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Dense Short Solution Segments from Monotonic Delayed Arguments

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Abstract
We construct a delay functional d on an open subset of the space $C_r^1 = C^1([-r, 0], \mathbb{R})$ and find $h \in (0, r)$ so that the equation

$$x'(t) = -x(t - d(x_t))$$

defines a continuous semiflow of continuously differentiable solution operators on the solution manifold

$$X = \{\phi \in C_r^1 : \phi'(0) = -\phi(-d(\phi))\},$$

and along each solution the delayed argument $t - d(x_t)$ is strictly increasing, and there exists a solution whose *short segments*

$$x_{t,short} = x(t + \cdot) \in C_h^2, \quad t \geq 0,$$

are dense in an infinite-dimensional subset of the space C_h^2. The result supplements earlier work on complicated motion caused by state-dependent delay with oscillatory delayed arguments.

Keywords Delay differential equation · State-dependent delay · Complicated motion

AMS Subject Classification 34 K 23

1 Introduction

The present paper continues the studies [6,10–14] of how time lags which are state-dependent affect the behaviour of feedback systems. The basic equation considered is

$$x'(t) = -\alpha\, x(t - r) \qquad (\alpha, r)$$

Dedicated to the memory of Geneviève Raugel.

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with $\alpha > 0$ and constant time lag $r > 0$. This is the simplest delay differential equation modelling negative feedback with respect to the zero solution. Let C^0_r denote the Banach space of continuous functions $[-r, 0] \to \mathbb{R}$ with the maximum norm, $|\phi|_{0,r} = \max_{-r \le t \le 0} |\phi(t)|$. The solutions $x : [-r, \infty) \to \mathbb{R}$ of Eq. (α, r), which are continuous and have differentiable restrictions to $[0, \infty)$ which satisfy Eq. (α, r), define a strongly continuous semigroup on C^0_r by the equations $T(t)x_0 = x_t$ with the solution segments

$$x_t : [-r, 0] \ni s \mapsto x(t + s) \in \mathbb{R} \quad \text{for} \quad t \ge 0,$$

see [2]. Except for $\alpha = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{N}_0$ the zero solution is hyperbolic [2,15].

Let C^1_r denote the Banach space of continuously differentiable functions $\phi : [-r, 0] \to \mathbb{R}$, with the norm given by $|\phi|_{1,r} = |\phi|_{0,r} + |\phi'|_{0,r}$. In [6,10–12] delay functionals $d : C^1_r \supset U \to [0, r]$ were constructed so that for certain $\alpha > 0$ the modified equation

$$x'(t) = -\alpha\, x(t - d(x_t)) \tag{α, d}$$

has homoclinic solutions, with chaotic motion nearby.

The results in [13,14] established another kind of complicated solution behaviour, namely, the existence of delay functionals d and parameters $\alpha > 0$ so that for a positive number $h < r$ there are solutions whose *short* solution segments

$$x_{t,short} : [-h, 0] \ni s \mapsto x(t + s) \in \mathbb{R}, t \ge 0,$$

are dense in open subsets of the space C^1_h.

In [13] density of short segments in the whole space C^1_h was achieved for a continuous delay functional on a set $Y \subset C^1_r$ which is large in some sense but not open, nor a differentiable submanifold. Because of this lack of regularity results from [8,9] on well-posedness of initial value problems and on differentiability of solutions with respect to initial data do not apply.

In [14] we constructed a continuously differentiable delay functional $d : U \to [0, r]$, $U \subset C^1_r$ open, so that the results from [8] apply, and found $h \in (0, r)$ so that the previous equation with $\alpha = 1$, namely,

$$x'(t) = -x(t - d(x_t)) \tag{1.1}$$

has a solution $x : [-r, \infty) \to \mathbb{R}$ whose short segments are dense in an open subset of the space C^1_h. The construction involves that the *delayed argument function*

$$[0, \infty) \ni t \mapsto t - d(x_t) \in \mathbb{R}$$

along the solution x is not monotonic, and this oscillatory behaviour seems crucial for density of short segments in an *open* subset of the space C^1_h.

Before stating the result of the present paper let us mention that equations with non-constant, state-dependent delay are not covered by the theory with state space C^0_r which is familiar from monographs on delay differential equations [1–3]. We recall what was shown in [8] for delay differential equations in the general form

$$x'(t) = f(x_t) \tag{f}$$

under hypotheses designed for applications to examples with state-dependent delay. Let $C^0_{r,n}$ and $C^1_{r,n}$ denote the analogues of the spaces C^0_r and C^1_r, for maps $[-r, 0] \to \mathbb{R}^n$. Assume $f : U \to \mathbb{R}^n$, $U \subset C^1_{r,n}$ open, is continuously differentiable so that

(e) *each derivative* $Df(\phi) : C^1_{r,n} \to \mathbb{R}^n, \phi \in U$, *has a linear extension* $D_e f(\phi) : C^0_{r,n} \to \mathbb{R}^n$ *and the map*

$$U \times C^0_{r,n} \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n$$

is continuous.

The extension property (e) is a variant of the notion of being *almost Fréchet differentiable* for maps $C^0_{r,n} \supset V \to \mathbb{R}^n$ which was introduced in [7].

Suppose also there exists $\phi \in U$ with $\phi'(0) = f(\phi)$. Then the nonempty set

$$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}$$

is a continuously differentiable submanifold with codimension n in $C^1_{r,n}$, and each initial value problem

$$x'(t) = f(x_t) \quad \text{for} \quad t > 0, \quad x_0 = \phi \in X_f,$$

has a unique maximal solution $x : [-r, t_\phi) \to \mathbb{R}^n$, $0 < t_\phi \le \infty$, which is continuously differentiable with $x'(t) = f(x_t)$ for all $t \in [0, t_\phi)$. The arrow

$$(t, \phi) \mapsto x^\phi_t,$$

with the said maximal solution $x = x^\phi$, defines a continuous semiflow of continuously differentiable *solution operators*

$$\{\phi \in X_f : t_\phi > t\} \ni \phi \mapsto x^\phi_t \in X_f, \quad t \ge 0.$$

In the present paper we prove the following result on complicated motion caused by a delay functional so that the delayed argument functions along solutions of Eq. (1.1) are monotonic.

Theorem 1.1 *There exist $r > h > 0$ and a continuously differentiable delay functional $d : N \to (0, r)$, $N \subset C^1_r$ open, and an open subset A of a closed affine subspace of codimension 6 in C^2_h so that Eq. (1.1) has a twice continuously differentiable solution $x^{(d)} : [-r, \infty) \to \mathbb{R}$ whose short segments $x^{(d)}_{t,short}$, $t \ge 0$, are dense in $A \cup (-A)$.*

The functional $f : N \ni \phi \mapsto -\phi(-d(\phi)) \in \mathbb{R}$ is continuously differentiable and has property (e), and for each $\phi \in X_f$ the delayed argument function

$$[0, t_\phi) \ni t \mapsto t - d(x^\phi_t) \in \mathbb{R}$$

along the maximal continuously differentiable solution $x^\phi : [-r, t_\phi) \to \mathbb{R}$ of the initial value problem

$$x'(t) = f(x_t) = -x(t - d(x_t)) \quad \textit{for} \quad t > 0, \quad x_0 = \phi \in X_f,$$

is strictly increasing.

Here C^2_h denotes the Banach space of twice continuously differentiable functions $\psi : [-h, 0] \to \mathbb{R}$, with the norm given by $|\psi|_{2,r} = \sum_{k=0}^2 \max_{-r \le t \le 0} |\psi^{(k)}(t)|$.

A different result on complicated motion caused by state-dependent delay with monotonic delayed argument functions has recently been obtained in [5].

The proof of Theorem 1.1 begins in Sect. 2 below with the choice of subsets $A = A_h \subset C^2_h$ as in the theorem, for arbitrary $h > 0$. For arbitrary $s > 0$ Sect. 3 prepares a sequence of twice continuously differentiable functions $\kappa_{s,n} : [-s, s] \to \mathbb{R}$ so that certain translates of $\kappa_{s,n}$ and $\kappa_{s,k}$, $n \ne k$, keep a minimal distance from each other, in the sense that there is a constant $a > 0$ with

$$|(\kappa_{s,n})'(t + u) - (\kappa_{s,k})'(u)| \ge \frac{a}{4}$$

for small t and some u.

Section 4 is the core of the proof of Theorem 1.1. For suitably chosen $t_b < 0 < t_5, h > 0$, $s > 0$, a sequence of continuously differentiable *delay functions* $\Delta_n : [0, t_5] \to (0, \infty)$ together with a sequence of twice continuously differentiable functions $x_{(n)} : [t_b, t_5] \to \mathbb{R}$ and a subset $A = A_h \subset C_h^2$ as in Sect. 2 are constructed so that for each $n \in \mathbb{N}$ - the linear nonautonomous equation

$$(x_{(n)})'(t) = -x_{(n)}(t - \Delta_n(t))$$

holds for $0 \le t \le t_5$,

- the delayed argument function $[0, t_5] \ni t \mapsto t - \Delta_n(t) \in \mathbb{R}$ along the delay function Δ_n is strictly increasing,
- on some subinterval of length h in $[0, t_5]$ the function $x_{(n)}$ coincides with a translate of a member p_n of a sequence which is dense in A,
- on some subinterval of length $2s$ in $[0, t_5]$ the function $x_{(n)}$ coincides with a translate of $\kappa_n = \kappa_{s,n}$.

In Sect. 5 shifted copies of the functions Δ_n and of the functions $\pm x_{(n)}$ are concatenated, respectively, and yield a twice continuously differentiable function $x : [t_b, \infty) \to \mathbb{R}$ and a continuously differentiable delay function Δ on $[0, \infty)$ which is bounded by some $r > \max\{h, -t_b\}$. A twice continuously differentiable extension of the function x to the ray $[-r, \infty) \to \mathbb{R}$ satisfies the linear equation

$$x'(t) = -x(t - \Delta(t)) \tag{1.2}$$

for all $t \ge 0$. Proposition 5.1 states that the curve $[r, \infty) \ni t \mapsto x_t \in C_r^1$ is injective, hence the equation

$$d(x_t) = \Delta(t)$$

converts the delay function into a delay functional d on the trace $\{x_t \in C_r^2 : t \ge r\}$.

Sections 6, 7, and 8 prepare the extension of this functional to an open neighbourhood N of the trace $\{x_t \in C_r^2 : (j_r - 1)t_5 \le t\}$ in the space C_r^1, with an integer $j_r \ge 2$ so that $r < (j_r - 1)t_5$. Section 6 contains an ingredient of the construction which will be used in the final Sect. 9, namely, separation of nonadjacent arcs

$$\{x_t \in C_r^2 : (n-1)t_5 \le t \le nt_5\} \text{ and } \{x_t \in C_r^2 : (j-1)t_5 \le t \le jt_5\},$$
$$2 \le n \in \mathbb{N} \quad \text{and} \quad 2 \le j \in \mathbb{N} \text{ with } |n - j| > 1,$$

in the space C_r^1. The separation result is based on the properties of the functions $\kappa_{s,n}$ from Sect. 3 whose translates appear as restrictions of x on a sequence of mutually disjoint intervals tending to infinity.

The constructions in Sects. 2, 3, 4, 5, and 6 are to some extent parallel to constructions in [14]. The next steps in Sects. 7 and 8 are rather different from their counterparts in [14]. The new tool, introduced in Sect. 7, is a bundle of transversal hyperplanes K_t, $t > 0$, along the curve $(0, \infty) \ni t \mapsto x_t \in C_r^0$. Working with the bundle allows for an extension of the delay functional from an arc $\{x_t \in C_r^2 : (k-1)t_5 \le t \le kt_5\}$, $j_r \le k \in \mathbb{N}$, to a kind of tubular neighbourhood $U_k \subset C_r^0$ (Sect. 8), and for the arrangement of compatibility relations on overlapping domains $U_k \cap U_{k+1}$, in ways which are simpler than corresponding procedures in [14].

Section 9 begins with the definition of the domain $N \subset C_r^1$ and the functional $d : N \to (0, r)$, and completes the proof of Theorem 1.1. The verification that the functional $f : N \to \mathbb{R}$ in Theorem 1.1 has property (e) uses that the delay functional $d : N \to (0, r)$ has property (e). The latter is achieved by means of the following proposition whose statement involves the injective linear continuous inclusion map

$$J : C^1_r \ni \phi \mapsto \phi \in C^0_r.$$

Proposition 1.2 *[14, Proposition 1.2] Suppose $d : C^1_r \supset N \to \mathbb{R}$ is continuously differentiable and for every $\phi \in N$ there exist an open neighbourhood V of $J\phi$ in C^0_r and a continuously differentiable map $d_V : C^0_r \supset V \to \mathbb{R}$ with $d(\psi) = d_V(J\psi)$ for all $\psi \in N \cap J^{-1}(V)$. Then d has property (e), with*

$$D_e d(\phi)\chi = Dd_V(J\phi)\chi \quad \textit{for all} \quad \phi \in N \cap J^{-1}(V) \quad \textit{and} \quad \chi \in C^0_r.$$

Notation, preliminaries. A sequence in a metric space is called dense if each point of the metric space is an accumulation point of the sequence. A metric space is called separable if it contains a dense sequence.

For $\epsilon > 0$ the open ϵ-neighbourhoods of a point x in a normed space X and of a subset $S \subset X$ are given by

$$U_\epsilon(x) = \{y \in X : |y - x| < \epsilon\}.$$

and

$$U_\epsilon(S) = \{y \in X : \mathrm{dist}(y, S) < \epsilon\},$$

respectively, with

$$\mathrm{dist}(y, S) = \inf_{x \in S} |y - x|.$$

For $a < b$ in $\mathbb{R}$ and $j \in \mathbb{N}$ let $C^j_{a,b}$ denote the Banach space of j times continuously differentiable functions $\phi : [a, b] \to \mathbb{R}$, with the norm given by

$$|\phi|_{j,a,b} = \sum_0^j \max_{a \le t \le b} |\phi^j(t)|,$$

and let $C^0_{a,b}$ denote the Banach space of continuous functions $\phi : [a, b] \to \mathbb{R}$, with the norm given by

$$|\phi|_{0,a,b} = \max_{a \le t \le b} |\phi(t)|.$$

In case $a = -r$ and $b = 0$, the abbreviations

$$C^j_r = C^j_{-r,0} \quad \text{and} \quad |\cdot|_{j,r} = |\cdot|_{j,-r,0}$$

are used. If functions $\phi \in C^2_r$ and $\phi \in C^1_r$ are considered as elements of the ambient space C^0_r then we use $\phi \in C^0_r$ or $J\phi \in C^0_r$, depending on which form makes an argument more transparent.

