THE CHANG - MARSHALL ALGEBRAS

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Contents

Preface

The purpose of this work is to study the ideal structure of the Chang-Marshall algebras QA_B and $C_B \cap H^\infty$ of analytic functions in the open unit disk $\mathbb{D}$. These were introduced by Chang [3] and later studied by Chang and Marshall [4] and also by Sundberg and Wolff [34]. In the first two parts of our work we deal with two special cases of Chang-Marshall algebras, namely the disk algebra

$A(\mathbb{D}) = \{f\colon \overline{\mathbb{D}} \to \mathbb{C},\ f \text{ continuous on } \overline{\mathbb{D}} \text{ and analytic in } \mathbb{D}\}$

and the algebra

$QA = \{f \in H^\infty\colon f(e^{it}) \text{ is of vanishing mean oscillation}\}$.

The third part will be devoted to the Chang-Marshall algebras in general.

In § 1 we answer two questions of F. Forelli ([7], [8]) concerning divisibility problems in $A(\mathbb{D})$. In particular we prove that every ideal of denominators in $A(\mathbb{D})$ has the Forelli property; i.e., there exists a function $f \in I$ such that its zero set $Z(f) = \{z \in \overline{\mathbb{D}}\colon f(z) = 0\}$ coincides with the zero set $Z(I) = \bigcap_{f \in I} Z(f)$ of the ideal.

In § 2 we study the relation between the ideals in $A(\mathbb{D})$ and those of the algebra $C(T)$ of all continuous complex-valued functions on the unit circle T. For instance, in Theorem 1 we give a complete characterization of those prime ideals in $A(\mathbb{D})$ which are traces of prime ideals in $C(T)$. Our characterization is based on an extension of Havin's notion of the F-property to ideals.

In § 3 we show that, in analogy to $A(\mathbb{D})$, the algebra QA is a Prebezout ring. Our proof is based on a generalization of the corona theorem for QA.

In § 4 we continue the research on closed ideals in QA, begun in the paper of Gorkin, Hedenmalm and the author [15]. Using the characterization theorem ([15], Theorem 2.5) for closed ideals in QA, we present a complete description of those closed ideals in QA which can be generated topologically by a single function. Whereas in $A(\mathbb{D})$ every closed ideal is the closure of a principal ideal, we shall see that in QA the situation is quite different.

In § 5 we use Hoffman's theory of the analytic structure of the maximal ideal space of H^∞ ([19]) to establish the precise relations between interpolating sequences and the Gleason part structure of the maximal ideal space $M(A)$ of an arbitrary Chang-Marshall algebra A.

One of the main themes of this work will be developed in § 6, where we examine the relations between the ideals in a Chang-Marshall algebra A and those in H^∞. In Theorem 6.5 we shall prove that every ideal I in A whose zero set

$$Z(I) = \{m \in M(A): \hat{f}(m) = 0 \text{ for all } f \in I\}$$

does not meet the Shilov boundary ∂A of A is the trace of a unique ideal in H^∞. The interesting feature of this result is that not only there exists an ideal J in H^∞ such that $J \cap A = I$, but that J is uniquely determined. Our proof is based on the fact that every such ideal I is generated algebraically by a set of inner functions. This result generalizes theorems of von Renteln [33] and Marshall [23]. As a useful corollary we obtain that the inner functions in any Chang-Marshall algebra A separate the points of $M(A) \setminus \partial A$.

Using the extension theorems of § 6 we shall be able to give in § 7 a complete characterization of the finitely generated prime ideals in an arbitrary Chang-Marshall algebra. This theorem extends the results for H^∞ proved by Gorkin [14] and the author [19].

§ 8 is devoted to the study of the closed ideals in Chang-Marshall algebras A. We present first a description of the countably generated closed ideals in QA_B. This result contains that for H^∞ obtained earlier by the author ([26] and [27]). The proofs, however, are different. Also we prove that any closed ideal in A which contains an interpolating Blaschke product is an intersection of maximal ideals.

We conclude this paper by exposing some peculiar properties of several Chang-Marshall algebras. For instance, we prove that there exists a Douglas algebra B such that in contrast to Axler's multiplier theorem for $B = L^\infty$ ([1]) not every function $f \in B \cap \bar{B}$ can be multiplied into $QA_B + C$ by a unimodular function $u \in L^\infty$.

I am deeply indebted to Professor Pamela Gorkin for many valuable discussions during her visits in Karlsruhe. Also I thank Professor Frank Forelli for having simplified some parts of the proof of Theorem 1.2. Finally, I want to thank Professor Michael von Renteln for his kind support during the preparation of this work.

Notations and conventions

Let $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ denote the open unit disk, $\overline{\mathbb{D}} = \{z \in \mathbb{C}: |z| \leq 1\}$ its closure and $T = \partial\mathbb{D} = \{z \in \mathbb{C}: |z| = 1\}$ the boundary of $\mathbb{D}$. As usual, H^∞ is the space of bounded analytic functions in $\mathbb{D}$, whereas L^∞ is the space of (equivalence classes) of essentially bounded, Lebesgue measurable functions on the unit circle T. Under the usual pointwise algebraic operations and the supremum norm (resp. essential supremum norm), H^∞ and L^∞ are commutative Banach algebras with identity element.

We assume that the reader is familiar with the theory of bounded analytic functions as presented in the books of Garnett [11] and Hoffman [18].

Throughout this paper a Blaschke product will be denoted by the letter b and an inner function by the letter u. We write Sing u for the set of boundary singularities of u, that is

$$\text{Sing } u = \{e^{it} \in T: u \text{ cannot be extended holomorphically to the point } e^{it}\} .$$

The zero set in $\mathbb{D}$ of an inner function u is the set

$$Z_{\mathbb{D}}(u) = \{z \in \mathbb{D}: u(z) = 0\} .$$

If the inner function u is a greatest common divisor of the inner parts of the functions in a set $I \subseteq H^\infty$, then u is called the inner factor of I. Observe that u is uniquely determined up to unimodular constants (see [18], p.85).

Let A be a commutative Banach algebra with identity element and let M(A) be its maximal ideal space. For an element

$f \in A$,

$$Z(f) = \{m \in M(A) : \hat{f}(m) = 0\}$$

denotes the zero set of the Gelfand transform $\hat{f}$ of f. Similarly,

$$Z(I) = \bigcap_{f \in I} Z(f)$$

is the zero set (or hull) of an ideal I in A. A^{-1} stands for the set of invertible elements in A. A uniform algebra A is a commutative Banach algebra with identity element such that the map $f \to \hat{f}$ is an isometry of A onto $\hat{A}$, the space of Gelfand transforms of A, supplied with the supremum norm on M(A) (see [11], p.185). Note that we can (and we will do this) identify in a uniform algebra a function f with its Gelfand transform (see [11], p.186).

Finitely generated ideals I in A are ideals of the form

$$I = (f_1, \ldots, f_N) = \{f \in A : f = \sum_{i=1}^{N} g_i f_i,\ g_i \in A\} .$$

Countably generated ideals are defined analogously:

$$I = (f_1, f_2, \ldots) = \{f \in A : f = \sum_{i=1}^{n} g_i f_i,\ g_i \in A,\ n \in \mathbb{N}\} .$$

An ideal is called principal, if it is generated by single element.

Let B be a commutative algebra which contains A and let I be an ideal in A. Then the ideal J generated by I in B will be denoted by $J = IB$. Note that $J = \{f \in B : f = \sum_{i=1}^{n} g_i f_i,\ g_i \in B,\ f_i \in I,\ n \in \mathbb{N}\}$. We shall say that an ideal $I \subseteq A$ can be lifted to an ideal $J \subseteq B$ if and only if $J \cap A = I$. The ideal I is then said to be the trace of J in A.

A prime ideal $P \subseteq A$ is a proper ideal such that $fg \in P$ implies $f \in P$ or $g \in P$. An ideal I in A is called (topologically) primary, if it is contained in a unique maximal ideal.

By an integral domain we understand as usual a commutative ring with identity element and without zero divisors. The term "greatest common divisor" will be abbreviated by "gcd".

Let ∂A denote the Shilov boundary of the uniform algebra A. A closed subset $E \subseteq \partial A$ is called a peak set for A if there exists a function $p \in A$ such that

$$p(x) = 1 \quad \text{for} \quad x \in E \quad \text{and}$$

$$|p(x)| < 1 \quad \text{for every } x \in \partial A \setminus E .$$

The function p is called a peak function for E. A weak peak set in A is an intersection of peak sets in A.

I. The disk algebra $A(\mathbb{D})$

Let $A(\mathbb{D})$ denote the disk algebra, that is the Banach algebra of all those continuous functions in $\overline{\mathbb{D}}$ which are analytic in $\mathbb{D}$, under the usual pointwise algebraic operations and the supremum norm. We may regard $A(\mathbb{D})$ as a subalgebra of the space $C = C(T)$ of all continuous functions on the Shilov boundary T of $A(\mathbb{D})$ (see [11], p.200).

For a compact set $E \subseteq T$, $I(E, A(\mathbb{D}))$ will be the ideal of all those functions in $A(\mathbb{D})$ which vanish on E, i.e.

$$I(E, A(\mathbb{D})) = \{f \in A(\mathbb{D}) : \; f = 0 \text{ on } E\} .$$

Let $K \subseteq T$ be a nonempty compact set of Lebesgue measure zero on T. Then it is well known that there exists a function $p_K \in A(\mathbb{D})$ such that

$$p_K(z) = 1 \quad \text{for } z \in K \text{ and}$$

$$|p_K(z)| < 1 \quad \text{for } z \in \overline{\mathbb{D}} \setminus K$$

(see [18], p.80,81). Hence K is a peak set for $A(\mathbb{D})$ and p_K is the associated peak function. If $K = \emptyset$, we put $p_K(z) \equiv 0$.

Finally, let us mention for later references the famous theorem of Beurling and Rudin on the characterization of the closed ideals in $A(\mathbb{D})$.

<u>Theorem</u> (Beurling and Rudin, see [18], p.82 ff.).

(1) Let $I \neq (0)$ be a closed ideal in $A(\mathbb{D})$ with inner factor u. Then there exists a compact set $E \subseteq T$ of Lebesgue measure zero such that

$$I = uI(E,A(\mathbb{D})). \qquad (1)$$

In particular, Sing $u \subseteq E$ and $E = Z(I) \cap T$.

(2) If u is an inner function whose boundary singularities are contained in a compact set $E \subseteq T$ of Lebesgue measure zero, then conversely every closed ideal of the form (1) is closed and non-zero.

(3) Every closed ideal I in $A(\mathbb{D})$ is the closure of a principal ideal. If $I \neq (0)$, the generator can be taken to be the function $u(1-p_E)$, where u is the inner factor of I and p_E is the peak function for the set $E = Z(I) \cap T$.

§ 1 Divisibility problems and ideals of denominators in the disk algebra $A(\mathbb{D})$

F. Forelli ([7], p.389) posed the problem of classifying those ideals I in $A(\mathbb{D})$ which have the property that there exists a function $f \in I$ such that the zero set of f agrees with the zero set of the ideal I; i.e., for which ideals I there is a function $f \in I$ with $Z(f) = Z(I)$? Such a property will be referred to as the "Forelli property". It is well known that every closed ideal I in $A(\mathbb{D})$ has this property, since by the Beurling-Rudin theorem I is the closure of a principal ideal. On the other hand, as we have shown in ([27], p.261), one cannot expect that finitely generated ideals in $A(\mathbb{D})$ have the Forelli property. Nevertheless, we could prove ([27], p.262) that finitely generated ideals I

have the weak Forelli property, i.e., there exists a function $f \in I$ such that $Z(f) \cap T = Z(I) \cap T$.

F. Forelli ([7], p.389) now asked in particular if in $A(\mathbb{D})$ the ideals of denominators, defined in the following way, have the Forelli property.

Definition 1. Let A denote either $A(\mathbb{D})$ or H^∞. Then an ideal $I \subseteq A$ is said to be an ideal of denominators, if there exists a quotient $\gamma = f_1/f_2$ of two functions f_1 and f_2 in $A \setminus \{0\}$ such that

$$I = Q(\gamma) = \{f \in A: f\gamma \in A\} .$$

The situation in H^∞, where in fact every ideal of denominators is principal ([8], p.397), gives us some hints in favour of a positive answer. Moreover, it was known that if an ideal of denominators in $A(\mathbb{D})$ has the weak Forelli property, then it already has the Forelli property (compare [8], p.396). In Theorem 1.2 we shall now give a positive answer to Forelli's question. Our proof is based on the following lemma.

Lemma 1.1. Let u be a continuous positive function on an open interval $I \subseteq \mathbb{R}$. Then there exists a continuously differentiable function v on I (for short we write $v \in C^1(I)$) such that

$$|u-v| \leq \frac{1}{2} u \quad \text{on } I .$$

Proof. We map the interval I by a continuously differentiable function bijectively onto $\mathbb{R}$ and apply, e.g., the approximation theorem of Carleman ([9], p.135).

Definition 2. Let I be an ideal in $A(\mathbb{D})$. Then I is said to have the Forelli property, if there exists a function $f \in I$ such that $Z(f) = Z(I)$.

Definition 3. Let E be an open subset of the unit circle T. Then we denote by H_E^∞ the algebra of bounded analytic functions in $\mathbb{D}$ which are continuously extendable to E (compare [11], p.399). In particular, $H_T^\infty = A(\mathbb{D})$ and $H_\emptyset^\infty = H^\infty$.

Theorem 1.2. Every ideal of denominators in $A(\mathbb{D})$ has the Forelli property.

Remark. First we present the main ideas of the proof. Let $\gamma = F_1/F_2$ be a quotient of two outer functions in $A(\mathbb{D})$ and let $d \in H^\infty$ be a gcd of the functions F_1 and F_2 (with respect to the algebra H^∞). We may assume that d has the form

$$d(z) = \exp \frac{1}{2\pi} \int_T \frac{e^{it}+z}{e^{it}-z} \log\,(|F_1|+|F_2|)(e^{it})dt$$

(see [33], p.519). Then $\gamma = \left(\frac{F_1}{d}\right) / \left(\frac{F_2}{d}\right)$ is a quotient of two relatively prime factors in H^∞. In particular we have

$$\left|\frac{F_2}{d}\right| = \frac{|F_2|}{|F_1|+|F_2|} = \frac{1}{1+|\gamma|} \quad \text{a.e. on } T\,.$$

Let E be the largest open subset of T to which γ is continously extendable. The clue of the proof is now to construct a function $G_2 \in H_E^\infty$, which differs from F_2/d only through multiplication by an invertible function in H^∞. Then the function

$$f = (1-p_{T\setminus E})G_2$$

satisfies $Z(f) = Z(Q(\gamma))$, where $Q(\gamma) \subseteq A(\mathbb{D})$ is the ideal of denominators associated to γ. Also observe that in the algebra H^∞ the ideal of denominators associated to γ is the principal ideal generated by the function G_2 (or equivalently by F_2/d).

