On instability, ω-limit sets and periodic solutions of nonlinear autonomous differential delay equations

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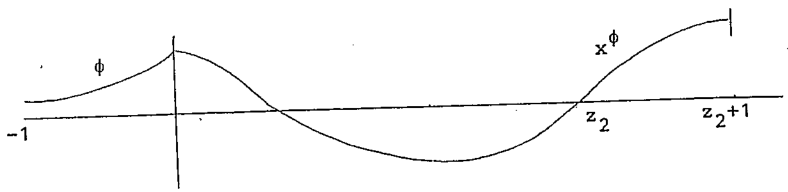
1. Introduction. This paper deals with slowly oscillating solutions of equation

$$(f) \qquad \dot{x}(t) = -f(x(t-1))$$

for continuous functions $f: R \to R$ which are differentiable at $\xi = 0$ and satisfy $\xi f(\xi) > 0$ for $\xi \neq 0$. We are interested in the behaviour of the trajectories $(x_t)_{t \geq 0}$ in the state space of continuous functions on the initial interval.

We prove that in case $f'(0) > \pi/2$ and f bounded below trajectories of slowly oscillating solutions are attracted by a set which may be viewed as a solid torus in function space (Theorem 2). - For a related result on $(x(t), \dot{x}(t))$-trajectories in the plane, see Kaplan and Yorke. - Theorem 2 implies the existence of solutions on the whole real line which are not necessarily periodic but regularly oscillating in some sense, with the set of zeros unbounded for $t > 0$ as well as for $t < 0$ (Corollary 2).

Slowly oscillating periodic solutions with minimal period correspond to fixed points of a Poincare operator on a cone of initial functions. This is Jones' well known idea to obtain periodic solutions. In case of equation (f), consider the operator $T:\phi \to x^{\phi}_{z_2+1}$ on the cone K of continuous increasing functions $\phi:[-1,0] \to R^+_o$ which are not identically zero.



Our method shows that it is possible to get by with an application of the Schauder fixed point theorem if $f'(0) > \pi/2$ and, say, f bounded below. We construct a suitable closed subset $D \subset K$ with $T(D) \subset D$ (proof of Theorem 3). In particular D is bounded away from the critical point $\phi = 0$. Former proofs which use Jones' idea have in common that they use restrictions of the Poincare operator to sets with $\phi = 0$ in the closure. In order to get a nonzero fixed point they need fixed point theorems of expansion-compression type or the theorem of Browder on nonejective fixed points $[G,N_1,N_2,C,A]$.

Our proofs make use of the Liapunov functional of Hale and Perello for unstable behaviour of functional differential equations. V is defined by a projection P onto an eigenspace of increasing exponential solutions of the linearized delay equation. A consequence of the result of Hale and Perello is our Theorem 1 saying that trajectories leave a neighbourhood of zero and, what is important, then keep away from it provided they lie in a set S' where an inequality

(1) $$c\|\phi\| \leq \|P\phi\| \quad \text{for all } \phi \in S'$$

with $c > 0$ is valid. The essential step is then to find sets which both satisfy an inequality of type (1) and contain trajectories of solutions with initial value in K. This is done in sections 5 - 6, beginning with Lemma 3 which exhibits the basic relation between characteristic values of the linear equation and slow oscillations. Due to Lemma 4, our method works without the "integral representation of the projection by the formal adjoint equation" [H, chapter 7]. It applies to equations with distributed delay too, see the proof of ejectivity in [Wa].

Finally, let us point out one difference to other proofs of instability properties which employ the functional of Hale and Perello. In [G,C,A] the integral representation just mentioned is used to deduce an inequality like (1) on the domain of definition of the Poincare operator. Note that these domains are invariant with respect to this operator but not for trajectories $(x_t)_{t\geq 0}$. In fact, the only segments x_t which they contain are those defined by the iterates of the Poincare operator.

2. <u>Notation</u>. X denotes the Banach space of continuous functions $\phi:[-\tau,0] \to R^n$. We assume $\tau > 0$ and $n \in N$ arbitrary in section 3, $\tau = 1 = n$ in sections 4 - 8. Y denotes the Banach space of continuous functions $\psi:[-\tau,0] \to C^n$.

