Density of Slowly Oscillating Solutions of $\dot{x}(t) = -f(x(t-1))$*

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INTRODUCTION

The most interesting solutions of equation

$$\dot{x}(t) = -f(x(t-1)), \tag{f}$$

with $f: \mathbb{R} \to \mathbb{R}$ continuous and $f(\xi)\xi > 0$ for all $\xi \neq 0$, and of its generalizations, are those which are slowly oscillating. This notion is related to the particular type of delay in the equation considered. In our case it is appropriate to call a function $x: [-1, \infty) \to \mathbb{R}$ slowly oscillating iff there exists $t \geqslant -1$ such that $z > z' + 1$ for every pair of zeros $z > z'$ of x in $[t, \infty)$. Slowly oscillating solutions may be undamped (see [4, 9, 10]), and results on periodicity are well known. We refer to Nussbaum's survey [5], and to the author's paper [9].

Here we are concerned with the domain of slow oscillations—that is, the set S_f of continuous functions $\phi: [-1, 0] \to \mathbb{R}$ such that the solution $x = x^\phi$ of $\dot{x}(t) = f(x(t-1))$ for $t > 0$, $x | [-1, 0] = \phi$, is slowly oscillating. In general, $S_f \cup \{0\}$ is a proper subset of the state space X of continuous real functions on $[-1, 0]$. Examples are provided by the case f linear, where $X \backslash S_f$ always contains an infinite-dimensional subspace, and by results on bifurcation of "rapidly oscillating" periodic solutions [1, 5]. But numerical experiments suggest that these "rapidly oscillating" solutions are unstable or rare in some sense [7]. The natural conjecture that S_f is open and dense in X was already stated by Kaplan and Yorke [4].

In case f is linear, this follows from our Lemma 3 in [9]; see Corollary 2 below. The main result of this paper deals with density for nonlinear equations, openness of S_f being immediate.

* This work was supported by Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 72, Universität Bonn.

THEOREM 1. *Let a continuous function $f: \mathbb{R} \to \mathbb{R}$ be given with $f(\xi)\,\xi > 0$ for $\xi \neq 0$. Assume f is strictly increasing on $\mathbb{R}$ and continuously differentiable on a neighborhood of $0 \in \mathbb{R}$ with $f'(0) > \pi/2$.*

Then there exists $a > 0$ such that every $\phi \in X$ with $\limsup_{t\to\infty} |x^\phi(t)| < a$ *is in the closure of S_f.*

$f'(0) > \pi/2$ guarantees undamped slowly oscillating solutions, so we are in one of the more interesting cases. A first consequence is density of the domain of undamped oscillations (Corollary 3). If in addition $|f| \leqslant c\,|\mathrm{id}|$, $c < \sqrt{2} + \frac{1}{2}$, then the conjecture of Kaplan and Yorke holds true (Corollary 4). If f also satisfies Nussbaum's conditions for uniqueness of the $(x(t), \dot{x}(t))$-orbit of slowly oscillating periodic solutions [6] then we can state that the domain of attraction of this orbit is open and dense (Corollary 5).

The proof of Theorem 1 involves nonautonomous delay equations. It is based on the following observation. Consider two solutions w, y of (f), say, w is "rapidly oscillating" and the initial value of y is close to $w\,|\,[-1, 0]$. The difference, or perturbation, $x = y - w$ satisfies the nonautonomous equation

$$\dot{x}(t) = -F_w(t, x(t-1)), \tag{$\mathrm{F_w}$}$$

where $F_w(t, \xi) = f(\xi + w(t-1)) - f(w(t-1))$ for $t \geqslant 0$, $\xi \in \mathbb{R}$. f strictly increasing yields $F_w(t, \xi)\ \xi > 0$ for all $t \geqslant 0$, $\xi \in \mathbb{R}$, $\xi \neq 0$. Therefore, if the initial value $x\,|\,[-1, 0]$ is in the cone

$$K = \{\phi \in X: \phi(-1) = 0,\ \phi \text{ strictly increasing}\}$$

then the same elementary argument as that in the autonomous case shows that the perturbation x is a slowly oscillating solution of $(\mathrm{F_w})$.

The first step towards Theorem 1 is now to extend the instability proof of [9] to nonautonomous equations. This is done in Sections 1 and 2, starting with a suitable reformulation of the result of Hale and Perello on unstable behavior of functional differential equations [3]. In Theorem 2 we obtain that $\limsup_{t\to\infty} |x(t)|$ e ceeds some positive constant δ for all slowly oscillating solutions of a class of equations of type $(\mathrm{F_w})$. All statements and proofs in Sections 1 and 2 are such that this constant δ can be made uniform with respect to solutions w of (f) which become sufficiently small at infinity. This is essential for the proof of Theorem 1 in Section 3. As above, consider a solution w of (f) and another solution, y, which starts close to w such that $x = y - w$ has initial value in K. For w sufficiently small at infinity, Theorem 2 applies to the slowly oscillating solution x of $(\mathrm{F_w})$. Imposing a further smallness condition on w, we can interchange the roles of x and w and regard w as a "small" perturbation of x so that $y = x + w$ becomes slowly oscillating as well.

Notation. Let $\tau > 0$, $n \in \mathbb{N}$. X and Y denote the Banach spaces of continuous functions $\phi\colon [-\tau, 0] \to \mathbb{R}^n$ and $\psi\colon [-\tau, 0] \to \mathbb{C}^n$, respectively, with supremum-norms. For functions x on $[-\tau, \infty)$ with values in $\mathbb{R}^n$ or $\mathbb{C}^n$, segments x_t, $t \geqslant 0$, are defined by $x_t(a) := x(t+a)$, $a \in [-\tau, 0]$. For $G\colon \mathbb{R}_0^+ \times X \to \mathbb{R}^n$ or $G\colon \mathbb{R}_0^+ \times Y \to \mathbb{C}^n$, solutions of

$$\dot{x}(t) = G(t, x_t) \tag{G}$$

are defined to be continuous functions $x\colon [-\tau, \infty) \to \mathbb{R}^n$ (or $\to \mathbb{C}^n$) which are differentiable and satisfy (G) for $t > 0$.

