HOMOCLINIC SOLUTION AND CHAOS IN $\dot{x}(t) = f(x(t-1))$

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1. INTRODUCTION

IN DIFFERENTIAL delay equations

$$\dot{x}(t) = f(x(t-1)) \tag{1}$$

and

$$\dot{x}(t) = -\sigma x(t) + f(x(t-1)), \quad \sigma > 0, \tag{2}$$

with a continuous function $f\colon R \to R$, the condition

$$xf(x) < 0 \quad \text{for all } x \neq 0 \tag{3}$$

makes it possible to single out a class of solutions which play a major role—"slowly oscillating" solutions. These are solutions which have all their zeros in some unbounded interval spaced at distances larger than the delay $d = 1$. In some cases [13] it can be proved that initial functions $\varphi\colon [-1, 0] \to R$ of slowly oscillating solutions form an open and dense set in the space C of continuous functions on $[-1, 0]$. For properties of slowly oscillating periodic solutions, see, e.g., Nussbaum's survey [10], and [12]. Slowly oscillating solutions may also be turbulent, or "chaotic". A first example, with f discontinuous, was found by Peters [11].

In the case of equation (1), condition (3) represents negative feedback: A deviation from the equilibrium $x = 0$, say $x(t-1) > 0$, is followed by a move in the opposite direction $(\dot{x}(t) < 0)$.

However, there are equations (1, 2) which are closely related to applications, and where the negative feedback condition (3) is only true in a certain neighbourhood of $x = 0$. Furumochi [3] studied equation

$$\dot{x}(t) = \delta - \sin(x(t-1) + \omega) \tag{4}$$

with $0 \leqslant \delta = \sin\omega$, $0 \leqslant \omega < \pi$, which models phase-locked loop control of high frequency generators, see, e.g., [2].

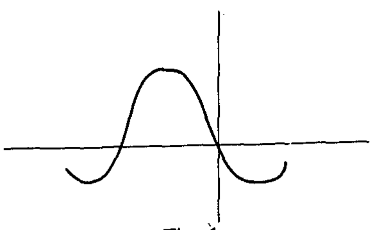


Fig. 1.

Equations of Lasota and Wazewska [5, 6], Mackey and Glass [4, 8] for the density of red blood cells transform into type (2) with hump functions as nonlinearities. For example, $f(x) = g(x + x_0)$ where $g(x) = (cx)^8 e^{-x}$, $c > 0$, $x_0 = g(x_0)$, see [6].

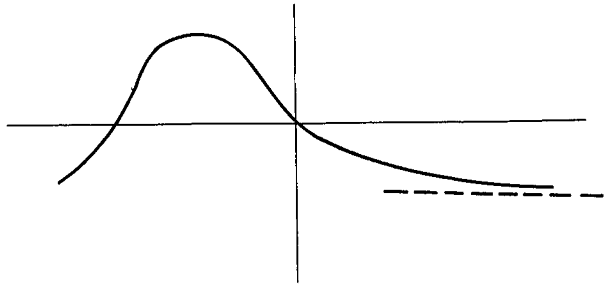


Fig. 2.

The same type of equation may be used to model the respiratory system [4, 8], and simple neuronal networks with delayed inhibitory effects [1].

All these examples have in common that there are large domains in parameter space where no workable class of solutions (like the slowly oscillating ones above) has been found, and where nothing is known about specific solutions—periodic ones or "periodic solutions of the second kind" as defined in [3]. Numerical experiments indicate that solutions become highly irregular, certainly not slowly oscillating [4, 8]. An example of an der Heiden, with f discontinuous, shows periodic solutions with more complicated oscillatory behavior; compare also [1].

In this paper we study equation (1) in a case where condition (3) does not hold globally: f will be a function with zeros at $x = -1$ and $x = 0$, with $f < 0$ for $x < -1$, $0 < f$ in $(-1, 0)$, $f < 0$ for $x > 0$. All solutions considered are bounded, $x(t) \in (-3, 2)$, so that our example may be regarded as close to equation (4).

We prove chaotic behavior in the sense of Li and Yorke [7] for a Poincare map which describes the behavior of solutions in the state space C. Chaos is caused by a periodic orbit

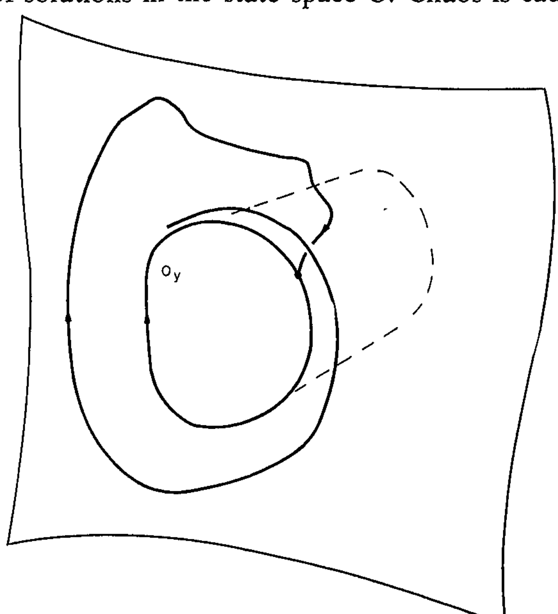


Fig. 3.

$o_y \subset C$ around the equilibrium $x = -1$ where (3) is violated, and by a solution which is homoclinic with respect to o_y and hits o_y precisely at a finite time t.

One may think of this situation in R^3, as depicted in figure 3.

A similar structure should also exist in case of the original equation (4), and for the red blood cell equation.

