# Dynamical behavior of a parametrized family of one-dimensional maps

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## Erklärung

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Gießen, June 2018........................................................Erkan Muştu.

Thanks

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## SUMMARY

**Dynamical behavior of a parametrized family of one-dimensional maps**

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We investigate the dynamics of the maps $f_{\mu,\omega}(x) := x^{\mu}\sin(\omega\ln(x))$ with $\mu > 1$ (and odd continuation). The first chapter describes how a family of one-dimensional maps $f_{\mu,\omega}$ appears in the context of return maps associated to homoclinic orbits for ODEs. Corresponding to the shape of graph of $f_{\mu,\omega}$, we introduce so-called 'flat' intervals containing exactly one maximum or minimum. We shall also use the expression 'steep' for intervals containing exactly one zero point of $f_{\mu,\omega}$. Then we construct an open set of points with orbits staying entirely in the 'flat' intervals in chapter three. In the fourth chapter, it is proved that there exist some points whose orbits stay totally within the 'steep' intervals. Then, to orbits $(f^{j}(x))$ of $f_{\mu,\omega}$ we associate a symbol sequence $(s_j) = (\mathrm{sign} f^{j}(x)) = (+1, -1, -1, +1, ...)$, and we show that the measure of the set of points which follow such symbol sequences is zero. In the last chapter, it is shown that there exist some points whose orbits travel regularly from 'flat' intervals to 'steep' intervals, then from 'steep' to 'flat' intervals and so on. To such orbits of $f_{\mu,\omega}$ we associate a symbol sequence $(L, R, R, L, ...)$, indicating whether the iterates of points are to the left or to the right of corresponding maxima of $f_{\mu,\omega}$, and finally the Lebesgue measure of the set of these points is shown to be zero.



**KEY WORDS:** Dynamical systems, one-dimensional maps, symbolic dynamics, measure

# 1. Introduction

In dynamical systems theory one-dimensional dynamical systems play an important role. Although they may seem very simple at the first glance, they can have very complicated dynamics. For instance, the problems in smooth dynamical systems, especially in low dimensions, can sometimes be reduced to the study of one-dimensional maps $f : I \to I$, where $f$ is a smooth function and $I$ is a circle or an interval. We try to understand the behavior of the orbits of given points in $I$. The orbit of a point $x$ is the sequence

$$x, f(x), f(f(x)), f(f(f(x))), ...$$

Our aim in this paper is to analyze the dynamics of certain parametrized families $f_{\mu,\omega}$ of one-dimensional maps. These arise in the dynamics of flows in three dimensions of saddle-focus homoclinic connections which were studied by P. Holmes [2]. Holmes considered maps $f$ similar to $f_{\mu,\omega} : x \to x^{\mu} \sin(\omega \ln(x))$ for $\mu > 1$, $\omega > 0$ (and odd continuation). The property $\mu > 1$ implies that all points $0 < x < 1$ approach 0 under $f^n$ as $n \to \infty$ . Holmes claimed that the set $Z$ of points $x$ for which there exists an $n_x \in \mathbb{N}$ such that $f^{n_x}(x) = 0$ can be a dense subset of $[0, 1]$, but it seems that this proof is not conclusive. In chapter four, we are interested in the orbit $x, f(x), f(f(x))....$ We first assign to $x$ a symbolic trajectory $s_0, s_1, s_2, ...$ where $s_n$ is $-1$ or $+1$ according as $f^n_{\mu,\omega}$ is in $(-1, 0)$ or $(0, 1)$ respectively. Then we construct sets $\Omega^c_n$ (depending on a parameter $c$ and $n \in \mathbb{N}$) of points with the first $n$ iterates contained in certain 'steep' intervals and following arbitrary symbol sequences. We show that $\Omega^c_n$ is contained in the closure of the set $Z$, but $\Omega^c_\infty = \bigcap_{n\in\mathbb{N}} \Omega^c_n$ has measure zero. The remark on the bottom of the page 395 of [2] conjectures, that open sets of points with orbit only in the 'flat' intervals can exist for certain parameters. (These 'flat' intervals are disjoint to $Z$.) We prove this in chapter 3.

In the last chapter, we focus on constructing another type of orbit whose points travel regularly from a 'flat' interval to a 'steep' interval, then again from the 'steep' interval to a 'flat' interval. These points form a Cantor type set and are described by sequences of the type $(L, R, R, L, ...)$, indicating whether iterates of the initial points are to the left or to the right of corresponding maxima of $f_{\mu,\omega}$. Taking counter images $f^{-1}(J)$ of intervals $J$ with $f^{-1}(J)$ close to a quadratic maximum of $f$ involves inversion of the second order Taylor expansion and thus taking square roots. We also show that, despite the expanding effect of the square root, the measure of the points with such orbits (and thus the measure of the Cantor set) is also zero.

## 1.1. Motivation of the map

In this section we briefly define the class of three-dimensional differential equations where the maps that we will study arrise. We consider the differential equation

$$\begin{aligned} \dot{x} &= sx - \omega y + F_1(x, y, z) \\ \dot{y} &= \omega x - sy + F_2(x, y, z) \\ \dot{z} &= \lambda z \quad + F_3(x, y, z) \end{aligned} \qquad \text{or} \qquad \dot{X} = F(X), \qquad (1.1.1)$$

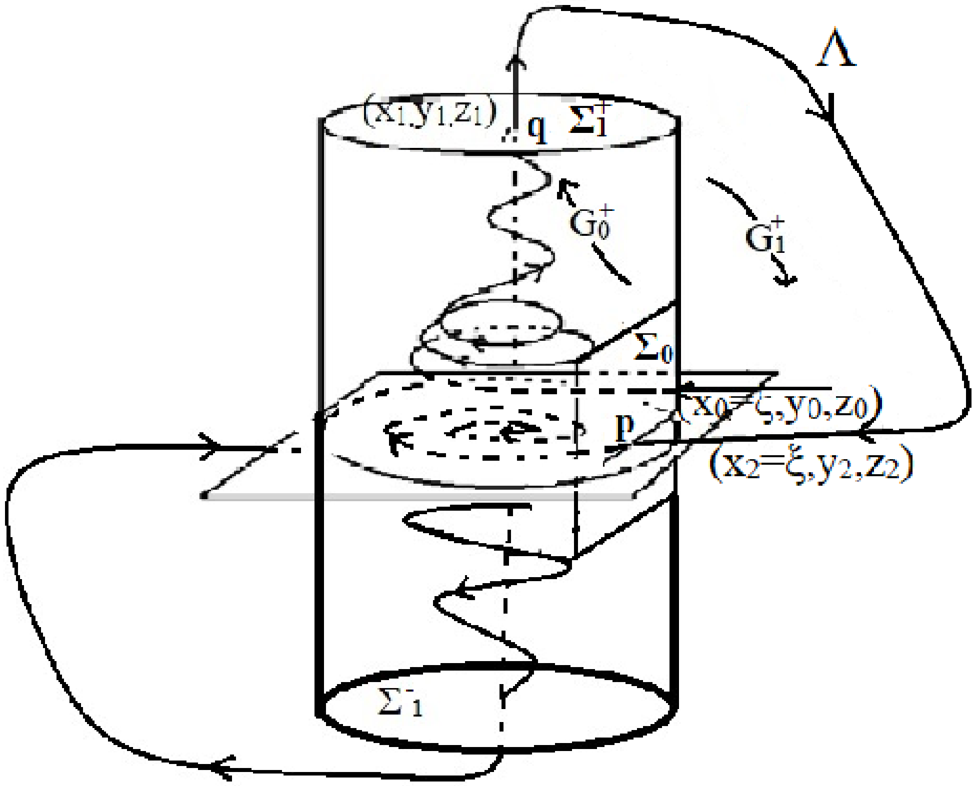


Figure 1: Cross sections $\Sigma_0$, $\Sigma_1$, and homoclinic orbit $\Lambda$

with smooth functions $F_1$, $F_2$, $F_3$ which vanish at the origin together with their derivatives. We assume that there exists a doubly homoclinic connection associated to a saddle-focus singularity at the origin $(0,0,0)$ with eigenvalues $s \pm i\omega$, $s < 0$, $\omega \neq 0$, $\lambda > 0$. We also assume that the saddle value $s+\lambda < 0$ and $F$ possesses symmetry under the change of sign, $F(X) = -F(-X)$. Here, note that while the stable manifold $W^s(0)$ is two-dimensional, the unstable manifold $W^u(0)$ is one-dimensional. The global unstable manifold $W^u(0)$ consists of the homoclinic loops and is contained in $W^s(0)$ (see Figure 1). Note also that in case $s + \lambda < 0$ stable periodic orbits bifurcate from the homoclinic loop as described by L. P. Šil'nikov in reference [4].

Furthermore, we derive expressions for a Poincaré first return map defined by the tracjectories close to the homoclinic loop $\Lambda$. For the sake of simplicity, we assume that the vector field is linear (i.e. $F_1 = F_2 = F_3 = 0$) in a neighborhood of $(0,0,0)$. First, in a neighborhood of $(0,0,0)$ we introduce a cross section $\Sigma_0$ that is transversal to $\Lambda$ and has a nonzero projection to the unstable direction. The second property is an automatic consequence of the first in three dimensions. The stable manifold $W^s_{loc}$ splits $\Sigma_0$ into the upper and lower components $\Sigma_0^+$ and $\Sigma_0^-$ respectively, and the homoclinic loop intersects $\Sigma_0$ at some point $p = (\xi, 0, 0) \in \Lambda \cap \Sigma_0$ on $W^s_{loc}$. We next introduce two cross-sections $\Sigma_1^\mp$ transversal to $W^u_{loc}$. Using the trajectories which travel from $\Sigma_0^+$ to $\Sigma_1^+$ we aim at computing local maps $G_0^+ : \Sigma_0^+ \to \Sigma_1^+$ and $G_0^- : \Sigma_0^- \to \Sigma_1^-$. These local maps assosiate to each point $p \in \Sigma_0$ the first intersection with $\Sigma_1$ of the trajectory which starts at $p$. Thus, a local map $G_0$ is defined by the flow on subsets $\Sigma_0^{\mp}$ of $\Sigma_0$. Note that since the upper and lower homoclinic orbit of the system have analogous behavior, we shall continue with one (the upper homoclinic loop) of them. For simplification we assume that there exist $\xi > 0$, $\zeta > 0$ such that $\Sigma_0^+ \subset \{(\xi, y, z) : (y, z) \in \mathbb{R}^2\}$ and $\Sigma_1^+ \subset \{(x, y, \zeta) : (x, y) \in \mathbb{R}^2\}$.

The solution $(x(t), y(t), z(t))$ of (1.1.1), which starts from a point $(x_0 = \xi, 0, z_0) \in \Sigma_0^+$ close to the origin at the time $t = 0$ and ends up at the point $(x_1, y_1, z_1 = \zeta) \in \Sigma_1^+$ at the time $t = \tau$, is written (taking into account only the linear terms in (1.1.1)) as follows:

$$\begin{aligned} (x(t), y(t)) &= e^{st}\begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ z(t) &= z_0 e^{\lambda t}. \end{aligned} \tag{1.1.2}$$

The flight time $\tau$ that the trajectory takes from $\Sigma_0^+$ to $\Sigma_1^+$ is given by $\tau = \dfrac{1}{\lambda}\ln\left(\dfrac{\zeta}{z_0}\right)$. Substituting $\tau$ and $\xi$ into formula (1.1.2), we get the following expression for the local map $G_0^+$, in complex notation:

$$x_1 + iy_1 = e^{(s+i\omega)\tau(z_0)}(x_0 + iy_0) = e^{(s+i\omega)\tau(z_0)}(\xi + iy_0). \tag{1.1.3}$$

On the other hand, due to the existence of the homoclinic connection and its transversal intersection with $\Sigma_0^+$ and $\Sigma_1^+$, we also have a Poincaré type map

$$G_1^+ : \Sigma_1^+ \to \Sigma_0$$

Hence, for $(x_1, y_1, z_1 = \zeta) \in \Sigma_1^+$ we have $G_1^+(x_1, y_1, z_1 = \zeta) = (\xi, y_2, z_2) \in \Sigma_0$. With $DG_1^+(0, 0, \zeta)$ represented by the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial y_1} \\ \dfrac{\partial z_2}{\partial x_1} & \dfrac{\partial z_2}{\partial y_1} \end{pmatrix}(0, 0), \text{ we have}$$

$$\begin{pmatrix} y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + h.o.t. \text{ (higher order terms).}$$

In this approximation, we obtain for the composite map

$$\left(G_1^+ \circ G_0^+\right) : (\xi, y_0, z_0) \to (\xi, y_2, z_2),$$

$$\begin{pmatrix} y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \gamma x_1 + \delta y_1 \end{pmatrix},$$

and finally we get

$$z_2 = \gamma x_1 + \delta y_1. \tag{1.1.4}$$

Substituting the value of $\tau(z_0)$, $x_1$ and $y_1$ in (1.1.4), in particular for $y_0 = 0$, one obtains

$$z_2 = \xi e^{s\tau(z_0)}\left[\gamma\cos + \delta\sin\right](\omega\tau(z_0)) \tag{1.1.5}$$

Setting $c := \xi\sqrt{\gamma^2+\delta^2}$ and choosing $\varphi$ with $\dfrac{\gamma}{\sqrt{\gamma^2+\delta^2}} = \sin\varphi$, $\dfrac{\delta}{\sqrt{\gamma^2+\delta^2}} = \cos\varphi$ in (1.1.5), we finally have

$$\begin{aligned} z_2 &= \xi e^{s\tau(z_0)}\sqrt{\gamma^2+\delta^2}\left[\sin(\varphi)\cos(w\tau(z_0))+\cos(\varphi)\sin(w\tau(z_0))\right] \\ &= c\left(\frac{\zeta}{z_0}\right)^{\dfrac{s}{\lambda}}\left[\sin\left(\frac{\omega}{\lambda}\left(\ln\frac{\zeta}{z_0}\right)+\varphi\right)\right]. \end{aligned} \tag{1.1.5}$$

Hence, the $z-$component after one return is approximately given by

$$z_0 \to z_2 = c\left(\frac{\zeta}{z_0}\right)^{\dfrac{s}{\lambda}}\left[\sin\left(\frac{\omega}{\lambda}\left(\ln\frac{\zeta}{z_0}\right)+\varphi\right)\right].$$

Note that $\dfrac{s}{\lambda} < -1$ so $\mu := -\dfrac{s}{\lambda} > 1$, and with $x := \dfrac{z_0}{\zeta}$ we can rewrite the last equation as

$$z_2 = cx^{\mu}\left[\sin\left(-\frac{\omega}{\lambda}(\ln x)+\varphi\right)\right].$$

This motivates the study of the one-dimensional map $f_{\omega,\mu} : [-1,1] \to [-1,1]$ given by the following simpler expression

$$f_{\mu,\omega}(x) = \begin{cases} x^{\mu}\sin(\omega\ln(x)), & x>0 \\ 0 & x=0 \\ -f(-x) & x<0 \end{cases}$$

where we use $x$ instead of $z$ from now on. Here, note that odd continuation in the definition of $f_{\mu,\omega}$ is motivated by the corresponding symmetry of vector field. The above process shows how to arrive at this map starting from homoclinic orbits, which is studied by P. Holmes [2, *p.* 388], or J. Guckenheimer/P. Holmes [1, *p.* 320]. The maps of this kind (see Figure 2) were also studied by M.J. Pasifico, A. Rovella and M. Vianna [3], but for $\mu < 1$ which has expansion properties of $f_{\mu,\omega}$ as a consequence. Briefly, they proved that a family of one dimensional maps with infinitely many critical points exhibit global chaotic behavior in a persistent way: For a positive Lebesgue measure set of values $\mu$, the map $f$ has positive Lyapunov exponent at every critical value and at Lebesgue almost all points in its domain; morover, $f$ is topologically transitive, i.e. has dense orbits [3].

After giving some preparatory calculations for the following chapters, we are going to study the orbit $f^n_{\omega,\mu}(x) = f^n(x)$; $n = 1,2,3,...$ of a typical point $x \in (0,1)$. If $f^n(x) = 0$ for some $n < \infty$, then it is clear that all $(f^j(x))_{j\geq n}$ will equal to 0. To orbits of $f$ we can associate symbol sequences

$(s_j) = (\text{sign} f^j(x))_{j\geq 0} = (+1,+1,-1,...)$. $f^n(x) = 0$ implies that $s_n = 0$, then $s_k = 0$ for all $k \geq n$. Here $+1,-1$ and 0 correspond to the upper, to the lower homoclinic branch or to the stable manifold $W^s(0)$ in terms of the original motivation. Consequently, the following questions arise:

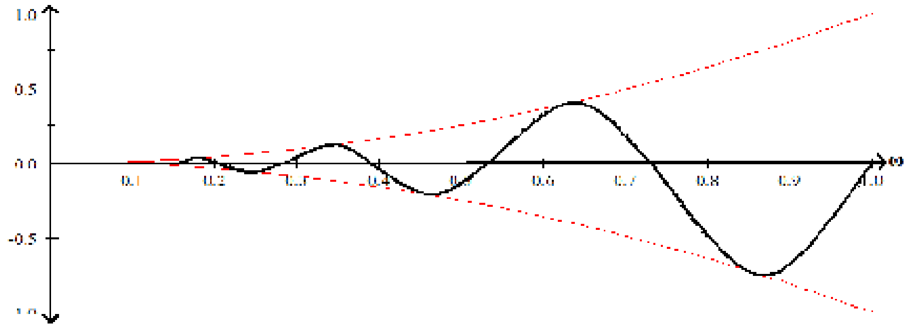


Figure 2: Graph of $f$ for $\mu = 2$, $\omega = 10$

1. Are all symbol sequences possible or not?
2. Does the symbol sequence change in every interval? (Is there chaotic motion?)
3. Is it possible to construct open intervals where the symbol sequence does not change?

In the fifth chapter, we shall also consider symbol sequences different from $(\text{sign} f^j(x))$, describing whether $f^n(x)$ is to the left or to the right hand side of maximum points of $f$.