1. Instability for Nonautonomous Equations

We assume the reader to be familiar with the theory of linear retarded autonomous functional differential equations as given in [2, Chap. 7]. Reference [8] contains a somewhat different approach to the results needed here, which does not make use of adjoint equations. We recall the definition of the Liapunov functional which was used by Hale and Perello [3] to describe unstable behavior:

Let a continuous linear map $L\colon Y \to \mathbb{C}^n$ be given. Let A denote the generator of the semigroup $(T_t)_{t \geqslant 0}$ defined by the solutions of the initial value problem $\dot{y}(t) = Ly_t$, $y_0 = \phi \in Y$. Let a nonempty set E of eigenvalues of A with positive real part be given. Consider the projection P of Y onto the generalized eigenspace which is associated with E. We have $d := \dim PY < \infty$. Choose a basis $\phi_1,..., \phi_d$ of PY. With respect to this basis $A|PY$ is represented by a $d \times d$-matrix B with the set of eigenvalues equal to E. There is a positive definite $d \times d$-matrix D with $B^{\mathrm{tr}}D + DB =$ identity matrix. Define the functional $V\colon X \to \mathbb{R}_0^+$ by $V\phi = y^{\mathrm{tr}}Dy$ and $P\phi = \sum_1^d y_i \phi_i$. Obviously, there are $c_1 > 0$, $c_2 > 0$ with

$$c_1 \|P\phi\|^2 \leqslant V\phi \leqslant c_2 \|P\phi\|^2 \qquad \text{for all} \quad \phi \in X. \tag{1}$$

Now consider nonautonomous equations

$$\dot{x}(t) = N(t, x_t), \tag{N}$$

with $N\colon \mathbb{R}_0^+ \times X \to \mathbb{R}^n$ continuous. The argument of [3] also implies

Lemma 1. *There is a constant $c_3 > 0$ such that for every $p > 0$ there exists $\varepsilon = \varepsilon_p > 0$ with the following property:*

For every triple (N, T, δ), $N\colon \mathbb{R}_0^+ \times X \to \mathbb{R}^n$ continuous, $T \geqslant 0$, $\delta > 0$, which satisfies

$$|N(t, \phi) - L\phi| \leqslant \varepsilon \|\phi\| \qquad \textit{on} \quad (T, \infty) \times \{\phi \in X\colon \|\phi\| < \delta\}, \tag{ε}$$

we have

$$(t \geqslant T \wedge \|x_t\| \leqslant \delta \wedge p^2 \|x_t\|^2 \leqslant Vx_t \Rightarrow c_3 Vx_t \leqslant \dot{V}x_t)$$

for every solution x of (N).

$\dot{V}x_t$ stands for the derivative of $t \to Vx_t$.

Proof (compare proof of Lemma 10.1.1 in [2]). By the decomposition in the variation-of-constants formula, we have, for any solution x of an equation of type (N), that the function $0 \leqslant t \to y(t) \in \mathbb{C}^d$ defined by $Px_t = \sum_1^d y_i(t)\,\phi_i$ satisfies

$$\dot{y}(t) = By(t) + C(N(t, x_t) - Lx_t)$$

with a constant $d \times n$-matrix C independent from N and x.

Set $\gamma := \sup_{|y|=1} y^{\mathrm{tr}} Dy > 0$, $\beta := \inf_{|y|=1} y^{\mathrm{tr}} Dy > 0$, $c_3 := 1/2\gamma$. Let $p > 0$. Choose $\varepsilon < p\sqrt{\beta}/4\gamma \|C\| \|D\|$. Assume (N, T, δ) satisfies (ε). For every solution x of (N) and all $t \geqslant 0$,

$$\dot{V}x_t = y(t)^{\mathrm{tr}}\, y(t) + g(t)^{\mathrm{tr}}\, Dy(t) + y(t)^{\mathrm{tr}}\, Dg(t),$$

where $g(t) := C(N(t, x_t) - Lx_t)$. $t \geqslant T$ and $\|x_t\| \leqslant \delta$ yield

$$\dot{V}x_t \geqslant Vx_t/\gamma - 2\,\|C\|\, \varepsilon\, \|x_t\|\, \|D\|\, \sqrt{Vx_t}/\sqrt{\beta}.$$

By the choice of ε and by $p^2 \|x_t\|^2 \leqslant Vx_t$,

$$\dot{V}x_t \geqslant Vx_t/\gamma - Vx_t/2\gamma.$$

COROLLARY 1. *Assume $L\colon Y \to \mathbb{C}^n$ is a linear continuous map. Let a nonempty set E of eigenvalues of the infinitesimal generator of the semigroup defined by $\dot{y}(t) = Ly_t$ be given such that* $\operatorname{Re} \lambda > 0$ *for all $\lambda \in E$. Let P denote the projection operator on Y associated with E.*

Let a set $S_* \subset X \backslash \{0\}$ and a constant $c_0 > 0$ be given with

$$c_0 \|\phi\| \leqslant \|P\phi\| \qquad \textit{for all} \quad \phi \in S_*. \tag{P}$$

Then there exists $\varepsilon > 0$ such that for every triple (N, T, δ) satisfying (ε) we have

$$\limsup_{t \to \infty} |x(t)| \geqslant \delta$$

for every solution x of (N) *with $\{x_t : t \geqslant t_x\} \subset S_*$ for some $t_x \geqslant 0$.*

