Periodic solutions of the N point-vortex problem in planar domains

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Chapter 1

Introduction

1.1 Point vortex model

The concept of point vortex model, introduced by Helmholtz [45] in 1858, is a classical and active topic in the field of incompressible fluid mechanics. Various practical applications of point vortex dynamics, e.g. in the study of large scale weather pattern, turbulent flow, atmosphere of planets, wake of boats, etc, make it worthwhile to deeply explore the point vortex motions.

Aside from its physical relevance, this model attracts much attention as a mathematical entity as well. Surprisingly, research of it has touched upon a large number of areas of Mathematics, not only the theory of dynamical systems, Hamiltonian methods, singular perturbation theory, stability techniques and numerical methods which appear expectedly, but also some others that seem irrelevant at the fist sight, like knots theory, Lie group, polynomial theory, etc. Over 150 years, considerable progress has been achieved both theoretically and numerically. For the set-up of point vortex system in mathematical aspect we refer to the work of Kirchhoff [47], Routh [68], Lin [54, 55], Saffman [70], and Gustafsson [38, 42], to name just a few.

In order to better understand this highly idealized system, we will show how it arises from the Euler equation of the fluid motion and has the Hamiltonian nature. We start with a two dimensional flow domain $\Omega \subset \mathbb{C}$. Particularly, the fluid is assumed to be ideal, i.e. incompressible and non-viscous with constant density $\rho = 1$. The velocity field of the fluid, denoted by v, satisfies the Euler equations

$$\begin{cases} v_t + (v \cdot \nabla)v = -\nabla P \\ \nabla \cdot v = 0 \end{cases} \tag{1.1.1}$$

where P is the pressure of the fluid. The first equation represents Newton's law for the fluid, with the left hand side describing the acceleration of the fluid and the right hand side the force acting on it. Here we assume that the only force is the pressure force. The second equation comes from the incompressible assumption, showing that the velocity field is divergence-free.

Let us define the *vorticity* as the curl of velocity:

$$\omega = \nabla \times v = \partial_1 v^2 - \partial_2 v^1.$$

Then taking the curl of the Euler equation, and using the fact that the curl of a gradient is always zero yield the transport equation for the vorticity:

$$\frac{D\omega}{Dt} = \omega_t + (v \cdot \nabla)\omega = 0, \tag{1.1.2}$$

from which we can say that the vorticity of a fluid particle is conserved in the co-moving frame of the fluid.

The velocity field of incompressible fluid in two dimensions is determined by a scalar stream function ψ. At a point z, the velocity generated is then computed according to a Hamiltonian system

$$v(z) = -i\nabla\psi(z)$$

with the stream function as a Hamiltonian function. Thus by the definition of the vorticity, the stream function ψ solves the Poisson equation

$$\begin{cases} -\Delta\psi = \omega & \text{in } \Omega, \\ \partial_{J\nu}\psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1.3}$$

where $\partial_{J\nu}$ denotes the tangential derivative.

We make an ansatz that the vorticity field is zero except a finite number of concentrated points. In this consideration, the vorticity could be written as a finite sum of fluid particles represented by a collection of discrete Dirac delta functions

$$\omega = \sum_{k=1}^{N} \Gamma_k \delta_{z_k}, \quad z_k \in \Omega.$$

Γ_k is referred to as the *strength* or *circulation* of the k^{th} vortex at position z_k, which may be positive or negative according to its orientation. The N fluid particles with nonzero vorticities are called *point vortices*. According to the vorticity equation (1.1.2) the circulation of a point vortex does not change with time as the vortex moves through the fluid.

Now given a vorticity field, the existence of solution to the Poisson equation allows us to use an appropriate Green's function to solve for the stream function, and therefore the velocity. If $\Omega = \mathbb{C}$ is the plane without boundary, due to the work of Kirchhoff [38], the stream function is

$$\psi_{\mathbb{C}}(z; z_1, \cdots, z_N) = -\frac{1}{2\pi}\sum_{k=1}^{N} \Gamma_k \log|z - z_k|.$$

If $\Omega \neq \mathbb{C}$ is a domain, observed by Routh [68], one has to take the additional influence of the boundary into account. If Ω is a bounded, simply or multiply connected domain with boundary $\partial\Omega$, the solution of the Dirichlet problem

$$\begin{cases} \Delta_z G(z,w) + \delta(|z-w|) = 0 & \text{in } \Omega, \\ G(z,w) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1.4}$$

is called *Green's function of the first kind* on Ω. G can be decomposed into two parts:

$$G(z,w) = -\frac{1}{2\pi}\log|z-w| - g(z,w) \qquad \text{for } z,w\in\Omega,\ z\neq w. \tag{1.1.5}$$

The radially symmetric singular part is just the Green's function in the unbounded plane, associated with the vortex-vortex interaction, while the regular part g is harmonic in Ω and enforced to satisfy the boundary condition, representing the vortex-boundary interaction. g and G are both symmetric. The function

$$h:\Omega\to\mathbb{R},\ h(z)=g(z,z)$$

is called the hydrodynamic Robin function (see [38]). $h(z)\to\infty$ when $z\to\partial\Omega$.

If Ω is unbounded, like the upper half-plane, G is no longer unique. Then additional condition on the behavior of the Green's function at infinity must be imposed to ensure its uniqueness.

Kirchhoff first recognized that since each point vortex moves with the local velocity of the fluid and the vorticity is conserved, motion of the point vortices is itself governed by a Hamiltonian system. One can then track the motion of the fluid just at these point vortices. Until 1943, the motion equations of N point-vortex in general domain Ω were finally shown by C. C. Lin in [54], stated as follows:

Theorem 1.1.1. *[54] Assume $\Omega\subset\mathbb{C}$ is a bounded domain, simply or multiply connected. Let $\psi_\Omega(z;z_1,\cdots,z_N)$ be the stream function for the fluid motion on Ω determined by N point vortices $\{z_k|\,k=1,\cdots,N\}\subset\Omega$ with strengths $\{\Gamma_k|\,k=1,\cdots,N\}$. Then*

(i)

$$\begin{aligned}\psi_\Omega(z;z_1,\cdots,z_N) &= \sum_{k=1}^N \Gamma_k G(z,z_k)\\ &= -\frac{1}{2\pi}\sum_{k=1}^N \Gamma_k\log|z-z_k| - \sum_{k=1}^N \Gamma_k g(z,z_k)\end{aligned} \tag{1.1.6}$$

with the corresponding Green's function G and its regular part g defined as in (1.1.4) *and* (1.1.5)*;*

(ii) the motions of these N vortices are governed by a Hamiltonian system

$$\Gamma_k\dot z_k = -i\nabla_{z_k}H_\Omega(z), \quad k=1,\cdots,N, \tag{1.1.7}$$

where

$$H_\Omega(z_1,\cdots,z_N) = \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^N \Gamma_j\Gamma_k G(z_j,z_k) - \frac{1}{2}\sum_{k=1}^N \Gamma_k^2 g(z_k,z_k) \tag{1.1.8}$$

with phase space $\mathcal{F}_N\Omega=\{z\in\Omega^N : z_j\neq z_k \text{ for } j\neq k\}$.

We see that the stream function is a linear superposition of Green's functions with respect to each source z_k and the Hamiltonian function is derived by contracting the stream function with the singular vorticity distribution. In the classical literature, the Hamiltonian is referred to as the *Kirchhoff-Routh path function*.

The construction of G is intimately related to the details of the boundary shape. For vortices in a domain possessing special symmetries, such as vortices in the upper bounded plane, inside or outside a circle, in the positive quadrant, the Green's functions can be obtained using a standard "method of images" (see [30]). A more general technique based on conformal mapping is given by C.C. Lin [55] to deal with point vortex in domains where the "method of images" becomes complicated or unclear.

1.2 Stationary and periodic solutions

It is natural then to ask: what types of solutions are allowed by these motion equations? Here we are only interested in periodic solutions (including stationary solutions), which repeat after a finite time.

Stationary configuration is certainly the simplest pattern of vortex periodic motions, determined by the critical points of H_Ω. However, it is already a highly nontrivial problem in consideration of the complexity of H_Ω for general domain Ω. The configuration space $\mathcal{F}_N\Omega$ is noncompact in $\mathbb{C}^N$ and H_Ω is unbounded from both above and below. Indeed, generally $H_\Omega(z)$ may approach any value in $\mathbb{R} \cup \{\pm\infty\}$ if some of the z_k's approach the boundary $\partial\Omega$. This implies there is no definite behavior of $H_\Omega(z)$ as $z \to \partial\mathcal{F}_N\Omega$, thus the energy surfaces are noncompact and the Palais-Smale condition fails. The only exception is the case $N = 2$ and $\Gamma_1\Gamma_2 < 0$ when H_Ω is bounded above and $H_\Omega(z_1, z_2) \to -\infty$ if $z_k \to \partial\Omega$ or $z_1 - z_2 \to 0$. Thus energy surfaces are compact and Palais-Smale condition holds.

In the paper [15], which mostly deals with the case $N = 2, \Gamma_1 + \Gamma_2 = 0$, the existence of at least two critical points of H_Ω has been proven for any smooth bounded domain. Later, Bartsch, Pistoia and Weth [17] proved when $N = 3$ or 4, $\Gamma_k = (-1)^k$, H_Ω processes a critical point for arbitrary $\mathcal{C}^2$-bounded domain. In their proof, a min-max argument yields a Palais-Smale sequence for H_Ω and a careful investigation of the behavior of H_Ω near $\partial\mathcal{F}_N\Omega$ shows H_Ω is bounded from above along a flow line of the gradient flow of it. Then H_Ω satisfies the Palais-Smale condition and the Palais-Smale sequence gives rise to a non-collision critical point of H_Ω. Moreover, If Ω is symmetric with respect to a line through it, for any $N \in \mathbb{N}$, $\Gamma_k = (-1)^k$, H_Ω has a critical point with all components lying on this line. The result without symmetry is improved in [16] where the existence of a critical point of H_Ω is obtained for $2 \leqslant N \leqslant 4$ and the Γ_k's having alternating signs and satisfying a certain condition. Under their hypothesis, any $\mathcal{C}^1$-function $F : \mathcal{F}_N\Omega \to \mathbb{R}$ which is $\mathcal{C}^1$-close to H_Ω on certain compact subset of $\mathcal{F}_N\Omega$ has a critical point in this subset. When Ω is bounded, not simply connected, and all the Γ_k's are equal, del Pino, Kowalczyk and Musso [35] proved that a critical point of H_Ω exists.

It is a basic problem in classical or celestial mechanics to study the periodic solutions of the

Hamiltonian system

$$\dot{z} = \mathcal{J} H_z(z). \tag{1.2.1}$$

These solutions can be found as critical points of the action functional

$$J(z) = \frac{1}{2}\int_0^T \mathcal{J}\dot{z}\cdot z\,dt + \int_0^T H(z)\,dt \tag{1.2.2}$$

on a suitable space of T-periodic functions.

This variational structure suggests results should be obtained through variational approach. However in the beginning, except in second order cases, the direct method of the calculus of variations which deals with absolute minima is unapplicable. This is due to the fact that the dominant integral of J is strongly indefinite, i.e., unbounded from below and from above. Until 1978, it was Rabinowitz [66] who successfully obtained for the first time periodic solutions of the first order system (1.2.1) by a variational principle. Since then general critical point theory for indefinite functionals was highly developed.

When the energy hypersurfaces are compact, this problem is well understood and plenty of famous results are found in the literature. For some pioneering work dealing with Hamiltonian systems via variational methods, we refer to the monographs [1, 56, 67] and the references therein. It is worth mentioning that Weinstein [80] conjectured that any contact type energy hypersurface admits a periodic orbit. This conjecture was first solved by Viterbo [77] in 1987 and generalized to the case of compact hypersurfaces in the cotangent bundle of a compact manifold by Hofer and Viterbo [46, 78].

If the Hamiltonian is singular, in general the energy levels are not compact any more, which is the case we discuss in this thesis, and then the situation turns out to be more complicated. There are two main features of the functional related to singular Hamiltonians, namely, that they are defined on an open subset of a Sobolev space and that they do not satisfy the usual compactness condition. Although there exists a large literature on periodic solutions of singular Hamiltonian systems, most are dealing with second order (Lagrangian) systems related to Kepler or N-body problems:

$$\ddot{q} + \nabla V(q) = 0 \tag{1.2.3}$$

with a singular potential $V \in \mathcal{C}^1(\mathbb{R}^n\backslash\{0\}, \mathbb{R})$. In this case, the Hamiltonian has the form $H(q,\dot{q}) = \frac{1}{2}|\dot{q}|^2 + V(q)$. The dominant integral in the corresponding action functional is $1/2\int_0^T |\dot{q}|^2\,dt$ defined on $H^1(\mathbb{R}/T\mathbb{Z},\mathbb{R}^n)$. Since $H^1(\mathbb{R}/T\mathbb{Z},\mathbb{R}^n)$ embeds into $L^\infty(\mathbb{R}/T\mathbb{Z},\mathbb{R}^n)$, $H^1(\mathbb{R}/T\mathbb{Z},\mathbb{R}^n\backslash\{0\})$ defines an open subset of $H^1(\mathbb{R}/T\mathbb{Z},\mathbb{R}^n)$. To avoid collisions, the rate of growth of V near the singularity plays a key role. Consider the model case $V(q) = -|q|^{-\alpha}, \alpha > 0$. If $\alpha \geqslant 2$, known as the "strong force condition" first used by Gordon in [39], (1.2.3) possesses infinitely many non-collision periodic solutions of prescribed period and periodic orbit exists on every positive energy surface. If $\alpha \in (0,2)$, which is called the "weak force condition", existence results are also explored in several directions including both prescribed period and prescribed energy problems. Many authors generalized the cases for general potential $V(q) \sim -|q|^{-\alpha}$, see [39, 65] for strong force case and [82, 74] for weak force case. Moreover, when V is singular not simply on $\{0\}$ but on a compact subset in $\mathbb{R}^n$, an analogous result holds, see [2]. We also refer to the book [3] of Ambrosetti and Coti Zelati for an overall view of this subject.

However, the investigation on periodic solutions of first order Hamiltonian systems is not so developed as second order ones. Nearly all existing papers in this direction are still concerned with a class of systems in the form of N-body type, that is, $H(p,q) \sim \frac{1}{\beta}|p|^\beta - \frac{1}{|q|^\alpha}, \alpha, \beta > 0$ with singularity at $q = 0$. In [75], Tanaka presented a minimax argument and obtained the existence of classical periodic solutions under the condition $\alpha \geqslant \beta > 1$. $\beta = 2, \alpha \geqslant 2$ corresponds to the strong force case of second order system. The function space is chosen to be the product of $L^\beta(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n)$ for p and $W^{1,\beta/(\beta-1)}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^n)$ for q so that the action functional is well defined and the (PS) condition is fulfilled as well. The case $\beta > \alpha > 1$ was treated by Boughariou in [21] where the existence of generalized (collision set has zero measure) periodic solutions has been proved. $\beta = 2$, $\alpha < 2$ corresponds to the weak force case of second order system. The same applies to [25] where periodic solutions on fixed energy surfaces have been found. The energy surface has to be of contact type, and the existence is obtained by reduction to a theorem of [46] on the Weinstein conjecture in cotangent bundles of manifolds.

Clearly, in our system (1.1.7)-(1.1.8) the singularity set $\{z \in \Omega^N : \exists\, j \neq k, \text{such that}\, z_j = z_k\}$ is much more complicated and the behavior of the Hamiltonian with respect to the conjugated variables x and y is completely different from the p and q in [75, 25]. In fact, neither the results nor the techniques of the existing papers on singular Hamiltonian systems apply here. In addition to the well-known technical problems due to the strong indefiniteness of the action functional for T-periodic solutions

$$J(z) = \frac{1}{2}\sum_{k=1}^{N}\int_0^T (i\dot{z}_k)\cdot z_k\,dt - \int_0^T H_\Omega(z)\,dt,$$

new difficulty arises. The first integral in the action functional is defined on $H^{1/2}(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$, whereas the second integral prefers $z(t) \in \mathcal{F}_N\Omega$. Since $H^{1/2}(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$ does not embed into L^∞ the condition $z(t) \in \mathcal{F}_N\Omega$ does not define an open subset of $H^{1/2}(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$. Working in $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$, or other spaces which embed into L^∞, will cause compactness problems. Compactness problems appear anyway because there is no definite behavior of $H_\Omega(z)$ as $z \to \partial\mathcal{F}_N\Omega \subset \mathbb{C}^N$.

1.3 Scope of the thesis

In this thesis we address two issues involving periodic solutions to the point vortex systems in planar domains.

The first problem concerns the relative equilibria of the vortex equations in a specific domain, the disk. In this case, the angular momentum is another conserved quantity beside the energy conservation. The system for two vortices is thus completely integrable in the Liouville sense. In Section 2.1 we introduce a set of appropriate canonical transformations to reduce the dimensionality of the two point-vortex system in the unit disk. As an example we particularly describe the classification of two identical vortices motion. The final (integrable) one-degree-of-freedom Hamiltonian, with angular momentum being a parameter, is written in conjugated coordinates which represent the relative positions of the two vortices. The structure in the phase space changes according to the value of the angular momentum. Numerically we plot some trajectories of the reduced system. Fixed points (periodic solutions of the initial system) appear in these figures which agree with our analytical computation. Section 2.2 is devoted to "vortex crystals" in the disk. We proved the existence of open or centered n-gon, symmetric two n-gons and alternate two n-gons with generic choice of radii or strengths.

The second part of this thesis deals with the N-vortex problem in general domains. we consider the problem whether (1.1.7)-(1.1.8) has nonconstant periodic solutions in a domain $\Omega \neq \mathbb{C}$. As a basic question about any Hamiltonian system, however this has not been addressed for the N-vortex problem in a general domain. As mentioned, the difficulty lies in the fact that in general, this Hamiltonian is singular, not integrable, and energy levels are not compact and also not known to be of contact type, so standard methods do not apply. Therefore it is not surprising that there are no results on the existence of nonconstant periodic solutions except when the domain is radial. In that case, and when $\Gamma_k = \Gamma_1$ for all k, it is not difficult to find periodic solutions where the N vortices are arranged symmetrically as shown in Section 2.2.

A central and perhaps most interesting result of the thesis is Theorem 3.1.4. For any $N \in \mathbb{N}$, when all the strengths are the same, we prove the existence of a family of periodic solutions $z^r(t)$, $0 < r < r_0$, of (1.1.7) and (1.1.8) with arbitrarily small minimal period $T_r \to 0$ as $r \to 0$. Instead of obtaining periodic solutions near an equilibrium we obtain periodic solutions near a singularity of H_Ω. The solutions we obtain are simple choreographies, i.e., the N point vortices travel on the same curve and exchange their mutual positions after a fixed time. The dynamics of a single vortex in Ω is completely described by the Robin function because h coincides with the Hamiltonian in that case; see [42]. Theorem 3.1.4 shows that the Robin function also plays a fundamental role in the analysis of the dynamics of $N \geq 2$ point vortices. Sections 3.2 to 3.4 constitute the proof of the main result. Lastly in Section 3.5 we generalize Theorem 3.1.4 to the case of two vortices with arbitrarily non-opposite strengths and also obtain a result when the logarithm function in H is replaced by $\frac{1}{|z_1 - z_2|^\alpha}$ with $\alpha > 0$.

Chapter 2

Periodic solutions of systems in disk

As mentioned before, despite in general it is hard to know the explicit Green's function when the planar domain $\Omega \neq \mathbb{C}$, we still have hope to solve (1.1.4) whenever Ω possesses a certain shape. The classical method of images was firstly introduced by Thomson [76] in 1845 and it became a powerful tool for solving boundary value problems. In particular, for some simple symmetric domains this method greatly facilitates the solution of the problem. The spirit of the method of images is: for a given vortex located in a domain with some symmetry, one places an "image" vortex with an opposite strength outside the domain, reflected across the boundary. In such a way the boundary condition is satisfied.

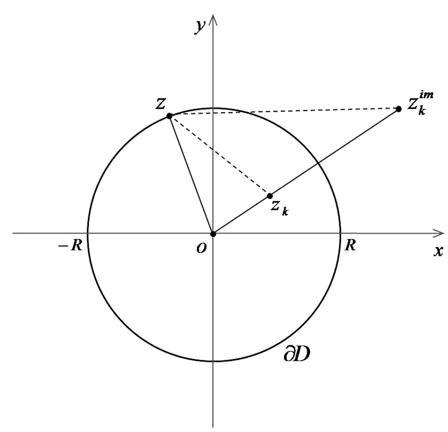
Here we take the disk as an example to explore periodic motions, including relative equilibria, of point vortices moving in it.

Consider a vortex of strength Γ_k located at z_k inside the disk $D = \{z \in \mathbb{C} : |z| < R\}$ of radius R. We place an image vortex with strength $-\Gamma_k$ at position z_k^{im}, where

$$z_k^{im} = \frac{R^2 z_k}{|z_k|^2}\,.$$

Then we find that for any $z \in \partial D$,

$$|z_k^{im}| \cdot |z_k| = R^2 = |z|^2,$$



From an easy geometric observation,

$$\triangle z o z_k^{im} \sim \triangle z_k o z,$$

which leads to

$$\frac{|z - z_k|}{|z - z_k^{im}|} = \frac{|z_k|}{|z|} = \frac{|z_k|}{R}$$

is a constant, so that the radial velocity components on the circle generated by z_k and z_k^{im} are canceled. Notice that the above equation implies $|z - z_k| = \frac{|z\bar{z}_k - R^2|}{R}$ for all z on ∂D, thus when we define the Green's function associated with z_k of strength Γ_k as

$$G(z, z_k) = -\frac{1}{2\pi}\log|z - z_k| + \frac{1}{2\pi}\log\frac{|R^2 - z\bar{z}_k|}{R},$$

the boundary condition $G(z, z_k) = 0$ on ∂D is satisfied. Then for n distinct vortices located at $z_k \in D$, $k = 1, \cdots, n$, each with strength Γ_k, the stream function governing the fluid motion is given by

$$\psi_D(z; z_1, \cdots, z_n) = \sum_{k=1}^{n}\Gamma_k G(z; z_k),$$

and hence the Hamiltonian function (remove the constant term $\log R$) is derived as

$$\begin{aligned} H_D(z_1, \cdots, z_n) &= -\frac{1}{4\pi}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\Gamma_j\Gamma_k\log\frac{|z_j - z_k|}{|R^2 - z_j\bar{z}_k|} + \frac{1}{4\pi}\sum_{k=1}^{n}\Gamma_k^2\log(R^2 - |z_k|^2) \\ &= \frac{1}{8\pi}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\Gamma_j\Gamma_k\log(R^2 + \frac{(R^2 - |z_j|^2)(R^2 - |z_k|^2)}{|z_j - z_k|^2}) + \frac{1}{4\pi}\sum_{k=1}^{n}\Gamma_k^2\log(R^2 - |z_k|^2) \end{aligned} \tag{2.0.1}$$

2.1 Motion of two vortices in disk

We begin with some definitions and well known facts in Hamiltonian dynamics for review.

Definition 2.1.1. A *Hamiltonian system* is a dynamical system described by a scalar function $H(z) = H(\mathbf{q}, \mathbf{p})$ of $2n$ coordinates $\mathbf{q} = (q_1, \cdots, q_n)$, $\mathbf{p} = (p_1, \cdots, p_n) \in \mathbb{R}^n$ and the evolution equation

$$\dot{z} = \mathcal{J}\nabla H(z),$$

known as Hamiltonian equation. Here, $\mathcal{J}$ is the $2n \times 2n$ symplectic matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ unit matrix.

Also recall that

Definition 2.1.2. A *Poisson bracket* is a binary operation on two functions $f(\mathbf{q}, \mathbf{p})$, $g(\mathbf{q}, \mathbf{p})$ defined as

$$\{f(\mathbf{q}, \mathbf{p}), g(\mathbf{q}, \mathbf{p})\} = \sum_{j=1}^{n}\Big(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\Big).$$

Definition 2.1.3. A transformation of coordinates is *canonical* if it preserves the form of Hamiltonian equations. In other words, under a canonical transformation the new Hamiltonian equation is simply obtained by substituting the new coordinates for the old ones.

To characterize a canonical transformation, several ways are developed:

Proposition 2.1.4. *A coordinate transformation*

$$\boldsymbol{Q}: \; q_j \to Q_j(\boldsymbol{q},\boldsymbol{p}), \quad \boldsymbol{P}: \; p_j \to P_j(\boldsymbol{q},\boldsymbol{p})$$

is canonical as long as one of the following holds:

1. *its Jacobian* $J = \begin{pmatrix} \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{q}} & \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{p}} \\ \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{q}} & \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{p}} \end{pmatrix}$ *satisfies* $J\mathcal{J}J^T = \mathcal{J}$. *Such a matrix* J *is called symplectic.*
2. $\{Q_j, Q_k\} = 0$, $\{P_j, P_k\} = 0$, $\{Q_j, P_k\} = \delta_{jk}$ *for any* $j, k = 1, \cdots, n$.
3. *For any* $j = 1, \cdots, n$,
$$\frac{\partial Q_j}{\partial \boldsymbol{q}} = \frac{\partial \boldsymbol{p}}{\partial P_j}, \quad \frac{\partial Q_j}{\partial \boldsymbol{p}} = -\frac{\partial \boldsymbol{p}}{\partial Q_j},$$
$$\frac{\partial P_j}{\partial \boldsymbol{p}} = \frac{\partial \boldsymbol{q}}{\partial Q_j}, \quad \frac{\partial P_j}{\partial \boldsymbol{q}} = -\frac{\partial \boldsymbol{q}}{\partial P_j}.$$

The Poisson bracket characterizes the first integrals of motion as well.

Definition 2.1.5. f is called a *first integral* of the Hamiltonian X_H, if it is a conserved quantity of the motion. An equivalent way to state this is that f is a first integral if $\{f, H\} = 0$ since

$$\{f, H\} = \sum_{j=1}^{n} \Big(\frac{\partial f}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial H}{\partial q_j} \Big) = \sum_{j=1}^{n} \Big(\frac{\partial f}{\partial q_j}\dot{q}_j + \frac{\partial f}{\partial p_j}\dot{p}_j \Big) = \frac{df}{dt}.$$

Let $f_1, \cdots, f_n$ be first integrals of X_H. They are said to be *independent* or *involutive* if $\{f_i, f_j\} = 0$ for any $i \neq j$.

Remark 2.1.6. Obviously $\{H, H\} = 0$. Hence the Hamiltonian is a first integral for an autonomous Hamiltonian system, as the sum of the kinetic and potential energies of the system.

A notion of complete integrability in the sense of Liouville is then defined based on the number of first integrals of the system:

Definition 2.1.7. For a Hamiltonian system defined on a $2n$-dimensional phase space, if there exist k independent first integrals, one can reduce the dimension of the phase space to $2(n-k)$. When $k = n$, the system is called *completely integrable*.

The knowledge of the first integral is of particular interest in both mathematics and physics because of the possibility of using it to get explicit expressions for the solutions of the system. For a completely integrable system, the Liouville-Arnold theorem ensures that in principle, using the conserved quantities one can find a special coordinate transformation such that the new Hamiltonian equations can be solved, or to say, "integrated". However, even if all the conserved quantities are known, the reduction procedure can be complicated and skill-intensive. Lichtenberg and Lieberman [53], Whittaker [81], or Arnold [12] have a nice discussion of this topic.

It is well known that in addition to the energy H, the n point-vortex system in the unbounded plane admits three independent first integrals of motion. Besides the energy H, the other two, the center of vorticity $Q + iP = \sum_{k=1}^{n} \Gamma_k z_k$ and the angular momentum $I = \sum_{k=1}^{n} \Gamma_k |z_k|^2$, correspond to the rotation and the translation invariances of the system in the two-dimensional plane. The integrability for the three point-vortex problem in $\mathbb{C}$ was studied by Synge [73] with the trilinear variables applied. The problem was further re-discussed by Novikov [62] for equal strengths and Aref [6] for general strengths. For $n = 4$, The system in $\mathbb{C}$ is integrable if the total vorticity $\Gamma = \sum_{k=1}^{4} \Gamma_k = 0$ (see Aref and Pomphey [9, 10], Aref [7] and Eckhardt [36]). This is because the four first integrals H, P, Q, I are mutually independent in this case. When $\Gamma \neq 0$, the four vortex system is not integrable anymore, proven in Ziglin [83], Koiller and Carvalho [48], and also Oliva [63], all of which are based on the Melnikov method.

