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Research paper

Some rigidity results and asymptotic properties for solutions to semilinear elliptic P.D.E.

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ABSTRACT

We will present some rigidity results for solutions to semilinear elliptic equations of the form $\Delta u = W'(u)$, where W is a quite general potential with a local minimum and a local maximum. We are particularly interested in Liouville-type theorems and symmetry results, which generalise some known facts about the Cahn–Hilliard equation.

1. Introduction and main results

Our aim is to revisit some asymptotic properties and rigidity results for solutions u to the semilinear elliptic P.D.E.

$$\Delta u(x) = W'(u(x)),\ u \in C^2(D),\ D \subset \mathbb{R}^n,\ n \geq 2, \tag{1.1}$$

with a potential $W \in C^{1,1}_{loc}(\mathbb{R})$. The domains $D \subset \mathbb{R}^n$ we consider are connected open unbounded sets such that

$$\forall R > 0,\ D \text{ contains a ball of radius } R. \tag{1.2}$$

We shall also assume that on D the solution u takes its values in a bounded interval where W is monotone. For instance, one may consider the Cahn–Hilliard equation

$$\Delta u = W'(u) = u^3 - u + \delta \qquad \text{in } \mathbb{R}^n, \tag{1.3}$$

with $|\delta| < \frac{2}{3\sqrt{3}}$, so that the polynomial $f(t) = t^3 - t + \delta$ admits exactly three real roots

$$z_1(\delta) < -\frac{1}{\sqrt{3}} < z_2(\delta) < \frac{1}{\sqrt{3}} < z_3(\delta),$$

and is negative on the interval $(z_2(\delta), z_3(\delta))$. This equation was largely studied in the literature. For example, some particular solutions were constructed in [15,17], while some results about radial and cylindrical symmetry of solutions and Liouville type results can be found in [21]. Our starting point is the following theorem.

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Theorem 1.1 (*[21]*). *Let $n \geq 2$, $\delta \in (-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}})$ and let u_δ be a solution to (1.3) such that*

$$u_\delta > z_2(\delta) \qquad \text{outside a ball } B_R \subset \mathbb{R}^N. \tag{1.4}$$

(1) *If $\delta \in (-\frac{2}{3\sqrt{3}}, 0]$, then $u \equiv z_3(\delta)$.*
(2) *If $\delta \in (0, \frac{2}{3\sqrt{3}})$, then u_δ is radially symmetric (not necessarily constant).*

Similar results in the case $\delta = 0$ can be found in [9,10].

The purpose here is to extend Theorem 1.1 to more general non linearities. The proofs in [21] are based on some known symmetry results (see [12]) which rely on the moving planes method. A key tool in these methods is the maximum principle, even for unbounded domains (see [1]).

1.1. *Asymptotic properties of solutions*

We shall first examine the asymptotic behaviour of solutions lying between two critical points of the potential W. The first result in this direction is the following.

Theorem 1.2. *Let $n \geq 2$, let $D \subset \mathbb{R}^n$ be a domain satisfying (1.2) and let u be a solution of (1.1) ($W \in C^{1,\alpha}_{loc}(\mathbb{R})$, $\alpha \in (0,1)$). Assume also that $u(D) \subset [a,b]$, and $W' < 0$ on the interval $[a,b)$ (with $a, b \in \mathbb{R}$). Then,*

$$\lim_{d(x,\partial D)\to\infty} u(x) = b \tag{1.5}$$

and $W'(b) = 0$ hold. If in addition $D = \mathbb{R}^n$, then we have $u \equiv b$.

Theorem 1.2 will be proved in Section 2.

After that, we shall focus on the more involved problem where

$$u(D) \subset (a,b], \tag{1.6a}$$

$$W'(a) = W'(b) = 0, \text{ and } W' < 0, \text{ on } (a,b) \text{ (with } a, b \in \mathbb{R}). \tag{1.6b}$$

If we assume in addition to (1.6), the nondegeneracy condition:

$$W'(s) \leq -C_0(s-a) \text{ on } [a, s_0], \text{ for some } C_0 > 0 \text{ and } s_0 \in (a,b), \tag{1.7}$$

we can apply comparison arguments of Berestycki, Caffarelli, and Nirenberg [1, Lemma 3.2] to deduce that

$$d(x,\partial D) > \eta \Rightarrow u(x) \geq a + \epsilon, \text{ for some constants } \eta, \epsilon > 0, \tag{1.8}$$

As a consequence, by Theorem 1.2 we have an alternative proof of the following result.

Theorem 1.3 (*[1]*). *Let $n \geq 2$, let $D \subset \mathbb{R}^n$ be a domain satisfying (1.2) and let u be a solution of (1.1) ($W \in C^{1,\alpha}_{loc}(\mathbb{R})$, $\alpha \in (0,1)$). Assume that (1.7) and (1.6b) are fulfilled. Then*

$$\lim_{|x|\to\infty} u(x) = b$$

and $W'(b) = 0$.

On the other hand, in the degenerate case where (1.7) does not hold, the asymptotic behaviour of the solutions may be more involved. In the case where D is the complement of a ball, we can relax condition (1.7) by assuming that

$$W'(s) \leq -C_0(s-a)^p \text{ on } [a, s_0], \text{ for some } C_0 > 0,\ p < \frac{n}{n-2}, \text{ and } s_0 \in (a,b), \tag{1.9}$$

Under assumption (1.9), we can still prove the asymptotic property (1.5) for solutions provided (1.6) holds.

Theorem 1.4. *Let $n \geq 3$, let $B_\rho \subset \mathbb{R}^n$ be the open ball of radius ρ centred at the origin, and let $W \in C^{1,1}_{loc}(\mathbb{R})$ be a potential fulfilling (1.6b), and (1.9). Then, every solution $u \in C^2(\mathbb{R}^n \setminus B_\rho)$ to (1.1) such that $u(\mathbb{R}^n \setminus B_\rho) \subset (a,b)$, satisfies $\lim_{|x|\to\infty} u(x) = b$.*

However, in the case of potentials such that

$$\lim_{u\to a^+} \frac{W'(u)}{(u-a)^p} = -\lambda$$

for some $\lambda > 0$ and $p > \frac{n}{n-2}$, radial solutions $u : \mathbb{R}^n \to (a,b)$ of (1.1) satisfying

$$\lim_{|x|\to\infty} u_0(x) = a \tag{1.10}$$

may exist in dimensions $n \geq 3$ (cf. Lemma 2.1). Therefore, condition (1.9) is optimal to derive (1.5), when D is the complement of a ball. For general domains, condition (1.9) is not sufficient to deduce the asymptotic behaviour of the solution. In Proposition 2.4, we construct a solution of (1.1) in a dumbbell shaped domain $D \subset \mathbb{R}^2$, such that $u \approx a$ on the one side of the neck, while $u \approx b$ on the other side.

We also point out that for general domains, condition (1.9) is not sufficient to deduce the asymptotic behaviour of the solution. In Proposition 2.4, we construct a solution of (1.1) in a dumbbell shaped domain $D \subset \mathbb{R}^2$, such that $u \approx a$ on the one side of the neck, while $u \approx b$ on the other side.

To sum up these results, we now state

Theorem 1.5. *Let $W \in C^{1,1}_{loc}(\mathbb{R})$ be a potential satisfying (1.6b).*

(i) *Assume $u \in C^2(\mathbb{R}^n)$ is a solution of (1.1) such that $u(\mathbb{R}^n) \subset [a,b]$. Then, when $n = 2$, or $n \geq 3$ and (1.9) holds, we have either $u \equiv a$, or $u \equiv b$. Otherwise (when $n \geq 3$ and (1.9) does not hold), we have either $u \equiv a$, or $u \equiv b$, or*[1]

$$\begin{cases} u(\mathbb{R}^n) \subset (a, C_W], \text{ for a constant } C_W \in (a,b) \text{ depending only on } W, \\ \liminf_{|x|\to\infty} u(x) = a. \end{cases} \tag{1.11}$$

(ii) *Assume the domain D satisfies (1.2), and $u \in C^2(D)$ is a solution of (1.1) such that $u(D) \subset (a,b]$. Then, we have $\lim_{d(x,\partial D)\to\infty} u(x) = b$, provided that (1.7) holds.*

The asymptotic properties we discussed here, will be proved in Section 2.

1.2. A Liouville type result

Next, in Theorem 1.6 below, we derive a Liouville type result extending Theorem 1.1 (1) (cf. [21]) to general nonlinearities under optimal assumptions. Indeed, the necessity of condition (1.9) (when $n \geq 3$) for Theorem 1.6 to hold, is clear in view of Lemma 2.1.

Theorem 1.6. *Let $W \in C^{1,1}_{loc}(\mathbb{R})$ be a nonnegative potential satisfying (1.6b), and $W(b) = 0$. Assume $u \in C^2(\mathbb{R}^n)$ is an entire solution of (1.1) such that*

$$u(x) \in (a,b] \qquad \forall\, x \in \mathbb{R}^n \setminus B_R, \tag{1.12}$$

for some $R > 0$. Then, $u \equiv b$, provided that $n = 2$, or $n \geq 3$ and (1.9) holds.