Proof. Let $p < c_0\sqrt{c_1}$. By (1), $p^2\|\phi\|^2 \leqslant V\phi$ for all $\phi \in S_*$. Choose $\varepsilon = \varepsilon_p$ according to Lemma 1. Let (N, T, δ) satisfy (ε). For every solution x of (N) with $x_t \in S_*$ for $t \geqslant t_x$ we have

$$p^2\|x_t\|^2 \leqslant Vx_t. \tag{2}$$

Assume $\sup_{t\geqslant s}\|x_t\| \leqslant \delta$ for some $s \geqslant \max\{t_x, T\}$. Then $c_3 Vx_t \leqslant \dot{V}x_t$ on $[s, \infty)$ by Lemma 1. Hence $0 < Vx_s e^{c_3 t} \leqslant Vx_t \leqslant c_2\|Px_t\|^2 \leqslant c_2\|x_t\|^2 \leqslant c_2\delta^2$, which contradicts $c_3 > 0$ and $Vx_s > 0$ ($Vx_s > 0$ by (2) and $0 \notin S_*$).

Remark. Obviously, if (N, T, δ) satisfies (ε') with $0 < \varepsilon' < \varepsilon$, ε such that the assertion of Corollary 1 holds, then we have (ε) too, and therefore $\limsup_{t\to\infty}|x(t)| \geqslant \delta$ for every solution x of (N) with $x_t \in S_*$ for t in some unbounded interval.

2. Instability and Slowly Oscillating Solutions for Nonautonomous Equations

We specialize $\tau = 1$, $n = 1$, and consider equation

$$\dot{x}(t) = -F(t, x(t-1)) \tag{F}$$

for continuous functions $F\colon \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ with

$$F(t, \xi)\,\xi > 0 \qquad \text{for all} \quad t \geqslant 0 \quad \text{and} \quad \xi \neq 0. \tag{H}$$

Lemma 2. *Every solution $x\colon [-1, \infty) \to R$ of equation* (F) *with x_0 in $K := \{\phi \in X\colon \phi(-1) = 0,\ \phi$ strictly increasing$\}$ is slowly oscillating with $z > z' + 1$ for every pair of zeros $z > z'$ of x. We have*

(i) $|x| > 0$ *and* $|x|$ *decreasing on some interval* $[t, \infty)$, *or*

(ii) *the zeros of x form a sequence $(z_j)_{j\in\mathbb{N}_0}$ such that*

$$\dot{x}\begin{Bmatrix} < \\ > \end{Bmatrix} 0 \quad in \quad (z_j + 1, z_{j+1} + 1) \quad for \quad j\begin{cases} even \\ odd \end{cases}.$$

Proof. If the zeros of x are unbounded set $z_0 := -1$ and let z_1 be the first zero in $(-1, \infty)$. By $x_0 \in K$, $z_1 > 0$. $x > 0$ in $(-1, z_1)$ and (F) imply $\dot{x} < 0$ in $(0, z_1 + 1) = (z_0 + 1, z_1 + 1)$. Hence $x < 0$ and $\dot{x} < 0$ in $(z_1, z_1 + 1)$. Denote the first zero in (z_1, ∞) by z_2. Then $z_2 > z_1 + 1$, and the same argument gives $0 < \dot{x}$ in $(z_1 + 1,\ z_2 + 1)$. Induction yields (ii). The argument also shows $|z - z'| > 1$ for every pair of zeros $z \neq z'$ if the zeros are bounded. In this case, (F) implies $|x|$ decreasing on some unbounded interval.

All segments of solutions with initial value in K lie in the set

$$S := \{\psi \in X \backslash \{0\} \colon \psi \text{ or } -\psi \text{ satisfies (S)}\},$$

$$\begin{aligned}&\psi \geqslant 0, \text{ or there exists } a \in [-1,0] \text{ with } \psi \geqslant 0 \text{ in } [-1,a] \text{ and} \\ &0 \geqslant \psi \text{ in } [a,0].\end{aligned} \tag{S}$$

S is the set of functions with at most one change of sign.

As in [9], we shall apply Corollary 1 to the linear equation

$$\dot{y}(t) = -\alpha y(t-1) \tag{α}$$

with $\alpha > 0$, to the projection P_α associated with the unique pair of eigenvalues $\lambda_\alpha, \bar{\lambda}_\alpha$ with $|\operatorname{Im} \lambda_\alpha| < \pi$, and to a subset of S. The next two lemmas are taken from [9]. They prepare the construction of a suitable subset. The basic relation between P_α and the notion of slowly oscillating is given by

Lemma 3. *For every $\phi \in X$, ϕ defines a slowly oscillating solution of* (α) *iff $P_\alpha \phi \neq 0$.*

Corollary 2. *$S_{\alpha \mathrm{id}}$ is open and dense in X.*

Proof. Open is obvious. If ψ does not define a slowly oscillating solution, $P_\alpha \psi = 0$. For every real valued $\phi \in P_\alpha Y \backslash \{0\}$ (or for every $\phi \in K$), we obtain $P_\alpha(\psi + \phi) = P_\alpha \phi \neq 0$, and $\psi + \phi$ defines a slowly oscillating solution. This implies density.

