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The Exceptional Tits Quadrangles Revisited

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Abstract
Tits polygons are generalizations of Moufang polygons in which the neighborhood of each vertex is endowed with an "opposition relation." There is a standard construction that produces a Tits polygon from an arbitrary irreducible spherical building of rank at least 3 when paired with a suitable Tits index. In this note, we complete the proof of a characterization of the Tits quadrangles that arise in this way from the spherical building associated to an exceptional algebraic group.

Keywords Building · Tits polygon · Exceptional group

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1 Introduction

A Tits polygon is a bipartite graph in which the neighborhood of each vertex is endowed with an "opposition relation" satisfying certain axioms which reduce to the axioms of a Moufang polygon in the case that the opposition relations are all trivial. There is a standard construction (described in [3, §3]) that produces a Tits polygon from a pair (Δ, T), where Δ is an arbitrary irreducible spherical building of rank at least 3 and T is a suitable Tits index. We say that a Tits polygon is of *index type* if it arises in this way. We call a Tits polygon *exceptional* if it arises from such a pair (Δ, T) for Δ the spherical building associated to the group of rational points of an exceptional algebraic group.

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Every Tits polygon has a distinguished set of circuits. A Tits quadrangle is a Tits polygon in which these circuits all have length 8. In [3], we characterized the exceptional Tits quadrangles as extensions of orthogonal Tits quadrangles.

The crucial step in the proof of this characterization is the result [3, 8.2], in which it is shown that the commutator relations of the Tits quadrangle in question are parametrized by a quadrangular algebra (defined in [3, §7]). At the time, we were not able to prove that the algebraic system we produce satisfies the last axiom, D1, of a quadrangular algebra without the hypothesis [3, 8.2(b)] introduced just for this purpose. The goal of this note is to prove that the hypothesis [3, 8.2(b)] is, in fact, unnecessary.

Our main result is formulated in Proposition 7.1. The quadratic space Λ that appears in the hypothesis [3, 8.2(a)] is either anisotropic or isotropic. To prove Proposition 7.1 in the first case, we apply recent results in [6] involving Veldkamp quadrangles. These results are themselves applications of the classification of non-degenerate polar spaces. In the case that Λ is isotropic, we reduce the problem to a simple calculation in the subgraph opposite a fixed edge. This strategy ought, in our opinion, to have other applications.

In Section 8, we recap the main results of [3]. With these results, the classification of Tits polygons having suitable versions of the properties "sharp" and "plump" defined in [3, 2.13 and 2.19] is almost complete. The only problem remaining is to find a suitable characterization of the pseudo-quadratic Tits quadrangles defined in Definition 6.7 below. Conjecturally, all sufficiently sharp and plump Tits polygons are either of index type or are determined by algebraic data (for example, a quadratic space) that is in some sense infinite dimensional.

Conventions 1.1 Let G be a group. As in [8], we set $a^b = b^{-1}ab$ and

$$[a, b] = a^{-1}b^{-1}ab$$

for all $a, b \in G$. With these definitions, we have

(i) $[ab, c] = [a, c]^b \cdot [b, c]$ and
(ii) $[a, bc] = [a, c] \cdot [a, b]^c$.

for all $a, b, c \in G$.

2 Tits Polygons

In this section, we assemble the basic properties of Tits polygons, in addition to those listed in [3, 2.5–2.19], that we will require. Let $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ be a Tits n-gon for some $n \geq 3$, let $(\gamma, i \mapsto w_i)$ be a coordinate system of X, let $i \mapsto U_i$ be the corresponding root group labeling as defined in [3, 2.2 and 2.3], and let $G = \mathrm{Aut}(X)$.

We recall that vertices u, v both in Γ_w for some vertex w of Γ are *opposite at* w if $u \equiv_w v$. A path $(v_0, v_1, \ldots, v_k)$ is *straight* if v_{i-1} is opposite v_{i+1} at v_i for all i in the interval $[1, k-1]$ and a *root* is a straight path of length n.

Proposition 2.1 *Every straight path of length $n+1$ is contained in a unique element of $\mathcal{A}$.*

Proof This holds by [5, 1.3.11]. □

Definition 2.2 Two vertices, respectively edges, of Γ are *opposite* if they are opposite vertices, respectively edges, of some element of $\mathcal{A}$. Thus by Proposition 2.1, two vertices u, v are opposite if there is a root from u to v and two edges are opposite if and only if they are the first and last edges on a straight path of length $n+1$.

Proposition 2.3 *Let u and v be a pair of opposite vertices. Then for each $w \in \Gamma_u$, there exists a unique root $(x_0, x_1, \ldots, x_n)$ such that $x_0 = u$, $x_1 = w$ and $x_n = v$.*

Proof This holds by [5, 1.3.18]. □

Proposition 2.4 *Let $G^\dagger$ denote the subgroup of G generated by all the root groups of X and let $U_{[1,n]}$ be as in* [3, 2.5(i)]. *Then, the following hold:*

(i) *$G^\dagger$ acts transitively on the set of edges of Γ.*
(ii) *$U_{[1,n]}$ acts sharply transitive on the set of edges opposite the edge $\{w_n, w_{n+1}\}$.*
(iii) *$G^\dagger$ acts transitively on the set of all ordered pairs of opposite edges and on all pairs of opposite vertices of Γ.*

Proof By [5, 1.3.6 and 1.3.40], (i) holds. By [5, 1.3.36(i) and 1.3.37], (ii) holds. By (i) and (ii), (iii) holds as well. □

Proposition 2.5 *Let $e = \{u, v\}$ and $f = \{x, y\}$ be two edges of Γ and suppose that u is opposite y and that v is opposite x. Then, e is opposite f.*

Proof By Proposition 2.3, there exists a root $\alpha = (u, v, \ldots, y)$ from u to y containing v and a root $\beta = (v, \ldots, y, x)$ from v to x containing y. By [3, 2.2(ii)], the subpath of α from v to y coincides with the subpath of β from v to y. Hence, the $(n+1)$-path $(u, v, \ldots, y, x)$ is straight. As we observed in Definition 2.2, it follows that the edges e and f are opposite. □

Definition 2.6 We recall that X is p-plump for some $p \geq 2$ if for every vertex v and every subset S of Γ_v of cardinality at most p, there exists a vertex in Γ_v that is opposite every vertex in S at v. As we observed in [5, 1.4.22], the Tits polygon $X_{\Delta,T}$ for a pair (Δ, T) (see [3, §3] for the definition) is p-plump whenever the field of definition of the building Δ contains at least p elements.