The approach presented here extends to equations without any nontrivial slowly oscillating solution, at the cost of more tedious calculations. It can also be modified as to apply to examples with decay term ($\sigma > 0$ in equation (2)) which correspond to the red blood cell equation. A joint paper with U. an der Heiden on this is in preparation.

2. NOTATION

C denotes the space of continuous functions $\varphi: [-1, 0] \to R$, with supremum-norm. A solution of equation (1) is either a continuous function $x: [-1, \infty) \to R$ which is differentiable for $t > 0$ and satisfies (1) for $t > 0$, or a differentiable function $x: R \to R$ which satisfies (1) everywhere. For $t - 1$ in the domain of a solution x, x_t denotes the element in C given by $x_t(s) := x(t+s)$, $-1 \leq s \leq 0$.

3. THE EQUATION (f)

We consider equation

$$\dot{x}(t) = f(x(t-1)) \tag{f}$$

for a smooth function $f = f_\delta$, $0 < \delta < 10^{-2}$, with the following properties:

$$f(x) \begin{cases} = -2 \quad \text{for } x < -1 - \delta \\ \text{strictly increasing from } -2 \text{ to } 2 \text{ on } [-1-\delta, -1+\delta] \\ \text{with } f(-1+x) = -f(-1-x) \quad \text{for } |x| < \delta \\ = 2 \quad \text{for } -1-\delta < x < -\delta \\ = -f(-1+x) \quad \text{for } |x| < \delta \\ = -2 \quad \text{for } \delta < x < 1 - \delta \\ \text{strictly increasing from } -2 \text{ to } -1 \text{ on } [1-\delta, 1+\delta] \\ \text{with } f(1+x) - 3/2 = 3/2 - f(1-x) \quad \text{for } |x| < \delta \\ = -1 \quad \text{for } x > 1 + \delta \end{cases}$$

Existence, uniqueness and continuous dependence on bounded intervals for the initial value problem

$$\dot{x}(t) = f(x(t-1)) \quad \text{for } t > 0, \quad x_0 = \varphi \in C$$

follow from the formula

$$x(t) - x(n) = \int_{n-1}^{t-1} f \circ x \quad \text{for } t \in [n, n+1], \quad n \in N_0.$$

The choice of a nonlinearity f which is constant on long intervals (compared to the δ-neighborhoods where f is not constant) is influenced by an observation which goes back to, at least, Glass and Mackey [4]. For step functions in (1) or (2), periodic solutions are easy

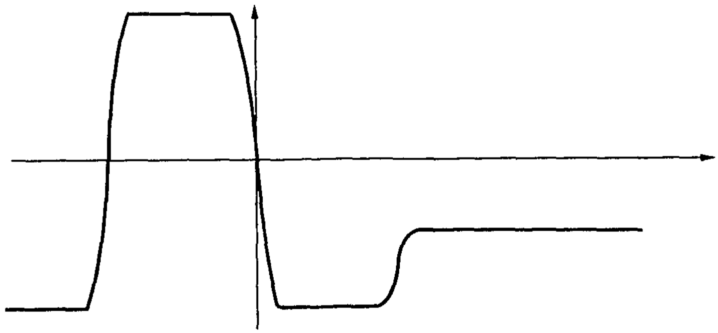


Fig. 4.

to find and to compute. For our smooth nonlinearity it will also be possible to explicitly compute certain solutions. Moreover we shall describe solutions in terms of just one real coordinate. This idea—how to introduce coordinates and how to get finite-dimensional Poincare maps for some differential delay equations—comes from an unpublished study (1978) of the equation $\dot{x}(t) = -\operatorname{sgn} x(t-1)$ where we showed that initial functions of slowly oscillating solutions are dense in C.

4. PERIODIC SOLUTION AND POINCARE MAP

Let us look at initial functions which define solutions oscillating around the equilibrium $x(t) \equiv -1$ for some time. Define D to be the set of all $\varphi \in C$ with the following properties: $\varphi(0) = -1$, and $\varphi(-v) = -1$ for exactly one $-v < 0$, $-1 + \delta/2 \leqslant -v \leqslant -\delta$; $\dot{\varphi} = 2$ in $(-\delta/2, 0)$ and $\dot{\varphi} = -2$ in $(-v - \delta/2, -v + \delta/2)$, $\varphi \leqslant -1 - \delta$ in $(-v + \delta/2, -\delta/2)$, $-1 + \delta \leqslant \varphi \leqslant -\delta$ in $(-1, -v - \delta/2)$.

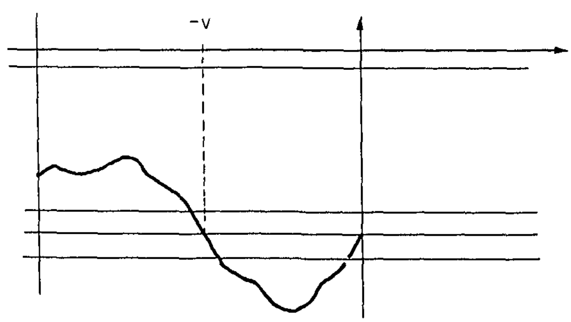


Fig. 5.

The projection V: $\varphi \to v$ maps D continuously onto $[\delta, 1 - \delta/2]$ but is not injective.

By our choice of the function f we see that the solution x with $x_0 = \varphi \in D$ increases with slope 2 on $(0, 1 - v - \delta/2)$.

