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# Exotic buildings, lattices and non-residual finiteness

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# *Abstract*

### Exotic buildings, lattices and non-residual finiteness

We provide the first examples of lattices on irreducible buildings that are not residually finite. Assuming that the normal subgroup property holds for them, which is expected, they are virtually simple. Our examples fall into five quasi-isometry classes, corresponding to their actions on five distinct exotic $\tilde{C}_2$-buildings. Each class contains a smallest lattice, which has no proper finite index subgroups. Consequently, if the normal subgroup property holds, these are in fact simple. As exotic lattices, our examples are $\mathrm{CAT}(0)$-groups, enjoy Kazhdan's property (T), and are quasi-isometrically rigid. If the smallest lattices turn out to be simple, then, to the author's knowledge, they constitute the first infinite examples of quasi-isometrically rigid simple groups, as well as the first infinite examples of simple $\mathrm{CAT}(0)$-groups with Kazhdan's property (T).

The lattices were found through a computer search, and the methods developed proved to be fruitful; as a direct application, we classify type-preserving, vertex-regular $\tilde{A}_2$-lattices acting on buildings of thickness three.

# *Zusammenfassung*

### Exotische Gebäude, Gitter und nicht-residuelle Endlichkeit

Wir konstruieren die ersten Beispiele von Gittern auf irreduziblen Gebäuden, die nicht residuell endlich sind. Unter der erwarteten Annahme, dass jeder ihrer echten Quotienten endlich ist, sind sie virtuell einfach.

Unsere Beispiele zerfallen in fünf Quasi-Isometrieklassen, die Wirkungen auf fünf verschiedenen exotischen $\tilde{C}_2$-Gebäuden entsprechen. Jede Klasse enthält ein kleinstes Gitter, das keine echten Untergruppen von endlichem Index besitzt. Falls diese Gitter nur endliche echte Quotienten haben, sind sie tatsächlich einfach.

Als exotische Gitter sind unsere Beispiele $\mathrm{CAT}(0)$-Gruppen, besitzen Kazhdans Eigenschaft (T) und sind quasi-isometrisch starr. Sollten sich die kleinsten Gitter als einfach erweisen, sind sie nach Kenntnis des Autors die ersten unendlichen Beispiele quasi-isometrisch starrer einfacher Gruppen, sowie die ersten unendlichen Beispiele einfacher $\mathrm{CAT}(0)$-Gruppen mit Kazhdans Eigenschaft (T).

Die Gitter wurden mittels einer Computersuche gefunden und die dabei entwickelten Methoden erwiesen sich als fruchtbar; als direkte Anwendung klassifizieren wir typerhaltende, eckenreguläre $\tilde{A}_2$-Gitter, die auf Gebäuden der Dicke drei wirken.

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# Chapter 1

# Introduction

## 1.1 On the main theorem and related material

Euclidean buildings (also called affine buildings) were introduced as geometric objects encoding the structure of algebraic groups over non-Archimedean local fields: in a seminal series of articles, Bruhat and Tits constructed and studied Euclidean buildings associated with the $\mathbb{K}$-points of a semisimple algebraic group $\mathbf{G}$ defined over a discretely[1] valued field $\mathbb{K}$ [BT72; BT84a; BT84b; BT87b; BT87a]. We will refer to these as *Bruhat–Tits buildings*. In general, a Euclidean building is a metric space obtained by gluing together Euclidean spaces along a cell structure induced by a Coxeter group.

Any Euclidean building has a combinatorial rank, which we refer to as the Coxeter rank (see 2.4.3). The Coxeter rank of a Euclidean building is at least two, and a Euclidean building of Coxeter rank two is just a tree. By celebrated work of Tits and Weiss [Tit86; Wei09], irreducible Euclidean buildings of Coxeter rank at least four are classified: every such building is a Bruhat–Tits building. Therefore, the focus lies naturally on irreducible Euclidean buildings of Coxeter rank three. In this rank, besides the Bruhat–Tits buildings, there exist other irreducible buildings, which we call *exotic buildings*.

From now on we only consider locally compact buildings; or equivalently locally finite buildings. One way to study (exotic) Euclidean buildings is through groups acting geometrically, meaning that the action is properly discontinuous and cocompact. The classical examples of geometric actions on Euclidean buildings come from $S$-arithmetic groups which admit a rich algebraic structure. Several deep results for higher rank $S$-arithmetic groups have been established in the 20th century with proofs relying heavily on the ambient algebraic group, we refer to [Mar91]. It is natural to ask what generalizes to groups acting geometrically on exotic buildings. We call a group that acts faithfully and geometrically on an exotic building a *uniform exotic lattice*.

Let us focus on three properties which are established for higher rank $S$-arithmetic groups: they enjoy Kazhdan's property (T) [Kaz67], after factoring out the center[2] they are hereditarily just infinite by Margulis' normal subgroup theorem [Mar79], and they are residually finite. The first two properties are theorems, and residual finiteness holds essentially by definition. We now explain the current state of research for exotic lattices; we also make brief remarks on the properties themselves.

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[1] The constructions of Bruhat and Tits do not require the valuation to be discrete. However, in the non-discrete case, the associated building is no longer a polyhedral complex and such buildings will not be considered in this thesis.

[2] If the group acts faithfully on a building, the center is necessarily trivial.

We will define property (T) later, for now it suffices to say that it is a rigidity property formulated in terms of representation theory. It was introduced by Kazhdan in the 60s. Very recently Oppenheim proved that any uniform exotic lattice enjoys property (T) [Opp25]. We point out that the $\tilde{A}_2$-case was already known [CM94; Pan98; Żuk96]. Also for the remaining types, $\tilde{C}_2$ and $\tilde{G}_2$, the mentioned work by Żuk [Żuk96] and a theorem by Oppenheim [Opp15] establish property (T) for groups acting on sufficiently thick buildings, with Oppenheim's theorem being a strengthening of Żuk's theorem.

We now recall the definition of just-infiniteness: a group $\Gamma$ is *just infinite* if it is infinite and every non-trivial normal subgroup has finite index; it is *hereditarily just infinite* if every finite index subgroup is just infinite. Theorems establishing that groups are (hereditarily) just infinite are also called normal subgroup theorems. Recently, Bader–Furman–Lécureux proved that uniform exotic $\tilde{A}_2$-lattices are hereditarily just infinite [BFL23]. Note that the adjective "hereditarily" is automatic, since a finite index subgroup of a lattice is again a lattice. The other types remain open, but it is expected that the normal subgroup theorem generalizes. We also point out that in both Margulis' proof of the normal subgroup theorem and in the work of Bader–Furman–Lécureux, it is essential that the groups involved have property (T).

The normal subgroup theorems motivate the discussion on the last property: residual finiteness. We recall the definition: the *finite residual* $\Gamma^{(\infty)}$ of a group $\Gamma$ is the intersection of all finite index subgroups; $\Gamma$ is *residually finite* if its finite residual is trivial. It follows immediately that a hereditarily just infinite group $\Gamma$ satisfies the following dichotomy: either $\Gamma$ is residually finite, or its finite residual has finite index and is simple. For $S$-arithmetic groups the situation is clear: they are residually finite as finitely generated, linear groups. For exotic lattices residual finiteness (or non-residual finiteness) is widely open. In type $\tilde{A}_2$, we know that the same argumentation for residual finiteness as for $S$-arithmetic lattices cannot work: Bader–Caprace–Lécureux proved that uniform exotic lattices on $\tilde{A}_2$-buildings are non-linear [BCL19]. They also conjecture in their work that these groups are virtually simple. Unfortunately, there is no example of an exotic $\tilde{A}_2$-lattice for which virtual simplicity could be established. However, in type $\tilde{C}_2$ we provide the first examples of exotic lattices that are not residually finite and these are the main subject of this thesis.

**Theorem A** (Main theorem). *There are five finite triangle complexes* $Y_1^2, Y_1^3, \dots Y_4^3$ *whose universal covers* $X_i^q := \tilde{Y}_i^q$ *are exotic buildings of type* $\tilde{C}_2$ *and whose fundamental groups* $\Gamma_i^q := \pi_1(Y_i^q)$ *are not residually finite. These five buildings are pairwise not isomorphic and the five fundamental groups are pairwise not quasi-isometric. Since the* $\Gamma_i^q$ *are uniform building lattices they are* $\mathrm{CAT}(0)$ *and have Kazhdan's property (T).*

The building $X_i^q$ is of thickness $q+1$ and $\Gamma_i^q$ acts on it preserving types. The action is regular on vertices of each special type. Each $\Gamma_i^q$ admits an extension $\bar{\Gamma}_i^q = \Gamma_i^q \rtimes C_2$ by a non-type-preserving, involutory building automorphism. The groups $\bar{\Gamma}_i^q$ act regularly on the special vertices of their buildings.

For $q = 2$ the group $\Gamma_1^2$ is its own finite residual $(\Gamma_1^2)^{(\infty)}$ and $\bar{\Gamma}_1^2$ is the full automorphism group $\mathrm{Aut}(X_1^2)$ of the building. For $q = 3$ the extension $\bar{\Gamma}_i^3$ is not unique and the full automorphism group is bigger; the finite residual is smaller. More precisely, letting $\hat{\Gamma}_i^q := \mathrm{Aut}(X_i^q)$ denote the automorphism group of the building and

$\check{\Gamma}_i^q := (\Gamma_i^q)^{(\infty)}$ the finite residual we have:

$$[\hat{\Gamma}_i^q : \Gamma_i^q] = \begin{cases} 2 & q = 2, \\ 8 & q = 3, \end{cases} \qquad [\Gamma_i^q : \check{\Gamma}_i^q] = \begin{cases} 1 & q = 2, \\ 4 & q = 3, i = 1, 2, \\ 8 & q = 3, i = 3, 4. \end{cases}$$

In particular, the $\check{\Gamma}_i^q$ are $\tilde{C}_2$-lattices with no finite index subgroups. If the normal subgroup theorem generalizes to them, then they are abstractly simple.

The lattice $\bar{\Gamma}_1^2$ is presented by generators $g_1, \ldots, g_{15}$ subject to the relations $\mathcal{R}$ indicated below. The Cayley graph of this presentation is the subgraph of the 1-skeleton of $X_1^2$ generated by special vertices.

$$\begin{array}{llllll} \mathcal{R} := \{ & g_1^2, & g_2^2, & g_3 g_6, & g_4^2, & g_5 g_8, \\ & g_7 g_9, & g_{10}^2 & g_{11}^2 & g_{12} g_{14}, & g_{13}^2, \\ & g_{15}^2, & g_1 g_4 g_5 g_2, & g_1 g_4 g_6 g_6, & g_1 g_7 g_7 g_6, & g_1 g_7 g_8 g_2, \\ & g_3 g_{11} g_{12} g_{13}, & g_3 g_{12} g_4 g_{13}, & g_3 g_{12} g_{13} g_{15}, & g_2 g_{14} g_7 g_{14}, & g_7 g_{10} g_{14} g_{14}, \\ & g_7 g_{14} g_{14} g_{15}, & g_1 g_{10} g_5 g_{11}, & g_5 g_{11} g_{10} g_{15}, & g_5 g_{13} g_{11} g_{10} \}. & \end{array}$$

The lattices $\bar{\Gamma}_i^3$ admit similar presentations but with 40 generators instead of 15. We give presentations for these groups in Appendix A.

The complexes $Y_i^q$ were found by a computationally expensive computer search. Now they are found, the main theorem can be verified by hand with the following exceptions: 1. Our verification that $\check{\Gamma}_i^q$ is the finite residual consists in proving that a certain group presentation presents a small finite group, which we verified using a computer in all cases. For $\check{\Gamma}_1^2$, a proof by hand is possible, but it would be long and tedious. 2. We show that $\hat{\Gamma}_i^q$ is the full automorphism group of the building by reconstructing certain balls and showing that they are rigid. 3. We show that the buildings for $q = 3$ are pairwise non-isomorphic by showing that the quotients $\check{\Gamma}_i^q \backslash X_i^q$ are non-isomorphic.

Another important source of non-residually finite CAT(0)-groups is given by irreducible lattices in products of trees, an area pioneered by Wise, Burger, and Mozes [Wis96; BM00]. Nowadays, several explicit examples are known, including many simple groups. Note that for these examples, the decomposition as a product of trees is crucial in order to construct non-residually finite lattices.

In constructing non-residually finite $\tilde{C}_2$-lattices, we make use of non-residually finite lattices on products of trees. Indeed, in each case we start with a non-residually finite lattice $\Lambda$ on a product of trees $T_1 \times T_2$ and build the irreducible building $X$ around $T_1 \times T_2$ in such a way that the action of $\Lambda$ extends to a cocompact lattice $\Gamma$ on $X$. So the fact that $\Gamma$ is not residually finite is built into the construction. It remains an open problem how, given a random exotic lattice $\Gamma \leq X$, one can identify a non-trivial element of the finite residual.

Note that from a geometric perspective, all uniform lattices on products of trees represent a single quasi-isometry class, whereas lattices on exotic buildings are quasi-isometrically rigid:

**Proposition B.** *Let $\Gamma$ be one of $\check{\Gamma}_1^2, \check{\Gamma}_1^3, \ldots, \check{\Gamma}_4^3$. If $\Lambda$ is quasi-isometric to $\Gamma$, then there is a finite index subgroup $\Lambda' \leq \Lambda$ and a finite normal subgroup $N \leq \Lambda'$ and such that $\Lambda'/N \cong \Gamma$.*

We also observed the following exceptional graph-theoretic phenomenon for $\bar{\Gamma}_1^2$.

**Proposition C.** *The Cayley graph of $\bar{\Gamma}_1^2$ has no perfect finite $r$-local model for $r \geq 5$.*

## 1.2 Higher rank geometries

Euclidean buildings arise naturally in the study of CAT$(0)$ spaces of higher metric rank. In what follows, we justify the study of exotic lattices from this perspective.

A geodesic metric space is said to have *metric rank n* if every geodesic segment is contained in an isometrically embedded Euclidean space $\mathbb{E}^n$ of dimension $n$, and $n$ is maximal with this property. We say that a geodesic space is of *higher metric rank* if its metric rank is at least two.

For an irreducible Euclidean building, the Coxeter rank is one greater than its metric rank. Therefore, irreducible Euclidean buildings of Coxeter rank at least three, and in particular exotic buildings are of higher metric rank.

The higher rank rigidity conjecture, formulated by Ballmann and Buyalo [BB08, Conjecture B][3], asserts that if a proper CAT$(0)$ space $X$ is of higher metric rank and admits a geometric group action, then $X$ is either a symmetric space, a Euclidean building, or splits as a non-trivial metric product. Recently there has been significant progress on this conjecture by Stadler [Sta24a; Sta24b].

Let us assume the higher rank rigidity conjecture holds for now and let $X$ be a proper CAT$(0)$-space of higher metric rank admitting a geometric group action. In order to understand geometric actions on $X$, we recall some facts about its isometry group. If $X$ is irreducible and a symmetric space or a Bruhat–Tits building, then it arises from a simple adjoint algebraic group $\mathbf{G}$ over a local field $\mathbb{K}$. In that case a finite index subgroup of $\mathrm{Isom}(X)$ is a semi-direct product $\mathbf{G}(\mathbb{K}) \rtimes C$ with $C \leq \mathrm{Aut}(\mathbb{K})$; if $\mathbf{G}$ is $\mathbb{K}$-split then $C = \mathrm{Aut}(\mathbb{K})$. Note that $\mathrm{Aut}(\mathbb{K})$ is finite when $\mathbb{K}$ is of characteristic zero. For exotic buildings there is no analogous structure theorem for the isometry group. For reducible spaces $X$ there are reasonable conditions ensuring a canonical splitting of the space $X$ such that the isometry group is essentially the product of the isometry groups of the factors [CM09].

Let $\Gamma \leq \mathrm{Isom}(X)$ be a group acting geometrically. First, suppose that $X$ is irreducible and a symmetric space or a Bruhat–Tits building. A finite index subgroup $\Gamma'$ of $\Gamma$ lies in $\mathbf{G}(\mathbb{K}) \rtimes C$, so we can project $\Gamma'$ to $C \leq \mathrm{Aut}(\mathbb{K})$. If the image is finite, which necessarily holds when $\mathrm{char}(\mathbb{K}) = 0$, then the kernel $\Gamma''$ is of finite index in $\Gamma$ and lies in $\mathbf{G}(\mathbb{K})$. In that case, the seminal arithmeticity theorems of Margulis [Mar84] and Ventankaramana [Ven88] imply that $\Gamma''$ is an $S$-arithmetic group. Thus $\Gamma''$ and $\Gamma$ are relatively well understood; we refer to the discussion at the beginning of this introduction. If $X$ is a Bruhat–Tits buildings defined in positive characteristic, the projection might have infinite image and in that case $\Gamma$ is called a *uniform Galois lattice*. It is open whether Galois lattices exist in higher rank, but any uniform Galois lattice has the following properties[4]: it has property (T)[5], it is non-linear [BCL19] and if it is of type $\tilde{A}_2$ then it is hereditarily just infinite [BFL23]. The properties for exotic lattices have already been discussed; we move on to the reducible case. If $X$ splits as a product, it is natural to introduce the notion of an irreducible lattice. This means that the action is not virtually a direct product of actions on the factors of a non-trivial decomposition. Irreducibility also admits a topological characterization. For irreducible lattices in products, different methods can be applied. We mention

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[3] The conjecture was originally called the diameter conjecture, since higher rank can be characterized in terms of the diameter of the visual boundary. For simplicity, we state here a slightly weaker form.

[4] To deduce property (T) and non-linearity, the assumption of uniformness is not necessary; however, in this discussion we do not address non-uniform lattices.

[5] If $X$ is a Bruhat–Tits building of higher rank, then $\mathrm{Aut}(X)$ has property (T) since it contains $\mathbf{G}(\mathbb{K})$ as cocompact subgroup. It follows that any Galois lattice has property (T).

in particular the celebrated Bader–Shalom normal subgroup theorem [BS06] and a necessary condition for residual finiteness due to Caprace–Monod [CM12, Proposition 2.4]. Important examples of irreducible lattices are $S$-arithmetic groups and lattices on tree products.

In summary, when we assume the higher rank rigidity conjecture, up to finite index, the groups acting faithfully and geometrically on spaces of higher metric rank are covered by the following classes: $S$-arithmetic groups, Galois lattices, exotic lattices and lattices on products.

## 1.3 Organization

This thesis is organized as follows. In Chapter 2 we collect preliminary material and definitions that will be used throughout the text, together with background results that are generally known. We provide appropriate references for each topic. The material is standard, with a few minor exceptions: in the first section, devoted to actions on metric spaces, we introduce a property that we call *G-locally determining*, which ensures for open stabilizers in group actions. We also introduce the notion of a *lattice on a space*, which for proper spaces coincides with a faithful geometric action. In the third section we define *buildings*, more precisely *chamber complexes*, using a metric approach rather than a purely combinatorial one.

Chapter 3 is the core of the thesis. In its first section we collect results on lattices acting on products of trees that will be needed later on. We then analyze the buildings $X_i^q$ together with their non-residually finite lattices in order to prove the main theorem A and further results. The complexes $Y_i^q$ were found via a computer search, which is described in the final section of the chapter.

In Chapter 4 we classify type-preserving, vertex-regular lattices acting on $\tilde{A}_2$-buildings of thickness three. This classification can be viewed as an application of the methods developed in the computer search.

Appendix A contains descriptions of the complexes $Y_i^3$ and presentations for the lattices $\Gamma_i^3$. Appendix B describes the complexes giving rise to the $\tilde{A}_2$-lattices in Chapter 4.

## 1.4 Computer results

Some results in this thesis were obtained with the assistance of a computer. For each such result, we describe explicitly how it was obtained, so that the reader can implement independent code to verify the arguments. We also provide code for verifying these results in the GitHub repository [Tit26]. The code is written in GAP [22].

# Chapter 2

# Preliminaries

## 2.1 Metric spaces and isometries

We introduce the notions of proper actions, proper spaces and other basic notions for isometric actions on metric spaces. We refer the reader to [Kra22] for a detailed introduction. All topological spaces and groups are assumed to be Hausdorff. When we consider a group action on a topological space or a metric space, we implicitly assume that it is by homeomorphism or isometries, respectively.

**Definition 1.**

1. Let $X, Y$ be topological spaces. We call a map $X \to Y$ *proper* if it is continuous, closed and preimages of compact sets are compact.
2. We call an action of a topological group $G$ on a topological space $X$ *proper* if the map $G \times X \to X \times X$, $(g, x) \mapsto (g.x, x)$ is proper.

**Lemma 2** ([Kra22, Corollary 1.8])**.** *Let $G$ be a topological group acting properly on a topological space $X$. If $H \leq G$ is closed then the action of $H$ on $X$ is proper.*

**Definition 3.**

1. We call a topological space *locally compact* if every point has a compact neighborhood.
2. We call a metric space *proper* if every closed ball is compact.
3. Let $X$ be a metric space and let $G \leq \mathrm{Isom}(X)$. The *compact-open topology* on $G$ is the smallest topology in which every set of the form $V_{K,U}$ is open, where $V_{K,U} := \{g \in G \mid g.K \subseteq U\}$ for $K$ compact and $U$ open.

In what follows, groups of isometries are always equipped with the compact-open topology.

**Proposition 4** ( [Kra22, Theorem 2.2][1])**.** *Let $X$ be a proper metric space.*

1. $\mathrm{Isom}(X)$ *is a locally compact topological group.*
2. *The action of* $\mathrm{Isom}(X)$ *on $X$ is proper. In particular, the action of any closed subgroup is proper as well.*

**Lemma 5.** *Let $G$ be a topological group acting properly on a topological space $X$. Then the stabilizer $G_x$ of any point $x$ is compact. In particular, if $G$ is discrete, then point stabilizers in $G$ are finite.*

*Proof.* The map $G \times X \to X \times X, (g, x) \mapsto (g.x, x)$ is proper. Hence, the preimage of $\{v\} \times \{v\}$ which is $G_x \times \{x\}$ is compact. Now projecting to the first factor yields the first claim. The second claim is immediate. □

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[1] Kramer works with the topology of pointwise convergence rather than with the compact-open topology. For isometry groups of metric spaces these topologies coincide [Kra22, Lemma 2.1].

Conversely, finite stabilizers do not imply discreteness of the group acting: the proper action of the reals $\mathbb{R}$ on itself by translations is a counterexample. We make the following observation.

*Observation* 6. Let $G$ be a topological group. If $G$ has an open finite subgroup, then $G$ is discrete.

*Proof.* Let $F$ be open and finite. Since $G$ is Hausdorff, for every non-identity element $g \in F$ there exists an open neighborhood $U_g$ of 1 in $G$ such that $g \notin U_g$. The intersection of the finitely many open sets $U_g$ intersects the open set $F$ in $\{1\}$. Therefore, $\{1\}$ is open and $G$ is discrete. $\square$

We continue with conditions that ensure that stabilizers are open in the compact-open topology. The next definition is partially non-standard.

**Definition 7.** Let $X$ be a metric space and let $G \leq \mathrm{Isom}(X)$. Let $V \subseteq X$ be a subset.

1. We say that $V$ is *uniformly discrete with constant* $\epsilon$ if there exists an $\epsilon > 0$ such that $d(x, y) > \epsilon$ for all $x, y \in V$ with $x \neq y$.
2. We say that $V$ is *G-globally determining*, if $V$ is $G$-invariant and the action of $G$ on $V$ is faithful.
3. We say that $V$ is *G-locally determining*, if $V$ is $G$-invariant and for every $x \in X$ there exists a finite set $V_x \subseteq V$ such that every isometry that fixes $V_x$ pointwise also fixes $x$.

**Lemma 8.** *Let $X$ be a metric space and let $G \leq \mathrm{Isom}(X)$. Let $V$ be a $G$-invariant uniformly discrete subset with constant $\epsilon$.*

1. *For any point in $V$ its stabilizer in $G$ is open.*
2. *If $V$ is $G$-globally determining then $G$ is totally disconnected. In particular, if $G$ is in addition locally compact, then it is a tdlc group.*
3. *If $V$ is $G$-locally determining then for any point in $X$ its stabilizer in $G$ is open.*

*Proof.* Let $x \in V$ be a point. Then the stabilizer of $x$ consists of all isometries that map $x$ into the open $(\epsilon/2)$-ball centered at $x$ and is therefore open.

Now assume that $V$ is $G$-globally determining. Let $x \in V$ and define the evaluation map $\mathrm{ev}_x : G \to X$ by $g \mapsto g.x$. Note that for an open set $U \in X$ the preimage under the evaluation map consists of all isometries that map $\{x\}$ into $U$, which is open. Hence $\mathrm{ev}_x$ is continuous. Let $G^\circ$ be the connected component of the identity. The image of $\mathrm{ev}_x(G^\circ)$ is connected, lies in $V$ and contains $x$. Hence $\mathrm{ev}_x(G^\circ) = \{x\}$. So $G^\circ$ fixes $x$. But $x$ was arbitrary so $G^\circ$ fixes $V$ pointwise and therefore $G^\circ = \{1\}$.

Now assume that $V$ is $G$-locally determining and let $x \in X$. Let $V_x \subseteq V$ be such that every isometry in $G$ that fixes $V_x$ pointwise also fixes $x$. Denote the stabilizer of $v \in V_x$ in $G$ by $G_v$. Then $G_v$ is open. The pointwise stabilizer $H$ of the set $V_x$ is the intersection of the $G_v$ and is therefore open as an intersection of finitely many open subgroups. Let $G_x$ be the stabilizer of $x$ which contains the open subgroup $H$. Then $G_x = \cup_{f \in G_x} fH$ is a union of open sets and is therefore open. $\square$

An example of a space with a uniformly discrete, $G$-locally determining subset is a building $X$ with $G = \mathrm{Aut}(X)$ and $V$ being the vertex set, see Subsection 2.4.3.

**Definition 9.** Let $X$ be a proper metric space.

1. Let $G \to \mathrm{Isom}(X)$ be an action on $X$ with closed image in $\mathrm{Isom}(X)$[2]. We say that $G$ acts *cocompactly* if the quotient $G\backslash X$ is compact.
2. Let $G \leq \mathrm{Isom}(X)$. If $G$ is discrete and acts cocompactly, we call it a *uniform lattice on* $X$.

*Observation* 10. Let $X$ be a proper space and let $G \leq \mathrm{Isom}(X)$. Assume that $G$ acts cocompactly and that there is a $G$-locally determining set as in Lemma 8. If there exists an $x \in X$ with finite stabilizer in $G$, then $G$ is a uniform lattice on $X$.

*Proof.* Immediate from Lemma 8 and Observation 6. □

**Definition 11.** Let $G$ be a group acting on a topological space $X$. The action is *properly discontinuous* if for all compact sets $K \subseteq X$ the set $\{g \in G \mid K \cap g.K \neq \emptyset\}$ is finite.

**Lemma 12.** *Let $X$ be a proper space. Let $G$ be a group acting properly discontinuously on $X$. Then the kernel $N$ of the action is finite and $G/N \leq \mathrm{Isom}(X)$ is discrete. On the other hand, if $G \leq \mathrm{Isom}(X)$ is discrete, then it acts properly discontinuously.*

*Proof.* Assume $G$ acts properly discontinuously. It is clear that the kernel $N$ of the action is finite. Let $H := G/N$, so that $H$ acts faithfully on $X$. Let $x \in X$ and $\epsilon > 0$. Then $U := \{h \in H \mid d(x, h.x) < \epsilon\}$ is an open set. Let $K$ be the closed $\epsilon$-ball centered at $x$. Since for every element $h \in U$ we have that $x \in K \cap h.K$, the set $U$ is finite. It follows that $H$ is discrete.

Now assume that $G \leq \mathrm{Isom}(X)$ is discrete. Let $K \subseteq X$ be compact. By properness of the action, the preimage of the compact set $K \times K$ under the map $G \times X \to X \times X$, $(g, x) \mapsto (g.x, x)$ is compact as well. It is

$$\{(g, y) \mid g.K \cap K \neq \emptyset, y \in K\}.$$

Projecting to the first factor yields that $\{g \mid g.K \cap K \neq \emptyset\}$ is compact. Since $G$ is discrete this set is finite. □

**Definition 13.** Let $G$ be a group acting properly discontinuously on a proper space $X$. We say that $G$ *acts geometrically* if it acts cocompactly.

*Observation* 14. Let $X$ be a proper space. Any uniform lattice on $X$ acts geometrically. On the other hand, if $\phi : G \to \mathrm{Isom}(X)$ is a geometric action, then the kernel $N$ is finite and $G/N \leq \mathrm{Isom}(X)$ is a uniform lattice on $X$.

## 2.2 Locally compact groups and lattices

In this subsection, we introduce the Haar measure and define lattices in locally compact groups. The theory is classical, and so are the references we cite. However, we refer the reader to a preprint by Tornier [Tor20], which we find particularly accessible. Throughout this subsection $G$ always denotes a locally compact group.

**Definition 15.** Let $X$ be a locally compact space. We equip $X$ with its Borel $\sigma$-algebra and it becomes a measurable space.

---

[2] This is to ensure that the quotient is Hausdorff.

1. Assume $G$ acts on $X$ from the left preserving measurable sets. We call a non-trivial measure $\mu$ on $X$ a *left invariant regular Borel measure with respect to the action of $G$ on $X$* if it satisfies the following.
   (a) If $K$ is compact, then $\mu(K) < \infty$.
   (b) If $A$ is measurable, then $\mu(A) = \inf\{\mu(U) \mid A \subseteq U,\ U \text{ open}\}$.
   (c) If $U$ is open, then $\mu(U) = \sup\{\mu(K) \mid K \subseteq U,\ K \text{ compact}\}$.
   (d) For $A$ measurable and $g \in G$ we have $\mu(A) = \mu(g.A)$.

   Analogously, we define *right invariant regular Borel measures* with respect to right actions.

2. Let $\mu$ be a left invariant regular Borel measure with respect to left multiplication of $G$ on itself. Then we call $\mu$ a *left Haar measure* on $G$. Analogously we define *right Haar measures*.

**Theorem 16** ([Bou04, Theorem 1 in VII.§1.2][3])**.** *Up to scaling, there exists a unique left (respectively right) Haar measure on $G$.*

**Definition 17.** The group $G$ is called *unimodular* if every left Haar measure is a right Haar measure.

**Proposition 18** ([Bou04, Theorem 3 in VII.§2.6])**.** *Assume that $G$ is unimodular and let $\Gamma \leq G$ be a discrete subgroup. Up to scaling there exists a unique left invariant regular Borel measure with respect to left multiplication of $G$ on $G/\Gamma$.*

**Definition 19.** Let $G$ be unimodular and $\Gamma$ be a discrete subgroup. Let $\mu$ be the measure from Proposition 18.

1. We call $\Gamma$ a *lattice in $G$* if $\mu(G/\Gamma) < \infty$.
2. If $G/\Gamma$ is compact, which implies that $\Gamma$ is a lattice, then we call $\Gamma$ a *uniform lattice in $G$*.

The following proposition justifies our notion for a lattice on a proper space.

**Proposition 20.** *Let $X$ be a proper metric space and let $G \leq \mathrm{Isom}(X)$ be a closed unimodular subgroup acting cocompactly. Then a subgroup $\Gamma \leq G$ is a uniform lattice in $G$ if and only if it is a uniform lattice on $X$.*

To prove the proposition we need the following lemma.

**Lemma 21.** *Let $X$ be a locally compact space and let $G \leq \mathrm{Homeo}(X)$. The quotient $G\backslash X$ is compact if and only if there exists a compact subset $C \subseteq X$ such that $G.C = X$.*[4]

*Proof.* Let $q : X \to G\backslash X$ be the quotient map. If there exists $C$ as in the lemma, then the quotient space $G\backslash X$ is compact as an image of a compact set. In order to show the converse, we show that $q$ is an open map. Let $U \subseteq X$ be open and consider its preimage $V := q^{-1}(q(U))$. We have

$$\begin{aligned} V &= \{x \in X \mid g.x = u \text{ for some } g \in G, u \in U\} \\ &= \{x \in X \mid x = g.u \text{ for some } g \in G, u \in U\} = \bigcup_{u \in U, g \in G} g.U. \end{aligned}$$

---

[3] In [Bou04], a measure on $X$ is defined as a linear functional on the space of continuous complex-valued functions on $X$ with compact support. By the Riesz representation theorem [Rud87, Theorem 2.14], every positive such functional corresponds uniquely to a regular Borel measure on $X$.

[4] The assumptions of this lemma do not ensure that the quotient is Hausdorff. However, we will only consider actions with Hausdorff quotients.

As a union of open sets $V$ is open and therefore $q$ is open. Now choose for every point in $x$ an open neighborhood $U_x$ which is contained in a compact neighborhood $C_x$. Then $\{q(U_x)\}_x$ is an open cover of the compact space $G\backslash X$ and we can choose finitely many points $x_1, \dots, x_n$ such that $\{q(U_{x_i})\}_i$ covers $G\backslash X$. Now the set $C := \bigcup_{i=1}^{n} C_{x_i}$ is compact and satisfies $G.C = X$. □

*Proof of Proposition 20.* We will use Lemma 21 implicitly throughout the proof. Assume that $\Gamma$ acts cocompactly on $X$. Let $C \subseteq X$ be such that $\Gamma.C = X$ and let $x_0 \in X$. By the properness of $G \times X \to X \times X, (g,x) \mapsto (g.x, x)$ the preimage of $C \times \{x_0\}$ is compact. By projecting this preimage to the first factor we get that $D := \{g \in G \mid g.x_0 \in C\}$ is compact. Now let $g \in G$. We can write $g.x_0 = h.c$ for some $h \in \Gamma, c \in C$. In particular, $h^{-1}g.x_0 \in C$ and therefore $h^{-1}g \in D$. Hence $g \in hD$, so $G = \Gamma.D$ and $\Gamma$ is a uniform lattice in $G$.

