Dynamics of Feedback Systems with Time Lag

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1. Here are four examples of autonomous functional differential equations with time lags which have been used for modelling. First, the logistic equation with delay in the growth-limiting factor

$$(dn/dt)(t) \equiv \dot{n}(t) = rn(t)[1 - n(t-\tau)/K] \tag{1}$$

proposed by the biologist HUTCHINSON [17] in 1948 in order to explain oscillations which had been observed in population growth experiments. If we set $x(t) := \log(n(\tau t)/K)$ and $\alpha = r\tau$ then positive solutions transform into solutions of

$$\dot{x}(t) = \alpha f(x(t-1)) \tag{αf}$$

with $f(\xi) = 1 - e^{\xi}$, $\xi \in \mathbb{R}$. f is monotonous and has the property

$$\xi f(\xi) < 0 \tag{NF}$$

for all $\xi \neq 0$. – In equations for the density of red blood cells as

$$\dot{x}(t) = [cx(t-\tau)]^8 e^{-x(t-\tau)} - sx(t) \tag{2}$$

the delay stands for the time between start of cell production and release of the platelets into the blood stream. Models of this type were constructed by MACKEY and GLASS, LASOTA and WAZEWSKA-CZYZEWSKA [25,23,24]. Nonlinear functions which are not monotonous as in (2) will make the dynamics more complicated. See also [7,11,13,22,27,35]. BANKS and MAHAFFY [2,26] developed a model for protein synthesis,

$$\begin{aligned}
\dot{m}(t) &= a(1 + kr(t-\tau_1)e(t-\tau_1))^{-1} - \beta_1 m(t) \\
\dot{e}(t) &= \alpha_2 m(t-\tau_2) - \beta_2 e(t) \\
\dot{r}(t) &= \alpha_3 e(t-\tau_3) - \beta_3 r(t)
\end{aligned} \tag{3}$$

The lags take into account times needed for transcription from DNA to messenger-RNA, and for transport to the ribosomes where substances react. See also [15]. – A model from a different field is

$$\dot{x}(t) = \delta - \sin(x(t-\tau) + \omega), \quad 0 \le \delta < 1 \tag{4}$$

The equation describes the phase difference x between a frequency generator in the high frequency range and a controlling oscillator in a phase-locked loop. The delay is caused by a low pass filter. Such models were used since the fifties [18]. See also [10].
If time is rescaled and if we perform a shift of the co-

ordinate system, (4) becomes an equation of type (αf) with periodic function f so that (NF) is only satisfied for ξ close enough to zero.

Already the most simple-looking scalar equation (αf) may have a rich structure in its state space, depending on f. This is much in contrast to ordinary differential equations (ODEs), where the more interesting behaviour is found only in systems of 3 or more equations. In the following, I want to describe typical properties of (αf) which have been proved. For more general equations like (2) and (3), and also for equations with distributed delay, much less could be established rigorously up to now.

2. Let $f:\mathbb{R} \to \mathbb{R}$ be smooth with $f(0) = 0$, with property (NF) and $f'(0) = -1$. Then $t \mapsto 0$ is the only constant solution of (αf), $\alpha > 0$. Equation (αf) describes delayed negative feedback with respect to zero: If $x(t-1) > 0$ then this deviation from equilibrium is followed by $\dot{x}(t) < 0$, a move in the opposite direction after a reaction lag, because of the sign condition (NF). - Equation (αf) is easily solved in forward time when, as an initial value, a continuous function ϕ on $[-1,0]$ is prescribed: For $t \in [0,1]$,

$$x(t) = x(0) + \int_0^t \dot{x}(s)\,ds = \phi(0) + \int_{-1}^{t-1} \alpha f(\phi(s))\,ds,$$