Let $F:X \to R^n$ be given. A solution of the equation

(F) $$\dot{x}(t) = F(x_t)$$

is either a continuous function $x:[-\tau,\infty) \to R^n$ which is differentiable for $t > 0$ and satisfies (F) for $t > 0$ or a differentiable function $x:R \to R^n$ which satisfies (F) for all $t \in R$. The segments $x_t \in X$ are defined by $x_t(a) := x(t+a)$, $a \in [-\tau,0]$. The trajectory of a solution x is the family $(x_t)_{t\geq 0}$, or $(x_t)_{t\in R}$ respectively.

3. Instability. We consider equation (F) for $F:X \to R^n$ continuous, $F(0) = 0$, F differentiable at $\phi = 0$. We assume that there is a non-empty set E of eigenvalues with positive real part of the infinitesimal generator of the semigroup in Y defined by the linear equation $(F'(0))$. Let P denote the eigenprojection which belongs to E. We state a variant of Theorem 3 [HP] as

Lemma 1: There exist a continuous quadratic functional $V:X \to R_0^+$ and positive constants c_1, c_2, c_3 with

(i) $c_1\|P\phi\|^2 \le V(\phi) \le c_2\|P\phi\|^2$ for all $\phi \in X$,

(ii) $\forall\, p > 0\ \exists\, \delta_p > 0\ \forall$ (x solution of (F)) $\forall\, t \ge 0$:

$$\|x_t\| \le \delta_p \wedge p^2\|x_t\|^2 \le V(x_t) \Rightarrow c_3 V(x_t) \le \dot{V}(x,t)$$

(with $\dot{V}(x,t) := \lim\limits_{\substack{0 \neq h \to 0\\ 0<t+h}} (V(x_{t+h}) - V(x_t))/h$ for $t \ge 0$).

Theorem 1: Let $S' \subset X \setminus \{0\}$ and $c > 0$ be given with

(1) $$c\|\phi\| \le \|P\phi\| \quad \text{for all } \phi \in S'.$$

Let V be a functional as in Lemma 1. Conclusions:

(i) There exists $p > 0$ with

(V) $$p^2\|\phi\|^2 \le V(\phi) \quad \text{for all } \phi \in S'.$$

(ii) There exists $a > 0$ such that for every solution x of (F) with trajectory in S' there is a real s with

$$a \le \|x_t\| \quad \text{for all } t \ge s.$$

(iii) Assume δ_p is chosen according to part (ii) of Lemma 1. Then we have

$$p^2\delta_p^2 \le V(x_t) \quad \text{for all } t \ge 0$$

for every solution x of (F) with trajectory in S' and with $p^2\delta_p^2 \le V(x_0)$.

Proof: $0 < p < c\sqrt{c_1}$ yields $p^2\|\phi\|^2 \le c^2c_1\|\phi\|^2 \le c_1\|P\phi\|^2 \le V(\phi)$ for all $\phi \in S'$, by (1) and Lemma 1 (i). Choose δ_p as in part (ii) of Lemma 1. Consider a solution x of (F) with $x_t \in S'$ for all $t \ge 0$.

a) In case $\|x_0\| < \delta_p$ there is $u > 0$ with $\|x_u\| \ge \delta_p$. Proof: Assume $\|x_t\| < \delta_p$ for all $t \ge 0$. By Lemma 1 (ii), by (V) and $0 \notin S'$, $0 < c_3V(x_t) \le \dot{V}(x,t)$, hence $0 < V(x_0)e^{c_3 t} \le V(x_t) \le c_2\|Px_t\|^2 \le$ $\le c_2\|x_t\|^2 < c_2\delta_p^2$ for all $t \ge 0$, contradiction.

b) Suppose $\|x_u\| \ge \delta_p$, $u \ge 0$. For every $v > u$ with $\|x_v\| < \delta_p$ there exists $w \in [u,v)$ with $\|x_w\| = \delta_p$ and $\|x_t\| < \delta_p$ for all $t \in (w,v]$. By Lemma 1 (ii) and by (V), $V(x_t)$ increases on $[w,v]$. Hence $p^2\delta_p^2 = p^2\|x_w\|^2 \le V(x_w) \le V(x_v) \le c_2\|Px_v\|^2 \le c_2\|x_v\|^2$.