On $(1-v-\delta/2, 1-v+\delta/2)$, we have

$$\begin{aligned} x(t) &= x(1-v-\delta/2) + \int_{1-v-\delta/2}^{t} f(x(s-1))\,ds \\ &= \ldots \qquad\qquad + \int_{-v-\delta/2}^{t-1} f(\varphi(s))\,ds \\ &= \ldots \qquad\qquad + \int_{-v-\delta/2}^{t-1} f(-2(s+v)-1)\,ds \\ &= \ldots \qquad\qquad + p(t-1+v) \end{aligned}$$

where

$$p(t):=\int_{-\delta/2}^{t} f(-2s-1)\,ds \quad \text{for } |t|\leqslant \delta/2.$$

Obviously, p does not depend on $\varphi\in D$. We have $0\leqslant p\leqslant \delta$.

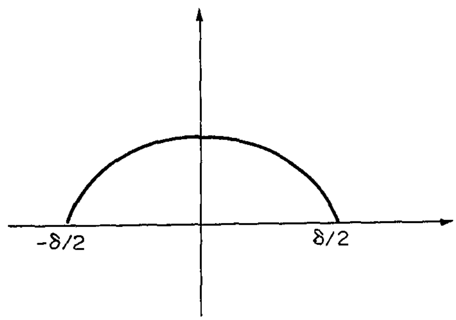


Fig. 6.

On $(1-v+\delta/2, 1-\delta/2)$, x decreases with slope -2 from

$$x(1-v+\delta/2)=x(1-v-\delta/2)+p(\delta/2)=x(1-v-\delta/2).$$

After that, $x(t)=x(1-\delta/2)-p(t-1)$ on $(1-\delta/2, 1)$. This follows from $\dot{x}(t)=f(2(t-1)-1)$ in $(1-\delta/2, 1)$, with $f(-1+x)=-f(-1-x)$ for $|x|<\delta$.

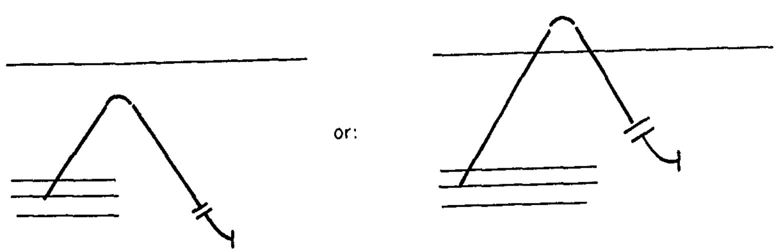


Fig. 7.

If we assume that $V(\varphi)=v\in[1/2+\delta/2, 3/4-\delta/8]=:I$, then we can say more: $1-v-\delta/2>\delta/2$ so that in particular x passes through the $\delta/2$-neighborhood of $t=0$ with slope 2.

Also, $\max |x|[0,1] = x(1-v) \leqslant -\delta$, and $x(1-\delta/2) < -1-\delta$ so that there is exactly one $a \in (0,1)$ with $x(a) = -1$, and x passes with slope -2 through the $\delta/2$-neighborhood of a. In fact, $a = 2-2v$ for obvious symmetry reasons.

We can now compute x on $[1,2]$ and find that $x(t) = x(1) - p(t-1)$ in $[1, 1+\delta/2]$, $x(1+\delta/2) < -1-\delta$ and $\dot{x} = 2$ on $[1+\delta/2, 1+a-\delta]$. By $x(1) = -1 - 2(1-a-\delta/2) - p(0)$ and $p \leqslant \delta$, it follows that there is a first $s > 1$ (namely $s = 1 + (1-a)$) such that $x(s) = -1$. We have $s > 1 + \delta/2$.

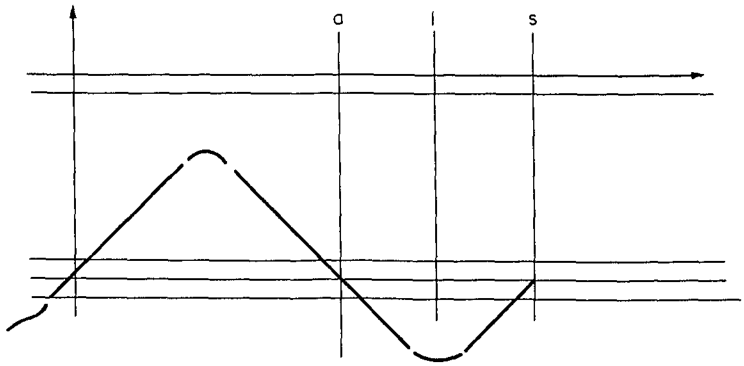


Fig. 8.

It is easily seen that $x_s \in D$ again, and $V(x_s) = s - a = 2 - 2a = 2 - 2(2-2v) = 4V(\varphi) - 2$.

The calculation shows in particular that for $\varphi \in D$ with $V(\varphi) = 2/3$ and

$$\begin{array}{ll}
\varphi(t) = -1/3 - \delta + p(t+1), & -1 \leqslant t \leqslant -1 + \delta/2 \\
\dot{\varphi}(t) = -2, & -1 + \delta/2 \leqslant t \leqslant -1/3 - \delta/2 \\
\varphi(t) = -5/3 + \delta - p(t+1/3), & -1/3 - \delta/2 \leqslant t \leqslant -1/3 + \delta/2 \\
\dot{\varphi}(t) = 2, & -1/3 + \delta/2 \leqslant t \leqslant 0
\end{array}$$

we have $x_s = \varphi$ so that we obtain a periodic solution of equation (*f*) with period 4/3 and values in $(-5/3, -1/3)$. From now on we denote this periodic solution by y.

