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3-spherical twin buildings



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ABSTRACT

We classify thick irreducible 3-spherical twin buildings of rank at least 3 in which every panel contains at least 6 chambers. Together with the Main result of [11] we obtain a classification of thick irreducible 3-spherical twin buildings.

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1. Introduction

In [20] Tits gave a complete classification of all thick irreducible spherical buildings of rank at least 3. The decisive step in this classification is the extension theorem for isometries (Theorem 4.1.2 in [20]). It implies that a thick spherical building is uniquely determined by its local structure. Inspired by the paper [21] on Kac-Moody groups over fields, Ronan and Tits introduced twin buildings. These combinatorial objects appear to be natural generalizations of spherical buildings, as they come equipped with an opposition relation which shares many important properties with the opposition relation in a spherical building. In [15] Mühlherr and Ronan gave a proof of the extension theorem

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for 2-spherical twin buildings satisfying an additional condition they call (co). Condition (co) turns out to be rather mild (see [15, Introduction]). In order to classify such twin buildings it suffices to determine all possible local structures which appear in twin buildings.

The local structure of a building mentioned before is essentially the union of all rank 2 residues of a chamber. In [18] Ronan and Tits introduced the notion of a *foundation* which is designed to axiomatize geometric structures that may occur as local structures of a building. Roughly speaking, it is a union of rank 2 buildings which are glued along certain rank 1 residues (for the precise definition we refer to Section 3). A foundation is called *integrable*, if it is the local structure of a twin building. One can associate with each twin building of 2-spherical type a foundation and all such are *Moufang foundations*, i.e. the irreducible rank 2 buildings are Moufang and the glueings are compatible with the Moufang structures induced on the rank 1 residues.

Inspired by [18], Tits conjectured that a Moufang foundation is integrable if and only if each of its spherical rank 3 restrictions is integrable (cf. [22, Conjecture 2]). It turned out that this conjecture was too optimistic and he gave a reformulation by omitting the word “spherical” in his earlier conjecture (cf. [23]). A proof of this conjecture would reduce the classification of 2-spherical twin buildings to the rank 3 case. In [12] and [13] a strategy for the classification of 2-spherical twin buildings is outlined that does not make use of this conjecture. Several important steps in this classification program have been carried out in the meantime (cf. [11], [25], [26]). The results obtained up until now suggest that the conjecture will be a consequence of the classification once it will be accomplished. However, it would be most desirable to have a proof of this conjecture that is independent of the classification. Our main result is a conceptual proof of this conjecture in the 3-spherical case. It turned out that we only need the integrability for the restriction to irreducible types (cf. Corollary (5.3)):

Theorem A. *Let \mathcal{F} be a Moufang foundation of irreducible 3-spherical type and of rank at least 3 such that every panel contains at least 6 chambers. Then the following are equivalent:*

- (i) \mathcal{F} is integrable.
- (ii) Each irreducible rank 3 restriction of \mathcal{F} is integrable.

A consequence of Theorem A together with [11,4] is the classification of thick irreducible 3-spherical twin buildings (cf. Theorem (5.4)):

Corollary B. *Let Δ be a thick irreducible 3-spherical twin building of rank at least 3. Then Δ is known.*

Let (W, S) be a Coxeter system and let Φ be the associated set of roots (viewed as half-spaces). An *RGD-system of type (W, S)* is a pair $(G, (U_\alpha)_{\alpha \in \Phi})$ consisting of a group

G together with a family of subgroups U_α indexed by Φ satisfying a few axioms (for the precise definition see Section 2). Let \mathcal{F} be a Moufang foundation of type (W, S) satisfying a certain condition (lco) (e.g. this is satisfied if every panel contains at least 5 chambers). In Section 3 we construct for all $s \neq t \in S$ RGD-systems $X_{s,t}$ acting on the corresponding building of the foundation. Moreover, we construct RGD-systems X_s and canonical homomorphisms $X_s \rightarrow X_{s,t}$. Let G be the direct limit of the inductive system formed by the groups X_s and $X_{s,t}$. Then there is a natural way of defining a family of root groups $(U_\alpha)_{\alpha \in \Phi}$ inside G . Let $\mathcal{D}_{\mathcal{F}} := (G, (U_\alpha)_{\alpha \in \Phi})$. We prove the following result (cf. Corollary (3.14)):

Theorem C. *Let \mathcal{F} be a Moufang foundation of 2-spherical type satisfying condition (lco). If the canonical mappings $U_{\pm\alpha_s} \rightarrow G$ are injective and if $\mathcal{D}_{\mathcal{F}}$ satisfies (RGD3), then $\mathcal{D}_{\mathcal{F}}$ is an RGD-system and \mathcal{F} is integrated in the twin building associated to $\mathcal{D}_{\mathcal{F}}$.*

We use Theorem C to prove Theorem A. Our strategy is to let G act on a building and deduce that the hypotheses of Theorem C are satisfied. Thus we have a twin building $\Delta(\mathcal{D}_{\mathcal{F}})$ in which \mathcal{F} is integrated. In particular, we have the following corollary (cf. Theorem (5.2)):

Corollary D. *Let \mathcal{F} be an irreducible Moufang foundation of 3-spherical type such that every panel contains at least 6 chambers. If each irreducible rank 3 restriction of \mathcal{F} is integrable, then $\mathcal{D}_{\mathcal{F}}$ is an RGD-system.*

Remarks. 1. In [11] Mühlherr accomplished the classification of thick locally finite twin buildings of irreducible 2-spherical type and $m_{st} < 8$ without residues associated with one of the groups $B_2(2), G_2(2), G_2(3)$. In particular, the thick locally finite twin buildings of irreducible 3-spherical type without residues associated with $B_2(2)$ are already known. As we will see in the proof of Theorem (5.4), the assumption about the residues can be dropped in the 3-spherical case.

2. By Corollary B we have a classification of 3-spherical simply-laced (i.e. $m_{st} \in \{2, 3\}$) twin buildings. We note that in this case the integrability of those Moufang foundations is already established by different methods: In a 3-spherical simply-laced Moufang foundation, the Moufang triangles are parameterized by skew-fields and all such are isomorphic. If there exists $s \in S$ with three neighbors in the Coxeter diagram, then there is a D_4 subdiagram (as there is no \tilde{A}_2 subdiagram) and hence the Moufang triangles are parameterized by a field. The existence of a twin building with the prescribed foundation follows now from [11]. If each $s \in S$ has at most 2 neighbors, the diagram is either A_n or \tilde{A}_n . In the first case the existence follows from projective geometry and in the second case from Kac-Moody theory.

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2. Preliminaries

2.1. Direct limits

This subsection is based on [19].

Let I be a set and let $(G_i)_{i \in I}$ be a family of groups. Furthermore, let $F_{i,j}$ be a set of homomorphisms of G_i into G_j . Then a group $G := \varinjlim G_i$ together with a family of homomorphisms $f_i : G_i \rightarrow G$ such that $f_j \circ f = f_i$ holds for all $f \in F_{i,j}$ is called *direct limit* of the groups G_i , relative to the $F_{i,j}$ if it satisfies the following condition:

If H is a group and if $h_i : G_i \rightarrow H$ is a family of homomorphisms such that $h_j \circ f = h_i$ holds for all $f \in F_{i,j}$, then there exists exactly one homomorphism $h : G \rightarrow H$ such that $h_i = h \circ f_i$.

In this paper we only consider the case where X_i and $X_{i,j} := \langle X_i, X_j \rangle$ are groups for $i \neq j \in I$ and $f : X_i \rightarrow X_{i,j}$ are the canonical homomorphisms. We call this direct limit the *2-amalgam of the groups X_i* .

2.2. Coxeter systems

Let (W, S) be a Coxeter system and let ℓ denote the corresponding length function. For $s, t \in S$ we denote the order of st in W by m_{st} . The *rank* of a Coxeter system is the cardinality of the set S . The *Coxeter diagram* corresponding to (W, S) is the labeled graph $(S, E(S))$, where $E(S) = \{\{s, t\} \mid m_{st} > 2\}$ and where each edge $\{s, t\}$ is labeled by m_{st} for all $s, t \in S$. We call a Coxeter system *irreducible*, if the underlying graph is connected; otherwise we call it *reducible*. It is well-known that the pair $(\langle J \rangle, J)$ is a Coxeter system (cf. [5, Ch. IV, §1 Theorem 2]). A subset $J \subseteq S$ is called *spherical* if $\langle J \rangle$ is finite; for $k \in \mathbb{N}$ the Coxeter system is called *k-spherical*, if $\langle J \rangle$ is spherical for each subset J of S with $|J| \leq k$. Given a spherical subset J of S , there exists a unique element of maximal length in $\langle J \rangle$, which we denote by r_J (cf. [2, Corollary 2.19]). For $s_1, \dots, s_k \in S$ we call $s_1 \cdots s_k$ *reduced*, if $\ell(s_1 \cdots s_k) = k$. Similar as in [16, Ch. 2.2] we define for distinct $s, t \in S$ with $m_{st} < \infty$ the element $p(s, t)$ to mean $stst \dots$ with $\ell(p(s, t)) = m_{st}$; e.g. if $m_{st} = 3$, we have $p(s, t) = sts$. It is well-known that for $w \in W, s \in S$ with $\ell(sw) < \ell(w)$, there exists $w' \in W$ such that $\ell(w') = \ell(w) - 1$ and $w = sw'$ (cf. [2, exchange condition on p.79]).

(2.1) Convention. For the rest of this paper we let (W, S) be a Coxeter system.

2.3. Chamber systems

Let I be a set. A *chamber system over I* is a pair $(\mathcal{C}, (\sim_i)_{i \in I})$ consisting of a non-empty set \mathcal{C} whose elements are called *chambers* and where \sim_i is an equivalence relation on the set of chambers for each $i \in I$. Given $i \in I$ and $c, d \in \mathcal{C}$, then c is called *i -adjacent* to d if $c \sim_i d$. The chambers c, d are called *adjacent*, if they are i -adjacent for some $i \in I$. If we restrict \sim_i to $\emptyset \neq \mathcal{X} \subseteq \mathcal{C}$ for each $i \in I$, then $(\mathcal{X}, (\sim_i)_{i \in I})$ is a chamber system over I and we call it the *induced* chamber system.

A *gallery* in $(\mathcal{C}, (\sim_i)_{i \in I})$ is a sequence (c_0, \dots, c_k) such that $c_\mu \in \mathcal{C}$ for all $0 \leq \mu \leq k$ and $c_{\mu-1}, c_\mu$ are adjacent for all $1 \leq \mu \leq k$. Given a gallery $G = (c_0, \dots, c_k)$, then we put $\beta(G) := c_0$ and $\varepsilon(G) := c_k$. If G is a gallery and if $c, d \in \mathcal{C}$ are such that $\beta(G) = c$ and $\varepsilon(G) = d$ then we say that G is a *gallery from c to d* or G *joins c and d* . The chamber system $(\mathcal{C}, (\sim_i)_{i \in I})$ is said to be *connected*, if for any two chambers there exists a gallery joining them. A gallery G will be called *closed* if $\beta(G) = \varepsilon(G)$. Given two galleries $G = (c_0, \dots, c_k)$ and $H = (d_0, \dots, d_l)$ such that $\varepsilon(G) = \beta(H)$, then GH denotes the gallery $(c_0, \dots, c_k = d_0, \dots, d_l)$.

Let J be a subset of I . A *J -gallery* is a gallery (c_0, \dots, c_k) such that for each $1 \leq \mu \leq k$ there exists an index $j \in J$ with $c_{\mu-1} \sim_j c_\mu$.

2.4. Homotopy of galleries and simple connectedness

In the context of chamber systems there is the notion of m -homotopy and m -simple connectedness for each $m \in \mathbb{N}$. In this paper we are only concerned with the case $m = 2$. Therefore our definitions are always to be understood as a specialization of the general theory to the case $m = 2$.

Let $(\mathcal{C}, (\sim_i)_{i \in I})$ be a chamber system over a set I . Two galleries G and H are said to be *elementary homotopic* if there exist two galleries X, Y and two J -galleries G', H' for some $J \subseteq I$ of cardinality at most 2 such that $G = XG'Y, H = XH'Y$. Two galleries G, H are said to be *homotopic* if there exists a finite sequence G_0, \dots, G_l of galleries such that $G_0 = G, G_l = H$ and such that $G_{\mu-1}$ is elementary homotopic to G_μ for all $1 \leq \mu \leq l$.

If two galleries G, H are homotopic, then it follows by definition that $\beta(G) = \beta(H)$ and $\varepsilon(G) = \varepsilon(H)$. A closed gallery G is said to be *null-homotopic* if it is homotopic to the gallery $(\beta(G))$. The chamber system $(\mathcal{C}, (\sim_i)_{i \in I})$ is called *simply connected* if it is connected and if each closed gallery is null-homotopic.

2.5. Buildings

A *building of type (W, S)* is a pair $\Delta = (\mathcal{C}, \delta)$ where \mathcal{C} is a non-empty set and where $\delta : \mathcal{C} \times \mathcal{C} \rightarrow W$ is a *distance function* satisfying the following axioms, where $x, y \in \mathcal{C}$ and $w = \delta(x, y)$:

- (WD1) $w = 1_W$ if and only if $x = y$.
- (WD2) if $z \in \mathcal{C}$ satisfies $s := \delta(y, z) \in S$, then $\delta(x, z) \in \{w, ws\}$, and if, furthermore, $\ell(ws) = \ell(w) + 1$, then $\delta(x, z) = ws$.
- (WD3) if $s \in S$, there exists $z \in \mathcal{C}$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

The *rank* of Δ is the rank of the underlying Coxeter system. The elements of \mathcal{C} are called *chambers*. Given $s \in S$ and $x, y \in \mathcal{C}$, then x is called *s-adjacent* to y , if $\delta(x, y) = s$. The chambers x, y are called *adjacent*, if they are *s-adjacent* for some $s \in S$. A *gallery* from x to y is a sequence $(x = x_0, \dots, x_k = y)$ such that x_{l-1} and x_l are adjacent for every $1 \leq l \leq k$; the number k is called the *length* of the gallery. Let (x_0, \dots, x_k) be a gallery and suppose $s_i \in S$ such that $\delta(x_{i-1}, x_i) = s_i$. Then (s_1, \dots, s_k) is called the *type* of the gallery. A gallery from x to y of length k is called *minimal* if there is no gallery from x to y of length $< k$.

Given a subset $J \subseteq S$ and $x \in \mathcal{C}$, the *J-residue* of x is the set $R_J(x) := \{y \in \mathcal{C} \mid \delta(x, y) \in \langle J \rangle\}$. A *residue* is a subset R of \mathcal{C} such that there exist $J \subseteq S$ and $x \in \mathcal{C}$ with $R = R_J(x)$. Since the subset J is uniquely determined by R , the set J is called the *type* of R and the *rank* of R is defined to be the cardinality of J . Each *J-residue* is a building of type $(\langle J \rangle, J)$ with the distance function induced by δ (cf. [2, Corollary 5.30]). Given $x \in \mathcal{C}$ and a *J-residue* $R \subseteq \mathcal{C}$, then there exists a unique chamber $z \in R$ such that $\ell(\delta(x, y)) = \ell(\delta(x, z)) + \ell(\delta(z, y))$ holds for every $y \in R$ (cf. [2, Proposition 5.34]). The chamber z is called the *projection of x onto R* and is denoted by $\text{proj}_R x$. Moreover, if $z = \text{proj}_R x$ we have $\delta(x, y) = \delta(x, z)\delta(z, y)$ for each $y \in R$. A residue is called *spherical* if its type is a spherical subset of S . A building is *spherical* if its type is spherical; for $k \in \mathbb{N}$ it is called *k-spherical* if (W, S) is *k-spherical*. Let R be a spherical *J-residue*. Then $x, y \in R$ are called *opposite in R* if $\delta(x, y) = r_J$. A *panel* is a residue of rank 1. An *s-panel* is a panel of type $\{s\}$ for $s \in S$. For $c \in \mathcal{C}$ and $s \in S$ we denote the *s-panel* containing c by $\mathcal{P}_s(c)$. The building Δ is called *thick*, if each panel of Δ contains at least three chambers. For an *s-panel* P we will also write P_s instead of P to underline the type of P . We denote the set of all panels in a given building Δ by \mathcal{P}_Δ . For $c \in \mathcal{C}$ and $k \in \mathbb{N}$ we denote the union of all residues of rank at most k containing c by $E_k(c)$. Let $\Delta = (\mathcal{C}, \delta), \Delta' = (\mathcal{C}', \delta')$ be two buildings of type (W, S) and let $X \subseteq \mathcal{C}, X' \subseteq \mathcal{C}'$. Then a map $\varphi : X \rightarrow X'$ is called *isomorphism* if it is bijective and preserves the distance functions. In this case we will write $X \cong X'$. We denote the set of all isomorphisms from a building Δ to itself by $\text{Aut}(\Delta)$.

(2.2) Remark. Our definition of a building agrees with the definition given in [16, Ch. 3] (cf. [2, Proposition 5.23]).

2.6. Coxeter buildings

(2.3) Example. We define $\delta : W \times W \rightarrow W, (x, y) \mapsto x^{-1}y$. Then $\Sigma(W, S) := (W, \delta)$ is a building of type (W, S) . Moreover, W acts on $\Sigma(W, S)$ by left multiplication.

A *reflection* is an element of W that is conjugate to an element of S . For $s \in S$ we let $\alpha_s := \{w \in W \mid \ell(sw) > \ell(w)\}$ be the *simple root* corresponding to s . A *root* is a subset $\alpha \subseteq W$ such that $\alpha = v\alpha_s$ for some $v \in W$ and $s \in S$. We denote the set of all roots by $\Phi(W, S)$. The set $\Phi(W, S)_+ = \{\alpha \in \Phi(W, S) \mid 1_W \in \alpha\}$ is the set of all *positive roots* and $\Phi(W, S)_- = \{\alpha \in \Phi(W, S) \mid 1_W \notin \alpha\}$ is the set of all *negative roots*. For $J \subseteq S$ we put $\Phi(W, S)^J := \Phi(\langle J \rangle, J)$ (resp. $\Phi(W, S)^J_+, \Phi(W, S)^J_-$). For each root $\alpha \in \Phi(W, S)$ we denote the *opposite* root by $-\alpha$ and we let r_α be the unique reflection which interchanges these two roots. A pair of distinct roots $\{\alpha, \beta\} \subseteq \Phi(W, S)$ is called *prenilpotent* if both $\alpha \cap \beta$ and $(-\alpha) \cap (-\beta)$ are non-empty sets. Given such a pair $\{\alpha, \beta\}$ we will write $[\alpha, \beta] := \{\gamma \in \Phi(W, S) \mid \alpha \cap \beta \subseteq \gamma \text{ and } (-\alpha) \cap (-\beta) \subseteq -\gamma\}$ and $(\alpha, \beta) := [\alpha, \beta] \setminus \{\alpha, \beta\}$. For a pair $\{\alpha, \beta\} \subseteq \Phi$ of prenilpotent roots there are two possibilities: either $o(r_\alpha r_\beta) < \infty$ or $o(r_\alpha r_\beta) = \infty$. The second case implies $\alpha \not\subseteq \beta$ or $\beta \not\subseteq \alpha$. In the case $\alpha \not\subseteq \beta$, the open interval (α, β) consists of all roots γ such that $\alpha \not\subseteq \gamma \not\subseteq \beta$.

(2.4) Convention. For the rest of this paper we let (W, S) be a Coxeter system of finite rank and $\Phi := \Phi(W, S)$ (resp. Φ_+, Φ_-) be the set of roots (resp. positive roots, negative roots).

For $\alpha \in \Phi$ we denote by $\partial\alpha$ (resp. $\partial^2\alpha$) the set of all panels (resp. spherical residues of rank 2) stabilized by r_α . The set $\partial\alpha$ is called the *wall* associated to α . For a gallery $G = (c_0, \dots, c_k)$ we say that G *crosses the wall* $\partial\alpha$ if $\{c_{i-1}, c_i\} \in \partial\alpha$ for some $1 \leq i \leq k$.

(2.5) Lemma. *Let $\alpha \in \Phi$ and let $P, Q \in \partial\alpha$. Then there exist a sequence $P_0 = P, \dots, P_n = Q$ of panels in $\partial\alpha$ and a sequence R_1, \dots, R_n of spherical rank 2 residues in $\partial^2\alpha$ such that P_{i-1}, P_i are distinct and contained in R_i . Moreover, $\{\text{proj}_{R_i} x \mid x \in P_0\} = P_{i-1}$ for all $1 \leq i \leq n$.*

Proof. This is a consequence of [7, Proposition 2.7]. The fact that $P_0, \dots, P_n \in \partial\alpha$ follows from the implication (iii) \Rightarrow (ii) in [7] Since $P_i \subseteq R_i$, we infer $R_i \in \partial^2\alpha$. \square

Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S) . A subset $\Sigma \subseteq \mathcal{C}$ is called *apartment* if it is isomorphic to W . A subset $\alpha \subseteq \mathcal{C}$ is called a *root* if it is isomorphic to α_s for some $s \in S$.

