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# Functional Itō-Calculus for Superprocesses and the Historical Martingale Representation

## Dissertation

Submitted by

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## Abstract

We derive an Itō-formula for the Dawson-Watanabe superprocess, a well-known class of measure-valued processes, extending the classical Itō-formula with respect to two aspects. Firstly, we extend the state-space of the underlying process $(X(t))_{t\in[0,T]}$ to an infinite-dimensional one – the space of finite measures. Secondly, we extend the formula to functionals $F(t, X_t)$ depending on the entire stopped paths $X_t = (X(s \wedge t))_{s\in[0,T]}$, $t \in [0,T]$. This later extension is usually called functional Itō-formula.

Given the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by an underlying superprocess, we show that by extending the functional derivative used in the functional Itō-formula we obtain the integrand in the martingale representation formula for square-integrable $(\mathcal{F}_t)_t$-martingales. This result is finally extended to square-integrable historical martingales. These are $(\mathcal{H}_t)_t$-martingales, where $(\mathcal{H}_t)_{t\in[\tau,T]}$ is the filtration generated by a historical Brownian motion, an enriched version of a Dawson-Watanabe superprocess.



## Kurzfassung

Wir leiten die funktionale Itō-Formel für eine bestimmte Klasse maßwertiger Prozesse, die Dawson-Watabe Superprozesse, her. Diese erweitert die klassiche Itō-Formel wie folgt: Während die klassiche Itō-Formel für $\mathbb{R}^d$-wertige Prozesse gilt, betrachten wird Prozesse $(X(t))_{t\in[0,T]}$ mit einem unendlichdimensionalen Zustandsraum. Zudem betrachten wir Funktionale von $X_t = (X(s \wedge t))_{s\in[0,T]}$, dem zur Zeit $t$ gestoppten Pfad von $X$, anstelle von Funktionen von $X(t)$, dem Zustand von $X$ zur Zeit $t$.

Weiter zeigen wir, dass, gegeben der von einem Superprozess erzeugten Filtration $(\mathcal{F}_t)_{t\in[0,T]}$, die funktionale Ableitung, welche in der funktionalen Itō-Formel auftaucht, genutzt werden kann, um den Integranden in der Martingaldarstellung für quadratintegrierbare $(\mathcal{F}_t)_t$-Martingale zu bestimmen. Zum Schluss erweitern wir dieses Resultat auf quadratintegrierbare historische Martingale. Dies sind $(\mathcal{H}_t)_t$-Martingale, wobei die Filtration $(\mathcal{H}_t)_{t\in[\tau,T]}$ von einer historischen Brownschen Bewegung erzeugt wird.

# Contents

# List of Figures

# A Remark on Notations

While we try to introduce most of the notations used throughout this monograph when they first appear, we have included some basic notations at the outset of this monograph to provide for an easier read. In addition, an index of all notations, regardless of initial introduction, appears after the bibliography.

As usual, the set of natural numbers is denoted by $\mathbb{N}$. We write $d \in \mathbb{N}_0$ if we allow $d$ to be a natural number or zero. The space of real numbers is denoted by $\mathbb{R}$ and, for $d \in \mathbb{N}$, its $d$-dimensional counterpart is denoted by $\mathbb{R}^d$.

Further, we write $\mathcal{B}(\cdot)$ for a Borel-$\sigma$-algebra, that is $\mathcal{B}([0,T])$, $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R}^d)$ stand for the Borel-$\sigma$-algebra on $[0,T]$, $\mathbb{R}$ and $\mathbb{R}^d$, respectively.

For a majority of this monograph, we consider an abstract metric space $E$ and denote its Borel-$\sigma$-algebra by $\mathcal{E}$. We write $\delta_y$, $y \in E$, for the Dirac measure on $(E, \mathcal{E})$, which is given by

$$\delta_y(x) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y, \end{cases}$$

for all $x \in E$. In addition, for any $B \subset E$, the indicator function $1_B$ is defined for all $x \in E$ by

$$1_B(x) = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \notin B. \end{cases}$$

Now, let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. The partial derivative of $f$ at $x \in \mathbb{R}^d$ in direction $i$, $i = 1, \ldots, d$, is given by

$$\partial_i f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon e_i) - f(x)}{\varepsilon}$$

if the limit exists, where $e_i$ denotes the $d$-dimensional basis vector with a 1 in the $i$-th coordinate and all other entries being 0's. The second order partial derivative of $f$ in directions

$i$ and $j$, $i$, $j = 1, \dots, d$, denoted by $\partial_{ij}$, is defined iteratively and the Laplacian $\Delta$ of $f$ is defined as

$$\Delta f = \sum_{i=1}^{d} \partial_{ii} f.$$

If $f$ can be interpreted as a function of time and space, i.e.

$$f : \mathbb{R} \times \mathbb{R}^d \ni (s, x) \mapsto f(s, x) \in \mathbb{R},$$

the partial derivative in direction of the first coordinate is denoted by $\partial_s$.

Finally, let $\mathcal{T}$ be an interval on the real line and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbb{P})$ a filtered probability space. Then, we denote by $([X]_t)_{t\in\mathcal{T}}$ the quadratic variation process of a continuous $(\mathcal{F}_t)_t$-local martingale $(X(t))_{t\in\mathcal{T}}$. The quadratic variation process is the unique continuous and adapted process such that

$$(X^2(t) - [X]_t)_{t\in\mathcal{T}}$$

is a $(\mathcal{F}_t)_t$-local martingale. Analogously, we write $([X, Y]_t)_{t\in\mathcal{T}}$ for the quadratic covariation process of two continuous $(\mathcal{F}_t)_t$-local martingales $X$ and $Y$, which is the unique continuous and adapted process such that

$$(X(t)Y(t) - [X, Y]_t)_{t\in\mathcal{T}}$$

is a $(\mathcal{F}_t)_t$-local martingale.

# Introduction

In order to introduce superprocesses, it is reasonable to start with a simple branching diffusion process on $\mathbb{R}^n$. Thus, consider a number $N(0) \in \mathbb{N}$ of particles moving around independently in $\mathbb{R}^n$. At independent times, the particles die and leave behind a random number of descendants that behave analogously. The number of descendants is determined by independent draws from a common probability distribution on $\mathbb{N}_0$. If we denote by $N(t)$ the number of particles alive at time $t$ and denote by $Z_1(t), \ldots, Z_{N(t)}(t)$ their locations in $\mathbb{R}^n$ at time $t$, the process

$$X(t) = \sum_{i=1}^{N(t)} w\delta_{Z_i(t)},$$

where $\delta_x$ denotes the Dirac measure with unit mass at $x \in \mathbb{R}^d$ and $w > 0$ is some weight, takes values in the space of finite measures on $\mathbb{R}^n$. Watanabe ([Watanabe, 1968]) was the first to show that

> *"when we change the scale of time and mass in an appropriate way,* [the process $X$ converges] *to a continuous random motion on the space of mass distributions on $\mathbb{R}^n$."*

The resulting limit process is a superprocess. To pay homage to Watanabe's findings as well as to Dawson's work on these processes in the following years (e.g. [Dawson, 1977] and [Dawson and Hochberg, 1979]), superprocesses are also often called *Dawson-Watanabe superprocesses* (see [Etheridge, 2000]).

Since the introduction of superprocesses, it has been shown that superprocesses arise as a scaling limit of numerous so-called *branching particle systems* (see e.g. [Dynkin, 1991a]) including contact processes (see e.g. [Durrett and Perkins, 1999]) and other interacting particle systems (see e.g. [Cox et al., 2000] or [Durrett et al., 2005]).

Superprocesses have attracted particular interest in stochastic analysis due to their connection to non-linear (partial) differential equations (see e.g. [Dynkin, 1991b], [Dynkin, 1992], [Dynkin, 1993] or [Le Gall, 1999]). Chapter 8 in [Etheridge, 2000] provides an extended overview of the research on this relation, which allows for a fruitful interplay between stochastic analysis and the traditional analysis of partial differential equations and, more recently, has led to some applications of the theory of superprocesses in mathematical finance as in [Guyon and Henry-Labordere, 2013] and [Schied, 2013].

More typical applications of superprocesses can be found in population genetics. These are often driven by the close link between superprocesses and a second class of measure-valued processes, the so-called *Fleming-Viot processes* (see e.g. [Etheridge and March, 1991], [Perkins, 1992] or [Ethier and Krone, 1995]). Note that, in the literature, Fleming-Viot processes are often also referred to as superprocesses. To avoid confusion, we only refer to Dawson-Watanabe superprocesses as superprocesses.

Thorough introductions to superprocesses in general can be found in [Dawson, 1993] and [Perkins, 2002]. In this work, we focus on a particular subclass of superprocesses, the so-called $B(A,c)$*-superprocesses*, which are also intensively studied in [Dawson, 1993] and introduced in Section 1.1.1 of this monograph. Processes in this subclass can be characterized as limits of branching diffusion processes with a specific motion and branching mechanism and come with two favorable properties that are essential for the proofs of our result: Firstly, $B(A,c)$-superprocesses have continuous sample paths and, secondly, they give rise to a continuous orthogonal martingale measure in the sense of [Walsh, 1986] (see Section 1.3.2).

As measure-valued Markov processes, superprocesses take values in infinite-dimensional spaces. Therefore, a fundamental tool in stochastic analysis, the traditional Itō-formula, is not directly applicable to functions of such processes. The first main result presented in this monograph is the Itō-formula for functions of $B(A,c)$-superprocesses. Dawson ([Dawson, 1978]) was the first to prove an Itō-formula for measure-valued processes, but his result is limited to what we call finitely based functions and compares to Theorem 2.4 in this monograph. Our result (Theorem 2.9), which is obtained by rewriting a result in [Jacka and Tribe, 2003], extends the Itō-formula to a much wider class of functions.

Given a Wiener process $X$, the traditional Itō-formula states that, for suitable functions $f:\mathbb{R}\to\mathbb{R}$, the value of $f$ at $X(t)$, the value of $X$ at time $t$, is given by

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX(s) + \frac{1}{2}\int_0^t f''(X(s))d[X]_s.$$

In many application, however, it is necessary to consider functionals of the whole path of $X$ up to time $t$, given by $X_t = \{X(t\wedge s) : s\in[0,T]\}$, instead of functions of the value of $X$ at time $t$. In [Dupire, 2009], Dupire writes

> *"[...] in many cases, uncertainty affects the current situation not only through the current state of the process but through its whole history. For instance, the quality of a harvest does not only depend on the current temperature, but also on the whole pattern of past temperatures; the price of a path dependent option may depend on the whole history of the underlying price; [...]".*

Motivated by this fact, Dupire develops the so-called *functional Itō-calculus* for $\mathbb{R}$-valued Itō-processes in [Dupire, 2009]. For suitable functionals $f$, the functional Itō-formula is given by

$$f(X_t) = f(X_0) + \int_0^t \Delta_s f(X_s) ds + \int_0^t \Delta_x f(X_s) dX(s) + \frac{1}{2} \int_0^t \Delta_{xx} f(X_s) d[X]_s,$$

where $\Delta$ denotes the so-called *functional derivatives.* We skip over details like the domain of the functionals or the definition of the functional derivatives at this point and deal with them in Section 1.2. Also in this section, we introduce two different approaches to formalize and extend Dupire's result. The first of the two is due to Levental and co-authors ([Levental et al., 2013]). The second one is by Cont and Fournié, who address the topic in a series of publications ([Cont and Fournié, 2010], [Cont and Fournié, 2013], [Cont, 2016]). In addition to deriving the functional Itō-formula, Cont and Fournié use the functional derivatives to derive the martingale representation formula for square-integrable $(\sigma(X(s) : s \leq t))_t$-martingales.

The martingale representation formula expresses a martingale as the sum of its expectation and a stochastic integral term. While the integrator of the stochastic integral is given by the underlying setting, obtaining the integrand requires more work. The traditional approach to obtain the integrand is the so-called *Clark-Ocone-Haussmann formula* ([Clark, 1970], [Clark, 1971], [Haussmann, 1978], [Haussmann, 1979], [Karatzas et al., 1991], [Ocone, 1984]), in which the integrand is obtained using Malliavin calculus.

The derivation of the functional Itō-formula for functionals of superprocesses is our second main result (Theorem 2.14). The proof of this result follows the approach by Cont and Fournié and, as in [Cont and Fournié, 2013] and [Cont, 2016], we also show that we can work with the functional derivative used in the functional Itō-formula to obtain our third main result, the martingale representation formula for square-integrable $(\mathcal{F}_t)_t$-martingales (Theorem 3.10), where $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration of the considered superprocess.

The martingale representation formula in the context of superprocesses was first studied in [Evans and Perkins, 1994], [Evans and Perkins, 1995] as well as [Overbeck, 1995]. While the uniqueness of the representation is proved in [Evans and Perkins, 1994] and [Overbeck, 1995], Evans and Perkins derive the explicit form of the integrand in [Evans and Perkins, 1995] using an approach following the ideas of Malliavin calculus. Strictly speaking, in [Evans and Perkins, 1995] the authors consider historical processes instead of superprocesses but note that one can obtain the result for superprocesses by projection.

A historical process is a time-inhomogeneous Markov process that can be viewed as an enriched version of a superprocess that contains information on genealogy (see [Perkins, 1992]). In Section 3.2, we consider a specific historical process, the so-called *historical Brownian motion* and derive the martingale representation formula for square-integrable $(\mathcal{H}_t)_t$-martingales, where $(\mathcal{H}_t)_{t\in[\tau,T]}$ is the natural filtration of the underlying historical Brownian motion. This representation (Theorem 3.24) is our fourth main result.

The structure of this monograph is as follows. We start by introducing three concepts from stochastic analysis, namely superprocesses and the historical Brownian motion, the functional Itō-calculus as well as martingale measures, that underlie all of the presented main results. This is subject of Chapter 1. In Chapter 2, we derive the Itō-formula as well as the functional Itō-formula for functions, respectively functionals, of superprocesses. The next chapter,

Chapter 3, covers our results on the martingale representation in both scenarios, the one considering superprocesses as well as the one considering the historical Brownian motion. In the final chapter, we conclude this monograph by outlining ongoing research and discussing potential future research question related to our work.

# Chapter 1

# Preliminaries

In this first chapter we lay the foundation for the remainder of this monograph by introducing the three central underlying mathematical concepts. The first one of these three concepts is a class of measure-valued Markov processes called $B(A, c)$*-superprocesses*. These processes arise as a scaling limit of branching diffusion processes and are introduced in the first section of this chapter. While the results in Chapter 2 hold for any $B(A, c)$-superprocess as defined in Definition 1.9, the results in Chapter 3 only hold for a specific $B(A, c)$-superprocess, the so-called *super-Brownian motion*, as well as the so-called *historical Brownian motion*. While not a $B(A, c)$-superprocess, the later one is an enriched version of the earlier one and both processes are also introduced in the first part of this chapter.

The second concept, the functional Itō-calculus, is a relatively new mathematical concept and introduced in the second part of this chapter. It is based on the work by Dupire ([Dupire, 2009]), which is why the functional derivatives used in the functional Itō-formula are often referred to as *Dupire derivatives*. However, instead of presenting the original approach by Dupire in more detail, we introduce the approaches by Levental as well as Cont and their respective co-authors, who formalized Dupire's original ideas in slightly different ways.

In the final section of this chapter, we introduce the concept of martingale measures. Introduced by Walsh ([Walsh, 1986]), martingale measures are a particular class of measure-valued martingales that play a crucial role in the formulation of the results in both, Chapter 2 as well as Chapter 3.

While in the first two sections of this chapter we mostly skip the proofs of the results stated to keep the introduction brief, we deviate from this principle in the final part. The reason for this is twofold. The fact that there exists a martingale measure associated with a $B(A, c)$-superprocess as introduced in the first part of this chapter is well known (see e.g. Example 7.1.3 in [Dawson, 1993]). However, as we could not find a detailed proof of this fact that we can refer to, fwe carry out the proof in Section 1.3.2 for the sake of completeness. In addition, we extend the class of valid integrands for the integral with respect to the martingale measure associated with a $B(A, c)$-superprocess in Section 1.3.3. While this result can be proved following standard arguments, the result seems to be new and is thus proved.

## 1.1 Superprocesses and the Historical Brownian Motion

The stochastic processes underlying almost all results in this monograph are superprocesses, more precisely $B(A, c)$-superprocesses. As a brief review of the history of superprocesses as well as their connections to other fields of research is provided in the introduction, the focus of this section is on introducing the $B(A, c)$-superprocess, stating some of its properties and explaining how one can obtain the super-Brownian motion by choosing $A$ and $c$ accordingly.

In some applications, instead of working with superprocesses, one has to work with enriched versions of superprocesses that keep track of the underlying genealogy. These processes are known as historical processes and go back to Perkins and his co-authors. In the second part of this section, we introduce a specific example of a historical process, namely the historical Brownian motion, which is the stochastic process studied in Section 3.2.

### 1.1.1 Superprocesses

While there are multiple equivalent ways to define a $B(A, c)$-superprocess, we define it via its martingale problem as this turns out to be advantageous in the later sections. However, we also briefly mention the derivation of superprocesses as scaling limits of branching diffusion processes to provide an intuitive interpretation of superprocesses and state its Laplace transform to highlight the connection of $B(A, c)$-superprocesses to critical Feller continuous state branching processes.

We do not attempt to provide a full introduction to the wide field of superprocesses and thus omit almost all proofs of the results presented in this brief introduction. Most of the proofs as well as detailed introductions to the topic can be found in the lecture notes by Dawson ([Dawson, 1993]) and Perkins ([Perkins, 2002]).

Now, let $(E, d)$ be a separable metric space, which we assume to be either compact or locally compact, and $\mathcal{E}$ its Borel-$\sigma$-algebra. Further, denote by $M_1(E)$ the space of probability measures on $(E, \mathcal{E})$ and by $M_F(E)$ the space of finite measure on the same measure space. We equip these spaces with the topology of weak convergence. If we write $\langle \mu, f \rangle = \int_E f d\mu$ for a function $f : E \to \mathbb{R}$, a series $\mu_n \subset M_1(E)$ ($\mu_n \subset M_F(E)$) converges to $\mu \in M_1(E)$ ($\mu \in M_F(E)$) in the weak topology if and only if $\langle \mu_n, f \rangle \to \langle \mu, f \rangle$ for $n \to \infty$ for all bounded and continuous $f$.

Denote by $C_0(E, \mathbb{R}) = C_0(E)$ the space of continuous functions from $E$ to $\mathbb{R}$ which satisfy $f(x) \to 0$ if $d(x, 0) \to \infty$. If $E$ is compact, $C_0(E, \mathbb{R}) = C(E, \mathbb{R})$, the space of continuous functions from $E$ to $\mathbb{R}$, also denoted by $C(E)$. By equipping it with the sup-norm $\| \cdot \|$, $\|f\| = \sup_{x \in E} |f(x)|$, the space $C_0(E)$ becomes a Banach space.

**Definition 1.1** (Feller semigroup). *A* Feller semigroup *is a conservative, positive, contraction semigroup $(S_t)_{t \in [0,T]}$ on $C_0(E)$, i.e. a linear map which satisfies*

*(i) $S_{t+s} = S_t S_s$ for all $s$, $t \in [0, T]$ such that $s + t \in [0, T]$ (semigroup property),*

*(ii) $S_t 1 = 1$ for all $t \in [0, T]$ (conservativeness),*

*(iii) $S_t f \geq 0$ for all $f \geq 0$ and $t \in [0, T]$ (positivity),*

*(iv)* $\|S_t f\| \leq \|f\|$ *for all* $t \in [0,T]$ *(contraction),*

*which also satisfies*

*(v)* $S_t : C_0(E) \to C_0(E)$ *for all* $t \in [0,T]$*,*

*(vi)* $S_t f(x) \to f(x)$ *as* $t \to 0$*, for all* $f \in C_0(E)$*,* $x \in E$ *(weakly continuous).*

*Remark* 1.2. There are various definitions of Feller processes in the literature that slightly deviate from each other. The one presented in Definition 1.1 is based on the one found in [Kallenberg, 2002]. The conservativeness of the Feller semigroup is not part of the original definition in [Kallenberg, 2002] but an additional assumption repeatedly used by the author. For simplicity and as it is part of the definition of other authors (see e.g. [Ethier and Kurtz, 1986]), we assume all Feller semigroups to be conservative.

**Proposition 1.3.** *Regardless of whether the conservativeness is included in its definition, a Feller semigroup is always strongly continuous, i.e. it holds*

$$S_t f \to f \text{ as } t \to 0 \text{ for all } f \in C_0(E).$$

*Proof.* See Theorem 19.6 in [Kallenberg, 2002]. □

**Definition 1.4** (Feller process)**.** *A* Feller process *is a Markov process whose transition semigroup is a Feller semigroup.*

In order to formulate the martingale problem defining $B(A,c)$-superprocesses, we have to introduce the notion of generators of Feller processes. These generators uniquely determine a Feller process. Thus, we can characterize Feller processes solely by their generator.

**Definition 1.5** (Generator)**.** *Let* $(S_t)_{t\in[0,T]}$ *be a Feller semigroup. The* (infinitesimal) generator $A$ *of* $(S_t)_{t\in[0,T]}$ *is defined by*

$$Af = \lim_{t\to 0} \frac{S_t f - f}{t}$$

*if the limit exists. Its* domain $D(A)$ *is the space of functions* $f$ *in* $C_0(E)$ *for which the limit exists.*

**Proposition 1.6.** *A Feller semigroup is uniquely characterized by its generator.*

*Proof.* See Lemma 19.5 in [Kallenberg, 2002]. □

**Proposition 1.7.** *Let* $(S_t)_{t\in[0,T]}$ *be a Feller semigroup with generator* $A$*. Then, for all* $f \in D(A)$*,*

$$S_t f - f = \int_0^t S_s A f ds = \int_0^t A S_s f ds \tag{1.1}$$

*holds.*

*Proof.* See Proposition 1.5 (Chapter 1) in [Ethier and Kurtz, 1986]. □

**Proposition 1.8.** *Let* $A$ *be the generator of a strongly continuous semigroup on* $C_0(E)$ *with domain* $D(A)$*. Then, the domain* $D(A)$ *is dense in* $C_0(E)$*.*

*Proof.* See Corollary 1.6 (Chapter 1) in [Ethier and Kurtz, 1986]. □

We can now define a $B(A,c)$-superprocess via its martingale problem. To do so, consider the process $(X(t))_{t\in[0,T]}$ given by $X(\omega)(t) = \omega(t)$ on the filtered probability space $\tilde{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}_m)$ with $\Omega = C([0,T], M_F(E))$, the space of continuous functions from $[0,T]$ to $M_F(E)$ equipped with the sup-norm, $\mathcal{F}$ the corresponding Borel-$\sigma$-algebra, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s^o$ with $\mathcal{F}_s^o = \sigma(X(r) : r \leq s)$ and $\mathbb{P}_m$ being the law of $X$. Further, let $A$ be the generator of a Feller process on $E$ with domain $D(A)$ and $c > 0$.

**Definition 1.9** ($B(A,c)$-superprocess)**.** *The process $X$ on $\tilde{\Omega}$ is called a $B(A,c)$-*superprocess *if its law $\mathbb{P}_m$, for a fixed $m \in M_F(E)$, is the unique solution of the martingale problem*

$$\begin{gathered}
\mathbb{P}_m(X(0)=m) = 1 \textit{ and for all } \phi \in D(A) \textit{ the process} \\
M(t)(\phi) = \langle X(t), \phi\rangle - \langle X(0), \phi\rangle - \int_0^t \langle X(s), A\phi\rangle ds, \quad t \in [0,T] \\
\textit{is a } (\mathcal{F}_t)_t\textit{-local martingale with respect to } \mathbb{P}_m \\
\textit{and has quadratic variation } [M(\phi)]_t = \int_0^t \langle X(s), c\phi^2\rangle ds.
\end{gathered} \tag{MP}$$

The resulting martingale $M(t)(\phi)$ is a true martingale if $\langle m, 1\rangle < \infty$. As we require that $m \in M_F(E)$, this is always the case in this monograph. In addition, it satisfies

$$[M(\phi), M(\psi)]_t = c\int_0^t \langle X(s), \phi\psi\rangle ds \quad \text{for all } \phi,\, \psi \in D(A) \tag{1.2}$$

and induces a martingale measure (see Section 1.3). Note that, in the literature, it is not uncommon to write that $X$ solves the martingale problem instead of being more specific and writing that its distribution $\mathbb{P}_m$ is a solution of the martingale problem.

An alternative way to define a $B(A,c)$-superprocess is given in the following theorem.

**Theorem 1.10.** *The $B(A,c)$-superprocess $X$ can also be characterized via its Laplace transform*

$$\mathbb{E}[\exp(-\langle X(t), \phi\rangle)|X(0)=m] = \exp\left(-\langle m, V_t\phi\rangle\right), \tag{1.3}$$

*where $\phi \in bp\mathcal{E}$, the space of non-negative, bounded, $\mathcal{E}$-measurable functions, and $V_t$ satisfies the log-Laplace equation*

$$V_t\phi = S_t\phi - \frac{1}{2}c\int_0^t S_{t-s}(V_s\phi)^2 ds.$$

*This characterization is equivalent to the characterization in Definition 1.9.*

*Proof.* See Chapter 4 in [Dawson, 1993]. □

By setting $V_t\phi(x) = u(t,x)$ in (1.3), we get that

$$\mathbb{E}[\exp(-\langle X(t), \phi\rangle)|X(0)=m] = \exp\left(-\langle m, u(t,\cdot)\rangle\right), \tag{1.4}$$

holds with $u$ being the unique solution to

$$\frac{\partial u}{\partial t} = Au - \frac{1}{2}cu^2, \quad u(0) = \phi.$$

This allows us to prove the following result, which is an immediate consequence of the above and needed in the subsequent chapters. The proof of this result is the motivation for introducing the Laplace approach at this point. For more on the this characterization of a $B(A,c)$-superprocess as well as Laplace transforms and log-Laplace equations for measure-valued processes, we refer to Chapter 4 in [Dawson, 1993].

**Proposition 1.11.** *If $X$ is a $B(A,c)$-superprocess, its total mass process $\langle X(t),1\rangle$ satisfies*

$$d\langle X(t),1\rangle = \sqrt{c\langle X(t),1\rangle}dW(t), \quad t \in (0,T], \tag{1.5}$$

*with $\langle X(0),1\rangle = \langle m,1\rangle$ and $W$ being a standard Brownian motion independent of $X$, i.e. $\langle X(t),1\rangle$ is a critical Feller continuous state branching process, and it holds for all $t \in [0,T]$*

$$\mathbb{E}[\langle X(t),1\rangle] = \langle m,1\rangle < \infty$$

*as well as*

$$\mathbb{E}[\langle X(t),1\rangle^2] = ct\langle m,1\rangle + \langle m,1\rangle^2 < \infty.$$

*Proof.* A Critical Feller continuous state branching process $\tilde{X}$ is given by the Laplace transform (see e.g. Section 4.3 in [Dawson, 2017])

$$\mathbb{E}[\exp(-\theta\tilde{X}(t))|\tilde{X}(0) = x] = \exp(-v_\theta(t)x)$$

with $v_\theta$ given by

$$v_\theta(t) = \frac{\theta}{1+\frac{c}{2}\theta t}, \quad c > 0.$$

For simplicity set $Y(t) = \langle X(t),1\rangle$, which implies $Y(0) = \langle m,1\rangle$ by (MP). Then, by (1.4), it holds for $\alpha \in \mathbb{R}$ and $t \in [0,T]$

$$\begin{aligned}\mathbb{E}[\exp(-\alpha Y(t))|Y(0) = \langle m,1\rangle] &= \mathbb{E}[\exp(-\langle X(t),\alpha\rangle)|X(0) = m] \\ &= \exp(-\langle m, u(t,\cdot)\rangle)\end{aligned}$$

with $u$ satisfying

$$\frac{\partial u}{\partial t} = Au - \frac{1}{2}cu^2, \quad u(0) = \alpha. \tag{1.6}$$

As $Au = 0$ if $u$ is constant in the $x$-argument, $u(t,x) = v_\alpha(t)$ satisfies (1.6) and it holds

$$\exp(-\langle m, u(t,\cdot)\rangle) = \exp(-u(t)\langle m,1\rangle) = \exp(-u(t)Y(0))$$

Consequently, the Laplace transform of $Y(t)$ coincides with the Laplace transform of a critical Feller continuous state branching process, which proves the first part.

To prove the second part, note that we get from (1.5) that $Y(t)$ is a martingale. Thus, we obtain $\mathbb{E}[Y(t)]$ from

$$\mathbb{E}[Y(t)] = \mathbb{E}[Y(t)|\mathcal{F}_0] = Y(0) = \langle m,1\rangle,$$

which is finite as $m \in M_F(E)$.

To obtain the second moment of $Y(t)$, note that from the above we get

$$\mathbb{E}[\exp(\theta Y(t))|Y(0) = \langle m, 1\rangle] = \exp\left(\frac{\theta}{1 - \frac{c}{2}\theta t}\langle m, 1\rangle\right),$$

which is the moment-generating function of $Y(t)$. Consequently, for all $t \in [0, T]$, the second moment of $Y(t)$ is given by

$$\begin{aligned}\mathbb{E}[Y(t)^2] &= \frac{d^2}{d\theta^2}\left(\frac{\theta}{1 - \frac{c}{2}\theta t}\langle m, 1\rangle\right)\bigg|_{\theta=0} \\ &= ct\langle m, 1\rangle + \langle m, 1\rangle^2,\end{aligned}$$

which is also finite as $m \in M_F(E)$. □

The interpretation of superprocesses as scaling limits of critical branching processes has already been briefly outlined in the introduction. In the following, more details are provided.

Let $\varepsilon > 0$ and consider the following branching diffusion process. At time zero, a random number of particles is placed in $E$ according to a Poisson random measure with intensity $\frac{m}{\varepsilon}$. As time goes on, the particles move around independently in $E$ with the motion given by a Feller motion process with generator $A$. The lifetime of each particle follows an independent exponential distribution with rate $\frac{c}{\varepsilon}$. At the time of death, a particle leaves behind either zero or two descendants, each with probability one half. The descendants start their independent motion at the place of death of the parent particle and act like their parent particle.

Denote by $N(t)$ the number of particles alive at time $t$ and denote their locations by $Z_i(t)$, $i = 1, \ldots, N(t)$. Further, denote the Dirac measure at $x \in E$ by $\delta_x$. The process

$$X^\varepsilon(t) = \varepsilon \sum_{i=1}^{N(t)} \delta_{Z_i(t)} \in M_F(E)$$

is called a *measure-valued branching process* and assigns mass $\varepsilon$ to each particle alive at time $t$. Now, if $\varepsilon$ goes to zero, the process $X^\varepsilon$ converges weakly to the $B(A, c)$-superprocess (see e.g. [Dawson, 1993]).

A special case of branching diffusions are binary branching Brownian motions, obtained by replacing the general Feller motion process on E in the above branching diffusion process by a Brownian motion on $\mathbb{R}^d$. The scaling limit of branching Brownian motions are the so-called *super-Brownian motion*, which are $B(\frac{1}{2}\Delta, 1)$-superprocesses, as $\frac{1}{2}\Delta$ is the generator of a Brownian motion.