By Lemma 3, P_α does not vanish on S. In [9] we used the simpler operator $T_\alpha(1)$, $T_\alpha(1)\,\phi(a) = \phi(0) - \alpha \int_{-1}^{a} \phi(a)\, da$ for $a \in [-1,0]$, to characterize subsets of S where an estimate like (P) in Corollary 1 for $P = P_\alpha$ is valid:

Lemma 4. *For every cone $S_* \subset S$ there exists $c_0 > 0$ with $c_0 \|\phi\| \leqslant \|P_\alpha \phi\|$ for all $\phi \in S_*$ iff there is $k > 0$ with $\|\phi\| \leqslant k \, \|T_\alpha(1)\, \phi\|$ for all $\phi \in S_*$.*

In Lemma 5 and Theorem 2 we shall consider triples (F, T, r), $F \colon \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ a continuous function with (H), $T \geqslant 0$ and $r > 0$ reals.

Lemma 5. *For every given set of constants $\alpha > 0$, $c > d > 1$, there exists a constant $k = k(c, d, \alpha) > 0$ with the following property. For every triple (F, T, r) such that*

$$d\,|\xi| \leqslant |F(t, \xi)| \leqslant c\,|\xi| \qquad \text{on} \quad [T, \infty) \times (-r, r), \tag{3}$$

we have, for every solution x of (F) *with $x_0 \in K$ and* $\limsup_{t\to\infty} |x(t)| < r$,

$$\|x_t\| \leqslant k \, \|T_\alpha(1)\, x_t\|$$

for all t in some unbounded interval.

Proof. Let (F, T, r) be given such that (3) holds. Assume x is a solution of (F) with $x_0 \in K$ and $|x(t)| < r$ for all t in some unbounded interval $[u, \infty)$, $u \geqslant 0$.

For x we have assertion (ii) of Lemma 2: Suppose $x > 0$ in some interval (t, ∞), $t > u$. Then $\dot{x} < 0$ in $(t+1, \infty)$. Together with $d > 1$, this implies $x(t+3) - x(t+2) = \int_{t+2}^{t+3} \dot{x}(s)\, ds = -\int_{t+2}^{t+3} F(s, x(s-1))\, ds \leqslant -d \int_{t+2}^{t+3} x(s-1)\, ds < -x(t+2)$, hence $x(t+3) < 0$, contradiction. The same proof applies in case $x < 0$ on some unbounded interval.

Now we can follow the proof of Lemma 5 in [9]. We choose a zero $z_j > u+1$ such that $x_{z_j+1} \in K$. The local extrema of x on $[z_j, \infty)$ are given by $m_k := z_k + 1$, $k \geqslant j$.

On the intervals $[m_k, m_k + 1/c]$ we have $|x| \geqslant |g_k|$, g_k the affine function given by $g_k(m_k) = x(m_k)$, $\dot{g}_k = -cx(m_k)$. This follows from $|\dot{x}(t)| = |F(t, x(t-1))| \leqslant c\,|x(t-1)| \leqslant c\,|x(m_k)|$ for $m_k < t \leqslant m_k + 1$, and from $c > 1$.

For every $t \geqslant 0$, we have

$$t - 1 \leqslant v < w \leqslant t \Rightarrow \alpha \left| \int_v^w x(s)\, ds \right| \leqslant 2\, \|T_\alpha(1)\, x_t\|. \tag{4}$$

Proof. $2\,\|T_\alpha(1)\, x_t\| \geqslant |(T_\alpha(1)\, x_t)(w-t) - (T_\alpha(1)\, x_t)(v-t)| = \alpha\,|\int_{-1}^{w-t} x_t(a)\, da - \int_{-1}^{v-t} x_t(a)\, da| = \alpha\,|\int_{v-t}^{w-t} x(t+a)\, da|$.

Let $t \geqslant z_j + 1$.

Case I

There exists $k \geqslant j$ with $t - 1 \leqslant m_k \leqslant t$.

Subcase 1. $\|x_t\| = |x(m_k)|$ and $t \leqslant m_k + 1/2c$. Then $\|T_\alpha(1)\, x_t\| \geqslant |(T_\alpha(1)\, x_t)(-1)| = |x(t)| \geqslant |g_k(t)| \geqslant |g_k(m_k + 1/2c)| = |x(m_k)|/2$.

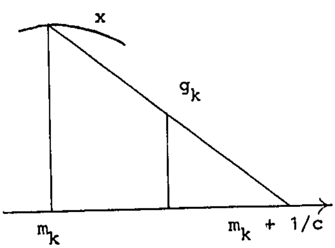

FIGURE 1

Subcase 2. $\|x_t\| = |x(m_k)|$ and $m_k + 1/2c < t$. On $[m_k, m_k + 1/2c] \subset [t-1, t]$ we have $|x| \geqslant |g_k| \geqslant |x(m_k)|/2 = \|x_t\|/2 > 0$. Equation (4) with $v = m_k$, $w = m_k + 1/2c$ yields $2\|T_\alpha(1)\,x_t\| \geqslant \|x_t\|/4c$.

Subcase 3. $|x(m_k)| < \|x_t\|$. $|x|$ increases on $[m_k - 1, m_k]$, decreases on $[m_k, z_{k+1}]$ and increases on $[z_{k+1}, z_{k+1} + 1]$. This implies $z_{k+1} \leqslant t$ and $|x(t)| = \|x_t\|$, hence $\|T_\alpha(1)\,x_t\| \geqslant |x(t)| = \|x_t\|$.

Case II

For every $k \geqslant j$, $m_k \notin [t-1, t]$. Then $z_j < t - 2$, and x is monotone on $[t-1, t]$. Hence $\|x_t\| = |x(t)|$ or $\|x_t\| = |x(t-1)|$.

Subcase 1. $\|x_t\| = |x(t)|$. $\|T_\alpha(1)\,x_t\| \geqslant \|x_t\|$ is obvious.