a) and b) imply the existence of $s \ge 0$ with

$$\min\{\delta_p, p\delta_p/\sqrt{c_2}\} \le \|x_t\| \quad \text{for all } t \ge s.$$

c) Assume $p^2\delta_p^2 \le V(x_0)$. Let $t > 0$. $\delta_p \le \|x_t\|$ implies $p^2\delta_p^2 \le$ $\le p^2\|x_t\|^2 \le V(x_t)$, by (V). $\|x_t\| < \delta_p$ and $\|x_s\| \le \delta_p$ for all $s \in [0,t)$ imply $p^2\delta_p^2 \le V(x_0) \le V(x_t)$, by (V) and Lemma 1 (ii). In case $\|x_t\| < \delta_p < \|x_s\|$ with $0 \le s < t$ there is $u \in (s,t)$ with $\|x_u\| = \delta_p$ and $\|x_v\| < \delta_p$ for all $v \in (u,t]$, hence $p^2\delta_p^2 = p^2\|x_u\|^2 \le V(x_u) \le$ $\le V(x_t)$.

4. Slowly oscillating solutions. In the following we consider equation (f) for continuous functions $f:R \to R$ which are differentiable at $\xi = 0$ and satisfy $\xi f(\xi) > 0$ for $\xi \neq 0$. For every $\phi \in X$ there exists a unique solution $x:[-1,\infty) \to R$ of (f) with $x_0 = \phi$. This solution will also be denoted by x^ϕ. On compact intervals solutions depend continuously on the initial values with respect to uniform convergence.

Lemma 2: Assume $f'(0) > 1$. Then we have:

(i) For every $\phi \in X \setminus \{0\}$ with $0 \le \phi$ there is a sequence of non-negative zeros $z_j = z_j(\phi)$ of $x = x^\phi$ with $0 \le x$ in $[-1, z_1]$, $x < 0$ in $(z_1, z_1+1]$, $\dot{x} \le 0$ in $(0, z_1+1]$ and

(Z) $\left\{\begin{matrix} 0 < \dot{x} \\ \dot{x} < 0 \end{matrix}\right\}$ in $(z_j+1, z_{j+1}+1)$ for all $\left\{\begin{matrix} \text{odd} \\ \text{even} \end{matrix}\right\}$ j.

(ii) For every $r > 0$ there exists $d > 0$ with $z_1(\phi) < d$ for all $\phi \in X \setminus \{0\}$ with $0 \le \phi \le r$.

Assume in addition $\inf f > -\infty$. Set $r_1 := -\inf f$, $r_2 := \sup_{[0,r_1]} f$. Then we have

(iii) $-r_2 \le x^\phi(t) \le r_1$ for all $\begin{cases} t \ge z_2(\phi) \text{ if } \phi \in X \setminus \{0\},\ 0 \le \phi. \\ t \ge -1 \text{ if } \phi \in X,\ 0 \le \phi \le r_1. \end{cases}$

Proof: See e. g. the proofs of Lemmas 2.3 and 2.2 in $[N_1]$.

Remark 1: Assertion (i) with signs of x and $\dot{x}$ reversed and assertion (ii) hold if $\phi \ne 0$ and $\phi \le 0$, or $-r \le \phi \le 0$ respectively.

Definition: A solution x of (f) is called slowly oscillating iff there exists a sequence of zeros of x with property (Z).

Solutions which start in K are slowly oscillating, and their trajectories lie in the set of functions with at most one change of sign, that is in the set S of $\phi \in X \setminus \{0\}$ such that ϕ or $-\phi$ satisfies

$0 \le \tilde{\phi}$ or

$\exists\, z \in [-1,0]: -1 \le a < z < a' \le 0 \Rightarrow \tilde{\phi}(a) \le 0 \le \tilde{\phi}(a')$.

S is a cone in the sense that we have $t\phi \in S$ whenever $t > 0$, $\phi \in S$. But it is not convex.

Proposition:

(i) For every slowly oscillating solution x of (f) there exists $s \in R$ with $x_t \in S$ for all $t \ge s$.

(ii) For every $\phi \in S$ there exists $t \in [0,1]$ with $0 \le x_t^\phi$ or $x_t^\phi \le 0$.