In the remainder of this work, both, the more general class of $B(A, c)$-superprocesses and super-Brownian motions are of interest. While the results in Chapter 2 are proved for any $B(A, c)$-superprocess satisfying some additional requirements, in Chapter 3 we restrict our results to super-Brownian motions to make use of a particular property of the domain of $\frac{1}{2}\Delta$.

### 1.1.2 Historical Brownian Motion

As mentioned previously, historical processes are enriched versions of superprocesses. In this section, we introduce a specific historical process, namely the historical Brownian motion, which is an enriched version of the super-Brownian motion. We once again introduce the process via its martingale problem as this turns out to be advantageous in Section 3.2. However, it should be mentioned that historical processes can also be obtained as weak limits of enriched branching processes, for which the particle motion is given by a motion on the path space, as well as via their Laplace transform. We present the Laplace transform at a later point in Section 3.2 but refer to Section II.3 in [Perkins, 2002] or Section 12 in [Dawson, 1993] for details on the branching process approach.

Before we can state the martingale problem, some preparatory work is necessary. For this, we mostly follow the notation introduced by Perkins in [Perkins, 1995]. In numerous publications, Perkins and his co-authors developed the theory of historical processes. For a thorough introduction to historical processes in general and the historical Brownian motion in particular, we refer to [Dawson and Perkins, 1991], [Perkins, 1995] as well as [Perkins, 2002].

Now, let $C = C([0,T], \mathbb{R}^d)$ be the space of continuous functions mapping $[0,T]$ to $\mathbb{R}^d$. Following the approach in [Perkins, 1995], we equip the space with the compact-open topology. However, since $[0,T]$ is compact and $\mathbb{R}^d$ is a metric space, this topology coincides with the topology of uniform convergence (see e.g. Chapter 7 in [Kelley, 1975]). Denote by $\mathcal{C}$ the Borel-$\sigma$-algebra of $C$ and let $(\mathcal{C}_t)_{t\in[0,T]}$ be the canonical filtration, which is given by

$$\mathcal{C}_t = \sigma(y(s) : s \leq t,\, y \in C).$$

Next, for $y$, $w \in C$ and $s \in [0,T]$, define $y_s(t) = y(s \wedge t)$ and

$$(y/s/w)(t) = \begin{cases} y(t), & \text{if } t < s, \\ w(t-s), & \text{if } t \geq s. \end{cases}$$

An element $y \in C$ can also be viewed as a continuous path in $\mathbb{R}^d$. Thus, the object $y_s$ is the stopped path of $y$, a notion thoroughly studied in the next section. From Section V.2 in [Perkins, 2002] we know that a function $\Phi : [0,T] \times C \to \mathbb{R}$ is $(\mathcal{C}_t)_t$-predictable if and only if it is Borel-measurable and it holds $\Phi(t,y) = \Phi(t,y_t)$ for all $t \in [\tau,T]$ and $y \in C$.

As before, denote by $M_F(C)$ the space of finite measures on $C$ equipped with the topology of weak convergence. For a $t \in [0,T]$ define

$$M_F(C)^t = \{m \in M_F(C) : y = y_t \text{ for } m\text{-almost all } y\}.$$

Further, define a measure $P_{\tau,m} \in M_F(C)$ by

$$P_{\tau,m}(A) = \int_C P_{y(\tau)}(\{w : y/\tau/w \in A\})dm(y)$$

for all $A \in \mathcal{C}$, where $P_x$ is the Wiener measure on $(C, \mathcal{C})$ starting at $x \in \mathbb{R}^d$, $\tau \in [0,T]$ and $m \in M_F(C)^\tau$.

Let

$$\Omega_H = \{H \in C([\tau,T], M_F(C)) : H(t) \in M_F(C)^t \text{ for all } t \in [\tau,T]\}$$

and let $\tilde{S}$ be the space of all starting points of the historical Brownian motion,

$$\tilde{S} = \{(\tau, m) : \tau \in [0,T],\, m \in M_F(C)^\tau\}.$$

Finally, let

$$\begin{aligned} F_{\tau,m} = \{\Phi : [\tau,T]\times C \to \mathbb{R} : \Phi \text{ is } (\mathcal{C}_t)_t\text{-predictable, } P_{\tau,m}\text{-a.s. right-continuous and} \\ \sup_{s\geq\tau} |\Phi(s,y)| \leq K \text{ holds } P_{\tau,m}\text{-a.s. for some } K\} \end{aligned}$$

and

$$\begin{aligned} D(A_{\tau,m}) = \{\Phi \in F_{\tau,m} : \text{there exists a } A_{\tau,m}\Phi \in F_{\tau,m} \text{ such that} \\ \Phi(t,y) - \Phi(\tau,y) - \int_\tau^t A_{\tau,m}\Phi(s,y)ds \\ \text{is a } (\mathcal{C}_t)_{t\in[\tau,T]}\text{-martingale under } P_{\tau,m}\}. \end{aligned}$$

**Definition 1.12** (Historical Brownian motion)**.** *A predictable process $H(t)$, $t \in [\tau,T]$, on a filtered probability space $\bar{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[\tau,T]}, \mathbb{P})$ and with sample paths almost surely in $\Omega_H$ is a* historical Brownian motion *with branching rate $\gamma > 0$ and starting at $(\tau,m) \in \tilde{S}$ if and only if its law $\mathbb{P}_{\tau,m}$ solves the martingale problem*

$$\begin{aligned} &\mathbb{P}_{\tau,m}(X(\tau) = m) = 1 \text{ and for all } \Phi \in D(A_{\tau,m}) \\ &Z(t)(\Phi) = \langle H(t), \Phi(t,\cdot)\rangle - \langle m, \Phi(\tau,\cdot)\rangle - \int_\tau^t \langle H(s), A_{\tau,m}\Phi(s,\cdot)\rangle ds, \quad t \in [\tau,T], \\ &\text{is a continuous } (\mathcal{F}_t)_t\text{-martingale with respect to } \mathbb{P}_{\tau,m} \\ &\text{and has quadratic variation } [Z(\Phi)]_t = \int_\tau^t \langle H(s), \gamma\Phi(s,\cdot)^2\rangle ds. \end{aligned} \tag{MP$_{HBM}$}$$

From [Perkins, 1995] we get that the historical Brownian motion can also be defined via a more explicit martingale problem. To introduce this result, denote by $C_0^\infty(\mathbb{R}^d)$ the space of infinitely continuously differentiable functions with compact support mapping $\mathbb{R}^d$ to $\mathbb{R}$ and define

$$\begin{aligned} D_{fd} = \{\Phi : C \to \mathbb{R} : \Phi(y) = \Psi(y(t_1), \ldots, y(t_n)), \\ 0 \leq t_1 \leq \ldots \leq t_n \leq T,\, \Psi \in C_0^\infty(\mathbb{R}^{nd}),\, n \in \mathbb{N}\}. \end{aligned}$$

Thus, the space $D_{fd}$ consists of functions mapping $C$ to $\mathbb{R}^d$ that only take the values of $y \in C$ at a finite number of times into account. Next, set

$$\tilde{D}_{fd} = \{\Phi : \Phi(t,y) = \tilde{\Phi}(y_t) \text{ for some } \tilde{\Phi} \in D_{fd}\}$$

and for $\Psi \in C_0^\infty(\mathbb{R}^{nd})$ let $\Psi_{i,j}$ be the second order partial derivative of $\Psi$. For $1 \leq i,j \leq d$ and $0 \leq t_1 \leq \ldots \leq t_n \leq T$ define the $(\mathcal{C}_t)_t$-predictable process $\bar{\Psi}$ by

$$\bar{\Psi}_{i,j}(t,y) = \sum_{k=1}^n \sum_{\ell=1}^n 1_{t\leq t_k\wedge t_\ell} \Psi_{(k-1)d+i,(\ell-1)+j}(y(t_1\wedge t), \ldots, y(t_n \wedge t)).$$

Using this process, we define

$$\bar{\Delta}\Psi(t, y) = \sum_{i=1}^{d} \bar{\Psi}_{i,i}(t, y),$$

which now allows us to formulate the following result.

**Theorem 1.13** ([Perkins, 1995]). *A $(\mathcal{C}_t)_t$-predictable process $H(t)$, $t \in [\tau, T]$ on $\bar{\Omega}$ is a historical Brownian motion starting at $(\tau, m) \in \tilde{S}$ and with branching rate $\gamma > 0$ if and only if $H(t) \in M_F(C)^t$ for all $t \in [\tau, T]$ and the law $\mathbb{P}_{\tau,m}$ of $H$ is a solution to the following martingale problem*

$$\begin{gathered} \mathbb{P}_{\tau,m}(X(\tau) = m) = 1 \textit{ and for all } \Psi \in D_{fd} \\ Z(t)(\Psi) = \langle H(t), \Psi\rangle - \langle m, \Psi\rangle - \int_\tau^t \langle H(s), \frac{1}{2}\bar{\Delta}\Psi(s, \cdot)\rangle ds, \quad t \in [\tau, T], \\ \textit{is a continuous } (\mathcal{F}_t)_t\textit{-martingale with respect to } \mathbb{P}_{\tau,m} \\ \textit{and has quadratic variation } [Z(\Phi)]_t = \int_\tau^t \langle H(s), \gamma\Psi^2\rangle ds. \end{gathered} \tag{MP$_{HBM-fd}$}$$

Finally, let $\mathbb{P}_{\tau,m}$ denote the law of the historical Brownian $H$ motion starting at $(\tau, m) \in \tilde{S}$ and set

$$\tilde{\mathcal{H}}[s, t] = \sigma(H(u) : s \le u \le t).$$

By denoting the $\mathbb{P}_{\tau,m}$-completion of $\tilde{\mathcal{H}}[\tau, T]$ by $\mathcal{H}[\tau, T]$ and setting

$$\mathcal{H}_t = \left(\bigcap_{s>t} \tilde{\mathcal{H}}[\tau, s]\right) \wedge \{\mathbb{P}_{\tau,m}\text{-null sets}\},$$

we obtain a filtered probability space $(\Omega_H, \mathcal{H}[\tau, T], (\mathcal{H}_t)_{t\in[\tau,T]}, \mathbb{P}_{\tau,m})$ on which the historical Brownian motion is given by $H(t)(\omega) = \omega(t)$ (see [Perkins, 1995]).

As the historical Brownian motion only comes into play in the final sections of this monograph, we keep this introductory section on this process brief and thus conclude it with the above result on the representation of the historical Brownian motion as a canonical process. Nevertheless, some further results, like the form of the Laplace transform of a historical Brownian motion, are presented in Section 3.2, when we encounter this process for the first time.

## 1.2 Functional Itō-Calculus

In his landmark paper [Dupire, 2009], Dupire derives a functional version of Itō's lemma to model events that do not only depend on the current state $X(t)$ of a stochastic process $X$ but on its whole past $\{X(s) : s \le t\}$. His approach has since been formalized in a series of publications by Cont and Fournié ([Cont and Fournié, 2010], [Cont and Fournié, 2013], [Cont, 2016]) as well as Levental et al. ([Levental et al., 2013]).

Dupire defines the path process $X_t$ of a process $X$ by $X_t(s) = X(s)$ for all $s \in [0, t]$. It is assumed that the process $X$ is such that the process $X_t$ is an element in the space of bounded right continuous functions from $[0, t]$ to $\mathbb{R}$ with left limits, denoted by $\Lambda_t$. The functionals for which the functional Itō-formula is derived [Dupire, 2009] map paths in $\Lambda = \bigcup_{t\in[0,T]} \Lambda_t$ to $\mathbb{R}$.

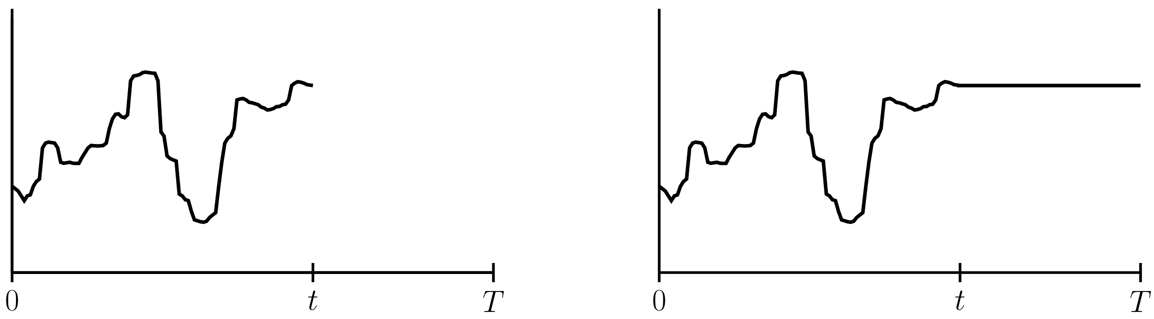


Figure 1.1: Examples of path processes on $\mathbb{R}$. Left: A path in the vector bundle considered by Dupire, which is only defined on $[0, t]$. Right: A stopped path as it is considered by Cont as well as Levental and their respective co-authors, which is defined on the whole interval $[0, T]$.

The space $\Lambda$ is often referred to as a *vector bundle* and is not a vector space.

The main difference between the two versions of the functional Itō-formula introduced below and the original work by Dupire is the underlying space of paths. Instead of considering the vector bundle, Cont and Levental and their respective co-authors modify the notion of paths of a process such that they are elements in $D([0, T], \mathbb{R}^d)$, the space of right continuous functions from $[0, T]$ to $\mathbb{R}^d$ with left limits, which is equipped with the sup-norm. More precisely, the authors consider *stopped paths*. In contrast to the paths defined in [Dupire, 2009], the path stopped at $t$ is always a function from the whole time interval $[0, T]$ to $\mathbb{R}^d$.

For simplicity, in the following we always assume that the process $X$ has continuous paths. However, to derive the functional Itō-formula, functionals defined on $D([0, T], \mathbb{R}^d)$ have to be considered. The reason for this is pointed out when we present the two versions of functional derivatives below. Both versions of the functional Itō-formula introduced below also hold for right continuous paths with left limits and we refer to the original works for the more general versions and proofs.

### 1.2.1 The Approach by Levental et al.

Let $X$ be a continuous process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote its value at time $t \in [0, T]$ by $X(t) \in \mathbb{R}^d$. Levental, Schroder and Sinha define the path of $X$ stopped at time $t \in [0, T]$ by

$$X_t(\cdot) = X(t \wedge \cdot).$$

Consequently, it holds for all $s \in [0, T]$

$$X_t(s) = \begin{cases} X(s), & \text{if } s < t, \\ X(t), & \text{if } s \geq t. \end{cases}$$

The directional functional derivatives of functionals $F : D([0, T], \mathbb{R}^d) \to \mathbb{R}$ introduced by the authors are defined for all paths in $D([0, T], \mathbb{R}^d)$ and not just stopped paths.

**Definition 1.14** ([Levental et al., 2013])**.** *Let $e_i$ be the $d$-dimensional vector with a one in the $i$th coordinate and zeros everywhere else. Let $1 \leq i,\ j \leq d$, $\omega \in C([0, T], \mathbb{R}^d)$ and*

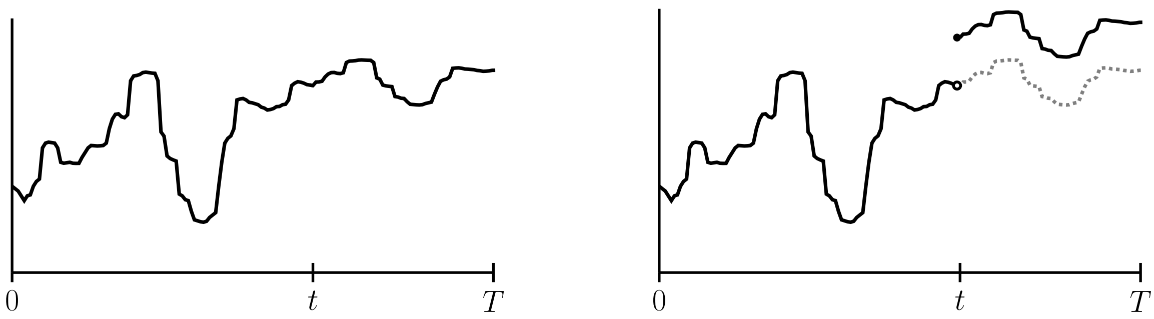


Figure 1.2: The two path processes playing a role in the definition of the functional derivative by Levental and co-authors. Left: The original path $\omega$. Right: The shifted path $\omega + \varepsilon 1_{[t,T]}$.

$F : D([0,T], \mathbb{R}^d) \to \mathbb{R}$. *Then the directional derivative of $F$ in direction $e_i 1_{[t,T]}$ is given by*

$$D_i F(\omega; [t,T]) = \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon e_i 1_{[t,T]}) - F(\omega)}{\varepsilon}$$

*if the limit exists. The second order directional derivative in directions $e_i 1_{[t,T]}$ and $e_j 1_{[t,T]}$ is given by*

$$D_{ij} F(\omega; [t,T]) = \lim_{\varepsilon \to 0} \frac{D_i F(\omega + \varepsilon e_j 1_{[t,T]}; [t,T]) - D_i F(\omega; [t,T])}{\varepsilon}$$

*if the limit exists.*

At this point it becomes clear why $F$ has to be defined on $D([0,T], \mathbb{R}^d)$. While the path $\omega$ is continuous, the *shifted path* $\omega + \varepsilon e_i 1_{[t,T]}$ is no longer continuous but only right continuous with left limits.

Next, the authors define a metric $\tilde{d}$ on $[0,T] \times D([0,T], \mathbb{R}^d)$ by

$$\tilde{d}((t,\omega),(t',\omega')) = |t - t'| + \sup_{s \in [0,T]} \|\omega(s) - \omega'(s)\|.$$

The definition of the directional functional derivatives as well as the metric $\tilde{d}$ result in the following version of the functional Itō-formula.

**Theorem 1.15** ([Levental et al., 2013])**.** *Assume the functional $F : D([0,T], \mathbb{R}^d) \to \mathbb{R}$ as well as its first and second order directional derivatives are continuous in $t$ and $\omega$ with respect to the metric $\tilde{d}$. Further, let $X$ be a continuous semimartingale. Then*

$$\begin{aligned} F(X_t) = F(X_0) &+ \sum_{i=1}^{d} \int_0^t D_i F(X_s; [s,T]) dX^i(s) \\ &+ \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t D_{ij} F(X_s; [t,T]) d[X^i, X^j](s). \end{aligned} \tag{1.7}$$

### 1.2.2 The Approach by Cont and Fournié

In their first work on the functional Itō-formula ([Cont and Fournié, 2010]), Cont and Fournié are still working with the vector bundle approach used in [Dupire, 2009]. In later publications,

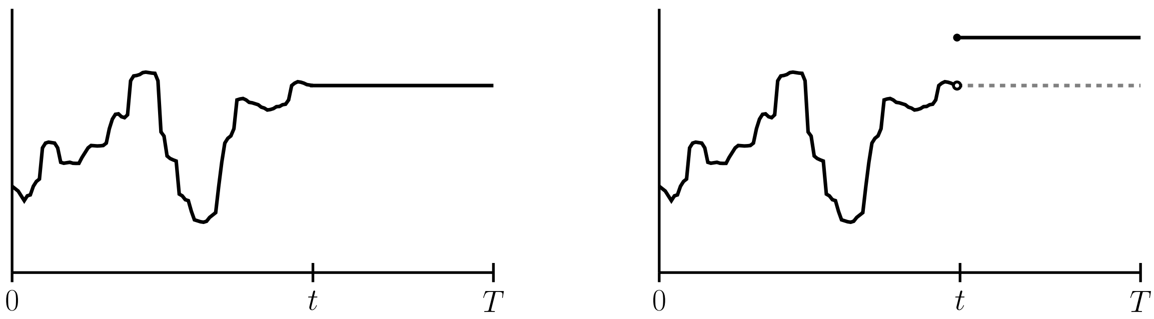


Figure 1.3: The two path processes playing a role in the definition of the functional derivative by Cont and Fournié. Left: The original stopped path $\omega_t$. Right: The shifted stopped path $\omega_t + \varepsilon 1_{[t,T]}$.

summarized in [Cont, 2016], the authors no longer use the vector bundle approach but work on a quotient space defined as follows.

Once again, for $\omega \in D([0,T], \mathbb{R}^d)$ set $\omega_t(\cdot) = \omega(t \wedge \cdot)$. Then, the space of stopped paths is defined as the quotient space

$$\Lambda^d = \{(t, \omega_t) : (t, \omega) \in [0,T] \times D([0,T], \mathbb{R}^d)\} = [0,T] \times D([0,T], \mathbb{R}^d)/\sim$$

with the equivalence relation given by

$$(t, \omega) \sim (t', \omega) \quad \Leftrightarrow \quad \{t = t' \text{ and } \omega_t = \omega'_{t'}\}.$$

This space is equipped with a metric $d_\infty$ defined by

$$d_\infty((t,\omega),(t',\omega')) = |t - t'| + \sup_{s \in [0,T]} \|\omega_t(s) - \omega'_{t'}(s)\|.$$

Using these definitions, two kinds of derivatives are defined for non-anticipative functionals on $[0,T] \times D([0,T], \mathbb{R}^d)$, i.e. measurable maps $F : (\Lambda^d, d_\infty) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The first kind of derivative is with respect to time $t$, called *horizontal* derivative in [Cont, 2016].

**Definition 1.16** ([Cont, 2016]). *A non-anticipative functional $F : \Lambda^d \to \mathbb{R}$ is said to be* horizontally differentiable *at $(t, \omega) \in \Lambda^d$ if the limit*

$$\mathcal{D}F(t, \omega) = \lim_{\varepsilon \downarrow 0} \frac{F(t + \varepsilon, w_t) - F(t, w_t)}{\varepsilon}$$

*exists. If $F$ is horizontally differentiable for all $(t, \omega) \in \Lambda^d$, the functional $\mathcal{D}F$ is called the* horizontal derivative *of $F$.*

The second kind of derivative is the actual functional derivative, called *vertical* derivative in [Cont, 2016]. It compares to the derivative introduced in [Levental et al., 2013] with the major difference being that the definition in [Cont, 2016] is only considering stopped paths.

**Definition 1.17** ([Cont, 2016]). *Let $e_i$ be the $d$-dimensional vector with a one in the $i$th coordinate and zeros everywhere else. A non-anticipative functional $F$ is said to be* vertically differentiable *at $(t, \omega) \in \Lambda^d$ if the limit*

$$\partial_i F(t, \omega) = \lim_{\varepsilon \to 0} \frac{F(t, \omega_t + \varepsilon e_i 1_{[t,T]}) - F(t, \omega_t)}{\varepsilon}, \quad i = 1, \ldots, d,$$

*exists. If $F$ is vertically differentiable for all $(t,\omega) \in \Lambda^d$, the vector $\nabla_\omega F(t,\omega) = (\partial_i F(t,\omega))_{i=1,\dots,d}$ is called the* vertical derivative *of $F$. The second order directional vertical derivative is given by*

$$\partial_i \partial_j F(t,\omega) = \lim_{\varepsilon \to 0} \frac{\partial_j F(t, \omega_t + \varepsilon e_i 1_{[t,T]}) - \partial_j F(t,\omega_t)}{\varepsilon}, \quad i,\, j = 1, \dots, d,$$

*if the limit exists. If all derivatives exist, set $\nabla^2_\omega F(t,\omega) = (\partial_i \partial_j F(t,\omega))_{i,j=1,\dots,d}$.*

As in the approach in [Levental et al., 2013], the definition of vertical derivatives requires the definition of $F$ for paths in $D([0,T],\mathbb{R}^d)$. By comparing Figure 1.3 to Figure 1.2, the differences in the paths considered in the definition of the derivatives becomes clear.

The following version of the functional Itō-formula is a slight simplification of the actual formulation found in [Cont, 2016]. To formulate it, let $X$ be a continuous process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote its value at time $t \in [0,T]$ by $X(t) \in \mathbb{R}^d$.

**Theorem 1.18** ([Cont, 2016])**.** *Assume the non-anticipative functional $F : \Lambda^d \to \mathbb{R}$ as well as the processes $\mathcal{D}F$, $\nabla_\omega F$ and $\nabla^2_\omega F$ are continuous with respect to the metric $d_\infty$ and bounded. Further, let $X$ be a continuous semimartingale. Then*

$$\begin{aligned} F(t, X_t) = F(0, X_0) + \int_0^t \mathcal{D}F(s, X_s) ds + \sum_{i=1}^d \int_0^t \partial_i F(s, X_s) dX(s) \\ + \frac{1}{2} \sum_{i,\, j=1}^d \int_0^t \partial_i \partial_j F(s, X_s) d[X^i, X^j](s). \end{aligned} \tag{1.8}$$

Note that the only difference between the two versions (1.7) and (1.8) is the addition of the time argument in the functional $F$ and the resulting $\mathcal{D}F$ term in (1.8). The difference in the definition of the derivatives vanishes as in (1.7) the derivatives are only computed for stopped paths.

## 1.3 Martingale Measures

The concept of martingale measures is introduced in [Walsh, 1986] as a measure-valued counterpart to the traditional stochastic white noise and is used to study stochastic partial differential equations. In our context, martingale measures play a fundamental role in the formulation of the Itō-formulae for $B(A,c)$-superprocesses in Chapter 2 as well as the martingale representation formulae in Chapter 3.

Both, a $B(A,c)$-superprocess as well as a historical Brownian motion give rise to a martingale measure. While we prove this result for $B(A,c)$-superprocesses in the second part of this section, we refer to Chapter 2 in [Perkins, 1995] for the derivation of the martingale measure corresponding to a historical Brownian motion.

We start this section with a summary of the relevant parts of the introduction of martingale measures and the integration with respect to such measures in [Walsh, 1986]. The definition of the stochastic integral with respect to a martingale measure in [Walsh, 1986] is restricted to predictable integrands. However, to prove the results in Chapter 2, we have to compute the integral for optional integrands. Therefore, we conclude this introductory chapter by proving that we can extend the class of valid integrands to include such optional functions.

### 1.3.1 Martingale Measures and Integration with respect to Martingale Measures

For the sake of a brief introduction to martingale measures, we only consider the scenario relevant for the remainder of this monograph. Among others restrictions, this implies a restriction to a locally compact separable metric space $E$ and the definition of stochastic integrals with respect to orthogonal martingale measures. For the more general setting in which $E$ is a Lusin space and the stochastic integral with respect to a more general worthy martingale measure as well as the proofs of the results stated, we refer to [Walsh, 1986].

Let $E$ be a locally compact separable metric space with Borel-$\sigma$-algebra $\mathcal{E}$. Further, assume $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ is a filtered probability space with a right continuous filtration, set $L^2 = L^2(\Omega, \mathcal{F}, \mathbb{P})$ and define the $L^2$-norm by $\|f\|_2 = \mathbb{E}[f^2]^{\frac{1}{2}}$.

Next, consider a function $U$ defined on $\mathcal{A} \times \Omega$ with $\mathcal{A}$ being a subalgebra of $\mathcal{E}$ that satisfies $\|U(B)\|_2 < \infty$ for all $B \in \mathcal{A}$ as well as $U(B_1 \cup B_2) = U(B_1) + U(B_2)$ for all $B_1$, $B_2 \in \mathcal{A}$ with $B_1 \cap B_2 = \emptyset$. Additionally, define a set function $\mu$ by

$$\mu(B) = \|U(B)\|_2^2.$$

The function $U$ is called *$\sigma$-finite* if there exists an increasing sequence $(E_n)_n \subset \mathcal{E}$ with $\bigcup_n E_n = E$ and such that $\mathcal{E}_n = \mathcal{E}|_{E_n} \subset \mathcal{A}$ as well as $\sup_{B\in\mathcal{E}_n} \|U(B)\|_2 < \infty$ for all $n \in \mathbb{N}$. It is called *countably additive* on $(\mathcal{E}_n)_n$ if , in addition, for any sequence $(B_j)_j$

$$B_j \in \mathcal{E}_n \text{ for all } n \text{ and } B_j \downarrow \emptyset \text{ implies } \lim_{j\to\infty} \mu(B_j) = 0.$$

Further, if $U$ is countably additive on $(\mathcal{E}_n)_n$, it can be extended to $\mathcal{E}$ by setting

$$U(B) = \begin{cases} \lim_{n\to\infty} U(B \cap E_n), & \text{if the limit exists,} \\ \text{undefined,} & \text{otherwise} \end{cases} \tag{1.9}$$

for any $B \in \mathbb{E}$.

**Definition 1.19** ($\sigma$-finite $L^2$-valued measure)**.** *A countably additive function $U$ is called a* $\sigma$-finite $L^2$-valued measure *if it has been extended as in* (1.9)*.*

**Definition 1.20** (Martingale measure)**.** *A process $M_t(B)$, $t \in [0,T]$, $B \in \mathcal{A}$, is called a* martingale measure *if*

*(i)* $M_0(B) = 0$ *for all* $B \in \mathcal{A}$*,*

*(ii)* $M_t$ *is a $\sigma$-finite $L^2$-valued measure for all* $t \in (0,T]$*,*

*(iii) the process* $(M_t(B))_{t\in[0,T]}$ *is a* $(\mathcal{F}_t)_t$*-martingale for all* $B \in \mathcal{A}$*.*

*A martingale measure is called* continuous *if for all $B \in \mathcal{A}$ the mapping $t \mapsto M_t(B)$ is continuous.*

In order to define the stochastic integral with respect to a martingale measure $M$, further conditions on $M$ have to be imposed. One such condition is the following.

**Definition 1.21** (Orthogonal martingale measure)**.** *A martingale measure $M$ is called* orthogonal *if $B_1$, $B_2 \in \mathcal{A}$, $B_1 \cap B_2 = \emptyset$ implies that the martingales $\{M_t(B_1)\}_{t\in[0,T]}$ and $\{M_t(B_2)\}_{t\in[0,T]}$ are orthogonal, i.e. $\{M_t(B_1)M_t(B_2)\}_{t\in[0,T]}$ is a martingale.*

The definition of the stochastic integral with respect to an orthogonal martingale measure relies on the fact that every orthogonal martingale measure is a worthy martingale measure. Worthy martingale measures are martingale measures for which a dominating measure exists. To define dominating measures, we first have to introduce the *(co)variation* $Q$ of an orthogonal martingale measure $M$. For such a martingale measure, define the set function $Q$ for $(s,t] \subset [0,T]$ and $B \in \mathcal{E}$ by

$$Q((s,t] \times B) = [M(B)]_t - [M(B)]_s$$

and extend $Q$ by additivitiy to finite unions of disjoint sets $(s_i,t_i] \times B_i$, $i = 1,\dots,n$, by

$$Q\left(\bigcup_{i=1}^{n}(s_i,t_i] \times B_i\right) = \sum_{i=1}^{n}([M(B_i)]_{t_i} - [M(B_i)]_{s_i}).$$

**Definition 1.22** (Dominating measure)**.** *A random $\sigma$-finite measure $K$ defined on $\mathcal{B}([0,T]) \times \mathcal{E} \times \Omega$ is called* dominating measure *of an orthogonal martingale measure $M$ if*

*(i) $K$ is positive definite,*

*(ii) for fixed $B \in \mathcal{E}$, $\{K((0,t] \times B)\}_{t\in[0,T]}$ is predictable,*

*(iii) for all $n$ it holds $\mathbb{E}[K([0,T] \times E_n)] < \infty$ ,*

*(iv) for any $(s,t] \times B \subset [0,T] \times E$ it holds $|Q((s,t] \times B)| \leq K((s,t] \times B)$ almost surely.*

The above definition is not the original version of the definition of dominating measures as it can be found in [Walsh, 1986]. Instead, it has already been adjusted to account for the fact that we only consider orthogonal martingale measures. For such martingale measures, as the following proposition states, we immediately get the dominating measure from the covariation $Q$ defined above.

