

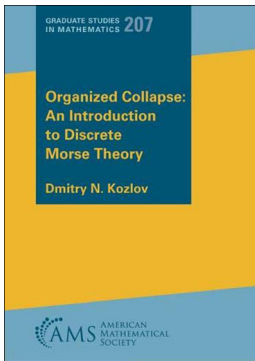


Dmitry N. Kozlov: “Organized Collapse: An Introduction to Discrete Morse Theory”

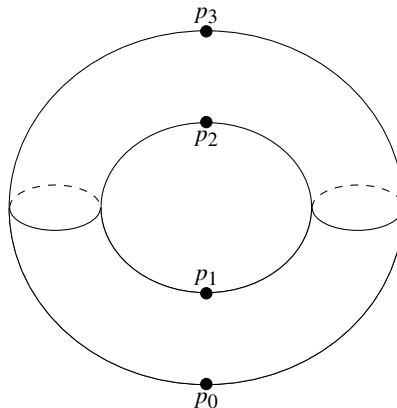
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Morse theory. The one thing almost everyone knows about Morse theory is that it’s the thing where you dip a doughnut in a mug of coffee, referring to the following example that Milnor’s seminal book [5] starts with. A torus T (the surface of the doughnut) is embedded in \mathbb{R}^3 as in the picture. For $\ell \in \mathbb{R}$ we consider the halfspace $H_\ell = \{(x, y, z) \mid z \leq \ell\}$ (the space below coffee level). We are wondering what the topology is of the part of the torus below the coffee level, i.e. of $T \cap H_\ell$. It turns out that the homotopy type of H_ℓ only changes when the level ℓ traverses the level ℓ_i of one of the *critical points* p_i : for $\ell < \ell_0$ it is empty; for $\ell_0 \leq \ell < \ell_1$ it is contractible; for $\ell_1 \leq \ell < \ell_2$ it is a circle; for $\ell_2 \leq \ell < \ell_3$ it is a wedge sum of two circles (a punctured torus); for $\ell_3 \leq \ell$ it is the full torus.



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This is explained by Morse theory as follows. The torus is a closed smooth manifold. The subspaces are the sublevel sets $T \cap H_\ell = h^{-1}((-\infty, \ell])$ of the height function

$$h: T \rightarrow \mathbb{R}$$

$$(x, y, z) \mapsto z.$$

A *critical point* is a point where the first derivative of h vanishes and it is *non-degenerate* if the Hesse matrix of second derivatives is regular. The *index* of a non-degenerate critical point is the dimension of the space of downward pointing directions or, more formally, the negative index of inertia of the Hessian. The fact that every critical point of h is non-degenerate makes it a *Morse function*. In general, the heights of the critical points can be taken to be distinct after a small perturbation and are called the *critical levels*.

Now Morse theory asserts that if ℓ is not a critical level then H_ℓ deformation retracts onto some $H_{\ell-\varepsilon}$. More precisely, the gradient of h is a vector field away from the critical points and the deformation retraction consists in flowing along this field. And if $\ell = \ell_i$ is a critical level then H_ℓ is obtained from $H_{\ell-\varepsilon}$, up to homotopy equivalence, by coning off a sphere whose dimension is the index of p_i . In particular, the manifold has the homotopy type of a CW-complex with one i -cell for each critical point of index i ¹.

In the torus example p_0 has index 0, p_1 and p_2 have index 1 and p_3 has index two and one recovers, that a torus is homotopy equivalent (and in this case even homeomorphic) to a CW-complex with one 0-cell, two 1-cells and one 2-cell. (To infer how the cells are glued together requires a more detailed analysis of the situation.)