$x|[0,1]$ defines x on $[1,2]$ in the same way, and so on. The result is a unique continuous function, defined for $t \geq -1$, differentiable for $t > 0$ and satisfying (αf) for $t > 0$. - In order to get solution curves in the space C of initial values, i.e. of continuous functions on $[-1,0]$, one sets $x_t(a) := x(t+a)$ for $a \in [-1,0]$, $t \geq 0$. The map $0 \leq t \mapsto x_t \in C$ becomes continuous, and the formula $T(t,\phi) = x_t$ with the solution x of (αf) with $x_0 = \phi$ defines a continuous semiflow, or semidynamical system, $T:[0,\infty) \times C \to C$ [14]. T is not everywhere differentiable and does not continue to a local flow. E.g. if ϕ has no derivative at zero, then T has no partial derivative with respect to t at points (t,ϕ), $t \in (0,1)$. Also, there is no solution to this initial value problem in backward time - no function x on $[t_0-1,\infty)$, $t_0 < 0$, which is differentiable and satisfies (αf) for $t > t_0$, $x_0 = \phi$. - Even if the backward initial value problem can be solved there is no uniqueness in general. Or, two solutions with different initial values may become identical after finite time: Suppose f is decreasing on $(0,\xi)$ and increasing on (ξ,∞). Then, one can choose different initial functions ϕ, ψ, $\phi \leq \xi \leq \psi$, with $\phi(0) = \xi = \psi(0)$ and $f \circ \phi = f \circ \psi$. It follows that the corresponding solutions coincide on $[0,1]$, and consequently for all $t \geq 0$.

The next important difference between (αf) and scalar ODEs is the presence of oscillating solutions. Let K denote the cone of increasing functions $\phi \in C$ with $\phi(-1) = 0$. For $\phi(0) > 0$ condition (NF) forces the solution x with $x_0 = \phi$ to decrease as long as $x(t-1) > 0$. In case $\alpha > 1$ it is not hard to see that x has a first zero $z_1 = z_1(\phi) > 0$. Time lag and condition (NF) make x decrease until time $z_1 + 1$ when the solution realizes that it is no longer positive. Now x starts to increase, reaches a second zero z_2, increases until time $z_2 + 1, \ldots$ [28,45]. We get solutions which are "slowly oscillating" in the sense that the distance between any pair of zeros in $[0,\infty)$ is greater than the delay $\tau = 1$. - By the way, a sharp bound for all solutions of (αf) to oscillate is $\alpha = 1/e$, as one can prove.

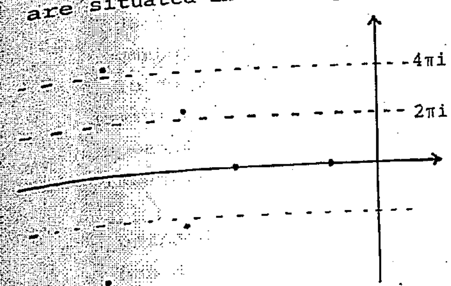
Oscillating solutions do not always damp out. For $0 < \alpha < \pi/2$ the zero solution is asymptotically stable but for $\alpha > \pi/2$ it becomes unstable, and no slowly oscillating solution tends to 0 as $t \to \infty$ [28,45]. The critical value $\pi/2$, and $1/e$ above too, is determined by the linearized equation

$$\dot{y}(t) = -\alpha y(t-1)$$

with f replaced by its derivative at $\xi = 0$. As for ODEs, the Ansatz $y(t) = e^{\lambda t}$ leads to a "characteristic equation"

$$\lambda + \alpha e^{-\lambda} = 0$$

the solutions of which decide about stability [14]. There are countably many of these "characteristic values". For $0 < \alpha < 1/e$, they are situated in the complex plane as indicated in the figure below.



If α passes the value $1/e$ the real characteristic values join and bifurcate into a complex conjugate pair; the other pairs move to the right [45]. If α increases beyond $\pi/2$ the pair with smallest imaginary parts crosses the imaginary axis at $\pm i\pi/2$. At $\alpha = 5\pi/2$, the next pair crosses at $\pm 5i\pi/2, \ldots$ The functions $t \mapsto ce^{t\operatorname{Re}\lambda}\sin(t\operatorname{Im}\lambda)$, $c \neq 0$, $|\operatorname{Im}\lambda| < \pi$, are now slowly oscillating solutions of the linear equation with increasing amplitude. We see that there are also many other, more rapidly oscillating solutions, given by the characteristic values with $2\pi < |\operatorname{Im}\lambda|$. For the nonlinear equations, one may now use a Hopf bifurcation theorem in order to obtain small amplitude rapidly oscillating solutions for α close to $5\pi/2$, $9\pi/2, \ldots$ and small amplitude slowly oscillating periodic solutions for α close to $\pi/2$ with periods close to 4 [4,8,9,31].