Proposition 2.7 *Suppose that X is p-plump for some $p \geq 2$. Then for each $k \in [1, 2p-2]$ and for each k-path $(x_0, \ldots, x_k)$, there exists an edge that is opposite the edge $\{x_{i-1}, x_i\}$ for all $i \in [1, k]$.*

Proof We proceed by induction with respect to k. By Proposition 2.4(i), the claim holds for $k = 1$. Suppose that $(x_0, \dots, x_k)$ is a k-path in Γ for some $k \in [2, 2p-2]$ and that f is an edge that is opposite $c_i := \{x_{i-1}, x_i\}$ for all $i \in [1, k-1]$. Let y be the unique vertex of f that is opposite x_{k-1}, let $v_i = x_{k-1-2i}$ for all $i \in [0, m]$, where m is the largest integer such that $k - 1 - 2m \geq 0$, and let $z_i = x_{k-2i}$ for all $i \geq 0$ such that $k - 2i \geq 0$. Thus for each $j \in [1, k]$, there is a unique $i \in [0, m]$ such that c_j is either $\{v_i, z_i\}$ or $\{v_i, z_{i+1}\}$.

Since f is opposite c_i for all $i \in [1, k-1]$, the vertex y is opposite v_i for all $i \in [0, m]$. By Proposition 2.3, there exists for each $i \in [0, m]$ a root $\alpha_i = (v_i, z_i, \dots, y)$ from v_i to y containing z_i and if k is even, there exists a root $\beta = (v_m, z_{m+1}, \dots, y)$ from v_m to y containing z_{m+1}. Let d_i be the unique vertex in Γ_y on α_i for each $i \in [0, m]$ and if k is even, let b be the unique vertex in Γ_y on β. We have $m \leq (k-1)/2$ and thus $m \leq p - 2$. Since X is p-plump, it follows that we can choose a vertex $e \in \Gamma_y$ that is opposite d_i at y for all $i \in [0, m]$ and, if k is even, also opposite b at y. Since the $(n+1)$-path (α_i, e) is straight, e is opposite z_i for all $i \in [0, m]$. If k is even, then the $(n+1)$-path (β, e) is straight, so e is also opposite z_{m+1}. By Proposition 2.5, it follows that f is opposite c_i for all $i \in [0, m]$. □

The following result refers to vertices w_1, w_2, etc., on γ in the coordinate system $(\gamma, i \mapsto w_i)$ that we have chosen. Note, in particular, that $\{w_1, w_{2n}\}$ and $\{w_n, w_{n+1}\}$ are opposite edges of Γ. Note, too, that by [3, 2.3], U_i fixes $\{w_n, w_{n+1}\}$ for all $i \in [1, n]$.

Proposition 2.8 *Let $u \in \Gamma_{w_1} \backslash \{w_2\}$ and $v \in \Gamma_{w_{2n}} \backslash \{w_{2n-1}\}$ and suppose that the edges $\{u, w_1\}$ and $\{v, w_{2n}\}$ are both opposite $\{w_n, w_{n+1}\}$. Then, the following hold:*

- (i) *There exist unique $a_1 \in U_1$ and $a_n \in U_n$ such that $u = w_{2n}^{a_1}$ and $v = w_1^{a_n}$.*
- (ii) *The 3-tuple (w_{2n}, w_1, u) is a path if and only if $a_1 \neq 1$ and it is a straight path if and only if $a_1 \in U_1^\sharp$. The 3-tuple (w_1, w_{2n}, v) is a path if and only if $a_n \neq 1$ and it is a straight path if and only if $a_n \in U_n^\sharp$.*

Proof The vertices u and w_n are opposite. By Proposition 2.3, therefore there exists a root

$$\alpha = (u, w_1, \dots, w_n)$$

from u to w_n containing w_1. By [3, 2.2(ii)], $\alpha = (u, w_1, w_2, \dots, w_{n-1}, w_n)$. In particular, u is opposite w_2 at w_1. By [3, 2.7], there exists a unique element $a_1 \in U_1$ mapping w_{2n} to u. Thus (w_{2n}, w_1, u) is a path if and only if $a_1 \neq 1$ and by [3, 2.8], $a_1 \in U_1^\sharp$ if and only if (w_{2n}, w_1, u) is a straight path. Thus (i) and (ii) hold for u. The remaining claims hold by a similar argument. □

Proposition 2.9 *The stabilizer of w_{2n} in $U_{[1,n]}$ is U_n and the stabilizer of w_1 in $U_{[1,n]}$ is U_1.*

Proof Let $u = w_1^g$ for some $g \in U_{[1,n]} \cap G_{w_{2n}}$. Then, $\{u, w_{2n}\}$ is an edge opposite $\{w_n, w_{n+1}\}$. By Proposition 2.8(i), therefore there exists $a_n \in U_n$ such that ga_n^{-1}

stabilizes $\{w_1, w_{2n}\}$. Hence, $g = a_n \in U_n$ by Proposition 2.4(ii). It follows that $U_{[1,n]} \cap G_{w_{2n}} \subset U_n$. By [3, 2.3], on the other hand, U_n fixes w_{2n}. Hence, the first claim holds. The second claim holds by a similar argument. □

Proposition 2.10 *Suppose that X is 3-plump and let $\delta = (x, y, z)$ be a straight 2-path. Then, δ is the unique 2-path from x to z.*

Proof Let $\xi = (x, u, z)$ be an arbitrary 2-path from x to z. By Propositions 2.4(iii) and 2.7, we can assume that the coordinate system $(\gamma, i \mapsto w_i)$ is chosen so that $x = w_1$ and $y = w_{2n}$ and that the edges on δ and the edges on ξ are all opposite $f := \{w_n, w_{n+1}\}$. By Proposition 2.8, therefore there exist $a_1 \in U_1$ and $a_n \in U_n^\sharp$ such that $w_{2n} = u^{a_1}$ and $z = w_1^{a_n}$. The element a_1 fixes f and $\{z^{a_1}, w_{2n}\} = \{z, u\}^{a_1}$. Hence, $\{z^{a_1}, w_{2n}\}$ is an edge opposite f. Thus by one more application of Proposition 2.8, there exists $b_n \in U_n$ such that $z^{a_1} = w_1^{b_n}$. Therefore,

$$w_1^{a_n} = z = w_1^{b_n a_1^{-1}},$$

so by Proposition 2.9, $a_n a_1 b_n^{-1} \in U_1$. By Conventions 1.1 and [3, 2.5(i)], we have $a_n a_1 b_n^{-1} = a_1 \cdot [a_1, a_n^{-1}] \cdot a_n b_n^{-1}$. Hence, $[a_1, a_n^{-1}] = 1$ by [3, 2.5(ii)]. By [3, 2.16(ii)], it follows that $a_1 = 1$ (since $a_n \in U_n^\sharp$). Hence, $u = w_{2n} = y$. □

3 Tits Quadrangles

We continue with X, n, $(\gamma, i \mapsto w_i)$, $i \mapsto U_i$ and $G = \mathrm{Aut}(X)$ as in the previous section, but now we assume that $n = 4$ and that X is sharp as defined in [3, 2.13].