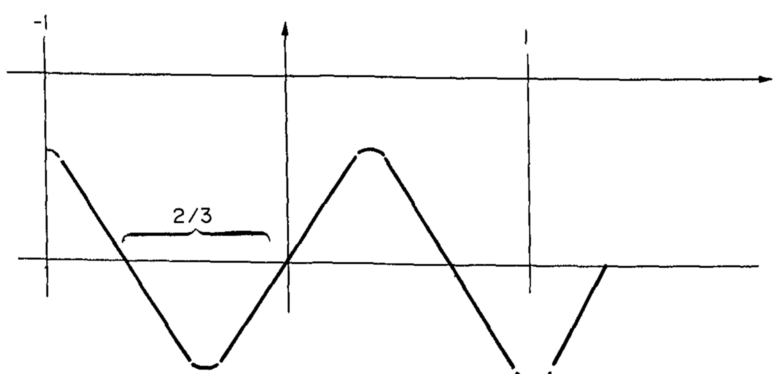


Fig. 9.

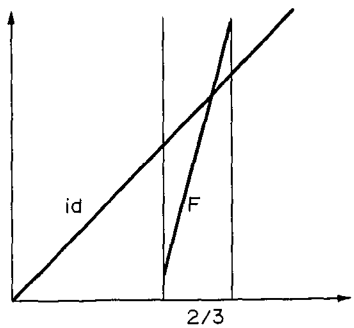
We have also proved that on $D \cap V^{-1}(I)$ the "Poincare map" given by $P\varphi = x_s$,

s the first positive time where $x_s \in D$

for the solution x of equation (f) with $x_0 = \varphi$,

is well defined. Moreover we have

LEMMA 1. For $F: I \ni v \to 4v - 2 \in R$, $V \circ P = F \circ V$.



$$\begin{tikzcd} D \cap V^{-1}(I) \arrow[r, "P"] \arrow[d, "V"'] & D \arrow[d, "V"] \\ I \arrow[r, "F"'] & R \end{tikzcd}$$

Fig. 10.

In order to use V as a coordinate we have to restrict P to a one-dimensional subset in C.

LEMMA 2. V is injective on $P(D \cap V^{-1}(I)) =: D_P$. We have $V(D_P) = F(I) = [2\delta, 1 - \delta/2] \supset I$. $(V|D_P)^{-1}$ is continuous on $V(D_P)$.

Proof. Injectivity: Let $v = V(\varphi)$, $\varphi = P\psi$ for some $\psi \in D \cap V^{-1}(I)$. Then $F(V(\psi)) = V(\varphi)$, or $V(\psi) = F^{-1}v$. Note that on $F(I)$ all iterated inverses F^{-n}, $n \in N$, exist. But for all $\tilde{\psi} \in D \cap V^{-1}(I)$ with $V(\tilde{\psi}) = F^{-1}v$, $\varphi = P\tilde{\psi}$. Hence φ is uniquely determined by v.—Next, $V(D_P) = V \circ P(D \cap V^{-1}(I)) = F(V(D \cap V^{-1}(I)) = F(I)$. —Continuity follows from $(V|D_P)^{-1}(v) = \varphi = P \circ \rho \circ F^{-1}(v)$ where $\rho : I \to D \cap V^{-1}(I)$ is defined by $\rho(w) = \psi$, $\psi = -1 + \delta$ in $[-1, -w - \delta/2]$, $\psi(t) = -2(t + w) - 1$ in $[-w - \delta/2, -w + \delta/2]$, $\psi = -1 - \delta$ in $[-w + \delta/2, -\delta/2]$, $\psi(t) = 2t - 1$ in $[-\delta/2, 0]$; compare the first part of this proof.

It follows that $V_P := V|D_P$ is a homeomorphism of $D_P \cap V^{-1}(I) = (V|D_P)^{-1}(I) = V_P^{-1}(I)$ onto I, and injective on D_P. We get the following commutative diagram:

$$\begin{tikzcd} V_P^{-1}(I) \arrow[r, "P"] \arrow[d, equal, "\wr"'] & D_P \arrow[d, hook] \\ I \arrow[r, "F"'] & R \end{tikzcd}$$

Fig. 11.

Of course, $V_P^{-1}(I)$ contains the initial value y_0 of the periodic solution which is a fixed point of P.

The iterates F^{-n}, $n \in N$, on $F(I)$ permit to continue all $\varphi \in D_P$ to functions on R_0^- which satisfy equation (f) for $t \leq 0$, and with $x_t \to o_y := \{y_t | t \in R\}$ as $t \to -\infty$. We shall not use such backward extensions explicitly. But they show that the periodic orbit o_y is unstable, and that a special solution which we shall obtain in the next section is in fact homoclinic with respect to o_y.

They are constructed as follows: Let $\varphi \in D_P$ be given. Then $\varphi = P\tilde{\varphi}$ where $\tilde{\varphi} \in D \cap V^{-1}(I)$, and $V(\tilde{\varphi}) = F^{-1}(V(\varphi)) \in I$. By Lemma 2 we may replace $\tilde{\varphi}$ by an element $\varphi_1 \in D_P$ such that $V(\varphi_1) = V(\tilde{\varphi})$. Then $\varphi = Px_{s_1}^{(1)}$, $x^{(1)}$ the solution with initial value φ_1 and s_1 the first positive time such that $x_{s_1}^{(1)} \in D$. We have $s_1 > 1$. Set $x(t) := x^{(1)}(t + s_1)$ on $[-s_1 - 1, 0]$, $s_{-1} := -s_1$, and proceed by induction. This yields a solution $x : R \to R$, $x_0 = \varphi$, and a decreasing sequence $s_{-n} \to -\infty$ as $n \to \infty$ such that $x_{s_{-n}} \in D_P$, $Px_{s_{-n}} = x_{s_{-n+1}}$, $V(x_{s_{-n}}) = F^{-n}(V(\varphi))$ for all $n \in N$. Since $F^{-n}v \to 2/3 = V(y_0)$ as $n \to \infty$ for all $v \in F(I)$, continuity of V_P^{-1} implies $x_{s_{-n}} \to y_0$ as $n \to \infty$, and $x_t \to o_y$ as $t \to -\infty$.