But for our system in a disk, because of the lack of invariance under translations, only the energy and the angular momentum survive. This tells that the system consisting of two vortices in a disk is completely integrable and chaos may occur when more vortices are present. In this section we are concerned with the complete integrable motion of two vortices in a disk. Our work here is similar to a part of [19], but carried out in different variables and we give detailed calculations for seeking relative equilibria.

For convenience of computation, we shall let the radius R of the disk be 1. Consider the motion of two point vortices on the unit disk, $z_1 = x_1 + iy_1$ with strength Γ_1 and $z_2 = x_2 + iy_2$ with strength Γ_2, determined by a Hamiltonian system according to (2.0.1):

$$\begin{aligned} \Gamma_k \dot{x_k} &= \frac{\partial H}{\partial y_k}(z_1, z_2), \\ \Gamma_k \dot{y_k} &= -\frac{\partial H}{\partial x_k}(z_1, z_2), \end{aligned} \qquad k = 1, 2, \tag{2.1.1}$$

with

$$H(z_1, z_2) = \frac{\Gamma_1^2}{4\pi} \log(1 - |z_1|^2) + \frac{\Gamma_2^2}{4\pi} \log(1 - |z_2|^2) + \frac{\Gamma_1 \Gamma_2}{4\pi} \log\left(1 + \frac{(1 - |z_1|^2)(1 - |z_2|^2)}{|z_1 - z_2|^2}\right)$$

defined on the phase space $\mathcal{F}_2 D := \{(z_1, z_2) \in D \times D : z_1 \neq z_2\}$, an open subset of D^2.

The angular momentum of vorticity, in this case is written as

$$I = \Gamma_1 |z_1|^2 + \Gamma_2 |z_2|^2.$$

The existence of this conserved quantity is a consequence of invariance of the motion equations under the continuous transformation group of the coordinates' rotations. Equations (2.1.1) are also invariant under certain discrete transformations, for instant, if (z_k, Γ_k) solves (2.1.1), so do $(\bar{z}_k, \Gamma_k)$ and $(-z_k, \Gamma_k)$. Although these discrete symmetries do not lead to first integrals like I, they do possess the property that as long as they are initially satisfied, they are preserved for all future times. Direct calculus shows that H and I are two independent first integrals. (2.1.1) is thus completely integrable and it allows us to reduce the freedom degrees of the system and to investigate its phase space.

First of all, we see that (2.1.1) is not in a standard canonical form. To put it canonically, we introduce a new set of variables

$$q_k = \sqrt{|\Gamma_k|}sgn(\Gamma_k)x_k, \quad p_k = \sqrt{|\Gamma_k|}y_k, \quad k = 1, 2.$$

In this new variables we can get the Hamiltonian system in standard canonical form

$$\begin{aligned} \dot{q_k} &= \frac{\partial H'}{\partial p_k}(\mathbf{q}, \mathbf{p}) \\ \dot{p_k} &= -\frac{\partial H'}{\partial q_k}(\mathbf{q}, \mathbf{p}), \end{aligned} \qquad k = 1, 2, \tag{2.1.2}$$

with

$$\begin{aligned} H'(\mathbf{q}, \mathbf{p}) =& \frac{\Gamma_1^2}{4\pi} \log \left(1 - \frac{q_1^2 + p_1^2}{|\Gamma_1|}\right) + \frac{\Gamma_2^2}{4\pi} \log \left(1 - \frac{q_2^2 + p_2^2}{|\Gamma_2|}\right) \\ &+ \frac{\Gamma_1 \Gamma_2}{4\pi} \log \left[\frac{(q_1^2 + p_1^2)(q_2^2 + p_2^2)}{|\Gamma_1 \Gamma_2|} - \frac{2(s_1 s_2 q_1 q_2 + p_1 p_2)}{\sqrt{|\Gamma_1 \Gamma_2|}} + 1\right] \\ &- \frac{\Gamma_1 \Gamma_2}{4\pi} \log \left[\frac{q_1^2 + p_1^2}{|\Gamma_1|} + \frac{q_2^2 + p_2^2}{|\Gamma_2|} - \frac{2(s_1 s_2 q_1 q_2 + p_1 p_2)}{\sqrt{|\Gamma_1 \Gamma_2|}}\right] \end{aligned}$$

where $s_k = sgn(\Gamma_k)$.

Now the polar coordinates are applied:

$$q_k = \sqrt{2\rho_k} \cos \theta_k, \quad p_k = \sqrt{2\rho_k} \sin \theta_k, \quad k = 1, 2,$$

namely, $q_k^2 + p_k^2 = 2\rho_k$ and $\theta_k = \arg(q_k + ip_k)$. This is a canonical transformation as $\{\rho_j, \rho_k\} = \{\theta_j, \theta_k\} = 0$ and $\{\rho_j, \theta_k\} = \delta_{jk}$. Observe that

$$\begin{aligned} s_1 s_2 q_1 q_2 + p_1 p_2 &= s_1 s_2 \sqrt{4\rho_1 \rho_2} \cos \theta_1 \cos \theta_2 + \sqrt{4\rho_1 \rho_2} \sin \theta_1 \sin \theta_2 \\ &= s_1 s_2 \sqrt{4\rho_1 \rho_2} (\cos \theta_1 \cos \theta_2 + s_1 s_2 \sin \theta_1 \sin \theta_2) \\ &= s_1 s_2 \sqrt{4\rho_1 \rho_2} \cos(s_1 \theta_1 - s_2 \theta_2) \\ &= s_1 s_2 \sqrt{4\rho_1 \rho_2} \cos(\theta_1 - s_1 s_2 \theta_2), \end{aligned}$$

then the equations are converted into

$$\begin{aligned} \dot{\rho_k} &= \frac{\partial H''}{\partial \theta_k}(\rho, \theta) \\ \dot{\theta_k} &= -\frac{\partial H''}{\partial \rho_k}(\rho, \theta), \end{aligned} \qquad k = 1, 2, \tag{2.1.3}$$

with

$$\begin{aligned} H''(\rho,\theta) =& \frac{\Gamma_1^2}{4\pi}\log\big(1-\frac{2\rho_1}{|\Gamma_1|}\big)+\frac{\Gamma_2^2}{4\pi}\log\big(1-\frac{2\rho_2}{|\Gamma_2|}\big) \\ &+\frac{\Gamma_1\Gamma_2}{4\pi}\log\left[\frac{4\rho_1\rho_2}{|\Gamma_1\Gamma_2|}-\frac{4s_1s_2\sqrt{\rho_1\rho_2}}{\sqrt{|\Gamma_1\Gamma_2|}}\cos(\theta_1-s_1s_2\theta_2)+1\right] \\ &-\frac{\Gamma_1\Gamma_2}{4\pi}\log\left[\frac{2\rho_1}{|\Gamma_1|}+\frac{2\rho_2}{|\Gamma_2|}-\frac{4s_1s_2\sqrt{\rho_1\rho_2}}{\sqrt{|\Gamma_1\Gamma_2|}}\cos(\theta_1-s_1s_2\theta_2)\right] \end{aligned}$$

Next, we introduce a set of canonical (checked as before) conjugated variables

$$\begin{aligned} G_1 &= \rho_1, & G_2 &= \rho_1+s_1s_2\rho_2, \\ Q_1 &= \theta_1-s_1s_2\theta_2, & Q_2 &= s_1s_2\theta_2, \end{aligned}$$

under which the equations of motion become

$$\begin{aligned} \dot G_1 &= \frac{\partial \widehat H}{\partial Q_1}(\mathbf{G},\mathbf{Q}), & \dot Q_1 &= -\frac{\partial \widehat H}{\partial G_1}(\mathbf{G},\mathbf{Q}), \\ \dot Q_2 &= -\frac{\partial \widehat H}{\partial G_2}(\mathbf{G},\mathbf{Q}), & \dot G_2 &= \frac{\partial \widehat H}{\partial Q_2}(\mathbf{G},\mathbf{Q}) = 0, \end{aligned} \tag{2.1.4}$$

with the new Hamiltonian

$$\begin{aligned} \widehat H(\mathbf{G},\mathbf{Q}) =& \ \frac{\Gamma_1^2}{4\pi}\log\big(1-\frac{2G_1}{|\Gamma_1|}\big)+\frac{\Gamma_2^2}{4\pi}\log\big(1-\frac{2s_1s_2(G_2-G_1)}{|\Gamma_2|}\big) \\ &+\frac{\Gamma_1\Gamma_2}{4\pi}\log\left[\frac{4s_1s_2G_1(G_2-G_1)}{|\Gamma_1\Gamma_2|}-\frac{4s_1s_2\sqrt{s_1s_2G_1(G_2-G_1)}}{\sqrt{|\Gamma_1\Gamma_2|}}\cos Q_1+1\right] \\ &-\frac{\Gamma_1\Gamma_2}{4\pi}\log\left[\frac{2G_1}{|\Gamma_1|}+\frac{2s_1s_2(G_2-G_1)}{|\Gamma_2|}-\frac{4s_1s_2\sqrt{s_1s_2G_1(G_2-G_1)}}{\sqrt{|\Gamma_1\Gamma_2|}}\cos Q_1\right]. \end{aligned}$$

Observe that G_2 is a multiple of the angular momentum:

$$\begin{aligned} G_2 &= \frac{1}{2}(q_1^2+p_1^2)+\frac{1}{2}s_1s_2(q_2^2+p_2^2) \\ &= \frac{1}{2}|\Gamma_1||z_1|^2+\frac{1}{2}s_1s_2|\Gamma_2||z_2|^2 \\ &= \frac{1}{2}s_1[s_1|\Gamma_1||z_1|^2+s_2|\Gamma_2||z_2|^2] \\ &= \frac{1}{2}s_1 I \end{aligned}$$

thus a constant of the motion. The Hamilton function $\widehat H$ is independent of the variable Q_2. Hence, we have carried out a reduction from the initial system to a one-degree of freedom Hamiltonian system (2.1.4) parametrically depending upon the value of the first integral G_2.

In the end, to describe the phase space, we make use of the new canonical conjugated coordinates

$$\begin{aligned} X(t) &= \sqrt{2G_1}\cos Q_1, \\ Y(t) &= \sqrt{2G_1}\sin Q_1, \end{aligned} \tag{2.1.5}$$

to obtain a new Hamiltonian system

$$\dot{X} = \frac{\partial \widetilde{H}}{\partial Y}(X,Y;G_2), \quad \dot{Y} = -\frac{\partial \widetilde{H}}{\partial X}(X,Y;G_2), \tag{2.1.6}$$

where the Hamiltonian is rewritten as

$$\begin{aligned} \widetilde{H}(X,Y;G_2) = &\ \frac{\Gamma_1^2}{4\pi}\log\big(1 - \frac{X^2+Y^2}{|\Gamma_1|}\big) + \frac{\Gamma_2^2}{4\pi}\log\big(1 - \frac{s_1 s_2(2G_2 - X^2 - Y^2)}{|\Gamma_2|}\big) \\ &+ \frac{\Gamma_1\Gamma_2}{4\pi}\log\left[\frac{s_1 s_2(X^2+Y^2)(2G_2 - X^2 - Y^2)}{|\Gamma_1\Gamma_2|} - \frac{2s_1 s_2 X\sqrt{s_1 s_2(2G_2 - X^2 - Y^2)}}{\sqrt{|\Gamma_1\Gamma_2|}} + 1\right] \\ &- \frac{\Gamma_1\Gamma_2}{4\pi}\log\left[\frac{X^2+Y^2}{|\Gamma_1|} + \frac{s_1 s_2(2G_2 - X^2 - Y^2)}{|\Gamma_2|} - \frac{2s_1 s_2 X\sqrt{s_1 s_2(2G_2 - X^2 - Y^2)}}{\sqrt{|\Gamma_1\Gamma_2|}}\right] \end{aligned}$$

such that the trajectories can be plotted at different values of the parameter G_2.

Example. $\Gamma_1 = \Gamma_2 = 1$.

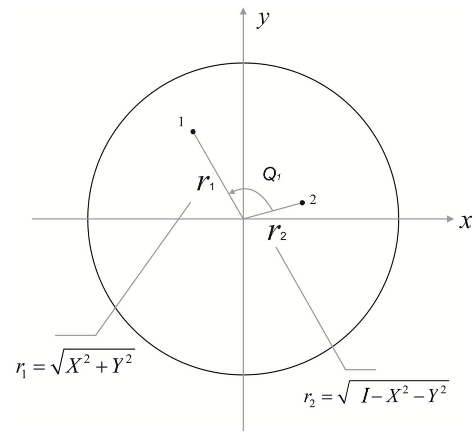
In the following as an example, let us consider the motion of two identical point vortices on the disk, i.e. $\Gamma_1 = \Gamma_2 = 1$, $s_1 = s_2 = 1$. The only effect of the strengths of the vortices is to change the ranges for the parameter and the variables of the system. We shall plot the constant $\widetilde{H}$ contours to demonstrate the integrability of the system and investigate some particular solutions corresponding to fixed points or periodic orbits. In this case $2G_2 = I$ is just the angular momentum. The range of the momentum is given by the inequalities $0 < I < 2$. The left boundary corresponds to the case that both vortices approach the center of the disk, and the right boundary refers to the situation that the two vortices both touch the outer wall of the disk. Our system is now in the form (2.1.6) specified by

$$\begin{aligned} \widetilde{H} = &\ \frac{1}{4\pi}\log(1 - X^2 - Y^2) + \frac{1}{4\pi}\log(1 + X^2 + Y^2 - I) \\ &+ \frac{1}{4\pi}\log\left[(X^2+Y^2)(I - X^2 - Y^2) - 2X\sqrt{I - X^2 - Y^2} + 1\right] \\ &- \frac{1}{4\pi}\log\left[I - 2X\sqrt{I - X^2 - Y^2}\right]. \end{aligned}$$

The variables (X, Y) physically represent the Cartesian coordinates of vortex 1 in a reference frame which rotates with vortex 2. $\sqrt{X^2+Y^2}$ is the Euclidean norm of z_1 and Q_1 refers to the angle difference between z_1 and z_2. The domain of X and Y is found from the relations:

$$\begin{aligned} 0 \leqslant X^2+Y^2 \leqslant I, &\quad \text{if } 0<I<1; \\ I-1 < X^2+Y^2 < 1, &\quad \text{if } 1 \leqslant I < 2. \end{aligned}$$

The trajectories of (2.1.6) describe the relative positions of these two vortices.



For a given angular momentum, the motion of the vortices is confined to be the energy surface of $\widetilde{H}$. $(\sqrt{I/2}, 0)$ is a singular point of $\widetilde{H}$, i.e., a merging of vortices. There is an interesting geometric interpretation of the absolute vortices' motion:

Proposition 2.1.8. *Any nonasymptotic motion of two identical vortices in disk is a superposition of a uniform rotation of the coordinate system and a periodic motion in this rotating system. More precisely, if $\xi(t)$ is a T-periodic orbit of the reduced system* (2.1.6)*, there is a frame of reference, uniformly rotating around the center of disk with angular velocity*

$$q_0 = \frac{1}{T}\int_0^T \dot{Q}_2(t)dt$$

such that in this frame, each vortex moves along a closed curve. Particularly, when q_0 and $\frac{2\pi}{T}$ are commensurable, i.e. $\frac{2\pi}{q_0 T} = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$, the vortex motion in the fixed frame is periodic.

Proof. For a given angular momentum I, the energy levels of the reduced $\widetilde{H}$ are all compact, so each nonasymptotic orbit in the (X, Y)-plane is closed. Here the nonasymptotic orbit means that it does not converge to any fixed point or periodic orbit. The nonasymptotic motion in the statement then refers to the absolute motion corresponding to a nonasymptotic orbit in the (X, Y)-plane. Thus since the reduced system is with one degree of freedom, along each nonasymptotic orbit the variables X and Y are periodic functions with the same period, assumed to be T:

$$X(t) = X(t+T) \quad \text{and} \quad Y(t) = Y(t+T).$$

T differs for each trajectory. Given such $X(t)$, $Y(t)$, we have

$$\begin{aligned} G_1(t) &= \frac{1}{2}(X(t)^2+Y(t)^2), \\ \tan Q_1(t) &= \frac{Y(t)}{X(t)}. \end{aligned}$$

As a result, $G_1(t)$ and $Q_1(t)$ are T-periodic functions as well. Now let us turn back to (2.1.4). The variable Q_2 satisfies

$$\dot{Q}_2(t) = -\frac{\partial \widehat{H}(G_2; G_1(t), Q_1(t))}{\partial G_2} =: Q(t). \tag{2.1.7}$$

Then we see that the right hand side term which we denote by $Q(t)$ is also a T-periodic function of time because G_2 is time-independent and $G_1(t)$ and $Q_1(t)$ are T-periodic. As a consequence, we can expand $Q(t)$ in Fourier series:

$$Q(t) = \sum_{m\in\mathbb{Z}} q_m e^{i\cdot 2\pi \frac{mt}{T}}.$$

Substituting it into (2.1.7) and integrating over time, we find that Q_2 can be represented as

$$Q_2(t) = q_0 t + \frac{T}{2\pi i} \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{q_m}{m} e^{i\cdot 2\pi \frac{mt}{T}} + c_0$$

with constants $q_0, c_0 \in \mathbb{R}$. Observe that

$$\overline{Q}_2(t) := \frac{T}{2\pi i} \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{q_m}{m} e^{i\cdot 2\pi \frac{mt}{T}} + c_0$$

is a function of period T. This means $Q_2(t) = q_0 t + \overline{Q}_2(t)$ is a combination of a linear function and a T-periodic function of time. Substituting these into the expressions of Cartesian coordinates of the two vortices yields

$$\begin{aligned}
x_1(t) &= \sqrt{2G_1(t)}\cos(Q_1(t) + \overline{Q}_2(t) + q_0 t),\\
y_1(t) &= \sqrt{2G_1(t)}\sin(Q_1(t) + \overline{Q}_2(t) + q_0 t);\\
x_2(t) &= \sqrt{2(G_2 - G_1(t))}\cos(\overline{Q}_2(t) + q_0 t),\\
y_2(t) &= \sqrt{2(G_2 - G_1(t))}\sin(\overline{Q}_2(t) + q_0 t).
\end{aligned}$$

Decomposing the trigonometric functions we obtain

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} \cos q_0 t & -\sin q_0 t \\ \sin q_0 t & \cos q_0 t \end{pmatrix} \begin{pmatrix} \sqrt{2G_1(t)}\cos(Q_1(t) + \overline{Q}_2(t)) \\ \sqrt{2G_1(t)}\sin(Q_1(t) + \overline{Q}_2(t)) \end{pmatrix}$$

and

$$\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \cos q_0 t & -\sin q_0 t \\ \sin q_0 t & \cos q_0 t \end{pmatrix} \begin{pmatrix} \sqrt{2(G_2 - G_1(t))}\cos \overline{Q}_2(t) \\ \sqrt{2(G_2 - G_1(t))}\sin \overline{Q}_2(t) \end{pmatrix}.$$

Denote

$$\overline{x}_1 = \sqrt{2G_1}\cos(Q_1 + \overline{Q}_2), \quad \overline{y}_1 = \sqrt{2G_1}\sin(Q_1 + \overline{Q}_2),$$

then $\overline{x}_1(t)=\overline{x}_1(t+T)$, $\overline{y}_1(t)=\overline{y}_1(t+T)$ and

$$z_1(t)=x_1(t)+iy_1(t)=e^{iq_0t}(\overline{x}_1(t)+i\overline{y}_1(t)).$$

Analogously,

$$z_2(t)=x_2(t)+iy_2(t)=e^{iq_0t}(\overline{x}_2(t)+i\overline{y}_2(t)),$$

where $\overline{x}_2(t)=\sqrt{2(G_2-G_1(t))}\cos\overline{Q}_2(t)$ and $\overline{y}_2(t)=\sqrt{2(G_2-G_1(t))}\sin\overline{Q}_2(t)$ are also T-periodic functions. From this we find that the two vortices move periodically in a rotating system with angular velocity q_0. □

For asymptotic orbits of the reduced system, they converge to fixed points or periodic orbits as $t\to-\infty$ or $+\infty$, thus they are not periodic. So the corresponding absolute motions can not be periodic in any uniformly rotating frame.

We shall refer to configurations that move without changing shape or size as *vortex equilibria*. These configurations are typically in a state of uniform rotation or translation and so are also called *relative equilibria*. In particular, a configuration where all the vortices remain in place is said to be *stationary*.

Fixed points of the reduced system (2.1.6) given by the critical points of the Hamiltonian $\widetilde{H}$ are relative equilibria of system (2.0.1). Indeed, for any fixed point of system (2.1.6), X and Y are constant, thus $|z_1|^2=X^2+Y^2$, $|z_2|^2=I-X^2-Y^2$ and the relative angle between z_1, z_2 does not change along the motion.

The equations defining the fixed points of system (2.1.6) are

$$\begin{aligned}4\pi\frac{\partial\widetilde{H}}{\partial X}=&-\frac{2X}{1-X^2-Y^2}+\frac{2X}{1+X^2+Y^2-I}\\&+\frac{2X(I-2X^2-2Y^2)+\frac{4X^2+2Y^2-2I}{\sqrt{I-X^2-Y^2}}}{1+(X^2+Y^2)(I-X^2-Y^2)-2X\sqrt{I-X^2-Y^2}}\\&-\frac{4X^2+2Y^2-2I}{\sqrt{I-X^2-Y^2}(I-2X\sqrt{I-X^2-Y^2})}=0\end{aligned}$$

and

$$\begin{aligned}4\pi\frac{\partial\widetilde{H}}{\partial Y}=&-\frac{2Y}{1-X^2-Y^2}+\frac{2Y}{1+X^2+Y^2-I}\\&+\frac{2Y(I-2X^2-2Y^2)+\frac{2XY}{\sqrt{I-X^2-Y^2}}}{1+(X^2+Y^2)(I-X^2-Y^2)-2X\sqrt{I-X^2-Y^2}}\\&-\frac{2XY}{\sqrt{I-X^2-Y^2}(I-2X\sqrt{I-X^2-Y^2})}=0.\end{aligned}$$

They lead to

$$\begin{aligned}&4\pi\frac{\partial\widetilde{H}}{\partial X}\cdot Y-4\pi\frac{\partial\widetilde{H}}{\partial Y}\cdot X\\=&\frac{2Y\sqrt{I-X^2-Y^2}(1-I+(X^2+Y^2)(I-X^2-Y^2))}{(I-2X\sqrt{I-X^2-Y^2})(1+(X^2+Y^2)(I-X^2-Y^2)-2X\sqrt{I-X^2-Y^2})}\\=&0.\end{aligned}$$

From this we see that all the fixed points must satisfy $Y = 0$, which means the configurations have the feature that the two vortices lie on the line across the origin. Moreover, with $Y = 0$, it follows $\frac{\partial \widetilde{H}}{\partial Y}|_{Y=0} = 0$ and

$$4\pi \frac{\partial \widetilde{H}}{\partial X}|_{Y=0} = (2I - 4X^2) \cdot \Big[\frac{X}{(1-X^2)(1+X^2-I)} - \frac{1}{\sqrt{I-X^2}(1-X\sqrt{I-X^2})} + \frac{1}{\sqrt{I-X^2}(I-2X\sqrt{I-X^2})} \Big]$$

Therefore, $\frac{\partial \widetilde{H}}{\partial X}|_{Y=0} = 0$ implies

$$I - 2X^2 = 0,$$

which gives $X = -\sqrt{I/2}$, or,

$$0 = \frac{X}{(1-X^2)(1+X^2-I)} - \frac{1}{\sqrt{I-X^2}(1-X\sqrt{I-X^2})} + \frac{1}{\sqrt{I-X^2}(I-2X\sqrt{I-X^2})}. \tag{2.1.8}$$

It shows that $(-\sqrt{I/2}, 0)$ is always a fixed point regardless of the choice of I, which corresponds to the configuration of uniform rotation of z_1 and z_2 along the same circle centered at 0. Meanwhile, from (2.1.8) we derive

$$\begin{aligned} 0 = &X\sqrt{I-X^2}(1-X\sqrt{I-X^2})(I-2X\sqrt{I-X^2}) \\ &+ (1-I+X^2(I-X^2))(1-I+X\sqrt{I-X^2}). \end{aligned} \tag{2.1.9}$$

Let $m := X\sqrt{I-X^2}$, then $m \in [-\frac{I}{2}, \frac{I}{2}]$ since $m^2 = X^2(I-X^2) \leqslant I^2/4$. To solve (2.1.9) we define a continuous map

$$f : [-\frac{I}{2}, \frac{I}{2}] \to \mathbb{R}$$

with

$$\begin{aligned} f(m) := &m(1-m)(I-2m) + (1-I+m^2)(1-I+m) \\ = &3m^3 - (1+2I)m^2 + m + (I-1)^2. \end{aligned}$$

The roots of f give rise to critical points of $\widetilde{H}$. Notice that

$$\begin{aligned} f'(m) = &\ 9m^2 - 2(1+2I)m + 1 \\ = &\ 9\big(m - \frac{1}{9}(1+2I)\big)^2 + 1 - \frac{1}{9}(1+2I)^2. \end{aligned}$$

Depending on the value of the first integral I, different situations are possible.

- $\mathbf{0 < I < 1}$

We have $1-\frac{1}{9}(1+2I)^2>0$, then f is strictly monotone increasing on $[-\frac{I}{2},\frac{I}{2}]$.

$$f(\frac{I}{2})=-\frac{1}{8}I^3+\frac{3}{4}I^2-\frac{3}{2}I+1=-\frac{1}{8}(I-2)^3>0,$$

and by numerical computation,

$$\begin{aligned}f(-\frac{I}{2})=&-\frac{7}{8}I^3+\frac{3}{4}I^2-\frac{5}{2}I+1\\ \approx&-\frac{7}{8}(I-I_1)(I-a)(I-\bar{a})\\ =&-\frac{7}{8}(I-I_1)|I-a|^2,\end{aligned}$$

where $I_1\approx 0.4275$ and a is a complex number with $\operatorname{Im} a\neq 0$. Thus we conclude
(i) $0<I<I_1\approx 0.4275$. In this case $f'(m)>0$ on $[-\frac{I}{2},\frac{I}{2}]$ and $f(-\frac{I}{2}),f(\frac{I}{2})$ are both positive. f has no zero point on interval $[-\frac{I}{2},\frac{I}{2}]$. Hence, there is no fixed point of the system other than $(-\sqrt{\frac{I}{2}},0)$.
(ii) $I_1<I<1$. As I increases, $f(-\frac{I}{2})<0$, $f(0)=(I-1)^2>0$, and f is strictly increasing. Consequently there exists exactly a number $m_0\in(-\frac{I}{2},0]$ such that $f(m_0)=0$. Moreover, it is easy to show

$$\begin{aligned}m_0&\to-\frac{I}{2}, &&\text{as } I\to I_1;\\ m_0&\to 0, &&\text{as } I\to 1.\end{aligned}$$

The corresponding fixed point satisfies $X\sqrt{I-X^2}=m_0\leqslant 0$, thus two new fixed points are born with coordinates $X_{1,2}=-\sqrt{\frac{I\pm\sqrt{I^2-4m_0^2}}{2}}$. They correspond to the same configuration where two vortices move on two different concentric circles centered at 0 of radii r_1, r_2 satisfying $r_1^2+r_2^2=I$ and $r_1^2\cdot r_2^2=m_0^2$. We see that

$$\begin{aligned}X_{1,2}&\to-\sqrt{\frac{I}{2}}, &&\text{as } I\to I_1;\\ X_{1,2}&\to\{-\sqrt{I},0\}, &&\text{as } I\to 1.\end{aligned}$$

This means they depart from $-\sqrt{\frac{I}{2}}$ at I near I_1 to the border $-\sqrt{I}$ and the origin at I up to 1.

- **$\mathbf{1\leqslant I<2}$**

In this case, owing to the constraints $X^2+Y^2<1$ and $I-X^2-Y^2<1$, the phase space becomes an annulus $I-1<X^2+Y^2<1$. As I goes to 2, the domain shrinks to the singular circle $X^2+Y^2=1$. When $Y=0$, $I-1<X^2<1$, then $I-1<|m|=|X|\sqrt{I-X^2}\leqslant\frac{I}{2}$.