For $r > 0$ the evaluation map

$$C^0_r \times [-r, 0] \ni (\phi, t) \mapsto \phi(t) \in \mathbb{R}$$

is continuous but not locally Lipschitz continuous, and the evaluation map

$$ev^1_r : C^1_r \times (-r, 0) \ni (\phi, t) \mapsto \phi(t) \in \mathbb{R}$$

is continuously differentiable with

$$D\,ev_r^1(\phi,t)(\hat{\phi},\hat{t}) = D_1 ev_h^1(\phi,t)\hat{\phi} + D_2 ev_r^1(\phi,t)\hat{t} = \hat{\phi}(t) + \hat{t}\phi'(t),$$

see e. g. [4,8].

In Sect. 8 below the following is used.

Proposition 1.3 *Let B be a Banach space. Let reals $a < b$, a continuous injective map $c : [a,b] \to B$, some $t \in (a,b)$, and $\epsilon > 0$ be given. Then there exists $\rho > 0$ with*

$$U_\rho(c([a,t]) \cap U_\rho(c([t,b]) \subset U_\epsilon(c(t)).$$

Proof By continuity there exists $t_a \in (a,t)$ with $c([t_a,t]) \subset U_{\epsilon/2}(c(t))$. The compact sets $c([a,t_a])$ and $c([t,b])$ are disjoint, which gives

$$0 < \min_{a \le u \le t_a} dist(c(u), c([t,b])).$$

Choose $\rho \in \left(0, \frac{\epsilon}{2}\right)$ with

$$2\,\rho < \min_{a \le u \le t_a} dist(c(u), c([t,b])).$$

Consider $z \in U_\rho(c([a,t])) \cap U_\rho(c([t,b]))$. There exist $u_a \in [a,t]$ and $u_b \in [t,b]$ with

$$|z - c(u_a)| < \rho \quad \text{and} \quad |z - c(u_b)| < \rho,$$

hence $|c(u_a) - c(u_b)| < 2\rho$. The assumption $u_a < t_a$ yields a contradiction to the inequality $2\rho < \min_{a \le u \le t_a} dist(c(u), c([t,b]))$. It follows that $u_a \in [t_a, t_b]$. Consequently,

$$|z - c(t)| \le |z - c(u_a)| + |c(u_a) - c(t) < \rho + \frac{\epsilon}{2} < \epsilon,$$

which means $z \in U_\epsilon(c(t))$.

2 Separability

Let $h > 0$ be given. The restrictions of polynomials $\mathbb{R} \to \mathbb{R}$ to the interval $[-h,0]$ are dense in C_h^2, which is an easy consequence of the Weierstraß approximation theorem. Let $P_5 \subset C_h^2$ denote the subspace of restrictions of polynomials of degree not larger than 5 and let $C_{h-0}^2 \subset C_h^2$ denote the closed subspace given by the equations

$$\phi^{(j)}(-h) = 0 = \phi^{(j)}(0) \quad \text{for} \quad j \in \{0,1,2\}.$$

Then dim $P_5 = 6$ and

$$C_h^2 = C_{h-0}^2 \oplus P_5,$$

which follows from the fact that given $\phi \in C_h^2$ there exists a unique $p \in P_5$ satisfying

$$p^{(j)}(-h) = \phi^{(j)}(-h) \text{ and } p^{(j)}(0) = \phi^{(j)}(0) \quad \text{for} \quad j \in \{0,1,2\},$$

or, $\phi - p \in C_{h-0}^2$.

Proposition 2.1 *Let an open set $U \subset C_h^2$ and $p_* \in C_h^2$ with $A = U \cap (p_* + C_{h-0}^2) \neq \emptyset$ be given. The open subset A of the affine space $p_* + C_{h-0}^2$ contains a sequence which is dense in A.*

Proof The restricted polynomials with rational coefficients form a sequence which is dense in C_h^2. Projection along P_5 onto C_{h-0}^2 yields a sequence which is dense in C_{h-0}^2, and translation by adding p_* results in a sequence which is dense in $p_*+C_{h-0}^2$. The members of this sequence which belong to U form a sequence which is dense in A.

Example 2.2 For given reals $w_0 < u_0 < 0$, $u_1 < w_1 < 0$, $u_2 > 0$, $w_2 > 0$ let $p_* \in P_5$ denote the unique restricted polynomial which satisfies

$$p_*^{(j)}(-h) = u_j, \ \ p_*^{(j)}(0) = w_j \quad \text{for} \quad j \in \{0, 1, 2\},$$

and take

$$U = \{\phi \in C_h^2 : 0 < \phi''(t) \text{ on } [-h, 0]\}.$$

Notice that

$$\begin{aligned} A &= U \cap (p_* + C_{h-0}^2) \\ &= \{\phi \in C_h^2 : 0 < \phi''(t) \text{ on } [-h, 0], \\ &\quad \phi^{(j)}(-h) = u_j, \ \phi^{(j)}(0) = w_j \quad \text{for} \quad j \in \{0, 1, 2\}\}. \end{aligned}$$

We add the obvious fact that the dense sequence provided by Proposition 2.1 is dense in $A \subset C_h^2 \subset C_h^1$ also with respect to the norm $|\cdot|_{1,h}$.

3 Differentiable Functions with Separated Shifted Copies

Let $s > 0$ be given. We construct a sequence of functions $\kappa_n \in C_{-s,s}^2$, $n \in \mathbb{N}$, so that shifted copies of these functions keep a positive minimal distance from each other with espect to the norm $|\cdot|_{1,-s,s}$.

Let also positive reals a, ξ, η be given and choose $\epsilon \in \left(0, \frac{a}{4}\right)$. There exists $\chi \in C_{-s,0}^1$ with

$$\begin{aligned} \chi(-s) &= -a, \\ \chi([-s, 0]) &\subset [-a, -a + \epsilon], \\ \chi'(-s) &= \eta, \\ \chi'(t) &> 0 \quad \text{on} \quad [-s, 0]. \end{aligned}$$

For every $n \in \mathbb{N}$ there exists $\rho_n \in C_{-s,s}^1$ with

$$\rho_n(t) = -\rho_n(-t) \quad \text{on} \quad [-s, s]$$

and

$$\begin{aligned} \rho_n(t) &= \chi(t) \quad \text{on} \quad \left[-s, -\frac{s}{2^n}\right], \\ \rho_n\left(-\frac{s}{2^{n+1}}\right) &= -\frac{a}{2}, \\ \rho_n(0) &= 0, \\ (\rho_n)'(t) &= (\rho_n)'(0) \text{ constant on } \left[-\frac{s}{2^{n+1}}, 0\right], \\ (\rho_n)'(t) &> 0 \quad \text{on} \quad [-s, 0]. \end{aligned}$$

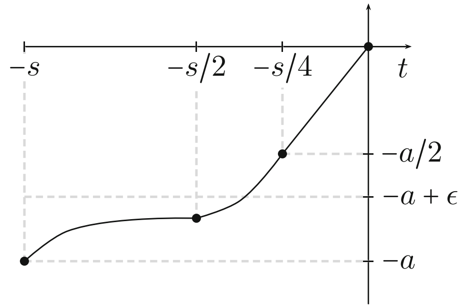


Fig. 1 The function ρ_1 for $-s \le t \le 0$

Proposition 3.1 *For all integers $n \neq k$ in $\mathbb{N}$ and for each $t \in \left[-\frac{s}{2}, 0\right]$ there exists $u \in [-s, s]$ with $t + u \in [-s, s]$ and*

$$|\rho_n(t+u) - \rho_k(u)| \ge \frac{a}{2} - \epsilon.$$

Proof Let positive integers $n \neq k$ and $t \in \left[-\frac{s}{2}, 0\right]$ be given. In case $n > k$ consider $u = -\frac{s}{2^{k+1}}$. Then $u \in \left[-\frac{s}{4}, 0\right]$ and

$$-s \le -\frac{s}{2} - \frac{s}{2^{k+1}} \le t + u \le u = -\frac{s}{2^{k+1}} \le -\frac{s}{2^n},$$

hence

$$\begin{aligned} \rho_n(t+u) - \rho_k(u) &= \chi(t+u) - \left(-\frac{a}{2}\right) \\ &\in [-a, -a+\epsilon] + \frac{a}{2} = \left[-\frac{a}{2}, -\frac{a}{2} + \epsilon\right]. \end{aligned}$$

In case $k > n$ set $u = -t + \frac{s}{2^{n+1}}$. Then

$$0 < \frac{s}{2^k} \le \frac{s}{2^{n+1}} = u + t \le u \ \left(\le \frac{s}{2} + \frac{s}{2^{n+1}} \le s\right),$$

hence

$$\begin{aligned} |\rho_n(t+u) - \rho_k(u)| &= \left|\rho_n\left(\frac{s}{2^{n+1}}\right) - \rho_k\left(-t + \frac{s}{2^{n+1}}\right)\right| \\ &= \left|-\rho_n\left(-\frac{s}{2^{n+1}}\right) + \rho_k\left(-\frac{s}{2^{n+1}} + t\right)\right| \\ &= \left|-\left(-\frac{a}{2}\right) + \chi\left(-\frac{s}{2^{n+1}} + t\right)\right| \\ &\ge \left|\chi\left(-\frac{s}{2^{n+1}} + t\right)\right| - \frac{a}{2} \ge a - \epsilon - \frac{a}{2} \\ &= \frac{a}{2} - \epsilon. \end{aligned}$$

For $n \in \mathbb{N}$ define $\kappa_n \in C^2_{-s,s}$ by

$$\kappa_n(t) = -\xi + \int_{-s}^{t} \rho_n(u)du$$

and observe that

$$\kappa_n(-t) = \kappa_n(t) \quad \text{on} \quad [-s, s],$$

$$\begin{aligned} \kappa_n(-s) = -\xi &= \kappa_n(s), \\ (\kappa_n)'(t) &< 0 \quad \text{on} \quad [-s, 0), \\ (\kappa_n)'(t) &> 0 \quad \text{on} \quad (0, s], \\ (\kappa_n)'(-s) &= -a, \\ (\kappa_n)'(s) &= a, \\ (\kappa_n)''(t) &> 0 \quad \text{on} \quad [-s, s], \\ (\kappa_n)''(-s) &= \eta. \end{aligned}$$

Using Proposition 3.1 and $\epsilon < \frac{a}{4}$ we get the following result.

Corollary 3.2 *For all integers $n \neq k$ in $\mathbb{N}$ and for each $t \in \left[-\frac{s}{2}, 0\right]$ there exists $u \in [-s, s]$ with $t + u \in [-s, s]$ and*

$$|(\kappa_n)'(t + u) - (\kappa_k)'(u)| \geq \frac{a}{4}.$$

4 The Delay Function on a Compact Interval

In this section we find $h > 0$, a set $A \subset C^2_h$, constants $t_b < 0$ and $t_5 < -t_b$, and functions

$$\Delta_n : [0, t_5] \to (0, \infty) \quad \text{and} \quad x_{(n)} : [t_b, t_5] \to \mathbb{R}, \quad n \in \mathbb{N},$$

which in the next section will be used to form a solution of Eq. (1.2) whose short segments are dense in the set $A \cup (-A)$. Choose reals

$$\xi > b > a > 0 \quad \text{with} \quad \xi - a > b$$

such that there exists $t_2 > 1$ with

$$bt_2 > \xi > at_2,$$

and choose $t_b \in (-1, 0)$ with

$$b < (-t_b)\xi.$$

Choose $v \in C^1_{t_b,0}$ with

$$\begin{aligned} v(t) &< 0 \quad \text{on} \quad [t_b, 0], \\ v'(t) &> 0 \quad \text{on} \quad [t_b, 0], \\ v(t_b) &= -\xi, \\ v(0) &= -b, \\ v'(t_b) &= \frac{a}{2}, \end{aligned}$$

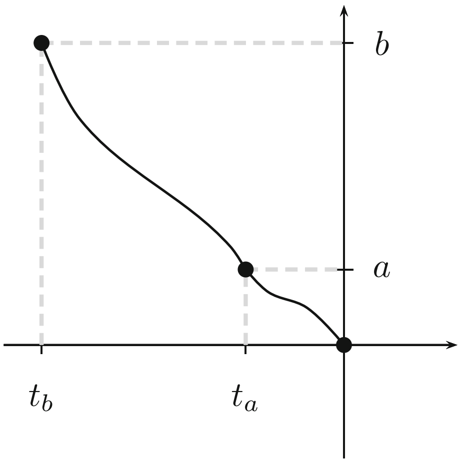
Because of $v([t_b, 0]) = [-\xi, -b]$ and

$$b + t_b b > 0 > b + t_b \xi$$

we can choose v in such a way that also

$$b + \int_{t_b}^{0} v(t)dt = 0.$$

Fig. 2 The function $x \in C^2_{t_b,0}$



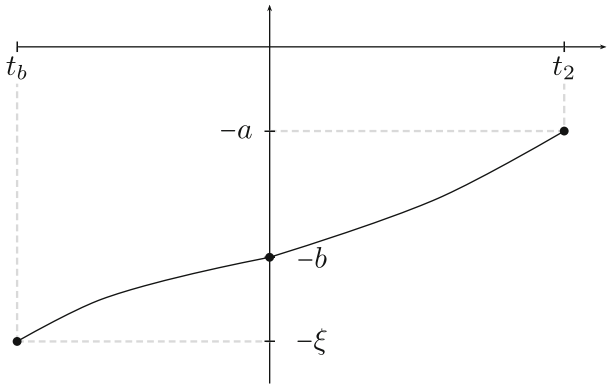


Fig. 3 The function $v \in C^1_{t_b,t_2}$

The equation

$$x(t) = b + \int_{t_b}^{t} v(u)du$$

defines a strictly decreasing function $x \in C^2_{t_b,0}$ with

$$x(t_b) = b \quad \text{and} \quad x(0) = 0.$$

Let $t_a \in (t_b, 0)$ be given by $x(t_a) = a$.