Notation 3.1 As in [3, 4.4 and 4.6], we let Y_i denote the pointwise stabilizer in G of the set of vertices of Γ at distance at most 2 from w_{i+2}. The group Y_i is a normal subgroup of U_i and it is normalized by the pointwise stabilizer G_γ for all i. Since X is sharp, it follows that for each i, either $Y_i = 1$ or $Y_i^\sharp := Y_i \cap U_i^\sharp \neq \emptyset$.

Proposition 3.2 *Suppose that X is* 3*-plump and let $\delta = (v, x, y, z)$ be a straight* 3*-path. Then δ is the unique path of length at most 3 from v to z.*

Proof By Proposition 2.10, the distance from v to z is 3. It will suffice, therefore to show that δ is the unique path of length 3 from v to z. By [3, 4.8], we can assume that $Y_1 \neq 1$. Thus $Y_1^\sharp \neq \emptyset$ by Notation 3.1. Choose $a \in Y_1^\sharp$. Then, (w_0, w_1, w_0^a) is a straight 2-path. By Proposition 2.4(i) and [3, 2.7], every straight path of length 3 is in the same G-orbit as (w_0, w_1, w_2, w_3) or (w_3, w_2, w_1, w_0). Replacing δ by (z, y, x, v) if necessary, we can assume, therefore that $\delta = (w_0, w_1, w_2, w_3)$. Now let $\xi = (v, x', y', z)$ be an arbitrary 3-path from v to z. Then, x' is at distance 2 from $z = w_3$ and $a \in Y_1$, so a fixes x'. Thus $\{x', w_0^a\} = \{x', v\}^a$ is an edge and hence

(w_0, x', w_0^a) is a path of length 2 from w_0 to w_0^a. Therefore, $x' = w_1 = x$ by Proposition 2.10 applied to (w_0, w_1, w_0^a). By Proposition 2.10 applied to (x, y, z), it follows that $y' = w_2 = y$ and thus $\xi = \delta$. □

Proposition 3.3 *Suppose that X is 3-plump, let v and z be opposite vertices and let $y \in \Gamma_z$. Then, there exists a root α from v to z containing y and α is the unique path of length 4 from v to z containing y.*

Proof The first claim holds by Proposition 2.3 and the second by Proposition 3.2. □

Proposition 3.4 *Suppose that X is 3-plump and let u, v be opposite vertices. Then, the distance from u to v in Γ is* 4.

Proof It suffices to observe that by [3, 2.2(ii)], $u \neq v$ and that by Proposition 3.2, there is no vertex z such that $u, v \in \Gamma_z$. □

Proposition 3.5 *Suppose that X is 3-plump. Let $a_i, b_i, c_i, d_i \in U_i$ for $i = 1$ and 4 and suppose that*

$$a_4a_1b_4 = d_1d_4c_1c_4b_1 \tag{3.6}$$

Suppose, too, that $a_1 \in U_1^\sharp$ and $a_4, b_4 \in U_4^\sharp$. Then, also $c_1 \in U_1^\sharp$ and $c_4, d_4 \in U_4^\sharp$.

Proof Let $v = w_1^{a_4a_1b_4b_1^{-1}}$ and let

$$\alpha = (w_1, w_8^{b_1^{-1}}, w_1^{b_4b_1^{-1}}, w_8^{a_1b_4b_1^{-1}}, v).$$

Since $a_1 \in U_1^\sharp$ and $a_4, b_4 \in U_4^\sharp$, α is a straight path of length 4 from w_1 to v by Proposition 2.8(ii). Therefore, w_1 and v are opposite. By Eq. 3.6, we have

$$v = w_1^{d_1d_4c_1c_4} = w_1^{d_4c_1c_4}.$$

Let

$$\beta = (w_1, w_8, w_1^{c_4}, w_8^{c_1c_4}, v).$$

By Propositions 2.8(ii) and 3.4, β is also a path of length 4 from w_1 to v, or equivalently, c_4, c_1, and d_4 are all non-trivial. By Proposition 3.3, β is also straight. By Proposition 2.8(ii), therefore $c_4, d_4 \in U_4^\sharp$ and $c_1 \in U_1^\sharp$. □

Proposition 3.7 *Let $a_i, b_i, c_i, d_i \in U_i$ for $i = 1$ and 4 and suppose that $a_1 = d_1c_1b_1$ and $a_4b_4 = d_4c_4$. Then,* Eq. 3.6 *holds if and only if*

$$\begin{aligned}[a_1, a_4^{-1}] = [b_1, [c_1, d_4^{-1}]_3^{-1}] \cdot [c_1,\ d_4^{-1}] \cdot [b_1, d_4^{-1}]\\ \cdot [b_1, c_4^{-1}] \cdot [[b_1, c_4^{-1}]_2, d_4^{-1}],\end{aligned} \tag{3.8}$$

where $[c_1, d_4^{-1}]_3$ and $[b_1, c_4^{-1}]_2$ are as in [3, 2.6].

Proof By [3, 2.5(ii)], it suffices to observe that

$$a_4a_1b_4 = a_1 \cdot [a_1, a_4^{-1}] \cdot a_4b_4$$

and

$$\begin{aligned}
d_1d_4c_1c_4b_1 &= d_1d_4c_1b_1 \cdot [b_1, c_4^{-1}] \cdot c_4 \\
&= d_1c_1b_1 \cdot [c_1b_1, d_4^{-1}] \cdot d_4 \cdot [b_1, c_4^{-1}] \cdot c_4 \\
&= d_1c_1b_1 \cdot [c_1, d_4^{-1}]^{b_1} \cdot [b_1, d_4^{-1}] \cdot [b_1, c_4^{-1}] \cdot [[b_1, c_4^{-1}]_2, d_4^{-1}] \cdot d_4c_4 \\
&= d_1c_1b_1 \cdot [b_1, [c_1, d_4^{-1}]_3^{-1}] \cdot [c_1, d_4^{-1}] \cdot [b_1, d_4^{-1}] \\
&\qquad \cdot [b_1, c_4^{-1}] \cdot [[b_1, c_4^{-1}]_2, d_4^{-1}] \cdot d_4c_4
\end{aligned}$$

by Conventions 1.1(ii) and [3, 2.5(i)]. □

4 Linear Forms

We pause now to prove two elementary results about linear forms that we will require in the next section.