Subcase 2. $\|x_t\| = |x(t-1)| > |x(t)|$ and $|x| \geqslant |x(t-1)|$ in $[t - 1 - 1/2c, t-1]$. For $s \in [t - 1/2c, t]$, we infer $|\dot{x}(s)| = |F(s, x(s-1))| \geqslant d\,|x(s-1)| \geqslant d\,\|x_t\| > 0$, hence $|x(t) - x(t - 1/2c)| \geqslant d\,\|x_t\|/2c$, and therefore

$$d\,\|x_t\|/4c \leqslant |x(t)| \leqslant \|T_\alpha(1)\,x_t\| \tag{5}$$

or

$$d\,\|x_t\|/4c \leqslant |x(t - 1/2c)|. \tag{6}$$

In case (6) and $|x(t - 1/2c)| \leqslant |x(t)|$ we obtain (5) once again. In case (6) and $|x(t - 1/2c)| > |x(t)|$, the monotonicity of x in $[t-1, t]$ implies $|x| \geqslant |x(t - 1/2c)|$ in $[t-1, t - 1/2c]$. Then (6) and (4) yield $2\|T_\alpha(1)\,x_t\| \geqslant \alpha(1 - 1/2c)\,d\,\|x_t\|/4c$.

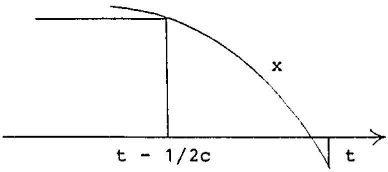

FIGURE 2

Subcase 3. $\|x_t\| = |x(t-1)| > |x(t)|$ and $|x(s)| < |x(t-1)|$ for some s in $[t - 1 - 1/2c, t-1]$. Lemma 2 implies the existence of $k \geqslant j$ with $m_k \in [t - 1 - 1/2c, t - 1)$ and $|x(m_k)| \geqslant |x(t-1)|$. On the interval $[t-1,$

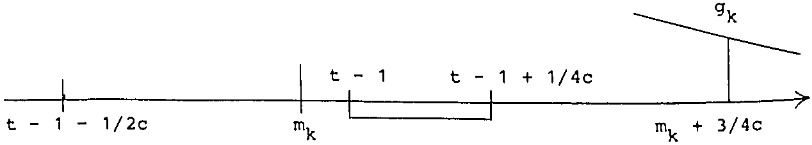

FIGURE 3

$t-1+1/4c] \subset [m_k, m_k+3/4c]$ we obtain $|x| \geqslant |g_k| \geqslant |g_k(m_k+3/4c)| = |x(m_k)|/4 \geqslant |x(t-1)|/4 = \|x_t\|/4$, and (4) yields $2\|T_\alpha(1)x_t\| \geqslant \alpha\|x_t\|/16c$.

Set $k(c, d, \alpha) := \max\{2, 32c/\alpha, 8c/(\alpha d(1-1/2c))\}$.

THEOREM 2. *Let $\alpha > \pi/2$. There exists $\varepsilon > 0$ such that for every $\delta > 0$ we have: For every triple (F, T, δ) satisfying*

$$|F(t,\xi)-\alpha\xi| \leqslant \varepsilon|\xi| \quad \text{on} \quad [T,\infty)\times(-\delta,\delta) \tag{(ε)}$$

and for every solution x of (F) with $x_0 \in K$,

$$\limsup_{t\to\infty} |x(t)| \geqslant \delta.$$

Proof. For $\alpha > \pi/2$, $\operatorname{Re}\lambda_\alpha > 0$ [10]. Let $\varepsilon_1 > 0$ such that $d := \alpha - \varepsilon_1 > 1$. Set $c := \alpha + \varepsilon_1$. Choose $k = k(c, d, \alpha) > 0$ according to Lemma 5 and define $S_k := \{\phi \in S: \|\phi\| \leqslant k\|T_\alpha(1)\phi\|\}$. By Lemma 4 there is $c_0 > 0$ such that $c_0\|\phi\| \leqslant \|P_\alpha\phi\|$ for all $\phi \in S_k$. We apply Corollary 1 and the subsequent remark to P_α, S_k, L_α: $\phi \to -\alpha\phi(-1)$, $E_\alpha := \{\lambda_\alpha, \bar\lambda_\alpha\}$ and to mappings N: $(t,\phi) \to -F(t,\phi(-1))$: There is $\varepsilon^* > 0$ such that for every ε in $(0, \varepsilon^*)$, for every triple (F, T, δ) with ((ε)) and for every solution x of (F) with $x_t \in S_k$ for t in some unbounded interval, we have $\limsup_{t\to\infty}|x(t)| \geqslant \delta$.

Choose ε in $(0, \min\{\varepsilon_1, \varepsilon^*\})$. Let (F, T, δ) with ((ε)) be given. Equation ((ε)) implies ((ε_1)), hence $|F(t,\xi)| - \alpha|\xi| \leqslant \varepsilon_1|\xi|$ and $\alpha|\xi| - |F(t,\xi)| \leqslant \varepsilon_1|\xi|$ on $[T,\infty)\times(-\delta,\delta)$, and the definition of c and d above gives

$$d|\xi| \leqslant |F(t,\xi)| \leqslant c|\xi| \quad \text{on} \quad [T,\infty)\times(-\delta,\delta) \tag{(3)}$$

for the triple (F, T, δ).

Now suppose $x_0 \in K$ and $\limsup_{t\to\infty}|x(t)| < \delta$ for some solution x of (F). Because of ((3)), Lemma 5 applies with $r = \delta$, and by our definition of S_k we obtain $x_t \in S_k$ for t in some unbounded interval.

By $\varepsilon < \varepsilon^*$, the first part of the proof yields $\limsup_{t\to\infty}|x(t)| \geqslant \delta$, a contradiction.