Proof of the theorem. Let I be an ideal of denominators in $A(\mathbb{D})$. Then there exists a quotient $\gamma = f_1/f_2$ of two functions in $A(\mathbb{D})$ such that $I = Q(\gamma)$. Let $u_1F_1 = f_1$ and $u_2F_2 = f_2$ be the inner outer factorization of the functions f_1 and f_2. We may assume without loss of generality that the inner factors u_1 and u_2 are relatively prime, since this does not change the values of γ.

Step 1. Put $\gamma_1 = f_1/F_2$ and let $Q(\gamma_1)$ be the associated ideal of denominators in $A(\mathbb{D})$. We put $E = T \setminus Z(Q(\gamma_1))$. Observe that $Z(Q(\gamma_1)) \subseteq T$, because $F_2 \in Q(\gamma_1)$. If $a = e^{it} \in E$, then there exists a function $f \in Q(\gamma_1)$ with $f(a) \neq 0$. In particular we have $\gamma_f = g/f$ for some $g \in A(\mathbb{D})$. Since a was an arbitrary point of E, we can thus conclude that γ_1 is continuous in $\mathbb{D} \cup E$.

Let us now consider the function $u = 1/1+|\gamma_1|$ which is defined and continuous on E. Because $Z(Q(\gamma_1))$ has Lebesgue measure zero, we can decompose the open set $E \subseteq T$ into a (at most) countable union of pairwise disjoint, open arcs $I_j = (a_j, b_j)$ such that $\bigcup_{j=1}^{\infty} \bar{I}_j = T$. By Lemma 1.1 there exists on each interval I_j a C^1-function v_j such that

$$|v_j - u| \le \tfrac{1}{2}|u| \quad \text{on } I_j \quad (j = 1, 2, \dots) \,. \tag{1}$$

Let v be defined on E by $v|I_j = v_j$. Then we obtain

$$0 < \tfrac{1}{2}\, u \le v \le \tfrac{3}{2}\, u \qquad \text{on } E \,. \tag{2}$$

In particular, $\log v \in L^1(T)$. Hence the function

$$G_2(z) = \exp \frac{1}{2\pi} \int \frac{e^{it}+z}{e^{it}-z} \log v(e^{it})\, dt$$

is an outer function in H^∞ satisfying $|G_2| = v$ a.e. on T. Because $\log v \in C^1(E)$, it follows from ([11], p.107) that the function G_2 is continuous on $\mathbb{D} \cup E$, i.e. $G_2 \in H^\infty_E$. By the left inequality in (2), G_2 vanishes nowhere on E. By the right inequality in (2) we obtain

$$\left| G_2 \frac{f_1}{F_2} \right| = |G_2 \gamma_1| \le |G_2|(1+|\gamma_1|) = |v|(1+|\gamma_1|) \le \tfrac{3}{2} \quad \text{a.e. on } T \,.$$

Since F_2 is outer, the extremal property for outer functions (see [18], p.62) yields that

$$|G_2 f_1| \le \tfrac{3}{2}|F_2| \text{ holds on } \mathbb{D} \,.$$

Hence $g_1 = G_2 f_1 / F_2$ is a bounded analytic function, and we have $\gamma_1 = g_1/G_2$. Moreover, $g_1 \in H^\infty_E$.

Now let $f = (1-p_{T\setminus E})G_2$, where $p_{T\setminus E}$ is the peak function associated to $T \setminus E$. Then $f \in A(\mathbb{D})$ and $f\gamma_1 = fg_1/G_2 = (1-p_{T\setminus E})g_1 \in A(\mathbb{D})$. Hence $f \in Q(\gamma_1)$. Since G_2 does not vanish on E, we have $Z(f) = T \setminus E = Z(Q(\gamma_1))$.

Step 2. Since $\gamma = \gamma_1/u_2$, it is clear that we have the following relation:

$g \in Q(\gamma)$ if and only if there exists $h \in Q(\gamma_1)$ such that $g = u_2 h$ and $\mathrm{Sing}\, u_2 \subseteq Z(h)$.

Hence the zero set $Z(Q(\gamma))$ contains $Z(Q(\gamma_1))$, $Z_{\mathbb{D}}(u_2)$ and $\mathrm{Sing}\, u_2$. By the first step there exists a function $f \in Q(\gamma_1)$ such that $Z(f) = Z(Q(\gamma_1))$. Put $g = u_2(1-p_{Z(Q(\gamma))\cap T})f$. Then $g \in Q(\gamma)$ and $Z(g) = Z(Q(\gamma))$. Moreover, we see that the inner factor of $Q(\gamma)$ is u_2. $\square$

An analysis of the proof shows that we also obtain the following information about the zero set of $Q(\gamma)$.

Corollary 1.3. Let $Q(\gamma)$ be an ideal of denominators in $A(\mathbb{D})$ with inner factor 1. Then a point $z \in \overline{\mathbb{D}}$ belongs to $\overline{\mathbb{D}} \setminus Z(Q(\gamma))$ if and only if there exists in $\overline{\mathbb{D}}$ a neighbourhood U of z such that $|\gamma|$ has continuous extension to U.

We want to derive now several interesting conclusions from Theorem 1.2.

Let $\emptyset \neq X \subseteq \overline{\mathbb{D}}$ be a subset of the maximal ideal space of $A(\mathbb{D})$. Then we define the ring A_X of fractions by

$$A_X = \{\frac{f}{g}: f, g \in A(\mathbb{D}),\ g \text{ vanishes nowhere in } X\} .$$

If $\xi \in \overline{\mathbb{D}}$, then the ring

$$A_\xi = \{\frac{f}{g}: f, g \in A(\mathbb{D}),\ g(\xi) \neq 0\}$$

is called the local ring of fractions in $A(\mathbb{D})$. These rings play an important rôle in ideal theory (see [22], p.22 f.). F. Forelli now asked in ([7], p.389) and ([8], p.396) for a characterization of those sets X for which

$$A_X = \bigcap_{\xi\in X} A_\xi . \tag{3}$$

In the algebra H^∞ every ring of fractions is the intersection of its local rings of fractions ([8], p.397). In the disk algebra $A(\mathbb{D})$ one could prove the relation (3) only for special sets X: Forelli ([8], p.396) has proven, e.g., that (3) holds whenever $X \cap T$ is closed. Using Theorem 1.2, we are now in a position to give a complete solution to this problem.

Theorem 1.4. *If $X \neq \emptyset$ is an arbitrary subset of the closed unit disk, then the ring A_X of fractions in $A(\mathbb{D})$ is the intersection of its local rings of fractions, i.e.,*

$$A_X = \bigcap_{\xi\in X} A_\xi .$$

Proof. Let $\gamma \in \bigcap_{\xi\in X} A_\xi$. Then for every $\xi \in X$, γ can be represented in the form $\gamma = f/g$, where $g(\xi) \neq 0$. Thus $g \in Q(\gamma)$, the ideal of denominators associated to γ, and $Z(Q(\gamma)) \cap X = \emptyset$. Choose by Theorem 1.2 a function $f \in Q(\gamma)$ such that $Z(f) = Z(Q(\gamma))$. Then $\gamma = f\gamma/f \in A_X$. Hence $\bigcap_{\xi\in X} A_\xi \subseteq A_X$. The reverse inclusion is trivial. □

It is well known that not every ideal of denominators in $A(\mathbb{D})$ is principal (see [8], p.397); e.g. let $\gamma = f_1/f_2$, where $f_2(z) = 1-z$ and $f_1(z) = (1-z)\exp\left(-\frac{1+z}{1-z}\right)$. Then $Q(\gamma)$ is the maximal ideal $M(1) = \{f \in A(\mathbb{D}): f(1) = 0\}$, which is surely not principal.

If, more generally, γ has the form $\gamma = u_1(1-p_E)/u_2(1-p_E)$, where the inner factors u_1 and u_2 are relatively prime and where $\operatorname{Sing} u_1 \cup \operatorname{Sing} u_2 = E$, then $Q(\gamma) = u_2 I(E, A(\mathbb{D}))$. Hence every closed ideal in $A(\mathbb{D})$ is an ideal of denominators, in contrast to H^∞, where a closed ideal I is an ideal of denominators if and only if I is a principal ideal generated by an inner function. This follows from ([28], p.546) and the fact that in H^∞ every ideal of denominators is principal.

It is now of interest to characterize in $A(\mathbb{D})$ those ideals of denominators which are closed. The following proposition gives us a complete answer.

<u>Proposition 1.5.</u> An ideal of denominators $Q(\gamma)$ in $A(\mathbb{D})$ is closed if and only if $\gamma \in L^\infty$.

<u>Proof.</u> If $\gamma \in L^\infty$, then $Q(\gamma)$ is trivially closed. Conversely, let $Q(\gamma)$ be closed and let u be the inner factor of $Q(\gamma)$. Then by the Beurling-Rudin theorem $Q(\gamma) = uI(E, A(\mathbb{D}))$, where $E = Z(Q(\gamma)) \cap T$.

Put $\gamma_1 = \gamma u$. Then

$$I(E, A(\mathbb{D})) = \frac{1}{u} Q(\gamma) \subseteq \{f \in H^\infty : f\gamma_1 \in H^\infty\} . \qquad (4)$$

Let $Q = \{f \in H^\infty : f\gamma_1 \in H^\infty\}$. Since Q is an ideal of denominators in H^∞, it is principal. But by (4) the gcd of the functions in Q is 1. Hence $Q = H^\infty$, so that $\gamma_1 \in H^\infty$. This implies that $\gamma = \bar{u}\gamma_1 \in L^\infty$. $\square$

§ 2 Relations between ideals in $A(\mathbb{D})$ and $C(T)$

In this section we consider for the pair $(R,S) = (C(T),A(\mathbb{D}))$ the following well known problem in ideal theory (see [13], § 11). Let R be a ring, S a subring and I a prime (resp. closed) ideal in S. Under what conditions does there exist a prime (resp. closed) ideal J in R such that $J \cap S = I$? By the Beurling-Rudin theorem it is easy to see that a closed ideal I in $A(\mathbb{D})$ is the trace of a closed ideal J in $C(T)$, i.e. $I = J \cap A(\mathbb{D})$, if and only if the inner factor of I is 1. On the other hand, since the inner factor of a nonmaximal prime ideal $P \subseteq A(\mathbb{D})$ is always invertible ([26], p.23 f.), one cannot expect that this property is sufficient for a prime ideal in $A(\mathbb{D})$ to be the trace of a prime ideal in $C(T)$. Indeed we have the following example.

<u>Example 2.1.</u> Let I be the ideal in $A(\mathbb{D})$ generated by the function $S(z)(1-z)$, where $S(z) = \exp\left(-\frac{1+z}{1-z}\right)$, and let $M = \{F \in A(\mathbb{D}):\ F \text{ outer}\}$. Then I is disjoint from the multiplicatively closed set M. Hence there exists by Zorn's lemma an ideal $P \supseteq I$ which is maximal with respect to $P \cap M = \emptyset$. By Krull's lemma P is prime. This ideal cannot occur as the trace of any ideal J in C, since otherwise J (and hence $P = J \cap A(\mathbb{D})$) would contain the outer function $(1-z)^2 = [\bar{S}(z)(1-z)]S(z)(1-z)$, in contradiction to the choice of P. $\square$

A more careful examination of the structure of the closed ideals in $A(\mathbb{D})$ yields the following observation. Every closed ideal I in $A(\mathbb{D})$ with inner factor 1 has the property that whenever $f \in I$ then the outer part of f also belongs to I. This leads to the following definition.

Definition 1. Let I be an ideal in $A(\mathbb{D})$. Then I is said to have the F-property if whenever $f \in I$ the outer part of f also belongs to I. Such an ideal is called an F-ideal.

Note that the concept of F-ideals extends Havin's notion of the F-property for subrings of H^1 (see [25]) to ideals. The following theorem now shows that this is just the right idea which enables us to give a complete answer to the problem mentioned above.

Theorem 2.1. A prime ideal P in $A(\mathbb{D})$ is the trace of a prime ideal Q in C(T) if and only if P has the F-property.

Proof. Let $P \subseteq A(\mathbb{D})$ be the trace of a prime ideal Q in C(T) and let f be a nonzero function in P. If $f = uF$ is the inner-outer factorization of f, then $\bar{u}F \in C(T)$. Hence $F^2 = (\bar{u}F)uF \in Q \cap A(\mathbb{D}) = P$. Since P is prime, we see that $F \in P$.

Conversely, let P be a prime F-ideal in $A(\mathbb{D})$ and let $J = PC(T)$ be the ideal generated by P in C(T). We claim that $J \cap A(\mathbb{D}) = P$. Let $f \in J \cap A(\mathbb{D})$. Then there exist functions $f_i \in P$ and $q_i \in C(T)$ such that $f = \sum_{i=1}^{n} q_i f_i$. In particular, $|f| \leq C \sum_{i=1}^{n} |f_i|$ on T for some constant $C > 0$. By ([27], p.262) there exists a function $h \in (f_1, \ldots, f_n) \subseteq P$ with

$$\sum_{i=1}^{n} |f_i|^2 \le |h| \quad \text{on } T \, .$$

The Cauchy-Schwarz inequality yields

$$|f|^2 \le C^2 \left(\sum_{i=1}^{n} |f_i| \right)^2 \le C^2 n^2 \sum_{i=1}^{n} |f_i|^2 \le C^2 n^2 |h| \quad \text{on } T \, .$$

Let $uH = h$ be the inner-outer factorization of h. Since P is an F-ideal and $h \in P$, the function H also belongs to P. Because H is outer, we have

$$|f|^2 \le C^2 n^2 |H| \quad \text{on } \overline{\mathbb{D}}$$

(see [18], p.62). Hence $|f|^4/|H| \le C^4 n^4 |H|$ on $\overline{\mathbb{D}}$, from which we can conclude that $f^4 = gH$ for some $g \in A(\mathbb{D})$. Thus $f^4 \in P$. Since P is prime, $f \in P$; consequently, $J \cap A(\mathbb{D}) = P$.

Let $S = A(\mathbb{D}) \setminus P$. Then S is a multiplicatively closed subset of $C(T)$ which is disjoint from the ideal J. Hence there exists by Zorn's lemma and Krull's lemma a prime ideal $Q \supseteq J$ with $Q \cap S = \emptyset$. Moreover, we have $P \subseteq Q \cap A(\mathbb{D}) \subseteq P$. Thus Q is the desired prime ideal. □

<u>Definition 2.</u> Let R be a commutative ring and let I be an ideal in R. The ideal I is called radical if it is the set of all elements in R having some power in I, i.e., if $f \in I$ whenever $f^n \in I$ for some $n \in \mathbb{N}$ (see [22], p.17).