**Proposition 1.23.** *For an orthogonal martingale measure, it holds for any $t \in [0,T]$ and $B \in \mathcal{E}$*

$$\mathbb{P}(Q((0,t] \times B) = K((0,t] \times B)) = 1.$$

We now have everything on hand to define the stochastic integral with respect to an orthogonal martingale measure. The construction follows the standard steps known from the construction of the regular Itō-integral.

**Definition 1.24** (Elementary and Simple functions)**.** *A function $f : \Omega \times [0,T] \times E \to \mathbb{R}$ is called* elementary *if it can be written as*

$$f(\omega,s,x) = X(\omega)1_{(a,b]}(s)1_B(x)$$

*with $0 \leq a < b \leq T$, $X$ a bounded, $\mathcal{F}_a$-measurable random variable and $B \in \mathcal{E}$. Functions which can be written as a linear combination of elementary functions are called* simple *and the class of simple functions is denoted by $\mathcal{S}$.*

**Definition 1.25** (Predictable functions)**.** *Denote by $\mathcal{P}$ the $\sigma$-algebra on $\Omega \times E \times [0,T]$ generated by $\mathcal{S}$. The $\sigma$-algebra $\mathcal{P}$ is called the* predictable $\sigma$-algebra *and functions that are measurable with respect to $\mathcal{P}$ are called* predictable functions*.*

**Definition 1.26** ($\|\cdot\|_M$, $\mathcal{P}_M$)**.** *For an orthogonal martingale measure with dominating measure $K$, define a norm on the set of predictable functions by*

$$\|f\|_M = \mathbb{E}\left[\int_0^T \int_E |f(s,x)|^2 K(dx,ds)\right]^{\frac{1}{2}}$$

*and denote by $\mathcal{P}_M$ the set of predictable functions with finite $\|\cdot\|_M$-norm.*

For an elementary function $f(\omega,s,x) = X(\omega)1_{(a,b]}(s)1_{\tilde{B}}(x)$, the stochastic integral with respect to an orthogonal martingale measure $M$, denoted by $f \bullet M$, is defined by

$$f \bullet M_t(B) = X(\omega)(M_{t\wedge b}(\tilde{B} \cap B) - M_{t\wedge a}(\tilde{B} \cap B))$$

and the definition can be extend to $f \in \mathcal{S}$ by linearity.

**Proposition 1.27.** *It holds for all $f \in \mathcal{S}$ and all orthogonal martingale measures $M$ that*

*(i)* $f \bullet M$ *is an orthogonal martingale measure,*

*(ii)* $\mathbb{E}[(f \bullet M_t(B))^2] \leq \|f\|_M^2$ *for all $B \in \mathcal{E}$ and $t \in [0,T]$.*

In order to extend the definition of the stochastic integral with respect to an orthogonal martingale measure to function in $\mathcal{P}_M$, we need the following result.

**Proposition 1.28.** *The class $\mathcal{S}$ is dense in $\mathcal{P}_M$ with respect to the norm $\|\cdot\|_M$.*

The above proposition allows us to find, for every $f \in \mathcal{P}_M$, a sequence $(f_n)_n \subset \mathcal{S}$ such that $\|f_n - f\|_M \to 0$ as $n \to \infty$. From Proposition 1.27 we further get

$$\mathbb{E}[(f_m \bullet M_t(B) - f_n \bullet M_t(B))^2] \leq \|f_m - f_n\|_M^2.$$

As the series $(f_n)_n$ converges to $f$ with respect to $\|\cdot\|_M$, we get that

$$\mathbb{E}[(f_m \bullet M_t(B) - f_n \bullet M_t(B))^2] \to 0 \quad \text{as } m,\, n \to \infty.$$

Consequently, the sequence $(f_n \bullet M_t(B))_n$ is Cauchy and as $L^2(\Omega,\mathcal{F},\mathbb{P})$ is complete, the $L^2$-limit $f \bullet M_t(B)$ exists. This completes the construction of the integral with respect to a martingale measure for functions in $\mathcal{P}_M$.

**Proposition 1.29.** *It holds for all $f \in \mathcal{P}_M$ and all orthogonal martingale measures $M$ that*

*(i)* $f \bullet M$ *is an orthogonal martingale measure,*

*(ii)* $\mathbb{E}[(f \bullet M_t(B))^2] \leq \|f\|_M^2$ *for all $B \in \mathcal{E}$ and $t \in [0,T]$.*

To conclude the introductory part on martingale measures and integrals with respect to an orthogonal martingale measure, we introduce the following notion which is in line with the familiar notation of integrals and thus simplifies the representation of the results in Chapter 2 and Chapter 3:

$$f \bullet M_t(B) = \int_0^t \int_B f(s,x) M(ds,dx). \tag{1.10}$$

### 1.3.2 The Martingale Measure of the $B(A,c)$-Superprocess

We previously mentioned that there are different ways to define $B(A,c)$-superprocesses. In this section, we prove the existence of a martingale measure associated with a $B(A,c)$-superprocess, which is based on the process $M(t)(\phi)$ in (MP). This connection between the martingale problem of a $B(A,c)$-superprocess and the martingale measure induced by it is part of the motivation for defining $B(A,c)$-superprocesses via their martingale problems.

Consider the setting in Section 1.1.1 with $E$ being a locally compact separable metric space with Borel-$\sigma$-algebra $\mathcal{E}$. Before we derive the martingale measure, recall the following concept.

**Definition 1.30.** *A sequence $(f_n)_n$ of functions from $E$ to $\mathbb{R}$ converges* bounded pointwise *(bp) to $f$ if the sequence $(f_n)_n$ converges pointwise to $f$ and there exists a constant $C \in \mathbb{R}$ such that $|f_n(x)| < C$ for all $x \in E$ and $n \in \mathbb{N}$.*

As the domain $D(A)$ of the generator $A$ is dense in $C(E)$ (see Proposition 1.8) and as the bounded pointwise closure of $C(E)$ is the set of bounded $\mathcal{E}$-measurable functions, denoted by $b\mathcal{E}$, $D(A)$ is bp-dense in $b\mathcal{E}$. Consequently, as $1_B \in b\mathcal{E}$, $B \in \mathcal{E}$, there exists a sequence $(f_n)_n \subset D(A)$ such that $f_n \xrightarrow{bp} 1_B$. By choosing the sequence $(f_n)_n$ such that $|f_n| \leq 1$ for all $n$, this allows us to define the following $L^2$-limit

$$M_t(B) := M(t)(1_B) := \lim_{n\to\infty} M(t)(f_n), \tag{1.11}$$

where $M(t)(\cdot)$ is the martingale arising from the martingale problem (MP) and the limit exists by the dominated convergence theorem.

**Theorem 1.31.** *The $L^2$-limit $M$ defined by* (1.11) *is a continuous orthogonal martingale measure with dominating measure*[1] *given by*

$$\nu((s,t] \times B) = \int_s^t \langle X(s), 1_B\rangle ds \quad \textit{for all } 0 \leq s < t \leq T \textit{ and } B \in \mathcal{E}.$$

*Proof.* To prove that $M$ is a martingale measure, we have to show that $M$ satisfies the three properties in Definition 1.20. The first property is trivial as by definition

$$\begin{aligned} M(0)(B) &= \lim_{n\to\infty} \left( \langle X(0), f_n\rangle - \langle X(0), f_n\rangle - \int_0^0 \langle X(s), Af_n\rangle ds \right) \\ &= 0 \end{aligned}$$

holds for every suitable sequence $(f_n)_n$.

To prove the second property, recall that we get from Proposition 1.11 that, for all $t \in [0,T]$, $\mathbb{E}[(M(t)(E))^2] < \infty$ holds. Thus, $\mathbb{E}[(M(t)(B))^2] < \infty$ for all $B \in \mathcal{E}$ and we can pick the subalgebra $\mathcal{A}$ to be $\mathcal{E}$.

Now, let $(B_j)_j \subset E$ be a sequence with $B_j \to \emptyset$ for $j \to \infty$ and set $f_j(\omega, s) = \int_{B_j} X(\omega, s)(dx)$. Then, $(f_j)_j$ is almost surely monotonically decreasing to zero and non-negative for all $s \in$

[1] In the context of superprocesses, it is common to denote the dominating measure of the martingale measure (and thus the covariation process if the martingale measure is orthogonal) by $\nu$ instead of $K$ (or $Q$).

$[0,T]$. As, in addition, $\int_0^t f(\omega,s)ds < \infty$ holds almost surely for any choice of $B_1 \in \mathcal{E}$, since $X(s) \in M_F(E)$ for all $s \in [0,T]$, we can apply the monotone convergence theorem to obtain

$$\lim_{j\to\infty} \int_0^t f_j(\omega,s)ds = \int_0^t \lim_{j\to\infty} f_j(\omega,s)ds = \int_0^t 0ds = 0.$$

Next, set $g_j^t(\omega) = \int_0^t f_j(\omega,s)ds$. The sequence $(g_j^t)_j$ is also almost surely monotonically decreasing to zero for all $t \in [0,T]$. In addition, as

$$\mathbb{E}[g_1^t] = \mathbb{E}\left[\int_0^t \int_{B_1} X(s)(dx)ds\right] = \mathbb{E}[(M(t)(B_1))^2] < \infty,$$

the monotone convergence theorem can be applied a second time to obtain

$$\lim_{j\to\infty} \mathbb{E}[g_j^t] = \mathbb{E}[\lim_{j\to\infty} g_j^t] = \mathbb{E}[0] = 0.$$

Combining the above, we have

$$\begin{aligned}
\lim_{j\to\infty} \mu(B_j) &= \lim_{j\to\infty} \mathbb{E}[(M(t)(B_j))^2] \\
&= \lim_{j\to\infty} \mathbb{E}\left[c\int_0^t \langle X(s), (1_{B_j})^2\rangle ds\right] \\
&= \lim_{j\to\infty} \mathbb{E}\left[c\int_0^t \langle X(s), 1_{B_j}\rangle ds\right] \\
&= 0.
\end{aligned}$$

By setting $E_n = E$ for all $n \in \mathbb{N}$, we obtain that $M$ is $\sigma$-finite. As it is also finitely additive and

$$\lim_{n\to\infty} M(t)(B \cap E_n) = \lim_{n\to\infty} M(t)(B \cap E) = \lim_{n\to\infty} M(t)(B) = M(t)(B)$$

holds for all $t \in [0,T]$, $M$ is a $\sigma$-finite $L^2$-valued measure.

The third property in Definition 1.20 follows from the fact that $M(t)(B)$ is defined as a $L^2$-limit of processes $M(t)(f_n)$ with $(f_n)_n \subset D(A)$. These processes are martingales by (MP) and thus the $L^2$-limit is also a martingale. As the processes $M(t)(f_n)$ are continuous, this also yields the continuity of the $L^2$-limit $M(t)(B)$. Hence, $M$ is a continuous martingale measure.

The martingale measure is also orthogonal as for any $B_1$, $B_2 \in \mathcal{E}$ with $B_1 \cap B_2 = \emptyset$ it holds

$$[M(B_1), M(B_2)]_t = c\int_0^t \langle X(s), 1_{B_1}1_{B_2}\rangle ds = 0$$

by (1.2). From the above, we also get that the dominating measure $\nu$ has to be of the form presented in the statement of the theorem, which completes the proof. □

Notation (1.10) allows us to highlight that $M$ is the martingale measure associated with a $B(A,c)$-superprocess $X$ by writing $M_X$ instead of $M$. Therefore,

$$\int_0^t \int_E f(s,x) M_X(ds,dx)$$

is the stochastic integral of $f$ with respect to the martingale measure associated with the $B(A,c)$-superprocess $X$. In particular, we get that the process $M(t)(\phi)$ in (MP) can be written as

$$M(t)(\phi) = \int_0^t \int_E \phi(s,x) M_X(ds,dx).$$

### 1.3.3 Extending the Class of Integrands

The results in Section 1.3.1 and Section 1.3.2 allow us to define the stochastic integral of a function $f \in \mathcal{P}_M$ with respect to the martingale measure associated with a $B(A,c)$-superprocess. However, some functions considered in Chapter 2 are not predictable but right continuous with left limits and thus not in the class of functions for which Walsh defines the stochastic integral with respect to a martingale measure. In this section, we extend the definition of the integral to a wider class of functions to include all the functions we consider in this monograph.

More precisely, we introduce a class of integrands $\mathcal{I}_{M_X}$ for which we can define the stochastic integral with respect to the martingale measure associated with a $B(A,c)$-superprocess. This class includes the class $\mathcal{P}_M$ introduced in Section 1.3.1 but also bounded optional functions. As bounded right continuous functions with left limits are optional (see Proposition 1.35), this completion is sufficient to prove the results in the following chapters.

Once again, consider the setting in Section 1.1.1, a $B(A,c)$-superprocess $X$ and denote the distribution of $X$ by $\mathbb{P}$.

**Definition 1.32** (Optional functions)**.** *Denote by $\mathcal{O}$ the $\sigma$-algebra generated by linear combinations of functions of the form*

$$f(\omega,s,x) = Y(\omega)1_{[a,b)}(s)1_B(x)$$

*with $0 \leq a < b \leq T$, $Y$ a bounded, $\mathcal{F}_a$-measurable random variable and $B \in \mathcal{E}$. The $\sigma$-algebra $\mathcal{O}$ is called* optional $\sigma$-algebra *and a function is called* optional *if it is $\mathcal{O}$-measurable.*

Futher, let $M_X$ be the martingale measure associated with $X$ and define a measure $\mu_{M_X}$ on $\mathcal{F} \times \mathcal{B}([0,T]) \times \mathcal{E}$ by

$$\mu_{M_X}(B_1 \times B_2 \times B_3) = \mathbb{E}\left[c\int_0^T \int_E 1_{B_1\times B_2\times B_3}(\omega,s,x)X(\omega,s)(dx)ds\right].$$

This is the extension of the so-called *Doléans measure* of $M_X$ to $\mathcal{F} \times \mathcal{B}([0,T]) \times \mathcal{E}$. Denote by $\mathcal{L}^2_{\mathcal{P}}$ the space $L^2(\Omega \times [0,T] \times E, \mathcal{P}, \mu_{M_X})$, i.e. the space of $\mathcal{P}$-measurable functions satisfying $\int f^2 d\mu_{M_X} < \infty$, and note that it coincides with the space $\mathcal{P}_{M_X}$ introduced in Section 1.3.1. Finally, denote the class of $\mu_{M_X}$-null sets in $\mathcal{F} \times \mathcal{B}([0,T]) \times \mathcal{E}$ by $\mathcal{N}$ and set $\tilde{\mathcal{P}} = \mathcal{P} \wedge \mathcal{N}$. In the remainder of this section we show that $\mathcal{L}^2_{\mathcal{P}} = \mathcal{L}^2_{\tilde{\mathcal{P}}}$ holds by following standard arguments that can for example be found in [Chung and Williams, 2014]. This allows us to extend the stochastic integral with respect $M_X$ to functions in $\mathcal{L}^2_{\tilde{\mathcal{P}}}$. For convenience we later denote this space of functions (for which we can define the stochastic integral with respect to $M_X$) by $\mathcal{I}_{M_X}$.

**Proposition 1.33.** *(i) A subset $\tilde{B}$ of $\Omega \times [0,T] \times E$ belongs to $\tilde{\mathcal{P}}$ if and only if there exists a $B \in \mathcal{P}$ such that*

$$\tilde{B}\Delta B = (\tilde{B} \setminus B) \cup (B \setminus \tilde{B}) \in \mathcal{N}.$$

*(ii) If $f : \Omega \times [0,T] \times E \to \mathbb{R}$ is $\mathcal{F} \times \mathcal{B}([0,T]) \times \mathcal{E}$-measurable, it is $\tilde{\mathcal{P}}$-measurable if and only if there exists a predictable function $g$ such that*

$$\{f \neq g\} \in \mathcal{N}.$$

*Proof.* (i) Set

$$\mathcal{A} = \{\tilde{B} : \exists B \in \mathcal{P} \text{ s.t. } \tilde{B}\Delta B \in \mathcal{N}\}.$$

The proof is complete if we can prove that $\mathcal{A} = \tilde{\mathcal{P}}$. Thus, consider $C = \tilde{B}\Delta B$ and observe that $\tilde{B} = C\Delta B$ holds. Now, if $C \in \mathcal{N} \subset \tilde{\mathcal{P}}$ and $\tilde{B} \in \mathcal{P} \subset \tilde{\mathcal{P}}$, this immediately yields $\tilde{B} \in \tilde{\mathcal{P}}$. Consequently $\mathcal{A} \subset \tilde{\mathcal{P}}$.

As $\mathcal{A}$ contains $\mathcal{N}$ and $\mathcal{P}$, to prove that $\tilde{\mathcal{P}} \subset \mathcal{A}$ also holds, it suffices to show that $\mathcal{A}$ is a $\sigma$-algebra. This is the case as

- $\emptyset \in \mathcal{P}$ and therefore $\emptyset\Delta\emptyset = \emptyset \in \mathcal{N}$, which implies $\emptyset \in \mathcal{A}$,
- $\tilde{B}^c\Delta B^c = \tilde{B}\Delta B$ and therefore $\tilde{B}^c \in \mathcal{A}$ for all $\tilde{B} \in \mathcal{A}$,
- $(\bigcup_n \tilde{B}_n)\Delta(\bigcup_n B_n) \subset \bigcup_n(\tilde{B}_n\Delta B_n)$ and therefore $\bigcup_n \tilde{B}_n \in \mathcal{A}$ for all $(\tilde{B}_n)_n \subset \mathcal{A}$.

Thus, $\tilde{\mathcal{P}} \subset \mathcal{A}$, which yields $\tilde{\mathcal{P}} = \mathcal{A}$.

(ii) Let $g$ be a predictable function and assume $\{f \neq g\} \in \mathcal{N}$. Then, we have

$$\{f \in S\}\Delta\{g \in S\} \subset \{f \neq g\} \quad \text{for all } S \in \mathcal{B}(\mathbb{R}).$$

By the first part of this proposition, as $\{g \in S\} \in \mathcal{P}$, we get $\{f \in S\} \in \tilde{\mathcal{P}}$ and consequently $f$ is $\tilde{\mathcal{P}}$-measurable.

Now assume $f$ is $\tilde{\mathcal{P}}$-measurable and $f = \sum_{j=1}^{\infty} c_j 1_{\tilde{B}_j}$ for disjoint $\tilde{B}_j \in \tilde{\mathcal{P}}$ and $c_j \in \mathbb{R}$. For each $\tilde{B}_j$ there exists a $B_j \in \mathcal{P}$ such that $\tilde{B}_j\Delta B_j \in \mathcal{N}$ by the first part of this proposition. As the $\tilde{B}_j$'s are disjoint, $\tilde{B}_j \cap B_j \subset \tilde{B}_j$ and $B_j \cap \tilde{B}_j^c = B_j \setminus \tilde{B}_j \subset \tilde{B}_j\Delta B_j$, we have

$$B_i \cap B_j = \Big((B_i \cap \tilde{B}_i) \cup (B_i \cap \tilde{B}_i^c)\Big) \cap \Big((B_j \cap \tilde{B}_j) \cup (B_j \cap \tilde{B}_j^c)\Big) \in \mathcal{N} \tag{1.12}$$

if $i \neq j$. To obtain disjoint sets, set

$$B_1' = B_1 \quad \text{and} \quad B_j' = \bigcap_{i=1}^{j-1} B_i^c \cap B_j \text{ for } j \geq 2.$$

Then, as $B_j\Delta B_j' = (B_j \cup B_j') \setminus (B_j \cap B_j')$, we get from (1.12) that

$$B_j\Delta B_j' = \left(B_j \cup \left(\bigcap_{i=1}^{j-1} B_i^c \cap B_j\right)\right) \setminus \left(B_j \cap \left(\bigcap_{i=1}^{j-1} B_i^c \cap B_j\right)\right) = B_j \setminus \left(\bigcap_{i=1}^{j-1} B_i^c\right) \in \mathcal{N}$$

holds. In addition, for $i \neq j$,

$$\begin{aligned}
\tilde{B}_j\Delta B_j' &= \left(\tilde{B}_j \setminus \left(B_j \cap \left(\bigcap_{i=1}^{j-1} B_i^c\right)\right)\right) \cup \left(\left(B_j \cap \left(\bigcap_{i=1}^{j-1} B_i^c\right)\right) \setminus \tilde{B}_j\right) \\
&= (\tilde{B}_j \setminus B_j) \cup \left(\tilde{B}_j \setminus \bigcap_{i=1}^{j-1} B_i^c\right) \cup \left((B_j \setminus \tilde{B}_j) \cap \left(\bigcap_{i=1}^{j-1} B_i^c\right)\right) \in \mathcal{N}
\end{aligned}$$

holds, as the union of the first and third term is a subset of $\tilde{B}_j \Delta B_j$ and the second term can be written as

$$\tilde{B}_j \setminus \bigcap_{i=1}^{j-1} B_i^c = \bigcup_{i=1}^{j-1} B_i \cap \tilde{B}_j$$

and, if $i \neq j$, $B_i \cap \tilde{B}_j \in \mathcal{N}$ holds as the $\tilde{B}_j$'s are disjoint and $(\tilde{B}_j \Delta B_j) \in \mathcal{N}$.

Next, set $g = \sum_{j=1}^{\infty} c_j 1_{B'_j}$. Then $g$ is $\mathcal{P}$-measurable and

$$\{f \neq g\} \subset \bigcup_{j=1}^{\infty} (\tilde{B}_j \Delta B'_j) \in \mathcal{N}.$$

For general $\tilde{\mathcal{P}}$-measurable $f$, there exists a sequence $(f_n)_n$ of $\tilde{\mathcal{P}}$-measurable functions of the above form such that $f_n \to f$ $\mu_{M_X}$-almost surely. Pick $\mathcal{P}$-measurable $g_n$'s such that $\{f_n \neq g_n\} \in \mathcal{N}$ for all $n \in \mathbb{N}$ and set $g = \liminf_{n\to\infty} g_n$. Then $g$ is also $\mathcal{P}$-measurable and

$$\bigcup_{n=1}^{\infty} \{f_n \neq g_n\} \cup \{\lim_{n\to\infty} f_n \neq f\} \in \mathcal{N}.$$

Set

$$\Sigma = (\Omega \times [0,T] \times E) \setminus \left( \bigcup_{n=1}^{\infty} \{f_n \neq g_n\} \cup \{\lim_{n\to\infty} f_n \neq f\} \right).$$

Obviously, the set $\Sigma$ has full mass, i.e. $(\Omega \times [0,T] \times E) \setminus \Sigma \in \mathcal{N}$, and on $\Sigma$ we have $f_n(\omega, s, x) = g_n(\omega, s, x)$ for all $n \in \mathbb{N}$. Therefore

$$\liminf_{n\to\infty} f_n(\omega, s, x) = \liminf_{n\to\infty} g_n(\omega, s, x)$$

on $\Sigma$ and

$$\begin{aligned}
\emptyset &= \{(\omega, s, x) \in \Sigma : f(\omega, s, x) \neq \lim_{n\to\infty} f_n(\omega, s, x)\} \\
&= \{(\omega, s, x) \in \Sigma : f(\omega, s, x) \neq \liminf_{n\to\infty} f_n(\omega, s, x)\} \\
&= \{(\omega, s, x) \in \Sigma : f(\omega, s, x) \neq \liminf_{n\to\infty} g_n(\omega, s, x)\}.
\end{aligned}$$

Thus, $\{(\omega, s, x) \in \Sigma : f(\omega, s, x) \neq \liminf_{n\to\infty} g_n(\omega, s, x)\} \in \mathcal{N}$ and we obtain

$$\begin{aligned}
&\{(\omega, s, x) \in \Omega \times [0,T] \times E : f(\omega, s, x) \neq \liminf_{n\to\infty} g_n(\omega, s, x)\} \\
&\subset \{(\omega, s, x) \in \Sigma : f(\omega, s, x) \neq \liminf_{n\to\infty} g_n(\omega, s, x)\} \cup (\Omega \times [0,T] \times E) \setminus \Sigma.
\end{aligned}$$

As both sets on the right hand side are null sets, we get

$$\{f \neq g\} \in \mathcal{N},$$

which completes the proof.

□

The previous proposition allows us to prove the following result which establishes the connection between $\tilde{\mathcal{P}}$ and optional functions.

**Proposition 1.34.** *Any bounded optional function is $\tilde{\mathcal{P}}$-measurable.*

*Proof.* Let $f(\omega,s,x) = Y(\omega)1_{[a,b)}(s)1_B(x)$ and $g(\omega,s,x) = Y(\omega)1_{(a,b]}(s)1_B(x)$ with $0 \le a < b \le T$, $Y$ bounded and $\mathcal{F}_a$-measurable and $B \in \mathcal{E}$. As $g$ is predictable, we get from Proposition 1.33 that $f$ is $\tilde{\mathcal{P}}$-measurable if $\{f \neq g\} \in \mathcal{N}$. This is the case as

$$\begin{aligned}
\mu_M(\{f \neq g\}) &= \mathbb{E}\left[c\int_0^T \int_E 1_{\{f\neq g\}} X(s)(dx)ds\right] \\
&= \mathbb{E}\left[c\int_0^T \int_E 1_{\{Y1_{[a,b)}(s)1_B(x)\neq Y1_{(a,b]}(s)1_B(s)\}} X(s)(dx)ds\right] \\
&= \mathbb{E}\left[c\int_0^T \int_E 1_{\{\Omega\times[a]\times B\}} X(s)(dx)ds\right] \\
&= \mathbb{E}\left[c\int_0^T 1_{[a]} X(s)(B)ds\right] \\
&= 0.
\end{aligned}$$

The last equality holds as $X(s)(B)$ is almost surely finite. Consequently $f$ is $\tilde{\mathcal{P}}$-measurable and as functions of this form generate $\mathcal{O}$, we get that every optional function is $\tilde{\mathcal{P}}$-measurable. □

Since, by definition, $\mathcal{P} \subset \tilde{\mathcal{P}}$ holds, we have $\mathcal{L}^2_{\mathcal{P}} \subset \mathcal{L}^2_{\tilde{\mathcal{P}}}$. To see that the two are actually equal, recall that, by Proposition 1.33, there exists a function $g \in \mathcal{L}^2_{\mathcal{P}}$ for any $f \in \mathcal{L}^2_{\tilde{\mathcal{P}}}$ such that $g = f$ $\mu_{M_X}$-almost surely. This allows us to extend the stochastic integral with respect to the martingale measure $M_X$ associated with the $B(A,c)$-superprocess $X$ to square-integrable, $\tilde{\mathcal{P}}$-measurable integrands. This class of integrands includes bounded optional functions and we denote it by $\mathcal{I}_{M_X}$.

As mentioned above, the extension of the class of integrands is necessary because some of the functions we integrate with respect to the martingale measure $M_X$ in the later parts of this monograph are not predictable but right continuous with left limits. To complete this section on the extension of the class of valid integrands, we still have to prove that right continuous functions with left limits are optional. In fact, it holds $\mathcal{O} = \sigma(r.c.l.l.)$, where *r.c.l.l.* is the set of adapted right continuous functions with left limit. However, as we are only interested in the inclusion $\sigma(r.c.l.l.) \subset \mathcal{O}$ and as the second inclusion follows from the definition of $\mathcal{O}$, we only prove the following.

**Proposition 1.35.** *It holds:*

*(i) If $f$ is $\mathcal{F}_a \times \mathcal{E}$ measurable, then $f1_{[a,b)}$ is optional for all $0 \le a < b \le T$.*

*(ii) $\sigma(r.c.l.l.) \subset \mathcal{O}$.*

*Proof.* (i) As $f$ is $\mathcal{F}_a \times \mathcal{E}$ measurable, we can approximate it pointwise by sums of indicator functions $1_{F\times B}$, $F \in \mathcal{F}_a$, $B \in \mathcal{E}$. Therefore, it is enough to consider $f = 1_{F\times B}$. Since we can write

$$(f1_{[a,b)})(\omega,s,x) = 1_F(\omega)1_{[a,b)}(s)1_B(x),$$

the function $f1_{[a,b)}$ is optional as $Y(\omega) = 1_F(\omega)$ is $\mathcal{F}_a$-measurable.

(ii) Let $f$ be an adapted right continuous function with left limits. Consider the partition $\{t_i^n : 0 \leq i \leq n, 1 \leq n \leq \infty\}$ with $t_0^n = 0$, $t_n^n = T$, $t_i^n < t_{i+1}^n$ for all $n$ and $\max_i |t_{i+1}^n - t_i^n| \to 0$ as $n \to \infty$. Further, define an approximation of the function $f$ by

$$App^n(f)(\omega, s, x) = \sum_{i=0}^{n-1} f(\omega, t_i^n, x) 1_{[t_i^n, t_{i+1}^n)}(s).$$

This approximation converges pointwise to $f$ and as every summand of the above sum is optional by the first part of this proposition, so is the approximation and thus the limit $f$ is also optional.

□

# Chapter 2

# Functional Itō-Calculus for Superprocesses

In this chapter, we derive the Itō-formula as well as the functional Itō-formula for functions, respectively functionals, of $B(A,c)$-superprocesses. One of the main steps towards deriving the two formulae is the definition of the necessary derivatives of functions from $[0,T]\times M_F(E)$ to $\mathbb{R}$ as well as the functional derivatives of functionals from $[0,T]\times D([0,T],M_F(E))$ to $\mathbb{R}$. For the later one, we choose to adapt the concept of horizontal and vertical derivatives as introduced by Cont and Fournié. However, we also note that a result equivalent to Theorem 2.14 can be obtained if one uses the approach by Levental and co-authors.

In Section 2.1 and Section 2.2 the Itō-formula and the functional Itō-formula, respectively, are obtained under the assumption that the underlying space $E$ is compact. In the final section of this chapter, we expand on how the two results can be extended to a setting with locally compact $E$.

## 2.1 The Itō-Formula for Superprocesses

To derive the Itō-formula for a wide class of functions of $B(A,c)$-superprocesses, we use the martingale measure associated with the underlying $B(A,c)$-superprocess to reformulate a result in [Jacka and Tribe, 2003]. Before summarizing the relevant results in [Jacka and Tribe, 2003], we introduce the class of finitely based functions of measure-valued processes – a class of basic functions for which one can easily compute the Itō-formula (see Theorem 2.4) using the traditional Itō-formula for $\mathbb{R}^d$-valued processes.