3. DISCUSSION OF MAIN EQUATION

Consider the equation

$$\dot x(t) = -f(x(t-1)) \tag{f}$$

with $f: \mathbb{R} \to \mathbb{R}$ continuous, $f(\xi)\xi > 0$ for all $\xi \neq 0$. Integration of the right-hand side for $t \in [0, 1]$ and iteration shows that every $\phi \in X$ defines a unique solution x^ϕ of (f) with $x_0 = \phi$. For every $t \geqslant 0$, segments x_t^ϕ depend continuously on ϕ. Arguments as in the proof of Lemma 2 give

PROPOSITION 1. *Every solution x of* (f) *with* $|x| > 0$ *on some interval* $(t-1, t)$, $t \geqslant 0$, *is slowly oscillating.*

LEMMA 6. *For every* $m > 0$ *there exists* $a = a(f) > 0$ *such that* $\phi \in K \cup (-K)$, $|\phi(0)| > m$ *and* $\psi \in X$, $\|\psi\| < a$ *imply* $\phi + \psi \in S_f$.

Proof. For $m > 0$, choose $a > 0$ with $a + \max_{[-2a,2a]} |f| < m$. Let $\phi \in K \cup (-K)$, $|\phi(0)| > m$. Set $x := x^{\phi+\psi}$. Choose $b \in (-1, 0)$ with $|\phi(b)| = a$. $\|\psi\| < a$ implies $|x| = |\phi + \psi| \geqslant |\phi| - |\psi| > a - a = 0$ in $(b, 0]$. For $t \in (0, b+1]$, we have $t \leqslant 1$ and $t - 1 \leqslant b$, hence $|x(t)| = |\phi(0) + \psi(0) + \int_0^t \dot{x}(s)\, ds| \geqslant |\phi(0)| - |\psi(0)| - \int_0^t |f(\phi(s-1) + \psi(s-1))|\, ds > m - a - \max_{[-2a,2a]} |f| > 0$. Therefore $|x| > 0$ on $(b, b+1)$, and Proposition 1 applies.

Proof of Theorem 1. Set $\alpha := f'(0) > \pi/2$ and choose $\varepsilon > 0$ such that the assertion of Theorem 2 holds. For continuous functions $w\colon [-1, \infty) \to R$, define $F_w(t, \xi) := f(\xi + w(t-1)) - f(w(t-1))$ for $t \geqslant 0$, $\xi \in \mathbb{R}$. $F_w\colon \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies (H) since f is strictly increasing. f continuously differentiable on a neighborhood of $0 \in \mathbb{R}$ implies the existence of $\delta > 0$ such that $|F_w(t, \xi) - \alpha\xi| = |\int_{w(t-1)}^{w(t-1)+\xi} (f'(\eta) - \alpha)\, d\eta| \leqslant \varepsilon |\xi|$ whenever $|w(t-1)| < \delta$ and $|\xi| < \delta$. We infer that for all continuous functions $w\colon [-1, \infty) \to R$ with $\limsup_{t\to\infty} |w(t)| < \delta$, there exists $T_w \geqslant 0$ such that (F_w, T_w, δ) satisfies $((\varepsilon))$. By Theorem 2, $\limsup_{t\to\infty} |x(t)| \geqslant \delta$ whenever x is a solution of (F_w) with $\limsup_{t\to\infty} |w(t)| < \delta$, $x_0 \in K$.

Now choose $a \in (0, \delta/2)$ according to Lemma 6. Let w be a solution of (f) with $\limsup_{t\to\infty} |w(t)| < a$. Let W be a neighborhood of w_0 in X. Choose $\psi \in W$ such that $\psi - w_0 \in K$. We show $\psi \in S_f$: Let y denote the solution of (f) defined by ψ. $x := y - w$ is a solution of (F_w) which $x_0 \in K$, hence $\limsup_{t\to\infty} |x(t)| \geqslant \delta$ (see the first part of the proof). In case $|x| > 0$ on some unbounded interval $[t, \infty)$, (H) implies that $|x|$ decreases on $(t+1, \infty)$ to $\limsup_{t\to\infty} |x(t)| \geqslant \delta$. Therefore we have $s \geqslant t+2$ with $|y| = |x + w| \geqslant \delta - a > 0$ on $(s-1, s)$, and y is slowly oscillating (Proposition 1).

In case the zeros of x are unbounded, there is a zero z_j of x such that $x_{z_j+1} \in K \cup (-K)$, $|x_{z_j+1}(0)| = |x(z_j + 1)| > \delta/2$, and $\|-w_{z_j+1}\| < a$. By Lemma 6, the solution $-1 \leqslant t \to y(t + z_j + 1)$ with initial value $y_{z_j+1} = x_{z_j+1} + w_{z_j+1}$ is slowly oscillating, and the assertion follows.

4. SOME CONSEQUENCES

COROLLARY 3. *Assume f satisfies the hypotheses of Theorem* 1. *Then there exists $a' > 0$ such that the set* $\{\phi \in X\colon \limsup_{t\to\infty} |x^\phi(t)| \geqslant a'\}$ *is dense.*

Proof. Set $\alpha := f'(0) > \pi/2$. For $\varepsilon > 0$ as in Theorem 2 there exists $\delta > 0$ with $|f(\xi) - \alpha\xi| \leqslant \varepsilon|\xi|$ for $|\xi| < \delta$. Therefore Theorem 2 yields $\limsup_{t\to\infty} |x(t)| \geqslant \delta$ for every solution x of (f) with $x_0 \in K$.

$f'(0) > 1$ implies that for every slowly oscillating solution x of (f) the zeros are unbounded (this follows by an argument similar to the first part of proof of Lemma 5), and $x_s \in K$ for some $s \geqslant 0$. For the solution $x: -1 \leqslant t \to x(t+s)$ of (f) with initial value x_s we have $\limsup_{t\to\infty} |x(t+s)| \geqslant \delta$, hence $\limsup_{t\to\infty} |x(t)| \geqslant \delta$, also.