On $[-\frac{I}{2},1-I)$, f' is everywhere positive, thus f is strictly increasing. Therefore, $f(1-I)=(1-I)(5I^2-8I+4)\leqslant 0$ proves non-existence of root of f on $[-\frac{I}{2},1-I)$.

On $(I-1,\frac{I}{2}]$, f' has a root $m_1=\frac{1+2I+2\sqrt{I^2+I-2}}{9}$. Observe that the minimum of $f|_{(I-1,\frac{I}{2}]}$ is either $f(I-1)$ or $f(m_1)$. However, $f(I-1)=(I-1)(I-2)^2>0$, and by numerical computation, $f(m_1)$ is also positive provided $I \in [1, 2)$, so f has no root on this interval. Hence, the system has exactly one fixed point $(-\sqrt{\frac{I}{2}}, 0)$.

The classification to the stability of the relative equilibria is determined by the Jacobian matrix of the reduced Hamiltonian vector field $(\frac{\partial \widetilde{H}}{\partial Y}, -\frac{\partial \widetilde{H}}{\partial X})$ evaluated in the corresponding fixed point p,

$$J_p(\widetilde{H}) = \left(\begin{array}{cc} \frac{\partial^2 \widetilde{H}}{\partial X \partial Y} & \frac{\partial^2 \widetilde{H}}{\partial Y^2} \\ -\frac{\partial^2 \widetilde{H}}{\partial X^2} & -\frac{\partial^2 \widetilde{H}}{\partial Y \partial X} \end{array} \right)_p .$$

It is called the stability matrix. The linear stability theory tells us that the eigenvalues of $J_p(\widetilde{H})$ characterize the type of the fixed point. When $J_p(\widetilde{H})$ has purely imaginary eigenvalues, p is an *elliptic fixed point*. p is an unstable *hyperbolic fixed point*, also called a *saddle point* when $J_p(\widetilde{H})$ has two real non-zero eigenvalues with different signs.

At the point $(-\sqrt{\frac{I}{2}}, 0)$ we compute

$$\begin{aligned} -\frac{\partial^2 \widetilde{H}}{\partial X^2}\Big|_{X=-\sqrt{\frac{I}{2}}, Y=0} &= \frac{1}{4\pi} \cdot \frac{(I+2)(7I^3 - 6I^2 + 20I - 8)}{4I(1-\frac{I^2}{4})^2} \\ &= \frac{1}{4\pi} \cdot \frac{7(I+2)(I-I_1)|I-a|^2}{4I(1-\frac{I^2}{4})^2}; \\ \frac{\partial^2 \widetilde{H}}{\partial Y^2}\Big|_{X=-\sqrt{\frac{I}{2}}, Y=0} &= \frac{1}{4\pi} \cdot \frac{(1-\frac{I}{2})^2}{I(1+\frac{I}{2})^2} > 0; \\ \frac{\partial^2 \widetilde{H}}{\partial X \partial Y}\Big|_{X=-\sqrt{\frac{I}{2}}, Y=0} &= \frac{\partial^2 \widetilde{H}}{\partial Y \partial X}\Big|_{X=-\sqrt{\frac{I}{2}}, Y=0} = 0. \end{aligned}$$

Hence,
(i) if $0 < I < I_1 \approx 0.4275$, $-\frac{\partial^2 \widetilde{H}}{\partial X^2}|_{X=-\sqrt{\frac{I}{2}}, Y=0} < 0$, thus $J(\widetilde{H})$ has two pure imaginary eigenvalues and then $(-\sqrt{\frac{I}{2}}, 0)$ is an elliptic fixed point;
(ii) if $I_1 < I < 2$, $-\frac{\partial^2 \widetilde{H}}{\partial X^2}|_{X=-\sqrt{\frac{I}{2}}, Y=0} > 0. J(\widetilde{H})$ has two real eigenvalues in different signs. $(-\sqrt{\frac{I}{2}}, 0)$ is then an unstable hyperbolic fixed point.

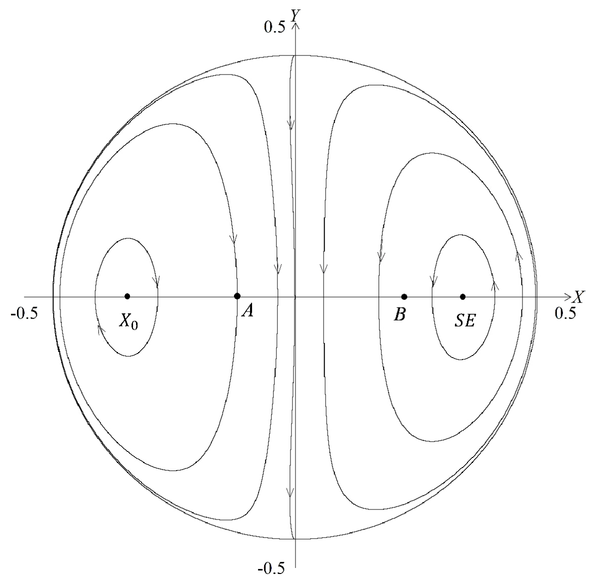
According to the analysis above, we have already grasped some information about the vortex behavior in the disk. From the aspect of numerical calculus, we investigate some examples to see how the absolute motions of the vortices look like. Based on our analytic research, four different feature types of phase portraits of the reduced system are possible according to the choice of I. In the (a) parts of Fig.2.1-Fig.2.4 , the constant $\widetilde{H}$ contours are plotted to demonstrate the characteristics of each type. Since the coordinates in the XY-plane do not represent the physical ones, we choose some points in the level curve maps and calculate their corresponding absolute trajectories of vortices.

It is worth mentioning that when $0 < I < 1$, the outer circle in XY-plane corresponds to the arrangement that $|z_1|^2 = I$ and $z_2 = 0$. This circle, together with a curve across the origin (referring to $z_1 = 0$, $|z_2|^2 = I$), constitutes the energy isoline $\widetilde{H}^{-1}(\frac{1}{4\pi}\log\frac{1-I}{I})$. However, the variables (2.1.5) are unsuitable for these configurations because of the singularity at 0 when polar coordinates are determined. Actually since the phase portraits in XY-plane describe the relative vortex positions, all the points on the outer circle represent the same one ($|z_1|^2 = I$ and $z_2 = 0$). Thus the entire circle can be regarded as one point. In such a way, we see that the orbit on $\widetilde{H}^{-1}(\frac{1}{4\pi}\log\frac{1-I}{I})$ is also closed and periodic.

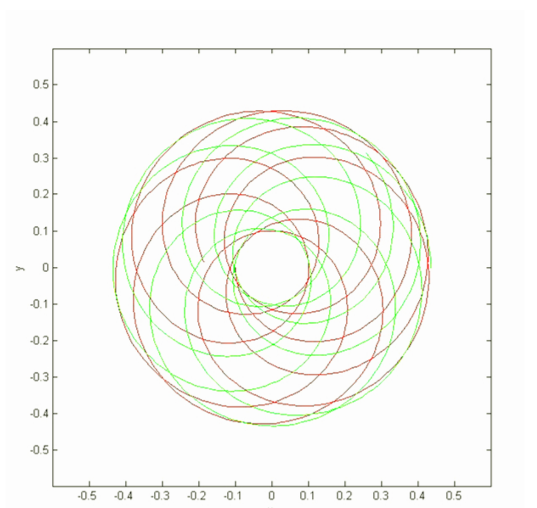
Another phenomenon is that some energy level sets are disconnected. For example for the case $I = 0.7$, $\widetilde{H}^{-1}(\frac{1}{4\pi}\log(0.42))$ is the disjoint union of a curve of new moon shape surrounding X_1 and a circle around X_2. Which relative orbit the vortices move along relies on which component the initial configuration lies on.

In each phase space figure, $X_0 = (-\sqrt{I/2}, 0)$ is a fixed point corresponding to a rotation of vortices in one circle such that they are always on the same straight line as the center of the disk and on the opposite sides of it. As I grows from 0 to 2, the radius of this circle increases from 0 to 1. During this procedure for small values of I, X_0 is the unique fixed point and stable (see Fig.2.1). As I reaches $I_1 \approx 0.4275$, X_0 becomes an unstable hyperbolic fixed point and a pair of elliptic fixed points X_1, X_2 arise in its neighborhood (see Fig.2.2, 2.3). These two new points correspond to a rotation of vortices in different circles, as before collinear with the center of the disk and on opposite sides of it. X_2 is just given by the interchanging of vortices in X_1. Two homoclinic orbits connect X_0 to itself. When $I < 0.5808$, they surround X_1 and X_2 respectively as in Fig.2.2. When I goes above this value, they turn to surround X_2 and the singular point SE, see Fig.2.3. The positions of $X_{1,2}$ tend to 0 and $(-I, 0)$ as I grows and they disappear until I is over 1. For $I > 1$, X_0 becomes the only fixed points again, but is still unstable.

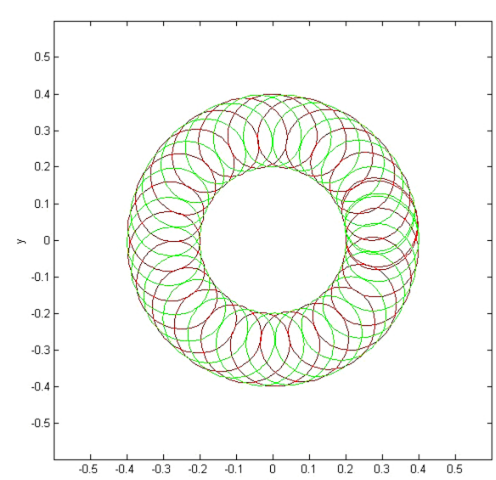
In some regions of the phase portrait, vortices move without affecting each other, such as point D in Fig.2.3 and point F in Fig.2.4. In some regions, like point A, B in Fig.2.1 and O, C in Fig.2.2, vortices rotate around the center of the disk and also around each other.



(a) Relative vortex trajectories at $I = 0.2$ in the XY-plane. X_0 represents the elliptic fixed point $(-\sqrt{I/2}, 0)$. SE denotes the singular configuration.

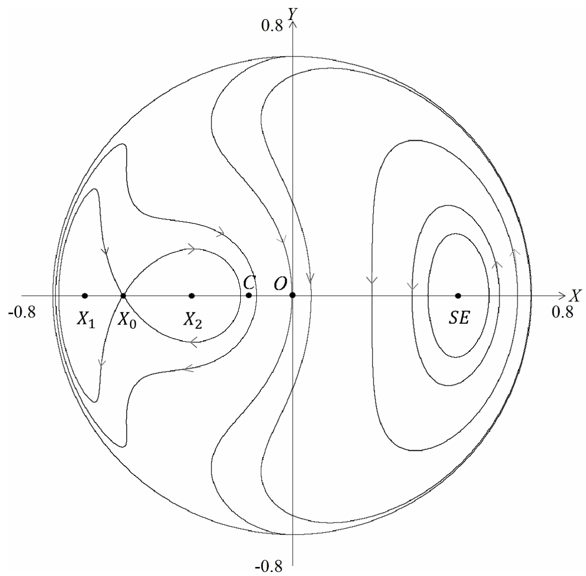


(b) Absolute motion of vortices with initial positions $(-0.1, 0)$ (red line) and $(0.435, 0)$ (green line) corresponding to the point A in the XY-plane.

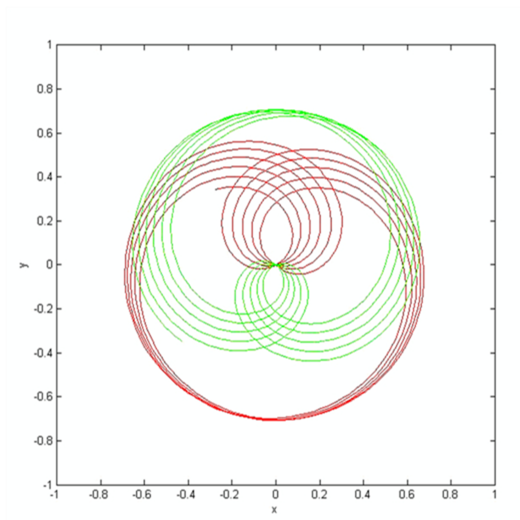


(c) Absolute motion of vortices with initial positions $(0.2, 0)$ (red line) and $(0.4, 0)$ (green line) corresponding to the point B in the XY-plane.

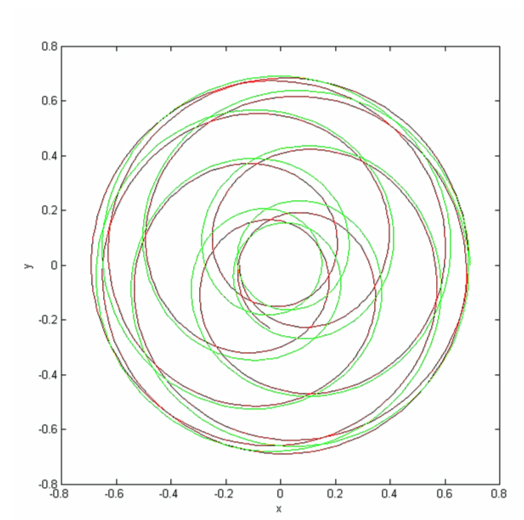
Figure 2.1: $I = 0.2$



(a) Relative vortex trajectories at $I = 0.5$ in the XY-plane. X_0 is the unstable hyperbolic fixed point $(-\sqrt{I/2}, 0)$. X_1 and X_2 are two elliptic fixed points.

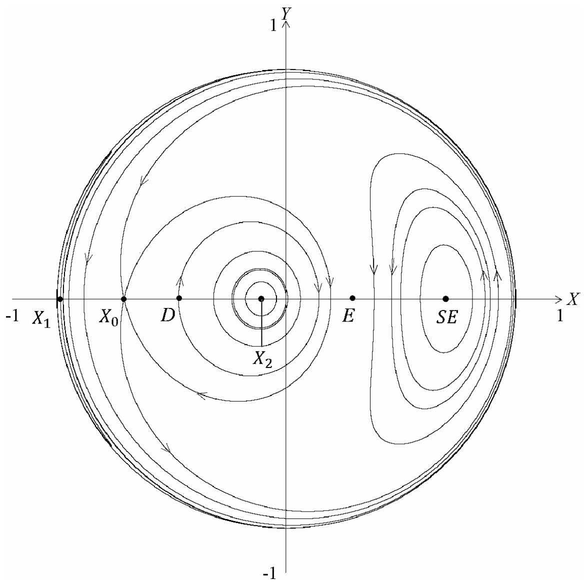


(b) Absolute motion of vortices with initial positions $(0, 0)$ (red line) and $(\sqrt{0.5}, 0)$ (green line) corresponding to the point O.

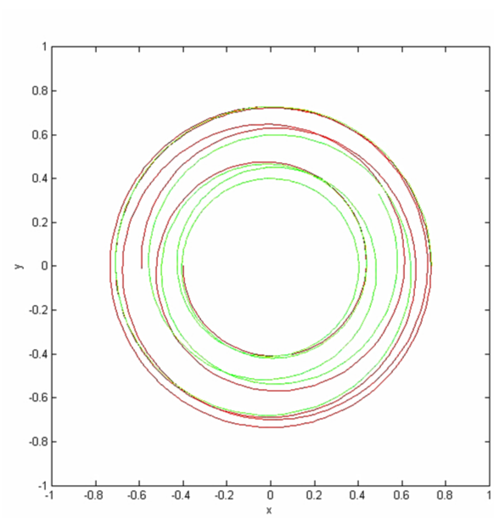


(c) Absolute motion of vortices with initial positions $(-0.15, 0)$ (red line) and $(0.691, 0)$ (green line) corresponding to the point C.

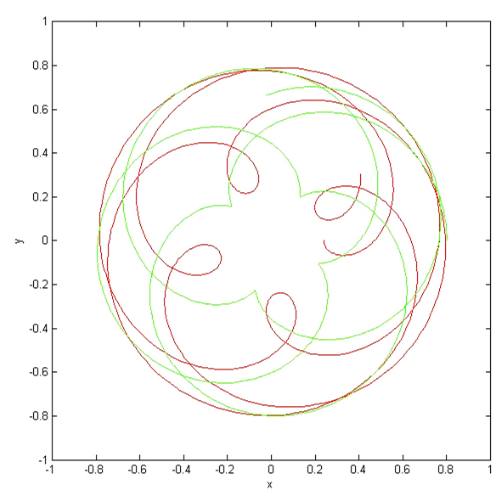
Figure 2.2: $I = 0.5$



(a) Relative vortex trajectories at $I = 0.7$ in the XY-plane. X_0 represents the unstable hyperbolic fixed point $(-\sqrt{I/2}, 0)$. X_1 and X_2 are elliptic fixed points.

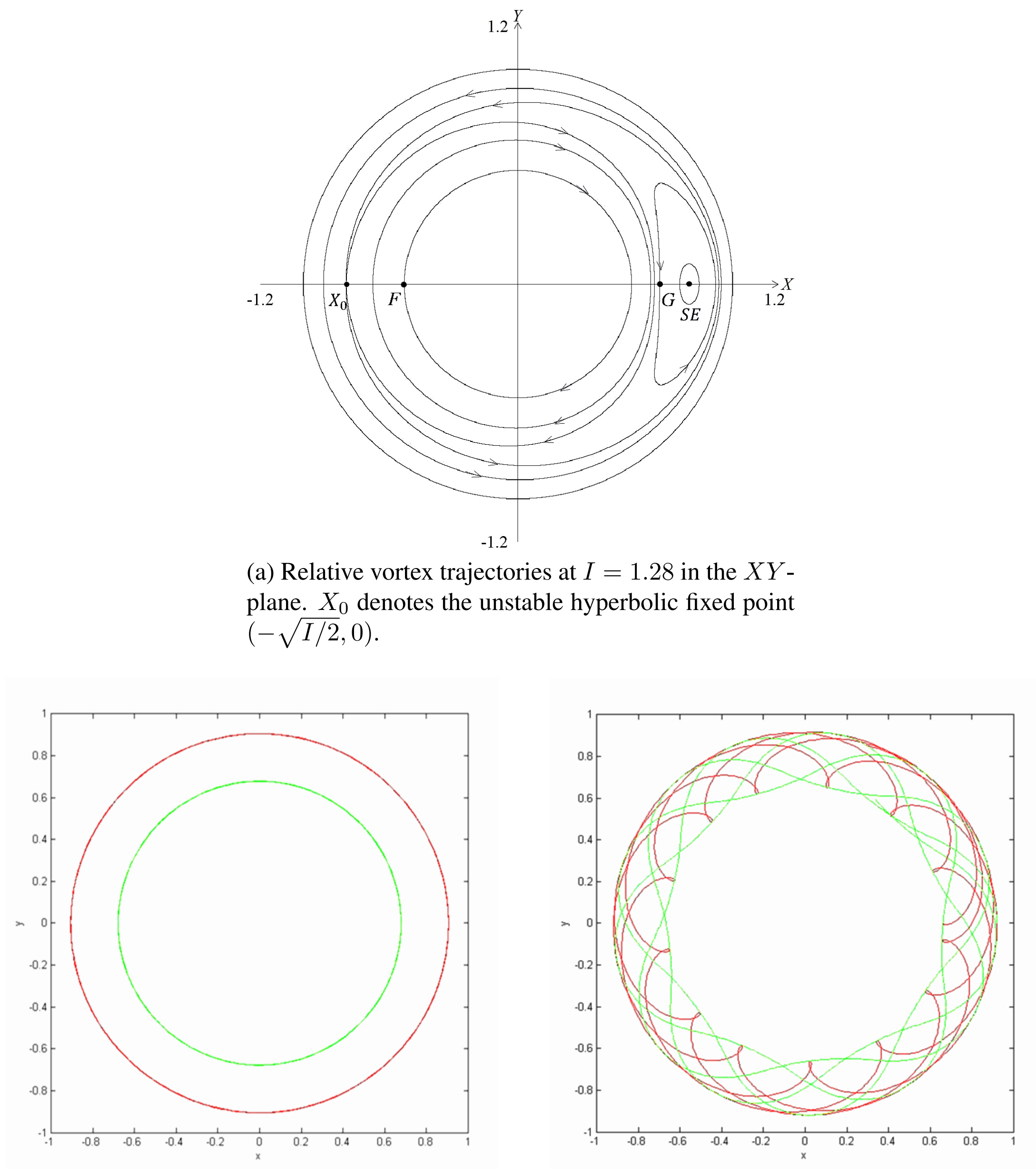


(b) Absolute motion of vortices with initial positions $(-0.4, 0)$ (red line) and $(0.7348, 0)$ (green line) corresponding to the point D.



(c) Absolute motion of vortices with initial positions $(0.24, 0)$ (red line) and $(0.8015, 0)$ (green line) corresponding to the point E.

Figure 2.3: $I = 0.7$



(a) Relative vortex trajectories at $I = 1.28$ in the XY-plane. X_0 denotes the unstable hyperbolic fixed point $(-\sqrt{I/2}, 0)$.

(b) Absolute motion of vortices with initial positions $(-0.9059, 0)$ (red line) and $(0.6778, 0)$ (green line) corresponding to the point F.

(c) Absolute motion of vortices with initial positions $(0.6588, 0)$ (red line) and $(0.9198, 0)$ (green line) corresponding to the point G.

Figure 2.4: I=1.28

2.2 Vortex polygon

Since the system of four or more point vortices in the unbounded plane is not integrable, chaotic behavior may occur. However, for some particular initial configurations of vortices, regular periodic motions are possible. "Vortex crystals", a typical type of relative equilibria, attract many efforts. Thomson studied the regular n-polygon configurations of identical vortices in $\mathbb{C}$ in his Adams Prize work [76]. Havelock [44] discussed the existence of double rings, i.e., configurations consisting of two groups of n vortices with opposite strengths. Furthermore, general studies of k n-polygons configurations have been carried out by several authors. See Aref [7], Koiller et al.[49], or Lewis and Ratiu [52], and references therein for detailed discussions. These beautiful vortex crystals provide a series of discrete symmetric periodic solutions. Related results are collected in the survey article [11]. For discussions on stability of relative equilibria and chaotic behaviors we refer to [50, 51].

In this section, we investigate these highly symmetric polygonal configurations in the disk.

For later use, we write down the equations of motions of n interacting vortices $k = 1, 2, \cdots, \mathrm{n}$ with strengths Γ_k and positions z_k in the unit disk $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$ in a complex form:

$$\overline{\frac{dz_k}{dt}} = \frac{1}{2\pi i}{\sum_{j=1}^{n}}' \frac{\Gamma_j}{z_k - z_j} - \frac{1}{2\pi i}\sum_{j=1}^{n}\frac{\Gamma_j}{z_k - \hat{z}_j}, \quad k = 1, 2, \cdots, n. \tag{2.2.1}$$

The overbar denotes complex conjugation. The prime means omitting the singular term $j = k$ in the summation. And $\hat{z}_j = \frac{1}{\bar{z}_j}$. In the sequel, we always assume $n \geqslant 2$.

Case I. **n-polygon**

The basic configuration of a relative equilibrium has vortices of the same strength $\Gamma_k = \Gamma$ at the corners of a regular n-polygon centered at the origin in the unit disk. We make the *Ansatz*

$$z_k(t) = re^{i\omega t}e^{i\frac{2\pi k}{n}}, \quad k = 0, \cdots, n-1$$

with constants $r \in (0, 1)$ and $\omega \in \mathbb{R}$. Substituting these into (2.2.1), we obtain

$$\frac{2\pi r^2\omega}{\Gamma} = \frac{r^2}{1-r^2} + \sum_{j\neq k}\Big[\frac{1}{1-e^{i\frac{2\pi}{n}(j-k)}} - \frac{1}{1-\frac{1}{r^2}e^{i\frac{2\pi}{n}(j-k)}}\Big], \quad k = 0, \cdots, n-1.$$

Since these equations are equivalent for each k, it is sufficient to solve

$$\frac{2\pi r^2\omega}{\Gamma} = \frac{r^2}{1-r^2} + \sum_{j=1}^{n-1}\Big[\frac{1}{1-e^{i\frac{2\pi j}{n}}} - \frac{1}{1-\frac{1}{r^2}e^{i\frac{2\pi j}{n}}}\Big]. \tag{2.2.2}$$

To evaluate the sum, we consider a polynomial of degree n in the complex variable x with distinct roots $x_1, \cdots, x_n$:

$$P(x) = (x - x_1)\cdots(x - x_n).$$

Then

$$P'(x) = P(x)\sum_{j=1}^{n}\frac{1}{x-x_j}.$$

Now we take $P(x) = x^n - \gamma^n$, where γ is a complex number satisfying $\gamma^n \neq 1$. The roots of P are $x_1 = \gamma, x_2 = \gamma\varepsilon, \cdots, x_n = \gamma\varepsilon^{n-1}$ with $\varepsilon = e^{i\frac{2\pi}{n}}$. So we derive

$$nx^{n-1} = (x^n - \gamma^n)\sum_{j=1}^{n}\frac{1}{x-\gamma\cdot\varepsilon^j}.$$

Let $x = 1$, we get

$$\sum_{j=1}^{n}\frac{1}{1-\gamma\cdot\varepsilon^j} = \frac{n}{1-\gamma^n} \quad \text{for } \gamma^n \neq 1. \tag{2.2.3}$$

On the other hand, for the case $\gamma = 1$, instead we work on

$$Q(x) = (x-\varepsilon)\cdots(x-\varepsilon^{n-1}).$$

Then

$$\begin{aligned} Q(x) &= \frac{x^n-1}{x-1} = 1 + x + \cdots + x^{n-1}, \\ Q'(x) &= 1 + 2x + \cdots + (n-1)x^{n-2}. \end{aligned}$$

Thus from $Q(1) = n$ and $Q'(1) = 1 + 2 + \cdots + n - 1 = \frac{1}{2}n(n-1)$ we have

$$\sum_{j=1}^{n-1}\frac{1}{1-\varepsilon^j} = \frac{n-1}{2}. \tag{2.2.4}$$

Combining (2.2.2)-(2.2.4) we obtain

$$\begin{aligned} \frac{2\pi r^2\omega}{\Gamma} &= \frac{r^2}{1-r^2} + \frac{n-1}{2} - \big[\frac{n}{1-\frac{1}{r^{2n}}} - \frac{1}{1-\frac{1}{r^2}}\big] \\ &= \frac{n-1}{2} + \frac{nr^{2n}}{1-r^{2n}}, \end{aligned}$$

so the angular velocity is

$$\omega = \frac{\Gamma}{2\pi r^2}\Big[\frac{n-1}{2} + \frac{nr^{2n}}{1-r^{2n}}\Big]. \tag{2.2.5}$$

This is a well known result. Stability analysis of such n-gons in disk in terms of n and r can be found in many literatures, here we refer to Kurakin [50, 51].

Case II. **Centered n-polygon**

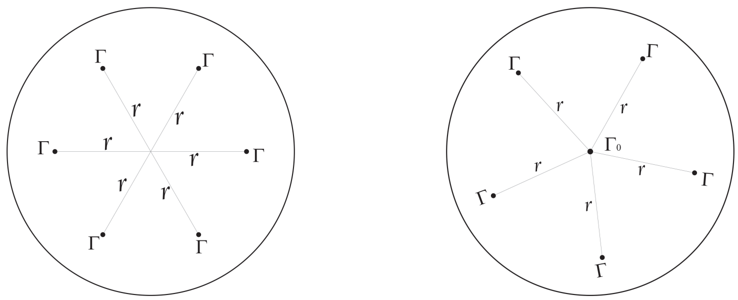


Figure 2.5: 6-Polygon and 5+1-centered polygon.

We locate a vortex of strength Γ_0 at the center of the regular n-gon. Since $\sum_{j=0}^{n-1} \frac{1}{e^{i\frac{2\pi j}{n}}} = 0$, the central vortex is stationary. For the other vortices at vertices of the polygon, the resulting equation becomes

$$2\pi r^2 \omega' = \Gamma\Big[\frac{n-1}{2} + \frac{nr^{2n}}{1-r^{2n}}\Big] + \Gamma_0.$$

Hence, vortices at corners rotate around the center in a uniform angular velocity augmented by the presence of the central vortex:

$$\omega' = \omega + \frac{\Gamma_0}{2\pi r^2}. \tag{2.2.6}$$

with ω as in (2.2.5). The configuration is stationary when $\Gamma_0 = -\Gamma\big[\frac{n-1}{2} + \frac{nr^{2n}}{1-r^{2n}}\big]$.

