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Strong Rigidity of Arithmetic Kac-Moody Groups

Thesis

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INTRODUCTION

The present thesis is devoted to the study of the rigidity phenomenon for arithmetic subgroups of real split Kac-Moody groups of certain types which are infinite-dimensional generalizations of Chevalley groups. There exists a rich body of literature for the rigidity phenomenon in the Chevalley context which has been developed throughout the 20th century with various branches and applications in a wide range of mathematics and physics. Therefore, we begin this introduction with a historical narrative about classical rigidity. Then we introduce the main ingredients of the thesis which are real split Kac-Moody groups of finite rank and their arithmetic subgroups. Then, we present the main results of the present thesis along with the main ideas how to prove them. At the end, follows an overview of the present thesis based on its chapters. To keep this introduction as brief as possible we do not define all the well-known mathematical concepts here. We assume that the reader is familiar with general concepts in the theory of algebraic groups, geometric group theory, ergodic theory and manifold theory. For geometric group theory we reference to [Löh11], for the theory of algebraic groups and ergodic theory one can reference to [Zim84] (see also [Mar91]) and for manifold theory a standard reference is [KN96].

Historical Narrative of Rigidity Phenomenon

In this section we present a historical narrative of the classical rigidity phenomenon from 1936 on. This concise overview is mainly extracted from [GP91] and [Spa04].

According to [Spa04] the study of the rigidity phenomenon started by A. Selberg's discovery in 1960 that up to conjugation the fundamental groups of certain compact locally symmetric spaces (i.e., spaces in which geodesic symmetries are local isometries) are always defined over algebraic numbers (see [Sel60]). This led to the study of deformation of such groups. A subgroup Γ of a group G is **deformation rigid** in G provided for any continuous path ρ_t of embeddings of Γ into G starting with $\rho_0 = \mathrm{id}$, ρ_t is conjugate to ρ_0. This can be seen as a special case of ergodic theoretic questions. In this way **rigidity theory** emerged and formed an area in mathematics.

In [GP91], the approaches towards rigidity theory are generally categorized into two main categories; the first is by means of the study of geometry and dynamics of actions. The approaches in the second category are through analytical methods.

According to [Spa04], E. Calabi and E. Vesentini in [CV60] and then A. Weil in [Wei60] and [Wei62] obtained the following full generality for the **local rigidity theorem**.

Local Rigidity Theorem. *(cf. [CV60, Introduction and summary of results], [Wei62, Section 16] and [Spa04]) Let Γ be a cocompact discrete subgroup of a semisimple Lie group*

G (i.e., G/Γ is compact). Assume that G has no compact nor $\mathrm{SL}(2,\mathbb{R})$ *nor* $\mathrm{SL}(2,\mathbb{C})$ *local factors. Then Γ is deformation rigid.*

The methods used in the above references for the local rigidity theorem are more analytic in nature (see [GP91]).

Later G. D. Mostow and H. Fürstenberg in a series of works: [Mos68], [Mos73] and [Für67] used geometric and dynamical methods and obtained the global form of the local rigidity theorem which is also known as the **strong (or Mostow) rigidity theorem** and has the following two equivalent versions (see [GP91] and [Spa04]).

Strong Rigidity Theorem - Geometric Form. *(cf. [Mos73, Theorem 12.1] and [Spa04]) If two closed manifolds (i.e., compact manifolds without boundary) of constant negative curvature and dimension at least 3 have isomorphic fundamental group, then they are isometric.*

Strong Rigidity Theorem - Algebraic Form. *[Spa04] Let Γ be a cocompact discrete subgroup of* $\mathrm{PSO}(1,n)$*, $n \geq 3$. Suppose $\Phi : \Gamma \to \mathrm{PSO}(1,n)$ is a homomorphism whose image is again a cocompact discrete subgroup of* $\mathrm{PSO}(1,n)$*. Then Φ extends to an automorphism of* $\mathrm{PSO}(1,n)$*.*

A discrete subgroup Γ of a locally compact group G is called a **lattice** in G if G/Γ has finite Haar measure. In 1973 G. Prasad obtained a generalization of the strong rigidity theorem for locally symmetric spaces of negative curvature with finite volume and equivalently for lattices in the corresponding isometry groups of such locally symmetric spaces (see [Pra73, Theorem B] and [Spa04]).

A lattice H in a linear algebraic $\mathbb{R}$-group G is called **irreducible** if G can **not** be decomposed into a product of two connected infinite normal subgroups G_1 and G_2 such that $G = G_1.G_2$, $G_1 \cap G_2 \subset Z(G)$ and the subgroup $(H \cap G_1).(H \cap G_2)$ is of finite index in H where $Z(G)$ denotes the center of G. In 1973, G. A. Margulis obtained a classification of all finite dimensional representations of irreducible lattices in semisimple groups which contain a 2-dimensional abelian subgroup diagonalizable over $\mathbb{R}$. According to [GP91] the methods used by G. A. Margulis are geometric and dynamical. This result is known as the **super rigidity theorem** and can be summarized in the form of the following theorem (see [Spa04], [Mar91]).

Super Rigidity Theorem. *(cf. [Mar91, Theorem (IX)6.16] or [Spa04]) Let Γ be an irreducible lattice in a connected semisimple Lie group G of* $\mathbb{R}$-rank *at least 2 with trivial center. Assume that G has no compact factors. Suppose $\Bbbk$ is a local field. Then any homomorphism π of Γ into a noncompact $\Bbbk$-simple group over $\Bbbk$ with Zariski dense image either has a precompact image or π extends to a homomorphism of the ambient groups.*

After this major result there has been a lot of development for the rigidity phenomenon in this direction which has been narrowed down to more specific groups for special applications in different branches of mathematics. For instance, N. Mok in [MoK89] gave an alternative proof of the above super rigidity theorem for (lattices in the isometry groups of) the Hermitian symmetric spaces (cf. [MoK89, Section 1.2]). Also, K. Corlette in [Cor92] proved the above super rigidity for discrete groups of isometries of the quaternionic hyperbolic spaces as well as for the Cayley plane (cf. [Cor92, Theorem 4.3]). We should mention here that the methods used in the two latter super rigidity results are analytic in nature (see [GP91]). For more recent works we reference to the works by U. Bader (see [BFGM07], [BM11] and [BF12]), N. Monod (see [BM02] and [Mon06]) and D. W. Morris (see [Mor89], [Mor95], [Mor97] and

[MS14]).

Main Results and Ideas

The main objects of the present thesis are real **split Kac-Moody groups** (of finite rank) and their arithmetic subgroups. Such split Kac-Moody groups can be seen as an infinite-dimensional generalization of Chevalley groups. Such split Kac-Moody groups (in some references they are called the minimal or smallest split Kac-Moody groups (see [KP83, page 141] or [Mar13, Chapter (II)5])) can also be regarded as the image of a group functor called the (split) **Tits functor** on the category of unital commutative rings which are uniquely determined over fields with similar arguments as for Chevalley group functors (see [Tit87]). Therefore, by means of the above functorial property of split Kac-Moody groups, it is plausible to define functorial arithmetic subgroups (i.e., the image of the group functor over integers) of real split Kac-Moody groups and study the rigidity of them.
Moreover, similar to Chevalley groups, every split Kac-Moody group has a root system, although it need not be finite. In the case that the root system is finite, the underlying split Kac-Moody group coincides with a Chevalley group (see for instance [KP83, Example 1.1]). Hence, one can define root subgroups as well as subgroups of finite rank. An important property of certain split Kac-Moody groups is that they can also be seen as a universal amalgamation of their rank one and rank two subgroups (see [AM97], see also [HKM13, Theorem 7.22]). Intuitively, a group G is a universal amalgamation of its proper subgroups if any other amalgamation of such subgroups is isomorphic to a quotient of G over a normal subgroup. We study the special case of two-spherical simply connected centered real split Kac-Moody groups. We see that for such split Kac-Moody groups, every fundamental subgroup of rank two (i.e., subgroups corresponding to a pair of simple roots), in turn, is a universal Chevalley group (i.e., a connected algebraically simply connected Chevalley group). This enables us to define **arithmetic subgroups** of real split Kac-Moody groups. An arithmetic subgroup of a real split Kac-Moody group G is the group generated by the $\mathbb{Z}$-points of the real root subgroups. We denote such a subgroup with $G(\mathbb{Z})$. Note that when G is a Chevalley group, the functorial arithmetic subgroup coincides with the arithmetic subgroup. But as we observe in Chapter 5, in general, arithmetic subgroup is only a subgroup of the functorial arithmetic subgroup. In the same fashion one can define the subgroup of G generated by the $n\mathbb{Z}$-point of the real root subgroups denoted by $G(n\mathbb{Z})$ for any $n \in \mathbb{N}$.
Using the functorial property of split Tits functors, one can define **functorial principal congruence subgroups** similar to the Chevalley context and we see in Section 5.2 that $G(n\mathbb{Z})$ is actually a subgroup of such functorial principal congruence subgroup of degree n and also a subgroup of an arithmetic subgroup of G. More importantly, the difference between the functorial principal congruence subgroup and $G(n\mathbb{Z})$ encodes the difference between the functorial arithmetic subgroup and the arithmetic subgroup of G when n is a prime number (see (5.129)).
Another important feature of real split Kac-Moody groups is that one can associate an algebraic-geometric structure to them, a **(twin) building**. Roughly speaking, a (twin) building associated to a real split Kac-Moody group G is a (double) coset of G over a special subgroup, called the **Borel** subgroup (e.g., the upper triangular matrices in $\mathrm{SL}_n(\mathbb{R})$) along with

an adjacency relation between the (double) coset elements which gives rise to a geometric realization of the associated building in the form of a (simplicial) polyhedral complex and enables us to apply geometric group theory in the Kac-Moody context (see [Dav08, Chapter 18]).

The goal of the present thesis is to obtain a counterpart for the rigidity phenomenon in the Kac-Moody context. To study the rigidity phenomenon for arithmetic subgroups of real split Kac-Moody groups we need to endow our real split Kac-Moody groups with a topology which is compatible with Lie topology on its rank one and rank two subgroups and the universal topology as well. This is achieved in [HKM13] by introducing a topology, called the **Kac-Peterson** topology, on certain split Kac-Moody groups. However, by [HKM13, Remark 7.28] our split Kac-Moody groups are **not** locally compact with respect to the Kac-Peterson topology and hence they do not carry a Haar measure. Therefore, it is not directly possible to define lattices or construct theories similar to ergodic theory.

However, we would like to obtain similar rigidity results as in the classical context for arithmetic subgroups of real split Kac-Moody groups. More precisely, we would like to find a unique extension of an abstract group homomorphism $\Phi : G(\mathbb{Z}) \to H(\mathbb{Q})$, namely $\tilde{\Phi} : G(\mathbb{Q}) \to H(\mathbb{Q})$. The strategy is to use the classical rigidity results in our situation. For this we need to localize our arguments and study the restriction of Φ into arithmetic subgroups of the irreducible fundamental rank two subgroups of G and obtain compatible local extensions. Then we can apply the universality property of the underlying real split Kac-Moody group in order to obtain a global extension. In Chapter 6 we elaborate on the proof of a fixed point result by P-E. Caprace and N. Monod in a more detailed level. This result, [CM09, Lemma 8.1], presents a fixed point theorem for a particular action of Chevalley groups (i.e., the action which preserves the polyhedral structure) on certain polyhedral complexes. More precisely, we present the following fixed point theorem.

Fixed Point Theorem. *(cf. [CM09, Lemma 8.1]) Let Δ be a complete* $\mathrm{CAT}(0)$ *polyhedral complex such that* $\mathrm{Shapes}(\Delta)$ *is finite (see Definition 3.2.2) and let G be an irreducible universal Chevalley group of rank at least two. Assume that the arithmetic subgroup $G(\mathbb{Z})$ of G acts on Δ by cellular isometries. Then $G(\mathbb{Z})$ has a fixed point.*

In our case, for the local homomorphisms induced by the restriction of Φ into the arithmetic subgroups of the rank two fundamental subgroups of $G(\mathbb{Q})$, we can apply the above fixed point theorem and the arguments in our case narrow down to the classical rigidity situation. For any $n \in \mathbb{N}$, define $\Gamma(n)$ to be the subgroup of $G(n\mathbb{Z})$ generated by all $n\mathbb{Z}$-points of real root subgroups of the fundamental irreducible rank two subgroups $G_{\alpha,\beta}(\mathbb{Q})$ of $G(\mathbb{Q})$ such that α and β are non-orthogonal simple roots of G. Now we can present the following super rigidity for the arithmetic subgroups of split Kac-Moody groups.

Super Rigidity Theorem - Kac-Moody context. *Let G and H be two irreducible two-spherical simply connected centered split Kac-Moody groups of rank at least two. Let $\Phi : G(\mathbb{Z}) \to H(\mathbb{Q})$ be an abstract group homomorphism whose restrictions to the $\mathbb{Z}$-points of any rank one fundamental subgroup of $G(\mathbb{Q})$ has infinite image. Then there exist an $n \in \mathbb{N}$ and a (uniquely determined) group homomorphism $\tilde{\Phi} : G(\mathbb{Q}) \to H(\mathbb{Q})$ such that $\tilde{\Phi}|_{\Gamma(n)} = \Phi|_{\Gamma(n)}$.*

For a more precise statement see Proposition 7.1.3. We also obtain the following version of the strong rigidity theorem in this context.

Strong Rigidity Theorem - Kac-Moody context. *Let G and H be two irreducible simply connected two-spherical centered split Kac-Moody groups of rank at least two whose associated*

Dynkin diagram do not contain any sub-diagram of type C_2. If two arithmetic subgroups $G(\mathbb{Z})$ and $H(\mathbb{Z})$ are strongly rigid (see Definition 7.2.1) and isomorphic, then $G(\mathbb{Q})$ and $H(\mathbb{Q})$ are isomorphic and have isomorphic root systems.

For a more detailed version of this strong rigidity result see Proposition 7.2.3.

Structure of the Thesis

As mentioned before, we assume that the reader is familiar with general concepts in the theory of algebraic groups, geometric group theory, ergodic theory and manifold theory. Nevertheless, Chapter 1 is devoted to the theory of algebraic groups and the classical rigidity results. The main aim is to fix the terminology and give precise statements which fit most to our context later on this thesis. In this chapter, Section 1.6 contains almost all details needed for our rigidity results concerning algebraic groups and we put it in the first chapter since it does not require any knowledge from Kac-Moody theory. We also cover preliminary definitions for amalgamation of groups in Section 1.7 of this chapter.

Chapter 2 contains all necessary knowledge in building theory which we need in order to apply geometric group theory to Kac-Moody theory. Chapter 3 is designed to give a brief introduction to CAT(0) geometry and its application to the geometric realization of a (twin) building. There is no original result in these two chapters and the reader who is familiar with algebraic (twin) buildings and their CAT(0) geometric realization may skip them.

In Chapter 4 we define split Kac-Moody groups in detail along with important results in Kac-Moody theory which will be needed later on in this thesis. Main features of this chapter are the split Tits functor, the adjoint representation of Kac-Moody groups and two topologies on such groups, namely the Zariski and Kac-Peterson topologies. Note that this chapter does not contain any original results.

In Chapter 5 we study (functorial) arithmetic subgroups of certain split Kac-Moody groups along with their (principal) congruence subgroups. There exist some original results in this chapter which are mostly combinatorial in nature and they provide calculations inside arithmetic subgroups which have some important use in the proof of the rigidity results in Chapter 7. More precisely, apart from Proposition 5.1.3 there exists no original result in Section 5.1. Section 5.2 is mainly designed to obtain Lemma 5.2.3 and Lemma 5.2.12 which are used in the proof of our rigidity results in Chapter 7. The main result of Section 5.3 is the equation (5.129).

Chapter 6 provides a brief introduction to geometric group theory and contains an elaboration on the proof of a fixed point theorem, [CM09, Lemma 8.1], for certain simplicial complexes which is a key element in our strategy towards the rigidity in the Kac-Moody context as explained in the preceding section of this introduction. This chapter does not contain any original result.

Finally in Chapter 7 we present our main results which gives a counterpart for the classical rigidity phenomenon in the Kac-Moody context. Namely, we prove the super rigidity (Proposition 7.1.3) and the strong rigidity (Proposition 7.2.3) for arithmetic subgroups of certain real split Kac-Moody groups.

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CHAPTER

ONE

CLASSICAL RESULTS

The main objects of the present thesis are arithmetic subgroups of certain real split Kac-Moody groups which are natural generalizations of Chevalley groups (see Chapter 4). Therefore, it is appropriate to define Chevalley groups and cite the known results which will be used later on in the Kac-Moody context. Moreover, since the main result of this thesis is on rigidity of arithmetic subgroups of certain real split Kac-Moody groups, the direction of our introduction to Chevalley groups is towards rigidity results for Chevalley groups. In addition, a way to treat split Kac-Moody groups is to see them as amalgamations of some certain subgroups. Thus we also spend a section on basic definitions and properties of amalgamation of groups.

More precisely, in this chapter, we start with the affine varieties and go on till we construct Chevalley groups and present thier adjoint representation. Also, the classical key theorems used in the next chapters are cited here. We end this chapter by defining basic concepts about amalgamation of groups. There is no original result in this chapter and we duly cite references for all well-known results. The main references are [Hum98], [Hum70], [Ste68], [Zim84] and [Mar91].

1.1 Linear Algebraic Groups

In this section we introduce linear algebraic groups and their basic properties. For this, we observe that the general linear group $\mathrm{GL}(n,\mathbb{C})$ can be seen as an affine variety with the Zariski topology. Then we investigate closed subgroups of $\mathrm{GL}(n,\mathbb{C})$ for some $n \in \mathbb{N} = \{1,2,...\}$ with respect to the Zariski topology.

Let $\Bbbk \subset \mathbb{K}$ be fields of characteristic 0, where $\mathbb{K}$ is, in addition, algebraically closed. We call the set $\mathbb{K}^n = \mathbb{K} \times ... \times \mathbb{K}$ an **affine n-space** and denote it by $\mathcal{A}^n$. If $X \subset \mathcal{A}^n$ is the set of common zeros of some collection of polynomials in $\mathbb{K}[T] = \mathbb{K}[T_1,...,T_n]$, it is called an **affine variety**. An affine variety X is said to be defined over $\Bbbk$ or to be an **affine $\Bbbk$-variety** if it is the set of zeros of a polynomial ideal in $\Bbbk[T_1,...,T_n]$ (see [Hum98, Section 1.1]).

Define **closed subsets** of $\mathcal{A}^n$ to be the set of common zeros of some collection of polynomials in $\mathbb{K}[T] = \mathbb{K}[T_1, ..., T_n]$. Therefore, the affine ($\Bbbk$-)varieties contained in X are closed subsets of X. Now topologizing the affine n-space $\mathcal{A}^n$ by declaring that the closed sets are to be precisely the affine varieties gives us a topology on $\mathcal{A}^n$ called the **Zariski topology**. Now we endow X with the induced topology from $\mathcal{A}^n$. An affine variety is called **irreducible** if it can not be written as the union of two proper non-empty closed subsets. Any affine variety can be written as a finite union $X = \cup X_i$ where X_i are irreducible affine varieties. Each X_i is called an **irreducible component** of X (see [Hum98, Section 1.2] and [Zim84, Section 3.1]).

Furthermore, let $X \subseteq \mathcal{A}^n$ be an arbitrary affine variety. Then we call the ring $\mathbb{K}[X] := \mathbb{K}[T]/I(X)$ the **affine algebra** (or **of regular functions**) of X where $I(X)$ is the ideal in $\mathbb{K}[T]$ generated by the collection of polynomials whose zero points form X. Similarly, $\Bbbk[X] := \Bbbk[T]/I_{\Bbbk}(X)$ is the affine $\Bbbk$-algebra corresponding to X where $I_{\Bbbk}(X)$ is the corresponding polynomial ideal over $\Bbbk$. If X is irreducible, we set $\mathbb{K}(X)$ to be the field of fractions of $\mathbb{K}[X]$ so called the **field of rational functions** on X. Similarly, if X is an irreducible affine $\Bbbk$-variety, $\Bbbk(X)$ is the field of rational functions over $\Bbbk$. A rational point is defined at all points of X except on a closed subset of X. One can show that a rational function defined over all points of X is a regular function i.e. it belongs to the affine algebra of X. Now if X is an affine ($\Bbbk$-) variety and $f \in \mathbb{K}[X]$ ($f \in \Bbbk[X]$) then $U_f := \{x \in X : f(x) \neq 0\}$ is called a **principal** ($\Bbbk$-)**open** set in X. These open subsets define a basis for the Zariski topology on X. Then the mapping

$$(x_1, ..., x_n) \to (x_1, ..., x_n, 1/f(x_1, ..., x_n))$$

defines a bijection from U_f onto an affine ($\Bbbk$-) variety $\widetilde{U}_f$ in $\mathcal{A}^{n+1}$, and $\mathbb{K}[\widetilde{U}_f] \cong (\mathbb{K}[X])[1/f]$ (see [Zim84, Section 3.1]).

Now we can define regular or rational morphisms (over $\Bbbk$) between affine $\Bbbk$-varieties.

Let $X \subseteq \mathcal{A}^n$ and $Y \subset \mathcal{A}^m$ be arbitrary affine varieties. A **regular (rational) morphism** of affine varieties is a (partial) mapping $\phi : X \to Y$ such that $\phi(x_1, ..., x_n) = (\psi_1(x), ..., \psi_m(x))$ where $\psi_i \in \mathbb{K}[X]$ ($\psi_i \in \mathbb{K}(X)$). Set $X_{\Bbbk} := X \cap \Bbbk^n$ which is exactly the set of points on X fixed by the Galois group $Gal(\mathbb{K}/\Bbbk)$. For a rational morphism of affine varieties $f : X \to Y$ which is defined over $\Bbbk$ we have $f(X_{\Bbbk}) \subset Y_{\Bbbk}$. By a ($\Bbbk$-) **isomorphism** of affine varieties we mean a bijective bi-regular morphism (defined over $\Bbbk$)(see [Zim84, Section 3.1]).

Definition 1.1.1. [Zim84, page 33] A topological space X which is a finite union of subspaces $X = \cup X_i$ is called **pre-variety** if

- each X_i is an open set which is itself an affine variety;
- $X_i \cap X_j$ is a principal open set in both X_i and X_j, and the two affine structures on X_i and X_j have the same affine structure on $X_i \cap X_j$ (i.e., $X_i \cap X_j$ is a closed subset in both X_i and X_j).

Moreover, a pre-variety is a **variety** if the diagonal set $\{(x, x) : x \in X\}$ is closed in $X \times X$.

A **morphism of pre-varieties** is defined to be locally a morphism of affine varieties.

Proposition 1.1.2. *[Hum98, Proposition 2.5] Let X be a variety and Y be a pre-variety.*

(1) If $\phi : X \to Y$ is a morphism then the graph $\Gamma_\phi := \{(x, \phi(x)) : x \in X\}$ is closed in $X \times Y$

(2) If $\phi, \psi : X \to Y$ are morphisms such that they agree on a dense subset of X then $\phi = \psi$.

If we look at the determinant operator as a polynomial function on $\mathcal{A}^{n^2}$ for any $n \in \mathbb{N}$, then the general linear group $\mathrm{GL}(n, \mathbb{K})$ is a principal open set of $\mathcal{A}^{n^2}$ defined by the non-vanishing set of det. Hence, $\mathrm{GL}(n, \mathbb{K})$ is a variety defined over $\mathbb{Q}$. Note that $\mathbb{Q} \subset \mathbb{K}$ since $\mathbb{K}$ is of characteristic 0.

Definition 1.1.3. A **linear algebraic** ($\Bbbk$-) **group** is a Zariski ($\Bbbk$-)closed subgroup of $\mathrm{GL}(n, \mathbb{K})$ for some $n \in \mathbb{N}$.

For example, $\mathrm{SL}(n, \mathbb{C})$ forms a linear algebraic $\mathbb{Q}$-group.

For any subgroup Γ in a linear algebraic $\Bbbk$-group G we define $\overline{\Gamma}$ to be the Zariski closure of Γ in G which is a linear algebraic $\Bbbk$-group by Definition 1.1.3. A linear algebraic group is called **connected** if there are no proper open subgroups. Assume that G° is the largest connected subgroup of G, then G/G° is finite (see [Mar91, I(0.13)]).

Proposition 1.1.4. *[Zim84, Corollary 3.1.2] If $\phi : G \to H$ is a regular morphism of linear algebraic groups then $\phi(G)$ is a linear algebraic group.*

Terminology 1.1.5. *For any ring $A \subset \mathbb{K}$ and a fixed embedding $G \subseteq \mathrm{GL}(n, \mathbb{K})$ we set*

$$G_A := G \cap \mathrm{GL}(n, A), \tag{1.1}$$

where G is a linear algebraic group.

(a) Throughout the present thesis, whenever we use the notation above, as a preassumption, we fix an embedding of G in some $\mathrm{GL}(n, \mathbb{K})$ without mentioning it unless specification is necessary.

(b) When G is a linear algebraic $\mathbb{Q}$-group, whenever we write $G_\mathbb{Z}$ we always consider embeddings of $G \subseteq \mathrm{GL}(n, \mathbb{C})$ which have the same $\mathbb{Q}$-rational points.

Definition 1.1.6. Let H be a subgroup of a group X. Two subgroups H and H' of a group X are called **commensurable** if the intersection $H \cap H'$ has finite index in both H and H'. Set

$$\mathrm{Comm}(H) := \{g \in X \mid gHg^{-1} \textit{ and } H \textit{ are commensurable } \}.$$

We call $\mathrm{Comm}(H)$ the **commensurability subgroup** of H in X.

$\mathrm{Comm}(H)$ in Definition 1.1.6 actually forms a subgroup of X and we have $H \subset N_X(H) \subset \mathrm{Comm}(H)$. Moreover, if H and H' are commensurable then $\mathrm{Comm}(H) = \mathrm{Comm}(H')$ (see [Mar91, page 8]).

Moreover, it is well-known to experts that different faithful representations of a linear algebraic $\mathbb{Q}$-group G with the same $\mathbb{Q}$-points have commensurable $\mathbb{Z}$-points (see [Mar91, Lemma I(3.1.3)]). In other words

$$G_\mathbb{Q} \subset \mathrm{Comm}(G_\mathbb{Z}). \tag{1.2}$$

Remark 1.1.7.

- In Terminology 1.1.5(b), since G is a linear algebraic $\mathbb{Q}$-group, all other embeddings of G are isomorphic to any of the chosen embeddings in Terminology 1.1.5(b). Hence, up to isomorphism, there exists no ambiguity between an arbitrary embedding of G and one chosen above.
- Moreover, in Terminology 1.1.5(b), by (1.2), up to commensurability, the notation $G_{\mathbb{Z}}$ is independent of the choice of an embedding in Terminology 1.1.5(b).

A commutative linear algebraic group T is said to be a **torus** if it is isomorphic to the product of $\dim T$ many copies of the group GL_1. A torus T is called $\Bbbk$-**split** if it is defined over $\Bbbk$ and is $\Bbbk$-isomorphic to the product of $\dim T$ many copies of the group GL_1. Every connected linear algebraic $\Bbbk$-group G contains a maximal $\Bbbk$-split torus (see [Mar91, I(0.22)]).

Now let G be a linear algebraic $\Bbbk$-group. The **radical** $\mathrm{Rad}(G)$ (resp. **unipotent radical** $\mathrm{Rad}_u(G)$) of G is the maximal connected linear algebraic solvable (resp. unipotent) normal subgroup in G. Such subgroups are $\Bbbk$-closed. Moreover, when $\Bbbk$ is a perfect field, the radical and the unipotent radical are defined over $\Bbbk$. We say G is **semisimple** (**reductive**) if its radical (unipotent radical) is the identity. The factor group of every linear algebraic group modulo its radical (unipotent radical) is semisimple (reductive). A linear algebraic group G is called (absolutely) **simple** ((absolutely) **almost simple**) if the identity is the only proper algebraic normal subgroup of G (all such subgroups are finite) and G is called $\Bbbk$-**simple** or simple over $\Bbbk$ (**almost** $\Bbbk$-**simple** or **almost simple over** $\Bbbk$) if this condition holds for $\Bbbk$-closed normal subgroups (see [Mar91, I(0.23) and I(.24)]).

Proposition 1.1.8. *[Mar91, page 22] Let G be a connected reductive linear algebraic $\Bbbk$-group. Then*

(a) if $\Bbbk$ is infinite, then the subgroup $G_{\Bbbk}$ is Zariski dense in G.

(b) the center of the group G is defined over $\Bbbk$.

(c) if $\Bbbk$ is infinite, then $Z(G_{\Bbbk}) = Z(G)_{\Bbbk}$.

A surjective group homomorphism is called an **isogeny** if it has finite kernel. An isogeny is called **central** if the kernel is central, i.e., the kernel is contained in the center. Every finite normal subgroup of a connected linear algebraic group is central, hence the kernel of every isogeny of connected linear algebraic groups is central. In the same way one can define $\Bbbk$-isogenies for $\Bbbk$-morphisms (see [Mar91, I(0.18)]).
Two linear algebraic $\Bbbk$-groups G and G' are called **isogenous** (reps. **strictly isogenous**), if there exists a linear algebraic $\Bbbk$-group H and two $\Bbbk$-isogenies (resp. central isogenies) $H \to G$ and $H \to G'$ (see [Mar91, I(1.4.4)]).

By definition a linear algebraic group G is semisimple (reductive) if the identity component of G is semisimple (reductive). If G is connected, non-commutative and almost $\Bbbk$-simple then G is semisimple. If G is an almost direct product of semisimple (reductive) subgroups then G is semisimple (reductive). If G is non-trivial, connected and semisimple then it decomposes uniquely (up to permutation of the factors) into an almost direct product of connected non-commutative almost simple algebraic subgroups G_1, ..., G_i and also into an almost direct product of connected non-commutative almost $\Bbbk$-simple $\Bbbk$-subgroups G'_1, ..., G'_j. These groups G_1, ..., G_i are called **almost simple factors** of the group G and the groups G'_1, ..., G'_j are called **almost $\Bbbk$-simple factors** of G (see [Mar91, page 21]).

Let G be a connected linear algebraic $\Bbbk$-group. If G is reductive or the field $\Bbbk$ is perfect, then the maximal $\Bbbk$-split tori of G are conjugate by elements of $G_{\Bbbk}$ and hence, all have the same dimension. In these two cases (i.e., if G is reductive or the filed $\Bbbk$ is perfect) the common dimension of the $\Bbbk$-split tori in G is called the **$\Bbbk$-rank** of G and is denoted by $\mathrm{rank}_{\Bbbk}(G)$. If $\mathrm{rank}_{\Bbbk}(G) > 0$, then G is said to be **$\Bbbk$-isotropic**. Otherwise, G is said to be $\Bbbk$-anisotropic (see [Mar91, I(0.25)]).

Definition 1.1.9. [Mar91, Definition I(1.4.9)] A connected semisimple linear algebraic group G is called **simply connected** (resp. **adjoint**) if every central isogeny $\phi : G' \to G$ (resp. $\phi : G \to G'$), for G' connected, is a linear algebraic group isomorphism.

Proposition 1.1.10. *Strictly isogenous semisimple linear algebraic $\Bbbk$-groups have common $\Bbbk$-rank.*

Proof. It follows from [Mar91, Corollary I(1.4.6)(a)]. □

The special linear group SL_n and the symplectic group Sp_{2n} are examples of simply connected linear algebraic groups.

Let G be a linear algebraic $\Bbbk$-group. A **Levi subgroup** of G is any linear algebraic reductive subgroup $L \subset G$ such that G is the semi-direct product of L and the unipotent radical $\mathrm{Rad}_u(G)$. The decomposition

$$G = L \ltimes \mathrm{Rad}_u(G), \tag{1.3}$$

is called a **Levi decomposition** of G. If $\mathrm{char}\,\Bbbk = 0$ then G always has a Levi $\Bbbk$-subgroup L (see [Mar91, I(0.28)]).

For the extension of local fields $\mathbb{R} \subset \mathbb{C}$, with the usual topology on $\mathrm{GL}(n, \mathbb{C})$, every linear algebraic $\mathbb{R}$-group G is a locally compact topological group. By looking at the **Borel algebra** of G, one can produce measure theory on G. In this setting, a non-Zero Borel measure μ on G is called a left (or right) **Haar measure** if $\mu(gA) = \mu(A)$ ($\mu(Ag) = \mu(A)$) for every Borel set A of G and every $g \in G$ (see [Hal50, Chapter XI]).

Theorem 1.1.11. *Let G be a locally compact topological group. Then there exists a left (or right) Haar measure on G which is unique up to multiplication by a positive constant.*

Proof. It follows from [Hal50, Theorem (XI)58.B] and [Hal50, Theorem (XI)60.C]. □

A discrete subgroup of a linear algebraic $\mathbb{R}$-group $H \subset G_{\mathbb{R}}$ is called a **lattice** if the quotient space $G_{\mathbb{R}}/H$ has a finite volume with respect to the Haar measure. A lattice H in a linear algebraic $\mathbb{R}$-group G is called **irreducible** if G can **not** be decomposed into a product of two connected infinite normal subgroups G_1 and G_2 such that $G = G_1.G_2$, $G_1 \cap G_2 \subset Z(G)$ and the subgroup $(H \cap G_1).(H \cap G_2)$ is of finite index in H where $Z(G)$ denotes the center of G. Note that every finite index subgroup of a lattice is again a lattice.

Theorem 1.1.12. *[BH62, Theorem 7.8] Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a semisimple linear algebraic $\mathbb{Q}$-group. Then $G_{\mathbb{Z}}$ is a lattice in $G_{\mathbb{R}}$.*

Example 1.1.13. By Theorem 1.1.12, it is evident that $G_{\mathbb{Z}} := \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ is a lattice in $G_{\mathbb{R}} := \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. However, this lattice is **not** irreducible since $G_1 := \mathrm{SL}_2(\mathbb{R}) \times \{\mathrm{Id}\}$ and $G_2 := \{\mathrm{Id}\} \times \mathrm{SL}_2(\mathbb{R})$ are two connected infinite normal subgroups of $G_{\mathbb{R}}$ which provide the decomposition required for $G_{\mathbb{Z}}$ not to be irreducible.
This must **not** lead to a misunderstanding that $G_{\mathbb{R}}$ does not contain any irreducible lattice. For if, $\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$ embedded in $G_{\mathbb{R}}$ via the following homomorphism forms an irreducible lattice in $G_{\mathbb{R}}$:

$$\Delta : \mathrm{SL}_2(\mathbb{Z}[\sqrt{2}]) \to \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}),$$
$$\Delta(\gamma) := (\gamma, \sigma(\gamma)),$$

where σ is the conjugation on $\mathbb{Q}[\sqrt{2}]$.
Note that

$$\Delta(\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])) \cap G_1 = \{\mathrm{Id}\}$$

also,

$$\Delta(\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])) \cap G_2 = \{\mathrm{Id}\}$$

and these are the only possible candidates for infinite proper normal subgroups of $G_{\mathbb{R}}$ which fit to the conditions of an irreducible lattice in $G_{\mathbb{R}}$. For the details why $\Delta(\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}]))$ is a lattice in $G_{\mathbb{R}}$ see [Mor15, Example 5.5.3].

For a linear algebraic group G, a discrete subgroup $H \subset G_{\mathbb{R}}$ with respect to the Lie topology on $G_{\mathbb{R}}$ is called **arithmetic** if it is commensurable with $G_{\mathbb{Z}}$. Therefore by Theorem 1.1.12 every arithmetic subgroup of a semisimple linear algebraic $\mathbb{Q}$-group G is a lattice in $G_{\mathbb{R}}$. The inverse of the latter statement is not true in general. Nonetheless, irreducible lattices in certain semisimple linear algebraic groups are arithmetic (see [Mar91, Theorem (IX)6.5]).

Theorem 1.1.14 (**Borel Density Theorem**)**.** *[Mar91, Proposition I.3.2.11] Let G be a connected semisimple linear algebraic $\mathbb{Q}$-group. Assume $\mathrm{rank}_{\mathbb{R}}\, G' > 0$ for each almost $\mathbb{Q}$-simple factor G' of G. Then every arithmetic subgroup of G is Zariski dense in G. In particular, $G_{\mathbb{Z}}$ is Zariski dense in $G_{\mathbb{R}}$.*

1.2 Finite Root System

This section is to introduce finite root systems and their basic properties. At the end, we cite a classification of such root systems. Note that since the Kac-Moody root systems are generalization of finite root systems and we use spherical subgroups of Kac-Moody groups which are one of the finite types presented in this section, it is essential to have some general knowledge about finite root systems (see Section 4.1). This section is a concise extraction of [Hum70, Chapter III].

We fix a Euclidean space E with a positive definite symmetric bilinear form (.) throughout this section. Any non-zero vector $\alpha \in E$ determines a **reflection** σ_α by reflecting over the hyperplane $P_\alpha := \{\beta \in E \mid (\beta, \alpha) = 0\}$. More precisely,

$$\forall \beta \in E \qquad \sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha. \tag{1.4}$$

Definition 1.2.1. [Hum70, 9.2] A subset $\Phi \subset E$ is called a **root system** in E if the following axioms hold:

(R1) Φ is finite, spans E, and does not contain zero.

(R2) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.

(R3) If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.

(R4) If $\alpha, \beta \in \Phi$, then $\langle \alpha, \beta \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Define $\mathrm{rank}(\Phi) := \dim E$ to be the **rank** of Φ. For a root $\alpha \in \Phi$, we call $l(\alpha) := (\alpha, \alpha)$ the **length** of α.
Now let W be the subgroup of $\mathrm{GL}(E)$ generated by reflections σ_α where $\alpha \in \Phi$. By (R3), W can be identified with a subgroup of the symmetric group on Φ which, in view of (R1), implies that W is finite. W is called the **Weyl group** of Φ.
Two root systems (Φ, E) and (Φ', E') are **isomorphic** if there exists a vector space isomorphism $\phi : E \to E'$ sending Φ onto Φ' such that $\langle \phi(\beta), \phi(\alpha) \rangle = \langle \alpha, \beta \rangle$ for each pair of roots $\alpha, \beta \in \Phi$. An isomorphism of root systems ϕ induces a natural isomorphism $\sigma \mapsto \phi \circ \sigma \circ \phi^{-1}$ of Weyl groups.

Define $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$. We call $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$ the **dual** (or **inverse**) of Φ which is also a root system in E whose Weyl group is canonically isomorphic to W (see [Hum70, Subsection 9.2]).
Definition 1.2.2. [Hum70, 10.1] A subset $\Phi^\circ \subset \Phi$ is called a **base** if:

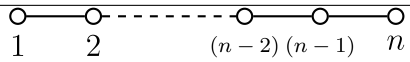
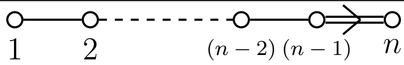
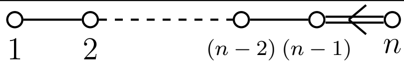
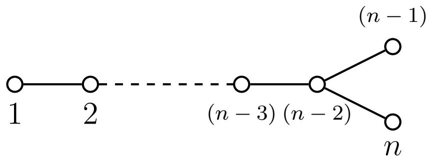
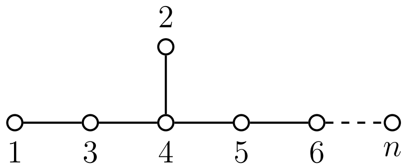
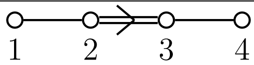
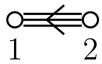
(B1) Φ° is a basis of E;

(B2) each root β can be written as $\beta = \sum k_\alpha \alpha$ where $\alpha \in \Phi^\circ$ and k_α are integral, all non-negative or all non-positive.

The roots in Φ° are called **simple**. By (B1), we have $\mathrm{rank}(\Phi) = \mathrm{Card}(\Phi^\circ)$ and the expression of β in (B2) is unique. This enables us to define **height** of a root relative to a base by $\mathrm{ht}(\beta) := \sum_{\alpha \in \Phi^\circ} k_\alpha$. This way we can use the notion of positive or negative roots with respect to a base. The collection of positive (negative) roots (relative to Φ°) is denoted by Φ^+ (Φ^-). It is known that every root system Φ has a base (see [Hum70, Subsection 10.1]).
A root system Φ in E is called **irreducible** if it cannot be partitioned into the union of two proper subsets such that each root in one set is orthogonal to each root in the other. This is equivalent to say: for a base Φ°, it cannot be partitioned in the above mentioned way (see [Hum70, Subsection 10.4]).
For a fixed ordering of elements $(\alpha_1, \cdots, \alpha_l)$ of a base Φ° where $\mathrm{rank}(\Phi) = l$ define $A := [A_{i,j}]$ where $A_{i,j} = \langle \alpha_i, \alpha_j \rangle$. A is called the **Cartan matrix** of Φ. Integers $A_{i,j}$ are called **Cartan integers**. For distinct positive roots $\alpha, \beta \in \Phi^+$ we have $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0, 1, 2, 3$. Define the **Coxeter graph** of Φ to be a graph with l vertices whose i-th vertex is joined to the j-th vertex ($i \neq j$) by $A_{i,j}$ edges. In case $A_{i,j} \geq 2$, if we add to the Coxeter graph of Φ an arrow pointing to the vertex corresponding to the shorter root of the two adjacent vertices, we

obtain a diagram which is called the **Dynkin diagram** of Φ. Moreover, Φ is irreducible if and only if its Dynkin diagram is connected (see [Hum70, Subsections 11.1, 11.2, 11.3 and 9.4]).
Here, we present a classification of root systems via the Dynkin diagram.
Theorem 1.2.3. *[Hum70, Theorem 11.4] If Φ is an irreducible root system of rank n, its Dynkin diagram is one of the following (n vertices in each case) on Table 1.1.*

Table 1.1: Classification of Spherical Dynkin Diagrams

Type	Dynkin Diagram
A_n	
B_n	
C_n $(2 \leq n)$	
D_n $(4 \leq n)$	
E_n $(6 \leq n \leq 8)$	
F_4	
G_2	

Theorem 1.2.4. *[Hum70, Theorem 12.1] For each Dynkin diagram (or Cartan matrix) of type A-G, there exists an irreducible root system having the given diagram.*

Note that the simple Lie algebras corresponding to the cases A_n, B_n, C_n and D_n are the classical simple algebras $\mathfrak{sl}_n$, $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n}$ respectively.

Next we observe how it is possible to express a finite dimensional simple Lie algebra over an algebraically closed field of characteristic 0 by associating it to a unique irreducible root system presented in Theorem 1.2.3. The material here is extracted from [Hum70, Sections 8 and 14].

Let L denote a finite dimensional semisimple Lie algebra over an algebraically closed field $\mathbb{K}$. For a **Cartan subalgebra** (abbreviated **CSA**) $H \subset L$, define $L_\alpha := \{x \in L \mid [h,x] = \alpha(h)x \ \forall\ h \in H\}$ where α runs through H^*. By abuse of notation we denote by Φ, the set of all non-zero $\alpha \in H^*$ for which $L_\alpha \neq 0$. Then we have the following **Cartan decomposition** of L ([Hum70, Subsection 8.1]):

$$L = H \oplus \coprod_{\alpha \in \Phi} L_\alpha. \tag{1.5}$$

Theorem 1.2.5. *Let L be a finite dimensional simple Lie algebra. Let H and Φ be as above. Then Φ is an irreducible root system and determines L uniquely up to isomorphism.*

Proof. Follows from [Hum70, Proposition 14.1] and [Hum70, Theorem 14.2]. $\square$

Note that Theorem 1.2.5 remains valid for finite dimensional semisimple Lie algebras where Φ need not be irreducible. Nonetheless, Φ is still a root system (see [Hum70, Section 14]).

1.3 Universal Enveloping Algebras

This section is comprised of basic concepts and a fundamental theorem about the universal enveloping algebra of a Lie algebra extracted from [Hum70, Chapter 17]. We need to define the concept of the universal enveloping algebra in order to present Chevalley groups in the next section as well as an adjoint representation for split Kac-Moody groups in Section 4.2. Note that to define the universal enveloping algebra of a Lie algebra we impose no condition on the dimension of these spaces.

Definition 1.3.1. Let L be an arbitrary Lie algebra over a field $\Bbbk$. A **universal enveloping algebra** of L is a pair $(\mathfrak{U}, i)$ where $\mathfrak{U}$ is a unital associative algebra over $\Bbbk$ and $i : L \to \mathfrak{U}$ a linear map satisfying the following condition:

$$i([x, y]) = i(x)i(y) - i(y)i(x), \tag{1.6}$$

for $x, y \in L$, and the following condition holds: for any other unital associative $\Bbbk$-algebra $\mathfrak{A}$ and any linear map $j : L \to \mathfrak{A}$ satisfying (1.6) there exists a unique morphism of unital algebras, namely $\phi : \mathfrak{U} \to \mathfrak{A}$, such that $\phi \circ i = j$. So we have the following commutative diagram:

$$\begin{tikzcd} & \mathfrak{U} . \arrow[d, dotted, "\phi"] \\ L \arrow[ur, "i"] \arrow[r, "j"'] & \mathfrak{A} \end{tikzcd}$$

Note that such a pair $(\mathfrak{U}, i)$ is unique up to isomorphism of unital associative $\Bbbk$-algebras, in the sense that, if there exists another pair $(\mathfrak{B}, i')$ with the above property, then $\mathfrak{U} \cong \mathfrak{B}$ (see [Hum70, Section 17.2]).

Now for any Lie algebra L one can construct a universal enveloping algebra $(\mathfrak{U}(L), i)$ for L as follows:

Let $\mathfrak{T}(L)$ be the tensor algebra on L and let J be the two-sided ideal in $\mathfrak{T}(L)$ generated by all $x \otimes y - y \otimes x - [x, y]$ for all $x, y \in L$. Define $\mathfrak{U}(L) := \mathfrak{T}(L)/J$, and let $\pi : \mathfrak{T}(L) \to \mathfrak{U}(L)$ be the canonical homomorphism. Let $i : L \to \mathfrak{U}(L)$ be the restriction of π to L. By the following theorem, $(\mathfrak{U}(L), i)$ forms a universal enveloping algebra of L.

Theorem 1.3.2 (**Poincaré-Birkhoff-Witt Theorem**). *[Hum70, Theorem 17.3]* $(\mathfrak{U}(L), i)$ *as defined above is a universal enveloping algebra of* L. *Moreover, let* $(x_1, x_2, ...)$ *be any ordered basis of* L. *Then the elements* $x_{i(1)}...x_{i(m)} = \pi(x_{i(1)} \otimes ... \otimes x_{i(m)})$ *where* $m \in \mathbb{Z}^+$ *and* $i(1) \leq ... \leq i(m)$ *along with 1 form a basis for* $\mathfrak{U}(L)$.

1.4 Chevalley Groups and Adjoint Representation

In this section we construct **adjoint Chevalley groups** by following [Hum70, Chapter VII]. Then we present Chevalley groups in general by following [Ste68]. Note that obtaining basic knowledge about Chevalley groups is essential in this thesis as split Kac-Moody groups are natural generalizations of Chevalley groups (see Section 4.1).

Let L be a finite dimensional semisimple Lie algebra defined over an algebraically closed field $\mathbb{K}$ of characteristic 0, H a Cartan subalgebra and Φ its root system. For two linearly independent roots $\alpha, \beta \in \Phi$ we look at all roots of the form $\beta + i\alpha$ ($i \in \mathbb{Z}$) and we call it the α-**string through** β. It turns out that the α-string through β is of the form

$$\beta - r\alpha, ..., \beta, ..., \beta + q\alpha \tag{1.7}$$

where $r, q \in \mathbb{N}$ and $\langle \beta, \alpha \rangle = r - q$ ([Hum70, Subsection (III)9.4]).

Proposition 1.4.1 (**Existence of Chevalley Basis**). *[Hum70, Proposition 25.2] Let* L *be a finite dimensional semisimple Lie algebra defined over an algebraically closed field* $\mathbb{K}$ *of characteristic 0,* H *a Cartan subalgebra and* Φ *its root system. It is possible to choose root vectors* $x_\alpha \in L_\alpha$ *for* $\alpha \in \Phi$ *satisfying:*

(a) $[x_\alpha, x_{-\alpha}] = h_\alpha$;

(b) If $\alpha,\ \beta,\ \alpha + \beta \in \Phi$, $[x_\alpha, x_\beta] = c_{\alpha\beta} x_{\alpha+\beta}$ *then* $c_{\alpha\beta} = -c_{-\alpha,-\beta}$. *For any such choice of root vectors the scalars* $c_{\alpha\beta}$ $(\alpha,\ \beta,\ \alpha + \beta \in \Phi)$ *automatically satisfying:*

(c) $c_{\alpha\beta}^2 = q(r+1)(\alpha+\beta, \alpha+\beta)/(\beta, \beta)$ *where* $\beta - r\alpha, ..., \beta, ..., \beta + q\alpha$ *is the* α*-string through* β.

Definition 1.4.2. Let L be a semisimple Lie algebra defined over $\mathbb{K}$, an algebraically closed field. Any basis $\{x_\alpha, \alpha \in \Phi;\ h_i, 1 \leq i \leq l\}$ satisfying the conditions (a) and (b) in Proposition 1.4.1 and $h_i = h_{\alpha_i}$ for some base $\Pi := \{\alpha_1, ..., \alpha_l\}$ of Φ, is called a **Chevalley basis** for L.

Theorem 1.4.3 (**Chevalley**). *[Hum70, Theorem 25.2] Let* $\{x_\alpha, \alpha \in \Phi;\ h_i, 1 \leq i \leq l\}$ *be a Chevalley basis. Then the resulting structure constants lie in* $\mathbb{Z}$. *More precisely:*

(a) $[h_i, h_j] = 0$ *where* $1 \leq i, j \leq l$;

(b) $[h_i, x_\alpha] = \langle \alpha, \alpha_i \rangle x_\alpha$ *where* $1 \leq i \leq l$ *and* $\alpha \in \Phi$;

(c) $[x_\alpha, x_{-\alpha}] = h_\alpha$ *is a* $\mathbb{Z}$*-linear combination of* $\{h_1, ..., h_l\}$;

(d) If $\alpha, \beta \in \Phi$ *are independent roots,* $\beta - r\alpha, ..., \beta, ..., \beta + q\alpha$ *the* α*-string through* β *then* $[x_\alpha, x_\beta] = 0$ *if* $q = 0$ *while* $[x_\alpha, x_\beta] = \pm(r+1)x_{\alpha+\beta}$ *if* $\alpha + \beta \in \Phi$.

A $\mathbb{Z}$-span of a Chevalley basis $L(\mathbb{Z})$ is a lattice in L, independent of the choice of Π. It is actually a Lie algebra over $\mathbb{Z}$ under the Lie bracket inherited from L. If $\mathbb{F}_p$ is a prime field of characteristic p then the tensor product $L(\mathbb{F}_p) := L(\mathbb{Z}) \bigotimes_{\mathbb{Z}} \mathbb{F}_p$ is defined:
$L(\mathbb{F}_p)$ is a vector space defined over $\mathbb{F}_p$ with basis $\{x_\alpha \otimes 1, h_i \otimes 1\}$ where $1 \leq i \leq l$ and $\alpha \in \Phi$. Moreover, the Lie bracket in $L(\mathbb{Z})$ induces a Lie bracket on $L(\mathbb{F}_p)$ which makes $L(\mathbb{F}_p)$ a Lie algebra over $\mathbb{F}_p$.