Now assume that $\Gamma$ is a uniform lattice in $G$. Let $C \subseteq X$ be such that $G.C = X$ and let $D \subseteq G$ be such that $\Gamma.D = G$. The image of $D \times C$ under $(g,x) \mapsto (g.x, x)$ is compact and when we project to the first factor we still have a compact set, so $E := D.C$ is compact. We have that $\Gamma.E = X$, so $\Gamma$ acts cocompactly. □

## 2.3 CAT($\kappa$)-spaces

In this subsection we introduce the notion of a CAT($\kappa$)-space. An excellent introduction to the topic is provided by the monograph [BH99].

**Definition 22.** Let $(X,d)$ be a metric space. Let $x, y \in X$ with $s = d(x,y)$. A *geodesic segment* from $x$ to $y$ is an isometric embedding $\gamma$ of the real interval $[0,s]$ to $X$ with $\gamma(0) = x$ and $\gamma(s) = y$. Given a geodesic segment we denote by $|\gamma| := s$ its length. A *geodesic ray starting at* $x \in X$ is an isometric embedding $[0,\infty) \to X$ with $\gamma(0) = x$. A *bi-infinite geodesic line* is an isometric embedding $\gamma : \mathbb{R} \to X$. We call $X$ *(uniquely) geodesic* if for all $x, y \in X$ there exists a (unique) geodesic segment from $x$ to $y$.

**Proposition 23** (Hopf–Rinow Theorem, [BH99, Proposition I.3.8])**.** *Let $X$ be a geodesic space. Then $X$ is proper if and only if it is complete and locally compact.*

In order to define CAT($\kappa$)-spaces we need the model spaces $M^n_\kappa$ of curvature $\kappa \in \mathbb{R}$ and dimension $n$. The model space $M^n_\kappa$ could be defined as the unique complete, simply connected Riemannian manifold of dimension $n$ and constant sectional curvature $\kappa$, but we rather give a direct construction. Below we define the model spaces of curvature $0, 1$ and $-1$ which are the Euclidean space, the sphere and the hyperbolic space (equipped with their standard metrics), respectively. We also define hyperplanes and reflections. Any other model space of positive or negative curvature is just the sphere or the hyperbolic space with the metric scaled appropriately.

**Definition 24.** We realize the model spaces as subsets of a real vector space equipped with a suitable metric. In the subsequent definitions, for convenience, we make use of the ambient vector space structure and an associated bilinear form. We emphasize that all objects introduced depend only on the underlying metric space.

1. Consider the vector space $\mathbb{R}^n$ equipped with its standard scalar product $\langle \cdot, \cdot \rangle$. The scalar product induces the Euclidean norm and the Euclidean metric. The *Euclidean space of dimension $n$* is the metric space with underlying set $\mathbb{R}^n$ and the Euclidean metric; we denote it by $\mathbb{E}^n$. Let $u \in \mathbb{R}^n$ be a unit vector and let $c \in \mathbb{R}$. The set $\{x \in \mathbb{R}^n \mid \langle u, x \rangle = c\}$ is called a *hyperplane orthogonal to $u$*.

Let $H$ be a hyperplane defined by $u$ and $c$, the sets $\{x \in \mathbb{R}^n \mid \langle u, x\rangle \geq c\}$ and $\{x \in \mathbb{R}^n \mid \langle u, x\rangle \leq c\}$ are called the *half-spaces bounded by $H$*. The *orthogonal reflection $r_H$ across $H$* is given by the formula below. It is an isometry of $\mathbb{E}^n$.

$$r_H(x) = x - 2(\langle u, x\rangle - c)u.$$

Let $F_1, F_2$ be two half-spaces defined by $\langle u_i, x\rangle \geq c_i$ with $u_i$ being a unit vector for $i \in \{1, 2\}$. The *dihedral angle $\alpha$* between $F_1$ and $F_2$ is the unique angle in $[0, \pi]$ with $\cos(\alpha) = -\langle u_1, u_2\rangle$.

2. In order to define the *sphere $\mathbb{S}^n$ of dimension $n$*, we consider the vector space $\mathbb{R}^{n+1}$ equipped with its standard scalar product and the Euclidean norm. The underlying set of $\mathbb{S}^n$ is the set of unit vectors. We define the *standard spherical metric* as follows: for $x, y \in \mathbb{S}^n$ their distance is the unique real number $\alpha \in [0, \pi]$ such that $\cos(\alpha) = \langle x, y\rangle$. Given $u \in \mathbb{S}^n$ we define the *hyperplane orthogonal to $u$* as the set $\{x \in \mathbb{S}^n \mid \langle u, x\rangle = 0\}$. Let $H$ be the hyperplane orthogonal to $u$, we call the sets $\{x \in \mathbb{S}^n \mid \langle u, x\rangle \geq 0\}$ and $\{x \in \mathbb{S}^n \mid \langle u, x\rangle \leq 0\}$ the *half-spaces bounded by $H$*. The *spherical reflection across $H$* is given by the formula below. It is an isometry of $\mathbb{S}^n$.

$$r_H(x) = x - 2\langle u, x\rangle u.$$

Let $F_1, F_2$ be two half-spaces defined by $\langle u_i, x\rangle \geq 0$ with $u_i \in \mathbb{S}^n$ for $i \in \{1, 2\}$. The *dihedral angle $\alpha$* between $F_1$ and $F_2$ is the unique angle $\alpha$ in $[0, \pi]$ such that $\cos(\alpha) = -\langle u_1, u_2\rangle$.

For any positive $\kappa$ we obtain the model space $M^n_\kappa$ by equipping $\mathbb{S}^n$ with the standard spherical metric scaled by the factor $(1/\sqrt{\kappa})$.

3. To define the *hyperbolic space $\mathbb{H}^n$ of dimension $n$* we consider the vector space $\mathbb{R}^{n+1}$ with the Lorentzian inner product $\langle x, y\rangle_L := (\sum_{i=1}^{n} x_i y_i) - x_{n+1}y_{n+1}$. Set $\mathbb{H}^n := \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle_L = -1, x_{n+1} > 0\}$ and define the *standard hyperbolic metric* by $\cosh(d_{\mathbb{H}}(x, y)) = -\langle x, y\rangle_L$ for $x, y \in \mathbb{H}^n$. Let $u \in \mathbb{R}^{n+1}$ be such that $\langle u, u\rangle_L = 1$. The *hyperplane orthogonal to $u$* is the set $\{x \in \mathbb{H}^n \mid \langle u, x\rangle_L = 0\}$. Let $H$ be the hyperplane orthogonal to $u$, we call the sets $\{x \in \mathbb{H}^n \mid \langle u, x\rangle_L \geq 0\}$ and $\{x \in \mathbb{H}^n \mid \langle u, x\rangle_L \leq 0\}$ the *half-spaces bounded by $H$*. The *hyperbolic reflection across $H$* is given by the formula below. It is an isometry of $\mathbb{H}^n$.

$$r_H(x) = x - 2\langle u, x\rangle_L u.$$

Let $u_1, u_2 \in \mathbb{R}^{n+1}$ be such that $\langle u_i, u_i\rangle_L = 1$ for $i \in \{1, 2\}$. Assume that the hyperplanes orthogonal to $u_1$ and $u_2$ intersect. Let $F_1, F_2$ be the two half-spaces defined by $\langle u_i, x\rangle_L \geq 0$ for $i \in \{1, 2\}$. The *dihedral angle $\alpha$* between $F_1$ and $F_2$ is the unique angle in $[0, \pi]$ with $\cos(\alpha) = -\langle u_1, u_2\rangle_L$.

For any negative $\kappa$ we obtain the model space $M^n_\kappa$ by equipping $\mathbb{H}^n$ with the standard hyperbolic metric scaled by the factor $(1/\sqrt{-\kappa})$.

*Remark* 25. For the sphere $\mathbb{S}^n$, hyperplanes are commonly referred to as *great spheres*, and the half-spaces as *hemispheres*. We decided to use the terms hyperplane and half-space to keep the terminology consistent across the different model spaces.

**Proposition 26** ([BH99, Proposition I.2.11]). *Denote the metric of $M^n_\kappa$ by $d$. The model space $M^n_\kappa$ is a geodesic space. If $\kappa \leq 0$, then $M^n_\kappa$ is uniquely geodesic. If $\kappa > 0$, then for all $x, y \in M^n_\kappa$ with distance $d(x, y) < (\pi/\sqrt{\kappa})$ there is a unique geodesic segment from $x$ to $y$.*

**Definition 27.** Let $(X, d)$ be a geodesic space. Let $(x_1, x_2, x_3) \in X$ be a triple of points in $X$ and let $\gamma_{ij}$ be a geodesic segment from $x_i$ to $x_j$ for $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$. We call the datum $((x_i)_i, (\gamma_{ij})_{i,j})$ a *geodesic triangle* and denote it with $\Delta((x_i)_i, (\gamma_{ij})_{i,j})$. Let $\Delta := \Delta((x_i)_i, (\gamma_{ij})_{i,j})$. The triple $(x_1, x_2, x_3)$ is called the *vertex triple* of $\Delta$. The triple $(|\gamma_{23}|, |\gamma_{31}|, |\gamma_{12}|)$ is called the *side length triple* of $\Delta$. The union of the images of the geodesics in $\Delta$ is called the *image* of $\Delta$. The sum of the lengths in the side length triple of $\Delta$ is called the *perimeter* of $\Delta$.

**Lemma 28** ([BH99, Lemma I.2.14])**.** *Let $\kappa \in \mathbb{R}$. Let $a \leq b \leq c$ be non-negative real numbers with $c \leq a + b$. If $\kappa > 0$ assume in addition that $a + b + c < (2\pi/\sqrt{\kappa})$. Then there exists a geodesic triangle in $M^2_\kappa$ with side length triple $(a, b, c)$. Moreover, if $(x, y, z)$ is the vertex triple of such a triangle, the vertex triple of any other geodesic triangle with the same side length triple is an isometric image of $(x, y, z)$.*

**Definition 29.** Let $(X, d)$ be a geodesic space. Let $\kappa \in \mathbb{R}$ and denote the metric in $M^2_\kappa$ also by $d$.

1. Let $\Delta := \Delta((x_i)_i, (\gamma_{ij})_{i,j})$ be a geodesic triangle in $X$. If $\kappa > 0$, assume in addition that the perimeter of $\Delta$ is less than $(2\pi/\sqrt{\kappa})$. Up to isometry there is a unique geodesic triangle $\Delta' := \Delta((x'_i)_i, (\gamma'_{ij})_{i,j})$ in $M^2_\kappa$ with the same side length triple. There is a unique bijection $f_{\Delta,\kappa}$ between the images of $\Delta$ and $\Delta'$, which maps $x_i$ to $x'_i$ and is an isometry when restricted to the image of one of the geodesic segments $\gamma_{ij}$. We say that $X$ is a CAT($\kappa$)*-space* if for every geodesic triangle $\Delta$, we have: $d(y, z) \leq d(f_{\Delta,\kappa}(y), f_{\Delta,\kappa}(z))$ for all $y, z \in \Delta$. This property can be interpreted as follows: Every triangle in $X$ is at most as thick as its unique comparison triangle in $M^2_\kappa$.
2. We say that $X$ is *locally* CAT($\kappa$) if for every $x \in X$ there exists a radius $r$ such that the open ball $B_r(x)$ endowed with the induced metric is a CAT($\kappa$)-space. We say that $X$ is *non-positively curved* if it is locally CAT(0).
3. A finitely generated group that acts geometrically on a proper CAT(0)-space is called a CAT(0)*-group*.

We recall some important facts about CAT($\kappa$)-spaces.

**Proposition 30** ([BH99, Chapter II.1])**.** *Let $(X, d)$ be a* CAT($\kappa$)*-space.*

1. *If $\kappa \leq 0$, then $X$ is a uniquely geodesic space. If $\kappa > 0$ and $x, y \in X$ are such that $d(x, y) < (1/\sqrt{\kappa})$, then there exists a unique geodesic from $x$ to $y$.*
2. *If $\kappa \leq 0$, then $X$ is contractible.*
3. *$X$ is* CAT($\kappa'$) *for every $\kappa' \geq \kappa$.*

**Theorem 31** (Cartan–Hadamard Theorem, [BH99, Theorem II.4.1])**.** *Let $X$ be a complete, connected, locally* CAT($\kappa$)*-space with $\kappa \leq 0$. Then its universal cover $\tilde{X}$ equipped with the length metric[5] is a* CAT($\kappa$)*-space.*

**Proposition 32** (Bruhat–Tits Fixed Point Theorem, [BH99, Theorem II.2.8])**.** *Let $X$ be a complete* CAT(0)*-space and let $F \leq$* Isom($X$) *be finite. Then $F$ fixes a point in $X$.*

**Proposition 33** ([BH99, Theorem II.4.13])**.** *Let $X$ be a complete, connected, locally* CAT(0)*-space. Then the fundamental group $\pi_1(X)$ is torsion-free. In particular, its action on the universal cover $\tilde{X}$ is free.*

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[5] For a definition of the length metric see [BH99, Definition I.3.24].

## 2.4 Coxeter polytopes, chamber complexes and buildings

In this section we define *Coxeter polytopes*, *chamber complexes* and *buildings* from a metric perspective. For our purposes, there is no loss of generality in restricting to the model spaces of curvature $\kappa = 0, 1$, or $-1$, and we impose this assumption throughout this section. So any considered model space is either the Euclidean space $\mathbb{E}^n$, the sphere $\mathbb{S}^n$ with diameter $\pi$ or the hyperbolic space $\mathbb{H}^n$ with its standard metric. We refer to [Dav08, Chapter 6] for more details on Coxeter polytopes. A standard reference for buildings is [AB08].

### 2.4.1 Definition of a Coxeter polytope

Let $M^n_\kappa$ be a model space. Let $F_1, \ldots, F_k$ be a finite number of half-spaces. Denote their intersection with $P$ and assume that $P$ has non-empty interior. If $M^n_\kappa = \mathbb{S}^n$ and $P$ contains no antipodal point, we call $P$ a *spherical convex polytope*. If $M^n_\kappa \in \{\mathbb{E}^n, \mathbb{H}^n\}$ and $P$ is bounded, we call $P$ a *Euclidean convex polytope* or *hyperbolic convex polytope*, respectively. Given a convex polytope $P$ there is a unique minimal set of half-spaces defining it and we call them the *defining half-spaces of P*. The hyperplanes that bound the defining half-spaces are called *bounding hyperplanes*. A *facet* of $P$ is a non-empty intersection of $P$ with a bounding hyperplane. A *face* of $P$ is an intersection of facets. The minimal non-empty faces are singletons and are called *vertices*; we will also refer to their unique elements as *vertices*. The *dihedral angle* between two intersecting facets is the dihedral angle between the defining half-spaces of $P$ corresponding to the hyperplanes supporting the two facets. Let $P$ be a convex polytope in $M^n_\kappa$ with bounding hyperplanes $H_1, \ldots, H_k$ supporting the facets $p_1, \ldots, p_k$, respectively. We say that $P$ is a *Coxeter polytope* if the the dihedral angle between any two intersecting facets is $(\pi/m)$ for some $2 \leq m \in \mathbb{N}$. The *Coxeter rank* of $P$ is the number of bounding hyperplanes. The *Coxeter matrix* of $P$ is the map $m_P : \{1, \ldots, k\}^2 \to \mathbb{N} \cup \{\infty\}$,

$$(i,j) \mapsto \begin{cases} 1 & , i = j, \\ m_{ij} & , p_i \cap p_j \neq \emptyset, \text{ the dihedral angle between } p_i \text{ and } p_j \text{ is } (\pi/m_{ij}), \\ \infty & , p_i \cap p_j = \emptyset. \end{cases}$$

The *Coxeter group* of $P \leq \mathrm{Isom}(M^n_\kappa)$ is the group generated by the reflections across its bounding hyperplanes. We say that $P$ is *reducible* if there exists a partition of its facets into two non-empty sets $\mathcal{P}_1, \mathcal{P}_2$ such that for all $p_i \in \mathcal{P}_1, p_j \in \mathcal{P}_2$ the facets $p_i$ and $p_j$ intersect and the dihedral angle between them is $(\pi/2)$. In that case the reflections across $H_i$ and $H_j$ commute and the Coxeter group splits as a direct product. We call $P$ *irreducible* if it is not reducible. Enumerate the vertices of a Coxeter polytope $v_1, \ldots, v_m$ by $I := \{1, \ldots, m\}$. This induces a type function on faces with types in $\mathcal{P}(I)$ as follows: $f \mapsto \{i \mid v_i \in f\}$; sometimes we refer to the type of a vertex with $i$ instead of $\{i\}$. We will assume that a Coxeter polytope always comes equipped with a type function.

Spherical and Euclidean Coxeter polytopes are classified [Cox34]. Hyperbolic Coxeter polytopes are classified in dimension two [Poi82], three [And70] and in the simplicial case [Lan50]. By a theorem of Vinberg a hyperbolic Coxeter polytope can only exist in dimension $n \leq 29$ [Vin85]. For further details we refer to [Dav08, Chapter 6], which also contains the following proposition.

**Proposition 34** ([Dav08, Theorem 6.4.3]). *Let $P$ be a Coxeter polytope defined in the model space $M^n_\kappa$. Let $H_1, \dots, H_k$ be its bounding hyperplanes and denote by $s_i$ the reflection across $H_i$ for $1 \leq i \leq k$. Let $W$ be the Coxeter group of $P$. Then $P$ is a strict fundamental domain for the action of $W$ on $M^n_\kappa$. Let $m_P$ be the Coxeter matrix of $P$ and let $F$ be the free group over $(s_i)_i$. Then the obvious homomorphism $F \to W$ descends to an isomorphism:*

$$\langle s_1, \dots, s_k \mid s_i^2, (s_i s_j)^{m_P(i,j)} \rangle \cong W.$$

### 2.4.2 Definition of a chamber complex

Let $P$ be a typed Coxeter polytope. Let $X$ be a complex that is obtained by gluing copies of $P$ along their faces such that the gluing respects the type function and is an isometry when restricted to faces. This ensures that the glued chambers inject in $X$, hence we have a natural notion of *chambers*, *faces*, *facets* and *vertices* in $X$. The type function on $P$ induces a type function on the faces in $X$. We call $X$ a *chamber complex with chambers of shape $P$* if every facet is contained in at least two chambers and for any two chambers $C, D$ there exists a chain of chambers $C = C_0, C_1, \dots, C_n = D$, such that $C_i$ and $C_{i+1}$ share a facet. To define a metric on a chamber complex $X$ we need $m$-strings: for $x, y \in X$ we call a finite sequence of points $x = x_0, \dots, x_n = y$ an *$m$-string from $x$ to $y$* if $x_i$ and $x_{i+1}$ are supported in a common chamber. The *length* of an $m$-string is the sum over the distances between $x_i$ and $x_{i+1}$ measured in a chamber. For $x, y \in X$ we define $d(x, y)$ as the infimum over the lengths of all $m$-strings connecting $x$ and $y$. Then $d$ is a metric on $X$, turning it into a complete geodesic space [BH99, Theorem I.7.50]. We call $X$ *finite* if it consists of finitely many chambers; we call $X$ *locally finite* if every vertex is contained in finitely many chambers. If $X$ is finite, it is compact, and if $X$ is locally finite, it is locally compact and proper. The *automorphism group* $\mathrm{Aut}(X)$ of $X$ is the group of isometries that preserve the cell structure; the *type-preserving automorphism group* is the subgroup respecting types. In particular, the vertex set of a chamber complex is $\mathrm{Aut}(X)$-locally determining in the sense of Lemma 8. Therefore, $\mathrm{Aut}(X)$ is totally disconnected and every point stabilizer in $\mathrm{Aut}(X)$ is open. By [Kra22, Proposition 3.12][6], $\mathrm{Aut}(X)$ is closed in $\mathrm{Isom}(X)$ and therefore the action of $\mathrm{Aut}(X)$ on $X$ is proper. If we assume in addition that $X$ is locally finite, then $\mathrm{Aut}(X)$ is locally compact.

### 2.4.3 Definition of a building

Given a Coxeter polytope $P$, the action on its Coxeter group defines a tessellation of the ambient model space, see Proposition 34. This tessellation is a chamber complex and we call it the *model apartment with chambers of shape $P$*.

Let $P$ be a Coxeter polytope and let $X$ be a chamber complex with chambers of shape $P$. Call a sub-chamber complex an *apartment* if it is isomorphic to the model apartment with chambers of shape $P$. Then $X$ is called a *building with chambers of shape $P$* if the following two axioms hold:

1. For every two points $x, y \in X$ there exist an apartment containing both.
2. If $A_1, A_2$ are apartments, then there exists an isomorphism $A_1 \to A_2$ that is the identity on the (possibly empty) intersection.

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[6] Kramer only considers simplicial complexes, but this is no restriction since we can always subdivide a chamber complex to a simplicial complex.

The facets of a building are also called *panels*. The building $X$ is *thick* if every panel is contained in at least three chambers. It is *reducible* (respectively *irreducible*) if $P$ is reducible (respectively irreducible). The *Coxeter rank* of $X$ is the Coxeter rank of $P$.

We say that a building is *spherical, Euclidean, hyperbolic* if its defining polytope is spherical, Euclidean or hyperbolic, respectively. This terminology is justified by the following.

**Proposition 35** ([AB08, Proposition 12.29]). *Euclidean buildings, spherical buildings and hyperbolic buildings are* $\mathrm{CAT}(0)$*-spaces,* $\mathrm{CAT}(1)$*-spaces and* $\mathrm{CAT}(-1)$*-spaces, respectively.*

The combinatorial properties of model apartments with chambers of shape $P$ depend only on the Coxeter matrix of $P$, and not on its specific geometric realization. When working with buildings, it is therefore standard not to specify the polytope itself, but rather its Coxeter matrix $m$.

The *type* of a Coxeter polytope is its Coxeter matrix, and the *type* of a building is the type of one of its chambers. Spherical and Euclidean types are usually referred to by short labels rather than by the Coxeter matrices. In the Euclidean plane $\mathbb{E}^2$, there are exactly four types. Triangles with angles

$$(\tfrac{\pi}{3}, \tfrac{\pi}{3}, \tfrac{\pi}{3}), \quad (\tfrac{\pi}{4}, \tfrac{\pi}{4}, \tfrac{\pi}{2}), \quad (\tfrac{\pi}{6}, \tfrac{\pi}{3}, \tfrac{\pi}{2})$$

are of types $\tilde{A}_2$, $\tilde{C}_2$, and $\tilde{G}_2$, respectively, and are irreducible. Any rectangle is of type $\tilde{A}_1 \times \tilde{A}_1$ and is reducible. However, for convenience, we always assume that a Coxeter polytope of this type is a square.

### 2.4.4 Exotic buildings and the local approach

The definition of an exotic building requires preparation, we refer to [Wei09] for the following claims. Further references are [BT72], [BT84a], [Tit86]. Given a semisimple algebraic group $\mathbf{G}$ defined over a non-Archimedean local field $\mathbb{K}$, there is a locally finite Euclidean building associated to $\mathbf{G}(\mathbb{K})$ with a highly transitive $\mathbf{G}(\mathbb{K})$-action. Buildings arising in this way are called *Bruhat–Tits buildings*. Any thick, irreducible Euclidean building of Coxeter rank at least 4 is Bruhat–Tits.

However, an irreducible Euclidean building with Coxeter rank 3, i.e. a building of type $\tilde{A}_2$, $\tilde{C}_2$ or $\tilde{G}_2$, is potentially not Bruhat–Tits:

**Definition 36.** A thick Euclidean building of type $\tilde{A}_2$, $\tilde{C}_2$ or $\tilde{G}_2$ that is not Bruhat–Tits is called *exotic*.

A first challenge in the study of exotic buildings is to construct examples (with automorphisms), as the classical methods give Bruhat–Tits buildings. An approach that has turned out to be fruitful is the local approach: one constructs buildings as universal covers of finite chamber complexes that are locally buildings. We restrict to the two-dimensional case, as our constructions are two-dimensional and the theory is less technical in this setting.

**Definition 37.** Let $X$ be a chamber complex with two-dimensional chambers of shape $P$. The facets in $P$ are edges, and every vertex $x$ in $P$ is the intersection of a unique pair of edges in $P$. We define the *angle* of $x$ in $P$ to be the dihedral angle between those edges.

Let $x \in X$ be a vertex. We define its *geometric link* denoted by $\mathrm{lk}(x)$ as the following metric graph. For every edge $e \subseteq X$ containing $x$ we have a vertex $\vec{e}$ in $\mathrm{lk}(x)$

and for every chamber $c \subseteq X$ containing $x$ we have an edge $\vec{c}$ in $\mathrm{lk}(x)$. A vertex $\vec{e}$ is incident with an edge $\vec{c}$ if $e \subseteq c$ for the corresponding faces in $X$. The length of an edge in $\mathrm{lk}(x)$ is defined as the angle at the vertex in $P$ with the same type as $x$.

In a two-dimensional building the link of every vertex is a metric generalized polygon, which are one-dimensional spherical buildings:

**Definition 38.**
1. A simplicial graph (without metric) is called *combinatorial generalized polygon* if its girth equals twice its diameter.
2. A metric graph is called *metric generalized polygon* if the diameter is $\pi$ and the girth is $2\pi$.

Note that given a combinatorial generalized polygon with diameter $m$ we can turn it into a metric generalized polygon by assigning every edge the length $(\pi/m)$.

The following theorem plays a central role in our strategy; all our examples are constructed through it. The first statement is a consequence of the Cartan–Hadamard theorem 31, the second can be interpreted as a strengthening. Note that when we apply the theorem to a finite chamber complex $Y$, the torsion-free fundamental group $\pi_1(Y)$ acts freely and cocompactly (in particular as a uniform lattice) on the universal cover $\tilde{Y}$ which is a proper $\mathrm{CAT}(0)$ or $\mathrm{CAT}(-1)$.

**Theorem 39.** *Let $Y$ be a chamber complex with two-dimensional, Euclidean or hyperbolic chambers of shape $P \subseteq M^2_\kappa$.*

1. *Every vertex link in $Y$ has girth at least $2\pi$ if and only if the universal cover of $Y$ is $\mathrm{CAT}(\kappa)$.*
2. *If $P$ is hyperbolic, assume in addition that $P$ is a triangle. Every vertex link in $Y$ is a metric generalized polygon if and only if the universal cover of $Y$ is a building.*

*Proof.* The first statement is a combination of [BH99, Theorem I.5.2] and the Cartan–Hadamard theorem 31. The second statement in the Euclidean case is established in [CL01, Theorem 7.3]. The result for hyperbolic triangle complexes follows from [Tit81]: By 1., the universal cover $\tilde{Y}$ is $\mathrm{CAT}(-1)$. In particular, it is simply connected. In the terminology of Tits, $\tilde{Y}$ corresponds to a geometry $\Gamma$ of type $M$ and rank 3. Conversely, given a geometry of type $M$, one can associate to it a chamber complex in our sense by gluing triangles of the appropriate shape according to the incidence in the geometry of type $M$. Tits also introduces a notion of coverings for geometries of type $M$ and defines simple connectedness in this framework. With the standing assumptions, it is easy to see that simple connectedness of $\tilde{Y}$ implies simple connectedness of $\Gamma$ in Tits' sense. Applying [Tit81, Theorem 1], we conclude that $\Gamma$ is a building. Consequently, $\tilde{Y}$ is a building as well. □

In the previous theorem, we restricted our assumptions to dimension two only because we have not defined links for higher-dimensional complexes. In the hyperbolic case, we further limited our consideration to triangles, since adapting Tits' results to other polygons becomes more involved. Nevertheless, the theorems of Charney–Lytchak hold in arbitrary dimension and rely on metric rather than combinatorial assumptions, while the results of Tits hold in full generality.

**Definition 40.** We call a chamber complex whose universal cover is a building a *geometry that is almost a building*, or *GAB* for short. If we want to specify the chambers, we speak of a *GAB with chambers of shape P*. If the chambers are of a given type, we say that the GAB is of that type.

We need the following proposition later on. It is essentially a consequence of [BH99, Proposition II.4.14].

**Proposition 41.** *Let $Y, Y'$ be GABs with chambers of the same shape and let $f : Y \to Y'$ be an injective map respecting the cell structure and the types. Then $f$ is locally convex in the following sense: for every $y \in Y$ there is neighborhood $U$ such that $f|_U$ is an isometric embedding. Moreover, the canonical homomorphism $\pi_1(Y, y) \to \pi_1(Y', f(y))$ is injective and every continuous lifting $\tilde{f} : \tilde{Y} \to \tilde{Y}'$ of $f$ is an isometric embedding.*

*Proof.* The statement on local convexity is clear for $y \in Y$ contained in the interior of a chamber or of an edge. So let $y \in Y$ be a vertex. The image $f(\mathrm{lk}(y))$ of the link of $y$ is naturally a subgraph of the link $\mathrm{lk}(f(y))$ in $Y'$. Both being generalized polygons we deduce that $f(\mathrm{lk}(y))$ is convex in $\mathrm{lk}(f(y))$. Now local convexity of $f$ follows, since balls of sufficiently small radius around vertices are metric cones over the links, and convexity in the links implies convexity in the corresponding cones. Applying [BH99, Proposition II.4.14] yields the full proposition. □

## 2.5 Quasi-isometries, property (T) and properties on normal subgroups

This section consists of three short subsections, on quasi-isometries, property (T), and the notions of residual finiteness and just infiniteness.

### 2.5.1 Quasi-isometries

The notion of a quasi-isometry is a fundamental concept in geometric group theory. We refer to [BH99, Chapter I.8] for further details.

**Definition 42.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A map $f : X \to Y$ is called an *$(A, B)$-quasi-isometric embedding* if there exist constants $A \geq 1$ and $B \geq 0$ such that for all $x_1, x_2 \in X$ we have

$$\frac{1}{A} d_X(x_1, x_2) - B \leq d_Y(f(x_1), f(x_2)) \leq A d_X(x_1, x_2) + B.$$

Let $f$ be a quasi-isometric embedding. If there exists a constant $C \geq 0$ such that for every point $y \in Y$ there exists a point $x \in X$ with $d_Y(f(x), y) \leq C$, then we call $f$ an *$(A, B)$-quasi-isometry*. We say that two spaces are *quasi-isometric* if there exists a quasi-isometry between them. Quasi-isometry defines an equivalence relation on the class of metric spaces.

The following is well known; see [BH99, Example I.8.17(2)] for a proof sketch.

**Proposition 43.** *Let $G$ be a finitely generated group. For any two finite generating sets, the corresponding Cayley graphs equipped with the word metric are quasi-isometric. In particular, there is a well-defined quasi-isometry type of a finitely generated group.*

**Proposition 44** (Svarc–Milnor Lemma)**.** *Let $X$ be a proper, geodesic space. If $\Gamma$ acts geometrically on $X$, then $\Gamma$ is finitely generated and for every $x \in X$ the orbit map $\Gamma \to X$, $g \mapsto g.x$ is a quasi-isometry. On the other hand, if $\Gamma$ is finitely generated, acts on $X$ by isometries, and there is an $x_0 \in X$ such that the orbit map $g \mapsto g.x_0$ is a quasi-isometry, then the action is geometric.*

*Proof.* The first implication is [BH99, p. I.8.19]; we prove its converse. Take a finite generating set, denote the corresponding word metric by $d$, and also denote the metric in $X$ with $d$. Let $x_0$ be such that the orbit map $g \mapsto g.x_0$ is an $(A, B)$-quasi-isometry. Let $C \geq 0$ be such that for all $x \in X$ there exists $g \in \Gamma$ with $d(g.x_0, x) \leq C$. Let $K \subseteq X$ be compact and let $g \in \Gamma$ such that $K \cap g.K \neq \emptyset$. As a compact set $K$ is contained in the closed $R$-ball centered at $x_0$ for some radius $R$. In particular, we have that $d(x_0, g.x_0) \leq 2R$. Using the inequalities we deduce that

$$\frac{d(1,g)}{A} - B \leq 2R.$$

Therefore, there are only finitely many such $g$ and the action is properly discontinuous. Let $D$ be the closed $C$-ball in $X$ centered at $x_0$. Then $\Gamma.D = X$ and $\Gamma$ acts cocompactly by Lemma 21. □

Later, when working with quasi-isometries, we will need the following notion.

**Definition 45.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Two maps $f, g\colon X \to Y$ are *at bounded distance* if there exists a constant $C \geq 0$ such that $d_Y\big(f(x), g(x)\big) \leq C$ for all $x \in X$.

### 2.5.2 Property (T)

In this subsection we define Kazhdan's property (T) for a locally compact group $G$. For a comprehensive treatment of property (T), we refer to [BHV08].

**Definition 46.**

1. Let $\mathcal{H}$ be a complex Hilbert space with unitary group $\mathcal{U}(\mathcal{H})$. A *unitary representation* of $G$ in $\mathcal{H}$ is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$, which is *strongly continuous* in the following sense: for every $\zeta \in \mathcal{H}$ the map $G \to \mathcal{H}$, $g \mapsto \pi(g).\zeta$ is continuous.

2. Let $Q \subseteq G$ be a compact subset and let $\epsilon > 0$. Let $(\pi, \mathcal{H})$ be a unitary representation of $G$. We say a vector $\zeta \in \mathcal{H}$ is *$(Q, \epsilon)$-invariant* if we have

$$\max_{x \in Q} ||\pi(x).\zeta - \zeta|| < \epsilon ||\zeta||.[7]$$

3. Let $(\pi, \mathcal{H})$ be a unitary representation of $G$. We say that $(\pi, \mathcal{H})$ *has almost invariant vectors* if it has $(Q, \epsilon)$-invariant vectors for every compact subset $Q \subseteq G$ and every $\epsilon > 0$. We say that $(\pi, \mathcal{H})$ has *invariant vectors* if there is a non-zero vector $\zeta$ with $\pi(g).\zeta = \zeta$ for all $g \in G$.

4. We say the group $G$ has *Kazhdan's property (T)* if every unitary representation with almost invariant vectors also has invariant vectors.