Consider the setting in Section 1.1.1 with $E$ compact and let $X$ be a $B(A,c)$-superprocess with associated martingale measure $M_X$. When applying the generator $A$ to a function $G$ with multiple arguments, we write $A^{(x)}G(x,y,z)$ to highlight that the generator is applied in the $x$-coordinate, i.e. $A^{(x)}G(x,y,z) = A(G(\cdot,y,z))(x)$. Finally, denote by $C^{1,2}([0,T]\times\mathbb{R}^d) = C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$ the space of functions from $[0,T]\times\mathbb{R}^d$ to $\mathbb{R}$ which are continuous with one partial derivative with respect to the first argument as well as two partial derivatives with respect to the second argument, which are also continuous.

**Definition 2.1** (Finitely based functions)**.** *Given a generator $A$, a function $F : [0,T] \times M_F(E) \to \mathbb{R}$ is called* finitely based *if a function $f \in C^{1,2}([0,T] \times \mathbb{R}^d)$ as well as $\phi_1, \dots, \phi_d \in D(A)$ exist such that*

$$F(t, \mu) = f(t, \langle \mu, \phi_1 \rangle, \dots, \langle \mu, \phi_d \rangle) \tag{2.1}$$

*holds for all $t \in [0,T]$, $\mu \in M_F(C)$.*

Before we can formulate the Itō-formula for finitely based functions of a $B(A,c)$-superprocess, we have to introduce the two following types of derivatives of a functions on finite measures.

**Definition 2.2.** *A continuous function $F : [0,T] \times M_F(E) \to \mathbb{R}$ is differentiable with respect to time if the limit*

$$D^* F(s, \mu) = \lim_{\varepsilon \to 0} \frac{F(s+\varepsilon, \mu) - F(s, \mu)}{\varepsilon}$$

*exists.*

**Definition 2.3** (Directional derivatives)**.** *A continuous function $F : [0,T] \times M_F(E) \to \mathbb{R}$ is differentiable in direction $\delta_x$, $x \in E$, if the limit*

$$D_x F(s, \mu) = \lim_{\varepsilon \to 0} \frac{F(s, \mu + \varepsilon \delta_x) - F(s, \mu)}{\varepsilon}$$

*exists. We call $D_x F$ the* directional derivative *of $F$. Higher order directional derivatives are defined iteratively.*

**Notation.** *We set $D_{xy} F(s,\mu) = D_x D_y F(s, \mu)$ and, if the derivative is continuous in all arguments, we have $D_{xy} F(s, \mu) = D_{yx} F(s, \mu)$. Further, we write $D^*_x F(s, \mu)$ instead of $D^* D_x F(s, \mu)$ and deal with higher order mixed derivatives alike.*

We can now formulate the Itō-formula for finitely based functions of a $B(A,c)$-superprocess $X$. In the proof of this formula, the definition of $X$ via its martingale problem comes in handy once again.

**Theorem 2.4** (Itō-formula for finitely based functions)**.** *Let $X$ be a $B(A,c)$-superprocess and $F : [0,T] \times M_F(E) \to \mathbb{R}$ finitely based. Then, for $t \in [0,T]$, the following holds:*

$$\begin{aligned} F(t, X(t)) = F(0, X(0)) + & \int_0^t D^* F(s, X(s)) ds \\ + & \int_0^t \int_E A^{(x)} D_x F(s, X(s)) X(s)(dx) ds \\ + \frac{1}{2} & \int_0^t \int_E c D_{xx} F(s, X(s)) X(s)(dx) ds \\ + & \int_0^t \int_E D_x F(s, X(s)) M_X(ds, dx). \end{aligned} \tag{2.2}$$

*Proof.* As $F$ is finitely based, it is of form (2.1). As the functions $\phi_i$ are in $D(A)$, we get from the martingale problem (MP) that the $\langle X(t), \phi_i \rangle$'s are semimartingales. Thus, the traditional

Itō-formula for $\mathbb{R}^d$-valued semimartingales yields

$$\begin{aligned}
& f(t, \langle X(t), \phi_1\rangle, \dots, \langle X(t), \phi_d\rangle) \\
= \quad & f(0, \langle X(0), \phi_1\rangle, \dots, \langle X(0), \phi_d\rangle) \\
+ \quad & \int_0^t \partial_s f(s, \langle X(s), \phi_1\rangle, \dots, \langle X(s), \phi_d\rangle) ds \\
+ \quad & \int_0^t \sum_{i=1}^d \partial_i f(s, \langle X(s), \phi_1\rangle, \dots, \langle X(s), \phi_d\rangle) d\langle X(s), \phi_i\rangle \\
+ \frac{1}{2} & \int_0^t \sum_{i,j=1}^d \partial_{ij} f(s, \langle X(s), \phi_1\rangle, \dots, \langle X(s), \phi_d\rangle) d[\langle X, \phi_i\rangle, \langle X, \phi_j\rangle]_s,
\end{aligned} \tag{2.3}$$

where $\partial_s f$, $\partial_i f$ and $\partial_{ij} f$ are the partial derivative of $f$.

From (MP) we further get

$$\langle X(s), \phi_i\rangle = M(s)(\phi_i) + \langle X(0), \phi_i\rangle + \int_0^s \langle X(r), A\phi_i\rangle dr.$$

As $\langle X(0), \phi_i\rangle$ is constant and $M(s)(\phi_i) = \int_0^s \int_E \phi_i(r,x) M_X(dr, dx)$ (see Section 1.3.2), the above yields

$$d\langle X(s), \phi_i\rangle = \int_E \phi_i M(ds, dx) + \langle X(s), A\phi_i\rangle ds.$$

In addition, as

$$[M(\phi_1), M(\phi_2)]_t = c \int_0^t \langle X(s), \phi_1\phi_2\rangle ds,$$

we have

$$d[M(\phi_1), M(\phi_2)]_t = c\langle X(s), \phi_1\phi_2\rangle dt.$$

Plugging these terms into (2.3) yields

$$\begin{aligned}
& f(t, \langle X(t), \phi_1\rangle, \dots, \langle X(t), \phi_d\rangle) \\
= \quad & f(0, \langle X(0), \phi_1\rangle, \dots, \langle X(0), \phi_d\rangle) \\
+ \quad & \int_0^t \partial_s f(s, \langle X(s), \phi_1\rangle, \dots, \langle X(s), \phi_d\rangle) ds \\
+ \quad & \int_0^t \sum_{i=1}^d \partial_i f(s, \langle X(s), \phi_1\rangle, \dots, \langle X(s), \phi_d\rangle) \int_E \phi_i(x) M(ds, dx) \\
+ \quad & \int_0^t \sum_{i=1}^d \partial_i f(s, \langle X(s), \phi_1\rangle, \dots, \langle X(s), \phi_d\rangle) \langle X(s), A\phi_i\rangle ds \\
+ \frac{1}{2} & \int_0^t \sum_{i,j=1}^d \partial_{ij} f(s, \langle X(s), \phi_1\rangle, \dots, \langle X(s), \phi_d\rangle) c\langle X(s), \phi_i\phi_j\rangle ds.
\end{aligned} \tag{2.4}$$

The result now follows by computing the directional derivatives of finitely based functions and identification with the expressions above. As $\partial_s f = D^* F$ holds, we get the equality of the first integral in (2.4) and the first integral in (2.2). Further, as $D_x\langle \mu, \phi\rangle = \phi(x)$, the chain rule of ordinary differentiation yields

$$D_x F(t,\mu) = \sum_{i=1}^{d} \partial_i f(y_1,\dots,y_d)|_{y_1=\langle\mu,\phi_1\rangle,\dots,y_d=\langle\mu,\phi_d\rangle}\phi_i(x).$$

Thus,

$$\begin{aligned}
&\int_0^t \int_E D_x F(s,X(s))M(dx,ds)\\
&=\int_0^t \int_E \sum_{i=1}^{d} \partial_i f(s,\langle X(s),\phi_1\rangle,\dots,\langle X(s),\phi_d\rangle)\phi_i(x)M(ds,dx)\\
&=\int_0^t \sum_{i=1}^{d} \partial_i f(s,\langle X(s),\phi_1\rangle,\dots,\langle X(s),\phi_d\rangle)\int_E \phi_i(x)M(ds,dx),
\end{aligned}$$

which is well-defined as $\phi$, $D.F \in \mathcal{I}_{M_X}$, and further

$$\begin{aligned}
&\int_0^t \int_E A^{(x)} D_x F(s,X(s))X(s)(dx)ds\\
&=\int_0^t \int_E A\left(\sum_{i=1}^{d} \partial_i f(s,\langle X(s),\phi_1\rangle,\dots,\langle X(s),\phi_d\rangle)D.\langle X(s),\phi_i\rangle\right)(x)X(s)(dx)ds\\
&=\int_0^t \sum_{i=1}^{d} \partial_i f(s,\langle X(s),\phi_i\rangle,\dots,\langle X(s),\phi_n\rangle)\int_E A\phi_i(x)X(s)(dx)ds\\
&=\int_0^t \sum_{i=1}^{d} \partial_i f(s,\langle X(s),\phi_i\rangle,\dots,\langle X(s),\phi_n\rangle)\langle X(s),A\phi_i\rangle ds.
\end{aligned}$$

Finally, as

$$D_{xx}F(t,\mu) = \sum_{i,j=1}^{d} \partial_{ij} f(s,\langle\mu,\phi_i\rangle,\dots,\langle\mu,\phi_d\rangle)\phi_i(x)\phi_j(x),$$

we obtain

$$\begin{aligned}
&\int_0^t \int_E cD_{xx}F(s,X(s))X(s)(dx)ds\\
&=\int_0^t \int_E c\sum_{i,j=1}^{d} \partial_{ij} f(s,\langle X(s),\phi_1\rangle,\dots,\langle X(s),\phi_d\rangle)\phi_i(x)\phi_j(x)X(s)(dx)ds\\
&=\int_0^t \sum_{i,j=1}^{d} \partial_{ij} f(s,\langle X(s),\phi_1\rangle,\dots,\langle X(s),\phi_d\rangle)\langle X(s),c\phi_i\phi_j\rangle ds,
\end{aligned}$$

which completes the proof. □

Note that (2.2) resembles the Itō-formula Dawson proved for finitely based function of measure-valued processes in [Dawson, 1978]. The introduction of martingale measures by Walsh eight years later allows us to write the Itō-formula for finitely based functions of $B(A,c)$-superprocesses in the presented form.

As mentioned above, the Itō-formula for a more general class of functions of the $B(A,c)$-superprocess (Theorem 2.9) is based on results in [Jacka and Tribe, 2003]. In the following, the relevant parts in [Jacka and Tribe, 2003] are introduced and one of the main theorems (see Theorem 2.7) and an outline of its proof are presented.

**Definition 2.5** (Good generator). *Let $(S_t)_{t\in[0,T]}$ be the semigroup of the generator $A$. The generator $A$ is called a* good generator *if a dense linear subspace $D_0$ of $C(E)$ that is an algebra exists and $S_t : D_0 \to D_0$ holds for all $t \in [0,T]$.*

*Remark* 2.6. If $A$ is a good generator, $D_0$ is a core of $A$.

The following set of conditions, introduced in [Jacka and Tribe, 2003], is crucial for the remainder of this chapter as it characterizes the class of functions for which we can formulate the Itō-formula in Theorem 2.9.

**Condition 1.** *The function $F : [0,T] \times M_F(E) \to \mathbb{R}$ satisfies*

*(i) $F(s,\mu)$, $D_xF(s,\mu)$, $D_{xy}F(s,\mu)$, $D_{xyz}F(s,\mu)$, $D^*F(s,\mu)$, $D^*_xF(s,\mu)$, $D^*_{xy}F(s,\mu)$ and $D^*_{xyz}F(s,\mu)$ exist and are continuous in $s \in [0,T]$, $x$, $y$, $z \in E$ and $\mu \in M_F(E)$,*

*(ii) the maps $x \mapsto D_xF(s,\mu)$, $x \mapsto D_{xy}F(s,\mu)$ and $x \mapsto D_{xyz}F(s,\mu)$ are in the domain of $A$ for fixed $s \in [0,T]$, $y$, $z \in E$ and $\mu \in M_F(E)$,*

*(iii) $A^{(x)}D_xF(s,\mu)$, $A^{(x)}D_{xy}F(s,\mu)$ and $A^{(x)}D_{xyz}F(s,\mu)$ are continuous in $s \in [0,T]$, $x$, $y$, $z \in E$ and $\mu \in M_F(E)$.*

With all the preparatory work concluded, we can now introduce the main result from [Jacka and Tribe, 2003], which is essential for the proof of Theorem 2.9. The class of processes considered in [Jacka and Tribe, 2003] is a slightly more general class of measure-valued processes but contains the class of $B(A,c)$-superprocesses. In the following, we state the result and present an outline of the proof.

**Theorem 2.7** ([Jacka and Tribe, 2003]). *Suppose $F : [0,T]\times M_F(E) \to \mathbb{R}$ satisfies Condition 1, $A$ is a good generator and $X$ is a $M_F(E)$-valued process with its law $\mathbb{P}$ being the solution of the martingale problem*

$$\begin{gathered}
\textit{for all } \phi \in D(A) \textit{ the process}\\
M(t)(\phi) = \langle X(t),\phi\rangle - \langle X(0),\phi\rangle - \int_0^t \langle X(s), A\phi\rangle ds, \quad t \in [0,T]\\
\textit{is a } (\mathcal{F}_t)_t\textit{-local martingale with respect to } \mathbb{P}\\
\textit{and has quadratic variation } [M(\phi)]_t = \int_0^t \langle X(s), \sigma(s)\phi^2\rangle ds,
\end{gathered}$$

*where $\sigma : \Omega \times [0,T] \times E \to \mathbb{R}$ is predictable and locally bounded. Then*

$$\begin{aligned}
&F(t,X(t)) - \int_0^t D^*F(s,X(s))ds\\
&\qquad - \int_0^t \int_E A^{(x)}D_xF(s,X(s)) + \frac{1}{2}\sigma(s,x)D_{xx}F(s,X(s))X(s)(dx)ds
\end{aligned} \tag{2.5}$$

*is a $(\mathcal{F}_t)_t$-local martingale.*

*Outline of the proof.* The proof can be broken down into the six following steps.

Step 1. To prove that (2.5) is a local martingale, let $K > 0$ and define the stopping times $\tau_K^1$ such that $\sigma(t, \cdot)1_{t<\tau_K^1}$ is bounded by $K$ as well as

$$\tau_K^2 = \begin{cases} 0, & \text{if } \langle X(0), 1\rangle \geq K, \\ \inf\{t : \langle X(t), 1\rangle \leq K\}, & \text{if } \langle X(0), 1\rangle < K, \end{cases} \tag{2.6}$$

with $\inf \emptyset = \infty$. Now, set $\tau_K = \tau_K^1 \wedge \tau_K^2$. As $\tau_K$ is increasing with $\tau_K \to \infty$ as $K \to \infty$, the proof is complete if

$$\begin{aligned} F(t \wedge \tau_K, X(t \wedge \tau_K)) &- \int_0^{t\wedge\tau_K} D^* F(s, X(s)) ds \\ &- \int_0^{t\wedge\tau_K} \int_E A^{(x)} D_x F(s, X(s)) + \frac{1}{2} c D_{xx} F(s, X(s)) X(s)(dx) ds \end{aligned}$$

is a martingale. To prove this, it is enough to consider the scenario in which there exists a $K > 0$ such that

$$\langle X(t), 1\rangle \leq K \quad \text{for all } t \in [0, T].$$

Step 2. Let $K > 0$ and define the subspace $M_K(E)$ of $M_F(E)$ by

$$M_K(E) = \{\mu \in M_F(E) : \langle \mu, 1\rangle \leq K\}.$$

In contrast to the space $M_F(E)$, the space $M_K(E)$ is compact. Next, for a function $\phi : E^k \to \mathbb{R}$, define its symmetrization $\phi^{sym}$ by

$$\phi^{sym}(x_1, \ldots, x_k) = \frac{1}{k!} \sum_{\pi \in S_k} \phi(x_{\pi 1}, \ldots, x_{\pi k}),$$

where $S_k$ is the space of permutations on $\{1, \ldots, k\}$. Further, consider functions

$$\phi(x_1, \ldots, x_k) = \prod_{i=1}^{k} \phi_i(x_i)$$

with $\phi_i \in D_0$ for $i = 1, \ldots, k$ and denote their span by $D_0^{prod}(E^k)$. Finally, consider the space of functions

$$D_0^{sym}(E^k) = \{\phi^{sym} : \phi \in D_0^{prod}(E^k)\},$$

which allows us to define the following set of functions on $[0, T] \times E^n \times M_K(E)$:

$$\mathscr{A}_n^{sym} = \left\{ \sum_{i=1}^{m} \int_{E^{k_i}} \psi_i(t) \phi_i(x, z) \mu^{k_i}(dz) : \psi_i \in C^1([0, T]),\ \phi_i \in D_0^{sym}(E^{k_i+n}),\ k_i, m \geq 0 \right\}.$$

If $F \in \mathscr{A}_0^{sym}$, it is a function from $[0, T] \times M_K(E)$ to $\mathbb{R}$ and finitely based as it consists of elements of form

$$\begin{aligned} \psi(t) \int_{E^k} \frac{1}{k!} \sum_{\pi \in S_k} \prod_{i=1}^{k} \phi_i(z_{\pi i}) \mu^k(dz) &= \psi(t) \frac{1}{k!} \sum_{\pi \in S_k} \prod_{i=1}^{k} \left( \int_E \phi_i(z_{\pi i}) \mu(dz_{\pi i}) \right) \\ &= \psi(t) \frac{1}{k!} \sum_{\pi \in S_k} \prod_{i=1}^{k} \langle \mu, \phi_i \rangle \\ &= \psi(t) \prod_{i=1}^{k} \langle \mu, \phi_i \rangle \end{aligned}$$

and as linear combinations of finitely based functions are finitely based.

Step 3. The key part of the proof of the theorem is the fact that the space $\mathscr{A}_n^{sym}$ is a dense subset of $\{D_{x_1\cdots x_n}F(t,\mu) : F \in C^n([0,T]\times M_K(E))\}$. In particular this yields that $\mathscr{A}_0^{sym}$ is dense in $C([0,T]\times M_K(E))$.

This is by far the most involved step of the proof and includes the proof of the strong continuity of the transition semigroup $(U_t)_{t\in[0,T]}$, given by

$$U_t\Phi(\mu) = \mathbb{E}[\Phi(X(t))|X(0)=\mu]$$

for suitable $\Phi$, as well as the existence and continuity of the derivatives $D_{x_1\cdots x_n}U_t\Phi(\mu)$. The authors refer to this as the *smoothing property of the Dawson-Watanabe semigroup*. The proof of these properties relies on the branching structure of the $B(A,c)$-superprocess, including its Poisson cluster representation.

Step 4. From the previous step, we know that $F$ as well as $D_{xy}F$ can be approximated by functions in $\mathscr{A}_0^{sym}$ and $\mathscr{A}_2^{sym}$, respectively. In order to find approximations for the remaining terms in (2.5), define a semigroup $(V_t^n)_{t\in[0,T]}$ on $[0,T]\times E^n\times M_K(E)$ by

$$V_s^nF(t,x_1,\dots,x_n,\mu) = \begin{cases} S_s^{(x_1)}\cdots S_s^{(x_n)}F(t+s,x_1,\dots,x_n,S_s^*\mu,) & \text{if } s+t\le T,\\ S_{T-t}^{(x_1)}\cdots S_{T-t}^{(x_n)}F(T,x_1,\dots,x_n,S_{T-t}^*\mu), & \text{if } s+t>T,\end{cases}$$

where $(S_t^*)_{t\in[0,T]}$ denotes the dual semigroup of $(S_t)_{t\in[0,T]}$. The authors show that the corresponding generator is given by

$$\begin{aligned}(D^*+Q^n)F(s,x_1,\dots,x_n,\mu) = D^*F(s,x_1,\dots,x_n,\mu) &+ \sum_{i=1}^n A^{(x_i)}F(t,x_1,\dots,x_n,\mu)\\ &+\int_E A^{(z)}D_zF(t,x_1,\dots,x_n,\mu)\mu(dz).\end{aligned}$$

Next, the authors prove that $\mathscr{A}_n^{sym}$ is a core for the generator $D^*+Q^n$, which yields the approximation for the remaining terms in (2.5).

Step 5. The two previous steps yield individual approximations of the different terms in (2.5). However, the existence of a sequence $(F^n)_n\subset\mathscr{A}_0^{sym}$ approximating $F$ that also satisfies $D_xF^n\to D_xF$ as well as $D_{xy}F^n\to D_{xy}F$ as $n\to\infty$ is not given. As the existence of such an approximation is required to complete the proof, the authors prove the following result.

Denote by $\|\cdot\|_\Theta$ the sup-norm on the space $C(\Theta)$. Then, for any $n>0$, $K>0$, there exists a $F^n\in\mathscr{A}_0^{sym}$ such that

$$\begin{aligned}\|F-F^n\|_{[0,T]\times M_K(E)} &\le \frac{1}{n},\\ \|D_{xy}F-D_{xy}F^n\|_{[0,T]\times E^2\times M_K(E)} &\le \frac{1}{n},\\ \|(D^*+Q^0)F-(D^*+Q^0)F^n\|_{[0,T]\times M_K(E)} &\le \frac{1}{n}.\end{aligned}$$

Step 6. Let $\langle X(0),1\rangle < K$[1] and consider the function $F^n\in\mathscr{A}_0^{sym}$ from the previous step.

[1]Recall that $X(0)=m$ is a finite, deterministic measure.

From the second step we know that $F^n$ is finitely based and thus an Itō-formula like the one in Theorem 2.4 can be derived. This yields, as $X(t \wedge \tau_K) \in M_K(E)$,

$$
\begin{aligned}
& F^n(0, X(0)) + \mathcal{M} \\
= \ & F^n(t \wedge \tau_K, X(t \wedge \tau_K)) - \int_0^{t\wedge\tau_K} D^* F^n(s, X(s)) ds \\
& - \int_0^{t\wedge\tau_K} \int_E A^{(x)} D_x F^n(s, X(s)) + \frac{1}{2} c D_{xx} F^n(s, X(s)) X(s)(dx) ds,
\end{aligned}
$$

where $\mathcal{M}$ is a $(\mathcal{F}_t)_t$-martingale. Using the uniform approximations from the previous step and letting $n$ go to infinity then yields that

$$
\begin{aligned}
& F(t \wedge \tau_K, X(t \wedge \tau_K)) - \int_0^{t\wedge\tau_K} D^* F(s, X(s)) ds \\
& - \int_0^{t\wedge\tau_K} \int_E A^{(x)} D_x F(s, X(s)) + \frac{1}{2} c D_{xx} F(s, X(s)) X(s)(dx) ds
\end{aligned}
$$

is a martingale, which, by the first step, completes the proof.

□

By considering a deterministic instead of random branching rate in Theorem 2.7, i.e. by setting $\sigma \equiv c$, we obtain the corresponding result for $B(A, c)$-superprocesses. To formulate the Itō-formula , we have to find an explicit representation of the martingale. To derive this representation, which is done in Theorem 2.9, we need the following proposition.

**Proposition 2.8.** *Let $F : M_F(E) \to \mathbb{R}$ be continuous and with a continuous derivative $D_x F$. Then*

$$
F(\mu) = F(0) + \int_0^1 \int_E D_x F(\theta\mu) \mu(dx) d\theta.
$$

*Proof.* See Lemma 4 in [Jacka and Tribe, 2003]. □

**Theorem 2.9** (Itō-formula)**.** *Let $X$ be a $B(A, c)$-superprocess with good generator $A$ and assume $F : [0, T] \times M_F(E) \to \mathbb{R}$ satisfies Condition 1. Then, for all $t \in [0, T]$, it holds*

$$
\begin{aligned}
F(t, X(t)) = F(0, X(0)) + & \int_0^t D^* F(s, X(s)) ds \\
+ & \int_0^t \int_E A^{(x)} D_x F(s, X(s)) X(s)(dx) ds \\
+ \frac{1}{2} & \int_0^t \int_E c D_{xx} F(s, X(s)) X(s)(dx) ds \\
+ & \int_0^t \int_E D_x F(s, X(s)) M_X(ds, dx).
\end{aligned}
$$

*Proof.* As in the outline of the proof of Theorem 2.7 and in the proof of the traditional Itō-formula (see e.g. [Karatzas and Shreve, 1998]), we have to localize the underlying process $X$. Thus, consider a stopping time $\tau_K$ which we define as in (2.6).

Fix a $K > 0$. From the proof of Theorem 2.7, we get the existence of a function $F^n \in \mathscr{A}_0^{sym}$ such that

$$\begin{aligned}
&\sup_{t\in[0,T],\,\mu\in M_K(E)} |F(t,\mu) - F^n(t,\mu)| \to 0,\\
&\sup_{\substack{t\in[0,T],\,x,y\in E,\\ \mu\in M_K(E)}} |D_{xy}F(t,\mu) - D_{xy}F^n(t,\mu)| \to 0,\\
&\sup_{t\in[0,T],\,\mu\in M_K(E)} |D^*F(t,\mu) + \int_E A^{(x)} D_x F(t,\mu)\mu(dx)\\
&\qquad - D^*F^n(t,\mu) - \int_E A^{(x)} D_z F^n(t,\mu)\mu(dx)| \to 0
\end{aligned} \tag{2.7}$$

as $n \to \infty$. As functions in $\mathscr{A}_0^{sym}$ are finitely based and $X(t\wedge\tau_K) \in M_K(E)$ for all $t \in [0,T]$, Theorem 2.4 yields

$$\begin{aligned}
F^n(t\wedge\tau_K, X(t\wedge\tau_K)) = F^n(0,X(0)) +& \int_0^{t\wedge\tau_K} D^*F^n(s,X(s))ds\\
+& \int_0^{t\wedge\tau_K}\int_E A^{(x)} D_x F^n(s,X(s))X(s)(dx)ds\\
+&\frac{1}{2}\int_0^{t\wedge\tau_K}\int_E cD_{xx}F^n(s,X(s))X(s)(dx)ds\\
+& \int_0^{t\wedge\tau_K}\int_E D_x F^n(s,X(s))M_X(ds,dx).
\end{aligned}$$

Combing the limits in (2.7) with the above equation, we obtain

$$\begin{aligned}
F(t\wedge\tau_K, X(t\wedge\tau_K)) &= \lim_{n\to\infty} F^n(t\wedge\tau_K, X(t\wedge\tau_K))\\
&= \lim_{n\to\infty}\Big(F^n(0,X(0))\\
&\qquad + \int_0^{t\wedge\tau_K} D^*F^n(s,X(s))ds\\
&\qquad + \int_0^{t\wedge\tau_K}\int_E A^{(x)} D_x F^n(s,X(s))X(s)(dx)ds\\
&\qquad + \frac{1}{2}\int_0^{t\wedge\tau_K}\int_E cD_{xx}F^n(s,X(s))X(s)(dx)ds\\
&\qquad + \int_0^{t\wedge\tau_K}\int_E D_x F^n(s,X(s))M_X(ds,dx)\Big)\\
&= F(0,X(0))\\
&\qquad + \int_0^{t\wedge\tau_K} D^*F(s,X(s))ds\\
&\qquad + \int_0^{t\wedge\tau_K}\int_E A^{(x)} D_x F(s,X(s))X(s)(dx)ds\\
&\qquad + \frac{1}{2}\int_0^{t\wedge\tau_K}\int_E cD_{xx}F(s,X(s))X(s)(dx)ds\\
&\qquad + \lim_{n\to\infty}\int_0^{t\wedge\tau_K}\int_E D_x F^n(s,X(s))M_X(ds,dx).
\end{aligned}$$

By Proposition 2.8, the convergence of the second order derivatives of $F^n$ yields

$$\begin{aligned}
&\sup_{\substack{t\in[0,T],\,x\in E\\ \mu\in M_K(E)}} |D_xF(t,\mu) - D_xF^n(t,\mu)| \\
\le &\sup_{\substack{t\in[0,T],\,x\in E\\ \mu\in M_K(E)}} |\int_0^1\int_E D_{xy}F(t,\theta\mu)\mu(dy)d\theta - \int_0^t\int_E D_{xy}F^n(t,\theta\mu)\mu(dy)d\theta| \\
&\quad + \sup_{\substack{t\in[0,T],\,x\in E\\ \mu\in M_K(E)}} |D_xF(t,0) - D_xF^n(t,0)| \\
= &\sup_{\substack{t\in[0,T],\,x\in E\\ \mu\in M_K(E)}} |\int_0^1\int_E D_{xy}F(t,\theta\mu) - D_{xy}F^n(t,\theta\mu)\mu(dy)d\theta| \\
&\quad + \sup_{\substack{t\in[0,T],\,x\in E\\ \mu\in M_K(E)}} |D_xF(t,0) - D_xF^n(t,0)| \\
\le &\int_0^1\int_E \sup_{\substack{t\in[0,T],\,x,\,y\in E\\ \mu\in M_K(E)}} |D_{xy}F(t,\theta\mu) - D_{xy}F^n(t,\theta\mu)|\mu(dy)d\theta \\
&\quad + \sup_{\substack{t\in[0,T],\,x\in E\\ \mu\in M_K(E)}} |D_xF(t,0) - D_xF^n(t,0)|.
\end{aligned}$$

Because of (2.7), the upper bound on the right hand side of the equation goes to zero if $n$ goes to infinity. Thus, $D_xF^n(t,\mu)$ converges to $D_xF(t,\mu)$ in the sup-norm, which yields the convergence with respect to $\|\cdot\|_M$. Therefore, by the definition of the stochastic integral with respect to a martingale measure,

$$\lim_{n\to\infty}\int_0^{t\wedge\tau_K}\int_E D_xF^n(s,X(s))M_X(ds,dx) = \int_0^{t\wedge\tau_K}\int_E D_xF(s,X(s))M_X(ds,dx).$$

Letting $K$ go to infinity completes the proof. □

The following elementary example illustrates how the Itō-formula can be applied.

**Example 2.10.** *Let $X$ be a $B(A,c)$-superprocess with good generator $A$ and consider the function $F(t,\mu) = \psi(t) + \langle\mu,\phi\rangle$ with $\psi\in C^1([0,T])$ and $\phi\in D(A)$ such that $A\phi\in D(A)$. Then*

$$D^*F(s,\mu) = \psi'(s), \quad D_xF(s,\mu) = \phi(x) \quad \text{and} \quad D_{xx}F(s,\mu) = 0.$$

*Therefore, Condition 1 is satisfied and the Itō-formula in Theorem 2.9 yields*

$$\begin{aligned}
F(t,X(t)) &= \psi(0) + \langle X(0),\phi\rangle + \int_0^t \psi'(s)ds \\
&\quad + \int_0^t\int_E A\phi(x)X(s)(dx)ds \\
&\quad + \int_0^t\int_E \phi(x)M_X(ds,dx) \\
&= \psi(t) + \langle X(0),\phi\rangle + \int_0^t \langle X(s),A\phi\rangle ds + M(t)(\phi) \\
&= \psi(t) + \langle X(t),\phi\rangle,
\end{aligned}$$

*with the last equation following from the martingale problem* (MP).