Extend $v \in C^1_{t_b,0}$ to a function in $C^1_{t_b,t_2}$ with

$$\begin{aligned} v(t) &< 0 \quad \text{on} \quad [0, t_2], \\ v(t_2) &= -a, \\ v'(t) &> 0 \quad \text{on} \quad [0, t_2]. \end{aligned}$$

Because of $v([0, t_2]) = [-b, -a]$ and

$$-bt_2 < -\xi < -at_2$$

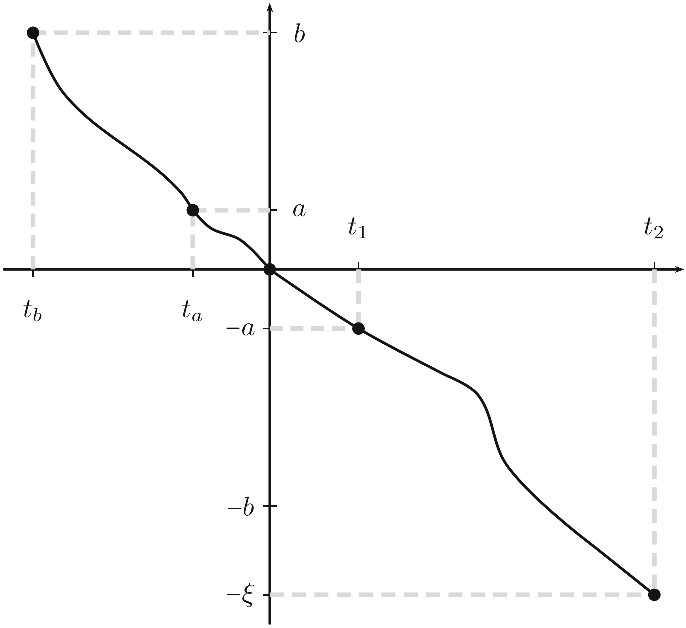


Fig. 4 The function $x \in C^2_{t_b,t_2}$

we can choose $v \in C^1_{t_a,t_2}$ in such a way that also

$$\int_0^{t_2} v(t)dt = -\xi.$$

Set

$$\eta = v'(t_2) > 0.$$

Extend $x \in C^2_{t_b,0}$ to a strictly decreasing function in $C^2_{t_b,t_2}$ by

$$x(t) = \int_0^t v(u)du = b + \int_{t_b}^t v(u)du \quad \text{on} \quad (0, t_2],$$

so that $x(t_2) = -\xi$, and let $t_1 \in (0, t_2)$ be given by $x(t_1) = -a$.

Fix $t_d \in (t_1, t_2)$ and

$$h > 0 \quad \text{with} \quad t_1 < t_d - h$$

and define

$$u_j = x^{(j)}(t_d - h) \text{ and } w_j = x^{(j)}(t_d) \text{ for } j \in \{0, 1, 2\}.$$

Then $0 > u_0 > w_0, u_1 < w_1 < 0, 0 < u_2, 0 < w_2$. Consider the set $A \subset C^2_h$ from Example 2.2. The functions in A are negative and strictly decreasing, with the derivative strictly increasing. Proposition 2.1 guarantees a sequence $(p_n)_{n\in\mathbb{N}}$ in A which is dense in A. For $n \in \mathbb{N}$ define $x_{(n)} \in C^2_{t_b,t_2}$ by

$$x_{(n)}(t) = x(t) \quad \text{on} \quad [t_b, t_d - h] \cup [t_d, t_2],$$

$$
\begin{aligned}
x_{(n)}(t) &= p_{\frac{n+1}{2}}(t-t_d) \quad \text{on} \quad [t_d-h, t_d] \\
&\quad \text{in case} \quad n \quad \text{odd}, \\
x_{(n)}(t) &= p_{\frac{n}{2}}(t-t_d) \quad \text{on} \quad [t_d-h, t_d] \\
&\quad \text{in case} \quad n \quad \text{even}.
\end{aligned}
$$

Notice that $x_{(n)}$ is strictly decreasing on $[t_b, t_2]$ with

$$
\begin{aligned}
x_{(n)}(t_b) &= b,\ x_{(n)}(0) = 0,\ x_{(n)}(t_1) = -a,\ x_{(n)}(t_2) = -\xi, \\
(x_{(n)})'(t) &< 0 \quad \text{on} \quad [t_b, t_2], \\
(x_{(n)})'(t_2) &= x'(t_2) = v(t_2) = -a, \\
(x_{(n)})''(t) &> 0 \quad \text{on} \quad [t_b, t_2], \\
(x_{(n)})''(t_2) &= v'(t_2) = \eta.
\end{aligned}
$$

The inverse $y_n = (x_{(n)})^{-1} \in C^2_{-\xi,b}$ maps its domain $[-\xi, b]$ onto the interval $[t_b, t_2]$, with

$$
(y_n)'(u) = \frac{1}{(x_{(n)})'(y_n(u))} < 0 \quad \text{for all} \quad u \in [-\xi, b].
$$

Obviously,

$$
\begin{aligned}
(x_{(n)})'([0, t_2]) &= [(x_{(n)})'(0), (x_{(n)})'(t_2)] = [-b, -a], \\
-(x_{(n)})'([0, t_2]) &= [a, b] \subset [-\xi, b].
\end{aligned}
$$

It follows that the equation

$$
y_n(-(x_{(n)})'(t)) = t - \Delta_n(t)
$$

defines a function $\Delta_n \in C^1_{0,t_2}$ with

$$
\begin{aligned}
1 - (\Delta_n)'(t) &= (y_n)'(-(x_{(n)})'(t))[-(x_{(n)})''(t)] > 0 \quad \text{on} \quad [0, t_2], \\
0 - \Delta_n(0) &= y_n(-(x_{(n)})'(0)) = y_n(b) = t_b, \\
t_2 - \Delta_n(t_2) &= y_n(-(x_{(n)})'(t_2)) = y_n(a) = t_a, \\
(x_{(n)})'(t) &= -x_{(n)}(t - \Delta_n(t)) \quad \text{on} \quad [0, t_2].
\end{aligned}
$$

In particular,

$$
(id - \Delta_n)([0, t_2]) = [t_b, t_a].
$$

The estimate $t - \Delta_n(t) \le t_a$ on $[0, t_2]$ yields

$$
\Delta_n(t) \ge t - t_a \ge t \ge 0 \quad \text{on} \quad [0, t_2].
$$

Fix some $s > 0$ and recall $\kappa_n \in C^2_{-s,s}$ from Sect. 3, with a, ξ, η from the present section. Then

$$
(\kappa_n)^{(j)}(-s) = (x_{(n)})^{(j)}(t_2) \quad \text{for} \quad j \in \{0, 1, 2\}.
$$

Set

$$
t_3 = t_2 + 2s
$$

and define an extension of $x_{(n)}$ to a map in $C^2_{t_b,t_3}$ by

$$
x_{(n)}(t) = \kappa_n(t - t_3 + s) \quad \text{on} \quad [t_2, t_3].
$$

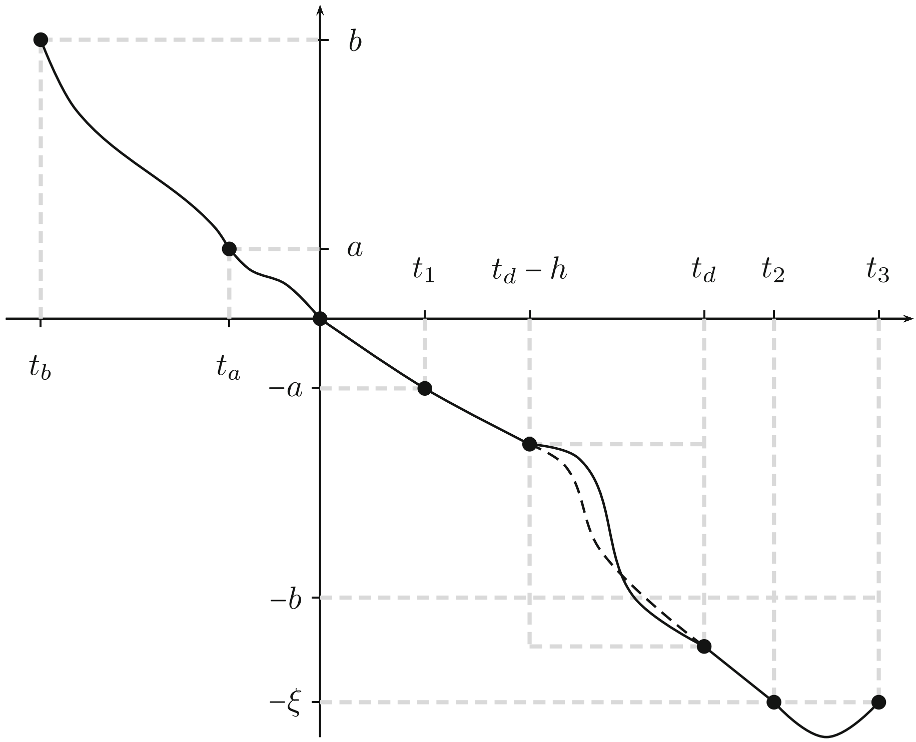


Fig. 5 The function $x_{(n)} \in C^2_{t_b,t_3}$

By the symmetry of κ_n,

$$
\begin{aligned}
x_{(n)}(t_3) &= x_{(n)}(t_2) = -\xi, \\
(x_{(n)})'(t_3) &= -(x_{(n)})'(t_2) = a, \\
(x_{(n)})''(t_3) &= (x_{(n)})''(t_2) = \eta,
\end{aligned}
$$

and

$$
(x_{(n)})'([t_2, t_3]) = (\kappa_n)'([-s, s]) = [-a, a] = x_{(n)}([t_a, t_1]).
$$

It follows that the equation

$$
y_n(-(x_{(n)})'(t)) = t - \delta_n(t) \quad \text{on} \quad [t_2, t_3]
$$

defines a map $\delta_n \in C^1_{t_2,t_3}$, with

$$
\begin{aligned}
t_2 - \delta_n(t_2) &= y_n(-(x_{(n)})'(t_2)) = y_n(a) = t_a, \\
t_3 - \delta_n(t_3) &= y_n(-(x_{(n)})'(t_3)) = y_n((x_{(n)})'(t_2)) = y_n(-a) = t_1, \\
1 - (\delta_n)'(t) &= (y_n)'(-(x_{(n)})'(t))[-(x_{(n)})''(t)] > 0 \text{ on } [t_2, t_3], \\
(x_{(n)})'(t) &= -x_{(n)}(t - \delta_n(t)) \quad \text{on} \quad [t_2, t_3].
\end{aligned}
$$

Notice that $\delta_n(t_2) = t_2 - t_a = \Delta_n(t_2)$ and

$$
1 - (\delta_n)'(t_2) = (y_n)'(-(x_{(n)})'(t_2))[-(x_{(n)})''(t_2)] = 1 - (\Delta_n)'(t_2).
$$

The estimate $t - \delta_n(t) \le t_1$ on $[t_2, t_3]$ yields

$$\delta_n(t) \ge t - t_1 \ge t_2 - t_1 > 0 \quad \text{on} \quad [t_2, t_3].$$

Setting

$$\Delta_n(t) = \delta_n(t) \quad \text{on} \quad [t_2, t_3]$$

we get an extension of $\Delta_n \in C^1_{0,t_2}$ to a nonnegative map in C^1_{0,t_3}, with

$$1 - (\Delta_n)'(t) > 0 \text{ on } [0, t_3] \text{ and } (id - \Delta_n)([t_2, t_3]) = [t_a, t_1].$$

Because of $a < \xi - b$ there exists $t_4 > t_3$ with

$$a(t_4 - t_3) < \xi - b < \xi(t_4 - t_3),$$

for example, $t_4 = t_3 + 1$.

Proposition 4.1 *There exists $\delta_{n*} \in C^1_{t_3,t_4}$ with*

$$\begin{aligned} 1 - (\delta_{n*})'(t) &> 0 \quad \textit{in} \quad [t_3, t_4], \\ t_3 - \delta_{n*}(t_3) &= t_1, \\ t_4 - \delta_{n*}(t_4) &= t_3, \\ 1 - (\delta_{n*})'(t_3) &= 1 - (\Delta_n)'(t_3), \\ 1 - (\delta_{n*})'(t_4) &= \frac{1}{2}, \quad \textit{and} \end{aligned}$$

$$-\xi + \int_{t_3}^{t_4} x_{(n)}(t - \delta_{n*}(t))dt = -b.$$

Proof Consider the discontinuous function $g_0 : [t_3, t_4] \to \mathbb{R}$ given by $g_0(t_3) = t_1$ and $g_0(t) = t_3$ for $t_3 < t \le t_4$. There is a sequence of functions $g_j \in C^1_{t_3,t_4}$, $j \in \mathbb{N}$, with

$$\begin{aligned} (g_j)'(t) &> 0 \quad \text{on} \quad [t_3, t_4], \\ g_j(t_3) &= t_1, \\ g_j(t_4) &= t_3, \\ (g_j)'(t_3) &= 1 - (\Delta_n)'(t_3), \\ (g_j)'(t_4) &= \frac{1}{2}. \end{aligned}$$

which converge pointwise to g_0. For every $j \in \mathbb{N}$, $g_j([t_3, t_4]) = [t_1, t_3]$, and the Lebesgue dominated convergence theorem yields

$$G_j = \int_{t_3}^{t_4} [-x_{(n)}(g_j(t))]dt \to -\int_{t_3}^{t_4} x_{(n)}(t_3)dt = \xi(t_4 - t_3) \quad \text{as} \quad j \to \infty.$$

Similarly there is a sequence of functions $h_j \in C^1_{t_3,t_4}$ with the same properties as g_j which converge pointwise to $h_0 : [t_3, t_4] \to \mathbb{R}$ given by $h_0(t_4) = t_3$ and $h_0(t) = t_1$ for $t_3 \le t < t_4$, and

$$H_j = \int_{t_3}^{t_4} [-x_{(n)}(h_j(t))]dt \to -\int_{t_3}^{t_4} x_{(n)}(t_1)dt = a(t_4 - t_3) \quad \text{as} \quad j \to \infty.$$

The limits satisfy

$$a(t_4 - t_3) < \xi - b < \xi(t_4 - t_3),$$

due to the choice of t_4. So there exists $j \in \mathbb{N}$ with

$$H_j < \xi - b < G_j.$$

The function

$$k : [0, 1] \times [t_3, t_4] \ni (\theta, t) \mapsto g_j(t) + \theta(h_j(t) - g_j(t)) \in \mathbb{R}$$

is continuous. Using the intermediate value theorem we find some $\theta \in (0, 1)$ with

$$\int_{t_3}^{t_4} x_{(n)}(k(\theta, t))dt = (1-\theta)G_j + \theta H_j = \xi - b.$$