2.7. Two conditions for buildings

(2.6) Example. Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S) . We define $x \sim_s y$ if $\delta(x, y) \in \langle s \rangle$. Then \sim_s is an equivalence relation and $(\mathcal{C}, (\sim_s)_{s \in S})$ is a chamber system.

We now introduce two conditions for a building:

- (lco) A building Δ satisfies condition (lco) if it is 2-spherical and if R is a rank 2 residue of Δ containing a chamber c , then the induced chamber system defined by the set of chambers opposite c inside R is connected.
- (lsco) A building Δ satisfies condition (lsco) if it is 3-spherical and if R is a rank 3 residue of Δ containing a chamber c , then the induced chamber system defined by the set of chambers opposite c inside R is simply connected.

Buildings which satisfy both conditions are discussed in [9, Chapter 9] and [10, Introduction]. In [9] it is stated that a 3-spherical building of simply-laced type (i.e. $m_{st} \in \{2, 3\}$ for all $s, t \in S$) in which every panel contains at least 4 chambers satisfies both conditions (lco) and (lsco). Moreover, (following [9] and [10]) a (general) 3-spherical building satisfies the conditions (lco) and (lsco) if every panel contains at least 6 chambers.

2.8. Spherical Moufang buildings

Let $\Delta = (\mathcal{C}, \delta)$ be a thick irreducible spherical building of type (W, S) and of rank at least 2. For a root α of Δ we define the *root group* U_α as the set of all automorphisms fixing α pointwise and fixing every panel P pointwise, where $|P \cap \alpha| = 2$. The building Δ is called *Moufang* if for every root α of Δ the root group U_α acts simply transitive on the set of apartments containing α .

Let $\Delta = (\mathcal{C}, \delta)$ be a Moufang building of type (W, S) , let Σ be an apartment of Δ and let $c \in \Sigma$. Identify the set of roots in Σ with Φ and let $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$. Let $H = \text{Fix}_G(\Sigma)$ and $B_+ = \text{Stab}_G(c) = H \langle U_\alpha \mid c \in \alpha \in \Phi \rangle$. Then $\Delta(G, B_+) = (G/B_+, \delta)$ is a building of type (W, S) , where $\delta : G/B_+ \times G/B_+ \rightarrow W, (gB_+, hB_+) \mapsto w$, where $g^{-1}h \in B_+wB_+$ (for more information we refer to [2, Section 6, 7]). Moreover, the buildings Δ and $\Delta(G, B_+)$ are isomorphic.

2.9. Twin buildings

Let $\Delta_+ = (\mathcal{C}_+, \delta_+), \Delta_- = (\mathcal{C}_-, \delta_-)$ be two buildings of the same type (W, S) . A *codistance* (or *twinning*) between Δ_+ and Δ_- is a mapping $\delta_* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \rightarrow W$ satisfying the following axioms, where $\varepsilon \in \{+, -\}, x \in \mathcal{C}_\varepsilon, y \in \mathcal{C}_{-\varepsilon}$ and $w = \delta_*(x, y)$:

- (Tw1) $\delta_*(y, x) = w^{-1}$;
- (Tw2) if $z \in \mathcal{C}_{-\varepsilon}$ is such that $s := \delta_{-\varepsilon}(y, z) \in S$ and $\ell(ws) = \ell(w) - 1$, then $\delta_*(x, z) = ws$;
- (Tw3) if $s \in S$, there exists $z \in \mathcal{C}_{-\varepsilon}$ such that $\delta_{-\varepsilon}(y, z) = s$ and $\delta_*(x, z) = ws$.

A *twin building of type* (W, S) is a triple $\Delta = (\Delta_+, \Delta_-, \delta_*)$, where $\Delta_+ = (\mathcal{C}_+, \delta_+), \Delta_- = (\mathcal{C}_-, \delta_-)$ are buildings of type (W, S) and where δ_* is a twinning between Δ_+ and Δ_- . The twin building is called *thick*, if Δ_+ and Δ_- are thick. A *panel* of a twin building is a panel of one of its two buildings, i.e. for $\varepsilon \in \{+, -\}, c \in \mathcal{C}_\varepsilon$ and $s \in S$ we have

$\mathcal{P}_s(c) = \{d \in \mathcal{C}_\varepsilon \mid \delta_\varepsilon(c, d) \in \langle s \rangle\}$. The twin building Δ satisfies condition (lco) if both buildings Δ_+, Δ_- satisfy condition (lco). For $\varepsilon \in \{+, -\}$ and $c \in \mathcal{C}_\varepsilon$ we define $c^{\text{op}} = \{d \in \mathcal{C}_{-\varepsilon} \mid \delta_*(c, d) = 1_W\}$. Let $\Sigma_+ \subseteq \mathcal{C}_+$ and $\Sigma_- \subseteq \mathcal{C}_-$ be apartments of Δ_+ and Δ_- , respectively. Then the set $\Sigma := \Sigma_+ \cup \Sigma_-$ is called a *twin apartment* if for all $\varepsilon \in \{+, -\}$ and $x \in \Sigma_\varepsilon$ there exists exactly one $y \in \Sigma_{-\varepsilon}$ with $\delta_*(x, y) = 1_W$. Moreover, for any $c_+ \in \mathcal{C}_+, c_- \in \mathcal{C}_-$ with $\delta_*(c_+, c_-) = 1_W$ there exists a unique twin apartment Σ containing c_+ and c_- (cf. [2, Proposition 5.179(1)]). Let $\Delta' = (\Delta'_+, \Delta'_-, \delta'_*)$ be another twin building of type (W, S) . A mapping $\varphi : \mathcal{C}_+ \cup \mathcal{C}_- \rightarrow \mathcal{C}'_+ \cup \mathcal{C}'_-$ is called *isomorphism*, if it preserves the sign, the distance and the codistance.

(2.7) Lemma. *Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a twin building of type (W, S) and let c be a chamber of Δ such that $|\mathcal{P}_s(c)| \geq 3$ holds for all $s \in S$. Then Δ is thick.*

Proof. Let $\varepsilon \in \{+, -\}$ and let $c \in \mathcal{C}_\varepsilon$. For each $d \in c^{\text{op}}$ and each $s \in S$ the s -panel containing d contains at least three elements by [2, Corollary 5.153]. Let $c' \in \mathcal{C}_\varepsilon$ and $d \in c^{\text{op}}$. If $c' \in d^{\text{op}}$ the s -panel containing c' contains at least three chambers by [2, Corollary 5.153]. Otherwise, there exists a chamber $d' \in E_1(d)$ such that $d' \in c^{\text{op}}$ and $\ell(\delta(c', d')) = \ell(\delta(c', d)) - 1$. Using induction we obtain a chamber $d'' \in \mathcal{C}_{-\varepsilon}$ such that $d'' \in c^{\text{op}} \cap (c')^{\text{op}}$. Applying [2, Corollary 5.153] twice, we obtain that the s -panel containing c' contains at least three chambers for each $s \in S$. Thus Δ_ε is a thick building. The thickness of $\Delta_{-\varepsilon}$ follows similarly. \square

(2.8) Lemma. *Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a thick 2-spherical twin building of type (W, S) , let $\varepsilon \in \{+, -\}, c \in \mathcal{C}_\varepsilon$ and for each $J \subseteq S$ with $|J| = 2$ we let Σ_J be an apartment of $R_J(c)$ containing c and such that $\Sigma_{\{r,s\}} \cap \mathcal{P}_s(c) = \Sigma_{\{s,t\}} \cap \mathcal{P}_s(c)$ for all $r \neq s \neq t \in S$. Then there exists a twin apartment Σ such that $\Sigma_J \subseteq \Sigma$.*

Proof. This follows from [26, Theorem 6.3.6]. \square

2.10. RGD-systems

An *RGD-system of type (W, S)* is a pair $(G, (U_\alpha)_{\alpha \in \Phi})$ consisting of a group G together with a family of subgroups U_α (called *root groups*) indexed by the set of roots Φ , which satisfies the following axioms, where $H := \bigcap_{\alpha \in \Phi} N_G(U_\alpha)$ and $U_\varepsilon := \langle U_\alpha \mid \alpha \in \Phi_\varepsilon \rangle$ for $\varepsilon \in \{+, -\}$:

- (RGD0) For each $\alpha \in \Phi$, we have $U_\alpha \neq \{1\}$.
- (RGD1) For each prenilpotent pair $\{\alpha, \beta\} \subseteq \Phi$, the commutator group $[U_\alpha, U_\beta]$ is contained in the group $U_{(\alpha, \beta)} := \langle U_\gamma \mid \gamma \in (\alpha, \beta) \rangle$.
- (RGD2) For each $s \in S$ and each $u \in U_{\alpha_s} \setminus \{1\}$, there exist $u', u'' \in U_{-\alpha_s}$ such that the product $m(u) := u'uu''$ conjugates U_β onto $U_{s\beta}$ for each $\beta \in \Phi$.
- (RGD3) For each $s \in S$, the group $U_{-\alpha_s}$ is not contained in U_+ .

(RGD4) $G = H\langle U_\alpha \mid \alpha \in \Phi \rangle$.

Let $\mathcal{D} = (G, (U_\alpha)_{\alpha \in \Phi})$ be an RGD-system of type (W, S) and let $H := \bigcap_{\alpha \in \Phi} N_G(U_\alpha)$ and $B_\varepsilon := H\langle U_\alpha \mid \alpha \in \Phi_\varepsilon \rangle$ for $\varepsilon \in \{+, -\}$. It follows from [2, Theorem 8.80] that there exists an *associated* twin building $\Delta(\mathcal{D}) = (\Delta_+, \Delta_-, \delta_*)$ of type (W, S) such that $\Delta_+ = (G/B_+, \delta_+)$ and $\Delta_- = (G/B_-, \delta_-)$ on which G acts by multiplication from the left.

We define $X_s := \langle U_{\alpha_s} \cup U_{-\alpha_s} \rangle$ and $X_{s,t} := \langle X_s \cup X_t \rangle$ for all $s \neq t \in S$. If (W, S) is 2-spherical, $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$ and $\Delta(\mathcal{D})$ satisfies condition (lco), it follows from the Main result of [3] that G is isomorphic to the direct limit of the inductive system formed by the groups $L_s := HX_s$ and $L_{s,t} := HX_{s,t}$ for all $s \neq t \in S$. Furthermore, the direct limit of the inductive system formed by the groups X_s and $X_{s,t}$ ($s \neq t \in S$) can be naturally endowed with an RGD-system and is a central extension of G (cf. [6, Theorem 3.7]).

For $s \in S$ we define $H_s := \langle m(u)m(v) \mid u, v \in U_{\alpha_s} \setminus \{1\} \rangle$. By [6, Lemma 3.3] we have $H = \prod_{s \in S} H_s$, if $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$ and if (W, S) is 2-spherical. By [2, Consequence (6) on p.415] we have $H_t^{m(u)} \subseteq H_s H_t$ for all $s, t \in S$ and $1 \neq u \in U_{\alpha_s}$. For the rest of this subsection we fix $1 \neq e_s \in U_{\alpha_s}$ and put $n_s := m(e_s)$ for all $s \in S$. By [2, Consequence (11) on p.416] there exist for every $1 \neq u_s \in U_{\alpha_s}$ elements $\bar{u}_s \in U_{\alpha_s}$ and $b(u_s) \in \langle U_{\alpha_s} \cup H_s \rangle$ such that $n_s u_s n_s = \bar{u}_s n_s b(u_s)$ (note that $\langle U_{\alpha_s} \cup H_s \rangle \cap U_{-\alpha_s} = \{1\}$). As in the case of a Coxeter system, we define $p(n_s, n_t)$ to mean $n_s n_t n_s n_t \dots$, where n_s, n_t appear m_{st} times, e.g. if $m_{st} = 3$, we have $p(n_s, n_t) = n_s n_t n_s$. By [16, Lemma 7.3] we have $p(n_s, n_t) = p(n_t, n_s)$.

(2.9) Theorem. *Let $(G, (U_\alpha)_{\alpha \in \Phi})$ be an RGD-system of spherical type (W, S) such that $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$. Let $B := \langle U_+, H \rangle$. Then G has the following presentation: as generators we have $\bigcup_{s \in S} H_s \cup \bigcup_{\alpha \in \Phi_+} U_\alpha$ and $\{n_s \mid s \in S\}$ and as relations we have all relations in B and for $s, t \in S, \alpha_s \neq \alpha \in \Phi_+, h \in H_t, u_s \in U_s$ we have the relations*

$$n_s h n_s = n_s^2 h^{n_s}, \quad n_s u_\alpha n_s = n_s^2 u_\alpha^{n_s}, \quad n_s u_s n_s = \bar{u}_s n_s b(u_s), \quad p(n_s, n_t) = p(n_t, n_s),$$

where $u_\alpha^{n_s} \in U_{s\alpha}, n_s^2 \in H_s, h^{n_s} \in H_s H_t$.

Proof. All these relations are relations in G . We observe that G has a *BN-pair* (cf. [2, Theorem 7.115]). Thus it suffices to show that all relations in [20, Corollary (13.4)] can be deduced from the relations in the statement. \square

(2.10) Example. Let Δ be an irreducible spherical Moufang building of rank at least 2 and let Σ be an apartment of Δ . Identifying the set of roots Φ with the set of roots in Σ , we deduce that $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$ is an RGD-system.

(2.11) Lemma. *Let $(G, (U_\alpha)_{\alpha \in \Phi})$ be an RGD-system of 2-spherical type (W, S) such that $\Delta(\mathcal{D})$ satisfies condition (lco) (e.g. each root group contains at least four elements). Then*

we have $\langle U_\gamma \mid \gamma \in [\alpha_s, \alpha_t] \rangle = \langle U_{\alpha_s} \cup U_{\alpha_t} \rangle$ or equivalently $[U_{\alpha_s}, U_{\alpha_t}] = U_{(\alpha_s, \alpha_t)}$ for all $s \neq t \in S$.

Proof. This is a consequence of [1, Lemma 18 and Proposition 7]. \square

(2.12) Remark. Let $(G, (U_\alpha)_{\alpha \in \Phi}), (H, (V_\alpha)_{\alpha \in \Phi})$ be two RGD-systems of the same irreducible spherical type (W, S) and of rank at least 2. Assume that $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$, the root groups U_α are nilpotent and that both twin buildings associated with the RGD-systems satisfy condition (lco). Then G, H are perfect by the previous lemma. Assume that there exists a homomorphism $\varphi : G \rightarrow H$ such that $\varphi(U_\alpha) = V_\alpha$. As $\ker \varphi \neq G$, we have $\ker \varphi \leq Z(G)$ by [2, Proposition 7.132].

(2.13) Lemma. Let $\mathcal{D} = (G, (U_\alpha)_{\alpha \in \Phi})$ be an RGD-system of 3-spherical type (W, S) such that $\Delta(\mathcal{D})_\pm$ satisfy the conditions (lco) and (lsco) (e.g. every root group contains at least five elements). Then B_+ is the 2-amalgam of the groups HU_{α_s} .

Proof. This is a consequence of [8, Corollary 1.2]. \square

(2.14) Lemma. Let $(G, (U_\alpha)_{\alpha \in \Phi})$ be an RGD-system of type (W, S) and let $X_{s,t} = \langle U_\alpha \mid \alpha \in \Phi^{\{s,t\}} \rangle, B_{\{s,t\}} = \langle H_s \cup H_t \cup U_\alpha \mid \alpha \in \Phi_+^{\{s,t\}} \rangle$. Then $(X_{s,t}, (U_\alpha)_{\alpha \in \Phi^{\{s,t\}}})$ is an RGD-system and $X_{s,t}/B_{s,t} \rightarrow R_{\{s,t\}}(B_+), gB_{s,t} \mapsto gB_+$ is an isomorphism of buildings.

Proof. As $X_{s,t}$ stabilizes $R_{\{s,t\}}(B_+)$, we deduce $\{gB_+ \mid g \in X_{s,t}\} \subseteq R_{\{s,t\}}(B_+)$. Let $g \in G$ be such that $w := \delta_+(B_+, gB_+) \in \langle s, t \rangle$. By [2, Proposition 8.59(2)] there exists $h \in U_w$ such that $hwB_+ = gB_+$ and hence $gB_+ \in \{hB_+ \mid h \in X_{s,t}\}$. Thus $R_{\{s,t\}}(B_+) = \{gB_+ \mid g \in X_{s,t}\}$. Since $B_{\{s,t\}} \subseteq B_+$, the map is well-defined. Suppose $g, h \in X_{s,t}$ such that $gB_+ = hB_+$. Then $g^{-1}h \in B_+ \cap X_{s,t} = B_{\{s,t\}}$ as $B_+ = \text{Stab}_G(B_+)$ and $B_{\{s,t\}} = \text{Stab}_{X_{s,t}}(B_+)$ and thus the mapping is injective. Since $R_{\{s,t\}}(B_+) = \{gB_+ \mid g \in X_{s,t}\}$ the mapping is also surjective. Furthermore, it preserves adjacency. The claim follows now from [2, Lemma 5.61]. \square

2.11. Blueprints

This subsection is based on [16, Ch. 7].

A parameter system will mean a family of disjoint parameter sets $(U'_s)_{s \in S}$, each having a distinguished element $\infty_s \in U'_s$. We shall write $U_s := U'_s \setminus \{\infty_s\}$.

Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S) and let $(U'_s)_{s \in S}$ be a parameter system. A labeling of Δ of type $(U'_s)_{s \in S}$, based at $c \in \mathcal{C}$, is a family $(\varphi_{P_s} : U'_s \rightarrow P_s)_{P_s \in \mathcal{P}_\Delta}$ of bijections such that $\varphi_{P_s}(\infty_s) = \text{proj}_{P_s} c$. We call $\varphi_{P_s}^{-1}(x) \in U'_s$ the s -label of x .

(2.15) Example. Let $\Delta = (\mathcal{C}, \delta)$ be a building of type $I_2(m)$, $m \geq 2$, with a labeling of type (U'_s, U'_t) , where $S = \{s, t\}$, based at some chamber $c \in \mathcal{C}$. Given any chamber $x \in \mathcal{C}$ at distance d from c one has a minimal gallery $(c = c_0, \dots, c_d = x)$. Let u_i be

the label attached to c_i in the unique panel containing c_{i-1} and c_i . The gallery thus determines the sequence (u_1, \dots, u_d) where the u_i lie alternately in U_s and U_t , and any such sequence obviously determines a unique gallery starting at c , and hence a unique chamber at the end of this gallery. If two sequences determine the same chamber they are called *equivalent*.

A *blueprint of type* (W, S) is a tuple $\left((U'_s)_{s \in S}, (\Delta_{st})_{s \neq t \in S} \right)$ consisting of a parameter system $(U'_s)_{s \in S}$ and buildings $\Delta_{st} = (\mathcal{C}_{st}, \delta_{st})$ of type $I_2(m_{st})$ for each $s \neq t \in S$, with a given labeling of type (U'_s, U'_t) , based at some chamber $\infty_{st} \in \mathcal{C}_{st}$.

A building Δ of type (W, S) will be said to *conform* to the blueprint $\left((U'_s)_{s \in S}, (\Delta_{st})_{s \neq t \in S} \right)$ if there exists a labeling of Δ of type $(U'_s)_{s \in S}$, based at some chamber $c \in \mathcal{C}$, such that for each $\{s, t\}$ -residue R of Δ there is an isomorphism $\varphi_R : \Delta_{st} \rightarrow R$ with the property that x and $\varphi_R(x)$ have the same s - and t -labels for each chamber x of Δ_{st} . We call a blueprint \mathcal{B} *realizable* if there exists a building which conforms to it.

2.12. A realizable criterion

Let $\mathcal{B} = \left((U'_s)_{s \in S}, (\Delta_{st})_{s \neq t \in S} \right)$ be a blueprint of type (W, S) . We want to construct a building which conforms to \mathcal{B} . As a first step we construct a chamber system $\mathbf{C}(\mathcal{B})$ as follows: The chambers are sequences $\bar{u} := (u_1, \dots, u_k)$, where $u_i \in U_{s_i}$ and $s_1 \cdots s_k$ is reduced. We call (s_1, \dots, s_k) the *type* of \bar{u} . We define s -adjacency via

$$(u_1, \dots, u_k) \sim_s (u_1, \dots, u_k, u_{k+1}) \sim_s (u_1, \dots, u_k, u'_{k+1}),$$

where $u_{k+1}, u'_{k+1} \in U_s$; this is evidently an equivalence relation, so $\mathbf{C}(\mathcal{B})$ is a chamber system. We remark that in the definition of s -adjacency, we have $\ell(s_1 \cdots s_k s) = k + 1$ (otherwise it is not a chamber). If $\ell(s_1 \cdots s_k s) = k - 1$ and $s \neq s_k$, then (u_1, \dots, u_k) is only s -adjacent to itself.