## 2. Formulas for the derivatives of $f_{\mu,\omega}$, for $\mu > 2$, $\omega > 0$.

**Lemma 2.1.** Assume $\mu \in (2,\infty)$, $\omega > 0$. Set $\varphi_j := \arctan\left(\frac{\omega}{\mu+1-j}\right) \in \left(0,\frac{\pi}{2}\right)$ and

$$g_{\omega,\mu+1-j} := \sqrt{(\mu+1-j)^2+\omega^2}$$

for $j \in \{1,2,3\}$. Consider the map

$$f_{\mu,\omega}(x) = \begin{cases} x^\mu \sin(\omega \ln(x)), & x > 0 \\ 0 & x = 0 \\ -f(-x) & x < 0 \end{cases}.$$

Then, the following formulas hold for $x \in \mathbb{R}$:

**(1).**

$$f'_{\mu,\omega}(x) = g_{\omega,\mu} \cdot f_{\mu-1,\omega,\varphi_1}(x), \tag{2.1.1}$$

$$\cos(\varphi_1) = \frac{\mu}{\sqrt{\mu^2+\omega^2}} = \frac{\mu}{g_{\omega,\mu}}, \tag{2.1.2}$$

$$\sin(\varphi_1) = \frac{\omega}{\sqrt{\mu^2+\omega^2}} = \frac{\omega}{g_{\omega,\mu}}. \tag{2.1.3}$$

**(2).**

$$f''_{\mu,\omega}(x) = g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot f_{\mu-2,\omega,\varphi_1+\varphi_2}(x). \tag{2.1.4}$$

**(3).**

$$f'''_{\mu,\omega}(x) = g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu-2} \cdot f_{\mu-3,\omega,\varphi_1+\varphi_2+\varphi_3}(x). \tag{2.1.5}$$

**Proof.**

**(1)**. From the definiton of $\varphi_j$, we have $\varphi_1 = \arctan\left(\frac{\omega}{\mu}\right)$, and also from the definition of $g_{\omega,\mu+1-j}$, we have $g_{\omega,\mu} = \sqrt{\mu^2+\omega^2}$. It follows that

$$\cos(\varphi_1) = \frac{\mu}{g_{\omega,\mu}} \text{ and } \sin(\varphi_1) = \frac{\omega}{g_{\omega,\mu}}.$$

This proves (2.1.2) and (2.1.3). Now, it is convenient to define a class of functions

$$f_{\mu,\omega,\varphi}(x) := x^\mu \sin(\omega \ln(x) + \varphi)$$

which is slightly more general than $f_{\mu,\omega}(x) = x^\mu \sin(\omega \ln(x))$. For $x > 0$, we have

$$\begin{aligned} f'_{\mu,\omega,\varphi}(x) &= x^\mu \cos(\omega \ln(x) + \varphi)\left(\frac{1}{x}\omega\right) + x^{\mu-1}\mu \sin(\omega \ln(x) + \varphi) \\ &= x^{\mu-1}\left(\mu \sin(\omega \ln(x) + \varphi) + \omega \cos(\omega \ln(x) + \varphi)\right). \end{aligned}$$

By multiplying and dividing the last equation with $g_{\omega,\mu}$, we have

$$f'_{\mu,\omega,\varphi}(x) = g_{\omega,\mu} \cdot x^{\mu-1} \left( \frac{\mu}{g_{\omega,\mu}} \sin\left(\omega \ln(x) + \varphi\right) + \frac{\omega}{g_{\omega,\mu}} \cos\left(\omega \ln(x) + \varphi\right) \right). \tag{2.1.6}$$

Putting (2.1.2) and (2.1.3) in (2.1.6), we finally obtain

$$\begin{aligned} f'_{\mu,\omega,\varphi}(x) &= g_{\omega,\mu} \cdot x^{\mu-1} \left(\cos(\varphi_1) \cdot \sin(\omega \ln(x) + \varphi) + \sin(\varphi_1) \cdot \cos(\omega \ln(x) + \varphi)\right) \\ &= g_{\omega,\mu} \cdot x^{\mu-1} \sin(\omega \ln(x) + \varphi + \varphi_1) \\ &= g_{\omega,\mu} \cdot f_{\mu-1,\omega,\varphi+\varphi_1}(x). \end{aligned} \tag{2.1.7}$$

**(2)**. Further, using (2.1.7) with $\varphi + \varphi_1$ instead of $\varphi$, and $\mu - 1$ instead of $\mu$, we see that

$$\begin{aligned} f''_{\mu,\omega}(x) &= f''_{\mu,\omega,0}(x) = \left(g_{\omega,\mu} \cdot f_{\mu-1,\omega,\varphi+\varphi_1}\right)'(x) \\ &= g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot f_{\mu-2,\omega,\varphi_1+\varphi_2}(x), \end{aligned}$$

which proves (2.1.4).

**(3).** Using (2.1.7) we obtain (2.1.5) analogously. ■

**Lemma 2.2.** Let $\mu > 2$ and $\omega > 0$ be given. Define $q := e^{-\frac{\pi}{\omega}}$ and $\varphi_j$ as in Lemma 2.1. Then, the following properties are satisfied in $(0,1]$:

**(1).** $f_{\mu,\omega}$ has the zero points

$$q^k = e^{-\frac{k\pi}{\omega}}, \tag{2.2.1}$$

$(k \in \mathbb{N})$ and

$$f'_{\mu,\omega}(q^k) = (-1)^k \omega q^{k(\mu-1)}. \tag{2.2.2}$$

**(2).** $f_{\mu,\omega}$ has the extremal points

$$m_k = q^k e^{\frac{-\varphi_1}{\omega}} \tag{2.2.3}$$

and

$$f_{\mu,\omega}(m_k) = (-1)^{k+1} \cdot \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \cdot \sin(\varphi_1). \tag{2.2.4}$$

**(3).** If $\mu$ is an even integer, and $\beta \in \mathbb{N}$ is odd and $l(k) := k\mu + \beta$, then $f_{\mu,\omega}$ has a maximum at

$$m_{l(k)} = q^{l(k)} e^{\frac{-\varphi_1}{\omega}}. \tag{2.2.5}$$

**Proof.**

**(1)**. We first find the zeros of $f_{\mu,\omega}$. For $x \in (0,1)$ one has

$$\sin(\omega \ln(x)) = 0 \Leftrightarrow \exists k \in \mathbb{N}\ \omega \ln(x) = -k\pi \Leftrightarrow \exists k \in \mathbb{N}\ \ln(x) = \frac{-k\pi}{\omega},$$

and hence $x = e^{\frac{-k\pi}{\omega}}$. With $q = e^{-\frac{\pi}{\omega}}$, the zeros of $f_{\mu,\omega}$ in $(0, q]$ are given by $x = e^{\frac{-k\pi}{\omega}} = q^k$. Therefore, by inserting $q^k$ in (2.1.1), we have

$$\begin{aligned} f'_{\mu,\omega}(q^k) &= q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \sin(\omega \left(\ln q^k\right) + \varphi_1) \\ &= q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \sin(\omega \left(\ln e^{\frac{-k\pi}{\omega}}\right) + \varphi_1) \\ &= (-1)^k \, q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \sin(\varphi_1) \end{aligned}$$

Using (2.1.3) we obtain

$$\begin{aligned} f'_{\mu,\omega}(q^k) &= (-1)^k \, q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \frac{\omega}{g_{\omega,\mu}} \\ &= (-1)^k \, \omega q^{k(\mu-1)}. \end{aligned}$$

Hence, assertion **(1)** is proved.

**(2)**. Let $k \in \mathbb{N}$. We find the extremum points of $f_{\mu,\omega}$ in the interval $I_k = \left[q^{k+1}, q^k\right]$ by solving $f'_{\mu,\omega}(x) = 0$ for $x \in I_k$. Since $x > 0$, $x^{\mu-1} \neq 0$. So, we have

$$\sin(\omega \left(\ln x\right) + \varphi_1) = 0,$$

and hence $x = e^{\frac{-k\pi-\varphi_1}{\omega}}$. The last expression equals to $q^k e^{\frac{-\varphi_1}{\omega}} = m_k$, which proves (2.2.3). Furthermore, for the extremum point $m_k$ of $f_{\mu,\omega}$ in the interval $\left(q^{k+1}, q^k\right)$ we have

$$\begin{aligned} f_{\mu,\omega}\left(m_k\right) &= m_k^{\mu} \sin(\omega \ln(m_k)) \\ &= \left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^{\mu} \sin(\omega \ln \left(q^k e^{-\frac{\varphi_1}{\omega}}\right)) \\ &= \left(e^{-\frac{k\pi}{\omega}} e^{-\frac{\varphi_1}{\omega}}\right)^{\mu} \sin(\omega \ln \left(e^{-\frac{k\pi}{\omega}} e^{-\frac{\varphi_1}{\omega}}\right)) \\ &= \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \sin(\omega \frac{-k\pi - \varphi_1}{\omega}) \\ &= (-1)^{k+1} \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \sin(\varphi_1). \end{aligned}$$

**(3)**. Substituting $l\left(k\right)$ instead of $k$ in (2.2.4), we have

$$\begin{aligned} f_{\mu,\omega}\left(m_{l(k)}\right) &= (-1)^{l(k)+1} \exp\left(-\frac{l\left(k\right)\pi\mu + \varphi_1\mu}{\omega}\right) \sin(\varphi_1) \\ &= \exp\left(-\frac{l\left(k\right)\pi\mu + \varphi_1\mu}{\omega}\right) (-1)^{k\mu+\beta+1} \sin(\varphi_1) \end{aligned}$$

Therefore, it is clear that $f_{\mu,\omega}\left(m_{l(k)}\right) > 0$ (and hence $f_{\mu,\omega}$ has a maximum at $m_{l(k)}$), if $\mu$ is even and $\beta$ is odd. ■

We shall frequently use the simple lemma below.

**Lemma 2.3.** Assume $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$. If $|f'| \geq c$, or $|f'| \leq d$ ($c$ and $d$ are constant), then we have

$$c\left|b - a\right| \leq \left|f\left(b\right) - f\left(a\right)\right| \leq d\left|b - a\right| \; . \tag{2.3.1}$$

**Proof.** (Follows from the mean value theorem.)

## 3. The behavior of orbits remaining in some 'flat' intervals

In this part we find some parameters $\mu$ and $\omega$ such that $f_{\mu,\omega}$ maps some extremal points $m_k$ to some other extremal points $m_{\ell(k)}$ (see Figure 3). Then, we construct some open intervals $U_k$ around $m_k$ and orbits of $f_{\mu,\omega} = f$ which are entirely contained in $\bigcup_{k\in\mathbb{N}} U_k$.

**Theorem 3.1.** For $k \in \mathbb{N}$, $\omega > 0$, and even integer $\mu > 2$, define

$$\eta := \min \left\{ \frac{q}{g_{\omega,\mu} \cdot g_{\omega,\mu-1}}, \frac{e^{-\frac{\varphi_1}{\omega}} - q}{2}, \frac{1 - e^{-\frac{\varphi_1}{\omega}}}{2} \right\}, \tag{3.1.1}$$

and set $\ell(k) := k\mu + 1$ (which corresponds to $\beta = 1$ in assertion (3) of Lemma 2.2), $\delta_k := \eta q^k$ , $\delta_{\ell(k)} := \eta q^{\ell(k)}$. Then, for every large even integer enough $\mu$ there exists a corresponding $\omega$ such that the following properties are satisfied:

**(1).** With the intervals $U_k = (m_k - \delta_k, m_k + \delta_k)$ one has $f(U_k) \subset U_{\ell(k)}$ and

$$\forall k \in \mathbb{N} : f^{-1}(\{0\}) \cap U_k = \emptyset.$$

**(2).** If $k$ is odd, then for $x \in U_k$, the orbits $(f^j(x))_{j\in\mathbb{N}}$ all have the symbol sequence $(s_j) = (\text{sign} f^j(x))_{j\in\mathbb{N}} = (+1, +1, +1, ...)$.

**(3).** The set

$$Z = \{x \mid \exists n \in \mathbb{N} : f^n(x) = 0\} \tag{3.1.2}$$

is disjoint to $\bigcup_k U_k$ and, in particular, is not dense in $[-1, 1]$.

The proof is divided into several lemmas.

**Lemma 3.2.** Let $k \in \mathbb{N}$ and define $\varphi_1$ as in Lemma 2.1. Define $\eta$ and $\delta_k$ as in Theorem 3.1, and

$$\overline{\eta} := \min \left\{ \frac{e^{-\frac{\varphi_1}{\omega}} - q}{2}, \frac{1 - e^{-\frac{\varphi_1}{\omega}}}{2} \right\}.$$

Then we have

$$(m_k - \delta_k, m_k + \delta_k) \subset [m_k - \delta_k, m_k + \delta_k] \subset \left[m_k - \overline{\eta} q^k, m_k - \overline{\eta} q^k\right] \subset \left(q^{k+1}, q^k\right).$$

**Proof.** From (3.1.1) we have $\eta \leq \overline{\eta}$. Multiplying both sides with $q^k$, and using (2.2.3) we have

$$\delta_k \leq \overline{\eta} q^k = \min \left\{ \frac{q^k e^{-\frac{\varphi_1}{\omega}} - q^{k+1}}{2}, \frac{q^k - q^k e^{-\frac{\varphi_1}{\omega}}}{2} \right\} = \min \left\{ \frac{m_k - q^{k+1}}{2}, \frac{q^k - m_k}{2} \right\},$$

it follows that $(m_k - \delta_k, m_k + \delta_k) \subset [m_k - \delta_k, m_k + \delta_k] \subset \left[m_k - \overline{\eta} q^k, m_k - \overline{\eta} q^k\right] \subset \left(q^{k+1}, q^k\right)$.
■

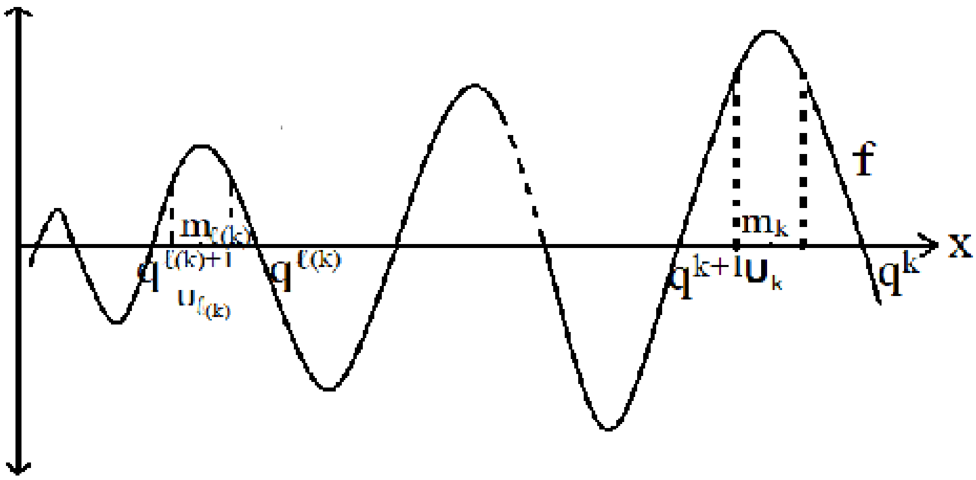


Figure 3: $f\,(m_k) = m_{\ell(k)}$

**Lemma 3.3.** Define $\varphi_1$ as in Lemma 2.1, and define $\ell\,(k)$ as in Theorem 3.1. Then the following statements are true.

**(a)** For every even integer $\mu \geq 32$, there exists $\omega \in (0,1)$ such that for all $k \in \mathbb{N}$ $f$ has the property

$$f\,(m_k) = m_{\ell(k)}.$$

**(b)** For any choice of $\omega$ as in assertion **(a)**, one has $\omega \to 0$ as $\mu \to \infty$.

**Proof.**
**(a)** From (2.2.4) we have for all $k \in \mathbb{N}$

$$|f\,(m_k)| = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right)\sin(\varphi_1). \tag{3.3.1}$$

On the other hand, from the third assertion of Lemma 2.2 we know that for even $\mu$, $f$ has a maximum at the point

$$m_{\ell(k)} = \exp\left(-\frac{\pi\ell\,(k) + \varphi_1}{\omega}\right). \tag{3.3.2}$$

Using (2.1.3), (3.3.1) and (3.3.2), we obtain the following equivalences:

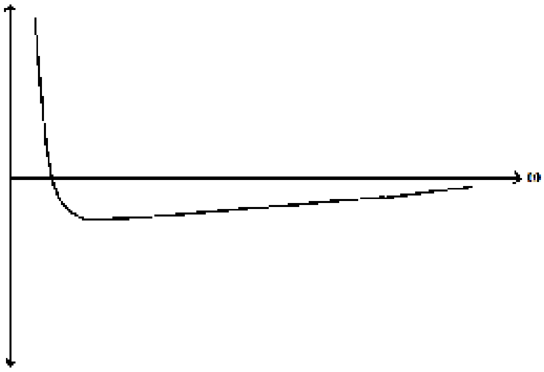


Figure 4: Graph of $F(\omega,\mu)$, for $\mu = 32$

$$\begin{aligned}
m_{\ell(k)} &= f(m_k) \\
&\Leftrightarrow \exp\left(-\frac{\pi\ell(k)+\varphi_1}{\omega}\right) = \exp\left(-\frac{k\pi\mu+\varphi_1\mu}{\omega}\right)\cdot\sin(\varphi_1) \\
&\Leftrightarrow \exp\left(-\frac{\pi\ell(k)+\varphi_1}{\omega}\right) = \exp\left(-\frac{k\pi\mu+\varphi_1\mu}{\omega}\right)\cdot\frac{1}{\sqrt{1+\frac{\mu^2}{\omega^2}}} \\
&\Leftrightarrow \exp\left(\frac{-\pi}{\omega}\left[k\mu-\ell(k)\right]+\frac{\varphi_1(1-\mu)}{\omega}\right) = \sqrt{1+\frac{\mu^2}{\omega^2}}\ . \qquad (3.3.3)
\end{aligned}$$

Substituting $\ell(k) = k\mu + 1$ in (3.3.3), we have

$$\exp\left(\frac{\pi+\varphi_1(1-\mu)}{\omega}\right) = \sqrt{1+\frac{\mu^2}{\omega^2}}$$

or, using the definition of $\varphi_1$,

$$\exp\left(\frac{\pi-(\mu-1)\arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right) = \sqrt{1+\frac{\mu^2}{\omega^2}}\ . \qquad (3.3.4)$$

In view of (3.3.4), we define

$$F(\omega,\mu) = \exp\left(\frac{\pi-(\mu-1)\arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right) - \sqrt{1+\frac{\mu^2}{\omega^2}}. \qquad (3.3.5)$$

We try to find $(\omega, \mu)$ with $F(\omega, \mu) = 0$ (see Figure 4). Noting that for fixed $\mu$, $\lim_{\omega \to 0} F(\omega, \mu) = +\infty$, it is enough to find at least one pair $(\omega, \mu)$ with $F(\omega, \mu) < 0$. For $\omega = 1$, we have

$$\begin{aligned} F(1, \mu) &= \exp\left(\pi - (\mu - 1)\arctan\left(\frac{1}{\mu}\right)\right) - \sqrt{1 + \mu^2} \\ &= \exp\left(\pi - \mu \arctan\left(\frac{1}{\mu}\right) + \arctan\left(\frac{1}{\mu}\right)\right) - \sqrt{1 + \mu^2}. \end{aligned} \tag{3.3.6}$$

Since $\arctan'(x) = \dfrac{1}{1 + x^2}$, we have $\arctan'(x) \geq \dfrac{1}{2}$ for $|x| \leq 1$. Hence, (2.3.1) shows $\arctan(x) \geq \dfrac{1}{2}x$ for $x \in [0, 1]$. It follows that for $\mu > 1$,

$$\mu \arctan\left(\frac{1}{\mu}\right) \geq \frac{1}{2}. \tag{3.3.7}$$

Using (3.3.7) and $\arctan\left(\dfrac{1}{\mu}\right) < \dfrac{\pi}{4}$ for $\mu > 1$ in (3.3.6), we have

$$F(1, \mu) \leq \exp\left(\pi - \frac{1}{2} + \frac{\pi}{4}\right) - \sqrt{1 + \mu^2} = \exp\left(\frac{5\pi}{4} - \frac{1}{2}\right) - \sqrt{1 + \mu^2}.$$

From the fact that $\exp\left(\dfrac{5\pi}{4} - \dfrac{1}{2}\right) < 32$, we have $F(\omega, \mu) < 0$, if we set $\omega = 1$ and $\mu \geq 32$. With the intermediate value theorem, it is trivial that $F(\omega, \mu)$ has at least one zero point $\omega \in (0, 1)$. It follows that (3.3.5) is satisfied with this $\omega$ depending on the even integer $\mu \geq 32$. Hence, the proof of assertion **(a)** is completed.