Let us already mention here that backward continuation of solutions is in no way unique for equation (f). This will be crucial for proving chaos in the sense of Li and Yorke.

5. HOMOCLINIC SOLUTION AND EXTENSION OF POINCARE MAP

Let J denote the closed ε-neighborhood of $v = 1/2 + 1/48$, with $0 < \varepsilon < 10^{-3}$. J is contained in I, and P maps $V_P^{-1}(J)$ onto $V_P^{-1}(K)$ where K denotes the 4ε-neighborhood of $1/12$; $K \subset F(I)$. This follows from the commutative diagrams in the last section. We have $K \cap I = \emptyset$.

LEMMA 3. For all $\varphi \in V_P^{-1}(K)$ there is a first $s > 0$ such that for the solution x of equation (f) with $x_0 = \varphi$, $x_s \in D_P$. Furthermore, $V(x_s) = 2 - 16\,V(\varphi)$.

Before proving this let us note some consequences: The solution x of equation (f) with $x_0 = V_P^{-1}(1/12)$ satisfies $V(x_s) = 2/3$ (and $x_s \in D_P$), therefore $x_s = y_0$. Since x can be continued to a solution on R with $x_t \to o_y$ as $t \to -\infty$, it is homoclinic with respect to the periodic orbit o_y.

Lemma 3 says that the commutative diagram of the last section extends to the one below if we use the old definition of P also on $V_P^{-1}(K)$ and if we set $F(v) := 2 - 16v$ on K:

$$\begin{tikzcd} V_P^{-1}(I) \cup V_P^{-1}(K) \arrow[r, "P"] \arrow[d, "\wr\wr"'] & D_P \arrow[d, hook] \\ I \cup K \arrow[r, "F"'] & R \end{tikzcd}$$

Fig. 12.

F maps K onto a compact interval $L \subset I$ which contains $V(y_0) = 2/3$ in the interior and satisfies $L \cap J = \emptyset$, see figure 13.

Proof of Lemma 3. We compute solutions which start in $V_P^{-1}(K)$. Let $\varphi \in D_P$, $V(\varphi) = v$

$\in K$ be given. Consider the solution x of equation (f) with $x_0 = \varphi$. We begin as in Section 4: Since $v \in K$, $1 - v - \delta/2 > 1/2$, and x increases with slope 2 from $x(0) = -1$ to $-1 + 2(1 - v - \delta/2) > \delta$ in the time interval $(0, 1 - v - \delta/2)$. In particular $x(1/2) = 0$. For t in the

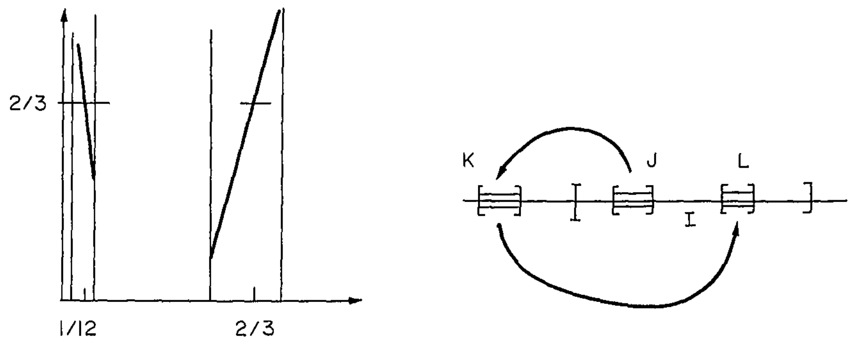


Fig. 13.

interval of length δ following $1 - v - \delta/2$, $x(t) = x(1 - v - \delta/2) + p(t - 1 + v)$. Further, $\dot{x} = -2$ in $(1 - v + \delta/2, 1 - \delta/2)$ and $x(t) = x(1 - \delta/2) - p(t - 1)$ in the $\delta/2$-neighborhood of $\bar{t} = 1$. We have $x(1 + \delta/2) = x(1 - v - \delta/2) - 2(v - \delta) = 1 - 4v + \delta$ and $\delta \leqslant x \leqslant 1 - \delta$ in $(1/2 + \delta/2, 1 + \delta/2)$ as is easily checked.

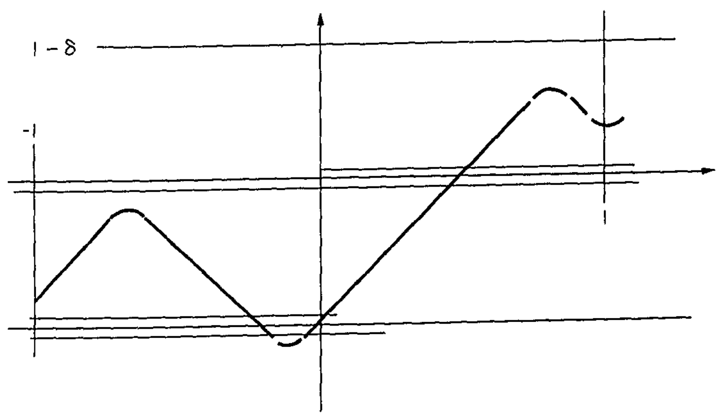


Fig. 14.

