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Dissertation

Construction of RGD-systems of type $(4, 4, 4)$ over $\mathbb{F}_2$

Eine Thesis zur Erlangung des akademischen Grades
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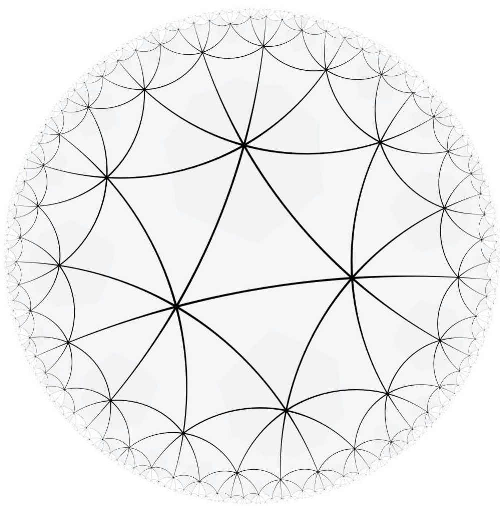

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Abstract

We investigate the structure of RGD-systems over $\mathbb{F}_2$. For this purpose we introduce the notion of *commutator blueprints* which prescribe the commutator relations between prenilpotent pairs of positive roots. To each RGD-system one can associate a commutator blueprint and such a commutator blueprint will be called *integrable*. We give necessary and sufficient conditions of an integrable commutator blueprint. Moreover, we construct uncountably many different integrable commutator blueprints of type $(4, 4, 4)$.

The existence of these integrable commutator blueprints disproves the general validity of the extension theorem for isometries of 2-spherical thick twin buildings. Additionally, we obtain the first example of a 2-spherical Kac-Moody group over a finite field which is not finitely presented. Furthermore, we construct the first example of a 2-spherical RGD-system with finite root groups which does not have property (FPRS).

Deutsche Zusammenfassung

Wir untersuchen die Struktur von RGD-systemen über $\mathbb{F}_2$. Aus diesem Grund führen wir den Begriff von *Kommutatorbauplänen* ein, welche die Kommutatorrelationen zwischen prenilpotenten Paaren von positiven Wurzeln vorschreiben. Zu jedem RGD-System kann man einen Kommutatorbauplan assoziieren und solch einen Kommutatorbauplan nennen wir *integrabel*. Wir geben notwendige und hinreichende Bedingungen eines integrablen Kommutatorbauplans an. Außerdem konstruieren wir überabzählbar viele verschiedene integrable Kommutatorbaupläne vom Typ $(4, 4, 4)$.

Die Existenz dieser integrablen Kommutatorbaupläne widerlegt die Allgemeingültigkeit des Erweiterungssatzes für Isometrien von 2-spärischen dicken Zwillingsgebäuden. Zusätzlich erhalten wir das erste Beispiel einer 2-sphärischen Kac-Moody Gruppe über einem endlichen Körper, welche nicht endlich präsentiert ist. Zudem konstruieren wir das erste Beispiel eines 2-sphärischen RGD-Systems mit endlichen Wurzelgruppen, welches nicht die Eigenschaft (FPRS) besitzt.

Contents

Introduction

Historical context

This is based on [25], [8] and [9].

Twin buildings

Buildings have been introduced by Tits in order to study semi-simple algebraic groups from a combinatorial point of view. One of the most celebrated results in the theory of abstract buildings is Tits' classification result for irreducible spherical buildings of rank at least 3. The decisive step in this classification is a local-to-global result for isometries of spherical buildings.

Twin buildings were introduced by Ronan and Tits in the late 1980s in order to study groups of Kac-Moody type. Their definition was motivated by the theory of Kac-Moody groups over fields. Each such group acts naturally on a pair of two buildings and the action preserves an opposition relation between the chambers of the buildings. This opposition relation shares many important information with the opposition relation of spherical buildings. Thus, twin buildings appear to be natural generalizations of spherical buildings.

Extension problem

Ronan and Tits conjectured in the 1990s that there exists a similar local-to-global result for isometries of twin buildings. To be more precise: let $\Delta = (\Delta_+, \Delta_-, \delta_*), \Delta' = (\Delta'_+, \Delta'_-, \delta'_*)$ be two twin buildings of the same type (W, S). An *isometry* is a bijection from $\mathcal{X} \subseteq \Delta$ to $\mathcal{X}' \subseteq \Delta'$ which preserves the sign, the distance and the codistance. For $c \in \Delta_+$ we denote by $E_2(c)$ the union of all residues of rank at most 2 containing c.

Extension theorem: Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ and $\Delta' = (\Delta'_+, \Delta'_-, \delta'_*)$ be two twin buildings of type (W, S). We say that the the extension theorem *holds* for Δ, if for all $c \in \Delta_+$ and $c' \in \Delta'_+$, every isometry $E_2(c) \to E_2(c')$ extends to an isometry $\Delta \to \Delta'$.

If the extension theorem holds for a subclass of twin buildings, then the classification of twin buildings contained in this subclass reduces to the classification of *foundations*, i.e. the local structure $E_2(c)$. First the extension theorem seems only be feasible under the additional assumption that (W, S) is 2-spherical. Tits observed that a proof of the extension theorem splits *roughly* into two parts:

Part 1 (first half): Any isometry $E_2(c) \to E_2(c')$ extends to an isometry $\Delta_+ \to \Delta'_+$.

Tits proved part 1 in [34] under the additional assumption that each panel is sufficiently large.

Part 2 (second half): Any isometry $\Delta_+ \to \Delta'_+$ extends to an isometry $\Delta \to \Delta'$.

The first contribution to part 2 is a result of Mühlherr and Ronan from 1995 published in [25] satisfying an additional condition (co).

Condition (co)

A twin building $\Delta = (\Delta_+, \Delta_-, \delta_*)$ satisfies condition (co) if for every $c \in \mathcal{C}_\varepsilon$ the set $\{d \in \mathcal{C}_{-\varepsilon} \mid \delta_*(c, d) = 1_W\}$ of chambers opposite c is connected. Mühlherr and Ronan have shown in [25] the following condition on the rank 2 residues implies condition (co):

(lco) No rank 2 residue of Δ is associated with one of the groups $B_2(2), G_2(2), G_2(3), {}^2F_4(2)$.

Condition (lco) shows that difficulties arise only if the size of panels is too small. In particular, they proved the following theorem:

Theorem (Mühlherr, Ronan, [25]): The second half of the extension theorem holds for twin buildings

- in which every panel contains at least 5 chambers.
- of simply-laced type.

In [31] Ronan generalized the first half of the extension theorem to twin buildings satisfying (co). In particular, the extension theorem holds for twin buildings satisfying (co). Recently, Chosson, Mühlherr and the author have shown in [8] that the first half of the extension theorem is true for any two 2-spherical thick twin buildings. Thus, the extension theorem holds if the second part of the extension theorem holds.

Condition (wc)

What is a bit unsatisfying about condition (co) is that not all affine twin buildings satisfy this condition. In [9] Mühlherr and the author introduced condition (wc) in order to prove the second half of the extension theorem for affine twin buildings. As a consequence, the second half of the extension theorem is true for 3-spherical thick twin buildings. This rather technical condition (wc) is a weaker condition than (co) and has a nice interpretation for RGD-systems. We do not give the definition here, but we refer to [9] for details.

Counterexamples coming from RGD-systems

The main motivation of this thesis is to construct two thick twin buildings of 2-spherical type for which the extension theorem does not hold. The twin buildings are associated with RGD-systems. In particular, we have constructed RGD-systems over $\mathbb{F}_2$ (i.e. every root group contains exactly two elements) with prescribed commutator relations.

Let $(G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD-system of type (W, S) over $\mathbb{F}_2$, where G is generated by the groups U_α. We let U_+ be the group generated by the set of root groups corresponding to positive roots. For each $s \in S$ we let P_s be the group generated by U_+ and $U_{-\alpha_s}$ and we let $\tau_s := u_{-s}u_s u_{-s}$, where $u_{\pm s} \in U_{\pm\alpha_s}\backslash\{1\}$. Then we have the following two well-known theorems:

Theorem 1: The group G is isomorphic to the direct limit of the inductive system formed by the groups $U_+, (P_s)_{s\in S}, \mathbb{Z}_2, W$ together with the natural inclusions

$$U_+ \longrightarrow P_s \xleftarrow[1\mapsto\tau_s]{} \mathbb{Z}_2 \xrightarrow{1\mapsto s} W \qquad \text{for all } s \in S.$$

Theorem 2: The group U_+ is isomorphic to the direct limit of the inductive system formed by the groups U_w, together with the canonical inclusions $U_w \to U_{ws}$ for every $w \in W, s \in S$ with $\ell(ws) = \ell(w) + 1$. Moreover, the group U_w has cardinality $2^{\ell(w)}$ for each $w \in W$.

Construction of RGD-systems over $\mathbb{F}_2$

In this thesis we follow the ideas of Theorem 1 and 2. In the following will describe the main strategy. In order to explain the main idea in a comprehensible way, we will not be formally mathematically correct at the one or other point in this description.

We first need to find a way of producing groups U_w having cardinality $2^{\ell(w)}$ for $w \in W$. These groups should come equipped with canonical homomorphisms $U_w \to U_{ws}$, whenever $\ell(ws) = \ell(w)+1$. Following Theorem 2, we define U_+ as the direct limit of the inductive system formed by the groups U_w, together with the homomorphisms $U_w \to U_{ws}$. It is not clear at this stage whether the homomorphisms $U_w \to U_+$ are injective, but for the moment we assume that they are. Next we have to construct the groups P_s for each $s \in S$. Therefore, we note that $U_+ \cong U_s \ltimes N_s$ splits as a semi-direct product. We will construct another automorphism $\tau_s \in \mathrm{Aut}(N_s)$ with the property $\tau_s(u_\alpha) = u_{s\alpha}$. Now we define $P_s := \langle u_s, \tau_s \rangle \ltimes N_s$. Moreover, we can define the direct limit of the inductive system formed by the groups $U_+, (P_s)_{s\in S}, \mathbb{Z}_2, W$ as in Theorem 1. We will work out sufficient conditions in order to show that G can be endowed with an RGD-system.

Overview

In Chapter 1 we introduce the basic definitions of the theory of buildings. Moreover, we state known results and prove some auxiliary results which will be needed later. In the second chapter we introduce the notion of *commutator blueprints*, the main objects of this thesis. They can be seen as a prescription of commutator relations. Each commutator blueprint provides the groups U_w having cardinality $2^{\ell(w)}$ for $w \in W$. To each RGD-system one can associate a commutator blueprint and such commutator blueprints are called *integrable*. Additionally, we define *faithful* and *Weyl-invariant* commutator blueprint: Faithfulness implies that the canonical homomorphisms $U_w \to U_+$ are injective. Moreover, we obtain the decomposition $U_+ \cong U_s \ltimes N_s$ and hence $u_s \in \mathrm{Aut}(N_s)$. Weyl-invariance allows to construct an automorphism $\tau_s \in \mathrm{Aut}(N_s)$ such that $\tau_s(u_\alpha) = u_{s\alpha}$. Moreover, if G denotes the direct limit of the inductive system formed by the groups $U_+, (P_s)_{s\in S}, \mathbb{Z}_2, W$ as in Theorem 1, we prove the following theorem:

Theorem (Theorem (2.4.3)): If $P_s \to G$ is injective for each $s \in S$, then the commutator blueprint is integrable.

In order to show the the group G can be endowed with an RGD-system, we have to show that the homomorphisms $P_s \to G$ are injective. We consider the chamber system $\mathbf{C}$, where each chamber is a coset contained in U_+/U_w for some $w \in W$. We define an action of P_s on $\mathbf{C}$ and deduce that this action is faithful. We are done, if the braid relations $(\tau_s\tau_t)^{m_{st}}$ act trivially on the chamber system $\mathbf{C}$ for all $s \neq t \in S$ with $m_{st} < \infty$. This is what we do in Chapter 3. We restrict to the cases $m_{st} \neq 6$, i.e. $m_{st} \in \{2,3,4\}$. As it is our main motivation to construct RGD-systems of type $(4,4,4)$, this is an acceptable restriction. It turns out that the braid relations act trivial in the case $m_{st} = 2$. We introduce two further conditions of the commutator blueprint (called (CR1) and (CR2)), and it turns out that if the groups U_w are of nilpotency class at most 2 and if (CR1) and (CR2) are satisfied, then the braid relations act trivial in the cases $m_{st} \in \{3,4\}$. In Chapter 4 we show that if the commutator relations are chosen in a way that they are somehow *of nilpotency class* 2 then the groups U_w have automatically cardinality $2^{\ell(w)}$.

In Part 3 we discuss faithful commutator blueprints of type $(4,4,4)$. Therefore, we analyze the geometry of the Coxeter system of type $(4,4,4)$ and its set of roots. Moreover, we prove that any RGD-system of type $(4,4,4)$ over $\mathbb{F}_2$ contains suitable tree products (called $V_{R,s}$) as subgroups. These groups will be needed in Chapter 6. The fact that $V_{R,s}$ is a subgroup is obtained by considering the action of the group on its associated twin building. In Chapter 6 we introduce several tree products and prove many subgroup and isomorphism properties of those. In Section 6.9 and 6.10 we construct the group U_+ successively as a tree product.

Here we need on the one hand the subgroups $V_{R,s}$ and, one the other hand the tree products constructed in Chapter 6. We remark that this construction does only work because we already have one example of an RGD-system of type $(4,4,4)$ over $\mathbb{F}_2$, namely the Kac-Moody group. This implies that a Weyl-invariant commutator blueprint of type $(4,4,4)$ is faithful. The main result of this thesis will be the following:

Main result (Corollary (6.10.7)): Any Weyl-invariant commutator blueprint of type $(4,4,4)$ satisfying the conditions (CR1) and (CR2) and such that the groups U_w are of nilpotency class at most 2 is integrable.

We remark that Weyl-invariance, (CR1), (CR2) and the nilpotency class assumption can be checked by only considering the commutator blueprint. In the last chapter we discuss several applications of the main result, which we will explain below. In the appendix we reproduce for convenience all figures from Chapter 6.

Applications of the Main result

First we construct uncountably many different Weyl-invariant commutator blueprints, which are integrable. The existence of these has itself two applications. The first concerns an answer of a 30 year-old question of Ronan and Tits about the extension problem. We obtain the following result:

Extension problem (Theorem (7.2.1)): The extension theorem does not hold for all thick 2-spherical twin buildings.

The second application answers a question about finiteness properties of groups acting on twin buildings. Abremenko and Mühlherr have shown in [3] that almost all 2-spherical Kac-Moody groups over finite fields are finitely presented. As a consequence of our construction we obtain the first 2-spherical, non-finitely presented Kac-Moody group over a finite field:

Theorem (Theorem (7.3.3)): Let G be the Kac-Moody group (in the sense of [34]) of type $(4,4,4)$ over $\mathbb{F}_2$. Then G is not finitely presented.

Moreover, Abramenko considered finiteness properties of the stabilizer of a chamber in a Kac-Moody groups and he proved (unpublished, cf. [1, Counter-Example 1(2)]) the following result, which is also a consequence of our construction:

Theorem (Theorem (7.3.4), Lemma (7.4.6)) Let $\mathcal{D}$ be an RGD-system of type $(4,4,4)$ over $\mathbb{F}_2$. Then group U_+ is not finitely generated. In particular, the automorphism group of the building associated with an RGD-system of type $(4,4,4)$ over $\mathbb{F}_2$ does not have property (T).

The last application concerns property (FPRS), which makes a statement about fixed points of the root groups on the associated building. Caprace and Rémy have shown in [16] that (almost) split Kac-Moody groups satisfy this property. We have shown that many (but not all) of the new examples satisfy this property (cf. Corollary (7.5.4)). In particular, we obtain the first example of a 2-spherical RGD-system which does not satisfy property (FPRS):

Theorem (Theorem (7.5.5)): There exists an RGD-system of 2-spherical type which does not satisfy property (FPRS).

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Part I.

Preliminaries

1. Basic definitions

In this chapter we introduce basic definitions of the theory of buildings. Moreover, we state some known results and prove some auxiliary results which we will need later.

1.1. Coxeter systems

Let (W, S) be a Coxeter system and let ℓ denote the corresponding length function. For $s, t \in S$ we denote the order of st in W by m_{st}. The *Coxeter diagram* corresponding to (W, S) is the labeled graph $(S, E(S))$, where $E(S) = \{\{s, t\} \mid m_{st} > 2\}$ and where each edge $\{s, t\}$ is labeled by m_{st} for all $s, t \in S$. The *rank* of the Coxeter system is the cardinality of the set S. Let (W, S) be of rank 3 and let $S = \{r, s, t\}$. Sometimes we will also call (m_{rs}, m_{rt}, m_{st}) the *type* of (W, S). If (W, S) is of type (m_{rs}, m_{rt}, m_{st}), then it is called *cyclic hyperbolic* if $m_{rs}, m_{rt}, m_{st} \geq 3$ and $\frac{1}{m_{rs}} + \frac{1}{m_{rt}} + \frac{1}{m_{st}} < 1$.

If $m_{st} \in \{2, 3, 4, 6, \infty\}$ for all $s \neq t \in S$, we call the Coxeter system *crystallographic.* In this case we define the *crystallographic Dynkin diagram* $\mathrm{Dyn}(W, S)$ corresponding to (W, S) as the Coxeter diagram, where each edge has a direction. We note that this is not exactly the notion of a Dynkin diagram in the literature.

It is well-known that for each $J \subseteq S$ the pair $(\langle J \rangle, J)$ is a Coxeter system (cf. [10, Ch. IV, §1 Theorem 2]). A subset $J \subseteq S$ is called *spherical* if $\langle J \rangle$ is finite. The Coxeter system is called *spherical* if S is spherical; it is called 2-*spherical* if $\langle J \rangle$ is finite for all $J \subseteq S$ containing at most 2 elements (i.e. $m_{st} < \infty$ for all $s, t \in S$). Given a spherical subset J of S, there exists a unique element of maximal length in $\langle J \rangle$, which we denote by r_J (cf. [2, Corollary 2.19]).

(1.1.1) Lemma. *Let $\varepsilon \in \{+, -\}$ and let (W, S) be a Coxeter system. Suppose $s, t \in S, w \in W$ with $\ell(sw) = \ell(w)\varepsilon 1 = \ell(wt)$. Then either $\ell(swt) = \ell(w)\varepsilon 2$ or else $swt = w$.*

Proof. The case $\varepsilon = +$ is [2, Condition (**F**) on p. 79]. Thus we consider the case $\varepsilon = -$. We put $w' := sw$. Then $\ell(sw') = \ell(w) = \ell(w') + 1$. We assume that $\ell(swt) \neq \ell(w) - 2$. Then $\ell(swt) = \ell(w)$ and hence $\ell(w't) = \ell(swt) = \ell(w) = \ell(w') + 1$. Using [2, Condition (**F**) on p. 79] we obtain either $\ell(sw't) = \ell(w') + 2$ or $sw't = w'$. Since $\ell(sw't) = \ell(wt) = \ell(sw) = \ell(w')$ we have $wt = sw't = w' = sw$ and the claim follows. □

1.2. Buildings

Let (W, S) be a Coxeter system. A *building of type* (W, S) is a pair $\Delta = (\mathcal{C}, \delta)$ where $\mathcal{C}$ is a non-empty set and where $\delta : \mathcal{C} \times \mathcal{C} \to W$ is a *distance function* satisfying the following axioms, where $x, y \in \mathcal{C}$ and $w = \delta(x, y)$:

(Bu1) $w = 1_W$ if and only if $x = y$;

(Bu2) if $z \in \mathcal{C}$ satisfies $s := \delta(y, z) \in S$, then $\delta(x, z) \in \{w, ws\}$, and if, furthermore, $\ell(ws) = \ell(w) + 1$, then $\delta(x, z) = ws$;

(Bu3) if $s \in S$, there exists $z \in \mathcal{C}$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

The *rank* of Δ is the rank of the underlying Coxeter system. The elements of $\mathcal{C}$ are called *chambers*. Given $s \in S$ and $x, y \in \mathcal{C}$, then x is called *s-adjacent* to y, if $\delta(x, y) = s$. The chambers x, y are called *adjacent*, if they are s-adjacent for some $s \in S$. A *gallery* from x to y is a sequence $(x = x_0, \ldots, x_k = y)$ such that x_{l-1} and x_l are adjacent for all $1 \leq l \leq k$; the number k is called the *length* of the gallery. Let $(x_0, \ldots, x_k)$ be a gallery and suppose $s_i \in S$ with $\delta(x_{i-1}, x_i) = s_i$. Then $(s_1, \ldots, s_k)$ is called the *type* of the gallery. A gallery from x to y of length k is called *minimal* if there is no gallery from x to y of length $< k$. In this case we have $\ell(\delta(x, y)) = k$ (cf. [2, Corollary 5.17(1)]). Let $x, y, z \in \mathcal{C}$ be chambers such that $\ell(\delta(x, y)) = \ell(\delta(x, z)) + \ell(\delta(z, y))$. Then the concatenation of a minimal gallery from x to z and a minimal gallery from z to y yields a minimal gallery from x to y.