Theorem 1.6 follows from the asymptotic property established in Theorem 1.4 and from a general Liouville type theorem (cf. Theorem 3.1) proved in Section 3. Other Liouville type results for stable solutions to semilinear PDEs were established in [8]. Here there is no stability assumption.

1.3. Radial symmetry

After that, we will address the issue of radial symmetry. In [13], the authors prove radial symmetry of solutions to fully nonlinear equations of very general form, provided these solutions have a suitable asymptotic polynomial decay at infinity (see [13, Theorem 4] and its application in [13, Proposition 1]).

Here we are interested in radial symmetry of solutions to (1.1) with W satisfying (1.6b), assuming that $\lim_{|x|\to\infty} u(x) = b$. If W is convex in an interval $(b-\delta, b)$, then the symmetry result follows from [12, Theorem 2] in any dimension $n \geq 2$. On the other hand, by combining [12, Theorem 2] with [13, Proposition 1], we derive the radial symmetry in dimensions $n \geq 3$, provided that the potential is sufficiently smooth. So we obtain the following generalisation of Theorem 1.1 (2):

Theorem 1.7. *Let $W \in C^2(\mathbb{R})$ be a potential such that $W'(t) < 0$ for any $t \in (a,b)$, $W'(a) = 0$, and $W(t) \geq W(b)$ for any $t > b$. In addition, we suppose that one of the following is true:*

(i) *$n \geq 3$, and $W \in C^{6,\alpha}(b-\delta, b+\delta)$, for some $\delta > 0$ and $\alpha \in (0,1)$.*
(ii) *$n \geq 2$, and W is convex in $(b-\delta, b)$, for some $\delta > 0$.*

Assume also that $u : \mathbb{R}^n \to \mathbb{R}$ is a solution to $\Delta u = W'(u)$ such that $u(\mathbb{R}^n \setminus B_R) \subset (a,b)$ and $\lim_{|x|\to\infty} u(x) = b$. Then $u < b$ in $\mathbb{R}^n$ and it is radially symmetric.

The regularity condition $W \in C^{6,\alpha}$ comes from Proposition 4.1. It is useful because the proof relies on some Taylor expansion.

Moreover, this regularity assumption can be relaxed in higher dimension. For example, it is enough to assume $W \in C^{4,\alpha}(b-\delta, b+\delta)$ if $n \geq 4$ and $W \in C^{3,\alpha}(b-\delta, b+\delta)$ if $n \geq 6$, for some $\delta > 0$.

Remark 1.8.

(1) If a potential W satisfies the assumptions of Theorem 1.7, then it has a local minimum at $t = b$, so that $W'(b) = 0$. However, this minimum is not required to be a global one.

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[1] For instance, let u_0 be the radial solution provided by *Lemma* 2.1. Then, by taking $u(x_1, \dots, x_n, x_{n+1}) = u_0(x_1, \dots, x_n)$, we can see that (1.11) holds.

(2) If $W'(t) > 0$ for $t > b$, it follows from the maximum principle that any bounded solutions u of (1.1) in $\mathbb{R}^n$, satisfies the bound $u \le b$.
(3) Let u be a solution of (1.1) in $\mathbb{R}^n$. Then, if $n = 2$ or $n \ge 3$ and (1.9) holds, the condition $u(\mathbb{R}^n \setminus B_R) \subset (a, b)$, implies that $\lim_{|x|\to\infty} u(x) = b$ in view of Lemma B.2 and Theorem 1.2 (resp. Theorem 1.4).
(4) For the existence of radial solutions satisfying $\lim_{|x|\to\infty} u(x) = b$, we refer to [2, Theorem 1, Theorem 4] and [16, Theorem 1.3].

We will prove these symmetry results in Section 4.

1.4. An open problem

Assuming again that W satisfies (1.6b), the description of entire solutions to (1.1) converging to a at infinity is a much more difficult task. In that case, only a few symmetry results are available, under somewhat restrictive hypotheses on the solution and the nonlinearity. Some results can be found in [5], where a monotonicity assumption is required. As a particular case, their results apply to bounded solutions to the Lane–Emden equation

$$-\Delta u = |u|^{p-1}u$$

in $\mathbb{R}^n$, for which several Liouville type results are known (see for example [3,11,18]). For future purposes, the main difficulty is to remove the monotonicity and convexity assumption about the non-linearity.

By Theorem 1.4, we know that non trivial solutions can exist only if condition (1.9) is violated. However, the fact that

$$u(\mathbb{R}^n \setminus B_R) \subset (a, b) \tag{1.13}$$

cannot guarantee a Liouville type result (cf. Lemma 2.1), or even radial symmetry under the assumption that

$$\lim_{|x|\to\infty} u(x) = a. \tag{1.14}$$

In Section 5, we check that the solutions constructed in [6], provide examples of nonradial solutions to (1.1), such that $u(x) - a$ changes sign in a compact set, and (1.13) as well as (1.14) hold. It would be interesting to see if a nonradial solution satisfying $u(\mathbb{R}^n) \subset (a, b)$ and $\lim_{|x|\to\infty} u(x) = a$, may also exist for a potential W having a negative derivative on the range of u. To the best of our knowledge, this is a difficult open problem.

2. Asymptotic properties of solutions

We first prove Theorem 1.2.

Proof of Theorem 1.2. We first recall that, for fixed $R > 0$, the solution u is uniformly bounded in $C^{2,\alpha}$ (for some $\alpha \in (0, 1)$) on the balls $B_R(x)$ satisfying $d(x, \partial D) > R+1$ with $x \in D$. This follows from the elliptic estimates (see [14]), the regularity of W and the fact that u is bounded. Let $l := \liminf_{d(x,\partial D)\to\infty} u(x)$, and let $\{x_k\} \subset D$ be a sequence such that $\lim_{k\to\infty} d(x_k, \partial D) = \infty$, and $\lim_{k\to\infty} u(x_k) = l$. We set $v_k(y) = u(x_k + y)$. In view of the previous estimates, we can apply the Ascoli theorem via a diagonal argument to the sequence $\{v_k\}$, and deduce that up to subsequence, v_k converges in $C^2_{\rm loc}(\mathbb{R}^n)$ to an entire solution v_∞ of (1.1). Moreover, we have

$$v_\infty(0) = l = \min_{y\in\mathbb{R}^n} v_\infty(y),$$

and

$$0 \le \Delta v_\infty(0) = W'(l) \le 0,$$

so that, $l = b$, $W'(b) = 0$, and $v_\infty \equiv b$. This proves that $\lim_{d(x,\partial D)\to\infty} u(x) = b$, and $W'(b) = 0$ hold.

In the particular case where $D = \mathbb{R}^n$, we have $u \equiv b$, since otherwise u would attain its minimum at a point x_0 where $0 \le \Delta u(x_0) = W'(u(x_0)) < 0$, which is a contradiction. □

Next we prove Theorem 1.4.

Proof of Theorem 1.4. Without loss of generality, we may assume that $a = 0$.

First we claim that

$$\text{any solution} u \in C^2(\mathbb{R}^n \setminus B_\rho) \text{to} (1.1) \text{ such that} u(\mathbb{R}^n \setminus B_\rho) \subset (0, b), \text{satisfies } \sup_{\mathbb{R}^n\setminus B_\rho} u = b. \tag{2.15}$$

For this purpose, we assume by contradiction that

$$u(\mathbb{R}^n \setminus B_\rho) \subset (0, b - \eta], \text{ for some } \eta > 0 \text{ small.} \tag{2.16}$$

Then, we have

$$W'(u) \le -c_1 u^p, \ \forall u \in [0, b - \eta] \tag{2.17}$$

for a constant $0 < c_1 < C_0$. We first examine the case where u is radial, that is, $u(x) = v(|x|)$. As a consequence, v solves

$$v''(r) + \frac{n-1}{r} v'(r) = W'(v(r)), \ \forall r \in [\rho, \infty). \tag{2.18}$$