Next, equation (f) and $x(t-1) \in [-1 + \delta/2, -\delta/2]$ imply that x increases again with slope 2 on the interval $(1 + \delta/2, 3/2 - \delta/2)$ to the value $x(3/2 - \delta/2) = 2 - 4v - \delta > 1$. There exists exactly one time t_1 in $(1 + \delta/2, 3/2 - \delta/2)$ such that $x(t_1) = 1$. We find

$$t_1 = 2v + 1$$

and $\dot{x} = 2$ in the $\delta/2$-neighborhood of t_1.

In the $\delta/2$-neighborhood of $3/2$, $x(t) = 2 - 4v - \delta + p(t - 3/2)$.

Proof of this: For those t,

$$x(t) - x(3/2 - \delta/2) = \int_{3/2-\delta/2}^{t} \dot{x}(s)|ds = \int_{1/2-\delta/2}^{t-1} f \circ x$$

$$= \int_{1/2-\delta/2}^{t-1} f(2(s-1/2))|ds = \int_{-\delta/2}^{t-3/2} f(2s)|ds$$

$$= \int_{\delta}^{2t-3} f(s)/2|ds = -\int_{-\delta}^{2t-3} f(-1+s)/2|ds$$

$$= \int_{-\delta}^{2t-3} f(-1-s)/2\,|ds = \int_{-\delta/2}^{t-3/2} f(-1-2s)|ds$$

$$= p(t - 3/2).$$

Consequently, $x(3/2+\delta/2) = x(3/2-\delta/2) = 2 - 4v - \delta > 1 + \delta$. By equation (f) and by $x(t-1) \in (\delta, 1-\delta)$ for $t-1 \in (1/2+\delta/2, t_1 - \delta/2)$, $\dot{x} = -2$ in $(3/2 + \delta/2, t_1 + 1 - \delta/2)$. Hence $x(t_1 + 1 - \delta/2) = x(3/2+\delta/2) - 2(t_1 + 1 - 3/2 - \delta) = 1 - 8v + \delta < 1 - \delta$.

Therefore there is a unique $t_2 \in (3/2+\delta/2, t_1 + 1 - \delta/2)$ with $x(t_2) = 1$, and we have $\dot{x} = -2$ in the $\delta/2$-neighborhood of t_2.

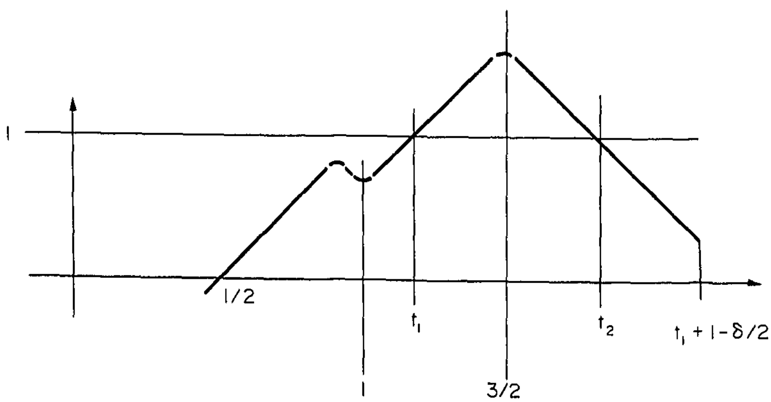


Fig. 15.

We find $t_2 = 2 - 2v$.
In the $\delta/2$-neighborhood of $t_1 + 1$,

$$x(t) - x(t_1+1) - \delta/2) = \int_{t_1+1-\delta/2}^{t} \dot{x} = \int_{t_1-\delta/2}^{t-1} f \circ x$$

$$= \int_{t_1-\delta/2}^{t-1} f(2(s-t_1)+1)\,ds = \int_{-\delta/2}^{t-1-t_1} f(2s+1)\,ds$$

$$= q(t - 1 - t_1)$$

where $q(t) := \int_{-\delta/2}^{t} f(2s+1)\, ds$ for $|t| \leqslant \delta/2$. By our assumption on f, $-2\delta \leqslant q \leqslant 0$.

Hence $x(t_1 + 1 + \delta/2) = 1 - 8v + \delta + q(\delta/2) \geqslant \delta > 0$.

Next, $\dot{x} = -1$ in $(t_1 + \delta/2 + 1, t_2 - \delta/2 + 1)$. Consequently $x(t_2 - \delta/2 + 1) = -4v + 2\delta + q(\delta/2)$.

From our assumptions on δ and ε, $x(t_2 - \delta/2 + 1) \leqslant -\delta$. So there is exactly on zero z of x in $(t_1 + \delta/2 + 1,\ t_2 - \delta/2 + 1)$, and $\dot{x} = -1$ in the δ-neighborhood of z. We find

$$z = 3 - 6v + 3\delta/2 + q(\delta/2).$$

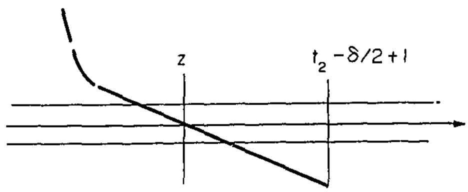


Fig. 16.