Given a subset $J \subseteq S$ and $x \in \mathcal{C}$, the *J-residue of* x is the set $R_J(x) := \{y \in \mathcal{C} \mid \delta(x, y) \in \langle J \rangle\}$. Each J-residue is a building of type $(\langle J \rangle, J)$ with the distance function induced by δ (cf. [2, Corollary 5.30]). A *residue* is a subset R of $\mathcal{C}$ such that there exist $J \subseteq S$ and $x \in \mathcal{C}$ with $R = R_J(x)$. Since the subset J is uniquely determined by R, the set J is called the *type* of R and the *rank* of R is defined to be the cardinality of J. A residue is called *spherical* if its type is a spherical subset of S. A building is called *spherical* if its type is spherical. Let R be a spherical J-residue. Then $x, y \in R$ are called *opposite in* R if $\delta(x, y) = r_J$. Two residues $P, Q \subseteq R$ are called *opposite in* R if for each $p \in P$ there exists $q \in Q$ such that p, q are opposite in R and if for each $q' \in Q$ there exists $p' \in P$ such that q', p' are opposite in R. A *panel* is a residue of rank 1. An *s-panel* is a panel of type $\{s\}$ for $s \in S$. The building Δ is called *thick*, if each panel of Δ contains at least three chambers; it is called *locally finite*, if each panel contains only finitely many chambers.

Given $x \in \mathcal{C}$ and a J-residue $R \subseteq \mathcal{C}$, then there exists a unique chamber $z \in R$ such that $\ell(\delta(x, y)) = \ell(\delta(x, z)) + \ell(\delta(z, y))$ holds for each $y \in R$ (cf. [2, Proposition 5.34]). The chamber z is called the *projection of* x *onto* R and is denoted by $\mathrm{proj}_R\, x$. Moreover, if $z = \mathrm{proj}_R\, x$ we have $\delta(x, y) = \delta(x, z)\delta(z, y)$ for each $y \in R$. Let $J \subseteq S$, let R be a J-residue and suppose $c \in \mathcal{C}, d \in R$ with $\ell(\delta(c, d)j) = \ell(\delta(c, d)) + 1$ for each $j \in J$. Then we have $d = \mathrm{proj}_R\, c$ (cf. [24, Lemma 21.6(iv)]). Let $R \subseteq T$ be two residues of Δ. Then $\mathrm{proj}_R\, c = \mathrm{proj}_R \mathrm{proj}_T\, c$ holds for every $c \in \mathcal{C}$ by [19, Proposition 2].

An *(type-preserving) automorphism* of a building $\Delta = (\mathcal{C}, \delta)$ is a bijection $\varphi : \mathcal{C} \to \mathcal{C}$ such that $\delta(\varphi(c), \varphi(d)) = \delta(c, d)$ holds for all chambers $c, d \in \mathcal{C}$. We remark that some authors distinguish between automorphisms and type-preserving automorphisms. An automorphism in our sense is type-preserving. We denote the set of all automorphisms of the building Δ by $\mathrm{Aut}(\Delta)$. It is a basic fact that the projection commutes with each automorphism. More precisely, let $c \in \mathcal{C}$, let R be a residue of Δ and let $\varphi \in \mathrm{Aut}(\Delta)$. It follows directly from the uniqueness of $\mathrm{proj}_R\, c$ that $\varphi(\mathrm{proj}_R\, c) = \mathrm{proj}_{\varphi(R)}\, \varphi(c)$.

(1.2.1) Example. We define $\delta : W \times W \to W, (x, y) \mapsto x^{-1}y$. Then $\Sigma(W, S) := (W, \delta)$ is a building of type (W, S). The group W acts faithful on $\Sigma(W, S)$ via left-multiplication, i.e. $W \leq \mathrm{Aut}(\Sigma(W, S))$.

A subset $\Sigma \subseteq \mathcal{C}$ is called *convex* if for any two chambers $c, d \in \Sigma$ and any minimal gallery $(c_0 = c, \ldots, c_k = d)$, we have $c_i \in \Sigma$ for all $0 \leq i \leq k$. Note that by [2, Example 5.44(b)] any residue of a building is convex. A subset $\Sigma \subseteq \mathcal{C}$ is called *thin* if $P \cap \Sigma$ contains exactly two chambers for every panel $P \subseteq \mathcal{C}$ which meets Σ. An *apartment* is a non-empty subset $\Sigma \subseteq \mathcal{C}$, which is convex and thin.

For two residues R and T we define $\mathrm{proj}_T\, R := \{\mathrm{proj}_T\, r \mid r \in R\}$. By [2, Lemma 5.36(2)] $\mathrm{proj}_T\, R$ is a residue contained in T. Two residues R and T are called *parallel* if $\mathrm{proj}_T\, R = T$ and $\mathrm{proj}_R\, T = R$. By [24, Proposition 21.8(i)] the residues $\mathrm{proj}_T\, R$ and $\mathrm{proj}_R\, T$ are parallel. If R and T are parallel, then it follows by [24, Proposition 21.8(ii), (iii)] that

$\text{proj}_R^T : T \to R, t \to \text{proj}_R\, t$ and $\text{proj}_T^R : R \to T, r \mapsto \text{proj}_T\, r$ are bijections inverse to each other and that the element $\delta(x, \text{proj}_T\, x) \in W$ is independent of the choice of $x \in R$.

(1.2.2) Lemma. *Let R be a spherical residue of rank 2 and let $P \neq Q \subseteq R$ be two parallel panels. Then P and Q are opposite in R.*

Proof. This is a consequence of [18, Lemma 18] and [2, Lemma 5.107]. □

(1.2.3) Theorem. *Let $\Delta = (\mathcal{C}, \delta)$ be a thick spherical building of type (W, S) and let $c, d \in \mathcal{C}$ be opposite chambers in $\mathcal{C}$. Then the only automorphism of Δ, which fixes $\bigcup_{s \in S} R_{\{s\}}(c) \cup \{d\}$ pointwise, is the identity.*

Proof. This is [2, Theorem 5.205]. □

1.3. Roots

Let (W, S) be a Coxeter system. A *reflection* is an element of W that is conjugate to an element of S. For $s \in S$ we let $\alpha_s := \{w \in W \mid \ell(sw) > \ell(w)\}$ be the *simple root* corresponding to s. A *root* is a subset $\alpha \subseteq W$ such that $\alpha = v\alpha_s$ for some $v \in W$ and $s \in S$. We denote the set of all roots by $\Phi(W, S)$. The set $\Phi(W, S)_+ = \{\alpha \in \Phi(W, S) \mid 1_W \in \alpha\}$ is the set of all *positive roots* and $\Phi(W, S)_- = \{\alpha \in \Phi(W, S) \mid 1_W \notin \alpha\}$ is the set of all *negative roots*. For each root $\alpha \in \Phi(W, S)$ we denote the *opposite* root by $-\alpha$ and we denote the unique reflection which interchanges these two roots by $r_\alpha \in W \leq \text{Aut}(\Sigma(W, S))$. Moreover, for each reflection r there exist two roots $\pm\beta_r$ which are interchanged by r. A pair $\{\alpha, \beta\}$ of distinct roots is called *prenilpotent* if both $\alpha \cap \beta$ and $(-\alpha) \cap (-\beta)$ are non-empty sets. For such a pair we will write $[\alpha, \beta] := \{\gamma \in \Phi(W, S) \mid \alpha \cap \beta \subseteq \gamma$ and $(-\alpha) \cap (-\beta) \subseteq -\gamma\}$ and $(\alpha, \beta) := [\alpha, \beta] \setminus \{\alpha, \beta\}$. A pair $\{\alpha, \beta\} \subseteq \Phi$ of two roots is called *nested*, if $\alpha \subsetneq \beta$ or $\beta \subsetneq \alpha$.

(1.3.1) Convention. For the rest of this paper we let (W, S) be a Coxeter system of finite rank and we define $\Phi := \Phi(W, S)$ (resp. Φ_+, Φ_-).

(1.3.2) Lemma. *For $s \neq t \in S$ we have $\alpha_t \subseteq (-\alpha_s) \cup t\alpha_s$.*

Proof. Let $w \in \alpha_t$. If $\ell(sw) < \ell(w)$, then $w \in (-\alpha_s)$ and we are done. Thus we can assume $\ell(sw) > \ell(w)$. As $w \in \alpha_t$, we have $\ell(tw) > \ell(w)$ and hence $\ell(stw) = \ell(w) + 2 > \ell(tw)$. Thus $tw \in \alpha_s$ and hence $w \in t\alpha_s$. □

(1.3.3) Remark. Let $s \neq t \in S$ and let $\beta \in (\alpha_s, \alpha_t)$. Then we have $\alpha_s \cap \alpha_t \subseteq \beta$ and hence $(-\beta) \subseteq (-\alpha_s) \cup (-\alpha_t)$. Moreover, we have $(-\alpha_s) \cap (-\alpha_t) \subseteq (-\beta)$ and hence $\beta \subseteq \alpha_s \cup \alpha_t$.

1.4. Coxeter buildings

In this section we consider the Coxeter building $\Sigma(W, S)$. At first we note that roots are convex (cf. [2, Lemma 3.44]). For $\alpha \in \Phi$ we denote by $\partial\alpha$ (resp. $\partial^2\alpha$) the set of all panels (resp. spherical residues of rank 2) stabilized by r_α. Furthermore, we define $\mathcal{C}(\partial\alpha) := \bigcup_{P \in \partial\alpha} P$ and $\mathcal{C}(\partial^2\alpha) := \bigcup_{R \in \partial^2\alpha} R$. The set $\partial\alpha$ is called the *wall* associated with α. Let $G = (c_0, \ldots, c_k)$ be a gallery. We say that *G crosses the wall $\partial\alpha$* if there exists $1 \leq i \leq k$ such that $\{c_{i-1}, c_i\} \in \partial\alpha$. It is a basic fact that a minimal gallery crosses a wall at most once (cf. [2, Lemma 3.69]). Let $(c_0, \ldots, c_k)$ and $(d_0 = c_0, \ldots, d_k = c_k)$ be two minimal galleries from c_0 to c_k and let $\alpha \in \Phi$. Then $\partial\alpha$ is crossed by the minimal gallery $(c_0, \ldots, c_k)$ if and only if it is crossed by the minimal gallery $(d_0, \ldots, d_k)$. Moreover, a gallery which crosses each wall at most once is already minimal. For a minimal gallery $G = (c_0, \ldots, c_k), k \geq 1$, we denote the unique root containing c_{k-1} but not c_k by α_G. For $\alpha_1, \ldots, \alpha_k \in \Phi$ we say that a minimal gallery $G = (c_0, \ldots, c_k)$ *crosses the sequence of roots* $(\alpha_1, \ldots, \alpha_k)$, if $c_{i-1} \in \alpha_i$ and $c_i \notin \alpha_i$ all $1 \leq i \leq k$.

We denote the set of all minimal galleries $G = (c_0 = 1_W, \ldots, c_k)$ by Min. For $w \in W$ we denote the set of all $G \in \text{Min}$ of type $(s_1, \ldots, s_k)$ with $w = s_1 \cdots s_k$ by $\text{Min}(w)$. For $w \in W$ with $\ell(sw) = \ell(w) - 1$ we let $\text{Min}_s(w)$ be the set of all $G \in \text{Min}(w)$ of type $(s, s_2, \ldots, s_k)$.

For a positive root $\alpha \in \Phi_+$ we define $k_\alpha := \min\{k \in \mathbb{N} \mid \exists G = (c_0, \ldots, c_k) \in \text{Min} : \alpha_G = \alpha\}$. We remark that $k_\alpha = 1$ if and only if α is a simple root. Furthermore, we define $\Phi(k) := \{\alpha \in \Phi_+ \mid k_\alpha \leq k\}$ for $k \in \mathbb{N}$. Let R be a residue and let $\alpha \in \Phi_+$. Then we call α a *simple root of* R if there exists $P \in \partial\alpha$ such that $P \subseteq R$ and $\text{proj}_R 1_W = \text{proj}_P 1_W$. In this case R is also stabilized by r_α and hence $R \in \partial^2\alpha$.

(1.4.1) Lemma. *Let R be a spherical residue of $\Sigma(W, S)$ of rank 2 and let $\alpha \in \Phi$. Then exactly one of the following hold:*

(a) $R \subseteq \alpha$;

(b) $R \subseteq (-\alpha)$;

(c) $R \in \partial^2\alpha$;

Proof. It is clear, that the three cases are exclusive. Suppose that $R \not\subseteq \alpha$ and $R \not\subseteq (-\alpha)$. Then there exist $c \in R \cap (-\alpha)$ and $d \in R \cap \alpha$. Let $(c_0 = c, \ldots, c_k = d)$ be a minimal gallery. As residues are convex, we have $c_i \in R$ for every $0 \leq i \leq k$. As $c \in (-\alpha), d \in \alpha$, there exists $1 \leq i \leq k$ with $c_{i-1} \in (-\alpha), c_i \in \alpha$. In particular, $\{c_{i-1}, c_i\} \in \partial\alpha$ and hence $R \in \partial^2\alpha$. □

(1.4.2) Lemma. *Let R, T be two residues of $\Sigma(W, S)$. Then the following are equivalent*

(i) *R, T are parallel;*

(ii) *a reflection of $\Sigma(W, S)$ stabilizes R if and only if it stabilizes T;*

(iii) *there exist two sequences $R_0 = R, \ldots, R_n = T$ and $T_1, \ldots, T_n$ of residues of spherical type such that for each $1 \leq i \leq n$ the rank of T_i is equal to $1 + \text{rank}(R)$, the residues R_{i-1}, R_i are contained and opposite in T_i and moreover, we have $\text{proj}_{T_i} R = R_{i-1}$ and $\text{proj}_{T_i} T = R_i$.*

Proof. This is [13, Proposition 2.7]. □

(1.4.3) Lemma. *Let $\alpha \in \Phi$ be a root and let $x, y \in \alpha \cap \mathcal{C}(\partial\alpha)$. Then there exists a minimal gallery $(c_0 = x, \ldots, c_k = y)$ such that $c_i \in \mathcal{C}(\partial^2\alpha)$ for each $0 \leq i \leq k$. Moreover, for every $1 \leq i \leq k$ there exists $L_i \in \partial^2\alpha$ with $\{c_{i-1}, c_i\} \subseteq L_i$.*

Proof. This is a consequence of [12, Lemma 2.3] and its proof. □

(1.4.4) Remark. Let $\alpha \in \Phi$ be a root and let $R \in \partial^2\alpha$. Then there exist $c \in \alpha \cap R$ and $d \in (-\alpha) \cap R$. By considering a minimal gallery from c to d, there exist adjacent chambers $c' \in \alpha \cap R$ and $d' \in (-\alpha) \cap R$. In particular, $\{c', d'\} \in \partial\alpha$. This shows that for all $R \in \partial^2\alpha$ there exists $P \in \partial\alpha$ such that $P \subseteq R$.

(1.4.5) Lemma. *Let $\alpha \neq \beta \in \Phi$ be two non-opposite roots and let $R \neq T \in \partial^2\alpha \cap \partial^2\beta$. Then R and T are parallel.*

Proof. As $R, T \in \partial^2\alpha \cap \partial^2\beta$, there exist panels $P_1, Q_1 \in \partial\alpha$ and $P_2, Q_2 \in \partial\beta$ such that $P_1, P_2 \subseteq R$ and $Q_1, Q_2 \subseteq T$ by the previous remark. By Lemma (1.4.2) the panels P_i, Q_i are parallel for each $i \in \{1,2\}$. [18, Lemma 17] yields that $P_i, \operatorname{proj}_T P_i$ are parallel and hence $\operatorname{proj}_T P_1 \in \partial\alpha, \operatorname{proj}_T P_2 \in \partial\beta$ by Lemma (1.4.2). As $\alpha \neq \pm\beta$, we deduce $\operatorname{proj}_T P_1 \neq \operatorname{proj}_T P_2$ and hence $\operatorname{proj}_T P_i \subseteq \operatorname{proj}_T R$ for each $i \in \{1,2\}$. Since $\operatorname{proj}_T R$ is a residue contained in T containing two different panels, we deduce that $\operatorname{proj}_T R$ is not a panel and hence $\operatorname{proj}_T R = T$. Using similar arguments, we obtain $\operatorname{proj}_R T = R$ and R, T are parallel. □

(1.4.6) Lemma. *Assume that $\langle J \rangle = \infty$ for all $J \subseteq S$ containing three elements and let $\alpha \neq \beta \in \Phi$ be two non-opposite roots. Then we have $|\partial^2\alpha \cap \partial^2\beta| \leq 1$.*

Proof. Assume that there exist $R \neq T \in \partial^2\alpha \cap \partial\beta$. By the previous lemma, R and T are parallel. But this is a contradiction to Lemma (1.4.2), as there exist no spherical residues of rank 3 by assumption and the claim follows. □

(1.4.7) Lemma. *Let $\alpha \neq \beta \in \Phi$ be two non-opposite roots. Then the following are equivalent:*

(i) $\{\alpha, \beta\}$ or $\{-\alpha, \beta\}$ is nested.

(ii) We have $o(r_\alpha r_\beta) = \infty$.

(iii) We have $\partial^2\alpha \cap \partial^2\beta = \emptyset$.

Proof. The implication $(i) \Rightarrow (ii)$ follows exactly as in [2, Proposition 3.165]. Now suppose (ii) and assume that there exists $R \in \partial^2\alpha \cap \partial^2\beta$. As R is finite, there exists $k \in \mathbb{N}$ such that $(r_\alpha r_\beta)^k$ fixes a chamber, i.e. $(r_\alpha r_\beta)^k w = (r_\alpha r_\beta)^k(w) = w$ for some $w \in W$. But this implies $(r_\alpha r_\beta)^k = 1$. As $o(r_\alpha r_\beta) = \infty$, we obtain a contradiction. Now suppose that non of $\{\alpha, \beta\}, \{-\alpha, \beta\}$ is nested. In particular, we have $\alpha \not\subseteq \beta, (-\alpha) \not\subseteq (-\beta)$ as well as $(-\alpha) \not\subseteq \beta, \alpha \not\subseteq (-\beta)$. This implies that non of $\alpha \cap (-\beta), (-\alpha) \cap \beta, (-\alpha) \cap (-\beta), \alpha \cap \beta$ is the empty set. By [37, Proposition 29.24] there exists $R \in \partial^2\alpha \cap \partial^2\beta$ and we are done. □

(1.4.8) Lemma. *Let $\alpha, \beta, \gamma \in \Phi$ be three pairwise distinct and pairwise non-opposite roots. Suppose that $\partial^2\alpha \cap \partial^2\beta \cap \partial^2\gamma \neq \emptyset$. Then the following hold:*

(a) $\partial^2\alpha \cap \partial^2\beta = \partial^2\alpha \cap \partial^2\gamma$;

(b) $((\alpha, \beta) \cup (-\alpha, \beta)) \cap \{\gamma, -\gamma\} \neq \emptyset$.

Proof. Let $R \in \partial^2\alpha \cap \partial^2\beta \cap \partial^2\gamma$ be a residue and let $\delta \in \{\beta, \gamma\}$. It suffices to show that for each $R \neq T \in \partial^2\alpha \cap \partial^2\delta$ we have $T \in \partial^2\alpha \cap \partial^2\beta \cap \partial^2\gamma$. Let $R \neq T \in \partial^2\alpha \cap \partial^2\delta$. Using Lemma (1.4.5), we deduce that R and T are parallel. Then Lemma (1.4.2) implies that a reflection of $\Sigma(W, S)$ stabilizes R if and only if it stabilizes T. As $r_\alpha, r_\beta, r_\gamma$ stabilize R, they also stabilize T and Assertion (a) follows.

Assume $(\alpha, \beta) \cap \{\gamma, -\gamma\} = \emptyset = (-\alpha, \beta) \cap \{\gamma, -\gamma\}$. This implies that non of $\alpha \cap \beta$ and $(-\alpha) \cap \beta$ is contained in γ or $-\gamma$, respectively. This implies that there exist $x, x' \in \alpha \cap \beta$ with $x \in (-\gamma), x' \in \gamma$. As roots as convex, [2, Lemma 5.45] yields $\operatorname{proj}_R x \in \alpha \cap \beta \cap (-\gamma)$ and $\operatorname{proj}_R x' \in \alpha \cap \beta \cap \gamma$. Similarly, there exist $y, y' \in (-\alpha) \cap \beta$ with $y \in (-\gamma), y' \in \gamma$ and $\operatorname{proj}_R y \in (-\alpha) \cap \beta \cap (-\gamma), \operatorname{proj}_R y' \in (-\alpha) \cap \beta \cap \gamma$. As residues and roots are convex, there exist $P, Q \in \partial\gamma$ such that $P, Q \subseteq R, P \subseteq \alpha \cap \beta$ and $Q \subseteq (-\alpha) \cap \beta$. As $P \subseteq \alpha$ and $Q \subseteq (-\alpha)$, we have $P \neq Q$ and Lemma (1.2.2) implies that there exist $p \in P, q \in Q$ which are opposite in R. Using [36, Proposition 5.4], every chamber in R lies on a minimal gallery from p to q. As roots are convex and $p, q \in \beta$, we infer $R \subseteq \beta$, which is a contradiction to $R \in \partial^2\beta$. □

1.5. Reflection and combinatorial triangles in $\Sigma(W, S)$

A *reflection triangle* is a set T of three reflections such that the order of tt' is finite for all $t, t' \in T$ and such that $\bigcap_{t \in T} \partial^2 \beta_t = \emptyset$, where β_t is one of the two roots associated with the reflection t. Note that $\partial^2 \beta_t = \partial^2(-\beta_t)$. A set of three roots T is called *combinatorial triangle* (or simply *triangle*) if the following hold:

(CT1) The set $\{r_\alpha \mid \alpha \in T\}$ is a reflection triangle.

(CT2) For each $\alpha \in T$, there exists $\sigma \in \partial^2 \beta \cap \partial^2 \gamma$ such that $\sigma \subseteq \alpha$, where $\{\beta, \gamma\} = T \backslash \{\alpha\}$.