Our claim is that $v'(\rho_0) \leq 0$ holds for some $\rho_0 \geq \rho$. Indeed, otherwise, on the one hand we would have

$$\forall r \in [\rho, \infty) : \ v'(r) > 0,$$

on the other hand

$$\text{and } v''(r) \leq \kappa := \max_{[v(\rho), b-\eta]} W' < 0,$$

which yields that

$$v'(r) = v'(\rho_0) + \int_{\rho_0}^{r} v''(s) ds \leq v'(\rho_0) + \kappa(r - \rho_0) \to -\infty, \qquad \text{as } r \to \infty,$$

a contradiction. So far, we have proved that $v'(\rho_0) \leq 0$ for some $\rho_0 \geq \rho$. By noticing that $v'(\rho_0) = 0 \Rightarrow v''(\rho_0) < 0$ in view of (2.18), one can see that $v' < 0$ holds on an interval $(\rho_0, \rho_0 + \epsilon)$, for small $\epsilon > 0$. Let $l := \sup\{r > \rho_0 : v' < 0 \text{ on } (\rho_0, r)\}$. It is clear that $l = \infty$, since otherwise we would deduce that $v'(l) = 0$ and $v''(l) < 0$, which is a contradiction. This establishes that $v' < 0$ on (ρ_0, ∞). Now, it follows from (2.18) that

$$\begin{aligned} \forall r > \rho_1 : \ r^{n-1} v'(r) &\leq r^{n-1} v'(r) - \rho_0^{n-1} v'(\rho_0) \\ &= \int_{\rho_0}^{r} s^{n-1} W'(v(s)) ds \leq -c_1 v^p(r) \int_{\rho_0}^{r} s^{n-1} ds \\ &\leq -k v^p(r) r^n, \end{aligned}$$

for a constant $k > 0$, and for $\rho_1 > \rho_0$ large enough. Next, an integration of the previous inequality gives

$$\begin{aligned} \forall r > \rho_1 : v^{1-p}(r) &\geq v^{1-p}(r) - v^{1-p}(\rho_1) \\ &\geq \frac{k(p-1)}{2}(r^2 - \rho_1^2), \end{aligned}$$

from which we deduce that $v(r) \leq \tilde{k} r^{-\frac{2}{p-1}}$, for a constant $\tilde{k} > 0$, and for $r > \rho_2 > \rho_1$ large enough. Since $p < \frac{n}{n-2} \Leftrightarrow \frac{2}{p-1} > n-2$, this contradicts the lower bound for superharmonic functions provided by Lemma B.1. Therefore the existence of a radial solution satisfying (2.16) is ruled out.

Now we will prove that (2.15) holds without the assumption that u is radially symmetric. For this purpose, assume by contradiction that $u \in C^2(\mathbb{R}^n \setminus B_\rho)$ is a (non necessarily radial) solution of (1.1) satisfying (2.16). In view of Lemma B.1, u satisfies the lower bound

$$u(x) > \phi_*(x) = c|x|^{2-n}, \ c > 0, \tag{2.19}$$

where ϕ_* is a subsolution of (1.1), that is, $\Delta\phi_* = 0 \geq W'(\phi_*)$. Starting from u, we shall construct a radial supersolution ϕ^* of (1.1), such that $\phi_* \leq \phi^*$. Let $\rho_{i,m}$ be the rotation of angle $\frac{\pi}{2^m}$ around the x_i coordinate axis of $\mathbb{R}^n$ ($m \geq 1$, $i = 1, \dots, n-1$), and let

$$G_m := \{\rho_{1,m}^{k_1} \circ \dots \circ \rho_{n-1,m}^{k_{n-1}} : 0 \leq k_i \leq 2^{m+1} - 1, \ i = 1, \dots, n-1\}.$$

Using spherical coordinates, one can see that given $|x_0| \geq \rho$, the set $\cup_{m\geq 1} G_m x_0$ is dense in the sphere $\{x \in \mathbb{R}^n : |x| = |x_0|\}$. In particular, we have

$$\lim_{m\to\infty} \min_{g \in G_m} u(g x_0) = \min_{|x|=|x_0|} u(x). \tag{2.20}$$

Next, we notice that for every $g \in G_m$, $x \mapsto u(gx)$ solves (1.1). On the other hand, in view of the Kato inequality, $\phi_m(x) := \min_{g\in G_m} u(gx)$ is a supersolution of (1.1), satisfying $\phi_* \leq \phi_m \leq u$. In addition, it follows from (2.20) that $\phi^*(x) := \lim_{m\to\infty} \phi_m(x) = \min\{u(y) : |y| = |x|\}$. Finally, since $|\nabla\phi_m|$ is uniformly bounded on $\mathbb{R}^n \setminus B_\rho$, we obtain that (up to subsequence) ϕ_m converges weakly to ϕ^* in $W^{1,2}(B_R \setminus \overline{B_\rho})$, for every $R > \rho$. This implies that ϕ^* (which belongs to $W^{1,2}(B_R \setminus \overline{B_\rho})$, for every $R > \rho$) is a radial supersolution of (1.1) satisfying $\phi_* \leq \phi^* \leq u$. To conclude, we deduce from the method of sub- and supersolutions (cf. Appendix, and for instance [7, Lemma 1.1.1]), the existence of a radial solution $v \in C^2(\mathbb{R}^n \setminus B_\rho)$, satisfying $0 < \phi_* \leq v \leq \phi^* \leq b - \eta$. In view of the first part of the proof, this is a contradiction. This concludes the proof of (2.15).

By (2.15), there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} |x_k| = \infty, \text{ and } \lim_{k\to\infty} u(x_k) = b. \tag{2.21}$$

Setting $v_k(y) := u(x_k + y)$, and proceeding as in Theorem 1.2, we obtain that (up to subsequence) v_k converges in $C^2_{\text{loc}}(\mathbb{R}^n)$ to an entire solution v_∞ of (1.1). Furthermore, since $v_\infty(0) = b$, the maximum principle implies that $v_\infty \equiv b$. At this stage we consider a minimiser $\phi_R \in H^1_{0,rad}(B_R(0))$ of the energy functional

$$\tilde{E}(v) = \int_{B_R(0)} \Big(\frac{1}{2}|\nabla v(x)|^2 + \tilde{W}(v(x))\Big) dx, \tag{2.22a}$$

in the space $H^1_{0,rad}(B_R(0))$ of radial $H^1_0(B_R(0))$ functions, where

$$\tilde{W}(v) = \begin{cases} W(0) & \text{for } v \leq 0 \\ W(v) & \text{for } 0 \leq v \leq b \\ W(b) & \text{for } v \geq b. \end{cases} \tag{2.22b}$$

It is known that ϕ_R is a smooth radial solution of (1.1) in $B_R(0)$, such that $0 \leq \phi_R \leq \max_{B_R(0)} \phi_R := b - \delta_R$ on $B_R(0)$, for some $\delta_R > 0$. In addition, we have $\lim_{R\to\infty} \delta_R = 0$. Thus, given $\epsilon > 0$, we can ensure that

- $\delta_R < \epsilon$ for some $R > 0$ large enough,
- and $\phi_R \leq b - \delta_R \leq v_k$ holds on $B_R(0)$, for $k \geq k_R$ large enough.

In other words, $u(x) \geq b - \delta_R \geq \phi_R(x - x_k)$ for any $x \in B_R(x_k)$. As a consequence, by applying the sliding method of Berestycki, Caffarelli, and Nirenberg [1, Lemma 3.1], we deduce that $u(x) \geq \phi_R(x - \bar{x})$ for any $x \in B_R(\bar{x})$, provided that $B_R(\bar{x}) \subset \mathbb{R}^n \setminus B_\rho$. This yields that $u(x) \geq b - \delta_R \geq b - \epsilon$ for any $x \in \mathbb{R}^n \setminus B_{\rho+R}$, which completes the proof of Theorem 1.4. ☐

In the subcritical case where $W'(u) \sim -\lambda|u-a|^p$ in a right neighbourhood of a, with $\lambda > 0$ and $p \in (\frac{n}{n-2}, \frac{n+2}{n-2})$, we shall see in Lemmas 2.1 and 2.2 below, that depending on the potential, there may or may not exist a radial solution such that $u(\mathbb{R}^n) \subset (a, b)$.

Lemma 2.1. *Given any $n \geq 3$, $p > \frac{n}{n-2}$ and $\lambda > 0$, there exists a potential $W \in C^2(\mathbb{R})$ fulfilling (1.6b), and a solution $u \in C^\infty(\mathbb{R}^n)$ to (1.1), such that*

(a) $\lim_{u\to a^+} \frac{W'(u)}{|u-a|^p} = -\lambda$,
(b) *u is radial and radially decreasing (i.e. $u(x) = \tilde{u}(|x|)$, for a smooth decreasing function $\tilde{u} : [0, \infty) \to (a, b)$),*
(c) *$u(\mathbb{R}^n) \subset (a, b)$, and $\lim_{|x|\to\infty} u(x) = a$,*
(d) $W''(u(0)) > 0$.