Notation 4.1 Let K be a field, let V be a vector space over K and let $f, g\colon V \to K$ be non-zero linear forms on V. Then, f and g are *proportional* if each is a scalar multiple of the other.

Proposition 4.2 *Let $c \in K$ and let f, g, h be linear forms on $V := K \oplus K$ such that f and g are non-zero and not proportional to each other. Then, there is an injection from K^* to the set $\{v \in V \mid f(v)g(v) + h(v) = c\}$.*

Proof Since f and g are non-zero and not proportional to each other, there exist $a, b \in K$ and $u \in V$ such that $h = af + bg$, $f(u) = -b$ and $g(u) = -a$. Let

$$q(v) = f(v+u)g(v+u) + h(v+u) - c$$

for all $v \in V$ and let $d = c - f(u)g(u) - h(u)$. Then,

$$\begin{aligned}
q(v) &= f(v)g(v) + f(v)g(u) + f(u)g(v) + f(u)g(u) + h(v) + h(u) - c \\
&= f(v)g(v) - af(v) - bg(v) + h(v) - d \\
&= f(v)g(v) - d
\end{aligned}$$

for all $v \in V$. For each $e \in K^*$, there is a unique $v_e \in V$ such that $f(v_e) = e$ and $g(v_e) = de^{-1}$. We have $q(v_e) = 0$ for each $e \in K^*$ and hence $e \mapsto v_e + u$ is an injection from K^* to the set $\{v \in V \mid f(v)g(v) + h(v) = c\}$. (This map is a bijection if $d \neq 0$. If $d = 0$, there are also solutions $w + u$ for $w \in V$ such that $f(w) = 0$ and $g(w)$ is arbitrary.) □

Proposition 4.3 *Let $c, d \in K$ and let f, g, h, j be linear forms on $V := K \oplus K$ such that f and g are non-zero and not proportional to each other and j is non-zero. Suppose that*

$$|\Omega| \geq 3, \tag{4.4}$$

where $\Omega := \{v \mid f(v)g(v) + h(v) = c$ *and* $j(v) = d\}$. *Then,* j *is proportional to* f *or* g.

Proof There exists $z \in V$ such that $j(z) = d$. Replacing the variable v by $v - z$, we can assume that $d = 0$. Suppose that u, v are arbitrary distinct non-zero elements of Ω. Since $j \neq 0$ and $d = 0$, we have $u = \lambda v$ for some $\lambda \in K\backslash\{0, 1\}$. Hence,

$$f(v)g(v) + h(v) = c = f(u)g(u) + h(u) = \lambda^2 f(v)g(v) + \lambda h(v)$$

and thus $(\lambda^2 - 1)f(v)g(v) + (\lambda - 1)h(v) = 0$. Since $\lambda \neq 1$, it follows that

$$(\lambda + 1)f(v)g(v) + h(v) = 0. \tag{4.5}$$

Therefore,

$$\lambda f(v)g(v) + c = 0. \tag{4.6}$$

Suppose $c \neq 0$. Then, $0 \notin \Omega$ and by Eq. 4.6, $f(v)g(v) \neq 0$. Hence by Eq. 4.5, λ is uniquely determined by $f(v)$, $g(v)$, and $h(v)$. Since u and v are arbitrary, it follows that there are no other non-zero elements in Ω. By Eq. 4.4, we conclude that $c = 0$. Thus by Eq. 4.6, $\lambda f(v)g(v) = 0$. Since $\lambda \neq 0$, we have $f(v)g(v) = 0$ and thus $f(v) = 0$ or $g(v) = 0$. Since $j(v) = 0$ and $v \neq 0$, it follows that j is proportional to f or g. □

5 The Exceptional Tits Quadrangles

Let $X, n, (\gamma, i \mapsto w_i)$ and $i \mapsto U_i$ be as in Section 2. Again we assume that $n = 4$. The main result of this section is Proposition 5.15.

Notation 5.1 Suppose that $\Xi = (K, L, q, f, \varepsilon, \mathcal{X}, \cdot, h, \theta)$ is an 8-tuple of algebraic data satisfying all the axioms of a quadrangular algebra in [3, 7.1] except possibly D1. Let g be as in [3, 7.1(C3)] and let $S_{\Xi,g}$ denote the group defined in [3, 8.1]. Thus the underlying set of $S_{\Xi,g}$ is $\mathcal{X} \times K$ and multiplication is given by the formula

$$(a, t) \cdot (b, s) = (a + b, t + s + g(b, a)). \tag{5.2}$$

Let $\pi(a) = \theta(a, \varepsilon)$ for all $a \in \mathcal{X}$ as in [3, 7.1(D1)].

Hypothesis 5.3 Let $\Lambda = (K, L, q)$ be a quadratic space, let f be the associated bilinear form and suppose that X, $(\gamma, i \mapsto w_i)$, Λ, and ε satisfy all the hypotheses of [3, 8.2] except possibly [3, 8.2(b)]. In particular, $|K| > 4$. In addition, we assume that X is 3-plump as defined in [3, 2.19]. As indicated in [3, 8.59], there exist algebraic data $\mathcal{X}$, $\cdot$, h, and θ such that the 8-tuple

$$\Xi = (K, L, q, f, \varepsilon, \mathcal{X}, \cdot, h, \theta)$$

satisfies all the axioms of a quadrangular algebra in [3, 7.1] except possibly D1 as well as isomorphisms x_i from the group $S_{\Xi,g}$ to U_i for i odd (where g and $S_{\Xi,g}$ are as in Notation 5.1) and from the additive group of L to U_i for i even such that the following commutator relations hold:

$$\begin{aligned}
[x_1(a,t), x_3(b,s)^{-1}] &= x_2(h(a,b)) \\
[x_2(v), x_4(w)^{-1}] &= x_3(0, f(v,w)) \\
[x_1(a,t), x_4(v)^{-1}] &= x_2(\theta(a,v)+tv)x_3(av, tq(v)+\phi(a,v))
\end{aligned} \tag{5.4}$$

for all $(a,t), (b,s) \in S_{\Xi,g}$ and all $v, w \in L$, where $av = a \cdot v$ and ϕ is as in [3, 7.1(C4)].