Definition 1.4.4. For any field extension $\mathbb{K}$ of $\mathbb{F}_p$, $L(\mathbb{K}) := L(\mathbb{F}_p) \bigotimes_{\mathbb{F}_p} \mathbb{K} \cong L(\mathbb{Z}) \bigotimes_{\mathbb{Z}} \mathbb{K}$ inherits both basis and Lie structure from $L(\mathbb{F}_p)$. This way for a pair $(L, \mathbb{K})$ we associate a Lie algebra $L(\mathbb{K})$ which is called a **Chevalley algebra**.

Example 1.4.5. For the pair $(\mathfrak{sl}_l, \mathbb{K})$ the construction above shows that $L(\mathbb{K}) = \mathfrak{sl}(l, \mathbb{K})$ for the standard basis of $\mathfrak{sl}_l$.

Proposition 1.4.6. *[Hum70, Proposition 25.5] Let $\alpha \in \Phi$, $m \in \mathbb{N}$. Define* $\operatorname{ad} x := [x, \ ]$ *for any $x \in L$. Then $(\operatorname{ad} x_\alpha)^m/(m!)$ leaves $L(\mathbb{Z})$ invariant.*

This proposition implies that $\exp(\operatorname{ad} x_\alpha) := \sum (\operatorname{ad} x_\alpha)^m/(m!)$ (which is well-defined since $\operatorname{ad} x_\alpha$ is nilpotent) leaves $L(\mathbb{Z})$ invariant. This way we can integrate L to construct a matrix group.

Definition 1.4.7. The matrix group $G_L(\mathbb{K})$ generated by all $exp(ad\ tx_\alpha)$ for $t \in \mathbb{K}$ in the automorphism group of $L(\mathbb{K})$ is called a **Chevalley group** of **adjoint type** associated to L.

Proposition 1.4.8. *[Car72, Proposition 4.4.3] Let L by a simple algebra over $\mathbb{C}$. The adjoint Chevalley group $G_L(\mathbb{K})$ in Definition 1.4.7 is determined up to isomorphism by the simple algebra L over $\mathbb{C}$ and the field $\mathbb{K}$.*

Next we give a construction of Chevalley groups in the general form.

Let L be a semisimple Lie algebra over $\mathbb{C}$ with a Cartan subalgebra H. Let V be a representation space for L. A vector $v \in V$ is called a **weight vector** if there is a linear function γ on H such that $hv = \gamma(h)v$ for all $h \in H$. If such a vector $v \neq 0$ exists, we call the corresponding γ a **weight** of the representation.

Let $\{x_\alpha, \alpha \in \Phi;\ h_i, 1 \leq i \leq l\}$ be a Chevalley basis for L as in Definition 1.4.2. Let $(\mathfrak{U}(L), i)$ be the universal enveloping algebra of L defined in Section 1.3. For any element $u \in \mathfrak{U}(L)$, $n \in \mathbb{Z}_{\geq 0}$ define

$$u^{[n]} := (n!)^{-1} u^n,$$

and

$$\binom{u}{n} := (n!)^{-1}.u.(u-1)...(u-n+1).$$

Theorem 1.4.9. *[Ste68, Theorem 2.2] In the above setting, assume that the Chevalley basis elements in $\{x_\alpha, \alpha \in \Phi;\ h_i, 1 \leq i \leq l\}$ are arranged in some order. For each choice of numbers $n_i, m_\alpha \in \mathbb{Z}_{\geq 0}$ ($i = 1, \cdots, l$; $\alpha \in \Phi$) form the product in $\mathfrak{U}(L)$, of all $\binom{h_i}{n_i}$ and $x_\alpha^{[m_\alpha]}$ according to the given order. The resulting collection is a basis for the $\mathbb{Z}$-algebra $\mathfrak{U}(L)_{\mathbb{Z}}$ generated by all $x_\alpha^{[m]}$ ($m \in \mathbb{Z}_{\geq 0}$; $\alpha \in \Phi$).*
Moreover, the above collection is a $\mathbb{C}$-basis for $\mathfrak{U}(L)$.

Let V be an irreducible representation of L. In this setting, if V_γ denotes the subspace of V consisting of weight vectors belonging to a weight γ, then $\dim V_\gamma = 1$. Moreover, every weight μ has the form $\gamma - \sum \alpha$, where α's are positive roots. Also, $V = \sum V_\mu$ where μ is a weight (see [Ste68, Theorem 2.3]).

Proposition 1.4.10. *In the setting of Theorem 1.4.9, every finite dimensional representation V of L contains a lattice (as a free Abelian subgroup) M invariant under all $x_\alpha^{[m]}$ ($m \in \mathbb{Z}_{\geq 0}$; $\alpha \in \Phi$). I.e., M is invariant under $\mathfrak{U}(L)_\mathbb{Z}$.*

Proof. Follows from [Ste68, Corollary 2.1] □

Proposition 1.4.11. *[Ste68, Corollary 2.2] Let V be a faithful representation space of L. Let M be a lattice in V invariant under $\mathfrak{U}(L)_\mathbb{Z}$. Let $L_\mathbb{Z}^V$ be the part of L which preserves M. Then $L_\mathbb{Z}^V$ is a lattice, and*

$$L_\mathbb{Z}^V = \sum_\alpha \mathbb{Z}x_\alpha + H_\mathbb{Z}^V, \tag{1.8}$$

where

$$H_\mathbb{Z}^V = \{h \in H \mid \gamma(h) \in \mathbb{Z} \text{ for all weights } \gamma \text{ of the given representation}\}. \tag{1.9}$$

In particular, $L_\mathbb{Z}^V$ is independent of M (but, $L_\mathbb{Z}^V$ is ***not*** *independent of the representation).*

Remark 1.4.12. Although the adjoint representation given in the first part of the present section is **not** faithful, but the notions above are compatible with the notions given for the adjoint form. For instance, $L(\mathbb{Z})$ plays the role of M here, in view of Proposition 1.4.6. For the details see [Ste68, Chapter 2].

Let V be a faithful representation space of L. Let M be a lattice in V invariant under $\mathfrak{U}(L)_\mathbb{Z}$. Let $L_\mathbb{Z}^V$ and $H_\mathbb{Z}^V$ be as in Proposition 1.4.11. For a weight γ, define $M_\gamma := V_\gamma \cap M$. Let $\mathbb{K}$ be an arbitrary field considered as a $\mathbb{Z}$-module. Define $V^\mathbb{K} := M \otimes_\mathbb{Z} \mathbb{K}$, $L^\mathbb{K} := L_\mathbb{Z}^V \otimes_\mathbb{Z} \mathbb{K}$, $H^\mathbb{K} := H_\mathbb{Z}^V \otimes_\mathbb{Z} \mathbb{K}$, $V_\gamma^\mathbb{K} := M_\gamma \otimes_\mathbb{Z} \mathbb{K}$ and $\mathbb{K}x_\alpha^\mathbb{K} := \mathbb{Z}x_\alpha \otimes_\mathbb{Z} \mathbb{K}$. Then we have (see [Ste68, Corollary 2.3])

$$V^\mathbb{K} = \sum V_\gamma^\mathbb{K} \text{ and } \dim_\mathbb{K} V_\gamma^\mathbb{K} = \dim_\mathbb{C} V_\gamma, \tag{1.10}$$

moreover,

$$L^\mathbb{K} = \sum \mathbb{K}x_\alpha^\mathbb{K} + H^\mathbb{K}, \text{ each } x_\alpha^\mathbb{K} \neq 0, \; \dim_\mathbb{K} H^\mathbb{K} = \dim_\mathbb{C} H, \text{ and } \dim_\mathbb{K} L^\mathbb{K} = \dim_\mathbb{C} L. \tag{1.11}$$

In the above setting, since for each root α, $x_\alpha^{[n]} \in \mathfrak{U}(L)_\mathbb{Z}$, we have an action of $x_\alpha^{[n]}$ on M. Hence, for each $\lambda \in \mathbb{K}$, we get an action of $\lambda^n x_\alpha^{[n]}$ on $M \otimes_\mathbb{Z} \mathbb{Z}[\lambda]$. As x_α^n acts trivially for sufficiently large n, we conclude that $\sum_{n=0}^\infty \lambda^n x_\alpha^{[n]}$ acts on $M \otimes_\mathbb{Z} \mathbb{Z}[\lambda]$ and hence on $M \otimes_\mathbb{Z} \mathbb{Z}[\lambda] \otimes_\mathbb{Z} \mathbb{K}$. Now by the embedding of $M \otimes_\mathbb{Z} \mathbb{Z}[\lambda] \otimes_\mathbb{Z} \mathbb{K}$ into $V^\mathbb{K} = M \otimes_\mathbb{Z} \mathbb{K}$ via $\lambda \mapsto t$, we get an action of $\exp tx_\alpha := \sum_{n=0}^\infty t^n x_\alpha^{[n]}$ on $V^\mathbb{K}$. For simplicity, define $x_\alpha(t) := \exp tx_\alpha$ and $U_\alpha := \{x_\alpha(t) \mid t \in \mathbb{K}\}$. Note that U_α is an additive group with respect to the parameter t (for details, see [Ste68, Chapter 3]).

Definition 1.4.13. In the above setting, the group G generated by all U_α ($\alpha \in \Phi$) is called a **Chevalley group** of type L. Furthermore, by an **irreducible** Chevalley group we understand a Chevalley group with an irreducible root system.

Remark 1.4.14.

(a) In view of Theorem 1.2.5, instead of using the term: "a Chevalley group of type L", we might say: "a Chevalley group of type Φ" to emphasize on the corresponding root system.

(b) Definition 1.4.13 suggests that once we have a fixed faithful representation of L, one can regard Chevalley groups as functors from the category of commutative unital rings into the category of matrix groups. Therefore, it makes sense to write $G(\mathbb{K})$ for an arbitrary field $\mathbb{K}$.

(c) By [Ste68, Corollary 3.5], if the additive groups generated by all weights of two faithful representations of L (with the same underlying fields) are the same then the resulting Chevalley groups with respect to these representations are isomorphic.

(d) When $\mathbb{K}$ is an algebraically closed field, then $G(\mathbb{K})$ can be viewed as a linear algebraic group defined over $\mathbb{K}_{\text{prime}}$ where $\mathbb{K}_{\text{prime}}$ is the prime subfield of $\mathbb{K}$. In the special case $\mathbb{K} = \mathbb{C}$, we can view G as a linear algebraic group defined over $\mathbb{Q}$ (see [Ste68, Chapter 5]).

(e) In (c), let $\Bbbk$ be a field embedded in $\mathbb{K}$, then $G(\Bbbk)$ is the group of the $\Bbbk$-rational points of G as an algebraic group. Thus we have $G_{\Bbbk} = G(\Bbbk)$ in view of Terminology 1.1.5. Moreover, in the special case $\mathbb{K} = \mathbb{C}$, we view $G(\mathbb{Z})$ as a subgroup of $G(\mathbb{Q})$. Then we have $G_{\mathbb{Z}} = G(\mathbb{Z})$.

Next we define simply connected (also called universal) Chevalley groups which play a major role in our rigidity results. For this, we need some definitions.
Let G be a Chevalley group of type L with a Cartan subalgebra H. For any root α and any $t \in \mathbb{K}^\times$ define

$$w_\alpha(t) := x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t), \text{ and } h_\alpha(t) := w_\alpha(t)w_\alpha(1)^{-1}. \tag{1.12}$$

It turns out that for each root α, $h_\alpha(t)$ acts on the weight space V_γ as multiplication by $t^{\gamma(H_\alpha)}$ where H_α is as in Theorem 1.4.3(c) (see [Ste68, Lemma 2.19]).
Let $\mathcal{T}$ denote the group generated by all $h_\alpha(t)$ ($\alpha \in \Phi$, $t \in \mathbb{K}$). Define γ_i for $i = 1, \cdots, l$ by $\gamma_i(\alpha_j) := \delta_{ij}$ for all simple roots α_j in Φ. We call γ_i a **fundamental weight**.
It turns out that the additive group generated by all the weights of all representations forms a lattice L_1 having $\{\gamma_i\}$ as a basis. Moreover, the additive group generated by all roots is a sublattice L_0 of L_1. In addition, the additive group generated by all weights of a faithful representation on V forms a lattice L_V such that $L_0 \subseteq L_V \subseteq L_1$. Also, all lattices between L_0 and L_1 can be realized as above by an appropriate choice of V. In particular, $L_V = L_0$ if V corresponds to the adjoint presentation discussed for Chevalley groups of adjoint type, and $L_V = L_1$ if V corresponds to the sum of the representations having the fundamental weights with the property that $\gamma_i(H_\alpha) \in \mathbb{Z}_{\geq 0}$ for all $i = 1, \cdots, l$ and all $\alpha \in \Phi$ (see [Ste68, Lemma 3.27 and page 43]).

It turns out that for each $\alpha \in \Phi$, $h_\alpha(t)$ is multiplicative as a function of t and $\mathcal{T}$ is an Abelian group generated by $h_{\alpha_i}(t)$'s where α_i's are simple roots. Moreover, if $L_V = L_1$, then every $h \in \mathcal{T}$ can be written uniquely as $h = \prod_{i=1}^{l} h_{\alpha_i}(t_i)$, $t_i \in \mathbb{K}^\times$. In addition, if $L_V = L_0$, then G has trivial center (see [Ste68, Lemma 3.28 and page 44]).

Definition 1.4.15. Let L be a semisimple Lie algebra over $\mathbb{C}$ with a Cartan subalgebra H and a Chevalley basis $\{x_\alpha, \alpha \in \Phi \ ; \ h_i \ , 1 \leq i \leq l\}$. The Chevalley group G_1 corresponding to L_1 is called the **simply connected** or **universal** Chevalley group of type Φ.

Note that it makes sense to talk about **the** simply connected Chevalley group of type Φ since it is unique up to isomorphism by Remark 1.4.14(c). One should also note that in some literature this simple connectedness is called **algebraic simple connectedness** in order to avoid confusion with the similar terminology in algebraic topology. However, throughout this thesis, by a simply connected Chevalley group we always mean the algebraic form of it unless we say otherwise. Moreover, a universal Chevalley group is simply connected in the sense of Definition 1.1.9 (see [Ste68, page 45]).
Here we mention a concise list of general properties of universal Chevalley groups.
Remark 1.4.16. (see [Mar91, Section I(0.27)])

(i) All universal Chevalley groups are semisimple linear algebraic groups.

(ii) A Chevalley group as a linear algebraic $\mathbb{Q}$-group is absolutely almost simple if and only if its root system is irreducible.

(iii) Two connected universal Chevalley groups over $\Bbbk$ are strictly isogenous if and only if they have the same Dynkin diagram. And hence they have the same type.

Now we give an equivalent definition of the universal Chevalley groups in terms of generators and relations.
Definition 1.4.17. ([Gra02, Definition C.3.1], cf. [Ste68, Theorem 8]) Let Φ be an irreducible root system of rank at least two, and let $\mathbb{K}$ be a field. Let G be the group generated by the collection of elements $\{x_\alpha(t) \mid \alpha \in \Phi, t \in \mathbb{K}\}$ subjected to the following relations:

(i) For each $\alpha \in \Phi$, $x_\alpha(t)$ is additive with respect to $t \in \mathbb{K}$.

(ii) If α and β are two linearly independent roots and $\alpha + \beta \neq 0$, then

$$[x_\alpha(t), x_\beta(u)] = \prod_{\lambda = m\alpha + n\beta} x_\lambda(k(\alpha, \beta, \lambda) t^m u^n),$$

with $m, n > 0$, $\lambda \in \Phi$ (if there are no such numbers, then $[x_\alpha(t), x_\beta(u)] = 1$), and the structure constants $k(\alpha, \beta, \lambda) \in \{\pm 1, \pm 2, \pm 3\}$.

(iii) $h_\alpha(t)$ is multiplicative in t, where

$$h_\alpha(t) := w_\alpha(t) w_\alpha(1)^{-1},$$

and

$$w_\alpha(t) := x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t).$$

For a certain choice of the structure constants $k(\alpha, \beta, \lambda)$, the group G is called **the Chevalley group constructed from** Φ and $\mathbb{K}$ (see [GLS98, Theorem 1.12.1]).

1.5 Congruence Subgroups

This section presents the congruence subgroup problem along with some well-known facts about them which will be used later on this thesis. The references for this subsection are [Sur03] and [Mat69]. Note that in Section 5.3 we see that the congruence subgroups play an important role in understanding of arithmetic subgroups of real split Kac-Moody groups. Thus it is important to introduce the congruence subgroups in the classical context.
For an ideal $\mathfrak{a}$ in $\mathbb{Z}$, one can define

$$\mathrm{SL}(n, \mathfrak{a}) := \{A \in \mathrm{SL}(n, \mathbb{Z}) \mid A \equiv I (\mathrm{mod}\ \ \mathfrak{a})\}. \tag{1.13}$$

More precisely, $A = [A_{i,j}] \in \mathrm{SL}(n, \mathfrak{a})$ is a matrix such that $A_{i,i} \equiv 1(\mathrm{mod}\ \ \mathfrak{a})$ and $A_{i,j} \equiv 0(\mathrm{mod}\ \ \mathfrak{a})$ for $i \neq j$. Moreover, the natural epimorphism $\pi : \mathbb{Z} \to \mathbb{Z}/\mathfrak{a}$ induces a group homomorphism

$$\pi : \mathrm{SL}(n, \mathbb{Z}) \to \mathrm{SL}(n, \mathbb{Z}/\mathfrak{a}) \tag{1.14}$$

whose kernel is $\mathrm{SL}(n, \mathfrak{a})$. In the same way as in (1.14), one can define $\mathrm{GL}(n, \mathfrak{a})$. More generally, if G is a connected linear algebraic $\mathbb{Q}$-group of rank n then we can define $G(\mathfrak{a}) := G \cap \mathrm{GL}(n, \mathfrak{a})$, by means of Terminology 1.1.5, which is a normal subgroup of $G_{\mathbb{Z}}$ since $\mathrm{GL}(n, \mathfrak{a})$ is a normal subgroup of $\mathrm{GL}(n, \mathbb{Z})$. The subgroups of the form $G(\mathfrak{a})$ are called **principal congruence subgroups** of $G_{\mathbb{Z}}$. A **congruence subgroup** of $G_{\mathbb{Z}}$ is a subgroup containing a principal congruence subgroup.

Now we can present a special case of the congruence subgroup problem:

Let G be a linear algebraic $\mathbb{Q}$-group. Is every subgroup of finite index in $G_{\mathbb{Z}}$ a congruence subgroup?

There are several results for this problem. For instance, the answer is negative for universal Chevalley groups of type SL_2 (i.e., of type A_1) (see [Fri87]). There are also some necessary conditions for an affirmative answer which are discussed in [Sur03, Chapter 6] in detail. Here we cite a well-known result for certain Chevalley groups. Note that this result will be used frequently in Chapter 7.

Theorem 1.5.1. *[Mat69, Corollary 12.6] Let G be a simply connected Chevalley group which is not of type* SL_2*. Then every finite index subgroup of $G(\mathbb{Z})$ is a congruence subgroup.*

1.6 Classical Rigidity Results

In this section we collect some well-known rigidity results on certain linear algebraic groups. For more details and proofs one can see [Mar91] or [Zim84]. We use these rigidity results in Chapter 7. We retain the notations in the previous sections of this chapter. Recall from Section 1.4 that, especially when the underlying algebraic group G is a Chevalley group we see G as a group scheme functor. Therefore, $G(\Bbbk)$ denotes the $\Bbbk$-points of G as an algebraic group (see Remark 1.4.14). In the case G is just assumed to be a linear algebraic group then we use the notation $G_{\Bbbk}$ given in (1.1) for the $\Bbbk$-points of G.

Theorem 1.6.1. *Let G be a connected semisimple linear algebraic $\mathbb{Q}$-group. Suppose that* $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$*, G has no non-trivial $\mathbb{R}$-anisotropic factors and its center $Z(G)$ is finite. Let*

Γ be an irreducible lattice in $G_{\mathbb{R}}$. Let H be a linear algebraic $\mathbb{Q}$-group and $\phi : \Gamma \to H_{\mathbb{Q}}$ a homomorphism. Let $\overline{\phi(\Gamma)}$ be the Zariski closure of $\phi(\Gamma)$ in H. Then $\overline{\phi(\Gamma)}$ is a semisimple linear algebraic $\mathbb{Q}$-group.

Proof. By Theorem 1.1.14, Γ is dense in G. Therefore, (a) in [Mar91, Theorem (IX)6.15] implies that $\overline{\phi(\Gamma)}$ is a semisimple linear algebraic $\mathbb{Q}$-group. □

Remark 1.6.2.

(i) Note that when G is an irreducible universal Chevalley group functor as introduced in Section 1.4, it is evident that $G(\mathbb{Z}) = G_{\mathbb{Z}}$ and similarly $G(\mathbb{R}) = G_{\mathbb{R}}$(see Remark 1.4.14). Therefore, Theorem 1.6.1 holds for $G(\mathbb{Z})$ when G is an irreducible universal Chevalley group functor by Theorem 1.1.12 and Remark 1.4.16.

(ii) In view of (i), since every subgroup commensurable with a (an irreducible) lattice is again a (an irreducible) lattice, Theorem 1.6.1 holds actually for any irreducible arithmetic subgroup of G where by an irreducible arithmetic subgroup we understand an arithmetic subgroup corresponding to an irreducible lattice.

Definition 1.6.3. Let H be a linear algebraic $\mathbb{Q}$-group and Λ be an arithmetic subgroup of G. We say a homomorphism $\phi : \Lambda \to H$ **virtually extends** to a homomorphism $\tilde{\phi} : G \to H$ if $\tilde{\phi}$ coincides with ϕ on a finite index subgroup Λ' of Λ. Moreover, $\tilde{\phi}$ is called a **virtual extension** of ϕ **on** Λ' in this situation.

Theorem 1.6.4 (Super Rigidity)**.** *Let G be a connected simply connected almost $\mathbb{Q}$-simple linear algebraic $\mathbb{Q}$-group of rank at least 2. Let H be a linear algebraic $\mathbb{Q}$-group, Λ be an irreducible arithmetic subgroup of G, and $\phi : \Lambda \to H_{\mathbb{Q}}$ be an abstract morphism of groups. Then*

(a) there exists (uniquely determined) a morphism $\tilde{\phi} : G \to H$ of linear algebraic groups and a morphism $\nu : \Lambda \to H$ such that the subgroup $\nu(\Lambda)$ is finite and commutes with $\tilde{\phi}(G)$ and

$$\phi(\lambda) = \nu(\lambda).\tilde{\phi}(\lambda) \ for\ all\ \lambda \in \Lambda \tag{1.15}$$

(b) the morphism $\tilde{\phi}$ is defined over $\mathbb{Q}$.

Proof. This theorem follows from [Mar91, Theorem (VIII)3.12]. □

Remark 1.6.5.

(i) By (b) in Theorem 1.6.4, we have the following $\mathbb{Q}$-morphism of algebraic $\mathbb{Q}$-groups

$$\tilde{\phi} : G_{\mathbb{Q}} \to H_{\mathbb{Q}}. \tag{1.16}$$

This yields that $\tilde{\phi}$ is defined by a regular $\mathbb{Q}$-morphism (i.e., the coordinate functions of $\tilde{\phi}$ are polynomials with coefficients in $\mathbb{Q}$), and hence it is also defined over $\mathbb{R}$. Moreover, this implies that $\tilde{\phi}$ is continuous with respect to the Lie topology on $G_{\mathbb{R}}$ and $H_{\mathbb{R}}$. Therefore, we have the following continuous homomorphism of Lie groups:

$$\tilde{\phi} : G_{\mathbb{R}} \to H_{\mathbb{R}}. \tag{1.17}$$

(ii) In view of Remark 1.6.2 and Theorem 1.6.1, if G is a connected irreducible universal Chevalley group in Theorem 1.6.4 then we can assume that the morphism ϕ in Theorem 1.6.4 is mapped to a semisimple $\mathbb{Q}$-subgroup $\overline{\phi(G(\mathbb{Z}))}$ of H. Hence by going to the semisimple $\mathbb{Q}$-subgroup $\overline{\phi(G(\mathbb{Z}))}$ of H if necessary one can always assume that the image of ϕ is Zariski dense in H without loss of generality.

(iii) In (ii), G is connected and absolutely almost simple by the assumption and Remark 1.4.16. Moreover, we assume that H is infinite and non-commutative. Therefore, the pre-image of the connected component of H under the extension $\tilde{\phi}$ has to be the whole G. If we denote the connected component of H by H° then

$$\tilde{\phi} : G \to H^\circ, \tag{1.18}$$

forms a central $\mathbb{R}$-isogeny.

(iv) The central $\mathbb{R}$-isogeny obtained in (1.18) implies that the $\mathbb{R}$-rank of H° equals the $\mathbb{R}$-rank of G which is at least two. Moreover, the root system of H° is the same as the root system of G, hence irreducible. Therefore, H° is absolutely almost $\mathbb{Q}$-simple (see Proposition 1.1.10 and Remark 1.4.16).

(v) In (iii), $\tilde{\phi} : G(\mathbb{R}) \to H^\circ_{\mathbb{R}}$ forms a continuous Lie group epimorphism and $\ker(\tilde{\phi}) \subset Z(G(\mathbb{R}))$. Moreover, $G(\mathbb{Z})$ is an irreducible lattice in $G(\mathbb{R})$ (in particular, an (irreducible) arithmetic subgroup). Hence by [Mar91, Remark IX(6.4)(ii)], $\tilde{\phi}(G(\mathbb{Z}))$ is an (irreducible) arithmetic subgroup of H°.

Remark 1.6.6.

(i) In Theorem 1.6.4 since $\nu(\Lambda)$ is finite, the kernel of ν has finite index in Λ, therefore $\tilde{\phi}$ is actually a virtual extension of ϕ.

(ii) It follows from (i) that in the situation of Remark 1.6.5(iii), we can always assume that the virtual extension in Theorem 1.6.4 is on a principal congruence subgroup because, on the one hand, since G is a connected irreducible universal Chevalley group, the congruence subgroup property, Theorem 1.5.1, guarantees existence of a principal congruence subgroup inside any finite index subgroup of $G(\mathbb{Z})$.
On the other hand, the Borel density theorem, Theorem 1.1.14, implies that every finite index subgroup of $G(\mathbb{Z})$, in particular, every principal congruence subgroup of $G(\mathbb{Z})$ is Zariski dense in $G(\mathbb{R})$. Therefore, (2) in Proposition 1.1.2 yields that the virtual extension is uniquely determined by a principal congruence subgroup in $G(\mathbb{Z})$.

(iii) Moreover, in (ii) assume that the virtual extension is determined by a principal congruence subgroup $G^{con}(n\mathbb{Z})$ corresponding to an ideal $n\mathbb{Z}$ in $\mathbb{Z}$ for some $n \in \mathbb{N}$. If $G(n\mathbb{Z})$ denotes the subgroup of $G(\mathbb{Z})$ generated by the $n\mathbb{Z}$-points of the root subgroups of G, then $G(n\mathbb{Z})$ is contained in $G^{con}(n\mathbb{Z})$. Moreover, $G(n\mathbb{Z})$ is Zariski dense in $G(\mathbb{R})$. In this situation, by the same arguments as in (ii) we conclude that the extension is also uniquely determined by any subgroup commensurable with $G(n\mathbb{Z})$. In particular, the virtual extension is uniquely determined by any multiple of n by an integer, namely; $G(rn\mathbb{Z})$ for any $r \in \mathbb{Z}$.

Corollary 1.6.7. *In Theorem 1.6.4 assume that G is a connected irreducible universal Chevalley group of rank at least two. In addition, assume that H is a simply connected linear algebraic $\mathbb{Q}$-group. If the image $\phi(\Lambda)$ is Zariski dense in H and other than $\{e\}$. Then the virtual extension*

$$\tilde{\phi} : G \to H, \tag{1.19}$$

is an isomorphism of linear algebraic $\mathbb{R}$-groups.

Proof. By (1.18) in Remark 1.6.5 we have the following $\mathbb{R}$-isogeny:

$$\tilde{\phi} : G \to H, \tag{1.20}$$

since in the definition of a simply connected linear algebraic group we assume that the underlying linear algebraic group is connected and semisimple (see Definition 1.1.9).
Now by definition, the $\mathbb{R}$-isogeny in (1.20) has to be an isomorphism since H is simply connected. □

There are many variations of the **strong rigidity theorem** (also known as the **Mostow-Margulis rigidity theorem**). Here we mention a version which suits best to our situation in Chapter 7.

Theorem 1.6.8 (Mostow-Margulis/Strong Rigidity)**.** *Let G be a connected semisimple linear algebraic $\mathbb{Q}$-group of $\mathbb{R}$-rank at least 2 whit no non-trivial $\mathbb{R}$-anisotropic factors and finite center $Z(G(\mathbb{R}))$. Let Λ be an irreducible arithmetic subgroup of $G(\mathbb{R})$. Let H be a connected semisimple linear algebraic $\mathbb{R}$-group. Let $\phi : \Lambda \to H$ be a non-trivial homomorphism with Zariski dense image in H. Moreover, assume that $\phi(\Lambda)$ is discrete in $H_{\mathbb{R}}$. If H is adjoint with no $\mathbb{R}$-anisotropic factors, then ϕ extends uniquely to a continuous homomorphism $\tilde{\phi}$: $G(\mathbb{R}) \to (H_{\mathbb{R}})^{\circ}$.*
Moreover, if $G(\mathbb{R})$ has trivial center with no non-trivial compact factors then $\tilde{\phi} : G(\mathbb{R}) \to (H_{\mathbb{R}})^{\circ}$ is a continuous isomorphism.

Proof. By the Borel density theorem, Theorem 1.1.14, we know that $G(\mathbb{Z})$ is Zariski dense in G. Hence the theorem follows from [Mar91, Theorem IX(6.16)(c)] and [Mar91, Remark IX(6.17)(iii)] □

In Chaper 7, two-spherical split Kac-Moody groups are of our main interest therefore it is important to give a refinement of the rigidity results above in the special case of connected irreducible universal Chevalley groups of rank two. We shall do this in the form of the following remark.

Remark 1.6.9. Let G be a connected irreducible universal Chevalley group of rank two. Let $\phi : G(\mathbb{Z}) \to H_{\mathbb{Q}}$ be an abstract homomorphism where H is a connected semisimple linear algebraic $\mathbb{Q}$-group. Moreover, assume that the image of ϕ is discrete in $H_{\mathbb{R}}$ with respect to the Lie topology and dense in H. In addition, assume that ϕ is **injective**. By (1.18) in Remark 1.6.5 there exists the following central $\mathbb{R}$-isogeny

$$\tilde{\phi} : G \to H, \tag{1.21}$$

which agrees with ϕ on a congruence subgroup $G(n\mathbb{Z})$ for some $n \in \mathbb{Z}$. Moreover, $\tilde{\phi}$ is also a $\mathbb{Q}$-morphism of algebraic $\mathbb{Q}$-groups.

Note that H is strictly isogenous with G. Now because G is irreducible of rank two, its root system is one of the following cases: A_2, $C_2(\cong B_2)$ and G_2. Therefore, H is a Chevalley group of one of the following types: A_2, $C_2(\cong B_2)$ and G_2. Hence one of the following cases occurs:

G_2: In this case since G is absolutely simple (hence adjoint), the central isogeny (1.21) is an isomorphism of linear algebraic $\mathbb{Q}$-groups. Moreover, by the strong rigidity theorem, Theorem 1.6.8, $\tilde{\phi} : G(\mathbb{R}) \to (H_{\mathbb{R}})^{\circ}$ is actually the unique continuous extension of the whole $G(\mathbb{Z})$ and, in particular, $\tilde{\phi} : G(\mathbb{R}) \to (H_{\mathbb{R}})^{\circ}$ is an isomorphism.

A_2: For A_2, since $Z(G(\mathbb{R})) = Z(G)_{\mathbb{R}} = \{e\}$ (see Proposition 1.1.8) and (1.21) is a central $\mathbb{R}$-isogeny, we have the following isomorphism of Lie groups (see [Mar91, Remark IX(6.17)(ii-iii)])

$$\tilde{\phi} : G(\mathbb{R}) \to (H_{\mathbb{R}})^{\circ}, \tag{1.22}$$

where $(H_{\mathbb{R}})^{\circ}$ denotes the connected component of $H_{\mathbb{R}}$.

C_2: In this case we know that the structure of the center of G is $\mathbb{Z}_2$ and $Z(G) = Z(G(\mathbb{R}))$. Hence the central isogeny in (1.21) is either an isomorphism or (by the first homomorphism theorem)

$$\mathrm{PSp}_4 \cong G/Z(G) \cong H. \tag{1.23}$$

Claim: (1.23) does not hold.
If (1.23) holds then H is absolutely simple and in particular adjoint and hence it is with trivial center. In this situation, we apply Theorem 1.6.8 to the morphism

$$\phi : G(\mathbb{Z}) \to H_{\mathbb{Q}}. \tag{1.24}$$

Use of Theorem 1.6.8 is valid because it is evident that $H \cong \mathrm{PSp}_4$ is an absolutely simple linear algebraic $\mathbb{Q}$-group with no $\mathbb{R}$-anisotropic factors. Hence, in this situation, H satisfies the conditions in the strong rigidity theorem, Theorem 1.6.8.
Let $\bar{\phi}$ be the unique extension of ϕ on the whole $G(\mathbb{Z})$ obtained by Theorem 1.6.8. Since $\bar{\phi}$ coincides with $\tilde{\phi}$ on $G(n\mathbb{Z})$, which is dense in G, uniqueness of $\tilde{\phi}$ implies that $\bar{\phi} = \tilde{\phi}$. Hence we have the following continuous extension:

$$\tilde{\phi} : G(\mathbb{R}) \to (H_{\mathbb{R}})^{\circ}. \tag{1.25}$$

Moreover, $Z(G) = Z(G(\mathbb{R}))$ in this case. This leads to a contradiction with (1.23) since ϕ is injective and contains $Z(G(\mathbb{R}))$. Hence (1.23) can **not** hold and (1.21) is actually an isomorphism.

Conclusion: At the end we conclude that in all of the above cases, $(H_{\mathbb{R}})^{\circ}$ is a simply connected linear algebraic $\mathbb{R}$-group. And $\tilde{\phi}(G(n\mathbb{Z}))$ forms an irreducible lattice in $(H_{\mathbb{R}})^{\circ}$.

Here we state a result which is so-called the **normal subgroup theorem** (or the **Margulis' finiteness Theorem**) which implies that any irreducible lattice in a connected simply connected semisimple real Chevalley group of rank at least two is almost simple meaning; every normal subgroups is either finite or of finite index.

Theorem 1.6.10 (Margulis-Kazhdan)**.** *[Zim84, Theorem 8.1.2] Let G be a connected semisimple Lie group with finite center and no compact factors, $\Gamma \subset G$ an irreducible lattice. Assume* $\mathbb{R}$-rank$(G) \geq 2$*. Let $N \subset \Gamma$ be a normal subgroup. Then either*

(a) $N \subset Z(G)$*, and so N is finite; or*

(b) Γ/N *is finite.*

1.7 Amalgams

In this section we present a method to construct abstract groups by amalgamating a family of groups with some conditions. Our main references are [BS04, Section 4] and [RS05, Chapter 1]. We should mention here that a comprehensive reference in this direction is [Ser03]. Note that, as we see in Theorem 4.4.7, a two-spherical simply connected split Kac-Moody group can be recovered by amalgamating its fundamental rank two subgroups over its fundamental rank on subgroups in a natural way. This highlights the importance of this section. In addition, this property is used in the proof of Proposition 7.1.3.

An **amalgam** $\mathcal{G}$ is a set equipped with a partial operation of multiplication which consists of a collection of subsets $\{G_i\}_{i\in I}$ for some index set I such that:

(A1) $\mathcal{G} = \bigcup_{i\in I} G_i$;

(A2) the product ab for $a, b \in \mathcal{G}$ is defined if and only if $a, b \in G_i$ for some $i \in I$;

(A3) the restriction of the multiplication on $\mathcal{G}$ to each G_i gives rise to a group structure on G_i; and

(A4) $G_i \cap G_j$ is a subgroup in both G_i and G_j for all $i, j \in I$.

The subsets G_i are called the **members** of the amalgam $\mathcal{G}$. For two amalgams $\mathcal{G} = \bigcup_{i\in I} G_i$ and $\mathcal{G}' = \bigcup_{i\in I} G'_i$ a mapping $\phi : \mathcal{G} \to \mathcal{G}'$ is an **amalgam morphism** provided for every $i \in I$ the restriction of ϕ to G_i is a group morphism from G_i to G'_i. If ϕ is bijective and the restrictions to G_i's are isomorphism then we call ϕ an **amalgam isomorphism**. An isomorphism of an amalgam onto itself is called an **amalgam automorphism**. The group of amalgam automorphisms of $\mathcal{G}$ is denoted by $\mathrm{Aut}(\mathcal{G})$.
An amalgam $\mathcal{G}' = \bigcup_{i\in I} G'_i$ is a **quotient** of an amalgam $\mathcal{G} = \bigcup_{i\in I} G_i$ if there is a morphism of amalgams $\phi : \mathcal{G} \to \mathcal{G}'$ such that the restriction of ϕ to each G_i is a surjective homomorphism from G_i to G'_i for all $i \in I$. A group $G_{\mathcal{G}}$ is called a **completion** of $\mathcal{G}$ if there exists a mapping $\pi : \mathcal{G} \to G_{\mathcal{G}}$ such that

(CA1) for all $i \in I$, $\pi|_{G_i} : G_i \to G_{\mathcal{G}}$ is a group homomorphism; and

(CA2) $G_{\mathcal{G}} = \langle \pi(\mathcal{G}) \rangle$.

Now consider the following group:

$$\mathcal{U}(\mathcal{G}) := \langle u_g \ : \ g \in \mathcal{G} \mid u_{g_1} u_{g_2} = u_{g_3}, \ \textit{whenever } g_1 g_2 = g_3 \textit{ in } \mathcal{G} \rangle. \tag{1.26}$$

We can define a map $\Psi : \mathcal{G} \to \mathcal{U}(\mathcal{G})$ which sends $g \mapsto u_g$. This implies that $\mathcal{U}(\mathcal{G})$ is a completion of $\mathcal{G}$. Moreover, for any completion $\pi : \mathcal{G} \to G_{\mathcal{G}}$ of $\mathcal{G}$ define $\hat{\pi} : \mathcal{U}(\mathcal{G}) \to G_{\mathcal{G}}$ by $u_g \mapsto \pi(g)$. $\hat{\pi}$ is a surjective group morphism, thus every completion of $\mathcal{G}$ is a quotient of $\mathcal{U}(\mathcal{G})$. Because of this, we call $\mathcal{U}(\mathcal{G})$ the **universal completion** (or **universal enveloping group**) of $\mathcal{G}$.

There is a more general way to define amalgams by means of the concept of direct limit of systems.

Definition 1.7.1. [RS05, Definition 1.1] Let $\{G_i\}_{i\in I}$ be a collection of groups for some index set I. For any $i, j \in I$, let F_{ij} be a subset of $\mathrm{Hom}_{gr}(G_i, G_j)$. The **direct limit** of this system, namely, $G = \varinjlim G_i$, is a group G equipped with group morphisms $\phi_i \in \mathrm{Hom}_{gr}(G_i, G)$ such that for any given group H and group morphisms $\psi_i \in \mathrm{Hom}_{gr}(G_i, G)$ with the condition $\psi_j \circ f_{ij} = \psi_i$ for all $i, j \in I$ and for all $f_{ij} \in F_{ij}$, there exists a unique group morphism ψ such that the following diagram commutes:

$$\begin{tikzcd}
G_i \arrow[rr, "f_{ij}"] \arrow[dr, "\phi_i"] \arrow[ddr, "\psi_i"'] & & G_j . \arrow[dl, "\phi_j"'] \arrow[ddl, "\psi_j"] \\
& G \arrow[d, dotted, "\psi"] & \\
& H &
\end{tikzcd} \tag{1.27}$$

Proposition 1.7.2. *[RS05, Proposition 1.3] Given a collection of groups $\{G_i\}_{i\in I}$ and a collection of group morphisms F_{ij} as in (1.27), the direct limit $\varinjlim G_i$ exists and is unique up to unique isomorphism.*

Note that for a given collection of groups $\{G_i\}_{i\in I}$ if we define $f_i : G_i \cap G_j \to G_i$ and $f_j : G_i \cap G_j \to G_j$ to be the inclusion map then it is straightforward to check that for the amalgam $\mathcal{G} = \bigcup_{i\in I} G_i$, the universal enveloping group $\mathcal{U}(\mathcal{G})$ is indeed the direct limit of the system $\{G_i \cap G_j, G_i : f_i\}_{i,j\in I}$.

CHAPTER

TWO

TITS BUILDINGS

This chapter is devoted to building theory which provides combinatorial and geometric features inside a group with a (twin) BN-pair. In particular, split Kac-Moody groups have (twin) BN-pairs (see Section 4.1). This allows us to benefit from building theory in Kac-Moody theory. Therefore, in the present chapter we give a concise introduction to building theory in the direction of Kac-Moody theory. We assume that the reader is familiar with Coxeter systems and their corresponding root systems. [Dav08] is a comprehensive reference in the study of Coxeter systems.
The main reference of this chapter is [AB08]. For the details we follow [Cap05]. This chapter does not contain any original results.

2.1 Root Data

A group W with a presentation

$$W = \langle s_1, ..., s_k \mid (s_i s_j)^{m_{ij}} \rangle, \tag{2.1}$$

where $m_{ii} = 1$, $m_{ij} \geq 2$ and $m_{ij} = m_{ji}$ for $i \neq j$, is called a **Coxeter group**. Note that $m_{ij} = \infty$ means there is no relation between s_i and s_j. The set of generators $S = \{s_1, \cdots, s_k\}$ is always assumed to be finite. A Coxeter group W with a fixed set of generators S is called a **Coxeter system** and is denoted by (W, S). We denote by l the **length function** of (W, S) (with respect to the generating set S).
Now let (W, S) be a Coxeter system. One can associate a root system Φ to (W, S) with a basis $\Phi^\circ = \{\alpha_{s_1}, \cdots, \alpha_{s_k}\}$ where α_{s_i} corresponds to the generator s_i for $1 \leq i \leq k$. We denote by Φ_+ (resp. Φ_-) the positive (resp. negative) roots in Φ.
Similar to the notion of the Coxeter graph in Section 1.2, we can construct a Coxeter graph (and a Dynkin diagram) for any Coxeter system. Moreover, this notion is compatible with the notion of finite root systems presented in Section 1.2. For if we choose a Dynkin diagram on Table 1.1, the above root system is isomorphic to the root system on Table 1.1 corresponding to the chosen Dynkin diagram. In other words, the notion of root systems here is a generalization of finite root systems (see [Dav08] and [Hum90]).

We call a pair of roots $\{\alpha, \beta\} \subset \Phi$, **prenilpotent** if there exist $w, w' \in W$ such that $\{w(\alpha), w(\beta)\} \subset \Phi_+$ and $\{w'(\alpha), w'(\beta)\} \subset \Phi_-$. For such a pair, we define

$$[\alpha, \beta] := \bigcap_{\substack{w \in W \\ \epsilon \in \{1,-1\}}} \{\gamma \in \Phi \mid \{w(\alpha), w(\beta)\} \subset \Phi_\epsilon \Rightarrow \ w(\gamma) \in \Phi_\epsilon\}$$

and

$$]\alpha, \beta[:= [\alpha, \beta] \setminus \{\alpha, \beta\}.$$

Definition 2.1.1. A **twin root datum** of type (W, S) is a system $\mathcal{Z} = (G, (U_\alpha)_{\alpha \in \Phi})$ where G is a group and $\{U_\alpha\}_{\alpha \in \Phi}$ is a family of subgroups of G indexed by Φ satisfying the following axioms, where $T := \bigcap_{\alpha \in \Phi} N_G(U_\alpha)$, $U_+ := \langle U_\gamma \mid \gamma \in \Phi_+ \rangle$ and $U_- := \langle U_\gamma \mid \gamma \in \Phi_- \rangle$:

(TRD0) For each $\alpha \in \Phi$, $U_\alpha \neq \{1\}$.

(TRD1) For each prenilpotent pair $\{\alpha, \beta\} \subset \Phi$, the commutator group $[U_\alpha, U_\beta]$ is contained in the group $U_{]\alpha,\beta[} := \langle U_\gamma \mid \gamma \in]\alpha, \beta[\rangle$.

(TRD2) For each $\alpha_s \in \Phi^\circ$ and each $u \in U_{\alpha_s} \setminus \{1\}$ there exist elements $u', u'' \in U_{-\alpha_s}$ such that the product $\mu(u) := u'.u.u''$ conjugates U_β to $U_{s(\beta)}$ for every $\beta \in \Phi$.

(TRD3) For each $\alpha_s \in \Phi^\circ$, the group $U_{-\alpha_s}$ is not contained in U_+ and the group U_{α_s} is not contained in U_-.

(TRD4) $G = T\langle U_\alpha \mid \alpha \in \Phi \rangle$.

Moreover, we call $\mathcal{Z}$ of **finite rank** if S is finite. A twin root datum is called **centered** provided $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$.

In the literature, such twin root data are also called **RGD systems** (see [HKM13, Section 2.2]).

Now let $\{\mathbb{K}_\alpha\}_{\alpha \in \Phi}$ be a collection of fields indexed by Φ. A twin root datum $\mathcal{Z} = (G, (U_\alpha)_{\alpha \in \Phi})$ of type (W, S) is called **locally split** over $\{\mathbb{K}_\alpha\}_{\alpha \in \Phi}$ (or over $\mathbb{K}$ if $\mathbb{K} \cong \mathbb{K}_\alpha$ for all $\alpha \in \Phi$) if the following conditions hold:

(LS1) The group $T := \bigcap_{\alpha \in \Phi} N_G(U_\alpha)$ is abelian.

(LS2) For each $\alpha \in \Phi$ there exists a homomorphism

$$\phi_\alpha : SL(2, \mathbb{K}_\alpha) \to \langle U_\alpha, U_{-\alpha} \rangle$$

which maps the subgroup of upper (resp. lower) triangular unipotent matrices onto U_α (resp. $U_{-\alpha}$).

Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha \in \Phi})$ and $\mathcal{Z}' = (G', (U'_{\alpha'})_{\alpha' \in \Phi'})$ be two twin root data of type (W, S) and (W', S') respectively. Moreover, let $S = S_1 \cup S_2 \cup ...S_n$ be the finest partition of S such that $[S_i, S_j] = 1$ whenever $1 \leq i \leq j \leq n$. We call S_i the **irreducible subset** of S. An ordered pair (ϕ, π) consisting of an isomorphism $\phi : G \to G'$ and an isomorphism $\pi : W \to W'$ is called an **isomorphism of twin root data** from $\mathcal{Z}$ to $\mathcal{Z}'$ if the following conditions hold:

(ITRD1) $\pi(S) = (S')$, hence, it induces an equivariant bijection $\Phi \to \Phi'$ denoted again by π.

(ITRD2) There exist an $x \in G'$ and a sign ϵ_i for each $1 \leq i \leq n$ such that

$$x\phi(U_\alpha)x^{-1} = U'_{\epsilon_i \pi(\alpha)}$$

for every $\alpha \in \Phi$ such that $s_\alpha \in W_{S_i}$ where s_α is the reflection corresponding to α in W.

Thus if S is irreducible then either $x\phi(U_\alpha)x^{-1} = U'_{\pi(\alpha)}$ or $x\phi(U_\alpha)x^{-1} = U'_{-\pi(\alpha)}$ for all $\alpha \in \Phi$. In particular, ϕ maps the union of conjugacy classes

$$\{gU_+g^{-1} \mid g \in G\} \cup \{gU_-g^{-1} \mid g \in G\}$$

to

$$\{gU'_+g^{-1} \mid g \in G'\} \cup \{gU'_-g^{-1} \mid g \in G'\}.$$

Theorem 2.1.2. *[CM05, Theorem 2.2] Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha \in \Phi})$ and $\mathcal{Z}' = (G', (U'_{\alpha'})_{\alpha' \in \Phi'})$ be two twin root data of finite rank. Let $\phi : G \to G'$ be an isomorphism such that*

$$\{\phi(U_\alpha) \mid \alpha \in \Phi\} = \{x.U'_\alpha.x^{-1} \mid \alpha \in \Phi'\}, \tag{2.2}$$

for some $x \in G'$. Then there exists an isomorphism $\pi : W \to W'$ such that the pair (ϕ, π) is an isomorphism of twin root data from $\mathcal{Z}$ to $\mathcal{Z}'$.

2.2 Buildings

In this section we introduce buildings and twin buildings along with some basic properties and facts about them. Then we present the Moufang property of buildings.

2.2.1 Buildings and Twin Buildings

Definition 2.2.1. Let (W, S) be a Coxeter system. A **building** of type (W, S) is a pair $\mathcal{B} = (\mathcal{C}, \delta)$ consisting of a non-empty set $\mathcal{C}$ and a map $\delta : \mathcal{C} \times \mathcal{C} \to W$ called the **distance function** (or the **Weyl distance**) subjected to the following axioms, where $x, y \in \mathcal{C}$ and $w = \delta(x, y)$:

(Bu1) $w = 1$ if and only if $x = y$.

(Bu2) If $z \in \mathcal{C}$ such that $\delta(y, z) = s$ for some $s \in S$, then $\delta(x, z) = w$ or ws, and if furthermore, $l(ws) = l(w) + 1$, then $\delta(x, z) = ws$.

(Bu3) For every $s \in S$ there exists $z \in \mathcal{C}$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

The Coxeter group W is called the **Weyl group** of $\mathcal{B}$ and elements in $\mathcal{C}$ are called **chambers** of $\mathcal{B}$ (see [AB08, Definition 5.1]).

Two chambers x, y are called s-**adjacent** if $\delta(x, y) \in \{1, s\}$. Two chambers are called **adjacent** if they are s-adjacent for some $s \in S$. A building of type (W, S) is called **thick** if for every chamber $x \in \mathcal{C}$ and every $s \in S$ there exist at least three distinct chambers s-adjacent to x, otherwise it is called **thin**.

Definition 2.2.2. Let $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+)$ and $\mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be two buildings of the same type (W, S). A **co-distance** (or a **twinning**) between $\mathcal{B}_+$ and $\mathcal{B}_-$ is a mapping $\delta_* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_+ \times \mathcal{C}_-) \to W$ such that it satisfies the following conditions, where $\epsilon \in \{+, -\}$, $x \in \mathcal{C}_\epsilon$, $y \in \mathcal{C}_{-\epsilon}$ and $w = \delta_*(x, y)$:

(Tw1) $\delta_*(y, x) = w^{-1}$;

(Tw2) if $z \in \mathcal{C}_{-\epsilon}$ such that $\delta_{-\epsilon}(y, z) = s$ for some $s \in S$ and $l(ws) = l(w) - 1$, then $\delta_*(x, z) = ws$;

(Tw3) for every $s \in S$ there exists $z \in \mathcal{C}_{-\epsilon}$ such that $\delta_{-\epsilon}(y, z) = s$ and $\delta_*(x, z) = ws$.