Forman [1] has developed a combinatorial (or discrete) version of Morse theory that applies to CW-complexes. Although this theory is different on a technical level, one can recover the main conclusions of Morse theory using it. As in the book under review we restrict to simplicial complexes here since this is the most interesting case. A *discrete Morse function* in the sense of Forman on a finite simplicial complex is a function

$$f: K \rightarrow \mathbb{R}$$

on the set of simplices K such that for every simplex σ the two conditions

$$\#\{\tau > \sigma \mid f(\tau) \leq f(\sigma)\} \leq 1 \quad \text{and} \quad \#\{\tau < \sigma \mid f(\tau) \geq f(\sigma)\} \leq 1 \quad (1)$$

are met. The notation $\sigma < \tau$ is supposed to express that σ is a face of codimension 1 of τ , that is, σ is a *facet* of τ and τ is a *cofacet* of σ . Calling a (co)facet τ *ascending* or *descending* for σ respectively if its f -value is strictly larger or strictly smaller than that of σ , we see that facets should typically be descending and cofacets should typically be ascending. More precisely, conditions (1) express that every simplex σ

¹A more precise description is a handlebody decomposition but we skip it because it has no counterpart in discrete Morse theory.

has at most one non-ascending cofacet and at most one non-descending facet. It is an elementary exercise to see that, in fact, it can only have either a non-ascending cofacet or a non-descending facet but not both (if f is a discrete Morse function).

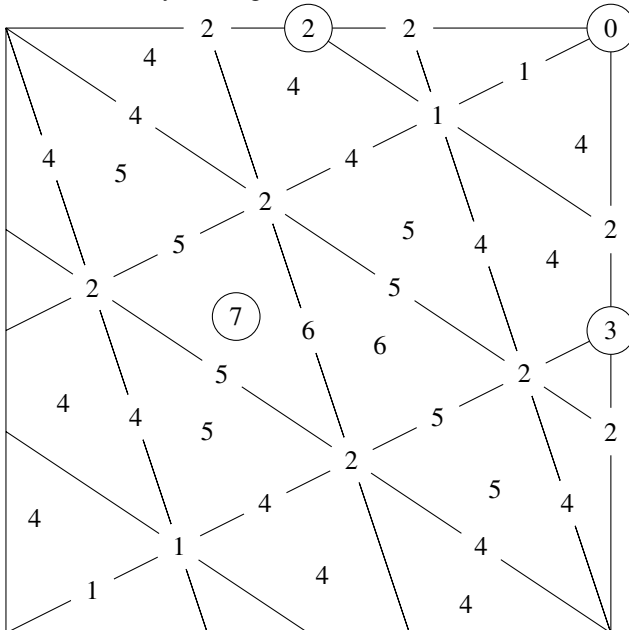
A simplex σ is *critical* if it has neither a non-ascending cofacet nor a non-descending facet, i.e. it satisfies

$$\#\{\tau > \sigma \mid f(\tau) \leq f(\sigma)\} = 0 \quad \text{and} \quad \#\{\tau < \sigma \mid f(\tau) \geq f(\sigma)\} = 0.$$

The main theorem of discrete Morse theory now asserts that the simplicial complex is homotopy equivalent to a cw-complex with one p -cell for every critical p -simplex.

The following picture shows the doughnut-coffee example using discrete Morse theory. The top and bottom edge of the square are identified as are the left and right, so as to form a torus. The torus is triangulated into 7 vertices, 21 edges and 14 triangles (the edges of the boundary square are not part of the triangulation) and the center of each simplex shows the value of the Morse function (the “height” of the simplex). The heights of critical simplices are emphasized. There is one critical simplex of dimension 0 (and height 0; the corners of the square), two critical simplices of dimension 1 (and height 2 respectively 3) and one critical simplex of dimension 2 (and height 7). As before, the subcomplexes that are formed of faces of simplices of heights at most 0, 2, 3 and 7 are contractible, a circle, a wedge sum of two circles, and the full torus, respectively.

The elementary exercise posed earlier means that the non-critical simplices can be paired up (or *matched*) producing pairs (σ, τ) where σ is the unique non-descending facet of τ and τ the unique non-ascending cofacet of σ . This acyclic matching is the analogue in Forman’s theory of the gradient field in classical Morse theory.