Notice that the convex combination $k(\theta, \cdot) \in C^1_{t_3,t_4}$ shares the properties of g_j and h_j. Define δ_{n*} by

$$t - \delta_{n*}(t) = k(\theta, t).$$

The estimate $t - \delta_{n*}(t) \le t_3$ on $[t_3, t_4]$ yields

$$\delta_{n*}(t) \ge t - t_3 \ge 0 \quad \text{on} \quad [t_3, t_4].$$

It follows that the equation

$$\Delta_n(t) = \delta_{n*}(t) \quad \text{for} \quad t_3 < t \le t_4$$

extends $\Delta_n \in C^1_{0,t_3}$ to a nonnegative function in C^1_{0,t_4} which satisfies

$$\begin{aligned}
1 - (\Delta_n)'(t) &> 0 \quad \text{on} \quad [0, t_4],\\
t_4 - \Delta_n(t_4) &= t_3,\\
t - \Delta_n(t) &\in [t_1, t_3] \quad \text{for} \quad t_3 \le t \le t_4,\\
1 - (\Delta_n)'(t_4) &= \frac{1}{2}.
\end{aligned}$$

The function $x_{n*} \in C^2_{t_3,t_4}$ given by

$$x_{n*}(t) = -\xi + \int_{t_3}^{t} [-x_{(n)}(u - \Delta_n(u))]du$$

satisfies

$$\begin{aligned}
x_{n*}(t_3) &= -\xi = x_{(n)}(t_3),\\
x_{n*}(t_4) &= -b,\\
(x_{n*})'(t) &= -x_{(n)}(t - \Delta_n(t)) \quad \text{on} \quad [t_3, t_4],\\
(x_{n*})'(t_3) &= -x_{(n)}(t_3 - \Delta_n(t_3)) = -x_{(n)}(t_1) = a = x'_{(n)}(t_3),\\
(x_{n*})'(t_4) &= -x_{(n)}(t_4 - \Delta_n(t_4)) = -x_{(n)}(t_3) = \xi,\\
(x_{n*})''(t_3) &= -(x_{(n)})'(t_3 - \Delta_n(t_3))[1 - (\Delta_n)'(t_3)] = (x_{(n)})''(t_3)\\
(x_{n*})''(t_4) &= -(x_{(n)})'(t_4 - \Delta_n(t_4))[1 - (\Delta_n)'(t_4)]\\
&\quad -(x_{(n)})'(t_3)\frac{1}{2} = -\frac{a}{2}.
\end{aligned}$$

Therefore the equation

$$x_{(n)}(t) = x_{n*}(t) \quad \text{for} \quad t_3 < t \le t_4$$

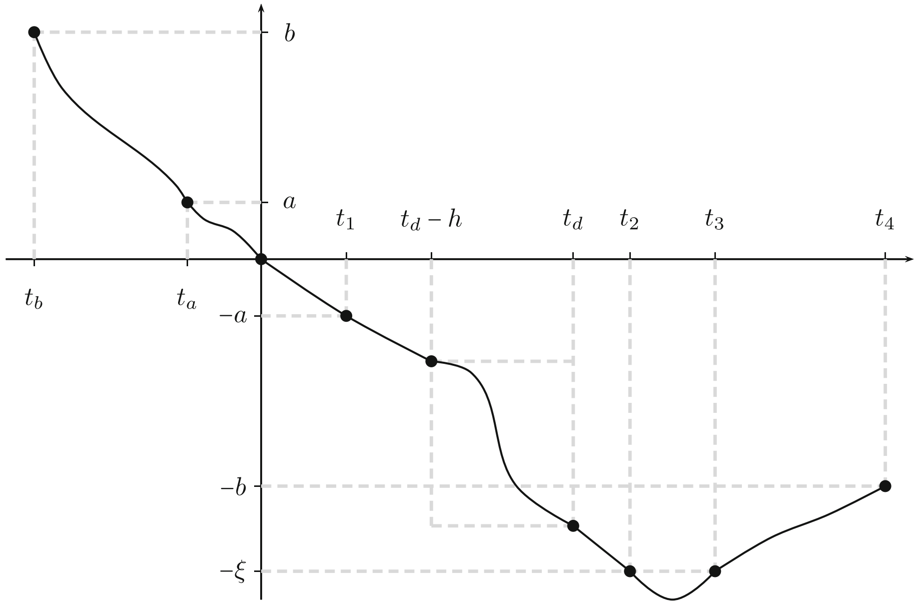


Fig. 6 The function $x_{(n)} \in C^2_{t_b,t_4}$

defines a continuation of $x_{(n)} \in C^2_{t_b,t_3}$ to a function in $C^2_{t_b,t_4}$ which satisfies Eq. (1.2) on $[0, t_4]$ and maps the interval $[t_3, t_4]$ onto $[-\xi, -b]$, with positive derivative and

$$\begin{aligned} x_{(n)}(t_4) &= -b = -x_{(n)}(t_b), \\ (x_{(n)})'(t_4) &= \xi = -v(t_b) = -(x_{(n)})'(t_b), \\ (x_{(n)})''(t_4) &= -\frac{a}{2} = -v'(t_b) = -(x_{(n)})''(t_b). \end{aligned}$$

We set $t_5 = t_4 - t_b$ and extend $x_{(n)} \in C^2_{t_b,t_4}$ to a function in $C^2_{t_b,t_5}$ by

$$x_{(n)}(t) = -x_n(t - t_5) \quad \text{on} \quad [t_4, t_5].$$

Then

$$-(x_{(n)})'([t_4, t_5]) = (x_{(n)})'([t_b, 0]) = [-\xi, -b] = x_{(n)}([t_3, t_4]).$$

The derivative of the function

$$y_{n,5} = (x_{(n)}|_{[t_3,t_4]})^{-1} \in C^2_{-\xi,-b}$$

is strictly positive, due to $(x_{(n)})'(t) > 0$ on $[t_3, t_4]$. The equation

$$y_{n,5}(-(x_{(n)})'(t)) = t - \delta_{n,5}(t) \quad \text{for} \quad t_4 \le t \le t_5$$

defines a function $\delta_{n,5} \in C^1_{t_4,t_5}$ which satisfies

$$\begin{aligned} t_4 - \delta_{n,5}(t_4) &= y_{n,5}(-(x_{(n)})'(t_4)) = y_{n,5}((x_{(n)})'(t_4 - t_5)) \\ &= y_{n,5}((x_{(n)})'(t_b)) = y_{n,5}(-\xi) = t_3 = t_4 - \Delta_n(t_4), \\ t_5 - \delta_{n,5}(t_5) &= y_n(-(x_{(n)})'(t_5)) = y_n((x_{(n)})'(0)) = y_n(-b) = t_4, \\ 1 - (\delta_{n,5})'(t) &= (y_{n,5})'(\ldots)[-(x_{(n)})''(t)] \end{aligned}$$

$$
\begin{aligned}
&= (y_{n,5})'(\ldots)[(x_{(n)})''(t-t_5)] > 0 \quad \text{on} \quad [t_4, t_5],\\
1-(\delta_{n,5})'(t_4) &= (y_{n,5})'(-(x_{(n)})'(t_4))[-(x_{(n)})''(t_4)]\\
&= (y_{n,5})'((x_{(n)})'(t_b))[(x_{(n)})''(t_b)] = (y_{n,5})'(-\xi)\frac{a}{2}\\
&= (y_{n,5})'(x_{(n)}(t_3))\frac{a}{2} = \frac{1}{(x_{(n)})'(t_3)}\frac{a}{2} = \frac{1}{a}\frac{a}{2}\\
&= \frac{1}{2} = 1-(\Delta_n)'(t_4).
\end{aligned}
$$

The estimate $t - \delta_{n,5}(t) \le t_4$ on $[t_4, t_5]$ yields

$$\delta_{n,5}(t) \ge t - t_4 \ge 0 \quad \text{on} \quad [t_4, t_5].$$

It follows that the equation

$$\Delta_n(t) = \delta_{n,5}(t) \quad \text{for} \quad t_4 < t \le t_5$$

defines a continuation of $\Delta_n \in C^1_{0,t_4}$ to a nonnegative function in C^1_{0,t_5} so that we have

$$
\begin{aligned}
t_4 - \Delta_n(t_4) &= t_3,\\
t_5 - \Delta_n(t_5) &= t_4, \quad \text{or equivalently,}\\
\Delta_n(t_5) &= t_5 - t_4 = -t_b = \Delta_n(0),\\
1-(\Delta_n)'(t) &> 0 \quad \text{on} \quad [0, t_5],\\
(x_{(n)})'(t) &= -x_{(n)}(t-\Delta_n(t)) \quad \text{on} \quad [0, t_5].
\end{aligned}
$$

Also,

$$(\Delta_n)'(t_5) = (\Delta_n)'(0)$$

because of

$$
\begin{aligned}
1-(\Delta_n)'(t_5) &= 1-(\delta_{n,5})'(t_5) = (y_{n,5})'(-(x_{(n)})'(t_5))[-(x_{(n)})''(t_5)]\\
&= (y_{n,5})'((x_{(n)})'(0))(x_{(n)})''(0) = (y_{n,5})'(-b)(x_{(n)})''(0)\\
&= \frac{1}{(x_{(n)})'(t_4)}(x_{(n)})''(0) = \frac{1}{-(x_{(n)})'(t_b)}(x_{(n)})''(0)
\end{aligned}
$$

and

$$(x_{(n)})''(0) = -(x_{(n)})'(0-\Delta_n(0)[1-(\Delta_n)'(0)] = -(x_{(n)})'(t_b)[1-(\Delta_n)'(0)].$$

5 Concatenation

All functions $x_{(n)} \in C^2_{t_b,t_5}$, $n \in \mathbb{N}$, coincide on the set

$$[t_b, t_d - h] \cup [t_d, t_2] \cup [t_4, t_5],$$

we have $t_4 = t_5 + t_b$, and for every $n \in \mathbb{N}$,

$$x_{(n)}(t) = -x_{(n)}(t - t_5) \quad \text{for all} \quad t \in [t_4, t_5].$$

Moreover, for every $n \in \mathbb{N}$ the nonnegative function $\Delta_n \in C^1_{0,t_5}$ satisfies

$$\Delta_n(t_5) = \Delta_n(0) = -t_b,$$

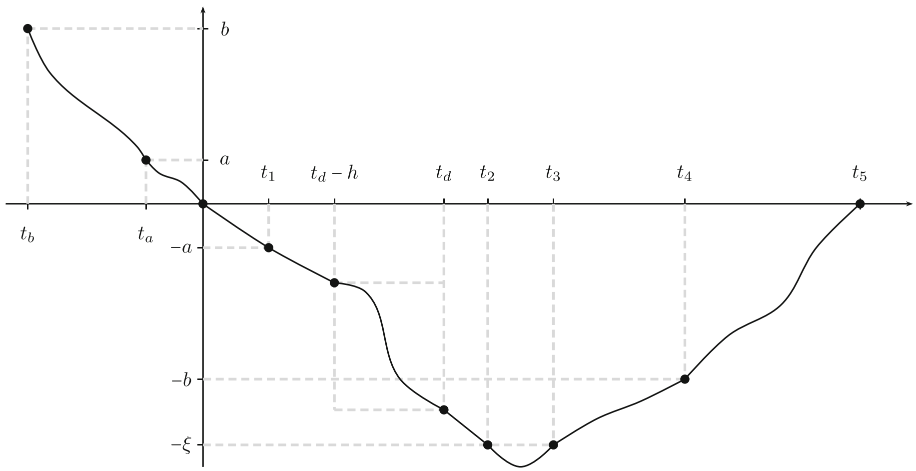


Fig. 7 The function $x_{(n)} \in C^2_{t_b,t_5}$

$$(\Delta_n)'(t_5) = (\Delta_n)'(0),$$
$$1 - (\Delta_n)'(t) > 0 \quad \text{for all} \quad t \in [0, t_5],$$

and we have

$$(x_{(n)})'(t) = -x_{(n)}(t - \Delta_n(t)) \quad \text{for all} \quad t \in [0, t_5].$$

Therefore the relations

$$x(t) = (-1)^{n-1} x_{(n)}(t - (n-1)t_5) \text{ for } n \in \mathbb{N}, \ (n-1)t_5 + t_b \leq t \leq nt_5,$$
$$\Delta(t) = \Delta_n(t - (n-1)t_5) \text{ for } n \in \mathbb{N}, \ (n-1)t_5 \leq t \leq nt_5$$

define a twice continuously differentiable function $x : [t_b, \infty) \to \mathbb{R}$ and a continuously differentiable nonnegative function $\Delta : [0, \infty) \to \mathbb{R}$ so that Eq. (1.2) holds for all $t \geq 0$, $\Delta(0) = -t_b$, and

$$1 - \Delta'(t) > 0 \quad \text{for all} \quad t \geq 0.$$

The *short segments* $x_{(n-1)t_5 + t_d, short} = p_{\frac{n+1}{2}} \in C^2_h$, $n \in \mathbb{N}$ odd, which are given by

$$x_{(n-1)t_5 + t_d, short}(u) = x((n-1)t_5 + t_d + u) \quad \text{for} \quad -h \leq u \leq 0,$$

are dense in the infinite-dimensional set $A \subset C^2_h \subset C^1_h$ with respect to the norm $|\cdot|_{1,h}$.