(iii) $x_t^\phi \in S$ for all $t \ge 0$ and all $\phi \in S$.

(iv) In case $f'(0) > 1$ x^ϕ is slowly oscillating for all $\phi \in S$.

(v) $\mathrm{cl}\, S = S \cup \{0\}$.

Proof of (ii): Let $\phi \le 0$ in $[-1, z)$ and $0 \le \phi$ in $(z, 0]$. (f) implies $0 \le \dot{x}^\phi$ in $(0, z+1]$, hence $0 \le x_{z+1}^\phi$.

Proof of (iv): By (ii), $0 \le x_t^\phi$ or $x_t^\phi \le 0$ for some $t \ge 0$. We have $x_t^\phi \ne 0$. Apply Lemma 2 (i) and Remark 1 to the solution $-1 \le u \to x^\phi(t+u)$ of (f).

5. Linear slow oscillations and inequality (1). Linearization of (f) near $x = 0$ yields equation

(α) $$\dot{y}(t) = -\alpha y(t-1)$$

with $\alpha = f'(0)$, a special case of the type considered. Equation (α) defines a semigroup $(T_t)_{t \ge 0}$ of bounded linear operators in Y by

$$T_t\phi := (y^{\mathrm{Re}\,\phi})_t + i(y^{\mathrm{Im}\,\phi})_t.$$

The eigenvalues of the infinitesimal generator are the zeros of the entire function $\lambda \to \lambda + \alpha e^{-\lambda}$.

For $\alpha > 1/e$ they are all simple, and they form a sequence of pairs $\lambda_j, \bar{\lambda}_j$; $j \in N_o$, with

(2) $u_{j+1} < u_j$ and $2\pi j < v_j < 2\pi j + \pi$ for all $j \in N_o$

$(u_j := \mathrm{Re}\,\lambda_j,\ v_j := \mathrm{Im}\,\lambda_j)$. We have: $0 < u_o \Leftrightarrow \alpha > \pi/2$.

For a proof of these assertions, see [Wr].

We consider the projection P_o which belongs to λ_o and $\bar{\lambda}_o$. With regard to (1) let us first determine the maximal subset in X where P_o does not vanish.

Lemma 3: Assume $\alpha > 1$. For all $\phi \in X$ we have

$$P_o\phi \neq 0 \iff (\text{Solution } y^\phi \text{ of } (\alpha) \text{ is slowly oscillating}).$$

Proof "<=": Suppose y^ϕ is a slowly oscillating solution of (α) with $P_o\phi = 0$. Let P_1 denote the projection which belongs to λ_1 and $\overline{\lambda}_1$.

a) Assume $\mathrm{Re}\, P_1\phi \neq 0$. There are $\varepsilon > 0$, $c > 0$ with

(3) $\|T_t(\phi - (P_o + P_1)\phi)\| \leq ce^{(u_1-\varepsilon)t}\|\phi - (P_o + P_1)\phi\|$ for all $t \geq 0$

[H, ch. 7]. The solution with initial value $\mathrm{Re}\,(P_o + P_1)\phi = \mathrm{Re}\, P_1\phi$ has the form

$$t \to e^{u_1 t}(a \cos v_1 t + b \sin v_1 t)$$

with a, b real, $|a| + |b| > 0$. By (3),

$$y^\phi(t)e^{-u_1 t} - a \cos v_1 t - b \sin v_1 t \to 0 \quad \text{as } t \to \infty.$$

$2\pi < v_1$ implies a contradiction to y^ϕ slowly oscillating.

b) Assume $\mathrm{Re}\, P_1\phi = 0$. There is a neighbourhhod U of ϕ in X with y^ψ a slowly oscillating solution of (α) for every $\psi \in U$. (Proof: By $y_t^\phi > 0$ for some $t > 0$ and by continuous dependence, there is a neighbourhood U with $y_t^\psi > 0$ for all $\psi \in U$. The solution $-1 \leq s \to y^\psi(t+s)$ with initial value y_t^ψ is slowly oscillating, hence y^ψ too.)

We choose a real-valued $\chi \neq 0$ in P_1Y with $\psi := \phi + \chi \in U$. Then $P_o\psi = 0$, $\mathrm{Re}\, P_1\psi = \chi \neq 0$. As in a), we derive a contradiction.