A **twin building** of type (W, S) is a triple $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ with the aforementioned description.

Elements in $\mathcal{C}_+ \cup \mathcal{C}_-$ are called **chambers** of $\mathcal{B}$. A twin building $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ is called **thick** (**thin**) if each of $\mathcal{B}_\pm$ is a thick (thin) building. Two chambers, $x, y \in \mathcal{B}$ are called **opposite** if $\delta_*(x, y) = 1$.

Here we collect basic concepts and facts about buildings extracted from [AB08, Chapter 5].

Definition 2.2.3. [AB08, Definition 5.59] Let $\mathcal{B} = (\mathcal{C}, \delta)$ and $\mathcal{B}' = (\mathcal{C}', \delta')$ be two buildings of type (W, S) and (W', S') receptively. Let $\sigma : (W, S) \to (W', S')$ be an isomorphism of Coxeter systems. A σ-**isometry** from $\mathcal{C}$ to $\mathcal{C}'$ is a map $\phi : \mathcal{C} \to \mathcal{C}'$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, "\delta"] \arrow[d, "{(\phi,\phi)}"'] & W \arrow[d, "\sigma"] \\ \mathcal{C}' \times \mathcal{C}' \arrow[r, "\delta'"'] & W' \end{tikzcd}$$

in other words,

$$\delta'(\phi(x), \phi(y)) = \sigma(\delta(x, y))$$

for all $x, y \in \mathcal{C}$. If $(W, S) = (W', S')$ and $\sigma = id_W$, we simply call ϕ an **isometry**.

Let $\delta : W \times W \to W : (x, y) \mapsto x^{-1}y$. It is straightforward to check that δ is a distance function and $\mathcal{A}(W, S) := (W, S)$ is a building of type (W, S).

Let $\mathcal{B}$ be a building of type (W, S). An **apartment** of $\mathcal{B}$ is a set of chambers isometric to $\mathcal{A}(W, S)$. Any pair of chambers in a building is contained in an apartment (see [AB08, Corollary 5.74]).

For a subset $J \subset S$, we write W_J for the subgroup of W generated by J. J is called **spherical** if W_J is finite.

Definition 2.2.4. Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S). Given a subset $J \subset S$ and a chamber $c \in \mathcal{C}$, the **residue** of type J (or the J-**residue**) containing c is the set

$$\mathrm{Res}_J(c) := \{x \in \mathcal{C} |\ \delta(c, x) \in W_J\}.$$

Moreover, the cardinality $|J|$ is called the **rank** of $\mathrm{Res}_J(c)$.

We call a J-residue, **spherical** if J is spherical in W. A subset $\mathcal{R} \subset \mathcal{C}$ is called a **residue** if it is a J-residue for some $J \subset S$. By Definition 2.2.4, a residue of rank 0 is just a chamber in the building. Moreover, the $\{s\}$-residue at $c \in \mathcal{C}$ for some $s \in S$ is the set of all s-adjacent chambers to c. This residue is called the s-**panel** containing c. A J-residue $\mathcal{R}$ of a building $\mathcal{B}$ is a building itself of type (W_J, J) (see [AB08, Corollary 5.30]). Moreover, if $\mathcal{A}$ is an apartment of $\mathcal{B}$, then $\mathcal{R} \cap \mathcal{A}$ is either empty or an apartment of $\mathcal{R}$. Furthermore, all apartments of $\mathcal{R}$ can be obtained in this way (a consequence of the three characterizations of apartments given in [AB08, page 241]; see also [AB08, Theorem 5.73]).
Let $\mathcal{B} = (\mathcal{C}, \delta)$ be a building of type (W, S). We call the function $d := l \circ \delta : \mathcal{C} \to \mathbb{Z}_{\geq 0}$ the **numerical distance** of $\mathcal{B}$. This numerical distance function defines a (discrete) metric on $\mathcal{C}$ in the usual sense.
For a spherical J-residue $\mathcal{R}$, two chambers $a, b, \in \mathcal{R}$ are called **opposite** in $\mathcal{R}$ if $d(a, b) = \max\{d(x, y) | x, y \in \mathcal{R}\}$ and we denote it by $a \text{ } opp_{\ \mathcal{R}} b$. Using the numerical function, we can define distances between two residues. Let $\mathcal{R}$ and $\mathcal{S}$ be two residues of type J and K respectively. Define

$$d(\mathcal{R}, \mathcal{S}) := \min\{d(x, y) \mid x \in \mathcal{R}, y \in \mathcal{S}\}$$

to be the **distance** between $\mathcal{R}$ and $\mathcal{S}$ (see [AB08, Section 5.3.2]).
Definition 2.2.5. Let $\mathcal{R}$ and $\mathcal{S}$ be two residues of a building $\mathcal{B}$ of type (W, S). The set

$$\mathrm{proj}_{\mathcal{R}}(\mathcal{S}) := \{x \in \mathcal{R} \mid d(x, \mathcal{S}) = d(\mathcal{R}, \mathcal{S})\}$$

is called the **projection** of $\mathcal{S}$ to $\mathcal{R}$.

Note that; first, $\mathrm{proj}_{\mathcal{R}}(\mathcal{S})$ consists of all chambers of $\mathcal{R}$ at minimal numerical distance from $\mathcal{S}$. Second, $\mathrm{proj}_{\mathcal{R}}(\mathcal{S})$ is a residue itself whose rank is bounded from above by the ranks of $\mathcal{R}$ and $\mathcal{S}$. In particular, for every $c \in \mathcal{B}$, $\mathrm{proj}_{\mathcal{R}}(c)$ is a unique chamber of $\mathcal{R}$.
Definition 2.2.6. Let σ be an s-panel of a building $\mathcal{B} = (\mathcal{C}, \delta)$ of type (W, S) containing $c \in \mathcal{C}$. Let $\mathcal{A}$ be an apartment of $\mathcal{B}$ such that $\sigma \cap \mathcal{A}$ is non-empty. In this case, the intersection $\sigma \cap \mathcal{A}$ consists of exactly two chambers, say c and d, and $\mathcal{A}$ is the disjoint union of

$$\phi_{\mathcal{A}}(\sigma, c) := \{x \in \mathcal{A} \mid \mathrm{proj}_{\sigma}(x) = c\}$$

and

$$\phi_{\mathcal{A}}(\sigma, d) := \{x \in \mathcal{A} \mid \mathrm{proj}_{\sigma}(x) = d\}.$$

The sets $\phi_{\mathcal{A}}(\sigma, c)$ and $\phi_{\mathcal{A}}(\sigma, d)$ are called **roots** (or **half-apartments**) of $\phi_{\mathcal{A}}(\sigma, c)$.

Let ϕ be a root in an apartment $\mathcal{A}$. The set of all panels meeting both ϕ and its complement in $\mathcal{A}$ is called the **wall** determined by ϕ and is denoted by $\partial\phi$.
Now we present similar concepts for twin buildings (see [AB08, Chapter 5, 8]).
Definition 2.2.7. Let $\mathcal{B}_+ = (\mathcal{C}_+, \delta_+)$ and $\mathcal{B}_- = (\mathcal{C}_-, \delta_-)$ be two buildings of the same type (W, S) and let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a twin building. For apartments $\mathcal{A}_+$ and $\mathcal{A}_-$ in $\mathcal{B}_+$ and $\mathcal{B}_-$ respectively, we say that $\mathcal{A}_+$ and $\mathcal{A}_-$ are **twins** if every chamber of $\mathcal{A}_+$ has a unique opposite in $\mathcal{A}_-$ (or vice versa). In that case, the pair $(\mathcal{A}_+, \mathcal{A}_-)$ is called a **twin apartment** of $\mathcal{B}$.

Any given pair (x_+, x_-) of opposite chambers is contained in a unique twin apartment $(\mathcal{A}_+, \mathcal{A}_-)$ which is obtained as follows ($\epsilon \in \{-, +\}$)(see [AB08, Section 5.8.4]):

$$\mathcal{A}_\epsilon := \{c \in \mathcal{C}_\epsilon \mid \delta_\epsilon(x_\epsilon, c) = \delta_*(x_{-\epsilon}, c)\}.$$

A **residue** $\mathcal{R}_\epsilon$ of a twin building $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ is simply a residue of the building $\mathcal{B}_\epsilon$ where $\epsilon \in \{-,+\}$. Two residues (of opposite sign) are called **opposite** if they are of the same type and contain opposite chambers.
The function $d_* := l \circ \delta_*$ is called the **numerical co-distance** of a twin building $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ of type (W, S). Therefore, two chambers are opposite if and only if their numerical co-distance is 0.
Definition 2.2.8. Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a twin building of type (W, S). Let $\mathcal{R}_\epsilon$ and $\mathcal{R}_{-\epsilon}$ be two spherical residues of $\mathcal{B}$ where $\epsilon \in \{-,+\}$. The set

$$\mathrm{proj}_{\mathcal{R}_\epsilon}(\mathcal{R}_{-\epsilon}) := \{x \in \mathcal{R}_\epsilon \mid \exists y \in \mathcal{R}_{-\epsilon} \ : \ d_*(x,y) \geq d_*(x',y') \ \forall x' \in \mathcal{R}_\epsilon, y' \in \mathcal{R}_{-\epsilon}\}$$

consisting of all chambers of $\mathcal{R}_\epsilon$ with maximal co-distance from $\mathcal{R}_{-\epsilon}$ is called the **projection** of $\mathcal{R}_{-\epsilon}$ to $\mathcal{R}_\epsilon$.

Note that $\mathrm{proj}_{\mathcal{R}_\epsilon}(\mathcal{R}_{-\epsilon})$ is a residue itself with its rank bounded from above by the ranks of $\mathcal{R}_\epsilon$ and $\mathcal{R}_{-\epsilon}$. In particular, for a chamber $c \in \mathcal{B}_{-\epsilon}$, $\mathrm{proj}_{\mathcal{R}_\epsilon}(c)$ is a unique chamber of $\mathcal{R}_\epsilon$.
Definition 2.2.9. Let $\mathcal{B} = ((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta_*)$ be a thick twin building of type (W, S). Given a chamber $c_\epsilon \in \mathcal{C}_\epsilon$ and a $k \in \mathbb{N}$, we define

$$E_{\epsilon,k}(c_\epsilon) := \bigcup_{\substack{J \subset S \\ |J| \leq k}} \mathrm{Res}_{\epsilon,J}(c_\epsilon)$$

where $\epsilon \in \{-,+\}$.
Theorem 2.2.10 (**Rigidity of Thick Twin Buildings**)**.** *Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a thick twin building and let c_+ and c_- be two opposite chambers of $\mathcal{B}$. Any two automorphisms of $\mathcal{B}$ coincide if and only if their restrictions to $E_{\epsilon,1}(c_\epsilon) \cup \{c_{-\epsilon}\}$ coincide for $\epsilon \in \{+,-\}$.*

Proof. See [AB08, Theorem 5.205] and [AB08, Remark 5.208] □

Note that the isometries between twin buildings are defined in the same way as we define isometries of buildings in Definition 2.2.3.

2.2.2 Moufang Buildings

In this subsection we introduce the Moufang property of buildings. The main references are [Cap05, Section 1.3.4] and [AB08, Chapters 7 and 8].
Definition 2.2.11. Let $\mathcal{B} = ((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta_*)$ be a thick twin building of type (W, S). Let $\mathcal{A} = (\mathcal{A}_+, \mathcal{A}_-)$ be a twin apartment. A subset $\phi \subset \mathcal{A}_+ \cup \mathcal{A}_-$ is called a **twin root** if $\phi_\epsilon := \phi \cap \mathcal{A}_\epsilon$ is a root of $\mathcal{A}_\epsilon$ and if

$$opp_{\ \mathcal{A}}(\phi_\epsilon) = -\phi_{-\epsilon},$$

where $\epsilon \in \{-,+\}$. In other words, the set of chambers of $\mathcal{A}_{-\epsilon}$ opposite a chamber of ϕ_ϵ coincides with the complement of $\phi_{-\epsilon}$ in $\mathcal{A}_{-\epsilon}$ (See [AB08, Definition 5.190]).

Any two chambers $x_+ \in \mathcal{C}_+$ and $x_- \in \mathcal{C}_-$ such that $\delta_*(x_+, x_-) = s \in S$ are contained in a unique twin root, denoted by $\phi(x_+, x_-)$, and all twin roots can be constructed this way (see [AB08, Corollary 5.194]).

Note that $\phi(x_+, x_-) = \phi_+(x_+, x_-) \cup \phi_-(x_+, x_-)$ where $\phi_\epsilon(x_+, x_-)$ consists of all chambers $c_\epsilon \in \mathcal{C}_\epsilon$ such that

$$\delta_\epsilon(x_\epsilon, c_\epsilon) = s\delta_*(x_{-\epsilon}, c_\epsilon)$$

and

$$l(\delta_\epsilon(x_\epsilon, c_\epsilon)) \leq l(\delta_*(x_\epsilon, c_\epsilon)).$$

Definition 2.2.12. Let $\mathcal{B} = ((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta_*)$ be a twin building of type (W, S). Let ϕ be a twin root of $\mathcal{B}$. The subset U_ϕ of $\mathrm{Aut}(\mathcal{B})$ consisting of all $g \in \mathrm{Aut}(\mathcal{B})$ which fix point-wise each panel ρ such that $|\rho \cap \phi| = 2$ is called the **root subgroup** associated with ϕ (see [AB08, Definition 8.16]).

Now with the assumption that (W, S) has no direct factor of type A_1, for each twin root ϕ and each panel ρ such that $|\rho \cap \phi| = 1$, the root group U_ϕ acts freely on the set $\rho \backslash \phi$ (see [AB08, Lemma 8.17(3)]). We say that the twin building $\mathcal{B}$ has the **Moufang property** if the action of U_ϕ is transitive on $\rho\backslash\phi$ for each twin root ϕ and each panel ρ provided that $|\rho \cap \phi| = 1$. A building is called **Moufang** if it has the Moufang property (see [AB08, Definition 8.18]).

2.3 Twin BN-Pairs and Twin Buildings

In this section we introduce the action of groups on twin buildings, basic definitions, concepts and facts. Then we define (twin) BN-pairs of groups, give twin BN-pairs for some certain twin buildings and vice-versa . Throughout this section we assume that $\mathcal{C} = ((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta_*)$ is a twin building of type (W, S) and G an abstract group. We follow closely [AB08, Section 6] in this section.

2.3.1 Strongly Transitive Actions and Birkhoff Decomposition

In this subsection we present strongly transitive action of groups on (twin) buildings which leads to a decomposition of the underlying group so called the Birkhoff decomposition.
Definition 2.3.1. We say that G **acts** on $\mathcal{C}$ if it acts simultaneously on the two sets $\mathcal{C}_+$ and $\mathcal{C}_-$ and preserves the Weyl distances and the co-distance. In other words:

(1) $\delta_+(g.x_+, g.y_+) = \delta_+(x_+, y_+)$

(2) $\delta_-(g.x_-, g.y_-) = \delta_-(x_-, y_-)$

(3) $\delta_*(g.x_+, g.x_-) = \delta_*(x_+, x_-)$

for all $g \in G$, $x_+, y_+ \in \mathcal{C}_+$ and $x_-, y_- \in \mathcal{C}_-$.

Note that Condition (3) means that G sends opposite chambers to opposite chambers. Moreover, one can define the action of a group on a building in the same way. Now assume that $x_+ \in \mathcal{C}_+$ and $x_- \in \mathcal{C}_-$ are opposite chambers in $\mathcal{C}$. The pair (x_+, x_-) gives us a unique twin apartment $\mathcal{A}(x_+, x_-)$ containing x_+ and x_- (see the paragraph after Definition 2.2.7). We define $B_\pm$ as the stabilizer of $x_\pm$:

$$B_\pm := \{g \in G \mid g.x_\pm = x_\pm\}, \tag{2.3}$$

and

$$N := \{g \in G \mid g.\mathcal{A}(x_+, x_-) = \mathcal{A}(x_+, x_-)\}, \tag{2.4}$$

moreover

$$T := \{g \in G \mid g.x = x \text{ for all } x \in \mathcal{A}(x_+, x_-)\}. \tag{2.5}$$

Note that N stabilizes $\mathcal{A}_+(x_+, x_-)$ as well as $\mathcal{A}_-(x_+, x_-)$. Moreover, T is the point-wise fixer of $\mathcal{A}_\pm(x_+, x_-)$. Furthermore, $B_+ \cap B_- = B_+ \cap N = B_- \cap N = T$ (see [AB08, Lemma 6.69]).

Lemma 2.3.2. *[AB08, Lemma 6.70] With the notations above. The following conditions are equivalent:*

(i) For any $w \in W$, G acts transitively on

$$\{(y_+, y_-) \in \mathcal{C}_+ \times \mathcal{C}_- \mid \delta_*(y_+, y_-) = w\}.$$

(ii) For $\epsilon = \pm$, G acts transitively on $\mathcal{C}_\epsilon$, and B_ϵ acts transitively on $\{y_{-\epsilon} \in \mathcal{C}_{-\epsilon} \mid \delta_*(x_\epsilon, y_{-\epsilon}) = w\}$ for each $w \in W$.

(iii) G acts transitively on all pairs of opposite chambers $(y_+, y_-) \in \mathcal{C}$.

(iv) G acts transitively on $\mathcal{A}(y_+, y_-)$ for all pairs of opposite chambers $(y_+, y_-) \in \mathcal{C}$, and N acts transitively on all pairs of opposite chambers $\{(y_+, y_-) \in \mathcal{A}_+(x_+, x_-) \times \mathcal{A}_-(x_+, x_-)\}$.

(v) For $\epsilon = \pm$, G acts transitively on $\mathcal{A}_\epsilon(y_+, y_-)$ for all pairs of opposite chambers $(y_+, y_-) \in \mathcal{C}$, and N acts transitively on the set of chambers of $\mathcal{A}_\epsilon(x_+, x_-)$.

Definition 2.3.3. [AB08, Definition 6.71] We say the action of G on $\mathcal{C}$ is **strongly transitive** if it satisfies one of the five equivalent conditions in Lemma 2.3.2.

Note that one can define strongly transitive actions on buildings in the following way: Action of a group G on a building $\mathcal{B}$ is **strongly transitive** if G acts transitively on all pairs $(\mathcal{A}, x)$ where $\mathcal{A}$ is an apartment in $\mathcal{B}$ containing x, i.e., given a pair $(\mathcal{A}, x)$, for any other pair $(\mathcal{A}', x')$, there exists a $g \in G$ such that $(\mathcal{A}', x') = (g.\mathcal{A}, g.x)$. Moreover, one can rewrite the definition of the strongly transitive action of groups on twin buildings in this way.

Assume that $(x_+, x_-) \in \mathcal{C}$ is a pair of opposite chambers. For an element $n \in N$, suppose that its action on $\mathcal{A}_+(x_+, x_-)$ is given by $w \in W$. Then $\delta_+(x_+, n.x_+) = w$; hence

$$\delta_-(x_-, n.x_-) = \delta_*(x_+, n.x_-) = \delta_+(x_+, n.x_+) = w,$$

since all the chambers that occur are in $\mathcal{A}(x_+, x_-) = \mathcal{A}(n.x_+, n.x_-)$. Thus n also acts as w on $\mathcal{A}_-(x_+, x_-)$. This defines a canonical homomorphism $N \to W$ where the kernel is T as it is the point-wise fixer of $\mathcal{A}_\pm(x_+, x_-)$. Now if the action of G is strongly transitive then N acts transitively on $\mathcal{A}_\pm(x_+, x_-)$, hence the homomorphism $N \to W$ is surjective and $W \cong N/T$ (see [AB08, page 326]).

Definition 2.3.4. [AB08, Definition 6.55] A pair of subgroups B and N of a group G is called a **BN-Pair** of G if B and N generate G, the intersection $T := B \cap N$ is normal in N and the quotient $W := N/T$ admits a set of generators S such that the following two conditions hold:

(BN1) For $s \in S$ and $w \in W$,

$$sBw \subseteq BswB \cup BwB.$$

(BN2) For $s \in S$,

$$sBs^{-1} \nsubseteq B.$$

The group W is called the **Weyl group** associated to the BN-pair. Moreover, in this setting we also call the quadruple (G, B, N, S) a **Tits system**.

Theorem 2.3.5. *[AB08, Corollary 6.72(2)] With the notations as (2.3), (2.4) and (2.5), suppose that G acts strongly transitively on $\mathcal{C}$. If $\mathcal{C}$ is thick, then (B_+, N) and (B_-, N) are* BN*-pairs in G with the common Weyl group $N/T \cong W$.*

Theorem 2.3.5 implies that there are the following decompositions for G so called the **Bruhat decomposition** of G (see [AB08, Section 6.1.5]):

$$G = \coprod_{w \in W} B_\pm w B_\pm. \tag{2.6}$$

Given a BN-pair (B, N) of a group G, set $\mathcal{C} := G/B$ and define $\delta : \mathcal{C} \times \mathcal{C} \to W$ to be the composite

$$G/B \times G/B \to B\backslash G/B \to W \tag{2.7}$$

where the first map is $(g.B, h.B) \mapsto B.g^{-1}h.B$ and the second is given by the Bruhat decomposition (2.6). Thus

$$\delta(g.B, h.B) = w \iff g^{-1}h \in BwB. \tag{2.8}$$

This forms a building $\mathcal{C}(G, B)$ which is called the building associated to the BN-pair (B, N) of G (see [AB08, page 308]).

Theorem 2.3.6. *[AB08, Theorem 6.56(1)] Given a* BN*-pair in a group G, the generating set S is uniquely determined, and (W, S) is a Coxeter system. There is a thick building $\mathcal{B}(G, B)$ that admits a strongly transitive G-action such that B is the stabilizer of a fixed chamber and N stabilizes a fixed apartment containing the fixed chamber and is transitive on the chambers of the fixed apartment.*

Theorem 2.3.7. *[AB08, Proposition 6.75] If a group G acts strongly transitively on a twin building $\mathcal{C}$ of type (W, S), then, with the notations as above,*

$$G = \coprod_{w \in W} B_\epsilon w B_{-\epsilon} \tag{2.9}$$

for each $\epsilon = \pm$. Given $g \in G$ and $w \in W$, we have

$$g \in B_\epsilon w B_{-\epsilon} \Leftrightarrow \delta_*(x_\epsilon, g.x_{-\epsilon}) = w$$

where $(x_\epsilon, x_{-\epsilon})$ is a fixed pair of opposite chambers in $\mathcal{C}$.

Equation 2.9 is called the **Birkhoff decomposition**.

2.3.2 Twin BN-Pairs

In this subsection we introduce twin BN-pairs then construct a thick twin building with a given twin BN-pair (see [AB08, Section 6.3.3]).

Definition 2.3.8. [AB08, Definition 6.78] Let B_+, B_- and N be subgroups of a group G such that $B_+ \cap N = B_- \cap N =: T$ is a normal subgroup of N. Set $W := N/T$. The triple (B_+, B_-, N) is called a **twin** BN**-pair** with Weyl group W if W admits a set S of generators such that the following conditions hold for all $w \in W$ and $s \in S$ and each $\epsilon \in \{+, -\}$:

(TBN0) (G, B_ϵ, N, S) is a Tits system.

(TBN1) If $l(sw) < l(w)$, then $B_\epsilon s B_\epsilon w B_{-\epsilon} = B_\epsilon sw B_{-\epsilon}$.

(TBN2) $B_+ s \cap B_- = \emptyset$.

In this situation we also say that the quintuple (G, B_+, B_-, N, S) is a **twin Tits system**.

Theorem 2.3.9. *[AB08, Corollary 6.79] If G acts strongly transitively on $\mathcal{C}$, a thick twin building of type (W, S), then with the notations as (2.3), (2.4) and (2.5), the quintuple (G, B_+, B_-, N, S) is a twin Tits system.*

Proposition 2.3.10. *[AB08, Proposition 6.81] Let (G, B_+, B_-, N, S) be a twin Tits system with Weyl group W. Then*

$$G = \coprod_{w \in W} B_\epsilon w B_{-\epsilon} \tag{2.10}$$

for each $\epsilon = \pm$.

Given a twin Tits system (G, B_+, B_-, N, S) with Weyl group $W = N/T = \langle S \rangle$, we construct a thick twin building $\mathcal{C}$ of type (W, S) in the following way:

Define $\mathcal{C}_\epsilon := \mathcal{C}(G, B_\epsilon)$ for $\epsilon = \pm$ as constructed in Subsection 2.3.1 ((2.7) and (2.8)). We now set the co-distance

$$\delta_* : \mathcal{C}_+ \times \mathcal{C}_- \cup \mathcal{C}_+ \times \mathcal{C}_- \to W \tag{2.11}$$

to be that map defined in the following way:

$$\delta_*(g.B_\epsilon, h.B_{-\epsilon}) = w \Longleftrightarrow g^{-1}h \in B_\epsilon w B_{-\epsilon} \tag{2.12}$$

for $g, h \in G$, $w \in W$ which is induced by the Birkhoff decomposition (2.9). These form a (thick) twin building for the twin Tits system (G, B_+, B_-, N, S) on which G acts strongly transitively. We denote this twin building by $\mathcal{C}(G, B_+, B_-)$ and call it the **associated twin building** to the Tits system (G, B_+, B_-, N, S). Note that in $\mathcal{C}(G, B_+, B_-)$, (B_+, B_-) corresponds to a pair of opposite chambers (x_+, x_-) and these notations match with the notations, (2.3), (2.4) and (2.5). In this situation, the opposite chambers (x_+, x_-) are called the **fundamental chambers** and their corresponding twin apartment$\mathcal{A}(x_+, x_-)$ is call the **fundamental apartment** (see [AB08, pages 331 and 332]).

Definition 2.3.11. [AB08, Definition 6.84] With the notations as above. The twin BN-pair (B_+, B_-, N) is called **saturated** if it satisfies

$$T = B_+ \cap B_-$$

i.e., if $N = \mathrm{Stab}_G(\mathcal{A}(x_+, x_-))$.

2.4 Twin Buildings and Twin Root Data

In this section we construct a Moufang twin building with a given twin root datum. Then we define the notion of parabolic subgroups. The main reference is [AB08, Chapter 8] and we follow [Cap05, Section 1.4] for the details.

2.4.1 Twin Buildings of Twin Root Data

Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ be a twin root datum of type (W, S). Define

$$H := \bigcap_{\alpha\in\Phi} N_G(U_\alpha), \tag{2.13}$$

and

$$N := \langle H, \mu(u) \ for \ u \in U_\alpha \backslash \{1\}\rangle, \tag{2.14}$$

where $\mu(u)$ are as given in (TRD2) in Definition 2.1.1. For each $\epsilon = \pm$, set

$$U_\epsilon := \langle U_\alpha \mid \alpha \in \Phi_\epsilon\rangle, \tag{2.15}$$

and

$$B_\epsilon := H.U_\epsilon. \tag{2.16}$$

Proposition 2.4.1. *[AB08, Proposition 8.54(1)] (G, B_ϵ, N, S) is a Tits system with Weyl group W.*

This implies that we have the following Bruhat decompositions of G:

$$G = \coprod_{w\in W} B_\pm w B_\pm. \tag{2.17}$$

Theorem 2.4.2. *[AB08, Theorem 8.80] (G, B_+, B_-, N, S) is a saturated twin Tits system with Weyl group W.*

Therefore, by Proposition 2.3.10 G also has a Birkhoff decomposition:

$$G = \coprod_{w\in W} B_\epsilon w B_{-\epsilon} \tag{2.18}$$

for each $\epsilon = \pm$. Furthermore, as we saw in Subsection 2.3.2, G acts strongly transitively on $\mathcal{C}(G, B_+, B_-)$.

Theorem 2.4.3. *[AB08, Theorem 8.81] $\mathcal{C}(G, B_+, B_-)$ is a Moufang twin building.*

Now we give a well-known result which wraps up almost all the information we have presented so far.

Theorem 2.4.4. *[AB08, Theorem 6.58]*

(1) Let G be a group that acts strongly transitively on a thick twin building $\mathcal{C} = (\mathcal{C}_+, \mathcal{C}_-)$ of type (W, S). Let $(x_+, x_-) \in \mathcal{C}_+ \times \mathcal{C}_-$ be a pair of opposite chambers, and let $\mathcal{A} = \mathcal{A}(x_+, x_-)$ be the associated twin apartment. Then if we denote by B_+, B_- and N the stabilizers of x_+, x_- and $\mathcal{A}$ in G respectively, the triple (B_+, B_-, N) is a saturated BN*-pair in G with Weyl group W. The twin building $\mathcal{C}(G, B_+, B_-)$ is canonically isomorphic to $\mathcal{C}$.*

(2) *Let (G, B_+, B_-, N, S) be a twin Tits system with Weyl group W. Then $\mathcal{C}(G, B_+, B_-)$ is a thick twin building of type (W, S) on which G acts strongly transitively. If we set $x_+ := B_+$, $x_- := B_-$, and $\mathcal{A} := \mathcal{A}(x_+, x_-)$ then we recover $B_\pm$ as the stabilizer of $x_\pm$ in G. Moreover, $N(B_+, B_-)$ is the stabilizer of $\mathcal{A}$ in G and hence we recover N as this stabilizer if the twin* BN*-pair (B_+, B_-, N) is saturated.*

The next proposition depicts how one can have local to global arguments by means of building structures.

Proposition 2.4.5. *Let $\mathcal{C}$ be a building obtained from a* BN*-pair and G an abstract group acting on $\mathcal{C}$. The group G acts transitively on $\mathcal{C}$ if and only if there exists a chamber $c \in \mathcal{C}$ such that for each panel σ of $\mathcal{C}$ containing c the stabilizer of σ in G acts transitively on σ.*

Proof. Follows from [MGH09, Corollary 3.11]. □

We end this subsection by the main result in [AM97] which shows how it is possible to describe a group as an amalgamation of its subgroups by observing its action on a suitable twin building.

Theorem 2.4.6. *[AM97] Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta_*)$ be a thick twin building of type (W, S) where S is finite and the Dynkin diagram corresponding to W is two-spherical. In addition, assume that the following condition holds for each rank two residue $\mathcal{R}$ of $\mathcal{B}$:*

($\star$) *The residue $\mathcal{R}$ is not isomorphic to one of the buildings associated to the groups $B_2(2)$, $G_2(2)$, $G_2(3)$ or ${}^2F_4(2)$.*

Let G be a group acting strongly transitively on $\mathcal{B}$. For each subset $S' \subset S$, denote by $G_{S'}$ the intersection of the stabilizers of the two residues of type S' which contains the pair of fundamental opposite chambers respectively.
Then G is the universal enveloping group of the subgroups $\{G_{S'}\}_{\{S' \subset S\}}$ for $S' \subset S$ with cardinality at most two.

2.4.2 Parabolic Subgroups

In this subsection we introduce the parabolic subgroups of a given group G with a twin root datum $\mathcal{Z} = (G, (U_\alpha)_{\alpha\in\Phi})$ of type (W, S). Then we collect some basic facts concerning these subgroups. The notations are as in Subsection 2.4.1. The main reference is [AB08, Chapter 6] and we follow [Cap05, Section 1.4.2].

Definition 2.4.7. A subgroup P of G containing B_ϵ is called a **standard parabolic subgroup** of sign ϵ where $\epsilon = \pm$. The conjugates of P are called **parabolic subgroups** of sign ϵ.

There exists a subset $J \subset S$ for any standard parabolic subgroup P of sign ϵ such that it has the following Bruhat decomposition (see [AB08, Sections 6.2.3, 6.2.4]):

$$P = \coprod_{w\in W_J} B_\pm w B_\pm. \tag{2.19}$$

We call the set J, the **type** of P. If J is spherical, then P is said to be **of finite type** or **spherical**. Two parabolic subgroups P_+ and P_- of opposite signs are called **opposite** if

there exits a $g \in G$ such that gP_+g^{-1} and gP_-g^{-1} are two standard parabolic subgroups of type J of opposite signs. A **minimal** parabolic subgroup (i.e. of type $\emptyset$) is called a **Borel subgroup**. For instance, $B_\pm$ are Borel subgroups.
Note that with regard to Section 2.4.1 one can associate a twin building to the twin root datum $\mathcal{Z}$. We denote this twin building by $\mathcal{C}(G, B_+, B_-)$. In this setting, the subgroups $B_\pm$ fix unique chambers $c_\pm$ in $\mathcal{C}_\pm$ which are called **standard** (or **fundamental**) with respect to $\mathcal{Z}$ and we have $B_\pm = \mathrm{Stab}_G(c_\pm)$. Moreover, c_+ and c_- are opposite chambers in $\mathcal{C}$. The unique twin apartment $\mathcal{A}(c_+, c_-)$ containing these two opposite chambers is called the **standard** (**fundamental**) twin apartment with respect to $\mathcal{Z}$. We might drop the adjective "twin" when it is clear we are working on twin buildings. Furthermore, we have

$$N = \mathrm{Stab}_G(\mathcal{A}(c_+, c_-)), \tag{2.20}$$

$$H = B_\pm \cap N = \mathrm{Fix}_G(\mathcal{A}(c_+, c_-)), \tag{2.21}$$

and for $J \subset S$,

$$P_\pm^J = \mathrm{Stab}_G(\mathrm{Res}_J(c_\pm)) \tag{2.22}$$

where $P_\pm^J$ denotes the standard parabolic subgroup of type J containing $B_\pm$. If a subgroup of G is the stabilizer of a pair of opposite chambers in $\mathcal{C}(G, B_+, B_-)$ then it is called **diagonalizable**. Now since the action of G on $\mathcal{C}(G, B_+, B_-)$ is strongly transitive, its action on pairs of opposite chambers in $\mathcal{C}(G, B_+, B_-)$ is transitive therefore H is a maximal diagonalizable subgroup of G and all other such subgroups are conjugate to H.
Let $\Phi(\mathcal{A}(c_+, c_-))$ be the set of all twin roots contained in $\mathcal{A}(c_+, c_-)$. There exists a canonical one-to-one correspondence $\xi : \Phi \to \Phi(\mathcal{A}(c_+, c_-))$ such that for all $\alpha \in \Phi$, the root group U_α fixes point-wise each panel σ with $|\sigma \cap \xi(\alpha)| = 2$ and acts regularly on $\sigma \backslash \xi(\alpha)$ for each panel σ with $|\sigma \cap \xi(\alpha)| = 1$. Hence, U_α coincides with the root group $U_{\xi(\alpha)}$ and if (W, S) has no direct factor of type A_1 then, by Theorem 2.4.3, the twin building $\mathcal{C}(G, B_+, B_-)$ is Moufang.
Conversely, let $\mathcal{C}$ be a Moufang twin building of type (W, S). Let $x_\pm \in \mathcal{C}_\pm$ be two opposite chambers and $\mathcal{A}(x_+, x_-)$ the unique apartment containing them. Let $\Phi(\mathcal{A}(x_+, x_-))$ be the set of all twin roots of $\mathcal{C}$ in $\mathcal{A}(x_+, x_-)$. Let $G(\mathcal{C})$ be the group generated by all root subgroups U_α such that $\alpha \in \Phi(\mathcal{A}(x_+, x_-))$. Then $(G(\mathcal{C}), (U_\alpha)_{\alpha \in \Phi(\mathcal{A}(x_+, x_-))})$ is a twin root datum of type (W, S) and the associated twin building is isomorphic to $\mathcal{C}$ (see [AB08, Chapter 8]).
Lemma 2.4.8. *[Cap05, Lemma 1.7] Let $\mathcal{Z} = (G, (U_\alpha)_{\alpha \in \Phi})$ be a twin root datum of type (W, S) and $H := \bigcap_{\alpha \in \Phi} N(U_\alpha)$. Then the* ***kernel*** *of the action of G on the associated twin building to $\mathcal{Z}$ coincides with $\bigcap_{\alpha \in \Phi} C_G(U_\alpha)$. In particular, if G is generated by the U_α's or if H is abelian, then this kernel is the center of G.*

CHAPTER

THREE

CAT(0) GEOMETRY

In this chapter we introduce CAT(0) polyhedral complexes and basic concepts in CAT(0) geometry along with some recent results that will be used in the next chapters, especially in Chapter 6. Then we observe that buildings are CAT(0). Throughout this Chapter we assume that (X, d) denotes a topological space X with a metric d which is compatible with the topology on X. The main references are [Dav08], [BH99] and [Cap05, Chapter 2].

3.1 CAT(0) Spaces

A **geodesic path** is a map $\gamma : I \to X$ which is an isometry onto its image where I is a closed interval in $\mathbb{R}$ with the natural metric. A **geodesic segment** is the image of a geodesic path. For a geodesic path $\gamma : [a, b] \to X$ where $\gamma(a) = p \in X$ and $\gamma(b) = q \in X$, we denote the geodesic segment connecting p to q in X by $[p, q]_\gamma$ or by I_γ in the case the end points are not specified. In the same way, a **geodesic line** is the image of a geodesic $\gamma : \mathbb{R} \to X$. The metric space (X, d) is called **geodesic** if every two points on X are connected by a geodesic segment. We call a geodesic space a **uniquely geodesic space** if the connecting geodesic path between every two points is unique. Moreover, a metric space is called **locally (uniquely) geodesic** if every point in the space is contained in a geodesic (uniquely geodesic) open neighborhood.

A **geodesic triangle** Δ in X consists of three points $p, q, r \in X$, which are called **vertices**, and a choice of three geodesic segments $[p, q]_\alpha$, $[q, r]_\beta$, $[r, p]_\gamma$ joining them which are called **edges** (or **sides**). Such a geodesic triangle will be denoted by $\Delta([p, q]_\alpha, [q, r]_\beta, [r, p]_\gamma)$. If a point $x \in X$ lies on the union of the sides of Δ, we write $x \in \Delta$.

Let $(\mathbb{E}^2, d_\mathbb{E})$ denote the Euclidean plane. Now for a triangle $\Delta([p, q]_\alpha, [q, r]_\beta, [r, p]_\gamma)$ in (X, d) let $\bar{\Delta}([\bar{p}, \bar{q}]_{\bar{\alpha}}, [\bar{q}, \bar{r}]_{\bar{\beta}}, [\bar{r}, \bar{p}]_{\bar{\gamma}})$ be a geodesic triangle in $(\mathbb{E}^2, d_\mathbb{E})$ such that $d(p, q) = d_\mathbb{E}(\bar{p}, \bar{q})$, $d(q, r) = d_\mathbb{E}(\bar{q}, \bar{r})$, $d(r, p) = d_\mathbb{E}(\bar{r}, \bar{p})$. Then Δ is said to satisfy the **CAT(0) inequality** if for all $\phi, \psi \in \{\alpha, \beta, \gamma\}$ and all $x \in I_\phi, y \in I_\psi$, we have

$$d(\phi(x), \psi(y)) \leq d_\mathbb{E}(\bar{\phi}(x), \bar{\psi}(y)). \tag{3.1}$$

In other words, Δ is at most as thick as its **comparison** Euclidean triangle $\bar{\Delta}$.
Definition 3.1.1. A **CAT(0) space** is a geodesic metric space in which all of geodesic triangles satisfy the CAT(0) inequality (3.1).

Note that by the CAT(0) inequality, a geodesic CAT(0) space is indeed a uniquely geodesic space. A metric space is called **locally** CAT(0) if for each point in this space there exists an open CAT(0) neighborhood containing the point.

Let (X, d) be a metric space and let Γ be a group of isometries of X. Then for $\gamma \in \Gamma$, the **displacement function** of γ is the function $d_\gamma : X \to \mathbb{R}_+$ defined by $d_\gamma(x) = d(\gamma(x), x)$. The **translation length** of γ is the number $|\gamma| := \inf\{d_\gamma(x) \mid x \in X\}$. The set of points where d_γ attains this infimum will be denoted by $\mathbf{Min}(\gamma)$ and more generally, we define $\mathbf{Min}(\Gamma) := \bigcap_{\gamma\in\Gamma} \mathbf{Min}(\gamma)$. Note that $\mathbf{Min}(\gamma)$ is γ-invariant and $\mathbf{Min}(\Gamma)$ is Γ-invariant. Also, if X is a CAT(0) space, then $\mathbf{Min}(\gamma)$ is a closed convex set and hence CAT(0) (see [BH99, Proposition 6.2]). In addition, we denote the **set of fixed points** of Γ on X by $\mathrm{Fix}(\Gamma, X)$. If $\mathrm{Fix}(\Gamma, X)$ is non-empty then it is contractible (see [Dav08, Proposition I.2.12]).
Furthermore, an isometry γ is called **semisimple** if $\mathbf{Min}(\gamma)$ is non-empty. An action of a group by isometries of X is called **semisimple** if all of its elements are semisimple (see [BH99, Definition II.6.1]).
Definition 3.1.2. [BH99, Definition II.6.3] Let (X, d) be a metric space. An isometry γ of X is called

1. **elliptic** if it has a fixed point,
2. **hyperbolic** (or **axial**) if d_γ attains a strictly positive minimum,
3. **parabolic** if d_γ does **not** attain its minimum, i.e., $\mathbf{Min}(\gamma) = \emptyset$.

Lemma 3.1.3. *Let X be a complete* CAT(0) *space and γ be an isometry of X. Then we have the following:*

(i) Given a closed convex γ-invariant subset C of X then, γ is elliptic (resp. hyperbolic) if and only if $\gamma|C$ is elliptic (resp. hyperbolic) and one has $\mathbf{Min}(\gamma|C) = \mathbf{Min}(\gamma) \cap C$.

(ii) If γ^n is elliptic (resp. hyperbolic) for some $n \in \mathbb{Z}_{>0}$ then, so is γ.

(iii) γ is hyperbolic if and only if there exists a geodesic line $c : \mathbb{R} \to X$ which is translated non-trivially by γ, namely $\gamma \cdot c(t) = c(t + a)$ for some $a > 0$. The set $c(\mathbb{R})$ is called an axis of γ. For any such axis, the number a is actually equal to $|\gamma|$.

(iv) γ is hyperbolic if and only if $\mathbf{Min}(\gamma)$ is a disjoint union of geodesic lines (axes) along each of which γ acts by translation.

(v) If γ is hyperbolic then γ^n is hyperbolic for any $n \in \mathbb{N}$ and we have $|\gamma^n| = n|\gamma|$. Conversely, if γ^m is hyperbolic for some $m \in \mathbb{N}$ then γ is hyperbolic.

Proof. For (i) see [BH99, Proposition II.6.2].
(ii) follows from [BH99, Proposition II.6.7].
(iii) is [BH99, Theorem II.6.8(1)] and (iv) follows from [BH99, Theorem II.6.8(3) and (4)].

(v): If γ is hyperbolic then by (iii) we know that there exists a geodesic line $c : \mathbb{R} \to X$ which is translated non-trivially by γ, namely $\gamma \cdot c(t) = c(t + |\gamma|)$. Therefore, the action of γ^n on c is $\gamma^n \cdot c(t) = c(t + n|\gamma|)$ which, in view of (iii), implies that γ^n is hyperbolic too. And in particular, we have $|\gamma^n| = n|\gamma|$. The inverse follows from [BH99, Theorem II.6.8(2)]. □

Next we present a well-known result to experts which was originally proved for Euclidean buildings by Bruhat and Tits (see [BT72]) then the idea of the proof was applied to CAT(0) spaces by Davis (see also [BH99, Chapter II.2]).

Theorem 3.1.4 (**Bruhat-Tits Fixed Point Theorem**). *[Dav08, Theorem I.2.11] Let Γ be a group of isometries of a* CAT(0) *space X. If Γ has a bounded orbit on X, then Γ has a fixed point on X.*

Definition 3.1.5. Let G be a group and $\{U_i\}_{i\in I}$ be a family of subgroups of G indexed by a set I. G is said to be **boundedly generated** by the family $\{U_i\}_{i\in I}$ if there exists a constant element $n \in \mathbb{Z}_{>0}$ such that every element $g \in G$ can be written as a product $g = g_1....g_{n_g}$ with $g_i \in \bigcup_{i\in I} U_i$ and $n_g \leq n$.

Theorem 3.1.6. *[Cap05, Corollary 2.5] Let (X, d) be a complete $CAT(0)$ space, let Γ be a group acting on X by isometries. Suppose that Γ is boundedly generated by a finite family of subgroups $\{U_i\}_{i\in I}$ each of which has a fixed point in X. Then Γ has a global fixed point.*

3.2 CAT(0) Polyhedral Complexes

In this section we introduce the **Modeling spaces** $\mathrm{M}^{\mathrm{n}}_{\kappa}$. Then we state some well-known results which will be used later on in this thesis. The main references are [BH99, Chapters I.2, I.7, II.5 and II.6] and [Cap05, Chapter 2].

Definition 3.2.1. [BH99, Definition I.2.10] Given a real number κ, we denote by $\mathrm{M}^{\mathrm{n}}_{\kappa}$ the following metric spaces:

1. if $\kappa = 0$ then, $\mathrm{M}^{\mathrm{n}}_{0}$ is the Euclidean space $\mathbb{E}^n$,
2. if $\kappa > 0$ then, $\mathrm{M}^{\mathrm{n}}_{\kappa}$ is obtained from the n-dimensional sphere $\mathbb{S}^n$ by multiplying the distance function by the constant $1/\sqrt{\kappa}$,
3. if $\kappa < 0$ then, $\mathrm{M}^{\mathrm{n}}_{\kappa}$ is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by $1/\sqrt{-\kappa}$.

For $\kappa \in \mathbb{R}$, a **convex** M_{κ}**-polyhedral cell** $C \in \mathrm{M}^{\mathrm{n}}_{\kappa}$ is the convex hull of a finite set of point $P \subset \mathrm{M}^{\mathrm{n}}_{\kappa}$; if $\kappa > 0$ then, P (hence C) is required to lie in an open ball of radius $D_\kappa/2$. The **dimension** of C is the dimension of the smallest m-plane containing it. The **interior** of C is the interior of C as a subset of this m-plane (see [BH99, Definition I.7.34]).
Let H be a hyperplane in $\mathrm{M}^{\mathrm{n}}_{\kappa}$. If C lies in one of the closed half-spaces bounded by H, and if $H \cap C \neq \emptyset$ then, $F := H \cap C$ is called a **face** of C; if $F \neq C$ then it is called a **proper face**. The **dimension of face** F is the dimension of the smallest m-plane containing it. The **interior** of F is the interior of F in this plane. The 0-dimensional faces of C are called its **vertices**. The **support** of $x \in C$, denoted by $\mathrm{Supp}(x)$, is the unique face containing x in its interior (see [BH99, page 113]).

Definition 3.2.2. Let $\{C_\lambda\}_{\lambda\in\Lambda}$ be a family of convex M_κ-polyhedral cells and let $X = \bigcup_{\lambda\in\Lambda}(C_\lambda \times \{\lambda\})$ denote their disjoint union. Let $\sim$ be an equivalence relation on X and define $K := X/\sim$. Let $p : X \to K$ be the natural projection and set $p_\lambda : C_\lambda \to K$ by $p_\lambda(x) := p(x,\lambda)$.
K is called an M_κ-**polyhedral complex** if:

(1) for every $\lambda \in \Lambda$, the restriction of p_λ to the interior of each face of C_λ is injective;

(2) for all $\lambda_1, \lambda_2 \in \Lambda$ and $x_1 \in C_{\lambda_1},\ x_2 \in C_{\lambda_2}$, if $p_{\lambda_1}(x_1) = p_{\lambda_2}(x_2)$, then there is an isometry $h : \mathrm{Supp}(x_1) \to \mathrm{Supp}(x_2)$ such that $p_{\lambda_1}(y) = p_{\lambda_2}(h(y))$ for all $y \in \mathrm{Supp}(x_1)$.

In other words, an M_κ-polyhedral complex is obtained by gluing a disjoint union of a family of convex polyhedral cells in $\mathrm{M}_\kappa^\mathrm{n}$ along their isometric faces. The set of isometry classes of the faces of the cells C_λ is denoted by $\mathrm{Shapes}(K)$ (see [BH99, Definition I.7.37] and [Cap05, Section 2.1.3]).
A subset $C \subset K$ is called an n-**cell** if it is the image $p_\lambda(F)$ of some n-dimensional face $F \subset C_\lambda$. Moreover, an isometry of K is called a **cellular isometry** if it preserves the cells of K.
Now let M_κ be a modeling space. A **simplex** σ in M_κ is the convex hull of a set $\{p_0, ..., p_n\}$ of affinely independent points (at p_0), i.e., σ is the set of all convex combinations of the p_i. The set $\{p_0, ..., p_n\}$ is called the set of **vertices** of σ and is denoted by $\mathrm{Vert}(\sigma)$. It follows that a simplex σ is a convex polyhedral of dimension n where $n = |\,\mathrm{Vert}(\sigma)| - 1$. Furthermore, if every cell of a polyhedral complex Δ is a simplex then, Δ is called a **simplicial complex**. A **cellulation** of a space X is a homeomorphism f from a polyhedral complex Δ onto X. f is a **triangulation** if Δ is a simplicial complex. The **standard** n-**simplex** Δ^n is the convex hull of the standard basis $\{e_1, ..., e_{n+1}\}$ in $\mathbb{R}^{n+1}$. More generally, for a set S, let $\mathbb{R}^S$ denote the real vector space of all finitely supported functions $S \to \mathbb{R}$ topologized as the direct limit of its finite-dimensional subspaces, i.e., a subset of $\mathbb{R}^S$ is closed if and only if its intersection with each finite-dimensional subspace is closed. Now, for each $s \in S$, let e_s be the characteristic function of $\{s\}$. Then $\{e_s\}_{s\in S}$ is the standard basis of $\mathbb{R}^S$. The **standard simplex** on S, denoted by Δ^S, is the convex hull of the standard basis in $\mathbb{R}^S$. In other words, it is the set of all convex combinations of the e_s (see [Dav08, Appendix A.1]).
Theorem 3.2.3. *[BH99, Theorem I.7.50] Let K be an M_κ-polyhedral complex. If* $\mathrm{Shapes}(K)$ *is finite, then K is a complete geodesic metric space.*
Theorem 3.2.4. *Let K be an M_κ-polyhedral complex with* $\mathrm{Shapes}(K)$ *finite and $\kappa \leq 0$. Then the following conditions are equivalent:*

(1) K is a $\mathrm{CAT}(\kappa)$ *space;*

(2) K is topologically simply connected and locally uniquely geodesic.