Using Zorn's and Krull's lemmas, it is easy to see that an ideal I is radical if and only if it is an intersection of prime ideals. As an immediate consequence of Theorem 2.1 we obtain thus the following corollary.

Corollary 2.2. A radical ideal $P \subseteq A(\mathbb{D})$ is the trace of a radical ideal Q in $C(T)$ if and only if P has the F-property.

The following example shows that there do exist non-maximal prime ideals in $A(\mathbb{D})$ having the F-property.

Example 2.2. Let $\alpha_n \in T$ be a sequence converging to $\alpha = 1$ and let $f \in A(\mathbb{D})$ be a function with $Z(f) = \{\alpha_n : n \in \mathbb{N}\} \cup \{1\}$. Define

$$S = \{f \in C(T) : \text{either } f(1) \neq 0 \text{ or } z = 1 \text{ is an isolated point of the zero set of } f\} .$$

Then the ideal $J = fC(T)$ is disjoint from the multiplicatively closed set S. Hence by Zorn's and Krull's lemmas there exists a prime ideal $Q \supseteq J$ disjoint from S. Let $P = Q \cap A(\mathbb{D})$. Then P is a non-maximal prime ideal in $A(\mathbb{D})$ which has the F-property by Theorem 2.1. Note that $0 \neq P \subseteq M(1)$ and that $1-z \notin P$. □

Remark. The concept of F-ideals will be studied in more detail in my common paper [16] with P. Gorkin. In particular, we present the generalization of Theorem 2.1 and its corollary to the algebras QA_B.

Another important problem in the analysis of the ideal structure of the pair $(A(\mathbb{D}), C(T))$ is to characterize those ideals J in $C(T)$ whose trace in $A(\mathbb{D})$ does not collapse to the trivial ideal $I = (0)$. Since by the theorem of Szegö ([11], p.64)

$$\int_T \log |f(e^{it})| dt > -\infty \tag{1}$$

for any nonzero function $f \in A(\mathbb{D})$, every such ideal J necessarily contains a function $f \in C(T)$ satisfying (1). The next proposition now shows that this is also sufficient.

Proposition 2.3. The trace in $A(\mathbb{D})$ of an ideal J in C(T) is non-trivial, i.e. $J \cap A(\mathbb{D}) \neq (0)$, if and only if J contains a function q satisfying

$$\int_T \log |q(e^{it})| dt > -\infty \ . \tag{2}$$

Proof. It only remains to show the sufficiency of (2). By (2) the zero set $Z(q) = \{z \in T: q(z) = 0\}$ is a compact set of Lebesgue measure zero. Hence we can decompose $T \setminus Z(q)$ into a (at most) countable union of disjoint open arcs $I_j = (a_j, b_j)$. By Lemma 1.1 there exist nonnegative functions $u_j \in C^1(I_j)$ such that

$$|u_j - q| \leq \tfrac{1}{2}|q| \quad \text{on} \quad I_j \tag{3}$$

and $u_j(a_j) = u_j(b_j) = 0$.

In particular,

$$\tfrac{1}{2}|q| \leq |u_j| \leq \tfrac{3}{2}|q| \quad \text{on} \quad I_j \ . \tag{4}$$

Let $u: T \to \mathbb{R}$ be defined by $u = u_j^2$ on I_j $(j = 1,2,\ldots)$. Then $u \in C(T)$ and by the left inequality in (4), $\log u \in L^1(T)$. Since $u \in C^1(T \setminus Z(q))$ and $u|Z(q) \equiv 0$, the outer function

$$f(z) = \exp \frac{1}{2\pi} \int_T \frac{e^{it}+z}{e^{it}-z} \log u(e^{it}) dt$$

belongs by ([32], p. 52) to the disk algebra $A(\mathbb{D})$. Moreover, we have $\left|\frac{f}{q}\right| = \frac{u}{|q|} \le \frac{9}{4}|q|$ on T. Thus $f/q \in C(T)$ from which we can conclude that $f \in qC(T) \cap A(\mathbb{D}) \subseteq J \cap A(\mathbb{D})$. □

Remark. If we merely assume that the ideal J contains a function $q \in C(T)$ whose zero set has Lebesgue measure zero, then the conclusion of Proposition 2.3 does not hold. To this end let

$$q(e^{it}) = \begin{cases} \exp\left(-\frac{1}{t^2}\right), & t \neq 0\ ,\ t \in [-\pi,\pi]\ , \\ 0\ , & t = 0\ . \end{cases}$$

Then the ideal $J = qC(T)$ does not contain any function f such that $\int_T \log|f(e^{it})|dt > -\infty$. Hence $J \cap A(\mathbb{D}) = (0)$, though $Z(q) = \{1\}$. □

To conclude this section, we want to state two closely related open problems.

Problem 1. Let $I = (f_1, \ldots, f_N)$ be a finitely generated ideal in $A(\mathbb{D})$. Under what conditions on the generators $f_1, \ldots, f_N$ can I be lifted to an ideal J in C(T) ?

Comment: We note that a necessary condition is that $uF \in I$ implies $F^2 \in I$, where uF is the inner-outer factorization of a function in $A(\mathbb{D})$.

Problem 2. Let $I = (f_1, \ldots, f_N)$ be a finitely generated ideal in $A(\mathbb{D})$ which has the F-property. Does this imply that I is a principal ideal generated by an outer function?

II. The algebra QA of bounded analytic functions of vanishing mean oscillation

It is well known that the Banach algebra H^∞, supplied with the supremum norm, may be regarded via radial limits as a subalgebra of L^∞, the space of all essentially bounded Lebesgue measurable functions on the unit circle T.

Sarason has shown that the space (under the usual norm)

$$H^\infty+C = \{f+g: f \in H^\infty, g \in C = C(T)\}$$

is the smallest uniformly closed subalgebra of L^∞ that contains H^∞ properly (see [11], § 9). In connection with $H^\infty+C$, he considered the Banach algebra

$$QC = \{f \in H^\infty+C: \bar{f} \in H^\infty+C\} ,$$

i.e., the largest C^*-algebra contained in $H^\infty+C$, known as the algebra of quasicontinuous functions, and its analytic subalgebra

$$QA = QC \cap H^\infty .$$

Using a famous theorem of Fefferman and Stein, Sarason proved that QC coincides with the space of essentially bounded functions of vanishing mean oscillation (VMO) on T (see [11], p.377), i.e. $QC = VMO \cap L^\infty$. Later on, T.H. Wolff [35] gave a detailed study of the spaces QC and QA. His main result is that every function $f \in L^\infty$ can be multiplied into QC by an outer function $F \in QA$ ([35], p.321). He also showed that in many situations QA behaves essentially in the same way as the disk algebra $A(\mathbb{D})$: e.g., QA is a Dirichlet algebra on the maximal ideal space M(QC) of QC ([35], p.325). Moreover, the analogue of the classical F. and M. Riesz theorem for $A(\mathbb{D})$ (see [18], p.47) holds in QA. This fact, proved by Wolff ([35], p.325) was used by Gorkin, Hedenmalm and the author ([15]) to give a complete characterization of the closed ideals in QA. Before we recall this result, we introduce the following notations.

Let λ be the normalized Lebesgue measure on T. Then σ will denote the Borel measure determined on M(QC) by the functional $L(f) = \int_T f\,d\lambda = \int_{M(QC)} \hat{f}\,d\sigma$ $(f \in QC)$. It is usually called the lifted Lebesgue measure. For a closed subset E of M(QC), let I(E,QA) be the ideal of all those functions in QA whose Gelfand transform $\hat{f}$, which we identify with f, vanishes on E, i.e.

$$I(E,QA) = \{f \in QA:\ f|_E = 0\}\ .$$

The QA-analogue of the Beurling-Rudin theorem now takes the following form:

<u>Theorem</u> ([15], Theorem 2.5). Let $I \neq (0)$ be a closed ideal in QA. Then there exists an inner function u and a closed subset E of M(QC) with $\sigma(E) = 0$ such that $I = uI(E,QA)$. In particular, $E = Z(I) \cap M(QC)$. The inner function u is called the inner factor of I.

In this chapter we now present the QA-analogues of some algebraic properties of the disk algebra $A(\mathbb{D})$, e.g., we prove in § 3 that QA has the rather striking property that it is a Prebezout ring (in the terminology of P.M. Cohn ([5], p.260)). But there are also some differences. For example, it is known that unlike $A(\mathbb{D})$ not every closed ideal in QA is the closure of a principal ideal ([15], § 2). In § 4 we shall give a complete characterization of this class of ideals.

§ 3 Divisibility in QA

It is well known that in QA the Corona Theorem is true ([34], p.563), i.e., the open unit disk is dense in M(QA), the maximal ideal space of QA. For the sake of completeness, we present a short proof of the following equivalent algebraic form. To this end recall that $QC = QA + C$, from which we can easily deduce that

$$M(QA) = M(QC) \cup \mathbb{D} \tag{1}$$

(see [35], p. 325).

Corona Theorem for QA ([34], p.563). The ideal $I = (f_1,\dots,f_N)$ generated by the functions $f_1,\dots,f_N$ in QA coincides with the whole algebra QA *if and only if*

$$\sum_{i=1}^{N} |f_i| \ge \delta > 0 \quad \text{holds in } \mathbb{D}$$

for some constant $\delta > 0$.

Proof. Let $\sum_{i=1}^{N} |f_i| \ge \delta > 0$ in $\mathbb{D}$. Using a result of Shilov on extensions of maximal ideals ([12], § 12) applied to the algebras QC and L^∞, it is easy to see that a function in QC is invertible if and only if it is essentially bounded away from zero on T. Hence

$$\sum_{i=1}^{N} \frac{\bar{h}_i}{\sum_{j=1}^{N} |h_j|^2} h_i$$

is a representation of the function 1 by means of QC functions. Consequently, $Z(I) \cap M(QC) = \emptyset$. Since by hypothesis, the functions f_i have no common zero in $\mathbb{D}$, we have also $Z(I) \cap \mathbb{D} = \emptyset$. Thus, by (1), $Z(I) \cap M(QA) = \emptyset$. Hence I is not a proper ideal. The converse assertion is trival. □

As a consequence of the corona theorem we obtain the fact that if $\sum_{i=1}^{N} |f_i| \geq \delta > 0$ holds in $\mathbb{D}$, then 1 is a greatest common divisor (gcd) of the functions $f_1, \ldots, f_N$ in QA. On the other hand, however, two functions in QA may not have a gcd, as it is the case in H^∞ ([33], p.519). Take, e.g., the functions $f_1(z) = 1-z$ and $f_2(z) = (1-z)\exp\left(-\frac{1+z}{1-z}\right)$. The surprising ingredient of the next theorem is now that, whenever the gcd of the functions $f_1, \ldots, f_N$ exists in QA, then it is a linear combination of the f_i and hence belongs to the ideal $I = (f_1, \ldots, f_N)$. This fact, which we have already encountered in $A(\mathbb{D})$ ([32], p.54), highly contrasts with the situation in H^∞. In order to prove this theorem, we need the following results of [16] and [15].

<u>Proposition 3.2</u> ([16], Lemma 2.2). Let $f, g \in QA$ with $|f| \leq |g|$ on $\mathbb{D}$. Then there exists a function $h \in QA$ such that $f^2 = hg$.

Proposition 3.3 ([16], Theorem 1.1). Let $f_1, \dots, f_N \in QA$. Then there exists a function $f \in I = (f_1, \dots, f_N)$ such that

$$|f| \geq \frac{1}{2N} \sum_{i=1}^{N} |f_i|^2 \quad \text{on} \quad M(QC) .$$

Proposition 3.4 ([15], Lemma 2.3). Let $f \in QA$ have the factorization $f = ug$, where u is an inner function and $g \in H^\infty$. Then $g \in QA$. Moreover, if $m(f) = 0$ for some $m \in M(QC)$, then $m(g) = 0$. In particular, QA has the F-property (in the sense of Havin).

We note that these propositions actually hold in the more general setting of the algebras QA_B (see [16]). We also observe that these are nontrivial generalizations of the corresponding results in the disk algebra $A(\mathbb{D})$ ([27]).

Lemma 3.5. Let $F_1, \dots, F_N$ be outer functions in QA. Then there exists an outer function $F \in QA$ which is a common divisor of the F_i's and satisfies $\bigcap_{i=1}^{N} Z(F_i) = Z(F)$.

Observe that by our conventions, $Z(F)$ denotes the zero set of the Gelfand transform of F. Otherwise, our lemma would be trivial (take $F = 1$).

Proof. According to Proposition 3.3 we construct a function $h \in (F_1, \dots, F_N)$ such that

$$\sum_{i=1}^{N} |F_i|^2 \leq |h| \quad \text{on} \quad M(QC) . \tag{1}$$

Let $uH = h$ be the inner-outer factorization of h. By Proposition 3.4 we have $H \in QA$ and $Z(H) = Z(h) \cap M(QC)$. Note that $M(QC) = M(QA) \setminus \mathbb{D}$. Obviously, $Z(h) \cap M(QC) = \bigcap_{i=1}^{N} Z(F_i)$. Therefore, $Z(H) = \bigcap_{i=1}^{N} Z(F_i)$. Let $F = \sqrt[4]{H}$. By ([16], Lemma 3.3), $F \in QA$ and $Z(H) = Z(F)$. Since u is inner, we obtain by (1):

$$|F_i|^2 \leq |uH| \leq |H| = |F|^4 \quad \text{on} \quad M(QC)\ .$$

Hence $|F_i| \leq |F|^2$ a.e. on T. Since F_i and F are outer functions, we get $|F_i| \leq |F|^2$ on $\mathbb{D}$. Proposition 3.2 now implies that $F_i^2 = h_i F^2$ for some (outer) functions $h_i \in QA$. Thus F divides all the F_i $(i = 1,\dots,N)$. □

We are now in a position to prove the main theorem of this paragraph; it is an algebraic generalization of the corona theorem.

Theorem 3.6. If the functions $f_1,\dots,f_N$ in QA have a gcd d, then there exist functions $g_1,\dots,g_N \in QA$ such that $d = \sum_{i=1}^{N} f_i g_i$.