**(b)** Consider a sequence $\mu_k$, $\mu_k \to \infty$ with corresponding $\omega_k \in (0, 1)$ such that $F(\mu_k, \omega_k) = 0$. Then $\sqrt{1 + \dfrac{\mu_k^2}{\omega_k^2}} \to \infty$. Further, $(\mu_k - 1)\arctan\left(\dfrac{\omega_k}{\mu_k}\right)$ is bounded. The exponential term in (3.3.5) must go to $+\infty$, so $\omega_k \to 0$ necessarily. This completes the proof of **(b)** and the proof of Lemma 3.3. ■

**Remark.** Consider the equality (3.3.3). Because $\mu > 1$, so $\dfrac{\varphi_1(1 - \mu)}{\omega} < 0$, and $\sqrt{1 + \dfrac{\mu^2}{\omega^2}} > 1$, the term $\dfrac{-\pi}{\omega}[k\mu - \ell(k)]$ must be positive, if we have a solution. Accordingly, $\ell(k) > k\mu$ must be satisfied. It means (3.3.4) has no solution for $\ell(k) \leq k\mu$. Thus $\ell(k) \geq k\mu + 1$ necessarily; we made the choice $\ell(k) = k\mu + 1$.

**Numerical observations.** In order to find a numerical solution we use two starting points where $F(\cdot, \mu)$ has opposite signs and at the 9 th step of a bisection method we obtained $\omega = 0.69895$ and $\mu = 24$ as an appropriate $F(\omega, \mu) = 0$. Although one can obtain some other solution points $\omega$, for some other the parameters $\mu$, we numerically found out that there is no solution for $\mu < 3.1$.

**Lemma 3.4.** Choose an even integer $\mu \geq 32$ and $\omega \in (0,1)$ with the properties as in Lemma 3.3. Define $\ell(k)$, $\eta$, $\delta_k$ and $\delta_{\ell(k)}$ as in Theorem 3.1. Then with the intervals $U_k = (m_k - \delta_k, m_k + \delta_k)$, we have $f(U_k) \subset U_{\ell(k)}$.

**Proof.** Let $\mu$ and $\omega$ be as in the assumption of the lemma, and $x \in U_k$. With $\ell(k) = k\mu + 1$ we claim that

$$\left|f(x) - m_{\ell(k)}\right| < \delta_{\ell(k)} = \eta q^{\ell(k)} . \tag{3.4.1}$$

From the second order Taylor expansion, we have

$$f(x) = f(m_k) + f'(m_k)(x - m_k) + \frac{f''(\xi)}{2}(x - m_k)^2 \tag{3.4.2}$$

with $\xi \in (m_k - \delta_k, m_k + \delta_k)$. Since $\mu > 2$, note that we also have

$$q^{(k+1)(\mu-2)} \leq |\xi|^{\mu-2} \leq q^{k(\mu-2)}. \tag{3.4.3}$$

Substituting the equality (3.4.2) in the left hand side of (3.4.1), we get

$$\left|f(x) - m_{\ell(k)}\right| = \left|f(m_k) + f'(m_k)(x - m_k) + f''(\xi)\frac{(x - m_k)^2}{2} - m_{\ell(k)}\right|.$$

From the fact that we now have fixed parameters $\mu$, $\omega$ with the property $f(m_k) = m_{\ell(k)}$ as in Lemma 3.3 and using $f'(m_k) = 0$ and $(x - m_k) < \delta_k$, the last equality gives

$$\left|f(x) - m_{\ell(k)}\right| \leq \left|f''(\xi)\frac{\delta_k^2}{2}\right|.$$

Using (2.1.4) in the last equality, we obtain

$$\left|f(x) - m_{\ell(k)}\right| = \left|g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot \sin(\omega \ln(x) + \varphi_1 + \varphi_2)\, |\xi|^{\mu-2}\frac{\delta_k^2}{2}\right|. \tag{3.4.4}$$

Using the upper estimate of (3.4.3) and substituting the value of $\delta_k$ in (3.4.4), we get

$$\begin{aligned}\left|f(x) - m_{\ell(k)}\right| &\leq \left|g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot \left|q^k\right|^{\mu-2}\frac{\eta^2 q^{2k}}{2}\right| \\ &= \left|g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{k\mu}\frac{\eta\eta}{2}\right|. \end{aligned}\tag{3.4.5}$$

Finally, using the definiton of $\eta$ from (3.1.1) in (3.4.5), we have

$$\begin{aligned}\left|f(x) - m_{\ell(k)}\right| &\leq \left|g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{k\mu}\frac{\eta}{2}\frac{q}{g_{\omega,\mu} \cdot g_{\omega,\mu-1}}\right| \\ &= q^{k\mu+1}\frac{\eta}{2} < \eta q^{k\mu+1} = \eta q^{\ell(k)} = \delta_{\ell(k)}. \ \blacksquare\end{aligned}$$

**Proof of Theorem 3.1.** Choose $\mu$, $\omega$ as in Lemma 3.3, and let $\ell(k)$ be as in Theorem 3.1.

**(1)**. Lemma 3.4 shows $f(U_k) \subset U_{\ell(k)}$ and the definition of $U_{\ell(k)}$ implies $0 \notin U_{\ell(k)}$, so $f^{-1}(\{0\}) \cap U_k = \emptyset$.

**(2).** If $k$ is odd and $\mu$ is as above (therefore even), then all $\ell^j(k)$ $(j \geq 0)$ are odd and all $U_{\ell^j(k)}$ are intervals around maxima of $f$, where $f$ is positive. Hence the assertion is proved.

**(3)**. For $k_0 \in \mathbb{N}$, $x \in U_{k_0}$ and $n \in \mathbb{N}_0$, $f^n(x) \in \bigcup\limits_{k\in\mathbb{N}} U_k$, in particular $f^n(x) \neq 0$, which proves assertion 3. ■

Note that the possible existence of the orbits which remain close to critical points, i.e implying non-density has been mentioned as a remark by Holmes P. J. in the bottom of the page 395 of [2] with only a vague indication of proof. With this section we gave a rigorous proof of that idea.

## 4. Behavior of the map $f_{\mu,\omega}$ in some 'steep' intervals

In this section we first construct some orbits whose points stay entirely in so-called 'steep' intervals, and then analyze the measure of the set of points which have such orbits. In contrast to chapters 3 and 5, where the parameters $\mu$ and $\omega$ are connected by the conditions given in assertion **(a)** of Lemma 3.3 and in (5.2.1), in this chapter both of them can be varied independently.

Consider the interval $(-m_k, -m_{k+1})$ or $(m_{k+1}, m_k)$. From Lemma 2.2 we have

$$\left|f'_{\mu,\omega}(q^{k+1})\right| = \omega\left(q^{k+1}\right)^{(\mu-1)}.$$

Since $f'_{\mu,\omega}(\mp m_k) = f'_{\mu,\omega}(\mp m_{k+1}) = 0$, continuity of $f'_{\mu,\omega}$ implies that we can choose a 'steep' interval $S_k$, either as a subset of $(m_{k+1}, m_k)$ or as a subset of $(-m_k, -m_{k+1})$, on which $\left|f'_{\mu,\omega}\right|$ satisfies a lower estimate. We begin by specifying the boundaries of the 'steep' interval $S_k$ and by giving some new notations.

We use the notation $|I|$ for the length of an interval $I$.

**Definition 4.1.** Let $k \in \mathbb{N}$ and $c \in (0,1)$. Define

$$a_k := \min\left\{x \in \left(m_{k+1}, q^{k+1}\right] : \left|f'_{\mu,\omega}(x)\right| \geq c\omega\left(q^{k+1}\right)^{(\mu-1)} \text{ on } \left[x, q^{k+1}\right]\right\}$$

and

$$b_k := \max\left\{x \in \left[q^{k+1}, m_k\right) : \left|f'_{\mu,\omega}(x)\right| \geq c\omega\left(q^{k+1}\right)^{(\mu-1)} \text{ on } \left[q^{k+1}, x\right]\right\}.$$

Note that $q^{k+2} < a_k < q^{k+1} < b_k < q^k$ (see Figure 5). Given a symbol sequence of the form

$$(s_j) = (+1, -1, +1, +1, +1, -1, ...),$$

where symbols represent the signs of $f^j_{\mu,\omega}(x)$ for some starting value $x$, we construct corresponding orbits of $f_{\mu,\omega}$. Note that in terms of the motivation by the three dimensional vector field, such orbits correspond to solutions converging to the doubly homoclinic loop, and taking turns along the upper and lower homoclinic orbit according to the symbol sequence. For $0 \leq a \leq b$, define

$$\begin{aligned}
[a,b]_{+1} &:= [a,b], \\
[a,b]_{-1} &:= [-b,-a],
\end{aligned}$$

and define 'steep' intervals by

$$S^c_{k,s} := [a_k, b_k]_s = \begin{cases} [a_k, b_k] & , \text{ if } s = +1 \\ [-b_k, -a_k] & , \text{ if } s = -1 \end{cases}.$$

So, we have

$$\left|f'_{\mu,\omega}(x)\right| \geq c\omega\left(q^{k+1}\right)^{(\mu-1)} \text{ for } x \in S^c_{k,s},\ s \in \{\pm 1\},\ k \in \mathbb{N}. \tag{4.1.1}$$

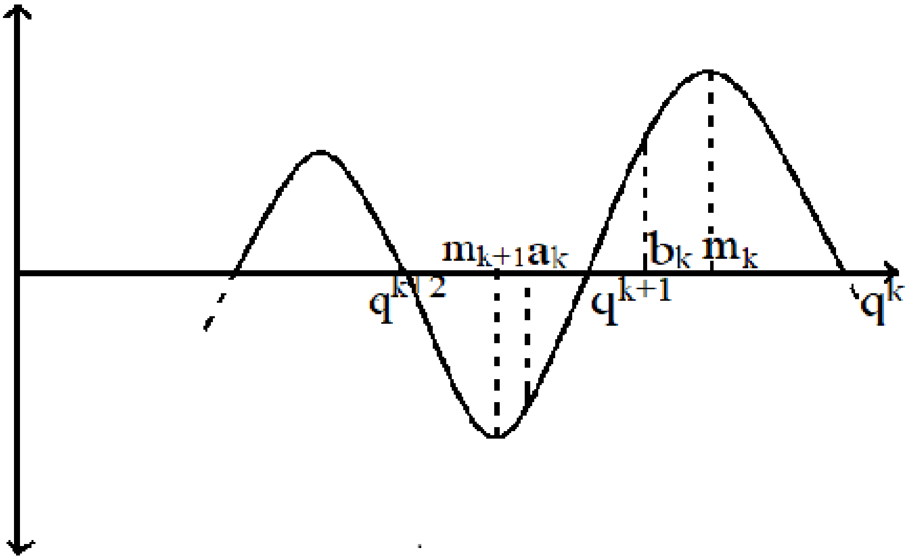


Figure 5: The interval $\left(q^{k+2}, q^{k}\right)$

We also define $S_{k,\pm1} := S^{c}_{k,+1} \cup S^{c}_{k,-1}$ and define the union of all 'steep' intervals by

$$\Psi^{c} = \bigcup_{k\in\mathbb{N}} S_{k,\pm1}.$$

Note that for $s \in \{\pm1\}$, $S^{c}_{k,s} \subset (m_{k+1}, m_{k})_{s}$, and hence

$$\left|S^{c}_{k,s}\right| \leq m_{k} - m_{k+1} = q^{k} e^{-\frac{\varphi_1}{\omega}} (1-q) \tag{4.1.2}$$

Setting $f' := f'_{\mu,\omega}$, we define sets of points with forward orbits which are contained in these 'steep' intervals (see Figure 6). Namely,

$$\Omega^{c}_{n} = \bigcap_{j=0}^{n} f^{-j}\left(\Psi^{c}\right); \ \Omega^{c}_{\infty} = \bigcap_{j=0}^{\infty} f^{-j}\left(\Psi^{c}\right).$$

**Theorem 4.2.** Let $c \in (0,1)$. Assume $\mu > 1$ and define $S^{c}_{k,\pm1}$ and $\Omega^{c}_{\infty}$ as above. Then for $k_0 \in \mathbb{N}$ the following statements are true:

**(1).** For every symbol sequence $\mathbf{s} = (s_0, s_1, s_2, ...)$ there exists a point $y_0 \in \left(S^{c}_{k_0,s_0} \cap \Omega^{c}_{\infty}\right)$ with the property that $\operatorname{sign} f^{j}(y_0)$ is given by $s_j \in \{\pm1\}$, where $j \in \mathbb{N}_0$.

**(2).** Let $\omega > \dfrac{1}{c} + \pi(\mu+1)$. Then with the set $Z$ from (3.1.2) we have $\left(S^{c}_{k_0,s_0} \cap \Omega^{c}_{\infty}\right) \subset \overline{Z}$.

**(3).** Let $c \in \left(\dfrac{2}{\pi}, 1\right)$ and $\omega > \dfrac{c\pi^{2}(2\mu+3)}{2(c\pi-1)}$. Then $\Omega^{c}_{\infty} \subset \overline{Z}$, but $\Omega^{c}_{\infty}$ has Lebesgue measure zero.

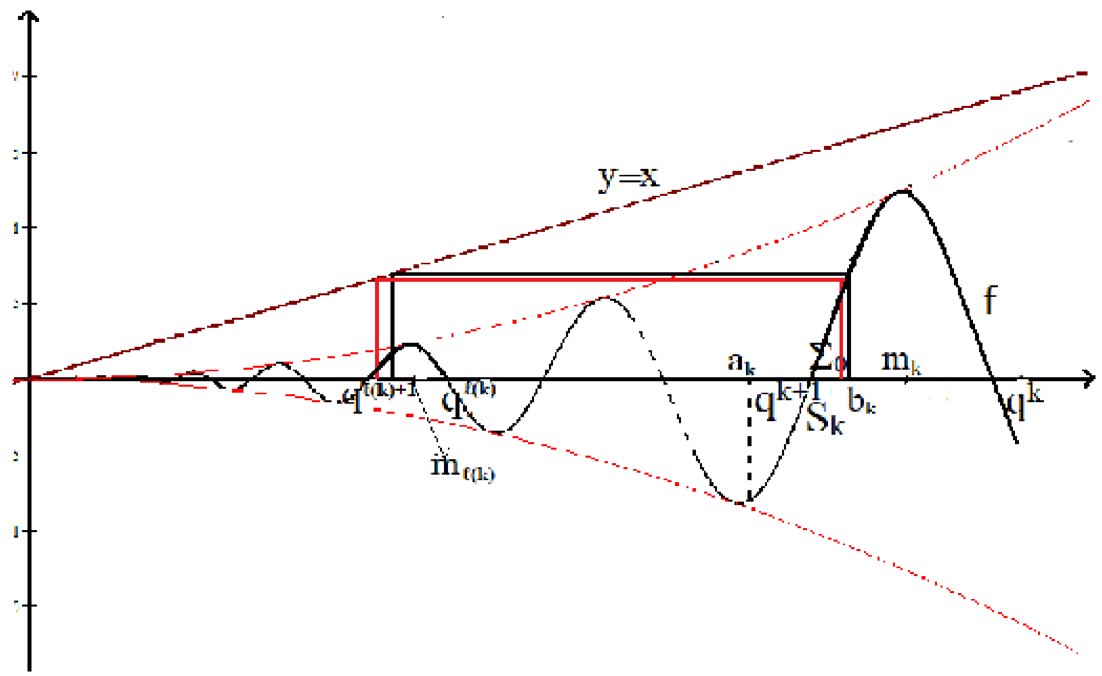


Figure 6: Graphical observations of orbits whose points stay in 'steep' intervals

**Remark.** A similar argument is sketched in the page 395 of [2], with the purpose to show that $Z$ can be dense, but it seems that the method gives density only in a set of measure zero.

The proof starts with the following lemma.

**Lemma 4.3.** Let $k_0 \in \mathbb{N}$, $c \in (0,1)$ and $\mathbf{s} = (s_0, s_1, s_2, \ldots) \in \{+1, -1\}^{\mathbb{N}_0}$ be given. Define $S^c_{k_0,s_j}$ as in the passage given before Theorem 4.2. Then the following statements are true:

**(a)** There exists a point $y_0 \in S^c_{k_0,s_0}$ and a sequence $k_0 < k_1 < k_2 < \ldots$ such that $\forall j \in \mathbb{N}_0$ $f^j(y_0) \in S^c_{k_j,s_j}$, in particular, $y_0 \in \Omega^c_\infty$.