Proof. See [BH99, Theorem II.5.4] and [BH99, Theorem II.5.5] □

In the following we present a well-known theorem by M. Bridson and A. Haefliger.
Corollary 3.2.5. *A locally* $\mathrm{CAT}(0)$ *topologically simply connected* M_κ*-polyhedral complex K with* $\mathrm{Shapes}(K)$ *finite, endowed with intrinsic metric, is a complete geodesic* $\mathrm{CAT}(0)$ *space.*

Proof. First note that by Theorem 3.2.3, K is a complete geodesic space.
Now since a locally CAT(0) space is locally uniquely geodesic (see Definition 3.1.1), by Theorem 3.2.4, K is a CAT(0) space as well. $\square$

Proposition 3.2.6. *[Bri99, Theorem A and Proposition] Let K be a connected M_κ-polyhedral complex with* $\mathrm{Shapes}(K)$ *finite and let Γ be a group acting on K by cellular isometries. Then the action of Γ is semisimple and the set $\{|\gamma| : \gamma \in \Gamma\}$ is a discrete subset of $\mathbb{R}$.*

3.3 Buildings are CAT(0)

In this section we associate a geometric realization to any building of finite rank which forms a CAT(0) polyhedral complex. The main references for the present section are [Dav08] and [BH99].

3.3.1 Abstract Simplicial Complex

A **poset** is a partially ordered set. Given a poset $\mathcal{P}$ and an element $p \in \mathcal{P}$, let

$$\mathcal{P}_{\leq p} := \{x \in \mathcal{P} \mid x \leq p\}. \tag{3.2}$$

In the same manner we define $\mathcal{P}_{\geq} p$, $\mathcal{P}_{<p}$ and $\mathcal{P}_{>p}$.
For $p, q \in \mathcal{P}$, define $\mathcal{P}_{[p,q]}$ by

$$\mathcal{P}_{[p,q]} := \{x \in \mathcal{P} \mid q \leq x \leq p\}. \tag{3.3}$$

which is called the **interval** from p to q. The **opposite** (or **dual**) poset to $\mathcal{P}$ is the poset $\mathcal{P}^{op}$ with the same underlying set and reversed order (see [Dav08, Appendix A.2]).
Definition 3.3.1. An **abstract simplicial complex** consists of a set S (the **vertex set**) and a collection $\mathcal{S}$ of finite subsets of S such that

(i) for each $s \in S$, $\{s\} \in \mathcal{S}$; and

(ii) if $T \in \mathcal{S}$ and if $T' \subset T$ then $T' \in \mathcal{S}$.

An abstract simplicial complex $\mathcal{S}$ is a poset ordered by inclusion. Condition (ii) in Definition 3.3.1 means that if $T \in \mathcal{S}$, then $\mathcal{S}_{\leq T}$ is the power set of T. An element of $\mathcal{S}$ is called a **simplex** of $\mathcal{S}$. If T is a simplex of $\mathcal{S}$ and $T' \leq T$, then T' is a **face** of T.
The **dimension** of a simplex T is $|T| - 1$ and denoted by $\dim T$. A simplex of dimension k is called a k-simplex. A 0-simplex is also called a **vertex**. A 1-simplex is an **edge**.
A subset $\mathcal{S}'$ of an abstract simplicial complex $\mathcal{S}$ is a **subcomplex** of $\mathcal{S}$ if it is also an abstract simplicial complex. A subcomplex $\mathcal{S}'$ of $\mathcal{S}$ is a **full subcomplex** if $T \in \mathcal{S}$ and $T \subset \mathrm{Vert}(\mathcal{S}')$ implies that $T \in \mathcal{S}'$.
The k-**skeleton** of an abstract simplicial complex $\mathcal{S}$ is the subcomplex $\mathcal{S}^k$ consisting of all simplices of dimension $\leq k$ (see [Dav08, Appendix A.2]).

3.3.2 Geometric Realization

For an abstract simplicial complex $\mathcal{S}$ with vertex set S, let Δ^S denote the standard simplex on S defined in Section 3.2. For each non-empty finite subset T of S, σ_T denotes the face of Δ^S spanned by T. Define a subcomplex $\mathrm{Geom}(\mathcal{S})$ of Δ^S by

$$\sigma_T \in \mathrm{Geom}(\mathcal{S}) \text{ if and only if } T \in \mathcal{S}_{>\emptyset}. \tag{3.4}$$

The convex cell complex $\mathrm{Geom}(\mathcal{S})$ is the **standard geometric realization** of $\mathcal{S}$ (see [Dav08, page 408]).
Suppose Δ is a simplicial complex. We associate an abstract simplicial complex $\mathcal{S}(\Delta)$ to Δ as follows.
For every $T \subset S := \mathrm{Vert}(\mathcal{S})$

$$T \in \mathcal{S}(\Delta) \text{ if and only if } \langle T \rangle =: \sigma_T \in \Delta, \tag{3.5}$$

where $\langle T \rangle$ denotes the convex hull of T.
Conversely, it can be shown that all abstract simplicial complexes arise from a polyhedral complex. That is, any abstract simplicial complex $\mathcal{S}$ is associated to a (geometric) simplicial complex Δ such that $\mathcal{S} = \mathcal{S}(\Delta)$. Such Δ is called a **geometric realization** of $\mathcal{S}$ (see [Dav08, pages 408,409]).
An **incident relation** on a set S is a symmetric and reflective relation. Now suppose S is a set with an incident relation on it. A **flag** in S is a subset of a pairwise incident elements. Let $\mathrm{Flag}(S)$ denote the set of all finite flags in S, partially ordered by inclusion. This forms an abstract simplicial complex with vertex set S. The **geometric realization** of a poset $\mathcal{P}$ is the geometric realization of the abstract simplicial complex $\mathrm{Flag}(\mathcal{P})$ denoted by $|\mathcal{P}| := \mathrm{Geom}(\mathrm{Flag}(\mathcal{P}))$ ([Dav08, Definition A.3.2]).

3.3.3 Geometric Realization of Buildings

Let $\mathcal{B}$ be a building of type (W, S). In this subsection we present a geometric realization for $\mathcal{B}$. To do this first we construct an abstract simplicial complex corresponding to $\mathcal{B}$. Then, as we have seen in previous subsection, one can construct a polyhedral complex corresponding to any abstract simplicial complex. This will imply that for any building $\mathcal{B}$ of finite rank k (i.e., $k = |S| < \infty$) there exists a k-simplicial complex $|\mathcal{B}|$ such that if a group that acts on $\mathcal{B}$ by type preserving automorphisms the action is by cellular isometries on $|\mathcal{B}|$ (see [Cap05, Subsection 2.1.4]).
Let $\mathcal{C}$ denote the set of all spherical residues in $\mathcal{B}$. With respect to the inclusion order, $\mathcal{C}$ is a poset. Therefore, $\mathrm{Flag}(\mathcal{C})$ forms an abstract simplicial complex. Define the **geometric realization** of $\mathcal{B}$ to be the geometric realization of $\mathrm{Flag}(\mathcal{C})$ which is denoted by $\mathrm{Geom}(\mathcal{B})$ (also denoted by $|\mathcal{B}|$). This defines a (simplicial) polyhedral complex so that for each apartment $\mathcal{A}$ in $\mathcal{B}$, the (simplicial) polyhedral subcomplex $\mathrm{Geom}(\mathcal{A})$ is isomorphic to the (simplicial) polyhedral complex corresponding to $\mathcal{A}$ as a thin building (see [Dav08, Section 18.3]).
Theorem 3.3.2. *Let $\mathcal{B}$ be a building. The polyhedral complex* $\mathrm{Geom}(\mathcal{B})$ *is a topologically simply connected complete uniquely geodesic* $\mathrm{CAT}(0)$ *space.*

Proof. This follows from [Dav08, Theorem 18.3.1, Lemma 18.3.4 and Corollary 18.3.6]. □

CHAPTER

FOUR

KAC-MOODY THEORY

In this chapter we define split Kac-Moody groups which are one of the main ingredients of this thesis. Then we introduce an adjoint representation of such split Kac-Moody groups along with some well-known results in Kac-Moody theory that will be of some use later on this thesis. At the end, we present two topologies and their basic properties on certain split Kac-Moody groups, namely, the Zariski and the Kac-Peterson topologies. This chapter contains no original results. The main references are [Rém02], [Cap05] and [HKM13].

4.1 Kac-Moody Groups

In this section we present split Kac-Moody groups as images of split Tits functors as well as some useful facts about them.

4.1.1 Tits Functor

In order to define a split Kac-Moody group via a split Tits functor one needs to introduce the Kac-Moody root data (systems). This subsection follows closely [HKM13, Section 7.1] and [Cap05, Section 1.1].

A **generalized Cartan matrix** is an integral matrix $A = (A_{ij})_{i,j\in I}$ where $I = \{1, 2, ..., n\}$ for some $n \in \mathbb{N}$ subjected to the following conditions:

$$A_{ii} = 2,\ A_{ij} \leq 0 \text{ for } i \neq j, \text{ and } A_{ij} = 0 \text{ if and only if } A_{ji} = 0. \tag{4.1}$$

A generalized Cartan matrix $A = (A_{ij})_{i,j\in I}$ is called **symmetrizable** if there exists an invertible diagonal matrix D and a symmetric matrix B such that $A = DB$.
The **Weyl group** of a generalized Cartan matrix $A = (A_{ij})_{i,j\in I}$ is the **Coxeter group**

$$W := \langle s_i \mid i \in I \text{ and } (s_i s_j)^{M_{ij}} = 1 \text{ whenever } M_{ij} < \infty \rangle$$

where $M_{ij} = 2, 3, 4, 6$ or ∞ according to whether $A_{ij}A_{ji} = 0, 1, 2, 3$ or > 3.
The matrix $M(A) := [M_{ij}]_{i,j\in I}$ obtained in this way is called the **Coxeter matrix associated** to A.

A generalized Cartan matrix $A = (A_{ij})_{i,j\in I}$ is called **affine** if the Dynkin diagram associated to $M(A)$ is one of the Dynkin diagrams listed on [Kac90, TABLE Aff 1, page 54]. Any such affine Cartan matrix can be seen as an extension of a Cartan matrix, denoted by A_{Fin} (see [Che96, page 3735]).

Example 4.1.1. For the generalized Cartan matrix $A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$, the associated Coxeter matrix is $M(A) = \begin{bmatrix} 2 & 6 \\ 6 & 2 \end{bmatrix}$ which produces the Coxeter group of type G_2.

Next we define the Kac-Moody root data (systems).

Definition 4.1.2. Let $A = (A_{ij})_{i,j\in I}$ be a generalized Cartan matrix. A quintuple $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ is called a **Kac-Moody root datum (system)** if Λ is a free $\mathbb{Z}$-module, $\{c_i\}_{i\in I} \subset \Lambda$ and $\{h_i\}_{i\in I} \subset \Lambda^\vee$ where $\Lambda^\vee$ denotes the $\mathbb{Z}$-dual of Λ with the property that for all $i, j \in I$ we have $h_i(c_j) = A_{ij}$.

Given a Kac-Moody root datum $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$, one can construct an infinite-dimensional Lie algebra which is call the **Kac-Moody algebra** associated to $\mathcal{D}$.

Definition 4.1.3. Let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a Kac-Moody root datum. The **primitive Kac-Moody algebra of type $\mathcal{D}$** over $\mathbb{C}$, which we denote by $\tilde{\mathfrak{g}}_{\mathcal{D}_\mathbb{C}}$, is the Lie $\mathbb{C}$-algebra generated by $\mathfrak{g}_{0_\mathbb{C}} := \Lambda^\vee \bigotimes_\mathbb{Z} \mathbb{C}$ and the set of generators $\{e_i, f_j\}_{i,j\in I}$ subjected to the following relations, where $h, h' \in \mathfrak{g}_{0_\mathbb{C}}$:

$$\begin{aligned}
[h, e_j] &= \langle c_i, h\rangle e_j, \\
[h, f_j] &= -\langle c_i, h\rangle f_j, \\
[h, h'] &= 0, \\
[e_i, f_i] &= -h_i \otimes 1, \\
[e_i, f_j] &= 0 \text{ for } i \neq j,
\end{aligned}$$

where

$$\begin{aligned}
\langle\ ,\ \rangle &: \Lambda \otimes_\mathbb{Z} \mathbb{C} \times \Lambda^\vee \otimes_\mathbb{Z} \mathbb{C} \to \mathbb{Z} \otimes_\mathbb{Z} \mathbb{C} \\
\langle x \otimes c, y \otimes c'\rangle &:= y(x) \otimes cc'. \qquad (4.2)
\end{aligned}$$

It turns out that among the ideals of $\tilde{\mathfrak{g}}_{\mathcal{D}_\mathbb{C}}$ intersecting $\mathfrak{g}_{0_\mathbb{C}}$ trivially, there exists a unique maximal ideal τ (see [Kac90, Theorem 1.2(e)]). The Lie algebra

$$\mathfrak{g}_{\mathcal{D}_\mathbb{C}} := \tilde{\mathfrak{g}}_{\mathcal{D}_\mathbb{C}}/\tau \qquad (4.3)$$

is called the **Kac-Moody algebra** of type $\mathcal{D}$ over $\mathbb{C}$ (see [Kac90, Section 1.3]).

By [Kac90, Theorem 9.11], when A is symmetrizable, the Kac-Moody algebra $\mathfrak{g}_{\mathcal{D}_\mathbb{C}}$ can be obtained by adding the following relations, known as the **Serre relations**, to the relations of the primitive Kac-Moody algebra $\tilde{\mathfrak{g}}_{\mathcal{D}_\mathbb{C}}$:

$$(\mathrm{ad} e_i)^{-A_{ij}+1}(e_j) = (\mathrm{ad} f_i)^{-A_{ij}+1}(f_j) = 0. \qquad (4.4)$$

Let $\mathfrak{g}_A$ denote the Kac-Moody algebra of type $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ with the property that $\Lambda^\vee$ is generated by $\{h_i\}_{i\in I}$ as a $\mathbb{Z}$-module. We call $\mathfrak{g}_A$ the **simply connected** Kac-Moody algebra of type $\mathcal{D}$.

Let R be a commutative unital ring R considered as a commutative $\mathbb{Z}$-algebra. Let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a Kac-Moody root datum and $\mathcal{T}_\Lambda$ denote the group functor on the category of commutative unital rings defined by

$$\mathcal{T}_\Lambda(R) := \mathrm{Hom}_{\mathrm{gr}}(\Lambda, R^\times) \cong \Lambda^\vee \otimes R \cong \mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[\Lambda], \mathrm{R}) \tag{4.5}$$

which is called the **split torus scheme**. Note that the first isomorphism in (4.5) is given via the following mapping (see [Mar13, Definition 5.5]):

$$h \otimes r \mapsto \{r^h : \Lambda \to R^\times : \ \lambda \mapsto r^{h(\lambda)}\}. \tag{4.6}$$

One can associate to $\mathcal{D}$ a triple $\mathcal{F} = (\mathcal{G}_\mathcal{D}, \{\phi_i\}_{i\in I}, \eta)$, where:

- $\mathcal{G}_\mathcal{D}$ is a group functor on the category of commutative unital rings,
- $\{\phi_i\}_{i\in I}$ is a collection of morphism of functors $\phi_i : \mathrm{SL}_2 \to \mathcal{G}_\mathcal{D}$ and
- $\eta : \mathcal{T}_\Lambda \to \mathcal{G}_\mathcal{D}$ is a morphism of functors,

such that the following conditions hold:

(KMG1) If $\Bbbk$ is a field, then $\mathcal{G}_\mathcal{D}(\Bbbk)$ is generated by the images $\{\phi_i(\mathrm{SL}_2(\Bbbk))\}_{i\in I}$ and $\eta(\mathcal{T}_\Lambda(\Bbbk))$.

(KMG2) For any commutative unital ring R, the homomorphism $\eta(R) : \mathcal{T}_\Lambda(R) \to \mathcal{G}_\mathcal{D}(R)$ is injective.

(KMG3) Given a commutative unital ring R, $i \in I$ and $r \in R^\times$, we have

$$\phi_i \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} = \eta \left(\lambda \mapsto r^{h_i(\lambda)} \right).$$

(KMG4) If $\iota : R \to \Bbbk$ is an injective homomorphism from a commutative unital ring R to a field $\Bbbk$, then $\mathcal{G}_\mathcal{D}(\iota) : \mathcal{G}_\mathcal{D}(R) \to \mathcal{G}_\mathcal{D}(\Bbbk)$ is an injective homomorphism as well.

(KMG5) There exists a homomorphism $\mathbf{Ad} : \mathcal{G}_\mathcal{D}(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{g}_A)$ with the kernel contained in $\mathcal{T}_\Lambda(\mathbb{C})$ such that, for $c \in \mathbb{C}$

$$\mathbf{Ad}\left(\phi_i \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\right) = \exp \mathrm{ad}\, ce_i$$

$$\mathbf{Ad}\left(\phi_i \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right) = \exp \mathrm{ad} -cf_i$$

and for $t \in \mathcal{T}_\Lambda(\mathbb{C})$,

$$\mathbf{Ad}(\eta(t))(e_i) = t(c_i).e_i, \qquad \mathbf{Ad}(\eta(t))(f_i) = t(-c_i).f_i.$$

The group functor $\mathcal{G}_\mathcal{D}$ is called a **split Tits functor** of type $\mathcal{D}$ and of **basis** $\mathcal{F}$. Moreover, for every commutative unital ring R, the group $\mathcal{G}_\mathcal{D}(R)$ is called a **split Kac-Moody group** of type $\mathcal{D}$ over R. $|I|$ is called the **rank** of $\mathcal{G}_\mathcal{D}$.
A Kac-Moody root datum $\mathcal{D}$ is **centered** provided for every field $\Bbbk$, the split Kac-Moody group $\mathcal{G}_\mathcal{D}(\Bbbk)$ is generated by the images of the split Tits functor basis $\{\phi_i\}_{i\in I}$. For a parallel but more constructive definition of split Tits functors see [Mar13, Definition 5.8]. Note that split Tits functor definition presented in [Mar13, Definition 5.8] coincides with our definition over large enough fields by its uniqueness property presented in [Tit87].

Next we associate a twin root datum to a split Kac-Moody group over a field which in view of Subsection 2.4.1 implies that one can associate a Moufang twin building to any split Kac-Moody group over a field on which the split Kac-Moody group acts strongly transitively.

Let $\Bbbk$ be a field, $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ a Kac-Moody root datum and $\mathcal{G}_\mathcal{D}(\mathbb{K})$ the associated split Kac-Moody group of type $\mathcal{D}$ over $\Bbbk$. Denote by $M(A)$ the associated Coxeter matrix of type (W, S) and $\Phi^\circ = \{\alpha_i \mid i \in I\}$ a set of simple roots such that the reflection corresponding to α_i is $s_i \in S$. Define the set of **real roots** $\Phi^{re} := W\Phi^\circ$. For $i \in I$, let U_{α_i} and $U_{-\alpha_i}$ be the respective images of the subgroups of strictly upper, respectively strictly lower triangular matrices of the matrix group $\mathrm{SL}_2(\Bbbk)$ under the map ϕ_i and let T denote the image $\eta(\mathcal{T}_\Lambda(\Bbbk))$ in $\mathcal{G}_\mathcal{D}(\mathbb{K})$. Then $T = \bigcap_{\alpha\in\Phi^{re}} N_{\mathcal{G}_\mathcal{D}(\mathbb{K})}(U_\alpha)$, $W \cong N_{\mathcal{G}_\mathcal{D}(\mathbb{K})}(T)/T$ and $\mathcal{Z}_\mathcal{D} := (\mathcal{G}_\mathcal{D}(\mathbb{K}), \{U_{\alpha\in\Phi^{re}}\}, T)$ is a twin root datum. Moreover, the concept of being centered here is compatible with the one in Definition 2.1.1 (see [HKM13, Proposition 7.2]).

In the sequel, let $\alpha, \beta \in \Phi^{re}$ be a pair of distinct real roots. We call $G_\alpha := \langle U_{\pm\alpha}\rangle$ and $G_{\alpha\beta} := \langle U_{\pm\alpha}, U_{\pm\beta}\rangle$, the **rank one** and **rank two subgroups** of $\mathcal{G}_\mathcal{D}(\mathbb{K})$ respectively. In particular, the subgroups $G_{\alpha_i} := \langle U_{\pm\alpha_i}\rangle$ and $G_{\alpha_i\alpha_j} := \langle U_{\pm\alpha_i}, U_{\pm\alpha_j}\rangle$ are called the **fundamental** rank one and rank two subgroups of $\mathcal{G}_\mathcal{D}(\mathbb{K})$ respectively.
Moreover, a split Tits functor $\mathcal{G}_\mathcal{D}$ of type $\mathcal{D}$ is called **two-spherical** if $M(A)$ is two-spherical, i.e., every residue of rank two in $\Delta_{\mathcal{G}_\mathcal{D}(\mathbb{K})}$, the twin building associated to $\mathcal{Z}_\mathcal{D}$, is spherical, i.e., every rank two fundamental subgroup of $\mathcal{G}_\mathcal{D}(\mathbb{K})$ is a Chevalley group. Therefore, for every centered two-spherical split Kac-Moody group $\mathcal{G}_\mathcal{D}(\mathbb{K})$, the corresponding rank two fundamental subgroups are Chevalley groups as defined in Section 1.4. Note that a split Tits functor $\mathcal{G}_\mathcal{D}$ is called **irreducible** if $M(A)$ is irreducible. A split Kac-Moody group (or a split Tits functor) is called **simply-laced** if the corresponding Dynkin diagram of $M(A)$ is simply-laced. We call a split Kac-Moody group $\mathcal{G}_\mathcal{D}(\mathbb{K})$ **simply connected** provided $\Lambda^\vee = \bigoplus_{m\in S} \mathbb{Z}h_m$, i.e., $\mathcal{T}_\Lambda(\mathbb{Z}) \cong \mathbb{Z}^k$ where k is the rank of $\mathcal{G}_\mathcal{D}$. Let $\mathbb{K}$ be an infinite field. By [Car92, Section 6], if $\mathcal{G}_\mathcal{D}(\mathbb{K})$ is a simply connected split Kac-Moody group then the associated Kac-Moody root datum $\mathcal{Z}_\mathcal{D} = (\mathcal{G}_\mathcal{D}(\mathbb{K}), \{U_{\alpha\in\Phi^{re}}\}, T)$ is centered.
Definition 4.1.4. Let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a Kac-Moody root datum where A is of affine type. Then for any filed $\mathbb{K}$, $\mathcal{G}_\mathcal{D}(\mathbb{K})$ is called an **affine** split Kac-Moody group.
Proposition 4.1.5. *[Che96, Proposition 2.4] Let $\mathbb{K}$ be a field of characteristic zero. Let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a Kac-Moody root datum where A is of affine type. If G be the Chevalley group of type A_{Fin} whose root system is compatible with $\mathcal{D}$. Then there exists a canonical central isogeny $\mathfrak{C} : \mathcal{G}_\mathcal{D}(\mathbb{K}) \to G(\mathbb{K}[t, t^{-1}])$ which maps root subgroups of $\mathcal{G}_\mathcal{D}(\mathbb{K})$ to root subgroups of $G(\mathbb{K}[t, t^{-1}])$. In particular, when $\mathcal{G}_\mathcal{D}(\mathbb{K})$ is simply connected then*

$\mathfrak{C} : \mathcal{G}_{\mathcal{D}}(\mathbb{K}) \to G(\mathbb{K}[t, t^{-1}])$ *is a canonical isomorphism.*

Proof. The first claim follows directly from [Che96, Proposition 2.4].
By [Che96, Lemma 2.3] and [Mar13, Definition 5.8]) $G(\mathbb{K}[t, t^{-1}])$ constitutes a Kac-Moody group. By the uniqueness of Kac-Moody groups over fields (see [Tit87]), we conclude that when $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ is simply connected then $\mathfrak{C}$ is an isomorphism (see also [AB08, Subsection 8.11.5]). □

4.1.2 Bounded Subgroups and Levi Decompositions

In this subsection we introduce bounded subgroups of split Kac-Moody groups along with their Levi decomposition which is similar to the Levi decomposition (1.3) stated in Section 1.1. Throughout this subsection we follow the notations of Subsection 4.1.1. The main references are [Cap05, Section 3.1] and [Rém02].

Definition 4.1.6. Let $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ be a split Kac-Moody group. A **bounded subgroup** of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ is a subgroup of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ contained in the intersection of stabilizers of two spherical residues of opposite signs in $\Delta^{\pm}_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}$, i.e., in the intersection of two spherical parabolic subgroups of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$. A bounded subgroup is called **maximal** if it is not properly contained in any other bounded subgroup.

Note that bounded subgroups are exactly those subgroups that have a fixed point in the geometric realization of $\Delta_{\mathcal{G}_{\mathcal{D}}(\mathbb{K})}$. This is a well-known result among experts and we mention a proof here for real split Kac-Moody groups.

Theorem 4.1.7. *Let $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ be a split Kac-Moody group. Then a subgroup H in $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ is bounded if and only if it has a fixed point in the geometric realization of each of buildings $\Delta^{\pm}_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}$.*

Proof. By definition, H is inside the stabilizers of two spherical residues with opposite signs in $\Delta_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}$. Moreover, H acts by cellular isometries on $|\Delta_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}|$ which is a topologically simply connected complete uniquely geodesic CAT(0) space by Theorem 3.3.2. Therefore, H stabilizes a bounded convex subset in each $|\Delta^{\pm}_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}|$. Hence, the Bruhat-Tits fixed theorem, Theorem 3.1.4, implies that H has a fixed point in each $|\Delta^{\pm}_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}|$.
Conversely, if H fixes a point in each $|\Delta^{\pm}_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}|$ then these fixed points are contained in a spherical residue in each $\Delta^{\pm}_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}$ as described in [CH09, page 16]. Now since H acts on $|\Delta^{\pm}_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}|$ by cellular isometries, it has to stabilize these spherical residues containing the fixed points. □

Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ denote the twin building $\Delta_{\mathcal{G}_{\mathcal{D}}(\mathbb{K})}$. For a given subset $J \subset S$, we set $\Phi_J := \{\alpha \in \Phi \mid r_\alpha \in W_J\}$ where r_α is the reflection corresponding to the root α in the fundamental apartment $\mathcal{A}$ of $\mathcal{B}$.
Define
$$L^J := T.\langle U_\alpha \mid \alpha \in \Phi_J\rangle. \tag{4.7}$$
Let $P^J_\pm$ denote the parabolic subgroup of type J containing $B_\pm$ and
$$U^J_\pm := \bigcap_{w \in W_J} wU_\pm w^{-1}. \tag{4.8}$$

Proposition 4.1.8. *[Rém02, Theorem 6.2.2] For a spherical subset $J \subset S$ we have*

$$P_{\pm}^{J} = L^{J} \ltimes U_{\pm}^{J}. \tag{4.9}$$

Moreover, if $R_{\pm}^{J}$ denotes the unique J-residue of $\mathcal{B}_{\pm}$ stabilized by $P_{\pm}^{J}$, then

$$L^{J} = P_{+}^{J} \cap P_{-}^{J} = \mathrm{Stab}_{\mathcal{G}_{\mathcal{D}}(\mathbb{K})}(R_{+}^{J}) \cap \mathrm{Stab}_{\mathcal{G}_{\mathcal{D}}(\mathbb{K})}(R_{-}^{J}) \tag{4.10}$$

and $U_{\pm}^{J}$ acts regularly on the J-residues opposite $R_{\pm}^{J}$ in $\mathcal{B}$.

The above decomposition is called the **Levi decomposition** of parabolic subgroups. Also, the subgroups $U_{\pm}^{J}$ are called the **unipotent radical** and L^{J} is called a **Levi factor** of $P_{\pm}^{J}$. Moreover, if in (4.10), $R_{+}^{J} \cup R_{-}^{J}$ contains the fundamental pair of opposite chambers then L^{J} is called a **fundamental** Levi factor. Furthermore, any subgroup arising as a (fundamental) Levi factor is called a **(fundamental) Levi subgroup** of type J. Also, a Levi subgroup is called **maximal** if the corresponding bounded subgroup is maximal.
Next we present a similar decomposition for the bounded subgroups which was studied first in [Rém02, Section 6] then [Cap05, Subsection 3.1.2]. To do this we need the following notations:
For subsets $J, K \subset S$ and $w \in W$ set

$$\Phi_{\epsilon}^{J,K,w} := (\Phi_{\epsilon} \cap w^{-1}.\Phi_{-\epsilon}) \backslash \Phi_{J \cap wKw^{-1}} \tag{4.11}$$

and

$$U_{\epsilon}^{J,K,w} := \langle U_{\alpha} \mid \alpha \in \Phi_{\epsilon}^{J,K,w} \rangle \tag{4.12}$$

where $\epsilon \in \{+,-\}$.
Proposition 4.1.9. *[Cap05, Proposition 3.2] For spherical subsets $J, K \subset S$ and $\epsilon \in \{+,-\}$, we have*

$$P_{\epsilon}^{J} \cap wP_{-\epsilon}^{K}w^{-1} = L^{J \cap wKw^{-1}} \ltimes U_{\epsilon}^{J,K,w} \tag{4.13}$$

and

$$P_{\epsilon}^{J} \cap wU_{-\epsilon}^{K}w^{-1} = U_{\epsilon}^{J,K,w}. \tag{4.14}$$

Moreover, there exists a prenilpotent pair of roots $\{\alpha, \beta\} \subset \Phi_{\epsilon}^{J,K,w}$ such that $\Phi_{\epsilon}^{J,K,w} = [\alpha, \beta]$ (in particular, $\Phi_{\epsilon}^{J,K,w}$ is finite). If R_J and R_K^w are the unique J-residue and K-residue stabilized by P_{ϵ}^{J} and $wP_{\epsilon}^{K}w^{-1}$ respectively, then

$$L^{J \cap wKw^{-1}} = \mathrm{Stab}_{\mathcal{G}_{\mathcal{D}}(\mathbb{K})}(\mathrm{proj}_{R_J}(R_K^w)) \cap \mathrm{Stab}_{\mathcal{G}_{\mathcal{D}}(\mathbb{K})}(\mathrm{proj}_{R_K^w}(R_J)) \tag{4.15}$$

and $\Phi_{\epsilon}^{J,K,w}$ coincides with the set of twin roots of the standard twin apartment which contain both $\mathrm{proj}_{R_J}(R_K^w)$ and $\mathrm{proj}_{R_K^w}(R_J)$.

We end this subsection by giving a proof for two known auxiliary lemmas.
Lemma 4.1.10. *Let $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ be a split Kac-Moody group. Then all spherical Levi subgroups of the same type are conjugate.*

Proof. As we saw in Subsection 4.1.1 and Section 2.4, $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ acts strongly transitively on $\Delta_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}$. Moreover, by (4.15) in Proposition 4.1.9 every Levi subgroup L of type K is the intersection of stabilizers of a pair of opposite spherical residues R_+ and R_- of type K. Let (w_+, w_-) be a pair of opposite chambers in $R_+ \cup R_-$ and (v_+, v_-) be another pair of opposite chambers corresponding to a fundamental Levi subgroup of type K. By the strongly transitivity of the action of $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ on its corresponding twin building, there exists a $g \in \mathcal{G}_{\mathcal{D}}(\mathbb{R})$ such that (w_+, w_-) is mapped to (v_+, v_-) via g. Since the action of $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ on its corresponding twin building is by cellular isometries, R_+ and R_- are mapped to a pair of opposite residues of type K containing (v_+, v_-) via g which correspond to a fundamental Levi subgroup of type K of $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$. □

Lemma 4.1.11. *Let $\mathcal{G}_{\mathcal{D}}$ be a split Tits functor with a Tits basis $\mathcal{F} = (\mathcal{G}_{\mathcal{D}}, \{\phi_i\}_{i\in I}, \eta)$. Then for every simple root $\alpha \in \Phi^\circ$ we have:*

$$G_\alpha(\mathbb{R}) \cong (P)\,\mathrm{SL}_2(\mathbb{R}). \tag{4.16}$$

In addition, if $\mathcal{G}_{\mathcal{D}}$ is a simply connected two-spherical split Tits functor then for every triple of pairwise different simple roots $\alpha, \beta, \gamma \in \Phi^\circ$ the following holds:

$$(P)\,\mathrm{SL}_2(\mathbb{R}) \cong G_\beta(\mathbb{R}) = G_{\alpha\beta}(\mathbb{R}) \cap G_{\beta\gamma}(\mathbb{R}). \tag{4.17}$$

Proof. First, by the definition of split Tits functors we know that there exists a morphism of functors

$$\phi_\alpha : \mathrm{SL}_2 \to \mathcal{G}_{\mathcal{D}}$$

for every simple root $\alpha \in \Phi^\circ$.
Second, let $\mathcal{Z}_{\mathcal{D}} := (\mathcal{G}_{\mathcal{D}}(\mathbb{R}), \{U_{\alpha\in\Phi^{re}}\}, T)$ be the twin root datum associated to $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ and $\Delta_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}$ be the associated twin building to $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ provided in Subsection 4.1.1. Then there exists the following surjective homomorphism (see also [HKM13, Proposition 7.2]):

$$\phi_\alpha : \mathrm{SL}_2(\mathbb{R}) \to G_\alpha(\mathbb{R}).$$

Therefore, (4.16) follows.
In addition, assume that $\mathcal{G}_{\mathcal{D}}$ is a simply connected two-spherical split Tits functor. For every triple of pairwise different simple roots $\alpha, \beta, \gamma \in \Phi^\circ$, let $r, s, t \in S$ be the corresponding generators respectively, where (W, S) is the Coxeter system of $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$. Let (x_+, x_-) be the fundamental pair of opposite chambers in $\Delta_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}$. By the description of the twin building $\Delta_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}$, it follows from (4.10) in Proposition 4.1.8 that

$$G_{\alpha\beta}(\mathbb{R}) \subset L^{\{r,s\}} = \mathrm{Stab}_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}(\mathrm{Res}_{\{r,s\}}(x_+)) \cap \mathrm{Stab}_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}(\mathrm{Res}_{\{r,s\}}(x_-)), \tag{4.18}$$

and similarly,

$$G_{\beta\gamma}(\mathbb{R}) \subset L^{\{s,t\}} = \mathrm{Stab}_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}(\mathrm{Res}_{\{s,t\}}(x_+)) \cap \mathrm{Stab}_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}(\mathrm{Res}_{\{s,t\}}(x_-)), \tag{4.19}$$

where $\mathrm{Res}_{\{r,s\}}(x_\epsilon)$ (resp. $\mathrm{Res}_{\{s,t\}}(x_\epsilon)$) denotes the spherical $\{r, s\}$-residue (resp. $\{s, t\}$-residue) containing x_ϵ for $\epsilon \in \{+, -\}$. Note that these residues are, indeed, spherical since $\mathcal{G}_{\mathcal{D}}$ is two-spherical.
We conclude from (4.18) and (4.19) the following:

$$G_\beta(\mathbb{R}) \subseteq\; G_{\alpha\beta}(\mathbb{R}) \cap G_{\beta\gamma}(\mathbb{R}) \subset L^{\{s\}} = \mathrm{Stab}_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}(\mathrm{Res}_{\{s\}}(x_+)) \cap \mathrm{Stab}_{\mathcal{G}_{\mathcal{D}}(\mathbb{R})}(\mathrm{Res}_{\{s\}}(x_-)). \tag{4.20}$$

Now, since $G_\beta(\mathbb{R})$ is the stabilizer of $\mathrm{Res}_{\{s\}}(x_+)$ and $\mathrm{Res}_{\{s\}}(x_-)$ in $\mathcal{G}_\mathcal{D}(\mathbb{R})$, it follows from (4.7) that $G_\beta(\mathbb{R})$ might **only** differ from $G_{\alpha\beta}(\mathbb{R}) \cap G_{\beta\gamma}(\mathbb{R})$ by some elements in the torus of $\mathcal{G}_\mathcal{D}(\mathbb{R})$.
But, on the one hand, since $\mathcal{G}_\mathcal{D}(\mathbb{R})$ is simply connected (hence centered), all torus elements in $G_{\alpha\beta}(\mathbb{R}) \cap G_{\beta\gamma}(\mathbb{R})$ are generated by the following (see (KMG3)):

$$\phi_\theta \begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix} = \eta\left(\lambda \mapsto l^{h_p(\lambda)}\right), \tag{4.21}$$

where $(\theta, p) \in \{(\alpha, r), (\beta, s), (\gamma, t)\}$ and h_p is the generator in $\Lambda^\vee$ corresponding to θ in $\mathcal{D} = (S, A, \Lambda, \{c_m\}_{m\in S}, \{h_m\}_{m\in S})$.
On the other hand, since $\mathcal{G}_\mathcal{D}(\mathbb{R})$ is simply connected, $\Lambda^\vee = \bigotimes_{m\in S} \mathbb{Z}h_m$, and hence, $\mathcal{T}_\Lambda(\mathbb{R}) \cong \mathbb{R}^k$ where k is the rank of $\mathcal{G}_\mathcal{D}(\mathbb{R})$.
Therefore, if there existed an element in the torus $\mathcal{T}_\Lambda(\mathbb{R})$ which was also in $G_{\alpha\beta}(\mathbb{R}) \cap G_{\beta\gamma}(\mathbb{R})$ but not in $G_\beta(\mathbb{R})$ it would induce a relation between the generators: h_r, h_s and h_t which reduces the dimension of $\mathcal{T}_\Lambda(\mathbb{R})$. This can not happen in the simply connected case because $\mathcal{T}_\Lambda(\mathbb{R}) \cong \mathbb{R}^k$. Thus, (4.17) follows (in view of (4.16)). □

4.2 Adjoint Representation

In this section we present the construction of Kac-Moody algebras over arbitrary fields. Then we give an adjoint representation of split Tits functors into their corresponding Kac-Moody algebras. Main references are [Cap05, Section 3.2], [Rém02, Chapters: 7 and 9] and [HKM13, Section 7.2].

4.2.1 $\mathbb{Z}$-Forms

In this subsection we define the complex Kac-Moody algebras and give a well-known construction of a $\mathbb{Z}$-form in the universal enveloping algebra of the underlying complex Kac-Moody algebra.

Let $A = (A_{ij})_{i,j\in I}$ be a generalized Cartan matrix. Let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a Kac-Moody root datum. The **Kac-Moody algebra of type $\mathcal{D}$ over a field $\mathbb{K}$**, which we denote by $\mathfrak{g}_{\mathcal{D}_\mathbb{K}}$, is the Lie $\mathbb{K}$-algebra generated by $\mathfrak{g}_{0_\mathbb{K}} := \Lambda^\vee \otimes_\mathbb{Z} \mathbb{K}$ and the set of generators $\{e_i, f_j\}_{i,j\in I}$ subjected to the following relations, where $h, h' \in \mathfrak{g}_{0_\mathbb{K}}$:

$$\begin{aligned}
[h, e_j] &= \langle c_i, h\rangle e_j, \\
[h, f_j] &= -\langle c_i, h\rangle f_j, \\
[h, h'] &= 0, \\
[e_i, f_i] &= -h_i \otimes 1, \\
[e_i, f_j] &= 0 \text{ for } i \neq j,
\end{aligned}$$

and the **Serre relations**

$$(\mathrm{ad} e_i)^{-\langle c_j, h_i\rangle + 1}(e_j) = (\mathrm{ad} f_i)^{-\langle c_j, h_i\rangle + 1}(f_j) = 0, \tag{4.22}$$

where

$$\langle\ ,\ \rangle : \Lambda \otimes_{\mathbb{Z}} \mathbb{K} \times \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{K} \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{K}$$
$$\langle x \otimes k, y \otimes k' \rangle := y(x) \otimes kk'. \tag{4.23}$$

For a Kac-Moody root datum $\mathcal{D}$, we denote by $\mathfrak{g}_{\mathcal{D}}$, the Kac-Moody algebra corresponding to $\mathcal{D}$ without specifying the field.

Remark 4.2.1. Note that the Kac-Moody algebra introduced here is similar to Definition 4.1.3 and coincides with it once $\mathbb{K} = \mathbb{C}$ and the generalized Cartan matrix A is symmetrizable. However, if we define $\mathfrak{g}_{\mathcal{D}_{\mathbb{K}}}$ as in Definition 4.1.3, the resulting Kac-Moody algebra is isomorphic to a quotient of the Kac-Moody algebra obtained here. And this difference does not affect the results we present in the thesis (see [Mar13, Remark 4.4]).

Definition 4.2.2. (cf. [Tit87, Subsection 4.4]) A subring $\mathfrak{X}$ of a $\mathbb{C}$-algebra $\mathfrak{X}_{\mathbb{C}}$ is called a $\mathbb{Z}$-form if the canonical map $\mathfrak{X} \otimes_{\mathbb{Z}} \mathbb{C} \to \mathfrak{X}_{\mathbb{C}}$ is bijective.

In the rest of this subsection we construct a $\mathbb{Z}$-form for the universal enveloping algebra of $\mathfrak{g}_{\mathcal{D}_{\mathbb{C}}}$.

Define the **fundamental reflections** $r_i \in \mathrm{Aut}_{\mathbb{K}}(\mathfrak{g}_{0_{\mathbb{K}}})$, $i \in I$, by $r_i(h) := h - \langle c_i, h\rangle h_i$. They generate the Weyl group (W, S) associated to the split Kac-Moody group $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ explained in Subsection 4.1.1. Hence, by assigning a one-to-one correspondence between the set I and the set of generators S of the Weyl group W we can use the elements in S as indexes. Moreover, if we denote the set of roots and the set of simple roots of (W, S) by Φ and Φ° respectively, then we can identify the simple root $\alpha_s \in \Phi^\circ$ with c_s for any $s \in S$ under the above one-to-one correspondence. Furthermore, under this identification, every non-trivial element α in Φ corresponds to an element $c_\alpha \in \Lambda \otimes_{\mathbb{Z}} \mathbb{K}$ such that $\mathfrak{g}_\alpha := \{x \in \mathfrak{g}_{\mathcal{D}} \mid [h, x] = \langle c_\alpha, h\rangle x \text{ for all } h \in \mathfrak{g}_{0_{\mathbb{K}}}\} \neq (0)$. The subspace $\mathfrak{g}_\alpha$ is called the **root space** attached to α. This way we obtain the following root decomposition (see [KP83, Section 1]):

$$\mathfrak{g}_{\mathcal{D}} = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha \tag{4.24}$$

By declaring the elements e_s (resp. f_s) to be of degree α_s (resp. $-\alpha_s$) and the elements in $\mathfrak{g}_{0_{\mathbb{K}}}$ to be of degree 0 there exists an **abstract Q-gradation** on $\mathfrak{g}_{\mathcal{D}}$. Each degree for which the corresponding subspace is non-trivial is called a **root** of $\mathfrak{g}_{\mathcal{D}}$ (see [Rém02, Section 7.3.1]). Let Δ denote the **set of all roots** and set $\Delta_+ := \Delta \cap Q_+$ where $Q_+ := \sum_{s\in\Pi} \mathbb{Z}_+\alpha_s$. We call Δ_+ the **set of positive roots**. For $\alpha = \sum k_s\alpha_s \in \Delta$, we write $\mathrm{ht}\,\alpha := \sum k_s$ which is called the **height** of α.

From the above description, it is evident that we have $\Phi \cong \Delta$. In particular, W preserves Δ. In the sequel, a **real root** is an element of $\Delta^{re} = \Phi^{re} := \{w(\alpha) \mid w \in W, \alpha \in \Pi\}$. If $\alpha \in \Delta^{re}$, then $\dim \mathfrak{g}_\alpha = 1$. Set $\Delta^{re}_+ := \Delta^{re} \cap \Delta_+$. For $\alpha \in \Delta^{re}$, write $\alpha = w(\alpha_s)$ for some $w \in W$ and $s \in S$; then $r_\alpha := w r_s w^{-1}$ depends only on α (see [KP83, Section 1]).

Now let $\mathcal{U}_{\mathfrak{g}_{\mathcal{D}_{\mathbb{C}}}}$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathcal{D}_{\mathbb{C}}}$. Note that the Q-gradation on $\mathfrak{g}_{\mathcal{D}_{\mathbb{C}}}$ induces a **filtration** on $\mathcal{U}_{\mathfrak{g}_{\mathcal{D}_{\mathbb{C}}}}$. Denote by $\mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathfrak{g}_{\mathcal{D}_{\mathbb{C}}}})$ the group of $\mathbb{C}$-linear automorphisms of $\mathcal{U}_{\mathfrak{g}_{\mathcal{D}_{\mathbb{C}}}}$ which preserve the above filtration.

For every $u \in \mathcal{U}_{\mathfrak{g}_{\mathcal{D}_{\mathbb{C}}}}$ define

$$u^{[n]} := (n!)^{-1}u^n, \tag{4.25}$$

$$\binom{u}{n} := (n!)^{-1}.u.(u-1).....(u-n+1). \tag{4.26}$$

Set $\mathcal{U}_0$ to be the subring of $\mathcal{U}_{\mathfrak{g}_{\mathcal{D}_\mathbb{C}}}$ generated by the (degree 0) elements of the form $\binom{h}{n}$ for $h \in \Lambda^\vee$ and $n \in \mathbb{N}$. Furthermore, define $\mathcal{U}_{\alpha_i}$ and $\mathcal{U}_{-\alpha_i}$ to be the subrings $\sum_{n\in\mathbb{N}} \mathbb{Z}e_i^{[n]}$ and $\sum_{n\in\mathbb{N}} \mathbb{Z}f_i^{[n]}$, respectively. Then $\mathcal{U}_\mathbb{Z}$ is a $\mathbb{Z}$-**form** of $\mathcal{U}_{\mathfrak{g}_{\mathcal{D}_\mathbb{C}}}$, i.e., the canonical map $\mathcal{U}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C} \to \mathcal{U}_{\mathfrak{g}_{\mathcal{D}_\mathbb{C}}}$ is a bijection. Moreover, $\mathcal{U}_\mathbb{Z}$ contains the ideal $\mathcal{U}_\mathbb{Z}^+ := \mathfrak{g}_\mathcal{D}.\mathcal{U}_{\mathfrak{g}_{\mathcal{D}_\mathbb{C}}} \cap \mathcal{U}_\mathbb{Z}$ (see [Tit87, Section 4] and [Rém02, Proposition 7.4.3]).

4.2.2 Adjoint Representation

Note that the construction of $\mathbb{Z}$-forms in Subsection 4.2.1 is very similar to the construction of $L(\mathbb{Z})$ in Section 1.4. Moreover, this allows us to plug in different commutative unital rings to the $\mathbb{Z}$-form: For an arbitrary commutative unital ring R define $(\mathcal{U}_\mathbb{Z})_R$ to the extension of scalars from $\mathbb{Z}$ to R of $\mathcal{U}_\mathbb{Z}$. This gives rise to a filtered R-algebra $(\mathcal{U}_\mathbb{Z})_R$.
In addition, define $\mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_\mathbb{Z})(R)$ to be the **group functor** on the category of commutative unital rings to the group of automorphisms of the R-algebra $(\mathcal{U}_\mathbb{Z})_R$ preserving its filtration arising from $\mathcal{U}_\mathbb{Z}$ and its ideal $\mathcal{U}_\mathbb{Z}^+ \otimes_\mathbb{Z} R$ (see [Rém02, 9.5.3]). In particular, for an arbitrary field $\mathbb{K}$ we have $(\mathcal{U}_\mathbb{Z})_\mathbb{K} := \mathcal{U}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{K}$ and $\mathrm{Aut}_{\mathrm{filt}}((\mathcal{U}_\mathbb{Z})_\mathbb{K})$ is the group of $\mathbb{K}$-linear automorphisms of $(\mathcal{U}_\mathbb{Z})_\mathbb{K}$ which preserve the above filtration.
Now for a given Kac-Moody root datum $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ and a commutative unital ring R, for every $i \in I$ define:

$$\forall r \in R \qquad x_i(r) := \phi_i \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \tag{4.27}$$

The image of x_i, $x_i(R)$ is also called the **one-parameter subgroup** of $\mathcal{G}_\mathcal{D}(R)$ which is nothing but the simple root subgroup $U_{\alpha_s}(R) \cong (R, +)$ where $s \in S$ corresponds to the index $i \in I$. In this case we denote x_i by x_s. Also, note that one can define the one-parameter subgroups for all roots $\alpha \in \Phi$.
Theorem 4.2.3. *[Rém02, Proposition 9.5.2] For every Kac-Moody root datum $\mathcal{D}$, there exists a morphism of group functors $\mathbf{Ad} : \mathcal{G}_\mathcal{D} \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_\mathbb{Z})$ characterized by:*

$$\mathbf{Ad}_R(x_s(r)) = \sum_{n\geq 0} \frac{(\mathrm{ad}\, e_s)^n}{n!} \otimes r^n, \tag{4.28}$$

$$\mathbf{Ad}_R(x_{-s}(r)) = \sum_{n\geq 0} \frac{(\mathrm{ad}\, f_s)^n}{n!} \otimes r^n, \tag{4.29}$$

$$\mathbf{Ad}_R(\mathcal{T}_\Lambda(R)) \text{ fixes } (\mathcal{U}_0)_R \text{ and} \tag{4.30}$$

$$\mathbf{Ad}_R(h)(e_s \otimes r) = \langle c_s, h\rangle(e_s \otimes r) \tag{4.31}$$

for $s \in S$, any commutative unital ring R, each h in $\mathcal{T}_\Lambda(R)$ and each $r \in R$.
Corollary 4.2.4. *For the ring morphism $\nu : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ the following diagram, induced by ν, is commutative.*

$$\begin{tikzcd}
\mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_\mathbb{Z}) \arrow[r, "\alpha"] & \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_\mathbb{Z} \bigotimes \mathbb{Z}/n\mathbb{Z}) \\
\mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Z}) \arrow[r, "\beta"] \arrow[u, "\mathbf{Ad}"] & \mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Z}/n\mathbb{Z}) \arrow[u, "\mathbf{Ad}"']
\end{tikzcd}$$

Proof. A direct consequent of the functorial property of the adjoint representation described in Theorem 4.2.3. □

Here we state a result similar to Proposition 1.4.6.

Proposition 4.2.5. *The $\mathbb{Z}$-form $\mathcal{U}_{\mathbb{Z}}$ is stable by $\frac{(\operatorname{ad} e_s)^n}{n!}$ for all $n \in \mathbb{N}$, hence by $\exp(\operatorname{ad} e_s) := \sum_{n\geq 0} \frac{(\operatorname{ad} e_s)^n}{n!}$ and, similarly, by $\exp(\operatorname{ad} f_s)$. Hence $(\mathcal{U}_{\mathbb{Z}})_R$ is preserved by $\mathbf{Ad}_R(x_s(r))$ and $\mathbf{Ad}_R(x_{-s}(r))$.*

Proof. It follows from [Tit87, (12)] and the definition of $(\mathcal{U}_{\mathbb{Z}})_R$ and (4.28) and (4.29) in Theorem 4.2.3. □

Proposition 4.2.6. *Let $\mathbb{K}$ be a field. Let $\mathcal{G}_{\mathcal{D}}$ be a split Tits functor such that the split Kac-Moody group $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ is centered. Then the kernel of the adjoint representation is the center of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ which coincides with the intersection of all centralizers of all root subgroups.*

Proof. Follows from [Rém02, Proposition 9.6.2]. □

For a Kac-Moody root datum $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$, define $\Lambda^{ad} := \bigotimes_{i\in I} \mathbb{Z}c_i$. In this case $\mathcal{D}^{ad} = (I, A, \Lambda^{ad}, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ is another Kac-Moody root datum denoted by $\mathcal{D}^{ad}$ which is called the **adjoint Kac-Moody root datum** associated to $\mathcal{D}$. Moreover, the group $\mathcal{G}_{\mathcal{D}^{ad}}(R)$ is called the **adjoint split Kac-Moody group** of type $\mathcal{D}$ over R.