Case III. **Two n-polygons**

Consider a collection of $2n$ point vortices in D, n at positions $z_1, \cdots, z_n$ with same strength Γ_1, n at positions $\zeta_1, \cdots, \zeta_n$ with strength Γ_2. Then the equations of their motions are

$$\begin{aligned} \dot{\bar{z}}_k &= \frac{1}{2\pi i}\Big[\Gamma_1 {\sum_{j=1}^{n}}' \frac{1}{z_k - z_j} + \Gamma_2 \sum_{j=1}^{n} \frac{1}{z_k - \zeta_j} - \Gamma_1 \sum_{j=1}^{n} \frac{1}{z_k - \hat{z}_j} - \Gamma_2 \sum_{j=1}^{n} \frac{1}{z_k - \hat{\zeta}_j}\Big] \\ \dot{\bar{\zeta}}_k &= \frac{1}{2\pi i}\Big[\Gamma_2 {\sum_{j=1}^{n}}' \frac{1}{\zeta_k - \zeta_j} + \Gamma_1 \sum_{j=1}^{n} \frac{1}{\zeta_k - z_j} - \Gamma_2 \sum_{j=1}^{n} \frac{1}{\zeta_k - \hat{\zeta}_j} - \Gamma_1 \sum_{j=1}^{n} \frac{1}{\zeta_k - \hat{z}_j}\Big] \end{aligned} \tag{2.2.7}$$

Proposition 2.2.1. *There are solutions of* (2.2.7) *of the form*

$$\begin{aligned} z_k(t) &= z(t)\exp(2\pi i k/n) \\ \zeta_k(t) &= \zeta(t)\exp(2\pi i k/n) \end{aligned} \tag{2.2.8}$$

for $k = 0, \cdots, n-1$.

Proof. Substituting (2.2.8) into (2.2.7) we obtain

$$\begin{aligned}\dot{\bar{z}} &= \frac{1}{2\pi i}\left[\frac{\Gamma_1(n-1)}{2z} + \frac{\Gamma_2 n z^{n-1}}{z^n - \zeta^n} + \frac{\Gamma_1 n z^{n-1}\bar{z}^n}{1 - z^n\bar{z}^n} + \frac{\Gamma_2 n z^{n-1}\bar{\zeta}^n}{1 - z^n\bar{\zeta}^n}\right] \\ \dot{\bar{\zeta}} &= \frac{1}{2\pi i}\left[\frac{\Gamma_2(n-1)}{2\zeta} + \frac{\Gamma_1 n \zeta^{n-1}}{\zeta^n - z^n} + \frac{\Gamma_2 n \zeta^{n-1}\bar{\zeta}^n}{1 - \zeta^n\bar{\zeta}^n} + \frac{\Gamma_1 n \zeta^{n-1}\bar{z}^n}{1 - \zeta^n\bar{z}^n}\right]\end{aligned} \tag{2.2.9}$$

Obviously if (2.2.8) holds at $t = 0$, they continue to hold for all future times when z and ζ evolve according to (2.2.9). □

Let us rewrite z, ζ in polar coordinates

$$z = re^{i\phi}, \quad \zeta = \rho e^{i\theta}.$$

Using (2.2.9) we can establish

Proposition 2.2.2. *The only possible solutions of Equations* (2.2.9) *that correspond to uniformly rotating configurations, i.e., satisfying*

$$\begin{aligned}\dot{r} &= \dot{\rho} = 0, \\ \dot{\phi} &= \dot{\theta} = \omega,\end{aligned}$$

where $\omega \in \mathbb{R}$ is a constant, are characterized by

(1) $\sin(n\psi) = 0$ *with* $\psi = \theta - \phi$*;*

(2) radii r and ρ satisfy

$$\frac{r^2}{\rho^2} = \frac{\frac{\Gamma_1(n-1)}{2} - \frac{\Gamma_1 n}{1-r^{-2n}} + \frac{\Gamma_2 n r^n}{r^n - \rho^n\cos(n\psi)} - \frac{\Gamma_2 n r^n\rho^n}{r^n\rho^n - \cos(n\psi)}}{\frac{\Gamma_2(n-1)}{2} - \frac{\Gamma_2 n}{1-\rho^{-2n}} + \frac{\Gamma_1 n\rho^n}{\rho^n - r^n\cos(n\psi)} - \frac{\Gamma_1 n r^n\rho^n}{r^n\rho^n - \cos(n\psi)}}. \tag{2.2.10}$$

Proof. Assume $\phi = \omega t + \alpha$ and $\theta = \omega t + \beta$, then

$$\begin{aligned}z(t) &= re^{i\alpha}e^{i\omega t}, \\ \zeta(t) &= \rho e^{i\beta}e^{i\omega t},\end{aligned}$$

and $\psi = \theta - \phi = \beta - \alpha$. Then (2.2.9) requires

$$\begin{aligned}2\pi\omega r^2 =& \frac{1}{2}(n-1)\Gamma_1 + \frac{n\Gamma_2}{1-\xi^n} - \frac{n\Gamma_1}{1-r^{-2n}} - \frac{n\Gamma_2}{1-\eta^n}; \\ 2\pi\omega\rho^2 =& \frac{1}{2}(n-1)\Gamma_2 + \frac{n\Gamma_1}{1-\xi^{-n}} - \frac{n\Gamma_2}{1-\rho^{-2n}} - \frac{n\Gamma_1}{1-\bar{\eta}^n},\end{aligned} \tag{2.2.11}$$

where the phases $\xi = \frac{\rho}{r}\exp(i\psi)$ and $\eta = \frac{\exp(i\psi)}{r\rho}$. As a consequence, it must hold that

$$\frac{1}{1-\xi^n} - \frac{1}{1-\eta^n} \in \mathbb{R}.$$

Suppose it equals a real value c, then we can derive

$$\frac{r^n}{r^n - \rho^n e^{in\psi}} - \frac{r^n}{r^n - \rho^{-n} e^{in\psi}} = c \in \mathbb{R},$$

that is

$$c(e^{in\psi})^2 - ((c+1)r^n\rho^n + (c-1)r^n\rho^{-n})e^{in\psi} + cr^{2n} = 0.$$

Thus we see that $e^{in\psi}$ solves a quadratic equation with real coefficients:

$$cz^2 - ((c+1)r^n\rho^n + (c-1)r^n\rho^{-n})z + cr^{2n} = 0. \tag{2.2.12}$$

Then so does its conjugation $e^{-in\psi}$. We find $c \neq 0$, otherwise $\rho = \rho^{-1}$ which can not occur. Hence, if $e^{in\psi} \notin \mathbb{R}$, $e^{in\psi}$ and $e^{-in\psi}$ are the two distinct roots of (2.2.12). There holds $r^{2n} = e^{in\psi} \cdot e^{-in\psi} = 1$. However this is impossible since we already know $r \in (0, 1)$. So we conclude that $e^{-in\psi}$ must be real, i.e. $\sin(n\psi) = 0$. Then (2.2.10) follows from (2.2.11) with $\xi = \frac{\rho}{r}\cos\psi$ and $\eta = \frac{\cos\psi}{r\rho}$. □

Remark 2.2.3. Actually for two nested, regular polygons to form an uniformly rotating configuration in the unit disk, they must have the same number of vortices. A study of vortex polygons in the unbounded plane can be found in [8].

Associated with two choices of ψ, 0 or $\frac{\pi}{n}$, two kinds of vortex configurations are obtained.

- $\psi = 0$ (**symmetric two n-gons**).

  We refer to the case $\psi = 0$ as the symmetric configuration, (2.2.10) now simplifies:

$$\frac{r^2}{\rho^2} = \frac{\frac{\Gamma_1(n-1)}{2} - \frac{\Gamma_1 n}{1-r^{-2n}} + \frac{\Gamma_2 n r^n}{r^n-\rho^n} - \frac{\Gamma_2 n r^n \rho^n}{r^n\rho^n-1}}{\frac{\Gamma_2(n-1)}{2} - \frac{\Gamma_2 n}{1-\rho^{-2n}} + \frac{\Gamma_1 n \rho^n}{\rho^n-r^n} - \frac{\Gamma_1 n r^n \rho^n}{r^n\rho^n-1}}. \tag{2.2.13}$$

Therefore, we need to verify the existence of proper values of $r, \rho \in (0, 1)$ with $r \neq \rho$ which satisfy (2.2.13). To solve this, we define a function $f(r, \rho) : (0, 1) \times (0, 1) \setminus \{r = \rho\} \to \mathbb{R}$ as

$$\begin{aligned} f(r,\rho) = & \frac{\Gamma_1(n-1)\rho^2}{2} - \frac{\Gamma_2(n-1)r^2}{2} + n\frac{\Gamma_1 r^2\rho^n + \Gamma_2 r^n \rho^2}{r^n - \rho^n} \\ & + n\frac{\Gamma_1 r^{2n}\rho^2}{1-r^{2n}} - n\frac{\Gamma_2 r^2 \rho^{2n}}{1-\rho^{2n}} + n\frac{r^n\rho^n(\Gamma_2\rho^2 - \Gamma_1 r^2)}{1 - r^n\rho^n}. \end{aligned}$$

$(r, \rho) \in (0, 1)^2 \setminus \{r = \rho\}$ satisfying (2.2.13) are zero points of f. We fix $r \in (0, 1)$. $f(r, \cdot)$ is continuous on $(0, r) \cup (r, 1)$. We see that

$$f(r,\rho) \to -\frac{1}{2}\Gamma_2(n-1)r^2 \begin{cases} < 0 & \text{if } \Gamma_2 > 0, \\ > 0 & \text{if } \Gamma_2 < 0, \end{cases} \quad \text{as } \rho \to 0^+,$$

$$f(r,\rho) = O(1) + n\frac{\Gamma_1 r^2 \rho^n + \Gamma_2 r^n \rho^2}{r^n - \rho^n} \to \begin{cases} +\infty & \text{if } \Gamma_1 + \Gamma_2 > 0, \\ -\infty & \text{if } \Gamma_1 + \Gamma_2 < 0, \end{cases} \quad \text{as } \rho \to r^-.$$

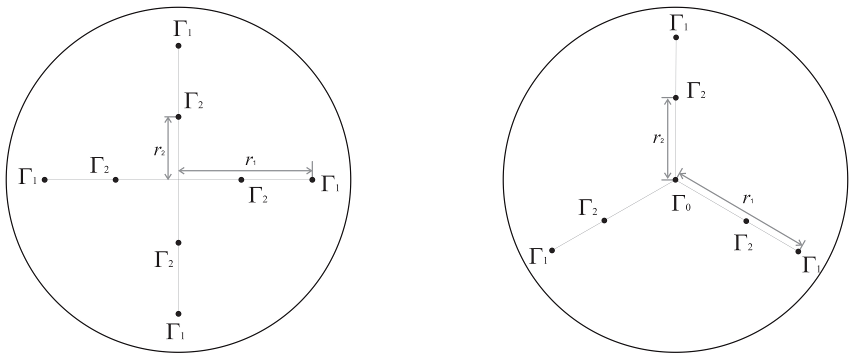


Figure 2.6: two 4-Polygons and centered two 3-polygon.

and

$$f(r,\rho) = O(1) + n\frac{\Gamma_1 r^2\rho^n + \Gamma_2 r^n \rho^2}{r^n - \rho^n} \to \begin{cases} -\infty & \text{if } \Gamma_1+\Gamma_2 > 0, \\ +\infty & \text{if } \Gamma_1+\Gamma_2 < 0, \end{cases} \quad \text{as } \rho \to r^+,$$

$$f(r,\rho) = O(1) - n\Gamma_2 r^2 \frac{\rho^{2n}}{1-\rho^{2n}} \to \begin{cases} -\infty & \text{if } \Gamma_2 > 0, \\ +\infty & \text{if } \Gamma_2 < 0, \end{cases} \quad \text{as } \rho \to 1^-.$$

Hence we obtain the following result:

Proposition 2.2.4. *Assume $\psi = 0$ and $\Gamma_1+\Gamma_2 \neq 0$. Then for any $r \in (0,1)$, there exists at least one value $\rho(r) \in (0,1)\setminus\{r\}$ such that (n,Γ_1,r) and $(n,\Gamma_2,\rho(r))$ form two uniformly rotating n-gons.*

Remark 2.2.5. Given n and Γ_k. For some r, there is only one proper ρ satisfying (2.2.13), while for some others, more than one solutions of (2.2.13) exist. In other words, 1 is a lower bound of the number of ρ for a given r. For instant, $n = 3$ and $\Gamma_1 = \Gamma_2 = 1$. Take $r = 0.5$, then $\rho \approx 0.2106$ is the unique solution of (2.2.13) on $\rho \in (0,1)$. However, letting $r = 0.3$, we can obtain three fitting values for ρ, approximately 0.1266, 0.7295 and 0.9240, to solve the equation.

- $\psi = \frac{\pi}{\mathbf{N}}$ **(alternate two n-gons)**.

This configuration is called *alternate* or *staggered*. In this case (2.2.10) becomes

$$\frac{r^2}{\rho^2} = \frac{\frac{\Gamma_1(n-1)}{2} - \frac{\Gamma_1 n}{1-r^{-2n}} + \frac{\Gamma_2 n r^n}{r^n+\rho^n} - \frac{\Gamma_2 n r^n\rho^n}{r^n\rho^n+1}}{\frac{\Gamma_2(n-1)}{2} - \frac{\Gamma_2 n}{1-\rho^{-2n}} + \frac{\Gamma_1 n\rho^n}{\rho^n+r^n} - \frac{\Gamma_1 n r^n\rho^n}{r^n\rho^n+1}}. \tag{2.2.14}$$

So possible (r,ρ) are zeros of the function $g : (0,1)\times(0,1) \to \mathbb{R}$ with

$$\begin{aligned} g(r,\rho) = {} & \frac{\Gamma_1(n-1)\rho^2}{2} - \frac{\Gamma_2(n-1)r^2}{2} + n\frac{\Gamma_2 r^n\rho^2 - \Gamma_1 r^2\rho^n}{r^n+\rho^n} \\ & + n\frac{\Gamma_1 r^{2n}\rho^2}{1-r^{2n}} - n\frac{\Gamma_2 r^2\rho^{2n}}{1-\rho^{2n}} + n\frac{r^n\rho^n(\Gamma_1 r^2 - \Gamma_2\rho^2)}{1+r^n\rho^n}. \end{aligned}$$

Again we fix $r \in (0,1)$. $g(r,\cdot)$ is continuous on $(0,1)$. Firstly we have

$$g(r,\rho) \to -\frac{1}{2}\Gamma_2(n-1)r^2 \begin{cases} <0 & \text{if } \Gamma_2 > 0, \\ >0 & \text{if } \Gamma_2 < 0, \end{cases} \quad \text{as } \rho \to 0^+,$$

$$g(r,\rho) = O(1) - n\Gamma_2 r^2 \frac{\rho^{2n}}{1-\rho^{2n}} \to \begin{cases} -\infty & \text{if } \Gamma_2 > 0, \\ +\infty & \text{if } \Gamma_2 < 0, \end{cases} \quad \text{as } \rho \to 1^-.$$

Take $\rho = r$, then computation shows

$$g(r,r) = (\Gamma_1 - \Gamma_2)r^2\Big[\frac{2nr^{2n}}{1-r^{4n}} - \frac{1}{2}\Big].$$

When $\Gamma_1 \neq \Gamma_2$, as a function of $r \in (0,1)$, $r \mapsto \frac{2nr^{2n}}{1-r^{4n}} - \frac{1}{2}$ is strictly increasing and has exactly one zero point

$$r_0 = (\sqrt{4n^2+1} - 2n)^{1/2n}. \tag{2.2.15}$$

If $r < r_0$,

$$g(r,r) \begin{cases} <0, & \text{if } \Gamma_1 - \Gamma_2 > 0, \\ >0, & \text{if } \Gamma_1 - \Gamma_2 < 0. \end{cases}$$

If $r > r_0$,

$$g(r,r) \begin{cases} >0, & \text{if } \Gamma_1 - \Gamma_2 > 0, \\ <0, & \text{if } \Gamma_1 - \Gamma_2 < 0. \end{cases}$$

Hence we can conclude that

Proposition 2.2.6. *Assume $\psi = \frac{\pi}{n}$ and let $r_0 = (\sqrt{4n^2+1} - 2n)^{1/2n}$.*

(1) If $\Gamma_2(\Gamma_1 - \Gamma_2) < 0$, then for any $r < r_0$, there exist at least two values ρ_1, ρ_2 with $0 < \rho_1 < r < \rho_2 < 1$, such that for $k = 1$ or 2, (n, Γ_1, r) and (n, Γ_2, ρ_k) form two uniformly rotating n-gons.

(2) If $\Gamma_2(\Gamma_1 - \Gamma_2) > 0$, then for any $r > r_0$, there exist at least two values ρ_1, ρ_2 with $0 < \rho_1 < r < \rho_2 < 1$, such that for $k = 1$ or 2, (n, Γ_1, r) and (n, Γ_2, ρ_k) form two uniformly rotating n-gons.

Remark 2.2.7. A special case is when $r = \rho$, i.e., these $2n$ vortices are located on one $2n$ polygon alternatively. Then the system attains a relative equilibrium if and only if $g(r,r) = 0$ which gives $\Gamma_1 = \Gamma_2$ (same as one $2n$-gon) or $r = r_0$. This implies for any pair of different strengths (Γ_1, Γ_2), if vortices (n, Γ_1), (n, Γ_2) are placed alternatively on the vertices of a $2n$-polygon, only when the radius of the polygon is $r_0 = (\sqrt{4n^2+1} - 2n)^{1/2n}$ can the vortices form a relative equilibrium. At this time, the uniform angular velocity of the vortices is

$$\omega = \frac{(4n+1-\sqrt{4n^2+1})(\Gamma_1+\Gamma_2)}{8\pi(\sqrt{4n^2+1} - 2n)^{1/n}},$$

which concises with (2.2.5) for the case $\Gamma_1 = \Gamma_2$. Moreover, we see that when $\Gamma_1 + \Gamma_2 = 0$, it is a stationary configuration.

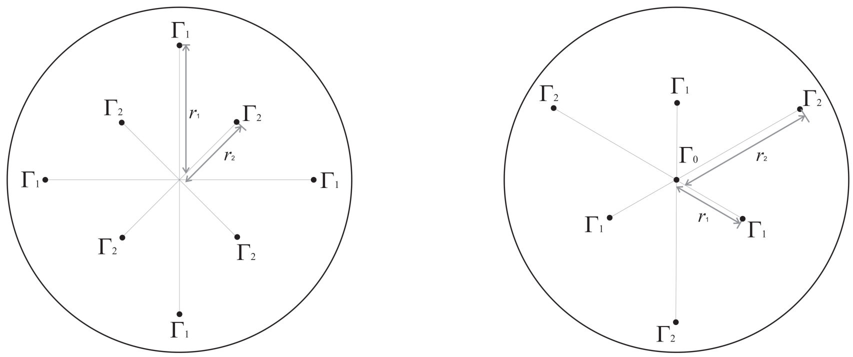


Figure 2.7: two 4-Polygons and centered two 3-polygon.

Formula (2.2.10) forms a complicated polynomial algebraic equation for the radii r and ρ, parametrized by the free constant vortex strengths Γ_1 and Γ_2. Although we obtained a general existence result for the solutions to it, it does not appear to possess a explicit expression of solutions. So we may reverse our sight and regard it as a linear equation for the vortex strengths. Then we can get an alternative form of (2.2.10):

$$\frac{\Gamma_1}{\Gamma_2} = \frac{r^2\big(\frac{n-1}{2} - \frac{n}{1-\rho^{-2n}}\big) - n\rho^2 r^n\big(\frac{1}{r^n-\rho^n\cos(n\psi)} - \frac{1}{r^n-\rho^{-n}\cos(n\psi)}\big)}{\rho^2\big(\frac{n-1}{2} - \frac{n}{1-r^{-2n}}\big) - nr^2\rho^n\big(\frac{1}{\rho^n-r^n\cos(n\psi)} - \frac{1}{\rho^n-r^{-n}\cos(n\psi)}\big)}.$$

It means the relative vortex strength $\frac{\Gamma_1}{\Gamma_2}$ of such a configuration can be uniquely determined as a function of radii r, ρ of the polygons and parameters n, ψ.

Case IV. **centered two n-polygons**

Place a vortex of strength Γ_0 at the the origin. These $2n+1$ vortices form a relative equilibrium if and only if the equation

$$\frac{r^2}{\rho^2} = \frac{\frac{\Gamma_1(n-1)}{2} - \frac{\Gamma_1 n}{1-r^{-2n}} + \frac{\Gamma_2 n r^n}{r^n-\rho^n\cos(n\psi)} - \frac{\Gamma_2 n r^n\rho^n}{r^n\rho^n-\cos(n\psi)} + \Gamma_0}{\frac{\Gamma_2(n-1)}{2} - \frac{\Gamma_2 n}{1-\rho^{-2n}} + \frac{\Gamma_1 n\rho^n}{\rho^n-r^n\cos(n\psi)} - \frac{\Gamma_1 n r^n\rho^n}{r^n\rho^n-\cos(n\psi)} + \Gamma_0}$$

is fulfilled. Thus for each pair of strengths Γ_1, Γ_2 and each pair of unequal radii r, ρ of polygons, as long as a vortex with strength

$$\begin{aligned}\Gamma_0 = &\Gamma_2\frac{r^2\big(\frac{n-1}{2} - \frac{n}{1-\rho^{-2n}}\big) - n\rho^2 r^n\big(\frac{1}{r^n-\rho^n\cos(n\psi)} - \frac{1}{r^n-\rho^{-n}\cos(n\psi)}\big)}{\rho^2 - r^2}\\ &- \Gamma_1\frac{\rho^2\big(\frac{n-1}{2} - \frac{n}{1-r^{-2n}}\big) - nr^2\rho^n\big(\frac{1}{\rho^n-r^n\cos(n\psi)} - \frac{1}{\rho^n-r^{-n}\cos(n\psi)}\big)}{\rho^2 - r^2}\end{aligned}$$

is placed at the center, the two polygons (n, Γ_1, r) and (n, Γ_2, ρ) rotate around the central vortex as a rigid body.

Chapter 3

Periodic solutions of N point-vortex systems in a general domain

3.1 Result statement

In this chapter we discuss periodic solutions to the Hamiltonian system modeled on the motion of N point vortices of equal strength $\Gamma_k = 1$ in a bounded planar domain Ω:

$$\begin{aligned} \dot{x}_k &= \frac{\partial H_\Omega}{\partial y_k}(z), \\ \dot{y}_k &= -\frac{\partial H_\Omega}{\partial x_k}(z), \end{aligned} \qquad k = 1, \cdots, N,$$

or in a more concise way,

$$\dot{z}_k = -i\nabla_{z_k} H_\Omega(z), \quad k = 1, \cdots, N. \tag{HS}$$

with

$$H_\Omega : \mathcal{F}_N\Omega = \{z \in \Omega^N : z_j \neq z_k \text{ for } j \neq k\} \to \mathbb{R}$$

of the form

$$\begin{aligned} H_\Omega(z) &= \frac{1}{4\pi} \sum_{\substack{j,k=1 \\ j\neq k}}^{N} \log \frac{1}{|z_j - z_k|} - F(z) \\ &= \frac{1}{4\pi} \sum_{\substack{j,k=1 \\ j\neq k}}^{N} \log \frac{1}{|z_j - z_k|} - \frac{1}{2} \sum_{\substack{j,k=1 \\ j\neq k}}^{N} g(z_j, z_k) - \frac{1}{2} \sum_{k=1}^{N} h(z_k). \end{aligned}$$

As mentioned in Section 1.1, $g : \overline{\Omega} \times \Omega \to \mathbb{R}$ is the regular part of the Green's function of the first kind on $\Omega \subset \mathbb{C}$, which is of class $\mathcal{C}^2$ and symmetric: $g(w, z) = g(z, w)$ for all $w, z \in \Omega$.

$h : \Omega \to \mathbb{R}$, $h(z) = g(z, z)$ is the Robin function. g is bounded below and $h(z) \to \infty$ as $z \to \partial\Omega$. In particular h achieves its minimum in Ω if Ω is bounded.

The local result (Theorem 3.1.4) we obtain here is based upon a combination of Lyapunov-Schmidt reduction arguments and perturbation theory. We treat the equations as a perturbed system of the N point-vortex system in the unbounded plane, which has a known series of periodic solutions. Working locally around these solutions, we do not need the compactness condition for the action functional, thus avoid the usual trouble encountered in studying of singular Hamiltonian systems. From the abstract point of view we are led to the investigation of the existence of critical points for perturbations of a functional whose critical points appear in manifold.

To state the theorem, we need some basic definitions. Let X be a $\mathcal{C}^2$ Hilbert manifold. $f \in \mathcal{C}^1(X, \mathbb{R})$. We denote the critical set of f by

$$K_f := \{x \in X \mid Df(x) = 0\}$$

and the sublevel sets by

$$f^c := \{x \in X \mid f(x) \leqslant c\}.$$

Definition 3.1.1. We call $p \in K_f$ an *isolated critical point* of f if there is a neighborhood B of p such that $B \cap K_f = \{p\}$.

For a given isolated critical point p, we shall assign to it a series of groups which describe the local properties of f on a neighborhood of p.

Definition 3.1.2. Let p be an isolated critical point of f. The *critical groups* are defined as

$$C_*(f, p) = H_*(f^c \cap B; (f^c \backslash \{p\}) \cap B; F)$$

where $c = f(p)$ and B is a neighborhood of p such that $B \cap K_f = \{p\}$. H_* stands for any kind of homology theory with Abelian coefficient group F satisfying the excision, homotopy and exactness properties.

The excision property of the homology theory ensures the critical groups are well defined, i.e., they do not depend on the choice of the neighborhood B.

Definition 3.1.3. A critical point $p \in X$ of the function $f : X \to \mathbb{R}$ is said to be *stable* if it is isolated and its critical group is not trivial.

Here comes our main result:

Theorem 3.1.4. *If $a_0 \in \Omega$ is a stable critical point of h, then there exists $r_0 > 0$, such that for each $0 < r \leq r_0$, (HS) has a periodic solution $z^r = (z_1^r, \ldots, z_N^r)$ with minimal period $T_r = 8\pi^2 r^2/(N-1)$ such that $z_1^r(t) \to a_0$ as $r \to 0$, and $z_k^r(t) = z_1^r(t + (k-1)T_r/N)$ for every $k = 1, \ldots, N$. In the limit $r \to 0$ the vortices z_k^r move on circles in the following sense. There exists $a_r \in \Omega$ with $a_r \to a_0$ such that the rescaled function*

$$u_1^r(t) := \frac{1}{r}\big(z_1^r(T_r t/2\pi) - a_r\big)$$

satisfies

$$u_1^r(t) \to u_0(t) := e^{it}.$$

The convergence $u_1^r \to u_0$ *as* $r \to 0$ *holds in* $H^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{C})$.

Remark 3.1.5. The domain Ω could be a bounded, simply or multiply connected region in the plane. Theorem 3.1.4 does not apply to the annulus because, due to its rotational symmetry, the minimum of h is not isolated but there is a circle of minima. On the other hand, perturbing the annulus one obtains domains where the Robin function has at least two critical points, a minimum and a saddle point. One can also construct simply connected domains, e. g. dumb-bell shaped, where the Robin function has arbitrarily many local minima, and many saddle points; see [37]. The function $r(z) = e^{-h(z)}$ is the inner radius (conformal radius for simply connected domains) from the theory of complex functions; see [43] where one can find a discussion of the geometric role of critical points of r, hence of h.

Remark 3.1.6. An isolated local minimum or maximum is stable as is a nondegenerate saddle point. If $h(a + z) = h(a) + \alpha\mathrm{Re}(z^k) + \beta\mathrm{Im}(z^k) + o(|z|^k)$ as $z \to 0$ for some $k \geq 2$ and $\alpha^2 + \beta^2 \neq 0$ then a is degenerate but stable.