**(b)** Let $y_0 \in \left(S^c_{k_0,s_0} \cap \Omega^c_\infty\right)$ be given and define the sequence $k_0 < k_1 < k_2 < \ldots$ by $f^j(y_0) = y_j \in S^c_{k_j,s_j}$ $(j \in \mathbb{N}_0)$. Then there exists a sequence $(\Sigma_j)$ of intervals in $S^c_{k_0,s_0}$ with $\Sigma_j \supset \Sigma_{j+1} \ni y_0$, $\left(f^j\right)' \neq 0$ on $\Sigma_j$ and

$$f^j(\Sigma_j) = \left[q^{k_j+1}, y_j\right]_{s_j} = \begin{cases} \left[q^{k_j+1}, y_j\right], & \text{if } s_j = +1 \\ \left[y_j, -q^{k_j+1}\right], & \text{if } s_j = -1 \end{cases} \subset S^c_{k_j,s_j} \text{ for } j \in \mathbb{N}_0, \tag{4.3.1}$$

in particular, $Z \cap \Sigma_j \neq \emptyset$ for all $j \in \mathbb{N}_0$.

**(c)** For $y_0 \in \left(S^c_{k_0,s_0} \cap \Omega^c_\infty\right)$ and $k_0$, $k_1$, $k_2$,... as in assertion **(b)** and all $j \in \mathbb{N}$ we have

$$\left|\left(f^j\right)'(y_0)\right| \geq (c\omega)^j \left(\prod_{n=0}^{j-1} q^{k_n+1}\right)^{\mu-1}. \tag{4.3.2}$$

**(d)** Let $y_0$ and the sequence $k_0 < k_1 < k_2 < ...$ be as in **(b)**. Then

$$\forall j \in \mathbb{N} : \; q^{k_j \mu} \geq q^{k_{j+1}+2}. \tag{4.3.3}$$

**(e)** Let $\omega > \dfrac{1}{c} + \pi(\mu+1)$. Let $y_0$ and the associated $\Sigma_j$ be as in assertion **(b)** and $\varphi_1$ be as in Lemma 2.1. Then $|\Sigma_j| \leq \dfrac{q^{k_0} e^{-\frac{\varphi_1}{\omega}}(1-q)}{(c\omega q^{\mu+1})^j}$ and $c\omega q^{\mu+1} > 1$; in particular, $|\Sigma_j| \to 0$, as $j \to \infty$.

**Proof.**

**(a)** Let $k_0 \in \mathbb{N}$ and $\mathbf{s} = (s_0, s_1, s_2, ...)$ be given. For $S^c_{k_0,s_0} = [a_{k_0}, b_{k_0}]_{s_0}$ it is clear that $f\left(S^c_{k_0,s_0}\right)$ is an interval which contains 0 in its interior, and since $a_k \to 0$, $b_k \to 0$ as $k \to \infty$, there exists $k_1 > k_0$ with $S^c_{k_1,s_1} \subset f\left(S^c_{k_0,s_0}\right)$. Further $f\mid_{S^c_{k_0,s_0}}$ is injective, and we set

$$J_1 := \left(f\mid_{S^c_{k_0,s_0}}\right)^{-1}\left(S^c_{k_1,s_1}\right) \; .$$

($f$ maps $J_1$ bijectively onto $S^c_{k_1,s_1}$.) Similarly, there exists $k_2 > k_1$ with $S^c_{k_2,s_2} \subset f\left(S^c_{k_1,s_1}\right)$, and a closed subinterval $J_2 \subset J_1$ such that $f^2\mid_{J_2}: J_2 \to S^c_{k_2,s_2}$ is bijective. Thus, we obtain a nested sequence

$$J_1 \supset J_2 \supset J_3 \supset ...$$

of closed intervals and sequence of numbers

$$k_0 < k_1 < k_2 < ...$$

with the property that $f^j(J_j) = S^c_{k_j,s_j}$, $j = 1, 2, 3, ....$ Furthermore, the intersection of nested closed intervals $\bigcap\limits_{j\in\mathbb{N}} J_j$ is not empty. It means that there exists a point $y_0 \in \bigcap\limits_{j\in\mathbb{N}} J_j$ which follows the symbol sequence $\mathbf{s}$, and this result completes the proof of assertion **(a)**.

**(b)** For the proof of this assertion we use a recursive construction. Define

$$\Sigma_0 := \left[q^{k_0+1}, y_0\right]_{s_0} = \begin{cases} \left[q^{k_0+1}, y_0\right], & \text{if } s_0 = +1 \\ \left[y_0, -q^{k_0+1}\right], & \text{if } s_0 = -1 \end{cases} \subset S^c_{k_0,s_0}.$$

Then $y_0 \in \Sigma_0$, and the definition of $S^c_{k_0,s_0}$ implies $f' \neq 0$ on $\Sigma_0$, so (4.3.1) holds for $j = 0$. Assume $\Sigma_j$ with the properties in (4.3.1) is constructed and we want to construct $\Sigma_{j+1} \subset \Sigma_j$ such that (4.3.1) is also satisfied for $j+1$. We have, observing that $\mathrm{sign}(y_j) = s_j$,

$$f\left(\left[q^{k_j+1}, y_j\right]_{s_j}\right) = \left[0, f(y_j)\right]_{s_{j+1}} = \left[0, y_{j+1}\right]_{s_{j+1}},$$

and $f^j\mid_{\Sigma_j}$ as well as $f\mid_{\left[q^{k_j+1}, y_j\right]_{s_j}}$ are invertible. Hence, we can define

$$\Sigma_{j+1} = \left(f^{-j}\mid_{\Sigma_j}\right)^{-1}\left(f\mid_{\left[q^{k_j+1}, y_j\right]_{s_j}}\right)^{-1}\left(\left[q^{k_{j+1}+1}, y_{j+1}\right]_{s_{j+1}}\right).$$

Then $y_0 \in \Sigma_{j+1} \subset \Sigma_j$, the chain rule shows $(f^{j+1})' \neq 0$ on $\Sigma_{j+1}$, and $(f^{j+1})(\Sigma_{j+1}) = \left[q^{k_{j+1}+1}, y_{j+1}\right]_{s_{j+1}} \subset S^c_{k_{j+1},s_{j+1}}$. Hence, the recursive construction is completed.

Note also that for $j \in \mathbb{N}$, $\Sigma_j$ contains a point $x_j$ with $f^j(x_j) = q^{k_j+1}$, so $f^{j+1}(x_j) = f\left(q^{k_j+1}\right) = 0$, hence $x_j \in \Sigma_j \cap Z$.

**(c)** By the chain rule the derivative $(f^j)'$ at $y_0 \in \bigcap\limits_{j\in\mathbb{N}} \Sigma_j$ can be calculated as the product of the derivatives of $f$ along the orbit

$$\left|\left(f^j\right)'(y_0)\right| = \left|f'(y_0) \cdot f'(y_1) \cdot \ldots \cdot f'(y_{j-2}) \cdot f'(y_{j-1})\right| = \prod_{n=0}^{j-1} \left|f'(y_n)\right|.$$

Using (4.1.1) for each derivative in the last equality, we have

$$\begin{aligned}\left|\left(f^j\right)'(y_0)\right| &= \prod_{n=0}^{j-1} \left|f'(y_n)\right| \geq (c\omega)^j \prod_{n=0}^{j-1} \left(q^{k_n+1}\right)^{\mu-1} \\ &= (c\omega)^j \left(\prod_{n=0}^{j-1} q^{k_n+1}\right)^{\mu-1}.\end{aligned}$$

This gives the proof of (4.3.2).

**(d)** Let now $y_0 \in S^c_{k_0,s_0}$ and sequence $k_0 < k_1 < k_2 < \ldots$ as in **(b)** be given. With $\Sigma_j$ from (4.3.1) we have $f^j(\Sigma_j) \subset S^c_{k_j,s_j}$, and so

$$f^{j+1}(y_0) \in S^c_{k_{j+1},s_{j+1}} \cap f^{j+1}(\Sigma_j) \subset f\left(f^j(\Sigma_j)\right) \subset f\left(S^c_{k_j,s_j}\right), \text{ for } j \in \mathbb{N}_0$$

which implies $f\left(S^c_{k_j,s_j}\right) \cap S^c_{k_{j+1},s_{j+1}} \neq \emptyset$. Morover, since $|f| \leq q^{k_j\mu}$ on $S^c_{k_j,s_j}$, we obviously have
$q^{k_j\mu} \geq \max\left\{|f(x)| : x \in S^c_{k_j,s_j}\right\}$. Together with

$$\max\left\{|f(x)| : x \in S^c_{k_j,s_j}\right\} \geq \min\left\{|y| : y \in S^c_{k_{j+1},s_{j+1}}\right\},$$

we conclude

$$q^{k_j\mu} \geq \max\left\{|f(x)| : x \in S^c_{k_j,s_j}\right\} \geq \min\left\{|y| : y \in S^c_{k_{j+1},s_{j+1}}\right\} = a_{k_{j+1}} \geq q^{k_{j+1}+2}.$$

Hence, the proof of **(d)** is also completed.

**(e)** Finally, from (2.3.1) we know that on $\Sigma_j$ we have

$$|\Sigma_j| \leq \frac{\left|(f^j)(\Sigma_j)\right|}{\min_{\Sigma_j}\left|(f^j)'\right|}. \tag{4.3.4}$$

From (4.3.1) we have $\left|(f^j)(\Sigma_j)\right| \leq \left|S^c_{k_j,s_j}\right|$, and from (4.1.2) we have $\left|S^c_{k_j,s_j}\right| \leq q^{k_j} e^{-\frac{\varphi_1}{\omega}}(1-q)$. Combining both inequalities, we get

$$\left|\left(f^j\right)(\Sigma_j)\right| \leq \left|S^c_{k_j,s_j}\right| \leq q^{k_j} e^{-\frac{\varphi_1}{\omega}} (1-q). \tag{4.3.5}$$

Using (4.3.5) and (4.3.2) in (4.3.4) , we obtain

$$|\Sigma_j| \leq \frac{q^{k_j} e^{-\frac{\varphi_1}{\omega}} (1-q)}{(c\omega)^j \left(\prod_{n=0}^{j-1} q^{k_n+1}\right)^{\mu-1}}. \tag{4.3.6}$$

By using (4.3.3) we can estimate the denominator of (4.3.6) as follows:

$$\begin{aligned}
(c\omega)^j \left(\prod_{n=0}^{j-1} q^{k_n+1}\right)^{\mu-1} &= (c\omega)^j \cdot \left(\prod_{n=0}^{j-1} q\right)^{\mu-1} \left(\prod_{n=0}^{j-1} q^{k_n}\right)^{\mu-1} \\
&= (c\omega)^j \cdot \frac{q^{j(\mu-1)} \prod_{n=0}^{j-1} q^{k_n \mu}}{\prod_{n=0}^{j-1} q^{k_n}} \geq (c\omega)^j \cdot q^{j(\mu-1)} \cdot \frac{\prod_{n=0}^{j-1} q^{k_{n+1}+2}}{\prod_{n=0}^{j-1} q^{k_n}} \\
&= (c\omega)^j \cdot q^{(\mu-1)j} \cdot \frac{q^{2j} \prod_{n=0}^{j-1} q^{k_{n+1}}}{\prod_{n=0}^{j-1} q^{k_n}} = \left(c\omega q^{\mu+1}\right)^j \cdot \frac{q^{k_j}}{q^{k_0}}.
\end{aligned}$$

Substituting this estimate in (4.3.6), we finally have

$$|\Sigma_j| \leq \frac{q^{k_j} e^{-\frac{\varphi_1}{\omega}} (1-q) q^{k_0}}{\left(c\omega q^{\mu+1}\right)^j q^{k_j}} = \frac{q^{k_0} e^{-\frac{\varphi_1}{\omega}} (1-q)}{\left(c\omega q^{\mu+1}\right)^j}.$$

To show that $|\Sigma_j| \to 0$ as $j \to \infty$, it is enough to show $\left(c\omega q^{\mu+1}\right) > 1$. Note that the first order Taylor expansion of $q^{\mu+1} = \exp\left(-\frac{\pi}{\omega}(\mu+1)\right)$ is

$$\exp\left(-\frac{\pi}{\omega}(\mu+1)\right) = 1 - \frac{\pi(\mu+1)}{\omega} + R_1(\xi),$$

where $R_1(\xi) = \frac{\exp''(\xi)}{2}\left(\frac{\pi(\mu+1)}{\omega}\right)^2 > 0$, and $\xi \in \left(-\frac{\pi(\mu+1)}{\omega}, 0\right)$. The assumption of $(e)$ gives us $\frac{1}{c} + \pi(\mu+1) < \omega$, and hence

$$1 < c\omega - c\pi(\mu+1) = c\omega\left(1 - \frac{\pi(\mu+1)}{\omega}\right).$$

Since $R_1(\xi) > 0$, we obtain

$$1 < c\omega\left(1-\frac{\pi(\mu+1)}{\omega}\right) < c\omega\left(1-\frac{\pi(\mu+1)}{\omega}+R_1(\xi)\right) = c\omega\exp\left(-\frac{\pi}{\omega}(\mu+1)\right) = c\omega q^{\mu+1},$$

and this completes the proof of **(e)**. ■

The next lemma estimates the measure of the points in the 'steep' interval $S^c_{k_0,+1}$ which have the first $n$ iterates in the union of all 'steep' intervals.

**Lemma 4.4.** Let $k_0 \in \mathbb{N}$, $c \in \left(\frac{1}{\pi}, 1\right)$. Let $\Psi^c$ and $S^c_{k_0,\pm 1}$ be as in the passage before Theorem 4.2. Define $\varphi_1$ as in Lemma 2.1. Then for $k_0 \in \mathbb{N}$ we have

$$\left|S^c_{k_0,+1} \cap \bigcap_{i=1}^{n} f^{-i}(\Psi^c)\right| \le \frac{q^{k_0} e^{-\frac{\varphi_1}{\omega}}(1-q)}{(c\omega q^{\mu+1}(1-q))^n}\ . \tag{4.4.1}$$

(The same estimate holds for $S^c_{k_0,-1}$)

**Proof.** Let $k_0 \in \mathbb{N}$ and $c \in \left(\frac{1}{\pi}, 1\right)$ be given. It is clear that $f\left(S^c_{k_0,+1}\right)$ contains infinitely many 'steep' intervals. Assume $\ell$, $i \in \mathbb{N}$ are such that $S^c_{k_0,+1} \cap f^{-i}\left(S^c_{\ell,\pm 1}\right) \neq \emptyset$. Since $\left|f^i(x)\right| \le |x|^{\mu^i}$ on $S^c_{k_0,\pm 1}$, one must have $q^{k_0\mu^i} \ge \min\left\{|y| : y \in S^c_{\ell,\pm 1}\right\} \ge q^{\ell+2}$. It follows that $\ell \ge k_0\mu^i - 2 \ge k_0\mu - 2$. Hence, the intersection in (4.4.1) equals $S^c_{k_0,+1} \cap \bigcap_{i=1}^{n} f^{-i}\left(\bigcup_{\ell \ge k_0\mu-2} S^c_{\ell,\pm 1}\right)$. We now prove (4.4.1) by induction over $n$. For $n = 1$,

$$\begin{aligned}\left|S^c_{k_0,+1} \cap f^{-1}(\Psi^c)\right| &= \left|S^c_{k_0,+1} \cap f^{-1}\left(\bigcup_{\ell \ge k_0\mu-2} S^c_{\ell,\pm 1}\right)\right| \\ &= \sum_{\ell \ge k_0\mu-2} \left|S^c_{k_0,+1} \cap f^{-1}\left(S^c_{\ell,\pm 1}\right)\right|\ .\end{aligned} \tag{4.4.2}$$

From (4.1.2) we have

$$\left|S^c_{\ell,\pm 1}\right| \le q^{\ell} e^{-\frac{\varphi_1}{\omega}}(1-q)\,. \tag{4.4.3}$$

Using (2.3.1), (4.1.1) and (4.4.3) in (4.4.2), we have

$$\begin{aligned}\left|S^c_{k_0,+1} \cap f^{-1}(\Psi^c)\right| &= \sum_{\ell \ge k_0\mu-2} \left|S^c_{k_0,+1} \cap f^{-1}\left(S^c_{\ell,\pm 1}\right)\right| \le \sum_{\ell \ge k_0\mu-2} \frac{1}{c\omega q^{(k_0+1)(\mu-1)}}\left|S^c_{\ell,\pm 1}\right| \\ &\le \frac{e^{-\frac{\varphi_1}{\omega}}(1-q)}{c\omega q^{(k_0+1)(\mu-1)}} \sum_{\ell \ge k_0\mu-2} q^{\ell}.\end{aligned} \tag{4.4.4}$$

Here, note that

$$\sum_{\ell \ge k_0\mu-2} q^{\ell} = \sum_{\ell \ge \lceil k_0\mu-2\rceil} q^{\ell} = q^{\lceil k_0\mu-2\rceil}\frac{1}{1-q}, \tag{4.4.5}$$

where $\lceil ... \rceil$ denotes the ceiling function. Setting $\varepsilon(k_0) := \lceil k_0\mu - 2\rceil - (k_0\mu - 1) \in [-1, 0)$ and using (4.4.5) in (4.4.4), we obtain

$$\begin{aligned}\left|S^c_{k_0,+1} \cap f^{-1}(\Psi^c)\right| &\leq \frac{e^{-\frac{\varphi_1}{\omega}}}{c\omega q^{(k_0+1)(\mu-1)}} q^{\lceil k_0\mu-2\rceil} = \frac{q^{k_0} e^{-\frac{\varphi_1}{\omega}}}{c\omega} \cdot \frac{q^{\lceil k_0\mu-2\rceil}}{q^{k_0\mu-1}} \cdot \frac{1}{q^\mu} \\ &= q^{\varepsilon(k_0)} \cdot \frac{q^{k_0} e^{-\frac{\varphi_1}{\omega}}}{c\omega q^\mu} \leq \frac{q^{-1} q^{k_0} e^{-\frac{\varphi_1}{\omega}}}{c\omega q^\mu} = \frac{q^{k_0} e^{-\frac{\varphi_1}{\omega}}(1-q)}{c\omega q^{\mu+1}(1-q)}\end{aligned}$$

which proves the case $n = 1$.