Proposition 4.2.7. *Let $\mathcal{D}$ be a Kac-Moody root datum. Then $\mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Q})$ is center free.*

Proof. On the one hand, in view of (4.5), we know that by (5.1) in [Mar13, Definition 5.8] for an element $t \in \mathcal{T}_\Lambda(\mathbb{Q})$ we have:

$$t.x_i(q).t^{-1} = x_i(t(c_i)q) \text{ for all } i \in I. \tag{4.32}$$

On the other hand, since the action of the central elements on the corresponding twin building of $\mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Q})$ is trivial, the center of $\mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Q})$ lies in all tori. This implies that (4.32) applies to any central element.
Moreover, (4.32) for a central element t yields the following identity:

$$x_i(q) = t.x_i(q).t^{-1} = x_i(t(c_i)q) \text{ for all } i \in I. \tag{4.33}$$

Therefore,

$$t(c_i) = 1 \text{ for all } i \in I. \tag{4.34}$$

Finally, this yields that every central element acts trivially on the generators of Λ and since $\mathcal{T}_\Lambda(\mathbb{Q})$ is defined to be $\operatorname{Hom}_{\mathrm{gr}}(\Lambda, \mathbb{Q}^\times)$ and in the adjoint case Λ is generated by c_i's, t is the identity element in $\mathcal{T}_\Lambda(\mathbb{Q})$. □

More generally, for a field $\mathbb{K}$ we have (see [Mar13, Subsection 5.1.3]):

$$\ker(\mathbf{Ad}) = Z(\mathcal{G}_{\mathcal{D}}(\mathbb{K})) = \{t \in \mathcal{T}_\Lambda(\mathbb{K}) \mid t(c_i) = 1 \text{ for all } i \in I\}, \tag{4.35}$$

where $Z(\mathcal{G}_{\mathcal{D}}(\mathbb{K}))$ denotes the center of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$.

Corollary 4.2.8. *Let $\mathcal{D}$ be a Kac-Moody root datum such that $\mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Q})$ is centered. Then following map is injective.*

$$\mathbf{Ad} : \mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Q}) \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathbb{Z}})(\mathbb{Q}), \tag{4.36}$$

Proof. Immediate consequence of Proposition 4.2.6 and Proposition 4.2.7. □

Here we reproduce the proof of [Mar11, Proposition 2.6.3] in the special case of integers.
Proposition 4.2.9. *Let $\mathcal{D}$ be a Kac-Moody root datum and $\mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Z})$ be the adjoint split Kac-Moody group of type $\mathcal{D}$ over $\mathbb{Z}$. Assume that $\mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Q})$ is centered. Then $\mathbf{Ad} : \mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Z}) \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathbb{Z}})$ is injective.*

Proof. By applying the functor morphism in Theorem 4.2.3 to $\mathbb{Z}$ under the canonical isomorphism $\mathcal{U}_{\mathbb{Z}} \cong \mathcal{U}_{\mathbb{Z}} \otimes \mathbb{Z}$ we obtain the following;

$$\mathbf{Ad} : \mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Z}) \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathbb{Z}}), \tag{4.37}$$

or more generally,

$$\mathbf{Ad} : \mathcal{G}_{\mathcal{D}}(\mathbb{Z}) \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathbb{Z}}). \tag{4.38}$$

On the other hand, by Corollary 4.2.8, $\mathbf{Ad} : \mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Q}) \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathbb{Z}})(\mathbb{Q})$ is injective. And hence $\mathbf{Ad} : \mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Z}) \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathbb{Z}})(\mathbb{Q})$ is injective as well. Therefore, it suffices to show that for any non-trivial element $x \in \mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Z})$, $\mathbf{Ad}(x)$ does not fix an element in $\mathcal{U}_{\mathbb{Z}}$ so that it cannot be a trivial automorphism.
By means of the notations introduced in Subsection 4.2.1, we know that an element $v \in (\mathcal{U}_{\mathbb{Z}})(\mathbb{Q})$ can be written in the form; $v = \sum_{\alpha\in Q} v_\alpha \otimes q_\alpha$ which has finitely many terms where $q_\alpha \in \mathbb{Q}$.
Now since $\mathbf{Ad} : \mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Z}) \to \mathrm{Aut}_{\mathrm{filt}}(\mathcal{U}_{\mathbb{Z}})(\mathbb{Q})$ is injective, for every non-trivial element $x \in \mathcal{G}_{\mathcal{D}^{ad}}(\mathbb{Z})$, there exists a $v \in (\mathcal{U}_{\mathbb{Z}})(\mathbb{Q})$ in the above form such that $\mathbf{Ad}(x)(v) \neq v$.
Set $n := [q_\alpha]_{\alpha\in Q}$ to be the least common multiple of the denominators of the q_α. Then $nv \in \mathcal{U}_{\mathbb{Z}}$ and we have

$$\mathbf{Ad}(x)(nv) = n\mathbf{Ad}(x)(v) \neq nv.$$

This, in view of (4.37), implies the proposition.

□

A subgroup H of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ is called **$\mathbf{Ad}_{\mathbb{K}}$-locally finite** if for every vector in the $\mathbb{K}$-vector space $(\mathcal{U}_{\mathbb{Z}})_{\mathbb{K}}$ there exists a finite-dimensional $\mathbf{Ad}_{\mathbb{K}}(H)$-invariant subspace V containing it. H is called **$\mathbf{Ad}_{\mathbb{K}}$-locally unipotent** if it is $\mathbf{Ad}_{\mathbb{K}}$-locally finite and in addition, $\mathbf{Ad}_{\mathbb{K}}(H)|_V$ is a unipotent subgroup of $\mathrm{GL}(V)$. A subgroup H of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ is called **$\mathbf{Ad}_{\mathbb{K}}$-semisimple** if $(\mathcal{U}_{\mathbb{Z}})_{\mathbb{K}}$ decomposes into a direct sum of finite-dimensional irreducible $\mathbf{Ad}_{\mathbb{K}}(H)$-modules invariant under $\mathbf{Ad}_{\mathbb{K}}(H)$. In this case, if the subspaces in the decomposition are one-dimensional, then H is called **$\mathbf{Ad}_{\mathbb{K}}(H)$-diagonalizable**. An element $g \in \mathcal{G}_{\mathcal{D}}(\mathbb{K})$ is called **$\mathbf{Ad}_{\mathbb{K}}$-locally finite** (respectively **$\mathbf{Ad}_{\mathbb{K}}$-locally unipotent**, **$\mathbf{Ad}_{\mathbb{K}}$-semisimple**, **$\mathbf{Ad}_{\mathbb{K}}(H)$-diagonalizable**) if $\langle g\rangle$ is so (see [Cap05, Subsection 3.2.3]).
Now let H be a bounded subgroup of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ with a Levi decomposition $H = L \ltimes U$ obtained from Proposition 4.1.9. The following theorem provides an algebraic structure on H by means of the adjoint representation.

Theorem 4.2.10. *[Cap05, Proposition 3.6] Let $H \leq \mathcal{G}_{\mathcal{D}}(\mathbb{K})$ be a bounded subgroup with a Levi decomposition $H = L \ltimes U$. Let T be a maximal diagonalizable subgroup of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ contained in L. Then there exists a finite-dimensional subspace V of $(\mathcal{U}_{\mathbb{Z}})_{\mathbb{K}}$ such that $H = \mathrm{Stab}_{\mathcal{G}_{\mathcal{D}}(\mathbb{K})}(V)$. Moreover, the following hold:*

(i) The Zariski closure $\bar{H}$ (resp. $\bar{L}$, $\bar{U}$, $\bar{T}$) of $\mathbf{Ad}_{\mathbb{K}}(H)|_V$ (resp. $\mathbf{Ad}_{\mathbb{K}}(L)|_V$, $\mathbf{Ad}_{\mathbb{K}}(U)|_V$, $\mathbf{Ad}_{\mathbb{K}}(T)|_V$) in $\mathrm{GL}(V_{\bar{\mathbb{K}}})$ is a connected $\mathbb{K}$-subgroup, where $V_{\bar{\mathbb{K}}} := V \otimes_{\mathbb{K}} \bar{\mathbb{K}} \subset (\mathcal{U}_{\mathbb{Z}})_{\bar{\mathbb{K}}}$. Moreover, if $\mathbb{K}$ is algebraically closed then $\bar{H} = \mathbf{Ad}_{\mathbb{K}}(H)|_V$ and the similar is true for L, U and T.

(ii) $\bar{L}$ is reductive, $\bar{T}$ is a maximal torus of $\bar{L}$, $\bar{U}$ is unipotent, and $\bar{H} = \bar{L} \ltimes \bar{U}$ is a Levi decomposition. Moreover, $\mathbf{Ad}_{\mathbb{K}}$ maps root subgroups of L (as in Definition 2.2.12) to root subgroups of $\bar{L}$ (in algebraic sense).

(iii) The kernel of the restriction $\mathbf{Ad}_{\mathbb{K}} : H \to \mathrm{GL}(V)$ is the center of H and is contained in the center of L, which is $\mathbf{Ad}_{\mathbb{K}}$-diagonalizable.

(iv) An element $g \in H$ is $\mathbf{Ad}_{\mathbb{K}}$-semisimple if and only if $\mathbf{Ad}_{\mathbb{K}}(g)|_V$ is semisimple in $\bar{H}$.

4.3 Isomorphism Theorem

In this section we state the isomorphism theorem for split Kac-Moody groups which will be used in our arguments later on this thesis. The reference is [Cap05, Section 4.1].

We start this section with a result which shows how homomorphisms from a simply connected Chevalley group to a split Kac-Moody group behave.

Theorem 4.3.1. *[Cap05, Theorem 5.11] Let G be a simply connected Chavalley group. Let $\mathcal{G}_{\mathcal{D}}$ be a split Tits functor and $\mathbb{K}$ be a field. Let $\phi : G(\mathbb{Q}) \to \mathcal{G}_{\mathcal{D}}(\mathbb{K})$ be a none-trivial homomorphism. Then $\mathbb{K}$ is of characteristic zero, the image $\phi(G(\mathbb{Q}))$ is contained in a spherical Levi subgroup of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ and diagonalizable subgroups of $G(\mathbb{Q})$ are mapped by ϕ to diagonalizable subgroups of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$.*

Let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ and $\mathcal{D}' = (I', A', \Lambda, \{c'_i\}_{i\in I'}, \{h'_i\}_{i\in I'})$ be two Kac-Moody root data and $\mathcal{F} = (\mathcal{G}, (\phi_i)_{i\in I}, \eta)$ and $\mathcal{F}' = (\mathcal{G}', (\phi'_i)_{i\in I}, \eta')$ the bases of two split Tits functors $\mathcal{G}$ and $\mathcal{G}'$ of type $\mathcal{D}$ and $\mathcal{D}'$. For any two fields $\mathbb{K}$ and $\mathbb{K}'$, let $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ and $\mathcal{G}'_{\mathcal{D}'}(\mathbb{K}')$ be the corresponding split Kac-Moody groups respectively. Then we have

Theorem 4.3.2 (Isomorphism Theorem)**.** *[Cap05, Theorem 4.1] Let $\phi : \mathcal{G}_{\mathcal{D}}(\mathbb{K}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{K}')$ be an isomorphism. Suppose that $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ is infinite and $|\mathbb{K}| \geq 4$. Then there exist an inner automorphism α of $\mathcal{G}'_{\mathcal{D}'}(\mathbb{K}')$, a bijection $\pi : I \to I'$ and for each $i \in I$, a field isomorphism $\xi_i : \mathbb{K} \to \mathbb{K}'$, a diagonal automorphism δ_i of $\mathrm{SL}_2(\mathbb{K}')$ and an automorphism ι_i of $\mathrm{SL}_2(\mathbb{K}')$ which is either trivial or transpose-inverse, such that the diagram*

$$\begin{tikzcd}
\mathrm{SL}_2(\mathbb{K}) \arrow[r, "\mathrm{SL}_2(\xi_i)"] \arrow[d, "\phi_i"'] & \mathrm{SL}_2(\mathbb{K}') \arrow[d, "\phi'_{\pi(i)}\circ\delta_i\circ\iota_i"] \\
\mathcal{G}_{\mathcal{D}}(\mathbb{K}) \arrow[r, "\alpha\circ\phi"'] & G'_{\mathcal{D}'}(\mathbb{K}')
\end{tikzcd}$$

commutes for every $i \in I$. In particular, the bijection π extends to an isomorphism, again denoted by π, of the Weyl groups of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ and $\mathcal{G}'_{\mathcal{D}'}(\mathbb{K}')$ such that (ϕ, π) is an isomorphism of the twin root data associated with $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ and $\mathcal{G}'_{\mathcal{D}'}(\mathbb{K}')$.

It is possible to glue together the local morphisms obtained in Theorem 4.3.2 in order to define global morphisms on the ambient split Kac-Moody groups. Notations as above, there exists an automorphism δ of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ such that $\delta \circ \phi_i = \phi_i \circ \delta_i$ for each $i \in I$. We call δ a **diagonal** automorphism of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$. Similarly, there exists an automorphism σ of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ whose restriction to $\phi_i(\mathrm{SL}_2(\mathbb{K}))$ is a transpose-inverse for each $i \in I$. Such automorphisms swap the two standard Borel subgroups of opposite signs of the underlying Kac-Moody group. This automorphism together with the identity are called the **sign** automorphisms of $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ (see [Cap05, Section 4.1]).

In the isomorphism theorem, Theorem 4.3.2, once the underlying split Kac-Moody groups are assumed to be irreducible, there exists the following global version.

Theorem 4.3.3. *[Cap05, Theorem 4.2] Let $\phi : \mathcal{G}_{\mathcal{D}}(\mathbb{K}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{K}')$ be an isomorphism. Suppose that $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ is infinite and $|\mathbb{K}| \geq 4$. Assume that $\mathcal{G}_D$ is irreducible. Then there exist a bijection $\pi : I \to I'$, an inner automorphism α of $\mathcal{G}'_{\mathcal{D}'}(\mathbb{K}')$, a diagonal automorphism δ and a sign automorphism σ of $\mathcal{G}'_{\mathcal{D}'}(\mathbb{K}')$ and for each $i \in I$, a field isomorphism $\xi_i : \mathbb{K} \to \mathbb{K}'$, such that the diagram*

$$\begin{tikzcd}
\mathrm{SL}_2(\mathbb{K}) \arrow[r, "\mathrm{SL}_2(\xi_i)"] \arrow[d, "\phi_i"'] & \mathrm{SL}_2(\mathbb{K}') \arrow[d, "\phi'_{\pi(i)}"] \\
\mathcal{G}_{\mathcal{D}}(\mathbb{K}) \arrow[r, "\sigma\circ\delta\circ\alpha\circ\phi"'] & \mathcal{G}'_{\mathcal{D}'}(\mathbb{K}')
\end{tikzcd}$$

commutes for every $i \in I$. Furthermore, if $\mathbb{K}$ is infinite then $A_{ij}A_{ji} = A'_{\pi(i)\pi(j)}A'_{\pi(j)\pi(i)}$ for all $i, j \in I$. Moreover, if $\mathrm{char}(\mathbb{K}) = 0$ or if $\mathrm{char}(\mathbb{K}) = p > 0$, $\mathbb{K}$ infinite and A_{ij} prime to p for all $i \neq j \in I$, then $A_{ij} = A'_{\pi(i)\pi(j)}$ and $\xi_i = \xi_j$ for all $i, j \in I$.

4.4 Topologies on Kac-Moody Groups

In this section we give two main topologies on split Kac-Moody groups and present some basic properties. We use these two topologies in the rigidity results of Chapter 7. The references for the present section are [KP83] and [HKM13].

4.4.1 Zariski Topology on Kac-Moody Groups

This subsection is to present a Zariski topology on Kac-Moody groups over a field $\mathbb{K}$ of characteristic zero. This topology is called the Zariski topology because it is compatible with the Zariski topology on the spherical parabolic subgroups of the underlying split Kac-Moody groups as linear algebraic groups (see Proposition 4.4.9). The main reference is [KP83].

Notations as in Subsection 4.2.1, let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a Kac-Moody root datum and $\mathfrak{g}_{\mathcal{D}_\mathbb{K}}$ the corresponding Kac-Moody algebra. Let $\mathfrak{g}'_{\mathcal{D}_\mathbb{K}}$ denote the derived algebra of $\mathfrak{g}_{\mathcal{D}_\mathbb{K}}$. Consider a $\mathfrak{g}'_{\mathcal{D}_\mathbb{K}}$-module V, or a pair (V, π) where $\pi : \mathfrak{g}'_{\mathcal{D}_\mathbb{K}} \to \mathrm{End}_\mathbb{K}(V)$. Define

$V_{\text{fin}} := \{v \in V \mid \forall \alpha \in \Phi^{re}\ \exists n \in \mathbb{N}\ s.t\ \pi(\mathfrak{g}_\alpha)^n(v) = 0\}$. Note that, in the definition of V_{fin} it suffices to take the root spaces corresponding to roots in $\pm\Phi^\circ$. V_{fin} is a $\mathfrak{g}'_{\mathcal{D}_\mathbb{K}}$-submodule of V. The $\mathfrak{g}'_{\mathcal{D}_\mathbb{K}}$-module V is called **integrable** if $V = V_{\text{fin}}$. Moreover, $(\mathfrak{g}'_{\mathcal{D}_\mathbb{K}}, \text{ad})$ is an example of an integrable $\mathfrak{g}'_{\mathcal{D}_\mathbb{K}}$-module ([KP83, 1B]).
Given an integrable $\mathfrak{g}'_{\mathcal{D}_\mathbb{K}}$-module (V, π), one can associate a homeomorphism (denoted again by π) $\mathcal{G}_\mathcal{D}(\mathbb{K}) \to \text{Aut}_\mathbb{K}(V)$ such that $\pi(\exp(e_\alpha)) = \exp(\pi(e_\alpha))$ for any $\alpha \in \Phi^{re}$. In particular, for the integrable $\mathfrak{g}'_{\mathcal{D}_\mathbb{K}}$-module $(\mathfrak{g}_{\mathcal{D}_\mathbb{K}}, \text{ad})$, we obtain a homeomorphism $\mathbf{Ad} : \mathcal{G}_\mathcal{D}(\mathbb{K}) \to \text{Aut}_\mathbb{K}(\mathfrak{g}_{\mathcal{D}_\mathbb{K}})$ which is compatible with the notion presented in Subsection 4.2.2 (see [KP83, page 144]).
With notations as in Subsection 4.1.1, for $\bar{\beta} = (\beta_1, ..., \beta_k) \in (\Phi^{re})^k$, define a map

$$x_{\bar{\beta}} : (\mathbb{K})^k \to \mathcal{G}_\mathcal{D}(\mathbb{K})$$

by

$$x_{\bar{\beta}}(t_1, ..., t_k) := x_{\beta_1}(t_1)...x_{\beta_k}(t_k), \tag{4.39}$$

and denote by $U_{\bar{\beta}}$ the image of $x_{\bar{\beta}}$.
Definition 4.4.1. [KP83, 2A] A function $f : \mathcal{G}_\mathcal{D}(\mathbb{K}) \to \mathbb{K}$ is called **weakly regular** if $f \circ x_{\bar{\beta}} : (\mathbb{K})^k \to \mathbb{K}$ is a polynomial function for all $\bar{\beta} \in (\Phi^{re})^k$ and $k \in \mathbb{N}$. Denote by $\mathbb{K}[\mathcal{G}_\mathcal{D}(\mathbb{K})]_{w.r}$ the algebra of all weakly regular functions.

The ring $\mathbb{K}[\mathcal{G}_\mathcal{D}(\mathbb{K})]_{w.r}$ is an integral domain and any $f \in \mathbb{K}[\mathcal{G}_\mathcal{D}(\mathbb{K})]_{w.r}$ is determined by its restriction to U_-TU_+ ([KP83, Lemma 2.1]).
A subgroup $U'_\pm$ of $U_\pm$ is called **large** if there exist $g_1, ..., g_m \in \mathcal{G}_\mathcal{D}(\mathbb{K})$ such that

$$\bigcap_{1 \le j \le m} g_j U_\pm g_j^{-1} \subset U'_\pm$$

for some $m \in \mathbb{N}$.
Definition 4.4.2. [KP83, 2C] A weakly regular function $f \in \mathbb{K}[\mathcal{G}_\mathcal{D}(\mathbb{K})]_{w.r}$ is called **strongly regular** if there exist large subgroups $U'_\pm \subset U_\pm$ such that $f(u_- g u_+) = f(g)$ for all $g \in \mathcal{G}_\mathcal{D}(\mathbb{K})$ and $u_\pm \in U'_\pm$.

Denote by $\mathbb{K}[\mathcal{G}_\mathcal{D}(\mathbb{K})]$, the algebra of all strongly regular functions on $\mathcal{G}_\mathcal{D}(\mathbb{K})$. If $\mathcal{G}_\mathcal{D}(\mathbb{K})$ is spherical then $\mathbb{K}[\mathcal{G}_\mathcal{D}(\mathbb{K})] = \mathbb{K}[\mathcal{G}_\mathcal{D}(\mathbb{K})]_{w.r}$ is the coordinate ring of the finite-dimensional affine variety $\mathcal{G}_\mathcal{D}(\mathbb{K})$ ([KP83, Example 2.1(a)]).
Definition 4.4.3. The **(weak) Zariski topology** on $\mathcal{G}_\mathcal{D}(\mathbb{K})$ is the topology induced by the set of zeros of ideals in $(\mathbb{K}[\mathcal{G}_\mathcal{D}(\mathbb{K})]_{w.r})$ $\mathbb{K}[\mathcal{G}_\mathcal{D}(\mathbb{K})]$ as closed subsets.
Remark 4.4.4. [KP83, 2E] By the Zariski topology on $\mathcal{G}_\mathcal{D}(\mathbb{K})$, $U_\pm$, $B_\pm$, T, spherical parabolic subgroups, and bounded subgroups are all closed and the topology on spherical parabolic subgroups and bounded subgroups is compatible with their Zariski topology as finite-dimensional affine varieties. Moreover, the basis functions $\phi_i : \text{SL}_2(\mathbb{K}) \to G_i$ for the split Tits functor $\mathcal{G}$ are Zariski homomorphisms. In addition, if $\mathcal{G}_\mathcal{D}(\mathbb{K})$ is a centered Kac-Moody group (i.e., generated by its fundamental root subgroups) then every element in $\text{Inn}(\mathcal{G}_\mathcal{D}(\mathbb{K}))$ is continuous with respect to the (weak) Zariski topology.

Next we show that every automorphism of a real split Kac-Moody group $\mathcal{G}_\mathcal{D}(\mathbb{R})$ is continuous with respect to the weak Zariski topology on $\mathcal{G}_\mathcal{D}(\mathbb{R})$.
Corollary 4.4.5. *Let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a Kac-Moody root datum and $\mathcal{G}_\mathcal{D}(\mathbb{R})$ be the associated real split Kac-Moody group. Let $\phi : \mathcal{G}_\mathcal{D}(\mathbb{R}) \to \mathcal{G}_\mathcal{D}(\mathbb{R})$ be an automorphism. Endow $\mathcal{G}_\mathcal{D}(\mathbb{R})$ with the weak Zariski topology. Then ϕ is a homeomorphism.*

Proof. By Theorem 4.3.3, since $\mathbb{R}$ is perfect, for every weakly regular function $f : \mathcal{G}_{\mathcal{D}}(\mathbb{R}) \to \mathbb{R}$, the composite $f \circ \phi : \mathcal{G}_{\mathcal{D}}(\mathbb{R}) \to \mathbb{R}$ is again a weakly regular function. Because composition of $f \circ \phi$ with any $x_{\bar{\beta}}$ defined in (4.39) induces only a permutation of the parameters of $f \circ x_{\bar{\beta}} : \mathcal{G}_{\mathcal{D}}(\mathbb{R}) \to \mathbb{R}$ by the commutative diagram in Theorem 4.3.3. And because $f \circ x_{\bar{\beta}}$ is a polynomial, permutation of its parameters results another polynomial. Hence by the definition of the weak Zariski topology, $\phi : \mathcal{G}_{\mathcal{D}}(\mathbb{R}) \to \mathcal{G}_{\mathcal{D}}(\mathbb{R})$ is a homeomorphism.

□

4.4.2 Kac-Peterson Topology on Kac-Moody Groups

In this subsection we present the Kac-Peterson topology on split Kac-Moody groups over local fields introduced in [HKM13]. This topology enables us to define the continuity of mappings between split Kac-Moody groups over local fields. In this subsection, we use the same notations as in the previous subsections of the present chapter. Moreover, throughout this subsection $\mathbb{K}$ denotes a local field.

For every k-tuple $\bar{\alpha} = (\alpha_1, ..., \alpha_k) \in (\Phi^{\circ})^k$ define

$$G_{\bar{\alpha}} := G_{\alpha_1}...G_{\alpha_k} \subset \mathcal{G}_{\mathcal{D}}(\mathbb{K}). \tag{4.40}$$

Define a partial order on the set of all k-tuples for all $k \in \mathbb{N}$ as follows:
$\bar{\alpha} \leq \bar{\beta}$ if $\bar{\alpha}$ is an ordered subtuple of $\bar{\beta}$. Hence we have the natural embedding $G_{\bar{\alpha}} \hookrightarrow G_{\bar{\beta}}$ (see [HKM13, page 49]).
Consider the torus T with the Lie group topology and the fundamental rank one subgroups G_{α_s} for $s \in S$ the natural topology coming from $\mathrm{GL}_2(\mathbb{K})$. Denote by $\tau_{\bar{\alpha}}$ the quotient topology on $TG_{\bar{\alpha}}$ with respect to the surjection

$$\pi_{\bar{\alpha}} : T \times G_{\alpha_1} \times ... \times G_{\alpha_k} \to TG_{\bar{\alpha}}.$$

Note that the set of topological spaces $(G_{\bar{\alpha}}, \tau_{\bar{\alpha}})$ forms a directed system with respect to subtuple order defined above. Moreover, $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ is a direct limit of the underlying directed system of sets (see [HKM13, Definition 7.7]).
Definition 4.4.6. [HKM13, Definition 7.8] The **Kac-Peterson topology** τ_{KP} on $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ is the direct limit topology with respect to the directed system $\{TG_{\bar{\alpha}}\}_{\bar{\alpha}}$.

If the Kac-Moody root datum $\mathcal{D}$ is centered, then the Kac-Peterson topology on $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ coincides with the final group topology with respect to the directed system $\{(G_{\bar{\alpha}}, \tau_{\bar{\alpha}}|_{G_{\bar{\alpha}}})\}_{\bar{\alpha}}$ ([HKM13, Remark 7.12]). Moreover, in $(\mathcal{G}_{\mathcal{D}}(\mathbb{K}), \tau_{KP})$ the Borel subgroups $B_{\pm}$ are closed ([HKM13, Proposition 7.15]). In addition, for every real root α, the induced topology on the rank one subgroup G_α by τ_{KP} coincides with the natural topology on G_α induced from $\mathrm{GL}_2(\mathbb{K})$. Furthermore, if H is a spherical subgroup, then the restriction of τ_{KP} to H coincides with its Lie topology and it is closed in $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ ([HKM13, Corollary 7.16]).
Theorem 4.4.7 (Topological Curtis-Tits Theorem)**.** *[HKM13, Theorem 7.22] Let $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ be a two-spherical simply connected split Kac-Moody group. For $\alpha, \beta \in \Phi^{\circ}$, let $\iota_{\alpha\beta} : G_\alpha \hookrightarrow G_{\alpha,\beta}$ be the canonical inclusion morphisms. Then the group $(\mathcal{G}_{\mathcal{D}}(\mathbb{K}), \tau_{KP})$ is a universal enveloping group of the amalgam*

$$\{G_\alpha, G_{\alpha,\beta} : \iota_{\alpha\beta}\}$$

in the category of abstract groups and Hausdorff topological groups.

Remark 4.4.8. Note that by the main result in [AM97], Theorem 2.4.6, even though $\mathbb{Q}$ is not a local field, the split Kac-Moody group $\mathcal{G}_{\mathcal{D}}(\mathbb{Q})$, as in Theorem 4.4.7, is a universal enveloping group of the amalgam

$$\{G_\alpha, G_{\alpha,\beta} : \iota_{\alpha\beta}\}$$

in the category of abstract groups.

With the Kac-Peterson topology, parabolic subgroups are closed, in particular bounded subgroups are closed ([HKM13, Proposition 7.29]). Next result depicts the topological properties of bounded subgroups.

Proposition 4.4.9. *[HKM13, Corollary 7.30] Bounded subgroups are algebraic Lie groups. Their Levi decomposition is a semi-direct product of closed subgroups.*

Remark 4.4.10. Let $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ be a (simply connected) real split Kac-Moody group of finite rank. By the definition of (simply connected) split Kac-Moody groups it is evident that maximal fundamental spherical Levi subgroups are finite-dimentional central extensions of (universal) Chevalley groups. Moreover, the derived subgroup of a maximal Levi subgroup modulo its center constitues an adjoint Chevalley group (see Equation 4.7, see also [Tit92, Section 3.3] and [CM06, Chapter 4]). In this case, since by Lemma 4.1.10 all spherical Levi subgroups of the same type are isomorphic we conclude that the above statement is true for any maximal Levi subgroup. In addition, if the underlying real split Kac-Moody group $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ is simply connected (hence centered) then irreducible fundamental subgroups of finite type are irreducible universal Chevalley groups.

Definition 4.4.11. [HKM13, Definition 7.19] Let $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ be a split Kac-Moody group. The **universal topology** τ on $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ is defined to be the final group topology with respect to the maps

$$\phi_\alpha : \mathrm{SL}_2(\mathbb{K}) \to \mathcal{G}_{\mathcal{D}}(\mathbb{K}), \quad \alpha \in \Phi^{re},$$

and

$$\eta(\mathbb{K}) : \mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[\Lambda], \mathbb{K}) \to \mathcal{G}_{\mathcal{D}}(\mathbb{K}),$$

where $\mathrm{SL}_2(\mathbb{K})$ and $\mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[\Lambda], \mathbb{K}) \cong (\mathbb{K}^\times)^{\mathrm{rank}(\Lambda)}$ are equipped with their Lie group topologies.

Proposition 4.4.12. *Let $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ be a simply connected split Kac-Moody group. Then the universal topology and the Kac-Peterson topology coincide.*

Proof. If we consider $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ as the trivial central quotient of itself, the proposition follows from [HKM13, Proposition 7.21]. □

4.4.3 Kac-Peterson Topology versus Zariski Topology

In this short subsection we mention a direct consequence of a Lemma 7.14 in [HKM13] which shows that on a split Kac-Moody group $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ the Kac-Peterson topology is coarser than the Zariski topology.

Lemma 4.4.13. *[HKM13, Lemma 7.14] Every weakly regular function is continuous with respect to the Kac-Peterson topology.*

Corollary 4.4.14. *Let $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ be a split Kac-Moody group. Then the weak Zariski topology is coarser than the Kac-Peterson topology on $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$.*

Proof. By definition, the weak Zariski topology on $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ is the topology generated by the elements of the following set as open sets:

$$\{\{x \in \mathcal{G}_{\mathcal{D}}(\mathbb{R}) \mid \ f(x) \neq 0\} \mid f \in \mathbb{K}[\mathcal{G}_{\mathcal{D}}(\mathbb{K})]\}. \tag{4.41}$$

It follows from Lemma 4.4.13 that every element of the set in (4.41) is open with respect to the Kac-Peterson topology on $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$. Therefore, the corollary follows. $\square$

CHAPTER

FIVE

ARITHMETIC SUBGROUPS OF KAC-MOODY GROUPS

In the present chapter we introduce arithmetic subgroups of certain real split Kac-Moody groups along with some combinatorial results for such subgroups which will be used in the proof of the super rigidity theorem in Chapter 7. To the knowledge of the author, apart from Proposition 5.1.3, Section 5.1 does not contain any original results. Section 5.2 is to obtain Lemma 5.2.3 and Lemma 5.2.12. At the end, in Section 5.3 we introduce (functorial) principal congruence subgroups of certain real split Kac-Moody groups and the major result is the equation (5.129). We retain the notations introduced in Chapter 4.

5.1 Arithmetic Subgroups

Let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be a Kac-Moody root datum and $\mathcal{G}_\mathcal{D}$ the associated split Tits functor with the Tits basis $\mathcal{F} = (\mathcal{G}_\mathcal{D}, \{\phi_i\}_{i\in I}, \eta)$, introduced in Subsection 4.1.1. By (KMG1) in Section 4.1 we know that the natural embeddings

$$\mathbb{Z} \hookrightarrow \mathbb{Q} \hookrightarrow \mathbb{R}$$

induce the following embeddings of split Kac-Moody groups:

$$\mathcal{G}_\mathcal{D}(\mathbb{Z}) \hookrightarrow \mathcal{G}_\mathcal{D}(\mathbb{Q}) \hookrightarrow \mathcal{G}_\mathcal{D}(\mathbb{R}). \tag{5.1}$$

We call $\mathcal{G}_\mathcal{D}(\mathbb{Z})$ obtained in this way the **functorial arithmetic subgroup** of $\mathcal{G}_\mathcal{D}(\mathbb{Q})$ (or $\mathcal{G}_\mathcal{D}(\mathbb{R})$) with respect to the Tits basis $\mathcal{F}$.
Any split Tits functor $\mathcal{G}_\mathcal{D}$ over large enough fields is unique up to isomorphism meaning if there is a group functor G whose image for a large enough field $\mathbb{K}$ satisfies the conditions (KMG1)-(KMG5) then $G(\mathbb{K})$ is isomorphic to $\mathcal{G}_\mathcal{D}(\mathbb{K})$. But in general, we lose this uniqueness property over commutative unital rings (see [Tit87]).
Therefore, by this construction, unlike the classical groups, it is **not** clear that $\mathcal{G}_\mathcal{D}(\mathbb{Z})$ is the group generated by the $\mathbb{Z}$-points of the rank one subgroups of $\mathcal{G}_\mathcal{D}(\mathbb{Q})$. Nevertheless, since the mappings

$$\phi_\alpha : \mathrm{SL}_2 \to \mathcal{G}_\mathcal{D} \tag{5.2}$$

for $\alpha \in \Phi^{re}$ are morphisms of functors, we have the following commutative diagrams:

$$\begin{tikzcd} \mathcal{G}_{\mathcal{D}}(\mathbb{Z}) \arrow[r, hook, "i"] & G_{\mathcal{D}}(\mathbb{Q}) \\ \mathrm{SL}_2(\mathbb{Z}) \arrow[u, "\phi_\alpha"'] \arrow[r, hook, "i"] & \mathrm{SL}_2(\mathbb{Q}) \arrow[u, "\phi_\alpha"'] \end{tikzcd} .$$

Note that the $\mathbb{Z}$-points of the rank one subgroups $G_\alpha(\mathbb{Q})$ are contained in the image of $\mathrm{SL}_2(\mathbb{Z})$ under ϕ_α which is embedded into the image of $\mathcal{G}_{\mathcal{D}}(\mathbb{Z})$ in $G_{\mathcal{D}}(\mathbb{Q})$ via $\phi_\alpha \circ i$ by the commutativity of the above diagram. This implies that the group generated by the $\mathbb{Z}$-points of the rank one subgroups of $\mathcal{G}_{\mathcal{D}}(\mathbb{Q})$ is indeed a subgroup of $\mathcal{G}_{\mathcal{D}}(\mathbb{Z})$. In particular, the $\mathbb{Z}$-points of the root subgroups of $\mathcal{G}_{\mathcal{D}}(\mathbb{Q})$ are precisely the root subgroups of $\mathcal{G}_{\mathcal{D}}(\mathbb{Z})$, which in this context means, the images of the upper and lower triangular matrices of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathcal{G}_{\mathcal{D}}(\mathbb{Z})$ via the Tits basis morphisms ϕ_α for all $\alpha \in \Phi^{re}$.

Now let $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ be centered. Since $\mathcal{G}_{\mathcal{D}}(\mathbb{Q})$ is generated by the image of the Tits basis ϕ_i, by (KMG3) we know that $\mathcal{T}_\Lambda(\mathbb{Q})$ is generated by

$$\phi_i \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} = \eta\left(\lambda \mapsto r^{h_i(\lambda)}\right), \tag{5.3}$$

for $q \in \mathbb{Q}$. Moreover, by the definition of split Tits functors we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{T}_\Lambda(\mathbb{Q}) \arrow[r, hook, "i"] & G_{\mathcal{D}}(\mathbb{Q}) \\ \mathcal{T}_\Lambda(\mathbb{Z}) \arrow[u, hook, "i"] \arrow[r, hook, "i"] & \mathcal{G}_{\mathcal{D}}(\mathbb{Z}) \arrow[u, hook, "i"'] \end{tikzcd} .$$

This shows that in the case of centered Kac-Moody root datum the torus $\mathcal{T}_\Lambda(\mathbb{Z})$ is generated by the elements of the form

$$\phi_i \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} = \eta\left(\lambda \mapsto (\pm 1)^{h_i(\lambda)}\right), \tag{5.4}$$

which already exist in the $\mathbb{Z}$-points of the rank one subgroups of $\mathcal{G}_{\mathcal{D}}(\mathbb{Q})$.
Recall from Subsection 4.1.1 that every simply connected Kac-Moody group is centered. Hence, the above result holds for simply connected split Kac-Moody groups as well. This implies the following lemma which is well-known to experts.

Lemma 5.1.1. *Let $\mathcal{G}_{\mathcal{D}}$ be a split Tits functor with a Tits basis $\mathcal{F} = (\mathcal{G}_{\mathcal{D}}, \{\phi_i\}_{i\in I}, \eta)$. Then the subgroup of $\mathcal{G}_{\mathcal{D}}(\mathbb{Q})$ generated by all $\phi_\alpha(\mathrm{SL}_2(\mathbb{Z}))$ for all $\alpha \in \Phi^{re}$ and $\mathcal{T}_\Lambda(\mathbb{Z})$ is a subgroup of the functorial arithmetic subgroup $\mathcal{G}_{\mathcal{D}}(\mathbb{Z})$ (with respect to $\mathcal{F}$) of $\mathcal{G}_{\mathcal{D}}(\mathbb{Q})$.*

Definition 5.1.2. Let $\mathcal{G}_{\mathcal{D}}$ be a split Tits functor with a Tits basis $\mathcal{F} = (\mathcal{G}_{\mathcal{D}}, \{\phi_i\}_{i\in I}, \eta)$. We call the subgroup of $\mathcal{G}_{\mathcal{D}}(\mathbb{Z})$ obtained as in Lemma 5.1.1 the **arithmetic subgroup** of $\mathcal{G}_{\mathcal{D}}(\mathbb{Q})$ (or $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$) with respect to $\mathcal{F}$ and denote it by $G_{\mathcal{D}}(\mathbb{Z})$. We use the same notation for any unital commutative ring R.

Here we show that if the generalized Cartan matrix is invertible then the **(standard) torus** $\mathcal{T}_\Lambda(\mathbb{Z})$ of $\mathcal{G}_\mathcal{D}(\mathbb{Z})$ can be recognized in $\mathcal{T}_\Lambda(\mathbb{Q})$ as those elements that preserve the $\mathbb{Z}$-form $\mathcal{U}_\mathbb{Z}$ via the action induced by the adjoint representation. This result is to give a better understanding of the standard torus and will not play any role in the final results of the present thesis.

Proposition 5.1.3. *Let $\mathcal{G}_\mathcal{D}(\mathbb{Q})$ be a simply connected two-spherical split Kac-Moody group obtained from a Kac-Moody root datum, namely $\mathcal{D} = (I, A, \Lambda, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$. Let $\mathcal{U}_\mathbb{Z}$ denote the corresponding $\mathbb{Z}$-form of $\mathcal{G}_\mathcal{D}(\mathbb{Q})$. If the generalized Cartan matrix A is invertible, then $\mathcal{T}_\Lambda(\mathbb{Z}) = \{g \in \mathcal{T}_\Lambda(\mathbb{Q}) \mid \mathbf{Ad}(g)\mathcal{U}_\mathbb{Z} \subseteq \mathcal{U}_\mathbb{Z}\}$.*

Proof. First note that by (4.5) we know that $\mathcal{T}_\Lambda(\mathbb{Q}) \cong \Lambda^\vee \otimes \mathbb{Q}$. Second, because $\mathcal{D}$ is assumed to be simply connected, we have $\Lambda^\vee = \bigotimes_{s\in S} \mathbb{Z}h_s$. Therefore, for every $h \in \mathcal{T}_\Lambda(\mathbb{Q})$ we have $h = \sum h_t \otimes q_t$ where $h_t \in \Lambda^\vee$and $q_t \in \mathbb{Q}$. And hence, (4.31) in Theorem 4.2.3 implies that

$$Ad_\mathbb{Q}(h)(e_s \otimes 1) = \langle \alpha_s, h\rangle(e_s \otimes 1)$$

where $e_s \otimes 1 \in \mathcal{U}_\mathbb{Z}$ and $s \in S$ for the corresponding Weyl group (W, S). Moreover, by (4.2) we have

$$\langle \alpha_s, h\rangle = \sum \langle h_t, c_s\rangle \otimes q_t = \sum_t A_{st} \otimes q_t = \sum_t 1 \otimes q_t^{A_{st}}$$

where A_{st}'s come from the generalized Cartan matrix A of $\mathcal{G}_\mathcal{D}(\mathbb{Q})$. Now we have

$$Ad_\mathbb{Q}(h)(e_s \otimes 1) = \langle \alpha_s, h\rangle(e_s \otimes 1) = (\sum_t 1 \otimes q_t^{A_{st}})(e_s \otimes 1) = e_s \otimes \prod_t q_t^{A_{st}}$$

Futhermore, if $\prod_t q_t^{A_{st}} \neq \pm 1$, then $\mathbf{Ad}_\mathbb{Q}(h)$ does not preserve $\mathcal{U}_\mathbb{Z}$ because

$$\mathbf{Ad}_\mathbb{Q}(h)(f_s \otimes 1) = f_s \otimes 1/(\prod_t q_t^{A_{st}})$$

and this holds for any $s \in S$.
Now we claim $\prod_t q_t^{A_{st}} = \pm 1$ if and only if all q_t are ± 1, hence $h \in \mathcal{T}_\Lambda(\mathbb{Z})$.
For if, we take the logarithm of the absolute value of each equation $\prod_t q_t^{A_{st}} = \pm 1$ we end up with the following linear system of equations where $\{s_i \mid i \in I\} = S$ and k is the rank of $\mathcal{G}_\mathcal{D}(\mathbb{Q})$:

$$A \cdot \begin{pmatrix} \ln(|q_{s_1}|) \\ \ln(|q_{s_2}|) \\ \vdots \\ \ln(|q_{s_k}|) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since A is invertible by our assumption, this implies that $\ln(|q_t|) = 0$ and hence $q_t = \pm 1$ for all $t \in S$. □

Next we present a decomposition of $\mathcal{G}_\mathcal{D}(\mathbb{Q})$ given by the action of $G_\mathcal{D}(\mathbb{Z})$ on the building corresponding to $\mathcal{G}_\mathcal{D}(\mathbb{Q})$. This is an unpublished result by Ralf Köhl (for the affine case, see also [CG12, Theorem 9.10]).

Proposition 5.1.4. *Let $\mathcal{G}_\mathcal{D}(\mathbb{Q})$ be an irreducible simply connected two-spherical split Kac-Moody group. Then $\mathcal{G}_\mathcal{D}(\mathbb{Q}) = G_\mathcal{D}(\mathbb{Z})\mathcal{T}_\Lambda(\mathbb{Q})U_\pm(\mathbb{Q})$.*

Proof. Note that $\mathcal{T}_\Lambda(\mathbb{Q})U_\pm(\mathbb{Q}) = B_\pm(\mathbb{Q})$ which correspond to the pair of the opposite fundamental chambers in $\Delta_{\mathcal{G}_\mathcal{D}(\mathbb{Q})}$, the twin building associated to $\mathcal{G}_\mathcal{D}(\mathbb{Q})$. So, to prove the proposition it suffices to show that the action of $G_\mathcal{D}(\mathbb{Z})$ on each of the buildings $\Delta^\pm_{\mathcal{G}_\mathcal{D}(\mathbb{Q})}$ is transitive.
By Proposition 2.4.5, it is enough to show that for the fundamental chambers $c_\pm$ in $\Delta^\pm_{\mathcal{G}_\mathcal{D}(\mathbb{Q})}$ the action of the arithmetic subgroup $G_{\alpha_s}(\mathbb{Z})$ of the fundamental rank one subgroup $G_{\alpha_s}(\mathbb{Q})$ on each s-panel $\sigma_s(c_\pm)$ is transitive where s runs through S.
Since $\mathcal{G}_\mathcal{D}(\mathbb{Q})$ is split we have (see Lemma 4.1.11)

$$G_\alpha(\mathbb{Q}) \cong (\mathrm{P})\,\mathrm{SL}_2(\mathbb{Q}), \tag{5.5}$$

and

$$G_\alpha(\mathbb{Z}) \cong (\mathrm{P})\,\mathrm{SL}_2(\mathbb{Z}). \tag{5.6}$$

Moreover, by the Moufang property, the action of the root subgroup U_{α_s} on $\sigma_s(c_\pm)\backslash\{c_\pm\}$ is sharply transitive. Now since $\mathcal{G}_\mathcal{D}(\mathbb{Q})$ is split and has the Moufang property, one can think of $\sigma_s(c_\pm)$ as the projective line over $\mathbb{Q}$ on which $G_{\alpha_s}(\mathbb{Q})$ acts via the Möbius transformations in view of (5.5) and so does $G_{\alpha_s}(\mathbb{Z}) \hookrightarrow G_{\alpha_s}(\mathbb{Q})$ in view of (5.6).
Hence, our argument narrows down to showing that the action of $\mathrm{SL}_2(\mathbb{Z})$ on the projective line over $\mathbb{Q}$ is transitive. For this, let $[x:y]$ be a point in $\mathbb{P}^1(\mathbb{Q})$, the projective line over $\mathbb{Q}$. The action of an element $\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ on $[x:y]$ is defined as follows

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.[x:y] = [a_{1,1}.x + a_{1,2} : a_{2,1}y + a_{2,2}].$$

A routine matrix calculation shows that for two arbitrary points $[x_1 : x_2]$ and $[y_1 : y_2]$ in $\mathbb{P}^1(\mathbb{Q})$, there exists an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that the following equation holds

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.[x_1 : x_2] = [y_1 : y_2].$$

Hence the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ is transitive. □

We devote the rest of this section to presenting some well-known results on commutation relations of root subgroups of split Kac-Moody groups. For this, first we need to know about the structure of the root strings defined below.
Let $\mathcal{G}_\mathcal{D}$ be a split Tits functor with a Tits basis $\mathcal{F} = (\mathcal{G}_\mathcal{D}, \{\phi_i\}_{i\in I}, \eta)$. Let $\beta \in \Phi$ and $\alpha \in \Phi^{re}$. There exist $p, q \in \mathbb{N}$ such that the α-string through β is (see [BP95, page 392])

$$\beta - p\alpha, ..., \beta, ..., \beta + q\alpha. \tag{5.7}$$

The next proposition describes the positions of real and imaginary roots in the above root string. We visualize this by attaching to the above string a series of nodes; black for real roots and white for imaginary roots.
Proposition 5.1.5. *[BP95, Proposition 1] Let $\beta \in \Phi$ and $\alpha \in \Phi^{re}$. Let $r(\alpha, \beta)$ denote the number of real roots of the α-string through β, (5.7). Assume $r(\alpha, \beta) > 0$. Then*

(i) The first and last roots of (5.7) are real.

(ii) $r(\alpha, \beta) = 1, 2, 3$ *or* 4. *Moreover,*

(a) If $r(\alpha, \beta) = 1$, *then the* α*-string through* β, *(5.7), contains only* β *and* β *is real.*

(b) If $r(\alpha, \beta) = 2$, *then the* α*-string through* β, *(5.7), is depicted by a diagram of shape*

$$\bullet \ \circ \cdots \circ \cdots \circ \ \bullet.$$

(c) If $r(\alpha, \beta) = 3$, *then the* α*-string through* β, *(5.7), is depicted by the diagram* $\bullet \ \bullet \ \bullet$ *and* $\{\alpha, \beta\}$ *generates a root system of type* C_2.

(d) If $r(\alpha, \beta) = 4$, *then the* α*-string through* β, *(5.7), is depicted by a diagram of shape*

$$\bullet \ \bullet \ \circ \cdots \circ \cdots \circ \ \bullet \ \bullet.$$

Furthermore, if the α*-string through* β, *(5.7), does not contain any imaginary roots, then* $\{\alpha, \beta\}$ *generates a root system of type* G_2.

Lemma 5.1.6. *Let* $\mathcal{G}_{\mathcal{D}}$ *be a simply-connected two-spherical split Tits functor with a Tits basis* $\mathcal{F} = (\mathcal{G}_{\mathcal{D}}, \{\phi_i\}_{i \in I}, \eta)$. *Let* $\alpha, \beta \in \Phi^{re}$ *be a pair of real roots such that* $\{\alpha, \beta\}$ *generates a finite root system. Then*

(a) If $r(\alpha, \beta) = 1$, *then* $\{\alpha, \beta\}$ *generates a root system of type* $A_1 \times A_1$.

(b) If $r(\alpha, \beta) = 2$, *then* $\{\alpha, \beta\}$ *generates a root system of type* A_2.

(c) If $r(\alpha, \beta) = 3$, *then* $\{\alpha, \beta\}$ *generates a root system of type* C_2.

(d) If $r(\alpha, \beta) = 4$, *then* $\{\alpha, \beta\}$ *generates a root system of type* G_2.

Proof. It follows from Proposition 5.1.5, in view of Lemma 4.1.10, Remark 4.4.10 and Theorem 1.2.3. □

The next aim is to give a formula for calculating the commutation relations of root subgroups. To do this first we present a geometric description of a pair of prenilpotent twin roots inside the associated fundamental twin apartment of the twin building associated to the underlying split Kac-Moody group (see Section 2.1).

Remark 5.1.7. [CM06, Subsection 2.3.1] A pair (α, β) of twin roots of the fundamental apartment $\mathcal{A}(\mathcal{A}_+, \mathcal{A}_-)$ is prenilpotent if $\alpha \cap \beta \cap \mathcal{A}_+$ and $(-\alpha) \cap (-\beta) \cap \mathcal{A}_+$ are both nonempty. Therefore, by notations as in Section 2.1, $[\alpha, \beta]$ is the (finite) set of all twin roots λ in $\mathcal{A}$ such that $\lambda \supseteq \alpha \cap \beta$ and $-\lambda \supseteq (-\alpha) \cap (-\beta)$. Note that in case (α, β) is **not** prenilpotent then $(\alpha, -\beta)$ forms a prenilpotent pair by the above geometric description.