Since a hydrodynamic Robin function satisfies $h(z) \to \infty$ as $z \to \partial\Omega$, the minimum is always achieved in a bounded domain. Caffarelli and Friedman [22] showed that the Robin function is strictly convex if Ω is convex but not an infinite strip. In the latter case the function is still convex and explicitely known (see [14]), but of course invariant under translations, so that it cannot have an isolated critical point. Thus in a bounded convex domain the Robin function has a unique critical point, the global minimum. This is in fact nondegenerate according to [22, Theorem 3.1]. If the domain is smooth, bounded, symmetric with respect to the origin, and convex in the direction of the two coordinates then Grossi [41] showed that the origin is a nondegenerate critical point. More precisely, they considered an arbitrary bounded smooth domain $\Omega \subset \mathbb{R}^n$ and diffeomorphisms of $\mathbb{R}^n$ of the form $\mathrm{id} + \Theta$ with $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ small in the $\mathcal{C}^k$-norm. They showed that the Robin function of the Dirichlet Laplacian of $(\mathrm{id} + \Theta)(\Omega)$ has only nondegenerate critical points for a residual set of Θ's. For a generic bounded smooth domain, Micheletti and Pistoia [59] proved that all critical points of the Robin function are nondegenerate. Thus in a generic domain Theorem 3.1.4 applies and yields periodic solutions with arbitrarily small minimal period oscillating around the minimum of the Robin function.

Remark 3.1.7. One may consider Theorem 3.1.4 as a kind of singular Lyapunov center theorem (see [5]). Instead of obtaining periodic solutions near an equilibrium we obtain periodic solutions near a singularity. The logarithmic growth of H_Ω as z approaches a singularity, $z_j - z_k \to 0$ or $z_k \to \partial\Omega$, is not essential. More general growth conditions can be dealt with, see Theorem 3.5.2. We concentrate on the N-vortex Hamiltonian since it is a most prominent example of a first order singular Hamiltonian system which appears in mathematical physics, but with a different type of singularity appearing in classical mechanics.

3.2 Rescaling and variational setting

For convenience in notations and without loss of generality we assume $a_0 = 0$, the generalization to generic $a_0 \in \Omega$ being straightforward (see Remark 3.4.15).

The motion of N identical point vortices in unbounded plane is governed by the singular first order Hamiltonian system

$$\dot{z_k} = -i\nabla_{z_k} H_{\mathbb{C}}(z), \quad k = 1, \cdots, N,$$

with

$$H_{\mathbb{C}}(z) = -\frac{1}{4\pi} \sum_{\substack{j,k=1 \\ j\neq k}}^{N} \log |z_j - z_k|.$$

For any $r > 0$, it owns a $T_r = \frac{8\pi^2 r^2}{N-1}$-periodic solution of the form

$$\begin{pmatrix} re^{i\omega_r t} \\ re^{i(\omega_r t + \frac{2\pi}{N})} \\ \vdots \\ re^{i(\omega_r t + \frac{2\pi(N-1)}{N})} \end{pmatrix},$$

meaning the vortices, located at vertices of a regular N-polygon of radius r, rotate in angular velocity $\omega_r = \frac{N-1}{4\pi r^2}$ as a rigid body. Inspired by this phenomenon, we wish to find T_r-periodic solutions of the N point-vortex motion equations in a domain $\Omega \subset \mathbb{C}$:

$$\dot{z_k} = -i\nabla_{z_k}(H_{\mathbb{C}}(z) - F(z)), \quad k = 1, \cdots, N, \tag{3.2.1}$$

as a perturbed system with

$$F(z) = \frac{1}{2} \sum_{\substack{j,k=1 \\ j\neq k}}^{N} g(z_j, z_k) + \frac{1}{2} \sum_{k=1}^{N} h(z_k).$$

This is a period prescribed problem. In the sequel, we shall rescale the problem and fix the period to be 2π. During such a procedure, r is converted into a parameter of the equations instead of a value related to the period. Before doing this, we notice the following fact:

Lemma 3.2.1. z is a T-periodic solution of $\dot{z} = -i\nabla_z H(z)$ is equivalent to $u(t) := \frac{1}{r} z(\frac{T}{2\pi} t)$ is a 2π-periodic solution of $\dot{u} = -i\nabla_u \widetilde{H}(u)$, where

$$\widetilde{H}(u) := \frac{T}{2\pi r^2} H(ru).$$

Proof. z is T-periodic iff u is 2π periodic. $\dot{z} = -i\nabla_z H(z)$ is equivalent to

$$\dot{z}(\frac{T}{2\pi}t) = -i\nabla_z H(z(\frac{T}{2\pi}t)), \quad \forall t \in \mathbb{R}.$$

For any $r \neq 0$, this is again equivalent to

$$\frac{T}{2\pi r}\dot{z}(\frac{T}{2\pi}t) = -i\nabla_z \frac{T}{2\pi r} H(z(\frac{T}{2\pi}t)), \quad \forall t \in \mathbb{R}.$$

With $u(t) := \frac{1}{r}z(\frac{T}{2\pi}t)$, this is exactly

$$\dot{u} = -i\nabla_u \widetilde{H}(u).$$

$\square$

Therefore, we compute

$$\begin{aligned}\frac{T_r}{2\pi r^2}H_\Omega(ru) &= \frac{4\pi}{N-1}\Big[-\frac{1}{4\pi}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\log|ru_j - ru_k| - F(ru)\Big]\\ &= -\frac{1}{N-1}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\log|u_j-u_k| - \frac{4\pi}{N-1}F(ru) - \frac{1}{N-1}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\log|r|\end{aligned}$$

Notice that the last term is a constant independent of u, thus vanishes in the Hamiltonian equations when we take its derivative with respect to the variable u. So we may define the rescaled Hamiltonian to be

$$\begin{aligned}H_r(u) &:= \frac{T_r}{2\pi r^2}H_\Omega(ru) + \frac{1}{N-1}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\log r\\ &= -\frac{1}{N-1}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\log|u_j-u_k| - \frac{4\pi}{N-1}F(ru).\end{aligned}$$

Then as a consequence of Lemma 3.2.1, T_r-periodic solutions of (3.2.1) exactly correspond to 2π-periodic solutions of

$$\dot{u_k} = -i\nabla_{u_k}H_r(u), \quad k = 1,\cdots,N \qquad (\mathrm{HS}_r)$$

for non-zero r. Observe that H_r defines a function

$$H : \mathcal{O} := \big\{(r,u) \in \mathbb{R}\times\mathbb{C}^N : u_j \neq u_k \text{ for } j\neq k,\ ru_k \in \Omega \text{ for all } k\big\} \to \mathbb{R}$$

which is also defined for $r = 0$. Moreover, for $r = 0$ there holds

$$H_0(u) = -\frac{1}{N-1}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\log|u_j-u_k| - \frac{4\pi}{N-1}F(0),$$

hence system (HS_0) is given by

$$\dot{u}_k = \frac{2i}{N-1}\sum_{\substack{j=1\\ j\neq k}}^{N} \frac{u_j - u_k}{|u_j - u_k|^2}, \quad k = 1, \cdots, N, \tag{HS_0}$$

which are the rescaled equations of motions of N vortices in the unbounded plane. Hereafter we denote $u_0(t) = e^{it}$. (HS_0) has a family of 2π-periodic solutions $\theta * U_a$ parametrized by $\theta \in S^1$ and $a \in \mathbb{C}$ where

$$U_a(t) = \begin{pmatrix} a + u_0(t) \\ a + u_0(t + \frac{2\pi}{N}) \\ \vdots \\ a + u_0(t + \frac{2\pi(N-1)}{N}) \end{pmatrix}.$$

generated from the translations and the rotations of $(e^{it}, e^{i(t+2\pi/N)}, \cdots, e^{i(t+2\pi(N-1)/N)})^{tr}$.

After explaining how the parameter r is involved, let us give the variational setting.

Let $L^2_{2\pi}(\mathbb{C}^N) = L^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{C}^N)$ be the Hilbert space of 2π-periodic square integrable functions with scalar product

$$\langle x, y\rangle_{L^2} = \int_0^{2\pi} \big\langle x(t), y(t)\big\rangle_{\mathbb{R}^{2N}}\, dt = \sum_{k=1}^{N} \int_0^{2\pi} \mathrm{Re}(\overline{x_k(t)} y_k(t))\, dt$$

and associated norm $\|\cdot\|_{L^2}$. The space $H^1_{2\pi}(\mathbb{C}^N) = H^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{C}^N)$ is the Sobolev space of 2π-periodic functions which are absolutely continuous with square integrable derivative with scalar product

$$\langle x, y\rangle = \langle x, y\rangle_{L^2} + \langle \dot{x}, \dot{y}\rangle_{L^2}$$

and associated norm $\|\cdot\|$. Without confusion we will use $\|\cdot\|$ for any dimension N. There is an essential fact (see [56]):

Lemma 3.2.2. *There exists a constant $\kappa = \kappa(N, T) > 0$, such that,*

$$\|u\|_\infty := \max_{t\in[0,T]} |u(t)| \leqslant \kappa\|u\| \quad \textit{for any } u \in H^1_T(\mathbb{C}^N).$$

Moreover, $H^1_T(\mathbb{C}^N)$ embeds into $\mathcal{C}^0_T(\mathbb{C}^N)$ compactly.

There are two group actions on $L^2_{2\pi}(\mathbb{C}^N)$ and $H^1_{2\pi}(\mathbb{C}^N)$. One is the action of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ given by time shift:

$$\theta * u(t) = u(t + \theta), \quad \theta \in S^1,$$

and the other is the action of Σ_N which permutes the components:

$$\sigma * u(t) = \left(u_{\sigma^{-1}(1)}(t), \ldots, u_{\sigma^{-1}(N)}(t)\right), \quad \sigma \in \Sigma_N,$$

for $u \in L^2_{2\pi}(\mathbb{C}^N)$ or $H^1_{2\pi}(\mathbb{C}^N)$. They combine to yield an isometric action of $S^1 \times \Sigma_N$, where without confusion we all use the notation "$*$", given by

$$(\theta, \sigma) * u(t) = \big(u_{\sigma^{-1}(1)}(t+\theta), \ldots, u_{\sigma^{-1}(N)}(t+\theta)\big)$$

for $(\theta, \sigma) \in S^1 \times \Sigma_N$. Since g is symmetric it holds

$$F(\sigma * u) = F(u), \quad \text{for any } \sigma \in \Sigma_N. \tag{3.2.2}$$

The action functional corresponding to (HS_r) is given by

$$\Phi_r(u) := \frac{1}{2}\sum_{k=1}^{N}\int_0^{2\pi} \langle i\dot{u}_k, u_k\rangle_{\mathbb{R}^2}\, dt - \int_0^{2\pi} H_r(u)\, dt, \quad u \in H^1_{2\pi}(\mathbb{C}^N).$$

Observe that $\Phi(r,u) = \Phi_r(u)$ is defined for (r,u) in the set

$$\Lambda := \big\{(r,u) \in \mathbb{R} \times H^1_{2\pi}(\mathbb{C}^N) : (r, u(t)) \in \mathcal{O} \text{ for all } t\big\}.$$

Critical points of Φ_r for $r > 0$ correspond to 2π-periodic solutions of (HS_r).

Proposition 3.2.3. *a) Λ is an open subset of $\mathbb{R} \times H^1_{2\pi}(\mathbb{C}^N)$.*
b) Φ_r is invariant under the action of $S^1 \times \Sigma_N$.

Proof. To prove a), we apply Lemma 3.2.2 and suppose $\|u\|_\infty \leqslant \kappa\|u\|$ for all $u \in H^1_{2\pi}(\mathbb{C})$. Let (r,u) be any element in Λ, that is, $u_k(t) \neq u_j(t)$ and $ru_k(t) \in \Omega$ for any t and $k \neq j$. Then for every $(r', u') \in \mathbb{R} \times H^1_{2\pi}(\mathbb{C}^N)$, a straightforward computation shows that

$$\begin{aligned}
\kappa\|u' - u\| &\geqslant \frac{\kappa}{2}(\|u'_k - u_k\| + \|u'_j - u_j\|) \\
&\geqslant \frac{\kappa}{2}\|u'_k - u_k - u'_j + u_j\| \\
&\geqslant \frac{1}{2}\|u'_k - u_k - u'_j + u_j\|_\infty \\
&\geqslant \frac{1}{2}\Big||u'_k(t) - u'_j(t)| - |u_k(t) - u_j(t)|\Big|
\end{aligned}$$

for all t. It follows that

$$|u'_k(t) - u'_j(t)| \geqslant |u_k(t) - u_j(t)| - 2\kappa\|u' - u\|, \quad \forall\, t.$$

u is periodic and non-collision, so $|u_k(t) - u_j(t)| > M > 0$ with some value M for all t and $k \neq j$. Thus

$$|u'_k(t) - u'_j(t)| \geqslant M - 2\kappa\|u' - u\|,$$

which implies $|u'_k(t) - u'_j(t)| > 0$ for all t as long as $\|u' - u\|$ is sufficiently small. Meanwhile,

$$\begin{aligned}
&|r'u'_k(t) - ru_k(t)| \\
&\qquad \leqslant |r'u'_k(t) - r'u_k(t) + r'u_k(t) - ru_k(t)| \\
&\qquad \leqslant |r'||u'_k(t) - u_k(t)| + |r' - r||u_k(t)| \\
&\qquad \leqslant \kappa|r'|\|u'_k - u_k\| + |r' - r|\|u_k\|_\infty \\
&\qquad \leqslant \kappa|r'|\|u' - u\| + |r' - r|\|u_k\|_\infty
\end{aligned}$$

$\|u_k\|_\infty$ is finite, hence, $r'u'_k(t) \in \Omega$ provided r' are closed to r and $\|u' - u\|$ is small. Therefore we see that Λ is open in $\mathbb{R} \times H^1_{2\pi}(\mathbb{C}^N)$. The autonomy of H_r and (3.2.2) lead to the invariance of Φ_r under the action of $S^1 \times \Sigma_N$. □

With these constructions, the principle of symmetric criticality given by Palais [64] helps us to make a symmetric reduction to Φ_r. That paper provided various versions of this principle in different settings. Here we choose the one applicable for our problem.

Theorem 3.2.4. *[64] Let G be a group of isometries of a Riemannian manifold M and let $f : M \to \mathbb{R}$ be a C^1 function invariant under G. M^G is the set of stationary points of M under the action of G. Then if $p \in M^G$ is a critical point of $f|_{M^G}$, p is in fact a critical point of f.*

As mentioned before, the set

$$\mathcal{M} = \left\{\theta * U_a : \theta \in S^1, a \in \mathbb{C}\right\}$$

is a 3-dimensional non-compact submanifold of $H^1_{2\pi}(\mathbb{C}^N)$ consisting of 2π-periodic solutions of (HS_0).

Let $\sigma = (1\ 2\ \ldots\ N) \in \Sigma_N$ be the right shift, and set $\tau := \left(\frac{2\pi}{N}, \sigma^{-1}\right) \in S^1 \times \Sigma_N$, hence

$$\tau * u(t) = \begin{pmatrix} u_N(t + \frac{2\pi}{N}) \\ u_1(t + \frac{2\pi}{N}) \\ \vdots \\ u_{N-1}(t + \frac{2\pi}{N}) \end{pmatrix}.$$

Since for all $u \in H^1_{2\pi}(\mathbb{C}^N)$,

$$\tau^N * u(t) = \begin{pmatrix} u_1(t + 2\pi) \\ u_2(t + 2\pi) \\ \vdots \\ u_N(t + 2\pi) \end{pmatrix} = u(t),$$

i.e. $\tau^N = id$, $\langle\tau\rangle \subset S^1 \times \Sigma_N$ is a cyclic subgroup of $S^1 \times \Sigma_N$ of order N. Proposition 3.2.3 tells for each r, the domain of Φ_r is an open subset of the Hilbert space $H^1_{2\pi}(\mathbb{C}^N)$, thus is a Hilbert manifold and a Riemannian manifold. Also Φ_r is $(S^1 \times \Sigma_N)$-invariant. Theorem 3.2.4 applies consequently and then it is sufficient to find critical points of Φ_r constrained to

$$\Lambda^\tau = \{(r, u) \in \Lambda : u = \tau * u\}.$$

Clearly for $(r, u) \in \Lambda$ we have $(r, u) \in \Lambda^\tau$ if, and only if, $u_k(t) = u_1\left(t + \frac{2\pi(k-1)}{N}\right)$ for all $k = 1, \ldots, N$. Thus the map

$$\begin{aligned} &H^1_{2\pi}(\mathbb{C}) \to H^1_{2\pi}(\mathbb{C}^N), \\ &\qquad v \mapsto \widehat{v} := \left(v, \frac{2\pi}{N} * v, \ldots, \frac{2\pi(N-1)}{N} * v\right)^{tr}, \end{aligned}$$

induces a diffeomorphism

$$\mathcal{M}_1 := \{\theta * u_a : \theta \in S^1, u_a = u_0 + a, a \in \mathbb{C}\} \to \mathcal{M} \subset \Lambda^\tau,$$

and a diffeomorphism

$$\begin{aligned} \Lambda_1 := \{(r, u_1) \in &\mathbb{R} \times H^1_{2\pi}(\mathbb{C}) : (r, \widehat{u}_1) \in \Lambda^\tau\} \to \Lambda^\tau, \\ &(r, u_1) \mapsto (r, \widehat{u}_1). \end{aligned}$$

Defining $\Psi : \Lambda_1 \to \mathbb{R}$ by

$$\Psi(r, u_1) = \Psi_r(u_1) := \Phi_r(\widehat{u}_1),$$

i. e.

$$\begin{aligned} \Psi_r(u_1) = \frac{N}{2}\int_0^{2\pi} \langle i\dot{u}_1, u_1\rangle_{\mathbb{R}^2}\, dt + \frac{N}{N-1}\sum_{k=1}^{N-1}\int_0^{2\pi} \log\left|u_1 - \frac{2k\pi}{N} * u_1\right| dt \\ + \frac{4\pi}{N-1}\int_0^{2\pi} F(r\widehat{u}_1)\, dt, \end{aligned}$$

it suffices to find critical points of Ψ_r. More precisely, if u_1 is a critical point of Ψ_r, that is $D\Psi_r(u_1)[v_1] = 0$ for all $v_1 \in H^1_{2\pi}(\mathbb{C})$, then $D\Phi_r(\widehat{u}_1)[\widehat{v}_1] = 0$ for all $v_1 \in H^1_{2\pi}(\mathbb{C})$, which implies $\widehat{u}_1$ is a critical point of Φ_r. Hence, so far our underlying space is concerned with only one complex component of u.

In the end of this section, we fix $\delta > 0$ such that the tubular δ-neighborhood

$$\mathcal{U}_\delta(\mathcal{M}_1) = \{u_1 \in H^1_{2\pi}(\mathbb{C}) : \operatorname{dist}_{H^1}(u_1, \mathcal{M}_1) \leqslant \delta\}$$

of $\mathcal{M}_1$ in $H^1_{2\pi}(\mathbb{C})$ is contained in the domain Λ_0 of Ψ_0:

$$\Lambda_0 := \left\{u_1 \in H^1_{2\pi}(\mathbb{C}) : u_1(t) \neq u_1(t + \frac{2k\pi}{N}) \text{ for all } t, \text{ all } k = 1, \ldots, N-1\right\}. \tag{3.2.3}$$

This can be seen as follows: for every $v \in \mathcal{M}_1$, all t and any $k = 1, \cdots, N-1$,

$$\begin{aligned} \kappa\|u_1 - v\| &\geqslant \|u_1 - v\|_\infty \\ &\geqslant \frac{1}{2}\big(\,|u_1(t) - v(t)| + |u_1(t + \frac{2k\pi}{N}) - v(t + \frac{2k\pi}{N})|\,\big) \\ &\geqslant \frac{1}{2}\Big|\,|u_1(t) - u_1(t + \frac{2k\pi}{N})| - |v(t) - v(t + \frac{2k\pi}{N})|\,\Big| \end{aligned}$$

so that

$$|u_1(t) - u_1(t + \frac{2k\pi}{N})| \geqslant |v(t) - v(t + \frac{2k\pi}{N})| - 2\kappa\|u_1 - v\|.$$

Observe that $|v(t) - v(t + \frac{2k\pi}{N})|$ is uniformly bounded below away from 0. A constant $\delta > 0$ always exists to guarantee $\mathcal{U}_\delta(\mathcal{M}_1) \subset \Lambda_0$.

3.3 Finite dimensional reduction

Before the main task of this section, we give some basic calculus.

Lemma 3.3.1. *For $(r, u_1) \in \Lambda_1$ we have*

$$\nabla\Psi_r(u_1) = N(Id - \Delta)^{-1}\left(i\dot{u}_1 + \frac{2}{N-1}\sum_{k=1}^{N-1}\frac{u_1 - \frac{2k\pi}{N} * u_1}{|u_1 - \frac{2k\pi}{N} * u_1|^2} + \frac{4\pi r}{N-1}\partial_1 F(r\widehat{u}_1)\right),$$

where $\Delta : H^2_{2\pi}(\mathbb{C}) \to L^2_{2\pi}(\mathbb{C})$, $\Delta v = \ddot{v}$, and ∂_1 means the gradient in the real sense with respect to the first complex component. Particularly,

$$\nabla\Psi_0(u_1) = N(Id - \Delta)^{-1}\left(i\dot{u}_1 + \frac{2}{N-1}\sum_{k=1}^{N-1}\frac{u_1 - \frac{2k\pi}{N} * u_1}{|u_1 - \frac{2k\pi}{N} * u_1|^2}\right).$$

Moreover,

$$\begin{aligned}\nabla^2\Psi_0(u_1)[v] = N(Id - \Delta)^{-1}\Bigg(& i\dot{v} + \frac{2}{N-1}\sum_{k=1}^{N-1}\frac{v - \frac{2k\pi}{N} * v}{|u_1 - \frac{2k\pi}{N} * u_1|^2} \\ & - \frac{4}{N-1}\sum_{k=1}^{N-1}\frac{\left\langle u_1 - \frac{2k\pi}{N} * u_1, v - \frac{2k\pi}{N} * v\right\rangle_{\mathbb{R}^2}}{|u_1 - \frac{2k\pi}{N} * u_1|^4}\left(u_1 - \frac{2k\pi}{N} * u_1\right)\Bigg).\end{aligned}$$

Proof. For any $v \in H^1_{2\pi}(\mathbb{C})$,

$$\begin{aligned}\langle\nabla\Psi_0(u_1), v\rangle &= D\Psi_0(u_1)[v] = \lim_{s\to 0}\frac{\Psi_0(u_1 + sv) - \Psi_0(u_1)}{s} \\ &= \frac{N}{2}\int_0^{2\pi}\lim_{s\to 0}\frac{1}{s}\left[\langle i(\dot{u}_1 + s\dot{v}), u_1 + sv\rangle_{\mathbb{R}^2} - \langle i\dot{u}_1, u_1\rangle_{\mathbb{R}^2}\right]dt \\ &\quad + \frac{N}{N-1}\sum_{k=1}^{N-1}\int_0^{2\pi}\lim_{s\to 0}\frac{1}{s}\Bigg[\log|u_1(t) - u_1(t + \frac{2\pi k}{N}) + sv(t) - sv(t + \frac{2\pi k}{N})| \\ &\qquad - \log|u_1(t) - u_1(t + \frac{2\pi k}{N})|\Bigg]dt \\ &= N\int_0^{2\pi}\langle i\dot{u}_1, v\rangle_{\mathbb{R}^2}dt + \frac{N}{N-1}\sum_{k=1}^{N-1}\int_0^{2\pi}\frac{\langle u_1(t) - u_1(t + \frac{2\pi k}{N}), v(t) - v(t + \frac{2\pi k}{N})\rangle_{\mathbb{R}^2}}{|u_1(t) - u_1(t + \frac{2\pi k}{N})|^2}dt.\end{aligned}$$

Observe that

$$
\begin{aligned}
\sum_{k=1}^{N-1} \int_0^{2\pi} & \frac{\langle u_1(t) - u_1(t+\frac{2\pi k}{N}), v(t+\frac{2\pi k}{N})\rangle_{\mathbb{R}^2}}{|u_1(t) - u_1(t+\frac{2\pi k}{N})|^2} dt \\
&= \sum_{k=1}^{N-1} \int_{-\frac{2\pi k}{N}}^{2\pi - \frac{2\pi k}{N}} \frac{\langle u_1(t-\frac{2\pi k}{N}) - u_1(t), v(t)\rangle_{\mathbb{R}^2}}{|u_1(t-\frac{2\pi k}{N}) - u_1(t)|^2}\, dt \\
&= \sum_{k=1}^{N-1} \int_0^{2\pi} \frac{\langle u_1(t-\frac{2\pi k}{N}) - u_1(t), v(t)\rangle_{\mathbb{R}^2}}{|u_1(t-\frac{2\pi k}{N}) - u_1(t)|^2}\, dt \\
&= \sum_{k=1}^{N-1} \int_0^{2\pi} \frac{\langle u_1(t+\frac{2\pi (N-k)}{N}) - u_1(t), v(t)\rangle_{\mathbb{R}^2}}{|u_1(t+\frac{2\pi (N-k)}{N}) - u_1(t)|^2}\, dt \\
&= \sum_{k'=1}^{N-1} \int_0^{2\pi} \frac{\langle u_1(t+\frac{2\pi k}{N}) - u_1(t), v(t)\rangle_{\mathbb{R}^2}}{|u_1(t+\frac{2\pi k'}{N}) - u_1(t)|^2}\, dt \\
&= -\sum_{k=1}^{N-1} \int_0^{2\pi} \frac{\langle u_1(t) - u_1(t+\frac{2\pi k}{N}), v(t)\rangle_{\mathbb{R}^2}}{|u_1(t) - u_1(t+\frac{2\pi k}{N})|^2}\, dt.
\end{aligned}
\tag{3.3.1}
$$

Consequently,

$$
\langle \nabla \Psi_0(u_1), v\rangle = N \int_0^{2\pi} \langle i\dot{u}_1, v\rangle_{\mathbb{R}^2}\, dt + \frac{2N}{N-1} \int_0^{2\pi} \Big\langle \sum_{k=1}^{N-1} \frac{u_1(t) - u_1(t+\frac{2\pi k}{N})}{|u_1(t) - u_1(t+\frac{2\pi k}{N})|^2}, v(t) \Big\rangle_{\mathbb{R}^2} dt,
$$

thus we get

$$
\nabla \Psi_0(u_1) = N(Id - \Delta)^{-1} \Big[i\dot{u}_1 + \frac{2}{N-1} \sum_{k=1}^{N-1} \frac{u_1(t) - u_1(t+\frac{2\pi k}{N})}{|u_1(t) - u_1(t+\frac{2\pi k}{N})|^2} \Big].
$$

Consider the perturbed functional

$$
\Psi_r(u_1) = \Psi_0(u_1) - \frac{8\pi^2}{N-1} F(0) + \frac{4\pi}{N-1} \int_0^{2\pi} F(r\widehat{u}_1)\, dt.
$$

Setting

$$
\mathcal{F}_r(u_1) := \int_0^{2\pi} F(r\widehat{u}_1)\, dt, \quad (r, u_1) \in \Lambda_1,
$$

we have

$$
\begin{aligned}
\langle \nabla \mathcal{F}_r(u_1), v\rangle &= r \int_0^{2\pi} \langle \nabla F(r\widehat{u}_1), \widehat{v}\rangle_{\mathbb{R}^{2N}}\, dt \\
&= r \sum_{k=1}^{N} \int_0^{2\pi} \Big\langle \partial_k F(r\widehat{u}_1), \frac{2\pi(k-1)}{N} * v \Big\rangle_{\mathbb{R}^2} dt.
\end{aligned}
$$