Assume the assertion is true for $n$, i.e, for all $k_0 \in \mathbb{N}$ we have

$$\left|S^c_{k_0,+1} \cap \bigcap_{i=1}^{n} f^{-i}(\Psi^c)\right| \leq q^{k_0} e^{-\frac{\varphi_1}{\omega}}(1-q)\left(\frac{1}{c\omega q^{\mu+1}(1-q)}\right)^n, \tag{4.4.6}$$

and now we show that it is true for $n+1$.

$$\begin{aligned}\left|S^c_{k_0,+1} \cap \bigcap_{i=1}^{n+1} f^{-i}(\Psi^c)\right| &= \left|S^c_{k_0,+1} \cap f^{-1}(\mathbf{\Psi}^c) \cap ... \cap f^{-n-1}(\mathbf{\Psi}^c)\right| \\ &= \left|S^c_{k_0,+1} \cap f^{-1}\left(\bigcap_{i=0}^{n} f^{-i}(\mathbf{\Psi}^c)\right)\right| \\ &= \left|S^c_{k_0,+1} \cap f^{-1}\left(\left(\bigcup_{\ell \geq k_0\mu-2} S^c_{\ell,\pm1}\right) \cap \bigcap_{i=0}^{n} f^{-i}(\mathbf{\Psi}^c)\right)\right|.\end{aligned}$$

Note that $S^c_{\ell,\pm1} \subset \mathbf{\Psi}^c$ implies

$$S^c_{\ell,\pm1} \cap \bigcap_{i=0}^{n} f^{-i}(\mathbf{\Psi}^c) = S^c_{\ell,\pm1} \cap \bigcap_{i=1}^{n} f^{-i}(\mathbf{\Psi}^c).$$

So, we obtain

$$\left|S^c_{k_0,+1} \cap \bigcap_{i=1}^{n+1} f^{-i}(\Psi^c)\right| = \left|S^c_{k_0,+1} \cap f^{-1}\left(\bigcup_{\ell \geq k_0\mu-2}\left(S^c_{\ell,\pm1} \cap \bigcap_{i=1}^{n} f^{-i}(\mathbf{\Psi}^c)\right)\right)\right| \tag{4.4.7}$$

Using (2.3.1), (4.1.1), (4.4.3), (4.4.5) and (4.4.6) in (4.4.7), we have

$$\begin{aligned}\left|S^c_{k_0,+1} \cap \bigcap_{i=1}^{n+1} f^{-i}(\Psi^c)\right| &\leq \frac{1}{(c\omega)\, q^{(k_0+1)(\mu-1)}} \sum_{\ell \geq k_0\mu-2} \left(\frac{1}{c\omega q^{\mu+1}(1-q)}\right)^n q^\ell e^{-\frac{\varphi_1}{\omega}}(1-q) \\ &= \frac{q^{k_0} e^{-\frac{\varphi_1}{\omega}}}{(c\omega)^{n+1}(q^{\mu+1})^n q^\mu q^{k_0\mu-1}} \left(\frac{1}{1-q}\right)^{n-1} \sum_{\ell \geq k_0\mu-2} q^\ell \\ &= \frac{q^{k_0} e^{-\frac{\varphi_1}{\omega}}}{(c\omega)^{n+1}(q^{\mu+1})^n q^\mu} \left(\frac{1}{1-q}\right)^{n-1} \frac{q^{\lceil k_0\mu-2\rceil}}{q^{k_0\mu-1}} \frac{1}{1-q}\end{aligned}$$

With $\varepsilon(k_0)$ as above, we obtain

$$\begin{aligned}\left|S^c_{k_0,+1}\cap\bigcap_{i=1}^{n+1}f^{-i}(\Psi^c)\right| &\le \frac{q^{k_0}e^{-\frac{\varphi_1}{\omega}}q^{\varepsilon(k_0)}}{(c\omega)^{n+1}(q^{\mu+1})^n q^{\mu}}\left(\frac{1}{1-q}\right)^n\\ &\le \frac{q^{k_0}e^{-\frac{\varphi_1}{\omega}}q^{-1}}{(c\omega)^{n+1}(q^{\mu+1})^n}\left(\frac{1}{1-q}\right)^n\\ &= q^{k_0}e^{-\frac{\varphi_1}{\omega}}(1-q)\left(\frac{1}{c\omega q^{\mu+1}(1-q)}\right)^{n+1},\end{aligned}$$

so the assertion is true for $n+1$ and hence, the proof of Lemma 4.4 is completed. ■

**Remark 4.5.** Let $c\in\left(\frac{2}{\pi},1\right)$ and $\mu>1$. Then $\frac{1}{c}+\pi(\mu+1)\le\frac{c\pi^2(2\mu+3)}{2(c\pi-1)}$.

**Proof.** Let $c\in\left(\frac{2}{\pi},1\right)$. Then

$$\frac{1}{c}+\pi(\mu+1)=\frac{1+c\pi\mu+c\pi}{c}=\frac{\pi+c\pi^2\mu+c\pi^2}{c\pi}\le\frac{2c\pi^2\mu+2c\pi^2+2\pi}{2(c\pi-1)}.$$

Since $c\pi>2$, we have $2\pi<c\pi^2$ and hence

$$\frac{1}{c}+\pi(\mu+1)\le\frac{2c\pi^2\mu+3c\pi^2}{2(c\pi-1)}=\frac{c\pi^2(2\mu+3)}{2(c\pi-1)}. \blacksquare$$

**4.6. Proof of Theorem 4.2**

**(1).** From assertion **(a)** in Lemma 4.3 we see that there exists a point $y_0\in\left(S^c_{k_0,s_0}\cap\Omega^c_\infty\right)$ with $\operatorname{sign} f^j(y_0)=s_j$, because $f^j(y_0)\in S^c_{k_j,s_j}$.

**(2).** Assume $y_0\in\left(S^c_{k_0,s_0}\cap\Omega^c_\infty\right)$. Assertion **(b)** of Lemma 4.3 shows that $\Sigma_j\ni y_0$ and $Z\cap\Sigma_j\neq\emptyset$. Further, assertion **(e)** of Lemma 4.3 shows that $|\Sigma_j|\to 0$ as $j\to\infty$. This means that there exists a sequence $(z_j)\subset Z$ with $z_j\to y_0$, and this completes the proof.

**(3).** Let $c\in\left(\frac{2}{\pi},1\right)$ be given. Remark 4.5 shows that the condition $\omega>\frac{c\pi^2(2\mu+3)}{2(c\pi-1)}$ from assertion **(3)** of Theorem 4.2 implies the condition $\omega>\frac{1}{c}+\pi(\mu+1)$ of assertion **(2)**. Hence, $\left(S^c_{k_0,\pm1}\cap\Omega^c_\infty\right)\subset\overline{Z}$ for all $k_0\in\mathbb{N}$. It follows that $\Omega^c_\infty=\bigcup_{k_0\in\mathbb{N}}\left(S^c_{k_0,\pm1}\cap\Omega^c_\infty\right)\subset\overline{Z}$, so $\Omega^c_\infty\subset\overline{Z}$. To prove that $\Omega^c_\infty$ has measure zero, we show $\lim_{n\to\infty}\left|\Omega^c_n\cap S^c_{k_0,\pm1}\right|=0$ for every $k_0\in\mathbb{N}$. For this purpose it is enough to show that under the conditions of assertion **(3)** of Theorem 4.2, $c\omega q^{\mu+1}(1-q)>1$ in (4.4.1). We use the second order Taylor expansion of $e^{-y}$ around 0 for $y>0$,

$$e^{-y}=1-y+\frac{y^2}{2}+R_3,$$

with $R_3=\frac{\exp'''(\xi)}{3!}(-y)^3<0$ for some $\xi\in(-y,0)$. Hence, since

$$q = e^{-\frac{\pi}{\omega}} = 1 - \frac{\pi}{\omega} + \frac{(\pi)^2}{2\omega^2} + R_3\left(\frac{\pi}{\omega}\right) < 1 - \frac{\pi}{\omega} + \frac{(\pi)^2}{2\omega^2},$$

we have

$$1 - q = 1 - e^{-\frac{\pi}{\omega}} = \frac{\pi}{\omega} - \frac{\pi^2}{2\omega^2} - R_3\left(\frac{\pi}{\omega}\right) > \frac{\pi}{\omega} - \frac{\pi^2}{2\omega^2}. \tag{4.6.1}$$

On the other hand, with appropriate $\xi$,

$$q^{\mu+1} = e^{-\frac{\pi}{\omega}(\mu+1)} = 1 - \frac{\pi(\mu+1)}{\omega} + \frac{\exp''(\xi)}{2!}\left(-\frac{\pi(\mu+1)}{\omega}\right)^2 > 1 - \frac{\pi(\mu+1)}{\omega}. \tag{4.6.2}$$

Using (4.6.1) and (4.6.2), we get

$$\begin{aligned} c\omega q^{\mu+1}(1-q) &> c\omega\left(1 - \frac{\pi(\mu+1)}{\omega}\right)\left(\frac{\pi}{\omega} - \frac{\pi^2}{2\omega^2}\right) = c\pi\left(1 - \frac{\pi\mu+\pi}{\omega}\right)\left(1 - \frac{\pi}{2\omega}\right) \\ &= c\pi\left(1 - \frac{(3\pi + 2\pi\mu)}{2\omega} + \frac{\pi^2(\mu+1)}{2\omega^2}\right) \\ &> c\pi\left(1 - \frac{\pi(3+2\mu)}{2\omega}\right). \end{aligned} \tag{4.6.3}$$

In view of Remark 4.5, and using the assumption which is given in the assertion **(3)** of Theorem 4.2 in (4.6.3), we finally obtain

$$c\omega q^{\mu+1}(1-q) > c\pi\left(1 - \frac{\pi(3+2\mu)}{2 \cdot \frac{c\pi^2(2\mu+3)}{2(c\pi-1)}}\right) = c\pi\left(1 - \frac{c\pi - 1}{c\pi}\right) = 1. \blacksquare$$

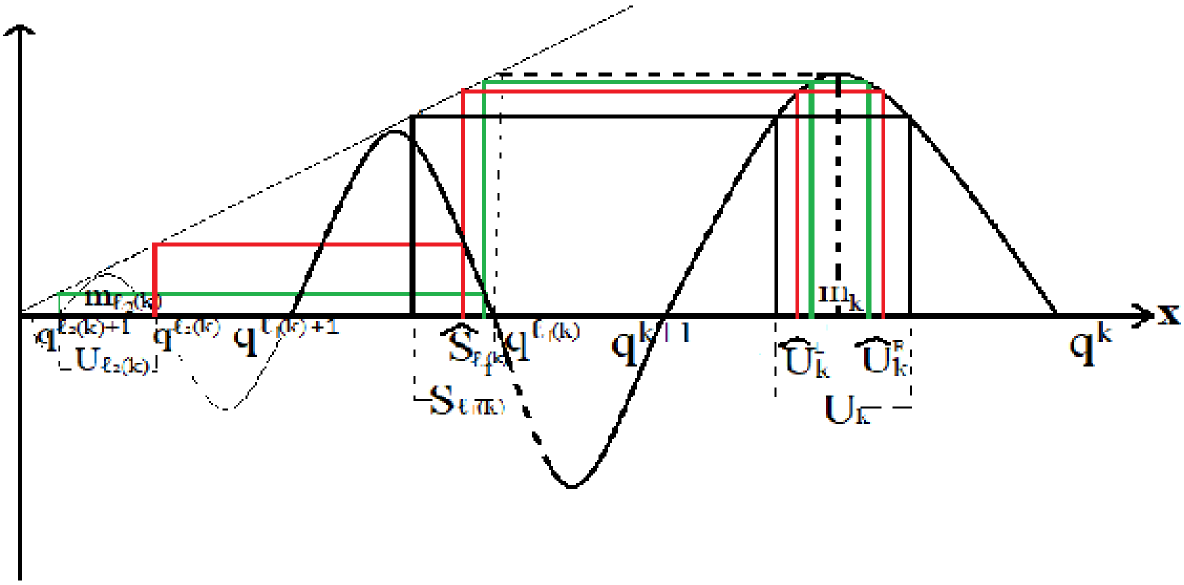


Figure 7: Graphical representation of $U_k^L$ and $U_k^R$

## 5. The behavior of the points whose orbits follow 'flat-steep-flat' intervals

In chapter three we analyzed the behavior of the points which are mapped from 'flat' intervals to some other 'flat' intervals, and in chapter four we studied the behavior of the points which are mapped from 'steep' intervals to some other 'steep' intervals. Finally in this chapter, as we briefly mentioned in the summary of this thesis, we first construct a specific type of orbit whose points travel from 'flat' intervals to 'steep' intervals, then from 'steep' intervals again to 'flat' intervals under the iteration (see Figure 7). Besides, to avoid repeating the same expression, we shall use $g_{\omega,\mu+1-j}$ as in Lemma 2.1 and $c \in (0,1)$ for the rest of the paper. For a specific choice of $\mu$, $\omega > 0$, maxima $m_k$ get mapped to zeros $q^{\ell_1(k)}$ of $f_{\mu,\omega}$. We shall first introduce 'flat' intervals of the form $U_k = [m_k - \delta_k, m_k + \delta_k]$ for odd $k$ and use the notations $U_k^R = [m_k, m_k + \delta_k]$ and $U_k^L = [m_k - \delta_k, m_k]$ for the right and left part of $U_k$ respectively, and we define $U = \bigcup\limits_{\substack{k\in\mathbb{N}\\ k \text{ odd}}} U_k$ , $S = \bigcup\limits_{\substack{k\in\mathbb{N}\\ k \text{ odd}}} S_k$. Then, we construct the orbit $\left(f^j(x)\right)_{j\in\mathbb{N}}$, with the properties

$$f^j(x) \in \begin{cases} U, & j \text{ is even} \\ S, & j \text{ is odd} \end{cases}.$$

Furthermore, for $k$, $\mu \in \mathbb{N}$, $\omega > 0$ define $\ell_1(k) = k\mu + 1$ and with $\varphi_1$ as in Lemma 2.1, we define

$$\ell_2(k) := \min\left\{\ell \in \mathbb{N} : q^\ell \le q^{\ell_1(k)\mu} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \frac{c(1-c)\,\omega^3}{4g_{\omega,\mu} \cdot g_{\omega,\mu-1}^2}\right\}.$$

We denote by $\ell_2^j(k)$ the $j$ $th$ iterate of the function $\ell_2$ applied to $k$. Then, given a symbol sequence of the form $(L, R, R, L, R, ..., R)$, where symbols represent the left '$L$' or right

'$R$' hand part of $U_k$ (that is $U_k^L, U_k^R$), we construct corresponding orbits of $f$. Given a finite sequence

$$\mathbf{s} = (s_0, s_1, s_2, ..., s_n) \in \{L, R\}^{n+1}$$

and $k \in \mathbb{N}$, we first construct the subset of points $x$ in $U_k$ which follow this symbol sequence in the sense that $f^{2j}(x) \in U^L_{\ell_2^j(k)}$ or $f^{2j}(x) \in U^R_{\ell_2^j(k)}$, $j = 0, 1, 2, ..., n$ depending on whether $s_j = L$ or $s_j = R$. Hence, we construct the set $I^n_{k,\mathbf{s}} = \bigcap_{j=0}^{n} f^{-2j}\left(U^{s_j}_{\ell_2^j(k)}\right)$ and the set $\Gamma_k^n = \bigcup_{s \in \{L,R\}^{n+1}} I^n_{k,s}$ which is the set of points following symbol sequences in the set $\{L, R\}^{\{0,1,2,\ldots,n\}}$. Finally, we analyze the Lebesgue measure of the set $\Gamma_k^n$, and consider the limit as $n \to \infty$.

Note that the 'steep' intervals $S_k$ that we use in our calculations in this chapter are some subintervals of $\left(m_k, q^k\right]$, whereas the 'steep' intervals which were used in the fourth chapter are some subintervals of $(m_{k+1}, m_k)$. In the theorem below we restrict ourselves to $\mu \in \mathbb{N}$ for simplicity.

**Theorem 5.1.** Let $k$ be a positive odd integer number. Let $c \in (0, 1)$, and $\mu \in \mathbb{N}$, $\mu \geq \max\left\{\left(\frac{30e}{7\pi}\right)^2\left(\frac{1-c}{2c}\right), 15\right\}$ be given. Then there exist an $\omega > 0$ (depending on $\mu$) and a set $\Gamma_n^k \subset U_k$ with the following properties:

**(1)**. Let a sequence of the form $\mathbf{s} = (L, R, R, R, L, R, L, R, ...) \in \{L, R\}^{\mathbb{N}_0}$ be given. Then, there exists exactly one point $x_{k,s} \in U_k$ with the property:

For all $n \in \mathbb{N}_0$, $f^{2n}(x_s) \in U_{\ell_2^n(k)}$, and $f^{2n}(x_s)$ is to the left of $m_{\ell_2^n(k)}$ or to the right of $m_{\ell_2^n(k)}$, depending on whether $s_n = L$ or $s_n = R$.

**(2)**. The measure of $\Gamma_k^n$ as defined above goes to zero, as $n \to \infty$.

The proof requires several lemmas and propositions. The proof of the following lemma is analogous to the proof of Lemma 3.3, but is included for completeness.