For $\bar{u} = (u_1, \dots, u_k)$ having type (s_1, \dots, s_k) and $\bar{v} = (v_1, \dots, v_n)$ having type (t_1, \dots, t_n) we define the sequence $\bar{u}\bar{v} := (u_1, \dots, u_k, v_1, \dots, v_n)$ if $s_1 \cdots s_k t_1 \cdots t_n$ is reduced. We now define an *elementary equivalence* to be an alteration from a sequence $\bar{u}_1 \bar{u}' \bar{u}_2$ of type $(f_1, p(s, t), f_2)$ to $\bar{u}_1 \bar{u}' \bar{u}_2$ of type $(f_1, p(t, s), f_2)$ where \bar{u} and \bar{u}' are equivalent in Δ_{st} . Two sequences \bar{u} and \bar{v} are called *equivalent*, written $\bar{u} \simeq \bar{v}$, if one can be transformed to the other by a finite sequence of elementary equivalences. We now consider $\mathbf{C}(\mathcal{B})$ modulo the equivalence relation. Notice that $[\bar{u}]$ determines a unique element $w \in W$ where (s_1, \dots, s_k) is the type of \bar{u} and $w = s_1 \cdots s_k$. We define $x \sim_s y$ if $x = [\bar{u}], y = [\bar{v}]$ with $\bar{u} \sim_s \bar{v}$.

(2.16) Theorem. *Assume that for any two sequences \bar{u}, \bar{v} of the same reduced type, $\bar{u} \simeq \bar{v}$ implies $\bar{u} = \bar{v}$. Then $\mathbf{C}_{\mathcal{B}} := ([\bar{u}], (\sim_s)_{s \in S})$ is a chamber system. The chambers are equivalence classes of sequences $\bar{u} := (u_1, \dots, u_k)$, denoted by $[\bar{u}]$ or $[u_1, \dots, u_k]$. Moreover, $\mathbf{C}_{\mathcal{B}}$ is a building which conforms to \mathcal{B} .*

Proof. This follows from the proof of [16, Theorem (7.1)]. \square

(2.17) Corollary. *A blueprint is realizable if and only if for any two sequences \bar{u}, \bar{v} of the same reduced type, $\bar{u} \simeq \bar{v}$ implies $\bar{u} = \bar{v}$. In particular, a blueprint is realizable if its restriction to each spherical rank 3 subdiagram is realizable.*

Proof. The first assertion is [16, Theorem (7.2)]; the second assertion follows from the first and [16, Step 1 of Theorem (7.1)]. \square

2.13. Natural labeling of Moufang buildings

This subsection is based on [16].

Let $\Delta = (\mathcal{C}, \delta)$ be a spherical Moufang building of type (W, S) and let $c \in \mathcal{C}$. For each $s \in S$ we fix $1 \neq e_s \in U_{\alpha_s}$ and put $n_s := m(e_s)$. By [2, Corollary 7.68] and [16, Lemma (7.4)] every chamber gB_+ can be written in the form $u_1 n_{s_1} \cdots u_k n_{s_k} B_+$ with $u_i \in U_{\alpha_{s_i}}$ and $s_1 \cdots s_k = \delta(B_+, gB_+)$ is reduced. If we fix the type (s_1, \dots, s_k) of $u_1 n_{s_1} \cdots u_k n_{s_k} B_+$, then the elements $u_i \in U_{\alpha_{s_i}}$ are uniquely determined. Sometimes, we will write for short n_i instead of n_{s_i} . This yields a natural labeling of the building Δ . More precisely let P be any s -panel of Δ , and let $\text{proj}_P c = d$ and $w = \delta(c, d)$. As cosets of B_+ the chambers of P may be written $u_1 n_{s_1} \cdots u_k n_{s_k} B_+$ (this is d), and $u_1 n_{s_1} \cdots u_k n_{s_k} v n_s B_+$ where $u_i \in U_{\alpha_{s_i}}$ and $v \in U_{\alpha_s}$. We assign them the s -labels ∞_s and v , using $U'_{\alpha_s} = U_{\alpha_s} \cup \{\infty_s\}$. If we let R_{st} be the $\{s, t\}$ -residue containing c , then R_{st} acquires a labeling and we have a blueprint given by the $(e_s)_{s \in S}$, namely $((U'_{\alpha_s})_{s \in S}, (R_{st})_{s \neq t \in S})$.

(2.18) Proposition. *Let Δ be a spherical Moufang building. Then Δ conforms to the blueprint given by the restriction to $E_2(c)$, i.e. $((U'_{\alpha_s})_{s \in S}, (R_{st})_{s \neq t \in S})$, and the natural labeling of Δ as above.*

Proof. This is [16, (7.5) Proposition]. \square

(2.19) Corollary. *Let Δ be a spherical Moufang building and let \mathcal{B} be the blueprint given by $E_2(c)$. Then (u_1, \dots, u_k) and (v_1, \dots, v_k) are equivalent if and only if $u_1 n_1 \cdots u_k n_k B_+ = v_1 n'_1 \cdots v_k n'_k B_+$. In particular, the map $\varphi : \mathbf{C}_{\mathcal{B}} \rightarrow \Delta, [(u_1, \dots, u_k)] \mapsto u_1 n_1 \cdots u_k n_k B_+$ is an isomorphism of buildings.*

Proof. If (u_1, \dots, u_k) and (v_1, \dots, v_k) are elementary equivalent, then $u_1 n_1 \cdots u_k n_k B_+ = v_1 n'_1 \cdots v_k n'_k B_+$. Thus φ is well-defined. Clearly, φ is surjective. To show that φ is injective, we suppose $[(u_1, \dots, u_k)], [(v_1, \dots, v_k)] \in \mathbf{C}_{\mathcal{B}}$ with $u_1 n_1 \cdots u_k n_k B_+ = \varphi[(u_1, \dots, u_k)] = \varphi[(v_1, \dots, v_k)] = v_1 n'_1 \cdots v_k n'_k B_+$. Let (u_1, \dots, u_k) (resp. (v_1, \dots, v_k)) be of type (s_1, \dots, s_k) (resp. (t_1, \dots, t_k)). Then $s_1 \cdots s_k = t_1 \cdots t_k$. Thus there exists a sequence $(w_1, \dots, w_k) \in [v_1, \dots, v_k]$ of type (s_1, \dots, s_k) and we have $u_1 n_1 \cdots u_k n_k B_+ = w_1 n_1 \cdots w_k n_k B_+$. The uniqueness of the decomposition in [16, Lemma (7.4)] implies

$u_i = w_i$ and hence $[u_1, \dots, u_k] = [w_1, \dots, w_k]$. Thus φ is a bijection. Clearly, φ preserves s -adjacency. Now the claim follows from [2, Lemma 5.61]. \square

2.14. Natural labeling of arbitrary buildings

We now extend the concept of a natural labeling to (arbitrary) buildings of type $A_1 \times A_1$, by defining a labeling of type (U'_1, U'_2) to be *natural* if (u_1, u_2) is equivalent to (u_2, u_1) for all $u_1 \in U_1, u_2 \in U_2$. If Δ is a spherical Moufang building with a natural labeling given by $1 \neq e_s \in U_{\alpha_s}$ then any $A_1 \times A_1$ residue acquires a natural labeling in this sense (because the appropriate root groups commute).

(2.20) Lemma. *Let (W, S) be a reducible 2-spherical Coxeter system of rank 3. Then a blueprint of type (W, S) is realizable if the labeling of the restriction to any $A_1 \times A_1$ residue is natural.*

Proof. Let $\mathcal{B} = ((U'_s)_{s \in S}, (\Delta_{st})_{s \neq t \in S})$ be a blueprint of type (W, S) . Let $S = \{r, s, t\}$ and assume $m_{sr} = 2 = m_{tr}$. By Corollary (2.17) it suffices to show that for any two sequences \bar{u}, \bar{v} of the same reduced type, $\bar{u} \simeq \bar{v}$ implies $\bar{u} = \bar{v}$. Therefore let $\bar{u} = (u_1, \dots, u_k)$ and $\bar{v} = (v_1, \dots, v_k)$ be two sequences of the same reduced type (s_1, \dots, s_k) such that $\bar{u} \simeq \bar{v}$ (note that $k \leq m_{st} + 1$). If (W, S) is of type $A_1 \times A_1 \times A_1$ the claim follows, because an elementary equivalence is just a permutation of the sequence and for each $s \in S$ there is at most one element of U_s in such a sequence. Thus we can assume that (W, S) is not of type $A_1 \times A_1 \times A_1$ and hence $m_{st} \geq 3$. Since $m_{sr} = 2 = m_{tr}$ we know that r occurs at most once in the reduced type. If u_i is in U_r , then the sequence with u_i, u_{i-1} (resp. u_i, u_{i+1}) reversed is equivalent to (u_1, \dots, u_k) , since u_i is the only element in U_r in the sequence \bar{u} . We also note that we can do an elementary equivalence in Δ_{st} only if u_i is at position 1 or k in the sequence. If we do an elementary equivalence in Δ_{st} we have to do this twice (because of the type) and get the same sequence as we started with. Thus the claim follows. \square

2.15. The action of G on Δ

Let $\Delta = (\mathcal{C}, \delta)$ be a spherical Moufang building of type (W, S) and let $G = \langle U_\alpha \mid \alpha \in \Phi \rangle \leq \text{Aut}(\Delta)$. For each $s \in S$ we fix $1 \neq e_s \in U_{\alpha_s}$ and put $n_s := m(e_s)$. As we have already mentioned, every chamber of Δ can be written in the form $u_1 n_1 \cdots u_k n_k B_+$, where $u_i \in U_{\alpha_{s_i}}$ and this decomposition is unique if one fixes the type (s_1, \dots, s_k) . Since G acts on Δ by multiplication from the left, we want to know how to calculate $g \cdot u_1 n_1 \cdots u_k n_k B$. Assume that Δ satisfies condition (lco). Since $U_+ = \langle U_{\alpha_s} \mid s \in S \rangle$ as a consequence of Lemma (2.11) and $U_{-\alpha_s}^{n_s} = U_{\alpha_s}$ it suffices to consider this action for U_{α_s}, H_s and $n_s^{\pm 1}$ for each $s \in S$. For $s_1, \dots, s_k \in S$ and $u_i \in U_{\alpha_{s_i}}$ we denote the chamber $u_1 n_1 \cdots u_k n_k B_+$ by (u_1, \dots, u_k) . We remark that we do not consider all chambers $g \cdot (u_1, \dots, u_k)$ for $g \in G$.

(2.21) Theorem. Let $t, s_1, \dots, s_k \in S$ and $u_i \in U_{\alpha_{s_i}}$. Let $\omega : G \times \Delta \rightarrow \Delta$ be the left multiplication of G on Δ . Then we have $\omega(g, ()) = ()$ for $g \in B_+$ and $\omega(g, (u_1, \dots, u_k))$ is given by the following:

$$\begin{aligned}
 & (u_1^{g^{-1}}, \omega(g^{n_{s_1}}, (u_2, \dots, u_k))) \text{ if } g \in H_s \text{ for some } s \in S, \\
 & (gu_1, u_2, \dots, u_k) \text{ if } g \in U_{\alpha_{s_1}}, \\
 & (u_1, \omega(g^{n_{s_1}}[g, u_1]^{n_{s_1}}, (u_2, \dots, u_k))) \text{ if } g \in U_{\alpha_t}, s_1 \neq t \in S, \\
 & \omega(n_s, \omega(n_s^{-2}, (u_1, \dots, u_k))) \text{ if } g = n_s^{-1} \\
 & \omega(n_{s_1}^2, (u_2, \dots, u_k)) \text{ if } g = n_{s_1}, u_1 = 1, \\
 & (\bar{u}_1, \omega(b(u_1), (u_2, \dots, u_k))) \text{ if } g = n_{s_1}, u_1 \neq 1, \\
 & (1_{U_t}, u_1, \dots, u_k) \text{ if } g = n_t, l(ts_1 \cdots s_k) = k + 1.
 \end{aligned}$$

Proof. This is a straightforward computation. \square

3. Foundations and enveloping groups

This subsection is based on [26, Ch. 11] and [11].

3.1. Foundations

Recall that $E(S) = \{\{s, t\} \subseteq S \mid m_{st} > 2\}$. A *foundation of type (W, S)* is a triple $\mathcal{F} := ((\Delta_J)_{J \in E(S)}, (c_J)_{J \in E(S)}, (\varphi_{rst})_{\{r,s\}, \{s,t\} \in E(S)})$ such that the following hold:

- (F1) $\Delta_J = (C_J, \delta_J)$ is a building of type $(\langle J \rangle, J)$ with $c_J \in C_J$ for each $J \in E(S)$.
- (F2) Each *glueing* $\varphi_{rst} : \mathcal{P}_s(c_{\{r,s\}}) \rightarrow \mathcal{P}_s(c_{\{s,t\}})$ is a base-point preserving bijection.
- (F3) The φ_{rst} satisfy the cocycle condition $\varphi_{tsu} \circ \varphi_{rst} = \varphi_{rsu}$.

It follows from the definition that $\varphi_{tst} = \text{id}$ (as $\varphi_{tst} \circ \varphi_{tst} = \varphi_{tst}$) and that $\varphi_{rst} = \varphi_{tsr}^{-1}$. We say that \mathcal{F} satisfies condition (lco) if Δ_J satisfies condition (lco) for each $J \in E(S)$. For each $J \subseteq S$ the *J-residue* of \mathcal{F} is the foundation $\mathcal{F}_J := ((\Delta_I)_{I \in E(J)}, (c_I)_{I \in E(J)}, (\varphi_{rst})_{\{r,s\}, \{s,t\} \in E(J)})$ of type $(\langle J \rangle, J)$. Two foundations $\mathcal{F} = ((\Delta_J)_{J \in E(S)}, (c_J)_{J \in E(S)}, (\varphi_{rst})_{\{r,s\}, \{s,t\} \in E(S)})$ and $\mathcal{F}' = ((\Delta'_J)_{J \in E(S)}, (c'_J)_{J \in E(S)}, (\varphi'_{rst})_{\{r,s\}, \{s,t\} \in E(S)})$ of the same type (W, S) are called *isomorphic* if there exist isomorphisms $\alpha_J : \Delta_J \rightarrow \Delta'_J$ for all $J \in E(S)$ such that $\alpha_J(c_J) = c'_J$ and for all $r, s, t \in S$ with $\{r, s\}, \{s, t\} \in E(S)$ we have $\varphi'_{rst} \circ \alpha_{\{r,s\}} = \alpha_{\{s,t\}} \circ \varphi_{rst}$.

(3.1) Remark. We remark that there is a notion of more general isomorphisms between foundations, which allow isomorphisms of the Coxeter system. In that sense our isomorphisms are called *special*.

Let $\mathcal{F} = ((\Delta_J)_{J \in E(S)}, (c_J)_{J \in E(S)}, (\varphi_{rst})_{\{r,s\},\{s,t\} \in E(S)})$ be a foundation of type (W, S) . An *apartment* of \mathcal{F} is a tuple $(\Sigma_J)_{J \in E(S)}$ such that the following hold:

- (FA1) Σ_J is an apartment of Δ_J containing c_J for each $J \in E(S)$.
- (FA2) Given $\{r, s\}, \{s, t\} \in E(S)$, then $\varphi_{rst}(\Sigma_{\{r,s\}} \cap \mathcal{P}_s(c_{\{r,s\}})) = \Sigma_{\{s,t\}} \cap \mathcal{P}_s(c_{\{s,t\}})$.

3.2. Moufang sets

A *Moufang set* is a pair $(X, (U_x)_{x \in X})$, where X is a set with $|X| \geq 3$ and for each $x \in X$, U_x is a subgroup of $\text{Sym}(X)$ (we compose from right to left) such that the following hold:

- (MS1) For each $x \in X$, U_x fixes x and acts simply transitive on $X \setminus \{x\}$.
- (MS2) For all $x, y \in X$ and each $g \in U_x$, $g \circ U_y \circ g^{-1} = U_{g(y)}$.

The groups U_x for $x \in X$ are called the *root groups* of the Moufang set. Let $(X, (U_x)_{x \in X})$ and $(X', (U_{x'})_{x' \in X'})$ be two Moufang sets and let $\varphi : X \rightarrow X'$ be a map. Then the Moufang sets are called *φ -isomorphic*, if φ is bijective and for all $x \in X$ we have $\varphi \circ U_x \circ \varphi^{-1} = U_{\varphi(x)}$.

(3.2) Example. Let Δ be a thick, irreducible, spherical Moufang building of rank at least 2. Let P be a panel, let $p \in P$ and let Σ be an apartment in Δ with $p \in \Sigma$. Let α denote the unique root in Σ containing p but not $P \cap \Sigma$. Let $U_p := \{g|_P \mid g \in U_\alpha\}$. Then the group U_p is independent of the choice of the apartment Σ and $\mathbb{M}(\Delta, P) := (P, (U_p)_{p \in P})$ is a Moufang set (cf. [14, Notation 1.19]).

3.3. Moufang foundations

A foundation $\mathcal{F} = ((\Delta_J)_{J \in E(S)}, (c_J)_{J \in E(S)}, (\varphi_{rst})_{\{r,s\},\{s,t\} \in E(S)})$ of 2-spherical type (W, S) is called *Moufang foundation*, if the following hold:

- (MF1) Δ_J is a Moufang building of type $(\langle J \rangle, J)$ for each $J \in E(S)$.
- (MF2) For all $r, s, t \in S$ with $\{r, s\}, \{s, t\} \in E(S)$, the Moufang sets $\mathbb{M}(\Delta_{\{r,s\}}, \mathcal{P}_s(c_{\{r,s\}}))$ and $\mathbb{M}(\Delta_{\{s,t\}}, \mathcal{P}_s(c_{\{s,t\}}))$ are φ_{rst} -isomorphic.

(3.3) Example. Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a thick twin building of irreducible 2-spherical type (W, S) and of rank at least 3. Let $\varepsilon \in \{+, -\}$ and $c \in \mathcal{C}_\varepsilon$. By [17, (8.3) Theorem 4] the residue $R_J(c)$ with the restriction of the distance function is a Moufang building for each $J \in E(S)$ and one can verify that $((\Delta_J)_{J \in E(S)}, (c_J)_{J \in E(S)}, (\varphi_{rst})_{\{r,s\},\{s,t\} \in E(S)})$ is a Moufang foundation of type (W, S) , where $\Delta_J = (R_J(c), \delta_\varepsilon)$, $c_J := c$ and $\varphi_{rst} = \text{id}$. We will denote this Moufang foundation by $\mathcal{F}(\Delta, c)$. It is a (non-trivial) fact that for any $\varepsilon \in \{+, -\}$ and $c, c' \in \mathcal{C}_\varepsilon$ we have $\mathcal{F}(\Delta, c) \cong \mathcal{F}(\Delta, c')$.

A foundation \mathcal{F} is called *integrable*, if there exists a twin building Δ of type (W, S) and a chamber c of Δ such that $\mathcal{F} \cong \mathcal{F}(\Delta, c)$. A foundation satisfies condition (lsc0) if it is 3-spherical and if there exists a twin building $\Delta = (\Delta_+, \Delta_-, \delta_*)$ and a chamber c of Δ such that $\mathcal{F} \cong \mathcal{F}(\Delta, c)$ and both buildings Δ_+, Δ_- satisfy condition (lsc0). In particular, if a foundation satisfies condition (lsc0), it is integrable.

Let \mathcal{F} be an integrable Moufang foundation. Then every panel contains at least 3 chambers. Since \mathcal{F} is integrable, there exists a twin building Δ and a chamber c of Δ such that $\mathcal{F} \cong \mathcal{F}(\Delta, c)$. By Lemma (2.7) the twin building Δ is thick. Moreover, every irreducible integrable Moufang foundation satisfying condition (lco) determines the isomorphism class of the corresponding twin building:

(3.4) Proposition. *Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$, $\Delta' = (\Delta'_+, \Delta'_-, \delta'_*)$ be two thick irreducible 2-spherical twin buildings of type (W, S) and of rank at least 3 satisfying condition (lco). Suppose $c \in \Delta_+, c' \in \Delta'_+$ such that $\mathcal{F}(\Delta, c) \cong \mathcal{F}(\Delta', c')$. Then $\Delta \cong \Delta'$.*

Proof. This is a consequence of [26, Theorem 11.1.12] and [15, Theorem 1.5]. \square

3.4. The Steinberg group associated with an RGD-system

Let $\mathcal{D} = (G, (U_\alpha)_{\alpha \in \Phi})$ be an RGD-system of irreducible spherical type (W, S) and rank at least 2. Following [21], the *Steinberg group* associated with \mathcal{D} is the direct limit \hat{G} of the inductive system formed by the groups U_α and $U_{[\alpha, \beta]} := \langle U_\gamma \mid \gamma \in [\alpha, \beta] \rangle$ for all prenilpotent pairs $\{\alpha, \beta\} \subseteq \Phi$. Then $U_\alpha, U_{[\alpha, \beta]} \rightarrow \hat{G}$ are injective and $(\hat{G}, (\hat{U}_\alpha)_{\alpha \in \Phi})$ is an RGD-system of type (W, S) , where \hat{U}_α denotes the image of U_α in \hat{G} . Let Δ be the associated spherical building, i.e. $\Delta = \Delta(\mathcal{D})_+$, where $\Delta(\mathcal{D}) = (\Delta(\mathcal{D})_+, \Delta(\mathcal{D})_-, \delta_*)$ is the associated twin building. Then the kernel of $\hat{G} \rightarrow \text{Aut}(\Delta)$ is equal to the center of \hat{G} by [2, Proposition 7.127(2)].

