi \notin pW$, and therefore

$$\chi \notin p \circ F(j\tau, \cdot) \circ M([a,0] \times [0,1]).$$

It follows that

$$0 \neq \mathrm{wind}(\chi, p \circ \eta) = \mathrm{wind}(\chi, p \circ F(j\tau, \cdot) \circ \eta) \quad \text{(by periodicity of} \quad \eta),$$

or

$$\chi \in \mathrm{int}(p \circ F(j\tau, \cdot) \circ \eta) = \mathrm{int}(p \circ F(j\tau, \cdot) \circ M(0, \cdot))$$

(with $|\eta| = |M(0, \cdot)|$). By homotopy invariance,

$$0 \neq \mathrm{wind}(\chi, p \circ F(j\tau, \cdot) \circ M(0, \cdot)) = \mathrm{wind}(p \circ F(j\tau, \cdot) \circ M(a, \cdot)),$$

hence

$$\chi \in \mathrm{int}(p \circ F(j\tau, \cdot) \circ M(a, \cdot)) = \mathrm{int}(p \circ F(j\tau, \cdot) \circ \xi_0)$$

(with $|\xi_0| = |M(a, \cdot)|$). ☐

PROPOSITION 14.2. *For every solution* $x^* : \mathbb{R} \longrightarrow \mathbb{R}$ *of eq.* (1.1) *with phase curve in* $W \cap p^{-1}(int(p \circ \eta))$,

$$x^*(t) \longrightarrow 0 \quad as \quad t \longrightarrow \infty.$$

PROOF. Suppose there exists a solution $x^* : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) with phase curve in $W \cap p^{-1}(\mathrm{int}(p \circ \eta))$ so that

$$0 < \limsup_{t \longrightarrow \infty} |x^*(t)|.$$

Then $J(x^*) = \infty$ (see Section 3), and

$$x^*_t \notin |\eta| \quad \text{for all} \quad t \in \mathbb{R}.$$

Let $(z^*_j)^{\infty}_{-\infty}$ denote the increasing sequence of zeros of x^*. We may assume

$$z^*_0 = -1 \quad \text{and} \quad x^*_0 = h(a^*) \quad \text{where} \quad \underline{a} < a^* < 0.$$

Then

$$\omega(x^*) \subset \overline{W} \setminus W \qquad \text{(Corollary 12.2)},$$

and

$$x^*_{z^*_{2j+1}+1} \in -K \quad \text{for all integers} \quad j.$$

There is a strictly increasing map $\nu : \mathbb{N} \longrightarrow \mathbb{N}$ such that the subsequence of points

$$x^*_{z^*_{2\nu(j)+1}+1}, \quad j \in \mathbb{N},$$

converges to some

$$\phi^* \in \overline{(-K)} \cap \overline{W} \cap \omega(x^*).$$

We have

$$\phi^* \neq 0$$

since otherwise, attractivity of 0 would imply $x^*_t \longrightarrow 0$ as $t \longrightarrow \infty$ (see Remark 4.1, part 2), a contradiction to our assumption. Using

$$x_t \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty, \quad x_t \longrightarrow |\eta| \subset W \quad \text{as} \quad t \longrightarrow -\infty,$$

$$x_t \in W \quad \text{for all} \quad t \in \mathbb{R}, \quad \text{and} \quad \phi^* \in \omega(x^*) \subset \overline{W} \setminus W,$$

we infer

$$0 < \mathrm{dist}(p\phi^*, \{px_t : t \in \mathbb{R}\}) =: d_x.$$

Furthermore,

$$0 < \mathrm{dist}(p\phi^*, |p \circ \eta|) =: d_\eta$$

and

$$p\phi^* \in \mathrm{int}(p \circ \eta),$$

as follows from

$$\phi^* \in \overline{W} \setminus W, \quad |\eta| \subset W; \quad p\phi^* \in p(\overline{W} \setminus W) = p\overline{W} \setminus pW, \quad |p \circ \eta| \subset pW,$$

$$p\phi^* = \lim_{j \longrightarrow \infty} px^*_{z^*_{2\nu(j)+1}+1} \in \overline{\mathrm{int}(p \circ \eta)} = (\mathrm{int}(p \circ \eta)) \cup |p \circ \eta|.$$