Proposition 5.5 *Let $u \in L$ and $t \in K$. Then, the following hold:*

(i) $x_1(0,t) \in U_1^\sharp$ *if and only if* $t \neq 0$.

(ii) $x_4(u) \in U_4^\sharp$ *if and only if* $q(u) \neq 0$.

Proof The subquadrangle $\hat{X}$ in [3, 8.2(a)] is orthogonal and satisfies the conclusions of [3, 5.2]. The claims hold, therefore by [3, 5.1(ii)–(iii) and 6.4(i)–(ii)]. □

Notation 5.6 Let $v^\sigma = f(\varepsilon, v)\varepsilon - v$ for all $v \in L$ as in [3, 7.1(A3)]. Then, $\sigma^2 = 1$ and $(av)v^\sigma = q(v)a$ for all $a \in \mathcal{X}$. As in [2, 2.4], we set $v^{-1} = v^\sigma/q(v)$ for all $v \in L$ such that $q(v) \neq 0$.

Notation 5.7 Suppose that the quadratic space $\Lambda = (K, L, q)$ is isotropic. By [3, 6.8], we can assume that the element ε in the hypothesis [3, 8.2(a)] lies in a hyperbolic plane P of Λ. Since $q(\varepsilon) = 1$, we can choose a basis z_1, z_2 of P such that $q(z_1) = q(z_2) = 0$, $f(z_1, z_2) = 1$ and $\varepsilon = z_1 + z_2$. Since $|K| > 4$, we can choose $r_1, r_2 \in K$ such that $r_1 \neq r_2$, $r_1 r_2 \neq 0$, and $r_i \neq 1$ for $i = 1$ and 2. Let $u = r_1 z_1 + r_2 z_2$. Then,

$$q(su + t\varepsilon) = (r_1 s + t)(r_2 s + t)$$

for all $s, t \in K$. In particular, $q(u) = r_1 r_2 \neq 0$ and $f(u, \varepsilon) = r_1 + r_2$.

Notation 5.8 Suppose that Λ is isotropic and let u be as in Notation 5.7. We choose an arbitrary element a of $\mathcal{X}$ and set $b = -au^{-1}$, where u^{-1} is as in Notation 5.6.

Remark 5.9 Let a, b, u be as in Notation 5.8 and let $v = -\varepsilon + u$. Then, $a + b + bv = 0$ (by [3, 7.1(A1)–(A3)]) and $q(v) + f(\varepsilon, v) = q(u) - 1$.

The proof of the following result involves a few minor calculations in Ξ. These calculations require only the axioms A1–A3 (for the first assertion in Remark 5.9) and C1. In particular, they do not require D1.

Proposition 5.10 *Suppose that $\Lambda = (K, L, q)$ is isotropic and that $|K| > 5$. Let a, b be as in Notation* 5.8, *let $v = -\varepsilon + u$, and let $a_1 = x_1(0, 1)$. Then, $a_1 \in U_1^\sharp$ and there exist $s, t \in K$ and $a_4, b_4 \in U_4^\sharp$ such that $a_4 b_4 = d_4 c_4$ and* Eq. 3.8 *holds with $c_4 = x_4(v)$, $d_4 = x_4(\varepsilon)$, $c_1 = x_1(a, t)$, and $b_1 = x_1(b, s)$.*

Proof By Proposition 5.5(i), $a_1 \in U_1^\sharp$. Let g be as in Hypothesis 5.3 and let π be as in Notation 5.1. Thus $\theta(b,v)+\pi(b)=\theta(b,u)$ by [3, 7.1(C1)]. Let $c_4=x_4(v)$, $d_4=x_4(\varepsilon)$, $c_1=x_1(a,t)$, and $b_1=x_1(b,s)$ for some $s,t\in K$. Using Eq. 5.2, Eq. 5.4, and Remark 5.9, we calculate that with these choices, the expression on the right-hand side of Eq. 3.8 is $x_2(w)x_3(0,p)$, where

$$\begin{aligned} w &= h(b,a)+\pi(a)+\pi(b)+t\varepsilon+s(v+\varepsilon)+\theta(b,v) \\ &= h(b,a)+\pi(a)+\theta(b,u)+t\varepsilon+su \end{aligned} \tag{5.11}$$

and

$$p=t+q(u)s+\beta \tag{5.12}$$

for

$$\beta := g(b,a)+\phi(b,v)+f(\varepsilon,\theta(b,v))+g(bv,a+b)+\phi(a,\varepsilon)+\phi(b,\varepsilon). \tag{5.13}$$

Let $a_4=x_4(w)$ and $b_4=x_4(u-w)$. Thus $a_4b_4=d_4c_4$. By Eq. 5.4, the expression on the left-hand side of Eq. 3.8 with our choices for a_1 and a_4 is

$$x_2(w)x_3(0,q(w)).$$

Thus $q(w)=p$ holds if and only if Eq. 3.8 holds.

The quantities $w\in L$ and $p\in K$ are functions of s and t. By Proposition 5.5(ii), $a_4=x_4(w)\in U_4^\sharp$ and $b_4=x_4(u-w)\in U_4^\sharp$ if and only if both $q(w)$ and $q(u-w)$ are non-zero. Our goal, therefore is to show that s and t can be chosen so that $q(w)=p$, $q(w)\neq 0$, and $q(u-w)\neq 0$.

By Eq. 5.11, we have

$$\begin{aligned} q(w) &= q(su+t\varepsilon)+f(u,z)s+f(\varepsilon,z)t+q(z) \\ &= (r_1s+t)(r_2s+t)+f(u,z)s+f(\varepsilon,z)t+q(z), \end{aligned}$$

where $z=h(b,a)+\pi(a)+\theta(b,u)$ and r_1,r_2 are as in Notation 5.7. Let $c=\beta-q(z)$, where β is as in Eq. 5.13. By Eq. 5.12,

$$q(w)-p=(r_1s+t)(r_2s+t)+(f(u,z)-q(u))s+(f(\varepsilon,z)-1)t-c.$$

Hence, $q(w)=p$ if and only if s,t is a solution to the equation

$$G_1G_2+H=c,$$

where $G_1=r_1s+t$, $G_2=r_2s+t$, and $H=(f(u,z)-q(u))s+(f(\varepsilon,z)-1)t$. Let S denote the solution set of this equation. Since $r_1\neq r_2$, the linear forms G_1 and G_2 are not equivalent. By Proposition 4.2, therefore $|S|\geq 5$ and for all $(s,t)\in S$, we have $q(w)=p$.