Proof "=>": The estimate $\|T_t(\phi - P_o\phi)\| \leq ce^{(u_0-\varepsilon)t}\|\phi - P_o\phi\|$ for $\phi \in X$ with $P_o\phi \neq 0$ implies $\mathrm{Re}\, P_o\phi \neq 0$. From $0 < v_o < \pi$ and from $\|T_t\phi - T_t \,\mathrm{Re}\, P_o\phi\|e^{-u_0 t} \to 0$ as $t \to \infty$ we infer the existence of a segment $T_t\phi > 0$. It follows that ϕ defines a slowly oscillating solution.

The hypothesis $\alpha > 1$ in Lemma 3 and Lemma 4 below can be weakened to $\alpha > 0$ (with the definition of "slowly oscillating" changed to include monotone solutions which exist for $\alpha \leq 1/e$).

Corollary 1: We have $P_o\phi \neq 0$ for all $\phi \in S$.

Proof: Apply part (iv) of the Proposition to $f = \alpha\, \mathrm{id}$, use Lemma 3.

We look for subsets of S where estimates of type (1) hold.

Lemma 4: Assume $\alpha > 1$. For every non-empty cone $S' \subset S$, estimate

$(1)_o$ $\quad \exists\, c > 0\ \forall\, \phi \in S': c\|\phi\| \leq \|P_o\phi\|$

is equivalent to

(4) $\quad \exists\, k > 0\ \forall\, \phi \in S': k\|\phi\| \leq \|T_1\phi\|$.

Proof: Set $S_1' := \{\phi \in S': \|\phi\| = 1\}$.

"(4) => $(1)_o$": T_1 completely continuous implies $\mathrm{cl}\, T_1 S_1'$ compact. This is a subset of $S \cup \{0\}$, by $T_1 S \subset S$ (Proposition (iii)) and by $\mathrm{cl}\, S = S \cup \{0\}$, and moreover a subset of S, by (4). Hence $P_o\psi \neq 0$ for all $\psi \in \mathrm{cl}\, T_1 S_1'$, and

$0 < \inf\{\|P_o\psi\|: \psi \in \mathrm{cl}\, T_1 S_1'\} \leq \inf\{\|P_o T_1\phi\|: \phi \in S_1'\} \leq$
$\leq \|T_1\| \inf\{\|P_o\phi\|: \phi \in S_1'\}$, by $P_o T_1 = T_1 P_o$. This yields $(1)_o$.

"$(1)_o$ => (4)": $(1)_o$ gives $0 \notin \mathrm{cl}\, P_o S_1'$. This is a closed bounded subset of the finite-dimensional space P_oY. $T_1\psi \neq 0$ for all ψ in $P_oY \setminus \{0\}$ implies $0 < \inf\{\|T_1\psi\|: \psi \in \mathrm{cl}\, P_o S_1'\}$, and the assertion follows as above.

Inequality (4) is easier to verify than $(1)_o$ since

$$T_1\phi(a) = \phi(0) - \alpha\int_{-1}^{a}\phi(t)\,dt \quad \text{for all } a \in [-1,0]$$

while integral formulas for P_o involve the functions $t \to e^{\lambda_0 t}$, see [H, ch. 7] and [Wa].

6. <u>Verification of inequality (4)</u>.

<u>Lemma 5</u>: Assume $f'(0) > 1$. Let $r > 0$ be given. Set $\alpha := f'(0)$ and consider the operator T_1 of the semigroup defined by equation (α). Then there exists a constant $k = k(r,f) > 0$ such that

$$k\|x_t\| \le \|T_1 x_t\| \quad \text{for all } t \ge 0$$

for every solution x of (f) with $|x| \le r$ and with x_0 in the convex cone $K = \{\phi \in X\!\!\!\!\cdot\;: 0 \le \phi \text{ increasing and } 0 < \phi(0)\}$.

<u>Proof</u>: Choose $a > 0$, $b > 1$ such that $a|\xi| < |f(\xi)| < b|\xi|$ for $0 < |\xi| \le r$. Let x be a solution as in the assertion. We consider its local extrema $m_0 := 0$ and $m_j := z_j+1$, $j \in N$ (see Lemma 2). On the intervals $[m_j, m_j + 1/b]$ we have $|x| \ge |g_j|$ for the affine functions g_j with $g_j(m_j) = x(m_j)$ and $\dot{g}_j = -bx(m_j)$. This follows from $|\dot{x}(t)| = |f(x(t-1))| \le b|x(t-1)| \le b|x(m_j)|$ for $t \in (m_j, m_j+1]$ and from $b > 1$.