## 2.2 The Functional Itō-Formula for Superprocesses

Based on the Itō-formula for $B(A,c)$-superprocesses in Theorem 2.9, we can now derive the functional Itō-formula for $B(A,c)$-superprocesses. The approach presented is based on the work by Cont and Fournié (see Section 1.2.2) as the functionals considered by the authors contain a time argument and thus the approach adopts more natural to our setting. However, if one prefers to define derivatives as in [Levental et al., 2013] (see Section 1.2.1), the result obtained is the same, as in the present setting, the two definitions of functional derivatives coincide. For more on this, check the remarks after Example 2.15 and the alternative formulation of the Itō-formula in Theorem 2.16.

As mentioned above, the setting in this section follows the ideas by Cont and Fournié. More precisely, we adjust the setting in [Cont, 2016] to measure-valued processes. Therefore, denote by $D([0,T],M_F(E))$ the space of right continuous functions with left limits from $[0,T]$ to $M_F(E)$ and equip the space with a metric $\tilde{d}$ given by

$$\tilde{d}(\omega,\omega') = \sup_{s\in[0,T]} d_P(\omega(s),\omega'(s))$$

for $\omega$, $\omega' \in D([0,T],M_F(E))$, where $d_P$ is the Prokhorov metric on $M_F(E)$. As in Section 1.2, the stopped path $\omega_t$ for $\omega \in D([0,T],M_F(E))$ is given by $\omega_t(s) = \omega(t\wedge s)$. Further, define for $s \in [0,T]$

$$\omega_{t-}(s) = \begin{cases} \omega(s), & \text{if } s\in[0,t), \\ \omega(t-), & \text{if } s \in [t,T]. \end{cases}$$

The notion of stopped paths allows us to define an equivalence relation on the space $[0,T]\times D([0,T],M_F(E))$ by

$$(t,\omega)\sim(t',\omega') \quad \Leftrightarrow \quad t=t' \text{ and } \omega_t = \omega'_{t'},$$

which gives rise to the quotient space

$$\Lambda_T := \{(t,\omega_t) : (t,\omega)\in[0,T]\times D([0,T],M_F(E))\} = [0,T]\times D([0,T],M_F(E))/\sim .$$

Next, define a metric $d_\infty$ on $\Lambda_T$ by

$$d_\infty((t,\omega),(t',\omega')) = \tilde{d}(\omega_t,\omega'_{t'}) + |t-t'| = \sup_{s\in[0,T]} d_P((\omega(t\wedge s),\omega'(t'\wedge s)) + |t-t'|.$$

**Definition 2.11** (Continuity with respect to $d_\infty$). *A functional $F:\Lambda_T\to\mathbb{R}$ is continuous with respect to $d_\infty$ if for all $(t,\omega)\in\Lambda_T$ and every $\varepsilon>0$ there exists an $\eta>0$ such that for all $(t',\omega')\in\Lambda_T$ with $d_\infty((t,\omega),(t',\omega'))<\eta$ we have*

$$|F(t,\omega)-F(t',\omega')|<\varepsilon.$$

**Definition 2.12** (Non-anticipative). *A measurable functional $F$ on $[0,T]\times D([0,T],M_F(E))$ is* non-anticipative *if*

$$F(t,\omega) = F(t,\omega_t) \quad \text{for all } \omega\in D([0,T],M_F(E)),$$

*which is the case if $F:\Lambda_T\to\mathbb{R}$.*

As the setting presented in Section 1.2.2 transfers nicely to real-valued functionals $F$ on $[0,T] \times D([0,T], M_F(E))$, we can define the two following types of derivatives.

**Definition 2.13** (Functional derivatives)**.** *A continuous non-anticipative functional* $F : \Lambda_T \to \mathbb{R}$ *is*

*(i)* horizontally differentiable *at* $(t,\omega) \in \Lambda_T$ *if the limit*

$$\mathcal{D}^* F(t,\omega) = \lim_{\varepsilon \to 0} \frac{F(t+\varepsilon, \omega_t) - F(t,\omega_t)}{\varepsilon}$$

*exists. If this is the case for all* $(t,\omega) \in \Lambda_T$*, we call* $\mathcal{D}^* F$ *the* horizontal derivative *of* $F$*.*

*(ii)* vertically differentiable *at* $(t,\omega) \in \Lambda_T$ *in direction* $\delta_x 1_{[t,T]}$*,* $x \in E$*, if the limit*

$$\mathcal{D}_x F(t,\omega) = \lim_{\varepsilon \to 0} \frac{F(t, \omega_t + \varepsilon \delta_x 1_{[t,T]}) - F(t,\omega_t)}{\varepsilon}$$

*exists. If this is the case for all* $(t,\omega) \in \Lambda_T$*, we call* $\mathcal{D}_x F$ *the* vertical derivative *of* $F$ in direction $\delta_x 1_{[t,T]}$*. Higher order vertical derivatives are defined iteratively.*

**Notation.** *As in Section 2.1, we set* $\mathcal{D}_{xy} F(t,\omega) = \mathcal{D}_x \mathcal{D}_y F(t,\omega)$*, write* $\mathcal{D}^*_x F(t,\omega)$ *instead of* $\mathcal{D}^* \mathcal{D}_x F(t,\omega)$ *and so on. Additionally, we denote by* $\mathcal{D}F$ *the functional*

$$\mathcal{D}F : [0,T] \times D([0,T], M_F(E)) \times E \ni (t,\omega,x) \mapsto \mathcal{D}_x F(t,\omega) \in \mathbb{R}.$$

The definition of horizontal and vertical derivatives allows us to define the following set of conditions on a functional $F$.

**Condition 2.** *The functional* $F : \Lambda_T \to \mathbb{R}$ *satisfies*

*(i)* $F$ *is bounded and continuous,*

*(ii) the horizontal derivative* $\mathcal{D}^* F(t,\omega)$ *is continuous and bounded in* $(t,\omega) \in \Lambda_T$*,*

*(iii) the vertical derivatives* $\mathcal{D}_{x_1} F(t,\omega)$*,* $\mathcal{D}_{x_1 x_2} F(t,\omega)$*,* $\mathcal{D}_{x_1 x_2 x_3} F(t,\omega)$ *and the mixed derivatives* $\mathcal{D}^*_{x_1} F(t,\omega)$*,* $\mathcal{D}^*_{x_1 x_2} F(t,\omega)$*,* $\mathcal{D}^*_{x_1 x_2 x_3} F(t,\omega)$ *are bounded and continuous in* $(t,\omega) \in \Lambda_T$ *and* $x_1$*,* $x_2$*,* $x_3 \in E$*,*

*(iv) for fixed* $(t,\omega) \in \Lambda_T$*,* $x_1$*,* $x_2 \in E$*, the maps* $x \mapsto \mathcal{D}_x F(t,\omega)$*,* $x \mapsto \mathcal{D}_{x x_1} F(t,\omega)$ *and* $x \mapsto \mathcal{D}_{x x_1 x_2} F(t,\omega)$ *are in the domain of* $A$*,*

*(v)* $A^{(x)} \mathcal{D}_{x_1} F(t,\omega)$*,* $A^{(x)} \mathcal{D}_{x_1 x_2} F(t,\omega)$ *and* $A^{(x)} \mathcal{D}_{x_1 x_2 x_3} F(t,\omega)$ *are continuous in* $(t,\omega) \in \Lambda_T$ *and* $x_1$*,* $x_2$*,* $x_3 \in E$*.*

For functionals $F$ satisfying the above condition, we can now formulate the functional Itō-formula for $B(A,c)$-superprocesses.

**Theorem 2.14** (Functional Itō-formula)**.** *Let $X$ be a $B(A,c)$-superprocess with good generator $A$ and assume $F : \Lambda_T \to \mathbb{R}$ satisfies Condition 2. Then, for all $t \in [0,T]$, it holds*

$$\begin{aligned}
F(t, X_t) = F(0, X_0) + & \int_0^t \mathcal{D}^* F(s, X_s) ds \\
+ & \int_0^t \int_E A^{(x)} \mathcal{D}_x F(s, X_s) X(s)(dx) ds \\
+ \frac{1}{2} & \int_0^t \int_E c \mathcal{D}_{xx} F(s, X_s) X(s)(dx) ds \\
+ & \int_0^t \int_E \mathcal{D}_x F(s, X_s) M_X(ds, dx).
\end{aligned}$$

*Proof.* As in the proof of Theorem 2.9, we can use the stopping time $\tau_K$ defined by (2.6) to localize $X$ such that $X(\tau_K \wedge t) \in M_K(E)$ for all $t \in [0,T]$. However, to keep the notation simple, we assume, without loss of generality, that there exists a $K > 0$ such that $X(t) \in M_K(E)$ for all $t \in [0,T]$.

We start by defining a mesh $\{\tau_k^n : k = 1, \ldots, k(n)\}$ on $[0,t]$ by

$$\tau_0^n = 0, \quad \tau_k^n = \inf\{s > \tau_{k-1}^n : 2^n s \in \mathbb{N}\} \wedge t$$

for all $n \in \mathbb{N}$ and use this mesh to define a stepwise approximation of the mapping $s \mapsto X_t(s)$ by

$$App^n(X_t)(s) = \sum_{i=1}^{k(n)} X(\tau_{i+1}^n) 1_{[\tau_i^n, \tau_{i+1}^n)}(s) + X(t) 1_{[t,T]}(s).$$

Note that, while $X$ itself has continuous path, $App^n(X_t)$ is a piecewise constant approximation of the path $X_t$ which is right continuous with left limits. It holds

$$\begin{aligned}
& F(\tau_{i+1}^n, App^n(X_t)_{\tau_{i+1}^n -}) - F(\tau_i^n, App^n(X_t)_{\tau_i^n -}) \\
= \; & F(\tau_{i+1}^n, App^n(X_t)_{\tau_{i+1}^n -}) - F(\tau_i^n, App^n(X_t)_{\tau_i^n}) \\
& + F(\tau_i^n, App^n(X_t)_{\tau_i^n}) - F(\tau_i^n, App^n(X_t)_{\tau_i^n -}).
\end{aligned} \tag{2.8}$$

To complete the proof, we proceed in two steps. In the first step, we consider the two differences on the right hand side of (2.8) and use the fundamental theorem of calculus as well as the Itō-formula in Theorem 2.9 to rewrite the two terms. In the second step, we let $n$ go to infinity and consider the limits of the individual terms.

By setting $h_i^n = \tau_{i+1}^n - \tau_i^n$ and $\psi(s) = F(\tau_i^n + s, App^n(X_t)_{\tau_i^n})$, we get that the first part of (2.8) equals $\psi(h^n) - \psi(0)$ as

$$\psi(h_i^n) - \psi(0) = F(\tau_i^n + h_i^n, App^n(X_t)_{\tau_i^n}) - F(\tau_i^n, App^n(X_t)_{\tau_i^n})$$

and, for all $s \in [0,T]$,

$$\begin{aligned}
App^n(X_t)_{\tau_{i+1}^n-}(u) &= \begin{cases} App^n(X_t)(u), & \text{if } u \in [0,\tau_{i+1}^n), \\ App^n(X_t)(\tau_{i+1}^n-), & \text{if } u \in [\tau_{i+1}^n, T], \end{cases} \\
&= \begin{cases} App^n(X_t)(u), & \text{if } u \in [0,\tau_{i+1}^n), \\ App^n(X_t)(\tau_i^n), & \text{if } u \in [\tau_{i+1}^n, T], \end{cases} \\
&= \begin{cases} App^n(X_t)(u), & \text{if } u \in [0,\tau_i^n), \\ App^n(X_t)(\tau_i^n), & \text{if } u \in [\tau_i^n, T], \end{cases} \\
&= App^n(X_t)_{\tau_i^n}(u).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&F(\tau_{i+1}^n, App^n(X_t)_{\tau_i^n}) - F(\tau_i^n, App^n(X_t)_{\tau_i^n}) \\
&= \int_0^{\tau_{i+1}^n - \tau_i^n} \mathcal{D}^* F(\tau_i^n + s, App^n(X_t)_{\tau_i^n}) ds \\
&= \int_{\tau_i^n}^{\tau_{i+1}^n} \mathcal{D}^* F(s, App^n(X_t)_{\tau_i^n}) ds
\end{aligned}$$

as $\psi(h_i^n) - \psi(0) = \int_0^{h_i^n} \psi'(s) ds$ and

$$\begin{aligned}
\psi'(u) &= \lim_{\varepsilon \to 0} \frac{\psi(u+\varepsilon) - \psi(u)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{F(\tau_i^n + u + \varepsilon, App^n(X_t)_{\tau_i^n}) - F(\tau_i^n + u, App^n(X_t)_{\tau_i^n})}{\varepsilon} \\
&= \mathcal{D}^* F(\tau_i^n + u, App^n(X_t)_{\tau_i^n}).
\end{aligned}$$

By setting $\phi(\mu) = \tilde{\phi}(\mu - X(\tau_i^n))$ with $\tilde{\phi}(\mu) = F(\tau_i^n, App^n(X_t)_{\tau_i^n-} + \mu 1_{[\tau_i^n, T]})$, we get that the second term on the right hand side of (2.8) is equal to $\phi(X(\tau_{i+1}^n)) - \phi(X(\tau_i^n))$ as

$$\begin{aligned}
&\phi(X(\tau_{i+1}^n)) - \phi(X(\tau_i^n)) \\
&= F(\tau_i^n, App^n(X_t)_{\tau_i^n-} + (X(\tau_{i+1}^n) - X(\tau_i^n))1_{[\tau_i^n, T]}) - F(\tau_i^n, App^n(X_t)_{\tau_i^n-})
\end{aligned}$$

and

$$\begin{aligned}
&App^n(X_t)_{\tau_i^n-} + (X(\tau_{i+1}^n) - X(\tau_i^n))1_{[\tau_i^n, T]}(u) \\
&= \begin{cases} App^n(X_t)(u), & \text{if } u \in [0,\tau_i^n), \\ App^n(X_t)(\tau_i^n-) + X(\tau_{i+1}^n) - X(\tau_i^n), & \text{if } u \in [\tau_i^n, T], \end{cases} \\
&= \begin{cases} App^n(X_t)(u), & \text{if } u \in [0,\tau_i^n), \\ X(\tau_{i+1}^n), & \text{if } u \in [\tau_i^n, T], \end{cases} \\
&= App^n(X_t)_{\tau_i^n}.
\end{aligned}$$

We now want to apply Theorem 2.9 to $\phi$. To do so, we have to check if $\phi$ satisfies Condition

1. From

$$\begin{aligned}
D_x\phi(\mu) &= \lim_{\varepsilon\to 0}\frac{\phi(\mu+\varepsilon\delta_x)-\phi(\mu)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(F(\tau_i^n, App^n(X_t)_{\tau_i^n-}+(\mu-X(\tau_i^n))1_{[\tau_i^n,T]}+\varepsilon\delta_x 1_{[\tau_i^n,T]})\\
&\qquad - F(\tau_i^n, App^n(X_t)_{\tau_i^n-}+(\mu-X(\tau_i^n))1_{[\tau_i^n,T]})\Big)\\
&= \mathcal{D}_x F(\tau_i^n, App^n(X_t)_{\tau_i^n-}+(\mu-X(\tau_i^n))1_{[\tau_i^n,T]})
\end{aligned}$$

we get the existence of $D_xF(s,\mu)$ as $F$ satisfies Condition 2. Analogously, we get

$$\begin{aligned}
D_{x_1x_2}\phi(\mu) &= \mathcal{D}_{x_1x_2}F(\tau_i^n, App^n(X_t)_{\tau_i^n-}+(\mu-X(\tau_i^n))1_{[\tau_i^n,T]})\\
D_{x_1x_2x_3}\phi(\mu) &= \mathcal{D}_{x_1x_2x_3}F(\tau_i^n, App^n(X_t)_{\tau_i^n-}+(\mu-X(\tau_i^n))1_{[\tau_i^n,T]})
\end{aligned}$$

and thus, as $F$ satisfies Condition 2, we get the existence of the higher order derivatives. To show that $\phi$ is continuous, consider

$$\begin{aligned}
&App^n(X_t)_{\tau_i^n-}(s)+(\mu-X(\tau_i^n))1_{[\tau_i^n,T]}(s)\\
&=\begin{cases} App^n(X_t)(s), & \text{if } s\in[0,\tau_i^n),\\ App^n(X_t)(\tau_i^n-)+\mu-X(\tau_i^n), & \text{if } s\in[\tau_i^n,T],\end{cases}\\
&=\begin{cases} App^n(X_t)_{\tau_i^n-}(s), & \text{if } s\in[0,\tau_i^n),\\ \mu, & \text{if } s\in[\tau_i^n,T].\end{cases}
\end{aligned}$$

Now, as

$$\begin{aligned}
&d_\infty((\tau_i^n, App^n(X_t)_{\tau_i^n-}+(\mu-X(\tau_i^n))1_{[\tau_i^n,T]}),\\
&\qquad(\tau_i^n, App^n(X_t)_{\tau_i^n-}+(\mu_m-X(\tau_i^n))1_{[\tau_i^n,T]}))\\
&=\sup_{u\in[0,T]} d_P(App^n(X_t)_{\tau_i^n-}(u)+(\mu-X(\tau_i^n))1_{[\tau_i^n,T]}(u),\\
&\qquad App^n(X_t)_{\tau_i^n-}(u)+(\mu_m-X(\tau_i^n))1_{[\tau_i^n,T]}(u))\\
&=d_P(\mu_m,\mu),
\end{aligned}$$

we get the continuity of $\phi$ with respect to $\mu$ from

$$\begin{aligned}
&\{\mu_m \xrightarrow{m\to\infty} \mu\}\\
\Rightarrow\ &\{d_P(\mu_m,\mu)\xrightarrow{m\to\infty} 0\}\\
\Rightarrow\ &\{d_\infty((\tau_i^n, App^n(X_t)_{\tau_i^n-}+(\mu-X(\tau_i^n))1_{[\tau_i^n,T]}),\\
&\qquad(\tau_i^n, App^n(X_t)_{\tau_i^n-}+(\mu_m-X(\tau_i^n))1_{[\tau_i^n,T]}))\xrightarrow{m\to\infty} 0\}\\
\Rightarrow\ &\{|F(\tau_i^n, App^n(X_t)_{\tau_i^n-}+(\mu-X(\tau_i^n))1_{[\tau_i^n,T]})\\
&\qquad - F(\tau_i^n, App^n(X_t)_{\tau_i^n-}+(\mu_m-X(\tau_i^n))1_{[\tau_i^n,T]})|\xrightarrow{m\to\infty} 0\},
\end{aligned}$$

where the last part follows from the continuity of $F$. Analogously, we obtain the continuity of the derivatives of $\phi$ with respect to $x_i$ and $\mu$ as well as the remaining conditions in Condition 1 from the conditions on $F$.

Consequently, we can apply Theorem 2.9 to $\phi$ and obtain, as $\phi$ has no time argument,

$$\begin{aligned}\phi(X(\tau_{i+1}^n)) - \phi(X(\tau_i^n)) = &\int_{\tau_i^n}^{\tau_{i+1}^n}\int_E A^{(x)}\mathcal{D}_x\phi(X(s))X(s)(dx)ds\\ &+ \frac{1}{2}\int_{\tau_i^n}^{\tau_{i+1}^n}\int_E c\mathcal{D}_{xx}\phi(X(s))X(s)(dx)ds\\ &+ \int_{\tau_i^n}^{\tau_{i+1}^n}\int_E \mathcal{D}_x\phi(X(s))M_X(ds,dx).\end{aligned}$$

Plugging in the definition of $\phi$, we end up with

$$\begin{aligned}&F(\tau_i^n, App^n(X_t)_{\tau_i^n}) - F(\tau_i^n, App^n(X_t)_{\tau_i^n-})\\ = &\int_{\tau_i^n}^{\tau_{i+1}^n}\int_E A^{(x)}\mathcal{D}_xF(\tau_i^n, App^n(X_t)_{\tau_i^n-} + (X(s) - X(\tau_i^n))1_{[\tau_i^n,T]})X(s)(dx)ds\\ + &\frac{1}{2}\int_{\tau_i^n}^{\tau_{i+1}^n}\int_E c\mathcal{D}_{xx}F(\tau_i^n, App^n(X_t)_{\tau_i^n-} + (X(s) - X(\tau_i^n))1_{[\tau_i^n,T]})X(s)(dx)ds\\ + &\int_{\tau_i^n}^{\tau_{i+1}^n}\int_E \mathcal{D}_xF(\tau_i^n, App^n(X_t)_{\tau_i^n-} + (X(s) - X(\tau_i^n))1_{[\tau_i^n,T]})M(ds,dx).\end{aligned}$$

Combining this with the result for the first part of the sum in (2.8) yields the following expression for the left hand side in (2.8):

$$\begin{aligned}&F(\tau_{i+1}^n, App^n(X_t)_{\tau_{i+1}^n-}) - F(\tau_i^n, App^n(X_t)_{\tau_i^n-})\\ = &\int_{\tau_i^n}^{\tau_{i+1}^n}\mathcal{D}^*F(s, App^n(X_t)_{\tau_i^n})ds\\ + &\int_{\tau_i^n}^{\tau_{i+1}^n}\int_E A^{(x)}\mathcal{D}_xF(\tau_i^n, App^n(X_t)_{\tau_i^n-} + (X(s) - X(\tau_i^n))1_{[\tau_i^n,T]})X(s)(dx)ds\\ + &\frac{1}{2}\int_{\tau_i^n}^{\tau_{i+1}^n}\int_E c\mathcal{D}_{xx}F(\tau_i^n, App^n(X_t)_{\tau_i^n-} + (X(s) - X(\tau_i^n))1_{[\tau_i^n,T]})X(s)(dx)ds\\ + &\int_{\tau_i^n}^{\tau_{i+1}^n}\int_E \mathcal{D}_xF(\tau_i^n, App^n(X_t)_{\tau_i^n-} + (X(s) - X(\tau_i^n))1_{[\tau_i^n,T]})M(ds,dx).\end{aligned}$$

Define the index $i_n(s)$ such that $s \in [\tau_{i_n(s)}^n, \tau_{i_n(s)+1}^n)$. Then, summation of the above terms over $i$ yields

$$\begin{aligned}&F(t, App^n(X_t)_{t-}) - F(0, X_0)\\ = &\int_0^t \mathcal{D}^*F(s, App^n(X_t)_{\tau_{i_n(s)}^n})ds\\ + &\int_0^t\int_E A^{(x)}\mathcal{D}_xF(\tau_{i_n(s)}^n, App^n(X_t)_{\tau_{i_n(s)}^n-} + (X(s) - X(\tau_{i_n(s)}^n))1_{[\tau_{i_n(s)}^n,T]})X(s)(dx)ds\\ + &\frac{1}{2}\int_0^t\int_E c\mathcal{D}_{xx}F(\tau_{i_n(s)}^n, App^n(X_t)_{\tau_{i_n(s)}^n-} + (X(s) - X(\tau_{i_n(s)}^n))1_{[\tau_{i_n(s)}^n,T]})X(s)(dx)ds\\ + &\int_0^t\int_E \mathcal{D}_xF(\tau_{i_n(s)}^n, App^n(X_t)_{\tau_{i_n(s)}^n-} + (X(s) - X(\tau_{i_n(s)}^n))1_{[\tau_{i_n(s)}^n,T]})M(ds,dx).\end{aligned} \tag{2.9}$$

Note that while the function

$$(\omega, s, x) \mapsto \mathcal{D}_x F(\tau^n_{i_n(s)}, App^n(X_t)_{\tau^n_{i_n(s)}-} + (X(s) - X(\tau^n_{i_n(s)}))1_{[\tau^n_{i_n(s)},T]})(\omega)$$

is not predictable, it is in $\mathcal{I}_{M_X}$ due to the results in Section 1.3.3. Hence, the integral with respect to the martingale measure $M_X$ in (2.9) is well-defined. This completes the first of the two steps.

For the convergence of the terms on the left hand side of (2.9) consider

$$\begin{aligned}
&d_\infty((s, X_s), (\tau^n_{i_n(s)}, App^n(X_t)_{\tau^n_{i_n(s)}})) \\
&= |s - \tau^n_{i_n(s)}| + \sup_{u\in[0,T]} d_P(X(s \wedge u), App^n(X_t)_{\tau^n_{i_n(s)}}(u)) \\
&\le \frac{1}{2^n} + \sup_{0\le i\le k(n)} \sup_{u\in[\tau^n_i,\tau^n_{i+1})} d_P(X(s \wedge u), X(\tau^n_{i_n(s)} \wedge \tau^n_{i+1})),
\end{aligned}$$

which goes to zero due to the continuity of the paths of $X$. In addition, the continuity of the paths of $X$ and the continuity of $F$ yield

$$\lim_{n\to\infty} F(t, App^n(X_t)_{t-}) = F(t, X_{t-}) = F(t, X_t).$$

From the continuity assumptions on $\mathcal{D}^*F$ and

$$d_\infty((s, X_s), (s, App^n(X_t)_{\tau^n_{i_n(s)}})) \to 0,$$

we get

$$\lim_{n\to\infty} \mathcal{D}^*F(s, App^n(X_t)_{\tau^n_{i_n(s)}}) = \mathcal{D}^*F(s, X_s).$$

In combination with the boundedness assumption on $\mathcal{D}^*F$ this allows us to apply the dominated convergence theorem to get the convergence of the first term on the right hand side of (2.9), namely

$$\lim_{n\to\infty} \int_0^t \mathcal{D}^*F(s, App^n(X_t)_{\tau^n_{i_n(s)}})ds = \int_0^t \mathcal{D}^*F(s, X_s)ds.$$

To prove the convergence of the second term on the right hand side, consider

$$\begin{aligned}
&d_\infty((s, X_s)(\tau^n_{i_n(s)}, App^n(X_t)_{\tau^n_{i_n(s)}-} + (X(s) - X(\tau^n_{i_n(s)}))1_{[\tau^n_{i_n(s)},T]})) \\
&= |s - \tau^n_{i_n(s)}| + \sup_{u\in[0,T]} d_P(X_s(u), App^n(X_t)_{\tau^n_{i_n(s)}-}(u) + (X(s) - X(\tau^n_{i_n(s)}))1_{[\tau^n_{i_n(s)},T]}(u)).
\end{aligned} \tag{2.10}$$

By the triangle inequality

$$\begin{aligned}
&\sup_{u\in[0,T]} d_P(X_s(u), App^n(X_t)_{\tau^n_{i_n(s)}-}(u) + (X(s) - X(\tau^n_{i_n(s)}))1_{[\tau^n_{i_n(s)},T]}(u)) \\
\le &\sup_{u\in[0,T]} d_P(X_s(u), App^n(X_t)_{\tau^n_{i_n(s)}-}(u)) \\
&+ \sup_{u\in[0,T]} d_P(App^n(X_t)_{\tau^n_{i_n(s)}-}(u), App^n(X_t)_{\tau^n_{i_n(s)}-}(u) + (X(s) - X(\tau^n_{i_n(s)}))1_{[\tau^n_{i_n(s)},T]}(u))
\end{aligned}$$

holds. Further, it holds

$$\begin{aligned}
&\sup_{u\in[0,T]} d_P(App^n(X_t)_{\tau^n_{i_n(s)}-}(u), App^n(X_t)_{\tau^n_{i_n(s)}-}(u) + (X(s) - X(\tau^n_{i_n(s)}))1_{[\tau^n_{i_n(s)},T]}(u)) \\
&= d_P(X(\tau^n_{i_n(s)}), X(\tau^n_{i_n(s)}) + X(s) - X(\tau^n_{i_n(s)})) \\
&= d_P(X(\tau^n_{i_n(s)}), X(s)),
\end{aligned}$$

which goes to zero as $n$ goes to infinity. In combination with

$$\sup_{u\in[0,T]} d_P(X_s(u), App^n(X_t)_{\tau^n_{i_n(s)}-}(u)) \to 0 \quad \text{and} \quad |s - \tau^n_{i_n(s)}| \to 0 \quad \text{as } n \to \infty,$$

this implies that (2.10) goes to zero as $n$ goes to inifnity. The continuity assumption on $A\mathcal{D}F$ then yields

$$\lim_{n\to\infty} A^{(x)}\mathcal{D}_x F(\tau^n_{i_n(s)}, App^n(X_t)_{\tau^n_{i_n(s)}-} + (X(s) - X(\tau^n_{i_n(s)}))1_{[\tau^n_{i_n(s)},T]}) = A^{(x)}\mathcal{D}_x F(s, X_s).$$

To get the convergence of the integrals, set

$$\alpha^n(s) = \tau^n_{i_n(s)} \quad \text{and} \quad \beta^n(s) = App^n(X_t)_{\tau^n_{i_n(s)}-} + (X(s) - X(\tau^n_{i_n(s)}))1_{[\tau^n_{i_n(s)},T]})$$

and assume the bound of $A\mathcal{D}F$ is given by $0 < B < \infty$. Then

$$\int_E |A^{(x)}\mathcal{D}_x F(\alpha^n(s), \beta^n(s))|X(s)(dx) \le \int_E BX(s)(dx) \le BK < \infty$$

for all $n$ and thus we can apply the dominated convergence theorem to obtain

$$\lim_{n\to\infty} \int_E A^{(x)}\mathcal{D}_x F(\alpha^n(s), \beta^n(s))X(s)(dx) = \int_E A^{(x)}\mathcal{D}_x F(s, X_s)X(s)(dx)$$

for all $s \in [0, T]$. Combining this with the fact that

$$|\int_E A^{(x)}\mathcal{D}_x F(\alpha^n(s), \beta^n(s))X(s)(dx)| \le \int_E |A^{(x)}\mathcal{D}_x F(\alpha^n(s), \beta^n(s))|X(s)(dx) \le BK < \infty$$

holds for all $n$ allows us to apply the dominated convergence theorem once again to end up with

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^t \int_E A^{(x)}\mathcal{D}_x F(\alpha^n(s), \beta^n(s))X(s)(dx)ds \\
&= \lim_{n\to\infty} \int_0^t \int_E A^{(x)}\mathcal{D}_x F(\tau^n_{i_n(s)}, App^n(X_t)_{\tau^n_{i_n(s)}-} + (X(s) - X(\tau^n_{i_n(s)}))1_{[\tau^n_{i_n(s)},T]})X(s)(dx)ds \\
&= \int_0^t \int_E A^{(x)}\mathcal{D}_x F(s, X_s)X(s)(dx)ds
\end{aligned}$$

and thus get the convergence of the second term on the right hand side of (2.9).

By using the same arguments, we get

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^t \int_E \mathcal{D}_{xx} F(\tau^n_{i_n(s)}, App^n(X_t)_{\tau^n_{i_n(s)}-} + (X(s) - X(\tau^n_{i_n(s)}))1_{[\tau^n_{i_n(s)},T]})X(s)(dx)ds \\
&= \int_0^t \int_E \mathcal{D}_{xx} F(s, X_s)X(s)(dx)ds,
\end{aligned}$$

i.e. the convergence of the third term on the right hand side of (2.9).