5. Assume there exists a compact subset $Q \subseteq G$ and $\epsilon > 0$ such that every unitary representation of $G$ with $(Q, \epsilon)$-invariant vectors also has invariant vectors. Then we call $Q$ a *Kazhdan set* and $(Q, \epsilon)$ a *Kazhdan pair* for $G$. Given a Kazhdan set $Q$ the maximal[8] $\epsilon$ such that $(Q, \epsilon)$ is a Kazhdan pair is called the *Kazhdan constant* with respect to $Q$.

---

[7] Since $Q$ is compact we can take the maximum here.

[8] We argue that the maximum is indeed attained. Given a unitary representation $\pi$, denote the supremal $\epsilon$ such that the existence of a $(Q, \epsilon)$-invariant vector implies the existence of an invariant vector by $\epsilon(\pi)$. Given a family of unitary representations $(\pi_i)_i$, we consider its product $\pi := \oplus_i \pi_i$ and we have that $\epsilon(\pi) = \sup\{\epsilon(\pi_i) \mid i\}$.

**Proposition 47** ([BHV08, Theorem 1.3.1, Proposition 1.3.2]). *Let $G$ be a locally compact group with property (T). Then $G$ is compactly generated. Furthermore, every compact generating set of $G$ is a Kazhdan set. In particular, if $G$ is discrete it is finitely generated. In that case every Kazhdan set is a finite generating set.*

### 2.5.3 Residual finiteness, just infiniteness and simplicity

In this subsection we define residual finiteness and just infiniteness. Throughout, $G$ denotes a group.

**Definition 48.**

1. The *finite residual* of $G$, denoted by $G^{(\infty)}$ is the intersection of its finite index subgroups; or equivalently the intersection of its finite index normal subgroups; or equivalently the collection of elements that vanish in every finite quotient.
2. We call $G$ *residually finite* if its finite residual is trivial.
3. We call $G$ *just infinite* if it is infinite and every non-trivial normal subgroup is of finite index. We call $G$ *hereditarily just infinite* if every finite index subgroup is just infinite.

*Observation* 49. Let $G$ be a hereditarily just infinite group, then $G$ is either residually finite or its finite residual is of finite index and simple.

*Proof.* Assume $G$ is not residually finite. Its finite residual is normal and therefore of finite index. Furthermore, it contains no proper finite index subgroups. Since it is just infinite it is simple. ☐

# Chapter 3

# Constructing $\tilde{C}_2$-lattices with no finite index subgroups

*The results of this chapter are based on a joint work with Stefan Witzel [TW25]. The text has been revised and refined for consistency. All results in this chapter are due to the author of this thesis or were obtained with significant contributions by the author.*

This chapter is organized as follows. Section 3.1 collects results and examples on products of trees needed later on. In Section 3.2 we analyze the buildings $X_i^q$ and the lattices $\Gamma_i^q$, and prove the main theorem A. The complexes $Y_i^q$ where found via a computer search, that has proven to be useful for other purposes; we describe it in Section 3.3.

## 3.1 Lattices on products of trees

The groups in the main theorem are not residually finite because they contain subgroups that are not residually finite. These subgroups are irreducible lattices on products of trees and arise from the celebrated work of Burger–Mozes [BM00] and Wise [Wis96]. The specific groups we will be working with are called Burger–Mozes–Wise (BMW) groups by Caprace and we follow this convention. A more detailed treatment can be found in [Cap19, Section 4].

**Definition 50.** Let $T_1$ and $T_2$ be regular trees of finite degrees $d_1$ and $d_2$, respectively. If $\Gamma < \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ acts regularly on the vertices of $T_1 \times T_2$, then the action of $\Gamma$ on $T_1 \times T_2$ is called *BMW-action* and $\Gamma$ is called a *BMW-group* of *degree* $(d_1, d_2)$.

A BMW-group is *reducible* if it has a finite index subgroup that decomposes as a direct product or, equivalently[1], its image in $\mathrm{Aut}(T_1)$ or $\mathrm{Aut}(T_2)$ is discrete. Otherwise it is *irreducible*.

A product of two trees is a Euclidean building of type $\tilde{A}_1 \times \tilde{A}_1$, whose chambers are Euclidean squares. The vertex links are complete bipartite graphs. Given a BMW-action $\Gamma \curvearrowright T_1 \times T_2$ of degree $(d_1, d_2)$, the subgroup $\Gamma^+$ preserving the types acts regularly on vertices of each type and the quotient group is the Klein four group $V_4$. The quotient complex $S \colon = \Gamma^+ \backslash T_1 \times T_2$ is an $\left(\tilde{A}_1 \times \tilde{A}_1\right)$-GAB or in other words a Euclidean square complex with each vertex link being the complete bipartite graph. Moreover, $S$ has four vertices and admits a vertex-regular action of $V_4$.

We call a finite $\left(\tilde{A}_1 \times \tilde{A}_1\right)$-GAB with four vertices and a vertex-regular $V_4$-action a *BMW-complex*. The fundamental group $\Gamma^+ := \pi_1(S)$ acts on the universal cover which is a product of trees $\tilde{S} = T_1 \times T_2$ by Theorem 39.2. The deck transformations of $T_1 \times T_2$ that cover $V_4$ form an extension $\Gamma$ of $\Gamma^+$ by $V_4$ that is a BMW-group.

---

[1] See [BM00, Theorem 1.2] for this claim.

In what follows we will take both perspectives on BMW-groups: as groups acting on products of trees, and as $(\tilde{A}_1 \times \tilde{A}_1)$-GABs with a $V_4$-action.

In order to compute presentations of BMW-groups and later $\tilde{C}_2$-lattices, we use the following special case of [Bro84, Theorem 1]. Let $X$ be a simply-connected CW-complex and assume that the 1-skeleton of $X$ does not contain loops. Assume that $G$ acts on $X$ permuting cells and regularly on vertices. Pick a vertex $x_0 \in X$ and let $E$ be the set of edges of $X$ incident with $x_0$. For $e \in E$ let $t(e, x_0)$ be the vertex of $e$, that is not $x_0$ and let $g_e \in G$ be the unique element that maps $x_0$ to $t(e, x_0)$. Let $F = F(g_e, e \in E)$ be the free group generated by the $g_e$.

Note that $g_e^{-1}$ maps $e \in E$ to an edge $f \in E$, possibly $e = f$. We define

$$R_1 := \{g_e g_f^{-1} \mid g_e^{-1}(e) = f\} \subseteq F.$$

If $\gamma = (e_1, \ldots, e_k)$ is a combinatorial edge path starting at $x_0$, we define the word $W(\gamma) \in F$ recursively by

$$W(\gamma) := \begin{cases} g_e & \text{if } \gamma = (e), \\ g_e W(g_e^{-1}(\gamma')) & \text{if } \gamma = (e) * \gamma'. \end{cases}$$

For a 2-cell $A$ incident with $x_0$ (pick an orientation for definedness and) let $\gamma_A \in F$ be the edge loop starting at $x_0$ along the boundary of $A$.

We define

$$R_2 := \{W(\gamma_A) \mid A \text{ is a 2-cell incident with } x_0\} \subseteq F.$$

**Proposition 51.** *Let $X$ be a simply-connected CW-complex and assume that the 1-skeleton of $X$ does not contain loops. Assume that $G$ acts on $X$ permuting cells and regularly on vertices, and let $E$, $R_1$, and $R_2$ be as above. Then the homomorphism $F \to G$ that takes $g_e$ to $g_e$ descends to an isomorphism $\langle g_e, e \in E \mid R_1 \cup R_2\rangle \cong G$.*

The following example of a BMW-group was studied by Radu [Rad20]. It will be our source of non-residual finiteness in the $q = 2$ case of the main theorem.

**Example 52.** Consider the $(\tilde{A}_1 \times \tilde{A}_1)$-GAB $S_{\mathrm{R}}$ indicated in Figure 3.1. If we equip it with the action by $V_4 = \langle \sigma_A, \sigma_X \rangle$ described below, it becomes a BMW-complex.

$$\begin{array}{llllll}
\sigma_A : & v_{00} \leftrightarrow v_{10}, & v_{01} \leftrightarrow v_{11}, & \sigma_X : & v_{00} \leftrightarrow v_{01}, & v_{10} \leftrightarrow v_{11}, \\
& a \leftrightarrow a^{-1}, & b \leftrightarrow b^{-1}, & & a \leftrightarrow a', & b \leftrightarrow b', \\
& c \leftrightarrow c^{-1}, & a' \leftrightarrow (a')^{-1}, & & c \leftrightarrow c', & x \leftrightarrow x^{-1}, \\
& b' \leftrightarrow (b')^{-1}, & c' \leftrightarrow (c')^{-1}, & & y \leftrightarrow y^{-1}, & z \leftrightarrow z^{-1}, \\
& x \leftrightarrow x', & y \leftrightarrow y', & & x' \leftrightarrow (x')^{-1}, & y' \leftrightarrow (y')^{-1}, \\
& z \leftrightarrow z'. & & & z' \leftrightarrow (z')^{-1}. &
\end{array}$$

Consider the universal cover $\tilde{S}_{\mathrm{R}}$ of $S_{\mathrm{R}}$ as homotopy classes of paths in $S_{\mathrm{R}}$ starting at $v_{00}$. Let $\Gamma_{\mathrm{R}} \leq \mathrm{Aut}(\tilde{S}_{\mathrm{R}})$ denote the BMW-group arising from Example 52, i.e. the extension of $\pi_1(S_{\mathrm{R}})$ covering $V_4 = \langle \sigma_A, \sigma_X \rangle$. Define the following automorphisms of $\tilde{S}_{\mathrm{R}}$ in $\Gamma_{\mathrm{R}}$:

$$\begin{array}{lll}
a : [\gamma] \mapsto [a^{-1} * \sigma_A(\gamma)], & b : [\gamma] \mapsto [b^{-1} * \sigma_A(\gamma)], & c : [\gamma] \mapsto [c^{-1} * \sigma_A(\gamma)], \\
x : [\gamma] \mapsto [x^{-1} * \sigma_X(\gamma)], & y : [\gamma] \mapsto [y^{-1} * \sigma_X(\gamma)], & z : [\gamma] \mapsto [z^{-1} * \sigma_X(\gamma)].
\end{array}$$

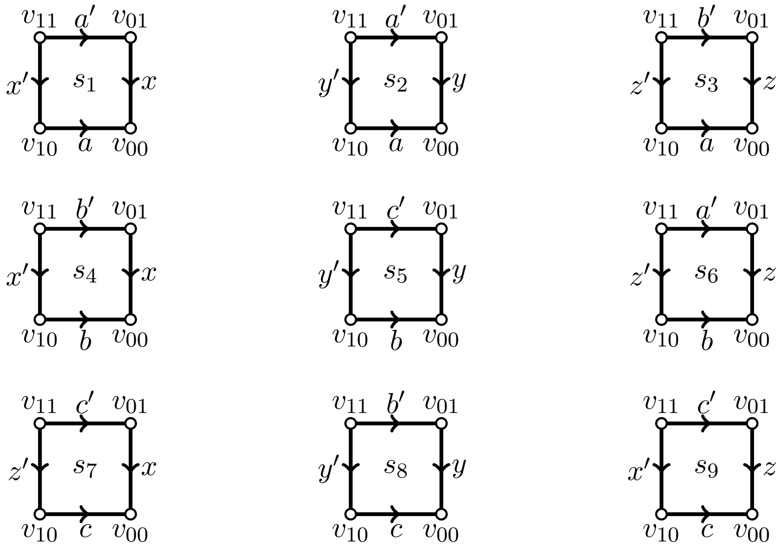

FIGURE 3.1: The $(\tilde{A}_1 \times \tilde{A}_1)$-GAB $S_{\mathrm{R}}$.

**Proposition 53** (Radu)**.**

1. *The obvious homomorphism $F(a,b,c,x,y,z) \to \Gamma_R$ descends to an isomorphism:*
$$\langle a,b,c,x,y,z \mid a^2,b^2,c^2,x^2,y^2,z^2,axax,ayay,azbz,bxbx,bycy,cxcz\rangle \cong \Gamma_R.$$
2. *The BMW-action $(\Gamma_{\mathrm{R}}, \tilde{S}_R)$ is irreducible and the group $\Gamma_{\mathrm{R}}$ is not residually finite.*
3. *The element $xz$ lies in the profinite closure of $A := \langle a,b,c\rangle$, i.e. $\phi(zx) \in \phi(A)$ for every homomorphism $\phi$ to a finite group.*
4. *At least one of the following elements is in the finite residual of $\Gamma_{\mathrm{R}}$.*
$$[y(xz)^2y, xz], \qquad\qquad [y(xz)^2y, xzb].$$

*Proof.* The presentation and its geometric interpretation can be deduced from Proposition 51. For the other statements we refer to [Rad20, Proposition 5.4]. □

Building on the previous proposition, we can prove the following.

**Lemma 54.** *The element $(xz)^4$ lies in the finite residual of $\Gamma_{\mathrm{R}}$. In particular, it lies in the finite residual of $\pi_1(S_{\mathrm{R}}, v_{00})$ and is represented by the path $(x^{-1} * z)^4$.*

*Proof.* Let $\delta := xz$ and let $\phi$ be an epimorphism from $\Gamma$ to a finite group. We observe that $\delta a = b\delta$, $\delta b = a\delta$ and $\delta c = c\delta^{-1}$. In particular, we have $\delta^2 c = c\delta^{-2}$. Now conjugating this equation with $y$ yields
$$y\delta^2 yb = by\delta^{-2}y. \tag{$*$}$$

Assume that $\phi(y\delta^2y)$ and $\phi(\delta)$ commute. Then we can conjugate $(*)$ with $\phi(\delta)$ and $\phi(y)$, and we obtain

$$\begin{aligned}\phi(\delta y\delta^2 y\delta^{-1}a) &= \phi(a\delta y\delta^{-2}y\delta^{-1}),\\ \phi(y\delta^2ya) &= \phi(ay\delta^{-2}y),\\ \phi(\delta^2 a) &= \phi(a\delta^{-2}),\\ \phi(\delta^2) &= \phi(\delta^{-2}).\end{aligned}$$

In particular, $\phi(\delta^4) = 1$ in this case. If $\phi(y\delta^2y)$ and $\phi(\delta b)$ commute, then we conjugate $(*)$ with $\phi(\delta b)$ and $\phi(y)$, and we obtain

$$\begin{aligned}\phi(\delta by\delta^2 yb\delta^{-1}a) &= \phi(a\delta b\delta y\delta^{-2}yb\delta^{-1}),\\ \phi(y\delta^2ya) &= \phi(ay\delta^{-2}y),\\ \phi(\delta^2) &= \phi(\delta^{-2}).\end{aligned}$$

Since by Proposition 53.4 at least one of these two cases occurs, we deduce that $\delta^4$ lies in the finite residual of $\Gamma_{\mathrm{R}}$. □

*Remark* 55.
1. The previous lemma also implies that $[y(xz)^2y, xz]$ lies in the finite residual of $\Gamma_{\mathrm{R}}$. Indeed, since $\delta^4$ vanishes in every finite quotient, we get that $\phi(y(xz)^2y)$ centralizes $\phi(A)$ for every map $\phi$ to a finite group. Since $xz$ lies in the profinite closure of $A$, we get that $\phi\left([y(xz)^2y, xz]\right) = 1$.
2. The elements $\delta^4$ and $\delta^{-4}$ are the shortest elements in the finite residual of $\Gamma_{\mathrm{R}}$. This can be seen as follows. Let $\mathbb{K}$ be a field of characteristic $\neq 2$ and let $\alpha, \beta, \gamma$ in $\mathbb{K}$ be such that $\alpha^2 - \alpha - 4 = 0$, $\beta^2 - \beta + 4 = 0$ and $\gamma^2 - \alpha + 12 = 0$. Then the following assignment extends to a homomorphism $\Phi := \Phi_{(\alpha,\beta,\gamma)}$ from the type-preserving subgroup of $\Gamma_{\mathrm{R}}$ to $\mathrm{GL}_2(\mathbb{K})$.

$$\begin{aligned} ab &\mapsto \begin{pmatrix} \frac{-\alpha+\gamma}{4} & 0 \\ 0 & \frac{-\alpha-\gamma}{4}\end{pmatrix},\\ yx &\mapsto \begin{pmatrix} 1 & -1+\frac{\alpha}{2}+\frac{\beta}{4}-\frac{3\alpha\beta}{16}+\frac{\beta\gamma}{2}+\frac{\beta\gamma^3}{16} \\ -1+\frac{\alpha}{2}+\frac{\beta}{4}-\frac{3\alpha\beta}{16}-\frac{\beta\gamma}{2}-\frac{\beta\gamma^3}{16} & 1\end{pmatrix}.\end{aligned}$$

We take $\mathbb{K}$ to be an appropriate extension of the rationals $\mathbb{Q}$ and the shortest elements in the kernel of $\Phi$ are $\delta^4, \delta^{-4}$.

The next example is due to Janzen–Wise [JW09] and will be our source of non-residual finiteness in the case $q = 3$ of the main theorem.

**Example 56.** Consider the $(\tilde{A}_1 \times \tilde{A}_1)$-GAB $S_{\mathrm{JW}}$ indicated in Figure 3.2. If we equip it with the action of $V_4 = \langle \sigma_A, \sigma_X\rangle$ described below, it becomes a BMW-complex.

$$\begin{array}{lllllll} \sigma_A\colon & v_{00} \leftrightarrow v_{10}, & v_{10} \leftrightarrow v_{11}, & \sigma_X\colon & v_{00} \leftrightarrow v_{01}, & v_{01} \leftrightarrow v_{11}, \\ & a \leftrightarrow \bar{a}^{-1}, & b \leftrightarrow \bar{b}^{-1}, & & a \leftrightarrow a', & b \leftrightarrow b', \\ & a' \leftrightarrow (\bar{a}')^{-1}, & b' \leftrightarrow (\bar{b}')^{-1}, & & \bar{a} \leftrightarrow \bar{a}', & \bar{b} \leftrightarrow \bar{b}', \\ & x \leftrightarrow x', & y \leftrightarrow y', & & x \leftrightarrow \bar{x}^{-1}, & y \leftrightarrow \bar{y}^{-1}, \\ & \bar{x} \leftrightarrow \bar{x}', & \bar{y} \leftrightarrow \bar{y}'. & & x' \leftrightarrow (\bar{x}')^{-1}, & y' \leftrightarrow (\bar{y}')^{-1}. \end{array}$$

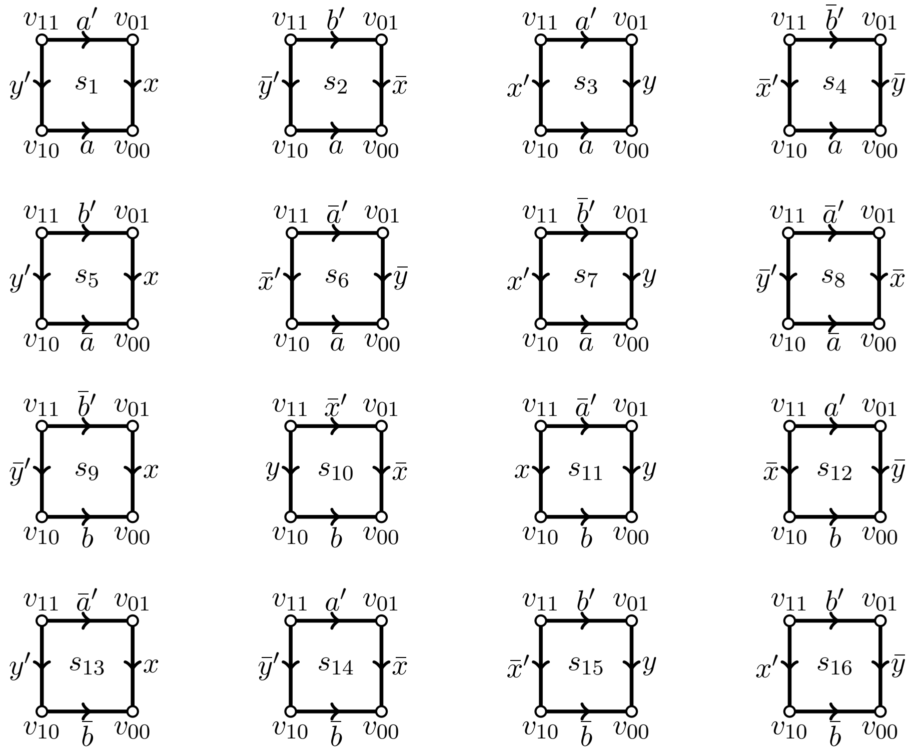

FIGURE 3.2: The complex $S_{\mathrm{JW}}$.

Consider the universal cover $\tilde{S}_{\mathrm{JW}}$ of $S_{\mathrm{JW}}$ as homotopy classes of paths in $S_{\mathrm{JW}}$ starting at $v_{00}$. Let $\Gamma_{\mathrm{JW}} \leq \mathrm{Aut}(\tilde{S}_{\mathrm{JW}})$ be the extension of $\pi_1(S_{\mathrm{JW}})$ covering $\langle \sigma_A, \sigma_X \rangle$. Define the following automorphisms of $\tilde{S}_{\mathrm{JW}}$ in $\Gamma_{\mathrm{JW}}$:

$$\begin{aligned} a &: [\gamma] \mapsto [a^{-1} * \sigma_A(\gamma)], & b &: [\gamma] \mapsto [b^{-1} * \sigma_A(\gamma)], \\ x &: [\gamma] \mapsto [x^{-1} * \sigma_X(\gamma)], & y &: [\gamma] \mapsto [y^{-1} * \sigma_X(\gamma)]. \end{aligned}$$

**Proposition 57** (Janzen–Wise, Caprace)**.**

1. *The obvious homomorphism* $F(a,b,x,y) \to \Gamma_{\mathrm{JW}}$ *descends to an isomorphism:*

$$\langle a,b,x,y \mid axay, ax^{-1}by^{-1}, ay^{-1}b^{-1}x^{-1}, bxb^{-1}y^{-1} \rangle \cong \Gamma_{\mathrm{JW}}.$$

2. *The BMW-action* $(\Gamma_{\mathrm{JW}}, S_{\mathrm{JW}})$ *is irreducible and the group* $\Gamma_{\mathrm{JW}}$ *is not residually finite.*

3. *The elements* $[x^3, y^4]$ *and* $[y^3, x^4]$ *are both in the finite residual of* $\Gamma_{\mathrm{JW}}$*. In particular, the homotopy classes of the following loops are contained in the finite residual of* $\pi_1(\mathcal{S}_{\mathrm{JW}}, v_{00})$*.*

$$x^{-1} * \bar{x} * x^{-1} * \bar{y} * y^{-1} * \bar{y} * y^{-1} * x * \bar{x}^{-1} * x * \bar{y}^{-1} * y * \bar{y}^{-1} * y,$$
$$y^{-1} * \bar{y} * y^{-1} * \bar{x} * x^{-1} * \bar{x} * x^{-1} * y * \bar{y}^{-1} * y * \bar{x}^{-1} * x * \bar{x}^{-1} * x.$$

*Proof.* Again the presentation can be deduced from Proposition 51. The irreducibility has been established by Janzen–Wise [JW09]. The computation of the explicit elements in the finite residual is due to Caprace [Cap19, Remark 4.20]. □

## 3.2 Non-residually finite $\tilde{C}_2$-lattices

In this section we present the complexes $Y_i^q$ from the main theorem A. The complexes have been found with computer assistance, and we present our approach to find these complexes in Section 3.3.

We begin with our example for $q = 2$.

**Theorem 58.** *Let $Y_1^2$ be the Euclidean chamber complex indicated in Figure 3.3. Its universal cover $X_1^2 = \tilde{Y}_1^2$ is a $\tilde{C}_2$-building of thickness three. Its fundamental group $\Gamma_1^2 = \pi_1(Y_1^2)$ is not residually finite. In fact, $\Gamma_1^2$ does not have any finite index subgroups.*

All assertions are straightforward to verify, except for the last statement. The proof for the claim that $\Gamma_1^2$ has no finite-index subgroups is postponed to the end of Subsection 3.2.2; the remaining statements are proved below.

*Proof of Theorem 58, Part 1.* By considering the links of $Y_1^2$, indicated in Figure 3.4, we immediately conclude that it is a GAB of type $\tilde{C}_2$. Therefore, $X_1^2$ is a building of type $\tilde{C}_2$ by Theorem 39.2. If we subdivide the complex $S_{\mathrm{R}}$ along the diagonals from $v_{00}$ to $v_{11}$ and label the new diagonals with $s_i$, we obtain a $\tilde{C}_2$-GAB that embeds in $Y_k^2$ via the following assignment.

$$\begin{array}{ccccccc} v_{00} \mapsto v, & v_{11} \mapsto w, & v_{10} \mapsto u_1, & v_{01} \mapsto u_2, & a \mapsto f_4, & b \mapsto f_5, & c \mapsto f_6, \\ x \mapsto f_1, & y \mapsto f_2, & z \mapsto f_3, & a' \mapsto e_4, & b' \mapsto e_5, & c' \mapsto e_6, & x' \mapsto e_1, \\ y' \mapsto e_2, & z' \mapsto e_3, & s_i \mapsto g_i. & & & & \end{array}$$

In particular, $\pi_1(S_{\mathrm{R}})$ embeds into $\Gamma_1^2$ by Proposition 41, showing that $\Gamma_1^2$ is not residually finite by Proposition 53.2.

□

The complex $Y_1^2$ admits an automorphism $\rho$ of order 2 mapping the vertices and edges as follows:

$$\begin{array}{cccccc} v \leftrightarrow w, & u_1 \leftrightarrow u_2, & u_3 \circlearrowleft, & u_4 \circlearrowleft, & u_5 \circlearrowleft, & e_i \leftrightarrow f_i^{-1}, \\ g_1 \leftrightarrow g_1^{-1}, & g_2 \leftrightarrow g_2^{-1}, & g_4 \leftrightarrow g_4^{-1}, & g_{15} \leftrightarrow g_{15}^{-1}, & g_3 \leftrightarrow g_6^{-1}, & g_5 \leftrightarrow g_8^{-1}, \\ g_7 \leftrightarrow g_9^{-1}, & g_{10} \leftrightarrow g_{10}^{-1}, & g_{11} \leftrightarrow g_{11}^{-1}, & g_{12} \leftrightarrow g_{14}^{-1}, & g_{13} \leftrightarrow g_{13}^{-1}. & \end{array}$$

By considering deck transformations, we obtain a split extension $\bar{\Gamma}_1^2 = \Gamma_1^2 \rtimes C_2$ that acts on $X_1^2$ such that the action on special vertices is regular.

For completeness, we recall the definition of a special vertex:

**Definition 59.** In a Coxeter polytope $P$ of type $\tilde{C}_2$, the two vertices with angle $(\pi/4)$ are called *special*. This definition lifts to $\tilde{C}_2$-GABs, in particular to $\tilde{C}_2$-buildings.

### 3.2.1 Geometric presentations

Our next goal is to use Proposition 51 to derive presentations for $\tilde{C}_2$-lattices acting regularly on special vertices.

Let $Y$ be a $\tilde{C}_2$-GAB equipped with two special vertices $v$ and $w$, one of each special type, and assume that $Y$ admits an involutory automorphism $\rho$ that interchanges $v$ and $w$. Let $X = \tilde{Y}$ be the universal cover and let $\Gamma$ be the group of deck transformations covering $\langle \rho \rangle$. Then $\Gamma$ acts regularly on special vertices and contains $\pi_1(Y)$ with index 2.

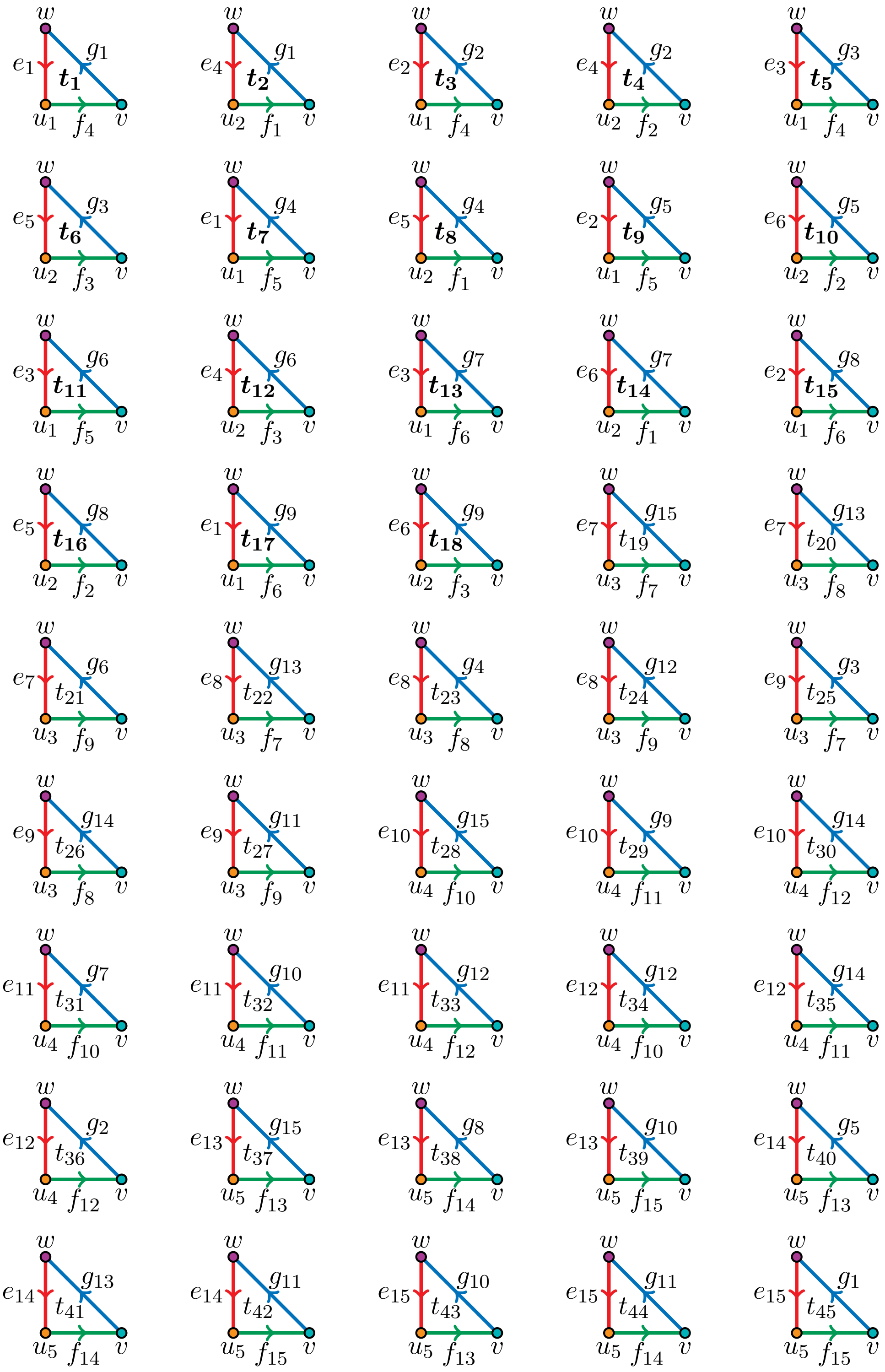

FIGURE 3.3: The $\tilde{C}_2$-GAB $Y_1^2$. The subcomplex generated by the triangles $\boldsymbol{t_1}, \dots, \boldsymbol{t_{18}}$ is isomorphic to a subdivision of $S_R$.

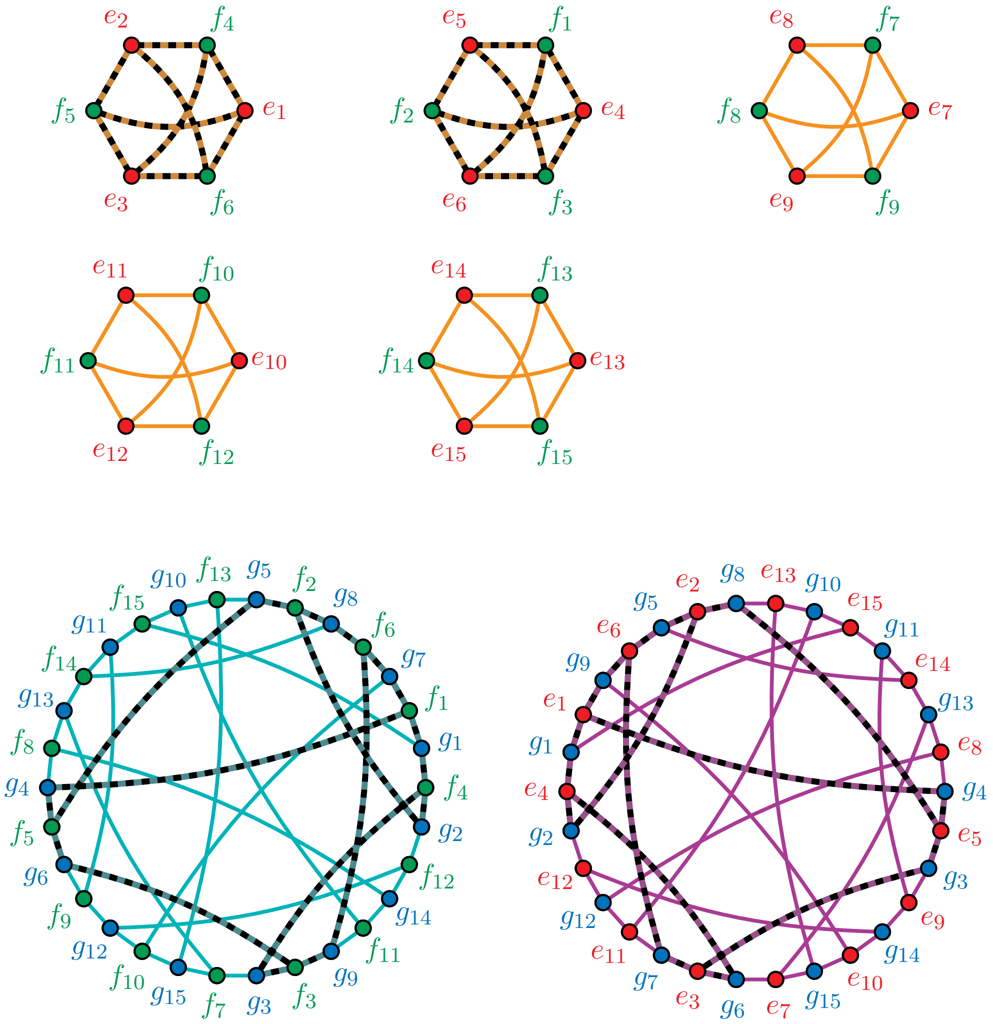

FIGURE 3.4: The links in $Y_1^2$ of the vertices $u_1$, $u_2$, $u_3$, $u_4$, $u_5$, $v$, and $w$.