Discrete Morse theory has an important role to play in computational topology such as software to compute the homology of a given simplicial complex. Computation of homology consists basically in reducing the matrices of the boundary maps to echelon form (in the case of field coefficients) or to nearly Smith normal form (for integral coefficients). These eliminations therefore need to be highly optimized but may still take long if the complex has many simplices, so that the matrices are very large. As above such a simplicial complex is typically homotopy equivalent to a CW-complex with way fewer cells and computing its homology instead would be much faster. Even without achieving a complex of minimal size, trading cells in adjacent dimensions has the potential of drastically reducing the size of the matrices thus speeding up calculations. This trading of cells can be done via discrete Morse theory.

An important application of computational topology is topological data analysis: a point cloud X (finite subset) in \mathbb{R}^n is considered as a coarse model of some topological space, such as its r -neighborhood for some $r > 0$. In practice, one considers the Vietoris–Rips complex $\text{VR}_r(X)$ instead, whose simplices are spanned by subsets of X of diameter $< r$. A nerve cover argument shows that, up to rescaling r , the Vietoris–Rips complex is not too far from the r -neighborhood in a certain sense. Computing the homology of $\text{VR}_r(X)$ would be a typical use case for discrete Morse theory. We mention in passing that the basic idea in topological data analysis is to study persistence, namely to let r vary and see which homology classes are long-living (likely feature) and which short-living (likely noise).

The book is written for people who care about the practical uses of discrete Morse theory described last but not about the origins sketched before (unlike Knudson’s [3]). In fact, the typical reader might not even be a mathematician but someone “in computer science or engineering who would like to learn about applications of topology in their fields” (from the Preamble). Accordingly, the book is not just an introduction to discrete Morse theory but starts as a self-contained introduction to algebraic topology, restricted to combinatorial and applied topics: all topological spaces are CW-complexes, mostly (finite) simplicial complexes; topological invariance of homology is only briefly sketched; and most homotopies are induced by collapses (thus implicitly the underlying equivalence is that of simple homotopy equivalence). After the general introduction the book introduces discrete Morse theory starting where we ended: an *acyclic matching* of a set of simplices is a matching of facet-cofacet pairs that does not admit a cycle $f_0 < g_0 > f_1 < g_1 \dots > f_0$ where each pair (f_i, g_i) is matched. For instance, a discrete Morse function provides an acyclic matching of all non-critical simplices.

The book consists of four parts. The first part gives a very gentle introduction to homology, starting with discrete sets and graphs before introducing finite ordered simplicial complexes and defining their homology (with coefficients in a ring). Various simplicial constructions (wedge, join, suspension, subdivision) and their effect on homology are also discussed. The part ends with an outlook on topics not properly covered by the book such as infinite complexes, CW-complexes, semisimplicial sets. Throughout the book the reader finds several of these outlooks.

The second part treats homological algebra: chain complexes, chain homotopy, and relative homology are thoroughly introduced. The part also has two chapters

whose aim is rather to sketch ideas and provide an outlook: one on singular and one on cellular homology.

The third part is the core of the book leading to discrete Morse theory. It starts off (in Chap. 9) by thoroughly discussing *simplicial collapse*, namely the removal of a simplex together with a facet that is contained in no other simplex. This is the atomic move not only of the deformation retractions of sublevel sets of discrete Morse functions (for instance, in our example one gets from the 6-sublevel set to the 5-sublevel set by such a move). Rather it generates what generally plays the role of a deformation retraction in the scope of the book. Accordingly an outlook on simple homotopy theory is provided.