(1.5.1) Remark. Let R be a reflection triangle. Then there exist three roots $\beta_1, \beta_2, \beta_3 \in \Phi$ such that $R = \{r_{\beta_1}, r_{\beta_2}, r_{\beta_3}\}$. Let $\{i, j, k\} = \{1, 2, 3\}$. As $o(r_{\beta_i} r_{\beta_j}) < \infty$, there exists $\sigma_k \in \partial^2 \beta_i \cap \partial^2 \beta_j$ by Lemma (1.4.7). Since R is a reflection triangle, we have $\sigma_k \notin \partial^2 \beta_k$ and Lemma (1.4.1) yields $\sigma_k \subseteq \beta_k$ or $\sigma_k \subseteq -\beta_k$. Define $\alpha_k := \varepsilon_k \beta_k$, where $\varepsilon_k \in \{+, -\}$ and $\sigma_k \subseteq \varepsilon_k \beta_k$. Then $\{\alpha_1, \alpha_2, \alpha_3\}$ is a triangle, which induces the reflection triangle R.

(1.5.2) Lemma. *Let $\alpha \neq \beta \in \Phi$ be two non-opposite roots such that $o(r_\alpha r_\beta) < \infty$ and let $\gamma \in (\alpha, \beta)$. Then $\partial^2 \alpha \cap \partial^2 \beta \cap \partial^2 \gamma \neq \emptyset$ and $o(r_\alpha r_\gamma), o(r_\beta r_\gamma) < \infty$.*

Proof. By Lemma (1.4.7) there exists $R \in \partial^2 \alpha \cap \partial^2 \beta$. We deduce $\emptyset \neq R \cap \alpha \cap \beta \subseteq \gamma$ and $\emptyset \neq R \cap (-\alpha) \cap (-\beta) \subseteq (-\gamma)$. If follows from Lemma (1.4.1) that $R \in \partial^2 \gamma$. In particular, $R \in \partial^2 \alpha \cap \partial^2 \beta \cap \partial^2 \gamma$. We deduce $o(r_\alpha r_\gamma), o(r_\beta r_\gamma) < \infty$ from Lemma (1.4.7). □

(1.5.3) Lemma. *Assume that (W, S) is 2-spherical and cyclic hyperbolic. Then any triangle T is a chamber, i.e. $|\bigcap_{\alpha \in T} \alpha| = 1$. In particular, $(-\alpha, \beta) = \emptyset$ for all $\alpha \neq \beta \in T$.*

Proof. Let $T = \{\alpha, \beta, \gamma\}$ and let $R \in \partial^2 \alpha \cap \partial^2 \beta$ be a residue such that $R \subseteq \gamma$. Suppose that $R \cap \alpha \cap \beta$ contains more than one chamber. Let c, d be adjacent and contained in $\alpha \cap \beta \cap R$ and let $\delta \in \Phi$ be a root with $\{c, d\} \in \partial \delta$. Then Lemma (1.4.8)(*b*) implies $(-\alpha, \beta) \cap \{\delta, -\delta\} \neq \emptyset$ and $\{r_\alpha, r_\beta, r_\gamma\}, \{r_\alpha, r_\gamma, r_\delta\}, \{r_\beta, r_\gamma, r_\delta\}$ are reflection triangles. But this is a contradiction to the classification in [20, Figure 8 in §5.1]. Thus $R \cap \alpha \cap \beta$ does only contain one chamber c. Assume that $|\bigcap_{\alpha \in T} \alpha| > 1$. Then there exists $\delta \in \Phi$ which contains c but not a neighbour contained in $\bigcap_{\alpha \in T} \alpha$. Again, this is a contradiction to the classification in [20, Figure 8 in §5.1]. This implies $|\bigcap_{\alpha \in T} \alpha| = 1$. Let $c \in \bigcap_{\alpha \in T} \alpha$, let $s \neq t \in S$, let $\alpha \in T$ be the root which does not contain the s-neighbour of c and let $\beta \in T$ be the root which does not contain the t-neighbour of c. Then $R := R_{\{s,t\}}(c) \in \partial^2 \alpha \cap \partial^2 \beta$. As $T = \{\alpha, \beta, \gamma\}$ is a triangle, we have $R \notin \partial^2 \gamma$. We deduce from $c \in R \cap \gamma$ that $R \subseteq \gamma$. This shows $(-\alpha, \beta) = \emptyset$ for all $\alpha, \beta \in T$. □

(1.5.4) Proposition. *Assume that (W, S) is 2-spherical and cyclic hyperbolic. Let $R \neq T$ be two residues of rank 2 such that $P := R \cap T$ is a panel. If $\ell(1_W, \operatorname{proj}_R 1_W) < \ell(1_W, \operatorname{proj}_T 1_W)$, then $\operatorname{proj}_T 1_W = \operatorname{proj}_P 1_W$.*

Proof. We let $\alpha \in \Phi_+$ be the root with $P \in \partial \alpha$. Let $(c_0 = 1_W, \ldots, c_k = \operatorname{proj}_R c_0, \ldots, c_{k'} = \operatorname{proj}_P c_0)$ be a minimal gallery from c_0 to $\operatorname{proj}_P c_0$ with $c_k, \ldots, c_{k'} \in R$ and we assume $\operatorname{proj}_T c_0 \neq \operatorname{proj}_P c_0$. Then we have $k' > \ell(1_W, \operatorname{proj}_T 1_W) > \ell(1_W, \operatorname{proj}_R 1_W) = k$. Let $(d_0 = c_0, \ldots, d_m = \operatorname{proj}_T c_0, \ldots, d_{m'} = \operatorname{proj}_P c_0)$ be a minimal gallery from c_0 to $\operatorname{proj}_P c_0$ with $d_m, \ldots, d_{m'} \in T$. We define $H := (d_0, \ldots, d_{m+1})$ and $\beta := \alpha_H$. Then we have $T \in \partial^2 \alpha \cap \partial^2 \beta$ and, as a minimal gallery crosses a wall at most once, we deduce $\alpha \neq \beta$. Note that the wall $\partial \beta$ is crossed by the minimal gallery $(c_0, \ldots, c_{k'})$. Since $R \neq T, T \in \partial^2 \alpha \cap \partial^2 \beta, R \in \partial^2 \alpha$ and $\alpha \neq \pm \beta$, Lemma (1.4.6) implies $R \notin \partial^2 \beta$. We define $\gamma := \alpha_{(c_0, \ldots, c_{k+1})}$. As $R \notin \partial^2 \beta$, we obtain that $\partial \beta$ is crossed by $(c_0, \ldots, c_k)$. As $k < k'$, we have $\operatorname{proj}_R 1_W \neq \operatorname{proj}_P 1_W$ and hence $\alpha \neq \gamma$. As $\alpha, \gamma \in \Phi_+$, we have $\alpha \neq \pm \gamma$. Assume that $o(r_\beta r_\gamma) = \infty$. We deduce $\beta \subseteq \gamma$.

But $\partial\gamma$ has to be crossed by the gallery $(d_0,\dots,d_{m'})$. Since $R \in \partial^2\alpha \cap \partial^2\gamma, T \in \partial^2\alpha$ and $\alpha \neq \pm\gamma$, we have $T \notin \partial\gamma^2$ by Lemma (1.4.6) as before. This implies that $(d_0,\dots,d_m)$ crosses the wall $\partial\beta$ and hence $\gamma \subseteq \beta$. This yields a contradiction and we have $o(r_\beta r_\gamma) < \infty$. As $R \in \partial^2\alpha \cap \partial^2\gamma, R \notin \partial^2\beta$, Lemma (1.4.8)$(a)$ implies $\partial^2\alpha \cap \partial^2\beta \cap \partial^2\gamma = \emptyset$ and hence $\{r_\alpha, r_\beta, r_\gamma\}$ is a reflection triangle.

As $T \in \partial^2\alpha \cap \partial^2\beta, T \notin \partial^2\gamma$ and $\operatorname{proj}_P 1_W \in T \cap (-\gamma)$, we have $T \subseteq (-\gamma)$. As $R \in \partial^2\alpha \cap \partial^2\gamma, R \notin \partial^2\beta$ and $\operatorname{proj}_P 1_W \in R \cap (-\beta)$, we have $R \subseteq (-\beta)$. Let $1 \leq i \leq k$ be such that $\{c_{i-1}, c_i\} \in \partial\beta$. Note that $\{d_m, d_{m+1}\} \in \partial\beta, d_{m+1} \in (-\beta) \cap T \cap \alpha \subseteq (-\gamma)$ and $c_i \in (-\beta) \cap \gamma$. By Lemma (1.4.3) there exists a minimal gallery $(e_0 = d_{m+1}, \dots, e_z = c_i)$ such that $e_j \in \mathcal{C}(\partial^2\beta)$. As $d_{m+1} \in (-\gamma)$ and $c_i \in \gamma$, there exists $1 \leq p \leq z$ such that $e_{p-1} \in (-\gamma)$ and $e_p \in \gamma$. Again by Lemma (1.4.3) there exists $L \in \partial^2\beta$ such that $\{e_{p-1}, e_p\} \in L$. Then $L \in \partial^2\beta \cap \partial^2\gamma$ and as a minimal gallery crosses a wall at most once, we have $e_{p-1} \in L \cap \alpha$ and, as $\{r_\alpha, r_\beta, r_\gamma\}$ is a reflection triangle and $L \notin \partial^2\alpha$, we obtain $L \subseteq \alpha$. This implies that $\{\alpha, -\beta, -\gamma\}$ is a triangle and hence $(\alpha, \gamma) = \emptyset$ by Lemma (1.5.3). In particular, $k+1 = k'$ and $\ell(1_W, \operatorname{proj}_R 1_W) = \ell(1_W, \operatorname{proj}_P 1_W) - 1 \geq \ell(1_W, \operatorname{proj}_T 1_W)$. Since $\ell(1_W, \operatorname{proj}_R 1_W) < \ell(1_W, \operatorname{proj}_T 1_W)$ holds by assumption, this yields a contradiction and we have $\operatorname{proj}_T 1_W = \operatorname{proj}_P 1_W$. □

(1.5.5) Corollary. *Assume that (W,S) is 2-spherical and cyclic hyperbolic. Let $\alpha \in \Phi_+$ be a root and let $P, Q \in \partial\alpha$. Let $P_0 = P, \dots, P_n = Q$ and $R_1, \dots, R_n$ as in Lemma (1.4.2). If* $\operatorname{proj}_{R_i} 1_W = \operatorname{proj}_{P_{i-1}} 1_W$ *for some $1 \leq i \leq n$, then* $\operatorname{proj}_{R_n} 1_W = \operatorname{proj}_{P_{n-1}} 1_W$.

Proof. We will show the hypothesis by induction on $n-i$. If $n-i = 0$ there is nothing to show. Thus we suppose $n-i > 0$. Let $(d_0 = 1_W, \dots, d_m = \operatorname{proj}_{R_i} d_0)$ be a minimal gallery of type $(t_1, \dots, t_m)$, let $w := t_1 \cdots t_m$ and let J_i be the type of R_i. Then $w = \operatorname{proj}_{R_i} 1_W = \operatorname{proj}_{P_{i-1}} 1_W \in P_{i-1}$. As $P_{i-1} \neq P_i$ are contained and opposite in R_i by Lemma (1.4.2), there exists $w' \in P_i$ such that $w \in P_{i-1}, w'$ are opposite in R_i, i.e. $w' = wr_{J_i}$. Let $s \in J_{i+1} \backslash J_i$. As $w = \operatorname{proj}_{R_i} 1_W$, we deduce $\ell(wr_{J_i}) = \ell(w) + \ell(r_{J_i})$. Since W is not of spherical type, we obtain $\ell(wr_{J_i}s) = \ell(w) + \ell(r_{J_i}) + 1$. Let $t \in S$ be such that $J_i \cap J_{i+1} = \{t\}$. Then $R_i \cap R_{i+1} = P_i = \mathcal{P}_t(w') = \mathcal{P}_t(wr_{J_i})$. Assume that $\ell((\operatorname{proj}_{P_i} 1_W)s) = \ell(\operatorname{proj}_{P_i} 1_W) - 1$. Let $(c_0 = w, \dots, c_k = \operatorname{proj}_{P_i} 1_W)$ be a minimal gallery contained in R_i. We deduce that $\ell(c_i s) = \ell(c_i) - 1$ for each $0 \leq i \leq k$. Let $r \in S$ be such that $\delta(c_1, c_2) = r$. As $m_{uv} \neq 2$ for all $u \neq v \in S$, we deduce $\ell(\operatorname{proj}_{R_i} 1_W) > \ell(\operatorname{proj}_{R_{\{r,s\}}(c_1)} 1_W)$. Applying the previous proposition to R_i and $R_{\{r,s\}}(c_1)$ we obtain a contradiction. Thus $\ell((\operatorname{proj}_{P_i} 1_W)s) = \ell(\operatorname{proj}_{P_i} 1_W) + 1$ and hence $\ell(1_W, \operatorname{proj}_{R_i} 1_W) < \ell(1_W, \operatorname{proj}_{R_{i+1}} 1_W)$. By Proposition (1.5.4) we infer $\operatorname{proj}_{R_{i+1}} 1_W = \operatorname{proj}_{P_i} 1_W$. Using induction the claim follows. □

(1.5.6) Lemma. *Assume that (W,S) is of type $(4,4,4)$. Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a triangle and let $\beta \in (\alpha_1, \alpha_2)$. Then $o(r_\beta r_{\alpha_3}) = \infty$. In particular, we have $-\beta \subseteq \alpha_3, -\alpha_3 \subseteq \beta$ and $(-\beta, \alpha_3) = \emptyset = (-\alpha_3, \beta)$.*

Proof. Since $\{\alpha_1, \alpha_2, \alpha_3\}$ is a triangle, there exist $R_1 \in \partial^2\alpha_2 \cap \partial^2\alpha_3$ with $R_1 \subseteq \alpha_1$, $R_2 \in \partial^2\alpha_1 \cap \partial^2\alpha_3$ with $R_2 \subseteq \alpha_2$ and $R_3 \in \partial^2\alpha_1 \cap \partial^2\alpha_2$ with $R_3 \subseteq \alpha_3$. Let $\beta \in (\alpha_1, \alpha_2)$ be a root and let $\{i, j\} = \{1, 2\}$. Then Lemma (1.5.2) implies $\partial^2\alpha_1 \cap \partial^2\alpha_2 \cap \partial^2\beta \neq \emptyset$ and $o(r_{\alpha_i} r_\beta) < \infty$. Lemma (1.4.8)$(a)$ yields $\partial^2\alpha_1 \cap \partial\alpha_2 = \partial^2\alpha_i \cap \partial^2\beta$ and $R_j \notin \partial^2\beta$ (note that $R_j \in \partial^2\alpha_i$ but $R_j \notin \partial^2\alpha_j$).

We assume that $o(r_\beta r_{\alpha_3}) < \infty$. Then $\{r_{\alpha_i}, r_\beta, r_{\alpha_3}\}$ is a reflection triangle. Since $\emptyset \neq \alpha_i \cap R_j \subseteq \alpha_i \cap \alpha_j \subseteq \beta$ and $R_j \notin \partial^2\beta$, we deduce $R_j \subseteq \beta$. Thus $\{\alpha_i, \alpha_3, \beta\}$ or $\{-\alpha_i, \alpha_3, \beta\}$ is a triangle. Assume that $\{\alpha_i, \alpha_3, \beta\}$ is a triangle. Then $\alpha_1 \cap \alpha_2 \cap \alpha_3 \subseteq \alpha_i \cap \beta \cap \alpha_3$. Since $\beta \neq \alpha_j$, Lemma (1.5.3) yields a contradiction. Thus $\{-\alpha_i, \alpha_3, \beta\}$ is a triangle, i.e. $\{-\alpha_1, \alpha_3, \beta\}$ and

$\{-\alpha_2, \alpha_3, \beta\}$ are triangles. But then $(\alpha_1, \beta) = \emptyset = (\beta, \alpha_2)$, which is a contradiction as the type is $(4, 4, 4)$.

Thus $o(r_\beta r_{\alpha_3}) = \infty$. As $\emptyset \neq R_3 \cap (-\beta) \subseteq \alpha_3$, we have $(-\beta) \cap \alpha_3 \neq \emptyset$. As $\emptyset \neq R_2 \cap \alpha_1 \cap (-\alpha_3) \subseteq \alpha_1 \cap \alpha_2 \subseteq \beta$, we have $(-\alpha_3) \cap \beta \neq \emptyset$ and $\{-\alpha_3, \beta\}$ is a prenilpotent pair. As $\bigcap_{i=1}^3 \alpha_i \subseteq \alpha_1 \cap \alpha_2 \subseteq \beta$ and $\bigcap_{i=1}^3 \alpha_i \not\subseteq (-\alpha_3)$, we deduce $(-\alpha_3) \subseteq \beta$ and hence also $(-\beta) \subseteq \alpha_3$. Let $\{x, y\} \in \partial\alpha_3$ be such that $\bigcap_{i=1}^3 \alpha_i = \{y\}$ and let $R \in \partial^2\alpha_1 \cap \partial^2\alpha_2$ be the residue containing y. Let $d \in R$ be opposite to y in R and let $(c_0 = x, c_1 = y, \dots, c_n = d)$ be a minimal gallery. Then $c_i \in R$ for each $1 \leq i \leq n$. Let $(\beta_1, \dots, \beta_n)$ be the sequence of roots crossed by $(c_0, \dots, c_n)$. Then $\beta_1 = -\alpha_3$ and $o(r_{\beta_i} r_\beta) < \infty$ for each $2 \leq i \leq n$ by Lemma (1.4.7). Assume $(-\alpha_3, \beta) \neq \emptyset$. [2, Lemma 3.69] implies that for each $\gamma \in (-\alpha_3, \beta)$ there exists $2 \leq i \leq n$ with $\gamma = \beta_i$. As $\gamma \subsetneq \beta$, this is a contradiction and hence $(-\alpha_3, \beta) = \emptyset$. $\square$

1.6. Twin buildings

Let $\Delta_+ = (\mathcal{C}_+, \delta_+), \Delta_- = (\mathcal{C}_-, \delta_-)$ be two buildings of the same type (W, S). A *codistance* (or a *twinning*) between Δ_+ and Δ_- is a mapping $\delta_* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$ satisfying the following axioms, where $\varepsilon \in \{+, -\}, x \in \mathcal{C}_\varepsilon, y \in \mathcal{C}_{-\varepsilon}$ and $w = \delta_*(x, y)$:

(Tw1) $\delta_*(y, x) = w^{-1}$;

(Tw2) if $z \in \mathcal{C}_{-\varepsilon}$ is such that $s := \delta_{-\varepsilon}(y, z) \in S$ and $\ell(ws) = \ell(w) - 1$, then $\delta_*(x, z) = ws$;

(Tw3) if $s \in S$, there exists $z \in \mathcal{C}_{-\varepsilon}$ such that $\delta_{-\varepsilon}(y, z) = s$ and $\delta_*(x, z) = ws$.

A *twin building of type* (W, S) is a triple $\Delta = (\Delta_+, \Delta_-, \delta_*)$ where $\Delta_+ = (\mathcal{C}_+, \delta_+), \Delta_- = (\mathcal{C}_-, \delta_-)$ are buildings of type (W, S) and where δ_* is a twinning between Δ_+ and Δ_-.

We put $\mathcal{C} := \mathcal{C}_+ \cup \mathcal{C}_-$ and define the distance function $\delta : \mathcal{C} \times \mathcal{C} \to W$ by setting $\delta(x, y) := \delta_+(x, y)$ (resp. $\delta_-(x, y), \delta_*(x, y)$) if $x, y \in \mathcal{C}_+$ (resp. $x, y \in \mathcal{C}_-, (x, y) \in \mathcal{C}_\varepsilon \times \mathcal{C}_{-\varepsilon}$ for some $\varepsilon \in \{+, -\}$).

Given $x, y \in \mathcal{C}$, we put $\ell(x, y) := \ell(\delta(x, y))$. If $\varepsilon \in \{+, -\}$ and $x, y \in \mathcal{C}_\varepsilon$, then we put $\ell_\varepsilon(x, y) := \ell(\delta_\varepsilon(x, y))$ and for $(x, y) \in \mathcal{C}_\varepsilon \times \mathcal{C}_{-\varepsilon}$ we put $\ell_\star(x, y) := \ell(\delta_*(x, y))$.

Let $\varepsilon \in \{+, -\}$. For $x \in \mathcal{C}_\varepsilon$ we put $x^{\mathrm{op}} := \{y \in \mathcal{C}_{-\varepsilon} \mid \delta_*(x, y) = 1_W\}$. It is a direct consequence of (Tw1) that $y \in x^{\mathrm{op}}$ if and only if $x \in y^{\mathrm{op}}$ for any pair $(x, y) \in \mathcal{C}_\varepsilon \times \mathcal{C}_{-\varepsilon}$. If $y \in x^{\mathrm{op}}$ then we say that y is *opposite* to x or that (x, y) *is a pair of opposite chambers*.

A *residue* (resp. *panel*) of Δ is a residue (resp. panel) of Δ_+ or Δ_-; given a residue $R \subseteq \mathcal{C}$ then we define its type and rank as before. The twin building Δ is called *thick* if Δ_+ and Δ_- are thick.