Set

$$d := \frac{1}{2} \cdot \min\{d_x, d_\eta\}.$$

Choose an integer j so large that for

$$j_1 := 2\nu(j) + 1 \quad \text{and} \quad j_2 := 2\nu(j+1) + 1,$$

$j_2 > j_1$, both points

$$px^*_{z^*_{j_1}+1} \quad \text{and} \quad px^*_{z^*_{j_2}+1} \quad \text{in} \quad p(-K) = -pK$$

belong to the open ball $N_L \subset L$ with center $p\phi^*$ and radius d. We define a closed curve

$$\xi^* : [-1, z^*_{j_2} - z^*_{j_1}] \longrightarrow C$$

by

$$\xi^*(t) := (-t) \cdot x^*_{z^*_{j_2}+1} + (1+t) \cdot x^*_{z^*_{j_1}+1} \quad \text{for} \quad -1 \leq t \leq 0,$$
$$\xi^*(t) := x^*_{t+z^*_{j_1}+1} \quad \text{for} \quad 0 < t \leq z^*_{j_2} - z^*_{j_1}.$$

Now, ξ^* is piecewise smooth. Observe that

$$\xi^*([-1,0]) \subset -K$$

since K is convex.

Recall from the proof of Theorem 5.1 the construction of the homotopy of closed curves in S which connects a reparameterization of the curve $\eta : [0, z_{2n}+1] \longrightarrow C$ given by the periodic solution y to the curve $n \cdot c_r$ in $L \setminus \{0\}$;

$$\text{wind}(0, n \cdot c_r) = n.$$

Using that for $-1 \leq t \leq z^*_{j_1+1} - 1 - z^*_{j_1}$,

$$\xi^*(t) \in \{\phi \in C : \phi < 0 \quad \text{in} \quad (-1,0)\}$$

(this set corresponds to $-K_{k0}$ in the proof of Theorem 5.1) we obtain, by an obvious modification of the construction in the proof of Theorem 5.1, a homotopy of closed curves in S from a reparameterization of ξ^* to the closed curve

$$n \cdot (-c_r) \quad \text{where} \quad n := \frac{1}{2} \cdot (j_2 - j_1) \geq 1.$$

Applying the projection p we arrive at a homotopy in $L \setminus \{0\}$ from a reparameterization of $p \circ \xi^*$ to the curve $n \cdot (-c_r)$ in L. It follows that

$$\text{wind}(0, p \circ \xi^*) = n \cdot \text{wind}(0, -c_r) = n \neq 0;$$

$$0 \in \text{int}(p \circ \xi^*);$$

$\text{int}(p \circ \xi^*)$ is an open neighborhood of 0 in L. Consequently, for $t > 0$ sufficiently large,

$$px_t \in \text{int}(p \circ \xi^*). \tag{14.4}$$

We have

$$|p \circ \xi^*| \subset \text{int}(p \circ \eta)$$

since $px^*_t \in \text{int}(p \circ \eta)$ for all $t \in \mathbb{R}$, and $p \circ \xi^*([-1,0])$ is the line segment from

$$px^*_{z^*_{j_2}+1} \quad \text{to} \quad px^*_{z^*_{j_1}+1}$$

in the convex set N_L which is disjoint with $|p \circ \eta|$. This implies

$$|p \circ \eta| \subset \text{ext}(p \circ \xi^*).$$

As $x_{-t} \longrightarrow |\eta|$ as $t \longrightarrow \infty$,

$$px_{-t} \in \text{ext}(p \circ \xi^*) \tag{14.5}$$

for $t > 0$ sufficiently large. From (14.4) and (14.5) we obtain that for some $t \in \mathbb{R}$