The next chapter studies collapsible complexes, namely those that can be reduced to a vertex via a sequence of collapses. In fact, performing a further collapse of that vertex from the empty simplex reduces it to the *void* simplicial complex, which differs from the empty simplicial complex in that it does not even contain the empty simplex. A first central result proven here is that a simplicial complex is collapsible if and only if it admits a complete acyclic matching (an acyclic matching of all simplices). For instance, a discrete Morse function without critical simplices gives rise to a complete acyclic matching and therefore proves the complex collapsible. This is analogous to the fact that a smooth function with a single critical point that is non-degenerate of index 0 proves the underlying Manifold (with boundary) contractible.

The next natural step are *internal collapses* which are treated in Chap. 11. For instance, in our torus example, the two simplices of height 6 cannot be collapsed away in the presence of the simplex of height 7. Rather, the edge of height 6 should be removed and the two triangles of heights 6 and 7 merged. Visibly this leaves the realm of simplicial complexes. The author deals with this challenge by proving the main theorem with input a simplicial complex and output a CW-complex: a simplicial complex with a (partial) acyclic matching is homotopy equivalent to a CW-complex with one cell for every unmatched (critical) simplex. Modulo the replacement of Morse function by acyclic matching this is how we formulated the main theorem of discrete Morse theory above.

Knowing that a simplicial complex is homotopy equivalent to a CW-complex with certain cells restricts its possible homology but does not, in general, determine it uniquely. However, a more careful analysis of how the critical cells fit together after the collapsing allows one to compute homology. This takes the form of a *Morse complex* that is generated by the critical cells and whose homology is that of the original simplicial complex. This procedure is explained in Chaps. 12 and 13. It is the formal basis for the application of speeding up homology computation mentioned earlier (although this is not explained).

Chapter 14 is again an outlook, establishing among other things the equivalence of the approaches via matchings and discrete Morse functions: every partial acyclic matching comes from a Morse function whose critical simplices are precisely the unmatched ones.

The fourth part carries the results of the third to other contexts, specifically to contexts where the main theorem has input and output in the same category. The first is chain complexes, which is the most general and the most important for practical implementations. The second is CW-complexes. Another chapter is concerned with a

structural investigation of acyclic matchings, specifically: given an acyclic matching what ways are there to extend it to a linear order; and how can two acyclic matchings be patched together? The last chapter briefly discusses the interplay with persistence (i.e. filtered) complexes. This is relevant for topological data analysis, but the reader would have to know about that from elsewhere.

Conclusion. The present book is neither (intended as) a general textbook on algebraic topology (like Hatcher's [2]) nor a (near) complete account of combinatorial topology (like the author's previous book [4]). Rather it introduces the reader to the basic concepts and methods of combinatorial topology as quick as possible. The challenge in such a book is what to omit and it is not a small one. The author manages extremely well to reduce the exposition of algebraic topology to the scope that combinatorial methods can deal with. This conveys the reassuring impression that topology can be mastered combinatorially (like graphs, only higher-dimensional) which is not true (as explained in the outlooks) but is good for the purposes of applied topology. First, the depths of topology (such as failure of the Hauptvermutung, Whitehead torsion) may be confusing and intimidating, and second, these phenomena tend to be computationally intractable anyway (e.g. deciding whether a polyhedron is simply connected).

For these reasons I think this is an excellent book for anyone who wants to know how to compute the homology of finite simplicial complexes using combinatorial methods without caring much about topology in general. Such a person is likely not to care about homological algebra either and might decide to skip the second part. Also, someone more seriously interested in optimizing homology computation in software will profit from the book, but would have to read the more advanced parts two and four and make the transfer themselves (this is a maths book, not computer science). In either case the book seems suited both for self-study and to teach from.

I personally would not recommend the book to mathematics students looking for a path into algebraic topology: they deserve to be confused and intimidated by the depths of topology and to learn and appreciate that homology is a topological invariant nonetheless. Also methods of computation (in the traditional sense of long exact sequences and such) and standard examples and applications would take more room for that purpose.

The book under review appears at a time where the importance of applied topology becomes more and more apparent and it provides a straight introduction to the underlying mathematics.

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