Next, we have for $t \in (t_2 - \delta/2 + 1, t_2 + \delta/2 + 1)$,

$$x(t) - x(t_2 - \delta/2 + 1) = \int_{t_2 - \delta/2}^{t-1} f(1 - 2(s - t_2))\, ds$$

$$= \int_{-\delta/2}^{t-1-t_2} f(1 - 2s)\, ds$$

from which it follows that

$$x(t_2 + \delta/2 + 1) = q(\delta/2) + x(t_2 - \delta/2 + 1) = -4v + 2\delta + 2q(\delta/2) > -1 + \delta.$$

$x(t-1) \in [\delta, 1 - \delta]$ for $t \in [t_2 + \delta/2 + 1, z - \delta + 1]$ gives $\dot{x} = -2$ in $[t_2 + \delta/2 + 1,\ z - \delta + 1]$. Therefore

$$x(z - \delta + 1) = 4v + 2\delta - 2 < -1 - \delta.$$

This implies the existence of a unique $t_3 \in (z, z - \delta + 1)$ such that $x(t_3) = -1$. We compute

$$t_3 = 7/2 - 4v + 3\delta/2 + q(\delta/2).$$

From $f(x) = -f(-x)$ for $|x| < \delta$ and $\dot{x} = -1$ in the δ-neighborhood of z we infer $x(z + 1 + \delta) = x(z + 1 - \delta) = 4v - 2 + 2\delta < -1 - \delta$ (Fig. 17 below).

$x(t-1) \in [-1 + \delta, -\delta]$ gives $\dot{x} = 2$ on $[z + 1 + \delta, t_3 - \delta/2 + 1]$. Hence

$$x(t_3 - \delta/2 + 1) = 4v + 2\delta - 2 + 2(t_3 - z - 3\delta/2) = -1 + 8v - \delta > -1 + \delta.$$

It follows that there is a unique $s^* \in [z + 1 + \delta, t_3 - \delta/2 + 1]$ such that $x(s^*) = -1$, and we have $\dot{x} = 2$ in the $\delta/2$-neighborhood of s^*. Moreover, $x(z + 1 + t) = x(z + 1 - t)$ for $|t| \leqslant z + 1 - t_3$, and consequently $s^* - t_3 = 2(z + 1 - t_3) = 1 - 4v$, compare figure 18 below.

x_{s^*} is not in D_P since $|\dot{x}| < 2$ in the whole δ-neighborhood of z + 1. But $x_{s^*} \in D$ (*Proof*:

Show $s^* - 1 \geqslant z + \delta$, $t_3 - \delta/2 \geqslant s^* - 1)$, and $V(x_{s^*}) = s^* - t_3 \in I$. So we find a first $\tilde{s} > 0$ with $\tilde{x}_{\tilde{s}} \in D_P$ for the solution $\tilde{x}$ with $\tilde{x}_0 = x_{s^*}$. Set $s := s^* + \tilde{s}$ and observe

$$V(P(x_{s^*})) = F(V(x_{s^*})) = 4(1 - 4v) - 2 = 2 - 16v.$$

Remark. One might ask why our nonlinearity f is not simply constant for $x > \delta$. The reason is that in examples of that kind, where calculations do not become too lengthy, the Poincare

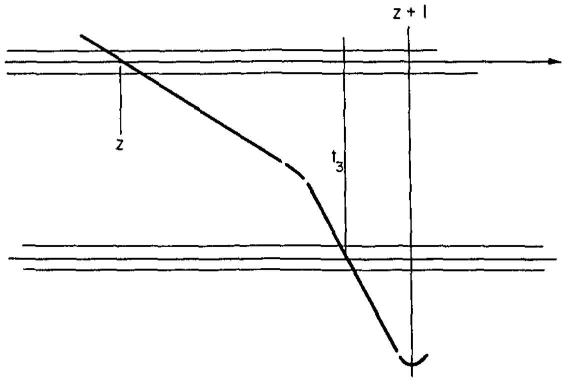


Fig. 17.

operator is not one-to-one on the analogue of $V_P^{-1}(K)$ but maps such set to the single point y_0. This means that o_y attracts the whole domain of the Poincare map—a situation which should not be called chaotic.

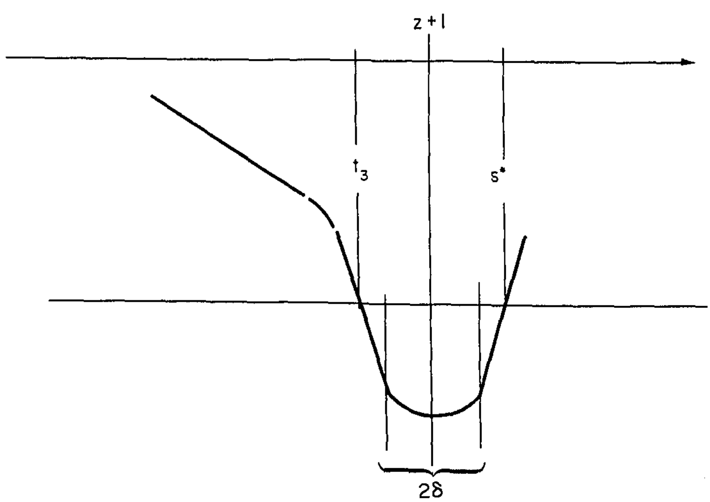


Fig. 18.

Let $g: A \to B$ be a map, $A \subset B$. A point $x \in A$ is called a periodic point of g with period p iff $g^j x$ is defined for $j = 1, \dots, p$; $g^j x \neq x$ for $j = 1, \dots, p-1$ and $g^p x = x$.

THEOREM. There is a number $n = n(P)$ such that for all $p > n$, P has periodic points of period p. In particular, there are countably many different periodic orbits of equation (f).