Proof. Without loss of generality, we may assume that $a = 0$. First, we note that the function $v(x) = \left(\frac{2((n-2)p-n)}{\lambda(p-1)^2}\right)^{\frac{1}{p-1}} |x|^{-\frac{2}{p-1}}$ solves the equation

$$\Delta v = -\lambda v^p \qquad \text{in } \mathbb{R}^n \setminus \{0\}.$$

Next, in order to eliminate the singularity at the origin, we take a smooth cutoff function $\xi : \mathbb{R} \to [0, 1]$ such that

$$\begin{cases} \xi = 1 \text{ in } [3, \infty), \\ 0 < \xi < 1 \text{ and } \xi' > 0 \text{ in } (2, 3), \\ \xi = 0 \text{ in } (-\infty, 2], \end{cases}$$

and we consider a function $\tilde{u} : (1, \infty) \to \mathbb{R}$ such that

$$\begin{cases} \tilde{u}''(r) = \xi(r)\tilde{v}''(r) & \forall\, r \in [1, \infty), \\ \tilde{u}(r) = \tilde{v}(r) & \forall r \geq 3. \end{cases}$$

where $v(x) =: \tilde{v}(|x|)$. One can see that

$$\tilde{u}'' + \frac{n-1}{r}\tilde{u}' < 0 \qquad \text{in } [1, \infty). \tag{2.23}$$

The latter inequality is clear if $r \geq 3$. In order to prove that (2.23) holds in $[1, 3)$ too, we note that

$$\begin{aligned} \tilde{u}'(r) &= -\int_r^\infty \tilde{u}''(t)dt = -\int_r^\infty \xi(t)\tilde{v}''(t)dt \\ &= \xi(r)\tilde{v}'(r) + \int_r^\infty \xi'(t)\tilde{v}'(t)dt < \xi(r)\tilde{v}'(r) \leq 0, \qquad \forall\, r \in [1, 3), \end{aligned} \tag{2.24}$$

so that

$$\tilde{u}'' + \frac{n-1}{r}\tilde{u}' < \xi\left(\tilde{v}'' + \frac{n-1}{r}\tilde{v}'\right) \leq 0, \qquad \forall\, r \in [1, 3).$$

Now, we extend $\tilde{u}$ to a smooth even positive function on the whole $\mathbb{R}$, still denoted by $\tilde{u}$, fulfilling $\tilde{u}' < 0$ in $(0, \infty)$, $\tilde{u}'' < 0$ in $[0, 1)$, so that $\tilde{u}'' + \frac{n-1}{r}\tilde{u}' < 0$ holds in $[0, \infty)$, $\tilde{u}'''(0) = 0$ and $\tilde{u}^{(4)}(0) < 0$. This can easily be done if we recall that $\tilde{u}$ is affine and decreasing on $[1, 2]$. Since $\tilde{u}$ is monotone in $[0, \infty)$, then it is invertible in this interval with inverse function $\beta : (0, \tilde{u}(0)] \to [0, \infty)$. Finally, setting

$$\varphi(r) := \tilde{u}''(r) + \frac{n-1}{r}\tilde{u}'(r), \qquad \forall\, r > 0,$$

and $H(s) := \varphi(\beta(s))$, for $s \in (0, \tilde{u}(0)]$, one can see that $u(x) := \tilde{u}(|x|)$ satisfies the equation $\Delta u = H(u)$ in $\mathbb{R}^n$. We also notice that $H(\tilde{u}(0)) = n\tilde{u}''(0) < 0$ and $H'(\tilde{u}(0)) = \frac{(n+2)\tilde{u}^{(4)}(0)}{3\tilde{u}''(0)} > 0$. Thus, one can find a C^1 extension of H to the whole $\mathbb{R}$, still denoted by H, such that $H < 0$ in $(0, b)$, for some $b > \tilde{u}(0)$, and $H(b) = 0$. By construction, we have $H(u) = -\lambda u^p$ in $(0, \tilde{u}(3))$, so that $H(0) = H'(0) = 0$. In order to conclude the proof it is enough to define W to be the primitive of H. ☐

Lemma 2.2. *Given any $n \geq 3$, $p \in (\frac{n}{n-2}, \frac{n+2}{n-2})$, and $\lambda > 0$, there exists a potential $W \in C^2(\mathbb{R})$ fulfilling (1.6b) and $\lim_{u \to a^+} \frac{W'(u)}{|u-a|^p} = -\lambda$, for which there are no radial solutions $u \in C^2(\mathbb{R}^n)$ of (1.1) such that $u(\mathbb{R}^n) \subset (a,b)$.*

Proof. Without loss of generality, we may assume that $a = 0$. We consider the function $H(u) = -\lambda u^p$ on an interval $[0,\beta]$, and since $p \in (\frac{n}{n-2}, \frac{n+2}{n-2})$, we set $\epsilon = \frac{n}{p+1} - \frac{n-2}{2} > 0$. One can find a C^1 extension of H to the whole $\mathbb{R}$, still denoted by H, such that

- $H < 0$ in $(0,b)$, and $H(b) = 0$, for some $b > \beta$. Let $b = \kappa\beta$, with $\kappa > 1$.
- $H([0,b]) = [-\lambda\mu\beta^p, 0]$ for some $\mu > 1$, such that $\kappa\mu < 1 + \frac{2\epsilon}{n-2}$.

Next, define $W \in C^2(\mathbb{R})$ to be the primitive of H vanishing at 0. We claim that

$$\frac{n-2}{2} W'(u)u - nW(u) > 0 \text{ on } (0,b]. \tag{2.25}$$

Indeed, we have $\frac{n-2}{2} W'(u)u - nW(u) = \epsilon\lambda u^{p+1}$ on $[0,\beta]$. On the other hand, if $u \in [\beta,b]$, then it follows that $\frac{n-2}{2} W'(u)u - nW(u) \geq \frac{n-2}{2} W'(u)u - nW(\beta) \geq (\frac{n}{p+1} - \frac{n-2}{2}\kappa\mu)\lambda\beta^{p+1} > 0$. Now that (2.25) is established, we consider a radial solution $u \in C^2(\mathbb{R}^n)$ of (1.1) such that $u(\mathbb{R}^n) \subset (0,b)$. Setting $v(|x|) = u(x)$ and proceeding as in the proof of Theorem 1.4, one can see that v satisfies the standard estimates $v(r) = O(r^{-\frac{2}{p-1}})$, $v'(r) = O(r^{-\frac{p+1}{p-1}})$, and $W(v(r)) = O(r^{-\frac{2(p+1)}{p-1}})$. To conclude we use the well-known Pohozaev identity:

$$\int_0^r s^{n-1}\Big(\frac{n-2}{2} W'(v(s))v(s) - nW(v(s))\Big)ds = \frac{n-2}{2} r^{n-1} v(r)v'(r) + r^n\Big(\frac{|v'(r)|^2}{2} - W(v(r))\Big). \tag{2.26}$$

We notice that since $p \in (\frac{n}{n-2}, \frac{n+2}{n-2})$, the right hand side of (2.26) goes to 0, as $r \to \infty$. On the other hand, the left hand side of (2.26) is strictly positive in view of (2.25). This rules out the existence of radial solutions such that $u(\mathbb{R}^n) \subset (0,b)$. ☐

The next Proposition examines the existence of radial solutions in the different regimes.

Proposition 2.3. *Let $n \geq 3$, and let $W \in C^{1,1}_{loc}(\mathbb{R})$ be a potential satisfying (1.6b).*

(i) *If (1.9) holds, there are no solutions $u \in C^2(\mathbb{R}^n)$ of (1.1) such that $u(\mathbb{R}^n) \subset (a,b)$.*
(ii) *If $\limsup_{u \to a^+} \frac{|W'(u)|}{|u-a|^{\frac{n+2}{n-2}}} = 0$ holds, there exists a radial solution $u \in C^2(\mathbb{R}^n)$ of (1.1) such that $u(\mathbb{R}^n) \subset (a,b)$.*
(iii) *Otherwise, if neither (1.9) nor $\limsup_{u \to a^+} \frac{|W'(u)|}{|u-a|^{\frac{n+2}{n-2}}} = 0$ hold, depending on W, there may or may not exist a radial solution $u \in C^2(\mathbb{R}^n)$ of (1.1) such that $u(\mathbb{R}^n) \subset (a,b)$.*

Proof. (i) By Theorem 1.4, a solution $u \in C^2(\mathbb{R}^n)$ of (1.1) such that $u(\mathbb{R}^n) \subset (a,b)$ fulfils $\lim_{|x| \to \infty} u(x) = b$. As a consequence it has a minimum point x_0. On the other hand

$$\Delta u(x_0) = W'(u(x_0)) < 0,$$

which is a contradiction.

(ii) Now, assume that $\limsup_{u \to a^+} \frac{|W'(u)|}{|u-a|^{\frac{n+2}{n-2}}} = 0$ holds, and define

$$\tilde{W}(v) = \begin{cases} W(v) & \text{for } v \leq b \\ W(b) & \text{for } v \geq b. \end{cases} \tag{2.27}$$

Theorem 4 of [2] provides the existence of a radial solution $u \in C^2(\mathbb{R}^n)$ of $\Delta u = \tilde{W}'(u)$, such that $u > a$, and $\lim_{|x| \to \infty} u(x) = a$. By the maximum principle, we have $u(\mathbb{R}^n) \subset (a,b)$, and thus u solves $\Delta u = W'(u)$.

Finally, (iii) follows from Lemmas 2.1 and 2.2. ☐

As we mentioned in the Introduction, for general domains, condition (1.9) is not sufficient to conclude that our solution fulfils

$$\lim_{|x| \to \infty} u(x) = b.$$

Proposition 2.4 below, provides examples of solutions having a different asymptotic behaviour.