Proof. Let $f_i = B_i S_i F_i$ be the canonical factorization of the functions f_i, where B_i are the Blaschke products, S_i the singular inner parts and F_i the outer parts associated to f_i. Let $I = (f_1,\dots,f_N\rangle$. We claim that $Z(d) = Z(I)$. Let B be the Blaschke factor associated to d. Since d is a gcd of the f_i, it is clear that B is that Blaschke product formed by the common zeros of the B_i (including multiplicities). Hence

$$Z(d) \cap \mathbb{D} = Z(I) \cap \mathbb{D}. \tag{1}$$

According to Lemma 3.5, we construct an outer function $F \in QA$

dividing all the F_i's such that $\bigcap_{i=1}^{N} Z(F_i) = Z(F)$ and $|F_i| \le |F|^2$ on $\mathbb{D}$. By Proposition 3.4, $Z(f_i) \setminus \mathbb{D} = Z(F_i)$. Hence $Z(F) = \bigcap_{i=1}^{N} Z(F_i) = Z(I) \setminus \mathbb{D}$. Moreover,

$$|f_i| \le |F_i| \le |F|^2 \quad \text{in } \mathbb{D}.$$

Hence by Proposition 3.2 we have $f_i^2 = h_i F^2$ for some $h_i \in QA$. Thus F is a common divisor of the f_i $(i = 1,\dots,N)$. Since d is a gcd, F divides d. Consequently,

$$Z(I) \cap M(QC) = Z(F) \subseteq Z(d) \cap M(QC) \subseteq Z(I) \cap M(QC) . \qquad (2)$$

(1) and (2) together now prove the claim $Z(I) = Z(d)$ (including multiplicities in $\mathbb{D}$).

Let $J = \left(\frac{f_1}{d},\dots,\frac{f_N}{d}\right)$. From the hypothesis it follows that a gcd $\tilde{d}$ of $\frac{f_i}{d}$ $(i = 1,\dots,N)$ is 1. Hence $Z(J) = Z(\tilde{d}) = \emptyset$. Consequently J is not contained in any maximal ideal of QA, i.e., $1 \in J$. Thus we can conclude that $d \in I$. □

According to P.M. Cohn ([5], p.260), we call an integral domain a Prebezout ring if the gcd of any two elements is a linear combination of these whenever it exists.

Theorem 3.6 has now the following corollary.

Corollary 3.7. The algebra QA is a Prebezout ring.

Remark. The proof of Theorem 3.6 is a nontrivial generalization of that of ([32], p.54), where it was proved that the disk algebra is a Prebezout ring.

§ 4 Closed ideals in QA

One of the main results in [15] is the description of the closed ideals in QA. Moreover, we observed that the maximal ideals of the Shilov boundary of QA are not topologically generated by single elements, i.e. are not the closures of principal ideals. This contrasts to the situation in $A(\mathbb{D})$. The purpose of this paragraph is to give a complete characterization of the closed ideals in QA which can be represented as closures of principal ideals.

In analogy to § 1 we give the following definition.

Definition. An ideal $I \subseteq QA$ is said to have the Forelli property, if there exists a function $f \in I$ such that $Z(f) = Z(I)$.

Theorem 4.1. Let I be a closed ideal in QA. Then the following assertions are equivalent:

(1) I has the Forelli property.

(2) I is the closure of a principal ideal.

(3) $Z(I) \cap M(QC)$ is a peak set or empty.

Proof. (1) $\Rightarrow$ (2): Let I be a closed ideal in QA. By ([15], Theorem 2.5) I has the form $I = uI(E,QA)$, where $E \subseteq M(QC)$ and u is the inner factor of I. By assumption there is a function $f \in I$ such that $Z(f) = Z(I)$. Using Proposition 3.4, we may assume without loss of generality that $f = uF$, where F is the outer part of f. Since closed ideals in QA are uniquely determined by the inner factor u and the zero set E, we see that $I = \overline{(f)}$. This gives (2).

(2) => (3): Let $I = \overline{(f)}$ for some $f \in QA$. Since $E :=$ $Z(I) \cap M(QC) = Z(f) \cap M(QC)$ (by Proposition 3.4), E is a G_δ-set with σ-measure zero ([35], p.325). By ([35], p.326) every closed set of σ-measure zero is a weak peak set for QA. But by ([2], p.96) every G_δ weak peak set is a peak set. This yields (3).

(3) => (1): Let $E = Z(I) \cap M(QC)$ and let $p_E \in QA$ be the peak function associated to the peak set E. Then $f = 1-p_E$ vanishes exactly on E. (Note that by the maximum modulus theorem $|p_E(z)| < 1$ for every $z \in \mathbb{D}$.) Thus by ([15], Theorem 2.5) $I = \overline{(uf)}$, where u is the inner factor of I. Hence $Z(I) = Z(uf)$. □

Remark. Since by ([20], p.298) no point in M(QC) can be a peak point for QA, we obtain the result that a maximal ideal M in QA has the Forelli property if and only if it is of the form $M = M(z_o) = \{f \in QA: f(z_o) = 0\}$ for some $z_o \in \mathbb{D}$.

Another fundamental difference between $A(\mathbb{D})$ and QA will arise when we consider the structure of the closed primary ideals. Whereas in $A(\mathbb{D})$ every maximal ideal contains an infinite chain of closed primary ideals (see [18], p.88), the situation in QA is completely different.

Theorem 4.2. A closed ideal in QA is primary if and only if it is a maximal ideal or if it is an ideal of the form $I = (z-z_o)^n$, for some $z_o \in \mathbb{D}$ and $n \in \mathbb{N}$.

Proof. Let I be a closed primary ideal in QA. If $Z(I) = \{z_o\}$, where $z_o \in \mathbb{D}$, then I has trivially the form $(z-z_o)^n$ for some $n \in \mathbb{N}$. Therefore, let $Z(I) = \{m\}$ with $m \in M(QC)$. By ([15], Theorem 2.5) I has the form $I = uI(E,QA)$, where $E = Z(I) \cap M(QC) = \{m\}$. So it remains to show that the inner factor u of I is invertible. Suppose not. Then there exists a sequence (z_n) in $\mathbb{D}$ such that $u(z_n) \to 0$ for $n \to \infty$. Without loss of generality we may choose (z_n) to be thin, i.e.,

$$\lim_{n\to\infty} \prod_{k \neq n} \left| \frac{z_n - z_k}{1-\bar{z}_n z_k} \right| = 1 .$$

Hence $S = \{z_n\}$ is an interpolating sequence for QA ([34], p.553). So its closure $\bar{S}$ in M(QA) is homeomorphic to $\beta\mathbb{N}$, the Stone-Čech compactification of the set of integers. Choose an arbitrary $f \in I = uI(E,QA)$. Then by continuity of the Gelfand transform, f vanishes on $\bar{S} \setminus S$. Thus $\bar{S} - S \subseteq Z(I)$. Hence I cannot be primary, since $\bar{S} - S$ is homeomorphic to $\beta\mathbb{N} \setminus \mathbb{N}$, which is very huge ([11], p.187). □

Remark. We observe that also in H^∞ every closed primary ideal contained in a maximal ideal of the Shilov boundary is in fact maximal ([30], p.221).

III. The Chang-Marshall algebras QA_B and $C_B \cap H^\infty$

Let B be a Douglas algebra, that is, a uniformly closed subalgebra of L^∞ containing H^∞. Sarason proved that the smallest Douglas algebra containing H^∞ properly, namely $H^\infty+C$, is generated as an algebra by H^∞ and the complex conjugate $\bar{z}$ of the inner function z; more precisely, $H^\infty+C$ is the (essential supremum) norm closure (denoted by $[H^\infty,\bar{z}]$) of the set

$$\{f\bar{z}^n : f \in H^\infty,\ n \in \mathbb{N} \cup \{0\}\}$$

(see [11], § 9).

In their famous theorem Chang and Marshall could generalize this to arbitrary Douglas algebras:

<u>Chang-Marshall Theorem.</u> Let B be a Douglas algebra. Then B is generated by H^∞ and the set

$\bar{\mathcal{B}} = \{\bar{b}$: b interpolating Blaschke product and b invertible in B$\}$,

i.e., B is the norm closure of the set

$$\{f\bar{b}_1^{n_1}\bar{b}_2^{n_2}\cdots\bar{b}_m^{n_m} : f \in H^\infty,\ \bar{b}_j \in \bar{\mathcal{B}},\ n_j \in \mathbb{N},\ m \in \mathbb{N}\}.$$

For short, $B = [H^\infty,\bar{\mathcal{B}}]$.

For the proof the reader is referred to the book of J. Garnett ([11], § 9). Recall that an interpolating Blaschke product is a Blaschke product b with zeros z_n satisfying

$$(1-|z_n|^2)|b'(z_n)| \geq \delta > 0 \quad \text{for all} \quad n \in \mathbb{N}$$

and some $\delta > 0$.

Associated to each Douglas algebra B are several natural subalgebras. These are

1. the largest C^*-algebra contained in B (with respect to the complex conjugate operation), denoted by QB,
2. the smallest C^*-algebra generated by the set U of invertible inner functions in B, denoted by C_B,

and their analytic counterparts QA_B resp. $C_B \cap H^\infty$.

In other words,

$$QB = \bar{B} \cap B = \{f \in B : \bar{f} \in B\} ,$$

$$C_B = [U,\bar{U}] = \mathrm{cl}\ \{\sum_{i=1}^{n} \lambda_i u_i \bar{u} : \lambda_i \in \mathbb{C};\ u_i, u \in B^{-1};\ n \in \mathbb{N}\} ,$$

$$QA_B = QB \cap H^\infty ,$$

$$C_B \cap H^\infty = [U,\bar{U}] \cap H^\infty ,$$

where cl {...} denotes the norm closure.

Note that by a theorem of Frostman ([11], p.79) we may define the algebra C_B also in the following way:

$C_B = [\mathcal{B},\bar{\mathcal{B}}]$, where $\mathcal{B}$ is the set of all invertible Blaschke products in B.

These algebras were introduced by Chang [3] and studied later by Chang and Marshall [4] resp. by Sundberg and Wolff [34]. Therefore, we shall call in the sequel the algebras QA_B and $C_B \cap H^\infty$ "the Chang-Marshall algebras".

If especially $B = H^\infty + C$, then we obtain the following algebras studied in Sections I and II:

$$QA_{H^\infty+C} = QA , \qquad C_{H^\infty+C} \cap H^\infty = A(\mathbb{D}) .$$

If $B = L^\infty$, then $QA_B = C_B \cap H^\infty = H^\infty$.

Finally, let us mention that every Douglas algebra B has the representation $B = H^{\infty} + C_B$ (see [11], p.386). From this we immediately obtain the fact that $QB = QA_B + C_B$. Hence every function $g \in QB$ can be approximated (in the norm) by functions of the form $f\bar{b}$, where $f \in QA_B$ and b is an invertible Blaschke product in B (see also [31]).

§ 5 The maximal ideal space of the Chang-Marshall algebras

It is well known that the corona theorem is true in any Chang-Marshall algebra A; this means that the open unit disk is dense in the maximal ideal space of A (see [4], p.18 for $C_B \cap H^{\infty}$ and ([34], p.563) for QA_B). As an obvious corollary we obtain the fact that every multiplicative linear functional in A can be extended to a multiplicative linear functional in H^{∞}. In other words, we have the following proposition.

Proposition 5.1 ([4], p.18, [16], Lemma 3.4). Let A be a Chang-Marshall algebra. Then the restriction map $\Gamma: M(H^{\infty}) \to M(A)$ defined by $\Gamma(m) = m|_A$ is surjective. Moreover, Γ is a continuous open map.

It is easy to see that Proposition 5.1 actually is equivalent to the corona theorem in A (see [4], p.18).

As a useful corollary we obtain:

<u>Corollary 5.2.</u> The maximal ideal space of any Chang-Marshall algebra is connected.

<u>Proof.</u> Since Γ is continuous, the assertion follows from the fact that $M(H^\infty)$ is connected ([18], p.188). □

As a counterpart to Proposition 5.1, Chang and Marshall proved in [4] the result that exactly the maximal ideals of the Shilov boundary of $C_B \cap H^\infty$ can be lifted to maximal ideals of C_B. In the following we shall now generalize this to the whole class of Chang-Marshall algebras.

A. The characterization of the Shilov boundary of QA_B and $C_B \cap H^\infty$

A former result of Chang and Marshall ([4], p.15) tells us that the algebra $C_B \cap H^\infty$ is logmodular on $M(C_B)$, i.e., that the set $\{\log |f|: f \text{ invertible in } C_B \cap H^\infty\}$ is norm dense in the space of all real valued continuous functions on $M(C_B)$. In [31] we proved that the algebras QA_B are even strongly logmodular on $M(QB)$, i.e., that every real valued continuous function q on $M(QB)$ can be represented in the form $q = \log |f|$, for some function f invertible in QA_B. Moreover, it is known that by the Stone-Weierstraß theorem the C^*-algebras QB resp. C_B are isometrically isomorphic to the spaces $C(M(QB))$ resp. $C(M(C_B))$ of all complex valued continuous functions on $M(QB)$ resp. $M(C_B)$.

Hence by a well known theorem (see [10], p.38), the Shilov boundary ∂QA_B of QA_B (resp. $\partial(C_B \cap H^\infty)$ of $C_B \cap H^\infty$) can be identified with the maximal ideal space of QB resp. C_B.

We are now in a position to present for the Chang-Marshall algebras the analogon of D.J. Newman's characterization of the Shilov boundary of H^∞. Note that our result contains as a special case the corresponding result of Chang and Marshall for $C_B \cap H^\infty$ ([4], p.18).

Theorem 5.3. Let A be any Chang-Marshall algebra and let $m \in M(A)$. Then the following assertions are equivalent:

(1) $m \in M(QB)$ resp. $m \in M(C_B)$;

(2) $m \in \partial A$;

(3) $|u(m)| = 1$ for every inner function $u \in A$;

(4) $|u(m)| > 0$ for every inner function $u \in A$;

(5) $|b(m)| = 1$ for every Blaschke product $b \in A$.

Proof. First let us observe that the algebras QA_B and $C_B \cap H^\infty$ contain exactly the same inner functions. The equivalence of (1) and (2) now follows from the remarks above.

(1) => (3): Let $m \in M(QB)$ and let $u \in QA_B$, u inner. Then $\bar{u} \in QB$; hence u is invertible in QB because $u\bar{u} = 1$ a.e. Thus $u(m)\bar{u}(m) = 1$. Because $\|u\| = 1$, we have $|u(m)| = 1$. The same proof works of course for $m \in M(C_B)$.

(3) => (4) is trivial.

(4) => (5): Suppose there exists a Blaschke product $b \in A$ such that $|b(m)| < 1$. Then the function

$$u = \frac{b - b(m)}{1 - \overline{b(m)}b}$$

is inner and belongs to A. But $u(m) = 0$, which contradicts the hypothesis.

(5) => (1): Let $m \in M(QA_B)$. We have to show that under the hypothesis (5) m can be extended to a multiplicative linear functional on QB.