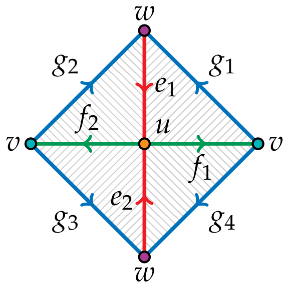

FIGURE 3.5: The edge path $(g_1 * g_2^{-1} * g_3 * g_4^{-1})$ is an $(A_1 \times A_1)$-boundary.

We cannot apply Proposition 51 directly, so we modify $X$ as follows. When we remove from $X$ the non-special vertices together with their stars, we are also removing all short edges. Thus we are left with the full subgraph of special vertices and long edges, which we denote by $X^{(s)}$. In $X^{(s)}$ there are many four-cycles, all of which are apartments in links of non-special vertices of $X$. We call such a four-cycle an $(A_1 \times A_1)$*-boundary* and construct a complex $X'$ by filling in all $(A_1 \times A_1)$-boundaries in $X^{(s)}$. Note that $\Gamma$ acts regularly on the vertices of $X'$.

Now let $G$ be the set of edges in $Y$ from $v$ to $w$ (oriented in this way) and call edge paths in $Y$ that lift to an $(A_1 \times A_1)$-boundary also $(A_1 \times A_1)$-boundaries. Consider the following words in the free group $F(G)$ over $G$:

$$R_1 := \{gh \mid g \in G \text{ and } \rho(g)^{-1} = h\},$$
$$R_2 := \{g_1 g_2^{-1} g_3 g_4^{-1} \mid g_1 * g_2^{-1} * g_3 * g_4^{-1} \text{ is an } (A_1 \times A_1)\text{-boundary}\}.$$

**Lemma 60.** *In the above situation the space $X'$ is simply connected and the obvious homomorphism $F(G) \to \Gamma$ descends to an isomorphism:*

$$\langle G \mid R_1 \cup R_2 \rangle \cong \Gamma.$$

*More precisely, if we consider $X$ as homotopy classes of paths in $Y$ starting at $v$ then a generator $g$ of this presentation acts on $X$ as follows*

$$[\gamma] \mapsto [g * \rho(\gamma)].$$

*In particular, $g \mapsto g.v$ is a $\Gamma$-equivariant isomorphism* $\mathrm{Cay}(\Gamma, G) \to X^{(s)}$.

*Proof.* If $u \in X$ is a non-special vertex and $g$ is an edge in its link (a complete bipartite graph), then $g$ together with the edges that meet it in a vertex form a spanning tree $T$ of $\operatorname{lk} u$, and $\pi_1(\operatorname{lk} u) \cong \pi_1(\operatorname{lk} u/T, T)$ is freely generated by edges opposite to $g$ (this is a very special case of the Solomon–Tits theorem). It follows that gluing in all four-cycles that contain $g$ produces a simply-connected space. Gluing in all four-cycles, rather than only the ones that contain $g$, certainly results in a simply connected space. Since $X$ is simply connected, it follows that removing the star of a non-special vertex and gluing in all four-cycles in its link produces a simply connected space. Thus $X'$ is simply connected and we can apply Proposition 51. □

Applying Lemma 60 to the lattice $\bar{\Gamma}_1^2$ gives:

**Proposition 61.** *1. The lattice $\bar{\Gamma}_1^2$ is presented by generators $g_1, \dots, g_{15}$ subject to the relations*

$$\begin{array}{lllll}
g_1^2, & g_2^2, & g_3g_6, & g_4^2, & g_5g_8, \\
g_7g_9, & g_{10}^2 & g_{11}^2 & g_{12}g_{14}, & g_{13}^2, \\
g_{15}^2, & g_1g_4g_5g_2, & g_1g_4g_6g_6, & g_1g_7g_7g_6, & g_1g_7g_8g_2, \\
g_3g_{11}g_{12}g_{13}, & g_3g_{12}g_4g_{13}, & g_3g_{12}g_{13}g_{15}, & g_2g_{14}g_7g_{14}, & g_7g_{10}g_{14}g_{14}, \\
g_7g_{14}g_{14}g_{15}, & g_1g_{10}g_5g_{11}, & g_5g_{11}g_{10}g_{15}, & g_5g_{13}g_{11}g_{10}. &
\end{array}$$

2. *If we regard $X_1^2$ as homotopy classes of paths in $Y_1^2$ starting at $v$, then the generator $g_i$ acts on $X_1^2$ via*

$$[\gamma] \mapsto [g_i * \rho(\gamma)].$$

*In particular, the full subgraph of the 1-skeleton of $X_1^2$ on the set of special vertices is $\bar{\Gamma}_1^2$-equivariantly identified with the Cayley graph of $\bar{\Gamma}_1^2$ with respect to the generators $g_1, \dots, g_{15}$.*

*Proof.* We apply Lemma 60 and obtain the following relations over the generating set $(g_i)^{\pm 1}$.

$$\begin{aligned}
R_1 &= \{g_1^2,\ g_2^2,\ g_3g_6,\ g_4^2,\ g_5g_8,\ g_7g_9,\ g_{10}^2,\ g_{11}^2,\ g_{12}g_{14},\ g_{15}^2\}, \\
R_2 &= \{(A_1 \times A_1)\text{-boundary relations}\}.
\end{aligned}$$

Up to cyclic permutation there are 45 relations in $R_2$, nine for each vertex of non-special type in $Y_1^2$. Since $\rho$ swaps $u_1$ and $u_2$, the relations arising from $(A_1 \times A_1)$-boundaries around these vertices are equivalent. We now list the 36 relations arising from boundaries around $u_1, u_3, u_4, u_5$. Note that these can be read out easily from Figure 3.6.

$$\begin{array}{llll}
r_{1,1} = g_1g_4^{-1}g_5g_2^{-1}, & r_{1,2} = g_1g_4^{-1}g_6g_3^{-1}, & r_{1,3} = g_1g_9^{-1}g_7g_3^{-1}, & r_{1,4} = g_1g_9^{-1}g_8g_2^{-1}, \\
r_{1,5} = g_2g_5^{-1}g_6g_3^{-1}, & r_{1,6} = g_2g_8^{-1}g_7g_3^{-1}, & r_{1,7} = g_4g_9^{-1}g_8g_5^{-1}, & r_{1,8} = g_4g_9^{-1}g_7g_6^{-1}, \\
r_{1,9} = g_5g_8^{-1}g_7g_6^{-1}, & r_{3,1} = g_3g_{11}^{-1}g_6g_{15}^{-1}, & r_{3,2} = g_3g_{11}^{-1}g_{12}g_{13}^{-1}, & r_{3,3} = g_3g_{14}^{-1}g_4g_{13}^{-1}, \\
r_{3,4} = g_3g_{14}^{-1}g_{13}g_{15}^{-1}, & r_{3,5} = g_4g_{12}^{-1}g_6g_{13}^{-1}, & r_{3,6} = g_4g_{12}^{-1}g_{11}g_{14}^{-1}, & r_{3,7} = g_4g_{13}^{-1}g_{15}g_{13}^{-1}, \\
r_{3,8} = g_6g_{13}^{-1}g_{14}g_{11}^{-1}, & r_{3,9} = g_6g_{15}^{-1}g_{13}g_{12}^{-1}, & r_{4,1} = g_2g_{12}^{-1}g_7g_{12}^{-1}, & r_{4,2} = g_2g_{12}^{-1}g_{15}g_{14}^{-1}, \\
r_{4,3} = g_2g_{14}^{-1}g_9g_{14}^{-1}, & r_{4,4} = g_2g_{14}^{-1}g_{10}g_{12}^{-1}, & r_{4,5} = g_7g_{10}^{-1}g_9g_{15}^{-1}, & r_{4,6} = g_7g_{10}^{-1}g_{14}g_{12}^{-1}, \\
r_{4,7} = g_7g_{12}^{-1}g_{14}g_{15}^{-1}, & r_{4,8} = g_9g_{14}^{-1}g_{12}g_{10}^{-1}, & r_{4,9} = g_9g_{15}^{-1}g_{12}g_{14}^{-1}, & r_{5,1} = g_1g_{10}^{-1}g_5g_{11}^{-1}, \\
r_{5,2} = g_1g_{10}^{-1}g_{15}g_{10}^{-1}, & r_{5,3} = g_1g_{11}^{-1}g_8g_{10}^{-1}, & r_{5,4} = g_1g_{11}^{-1}g_{13}g_{11}^{-1}, & r_{5,5} = g_5g_{11}^{-1}g_{10}g_{15}^{-1}, \\
r_{5,6} = g_5g_{13}^{-1}g_8g_{15}^{-1}, & r_{5,7} = g_5g_{13}^{-1}g_{11}g_{10}^{-1}, & r_{5,8} = g_8g_{10}^{-1}g_{11}g_{13}^{-1}, & r_{5,9} = g_8g_{15}^{-1}g_{10}g_{11}^{-1}.
\end{array}$$

The relations in the presentation in the proposition which are not in $R_1$ are equivalent to the relations $r_{1,1}, r_{1,2}, r_{1,3}, r_{1,4}, r_{3,2}, r_{3,3}, r_{3,4}, r_{4,1}, r_{4,6}, r_{4,7}, r_{5,1}, r_{5,5}$ and $r_{5,7}$.

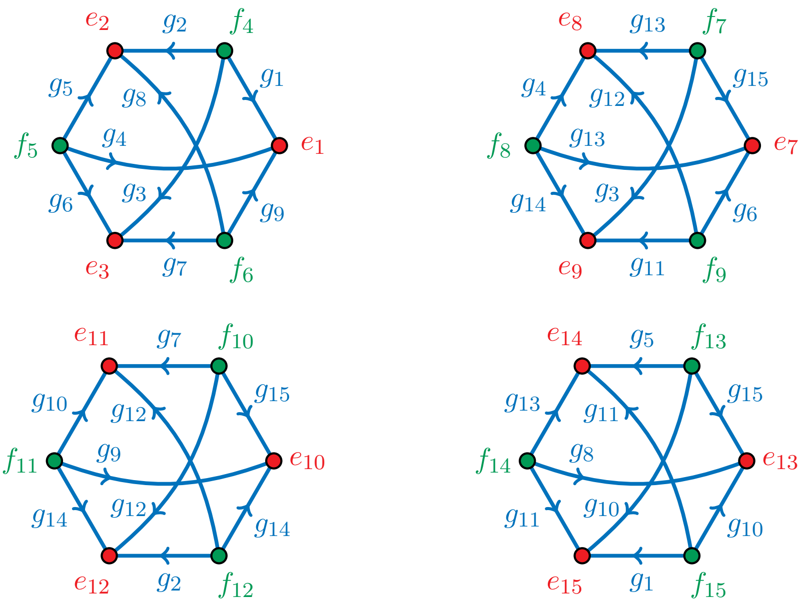

FIGURE 3.6: The $(A_1 \times A_1)$-boundaries around $u_1, u_3, u_4$ and $u_5$ are cycles of length four in the indicated graphs.

The remaining relations can be deduced as follows.

$$\begin{array}{rrrr}
r_{1,1} \wedge r_{1,2} \rightsquigarrow r_{1,5}, & r_{1,3} \wedge r_{1,4} \rightsquigarrow r_{1,6}, & r_{1,1} \wedge r_{1,4} \rightsquigarrow r_{1,7}, & r_{1,2} \wedge r_{1,3} \rightsquigarrow r_{1,8}, \\
r_{1,7} \wedge r_{1,8} \rightsquigarrow r_{1,9}, & r_{3,2} \rightsquigarrow r_{3,8}, & r_{3,3} \rightsquigarrow r_{3,5}, & r_{3,4} \rightsquigarrow r_{3,9}, \\
r_{3,2} \wedge r_{3,3} \rightsquigarrow r_{3,6}, & r_{3,3} \wedge r_{3,4} \rightsquigarrow r_{3,7}, & r_{3,8} \wedge r_{3,9} \rightsquigarrow r_{3,1}, & r_{4,1} \rightsquigarrow r_{4,3}, \\
r_{4,6} \rightsquigarrow r_{4,8}, & r_{4,7} \rightsquigarrow r_{4,9}, & r_{4,1} \wedge r_{4,6} \rightsquigarrow r_{4,4}, & r_{4,1} \wedge r_{4,7} \rightsquigarrow r_{4,2}, \\
r_{4,8} \wedge r_{4,9} \rightsquigarrow r_{4,5}, & r_{5,1} \rightsquigarrow r_{5,3}, & r_{5,5} \rightsquigarrow r_{5,9}, & r_{5,7} \rightsquigarrow r_{5,8}, \\
r_{5,1} \wedge r_{5,5} \rightsquigarrow r_{5,2}, & r_{5,3} \wedge r_{5,8} \rightsquigarrow r_{5,4}, & r_{5,8} \wedge r_{5,9} \rightsquigarrow r_{5,6}. &
\end{array}$$

Lemma 60 also yields that the generators act on $X_1^2$ as described in the proposition. □

Applying the Reidemeister–Schreier procedure to the presentation of $\bar{\Gamma}_1^2$ we obtain a presentation for $\Gamma_1^2$:

**Corollary 62.** *1. The lattice $\Gamma_1^2$ is presented by generators $h_2, \ldots, h_{15}$ subject to the relations*

$$\begin{array}{llll}
h_4 h_8^{-1} h_2, & h_4^{-1} h_5 h_2^{-1}, & h_4 h_3^{-1} h_6, & h_4^{-1} h_6 h_3^{-1}, \\
h_7 h_9^{-1} h_6, & h_9^{-1} h_7 h_3^{-1}, & h_7 h_5^{-1} h_2, & h_9^{-1} h_7 h_3^{-1}, \\
h_3 h_{11}^{-1} h_{12} h_{13}^{-1}, & h_6^{-1} h_{11} h_{14}^{-1} h_{13}, & h_3 h_{14}^{-1} h_4 h_{13}^{-1}, & h_6^{-1} h_{12} h_4^{-1} h_{13}, \\
h_3 h_{14}^{-1} h_{13} h_{15}^{-1}, & h_6^{-1} h_{12} h_{13}^{-1} h_{15}, & h_2 h_{12}^{-1} h_7 h_{12}^{-1}, & h_2^{-1} h_{14} h_9^{-1} h_{14}, \\
h_7 h_{10}^{-1} h_{14} h_{12}^{-1}, & h_9^{-1} h_{10} h_{12}^{-1} h_{14}, & h_7 h_{12}^{-1} h_{14} h_{15}^{-1}, & h_9^{-1} h_{14} h_{12}^{-1} h_{15}, \\
h_{10} h_8^{-1} h_{11}, & h_{10}^{-1} h_5 h_{11}^{-1}, & h_5 h_{11}^{-1} h_{10} h_{15}^{-1}, & h_8^{-1} h_{11} h_{10}^{-1} h_{15}, \\
h_5 h_{13}^{-1} h_{11} h_{10}^{-1}, & h_8^{-1} h_{13} h_{11}^{-1} h_{10}. & &
\end{array}$$

2. *The inclusion* $\iota : \Gamma_1^2 \to \bar{\Gamma}_1^2$ *is given by* $h_i \mapsto g_1 g_i$.

3. *Identify* $\Gamma_1^2$ *as a subgroup* $\bar{\Gamma}_1^2$ *via* $\iota$. *The conjugation with* $g_1$ *in* $\bar{\Gamma}_1^2$ *acts on* $\Gamma_1^2$ *as follows:*

$$\begin{array}{lllll} h_2 \leftrightarrow h_2^{-1}, & h_3 \leftrightarrow h_6^{-1}, & h_4 \leftrightarrow h_4^{-1}, & h_5 \leftrightarrow h_8^{-1}, & h_7 \leftrightarrow h_9^{-1}, \\ h_{10} \leftrightarrow h_{10}^{-1}, & h_{11} \leftrightarrow h_{11}^{-1}, & h_{12} \leftrightarrow h_{14}^{-1}, & h_{13} \leftrightarrow h_{13}^{-1}, & h_{15} \leftrightarrow h_{15}^{-1}. \end{array}$$

### 3.2.2 Normal forms

Any uniform lattice on a locally finite building is biautomatic by [Świ06, Theorem 6.7] and the main theorem of [OP22]. However, extracting explicit automatic structures from this proof is not immediate. In this subsection, we develop an easy algorithm to compute normal forms for presentations arising from Lemma 60. We will use it in order to prove that $\Gamma_2^1$ contains no proper finite index subgroups and to compute the full automorphism groups of the buildings $X_i^q$. Moreover, the algorithm allows us to perform computations in group algebras of the lattices $\bar{\Gamma}_i^q$ which was used to establish property (T) for $\bar{\Gamma}_1^2$ using Ozawa's method [Oza16] before [Opp25] was available.

Throughout this subsection, let $X$ be a $\tilde{C}_2$-building and $\Gamma$ be a lattice acting freely on $X$ and regularly on vertices of each special type. Let $Y = \Gamma \backslash X$ be the quotient, which is a $\tilde{C}_2$-GAB with two special vertices $v, w$. Let $\bar{\Gamma}$ be an extension of $\Gamma$ that acts regularly on special vertices of $X$. This determines an involutory automorphism $\rho$ on $Y$ interchanging the two special vertices. We apply Lemma 60 to obtain a presentation of $\bar{\Gamma} = \langle G \mid R \rangle$ with relations coming from squares as well as from edges paired by the action of $\rho$. Note that we take into account all squares, not just a sufficient number to present the group. When we say that $g_{i-1}g_i = g'_{i-1}g'_i$ is a relation, we mean of course that $g_i^{-1}g_{i-1}^{-1}g'_{i-1}g'_i$ is a relator taking cyclic permutations and inverses into account. We want to describe normal forms for $\bar{\Gamma}$.

The geometric starting point is:

**Lemma 63.** *Let $X$ be a $\tilde{C}_2$-building and let $v_0$ and $v_1$ be special vertices of $X$. There is a unique path of edges $f_1, \ldots, f_{2k}, g_1, \ldots, g_m$ (possibly $k = 0$ or $m = 0$) from $v_0$ to $v_1$ such that the $f_i$ are short edges, the $g_j$ are long edges, $f_{i-1}$ and $f_i$ as well as $g_{j-1}$ and $g_j$ meet in a vertex in which they form an angle of $\pi$ for $1 < i \leq 2k$ and $1 < j \leq m$, and $f_{2k}$ and $g_1$ meet in a vertex in which they form an angle of $(3\pi/4)$.*

*Proof.* The least convex subcomplex of $X$ containing $v_0$ and $v_1$ is a parallelogram with a $\tilde{C}_2$-tiling (see Figure 3.7): it is the intersection of all apartments containing $v_0$ and $v_1$ and therefore is the combinatorial convex hull of the two in any such apartment. For existence, take the path along the boundary of this convex hull. For uniqueness note that if two vertices are connected by such a path, the path runs along the boundary of the convex hull of the two vertices. □

**Corollary 64.** *Let $X$ be a $\tilde{C}_2$ building and let $v_0$ and $v_1$ be special vertices of $X$. There is a path of long edges $g_1, \ldots, g_\ell$ from $v_0$ to $v_1$ and a $k$ such that the edges $g_{j-1}$ and $g_j$ meet in a vertex in which they form an angle of $(\pi/2)$ for $1 < j \leq k$ and an angle of $\pi$ for $k < j \leq \ell$, and no three consecutive edges lie in the link of a non-special vertex. Every other path of this form is obtained by a finite sequence of replacements of subpaths $g_{2i+1}g_{2i+2}$ by two long edges $g'_{2i+1}g'_{2i+2}$ connecting the same pair of special vertices. For the resulting path, the value $k$ might have changed by 1.*

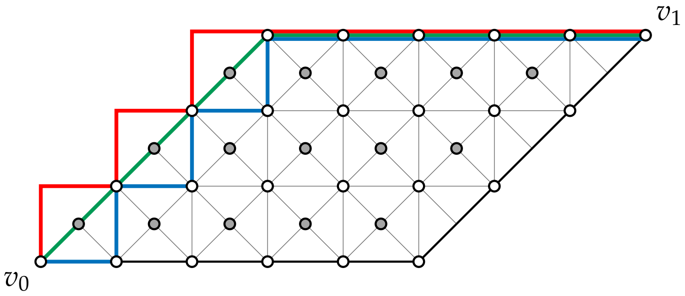

FIGURE 3.7: The combinatorial convex hull of two vertices $v$ and $w$ together with a canonical path of edges connecting them (green) and two semi-canonical paths of long edges connecting them (blue, red). The blue path is distinguished among the long-edge paths but we make no use of this.

*Remark* 65. There is in fact a canonical path even among the ones formed of long edges, namely the one that lies in the convex hull of $v_0$ and $v_1$, which is distinguished by having one more turn than the other ones, see Figure 3.7. However this is less easy to identify by local conditions.

Let $\omega = g_1 \cdots g_k$ be a sequence of oriented edges in $Y$ that start in $v$ and end in $w$. This sequence represents an element in $\bar{\Gamma}$. We say that $g_{i-1}$ and $g_i$ *cancel* if $g_i = g_{i-1}^{-1}$ in $\bar{\Gamma}$, meaning that $\rho(g_i) = g_{i-1}^{-1}$; that they form a *turn* if $g_{i-1}g_i$ is part of a relation; and that they are *straight* otherwise. We say that $g_{i-1}g_ig_{i+1}$ can be *shortened* if $g_{i-1}g_ig_{i+1} = g'$ is a relation.

From Corollary 64 we get:

**Corollary 66.** *Every element of $\bar{\Gamma}$ is represented by a word $\omega = g_1 \ldots g_\ell$ for which there is a $k$ such that $g_{i-1}$ and $g_i$ form a turn for $i \leq k$ and are straight for $i > k$, and no shortening is possible. Every other word of this form is obtained from a given one by applying a sequence of rewritings $g_{2i+1}g_{2i+2} \rightsquigarrow g'_{2i+1}g'_{2i+2}$ where $g_{2i+1}g_{2i+2} = g'_{2i+1}g'_{2i+2}$ is a relation.*

**Definition 67.** We say that a word $\omega \in F(G)$ is *reduced* if its length is minimal among all words representing the same element of $\bar{\Gamma}$. We say that it is in *semi-normal form* if it is of the form described in Corollary 66. We say that it is in *normal form* if it is minimal in lexicographic order among all words in semi-normal form representing the same element of $\bar{\Gamma}$.

**Proposition 68.**

1. *Any word in semi-normal form is reduced. Two words in normal form represent the same element of $\bar{\Gamma}$ if and only if they are equal.*
2. *Any word of length $\ell$ can be brought into semi-normal form by applying $O(\ell^2)$ rewritings.*
3. *Any word in semi-normal form of length $\ell$ can be brought into normal form by applying $O(\ell)$ rewritings.*

*Proof.* Let $v_0$ be a lift of $v$. Every word $\omega$ in $F(G)$ represents an edge path in $X$ along long edges starting in $v_0$ ending in $g.v_0$ where $g \in \bar{\Gamma}$ is the element represented by $\omega$. Note that the graph distance of two special vertices in $X$ is the same whether measured with respect to all edges or with respect to only long edges. If $\gamma$ is an edge path from a special vertex $v_0$ to a special vertex $v_1$, let $\Sigma$ be an apartment containing $v_0$ and $v_1$ and let $\rho_{\Sigma,c} \colon X \to \Sigma$ be the retraction centered at any chamber $c$. Then $\rho_{\Sigma,c}(\gamma)$ is an edge path from $v_0$ to $v_1$ in $\Sigma$. This shows that the minimal edge distance between $v_0$ and $v_1$ is realized inside $\Sigma$. It is easy to see that semi-normal paths are of minimal length among paths in $\Sigma$. Therefore, semi-normal forms are reduced. That words in normal form represent elements uniquely is clear from the definition.

Let $\omega = g_1, \dots, g_\ell$ be a word and without loss of generality assume that no cancellation or shortening in $\omega$ is possible. Let $i$ be the first index such that $g_{i-1}g_i$ is straight, let $a = \ell - i$. If no such $i$ exists, let $a := 0$ for definedness. If there is a $k > i$ such that $g_{k-1}g_k$ is a turn, take the minimal one and let $b = k - i$. If no such $k$ exists, let $b := 0$ for definedness. Note that $b \leq a \leq \ell$ and that if $b = 0$, then $\omega$ is in semi-normal form. We order the set of $(\ell, a, b)$ lexicographically and proceed by induction. There is a relation $g_{k-1}g_k \rightsquigarrow g'_{k-1}g'_k$ such that $g_{k-2}g'_{k-1}$ is a turn and we apply this rewriting at the positions $k-1$ and $k$ to obtain the word $\omega'$. If $k = i+1$ it might happen that $g_{i-2}g_{i-1}g'_{k-1}$ can be shortened to $g'_{i-2}$ and then we apply this shortening. The resulting word is of length $\ell - 2$ and the index indicating the first straight pair of edges is reduced by two. Since the subword $g_1 \dots g_{i-3}g'_{i-2}$ is reduced, further cancellations or shortenings might decrease the index that indicates the first straight pair, but at by most two per operation. In particular, we can apply cancellations or shortenings as long as possible and $a$ never increases. However, $b$ might increase. If $k = i+1$ but we cannot shorten $g_{i-2}g_{i-1}g'_{k-1}$, then $\omega'$ is of length $\ell$ and the index indicating the first straight pair is at least $i+2$. Furthermore, the subword $g_1, \dots g_{i+1}$ is reduced. Applying further cancellations and shortenings as long as possible, might decrease the index indicating the first straight pair below $i+2$ but only if they reduce the word length by at least the same amount. In particular, $\ell$ does not increase, $a$ decreases and $b$ might increase. If $k > i+1$, then $\omega'$ is of length $\ell$, the index indicating the first straight pair is $i$, but the distance to the next turn has decreased, i.e. $b$ has decreased. Now performing cancellations and shortenings as long as possible will not increase $a$ by the same argumentation as before. In particular, $a$ and $\ell$ never increase. If they do not decrease, $b$ decreases. Therefore, repeating this procedure will eventually yield a word in semi-normal form. Since $b \leq a \leq \ell$ the number of applications of relations is quadratic in $\ell$.

If $g_1 \dots g_\ell$ is in semi-normal form, then to obtain normal form we replace the pairs $g_{2i+1}g_{2i+2}$ successively from left to right by $g'_{2i+1}g'_{2i+2}$, choosing $g'_{2i+1}$ minimal among the admissible generators in the given order. This requires only a linear number of replacements, with a constant number of relations available at each step. ☐

We now complete the proof of Theorem 58.

*Proof of Theorem 58, Part 2.* The subcomplex of $Y$ consisting of the triangles $t_1$ to $t_{18}$ (indicated in bold in Figure 3.3) is isomorphic to a subdivision $\dot{S}_R$ of the square complex $S_R$. The assignment indicated in Part 1 of this proof, induces a locally convex embedding of $\dot{S}_R$ into $Y_1^2$ by Proposition 41. Moreover, the induced homomorphism

$$\iota : \pi_1(S_R, x_{00}) = \pi_1(\dot{S}_R, x_{00}) \to \pi_1(Y_1^2, v)$$

is injective. Let $\delta \in \pi_1(S_R, x_{00})$ be as in Lemma 54 such that $\delta^4$ lies in the finite residual of $\pi_1(S_R, x_{00})$. Then $\bar{\delta}^4 := \iota(\delta)^4$ lies in the finite residual of $\pi_1(Y_1^2, v)$. Tracing the maps we find that the element $\bar{\delta}$ is represented by the loop $f_1^{-1} * f_3$ in $\pi_1(Y_1^2, v)$. Using the triangles $t_2$ and $t_{12}$ one sees that this loop is homotopic to $g_1 * g_6^{-1}$, thus in the presentation of Proposition 61 we have $\bar{\delta} = g_1 g_6^{-1}$.

We now show that the normal closure of $\bar{\delta}^4$ in $\Gamma_2^1$ is $\Gamma_2^1$, which is therefore the finite residual. We claim the following verifications are human-verifiable; however, they are long and tedious and the assistance of a computer is advantageous. To be more precise, we write $g_7 g_4$ as a concrete product of elements which lie in the normal closure of $\bar{\delta}^4$. From there it is easy to see that the normal closure equals $\Gamma_2^1$. To perform computations in the presentation we make excessive use of our algorithm to compute normal forms. In summary, it requires 695 elementary operations, such as cancellations, shortenings or rewritings to deduce that $g_7g_4$ lies in the normal closure of $\bar{\delta}^4$.

Let $\nu_0 := \bar{\delta}^4$, let $\gamma_1 := g_2g_{11}$, and define $\nu_1 := \nu_0\gamma_1\nu_0\gamma_1^{-1}$. Computing the following normal form for $\nu_1$ requires 19 elementary operations.

$$\nu_1 := g_1g_3g_4g_{13}g_4g_2g_6g_{11}g_3g_1g_9g_6g_{11}g_2.$$

Let $\gamma_2$ and $\gamma_3$ be defined as below. If we define $\nu_2 := \gamma_2\nu_1\gamma_2^{-1}$, $\nu_3 := \gamma_3\nu_1^{-1}\gamma_3^{-1}$ and $\nu_4 := \nu_2\nu_3$, then one computes their normal forms using 30, 211 and 44 elementary operations, respectively.

$$\begin{aligned}
\gamma_2 &:= g_3g_6g_{12}g_7g_{11}g_2,\\
\gamma_3 &:= g_3g_2g_6g_6g_{14}g_5g_{12}g_6g_{11}g_9g_3g_{14}g_{13}g_2g_{11}g_{15}g_{14}g_8g_{13}g_5g_{12}g_5g_{14}g_4g_3g_1,\\
\nu_2 &:= g_7g_{10}g_{15}g_7g_6g_{13}g_3g_4g_6g_{11}g_3g_{14}g_{13}g_2,\\
\nu_3 &:= g_1g_4g_{14}g_2g_8g_8g_{10}g_{15}g_8g_7g_{12}g_5g_3g_{15},\\
\nu_4 &:= g_7g_6g_{13}g_{15}g_7g_{13}g_9g_5g_4g_{15}.
\end{aligned}$$

Define $\gamma_4, \gamma_5$ as below. The product of the conjugates $\nu_5 := \gamma_4\nu_4\gamma_4^{-1}$, $\nu_6 := \gamma_5\nu_4\gamma_5^{-1}$ is is the short word $g_7g_4 = \nu_5\nu_6$. One computes the normal forms of $\nu_5$ and $\nu_6$ indicated below using 192 and 188 elementary operations. Then computing the normal form of their product requires 11 elementary operations, which sums up to 695 elementary operations in total.

$$\begin{aligned}
\gamma_4 :=&g_1g_{12}g_5g_{15}g_1g_{12}g_5g_{12}g_{11}g_2g_{10}g_7g_2g_{11}g_9g_{14}g_8g_{12}g_8g_9g_{11}g_{15}g_{14}g_4g_2g_6g_7g_1g_{12}g_5\\
&g_{15}g_{11}g_9,\\
\gamma_5 :=&g_1g_3g_5g_{13}g_4g_5g_{11}g_{12}g_{10}g_{11}g_2g_{10}g_7g_2g_{11}g_9g_{14}g_8g_{12}g_8g_9g_{11}g_{15}g_{14}g_4g_2g_6g_7g_1g_{12}\\
&g_5g_{15}g_{11}g_9,\\
\nu_5 :=&g_7g_{14}g_7g_6g_9g_{12}g_{13}g_5g_3g_1,\\
\nu_6 :=&g_1g_{11}g_{12}g_9g_1g_{10}g_{11}g_3g_2g_4.
\end{aligned}$$

Let $Q := \bar{\Gamma}_2^1/\langle\langle g_7g_4\rangle\rangle$ and denote its generators by $g_i$. It easily follows that in $Q$ we have $x := g_1 = g_3 = g_4 = g_6 = g_7 = g_9$ and $y := g_2 = g_5 = g_8$. We observe that $g_{12}$ and $g_{13}$ are conjugated by $x$, so $g_{12}^2 = 1$ and therefore $z := g_{12} = g_{14}$. Using the relation $g_7g_{10}g_{14}g_{14}$ yields $x = g_{10}$. Now the relation $g_1g_{10}g_5g_{11}$ implies $y = g_{11}$. We also have that $x = g_{15}$ since $g_7g_{14}g_{14}g_{15}$ is a relation. Now using the relation $g_3g_{12}g_{13}g_{15}$ yields $z = g_{13}$. Finally $g_3g_{11}g_{12}g_{13}$ yields $x = y$ and $g_5g_{13}g_{11}g_{10}$ yields $x = z$. In particular, $Q$ is the cyclic group of order 2. □

### 3.2.3 The full automorphism group of the building

Implementing the algorithm to compute normal forms in $\bar{\Gamma}_1^2$ on a computer, one can reconstruct finite balls in the Cayley graph of $\bar{\Gamma}_1^2$ and study their automorphism groups. It turns out that the automorphism group of a ball of radius 4 fixes the ball of radius 2 pointwise, and in particular fixes all special vertices adjacent to the center. Since automorphisms of the Cayley graph induce automorphisms of $X_1^2$ and vice versa, it follows by induction that the stabilizer of a special vertex is trivial. From this we conclude:

**Proposition 69.** *The lattice* $\bar{\Gamma}_1^2$ *is the full automorphism group and* $\Gamma_1^2$ *is the type-preserving automorphism group of the building* $X_1^2$*. In particular, the building* $X_1^2$ *is exotic, since its automorphism group is discrete.*

By combining the rigidity of balls in the Cayley graph of $\bar{\Gamma}_1^2$ with its non-residual finiteness, we obtain the following graph-theoretic consequence. If $Z$ is a locally finite, vertex-transitive graph, we call finite graph $Z_0$ a *perfect finite r-local model for Z* if for any vertex $v_0 \in Z_0$ the $r$-ball around $v_0$ is isomorphic to the $r$-ball in $Z$ (around any vertex). Following the proof of [Tho25, Theorem 4] we can show:

**Proposition 70.** *For* $r \geq 5$ *there exists no perfect finite r-local model for the Cayley graph of* $\bar{\Gamma}_1^2$ *with respect to its generating set from Proposition 61.*

*Proof.* Denote the generating set of $\bar{\Gamma}_1^2$ by $S$ and the Cayley graph by $Z$. When considering balls of radius $r$ in $Z$, we will suppress the choice of the center and simply speak of "the $r$-ball". Assume $Z_0$ is a perfect finite 5-local model for $Z$. Let $v_0 \in Z_0$ be a vertex. Any rooted isomorphism from the 4-ball in $Z$ to the 4-ball in $Z_0$ around $v_0$ induces an $S$-labeling on the 4-ball around $v_0$. Since any automorphism of a 4-ball in $Z$ fixes the centered 2-ball, the induced labeling on the 2-ball around $v_0$ does not depend on the chosen isomorphism. In particular, we have a canonical labeling for every 2-ball in $Z_0$.