Proposition 2.4. *Let $p > 1$, and let $W \in C^{1,1}_{loc}(\mathbb{R})$ be a potential fulfilling (1.6b), as well as*

$$\forall u \in [a,b]: \ W'(u) \geq -c(u-a)^p, \text{ for a constant } c > 0. \tag{2.28}$$

Let $D = \{x \in \mathbb{R}^2 : |x_2| < \psi(x_1)\}$, where $\psi \in C^\infty(\mathbb{R})$ is a positive function such that $\psi(s) = \lambda|s|$, for $|s| > \epsilon$ (with $\lambda, \epsilon > 0$ sufficiently small, depending on W). Then, there exists a solution $u \in C^2(D)$ of (1.1) such that $u(D) \subset (a,b)$, and

$$\lim_{x_1 \to +\infty} u(x) = a \text{ and } \lim_{x_1 \to -\infty} u(x) = b. \tag{2.29}$$

Proof. Without loss of generality we may assume that $a = 0$. We shall first construct a supersolution ϕ^* of (1.1) in D. We define the auxiliary functions

$$f(re^{i\theta}) = r^{-\frac{2}{p-1}} g(\theta), \tag{2.30a}$$

with $g : [-\theta_0, \theta_0] \to (0, \infty)$ ($\theta_0 < \frac{\pi}{2}$), a positive solution of the O.D.E.:

$$g''(\theta) = -c g^p(\theta) - \frac{4}{(p-1)^2} g(\theta). \tag{2.30b}$$

Next, setting $\lambda = \tan(\theta_0)$, one can check that

$$\Delta f(x) = -c(f(x))^p \text{ in the sector } S = \{x_1 > 0, |x_2| < \lambda x_1\}. \tag{2.31}$$

In addition, we have $f(x) > b$ in the set $\{0 < x_1 \le \epsilon, |x_2| < \lambda x_1\}$, provided that $\epsilon > 0$ is sufficiently small. Finally, we take

$$\phi^*(x) = \begin{cases} \min(f(x), b) & \text{when } x_1 > \epsilon, \text{ and } |x_2| < \lambda x_1. \\ b & \text{when } x_1 \le \epsilon, \text{ and } |x_2| < \psi(x_1). \end{cases} \tag{2.32}$$

Using the Kato inequality, one can see that ϕ^* is a supersolution of (1.1) in D. Indeed, in view of (2.28) and (2.31), we have

$$\Delta\phi^* \le -c f^p \chi_{\{f<b\}} \le W'(\phi^*) \text{ in } H^1_{loc}(D), \tag{2.33}$$

where χ is the characteristic function.

To construct a subsolution ϕ_* of (1.1) in D, we take

$$\phi_*(x) = \begin{cases} e(x_1) & \text{when } x_1 < 0, \text{ and } |x_2| < \psi(x_1) \\ 0 & \text{when } x_1 \ge 0, \text{ and } |x_2| < \psi(x_1), \end{cases} \tag{2.34}$$

where $e : (-\infty, 0] \to [0, b)$ is the heteroclinic orbit, solving

$$e''(s) = W'(e(s)), \ e(0) = 0, \ \lim_{s\to-\infty} e(s) = b. \tag{2.35}$$

The existence of such a heteroclinic orbit is proved by extending W to an even $C^{1,1}(\mathbb{R})$ function $\tilde{W}$ such that $\tilde{W} = W$ in $(0, b)$, $\tilde{W}' > 0$ in (b, ∞) and considering the phase plane for the ODE $v'' = \tilde{W}'(v)$. The situation is analogue to the one we have for the classical double well potential $\frac{1}{4}(1-t^2)^2$.

It follows again from the Kato inequality that $\Delta\phi_* \ge W'(\phi_*)$ holds in $H^1_{loc}(D)$. In addition, it is clear that $\phi_* < \phi^*$ holds in D. Therefore, we deduce from the method of sub- and supersolutions (cf. Appendix, and for instance [7, Lemma 1.1.1]), the existence of a solution $u \in C^2(D)$ of (1.1) satisfying $\phi_* \le u \le \phi^*$. Since $0 < u < b$ by the maximum principle, the solution u has all the desired properties. ☐

Now, we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Assume $u \in C^2(\mathbb{R}^n)$ is an entire solution of (1.1) such that $u(\mathbb{R}^n) \subset [a, b]$. When $n = 2$, u is a bounded superharmonic function defined on $\mathbb{R}^2$. Thus, u is constant and equal to a critical point of W. That is, $u \equiv a$ or $u \equiv b$.

In higher dimensions $n \ge 3$, we have by the maximum principle either $u \equiv a$, or $a < u \le b$ on $\mathbb{R}^n$. We shall first assume that $a < u \le b$ as well as (1.9) hold, and we shall prove that $u \equiv b$. In view of (1.9), Theorem 1.4 implies that $\lim_{|x|\to\infty} u(x) = b$, and $a + \epsilon \le u \le b$ holds on $\mathbb{R}^n$, for some $\epsilon > 0$. Thus, $u \equiv b$, by Theorem 1.2.

Next, we consider again an entire solution u of (1.1) satisfying $u(\mathbb{R}^n) \subset [a, b]$ in dimensions $n \ge 3$, but without assuming (1.9). By the maximum principle, we have either $u \equiv a$, or $u \equiv b$, or $a < u < b$. Let

$$\mathcal{F} = \{u \text{is a solution of(1.1) such that } u(\mathbb{R}^n) \subset (a, b)\},$$

$$C_W = \sup\{u(x) : x \in \mathbb{R}^n, u \in \mathcal{F}\}.$$

Our first claim is that

$$C_W < b. \tag{2.36}$$

Indeed, assume by contradiction that there exists a sequence $\{u_k\} \subset \mathcal{F}$, and a sequence $\{x_k\} \subset \mathbb{R}^n$, such that $\lim_{k\to\infty} u_k(x_k) = b$. Setting $v_k(y) = u_k(x_k + y)$, and proceeding as in Theorem 1.2, we obtain that (up to subsequence) v_k converges in $C^2_{\text{loc}}(\mathbb{R}^n)$ to an entire solution v_∞ of (1.1). Furthermore, since $v_\infty(0) = b$, the maximum principle implies that $v_\infty \equiv b$. At this stage we consider the minimiser $\phi_R \in H^1(B_R(0))$ defined in (2.22). It is known that ϕ_R is a smooth radial solution of (1.1) in $B_R(0)$, such that $a \le \phi_R \le b - \delta_R$ on $B_R(0)$, for some $\delta_R > 0$. In addition, by taking $R > R_0$ large enough, we have $a < \phi_R \le b - \delta_R$ on $B_R(0)$. Thus, for fixed $R > R_0$, we can ensure that $a < \phi_R \le b - \delta_R \le v_k$ holds on $B_R(0)$, provided that $k \ge k_R$ is large enough. Finally, by applying the sliding method of Berestycki, Caffarelli, and Nirenberg [1, Lemma 3.1], we deduce that for $k \ge k_R$, v_k as well as u_k are entire solutions of (1.1) satisfying respectively $v_k(\mathbb{R}^n) \subset [a + \epsilon_R, b]$, and $u_k(\mathbb{R}^n) \subset [a + \epsilon_R, b]$, with $\epsilon_R := \phi_R(0) - a > 0$. In view of Theorem 1.2, this implies that $u_k \equiv b$, for $k \ge k_R$, which is a contradiction. This proves (2.36).

The fact that $\liminf_{|x|\to\infty} u(x) = a$ holds for every $u \in \mathcal{F}$ also follows from Theorem 1.2. Indeed, assuming by contradiction that $\liminf_{|x|\to\infty} u(x) > a$ we would obtain that $u(\mathbb{R}^n) \subset [a + \epsilon, b]$, for some $\epsilon > 0$. Therefore, using Theorem 1.2, we conclude that $u \equiv b$, which is a contradiction. ☐

Remark 2.5. Let D be a domain satisfying (1.2), and let $u \in C^2(D)$ be a solution of (1.1) such that $u(D) \subset (a, b]$. In the nondegenerate case where (1.7) holds, [1, Lemma 3.2] implies that $a + \epsilon < u(x) \le b$ holds for some $\epsilon > 0$, provided that $d(x, \partial D) > \eta$, for some $\eta > 0$. Thus, in view of Theorem 1.2, we have $\lim_{d(x,\partial D)\to\infty} u(x) = b$.

3. Some Liouville type results

In this section, we prove a general Liouville type theorem (Theorem 3.1 below) from which we derive two corollaries. One the one hand, Corollary 3.3 holds in the nondegenerate case where (1.7) holds. On the other hand, Theorem 1.6 that we already mentioned in the Introduction, is applicable in the complement of a ball, under the weaker assumption (1.9).

Theorem 3.1. *Let $W \in C^{1,1}_{loc}(\mathbb{R})$ be a nonnegative potential satisfying (1.6b), as well as $W(b) = 0$, and let $D \subset \mathbb{R}^n$ be a domain such that*

$$\textit{the radii of the balls contained in } \mathbb{R}^n \setminus D \textit{ are uniformly bounded by a constant } \Lambda > 0. \tag{3.37}$$

We also assume that $u \in C^2(\mathbb{R}^n)$ is a bounded entire solution of (1.1) such that $\sup_{\mathbb{R}^n} u = b$, and $u(D) \subset (a, b]$. Then, $u \equiv b$.

Proof. On the one hand, since

$$\sup_{\mathbb{R}^n} u = b,$$

let $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ be a sequence such that $\lim_{k\to\infty} u(x_k) = b$, and set $v_k(y) = u(x_k + y)$. Proceeding as in the proof of Theorem 1.5, one can see that (up to subsequence), v_k converges in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ to an entire solution $v_\infty \equiv b$. In particular, given $R > 0$ and $\delta > 0$, we have $u(x) \in [b - \delta, b]$, provided that $x \in B_R(x_k)$, and $k \ge K(R, \delta)$ is large enough.