For the convergence of the last term, the integral with respect to the martingale measure $M_X$, assume $(\omega_n)_n \subset D([0,T], M_F(E))$ with $\omega_n \to \omega$ as $n$ goes to infinity. The continuity assumptions on $\mathcal{D}F$ then yield

$$\sup_{x\in E,\, s\in[0,T]} |\mathcal{D}_x F(\tau^n_{i_n(s)}, \omega_n) - \mathcal{D}_x F(s,\omega)| \to 0. \tag{2.11}$$

To see this, assume (2.11) does not hold. Then, we can find an $\varepsilon > 0$ and sequences $(s_n) \subset [0,T]$ and $(x_n) \subset E$ such that

$$|\mathcal{D}_{x_n} F(\tau^n_{i_n(s)}, \omega_n) - \mathcal{D}_{x_n} F(s_n,\omega)| \geq \varepsilon \quad \forall n \in \mathbb{N}. \tag{2.12}$$

As $[0,T]$ and $E$ are compact metric spaces, there exist convergent subsequences $(\tilde{s}_n)$ and $(\tilde{x}_n)$ of $(s_n)$ and $(x_n)$. Assume the limits of these subsequences are given by $\tilde{s}$ and $\tilde{x}$. The existence of these subsequences and the continuity assumptions then yield

$$|\mathcal{D}_{\tilde{x}_n} F(\tau^n_{i_n(\tilde{s}_n)}, \omega_n) - \mathcal{D}_{\tilde{x}} F(\tilde{s},\omega)| \to 0$$

and

$$|\mathcal{D}_{\tilde{x}_n} F(\tilde{s}_n, \omega) - \mathcal{D}_{\tilde{x}} F(\tilde{s},\omega)| \to 0$$

which contradicts (2.12). Therefore (2.11) holds.

Considering

$$d_\infty((s,X_s)(\tau^n_{i_n(s)}, App^n(X_t)_{\tau^n_{i_n(s)}-} + (X(s) - X(\tau^n_{i_n(s)}))1_{[\tau^n_{i_n(s)},T]})) \to 0 \quad \text{as } n \to \infty,$$

we get from (2.11) that for $n \to \infty$

$$\mathbb{E}\Bigg[c\int_0^t \int_E |\mathcal{D}_x F(\tau^n_{i_n(s)}, App^n(X_t)_{\tau^n_{i_n(s)}-} + (X(s) - X(\tau^n_{i_n(s)}))1_{[\tau^n_{i_n(s)},T]}) \\ - \mathcal{D}_x F(s,X_s)|^2 X(s)(dx)ds\Bigg] \to \infty.$$

By the definition of the integral with respect to the martingale measure $M_X$ (see Section 1.3.1) we therefore end up with

$$\lim_{n\to\infty} \int_0^t \int_E \mathcal{D}_x F(\tau^n_{i_n(s)}, App^n(X_t)_{\tau^n_{i_n(s)}-} + (X(s) - X(\tau^n_{i_n(s)}))1_{[\tau^n_{i_n(s)},T]}) M(ds,dx) \\ = \int_0^t \int_E \mathcal{D}_x F(s,X_s) M(ds,dx),$$

which completes the second step and thus the proof. □

**Example 2.15.** *Let $A$ be a good generator and $h \in D_0$, the dense subspace in Definition 2.5. Consider the functional given by*

$$F(t,\omega) = \int_0^t \int_E Ah(y)\omega(s)(dy)ds.$$

*For this functional, it holds $F(0,\omega)=0$ as well as*

$$\mathcal{D}^*F(s,\omega)=\int_E Ah(y)\omega(s)(dy)$$

*and all vertical and mixed derivatives are zero. Consequently, Condition 2 is satisfied and, if $X$ is a $B(A,c)$-superprocess, we obtain from the functional Itō-formula (Theorem 2.14)*

$$F(t,X_t)=\int_0^t\int_E Ah(y)X(s)(dy)ds.$$

Instead of working with the vertical derivatives $\mathcal{D}F$ introduced by Cont and Fournié, we can follow the approach by Levental and co-authors and define functional derivatives by

$$D_x^pF(t,\omega;[t,T])=\lim_{\varepsilon\to 0}\frac{F(t,\omega+\varepsilon\delta_x 1_{[t,T]})-F(t,\omega)}{\varepsilon}$$

for $x\in E$, $\omega\in D([0,T],M_F(E))$ and $t\in[0,T]$. After adjusting Condition 2 accordingly, the proof of Theorem 2.14 also works when using the alternative functional derivative as

$$\mathcal{D}_xF(t,\omega_t)=D_x^pF(t,\omega_t;[t,T]) \tag{2.13}$$

for all $x\in E$, $\omega\in D([0,T],M_F(E))$ and $t\in[0,T]$. By defining mixed and higher order functional derivatives accordingly and by writing $D_x^{p*}F(t,\omega;[t,T])$ for $\mathcal{D}^*D_x^pF(t,\omega;[t,T])$, an alternative version of the Itō-formula can be formulated as follows.

**Theorem 2.16** (Functional Itō-formula (alternative form)). *Let $X$ be a $B(A,c)$-superprocess with good generator $A$ and assume $F:\Lambda_T\to\mathbb{R}$ satisfies*

*(i) $F$ is bounded and continuous,*

*(ii) the horizontal derivative $\mathcal{D}^*F(t,\omega)$ is continuous and bounded in $(t,\omega)\in\Lambda_T$,*

*(iii) the vertical derivatives $D_{x_1}^pF(t,\omega;[t,T])$, $D_{x_1x_2}^pF(t,\omega;[t,T])$, $D_{x_1x_2x_3}^pF(t,\omega;[t,T])$ and the mixed derivatives $D_{x_1}^{p*}F(t,\omega;[t,T])$, $D_{x_1x_2}^{p*}F(t,\omega;[t,T])$, $D_{x_1x_2x_3}^{p*}F(t,\omega;[t,T])$ are bounded and continuous in $(t,\omega)\in\Lambda_T$ and $x_1,\,x_2,\,x_3\in E$,*

*(iv) for fixed $(t,\omega)\in\Lambda_T$, $x_1,\,x_2\in E$ the maps $x\mapsto D_x^pF(t,\omega;[t,T])$, $x\mapsto D_{xx_1}^pF(t,\omega;[t,T])$ and $x\mapsto D_{xx_1x_2}^pF(t,\omega;[t,T])$ are in the domain of $A$,*

*(v) $A^{(x)}D_x^pF(t,\omega;[t,T])$, $A^{(x)}D_{xx_1}^pF(t,\omega;[t,T])$ and $A^{(x)}D_{xx_1x_2}^pF(t,\omega;[t,T])$ are continuous in $(t,\omega)\in\Lambda_T$ and $x,\,x_1,\,x_2\in E$.*

*Then, for all $t\in[0,T]$, it holds*

$$\begin{aligned}F(t,X_t)=F(0,X_0)+&\int_0^t\mathcal{D}^*F(s,X_s)ds\\+&\int_0^t\int_E A^{(x)}D_x^pF(s,X_s;[t,T])X(s)(dx)ds\\+\frac{1}{2}&\int_0^t\int_E cD_{xx}^pF(s,X_s;[t,T])X(s)(dx)ds\\+&\int_0^t\int_E D_x^pF(s,X_s;[t,T])M_X(ds,dx).\end{aligned}$$

From (2.13) we obtain that the two formulations of derivatives coincide for functionals $F:\Lambda_T\to\mathbb{R}$ and thus the Itō-formulae in Theorem 2.14 and Theorem 2.16 are equivalent.

## 2.3 Extension to the Locally Compact Case

While in the two previous section we assumed that $E$ is a compact space, the results can be extended to locally compact spaces $E$, which is for example essential if one considers the super-Brownian motion, i.e. $E = \mathbb{R}^d$. The extension is based on the one point compactification of $E$ and follows the ideas presented in the example section in [Jacka and Tribe, 2003].

Now, let $E$ be a locally compact space, $(S_t)_{t\in[t,T]}$ be the Feller semigroup on $C_0(E)$ associated with the generator $A$ and $X$ a $B(A,c)$-superprocess. Denote by $\bar{E}$ the one point compactification of $E$, which is obtained by including a point at infinity, and define a new semigroup $(\bar{S}_t)_{t\in[0,T]}$ by

$$\bar{S}_t\bar{f}(x) = \begin{cases} S_tf(x), & \text{if } x \neq \infty, \\ \bar{f}(x), & \text{if } x = \infty, \end{cases} \tag{2.14}$$

where $\bar{f} \in C(\bar{E})$ and $f = \bar{f}|_E$.

**Proposition 2.17.** *The semigroup* $(\bar{S}_t)_{t\in[0,T]}$ *defined in* (2.14) *is a Feller semigroup.*

*Proof.* Let $\bar{f} \in C(\bar{E})$. To prove that $(\bar{S}_t)_{t\in[0,T]}$ is indeed a Feller semigroup, we have to show that it satisfies the six properties in Defintion 1.1. Most of them follow directly from the fact that $(S_t)_{t\in[0,T]}$ is a Feller semigroup:

(i) As $\bar{S}_t\bar{f}(\infty) = \bar{f}(\infty)$ for $t \in [0,T]$, we get for $s$, $t \in [0,T]$ with $t+s \in [0,T]$ and $x \in \bar{E}$

$$\begin{aligned} \bar{S}_{t+s}\bar{f}(x) &= \begin{cases} S_{t+s}f(x), & \text{if } x \neq \infty, \\ \bar{f}(x), & \text{if } x = \infty, \end{cases} \\ &= \begin{cases} S_tS_sf(x), & \text{if } x \neq \infty, \\ \bar{S}_s\bar{f}(x), & \text{if } x = \infty, \end{cases} \\ &= \bar{S}_t\bar{S}_s\bar{f}(x). \end{aligned}$$

(ii) For $\bar{f} \equiv 1$, it holds $\bar{f}(\infty) = 1$ and thus we get that $\bar{S}_t1 = 1$ holds for all $t \in [0,T]$.

(iii) If $\bar{f} > 0$, we get for $t \in [0,T]$

$$\bar{S}_t\bar{f}(x) = \begin{cases} S_tf(x) > 0, & \text{if } x \neq \infty, \\ \bar{f}(x) > 0, & \text{if } x = \infty. \end{cases}$$

(iv) As $|\bar{f}(\infty)| \leq \sup_{x\in\bar{E}} |f(x)| = \|f\|$, we have for $t \in [0,T]$

$$\|\bar{S}_tf\| = \max\{\sup_{x\in E} |S_tf(x)|, |\bar{f}(\infty)|\} \leq \|f\|.$$

(v) As $\bar{E}$ is compact, we get $C_0(\bar{E}) = C(\bar{E})$. Further, as $\lim_{x\to\infty} \bar{S}_t\bar{f}(x) = \bar{\phi}(\infty)$ for $t \in [0,T]$ by definition of $\bar{S}_t$, we get that $\bar{S}_t\bar{f} \in C(\bar{E})$ and thus for $t \in [0,T]$

$$\bar{S}_t : C(\bar{E}) \to C(\bar{E}).$$

(vi) As $S_t f \to f$ for $t \to 0$ and as the value of $\bar{S}_t f$ at the point at infinity is independent of $t$, we get for all $x \in \bar{E}$

$$\bar{S}_t \bar{f}(x) \to \bar{f}(x) \quad \text{as } t \to 0.$$

□

The generator of the semigroup $(\bar{S}_t)_{t\in[0,T]}$ is denoted by $\bar{A}$ and given by

$$\bar{A}f(x) = \lim_{t\to 0} \frac{\bar{S}_t \bar{f}(x) - \bar{f}(x)}{t} = \begin{cases} \lim_{t\to 0} \frac{S_t f(x) - f(x)}{t} = Af(x), & \text{if } x \neq \infty, \\ \lim_{t\to 0} \frac{\bar{f}(\infty) - \bar{f}(\infty)}{t} = 0, & \text{if } x = \infty. \end{cases}$$

Its domain is given by

$$D(\bar{A}) = \{\bar{f} \in C(\bar{E}) : \bar{f}|_E - \bar{f}(\infty) \in D(A)\}.$$

Next, we modify the martingale problem (MP) such that the martingale $M(t)(\bar{\phi})$ is given for all $\bar{\phi} \in D(\bar{A})$ by

$$M(t)(\bar{\phi}) = \langle \tilde{X}(t), \bar{\phi} \rangle - \langle \tilde{X}(0), \bar{\phi} \rangle - \int_0 \langle \tilde{X}(s), \bar{A}\bar{\phi} \rangle ds$$

and

$$[M(\bar{\phi})]_t = \int_0^t \langle \tilde{X}(s), c1_{\{x\neq\infty\}} \bar{\phi}^2 \rangle ds.$$

The process $\tilde{X}$, whose law is the solution of the martingale problem, takes values in $M_F(\bar{E})$ and gives zero mass to the point at infinity. As $\bar{E}$ is compact and the modified martingale problem still allows us to apply the result in [Jacka and Tribe, 2003] by setting $\sigma(s, x) = c1_{x\neq\infty}$, we can apply Theorem 2.9 and Theorem 2.14 with the $B(A, c)$-superprocess replaced by $\tilde{X}$.

As $\tilde{X}$ coincides with the $B(A, c)$-sperpocess $X$ on the localy compact space $E$ and has no mass outside of $E$, applying Theorem 2.9 and Theorem 2.14 to $\tilde{X}$ yields the fomulae for $X$ when the underlying space $E$ is locally compact.

# Chapter 3

# Martingale Representation

The traditional martingale representation formula states the following. Let $(\mathcal{G}_t)_{t\in[0,\infty)}$ be the filtration generated by a Brownian motion $W$. Then, for any square-integrable $\mathcal{G}_\infty$-measurable random variable $Y$ there exists a predictable, $(\mathcal{G}_t)_t$-adapted process $Z$ such that

$$Y = \mathbb{E}[Y] + \int_0^\infty Z(s)dW(s)$$

holds.

Evans and Perkins studied the martingale representation formula in the scenario where the Brownian motion is replaced by a superprocess or a historical process. In [Evans and Perkins, 1994], the authors prove that the martingale representation is unique if the Brownian motion is replaced by a superprocess. In this case, the stochastic integral is with respect to the martingale measure associated with the underlying superprocess. In a subsequent article ([Evans and Perkins, 1995]), the uniqueness as well as the explicit form of the integrand in the scenario with the Brownian motion replaced by a historical Brownian motion are obtained. Once again, the stochastic integral is with respect to a martingale measure, this time the martingale measure associated with the underlying historical Brownian motion.

In [Evans and Perkins, 1995] the explicit form of the integrand is derived from the cluster representation of the underlying historical Brownian motion using a technique that resembles the traditional approach to obtain the integrand, which is based on Malliavin calculus. While the authors only derive the explicit form in the scenario where the underlying process is a historical process, they note that one can obtain the result for the scenario where the underlying process is a superprocess by projection.

A different approach to the subject is used in this chapter as we derive an alternative form of the martingale representation formula in both scenarios. The approach is motivated by the work in [Cont and Fournié, 2013] as well as [Cont, 2016], where a weak extension of the vertical derivative $\mathcal{D}F$ is used to derive a functional version of the representation formula for

Brownian martingales.

In each of the two following sections, we first consider specific scenarios in which we can directly obtain the explicit form of the integrand from known results before we derive our actual, more general representation formula. While in Section 3.1 we obtain the representation for a wide class of processes by applying the Itō-formula from Chapter 2, the classes of processes considered in the two examples at the beginning of Section 3.2 are much smaller as there is no Itō-formula that we can take into account.

## 3.1 The Representation Formula for Square-Integrable $(\mathcal{F}_t)_t$-Martingales

While the focus of this section is on the scenario with the underlying process being a super-Brownian motion, let us first take a step back and reconsider the results in Chapter 2. If the process $Y$ given by $Y(t) = F(t, X_t)$ is a martingale and $F$ is a sufficiently smooth functional, we obtain the martingale representation formula as an immediate consequence of the functional Itō-formula in Theorem 2.14 as the following corollary shows.

**Corollary 3.1.** *Let $F$ satisfy Condition 2, $E$ be compact, $X$ be a $B(A, c)$-superprocess with good generator $A$ and assume $(F(t, X_t))_{t\in[0,T]}$ is a martingale. Then*

$$F(t, X_t) = F(0, X_0) + \int_0^t \int_E \mathcal{D}_x F(s, X_s) M_X(ds, dx).$$

*Proof.* From Theorem 2.14 we get that

$$\begin{aligned} F(t, X_t) = F(0, X_0) + &\int_0^t \mathcal{D}^* F(s, X_s) ds \\ + &\int_0^t \int_E A^{(x)} \mathcal{D}_x F(s, X_s) X(s)(dx) ds \\ +\frac{1}{2} &\int_0^t \int_E c\mathcal{D}_{xx} F(s, X_s) X(s)(dx) ds \\ + &\int_0^t \int_E \mathcal{D}_x F(s, X_s) M_X(ds, dx) \end{aligned}$$

holds. As $F(t, X_t)$ is a martingale, all but the last integral on the right hand side vanish and thus we obtain

$$F(t, X_t) = F(0, X_0) + \int_0^t \int_E \mathcal{D}_x F(s, X_s) M_X(ds, dx).$$

□

Now, consider $\Omega = C([0, T], M_F(\mathbb{R}^d))$ with $d \geq 1$. Let $\mathbb{P}_m$ be the law of a super-Brownian motion $X$ with initial law $m \in M_F(\mathbb{R}^d)$ and denote by $\mathcal{F}$ the Borel-$\sigma$-algebra of $\Omega$ and by $(\mathcal{F}_t)_{t\in[0,T]}$ the canonical filtration generated by $X$ that is assumed to satisfy the usual conditions. From Section 1.1.1 we know that $\mathbb{P}_m$ is the unique solution to the martingale problem

$$\begin{aligned} &\mathbb{P}_m(X(0)=m)=1 \text{ and for all } \phi\in D(\tfrac{1}{2}\Delta) \text{ the process}\\ &M(t)(\phi)=\langle X(t),\phi\rangle-\langle X(0),\phi\rangle-\int_0^t\langle X(s),\frac{1}{2}\Delta\phi\rangle ds,\quad t\in[0,T]\\ &\text{is a } (\mathcal{F}_t)_t\text{-local martingale with respect to } \mathbb{P}_m\\ &\text{and has quadratic variation } [M(\phi)]_t=\int_0^t\langle X(s),\phi^2\rangle ds. \end{aligned}\tag{MP$_{\mathrm{SBM}}$}$$

In this section, we denote the expectation with respect to $\mathbb{P}_m$ by $\mathbb{E}[\,\cdot\,]$ to keep notations simple. Further, we denote by $\mathbb{S}(\mathbb{R}^d)$ the Schwartz space on $\mathbb{R}^d$. Functions in $\mathbb{S}(\mathbb{R}^d)$ are infinitely continuously differentiable and rapidly decaying (see e.g. Section 2.2.1 in [Grafakos, 2014]). Thus $\mathbb{S}(\mathbb{R}^d)\subset D(\frac{1}{2}\Delta)$ and for any $h\in\mathbb{S}(\mathbb{R}^d)$ it holds $\frac{1}{2}\Delta h\in D(\frac{1}{2}\Delta)$. In the following, $\mathbb{S}(\mathbb{R}^d)$ plays the role of $D_0$ in the definition of a good generator and thus $\frac{1}{2}\Delta$ is a good generator (see [Jacka and Tribe, 2003]).

**Definition 3.2** ($\|\cdot\|^2_{\mathcal{L}^2(M_X)}$, $\mathcal{L}^2(M_X)$)**.** *Denote by $\mathcal{L}^2(M_X)$ the space of predictable functions $\phi:\Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}$ satisfying*

$$\|\phi\|^2_{\mathcal{L}^2(M_X)}=\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^d}\phi^2(s,x)X(s)(dx)ds\right]<\infty.$$

The space $\mathcal{L}^2(M_X)$ coincides with the space $\mathcal{P}_M$ introduced in Section 1.3.1 but the new notation is introduced as it is more natural in the present context.

**Definition 3.3** ($\|\cdot\|^2_{\mathcal{M}^2}$, $\mathcal{M}^2$)**.** *Denote by $\mathcal{M}^2$ the space of square-integrable $(\mathcal{F}_t)_t$-martingales with initial value zero and with norm*

$$\|Y\|^2_{\mathcal{M}^2}=\mathbb{E}[Y(T)^2].$$

Next, consider functions $\phi:\Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}$ of form

$$\phi(\omega,t,x)=\phi_{\Gamma,a,h}(\omega,t,x)=\Gamma(\omega)h(x)1_{(a,T]}(t)$$

with $\Gamma$ being $\mathcal{F}_a$-measurable and bounded, $h\in\mathbb{S}(\mathbb{R}^d)$ and $0\le a\le T$. Denote the linear span of such functions by $\mathcal{U}$. Then the following holds.

**Proposition 3.4.** *The space $\mathcal{U}$ is a dense subspace of $\mathcal{L}^2(M_X)$.*

*Proof.* As the functions $\phi_{\Gamma,a,h}$ can be expressed as pointwise limit of functions in $\mathcal{S}$, functions in $\mathcal{U}$ are predictable. Next, assume that the bounds of $\Gamma^2$ and $h^2$ are given by $C_{\Gamma^2}$ and $C_{h^2}$, respectively. Then

$$\begin{aligned}\|\phi_{\Gamma,a,h}\|^2_{\mathcal{L}^2(M_X)}&=\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^d}(\Gamma h(x)1_{(a,T]}(s))^2X(s)(dx)ds\right]\\ &\le C_{\Gamma^2}\mathbb{E}\left[\int_0^T\int_{\mathbb{R}^d}h^2(x)1_{(a,T]}(s)X(s)(dx)ds\right]\\ &\le C_{\Gamma^2}\,C_{h^2}\mathbb{E}\left[\int_a^T X(s)(\mathbb{R}^d)ds\right]\\ &=C_{\Gamma^2}\,C_{h^2}\int_a^T\mathbb{E}[X(s)(\mathbb{R}^d)]ds\\ &\le C_{\Gamma^2}\,C_{h^2}\,(T-a)\max_{t\in[a,T]}\mathbb{E}\left[X(t)(\mathbb{R}^d)\right].\end{aligned}$$

As the total mass process of $X$ is a critical Feller continuous state branching process (see Proposition 1.11), we know that

$$\mathbb{E}[X(t)(\mathbb{R}^d)] = X(0)(\mathbb{R}^d) < \infty$$

holds for every $t \in [0, T]$, which yields that every $\phi_{\Gamma,a,h}$ has finite $\| \cdot \|_{\mathcal{L}^2(M_X)}$-norm. Consequently, $\phi_{\Gamma,a,h} \in \mathcal{L}^2(M_X)$ and thus the linear span $\mathcal{U}$ is a subspace of $\mathcal{L}^2(M_X)$.

Now, note that the space of simple functions $\mathcal{S}$ is dense in $\mathcal{L}^2(M_X)$, i.e. $\bar{\mathcal{S}} = \mathcal{L}^2(M_X)$ holds. However, as the closures $\bar{\mathcal{S}}$ and $\bar{\mathcal{U}}$ coincide, we obtain $\bar{\mathcal{U}} = \mathcal{L}^2(M_X)$. Thus, $\mathcal{U}$ is a dense subspace of $\mathcal{L}^2(M_X)$. □

**Definition 3.5.** *A linear operator* $\Pi$ *mapping from its domain* $D(\Pi)$ *into a Hilbert space* $\mathcal{H}$ *is called an* extension *of the linear operator* $\tilde{\Pi} : D(\tilde{\Pi}) \to \mathcal{H}$ *if* $D(\tilde{\Pi}) \subset D(\Pi)$ *and* $\tilde{\Pi}v = \Pi v$ *for all* $v \in D(\tilde{\Pi})$.

From Corollary 3.1 we obtain the martingale representation for a certain class of $(\mathcal{F}_t)_t$-martingales with the integrand in the representation given by the vertical derivative of the martingale. In the following, we derive such a representation formula for processes in the wider class $\mathcal{M}^2$. We do so by using the vertical derivative to first define an operator $\nabla_M$ on a subset of $\mathcal{M}^2$. This operator is then used to derive the martingale representation formula for elements in the subspace. As the considered subspace is dense in $\mathcal{M}^2$, we can extend the operator as well as the martingale representation formula to all $Y \in \mathcal{M}^2$.

**Proposition 3.6.** *The mapping*

$$\begin{array}{rccl} I_{M_X} : & \mathcal{L}^2(M_X) & \to & \mathcal{M}^2 \\ & \phi & \mapsto & \int_0^{\cdot} \int_{\mathbb{R}^d} \phi(s,x) M_X(ds,dx) \end{array} \tag{3.1}$$

*is an isometry.*

*Proof.* Let $Q_{M_X}$ be the covariation of $M_X$ and $\phi$, $\psi \in \mathcal{L}^2(M_X)$. Then, by adapting the proof of Theorem 2.5 in [Walsh, 1986] to our setting where the underlying martingale measure is orthogonal, we get that

$$\begin{aligned} \Bigg( \int_0^t \int_{\mathbb{R}^d} \phi(s,x) M_X(ds,dx) \int_0^t \int_{\mathbb{R}^d} \psi(s,x) M_X(ds,dx) \\ - \int_0^t \int_{\mathbb{R}^d} \phi(s,x)\psi(s,x) Q_{M_X}(ds,dx) \Bigg)_{t \in [0,T]} \end{aligned}$$

is a martingale for all $\phi$, $\psi \in \mathcal{L}^2(M_X)$ and thus

$$\begin{aligned} & \mathbb{E}\left[ \int_0^t \int_{\mathbb{R}^d} \phi(s,x) M_X(ds,dx) \int_0^t \int_{\mathbb{R}^d} \psi(s,x) M_X(ds,dx) \right] \\ = \; & \mathbb{E}\left[ \int_0^t \int_{\mathbb{R}^d} \phi(s,x)\psi(s,x) Q_{M_X}(ds,dx) \right] \end{aligned}$$

for all $t \in [0, T]$.

As the martingale measure $M_X$ is orthogonal, it holds $Q((0,t],B) = \nu((0,t],B)$ with $\nu$ being the dominating measure defined on $\mathbb{R}^d \times [0,T]$. From Example 7.1.3 in [Dawson, 1993] or Theorem 1.31 we know that $\nu$ has the following form if $M_X$ is the martingale measure associated with a $B(A,c)$-superprocess:

$$\nu(ds,dx) = cX(s)(dx)ds.$$

As $c = 1$ in our setting, this yields

$$\begin{aligned} & \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \phi(s,x) M_X(ds,dx) \int_0^t \int_{\mathbb{R}^d} \psi(s,x) M_X(ds,dx)\right] \\ = \; & \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \phi(s,x)\psi(s,x) X(s)(dx)ds\right] \end{aligned} \tag{3.2}$$

for all $t \in [0,T]$ and therefore

$$\begin{aligned} \|I_{M_X}(\phi)\|^2_{\mathcal{M}^2} &= \mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}^d} \phi(s,x) M_X(ds,dx)\right)^2\right] \\ &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \phi^2(s,x) X(s)(dx)ds\right] \\ &= \|\phi\|^2_{\mathcal{L}^2(M_X)}. \end{aligned}$$

□

As the stochastic integral with respect to the martingale measure $M_X$ is well-defined for all $\phi \in \mathcal{U}$, we can define the space

$$I_{M_X}(\mathcal{U}) = \{Y : Y(t) = \int_0^t \int_{\mathbb{R}^d} \phi(s,x) M_X(ds,dx),\ \phi \in \mathcal{U},\ t \in [0,T]\}.$$

For any function $\phi = \phi_{\Gamma,a,h} \in \mathcal{U}$ we have

$$\begin{aligned} I_{M_X}(\phi)(t) &= \int_0^t \int_{\mathbb{R}^d} \phi_{\Gamma,a,h}(s,x) M_X(ds,dx) \\ &= \int_0^t \int_{\mathbb{R}^d} \Gamma \cdot h(x) 1_{(a,T]}(s) M_X(ds,dx) \\ &= \Gamma \cdot (M(t)(h) - M(a)(h)) 1_{t>a} \\ &= \Gamma \cdot \left(\langle X(t),h\rangle - \langle X(a),h\rangle - \int_a^t \int_{\mathbb{R}^d} \frac{1}{2}\Delta h(y) X(s)(dy)ds\right) 1_{t>a} \end{aligned}$$

by the definition of $M(t)(\phi)$. As $h \in \mathbb{S}(\mathbb{R}^d) \subset D(\frac{1}{2}\Delta)$, we get from the martingale problem ($\mathrm{MP}_{\mathrm{SBM}}$) that $(I_{M_X}(\phi)(t))_{t\in[0,T]}$ is a $(\mathcal{F}_t)_t$-martingale for any $\phi \in \mathcal{U}$. In addition, as $\mathcal{U} \subset \mathcal{L}^2(M_X)$, we get that the process $(I_{M_X}(\phi)(t))_{t\in[0,T]}$ is square-integrable since

$$\begin{aligned} \mathbb{E}[(I_{M_X}(\phi)(t))^2] &= \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \phi(s,x) M_X(ds,dx)\right)^2\right] \\ &= \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \phi^2(s,x) X(s)(dx)ds\right] \\ &\le \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \phi^2(s,x) X(s)(dx)ds\right] \\ &< \infty \end{aligned}$$

holds for all $t \in [0,T]$. Combining the fact that $(I_{M_X}(\phi)(t))_{t\in[0,T]}$ is a $(\mathcal{F}_t)_t$-martingale for all $\phi \in \mathcal{U}$ and the square-integrability, we get that $I_{M_X}(\mathcal{U})$ is a subspace of $\mathcal{M}^2$.

Next, consider the function $F$ of form

$$\begin{array}{rccl} F: & \Lambda_T & \to & \mathbb{R} \\ & (t,\omega) & \mapsto & \Gamma(\omega)\left(\langle \omega(t), h\rangle - \langle \omega(a), h\rangle - \int_a^t \int_{\mathbb{R}^d} \frac{1}{2}\Delta h(y)\omega(s)(dy)ds\right) 1_{t>a}. \end{array}$$

Plugging $X_t$ into $F$ for $\omega$ yields

$$\begin{aligned} F(t,X_t) &= \Gamma(X_t)\left(\langle X_t(t), h\rangle - \langle X_t(a), h\rangle - \int_a^t \int_{\mathbb{R}^d} \frac{1}{2}\Delta h(x)X_t(s)(dx)ds\right) 1_{t>a} \\ &= \Gamma(X_t)\left(\langle X(t), h\rangle - \langle X(a), h\rangle - \int_a^t \int_{\mathbb{R}^d} \frac{1}{2}\Delta h(x)X(s)(dx)ds\right) 1_{t>a} \end{aligned}$$

and, as $X_t(\omega) = \omega_t$ and $X(t)(\omega) = \omega(t)$, we get

$$F(t,X_t)(\omega) = \Gamma(\omega_a)\left(\langle \omega(t), h\rangle - \langle \omega(a), h\rangle - \int_a^t \int_{\mathbb{R}^d} \langle \omega(s), \frac{1}{2}\Delta h(x)\omega(s)(dx)ds\right),$$

from which we get, as $\Gamma$ is $\mathcal{F}_a$-measurable, that $F(t,X_t) = I_{M_X}(\phi_{\Gamma,a,h})(t)$.