To see that these labelings are compatible, consider adjacent vertices $v_0, w_0 \in Z_0$. Choose a rooted isomorphism from the 5-ball in $Z$ to the 5-ball in $Z_0$ around $v_0$ and use it to label the 5-ball around $v_0$. In particular, the 4-ball around $w_0$ gets labeled. Since this labeling arises from $Z$ we can find a rooted isomorphism from the 4-ball in $Z$ to the 4-ball around $w_0$ preserving the labels. In particular, the labeling on the 2-ball around $w_0$ induced by the labeling of the 5-ball around $v_0$ is the canonical one.

Therefore, the local labelings are compatible and $Z_0$ admits a canonical $S$-labeling. This defines a vertex-transitive action of the free group over $S$ on $Z_0$. Since any relation of $\Gamma_1^2$ is witnessed in the ball of radius two, the action factors through $\Gamma_1^2$. But its finite residual is of index two, so $Z_0$ would have to be infinite or have at most two vertices, which is impossible. □

### 3.2.4 The complexes $Y_k^3$

The complexes $Y_k^3$ are similar to the complex $Y_1^2$, but they are bigger. Each of them consists of 160 triangles, 120 edges and 12 vertices. We provide an explicit description of these complexes and further details in Appendix A. Given the complexes it is easy to verify that their universal covers $X_k^3$ are $\tilde{C}_2$-buildings of thickness four and that they all contain a subdivision of $S_{\text{JW}}$ as subcomplex. In particular, their fundamental groups $\Gamma_k^3$ are non-residually finite $\tilde{C}_2$-lattices by Proposition 41 and Proposition 57. Each of the four complexes admits an involutory automorphism that swaps its two special vertices, which allows us to apply Lemma 60 and obtain a geometric presentation for the extension $\bar{\Gamma}_k^3$. Recall that Proposition 57 provides explicit elements in the finite residual of $\pi_1(S_{\text{JW}})$, and as in the proof of Theorem 58, we compute the images of these explicit elements in $\bar{\Gamma}_k^3$. With the help of a computer, we verify that their normal closure is of finite index in $\bar{\Gamma}_k^3$, and therefore this normal closure equals the finite residual of $\Gamma_k^3$ and $\bar{\Gamma}_k^3$. To be more precise, if $\check{\Gamma}_k^3$ denotes the finite residual then $[\Gamma_k^3 : \check{\Gamma}_k^3] = 4$ if $k = 1, 2$ and $[\Gamma_k^3 : \check{\Gamma}_k^3] = 8$ if $k = 3, 4$ (we indicate the index in $\Gamma_k^3$ for consistency with the main theorem A). As for $X_1^2$ we reconstruct finite balls in the Cayley complex for each case to compute the full automorphism group $\hat{\Gamma}_k^3$ of the buildings $X_k^3$. It turns out that we always have $[\hat{\Gamma}_k^3 : \Gamma_k^3] = 8$. In particular, the automorphism group is discrete and $\check{\Gamma}_k^3 \backslash X_k^3$ is the maximal finite quotient of $X_k^3$ and therefore an invariant of the building. One can compute these quotients which are of course finite covers of the complexes $Y_k^3$, and as it turns out they are all different. In particular, the buildings $X_k^3$ are pairwise not isomorphic.

### 3.2.5 Quasi-isometric rigidity

We recall quasi-isometric rigidity of Euclidean buildings in order to draw conclusions for lattices. The following is [KL97, Theorem 1.1.3] where it is asserted that the Moufang condition can be removed using [Lee00]. Alternatively, one may combine [KW14, Theorem III] with the fact that the distance between a quasi-isometry of buildings and an isometry is bounded in terms of the quasi-isometry constants [KKL98, Lemma 2.4].

**Theorem 71.** *Let $X$ and $Y$ be thick, irreducible Euclidean buildings of dimension $\geq 2$ and let $f\colon X \to Y$ be an $(A, B)$-quasi isometry. Then there is a unique isomorphism $\bar{f}\colon X \to Y$ at distance $\leq D$ from $f$, where $D$ depends only on $A$ and $B$.*

We immediately conclude.

**Corollary 72.** *The lattices $\Gamma_1^2, \Gamma_1^3, \dots, \Gamma_4^3$ are pairwise not quasi-isometric.*

*Proof.* As a uniform lattice $\Gamma_i^q$ is quasi-isometric to $X_i^q$ by the Svarc–Milnor lemma 44. Theorem 71 translates quasi-isometries of the $\Gamma_i^q$ into isomorphisms of the $X_i^q$. As we have seen before the buildings $X_1^2, X_1^3, \dots, X_4^3$ are pairwise non-isomorphic. □

Another consequence of Theorem 71 is quasi-isometric rigidity for lattices on buildings with a discrete automorphism group.

**Proposition 73.** *Let $X$ be a thick, irreducible 2-dimensional Euclidean building and let $\Gamma < \operatorname{Aut}(X)$ be a uniform lattice on $X$. Let $\Lambda$ be a finitely generated group quasi-isometric to $\Gamma$. Then there exists a homomorphism $\Lambda \to \operatorname{Aut}(X)$ with finite kernel $N$ such that $\Lambda/N$ is a uniform lattice on $X$. In particular, if $\operatorname{Aut}(X)$ is discrete, then there is a finite index subgroup $\Lambda' < \Lambda$ and a homomorphism $\Lambda' \to \Gamma$ with finite kernel and finite index image.*

*Proof.* Let $X$, $\Gamma$ and $\Lambda$ be as in the statement and pick a basepoint $o \in X$. The orbit map $\tau\colon \Gamma \to X, g \mapsto g.o$ is a quasi-isometry by the Svarc–Milnor lemma 44. By assumption there is also a quasi-isometry $\iota\colon \Lambda \to \Gamma$, and without loss of generality assume that $\iota(1) = \mathrm{id}_X$. So $\tau \circ \iota\colon \Lambda \to X$, $g \mapsto \iota(g).o$ is a quasi-isometry. By slightly perturbing $\tau \circ \iota$, we obtain an injective quasi-isometry $\kappa\colon \Lambda \to X$ with $\kappa(1) = o$ and such that $\kappa(g)$ and $\iota(g).o$ have distance at most $\epsilon$ for some small fixed $\epsilon$. Let $\bar{\kappa}\colon X \to \Lambda$ be a quasi-inverse satisfying $\bar{\kappa} \circ \kappa = \mathrm{id}$. For $g \in \Lambda$ let $\lambda_g\colon \Lambda \to \Lambda$ be left-multiplication, which is an isometry. Then $\tilde{\mu}_g = \kappa \circ \lambda_g \circ \bar{\kappa}$ is a quasi-isometry, and the constants are uniform in $g$. For every $g \in \Lambda$ we have that $\tilde{\mu}_g.o = \kappa(g)$ which is $\epsilon$-close to $\iota(g).o$. In particular, $g \mapsto \tilde{\mu}_g.o$ is a quasi-isometry. Furthermore, $g \mapsto \tilde{\mu}_g$ is a quasi-action, indeed $\tilde{\mu}_1$ has bounded distance from the identity and $\tilde{\mu}_g \circ \tilde{\mu}_h = \tilde{\mu}_{gh}$. By Theorem 71 there is a unique automorphism $\mu_g$ at bounded distance from $\tilde{\mu}$. Moreover, $\mu_g \circ \mu_h$ and $\mu_{gh}$ are at bounded distance so they are equal and therefore $\Lambda \to \operatorname{Aut}(X)$, $g \mapsto \mu_g$ is a homomorphism. Since the quasi-isometry constants of $\tilde{\mu}_g$ are uniform, the distance from $\tilde{\mu}_g(o)$ to $\mu_g(o)$ is uniformly bounded, and $\Lambda \to X, g \mapsto \mu_g.o$ is a quasi-isometry. Applying (the converse of) the Svarc–Milnor lemma 44 shows the claim.

If $\operatorname{Aut}(X)$ is discrete, $\Gamma$ has finite index in it. Let $\Lambda' = \mu^{-1}(\Gamma)$. Then the restriction of $\mu$ to $\Lambda'$ is the desired homomorphism. □

Applying this to our lattices without proper finite index subgroups we get:

**Corollary 74.** *Let* $\Gamma$ *be one of* $\check{\Gamma}^2_1, \check{\Gamma}^3_1, \dots, \check{\Gamma}^3_4$. *If* $\Lambda$ *is quasi-isometric to* $\Gamma$*, then there is a finite index subgroup* $\Lambda' \leq \Lambda$ *and a finite normal subgroup* $N \leq \Lambda'$ *and such that* $\Lambda'/N \cong \Gamma$.

### 3.2.6 Property (T) using Ozawa's method

Uniform lattices on thick, irreducible Euclidean buildings enjoy Kazhdan's property (T) by [Opp25]. This is particularly relevant as it represents half of the proof of the normal subgroup property, the other half being amenability of proper quotients. Previously, the best known bound [Opp15] for lattices on $\tilde{C}_2$-buildings of thickness $q+1$ to have (T) was $q \geq 4$, which does not apply to our lattices.

We therefore established property (T) for $\bar{\Gamma}^2_1$ using Ozawa's method [Oza16]. Now, besides showing that Ozawa's method can be applied, the proof has the extra merit of providing quantitative information. Namely, $\bar{\Gamma}^2_1$ with respect to the generating set from Proposition 61 has Kazhdan radius at most 2 and Kazhdan constant at least 0.4147.

We mimic the strategy in [NT15]. In what follows $\Gamma$ denotes a finitely generated group and we fix a finite, symmetric generating set $S$ that does not contain 1. Several of the upcoming definitions depend on $S$ which is not reflected in our notation. We denote by $\mathbb{R}\,\Gamma$ the real group algebra of $\Gamma$, equipped with the anti-automorphism $*$ that takes $\gamma$ to $\gamma^{-1}$. Its fixed elements are called *hermitian*. The (unnormalized) *Laplacian* is

$$\Delta = |S| - \sum_{s \in S} s \in \mathbb{R}\,\Gamma.$$

The *augmentation ideal* $I\Gamma$ is the kernel of $\chi\colon \mathbb{R}\,\Gamma \to \mathbb{R}, \gamma \mapsto 1$. The *support* of $x \in \mathbb{R}\,\Gamma$ is the subset of $\Gamma$ with non-zero coefficient. The *square sum* of $q_1, \dots, q_n \in \mathbb{R}\,\Gamma$ is $x = \sum_i q_i^* q_i$ and this is a *sum of squares decomposition* of $x$. It is *supported* on a set $B \subseteq \Gamma$ if all $q_i$ have support in $B$.

Note that $\Delta \in I\Gamma$, and if $x$ is a sum of squares as above, then

$$\chi(x) = \sum_i \chi(q_i^*)\chi(q_i) = \sum_i \chi(q_i)^2,$$

so $x \in I\Gamma$ if and only if $q_i \in I\Gamma$ for all $i$.

**Theorem 75** (Ozawa, [Oza16])**.** *The following are equivalent.*

1. $\Gamma$ *has property (T).*
2. *There is an* $\epsilon > 0$ *such that* $\Delta^2 - \epsilon\Delta$ *admits a sum of squares decomposition in* $\mathbb{R}\,\Gamma$.

The *Kazhdan radius* is the infimal $R$ such that there is an $\epsilon > 0$ such that $\Delta^2 - \epsilon\Delta$ admits a sum of squares decomposition supported on the ball of radius $R$.

We now discuss how the existence of a sum of squares decomposition can be approached via semidefinite programming. Let $b_1, \dots, b_m \in \mathbb{R}\,\Gamma$ be linearly independent and let $c_1, \dots, c_n \in \langle b_i \rangle_{\mathbb{R}}$ be written as $c_i = \sum_j q_{ij} b_j$ with $Q = (q_{ij})_{ij} \in \mathbb{R}^{n \times m}$. If $x \in \mathbb{R}\,\Gamma$ is the square sum

$$x = \sum_i c_i^* c_i, \tag{3.1}$$

then

$$x = \sum_j b_j^* p_{jk} b_k, \tag{3.2}$$

where $P = (p_{jk})_{jk} \in \mathbb{R}^{m \times m}$ is the symmetric positive semidefinite matrix $P = Q^T Q$.

On the other hand, if $P \in \mathbb{R}^{m \times m}$ is a symmetric positive semidefinite matrix satisfying (3.2), writing $P = Q^T Q$ and $c_i = \sum_j q_{ij} b_j$ produces a sum of squares decomposition as in (3.1). We call such a matrix $P$ a *Gram matrix* for $x$ with respect to $(b_1, \dots, b_m)$.

Solving (3.2) is a semidefinite programming problem, for which solvers exist. The solution will usually involve some numerical error, so it is important to take this into account, and the following lemma does so.

**Lemma 76** ([KKN21, Lemma 4.10]). *Assume that $x \in \mathbb{R}\Gamma$ belongs to the augmentation ideal, admits a sum of squares decomposition and is supported on the ball of radius $R$. Let $\epsilon > 0$ and let $\nu := ||\Delta^2 - \epsilon\Delta - x||_1$ be the 1-norm of this difference. Let $C = 2^{2\lceil \log_2 R \rceil}$. Then $\Delta^2 - (\epsilon - \omega)\Delta$ admits a sum of squares decomposition supported in the ball of radius $R$ for every $\omega \geq C\nu$.*

In order to compute a sum of squares decomposition, we proceeded as follows. Using the normal forms from Subsection 3.2.2, we computed all 12526 elements in the ball of radius 4 in $\bar{\Gamma}_1^2$ and the multiplication table on the 166 elements in ball of radius 2. Denote these by $\gamma_1, \dots, \gamma_{166}$. We used the Python-package Cvxpy [DB16] and the solver Mosek [ApS24] to obtain an approximation to a Gram matrix $P$ for $\Delta^2 - \epsilon\Delta$ with respect to $(\gamma_1, \dots, \gamma_{166})$ and $\epsilon = 1.29$. We computed a Cholesky decomposition $Q^T Q = P$. In order to do exact calculations, we approximated $Q$ by $10^{-12}Q'$ where $Q'$ is an integer matrix. Since $\Delta^2 - \epsilon\Delta$ lies in the augmentation ideal, we know that the elements represented by rows of $Q$ lie close to it, meaning that they sum to nearly zero. We adjusted $Q'$ slightly to obtain $Q''$ in which the sum of each row is zero. We computed the sum of squares $x$, indicated below, which also lies in the augmentation ideal.

$$q_i = \sum_{j=1}^{166} Q''_{ij}\gamma_j \qquad \text{and} \qquad x = \sum_j q_j^* q_j$$

and the norm

$$||\Delta^2 - \epsilon\Delta - 10^{-24}x||_1 = 7589138977410503812 \cdot 10^{-24} \approx 7.59 \cdot 10^{-6}.$$

Applying Lemma 76 we obtain:

**Proposition 77.** *The lattice $\bar{\Gamma}_1^2$ has property (T) with Kazhdan radius at most 2 with respect to its generating set in Proposition 61. If $\Delta \in \mathbb{R}\bar{\Gamma}_1^2$ is the Laplacian, then we have that $(\Delta^2 - 1.2899\Delta)$ admits a sum of squares decomposition. The Kazhdan constant of $\bar{\Gamma}_1^2$ with respect to $S$ is bounded below by*

$$0.4147 < \sqrt{1.2899 \cdot \frac{2}{15}}.$$

*Proof.* The condition is that $\omega \geq 4 \cdot 7589138977410503812 \cdot 10^{-24} \approx 3.036 \cdot 10^{-5}$. Choosing $\omega = 0.0001$ gives the sum of squares decomposition. The Kazhdan constant follows from [BHV08, Remark 5.4.7], taking into account that the Laplace operator there is normalized by $(1/|S|)$. ☐

## 3.3 Searching for non-positively curved chamber complexes

In this section, we describe the algorithm that was used to produce the examples in Section 3.2. It can be used more generally to search for finite, non-positively curved chamber complexes whose chambers are triangles. For convenience we just write *f.n.p.c. triangle complex* for such a complex. Recall that applying Theorem 39.1 to a f.n.p.c. triangle complex provides a torsion-free uniform lattice acting freely on a proper CAT(0)-space. The algorithm is a variation of the search algorithm that was used in [Rad17] to find a triangle presentation for the Hughes plane.

**Definition 78.** Let $\Sigma$ be a connected, finite simplicial graph. Assume it is equipped with a type function that assigns to a vertex one of three types in $I$, and maps adjacent vertices to different types. For two types $i, j \in I$, we call the subgraph generated by the vertices of types $i$ and $j$ the *local geometry* of type $\{i, j\}$; we denote it with $\Sigma_{ij}$. The *angle* of the local geometry $\Sigma_{ij}$ is $\theta_{ij} = (2\pi/g_{ij})$, where $g_{ij}$ is the girth of $\Sigma_{ij}$. We call $\Sigma$ a *Radu graph* if

(a) every local geometry has the same number of edges,

(b) every vertex in a local geometry has valency at least 2,

(c) $\theta_{12} + \theta_{23} + \theta_{31} \leq \pi$.

Given a Radu graph, we extend the type function to edges by assigning to each edge the type $\{i, j\}$, where $i$ and $j$ are the types of its vertices.

For a three-element typeset $I = \{i, j, k\}$ we introduce the complementary type $\overline{\cdot}$ which assigns to $\{i, j\}$ the type $\overline{\{i, j\}} := k$.

**Lemma 79.** *Let $Y$ be a f.n.p.c. triangle complex with typeset $I$. Let $\Sigma$ be the graph defined as follows. The vertex set of $\Sigma$ is the set of edges of $Y$. We connect two vertices in $\Sigma$ if the corresponding edges are incident with a common triangle. The type function is given by the complementary type in $I$. Then $\Sigma$ is a Radu graph. Furthermore, a local geometry $\Sigma$ is the union of the vertex links in $Y$ of a given type.*

*Proof.* It is clear that the type function on $\Sigma$ is indeed a type function. As for a Radu graph, we call the subgraphs of $\Sigma$ generated by two types of vertices local geometries. These are finite bipartite graphs. Since every edge in $Y$ is contained in at least two triangles, the valency of a vertex in a local geometry of $\Sigma$ is at least 2. In particular, a local geometry contains a non-trivial cycle and has finite girth. Now let $i$ be a vertex type of $Y$. Then there are two edge types of $Y$ that contain $i$. These two edge types of $Y$ now correspond to two vertex types of $\Sigma$ via the complementary type. By construction, the local geometry in $\Sigma$ generated by vertices of these two types is isomorphic to the union of the vertex links of vertices of type $i$ in $Y$. By non-positive curvature the links must not contain double edges. In particular, $\Sigma$ is simplicial. The connectedness of $\Sigma$ follows from the fact that any two triangles in $Y$ can be connected by a sequence of triangles that share an edge. Every triangle in $Y$ contributes exactly one edge to a vertex link of each vertex type of $Y$. In particular, the number of edges in a local geometry is the number of triangles of $Y$. Recall that in a non-positively curved triangle, the angle sum is at most $\pi$. With Theorem 39.1 we deduce that the angle sum of $\Sigma$ is at most $\pi$. $\square$

**Definition 80.** Let $\Sigma$ be a Radu graph. A *triangle* in $\Sigma$ is a circle of length 3 in $\Sigma$. Note that the vertices in a triangle are necessarily of three different types. A *partial triangle cover* is a set of triangles such that every edge of $\Sigma$ is contained in at most one triangle of the family. It is an *exact triangle cover* if every edge in $\Sigma$ is contained in exactly one triangle. A Radu graph is *perfect* if it admits an exact triangle cover.

**Proposition 81.** *A Radu graph is perfect if and only if it is the Radu graph of a f.n.p.c. triangle complex.*

*Proof.* Let $Y$ be a f.n.p.c triangle complex and let $\Sigma$ be the Radu graph associated to it. We obtain a family of triangles in $\Sigma$ as follows. For every triangle in $Y$ its boundary consists of three edges. These three edges in $Y$ correspond to three vertices in $\Sigma$ that form a triangle. Since edges in $\Sigma$ arise from boundaries of triangles, this triangle family is an exact cover.

Now assume that we have a perfect Radu graph $\Sigma$. We will construct a f.n.p.c. triangle complex $Y$ out of it as follows. Let $\Sigma_{ij}$ be the local geometry in $\Sigma$ generated by vertices of type $i$ and $j$. We call a connected component of $\Sigma_{ij}$ a link of type $\{i, j\}$. For every link of type $\{i, j\}$, we have a vertex in $Y$ of type $\overline{\{i, j\}}$. For every vertex $e$ in $\Sigma$ of type $i$ we now glue in an edge in $Y$ between the pair of vertices in $Y$, whose corresponding links in $\Sigma$ contain $e$. Note that the two types of these vertices in $Y$ are the two types not equal to $i$. In particular, the 1-skeleton of $Y$ does not contain any loops. Let $T$ be an exact triangle cover for $\Sigma$. For a triangle $t$ in $T$, we glue in a triangle along the edges in $Y$ corresponding to three vertices in $t$. We now verify the defining properties a of chamber complex starting with the combinatorial properties. By construction, the vertex links of $Y$ correspond to the links in $\Sigma$. Indeed, for a vertex $u$ in $Y$, two edges incident with $u$ share a triangle if and only if the corresponding vertices $e, f$ in $\Sigma$ are contained in a triangle. Since $T$ is an exact cover, this happens exactly when $e$ and $f$ are connected in $\Sigma$. We deduce that every edge is contained in at least two triangles, since every vertex in a vertex link has valency at least 2. Moreover, since $\Gamma$ is connected, any two triangles can be connected by a chain of triangles with consecutive triangles sharing an edge. Finally we equip the triangles of $Y$ with the metric of a non-positively curved triangle with angles according to the angles of the local geometries of $\Sigma$. By Theorem 39.1 the metric on $Y$ is non-positively curved. □

*Remark* 82. Many of the perfect Radu graphs we encountered during our experiments admit a unique triangle cover. In general a perfect Radu graph may admit more than one perfect triangle cover and if it does the associated triangle complexes may be non-isomorphic. Figure 3.8 shows an example of a Radu graph that admits two triangle covers. For this example the two corresponding complexes are isomorphic. An example of a Radu graph arising from triangle complexes that are not isomorphic appears in [Lou24].

### 3.3.1 Acting on Radu graphs

In this subsection, we define an action on the space of Radu graphs with fixed vertex set and type function, which will later be used to construct perfect Radu graphs.

Let $V$ be a set of vertices with a type function TYP onto a three-element set and for $i \in I$ let $V_i$ be the set of vertices of type $i$. Let $G = \prod_i \mathrm{Sym}(V_i)$ be the group of type-preserving permutations of $V$. We will implicitly identify elements in $\mathrm{Sym}(V_i)$ with elements in $G$; in particular, we may omit writing elements as explicit triples.

Let $X$ be the set of Radu graphs with vertex set $V$ and type function TYP. We say that $\Sigma, \Sigma' \in X$ are *isomorphic*, if there exists a type-preserving isomorphism between

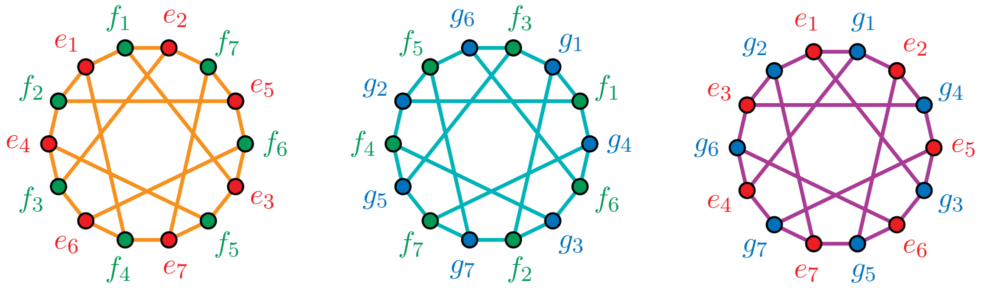

FIGURE 3.8: A perfect Radu graph admitting two exact covers.

them; we say that $\Sigma, \Sigma'$ are *locally isomorphic* if there exist type-preserving isomorphisms $\phi_{i,j} \colon \Sigma_{ij} \to \Sigma'_{ij}$ between the three local geometries. We will often consider isomorphisms between local geometries as elements in $\mathrm{Sym}(V_i) \times \mathrm{Sym}(V_j)$.

We now define an action of $G$ on $X$. We fix a cyclic ordering on $I$, represented by a cycle $\rho \in \mathrm{Sym}(I)$. Let $\Sigma \in X$, let $i \in I$ and let $g \in \mathrm{Sym}(V_i)$. Let $j := \rho(i)$ and $k := \rho^2(i)$. We define $g.\Sigma$ as the Radu graph in $X$ with the same local geometries of type $\{j,k\}$ and $\{k,i\}$ as $\Sigma$, and with the following edges in the local geometry of type $\{i,j\}$: let $e \in V_i$ and $f \in V_j$ then $\{g(e), f\}$ is an edge in $g.\Sigma$ if and only if $\{e, f\}$ is an edge in $\Sigma$. This induces a type-preserving action on graphs with vertex set $V$, which we denote by $(g, \Sigma) \mapsto g.\Sigma$.

Note that isomorphisms between Radu graphs in $X$ are also naturally encoded by elements in $G$. We will be explicit throughout to avoid any ambiguity.

**Proposition 83.** *The group $G$ acts on $X$. If two Radu graphs in $X$ lie in the same orbit, then they are locally isomorphic. If two Radu graphs $\Sigma, \Sigma'$ in $X$ are locally isomorphic, then an isomorphic copy of $\Sigma'$ lies in the orbit of $\Sigma$.*

*Proof.* It is clear that $G$ acts on $X$ and it is clear that two Radu graphs in the same orbit are locally isomorphic. On the other hand let $\Sigma, \Sigma'$ be locally isomorphic Radu graphs. For $i \in I$ there is an isomorphism

$$\left(\phi_i^{i\rho(i)}, \phi_{\rho(i)}^{i\rho(i)}\right) \in \mathrm{Sym}(V_i) \times \mathrm{Sym}(V_{\rho(i)})$$

between the two local geometries $\Sigma_{i\rho(i)}$ and $(\Sigma')_{i\rho(i)}$. Define $g$ as below. We claim that the permutation $h$, defined below, induces a graph isomorphism from $g.\Sigma$ to $\Sigma'$.

$$g := \left(\left(\phi_i^{\rho^2(i)i}\right)^{-1} \phi_i^{i\rho(i)}\right)_i, \qquad h := \left(\phi_i^{\rho^2(i)i}\right)_i.$$

Indeed, let $i$ be a type, let $j := \rho(i)$ and $k := \rho^2(i)$. Let $e \in V_i$ and $f \in V_j$. We have the following equivalences.

$$\left\{\left(\phi_i^{ki}\right)^{-1} \phi_i^{ij}(e), f\right\} \text{ is an edge in } (g.\Sigma)_{ij} \tag{3.3}$$

$$\iff \{e, f\} \text{ is an edge in } \Sigma_{ij}$$

$$\iff \left\{\phi_i^{ij}(e), \phi_j^{ij}(f)\right\} \text{ is an edge in } \Sigma'_{ij}. \tag{3.4}$$

The permutation $h$ maps the edges in (3.3) to the edges in (3.4) and therefore induces a well-defined graph isomorphism from $g.\Sigma$ to $\Sigma'$. □

Thus looking for a triangle complex with prescribed local geometries $\Sigma_{ij}$ amounts to picking an arbitrary Radu graph $\Sigma$ with these local geometries and looking for a perfect one within the orbit $G.\Sigma$.

### 3.3.2 The score of a Radu graph

Obviously $G$ is too large to do an exhaustive search. Rather we do a greedy search using the score function

$$\operatorname{score}(\Sigma) := \max\left\{3 \cdot |T| \mid T \text{ a partial triangle cover of } \Sigma\right\}.$$

Note that a Radu graph is perfect if and only if its score equals its number of edges.

An elementary but crucial observation is that small variations in $G$ (with respect to the generating set of transpositions) lead to small variations in score:

**Lemma 84.** *Let $\Sigma$ be a Radu graph on $X$. Let $\lambda_i$ be a transposition in $\operatorname{Sym}(V_i)$. Let $d_i$ be the maximal degree of a vertex of type $i$ in $\Sigma$. Then we have*

$$|\operatorname{score}(\Sigma) - \operatorname{score}(\lambda_i.\Sigma)| \leq 3d_i.$$

*Proof.* Let $e, e'$ be the two vertices swapped by the transposition. Let $T$ be a partial triangle cover of $\Sigma$. At most $(d_i/2)$ triangles contain $e$, and at most $(d_i/2)$ triangles contain $e'$. Removing these triangles from $T$ yields a partial triangle cover for $\lambda_i.\Sigma$. The lemma follows from the symmetry of the argument. □

Starting with a Radu graph $\Sigma$, we are looking for a f.n.p.c. triangle complex $Y$ whose Radu graph $\Sigma_Y$ is locally isomorphic to $\Sigma$. We employ the following strategy: We compute the score of $\Sigma$ and all Radu graphs obtained by acting with transpositions. Out of these we take the one(s) with the highest score and repeat the procedure. To avoid loops, we keep record of the Radu graphs we already checked. This search leads us to local maxima in the $G$-orbit of $\Sigma$. To avoid circulating around local maxima that are not perfect Radu graphs, we restart the search at a different Radu graph after a fixed number of steps.

In fact, the score is expensive to compute and generally not worth knowing exactly. Rather we follow [Rad17] in computing an estimate of the score in a greedy fashion, that has the important property that it equals the score for perfect Radu graphs. It is computed by first looking for an edge that is contained in a unique triangle (which therefore has to cover it in an exact cover) and adding that triangle to the cover. This is done as long as there are such edges. Finally, the exact maximal number of triangles covering the remaining edges is computed. There may be larger partial covers not using all of the initial triangles but if an exact cover exists, it has to use them.

The above search strategy using the score estimate has been implemented in GAP [22] and succeeded in producing the examples in Section 3.2. The examples in Chapter 4 were found using the estimated score function.

### 3.3.3 Stabilizers of isomorphism types

Certain elements $g \in G$ act on a Radu graph while preserving its isomorphism class. When classifying Radu graphs, it is therefore natural to quotient by the stabilizers of these isomorphism classes. This is the purpose of the following lemma.

**Lemma 85.** *Let $\Sigma$ be a Radu graph in $X$ and for each type $i \in I$ let*

$$\left(\phi_i^{i\rho(i)}, \phi_{\rho(i)}^{i\rho(i)}\right) \in \mathrm{Sym}(V_i) \times \mathrm{Sym}(V_{\rho(i)})$$

*be an automorphism of the local geometry of type $\{i, \rho(i)\}$. Let $g = (g_i)_i \in G$ and define $h \in G$ as below. Then $h.\Sigma$ and $g.\Sigma$ are isomorphic Radu graphs via the permutation $k$ defined below.*

$$h := \left(\left(\phi_i^{\rho^2(i)i}\right)^{-1} \circ g_i \circ \phi_i^{i\rho(i)}\right)_i, \qquad k := \left(\phi_i^{\rho^2(i)i}\right)_i.$$

*Proof.* Let $i$ be a type, let $j := \rho(i)$ and $k := \rho^2(i)$. Let $e \in V_i$ and $f \in V_j$. We have:

$$\{e, f\} \text{ is an edge in } \Sigma_{ij} \iff \left\{\phi_i^{ij}(e), \phi_j^{ij}(f)\right\} \text{ is an edge in } \Sigma_{ij}.$$

We use the above equivalence to characterize edges in $(g.\Sigma)_{ij}$.

$$\begin{aligned} &\left\{\left(\phi_i^{ki}\right)^{-1} \circ g_i \circ \phi_i^{ij}(e), f\right\} \text{ is an edge in } (h.\Sigma)_{ij} && (3.5)\\ \iff &\{e, f\} \text{ is an edge in } \Sigma_{ij} \\ \iff &\left\{g_i \circ \phi_i^{ij}(e), \phi_j^{ij}(f)\right\} \text{ is an edge in } (g.\Sigma)_{ij}. && (3.6)\end{aligned}$$

The permutation $k$ maps the edges in (3.5) to the edges in (3.6) and therefore induces a well defined graph isomorphism from $h.\Sigma$ to $g.\Sigma'$. □

# Chapter 4

# $\tilde{A}_2$-lattices in thickness three

*The results of this chapter are based on a joint work with Stefan Witzel [TW25]. The text has been revised and refined for consistency. All results in this chapter are due to the author of this thesis or were obtained with significant contributions by the author.*

To find the lattices in Section 3.2 we employed the strategy described in Section 3.3 on a search space that was too big to exhaust without using a score function as a guide. In this section we provide the results of an exhaustive search of certain $\tilde{A}_2$-lattices. Before we embark on the actual discussion of these, we briefly discuss important classes of previously known $\tilde{A}_2$-lattices.