3.5. The direct limit of a foundation

In this subsection we let $\mathcal{F} = ((\Delta_J)_{J \in E(S)}, (c_J)_{J \in E(S)}, (\varphi_{rst})_{\{r,s\}, \{s,t\} \in E(S)})$ be a Moufang foundation of type (W, S) satisfying condition (lco) and let $(\Sigma_J)_{J \in E(S)}$ be an apartment of \mathcal{F} . As \mathcal{F} is a Moufang foundation, the buildings Δ_J are Moufang buildings. We identify the roots in Σ_J with Φ^J . For $\alpha \in \Phi^J$ we let U_α^J be the root group associated with $\alpha \subseteq \Sigma_J$ and we let $H_J = \langle U_\alpha^J \mid \alpha \in \Phi^J \rangle \leq \text{Aut}(\Delta_J)$. As $J \in E(S)$, it follows from [2, Remark 7.107(a)] that the root groups U_α^J are nilpotent.

We note that for each $\{s, t\} \in E(S)$ the restriction $U_{\pm\alpha_s}^{\{s,t\}} \rightarrow U_{\pm\alpha_s}^{\{s,t\}}|_{\mathcal{P}_s(c_{\{s,t\}})}$ is an isomorphism. For each $s \in S$ we fix $s_0 \in S$ such that $m_{ss_0} > 2$ and we define $U_{\pm s} := U_{\pm\alpha_s}^{\{s,s_0\}}|_{\mathcal{P}_s(c_{\{s,s_0\}})}$. Using (MF2) we know that for each $t \in S$ with $\{s, t\} \in E(S)$ the mapping $U_{\pm s} \rightarrow U_{\pm\alpha_s}^{\{s,t\}}|_{\mathcal{P}_s(c_{\{s,t\}})}: g \mapsto \varphi_{s_0st} \circ g \circ \varphi_{s_0st}^{-1}$ is an isomorphism. Thus we have canonical isomorphisms $U_{\pm s} \rightarrow U_{\pm\alpha_s}^{\{s,t\}}$.

For each K with $s \in K \in E(S)$ we denote the image of $u \in U_{\pm s}$ in $\text{Aut}(\Delta_K)$ by u_K . Then for each $1 \neq u \in U_s$ there exist $u', u'' \in U_{-s}$ such that $(u')_K u_K (u'')_K$ stabilizes Σ_K and acts on Σ_K as the reflection r_{α_s} . By [2, Consequence (3) on p.415] u', u'' are unique for a given K . Moreover, $(u')_J u_J (u'')_J$ interchanges the elements in $\Sigma_J \cap \mathcal{P}_s(c_J)$ for all J with $s \in J \in E(S)$. There exist also $v', v'' \in U_{-s}$ such that $(v')_J u_J (v'')_J$ stabilizes Σ_J and acts on Σ_J as the reflection r_{α_s} . The uniqueness implies $v' = u'$ and $v'' = u''$ and hence u' and u'' do not depend on K . We define for each $1 \neq u \in U_s$ the element $m(u) := u' u u''$, where $u', u'' \in U_{-s}$ are as above.

(3.5) Remark. For two groups G, H we denote their free product by $G * H$.

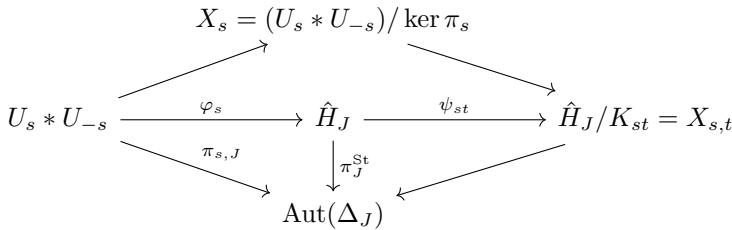
(3.6) Lemma. Let $s \in S$, let $\pi_s : U_s * U_{-s} \rightarrow \prod_{s \in J \in E(S)} \text{Aut}(\Delta_J)$ be the canonical homomorphism, let $K_s := \ker \pi_s$ and let $X_s := (U_s * U_{-s}) / K_s$. Then $U_{\pm s} \rightarrow X_s$ is injective and $(X_s, (U_{\pm s}))$ is an RGD-system of type A_1 , where we identify $U_{\pm s}$ with its image in X_s .

Proof. Note that $U_{\pm s} \rightarrow \text{Aut}(\Delta_J)$ is injective for each J with $s \in J \in E(S)$ and $U_{-\alpha_s}^J \not\leq U_{\alpha_s}^J$. Thus $U_{\pm s} \rightarrow X_s$ is injective and $U_{-s} \not\leq U_s$ in X_s . It suffices to show (RGD2). We show that $m(u) = u' u u''$ conjugates $U_{\pm s}$ to $U_{\mp s}$ for every $1 \neq u \in U_s$. Let $J = \{s, s_0\}$ and $x \in U_{\pm s}$. Then $m(u)_J^{-1} x m(u)_J \in U_{\mp \alpha_s}^J$ and hence $m(u)_J^{-1} x m(u)_J = (x')_J$ for some $x' \in U_{\mp s}$. By the construction of $U_{\pm s} \rightarrow \text{Aut}(\Delta_K)$ for K with $s \in K \in E(S)$, we infer $m(u)_K^{-1} x m(u)_K = (x')_K$. Thus $(x')^{-1} m(u)^{-1} x m(u) \in K_s$ for each $x \in U_{\pm s}$ and hence $m(u)^{-1} x m(u) K_s = x' K_s$ in X_s . This implies that $(X_s, (U_{\pm s}))$ is an RGD-system of type A_1 . \square

(3.7) Example. Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be a thick irreducible twin building satisfying condition (lco). Using [15, Theorem 1.5] and [2, Theorem 8.27], Δ is a Moufang twin building. Let $c_+ \in \Delta_+$ and $c_- \in \Delta_-$ be opposite chambers in Δ and let Σ be the twin apartment containing c_+ and c_- . Note that Σ is isomorphic to the standard thin twin building (cf. [2, Example 5.136(b)]). The set of chambers of each half is a copy of W and the set of roots Φ can be identified with the set of roots in a twin apartment (cf. [2, Remark 8.11]). Let U_α be the corresponding root groups and let $G = \langle U_\alpha \mid \alpha \in \Phi \rangle \leq \text{Aut}(\Delta)$. Then $(G, (U_\alpha)_{\alpha \in \Phi})$ is an RGD-system by [2, Example 8.47(a)].

Let $\mathcal{F} = \mathcal{F}(\Delta, c_+)$ and $\Sigma_J := R_J(c_+) \cap \Sigma$. Then $(\Sigma_J)_{J \in E(S)}$ is an apartment of \mathcal{F} . Let $\varphi : U_s * U_{-s} \rightarrow \text{Aut}(\Delta)$ be the canonical homomorphism. Let $g \in \ker \varphi$. Then g fixes every rank 2 residue and hence $g \in \ker \pi_s$. Now let $g \in \ker \pi_s$. Then $\varphi(g) \in \langle U_{\alpha_s} \cup U_{-\alpha_s} \rangle$. As $\varphi(g)$ fixes $\mathcal{P}_s(c_+)$, we deduce $\varphi(g) \in H_{\alpha_s}$ and hence $\varphi(g)$ fixes Σ . If $m_{st} > 2$, then $\varphi(g)$ fixes $R_{\{s,t\}}(c_+)$ by assumption. If $m_{st} = 2$, then $\varphi(g)$ fixes $\mathcal{P}_t(c_+)$, as the corresponding root groups commute. In particular, $\varphi(g)$ fixes a twin apartment and all neighbors of c_+ . Thus $\varphi(g) = 1$ by [17, Theorem 1] and we have $\ker \pi_s = \ker \varphi$. In particular, $X_s \rightarrow \text{Aut}(\Delta)$ is injective, where X_s is as in the previous lemma.

Let $J = \{s, t\} \in E(S)$ and let \hat{H}_J be the Steinberg group associated with $(\hat{H}_J, (U_\alpha^J)_{\alpha \in \Phi^J})$. Let $\pi_J^{\text{St}} : \hat{H}_J \rightarrow \text{Aut}(\Delta_J), \pi_{s,J} : U_s * U_{-s} \rightarrow \text{Aut}(\Delta_J), \varphi_s : U_s * U_{-s} \rightarrow \hat{H}_J$ be the canonical homomorphisms. As $\pi_s : U_s * U_{-s} \rightarrow \prod_{s \in K \in E(S)} \text{Aut}(\Delta_K)$, we deduce $\ker \pi_s \leq \ker \pi_{s,J}$. As $\pi_{s,J} = \pi_J^{\text{St}} \circ \varphi_s$, we have $\varphi_s(\ker \pi_s) \leq \varphi_s(\ker \pi_{s,J}) \leq \ker \pi_J^{\text{St}}$. Since $\ker \pi_J^{\text{St}} = Z(\hat{H}_J)$, we deduce $K_{st} := \varphi_s(\ker \pi_s) \varphi_t(\ker \pi_t) \trianglelefteq \hat{H}_J$. Now \hat{H}_J/K_{st} is again an RGD-system by [2, 7.131]. Thus we let $\mathcal{D}_J = (\hat{H}_J/K_{st}, (\hat{U}_\alpha K_{st}/K_{st})_{\alpha \in \Phi^J})$ and write $X_{s,t} := \hat{H}_J/K_{st}$. Let $\psi_{st} : \hat{H}_J \rightarrow X_{s,t}$ be the canonical homomorphism. Then $(\psi_{st} \circ \varphi_s)(\ker \pi_s) \leq \psi_{st}(K_{st}) = 1$ and hence $\psi_{st} \circ \varphi_s$ factors through $X_s \rightarrow \hat{H}_J/K_{st}$.



At this point it is not clear, whether $X_s \rightarrow X_{s,t}$ is injective (cf. Lemma (3.18)). If $s \neq t \in S$ are such that $m_{st} = 2$, we define $X_{s,t} := X_s \times X_t$. We define G to be the direct limit of the inductive system formed by $X_s, X_{s,t}$ for all $s \neq t \in S$. Note that $X_{s,t}$ is generated by the canonical image of X_s, X_t in $X_{s,t}$ and hence $G = \langle X_s \mid s \in S \rangle$. We note that it is not clear whether $X_s \rightarrow G$ is injective. If we write $1 \neq u \in U_s$ we simply mean that $u \neq 1$ in the group U_s .

(3.8) Lemma. *Let $s_1, \dots, s_k, s \in S$ and let $u_i, v_i \in U_{s_i}$ be all non-trivial. Then we have $U_s^{m(u_1) \cdots m(u_k)} = U_s^{m(v_1) \cdots m(v_k)}$.*

Proof. We show the hypothesis by induction on k . For $k = 1$ we have $m(v_1)^{-1}m(u_1) \in H_{s_1} \leq N_{X_{s_1,s}}(U_s)$. Thus we assume $k > 1$. Using [2, Consequences (4) and (5) on p.415] we deduce $hm(u_i)h^{-1} = m(hu_i h^{-1})$ and $hm(u_i)^{-1}h^{-1} = m(hu_i h^{-1})^{-1}$ for each $h \in H_t$ with $t \in S$ and $u_i \in U_{s_i}$. Note that $hu_i h^{-1} \in U_{s_i}$. Using induction, this implies for $h := m(v_k)m(u_k)^{-1} \in H_{s_k}$:

$$\begin{aligned}
 & hm(u_{k-1})^{-1} \cdots m(u_1)^{-1} U_s m(u_1) \cdots m(u_{k-1}) h^{-1} \\
 &= m(hu_{k-1} h^{-1})^{-1} \cdots m(hu_1 h^{-1})^{-1} h U_s h^{-1} m(hu_1 h^{-1}) \cdots m(hu_{k-1} h^{-1}) \\
 &= m(hu_{k-1} h^{-1})^{-1} \cdots m(hu_1 h^{-1})^{-1} U_s m(hu_1 h^{-1}) \cdots m(hu_{k-1} h^{-1}) \\
 &= m(v_{k-1})^{-1} \cdots m(v_1)^{-1} U_s m(v_1) \cdots m(v_{k-1}) \quad \square
 \end{aligned}$$

(3.9) Lemma. *Suppose $s, t, s_1, \dots, s_k, t_1, \dots, t_l \in S$ such that $s_1 \cdots s_k \alpha_s = t_1 \cdots t_l \alpha_t$. Let $1 \neq u_i \in U_{s_i}, 1 \neq v_i \in U_{t_i}$. Then $U_s^{m(u_k) \cdots m(u_1)} = U_t^{m(v_l) \cdots m(v_1)}$.*

Proof. At first we show that we have an action of W on the conjugacy class of U_s , where $s \in S$ acts on every conjugacy class by conjugation with $m(u)$ for some $1 \neq u \in U_s$. By

the previous lemma, conjugation with $m(u)$ does not depend on $1 \neq u \in U_s$ and $m(u)^2$ fixes every conjugacy class. Moreover, $(st)^{m_{st}}$ acts as $(m(u_s)m(u_t))^{m_{st}} \in \langle H_s \cup H_t \rangle$ (cf. [6, Lemma 3.3]) and hence trivially.

We are now in the position to prove the claim. It suffices to show that for $w \in W$ and $s, t \in S$ with $w\alpha_s = \alpha_t$, we have $wU_s w^{-1} = U_t$. We show that claim via induction on $\ell(w)$. For $\ell(w) = 0$ there is nothing to show. Thus we assume $\ell(w) > 0$. Note that $w \in w\alpha_s = \alpha_t$ and $ws \notin w\alpha_s = \alpha_t$. This implies $\{1_W, t\}, \{w, ws\} \in \partial\alpha_t$. By Lemma (2.5) there exist a sequence of panels $P_0 = \{1_W, t\}, \dots, P_n = \{w, ws\}$ contained in $\partial\alpha_t$ and a sequence of spherical rank 2 residues R_1, \dots, R_n contained in $\partial^2\alpha_t$ such that P_{i-1}, P_i are distinct and contained in R_i . Let $x \in P_{n-1} \cap \alpha_t$. Then $x^{-1}w \in \langle r, s \rangle$ for some $r \in S$ and $(x^{-1}w)\alpha_s = \alpha_{s'}$, where $s' = r$ if $m_{rs} = 3$ and $s' = s$ if m_{rs} is even. It follows from Lemma (2.5) that $x = \text{proj}_{P_{n-1}} 1_W = \text{proj}_{R_n} 1_W$ and hence $\ell(x) < \ell(w)$. Using induction, we deduce that $wU_s w^{-1} = x(x^{-1}w)U_s(x^{-1}w)^{-1}x^{-1} = xU_{s'}x^{-1} = U_t$. \square

For $s \in S$ we define $U_{\alpha_s} := U_s$ (recall that $\alpha_s \in \Phi$ is the simple root corresponding to s) and for $\alpha \in \Phi$ we define U_α as a conjugate of U_s as in (RGD2). This is well-defined by the previous lemma. We put $\mathcal{D}_{\mathcal{F}} := (G, (U_\alpha)_{\alpha \in \Phi})$.

(3.10) Lemma. *The system $\mathcal{D}_{\mathcal{F}}$ satisfies (RGD2) and (RGD4). Furthermore, (RGD1) holds for each pair $\{\alpha, \beta\}$ of prenilpotent roots with $o(r_\alpha r_\beta) < \infty$.*

Proof. It follows by the definition of G and U_α that (RGD2) and (RGD4) are satisfied. Let $\{\alpha, \beta\}$ be a prenilpotent pair with $o(r_\alpha r_\beta) < \infty$. Then there exists $R \in \partial^2\alpha \cap \partial^2\beta$. Let $w = \text{proj}_R 1_W$. Then $w^{-1}\alpha, w^{-1}\beta \in \Phi^{\{s,t\}}$ for some $s \neq t \in S$. As $[U_{w^{-1}\alpha}, U_{w^{-1}\beta}] \leq U_{(w^{-1}\alpha, w^{-1}\beta)}$ holds in $X_{s,t}$, it also holds in G . This implies $[U_\alpha, U_\beta]^w = [U_{w^{-1}\alpha}, U_{w^{-1}\beta}] \leq U_{(w^{-1}\alpha, w^{-1}\beta)}$ and hence $[U_\alpha, U_\beta] \leq U_{(w^{-1}\alpha, w^{-1}\beta)}^w = \langle U_{w\gamma} \mid \gamma \in (w^{-1}\alpha, w^{-1}\beta) \rangle = U_{(\alpha, \beta)}$. \square

(3.11) Lemma. *Let (c_0, \dots, c_k) be a minimal gallery in $\Sigma(W, S)$ for some $k \geq 2$ and let α_i be the root containing c_{i-1} but not c_i for each $i \in \{1, \dots, k\}$. Then we have $[U_{\alpha_1}, U_{\alpha_k}] = 1$ if $k = 2$ and $[U_{\alpha_1}, U_{\alpha_k}] \leq \langle U_{\alpha_2}, \dots, U_{\alpha_{k-1}} \rangle$ if $k > 2$. In particular, $U_{\alpha_1} \cdots U_{\alpha_k}$ is a nilpotent group.*

Proof. For $k = 2$ we have $o(r_{\alpha_1} r_{\alpha_2}) < \infty$ and $(\alpha_1, \alpha_2) = \emptyset$. Thus the claim follows from the previous lemma. Thus we can assume $k > 2$ and proceed by induction. If $o(r_{\alpha_1} r_{\alpha_k}) < \infty$, then the claim follows from the previous lemma, as $(\alpha_1, \alpha_k) \subseteq \{\alpha_2, \dots, \alpha_{k-1}\}$. Thus we can assume $\alpha_1 \not\subseteq \alpha_k$. Let R be the residue containing c_{k-2}, c_{k-1}, c_k . Note that $c_0 \notin R$. Then there exists a minimal gallery (d_0, \dots, d_k) in $\Sigma(W, S)$ with $d_0 = c_0, d_k = c_k$ and $d_{p-1} = \text{proj}_R c_0$ for some $p \in \{2, \dots, k-1\}$. Let β_i be the root containing d_{i-1} but not d_i for each $i \in \{1, \dots, k\}$. We note that the sets $\{\beta_1, \dots, \beta_k\}$ and $\{\alpha_1, \dots, \alpha_k\}$ coincide. Thus there exists $1 \leq j \leq k$ with $\beta_j = \alpha_1$. As $\beta_j \cap R = \emptyset$, we have $j < p$. Let d'_p be the neighbor of d_{p-1} in R distinct from d_p and let γ be the root containing d_{p-1} but

not d'_p . Since the set $\{\beta_{j+1}, \dots, \beta_{p-1}\}$ is contained in the set $\{\alpha_2, \dots, \alpha_{k-1}\}$, it suffices to show $[U_{\alpha_1}, U_{\alpha_k}] \leq \langle U_{\beta_{j+1}}, \dots, U_{\beta_{p-1}} \rangle$. Since α_k contains c_0 and c_{k-1} , it contains $\text{proj}_R c_0 = d_{p-1}$. Since α_k does not contain c_k , the intersection of $\alpha_k \cap R$ is a root of R . By Lemma (2.11), therefore, $U_{\alpha_k} \leq \langle U_\gamma \cup U_{\beta_p} \rangle$.

Let $u_k \in U_{\alpha_k}$. Then there exist $a_1, \dots, a_n \in U_\gamma$ and $b_1, \dots, b_n \in U_{\beta_p}$ such that $u_k = a_1 b_1 \cdots a_n b_n$. Using induction, we know that $[U_{\alpha_1}, U_\gamma], [U_{\alpha_1}, U_{\beta_p}] \leq \langle U_{\beta_{j+1}}, \dots, U_{\beta_{p-1}} \rangle$. Let $u_1 \in U_{\alpha_1}$. We deduce

$$[u_1, u_k] = [u_1, a_1 b_1 \cdots a_n b_n] = [u_1, b_n][u_1, a_1 b_1 \cdots a_n]^{b_n} \in \langle U_{\beta_{j+1}}, \dots, U_{\beta_{i-1}} \rangle$$

Let $V := U_{\alpha_1} \cdots U_{\alpha_{k-1}}$. Using induction, $V \leq G$. Since $[U_{\alpha_i}, U_{\alpha_k}] \leq \langle U_{\alpha_{i+1}}, \dots, U_{\alpha_{k-1}} \rangle \leq V$ for $1 \leq i \leq k-1$, we have $VU_{\alpha_k} = U_{\alpha_k}V$. It follows that $VU_{\alpha_k} = U_{\alpha_1} \cdots U_{\alpha_k}$ is a subgroup of G . In particular, the subgroup is nilpotent because of the commutator relations and the fact that the groups U_δ are nilpotent. \square

(3.12) Lemma. *Let (c_0, \dots, c_k) be a minimal gallery in $\Sigma(W, S)$ for some $k \geq 1$ and let α_i be the root containing c_{i-1} but not c_i for each $i \in \{1, \dots, k\}$. Then the product map $U_{\alpha_1} \times \cdots \times U_{\alpha_k} \rightarrow \langle U_{\alpha_i} \mid 1 \leq i \leq k \rangle$ is a bijection, if $\mathcal{D}_\mathcal{F}$ satisfies (RGD3).*

Proof. As in the definition of an RGD-system we let $U_+ := \langle U_\alpha \mid \alpha \in \Phi_+ \rangle \leq G$. At first we assume that $U_{-\alpha_s} \cap U_+ \neq \{1\}$. Then [2, Consequence (1) on p.414] would imply $U_{-\alpha_s} \leq U_+$, which is a contradiction to (RGD3). Thus $U_{-\alpha_s} \cap U_+ = \{1\}$.