At this point I have to remark that slowly oscillating solutions are the only ones which seem to be relevant for applications. A very natural conjecture [21] says that the set of initial conditions which produce (eventually) slowly oscillating solutions is open and dense in the state space C. This is easily seen in the linear case; it could be proved for some nonlinear functions αf [41], but so far no proof is available for all equations (αf). The conjecture implies that any rapidly oscillating periodic solution is unstable, stability can only be expected from slowly oscillating periodic solutions. Let me write SO for slowly oscillating from now on.

The basic global bifurcation theorem for SO periodic solutions is due to NUSSBAUM [29]. Note first that every SO periodic solution contains elements $\phi \in K$ on its trajectory in C. So one may consider the set S of pairs (α,ϕ), $\alpha > 0$ and $\phi \in K$, such that ϕ defines a SO periodic solution of (αf) with period $z_2 + 1$. Let f be bounded from below or from above. Then NUSSBAUM's result is that there exists a maximal connected subset $S' \subset S$ with $(\pi/2,0)$ in its closure and with points (α,ϕ) for all $\alpha > \pi/2$; S' does not return to the trivial axis $(\alpha,0)$, $\alpha > 0$, for $\alpha \neq \pi/2$.

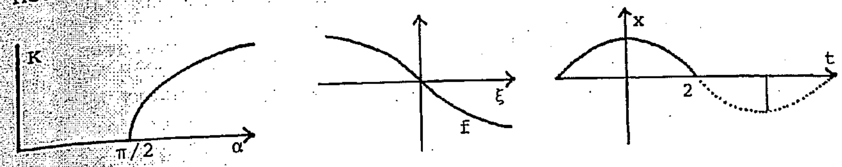
In the proof, NUSSBAUM employed tools from asymptotic fixed point theory, notably his own result on the index of an ejective fixed point [28]. This is not necessary, and moreover the concept of ejectivity is not really appropriate. It goes back to BROWDER [3] who developed it as an abstraction from what was known about undamped behaviour of SO solutions for $\alpha > \pi/2$ since a beautiful paper of WRIGHT (1955,[45]). However, the unstable behaviour of SO solutions is much more regular than ejectivity predicts. One of the consequences is that NUSSBAUM's result can be proved without any asymptotic fixed point theory [39,40].

So there are SO periodic solutions for every $\alpha > \pi/2$. Nothing can be said about the finer structure of the set S in general. S and S' do not necessarily decompose into smooth curves. Many stable and unstable SO periodic solutions may coexist for fixed values of α. SO periodic solutions for $\alpha \leq \pi/2$ are not excluded. Continua in S disjoint from S' are possible, as well as initial values in K which produce periodic solutions of (αf) with periods $z_4, z_6, \ldots$ so that $(\alpha,\phi) \notin S$. Numerical results of JÜRGENS, PEITGEN, SAUPE [19], HADELER [12], SAUPE [37] provide evidence that such possibilities are realized by functions f which are related to models like (1) and (4).

What is proved ? For a set of decreasing <u>odd</u> functions, $f(\xi) = -f(-\xi)$ for all ξ, NUSSBAUM [32] showed uniqueness for SO periodic solutions with x_0 in K, $\alpha > \pi/2$. Under his conditions there are no SO periodic solutions for $\alpha \leq \pi/2$; the SO periodic orbits for $\alpha > \pi/2$ are all asymptotically stable, all periods are 4, and the symmetry property

$$x(t) = -x(t-(z_1+1)) \quad \text{for all } t \in \mathbb{R} \tag{σ}$$

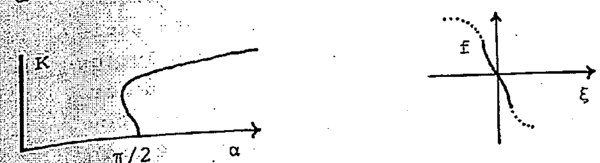
holds with $z_1 = 1$. Let Sp denote the set of pairs (α,ϕ) so that ϕ is



in K and defines such a "special periodic solution" with period 4 and symmetry (σ). Existence of special periodic solutions for odd f had been proved earlier by KAPLAN and YORKE [20]. The uniqueness proof is by refinements of a phase-plane method for monotonous f also due to KAPLAN and YORKE [21,22].