**Lemma 5.2.** Define $\varphi_1$ as in Lemma 2.1. Then the following statements are true.

**(a)** Assume $\mu \in \mathbb{N}$, $\mu \geq 15$ and define $\ell_1(k)$ as in the passage before Theorem 5.1. Then there exists an $\omega \in (0, 1)$ such that for all $k \in \mathbb{N}$, $f$ has the property

$$|f(m_k)| = q^{\ell_1(k)}, \tag{5.2.1}$$

which is eqivalent to

$$\exp\left(\frac{\pi - \mu \arctan\left(\dfrac{\omega}{\mu}\right)}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}\ . \tag{5.2.2}$$

**(b)** For any choice of $\omega$ as in assertion **(a)**, we have $\omega \to 0$ as $\mu \to \infty$.

**Proof.**
**(a)** Let $k \in \mathbb{N}$ be given. With $m_k$ from (2.2.3), we have from (2.2.4)

$$|f(m_k)| = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \cdot \sin(\varphi_1).$$

Using (2.1.3) we obtain

$$|f(m_k)| = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \cdot \frac{1}{\sqrt{1 + \dfrac{\mu^2}{\omega^2}}}. \tag{5.2.3}$$

On the other hand, from (2.2.1) we have

$$q^{\ell_1(k)} = \exp\left(-\frac{\pi\ell_1(k)}{\omega}\right). \tag{5.2.4}$$

With (5.2.3) and (5.2.4) together, we see that (5.2.1) is equivalent to

$$\exp\left(-\frac{\pi\ell_1(k)}{\omega}\right) = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \frac{1}{\sqrt{1 + \dfrac{\mu^2}{\omega^2}}}$$

and hence to

$$\exp\left(\frac{\pi(\ell_1(k) - k\mu) - \varphi_1\mu}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}. \tag{5.2.5}$$

So, if we substitute $\ell_1(k) = k\mu + 1$ and the value of $\varphi_1$ given by Lemma 2.1 in (5.2.5), we finally get that (5.2.1) is equivalent to

$$\exp\left(\frac{\pi - \mu \arctan\left(\dfrac{\omega}{\mu}\right)}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}},$$

which proves the equivalence of (5.2.1) and (5.2.2). Now, we want to find $\omega$ and $\mu$ such that
$|f(m_k)| = q^{\ell_1(k)}$. Define

$$F(\omega, \mu) = \exp\left(\frac{\pi - \mu \arctan\left(\dfrac{\omega}{\mu}\right)}{\omega}\right) - \sqrt{1 + \frac{\mu^2}{\omega^2}}$$

and try to find $F(\omega, \mu) = 0$ at least for a special pair of $(\omega, \mu)$ (See Figure 8). On the one hand, for a fixed $\mu > 2$, $\arctan\left(\dfrac{\omega}{\mu}\right) \to 0$ as $\omega \to 0$. Hence, due to the exponential growth, $F(\omega, \mu) \to \infty$ as $\omega \to 0$. On the other hand, for $\omega = 1$ we have

$$F(1, \mu) = \exp\left(\pi - \mu \arctan\left(\frac{1}{\mu}\right)\right) - \sqrt{1 + \mu^2}. \tag{5.2.6}$$

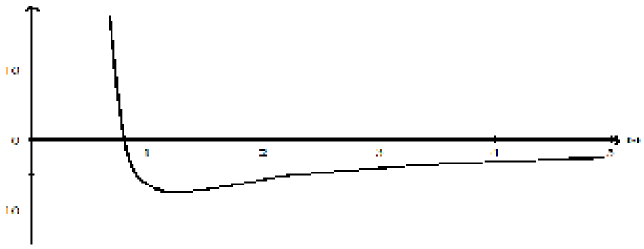

Figure 8: Graph of $F(\cdot, \mu)$ for $\mu = 15$

From (3.3.7) we have $\mu \arctan\left(\frac{1}{\mu}\right) \geq \frac{1}{2}$ for $\mu > 2$, and using this estimate in (5.2.6), we finally have $F(1, \mu) < e^{\pi - \frac{1}{2}} - \sqrt{1 + \mu^2}$. From the fact that $e^{\pi - \frac{1}{2}} < 15$, we finally have $F(1, \mu) < 0$, if we choose $\mu \geq 15$. With the intermediate value theorem, it is clear that there exists at least one $\omega \in (0, 1)$ which satisfies $F(\omega, \mu) = 0$ for fixed $\mu$. This gives the proof of assertion **(a).**

**(b)** The proof is analogous to the proof of the assertion **(b)** of Lemma 3.3. ■

In order to find a numerical solution, one can use the bisection method, and we found numerically that there is no solution for $\mu < 2.3$.

The next three propositions (5.3, 5.4, 5.5 ) give some preparatory calculations.

**Proposition 5.3.** Let $\varphi_1$ be as in Lemma 2.1. Set $\alpha(\omega, \mu, c) := \dfrac{\exp\left(\dfrac{(\mu - 2)\varphi_1}{2\omega}\right)}{g_{\omega, \mu - 1}} \sqrt{\dfrac{1 - c}{2c\omega}}$. If $\mu \in \mathbb{N}$,

$$\mu \geq \max\left\{\left(\frac{30e}{7\pi}\right)^2 \left(\frac{1 - c}{2c}\right), 15\right\},$$

and $\omega$ is a corresponding value obtained as in Lemma 5.2, then we have $\alpha(\omega, \mu, c) < \frac{1}{2}$.

**Proof.** Let $\mu$ and $\omega \in (0, 1)$ be as in the assumption. Then, it is clear that $\frac{3\mu^2}{\omega^2} \geq 1$, and in view of (5.2.2) we have

$$\frac{2\mu}{\omega} = \sqrt{\frac{3\mu^2}{\omega^2} + \frac{\mu^2}{\omega^2}} \geq \sqrt{1 + \frac{\mu^2}{\omega^2}} = \exp\left(\frac{\pi - \mu \arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right).$$

Since $\arctan\left(\dfrac{\omega}{\mu}\right) \leq \dfrac{\omega}{\mu}$, we get

$$\frac{2\mu}{\omega} \geq \exp\left(\frac{\pi - \mu \cdot \dfrac{\omega}{\mu}}{\omega}\right) = \exp\left(\frac{\pi}{\omega} - 1\right),$$

and hence we have $2\mu e \geq \omega e^{\frac{\pi}{\omega}}$. Using the second order Taylor expansion of $e^{\frac{\pi}{\omega}}$ in the last inequality, we obtain

$$\begin{aligned} 2\mu e &\geq \omega\left(1 + \frac{\pi}{\omega} + \frac{1}{2}\frac{\pi^2}{\omega^2}\right) \geq \frac{1}{2}\frac{\pi^2}{\omega}, \text{ or} \\ 4e\mu &\geq \frac{\pi^2}{\omega}. \end{aligned} \tag{5.3.1}$$

On the other hand, we know that $\mu \geq \left(\dfrac{30e}{7\pi}\right)^2 \left(\dfrac{1-c}{2c}\right)$, so $\sqrt{\mu} \geq \dfrac{30e}{7\pi}\sqrt{\dfrac{1-c}{2c}}$ which implies $\dfrac{1}{2} \geq e^{\frac{1}{2}}\sqrt{\dfrac{1-c}{2c}}\dfrac{30e^{\frac{1}{2}}}{14\pi\sqrt{\mu}}$. Since $e^{\frac{1}{2}} > e^{\frac{1}{2}-\frac{1}{\mu}}$, it follows that

$$\frac{1}{2} > \exp\left(\frac{1}{2} - \frac{1}{\mu}\right) \cdot \sqrt{\frac{1-c}{2c}} \cdot \frac{15\sqrt{4e}}{14\pi\sqrt{\mu}} = \exp\left(\frac{\omega}{2\mu\omega}(\mu - 2)\right) \cdot \sqrt{\frac{1-c}{2c}} \cdot \frac{15\sqrt{4e\mu}}{14\pi\mu}. \tag{5.3.2}$$

On the other hand, since $\mu \geq 15$, we have $\dfrac{1}{\mu - 1} \leq \dfrac{15}{14\mu}$, and with the fact that $\arctan\left(\dfrac{\omega}{\mu}\right) \leq \dfrac{\omega}{\mu}$ we can conclude from (5.3.2)

$$\frac{1}{2} > \exp\left(\frac{(\mu-2)\arctan\left(\dfrac{\omega}{\mu}\right)}{2\omega}\right) \cdot \sqrt{\frac{1-c}{2c}} \cdot \frac{\sqrt{4e\mu}}{\pi(\mu-1)}.$$

Finally, using (5.3.1) and the definition of $\varphi_1$, we obtain

$$\begin{aligned} \frac{1}{2} &> \exp\left(\frac{(\mu-2)\varphi_1}{2\omega}\right) \cdot \sqrt{\frac{1-c}{2c\omega}} \cdot \sqrt{\frac{1}{(\mu-1)^2}} \\ &\geq \exp\left(\frac{(\mu-2)\varphi_1}{2\omega}\right) \cdot \sqrt{\frac{1-c}{2c\omega}} \cdot \sqrt{\frac{1}{\omega^2 + (\mu-1)^2}} \\ &= \frac{\exp\left(\dfrac{(\mu-2)\varphi_1}{2\omega}\right)}{g_{\omega,\mu-1}} \cdot \sqrt{\frac{1-c}{2c\omega}} = \alpha(\omega, \mu, c) \end{aligned}$$

and this completes the proof. ■

**Proposition 5.4.** Let $\varphi_1$ be as in Lemma 2.1 and $c \in (0,1)$ be given. Set $\widetilde{\eta}_1(\omega,\mu) := \dfrac{\omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2g_{\omega,\mu-1} \cdot g_{\omega,\mu-2}}$, $\widetilde{\eta}_2(\omega,\mu) := \dfrac{e^{-\frac{\varphi_1}{\omega}} - q}{2}$, $\widetilde{\eta}_3(\omega,\mu) := \dfrac{1-e^{-\frac{\varphi_1}{\omega}}}{2}$ and $\widetilde{\eta}_4(\omega,\mu) := \dfrac{\sqrt{(1-c)\,q\omega}}{g_{\omega,\mu} \cdot g_{\omega,\mu-1}}$. Then for small enough $\omega > 0$ and large enough $\mu \in \mathbb{N}$, $\mu \geq 3$, we have

$$\widetilde{\eta} := \min\left\{\widetilde{\eta}_1(\omega,\mu),\ \widetilde{\eta}_2(\omega,\mu),\ \widetilde{\eta}_3(\omega,\mu),\ \widetilde{\eta}_4(\omega,\mu)\right\} = \widetilde{\eta}_4(\omega,\mu). \tag{5.4.1}$$

**Proof.** We prove that $\widetilde{\eta}_4(\omega,\mu) \leq \widetilde{\eta}_1(\omega,\mu) \leq \min\{\widetilde{\eta}_2(\omega,\mu), \widetilde{\eta}_3(\omega,\mu)\}$ for $\mu$ large enough, $\omega$ small enough. For $\omega > 0$, we have $g_{\omega,\mu-1} \geq \sqrt{(\mu-1)^2}$, $g_{\omega,\mu-2} \geq \sqrt{(\mu-2)^2}$ and using these simplifications, we obtain

$$\widetilde{\eta}_1(\omega,\mu) \leq \frac{\omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2\sqrt{(\mu-1)^2(\mu-2)^2}} = \frac{\omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2(\mu-1)(\mu-2)}. \tag{5.4.2}$$

We have already defined

$$\widetilde{\eta}_2(\omega,\mu) = \frac{e^{-\frac{\varphi_1}{\omega}} - q}{2} = \frac{1}{2}e^{-\frac{\varphi_1}{\omega}}\left(1 - e^{\frac{\varphi_1-\pi}{\omega}}\right),$$

and for $\omega$ small, $\varphi_1 - \pi < 0$ implies $\left(1 - e^{\frac{\varphi_1-\pi}{\omega}}\right) > \dfrac{1}{2}$. Hence, we have

$$\widetilde{\eta}_2(\omega,\mu) \geq \frac{1}{4}e^{-\frac{\varphi_1}{\omega}}. \tag{5.4.3}$$

From (5.4.2) and (5.4.3) it is obvious that $\widetilde{\eta}_1(\omega,\mu) \leq \widetilde{\eta}_2(\omega,\mu)$ for $\mu \geq 3$ and small enough $\omega$. The proof of $\widetilde{\eta}_1(\omega,\mu) \leq \widetilde{\eta}_3(\omega,\mu)$ for small $\omega$ and large $\mu$ is analogous; observe $\widetilde{\eta}_3(\omega,\mu) \to \dfrac{1}{2}$ as $\omega \to 0$. There exist $c_1, c_2 > 0$, and $\mu_0 > 0$ for $\mu \geq \mu_0$ such that $\widetilde{\eta}_1(\omega,\mu) \geq c_1 \dfrac{\omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{\mu^2}$ and $\widetilde{\eta}_4(\omega,\mu_0) \leq c_2 \dfrac{\sqrt{\omega e^{-\frac{\pi}{\omega}}}}{\mu^2}$. Hence, we have for $\mu \geq \mu_0$

$$\frac{\widetilde{\eta}_4(\omega,\mu)}{\widetilde{\eta}_1(\omega,\mu)} \leq \frac{c_2\sqrt{\omega e^{-\frac{\pi}{\omega}}}}{c_1 \omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}} = \frac{c_2}{c_1}\frac{1}{\sqrt{\omega}} \exp\left(\frac{\varphi_1(\mu-2) - \frac{\pi}{2}}{\omega}\right)$$

Substituting the explicit form of $\varphi_1$ as in Lemma 2.1, the last equality turns to

$$\frac{\widetilde{\eta}_4(\omega,\mu)}{\widetilde{\eta}_1(\omega,\mu)} \leq \frac{c_2}{c_1}\frac{1}{\sqrt{\omega}} \exp\left(\frac{(\mu-2)\arctan\left(\frac{\omega}{\mu}\right) - \frac{\pi}{2}}{\omega}\right). \tag{5.4.4}$$

Using the fact that

$$\lim_{\mu\to\infty,\omega\to 0}\left(\frac{(\mu-2)\arctan\left(\dfrac{\omega}{\mu}\right)}{\omega}\right)=\lim_{\mu\to\infty,\omega\to 0}\left(\frac{(\mu-2)\dfrac{\omega}{\mu}}{\omega}\right)=1,\ \text{and}$$

$\lim_{\omega\to 0}\frac{1}{\sqrt{\omega}}\exp\left(\frac{-\pi}{2\omega}\right)=0$ in (5.4.4), we finally have

$$\lim_{\mu\to\infty,\omega\to 0}\left(\frac{\widetilde{\eta}_4(\omega,\mu)}{\widetilde{\eta}_1(\omega,\mu)}\right)\le\lim_{\mu\to\infty,\omega\to 0}\frac{c_2}{c_1}\frac{1}{\sqrt{\omega}}\exp\left(1-\frac{\pi}{2\omega}\right)=0$$

and that shows for large enough $\mu$ and small enough $\omega$ one has $\widetilde{\eta}_4(\omega,\mu)\le\widetilde{\eta}_1(\omega,\mu)$. ■

Now, we aim at finding an interval $U_k:=[m_k-\delta_k,m_k+\delta_k]$ as indicated in the passage before Theorem 5.1, which gets mapped to a 'steep' interval $S_{\ell_1(k)}$, but we first provide upper and lower estimates for the second derivative $f''_{\mu,\omega}$ of $f$.

**Proposition 5.5.** Let $k\in\mathbb{N}$. Assume $\mu$ and $\omega$ are as in Proposition 5.4. Define $\widetilde{\eta}$ as in Proposition 5.4 and set $\delta_k:=\widetilde{\eta}q^k$, $J_k:=\left[q^{k+1},q^k\right]$. Then

$$U_k:=[m_k-\delta_k,m_k+\delta_k]\subset\left[q^{k+1},q^k\right]=J_k$$

and the following estimates hold:

$$\forall x\in[m_k-\delta_k,m_k+\delta_k]:g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k(\mu-2)}\ge|f''(x)|\ge\frac{\left(q^k e^{-\frac{\varphi}{\omega}}\right)^{\mu-2}\cdot\omega\cdot g_{\omega,\mu}}{2}.\quad(5.5.1)$$

**Proof.** Let $k\in\mathbb{N}$. With $\overline{\eta}$ from Lemma 3.2, the definition of $\widetilde{\eta}$ given in Proposition 5.4 shows $\widetilde{\eta}\le\min\{\widetilde{\eta}_2,\ \widetilde{\eta}_3\}=\overline{\eta}$. Hence, in view of Lemma 3.2, we see that

$$U_k=[m_k-\delta_k,m_k+\delta_k]\subset\left[m_k-\overline{\eta}q^k,m_k+\overline{\eta}q^k\right]\subset\left[q^{k+1},q^k\right]=J_k.$$

Further, inserting $m_k$ from (2.2.3) in (2.1.4) we have

$$\begin{aligned}|f''(m_k)| &= g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot|m_k|^{\mu-2}\left|\sin\left((\omega\ln(m_k)+\varphi_1)+\varphi_2\right)\right|\\ &= g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot|m_k|^{\mu-2}\left|\sin(-k\pi+\varphi_2)\right|.\end{aligned}$$

Using $\varphi_2=\arctan\left(\dfrac{\omega}{\mu-1}\right)$ from Lemma 2.1, we have $\sin(\varphi_2)=\dfrac{\omega}{g_{\omega,\mu-1}}$ and, inserting this value in the last equality, we have

$$|f''(m_k)|=g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot|m_k|^{\mu-2}\frac{\omega}{g_{\omega,\mu-1}}=|m_k|^{\mu-2}\omega\cdot g_{\omega,\mu}.\qquad(5.5.2)$$

From (2.1.5) we have on $\left[q^{k+1}, q^k\right]$

$$\begin{aligned} |f'''(x)| &= \left|g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu-2} \cdot x^{\mu-3} \sin(\omega \ln(x) + (\varphi_1 + \varphi_2 + \varphi_3))\right| \\ &\leq g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu-2} \cdot q^{k(\mu-3)}. \end{aligned} \tag{5.5.3}$$

From (2.3.1) for $x \in [m_k - \delta_k, m_k + \delta_k]$ and with the definition of $\delta_k$, we also have

$$\begin{aligned} |f''(x)| &\geq |f''(m_k)| - \delta_k \cdot \max_{[m_k-\delta_k, m_k+\delta_k]} |f'''| \\ &= |f''(m_k)| - \widetilde{\eta} q^k \cdot \max_{[m_k-\delta_k, m_k+\delta_k]} |f'''| \\ &\geq |f''(m_k)| - \widetilde{\eta}_1 q^k \cdot \max_{[m_k-\delta_k, m_k+\delta_k]} |f'''|. \end{aligned} \tag{5.5.4}$$

With the definition of $\widetilde{\eta}_1$, using (2.2.3), (5.5.2) and (5.5.3) in (5.5.4), we finally have

$$\begin{aligned} |f''(x)| &\geq m_k^{\mu-2} \omega \cdot g_{\omega,\mu} - \frac{q^k \omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2 g_{\omega,\mu-1} \cdot g_{\omega,\mu-2}} g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu-2} \cdot q^{k(\mu-3)} \\ &= \left(q^k e^{-\frac{\varphi}{\omega}}\right)^{\mu-2} \cdot \omega \cdot g_{\omega,\mu} - \frac{q^{k(\mu-2)} \omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2} g_{\omega,\mu} \\ &= \left(q^k e^{-\frac{\varphi}{\omega}}\right)^{\mu-2} \cdot \omega \cdot g_{\omega,\mu} - \frac{\left(q^k e^{-\frac{\varphi}{\omega}}\right)^{\mu-2} \cdot \omega \cdot g_{\omega,\mu}}{2} \\ &= \frac{\left(q^k e^{-\frac{\varphi}{\omega}}\right)^{\mu-2} \cdot \omega \cdot g_{\omega,\mu}}{2}. \end{aligned}$$

This is the lower estimate for $|f''(x)|$; the upper estimate even on the interval $\left[q^{k+1}, q^k\right]$ follows with the formula for $f''$ in (2.1.4). ■

For $k \in \mathbb{N}$, we specify the boundaries of an associated 'steep' interval $S_{\ell_1(k)}$ with the next proposition.