Observe for each k,

$$
\begin{aligned}
&\int_0^{2\pi} \big\langle \partial_k F(r\widehat{u}_1), \frac{2\pi(k-1)}{N} * v\big\rangle_{\mathbb{R}^2}\, dt \\
=&\int_0^{2\pi} \big\langle \partial_k F(ru_1(t), ru_1(t+\frac{2\pi}{N}), \cdots, ru_1(t+\frac{2\pi(N-1)}{N})), v(t+\frac{2\pi(k-1)}{N})\big\rangle_{\mathbb{R}^2}\, dt \\
=&\int_{-\frac{2\pi(k-1)}{N}}^{2\pi-\frac{2\pi(k-1)}{N}} \big\langle \partial_k F(ru_1(t+\frac{2\pi(N-k+1)}{N}), ru_1(t+\frac{2\pi(N-k+2)}{N}), \cdots, ru_1(t+\frac{2\pi(N-1)}{N}), \\
&\qquad ru_1(t), ru_1(t+\frac{2\pi}{N}), \cdots, ru_1(t+\frac{2\pi(N-k)}{N})), v(t)\big\rangle_{\mathbb{R}^2}\, dt \\
=&\int_0^{2\pi} \big\langle \partial_k F(ru_1(t+\frac{2\pi(N-k+1)}{N}), ru_1(t+\frac{2\pi(N-k+2)}{N}), \cdots, ru_1(t+\frac{2\pi(N-1)}{N}), \\
&\qquad ru_1(t), ru_1(t+\frac{2\pi}{N}), \cdots, ru_1(t+\frac{2\pi(N-k)}{N})), v(t)\big\rangle_{\mathbb{R}^2}\, dt \\
=&\int_0^{2\pi} \big\langle \partial_1 F(ru_1(t), ru_1(t+\frac{2\pi}{N}), \cdots, ru_1(t+\frac{2\pi(N-1)}{N})), v(t)\big\rangle_{\mathbb{R}^2}\, dt,
\end{aligned}
$$

here we used the invariance of F under permutation. Therefore, we conclude

$$
\nabla\mathcal{F}_r(u_1) = (Id-\Delta)^{-1}\Big[rN\partial_1 F(r\widehat{u}_1)\Big],
$$

and hence,

$$
\nabla\Psi_r(u_1) = N(Id-\Delta)^{-1}\Big[i\dot{u}_1 + \frac{2}{N-1}\sum_{k=1}^{N-1}\frac{u_1(t)-u_1(t+\frac{2\pi k}{N})}{|u_1(t)-u_1(t+\frac{2\pi k}{N})|^2} + \frac{4\pi r}{N-1}\partial_1 F(r\widehat{u}_1)\Big].
$$

Furthermore, for $v, w \in H^1_{2\pi}(\mathbb{C})$, we continue to compute that

$$
\begin{aligned}
&\langle \nabla^2\Psi_0(u_1)[v], w\rangle = \lim_{s\to 0}\frac{1}{s}\big[\langle \nabla\Psi_0(u_1+sw), v\rangle - \langle \nabla\Psi_0(u_1), v\rangle\big] \\
=&N\int_0^{2\pi} \lim_{s\to 0}\frac{1}{s}\big[\langle i(\dot{u}_1+s\dot{w}), v\rangle_{\mathbb{R}^2} - \langle i\dot{u}_1, v\rangle_{\mathbb{R}^2}\big]\, dt \\
&+\frac{2N}{N-1}\sum_{k=1}^{N-1}\int_0^{2\pi}\lim_{s\to 0}\frac{1}{s}\Big\langle \frac{u_1(t)-u_1(t+\frac{2\pi k}{N}) + s\big(w(t)-w(t+\frac{2\pi k}{N})\big)}{|u_1(t)-u_1(t+\frac{2\pi k}{N}) + s\big(w(t)-w(t+\frac{2\pi k}{N})\big)|^2} \\
&\qquad - \frac{u_1(t)-u_1(t+\frac{2\pi k}{N})}{|u_1(t)-u_1(t+\frac{2\pi k}{N})|^2}, v(t)\Big\rangle_{\mathbb{R}^2}\, dt \\
=&N\int_0^{2\pi}\langle i\dot{w}, v\rangle_{\mathbb{R}^2}\, dt + \frac{2N}{N-1}\sum_{k=1}^{N-1}\int_0^{2\pi}\frac{\langle w(t)-w(t+\frac{2\pi k}{N}), v(t)\rangle_{\mathbb{R}^2}}{|u_1(t)-u_1(t+\frac{2\pi k}{N})|^2}\, dt \\
&-\frac{4N}{N-1}\sum_{k=1}^{N-1}\int_0^{2\pi}\frac{\langle u_1(t)-u_1(t+\frac{2\pi k}{N}), v(t)\rangle_{\mathbb{R}^2}\cdot\langle u_1(t)-u_1(t+\frac{2\pi k}{N}), w(t)-w(t+\frac{2\pi k}{N})\rangle_{\mathbb{R}^2}}{|u_1(t)-u_1(t+\frac{2\pi k}{N})|^4}\, dt
\end{aligned}
\tag{3.3.2}
$$

As in (3.3.1) there hold

$$\sum_{k=1}^{N-1}\int_0^{2\pi}\frac{\langle w(t+\frac{2\pi k}{N}),v(t)\rangle_{\mathbb{R}^2}}{|u_1(t)-u_1(t+\frac{2\pi k}{N})|^2}\,dt=\sum_{k=1}^{N-1}\int_0^{2\pi}\frac{\langle w(t),v(t+\frac{2\pi k}{N})\rangle_{\mathbb{R}^2}}{|u_1(t)-u_1(t+\frac{2\pi k}{N})|^2}\,dt \tag{3.3.3}$$

and

$$\begin{aligned}&\sum_{k=1}^{N-1}\int_0^{2\pi}\frac{\langle u_1(t)-u_1(t+\frac{2\pi k}{N}),v(t)\rangle_{\mathbb{R}^2}\cdot\langle u_1(t)-u_1(t+\frac{2\pi k}{N}),w(t+\frac{2\pi k}{N})\rangle_{\mathbb{R}^2}}{|u_1(t)-u_1(t+\frac{2\pi k}{N})|^4}\,dt\\ &\quad=\sum_{k=1}^{N-1}\int_0^{2\pi}\frac{\langle u_1(t)-u_1(t+\frac{2\pi k}{N}),v(t+\frac{2\pi k}{N})\rangle_{\mathbb{R}^2}\cdot\langle u_1(t)-u_1(t+\frac{2\pi k}{N}),w(t)\rangle_{\mathbb{R}^2}}{|u_1(t)-u_1(t+\frac{2\pi k}{N})|^4}\,dt\end{aligned} \tag{3.3.4}$$

Thus combining (3.3.2)-(3.3.4) we have

$$\begin{aligned}\langle\nabla^2\Psi_0(u_1)[v],w\rangle&=N\int_0^{2\pi}\langle i\dot v,w\rangle_{\mathbb{R}^2}\,dt+\frac{2N}{N-1}\sum_{k=1}^{N-1}\int_0^{2\pi}\frac{\langle v(t)-v(t+\frac{2\pi k}{N}),w(t)\rangle_{\mathbb{R}^2}}{|u_1(t)-u_1(t+\frac{2\pi k}{N})|^2}\,dt\\ &-\frac{4N}{N-1}\sum_{k=1}^{N-1}\int_0^{2\pi}\frac{\langle u_1(t)-u_1(t+\frac{2\pi k}{N}),v(t)-v(t+\frac{2\pi k}{N})\rangle_{\mathbb{R}^2}\cdot\langle u_1(t)-u_1(t+\frac{2\pi k}{N}),w(t)\rangle_{\mathbb{R}^2}}{|u_1(t)-u_1(t+\frac{2\pi k}{N})|^4}\,dt.\end{aligned}$$

As a result,

$$\begin{aligned}\nabla^2\Psi_0(u_1)[v]=&N(Id-\Delta)^{-1}\Big[i\dot v+\frac{2}{N-1}\sum_{k=1}^{N-1}\frac{v(t)-v(t+\frac{2\pi k}{N})}{|u_1(t)-u_1(t+\frac{2\pi k}{N})|^2}\\ &-\frac{4}{N-1}\sum_{k=1}^{N-1}\frac{\big\langle u_1(t)-u_1(t+\frac{2\pi k}{N}),v(t)-v(t+\frac{2\pi k}{N})\big\rangle_{\mathbb{R}^2}}{|u_1(t)-u_1(t+\frac{2\pi k}{N})|^4}\big(u_1(t)-u_1(t+\frac{2\pi k}{N})\big)\Big]\end{aligned}$$

is obtained. □

From the formula above we have

$$\nabla\Psi_0(u_1+a)=\nabla\Psi_0(u_1)\quad\text{and}\quad\nabla^2\Psi_0(u_1+a)=\nabla^2\Psi_0(u_1) \tag{3.3.5}$$

for any $u_1\in\Lambda_0$ and $a\in\mathbb{C}$. A direct computation shows for $u_0(t)=e^{it}$ that

$$\nabla^2\Psi_0(u_0)[v]=N(Id-\Delta)^{-1}\left(i\dot v-\frac{2}{N-1}\sum_{k=1}^{N-1}\frac{(u_0-\frac{2k\pi}{N}*u_0)^2}{|1-e^{2k\pi i/N}|^4}\left(\overline{v}-\frac{2k\pi}{N}*\overline{v}\right)\right).$$

When a functional is invariant under a continuous group action, its critical points, which appear as orbits, are degenerate in general. Like here u_0 is degenerate as a critical point of Ψ_0. Therefore, for such problems with symmetries, Bott [20] introduced the concept of nondegenerate critical manifold to extend the notion of nondegenerate critical point.

Definition 3.3.2. Let Z be a connected $\mathcal{C}^2$-submanifold of a Hilbert manifold X and let $f \in \mathcal{C}^2(X, \mathbb{R})$. We say that Z is a *nondegenerate critical manifold* of f if

(1) all points of Z are critical points of f;

(2) the nullity of each $z \in Z$ is equal to the dimension of Z, i.e. dim Ker$f''(z) = \dim Z$;

(3) $f''(z)$ is a Fredholm operator for each $z \in Z$.

Remark 3.3.3. 1. Assumption (2) implies Ker$f''(z) = T_z Z$ for each $z \in Z$, herein $T_z Z$ is the tangent space of Z at z. In other words, z is a nondegenerate critical point of f constricted to the normal space of Z at z.

2. This is slightly different from the definition 10.3 of [26] in which Z is required to be compact. Since the compactness condition does not make any trouble to our problem, we just ignore it for convenience.

Then we can prove

Lemma 3.3.4. *$\mathcal{M}_1$ is a nondegenerate critical manifold of Ψ_0, in particular Ker $\nabla^2\Psi_0(u_0) = T_{u_0}\mathcal{M}_1$.*

Proof. Since $\mathcal{M}_1 = S^1 * (u_0 + \mathbb{C})$ is the homogeneous space obtained from u_0 via the translations $u_0 \mapsto u_0 + a$ and via the S^1-action, and since Ψ_0 is invariant under these actions, it is sufficient to show that Ker $\nabla^2\Psi_0(u_0) = T_{u_0}\mathcal{M}_1$. Clearly $T_{u_0}\mathcal{M}_1 \subset$ Ker $\nabla^2\Psi_0(u_0)$, hence we only need to prove that

$$\mathrm{Ker}\, \nabla^2\Psi_0(u_0) \subset T_{u_0}\mathcal{M}_1 = \{a + ic \cdot u_0 \,:\, a \in \mathbb{C},\, c \in \mathbb{R}\,\}.$$

Consider an element $v \in$ Ker $\nabla^2\Psi_0(u_0)$, so that

$$i\dot{v}(t) - \frac{2}{N-1}\sum_{k=1}^{N-1} \frac{(e^{it} - e^{i(t+\frac{2k\pi}{N})})^2}{|1 - e^{i\cdot\frac{2k\pi}{N}}|^4}\left(\overline{v(t)} - \overline{v(t+\frac{2k\pi}{N})}\right) = 0. \tag{3.3.6}$$

We write v in its Fourier expansion, $v(t) = \sum_{n\in\mathbb{Z}} \alpha_n e^{int}$ with coefficients $\alpha_n \in \mathbb{C}$, and substitute it into (3.3.6) obtaining

$$\sum_{n\in\mathbb{Z}} n\alpha_n e^{int} + \frac{2}{N-1}\sum_{n\in\mathbb{Z}}\sum_{k=1}^{N-1} \frac{1 - e^{-i\cdot\frac{2kn\pi}{N}}}{(1 - e^{-i\cdot\frac{2k\pi}{N}})^2} \cdot \overline{\alpha}_n e^{i(2-n)t} = 0.$$

Setting

$$\xi_n := \sum_{k=1}^{N-1} \frac{1 - e^{i\cdot\frac{2\pi k(n-2)}{N}}}{(1 - e^{-i\cdot\frac{2\pi k}{N}})^2},$$

a comparison of the coefficients yields for each $n \in \mathbb{Z}$:

$$n\alpha_n + \frac{2}{N-1}\xi_n \cdot \overline{\alpha}_{2-n} = 0, \tag{3.3.7}$$

and, replacing n by $2-n$:

$$(2-n)\overline{\alpha}_{2-n} + \frac{2}{N-1}\overline{\xi}_{2-n}\cdot\alpha_n = 0. \tag{3.3.8}$$

Observe that

$$\overline{\xi_n} = \sum_{k=1}^{N-1}\frac{1-e^{-i\cdot\frac{2k(n-2)\pi}{N}}}{(1-e^{i\cdot\frac{2k\pi}{N}})^2} = \sum_{k=1}^{N-1}\frac{1-e^{i\cdot\frac{2(N-k)(n-2)\pi}{N}}}{(1-e^{-i\cdot\frac{2(N-k)\pi}{N}})^2} = \sum_{k=1}^{N-1}\frac{1-e^{i\cdot\frac{2k(n-2)\pi}{N}}}{(1-e^{-i\cdot\frac{2k\pi}{N}})^2} = \xi_n,$$

hence $\xi_n \in \mathbb{R}$ for any $n \in \mathbb{Z}$. On the other hand, the computation

$$\begin{aligned}\xi_n - \xi_{2-n} &= \sum_{k=1}^{N-1}\left(\frac{1-e^{i\cdot\frac{2k(n-2)\pi}{N}}}{(1-e^{-i\cdot\frac{2k\pi}{N}})^2} - \frac{1-e^{i\cdot\frac{2kn\pi}{N}}}{(1-e^{i\cdot\frac{2k\pi}{N}})^2}\right)\\ &= \sum_{k=1}^{N-1}\frac{e^{i\cdot\frac{4k\pi}{N}} - e^{-i\cdot\frac{4k\pi}{N}} + 2e^{-i\cdot\frac{2k\pi}{N}} - 2e^{i\cdot\frac{2k\pi}{N}}}{|1-e^{i\cdot\frac{2k\pi}{N}}|^4}\end{aligned}$$

shows that $\overline{\xi}_n - \overline{\xi}_{2-n} = -(\xi_n - \xi_{2-n})$, i.e. $\xi_n - \xi_{2-n} \in i\mathbb{R}$ and therefore $\xi_n = \xi_{2-n} \in \mathbb{R}$. Combining this with (3.3.7) and (3.3.8) we deduce

$$n(2-n)\alpha_n = \frac{4}{(N-1)^2}\,\xi_n^2\,\alpha_n \quad \text{for all } n \in \mathbb{Z},$$

which immediately implies

$$\alpha_n = 0, \quad \text{for all } n \neq 0, 1, 2. \tag{3.3.9}$$

Next we take $n = 1$ in (3.3.7) and obtain, using the equality $\xi_1 = \frac{N-1}{2}$:

$$0 = \alpha_1 + \frac{2}{N-1}\xi_1\cdot\overline{\alpha}_1 = \alpha_1 + \overline{\alpha}_1, \quad \text{thus } \alpha_1 \in i\mathbb{R}. \tag{3.3.10}$$

Finally, considering $n = 0$ in (3.3.8) yields

$$\alpha_2 = 0 \tag{3.3.11}$$

because $\xi_2 = 0$. Now (3.3.9)-(3.3.11) imply $v \in T_{u_0}\mathcal{M}_1$. $\nabla^2\Psi_0(u_0)$ is Fredholm with

$$\dim \mathrm{Ker}\nabla^2\Psi_0(u_0) = \mathrm{codim}\,\mathrm{Ran}\nabla^2\Psi_0(u_0) = 3.$$

□

After proving $\mathcal{M}_1$ is a nondegenerate critical manifold of Ψ_0, we are in the position to seek critical points of the functional

$$\Psi_r(u_1) = \Psi_0 - \frac{8\pi^2}{N-1}F(0) + \frac{4\pi}{N-1}\int_0^{2\pi} F(r\widehat{u}_1)\,dt.$$

Remark 3.3.5. In the work [4] of Ambrosetti, Coti Zelati and Ekeland, they applied the celebrated Lusternik-Schnirelmann theory to obtain a perturbation result by means of which a class of first order singular Hamiltonian systems can be addressed. Their theorem states: let E be a Hilbert space, Λ an open subset of E and $f_\epsilon \in C^2(\Lambda, \mathbb{R})$ is smooth in ϵ. If Z is a compact nondegenerate critical manifold of f_0 and f_0'' is a Fredholm operator of index 0 for all $z \in Z$, then there exists an $\epsilon_0 > 0$ and a neighborhood U of Z such that f_ϵ has at least cat(Z) critical points in U for any $0 < |\epsilon| < \epsilon_0$. In the proof, equivalently they looked for critical point of f_ϵ constrained on a neighborhood $Z_\epsilon \subset U$ of Z. The compactness of Z is of great importance. It yields the compactness of Z_ϵ and then Lusternik-Schnirelmann theory is applicable since the Palais-Smale condition is then satisfied naturally. However, in our case $\mathcal{M}_1$ is not compact and the Palais-Smale condition fails. Without the (PS) condition the Lusternik-Schnirelmann theory is no longer true, thus a result like in [4] does not hold in our setting. In contrast with their approach, we turn to the continuity method and make use of the property of the Robin's function.

With the preceding lemma, we come to the procedure of the Lyapunov-Schmidt reduction, which is a sophisticated technique widely used to study the solutions of nonlinear equations when the implicit function theorem does not work. It reduces infinite dimensional equations in Banach spaces to finite dimensional equations.

For a given $v \in \mathcal{M}_1$ we denote $P_v : H^1_{2\pi}(\mathbb{C}) \to T_v\mathcal{M}_1$ the orthogonal projection. The equation (HS$_r$) is equivalent to the system

$$\begin{cases} P_v\big(\nabla\Psi_r(u_1)\big) = 0, \\ (Id - P_v)\big(\nabla\Psi_r(u_1)\big) = 0. \end{cases} \tag{3.3.12}$$

Since $\nabla^2\Psi_0(v)$ is self-adjoint and Fredholm, we have the orthogonal direct sum decomposition

$$H^1_{2\pi}(\mathbb{C}) = T_v\mathcal{M}_1 \oplus N_v\mathcal{M}_1$$

with $T_v\mathcal{M}_1 = \operatorname{Ker}\nabla^2\Psi_0(v) = \operatorname{Ran} P_v$ and $N_v\mathcal{M}_1 = \operatorname{Ran}\nabla^2\Psi_0(v) = \operatorname{Ker} P_v$. We try to find solutions of the form $u_1 = v + w$ with $v \in \mathcal{M}_1$ and $w \in N_v\mathcal{M}_1$ small. Technically, we apply a Lyapunov-Schmidt reduction to the system

$$\begin{cases} P_v\big(\nabla\Psi_r(v+w)\big) = 0, \\ (Id - P_v)\big(\nabla\Psi_r(v+w)\big) = 0. \end{cases} \tag{3.3.13}$$

More precisely, for fixed $v \in \mathcal{M}_1$ and $r \sim 0$ we first solve the second equation in (3.3.13), using the contraction mapping principle in a suitable neighborhood of $0 \in N_v\mathcal{M}_1$. This yields a solution $w = W(r, v) \in N_v\mathcal{M}_1$ which in turn will be substituted into the first equation of (3.3.13).

In order to do this, we fix a constant $\rho > 0$ such that $B_{2\rho}(0) \subset \Omega$. Then $\Psi_r(u_1)$ is well-defined provided $u_1 \in \mathcal{U}_\delta(\mathcal{M}_1)$ and $|ru_1(t)| \le 2\rho$ for all t; here δ is from (3.2.3).

Lemma 3.3.6. *There exists a constant $r_0 = r_0(\rho, \delta) > 0$ and an S^1-equivariant map*

$$W : \mathcal{U} := \big\{(r, v) \in \mathbb{R} \times \mathcal{M}_1 : |r| \le r_0, v = \theta * u_a, a \in \mathbb{C}, |ra| \le \rho\big\} \to H^1_{2\pi}(\mathbb{C})$$

such that $W(r, v) \in N_v\mathcal{M}_1$, satisfying $\|W(r, v)\| \le \delta$ and solving the equation

$$(Id - P_v)\Big(\nabla\Psi_r\big(v + W(r, v)\big)\Big) = 0.$$

Proof. The proof is based on an application of the contraction mapping principle, and consists of four steps.

STEP 1. Reduction to a fixed point problem
We fix $R > 0$ and define

$$\mathcal{M}_1^R := \big\{v = \theta * u_a \in \mathcal{M}_1 : \theta \in S^1,\ a \in \mathbb{C},\ |a| \le R\big\}.$$

We shall define $W(r, v) \in N_v\mathcal{M}_1$ for $|r|$ small and $v \in \mathcal{M}_1^R$. First of all, there exists a constant $\kappa > 0$ such that

$$\|u\|_\infty \le \kappa\|u\|, \quad \text{for all } u \in H^1_{2\pi}(\mathbb{C}).$$

Given $r \in \mathbb{R}$ with $|r| \le \frac{\rho}{R}$ and $|r| \le r_1 := \frac{\rho}{1+\kappa\delta}$, it follows for any $v = \theta * u_a \in \mathcal{M}_1^R$ and $\|w\| \le \delta$ that $v + w \in \mathcal{U}_\delta(\mathcal{M}_1)$ and

$$\begin{aligned} |rv(t) + rw(t)| &\le |ra| + |ru_0(t)| + |rw(t)| \\ &\le |r|R + |r| + |r|\kappa\delta \\ &\le 2\rho \end{aligned}$$

for all t so that $\Psi_r(v + w)$ is well-defined.

The second equation in (3.3.13) is equivalent to

$$\begin{aligned} &(Id - P_v) \circ \nabla^2\Psi_0(v)[w] \\ &\qquad = -(Id - P_v)\Big[\nabla\Psi_0(v + w) - \nabla^2\Psi_0(v)[w] + \frac{4\pi rN}{N-1}(Id - \Delta)^{-1}\big(\partial_1 F(r\widehat{v} + r\widehat{w})\big)\Big]. \end{aligned}$$

As a consequence of Lemma 3.3.4 the operator $\mathcal{L}_v := (Id - P_v) \circ \nabla^2\Psi_0(v)$ induces an isomorphism $\mathcal{L}_v|_{N_v\mathcal{M}_1}$ on $N_v\mathcal{M}_1$. Also notice that if $v = \theta * u_a$ then $\mathcal{L}_v = (Id - P_{\theta*u_0}) \circ \nabla^2\Psi_0(\theta * u_0)$ is actually independent of a because of (3.3.5). Thus $(\mathcal{L}_v|_{N_v\mathcal{M}_1})^{-1}$ exists and there is a constant $\gamma > 0$ independent of v, such that

$$\|(\mathcal{L}_v|_{N_v\mathcal{M}_1})^{-1}(w)\| \le \gamma\|w\|, \quad \text{for all } v \in \mathcal{M}_1 \text{ and } w \in N_v\mathcal{M}_1. \tag{3.3.14}$$

Next we define the operator $T(r, v, \cdot) : N_v\mathcal{M}_1 \to N_v\mathcal{M}_1$ by

$$\begin{aligned} T(r, v, w) = -(\mathcal{L}_v|_{N_v\mathcal{M}_1})^{-1} \circ (Id - P_v)\Big[&\nabla\Psi_0(v + w) - \nabla^2\Psi_0(v)[w] \\ &+ \frac{4\pi rN}{N-1}(Id - \Delta)^{-1}\big(\partial_1 F(r\widehat{v} + r\widehat{w})\big)\Big]. \end{aligned}$$

Then $w \in N_v\mathcal{M}_1$ solving the second equation in (3.3.13) is equivalent to being a solution of the fixed point equation $w = T(r, v, w)$. In the following, we will prove that for $r \sim 0$ and $v \in \mathcal{M}_1^R$ arbitrary, $T(r, v, \cdot)$ is a contraction on a suitable neighborhood of 0 in $N_v\mathcal{M}_1$.

STEP 2. We prove that there exist constants $0 < r_2 < r_1 = \frac{\rho}{1+\kappa\delta}$ and $0 < \delta_1 < \delta$, such that for any $|r| \le \min\{r_2, \frac{\rho}{R}\}$ and $w, w' \in N_v\mathcal{M}_1$ with $\|w\|, \|w'\| \le \delta_1$, there holds

$$\|T(r, v, w) - T(r, v, w')\| \leqslant \frac{1}{2}\|w - w'\|. \tag{3.3.15}$$

In order to see this we first observe that (3.3.14) implies for $w, w' \in N_v\mathcal{M}_1$ with $\|w\|, \|w'\| \leqslant \delta$, that

$$\begin{aligned}\|T(r, v, w) - T(r, v, w')\| \le \gamma\Big(&\big\|\nabla\Psi_0(v + w) - \nabla\Psi_0(v + w') - \nabla^2\Psi_0(v)[w - w']\big\| \\ &+ \frac{4\pi|r|N}{N-1}\big\|(Id - \Delta)^{-1}\big(\partial_1 F(r\widehat{v} + r\widehat{w}) - \partial_1 F(r\widehat{v} + r\widehat{w'})\big)\big\|\Big)\end{aligned} \tag{3.3.16}$$

Next observe that due to (3.3.5) there exists a constant $0 < \delta_1 < \delta$, such that for any $v \in \mathcal{M}_1$, $w \in N_v\mathcal{M}_1$ with $\|w\| \le \delta_1$,

$$\big\|\nabla^2\Psi_0(v + w) - \nabla^2\Psi_0(v)\big\| \le \frac{1}{4\gamma} \tag{3.3.17}$$

in operator norm. Hence, there holds for any $w, w' \in N_v\mathcal{M}_1$ with $\|w\|, \|w'\| \le \delta_1$:

$$\begin{aligned}&\big\|\nabla\Psi_0(v + w) - \nabla\Psi_0(v + w') - \nabla^2\Psi_0(v)[w - w']\big\| \\ &\quad = \left\|\int_0^1 \big(\nabla^2\Psi_0\big(v + sw + (1 - s)w'\big) - \nabla^2\Psi_0(v)\big)[w - w']\, ds\right\| \\ &\quad \le \frac{1}{4\gamma}\|w - w'\|.\end{aligned} \tag{3.3.18}$$

In addition, by the uniform boundness of $|rv(t)|$ and the smoothness of F, there exists $0 < r_2 < r_1$, such that for any $|r| \le \frac{\rho}{R}$ and $|r| \le r_2$, $\|w\|, \|w'\| \le \delta$,

$$\begin{aligned}&\frac{4\pi|r|N}{N-1}\big\|(Id - \Delta)^{-1}\big(\partial_1 F(r\widehat{v} + r\widehat{w}) - \partial_1 F(r\widehat{v} + r\widehat{w'})\big)\big\| \\ &\qquad \le \frac{4\pi|r|N}{N-1}\big\|\partial_1 F(r\widehat{v} + r\widehat{w}) - \partial_1 F(r\widehat{v} + r\widehat{w'})\big\|_{L^2} \\ &\qquad \le \frac{1}{4\gamma}\|w - w'\|.\end{aligned} \tag{3.3.19}$$

Substituting (3.3.18) and (3.3.19) into (3.3.16) yields (3.3.15).

STEP 3. We shall verify that there is a constant $0 < r_3 < r_1$, such that $T(r, v, \cdot)$ maps $\big\{w \in N_v\mathcal{M}_1 : \|w\| \le \delta_1\big\}$ into itself provided $|r| \le \min\{r_3, \rho/R\}$.

Indeed,

$$\|T(r,v,w)\| \leq \gamma \cdot \left(\left\| \nabla\Psi_0(v+w) - \nabla^2\Psi_0(v)[w] \right\| + \frac{4\pi|r|N}{N-1} \left\| \partial_1 F(r\widehat{v} + r\widehat{w}) \right\|_{L^2} \right).$$

Similarly, $r\partial_1 F(r\widehat{v} + r\widehat{w})$ converges to 0 uniformly as $r \to 0$, so there exists $0 < r_3 < r_1$, such that for $|r| \leq \min\{r_3, \rho/R\}$,

$$\frac{4\pi|r|N}{N-1} \left\| \partial_1 F(r\widehat{v} + r\widehat{w}) \right\|_{L^2} \leq \frac{\delta_1}{4\gamma}, \quad \text{for all } v \in \mathcal{M}_1^R,\ \|w\| \leq \delta_1.$$

Moreover, (3.3.17) implies

$$\begin{aligned} \left\| \nabla\Psi_0(v+w) - \nabla^2\Psi_0(v)[w] \right\| &= \left\| \nabla\Psi_0(v+w) - \nabla\Psi_0(v) - \nabla^2\Psi_0(v)[w] \right\| \\ &= \left\| \int_0^1 \left(\nabla^2\Psi_0(v+sw) - \nabla^2\Psi_0(v) \right)[w]\, ds \right\| \\ &\leq \frac{1}{4\gamma}\|w\|. \end{aligned} \tag{3.3.20}$$

Consequently,

$$\|T(r,v,w)\| \leq \gamma \cdot \left[\frac{1}{4\gamma}\|w\| + \frac{\delta_1}{4\gamma} \right] \leq \frac{\delta_1}{2}$$

as long as $|r| \leq \frac{\rho}{R}$, $|r| \leq r_3$ and $\|w\| \leq \delta_1$.