As, in addition, for any path $\omega \in C([0,T], M_F(E))$, it holds

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\Bigg(\langle(\omega + \varepsilon\delta_x 1_{[t,T]})(t), h\rangle - \langle(\omega + \varepsilon\delta_x 1_{[t,T]})(a), h\rangle \\ &\qquad\qquad - \int_a^t \int_{\mathbb{R}^d} \frac{1}{2}\Delta h(y)(\omega + \varepsilon\delta_x 1_{[t,T]})(r)(dy)dr \\ &\qquad - \langle \omega(t), h\rangle + \langle \omega(a), h\rangle + \int_a^t \int_{\mathbb{R}^d} \frac{1}{2}\Delta h(y)\omega(r)(dy)dr\Bigg) \\ &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\left(\varepsilon h(x) - \varepsilon \int_a^t \int_{\mathbb{R}^d} \frac{1}{2}\Delta h(y)1_{[t,T]}(r)\delta_x(dy)dr\right) \\ &= h(x) \end{aligned}$$

and, as $\Gamma$ is $\mathcal{F}_a$-measurable, for $t \in (a,T]$ it holds

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\Gamma(\omega + \varepsilon\delta_x 1_{[t,T]}) - \Gamma(\omega)\right) = 0,$$

we obtain, for all $(t,\omega) \in \Lambda_T$,

$$\mathcal{D}_x F(t,\omega) = \Gamma(\omega)1_{(a,T]}(t)h(x) = \phi_{\Gamma,a,h}(t,x). \tag{3.3}$$

Now, for a process $Y$ defined by

$$Y(t) = I_{M_X}(\phi_{\Gamma,a,h})(t) = F(t,X_t)$$

for $t \in [0,T]$, we can define the operator $\nabla_M Y$ of form

$$\begin{array}{rccl} \nabla_M: & I_{M_X}(\mathcal{U}) & \to & \mathcal{L}^2(M_X) \\ & Y & \mapsto & \nabla_M Y, \end{array} \tag{3.4}$$

where $\nabla_M Y$ is given by the vertical derivative of $F$:

$$\nabla_M Y : (\omega, t, x) \mapsto \nabla_M Y(\omega, t, x) := \mathcal{D}_x F(t, X_t(\omega)) = \mathcal{D}_x F(t, \theta)|_{\theta = X_t(\omega)}.$$

Further, from (3.3) and the definition of $Y$, we get the following representation of $Y$:

$$Y(t) = \int_0^t \int_{\mathbb{R}^d} \nabla_M Y(s, x) M_X(ds, dx). \tag{3.5}$$

Due to the connection of $F$ and $Y$ as well as the definition of $\nabla_M Y$, this is equal to

$$F(t, X_t) = \int_0^t \int_{\mathbb{R}^d} \mathcal{D}_x F(s, X_s) M_X(ds, dx), \tag{3.6}$$

which coincides with the representation obtained in Corollary 3.1 as $F(0, X_0) = 0$ holds.

The representation in (3.5) is the martingale representation formula for processes in the subspace $I_{M_X}(\mathcal{U})$ of $\mathcal{M}^2$ and based on the operator $\nabla_M$ defined by (3.4) on $I_{M_X}(\mathcal{U})$. In [Evans and Perkins, 1994], the authors prove that, if $Y$ belongs to $\mathcal{M}^2$, there exists a unique $\rho \in \mathcal{L}^2(M_X)$ such that

$$Y(t) = \int_0^t \int_{\mathbb{R}^d} \rho(s, x) M_X(ds, dx) \quad \forall t \geq 0 \tag{3.7}$$

holds $\mathbb{P}_m$-almost surely. Consequently, the representation of $Y$ in (3.5) is unique. The result by Evans and Perkins further yields that the mapping $I_{M_X}$ defined by (3.1) is a bijection, which allows us to prove the following proposition.

**Proposition 3.7.** *The space $\{\nabla_M Y : Y \in I_{M_X}(\mathcal{U})\}$ is dense in $\mathcal{L}^2(M_X)$ and the space $I_{M_X}(\mathcal{U})$ is dense in $\mathcal{M}^2$.*

*Proof.* From Proposition 3.4 we know that $\mathcal{U}$ is dense in $\mathcal{L}^2(M_X)$. As

$$\mathcal{U} = \{\nabla_M Y : Y \in I_{M_X}(\mathcal{U})\} \subset \mathcal{L}^2(M_X)$$

holds, we immediately get the density of $\{\nabla_M Y : Y \in I_{M_X}(\mathcal{U})\}$ in $\mathcal{L}^2(M_X)$. Further, as $I_{M_X}$ is a bijective isometry (see Proposition 3.6), we get the density of $I_{M_X}(\mathcal{U})$ in $\mathcal{M}^2$ from the density of $\mathcal{U}$ in $\mathcal{L}^2(M_X)$. $\square$

The density results above allow us to prove the following proposition that plays an essential role in the extension of the operator $\nabla_M$ from $I_{M_X}(\mathcal{U})$ to $\mathcal{M}^2$.

**Proposition 3.8.** *If $Y \in I_{M_X}(\mathcal{U})$, then $\nabla_M Y$ is the unique element in $\mathcal{L}^2(M_X)$ such that*

$$\mathbb{E}\left[Y(T)Z(T)\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s, x) \nabla_M Z(s, x) X(s)(dx) ds\right] \tag{3.8}$$

*holds for all $Z \in I_{M_X}(\mathcal{U})$.*

*Proof.* Let $Y$, $Z \in I_{M_X}(\mathcal{U})$. Then $Y$ and $Z$ have a representation of form (3.5) and from (3.2) we get that

$$\mathbb{E}\left[Y(T)Z(T)\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s,x) \nabla_M Z(s,x) X(s)(dx)ds\right] \tag{3.9}$$

holds.

To prove the uniqueness of the representation, assume $\psi \in \mathcal{L}^2(M_X)$ is another process such that, for all $Z \in I_{M_X}(\mathcal{U})$, it holds

$$\mathbb{E}\left[Y(T)Z(T)\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \psi(s,x) \nabla_M Z(s,x) X(s)(dx)ds\right].$$

In this case, by subtraction we obtain

$$0 = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} (\psi(s,x) - \nabla_M Y(s,x)) \nabla_M Z(s,x) X(s)(dx)ds\right]$$

for all $Z \in I_{M_X}(\mathcal{U})$. As $\{\nabla_M Z : Z \in I_{M_X}(\mathcal{U})\}$ is dense in $\mathcal{L}^2(M_X)$, this yields that $\psi = \nabla_M Y$ in $\mathcal{L}^2(M_X)$ and thus the uniqueness. □

Equation (3.8) can be interpreted as an integration by parts formula. This becomes clear by considering the following alternative form, which holds for all $\phi \in \mathcal{L}^2(M_X)$:

$$\mathbb{E}\left[Y(T)\int_0^T \int_{\mathbb{R}^d} \phi(s,x) M_X(ds,dx)\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s,x) \phi(s,x) X(s)(dx)ds\right].$$

**Theorem 3.9.** *The operator defined in* (3.4) *can be extended to an operator*

$$\begin{array}{rccc} \nabla_M : & \mathcal{M}^2 & \to & \mathcal{L}^2(M_X) \\ & Y & \mapsto & \nabla_M Y. \end{array}$$

*This operator, which is a bijection and the unique continuous extension of the operator defined in* (3.4)*, is given by the following: For a given $Y \in \mathcal{M}^2$, $\nabla_M Y$ is the unique element in $\mathcal{L}^2(M_X)$ such that*

$$\mathbb{E}[Y(T)Z(T)] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s,x) \nabla_M Z(s,x) X(s)(dx)ds\right] \tag{3.10}$$

*holds for all $Z \in I_{M_X}(\mathcal{U})$.*

*Proof.* As

$$\nabla_M : I_{M_X}(\mathcal{U}) \to \mathcal{L}^2(M_X)$$

is a bounded linear operator, $\mathcal{L}^2(M_X)$ is a Hilbert space and $I_{M_X}(\mathcal{U})$ is dense in $\mathcal{M}^2$, the BLT theorem (bounded linear transformation theorem; see e.g. Theorem 5.19 in [Hunter and Nachtergaele, 2001]) yields the existence of a unique continuous extension

$$\nabla_M : \mathcal{M}^2 \to \mathcal{L}^2(M_X).$$

To prove that (3.10) uniquely characterizes the extension, we have to prove that the operator's restriction to $I_{M_X}(\mathcal{U})$ is equal to the initial operator. As we immediately get this from Proposition 3.8, the unique continuous extension is given by (3.10).

For $Y \in \mathcal{M}^2$, there exists a unique $\rho \in \mathcal{L}^2(M_X)$ such that (3.7) holds. By combining (3.7) with (3.5) and (3.2), we get that $\rho$ satisfies (3.10) as

$$\begin{aligned}\mathbb{E}[Y(T)Z(T)] &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \rho(s,x) M_X(ds,dx) Z(T)\right] \\ &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \rho(s,x) \nabla_M Z(s,x) X(s)(dx) ds\right]\end{aligned}$$

holds for all $Z \in I_{M_X}(\mathcal{U})$. Thus, by the uniqueness of the integrand in (3.7), $\rho$ and $\nabla_M Y$ coincide in $\mathcal{L}^2(M_X)$.

Now assume $Y$, $Y' \in \mathcal{M}^2$ with $\nabla_M Y = \nabla_M Y'$ for $\nabla_M Y$, $\nabla_M Y' \in \mathcal{L}^2(M_X)$. Then, for all $Z \in I_{M_X}(\mathcal{U})$,

$$\begin{aligned}0 &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} (\nabla_M Y(s,x) - \nabla_M Y'(s,x)) \nabla_M Z(s,x) X(s)(dx) ds\right] \\ &= \mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s,x) M_X(ds,dx) ds - \int_0^T \int_{\mathbb{R}^d} \nabla_M Y'(s,x) M_x(ds,dx)\right) Z(T)\right] \\ &= \mathbb{E}[(Y(T) - Y'(T))Z(T)],\end{aligned}$$

which implies that $Y = Y'$ in $\mathcal{M}^2$ as $I_{M_X}(\mathcal{U})$ is dense in $\mathcal{M}^2$. Therefore, the operator $\nabla_M$ is injective. In addition, the operator is surjective as for every $\phi \in \mathcal{L}^2(M_X)$ there exists the process

$$Y = \int_0^{\cdot} \int_{\mathbb{R}^d} \phi(s,x) M_X(ds,dx) \in \mathcal{M}^2$$

for which $\nabla_M Y = \phi$ holds. Consequently, the operator is bijective. □

Combining the above results allows us to formulate a version of (3.5) for all processes in $\mathcal{M}^2$ and therefore for all square-integrable $(\mathcal{F}_t)_t$-martingales.

**Theorem 3.10.** *For any square-integrable $(\mathcal{F}_t)_t$-martingale $Y$ and every $t \in [0,T]$ it holds*

$$Y(t) = Y(0) + \int_0^t \int_{\mathbb{R}^d} \nabla_M Y(s,x) M_X(ds,dx) \quad \mathbb{P}_m\text{-}a.s. \ . \tag{3.11}$$

*Proof.* First, consider $Y \in \mathcal{M}^2$. We know that for every such $Y$, there exists a unique $\rho \in \mathcal{L}^2(M_X)$ such that (3.7) holds. From the proof of Theorem 3.9, we also know that this $\rho$ satisfies (3.10) and that $\rho = \nabla_M Y$ holds in $\mathcal{L}^2(M_X)$. Consequently, for $Y \in \mathcal{M}^2$ and all $t \in [0,T]$,

$$Y(t) = \int_0^t \int_{\mathbb{R}^d} \nabla_M Y(s,x) M_X(ds,dx)$$

holds $\mathbb{P}_m$-almost surely. As the integrand $\rho$ in (3.7) is unique, this completes the proof for $Y \in \mathcal{M}^2$.

To obtain the result for all square-integrable $(\mathcal{F}_t)_t$-martingales $Y$, note that, for any such $Y$, $\tilde{Y} = Y - Y(0)$ is an element in $\mathcal{M}^2$. Therefore, we can apply the above to $\tilde{Y}$ and then add $Y(0)$ to both sides to obtain (3.11). □

Besides its role in the martingale representation, the operator $\nabla_M$ defined on $\mathcal{M}^2$ has the following properties which are worth mentioning.

**Proposition 3.11.** *The operator $\nabla_M$ defined on $\mathcal{M}^2$ is an isometry and the adjoint operator of $I_{M_X}$, the stochastic integral with respect to the martingale measure $M_X$.*

*Proof.* Let $Y \in \mathcal{M}^2$. By following the arguments in the proof of Proposition 3.6, we get the isometry property from

$$\begin{aligned}
\|\nabla_M Y\|^2_{\mathcal{L}^2(M_X)} &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} (\nabla_M Y(s,x))^2 X(s)(dx)ds\right] \\
&= \mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s,x) M_X(ds,dx)\right)^2\right] \\
&= \left\|\int_0^{\cdot} \int_{\mathbb{R}^d} \nabla_M Y(s,x) M_X(ds,dx)\right\|^2_{\mathcal{M}^2} \\
&= \|Y\|^2_{\mathcal{M}^2}.
\end{aligned}$$

To show that $\nabla_M$ is the adjoint operator of $I_{M_X}$, let $\phi \in \mathcal{L}^2(M_X)$. Then the result follows from

$$\begin{aligned}
\langle I_{M_X}(\phi), Y\rangle_{\mathcal{M}^2} &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \phi(s,x) M_X(ds,dx) Y(T)\right] \\
&= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \phi(s,x) M_X(ds,dx) \int_0^T \int_{\mathbb{R}^d} \nabla_M Y(s,x) M_X(ds,dx)\right] \\
&= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \phi(s,x) \nabla_M Y(s,x) X(s)(dx)ds\right] \\
&= \langle \phi, \nabla_M Y\rangle_{\mathcal{L}^2(M_X)}.
\end{aligned}$$

□

## 3.2 The Representation Formula for Square-Integrable $(\mathcal{H}_t)_t$-Martingales

If we replace the super-Brownian motion by a historical Brownian motion, the derivation of the martingale representation formula is analogous to the derivation of the representation in the previous section. However, while concepts like the functional derivatives used in Section 3.1 have already been used in Chapter 2, we still have to introduce these concepts in the setting underlying the historical Brownian motion. For the sake of simplicity, we reuse most of the notation from the previous section. As we are only concerned with the historical Brownian motion from here on, this reuse should not lead to any confusion.

Let $D([\tau,T],M_F(C))$ be the space of right continuous functions from $[\tau,T]$ to the space of finite measures on $C$, $M_F(C)$, with left limits and equip this space with the metric given by

$$\tilde{d}(\omega,\omega') = \sup_{s\in[\tau,T]} d_P(\omega(s),\omega'(s))$$

for all $\omega$, $\omega' \in D([\tau,T],M_F(C))$, where $d_P$ is the Prokhorov metric on $M_F(C)$. Using this metric, we can define an equivalence relation on the space $[\tau,T]\times D([\tau,T],M_F(C))$ by

$$(t,\omega)\sim(t',\omega') \quad \Leftrightarrow \quad t=t' \text{ and } \omega_t=\omega'_{t'}.$$

This relation gives rise to the quotient space

$$\Lambda_T = \{(t,\omega_t) : (t,\omega)\in[\tau,T]\times D([\tau,T],M_F(C))\} = [\tau,T]\times D([\tau,T],M_F(C))/\sim,$$

which we equip with the metric given by

$$d_\infty((t,\omega),(t',\omega')) = \tilde{d}(\omega_t,\omega'_{t'}) + |t-t'| = \sup_{s\in[\tau,T]} d_P(\omega(t\wedge s),\omega'(t'\wedge s)) + |t-t'|.$$

This leads to the following definitions of continuous and non-anticipative functionals as well as to the definition of functional derivatives, which are equivalent to Definitions 2.11, 2.12 and 2.13.

**Definition 3.12** (Continuity with respect to $d_\infty$)**.** *A functional $F:\Lambda_T\to\mathbb{R}$ is continuous with respect to $d_\infty$ if for all $(t,\omega)\in\Lambda_T$ and every $\varepsilon>0$ there exists an $\eta>0$ such that for all $(t',\omega')\in\Lambda_T$ with $d_\infty((t,\omega),(t',\omega'))<\eta$ we have*

$$|F(t,\omega)-F(t',\omega')|<\varepsilon.$$

**Definition 3.13** (Non-anticipative)**.** *A measurable functional $F$ on $[\tau,T]\times D([\tau,T],M_F(C))$ is* non-anticipative *if*

$$F(t,\omega)=F(t,\omega_t) \quad \textit{for all } \omega\in D([\tau,T],M_F(C)),$$

*which is the case if $F:\Lambda_T\to\mathbb{R}$.*

**Definition 3.14** (Functional derivatives)**.** *A continuous non-anticipative functional $F:\Lambda_T\to\mathbb{R}$ is*

*(i)* horizontally differentiable *at $(t,\omega)\in\Lambda_T$ if the limit*

$$\mathcal{D}^*F(t,\omega)=\lim_{\varepsilon\to 0}\frac{F(t+\varepsilon,\omega_t)-F(t,\omega_t)}{\varepsilon}$$

*exists. If this is the case for all $(t,\omega)\in\Lambda_T$, we call $\mathcal{D}^*F$ the* horizontal derivative *of $F$.*

*(ii)* vertically differentiable *at $(t,\omega)\in\Lambda_T$ in direction $\delta_y 1_{[t,T]}$, $y\in C$, if the limit*

$$\mathcal{D}_yF(t,\omega)=\lim_{\varepsilon\to 0}\frac{F(t,\omega_t+\varepsilon\delta_y 1_{[t,T]})-F(t,\omega_t)}{\varepsilon}$$

*exists. If this is the case for all $(t,\omega)\in\Lambda_T$, we call $\mathcal{D}_yF$ the* vertical derivative *of $F$* in direction $\delta_y 1_{[t,T]}$. *Higher order vertical derivatives are defined iteratively.*

As before, a function is called predictable, if it measurable with respect to the predictable $\sigma$-algebra. In the present scenario, the predictable $\sigma$-algebra is the $\sigma$-algebra on $\Omega_H \times [\tau, T] \times C$ generated by the space of simple functions $\mathcal{S}$, which are given by linear combinations of functions on $\Omega_H \times [\tau, T] \times C$ of form

$$\Phi_{\Gamma,B,a}(\omega, t, y) = \Gamma(\omega) 1_B(y) 1_{(a,T]}(t),$$

where $\Gamma$ is a bounded, $\mathcal{H}_a$-measurable random variable, $B \in \mathcal{C}$ and $\tau \leq a \leq T$.

Now, let $H$ be a historical Brownian motion on $(\Omega_H, \mathcal{H}[\tau, T], (\mathcal{H}_t)_{t\in[\tau,T]}, \mathbb{P}_{\tau,m})$ with branching rate $\gamma = 1$. To keep notations simple, we denote the expectation with respect to $\mathbb{P}_{\tau,m}$ by $\mathbb{E}[\,\cdot\,]$. In Section 1.3 we briefly mention the existence of a martingale measure associated with the historical Brownian motion, which we now denote by $M_H$. The covariation of this martingale measure is given by

$$\nu((s, t] \times B) = \int_s^t \langle H(s), 1_B\rangle ds \quad \text{for all } \tau \leq s < t \leq T \text{ and } B \in \mathcal{C}.$$

Using the martingale measure $M_H$, we can write the martingale $Z(t)(\Phi)$ in (MP$_{HBM}$) and (MP$_{HBM-fd}$) as

$$Z(t)(\Phi) = \int_\tau^t \int_C \Phi(s, y) M_H(ds, dy)$$

for every $\Phi \in D(A_{\tau,m})$ or $\Phi \in D_{fd}$, respectively.

Martingales with respect to the filtration $(\mathcal{H}_t)_t$ are sometimes also called *historical martingales*. Before we follow the steps from the previous sections to obtain the martingale representation formula for all square-integrable historical martingales, we provide the two examples to illustrate that there are specific scenarios in which we can directly compute the martingale representation even without using the Itō-formula. However, before we can present the example, we have to introduce the Laplace transform for a historical Brownian motion, which goes along with introducing the following concepts from Section 12 in [Dawson, 1993].

Let $W$ be a Brownian motion on $\mathbb{R}^d$. Then, for $(s, y) \in \{(s, y) : y = y_s\}$, $\tau \leq s \leq t \leq T$, we can define the semingroup

$$S_{s,t} f(y) = \mathbb{E}[f(y/s/W_{t-s})|W(0) = y(s)] \tag{3.12}$$

for all $f \in b\mathcal{C}$, the space of bounded, $\mathcal{C}$-measurable functions, with the expectation being with respect to the law of the underlying Brownian motion. Another way of characterizing $S_{s,t}$ is given by

$$S_{s,t} f(y) = \mathbb{E}[f(W_t)|W_s = y].$$

Thus, $S_{s,t}$ is the transition semigroup of the path process of a Brownian motion. This leads to our first example.

**Example 3.15.** *Let $f : C \to \mathbb{R}$ be such that $\Phi \in D(A_{\tau,m})$ with $\Phi$ given by $\Phi(s, y) = S_{s,T} f(y)$. Then, we get from the* (MP$_{HBM}$) *that for all $t \in [\tau, T]$*

$$\langle H(t), \Phi(t, \cdot)\rangle - \langle H(\tau), \Phi(\tau, \cdot)\rangle - \int_\tau^T \langle H(s), A_{\tau,m}\Phi(s, \cdot)\rangle ds = \int_\tau^t \int_C \Phi(s, y) M_H(ds, dy) \tag{3.13}$$

*holds.*

*As*

$$\Phi(s,y) = S_{s,t}f(y) = \mathbb{E}[f(W_t)|W_s = y]$$

*is a* $(\mathcal{C}_t)_t$*-martingale, we get* $A_{\tau,m} \equiv 0$ *and thus* (3.13) *becomes*

$$\langle H(t), \Phi(t,\cdot)\rangle - \langle H(\tau), \Phi(\tau,\cdot)\rangle = \int_\tau^t \int_C \Phi(s,y) M_H(ds,dy). \tag{3.14}$$

*Next, set* $Y(t) = \langle H(t), S_{t,T}f\rangle$*. We immediately get the martingale representation of* $Y$ *from* (3.14)*, which is given by*

$$Y(T) = \langle H(T), f\rangle = \langle H(\tau), S_{\tau,T}f\rangle + \int_\tau^T \int_C S_{s,T}f(y) M_H(ds,dy). \tag{3.15}$$

*If we think of* $Y(t)$ *as a function* $F$ *of* $t$ *and* $H(t)$ *with* $F$ *given by* $F(t,H(t)) = \langle H(t), S_{t,T}f\rangle$*, the integrand in* (3.15) *is given by the directional derivative of* $F$ *in direction* $\delta_y$ *as*

$$D_y F(t,H(t)) = \lim_{\varepsilon\to 0} \frac{\langle H(t) + \varepsilon\delta_y, S_{t,T}f\rangle - \langle H(t), S_{t,T}f\rangle}{\varepsilon} = S_{t,T}f(y) \tag{3.16}$$

*holds.*

*We can also think of* $Y(t)$ *as a functional* $\tilde{F}$ *of* $t$ *and* $H_t$ *with* $\tilde{F}$ *given by* $\tilde{F}(t,H_t) = \langle H_t(t), S_{t,T}f\rangle$*. Then, since*

$$\mathcal{D}_y F(t,H_t) = \lim_{\varepsilon\to 0} \frac{\langle (H_t + \varepsilon\delta_y 1_{[t,T]})(t), S_{t,T}f\rangle - \langle H_t(t), S_{t,T}f\rangle}{\varepsilon} = S_{t,T}f(y)$$

*holds, the integrand in the martingale representation* (3.15) *also coincides with the vertical derivative in direction* $\delta_y 1_{[t,T]}$ *of* $\tilde{F}$ *.*

Given the semigroup defined before Example 3.15, we can introduce the Laplace transform for the historical Brownian motion. For all $\Psi \in bp\mathcal{C}$, the space of non-negative, bounded, $\mathcal{C}$-measurable functions, we have (see e.g. Theorem 12.3.1.1 in [Dawson, 1993]) that

$$\mathbb{E}[\exp(-\langle H(t), \Psi\rangle)|H(s) = \mu] = \exp(-\langle \mu, V_{s,t}\Phi\rangle) \tag{3.17}$$

holds, where $V_{s,t}\Psi(y)$ is the unique solution of

$$V_{s,t}\Phi(y) = S_{s,t}\Psi(y) - \frac{1}{2}\int_s^t S_{s,r}((V_{r,t}\Psi)^2)(y)dr. \tag{3.18}$$

Further, by Theorem 12.3.1.1 in [Dawson, 1993] we get that (3.18) is bounded and Borel measurable in $(s,y,t) \in \{(s,y,t) : s \in [\tau,T], y = y_s, t \in [s,T]\}$. This finally allows us to introduce the second scenario in which we can compute the martingale representation.

**Example 3.16.** *Let* $\Psi \in bp\mathcal{C}$ *and set* $\Phi(s,y) = V_{s,T}\Psi(y)$*. Then, combining the above with the result on* $(\mathcal{C}_t)_t$*-predictability from Section V.2 in [Perkins, 2002] (which we already mentioned* *in Section 1.1.2) we get that* $\Phi$ *is* $(\mathcal{C}_t)_t$*-predictable. Further, as*

$$\begin{aligned}\frac{\partial}{\partial s}V_{s,t}\Psi(y) &= \frac{\partial}{\partial s}S_{s,t}\Psi(y) - \frac{1}{2}\left(\int_s^t \frac{\partial}{\partial s}S_{s,r}((V_{r,t}\Psi)^2)(y)dr - S_{s,s}((V_{s,t}\Psi)^2)(y)\right)\\ &= -A_sS_{s,t}\Psi(y) - \frac{1}{2}\left(\int_s^t -A_sS_{s,r}((V_{r,t}\Psi)^2)(y)dr + (V_{s,t}\Psi)^2(y)\right)\\ &= -A_s\left(S_{s,t}\Psi(y) - \frac{1}{2}\int_s^t S_{s,r}((V_{r,t}\Psi)^2)(y)dr\right) + \frac{1}{2}(V_{s,t}\Psi)^2(y)\\ &= -A_sV_{s,t}\Psi(y) - \frac{1}{2}(V_{s,t}\Psi(y))^2\end{aligned} \tag{3.19}$$

*holds with*

$$A_sf = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(S_{s,s+\varepsilon}f - f)$$

*for time-independent* $f$*, we get that* $\Phi$ *is continuous in* $s$ *and thus that* $\Phi \in F_{\tau,m}$*. By rearranging the terms in* (3.19)*, we get*

$$A_{\tau,m}\Phi(s,y) = \frac{\partial}{\partial s}\Phi(s,y) + A_s\Phi(s,y) = -\frac{1}{2}\Phi(s,y)^2.$$

*Consequently,* $\Phi \in D(A_{\tau,m})$*.*

*Now, set* $Y(t) = \exp(-\langle H(t), V_{t,T}\Psi\rangle)$ *for a* $\Psi \in bp\mathcal{C}$*. As* $\Phi \in D(A_{\tau,m})$*, we obtain from the martingale problem* (MP$_{HBM}$) *that*

$$\tilde{Y}(t) = \langle H(t), V_{t,T}\Psi\rangle = \langle H(\tau), V_{\tau,T}\Psi\rangle + \int_\tau^t \langle H(s), V_{s,T}\Psi\rangle ds + Z(t)(V_{\cdot,T}\Psi)$$

*holds and therefore, that* $\tilde{Y}(t)$ *is a* $\mathbb{R}$*-valued semimartingale. Thus, we can apply the Itō-formula for such processes, which yields for* $Y(t) = f(\tilde{Y}(t)) = \exp(-\tilde{Y}(t))$

$$\begin{aligned}f(\tilde{Y}(t)) &= f(\tilde{Y}(\tau)) + \int_\tau^t f'(\tilde{Y}(s))d\tilde{Y}(s) + \frac{1}{2}\int_\tau^t f''(\tilde{Y}(s))d[\tilde{Y}]_s\\ &= \exp(-\langle H(\tau), V_{\tau,T}\Psi\rangle) + \int_\tau^t\int_C -\exp(-\langle H(s), V_{s,T}\Psi\rangle)V_{s,T}\Psi(y)M_H(ds,dy)\\ &\quad + \int_\tau^t\int_C -\exp(-\langle H(s), V_{s,T}\Psi\rangle)A_{\tau,m}V_{s,T}\Psi(y)H(s)(dy)ds\\ &\quad + \frac{1}{2}\int_\tau^t\int_C \exp(-\langle H(s), V_{s,T}\Psi\rangle)(V_{s,T}\Psi(y))^2H(s)(dy)ds.\end{aligned}$$

*As* $Y(t)$ *is a martingale by* (3.17)*, the two last terms in the above equation cancel out, from which we obtain the following martingale representation of* $Y$*:*

$$Y(t) = Y(\tau) - \int_\tau^t\int_C \exp(-\langle H(s), V_{s,T}\Psi\rangle)V_{s,T}\Psi(y)M_H(ds,dy). \tag{3.20}$$

*As*

$$\begin{aligned}&\lim_{\varepsilon\to\infty}\frac{\exp(-\langle H(t)+\varepsilon\delta_y, V_{t,T}\Psi\rangle) - \exp(\langle H(t), V_{t,T}\Psi)}{\varepsilon}\\ =\ &\lim_{\varepsilon\to\infty}\frac{\exp(-\langle (H_t+\varepsilon\delta_y 1_{[t,T]})(t), V_{t,T}\Psi\rangle) - \exp(\langle H_t(t), V_{t,T}\Psi)}{\varepsilon}\\ =\ &\exp(-\langle H(t), V_{t,T}\Psi\rangle)(-V_{t,T}\Psi(y))\end{aligned}$$

*holds, the integrand in* (3.20) *coincides with the directional as well as the vertical directional derivative if we interpret $Y$ as a function of $H(t)$ or a functional of $H_t$, respectively.*

The process $Y$ studied in Example 3.15 is also mentioned in the introduction of [Evans and Perkins, 1995] as an example for a process for which the martingale representation is known. By artificially thinking of the process as a functional of the stopped path $H_t$, we observe that the integrand in the martingale representation formula for a process in this particular class is given by the vertical derivative of the process. The same phenomenon arises for the second class of processes studied in Example 3.16, which suggests a relationship between the integrand in the martingale representation formula and the vertical derivative of a process like in Section 3.1. In the following, we show that this relationship indeed exists and that, once again, by extending the vertical derivative operator, we obtain the integrand in the martingale representation for all square-integrable $(\mathcal{H}_t)_t$-martingales.