Let $\varepsilon \in \{+, -\}$, let J be a spherical subset of S and let R be a J-residue of Δ_ε. For every chamber $x \in \mathcal{C}_{-\varepsilon}$ there exists a unique chamber $z \in R$ such that $\ell_\star(x, y) = \ell_\star(x, z) - \ell_\varepsilon(z, y)$ holds for each chamber $y \in R$ (cf. [2, Lemma 5.149]). The chamber z is called the *projection of x onto R*; it will be denoted by $\mathrm{proj}_R\, x$. Moreover, if $z = \mathrm{proj}_R\, x$ we have $\delta_*(x, y) = \delta_*(x, z)\delta_\varepsilon(z, y)$ for each $y \in R$.

Let $\Sigma_+ \subseteq \mathcal{C}_+$ and $\Sigma_- \subseteq \mathcal{C}_-$ be apartments of Δ_+ and Δ_-, respectively. Then the set $\Sigma := \Sigma_+ \cup \Sigma_-$ is called *twin apartment* if $|x^{\mathrm{op}} \cap \Sigma| = 1$ holds for each $x \in \Sigma$. If (x, y) is a pair of opposite chambers, then there exists a unique twin apartment containing x and y. We will denote it by $A(x, y)$. It is a fact that $A(x, y) = \{z \in \mathcal{C} \mid \delta(x, z) = \delta(y, z)\}$ (cf. [2, Proposition 5.179(1)]).

An *automorphism* of Δ is a bijection $\varphi : \mathcal{C} \to \mathcal{C}$ such that φ preserves the sign, the distance functions δ_ε and the codistance δ_*.

1.7. Root group data

An *RGD-system of type* (W,S) is a pair $\mathcal{D} = \big(G, (U_\alpha)_{\alpha\in\Phi}\big)$ consisting of a group G together with a family of subgroups U_α (called *root groups*) indexed by the set of roots Φ, which satisfies the following axioms, where $H := \bigcap_{\alpha\in\Phi} N_G(U_\alpha)$ and $U_\varepsilon := \langle U_\alpha \mid \alpha \in \Phi_\varepsilon\rangle$ for $\varepsilon \in \{+,-\}$:

(RGD0) For each $\alpha \in \Phi$, we have $U_\alpha \neq \{1\}$.

(RGD1) For each prenilpotent pair $\{\alpha,\beta\} \subseteq \Phi$, the commutator group $[U_\alpha, U_\beta]$ is contained in the group $U_{(\alpha,\beta)} := \langle U_\gamma \mid \gamma \in (\alpha,\beta)\rangle$.

(RGD2) For every $s \in S$ and each $u \in U_{\alpha_s}\backslash\{1\}$, there exist $u', u'' \in U_{-\alpha_s}$ such that the product $m(u) := u'uu''$ conjugates U_β onto $U_{s\beta}$ for each $\beta \in \Phi$.

(RGD3) For each $s \in S$, the group $U_{-\alpha_s}$ is not contained in U_+.

(RGD4) $G = H\langle U_\alpha \mid \alpha \in \Phi\rangle$.

Let $w \in W, G = (c_0,\dots,c_k) \in \mathrm{Min}(w)$ and let $(\alpha_1,\dots,\alpha_k)$ be the sequence of roots crossed by G. Then we define the group $U_w := U_{\alpha_1}\cdots U_{\alpha_k}$. We note that the group U_w does not depend on $G \in \mathrm{Min}(w)$. Following [34, Remark (1) on p. 258] we have $m_{st} \in \{2,3,4,6,8,\infty\}$ for all $s \neq t \in S$. An RGD-system $\mathcal{D} = (G,(U_\alpha)_{\alpha\in\Phi})$ is said to be *over* $\mathbb{F}_2$ if every root group has cardinality 2.

Let $\mathcal{D} = (G,(U_\alpha)_{\alpha\in\Phi})$ be an RGD-system of type (W,S) and let $H := \bigcap_{\alpha\in\Phi} N_G(U_\alpha), B_\varepsilon = H\langle U_\alpha \mid \alpha \in \Phi_\varepsilon\rangle$ for $\varepsilon \in \{+,-\}$. It follows from [2, Theorem 8.80] that there exists an *associated* twin building $\Delta(\mathcal{D}) = (\Delta(\mathcal{D})_+, \Delta(\mathcal{D})_-, \delta_*)$ of type (W,S) such that $\Delta(\mathcal{D})_\varepsilon = (G/B_\varepsilon, \delta_\varepsilon)$ for $\varepsilon \in \{+,-\}$ and G acts on $\Delta(\mathcal{D})$ via left multiplication. There is a distinguished pair of opposite chambers in $\Delta(\mathcal{D})$ corresponding to the subgroups B_ε for $\varepsilon \in \{+,-\}$. We will denote this pair by (c_+, c_-).

(1.7.1) Example. Let (W,S) be spherical and of rank 2 and let $\mathcal{D} = (G,(U_\alpha)_{\alpha\in\Phi})$ be an RGD-system of type (W,S) over $\mathbb{F}_2$. For $S = \{s,t\}$ we deduce $m_{st} \in \{2,3,4,6\}$, since in an octagon there exists a root group of cardinality at least 4 (cf. [35, (16.9) and (17.7)]). Let $G \in \mathrm{Min}(r_S)$ and let $(\beta_1,\dots,\beta_m)$ be the sequence of roots crossed by G, where $m = m_{st}$. Then $\Phi_+ = \{\beta_1,\dots,\beta_m\}$ and β_1, β_m are the two simple roots. We let $U_{\beta_i} = \langle u_i\rangle$. For all $1 \leq i < j \leq m$ we will define subsets $M_{\{\beta_i,\beta_j\}} \subseteq (\beta_i,\beta_j)$ which correspond to the commutator relations. If $[u_i,u_j] = 1$, we put $M_{\{\beta_i,\beta_j\}} := \emptyset$. We now state all non-trivial commutator relations depending on the type (W,S) (cf. [35, Ch. 16, 17]):

- $A_1 \times A_1$: There are no non-trivial commutator relations.
- A_2: There is only one non-trivial commutator relation, namely $[u_1,u_3] = u_2$ (cf. [35, 16.1, 17.2]). We define $M_{\{\beta_1,\beta_3\}} = \{\beta_2\}$.
- $B_2 = C_2$: As in the case of A_2 there is only one non-trivial commutator relation, namely $[u_1,u_4] = u_2u_3$ (cf. [35, 16.2, 17.4] and [27, 5.2.3]). We define $M_{\{\beta_1,\beta_4\}} := \{\beta_2,\beta_3\}$.
- G_2: We have the following non-trivial commutator relations (cf. [35, 15.20, 16.8, 17.6]):

$$[u_1,u_3] = u_2, \quad [u_3,u_5] = u_4, \quad [u_1,u_5] = u_2u_4, \quad [u_2,u_6] = u_4, \quad [u_1,u_6] = u_2u_3u_4u_5$$

  We define $M_{\{\beta_1,\beta_3\}} := \{\beta_2\}, M_{\{\beta_3,\beta_5\}} := \{\beta_4\}, M_{\{\beta_1,\beta_5\}} := \{\beta_2,\beta_4\}, M_{\{\beta_2,\beta_6\}} := \{\beta_4\}$ and $M_{\{\beta_1,\beta_6\}} := \{\beta_2,\beta_3,\beta_4,\beta_5\}$.

Note that for $i < j$ we have $[u_i, u_j] = \prod_{\gamma \in M_{\{\beta_i,\beta_j\}}} u_\gamma$, where the order of the product is taken via the order of the indices. For $i > j$ we have $[u_i, u_j] = \prod_{\gamma \in M_{\{\beta_i,\beta_j\}}} u_\gamma$, where the order of the product is taken in the inverse order. Thus $M_{\{\beta_i,\beta_j\}}$ contains all information about the commutators $[u_i, u_j]$ and $[u_j, u_i]$.

Property (FPRS)

Let $(G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD-system and let $\Delta(\mathcal{D}) = (\Delta(\mathcal{D})_+, \Delta(\mathcal{D})_-, \delta_*)$ be the associated twin building. For $\Gamma \leq G$ we define $r(\Gamma)$ to be the supremum of the set of all non-negative real numbers r such that Γ fixes pointwise the *closed ball* $B(c_+, r) := \{d \in \mathcal{C}_+ \mid \ell_+(c_+, d) \leq r\}$, where $\mathcal{C}_+$ is the set of chambers of $\Delta(\mathcal{D})_+$. In [16], Caprace and Rémy have introduced the following property, where $\ell(1_W, \alpha) := \min\{k \in \mathbb{N} \mid \exists d \in \alpha : \ell(1_W, d) = k\}$ for all roots $\alpha \in \Phi$:

(FPRS) Given any sequence of roots $(\alpha_n)_{n\geq 0}$ of Φ such that $\lim_{n\to\infty} \ell(1_W, \alpha_n) = \infty$, we have $\lim_{n\to\infty} r(U_{-\alpha_n}) = \infty$.

1.8. Graphs of groups

This subsection is based on [22, Section 2] and [32].

Following Serre, a *graph* Γ consists of a vertex set $V\Gamma$, an edge set $E\Gamma$, the inverse function ${}^{-1} : E\Gamma \to E\Gamma$ and two edge endpoint functions $o : E\Gamma \to V\Gamma, t : E\Gamma \to V\Gamma$ satisfying the following axioms:

(i) The function ${}^{-1}$ is a fixed-point free involution on $E\Gamma$;

(ii) For each $e \in E\Gamma$ we have $o(e) = t(e^{-1})$.

For an edge $e \in E\Gamma$ we call e^{-1} the *inverse edge* of e.

A *tree of groups* is a triple $\mathbb{G} = (T, (G_v)_{v\in V\Gamma}, (G_e)_{e\in E\Gamma})$ consisting of a finite tree T (i.e. VT and ET are finite), a family of *vertex groups* $(G_v)_{v\in VT}$ and a family of *edge groups* $(G_e)_{e\in ET}$. Every edge $e \in ET$ comes equipped with two *boundary monomorphisms* $\alpha_e : G_e \to G_{o(e)}$ and $\omega_e : G_e \to G_{t(e)}$. We assume that for each $e \in ET$ we have $G_{e^{-1}} = G_e, \alpha_{e^{-1}} = \omega_e$ and $\omega_{e^{-1}} = \alpha_e$. We let $G_T := \varinjlim \mathbb{G}$ be the direct limit of the inductive system formed by the vertex groups, edge groups and boundary monomorphisms and call G_T a *tree product*. A *sequence of groups* is a tree of groups where the underlying graph is a sequence. If the tree T is an edge, i.e. $VT = \{v, w\}$ and $ET = \{e, e^{-1}\}$, we will write $G_T = G_v \star_{G_e} G_w$. We extend this notation to arbitrary sequences T: if $VT = \{v_0, \ldots, v_n\}, ET = \{e_i, e_i^{-1} \mid 1 \leq i \leq n\}$ and $o(e_i) = v_{i-1}, t(e_i) = v_i$, then we will write $G_T = G_{v_0} \star_{G_{e_1}} G_{v_1} \star_{G_{e_2}} \cdots \star_{G_{e_n}} G_{v_n}$.

(1.8.1) Proposition. *Let $\mathbb{G} = (T, (G_v)_{v\in VT}, (G_e)_{e\in ET})$ be a tree of groups. If T is partitioned into subtrees whose tree products are $G_1, \ldots, G_n$ and the subtrees are contracted to vertices, then G_T is isomorphic to the tree product of the tree of groups whose vertex groups are the G_i and the edge groups are the G_e, where e is the unique edge which joins two subtrees. Moreover, $G_i \to G_T$ is injective.*

Proof. This is [23, Theorem 1]. □

(1.8.2) Remark. The next proposition is a special case of a more general result (cf. [22, Proposition 4.3]). As we only need a special case, we have reformulated the claim and its proof.

(1.8.3) Proposition. *Let T be a tree and let T' be a subtree of T. Moreover, we let $\mathbb{G} = (T, (G_v)_{v\in VT}, (G_e)_{e\in ET})$ and $\mathbb{H} = (T', (H_v)_{v\in VT'}, (H_e)_{e\in ET'})$ be two trees of groups and suppose the following:*

(i) For each $v \in VT'$ we have $H_v \leq G_v$.

(ii) For each $e \in ET'$ we have $\alpha_e^{-1}(H_{o(e)}) = \omega_e^{-1}(H_{t(e)})$.

(iii) For each $e \in ET'$ we have $H_e = \alpha_e^{-1}(H_{o(e)}) = \omega_e^{-1}(H_{t(e)})$.

Then the canonical homomorphism $\nu : H_{T'} \to G_T$ between the tree product $H_{T'}$ and the tree product G_T is injective. In particular, we have $\nu(H_{T'}) \cap G_v = H_v$ for each $v \in VT'$.

Proof. We use the notations from [22]. Let $\mathcal{B}$ be the $\mathbb{G}$-graph (cf. [22, Definition 3.1]) defined as follows: As graph-morphism choose the inclusion mapping $T' \to T$. The associated groups are given by H_v and we let $f_\alpha = 1 = f_\omega$ for all $f \in ET'$. By [22, Convention 3.2] each edge $f \in ET'$ has label $(1, f, 1)$ and each vertex $u \in VT'$ has label (H_u, u). We show that $\mathcal{B}$ is folded (cf. [22, Definition 4.1]). Since the inclusion mapping $T' \to T$ is injective, [22, Definition 4.1(i)] does not hold. Moreover, let $f \in ET'$ be an edge. Then f has label $(1, f, 1)$, $o(f)$ has label $(H_{o(f)}, o(f))$ and $t(f)$ has label $(H_{t(f)}, t(f))$. By assumption we have $\alpha_f^{-1}(H_{o(f)}) = \omega_e^{-1}(H_{t(f)})$ and hence [22, Definition 4.1(ii)] does also not hold. In particular, $\mathcal{B}$ is folded. Now [22, Lemma 4.2] implies that any $\mathbb{H}$-reduced $\mathbb{H}$-path is also $\mathbb{G}$-reduced. Now the claim follows from the normal form theorem [22, Proposition 2.4].

We should remark that in [22] they work with fundamental groups instead of tree products. But the fundamental group $\pi(\mathbb{A}, v_0)$ in [22] is equal to the group $\pi(G, T, v_0)$ in [32] and by [32, Proposition 20] this group is isomorphic to the corresponding tree product. □

(1.8.4) Corollary. *Let $\mathbb{G} = (T, (G_v)_{v\in VT}, (G_e)_{e\in ET})$ be a tree of groups and let $H_v \leq G_v$ for each $v \in VT$. Assume that $H_e := \alpha_e^{-1}(H_{o(e)}) = \omega_e^{-1}(H_{t(e)})$ for all $e \in ET$ and let $\mathbb{H} = (T, (H_v)_{v\in VT}, (H_e)_{e\in ET})$ be the associated tree of groups. Let T' be a subtree of T and let $\mathbb{L} = (T', (G_v)_{v\in VT'}, (G_e)_{e\in ET'})$, $\mathbb{K} = (T', (H_v)_{v\in VT'}, (H_e)_{e\in ET'})$. Then $H_T \cap L_{T'} = K_{T'}$ in G_T.*

Proof. Using Proposition (1.8.1) we deduce that $L_{T'} \leq G_T$ and $K_{T'} \leq H_T$. Using Proposition (1.8.3) we deduce $H_T \leq G_T$ and $K_{T'} \leq L_{T'}$. Using Proposition (1.8.1) again, we can contract the tree T' to a vertex. Then $L_{T'}$ is a vertex group containing $K_{T'}$. Let $e \in ET$ be an edge joining T' with a vertex of $VT \backslash VT'$ and suppose $o(e) \in T'$. As $\alpha_e(G_e) \leq G_{o(e)}$, the previous proposition yields $\alpha_e(G_e) \cap K_{T'} \leq G_{o(e)} \cap K_{T'} = H_{o(e)}$. This implies $\alpha_e^{-1}(K_{T'}) \leq \alpha_e^{-1}(H_{o(e)})$. As $H_e \leq \alpha_e^{-1}(K_{T'}) \leq \alpha_e^{-1}(H_{o(e)}) = H_e$, we deduce $\alpha_e^{-1}(K_{T'}) = H_e = \alpha_e^{-1}(H_{o(e)})$. We denote the tree products of the trees of groups $\mathbb{G}$ and $\mathbb{H}$, where T' is contracted to a vertex, by G' and H'. Using Proposition (1.8.3) the canonical homomorphism $\nu' : H' \to G'$ is injective and we have $\nu'(H') \cap L_{T'} = K_{T'}$ (note that $L_{T'}$ is a vertex group of G'). This finishes the claim. □

(1.8.5) Corollary. *Let A, B, C be groups and let $C \to A, C \to B$ be two monomorphisms. Then $A \cap B = C$ in $A \star_C B$.*

Proof. Using Proposition (1.8.3) we have a monomorphism $A \cong A \star_C C \to A \star_C B$ and $A \cap B = C$. □

(1.8.6) Remark. Let A', A, B, C be groups, let $\alpha : C \to A, \beta : C \to B$ and $\alpha' : C \to A'$ be monomorphisms and let $\varphi : A \to A'$ be an isomorphism. If $\alpha' = \varphi \circ \alpha$, then the amalgamated products $A \star_C B$ and $A' \star_C B$ are isomorphic. One can prove this by constructing two unique

homomorphisms $A \star_C B \to A' \star_C B$ and $A' \star_C B \to A \star_C B$ such that the concatenation is the identity on A (resp. A') and on B.

(1.8.7) Lemma. *Let $\mathbb{G} = (T, (G_v)_{v \in VT}, (G_e)_{e \in ET})$ be a tree of groups. Let $e \in ET$ and $G_e \leq H_{o(e)} \leq G_{o(e)}$. Let $VT' = VT \cup \{x\}, ET' = \big(ET \backslash \{e, e^{-1}\}\big) \cup \{f, f^{-1}, h, h^{-1}\}$ with $o(f) = o(e), t(f) = x = o(h), t(h) = t(e)$, $G_x := H_{o(e)} =: G_f, G_h := G_e$. Then the two tree products of the trees of groups are isomorphic.*

Proof. Using Proposition (1.8.1), we contract the edge f to a vertex. The claim follows now from Remark (1.8.6) and the fact that $G_{o(e)} \star_{H_{o(e)}} H_{o(e)} \cong G_{o(e)}$. □

Part II.

RGD-systems over $\mathbb{F}_2$

2. Commutator blueprints

We introduce the notion of *commutator blueprints*, the main objects of this thesis. These objects will canonically provide the groups U_w. We first establish the decomposition $U_+ \cong U_s \ltimes N_s$ and show the existence of the automorphism $\tau_s \in \mathrm{Aut}(N_s)$ with $\tau_s(u_\alpha) = u_{s\alpha}$. In Definition (2.2.11) we define the group P_s by using the automorphisms $u_s, \tau_s \in \mathrm{Aut}(N_s)$. The main result of this chapter is Theorem (2.2.14), where we give a sufficient condition in order to show that a faithful and Weyl-invariant commutator blueprint is integrable.

2.1. Definition

(2.1.1) Convention. For the rest of this thesis we assume that (W, S) is crystallographic.

We let $\mathcal{P}$ be the set of prenilpotent pairs of positive roots. For $w \in W$ we define $\Phi(w) := \{\alpha \in \Phi_+ \mid w \notin \alpha\}$. Note that $\Phi(G) = \Phi(w)$ holds for every $G \in \mathrm{Min}(w)$. Let $G = (c_0, \ldots, c_k) \in \mathrm{Min}$ and let $(\alpha_1, \ldots, \alpha_k)$ be the set of roots crossed by G. We define $\Phi(G) := \{\alpha_i \mid 1 \leq i \leq k\}$. Using the indices we obtain an ordering $\leq_G$ on $\Phi(G)$ and, in particular, on $[\alpha, \beta] = [\beta, \alpha] \subseteq \Phi(G)$ for all $\alpha, \beta \in \Phi(G)$. We abbreviate $\mathcal{I} := \{(G, \alpha, \beta) \in \mathrm{Min} \times \Phi_+ \times \Phi_+ \mid \alpha, \beta \in \Phi(G), \alpha \leq_G \beta\}$.

Given a family $\left(M^G_{\alpha,\beta}\right)_{(G,\alpha,\beta)\in\mathcal{I}}$, where $M^G_{\alpha,\beta} \subseteq (\alpha, \beta)$ is ordered via $\leq_G$. For $w \in W$ we define the group U_w via the following presentation:

$$U_w := \langle \{u_\alpha \mid \alpha \in \Phi(w)\} \mid R_{\mathrm{inv}}, R_{\mathrm{cr}} \rangle,$$

where $R_{\mathrm{inv}} = \{u_\alpha^2 = 1 \mid \alpha \in \Phi(w)\}$ and $R_{\mathrm{cr}} = \{[u_\alpha, u_\beta] = \prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma \mid (G, \alpha, \beta) \in \mathcal{I}, G \in \mathrm{Min}(w)\}$. Here the product is understood to be ordered via the ordering $\leq_G$, i.e. if $G \in \mathrm{Min}(w), \alpha \leq_G \beta \in \Phi(G)$ and $M^G_{\alpha,\beta} = \{\gamma_1 \leq_G \ldots \leq_G \gamma_k\} \subseteq (\alpha, \beta) \subseteq \Phi(G)$, then $\prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma = u_{\gamma_1} \cdots u_{\gamma_k}$. Note that there could be $G, H \in \mathrm{Min}(w), \alpha, \beta \in \Phi(w)$ with $\alpha \leq_G \beta$ and $\beta \leq_H \alpha$. In this case we obtain two commutator relations, namely

$$[u_\alpha, u_\beta] = \prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma \qquad \text{and} \qquad [u_\beta, u_\alpha] = \prod_{\gamma \in M^H_{\beta,\alpha}} u_\gamma$$

From now on we will implicitly assume that each product $\prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma$ is ordered via the ordering $\leq_G$. Note that $\Phi(1_W) = \emptyset$ and hence $U_{1_W} = \langle \emptyset \mid \emptyset \rangle = \{1\}$.