**Proposition 5.6.** Let $k \in \mathbb{N}$. Assume $\mu$ and $\omega$ are as in Proposition 5.4 and define $\ell_1(k) = k\mu + 1$ as in the passage before the Theorem 5.1. Set $r_{\ell_1(k)} := \dfrac{(1-c)\,\omega}{g_{\omega,\mu} \cdot g_{\omega,\mu-1}} q^{\ell_1(k)}$ and $S_{\ell_1(k)} := \left[q^{\ell_1(k)} - r_{\ell_1(k)}, q^{\ell_1(k)}\right]$. Then, $S_{\ell_1(k)} \subset \left(m_{\ell_1(k)}, q^{\ell_1(k)}\right]$ and on $S_{\ell_1(k)}$ we have

$$|f'| \geq c\omega q^{\ell_1(k)(\mu-1)}. \tag{5.6.1}$$

**Proof.** Let $k \in \mathbb{N}$. From the upper estimate of (5.5.1), on $S_{\ell_1(k)}$ we have

$$\|f''\|_{\infty, S_{\ell_1(k)}} \leq g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{\ell_1(k)(\mu-2)}. \tag{5.6.2}$$

From (2.3.1) we have

$$\forall x \in S_{\ell_1(k)} : |f'(x)| \geq \left|f'\left(q^{\ell_1(k)}\right)\right| - \|f''\|_{\infty, S_{\ell_1(k)}} \cdot r_{\ell_1(k)}\ , \tag{5.6.3}$$

and from (2.2.2) we also have $\left|f'\left(q^{\ell_1(k)}\right)\right| = \omega q^{\ell_1(k)(\mu-1)}$. Using (5.6.2) and substituting the explicit values of both $\left|f'\left(q^{\ell_1(k)}\right)\right|$ and $r_{\ell_1(k)}$ in (5.6.3), we get

$$\begin{aligned}
\forall x &\in S_{\ell_1(k)} : |f'(x)| \geq \left|f'\left(q^{\ell_1(k)}\right)\right| - g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{\ell_1(k)(\mu-2)} \cdot r_{\ell_1(k)} \\
&= \omega q^{\ell_1(k)(\mu-1)} - g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{\ell_1(k)(\mu-2)} \cdot \frac{(1-c)\,\omega}{g_{\omega,\mu-1} \cdot g_{\omega,\mu}} q^{\ell_1(k)} \\
&= \omega q^{\ell_1(k)(\mu-1)} - (1-c)\,\omega q^{\ell_1(k)(\mu-1)} = c\omega q^{\ell_1(k)(\mu-1)} .
\end{aligned}$$

It follows now from $f'\left(m_{\ell_1(k)}\right) = 0$ that $m_{\ell_1(k)} < q^{\ell_1(k)} - r_{\ell_1(k)}$. ■

From the graph of the map one can understand that the image of $S_{\ell_1(k)}$ under $f_{\mu,\omega}$ includes many 'steep' and 'flat' intervals, but we continue our calculations with a subinterval $\widetilde{S_{\ell_1(k)}}$ of $S_{\ell_1(k)}$ which is contained in $f(U_k)$. The next lemma gives an estimate for the size of $f(U_k)$ with a relation between $\widetilde{S_{\ell_1(k)}}$ and $S_{\ell_1(k)}$.

Note that for the sake of simplicity we shall use $k$ as a positive odd integer number for the rest of the paper.

**Lemma 5.7.** Let $k$ be a positive odd integer number. Let $\omega$ and even integer $\mu$ be as in Proposition 5.4 and satisfying (5.2.2). Define $\widetilde{\eta}$ as in Propositon 5.4, $\delta_k$ and $U_k$ as in Proposition 5.5, and $U_k^{L\backslash R}$ as in the passage before Theorem 5.1.Then the following statements are true.

**(a)** Define $r_{\ell_1(k)}$ and $S_{\ell_1(k)}$ as in Proposition 5.6. Then we have $f(U_k) \subset S_{\ell_1(k)}$;
**(b)** Set

$$\widetilde{r_{\ell_1(k)}} := q^{\ell_1(k)} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \frac{(1-c)\cdot\omega^2}{4g_{\omega,\mu}\cdot g_{\omega,\mu-1}^2} \tag{5.7.1}$$

and $\widetilde{S_{\ell_1(k)}} = \left[q^{\ell_1(k)} - \widetilde{r_{\ell_1(k)}}, q^{\ell_1(k)}\right]$. Then we have $f\left(U_k^{L\backslash R}\right) \supset \widetilde{S_{\ell_1(k)}}$.

**Proof.** Note that due to (5.2.1), and since $k$ is odd (see (2.2.4)), $\max\{f(U_k)\} = f(m_k) = q^{\ell_1(k)}$. For the interval $U_k$ we have

$$\begin{aligned}
&\min\left\{\left|f(m_k-\delta_k) - q^{\ell_1(k)}\right|, \left|f(m_k+\delta_k) - q^{\ell_1(k)}\right|\right\} \\
\leq\ & |f(U_k)| \\
\leq\ & \max\left\{\left|f(m_k-\delta_k) - q^{\ell_1(k)}\right|, \left|f(m_k+\delta_k) - q^{\ell_1(k)}\right|\right\}.
\end{aligned}$$

It follows from second order Taylor expansion of $f$ around the extremum $m_k$ and from (5.2.1) that

$$\min_{\xi\in[m_k-\delta_k,m_k+\delta_k]} |f''(\xi)| \frac{\delta_k^2}{2} \leq |f(U_k)| \leq \max_{\xi\in[m_k-\delta_k,m_k+\delta_k]} |f''(\xi)| \frac{\delta_k^2}{2}. \tag{5.7.2}$$

Consequently, using (5.4.1) and inserting the upper estimate of $|f''|$ given in (5.5.1) and the value of $\delta_k$ in the upper estimate of (5.7.2), we finally get

$$\begin{aligned}
|f(U_k)| &\leq \max_{\xi\in[m_k-\delta_k,m_k+\delta_k]} |f''(\xi)| \frac{\delta_k^2}{2} \leq g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k(\mu-2)}\frac{\delta_k^2}{2} \\
&= g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k(\mu-2)}\widetilde{\eta}^2 q^{2k} \\
&\leq g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k(\mu-2)}\left(\widetilde{\eta}_4\right)^2 q^{2k} \\
&\leq g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k(\mu-2)}\left(\frac{\sqrt{(1-c)\,q\omega}}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}}\right)^2 q^{2k} \\
&= q^{k\mu+1}\frac{(1-c)\,\omega}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}} = \frac{(1-c)\,\omega}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}} q^{\ell_1(k)} = r_{\ell_1(k)} = \left|S_{\ell_1(k)}\right|.
\end{aligned}$$

From (5.2.1) we know that $f(m_k) = q^{\ell_1(k)}$. So, $f(U_k) = \left[\min f(U_k), q^{\ell_1(k)}\right]$ and the estimate $|f(U_k)| \leq r_{\ell_1(k)}$ shows $f(U_k) \subset \left[q^{\ell_1(k)} - r_{\ell_1(k)}, q^{\ell_1(k)}\right] = S_{\ell_1(k)}$ and this completes the proof of assertion **(a).**

Note that, although there is no symmetry between the graph of $f_{\mu,\omega}$ to the left and right hand side of $U_k$, we can estimate the size of $f\left(U_k^L\right)$ and $f\left(U_k^R\right)$ in a similar way. Substituting the lower bound of $|f''|$ given by (5.5.1), the value of $\delta_k$ in the analogue of the lower estimate of (5.7.2) for $U_k^{L\backslash R}$, and using (5.4.1) we obtain

$$\begin{aligned}
\left|f\left(U_k^{L\backslash R}\right)\right| &\geq \min_{\xi\in U_k} |f''(\xi)| \frac{\delta_k^2}{2} \geq \frac{\left(q^k e^{-\frac{\varphi}{\omega}}\right)^{\mu-2}\cdot\omega\cdot g_{\omega,\mu}}{2}\frac{\delta_k^2}{2} \\
&= \frac{\left(q^k e^{-\frac{\varphi}{\omega}}\right)^{\mu-2}\cdot\omega\cdot g_{\omega,\mu}}{4}\widetilde{\eta}^2 q^{2k} = \frac{q^{k\mu-2k}\cdot e^{-\frac{\varphi_1(\mu-2)}{\omega}}\omega\cdot g_{\omega,\mu}}{4}\left(\widetilde{\eta}_4\right)^2 q^{2k} \\
&= \frac{q^{k\mu-2k}\cdot e^{-\frac{\varphi_1(\mu-2)}{\omega}}\omega\cdot g_{\omega,\mu}}{4}\left(\frac{\sqrt{(1-c)\,q\omega}}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}}\right)^2 q^{2k} \\
&= \frac{q^{k\mu+1}\cdot e^{-\frac{\varphi_1(\mu-2)}{\omega}}\cdot\omega^2}{4}\frac{(1-c)}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}^2} \\
&= q^{\ell_1(k)}\cdot q^{\frac{\varphi_1(\mu-2)}{\pi}}\cdot\frac{(1-c)\cdot\omega^2}{4g_{\omega,\mu}\cdot g_{\omega,\mu-1}^2} = \widetilde{r_{\ell_1(k)}} = \left|\widetilde{S_{\ell_1(k)}}\right|,
\end{aligned}$$

and this completes the proof of assertion **(b)** and the proof of the lemma. ■

We continue analyzing the next 'flat' interval obtained by the second iteration of $f$.

**Lemma 5.8.** Let $k$ be a positive odd integer number. Let $\omega, \mu$ be as in Lemma 5.7. Define $\ell_1(k)$ and $\ell_2(k)$ as in the passage before Theorem 5.1. Then for $\widetilde{S_{\ell_1(k)}}$ as in Lemma 5.7, we have

$$f\left(\widetilde{S_{\ell_1(k)}}\right) \supset \left[0, q^{\ell_2(k)}\right].$$

**Proof.** Using (2.3.1) on $\widetilde{S_{\ell_1(k)}}$, we obtain

$$\left|f\left(\widetilde{S_{\ell_1(k)}}\right)\right| \geq \widetilde{r_{\ell_1(k)}}\cdot\min_{x\in\widetilde{S_{\ell_1(k)}}} |f'(x)|. \tag{5.8.1}$$

Using (5.6.1) and (5.7.1) in (5.8.1), and also the definition of $\ell_2(k)$ at the beginning of this section, we have

$$\begin{aligned}\left|f\left(\widetilde{S_{\ell_1(k)}}\right)\right| &\geq \widetilde{r_{\ell_1(k)}}\cdot\min_{x\in\widetilde{S_{\ell_1(k)}}}\left|f'(x)\right| \\ &= q^{\ell_1(k)}\cdot q^{\frac{\varphi_1(\mu-2)}{\pi}}\cdot\frac{(1-c)\cdot\omega^2}{4g_{\omega,\mu}\cdot g_{\omega,\mu-1}^2}\cdot c\omega q^{\ell_1(k)(\mu-1)} \\ &= q^{\ell_1(k)\mu}\cdot q^{\frac{\varphi_1(\mu-2)}{\pi}}\cdot\frac{c(1-c)\,\omega^3}{4g_{\omega,\mu}\cdot g_{\omega,\mu-1}^2}\geq q^{\ell_2(k)}.\end{aligned}$$

Note also that $\ell_1(k)=k\mu+1$ is odd, since $\mu$ is even. Hence, $f\geq 0$ on $\widetilde{S_{\ell_1(k)}}$ and since $f\left(q^{\ell_1(k)}\right)=0$, $f\left(\widetilde{S_{\ell_1(k)}}\right)=\left[0,\max f\left(\widetilde{S_{\ell_1(k)}}\right)\right]$. The estimate $\left|f\left(\widetilde{S_{\ell_1(k)}}\right)\right|\geq q^{\ell_2(k)}$ implies that $f\left(\widetilde{S_{\ell_1(k)}}\right)\supset\left[0,q^{\ell_2(k)}\right]$. ■

From Lemma 5.7 we know $f\left(U_k^{L\backslash R}\right)\supset\widetilde{S_{\ell_1(k)}}$. In Proposition 5.8 we showed that $f\left(\widetilde{S_{\ell_1(k)}}\right)\supset\left[0,q^{\ell_2(k)}\right]$. In particular, $U_{\ell_2(k)}\subset\left[q^{\ell_2(k)+1},q^{\ell_2(k)}\right]\subset f\left(\widetilde{S_{\ell_1(k)}}\right)$. Now, in the next lemma we estimate the counterimage of subsets of $U_{\ell_2(k)}$ under $\left(f^2\mid_{U_k^{L\backslash R}}\right)$.

**Lemma 5.9.** Let $k$ be a positive odd integer number. Assume $\mu$ is an even integer, $\mu\geq\max\left\{\left(\frac{30e}{7\pi}\right)^2\left(\frac{1-c}{2c}\right),15\right\}$ and $\omega\in(0,1)$ is a corresponding value satisfying (5.2.2) and such that the assertion of Proposition 5.4 is true (this is possible due to assertion **(b)** of Lemma 5.2). Define $\alpha(\omega,\mu,c)$ as in Proposition 5.3 and $J_k$ as in Proposition 5.5. Then, for $p\in(0,1]$ and any subinterval $\widehat{U_{\ell_2(k)}}$ of $U_{\ell_2(k)}$ with $\ell_2(k)$ as in the passage before Theorem 5.1, if

$$\left|\widehat{U_{\ell_2(k)}}\right|=p\left|J_{\ell_2(k)}\right|,\text{ then }\left(f\mid_{U_k}\right)^{-2}\left(\widehat{U_{\ell_2(k)}}\right)$$

has two parts of the form

$$\widehat{U_k^L}=\left[m_k-\delta_{k,2}^L,m_k-\delta_{k,1}^L\right]\subset U_k^L,\text{ and }\widehat{U_k^R}=\left[m_k+\delta_{k,1}^R,m_k+\delta_{k,2}^R\right]\subset U_k^R,$$

where $\delta_{k,1}^L,\delta_{k,2}^L\in\left(0,m_k-q^{k+1}\right)$ and $\delta_{k,1}^R,\delta_{k,2}^R\in\left(0,q^k-m_k\right)$, and each of them has the size

$$\left|\widehat{U_k^{L\backslash R}}\right|\leq\alpha\cdot p\cdot\left|J_k\right|. \tag{5.9.1}$$

**Proof.** Set $\widehat{S_{\ell_1(k)}}:=\left(f\mid_{\widetilde{S_{\ell_1(k)}}}\right)^{-1}\left(\widehat{U_{\ell_2(k)}}\right)$. Note that injectivity of $f\mid_{S_{\ell_1(k)}}$ and Lemma 5.8 imply that $\left(f\mid_{S_{\ell_1(k)}}\right)^{-1}\left(\widehat{U_{\ell_2(k)}}\right)=\left(f\mid_{\widetilde{S_{\ell_1(k)}}}\right)^{-1}\left(\widehat{U_{\ell_2(k)}}\right)$. Using (2.3.1) on $\widehat{S_{\ell_1(k)}}$, we have

$$\left|\left(f\mid_{\widetilde{S_{\ell_1(k)}}}\right)^{-1}\left(\widehat{U_{\ell_2(k)}}\right)\right| = \left|\widehat{S_{\ell_1(k)}}\right| \le \frac{\left|\widehat{U_{\ell_2(k)}}\right|}{\min_{\widehat{S_{\ell_1(k)}}} |f'|}.$$

On the other hand, from Proposition 5.6 we already know that on $S_{\ell_1(k)}$, $|f'| \ge c\omega q^{\ell_1(k)(\mu-1)}$. Because of $\widehat{S_{\ell_1(k)}} \subset \widetilde{S_{\ell_1(k)}} \subset S_{\ell_1(k)}$, this property also satisfied on $\widehat{S_{\ell_1(k)}}$. Hence, inserting both $\left|\widehat{U_{\ell_2(k)}}\right| = p\left|J_{\ell_2(k)}\right|$ and the estimate of $\min_{S_{\ell_1(k)}} |f'|$ in the last expression, we have

$$\left|\widehat{S_{\ell_1(k)}}\right| \le \frac{\left|\widehat{U_{\ell_2(k)}}\right|}{\min_{\widehat{S_{\ell_1(k)}}} |f'|} \le \frac{p\left|J_{\ell_2(k)}\right|}{\min_{S_{\ell_1(k)}} |f'|} \le \frac{p \cdot q^{\ell_2(k)}(1-q)}{c\omega q^{\ell_1(k)(\mu-1)}}. \tag{5.9.2}$$

Now, we calculate subintervals of $(m_k - \delta_k, m_k + \delta_k)$ which get mapped bijectively to $\widehat{S_{\ell_1(k)}}$. Note that the counterimage of $\widehat{S_{\ell_1(k)}}$ has two parts in the form $\widehat{U_k^L} \subset U_k^L$, and $\widehat{U_k^R} \subset U_k^R$. It follows from strict monotonicty of $f$ on $[m_k - \delta_k, m_k]$ and $[m_k, m_k + \delta_k]$ and from the fact that $f\left(U_k^{L\backslash R}\right) \supset \widetilde{S_{\ell_1(k)}}$ that there exist $\delta_{k,1}^{L\backslash R}, \delta_{k,2}^{L\backslash R}$ with

$$\begin{aligned}
\left|f\left(\widehat{U_k^R}\right)\right| &= \left|f\left(\left[m_k + \delta_{k,1}^R, m_k + \delta_{k,2}^R\right]\right)\right| \\
&= \left|f\left(\left[m_k - \delta_{k,2}^L, m_k - \delta_{k,1}^L\right]\right)\right| = \left|f\left(\widehat{U_k^L}\right)\right| = \left|\widehat{S_{\ell_1(k)}}\right|.
\end{aligned} \tag{5.9.3}$$