By Theorem 1, there is $a > 0$ such that every initial value ϕ with $\limsup_{t\to\infty} |x^\phi(t)| < a$ may be approximated by initial values in S_f. Set $a': \min\{a, \delta\}$.

COROLLARY 4. *Assume f satisfies the hypotheses of Theorem 1 and $|f| \leqslant c\,|\mathrm{id}|$ with $c < \sqrt{2} + \frac{1}{2}$. Then S_f is open and dense.*

This follows from the subsequent Propositions 2 and 3 and from Theorem 1.

PROPOSITION 2. *S_f is open for every continuous function $f: \mathbb{R} \to \mathbb{R}$ with $f(\xi)\,\xi > 0$ for $\xi \neq 0$.*

Proof. For $\phi \in S_f$ there exists $t \geqslant 0$ with $|x^\phi| > 0$ in $[t-1, t]$. By continuous dependence, $|x^\psi| > 0$ in $[t-1, t]$ for all ψ in some neighborhood of ϕ. Apply Proposition 1.

PROPOSITION 3. *Let $f: \mathbb{R} \to \mathbb{R}$ be continuous with $f(\xi)\,\xi > 0$ for $\xi \neq 0$ and $|f| \leqslant c\,|\mathrm{id}|$, $c < \sqrt{2} + \frac{1}{2}$. Then $\lim_{t\to\infty} x(t) = 0$ for all solutions x of* (f) *which are not slowly oscillating.*

Proof. Consider a solution x which is not slowly oscillating. By Proposition 1, every open interval of unit length contains zeros. Hence $u := \limsup_{t\to\infty} x(t) \geqslant 0$; $v := -\liminf_{t\to\infty} x(t) \geqslant 0$. $u = 0$ and $v = 0$ are equivalent: Let $u = 0$. For every negative local minimum $x(m)$, $m \geqslant 0$, we then have $x(m-1) = 0$ (by (f)); hence $0 > x(m) = \int_{m-1}^{m} \dot{x}(s)\,ds = -\int_{m-2}^{m-1} f(x(s))\,ds \geqslant -\max\{f(\xi)\colon\ 0 \leqslant \xi \leqslant \max_{[m-1,m-2]} x\}$, so $u = 0$ implies $v = 0$. Similarly $v = 0$ yields $u = 0$.

Therefore it is enough to derive a contradiction from the assumption $u > 0$ and $v > 0$.

Because of $c < \sqrt{2} + \frac{1}{2}$ there exists $\varepsilon > 0$ with $0 < \frac{1}{2} - \varepsilon/(u+\varepsilon) - \frac{1}{4}(c - \frac{1}{2})^2$ and $2\varepsilon/(v+\varepsilon) + (\sqrt{2} + \frac{1}{2})/2 < 1$.

Set $p := u + \varepsilon$, $q := v + \varepsilon$. Choose $s \geqslant -1$ with $-q < x < p$ on $[s, \infty)$. Equation (f) and $|f| \leqslant c\,|\mathrm{id}|$ imply

$$-cp \leqslant \dot{x} \leqslant cq \quad \text{on } [s+1, \infty). \tag{7}$$

Choose a local minimum $x(m) \leqslant -v + \varepsilon$, $m \geqslant s+4$. Then $x(z) = 0$ for $z = m-1$, and

$$
\begin{aligned}
-v + \varepsilon \geqslant x(m) &= \int_z^m \dot{x}(t)\,dt = -\int_{z-1}^z f(x(t))\,dt \\
&\geqslant -\int_{[z,z-1]\cap\{t \geqslant -1 : x(t) > 0\}} f(x(t))\,dt \\
&\geqslant -\int_{\ldots} cx(t)\,dt =: -cI. \qquad (8)
\end{aligned}
$$

We shall prove

$$
I \leqslant \int_{z-1}^{z} h(t)\,dt, \qquad (9)
$$

where $h\colon t \to \min\{p, h_1, h_2\}$ with the functions h_1, h_2 given by $h_1(z-1) = 0$, $\dot{h}_1 = cq$, $h_2(z) = 0$, $\dot{h}_2 = -cp$. Define t_1 and t_2 by $h_1(t_1) = p = h_2(t_2)$. The set

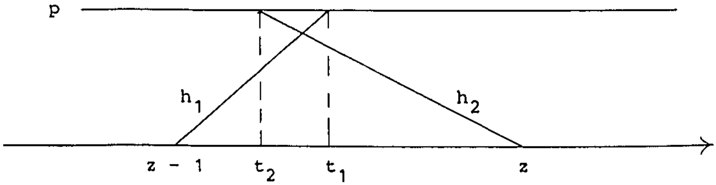

FIGURE 4

$\{(t, \xi) \in \mathbb{R}^2\colon z-1 \leqslant t \leqslant z,\ 0 \leqslant \xi \leqslant h(t)\}$ is a triangle if $t_2 \leqslant t_1$, a trapezoid if $t_1 < t_2$. In the triangle case we have

$$
z - 1/c = t_2 \leqslant t_1 = p/cq + z - 1, \quad \text{or} \quad c \leqslant 1 + p/q. \qquad (10)
$$

Proof of (9). For $x(z-1) = 0$, $z-1 \geqslant s+1$ and (7) and $x < p$ on $[s, \infty)$ imply $x \leqslant h$ on $[z-1, z]$, hence (9). If $x(z-1) \neq 0$, set $z' := \sup\{t > z-1\colon |x| > 0 \text{ on } [z, t)\}$. $z' \in (z-1, z)$ is a zero of x, and $[z-1, z')$ contains no zero. This implies the existence of a zero z'' of x in $(z'-1, z-1)$. Equation (7), $z'' > z'-1 > z-2 \geqslant s+1$, and $x < p$ on $[s, \infty)$ yield