Let j denote the linear form $q(u)s+t$. Thus $j=r_1r_2s+t$ and by Eq. 5.12, $p=j(s,t)+\beta$. By the choice of r_1 and r_2 in Notation 5.7, j is equivalent to neither G_1 nor G_2. By Proposition 4.3 with $d=-\beta$, therefore there are at most two elements (s,t) of S such that $p=j(s,t)+\beta=0$. Next we note that by Eq. 5.11,

$$q(w-u)=q(w)-f(w,u)+q(u)=q(w)-2q(u)s-f(\varepsilon,u)t+\delta \tag{5.14}$$

for all (s,t), where $\delta=q(u)-f(u,z)$. Let j_0 denote the linear form

$$t-q(u)s-f(\varepsilon,u)t.$$

By the choice of r_1 and r_2 in Notation 5.7, we have

$$j_0 = -\big(r_1 r_2 s + (r_1 + r_2 - 1)t\big).$$

and thus j_0 is equivalent to neither G_1 nor G_2. By another application of Proposition 4.3, therefore there are at most two elements (s,t) of S such that $j_0(s,t) = -\beta - \delta$. Since $|S| \geq 5$, we conclude that there exists $(s,t) \in S$ such that $p = j(s,t) + \beta \neq 0$ and $j_0(s,t) + \beta + \delta \neq 0$. For all such (s,t), we have $q(w) = p \neq 0$, hence

$$\begin{aligned} j_0(s,t) &= t + q(u)s - 2q(u)s - f(\varepsilon,u)t \\ &= p - \beta - 2q(u)s - f(\varepsilon,u)t && \text{by Eq. 5.12} \\ &= q(w) - 2q(u)s - f(\varepsilon,u)t - \beta \\ &= q(w-u) - \delta - \beta && \text{by Eq. 5.14} \end{aligned}$$

and thus $q(w-u) \neq 0$. □

Proposition 5.15 *Suppose that* $\Lambda = (K, L, q)$ *is isotropic and* $|K| > 5$. *Then for each* $a \in \mathcal{X}$, *there exists* $t \in K$ *such that* $x_1(a,t) \in U_1^\sharp$.

Proof Let $a \in \mathcal{X}$. By Proposition 5.10, we can choose $a_1 \in U_1^\sharp$, $b_1 \in U_1$, $a_4, b_4 \in U_4^\sharp$, $c_4, d_4 \in U_4$, and $t \in K$ such that $a_4 b_4 = d_4 c_4$ and the elements $a_1, b_1, c_1, a_4, b_4, c_4, d_4$ with $c_1 = x_1(a,t)$ satisfy Eq. 3.8. Let $d_1 = a_1 b_1^{-1} c_1^{-1}$. Then by Proposition 3.7, the elements $a_1, b_1, c_1, d_1, a_4, b_4, c_4, d_4$ satisfy Eq. 3.6. By Proposition 3.5, therefore $c_1 = x_1(a,t) \in U_1^\sharp$. □

6 Pseudo-quadratic Tits Quadrangles

The main result of this section is Proposition 6.10.

Definition 6.1 An *involutory ring* (K, K_0, σ) is an associative ring K endowed with an involution σ and an additive subgroup K_0 such that σ acts trivially on K_0 and $1 \in K_0$ as well as $a + a^\sigma \in K_0$ and $a^\sigma K_0 a \subset K_0$ for all $a \in K$.

Definition 6.2 A *pseudo-quadratic module* is a 6-tuple $\Lambda = (K, K_0, \sigma, L, q, f)$ such that (K, K_0, σ) is an involutory ring, L is a right K-module, f is a skew-hermitian form on L, and q is a map from L to K such that

$$\begin{aligned} q(a+b) &\equiv q(a) + q(b) + f(a,b) \pmod{K_0} \text{ and} \\ q(at) &\equiv t^\sigma q(a) t \pmod{K_0} \end{aligned} \tag{6.3}$$

for all $a, b \in L$ and all $t \in K$. We call Λ *non-degenerate* if f is non-degenerate, i.e., if $\{a \in L \mid f(a, L) = 0\} = \{0\}$.

Notation 6.4 Let $\Lambda = (K, K_0, \sigma, L, q, f)$ be a pseudo-quadratic module. Let T_Λ denote the subset

$$\{(a,t) \in L \times K \mid q(a) - t \in K_0\} \tag{6.5}$$

of $L \times K$. (If $K_0 = K$, in which case $\sigma = 1$ and f is alternating, then $T_\Lambda = L \times K$.) We define an associative multiplication on T_Λ by setting

$$(a, t) \cdot (b, s) = (a + b, t + s + f(b, a))$$

for all $(a, t), (b, s)$. Setting $b = a$ and $t = -1$ and 2 in Eq. 6.3, we obtain

$$q(-a) \equiv q(a) \text{ and } 2q(a) \equiv f(a, a) \pmod{K_0}$$

for all $a \in \mathcal{X}$ and hence

$$q(-a) + t - f(a, a) \in K_0$$

for all $(a, t) \in T$. It follows that T_Λ is a group with identity $(0, 0)$ and $(a, t)^{-1} = (-a, -t + f(a, a))$ for all $(a, t) \in T_\Lambda$. Note that $T_\Lambda = K_0$ if $L = 0$.

Proposition 6.6 *Let $(a, t) \in T_\Lambda$ and suppose that t has a right inverse $s \in K$. Then, $(as, -s) \in T_\Lambda$.*

Proof By Definition 6.1, $s + s^\sigma \in K_0$. Thus

$$q(as) + s \equiv s^\sigma q(a)s - s^\sigma \pmod{K_0}$$

by Eq. 6.3. Since $ts = 1$, it follows that

$$s^\sigma q(a)s - s^\sigma = s^\sigma (q(a) - t)s \in s^\sigma K_0 s \subset K_0$$

by Definition 6.1 and Eq. 6.5. Hence, $q(as) + s \in K_0$. □

Definition 6.7 Let $\Lambda = (K, K_0, \sigma, L, q, f)$ be a pseudo-quadratic module, let T_Λ be as in Notation 6.5, and let X be a Tits quadrangle. Then, X is *pseudo-quadratic* of type Λ if there exists a coordinate system $(\gamma, i \mapsto w_i)$ and isomorphisms x_i from T_Λ to U_i for $i = 1$ and 3 and from the additive group of K to U_i for $i = 2$ and 4 such that the following commutator relations hold:

$$\begin{aligned} [x_1(a, t), x_3(b, s)^{-1}] &= x_2(f(a, b)) \\ [x_2(v), x_4(w)^{-1}] &= x_3(0, v^\sigma w + w^\sigma v) \\ [x_1(a, t), x_4(v)^{-1}] &= x_2(tv)x_3(av, v^\sigma tv) \end{aligned} \tag{6.8}$$

for all $(a, t), (b, s) \in T_\Lambda$ and all $v, w \in K$. If $L = 0$, we say that X is *involutory* of type (K, K_0, σ).