By the previous lemma we have $\langle U_{\alpha_1}, \dots, U_{\alpha_k} \rangle = U_{\alpha_1} \cdots U_{\alpha_k}$ and hence the product map is surjective. We prove by induction on k that it is injective. If $k = 1$ there is nothing to show. Thus let $k \geq 2$ and let $u_i, v_i \in U_{\alpha_i}$ such that $u_1 \cdots u_k = v_1 \cdots v_k$. Then we have $v_1^{-1} u_1 = v_2 \cdots v_k (u_2 \cdots u_k)^{-1}$. Using (RGD2) we can arrange it that $\alpha_1 = -\alpha_s \in \Phi_-$ for some $s \in S$ and $\alpha_i \in \Phi_+$ for every $2 \leq i \leq k$. Since $U_{-\alpha_s} \cap U_+ = 1$ we obtain $u_1 = v_1$ and $u_2 \cdots u_k = v_2 \cdots v_k$. Using induction the claim follows. \square

(3.13) Theorem. *If $\mathcal{D}_\mathcal{F}$ satisfies (RGD0) and (RGD3), then $\mathcal{D}_\mathcal{F}$ is an RGD-system.*

Proof. By assumption and Lemma (3.10) it suffices to show (RGD2) for all roots $\alpha, \beta \in \Phi$ with $\alpha \subsetneq \beta$. Suppose $\alpha, \beta \in \Phi$ with $\alpha \subsetneq \beta$, let $G = (c_0, \dots, c_k)$ be a minimal gallery in $\Sigma(W, S)$, let α_i be the root containing c_{i-1} but not c_i for each $i \in \{1, \dots, k\}$ and let $\alpha_1 = \alpha$ and $\alpha_k = \beta$. Using Lemma (3.12) we obtain a unique set $I \subseteq \{2, \dots, k-1\}$ and unique elements $1 \neq u_{\alpha_i} \in U_{\alpha_i}$ such that $[u_\alpha, u_\beta] = \prod_{i \in I} u_{\alpha_i}$.

Let (d_0, \dots, d_m) be a minimal gallery in $\Sigma(W, S)$ and let β_i be the root containing d_{i-1} but not d_i for each $i \in \{1, \dots, m\}$. Suppose that $\beta_1 = \alpha$ and $\beta_m = \beta$. Assume that $c_0 = d_0$ or $c_k = d_m$. We claim that for all $p \in I$ we have $\alpha_p \in \{\beta_2, \dots, \beta_{m-1}\}$.

Proof of claim. We only prove the claim for $c_0 = d_0$. Similar arguments yield the claim for $c_k = d_m$. Let $P := \{c_{k-1}, c_k\}$ and $Q := \{d_{m-1}, d_m\}$. Then $P, Q \in \partial\beta$. Using Lemma (2.5) there exist a sequence $P_0 = P, \dots, P_n = Q$ of panels in $\partial\beta$ and a sequence R_1, \dots, R_n

of spherical rank 2 residues in $\partial^2\beta$ such that P_{i-1}, P_i are distinct and contained in R_i . We will show the claim by induction on n . For $n = 0$ there is nothing to show, as in this case we have $\{\alpha_2, \dots, \alpha_{k-1}\} = \{\beta_2, \dots, \beta_{m-1}\}$. Thus we assume $n > 0$. Note that by [2, Lemma 3.69] every minimal gallery from c_0 to a chamber which is not contained in β has to cross $\partial\alpha$ and $\partial\beta$. Let $(e_0 = c_0, \dots, e_l)$ be a minimal gallery such that $P_{n-1} = \{e_{l-1}, e_l\}$ and $e_z = \text{proj}_{R_n} e_0$ for some z . Moreover, we let γ_i be the root containing e_{i-1} but not e_i for each $i \in \{1, \dots, l\}$. Using induction, we have $\alpha_p \in \{\gamma_2, \dots, \gamma_{l-1}\}$ for each $p \in I$. Using Lemma (3.12) we obtain a unique set $J \subseteq \{2, \dots, l-1\}$ and unique elements $1 \neq v_{\gamma_j} \in U_{\gamma_j}$ such that $[u_\alpha, u_\beta] = \prod_{j \in J} v_{\gamma_j}$. Note that $\alpha \neq \gamma_1$ in general. Now we consider the minimal gallery $(e_0, \dots, e_z = \text{proj}_{R_n} e_0, \dots, e_m)$ with $e_z, \dots, e_m \in R_n$ and $Q = \{e_{m-1}, e_m\}$. Let δ_i be the root containing e_{i-1} but not e_i for each $i \in \{1, \dots, m\}$ and note that $\gamma_i = \delta_i$ for all $i \in \{1, \dots, z\}$. Using Lemma (3.12) again, there exist a unique set $F \subseteq \{2, \dots, m-1\}$ and unique elements $1 \neq x_{\delta_f} \in U_{\delta_f}$ such that $[u_\alpha, u_\beta] = \prod_{f \in F} x_{\delta_f}$. Extending both galleries to q , where e_z, q are opposite in R_n , we have $\prod_{j \in J} v_{\gamma_j} = [u_\alpha, u_\beta] = \prod_{f \in F} x_{\delta_f}$ and hence $\left(\prod_{z \geq f \in F} x_{\delta_f}\right)^{-1} \cdot \left(\prod_{z \geq j \in J} v_{\delta_j}\right) = \left(\prod_{z < f \in F} x_{\delta_f}\right) \cdot \left(\prod_{z < j \in J} v_{\delta_j}\right)^{-1}$. Lemma (3.12) implies that both sides are trivial and as $\langle U_{\gamma_x} \mid z < x < l \rangle \cap \langle U_{\delta_x} \mid z < x < m \rangle = \{1\}$, we deduce $J, F \subseteq \{2, \dots, z-1\}$. We have $\prod_{i \in I} u_{\alpha_i} = \prod_{j \in J} v_{\gamma_j}$. Assume $\gamma_{l-1} \in \{\alpha_i \mid i \in I\}$. Then $[u_\alpha, u_\beta] = uu_{\gamma_{l-1}}u'$ and hence $u_{\gamma_{l-1}} = u^{-1}(\prod_{j \in J} v_{\gamma_j})u'^{-1} \in \langle U_{\gamma_j} \mid 2 \leq j \leq l-2 \rangle$ and the previous lemma yields a contradiction. Thus $\gamma_{l-1} \notin \{\alpha_i \mid i \in I\}$ and $\alpha_p \in \{\gamma_2, \dots, \gamma_{l-2}\}$ for all $p \in I$. Repeating that argument we deduce $\alpha_p \in \{\gamma_2, \dots, \gamma_z\} = \{\delta_2, \dots, \delta_z\}$ for all $p \in I$. As $\{\beta_1, \dots, \beta_m\} = \{\delta_1, \dots, \delta_m\}$, the claim follows.

Now assume that for $p \in I$ we have $o(r_\beta r_{\alpha_p}) < \infty$. Then there exists $R \in \partial^2\beta \cap \partial^2\alpha_p$. Suppose $Q \in \partial\beta$ such that $Q \subseteq \alpha_p \cap R$ and let $\text{proj}_Q c_0 \neq q \in Q$. Then a minimal gallery from c_0 to q does not cross the root α_p . This yields a contradiction and we infer $\alpha_p \subsetneq \beta$ for every $p \in I$. Using similar arguments, we deduce $\alpha \subsetneq \alpha_p$ for every $p \in I$. Since $\gamma \in (\alpha, \beta)$ if and only if $\alpha \subsetneq \gamma \subsetneq \beta$, the claim follows. \square

(3.14) Corollary. *If the canonical mappings $U_{\pm s} \rightarrow G$ are injective and if $\mathcal{D}_{\mathcal{F}}$ satisfies (RGD3), then $\mathcal{D}_{\mathcal{F}}$ is an RGD-system and we have $\mathcal{F} \cong \mathcal{F}(\Delta(\mathcal{D}_{\mathcal{F}}), B_+)$.*

Proof. By Theorem (3.13) $\mathcal{D}_{\mathcal{F}}$ is an RGD-system and it suffices to show that the two foundations \mathcal{F} and $\mathcal{F}(\Delta(\mathcal{D}_{\mathcal{F}}), B_+)$ are isomorphic. Since $U_{\pm s} \rightarrow G$ are injective, we do not distinguish between them and their images in G .

Each $\Delta_{\{s,t\}}$ is a spherical Moufang building for $s \neq t$ with $\{s, t\} \in E(S)$. Thus there exists an isomorphism $\beta_{\{s,t\}} : X_{s,t}/B_{\{s,t\}} \rightarrow \Delta_{\{s,t\}}, gB_{\{s,t\}} \mapsto g(c_{\{s,t\}})$ by [2, Lemma 7.28], where $B_{\{s,t\}} := \langle H_s \cup H_t \cup U_\alpha \mid \alpha \in \Phi_{\pm}^{\{s,t\}} \rangle$.

By Remark (2.12) and Lemma (2.14), we know that $\alpha_{\{s,t\}} : X_{s,t}/B_{\{s,t\}} \rightarrow R_{\{s,t\}}(B_+), gB_{\{s,t\}} \mapsto gB_+$ is an isomorphism for every $\{s, t\} \in E(S)$. Thus we have an isomorphism $\gamma_{\{s,t\}} := \alpha_{\{s,t\}} \circ \beta_{\{s,t\}}^{-1} : \Delta_{\{s,t\}} \rightarrow R_{\{s,t\}}(B_+), d = g(c_{\{s,t\}}) \mapsto gB_+$.

It remains to show that $\gamma_{\{r,s\}}(d) = (\gamma_{\{s,t\}} \circ \varphi_{rst})(d)$ for every $d \in \mathcal{P}_s(c_{\{r,s\}})$. Let $d = g(c_{\{r,s\}})$ for some $g \in \langle U_{\alpha_s}^{\{r,s\}} \cup U_{-\alpha_s}^{\{r,s\}} \rangle$. Let $g_i \in U_{\pm\alpha_s}^{\{r,s\}}$ such that $g = g_1 \cdots g_n$ and let $g'_i \in U_{\pm\alpha_s}^{\{s,t\}}$ be the unique element such that $g'_i|_{\mathcal{P}_s(c_{\{s,t\}})} = \varphi_{rst} \circ g_i|_{\mathcal{P}_s(c_{\{r,s\}})} \circ \varphi_{rst}^{-1}$. Let $g' := g_1 \cdots g'_n$. Then

$$\begin{aligned} \varphi_{rst}(d) &= \varphi_{rst}(g(c_{\{r,s\}})) = \varphi_{rst} \left(g|_{\mathcal{P}_s(c_{\{r,s\}})} \left(\varphi_{rst}^{-1}(c_{\{s,t\}}) \right) \right) = g'_1 \cdots g'_n|_{\mathcal{P}_s(c_{\{s,t\}})}(c_{\{s,t\}}) \\ &= g'(c_{\{s,t\}}) \end{aligned}$$

Since $g'_i = g_i$ in G , we deduce $g' = g$ in G . As $\gamma_{\{r,s\}}(d) = \gamma_{\{r,s\}}(g(c_{\{r,s\}})) = gB_+$, we have

$$\begin{aligned} (\gamma_{\{s,t\}} \circ \varphi_{rst})(d) &= \gamma_{\{s,t\}}(\varphi_{rst}(d)) = \gamma_{\{s,t\}}(g'(c_{\{s,t\}})) = g'B_+ = gB_+ \\ &= \gamma_{\{r,s\}}(d) \quad \square \end{aligned}$$

For $J \subseteq S$ we define G_J to be the direct limit of the groups X_s and $X_{s,t}$ with the canonical homomorphisms as above, where $s \neq t \in J$. It follows directly from the definition of the direct limit that we have a homomorphism $G_J \rightarrow G$ extending $X_s, X_{s,t} \rightarrow G$. We define $(\mathcal{D}_{\mathcal{F}})_J = (G_J, (U_{\alpha})_{\alpha \in \Phi^J})$ and note that we have $\mathcal{D}_{(\mathcal{F}_J)} \neq (\mathcal{D}_{\mathcal{F}})_J$ in general, as the group X_s depends on \mathcal{F} .

(3.15) Lemma. *Let \mathcal{F} be an irreducible Moufang foundation of type (W, S) satisfying condition (1co). If $J \subseteq S$ is irreducible such that $|J| \geq 3$ and \mathcal{F}_J is integrable, then the following hold:*

- (a) *Let Δ_J be a twin building of type $(\langle J \rangle, J)$ and let c_J be a chamber of Δ_J such that $\mathcal{F} \cong \mathcal{F}(\Delta_J, c_J)$. Then we have a canonical homomorphism $G_J \rightarrow \text{Aut}(\Delta_J)$*
- (b) *The homomorphisms $U_{\pm s} \rightarrow G_J$ are injective for every $s \in J$ and $(\mathcal{D}_{\mathcal{F}})_J$ is an RGD-system.*

Proof. Since \mathcal{F}_J is integrable, there exists a thick twin building Δ_J of type $(\langle J \rangle, J)$ and a chamber c_J of Δ_J such that $\mathcal{F}_J \cong \mathcal{F}(\Delta_J, c_J)$. Using [15, Theorem 1.5] and [2, Theorem 8.27], Δ_J is a *Moufang twin building*. Let $\varepsilon \in \{+, -\}$ be such that $c_J \in \mathcal{C}_{\varepsilon}$. By Lemma (2.8) there exists a twin apartment Σ containing the image of the apartments $(\Sigma_K)_{K \in E(J)}$ of the foundation. As we have seen in Example (3.7), K_s fixes Δ_J and we have homomorphism $X_s \rightarrow \text{Aut}(\Delta_J)$. Clearly, $U_{\pm s} \rightarrow \text{Aut}(\Delta_J)$ are injective. Let $\{\alpha, \beta\} \subseteq \Phi^{\{s,t\}}$ be a prenilpotent pair. Note that for $s \neq t \in J$, restriction of an automorphism of Δ_J to $R_{\{s,t\}}(c_J)$ is an epimorphism. Using [2, Corollary 7.66] it is an isomorphism from $U_{[\alpha,\beta]}$ to its image in $\text{Aut}(R_{\{s,t\}}(c_J))$. Thus we have a homomorphism $\hat{H}_{\{s,t\}} \rightarrow \text{Aut}(\Delta_J)$ for $m_{st} > 2$. As K_s is contained in the kernel of this map, we obtain a homomorphism $X_{s,t} = \hat{H}_{\{s,t\}}/K_{st} \rightarrow \text{Aut}(\Delta_J)$ for $m_{st} > 2$. As $U_{\pm\alpha_s}$ commutes with $U_{\pm\alpha_t}$ in $\text{Aut}(\Delta_J)$ if $m_{st} = 2$, we also have a homomorphism $X_{s,t} = X_s \times X_t \rightarrow \text{Aut}(\Delta_J)$.

We conclude that there exists a homomorphism $G_J \rightarrow \text{Aut}(\Delta_J)$ mapping $U_{\pm s}$ onto $U_{\pm\alpha_s}$. Thus (a) follows. Note that $U_\alpha \leq \langle U_{\alpha_s} \mid s \in S \rangle$ fixes c_J for every $\alpha \in \Phi_+^J$, whereas $U_{-\alpha_s}$ does not. Thus (RGD3) holds and (b) follows from Theorem (3.13). \square

(3.16) Lemma. *Let \mathcal{F} be a Moufang foundation and let $J \subseteq S$ be reducible such that $|J| = 3$. Then $U_{\pm s} \rightarrow G_J$ is injective for all $s \in J$ and $(\mathcal{D}_{\mathcal{F}})_J$ is an RGD-system.*

Proof. Let $J = \{r, s, t\}$ and assume $m_{rs} = 2 = m_{rt}$. If $(\langle J \rangle, J)$ is of type $A_1 \times A_1 \times A_1$, then $G_{\{r,s,t\}} = X_r \times X_s \times X_t$. Otherwise, we have $G_{\{r,s,t\}} = X_r \times X_{s,t}$. In both cases, $(\mathcal{D}_{\mathcal{F}})_J$ is an RGD-system. \square

(3.17) Lemma. *Let \mathcal{F} be an irreducible Moufang foundation of type (W, S) satisfying condition (lco). Then \mathcal{F} is integrable if and only if $\mathcal{D}_{\mathcal{F}}$ is an RGD-system and $U_{\pm s} \rightarrow G$ is injective for all $s \in S$.*

Proof. One implication is Corollary (3.14); the other follows from Lemma (3.15)(b) applied to $J = S$. \square

(3.18) Lemma. *Assume that for each irreducible $J \subseteq S$ with $|J| = 3$ the J -residue \mathcal{F}_J is integrable. Then $X_s \rightarrow X_{s,t}$ is injective for all $s \neq t \in S$.*

Proof. Let $s \neq t \in S$. If $m_{st} = 2$ there is nothing to show, as $X_{s,t} = X_s \times X_t$. Thus we assume $m_{st} \neq 2$ and let $1 \neq g \in \ker(X_s \rightarrow X_{s,t})$. Since $\ker(X_s \rightarrow \prod_{s \in J \in E(S)} \text{Aut}(\Delta_J)) = \{1\}$, there exists K with $s \in K \in E(S)$ such that $g \notin \ker(X_s \rightarrow \text{Aut}(\Delta_K))$. As $g \in \ker(X_s \rightarrow X_{s,t})$, we have $g \in \ker(X_s \rightarrow \text{Aut}(\Delta_{\{s,t\}}))$ and hence $K \neq \{s, t\}$. Let $J = K \cup \{t\}$. Then J is irreducible. As \mathcal{F}_J is integrable, there exists a twin building Δ and a chamber c of Δ such that $\mathcal{F}_J \cong \mathcal{F}(\Delta, c)$. By Lemma (3.15)(a) we obtain that G_J acts on Δ . As g is trivial in G_J but not in $\text{Aut}(\Delta)$, this yields a contradiction and hence $X_s \rightarrow X_{s,t}$ is injective for all $s \neq t \in S$. \square

4. The 3-spherical case

In this section we let \mathcal{F} be a Moufang foundation of irreducible 3-spherical type (W, S) and of rank at least 3 satisfying condition (lco). Moreover, we assume that for each irreducible $J \subseteq S$ with $|J| = 3$ the J -residue \mathcal{F}_J is integrable and satisfies condition (lsc). Let $(\Sigma_J)_{J \in E(S)}$ be an apartment of \mathcal{F} and let $U_{\pm s}, X_s, X_{s,t}$ and $\mathcal{D}_{\mathcal{F}} = (G, (U_\alpha)_{\alpha \in \Phi})$ be as before.

Our goal is to show that \mathcal{F} is integrable. By Corollary (3.14) it suffices to show that the canonical mappings $U_{\pm s} \rightarrow G$ are injective and that $\mathcal{D}_{\mathcal{F}}$ satisfies (RGD3). We will show that G acts non-trivially as a permutation group of the set of chambers of a building and deduce both assertions from the action. For all $s \in S$ we fix $1 \neq e_s \in U_s$ and let $n_s := m(e_s)$.

For $s \neq t \in S$ with $m_{st} = 2$ we define $\Delta_{\{s,t\}}$ to be the spherical building associated with $X_s \times X_t$ (cf. [2, Proposition 7.116]). Using [2, Proposition 7.116, Corollary 7.68 and Remark 7.69]), we get (similar as for Moufang buildings) a natural labeling. In particular, $((U_s \cup \infty_s)_{s \in S}, (\Delta_{\{s,t\}})_{s \neq t \in S})$ is a blueprint. We will denote it by $\mathcal{B}_{\mathcal{F}}$ and remark that the restriction of $\mathcal{B}_{\mathcal{F}}$ to any $A_1 \times A_1$ residue is natural.