Now we give an explicit formula for the commutation relation of root subgroups corresponding to a prenilpotent pair of twin roots. This is due to Tits in [Tit87].

For every prenilpotent pair of (twin) roots (α, β), choose a total order on the finite set $[\alpha, \beta]\backslash\{\alpha, \beta\}$. Then, there exist well-defined integers $k(\alpha, \beta, \lambda)$ such that for any commutative unital ring R and any $r, r' \in R$, the following commutation relation holds:

$$[x_\alpha(r), x_\beta(r')] = \prod_{\lambda = m\alpha + n\beta} x_\lambda(k(\alpha, \beta, \lambda) r^m r'^n), \tag{5.8}$$

where $\lambda = m\alpha + n\beta$ runs through $[\alpha, \beta]\backslash\{\alpha, \beta\}$ with respect to the total order (see [Tit87, Section 3.6]). Note that in the above formula the commutator is defined by $[g, h] := ghg^{-1}h^{-1}$.

For any pair of real roots $\alpha, \beta \in \Phi^{re}$ such that $\alpha + \beta \in \Phi^{re}$ define

$$Q_{\alpha,\beta} := (\mathbb{Z}_{\geq 0}\alpha + \mathbb{Z}_{\geq 0}\beta) \cap \Phi^{re}. \tag{5.9}$$

Note that in this situation, we can look at the root system $\Phi_{\alpha,\beta}$ generated by $\{\alpha, \beta\}$ in Φ. Since $\Phi_{\alpha,\beta}$ is a root system of rank two, the corresponding generalized Cartan matrix $A_{\alpha,\beta}$ is symmetrizable. Hence, the exists a symmetric bilinear form $(\ |\ )$ on $\Phi_{\alpha,\beta}$ induced by $A_{\alpha,\beta}$.

Theorem 5.1.8. *[Mor87, Theorem 2] Let $\mathcal{G}_\mathcal{D}$ be a split Tits functor with a Tits basis $\mathcal{F} = (\mathcal{G}_\mathcal{D}, \{\phi_i\}_{i\in I}, \eta)$ and R a commutative unital ring. Let $\alpha, \beta \in \Phi^{re}$ be a pair of real roots such that $\alpha + \beta \in \Phi^{re}$. Assume that $(\mathbb{Z}_{\geq 0}\alpha + \mathbb{Z}_{\geq 0}\beta) \cap \Phi^{im} = \emptyset$ and $(\alpha \mid \alpha) \leq (\beta \mid \beta)$. For $s, t \in R$ we have:*

(1) If $Q_{\alpha,\beta} = \{\alpha, \beta\}$, then either $\beta = -\alpha$ or
$[x_\alpha(s), x_\beta(t)] = 1$.

(2) If $Q_{\alpha,\beta} = \{\alpha, \beta, \alpha + \beta\}$, then
$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm(p + 1)st)$
where p is obtained by the α-string through β: $\beta - p\alpha, ..., \beta, ..., \beta + q\alpha$.

(3) If $Q_{\alpha,\beta} = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$, then
$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm st)x_{2\alpha+\beta}(\pm s^2 t)$.

(4) If $Q_{\alpha,\beta} = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, \alpha + 2\beta\}$, then
$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm 2st)x_{2\alpha+\beta}(\pm 3s^2 r)x_{\alpha+2\beta}(\pm 3st^2)$.

(5) $Q_{\alpha,\beta} = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}$, then
$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm st)x_{2\alpha+\beta}(\pm s^2 t)x_{3\alpha+\beta}(\pm s^3 t)x_{3\alpha+2\beta}(\pm 2s^3 t^2)$.

Moreover, only one of these five cases happens.

Remark 5.1.9. Let $\mathcal{G}_\mathcal{D}$ be a two-spherical split Tits functor. By definition, the condition $(\mathbb{Z}_{\geq 0}\alpha + \mathbb{Z}_{\geq 0}\beta) \cap \Phi^{im} = \emptyset$ in Theorem 5.1.8 automatically holds for every pair of simple roots $\alpha, \beta \in \Phi^\circ$ by the two-spherical property of $\mathcal{G}_\mathcal{D}$. Moreover, in this situation, by Lemma 5.1.6, Case (4) of Theorem 5.1.8 does **not** occur. Furthermore, in Case (2) of Theorem 5.1.8, $p = 0$.

5.2 $n\mathbb{Z}$-Points of Arithmetic Subgroups

In this section we investigate some properties of the group generated by the $n\mathbb{Z}$-points of certain real split Kac-Moody groups. These properties will be essential in Chapter 7. Here we follow the notations introduced in the previous section.

Terminology 5.2.1. *Let $\mathcal{G}_{\mathcal{D}}$ be an irreducible simply connected two-spherical Tits functor with a Tits basis $\mathcal{F} = (\mathcal{G}_{\mathcal{D}}, \{\phi_i\}_{i\in I}, \eta)$.*

(1) For any $n \in \mathbb{N}$, denote by $G_{\mathcal{D}}(n\mathbb{Z})$ the subgroup of $G_{\mathcal{D}}(\mathbb{Z})$ generated by the $n\mathbb{Z}$-points of the real root subgroups of $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$.

(2) Denote every pair of simple roots $\alpha, \beta \in \Phi^\circ$ whose corresponding nodes on the Dynkin diagram of $\mathcal{D}$ are connected by $\alpha \approx \beta$.

(3) For a pair of simple roots $\alpha \approx \beta \in \Phi^\circ$, the rank two fundamental subgroup $G_{\alpha\beta}$ is an irreducible universal Chevalley group by Remark 4.4.10. One can observe that

$$G_{\alpha\beta}(\mathbb{Q}) = \{U_{\pm\lambda}(\mathbb{Q}) \mid \lambda \in Q_{\alpha,\beta}\}. \tag{5.10}$$

For any $n \in \mathbb{N}$, denote by $G_{\alpha\beta}(n\mathbb{Z})$ the subgroup of $G_{\alpha\beta}(\mathbb{Z})$ generated by the $n\mathbb{Z}$-points of the root subgroups of $G_{\alpha\beta}$.

(4) We denote by $\Gamma(n)$ (also by $\Gamma_{\Phi^\circ}(n)$) the subgroup of $G_{\mathcal{D}}(\mathbb{Z})$ (or $G_{\mathcal{D}}(n\mathbb{Z})$) generated by all $G_{\alpha,\beta}(n\mathbb{Z})$ where $\alpha \approx \beta \in \Phi^\circ$.

(5) Let $G^{con}_{\alpha\beta}(n\mathbb{Z})$ denote the principal congruence subgroup of $G_{\alpha\beta}(\mathbb{Z})$ with respect to the ideal $n\mathbb{Z}$ in $\mathbb{Z}$. We denote by $\Gamma^{con}(n)$ (also by $\Gamma^{con}_{\Phi^\circ}(n)$) the subgroup of $G_{\mathcal{D}}(\mathbb{Z})$ generated by all $G^{con}_{\alpha,\beta}(n\mathbb{Z})$ where $\alpha \approx \beta \in \Phi^\circ$.

Before stating the next result, recall that each element $w \in W$ corresponds to an element of N/T where N denotes the set-wise stabilizer of the fundamental (twin) apartment and T is the chamber-wise stabilizer of the fundamental (twin) apartment in the twin building associated to $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ as we discussed in Subsection 4.1.1.
More precisely, define

$$\xi : N \to W \qquad \tilde{s}_i := \phi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto s_i. \tag{5.11}$$

This assignment extends to a surjective homomorphism, denoted again by ξ, whose kernel is T (see [Cap05, Lemma 1.4]).
Define $\widetilde{W}$ to be the subgroup of N generated by the set $\widetilde{S} := \{\tilde{s}_i \mid i \in I\}$. Let $T_{(2)}$ be the subgroup of T generated by the set $\{\tilde{s}_i{}^2 \mid i \in I\}$. Then, $T_{(2)} = \{h \in T \mid h^2 = 1\}$ and the inclusion map $\widetilde{W} \hookrightarrow N$ induces an isomorphism from $\widetilde{W}/T_{(2)}$ onto W via ξ in (5.11). Moreover, there exists a unique map (which need **not** be a homomorphism) such that (see [KP85, Corollary 2.3])

$$\sim : (W, S) \to (\widetilde{W}, \widetilde{S}) \quad w \mapsto \widetilde{w}, \tag{5.12}$$

such that

(i) $\tilde{1}_W = 1_{\widetilde{W}}$;

(ii) $\tilde{s}_i := \phi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

(ii) $\widetilde{ww'} = \tilde{w}\tilde{w}'$ if $w, w' \in W$ and $l(ww') = l(w) + l(w')$.

In addition, if $\psi : \widetilde{W} \to W$ is the canonical map, then the map in (5.12) is a well-defined section of ψ (see [KP85, Corollary 2.3]).
Thus, for any reduced expression $w = s_{w_1} \cdots s_{w_l}$ of $w \in W$, we conclude that w is mapped to $\tilde{w} := \tilde{s}_{w_1} \cdots \tilde{s}_{w_l}$ where $\{s_{w_1}, \cdots, s_{w_l}\} \subset S$ and $l \in \mathbb{N}$.
Furthermore, this construction implies

$$\tilde{w} = \phi_{s_{w_1}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdots \phi_{s_{w_l}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5.13}$$

which lies in $G_{\mathcal{D}}(\mathbb{Z})$.
Now for each $w \in W$ define σ_w to be the action of $\tilde{w}$ on the twin building associated to $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$. Then it follows from [AB08, Lemma 8.17] that for any $s \in S$ we have

$$\sigma_w(U_{\alpha_s}) := \tilde{w} U_{\alpha_s} \tilde{w}^{-1} = U_{w(\alpha_s)}, \tag{5.14}$$

which depends only on $\alpha := w(\alpha_s)$ (see [Cap05, Lemma 1.4]).
Lemma 5.2.2. *Let $\mathcal{G}_{\mathcal{D}}$ be a simply connected two-spherical split Tits functor with a Tits basis $\mathcal{F} = (\mathcal{G}_{\mathcal{D}}, \{\phi_i\}_{i\in I}, \eta)$. Then*

$$\sigma_w(U_{\alpha_s}(n\mathbb{Z})) = U_{w(\alpha_s)}(n\mathbb{Z}) \tag{5.15}$$

for all $n \in \mathbb{N}$, $s \in S$ and $w \in W$.
Moreover, σ_w preserves $G_{\mathcal{D}}(n\mathbb{Z})$.

Proof. First, by (5.13) we have $\tilde{w} \in G_{\mathcal{D}}(\mathbb{Z})$. Hence by (5.14) and the functoriality of the root subgroups induced by the Tits basis $\mathcal{F}$ (see Subsection 4.1.1) we know that every generator (actually any of the two generators of $U_{\alpha_s}(\mathbb{Z})$, namely $U_{\alpha_s}(\pm 1)$), is mapped to a generator of $U_{w(\alpha_s)}(\mathbb{Z})$ (actually one of the two generators of $U_{w(\alpha_s)}(\mathbb{Z})$, namely $U_{w(\alpha_s)}(\pm 1)$).
And because σ_w is a group homomorphism, we conclude that n-the power of any generator of $U_{\alpha_s}(\mathbb{Z})$ is mapped to either $U_{w(\alpha_s)}(n)$ or $U_{w(\alpha_s)}(-n)$. But both $U_{\alpha_s}(n)$ and $U_{\alpha_s}(-n)$ are generators of $U_{w(\alpha_s)}(n\mathbb{Z})$. Therefore, (5.15) follows.
Moreover, because by definition $G_{\mathcal{D}}(n\mathbb{Z})$ is the group generated by the $n\mathbb{Z}$-points of the real root subgroups of $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ (see Terminology 5.2.1) it follows from (5.15) that σ_w preserves $G_{\mathcal{D}}(n\mathbb{Z})$. □

The next lemma shows how one can recover $G_{\mathcal{D}}(n\mathbb{Z})$ by acting on $\Gamma(n)$ via σ_w for all $w \in W$.
Lemma 5.2.3. *Let $\mathcal{G}_{\mathcal{D}}$ be an irreducible simply connected centered two-spherical split Tits functor with a Tits basis $\mathcal{F} = (\mathcal{G}_{\mathcal{D}}, \{\phi_i\}_{i\in I}, \eta)$. Then we have*

$$G_{\mathcal{D}}(n\mathbb{Z}) = \langle \sigma_w(\Gamma(n)) \mid w \in W \rangle \subset \langle \sigma_w(\Gamma^{con}(n)) \mid w \in W \rangle, \tag{5.16}$$

for all $n \in \mathbb{N}$. In particular,

$$G_{\mathcal{D}}(\mathbb{Z}) = \langle \sigma_w(\Gamma(1)) \mid w \in W \rangle = \langle \sigma_w(\Gamma^{con}(1)) \mid w \in W \rangle. \tag{5.17}$$

Proof. First note that, by definition, it is evident that

$$\langle \sigma_w(\Gamma(n)) \mid w \in W\rangle \subset \langle \sigma_w(\Gamma^{con}(n)) \mid w \in W\rangle$$

and, in particular,

$$\langle \sigma_w(\Gamma(1)) \mid w \in W\rangle = \langle \sigma_w(\Gamma^{con}(1)) \mid w \in W\rangle.$$

Now let α be a real root. By definition, there exist $s \in S$ and $w \in W$ such that $\alpha = w(\alpha_s)$. Lemma 5.2.2 implies that

$$\sigma_w(U_{\alpha_s}(n\mathbb{Z})) = U_\alpha(n\mathbb{Z}) \subseteq \langle \sigma_\omega(\Gamma(n)) \mid \omega \in W\rangle. \tag{5.18}$$

Therefore, $U_{\alpha_s}(n\mathbb{Z}) \leq \Gamma(n)$ implies

$$G_{\mathcal{D}}(n\mathbb{Z}) = \langle U_\alpha(n\mathbb{Z}) \mid \alpha \in \Phi^{re}\rangle \subseteq \langle \sigma_w(\Gamma(n)) \mid w \in W\rangle. \tag{5.19}$$

Conversely, by definition we have $\Gamma(n) \subseteq G_{\mathcal{D}}(n\mathbb{Z})$. Furthermore, since $\Gamma(n)$ is generated by $G_{\alpha_{s_1}\alpha_{s_2}}(n\mathbb{Z})$ for all pairs $\alpha_{s_1} \approx \alpha_{s_2} \in \Phi^\circ$, for any $w \in W$ the image $\sigma_w(\Gamma(n))$ is generated by $\sigma_w(G_{\alpha_{s_1}\alpha_{s_2}}(n\mathbb{Z}))$ for all pairs $\alpha_{s_1} \approx \alpha_{s_2} \in \Phi^\circ$ (see Terminology 5.2.1).
In addition, for any $w \in W$, $\sigma_w(G_{\alpha_{s_1}\alpha_{s_2}}(n\mathbb{Z}))$ is generated by $\sigma_w(G_\tau(n\mathbb{Z})) = G_{w(n\tau)}(\mathbb{Z})$ where τ runs through $Q_{\alpha_{s_1},\alpha_{s_2}}$. For if, $\mathcal{D}$ is two-spherical and hence all combinations of root α_{s_1} and α_{s_2} are real and generate the full (finite) root system of $G_{\alpha_{s_1}\alpha_{s_2}}$ by Lemma 5.1.6.
Now note that $\sigma_w(G_\tau(n\mathbb{Z})) = G_{w(\tau)}(n\mathbb{Z})$ is contained in $G_{\mathcal{D}}(n\mathbb{Z})$ because $w(\tau)$ is a real root in $\mathcal{D}$. Hence $G_{\mathcal{D}}(n\mathbb{Z})$ contains all generators of $\sigma_w(\Gamma(n))$ for all $w \in W$. Therefore,

$$\langle \sigma_w(\Gamma(n)) \mid w \in W\rangle \subseteq G_{\mathcal{D}}(n\mathbb{Z}) \tag{5.20}$$

which yields (5.16) in view of (5.19). □

Corollary 5.2.4. *Let $\mathcal{G}_{\mathcal{D}}$ be an irreducible simply connected two-spherical split Tits functor with a Tits basis $\mathcal{F} = (\mathcal{G}_{\mathcal{D}}, \{\phi_i\}_{i\in I}, \eta)$. Then*

$$\Gamma(1) = G_{\mathcal{D}}(\mathbb{Z}). \tag{5.21}$$

Proof. First note that for every $w \in W$ we have $\tilde{w} \in \Gamma(1)$ by (5.13). Therefore, by the definition of $\tilde{w}$ we have $\langle \sigma_w(\Gamma(1)) : w \in W\rangle = \Gamma(1)$. Now (5.17) in Lemma 5.2.3 implies the corollary. □

Corollary 5.2.5. *Let $\mathcal{G}_{\mathcal{D}}$ be an irreducible simply connected two-spherical split Tits functor with a Tits basis $\mathcal{F} = (\mathcal{G}_{\mathcal{D}}, \{\phi_i\}_{i\in I}, \eta)$. Then*

$$\langle x_{\pm\alpha_s}(1) \mid s \in S\rangle = G_{\mathcal{D}}(\mathbb{Z}). \tag{5.22}$$

Proof. On the one hand, by Corollary 5.2.4 we have $\Gamma(1) = G_{\mathcal{D}}(\mathbb{Z})$.
On the other hand, similar to the proof of Lemma 5.2.2, for every $\tau \in Q_{\alpha_s,\alpha_t}$ where $\alpha_s \approx \alpha_t \in \Phi^\circ$, $U_\tau(\mathbb{Z}) = \sigma_w(U_{\alpha_l})$ for some $w \in W$ and $l \in S$. Moreover, by (5.13) we know that for every $w \in W$, $\tilde{w} \in \langle x_{\pm\alpha_s}(1) \mid s \in S\rangle$. Therefore, by the definition of $\Gamma(1)$, we have

$$\langle x_{\pm\alpha_s}(1) \mid s \in S\rangle = \Gamma(1) = G_{\mathcal{D}}(\mathbb{Z}).$$

□

In the next result, Lemma 5.2.6, we consider the following formula for the commutation brackets:

$$[g, h] = g^{-1}h^{-1}gh. \tag{5.23}$$

Nonetheless, for the other formula $[g, h]' = ghg^{-1}h^{-1}$, which is used in some literature, we have $[g, h]' = [g^{-1}, h^{-1}]$. Now since in the proof of Lemma 5.2.6 we are only interested in relations of the generators and the fact that both $x_\alpha(n)$ and $x_\alpha(-n)$ generate $U_\alpha(n\mathbb{Z})$ yields that the proof of Lemma 5.2.6 is immune to the change that might occur by the different choices of the commutation brackets. Moreover, we use frequently the following formulas in the proof of Lemma 5.2.6:

$$[x, zy] = [x, y][x, z]^y \qquad [xz, y] = [x, y]^z[z, y] \qquad x^y = x[x, y], \tag{5.24}$$

where $x^y := x^{-1}yx$.
Furthermore, let $\mathcal{G}_\mathcal{D}(\mathbb{R})$ be a split Kac-Moody group. Recall that for any two real roots $\alpha, \beta \in \Phi^{re}$, we denote the set $(\mathbb{Z}_{\geq 0}\alpha + \mathbb{Z}_{\geq 0}\beta) \cap \Phi^{re}$ by $Q_{\alpha,\beta}$.
Lemma 5.2.6. *Let $\mathcal{G}_\mathcal{D}(\mathbb{R})$ be a split Kac-Moody group and $\alpha, \beta \in \Phi^{re}$. Assume $(\mathbb{Z}_{\geq 0}\alpha + \mathbb{Z}_{\geq 0}\beta) \cap \Phi^{im} = \emptyset$ and $(\alpha \mid \alpha) \leq (\beta \mid \beta)$. Then for any $n \in \mathbb{N}$ we have*

$$\forall \gamma \in Q_{\alpha,\beta}\ \exists\ j_\gamma \in \mathbb{N} \text{ such that } U_\gamma(j_\gamma n\mathbb{Z}) \leq \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.25}$$

Proof. We argue case by case as presented in Theorem 5.1.8.

Note that for **Case 1**(5.1.8(1)) the claim is obvious.

Case 2 (5.1.8(2)):
In this case, we have the following commutation relation:

$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm l_{\alpha,\beta}st), \tag{5.26}$$

where $l_{\alpha,\beta} := (p + 1)$ to simplify the notations. Therefore,

$$\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle \ni [x_\alpha(n), x_\beta(n)] = x_{\alpha+\beta}(\pm l_{\alpha,\beta}n^2) \tag{5.27}$$

and hence, the claim follows by setting $j_{\alpha+\beta} = l_{\alpha,\beta}n$, since $x_\theta(m)$ is a generator of $U_\theta(m\mathbb{Z})$ for any $\theta \in \Phi^{re}$ and any $m \in \mathbb{Z}$.

Case 3 (5.1.8(3)):
In this case by Equation (5.26) and again by Theorem 5.1.8(3) we have the following commutation relations

$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm st)x_{2\alpha+\beta}(\pm s^2t) \tag{5.28}$$

$$[x_\alpha(s), x_{\alpha+\beta}(t)] = x_{2\alpha+\beta}(\pm l_{\alpha,\alpha+\beta}st). \tag{5.29}$$

Therefore,

$$\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle \ni [x_\alpha(n), x_\beta(n)] = x_{\alpha+\beta}(\pm n^2)x_{2\alpha+\beta}(\pm n^3) \tag{5.30}$$

Note that since we are working with $\mathbb{Z}$ and only interested in the generators of $U_\alpha(n\mathbb{Z})$ for any $\alpha \in \Phi^{re}$, we only consider positive coefficients from now on. With similar arguments, one can obtain similar equations for other signs of the coefficients which appear in the process. Now Equation (5.30) implies that

$$\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle \ni [x_\alpha(n), x_{\alpha+\beta}(n^2)x_{2\alpha+\beta}(n^3)] \tag{5.31}$$

Now we use (5.24) to calculate (5.31). Hence

$$[x_\alpha(n), x_{\alpha+\beta}(n^2)x_{2\alpha+\beta}(n^3)] \overset{(5.24)}{=} [x_\alpha(n), x_{2\alpha+\beta}(n^3)][x_\alpha(n), x_{\alpha+\beta}(n^2)]^{x_{2\alpha+\beta}(n^3)} \tag{5.32}$$

$$\overset{(5.1.8(1),5.29,5.24)}{=} x_{2\alpha+\beta}(l_{\alpha,\alpha+\beta}n^3) \tag{5.33}$$

which, by (5.31), implies that

$$\text{for } j_{2\alpha+\beta} := l_{\alpha,\alpha+\beta}n^2 \quad : \quad U_{2\alpha+\beta}(j_{2\alpha+\beta}n\mathbb{Z}) \le \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.34}$$

On the other hand, we have

$$\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle \ni [x_\alpha(n), x_\beta(l_{\alpha,\alpha+\beta}n)] \overset{(5.28)}{=} x_{\alpha+\beta}(l_{\alpha,\alpha+\beta}n^2)x_{2\alpha+\beta}(l_{\alpha,\alpha+\beta}n^3) \tag{5.35}$$

which, by (5.34), implies that $x_{\alpha+\beta}(l_{\alpha,\alpha+\beta}n^2) \in \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle$.

Therefore

$$\text{for } j_{\alpha+\beta} := l_{\alpha,\alpha+\beta}n \quad : \quad U_{\alpha+\beta}(j_{\alpha+\beta}n\mathbb{Z}) \le \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.36}$$

Case 4 (5.1.8(4)):

For this case, again by Theorem 5.1.8(5) we have the following equation

$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm 2st)x_{2\alpha+\beta}(\pm 3s^2t)x_{\alpha+2\beta}(\pm 3st^2) \tag{5.37}$$

Moreover, $Q_{\alpha,\alpha+\beta} = \{\alpha, \alpha+\beta, 2\alpha+\beta\}$ and $Q_{\beta,\alpha+\beta} = \{\alpha, \alpha+\beta, \alpha+2\beta\}$ which are **Case** (2) and induce the following equations:

$$[x_\alpha(s), x_{\alpha+\beta}(t)] = x_{2\alpha+\beta}(\pm l_{\alpha,\alpha+\beta}st), \tag{5.38}$$

$$[x_\beta(s), x_{\alpha+\beta}(t)] = x_{\alpha+2\beta}(\pm l_{\beta,\alpha+\beta}st) \tag{5.39}$$

Now by (5.37) we have

$$\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle \ni [x_\alpha(n), x_\beta(n)] = x_{\alpha+\beta}(2n^2)x_{2\alpha+\beta}(3n^3)x_{\alpha+2\beta}(3n^3) \tag{5.40}$$

and hence

$$\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle \ni [x_\alpha(n), x_{\alpha+\beta}(2n^2)x_{2\alpha+\beta}(3n^3)x_{\alpha+2\beta}(3n^3)]. \tag{5.41}$$

We use equations in (5.24) to calculate (5.41):

$$(5.41) \overset{(5.24)}{=} [x_\alpha(n), x_{2\alpha+\beta}(3n^3)x_{\alpha+2\beta}(3n^3)][x_\alpha(n), x_{\alpha+\beta}(2n^2)]^{x_{2\alpha+\beta}(3n^3)x_{\alpha+2\beta}(3n^3)}. \tag{5.42}$$

Note that the first term is the identity by (5.24) and (5.1.8(1)). Again by (5.24) and (5.38) we have

$$(5.41) \overset{(5.24,5.38)}{=} x_{2\alpha+\beta}(2l_{\alpha,\alpha+\beta}n^3)[x_{2\alpha+\beta}(2l_{\alpha,\alpha+\beta}n^3), x_{2\alpha+\beta}(3n^3)x_{\alpha+2\beta}(3n^3)] \tag{5.43}$$

where the second term is identity by (5.1.8(1)), therefore

$$x_{2\alpha+\beta}(2l_{\alpha,\alpha+\beta}n^3) \in \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.44}$$

Hence for $j_{2\alpha+\beta} := 2l_{\alpha,\alpha+\beta}n^2$ we have

$$U_{2\alpha+\beta}(j_{2\alpha+\beta}n\mathbb{Z}) \leq \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.45}$$

With similar computations using (5.39) one can obtain the following equation:

$$\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle \ni [x_\beta(n), x_{\alpha+\beta}(2n^2)x_{2\alpha+\beta}(3n^3)x_{\alpha+2\beta}(3n^3)] = x_{\alpha+2\beta}(2l_{\beta,\alpha+\beta}n^3). \tag{5.46}$$

This implies that for $j_{\alpha+2\beta} := 2l_{\beta,\alpha+\beta}n^2$ we have

$$U_{\alpha+2\beta}(j_{\alpha+2\beta}n\mathbb{Z}) \leq \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.47}$$

Using Equation 5.37 we have

$$\begin{aligned}\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle &\ni [x_\alpha(l_{\beta,\alpha+\beta}n), x_\beta(2l_{\alpha,\alpha+\beta}n)] && (5.48)\\ &\overset{5.37}{=} x_{\alpha+\beta}(4l_{\beta,\alpha+\beta}l_{\alpha,\alpha+\beta}n^2)x_{2\alpha+\beta}(6l^2_{\beta,\alpha+\beta}l_{\alpha,\alpha+\beta}n^3) && (5.49)\\ &\quad x_{\alpha+2\beta}(12l^2_{\alpha,\alpha+\beta}l_{\beta,\alpha+\beta}n^3). && (5.50)\end{aligned}$$

Note that by (5.45) and (5.47) the last two terms are in $\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle$ therefore $x_{\alpha+\beta}(4l_{\beta,\alpha+\beta}l_{\alpha,\alpha+\beta}n^2)$ is in $\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle$.

Hence for $j_{\alpha+\beta} := 4l_{\beta,\alpha+\beta}l_{\alpha,\alpha+\beta}n$ we have

$$U_{\alpha+\beta}(j_{\alpha+\beta}n\mathbb{Z}) \leq \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.51}$$

Case 5 (5.1.8(5)):

In **Case** (5), Theorem 5.1.8(5) implies the following equation:

$$[x_\alpha(s), x_\beta(t)] = x_{\alpha+\beta}(\pm st)x_{2\alpha+\beta}(\pm s^2t)x_{3\alpha+\beta}(\pm s^3t)x_{3\alpha+2\beta}(\pm 2s^3t^2). \tag{5.52}$$

Moreover, note that $Q_{\alpha,\alpha+\beta} = \{\alpha, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$ which is **Case** (4). And hence, we have the following equations:

$$[x_\alpha(s), x_{\alpha+\beta}(t)] \overset{5.37}{=} x_{2\alpha+\beta}(\pm 2st)x_{3\alpha+\beta}(\pm 3s^2t)x_{3\alpha+2\beta}(\pm 3st^2), \tag{5.53}$$

$$[x_\alpha(s), x_{2\alpha+\beta}(t)] \overset{5.38}{=} x_{3\alpha+\beta}(\pm l_{\alpha,2\alpha+\beta}st), \tag{5.54}$$

$$[x_{\alpha+\beta}(s), x_{2\alpha+\beta}(t)] \overset{5.39}{=} x_{3\alpha+2\beta}(\pm l_{\alpha+\beta,2\alpha+\beta}st). \tag{5.55}$$

Furthermore, $Q_{3\alpha+\beta,\beta} = \{3\alpha+\beta, \beta, 3\alpha+2\beta\}$ which is **Case** (2) and implies the following equation:

$$[x_{3\alpha+\beta}(s), x_\beta(t)] \overset{5.26}{=} x_{3\alpha+2\beta}(\pm l_{3\alpha+\beta,\beta} st). \tag{5.56}$$

First by Equation (5.52) we have

$$[x_\alpha(n), x_\beta(n)] = x_{\alpha+\beta}(n^2)x_{2\alpha+\beta}(n^3)x_{3\alpha+\beta}(n^4)x_{3\alpha+2\beta}(2n^5). \tag{5.57}$$

This shows that

$$[x_\alpha(n), x_{\alpha+\beta}(n^2)x_{2\alpha+\beta}(n^3)x_{3\alpha+\beta}(n^4)x_{3\alpha+2\beta}(2n^5)] \in \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.58}$$

Using (5.24) one can rewrite (5.58) as follows:

$$[x_\alpha(n), x_{2\alpha+\beta}(n^3)x_{3\alpha+\beta}(n^4)x_{3\alpha+2\beta}(2n^5)][x_\alpha(n), x_{\alpha+\beta}(n^2)]^{x_{2\alpha+\beta}(n^3)x_{3\alpha+\beta}(n^4)x_{3\alpha+2\beta}(2n^5)}. \tag{5.59}$$

Again by (5.24) the fist term of (5.59) is equal to

$$\begin{aligned}
& [x_\alpha(n), x_{2\alpha+\beta}(n^3)x_{3\alpha+\beta}(n^4)x_{3\alpha+2\beta}(2n^5)] && (5.60)\\
\overset{5.24}{=} & [x_\alpha(n), x_{3\alpha+\beta}(n^4)x_{3\alpha+2\beta}(2n^5)][x_\alpha(n), x_{2\alpha+\beta}(n^3)]^{x_{3\alpha+\beta}(n^4)x_{3\alpha+2\beta}(2n^5)} && (5.61)\\
\overset{5.1.8(1)}{=} & [x_\alpha(n), x_{2\alpha+\beta}(n^3)]^{x_{3\alpha+\beta}(n^4)x_{3\alpha+2\beta}(2n^5)} && (5.62)\\
\overset{5.54,5.24}{=} & x_{3\alpha+\beta}(l_{\alpha,2\alpha+\beta}n^4)[x_{3\alpha+\beta}(l_{\alpha,2\alpha+\beta}n^4), x_{3\alpha+\beta}(n^4)x_{3\alpha+2\beta}(2n^5)] && (5.63)\\
\overset{5.1.8(1)}{=} & x_{3\alpha+\beta}(l_{\alpha,2\alpha+\beta}n^4). && (5.64)
\end{aligned}$$

Therefore, the first term of (5.59) is equal to $x_{3\alpha+\beta}(l_{\alpha,2\alpha+\beta}n^4)$. Now we compute the second term

$$\begin{aligned}
& [x_\alpha(n), x_{\alpha+\beta}(n^2)]^{x_{2\alpha+\beta}(n^3)x_{3\alpha+\beta}(n^4)x_{3\alpha+2\beta}(2n^5)} \overset{5.1.8(1),5.24,5.53}{=} && (5.65)\\
& \qquad x_{2\alpha+\beta}(2n^3)x_{3\alpha+\beta}(3n^4)x_{3\alpha+2\beta}(3n^5) \quad . && (5.66)
\end{aligned}$$

Hence (5.59) is equal to

$$\begin{aligned}
& x_{3\alpha+\beta}(l_{\alpha,2\alpha+\beta}n^4)x_{2\alpha+\beta}(2n^3)x_{3\alpha+\beta}(3n^4)x_{3\alpha+2\beta}(3n^5) && (5.67)\\
\overset{5.1.8(1)}{=} & x_{2\alpha+\beta}(2n^3)x_{3\alpha+\beta}((3+l_{\alpha,2\alpha+\beta})n^4)x_{3\alpha+2\beta}(3n^5). && (5.68)
\end{aligned}$$

Therefore

$$x_{2\alpha+\beta}(2n^3)x_{3\alpha+\beta}((3+l_{\alpha,2\alpha+\beta})n^4)x_{3\alpha+2\beta}(3n^5) \in \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle \tag{5.69}$$

This implies

$$[x_\alpha(n), x_{2\alpha+\beta}(2n^3)x_{3\alpha+\beta}((3+l_{\alpha,2\alpha+\beta})n^4)x_{3\alpha+2\beta}(3n^5)] \in \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.70}$$

Now we compute (5.70) by using (5.24) as follows:

$$\begin{aligned}
& [x_\alpha(n), x_{2\alpha+\beta}(2n^3)x_{3\alpha+\beta}((3+l_{\alpha,2\alpha+\beta})n^4)x_{3\alpha+2\beta}(3n^5)] && (5.71)\\
\overset{5.24}{=} & [x_\alpha(n), x_{3\alpha+\beta}((3+l_{\alpha,2\alpha+\beta})n^4)x_{3\alpha+2\beta}(3n^5)][x_\alpha(n), x_{2\alpha+\beta}(2n^3)]^{x_{3\alpha+\beta}((3+l_{\alpha,2\alpha+\beta})n^4)x_{3\alpha+2\beta}(3n^5)} && (5.72)\\
\overset{5.1.8(1)}{=} & [x_\alpha(n), x_{2\alpha+\beta}(2n^3)]^{x_{3\alpha+\beta}((3+l_{\alpha,2\alpha+\beta})n^4)x_{3\alpha+2\beta}(3n^5)} && (5.73)\\
\overset{5.24,5.54}{=} & x_{3\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^4)[x_{3\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^4), x_{3\alpha+\beta}((3+l_{\alpha,2\alpha+\beta})n^4)x_{3\alpha+2\beta}(3n^5)] && (5.74)\\
\overset{5.1.8(1)}{=} & x_{3\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^4). && (5.75)
\end{aligned}$$

Therefore, for $j_{3\alpha+\beta} := 2l_{\alpha,2\alpha+\beta}n^3$ we have

$$U_{3\alpha+\beta}(j_{3\alpha+\beta}n\mathbb{Z}) \leq \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.76}$$

Now we have

$$\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle \ni [x_\alpha(n), x_\beta(2l_{\alpha,2\alpha+\beta}n)] \overset{5.52}{=} \tag{5.77}$$
$$x_{\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^2)x_{2\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^4)x_{3\alpha+2\beta}(8l^2_{\alpha,2\alpha+\beta}n^5), \tag{5.78}$$

which, in view of (5.76) and (5.1.8(1)), implies that

$$x_{\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^2)x_{2\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+2\beta}(8l^2_{\alpha,2\alpha+\beta}n^5) \in \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.79}$$

This implies

$$\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle \ni [x_\alpha(n), x_{\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^2)x_{2\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+2\beta}(8l^2_{\alpha,2\alpha+\beta}n^5)] \tag{5.80}$$

$$\overset{5.24}{=} [x_\alpha(n), x_{2\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+2\beta}(8l^2_{\alpha,2\alpha+\beta}n^5)]\times \tag{5.81}$$
$$[x_\alpha(n), x_{\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^2)]^{x_{2\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+2\beta}(8l^2_{\alpha,2\alpha+\beta}n^5)} \tag{5.82}$$

By (5.24), the first term is equal to

$$(5.81) \overset{5.24}{=} [x_\alpha(n), x_{3\alpha+2\beta}(8l^2_{\alpha,2\alpha+\beta}n^5)][x_\alpha(n), x_{2\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^3)]^{x_{3\alpha+2\beta}(8l^2_{\alpha,2\alpha+\beta}n^5)} \tag{5.83}$$
$$\overset{5.1.8(1),5.54}{=} x_{3\alpha+2\beta}(2l^2_{\alpha,2\alpha+\beta}n^4)^{x_{3\alpha+2\beta}(8l^2_{\alpha,2\alpha+\beta}n^5)} \tag{5.84}$$
$$\overset{5.24,5.1.8(1)}{=} x_{3\alpha+2\beta}(2l^2_{\alpha,2\alpha+\beta}n^4). \tag{5.85}$$

Moreover,

$$(5.82) \overset{5.1.8(1),5.24,5.53}{=} x_{2\alpha+\beta}(4l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+\beta}(6l_{\alpha,2\alpha+\beta}n^4)x_{3\alpha+2\beta}(12l^2_{\alpha,2\alpha+\beta}n^5). \tag{5.86}$$

Hence

$$(5.80) \overset{(5.67),5.86,5.1.8(1)}{=} x_{2\alpha+\beta}(4l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+\beta}(2(3+l_{\alpha,2\alpha+\beta})l_{\alpha,2\alpha+\beta}n^4)x_{3\alpha+2\beta}(12l^2_{\alpha,2\alpha+\beta}n^5). \tag{5.87}$$

Therefore, by equations (5.87), (5.76) and (5.1.8(1)) we have

$$x_{2\alpha+\beta}(4l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+2\beta}(12l^2_{\alpha,2\alpha+\beta}n^5) \in \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle \tag{5.88}$$

Note that by (5.79) and (5.88) the following belongs to $\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle$

$$[x_{\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^2)x_{2\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+2\beta}(8l^2_{\alpha,2\alpha+\beta}n^5), x_{2\alpha+\beta}(4l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+2\beta}(12l^2_{\alpha,2\alpha+\beta}n^5)], \tag{5.89}$$

which, in view of (5.24), is the product of

$$[x_{\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^2)x_{2\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+2\beta}(8l^2_{\alpha,2\alpha+\beta}n^5), x_{3\alpha+2\beta}(12l^2_{\alpha,2\alpha+\beta}n^5)] \tag{5.90}$$

and

$$[x_{\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^2)x_{2\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+2\beta}(8l^2_{\alpha,2\alpha+\beta}n^5), x_{2\alpha+\beta}(4l_{\alpha,2\alpha+\beta}n^3)]^{x_{3\alpha+2\beta}(12l^2_{\alpha,2\alpha+\beta}n^5)}. \tag{5.91}$$

By (5.24) and (5.1.8(1)) the term (5.90) is equal to identity. Now we compute (5.91). By (5.24) the term

$$[x_{\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^2)x_{2\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+2\beta}(8l^2_{\alpha,2\alpha+\beta}n^5), x_{2\alpha+\beta}(4l_{\alpha,2\alpha+\beta}n^3)] \tag{5.92}$$

is the product of the following two terms

$$[x_{\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^2), x_{2\alpha+\beta}(4l_{\alpha,2\alpha+\beta}n^3)]^{x_{2\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+2\beta}(8l^2_{\alpha,2\alpha+\beta}n^5)}, \tag{5.93}$$

and

$$[x_{2\alpha+\beta}(2l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+2\beta}(8l^2_{\alpha,2\alpha+\beta}n^5), x_{2\alpha+\beta}(4l_{\alpha,2\alpha+\beta}n^3)]. \tag{5.94}$$

Again by (5.24) and (5.1.8(1)) the term (5.94) is equal to identity and by (5.55), (5.24) and (5.1.8(1)) the term (5.93) is equal to

$$(5.93) = x_{3\alpha+2\beta}(8l_{\alpha+\beta,2\alpha+\beta}l^2_{\alpha,2\alpha+\beta}n^5). \tag{5.95}$$

Hence

$$(5.89) = x_{3\alpha+2\beta}(8l_{\alpha+\beta,2\alpha+\beta}l^2_{\alpha,2\alpha+\beta}n^5)^{x_{3\alpha+2\beta}(12l^2_{\alpha,2\alpha+\beta}n^5)} \overset{5.24,5.1.8(1)}{=} x_{3\alpha+2\beta}(8l_{\alpha+\beta,2\alpha+\beta}l^2_{\alpha,2\alpha+\beta}n^5). \tag{5.96}$$

Therefore, for $j_{3\alpha+2\beta} := 8l_{\alpha+\beta,2\alpha+\beta}l^2_{\alpha,2\alpha+\beta}n^4$ we have

$$U_{3\alpha+2\beta}(j_{3\alpha+2\beta}n\mathbb{Z}) \leq \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.97}$$

By (5.52) we have

$$\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle \ni [x_\alpha(l_{\alpha+\beta,2\alpha+\beta}n), x_\beta(2l_{\alpha,2\alpha+\beta}n)] \tag{5.98}$$

$$\overset{5.52}{=} x_{\alpha+\beta}(2l_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^2)x_{2\alpha+\beta}(2l^2_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^3) \tag{5.99}$$

$$x_{3\alpha+\beta}(2l^3_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^4)x_{3\alpha+2\beta}(8l^3_{\alpha+\beta,2\alpha+\beta}l^2_{\alpha,2\alpha+\beta}n^5), \tag{5.100}$$

which by (5.76) and (5.97) implies

$$x_{\alpha+\beta}(2l_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^2)x_{2\alpha+\beta}(2l^2_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^3) \in \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.101}$$

This shows that the following is in $\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle$ as well

$$[x_\alpha(n), x_{\alpha+\beta}(2l_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^2)x_{2\alpha+\beta}(2l^2_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^3)], \tag{5.102}$$

which, by (5.24) is the product of the following terms

$$[x_\alpha(n), x_{2\alpha+\beta}(2l^2_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^3)] \tag{5.103}$$

and

$$[x_\alpha(n), x_{\alpha+\beta}(2l_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^2)]^{x_{2\alpha+\beta}(2l^2_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^3)}. \tag{5.104}$$

Now we have

$$(5.103) \overset{5.54}{=} x_{3\alpha+\beta}(2l^2_{\alpha+\beta,2\alpha+\beta}l^2_{\alpha,2\alpha+\beta}n^4) \tag{5.105}$$

and

$$(5.104) \overset{5.53,5.24,5.1.8(1)}{=} x_{2\alpha+\beta}(4l_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+\beta}(6l_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^4) \tag{5.106}$$

$$x_{3\alpha+2\beta}(12l^2_{\alpha+\beta,2\alpha+\beta}l^2_{\alpha,2\alpha+\beta}n^5). \tag{5.107}$$

Hence

$$(5.102) \overset{5.105,5.106,5.107}{=} x_{3\alpha+\beta}(2l^2_{\alpha+\beta,2\alpha+\beta}l^2_{\alpha,2\alpha+\beta}n^4)x_{2\alpha+\beta}(4l_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^3) \tag{5.108}$$

$$x_{3\alpha+\beta}(6l_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^4)x_{3\alpha+2\beta}(12l^2_{\alpha+\beta,2\alpha+\beta}l^2_{\alpha,2\alpha+\beta}n^5), \tag{5.109}$$

which by (5.97), (5.76) and (5.1.8(1)) implies that the following is in $\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle$:

$$x_{2\alpha+\beta}(8l_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^3)x_{3\alpha+2\beta}(24l^2_{\alpha+\beta,2\alpha+\beta}l^2_{\alpha,2\alpha+\beta}n^5). \tag{5.110}$$

Therefore, again by (5.97) we have

$$x_{2\alpha+\beta}(8l_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^3) \in \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.111}$$

Hence, for $j_{2\alpha+\beta} := 8l_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^2$ we have

$$U_{2\alpha+\beta}(j_{2\alpha+\beta}n\mathbb{Z}) \leq \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.112}$$

Note that

$$\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle \ni [x_\alpha(2l_{\alpha+\beta,2\alpha+\beta}n), x_\beta(8l_{\alpha,2\alpha+\beta}n)] \tag{5.113}$$

$$\overset{5.52}{=} x_{\alpha+\beta}(16l_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^2)x_{2\alpha+\beta}(32l^2_{\alpha+\beta,2\alpha+\beta}l^2_{\alpha,2\alpha+\beta}n^3) \tag{5.114}$$

$$x_{3\alpha+\beta}(64l^3_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^4)x_{3\alpha+2\beta}(864l^3_{\alpha+\beta,2\alpha+\beta}l^2_{\alpha,2\alpha+\beta}n^5). \tag{5.115}$$

Now note that by (5.112), (5.97) and (5.76) the last three terms of (5.114) and (5.115) are in $\langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle$ therefore

$$x_{\alpha+\beta}(16l_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n^2) \in \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.116}$$

Hence, for $j_{\alpha+\beta} := 16l_{\alpha+\beta,2\alpha+\beta}l_{\alpha,2\alpha+\beta}n$ we have

$$U_{\alpha+\beta}(j_{\alpha+\beta}n\mathbb{Z}) \leq \langle U_\alpha(n\mathbb{Z}), U_\beta(n\mathbb{Z})\rangle. \tag{5.117}$$

□

Remark 5.2.7. Let $\mathcal{G}_\mathcal{D}$ be a two-spherical split Tits functor. Then Lemma 5.2.6 is applicable to any pair of simple roots of $\mathcal{G}_\mathcal{D}(\mathbb{R})$ by Remark 5.1.9. More generally, Lemma 5.2.6 is valid for any pair of roots such that the corresponding generated root system is spherical by Lemma 5.1.6.

The next aim is to find a coefficient j_α for any $\alpha \in \Phi^{re}$ such that the subgroup $U_\alpha(j_\alpha n\mathbb{Z})$ is contained in the group generated by the $n\mathbb{Z}$-points of the fundamental rank one subgroups. To do this first we need to introduce some terminology.

Terminology 5.2.8. *Suppose that $\mathcal{A} = (\mathcal{A}_+, \mathcal{A}_-)$ denotes a fixed (fundamental) twin apartment in the corresponding twin building of an irreducible two-spherical simply connected split Kac-Moody group $\mathcal{G}_\mathcal{D}(\mathbb{R})$. Every s-wall (for all $s \in S$) of a pair of opposite chambers $C_\pm$ in $\mathcal{A}_\pm$ corresponds to two root subgroups of opposite signs, namely, $U_\alpha(\mathbb{R})$ and $U_{-\alpha}(\mathbb{R})$ of $\mathcal{G}_\mathcal{D}(\mathbb{R})$ where $U_\alpha(\mathbb{R})$ fixes $C_\pm$ and acts sharply transitively on the other chambers of the s-panel $\rho_s(C_\pm)$ containing $C_\pm$. We denote $U_\alpha(\mathbb{R})$ by $U_{C_\pm,s}(\mathbb{R})$. Hence $U_{-\alpha}(\mathbb{R})$ will be denoted by $U_{sC_\pm,s}(\mathbb{R})$ where $sC_\pm$ is the reflection of $C_\pm$ over its s-wall in $\mathcal{A}_\pm$. Moreover, in the above setting, we denote α also by $\alpha_{\{C_\pm,s\}}$ when it is needed to be specified in this way. Since the arguments work for each half of the fundamental twin apartment similarly and simultaneously, we may drop the sign of the chambers (and the twin apartments) and give arguments only on one half of the fundamental twin apartment unless specification is necessary.*

Remark 5.2.9. Note that in view of Terminology 5.2.8, by identifying the identity element in (W, S) with a chamber $1_\mathcal{A} \in \mathcal{A}$ (or more precisely, a pair of opposite chambers $1_{\pm\mathcal{A}}$ in $\pm\mathcal{A}$), every simple root α_s can be denoted by $\alpha_{\{1_\mathcal{A},s\}}$. Moreover, for any $\omega \in W$ we have $\omega\alpha_s = \omega\alpha_{\{1_\mathcal{A},s\}} = \alpha_{\{\omega 1_\mathcal{A},s\}}$. For if, we write down ω in a reduced expression $s_{\omega_1} \cdots s_{\omega_k}$ where $k \in \mathbb{N}$ and $\{s_{\omega_1}, \cdots, s_{\omega_k}\} \subset S$, then the action of ω on $\mathcal{A}$ is compatible with the action of σ_ω as described in (5.14) via the expression (5.13). Therefore, on the one hand, this action is type-preserving so the second component in the expression $\alpha_{\{1_\mathcal{A},s\}}$ is preserved. On the other hand, the galleries starting from a chamber $C \in \mathcal{A}$ are in one-to-one correspondence with words in the alphabet S (see[AB08, Subsection 1.5.1]). Thus the element $\omega := \omega 1_\mathcal{A}$ can be treated as a chamber in $\mathcal{A}$ by abuse of notations. In this regard, if we consider the chamber C in Terminology 5.2.8 as an element $\omega \in W$ such that $\omega 1_\mathcal{A} = C$ then we have $U_\alpha(\mathbb{R}) = U_{C,s}(\mathbb{R}) = U_{\omega,s}(\mathbb{R})$ and $U_{-\alpha}(\mathbb{R}) = U_{sC,s}(\mathbb{R}) = U_{\omega s 1_\mathcal{A},s}(\mathbb{R})$ which might be denoted again by $U_{\omega s,s}(\mathbb{R})$.

For example, $\alpha_{s_1} = \alpha_{\{1_\mathcal{A},s_1\}}$ in the half apartment of the fundamental twin apartment, in the twin building corresponding to a Kac-Moody root datum of type $\tilde{A}_2$, has the simplicial complex structure depicted in Figure 5.1.

Then $s_2\alpha_{s_1} = \alpha_{\{s_2,s_1\}}$ can be obtained via the global reflection of α_{s_1} via s_2 which is illustrated in Figure 5.2.

The next lemma is basically rewriting Lemma 5.2.6 by means of Terminology 5.2.8.

Lemma 5.2.10. *Let $\mathcal{G}_\mathcal{D}$ be an irreducible simply connected two-spherical split Tits functor and $\mathcal{A} = (\mathcal{A}_+, \mathcal{A}_-)$ be the fundamental twin apartment in $\Delta_{\mathcal{G}_\mathcal{D}(\mathbb{R})}$, the twin building corresponding to $\mathcal{G}_\mathcal{D}(\mathbb{R})$. Let $\omega 1_\mathcal{A}$ be a chamber in $\mathcal{A}$. Let s and t be two generators of (W, S), the Weyl group of $\mathcal{G}_\mathcal{D}(\mathbb{R})$. Then the residue of type $\{t, s\}$, R containing $\omega 1_\mathcal{A}$ is spherical and for every root subgroup U_α of $\mathcal{G}_\mathcal{D}(\mathbb{R})$ that the corresponding wall of α in $\mathcal{A}$ intersects non-trivially the interior of this residue, there exists a $j_\alpha \in \mathbb{N}$ such that*

$$U_\alpha(j_\alpha n\mathbb{Z}) \leq \langle U_{\omega 1_\mathcal{A},s}(n\mathbb{Z}), U_{\omega 1_\mathcal{A},t}(n\mathbb{Z}), U_{\omega s 1_\mathcal{A},s}(n\mathbb{Z}), U_{\omega t 1_\mathcal{A},t}(n\mathbb{Z})\rangle. \tag{5.118}$$

Proof. First note that the split Kac-Moody group $\mathcal{G}_\mathcal{D}(\mathbb{R})$ which we started with is two-spherical, hence Lemma 4.1.10 implies that the residue of type $\{t, s\}$ containing $\omega 1_\mathcal{A}$ is spherical.