Proposition 6.9 *Let X be a pseudo-quadratic Tits quadrangle of type Λ for some pseudo-quadratic module $\Lambda = (K, K_0, \sigma, L, q, f)$ and let T_Λ, $(\gamma, i \mapsto w_i)$ and $x_1, \ldots, x_4$ be as in Definition 6.7. Suppose that Λ is non-degenerate and that X is sharp as defined in* [3, 2.13]. *Let $U_1^\sharp$ and Y_1 be as in* [3, 2.8 and 4.6]. *Then, $x_1(0, K_0) = Y_1$ and if*

$$x_1(0, 1) \in U_1^\sharp, \tag{6.10}$$

then the following hold:

(i) *If $(a, t) \in T_\Lambda$, then $x_1(a, t) \in U_1^\sharp$ if and only if $t \in K^\times$.*

(ii) *If K is a field or skew-field and $a \in L$, then there exists $t \in K$ such that $(a,t) \in T_\Lambda$ and $x_1(a,t) \in U_1^\sharp$.*

Proof Since Λ is non-degenerate, it follows from [3, 4.9] and Eq. 6.8 that $x_1(0, K_0) = Y_1$. Suppose that Eq. 6.10 holds. Let $m = \mu_\gamma(x_1(0,1))$, where μ_γ is as in [3, 2.9], and let $x_5(a,t) = x_1(a,t)^m$ for all $(a,t) \in T_\Lambda$. Then, $[m, U_3] = 1$ by [3, 4.9 and 5.3] and $x_4(u)^m = x_2(u)$ for all $u \in L$ by Eq. 6.8 and [3, 2.16(i)]. Since X is sharp, we can apply [4, 4.12(i)]. It follows that $x_2(u)^m = x_4(u)^{m^2} = x_4(u)^{-1}$ for all $u \in L$. Conjugating the last commutator relation in Eq. 6.8 by m, we find that

$$[x_5(a,t), x_2(v)^{-1}] = x_4(-tv)x_3(av, v^\sigma tv)$$

for all $(a,t) \in T_\Lambda$ and all $v \in K$. Thus

$$\begin{aligned} x_2(v)^{x_5(a,t)} &= [x_5(a,t), x_2(v)^{-1}] \cdot x_2(v) \\ &= x_3(av, v^\sigma tv)x_4(-tv) \cdot x_2(v) \\ &= x_2(v)x_3(av, v^\sigma tv)x_3(0, -(tv)^\sigma v - v^\sigma tv)x_4(-tv) \\ &= x_2(v)x_3(av, -v^\sigma t^\sigma v)x_4(-tv) \end{aligned} \tag{6.11}$$

for all $(a,t) \in T_\Lambda$ and all $v \in K$.

Now choose $(a,t) \in T_\Lambda$ and suppose that $t \in K^\times$. Let $s = t^{-1}$. By Proposition 6.7, $(as, -s) \in T_\Lambda$. By Eq. 6.11, we have

$$x_2(v)^{x_5(as,-s)} = x_2(v)x_3(asv, v^\sigma s^\sigma v)x_4(sv)$$

for all $v \in K$ and by Eq. 6.8, we have

$$\begin{aligned} x_4(sv)^{x_1(a,t)} &= [x_1(a,t), x_4(sv)^{-1}] \cdot x_4(sv) \\ &= x_2(v)x_3(asv, v^\sigma s^\sigma v)x_4(sv) \end{aligned}$$

for all $v \in K$. Thus

$$x_4(sv)^{x_1(a,t)x_5(as,-s)^{-1}} = x_2(v)$$

for all $v \in K$. Since $s \in K^\times$, we have $\{x_4(sv) \mid v \in K\} = U_4$. Hence, $x_1(a,t) \in U_1^\sharp$ by [3, 2.17(i)]. Suppose, conversely, that $x_1(a,t) \in U_1^\sharp$. By [3, 2.16(i)] and Eq. 6.8, $x_2(v)^{\mu_\gamma(x_1(a,t))^{-1}} = x_4(tv)$ for all $v \in K$. By [3, (2.11)], $x_2(v)^{\mu_\gamma(x_1(a,t))^{-1}} = U_4$. Therefore, $v \mapsto tv$ is a bijection from the additive group of K to itself. Hence, $t \in K^\times$. Thus (i) holds.

Now suppose that K is a field or skew-field and let $a \in L$ be arbitrary. Let $t = q(a)$ if $q(a) \neq 0$ and let $t = 1$ if $q(a) = 0$. By Definition 6.1, both 0 and 1 are contained in K_0, so $(a,t) \in T_\Lambda$. By (i), $x_1(a,t) \in U_1^\sharp$. Thus (ii) holds. □

7 The Main Result

In [6], the notion of a Veldkamp n-gon was introduced (for all $n \geq 3$). Axioms for a Veldkamp n-gon are given in [6, 2.8]. Every Tits n-gon satisfies these axioms.

Now let X, n, $(\gamma, i \mapsto w_i)$ and $i \mapsto U_i$ be as in Section 2. Once again we assume that Hypothesis 5.3 holds with $n = 4$. Thus our Tits quadrangle X is, in particular, a Veldkamp quadrangle, so we can apply results in [6] to X. In this section, we use this observation to finish proving the following:

Proposition 7.1 *Suppose that X, $(\gamma, i \mapsto w_i)$ and $\Lambda = (K, L, q)$ satisfy all the hypotheses of* [3, 8.2] *except possibly* [3, 8.2(b)]. *Suppose, too, that X is 3-plump and that $|K| > 5$. Then, the hypothesis* [3, 8.2(b)] *holds as well.*

Proof Let $\Xi = (K, L, q, f, \varepsilon, \mathcal{X}, \cdot, h, \theta)$ be as in Hypothesis 5.3. Our goal is to show that

$$\text{for each } a \in \mathcal{X} \text{ there exists } t \in K \text{ such that } x_1(a, t) \in U_1^\sharp. \tag{7.2}$$

If X is a Moufang quadrangle, i.e., if all the local opposition relations are trivial, then $U_1^\sharp = U_1^*$, so $x_1(a, 1) \in U_1^\sharp$ and thus Eq. 7.2 holds. If X is of type E_7 as defined in [6, 6.15], then Eq. 7.2 holds by [3, 10.4(ii)]. We can thus assume that X is neither a Moufang quadrangle nor of type E_7. By Proposition 5.15, on the other hand, we can assume that Λ is anisotropic. By [3, 5.1(iii) and 6.4(ii)], this assumption implies that

$$U_i^\sharp = U_i^* \tag{7.3}$$

for all even i. We now think of X as a Veldkamp quadrangle. Let P denote the vertices of Γ at even distance from w_1 and let L denote the vertices at even distance from w_4. By [5, 1.4.15] and Eq. 7.3, the local opposition relation $\equiv_v$ is trivial for all $v \in L$. We declare the elements of P to be points and the elements of L to be lines. With this choice of point set and line set, X is green as defined in [6, 3.1].