Let $\mathrm{Dyn}(W, S)$ be a crystallographic Dynkin diagram. A *commutator blueprint of type* $\mathrm{Dyn}(W, S)$ is a family $\mathcal{M} = \left(M^G_{\alpha,\beta}\right)_{(G,\alpha,\beta)\in\mathcal{I}}$ of subsets $M^G_{\alpha,\beta} \subseteq (\alpha, \beta)$ ordered via $\leq_G$ satisfying the following axioms:

(CB1) Let $G = (c_0, \ldots, c_k) \in \mathrm{Min}$ and let $H = (c_0, \ldots, c_m)$ for some $1 \leq m \leq k$. Then we have $M^H_{\alpha,\beta} = M^G_{\alpha,\beta}$ for all $\alpha, \beta \in \Phi(H)$ with $\alpha \leq_H \beta$.

(CB2) Let $s \neq t \in S$ be with $m := m_{st} < \infty$ and assume that $(s, t) \in E(\mathrm{Dyn}(W, S))$. Let $G \in \mathrm{Min}_s(r_{\{s,t\}})$, let $(\alpha_1, \ldots, \alpha_m)$ be the sequence of roots crossed by G and let $1 \leq i < j \leq m$. Then $M^G_{\alpha_i,\alpha_j} = M_{\{\beta_i,\beta_j\}}$ as sets, where $M_{\{\beta_i,\beta_j\}}$ is given in Example (1.7.1).

(CB3) For each $w \in W$ we have $|U_w| = 2^{\ell(w)}$, where U_w is defined as above.

(2.1.2) Remark. (a) In (CB1) we have $\Phi(H) \subseteq \Phi(G)$ and the order $\leq_G$ restricted to elements in $\Phi(H)$ is precisely the order $\leq_H$. Thus the expression $M_{\alpha,\beta}^G$ is defined. In (CB2) we have $\Phi(G) = [\alpha_s, \alpha_t]$ and we only require that $M_{\alpha,\beta}^G = M_{\{\alpha,\beta\}}$ as sets. Note that $M_{\alpha,\beta}^G$ is an ordered set and the axiom only makes a statement about the underlying set. We also remark that it is a direct consequence of (CB3), that for all $G = (c_0, \ldots, c_k) \in \mathrm{Min}(w)$ and $\mathbb{Z}_2 \cong U_{\alpha_i} = \langle u_{\alpha_i} \rangle \leq U_w$ the product map $U_{\alpha_1} \times \cdots \times U_{\alpha_k} \to U_w, (u_1, \ldots, u_k) \mapsto u_1 \cdots u_k$ is a bijection.

(b) Suppose $m_{st} \neq 6$ for all $s, t \in S$. As (W, S) is crystallographic, we have $m_{st} \in \{2, 3, 4, \infty\}$. In this case (CR2) reduced to the following:

(CR2) Let $s \neq t \in S$ be with $m_{st} < \infty$, let $G \in \mathrm{Min}(r_{\{s,t\}})$ and let $\alpha \neq \beta \in \Phi(G) = [\alpha_s, \alpha_t]$ be such that $\alpha \leq_G \beta$. Then

$$M_{\alpha,\beta}^G = \begin{cases} (\alpha, \beta) & \text{if } \{\alpha, \beta\} = \{\alpha_s, \alpha_t\} \\ \emptyset & \text{if } \{\alpha, \beta\} \neq \{\alpha_s, \alpha_t\} \end{cases}$$

Note that all the information needed from $\mathrm{Dyn}(W, S)$ are already contained in the Coxeter system. Thus, if $m_{st} \neq 6$ for all $s, t \in S$, we will say for short *commutator blueprint of type* (W, S).

(2.1.3) Convention. For the rest of Chapter 2 we let $\mathrm{Dyn}(W, S)$ be a crystallographic Dynkin diagram and $\mathcal{M} = \left(M_{\alpha,\beta}^G \right)_{(G,\alpha,\beta) \in \mathcal{I}}$ be a commutator blueprint of type $\mathrm{Dyn}(W, S)$.

(2.1.4) Lemma. *Let $w \in W, G = (c_0, \ldots, c_k) \in \mathrm{Min}(w)$ and let $(\alpha_1, \ldots, \alpha_k)$ be the sequence of roots crossed by G. Then $\Phi(w) = \{\alpha_1, \ldots, \alpha_k\}$ and the group U_w has the following presentation:*

$$U_G := \left\langle u_{\alpha_1}, \ldots, u_{\alpha_k} \mid \begin{cases} \forall 1 \leq i \leq k : u_{\alpha_i}^2 = 1, \\ \forall 1 \leq i < j \leq k : [u_{\alpha_i}, u_{\alpha_j}] = \prod_{\gamma \in M_{\alpha_i,\alpha_j}^G} u_\gamma \end{cases} \right\rangle$$

Proof. Clearly, we have an epimorphism $U_G \to U_w$. Since each element in U_G is of the form $\prod_{i=1}^k u_{\alpha_i}^{\varepsilon_i}$, where $\varepsilon_i \in \{0, 1\}$, U_G has cardinality at most 2^k. As U_w has cardinality 2^k, the claim follows. ☐

Using the previous lemma, the axioms (CB1) and (CB3) imply that the canonical mapping $u_\alpha \mapsto u_\alpha$ induces a monomorphism from U_w to U_{ws} for all $w \in W, s \in S$ with $\ell(ws) = \ell(w)+1$. We let U_+ be the direct limit of the groups U_w with natural inclusions $U_w \to U_{ws}$ if $\ell(ws) = \ell(w) + 1$. Then $\mathcal{M}$ is called *faithful*, if the canonical homomorphisms $U_w \to U_+$ are injective.

We call the commutator blueprint $\mathcal{M}$ *(locally) Weyl-invariant* if for every $1 \neq w \in W, s \in S$ and $G = (c_0, \ldots, c_k) \in \mathrm{Min}(w)$ the following hold:

- If $\ell(sw) = \ell(w) + 1$, then $sG := (1_W, sc_0 = s, sc_1, \ldots, sc_k)$ is a minimal gallery and we have $M_{s\alpha,s\beta}^{sG} = sM_{\alpha,\beta}^G := \{s\gamma \mid \gamma \in M_{\alpha,\beta}^G\}$ for all $\alpha \leq_G \beta \in \Phi(G)$ (with $o(r_\alpha r_\beta) < \infty$).
- If $\ell(sw) = \ell(w) - 1$ and $G \in \mathrm{Min}_s(w)$, then $sG := (sc_1 = 1_W, sc_2, \ldots, sc_k)$ is a minimal gallery and we have $M_{s\alpha,s\beta}^{sG} = sM_{\alpha,\beta}^G$ for all $\alpha_s \neq \alpha \leq_G \beta \in \Phi(G)$ (with $o(r_\alpha r_\beta) < \infty$).

(2.1.5) Remark. Let $\mathcal{M}$ be Weyl-invariant and let $1 \neq w \in W, s \in S$.

(a) Suppose $\ell(sw) = \ell(w) + 1, G \in \mathrm{Min}(w)$ and $\alpha \neq \beta \in \Phi(G)$. Then $\alpha \leq_G \beta$ if and only if $s\alpha \leq_{sG} s\beta$. Moreover, we have the following relation in U_{sw}:

$$[u_{s\alpha}, u_{s\beta}] = \prod_{\gamma \in M^{sG}_{s\alpha,s\beta}} u_\gamma = \prod_{\gamma \in sM^G_{\alpha,\beta}} u_\gamma = \prod_{\gamma \in M^G_{\alpha,\beta}} u_{s\gamma}$$

(b) Suppose $\ell(sw) = \ell(w) - 1, G \in \mathrm{Min}_s(w)$ and $\alpha \neq \beta \in \Phi(G)\backslash\{\alpha_s\}$. Then again $\alpha \leq_G \beta$ if and only if $s\alpha \leq_{sG} s\beta$ and we have the following relation in U_{sw}:

$$[u_{s\alpha}, u_{s\beta}] = \prod_{\gamma \in M^{sG}_{s\alpha,s\beta}} u_\gamma = \prod_{\gamma \in sM^G_{\alpha,\beta}} u_\gamma = \prod_{\gamma \in M^G_{\alpha,\beta}} u_{s\gamma}$$

(2.1.6) Lemma. *For $w \in W, s \in S$ with $\ell(sw) = \ell(w) - 1$ we define the group $V_{w,s}$ as the subgroup of U_w generated by $\{u_\alpha \mid \alpha \in \Phi(w)\backslash\{\alpha_s\}\}$. Then $V_{w,s}$ is a normal subgroup of U_w and a presentation of $V_{w,s}$ is given by the presentation of U_w by deleting the generator u_{α_s} and all relations in which u_{α_s} appears.*

Proof. Using the commutator relations and the fact that $[u_{\alpha_s}, u_\alpha] = u_\alpha^{u_{\alpha_s}} u_\alpha$, the subgroup $V_{w,s}$ is a normal subgroup of U_w. Let $\tilde{V}_{w,s}$ be the group given by the presentation in the statement. Then we have a canonical homomorphism $\tilde{V}_{w,s} \to U_w$. Let $G = (c_0, \dots, c_k) \in \mathrm{Min}_s(w)$. Then $\alpha_1 = \alpha_s$ and each element of $\tilde{V}_{w,s}$ can be written in the form $\prod_{i=2}^k u_{\alpha_i}^{\varepsilon_i}$, where $\varepsilon_i \in \{0, 1\}$. Thus $\tilde{V}_{w,s}$ is a group of cardinality at most 2^{k-1}. Since the image of $\tilde{V}_{w,s}$ in U_w is $V_{w,s}$ and this group has cardinality 2^{k-1}, the homomorphism is an isomorphism and we are done. □

(2.1.7) Lemma. *Suppose $w \in W, s \in S$ with $\ell(sw) = \ell(w) - 1$, let $G = (c_0, \dots, c_k) \in \mathrm{Min}_s(w)$ and let $(\alpha_1 = \alpha_s, \dots, \alpha_k)$ be the sequence of roots crossed by G. Then we define the group*

$$V_G := \left\langle u_{\alpha_2}, \dots, u_{\alpha_k} \mid \begin{cases} \forall 2 \leq i \leq k : u_{\alpha_i}^2 = 1, \\ \forall 2 \leq i < j \leq k : [u_{\alpha_i}, u_{\alpha_j}] = \prod_{\gamma \in M^G_{\alpha_i,\alpha_j}} u_\gamma \end{cases} \right\rangle$$

and the canonical mapping $u_{\alpha_i} \mapsto u_{\alpha_i}$ extends to an isomorphism from V_G to $V_{w,s}$. Moreover, if $\mathcal{M}$ is Weyl-invariant, the mapping $u_\alpha \mapsto u_{s\alpha}$ extends to an isomorphism from $V_{w,s}$ to U_{sw}.

Proof. The first part follows similar as in Lemma (2.1.4). For the second part we note that $sG \in \mathrm{Min}(sw)$. Using Lemma (2.1.4) and Remark (2.1.5), we obtain that the mapping $u_\alpha \to u_{s\alpha}$ extends to an isomorphism. □

(2.1.8) Example. Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD-system of type (W, S) over $\mathbb{F}_2$, let $H = (c_0, \dots, c_k) \in \mathrm{Min}$ and let $(\alpha_1, \dots, \alpha_k)$ be the sequence of roots crossed by H. Then we have $\Phi(H) = \{\alpha_1 \leq_H \cdots \leq_H \alpha_k\}$. By [2, Corollary 8.34(1)] there exists for each $1 \leq m < i < n \leq k$ a unique $\varepsilon_i \in \{0, 1\}$ such that $[u_{\alpha_m}, u_{\alpha_n}] = \prod_{i=m+1}^{n-1} u_{\alpha_i}^{\varepsilon_i}$, and $\varepsilon_i = 1$ implies $\alpha_i \in (\alpha_m, \alpha_n)$. We define $M(\mathcal{D})^H_{\alpha_m,\alpha_n} := \{\alpha_i \in \Phi(H) \mid [u_{\alpha_m}, u_{\alpha_n}] = \prod_{i=m+1}^{n-1} u_{\alpha_i}^{\varepsilon_i}, \varepsilon_i = 1\} \subseteq (\alpha_m, \alpha_n)$ and $\mathcal{M}_\mathcal{D} := \left(M(\mathcal{D})^H_{\alpha,\beta}\right)_{(H,\alpha,\beta)\in\mathcal{I}}$.

For $s, t \in S$ with $m_{st} = 6$ we get a canonical direction of the edge $\{s, t\}$ via the commutator relations. For $s, t \in S$ with $m_{st} \in \{3, 4, \infty\}$ we choose any direction. This gives us a crystallographic Dynkin diagram $\mathrm{Dyn}(W, S)$. Clearly, (CB1) is satisfied. By Example (1.7.1) (CB2) holds and (CB3) is satisfies by [2, Corollary 8.34(1)]. Thus $\mathcal{M}_\mathcal{D}$ is a commutator blueprint of type $\mathrm{Dyn}(W, S)$, which is faithful (cf. [2, Theorem 8.85]) and Weyl-invariant.

The commutator blueprint $\mathcal{M}$ is called *integrable* if there exists an RGD-system $\mathcal{D}$ of type (W, S) over $\mathbb{F}_2$ such that $M^G_{\alpha,\beta} = M(\mathcal{D})^G_{\alpha,\beta}$ holds for every $(G, \alpha, \beta) \in \mathcal{I}$.

2.2. Integrability of certain commutator blueprints

(2.2.1) Convention. For the rest of Chapter 2 we assume that the commutator blueprint $\mathcal{M}$ is faithful and Weyl-invariant. Moreover, we fix $s \in S$ in this section, unless it is stated.

As we have seen in the previous example, an integrable commutator blueprint is necessarily faithful and Weyl-invariant. We will work out sufficient conditions in order to show that $\mathcal{M}$ is integrable. We will see (cf. Definition (2.2.12) and Theorem (2.2.14)) that under some conditions, there exists an RGD-system containing U_+. As a first step we construct the group P_s (mentioned in Theorem (2.2.14)), which contains U_+ as a subgroup.

Since $\mathcal{M}$ is faithful, we can identify U_w with its image in U_+. In particular, we have $u_\alpha \in U_+$ for all $\alpha \in \Phi_+$. We will write for short $u_s := u_{\alpha_s}$.

We define the subgroup $N_s := \langle x^{-1} u_\alpha x \mid \alpha \in \Phi_+ \backslash \{\alpha_s\}, x \in U_s \rangle \leq U_+$ (the idea of the definition of N_s is obtained from [29, 6.2.1]). Next, we will construct two automorphisms of N_s. Clearly, U_+ is generated by U_s and N_s, and N_s is a normal subgroup of U_+.

(2.2.2) Lemma. *We have $U_+ = U_s \ltimes N_s$.*

Proof. It suffices to show that $U_s \cap N_s = 1$. At first we will show that the assignment $u_\alpha \mapsto 1$ for $\alpha_s \neq \alpha \in \Phi_+$ and $u_s \mapsto u_s$ will extend to a homomorphism $U_w \to U_s$. In view of the definition of U_w it suffices to consider the relations $u_\alpha^2 = 1$ and $[u_\alpha, u_\beta] = u_{\gamma_1} \cdots u_{\gamma_k}$. Since $\alpha_s \notin (\alpha, \beta)$ for every $\{\alpha, \beta\} \in \mathcal{P}$, these relations are mapped to 1 and we obtain homomorphisms $U_w \to U_s$ for every $w \in W$. Since these homomorphisms respect the natural inclusions $U_w \to U_{wt}$, the universal property of direct limits yields a homomorphism $\varphi : U_+ \to U_s$ with $\varphi(u_\alpha) = 1$ for $\alpha_s \neq \alpha \in \Phi_+$ and $\varphi(u_s) = u_s$. Since $N_s \leq \ker\varphi$ and $U_s \cap \ker\varphi = 1$, the claim follows. □

(2.2.3) Remark. The next step is to construct an automorphism τ_s on N_s which maps u_α to $u_{s\alpha}$. The rough idea is that P_s should look like $\langle u_s, \tau_s \rangle \ltimes N_s$.

In the next lemma we will show that N_s has a suitable presentation. The elements v_α will play the role of the elements $u_s u_\alpha u_s$ for all $\alpha_s \neq \alpha \in \Phi_+$.

(2.2.4) Lemma. *We define the group M_s via the following presentation:*

$$\left\langle \{u_\alpha, v_\alpha \mid \alpha_s \neq \alpha \in \Phi_+\} \;\middle|\; \begin{cases} \forall \alpha_s \neq \alpha \in \Phi_+ : u_\alpha^2 = 1 = v_\alpha^2, \\ \forall w \in W, \ell(sw) = \ell(w) + 1, G \in \mathrm{Min}(w), \alpha \leq_G \beta \in \Phi(G) : \\ \qquad [u_\alpha, u_\beta] = \prod_{\gamma \in M_{\alpha,\beta}^G} u_\gamma, \quad [v_\alpha, v_\beta] = \prod_{\gamma \in M_{\alpha,\beta}^G} v_\gamma, \\ \forall w \in W, \ell(sw) = \ell(w) - 1, G \in \mathrm{Min}_s(w), \alpha_s \neq \alpha \leq_G \beta \in \Phi(G) : \\ \qquad [u_\alpha, u_\beta] = \prod_{\gamma \in M_{\alpha,\beta}^G} u_\gamma, \quad [v_\alpha, v_\beta] = \prod_{\gamma \in M_{\alpha,\beta}^G} v_\gamma, \\ \forall w \in W, \ell(sw) = \ell(w) - 1, G \in \mathrm{Min}_s(w), \alpha_s \neq \alpha \in \Phi(G) : \\ \qquad v_\alpha = \left(\prod_{\gamma \in M_{\alpha_s,\alpha}^G} u_\gamma \right) u_\alpha \end{cases} \right\rangle$$

Then we have $u_s \in \mathrm{Aut}(M_s)$ such that $u_s(u_\alpha) = v_\alpha$ and $u_s(v_\alpha) = u_\alpha$. In particular, $M_s \to N_s, \begin{cases} u_\alpha \mapsto u_\alpha \\ v_\alpha \mapsto u_s u_\alpha u_s \end{cases}$ is an isomorphism.

Proof. We show that the assignments $u_\alpha \mapsto v_\alpha$ and $v_\alpha \mapsto u_\alpha$ extend to an endomorphism of M_s. Therefore we have to show that every relation is mapped to a relation. For that it suffices to consider the relations of the form $v_\alpha = \left(\prod_{\gamma \in M_{\alpha_s,\alpha}^G} u_\gamma \right) u_\alpha$. Suppose $w \in W$ with that $\ell(sw) = \ell(w) - 1$ and let $G \in \mathrm{Min}_s(w)$. Let $V_{w,s}$ be the normal subgroup of U_w as in Lemma (2.1.6). Using Lemma (2.1.7) we deduce that the canonical assignment $u_\alpha \mapsto u_\alpha$ extends to a homomorphism from $V_{w,s} \cong V_G$ to M_s. Moreover, for $\alpha_s \neq \alpha \in \Phi(G)$ we have the following relation in U_w and, since both sides of the equation are contained in $V_{w,s}$, this yields also a relation in M_s (note that $\alpha \in \Phi(G)$ implies $\gamma \in \Phi(G)$ for all $\gamma \in (\alpha_s, \alpha)$):

$$\begin{aligned}\left(\prod_{\gamma\in M^G_{\alpha_s,\alpha}} \left(\prod_{\beta\in M^G_{\alpha_s,\gamma}} u_\beta\right) u_\gamma\right)\left(\prod_{\gamma\in M^G_{\alpha_s,\alpha}} u_\gamma\right) u_\alpha &= \left(\prod_{\gamma\in M^G_{\alpha_s,\alpha}} [u_s, u_\gamma]u_\gamma\right)[u_s,u_\alpha]u_\alpha \\ &= u_s\left(\prod_{\gamma\in M^G_{\alpha_s,\alpha}} u_\gamma\right) u_\alpha u_s \\ &= u_s[u_s,u_\alpha]u_\alpha u_s \\ &= u_\alpha\end{aligned}$$

Note that by definition we also have the relation $v_\delta = \left(\prod_{\varepsilon\in M^G_{\alpha_s,\delta}} u_\varepsilon\right) u_\delta$ for every $\alpha_s \neq \delta \in \Phi(G)$. Now we consider the discussed relation:

$$\left(\prod_{\gamma\in M^G_{\alpha_s,\alpha}} v_\gamma\right) v_\alpha = \left(\prod_{\gamma\in M^G_{\alpha_s,\alpha}} \left(\prod_{\beta\in M^G_{\alpha_s,\gamma}} u_\beta\right) u_\gamma\right)\left(\prod_{\gamma\in M^G_{\alpha_s,\alpha}} u_\gamma\right) u_\alpha = u_\alpha$$

Thus every relation is mapped to a relation and we have an endomorphism u_s of M_s interchanging u_α and v_α. Since $u_s^2 = \mathrm{id}$, it is an automorphism of M_s. Consider $U := \mathbb{Z}_2 \ltimes M_s$, where $\mathbb{Z}_2$ acts on M_s via u_s. Moreover, we denote the generator of $\mathbb{Z}_2$ by u_s. Then the assignment

$$\begin{aligned} u_s &\mapsto u_s \\ u_\alpha &\mapsto u_\alpha \\ v_\alpha &\mapsto u_s u_\alpha u_s \end{aligned}$$

extends to a homomorphism $U \to U_+$, since all relations in U do also hold in U_+. Now we will show that there does also exist a homomorphism $U_+ \to U$ mapping u_s onto u_s and u_α onto u_α. For this we consider $w \in W$. If $\ell(sw) = \ell(w) + 1$, then every relation in U_w is also a relation in M_s and hence in U. Thus we obtain a homomorphism $U_w \to U$ mapping u_α onto u_α. Assume that $\ell(sw) = \ell(w) - 1$ and let $G \in \mathrm{Min}_s(w)$. By Lemma (2.1.4) U_w is isomorphic to U_G and we have to show that $[u_s, u_\alpha] = \prod_{\gamma\in M^G_{\alpha_s,\alpha}} u_\gamma$ is a relation in U. Note that this is a relation if and only if $u_s u_\alpha u_s = \left(\prod_{\gamma\in M^G_{\alpha_s,\alpha}} u_\gamma\right) u_\alpha$ is a relation in U. But in U we have $u_s u_\alpha u_s = v_\alpha$ and hence it is a relation by definition. In particular, the mappings $U_w \to U$ preserve the inclusion mappings $U_w \to U_{wt}$ and by the universal property of direct limits there exists a homomorphism $U_+ \to U$. Since both concatenations are the identity on the generating sets, both homomorphisms are isomorphisms. In particular, M_s is isomorphic to N_s. □

(2.2.5) Lemma. *Let $w, w' \in W$ be such that $\ell(sw) = \ell(w) - 1$ and $\ell(sw') = \ell(w') - 1$. Let $G \in \mathrm{Min}_s(w), H \in \mathrm{Min}_s(w')$ and let $\alpha_s \neq \alpha \in \Phi(G) \cap \Phi(H)$. Then the following hold in M_s:*

(a) $\left(\prod_{\gamma\in M^G_{\alpha_s,\alpha}} u_{s\gamma}\right) u_{s\alpha} = \left(\prod_{\gamma\in M^H_{\alpha_s,\alpha}} u_{s\gamma}\right) u_{s\alpha}$;

(b) $\left(\prod_{\gamma\in M^G_{\alpha_s,\alpha}} v_{s\gamma}\right) v_{s\alpha} = \left(\prod_{\gamma\in M^H_{\alpha_s,\alpha}} v_{s\gamma}\right) v_{s\alpha}$.