## 4.1 Chamber-, vertex- and panel-regular lattices

Lattices on $\tilde{A}_2$-buildings that act regularly on chambers were first studied by Ronan and Tits; Timmesfeld classifies chamber-transitive lattices on $\tilde{A}_2$-buildings with Desarguesian vertex links, see [Ron84; Tit85; Tim89]. There exist four chamber-regular $\tilde{A}_2$-lattices in thickness three and 44 in thickness nine. The examples in thickness three are given by the following presentations. In each presentation, the cyclic groups generated by $a$, $b$ and $c$ are edge stabilizers, and the groups generated by two of them are vertex stabilizers isomorphic to the Frobenius groups of order 21. The building can be recovered from the group as a coset geometry. The lattice $\Omega_3$ is an arithmetic subgroup of $\mathrm{PGL}_3(\mathbb{F}_2((t)))$ [KMW84] and $\Omega_1$ is an arithmetic subgroup of $\mathrm{PGL}_3(\mathbb{Q}_2)$ [KMW85, Theorem 8]. The lattices $\Omega_2$ and $\Omega_4$ act on distinct exotic buildings.

$$\begin{aligned}
\Omega_1 &= \langle a,b,c \mid a^3 = b^3 = c^3 = 1, (ab)^2 = ba, (bc)^2 = cb, (ca)^2 = ac \rangle, \\
\Omega_2 &= \langle a,b,c \mid a^3 = b^3 = c^3 = 1, (ab)^2 = ba, (bc)^2 = cb, (ac)^2 = ca \rangle, \\
\Omega_3 &= \langle a,b,c \mid a^3 = b^3 = c^3 = 1, (ab)^2 = ba, (c^2b)^2 = bc^2, (ac)^2 = ca \rangle, \\
\Omega_4 &= \langle a,b,c \mid a^3 = b^3 = c^3 = 1, (ab)^2 = ba, (bc^2)^2 = c^2b, (ac)^2 = ca \rangle.
\end{aligned}$$

Lattices that act type-rotatingly and regularly on all vertices were studied by Cartwright, Mantero, Steger, and Zappa [Car+93]. The examples in thickness three are the groups given by the following presentations. In each presentation, the generators take some base vertex to its neighbors as do their inverses, and thus correspond to edges. The relations are products of generators that take the base vertex back to itself and correspond to triangles. The building is the Cayley complex of the presentation. The groups $\Gamma_{A.i}$ are arithmetic lattices in $\mathrm{PGL}_3(\mathbb{F}_2((t)))$, whereas the groups $\Lambda_{B.i}$ and $\Lambda_{C.i}$ are arithmetic lattices in $\mathrm{PGL}_3(\mathbb{Q}_2)$.

$$\begin{aligned}
\Lambda_{A.1} &:= \langle s_1, \dots, s_7 \mid s_1s_2s_4,\ s_2s_3s_5,\ s_3s_4s_6,\ s_4s_5s_7,\ s_5s_6s_1,\ s_6s_7s_2,\ s_7s_1s_3 \rangle, \\
\Lambda_{A.2} &:= \langle s_1, \dots, s_7 \mid s_1s_4s_2,\ s_3s_3s_1,\ s_5s_5s_4,\ s_6s_6s_2,\ s_7s_1s_6,\ s_7s_2s_5,\ s_7s_4s_3 \rangle, \\
\Lambda_{A.3} &:= \langle s_1, \dots, s_7 \mid s_1s_4s_3,\ s_2s_1s_6,\ s_3s_3s_3,\ s_4s_2s_5,\ s_5s_5s_5,\ s_6s_6s_6,\ s_7s_1s_3,\ s_7s_2s_6,\ s_7s_4s_5 \rangle, \\
\Lambda_{A.4} &:= \langle s_1, \dots, s_7 \mid s_1s_1s_1,\ s_1s_3s_2,\ s_2s_5s_4,\ s_3s_3s_5,\ s_4s_4s_4,\ s_6s_6s_2,\ s_7s_4s_1,\ s_7s_5s_6,\ s_7s_7s_7 \rangle, \\
\Lambda_{B.1} &:= \langle s_1, \dots, s_7 \mid s_1s_6s_3,\ s_2s_5s_6,\ s_3s_6s_5,\ s_4s_3s_5,\ s_7s_1s_2,\ s_7s_2s_4,\ s_7s_4s_1 \rangle, \\
\Lambda_{B.2} &:= \langle s_1, \dots, s_7 \mid s_3s_3s_2,\ s_3s_5s_6,\ s_6s_6s_4,\ s_5s_5s_1,\ s_7s_1s_4,\ s_7s_2s_1,\ s_7s_4s_2 \rangle, \\
\Lambda_{B.3} &:= \langle s_1, \dots, s_7 \mid s_1s_1s_7,\ s_1s_3s_6,\ s_2s_2s_7,\ s_2s_6s_5,\ s_3s_3s_3,\ s_4s_4s_7,\ s_4s_5s_3,\ s_5s_5s_5,\ s_6s_6s_6 \rangle, \\
\Lambda_{C.1} &:= \langle s_1, \dots, s_7 \mid s_1s_2s_6,\ s_1s_3s_5,\ s_1s_5s_4,\ s_2s_4s_5,\ s_3s_4s_6,\ s_7s_7s_6,\ s_7s_2s_3 \rangle.
\end{aligned}$$

Lattices that preserve types and act regularly on each type of panel were described by Essert and further studied by Witzel [Ess13; Wit17]. The examples in thickness three are given by the following presentations. The group $\Sigma_1$ embeds as an index 3 subgroup into $\Omega_3$ and is therefore arithmetic in $\mathrm{PGL}_3(\mathbb{F}_2((t)))$. Similarly, the group $\Sigma_2$ embeds into $\Omega_4$ and is therefore exotic.

$$\begin{aligned}
\Sigma_1 &:= \langle x, y, z \mid x^7 = y^7 = z^7 = xyz = x^3y^3z^3 = 1 \rangle, \\
\Sigma_2 &:= \langle x, y, z \mid x^7 = y^7 = z^7 = xyz^3 = x^3y^3z = 1 \rangle.
\end{aligned}$$

## 4.2 Type-preserving vertex-regular lattices

We classify lattices on $\tilde{A}_2$-buildings of thickness three that preserve types and act regularly on the vertices of each type. Every vertex regular lattice in [Car+93] contains such a lattice with index 3: the kernel of the action on types. Conversely, not every type-preserving lattice needs to have a type-transitive extension.

**Theorem 86.**

1. *Consider the $\tilde{A}_2$-GABs $\Delta_1, \dots, \Delta_{13}$ indicated in Appendix B. Let $\Gamma_i$ be the fundamental group and let $X_i$ be the universal cover of $\Delta_i$ for $1 \leq i \leq 13$. Then $\Gamma_i$ is a uniform lattice on $X_i$ that acts regularly on vertices of each type. On the other hand, if $\Gamma \curvearrowright X$ is such an action then it is equivariant to one of the actions $\Gamma_i \curvearrowright X_i$ for some $1 \leq i \leq 13$. These 13 lattices are pairwise not isomorphic.*

2. *The lattices $\Gamma_1, \Gamma_2, \Gamma_3$ embed as arithmetic lattices in $\mathrm{PGL}_3(\mathbb{F}_2((t)))$ and the lattices $\Gamma_4, \Gamma_5$ embed as arithmetic lattices in $\mathrm{PGL}_3(\mathbb{Q}_2)$. In particular, they act on the associated Bruhat–Tits buildings. The seven lattices $\Gamma_7, \dots, \Gamma_{13}$ act on different exotic $\tilde{A}_2$-buildings. The lattice $\Gamma_6$ is either arithmetic in $\mathrm{PGL}_3(\mathbb{Q}_2)$ or exotic; if it is exotic then its building is not isomorphic to any of the exotic buildings $X_7, \dots, X_{13}$. In particular, the 13 lattices $\Gamma_i$ fall into nine or ten quasi-isometry classes.*

We expect that the lattice $\Gamma_6$ is an arithmetic lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$. This expectation is supported by the fact that the combinatorial 4-balls in the building on which $\Gamma_6$ acts are all isomorphic to the combinatorial 4-ball in the Bruhat–Tits building associated to $\mathrm{PGL}_3(\mathbb{Q}_2)$. However, we have not been able to prove arithmeticity of $\Gamma_6$.

Before we prove the theorem, we provide some more information on the lattices, GABs and buildings in the Tables 4.1, 4.2, 4.3.

| Lattice | Ambient $\mathbf{G}(\mathbb{K})$ | $\mathrm{Aut}(\Delta)$ | $\mathrm{Aut}(\Delta)/$ $\mathrm{Aut}^+(\Delta)$ | Extensions |
|---|---|---|---|---|
| $\Gamma_1$ | $\mathrm{PGL}_3(\mathbb{F}_2((t)))$ | $(C_7 \rtimes C_3) \times C_3$ | $C_3$ | $\Sigma_1, \Omega_3,$<br>$\Lambda_{A.1}, \Lambda_{A.2}, \Lambda_{A.3}$ |
| $\Gamma_2$ | $\mathrm{PGL}_3(\mathbb{F}_2((t)))$ | $\mathrm{Sym}(3)$ | $\mathrm{Sym}(3)$ | $\Lambda_{A.4}$ |
| $\Gamma_3$ | $\mathrm{PGL}_3(\mathbb{F}_2((t)))$ | $(C_7 \rtimes C_6)$ | $C_2$ | $\Sigma_1, \Omega_3$ |
| $\Gamma_4$ | $\mathrm{PGL}_3(\mathbb{Q}_2)$ | $(C_3 \times C_3) \rtimes C_2$ | $\mathrm{Sym}(3)$ | $\Lambda_{B.1}, \Lambda_{B.2}, \Lambda_{B.3}$ |
| $\Gamma_5$ | $\mathrm{PGL}_3(\mathbb{Q}_2)$ | $\mathrm{Sym}(3)$ | $\mathrm{Sym}(3)$ | $\Lambda_{C.1}$ |
| $\Gamma_6$ | **?** | $\mathrm{Sym}(3)$ | $C_2$ | – |

TABLE 4.1: Arithmetic lattices $\Gamma_1, \dots, \Gamma_5$ and the lattice $\Gamma_6$. With $\Delta$ we mean the associated GAB and $\mathrm{Aut}^+$ is the type-preserving automorphism group.

| Lattice | $\mathrm{Aut}(\Delta)$ | $\lvert \mathrm{Aut}(X) : \hat{\Gamma} \rvert$ | $\mathrm{Aut}(X)/$ $\mathrm{Aut}^+(X)$ | Comments |
|---|---|---|---|---|
| $\Gamma_7$ | $C_2$ | 21 | $C_2$ | $\mathrm{Aut}(X) = \Omega_2 \rtimes C_2$,<br>biggest solvable quotient: $C_6$ |
| $\Gamma_8$ | $S_3$ | 1 | $C_2$ | perfect,<br>smallest quotient: $\mathrm{PSL}_2(11)$ |
| $\Gamma_9$ | $S_3$ | 1 | $C_2$ | perfect,<br>smallest quotient: $\mathrm{Alt}(5)$ |
| $\Gamma_{10}$ | $C_3$ | 1 | 1 | perfect,<br>no quotient of size $< 10^6$ |
| $\Gamma_{11}$ | $C_6$ | 1 | $C_2$ | biggest solvable quotient: $C_{13}$ |
| $\Gamma_{12}$ | $C_3$ | 1 | 1 | biggest solvable quotient: $C_2$ |
| $\Gamma_{13}$ | $C_3$ | 1 | 1 | perfect,<br>no quotient of size $< 10^6$ |

TABLE 4.2: Exotic lattices, automorphisms of their buildings and quotients. With $X$ we mean the associated building and with $\hat{\Gamma}$ we mean the deck transformation extension $\Gamma \rtimes \mathrm{Aut}(\Delta)$ of $\Gamma$ induced by $\mathrm{Aut}(\Delta)$.

| Building | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ | $X_{11}$ | $X_{12}$ | $X_{13}$ |
|---|---|---|---|---|---|---|---|---|
| 2-ball around $\tilde{u}$ | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 2 |
| 2-ball around $\tilde{v}$ | 0 | 2 | 2 | 2 | 2 | 0 | 0 | 0 |
| 2-ball around $\tilde{w}$ | 0 | 2 | 2 | 2 | 2 | 0 | 0 | 0 |

TABLE 4.3: Balls of radius two in $X_6, \dots, X_{13}$. There are two isomorphism types of such balls in $\tilde{A}_2$-buildings of thickness three. Representatives are the 2-balls in the Bruhat–Tits buildings associated to $\mathrm{PGL}(\mathbb{Q}_2)$ and $\mathrm{PGL}(\mathbb{F}_2((t)))$, respectively. The entries 0 and 2 in the table indicate the isomorphism type.

*Proof for Table 4.1.* In Table 4.1, we claim that the lattices $\Gamma_1, \dots, \Gamma_5$ are arithmetic lattices, we provide some information about cellular automorphisms of the quotient GAB, and we claim that our lattices admit extensions to the $\tilde{A}_2$-lattices mentioned at the beginning of this chapter. The claims about the cellular automorphism groups can be verified easily. In Table B.2 in Appendix B, we provide explicit generators. The extensions to other $\tilde{A}_2$-lattices in Table 4.1 all arise as deck transformations covering automorphisms of the quotient GAB. We provide the explicit automorphisms that give rise to the extensions in Table B.3. It now follows that the lattices $\Gamma_1, \dots, \Gamma_5$ are arithmetic, since they all embed as finite index subgroup in arithmetic groups. ☐

*Proof for Table 4.3.* In Table 4.3 we present the isomorphism types of the two balls in the buildings $X_6, \dots, X_{13}$. These balls have been constructed with the help of a computer and the computability of these balls follows from Section 4.3, where we present an explicit algorithm for computing normal forms of combinatorial paths in certain $\tilde{A}_2$-buildings. ☐

*Proof for Table 4.2.* In Table 4.2, we provide information about the lattices $\Gamma_7, \dots, \Gamma_{13}$, their buildings and their quotient GABs. By Tables 4.3, these are exotic lattices. Again the explicit automorphism groups of the GABs can be computed easily and in Table B.2 we provide explicit generators. Let $7 \leq i \leq 13$ and let $\Gamma := \Gamma_i$. Let $X$ and $\Delta$ be the associated building and GAB, respectively. Let $\hat{\Gamma}$ be the extension of $\Gamma$ by $\mathrm{Aut}(\Delta)$. Note that $\mathrm{Aut}(\Delta)$ fixes the vertex $u$ so the stabilizer in $\hat{\Gamma}$ of any lift $\tilde{u}$ is naturally isomorphic to $\mathrm{Aut}(\Delta)$ and it is a complement to $\Gamma$. Therefore, $\hat{\Gamma} = \Gamma \rtimes \mathrm{Aut}(\Delta)$. We need to show, that in every case, except for $i = 7$, the the full automorphism group of the building $X$ is $\hat{\Gamma}$. Fix a lift $\tilde{u}$ with stabilizer $D$ in $\hat{\Gamma}$. Let $B_4$ be the ball of radius four in $X$ around $\tilde{u}$. In Table B.4 we indicate the size of certain groups associated to $B_4$. In that table, the group $A(u)$ denotes the automorphism group of the ball $B_4$. If we denote by $B_1$ and $B_3$ the balls of radius one and three around $\tilde{u}$, then $A^1(u)$ denotes the image of the restriction map $\mathrm{Aut}(B_4) \to \mathrm{Aut}(B_1)$ and $A^3_{(1)}(u)$ denotes the image of the fixator of $B_1$ in $\mathrm{Aut}(B_4)$ under the restriction map $\mathrm{Aut}(B_4) \to \mathrm{Aut}(B_3)$. The entries of the table have been calculated with a computer and it turns out that $A^3_{(1)}(u)$ is trivial in every case. The size of $A^1(u)$ equals the size of $D$ in every case, except for $i = 7$. Now assume that $i \neq 7$. Let $\sigma$ be an automorphism of $X$ that fixes $\tilde{u}$. We can find an element $\delta \in D$ such that $\sigma \circ \delta$ fixes $B_1$ pointwise. Since $A^3_{(1)}(u)$ is trivial, we know that $\sigma \circ \delta$ fixes $B_3$ pointwise. In particular, $\sigma \circ \delta$ fixes all vertices of the same type as $\tilde{u}$ that are at distance two and furthermore $\sigma \circ \delta$ fixes the 1-ball around these vertices. By induction we deduce that $\sigma \circ \delta$ is trivial and hence $\sigma \in D$. By Table 4.3, there cannot exist an automorphism of $X$, that maps $\tilde{u}$ to a vertex of another type. In particular, $\mathrm{Aut}(X) = \mathrm{Aut}(X)_{\tilde{u}}\Gamma = D\Gamma = \hat{\Gamma}$. For the case $i = 7$ we refer to Proposition 88. The action on types of the full automorphism group of the building can be deduced by the previous discussion. The comments in the fifth column that concern quotients have been calculated with a computer. The comment regarding the automorphism group of $X_7$ follows again from Proposition 88. □

*Proof of Theorem 86.* The arithmeticity of $\Gamma_1, \dots, \Gamma_5$ follows from Table 4.1. To see that the groups $\Gamma_1, \dots, \Gamma_6$ are pairwise not isomorphic, we refer to Table B.5, which shows that the lattices $\Gamma_1, \dots, \Gamma_6$ can be distinguished by their quotients modulo their second derived subgroups. We now show that the buildings $X_6, \dots, X_{13}$ are pairwise not isomorphic. By the Tables 4.2, 4.3, we know that the only possible isomorphisms between these buildings could be between $X_8$ and $X_9$, or between $X_{12}$ and $X_{13}$. Now assume that $X_8$ and $X_9$ are isomorphic. Then the lattices $\Gamma_8$ and $\Gamma_9$ are both index 3 subgroups in the type-preserving automorphism group of $X_8$. Note that $\Gamma_8$ and $\Gamma_9$ are not isomorphic since they have different smallest simple quotients. In particular, their intersection is a proper subgroup of index at most 3 in $\Gamma_8$ which implies that $\Gamma_8$ contains a normal subgroup of index at most $6 = 3!$. But the smallest quotient of $\Gamma_8$ is $A_5$, so its biggest normal subgroup is of index 60. Therefore, $X_8$ and $X_9$ cannot be isomorphic. The same argument shows that $X_{12}$ and $X_{13}$ are not isomorphic.

We now argue that if $\Gamma_6$ is not exotic, then it is an arithmetic lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$ and acts on the associated Bruhat–Tits building. Let $X$ be a $\tilde{A}_2$–Bruhat–Tits building of thickness three defined over a non-Archimedean local field $\mathbb{K}$. Then $\mathbb{K} \cong \mathbb{F}_2((t))$ or $\mathbb{K}$ is a totally ramified extension of $\mathbb{Q}_2$. Let $\mathcal{O} \subseteq \mathbb{K}$ be the ring of integers with maximal ideal $(\pi)$. The combinatorial balls of radius two are parameterized by the ring $R := \mathcal{O}/(\pi^2)$, which has order four and is isomorphic to either $\mathbb{F}_2[t]/(t^2)$ or $\mathbb{Z}/4\mathbb{Z}$. If $\mathbb{K} = \mathbb{F}_2((t))$, then $R \cong \mathbb{F}_2[t]/(t^2)$, while if $\mathbb{K} = \mathbb{Q}_2$, then $R \cong \mathbb{Z}/4\mathbb{Z}$. Moreover, if $\mathbb{K}$ is a non-trivial, totally ramified extension of $\mathbb{Q}_2$, then $R \cong \mathbb{F}_2[t]/(t^2)$. Although this can be shown by elementary methods, we refer to [DT16, Lemma 1.3] for a broader perspective on balls in Bruhat–Tits buildings. In particular, the building on which $\Gamma_6$ acts on is the Bruhat–Tits building associated to $\mathrm{PGL}_3(\mathbb{Q}_2)$ by Table 4.3. For this building the type-preserving automorphism group agrees with the algebraic group and therefore $\Gamma_6$ is an arithmetic lattice $\mathrm{PGL}_3(\mathbb{Q}_2)$ by the Margulis arithmeticity theorem [Mar84].

In summary, we have shown that the lattices $\Gamma_1, \dots, \Gamma_{13}$ are pairwise not isomorphic and that we have either nine or ten isomorphism classes of buildings, or equivalently either nine or ten quasi-isometry classes of groups.

What remains to show is that any action of a lattice $\Gamma$ with the described properties is equivariant to one of the actions arising from the GABs in our list. So let $\Gamma$ be such a lattice and let $\Sigma_\Gamma$ be the Radu graph of the quotient GAB. Recall that in Subsection 3.3.1 we defined a canonical action on Radu graphs with the same vertex sets. Now Proposition 83 implies that an isomorphic copy of $\Sigma_\Gamma$ lies in the orbit of the Radu graph $\Sigma$ indicated in Figure 3.8. In particular, we can obtain every lattice with the desired properties by computing the orbit of $\Sigma$, which is of size $(7!)^3$. Lemma 85 indicates, that it is sufficient to consider a much smaller representative system of Radu graphs; a quantification of this is Lemma 87 below. By that lemma we only need to consider 604800 Radu graphs, check for perfectness, and obtain the equivariance classes of the lattices acting on their buildings. We performed this calculation on an ordinary office computer and it took a few minutes. □

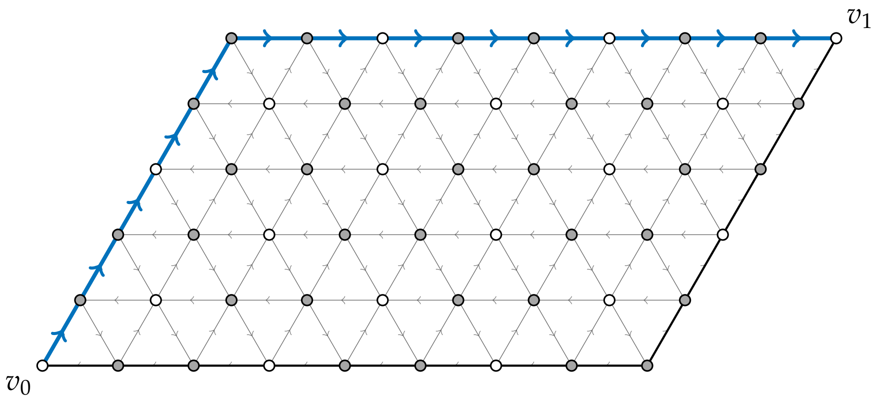

FIGURE 4.1: An edge path in normal form consisting of straight forward edges, a wide turn and straight reverse edges. The gray arrows indicate the preferred orientation, and the blue arrows indicate the orientation in which the edges are traversed.

**Lemma 87.** *Let $\Sigma$ be the Radu graph indicated in Figure 3.8. Let $I := \{1, 2, 3, \}$ be the set of types and denote the vertices of type $i$ by $V_i$. Let $\rho := (1\ 2\ 3) \in \mathrm{Sym}(I)$. For $i \in I$ let $A^i \leq \mathrm{Sym}(V_i) \times \mathrm{Sym}(V_{\rho(i)})$ be the type-preserving automorphism group of the local geometry $\Sigma_{i\rho(i)}$. Denote the restriction of $A^i$ to $V_j$ with $A^i_j$ for $j \in \{i, \rho(i)\}$. Now let $T_1$ be a double coset representative system for $A^3_1 \backslash \mathrm{Sym}(V_1) / A^1_1$ and let $T_3$ be a right transversal for $A^2_3 \backslash \mathrm{Sym}(V_3)$. Then for every Radu graph $\Sigma'$, that is locally isomorphic to $\Sigma$ there exist $\lambda_1 \in T_1$, $\lambda_2 \in \mathrm{Sym}(V_2)$ and $\lambda_3 \in T_3$ such that $\Sigma'$ is isomorphic to $(\lambda_1, \lambda_2, \lambda_3).\Sigma$.*

*Proof.* This is a direct consequence of Lemma 85. ☐

**Proposition 88.** *The lattice $\Gamma_7$ embeds as an index 21 subgroup in the exotic lattice $\Omega_2$. Furthermore, the automorphism group of the corresponding building is $\Omega \rtimes C_2$, which therefore contains $\Gamma_7$ as an index 42 subgroup.*

*Proof.* The lattice $\Omega_2$ admits a quotient $Q$ that is a non-split extension of $C_6$ by $\mathrm{Alt}(7)$. This quotient can be presented as follows.

$$Q := \left\langle a, b, c \;\middle|\; \begin{array}{l} a^3 = b^3 = c^3 = 1, \quad (ab)^2 = ba, \quad (bc)^2 = cb, \\ (ac)^2 = ca, \quad cab^{-1}cabc^{-1}ba^{-1}b^{-1}c^{-1}ba^{-1}b \end{array} \right\rangle .$$

Let $K$ be the kernel of the corresponding epimorphism. If we denote the building that $\Omega_2$ acts on by $X$, then $Y := K \backslash X$ is a finite GAB and $Q$ acts on it by deck transformations. Now the subgroup $R = \langle bca, abcb \rangle \subseteq Q$ is of index 21 and acts regular on vertices of each type. The quotient geometry $R \backslash Y$ is again a GAB and is in fact isomorphic to $\Delta_7$. It follows that $\Gamma_7$ embeds as an index 21 subgroup in $\Omega_2$, and it turns out, that $\Gamma_7 \cong \langle bca, abcb \rangle \leq \Omega_2$. The fact that $\Omega_2 \rtimes C_2$ is the full automorphism group of $X$ follows from Table B.4 with similar arguments as in the proof for Table 4.2. ☐

## 4.3 Normal forms

Throughout this section let $X$ be a $\tilde{A}_2$-building, let $\Gamma$ be a uniform lattice acting freely and type-preservingly on $X$, and let $Y = \Gamma \backslash X$.

This subsection is similar to Subsection 3.2.2, where we computed normal forms for words in $\tilde{C}_2$-lattices that act regularly on special vertices. Here, however, we take a slightly different approach. We embrace the fact that $\Gamma\backslash X$ has more than one vertex and give a normal form for edge paths up to homotopy. Conceptually, we work with the fundamental groupoid of $\Gamma\backslash X$ and regard $\Gamma$ as the subgroup of homotopy classes of paths that start and end at the same vertex.

We put a cyclic ordering $\rho \in \mathrm{Sym}(I)$ on the set of types $I$, thus giving each edge of type $\{i, j\}$ a preferred orientation $(i, j)$ such that $\rho(i) = j$. We say that an oriented edge is a *forward* edge if its orientation agrees with the preferred orientation, and is a *reverse* edge if it does not.

**Lemma 89.** *Let $v_0$ and $v_1$ be vertices in $X$ of the same type. There is a unique path of edges $f_1, \dots, f_k, g_1, \dots, g_m$ (possibly $k = 0$ or $m = 0$) from $v_0$ to $v_1$ such that the $f_i$ are forward edges, the $g_j$ are reverse edges, $f_{i-1}$ and $f_i$ as well as $g_{j-1}$ and $g_j$ meet in a vertex in which they form an angle of $\pi$ for $1 < i \leq k$, $1 < j \leq m$, $f_k$ and $g_1$ meet in a vertex in which they form an angle of $(2\pi/3)$.*

*Proof.* The least convex subcomplex of $X$ containing $v_0$ and $v_1$ is a parallelogram with a $\tilde{A}_2$-tiling (see Figure 4.1): it is the intersection of all apartments containing $v_0$ and $v_1$ and therefore is the combinatorial convex hull of the two in any such apartment. For existence, take the path along the boundary of this convex hull. For uniqueness note that if two vertices are connected by such a path, the path runs along the boundary of the convex hull of the two vertices. ☐

Let $\omega = g_1 \cdots g_k$ be a path of oriented edges in $Y$. We say that $g_{i-1}$ and $g_i$ *cancel* if $g_i = g_{i-1}^{-1}$; that they form a *sharp turn* if they form an angle of $(\pi/3)$; that they form a *wide turn* if they form an angle of $(2\pi/3)$; and are *straight* if they include an angle of $\pi$. Note that straight edges and edges forming a sharp turn are either both forward or both reverse while wide turns represent a change from forward to reverse or vice versa. We say that $\omega$ is *reduced* if its length is minimal among all paths with the same endpoints. We say that it is in *normal form* if it contains no cancellations or sharp turns, at most one wide turn, which then has to be from a forward edge to a reverse edge.

**Corollary 90.** *Every edge path in $Y$ is homotopic to a unique one in normal form.*

When manipulating edge paths we use the following rewritings: free cancellations; rewriting $g_{i-1}g_i \rightsquigarrow h$ when $g_{i-1}g_i$ is a sharp turn in the triangle $g_{i-1}g_ih$; and rewriting $g_{i-1}g_i \rightsquigarrow g'_{i-1}g'_i$ when $g_{i-1}g_i$ form a wide turn and $h$ is such that $(g'_{i-1})^{-1}g_{i-1}h$ and $g'_ig_i^{-1}h$ form triangles, and $g_{i-1}g_i$ and $g'_{i-1}g'_i$ form a parallelogram.

**Proposition 91.**

1. *Any edge path in normal form is reduced. Two paths in normal form are homotopic if and only if they are equal.*
2. *Any edge path of length $\ell$ can be brought into normal form by applying $O(\ell^2)$ rewritings.*

*Proof.* Let $\omega$ be an edge path in $X$ from a vertex $v$ to a vertex $w$. Let $\Sigma$ be an apartment containing both and let $\rho_{\Sigma,c}$ be the retraction centered at some chamber $c \subseteq \Sigma$. Then $\rho_{\Sigma,c}(\omega)$ is an edge path from $v$ to $w$, showing that the minimal edge distance is realized in $\Sigma$. In $\Sigma$ it is easy to see that paths in normal form are reduced. That homotopic paths in normal form are equal follows from Corollary 90.