Let $S = \{f\bar{b}: f \in QA_B,$ b Blaschke product, $b \in QA_B\}$. Then the norm closure $\bar{S}$ of S equals QB. Define $m_o: S \to \mathbb{C}$ by $m_o(f\bar{b}) = m(f)\overline{m(b)}$. Then m_o is a well defined map. In fact, let $f_1\bar{b}_1 = f_2\bar{b}_2 \in S$. Then $f_1 b_2 = f_2 b_1$. Hence $m(f_1 b_2) = m(f_2 b_1)$. Multiplication by the factor $\overline{m(b_1)}\,\overline{m(b_2)}$ yields

$$m(f_1)m(b_2)\overline{m(b_1)}\,\overline{m(b_2)} = m(f_2)m(b_1)\overline{m(b_1)}\,\overline{m(b_2)} .$$

Since $|m(b_i)|^2 = 1$, we have $m(f_1)\overline{m(b_1)} = m(f_2)\overline{m(b_2)}$. Thus by definition, $m_o(f_1\bar{b}_1) = m_o(f_2\bar{b}_2)$, which proves the claim. Moreover, m_o is continuous, linear and multiplicative on S. Since S is dense in QB, m_o has a unique extension m_1 to QB. Moreover, m_1 is again a multiplicative linear functional; hence $m_1 \in M(QB)$. Since $m_1|_{QA_B} = m$, m_1 is the desired extension. The same proof works again for $C_B \cap H^\infty$. □

<u>Remark.</u> We do not know if the condition $|\hat{b}(m)| > 0$ for every Blaschke product b in A is sufficient for m to belong to the Shilov boundary ∂A of A.

As a corollary we mention the well known fact that, in analogy to the disk algebra, the maximal ideal space of QA is the disjoint union of $\mathbb{D}$ and the Shilov boundary M(QC) of

QA. This fact is now characteristic for QA resp. $A(\mathbb{D})$, as the following proposition shows.

Proposition 5.4. Let B be a Douglas algebra different from $H^\infty+C$. Then the maximal ideal space of any Chang-Marshall algebra A associated to B contains a point which neither belongs to $\mathbb{D}$ nor to the Shilov boundary of A.

Proof. By the Chang-Marshall theorem there exists an infinite Blaschke product which is invertible in B. Now let $m \in M(A) \setminus \mathbb{D}$ be any point in the closure of the zero set of b in $\mathbb{D}$. Since the Gelfand transform is continuous, $b(m) = 0$. Thus, by Theorem 5.3, $b \notin \partial A$. □

B. The Gleason part structure of QA_B and $C_B \cap H^\infty$

It was K. Hoffman who first gave in his famous paper [19] a detailed study of the maximal ideal space of H^∞. He discovered the intrinsic relations between the Gleason parts of $M(H^\infty)$ and interpolating sequences. In this section, we shall use his results to describe the analytic structure in the maximal ideal space of any Chang-Marshall algebra A. Our main result is a characterization of the nontrivial Gleason parts in M(A) which is similar to that of Hoffman for $M(H^\infty)$.

Definition. Let A be a Chang-Marshall algebra and let

$$\rho_A(m_1,m_2) = \sup\{|f(m_1)|: f \in A,\ \|f\| \le 1,\ f(m_2) = 0\}$$

be the pseudohyperbolic distance of two points m_1 and m_2 in the maximal ideal space M(A) of A. Then the Gleason parts of M(A) may be defined as the equivalence classes of the relation

$$m_1 \sim m_2 \quad \Longleftrightarrow \quad \rho_A(m_1, m_2) < 1 .$$

We write

$$P(m) = \{m' \in M(A): \ \rho(m,m') < 1\}$$

for the Gleason part containing the point $m \in M(A)$. We note that ρ_A coincides on $\mathbb{D}$ with the usual pseuohyperbolic metric

$$\rho(z,w) = \left|\frac{z - w}{1 - \bar{z}w}\right| .$$

Other general facts about ρ_A may be found in ([11], § 10). A Gleason part is called trivial if P(m) reduces to {m}; otherwise, P(m) is called nontrivial.

As a generalization of a definition of Hoffman ([19], p.88) (see also [11], p.410, and [26], p.60) we define

$$G_A = \{m \in M(A): \text{ m contains an interpolating Blaschke product } b \in A\} .$$

It is important to note that in contrast to H^∞ the set G_A does not coincide in general with the set

$$H_A = \{m \in M(A): \text{ m lies in the closure of an A-interpolating sequence}\} ,$$

where as usual a sequence $\{z_n\}$ in $\mathbb{D}$ is called A-interpolating if for every bounded sequence (w_n) of complex numbers there exists a function $f \in A$ such that $f(z_n) = w_n$ for all $n \in \mathbb{N}$.

In fact, let $\{z_n\}$ be an infinite QA-interpolating sequence.

Then by ([34], p.553) $\{z_n\}$ is thin, i.e., $\lim\limits_{n\to\infty} \prod\limits_{k\neq n} \rho(z_k,z_n) = 1$ (see § 4).

Let $\bar{S}$ denote its closure in M(QA). Then no points $m \in \bar{S} \setminus \mathbb{D}$ contains an interpolating Blaschke product, since the only inner functions of QA are the finite Blaschke products; but those have modulus 1 on $M(QA) \setminus \mathbb{D}$ (by Theorem 5.3).

Combining this fact with the following proposition, we can thus conclude that in general the set G_A is a proper subset of H_A.

Proposition 5.5. Let $b \in A$ be an interpolating Blaschke product. If $b(m) = 0$ for some point $m \in M(A)$, then m lies in the (M(A)-) closure of the zero set $Z_{\mathbb{D}}(b) = \{z_n\}$ of b. Moreover, $\{z_n\}$ is an A-interpolating sequence.

Proof. Choose, according to Proposition 5.1, a point $x \in M(H^\infty)$ such that $x|_A = m$. Obviously $x(b) = 0$. By ([11], p.379), x lies in the closure of $\{z_n\}$ with respect to the topology in $M(H^\infty)$. Hence there exists a subnet $(z_{n(\alpha)})$ of the zero sequence of b such that $f(z_{n(\alpha)}) \to x(f) = m(f)$ for every $f \in A \subseteq H^\infty$. This yields the assertion. That $\{z_n\}$ is even an A-interpolating sequence follows from ([34], p.554, Remark 1). □

We are now able to state the main result of this paragraph.

Theorem 5.6. Let A be Chang-Marshall algebra and let $m \in G_A$. Then P(m) is an analytic disk (for a definition of this term, see [11], p.401).

Moreover, there exists a bijective map $L_m\colon \mathbb{D} \to P(m)$ such that, whenever (w_α) is a net in $\mathbb{D}$ converging to m, then $\lim L_{w_\alpha} = L_m$, where

$$L_{w_\alpha}(\xi) = \frac{\xi + w_\alpha}{1 + \bar{w}_\alpha \xi} \qquad \text{for} \quad \xi \in \mathbb{D}.$$

Proof. The proof works exactly in the same manner as that of the corresponding result for H^∞ (see [11], p.408, Theorem 1.7). □

The following theorem now completes the characterization of the nontrivial Gleason parts in A. It shows that all analytic structure in M(A) comes about in the manner described by Theorem 5.6.

Theorem 5.7. Let A be a Chang-Marshall algebra and let $m \in M(A)$. Then the following assertions are equivalent:

(1) $m \in M(A) \setminus G_A$;

(2) If u is an inner function in A and if $u(m) = 0$, then $u = u_1 u_2$, where $u_i(m) = 0$ $(i = 1,2)$;

(3) The Gleason part P(m) is trivial;

(4) If (w_j) is a net in $\mathbb{D}$ converging to m, then $\lim L_{w_j}$ exists in $M(A)^{\mathbb{D}}$ and is the constant map $L(\xi) = m$ $(\xi \in \mathbb{D})$.

Remark. $M(A)^{\mathbb{D}}$ is the space of all maps from $\mathbb{D}$ into $M(A)$.

Proof. (1) => (2): First we note that if an inner function $u \in A$ is the product of two inner functions u_1 and u_2, then u_1 and u_2 belong to A. So let $u(m) = 0$ and let $bS = u$ be the Riesz factorization of u, where b is a Blaschke product and S a singular inner function. If $S(m) = 0$, then $(b\sqrt{S})\sqrt{S} = u$ yields the desired factorization. If $S(m) \neq 0$, then $b(m) = 0$. Assume that b does not admit the factorization (2), then m is a single zero of u. Choose, by Proposition 5.1, $x \in M(H^\infty)$ such that $x|_A = m$. Then x itself is a simple zero of b in $M(H^\infty)$. By Theorem 5.3 of ([19], p.100), there exists an interpolating subsequence $\{z_k\}$ of the zero sequence of b in $\mathbb{D}$ such that x belongs to the closure in $M(H^\infty)$ of $\{z_k\}$. Hence $b = b_1 b_2$, where b_1 is the interpolating Blaschke product associated to $\{z_k\}$. Since $b_1, b_2 \in A$, we have $b_1(m) = b_1(x) = 0$. Thus $m \in G_A$, which contradicts the choice of m.

(2) => (3): If $m \in \partial A$, then P(m) is trivial. This follows from a general result of Hoffman (see [11], p.402) by observing that A is a logmodular algebra. So let $m \in M(A) \setminus \partial A$. Choose $m' \in M(A)$ and $m' \neq m$. We claim that $m' \notin P(m)$. If $m' \in \partial A$, we are done. So let $m' \in M(A) \setminus \partial A$. Using Corollary 6.4 of the next section, we see that there exists an inner function $u \in A$ such that $u(m) = 0$, but $u(m') \neq 0$. The hypothesis (2) now implies that u has the factorization $u = u_1 \cdots u_n$, where $u_j(m) = 0$ $(j = 1,2,\ldots,n;\ n \in \mathbb{N})$. Thus by Lemma 1.2 in ([11], p.403), u vanishes identically on P(m). Hence $m' \notin P(m)$. Since $m' \neq m$ was an arbitrary point of M(A), we can conclude that P(m) is trivial.

Clearly (3) implies (4), because any limit point of L_{w_j} in $M(A)^{\mathbb{D}}$ is an analytic map whose range is contained in $P(m)$ ([11], p.402, Lemma 1.1).

That (4) implies (1) follows from Theorem 5.6. □

Before we proceed, we want to mention the following open problem.

Open problem. Let $m \in M(A) \setminus G_A$ and $f(m) = 0$. Does, in analogy to H^∞, f admit the factorization $f = f_1 f_2$, where $f_1(m) = f_2(m) = 0$?

A positive answer can be given if m belongs to the Shilov boundary of A. This follows from the next proposition and the fact that every Chang-Marshall algebra is a logmodular algebra.

Proposition 5.8. Let A be a logmodular algebra and let m be a point of the Shilov boundary ∂A of A. Then every function $f \in A$ which vanishes in m has a factorization of the form $f = f_1 f_2$, where $f_1(m) = f_2(m) = 0$.

Proof. Since A is logmodular, every maximal ideal admits a unique representation measure on ∂A (see [11], p.201). Hence the Shilov boundary coincides with the Choquet boundary (see [2], p.87). Thus every maximal ideal contains a bounded approximate identity ([2], p.901 and p.99). Cohen's factorization theorem ([2], p.74) now yields the assertion. □

§ 6 A comparison between the ideal structures of H^∞, QA_B and $C_B \cap H^\infty$

Let A be a Chang-Marshall algebra. In § 5 we have seen that every maximal ideal in A can be lifted to a maximal ideal in H^∞. In [16] we proved that, more generally, every prime ideal in QA_B is the trace of a prime ideal in H^∞. Here we are now studying this lifting process for ideals whose zero set does not meet the Shilov boundary. We will encounter some new phenomena. In fact, if I is an ideal in A which can be lifted to an ideal in H^∞ (which is not always possible, see the remark after Corollary 6.6), then one cannot expect in general that the extension is unique. However, if the zero set of the ideal I in A does not meet the Shilov boundary of A, then I is the trace of a unique ideal in H^∞. This result will be proved in Theorem 6.5. Our proof is based on several generalizations of a result of Marshall ([23], p.20) which tells us that, if I is an ideal in H^∞ whose zero set is disjoint from the Shilov boundary of H^∞, then I does not only contain an inner function, but is even generated algebraically by the set of its inner functions (see also [33], p.521). We now present a first generalization. To this end we have to observe that the maximal ideal space M(B) of any Douglas algebra can be identified with the set

$$M(B) = \{m \in M(H^\infty) : |u(m)| = 1 \text{ for every inner function } u \text{ invertible in } B\}$$

([11], p.375).

<u>Theorem 6.1.</u> Let B be a Douglas algebra and let I be an ideal in H^∞ such that $Z(I) \cap M(B) = \emptyset$. Then I is generated (algebraically) by the set of inner functions invertible in B and contained in I; in other words,

$$I = \{\sum_{i=1}^{n} u_i h_i,\ u_i \in U_I,\ h_i \in H^\infty,\ n \in \mathbb{N}\},$$

where

$$U_I = \{u \in B:\ u \text{ inner},\ u \text{ invertible in } B,\ u \in I\}.$$

For short we write $I = [U_I]H^\infty$.

<u>Proof.</u> Step 1. We claim that I contains an inner function u which is invertible in B, i.e., for which $|u(m)| = 1$ holds for every $m \in M(B)$. By hypothesis the ideal IB generated by I in B is not proper, i.e. $IB = B$. Hence there exist functions $f_i \in I$ and $q_i \in B$ such that $1 = \sum_{i=1}^{n} q_i f_i$. Let $C = \sum_{i=1}^{n} \|f_i\|$. By the Chang-Marshall theorem choose $h_i \in H^\infty$ and $b_i \in B^{-1}$, b_i Blaschke products, such that

$$\|q_i - h_i \bar{b}_i\| \le \frac{1}{2C}.$$

Thus we have

$$\begin{aligned} |\sum_{i=1}^{n} h_i \bar{b}_i f_i| &\ge |\sum_{i=1}^{n} q_i f_i| - \sum_{i=1}^{n} |q_i - h_i \bar{b}_i|\,|f_i| \\ &\ge 1 - \frac{1}{2C} C = \frac{1}{2} \quad \text{on } M(B). \end{aligned} \tag{1}$$

Let $g_i = h_i \prod_{j \ne i} b_j$ and $b = \prod_{j=1}^{n} b_j$. Then $b \in B^{-1}$, i.e. $|b| = 1$ on M(B). Multiplying (1) by b, we obtain

$$|\sum_{i=1}^{n} g_i f_i| \ge \frac{1}{2}|b| = \frac{1}{2} \quad \text{on } M(B). \tag{2}$$

Let $f = \sum_{i=1}^{n} g_i f_i$. Then $f \in I$. Let $uF = f$ be the inner-outer factorization of f. Since $M(L^\infty) \subseteq M(B)$, F does not vanish by (2) on $M(L^\infty)$. Hence F is invertible in H^∞. So $u = \frac{1}{F}(uF) \in I$. Let $m \in M(B)$. Then $u(m) = \frac{1}{F}(m)f(m) \neq 0$. Hence u is invertible in B.