Proof. Assertion (b) is a direct consequence of Assertion (a) and the fact that u_s is an automorphism of M_s interchanging u_α and v_α. Thus it suffices to show Assertion (a).

By definition we have the following two equations in M_s:

$$\left(\prod_{\gamma\in M^G_{\alpha_s,\alpha}} u_\gamma\right) u_\alpha = v_\alpha = \left(\prod_{\gamma\in M^H_{\alpha_s,\alpha}} u_\gamma\right) u_\alpha$$

Using Lemma (2.2.4) we infer that $\left(\prod_{\gamma\in M^G_{\alpha_s,\alpha}} u_\gamma\right) u_\alpha = \left(\prod_{\gamma\in M^H_{\alpha_s,\alpha}} u_\gamma\right) u_\alpha$ is a relation in $N_s \le U_+$. We remark that $[\alpha_s,\alpha] \subseteq \Phi(G)\cap\Phi(H)$. Using the fact that $U_w \to U_+$ is injective and both sides of the relation are contained in U_w, we deduce that it is also a relation in U_w. Moreover, both sides are contained in the subgroup $V_{w,s} \le U_w$. Now we can apply Lemma (2.1.7) and the fact that $U_{sw} \to M_s$ is a homomorphism to show that

$$\left(\prod_{\gamma\in M^G_{\alpha_s,\alpha}} u_{s\gamma}\right) u_{s\alpha} = \left(\prod_{\gamma\in M^H_{\alpha_s,\alpha}} u_{s\gamma}\right) u_{s\alpha}$$

is a relation in M_s. This finishes the claim. □

(2.2.6) Remark. Let $R \in \partial^2\alpha_s$ and let $\Phi(R) = \{\alpha\in\Phi_+ \mid R\in\partial^2\alpha\}$. Then $[\alpha,\beta]\subseteq\Phi(R)$ for all $\alpha,\beta\in\Phi(R)$.

(2.2.7) Lemma. *Let $R \in \partial^2\alpha_s$ and let $\Phi(R) = \{\alpha\in\Phi_+ \mid R\in\partial^2\alpha\}$. We define the group U_R via the following presentation*

$$U_R := \left\langle \{u_\alpha \mid \alpha\in\Phi(R)\} \mid \begin{cases} \forall\alpha\in\Phi(R): u_\alpha^2 = 1, \\ \forall w\in W, \ell(sw)=\ell(w)+1, G\in\mathrm{Min}(w), \alpha,\beta\in\Phi(G)\cap\Phi(R), \alpha\le_G\beta: \\ \qquad [u_\alpha,u_\beta] = \prod_{\gamma\in M^G_{\alpha,\beta}} u_\gamma, \\ \forall w\in W, \ell(sw)=\ell(w)-1, G\in\mathrm{Min}_s(w), \alpha,\beta\in\Phi(G)\cap\Phi(R), \alpha\le_G\beta: \\ \qquad [u_\alpha,u_\beta] = \prod_{\gamma\in M^G_{\alpha,\beta}} u_\gamma \end{cases} \right\rangle$$

For $N_R := \langle u_\alpha \mid \alpha_s\neq\alpha\in\Phi(R)\rangle \le U_R$ we have $U_R \cong U_s \ltimes N_R$ and a presentation of N_R is given by the presentation of U_R by deleting the generator u_{α_s} and all relations in which u_{α_s} appears. Furthermore, there exists $\tau_s\in\mathrm{Aut}(N_R)$ such that $\tau_s(u_\alpha) = u_{s\alpha}$ holds for all $\alpha_s\neq\alpha\in\Phi(R)$, and we have $\tau_s^2 = 1 = (u_s\tau_s)^3$ in $\mathrm{Aut}(N_R)$.

Proof. Similarly as in Lemma (2.2.2) we deduce $U_R \cong U_s\ltimes N_R$. Suppose $w\in W$ with $\ell(sw)=\ell(w)-1$ and let $G\in\mathrm{Min}_s(w)$ be such that $\Phi(R)\subseteq\Phi(G)$. Then each element of U_R can be written in the form $\prod_{j=1}^m u_{\beta_j}^{\varepsilon_j}$, where $\varepsilon_j\in\{0,1\}$ and $\{\beta_1=\alpha_s\le_G\cdots\le_G\beta_m\} = \Phi(R)\subseteq\Phi(G)$. Since we have a homomorphism $U_R\to U_+$ and the image of U_R is contained in U_w, (CB3) implies that $U_R\to U_+$ is a monomorphism. Let $\tilde N_R$ be the group given by the presentation in the statement. Then again each element in $\tilde N_R$ can be written in the form $\prod_{j=2}^m u_{\beta_j}^{\varepsilon_j}$. Since we have a homomorphism $\tilde N_R\to U_R$ with image N_R, the cardinality of N_R implies that this homomorphism must be an isomorphism.

Now we will see that the assignment $u_\alpha\mapsto u_{s\alpha}$ extends to an endomorphism of N_R. First of all we note that for $\alpha_s\neq\alpha\in\Phi(R)$ we have $\alpha_s\neq s\alpha\in\Phi(R)$. We consider all three types of relations, where $u_\alpha^2=1$ is obvious. Suppose $w\in W$ with $\ell(sw)=\ell(w)+1$ and let $G\in\mathrm{Min}(w), \alpha,\beta\in\Phi(G)\cap\Phi(R)$ with $\alpha\le_G\beta$. Using the Weyl-invariance and the fact that $[u_{s\alpha},u_{s\beta}] = \prod_{\gamma\in M^{sG}_{s\alpha,s\beta}} u_\gamma$ is a relation, we deduce similar as in Remark (2.1.5) that

$$[u_{s\alpha},u_{s\beta}] = \prod_{\gamma\in M^{sG}_{s\alpha,s\beta}} u_\gamma = \prod_{\gamma\in sM^G_{\alpha,\beta}} u_\gamma = \prod_{\gamma\in M^G_{\alpha,\beta}} u_{s\gamma}$$

is a relation in N_R. Vice versa, we assume $\ell(sw) = \ell(w) - 1$ and we let $G \in \mathrm{Min}_s(w), \alpha \neq \alpha_s \neq \beta \in \Phi(G) \cap \Phi(R)$ with $\alpha \leq_G \beta$. Using the Weyl-invariance and the fact that $[u_{s\alpha}, u_{s\beta}] = \prod_{\gamma \in M^{sG}_{s\alpha, s\beta}} u_\gamma$ is a relation, we deduce similar as in Remark (2.1.5) that

$$[u_{s\alpha}, u_{s\beta}] = \prod_{\gamma \in M^{sG}_{s\alpha, s\beta}} u_\gamma = \prod_{\gamma \in sM^{G}_{\alpha, \beta}} u_\gamma = \prod_{\gamma \in M^{G}_{\alpha, \beta}} u_{s\gamma}$$

is a relation in N_R. Thus $\tau_s : N_R \to N_R, u_\alpha \mapsto u_{s\alpha}$ is an endomorphism. Since $\tau_s^2 = 1$, we infer $\tau_s \in \mathrm{Aut}(N_R)$.

To show the claim it suffices to show that $(u_s \tau_s)^3 = 1$. We do a case by case distinction on the type of the residue R (we will write for short $f.u_\beta := f(u_\beta)$):

- $A_1 \times A_1$: Let $\Phi(R) = \{\alpha_s, \beta\}$. Then $s\beta = \beta$. Since u_s, u_β commute by (CB2), Example (1.7.1) and the Weyl-invariance, we obtain

$$(u_s\tau_s)^3.u_\beta = (u_s\tau_s)^2.[u_s, u_\beta]u_\beta = (u_s\tau_s)^2.u_\beta = u_\beta$$

- A_2: Let $\Phi(R) = \{\alpha_s, \delta, \varepsilon\}$. Then $s\varepsilon = \delta$ and we assume that $\{\alpha_s, \varepsilon\}$ is a set of simple roots of R. Using (CB2), Example (1.7.1) and the Weyl-invariance, we obtain the following:

$$\begin{aligned}
(u_s\tau_s)^3.u_\varepsilon &= (u_s\tau_s)^2.u_\delta = (u_s\tau_s).u_\delta u_\varepsilon = u_\varepsilon \\
(u_s\tau_s)^3.u_\delta &= u_s\tau_s.u_\varepsilon = u_\delta
\end{aligned}$$

- $B_2 = C_2$: Let $\Phi(R) = \{\alpha_s, \delta, \gamma, \varepsilon\}$ and assume that $\{\alpha_s, \varepsilon\}$ is a set of simple roots of R. Furthermore, we assume that $s\gamma = \gamma$ and $s\varepsilon = \delta$. Using (CB2), Example (1.7.1) and the Weyl-invariance, we obtain that only u_s and u_ε do not commute. We compute the following:

$$\begin{aligned}
(u_s\tau_s)^3.u_\gamma &= (u_s\tau_s)^2.u_\gamma = u_\gamma \\
(u_s\tau_s)^3.u_\varepsilon &= (u_s\tau_s)^2.u_\delta = u_s\tau_s.u_\delta u_\gamma u_\varepsilon = u_\varepsilon \\
(u_s\tau_s)^3.u_\delta &= u_s\tau_s.u_\varepsilon = u_\delta
\end{aligned}$$

- G_2: Let $\Phi(R) = \{\beta_1, \ldots, \beta_6\}$ and we assume that $\{\beta_1, \beta_6\}$ is a set of simple roots of R and that the roots are ordered via their indices. Assume first that $\alpha_s = \beta_1$. Then $s\beta_2 = \beta_6, s\beta_3 = \beta_5$ and $s\beta_4 = \beta_4$. Let $u_i := u_{\beta_i} \in U^\star_{\beta_i}$. Using (CB2), Example (1.7.1) and the Weyl-invariance, we obtain

$$\begin{aligned}
(u_s\tau_s)^3.u_4 &= (u_s\tau_s)^2.u_4 = u_4 \\
(u_s\tau_s)^3.u_6 &= (u_s\tau_s)^2.u_2 = u_s\tau_s.[u_1, u_6]u_6 = u_s\tau_s.u_2u_3u_4u_5u_6 \\
&= [u_1, u_6]u_6[u_1, u_5]u_5[u_1, u_4]u_4[u_1, u_3]u_3[u_1, u_2]u_2 \\
&= u_2u_3u_4u_5u_6u_2u_4u_5u_4u_2u_3u_2 = u_2u_3u_4u_6u_3u_2 = u_6 \\
(u_s\tau_s)^3.u_2 &= u_s\tau_s.u_6 = u_2 \\
(u_s\tau_s)^3.u_5 &= (u_s\tau_s)^2.[u_1, u_3]u_3 = (u_s\tau_s)^2.u_2u_3 \\
&= u_s\tau_s.[u_1, u_6]u_6[u_1, u_5]u_5 \\
&= u_s\tau_s.u_2u_3u_4u_5u_6u_2u_4u_5 = u_s\tau_s.u_3u_4u_6 \\
&= [u_1, u_5]u_5[u_1, u_4]u_4[u_1, u_2]u_2 = u_2u_4u_5u_4u_2 = u_5
\end{aligned}$$

$$(u_s\tau_s)^3.u_3 = (u_s\tau_s)^2.[u_1,u_5]u_5 = (u_s\tau_s)^2.u_2u_4u_5 = u_6u_4u_3u_4u_6 = u_3$$

It is also possible that $\alpha_s = \beta_6$. In this case $s\beta_1 = \beta_5, s\beta_2 = \beta_4$ and $s\beta_3 = \beta_3$ and we compute the following:

$$\begin{aligned}
(u_s\tau_s)^3.u_3 &= (u_s\tau_s)^2.u_3 = u_3 \\
(u_s\tau_s)^3.u_1 &= (u_s\tau_s)^2.u_5 = u_s\tau_s.u_1[u_1,u_6] = u_s\tau_s.u_1u_2u_3u_4u_5 \\
&= u_5[u_5,u_6]u_4[u_4,u_6]u_3[u_3,u_6]u_2[u_2,u_6]u_1[u_1,u_6] \\
&= u_5u_4u_3u_2u_4u_1u_2u_3u_4u_5 = u_5u_4u_1u_2u_5 = u_4u_1[u_1,u_5]u_2 = u_1 \\
(u_s\tau_s)^3.u_5 &= u_s\tau_s.u_1 = u_5 \\
(u_s\tau_s)^3.u_2 &= (u_s\tau_s)^2.u_4[u_4,u_6] = (u_s\tau_s)^2.u_4 \\
&= u_s\tau_s.u_2[u_2,u_6] = u_s\tau_s.u_2u_4 \\
&= u_4[u_4,u_6]u_2[u_2,u_6] = u_4u_2u_4 = u_2 \\
(u_s\tau_s)^3.u_4 &= u_s\tau_s.u_2 = u_4[u_4,u_6] = u_4
\end{aligned}$$

- $I_2(8)$: This type does not occur since (W,S) is crystallographic.
- $I_2(\infty)$: Since R is a spherical rank 2 residue, R cannot be of type $I_2(\infty)$. □

(2.2.8) Remark. Let $-\alpha_s \subseteq \beta \in \Phi_+$, let $w, w' \in W$ such that $\ell(sw) = \ell(w) - 1$ and $\ell(sw') = \ell(w') - 1$, let $G \in \mathrm{Min}_s(w), H \in \mathrm{Min}_s(w')$ such that $s\beta \in \Phi(G) \cap \Phi(H)$. Note that $\alpha_s \in \Phi(G) \cap \Phi(H)$ as well. Then we have

$$\left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} u_{s\gamma} \right) u_\beta = \left(\prod_{\gamma \in M^H_{\alpha_s, s\beta}} u_{s\gamma} \right) u_\beta$$

in M_s by Lemma (2.2.5). Using the isomorphism $M_s \to N_s$ from Lemma (2.2.4), this is also a relation in N_s.

(2.2.9) Proposition. *There exists an endomorphism $\tau_s : N_s \to N_s$ such that $\tau_s(u_\alpha) = u_{s\alpha}$ for each $\alpha_s \neq \alpha \in \Phi_+$ and $\tau_s(u_s u_\beta u_s) = u_s \left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} u_{s\gamma} \right) u_\beta u_s$ for each $-\alpha_s \subseteq \beta \in \Phi_+$, where $w \in W$ is such that $\ell(sw) = \ell(w) - 1$ and $G \in \mathrm{Min}_s(w)$ with $s\beta \in \Phi(G)$.*

Proof. We will construct an endomorphism $\tau_s : M_s \to M_s$ and show that the induced endomorphism on N_s is as required. At first we will show that the following assignments (call it τ_s) extend to an endomorphism of M_s, where in the second case $G \in \mathrm{Min}_s(w)$ is such that $\{\alpha_s, \alpha\} \subseteq \Phi(G)$ for some $w \in W$ with $\ell(sw) = \ell(w) - 1$, and in the third case $G \in \mathrm{Min}_s(w)$ is such that $\{\alpha_s, s\alpha\} \subseteq \Phi(G)$ for some $w \in W$ with $\ell(sw) = \ell(w) - 1$ (note that by Lemma (2.2.5) the assignments do neither depend on $w \in W$ with $\ell(sw) = \ell(w) - 1$ nor on the gallery $G \in \mathrm{Min}_s(w)$):

$$\begin{aligned}
\forall \alpha_s \neq \alpha \in \Phi_+ : u_\alpha &\mapsto u_{s\alpha} \\
\forall \{\alpha_s, \alpha\} \in \mathcal{P} : v_\alpha &\mapsto \left(\prod_{\gamma \in M^G_{\alpha_s, \alpha}} u_{s\gamma} \right) u_{s\alpha} \\
-\alpha_s \subseteq \alpha : v_\alpha &\mapsto \left(\prod_{\gamma \in M^G_{\alpha_s, s\alpha}} v_{s\gamma} \right) v_\alpha
\end{aligned}$$

We distinguish all relations:

(i) $u_\alpha^2 = 1$: There is nothing to show.

(ii) $v_\alpha^2 = 1$: We distinguish the following cases:

(a) $\{\alpha_s, \alpha\} \in \mathcal{P}$: Suppose $w \in W$ with $\ell(sw) = \ell(w) - 1$ and $G \in \mathrm{Min}_s(w)$ with $\alpha_s, \alpha \in \Phi(G)$. Then we have $\left(\left(\prod_{\gamma \in M^G_{\alpha_s,\alpha}} u_\gamma\right) u_\alpha\right)^2 = ([u_s, u_\alpha]u_\alpha)^2 = 1$ in U_w and hence in $V_{w,s}$. This implies that

$$\left(\left(\prod_{\gamma \in M^G_{\alpha_s,\alpha}} u_{s\gamma}\right) u_{s\alpha}\right)^2$$

is a relation in U_{sw} by Lemma (2.1.7) and, using the homomorphism $U_{sw} \to M_s$, hence also in M_s. But this is exactly the image of v_α under the assignment τ_s.

(b) $-\alpha_s \subseteq \alpha$: Suppose $w \in W$ with $\ell(sw) = \ell(w) - 1$ and $G \in \mathrm{Min}_s(w)$ with $\alpha_s, s\alpha \in \Phi(G)$. We have to show that

$$\left(\left(\prod_{\gamma \in M^G_{\alpha_s,s\alpha}} v_{s\gamma}\right) v_\alpha\right)^2$$

is a relation. Clearly, $\alpha_s \neq s\alpha \in \Phi_+$ and $v_{s\alpha}^2$ is a relation by definition. Using Case (a), we already know that

$$\left(\left(\prod_{\gamma \in M^G_{\alpha_s,s\alpha}} u_{s\gamma}\right) u_\alpha\right)^2$$

is a relation in M_s. Since u_s is an automorphism of M_s interchanging u_α and v_α by Lemma (2.2.4), we obtain the relation

$$1 = u_s\left(\left(\left(\prod_{\gamma \in M^G_{\alpha_s,s\alpha}} u_{s\gamma}\right) u_\alpha\right)^2\right) = \left(\left(\prod_{\gamma \in M^G_{\alpha_s,s\alpha}} v_{s\gamma}\right) v_\alpha\right)^2$$

(iii) $[u_\alpha, u_\beta] = \prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma$: Suppose $w \in W, G \in \mathrm{Min}(w)$ and $\alpha \leq_G \beta \in \Phi(G)\backslash\{\alpha_s\}$. If $\ell(sw) = \ell(w) + 1$ (resp. $\ell(sw) = \ell(w) - 1$ and if $G \in \mathrm{Min}_s(w)$), the Weyl-invariance yields that

$$[u_{s\alpha}, u_{s\beta}] = \prod_{\gamma \in M^{sG}_{s\alpha,s\beta}} u_\gamma = \prod_{\gamma \in sM^G_{\alpha,\beta}} u_\gamma = \prod_{\gamma \in M^G_{\alpha,\beta}} u_{s\gamma}$$

is a relation (cf. Remark (2.1.5)).