Let $\omega = g_1 \dots g_\ell$ be an edge path. Without loss of generality assume that $\omega$ only involves straight pairs of edges and wide turns. Let $i$ be the first index such that $g_i$ is a reverse edge and define $a := \ell - i$. For definedness set $a := 0$ if the path contains no reverse edge. If the word is not in normal form there exists a minimal $k$ greater than $i$

such that $g_k$ is a forward edge. The path $g_i \dots g_k$ lies in an apartment and we can apply $k - i$ rewritings to obtain a path $g'_i \dots g'_k$ such that $g'_i$ is forward and $g'_{i+1} \dots g'_k$ only consists of reversed edges. The path $\omega' = g_1 \dots g_{i-1} g'_i \dots g'_k g_{k+1} \dots g_\ell$ has its first reverse edge at position $i + 1$. If $g_{i-1} g'_i$ is a sharp turn, we rewrite it and obtain a path with its first reverse edge at position $i - 1$ and length $\ell - 1$. In particular $\ell$ decreased and $a$ did not increase. If we now apply a cancellation or a rewrite a sharp turn, the position of the first reverse edge may decrease, but at most by the amount by which the word length decreases. If $g_{i-1} g'_i$ is straight, then $\ell$ remains the same and $a$ decreased. Again if we apply a cancellation or rewrite a sharp turn, the position of the first reverse edge may decrease but at most by the amount by which the word length decreases. In particular, performing these operations as long as possible results in both cases in a path $\omega''$ for which, compared to $\omega$, either $\ell$ decreased and $a$ did not increase, or $\ell$ did not increase and $a$ decreased. Repeating this procedure, we obtain a word in normal form after performing at most $(a + \ell)a$ rewritings that do not decrease the length and at most $\ell$ rewritings that decrease the length. □

## Appendix A

# The complexes $Y_k^3$

In this section we present the $\tilde{C}_2$-GABs $Y_1^3, \dots, Y_4^3$. They all consist of 12 vertices labeled by $v, w, u_1, \dots u_{10}$, 120 edges labeled by $e_1, \dots, e_{40}$, $f_1 \dots, f_{40}$, $g_1, \dots, g_{40}$, and 160 triangles. The links around the vertices $v, w$ are always the symplectic quadrangle of order 3 and the links around the vertices $u_1, \dots, u_{10}$ are always the complete bipartite graph $K_{4,4}$. The boundary of the edge $g_i$ is always $(v, w)$ and the boundary of the edge $e_i$ is always $(w, u_j)$ where $j = \lceil i/4 \rceil$. Let $\tau = (1,2)(3,4)(5,6) \in \mathrm{Sym}(10)$, then the boundary of the edge $f_i \in Y_k^3$ is always $(u_j, v)$ with $j = \tau_k(\lceil i/4 \rceil)$. For each complex the assignment $e_i \leftrightarrow f_i$ induces an involutory automorphism $\sigma_k$ of $Y_k^3$ that swaps the two vertices of special type. In particular, we can apply Lemma 60 to $(Y_k^3, \sigma_k)$ and obtain a presentation for the extension $\bar{\Gamma}_k^3$ of $\Gamma_k^3 = \pi_1(Y_k^3)$ induced by $\sigma_k$. We also provide these presentations. If we subdivide the complex $S_{\mathrm{JW}}$ along the diagonals from $v_{00}$ to $v_{11}$ and label the new diagonals with $s_i$, we obtain a complex that embeds in each of the complexes $Y_k^3$ via the following assignment.

$$
\begin{array}{llllllll}
a \mapsto f_5, & \bar{a} \mapsto f_6, & b \mapsto f_7, & \bar{b} \mapsto f_8, & x \mapsto f_1, & \bar{x} \mapsto f_2, & y \mapsto f_3, & \bar{y} \mapsto f_4, \\
a' \mapsto e_5, & \bar{a}' \mapsto e_6 & b' \mapsto e_7, & \bar{b}' \mapsto e_8, & x' \mapsto e_1, & \bar{x}' \mapsto e_2, & y' \mapsto e_3, & \bar{y}' \mapsto e_4, \\
s_i \mapsto g_i. & & & & & & &
\end{array}
$$

The image of these embeddings is always the complex consisting of the first 32 triangles of $Y_k^3$. In particular, the complexes $Y_k^3$ agree on their first 32 triangles and the boundaries of these triangles are the following ones.

$$
\begin{array}{llllll}
(e_3, f_5, g_1), & (e_5, f_1, g_1), & (e_4, f_5, g_2), & (e_7, f_2, g_2), & (e_1, f_5, g_3), & (e_5, f_3, g_3), \\
(e_2, f_5, g_4), & (e_8, f_4, g_4), & (e_3, f_6, g_5), & (e_7, f_1, g_5), & (e_4, f_6, g_6), & (e_6, f_2, g_6), \\
(e_1, f_6, g_7), & (e_8, f_3, g_7), & (e_2, f_6, g_8), & (e_6, f_4, g_8), & (e_4, f_7, g_9), & (e_8, f_1, g_9), \\
(e_3, f_7, g_{10}), & (e_8, f_2, g_{10}), & (e_1, f_7, g_{11}), & (e_6, f_3, g_{11}), & (e_2, f_7, g_{12}), & (e_5, f_4, g_{12}), \\
(e_3, f_8, g_{13}), & (e_6, f_1, g_{13}), & (e_4, f_8, g_{14}), & (e_5, f_2, g_{14}), & (e_2, f_8, g_{15}), & (e_7, f_3, g_{15}), \\
(e_1, f_8, g_{16}), & (e_7, f_4, g_{16}). & & & &
\end{array}
$$

Now we indicate the complexes $Y_k^3$ by indicating the boundaries of their remaining 128 triangles. All the properties claimed so far can be checked by hand or with a simple program. The boundaries of the remaining triangles in $Y_1^3$ are

$$
\begin{array}{lllll}
(e_9, f_{13}, g_1), & (e_9, f_{14}, g_{23}), & (e_9, f_{15}, g_{25}), & (e_9, f_{16}, g_{36}), & (e_{10}, f_{13}, g_{30}), \\
(e_{10}, f_{14}, g_{36}), & (e_{10}, f_{15}, g_8), & (e_{10}, f_{16}, g_{19}), & (e_{11}, f_{13}, g_{17}), & (e_{11}, f_{14}, g_{40}), \\
(e_{11}, f_{15}, g_{31}), & (e_{11}, f_{16}, g_6), & (e_{12}, f_{13}, g_{35}), & (e_{12}, f_{14}, g_3), & (e_{12}, f_{15}, g_{33}), \\
(e_{12}, f_{16}, g_{37}), & (e_{13}, f_9, g_3), & (e_{13}, f_{10}, g_{30}), & (e_{13}, f_{11}, g_{17}), & (e_{13}, f_{12}, g_{39}), \\
(e_{14}, f_9, g_{34}), & (e_{14}, f_{10}, g_{24}), & (e_{14}, f_{11}, g_{40}), & (e_{14}, f_{12}, g_1), & (e_{15}, f_9, g_{25}),
\end{array}
$$

$(e_{15}, f_{10}, g_{6})$, $(e_{15}, f_{11}, g_{22})$, $(e_{15}, f_{12}, g_{33})$, $(e_{16}, f_{9}, g_{24})$, $(e_{16}, f_{10}, g_{26})$,
$(e_{16}, f_{11}, g_{8})$, $(e_{16}, f_{12}, g_{37})$, $(e_{17}, f_{17}, g_{32})$, $(e_{17}, f_{18}, g_{11})$, $(e_{17}, f_{19}, g_{34})$,
$(e_{17}, f_{20}, g_{27})$, $(e_{18}, f_{17}, g_{5})$, $(e_{18}, f_{18}, g_{33})$, $(e_{18}, f_{19}, g_{21})$, $(e_{18}, f_{20}, g_{19})$,
$(e_{19}, f_{17}, g_{23})$, $(e_{19}, f_{18}, g_{21})$, $(e_{19}, f_{19}, g_{28})$, $(e_{19}, f_{20}, g_{11})$, $(e_{20}, f_{17}, g_{27})$,
$(e_{20}, f_{18}, g_{26})$, $(e_{20}, f_{19}, g_{5})$, $(e_{20}, f_{20}, g_{17})$, $(e_{21}, f_{21}, g_{18})$, $(e_{21}, f_{22}, g_{35})$,
$(e_{21}, f_{23}, g_{13})$, $(e_{21}, f_{24}, g_{20})$, $(e_{22}, f_{21}, g_{39})$, $(e_{22}, f_{22}, g_{38})$, $(e_{22}, f_{23}, g_{29})$,
$(e_{22}, f_{24}, g_{13})$, $(e_{23}, f_{21}, g_{7})$, $(e_{23}, f_{22}, g_{20})$, $(e_{23}, f_{23}, g_{25})$, $(e_{23}, f_{24}, g_{26})$,
$(e_{24}, f_{21}, g_{29})$, $(e_{24}, f_{22}, g_{7})$, $(e_{24}, f_{23}, g_{19})$, $(e_{24}, f_{24}, g_{40})$, $(e_{25}, f_{25}, g_{40})$,
$(e_{25}, f_{26}, g_{35})$, $(e_{25}, f_{27}, g_{28})$, $(e_{25}, f_{28}, g_{15})$, $(e_{26}, f_{25}, g_{39})$, $(e_{26}, f_{26}, g_{25})$,
$(e_{26}, f_{27}, g_{15})$, $(e_{26}, f_{28}, g_{32})$, $(e_{27}, f_{25}, g_{28})$, $(e_{27}, f_{26}, g_{10})$, $(e_{27}, f_{27}, g_{30})$,
$(e_{27}, f_{28}, g_{22})$, $(e_{28}, f_{25}, g_{10})$, $(e_{28}, f_{26}, g_{32})$, $(e_{28}, f_{27}, g_{31})$, $(e_{28}, f_{28}, g_{37})$,
$(e_{29}, f_{29}, g_{37})$, $(e_{29}, f_{30}, g_{2})$, $(e_{29}, f_{31}, g_{29})$, $(e_{29}, f_{32}, g_{23})$, $(e_{30}, f_{29}, g_{12})$,
$(e_{30}, f_{30}, g_{27})$, $(e_{30}, f_{31}, g_{22})$, $(e_{30}, f_{32}, g_{29})$, $(e_{31}, f_{29}, g_{20})$, $(e_{31}, f_{30}, g_{31})$,
$(e_{31}, f_{31}, g_{21})$, $(e_{31}, f_{32}, g_{12})$, $(e_{32}, f_{29}, g_{34})$, $(e_{32}, f_{30}, g_{20})$, $(e_{32}, f_{31}, g_{2})$,
$(e_{32}, f_{32}, g_{30})$, $(e_{33}, f_{33}, g_{17})$, $(e_{33}, f_{34}, g_{23})$, $(e_{33}, f_{35}, g_{18})$, $(e_{33}, f_{36}, g_{4})$,
$(e_{34}, f_{33}, g_{34})$, $(e_{34}, f_{34}, g_{33})$, $(e_{34}, f_{35}, g_{4})$, $(e_{34}, f_{36}, g_{38})$, $(e_{35}, f_{33}, g_{18})$,
$(e_{35}, f_{34}, g_{14})$, $(e_{35}, f_{35}, g_{32})$, $(e_{35}, f_{36}, g_{19})$, $(e_{36}, f_{33}, g_{14})$, $(e_{36}, f_{34}, g_{38})$,
$(e_{36}, f_{35}, g_{26})$, $(e_{36}, f_{36}, g_{28})$, $(e_{37}, f_{37}, g_{21})$, $(e_{37}, f_{38}, g_{24})$, $(e_{37}, f_{39}, g_{9})$,
$(e_{37}, f_{40}, g_{39})$, $(e_{38}, f_{37}, g_{36})$, $(e_{38}, f_{38}, g_{38})$, $(e_{38}, f_{39}, g_{31})$, $(e_{38}, f_{40}, g_{16})$,
$(e_{39}, f_{37}, g_{16})$, $(e_{39}, f_{38}, g_{22})$, $(e_{39}, f_{39}, g_{18})$, $(e_{39}, f_{40}, g_{24})$, $(e_{40}, f_{37}, g_{35})$,
$(e_{40}, f_{38}, g_{9})$, $(e_{40}, f_{39}, g_{36})$, $(e_{40}, f_{40}, g_{27})$.

The boundaries of the remaining triangles in $Y_2^3$ are

$(e_{9}, f_{13}, g_{8})$, $(e_{9}, f_{14}, g_{37})$, $(e_{9}, f_{15}, g_{24})$, $(e_{9}, f_{16}, g_{26})$, $(e_{10}, f_{13}, g_{40})$,
$(e_{10}, f_{14}, g_{1})$, $(e_{10}, f_{15}, g_{34})$, $(e_{10}, f_{16}, g_{24})$, $(e_{11}, f_{13}, g_{22})$, $(e_{11}, f_{14}, g_{33})$,
$(e_{11}, f_{15}, g_{25})$, $(e_{11}, f_{16}, g_{6})$, $(e_{12}, f_{13}, g_{17})$, $(e_{12}, f_{14}, g_{39})$, $(e_{12}, f_{15}, g_{3})$,
$(e_{12}, f_{16}, g_{30})$, $(e_{13}, f_{9}, g_{6})$, $(e_{13}, f_{10}, g_{40})$, $(e_{13}, f_{11}, g_{31})$, $(e_{13}, f_{12}, g_{17})$,
$(e_{14}, f_{9}, g_{37})$, $(e_{14}, f_{10}, g_{3})$, $(e_{14}, f_{11}, g_{33})$, $(e_{14}, f_{12}, g_{35})$, $(e_{15}, f_{9}, g_{36})$,
$(e_{15}, f_{10}, g_{23})$, $(e_{15}, f_{11}, g_{25})$, $(e_{15}, f_{12}, g_{1})$, $(e_{16}, f_{9}, g_{19})$, $(e_{16}, f_{10}, g_{36})$,
$(e_{16}, f_{11}, g_{8})$, $(e_{16}, f_{12}, g_{30})$, $(e_{17}, f_{17}, g_{30})$, $(e_{17}, f_{18}, g_{20})$, $(e_{17}, f_{19}, g_{34})$,
$(e_{17}, f_{20}, g_{2})$, $(e_{18}, f_{17}, g_{20})$, $(e_{18}, f_{18}, g_{21})$, $(e_{18}, f_{19}, g_{12})$, $(e_{18}, f_{20}, g_{31})$,
$(e_{19}, f_{17}, g_{23})$, $(e_{19}, f_{18}, g_{2})$, $(e_{19}, f_{19}, g_{37})$, $(e_{19}, f_{20}, g_{29})$, $(e_{20}, f_{17}, g_{12})$,
$(e_{20}, f_{18}, g_{22})$, $(e_{20}, f_{19}, g_{29})$, $(e_{20}, f_{20}, g_{27})$, $(e_{21}, f_{21}, g_{18})$, $(e_{21}, f_{22}, g_{13})$,
$(e_{21}, f_{23}, g_{20})$, $(e_{21}, f_{24}, g_{35})$, $(e_{22}, f_{21}, g_{7})$, $(e_{22}, f_{22}, g_{40})$, $(e_{22}, f_{23}, g_{19})$,
$(e_{22}, f_{24}, g_{29})$, $(e_{23}, f_{21}, g_{20})$, $(e_{23}, f_{22}, g_{26})$, $(e_{23}, f_{23}, g_{25})$, $(e_{23}, f_{24}, g_{7})$,
$(e_{24}, f_{21}, g_{39})$, $(e_{24}, f_{22}, g_{29})$, $(e_{24}, f_{23}, g_{13})$, $(e_{24}, f_{24}, g_{38})$, $(e_{25}, f_{25}, g_{21})$,
$(e_{25}, f_{26}, g_{24})$, $(e_{25}, f_{27}, g_{9})$, $(e_{25}, f_{28}, g_{39})$, $(e_{26}, f_{25}, g_{36})$, $(e_{26}, f_{26}, g_{38})$,
$(e_{26}, f_{27}, g_{31})$, $(e_{26}, f_{28}, g_{16})$, $(e_{27}, f_{25}, g_{16})$, $(e_{27}, f_{26}, g_{22})$, $(e_{27}, f_{27}, g_{18})$,
$(e_{27}, f_{28}, g_{24})$, $(e_{28}, f_{25}, g_{35})$, $(e_{28}, f_{26}, g_{9})$, $(e_{28}, f_{27}, g_{36})$, $(e_{28}, f_{28}, g_{27})$,
$(e_{29}, f_{29}, g_{40})$, $(e_{29}, f_{30}, g_{35})$, $(e_{29}, f_{31}, g_{28})$, $(e_{29}, f_{32}, g_{15})$, $(e_{30}, f_{29}, g_{39})$,
$(e_{30}, f_{30}, g_{25})$, $(e_{30}, f_{31}, g_{15})$, $(e_{30}, f_{32}, g_{32})$, $(e_{31}, f_{29}, g_{28})$, $(e_{31}, f_{30}, g_{10})$,
$(e_{31}, f_{31}, g_{30})$, $(e_{31}, f_{32}, g_{22})$, $(e_{32}, f_{29}, g_{10})$, $(e_{32}, f_{30}, g_{32})$, $(e_{32}, f_{31}, g_{31})$,

$$\begin{array}{lllll}
(e_{32}, f_{32}, g_{37}), & (e_{33}, f_{33}, g_{28}), & (e_{33}, f_{34}, g_{26}), & (e_{33}, f_{35}, g_{14}), & (e_{33}, f_{36}, g_{38}), \\
(e_{34}, f_{33}, g_{19}), & (e_{34}, f_{34}, g_{32}), & (e_{34}, f_{35}, g_{18}), & (e_{34}, f_{36}, g_{14}), & (e_{35}, f_{33}, g_{4}), \\
(e_{35}, f_{34}, g_{18}), & (e_{35}, f_{35}, g_{17}), & (e_{35}, f_{36}, g_{23}), & (e_{36}, f_{33}, g_{38}), & (e_{36}, f_{34}, g_{4}), \\
(e_{36}, f_{35}, g_{34}), & (e_{36}, f_{36}, g_{33}), & (e_{37}, f_{37}, g_{33}), & (e_{37}, f_{38}, g_{21}), & (e_{37}, f_{39}, g_{5}), \\
(e_{37}, f_{40}, g_{19}), & (e_{38}, f_{37}, g_{21}), & (e_{38}, f_{38}, g_{28}), & (e_{38}, f_{39}, g_{23}), & (e_{38}, f_{40}, g_{11}), \\
(e_{39}, f_{37}, g_{11}), & (e_{39}, f_{38}, g_{34}), & (e_{39}, f_{39}, g_{32}), & (e_{39}, f_{40}, g_{27}), & (e_{40}, f_{37}, g_{26}), \\
(e_{40}, f_{38}, g_{5}), & (e_{40}, f_{39}, g_{27}), & (e_{40}, f_{40}, g_{17}). & &
\end{array}$$

The boundaries of the remaining triangles in $Y_3^3$ are

$$\begin{array}{lllll}
(e_{9}, f_{13}, g_{4}), & (e_{9}, f_{14}, g_{33}), & (e_{9}, f_{15}, g_{38}), & (e_{9}, f_{16}, g_{34}), & (e_{10}, f_{13}, g_{28}), \\
(e_{10}, f_{14}, g_{11}), & (e_{10}, f_{15}, g_{23}), & (e_{10}, f_{16}, g_{21}), & (e_{11}, f_{13}, g_{26}), & (e_{11}, f_{14}, g_{27}), \\
(e_{11}, f_{15}, g_{17}), & (e_{11}, f_{16}, g_{5}), & (e_{12}, f_{13}, g_{18}), & (e_{12}, f_{14}, g_{19}), & (e_{12}, f_{15}, g_{14}), \\
(e_{12}, f_{16}, g_{32}), & (e_{13}, f_{9}, g_{14}), & (e_{13}, f_{10}, g_{28}), & (e_{13}, f_{11}, g_{26}), & (e_{13}, f_{12}, g_{38}), \\
(e_{14}, f_{9}, g_{33}), & (e_{14}, f_{10}, g_{5}), & (e_{14}, f_{11}, g_{21}), & (e_{14}, f_{12}, g_{19}), & (e_{15}, f_{9}, g_{18}), \\
(e_{15}, f_{10}, g_{23}), & (e_{15}, f_{11}, g_{17}), & (e_{15}, f_{12}, g_{4}), & (e_{16}, f_{9}, g_{34}), & (e_{16}, f_{10}, g_{27}), \\
(e_{16}, f_{11}, g_{11}), & (e_{16}, f_{12}, g_{32}), & (e_{17}, f_{17}, g_{35}), & (e_{17}, f_{18}, g_{20}), & (e_{17}, f_{19}, g_{13}), \\
(e_{17}, f_{20}, g_{18}), & (e_{18}, f_{17}, g_{20}), & (e_{18}, f_{18}, g_{26}), & (e_{18}, f_{19}, g_{25}), & (e_{18}, f_{20}, g_{7}), \\
(e_{19}, f_{17}, g_{7}), & (e_{19}, f_{18}, g_{40}), & (e_{19}, f_{19}, g_{19}), & (e_{19}, f_{20}, g_{29}), & (e_{20}, f_{17}, g_{38}), \\
(e_{20}, f_{18}, g_{13}), & (e_{20}, f_{19}, g_{29}), & (e_{20}, f_{20}, g_{39}), & (e_{21}, f_{21}, g_{23}), & (e_{21}, f_{22}, g_{29}), \\
(e_{21}, f_{23}, g_{37}), & (e_{21}, f_{24}, g_{2}), & (e_{22}, f_{21}, g_{29}), & (e_{22}, f_{22}, g_{22}), & (e_{22}, f_{23}, g_{12}), \\
(e_{22}, f_{24}, g_{27}), & (e_{23}, f_{21}, g_{30}), & (e_{23}, f_{22}, g_{2}), & (e_{23}, f_{23}, g_{34}), & (e_{23}, f_{24}, g_{20}), \\
(e_{24}, f_{21}, g_{12}), & (e_{24}, f_{22}, g_{21}), & (e_{24}, f_{23}, g_{20}), & (e_{24}, f_{24}, g_{31}), & (e_{25}, f_{25}, g_{39}), \\
(e_{25}, f_{26}, g_{15}), & (e_{25}, f_{27}, g_{32}), & (e_{25}, f_{28}, g_{25}), & (e_{26}, f_{25}, g_{10}), & (e_{26}, f_{26}, g_{22}), \\
(e_{26}, f_{27}, g_{30}), & (e_{26}, f_{28}, g_{28}), & (e_{27}, f_{25}, g_{32}), & (e_{27}, f_{26}, g_{37}), & (e_{27}, f_{27}, g_{31}), \\
(e_{27}, f_{28}, g_{10}), & (e_{28}, f_{25}, g_{40}), & (e_{28}, f_{26}, g_{28}), & (e_{28}, f_{27}, g_{15}), & (e_{28}, f_{28}, g_{35}), \\
(e_{29}, f_{29}, g_{39}), & (e_{29}, f_{30}, g_{17}), & (e_{29}, f_{31}, g_{3}), & (e_{29}, f_{32}, g_{30}), & (e_{30}, f_{29}, g_{17}), \\
(e_{30}, f_{30}, g_{31}), & (e_{30}, f_{31}, g_{40}), & (e_{30}, f_{32}, g_{6}), & (e_{31}, f_{29}, g_{1}), & (e_{31}, f_{30}, g_{25}), \\
(e_{31}, f_{31}, g_{23}), & (e_{31}, f_{32}, g_{36}), & (e_{32}, f_{29}, g_{37}), & (e_{32}, f_{30}, g_{8}), & (e_{32}, f_{31}, g_{24}), \\
(e_{32}, f_{32}, g_{26}), & (e_{33}, f_{33}, g_{22}), & (e_{33}, f_{34}, g_{33}), & (e_{33}, f_{35}, g_{6}), & (e_{33}, f_{36}, g_{25}), \\
(e_{34}, f_{33}, g_{33}), & (e_{34}, f_{34}, g_{35}), & (e_{34}, f_{35}, g_{37}), & (e_{34}, f_{36}, g_{3}), & (e_{35}, f_{33}, g_{8}), \\
(e_{35}, f_{34}, g_{30}), & (e_{35}, f_{35}, g_{19}), & (e_{35}, f_{36}, g_{36}), & (e_{36}, f_{33}, g_{40}), & (e_{36}, f_{34}, g_{1}), \\
(e_{36}, f_{35}, g_{24}), & (e_{36}, f_{36}, g_{34}), & (e_{37}, f_{37}, g_{31}), & (e_{37}, f_{38}, g_{16}), & (e_{37}, f_{39}, g_{36}), \\
(e_{37}, f_{40}, g_{38}), & (e_{38}, f_{37}, g_{9}), & (e_{38}, f_{38}, g_{35}), & (e_{38}, f_{39}, g_{27}), & (e_{38}, f_{40}, g_{36}), \\
(e_{39}, f_{37}, g_{24}), & (e_{39}, f_{38}, g_{21}), & (e_{39}, f_{39}, g_{39}), & (e_{39}, f_{40}, g_{9}), & (e_{40}, f_{37}, g_{18}), \\
(e_{40}, f_{38}, g_{24}), & (e_{40}, f_{39}, g_{16}), & (e_{40}, f_{40}, g_{22}). & &
\end{array}$$

The boundaries of the remaining triangles in $Y_4^3$ are

$$\begin{array}{lllll}
(e_{9}, f_{13}, g_{12}), & (e_{9}, f_{14}, g_{31}), & (e_{9}, f_{15}, g_{21}), & (e_{9}, f_{16}, g_{20}), & (e_{10}, f_{13}, g_{19}), \\
(e_{10}, f_{14}, g_{7}), & (e_{10}, f_{15}, g_{29}), & (e_{10}, f_{16}, g_{40}), & (e_{11}, f_{13}, g_{30}), & (e_{11}, f_{14}, g_{20}), \\
(e_{11}, f_{15}, g_{2}), & (e_{11}, f_{16}, g_{34}), & (e_{12}, f_{13}, g_{29}), & (e_{12}, f_{14}, g_{38}), & (e_{12}, f_{15}, g_{39}), \\
(e_{12}, f_{16}, g_{13}), & (e_{13}, f_{9}, g_{2}), & (e_{13}, f_{10}, g_{23}), & (e_{13}, f_{11}, g_{37}), & (e_{13}, f_{12}, g_{29}), \\
(e_{14}, f_{9}, g_{35}), & (e_{14}, f_{10}, g_{13}), & (e_{14}, f_{11}, g_{20}), & (e_{14}, f_{12}, g_{18}), & (e_{15}, f_{9}, g_{27}),
\end{array}$$

$$\begin{array}{lllll}
(e_{15}, f_{10}, g_{29}), & (e_{15}, f_{11}, g_{12}), & (e_{15}, f_{12}, g_{22}), & (e_{16}, f_{9}, g_{20}), & (e_{16}, f_{10}, g_{25}), \\
(e_{16}, f_{11}, g_{26}), & (e_{16}, f_{12}, g_{7}), & (e_{17}, f_{17}, g_{33}), & (e_{17}, f_{18}, g_{3}), & (e_{17}, f_{19}, g_{35}), \\
(e_{17}, f_{20}, g_{37}), & (e_{18}, f_{17}, g_{1}), & (e_{18}, f_{18}, g_{36}), & (e_{18}, f_{19}, g_{25}), & (e_{18}, f_{20}, g_{23}), \\
(e_{19}, f_{17}, g_{31}), & (e_{19}, f_{18}, g_{40}), & (e_{19}, f_{19}, g_{17}), & (e_{19}, f_{20}, g_{6}), & (e_{20}, f_{17}, g_{30}), \\
(e_{20}, f_{18}, g_{19}), & (e_{20}, f_{19}, g_{8}), & (e_{20}, f_{20}, g_{36}), & (e_{21}, f_{21}, g_{24}), & (e_{21}, f_{22}, g_{8}), \\
(e_{21}, f_{23}, g_{37}), & (e_{21}, f_{24}, g_{26}), & (e_{22}, f_{21}, g_{6}), & (e_{22}, f_{22}, g_{33}), & (e_{22}, f_{23}, g_{22}), \\
(e_{22}, f_{24}, g_{25}), & (e_{23}, f_{21}, g_{30}), & (e_{23}, f_{22}, g_{39}), & (e_{23}, f_{23}, g_{17}), & (e_{23}, f_{24}, g_{3}), \\
(e_{24}, f_{21}, g_{34}), & (e_{24}, f_{22}, g_{40}), & (e_{24}, f_{23}, g_{1}), & (e_{24}, f_{24}, g_{24}), & (e_{25}, f_{25}, g_{36}), \\
(e_{25}, f_{26}, g_{31}), & (e_{25}, f_{27}, g_{38}), & (e_{25}, f_{28}, g_{16}), & (e_{26}, f_{25}, g_{35}), & (e_{26}, f_{26}, g_{36}), \\
(e_{26}, f_{27}, g_{9}), & (e_{26}, f_{28}, g_{27}), & (e_{27}, f_{25}, g_{18}), & (e_{27}, f_{26}, g_{16}), & (e_{27}, f_{27}, g_{24}), \\
(e_{27}, f_{28}, g_{22}), & (e_{28}, f_{25}, g_{9}), & (e_{28}, f_{26}, g_{21}), & (e_{28}, f_{27}, g_{39}), & (e_{28}, f_{28}, g_{24}), \\
(e_{29}, f_{29}, g_{17}), & (e_{29}, f_{30}, g_{23}), & (e_{29}, f_{31}, g_{18}), & (e_{29}, f_{32}, g_{4}), & (e_{30}, f_{29}, g_{19}), \\
(e_{30}, f_{30}, g_{32}), & (e_{30}, f_{31}, g_{14}), & (e_{30}, f_{32}, g_{18}), & (e_{31}, f_{29}, g_{38}), & (e_{31}, f_{30}, g_{4}), \\
(e_{31}, f_{31}, g_{33}), & (e_{31}, f_{32}, g_{34}), & (e_{32}, f_{29}, g_{14}), & (e_{32}, f_{30}, g_{38}), & (e_{32}, f_{31}, g_{26}), \\
(e_{32}, f_{32}, g_{28}), & (e_{33}, f_{33}, g_{33}), & (e_{33}, f_{34}, g_{19}), & (e_{33}, f_{35}, g_{5}), & (e_{33}, f_{36}, g_{21}), \\
(e_{34}, f_{33}, g_{23}), & (e_{34}, f_{34}, g_{28}), & (e_{34}, f_{35}, g_{21}), & (e_{34}, f_{36}, g_{11}), & (e_{35}, f_{33}, g_{11}), \\
(e_{35}, f_{34}, g_{27}), & (e_{35}, f_{35}, g_{32}), & (e_{35}, f_{36}, g_{34}), & (e_{36}, f_{33}, g_{27}), & (e_{36}, f_{34}, g_{5}), \\
(e_{36}, f_{35}, g_{26}), & (e_{36}, f_{36}, g_{17}), & (e_{37}, f_{37}, g_{32}), & (e_{37}, f_{38}, g_{10}), & (e_{37}, f_{39}, g_{37}), \\
(e_{37}, f_{40}, g_{31}), & (e_{38}, f_{37}, g_{15}), & (e_{38}, f_{38}, g_{32}), & (e_{38}, f_{39}, g_{39}), & (e_{38}, f_{40}, g_{25}), \\
(e_{39}, f_{37}, g_{30}), & (e_{39}, f_{38}, g_{22}), & (e_{39}, f_{39}, g_{28}), & (e_{39}, f_{40}, g_{10}), & (e_{40}, f_{37}, g_{35}), \\
(e_{40}, f_{38}, g_{40}), & (e_{40}, f_{39}, g_{15}), & (e_{40}, f_{40}, g_{28}). & &
\end{array}$$

With Lemma 60 we computed presentations for the extensions $\bar{\Gamma}_k^3$ (and extracted a sufficiently big set of relations to present the group). In particular, the Cayley graphs of these are the subgraphs of the 1-skeletons of the buildings $X_k^3 = \tilde{Y}_k^3$ generated by special vertices. The generators of these presentations are $g_1, \ldots, g_{40}$ and since the complexes $Y_k^3$ share the first 32 triangles, the four presentation share the following relations.

$$\begin{array}{llllll}
g_1g_3, & g_2g_{12}, & g_4g_{14}, & g_5g_{11}, & g_6g_8, & g_7g_{13}, \\
g_9g_{16}, & g_{10}g_{15}, & g_1^2g_{15}g_{11}, & g_1g_7g_{14}g_{12}, & g_1g_{15}g_{12}g_{14}, & g_2g_8^2g_{14}, \\
g_2g_9g_{12}g_4, & g_2g_{11}g_9g_{15}, & g_5g_{10}g_7g_{16}, & g_5g_{10}g_{11}g_7, & g_5g_{15}g_9g_8. &
\end{array}$$

The remaining relations of the presentation for $\bar{\Gamma}_1^3$ are

$$\begin{array}{lllll}
g_{17}^2, & g_{18}^2, & g_{19}g_{26}, & g_{20}g_{29}, & g_{21}^2, \\
g_{22}g_{31}, & g_{23}g_{34}, & g_{24}g_{36}, & g_{25}^2, & g_{27}^2, \\
g_{28}^2, & g_{30}^2, & g_{32}^2, & g_{33}^2, & g_{35}g_{39}, \\
g_{37}^2, & g_{38}^2, & g_{40}^2, & g_1g_{23}g_{25}g_{33}, & g_1g_{24}g_6g_{17}, \\
g_1g_{24}g_{19}g_{30}, & g_1g_{25}g_8g_{30}, & g_1g_{34}g_{36}g_{30}, & g_1g_{40}g_{17}g_{35}, & g_2g_{20}g_{22}g_{27}, \\
g_2g_{23}g_{12}g_{31}, & g_2g_{30}g_{23}g_{20}, & g_2g_{34}g_{30}g_{29}, & g_2g_{37}g_{12}g_{27}, & g_4g_{17}g_{14}g_{28}, \\
g_4g_{23}g_{14}g_{19}, & g_4g_{33}g_{14}g_{32}, & g_4g_{34}g_{38}g_{28}, & g_5g_{19}g_{11}g_{23}, & g_5g_{21}g_{34}g_{32}, \\
g_5g_{26}g_{17}g_{27}, & g_5g_{27}g_{23}g_{28}, & g_6g_{22}g_{33}g_{37}, & g_6g_{33}g_{37}g_{19}, & g_6g_{40}g_{36}g_{26}, \\
g_7g_{19}g_{40}g_{20}, & g_7g_{20}g_{18}g_{39}, & g_7g_{26}g_{29}g_{38}, & g_9g_{21}g_{16}g_{18}, & g_9g_{24}g_{31}g_{38},
\end{array}$$

$$g_{9}g_{27}g_{24}g_{31},\quad g_{9}g_{35}g_{16}g_{22},\quad g_{9}g_{39}g_{21}g_{36},\quad g_{10}g_{28}g_{39}g_{25},\quad g_{10}g_{28}g_{40}g_{39},$$
$$g_{10}g_{30}g_{31}g_{32},\quad g_{10}g_{31}g_{37}g_{32},\quad g_{10}g_{37}g_{32}g_{35},\quad g_{17}g_{19}g_{33}g_{26},\quad g_{17}g_{34}g_{33}g_{23},$$
$$g_{18}g_{26}g_{38}g_{23},\quad g_{18}g_{29}g_{40}g_{20},\quad g_{19}g_{20}g_{39}g_{20},\quad g_{20}g_{25}g_{29}g_{38},\quad g_{21}g_{22}g_{27}g_{31},$$
$$g_{22}g_{36}g_{39}g_{36},\quad g_{23}g_{28}g_{34}g_{32},\quad g_{25}g_{32}g_{37}g_{32}.$$

The remaining relations of the presentation for $\bar{\Gamma}_2^3$ are

$$g_{17}^2,\quad g_{18}^2,\quad g_{19}g_{26},\quad g_{20}^2,\quad g_{21}^2,$$
$$g_{22}g_{31},\quad g_{23}g_{34},\quad g_{24}g_{36},\quad g_{25}^2,$$
$$g_{27}^2,\quad g_{28}^2,\quad g_{29}^2,\quad g_{30}^2,\quad g_{32}^2,$$
$$g_{33}^2,\quad g_{35}g_{39},\quad g_{37}^2,\quad g_{38}^2,\quad g_{40}^2,$$
$$g_{1}g_{25}g_{31}g_{17},\quad g_{1}g_{40}g_{8}g_{37},\quad g_{2}g_{20}g_{22}g_{27},\quad g_{2}g_{23}g_{12}g_{22},\quad g_{2}g_{34}g_{30}g_{20},$$
$$g_{2}g_{37}g_{29}g_{31},\quad g_{4}g_{17}g_{18}g_{26},\quad g_{4}g_{17}g_{34}g_{38},\quad g_{4}g_{23}g_{14}g_{19},\quad g_{4}g_{34}g_{38}g_{28},$$
$$g_{5}g_{19}g_{11}g_{23},\quad g_{5}g_{26}g_{17}g_{27},\quad g_{5}g_{27}g_{23}g_{28},\quad g_{5}g_{33}g_{11}g_{32},\quad g_{5}g_{33}g_{26}g_{27},$$
$$g_{6}g_{17}g_{35}g_{37},\quad g_{6}g_{22}g_{8}g_{26},\quad g_{6}g_{25}g_{34}g_{36},\quad g_{6}g_{31}g_{17}g_{30},\quad g_{7}g_{19}g_{13}g_{39},$$
$$g_{7}g_{20}g_{39}g_{38},\quad g_{7}g_{25}g_{19}g_{29},\quad g_{7}g_{26}g_{20}g_{18},\quad g_{7}g_{40}g_{29}g_{35},\quad g_{9}g_{24}g_{18}g_{31},$$
$$g_{9}g_{24}g_{31}g_{38},\quad g_{9}g_{27}g_{24}g_{31},\quad g_{9}g_{35}g_{16}g_{22},\quad g_{9}g_{36}g_{38}g_{22},\quad g_{9}g_{39}g_{21}g_{36},$$
$$g_{10}g_{28}g_{40}g_{39},\quad g_{10}g_{30}g_{28}g_{39},\quad g_{10}g_{31}g_{32}g_{25},\quad g_{10}g_{32}g_{35}g_{40},\quad g_{10}g_{37}g_{22}g_{28},$$
$$g_{10}g_{37}g_{32}g_{35},\quad g_{17}g_{18}g_{32}g_{18},\quad g_{17}g_{22}g_{33}g_{39},\quad g_{17}g_{34}g_{33}g_{23},\quad g_{18}g_{20}g_{25}g_{20},$$
$$g_{19}g_{21}g_{34}g_{27},\quad g_{19}g_{24}g_{23}g_{24},\quad g_{20}g_{21}g_{20}g_{30},\quad g_{22}g_{25}g_{34}g_{40},\quad g_{27}g_{29}g_{37}g_{29}.$$