Recall

$$t_b \leq t - \Delta_n(t) \quad \text{in} \quad [0, t_2],$$
$$t_a \leq t - \Delta_n(t) \quad \text{in} \quad [t_2, t_3],$$
$$t_1 \leq t - \Delta_n(t) \quad \text{in} \quad [t_3, t_4],$$
$$t_3 \leq t - \Delta_n(t) \quad \text{in} \quad [t_4, t_5] = [t_4, t_4 - t_b]$$

for each $n \in \mathbb{N}$ and set

$$r = \max\{t_2 - t_b, t_3 - t_a, t_4 - t_1, t_4 - t_b - t_3, t_5 + 3s\}.$$

Then

$$\Delta(t) \le r \quad \text{for all} \quad t \ge 0.$$

Extend $x : [t_b, \infty) \to \mathbb{R}$ backward to a twice continuously differentiable function $x : [-r, \infty) \to \mathbb{R}$, with *long segments* $x_t \in C_r^2 \subset C_r^1$, $t \ge 0$, given by

$$x_t(u) = x(t+u) \quad \text{for} \quad -r \le u \le 0.$$

The curve

$$\hat{x} : (0, \infty) \ni t \mapsto x_t \in C_r^1$$

is continuously differentiable with

$$D\hat{x}(t)1 = (x_t)' = (x')_t \in C_r^1 \quad \text{for all} \quad t > 0,$$

compare [13, Proposition 4.1]. As $\frac{t_2+t_3}{2}$ is the only zero of $(x_{(n)})' : [t_b, t_5] \to \mathbb{R}$, for any $n \in \mathbb{N}$, we have

$$D\hat{x}(t)1 = (x_t)' \neq 0 \quad \text{for all} \quad t > 0.$$

Proposition 5.1 *The restriction of the curve $\hat{x}$ to the ray $[r, \infty)$ is injective.*

Proof Assume $r \le t \le u$ and $\hat{x}(t) = \hat{x}(u)$. Then

$$x(t+v) = x(u+v) \quad \text{for all} \quad v \in [-r, 0].$$

There are $n \in \mathbb{N}$ and $k \in \mathbb{N}$ with

$$(n-1)t_5 \le t < nt_5 \quad \text{and} \quad (k-1)t_5 \le u < kt_5.$$

From $t_5 < r \le t$ we have $n \ge 2$, and from $t \le u$ we have $n \le k$.

1. Proof of $t - (n-1)t_5 = u - (k-1)t_5$. The argument $w = (n-1)t_5 - t$ is contained in $(-t_5, 0] \subset [-r, 0]$, and

$$0 = x((n-1)t_5) = x(t+w) = x(u+w).$$

As the interval $(u - t_5, u]$ contains exactly one zero of x, situated at $(k-1)t_5$, we get $u + w = (k-1)t_5$, hence

$$u - (k-1)t_5 = -w = t - (n-1)t_5.$$

2. The case $(n-1)t_5 + t_3 \le t$ $(< nt_5)$. Using Part 1 of the proof we get

$$(k-1)t_5 + t_3 \le u.$$

For every $w \in [-s, s]$ we obtain

$$\begin{aligned}
\kappa_n(w) &= (-1)^{n-1} x_{(n)}(t_3 - s + w) = x((n-1)t_5 + t_3 - s + w) \\
&= x(t + [-t + (n-1)t_5 + t_3 - s + w]) \\
&= x(u + [-t + (n-1)t_5 + t_3 - s + w]) \\
&\quad (\text{with} \quad [-t + (n-1)t_5 + t_3 - s + w] \in [-t_5, 0] \subset [-r, 0]) \\
&= x(u + [-u + (k-1)t_5 + t_3 - s + w]) \\
&\quad (\text{with Part 1}) \\
&= x((k-1)t_5 + t_3 - s + w) = (-1)^{k-1} x_{(k)}(t_3 - s + w) = \kappa_k(w),
\end{aligned}$$

and it follows that $n = k$. By Part 1, $t = u$.

3. The case $((n-1)t_5 \leq)\ t < (n-1)t_5 + t_3$. Using Part 1 of the proof we get

$$((k-1)t_5 \leq)\ u < (k-1)t_5 + t_3.$$

For every $w \in [-s, s]$ we have

$$\begin{aligned} -t + (n-2)t_5 + t_3 - s + w &> -[(n-1)t_5 + t_3] + (n-2)t_5 + t_3 - s + w \\ &= -t_5 - s + w \geq -t_5 - 2s \geq -r \\ &\quad \text{and} \\ -t + (n-2)t_5 + t_3 - s + w &\leq -(n-1)t_5 + (n-2)t_5 + t_3 - s + w \\ &\leq -t_5 + t_3 - s + s \leq 0, \end{aligned}$$

hence $[-t + (n-2)t_5 + t_3 - s + w] \in [-r, 0]$. It follows that

$$\begin{aligned} \kappa_{n-1}(w) &= (-1)^{n-2} x_{(n-1)}(t_3 - s + w) = x((n-2)t_5 + t_3 - s + w) \\ &= x(t + [-t + (n-2)t_5 + t_3 - s + w]) \\ &= x(u + [-t + (n-2)t_5 + t_3 - s + w]) \\ &\quad (\text{with} \quad [-t + (n-2)t_5 + t_3 - s + w] \in [-r, 0]) \\ &= x(u + [-u + (k-2)t_5 + t_3 - s + w]) \\ &\quad (\text{with Part 1}) \\ &= x((k-2)t_5 + t_3 - s + w) = (-1)^{k-2} x_{(k-1)}(t_3 - s + w) = \kappa_{k-1}(w). \end{aligned}$$

Hence $n - 1 = k - 1$, and by Part 1, $t = u$.

6 Separation of Arcs

Proposition 6.1 *There exists $\hat{a} > 0$ so that for all integers $n \geq 2$, $j \geq 2$ with $|n - j| > 1$ and for all $t \in [(n-1)t_5, nt_5]$, $u \in [(j-1)t_5, jt_5]$ we have*

$$|\hat{x}(t) - \hat{x}(u)|_{1,r} \geq \hat{a}.$$

Proof 1. Recall from Sect. 4 the function $v \in C^1_{t_b, t_2}$. Let $n \in \mathbb{N}$. Notice that

$$(x_{(n)})'(t) = v(t) < 0 \quad \text{on} \quad [t_a, t_1].$$

With $v_m = -\max_{t_a \leq t \leq t_1} v(t)$ and $x_{(n)}(0) = 0$ we obtain

$$|x_{(n)}(t)| \geq |t| v_m \quad \text{on} \quad [t_a, t_1].$$

On $[t_1, t_5 + t_a]$ we have $x_{(n)}(t) \leq -a$.

2. Let $n \in \mathbb{N}$, $j \in \mathbb{N}$ and $t \in [(n-1)t_5, nt_5]$, $u \in [(j-1)t_5, jt_5]$ be given. Then

$$t = (n-1)t_5 + t_* \quad \text{with} \quad 0 \leq t_* \leq t_5 \quad \text{and} \quad u = (j-1)t_5 + u_* \quad \text{with} \quad 0 \leq u_* \leq t_5.$$

We may assume $u_* \leq t_*$. Set $w = u_* - t_* \in [-t_5, 0]$.

3. In case $t_1 \leq -w \leq t_5 + t_a$ Part 1 yields the estimate

$$\begin{aligned} |\hat{x}(t) - \hat{x}(u)|_{1,r} &\geq |x_t(-u_*) - x_u(-u_*)| \\ &= |x((n-1)t_5 + t_* - u_*) - x((j-1)t_5 + u_* - u_*)| \\ &= |x((n-1)t_5 - w)| = |x_{(n)}(-w)| \geq a. \end{aligned}$$

4. In case $\min\left\{t_1, \frac{s}{2}\right\} \le -w \le t_1$ Part 1 yields the estimate

$$\begin{aligned}|\hat{x}(t) - \hat{x}(u)|_{1,r} &\ge |x_t(-u_*) - x_u(-u_*)| \\ &= |x((n-1)t_5 - w)| = |x_{(n)}(-w)| \ge (-w)v_m \ge v_m \cdot \min\left\{t_1, \frac{s}{2}\right\}.\end{aligned}$$

5. In case $t_5 + t_a \le -w \le t_5 - \min\left\{-t_a, \frac{s}{2}\right\}$ we have $t_a \le -w - t_5 \le -\min\left\{-t_a, \frac{s}{2}\right\}$. Using Part 1 we infer

$$\begin{aligned}|\hat{x}(t) - \hat{x}(u)|_{1,r} &\ge |x_t(-u_*) - x_u(-u_*)| \\ &= |x((n-1)t_5 - w)| = |x(nt_5 - w - t_5)| \\ &= |x_{(n+1)}(-w - t_5)| \ge |-w - t_5|v_m \ge v_m \cdot \min\{-t_a, s\}.\end{aligned}$$

6. The case $n - j \in 2\mathbb{Z} + 1$, $-w \le s$, and $t_3 \le u_*$.Then $t_2 + s - t_* \in [-t_5, 0] \subset [-r, 0]$ since

$$-t_5 \le -t_* \le t_2 + s - t_* \le t_3 - t_* \le t_3 - u_* \le 0.$$

Using $x_{(m)}(t) \le -\xi$ for all $m \in \mathbb{N}$ and all $t \in [t_2, t_3] = [t_2, t_2 + 2s]$ we infer

$$\begin{aligned}|\hat{x}(t) - \hat{x}(u)|_{1,r} &\ge |x_t(t_2 + s - t_*) - x_u(t_2 + s - t_*)| \\ &= |x((n-1)t_5 + t_* + t_2 + s - t_*) - \\ &\quad x((j-1)t_5 + u_* + t_2 + s - t_*)| \\ &= |x((n-1)t_5 + t_2 + s) - x((j-1))t_5 + t_2 + s + w)| \\ &= |(-1)^{n-1}x_{(n)}(t_2 + s) - (-1)^{j-1}x_{(j)}(t_2 + s + w)| \\ &= |(-1)^{n-j}x_{(n)}(t_2 + s) - x_{(j)}(t_2 + s + w)| \\ &\ge 2\xi.\end{aligned}$$

7. The case $0 \ne n - j \in 2\mathbb{Z}$, $-w \le \frac{s}{2}$, and $t_3 \le u_*$. Corollary 3.2 yields some $v \in [-s, s]$ so that $w + v \in [-s, s]$ and

$$|(\kappa_j)'(w + v) - (\kappa_n)'(v)| \ge \frac{a}{4}.$$

We have $t_2 + s + v - t_* \in [-t_5, 0] \subset [-r, 0]$ since

$$-t_5 \le -t_* \le t_2 + s + v - t_* \le t_3 - t_* \le t_3 - u_* \le 0.$$

Hence

$$\begin{aligned}|\hat{x}(t) - \hat{x}(u)|_{1,r} &\ge |(x_t)'(t_2 + s + v - t_*) - (x_u)'(t_2 + s + v - t_*)| \\ &= |x'((n-1)t_5 + t_* + t_2 + s + v - t_*) - \\ &\quad x'((j-1)t_5 + u_* + t_2 + s + v - t_*)| \\ &= |x'((n-1)t_5 + t_2 + s + v) - x'((j-1))t_5 + t_2 + s + w + v)| \\ &= |(-1)^{n-1}(x_{(n)})'(t_2 + s + v) - (-1)^{j-1}(x_{(j)})'(t_2 + s + w + v)| \\ &= |(-1)^{n-j}(x_{(n)})'(t_2 + s + v) - (x_{(j)})'(t_2 + s + w + v)| \\ &= |(\kappa_n)'(v) - (\kappa_j)'(w + v)| \ge \frac{a}{4}.\end{aligned}$$

8. The case $n - j \in 2\mathbb{Z} + 1$, $2 \le n$, $2 \le j$, $-w \le s$, and $u_* < t_3$. Then $t_5 + t_* - t_2 - s \in [0, t_5 + 2s] \subset [0, r]$ since

$$0 \le t_5 - t_3 + u_* \le t_5 - (t_2 + s) + t_* = t_5 + t_* - t_2 - s$$

$$\leq t_5 + (u_* + s) - t_2 - s \leq t_5 + t_3 - t_2 = t_5 + 2s \leq r.$$

Hence

$$\begin{aligned}
|\hat{x}(t) - \hat{x}(u)|_{1,r} &\geq |x_t(-t_5 - t_* + t_2 + s) - x_u(-t_5 - t_* + t_2 + s)| \\
&= |x((n-1)t_5 + t_* - t_5 - t_* + t_2 + s) - \\
&\quad x((j-1)t_5 + u_* - t_5 - t_* + t_2 + s)| \\
&= |x((n-2)t_5 + t_2 + s) - x((j-2)t_5 + w + t_2 + s)| \\
&= |(-1)^{n-2} x_{(n-1)}(t_2 + s) - (-1)^{j-2} x_{(j)}(w + t_2 + s)| \\
&= |(-1)^{n-j} x_{(n-1)}(t_2 + s) - x_{(j)}(w + t_2 + s)| \geq 2\xi.
\end{aligned}$$

9. The case $0 \neq n - j \in 2\mathbb{Z}$, $2 \leq n$, $2 \leq j$, $-w \leq \frac{s}{2}$, and $u_* < t_3$. Corollary 3.2 yields some $v \in [-s, s]$ so that $w + v \in [-s, s]$ and

$$|(\kappa_{j-1})'(w + v) - (\kappa_{n-1})'(v)| \geq \frac{a}{4}.$$

We have $t_5 + t_* - t_2 - s - v \in [0, t_5 + 3s] \subset [0, r]$ since

$$\begin{aligned}
0 &\leq t_5 - t_3 + u_* \leq t_5 - (t_2 + 2s) + t_* \leq t_5 + t_* - t_2 - s - v \\
&\leq t_5 + \left(u_* + \frac{s}{2}\right) - t_2 - s - v < t_5 + t_3 - t_2 - v = t_5 + 2s - v \leq r.
\end{aligned}$$

Hence

$$\begin{aligned}
|\hat{x}(t) - \hat{x}(u)|_{1,r} &\geq |(x_t)'(-t_5 - t_* + t_2 + s + v) - \\
&\quad (x_u)'(-t_5 - t_* + t_2 + s + v)| \\
&= |x'((n-1)t_5 + t_* - t_5 - t_* + t_2 + s + v) - \\
&\quad x'((j-1)t_5 + u_* - t_5 - t_* + t_2 + s + v)| \\
&= |x'((n-2)t_5 + t_2 + s + v) - x'((j-2))t_5 + t_2 + s + w + v)| \\
&= |(-1)^{n-2}(x_{(n-1)})'(t_2 + s + v) - (-1)^{j-2}(x_{(j-1)})'(t_2 + s + w + v)| \\
&= |(-1)^{n-j}(x_{(n-1)})'(t_2 + s + v) - (x_{(j-1)})'(t_2 + s + w + v)| \\
&= |(\kappa_{n-1})'(v) - (\kappa_{j-1})'(w + v)| \geq \frac{a}{4}.
\end{aligned}$$

10. The case $0 \neq n - j \in 2\mathbb{Z}$, $2 \leq j$, $t_5 - \min\{-t_a, s\} \leq -w = t_* - u_* \leq t_5$. Then

$$u_* \leq t_* - t_5 + s \leq s,$$

and $w_* = t_* - u_* - t_5$ satisfies $w_* \in [-s, 0]$. We have $t_5 + u_* - t_2 - s \in [0, t_5] \subset [0, r]$ since

$$0 \leq t_5 - t_3 \leq t_5 - t_2 - s \leq t_5 + u_* - t_2 - s \leq t_5 + s - t_2 - s \leq t_5 \leq r.$$