On the other hand, let $\iota := \inf_{\mathbb{R}^n} u \le b$, and assume by contradiction that $\iota < b$.

We distinguish two scenarios. If $W(\iota) > 0$, we define the auxiliary potential

$$\tilde{W}(u) = \begin{cases} W(u) & \text{for } u \ge \iota \\ W(\iota) & \text{for } u \le \iota, \end{cases}$$

and consider a minimiser $\phi_R \in H^1_{rad}(B_R(0))$ of the energy functional

$$\tilde{E}(v) = \int_{B_R(0)} \Big(\frac{1}{2}|\nabla v(x)|^2 + \tilde{W}(v(x))\Big)dx,$$

in the class $\mathcal{A} = \{v \in H^1_{rad}(B_R(0)),\ v = \iota \text{ on } \partial B_R(0)\}$ of radial functions in $H^1(B_R)$ whose boundary value is ι. Setting $\sigma := \min\{t \ge \iota : W(t) = 0\}$, and $\sigma_R := \sup_{B_R(0)} \phi_R$, one can see that

$$\iota \le \phi_R \le \sigma_R < \sigma$$

holds for every $R > 0$, since $W(\iota) > 0$. In addition, ϕ_R is a smooth radial solution of (1.1) in $B_R(0)$, such that $\lim_{R\to\infty} \sigma_R = \sigma$. Thus by taking $R > 0$ large enough, we can ensure that $\iota < \sigma_R < b$. As a consequence, we also have $\phi_R(y) \le u(y + x_k)$, provided that $y \in B_R(0)$, and $k \ge K(R, \delta)$. Finally, by applying the sliding method of Berestycki, Caffarelli, and Nirenberg [1, Lemma 3.1], we deduce that for $u \ge \sigma_R > \iota$, holds on $\mathbb{R}^n$, which is a contradiction.

It remains to treat the case $W(\iota) = 0$, or equivalently $\iota = \sigma$. To complete the proof of Theorem 3.1, it remains to show that $\iota = b$. Indeed, if $\iota < b$ and $W(\iota) = 0$, then in particular $\iota < a$, since $W' < 0$ on (a, b). Let $\{z_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ be a sequence such that $\lim_{k\to\infty} u(z_k) = \iota$, and set $w_k(y) = u(z_k + y)$. Proceeding as previously, we obtain that (up to subsequence), w_k converges in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ to an entire solution $w_\infty \equiv \iota$. In particular, w_k converges to ι uniformly on the ball centred at the origin of radius $R_0 = \Lambda + 1$ (cf. (3.37)). As a consequence, taking $\eta > 0$ such that $\iota + \eta < a$, we have $u(x) \in [\iota, \iota + \eta]$, provided that $x \in B_{R_0}(z_k) \cap D$, and $k \ge \tilde{K}(\eta)$ is large enough (we note that, in view of (3.37), $B_{R_0}(z_k) \cap D \neq \emptyset$). This is once again a contradiction, since $u(D) \subset (a, b]$ and $\eta + \iota < a$.

Therefore, we have proved that $\iota = b$, hence $u \equiv b$. ☐

Remark 3.2. Modica [19] proved that if $W \in C^2(\mathbb{R})$ is a non negative potential, and u is a bounded solution of (1.1) in $\mathbb{R}^n$, then the condition $W(u(x_0)) = 0$ for some $x_0 \in \mathbb{R}^n$ implies that u is constant. In the sequel, a new proof of this result which also applies to potentials $W \in C^{1,1}_{loc}(\mathbb{R})$ was proposed in [4]. Therefore, the hypothesis $\sup_{\mathbb{R}^n} u = b$ in Theorem 3.1 is not very strong, since the condition $u(D) \subset (a, b]$ yields that either $u < b$ in $\mathbb{R}^n$ or $u \equiv b$, so that $\sup_{\mathbb{R}^n} u \le b$.

Since the nondegeneracy condition (1.7) implies that $\lim_{d(x,\partial D)\to\infty} u(x) = b$, when D satisfies (1.2) (cf. Remark 2.5), we obtain a first corollary of Theorem 3.1:

Corollary 3.3. *Let $W \in C^{1,1}_{loc}(\mathbb{R})$ be a non negative potential satisfying (1.6b), (1.7) and $W(b) = 0$. Assume the domain D satisfies (1.2) as well as (3.37), and $u \in C^2(\mathbb{R}^n)$ is a bounded entire solution of (1.1) such that $u(D) \subset (a, b]$. Then, $u \equiv b$.*

Proof. Under the assumptions of Corollary 3.3, it follows from Remark 2.5 that $\lim_{d(x,\partial D)\to\infty} u(x) = b$. On the other hand, in view of Remark 3.2 we have $u \le b$, so that $\sup_{\mathbb{R}^n} u = b$. Therefore, Theorem 3.1 implies that $u \equiv b$. ☐

Next, we give the proof of Theorem 1.6.

Proof of Theorem 1.6. When $n \geq 3$, we first apply Theorem 1.4 to deduce that $\lim_{|x|\to\infty} u(x) = b$. Next, in view of Remark 3.2 we obtain that $\sup_{\mathbb{R}^n} u = b$. Finally, Theorem 3.1 implies that $u \equiv b$. On the other hand, when $n = 2$, u is superharmonic in $\{x \in \mathbb{R}^2 : |x| > R\}$. Setting $\gamma := \min_{|x|=R+1} u(x) \in (a, b)$, we deduce from the maximum principle in unbounded domains, stated in Lemma B.2, that $u(x) \in [\gamma, b)$, provided that $|x| > R+1$. Since $W' < 0$ in $[\gamma, b)$, in view of Theorem 1.2 we still have $\lim_{|x|\to\infty} u(x) = b$. By Remark 3.2, we still have $\sup_{\mathbb{R}^n} u = b$ so that finally, by Theorem 3.1, we conclude as previously that $u \equiv b$. ☐

4. Radial symmetry for solutions converging to the local minimum of the potential: Proof of Theorem 1.7

The following result that will be used in the proof of Theorem 1.7, is a consequence of [13, Proposition 1]:

Proposition 4.1. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a solution to $\Delta u = W'(u)$, $n \geq 3$, where $W \in C^2(\mathbb{R}^n)$ is a potential fulfilling (1.6b) and such that*

$$[b-\delta, b] \ni t \mapsto \frac{W'(t)}{|t-b|^p} \text{ is Hölder continuous for some } \delta > 0,\ p \geq \frac{n+2}{n-2}. \tag{4.38}$$

Assume that $u < b$ in $\mathbb{R}^n$ and $\lim_{|x|\to\infty} u(x) = b$. Then u is radially symmetric.

Proof. First we note that $v := b - u > 0$ is bounded and subharmonic outside B_R, in fact $-\Delta v = \Delta u = W'(b-v) \leq 0$ in $\mathbb{R}^n \backslash B_R$, hence by [12, Lemma 22] we have the decay estimate

$$v(x) \leq C|x|^{2-n} \qquad \forall,\ |x| \geq \rho. \tag{4.39}$$

Next, it follows from [13, Proposition 1 and Theorem 4] that v is radial. ☐

Proposition 4.1 together with [12, Theorem 2] are the main ingredients in the proof of Theorem 1.7, that we give below.

Proof of Theorem 1.7. First we show that $u < b$ in all $\mathbb{R}^n$. By the strong maximum principle, it is enough to prove that $u \leq b$ in $\mathbb{R}^n$. For this purpose, assume by contradiction that $c := \sup_{\mathbb{R}^n} u = \max_{\mathbb{R}^n} u > b$, and $W(c) > W(b)$. We note that u achieves its maximum since we are assuming that $c = \sup_{\mathbb{R}^n} u > b = \lim_{|x|\to\infty} u(x)$. Next, define the auxiliary potential

$$\tilde{W}(u) = \begin{cases} W(b) & \text{for } u \leq b \\ W(u) & \text{for } b \leq u \leq c \\ W(c) & \text{for } u \geq c, \end{cases}$$

and consider a minimiser $\phi_R \in H^1_{rad}(B_R(0))$ of the energy functional

$$\tilde{E}(v) = \int_{B_R(0)} \Big(\frac{1}{2}|\nabla v(x)|^2 + \tilde{W}(v(x))\Big)dx,$$

in the class $\mathcal{A} = \{v \in H^1_{rad}(B_R(0)),\ v = c \text{ on } \partial B_R(0)\}$ of radial functions in $H^1(B_R(0))$ whose boundary value is c. We know that ϕ_R is a radial solution to (1.1) such that $b < \min_{B_R(0)} \phi_R = \phi_R(0) < c$, for $R \geq R_0$ sufficiently large, since $\phi_R(0) \to b$ as $R \to \infty$. In addition, since $u(\mathbb{R}^n \setminus B_R(0)) \subset (a, b)$, we have $u(x+x_0) < \phi_{R_0}(x)$ on $B_{R_0}(0)$, provided that $|x_0| > R+R_0$. Finally, by applying the sliding method of Berestycki, Caffarelli, and Nirenberg [1, Lemma 3.1], we deduce that $u \leq \phi_{R_0}(0) < c$ holds on $\mathbb{R}^n$, which is a contradiction.