(iv) $[v_\alpha, v_\beta] = \prod_{\gamma \in M^G_{\alpha,\beta}} v_\gamma$: Suppose $w \in W, G \in \mathrm{Min}(w)$ and $\alpha \leq_G \beta \in \Phi(G)\backslash\{\alpha_s\}$. We distinguish the following cases:

(aa) $\ell(sw) = \ell(w) - 1$: Suppose $G \in \mathrm{Min}_s(w)$ and note that $\{\alpha_s, \delta\} \in \mathcal{P}$ for each $\alpha_s \neq \delta \in \Phi(G)$. We have to show that

$$\left[\left(\prod_{\gamma \in M^G_{\alpha_s,\alpha}} u_{s\gamma}\right) u_{s\alpha}, \left(\prod_{\gamma \in M^G_{\alpha_s,\beta}} u_{s\gamma}\right) u_{s\beta}\right] = \prod_{\gamma \in M^G_{\alpha,\beta}} \left(\prod_{\delta \in M^G_{\alpha_s,\gamma}} u_{s\delta}\right) u_{s\gamma}$$

is a relation in M_s. Note that $[u_\alpha, u_\beta] = \prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma$ is a relation in U_w and $V_{w,s}$ and hence also the u_s-conjugate, which is given by

$$\begin{aligned}\left[\left(\prod_{\gamma\in M^G_{\alpha_s,\alpha}} u_\gamma\right) u_\alpha, \left(\prod_{\gamma\in M^G_{\alpha_s,\beta}} u_\gamma\right) u_\beta\right] &= [u_s u_\alpha u_s, u_s u_\beta u_s] \\ &= u_s [u_\alpha, u_\beta] u_s \\ &= u_s \left(\prod_{\gamma\in M^G_{\alpha,\beta}} u_\gamma\right) u_s \\ &= \prod_{\gamma\in M^G_{\alpha,\beta}} \left(\prod_{\delta\in M^G_{\alpha_s,\gamma}} u_\delta\right) u_\gamma\end{aligned}$$

Using Lemma (2.1.7) and the homomorphism $U_{sw} \to M_s$ the claim follows.

(bb) $\ell(sw) = \ell(w) + 1$: Then $\alpha_s \notin \Phi(G)$. Let $\delta \in \Phi(G)$. Then either $-\alpha_s \subseteq \delta$ or $o(r_{\alpha_s} r_\delta) < \infty$. At first we observe the following: Suppose $o(r_{\alpha_s} r_\delta) < \infty$, let $R \in \partial^2\alpha_s \cap \partial^2\delta$ and let $H \in \mathrm{Min}_s(sw)$ be such that $\Phi(R) \subseteq \Phi(H)$. Then $\alpha_s \leq_H \beta$ for each $\alpha_s \neq \beta \in \Phi(H)$. Using Lemma (2.2.7) we deduce the following relation in N_R:

$$\begin{aligned}\left(\prod_{\gamma\in M^H_{\alpha_s,\delta}} u_{s\gamma}\right) u_{s\delta} &= \tau_s u_s . u_\delta \\ &= u_s \tau_s u_s \tau_s . u_\delta \\ &= u_s . \left(\prod_{\gamma\in M^H_{\alpha_s,s\delta}} u_{s\gamma}\right) u_\delta \\ &= \left(\prod_{\gamma\in M^H_{\alpha_s,s\delta}} \left(\prod_{\omega\in M^H_{\alpha_s,s\gamma}} u_\omega\right) u_{s\gamma}\right)\left(\prod_{\omega\in M^H_{\alpha_s,\delta}} u_\omega\right) u_\delta\end{aligned}$$

Since we have a canonical homomorphism $N_R \to M_s$, this is also a relation in M_s. In particular, we have the following relation in M_s (using Lemma (2.2.5) (b) and the fact that $v_\rho = \left(\prod_{\omega\in M^H_{\alpha_s,\rho}} u_\omega\right) u_\rho$ for both $\rho \in \{s\gamma, \delta\}$):

$$\begin{aligned}\left(\prod_{\gamma\in M^H_{\alpha_s,\delta}} u_{s\gamma}\right) u_{s\delta} &= \left(\prod_{\gamma\in M^H_{\alpha_s,s\delta}} \left(\prod_{\omega\in M^H_{\alpha_s,s\gamma}} u_\omega\right) u_{s\gamma}\right)\left(\prod_{\omega\in M^H_{\alpha_s,\delta}} u_\omega\right) u_\delta \\ &= \left(\prod_{\gamma\in M^H_{\alpha_s,s\delta}} v_{s\gamma}\right) v_\delta \\ &= \left(\prod_{\gamma\in M^{sG}_{\alpha_s,s\delta}} v_{s\gamma}\right) v_\delta\end{aligned}$$

This shows that v_δ is mapped onto $\left(\prod_{\gamma \in M^{sG}_{\alpha_s, s\delta}} v_{s\gamma}\right) v_\delta$ for each $\delta \in \Phi(G)$. In particular, this assignment does not depend on $o(r_{\alpha_s} r_\delta)$ for $\delta \in \Phi(G)$. We have to verify that

$$\left[\left(\prod_{\gamma \in M^{sG}_{\alpha_s, s\alpha}} v_{s\gamma}\right) v_\alpha, \left(\prod_{\gamma \in M^{sG}_{\alpha_s, s\beta}} v_{s\gamma}\right) v_\beta\right] = \prod_{\gamma \in M^G_{\alpha,\beta}} \left(\prod_{\delta \in M^{sG}_{\alpha_s, s\gamma}} v_{s\delta}\right) v_\gamma$$

is a relation in M_s. For that we observe the following:

- $[v_{s\alpha}, v_{s\beta}] = \prod_{\gamma \in M^{sG}_{s\alpha, s\beta}} v_\gamma$ is a relation in M_s.
- $\left[\left(\prod_{\gamma \in M^{sG}_{\alpha_s, s\alpha}} u_{s\gamma}\right) u_\alpha, \left(\prod_{\gamma \in M^{sG}_{\alpha_s, s\beta}} u_{s\gamma}\right) u_\beta\right] = \prod_{\gamma \in M^{sG}_{s\alpha, s\beta}} \left(\prod_{\delta \in M^{sG}_{\alpha_s, \gamma}} u_{s\delta}\right) u_{s\gamma}$ is a relation in M_s by (aa).
- Since u_s is an automorphism of M_s we deduce that the following is also a relation in M_s:

$$\left[\left(\prod_{\gamma \in M^{sG}_{\alpha_s, s\alpha}} v_{s\gamma}\right) v_\alpha, \left(\prod_{\gamma \in M^{sG}_{\alpha_s, s\beta}} v_{s\gamma}\right) v_\beta\right] = \prod_{\gamma \in M^{sG}_{s\alpha, s\beta}} \left(\prod_{\delta \in M^{sG}_{\alpha_s, \gamma}} v_{s\delta}\right) v_{s\gamma}$$

- Since $\mathcal{M}$ is Weyl-invariant, we have $M^{sG}_{s\alpha, s\beta} = sM^G_{\alpha,\beta}$. Using substitution, we deduce that

$$\left[\left(\prod_{\gamma \in M^{sG}_{\alpha_s, s\alpha}} v_{s\gamma}\right) v_\alpha, \left(\prod_{\gamma \in M^{sG}_{\alpha_s, s\beta}} v_{s\gamma}\right) v_\beta\right] = \prod_{\gamma \in M^G_{\alpha,\beta}} \left(\prod_{\delta \in M^{sG}_{\alpha_s, s\gamma}} v_{s\delta}\right) v_\gamma$$

  is also a relation in M_s.

(v) $v_\alpha = \left(\prod_{\gamma \in M^G_{\alpha_s, \alpha}} u_\gamma\right) u_\alpha$: This holds by definition.

This shows the existence of the endomorphism $\tau_s : M_s \to M_s$. Using the isomorphism $\varphi : M_s \to N_s$ from Lemma (2.2.4), we obtain an endomorphism $\tau_s : N_s \to N_s$ via $N_s \overset{\varphi^{-1}}{\to} M_s \overset{\tau_s}{\to} M_s \overset{\varphi}{\to} N_s$. Moreover, this endomorphism is as required. □

(2.2.10) Corollary. *We have* $\tau_s^2 = 1 = (u_s \tau_s)^3$. *In particular,* $\tau_s \in \mathrm{Aut}(N_s)$.

Proof. For short we will not specify a gallery G. If $M^G_{\alpha_s,\alpha}$ appears, we will implicitly assume that $G \in \mathrm{Min}_s(w)$ for some $w \in W$ with $\ell(sw) = \ell(w) - 1$ such that $\alpha \in \Phi(G)$.

By the previous proposition we have $\tau_s(u_\alpha) = u_{s\alpha}$ for each $\alpha_s \neq \alpha \in \Phi_+$ and $\tau_s(u_s u_\beta u_s) = u_s \left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} u_{s\gamma}\right) u_\beta u_s$ for each $-\alpha_s \subseteq \beta \in \Phi_+$. Using this we establish the claim. At first we will show $\tau_s^2 = 1$. Therefore, let $\alpha_s \neq \alpha \in \Phi_+$. Then $\alpha_s \neq s\alpha \in \Phi_+$ and we have $\tau_s^2(u_\alpha) = \tau_s(u_{s\alpha}) = u_\alpha$. Now let $-\alpha_s \subseteq \beta \in \Phi_+$. Note that for $\gamma \in M^G_{\alpha_s, s\beta}$ we have $-\alpha_s \subseteq s\gamma$. This implies

$$\tau_s^2(u_s u_\beta u_s) = \tau_s(u_s \left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} u_{s\gamma}\right) u_\beta u_s)$$

$$
\begin{aligned}
&= \left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} \tau_s(u_s u_{s\gamma} u_s) \right) \tau_s(u_s u_\beta u_s) \\
&= \left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} u_s \left(\prod_{\delta \in M^G_{\alpha_s, \gamma}} u_{s\delta} \right) u_{s\gamma} u_s \right) \left(u_s \left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} u_{s\gamma} \right) u_\beta u_s \right) \\
&= u_s \left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} \left(\prod_{\delta \in M^G_{\alpha_s, \gamma}} u_{s\delta} \right) u_{s\gamma} \right) \left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} u_{s\gamma} \right) u_\beta u_s
\end{aligned}
$$

Note that we have the following relation in U_w and hence in $V_{w,s}$:

$$
\begin{aligned}
\left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} \left(\prod_{\delta \in M^G_{\alpha_s, \gamma}} u_\delta \right) u_\gamma \right) \prod_{\gamma \in M^G_{\alpha_s, s\beta}} u_\gamma &= \left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} [u_s, u_\gamma] u_\gamma \right) [u_s, u_{s\beta}] \\
&= u_s [u_s, u_{s\beta}] u_s [u_s, u_{s\beta}] \\
&= (u_{s\beta} u_s u_{s\beta})^2 = 1
\end{aligned}
$$

Using Lemma (2.1.7), the following is a relation in U_{sw} and hence in N_s:

$$
\left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} \left(\prod_{\delta \in M^G_{\alpha_s, \gamma}} u_{s\delta} \right) u_{s\gamma} \right) \left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} u_{s\gamma} \right) = 1
$$

This shows $\tau_s^2(u_s u_\beta u_s) = u_s u_\beta u_s$ and hence $\tau_s^2 = 1$. In particular, τ_s is an automorphism. To show that $(u_s \tau_s)^3 = 1$, we distinguish the following cases. Let $\alpha_s \neq \alpha \in \Phi_+$. Assume that $o(r_{\alpha_s} r_\alpha) < \infty$ and let $R \in \partial^2 \alpha_s \cap \partial^2 \alpha$. Note that we have a homomorphism $N_R \to M_s \to N_s$. Lemma (2.2.7) yields

$$
\left(\prod_{\gamma \in M^G_{\alpha_s, s\alpha}} u_{s\gamma} \right) u_\alpha = \left(\prod_{\gamma \in M^G_{\alpha_s, \alpha}} \left(\prod_{\gamma' \in M^G_{\alpha_s, s\gamma}} u_{\gamma'} \right) u_{s\gamma} \right) \left(\prod_{\gamma \in M^G_{\alpha_s, s\alpha}} u_\gamma \right) u_{s\alpha}
$$

and hence $(u_s \tau_s)^3(u_\alpha) = u_\alpha$. Thus we assume $\alpha_s \subsetneq \alpha$. Then we have the following:

$$
\begin{aligned}
(u_s \tau_s)^3(u_\alpha) &= (u_s \tau_s)^2(u_s u_{s\alpha} u_s) \\
&= (u_s \tau_s u_s) \left(u_s \left(\prod_{\gamma \in M^G_{\alpha_s, \alpha}} u_{s\gamma} \right) u_{s\alpha} u_s \right) \\
&= (u_s \tau_s) \left(\left(\prod_{\gamma \in M^G_{\alpha_s, \alpha}} u_{s\gamma} \right) u_{s\alpha} \right) \\
&= u_s \left(\left(\prod_{\gamma \in M^G_{\alpha_s, \alpha}} u_\gamma \right) u_\alpha \right) \\
&= u_\alpha
\end{aligned}
$$

Now we assume $-\alpha_s \subseteq \alpha$. Using the previous case, we deduce the following:

$$\begin{aligned}(u_s\tau_s)^3(u_s u_\alpha u_s) &= (u_s\tau_s)(u_{s\alpha}) = u_s(u_\alpha) = u_s u_\alpha u_s \\ (u_s\tau_s)^3(u_\alpha) &= (u_s\tau_s)^2([u_s, u_{s\alpha}]u_{s\alpha}) = (u_s\tau_s)^{-1}([u_s, u_{s\alpha}]u_{s\alpha}) = u_\alpha\end{aligned}$$ □

(2.2.11) Definition. Note that $\varphi : \mathrm{Sym}(3) \to \langle u_s, \tau_s \rangle \leq \mathrm{Aut}(N_s), \begin{cases}(1\ \ 2) \mapsto u_s \\ (2\ \ 3) \mapsto \tau_s\end{cases}$ is an epimorphism. Thus we define the group $P_s := \mathrm{Sym}(3) \ltimes_\varphi N_s$. For short we will denote the elements in Sym(3) by their images in $\mathrm{Aut}(N_s)$. Note that $\tau_s n_s \tau_s = \tau_s(n_s) \in N_s$. In particular, we have $\tau_s u_\alpha \tau_s = u_{s\alpha}$ for each $\alpha_s \neq \alpha \in \Phi_+$. Note that $U_+ \cong \langle u_s \rangle \ltimes N_s \leq P_s$.

(2.2.12) Definition. We let G be the direct limit of the groups $U_+, (P_s)_{s\in S}, (\langle \tau_s \rangle)_{s\in S}, W$ with canonical inclusions $U_+ \hookrightarrow P_s, \langle \tau_s \rangle \hookrightarrow P_s, \langle \tau_s \rangle \hookrightarrow W, \tau_s \mapsto s$.

(2.2.13) Lemma. *Let $s_1, \dots, s_n, t_1, \dots, t_m, s, t \in S$ be such that $s_1 \cdots s_n \alpha_s = t_1 \cdots t_m \alpha_t$. Then $U_{\alpha_s}^{\tau_n \cdots \tau_1} = U_{\alpha_t}^{\tau'_m \cdots \tau'_1}$, where $\tau_i = \tau_{s_i}$ and $\tau'_j = \tau_{t_j}$.*

Proof. The claim follows if $U_{\alpha_s}^{\tau_n \cdots \tau_1 \tau'_1 \cdots \tau'_m} = U_{\alpha_t}$. Suppose $f_1, \dots, f_k \in S$ with $\ell(f_1 \cdots f_k) = k$ and $f_1 \cdots f_k = t_m \cdots t_1 s_1 \cdots s_n$. Then $f_k \cdots f_1 = s_n \cdots s_1 t_1 \cdots t_m$ and since every relation in W is a relation in G, we obtain

$$\tau_{f_k} \cdots \tau_{f_1} = \tau_{s_n} \cdots \tau_{s_1} \tau_{t_1} \cdots \tau_{t_m}$$

Now let $i = \max\{1, \dots, k \mid \exists r \in S : f_i \cdots f_k \alpha_s = \alpha_r\}$. For $g := f_1 \cdots f_k$ we have $g\alpha_s = \alpha_t$ and hence $g^{-1} \in \alpha_s$. This implies $\ell(gs) = \ell((gs)^{-1}) = \ell(sg^{-1}) > \ell(g^{-1}) = \ell(g)$. This implies $f_k \neq s$ and hence $f_k\alpha_s \in \Phi_+$. Thus the roots $\alpha_s, f_k\alpha_s, \dots, f_i \cdots f_k\alpha_s = \alpha_r$ are all positive roots and we obtain $U_{\alpha_s}^{\tau_{f_k} \cdots \tau_{f_i}} = U_{f_i \cdots f_k \alpha_s} = U_{\alpha_r}$ in G. If $i = 1$ we are done. Otherwise we repeat the argument with $g := f_1 \cdots f_{i-1}$. After finitely many steps we are done. □

(2.2.14) Theorem. *Assume that $P_s \to G$ is injective for every $s \in S$. Then $\mathcal{M}$ is integrable.*

Proof. Let $\alpha \in \Phi$ be a root. Then there exist $w \in W$ and $s \in S$ with $\alpha = w\alpha_s$. Let $s_1, \dots, s_k \in S$ be such that $w = s_1 \cdots s_k$ and let $\tau_i := \tau_{s_i}$. Then we define

$$U_\alpha := U_{\alpha_s}^{\tau_k \cdots \tau_1}$$

In view of the previous lemma, the group U_α is well-defined. We will show that $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ is an RGD-system of type (W, S).

(RGD0) The mappings $P_s \to G$ are injective and hence the groups U_α are non-trivial.

(RGD1) Let $\{\alpha, \beta\} \subseteq \Phi$ be a prenilpotent pair. Then there exists $w \in W$ such that $\{w\alpha, w\beta\} \in \mathcal{P}$. By definition of the root groups and the commutator blueprint we deduce (τ_w is a product of suitable τ_s)

$$\begin{aligned}[][U_\alpha, U_\beta] = [U_{w\alpha}, U_{w\beta}]^{\tau_w} &\leq \langle U_\gamma \mid \gamma \in (w\alpha, w\beta) \rangle^{\tau_w} \\ &= \langle U_{w^{-1}\gamma} \mid \gamma \in (w\alpha, w\beta) \rangle \\ &= \langle U_\gamma \mid \gamma \in (\alpha, \beta) \rangle\end{aligned}$$

(RGD2) For $s \in S$ we have $(u_s\tau_s)^3 = 1$ and hence $\tau_s = \tau_s(u_s\tau_s)^3 = u_{-s}u_s u_{-s}$ by Corollary (2.2.10). Let $\alpha \in \Phi$ be a root. Then there exist $w \in W, t \in S$ such that $\alpha = w\alpha_t$. Let $s_1, \dots, s_k \in S$ be such that $w = s_1 \cdots s_k$ and let $\tau_i := \tau_{s_i}$. Then $s\alpha = ss_1 \cdots s_k\alpha_t$ and we deduce

$$U_\alpha^{\tau_s} = \left(U_{\alpha_t}^{\tau_k \cdots \tau_1}\right)^{\tau_s} = U_{\alpha_t}^{\tau_k \cdots \tau_1 \tau_s} = U_{s\alpha}$$

(RGD3) Since $P_s \to G$ is injective, we have $\tau_s \notin U_+$. As $U_+^{u_s} = U_+$ and $(u_s\tau_s)^3 = 1$, we infer $u_{-s} = \tau_s u_s \tau_s = u_s \tau_s u_s \notin U_+^{u_s} = U_+$

(RGD4) Since G is generated by U_α and τ_s, it is generated by all root groups.

Note that $\mathcal{M}_\mathcal{D}$ is a commutator blueprint of type $\mathrm{Dyn}(W, S)$. By definition we have $M_{\alpha,\beta}^G = M(\mathcal{D})_{\alpha,\beta}^G$ for each $(G, \alpha, \beta) \in \mathcal{I}$. We deduce that $\mathcal{M}$ is integrable. □

(2.2.15) Corollary. *Assume that $m_{st} = \infty$ for all $s \neq t \in S$. Then every Weyl-invariant commutator blueprint is integrable.*

Proof. Let $\mathcal{M}$ be a commutator blueprint which is Weyl-invariant. Since $m_{st} = \infty$ for all $s \neq t \in S$ we deduce that $\Sigma(W, S)$ is a tree and hence the canonical homomorphisms $U_w \to U_+$ are injective (cf. [32, Ch. 4.4]). In particular, $\mathcal{M}$ is faithful. Since G is (isomorphic to) the direct limit of the groups U_+ and $(P_s)_{s\in S}$, i.e. the free amalgamated product of the $(P_s)_{s\in S}$ along the common subgroup U_+, the claim follows from the previous theorem. □

2.3. An action of the P_s

In this section we will show that the groups P_s act faithfully on a chamber system $\mathbf{C}$ over S for every $s \in S$. Moreover, we will give sufficient conditions in order to show that $W \cong \langle \tau_s \mid s \in S \rangle$ acts on $\mathbf{C}$. In particular, the action of the groups P_s extend to an action of G on $\mathbf{C}$. This will imply that the mappings $P_s \to G$ are injective. The sufficient conditions are rather mild and only depend on the commutator blueprint.

We start by defining the chamber system $\mathbf{C}$ over S. We let $U_{1_W} := \{1\} \leq U_+$. The set of chambers is given by $\mathcal{C} := \{gU_w \mid g \in U_+, w \in W\}$, and s-adjacency is defined as follows:

$$gU_w \sim_s hU_{w'} :\Leftrightarrow w' \in \{w, ws\} \text{ and } g^{-1}h \in U_w \cup U_{ws}$$

Then $\mathbf{C} = (\mathcal{C}, (\sim_s)_{s\in S})$ is a chamber system over S. The idea of considering this chamber system is not new (cf. [2, Section 8.7]). Before we define an action of P_s on the chamber system $\mathbf{C}$ we note that every element of U_+ can be written uniquely as nu with $n \in N_s$ and $u \in U_s$ by Lemma (2.2.2). Thus it suffices to define the action on cosets nuU_w with $n \in N_s, u \in U_s$ and $w \in W$. To show that our assignment will actually be an action we need the following auxiliary result.