Let $t \ge 0$. For $t-1 \le u < v \le t$ we have

(5) $\quad \alpha|\int_u^v x(s)ds| \le 2\|T_1 x_t\|$.

Proof: $2\|T_1 x_t\| \ge |T_1 x_t(v-t) - T_1 x_t(u-t)| = \alpha|\int_{u-t}^{v-t} x_t(s')ds'| = \alpha|\int_u^v x(s)ds|$.

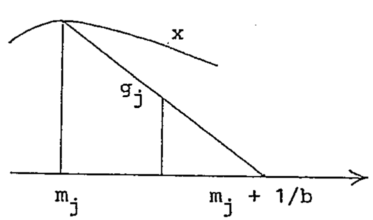
Case I: $t-1 \le m_j \le t$ for some $j \in N_0$.

Subcase 1: $\|x_t\| = |x(m_j)|$ and $t \le m_j + 1/2b$.

Then $\|T_1 x_t\| \ge |T_1 x_t(-1)| = |x(t)| \ge |g_j(t)| \ge |g_j(m_j + 1/2b)| = |x(m_j)|/2$.

Subcase 2: $\|x_t\| = |x(m_j)|$ and $m_j + 1/2b < t$.

On $[m_j, m_j + 1/2b] \subset [t-1,t]$ we have $|x| \ge |g_j| \ge |x(m_j)|/2 = \|x_t\|/2 > 0$. (5) with $u = m_j$, $v = m_j + 1/2b$ implies $2\|T_1 x_t\| \ge \alpha\|x_t\|/4b$.



Subcase 3: $|x(m_j)| < \|x_t\|$.

$|x|$ increases on $[m_j-1, m_j]$, decreases on $[m_j, z_{j+1}]$ and increases on $[z_{j+1}, z_{j+1} + 1]$. This yields $z_{j+1} \le t$ and $|x(t)| = \|x_t\|$, hence $\|T_1 x_t\| \ge |x(t)| = \|x_t\|$.

Case II: $\forall j \in N_0$: $m_j \notin [t-1,t]$.

Then $-1 < t-2$, and x is monotone in $[t-1,t]$. Therefore $\|x_t\| = |x(t)|$ or $\|x_t\| = |x(t-1)|$.

Subcase 1: $\|x_t\| = |x(t)|$ implies $\|T_1 x_t\| \ge \|x_t\|$.

Subcase 2: $\|x_t\| = |x(t-1)| > |x(t)|$ and $|x| \ge |x(t-1)|$ in $[t - 1 - 1/2b, t - 1]$.

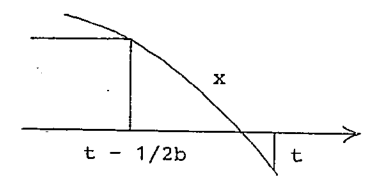
For $s \in [t - 1/2b, t]$ we infer $|\dot{x}(s)| = |f(x(s-1))| \ge a|x(s-1)| \ge a\|x_t\| > 0$, hence $|x(t) - x(t - 1/2b)| \ge a\|x_t\|/2b$, therefore

(6) $\quad a\|x_t\|/4b \le |x(t)| \le \|T_1 x_t\|$ or

(7) $\quad a\|x_t\|/4b \le |x(t - 1/2b)|$.

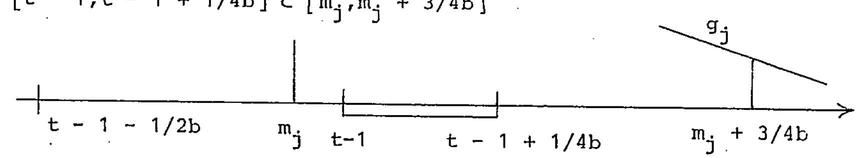
In case of (7) and $|x(t - 1/2b)| \le |x(t)|$ we obtain (6) once more.