So far we have established that $W(c) = W(b)$. To conclude that $u \leq b$, it remains to show that $c = b$. Indeed, if $c > b$ and $W(c) = W(b)$, then c is a local minimum of W satisfying $W'(c) = 0$, and there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) = c$. Thus, by the maximum principle, we obtain $u \equiv c$, which is excluded since $u(\mathbb{R}^n \backslash B_R) \subset (a, b)$.

To complete the proof of Theorem 1.7, we shall use Proposition 4.1, [12, Theorem 2], and the regularity of W. We first assume that hypothesis (i) holds, and distinguish the following cases.

(a) If $W''(b) > 0$, then $v := b - u > 0$ is a decaying entire solution to

$$-\Delta v = f(v) := W'(b-v),$$

therefore it is radial by [12, Theorem 2], since $f'(t) \leq 0$ for $t \in (0, \delta)$.

Otherwise, $W''(b) = 0$ implies that $W'''(b) = 0$, since $W \in C^6(\mathbb{R})$, and b is a local minimum. Thus we shall examine the sign of $\frac{d^4W}{du^4}(b)$.

(b) In the case where $\frac{d^4W}{du^4}(b) > 0$, the radial symmetry of u follows again from [12, Theorem 2], since $f'(t) = W''(b-t) \leq 0$ holds for $t \in (0, \delta)$.

(c) In the case where $\frac{d^4W}{du^4}(b) = 0$, arguing as above we have $\frac{d^5W}{du^4}(b) = 0$, and $[b-\delta, b] \ni t \mapsto \frac{W'(t)}{|t-b|^5}$ is Hölder continuous. Moreover, $5 \geq \frac{n+2}{n-2}$ holds for every $n \geq 3$, hence the result follows from Proposition 4.1.

Finally, in the case where hypothesis (ii) holds, the result is straightforward in view of [12, Theorem 2]. ☐

5. A nonradial solution converging to the local maximum of the potential

In this section we will provide an example of a potential W of the form (1.6b) for which Eq. (1.1) admits a solution u such that $u(x) > a$ for $|x| > R$ and $\lim_{|x|\to\infty} u(x) = a$, but u is not radial.

The counterexample can be found in [6] using the Yamabe equation

$$-\Delta u = \frac{n(n-2)}{4}|u|^{\frac{4}{n-2}}u \quad \text{in } \mathbb{R}^n,\ n \geq 3. \tag{5.40}$$

Eq. (5.40) is variational, in the sense that it is the Euler–Lagrange equation of the energy functional

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2 - \frac{(n-2)^2}{8}\int_{\mathbb{R}^n}|u|^{\frac{2n}{n-2}}.$$

It is known that the only finite energy positive solutions are given by

$$\mu^{-\frac{n-2}{2}}U(\mu^{-1}(x-\xi)), \qquad U(x) := \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}},\ \mu > 0,\ \xi \in \mathbb{R}^n.$$

These solutions which are called the *standard bubbles*, are also the only positive solutions of (5.40) (see [5]).

Using these bubbles, in [6] the authors construct a sequence of bounded entire solutions $\{u_k\}_{k\geq k_0}$ to (5.40) in $\mathbb{R}^n$ of the form

$$u_k := v_k + \phi_k, \tag{5.41}$$

where the approximate solution v_k is given by

$$\begin{aligned} v_k(x) &:= U(x) - \sum_{j=1}^{k}\mu_k^{-\frac{n-2}{2}}U(\mu_k^{-1}(x-\xi_{j,k})),\\ \mu_k &= c_n k^{-2} \text{ for } n \geq 4,\ \mu_k = c_3 k^{-2}(\log k)^{-2} \text{ for } n = 3\\ \xi_{j,k} &:= (\cos(\frac{2\pi j}{k}), \cos(\frac{2\pi j}{k}), 0, \dots, 0) \qquad 1 \leq j \leq k \end{aligned} \tag{5.42}$$

and the corrections ϕ_k fulfil

$$|\phi_k(x)| \leq \frac{c}{\log k(1+|x|)} \text{ if } n = 3, \qquad |\phi_k(x)| \leq \frac{c}{k^{\alpha_n}(1+|x|^{n-2})} \text{ if } n \geq 4, \text{ with } \alpha_n > 0. \tag{5.43}$$

As a consequence, these solutions v_k are $L^\infty(\mathbb{R}^n)$ close to a linear combination of $k+1$ rescaled bubbles. One of them is positive and centred at the origin, the other ones are negative and centred along the unit circle $S^1 \subset \mathbb{R}^2$. It particular, they are sign changing solutions. Moreover, it follows from (5.42) and (5.43) that

$$u_k(x) \to 0 \text{ as } |x| \to \infty, \text{ for any } k \geq k_0. \tag{5.44}$$

We are going to study the asymptotic behaviour of the solutions u_k constructed in [6] in order to prove that, for some k large enough, u_k is positive outside a ball.

Lemma 5.1. *There exist $\bar{r} > 0$ and $\bar{k} > 0$ such that $u_k(x) > 0$ if $|x| > \bar{r}$ and $k \geq \bar{k}$.*

Proof. We will show that, for k large enough, the approximate solution v_k fulfils

$$v_k(x) > \frac{2}{3}U(x) > 0 \qquad \text{if } |x| > \bar{r} \tag{5.45}$$

for some large $\bar{r} > 0$. In fact, (5.45), we note that, in dimension $n = 3$ we have

$$\begin{aligned} v_k(x) &= \left(\frac{2}{1+r^2}\right)^{\frac{1}{2}} - \sum_{j=1}^{k}\mu_k^{-\frac{1}{2}}U\left(\frac{x-\xi_{j,k}}{\mu_k}\right) \geq \left(\frac{2}{1+r^2}\right)^{\frac{1}{2}} - k\mu_k^{-\frac{1}{2}}\left(\frac{2\mu_k^2}{\mu_k^2+(r-1)^2}\right)^{\frac{1}{2}}\\ &\geq \left(\frac{2}{1+r^2}\right)^{\frac{1}{2}}\left(1 - k\mu_k^{\frac{1}{2}}\left(\frac{1+r^2}{(r-1)^2}\right)^{\frac{1}{2}}\right) = \left(\frac{2}{1+r^2}\right)^{\frac{1}{2}}\left(1 - \frac{\sqrt{c_3}}{\log k}\left(\frac{1+r^2}{(r-1)^2}\right)^{\frac{1}{2}}\right) > \frac{2}{3}U(x) \end{aligned}$$

where $r = |x|$ and k are large enough. Similarly, in higher dimension, we have

$$v_k(x) \geq \left(\frac{2}{1+r^2}\right)^{\frac{n-2}{2}}\left(1 - \frac{c_n^{\frac{n-2}{2}}}{k^{n-3}}\left(\frac{1+r^2}{(r-1)^2}\right)^{\frac{n-2}{2}}\right) > \frac{2}{3}U(x).$$

for r and k large enough.

Then we apply (5.43) and (5.45) to conclude that

$$u_k(x) = v_k(x) + \phi_k(x) > \frac{2}{3}U(x) - \frac{C}{(1+|x|)\log k} > \frac{1}{2}U(x) > 0$$

outside a large ball in dimension $n \geq 3$. Similarly, in higher dimension we have

$$u_k(x) = v_k(x) + \phi_k(x) > \frac{2}{3}U(x) - \frac{C}{k^{\alpha_n}(1+|x|^{n-2})} > \frac{1}{2}U(x) > 0$$

outside a large ball. ☐

Finally, we can take $b > \|u_{\bar{k}}\|_{L^\infty(\mathbb{R}^n)}$ and define a $C^1(\mathbb{R})$ function f such that $f(t) < 0$ for any $t \in (0, b)$, $f(t) = -\frac{n(n-2)}{4}|t|^{\frac{4}{n-2}}t$ for $|t| \le \|u_{\bar{k}}\|_{L^\infty(\mathbb{R}^n)}$, and $f(b) = 0$. Then $f'(0) = 0$ and $u_{\bar{k}}$ is a solution to $\Delta u = f(u)$. Taking W to be a primitive of f, we have the required counter example. In fact, we have $0 < u_{\bar{k}}(x) < b$ in $D := \mathbb{R}^n \backslash B_{\bar{r}}$, $u_{\bar{k}} \to 0$ as $|x| \to \infty$ but $u_{\bar{k}}$ is not radial.

Remark 5.2. We stress that the solution $u_{\bar{k}}$ to which we refer in Lemma 5.1 is actually sign changing and $a = 0$ for the equation that it satisfies. We do not know if entire solutions $u : \mathbb{R}^n \to (a, \infty)$ to (1.1) with W fulfilling (1.6b) and $\lim_{|x|\to\infty} u(x) = a$ are radial. This is a challenging open problem.