Step 2. Using an idea of Chang-Marshall [4], we are now constructing a set of inner functions which generates I as an ideal.

Let $g \in I$. Without loss of generality we may assume $\|g\| \leq \frac{1}{2}$. Since $u \in B^{-1}$, we have $u \in H^\infty \cap C_B$. Furthermore, dist $(g\bar{u}, H^\infty) = \inf_{h \in H^\infty} \|g\bar{u} - h\| \leq \|g\bar{u}\| = \|g\| \leq \frac{1}{2} < 1$. Hence the hypothesis of ([4], p.14, Corollary 2.2) is fulfilled. Therefore we have functions $h \in H^\infty$ and $u_1 \in C_B$, u_1 unimodular on T, such that $u_1 = g\bar{u} + h$. Let $v = v_g = g + uh$. Then $v \in H^\infty$ and $|v| = |uu_1| = 1$ a.e. on T. Hence v is an inner function. Because g and u belong to the ideal I, we have $v \in I$. Thus we may conclude that I is generated by u and the set of inner functions $\{v_g : g \in I\}$. $\square$

Our next theorem will generalize the result of Marshall mentioned above to the Chang-Marshall algebras QA_B and $C_B \cap H^\infty$.

<u>Theorem 6.2.</u> Let A be a Chang-Marshall algebra and let I be an ideal in A whose zero set is disjoint from the Shilov boundary of A. Then I is generated (algebraically) by a set of inner functions.

<u>Proof.</u> A combination of the results in [31] (Theorem 3, Theorem 1 and the second remark after Theorem 3) enables us to

conclude that I contains an inner function u. Let $g \in I$, $\|g\| \le \frac{1}{2}$. Construct, as in the second step of the proof of Theorem 6.1, an inner function $v = v_g \in H^\infty$ such that $v = g + uh = uu_1$, where $u_1 \in C_B$ and $h \in H^\infty$. In particular, $v \in C_B$. Hence v is an inner function in $C_B \cap H^\infty$. Thus $h = (v-g)\bar{u} \in QB \cap H^\infty$ resp. $C_B \cap H^\infty$. Hence $v \in I$. The set $\{v_g : g \in I\}$ and the function u now generate I. □

In case of the algebras $C_B \cap H^\infty$ we even obtain the following stronger result.

Theorem 6.3. Let I be an ideal in $C_B \cap H^\infty$ such that $Z(I) \cap M(C_B) = \emptyset$. Then finite linear combinations of inner functions in I are dense in I.

Proof. Let $f \in I$. Then the proof of Theorem 6.2 shows that there exist two inner functions u and v in I such that $f = v + uh$ for some $h \in C_B \cap H^\infty$. Approximating, according Theorem 4.1 ([4], p.14), the function h by finite linear combinations of inner functions in $C_B \cap H^\infty$, we see that f may be appoximated by linear combinations of inner functions in I. □

Remark. Theorem 6.3 contains as a special case the result of Marshall for the algebra H^∞ ([23], p.20, Corollary 3.11).

As a useful corollary of Theorem 6.2 we obtain the following separation property.

Corollary 6.4. The inner functions in any Chang-Marshall algebra A separate the points of $M(A) \setminus \partial A$.

Proof. Let $m_1 \neq m_2$ be two maximal ideals in $M(A) \setminus \partial A$. Since m_1 and m_2 are generated by sets of inner functions, there exists an inner function u such that $u(m_1) = 0$, but $u(m_2) \neq 0$. □

Remark. The inner functions in A do not (in general) separate the points of the Shilov boundary of A. Take, e.g., the algebra QA. Then the inner functions, which are the finite Blaschke products, are constant on the maximal ideals of any fibre $M_\alpha(QA) = \{m \in M(QA): z-\alpha \in m\}$ $(\alpha \in T)$.

We are now in a position to prove the main result of this section. It shows that the similarity in the proofs of Theorem 6.1 and 6.2 is not incidental, but has its source in the intimate relations between ideals in A and H^∞.

Theorem 6.5. Let A be a Chang-Marshall algebra and B the associated Douglas algebra. Then the ideals $I \subseteq A$ satisfying $Z(I) \cap \partial A = \emptyset$ are in a one-to-one correspondence with the ideals $J \subseteq H^\infty$ such that $Z(J) \cap M(B) = \emptyset$.

In particular, every ideal I in A whose zero set is disjoint from the Shilov boundary of A can be lifted to a unique ideal in H^∞.

Proof. We have to show that the map $J \to J \cap A$ is a bijection from the set of ideals in H^∞ satisfying $Z(J) \cap M(B) = \emptyset$ onto the set of ideals I in A satisfying $Z(I) \cap \partial A = \emptyset$. To this end let J be an ideal in H^∞ with $Z(J) \cap M(B) = \emptyset$. Then J contains by Theorem 6.1 an inner function u invertible in B. Hence $u \in J \cap A = I$. Therefore $Z(I) \cap \partial A = \emptyset$, since $|u| = 1$ on ∂A by Theorem 5.1.

Conversely, let I be an ideal in A with $Z(I) \cap \partial A = \emptyset$. We shall show that I can be lifted to an ideal J in H^∞. Let $J = IH^\infty$. We claim that $J \cap A = I$. Let $f \in J \cap A$. By Theorem 6.2 I is generated by a set of inner functions invertible in B. Hence there exist functions $g_i \in H^\infty$ and inner functions $u_i \in I$ such that $f = \sum_{i=1}^{n} g_i u_i$. Using an idea of Chang and Marshall ([4], p.18) we shall show that we have $f = \sum_{i=1}^{n} f_i u_i$ for some functions $f_i \in A$.

Let $u = \prod_{i=1}^{n} u_i$. By ([4], p.14, Corollary 2.2) there exist functions $h_i \in H^\infty$ and $v_i \in C_B$, $|v_i| = 2\|g_i\|$ a.e. on T, such that $\bar{u}u_i g_i = h_i + v_i$. So

$$f = \sum_{i=1}^{n} u_i g_i = \sum_{i=1}^{n} u_i(h_i + v_i)u\bar{u}_i$$
$$= u \sum_{i=1}^{n} h_i + \sum_{i=1}^{n} u_i(u\bar{u}_i v_i) .$$

Since u and $u\bar{u}_i v_i = g_i - h_i u\bar{u}_i \in C_B \cap H^\infty$ and $f \in A$, we see that $u \sum_{i=1}^{n} h_i \in A$. Moreover, $\sum_{i=1}^{n} h_i = \bar{u}\left(u \sum_{i=1}^{n} h_i\right) \in C_B$ if $A = C_B$ resp. $\sum_{i=1}^{n} h_i \in QB$ if $A = QB$. Hence $\sum_{i=1}^{n} h_i \in A$. Thus $f = \sum_{i=1}^{n} u_i f_i$, where $f_1 = \left(\sum_{i=1}^{n} h_i + v_1\right)u\bar{u}_1$ and $f_i = u\bar{u}_i v_i$ $(i = 2,\ldots,n)$ are functions in A. Hence $f \in I$, which proves the claim.

It only remains to show the uniqueness. Let J_1, J_2 be ideals in H^∞ such that $J_i \cap A = I$ $(i = 1,2)$. Since $Z(J_i) \cap M(B) = \emptyset$, there exist by Theorem 6.1 sets U_i of inner functions invertible in B such that $J_i = [U_i]H^\infty$ $(i = 1,2)$. Since $U_i \subseteq A$, the hypothesis $J_1 \cap A = I$ yields that $U_1 \subseteq I \subseteq [U_2]H^\infty$. Hence $[U_1]H^\infty \subseteq [U_2]H^\infty$, and vice versa. Thus $J_1 = J_2$. □

Theorem 6.5 has the following useful corollary.

Corollary 6.6. Let B be a Douglas algbra. Let A be one of the associated Chang-Marshall algebras and let $\Gamma: M(H^\infty) \to M(A)$ be the restriction map (of Proposition 5.1). Finally, let J be an ideal in H^∞ such that $Z(J) \cap M(B) = \emptyset$ and let $I = J \cap A$ be its trace in A. Then $Z(I) = \Gamma(Z(J))$.

Proof. The proof follows from the fact that by Theorem 6.5 the ideals J and I are generated by the same set of inner functions. □

Remark. Let A be a Chang-Marshall algebra with $A \nsupseteq H^\infty$. Then not every ideal in A can be lifted to an ideal in H^∞. If A is the disk algebra, we take the ideal I = (1-z).
If $A = QA_B$, let u be an inner function such that $u \notin A$. Choose by ([35], p.321) an outer function $F \in QA$ with $uF \in QA$. Then $uF \in F \cdot H^\infty$, but $uF \notin F \cdot QA_B$. Hence $F \cdot H^\infty \cap QA_B$ contains properly the ideal $I = F \cdot QA_B$.

If we specialize Theorem 6.5 to maximal ideals, we obtain the following information on the relations between the maximal ideal space of a Chang-Marshall algebra A and that of H^∞. Let us first give a definition.

Definition. Let B be a commutative Banach algebra with identity element and let A be closed subalgebra of B. Assume that the restriction map Γ: $M(B) \to M(A)$ is onto. Let $m \in M(A)$. Then the set

$$\Gamma^{-1}(m) = \{x \in M(B): m(f) = x(f) \text{ for all } f \in A\}$$

is called the A-level set of m in B.

Remark. Note that, since Γ is continuous, the A-level sets form a partition of M(B) into closed sets.

Corollary 6.7. Let A be a Chang-Marshall algebra and let Γ: $M(H^\infty) \to M(A)$ be the restriction map (of Proposition 5.1). Then the A-level sets in H^∞ of the maximal ideals m which do not belong to the Shilov boundary of A are trivial, i.e. consist only of one point in $M(H^\infty)$.

Remark. Note that in contrast to the above, the A-level sets in H^∞ associated to points of the Shilov boundary of A are in general very huge. For example, in case of the disk algebra $A(\mathbb{D})$, the $A(\mathbb{D})$-level set of the maximal ideal $M(\alpha) = \{f \in A(\mathbb{D}): f(\alpha) = 0\}$, $\alpha \in T$, is the whole fiber $M_\alpha(H^\infty) = \{m \in M(H^\infty): z-\alpha \in m\}$. In case of the algebra QA see the paper of K. Izuchi [20].

We also note that the A-level sets in H^∞ of maximal ideals of the Shilov boundary of A are entirely contained in the maximal ideal space M(B) of the associated Douglas algebra B. This follows from Theorem 6.5 and Theorem 5.3..

Since $\Gamma: M(H^{\infty}) \to M(A)$ is open, we obtain the following refinement of Corollary 6.7.

<u>Corollary 6.8.</u> The map Γ is a homeomorphism of $M(H^{\infty}) \setminus M(B)$ onto $M(A) \setminus \partial A$.

We shall now study the behaviour of Γ on Gleason parts. First we observe that if B is a Douglas algebra, then $x \in M(B)$ implies that its Gleason part $P(x)$ in H^{∞} is contained in $M(B)$ ([21], p.437).

<u>Proposition 6.9.</u> The map Γ sends Gleason parts onto Gleason parts. The Gleason parts in $M(H^{\infty}) \setminus M(B)$ are in a one-to-one correspondence with the Gleason parts in $M(A) \setminus \partial A$.

<u>Proof.</u> Let $x_1, x_2 \in M(H^{\infty})$ and $\rho_{H^\infty}(x_1, x_2) < 1$. Then

$$\begin{aligned}
\rho_A(\Gamma(x_1), \Gamma(x_2)) &= \sup_{\substack{f \in A \\ \|f\| < 1}} \rho(\Gamma(x_1)(f), \Gamma(x_2)(f)) \\
&= \sup_{\substack{f \in A \\ \|f\| < 1}} \rho(f(x_1), f(x_2)) \\
&\leq \sup_{\substack{f \in H^{\infty} \\ \|f\| < 1}} \rho(f(x_1), f(x_2)) \\
&= \rho_{H^\infty}(x_1, x_2) < 1 .
\end{aligned}$$

Hence $\Gamma(x_1)$ and $\Gamma(x_2)$ lie in the same Gleason part.

Let us now prove the "onto" assertion. Let $x \in M(H^{\infty}) \setminus M(B)$ be a point whose Gleason part is nontrivial and let $m \in M(A) \setminus \partial A$ such that $\rho_A(m, \Gamma(x)) < 1$. We claim that $\Gamma^{-1}(m) \in P(x)$. Since the Gleason part of $\Gamma(x)$ in A is nontrivial, there exists by Theorem 5.6 a point $z \in \mathbb{D}$ such that $L_{\Gamma(x)}(z) = m$. Choose an interpolating sequence $\{z_n\}$ such that x lies in the closure of $\{z_n\}$. Hence $\lim f \circ L_{z_\alpha}(z) = f \circ L_x(z)$ for every $f \in H^\infty$. On the other hand, if $f \in A$, $\lim f \circ L_{z_\alpha}(z) = f \circ L_{\Gamma(x)}(z) = f(m)$, since $z_\alpha \to \Gamma(x)$ in the topology of $M(A)$. Then $L_x(z)|_A = m$. This means that by Corollary 6.8 $\Gamma^{-1}(m) = L_x(z)$. This proves that $\Gamma^{-1}(m) \in P(x)$.

The other assertion of the proposition follows immediately. □

§ 7 Prime ideals in QA_B and $C_B \cap H^\infty$

In [29] we solved a problem of F. Forelli and J. Kelleher by showing that a prime ideal in the algebra H^∞ (resp. $A(\mathbb{D})$) is finitely generated if and only if it is a maximal ideal of the form $M(z_0) = \{f \in H^\infty \text{ (resp. } A(\mathbb{D})): f(z_0) = 0\}$ for some $z_0 \in \mathbb{D}$. In [14] P. Gorkin could also give a similar characterization of the finitely generated prime ideals in the algebra QA. The proofs of these results are different. The general case of an arbitrary Chang-Marshall algebra remained open. In the following we now settle this problem. First we present some well known lemmas.

<u>Lemma 7.1</u> ([13], p.104). **Let B be a commutative algebra and let A be a subalgebra of B. If J is an ideal in B such that $J \cap A$ is prime, then there exists a prime ideal Q in B such that $Q \cap A = J \cap A$.**

Proof. Let $S = A \setminus (J \cap A)$. Then S is a multiplicatively closed subset of B with $S \cap J = \emptyset$. By Zorn's lemma there exists an ideal $Q \supseteq J$ in B which is maximal with respect to $S \cap Q = \emptyset$. By Krull's lemma Q is prime. By construction we have $J \cap A \subseteq Q \cap A \subseteq J \cap A$. □

Lemma 7.2 (Nakayama's lemma ([22], p.50 ff., Theorem 76)). Let A be a commutative ring with identity element, I a finitely generated ideal in A and M an arbitrary ideal. Suppose that $IM = I$. Then there exists an element $f \in A$ of the form $f = 1+m$, $m \in M$, such that $fI = 0$.