STEP 4. Application of the contraction mapping principle

Taking $r_0 = r_0(\rho, \delta) := \min\{r_2, r_3\}$ the contraction mapping theorem applied to $T(r, v, \cdot) : \{w \in N_v\mathcal{M}_1 : \|w\| \leq \delta_1\}$ yields that for each $|r| \leq r_0$, $v \in \mathcal{M}_1^R$, there exists a unique $w = W_R(r,v) \in N_v\mathcal{M}_1$ with $\|w\| \leq \delta_1$ solving (3.3.13). Moreover, P_v is continuously differentiable in v (see 10.2 [56]), hence $W_R(r,\cdot)$ is also of class $\mathcal{C}^1$. In addition, since Ψ_r is autonomous and $T_v\mathcal{M}_1$ is S^1-equivariant also $W(r,\cdot)$ is S^1-equivariant: $W_R(r, \theta * v) = \theta * W_R(r,v)$. Observe that $W_R(r,v)$ is uniquely determined, thus $W_R(r,v) = W_{R'}(r,v)$ if (r,v) lies in the domains of both $\mathcal{M}_1^R$ and $\mathcal{M}_1^{R'}$. Hence we can simply write W instead of W_R and obtain a map

$$W : \mathcal{U} = \left\{ (r,v) \in \mathbb{R} \times \mathcal{M}_1 : v = \theta * u_a, |r| \leq r_0, |ra| \leq \rho \right\} \to H^1_{2\pi}(\mathbb{C})$$

as required.

□

Remark 3.3.7. The solution W can be obtained directly from the implicit function theorem in version of manifold, the proof of which is also achieved by the contraction mapping principle. The reason we do it in a complete way here is that the estimates in the following lemma demand for an explicit relation among the norm of W, r and some constants that appeared in the proof of Proposition 3.3.6 .

As a result of the construction of the solution map W, some properties about the behavior of $W(r,v)$ as $r \to 0$ can be proved, which are crucial for the proof of the main theorem:

Lemma 3.3.8. *The following holds uniformly on $\mathcal{U}$ as $r \to 0$:*

a) $\|W(r,v)\| = O(r)$

b) $\big\|P_v D_v W(r,v)\big\|_{\mathcal{L}(T_v\mathcal{M}_1)} = O(r)$.

Proof. The inequality (3.3.20) implies

$$\begin{aligned}\|W(r,v)\| &= \|T\big(r,v,W(r,v)\big)\| \\ &\le \frac{1}{4}\|W(r,v)\| + \frac{4\pi N}{N-1}\gamma|r| \cdot \big\|\partial_1 F(r\widehat{v} + r\widehat{W}(r,v))\big\|_{L^2}.\end{aligned} \tag{3.3.21}$$

Since $\|\partial_1 F(r\widehat{v}+r\widehat{W}(r,v))\|_{L^2}$ is uniformly bounded on $\mathcal{U}$, there exists a constant $M>0$, such that

$$4\pi\gamma\big\|\partial_1 F(r\widehat{v} + r\widehat{W}(r,v))\big\|_{L^2} \le M, \quad \text{for all } (r,v) \in \mathcal{U}.$$

This, substituted into (3.3.21), yields

$$\|W(r,v)\| \le \frac{4}{3}M|r|, \quad \text{for all } (r,v) \in \mathcal{U},$$

proving a).

Next, let $\{f_i(v)\}_{i=1}^3$ be an orthonormal basis of $T_v\mathcal{M}_1$ depending smoothly on $v \in \mathcal{M}_1$. In order to estimate $P_v D_v W(r,v)$ we differentiate the identity

$$P_v W(r,v) = \sum_{i=1}^{3} \langle W(r,v), f_i(v)\rangle f_i(v) = 0$$

with respect to v. This gives

$$\sum_{i=1}^{3} \big(\langle D_v W(r,v)\phi, f_i(v)\rangle f_i(v) + \langle W(r,v), f_i'(v)\phi\rangle f_i(v) + \langle W(r,v), f_i(v)\rangle f_i'(v)\phi\big) = 0$$

for any $\phi \in T_v\mathcal{M}_1$, and therefore,

$$P_v D_v W(r,v)\phi = -\sum_{i=1}^{3} \big(\langle W(r,v), f_i'(v)\phi\rangle f_i(v) + \langle W(r,v), f_i(v)\rangle f_i'(v)\phi\big).$$

The invariance of the tangent spaces along $\mathcal{M}_1$ under translations and the equivariance with respect to the S^1-action imply that $f_i(v)$ and $f_i'(v)$ are uniformly bounded for $v \in \mathcal{M}_1$. Then together with part a), we obtain

$$\big\|P_v D_v W(r,v)\big\|_{L(T_v\mathcal{M}_1)} = O(r) \quad \text{as } r \to 0 \text{ uniformly on } \mathcal{U}.$$

□

Following Proposition 3.3.6, it remains to solve

$$P_v\Big(\nabla\Psi_r\big(v+W(r,v)\big)\Big)=0$$

for $(r,v)\in\mathcal{U}$. This can be reformulated as a finite-dimensional variational problem using the function

$$\psi:\widetilde{\mathcal{U}}\to\mathbb{R},\quad \psi(r,a)=\psi_r(a):=\Psi_r\big(u_a+W(r,u_a)\big),$$

where

$$\widetilde{\mathcal{U}}:=\{(r,a)\in\mathbb{R}\times\mathbb{C}:\ |r|\leqslant r_0,|ra|\leqslant\rho\}.$$

Lemma 3.3.9. *There exists $\widetilde{r_0}>0$, such that if a is a critical point of ψ_r for some $|r|\le\widetilde{r_0}$, then $\nabla\Psi_r\big(u_a+W(r,u_a)\big)=0$.*

Proof. According to Lemma 3.3.8 b), there exists $\widetilde{r_0}>0$ such that

$$\big\|P_vD_vW(r,v)\big\|\le\frac{1}{2}\quad\text{for all }(r,v)\in\mathcal{U}\text{ with }|r|\le\widetilde{r_0}.\tag{3.3.22}$$

Suppose $(r,a)\in\widetilde{\mathcal{U}}$, $|r|\le\widetilde{r_0}$ and $\nabla\psi_r(a)=0$. Then

$$\big\langle\nabla\Psi_r\big(u_a+W(r,u_a)\big),\,a'+D_vW(r,u_a)a'\big\rangle=0\tag{3.3.23}$$

for any $a'\in\mathbb{C}\subset T_{u_a}\mathcal{M}_1$.

Since $\Psi_r\big(\theta*u_a+W(r,\theta*u_a)\big)$ is independent of $\theta\in S^1$, differentiating it at $\theta=0$ gives

$$\big\langle\nabla\Psi_r\big(u_a+W(r,u_a)\big),\,\big(Id+D_vW(r,u_a)\big)iu_0\big\rangle=0\,.\tag{3.3.24}$$

Combining (3.3.23) and (3.3.24) we obtain

$$\big\langle\nabla\Psi_r\big(u_a+W(r,u_a)\big),\big(Id+D_vW(r,u_a)\big)\phi\big\rangle=0$$

for any $\phi\in T_{u_a}\mathcal{M}_1$. Moreover, as a consequence of (3.3.22) the map $Id+P_{u_a}D_vW(r,u_a)$ is invertible on $T_{u_a}\mathcal{M}_1$, hence

$$\big\langle\nabla\Psi_r\big(u_a+W(r,u_a)\big),\phi\big\rangle=0,\quad\text{for all }\phi\in T_{u_a}\mathcal{M}_1.$$

This implies

$$P_{u_a}\nabla\Psi_r\big(u_a+W(r,u_a)\big)=0,$$

hence $u_a+W(r,u_a)$ is a critical point of Ψ_r. □

Now we make a first order Taylor expansion for ψ_r .

Lemma 3.3.10. *There holds*

$$\psi_r(a)=c_0+\frac{4\pi^2N^2}{N-1}h(ra)+\varphi_r(a)$$

with $\nabla\varphi_r(a)=o(r)$ as $r\to0$ uniformly on $\widetilde{\mathcal{U}}$.

Proof. We compute

$$\begin{aligned}\psi_r(a) &= \Psi_r\big(u_a + W(r,u_a)\big)\\ &= \Psi_r\big(u_a + W(r,u_a)\big) - \Psi_r(u_a) + \Psi_0(u_a) - \frac{8\pi^2}{N-1}F(0) + \frac{4\pi}{N-1}\int_0^{2\pi} F(r\widehat{u}_a)\,dt\end{aligned}$$

and

$$\begin{aligned}\int_0^{2\pi} F(r\widehat{u}_a)\,dt = \pi N^2 h(ra)&\\ +\frac{1}{2}\int_0^{2\pi}\sum_{\substack{j,k=1\\ j\neq k}}^{N}&\Big(g(ra+re^{i(t+\frac{2\pi(j-1)}{N})}, ra+re^{i(t+\frac{2\pi(k-1)}{N})}) - g(ra,ra)\Big)\,dt\\ +\frac{1}{2}\int_0^{2\pi}\sum_{k=1}^{N}&\Big(h(ra+re^{i(t+\frac{2\pi(k-1)}{N})}) - h(ra)\Big)\,dt\,.\end{aligned}$$

Setting $c_0 := \Psi_0(u_a) - \frac{8\pi^2}{N-1}F(0)$ and

$$\begin{aligned}\varphi_r(a) := \frac{2\pi}{N-1}\int_0^{2\pi}\sum_{\substack{j,k=1\\ j\neq k}}^{N}&\Big(g(ra+re^{i(t+\frac{2\pi(j-1)}{N})}, ra+re^{i(t+\frac{2\pi(k-1)}{N})}) - g(ra,ra)\Big)\,dt\\ +\frac{2\pi}{N-1}\int_0^{2\pi}\sum_{k=1}^{N}&\Big(h(ra+re^{i(t+\frac{2\pi(k-1)}{N})}) - h(ra)\Big)\,dt\\ +\Psi_r\big(u_a + W(r,u_a)\big) - \Psi_r(u_a),&\end{aligned}$$

we have $\psi_r(a) = c_0 + \frac{4\pi^2N^2}{N-1}h(ra) + \varphi_r(a)$. Now

$$\begin{aligned}\big|\nabla\varphi_r(a)\big| &\le \frac{2\pi|r|}{N-1}\int_0^{2\pi}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\sum_{m=1,2}\Big|\partial_m g(ra+re^{i(t+\frac{2\pi(j-1)}{N})}, ra+re^{i(t+\frac{2\pi(k-1)}{N})}) - \partial_m g(ra,ra)\Big|\,dt\\ &\quad+\frac{2\pi|r|}{N-1}\int_0^{2\pi}\sum_{k=1}^{N}\Big|h'(ra+re^{i(t+\frac{2\pi(k-1)}{N})}) - h'(ra)\Big|\,dt\\ &\quad+\big|\nabla_a\Psi_r\big(u_a+W(r,u_a)\big) - \nabla_a\Psi_r(u_a)\big|\\ &= o(r) + \big|\nabla_a\Psi_r\big(u_a+W(r,u_a)\big) - \nabla_a\Psi_r(u_a)\big|\end{aligned}$$

because g is of class $\mathcal{C}^1$ and $|ra|$ is uniformly bounded on $\mathcal{U}$.

Moreover, since $\nabla\Psi_r\big(u_a + W(r,u_a)\big) \in T_{u_a}\mathcal{M}_1$ and $D_vW(r,u_a)a' \in N_{u_a}\mathcal{M}_1$ for any

$a' \in \mathbb{C}$, we deduce for $r \to 0$:

$$
\begin{aligned}
&\nabla_a \Psi_r\big(u_a + W(r,u_a)\big)[a'] - \nabla_a \Psi_r(u_a)[a'] \\
&\quad = \Psi_r'\big(u_a + W(r,u_a)\big)\big[a' + D_v W(r,u_a)a'\big] - \Psi_r'(u_a)[a'] \\
&\quad = \Psi_r'\big(u_a + W(r,u_a)\big)[a'] - \Psi_r'(u_a)[a'] \\
&\quad = \Psi_0'\big(u_a + W(r,u_a)\big)[a'] - \Psi_0'(u_a)[a'] \\
&\qquad\quad + \frac{4\pi r N}{N-1}\int_0^{2\pi} \big(\partial_1 F(r\widehat{u}_a + r\widehat{W}(r,u_a)) - \partial_1 F(r\widehat{u}_a)\big)[a']\,dt \\
&\quad = \Psi_0'\big(u_0 + W(r,u_a)\big)[a'] - \Psi_0'(u_0)[a'] + o(r)\cdot|a'| \\
&\quad = \Psi_0''(u_0)\big[a', W(r,u_a)\big] + o(\|W(r,u_a)\|)\cdot|a'| + o(r)\cdot|a'| \\
&\quad = o(r)\cdot|a'|,
\end{aligned}
$$

uniformly on $\widetilde{\mathcal{U}}$. Here we applied $a' \in \operatorname{Ker}\Psi_0''(u_0)$ and Lemma 3.3.8 a). Summarizing we have proved that $\nabla\varphi_r(a) = o(r)$.

□

3.4 Proof of Theorem 3.1.4

This section is devoted to obtaining critical points of the reduced function. We simply apply some well-known facts from the Conley index theory as well as Morse index theory as presented in [26, 29, 40]. The index theory has a long history and can be found in different versions in the literature. Here we just adapt some theorems related to our problem for the information of the reader.

We consider a $\mathcal{C}^2$ Hilbert manifold X and $f \in \mathcal{C}^1(X,\mathbb{R})$. K_f, the critical set of f, and the critical groups $C_*(f,p)$ are introduced as in Section 3.1.

Definition 3.4.1. A continuous map $\eta : \mathbb{R}\times X \to X$ is called a *flow* on X if the following properties are satisfied:

(1) $\eta(0,x) = x$; (2) $\eta(s,\eta(t,x)) = \eta(s+t,x)$ for all $s,t\in\mathbb{R}$.

Given a locally Lipschitz continuous vector field V on X, we can associate a local flow η on X *with respect to* V as the maximal solution to the Cauchy problem

$$
\begin{aligned}
\dot\eta(x,t) &= V(\eta(x,t)),\\
\eta(x,0) &= x.
\end{aligned}
\tag{3.4.1}
$$

Definition 3.4.2. Let $f \in \mathcal{C}^1(X,\mathbb{R})$. A *pseudo-gradient vector field* for f (for short *p.g.v.f.*) is defined to be a locally Lipschitz continuous section V of the tangent bundle $T(X)$ satisfying, for all $x \in X$,

$$\langle V(x), df(x)\rangle \geqslant \|df(x)\|^2,$$

$$\|V(x)\| \leqslant \alpha \|df(x)\|$$

for some constant $\alpha > 0$.

Remark 3.4.3. 1. Any function $f \in \mathcal{C}^1(X, \mathbb{R})$ admits a p.g.v.f..

2. Since V is required to be locally Lipschitz continuous, the gradient ∇f of a function $f \in \mathcal{C}^1(X, \mathbb{R})$ is not necessarily a p.g.v.f. for f. But if $f \in \mathcal{C}^2(X, \mathbb{R})$, then ∇f is a p.g.v.f. for f.

Gromoll and Meyer [40] pointed out that the local homology of an isolated critical point can also be computed as the homology of a suitable closed neighborhood of the critical point relative to a part of its boundary. Using definition III.1 of Chang and Ghoussoub [27], the Gromoll-Meyer pair for a critical subset is defined as follows.

Definition 3.4.4. Let $f \in \mathcal{C}^1(X, \mathbb{R})$ and S be a subset of K_f. A pair of topological subsets (W, W^-) is said to be a *Gromoll-Meyer pair* for S associated with a p.g.v.f. V if, for the flow η associated with V via (3.4.1), the following conditions hold:

- W is a closed neighborhood of S with $W \cap K_f = S$ and $W \cap f_a = \emptyset$ for some a.
- W has the *Mean Value Property*, that is, if $t_0 < t_1$ are such that $\eta(x, t_0) \in W$ and $\eta(x, t_1) \in W$ for some $x \in X$, then $\eta(x, [t_0, t_1]) \subset W$.
- W^- is an *exit set* for W, i.e.,

$$W^- = \{x \in W \mid \max \{t \in \mathbb{R} \mid \eta(x, t) \in W\} = 0\}.$$

- W^- is a piecewise submanifold, and transversal to the flow η, that is, $\frac{d\eta}{dt}(0, x) \notin T_x W^-$ for all $x \in W^-$.

In [27] proposition 3.2, the authors gave the existence of Gromoll-Meyer pair of an isolated critical point.

Proposition 3.4.5. [27] Suppose $f \in \mathcal{C}^1(X, \mathbb{R})$ and satisfies the (PS) condition. p is an isolated critical point of f. Then for any open neighborhood $\mathcal{U}$ of p, there exists a Gromoll-Meyer pair (W, W^-) for p with $W \subset \mathcal{U}$, and in particular if f is $\mathcal{C}^2$, there exists one pair associated with the negative gradient vector field $-\nabla f$.

They also established a relationship between the critical groups of a functional and the homology of the Gromoll-Meyer pair at an isolated critical point.

Proposition 3.4.6. [27] Assume p is an isolated critical point of $f \in \mathcal{C}^1(X, \mathbb{R})$. Let (W, W^-) be a Gromoll-Meyer pair of p associated with a p.g.v.f. V. Then we have

$$C_*(f, p) = H_*(W, W^-; F).$$

The homology groups do not depend on the special choice of the Gromoll-Meyer pair (W, W^-) nor on the choice of the p.g.v.f. V for f.

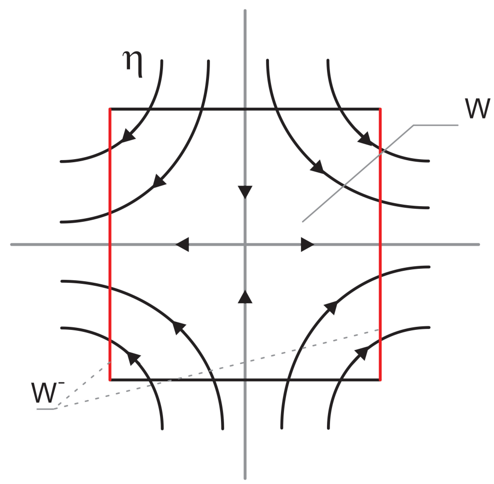


Figure 3.1: An example of Gromoll-Mayer pair.

Now we turn to the Conley index.

Definition 3.4.7. N is a subset of X. The *maximal η-invariant set* contained in N is denoted as

$$\text{Inv}(N, \eta) = \{x \in N : \eta(x,t) \in N \text{ for all } t \in I_x\},$$

where I_x is the maximal existence interval of the initial value problem. N is an *isolating neighborhood* if N is compact in X and

$$x \in \partial N \Rightarrow \eta(x, I_x) \nsubseteq N,$$

in other words,

$$\text{Inv}(N, \eta) \subset \text{int}(N).$$

A compact set $S \subset X$ is called an *isolated invariant set* if there exists an isolating neighborhood N such that $S = \text{Inv}(N, \eta)$. The *exit set* of N, denoted as N^-, is

$$N^- = \{x \in N \,|\, \exists \varepsilon > 0 : \eta(x,t) \notin N \text{ for } 0 < t < \varepsilon\}.$$

An isolating neighborhood N is an isolating block if N^- is closed. We say that $L \subset N$ is *positively invariant relative to* N, if for any $x \in L$, $\eta([0,t], x) \subset N$ implies $\eta([0,t], x) \subset L$.

Definition 3.4.8. $L \subset N \subset X$ and N, L are compact. (N, L) is called an *index pair* for $S = \text{Inv}(\text{Cl}(N \backslash L))$, if:

- $N \backslash L$ is a neighborhood of S, that is, $\text{Inv}(\text{Cl}(N \backslash L)) \subset \text{int}(N \backslash L)$;
- L is positively invariant relative to N;
- Every orbit which exits N goes through L first:

$$x \in N,\ \eta(t,x) \notin N \text{ for some } t > 0 \ \Rightarrow\ \eta(s,x) \in L \text{ for some } s \in [0,t].$$

Definition 3.4.9. The *homological Conley index* is defined by

$$CH_*(S) = H_*(N, L).$$

As noted by Conley-Zehnder and Salamon, the homological Conley index is a topological invariant for isolating neighborhoods, that is, if (N_1, L_1) and (N_2, L_2) are two index pairs for S, then $H_*(N_1, L_1) \cong H_*(N_2, L_2)$.

It satisfies *Wazewski property*:

Proposition 3.4.10. Let N be an isolating neighborhood and assume that $CH_*(S) \neq 0$. Then $\text{Inv}(N) \neq \emptyset$.

Definition 3.4.11. An equation is called *gradient-like* if there is some continuous real valued function which is strictly decreasing on nonconstant solutions. The associated function is named *Lyapunov function*.

Remark 3.4.12. 1. The simplest illustration is provided by gradient equations, namely, a system of the form $\dot{x} = \nabla f(x)$ with Lyapunov function $-f$.
2. Gradient-like equations and isolated invariant sets are related by the fact that an isolated rest point of a gradient-like equation is also an isolated invariant set. In general, a rest point may be isolated as a rest point but not as an invariant set. A "center" of an equation in the plane is an example: every neighborhood of the center contains a nonconstant periodic solution.

Then we have the following

Corollary 3.4.13. Suppose the local flow η is generated by a gradient-like vector field V and N is an isolating neighborhood relative to η. If $\text{Inv}(N, \eta) \neq \emptyset$, then there exists an $a \in N$ with $V(a) = 0$.

Proof. If V is gradient-like with respect to f and $\text{Inv}(N, \eta) \neq \emptyset$, then there is an $x \in N$ such that $\eta(x, t) \in N$ for all $t \in I_x$. By the compactness of N, $I_x = (-\infty, +\infty)$ and there exists a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} \eta(x, t_n) = a \in N$. Thus since f is continuous and nonincreasing we deduce that $f(a) = \inf\{f(\eta(x, t_n)) : n \in \mathbb{N}\}$. It follows that $f \circ \eta(a, \cdot)$ is constant so $\eta(a, \cdot)$ is constant. Hence, $\dot{\eta}(a, t) \equiv 0$ and so $0 = \dot{\eta}(a, 0) = V(\eta(a, 0)) = V(a)$. The corollary is proved. □

Remark 3.4.14. The significance of the above corollary lies in the fact that, in applications to variational problems, one is interested in finding critical points of a given $\mathcal{C}^1$-function $f : N \subset U \to \mathbb{R}$, i.e. solutions a of the equation $f'(a) = 0$. In this case $V = f'$ is gradient-like with respect to Lyapunov function $-f$. Thus, if $\text{Inv}(N) \neq \emptyset$ then the corollary implies the existence of a critical point of f contained in N.

End of proof of Theorem 3.1.4

Suppose 0 is an isolated stable critical point of h with $h(0) = c$. The Palais-Smale condition is satisfied automatically for h since $h(z) \to \infty$ as $z \to \partial\Omega$ as a smooth function defined on a finite dimensional space. Then for any fixed $0 < \varepsilon < \rho$, we can choose a Gromoll-Meyer pair (B, B^-) for 0 of h such that $B \subset B_\varepsilon(0)$ and

$$H_*(B, B^-) \cong H^*(h^c, h^c \setminus \{0\}) \neq 0.$$

In particular, $\partial B \subset M^1 \cup \cdots \cup M^k$ is contained in a finite union of submanifolds $M^j = (g^j)^{-1}(0)$, where $g^j \in \mathcal{C}^1(\mathbb{C}, \mathbb{R})$ with 0 being a regular value, and $\nabla g^j(a)$ being the exterior normal to B at $a \in \partial B$. By definition there holds

$$\big\langle \nabla h(a), \nabla g^j(a) \big\rangle_{\mathbb{R}^2} \neq 0, \quad \text{for all } a \in \partial B \cap M^j,\ j = 1, \ldots, k, \tag{3.4.2}$$

and

$$\big\langle \nabla h(a), \nabla g^j(a) \big\rangle_{\mathbb{R}^2} < 0, \quad \text{if, and only if, } a \in B^- \cap M^j,\ j = 1, \ldots, k. \tag{3.4.3}$$

Now we scale these sets and functions as

$$B_r := \frac{1}{r} B, \quad B_r^- := \frac{1}{r} B^-, \quad M_r^j := \frac{1}{r} M^j, \quad g_r^j(a) := g^j(ra),$$

so that $\partial B_r \subset M_r^1 \cup \cdots \cup M_r^k = (g_r^1)^{-1}(0) \cup \cdots \cup (g_r^k)^{-1}(0)$.

We consider only $|r| \leq \min\{r_0, \widetilde{r_0}\}$ so that the lemmas from the former sections make sense. Lemma 3.3.10 implies for $a \in \partial B_r \cap M_r^j$, i. e. $ra \in \partial B \cap M^j$:

$$\begin{aligned}
\big\langle \nabla \psi_r(a), \nabla g_r^j(a) \big\rangle_{\mathbb{R}^2} &= \left\langle \frac{4\pi^2 r N^2}{N-1} \nabla h(ra) + \nabla \varphi_r(a), \nabla g_r^j(a) \right\rangle_{\mathbb{R}^2} \\
&= \frac{4\pi^2 r^2 N^2}{N-1} \big\langle \nabla h(ra), \nabla g^j(ra) \big\rangle_{\mathbb{R}^2} + r \big\langle \nabla \varphi_r(a), \nabla g^j(ra) \big\rangle_{\mathbb{R}^2} \\
&= \frac{4\pi^2 r^2 N^2}{N-1} \big\langle \nabla h(ra), \nabla g^j(ra) \big\rangle_{\mathbb{R}^2} + o(r^2)
\end{aligned}$$

as $r \to 0$. Using the compactness of ∂B and (3.4.2) we see that

$$\big\langle \nabla \psi_r(a), \nabla g_r^j(a) \big\rangle \neq 0 \quad \text{for all } a \in \partial B_r \cap M_r^j,\ j = 1, \ldots, k.$$

for $|r| > 0$ small enough. This implies for $|r| > 0$ sufficiently small, that B_r is an isolating neighborhood for the negative gradient flow of ψ_r, and as a consequence of (3.4.3) the exit set is B_r^-. Since B and B_r, B^- and B_r^- are both homeomorphic,

$$H_*(B_r, B_r^-) \cong H_*(B, B^-) \neq 0.$$

So Proposition 3.4.10 and Corollary 3.4.13 imply that there exists a critical point b_r of ψ_r in B_r. Then $u_{b_r} + W(r, u_{b_r})$ is a critical point of Ψ_r. Rescaling back, we obtain a T_r-periodic solution

$$z^r(t) = r\big(\widehat{u}_{b_r} + \widehat{W}(r, u_{b_r})\big)\left(\frac{2\pi}{T_r} t\right) = r \begin{pmatrix} b_r + e^{i\frac{2\pi}{T_r} t} \\ b_r + e^{i(\frac{2\pi}{T_r} t + \frac{2\pi}{N})} \\ \vdots \\ b_r + e^{i(\frac{2\pi}{T_r} t + \frac{2\pi(N-1)}{N})} \end{pmatrix} + r\widehat{W}(r, u_{b_r})\left(\frac{2\pi}{T_r} t\right)$$

of (HS), proving Theorem 3.1.4 with $a_r = r \cdot b_r \in B \subset B_\varepsilon(0)$. □

Remark 3.4.15. For arbitrary a_0 in Theorem 3.1.4, we modify the settings as follows:

$$\begin{aligned} H_r(u) &:= \frac{T_r}{2\pi r^2}\left(H_\Omega(ru+\widehat{a_0}) + \frac{1}{4\pi}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\log r\right) \\ &= -\frac{1}{N-1}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\log|u_j-u_k| - \frac{4\pi}{N-1}F(ru+\widehat{a_0}), \end{aligned}$$

where $\widehat{a_0} = (a_0,\cdots,a_0)^{tr} \in \mathbb{C}^N$. Precisely, if a T_r-periodic function $u(t)$ solves $\dot{u}_k = -i\nabla_{u_k}H_r(u)$ for $k=1,\cdots,N$, then the 2π-periodic $z(t) = ru(2\pi t/T_r)+\widehat{a_0}$ solves $\dot{z}_k = -i\nabla_{z_k}H_\Omega(z)$. $H(r,u) := H_r(u)$ is defined on

$$\mathcal{O} := \left\{(r,u)\in\mathbb{R}\times\mathbb{C}^N : u_j\neq u_k \text{ for } j\neq k,\ ru_k+a_0\in\Omega \text{ for all } k\right\},$$

while Φ_r, Λ, $\mathcal{M}$ and the group action remain unchanged. This means that we just make a translation to the system with the "center" from a_0 back to 0.