Data availability

No data was used for the research described in the article.

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Appendix. The method of sub- and supersolutions

Let $\Omega \subset \mathbb{R}^n$ be a open set with Lipschitz boundary, and let $f \in C^\alpha_{loc}(\mathbb{R})$, for some $\alpha \in (0, 1)$. We say that $\underline{u} \in W^{1,2}(\Omega)$ is a subsolution (respectively $\bar{u} \in W^{1,2}(\Omega)$ is a supersolution) to

$$\Delta u = f(u), \tag{A.1}$$

if $\Delta \underline{u} \ge f(\underline{u})$ (respectively $\Delta \bar{u} \le f(\bar{u})$) holds in Ω in the weak sense.

Proposition A.1. *Let $\underline{u} \le \bar{u}$ be a couple of bounded $W^{1,2}(\Omega)$ sub- and supersolutions to (A.1). Then, there exists a solution $u \in C^2(\Omega) \cap W^{1,2}(\Omega)$ to (A.1), satisfying $\underline{u} \le u \le \bar{u}$.*

Proof. We introduce the nonlinearity

$$g(x, u) := \begin{cases} f(\underline{u}(x)) & \text{if } u < \underline{u}(x), \\ f(u) & \text{if } \underline{u}(x) \le u \le \bar{u}(x), \\ f(\bar{u}(x)) & \text{if } u > \bar{u}(x), \end{cases} \tag{A.2}$$

and set $G(x, u) = \int_0^u g(x, t)dt$. Next, we establish (exactly as in the proof of [7, Lemma 1.1.1]), the existence of a minimiser u of the energy functional:

$$\mathcal{E}(v) = \int_\Omega \Big(\frac{1}{2}|\nabla v(x)|^2 + G(x, v(x))\Big)dx, \tag{A.3}$$

in the class $\mathcal{A} = \underline{u} + W^{1,2}_0(\Omega)$. For the sake of simplicity, we consider in the definition of $\mathcal{A}$, the boundary condition $v = \underline{u}$ on $\partial\Omega$. However, we could also set $\mathcal{A} = \phi + W^{1,2}_0(\Omega)$, with any $\phi \in W^{1,2}(\Omega)$ such that $\underline{u} \le \phi \le \bar{u}$ holds on $\partial\Omega$. By construction, u solves the Euler–Lagrange equation

$$\Delta u = g(x, u), \ \ x \in \Omega. \tag{A.4}$$

Moreover, it follows from the maximum principle that $\underline{u} \le u \le \bar{u}$ in Ω, which yields that u is actually a $C^2(\Omega)$ solution to (A.1), satisfying $\underline{u} \le u \le \bar{u}$. □

Remark A.2. If in Proposition A.1, we consider a domain $\Omega = \{x \in \mathbb{R}^n : \rho_1 < |x| < \rho_2\}$, and a couple $\underline{u} \le \bar{u}$ of bounded *radial* sub- and supersolutions to (A.1), then we obtain the existence of a *radial* solution $u \in C^2(\Omega) \cap W^{1,2}(\Omega)$ to (A.1), satisfying $\underline{u} \le u \le \bar{u}$. Indeed, since the nonlinearity (A.2) and the energy functional (A.3) are invariant by the orthogonal group $O(n)$, we can look for a minimiser u in the class $\mathcal{A}_{O(n)} = \{v \in \mathcal{A} : v(\sigma x) = v(x), \forall \sigma \in O(n)\}$. By the principle of symmetric criticality [20], u is a smooth *radial* solution to (A.4), and the bounds $\underline{u} \le u \le \bar{u}$ follow as previously from the maximum principle.

The method of sub- and supersolutions is also applicable in unbounded domains. In Theorem 1.4, we apply it in $\Omega = \mathbb{R}^n \setminus B_\rho$, with a radial subsolution $\phi_*(x) = c|x|^{2-n}$, and a radial supersolution $\phi^* \ge \phi_*$, $\phi^* \in W^{1,2}(B_R \setminus \overline{B}_\rho)$, $\forall R > \rho$. As a consequence of Proposition A.1 and Remark A.2, we obtain for every $R > \rho$, a radial solution v_R to (1.1) in $\Omega_R := B_R \setminus \overline{B}_\rho$, satisfying

- $\phi_* \le v_R \le \phi^*$ in Ω_R,
- $v_R = \phi_*$ on $\partial\Omega_R$.

In addition, since for any $\alpha \in (0,1)$, the $C^{1,\alpha}$ norm of $\partial\Omega_R$ is uniformly bounded, and the $C^{1,\alpha}$ norm of ϕ_* is also bounded in $\overline{\Omega}$, we deduce that the $C^{1,\alpha}$ norm of v_R is uniformly bounded in $\overline{\Omega_R}$, $\forall R > \rho$ (cf. [14, Theorem 8.33]). Finally, we use the Theorem of Ascoli, via a diagonal argument, to prove that the limit $v = \lim_{R\to\infty} v_R$ exists (up to subsequence) and is a radial solution to (1.1) in Ω, satisfying $\phi_* \le v \le \phi^*$ in Ω.

In Proposition 2.4, we have a second application of the method of sub- and supersolutions in an unbounded domain D, such that ∂D is bounded for the $C^{1,\alpha}$ norm. Here again, we consider an increasing sequence of bounded domains D_k, such that $D = \cup_k D_k$, and the boundaries ∂D_k are uniformly bounded for the $C^{1,\alpha}$ norm. In view of Proposition A.1, we obtain in each domain D_k a solution u_k of (1.1), and then by taking the limit $u = \lim_{k\to\infty} u_k$ via the same diagonal argument, we construct the solution u in the whole domain D.

Appendix B. Two lemmas for superharmonic functions

Here we recall two classical results on superharmonic functions.

Lemma B.1. *Let $n \ge 3$, let $B_\rho \subset \mathbb{R}^n$ be the open ball of radius ρ centred at the origin, and let $u \in C^2(\mathbb{R}^n \setminus B_\rho)$ be a positive and bounded function, such that $\Delta u \le 0$ in $\mathbb{R}^n \setminus B_\rho$. Then, there exists a constant $c > 0$ such that $u(x) \ge c|x|^{2-n}$, for any $x \in \mathbb{R}^n \setminus B_\rho$.*

Proof. We fix $y \in \mathbb{R}^n \setminus \overline{B_\rho(0)}$, $\varepsilon > 0$ and we prove that $u(y) \ge c|y|^{2-n} - \varepsilon$, for some constant $c > 0$ independent of ε, so that the result follows by letting $\varepsilon \to 0$.

In order to do so, we note that

$$u(x) \ge \inf_{\partial B_\rho} u =: c\rho^{2-n} = c|x|^{2-n} > c|x|^{2-n} - \varepsilon \qquad \forall\, x \in \partial B_\rho.$$

Moreover, taking $R > |y|$ large enough, we have

$$c|x|^{2-n} - \varepsilon < 0 < u(x) \qquad \forall\, x \in \partial B_R.$$

As a consequence, using that $c|x|^{2-n} - \varepsilon$ is harmonic in the set $A := \{x \in \mathbb{R}^n : \rho < |x| < R\}$, the maximum principle yields that $u \ge c|x|^{2-n} - \varepsilon$ in A. In particular we have $u(y) \ge c|y|^{2-n} - \varepsilon$. ☐

Lemma B.2. *Let $B_r(0) \subset \mathbb{R}^2$ be the open ball of radius r centred at the origin, and let $\psi \in C(\mathbb{R}^2 \setminus B_r(0))$ be a function such that*

- $\psi \in W^{1,2}_{loc}(\mathbb{R}^2 \setminus \overline{B_r(0)})$,
- *ψ is bounded from below on $\mathbb{R}^2 \setminus B_r(0)$,*
- *$\Delta\psi \le 0$, on $\mathbb{R}^2 \setminus \overline{B_r(0)}$.*

Then, ψ attains its minimum on $\partial B_r(0)$.

Proof. Let $x_0 \in \partial B_r(0)$ be such that $\min_{\partial B_r(0)} \psi = \psi(x_0)$. For every $\epsilon > 0$ fixed, we consider the function $\zeta_\epsilon(x) = \psi(x) + \epsilon \ln(|x|/r)$ which is superharmonic on $\mathbb{R}^2 \setminus \overline{B_r(0)}$. In addition, we have $\zeta_\epsilon(x) > \zeta_\epsilon(x_0) = \psi(x_0)$, provided that $|x| \ge R_\epsilon$ (with R_ϵ sufficiently large). Thus, by the maximum principle, the minimum of ζ_ϵ in the annuli $r \le |x| \le R$, with $R \ge R_\epsilon$, is attained at x_0. This implies, that for every $\epsilon > 0$, and $x \in \mathbb{R}^2 \setminus B_r(0)$, we have $\zeta_\epsilon(x) \ge \psi(x_0) \Leftrightarrow \psi(x) \ge \psi(x_0) - \epsilon \ln(|x|/r)$. Finally, letting $\epsilon \to 0$, we obtain that $\psi(x) \ge \psi(x_0)$ holds for every $x \in \mathbb{R}^2 \setminus B_r(0)$. ☐

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