(4.1) Lemma. *Let $J \subseteq S$ be irreducible and of rank 3. Then the blueprint $\mathcal{B}_{(\mathcal{F}_J)}$ is realizable.*

Proof. Since \mathcal{F}_J is an integrable Moufang foundation, there exists a thick twin building $\Delta = (\Delta_+, \Delta_-, \delta_*)$ and a chamber c of the twin building Δ such that $\mathcal{F}_J \cong \mathcal{F}(\Delta, c)$. Without loss of generality let $c \in \mathcal{C}_+$. Let Σ be a twin apartment containing the images of the apartments $(\Sigma_K)_{K \in E(J)}$ of the foundation. We deduce from [15, Theorem 1.5] and [2, Theorem 8.27 and Proposition 8.21] that Δ_+ is a spherical Moufang building. By Proposition (2.18) the building Δ_+ conforms to the blueprint given by its restriction to $E_2(c)$ and the natural labeling of Δ_+ , i.e. $(U_{\alpha_s} \cup \{\infty_s\})_{s \in J}, (R_{\{s,t\}}(c))_{s \neq t \in J}$, where the $U_{\pm\alpha_s}$ are the root groups corresponding to the roots in $\Sigma \cap \Delta_+$. We will show that Δ_+ conforms to $\mathcal{B}_{(\mathcal{F}_J)}$. As we have a labeling of Δ of type $(U_{\alpha_s} \cup \{\infty_s\})_{s \in J}$ and $U_s \rightarrow U_{\alpha_s}$ are isomorphisms, we also have a labeling of type $(U_s \cup \{\infty_s\})_{s \in J}$. Let R be an $\{s, t\}$ -residue of Δ_+ . Then there exists an isomorphism $\varphi_R : R_{\{s,t\}}(c) \rightarrow R$ such that $x, \varphi_R(x)$ have the same s - and t -labels for each $x \in R_{\{s,t\}}(c)$. As $\mathcal{F}_J \cong \mathcal{F}(\Delta, c)$, there exist isomorphisms $\alpha_K : \Delta_K \rightarrow R_K(c)$ for each $K \in E(J)$. Then x and $\alpha_{\{s,t\}}(x)$ have the same s - and t -labels and hence Δ_+ conforms to the blueprint $\mathcal{B}_{\mathcal{F}_J}$. In particular, $\mathcal{B}_{(\mathcal{F}_J)}$ is realizable. \square

(4.2) Theorem. *The blueprint $\mathcal{B}_{\mathcal{F}}$ is realizable.*

Proof. Let $J \subseteq S$ be of rank 3. Then J is spherical by assumption. If J is irreducible, then \mathcal{F}_J is integrable and hence $\mathcal{B}_{(\mathcal{F}_J)}$ is realizable by Lemma (4.1). If J is reducible, then $\mathcal{B}_{(\mathcal{F}_J)}$ is realizable by Lemma (2.20). Thus each restriction to a spherical rank 3 subdiagram is realizable and hence the claim follows from Corollary (2.17). \square

Recall that $H_s = \langle m(u)m(v) \mid u, v \in U_s \setminus \{1\} \rangle$ and $B_J = \langle H_s, U_s \mid s \in J \rangle$ for $J \subseteq S$ seen as subgroups of X_s and $X_{s,t}$ (and not of G). Then we have $u_t^{h^{-1}} \in U_t$ for $u_t \in U_t, h \in H_s$ and $u_s^{n_t}, [u_s, u_t]^{n_t} \in B_{\{s,t\}}$ for $s \neq t \in S, u_s \in U_s, u_t \in U_t$.

4.1. An action associated with left multiplication

(4.3) Theorem. *Let $s \neq t \in S$. Then $B_{\{s,t\}}$ acts on the set of sequences of $\mathbf{C}(\mathcal{B}_{\mathcal{F}})$ as a permutation group as follows: Let $s_1, \dots, s_k \in S$ be such that $s_1 \cdots s_k$ is reduced and let $u_i \in U_{s_i}$. Let $r \in \{s, t\}$ and $g \in U_r \cup H_r$. Then $\omega_B(g, ()) := ()$ and for $k > 0$ we have*

$$\omega_B(g, (u_1, \dots, u_k)) := \begin{cases} (u_1^{g^{-1}}, \omega_B(g^{n_{s_1}}, (u_2, \dots, u_k))) & \text{if } g \in H_r, \\ (gu_1, u_2, \dots, u_k) & \text{if } r = s_1 \text{ and } g \in U_r, \\ (u_1, \omega_B(g^{n_{s_1}}[g, u_1]^{n_{s_1}}, (u_2, \dots, u_k))) & \text{if } r \neq s_1 \text{ and } g \in U_r. \end{cases}$$

Proof. We have to show that every relation in $B_{\{s,t\}}$ fixes (u_1, \dots, u_k) . We prove the hypothesis by induction on k . For $k = 0$ there is nothing to show. Thus we assume $k \geq 1$. We consider a relation in $B_{\{s,t\}} = \langle H_s \cup H_t \rangle \rtimes \langle U_s \cup U_t \rangle$ (note that H_s, H_t normalize U_s, U_t and their intersection is trivial by [2, Lemma 7.62]). We distinguish the following cases:

Relations in $\langle H_s \cup H_t \rangle$: Suppose $h_1, \dots, h_m \in \langle H_s \cup H_t \rangle$ with $h_1 \cdots h_m = 1$ in $\langle H_s \cup H_t \rangle$. Then $\omega_B(h_1 \cdots h_m, (u_1, \dots, u_k)) = (u_1^{(h_1 \cdots h_m)^{-1}}, \omega_B(\prod_{i=1}^m h_i^{n_{s_1}}, (u_2, \dots, u_k)))$. We have $u_1^{(h_1 \cdots h_m)^{-1}} = u_1$ and $\prod_{i=1}^m h_i^{n_{s_1}} = (h_1 \cdots h_m)^{n_{s_1}}$ is a relation in $G_{\{s,t,s_1\}}$. As the product is contained in $B_{\{s,t,s_1\}}$, it is a relation in $B_{\{s,t,s_1\}}$. Note that $(\mathcal{D}_{\mathcal{F}})_{\{s,t,s_1\}}$ is an RGD-system by Lemma (3.15)(b) or Lemma (3.16). Using Lemma (2.13) we know that $B_{\{s,t,s_1\}}$ is the 2-amalgam of the groups $B_{\{r\}}$ with $r \in \{s, t, s_1\}$. By the universal property of direct limits, we have

$$\prod_{i=1}^m h_i^{n_{s_1}} = \prod_{i=1}^n g_i^{-1} r_i g_i$$

for some $g_i \in B_{\{s,t,s_1\}}$ and relations $r_i \in B_{\{s,t\}} \cup B_{\{s,s_1\}} \cup B_{\{t,s_1\}}$ in $B_{\{s,t,s_1\}}$. Using induction, we already know that all relations in $B_{\{s,t\}}, B_{\{s,s_1\}}, B_{\{t,s_1\}}$ act trivial on sequences of length at most $k - 1$. In particular, the group $B_{\{s,t,s_1\}}$ acts on all sequences of length at most $k - 1$. Using induction, we obtain the following:

$$\omega_B(h_1 \cdots h_m, (u_1, \dots, u_k)) = (u_1, \omega_B(\prod_{i=1}^n g_i^{-1} r_i g_i, (u_2, \dots, u_k))) = (u_1, \dots, u_k)$$

Relations in $\langle U_s \cup U_t \rangle$: Suppose $v_1, \dots, v_m \in U_s, w_1, \dots, w_m \in U_t$ such that $\prod_{i=1}^m v_i w_i = 1$ in $\langle U_s \cup U_t \rangle$. We distinguish the following cases:

- (Case I) $s_1 \notin \{s, t\}$: Then $\omega_B(\prod_{i=1}^m v_i w_i, (u_1, \dots, u_k)) = (u_1, \omega_B(b, (u_2, \dots, u_k)))$, where $b = \prod_{i=1}^m v_i^{n_{s_1}} [v_i, u_1]^{n_{s_1}} w_i^{n_{s_1}} [w_i, u_1]^{n_{s_1}}$, and $b = (\prod_{i=1}^m v_i w_i)^{u_1^{n_{s_1}}}$ is a relation in $G_{\{s,t,s_1\}}$ and hence in $B_{\{s,t,s_1\}}$. As before, the claim follows by induction.
- (Case II) $s_1 \in \{s, t\}$: Without loss of generality we can assume $s_1 = t$. Then we compute $\omega_B(\prod_{i=1}^m v_i w_i, (u_1, \dots, u_k)) = (w_1 \cdots w_m u_1, \omega_B(\prod_{i=1}^m v_i^{n_{s_1}} [v_i, w_i \cdots w_m u_1]^{n_{s_1}}, (u_2, \dots, u_k)))$. Consider now the action of U_s, U_t on $\Delta_{\{s,t\}}$. Since $\prod_{i=1}^m v_i w_i$ is a relation in $\langle U_s \cup U_t \rangle$ and U_s fixes the t -panel containing a fundamental chamber, the element $w_1 \cdots w_m \in U_t$ acts also trivial on this t -panel. Since the action of U_t on this panel is simply transitive, we deduce $w_1 \cdots w_m = 1$. This yields that:

$$\begin{aligned} \prod_{i=1}^m v_i^{n_{s_1}} [v_i, w_i \cdots w_m u_1]^{n_{s_1}} &= \left(\prod_{i=1}^m (w_i \cdots w_m u_1)^{-1} v_i w_i \cdots w_m u_1 \right)^{n_{s_1}} \\ &= \left(\prod_{i=1}^m (w_i \cdots w_m)^{-1} v_i w_i \cdots w_m \right)^{u_1 n_{s_1}} \\ &= \left((w_1 \cdots w_m)^{-1} \prod_{i=1}^m v_i w_i \right)^{u_1 n_{s_1}} \end{aligned}$$

is a relation in $G_{\{s,t\}}$ and hence in $B_{\{s,t\}}$. The claim follows by induction.

It remains to show that for all $x, y \in \{s, t\}, h \in H_x, v \in U_y, v' := h^{-1}v^{-1}h = (h^{-1}vh)^{-1} \in U_y$ the element $h^{-1}vhv'$ fixes (u_1, \dots, u_k) . We distinguish the following cases:

- (Case I) $y = s_1$: Then $\omega_B(h^{-1}vhv', (u_1, \dots, u_k)) = \left((v(v'u_1)^{h^{-1}})^h, \omega_B((h^{-1})^{n_{s_1}} h^{n_{s_1}}, (u_2, \dots, u_k)) \right)$. As $(v(v'u_1)^{h^{-1}})^h = v^h v' u_1 = u_1$ in U_y and $(h^{-1})^{n_{s_1}} h^{n_{s_1}}$ is a relation in $G_{\{y\}}$ and hence in $B_{\{y\}}$, the claim follows by induction.
- (Case II) $y \neq s_1$: Then $\omega_B(h^{-1}vhv', (u_1, \dots, u_k)) = ((u_1^{h^{-1}})^h, \omega_B(b, (u_2, \dots, u_k)))$, where $b = (h^{-1})^{n_{s_1}} v^{n_{s_1}} [v, u_1^{h^{-1}}]^{n_{s_1}} h^{n_{s_1}} (v')^{n_{s_1}} [v', u_1]^{n_{s_1}}$, and $b = (h^{-1}vhv')^{u_1 n_{s_1}}$ is a relation in $G_{\{s,t,s_1\}}$ and hence in $B_{\{s,t,s_1\}}$. As before, the claim follows by induction. \square

(4.4) Lemma. *Let $J \subseteq S$ be such that $|J| \leq 3$. Let $u_i, u'_i \in \bigcup_{j \in J} U_j$.*

- (a) *Let $g \in B_J$ be such that $\omega_B(g, (u_1, \dots, u_k)) = (u'_1, \dots, u'_k)$. Then $g \cdot u_1 n_1 \cdots u_k n_k B_J = u'_1 n_1 \cdots u'_k n_k B_J$.*
- (b) *Let $g_1, \dots, g_m \in \bigcup_{j \in J} U_j$. Assume that the action ω_B does not stop after r steps, i.e. $\omega_B(g_1 \cdots g_m, (u_1, \dots, u_k)) = (u'_1, \dots, u'_r, \omega_B(g', (u_{r+1}, \dots, u_k)))$. Then we have $g' = (u'_1 n_1 \cdots u'_r n_r)^{-1} g (u_1 n_1 \cdots u_r n_r)$ in G_J , where $g = g_1 \cdots g_m$.*

Proof. We prove Assertion (a) by induction on k . For $k = 0$ the claim follows directly, as $\omega_B(g, ()) = ()$ and $g \cdot B_J = B_J$. Thus we assume $k \geq 1$. It suffices to show the claim for $g \in U_s \cup H_s$ with $s \in J$. If $g \in H_s$, we have $\omega_B(g, (u_1, \dots, u_k)) = (u_1^{g^{-1}}, \omega_B(g^{n_1}, (u_2, \dots, u_k)))$. Let $\omega_B(g^{n_1}, (u_2, \dots, u_k)) = (u'_2, \dots, u'_k)$. Using induction we have $g^{n_1} \cdot u_2 n_2 \cdots u_k n_k B_J = u'_2 n_2 \cdots u'_k n_k B_J$. This implies

$$g \cdot u_1 n_1 \cdots u_k n_k B_J = u_1^{g^{-1}} n_1 g^{n_1} u_2 n_2 \cdots u_k n_k B_J = u_1^{g^{-1}} n_1 u'_2 n_2 \cdots u'_k n_k B_J$$

As $\omega_B(g, (u_1, \dots, u_k)) = (u_1^{g^{-1}}, u'_2, \dots, u'_k)$, the claim follows. Now we suppose $g \in U_s$ for some $s \in S$. If $u_1 \in U_s$, then $\omega_B(g, (u_1, \dots, u_k)) = (g u_1, u_2, \dots, u_k)$ and $g \cdot u_1 n_1 \cdots u_k n_k B_J = (g u_1) n_1 u_2 n_2 \cdots u_k n_k B_J$. Thus we assume $u_1 \in U_t$ for some

$s \neq t \in S$. Then we have $\omega_B(g, (u_1, \dots, u_k)) = (u_1, \omega_B(g^{n_1}[g, u_1]^{n_1}, (u_2, \dots, u_k)))$. Let $\omega_B(g^{n_1}[g, u_1]^{n_1}, (u_2, \dots, u_k)) = (u'_2, \dots, u'_k)$. Using induction we have $g^{n_1}[g, u_1]^{n_1} \cdot u_2 n_2 \cdots u_k n_k B_J = u'_2 n_2 \cdots u'_k n_k B_J$. This implies

$$gu_1 n_1 \cdots u_k n_k B_J = u_1 n_1 g^{n_1}[g, u_1]^{n_1} u_2 n_2 \cdots u_k n_k B_J = u_1 n_1 u'_2 n_2 \cdots u'_k n_k B_J$$

As $\omega_B(g, (u_1, \dots, u_k)) = (u_1, u'_2, \dots, u'_k)$, the claim follows. We prove Assertion (b) by induction on r . Let $j \in J$ be such that $u_1 \in U_j$. Let $i_1 < \dots < i_n$ be all indices such that $g_{i_1}, \dots, g_{i_n} \in U_j$. Then we have the following in G_J :

$$\omega_B(g, (u_1, \dots, u_k)) = (g_{i_1} \cdots g_{i_n} u_1, \omega_B(g', (u_2, \dots, u_k)))$$

where $g' = \prod_{x=0}^n \prod_{y=i_{x+1}}^{i_{x+1}-1} g_y^{n_j} [g_y, g_{i_{x+1}} \cdots g_{i_n} u_1]^{n_j}$ and $i_0 := 0, i_{n+1} := m + 1$. We compute the following:

$$\begin{aligned} g' &= \prod_{x=0}^n \prod_{y=i_{x+1}}^{i_{x+1}-1} g_y^{n_j} [g_y, g_{i_{x+1}} \cdots g_{i_n} u_1]^{n_j} \\ &= \prod_{x=0}^n \prod_{y=i_{x+1}}^{i_{x+1}-1} ((g_{i_{x+1}} \cdots g_{i_n} u_1)^{-1} g_y (g_{i_{x+1}} \cdots g_{i_n} u_1))^{n_j} \\ &= \prod_{x=0}^n \left((g_{i_{x+1}} \cdots g_{i_n} u_1)^{-1} \left(\prod_{y=i_{x+1}}^{i_{x+1}-1} g_y \right) (g_{i_{x+1}} \cdots g_{i_n} u_1) \right)^{n_j} \\ &= \left((g_{i_1} \cdots g_{i_n} u_1)^{-1} \left(\prod_{y=1}^m g_y \right) u_1 \right)^{n_j} \\ &= (g_{i_1} \cdots g_{i_n} u_1 n_1)^{-1} g u_1 n_1 \end{aligned}$$

As $g' \in \langle U_j \mid j \in J \rangle$, the claim follows by induction. \square

(4.5) Lemma. *The action in Theorem (4.3) maps equivalent sequences to equivalent sequences. In particular, ω_B extends to an action on the set of chambers of the building \mathbf{C}_{B_F} as a permutation group.*

Proof. In this proof we use the notation $U_{s_1 \dots s_k}$ for $U_{s_1} \times \cdots \times U_{s_k}$. It suffices to show that every element in $\bigcup_{r \in S} U_r \cup H_r$ maps two elementary equivalent sequences to elementary equivalent sequences. Let $s \neq t \in S$, let $\bar{u} = \bar{u}_1 \bar{u}_0 \bar{u}_2$ be of type $(f_1, p(s, t), f_2)$ and $\bar{v} = \bar{v}_1 \bar{v}_0 \bar{v}_2$ be of type $(f_1, p(t, s), f_2)$, where \bar{u}_0 and \bar{v}_0 are equivalent in $\Delta_{\{s, t\}}$. It suffices to show the claim for the sequences $\bar{u}_0 \bar{u}_2$ of type $(p(s, t), f_2)$ and $\bar{v}_0 \bar{v}_2$ of type $(p(t, s), f_2)$. Suppose $w \in W$ and $s \neq t \in S$ with $\ell(sw) = \ell(w) + 1 = \ell(tw)$. Let $(u_1, \dots, u_m) \in U_{p(s, t)}, (v_1, \dots, v_m) \in U_{p(t, s)}$ be equivalent in $\Delta_{\{s, t\}}$, let $s_1, \dots, s_k \in S$ be such that $w = s_1 \cdots s_k$ is reduced and let $(w_1, \dots, w_k) \in U_{s_1 \dots s_k}$. Then $(u_1, \dots, u_m, w_1, \dots, w_k)$

and $(v_1, \dots, v_m, w_1, \dots, w_k)$ are elementary equivalent. First we assume that $g \in H_r$ for some $r \in S$. Then we have the following:

$$\begin{aligned} \omega_B(g, (u_1, \dots, u_m, w_1, \dots, w_k)) &= (u'_1, \dots, u'_m, \omega_B(g^{p(n_s, n_t)}, (w_1, \dots, w_k))) \\ \omega_B(g, (v_1, \dots, v_m, w_1, \dots, w_k)) &= (v'_1, \dots, v'_m, \omega_B(g^{p(n_t, n_s)}, (w_1, \dots, w_k))) \end{aligned}$$

Let $J := \{r, s, t\}$. We deduce from Lemma (4.4)(a) and Corollary (2.19) that

$$u'_1 n_1 \cdots u'_m n_m B_J = g \cdot u_1 n_1 \cdots u_m n_m B_J = g \cdot v_1 n'_1 \cdots v_m n'_m B_J = v'_1 n'_1 \cdots v'_m n'_m B_J$$

Using Corollary (2.19) again, we deduce that $(u'_1, \dots, u'_m) \simeq (v'_1, \dots, v'_m)$. Since $p(n_s, n_t) = p(n_t, n_s)$ is a relation in $G_{\{r, s, t\}}$, we obtain $g^{p(n_s, n_t)} = g^{p(n_t, n_s)}$ in $B_{\{r, s, t\}}$ and by Lemma (2.13) they act equally on (w_1, \dots, w_k) .