There exists an uncountable set S which is invariant with respect to P, contains no periodic point and has the following property:

$$\left.\begin{array}{l}\limsup\limits_{m\to\infty}\\ \liminf\limits_{m\to\infty}\\ \limsup\limits_{m\to\infty}\end{array}\right\} \|P^m\varphi - p^m\psi\| \left\{\begin{array}{ll}\left.\begin{array}{l}>0\\=0\end{array}\right\} & \text{for all } \varphi, \psi \in S;\ \varphi \neq \psi\\ >0 & \text{for all } \varphi \in S \text{ and all periodic points } \psi \text{ of } P.\end{array}\right.$$

Proof. (a) There exists $n = n(F)$ such that for all integers $p \geqslant n$ there is a periodic point $v \in I$ of F with period p. *Proof*: Choose n such that

$$(F|I)^{-p}(K) \subset L \quad \text{for all } p \geqslant n. \tag{5}$$

(Note that all interated inverses of $F|I$ are defined on $F(I) \supset K$.) Every map $(F|I)^{-p} \circ (F|K)^{-1}$ with $p \geqslant n$ has a fixed point $v_p \in L$. The period of v_p is $p+1$: Suppose it is $j < p$. By $F^{p-1}v_p \in J \subset I$ and $F' = 4$ in I, the orbit of v_p with respect to F is then contained in I and different from $v = 2/3$, contradiction.—If the period were p then $L \ni v_p = F^p v_p \in K$, contradicting $K \cap L = \emptyset$.

(b) We prepare the application of arguments due to Li and Yorke [7] and Marotto [9] which will imply chaos for our map F. The main difference to the situation considered in [7, 9] is that F is not a self-mapping of its domain.

Define X as the intersection of the decreasing sequence of compact sets given by $X_0 := I \cup K$ and $X_{m+1} := \{x \in X_m | F^{m+1}x \in I \cup K\}$ for m in N_0. Then

$$x \in X \text{ for all } x \in I \cup K \text{ with } F^m x \in X \text{ for some } m \in N_0. \tag{6}$$

We set $\tilde{L} := \bigcap_{m \in N_0} L_m$ and $\tilde{K} := \bigcap_{m \in N_0} K_m$ where $L_0 := L$, $L_{m+1} := \{x \in L_m | F^{m+1}x \in I \cup K\}$ for all $m \in N_0$, $K_0 := K$, $K_{m+1} := \{x \in K_m | F^{m+1}x \in I \cup K\}$ for all $m \in N_0$.

$\tilde{L}$ and $\tilde{K}$ are nonempty since $2/3 \in \tilde{L}$ and $1/12 \in \tilde{K}$. All sets $\tilde{L}$, $\tilde{K}$, X are compact. We have $\tilde{L} \cap \tilde{K} = \emptyset$, $\tilde{L} \cup \tilde{K} \subset X$, $F(X) \subset X$.

With n as in (5), $\tilde{L} \subset F^{n+2}(\tilde{K})$. *Proof*: $v \in \tilde{L}$ gives $v = Fw$ for some $w \in K$. Then $(F|I)^{-n}w \in L$, or $(F|I)^{-n}w = Fu$ with some $u \in K$. Hence $v = Fw = F^{n+2}u$. $v \in \tilde{L}$ implies $u \in \tilde{K}$, compare (6).

$\tilde{K} \subset F^{n+2}(\tilde{L})$. *Proof*: $v \in \tilde{K}$ implies $w = (F|I)^{-n-2}v \in L$, or $v = F^{n+2}w$. By $v \in \tilde{K}$, $w \in \tilde{L}$; hence $v \in F^{n+2}(\tilde{L})$.

$\tilde{L} \subset F^{n+2}(\tilde{L})$. *Proof*: $v \in \tilde{L}$ gives $v = Fu$ where $u \in K$. $u = F^{n+1}w$ with $w \in L$ gives $v = F^{n+2}w$. By $v \in \tilde{L}$, $w \in \tilde{L}$; consequently $v \in F^{n+2}(\tilde{L})$.

It follows that the continuous map $H := F^{n+2}: X \to X$ satisfies $\tilde{L} \subset H(\tilde{K})$, $\tilde{K} \subset H(\tilde{L})$, $\tilde{L} \subset H(\tilde{L})$. We can now use the argument in [9, p. 208]—which is a generalization of [7, p. 991]—to obtain the existence of an uncountable set $\tilde{S} \subset X$ which is invariant with respect to

F, contains no periodic point and has the property that

$$\left.\begin{array}{l}\limsup\limits_{m\to\infty}\\ \liminf\limits_{m\to\infty}\\ \limsup\limits_{m\to\infty}\end{array}\right\}|F^m v - F^m w|\left\{\begin{array}{ll}\left.\begin{array}{l}>0\\ =0\end{array}\right\} & \text{for all } v, w \in \tilde{S};\ v \neq w\\ >0 & \text{for all } v \in \tilde{S} \text{ and all}\\ & \text{periodic points } w \in X.\end{array}\right. \tag{7}$$

Since X contains all periodic points of F in $I \cup K$ we may replace "$w \in X$" in (7) by "$w \in I \cup K$".

The proof of (7) requires another version of Lemma 3.2 [9] which reads:

"Let $X \subset R$ be compact and nonempty, let $H : X \to X$ be continuous. Let $(C_k)_{k \in N_0}$ be a sequence of nonempty compact subsets of X with $H(C_k) \supset C_{k+1}$ for all $k \in N_0$. Then there is a nonempty compact set $C \subset C_0$ such that for all $x \in C$ and all $k \in N_0$, $H^k x \in C_k$."

This follows exactly as Lemma 3.2 [9], or as Lemmas 0 and 1 in [7].

(c) Application of V yields the assertions on P.

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