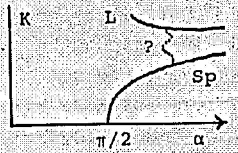
It might seem surprising that uniqueness and stability could not yet be proved for the monotonous function $f: \xi \to 1 - e^{\xi}$ from HUTCHINSON's equation (1).

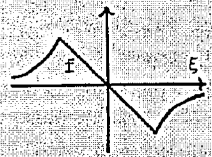
First examples of nonuniqueness came from results on the direction of bifurcation at $\alpha = \pi/2$. If, say, $f' < -1$ for all $\xi \neq 0$ close to zero then S' leaves the point $(\pi/2,0)$ in direction of decreasing α [38]. Work of CHOW and MALLET-PARET [8] leads to similar diagrams.

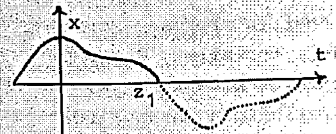


Further nonlinear monotonous functions with multiple SO periodic solutions were constructed by ANGELSTORF [1].

Numerical experiments with equations like (2) and (4) suggested that nonmonotonous functions f define very complicated sets S. For certain <u>odd</u> functions with hump-shaped graph NUSSBAUM constructed a continuum $L \subset S$ disjoint from Sp [32,33]. In cases considered Sp forms a connected set which bifurcates from $(\pi/2,0)$, so $Sp \subset S'$. Solutions which belong to L share the symmetry (σ) with the special periodic solutions but have minimal periods larger than 4.

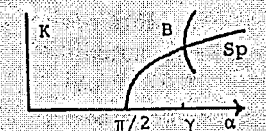






Note that Sp and L may or may not belong to the same connected component S' of S. This is an unsolved problem. SAUPE found a trick how to transform L into a third continuum, where periodic solutions have period less than 4 and symmetry (σ).

For another class of odd hump functions where Sp is a smooth curve emanating from $(\pi/2, 0)$, I could show existence of a secondary bifurcation [43]: There is a critical parameter $\gamma > \pi/2$ so that for $\pi/2 < \alpha < \gamma$, the (unique) special periodic solution $x = x^\alpha$ with x_0^α in K is asymptotically stable. For $\alpha > \gamma$ (and α not too large), x^α is unstable, and there is a nontrivial continuum $B \subset S$ with $B \cap Sp = \{(\gamma, x_0^\gamma)\}$ such that periodic solutions defined by points $(\alpha, \phi) \in B \setminus \{(\gamma, x_0^\gamma)\}$ do <u>not</u> have the symmetry (σ).



It is not clear whether for the functions f considered here an analogue of NUSSBAUM's continuum L exists. In any case, the bifurcating periodic solutions would not belong to it. - The bifurcation result applies to the model (4) with $\delta = \omega = 0$. Of course, $f = -\sin$ satisfies (NF) only for $0 < |\xi| < \pi$. But all relevant periodic solutions are bounded by π which allows to incorporate this application here.

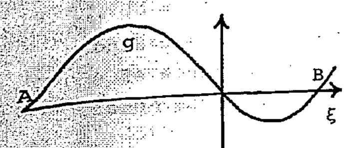
The complete list of hypotheses for the results described above creates the impression that bifurcation diagrams depend in a rather subtle way on the shape of f. This is confirmed by work of NUSSBAUM (and POTTER, CHAPIN) on the dependence of periods on α and f [30,32,33,34,5,6]. Therefore it is not always easy to make a correct guess about the dynamics on the basis of numerically obtained bifurcation diagrams, or from rigorous results for (αf) with f a step function. In this last case, one can calculate bifurcation diagrams completely (if there are not too many steps). Such a program was carried out by PETERS [36].