Second, since the wall corresponding to α intersects non-trivially the interior of the $\{t, s\}$-residue R containing $\omega 1_\mathcal{A}$, U_α can be described by means of an element $\omega' 1_\mathcal{A} \in R$ together with a panel which is determined by the wall corresponding to α. Therefore, the panel is of

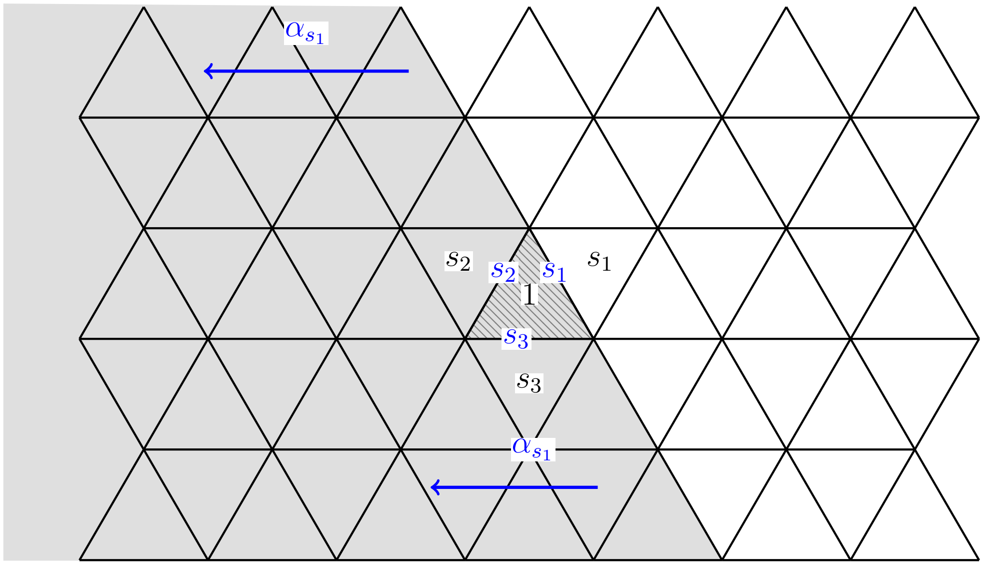


Figure 5.1: Illustration of α_{s_1} inside of an $\tilde{A}_2$ apartment.

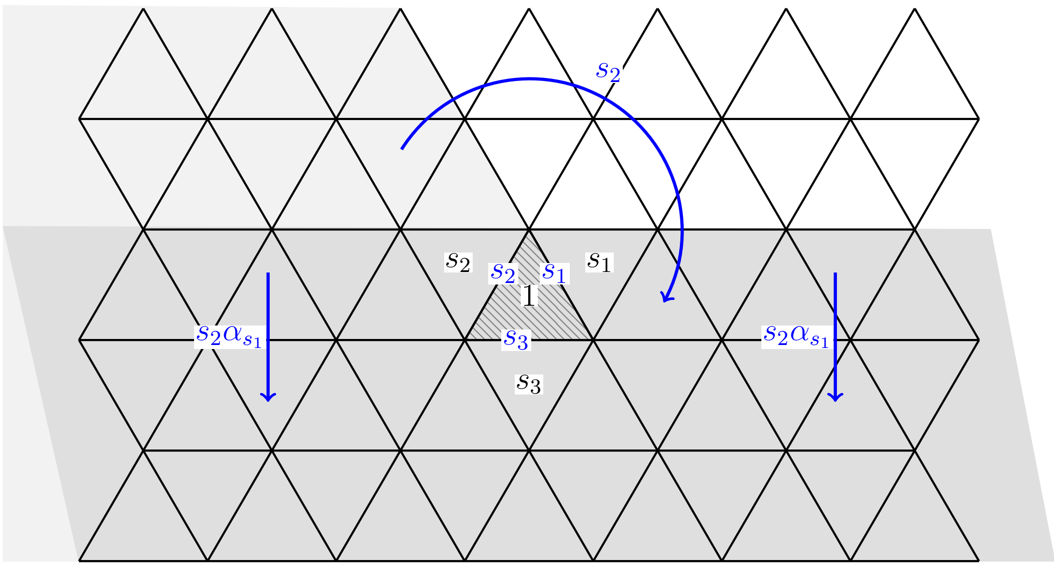


Figure 5.2: Global Reflection of α_{s_1} via s_2.

type either s or t. So by Terminology 5.2.8 we have

$$\exists\ \omega' \in W \cong \mathcal{A} \text{ such that } U_\alpha = U_{\omega',r} \tag{5.119}$$

where $r \in \{t, s\}$. Moreover, α is a $\mathbb{Z}$-linear combination of $\alpha_{\{\omega,s\}}$ and $\alpha_{\{\omega,t\}}$ (see Figure 5.3).
Now the claim follows from Lemma 5.2.6 by Terminology 5.2.8. □

Lemma 5.2.11. *Let $\mathcal{G}_\mathcal{D}$ be an irreducible simply connected two-spherical split Tits functor and $\mathcal{A} = (\mathcal{A}_+, \mathcal{A}_-)$ be the fundamental twin apartment in $\Delta_{\mathcal{G}_\mathcal{D}(\mathbb{R})}$, the twin building corresponding to $\mathcal{G}_\mathcal{D}(\mathbb{R})$. Let α be a real root of $\mathcal{G}_\mathcal{D}(\mathbb{R})$. Then there exist a subset $\{\alpha_1, ..., \alpha_l\} \subset \Phi^\circ$ of simple roots of $\mathcal{G}_\mathcal{D}$ and a $j_\alpha \in \mathbb{N}$ such that*

$$U_\alpha(j_\alpha n\mathbb{Z}) \le \langle U_{\pm\alpha_1}(n\mathbb{Z}), ..., U_{\pm\alpha_l}(n\mathbb{Z})\rangle. \tag{5.120}$$

Proof. Since α is a real root, there exist $\omega \in W$ and $s \in S$ such that $\alpha = \omega\alpha_s$. Identify ω with a pair of opposite chambers in $\mathcal{A}$. Terminology 5.2.8 and Remark 5.2.9 imply that $\alpha = \alpha_{\{\omega 1_\mathcal{A}, s\}}$. Now we prove the lemma by induction on the length of ω.
For $l(\omega) = 1 = 0$ we have $\alpha = \alpha_s$ and the claim follows.
Assume that $l(\omega) = 1$. Then $\omega = t$ for some $t \in S$ and we have $\alpha = t\alpha_s = \alpha_{t,s}$. If $t = s$ then $\alpha = t\alpha_s = \alpha_{s,s} = -\alpha_s$ in view of Remark 5.2.9 and we are done. If $t \ne s$ then the claim follows from Lemma 5.2.10 and Remark 5.2.9 considering the spherical residue of type $\{s, t\}$ containing $1_\mathcal{A}$.
Suppose the claim is true for $l(\omega) \le (n-1)$ when $n \ge 2$. If $l(\omega) = n$ then consider the reduced expression $\omega = s_{\alpha_1}...s_{\alpha_n}$ where $\{s_{\alpha_1}, ..., s_{\alpha_n}\}$ is a subset of S, the set of the generators of W. If $s_{\alpha_n} = s$ then $\alpha = \omega\alpha_s = s_{\alpha_1}...s_{\alpha_{(n-1)}}(\alpha_{\{s,s\}}) = s_{\alpha_1}...s_{\alpha_{(n-1)}}(-\alpha_s)$ by Remark 5.2.9 and the claim follows from the inductive hypothesis.
Now suppose $s_{\alpha_n} \ne s$. Set $\omega' := s_{\alpha_1}...s_{\alpha_{(n-1)}}$, α_i the simple root corresponding to s_i for $1 \le i \le n$ and β the simple root corresponding to s. Then by the inductive hypothesis there exist $j_{\pm\{\omega', s_{\alpha_n}\}}$ and $j_{\pm\{\omega',s\}}$ in $\mathbb{N}$ such that

$$U_{\pm\{\omega',s_{\alpha_n}\}}(j_{\pm\omega',s_{\alpha_n}} n\mathbb{Z}) \le \langle U_{\pm\eta_1}(n\mathbb{Z}), ..., U_{\pm\eta_k}(n\mathbb{Z})\rangle$$

and

$$U_{\pm\{\omega',s\}}(j_{\pm\{\omega',s\}} n\mathbb{Z}) \le \langle U_{\pm\nu_1}(n\mathbb{Z}), ..., U_{\pm\nu_m}(n\mathbb{Z})\rangle,$$

for finite subsets of simple roots

$$\{\eta_1, ..., \eta_k\}$$

and

$$\{\nu_1, ..., \nu_m\}.$$

Furthermore, for the triple $\{\omega = \omega' s_{\alpha_n}, \omega', \omega' s\}$ we are in the situation of Lemma 5.2.10 since the residue generated by $\omega' s_{\alpha_n}$ and $\omega' s$ is of type $\{s_{\alpha_n}, s\}$ and $\omega = \omega' s_{\alpha_n} 1_\mathcal{A}$ belongs to this residue, the wall corresponding to α intersects non-trivially the interior of this residue. Hence by Lemma 5.2.10, there exists a $j_\alpha \in \mathbb{N}$ such that

$$U_\alpha(j_\alpha n\mathbb{Z}) \le \langle U_{\pm\{\omega',s_{\alpha_n}\}}(d\mathbb{Z}), U_{\pm\{\omega',s\}}(d\mathbb{Z})\rangle,$$

where $d := lcm[j_{\pm\{\omega',s_{\alpha_n}\}} n, j_{\pm\{\omega',s\}} n]$. □

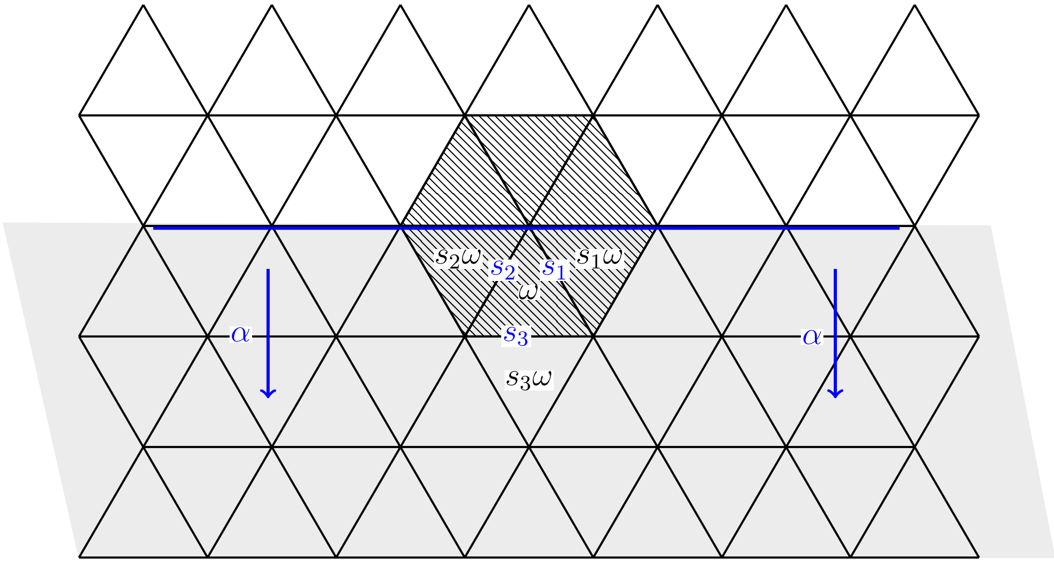


Figure 5.3: Root intersecting spherical residue.

Next we present a technical lemma which will be of some important use later in the proof of Corollary 7.1.6.

Lemma 5.2.12. *Let $\mathcal{G}_\mathcal{D}$ be an irreducible simply connected centered two-spherical split Tits functor and $\mathcal{A} = (\mathcal{A}_+, \mathcal{A}_-)$ be the fundamental twin apartment in $\Delta_{\mathcal{G}_\mathcal{D}(\mathbb{R})}$, the twin building corresponding to $\mathcal{G}_\mathcal{D}(\mathbb{R})$. Then*

$$\forall \omega \in W \ \exists j_\omega \in \mathbb{N} \text{ such that } \Gamma(j_\omega n) \subseteq \Gamma(n) \cap \sigma_\omega(\Gamma(n)). \tag{5.121}$$

Proof. First note that the rank of $\mathcal{G}_\mathcal{D}(\mathbb{R})$ is finite so the set of generators S of W is finite. Let $S = \{s_1, ..., s_k\}$. By Lemma 5.2.2, for any $\omega \in W$, σ_ω preserves the $n\mathbb{Z}$-points of the real root subgroups. Hence, $\sigma_\omega(\Gamma(n))$ is generated by $\sigma_\omega(G_\tau(n\mathbb{Z})) = G_{\omega\tau}(n\mathbb{Z})$ where τ runs through all possible combinations of α_{s_i} and α_{s_j} where $s_i, s_j \in S$ and $\alpha_{s_i} \approx \alpha_{s_j}$ (see Lemma 5.2.2 and the proof of Lemma 5.2.3).
Now, since $\mathcal{D}$ is two-spherical, by Lemma 5.2.11, for every $s_i \in S$ there exists a $j_{s_i} \in \mathbb{N}$ such that

$$U_{\alpha_{s_i}}(j_{s_i} n\mathbb{Z}) = U_{\omega^{-1}\omega\alpha_{s_i}}(j_{s_i} n\mathbb{Z}) \leq \langle U_{\pm\omega\alpha_{s_1}}(n\mathbb{Z}), ..., U_{\pm\omega\alpha_{s_k}}(n\mathbb{Z})\rangle$$

for $1 \leq i \leq k$. Note that here we actually applied Lemma 5.2.11 to the new basis $\{\omega\alpha_{s_1}, ..., \omega\alpha_{s_k}\}$ of the root system Φ i.e., considering $\omega 1_\mathcal{A}$ as the fundamental chamber in the corresponding twin building of $\mathcal{G}_\mathcal{D}(\mathbb{R})$.
This implies that for $j_\omega := lcm[j_{s_1}, ..., j_{s_k}]$ we have $\Gamma(j_\omega n) \subseteq \sigma_\omega(\Gamma(n))$. □

Corollary 5.2.13. *Let $\mathcal{G}_\mathcal{D}$ be an irreducible simply connected centered two-spherical split Tits functor and $\mathcal{A} = (\mathcal{A}_+, \mathcal{A}_-)$ be the fundamental twin apartment in $\Delta_{\mathcal{G}_\mathcal{D}(\mathbb{R})}$, the twin building corresponding to $\mathcal{G}_\mathcal{D}(\mathbb{R})$. Then*

$$\forall \omega \in W \ \exists j_\omega \in \mathbb{N} \text{ such that } \Gamma(j_\omega n) \subseteq \Gamma^{con}(n) \cap \sigma_\omega(\Gamma^{con}(n)). \tag{5.122}$$

Proof. Since by definition, $\Gamma(n) \subset \Gamma^{con}(n)$, the corollary is immediate by Lemma 5.2.12. □

5.3 Congruence Subgroups

Using the same method as we used to define the arithmetic subgroups, one can define congruence subgroups of split Kac-Moody groups similar to Section 1.5. In this section $\mathcal{G}_\mathcal{D}$ always denotes an irreducible spherical simply two-connected split Tits functor with a fixed Tits basis $\mathcal{F} = (\mathcal{G}_\mathcal{D}, \{\phi_i\}_{i\in I}, \eta)$.
Let $\mathfrak{a}$ be an ideal in $\mathbb{Z}$. For the following natural epimorphism

$$\mathbb{Z} \to \mathbb{Z}/\mathfrak{a}$$

by the functorial property of the split Tits functor $\mathcal{G}_\mathcal{D}$, stated in Subsection 4.1.1, there exists the following morphism of groups:

$$\mathcal{G}_\mathcal{D}(\mathbb{Z}) \xrightarrow{\pi_\mathfrak{a}} \mathcal{G}_\mathcal{D}(\mathbb{Z}/\mathfrak{a}). \tag{5.123}$$

We call the kernel of $\pi_\mathfrak{a}$ a **functorial principal congruence subgroup** of $\mathcal{G}_\mathcal{D}(\mathbb{Z})$ (with respect to $\mathcal{F}$) and denote it by $\mathrm{P}\mathcal{G}_\mathcal{D}(\mathfrak{a})$. Any subgroup of $\mathcal{G}_\mathcal{D}(\mathbb{Z})$ containing a functorial principal congruence subgroup is called **functorial congruence subgroup** denoted by $\mathcal{G}_\mathcal{D}(\mathfrak{a})$. Since $G_\mathcal{D}(\mathbb{Z}) \subseteq \mathcal{G}_\mathcal{D}(\mathbb{Z})$, by restricting $\pi_\mathfrak{a}$ to $G_\mathcal{D}(\mathbb{Z})$ one can obtain the following group morphism:

$$G_\mathcal{D}(\mathbb{Z}) \xrightarrow{\pi_\mathfrak{a}} \mathcal{G}_\mathcal{D}(\mathbb{Z}/\mathfrak{a}). \tag{5.124}$$

We denote the kernel of (5.124) by $\mathrm{PCon}_{\mathcal{G}_\mathcal{D}}(\mathfrak{a})$ and call it a **principal congruence subgroup** of $G_\mathcal{D}(\mathbb{Z})$ with respect to $\mathfrak{a}$ and $\mathcal{F}$. Moreover, a subgroup of $G_\mathcal{D}(\mathbb{Z})$ containing a principal congruence subgroup $\mathrm{PCon}_{\mathcal{G}_\mathcal{D}}(\mathfrak{a})$ is called a **congruence subgroup** for some ideal $\mathfrak{a}$ in $\mathbb{Z}$. We denote such subgroups by $\mathrm{Con}_{\mathcal{G}_\mathcal{D}}(\mathfrak{a})$.
Note that, unlike the classical case in Section 1.5, if $\mathfrak{a} = n\mathbb{Z}$ for some $n \in \mathbb{Z}$, it is **not** clear that $\mathrm{P}\mathcal{G}_\mathcal{D}(n\mathbb{Z}) \overset{?!}{=} G_\mathcal{D}(n\mathbb{Z})$ or even $\mathrm{PCon}_{\mathcal{G}_\mathcal{D}}(n\mathbb{Z}) \overset{?!}{=} G_\mathcal{D}(n\mathbb{Z})$. But again by using the morphism of functors as in (5.2) we have $G_\mathcal{D}(n\mathbb{Z}) \subset \mathrm{P}\mathcal{G}_\mathcal{D}(n\mathbb{Z})$ and similarly $G_\mathcal{D}(n\mathbb{Z}) \subset \mathrm{PCon}_{\mathcal{G}_\mathcal{D}}(n\mathbb{Z})$ since the following diagram is commutative for any real root $\alpha \in \Phi^{re}$:

$$\begin{tikzcd}
\mathcal{G}_\mathcal{D}(\mathbb{Z}) \arrow[r, "\pi_{n\mathbb{Z}}"] & \mathcal{G}_\mathcal{D}(\mathbb{Z}/n\mathbb{Z}) \\
\mathrm{SL}_2(\mathbb{Z}) \arrow[u, "\phi_\alpha"] \arrow[r, "\pi"] & \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) \arrow[u, "\phi_\alpha"']
\end{tikzcd} \tag{5.125}$$

Definition 5.3.1. Let $\mathcal{G}_\mathcal{D}$ be a centered two-spherical split Tits functor with a basis $\mathcal{F} = (\mathcal{G}_\mathcal{D}, \{\phi_i\}_{i\in I}, \eta)$. We call $G_\mathcal{D}(\mathbb{Z})$ **strongly rigid** if for any $n \in \mathbb{N}$, $G_\mathcal{D}(n\mathbb{Z})$ is a finite index subgroup of $\mathrm{PCon}_{\mathcal{G}_\mathcal{D}}(n\mathbb{Z})$.

Let $\mathcal{G}_\mathcal{D}$ be an irreducible simply connected centered split Tits functor of affine type with a Tits basis $\mathcal{F} = (\mathcal{G}_\mathcal{D}, \{\phi_i\}_{i\in I}, \eta)$. Let G be the Chevalley group associated to $\mathcal{G}_\mathcal{D}$ compatible with $\mathcal{F}$. Then $\mathcal{G}_\mathcal{D}(\mathbb{K})$ is a central extension of $G(\mathbb{K}[t, t^{-1}])$ (see Proposition 4.1.5, see also [Che96, Lemma 2.3]).
Lemma 5.3.2. *Let $\mathcal{G}_\mathcal{D}$ be an irreducible simply connected centered split Tits functor of affine type with a Tits basis $\mathcal{F} = (\mathcal{G}_\mathcal{D}, \{\phi_i\}_{i\in I}, \eta)$. Then $G_\mathcal{D}(n\mathbb{Z}) = \mathrm{PCon}_{\mathcal{G}_\mathcal{D}}(n\mathbb{Z})$. In particular, $G_\mathcal{D}(\mathbb{Z})$ is strongly rigid.*

Proof. Let G be the Chevalley group associated to $\mathcal{G}_{\mathcal{D}}$ compatible with $\mathcal{F}$. By Proposition 4.1.5, $\mathcal{G}_{\mathcal{D}}(\mathbb{Q}) \cong G(\mathbb{Q}[t, t^{-1}])$. Therefore, $G_{\mathcal{D}}(n\mathbb{Z}) = \mathrm{PCon}_{\mathcal{G}_{\mathcal{D}}}(n\mathbb{Z})$ by the definition of the principal congruence subgroups (with respect to $\mathcal{F}$). □

Now for a prime number p and a centered split Tits functor $\mathcal{G}_{\mathcal{D}}$, on the one hand, we know that $\mathcal{G}_{\mathcal{D}}(\mathbb{Z}/p\mathbb{Z})$ is generated by the root subgroups. On the other hand, in view of the commutativity of the above diagram, one can observe that for every element $x_\alpha(q)$ in $U_\alpha(\mathbb{Z}/p\mathbb{Z}) \subset \mathcal{G}_{\mathcal{D}}(\mathbb{Z}/p\mathbb{Z})$ there exists at least one element $x_\alpha(m) \in U_\alpha(\mathbb{Z})$ in $G_{\mathcal{D}}(\mathbb{Z})$ (and hence in $\mathcal{G}_{\mathcal{D}}(\mathbb{Z})$) such that $\pi(x_\alpha(m)) = x_\alpha(q)$. Therefore, π is an epimorphism on $G_{\mathcal{D}}(\mathbb{Z})$ (and hence on $\mathcal{G}_{\mathcal{D}}(\mathbb{Z})$). This implies the following equations

$$\mathcal{G}_{\mathcal{D}}(\mathbb{Z})/\, \mathrm{P}\mathcal{G}_{\mathcal{D}}(p\mathbb{Z}) \cong \mathcal{G}_{\mathcal{D}}(\mathbb{Z}/p\mathbb{Z}) \cong G_{\mathcal{D}}(\mathbb{Z})/\, \mathrm{PCon}_{\mathcal{G}_{\mathcal{D}}}(p\mathbb{Z}). \tag{5.126}$$

Moreover, by the following commutative diagram:

$$\begin{tikzcd} \mathcal{G}_{\mathcal{D}}(\mathbb{Z}) \arrow[r, "\pi"] & \mathcal{G}_{\mathcal{D}}(\mathbb{Z}/p\mathbb{Z}) \\ G_{\mathcal{D}}(\mathbb{Z}) \arrow[u, "i"] \arrow[ur, "\pi"'] & \end{tikzcd},$$

it is evident that

$$\mathrm{PCon}_{\mathcal{G}_{\mathcal{D}}}(p\mathbb{Z}) = \mathrm{P}\mathcal{G}_{\mathcal{D}}(p\mathbb{Z}) \cap G_{\mathcal{D}}(\mathbb{Z}). \tag{5.127}$$

Furthermore, (5.127) together with (5.126) yields the following equations:

$$\mathcal{G}_{\mathcal{D}}(\mathbb{Z})/\, \mathrm{P}\mathcal{G}_{\mathcal{D}}(p\mathbb{Z}) \cong G_{\mathcal{D}}(\mathbb{Z})/\, \mathrm{P}\mathcal{G}_{\mathcal{D}}(p\mathbb{Z}) \cap G_{\mathcal{D}}(\mathbb{Z}) \cong G_{\mathcal{D}}(\mathbb{Z})\, \mathrm{P}\mathcal{G}_{\mathcal{D}}(p\mathbb{Z})/\, \mathrm{P}\mathcal{G}_{\mathcal{D}}(p\mathbb{Z}), \tag{5.128}$$

where the later is obtained by the second isomorphism theorem of groups. Therefore,

$$\mathcal{G}_{\mathcal{D}}(\mathbb{Z}) \cong G_{\mathcal{D}}(\mathbb{Z})\, \mathrm{P}\mathcal{G}_{\mathcal{D}}(p\mathbb{Z}). \tag{5.129}$$

This narrows down understanding the difference between $\mathcal{G}_{\mathcal{D}}(\mathbb{Z})$ and $G_{\mathcal{D}}(\mathbb{Z})$ to understanding the difference between $\mathrm{P}\mathcal{G}_{\mathcal{D}}(p\mathbb{Z})$ and $\mathrm{PCon}_{\mathcal{G}_{\mathcal{D}}}(p\mathbb{Z})$.

CHAPTER

SIX

A FIXED POINT THEOREM

This chapter is to present a detailed elaboration on the proof of [CM09, Lemma 8.1]. Note that [CM09] contains a proof for [CM09, Lemma 8.1], nevertheless, since this result is a cornerstone of our approach towards the rigidity of arithmetic subgroups of certain real split Kac-Moody groups and the proof in [CM09] is very concise we devoted this chapter to elaborate more on it. We should mention here that the main ideas how to prove the fixed point theorem in detail were suggested by Pierre-Emmanuel Caprace and Ralf Köhl.
In the first section of this chapter we collect basic facts from geometric group theory and in the second section a detailed proof of the aforementioned result is given. There exists no original result in this section.

6.1 Geometric Group Theory

In this section we present some basic concepts along with some advanced results in geometric group theory which will be used in the next section. The main references are [Löh11], [Dav08], [LMR00] and [BH99].

6.1.1 Word Metric

In this brief subsection we define the word metric. Then we present the Cayley graph with a natural metric on it which can be associated to any group with a generating set. At the end, we observe that the word metric coincides with the metric on the Cayley graph of any group with a fixed generating set.

Definition 6.1.1. Let G be a group and let $S \subset G$ be a generating set. Define the **word metric** d_S on G with respect to S as follows:

$$d_S(g,h) := min\{n \in \mathbb{N} \mid \exists s_1, ..., s_n \in S \cup S^{-1} \ s.t \ g^{-1}h = s_1 \cdots s_n\} \tag{6.1}$$

for all $g, h \in G$.

Now let $\Gamma(V, E)$ be a connected graph. Then the map

$$(\ ,\ ) := \begin{cases} V \times V \to \mathbb{R}_{\geq 0}, \\ (v,w) \mapsto min\{n \in \mathbb{N} \mid \text{there is a path of length } n \text{ connecting } v \text{ and } w \text{ in } \Gamma\} \end{cases}$$

defines a metric on V. We call this map the **associated metric** on V.
Define the **Cayley graph** associated to a group G with a generating set $S \subset G$ to be the graph denoted by $\mathrm{Cay}(G, S)$ whose

- set of vertices is G, and whose
- set of edges is
$$\{\{g, g \cdot s\} \mid g \in G,\ s \in S \backslash \{e\}\}.$$

In other words, two vertices in a Cayley graph are adjacent if and only if they differ by an element of the generating set S.
It is evident that the above associated metric on a Cayley graph $\mathrm{Cay}(G, S)$ is the same as the word metric on G with respect to S.

6.1.2 U-Elements

Definition 6.1.2. Let G be a finitely generated group by a finite set S. Let $g \in G$ is an element of infinite order. We say g has property

(U1) if $l_S(g^n) := d_S(1, g^n) = \mathcal{O}(\log n)$ for $n \in \mathbb{N}$;

(U2) if $|\langle g \rangle \cap B_S(n)|$ grows exponentially with $n \in \mathbb{N}$, i.e., there exists a $c \geq 1$ such that for all large enough n, the ball of radius n around the identity in $\mathrm{Cay}(G, S)$ contains at least c^n elements from the cyclic group generated by g;

(U3) if $\liminf(\log(l_S(g^n))/\log(n)) = 0$.

Note that Property U_i for $i \in \{1, 2, 3\}$ depends only on G and g not on S. We say that $g \in G$ is a U_i-**element** of G if it has Property U_i. In the same fashion, $g \in G$ is said to be a U-**element** of G if it has at least one of these properties (see [LMR00, Section 2]).
Proposition 6.1.3. *[LMR00, Proposition (2.3)] Let G be a finitely generated group by a finite set S. Every U_1-element is a U_2-element. And every U_2-element is a U_3-element in G. In other words,*
$$U_1 \Longrightarrow U_2 \Longrightarrow U_3.$$
Theorem 6.1.4. *[LMR00, Theorem (2.15)] Let $G = \prod_{i=1}^{l} G_i(\Bbbk_i)$ be a semi-simple linear algebraic group where for $i = 1, ..., l$, $\Bbbk_i$ is a locally compact non-discrete field and G_i is a connected almost simple linear algebraic $\Bbbk_i$-group. Let Γ be an irreducible lattice in G and $\gamma \in \Gamma$. Then γ is a U-element of Γ if and only if the following four conditions hold:*

(a) *For every $i = 1, ..., l$, $char(\Bbbk_i) = 0$.*

(b) *For every $i = 1, ..., l$, $rank_{\Bbbk_i}(G_i) \geq 1$.*

(c) *$rank(G) = \sum rank_{\Bbbk_i}(G_i) \geq 2$.*

(d) *γ is a virtually unipotent element of infinite order.*

Note that an element is **virtually unipotent** if some power of it is unipotent. Moreover, an element of $G = \prod_{i=1}^{l} G_1(\Bbbk_i)$ is **unipotent** if all its components are unipotent.
Remark 6.1.5. [LMR00, Remark 2.16(i)] In the proof of Theorem 6.1.4, it is actually proven that $\gamma \in \Gamma$ is a U_1-element if and only if it is a U_2-element if and only if it is a U_3-element.

6.1.3 Bounded Generation

Let G be a group and $\{U_i\}_{i\in I}$ be a family of subgroups of G indexed by a set I. Recall from Definition 3.1.5 that G is said to be **boundedly generated** by a family $\{U_i\}_{i\in I}$ if there exists a constant element $n \in \mathbb{Z}_{>0}$ such that every $g \in G$ can be written a product $g = g_1 \cdots g_n$ with $g_i \in \bigcup_{i\in I} U_i$.

Theorem 6.1.6. *[Tav91] Let G be an irreducible universal Chevalley group of rank ≥ 2. Then $G(\mathbb{Z})$ is boundedly generated by the $\mathbb{Z}$-points of the root subgroups $U_{\pm\alpha_s}(\mathbb{R})$ where s runs through S, the generating set of the Weyl group W of G.*

6.2 A Fixed Point Theorem

This section is to give a detailed elaboration on the proof of the following fixed point theorem

Theorem 6.2.1 (**Fixed Point Theorem**). *[CM09, Lemma 8.1] Let Δ be a complete* CAT(0) *polyhedral complex such that* Shapes(Δ) *is finite (see Definition 3.2.2) and let G be an irreducible universal Chevalley group of rank at least two. Assume that $G(\mathbb{Z})$ acts on Δ by cellular isometries. Then $G(\mathbb{Z})$ has a fixed point.*

To prove the fixed point theorem first we show that each unit elementary matrix in $G(\mathbb{Z})$ (i.e., an element of the form $x_\alpha(1)$ where α is a root of the universal Chevalley group G as described in Section 1.4) has a fixed point in Δ with respect to the notations in Theorem 6.2.1. For this first note that in Theorem 6.2.1 if Σ denotes the set of unit elementary matrices in $G(\mathbb{Z})$, and d_Σ denotes the word metric on $G(\mathbb{Z})$ with respect to Σ (see Definition 6.1.1) then for any $u \in \Sigma$ it follows from Theorem 6.1.4 and Remark 6.1.5 that u is a U_1-element. Hence we have

$$l_\Sigma(u^n) := d_\Sigma(1, u^n) = \mathcal{O}(\log(n)). \tag{6.2}$$

Lemma 6.2.2. *Let $x \in \Delta$ and $\gamma \in G(\mathbb{Z})$. Then for all $n \in \mathbb{N}$*

$$d_\Delta(x, \gamma^n x) \leq l_\Sigma(\gamma^n). \max\{d_\Delta(x, sx) | s \in \Sigma\}. \tag{6.3}$$

Proof. Set $k := l_\Sigma(\gamma^n)$. Then there exist $s_1, ..., s_k \in \Sigma \cup \Sigma^{-1}$ such that

$$s_1 s_2 \cdots s_k = \gamma^n.$$

The triangle inequality applied to the metric d_Δ plus the fact that $G(\mathbb{Z}) \leq G$ acts via isometries on the metric space (Δ, d_Δ) implies that

$$\begin{aligned} d_\Delta(x, \gamma^n x) &\leq d_\Delta(x, s_1 x) + d_\Delta(s_1 x, s_1 s_2 x) + \cdots + d_\Delta(s_1 s_2 \cdots s_{k-1} x, s_1 s_2 \cdots s_k x) \\ &= d_\Delta(x, s_1 x) + d_\Delta(x, s_2 x) + \cdots + d_\Delta(x, s_k x) \\ &\leq l_\Sigma(\gamma^n). \max\{d_\Delta(x, sx) | s \in \Sigma\}. \end{aligned}$$

□

Recall from Section 3.1, for any $\gamma \in G(\mathbb{Z})$ the translation length of γ for the action of $G(\mathbb{Z})$ on Δ is defined as follows

$$|\gamma| := \inf\{d_\gamma(x) \mid x \in \Delta\}. \tag{6.4}$$

Lemma 6.2.3. *Let $\gamma \in G(\mathbb{Z})$ and $x \in \Delta$. Then*

$$|\gamma| = \lim_{n\to\infty} \frac{d_\Delta(x, \gamma^n x)}{n}, \tag{6.5}$$

where $|\gamma|$ is the translation length of γ.

Proof. First we show that the function $n \longmapsto d_\Delta(x, \gamma^n x)$ is sub-additive (i.e., $d_\Delta(x, \gamma^{n+m}x) \leq d_\Delta(x, \gamma^n x) + d_\Delta(x, \gamma^m x)$). And hence the limit on the right hand side of (6.5) exists.
But this is evident because

$$\begin{aligned} d_\Delta(x, \gamma^{n+m}x) &\leq d_\Delta(x, \gamma^n x) + d_\Delta(\gamma^n x, \gamma^{n+m}x) \\ &= d_\Delta(x, \gamma^n x) + d_\Delta(x, \gamma^m x). \end{aligned}$$

The latter equality follows from the fact that γ is an isometry.
Next we show that $\lim_{n\to\infty} \frac{1}{n} d_\Delta(x, \gamma^n x)$ is independent of the choice of x.
Let $x_1, x_2 \in \Delta$ then

$$\begin{aligned} d_\Delta(x_1, \gamma^n x_1) &\leq d_\Delta(x_1, x_2) + d_\Delta(x_2, \gamma^n x_1) \\ &\leq d_\Delta(x_1, x_2) + d_\Delta(x_2, \gamma^n x_2) + d_\Delta(\gamma^n x_2, \gamma^n x_1) \\ &= 2d_\Delta(x_1, x_2) + d_\Delta(x_2, \gamma^n x_2). \end{aligned}$$

Thus we have

$$\lim_{n\to\infty} \frac{d_\Delta(x_1, \gamma^n x_1)}{n} \leq \lim_{n\to\infty} \frac{d_\Delta(x_2, \gamma^n x_2)}{n}.$$

The claim follows by the symmetry of the above discussion.

Now we show that if γ is semisimple then the above limit is equal to the translation length of the isometry.
Assume that γ is semisimple, hence $\mathbf{Min}(\gamma) := \{x \in \Delta | d_\Delta(x, \gamma x) = |\gamma|\}$ is non-empty. Therefore, we can choose $x_0 \in \mathbf{Min}(\gamma)$ with the property $|\gamma| = d_\Delta(x_0, \gamma x_0)$.
On the other hand, subadditivity of the function $n \longmapsto d_\Delta(x, \gamma^n x)$ implies that

$$d_\Delta(x_0, \gamma^n x_0) \leq n d_\Delta(x_0, \gamma x_0)$$

Therefore,

$$\lim_{n\to\infty} \frac{d_\Delta(x_0, \gamma^n x_0)}{n} \leq |\gamma|. \tag{6.6}$$

Now since $|\gamma^n| = n|\gamma|$ by Lemma 3.1.3(v), (6.6) is actually an equality.
Finally, because $G(\mathbb{Z})$ acts by cellular isometries on Δ, by Proposition 3.2.6, the element γ cannot be parabolic. And hence γ is either elliptic or hyperbolic, i.e., semi-simple (see Definition 3.1.2). Therefore, the lemma follows. $\square$

Lemma 6.2.4. *Let $u \in \Sigma$. Then u has a fixed point in Δ.*

Proof. We compute

$$\begin{aligned}
|u| &\stackrel{6.2.3}{=} \lim_{n\to\infty} \frac{d_\Delta(x, u^n x)}{n} \\
&\stackrel{6.2.2}{\leq} \lim_{n\to\infty} \frac{l_\Sigma(u^n).max\{d_\Delta(x, sx)|s \in \Sigma\}}{n} \\
&\stackrel{6.2}{\leq} max\{d_\Delta(x, sx)|s \in \Sigma\}. \lim_{n\to\infty} \frac{O(log(n))}{n} \\
&= 0.
\end{aligned}$$

□

Proof of the fixed point theorem. By Theorem 6.1.6 we know that $G(\mathbb{Z})$ is boundedly generated by elementary matrices. Since every elementary matrix $x_\alpha(t)$ (where α is a root of the universal Chevalley group G and $t \in \mathbb{Z}$) is in the group generated by $x_\alpha(1)$ ($x_\alpha(t) = x_\alpha(1)^t$), $G(\mathbb{Z})$ is boundedly generated by the (finite) family of subgroups $U_\alpha := \langle x_\alpha(1)\rangle$.
Lemma 6.2.4 implies that each *unit* elementary matrix in $G(\mathbb{Z})$ has a fixed point. Therefore, each U_α has a fixed point.
Now, by Theorem 3.1.6, $G(\mathbb{Z})$ has a fixed point as well. □

Corollary 6.2.5. *Let $\mathcal{G}_\mathcal{D}$ be a simply connected two-spherical split Tits functor with a Tits basis $\mathcal{F} = (\mathcal{G}_\mathcal{D}, \{\phi_i\}_{i\in I}, \eta)$. Let $\mathcal{G}'_\mathcal{D}$ be another split Tits functor and $\phi : \mathcal{G}_\mathcal{D}(\mathbb{Z}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q})$ be an abstract homomorphism. Let $L(\mathbb{R})$ be a fundamental subgroup of rank at least two of $\mathcal{G}_\mathcal{D}(\mathbb{R})$. Let $\Gamma := L(\mathbb{Z})$ be the group of $\mathbb{Z}$-points of $L(\mathbb{R})$.
Then $\phi(\Gamma)$ is a bounded subgroup of $\mathcal{G}'_{\mathcal{D}'}(\mathbb{R})$ (or $\mathcal{G}'_{\mathcal{D}'}(\mathbb{Q})$).*

Proof. On the one hand, since $\mathcal{G}_\mathcal{D}(\mathbb{R})$ is simply connected and two-spherical (hence centered), the fundamental subgroups of $\mathcal{G}_\mathcal{D}(\mathbb{R})$ are universal (i.e., simply connected) Chevalley groups (see Remark 4.4.10).
On the other hand, the restriction of the homomorphism ϕ to Γ induces an action of Γ on the geometric realization of $\mathcal{G}'_{\mathcal{D}'}(\mathbb{R})$ (or $\mathcal{G}'_{\mathcal{D}'}(\mathbb{Q})$). This action is induced by $\phi(\Gamma)$ which is a subgroup of $\mathcal{G}'_{\mathcal{D}'}(\mathbb{R})$ (or $\mathcal{G}'_{\mathcal{D}'}(\mathbb{Q})$). Hence the action is by cellular isometries. Now the fixed point theorem, Theorem 6.2.1, along with Theorem 4.1.7 implies that the group $\phi(\Gamma)$ is a bounded subgroup of $\mathcal{G}'_{\mathcal{D}'}(\mathbb{R})$ (or $\mathcal{G}'_{\mathcal{D}'}(\mathbb{Q})$). □

CHAPTER
SEVEN

RIGIDITY OF ARITHMETIC KAC-MOODY GROUPS

In this chapter we investigate the rigidity phenomenon in Kac-Moody theory. Here by an **arithmetic Kac-Moody group** we understand an arithmetic subgroup of a real split Kac-Moody group as presented in Chapter 5. This chapter has two sections. The first section contains a counterpart of the classical super rigidity, Theorem 1.6.4, for certain arithmetic Kac-Moody groups which is formulated in Proposition 7.1.3. In the second section we prove a similar result to the strong rigidity theorem, Theorem 1.6.8, for some arithmetic Kac-Moody groups in the form of Proposition 7.2.3.
We follow notations given in Chapter 4 and Chapter 5 throughout this chapter. Recall from Chapter 1; when the underlying algebraic group G is a Chevalley group we see G as a group functor. Therefore, $G(\mathbb{k})$ denotes the $\mathbb{k}$-points of G as an algebraic group where $\mathbb{k}$ is a subfield of characteristic zero of $\mathbb{C}$. Moreover, when G is a linear algebraic $\mathbb{Q}$-group, we use the notation $G_{\mathbb{k}}$ given in (1.1) for the $\mathbb{k}$-points of G in a fixed representation of G in $\mathrm{GL}(n, \mathbb{C})$ for some $n \in \mathbb{N} = \{1, 2, ...\}$.
All results in this chapter are obtained by contribution of Max Horn and Ralf Köhl.

7.1 Super Rigidity

The aim of this section is to prove the super rigidity for arithmetic subgroups of irreducible simply connected two-spherical real split Kac-Moody groups. To do this, we start off with giving an auxiliary lemma which will be used in the proof of the super rigidity.

We should mention here that the use of Theorem 1.6.4 in the proof of the following lemma was suggested by Dave Witte Morris.
Lemma 7.1.1. *Let G be an irreducible simply connected Chevalley group functor of rank at least two. Let $\mathcal{G}_{\mathcal{D}}$ be a centered two-spherical split Tits functor.*
Let $\phi : G(\mathbb{Z}) \to \mathcal{G}_{\mathcal{D}}(\mathbb{Q})$ be a group homomorphism with infinite image. Then

(a) *there exist an $n \in \mathbb{N}$, a uniquely determined connected semisimple linear algebraic $\mathbb{Q}$-group H, a bounded subgroup $B_{\mathbb{Q}}$ of $\mathcal{G}_{\mathcal{D}}(\mathbb{Q})$, group homomorphisms*

$$\imath : B_{\mathbb{Q}} \to H_{\mathbb{Q}},$$

and

$$\phi_{\mathbb{Q}} : G(\mathbb{Q}) \to B_{\mathbb{Q}}$$

and a uniquely determined central $\mathbb{Q}$*-isogeny*

$$\psi : G \to H,$$

with the property that

$$\phi_{\mathbb{Q}}|_{G(n\mathbb{Z})} = \phi|_{G(n\mathbb{Z})}, \tag{7.1}$$

and making the following diagram commute:

$$\begin{tikzcd} G(\mathbb{Q}) \arrow[r, "\phi_{\mathbb{Q}}"] \arrow[dr, "\psi"'] & B_{\mathbb{Q}} \arrow[r, hook] \arrow[d, "\imath"] & \mathcal{G}_{\mathcal{D}}(\mathbb{Q}) \\ & H_{\mathbb{Q}} & \end{tikzcd} \tag{7.2}$$

(b) Endow $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ *with the Kac-Peterson topology. Then there exist a bounded subgroup* $B_{\mathbb{R}}$ *of* $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ *containing* $B_{\mathbb{Q}}$ *as the* $\mathbb{Q}$*-points of* $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ *(i.e.,* $B_{\mathbb{Q}} = B_{\mathbb{R}} \cap \mathcal{G}_{\mathcal{D}}(\mathbb{Q})$*) which is a semisimple Lie group with the induced topology from* $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$*, a uniquely determined continuous group homomorphism* $\phi_{\mathbb{R}} : G(\mathbb{R}) \to B_{\mathbb{R}}$*, and a uniquely determined continuous group homomorphism* $\imath_{\mathbb{R}} : B_{\mathbb{R}} \to H_{\mathbb{R}}$ *with the property that*

$$\phi_{\mathbb{R}}|_{G(\mathbb{Q})} = \phi_{\mathbb{Q}} \quad \textit{and} \quad \imath_{\mathbb{R}}|_{B_{\mathbb{Q}}} = \imath \tag{7.3}$$

and making the following diagram commute:

$$\begin{tikzcd} G(\mathbb{R}) \arrow[r, "\phi_{\mathbb{R}}"] \arrow[dr, "\psi"'] & B_{\mathbb{R}} \arrow[r, hook] \arrow[d, "\imath_{\mathbb{R}}"] & \mathcal{G}_{\mathcal{D}}(\mathbb{R}) \\ & H_{\mathbb{R}} & \end{tikzcd} \tag{7.4}$$

Proof. **(a)** By applying Corollary 6.2.5 to the homomorphism $\phi : G(\mathbb{Z}) \to \mathcal{G}_{\mathcal{D}}(\mathbb{Q})$ we know that $\phi(G(\mathbb{Z}))$ is a bounded subgroup of $\mathcal{G}'_{\mathcal{D}}(\mathbb{Q})$. Hence, by Definition 4.1.6, $\phi(G(\mathbb{Z}))$ lies in the intersection of two spherical parabolic subgroups of opposite signs $P_{\pm}$ of $\mathcal{G}_{\mathcal{D}}(\mathbb{Q})$, namely; $A := P_+ \bigcap P_-$.
Let $A = L \ltimes U$ be a Levi decomposition of A containing a torus T (see Proposition 4.1.9). We look at the adjoint representation of A (see Section 4.2). By Theorem 4.2.10, there exists a finite-dimensional subspace V of $(\mathcal{U}_{\mathbb{Z}})_Q$ such that $H = \mathrm{Stab}_{\mathcal{G}_{\mathcal{D}}(\mathbb{Q})}(V)$. By Theorem 4.2.10(i), the Zariski closure $\bar{A}$ (resp. $\bar{L}$, $\bar{U}$) of $\mathbf{Ad}_{\mathbb{Q}}(A)|_V$ (resp. $\mathbf{Ad}_{\mathbb{Q}}(L)|_V$, $\mathbf{Ad}_{\mathbb{Q}}(U)|_V$) in $\mathrm{GL}(V_{\overline{\mathbb{Q}}})$ is a connected linear algebraic $\mathbb{Q}$-group, where $V_{\overline{\mathbb{Q}}} := V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \subset (\mathcal{U}_{\mathbb{Z}})_{\overline{\mathbb{Q}}}$. Moreover, by Theorem 4.2.10(ii), $\bar{A} = \bar{L} \ltimes \bar{U}$ is a Levi decomposition in the sense of (1.3). Moreover, $\mathbf{Ad}_{\mathbb{Q}}$ maps root subgroups of L (as in Definition 2.2.12) to root subgroups of $\bar{L}$ (in algebraic sense).
Now we look at the following group homomorphism:

$$\mathbf{Ad}_{\mathbb{Q}}|_A \circ \phi : G(\mathbb{Z}) \to \bar{A}_{\mathbb{Q}}. \tag{7.5}$$

Let H_1 be the Zariski closure of $\mathbf{Ad}_{\mathbb{Q}}|_A \circ \phi(G(\mathbb{Z}))$ in $\bar{A}$. By Theorem 1.6.1, H_1 is a semisimple linear algebraic $\mathbb{Q}$-group. Since H_1 is semisimple, it is contained in $\bar{L}$ as a semisimple linear algebraic $\mathbb{Q}$-subgroup (see Remark 4.4.10).
Because G is simply connected by the hypotheses, we can apply the classical super rigidity theorem, Theorem 1.6.4, to the following homomorphism:

$$\mathbf{Ad}_{\mathbb{Q}}|_A \circ \phi : G(\mathbb{Z}) \to (H_1)_{\mathbb{Q}}. \tag{7.6}$$

Therefore, there exists a (uniquely determined) virtual extension $\psi_1 : G \to H_1$ of $\mathbf{Ad}_{\mathbb{Q}}|_A \circ \phi$ which is a morphism of linear algebraic $\mathbb{Q}$-groups.
Since G is connected, $\psi_1 : G \to H_1$ co-restricts to a morphism of linear algebraic $\mathbb{Q}$-groups $\psi_1 : G \to H$ where $H := (H_1)^{\circ}$. Note that by definition, H is a connected semisimple linear algebraic $\mathbb{Q}$-group.
By Remark 1.6.6, ψ is uniquely determined by $G(n\mathbb{Z})$ for some $n \in \mathbb{N}$ with the property that it coincides with $\mathbf{Ad}_{\mathbb{Q}}|_A \circ \phi$ on $G(n\mathbb{Z})$.
By Remark 1.6.5, ψ is a central $\mathbb{Q}$-isogeny. Hence, for every root α of G, the root subgroup $U_\alpha(\mathbb{Q})$ of $G(\mathbb{Q})$ is mapped **injectively** to a root subgroup of $H_{\mathbb{Q}}$ via ψ. In particular, $U_\alpha(n\mathbb{Z})$ is mapped to a root subgroup of $H_{\mathbb{Q}}$ via ψ (see [Mar91, Proposition I(1.4.7)]).
Moreover, since by Theorem 1.6.1(ii), $\mathbf{Ad}_{\mathbb{Q}}$ maps **injectively** any root subgroup of L to a root subgroup $\bar{L}$ we conclude that $U_\alpha(n\mathbb{Z})$ lies in a root subgroup U_L^α of L.
Let $x_\alpha(1)$ be a generator of $U_\alpha(\mathbb{Z})$. Then $x_\alpha(n) \in G(n\mathbb{Z})$ and hence, $\phi(x_\alpha(n)) \in U_L^\alpha$. Now since both $U_\alpha(\mathbb{Q})$ and U_L^α are isomorphic to the additive group $(\mathbb{Q}, +)$ which is a unique divisible group, for each root α of G we obtain a group homomorphism $\phi_{\mathbb{Q}}^\alpha : U_\alpha(\mathbb{Q}) \to U_L^\alpha$. Furthermore, by Theorem 1.6.1(iii), $\mathbf{Ad}_{\mathbb{Q}}$ is injective on root subgroups of L. Hence, by construction and the fact that ψ coincides with $\mathbf{Ad}_{\mathbb{Q}}|_A \circ \phi$ on $G(n\mathbb{Z})$, we have the following commutative diagram:

$$\begin{tikzcd} U_\alpha(\mathbb{Q}) \arrow[r, "\phi_{\mathbb{Q}}^\alpha"] \arrow[dr, "\psi"'] & U_L^\alpha \arrow[d, "\mathbf{Ad}_{\mathbb{Q}}|_{U_L^\alpha}"] \\ & H_{\mathbb{Q}} \end{tikzcd} \tag{7.7}$$

with the property

$$\phi_{\mathbb{Q}}^\alpha|_{U_\alpha(n\mathbb{Z})} = \phi|_{U_\alpha(n\mathbb{Z})}. \tag{7.8}$$

Note that the above construction implies that actually

$$\phi_{\mathbb{Q}}^\alpha(q) := (\mathbf{Ad}_{\mathbb{Q}}^{-1}|U_L^\alpha \circ \psi)(q), \quad \text{for all } q \in \mathbb{Q},$$

which is well-defined by the injectivity of $\mathbf{Ad}_{\mathbb{Q}}$ on root subgroups of L (see Theorem 1.6.1(iii)).
In other words, $\phi_{\mathbb{Q}}^\alpha$ is the lift of ψ which is uniquely determined by the $n\mathbb{Z}$-points of $U_\alpha(\mathbb{Q})$.
Define $B_{\mathbb{Q}} := \mathbf{Ad}_{\mathbb{Q}}^{-1}(H_{\mathbb{Q}})$ and $\imath := \mathbf{Ad}_{\mathbb{Q}}|_{B_{\mathbb{Q}}} : B_{\mathbb{Q}} \to H_{\mathbb{Q}}$. By definition, $B_{\mathbb{Q}}$ contains $\phi(G(n\mathbb{Z}))$.
Hence, we obtain the following commutative diagram for any root α of G:

$$\begin{tikzcd} U_\alpha(\mathbb{Q}) \arrow[r, "\phi_{\mathbb{Q}}^\alpha"] \arrow[dr, "\psi"'] & B_{\mathbb{Q}} \arrow[r, hook] \arrow[d, "\imath"] & \mathcal{G}_{\mathcal{D}}(\mathbb{Q}) \\ & H_{\mathbb{Q}} & \end{tikzcd} \tag{7.9}$$

with the property
$$\phi^{\alpha}_{\mathbb{Q}}|_{U_\alpha(n\mathbb{Z})} = \phi|_{U_\alpha(n\mathbb{Z})}. \tag{7.10}$$
Now, because G is simply connected, $G(\mathbb{Q}) = \langle U_{\pm\alpha}(\mathbb{Q}) \mid \alpha \text{ simple root of } G\rangle$ and it is uniquely determined, up to isomorphism, by its root subgroups and certain relations (see Definition 1.4.17, see also [Gra02, Section 3.1]). This induces the same relations in $B_{\mathbb{Q}}$ by all $\phi^{\alpha}_{\mathbb{Q}}$ for roots α of G. The induced relations in $B_{\mathbb{Q}}$ hold since by Theorem 1.6.1(iii), the kernel of $\mathbf{Ad}_{\mathbb{Q}}$ is in the center of L. Therefore, we obtain a canonical group homomorphism

$$\begin{tikzcd} G(\mathbb{Q}) \arrow[r, "\phi_{\mathbb{Q}}"] \arrow[dr, "\psi"'] & B_{\mathbb{Q}} \arrow[r, hook] \arrow[d, "\imath"] & \mathcal{G}_{\mathcal{D}}(\mathbb{Q}) \\ & H_{\mathbb{Q}} & \end{tikzcd} \tag{7.11}$$

induced by $\phi^{\alpha}_{\mathbb{Q}}$ with the property
$$\phi_{\mathbb{Q}}|_{G(n\mathbb{Z})} = \phi|_{G(n\mathbb{Z})}. \tag{7.12}$$
(b) Applying functoriality of the split Tits functor and the morphism of functors **Ad** to the injective unital rings homomorphism $i : \mathbb{Q} \hookrightarrow \mathbb{R}$, we obtain a bounded subgroup $B_{\mathbb{R}}$ of $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ such that the following diagram commutes (see Subsection 4.1.1 and Theorem 4.2.3):

$$\begin{tikzcd} G(\mathbb{Q}) \arrow[r, "\phi_{\mathbb{Q}}"] \arrow[dr, "\psi"'] & B_{\mathbb{Q}} \arrow[r, hook] \arrow[d, "\imath"] & B_R \arrow[r, hook] \arrow[d, "\imath_{\mathbb{R}}"] & \mathcal{G}_{\mathcal{D}}(\mathbb{R}) \\ & H_{\mathbb{Q}} \arrow[r, hook] & H_{\mathbb{R}} & \end{tikzcd} \tag{7.13}$$

where $\imath_{\mathbb{R}} := \mathbf{Ad}_{\mathbb{R}}|_{B_{\mathbb{R}}}$.
Note that the injection $B_{\mathbb{Q}} \hookrightarrow B_{\mathbb{R}}$ follows from (KMG4). In particular, $B_{\mathbb{Q}} = B_{\mathbb{R}} \cap \mathcal{G}_{\mathcal{D}}(\mathbb{Q})$. Moreover, from the above diagram and Proposition 4.4.9, $B_{\mathbb{R}}$ is a semisimple Lie group with the induced topology from $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$.
Now, by Remark 1.6.5, we actually have a continuous Lie group homomorphism $\psi : G(\mathbb{R}) \to H_{\mathbb{R}}$. Also, since $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$ with the Lie topology, we obtain a unique continuous extension $\phi_{\mathbb{R}} : G(\mathbb{R}) \to B_{\mathbb{R}}$ of $\phi_{\mathbb{Q}} : G(\mathbb{Q}) \to B_{\mathbb{Q}} \hookrightarrow B_{\mathbb{R}}$. Hence, we obtain the following commutative diagram which satisfies the conditions in (b):

$$\begin{tikzcd} G(\mathbb{R}) \arrow[r, "\phi_{\mathbb{R}}"] \arrow[dr, "\psi"'] & B_{\mathbb{R}} \arrow[r, hook] \arrow[d, "\imath_{\mathbb{R}}"] & \mathcal{G}_{\mathcal{D}}(\mathbb{R}) \\ & H_{\mathbb{R}} & \end{tikzcd} \tag{7.14}$$

□

Remark 7.1.2. In Lemma 7.1.1, in view of Remark 1.6.6 we have actually the following:
$$\phi_{\mathbb{Q}}|_{G^{con}(n\mathbb{Z})} = \phi|_{G^{con}(n\mathbb{Z})}, \tag{7.15}$$
where $G^{con}(n\mathbb{Z})$ is the principal congruence of $G(\mathbb{Z})$ corresponding to the ideal $n\mathbb{Z}$ in $\mathbb{Z}$.