We continue our calculations by using the boundaries of $\widehat{U_k^R}$. Note that for the interval $\left[m_k, m_k + \delta_{k,1}^R\right]$ we know that $f\left(m_k + \delta_{k,1}^R\right) = \max \widehat{S_{\ell_1(k)}}$ and $f(m_k) = q^{\ell_1(k)}$. Again from the monotonicity of the map it follows that $f\left(\left[m_k, m_k + \delta_{k,1}^R\right]\right) = \left[\max \widehat{S_{\ell_1(k)}}, q^{\ell_1(k)}\right]$. Consequently, since $f\left(q^{\ell_1(k)}\right) = 0$ and $f\left(\max \widehat{S_{\ell_1(k)}}\right) \in \left[q^{\ell_2(k)+1}, q^{\ell_2(k)}\right]$, from (2.3.1) we have

$$\left|\max \widehat{S_{\ell_1(k)}} - q^{\ell_1(k)}\right| \ge \frac{q^{\ell_2(k)+1}}{\|f'\|_{\infty, S_{\ell_1(k)}}}. \tag{5.9.4}$$

From (2.1.1) we also have that $\|f'\|_{\infty, S_{\ell_1(k)}} \le g_{\omega,\mu} \cdot q^{\ell_1(k)(\mu-1)}$. Inserting this estimate in (5.9.4), we obtain

$$\left|\max \widehat{S_{\ell_1(k)}} - q^{\ell_1(k)}\right| \ge \frac{q^{\ell_2(k)+1}}{g_{\omega,\mu} \cdot q^{\ell_1(k)(\mu-1)}}. \tag{5.9.5}$$

In addition, from (2.1.4) we know that

$$\|f''\|_{\infty, U_k} \le g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{k(\mu-2)}. \tag{5.9.6}$$

Now, using the second order Taylor expansion of $f\left(m_k + \delta_{k,1}^R\right)$, we have

$$\left|f\left(m_k + \delta_{k,1}^R\right) - f(m_k)\right| \le \left|f''(\xi)\frac{\left(\delta_{k,1}^R\right)^2}{2}\right|, \tag{5.9.7}$$

where $\xi \in \left(m_k, m_k + \delta_{k,1}^R\right)$. Substituting the values of $f\left(m_k + \delta_{k,1}^R\right)$ and $f\left(m_k\right)$ in (5.9.7), we have

$$\begin{aligned} \left|f\left(m_k+\delta_{k,1}^R\right)-f\left(m_k\right)\right| &= \left|\max \widehat{S_{\ell_1(k)}} - q^{\ell_1(k)}\right| \leq \|f''\|_{\infty,U_k} \frac{\left(\delta_{k,1}^R\right)^2}{2}, \text{ which implies} \\ \delta_{k,1}^R &\geq \sqrt{2\frac{\left|\max \widehat{S_{\ell_1(k)}} - q^{\ell_1(k)}\right|}{\|f''\|_{\infty,U_k}}}. \end{aligned} \tag{5.9.8}$$

Using both estimates (5.9.5) and (5.9.6) in (5.9.8), we finally get

$$\delta_{k,1}^R \geq \sqrt{2\frac{q^{\ell_2(k)+1}}{g_{\omega,\mu}^2 \cdot g_{\omega,\mu-1} \cdot q^{k(\mu-2)} \cdot q^{\ell_1(k)(\mu-1)}}}. \tag{5.9.9}$$

On the other hand, from Taylor's formula with the integral remainder term we have

$$\begin{aligned} f\left(m_k+\delta\right) &= f\left(m_k\right) + \int_{m_k}^{m_k+\delta} \left(m_k+\delta-t\right) f''(t)\, dt \\ &= f\left(m_k\right) + \int_0^{\delta} (\delta - t) f''\left(m_k+t\right) dt. \end{aligned} \tag{5.9.10}$$

Consequently, applying (5.9.10) for the boundaries of $\widehat{U_k^R}$, we have

$$\begin{aligned} \left|\widehat{S_{\ell_1(k)}}\right| &= \left|f\left(\widehat{U_k^R}\right)\right| = \left|f\left(m_k+\delta_{k,2}^R\right) - f\left(m_k+\delta_{k,1}^R\right)\right| \\ &= \left|\int_0^{\delta_{k,2}^R} \left(\delta_{k,2}^R - t\right) f''\left(m_k+t\right) dt - \int_0^{\delta_{k,1}^R} \left(\delta_{k,1}^R - t\right) f''\left(m_k+t\right) dt\right|. \end{aligned}$$

From (5.5.1) we already know that $M := \min_{x\in U_k} |f''(x)| \geq \dfrac{q^{k(\mu-2)} q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \omega \cdot g_{\omega,\mu}}{2}$. In particular, $f''$ has constant sign on $U_k$. Using the fact that $\delta_{k,1}^R < \delta_{k,2}^R$ in the last equality, we obtain

$$\begin{aligned} \left|\widehat{S_{\ell_1(k)}}\right| &= \left|\int_0^{\delta_{k,1}^R} \left(\delta_{k,2}^R - \delta_{k,1}^R\right) f''\left(m_k+t\right) dt + \int_{\delta_{k,1}^R}^{\delta_{k,2}^R} \left(\delta_{k_2}^R - t\right) f''\left(m_k+t\right) dt\right| \\ &\geq \left|\int_0^{\delta_{k,1}^R} \left(\delta_{k,2}^R - \delta_{k,1}^R\right) f''\left(m_k+t\right) dt\right| \geq \left|\delta_{k,2}^R - \delta_{k,1}^R\right| \cdot M \cdot \delta_{k,1}^R, \end{aligned}$$

so

$$\left|\delta_{k,2}^R - \delta_{k,1}^R\right| \leq \frac{\left|\widehat{S_{\ell_1(k)}}\right|}{M \cdot \delta_{k,1}^R}. \tag{5.9.11}$$

Substituting the estimate of $M$ and the estimate $\delta_{k,1}^R$ given by (5.9.9) in (5.9.11), we obtain

$$\left|\delta_{k,2}^R - \delta_{k,1}^R\right| \leq \frac{\left|\widehat{S_{\ell_1(k)}}\right|}{\dfrac{q^{k(\mu-2)} q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \omega \cdot g_{\omega,\mu}}{2} \cdot \sqrt{2\dfrac{q^{\ell_2(k)+1}}{q^{k(\mu-2)} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu}^2 \cdot q^{\ell_1(k)(\mu-1)}}}}. \tag{5.9.12}$$

Combining the estimate of $\widehat{S_{\ell_1(k)}}$ given by (5.9.2) with (5.9.12), we finally have

$$\begin{aligned}
\left|\widehat{U_k^R}\right| &= \left|\delta_{k,2}^R - \delta_{k,1}^R\right| \\
&\leq \frac{\sqrt{2}p \cdot q^{\ell_2(k)}(1-q)}{c\omega q^{\ell_1(k)(\mu-1)}} \frac{\sqrt{q^{k(\mu-2)} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu}^2 \cdot q^{\ell_1(k)(\mu-1)}}}{q^{k(\mu-2)} q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \omega \cdot g_{\omega,\mu} \cdot \sqrt{q^{\ell_2(k)+1}}} \\
&= \sqrt{2}p \frac{q^{\ell_2(k)}(1-q)}{c\omega q^{\ell_1(k)(\mu-1)}} \cdot \frac{\sqrt{q^{k(\mu-2)} g_{\omega,\mu-1}}}{q^{k(\mu-2)} q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \omega} \cdot \frac{\sqrt{q^{\ell_1(k)(\mu-1)}}}{\sqrt{q^{\ell_2(k)+1}}} \\
&= \sqrt{2}p \frac{(1-q)\sqrt{g_{\omega,\mu-1}}}{c\omega^2 \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \frac{q^{\ell_2(k)}}{q^{\frac{\ell_2(k)}{2}}} \cdot \frac{\sqrt{q^{k(\mu-2)}}}{q^{k(\mu-2)+\frac{1}{2}}} \cdot \frac{\sqrt{q^{\ell_1(k)(\mu-1)}}}{q^{\ell_1(k)(\mu-1)}} \\
&= \frac{\sqrt{2}p \cdot q^k (1-q) \cdot q^{\frac{\ell_2(k)}{2}}}{q^{\frac{\ell_1(k)(\mu-1)}{2}} \cdot q^{\frac{k\mu+1}{2}} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \frac{\sqrt{g_{\omega,\mu-1}}}{c\omega^2}.
\end{aligned}$$

Here, using the estimate of $q^{\ell_2(k)}$ given in the passage before Theorem 5.1 and $|J_k| = q^k(1-q)$, we obtain

$$\begin{aligned}
\left|\widehat{U_k^R}\right| &\leq \frac{\sqrt{2}p \cdot |J_k| \cdot \sqrt{q^{\ell_1(k)\mu} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \frac{c(1-c)\omega^3}{4g_{\omega,\mu} \cdot g_{\omega,\mu-1}^2}}}{q^{\frac{\ell_1(k)(\mu-1)}{2}} \cdot q^{\frac{k\mu+1}{2}} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \frac{\sqrt{g_{\omega,\mu-1}}}{c\omega^2} \\
&= \frac{\sqrt{2}}{2} p \cdot |J_k| \cdot \frac{\sqrt{q^{\ell_1(k)\mu}}}{q^{\frac{\ell_1(k)\mu - \ell_1(k)}{2}} \cdot q^{\frac{k\mu+1}{2}}} \cdot \frac{\sqrt{q^{\frac{\varphi_1(\mu-2)}{\pi}}}}{q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \sqrt{\frac{(1-c)}{c\omega g_{\omega,\mu} \cdot g_{\omega,\mu-1}}}.
\end{aligned}$$

Inserting $\ell_1(k) = k\mu + 1$ one gets

$$\begin{aligned}
\left|\widehat{U_k^R}\right| &\leq \frac{\sqrt{2}}{2} p \cdot |J_k| \cdot \frac{\sqrt{q^{(k\mu+1)\mu}}}{q^{\frac{(k\mu+1)\mu - (k\mu+1)}{2}} \cdot q^{\frac{k\mu+1}{2}}} \cdot \frac{\sqrt{q^{\frac{\varphi_1(\mu-2)}{\pi}}}}{q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \sqrt{\frac{(1-c)}{c\omega g_{\omega,\mu} \cdot g_{\omega,\mu-1}}} \\
&= p \cdot |J_k| \cdot q^{-\frac{\varphi_1(\mu-2)}{2\pi}} \cdot \sqrt{\frac{(1-c)}{2c\omega g_{\omega,\mu} \cdot g_{\omega,\mu-1}}}.
\end{aligned}$$

Since $g_{\omega,\mu-1} < g_{\omega,\mu}$, we can simplify the last inequality as follows:

$$\begin{aligned}
\left|\widehat{U_k^R}\right| &\leq p \cdot |J_k| \cdot q^{-\frac{\varphi_1(\mu-2)}{2\pi}} \cdot \sqrt{\frac{(1-c)}{2c\omega g_{\omega,\mu-1}^2}} \\
&= p \cdot |J_k| \cdot \frac{q^{-\frac{\varphi_1(\mu-2)}{2\pi}}}{g_{\omega,\mu-1}} \cdot \sqrt{\frac{1-c}{2c\omega}}.
\end{aligned}$$

Inserting $q = e^{-\frac{\pi}{\omega}}$ we have

$$\begin{aligned} \left|\widehat{U_k^R}\right| &\leq p \cdot |J_k| \cdot \frac{\exp\left(\dfrac{\pi}{\omega} \cdot \dfrac{(\mu - 2)\,\varphi_1}{2\pi}\right)}{g_{\omega,\mu-1}} \cdot \sqrt{\frac{1-c}{2c\omega}} \\ &= p \cdot |J_k| \cdot \frac{\exp\left(\dfrac{(\mu - 2)\,\varphi_1}{2\omega}\right)}{g_{\omega,\mu-1}} \cdot \sqrt{\frac{1-c}{2c\omega}}. \end{aligned}$$

Finally, using the definition of $\alpha(\mu, \omega, c)$, we get

$$\left|\widehat{U_k^R}\right| \leq \alpha \cdot p \cdot |J_k|,$$

and this completes the proof for $\widehat{U_k^R}$. The proof for $\widehat{U_k^L}$ is analogous.■

**Corollary 5.10.** If the set $\widehat{U_{\ell_2(k)}}$ in Lemma 5.9 is not only one interval, but a disjoint union of subintervals of $U_{\ell_2(k)}$, and $\left|\widehat{U_{\ell_2(k)}}\right|$ (the measure of $\widehat{U_{\ell_2(k)}}$) satisfies $\left|\widehat{U_{\ell_2(k)}}\right| = p\left|J_{\ell_2(k)}\right|$, then $(f\mid_{U_k})^{-2}\left(\widehat{U_{\ell_2(k)}}\right)$ has two parts (one in $U_k^L$ and the other in $U_k^R$) and each of them has measure less or equal $\alpha p\,|J_k|$.

**Proof.** (By summation over the subintervals.) ■

Now, we consider symbol sequences of the form $(L, R, R, L, R, ..., R)$ and construct corresponding orbits of $f$. For given a finite sequence

$$\mathbf{s} = (s_0, s_1, s_2, ..., s_n) \in \{L, R\}^{n+1}$$

and odd $k \in \mathbb{N}$, we now construct the subset of points $x$ in $U_k$ which follow this symbol sequence. Recall the set $I_{k,\mathbf{s}}^n = \bigcap_{j=0}^{n} f^{-2j}\left(U_{\ell_2^j(k)}^{s_j}\right)$ defined in the passage before Theorem 5.1. We estimate the size of $\left|I_{k,\mathbf{s}}^n\right|$.

**Corollary 5.11.** Let $\mathbf{s} = (s_0, s_1, s_2, ..., s_n)$ and an odd $k \in \mathbb{N}$ be given. Then, with $\omega, \mu$ as in Lemma 5.9 and $\alpha(\omega, \mu, c)$ as in Proposition 5.3 we have $\emptyset \neq I_{k,\mathbf{s}}^n$ and

$$\left|I_{k,\mathbf{s}}^n\right| \leq \alpha^n\,|J_k|.$$

**Proof.** We prove the corollary by induction over $n$. For $n = 0$, $I_{k,\mathbf{s}}^0 = U_k^{s_0} \neq \emptyset$, and

$$\left|I_{k,\mathbf{s}}^0\right| = \left|U_k^{s_0}\right| \leq |J_k|.$$

Now, we assume the result is true for $n$, and we verify it for $n+1$. Let $\mathbf{s} = (s_0, s_1, s_2, ..., s_{n+1})$ be given. Define $\widetilde{\mathbf{s}} = (s_1, s_2, ..., s_{n+1})$. From the induction hypothesis we have $I_{\ell_2(k),\widetilde{\mathbf{s}}}^n \neq \emptyset$, $I_{\ell_2(k),\widetilde{\mathbf{s}}}^n \subset U_{\ell_2(k)}$, and

$$\begin{aligned}\left|I^n_{\ell_2(k),\widetilde{\mathbf{s}}}\right| &= \left|\bigcap_{j=0}^{n} f^{-2j}\left(U^{s_{j+1}}_{\ell_2^j(\ell_2(k))}\right)\right| \\ &\leq \alpha^n \left|J_{\ell_2(k)}\right|.\end{aligned}$$

Note that $I^{n+1}_{k,\mathbf{s}} = f^{-2}\left(I^n_{\ell_2(k),\widetilde{\mathbf{s}}}\right) \cap U^{s_0}_k$. Hence, we have

$$\left|I^{n+1}_{k,\mathbf{s}}\right| = \left|f^{-2}\left(I^n_{\ell_2(k),\widetilde{\mathbf{s}}}\right) \cap U^{s_0}_k\right|. \tag{5.11.1}$$

Applying Corollary 5.10 with $p := \alpha^n$ and $I^n_{\ell_2(k),\widetilde{\mathbf{s}}}$ instead of $\widehat{U_{\ell_2(k)}}$ in (5.11.1), and using this $p$ together with (5.9.1), we finally obtain

$$\left|I^{n+1}_{k,\mathbf{s}}\right| = \left|f^{-2}\left(I^n_{\ell_2(k),\widetilde{\mathbf{s}}}\right) \cap U^{s_0}_k\right| \leq \alpha \cdot p \cdot |J_k| = \alpha^{n+1} |J_k|.$$

This completes the induction and the proof of corollary 5.11. ■

**Proof of Theorem 5.1.** Assume $k$, $c$ and $\mu$ are as in the assumptions of the Theorem 5.1 and, $\alpha = \alpha(\omega, \mu, c)$ be as in Proposition 5.3, so that $\alpha < \dfrac{1}{2}$. Choose $\omega \in (0, 1)$ as in Lemma 5.2.

**(1).** Let a symbol sequence $s = (s_0, s_1, s_2, ...) \in \{L, R\}^{\mathbb{N}_0}$ be given. From Corollary 5.11 one can see that for $n \in \mathbb{N}_0$ the the closed interval $I^n_{k,s}$ consists of the points $x \in U_k$ which follow the finite symbol sequence $s = (s_0, s_1, s_2, ..., s_n) \in \{L, R\}^{n+1}$. Further we have $I^{n+1}_{k,s} \subset I^n_{k,s}$. It follows that $\bigcap\limits_{n\in\mathbb{N}_0} I^n_{k,s} \neq \emptyset$. Since, in view of Corollary 5.11 and $\alpha < \dfrac{1}{2}$, we have $\left|I^n_{k,s}\right| \to 0$ for $n \to \infty$, the intersection $\bigcap\limits_{n\in\mathbb{N}_0} I^n_{k,s}$ contains exactly one point $x_{k,s}$. This point $x_{k,s}$ has the asserted properties. Any point in $U_k$ with these properties would also be contained in this intersection and thus equal $x_{k,s}$.

**(2).** The set $\{L, R\}^{\{0,1,\ldots,n\}}$ has $2^{n+1}$ elements and from Corollary 5.11 we know that each set corresponding to one $s \in \{L, R\}^{\{0,1,2,..,n\}}$ satisfies the estimate $\left|I^n_{k,\mathbf{s}}\right| \leq \alpha^n |J_k|$. It follows that $|\Gamma^n_k| \leq 2^{n+1}\alpha^n |J_k|$, and it turns out that the measure

$$\lim_{n\to\infty} |\Gamma^n_k| = \lim_{n\to\infty} \left|\bigcup_{s\in\{L,R\}^{\{0,1,2,..,n\}}} I^n_{k,\mathbf{s}}\right| \leq 2 \lim_{n\to\infty} 2^n\alpha^n |J_k| = 0$$

and this completes the proof. ■

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