Hence

$$\begin{aligned}
|\hat{x}(t) - \hat{x}(u)|_{1,r} &\geq |x_t(-t_5 - u_* + t_2 + s) - x_u(-t_5 - u_* + t_2 + s)| \\
&= |x((n-1)t_5 + t_* - t_5 - u_* + t_2 + s) - \\
&\quad x((j-1)t_5 + u_* - t_5 - u_* + t_2 + s)| \\
&= |x((n-1)t_5 + w_* + t_2 + s) - x((j-2)t_5 + t_2 + s)| \\
&= |(-1)^{n-1} x_{(n)}(w_* + t_2 + s) - (-1)^{j-2} x_{(j-1)}(t_2 + s)| \\
&= |(-1)^{n-j+1} x_{(n)}(w_* + t_2 + s) - x_{(j-1)}(t_2 + s)| \geq 2\xi.
\end{aligned}$$

11. The case $n-j \in 2\mathbb{Z}+1, 2 \le j, j-1 \ne n, t_5 - \min\left\{-t_a, \frac{s}{2}\right\} \le -w = t_* - u_* \le t_5$. Now

$$u_* \le t_* - t_5 + \frac{s}{2} \le \frac{s}{2},$$

and $w_* = t_* - u_* - t_5$ belongs to $\left[-\frac{s}{2}, 0\right]$. Corollary 3.2 yields $v \in [-s, s]$ so that $w_* + v \in [-s, s]$ and

$$|(\kappa_{j-1})'(v) - (\kappa_n)'(w_* + v)| \ge \frac{a}{4}.$$

We have $t_5 + u_* - t_2 - s - v \in [0, t_5 + s] \subset [0, r]$ since

$$\begin{aligned} 0 &\le t_5 - t_3 = t_5 - t_2 - 2s \le t_5 + u_* - t_2 - 2s \le t_5 + \frac{s}{2} - t_2 - s - v \\ &\le t_5 - v \le t_5 + s \le r. \end{aligned}$$

Hence

$$\begin{aligned} |\hat{x}(t) - \hat{x}(u)|_{1,r} &\ge |(x_t)'(-t_5 - u_* + t_2 + s + v) - \\ &\quad (x_u)'(-t_5 - u_* + t_2 + s + v)| \\ &= |x'((n-1)t_5 + t_* - t_5 - u_* + t_2 + s + v) - \\ &\quad x'((j-1)t_5 + u_* - t_5 - u_* + t_2 + s + v)| \\ &= |x'((n-1)t_5 + w_* + t_2 + s + v) - x'((j-2))t_5 + t_2 + s + v)| \\ &= |(-1)^{n-1}(x_{(n)})'(t_2 + s + w_* + v) - (-1)^{j-2}(x_{(j-1)})'(t_2 + s + v)| \\ &= |(-1)^{n-j+1}(x_{(n)})'(t_2 + s + w_* + v) - (x_{(j-1)})'(t_2 + s + v)| \\ &= |(\kappa_n)'(w_* + v) - (\kappa_{j-1})'(v)| \ge \frac{a}{4}. \end{aligned}$$

12. Combining the results of Parts 3-11 and the relation $\xi > a$ we arrive at the estimate

$$|\hat{x}(t) - \hat{x}(u)|_{1,r} \ge \min\left\{\frac{a}{4}, v_m \cdot \min\{t_1, s\}, v_m \cdot \min\{-t_a, s\}\right\}$$

for all integers $n \ge 2$, $j \ge 2$ with $|n-j| > 1$ and all $t \in [(n-1)t_5, nt_5]$, $u \in [(j-1)t_5, jt_5]$.

7 Delay Functionals on C_r^0-Neighbourhoods of Compact Arcs

For $t > 0$ define $x_t' \in C_r^0$ by $x_t'(u) = x'(t+u)$, $-r \le u \le 0$. Then

$$x_t' = J(x')_t = JD\hat{x}(t)1.$$

The curve

$$\hat{x}' : (0, \infty) \ni t \mapsto x_t' \in C_r^0$$

is continuously differentiable since the derivative $x' : [-r, \infty) \to \mathbb{R}$ is continuously differentiable, compare [13, Proposition 4.1]. Consider the map

$$L : (0, \infty) \times C_r^0 \to \mathbb{R}$$

given by

$$L(t, \phi) = \phi(0)x'(t) + \phi(t_b)x'(t + t_b).$$

We have

$$L = m \circ ((ev_0 \circ \hat{x}' \circ pr_1) \times (ev_0 \circ pr_2)) + m \circ ((ev_{t_b} \circ \hat{x}' \circ pr_1) \times (ev_{t_b} \circ pr_2))$$

with the projections

$$pr_1 : (0, \infty) \times C_r^0 \to \mathbb{R}, \quad pr_2 : (0, \infty) \times C_r^0 \to C_r^0$$

onto the first and second component, respectively, with the continuous linear evaluation maps

$$ev_0 : C_r^0 \ni \phi \mapsto \phi(0) \in \mathbb{R}, \quad ev_{t_b} : C_r^0 \ni \phi \mapsto \phi(t_b) \in \mathbb{R},$$

and with the multiplication $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. So L is continuously differentiable.

Each map $L(t, \cdot) : C_r^0 \to \mathbb{R}$, $t > 0$, is linear. For the nullspace

$$K_t = \{\phi \in C_r^0 : L(t, \phi) = 0\}$$

of $L(t, \cdot)$ we have

$$x_t' \notin K_t$$

since

$$L(t, \hat{x}'(t)) = (x'(t))^2 + (x'(t + t_b))^2 > 0,$$

which follows from the fact that the zeros of x' in $[t_b, \infty)$ are given by $\frac{1}{2}(t_2 + t_3) + jt_5$, $j \in \mathbb{N}_0$. We infer

$$C_r^0 = \mathbb{R}x_t' \oplus K_t \quad \text{for all} \quad t > 0.$$

In the sequel we show that every compact arc $J\hat{x}([u, v]) \subset C_r^0$, $r < u < v$, has a neighbourhood U in C_r^0 on which the representation

$$\phi = x_t + \kappa \text{ with } \kappa \in K_t, \; t \text{ close to } [u, v], \text{ and } \kappa = \phi - x_t \text{ small in } C_r^0$$

is unique. Knowing this we shall define a delay functional $d_U : C_r^0 \supset U \to \mathbb{R}$ by

$$d(\phi) = \Delta(x_t).$$

Then d is constant along each fibre $(x_t + K_t) \cap U$, with t close to $[u, v]$.

Obviously,

$$\phi - x_t \in K_t \quad \Leftrightarrow \quad L(t, \phi - x_t) = 0$$

for all $\phi \in C_r^0$ and all $\sigma > 0$.

Proposition 7.1 *[Local fibre representation] For every $t > 0$ there exist $\delta \in (0, t)$, $\epsilon \in (0, \delta]$, and a continuously differentiable map*

$$\tau : C_r^0 \supset U_\epsilon(x_t) \to (t - \delta, t + \delta) \subset \mathbb{R}$$

with $\tau(x_t) = t$ so that for every $(\sigma, \phi) \in (t - \delta, t + \delta) \times U_\epsilon(x_t)$,

$$L(\sigma, \phi - x_\sigma) = 0 \Leftrightarrow \sigma = \tau(\phi)\}.$$

For every $\phi \in U_\epsilon(x_t)$ and for $\sigma = \tau(\phi)$,

$$|\phi - x_\sigma|_{0,r} \leq \left(1 + \sup_{t-\delta \leq u \leq t+\delta} |x_u'|_{0,r}\right) \delta.$$

Proof Let $t > 0$ be given. The map

$$f : (0, \infty) \times C_r^0 \ni (\sigma, \phi) \mapsto L(\sigma, \phi - J\hat{x}(\sigma)) \in \mathbb{R}$$

is continuously differentiable and satisfies $f(t, x_t) = 0$. Using the formula defining the map L we infer

$$\begin{aligned} D_1 f(t, \phi)1 &= \phi(0)x''(t) + \phi(t_b)x''(t + t_b) \\ &\quad -((x'(t))^2 + x(t)x''(t)) - ((x'(t + t_b))^2 + x(t + t_b)x''(t + t_b)), \end{aligned}$$

hence

$$D_1 f(t, x_t)1 = -(x'(t))^2 - (x'(t + t_b))^2 < 0.$$

Apply the Implicit Function Theorem and obtain $\delta \in (0, t)$, $\epsilon > 0$, and a continuously differentiable map τ with the properties stated in the first sentence of the proposition. Notice that one can achieve $\epsilon \le \delta$. For $\phi \in U_\epsilon(x_t)$ and $\sigma = \tau(\phi)$ we get

$$\begin{aligned} |\phi - x_\sigma|_{0,r} &\le |\phi - x_t|_{0,r} + |x_t - x_\sigma|_{0,r} \\ &= |\phi - x_t|_{0,r} + |J\hat{x}(t) - J\hat{x}(\sigma)|_{0,r} \\ &\le \epsilon + \sup_{t-\delta \le u \le t+\delta} |DJ\hat{x}(u)1|_{0,r}|t - \sigma| \\ &= \epsilon + \sup_{t-\delta \le u \le t+\delta} |JD\hat{x}(u)1|_{0,r}|t - \sigma| \\ &= \epsilon + \sup_{t-\delta \le u \le t+\delta} |x'_u|_{0,r}|t - \sigma| \\ &\le (1 + \sup_{t-\delta \le u \le t+\delta} |x'_u|_{0,r})\delta. \end{aligned}$$

Proposition 7.2 **(Fibre representation along compact arcs)** *Let reals $u < v$ in (r, ∞) and $n \in \mathbb{N}$ be given. There exist positive $\rho = \rho(u, v, n) \le \frac{1}{n}$ so that for every $\phi \in U_\rho(J\hat{x}([u, v]))$ there is one and only one*

$$\sigma \in \left[u - \frac{1}{n}, v + \frac{1}{n}\right] \cap (0, \infty)$$

such that

$$L(\sigma, \phi - x_\sigma) = 0 \quad \text{and} \quad |\phi - x_\sigma|_{0,r} \le \frac{1}{n}.$$

In case $\phi = x_t$ with $t \in [u, v]$ we have $\sigma = t$.

Proof 1. Let reals $u < v$ in (r, ∞) be given. As the curve $J \circ \hat{x}$ is continuously differentiable with $DJ\hat{x}(w)1 = x'_w \in C_r^0$ for all $w > 0$ we obtain

$$|x_t - x_\sigma)|_{0,r} \le c|t - \sigma| \quad \text{for all} \quad t, \sigma \ \text{in} \ \left[\frac{u}{2}, v + 1\right]$$

with

$$c = \max_{\frac{u}{2} \le w \le v+1} |x'_w|_{0,r}.$$

2. Apply Proposition 7.1 to each $w \in [u, v]$, and obtain $\epsilon = \epsilon_w$ and $\delta = \delta_w$ and $\tau = \tau_w$ according to Proposition 7.1. Notice that one my assume

$$\frac{u}{2} \le w - \delta_w, \quad (1 + c)\delta_w \le \frac{1}{n}.$$

Using the compactness of $J\hat{x}([u,v]) \subset C_r^0$ one finds a strictly increasing finite sequence $(w_j)_1^{\bar{j}}$ in $[u,v]$ so that the associated neighbourhoods $U_{\epsilon_{w_j}}(\hat{x}(w_j))$, $j \in \{1,\ldots,\bar{j}\}$, form a covering of $J\hat{x}([u,v])$. There exists a positive real number

$$\rho = \rho(u,v,n) \le \min_{j=1,\ldots,\bar{j}} \epsilon_{w_j}$$

with

$$U_\rho(J\hat{x}([u,v]) \subset \bigcup_{j=1}^{\bar{j}} U_{\epsilon_{w_j}}(\hat{x}(w_j)).$$

Notice that

$$\rho \le \min_{j=1,\ldots,\bar{j}} \epsilon_{w_j} \le \max_{j=1,\ldots,\bar{j}} \delta_{w_j} \le \frac{1}{n}.$$

For every $\phi \in U_\rho(J\hat{x}([u,v])$ we obtain (at least one)

$$\begin{aligned}\sigma &\in \bigcup_{j=1}^{\bar{j}} (w_j - \delta_{w_j}, w_j + \delta_{w_j})\\ &\subset \left[\max\left\{\frac{u}{2}, u - \frac{1}{n}\right\}, v + \frac{1}{n}\right]\end{aligned}$$

with

$$L(\sigma, \phi - x_\sigma) = 0 \quad \text{and} \quad |\phi - x_\sigma|_{0,r} \le (1+c) \max_{j=1,\ldots,\bar{j}} \delta_{w_j} \le \frac{1}{n}.$$

Or, the set $R_n \subset (0,\infty)$ of all $\rho \in \left(0, \frac{1}{n}\right]$ such that for every $\phi \in U_\rho(J\hat{x}([u,v]))$ there exist $\sigma \in \left[u - \frac{1}{n}, v + \frac{1}{n}\right] \cap (0,\infty)$ with

$$L(\sigma, \phi - x_\sigma) = 0 \quad \text{and} \quad |\phi - x_\sigma|_{0,r} \le \frac{1}{n}$$

is nonempty. Observe that

$$\rho_n = \frac{1}{2} \sup\ R_n$$

belongs to R_n.