Let $\mathcal{G}_{\mathcal{D}}$ be an irreducible simply connected two-spherical split Tits functor with a Tits basis $\mathcal{F} = (\mathcal{G}_{\mathcal{D}}, \{\phi_i\}_{i\in I}, \eta)$. Recall from Terminology 5.2.1 that for any $n \in \mathbb{N}$ we denote by $\Gamma(n)$ the subgroup of $G_{\mathcal{D}}(\mathbb{Z})$ $(G_{\mathcal{D}}(n\mathbb{Z}))$ generated by all $G_{\alpha,\beta}(n\mathbb{Z})$ where $\alpha \approx \beta \in \Phi^{\circ}$.

Proposition 7.1.3 (Super Rigidity)**.** *Let $\mathcal{G}_{\mathcal{D}}$ be an irreducible simply connected centered two spherical split Tits functor with a Tits basis $\mathcal{F} = (\mathcal{G}_{\mathcal{D}}, \{\phi_i\}_{i\in I}, \eta)$. Assume that the rank of $\mathcal{G}_{\mathcal{D}}$ is at least two. Let $\mathcal{G}'_{\mathcal{D}'}$ be a centered two-spherical split Tits functor. Let $\phi : G_{\mathcal{D}}(\mathbb{Z}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q})$ be a group homomorphism such that the restriction of ϕ to the $\mathbb{Z}$-points of each simple root subgroup (with respect to $\mathcal{F}$) has infinite image. Then there exist an $n \in \mathbb{N}$ and a uniquely determined group homomorphism*

$$\phi_{\mathbb{Q}} : \mathcal{G}_{\mathcal{D}}(\mathbb{Q}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q}) \quad \textit{satisfying} \quad \phi_{\mathbb{Q}}|_{\Gamma(n)} = \phi|_{\Gamma(n)}.$$

Furthermore, there exists a uniquely determined continuous homomorphism with respect to the Kac-Peterson topology

$$\phi_{\mathbb{R}} : \mathcal{G}_{\mathcal{D}}(\mathbb{R}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{R}) \quad \textit{satisfying} \quad \phi_{\mathbb{R}}|_{\mathcal{G}_{\mathcal{D}}(\mathbb{Q})} = \phi_{\mathbb{Q}}.$$

Proof. If the rank of $\mathcal{G}_{\mathcal{D}}$ is 2, then the proposition follows from Lemma 7.1.1 and the following assumptions on $\mathcal{G}_{\mathcal{D}}$: irreducible, simply connected, centered and two-spherical.
Assume that the rank of $\mathcal{G}_{\mathcal{D}} > 2$. Consider a pair of simple roots $\alpha \approx \beta \in \Phi^{\circ}$ (where (Φ°) Φ denotes the set of (simple) roots of $\mathcal{G}_{\mathcal{D}}$ with respect to $\mathcal{F}$). We apply Lemma 7.1.1 to the following group homomorphism

$$\phi^{\alpha,\beta} := \phi|_{G_{\alpha,\beta}(\mathbb{Z})} : G_{\alpha,\beta}(\mathbb{Z}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q}),$$

where $G_{\alpha,\beta}(\mathbb{Z})$ denotes the group of $\mathbb{Z}$-points of the fundamental irreducible rank two subgroup of $\mathcal{G}_{\mathcal{D}}(\mathbb{Q})$ corresponding to the pair α and β. Hence we obtain the following uniquely determined local extension:

$$\begin{tikzcd} G_{\alpha,\beta}(\mathbb{Q}) \arrow[r, "\phi_{\mathbb{Q}}^{\alpha,\beta}"] \arrow[dr, "\psi^{\alpha,\beta}"'] & B_{\mathbb{Q}}^{\alpha,\beta} \arrow[d, "\iota_{\mathbb{Q}}^{\alpha,\beta}"] \\ & H_{\mathbb{Q}}^{\alpha,\beta} \end{tikzcd} \tag{7.16}$$

with the property

$$\phi_{\mathbb{Q}}^{\alpha,\beta}|_{G_{\alpha,\beta}(n_{\alpha,\beta}\mathbb{Z})} = \phi|_{G\alpha,\beta(n_{\alpha,\beta}\mathbb{Z})}, \tag{7.17}$$

for some $n_{\alpha,\beta} \in \mathbb{N}$.
Since the rank of $\mathcal{G}_{\mathcal{D}}$ is greater than two and it is irreducible, without loss of generality we can assume that there exists a simple root γ of $\mathcal{G}_{\mathcal{D}}$ (with respect to $\mathcal{F}$) distinct from α and β such that $\beta \approx \gamma$.
We apply Lemma 7.1.1 to the following group homomorphism:

$$\phi^{\beta,\gamma} := \phi|_{G_{\beta,\gamma}(\mathbb{Z})} : G_{\beta,\gamma}(\mathbb{Z}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q}).$$

Hence we obtain the following uniquely determined local extension:

$$\begin{tikzcd} G_{\beta,\gamma}(\mathbb{Q}) \arrow[r, "\phi_{\mathbb{Q}}^{\beta,\gamma}"] \arrow[dr, "\psi^{\beta,\gamma}"'] & B_{\mathbb{Q}}^{\beta,\gamma} \arrow[d, "\iota_{\mathbb{Q}}^{\beta,\gamma}"] \\ & H_{\mathbb{Q}}^{\beta,\gamma} \end{tikzcd} \tag{7.18}$$

with the property

$$\phi_{\mathbb{Q}}^{\beta,\gamma}|_{G_{\beta,\gamma}(n_{\beta,\gamma}\mathbb{Z})} = \phi|_{G\beta,\gamma(n_{\beta,\gamma}\mathbb{Z})}, \tag{7.19}$$

for some $n_{\beta,\gamma} \in \mathbb{N}$.
Since $\mathcal{G}_{\mathcal{D}}$ is simply connected and two-spherical, by Lemma 4.1.11 (with similar arguments for $\mathbb{Q}$) we have

$$G_{\alpha,\beta}(\mathbb{Q}) \cap G_{\beta,\gamma}(\mathbb{Q}) = G_\beta(\mathbb{Q}) \cong \mathrm{SL}_2(\mathbb{Q}).$$

By the universality of simply connected two-spherical split Kac-Moody groups over $\mathbb{Q}$ with respect to their fundamental subgroups of rank one and two (see Theorem 2.4.6), if we show that for any triple of simple roots (α,β,γ) as constructed above, the local extensions $\phi_{\mathbb{Q}}^{\alpha,\beta}$ and $\phi_{\mathbb{Q}}^{\beta,\gamma}$ coincide on $G_\beta(\mathbb{Q})$, then we obtain a global morphism

$$\phi_{\mathbb{Q}} : \mathcal{G}_{\mathcal{D}}(\mathbb{Q}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q}) \ \text{ satisfying } \ \phi_{\mathbb{Q}}|_{\Gamma(n)} = \phi|_{\Gamma(n)},$$

where n is the least common multiple of the $n_{\alpha,\beta}$ for all $\alpha \approx \beta \in \Phi^\circ$. Notice that n exists since $\mathcal{G}_{\mathcal{D}}$ is irreducible with finite rank. Moreover, going to the least common multiples does not influence the uniqueness of the local extensions (and hence the global extension) in view of Remark 1.6.6.
For this, we know that $\psi^{\alpha,\beta}(G_\beta(n\mathbb{Z})) = \psi^{\beta,\gamma}(G_\beta(n\mathbb{Z})) \subset H^{\alpha,\beta} \cap H^{\beta,\gamma}$. Now since, by construction, the local extensions $\phi_{\mathbb{Q}}^{\alpha,\beta}$ and $\phi_{\mathbb{Q}}^{\beta,\gamma}$ are uniquely determined by lifting $\psi^{\alpha,\beta}$ and $\psi^{\beta,\gamma}$ respectively, the lifting arguments in the proof of Lemma 7.1.1, which are unique with respect to $G_\beta(n\mathbb{Z})$, guarantee that $\phi_{\mathbb{Q}}^{\alpha,\beta}(G_\beta(\mathbb{Q})) \subset B_{\mathbb{Q}}^{\beta,\gamma}$ and $\phi_{\mathbb{Q}}^{\beta,\gamma}(G_\beta(\mathbb{Q})) \subset B_{\mathbb{Q}}^{\alpha,\beta}$. In particular, $\phi_{\mathbb{Q}}^{\alpha,\beta}|_{G_\beta(\mathbb{Q})} = \phi_{\mathbb{Q}}^{\beta,\gamma}|_{G_\beta(\mathbb{Q})}$.
Hence we obtain a global extension

$$\phi_{\mathbb{Q}} : \mathcal{G}_{\mathcal{D}}(\mathbb{Q}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q}),$$

which is the amalgamation of the local extensions $\phi_{\mathbb{Q}}^{\alpha,\beta}$ and, by construction, satisfies the following property:

$$\phi_{\mathbb{Q}}|_{\Gamma(n)} = \phi|_{\Gamma(n)}.$$

Moreover, it is uniquely determined by the above property by the uniqueness of the local extensions. The following diagram depicts the construction of $\phi_{\mathbb{Q}}$:

$$\begin{tikzcd}
G_\alpha(\mathbb{Q}) \arrow[d] & & G_\beta(\mathbb{Q}) \arrow[dll] \arrow[drr] & & G_\gamma(\mathbb{Q}) \arrow[d] \\
G_{\alpha,\beta}(\mathbb{Q}) \arrow[drr] \arrow[ddrr, "\phi_{\mathbb{Q}}^{\alpha,\beta}"'] & & & & G_{\beta,\gamma}(\mathbb{Q}) \arrow[dll] \arrow[ddll, "\phi_{\mathbb{Q}}^{\beta,\gamma}"] \\
& & \mathcal{G}_{\mathcal{D}}(\mathbb{Q}) \arrow[d, dotted, "\phi_{\mathbb{Q}}"'] & & \\
& & \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q}) & &
\end{tikzcd}$$

Furthermore, by Lemma 7.1.1(b), local extensions $\phi_{\mathbb{Q}}^{\alpha,\beta}$ above, have uniquely determined extensions $\phi_{\mathbb{R}}^{\alpha,\beta}$ on $G_{\alpha,\beta}(\mathbb{R})$. Moreover, for every triple of simple roots (α,β,γ) as above, the extensions $\phi_{\mathbb{R}}^{\alpha,\beta}$ and $\phi_{\mathbb{R}}^{\beta,\gamma}$ are compatible on $G_\beta(\mathbb{Q})$. Since $G_\beta(\mathbb{Q})$ is dense in $G_\beta(\mathbb{R})$ with respect to the Lie topology on $G_\beta(\mathbb{R})$ and the extensions $\phi_{\mathbb{R}}^{\alpha,\beta}$ and $\phi_{\mathbb{R}}^{\beta,\gamma}$ are unique continuous extension of $\phi_{\mathbb{Q}}^{\alpha,\beta}$ and $\phi_{\mathbb{Q}}^{\beta,\gamma}$ on $G_{\alpha,\beta}(\mathbb{Q})$ and $G_{\beta,\gamma}(\mathbb{Q})$ respectively, we conclude that $\phi_{\mathbb{R}}^{\alpha,\beta}|_{G_\beta(\mathbb{R})} = \phi_{\mathbb{R}}^{\beta,\gamma}|_{G_\beta(\mathbb{R})}$. Hence again by the universality of simply connected two-spherical real split Kac-Moody groups with respect to their fundamental subgroups of rank one and two (see Theorem 2.4.6), we conclude that there exists a uniquely determined group homomorphism

$$\phi_{\mathbb{R}} : \mathcal{G}_{\mathcal{D}}(\mathbb{R}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{R}) \ \text{ satisfying } \ \phi_{\mathbb{R}}|_{\mathcal{G}_{\mathcal{D}}(\mathbb{Q})} = \phi_{\mathbb{Q}}.$$

Moreover, since the local extensions $\phi_{\mathbb{R}}^{\alpha,\beta}$ are continuous with respect to the Lie topology on $G_{\alpha,\beta}(\mathbb{R})$ and the Kac-Peterson topology on $\mathcal{G}'_{\mathcal{D}'}(\mathbb{R})$, $\phi_{\mathbb{R}}$ is continuous with respect to the universal group topology on $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ and the Kac-Peterson topology on $\mathcal{G}'_{\mathcal{D}'}(\mathbb{R})$. But since $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ is simply connected, by Proposition 4.4.12, the universal group topology on $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$ coincides with the Kac-Peterson topology on $\mathcal{G}_{\mathcal{D}}(\mathbb{R})$. Hence, $\phi_{\mathbb{R}} : \mathcal{G}_{\mathcal{D}}(\mathbb{R}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{R})$ is a continuous group homomorphism with respect to the Kac-Peterson topology. □

Remark 7.1.4. In Proposition 7.1.3, since the global extension is uniquely determined by the local extensions, the uniqueness property holds for any multiples of n obtained in the statement, in view of Remark 1.6.6.

Remark 7.1.5. In Proposition 7.1.3, in view of Remark 7.1.2, we actually have

$$\phi_{\mathbb{Q}}|_{\Gamma^{con}(n)} = \phi|_{\Gamma^{con}(n)}, \tag{7.20}$$

where $\Gamma^{con}(n)$ as defined in Terminology 5.2.1.

Corollary 7.1.6. *Let $\mathcal{G}_{\mathcal{D}}$ be an irreducible simply connected centered two spherical split Tits functor with a Tits basis $\mathcal{F} = (\mathcal{G}_{\mathcal{D}}, \{\phi_i\}_{i\in I}, \eta)$. Assume that the rank of $\mathcal{G}_{\mathcal{D}}$ is at least two. Let $\mathcal{G}'_{\mathcal{D}'}$ be a centered two-spherical split Tits functor. Let $\phi : G_{\mathcal{D}}(\mathbb{Z}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q})$ be a group homomorphism such that the restriction of ϕ to the $\mathbb{Z}$-points of each simple root subgroup of $\mathcal{G}_{\mathcal{D}}$ (with respect to $\mathcal{F}$) has infinite image. Let Φ^{re} be the set of real roots of $\mathcal{G}_{\mathcal{D}}$ (with respect to $\mathcal{F}$), let $\{\Phi_i^\circ\}_{1\le i\le k}$ be a finite family of Weyl-conjugate systems of simple roots in Φ^{re}. Then there exist an $n \in \mathbb{N}$ and a uniquely determined group homomorphism*

$$\phi_{\mathbb{Q}} : \mathcal{G}_{\mathcal{D}}(\mathbb{Q}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q}) \ \ \textit{satisfying} \ \ \phi_{\mathbb{Q}}|_{\langle \Gamma_{\Phi_i^\circ}(n) | 1\le i\le k\rangle} = \phi|_{\langle \Gamma_{\Phi_i^\circ}(n) | 1\le i\le k\rangle}.$$

Furthermore, there exists a uniquely determined continuous group homomorphism with respect to the Kac-Peterson topology

$$\phi_{\mathbb{R}} : \mathcal{G}_{\mathcal{D}}(\mathbb{R}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{R}) \ \ \textit{satisfying} \ \ \phi_{\mathbb{R}}|_{\mathcal{G}_{\mathcal{D}}(\mathbb{Q})} = \phi_{\mathbb{Q}}.$$

Proof. Let $w \in W$ be an element of the Weyl group W associated to $\mathcal{G}_{\mathcal{D}}$. Let σ_w denote the action of the lift of w on $G_{\mathcal{D}}(\mathbb{Z})$ obtained in (5.14). We know that $\sigma_w \in \mathrm{Inn}(G_{\mathcal{D}}(\mathbb{Z}))$ in view of (5.13). Denote by Φ_w° the set of simple roots in Φ^{re}, w-conjugate to Φ°.
By applying twice Proposition 7.1.3 to the following group homomorphisms:

$$\phi : G_{\mathcal{D}}(\mathbb{Z}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q}) \quad \text{and} \quad \phi \circ \sigma_w : G_{\mathcal{D}}(\mathbb{Z}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q})$$

there exist an $n \in \mathbb{N}$ and uniquely determined group homomorphisms

$$\phi_{\mathbb{Q}}^1 : \mathcal{G}_{\mathcal{D}}(\mathbb{Q}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q}) \quad \text{and} \quad \phi_{\mathbb{Q}}^w : \mathcal{G}_{\mathcal{D}}(\mathbb{Q}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q})$$

satisfying (possibly by going to the least common multiple in view of Remark 7.1.4)

$$\phi^1_{\mathbb{Q}}|_{\Gamma_{\Phi^\circ}(n)} = \phi|_{\Gamma_{\Phi^\circ}(n)} \quad \text{and} \quad \phi^w_{\mathbb{Q}}|_{\Gamma_{\Phi^\circ}(n)} = \phi \circ \sigma_w|_{\Gamma_{\Phi^\circ}(n)}.$$

Define $\phi^2_{\mathbb{Q}} := \phi^w_{\mathbb{Q}} \circ \sigma_{w^{-1}}$. Then

$$\phi^2_{\mathbb{Q}}|_{\Gamma_{\Phi^\circ_w}(n)} = (\phi^w_{\mathbb{Q}} \circ \sigma_{w^{-1}})|_{\Gamma_{\Phi^\circ_w}(n)} = \phi^w_{\mathbb{Q}}|_{\Gamma_{\Phi^\circ}(n)} = \phi|_{\Gamma_{\Phi^\circ}(n)}. \tag{7.21}$$

By Lemma 5.2.12, there exists an $m_w \in \mathbb{N}$ such that

$$\Gamma_{\Phi^\circ}(m_w) \subseteq \Gamma_{\Phi^\circ}(n) \cap \Gamma_{\Phi^\circ_w}(n).$$

Hence by (7.21) we have

$$\phi^2_{\mathbb{Q}}|_{\Gamma_{\Phi^\circ}(m_w)} = \phi|_{\Gamma_{\Phi^\circ}(m_w)} = \phi^1_{\mathbb{Q}}|_{\Gamma_{\Phi^\circ}(m_w)}. \tag{7.22}$$

By going to the least common multiple of m_w and n (see Remark 7.1.4), the uniqueness in Proposition 7.1.3, in view of (7.22), implies

$$\phi^2_{\mathbb{Q}} = \phi^1_{\mathbb{Q}}.$$

Therefore, the map

$$\phi_{\mathbb{Q}} := \phi^2_{\mathbb{Q}} = \phi^1_{\mathbb{Q}} : \mathcal{G}_{\mathcal{D}}(\mathbb{Q}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q})$$

satisfies

$$\phi_{\mathbb{Q}}\big|_{\langle \Gamma_{\Phi^\circ}(n), \Gamma_{\Phi^\circ_w}(n)\rangle} = \phi\big|_{\langle \Gamma_{\Phi^\circ}(n), \Gamma_{\Phi^\circ_w}(n)\rangle}.$$

Therefore, the first claim follows by induction on the above argument and the second claim follows directly from Proposition 7.1.3. □

Remark 7.1.7. In Corollary 7.1.6, in view of Remark 7.1.5 and Corollary 5.2.13, we actually have

$$\phi_{\mathbb{Q}}\big|_{\langle \Gamma^{con}_{\Phi^\circ_i}(n) | 1 \le i \le k\rangle} = \phi\big|_{\langle \Gamma^{con}_{\Phi^\circ_i}(n) | 1 \le i \le k\rangle}. \tag{7.23}$$

7.2 Strong Rigidity

This section is to present a version of the classical Strong (Mostow) rigidity for arithmetic subgroups of certain real split Kac-Moody groups using the super rigidity obtained in the preceding section. Here we retain the notations in the previous section of this chapter.

Let $\mathcal{G}_{\mathcal{D}}$ be a centered two-spherical split Tits functor with a basis $\mathcal{F} = (\mathcal{G}_{\mathcal{D}}, \{\phi_i\}_{i \in I}, \eta)$. We call a subgroup $B_{\mathbb{Z}}$ of $G_{\mathcal{D}}(\mathbb{Z})$ **bounded** if it is contained in a bounded subgroup of $\mathcal{G}_{\mathcal{D}}(\mathbb{Q})$.

Definition 7.2.1. Let $\mathcal{G}_{\mathcal{D}}$ be a centered two-spherical split Tits functor with a basis $\mathcal{F} = (\mathcal{G}_{\mathcal{D}}, \{\phi_i\}_{i \in I}, \eta)$. We call $G_{\mathcal{D}}(\mathbb{Z})$ **strongly rigid** if for any bounded subgroup $B_{\mathbb{Z}}$ in $G_{\mathcal{D}}(\mathbb{Z})$, there exist an $n \in \mathbb{N}$ and a finite family $\{\Phi^\circ_i\}_{1 \le i \le k}$ of Weyl group-conjugate systems of simple roots for Φ^{re} such that $B_{\mathbb{Z}} \cap \langle \Gamma^{con}_{\Phi^\circ_i}(n) \mid 1 \le i \le k\rangle$ has finite index in $B_{\mathbb{Z}}$.

Example 7.2.2. Let G be an irreducible simply connected Chevalley group of rank at least two. G has the arithmeticity and the congruence subgroup property by [Mar91, Theorem (IX)6.5] and Theorem 1.5.1. Moreover, the Weyl group associated to W is finite. Hence, by the proof of the main result in [SV94], $\langle \Gamma^{con}_{\Phi^\circ_w}(n) \mid w \in W\rangle$ is a congruence subgroup of $G(\mathbb{Z})$ for any $n \in \mathbb{N}$. Therefore, $G(\mathbb{Z})$ is strongly rigid.

Proposition 7.2.3 (Strong Rigidity)**.** *Let $\mathcal{G}_\mathcal{D}$ and $\mathcal{G}'_{\mathcal{D}'}$ be irreducible simply connected centered two-spherical split Tits functors of rank at least two with Tits bases $\mathcal{F} = (\mathcal{G}_\mathcal{D}, \{\phi_i\}_{i\in I}, \eta)$ and $\mathcal{F}' = (\mathcal{G}'_{\mathcal{D}'}, \{\phi'_i\}_{i\in I'}, \eta')$. Assume that the Dynkin diagrams associated to $\mathcal{G}_\mathcal{D}$ and $\mathcal{G}'_{\mathcal{D}'}$ do not contain any sub-diagram of type C_2. Assume that $G_\mathcal{D}(\mathbb{Z})$ and $G'_{\mathcal{D}'}(\mathbb{Z})$ are strongly rigid. Let Φ^{re} be the set of real roots of $\mathcal{G}_\mathcal{D}$ and Φ° be the set of simple roots in Φ^{re} with respect to $\mathcal{F}$. Let $\phi : G_\mathcal{D}(\mathbb{Z}) \to G'_{\mathcal{D}'}(\mathbb{Z})$ be a group isomorphism of arithmetic Kac-Moody groups. Then there exist an $n \in \mathbb{N}$ and a uniquely determined group isomorphism*

$$\phi_\mathbb{Q} : \mathcal{G}_\mathcal{D}(\mathbb{Q}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q}) \quad \textit{satisfying} \quad \phi_\mathbb{Q}|_{\Gamma_{\Phi^\circ}(n)} = \phi|_{\Gamma_{\Phi^\circ}(n)}.$$

Furthermore, there exists a uniquely determined continuous group isomorphism with respect to the Kac-Peterson topology

$$\phi_\mathbb{R} : \mathcal{G}_\mathcal{D}(\mathbb{R}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{R}) \quad \textit{satisfying} \quad \phi_\mathbb{R}|_{\mathcal{G}_\mathcal{D}(\mathbb{Q})} = \phi_\mathbb{Q}.$$

Proof. Let $(\Phi^{re})'$ be the set of real roots of $\mathcal{G}'_{\mathcal{D}'}$ and $(\Phi^\circ)'$ be the set of simple roots in $(\Phi^{re})'$ with respect to $\mathcal{F}'$. Consider the following group homomorphisms

$$\phi : G_\mathcal{D}(\mathbb{Z}) \to G'_{\mathcal{D}'}(\mathbb{Z}) \hookrightarrow \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q}) \quad \text{and} \quad \phi^{-1} : G'_{\mathcal{D}'}(\mathbb{Z}) \to G_\mathcal{D}(\mathbb{Z}) \hookrightarrow \mathcal{G}_\mathcal{D}(\mathbb{Q}).$$

By Proposition 7.1.3 there exist $n, n' \in \mathbb{N}$ and uniquely determined group homomorphisms

$$\phi^1_\mathbb{Q} : \mathcal{G}_\mathcal{D}(\mathbb{Q}) \to \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q}) \quad \text{and} \quad \phi^2_\mathbb{Q} : \mathcal{G}'_{\mathcal{D}'}(\mathbb{Q}) \to \mathcal{G}_\mathcal{D}(\mathbb{Q})$$

satisfying

$$\phi^1_\mathbb{Q}|_{\Gamma_{\Phi^\circ}(n)} = \phi|_{\Gamma_{\Phi^\circ}(n)} \quad \text{and} \quad \phi^2_\mathbb{Q}|_{\Gamma_{(\Phi^\circ)'}(n')} = \phi^{-1}|_{\Gamma_{(\Phi^\circ)'}(n')}.$$

In view of Remark 7.1.4, by going to the least common multiple of n and n', we can assume that $n = n'$ without loss of generality.

Now we claim that $\phi^1_\mathbb{Q}$ is the inverse map of $\phi^2_\mathbb{Q}$. For this, by Theorem 2.4.6 and the symmetry of the arguments (in view of irreducibility of the underlying split Tits functors) it suffices to show that for each pair of simple roots $\alpha \approx \beta \in \Phi^\circ$, the following holds:

$$(\phi^2_\mathbb{Q} \circ \phi^1_\mathbb{Q})|_{G_{\alpha,\beta}(\mathbb{Q})} = \mathrm{id}\,. \tag{7.24}$$

Here we retain the notations introduced in the proof of Proposition 7.1.3. Hence

$$\phi^1_\mathbb{Q}|_{G_{\alpha,\beta}(\mathbb{Q})} = \phi^{1^{\alpha,\beta}}_\mathbb{Q} : G_{\alpha,\beta}(\mathbb{Q}) \to B^{\alpha,\beta}_\mathbb{Q}$$

with bounded subgroup $B^{\alpha,\beta}_\mathbb{Q} \leq \mathcal{G}_\mathcal{D}(\mathbb{Q})$ satisfying

$$\phi^{1^{\alpha,\beta}}_\mathbb{Q}|_{G_{\alpha,\beta}(n\mathbb{Z})} = \phi|_{G_{\alpha,\beta}(n\mathbb{Z})}.$$

Moreover, by Lemma 7.1.1, $\phi_{\mathbb{Q}}^{1^{\alpha,\beta}}$ has the following continuous extension:

$$\phi_{\mathbb{R}}^{1}|_{G_{\alpha,\beta}(\mathbb{R})} = \phi_{\mathbb{R}}^{1^{\alpha,\beta}} : G_{\alpha,\beta}(\mathbb{R}) \to B_{\mathbb{R}}^{\alpha,\beta},$$

where $B_{\mathbb{R}}^{\alpha,\beta} \leq \mathcal{G}_{\mathcal{D}}(\mathbb{R})$ is a semisimple Lie group. Furthermore, we have the following commutative diagram:

$$\begin{tikzcd} G_{\alpha,\beta}(\mathbb{R}) \arrow[r, "\phi_{\mathbb{R}}^{1^{\alpha,\beta}}"] \arrow[dr, "\psi^{\alpha,\beta}"'] & B_{\mathbb{R}}^{\alpha,\beta} \arrow[r, hook] \arrow[d, "\imath_{\mathbb{R}}^{\alpha,\beta}"] & \mathcal{G}_{\mathcal{D}}(\mathbb{R}) \\ & H_{\mathbb{R}}^{\alpha,\beta} & \end{tikzcd} \tag{7.25}$$

By hypotheses, $G_{\alpha,\beta}$ is a universal Chevalley group of type A_2 or G_2. In either case, $G_{\alpha,\beta}(\mathbb{R})$ is an absolutely simple group. On the other hand, by the proof of Lemma 7.1.1, $\phi_{\mathbb{R}}^{1^{\alpha,\beta}}$ is surjective. Hence we conclude that $\phi_{\mathbb{R}}^{1^{\alpha,\beta}}$ is a group isomorphism.
Because $\phi_{\mathbb{R}}^{1^{\alpha,\beta}}$ is a continuous extension of $\phi_{\mathbb{Q}}^{1^{\alpha,\beta}}$ and $G_{\alpha,\beta}(\mathbb{Z})$ is an irreducible lattice in $G_{\alpha,\beta}(\mathbb{R})$, $\phi_{\mathbb{R}}^{1^{\alpha,\beta}}(G_{\alpha,\beta}(\mathbb{Z}))$ is an irreducible arithmetic subgroup of $B_{\mathbb{Q}}^{\alpha,\beta}$.
Let $(\Phi')^{re}$ denote the set of real roots of $\mathcal{G}'_{\mathcal{D}'}$. Since $G'_{\mathcal{D}'}(\mathbb{Z})$ is strongly rigid by the hypotheses, there exist an $m \in \mathbb{N}$ and a finite family $\{(\Phi')_i^{\circ}\}_{1\leq i\leq k}$ of Weyl group-conjugate systems of simple roots for $(\Phi')^{re}$ such that $\phi_{\mathbb{R}}^{1^{\alpha,\beta}}(G_{\alpha,\beta}(\mathbb{Z})) \cap \langle \Gamma_{(\Phi')_i^{\circ}}^{con}(m) \mid 1 \leq i \leq k\rangle$ has finite index in $\phi_{\mathbb{R}}^{1^{\alpha,\beta}}(G_{\alpha,\beta}(\mathbb{Z}))$.
By Corollary 7.1.6 and its proof, in view of Remark 7.1.7, there exists a suitable multiple $l \in \mathbb{N}$ of m such that

$$\phi_{\mathbb{Q}}^{2}|_{\langle \Gamma_{(\Phi')_i^{\circ}}^{con}(l) | 1\leq i\leq k\rangle} = \phi^{-1}|_{\langle \Gamma_{(\Phi')_i^{\circ}}^{con}(l) | 1\leq i\leq k\rangle}. \tag{7.26}$$

By Remark 7.1.4, with out loss of generality, we can assume that $l = n$.
Now since $B^{\alpha,\beta}$ has the congruence subgroup property (see Theorem 1.5.1), $\phi_{\mathbb{R}}^{1^{\alpha,\beta}}(G_{\alpha,\beta}(\mathbb{Z})) \cap \langle \Gamma_{(\Phi')_i^{\circ}}^{con}(m) \mid 1 \leq i \leq k\rangle$ contains a principal congruence subgroup Γ of $B_{\mathbb{Q}}^{\alpha,\beta}$.
Therefore, the maps

$$(\phi_{\mathbb{Q}}^{1^{\alpha,\beta}})^{-1} : B_{\mathbb{Q}}^{\alpha,\beta} \to G_{\alpha,\beta}(\mathbb{Q}) \leq \mathcal{G}_{\mathcal{D}}(\mathbb{Q}) \quad \text{and} \quad \phi_{\mathbb{Q}}^{2}|_{B_{\mathbb{Q}}^{\alpha,\beta}} : B_{\mathbb{Q}}^{\alpha,\beta} \to \mathcal{G}_{\mathcal{D}}(\mathbb{Q})$$

satisfy

$$(\phi_{\mathbb{Q}}^{1^{\alpha,\beta}})^{-1}|_{\Gamma} = \phi^{-1} = \phi_{\mathbb{Q}}^{2}|_{\Gamma},$$

which by the uniqueness property in Lemma 7.1.1 results $(\phi_{\mathbb{Q}}^{2} \circ \phi_{\mathbb{Q}}^{1})|_{G_{\alpha,\beta}(\mathbb{Q})} = \mathrm{id})$. □

Theorem 7.2.4 (Solution of the isomorphism problem)**.** *Let $\mathcal{G}_{\mathcal{D}}$ and $\mathcal{G}'_{\mathcal{D}'}$ be two irreducible simply connected centered two-spherical split Tits functors of rank at least two whose associated Dynkin diagrams do not contain any sub-diagram of type C_2. If $\mathcal{G}_{\mathcal{D}}(\mathbb{Q})$ and $\mathcal{G}'_{\mathcal{D}'}(\mathbb{Q})$ contain isomorphic strongly rigid arithmetic subgroups, then $\mathcal{G}_{\mathcal{D}} = \mathcal{G}'_{\mathcal{D}'}$.*

Proof. It follows from Proposition 7.2.3 and Theorem 4.3.3. □

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INDEX

DEUTSCHE ZUSAMMENFASSUNG

Hauptgegenstand der vorliegenden Dissertation sind reelle zerfallende Kac-Moody-Gruppen (von endlichem Rang) und ihre arithmetischen Untergruppen; zerfallende Kac-Moody-Gruppen können als unendlichdimensionale Verallgemeinerung von Chevalley-Gruppen angesehen werden. Die zerfallenden Kac-Moody-Gruppen, die in der Literatur manchmal auch als minimale zerfallende Kac-Moody-Gruppen bezeichnet werden (vgl. [KP85, Seite 141], [Mar13, Kapitel (II)5]), können auch als Bild eines Gruppenfunktors angesehen werden, dem (zerfallenden) Tits-Funktor auf der Kategorie der kommutativen Ringe mit 1, welcher über Körpern aufgrund ähnlicher Überlegungen wie för Chevalley-Gruppenfunktoren eindeutig bestimmt ist (vgl. [Tit87]). Daher macht es Sinn, in Anlehnung an die oben genannte funktorielle Eigenschaft der zerfallenden Kac-Moody-Gruppen, die funktoriell arithmetischen Untergruppen (d. h. die Bilder des Gruppenfunktors Über den ganzen Zahlen) reeller zerfallender Kac-Moody-Gruppen zu definieren und ihre Starrheit zu untersuchen.
Des Weiteren hat, ähnliche wie bei Chevalley-Gruppen, jede zerfallende Kac-Moody-Gruppe ein Wurzelsystem, obgleich dieses nicht endlich sein muss. Im Fall, dass das Wurzelsystem endlich ist, stimmt die zugrunde liegende zerfallende Kac-Moody-Gruppe mit einer Chevalley-Gruppe überein (vgl. z.B. [KP85, Beispiel 1.1]). Folglich kann man sowohl Wurzeluntergruppen definieren, als auch Untergruppen von endlichem Rang. Eine wichtige Eigenschaft gewisser zerfallender Kac-Moody-Gruppen ist die Tatsache, dass sie das universelle Amalgam ihrer Rang-1- und Rang-2-Untergruppen sind (vgl. [AM97], [HKM13, Theorem 7.22]). Anschaulich ist eine Gruppe G das universelle Amalgam ihrer echten Untergruppen, falls jedes andere Amalgam dieser Untergruppen isomorph zur Faktorgruppe von G über einem Normalteiler ist. Wir untersuchen den Spezialfall von zentrierten, 2-sphärisch, reellen zerfallenden Kac-Moody-Gruppen. Dies führt dazu, dass für solche zerfallenden Kac-Moody-Gruppen jede fundamentale Untergruppe von Rang 2 (d. h. jede Untergruppe, die einem Paar einfacher Wurzeln entspricht) wiederum eine universelle Chevalley-Gruppe (d. h. eine zusammenhängende, algebraisch einfach zusammenhängende Chevalley-Gruppe) ist. Dies ermöglicht es uns, die arithmetischen Untergruppen von reellen zerfallenden Kac-Moody-Gruppen zu definieren.
Eine arithmetische Untergruppe einer reellen zerfallenden Kac-Moody-Gruppe G ist eine Gruppe, die von den $\mathbb{Z}$-Punkten der reellen Wurzeluntergruppen erzeugt wird. Wir bezeichnen eine solche Untergruppe mit $G(\mathbb{Z})$. Man beachte, dass im Fall, dass G eine Chevalley-Gruppe ist, die funktoriell arithmetische Untergruppe mit der arithmetischen Untergruppe übereinstimmt. Allerdings werden wir in Kapitel 5 feststellen, dass im Allgemeinen die arithmetische Untergruppe nur eine Untergruppe der funktoriell arithmetischen Untergruppe ist. Auf dieselbe Art kann man für n in $\mathbb{N}$ die Untergruppe von G, die von den $n\mathbb{Z}$-Punkten der reellen Wurzeluntergruppen erzeugt wird, definieren; wir werden sie mit $G(n\mathbb{Z})$ bezeichnen. Abermals kann man unter Verwendung der funktoriellen Eigenschaft des zerfallenden Tits-

Funktors die funktoriell principal congruence Untergruppen ähnlich wie im Fall von Chevalley-Gruppen definieren. In Abschnitt 5.2 wird gezeigt, dass $G(n\mathbb{Z})$ tatsächlich eine Untergruppe einer solchen funktoriell principal congruence Untergruppe vom Grad n ist, und zudem eine Untergruppe der arithmetischen Untergruppe von G. Bedeutungsvoll ist die Tatsache, dass der Unterschied zwischen der funktoriell principal congruence Untergruppe und $G(n\mathbb{Z})$ den Unterschied zwischen der funktoriell arithmetischen Untergruppe und der arithmetischen Untergruppe von G für den Fall, dass n eine Primzahl ist, festlegt (vgl. (5.129)).
Eine weitere wichtige Eigenschaft von reellen zerfallenden Kac-Moody-Gruppen ist diejenige, dass eine algebraisch-geometrische Struktur mit ihnen korrespondiert, nämlich die eines (Zwillings-) Gebäudes. Ein (Zwillings-) Gebäude, das zu einer reellen zerfallenden Kac-Moody-Gruppe gehört, ist eine (Doppel-) Nebenklasse von G über einer speziellen Untergruppe, die Borel-Untergruppe (d.h. z.B. die Gruppe der oberen Dreiecksmatrizen in $\mathrm{SL}_n(\mathbb{R})$) genannt wird, zusammen mit einer adjacency Relation zwischen den Elementen der (Doppel-) Nebenklasse, die uns eine geometrische Umsetzung des entsprechenden (Zwillings-) Gebäudes in Form eines (einfachen) polyhedrischen Komplexes liefert und uns erlaubt, geometrische Gruppentheorie im Zusammenhang mit Kac-Moody-Gruppen anzuwenden (vgl. [Dav08, Kapitel 18]).
Das Ziel der vorliegenden Dissertation ist es, ein Gegenstück des Starrheit-Phänomens zu präsentieren, das im Zusammenhang mit Kac-Moody-Gruppen existiert. Um das Starrheit-Phänomen für arithmetische Untergruppen reeller zerfallender Kac-Moody-Gruppen zu untersuchen, müssen wir die reellen zerfallenden Kac-Moody-Gruppen mit einer Topologie versehen, die sowohl verträglich mit der Lie-Topologie ihrer Rang-1- und Rang-2-Untergruppen, als auch mit der universellen Topologie ist. Dies wurde in [HKM13] durch Einführung einer Topologie, der Kac-Peterson-Topologie, auf gewissen zerfallenden Kac-Moody-Gruppen erzielt. Allerdings sind nach [HKM13, Bemerkung 7.28] zerfallende Kac-Moody-Gruppen nicht lokalkompakt bezüglich der Kac-Peterson-Topologie, und folglich haben sie kein Haar-Maß. Dementsprechend ist es nicht ohne Weiteres möglich, mit Gittern zu arbeiten oder der Ergodentheorie ähnliche Formeln zu entwickeln.
Trotzdem möchten wir ähnliche Starrheit-Ergebnisse wie im klassischen Zusammenhang für arithmetische Untergruppen reeller zerfallender Kac-Moody-Gruppen gewinnen. Insbesondere wollen wir eine eindeutige Fortsetzung eines abstrakten Gruppenhomomorphismus $\Phi : G(\mathbb{Z}) \to H(\mathbb{Q})$ finden, nämlich $\tilde{\Phi} : G(\mathbb{Q}) \to H(\mathbb{Q})$. Unsere Vorgehensweise ist es, das klassische Starrheit-Ergebnis in unserem Fall anzuwenden. Dafür müssen wir unsere Argumente lokalisieren und die Einschränkung von Phi auf die arithmetischen Untergruppen der irreduziblen fundamentalen Rang-2-Untegruppen von $G(\mathbb{Z})$ untersuchen, und verträgliche lokale Fortsetzungen erzielen. Dann können wir die universelle Eigenschaft der zugrunde liegenden reellen zerfallenden Kac-Moody-Gruppe verwenden, um eine globale Fortsetzung zu erhalten.
In Kapitel 6 arbeiten wir den Beweis eines Resultats über Fixpunkte von P-E. Caprace und N. Monod ausführlich aus. Dieses Ergebnis, [CM09, Lemma 8.1], liefert einen Fixpunksatz für eine spezielle Wirkung von Chevalley-Gruppen auf gewissen polyhedrischen Komplexen (d. h. für die Wirkung, die die polyhedrische Struktur erhält). Um genau zu sein, geben wir den folgenden Fixpunktsatz an:
Fixpunktsatz. *(vgl. [CM09, Lemma 8.1]) Sei* Δ *ein vollständiger* $\mathrm{CAT}(0)$ *polyedrischer Komplex so, dass* $\mathrm{Shapes}(\Delta)$ *endlich ist (vgl. Definition 3.2.2) und sei* G *eine universelle Chevalley-Gruppe von Rang mindestens 2. Wirkt die arithmetische Untergruppe* $G(\mathbb{Z})$ *von* G

auf Δ durch zelluläre Isometrien, so hat $G(\mathbb{Z})$ einen Fixpunkt.

In unserem Fall können wir für den lokalen Homomorphismus, der von der Einschränkung von Φ auf die arithmetischen Untergruppen der fundamentalen Rang-2-Untergruppen von $G(\mathbb{Q})$ induziert wird, den obigen Fixpunktsatz anwenden und die Argumente in unserer Situation auf die klassische Starrheit-Situation zurückführen. Nun können wir den folgenden Super-Starrheit-Satz für arithmetische Untergruppen von zerfallenden Kac-Moody-Gruppen formulieren:

Super-Starrheit-Satz (im Kac-Moody-Kontext). *Seien G und H zwei zentrierte, irreduzible, zwei-sphärische, einfache zusammenhängende, zerfallende und reelle Kac-Moody-Gruppen von endlichem Rang. Angenommen, die Einschränkung von ϕ auf die Z-Punkte einfacher Wurzelgruppen hat ein unendliches Bild. Sei $\Phi : G(\mathbb{Z}) \to H(\mathbb{Q})$ ein abstrakter Gruppenhomomorphismus. Dann existiert ein $n \in \mathbb{N}$ und ein (eindeutig bestimmter) Gruppenhomomorphismus $\tilde{\Phi} : G(\mathbb{Q}) \to H(\mathbb{Q})$ so, dass $\tilde{\Phi}|_{\Gamma(n)} = \Phi|_{\Gamma(n)}$.*

Für eine genauere Ausführung vergleiche man Proposition 7.1.3. Außerdem erhalten wir in diesem Zusammenhang die folgende Version des Strong-Starrheit-Satzes.

Strong-Starrheit-Satz (im Kac-Moody-Kontext). *Seien G und H zwei zentrierte, irreduzible, zwei-sphärische, einfache zusammenhängende, zerfallende und reelle Kac-Moody-Gruppen von endlichem Rang. Angenommen, die Dynkin-Diagramme von G und H enthalten kein Unterdiagramm des Typs C_2. Angenommen, $G(\mathbb{Z})$ und $H(\mathbb{Z})$ sind stark-starr. Sei $\Phi : G(\mathbb{Z}) \to H(\mathbb{Z})$ ein abstrakter Gruppenisomorphismus. Dann sind G und H isomorph und haben isomorphe Wurzelsysteme.*

Für eine genauere Ausführung vergleiche man Proposition 7.2.3.

SELBSTÄNDIGKEITSERKLÄRUNG

Ich erkläre:

Ich habe die vorgelegte Dissertation selbständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt, die ich in der Dissertation angegeben habe. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben, die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht. Bei den von mir durchgeführten und in der Dissertation erwähnten Untersuchungen habe ich die Grundsätze guter wissenschaftlicher Praxis, wie sie in der „Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis“ niedergelegt sind, eingehalten.

Gießen, 22 September 2015

Amir Farahmand Parsa