In case of (7) and $|x(t - 1/2b)| > |x(t)|$ the monotonicity of x in $[t-1,t]$ implies $|x| \ge |x(t - 1/2b)|$ in $[t-1, t - 1/2b]$. (7) and (5) yield $2\|T_1 x_t\| \ge \alpha(1 - 1/2b)a\|x_t\|/4b$.



Subcase 3: $\|x_t\| = |x(t-1)| > |x(t)|$ and $|x(s)| < |x(t-1)|$ for some s in $[t - 1 - 1/2b, t - 1]$.

Lemma 2 (i) implies the existence of $j \in N_0$ with m_j in $[t - 1 - 1/2b, t - 1]$ and $|x(m_j)| \ge |x(t-1)|$. On $[t - 1, t - 1 + 1/4b] \subset [m_j, m_j + 3/4b]$



we obtain $|x| \geq |g_j| \geq |g_j(m_j + 3/4b)| = |x(m_j)|/4 \geq |x(t-1)|/4 = \|x_t\|/4$, and (5) yields $2\|T_1 x_t\| \geq \alpha\|x_t\|/16b$.

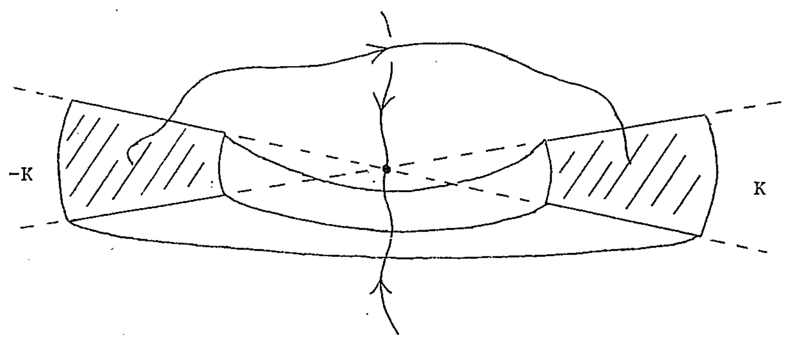
7. Attractor and ω-limit sets.

Theorem 2: Let a continuous function $f: R \to R$ be given which is differentiable at $\xi = 0$ and satisfies $\xi f(\xi) > 0$ for $\xi \neq 0$. Assume $f'(0) > \pi/2$ and $\inf f > -\infty$. Then there are constants $a > 0$, $r > 0$ such that for every slowly oscillating solution x of equation

(f) $$\dot{x}(t) = -f(x(t-1))$$

there is a number $s \geq 0$ with

$$x_t \in \{\phi \in S: a \leq \|\phi\| \leq r\} \quad \text{for all } t \geq s.$$



Proof: Set $r := \max\{r_1, r_2\}$. Define

$$S^f := \{\phi \in S: k\|\phi\| \leq \|T_1\phi\|\}$$

with $k = k(r,f)$ from Lemma 5. By Lemma 4 S^f satisfies $(1)_0$, and we obtain a constant $a > 0$ according to Theorem 1 (ii). Consider a solution x of (f) which has a sequence of zeros z_j with (Z). We have $x_{z_2+1} \in K$ and $|x(t)| \leq r$ for $t \geq z_4$, by Lemma 2 applied to the solution $-1 \leq t \to x(z_2 + 1 + t)$. Lemma 5 implies that the trajectory of the solution $\tilde{x}: -1 \leq t \to x(z_4 + 1 + t)$ is contained in S^f, and Theorem 1 (ii) yields $a \leq \|\tilde{x}_t\| = \|x_{z_4} + 1 + t\|$ in an unbounded interval in R^+.

For a bounded solution x of (f) the ω-limit set

$$\omega_x := \{\phi \in X: \exists (t_j)_{j\in N} \text{ in } R: \lim_{j\to\infty} t_j = \infty \text{ and } \lim_{j\to\infty} x_{t_j} = \phi\}$$

is non-empty. For every $\phi \in \omega_x$ there is a solution $y: R \to R$ of (f) with $y_0 = \phi$ and $y_t \in \omega_x$ for all $t \in R$ [H, p. 82, Corollary 2.1].