The remaining relations of the presentation for $\bar{\Gamma}_3^3$ are

$$g_{17}^2,\quad g_{18}g_{38},\quad g_{19}^2,\quad g_{20}^2,\quad g_{21}g_{27},$$
$$g_{22}^2,\quad g_{23}^2,\quad g_{24}g_{36},\quad g_{25}g_{40},\quad g_{26}^2,$$
$$g_{28}^2,\quad g_{29}^2,\quad g_{30}g_{37},\quad g_{31}^2,\quad g_{32}^2,$$
$$g_{33}^2,\quad g_{34}^2,\quad g_{35}^2,\quad g_{39}^2,\quad g_{1}g_{24}g_{30}g_{39},$$
$$g_{1}g_{34}g_{25}g_{33},\quad g_{1}g_{36}g_{37}g_{35},\quad g_{1}g_{40}g_{8}g_{30},\quad g_{1}g_{40}g_{17}g_{39},\quad g_{2}g_{23}g_{12}g_{31},$$
$$g_{2}g_{29}g_{21}g_{31},\quad g_{2}g_{30}g_{34}g_{20},\quad g_{2}g_{34}g_{37}g_{29},\quad g_{2}g_{37}g_{29}g_{22},\quad g_{4}g_{17}g_{21}g_{19},$$
$$g_{4}g_{18}g_{17}g_{26},\quad g_{4}g_{34}g_{32}g_{38},\quad g_{5}g_{19}g_{32}g_{21},\quad g_{5}g_{21}g_{11}g_{27},\quad g_{5}g_{27}g_{17}g_{23},$$
$$g_{5}g_{27}g_{26}g_{28},\quad g_{6}g_{17}g_{37}g_{26},\quad g_{6}g_{25}g_{23}g_{24},\quad g_{6}g_{31}g_{25}g_{24},\quad g_{6}g_{40}g_{36}g_{19},$$
$$g_{7}g_{20}g_{38}g_{39},\quad g_{7}g_{25}g_{26}g_{20},\quad g_{7}g_{29}g_{18}g_{35},\quad g_{7}g_{40}g_{19}g_{29},\quad g_{9}g_{24}g_{9}g_{36},$$
$$g_{9}g_{24}g_{22}g_{38},\quad g_{9}g_{24}g_{38}g_{31},\quad g_{9}g_{27}g_{24}g_{22},\quad g_{9}g_{39}g_{27}g_{24},\quad g_{10}g_{22}g_{15}g_{39},$$
$$g_{10}g_{28}g_{35}g_{25},\quad g_{10}g_{30}g_{15}g_{40},\quad g_{10}g_{31}g_{32}g_{40},\quad g_{10}g_{32}g_{39}g_{40},\quad g_{10}g_{32}g_{40}g_{35},$$
$$g_{17}g_{21}g_{33}g_{18},\quad g_{18}g_{20}g_{40}g_{29},\quad g_{18}g_{32}g_{21}g_{28},\quad g_{19}g_{24}g_{34}g_{36},\quad g_{19}g_{25}g_{26}g_{40},$$
$$g_{20}g_{27}g_{29}g_{30},\quad g_{22}g_{33}g_{35}g_{33},\quad g_{22}g_{36}g_{35}g_{24},\quad g_{24}g_{25}g_{33}g_{30}.$$

Then remaining relations of the presentation for $\bar{\Gamma}_4^3$ are

$$g_{17}^2,\quad g_{18}g_{38},\quad g_{19}g_{23},\quad g_{20}^2,\quad g_{21}g_{27},$$
$$g_{22}g_{39},\quad g_{24}^2,\quad g_{25}g_{40},\quad g_{26}g_{34},\quad g_{28}^2,$$
$$g_{29}^2,\quad g_{30}g_{37},\quad g_{31}g_{35},\quad g_{32}^2,\quad g_{33}^2,$$
$$g_{36}^2,\quad g_{1}g_{19}g_{36}g_{37},\quad g_{1}g_{24}g_{26}g_{30},\quad g_{1}g_{25}g_{33}g_{39},\quad g_{1}g_{36}g_{3}g_{33},$$
$$g_{1}g_{40}g_{8}g_{37},\quad g_{2}g_{26}g_{20}g_{27},\quad g_{2}g_{29}g_{22}g_{21},\quad g_{4}g_{17}g_{38}g_{26},\quad g_{4}g_{26}g_{18}g_{32},$$

$g4g33g26g18$, $g4g38g26g28$, $g5g17g21g23$, $g5g21g11g21$, $g5g21g23g28$,
$g5g23g27g32$, $g5g27g17g34$, $g5g33g11g32$, $g6g17g35g30$, $g6g25g36g19$,
$g6g39g37g24$, $g7g23g30g20$, $g7g40g23g29$, $g9g21g16g18$, $g9g22g38g36$,
$g9g27g36g31$, $g9g36g21g22$, $g10g28g39g40$, $g10g30g39g32$, $g10g32g30g39$,
$g10g37g32g35$, $g17g22g33g39$, $g17g25g36g40$, $g17g38g33g18$, $g18g19g38g34$,
$g18g20g37g29$, $g18g24g38g36$, $g18g31g27g39$, $g19g25g34g37$, $g19g29g39g29$,
$g20g26g20g35$, $g21g24g27g36$, $g22g37g35g25$, $g22g40g26g30$, $g24g26g24g34$,
$g25g32g40g28$.

Recall that by Proposition 57 the homotopy classes of the following loops lie in the finite residual of $\pi_1(S_{\mathrm{JW}})$.

$$x^{-1} * \bar{x} * x^{-1} * (\bar{y} * y^{-1})^2 * x * \bar{x}^{-1} * x * (\bar{y}^{-1} * y)^2,$$
$$y^{-1} * \bar{y} * y^{-1} * (\bar{x} * x^{-1})^2 * y * \bar{y}^{-1} * y * (\bar{x}^{-1} * x)^2.$$

By tracing the embedding we find that these loops correspond to the following loops in $\Gamma_k^3$.

$$f_1^{-1} * f_2 * f_1^{-1} * (f_4 * f_3^{-1})^2 * f_1 * f_2^{-1} * f_1 * (f_4^{-1} * f_3)^2,$$
$$f_3^{-1} * f_4 * f_3^{-1} * (f_2 * f_1^{-1})^2 * f_3 * f_4^{-1} * f_3 * (f_2^{-1} * f_1)^2.$$

And these loops are homotopy equivalent to the following ones.

$$\eta_1 := g_1 * g_{14}^{-1} * g_1 * (g_{12}^{-1} * g_3)^2 * g_1^{-1} * g_{14} * g_1^{-1} * (g_{12} * g_3^{-1})^2,$$
$$\eta_2 := g_3 * g_{12}^{-1} * g_3 * (g_{14}^{-1} * g_1)^2 * g_3^{-1} * g_{12} * g_3^{-1} * (g_{14} * g_1^{-1})^2.$$

We can interpret $\eta_1$ and $\eta_2$ as elements in $\bar{\Gamma}_k^3$ and deduce that they lie in the finite residual. With the help of a computer we compute the indexes of the normal closures $\langle\langle \eta_1, \eta_2 \rangle\rangle$ in the $\bar{\Gamma}_k^3$ and they are 8, 8, 16, 16, respectively. In particular, $\langle\langle \eta_1, \eta_2 \rangle\rangle$ is the finite residual in every case. Using the algorithm in Subsection 3.2.2, we compute balls in the Cayley graphs for the geometric presentation of $\Gamma_k^3$ and study their automorphism group. At the time writing an ordinary office machine is not capable of computing the automorphism group of the 4-ball in a reasonable time. However, we can compute the automorphism group of the 3-ball, and apply the following lemma.

**Lemma 92.** *Let $Z$ be a Cayley graph for a group $\Gamma$ with generating set $S$. For $g \in Z, r \in \mathbb{N}$ denote the ball of radius $r$ centered at $g$ with $B_r(g)$ and the automorphism group of $B_r(g)$ that fixes the center with $A^r(g)$. We denote the image of the projection of $A^r(g)$ to the automorphism group of the $m$-ball centered at $g$ with $A_m^r(g)$ for $m \leq r$. Define*

$$E = \{g \in B_1(1) \mid \mathrm{Stab}(A_1^3, g) = 1\} \quad \text{and} \quad F = B_1(1) \cup \bigcup_{g \in E} B_1(g^{-1}).$$

*If the pointwise stabilizer of $F$ in $A_2^3(1)$ is trivial, then the pointwise stabilizer of $B_1(1)$ in* $\mathrm{Aut}(Z)$ *is trivial and in particular the index of* $\Gamma$ *in* $\mathrm{Aut}(Z)$ *is at most* $|A_1^3(1)|$.

*Proof.* If we have $g \in E$, then any $\sigma \in \mathrm{Aut}(Z)$ that fixes 1 and $g$ fixes $B_1(1)$. Translating this phenomenon to an arbitrary $h \in Z$ yields that if $\sigma \in \mathrm{Aut}(Z)$ fixes $h$ and $hg$, then it fixes $B_1(h)$. In particular, every $\sigma \in \mathrm{Aut}(Z)$ that fixes $B_1(1)$ pointwise, fixes $F$ pointwise since it fixes $g^{-1}$ and $g^{-1}g = 1$. Now assume that the pointwise stabilizer of $F$ in $A_2^3(1)$ is trivial. Then every automorphism of $Z$ that fixes $B_1(1)$ pointwise, fixes $F$ pointwise by the previous discussion, and therefore fixes $B_2(1)$ pointwise. It follows by induction that $\sigma$ is trivial. Therefore, the projection of $\mathrm{Stab}(\mathrm{Aut}(Z), 1)$ to $A^1(1)$ is injective and the image lies in $A_1^3(1)$. $\square$

We checked that the lattices $\Gamma_k^3$ satisfy the assumptions of Lemma 92 and as it turns out the index of $\Gamma_k^3$ in the automorphism group of the Cayley graph (which equals $\mathrm{Aut}(X_k^3)$) is at most 4 in every case. On the other hand, the automorphism group of the complexes $Y_k^3$ is always 8. Therefore, the group of deck transformations covering $\mathrm{Aut}(Y_k^3)$ is $\mathrm{Aut}(X_k^3)$.

# Appendix B

# Small $\tilde{A}_2$-GABs and related data

In this section we list the 13 GABs from Theorem 86 and provide some more data on these GABs, their lattices and their buildings. The GABs all share the following combinatorial properties: they consists of three vertices $u, v, w$ and 21 edges $e_1, \dots, e_7, f_1, \dots, f_7, g_1, \dots, g_7$ and 21 triangles. The edges $e_i$ point from $w$ to $u$, the edges $f_i$ point from $u$ to $v$ and the edges $g_i$ point from $v$ to $w$. The GABs can be defined via the boundaries of their triangles and we do so in Table B.1.

| GAB | Triangles |
|---|---|
| $\Delta_1$ | $(e_1, f_1, g_1), (e_1, f_2, g_3), (e_1, f_4, g_2), (e_2, f_1, g_4), (e_2, f_3, g_1), (e_2, f_7, g_5), (e_3, f_1, g_2), (e_3, f_5, g_6), (e_3, f_6, g_4), (e_4, f_2, g_1), (e_4, f_3, g_6), (e_4, f_5, g_7), (e_5, f_2, g_7), (e_5, f_6, g_3), (e_5, f_7, g_4), (e_6, f_3, g_5), (e_6, f_4, g_3), (e_6, f_6, g_6), (e_7, f_4, g_5), (e_7, f_5, g_2), (e_7, f_7, g_7)$ |
| $\Delta_2$ | $(e_1, f_1, g_3), (e_1, f_2, g_1), (e_1, f_3, g_2), (e_2, f_1, g_4), (e_2, f_4, g_1), (e_2, f_5, g_5), (e_3, f_1, g_7), (e_3, f_6, g_1), (e_3, f_7, g_6), (e_4, f_2, g_6), (e_4, f_4, g_4), (e_4, f_7, g_2), (e_5, f_2, g_7), (e_5, f_5, g_4), (e_5, f_6, g_3), (e_6, f_3, g_7), (e_6, f_4, g_2), (e_6, f_6, g_5), (e_7, f_3, g_5), (e_7, f_5, g_6), (e_7, f_7, g_3)$ |
| $\Delta_3$ | $(e_1, f_1, g_2), (e_1, f_2, g_3), (e_1, f_3, g_1), (e_2, f_1, g_4), (e_2, f_4, g_5), (e_2, f_5, g_1), (e_3, f_1, g_7), (e_3, f_6, g_1), (e_3, f_7, g_6), (e_4, f_2, g_6), (e_4, f_4, g_2), (e_4, f_6, g_4), (e_5, f_2, g_7), (e_5, f_5, g_2), (e_5, f_7, g_5), (e_6, f_3, g_7), (e_6, f_4, g_3), (e_6, f_7, g_4), (e_7, f_3, g_5), (e_7, f_5, g_6), (e_7, f_6, g_3)$ |
| $\Delta_4$ | $(e_1, f_1, g_1), (e_1, f_2, g_2), (e_1, f_3, g_3), (e_2, f_1, g_5), (e_2, f_4, g_1), (e_2, f_7, g_4), (e_3, f_1, g_6), (e_3, f_5, g_4), (e_3, f_6, g_2), (e_4, f_2, g_5), (e_4, f_4, g_2), (e_4, f_5, g_7), (e_5, f_2, g_4), (e_5, f_6, g_7), (e_5, f_7, g_3), (e_6, f_3, g_5), (e_6, f_4, g_3), (e_6, f_6, g_6), (e_7, f_3, g_7), (e_7, f_5, g_1), (e_7, f_7, g_6)$ |
| $\Delta_5$ | $(e_1, f_1, g_2), (e_1, f_2, g_3), (e_1, f_3, g_1), (e_2, f_1, g_1), (e_2, f_4, g_5), (e_2, f_7, g_4), (e_3, f_1, g_4), (e_3, f_5, g_2), (e_3, f_6, g_6), (e_4, f_2, g_1), (e_4, f_4, g_7), (e_4, f_5, g_6), (e_5, f_2, g_7), (e_5, f_6, g_4), (e_5, f_7, g_3), (e_6, f_3, g_5), (e_6, f_4, g_2), (e_6, f_6, g_7), (e_7, f_3, g_6), (e_7, f_5, g_3), (e_7, f_7, g_5)$ |
| $\Delta_6$ | $(e_1, f_1, g_2), (e_1, f_2, g_1), (e_1, f_3, g_3), (e_2, f_1, g_1), (e_2, f_4, g_4), (e_2, f_5, g_5), (e_3, f_1, g_4), (e_3, f_6, g_2), (e_3, f_7, g_6), (e_4, f_2, g_6), (e_4, f_4, g_3), (e_4, f_6, g_5), (e_5, f_2, g_7), (e_5, f_5, g_4), (e_5, f_7, g_3), (e_6, f_3, g_5), (e_6, f_4, g_7), (e_6, f_7, g_2), (e_7, f_3, g_1), (e_7, f_5, g_6), (e_7, f_6, g_7)$ |
| $\Delta_7$ | $(e_1, f_1, g_2), (e_1, f_2, g_7), (e_1, f_3, g_5), (e_2, f_1, g_1), (e_2, f_4, g_4), (e_2, f_7, g_5), (e_3, f_1, g_3), (e_3, f_5, g_7), (e_3, f_6, g_4), (e_4, f_2, g_4), (e_4, f_4, g_2), (e_4, f_5, g_6), (e_5, f_2, g_1), (e_5, f_6, g_6), (e_5, f_7, g_7), (e_6, f_3, g_6), (e_6, f_4, g_5), (e_6, f_6, g_3), (e_7, f_3, g_1), (e_7, f_5, g_2), (e_7, f_7, g_3)$ |

| | |
|---|---|
| $\Delta_8$ | $(e_1, f_1, g_2), (e_1, f_2, g_1), (e_1, f_7, g_3), (e_2, f_1, g_3), (e_2, f_3, g_7), (e_2, f_6, g_4), (e_3, f_1, g_1),$<br>$(e_3, f_4, g_4), (e_3, f_5, g_5), (e_4, f_2, g_4), (e_4, f_3, g_6), (e_4, f_4, g_2), (e_5, f_2, g_5), (e_5, f_5, g_2),$<br>$(e_5, f_6, g_7), (e_6, f_3, g_1), (e_6, f_5, g_7), (e_6, f_7, g_6), (e_7, f_4, g_6), (e_7, f_6, g_3), (e_7, f_7, g_5)$ |
| $\Delta_9$ | $(e_1, f_2, g_1), (e_1, f_4, g_2), (e_1, f_7, g_3), (e_2, f_1, g_1), (e_2, f_6, g_4), (e_2, f_7, g_5), (e_3, f_3, g_1),$<br>$(e_3, f_5, g_6), (e_3, f_7, g_7), (e_4, f_2, g_4), (e_4, f_5, g_2), (e_4, f_6, g_6), (e_5, f_1, g_2), (e_5, f_2, g_7),$<br>$(e_5, f_3, g_5), (e_6, f_3, g_6), (e_6, f_4, g_5), (e_6, f_6, g_3), (e_7, f_1, g_3), (e_7, f_4, g_4), (e_7, f_5, g_7)$ |
| $\Delta_{10}$ | $(e_1, f_2, g_1), (e_1, f_5, g_2), (e_1, f_6, g_3), (e_2, f_3, g_1), (e_2, f_4, g_5), (e_2, f_5, g_4), (e_3, f_1, g_1),$<br>$(e_3, f_5, g_7), (e_3, f_7, g_6), (e_4, f_1, g_6), (e_4, f_2, g_4), (e_4, f_4, g_2), (e_5, f_2, g_3), (e_5, f_3, g_7),$<br>$(e_5, f_7, g_4), (e_6, f_4, g_3), (e_6, f_6, g_6), (e_6, f_7, g_5), (e_7, f_1, g_2), (e_7, f_3, g_5), (e_7, f_6, g_7)$ |
| $\Delta_{11}$ | $(e_1, f_1, g_3), (e_1, f_2, g_5), (e_1, f_3, g_6), (e_2, f_1, g_1), (e_2, f_4, g_2), (e_2, f_7, g_3), (e_3, f_1, g_2),$<br>$(e_3, f_5, g_6), (e_3, f_6, g_4), (e_4, f_2, g_1), (e_4, f_4, g_4), (e_4, f_5, g_5), (e_5, f_2, g_4), (e_5, f_6, g_3),$<br>$(e_5, f_7, g_7), (e_6, f_3, g_1), (e_6, f_4, g_7), (e_6, f_6, g_6), (e_7, f_3, g_7), (e_7, f_5, g_2), (e_7, f_7, g_5)$ |
| $\Delta_{12}$ | $(e_1, f_1, g_2), (e_1, f_2, g_1), (e_1, f_4, g_3), (e_2, f_1, g_1), (e_2, f_5, g_4), (e_2, f_7, g_5), (e_3, f_1, g_6),$<br>$(e_3, f_3, g_1), (e_3, f_6, g_7), (e_4, f_4, g_2), (e_4, f_6, g_6), (e_4, f_7, g_4), (e_5, f_3, g_7), (e_5, f_4, g_5),$<br>$(e_5, f_5, g_2), (e_6, f_2, g_3), (e_6, f_3, g_5), (e_6, f_7, g_6), (e_7, f_2, g_4), (e_7, f_5, g_7), (e_7, f_6, g_3)$ |
| $\Delta_{13}$ | $(e_1, f_1, g_2), (e_1, f_3, g_1), (e_1, f_6, g_3), (e_2, f_1, g_1), (e_2, f_4, g_5), (e_2, f_7, g_4), (e_3, f_1, g_7),$<br>$(e_3, f_2, g_1), (e_3, f_5, g_6), (e_4, f_2, g_4), (e_4, f_3, g_6), (e_4, f_4, g_2), (e_5, f_3, g_5), (e_5, f_5, g_2),$<br>$(e_5, f_7, g_7), (e_6, f_4, g_3), (e_6, f_5, g_4), (e_6, f_6, g_7), (e_7, f_2, g_3), (e_7, f_6, g_6), (e_7, f_7, g_5)$ |

TABLE B.1: The GABs $\Delta_1, \ldots, \Delta_{13}$.

| GAB | Generators of Aut($\Delta$) |
|---|---|
| $\Delta_1$ | $(e_2\, e_7\, e_5)(e_3\, e_6\, e_4)(f_1\, f_4\, f_2)(f_3\, f_5\, f_6)(g_1\, g_2\, g_3)(g_4\, g_5\, g_7),$<br>$(e_1\, g_3\, f_6\, e_3\, g_2\, f_4\, e_6\, g_6\, f_5\, e_7\, g_5\, f_3\, e_4\, g_7\, f_7\, e_2\, g_1\, f_2\, e_5\, g_4\, f_1)$ |
| $\Delta_2$ | $(e_1\, e_5)(e_2\, e_6)(e_4\, e_7)(f_1\, g_7)(f_2\, g_3)(f_3\, g_4)(f_4\, g_5)(f_5\, g_2)(f_6\, g_1)(f_7\, g_6),$<br>$(e_1\, g_1\, f_2)(e_2\, g_6\, f_3)(e_3\, g_7\, f_1)(e_4\, g_2\, f_4)(e_5\, g_3\, f_6)(e_6\, g_4\, f_7)(e_7\, g_5\, f_5)$ |
| $\Delta_3$ | $(e_1\, e_6\, e_5\, e_7\, e_3\, e_2\, e_4)(f_1\, f_4\, f_2\, f_3\, f_7\, f_5\, f_6)(g_1\, g_4\, g_2\, g_3\, g_7\, g_5\, g_6),$<br>$(e_1\, f_7)(e_2\, f_5\, e_7\, f_6\, e_4\, f_4)(e_3\, f_2\, e_6\, f_1\, e_5\, f_3)(g_1\, g_6\, g_3\, g_4\, g_2\, g_5)$ |
| $\Delta_4$ | $(e_3\, e_7)(e_4\, e_6)(f_1\, g_1)(f_2\, g_3)(f_3\, g_2)(f_4\, g_5)(f_5\, g_6)(f_6\, g_7)(f_7\, g_4),$<br>$(e_1\, g_5\, f_4)(e_2\, g_1\, f_1)(e_3\, g_4\, f_5)(e_4\, g_2\, f_2)(e_5\, g_7\, f_6)(e_6\, g_3\, f_3)(e_7\, g_6\, f_7),$<br>$(e_1\, g_5\, f_4)(e_2\, g_3\, f_2)(e_3\, g_6\, f_6)(e_4\, g_1\, f_3)(e_5\, g_4\, f_7)(e_6\, g_2\, f_1)(e_7\, g_7\, f_5)$ |

| | |
|---|---|
| $\Delta_5$ | $(e_1\, e_6)(e_3\, e_7)(f_1\, g_5)(f_2\, g_7)(f_3\, g_2)(f_4\, g_1)(f_5\, g_6)(f_6\, g_3)(f_7\, g_4)$,<br>$(e_1\, g_6\, f_4)(e_2\, g_3\, f_6)(e_3\, g_5\, f_2)(e_4\, g_2\, f_3)(e_5\, g_4\, f_7)(e_6\, g_1\, f_5)(e_7\, g_7\, f_1)$ |
| $\Delta_6$ | $(e_1\, e_2\, e_3)(e_5\, e_6\, e_7)(f_2\, f_4\, f_6)(f_3\, f_5\, f_7)(g_1\, g_4\, g_2)(g_3\, g_5\, g_6)$,<br>$(e_1\, g_1)(e_2\, g_2)(e_3\, g_4)(e_4\, g_7)(e_5\, g_6)(e_6\, g_5)(e_7\, g_3)(f_4\, f_6)(f_5\, f_7)$ |
| $\Delta_7$ | $(e_1\, f_7)(e_2\, f_3)(e_3\, f_5)(e_4\, f_6)(e_5\, f_2)(e_6\, f_4)(e_7\, f_1)(g_2\, g_3)(g_4\, g_6)$ |
| $\Delta_8$ | $(e_1\, e_4\, e_5)(e_2\, e_7\, e_6)(f_1\, f_4\, f_5)(f_3\, f_6\, f_7)(g_1\, g_4\, g_5)(g_3\, g_6\, g_7)$,<br>$(e_1\, f_1)(e_2\, f_7)(e_3\, f_2)(e_4\, f_5)(e_5\, f_4)(e_6\, f_3)(e_7\, f_6)(g_4\, g_5)(g_6\, g_7)$ |
| $\Delta_9$ | $(e_2\, e_5\, e_7)(e_3\, e_4\, e_6)(f_2\, f_4\, f_7)(f_3\, f_5\, f_6)(g_1\, g_2\, g_3)(g_4\, g_5\, g_7)$,<br>$(e_1\, f_1)(e_2\, f_2)(e_3\, f_3)(e_4\, f_6)(e_5\, f_7)(e_6\, f_5)(e_7\, f_4)(g_2\, g_3)(g_5\, g_7)$ |
| $\Delta_{10}$ | $(e_1\, e_2\, e_3)(e_4\, e_5\, e_7)(f_1\, f_2\, f_3)(f_4\, f_7\, f_6)(g_2\, g_4\, g_7)(g_3\, g_5\, g_6)$ |
| $\Delta_{11}$ | $(e_1\, f_4)(e_2\, f_2\, e_6\, f_1\, e_4\, f_3)(e_3\, f_5\, e_7\, f_7\, e_5\, f_6)(g_2\, g_5\, g_7\, g_3\, g_4\, g_6)$ |
| $\Delta_{12}$ | $(e_2\, e_4\, e_6)(e_3\, e_5\, e_7)(f_1\, f_4\, f_2)(f_3\, f_5\, f_6)(g_1\, g_2\, g_3)(g_4\, g_6\, g_5)$ |
| $\Delta_{13}$ | $(e_1\, e_2\, e_7)(e_3\, e_5\, e_6)(f_1\, f_7\, f_6)(f_2\, f_3\, f_4)(g_1\, g_5\, g_3)(g_2\, g_4\, g_6)$ |

TABLE B.2: Automorphism groups of the GABs $\Delta_1, \ldots, \Delta_{13}$.

| GAB | Corresponding automorphisms in Aut($\Delta$) | Extension |
|---|---|---|
| $\Delta_1$ | $\langle(e_1\, e_6\, e_4\, e_5\, e_3\, e_7\, e_2)(f_1\, f_4\, f_3\, f_2\, f_6\, f_5\, f_7)(g_1\, g_3\, g_6\, g_7\, g_4\, g_2\, g_5)\rangle$ | $\Sigma_1$ |
| $\Delta_1$ | $\langle(e_2\, e_7\, e_5)(e_3\, e_6\, e_4)(f_1\, f_4\, f_2)(f_3\, f_5\, f_6)(g_1\, g_2\, g_3)(g_4\, g_5\, g_7)$,<br>$(e_1\, e_6\, e_4\, e_5\, e_3\, e_7\, e_2)(f_1\, f_4\, f_3\, f_2\, f_6\, f_5\, f_7)(g_1\, g_3\, g_6\, g_7\, g_4\, g_2\, g_5)\rangle$ | $\Omega_3$ |
| $\Delta_1$ | $\langle(e_1\, g_6\, f_7)(e_2\, g_3\, f_5)(e_3\, g_5\, f_2)(e_4\, g_4\, f_4)(e_5\, g_2\, f_3)(e_6\, g_7\, f_1)(e_7\, g_1\, f_6)\rangle$ | $\Lambda_{A.1}$ |
| $\Delta_1$ | $\langle(e_1\, g_2\, f_1)(e_2\, g_3\, f_5)(e_3\, g_1\, f_4)(e_4\, g_5\, f_6)(e_5\, g_7\, f_7)(e_6\, g_6\, f_3)(e_7\, g_4\, f_2)\rangle$ | $\Lambda_{A.2}$ |
| $\Delta_1$ | $\langle(e_1\, g_1\, f_1)(e_2\, g_2\, f_2)(e_3\, g_3\, f_3)(e_4\, g_4\, f_4)(e_5\, g_5\, f_5)(e_6\, g_6\, f_6)(e_7\, g_7\, f_7)\rangle$ | $\Lambda_{A.3}$ |
| $\Delta_2$ | $\langle(e_1\, f_2\, g_1)(e_2\, f_3\, g_6)(e_3\, f_1\, g_7)(e_4\, f_4\, g_2)(e_5\, f_6\, g_3)(e_6\, f_7\, g_4)(e_7\, f_5\, g_5)\rangle$ | $\Lambda_{A.4}$ |
| $\Delta_3$ | $\langle(e_1\, e_6\, e_5\, e_7\, e_3\, e_2\, e_4)(f_1\, f_4\, f_2\, f_3\, f_7\, f_5\, f_6)(g_1\, g_4\, g_2\, g_3\, g_7\, g_5\, g_6)\rangle$ | $\Sigma_1$ |
| $\Delta_3$ | $\langle(e_2\, e_7\, e_4)(e_3\, e_6\, e_5)(f_1\, f_3\, f_2)(f_4\, f_5\, f_6)(g_1\, g_3\, g_2)(g_4\, g_5\, g_6)$,<br>$(e_1\, e_6\, e_5\, e_7\, e_3\, e_2\, e_4)(f_1\, f_4\, f_2\, f_3\, f_7\, f_5\, f_6)(g_1\, g_4\, g_2\, g_3\, g_7\, g_5\, g_6)\rangle$ | $\Omega_3$ |

| | | |
|---|---|---|
| $\Delta_4$ | $\langle (e_1\, g_5\, f_4)(e_2\, g_2\, f_3)(e_3\, g_7\, f_7)(e_4\, g_3\, f_1)(e_5\, g_6\, f_5)(e_6\, g_1\, f_2)(e_7\, g_4\, f_6) \rangle$ | $\Lambda_{B.1}$ |
| $\Delta_4$ | $\langle (e_1\, f_4\, g_5)(e_2\, f_2\, g_3)(e_3\, f_6\, g_6)(e_4\, f_3\, g_1)(e_5\, f_7\, g_4)(e_6\, f_1\, g_2)(e_7\, f_5\, g_7) \rangle$ | $\Lambda_{B.2}$ |
| $\Delta_4$ | $\langle (e_1\, f_4\, g_5)(e_2\, f_1\, g_1)(e_3\, f_5\, g_4)(e_4\, f_2\, g_2)(e_5\, f_6\, g_7)(e_6\, f_3\, g_3)(e_7\, f_7\, g_6) \rangle$ | $\Lambda_{B.3}$ |
| $\Delta_5$ | $\langle (e_1\, f_4\, g_6)(e_2\, f_6\, g_3)(e_3\, f_2\, g_5)(e_4\, f_3\, g_2)(e_5\, f_7\, g_4)(e_6\, f_5\, g_1)(e_7\, f_1\, g_7) \rangle$ | $\Lambda_{C.1}$ |

TABLE B.3: Extensions of the lattices $\Gamma_1, \ldots, \Gamma_5$. The extension is the group of deck transformations covering the automorphisms indicated.

| Building | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ | $X_{11}$ | $X_{12}$ | $X_{13}$ |
|---|---|---|---|---|---|---|---|
| $\lvert A(u))\rvert$ | 10752 | 1536 | 1536 | 1536 | 3072 | 1536 | 1536 |
| $\lvert A^1(u)\rvert$ | 42 | 6 | 6 | 3 | 6 | 3 | 3 |
| $\lvert A^3_{(1)}(u)\rvert$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

TABLE B.4: Balls of radius four in the exotic buildings $X_7, \ldots, X_{13}$. We denote by $A(u)$ the automorphism group of a ball of radius four in the building around a lift $\tilde{u}$ of $u$. We denote by $A^1(u)$ the group of automorphisms on the 1-ball around $\tilde{u}$ induced by $A(u)$. We denote by $A^3_{(1)}(u)$ the groups of automorphism of the 3-ball induced by the pointwise stabilizer of the 1-ball in $A(u)$.

| Lattice | $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ | $\Gamma_4$ | $\Gamma_5$ | $\Gamma_6$ |
|---|---|---|---|---|---|---|
| $\lvert\Gamma : \Gamma^{(2)}\rvert$ | 14336 | 441 | 448 | 8 | 16 | 1 |

TABLE B.5: The lattices $\Gamma_1, \ldots, \Gamma_6$ can be distinguished by their quotients modulo their second derived subgroups.

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# Declaration of Authorship

I declare that I have completed this dissertation single-handedly without the unauthorized help of a second party and only with the assistance acknowledged therein. I have appropriately acknowledged and cited all text passages that are derived verbatim from or are based on the content of published work of others, and all information relating to verbal communications. I consent to the use of an anti-plagiarism software to check my thesis. I have abided by the principles of good scientific conduct laid down in the charter of the Justus Liebig University Giessen „Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis“ in carrying out the investigations described in the dissertation.

Thomas Titz Mite