Now let $g \in U_r$ for some $r \in S$. We note that $(u_1, \dots, u_k) \simeq (v_1, \dots, v_k)$ for $u_i, v_i \in U_s \cup U_t$ implies $u_1 n_1 \cdots u_k n_k = v_1 n'_1 \cdots v_k n'_k$ (cf. Corollary (2.19) and [16, p. 87]). We consider the following two cases:

(Case I) $r \in \{s, t\}$: W.l.o.g. we assume that $r = s$. Then

$$\omega_B(g, (u_1, \dots, u_m, w_1, \dots, w_k)) = (gu_1, u_2, \dots, u_m, w_1, \dots, w_k).$$

Let v'_1, \dots, v'_m be such that $\omega_B(g, (v_1, \dots, v_k)) = (v'_1, \dots, v'_k)$. By Lemma (4.4)(a) and Corollary (2.19) we deduce $(gu_1, u_2, \dots, u_k) \simeq (v'_1, \dots, v'_k)$. Now there are two possibilities:

- (a) $\omega_B(g, (v_1, \dots, v_m, w_1, \dots, w_k)) = (v'_1, \dots, v'_m, w_1, \dots, w_k)$, i.e. the action stops after at most m steps. Then the claim follows directly.
- (b) $\omega_B(g, (v_1, \dots, v_k, w_1, \dots, w_k)) = (v'_1, \dots, v'_m, \omega_B(g', w_1, \dots, w_k))$: Then by Lemma (4.4)(b) we deduce $g' = (v'_1 n'_1 \cdots v'_m n'_m)^{-1} g (v_1 n_1 \cdots v_m n_m)$ in $G_{\{s, t\}}$. But then we have the following in $G_{\{s, t\}}$:

$$\begin{aligned} g' &= (v'_1 n'_1 \cdots v'_m n'_m)^{-1} g (v_1 n_1 \cdots v_m n_m) \\ &= (gu_1 n_1 u_2 n_2 \cdots u_m n_m)^{-1} g (u_1 n_1 \cdots u_k n_k) \\ &= 1 \end{aligned}$$

Thus $\omega_B(g', (w_1, \dots, w_k)) = (w_1, \dots, w_k)$ and the claim follows.

(Case II) $r \notin \{s, t\}$: Suppose that $\omega_B(g, (u_1, \dots, u_k)) = (u'_1, \dots, u'_k)$ and $\omega_B(g, (v_1, \dots, v_k)) = (v'_1, \dots, v'_k)$. Then Lemma (4.4)(a) implies

$$\begin{aligned} g \cdot u_1 n_1 \cdots u_m n_m B_{\{r, s, t\}} &= u'_1 n_1 \cdots u'_m n_m B_{\{r, s, t\}} \\ g \cdot v_1 n'_1 \cdots v_m n'_m B_{\{r, s, t\}} &= v'_1 n'_1 \cdots v'_m n'_m B_{\{r, s, t\}} \end{aligned}$$

In particular, we deduce that

$$g' := (u'_1 n_1 \cdots u'_m n_m)^{-1} g u_1 n_1 \cdots u_m n_m \in B_{\{r,s,t\}}$$

$$g'' := (v'_1 n'_1 \cdots v'_m n'_m)^{-1} g v_1 n'_1 \cdots v_m n'_m \in B_{\{r,s,t\}}$$

Assume that $\omega_B(g, (u_1, \dots, u_m, w_1, \dots, w_k)) = (u'_1, \dots, u'_m, w_1, \dots, w_k)$, i.e. the action stops after at most m steps. Then $g u_1 n_1 \cdots u_m n_m = u'_1 n_1 \cdots u'_m n_m$ and g' is a relation in $G_{\{r,s,t\}}$. We will show that this is a contradiction. Let $u, u' \in \langle U_{\alpha_s} \cup U_{\alpha_t} \rangle$ be such that $u_1 n_1 \cdots u_m n_m = un(w)$ and $u'_1 n_1 \cdots u'_m n_m = u'n(w)$ as in [16, Lemma (7.4)]. Then $(u')^{-1}gu$ is a relation as well and we deduce $U_{\alpha_r} \cap \langle U_{\alpha_s} \cup U_{\alpha_t} \rangle \neq \{1\}$, which is a contradiction. Thus g' cannot be a relation. Moreover, we have $u_1 n_1 \cdots u_m n_m = v_1 n'_1 \cdots v_m n'_m$, as $(u_1, \dots, u_m) \simeq (v_1, \dots, v_m)$, and $u'_1 n_1 \cdots u'_m n_m = v'_1 n'_1 \cdots v'_m n'_m$, as $(u'_1, \dots, u'_m) \simeq (v'_1, \dots, v'_m)$ by Corollary (2.19). In particular, we have $g' = g''$ in $B_{\{r,s,t\}}$ and g'' is no relation as well. In particular, the action does not stop after at most m steps. Together with Theorem (4.3) we obtain the following:

$$\begin{aligned} \omega_B(g, (u_1, \dots, u_m, w_1, \dots, w_k)) &= (u'_1, \dots, u'_m, \omega_B(g', (w_1, \dots, w_m))) \\ &\simeq (v'_1, \dots, v'_m, \omega_B(g'', (w_1, \dots, w_m))) \\ &= \omega_B(g, (v_1, \dots, v_m, w_1, \dots, w_k)). \quad \square \end{aligned}$$

(4.6) Lemma. *Let $s_1, \dots, s_k \in S$ be such that $s_1 \cdots s_k$ is reduced and let $u_i \in U_{s_i}$. Then we have for each $s \in S$ a well-defined mapping*

$$\omega(n_s, \cdot) : \mathbf{C}(\mathcal{B}_{\mathcal{F}}) \rightarrow \mathbf{C}_{\mathcal{B}_{\mathcal{F}}},$$

$$(u_1, \dots, u_k) \mapsto \begin{cases} \omega_B(n_{s_1}^2, [u_2, \dots, u_k]) & \text{if } s = s_1, u_1 = 1, \\ [\bar{u}_1, \omega_B(b(u_1), [u_2, \dots, u_k])] & \text{if } s = s_1, u_1 \neq 1, \\ [1_{U_s}, u_1, \dots, u_k] & \text{if } \ell(ss_1 \cdots s_k) = k + 1 \\ \omega(n_s, (v_1, \dots, v_k)) & \text{else,} \end{cases}$$

where (v_1, \dots, v_k) is of type (s, t_2, \dots, t_k) , $st_2 \cdots t_k = s_1 \cdots s_k$ and $(u_1, \dots, u_k) \simeq (v_1, \dots, v_k)$. Moreover, this mapping extends to a mapping on the set of chambers of $\mathbf{C}_{\mathcal{B}_{\mathcal{F}}}$. More precisely, for any two equivalent sequences $(u_1, \dots, u_k) \simeq (v_1, \dots, v_k)$ we have $\omega(n_s, (u_1, \dots, u_k)) = \omega(n_s, (v_1, \dots, v_k))$.

Proof. It suffices to show that the mapping maps equivalent sequences to the same chamber in the building. Suppose $s_1, \dots, s_k, t_1, \dots, t_k \in S$ such that $s_1 \cdots s_k = t_1 \cdots t_k$ is reduced. Let $u_i \in U_{s_i}, v_i \in U_{t_i}$ be such that $(u_1, \dots, u_k) \simeq (v_1, \dots, v_k)$. If $\ell(ss_1 \cdots s_k) = k + 1$, we have $(1, u_1, \dots, u_k) \simeq (1, v_1, \dots, v_k)$ and hence $[1, u_1, \dots, u_k] = [1, v_1, \dots, v_k]$. Now we assume $\ell(ss_1 \cdots s_k) = k - 1$ and distinguish the following cases:

- (a) $s_1 = s = t_1$: As $s_2 \cdots s_k = t_2 \cdots t_k$, there exists $v'_i \in U_{s_i}$ with $(v'_2, \dots, v'_k) \simeq (v_2, \dots, v_k)$. Moreover, we have $(u_1, \dots, u_k) \simeq (v_1, \dots, v_k) \simeq (v_1, v'_2, \dots, v'_k)$ and Corollary (2.17) implies $u_1 = v_1$ and $u_i = v'_i$ for each $i \in \{2, \dots, k\}$. In particular, $(u_2, \dots, u_k) = (v'_2, \dots, v'_k) \simeq (v_2, \dots, v_k)$. Using Lemma (4.5) we deduce

$$\begin{aligned} \omega(n_s, (v_1, \dots, v_k)) &= \begin{cases} \omega_B(n_{s_1}^2, [v_2, \dots, v_k]) & \text{if } v_1 = 1 \\ [\bar{v}_1, \omega_B(b(v_1), [v_2, \dots, v_k])] & \text{if } v_1 \neq 1 \end{cases} \\ &= \begin{cases} \omega_B(n_{s_1}^2, [w_2, \dots, w_k]) & \text{if } w_1 = 1 \\ [\bar{w}_1, \omega_B(b(w_1), [w_2, \dots, w_k])] & \text{if } w_1 \neq 1 \end{cases} \\ &= \omega(n_s, (w_1, \dots, w_k)) \end{aligned}$$

- (b) All other cases can be reduced to the previous case by choosing suitable equivalent sequences. \square

(4.7) Lemma. *Let $s \in S$ and $h_s := n_s^2 \in H_s$. Moreover, we let $s_1, \dots, s_k \in S$ be such that $s_1 \cdots s_k$ is reduced and we let $u_i \in U_{s_i}$. Then the following hold:*

$$\omega(n_s, \omega_B(h_s, [u_1, \dots, u_k])) = \omega_B(h_s, \omega(n_s, [u_1, \dots, u_k]))$$

Proof. We distinguish the following cases:

- (a) $s = s_1, u_1 = 1$: Then $h_s^{n_s} = h_s$ is a relation in $B_{\{s\}}$ and the following holds:

$$\begin{aligned} \omega(n_s, \omega_B(h_s, [u_1, \dots, u_k])) &= \omega(n_s, [1, \omega_B(h_s^{n_s}, [u_2, \dots, u_k])]) \\ &= \omega_B((n_s^2)h_s, [u_2, \dots, u_k]) \\ &= \omega_B(h_s, \omega_B(n_s^2, [u_2, \dots, u_k])) \\ &= \omega_B(h_s, \omega(n_s, [1, u_2, \dots, u_k])) \end{aligned}$$

- (b) $s = s_1, u_1 \neq 1$: Note that $\overline{u_1^{h_s^{-1}}} n_s B_{\{s\}} = \overline{u_1^{h_s^{-1}}} n_s b(u_1^{h_s^{-1}}) h_s B_{\{s\}} = n_s u_1^{h_s^{-1}} n_s h_s B_{\{s\}} = n_s h_s u_1 n_s B_{\{s\}} = h_s n_s u_1 n_1 B_{\{s\}} = h_s \overline{u_1} n_s b(u_1) B_{\{s\}} = \overline{u_1}^{h_s^{-1}} n_s h_s b(u_1) B_{\{s\}}$. Thus we have $\overline{u_1^{h_s^{-1}}} = \overline{u_1}^{h_s^{-1}}$ in U_s and $b(u_1^{h_s^{-1}}) h_s = h_s^{n_s} b(u_1)$ is a relation in $B_{\{s\}}$. Moreover, $h_s^{n_s} = h_s$ in $B_{\{s\}}$. Thus the following holds:

$$\begin{aligned} \omega(n_s, \omega_B(h_s, [u_1, \dots, u_k])) &= \omega(n_s, [u_1^{h_s^{-1}}, \omega_B(h_s^{n_s}, [u_2, \dots, u_k])]) \\ &= [\overline{u_1^{h_s^{-1}}}, \omega_B(b(u_1^{h_s^{-1}}) h_s, [u_2, \dots, u_k])] \\ &= [\overline{u_1}^{h_s^{-1}}, \omega_B(h_s^{n_s} b(u_1), [u_2, \dots, u_k])] \\ &= \omega_B(h_s, [\overline{u_1}, \omega_B(b(u_1), [u_2, \dots, u_k])]) \end{aligned}$$

$$= \omega_B(h_s, \omega(n_s, [u_1, \dots, u_k]))$$

(c) $\ell(ss_1 \cdots s_k) = k + 1$: Then $h_s^{n_s} = h_s$ is a relation in $B_{\{s\}}$ and the following holds:

$$\begin{aligned} \omega(n_s, \omega_B(h_s, [u_1, \dots, u_k])) &= [1_{U_s}, \omega_B(h_s^{n_s}, [u_1, \dots, u_k])] \\ &= \omega_B(h_s, [1_{U_s}, u_1, \dots, u_k]) \\ &= \omega_B(h_s, \omega(n_s, [u_1, \dots, u_k])) \end{aligned}$$

(d) $s \neq s_1$ and $\ell(ss_1 \cdots s_k) = k - 1$: Let $(u_1, \dots, u_k) \simeq (v_1, \dots, v_k)$ be with $v_1 \in U_s$. Using Lemma (4.5) we deduce $\omega_B(h_s, [u_1, \dots, u_k]) = \omega_B(h_s, [v_1, \dots, v_k])$. Using the cases (a) and (b) we have the following:

$$\begin{aligned} \omega(n_s, \omega_B(h_s, [u_1, \dots, u_k])) &= \omega(n_s, \omega_B(h_s, [v_1, \dots, v_k])) \\ &= \omega_B(h_s, \omega(n_s, [v_1, \dots, v_k])) \\ &= \omega_B(h_s, \omega(n_s, [u_1, \dots, u_k])) \quad \square \end{aligned}$$

(4.8) Lemma. For each $s \in S$ we have $n_s \in \text{Sym}(\mathbf{C}_{B_{\mathcal{F}}})$ via

$$\omega(g, [u_1, \dots, u_k]) := \begin{cases} \omega(n_s, \omega_B(n_s^{-2}, [u_1, \dots, u_k])) & \text{if } g = n_s^{-1} \\ \omega(n_s, [u_1, \dots, u_k]) & \text{if } g = n_s \end{cases}$$

Proof. It suffices to show that $n_s^{-1}n_s$ fixes every sequence. Once this is done, we deduce the following:

$$\begin{aligned} \omega(n_s n_s^{-1}, [u_1, \dots, u_k]) &= \omega(n_s n_s, \omega_B(n_s^{-2}, [u_1, \dots, u_k])) \\ &= \omega(n_s (n_s^2) (n_s^{-2}) n_s, \omega_B(n_s^{-2}, [u_1, \dots, u_k])) \\ &= \omega((n_s^2) n_s (n_s^{-2}) n_s, \omega_B(n_s^{-2}, [u_1, \dots, u_k])) \\ &= \omega((n_s^2), \omega_B(n_s^{-2}, [u_1, \dots, u_k])) \\ &= [u_1, \dots, u_k] \end{aligned}$$

Now we will show that $n_s^{-1}n_s$ fixes every sequence. Let $s_1, \dots, s_k \in S$ be such that $s_1 \cdots s_k$ is reduced and let $u_i \in U_{s_i}$. We distinguish the following cases:

(i) $s = s_1$ and $u_1 = 1$: Then $\omega(n_s^{-1}n_s, [u_1, \dots, u_k]) = [1, \omega_B(n_s^{-2}n_s^2, [u_2, \dots, u_k])] = [u_1, \dots, u_k]$.

(ii) $s = s_1$ and $u_1 \neq 1$: Then

$$\begin{aligned} \omega(n_s^{-1}n_s, [u_1, \dots, u_k]) &= \omega(n_s^{-1}, [\bar{u}_1, \omega_B(b(u_1), [u_2, \dots, u_k])]) \\ &= \omega(n_s, [\bar{u}_1^{n_s^2}, \omega_B((n_s^{-2})^{n_s} b(u_1), [u_2, \dots, u_k])]) \end{aligned}$$

$$= [\overline{u_1^{n_s^2}}, \omega_B(b(\overline{u_1^{n_s^2}})(n_s^{-2})^{n_s}b(u_1)), [u_2, \dots, u_k]]$$

We note that $u_1 n_s B_{\{s\}} = n_s^{-1} n_s u_1 n_s B_{\{s\}} = n_s n_s^{-2} \overline{u_1} n_s b(u_1) B_{\{s\}} = n_s \overline{u_1^{n_s^2}} n_s B_{\{s\}} = \overline{u_1^{n_s^2}} n_s B_{\{s\}}$. Thus $u_1 = \overline{u_1^{n_s^2}}$ and $b(\overline{u_1^{n_s^2}})(n_s^{-2})^{n_s}b(u_1)$ is a relation in $B_{\{s\}}$. The claim follows now from Theorem (4.3).

- (iii) $\ell(ss_1 \cdots s_k) = k + 1$: Then $\omega(n_s^{-1} n_s, [u_1, \dots, u_k]) = \omega_B(n_s^2 (n_s^{-2})^{n_s}, [u_1, \dots, u_k]) = [u_1, \dots, u_k]$.
- (iv) $\ell(ss_1 \cdots s_k) = k - 1$ and $s \neq s_1$: Let $(v_1, \dots, v_k) \in [u_1, \dots, u_k]$ with $v_1 \in U_s$. Then we have $\omega(n_s^{-1} n_s, [v_1, \dots, v_k]) = [v_1, \dots, v_k]$ as before and hence $\omega(n_s^{-1} n_s, [u_1, \dots, u_k]) = [v_1, \dots, v_k] = [u_1, \dots, u_k]$. \square

4.2. Some relations in $\text{Sym}(\mathbf{C}_{\mathcal{B}_{\mathcal{F}}})$

In this subsection we will show that the relations in Theorem (2.9) act trivial on the set of chambers of $\mathbf{C}_{\mathcal{B}_{\mathcal{F}}}$. This will imply that G acts on the set of chambers of the building $\mathbf{C}_{\mathcal{B}_{\mathcal{F}}}$.

(4.9) Lemma. *For $s \in S$ and $1 \neq u_s \in U_s$ We have $n_s u_s n_s = \overline{u_s} n_s b(u_s)$ in $\text{Sym}(\mathbf{C}_{\mathcal{B}_{\mathcal{F}}})$.*

Proof. Let (u_1, \dots, u_k) be a sequence of type (s_1, \dots, s_k) . We distinguish the following two cases:

- (a) $\ell(ss_1 \cdots s_k) = k + 1$. Then we have

$$\omega(n_s u_s n_s, [u_1, \dots, u_k]) = [\overline{u_s}, \omega_B(b(u_s), [u_1, \dots, u_k])] = \omega(\overline{u_s} n_s b(u_s), [u_1, \dots, u_k])$$

- (b) $\ell(ss_1 \cdots s_k) = k - 1$: W.l.o.g. we assume $s = s_1$. We let $b(u_s) = v_s h_s$ and distinguish the following cases:

- (i) $u_1 = 1$: Then we compute the following:

$$\begin{aligned} \omega(n_s u_s n_s, [1, u_2, \dots, u_k]) &= [1, \omega_B(u_s n_s^2, [u_2, \dots, u_k])] \\ \omega(\overline{u_s} n_s b(u_s), [1, u_2, \dots, u_k]) &= [\overline{u_s} \overline{v_s}, \omega_B(b(v_s) h_s^{n_s}, [u_2, \dots, u_k])] \end{aligned}$$

Comparing the cosets in the Moufang building we obtain $\overline{u_s} \overline{v_s} = 1$. Moreover, we have the following:

$$\begin{aligned} b(v_s) h_s^{n_s} &= n_s^{-1} \overline{v_s}^{-1} \overline{v_s} n_s b(v_s) n_s^{-1} h_s n_s = n_s^{-1} \overline{v_s}^{-1} \overline{u_s}^{-1} \overline{u_s} n_s v_s n_s n_s^{-1} h_s n_s \\ &= n_s^{-1} n_s u_s n_s n_s = u_s n_s^2 \end{aligned}$$

Thus $(u_s n_s^2)^{-1} b(v_s) h_s^{n_s}$ is a relation in $B_{\{s\}}$ and the claim follows.

(ii) $u_1 \neq 1$: Then we compute the following:

$$\begin{aligned} &\omega(n_s u_s n_s, [u_1, \dots, u_k]) \\ &= \begin{cases} \omega_B(n_s^2 b(u_1), [u_2, \dots, u_k]) & \text{if } u_s \bar{u}_1 = 1 \\ [\bar{u}_s \bar{u}_1, \omega_B(b(u_s \bar{u}_1) b(u_1), [u_2, \dots, u_k])] & \text{if } u_s \bar{u}_1 \neq 1 \end{cases} \\ &\omega(\bar{u}_s n_s b(u_s), [u_1, \dots, u_k]) \\ &= \begin{cases} \omega_B(\bar{u}_s n_s^2 h_s^{n_s}, [u_2, \dots, u_k]) & \text{if } v_s u_1^{h_s^{-1}} = 1 \\ [\bar{u}_s v_s u_1^{h_s^{-1}}, \omega_B(b(v_s u_1^{h_s^{-1}}) h_s^{n_s}, [u_2, \dots, u_k])] & \text{if } v_s u_1^{h_s^{-1}} \neq 1 \end{cases} \end{aligned}$$

Note that in X_s we have $n_s u_s \bar{u}_1 n_s b(u_1) = n_s u_s n_s u_1 n_s = \bar{u}_s n_s b(u_s) u_1 n_s = \bar{u}_s n_s v_s u_1^{h_s^{-1}} n_s h_s^{n_s}$. If $u_s \bar{u}_1 = 1$, then the left hand side is contained in $B_{\{s\}}$. If $v_s u_1^{h_s^{-1}} \neq 1$, then $U_{-\alpha_s} \cap B_{\{s\}} \neq \{1\}$, which is a contradiction. Thus $u_s \bar{u}_1 = 1$ implies $v_s u_1^{h_s^{-1}} = 1$. On the other hand if $v_s u_1^{h_s^{-1}} = 1$, then the right hand side is contained in $B_{\{s\}}$. If $u_s \bar{u}_1 \neq 1$, then again $U_{-\alpha_s} \cap B_{\{s\}} \neq \{1\}$, which is a contradiction. Thus $v_s u_1^{h_s^{-1}} = 1$ implies $u_s \bar{u}_1 = 1$.