Corollary 2:

(i) ω-limit sets of slowly oscillating solutions are contained in the set $\{\phi \in S: a \leq \|\phi\| \leq r\}$.

(ii) For every solution $y: R \to R$ with trajectory in the ω-limit set of a slowly oscillating solution the zeros of y form a family $(z_j)_{j\in Z}$ with property (Z). In particular: $\lim_{j\to-\infty} z_j = -\infty$.

Proof of (i): Use $\mathrm{cl}\, S = S \cup \{0\}$ and Theorem 2.

Proof of (ii): It is enough to show the existence of $d' > 0$ such that for every $s \in R$ there is a sequence of zeros w_j, $j \in N$, of y in (s,∞) with $w_1 < s + d'$, $w_j + 1 < w_{j+1}$, $0 < |y|$ in (w_j, w_{j+1}) for all j in N. Let $s \in R$ be given. $y_s \in S$ implies $0 \leq y_{s+t}$ or $y_{s+t} \leq 0$ for some $t \in [0,1]$ (Proposition (ii) applied to $-1 \leq v \to y(s+v)$). By $0 < \|y_{s+t}\| \leq r$, by Lemma 2 (ii) and Remark 1 there is a sequence of zeros $w'_j \geq 0$, $j \in N$, of the solution $-1 \leq u \to y(s+t+u)$ with $w'_1 < d$, $w'_j + 1 < w'_{j+1}$, $0 < |y(s+t+\cdot)|$ in (w'_j, w'_{j+1}) for all $j \in N$. Set $d' := d + 1$.

Remark 2: Without boundedness assumptions on f we obtain: For every $r > 0$ there exists $a_r > 0$ such that for every slowly oscillating solution x bounded by r there is $s \in R$ with $a_r \leq \|x_t\|$ for all $t \geq s$.

8. Periodic solutions by Schauder's theorem.

Theorem 3 [N_1]: Let a continuous function $f: R \to R$ be given which is differentiable at $\xi = 0$ and satisfies $\xi f(\xi) > 0$ for $\xi \neq 0$. Assume $f'(0) > \pi/2$ and $\inf f > -\infty$. Then there exists a slowly oscillating periodic solution of

(f) $$\dot{x}(t) = -f(x(t-1)).$$

Proof: Solutions $x: [-1,\infty) \to R$ with $x_0 \in K$ and $x_0 \leq r_1$ satisfy $-r_2 \leq x \leq r_1$ (Lemma 2), and their trajectories lie in the set S^f (see proof of Theorem 2) which together with the projection P_0 fulfills the hypotheses of Theorem 1. We choose $\delta_p < r_1$ according to parts (i) and (iii) of Theorem 1 and define

$$D := \{\phi \in K: p^2\delta_p^2 \leq V(\phi) \text{ and } \phi \leq r_1\}.$$

Theorem 1 (iii) and Lemma 2 (iii) show that the operator $T: \phi \to x^{\phi}_{z_2+1}$ maps D into D. T is completely continuous.

D is homeomorphic to a closed bounded convex set: First, we have $D \neq \emptyset$ ($\phi \in K$ and $\delta_p \leq \|\phi\| \leq r_1$ imply $\phi \in S^f$, hence $p^2\delta_p^2 \leq p^2\|\phi\|^2 \leq V(\phi)$) and $V(\phi) > 0$ for all $\phi \in D$; in particular $0 \notin D$, see e. g. Lemma 1 (i). Set $\rho_\phi := \|\phi\| p\delta_p/\sqrt{V(\phi)}$ for $\phi \in D$. Then $\phi \in D$ iff $\phi \in K$ and $\rho_\phi \leq \|\phi\| \leq r_1$. We have $\rho_\phi < r_1$ (otherwise $r_1 \leq \rho_\phi = \|\phi\| p\delta_p/\sqrt{V(\phi)} < r_1\|\phi\| p/\sqrt{V(\phi)}$ by the choice $\delta_p < r_1$, contradiction to $\phi \in D \subset S^f$ and (V)). The map $\phi \to (1 + (\|\phi\| - \rho_\phi)/(r_1 - \rho_\phi))\phi/\|\phi\|$ is a homeomorphism of D onto the set $\{\phi \in K: 1 \leq \phi(0) \leq 2\}$.

The fixed point of T in D defines the periodic solution.

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