$$px_t \in |p \circ \xi^*|; \quad x_t \in |\xi^*|,$$

which leads to

$$0 \neq \phi^* \in \omega(x);$$

a contradiction. □

Proof of (14.2). 1. Theorem 5.1 and

$$\{0\} = \omega(x) \subset \overline{W} \setminus W$$

give

$$0 \in (\overline{W} \setminus W) \cap p^{-1}(\text{int}(p \circ \eta)).$$

2. It remains to show that, given an open ball N_L in L with center 0, we have

$$p(\overline{W} \setminus W) \cap \text{int}(p \circ \eta) \subset N_L.$$

The stationary point $0 \in C$ is attractive in the sense of Remark 4.1, part 2:newline There exist positive ϵ, δ, c; $\epsilon \leq c$, so that for all $\phi \in C$ with $||\phi|| < \epsilon$ and for all $t \geq 0$,

$$||F(t,\phi)|| \leq c \cdot e^{-\delta t}.$$

For every $\psi \in |\xi_0|$ there are $t = t(\psi) > 0$ and a neighborhood U_ψ of ψ in C with

$$||F(t,\psi^*)|| < \epsilon \quad \text{for all} \quad \psi^* \in U_\psi.$$

It follows that for $\psi^* \in U_\psi$ and for all $t > t(\psi) + \frac{1}{\delta}\log\left(\frac{c}{\epsilon}\right)$,

$$||F(t,\psi^*)|| < \epsilon.$$

The compact set $|\xi_0|$ is covered by a finite collection $U_{\psi_1}, \ldots, U_{\psi_n}$ of such neighborhoods. Consequently,

$$\text{for} \quad t_0 := \max_{1,\ldots,n} t(\psi_n) \quad + \frac{1}{\delta}\log\left(\frac{c}{\epsilon}\right) \quad + 1 \quad \text{and for each} \quad \psi \in |\xi_0|,$$

$$||F(t_0,\psi)|| < \epsilon.$$

For $t \geq t_0$ and $\psi \in |\xi_0|$,

$$||F(t,\psi)|| \leq c \cdot e^{-\delta(t-t_0)}.$$

We infer that for $n \in \mathbb{N}$ sufficiently large,

$$|F(n\tau,\cdot) \circ \xi_0| \subset p^{-1}(N_L);$$

$$|p \circ F(n\tau,\cdot) \circ \xi_0| \subset N_L.$$

Proposition 14.1 gives

$$p(\overline{W} \setminus W) \cap \text{int}(p \circ \eta) \subset \text{int}(p \circ F(n\tau,\cdot) \circ \xi_0) \subset N_L. \quad □$$

15. Outside the Projected Periodic Orbit

The proof of part 2 of Theorem 5.4, i.e. that

$$E := (\overline{W} \setminus W) \cap p^{-1}(\mathrm{ext}(p \circ \eta))$$

is the orbit of a slowly oscillating periodic solution of eq. (1.1), is analogous but easier than the proof of part 3, because of the following simple observations.

Proposition 15.1. $0 \notin E$.

Proof. Otherwise, $0 = p0 \in \mathrm{ext}(p \circ \eta)$, a contradiction to Theorem 5.1. ☐

Proposition 15.2. *For every solution* $x : \mathbb{R} \longrightarrow \mathbb{R}$ *of eq.* (1.1) *with phase curve in* $W \cap p^{-1}(ext(p \circ \eta))$,

$$J(x) = \infty.$$

Proof. Suppose $J(x) < \infty$ for a solution $x : \mathbb{R} \longrightarrow \mathbb{R}$ of eq. (1.1) with phase curve in $W \cap p^{-1}(\mathrm{ext}(p \circ \eta))$. Then $x(t) \longrightarrow 0$ as $t \longrightarrow \infty$ (Section 3), hence

$$\mathrm{ext}(p \circ \eta) \ni px_t \longrightarrow p0 = 0 \quad \text{as} \quad t \longrightarrow \infty;$$

$$0 \in \overline{\mathrm{ext}(p \circ \eta)} = |p \circ \eta| \cup \mathrm{ext}(p \circ \eta),$$

a contradiction to Theorem 5.1. ☐

Constructions like those in Sections 11–13 now yield the desired result. Proposition 15.2 guarantees that we do not have to deal with eventually monotone solutions (as in Section 14), and Proposition 15.1 shows that the analogue of the case $b = 0$ in Section 13 does not occur. We omit the details.

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