Now we state the main result of this section.

Theorem 7.3. Let A be a Chang-Marshall algebra. A prime ideal $P \neq (0)$ in A is finitely generated if and only if it is a maximal ideal of the form $M(z_o) = \{f \in A: f(z_o) = 0\}$ $(z_o \in \mathbb{D})$.

Proof. Let $P = M(z_o)$. Then $P = (z-z_o)$ and hence P is finitely generated. Conversely, let $P \neq (0)$ be a finitely generated prime ideal in A. Assume there exists a point m of the Shilov boundary ∂A of A which contains P. By Proposition 5.8 every function $f \in m$ can be decomposed in a product $f = f_1 f_2$ of two functions f_1 and f_2 such that $f_1(m) = f_2(m) = 0$. Since P is prime we obtain the relation $P = Pm$. Because P is finitely generated and A has no zero divisors, Nakayama's lemma implies that $P = (0)$. This is a contradiction. Thus $Z(P) \cap \partial A = \emptyset$. By Theorem 6.5 P can be lifted to a unique ideal J in H^∞.

Obviously, J has the same generators, hence J is finitely generated. Lemma 7.1 implies that there exists a prime ideal Q in H^∞ such that $Q \cap A = P$. Since by Theorem 6.5 the extension of P is uniquely determined, we have $Q = J$. Thus Q is a finitely generated prime ideal in H^∞. By ([29], p.300, or [14], p.317) Q has the form $Q = \{f \in H^\infty : f(z_o) = 0\}$, $z_o \in \mathbb{D}$. Hence $P = Q \cap A = \{f \in A : f(z_o) = 0\}$. □

Remarks. We can avoid Nakayama's lemma in the proof by using the analytical method in ([29], p.300).

A main ingredient in the proof of the characterization of the finitely generated prime ideals in H^∞ was the fact that every prime ideal in H^∞ which contains an interpolating Blaschke product is primary ([29], p.298). The analogous result now holds for any Chang-Marshall algebra.

Theorem 7.4. Let A be a Chang-Marshall algebra. Then every prime ideal P in A that contains an interpolating Blaschke product b is primary.

Proof. Since P contains the inner function b, Theorem 5.3 implies that $Z(P) \cap \partial A = \emptyset$. Hence there exists by Theorem 6.5 and Lemma 7.1 a (unique) prime ideal $Q \subseteq H^\infty$ such that $Q \cap H^\infty = P$. By ([29], p.298) Q is primary. Corollary 6.6 now yields the assertion. □

The following proposition now guarantees the existence of non-maximal primary prime ideals.

Recall that G_A denotes the set of maximal ideals in A whose Gleason part is an analytic disk (see § 5 B).

<u>Proposition 7.5.</u> Let $B \neq H^\infty + C$ be a Douglas algebra and let A be one of the associated Chang-Marshall algebras. Then every maximal ideal $m \in G_A \setminus \mathbb{D}$ contains a non-maximal prime ideal which is primary.

<u>Proof.</u> The assumption $B \neq H^\infty + C$ implies that $A \neq QA$ and $A \neq A(\mathbb{D})$. Hence $G_A \setminus \mathbb{D} \neq \emptyset$ (see § 5 B). Choose $m \in G_A \setminus \mathbb{D}$ and $b \in m$, where b is an interpolating Blaschke product. Let

$$S = \{Fg: g, F \in A,\ F \text{ outer},\ g \notin m\}\ .$$

Then S is a multiplicatively closed subset of A such that $S \cap (b) = \emptyset$. Zorn's and Krull's lemmas now imply the existence of a prime ideal $P \subseteq m$ which contains b and satisfies $P \cap S = \emptyset$. In particular, $z - \alpha \notin P$ for every $\alpha \in \mathbb{T}$. Thus P is non-maximal. By Theorem 7.4, P is primary. □

<u>Remark.</u> 1. Proposition 7.5 generalizes ([26], p.61). The proof there cannot be used in the present general setting, since the inner part of a function $f \in A$ may not belong to A.

2. Whereas in the disk algebra every prime ideal is primary (see [26], p.22), we do not know if in QA there even exist non-maximal primary prime ideals. On the other hand, there exist many prime ideals in QA which are not primary. As a concrete example take $P = \{S^{1/n} f \in QA:\ n \in \mathbb{N}\}$, where $S(z) = \exp\left(-\frac{1+z}{1-z}\right)$. This ideal is prime, since $P = QA \cap Q$, where Q is

the (countably generated!) prime ideal $Q = (S^1, S^{1/2}, S^{1/3}, S^{1/4}, \ldots)$ in H^∞ (see [26], p.59). On the other hand, P is not primary. This follows in the same manner as the proof of Theorem 4.2.

The existence of non-primary prime ideals in A, $A \neq QA$ and $A \neq A(\mathbb{D})$, which are even closed, will be clear when using the Gleason part structure of A. Indeed, if $m \in G_A \setminus \mathbb{D}$, then the ideal

$$P = \{f \in A: f \text{ vanishes identically on } P(m)\}$$

is a closed prime ideal (see [26], p.62). Recall that in $A(\mathbb{D})$ and QA there do not exist any non-maximal closed prime ideals.

At the end of this section, we want to state an open problem. In [16] we have proved that every prime ideal in QA_B can be lifted to a prime ideal in H^∞. Question: Is this also true for the algebra $A = C_B \cap H^\infty$? Note that by Theorem 6.5 and Lemma 7.1 this can be done if $Z(P) \cap \partial A = \emptyset$.

§ 8 Closed ideals in QA_B and $C_B \cap H^\infty$

A major problem in the theory of Banach algebras is a characterization of the closed ideals. In the disk algebra $A(\mathbb{D})$ this problem was solved by Beurling and Rudin (see § 1). A similar characterization of the closed ideals in QA could be obtained in a joint work [15] of the author with P. Gorkin and H. Hedenmalm (see §§ 3, 4). However, the situation in the other Chang-Marshall algebras is much more difficult.

Only partial results are known (see [30] and [16]). The following theorem now provides us at least with a complete description of the countably generated closed ideals in QA_B. Since it seems that our proof remains valid for a larger class of algebras, we present it in this generalized form.

Theorem 8.1. Let A be closed subalgebra of H^∞ such that

(1) A contains the algebra QA,

(2) the corona theorem holds in A, i.e., $\mathbb{D}$ is dense in M(A),

(3) A has the F-property in the sense of Havin, i.e., if $uf \in A$, where u is inner, then $f \in A$.

Then a closed ideal $I \neq (0)$ in A is countably generated if and only if I is a pricipal ideal generated by an inner function u in A.

Proof. Let $0 \neq I = (f_1, f_2, \dots)$ be a countably generated, closed ideal in A. We may assume without loss of generality $\|f_i\| \leq 2^{-i}$. Let u ($u \in H^\infty$) be the inner factor of I, i.e. the gcd of the inner parts of the functions in I. By hypothesis (3) the ideal

$$J = \{f \in A: uf \in I\}$$

is also a closed countably generated ideal in A. Note that the inner factor of $J = (g_1, g_2, \dots)$, where $g_j = f_j/u$, is 1. We shall show that $J = A$, from which we can conclude that $I = uJ = (u)$. In particular, we have $u \in A$.

Suppose that J is a proper ideal in A. By ([35], p.321, Theorem 1) and (1), $J \cap QA \neq (0)$. According to ([15], Theorem 2.5), we have $J \cap QA = I(E,QA)$, where $\emptyset \neq E \subseteq M(QC)$. In particular, there exists an outer function $F \in I(E,QA)$ with $\|F\| = 1$. By ([16], Lemma 3.3), $\sqrt[n]{F} \in QA$. Hence $\sqrt[n]{F} \in I(E,QA) \subseteq J$.

Proposition 4.4 in ([26], p.44) implies that each function $f \in J$ can be represented in the form

$$f = \sum_{i=1}^{n} h_i g_i \qquad (n = n(f) \in \mathbb{N}),$$

where $h_i \in A$ and $\|h_i\| \le C\|f\|$ for a constant $C > 0$, which is independent of f. Hence $|\sqrt[n]{F}| \le C \sum_{i=1}^{n} |g_i|$ on $\mathbb{D}$ for all $n \in \mathbb{N}$. Choose $n_o \in \mathbb{N}$ so that $\sum_{i=n_o}^{\infty} 2^{-i} \le \frac{1}{6C}$. Since we assumed that J was proper, there exists a point $m \in M(A) \setminus \mathbb{D}$ such that $J \subseteq m$. Thus in a neighbourhood $U(m)$ of m in $M(A)$ we have

$$\sum_{i=1}^{n_o} |g_i| \le \frac{1}{6C} .$$

By hypothesis (2), the set $U(m) \cap \mathbb{D}$ is nonempty. Let $z \in U(m) \cap \mathbb{D}$. Then for n sufficiently large we have

$$\frac{1}{2} \le \sqrt[n]{|F(z)|} \le C \sum_{i=1}^{\infty} |g_i| \le C\left(\sum_{i=1}^{n_o} |g_i| + \sum_{i=n_o}^{\infty} |g_i| \right)$$

$$\le C\left(\frac{1}{6C} + \frac{1}{6C}\right) = \frac{1}{3} ,$$

which is a contradiction. Thus, $J = A$. □

Remarks. 1. Theorem 8.1 generalizes Theorem 3.8 of [16], where the finitely generated closed ideals in QA_B were characterized by essentially the same method. As a further special case we obtain Theorem 4.6 of ([26], p.47), see also ([28], p.548), where the countably generated closed ideals in H^{∞} were characterized. The proof in ([26], [28]) is based on the fact that H^{∞} is the smallest weak-*-closed subspace of L^{∞} containing the polynomials. Since the algebras QA_B, $B \neq L^{\infty}$, are proper subspaces of H^{∞}, we had thus to develop new methods to prove Theorem 8.1.

2. An analysis of the proof above shows that if A is a closed subalgebra of H^{∞} satisfying (1) and (2), then one can conclude that any closed ideal I in A whose inner factor is 1, is countably generated if and only if it is trivial, i.e., if $I = (0)$ or $I = A$. As an example of such an algebra we mention the algebra COP of all bounded analytic functions which are constant on the Gleason parts of $M(H^{\infty} + C)$ ([11], p.442). It is well known that this algebra can also be described in the following way:

$$COP = B_o \cap H^{\infty} = \{f \in H^{\infty}: \lim_{|z| \to 1} (1-|z|^2)|f'(z)| = 0\} .$$

Here, B_o is the little Bloch space.

Before we study the case of the algebras $C_B \cap H^{\infty}$, we want to present some general facts on countably generated closed ideals. W. Dietrich ([6], p.72), e.g., proved that for any such ideal in a uniform algebra A, its zero set restricted to the Shilov boundary ∂A of A is open-closed in ∂A. The following proposition is closely related to Dietrich's result.

<u>Proposition 8.2.</u> Let A be a uniform algebra and $I \neq 0$ a countably generated closed ideal in A whose zero set is nowhere dense in M(A). Assume that the Shilov boundary ∂A of A

coincides with the Choquet boundary of A. Then the zero set of I is disjoint from ∂A.

Proof. We may assume without loss of generality that $I = (f_1, f_2, \ldots)$, where $\|f_i\| \le 4^{-i}$ $(i = 1,2,\ldots)$. Suppose that $Z(I) \cap \partial A \neq \emptyset$. Let $m \in Z(I) \cap \partial A$. Since by hypothesis ∂A coincides with the Choquet boundary, the proof of Proposition 5.8 shows that every function $f \in I$ can be factorized in a product $\frac{f_i}{\|f_i\|} = g_i h_i$ of two functions g_i and h_i such that $g_i(m) = h_i(m) = 0$ $(i = 1,2,\ldots)$. Moreover, g_i may be chosen in the closure of the ideal generated by f_i with $\left\|\frac{f_i}{\|f_i\|} - g_i\right\| \le 1$ and with $\|h_i\| \le k$ for some constant k (see [2], p.76).

Let $G_i = \sqrt{\|f_i\|}\, g_i$ and $H_i = \sqrt{\|f_i\|}\, h_i$. Then $\|G_i\| \le 2\cdot 2^{-i}$ and $\|H_i\| \le k2^{-i}$. Moreover, $G_i \in I$ and $f_i = G_i H_i$.

Again by ([26], p.44, Proposition 4.4), every function $f \in I$ can be represented in the form $f = \sum_{i=1}^{n} q_i f_i$, where $q_i \in A$, $\|q_i\| \le C\|f\|$ for some $C > 0$, and where $n = n(f)$. Hence

$$\sum_{i=1}^{\infty} |G_i| \le \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} C\,\|G_i\|\,|f_j| = C \sum_{i=1}^{\infty} \|G_i\| \sum_{j=1}^{\infty} |f_j|$$

$$\le 2C \sum_{j=1}^{\infty} |f_j| \qquad \text{on} \quad M(A)\,.$$

The Cauchy-Schwarz inequality now yields

$$\sum_{i=1}^{\infty} |G_i|^2 \le \left(\sum_{i=1}^{\infty} |G_i|\right)^2 \le 4C^2\left(\sum_{i=1}^{\infty} |f_i|\right)^2 = 4C^2\left(\sum_{i=1}^{\infty} |G_i H_i|\right)^2$$

$$\le 4C^2 \sum_{i=1}^{\infty} |G_i|^2 \sum_{i=1}^{\infty} |H_i|^2 \quad \text{on} \quad M(A)\,.$$

Noting that $Z(I) = \{x \in M(A) : \sum_{i=1}^{\infty} |G_i(x)| = 0\}$, we obtain on $M(A) \setminus Z(I)$

$$\frac{1}{4C^2} \le \sum_{i=1}^{\infty} |H_i|^2\,.$$

Let Z be the zero set of the function $H = \sum_{i=1}^{\infty} |H_i|^2$. Note that $H \in C(M(A))$ and that $Z \subseteq Z(I)$. Since $Z(I)$ is nowhere dense, there exists a net $x_\alpha \in M(A) \setminus Z(I)$ such that $H(x_\alpha) \to 0$. But $H(x_\alpha) \geq \frac{1}{4C^2}$, which is a contradiction. Therefore,

$$Z(I) \cap \partial A = \emptyset . \qquad \square$$

Using the fact that the algebras $C_B \cap H^\infty$ are logmodular ([4], p.15), we see that the hypotheses of Proposition 8.2 are fulfilled. Hence we obtain the following corollary.

Corollary 8.3. Let I be a countably generated closed ideal in $C_B \cap H^\infty$ whose zero set meets the Shilov boundary. Then $I = (0)$.

Our next theorem gives in any Chang-Marshall algebra a characterization of the closed ideals which contain an interpolating Blaschke product. The proof is based on an idea of H. Hedenmalm.