3. Assume that the set I of all $n \in \mathbb{N}$ such that $U_{\rho_n}(J\hat{x}([u,v]))$ contains ϕ with

$$\begin{aligned}2 \le \#\Big\{\sigma \in \left[u - \frac{1}{n}, v + \frac{1}{n}\right] \cap (0,\infty) : L(\sigma, \phi - x_\sigma) = 0\\ \text{and } |\phi - x_\sigma|_{0,r} \le \frac{1}{n}\Big\}\end{aligned}$$

is unbounded. We derive a contradiction. The elements of I form a strictly increasing sequence $(n_k)_1^\infty$. For every $k \in \mathbb{N}$ select some ϕ_k in $U_\rho(J\hat{x}([u,v]))$ with $\rho = \rho_{n_k}$ and $\sigma_k^{(1)} < \sigma_k^{(2)}$ in $\left[u - \frac{1}{n_k}, v + \frac{1}{n_k}\right] \cap (0,\infty)$ with

$$L(\sigma_k^{(m)}, \phi_k - x_{\sigma_k^{(m)}}) = 0 \quad \text{and} \quad |\phi_k - x_{\sigma_k^{(m)}}|_{0,r} \le \frac{1}{n_k} \quad \text{for} \quad m \in \{1,2\}.$$

Using the compactness of, say, $[0, v+1]$, and successively choosing subsequences we find a strictly increasing sequence $(k_\kappa)_1^\infty$ so that the equations

$$z_\kappa^{(m)} = \sigma_{k_\kappa}^{(m)} \quad \text{for} \quad \kappa \in \mathbb{N} \quad \text{and} \quad m \in \{1, 2\}$$

define two sequences which converge to $z^{(1)} \leq z^{(2)}$ in $[0, v+1]$, respectively. Necessarily, $u \leq z^{(1)} \leq z^{(2)} \leq v$. The continuity of $J \circ \hat{x}$ yields $x_{z_\kappa^{(m)}} \to x_{z^{(m)}}$ in C_r^0 as $\kappa \to \infty$, for $m \in \{1, 2\}$. Using the inequalities

$$|\phi_k - x_{\sigma_k^{(m)}}|_{0,r} \leq \frac{1}{n_k} \quad \text{for} \quad m \in \{1, 2\} \quad \text{and} \quad k \in \mathbb{N}$$

we obtain $\phi_{k_\kappa} \to x_{z^{(1)}} = x_{z^{(2)}}$ as $\kappa \to \infty$. As $\hat{x}$ is injective on $[r, \infty) \supset [u, v]$, $z^{(1)} = z^{(2)}$. Apply Proposition 7.1 to $t = z^{(1)} = z^{(2)}$ and choose positive $\epsilon \leq \delta$ according to this proposition. For $\kappa \in \mathbb{N}$ sufficiently large we have

$$\phi_{k_\kappa} \in U_\epsilon(x_t),$$

both $z_\kappa^{(1)} < z_\kappa^{(2)}$ belong to $(t - \delta, t + \delta)$, and

$$L(\sigma, \phi_{k_\kappa} - x_\sigma) = 0 \text{ for } \sigma = z_\kappa^{(1)} \text{ and for } \sigma = z_\kappa^{(2)}.$$

This yields a contradiction to the first part of Proposition 7.1.

4. Combining the results of Parts 1 and 2 we obtain $n(u, v) \in \mathbb{N}$ such that for every integer $n \geq n(u, v)$ and for every $\phi \in U_{\rho_n}(J\hat{x}([u, v]))$ there exists one and only one $\sigma \in \left[u - \frac{1}{n}, v + \frac{1}{n}\right] \cap (0, \infty)$ with $L(\sigma, \phi - x_\sigma) = 0$ and $|\phi - x_\sigma|_{0,r} \leq \frac{1}{n}$. Now the assertion of Proposition 7.2 follows easily.

Proposition 7.2 yields that for $u < v$ in (r, ∞) and $n \in \mathbb{N}$ there exists $\rho \leq \frac{1}{n}$ so that the relations

$$\phi \in U_\rho(J\hat{x}([u, v])), \quad \sigma \in \left[u - \frac{1}{n}, v + \frac{1}{n}\right] \cap (0, \infty),$$
$$L(\sigma, \phi - x_\sigma) = 0, \quad |\phi - x_\sigma|_{0,r} \leq \frac{1}{n}$$

define a map

$$s_{u,v,\rho} : C_r^0 \supset U_\rho(J\hat{x}([u, v])) \to (0, \infty)$$

with

$$|\phi - x_{s_{u,v,\rho}(\phi)}|_{0,r} \leq \frac{1}{n} \quad \text{for all} \quad \phi \in U_\rho(J\hat{x}([u, v])).$$

Proposition 7.3 *Let reals $u < v$ in (r, ∞) and $n \in N$ be given and choose $\rho = \rho(u, v, n)$ according to Proposition 7.2. There exist $\eta = \eta(u, v, n) \in (0, \rho]$ so that the restriction $s_{u,v,\eta}$ of $s_{u,v,\rho}$ to $U_\eta(J\hat{x}([u, v]))$ is continuously differentiable.*

For every $\phi \in U_\eta(J\hat{x}([u, v]))$ and for every $\sigma \in \left[u - \frac{1}{n}, v + \frac{1}{n}\right] \cap (0, \infty)$,

$$\sigma = s_{u,v,\eta}(\phi) \quad \Leftrightarrow \quad \left(L(\sigma, \phi - x_\sigma) = 0 \textit{ and } |\phi - x_\sigma|_{0,r} \leq \frac{1}{n}\right).$$

For every $\sigma \in [u, v]$, $s_{u,v,\eta}(x_\sigma) = \sigma$.

Proof For each $t \in [u, v]$ choose $\epsilon = \epsilon_t \le \delta_t = \delta$ and $\tau = \tau_t$ according to Proposition 7.1. Observe that we may assume that δ_t satisfies

$$\max\left\{0, u - \frac{1}{n}\right\} < t - \delta_t, \quad t + \delta_t < v + \frac{1}{n}$$

and

$$(1 + \sup_{t-\delta_t \le w \le t+\delta_t} |x'_w|_{0,r})\delta_t < \frac{1}{n}.$$

For every $\phi \in U_\rho(J\hat{x}([u, v])) \cap U_{\epsilon_t}(x_t)$ we have that

$$\sigma = \tau_t(\phi) \in (t - \delta_t, t + \delta_t) \subset \left[u - \frac{1}{n}, v + \frac{1}{n}\right] \cap (0, \infty)$$

satisphies $L(\sigma, \phi - x_\sigma) = 0$ and

$$|\phi - x_\sigma|_{0,r} \le (1 + \sup_{t-\delta_t \le w \le t+\delta_t} |x'_w|_{0,r})\delta_t < \frac{1}{n}.$$

By the definition of $s_{u,v,\rho}$,

$$s_{u,v,\rho}(\phi) = \sigma = \tau_t(\phi).$$

It follows that the restriction of $s_{u,v,\rho}$ to $U_\rho(J\hat{x}([u, v])) \cap U_{\epsilon_t}(x_t)$ is continuously differentiable. There exists $\eta \in (0, \rho)$ with

$$U_\eta(J\hat{x}([u, v])) \subset \bigcup_{u \le t \le v} U_\rho(J\hat{x}([u, v])) \cap U_{\epsilon_t}(x_t).$$

The last statement in Proposition 7.3 is obvious from Proposition 7.2.

Using continuous differentiability of the delay function Δ we infer that the delay functional

$$d_{u,v,\eta} = \Delta \circ s_{u.v,\eta}$$

defined on the open neighbourhood $U_\eta(J\hat{x}([u, v]))$ of the arc $J\hat{x}([u, v])$ is continuously differentiable (with respect to the topology of C^0_r). For every $\sigma \in [u, v]$ we have $s_{u,v,\eta}(x_\sigma) = \sigma$, hence

$$d_{u,v,\eta}(x_\sigma) = \Delta(s_{u,v,\eta}(x_\sigma)) = \Delta(\sigma).$$

8 Compatibility on C^0_r-Neighbourhoods of Adjacent Arcs

Let $j = j_r \ge 2$ denote the smallest integer with $r < (j-1)t_5$. For $j \le k \in \mathbb{N}$ set

$$X_k = \hat{x}([(k-1)t_5, kt_5]) \subset C^1_r.$$

In the sequel we construct open neighbourhoods U_k of JX_k in C^0_r and continuously differentiable delay functionals $d_k : C^0_r \supset U_k \to (0, r)$ with $d_k(x_t) = \Delta(t)$ for all $t \in [(k-1)t_5, kt_5]$ so that for every integer $k \ge j$ we have

$$d_k(\phi) = d_{k+1}(\phi) \quad \text{for all} \quad \phi \in U_k \cap U_{k+1}. \tag{8.1}$$

The construction is iterative. We carry out the initial step and the step thereafter. This second step is the model for the step from statements for general $k \geq j$ to statements for $k+1$.

1. The initial step for $k = j$.

1.1. Apply Proposition 7.1 with $t = jt_5$ at $\hat{x}(t)$, choose $\delta = \delta(j) > 0$, $\epsilon = \epsilon(j) \in (0, \delta]$, and a map $\tau = \tau_j$ from $U_\epsilon(\hat{x}(t)) \subset C_r^0$ into $(t-\delta, t+\delta)$ accordingly. By continuity there are $n = n(j) \in \mathbb{N}$ with

$$\hat{x}\left(\left[t - \frac{1}{n}, t + \frac{1}{n}\right]\right) \subset U_\epsilon(\hat{x}(t)) \quad \text{and} \quad r < (j-1)t_5 - \frac{1}{n},$$

and $\epsilon_j \in (0, \epsilon(j)]$ with

$$\tau(U_{\epsilon_j}(\hat{x}(t))) \subset \left[t - \frac{1}{n}, t + \frac{1}{n}\right].$$

An application of Proposition 1.3 with $a = (j-1)t_5$, $b = (j+1)t_5$, $t = jt_5$ yields $\rho = \rho(j) > 0$ with

$$U_\rho(JX_j) \cap U_\rho(JX_{j+1}) \subset U_{\epsilon_j}(\hat{x}(t));$$

notice that $X_j = \hat{x}([a,t])$ and $X_{j+1} = \hat{x}([t,b])$.

1.2. We apply Proposition 7.3 twice, first with $u = (j-1)t_5$, $v = jt_5$, and $n = n(j)$. This yields $\eta > 0$ and a continuously differentiable map

$$s_{u,v,\eta} : U_\eta(JX_j) \to \left[u - \frac{1}{n}, v + \frac{1}{n}\right] \subset \mathbb{R}$$

so that for every $\phi \in U_\eta(JX_j)$ we have

$$\left(\sigma \in \left[u - \frac{1}{n}, v + \frac{1}{n}\right] \text{ and } L(\sigma, \phi - x_\sigma) = 0\right) \Leftrightarrow \sigma = s_{u,v,\eta}(\phi).$$

Also, $s_{u,v,\eta}(x_w) = w$ for all $w \in [u, v]$. We may assume

$$\eta < \rho = \rho(j).$$

Set

$$U_j = U_\eta(JX_j) \quad \text{and} \quad s_j = s_{u,v,\eta}.$$

The map

$$d_j : U_j \ni \phi \mapsto \Delta(s_j(\phi)) \in (0, r)$$

is continuously differentiable with $d_j(x_w) = \Delta(w)$ for all $w \in [(j-1)t_5, jt_5]$.

The second application of Proposition 7.3, with $\hat{u} = (j+1)-1)t_5 = jt_5$, $\hat{v} = (j+1)t_5$, and $n = n(j)$ yields $\hat{\eta} > 0$ and a continuously differentiable map $s_{\hat{u},\hat{v},\hat{\eta}} : U_{\hat{\eta}}(JX_{j+1}) \to \left[\hat{u} - \frac{1}{n}, \hat{v} + \frac{1}{n}\right] \subset \mathbb{R}$ such that for every $\phi \in U_{\hat{\eta}}(JX_{j+1})$ we have

$$\left(\sigma \in \left[\hat{u} - \frac{1}{n}, \hat{v} + \frac{1}{n}\right] \text{ and } L(\sigma, \phi - x_\sigma) = 0\right) \Leftrightarrow \sigma = s_{\hat{u},\hat{v},\hat{\eta}}(\phi).$$

Also, $s_{\hat{u},\hat{v},\hat{\eta}}(x_w) = w$ for all $w \in [\hat{u}, \hat{v}]$. We may assume

$$\hat{\eta} < \rho = \rho(j).$$

Set

$$\hat{U}_{j+1} = U_{\hat{\eta}}(JX_{j+1}) \quad \text{and} \quad \hat{s}_{j+1} = s_{\hat{u},\hat{v},\hat{\eta}}.$$

1.3. Let $\phi \in U_j \cap \hat{U}_{j+1}$. Proof of $s_j(\phi) = \hat{s}_{j+1}(\phi)$.

We have $\phi \in U_{\epsilon_j}(\hat{x}(t))$, due to Part 1.1 and to $\max\{\eta, \hat{\eta}\} \leq \rho(j)$. Hence

$$\tau(\phi) \in \left[t - \frac{1}{n}, t + \frac{1}{n}\right].$$

Notice that $t = v = \hat{u}$, and thereby

$$\left[t - \frac{1}{n}, t + \frac{1}{n}\right] \subset \left[u - \frac{1}{n}, v + \frac{1}{n}\right] \cap \left[\hat{u} - \frac{1}{n}, \hat{v} + \frac{1}{n}\right].$$

For $\sigma = \tau(\phi)$ we have $L(\sigma, \phi - x_\sigma) = 0$, see Proposition 7.1. Now the properties of s_j and of $s_{\hat{u},\hat{v},\hat{\eta}}$ from Part 1.2 yield

$$s_j(\phi) = \sigma = s_{\hat{u},\hat{v},\hat{\eta}}(\phi) = \hat{s}_{j+1}(\phi).$$

2. The second step, which includes the definitions of $U_{j+1} \subset \hat{U}_{j+1}$, of s_{j+1}, and of d_{j+1}, and contains the proof of $d_j(\phi) = d_{j+1}(\phi)$ on $U_j \cap U_{j+1}$.

2.1. Apply Proposition 7.1, now at $\hat{x}(t)$ with $t = (j+1)t_5$, and choose $\delta = \delta(j+1) > 0$, $\epsilon = \epsilon(j+1) \in (0, \delta]$, and a map $\tau = \tau_{j+1}$ from $U_\epsilon(\hat{x}(t))$ into $(t - \delta, t + \delta)$ accordingly. By continuity there is an integer $n = n(j+1) \geq n(j)$ with

$$J\hat{x}\left(\left[t - \frac{1}{n}, t + \frac{1}{n}\right]\right) \subset U_\epsilon(\hat{x}(t)) \quad \left(\text{and} \quad r < ((j+1)-1)t_5 - \frac{1}{n}\right),$$

and there exists $\epsilon_{j+1} \in (0, \epsilon(j+1)]$ with

$$\tau(U_{\epsilon_{j+1}}(\hat{x}(t))) \subset \left[t - \frac{1}{n}, t + \frac{1}{n}\right].$$

An application of Proposition 1.3 with $a = ((j+1)-1)t_5 = jt_5$, $b = ((j+1)+1)t_5 = (j+2)t_5$, $t = (j+1)t_5$ yields $\rho = \rho(j+1) > 0$ with

$$U_\rho(JX_{j+1}) \cap U_\rho(JX_{j+2}) \subset U_{\epsilon_{j+1}}(\hat{x}(t));$$

notice that $X_{j+1} = \hat{x}([a, t])$ and $X_{j+2} = \hat{x}([t, b])$.