Later we fix a neighborhood $B_{2\rho}(a_0)$ of a_0 in Ω. Still, the reduced functional $\Psi_r(u_1)$ is well-defined provided $u_1\in\mathcal{U}_\delta(\mathcal{M}_1)$ and $|ru_1(t)|\leqslant 2\rho$ for all t. Lemma 3.3.6 holds with a constant $r_0 = r_0(\rho,\delta,a_0)$ and the final Taylor expansion formula for ψ_r is

$$\psi_r(a) = c_0 + \frac{4\pi^2N^2}{N-1}h(ra+a_0)+\varphi_r(a)$$

with $\nabla\varphi_r(a) = o(r)$ as $r\to 0$ uniformly on $\widetilde{\mathcal{U}}$. Hence, analogously we can choose a Gromoll-Meyer pair (B,B^-) for a_0 of h such that $B\subset B_\varepsilon(a_0)\subset B_\rho(a_0)$ and $H_*(B,B^-)\neq 0$. M^j and g^j play a role as before. Naturally now we set

$$B_r := \frac{1}{r}(B-a_0),\quad B_r^- := \frac{1}{r}(B^--a_0),\quad M_r^j := \frac{1}{r}(M^j-a_0),\quad g_r^j(a) := g^j(ra+a_0),$$

then for $|r|>0$ small enough, B_r is an isolating neighborhood for the negative gradient flow of ψ_r with exit set B_r^-. Hence

$$H_*(B_r,B_r^-)\cong H_*(B,B^-)\neq 0$$

yields the existence of a critical point b_r of ψ_r in B_r. Thus Theorem 3.1.4 holds when setting $a_r = rb_r+a_0\in B$.

3.5 Generalizations

Parallel results for other forms of Hamiltonian systems in $\mathbb{C}^2$ with singularity at $z_1 = z_2$ can be obtained in the same manner.

The system governing the motion of two vortices (z_1, Γ_1) and (z_2, Γ_2) in a domain $\Omega \subset \mathbb{C}$ is

$$\begin{cases} \Gamma_1 \dot{z_1} = -i\nabla_{z_1} H(z); \\ \Gamma_2 \dot{z_2} = -i\nabla_{z_2} H(z), \end{cases} \tag{3.5.1}$$

with

$$\begin{aligned} H(z) &= -\frac{\Gamma_1\Gamma_2}{2\pi}\log|z_1 - z_2| - \Gamma_1\Gamma_2 g(z_1, z_2) - \frac{1}{2}\Gamma_1^2 h(z_1) - \frac{1}{2}\Gamma_2^2 h(z_2) \\ &=: -\frac{\Gamma_1\Gamma_2}{2\pi}\log|z_1 - z_2| - F(z) \end{aligned}$$

defined on $\mathcal{F}_2\Omega$.

Theorem 3.5.1. *Assume $\Gamma_1 + \Gamma_2 \neq 0$. If $a_0 \in \Omega$ is a stable critical point of h, then there exists $r_0 > 0$, such that for each $0 < r \leq r_0$,* (3.5.1) *has a periodic solution $z^r = (z_1^r, z_2^r)$ with minimal period $|T_r|$ for $T_r = 4\pi^2 r^2(\Gamma_1 + \Gamma_2)/\Gamma_2^2$ such that in the limit $r \to 0$ the vortices z_k^r move on circles in the following sense. There exists $a_r \in \Omega$ with $a_r \to a_0$ such that the rescaled functions*

$$u_k^r(t) := \frac{1}{r}\big(z_k^r(T_r t/2\pi) - a_r\big)$$

satisfy

$$u_1^r(t) \to e^{it}, \quad u_2^r(t) \to -\frac{\Gamma_1}{\Gamma_2}e^{it}.$$

The convergence holds in $H^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{C})$.

Proof. Suppose $a_0 = 0$. After rescaling, the problem becomes looking for critical points of the functional

$$\Phi_r(u) = \frac{1}{2}\int_0^{2\pi} \Gamma_1\langle i\dot{u}_1, u_1\rangle_{\mathbb{R}^2} + \Gamma_2\langle i\dot{u}_2, u_2\rangle_{\mathbb{R}^2}\, dt - \int_0^{2\pi} H_r(u)\, dt$$

with

$$\begin{aligned} H_r(u) &= \frac{T_r}{2\pi r^2}\left(H(ru) + \frac{\Gamma_1\Gamma_2}{2\pi}\log|r|\right) \\ &= -\frac{\Gamma_1(\Gamma_1+\Gamma_2)}{\Gamma_2}\log|u_1 - u_2| - \frac{2\pi(\Gamma_1+\Gamma_2)}{\Gamma_2^2}F(ru) \end{aligned}$$

$\Phi(r, u) := \Phi_r(u)$ is defined on an open subset of $H^1_{2\pi}(\mathbb{C}^2)$:

$$\Lambda := \{(r, u) \in \mathbb{R} \times H^1_{2\pi}(\mathbb{C}^2) : \ u_1(t) \neq u_2(t) \text{ and } ru_1(t), ru_2(t) \in \Omega \text{ for all } t\}.$$

Then

$$\nabla\Phi_r(u) = (Id-\Delta)^{-1}\Bigg[\begin{pmatrix} i\Gamma_1\dot{u}_1 \\ i\Gamma_2\dot{u}_2\end{pmatrix} + \frac{\Gamma_1(\Gamma_1+\Gamma_2)}{\Gamma_2|u_1-u_2|^2}\begin{pmatrix} u_1-u_2 \\ u_2-u_1\end{pmatrix} + \frac{2\pi(\Gamma_1+\Gamma_2)r}{\Gamma_2^2}F'(ru)\Bigg].$$

In particular for $r=0$,

$$\begin{aligned}\nabla\Phi_0(u) &= (Id-\Delta)^{-1}\Bigg[\begin{pmatrix} i\Gamma_1\dot{u}_1 \\ i\Gamma_2\dot{u}_2\end{pmatrix} + \frac{\Gamma_1(\Gamma_1+\Gamma_2)}{\Gamma_2|u_1-u_2|^2}\begin{pmatrix} u_1-u_2 \\ u_2-u_1\end{pmatrix}\Bigg],\\ \nabla^2\Phi_0(u)[v] &= (Id-\Delta)^{-1}\Bigg[\begin{pmatrix} i\Gamma_1\dot{v}_1 \\ i\Gamma_2\dot{v}_2\end{pmatrix} + \frac{\Gamma_1(\Gamma_1+\Gamma_2)}{\Gamma_2|u_1-u_2|^2}\begin{pmatrix} v_1-v_2 \\ v_2-v_1\end{pmatrix}\\ &\qquad - \frac{2\Gamma_1(\Gamma_1+\Gamma_2)}{\Gamma_2}\frac{\langle u_1-u_2, v_1-v_2\rangle_{\mathbb{R}^2}}{|u_1-u_2|^4}\begin{pmatrix} u_1-u_2 \\ u_2-u_1\end{pmatrix}\Bigg],\end{aligned}$$

and thus $\nabla\Phi_0(u+(a,a)^{tr}) = \nabla\Phi_0(u)$ and $\nabla^2\Phi_0(u+(a,a)^{tr}) = \nabla^2\Phi_0(u)$ for all $a\in\mathbb{C}$. Standardly as in Lemma 3.3.4, one can prove that for strengths Γ_1,Γ_2 with $\Gamma_1+\Gamma_2\neq 0$,

$$\mathcal{M} := \left\{\theta * \begin{pmatrix} e^{it} \\ -\frac{\Gamma_1}{\Gamma_2}e^{it}\end{pmatrix} + \begin{pmatrix} a \\ a\end{pmatrix} : \theta\in S^1, a\in\mathbb{C}\right\}$$

is a nondegenerate critical manifold of Φ_0. To see this, set $u_0 = (e^{it}, -\frac{\Gamma_1}{\Gamma_2}e^{it})^{tr}$.

$$\Phi_0''(u_0)[v] = (Id-\Delta)^{-1}\Bigg[\begin{pmatrix} i\Gamma_1\dot{v}_1 \\ i\Gamma_2\dot{v}_2\end{pmatrix} + \frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\begin{pmatrix} v_1-v_2 \\ v_2-v_1\end{pmatrix} - \frac{2\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\langle e^{it}, v_1-v_2\rangle_{\mathbb{R}^2}\begin{pmatrix} e^{it} \\ -e^{it}\end{pmatrix}\Bigg].$$

So $v\in\ker\Phi_0''(u_0)$ implies

$$\begin{cases} i\Gamma_1\dot{v}_1 + \frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}(v_1-v_2) - \frac{2\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\langle e^{it}, v_1-v_2\rangle_{\mathbb{R}^2}e^{it} = 0;\\ i\Gamma_2\dot{v}_2 + \frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}(v_2-v_1) + \frac{2\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}\langle e^{it}, v_1-v_2\rangle_{\mathbb{R}^2}e^{it} = 0.\end{cases}$$

Summing them up we get $\Gamma_1\dot{v}_1+\Gamma_2\dot{v}_2=0$. Write v_1, $v_2\in H^1_{2\pi}(\mathbb{C})$ in their Fourier expansions as $v_1=\sum_{k\in\mathbb{Z}}\alpha_k e^{ikt}$, $v_2=\sum_{k\in\mathbb{Z}}\beta_k e^{ikt}$ with α_k, $\beta_k\in\mathbb{C}$. Then it follows

$$\beta_k = -\frac{\Gamma_1}{\Gamma_2}\alpha_k, \quad ,\forall\, k\in\mathbb{Z}\backslash\{0\}.$$

Substituting these into one of the equations we derive

$$\frac{\Gamma_2}{\Gamma_1+\Gamma_2}(\bar{\alpha}_0-\bar{\beta}_0)e^{2it} + \sum_{k\in\mathbb{Z}}k\alpha_k e^{ikt} + \sum_{k\in\mathbb{Z}\backslash\{0\}}\bar{\alpha}_k e^{i(2-k)t} = 0. \tag{3.5.2}$$

Comparison of the coefficients gives

$$k\alpha_k + \bar{\alpha}_{2-k} = 0, \quad \text{for any } k\in\mathbb{Z}\backslash\{0,2\}.$$

Replace k by $2-k$, then there holds $\alpha_k + (2-k)\bar{\alpha}_{2-k} = 0$. Combining these we obtain $(k-1)^2\alpha_k = 0$ for any $k \neq 0, 2$, thus

$$\alpha_k = \beta_k = 0, \quad \text{for all } k \neq 0, 1, 2.$$

Considering the constant term in (3.5.2), we get $\alpha_2 = \beta_2 = 0$. Then from the e^{2it} term, we obtain $\alpha_0 = \beta_0$. For $k = 1$, there yields $\alpha_1 + \bar{\alpha}_1 = 0$, i.e., $\alpha_1 \in i\mathbb{R}$. In summary, $v \in \ker \Phi_0''(u_0)$ has the form

$$v = \begin{pmatrix} a \\ a \end{pmatrix} + ic \begin{pmatrix} e^{it} \\ -\frac{\Gamma_1}{\Gamma_2} e^{it} \end{pmatrix}$$

with some $a \in \mathbb{C}$ and $c \in \mathbb{R}$, which is an element in $T_{u_0}\mathcal{M}$.

We fix $\delta > 0$ such that the δ-neighborhood $\mathcal{U}_\delta(\mathcal{M})$ of $\mathcal{M}$ in $H^1_{2\pi}(\mathbb{C}^2)$ is contained in the domain of Φ_0, $\Lambda_0 := \{u \in H^1_{2\pi}(\mathbb{C}^2) : u_1(t) \neq u_2(t), \forall t\}$, and also $\rho > 0$ with $B_{2\rho}(0) \subset \Omega$. Then $\Phi_r(u)$ is well defined as long as $u \in \mathcal{U}_\delta(\mathcal{M})$ and $|ru_1(t)|$, $|ru_2(t)| \leqslant 2\rho$ for all t.

Given $v \in \mathcal{M}$, in order to solve the equation $(Id - P_v)(\nabla\Phi_r(v+w)) = 0$ for $w \in N_v\mathcal{M}$, we are led to finding a fixed point of the function

$$\begin{aligned} T(r, v, w) = -(\mathcal{L}_v|_{N_v\mathcal{M}})^{-1} \circ (Id - P_v) \Big[&\nabla\Psi_0(v+w) - \nabla^2\Psi_0(v)[w] \\ &+ \frac{2\pi(\Gamma_1+\Gamma_2)r}{\Gamma_2^2}(Id - \Delta)^{-1}\big(F'(rv + rw)\big)\Big]. \end{aligned}$$

Denote $u_a = u_0 + (a, a)^{tr}$. Again making use of the invariance of $\nabla\Phi_0$ and $\nabla^2\Phi_0$ under translations and the equivariance of them under S^1-action, as in Lemma 3.3.6 we obtain a constant $r_0 = r_0(\rho, \delta, \Gamma_1, \Gamma_2) > 0$ such that for any $|r| \leqslant r_0$ and arbitrary $v = \theta * u_a \in \mathcal{M}$ with $|ra| \leqslant \rho$, $T(r, v, \cdot)$ is a contraction on a small neighborhood of 0 in $N_v\mathcal{M}$ which gives rise to a unique fixed point $W(r, v)$. So this yields an S^1-equivariant map

$$W : \mathcal{U} := \big\{(r, v) \in \mathbb{R} \times \mathcal{M} : |r| \leq r_0, v = \theta * u_a, a \in \mathbb{C}, |ra| \leq \rho\big\} \to H^1_{2\pi}(\mathbb{C}^2)$$

such that $W(r, v) \in N_v\mathcal{M}$, satisfying $\|W(r, v)\| \leqslant \delta$ and $\nabla\Phi_r(v + W(r, v)) \in T_v\mathcal{M}$. Furthermore, the estimations

$$\|W(r, v)\| = O(r), \quad \|P_v D_v W(r, v)\|_{\mathcal{L}(T_v\mathcal{M})} = O(r)$$

hold uniformly on $\mathcal{U}$ as $r \to 0$. In the end, the problem is reduced to seeking critical points of

$$\psi_r(a) := \Phi_r\big(u_a + W(r, u_a)\big)$$

with $\psi(r, a) = \psi_r(a)$ defined on

$$\widetilde{\mathcal{U}} := \{(r, a) \in \mathbb{R} \times \mathbb{C} : |r| \leqslant \widetilde{r_0}, |ra| \leqslant \rho\},$$

where $\widetilde{r_0} \leqslant r_0$ depends upon ρ, δ, Γ_1 and Γ_2. We see that

$$\begin{aligned}\psi_r(a) &= \Phi_r\big(u_a + W(r,u_a)\big) - \Phi_r(u_a) + \Phi_r(u_a)\\ &= \Phi_r\big(u_a + W(r,u_a)\big) - \Phi_r(u_a) + \Phi_0(u_a) - \frac{4\pi^2(\Gamma_1+\Gamma_2)}{\Gamma_2^2}F(0)\\ &\qquad\qquad + \frac{2\pi(\Gamma_1+\Gamma_2)}{\Gamma_2^2}\int_0^{2\pi} F(ru_a)\,dt\end{aligned}$$

Let $c_0 := \Phi_0(u_a) - \frac{4\pi^2(\Gamma_1+\Gamma_2)}{\Gamma_2^2}F(0)$. It is a constant independent of a. Observe

$$\begin{aligned}\int_0^{2\pi} F(ru_a)\,dt = 2\pi(\Gamma_1\Gamma_2 + \frac{\Gamma_1^2}{2} + \frac{\Gamma_2^2}{2})h(ra)\\ + \int_0^{2\pi}\Gamma_1\Gamma_2\left(g(ra+re^{it}, ra - r\frac{\Gamma_1}{\Gamma_2}e^{it}) - g(ra,ra)\right)dt\\ + \int_0^{2\pi}\frac{\Gamma_1^2}{2}\big(h(ra+re^{it}) - h(ra)\big) + \frac{\Gamma_2^2}{2}\left(h(ra - r\frac{\Gamma_1}{\Gamma_2}e^{it}) - h(ra)\right)dt\,.\end{aligned}$$

Setting

$$\begin{aligned}\varphi_r(a) := \Phi_r\big(u_a + W(r,u_a)\big) - \Phi_r(u_a)\\ + \frac{2\pi(\Gamma_1+\Gamma_2)}{\Gamma_2^2}\int_0^{2\pi}\Gamma_1\Gamma_2\left(g(ra+re^{it}, ra - r\frac{\Gamma_1}{\Gamma_2}e^{it}) - g(ra,ra)\right)dt\\ + \frac{2\pi(\Gamma_1+\Gamma_2)}{\Gamma_2^2}\int_0^{2\pi}\frac{\Gamma_1^2}{2}\big(h(ra+re^{it}) - h(ra)\big) + \frac{\Gamma_2^2}{2}\left(h(ra - r\frac{\Gamma_1}{\Gamma_2}e^{it}) - h(ra)\right)dt\end{aligned}$$

we have $\psi_r(a) = c_0 + \frac{2\pi^2(\Gamma_1+\Gamma_2)^3}{\Gamma_2^2}h(ra) + \varphi_r(a)$. We find that

$$\begin{aligned}\big|\nabla\varphi_r(a)\big| \le \frac{2\pi|\Gamma_1+\Gamma_2||r|}{\Gamma_2^2}\int_0^{2\pi}|\Gamma_1\Gamma_2|\sum_{m=1,2}\left|\partial_m g(ra+re^{it}, ra - r\frac{\Gamma_1}{\Gamma_2}e^{it}) - \partial_m g(ra,ra)\right|dt\\ + \frac{2\pi|\Gamma_1+\Gamma_2||r|}{\Gamma_2^2}\int_0^{2\pi}\frac{\Gamma_1^2}{2}\big|h'(ra+re^{it}) - h'(ra)\big| + \frac{\Gamma_2^2}{2}\left|h'(ra - r\frac{\Gamma_1}{\Gamma_2}e^{it}) - h'(ra)\right|dt\\ + \big|\nabla_a\Phi_r\big(u_a + W(r,u_a)\big) - \nabla_a\Phi_r(u_a)\big|\\ = o(r) + \big|\nabla_a\Phi_r\big(u_a + W(r,u_a)\big) - \nabla_a\Phi_r(u_a)\big|\end{aligned}$$

from the facts g is of class $\mathcal{C}^1$ and $|ra|$ is uniformly bounded on $\widetilde{\mathcal{U}}$. Meanwhile, since $\nabla\Phi_r\big(u_a +$

$W(r,u_a)\big) \in T_{u_a}\mathcal{M}$, $D_vW(r,u_a)[(a',a')^{tr}] \in N_{u_a}\mathcal{M}$ for any $a' \in \mathbb{C}$, we deduce for $r \to 0$:

$$
\begin{aligned}
&\nabla_a\Phi_r\big(u_a+W(r,u_a)\big)[a'] - \nabla_a\Phi_r(u_a)[a'] \\
&\quad = \Phi_r'\big(u_a+W(r,u_a)\big)\Big[\begin{pmatrix} a' \\ a' \end{pmatrix} + D_vW(r,u_a)\begin{pmatrix} a' \\ a' \end{pmatrix}\Big] - \Phi_r'(u_a)[\begin{pmatrix} a' \\ a' \end{pmatrix}] \\
&\quad = \Phi_r'\big(u_a+W(r,u_a)\big)[\begin{pmatrix} a' \\ a' \end{pmatrix}] - \Phi_r'(u_a)[\begin{pmatrix} a' \\ a' \end{pmatrix}] \\
&\quad = \Phi_0'\big(u_a+W(r,u_a)\big)[\begin{pmatrix} a' \\ a' \end{pmatrix}] - \Phi_0'(u_a)[\begin{pmatrix} a' \\ a' \end{pmatrix}] \\
&\qquad + \frac{2\pi(\Gamma_1+\Gamma_2)r}{\Gamma_2^2}\int_0^{2\pi}\big(F'(ru_a+rW(r,u_a)) - F'(ru_a)\big)[\begin{pmatrix} a' \\ a' \end{pmatrix}]\,dt \\
&\quad = \Phi_0'\big(u_0+W(r,u_a)\big)[\begin{pmatrix} a' \\ a' \end{pmatrix}] - \Phi_0'(u_0)[\begin{pmatrix} a' \\ a' \end{pmatrix}] + o(r)\cdot|a'| \\
&\quad = \Phi_0''(u_0)\Big[\begin{pmatrix} a' \\ a' \end{pmatrix}, W(r,u_a)\Big] + o(\|W(r,u_a)\|)\cdot|a'| + o(r)\cdot|a'| \\
&\quad = o(r)\cdot|a'|,
\end{aligned}
$$

uniformly on $\widetilde{\mathcal{U}}$. Therefore we obtain $\nabla\varphi_r(a) = o(r)$. Then all the remaining arguments proceed in the same way as in Section 3.4 when $\Gamma_1+\Gamma_2 \neq 0$. However, since in this case we don't have a cyclic group action under which Φ_r is invariant, the periodic solutions we obtain may not fulfill the symmetry property $z_2^r = -\frac{\Gamma_1}{\Gamma_2}z_2^r$. $\square$

Another example is with a Hamiltonian which possesses a singular term $-|z_1-z_2|^{-\alpha}$ for some $\alpha > 0$. Precisely,

$$
H(z_1,z_2) = -\frac{1}{|z_1-z_2|^\alpha} + F(z_1,z_2) \tag{3.5.3}
$$

with $F \in C^2(\Omega^2,\mathbb{R})$. Carrying over similar arguments with minor changes we can obtain

Theorem 3.5.2. *If $a_0 \in \Omega$ is a stable critical point of $h : \Omega \to \mathbb{R}; z \mapsto F(z,z)$, then there exists $r_0 > 0$, such that for each $0 < r \leq r_0$, the system*

$$
z_k = -i\nabla_{z_k}H(z_1,z_2), \quad k = 1,2,
$$

with H as in (3.5.3) has a periodic solution $z^r = (z_1^r, z_2^r)$ with minimal period $T_r = \pi(2r)^{\alpha+2}/\alpha$ such that in the limit $r \to 0$ the points z_k^r move on circles in the following sense. There exists $a_r \in \Omega$ with $a_r \to a_0$ such that the rescaled functions

$$
u_k^r(t) := \frac{1}{r}\big(z_k^r(T_rt/2\pi) - a_r\big)
$$

satisfy

$$
u_1^r(t) \to e^{it}, \quad u_2^r(t) \to -e^{it}.
$$

The convergence holds in $H^1(\mathbb{R}/2\pi\mathbb{Z},\mathbb{C})$.

Proof. Assume $a_0 = 0$ and fix a neighborhood of 0, $B_{2\rho} \subset \Omega$. In this setting, the rescaled Hamiltonian function is

$$H_r(u) = \frac{2^{\alpha+1}}{\alpha} \cdot \frac{1}{|u_1 - u_2|^\alpha} - \frac{2^{\alpha+1} r^\alpha}{\alpha} F(ru)$$

and the usual perturbed action functional Φ_r is as in Section 3.2, defined on $H^1_{2\pi}(\mathbb{C}^2)$.

$$\begin{aligned}
\nabla \Phi_0(u) &= (Id - \Delta)^{-1} \Bigg(\begin{pmatrix} i\dot{u}_1 \\ i\dot{u}_2 \end{pmatrix} + \frac{2^{\alpha+1}}{|u_1 - u_2|^{\alpha+2}} \begin{pmatrix} u_1 - u_2 \\ u_2 - u_1 \end{pmatrix} \Bigg); \\
\nabla^2 \Phi_0(u)[v] &= (Id - \Delta)^{-1} \Bigg(\begin{pmatrix} i\dot{v}_1 \\ i\dot{v}_2 \end{pmatrix} + \frac{2^{\alpha+1}}{|u_1 - u_2|^{\alpha+2}} \begin{pmatrix} v_1 - v_2 \\ v_2 - v_1 \end{pmatrix} \\
&\qquad - \frac{2^{\alpha+1}(\alpha+2)}{|u_1 - u_2|^{\alpha+4}} \langle u_1 - u_2, v_1 - v_2 \rangle_{\mathbb{R}^2} \begin{pmatrix} u_1 - u_2 \\ u_2 - u_1 \end{pmatrix} \Bigg).
\end{aligned}$$

One can verify as in Section 3.3 that

$$\mathcal{M} := \left\{ \theta * u_a = \theta * \begin{pmatrix} e^{it} \\ -e^{it} \end{pmatrix} + \begin{pmatrix} a \\ a \end{pmatrix} : \theta \in S^1, a \in \mathbb{C} \right\}$$

is a nondegenerate critical manifold of Φ_0. For given $v \in \mathcal{M}$, to solve $(Id - P_v)(\nabla \Phi_r(v+w)) = 0$, we look for fixed point of $T(r, v, \cdot) : N_v\mathcal{M} \to N_v\mathcal{M}$ by

$$\begin{aligned}
T(r, v, w) = -(\mathcal{L}_v|_{N_v\mathcal{M}})^{-1} \circ (Id - P_v) \Big[&\nabla \Psi_0(v+w) - \nabla^2 \Psi_0(v)[w] \\
&+ \frac{2^{\alpha+1} r^{\alpha+1}}{\alpha} (Id - \Delta)^{-1} \big(F(rv + rw) \big) \Big]
\end{aligned}$$

where $\mathcal{L}_v := (Id - P_v) \circ \nabla^2 \Psi_0(v)$ is an isomorphism when restricted to $N_v\mathcal{M}$, and satisfies $\|\mathcal{L}_v|_{N_v\mathcal{M}})^{-1}\| \leqslant \gamma$ with constant $\gamma > 0$ for all $v \in \mathcal{M}$.

Then repeating the steps in Lemma 3.3.6, we claim that there exist constants $\delta > 0$ and $r_0 > 0$ independent of the choice of v, such that for any $|r| \leqslant r_0$ and $v = \theta * u_a \in \mathcal{M}$ with $|ra| \leqslant \rho$, $T(r, v, \cdot)$ is a contraction mapping from $\{w \in N_v\mathcal{M} : \|w\| \leqslant \delta\}$ to itself and particularly,

$$\|\nabla \Phi_0(v+w) - \nabla^2 \Phi_0(v)[w]\| \leqslant \frac{1}{4\gamma} \|w\|.$$

Thus a solution map

$$W : \mathcal{U} = \Big\{ (r, v) \in \mathbb{R} \times \mathcal{M} : v = \theta * u_a, |r| \leq r_0, |ra| \leq \rho \Big\} \to H^1_{2\pi}(\mathbb{C}^2)$$

is obtained. Moreover,

$$\|W(r,v)\| = \|T(r, v, W(r,v))\| \leqslant \frac{1}{4} \|W(r,v)\| + \frac{2^{\alpha+1} \gamma |r|^{\alpha+1}}{\alpha} \|F'(rv + rW(r,v))\|_{L^2}.$$

Therefore, since F' is bounded on $\mathcal{U}$, it follows that $\|W(r,v)\| = O(r^{\alpha+1})$ uniformly on $\mathcal{U}$ as $r \to 0$. This leads to

$$\Phi_r(u_a + W(r,u_a)) = c_0 + \frac{\pi 2^{\alpha+2} r^\alpha}{\alpha} h(ra) + \varphi_r(a)$$

with $\nabla\varphi_r(a) = o(r^{\alpha+1})$ uniformly. So the result can be proved as in Section 3.4. $\square$

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