**Definition 3.17** ($\|\cdot\|^2_{\mathcal{L}^2(M_H)}$, $\mathcal{L}^2(M_H)$)**.** *Denote by $\mathcal{L}^2(M_H)$ the space of predictable functions $\Phi : \Omega_H \times [\tau, T] \times C \to \mathbb{R}$ satisfying*

$$\|\Phi\|^2_{\mathcal{L}^2(M_H)} = \mathbb{E}\left[\int_\tau^T \int_C \Phi^2(s,y)H(s)(dy)ds\right] < \infty.$$

**Definition 3.18** ($\|\cdot\|^2_{\mathcal{M}^2}$, $\mathcal{M}^2$)**.** *Denote by $\mathcal{M}^2$ the space of square-integrable $(\mathcal{H}_t)_t$-martingales with initial value zero and with norm*

$$\|Y\|^2_{\mathcal{M}^2} = \mathbb{E}[Y(T)^2].$$

The space $\mathcal{U}$ consists of linear combinations of functions $\Phi : \Omega_H \times [\tau, T] \times C \to \mathbb{R}$ of form

$$\Phi(\omega, t, y) = \Phi_{\Gamma,\Psi,a}(\omega, t, y) = \Gamma(\omega)\Psi(y)1_{(a,T]}(t),$$

where $\Gamma$ is a bounded, $\mathcal{H}_a$-measurable random variable, $\Psi \in D_{fd}$ and $\tau \le a \le T$.

**Proposition 3.19.** *The space $\mathcal{U}$ is a dense subspace of $\mathcal{L}^2(M_H)$.*

*Proof.* Just as in the proof of Proposition 3.4, functions in $\mathcal{U}$ can be expressed as pointwise limits of functions in $\mathcal{S}$ and thus are predictable. Now, let $\Phi_{\Gamma,\Psi,a} \in \mathcal{U}$ and denote the bounds of $\Gamma^2$ and $\Psi^2$ by $C_{\Gamma^2}$ and $C_{\Psi^2}$, respectively. Then

$$\begin{aligned}\|\Phi_{\Gamma,\Psi,a}\|^2_{\mathcal{L}^2(M_H)} &= \mathbb{E}\left[\int_\tau^T \int_C (\Gamma\Psi(s,y)1_{(a,T]}(s))^2 H(s)(dy)ds\right] \\ &\le C_{\Gamma^2}C_{\Psi^2}\mathbb{E}\left[\int_a^T \int_C H(s)(dy)ds\right] \\ &< \infty,\end{aligned}$$

where we get that the expectation is finite from Corollary 2.2 in [Perkins, 1995]. This yields $\mathcal{U} \subset \mathcal{L}^2(M_H)$.

For the proof of the density, recall that $\mathcal{S}$ is dense in $\mathcal{L}^2(M_H)$. Thus, if $\bar{\mathcal{U}} = \bar{\mathcal{S}}$, we get that $\mathcal{U}$ is dense in $\mathcal{L}^2(M_H)$. As the inclusion $\mathcal{U} \subset \bar{\mathcal{S}}$ is obvious since $\mathcal{U} \subset \mathcal{L}^2(M_H)$, we only have to show that $\mathcal{S} \subset \bar{\mathcal{U}}$. However, this follows from the fact that $D_{fd}$ is bp-dense in $b\mathcal{C}$ (see [Perkins, 1995]) and $1_B \in b\mathcal{C}$, which completes the proof. □

For the integration with respect to the martingale measure $M_H$ associated with the historical Brownian motion $H$, the following result holds.

**Proposition 3.20.** *The mapping*

$$\begin{aligned} I_{M_H}: \quad \mathcal{L}^2(M_H) &\to \mathcal{M}^2 \\ \Phi &\mapsto \int_\tau^\cdot \int_C \Phi(s,y) M_H(ds,dy) \end{aligned}$$

*is an isometry.*

*Proof.* As $M_H$ is an orthogonal martingale measure with covariation given by

$$\nu(ds,dy) = H(s)(dy)ds$$

and from the proof of Theorem 2.5 in [Walsh, 1986] with 0 in the lower integral bound replaced by $\tau$, we get for all predictable $\Phi$ and $\Psi$ and all $t \in [\tau, T]$

$$\begin{aligned} &\mathbb{E}\left[\int_\tau^t \int_C \Phi(s,y) M_M(ds,dy) \int_\tau^t \int_C \Psi(s,y) M_H(ds,dy)\right] \\ &= \mathbb{E}\left[\int_\tau^t \int_C \Phi(s,y)\Psi(s,y) H(s)(dy)ds\right]. \end{aligned} \tag{3.21}$$

Setting $\Psi = \Phi$ in (3.21) yields the isometry property

$$\begin{aligned} \|I_{M_H}(\Phi)\|^2_{\mathcal{M}^2} &= \mathbb{E}\left[\left(\int_\tau^T \int_C \Phi(s,y) M_H(ds,dy)\right)^2\right] \\ &= \mathbb{E}\left[\int_\tau^T \int_C \Phi(s,y)^2 H(s)(dy)ds\right] \\ &= \|\Phi\|^2_{\mathcal{L}^2(M_H)}. \end{aligned}$$

□

Now, as in the previous section, we define the space

$$I_{M_H}(\mathcal{U}) = \{Y : Y(t) = \int_\tau^t \int_C \Phi(s,y) M_H(ds,dy), \Phi \in \mathcal{U},\, t \in [\tau, T]\},$$

on which we later define the initial operator $\nabla_M$. As for any $\Phi = \Phi_{\Gamma,\Psi,a} \in \mathcal{U}$

$$\begin{aligned} I_{M_H}(\Phi)(t) &= \int_\tau^t \int_C \Gamma\Psi(y) 1_{(a,T]}(s) M_H(ds,dy) \\ &= \Gamma \int_a^t \int_C \Psi(y) M_H(ds,dy) 1_{t>a} \\ &= \Gamma(M(t)(\Psi) - M(a)(\Psi)) 1_{t>a} \\ &= \Gamma\left(\langle H(t), \Psi\rangle - \langle H(a), \Psi\rangle - \int_a^t \langle H(s), \frac{1}{2}\bar{\Delta}\Psi\rangle ds\right) 1_{t>a}. \end{aligned}$$

holds, we get from the martingale problem ($\mathrm{MP}_{HBM-fd}$) that the term in parentheses is a martingale and thus, as $\Gamma$ is $\mathcal{H}_a$-measurable, that elements in $I_{M_H}(\mathcal{U})$ are martingales with initial value $I_{M_H}(\Phi)(\tau) = 0$. As

$$\begin{aligned}
\mathbb{E}[(I_{M_H}(\Phi)(t))^2] &= \mathbb{E}\left[\left(\int_\tau^t \int_C \Phi(s,y) M_H(ds,dy)\right)^2\right] \\
&= \mathbb{E}\left[\int_\tau^t \int_C \Phi(s,y)^2 H(s)(dy)ds\right] \\
&\leq \mathbb{E}\left[\int_\tau^T \int_C \Phi(s,y)^2 H(s)(dy)ds\right] \\
&< \infty
\end{aligned}$$

holds for all $\Phi \in \mathcal{U} \subset \mathcal{L}^2(M_H)$, elements in $I_{M_H}(\mathcal{U})$ are also square-integrable and thus $I_{M_H}(\mathcal{U})$ is a subspace of $\mathcal{M}^2$.

Next, define a functional $F = F_{\Phi_{\Gamma,\Psi,a}}$ by

$$\begin{aligned}
F : [\tau,T] \times D([\tau,T], M_F(C)) &\to \mathbb{R} \\
(t,\omega) &\mapsto \Gamma(\omega)\Big( \langle \omega(t), \Psi\rangle - \langle \omega(a), \Psi\rangle \\
&\qquad - \int_a^t \langle \omega(s), \frac{1}{2}\bar{\Delta}\Psi\rangle ds\Big) 1_{t>a}.
\end{aligned}$$

If we plug in $H_t$ for $\omega$, we obtain

$$\begin{aligned}
F(t,H_t) &= \Gamma(H_t)\left(\langle H_t(t), \Psi\rangle - \langle H_t(a), \Psi\rangle - \int_a^t \langle H_t(s), \frac{1}{2}\bar{\Delta}\Psi\rangle ds\right) 1_{t>a} \\
&= \Gamma(H_t)\left(\langle H(t), \Psi\rangle - \langle H(a), \Psi\rangle - \int_a^t \langle H(s), \frac{1}{2}\bar{\Delta}\Psi\rangle ds\right) 1_{t>a},
\end{aligned}$$

which yields, as $H(t)(\omega) = \omega(t)$ and $H_t(\omega) = \omega_t$,

$$F(t,H_t)(\omega) = \Gamma(\omega_a)\left(\langle \omega(t), \Psi\rangle - \langle \omega(a), \Psi\rangle - \int_a^t \langle \omega(s), \frac{1}{2}\bar{\Delta}\Psi\rangle ds\right) 1_{t>a}.$$

Thus, as $\Gamma$ is $\mathcal{H}_a$-measurable, we have that $F(t,H_t) = I_{M_H}(\Phi_{\Gamma,\Psi,a})(t)$ holds.

Additionally, for any $\omega \in D([\tau,T], M_F(C))$ we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big( \langle(\omega + \varepsilon\delta_y 1_{[t,T]})(t), \Psi\rangle - \langle(\omega + \varepsilon\delta_y 1_{[t,T]})(a), \Psi\rangle \\
&\qquad - \int_a^t \langle(\omega + \varepsilon\delta_y 1_{[t,T]})(s), \frac{1}{2}\bar{\Delta}\Psi\rangle ds \\
&\qquad - \langle \omega(t), \Psi\rangle + \langle \omega(a), \Psi\rangle\rangle + \int_\tau^t \langle \omega(s), \frac{1}{2}\bar{\Delta}\Psi\rangle ds\Big) \\
= &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\varepsilon\Psi(y) - \varepsilon\int_\tau^t \langle \delta_y 1_{[t,T]}, \frac{1}{2}\bar{\Delta}\Psi\rangle ds\right) \\
= &\Psi(y).
\end{aligned}$$

As $\Gamma$ is $\mathcal{H}_a$-measurable, it holds for any $t > a$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big( \Gamma(\omega + \varepsilon \delta_y 1_{[t,T]}) - \Gamma(\omega) \Big) = 0$$

and thus we get for $(t, \omega) \in \Lambda_T$

$$\mathcal{D}_y F(t, \omega) = \Gamma(\omega) \Psi(y) 1_{(a,T]}(t) = \Phi_{\Gamma,\Psi,a}. \tag{3.22}$$

This allows us to derive the martingale representation formula for elements in $I_{M_H}(\mathcal{U})$ as follows. If $Y \in I_{M_H}(\mathcal{U})$, we can set $Y(t) = F(t, H_t)$. As $(t, H_t(\omega)) \in \Lambda_T$, we can define an operator $\nabla_M$ on $I_{M_H}(\mathcal{U})$ by

$$\begin{array}{rccc} \nabla_M : & I_{M_H}(\mathcal{U}) & \to & \mathcal{L}^2(M_H) \\ & Y & \mapsto & \nabla_M Y, \end{array} \tag{3.23}$$

where

$$\nabla_M Y : (\omega, t, y) \mapsto \nabla_M Y(\omega, t, y) := \mathcal{D}_y F(t, H_t(\omega)).$$

By considering (3.22) as well as the definition of $Y \in I_{M_H}(\mathcal{U})$ as the integral of a function in $\mathcal{U}$, this immediately leads to martingale representation

$$Y(t) = \int_0^t \int_C \nabla_M Y(s, y) M_H(ds, dy). \tag{3.24}$$

The uniqueness of the representation follows from [Overbeck, 1995] or [Evans and Perkins, 1995], from which we get the existence of a unique $\rho \in \mathcal{L}^2(M_H)$ such that

$$Y(t) = \int_\tau^t \int_C \rho(s, y) M_H(ds, dy) \quad \text{for all } t \in [\tau, T] \tag{3.25}$$

holds $\mathbb{P}_{\tau,m}$-almost surely. In addition to the uniqueness of the representation (3.24) this yields that the mapping $I_{M_H}$ is a bijective isometry, which allows us to prove the following.

**Proposition 3.21.** *The space $\{\nabla_M Y : Y \in I_{M_H}(\mathcal{U})\}$ is dense in $\mathcal{L}^2(M_H)$ and the space $I_{M_H}(\mathcal{U})$ is dense in $\mathcal{M}^2$.*

*Proof.* As we have

$$\mathcal{U} = \{\nabla_M Y : Y \in I_{M_H}(\mathcal{U})\} \subset \mathcal{L}^2(M_H),$$

the fact that $\mathcal{U}$ is dense in $\mathcal{L}^2(M_H)$ (Proposition 3.19) yields that $\{\nabla_M Y : Y \in I_{M_H}(\mathcal{U})\}$ is dense in $\mathcal{L}^2(M_H)$. From Proposition 3.20 we know that the mapping $I_{M_H}$ is an isometry and as it is also bijective, we get the density of $I_{M_H}(\mathcal{U})$ in $\mathcal{M}^2$ from the density of $\mathcal{U}$ in $\mathcal{L}^2(M_H)$. $\square$

Taking the above density properties into account, we can prove the following result.

**Proposition 3.22.** *If $Y \in I_{M_H}(\mathcal{U})$, then $\nabla_M Y$ is the unique element in $\mathcal{L}^2(M_H)$ such that*

$$\mathbb{E}[Y(T) Z(T)] = \mathbb{E}\left[ \int_\tau^T \int_C \nabla_M Y(s, y) \nabla_M Z(s, y) H(s)(dy) ds \right] \tag{3.26}$$

*holds for all $Z \in I_{M_H}(\mathcal{U})$.*

*Proof.* Let $Y$, $Z \in I_{M_H}(\mathcal{U})$. Then, $Y$ and $Z$ have a representation of form (3.24) and applying (3.21) yields

$$\begin{aligned}\mathbb{E}[Y(T)Z(T)] &= \mathbb{E}\left[\int_\tau^T \int_C \nabla_M Y(s,y) M_H(ds,dy) \int_\tau^T \int_C \nabla_M Z(s,y) M_H(ds,dy)\right] \\ &= \mathbb{E}\left[\int_\tau^T \int_C \nabla_M Y(s,y) \nabla_M Z(s,y) H(s)(dy) ds\right].\end{aligned}$$

To prove the uniqueness of the element in $\mathcal{L}^2(M_H)$, assume there exists a $\Psi \in \mathcal{L}^2(M_H)$ such that

$$\mathbb{E}[Y(T)Z(T)] = \mathbb{E}\left[\int_\tau^T \int_C \Psi(s,y) \nabla_M Z(s,y) H(s)(dy) ds\right]$$

holds for all $Z \in I_{M_H}(\mathcal{U})$. If this is the case, subtraction yields

$$0 = \mathbb{E}\left[\int_\tau^T \int_C (\Psi(s,y) - \nabla_M Y(s,y)) \nabla_M Z(s,y) H(s)(dy) ds\right]$$

for all $Z \in I_{M_H}(\mathcal{U})$. As $\{\nabla_M Z : Z \in I_{M_H}(\mathcal{U})\}$ is dense in $\mathcal{L}^2(M_H)$, this yields that $\Psi$ is equal to $\nabla_M Y$ in $\mathcal{L}^2(M_H)$ and thus the uniqueness. $\square$

Like equation (3.8), equation (3.26) can be rewritten as

$$\mathbb{E}\left[Y(T) \int_\tau^T \int_C \Phi(s,y) M_H(ds,dy)\right] = \mathbb{E}\left[\int_\tau^T \int_C \nabla_M Y(s,y) \Phi(s,y) H(s)(dy) ds\right],$$

which holds for all $\Phi \in \mathcal{L}^2(M_H)$ and yields the interpretation of the result in Proposition 3.22 as an integration by parts formula.

The uniqueness of $\nabla_M$ in (3.26) allows us to extend the operator $\nabla_M$ from a subspace $I_{M_H}(\mathcal{U})$ of $\mathcal{M}^2$ to the whole space $\mathcal{M}^2$ by considering the following.

**Theorem 3.23.** *The operator defined in* (3.23) *can be extended to an operator*

$$\begin{array}{rccc}\nabla_M : & \mathcal{M}^2 & \to & \mathcal{L}^2(M_H) \\ & Y & \mapsto & \nabla_M Y.\end{array}$$

*This operator, which is a bijection and the unique continuous extension of the operator defined in* (3.23)*, is given by the following: For a given $Y \in \mathcal{M}^2$, $\nabla_M Y$ is the unique element in $\mathcal{L}^2(M_H)$ such that*

$$\mathbb{E}[Y(T)Z(T)] = \mathbb{E}\left[\int_\tau^T \int_C \nabla_M Y(s,y) \nabla_M Z(s,y) H(s)(dy) ds\right] \tag{3.27}$$

*holds for all $Z \in I_{M_H}(\mathcal{U})$.*

*Proof.* As in the proof of Theorem 3.9, the existence of a unique continuous extension

$$\nabla_M : \mathcal{M}^2 \to \mathcal{L}^2(M_H)$$

follows from the BLT theorem since

$$\nabla_M : I_{M_H}(\mathcal{U}) \to \mathcal{L}^2(M_H)$$

is a bounded linear operator, $\mathcal{L}^2(M_H)$ is a Hilbert space and $I_{M_H}(\mathcal{U})$ is dense in $\mathcal{M}^2$. From Proposition 3.22 we immediately get that the restriction of the operator defined by (3.27) to $I_{M_H}(\mathcal{U})$ coincides with the initial operator and thus have that the new operator is indeed the unique continuous extension.

As for every $Y \in \mathcal{M}^2$ there exists a unique $\rho$ such that (3.25) holds, we can combine (3.25) with (3.21) and (3.24) to get for all $Z \in I_{M_H}(\mathcal{U})$

$$\begin{aligned}\mathbb{E}[Y(T)Z(T)] &= \mathbb{E}\left[\int_\tau^T \int_C \rho(s,y) M_H(ds,dy) Z(T)\right] \\ &= \mathbb{E}\left[\int_\tau^T \int_C \rho(s,y) \nabla_M Z(s,y) H(s)(dy)ds\right].\end{aligned}$$

Thus, $\rho$ and $\nabla_M Y$ have to coincide in $\mathcal{L}^2(M_H)$ because of the uniqueness of the integrand in (3.25).

Using this, we can prove that the operator is bijective. To do so, let $Y$, $Y' \in \mathcal{M}^2$ with $\nabla_M Y = \nabla_M Y'$ and $\nabla_M Y$, $\nabla_M Y' \in \mathcal{L}^2(M_H)$. Then, as $I_{M_H}(\mathcal{U})$ is dense in $\mathcal{M}^2$, we get from

$$\begin{aligned}0 &= \mathbb{E}\left[\int_\tau^T \int_C (\nabla_M Y(s,y) - \nabla_M Y'(s,y)) \nabla_M Z(s,y) H(s)(dy)ds\right] \\ &= \mathbb{E}\left[\left(\int_\tau^T \int_C \nabla_M Y(s,y) M_H(ds,dy) - \int_\tau^T \int_C \nabla_M Y'(s,y) M_H(ds,dy)\right) Z(T)\right] \\ &= \mathbb{E}[(Y(T) - Y'(T))Z(T)]\end{aligned}$$

for all $Z \in I_{M_H}(\mathcal{H})$ that $Y = Y'$ in $\mathcal{M}^2$ holds. Consequently, the operator $\nabla_M$ is injective and as for every $\Phi \in \mathcal{L}^2(M_H)$ the process given by

$$Y = \int_\tau^\cdot \int_C \Phi(s,y) M_H(ds,dy)$$

is in $\mathcal{M}^2$ and satisfies $\nabla_M Y = \Phi$, the operator is also surjective. Therefore, the operator is bijective. □

As in Section 3.1, by combining the above results, we can now formulate the following martingale representation formula, which extends the representation in (3.24) to all square-integrable historical martingales.

**Theorem 3.24.** *For any square-integrable $(\mathcal{H}_t)_t$-martingale $Y$ and every $t \in [\tau, T]$ it holds*

$$Y(t) = Y(0) + \int_\tau^t \int_C \nabla_M Y(s,y) M_H(ds,dy) \quad \mathbb{P}_{\tau,m} - a.s.. \tag{3.28}$$

*Proof.* First, assume that $Y \in \mathcal{M}^2$. From the proof of Theorem 3.23 we know that the unique integrand $\rho$ in (3.25) is given by $\nabla_M Y$. Therefore, for $Y \in \mathcal{M}^2$ and $t \in [\tau, T]$, it holds

$$Y(t) = \int_\tau^t \int_C \nabla_M Y(s,y) M_H(ds,dy)$$

$\mathbb{P}_{\tau,m}$-almost surely.

To obtain the result for all square-integrable $(\mathcal{H}_t)_t$-martingales $Y$, we can once again get a process $\tilde{Y} \in \mathcal{M}^2$ by setting $\tilde{Y} = Y - Y(0)$. Then, applying the above to $\tilde{Y}$ and adding $Y(0)$ to both sides yields (3.28). □

Like the operator considered in Section 3.1, the operator $\nabla_M$ has to the following connection to the operator $I_{M_H}$ defined in Proposition 3.20.

**Proposition 3.25.** *The operator $\nabla_M$ defined on $\mathcal{M}^2$ is an isometry and the adjoint operator of $I_{M_H}$, the stochastic integral with respect to the martingale measure $M_H$.*

*Proof.* Let $Y \in \mathcal{M}^2$. Following the arguments in the proof of Proposition 3.20 yields

$$\begin{aligned}
\|\nabla_M Y\|^2_{\mathcal{L}^2(M_H)} &= \mathbb{E}\left[\int_\tau^T \int_C (\nabla_M Y(s,y))^2 H(s)(dy)ds\right] \\
&= \mathbb{E}\left[\left(\int_\tau^T \int_C \nabla_M Y(s,y) M_H(ds,dy)\right)^2\right] \\
&= \left\|\int_\tau^\cdot \int_C \nabla_M Y(s,y) M_H(ds,dy)\right\|^2_{\mathcal{M}^2} \\
&= \|Y\|^2_{\mathcal{M}^2},
\end{aligned}$$

which proves the isometry property.

To show that $\nabla_M$ is the adjoint operator of $I_{M_H}$, consider a $\Phi \in \mathcal{L}^2(M_H)$. We then get

$$\begin{aligned}
\langle I_{M_H}(\Phi), Y\rangle_{\mathcal{M}^2} &= \mathbb{E}\left[\int_\tau^T \int_C \Phi(s,y) M_H(ds,dy) Y(T)\right] \\
&= \mathbb{E}\left[\int_\tau^T \int_C \Phi(s,y) M_H(ds,dy) \int_\tau^T \int_C \nabla_M Y(s,y) M_H(ds,dy)\right] \\
&= \mathbb{E}\left[\int_\tau^T \int_C \Phi(s,y) \nabla_M Y(s,y) H(s)(dy)ds\right] \\
&= \langle \Phi, \nabla_M Y\rangle_{\mathcal{L}^2(M_H)},
\end{aligned}$$

which completes the proof. □

## 3.3 Comparison to the Results by Evans and Perkins

To conclude this chapter on martingale representation formulae, we compare our findings to the results by Evans and Perkins. We start with a brief summary of the results in [Evans and Perkins, 1994] and [Evans and Perkins, 1995], in which we slightly adjust the results to match our setting and notation and skip over most of the details for the sake of brevity.

In [Evans and Perkins, 1994], the setting used in Section 3.1 is considered. The authors prove that for any square-integrable $(\mathcal{F}_t)_t$-martingale $Y$ there exists a unique $f$ such that $Y$ can be written as

$$Y(t) = \mathbb{E}[Y(0)] + \int_0^t \int_E f(s,x) M_X(ds,dx).$$

This result is used multiple times in Section 3.1, most prominently to obtain (3.7).

In [Evans and Perkins, 1995], the authors consider the setting in Section 3.2 and study the representation for a square-integrable functional $F$ applied to the path of a historical Brownian motion $H$. Besides the existence and uniqueness of a $\phi^F$ such that

$$F(H_T) = \mathbb{E}[F(H_T)] + \int_\tau^T \int_C \phi^F(s,y) M_H(ds,dy)$$

holds, the authors also derive the explicit form of the integrand $\phi^F$ if $F$ satisfies some regularity conditions. In this case, the representation formula becomes

$$F(H_T) = \mathbb{E}[F(H_T)] + \int_\tau^T \int_C \mathcal{J}F(H_T)(s,y) M_H(ds,dy). \tag{3.29}$$

The integrand $\mathcal{J}F(H_T)(s,y)$ is given by a specific predictable projection of the process

$$JF(H_T)(s,y) = \int_{C([\tau,T],M_F(C))} F(H_T+h) - F(H_T)\mathbb{Q}^{s;y_{s-}}(dh),$$

with $\mathbb{Q}^{s;y_{s-}}$ playing the role of the canonical measure in the Poisson cluster representation of the path of $H$ from $s$ to $T$. For details on these concepts, we refer to the original work [Evans and Perkins, 1995].

In order to compare the representation in (3.29) to the result in Theorem 3.24, note that if we set $t = T$, our representation is of form

$$Y(T) = \mathbb{E}[Y(T)] + \int_\tau^T \int_C \nabla_M Y(s,y) M_H(ds,dy).$$

As $Y(T)$ is $\mathcal{H}_T$-measurable, there exists a functional $G$ such that $Y(T) = G(H_T)$, which allows us to reformulate our result as follows:

$$G(H_T) = \mathbb{E}[G(H_T)] + \int_\tau^T \int_C \nabla_M G(H_T)(s,y) M_H(ds,dy).$$

Given that $G$ satisfies the regularity conditions on $F$ in [Evans and Perkins, 1995], the uniqueness of the integrand in the martingale representation formula yields that

$$\nabla_M G(H_T) = \mathcal{J}F(H_T) \quad \text{with respect to } \|\cdot\|_{\mathcal{L}^2(M_H)}$$

holds.

The derivation of the integrand in [Evans and Perkins, 1995] is based on two step approach. In the first step, $JF(H_T)$ is computed. As $JF(H_T)$ is not predictable, computing the predictable projection $\mathcal{J}F(H_T)$ in the second step is necessary to obtain the integrand in the martingale representation formula. This is in line with the classical derivation of the integrand in the Clark-Ocone-Haussmann formula, which is based on Malliavin calculus.

In contrast to that, the approach presented in this monograph is based on a single step as the vertical derivative $\mathcal{D}$ as well as the extended operator $\nabla_M$ are already predictable. Additionally, while the proofs in [Evans and Perkins, 1995] rely, in large parts, on the cluster representation of the historical Brownian motion, our proofs are mostly based on properties derived from the martingale problem defining the historical Brownian motion as well as the martingale measure in combination with standard arguments from (funtional) stochastic calculus.

A more detailed comparison of the approach based on Malliavin calculus and the approach based on functional calculus in the case of $\mathbb{R}^d$-valued processes is presented in Chapter 7.3 in [Cont, 2016]. All in all, the new approach yields promising results in general and when working with superprocesses as well as the historical Brownian motion in particular. Thus, it constitute a valid alternative to the traditional approach used for example in [Evans and Perkins, 1995].

# Chapter 4

# Outlook

The previous chapters summarize our research up to now. In this final chapter, we briefly outline ongoing research, present preliminary results and discuss open problems that we came across during our research.

While, in Chapter 2, the Itō-formula as well as the functional Itō-formula for a wide class of functions, respectively functionals, of a $B(A,c)$-superprocess, respectively its path, are introduced, the derivation of an equivalent result for the historical Brownian motion is still subject of ongoing research.

By using the same arguments as in the proof of Theorem 2.4, we can derive the Itō-formula for finitely based functions of the historical Brownian motion, which is given by

$$
\begin{aligned}
F(t,H(t)) = F(\tau,H(\tau)) + & \int_\tau^t D^*F(s,H(s))ds \\
+ & \int_\tau^t \int_C A_{\tau,m} D_y F(s,H(s))H(s)(dy)ds \\
+ & \frac{1}{2}\int_\tau^t \int_C c D_{yy} F(s,H(s))H(s)(dy)ds \\
+ & \int_\tau^t \int_C D_y F(s,H(s)) M_H(ds,dy),
\end{aligned}
\tag{4.1}
$$

where the derivative $D^*$ is defined analogously to the horizontal derivative in Section 2.1 and the derivative $D_y$ is defined as in (3.16). The form of (4.1) and the results in Section 2.1 suggest that the Itō-formula for a wider class of functions of the historical Brownian motion is of similar form. However, the extension to a wider class of functions proves to be more challenging than it is the case for functions of the $B(A,c)$-superprocess. Since the underlying space $C$ is an infinite-dimensional space and thus not locally compact, many of the arguments used to obtain the result in [Jacka and Tribe, 2003] fail in this setting.

Once one derives the Itō-formula for a wider class of functions of the historical Brownian motion, it should be rather straightforward to extend it to a functional Itō-formula for functionals of the path of historical Brownian motions by following the steps in Section 2.2. The necessary step to extend the class of valid integrands for the integral with respect to the martingale measure $M_H$ should also follow directly from the steps in Section 1.3.3 as the arguments almost exclusively rely on the orthogonality of the martingale measure.

The orthogonality of the martingale measure is also crucial for the derivation of both martingale representation results in Chapter 3. The similarity of the steps in the proofs in Section 3.1 and Section 3.2 suggests that the results might be extended to a more general, abstract setting. To be more precise, we might be able to obtain the martingale representation formula for any underlying measure-valued process that yields an orthogonal martingale measure $M$ as long as some conditions, like the existence of a space like $D_0$ in Section 3.1 or $D_{fd}$ in Section 3.2 that yields a $\mathcal{U}$ that is dense in $\mathcal{L}^2(M)$, are satisfied.

Another open problem regarding the martingale representation formula concerns the weak derivative. Assume that, in the setting of Section 3.2, the martingale $Y \in \mathcal{M}^2$ is not in $I_{M_H}(\mathcal{U})$ but is such that we can find a functional $F$ with $Y(t) = F(t, H_t)$. In this case, if $\mathcal{D}_y F(t, H_t)$ exists, we are fairly certain that $\mathcal{D}_y F(t, H_t)$ and $\nabla_M Y(s, y)$ are equal with respect to $\| \cdot \|_{\mathcal{L}^2(M_H)}$ but the actual result has not yet been proved. The same applies in the setting of Section 3.1.

Finally, in addition to the extensions of the result on the martingale representation formula, applications of both, the Itō-formulae as well as the martingale representation formulae are of interest for potential further research. Part of this could be a study of the Fleming-Viot process in view of our results. However, while there exists a martingale measure associated with the Fleming-Viot process, the martingale measure is not orthogonal. In addition, the martingale representation formula is not unique (see [Overbeck, 1995]). In light of these properties, it is still an open question whether one can gain any insight from applying the techniques presented in this monograph to Fleming-Viot processes.

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# Index of Notations

## Declaration of Authorship

I declare that I have completed this dissertation single-handedly without the unauthorized help of a second party and only with the assistance acknowledged therein. I have appropriately acknowledged and cited all text passages that are derived verbatim from or are based on the content of published work of others, and all information relating to verbal communications. I consent to the use of an anti-plagiarism software to check my thesis. I have abided by the principles of good scientific conduct laid down in the charter of the Justus Liebig University Giessen „Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis" in carrying out the investigations described in the dissertation.

## Selbstständigkeitserklärung

Ich erkläre: Ich habe die vorgelegte Dissertation selbstständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt, die ich in der Dissertation angegeben habe. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben, die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht. Ich stimme einer evtl. Überprüfung meiner Dissertation durch eine Antiplagiat-Software zu. Bei den von mir durchgeführten und in der Dissertation erwähnten Untersuchungen habe ich die Grundsätze guter wissenschaftlicher Praxis, wie sie in der „Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis" niedergelegt sind, eingehalten.

Giessen, March 2021