2.2. First we restrict $\hat{s}_{j+1}$ from Part 1.2. As $\hat{s}_{j+1}$ maps JX_{j+1} onto $[(jt_5, (j+1)t_5]$ continuity yields $\tilde{\eta} \in (0, \rho(j+1)]$ such that

$$U_{j+1} = U_{\tilde{\eta}}(JX_{j+1})$$

is contained in $\hat{U}_{j+1}$ and

$$\hat{s}_{j+1}(U_{j+1}) \subset \left[jt_5 - \frac{1}{n}, (j+1)t_5 + \frac{1}{n}\right],$$

with $n = n(j+1)$. Set $s_{j+1} = \hat{s}_{j+1}|_{U_{j+1}}$. Part I.3 gives

$$s_{j+1}(\phi) = s_j(\phi) \quad \text{for all} \quad \phi \in U_{j+1} \cap U_j,$$

and it follows that the continuously differentiable map

$$d_{j+1} : U_{j+1} \ni \phi \mapsto \Delta(s_{j+1}(\phi)) \in (0, r)$$

satisfies $d_{j+1}(\phi) = \Delta(s_{j+1}(\phi)) = \Delta(s_j(\phi)) = d_j(\phi)$ for all $\phi \in U_{j+1} \cap U_j$. Also, $d_{j+1}(x_w) = \Delta(s_{j+1}(x_w)) = \Delta(w)$ for all $w \in [jt_5, (j+1)t_5]$.

Next we apply Proposition 7.3, with $\check{u} = (j+2) - 1)t_5 = (j+1)t_5$, $\check{v} = (j+2)t_5$, and $n = n(j+1)$. This yields $\check{\eta} > 0$ and a continuously differentiable map $s_{\check{u},\check{v},\check{\eta}} : U_{\check{\eta}}(JX_{j+2}) \to \left[\check{u} - \frac{1}{n}, \check{v} + \frac{1}{n}\right] \subset \mathbb{R}$ such that for every $\phi \in U_{\check{\eta}}(JX_{j+2})$ we have

$$\left(\sigma \in \left[\check{u} - \frac{1}{n}, \check{v} + \frac{1}{n}\right] \text{ and } L(\sigma, \phi - x_\sigma) = 0\right) \Leftrightarrow \sigma = s_{\check{u},\check{v},\check{\eta}}(\phi).$$

Also, $s_{\check{u},\check{v},\check{\eta}}(x_w) = w$ for all $w \in [\check{u}, \check{v}]$. Again we may assume

$$\check{\eta} < \rho = \rho(j+1).$$

Set

$$\hat{U}_{j+2} = U_{\check{\eta}}(JX_{j+2}) \quad \text{and} \quad \hat{s}_{j+2} = s_{\check{u},\check{v},\check{\eta}}.$$

2.3. Proof of $s_{j+1}(\phi) = \hat{s}_{j+2}(\phi)$ for all $\phi \in U_{j+1} \cap \hat{U}_{j+2}$. Such ϕ belong to $U_{\epsilon_{j+1}}(\hat{x}((j+1)t_5)$, due to Part 2.1 and to the inequality $\max\{\tilde{\eta}, \check{\eta}\} \leq \rho(j+1)$. Hence $\sigma = \tau(\phi)$ is contained in $\left[t - \frac{1}{n}, t + \frac{1}{n}\right]$, for $n = n(j+1)$. Notice that $t = (j+1)t_5 = \check{u}$, and thereby

$$\begin{aligned}\left[t - \frac{1}{n}, t + \frac{1}{n}\right] &\subset \left[jt_5 - \frac{1}{n}, (j+1)t_5 + \frac{1}{n}\right] \cap \left[\check{u} - \frac{1}{n}, \check{v} + \frac{1}{n}\right] \\ &\subset \left[jt_5 - \frac{1}{n(j)}, (j+1)t_5 + \frac{1}{n(j)}\right] \cap \left[\check{u} - \frac{1}{n(j+1)}, \check{v} + \frac{1}{n(j+1)}\right].\end{aligned}$$

We also have $L(\sigma, \phi - x_\sigma) = 0$, see Proposition 7.1. Now the properties of $\hat{s}_{j+1}$ from Part 1.2 and of $\hat{s}_{j+2} = s_{\check{u},\check{v},\check{\eta}}$ from Part 2.2 yield

$$\hat{s}_{j+1}(\phi) = \sigma = \hat{s}_{j+2}(\phi),$$

which is $s_{j+1}(\phi) = \hat{s}_{j+2}(\phi)$.

This ends the second step.

9 A Functional on a C_r^1-Neighbourhood of the Trace $\hat{x}([(j_r - 1)t_5, \infty))$

In this section the constructions from Sects. 2–8 are used to prove Theorem 1.1. Let an integer $k \geq j_r$ be given. On the open set of all reals $t > 0$ with $J\hat{x}(t) \in U_k$ we have that the map given by $t \mapsto d_k(J\hat{x}(t))$ is continuously differentiable, with the derivatives given by

$$Dd_k(Jx_t)JD\hat{x}(t)1 = Dd_k(Jx_t)x_t' \in \mathbb{R}.$$

On $[(k-1)t_5, kt_5]$ we have $\Delta(t) = d_k(J\hat{x}(t))$. It follows that on this interval,

$$1 > \Delta'(t) = Dd_k(Jx_t)x_t'.$$

Recall the constant $\hat{a}$ from Proposition 6.1. The subset

$$\begin{aligned}N_k = \{\phi \in C_r^1 \cap J^{-1}(U_k) : \quad & Dd_k(J\phi)\phi' < 1 \quad \text{and there exists} \\ & t \in [(k-1)t_5, kt_5] \quad \text{with} \quad |\phi - x_t|_{1,r} < \frac{\hat{a}}{2}\right\}\end{aligned}$$

of the space C_r^1 is open. Proposition 6.1 yields $N_k \cap N_m = \emptyset$ for all integers $k \geq j_r$ and $m \geq j_r$ with $|k - m| > 1$. Also, $N_k \cap N_{k+1} \subset J^{-1}(U_k) \cap J^{-1}(U_{k+1})$ for $j_r \leq k \in \mathbb{N}$. Using the relations (8.1) we obtain that on the open set

$$N = \bigcup_{k \geq j_r} N_k \quad \supset \hat{x}([(j_r - 1)t_5, \infty))$$

the equations

$$d(\phi) = d_k(J\phi) \quad \text{for} \quad \phi \in N_k \quad \text{and} \quad j_r \leq k \in \mathbb{N}$$

define a map $d : C_r^1 \supset N \to (0, r)$. It follows that

$$d(x_t) = \Delta(t) \quad \text{for all} \quad t \geq (j_r - 1)t_5 \tag{9.1}$$

since for such t there exists $k \geq j_r$ with $t \in [(k-1)t_5, kt_5]$, hence $x_t \in N_k$, and thereby $d(x_t) = d_k(Jx_t) = d_k(x_t) = \Delta(t)$, see Sect. 8.

Proposition 1.2 applies and yields that the functional d is continuously differentiable and has property (e).

Proposition 9.1 *The functional*

$$f : C_r^1 \supset N \ni \phi \mapsto -\phi(-d(\phi)) \in \mathbb{R}$$

is continuously differentiable and has the extension property (e).

This is analogous to [14, Proposition 11.1]. We include the proof for convenience.

Proof We have

$$f(\phi) = -ev_r^1(\phi, -d(\phi)) = -(ev_r^1 \circ (id \times (-d)))(\phi) \quad \text{for all} \quad \phi \in N,$$

which shows that f is continuously differentiable. Recall $D_1 ev_r^1(\phi, t)\hat{\phi} = \hat{\phi}(t)$ and $D_2 ev_r^1(\phi, t)\hat{t} = \hat{t}\phi'(t)$. The chain rule yields

$$Df(\phi)\hat{\phi} = -\hat{\phi}(-d(\phi)) - \phi'(-d(\phi))[-Dd(\phi)\hat{\phi}] = \phi'(-d(\phi))Dd(\phi)\hat{\phi} - \hat{\phi}(-d(\phi)).$$

For $\phi \in N$ the equation

$$D_e f(\phi)\chi = \phi'(-d(\phi))D_e d(\phi)\chi - \chi(-d(\phi)).$$

defines a linear extension $D_e f(\phi) : C_r^0 \to \mathbb{R}$ of the derivative $Df(\phi) : C_r^1 \to \mathbb{R}$. Using the continuity of the evaluation map $C_r^0 \times [-r, 0] \ni (\chi, t) \mapsto \chi(t) \in \mathbb{R}$ and property (e) of d one finds that the map $N \times C_r^0 \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}$ is continuous.

For $t \geq j_r t_5$ we have $x_t \in N$ and, due to Eq. (9.1),

$$x'(t) = -x(t - \Delta(t)) = -x(t - d(x_t)) = f(x_t).$$

This implies that the twice continuously differentiable function

$$x^{(d)} : [-r, \infty) \ni t \mapsto x(t + j_r t_5) \in \mathbb{R}$$

is a solution of the equation

$$y'(t) = f(y_t)$$

with the flowline $[0, \infty) \ni t \mapsto x_t^{(d)} \in C_r^1$ in the solution manifold

$$X_f = \{\phi \in N : \phi'(0) = f(\phi)\}.$$

Recall the non-empty set $A \subset C_h^2$ chosen in Sect. 4 as a special case of the sets from Example 2.2. The set A is open in the affine space $p_* + C_{h-0}^2$ of codimension 6 in C_{h-0}^2. Recall the choice of x on $[t_d - h, t_d] \subset [0, t_5]$ in Sect. 4. The short segments $x_{t_d+(n-1)t_5,short}^{(d)} \in C_h^2$, $n \in \mathbb{N}$, are dense in $A \cup (-A)$.

Finally we show that for each $\phi \in X_f$ the delayed argument function

$$[0, t_\phi) \ni t \mapsto t - d(x_t^\phi) \in \mathbb{R}$$

is strictly increasing. Let $\phi \in X_f$ and $t \in (0, t_\phi)$ be given and set $y = x^\phi$. As $y : [-r, t_\phi) \to \mathbb{R}$ is continuously differentiable the curve $\tilde{y} : [0, t_\phi) \ni t \mapsto Jy_t \in C_r^0$ is continuously differentiable with $D\tilde{y}(u)1 = y'_u$ for all $u > 0$, compare [13, Proposition 4.1]. The segment $y_t \in X_f \subset N$ is contained in N_k for some integer $k \geq j_r$. By continuity of the flowline $[0, t_\phi) \ni u \mapsto y_u \in X_f \subset N \subset C_r^1$, there is $\epsilon > 0$ with $y_u \in N_k$ for all $u \in (t - \epsilon, t + \epsilon)$. Then $d(y_u) = d_k(Jy_u) = d_k(\tilde{y}(u))$ on $(t - \epsilon, t + \epsilon)$. It follows that the curve

$$(t - \epsilon, t + \epsilon) \ni u \mapsto d(y_u) \in \mathbb{R}$$

is differentiable with derivatives given by $Dd_k(Jy_u)y'_u < 1$. This implies that on $(0, t_\phi)$ the delayed argument function is differentiable with positive derivative, from which the assertion follows.

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References

1. Diekmann, O., van Gils, S.A., Verduyn Lunel, S.M., Walther, H.O.: Delay Equations: Functional-, Complex- and Nonlinear Analysis. Springer, New York (1995)
2. Hale, J.K.: Functional Differential Equations. Springer, New York. (1971)
3. Hale, J.K., Verduyn Lunel, S.M.: Introduction to Functional Differential Equations. Springer, New York (1993)
4. Hartung, F., Krisztin, T., Wu, J., and H.O. Walther, *Functional differential equations with state-dependent delay: Theory and applications*. In: HANDBOOK OF DIFFERENTIAL EQUATIONS, Ordinary Differential Equations, Vol. 3, pp. 435-545, Canada, A., Drabek., P., and A. Fonda eds., Elsevier Science B. V., North Holland, Amsterdam, 2006
5. Kennedy, B., Mao, Y., Wendt, E.L.: A state-dependent delay equation with chaotic solutions. Electr J. Qual. Theory Dif. Eqs. **2019**(22), 1–20 (2019)
6. Lani-Wayda, B., Walther, H.O.: A Shilnikov phenomenon due to state-dependent delay, by means of the fixed point index. J. Dyn. Dif. Eqs. **28**, 627–688 (2016). https://doi.org/10.1007/s10884-014-9420-z
7. Mallet-Paret, J., Nussbaum, R.D., Paraskevopoulos, P.: Periodic solutions for functional differential equations with multiple state-dependent time lags. Topol. Methods Nonlinear Anal. **3**, 101–162 (1994)
8. Walther, H.O.: The solution manifold and C^1-smoothness of solution operators for differential equations with state-dependent delay. J. Dif. Eqs. **195**, 46–65 (2003)
9. Math, J.: Walther, H.O., Smoothness properties of semiflows for differential equations with state-dependent delay. In Proc. Int. Conf. Dif. and Functional Dif. Eqs. Moscow 2002, vol. 1, 40–55. Moscow State Aviation Institute (MAI), Moscow, 2003. English version. Sci. **124**, 5193–5207 (2004)
10. Walther, H.O.: A homoclinic loop generated by variable delay. J. Dyn. Dif. Eqs. **27**, 1101–1139 (2015)

11. Walther, H.O.: Complicated histories close to a homoclinic loop generated by variable delay. Adv. Dif. Eqs. **19**, 911–946 (2014)
12. Walther, H.O.: Merging homoclinic solutions due to state-dependent delay. J. Dif. Eqs. **259**, 473–509 (2015). https://doi.org/10.1016/j.jde.2015.02.009
13. Walther, H.O.: A delay differential equation with a solution whose shortened segments are dense. J. Dyn. Dif. Eqs. **31**, 1495–1523 (2019). https://doi.org/10.1007/s10884-018-9655-1
14. Walther, H.O.: Solutions with dense short segments from regular delays. J. Dif. Eqs. **268**, 6821–6871 (2020). https://doi.org/10.1016/j.jde.2019.11.079
15. Wright, E.M.: A non-linear difference-differential equation. J. Reine Angew. Math. **194**, 66–87 (1955)

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