Assume that $u_s \bar{u}_1 = 1 = v_s u_1^{h_s^{-1}}$. Then $n_s^2 b(u_1) = \bar{u}_s n_s^2 h_s^{n_s}$ in X_s and hence in $B_{\{s\}}$. Thus we are done. Now we assume that $u_s \bar{u}_1 \neq 1 \neq v_s u_1^{h_s^{-1}}$. Note that in X_s we have the following:

$$\begin{aligned} \overline{u_s \bar{u}_1} n_s b(u_s \bar{u}_1) b(u_1) &= n_s u_s \bar{u}_1 n_s b(u_1) \\ &= n_s u_s n_s u_1 n_s \\ &= \bar{u}_s n_s b(u_s) u_1 n_s \\ &= \bar{u}_s n_s v_s u_1^{h_s^{-1}} n_s h_s^{n_s} \\ &= \overline{u_s v_s u_1^{h_s^{-1}}} n_s b(v_s u_1^{h_s^{-1}}) h_s^{n_s} \end{aligned}$$

This implies $\overline{u_s \bar{u}_1} = \overline{u_s v_s u_1^{h_s^{-1}}}$ and hence $b(u_s \bar{u}_1) b(u_1) = b(v_s u_1^{h_s^{-1}}) h_s^{n_s}$. Thus we are done. \square

(4.10) Lemma. For $s, t \in S$ and $h \in H_t$ we have $n_s h n_s = n_s^2 h^{n_s}$ in $\text{Sym}(\mathbf{C}_{B_{\mathcal{F}}})$.

Proof. Let (u_1, \dots, u_k) be a sequence of type (s_1, \dots, s_k) . We distinguish the following cases:

- (a) $\ell(ss_1 \cdots s_k) = k + 1$: Then we have $\omega(n_s h n_s, [u_1, \dots, u_k]) = \omega_B(n_s^2 h^{n_s}, [u_1, \dots, u_k])$.
- (b) $\ell(ss_1 \cdots s_k) = k - 1$: Again we can assume $s = s_1$. We distinguish the cases $u_1 = 1$ and $u_1 \neq 1$ and compute the following:

$$\begin{aligned} \omega(n_s h n_s, [1, u_2, \dots, u_k]) &= [1, \omega_B(h n_s^2, [u_2, \dots, u_k])] \\ \omega(n_s^2 h^{n_s}, [1, u_2, \dots, u_k]) &= [1, \omega_B((n_s^2)^{n_s} h^{n_s^2}, [u_2, \dots, u_k])] \\ \omega(n_s h n_s, [u_1, \dots, u_k]) &= [\overline{u_1^{h^{-1}}}, \omega_B(b(\overline{u_1^{h^{-1}}}) h^{n_s} b(u_1), [u_2, \dots, u_k])] \\ \omega(n_s^2 h^{n_s}, [u_1, \dots, u_k]) &= [(u_1^{(h^{n_s})^{-1}})^{n_s^{-2}}, \omega_B((n_s^2)^{n_s} h^{n_s^2}, [u_2, \dots, u_k])] \end{aligned}$$

In the case $u_1 = 1$ we have $h n_s^2 = n_s^2 n_s^{-2} h n_s^2 = n_s^2 h^{n_s^2}$ and hence we are done. In the case $u_1 \neq 1$ we have $\overline{u_1^{h^{-1}}} = (u_1^{(h^{n_s})^{-1}})^{n_s^{-2}}$, as $\overline{u_1^{h^{-1}}} n_s B_{\{s\}} = (u_1^{(h^{n_s})^{-1}})^{n_s^{-2}} n_s B_{\{s\}}$ and hence $b(\overline{u_1^{h^{-1}}}) h^{n_s} b(u_1) = (n_s^2)^{n_s} h^{n_s^2}$. The claim follows now from Theorem (4.3). \square

(4.11) Lemma. For $s \neq t \in S$ and $\alpha_s \neq \alpha \in \Phi_+^{\{s,t\}}$ we have $n_s u_\alpha n_s = n_s^2 u_\alpha^{n_s}$ in $\text{Sym}(\mathbf{C}_{B_{\mathcal{F}}})$.

Proof. Let $\alpha_s \neq \alpha \in \Phi_+^{\{s,t\}}$. By Lemma (2.11) there exists $g_i \in U_t, h_i \in U_s$ such that $u_\alpha = \prod_{i=1}^n g_i h_i$. Note that $h_1 \cdots h_n = 1$ as in the proof of Theorem (4.3). We distinguish the following cases:

(a) $\ell(ss_1 \cdots s_k) = k + 1$: We deduce the following

$$\omega(n_s u_\alpha n_s, [u_1, \dots, u_k]) = \omega(n_s, [h_1 \cdots h_n, \omega_B(b, [u_1, \dots, u_k])]) = \omega_B(n_s^2 b, [u_1, \dots, u_k])$$

where $b = \prod_{i=1}^n g_i [g_i, h_i \cdots h_n]^{n_s}$. Note that $b = ((h_1 \cdots h_n)^{-1} \prod_{i=1}^n g_i h_i)^{n_s} = u_\alpha^{n_s}$ and hence the claim follows by Lemma (4.5).

(b) $\ell(ss_1 \cdots s_k) = k - 1$. W.l.o.g. we assume $s = s_1$. Let $g'_i \in U_t, h'_i \in U_s$ such that $u_\alpha^{n_s} = \prod_{i=1}^m g'_i h'_i$. Again, $h'_1 \cdots h'_m = 1$. We deduce the following:

(i) If $u_1 = 1$, we have:

$$\begin{aligned} \omega(n_s u_\alpha n_s, [u_1, \dots, u_k]) &= [1, \omega_B(u_\alpha n_s^2, [u_2, \dots, u_k])] \\ \omega(n_s^2 u_{s\alpha}, [u_1, \dots, u_k]) &= \omega(n_s^2, [h'_1 \cdots h'_m, \omega_B(b, [u_2, \dots, u_k])]) \\ &= [1, \omega_B((n_s^2)^{n_s} b, [u_2, \dots, u_k])] \end{aligned}$$

where $b = \prod_{i=1}^m (g'_i)^{n_s} [g'_i, h'_i \cdots h'_m]^{n_s}$. Note that $b = ((h'_1 \cdots h'_m)^{-1} \prod_{i=1}^m g'_i h'_i)^{n_s} = (u_\alpha^{n_s})^{n_s} = n_s^{-2} u_\alpha n_s^2$. Thus $(n_s^2)^{n_s} b = u_\alpha n_s^2$ in $B_{\{s,t\}}$ and the claim follows from Lemma (4.5).

(ii) If $u_1 \neq 1$, we compute the following:

$$\begin{aligned} \omega(n_s u_\alpha n_s, [u_1, \dots, u_k]) &= \omega(n_s u_\alpha, [\overline{u_1}, \omega_B(b(u_1), [u_2, \dots, u_k])]) \\ &= \omega(n_s, [h_1 \cdots h_n \overline{u_1}, \omega_B(b \cdot b(u_1), [u_2, \dots, u_k])]) \end{aligned}$$

$$\begin{aligned} &= [\bar{u}_1, \omega_B(b(\bar{u}_1) \cdot b \cdot b(u_1), [u_2, \dots, u_k])] \\ \omega(n_s^2 u_s \alpha, [u_1, \dots, u_k]) &= \omega(n_s^2, [h'_1 \cdots h'_m u_1, \omega_B(b', [u_2, \dots, u_k])]) \\ &= [u_1^{n_s-2}, \omega_B((n_s^2)^{n_s} b', [u_2, \dots, u_k])] \end{aligned}$$

where $b = \prod_{i=1}^n g_i^{n_s} [g_i, h_i \cdots h_m u_1]^{n_s}$ and $b' = \prod_{i=1}^m g_i'^{n_s} [g_i', h_i' \cdots h_m' u_1]^{n_s}$. As before, we have $\bar{u}_1 = u_1^{n_s-2}$ and $b(\bar{u}_1) \cdot b \cdot b(u_1) = (n_s^2)^{n_s} b'$ in $B_{\{s\}}$. Now the claim follows from Lemma (4.5). \square

(4.12) **Lemma.** For all $s \neq t \in S$ we have $p(n_s, n_t) = p(n_t, n_s)$ in $\text{Sym}(\mathbf{C}_{B_{\mathcal{F}}})$.

Proof. At first we transform some elements by only using relations which are already known to hold in $\text{Sym } \mathbf{C}_{B_{\mathcal{F}}}$, i.e. using Lemmas (4.9) - (4.11). We compute the following:

(a) $u = 1$ and $n_1 = n_t$: Then

$$\begin{aligned} n_s^{-1} n_t^{-1} n_s n_t n_t &= n_t n_t^{-1} n_s^{-1} n_t n_s (n_t^{-2})^{n_s} n_t^2 \\ n_s^{-1} n_t^{-1} n_s^{-1} n_t n_s n_t n_t &= n_t n_t^{-1} n_s^{-1} n_t^{-1} n_s n_t n_s (n_s^{-2})^{n_t n_s} n_t^2 \\ n_s^{-1} n_t^{-1} n_s^{-1} n_t^{-1} n_s n_t n_s n_t n_t &= n_t n_t^{-1} n_s^{-1} n_t^{-1} n_s^{-1} n_t n_s n_t n_s (n_t^{-2})^{n_s n_t n_s} n_t^2 \end{aligned}$$

(b) $U_t \ni u_t \neq 1$ and $n_1 = n_t$: Then

$$\begin{aligned} n_s^{-1} n_t^{-1} n_s n_t u_t n_t &= n_s^{-1} n_t^{-1} n_s \bar{u}_t n_s^{-1} n_s n_t b_t \\ &= n_s^{-1} n_t n_t^{-2} u'_t n_s n_t b_t \\ &= n_s^{-1} n_t u''_t n_t n_t^{-2} n_t^{-1} n_s n_t b_t \\ &= n_s^{-1} \bar{u}_t n_s n_s^{-1} n_t n_s n_s^{-1} b'_t n_t^{-2} n_s n_s^{-1} n_t^{-1} n_s n_t b_t \\ &= u'''_t n_t n_t^{-1} n_s^{-1} n_t n_s b''_t n_s^{-1} n_t^{-1} n_s n_t b_t \\ n_s^{-1} n_t^{-1} n_s^{-1} n_t n_s n_t u_t n_t &= n_s^{-1} n_t^{-1} n_s^{-1} n_t n_s \bar{u}_t n_s^{-1} n_t^{-1} n_t n_s n_t b_t \\ &= n_s^{-1} n_t^{-1} n_s n_s^{-2} u'_t n_s n_s^{-1} n_t n_s n_t b_t \quad (\text{note: } u'_t \in U_s) \\ &= n_s^{-1} n_t^{-1} n_s u''_t n_s n_s^{-2} n_s^{-1} n_t n_s n_t b_t \\ &= n_s^{-1} n_t^{-1} \bar{u}_t n_t n_s n_s^{-1} n_t^{-1} n_s b_s n_s^{-2} n_s^{-1} n_t n_s n_t b_t \\ &= u'''_t n_t n_t^{-1} n_s^{-1} n_t^{-1} n_s n_t n_s n_s^{-1} n_t^{-1} b_s n_s^{-2} n_t n_s n_s^{-1} n_t^{-1} n_s^{-1} n_t n_s n_t b_t \\ &= u'''_t n_t n_t^{-1} n_s^{-1} n_t^{-1} n_s n_t n_s b'_s n_s^{-1} n_t^{-1} n_s^{-1} n_t n_s n_t b_t \\ (n_s^{-1} n_t^{-1})^2 (n_s n_t)^2 u_t n_t &= (n_s^{-1} n_t^{-1})^2 n_s n_t n_s \bar{u}_t n_s^{-1} n_t^{-1} n_s^{-1} n_s n_t n_s n_t b_t \\ &= n_s^{-1} n_t^{-1} n_s^{-1} n_t n_t^{-2} u'_t n_s n_t n_s n_t b_t \\ &= n_s^{-1} n_t^{-1} n_s^{-1} n_t u''_t n_t n_t^{-2} n_t^{-1} n_s n_t n_s n_t b_t \end{aligned}$$

$$\begin{aligned}
 &= n_s^{-1}n_t^{-1}n_s^{-1}\overline{u_t''}n_t n_s n_t n_s n_s^{-1}n_t^{-1}n_s^{-1}b_t' n_t^{-2}n_s n_t n_s (n_s^{-1}n_t^{-1})^2(n_s n_t)^2 b_t \\
 &= n_s^{-1}n_t^{-1}n_s^{-1}\overline{u_t''}n_s n_t n_s n_s^{-1}n_t^{-1}n_s^{-1}(n_t n_s)^2 b_t'' (n_s^{-1}n_t^{-1})^2(n_s n_t)^2 b_t \\
 &= u_t''' n_t n_t^{-1}n_s^{-1}n_t^{-1}n_s^{-1}(n_t n_s)^2 b_t'' (n_s^{-1}n_t^{-1})^2(n_s n_t)^2 b_t \\
 &= u_t''' n_t (n_t^{-1}n_s^{-1})^2(n_t n_s)^2 b_t'' (n_s^{-1}n_t^{-1})^2(n_s n_t)^2 b_t
 \end{aligned}$$

Now we prove the claim. Let $s \neq t \in S, s_1, \dots, s_k \in S, w \in \langle s, t \rangle$ be such that $s_1 \cdots s_k$ and $ws_1 \cdots s_k$ are reduced. Moreover, we let $v_i \in U_{s_i}, u_i \in U_s \cup U_t$. We show by induction on $\ell(w)$ that $p(n_s, n_t)^{-1}p(n_t, n_s)$ fixes the equivalence class $[u_1, \dots, u_n, v_1, \dots, v_k]$, where $(u_1, \dots, u_n, v_1, \dots, v_k)$ is of type (w, s_1, \dots, s_k) . For $\ell(w) = 0$ we have $l(p(s, t)s_1 \cdots s_k) = m_{st} + k$. Since $(1, \dots, 1)$ of type $(p(s, t))$ is equivalent to $(1, \dots, 1)$ of type $(p(t, s))$ it follows that

$$\omega(p(n_s, n_t), [v_1, \dots, v_k]) = [1, \dots, 1, v_1, \dots, v_k] = \omega(p(n_t, n_s), [v_1, \dots, v_k])$$

and hence $\omega(p(n_s, n_t)^{-1}p(n_t, n_s), [v_1, \dots, v_k]) = [v_1, \dots, v_k]$. Now let $\ell(w) > 0$. Note that it suffices to show that one of $p(n_s, n_t)^{-1}p(n_t, n_s)$ and $p(n_t, n_s)^{-1}p(n_s, n_t)$ acts trivial, as the product of these two elements acts trivial. Using the previous computations and induction, we infer:

$$\begin{aligned}
 &\omega(p(n_s, n_t)^{-1}p(n_t, n_s), [(u_1, \dots, u_n, v_1, \dots, u_k)]) \\
 &= \omega(p(n_s, n_t)^{-1}p(n_t, n_s)u_1 n_1, [(u_2, \dots, u_n, v_1, \dots, u_k)]) \\
 &= [u_1, \dots, u_n, v_1, \dots, v_k]
 \end{aligned}$$

For example we consider the case $1 \neq u_1 \in U_s, n_1 = n_s$ and $m_{st} = 4$ explicitly. We have the following:

$$\begin{aligned}
 &\omega((n_s n_t)^{-2}(n_t n_s)^2, [u_1, \dots, u_k]) \\
 &= \omega((n_s n_t)^{-2}(n_t n_s)^2 u_1 n_s, [u_2, \dots, u_k]) \\
 &= \omega(u_s''' n_s (n_s^{-1}n_t^{-1})^2(n_s n_t)^2 b_s'' (n_t^{-1}n_s^{-1})^2(n_t n_s)^2 b_s, [u_2, \dots, u_k]) \\
 &= \omega(u_s''' n_s b_s'' b_s, [u_2, \dots, u_k])
 \end{aligned}$$

As before, we have $u_s''' = u_1$ and $b_s'' b_s = 1$ in $X_{s,t}$. This finishes the claim. \square

(4.13) Theorem. *The mapping $G \times \mathbf{C}_{\mathcal{B}_{\mathcal{F}}} \rightarrow \mathbf{C}_{\mathcal{B}_{\mathcal{F}}}, (g, [(u_1, \dots, u_k)]) \mapsto [\omega(g, (u_1, \dots, u_k))]$ is a well-defined action on the set of chambers of the building $\mathbf{C}_{\mathcal{B}_{\mathcal{F}}}$ as a permutation group.*

Proof. Note that G is the direct limit of the inductive system formed by the groups X_s and $X_{s,t}$ for all $s \neq t \in S$. Thus the mapping in the statement is an action if the restriction to the groups $X_{s,t}$ is an action. But this is a consequence of Theorem (2.9) and the Lemmas (4.5), (4.8) - (4.12). \square

5. Main results

(5.1) Remark. Let $\mathcal{F} := ((\Delta_J)_{J \in E(S)}, (c_J)_{J \in E(S)}, (\varphi_{rst})_{\{r,s\}, \{s,t\} \in E(S)})$ be an irreducible 3-spherical Moufang foundation such that every panel contains at least 6 chambers and such that each residue \mathcal{F}_J of rank 3 is integrable. Then \mathcal{F} satisfies condition (lco) and for each $J \subseteq S$ with $|J| = 3$ the J -residue \mathcal{F}_J satisfies the conditions (lco) and (lsc0).

(5.2) Theorem. *Let \mathcal{F} be an irreducible, 3-spherical Moufang foundation of rank at least 3 such that every panel contains at least 6 chambers. Assume that for each irreducible $J \subseteq S$ with $|J| = 3$ the J -residue \mathcal{F}_J is integrable. Then $\mathcal{D}_{\mathcal{F}}$ is an RGD-system and $\mathcal{F} \cong \mathcal{F}(\Delta(\mathcal{D}_{\mathcal{F}}), B_+)$. In particular, \mathcal{F} is integrable.*

Proof. By Theorem (4.13) we have an action of G on the set of chambers of $\mathbf{C}_{\mathcal{B}_{\mathcal{F}}}$ as a permutation group. Clearly, $U_s \rightarrow G$ is injective. Let $1 \neq u' \in U_{-s}$. Then $1 \neq u := n_s^{-1}u'n_s \in U_s$ and $\omega(n_sun_s^{-1}, [()]) = \omega(n_s, [(u)]) = [\bar{u}]$. Since $\bar{u} = 1$ if and only if $u = 1$, the element u' acts non-trivial on the set of chambers of the building and hence $U_{-s} \rightarrow G$ is injective as well. Since for each $\alpha \in \Phi_+$ we have $U_{\alpha} \leq \langle U_{\alpha_s} \mid s \in S \rangle$ and U_{α_s} fixes $[()]$, but $U_{-\alpha_s}$ does not fix $[()]$, $\mathcal{D}_{\mathcal{F}}$ satisfies (RGD3). Thus $\mathcal{D}_{\mathcal{F}}$ is an RGD-system of type (W, S) and the claim follows from Corollary (3.14). \square

(5.3) Corollary. *Let \mathcal{F} be an irreducible, 3-spherical Moufang foundation of rank at least 3 such that every panel contains at least 6 chambers. Then the following are equivalent:*

- (i) \mathcal{F} is integrable.
- (ii) For each irreducible $J \subseteq S$ with $|J| = 3$ the J -residue \mathcal{F}_J is integrable.

Proof. One implication follows directly by considering restriction of the twin building to the twin building $(R_J(c_+), R_J(c_-), \delta_*)$. The other follows from the previous theorem. \square

(5.4) Theorem. *Let Δ be a thick irreducible 3-spherical twin building of rank at least 3. Then Δ is known.*

Proof. At first we assume that every panel of Δ contains at least 6 chambers. Then Δ satisfies the conditions (lco) and (lsc0). Let c be a chamber of Δ and let $\mathcal{F} := \mathcal{F}(\Delta, c)$. As in the previous corollary, each rank 3-residue of \mathcal{F} is integrable. Theorem (5.2) implies that $\mathcal{D}_{\mathcal{F}}$ is an RGD-system and $\mathcal{F} \cong \mathcal{F}(\Delta(\mathcal{D}_{\mathcal{F}}), B_+)$. Using Proposition (3.4), we deduce $\Delta \cong \Delta(\mathcal{D}_{\mathcal{F}})$.

Now we assume that there exists a panel containing at most 5 chambers. Then [24, (34.5)] implies that every panel contains only finitely many chambers. But then Δ is known by the Main result of [11]. We note that the Main result of [11] uses the fact that no rank 2 residue is associated with $B_2(2)$ in order to use the extension theorem of [15].

But as it is shown in [4, Corollary 6.4], the extension theorem holds for arbitrary thick 3-spherical twin buildings. Thus every irreducible locally finite 3-spherical twin building is known by [4] and [11]. \square

Data availability

No data was used for the research described in the article.

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