Theorem 8.4. Let A be a Chang-Marshall algebra and let I be a closed ideal in A with inner factor 1. Assume that I contains an interpolating Blaschke product b. Then I is an intersection of maximal ideals, i.e., $I = I(Z(I),A)$. Hence I is saturated.

Proof. By ([34], p.554) the zero set $\{z_n\}$ of b in $\mathbb{D}$ is an A-interpolating sequence. Therefore the map $T: A/bA \to l^\infty$ defined by $T(f+bA) = (f(z_1), f(z_2), \ldots)$ is an algebra isomorphism of the quotient space A/bA onto l^∞. By ([12], p.17, § 2), the closed ideals in A that contain the closed ideal bA are in a one-to-one correspondence with the closed ideals in A/bA, hence in l^∞. Since l^∞ is isometrical isomorphic to $C(\beta\mathbb{N})$, where $\beta\mathbb{N}$ is the Stone-Cech compactification of the integers, every closed ideal in l^∞ is an intersection of maximal ideals (see [12], p.271, § 36). This yields the assertion. □

As a corollary we obtain a generalization of ([30], p.223, Theorem 3.1).

Corollary 8.5. Let P be a prime ideal in the Chang-Marshall algebra A. Assume that P contains an interpolating Blaschke product. Then P is dense in a unique maximal ideal.

Proof. The proof follows immediately from Theorem 7.4 and Theorem 8.4. □

Theorem 8.4 can also be applied to get more information on the countably generated closed ideals in $C_B \cap H^\infty$.

Corollary 8.6. Let I be a closed ideal in $C_B \cap H^\infty$ with inner factor 1. Assume that I contains an interpolating Blaschke product. Then I is not countably generated.

Proof. By Theorem 8.4, I is an intersection of maximal ideals. These are all contained in $M(C_B \cap H^\infty) \setminus M(C_B)$. Hence there exists by Theorem 6.5 a unique extension J to H^∞. Now it is easy to see that J itself is an intersection of maximal ideals of H^∞. Because J is closed, Theorem 4.6 of ([26], p.47) implies that J is not countably generated. Hence $I = J \cap C_B$ is not countably generated, since the extension was unique. □

Remark. We do not know a complete characterization of the countably generated closed ideals in $C_B \cap H^\infty$, but we guess that they are just the principal ideals generated by inner functions.

Using Theorem 6.5 and Corollary 8.3, we obtain nevertheless a characterization of the countably generated maximal ideals in $C_B \cap H^\infty$.

Proposition 8.7. A maximal ideal m in $C_B \cap H^\infty$ is countably generated if and only if

$$m = M(z_o) = \{f \in C_B \cap H^\infty : f(z_o) = 0\}$$

for some $z_o \in \mathbb{D}$.

Proof. If $m \in M(C_B)$, then m cannot be countably generated by Corollary 8.3. If $m \in M(C_B \cap H^\infty) \setminus M(C_B)$, then the proof of Corollary 8.6 shows that m is not countably generated unless $m = M(z_o)$. □

§ 9 Some unusual properties of QA_B

A. Algebraic properties of QA_B

It is well known that the algebra H^∞ is a Pseudobezout ring, i.e. an integral domain such that any two functions in H^∞ have a gcd (see [33], p.520). It is now a natural question to ask for a characterization of those algebras of type QA_B which have this property. Having in mind the results of § 2, we may also ask whether there exist, apart from QA, other algebras of the form QA_B which are Prebezout rings. The following theorem gives a complete answer to both questions.

Theorem 9.1. Let A be an algebra of type QA_B. Then we have:

(1) A is a Pseudobezout ring if and only if $A = H^\infty$,

(2) A is a Prebezout ring if and only if $A = QA$.

Proof. (1) In view of the previous discussion, we have only to show that if $A \neq H^\infty$, then there exist two functions in A whose gcd does not exist. In fact, let u be an inner function which is not contained in QA_B. We note that the Douglas-Rudin theorem ([11], p.192) implies that such a function exists, since otherwise A would coincide with H^∞. By a theorem of T. Wolff ([35], p.321), there is an outer function $F \in QA$ such that $uF \in QA$. It is now easy to check that the functions uF and F have no gcd in QA_B. Hence QA_B is not a Pseudobezout ring if $B \neq L^\infty$.

(2) In § 2 we have shown that QA is a Prebezout ring. Hence it remains to show that the algebras QA_B, $B \nsubseteq H^\infty + C$ are not Prebezout rings. To this end choose an infinite Blaschke product $b \in QA_B$; its existence is guaranteed by the Chang-Marshall theorem (see Proposition 5.4). Let $\alpha \in T$ be a cluster point of the zeros of b. It is obvious that the gcd of b and $z-\alpha$ is one. But on the other hand, the functions b and $z-\alpha$ generate a proper ideal in QA_B. Hence $1 \notin (b, z-\alpha)$. Thus, if $B \nsubseteq H^\infty + C$, the algebra QA_B cannot be a Prebezout ring. □

Next we turn to a problem of L. Rubel of characterizing subrings of H^∞ which are coherent. We call an integral domain coherent if and only if the intersection of any two finitely generated ideals is finitely generated again. McVoy and Rubel [24] proved that H^∞ is coherent. In contrast to this we prove that no algebra of the form QA_B, $B \nsubseteq L^\infty$, has this property. In fact, there exist in QA_B, $B \nsubseteq L^\infty$, two principal ideals whose intersection is not even countably generated, as the next proposition shows.

<u>Proposition 9.2.</u> No algebra of the form QA_B is coherent with the exception of $QA_B = H^\infty$.

<u>Proof.</u> Let $QA_B \neq H^\infty$ and let u be an inner function not contained in QA_B. Choose according to Wolff ([35], p.321) an outer function $F \in QA$ with $uF \in QA$. We claim that the intersection of the ideals $I_1 = (F)$ and $I_2 = (uF)$ is not countably generated. Noting that QA_B has the F-property, we obtain

$I_1 \cap I_2 = \{uFf\colon uf \in QA_B\}$. Since the ideal $J := \{uf\colon uf \in QA_B\}$ is the trace in QA_B of the closed ideal uH^∞, J itself is closed. Its inner factor is u. Since $u \notin QA_B$, we can conclude from Theorem 8.1 that J is not countably generated. Hence $I_1 \cap I_2 = FJ$ is not countably generated. □

B. Separating properties of QA_B

Let m be a nontrivial point in $M(H^\infty) \setminus \mathbb{D}$, i.e. a point whose Gleason part $P(m)$ is not trivial. Then the closure $\overline{P(m)}$ of $P(m)$ does not meet the Shilov boundary ([19], p.102). Surprisingly this cannot be generalized to arbitrary Chang-Marshall algebras, as we shall show in this section.

<u>Definition.</u> Let $\{z_n\}$ be an interpolating sequence. We call $\{z_n\}$ thin, if it is finite or if $\lim\limits_{n} \prod\limits_{k \neq n} \rho(z_k, z_n) = 1$ (see § 4). The Blaschke product b associated to a thin interpolating sequence is called a thin Blaschke product.

Let B_o denote the Douglas algebra generated by H^∞ and the complex conjugates of all thin interpolating Blaschke products. This algebra has been studied in recent years by many authors (Guillory, Izuchi, Sarason (see [15] and [17])). It is well known that an inner function u is invertible in B_o if and only if u is a finite product of thin interpolating Blaschke products (see [17], Theorem 2.6). Hence we have the rather curious situation that by Theorems 5.3, 5.6 and 5.7

the maximal ideal space of the associated Chang-Marshall algebra contains only nontrivial Gleason parts with the exception of those of the Shilov boundary. We are now in a position to present an example which shows that in the algebras $A = QA_{B_o}$ and $A = C_{B_o} \cap H^\infty$ the closure of any nontrivial Gleason part $m \in M(A) \setminus \mathbb{D}$ does not only meet the Shilov boundary of A, but that $\overline{P(m)} \setminus P(m)$ is entirely contained in ∂A.

<u>Proposition 9.3.</u> Let B_o be the Douglas algebra generated by H^∞ and the complex conjugates of the thin interpolating Blaschke products and let A_o be one of the associated Chang-Marshall algebras. If m is a maximal ideal in $M(A_o) \setminus \mathbb{D}$, then $\overline{P(m)} \setminus P(m)$ is contained in the Shilov boundary of A_o.

<u>Proof.</u> We have only to consider the case where $m \notin \partial A_o$. Then, by the remarks above, $P(m)$ is nontrivial. Let $m_1 \in \overline{P(m)} \setminus P(m)$. Assume there exists a thin Blaschke product b such that $|b(m_1)| < 1$. Because $|b(m)| = 1$ implies that b has modulus 1 on $\overline{P(m)}$ (see [19], p. 78), we have $|b(m)| < 1$. By ([17], Proposition 2.3) and Proposition 6.9 there exists a point $m_o \in P(m)$ such that $b(m_o) = 0$. Since $b \circ L_{m_o}(z) = e^{i\alpha} z$, where L_{m_o} denotes the corresponding analytic disk (see § 5 B) of $\mathbb{D}$ onto $P(m_o) = P(m)$, we see that $|b| \equiv 1$ on $\overline{P(m)} \setminus P(m)$. In particular, $|b(m_1)| = 1$, which contradicts the choice of b. Thus $|b(m_1)| = 1$ for every thin Blaschke product. Since every inner function in A_o is a finite product of thin Blaschke products, Theorem 5.3 implies that m_1 belongs to the Shilov boundary of A_o. □

Proposition 9.3 has now several unexpected consequences. First we give some definitions.

Definition. For $\alpha \in T$ let $M_\alpha(A) = \{m \in M(A): z-\alpha \in m\}$ be the fiber over α of the maximal ideal space of the Chang-Marshall algebra A. We shall say that A is regular on $\partial_\alpha A := M_\alpha(A) \cap \partial A$ if for every closed subset E of $\partial_\alpha A$ and for every point $x \in \partial_\alpha A \setminus E$ there exists a function $f \in A$ such that f vanishes on E but $f(x) \neq 0$.

K. Hoffman ([18], p.87) proved that H^∞ is regular on $\partial_\alpha H^\infty$ for every $\alpha \in T$. Our next proposition now shows that this cannot be generalized to an arbitrary Chang-Marshall algebra.

Proposition 9.4. Let A be an algebra of the form QA_B. Assume there exists a nontrivial Gleason part $P(m) \neq \mathbb{D}$ in A whose closure meets the Shilov boundary. Then there exists a point $\alpha \in T$ such that A is not regular on $\partial_\alpha A$.

Proof. Let $\alpha \in T$ be chosen so that $m \in M_\alpha(A)$. Then it is easy to check that $\overline{P(m)} \subseteq M_\alpha(A)$. Moreover, the support set supp μ_m of the representing measure for m is contained in $M_\alpha(A) \cap \partial A = \partial_\alpha A$ ([18], p.188). Also, $A|_{M_\alpha(A)}$ is a uniform closed subalgebra of $C(M_\alpha(A))$([18], p.187). Let $y \in \overline{P(m)} \cap \partial A$. Then $y \in \partial_\alpha A$. Since $m \notin \partial A$, the support set supp μ_m of the representing measure μ_m for m contains a point $x \in \partial_\alpha A$ different from y. Choose an open set U in the compact Hausdorff space $\partial_\alpha A$ such that $x \in U$, but $y \notin \bar{U}$. Let $E = \bar{U}$. We claim that every function which vanishes on E also vanishes in y. Hence A cannot be regular on $\partial_\alpha A$.

Let $f \in I(E,A)$ and let uF be its inner-outer factorization. Since A has the F-property, $F \in A$ ([16]). Moreover, $F \in I(E,A)$ (see the remarks after Proposition 3.4). By ([18], pp.190/191), $F(m) = 0$, since F vanishes on an open set U which intersects supp u_m. Since $\bar{F} \in A$ ([16], Lemma 3.3), Lemma 1.2 of ([11], p.403) implies that F vanishes on $\overline{P(m)}$. In particular, $f = uF$ vanishes in y, which proves our claim.

A famous theorem of S. Axler [1] shows that if $f \in L^\infty$, then there exists an inner function $u \in H^\infty$ such that $uf \in H^\infty + C$. In view of this result, one may ask whether it generalizes to the algebras QA_B.

<u>Problem.</u> Let $f \in QB$. Does there exist an inner function $u \in QA_B$ such that $uf \in QA_B + C$?

T. Wolff has shown that if $B = H^\infty + C$, even a much stronger relation holds; in fact, $QC = QA + C$ ([35], p.325), i.e., the inner function in the problem above can be taken to be the constant 1. Quite unexpected, P. Gorkin has now discovered a Douglas algebra B for which the answer is negative (private communication). The final proposition of this work now strengthens her result. It shows that if $A = QA_B$ fails to be regular on $\partial_\alpha A$, then there do not exist even unimodular functions *in* L^∞ which multiply some specified function $f \in QB$ into $QA_B + C$.

Proposition 9.5. Let B be a Douglas algebra so that $A = QA_B$ is not regular on $\partial_\alpha A$ for some $\alpha \in T$. Then there exists a function $q \in QB$ which cannot be multiplied into $QA_B + C$ by any unimodular function $u \in L^\infty$. (As usual we call $u \in L^\infty$ unimodular if $|u| = 1$ a.e. on T.)

Proof. By hypothesis there exist $\alpha \in T$, a closed subset E of $\partial_\alpha A$ and a point $x \in \partial_\alpha A \setminus E$ such that $f(x) = 0$ whenever $f \in QA_B$ and f vanishes on E. Choose $q \in QB$ so that q vanishes on E but $q(x) \neq 0$. This is possible, since $QB = C(M(QB))$ is a regular algebra.

Assume there exists a unimodular function $u \in L^\infty$ such that $uq \in QA_B + C$, then $uq|_{M_\alpha(A)} \in QA_B|_{M_\alpha(A)}$. Let $f = uq$ and let $m \in E$. Choose $x \in M_\alpha(L^\infty)$ such that $x|_{QB} = m$. Then $m(f) = x(f) = x(u)x(q) = x(u)m(q) = 0$. Hence f vanishes on E. By the choice of E, $f(x) = 0$. Now let $y \in M(L^\infty)$ so that $y|_{QB} = x$. Then

$$0 = x(f) = y(f) = y(uq) = y(u)y(q) .$$

Since $|y(u)| = 1$, $x(q) = y(q) = 0$. This is a contradiction to the choice of q. $\square$

Remark. We find this result rather surprising, since by Chang-Marshall's result $QB = QA_B + C_B$.

At the end of this work, we want to state the following open problem.

Open problem. Let B be a Douglas algebra. For which B every $f \in QB$ can be multiplied by unimodular functions $u \in QB$ into $QA_B + C$?

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