By [6, 6.19], X is also flat as defined in [6, 2.9]. We can thus invoke the classification of flat green Veldkamp quadrangles in [6, 6.16]. Since X is wide as defined in [3, 4.7], it is neither orthogonal as defined in [6, 6.11(i)] nor of type D_3 as defined in [6, 6.13]. We conclude that X is pseudo-quadratic. Since $Y_1^* = Y_1^\sharp$ by [3, 8.2(a)] and $Y_1^\sharp = Y_1 \cap U_1^\sharp$ by [3, 5.1(iii)], X satisfies Eq. 6.10. By Proposition 6.10(ii), therefore Eq. 7.2 holds also in this last case. □

8 Conclusions

We suppose one last time that X, n, $(\gamma, i \mapsto w_i)$ and $i \mapsto U_i$ are as in Section 2, that $n = 4$ and that Hypothesis 5.3 holds and let

$$\Xi = (K, L, q, f, \varepsilon, \mathcal{X}, \cdot, h, \theta)$$

be as in Hypothesis 5.3. We assume in this section that X is 4-plump and that $|K| > 5$. By Proposition 7.1, we know that X satisfies all the conclusions of [3, 8.2]. In Theorems 8.4 and 8.6, we summarize the main conclusions in [3] that follow from [3, 8.2].

Remark 8.1 By [3, 12.1], X is uniquely determined by the isotopy class of Ξ. (The isotopy class of a quadrangular algebra is defined in [2, 5.3].) We note that the hypothesis 4-plump is needed in the proof of [3, 12.1]; see [3, 4.1] and [4, 5.93].

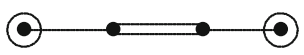
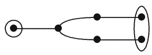
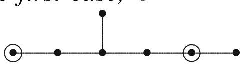
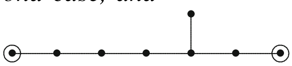
Remark 8.2 See [1, 15.3] for the notion of a pseudo-split building that appears in Theorem 8.4(ii). See [1, §6] for the notion of the fixed point building of a Galois group and the notion of a Tits index as they are being used in Theorem 8.4(ii)–(iii).

Remark 8.3 In Theorem 8.4(iii), we use the nomenclature for Moufang spherical buildings in [10, 30.15] when we refer to the buildings $C_3(C, K) := C_3^{\mathcal{I}}(C, K)$, $F_4(C, K)$, and $E_\ell(K)$. See Remark 8.5 below. The building $C_3(C, K)$ is the building associated with the non-embeddable polar space studied in [7, Chapter 9].

Theorem 8.4 *Suppose* $\dim_K L > 4$. *Then, the following hold:*

(i) *q is either similar to the norm n_C of an octonion division algebra (C, K) or to the quadratic form q_C for some composition algebra (C, K) as defined in* [3, 7.3] *or q is a quadratic form of type E_6, E_7, E_8, or F_4 as defined in* [9, 2.13 and 2.15].

(ii) *If q is a quadratic form of type Π for $\Pi = E_6$, E_7, E_8, or F_4, then X is a Moufang quadrangle. It arises as the fixed point building of a Galois group of order* 2 *acting on a split or pseudo-split building of type Π as described in* [1, §§11, 13, 14, and 17].

(iii) *If q is similar to the norm n_C, let Δ be the building $C_3(C, K)$. If q is similar to q_C and (C, K) is division, let Δ be the building $F_4(C, K)$. If q is similar to q_C, (C, K) is not division and* $\dim_K C = 2^k$, *let Δ be the building $E_{5+k}(K)$. Let T be the Tits index*  *in the first case,*  *in the second case, and*  *if $k = 1$,*  *if $k = 2$, and*  *if $k = 3$ in the third case. Then, X is isomorphic to the Tits quadrangle $X_{\Delta,T}$ defined in* [3, 3.3].

Proof This is a restatement of [3, 12.3] which is, in turn, a consequence of Remark 8.1 and the classification of quadrangular algebras $(K, L, q, f, \varepsilon, \mathcal{X}, \cdot, h, \theta)$ with $|K| > 5$ and h non-degenerate given in [2, 5.6 and 5.10(i)–(ii)]. □

Remark 8.5 Let (C, K) be a division composition algebra. If $C = K$ or $C^2 \subset K$, then the building $F_4(C, K)$ is split or pseudo-split. In the remaining cases, the building $F_4(C, K)$ arises as the fixed point building of a Galois group of order 2 acting on a building of type E_{5+k}, where $k = \log_2(\dim_K C)$, and when C is octonion, the building $C_3(C, K)$ arises as the fixed point building of a Galois group of order 2 acting on a building of type E_7. See [1, §15] for details.

Theorem 8.6 *Suppose that* $\dim_K L \leq 4$. *Then, L can be given the structure of a composition algebra over K whose reduced trace is not identically zero, whose standard involution σ is given by*

$$a \mapsto f(a, \varepsilon)\varepsilon - a$$

and whose identity is ε, there exist V, Q, and F such that

$$\Psi := (L, K, \sigma, V, Q, F)$$

is a non-degenerate pseudo-quadratic module as defined in Definition 6.2 *and* X *is pseudo-quadratic of type* Ψ *as defined in Definition* 6.7.

Proof This holds by [2, 5.10(iii)] (or, more precisely, [2, 5.14 and 11.16]). □

Remark 8.7 If $\dim_K L \leq 4$ and $\dim_K X < \infty$, then X is of index type; more precisely, X is isomorphic to a Tits quadrangle $X_{\Delta,T}$ for one of the pairs (Δ, T) in [3, 12.6].

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Declarations

Conflict of Interest The authors declare no competing interests.

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