3. A remark about functional differential equations and chaotic interval maps: When MACKEY and GLASS proposed equations like (3) as models for certain physiological control processes, they also expected chaotic trajectories for suitable parameters. This could be proved for a few rather special equations. PETERS [36] did it for step functions as nonlinearities, AN DER HEIDEN and I showed it for equations like (2) and (4) with nonlinearities which are smooth but constant on long intervals, like step functions [16,42]. The proofs establish and use that chaotic interval maps represent the dynamics of these special functional differential equations precisely, in a "thin" region in state space. This should be compared to the simple fact that chaotic interval maps with their hump-shaped graphs are never directly related to decent ODEs - Poincaré maps and time-one-maps of ODEs always being one-to-one.

4. Coexistence of stable equilibrium and periodic motion for the phase-locking equation. - If condition (NF) for (αf) does not hold globally, like in (4), or if equations like (2) are considered, then it is not clear how to single out a predominant class of solutions as the slowly oscillating ones above. Other phenomena come into play. Let me finally describe a result which I obtained recently for model (4). Consider

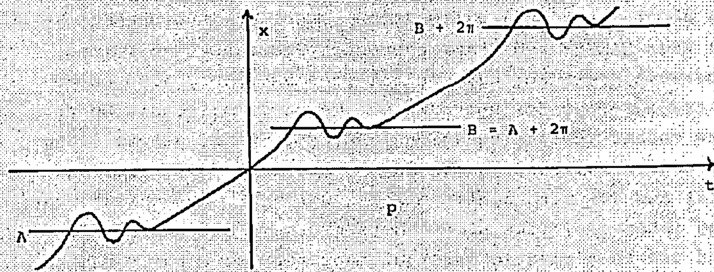
$$\dot{x}(t) = \alpha g(x(t-1))$$

with g a shifted sinus function, $g(0) = 0$, $g'(0) < 0$.



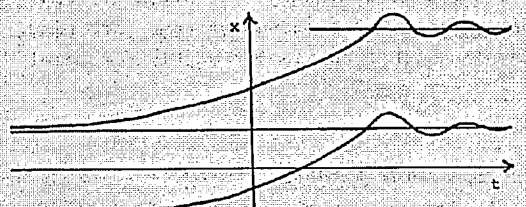
$B - A = 2\pi$

In a range of parameters $\alpha > 0$ where the zero solution is asymptotically stable there exists a critical parameter $\tilde{\alpha}$, such that for $\alpha > \tilde{\alpha}$ (and α not too large) periodic solutions of the second kind bifurcate [44]. - A periodic solution of the second kind with period $p > 0$ is a solution $x: R \to R$ where $x(t+p) = x(t) + 2\pi$ for all t in R; the graph looks like a staircase. If real numbers modulo 2π are identified, then x defines a periodic map with values on the circle $R/2\pi Z \cong S^1$; after one period p a full turn around the circle is completed. The words "of the second kind" are meant to distinguish this from "small" periodic oscillations on the circle which do not wind ...ound during a period. For example, SO periodic solutions with va-



lues in (A,B) would define periodic solutions "of the first kind" on the circle. – Recall also that for the model x is interpreted as a phase difference so that only values modulo 2π are of interest.

The periodic solutions of the second kind arise in a nonlocal bifurcation from a sequence of heteroclinic solutions which exist for the critical parameter $\alpha = \tilde{\alpha}$, i.e. from a solution $\tilde{x}$ with $\lim\limits_{t\to-\infty} \tilde{x}(t) = A$ and $\lim\limits_{t\to\infty} \tilde{x}(t) = B$ and its translates $\tilde{x} + 2\pi\mathbb{Z}$, and the periods tend to infinity for $\alpha \to \tilde{\alpha}$.



Most likely the bifurcating periodic solutions of the second kind are asymptotically stable, so that one would have coexistence of stable periodic motion with the stable equilibrium zero in the model. However, stability is not yet proved.

5. References. There are a large number of interesting contributions to the dynamics of autonomous differential delay equations. The list below only contains papers used for the preceding article.

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