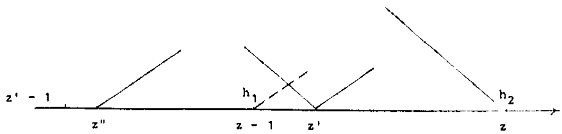

FIGURE 5

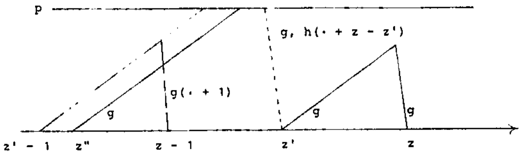

FIGURE 6

$x \leqslant \min\{p, h_1(\cdot + z - 1 - z''), h_2(\cdot + 1 - z'), h_1(\cdot + z - 1 - z'), h_2\} =: g$ on $[z'', z]$.

Hence $I \leqslant \int_{z-1}^{z} g(t)\,dt = \int_{z-1}^{z'} g(t)\,dt + \int_{z'-1}^{z-1} g(t+1)\,dt$. It is easily seen that $0 \leqslant g \leqslant h(\cdot + z - z')$ on $[z-1, z']$ and $0 \leqslant g(\cdot + 1) \leqslant h(\cdot + z - z')$ on $[z'-1, z-1]$. Figure 6 illustrates one of the combinations of triangles and trapezia to be considered.

We infer $I \leqslant \int_{z'-1}^{z'} h(t + z - z')\,dt = \int_{z-1}^{z} h(t)\,dt$. Computation of $\int_{z-1}^{z} h(t)\,dt$: In the trapezium case, $t_1 < t_2$, we have $\int_{z-1}^{z} h(t)\,dt = (1/2)\,p(1 + t_2 - t_1) = -p^2/2cq + p(1 - 1/2c)$. For $t_2 \leqslant t_1$ we obtain $\int_{z-1}^{z} h(t)\,dt = pq/2(p+q)$.

Now (8) and (9) imply

$$\begin{aligned} v - \varepsilon &\leqslant -p^2/2q + p(c - \tfrac{1}{2}) \qquad \text{if} \quad t_1 < t_2, \\ v - \varepsilon &\leqslant c^2 pq/2(p+q) \qquad \text{if} \quad t_2 \leqslant t_1. \end{aligned} \tag{11}$$

Let $q = bp$, $b > 0$. In case $t_1 < t_2$, (11) gives $bp \leqslant 2\varepsilon - p^2/2bp + p(c - \frac{1}{2})$, or $b^2 - (c - \frac{1}{2})\,b + (\frac{1}{2} - \varepsilon/p) \leqslant 0$. On the other hand, the function $\beta \to \beta^2 - (c - \frac{1}{2})\,\beta + (\frac{1}{2} - \varepsilon/p)$ has its absolute minimum at $\beta = (c - \frac{1}{2})/2$, with value $\frac{1}{2} - \varepsilon/p - (c - \frac{1}{2})^2/4 > 0$, by the choice of ε, contradiction. In case $t_2 \leqslant t_1$ we have $q \leqslant 2\varepsilon + c^2 pq/2(p+q)$, by (11). Hence $1 \leqslant 2\varepsilon/q + c^2/2(1+b)$. By (10), $1 < 2\varepsilon/q + c^2/2c = 2\varepsilon/q + c/2 < 2\varepsilon/q + (\sqrt{2} + \frac{1}{2})/2$, contradicting the choice of ε.

Kaplan and Yorke proved that for every slowly oscillating solution x of (f), $f\colon \mathbb{R} \to \mathbb{R}$ continuously differentiable with $f(0) = 0$, $f' > 0$, $f'(0) > \pi/2$ and f bounded below, the trajectory $-1 \leqslant t \to (x(t), \dot{x}(t)) \in \mathbb{R}^2$ tends to a closed annulus A between two orbits of slowly oscillating periodic solutions [4]. We have $(0, 0) \notin A$. By a result of Nussbaum [6], there is exactly one orbit of slowly oscillating periodic solutions, say, $o_f = \{(y(t), \dot{y}(t)) \in \mathbb{R}^2 : t \in \mathbb{R}\}$, provided f is odd and f' and $\xi \to f(\xi)/\xi$ are increasing for $\xi < 0$ and decreasing for $\xi > 0$ (and the hypotheses of Kaplan and Yorke are satisfied). Hence $(x(t), \dot{x}(t)) \to o_f$ as $t \to \infty$ for every slowly oscillating solution in this case.

If in addition $|f| \leqslant c\,|\mathrm{id}|$, $c < \sqrt{2} + \frac{1}{2}$, then Proposition 3 yields x slowly oscillating whenever $(x(t), \dot{x}(t)) \to o_f$ as $t \to \infty$ since trajectories of solutions not slowly oscillating tend to $(0, 0) \notin o_f$. Therefore $S_f = \{\phi \in X: (x(t), \dot{x}(t)) \to o_f \text{ as } t \to \infty\}$, and Corollary 4 implies

COROLLARY 5. *The domain of attraction of the unique periodic orbit o_f is open and dense provided f is odd, bounded, continuously differentiable with $f' > 0, f'(0) > \pi/2, f'$ and $\xi \to f(\xi)/\xi$ increasing for $\xi > 0$, and decreasing for $\xi > 0$, $|f| \leqslant c\,|\mathrm{id}|$ with $c < \sqrt{2} + \frac{1}{2}$.*

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