(2.3.1) Lemma. *For $n \in N_s$ the following hold:*

(a) If $n \in U_w$, then $n^{\tau_s} \in N_s \cap U_{sw}$;

(b) If $\ell(sw) = \ell(w) + 1$ and $n^{u_s} \in U_w$, then $n^{\tau_s u_s} \in N_s \cap U_w$.

Proof. Let $w \in W$ and $G = (c_0, \ldots, c_k) \in \mathrm{Min}(w)$ and let $(\alpha_1, \ldots, \alpha_k)$ be the sequence of roots crossed by G. Since $n \in U_w$, there exists $u_i \in U_{\alpha_i}$ such that $n = u_1 \cdots u_k$. If $\ell(sw) = \ell(w) + 1$, then $u_i^{\tau_s} \in U_{s\alpha_i} \leq U_{sw}$ and hence $n^{\tau_s} \in U_{sw}$. Thus we assume that $\ell(sw) = \ell(w) - 1$. Using Lemma (2.1.4) we can assume $G \in \mathrm{Min}_s(w)$ and hence $\alpha_1 = \alpha_s$. Since $U_{\alpha_i} \leq N_s$ for each $2 \leq i \leq k$, we have $u_1 = n(u_2 \cdots u_k)^{-1} \in N_s \cap U_s = \{1\}$. Thus $n^{\tau_s} \in U_{sw}$ and Assertion (a) follows. Now we assume that $\ell(sw) = \ell(w)+1$ and that $n^{u_s} \in U_w$. Note that $n^{u_s} \in N_s$. Then (a) provides $n^{u_s\tau_s} \in N_s \cap U_{sw}$. Since $\ell(ssw) = \ell(w) = \ell(sw) - 1$, we have $u_s \in U_{sw}$ and hence $n^{u_s\tau_s u_s} \in N_s \cap U_{sw}$. Using Corollary (2.2.10) and Assertion (a) we obtain $n^{\tau_s u_s} = n^{u_s \tau_s u_s \tau_s} \in N_s \cap U_w$. □

(2.3.2) Remark. Let $\langle G_s \mid R_s\rangle$ be a presentation of N_s. Then a presentation of P_s is given by $\langle u_s, \tau_s, G_s \mid u_s^2, \tau_s^2, (u_s\tau_s)^3, R_s, u_s n u_s = n^{u_s}, \tau_s n \tau_s = n^{\tau_s}$ for every $n \in G_s\rangle$.

(2.3.3) Proposition. *For $s \in S$ the group P_s acts on $\mathbf{C}$ as follows:*

$$g.nuU_w := \begin{cases} gnuU_w & g \in U_+ \\ n^{\tau_s}U_{sw} & g = \tau_s, \ell(sw) = \ell(w) - 1 \text{ or } u = 1 \\ n^{\tau_s}u_sU_w & g = \tau_s, \ell(sw) = \ell(w) + 1, u = u_s \end{cases}$$

Moreover, this action is faithful.

Proof. For $g \in U_+ \cup \{\tau_s\}$ we let $\varphi_g : \mathcal{C} \to \mathcal{C}, nuU_w \mapsto g.nuU_w$.

The mapping φ_g is well-defined: We note that $u_s.nuU_w = u_snuU_w = n^{u_s}u_suU_w$. At first we will show that the assignment is well-defined. Since the assignment of U_+ is via left multiplication, it suffices to consider the assignment of τ_s. Let $w \in W$ and $n, n' \in N_s, u, u' \in U_s$ such that $nuU_w = n'u'U_w$. Then $u^{-1}n^{-1}n'u' \in U_w$.

(Case I) $\ell(sw) = \ell(w) - 1$: Then $u_s \in U_w$ and hence $n^{-1}n' \in U_w$. Using Lemma (2.3.1)(a), we obtain $(n^{-1}n')^{\tau_s} \in U_{sw}$. This implies $\tau_s.nuU_w = n^{\tau_s}U_{sw} = (n')^{\tau_s}U_{sw} = \tau_s.n'u'U_w$.

(Case II) $\ell(sw) = \ell(w) + 1$: We distinguish the following three cases:

- $u = 1 = u'$: Then the claim follows as in Case I.
- $\{u, u'\} = \{1, u_s\}$: Assume $u \neq 1 = u'$. Then we have $u^{-1}n^{-1}n' \in U_w$. Since $\ell(sw) = \ell(w) + 1$, we have $U_w \leq N_s$ and hence $u_s = u^{-1} \in N_s$. This is a contradiction. The case $u = 1 \neq u'$ is similar.
- $u \neq 1 \neq u'$: Then $u = u_s = u'$ and $(n^{-1}n')^{u_s} \in N_s \cap U_w$. Using Lemma (2.3.1)$(b)$, we obtain $(n^{-1}n')^{\tau_s u_s} \in N_s \cap U_w$ and hence $\tau_s.nuU_w = n^{\tau_s}u_sU_w = (n')^{\tau_s}u_sU_w = \tau_s.n'u'U_w$.

Thus φ_g is well-defined.

φ_g is bijective for every $g \in U_+ \cup \{\tau_s\}$: We will show that $\varphi_{g^{-1}} \circ \varphi_g = \mathrm{id}$. If $g \in U_+$ there is nothing to show. Thus we consider $g = \tau_s$. By construction and Corollary (2.2.10) we have $\varphi_{\tau_s} \circ \varphi_{\tau_s} = \mathrm{id}$ and φ_g is bijective for every $g \in U_+ \cup \{\tau_s\}$.

$\varphi_g \in \mathrm{Aut}(\mathbf{C})$: As φ_g is bijective, it suffices to show that φ_g preserves t-adjacency for each $t \in S$. Let $n, n' \in N_s, u, u' \in U_s$ and $w, w' \in W$ such that $nuU_w \sim_t n'u'U_{w'}$. Then we have $w' \in \{w, wt\}$ and $u^{-1}n^{-1}n'u' \in U_w \cup U_{wt}$. Since for $g \in U_+$ the bijection φ_g is left multiplication by g, it preserves t-adjacency. Thus it suffices to consider φ_{τ_s}. We distinguish the following cases:

(Case I) $u = 1 = u'$: Then $\tau_s.nU_w = n^{\tau_s}U_{sw}$ and $\tau_s.n'U_{w'} = (n')^{\tau_s}U_{sw'}$. Because of the t-adjacency we have $n^{-1}n \in U_w \cup U_{wt}$ and Lemma (2.3.1)(a) implies $(n^{-1})^{\tau_s}(n')^{\tau_s} = (n^{-1}n')^{\tau_s} \in U_{sw} \cup U_{swt}$. Since $sw' \in \{sw, swt\}$, we deduce $\varphi_{\tau_s}(nU_w) \sim_t \varphi_{\tau_s}(n'U_{w'})$.

(Case II) $\ell(sw) = \ell(w) - 1$ and $\ell(sw') = \ell(w') - 1$: Then $nuU_w = nU_w$ and $n'u'U_{w'} = n'U_{w'}$ and the claim follows from Case I.

(Case III) $\ell(sw) = \ell(w) + 1$ and $\ell(sw') = \ell(w') + 1$: Recall that $w' \in \{w, wt\}$. If $u = 1 = u'$ the claim follows from Case I. If $u = u_s = u'$ we have $(n^{-1}n')^{u_s} \in U_w \cup U_{wt}$ and $\tau_s.nu_sU_w = n^{\tau_s}u_sU_w, \tau_s.n'u_sU_{w'} = (n')^{\tau_s}u_sU_{w'}$. If $\ell(swt) = \ell(wt) + 1$, then we have $(n^{-1}n')^{\tau_s u_s} \in N_s \cap (U_w \cup U_{wt})$ by Lemma (2.3.1)(b) and we deduce $\varphi_{\tau_s}(nuU_w) \sim_t \varphi_{\tau_s}(n'u'U_{w'})$. Thus we assume $\ell(swt) = \ell(wt) - 1$. Then $u_s \in U_{wt}$. Since $\ell(wt) - 1 = \ell(swt) \geq \ell(sw) - 1 = \ell(w)$, we have $\ell(wt) = \ell(w) + 1$ and thus $(n^{-1}n')^{u_s} \in U_w \cup U_{wt} = U_{wt}$. This implies $n^{-1}n' \in U_{wt}$. By Lemma (1.1.1) we infer $swt = w$. Now Lemma (2.3.1)(a) yields $(n^{-1}n')^{\tau_s} \in N_s \cap U_{swt} = N_s \cap U_w \leq N_s \cap U_{wt}$ and, as $u_s \in U_{wt}$, $(n^{-1}n')^{\tau_s u_s} \in U_{wt} = U_w \cup U_{wt}$. In particular, $\varphi_{\tau_s}(nuU_w) \sim_t \varphi_{\tau_s}(n'u'U_{w'})$.

If $u = 1 \neq u'$ we have $(n^{-1})n'u_s \in U_w \cup U_{wt}$ and $\tau_s.nU_w = n^{\tau_s}U_{sw}, \tau_s.n'u_sU_{w'} = (n')^{\tau_s}u_sU_{w'}$. If $\ell(swt) = \ell(wt) + 1$, we would have $U_w, U_{wt} \leq N_s$ and hence $u_s \in N_s$. Thus we have $\ell(swt) = \ell(wt) - 1$. Since $\ell(sw') = \ell(w') + 1$ and $w' \in \{w, wt\}$, we deduce $w = w'$. As $\ell(sw) = \ell(w) + 1$, we obtain $\ell(wt) - 1 = \ell(swt) \geq \ell(sw) - 1 = \ell(w)$. This yields $\ell(wt) = \ell(w) + 1$ and hence $swt = w$ as before. This implies $w' = w = swt \in \{sw, swt\}$ and $U_w \leq U_{wt}$. Thus we obtain $(n^{-1})n'u_s \in U_{wt}$ and hence $(n^{-1})n' \in U_{wt}$. Using Lemma (2.3.1)(a) we obtain $(n^{-1}n')^{\tau_s} \in U_{swt} \leq U_{sw}$ (since $\ell(swt) = \ell(sw) - 1$). This implies $(n^{-1}n')^{\tau_s}u_s \in U_{sw} = U_{sw} \cup U_{swt}$ and hence $\varphi_{\tau_s}(nU_w) \sim_t \varphi_{\tau_s}(n'u'U_{w'})$. The case $u \neq 1 = u'$ is similar.

(Case IV) Without loss of generality we assume $\ell(sw) = \ell(w) - 1$ and $\ell(sw') = \ell(w') + 1$. This implies $w \neq w'$ and hence $w' = wt$. Thus $\ell(wt) = \ell(w') = \ell(sw') - 1 \leq \ell(sw) = \ell(w) - 1$ and hence $\ell(wt) = \ell(w) - 1$. Since $\ell(swt) = \ell(w)$, Lemma (1.1.1) implies $w = swt$.

Now we have $nuU_w = nU_w$ and $\tau_s.nU_w = n^{\tau_s}U_{sw}$. If $u' = 1$, the claim follows from Case I. Thus we assume $u' = u_s$. Then $\tau_s.n'u_sU_{w'} = (n')^{\tau_s}u_sU_{w'}$. Since $w' = wt = sw \in \{sw, swt\}$ it suffices to show that $(n^{-1}n')^{\tau_s}u_s \in U_{sw} \cup U_{swt}$. As $\ell(wt) = \ell(w) - 1$, we have $U_{wt} \leq U_w$. Because $\ell(sw) = \ell(w) - 1$ and $n^{-1}n'u_s \in U_w \cup U_{wt} = U_w$ we have $u_s \in U_w$ and hence $n^{-1}n' \in U_w$. Using Lemma (2.3.1)(a) we deduce $(n^{-1}n')^{\tau_s} \in U_{sw}$. Since $\ell(swt) = \ell(w) = \ell(sw) + 1$, we obtain $U_{sw} \leq U_{swt}$. This implies $(n^{-1}n')^{\tau_s}u_s \in U_{swt} \subseteq U_{sw} \cup U_{swt}$ and we obtain $\varphi_{\tau_s}(nU_w) \sim_t \varphi_{\tau_s}(n'u'U_{w'})$.

The assignment $g \mapsto \varphi_g$ for $g \in U_+ \cup \{\tau_s\}$ extends to a homomorphism $P_s \to \mathrm{Aut}(\mathbf{C})$: For this we need to consider a presentation of P_s (cf. Remark (2.3.2)) and show that every relation of P_s acts trivial on the chamber system $\mathbf{C}$. Since the action of $U_+ \leq P_s$ is via left multiplication it suffices to consider relations concerning τ_s. As we have already seen before, τ_s^2 acts trivial. Let $m, m' \in N_s$ be such that $\tau_s m \tau_s = \tau_s(m) = (m')^{-1}$. Then

$$\tau_s m \tau_s m'.nuU_w = \tau_s m.(m'n)^{\tau_s}(\tau_s.uU_w) = (m(m'n)^{\tau_s})^{\tau_s}uU_w = m^{\tau_s}m'nuU_w = nuU_w$$

Thus it suffices to show that $(u_s\tau_s)^3$ acts trivial on $\mathbf{C}$. As $(u_s\tau_s)^3.nuU_w = n^{(\tau_s u_s)^3}\cdot(u_s\tau_s)^3.uU_w$, we can assume that $n = 1$, since $(u_s\tau_s)^3$ acts trivial on N_s by Corollary (2.2.10). If $\ell(sw) = \ell(w) - 1$, then $uU_w = U_w = u_sU_w$ and we obtain the following:

$$(u_s\tau_s)^3.uU_w = (u_s\tau_s)^2.u_sU_{sw} = u_s\tau_s.U_{sw} = u_sU_w = U_w$$

Thus we can assume that $\ell(sw) = \ell(w) + 1$. We distinguish the cases $u = 1$ and $u = u_s$:

$$(u_s\tau_s)^3.U_w = (u_s\tau_s)^2.U_{sw} = u_s\tau_s.u_sU_w = U_w$$
$$(u_s\tau_s)^3.u_sU_w = (u_s\tau_s)^2.U_w = u_s\tau_s.U_{sw} = u_sU_w$$

The homomorphism $P_s \to \mathrm{Aut}(\mathbf{C})$ is injective: We have to show that each $1 \neq g \in P_s$ induces a non-trivial automorphism of the chamber system. We first consider $1 \neq g \in \mathrm{Sym}(3) = \{1, u_s, u_s\tau_s, u_s\tau_s u_s, \tau_s u_s, \tau_s\}$. Then we have the following:

$$u_s.U_{1_W} = u_sU_{1_W}$$

$$\begin{aligned} u_s\tau_s.U_{1_W} &= U_s \\ u_s\tau_s u_s.u_s U_{1_W} &= U_s \\ \tau_s u_s.U_{1_W} &= u_s U_{1_W} \\ \tau_s.U_{1_W} &= U_s \end{aligned}$$

Thus each $1 \neq g \in \mathrm{Sym}(3)$ acts non-trivial. Now we consider the general case. Let $1 \neq g \in P_s$. Then there exist $x \in \mathrm{Sym}(3), n \in N_s$ such that $g = xn$. If $x = 1$, we have $g.n^{-1}U_{1_W} = U_{1_W} \neq n^{-1}U_{1_W}$. Otherwise the let $c \in \mathcal{C}$ be as above such that $x.c \neq c$. Then $g.n^{-1}c \neq n^{-1}c$ and the claim follows. □

2.4. Braid relations act trivially on suitable subset

For $J \subseteq S$ we define $\Phi^J := \{w\alpha_s \mid s \in J, w \in \langle J \rangle\}$ and $\Phi^J_\varepsilon := \Phi^J \cap \Phi_\varepsilon$ for $\varepsilon \in \{+,-\}$. Moreover, we define for all $s \neq t \in S$ the subgroup $U_{s,t} := \langle U_\alpha \mid \alpha \in \Phi_+^{\{s,t\}} \rangle$ and $N_{s,t} := \langle x^{-1}U_\alpha x \mid x \in U_{s,t}, \alpha \in \Phi_+ \backslash \Phi_+^{\{s,t\}} \rangle$. It is not hard to see that $N_{s,t}$ is a normal subgroup of U_+ and that $N_{s,t}$ is stabilized by τ_s and by τ_t.

(2.4.1) Lemma. *Let $s \neq t \in S$ be with $m_{st} < \infty$ and let $J := \{s,t\}$. Then the sub-chamber system $\mathbf{C}_J = (\mathcal{C}_J, (\sim_j)_{j \in J})$ with $\mathcal{C}_J = \{uU_w \mid u \in U_{s,t}, w \in \langle J \rangle\}$ is a spherical building of rank 2.*

Proof. Since $\mathcal{M}$ is faithful, the mapping $U_{r_J} \to U_+$ is injective. Considering the sub-chamber system $\mathbf{C}_J$ as in the statement, this is exactly the chamber system which we get from the RGD-system over $\mathbb{F}_2$ of type $I_2(m_{st})$. This chamber system is a building by [2, Exercise 8.36(b)]. □

(2.4.2) Lemma. *Let $s \neq t \in S$ be with $m_{st} < \infty$. Then we have $(\tau_s\tau_t)^{m_{st}}.u_{s,t}U_w = u_{s,t}U_w$ for all $w \in W$ and $u_{s,t} \in U_{s,t}$.*

Proof. We put $J := \{s,t\}$. For $w \in W$ we let $w' \in W, w_J \in \langle J \rangle$ be such that $w = w_J w'$ and $\ell(sw') = \ell(w') + 1 = \ell(tw')$. Then the action of τ_s on uU_w only depends on u and w_J and is independent on w', i.e. for $u, u' \in U_{s,t}$ and $w'_J \in \langle J \rangle$ with $\tau_s.uU_{w_J} = u'U_{w'_J}$, we have $\tau_s.uU_w = u'U_{w'_J w'}$. Thus it suffices to show the claim for $w \in \langle J \rangle$. We restrict the action of $(\tau_s\tau_t)^{m_{st}}$ to the chambers of the form uU_w with $u \in U_{s,t}$ and $w \in \langle J \rangle$.

Restricting τ_s, τ_t to the sub-chamber system, we infer that $(\tau_s\tau_t)^{m_{st}}$ is an automorphism of this sub-chamber system. By the previous lemma this chamber system is a building of type $(\langle J \rangle, J)$. Since this automorphism fixes all chambers U_w with $w \in \langle J \rangle$, it fixes the two opposite chambers U_{1_W} and U_{r_J}. Since every panel contains exactly three chambers, the automorphism fixes $R_{\{s\}}(U_{1_W})$ for all $s \in S$. Using Theorem (1.2.3), we obtain $(\tau_s\tau_t)^{m_{st}}.uU_w = uU_w$ for all $u \in U_{s,t}$ and $w \in \langle J \rangle$. This finishes the claim. □

(2.4.3) Theorem. *Assume that $[(\tau_s\tau_t)^{m_{st}}, n] = 1$ in $P_s \star_{U_+} P_t$ for all $s \neq t \in S$ with $m_{st} < \infty$ and $n \in N_{s,t}$. Then the natural mapping $P_s \to G$ is injective for all $s \in S$.*

Proof. Suppose $s \neq t \in S$ with $m_{st} < \infty$. By assumption $n^{(\tau_t\tau_s)^{m_{st}}} = n$ for all $n \in N_{s,t}$. Together with the previous lemma we deduce $(\tau_s\tau_t)^{m_{st}}.nuU_w = nuU_w$ for all $u \in U_{s,t}$ and hence $(\tau_s\tau_t)^{m_{st}}$ acts trivial on the chamber system. Thus G acts on $\mathbf{C}$ and since P_s acts faithfully on $\mathbf{C}$ by Proposition (2.3.3), the claim follows. □

3. Braid relations

In Chapter 3 we assume that $m_{st} \neq 6$ for all $s,t \in S$. Moreover, we let $\mathcal{M}$ be a faithful and Weyl-invariant commutator blueprint of type (W,S). We will compute the automorphisms $(\tau_s\tau_t)^{m_{st}} \in \mathrm{Aut}(\mathbf{C})$ for $m_{st} < \infty$ and give sufficient conditions of the commutator blueprint in order to achieve that this automorphisms is trivial. This is done by a case distinction on m_{st}.

3.1. Notations

In this chapter we will work out sufficient conditions of the commutator blueprint such that $[(\tau_s\tau_t)^{m_{st}}, n] = 1$ in $P_s \star_{U_+} P_t$ for all $s \neq t \in S$ with $m_{st} < \infty$ and all $n \in N_{s,t}$. It suffices to consider a generating set of $N_{s,t}$, i.e. $n \in \{u^{-1}u_\alpha u \mid u \in U_{s,t}, \alpha \in \Phi_+ \backslash \Phi_+^{\{s,t\}}\}$. We abbreviate $u_{ws} := u_{w\alpha_s} \in U_{ws}^\star$, i.e. $u_{ts} = u_{t\alpha_s}$. We will always assume that $-\beta \subseteq \alpha$, if u_β appears in u. Otherwise we can reduce u as we see in the next example.

(3.1.1) Example. Suppose $\alpha \in \Phi_+ \backslash \Phi_+^{\{s,t\}}$ with $-\alpha_s \not\subseteq \alpha$. Then $\{\alpha_s, \alpha\} \in \mathcal{P}$ by definition and we have $u_s u_\alpha u_s = \left(\prod_{\gamma \in M^G_{\alpha_s,\alpha}} u_\gamma\right) u_\alpha$ for some $G \in \mathrm{Min}$ with $\alpha_s, \alpha \in \Phi(G)$.

For short we will write $u_s.n := u_s n u_s$ and $\tau_s.n := \tau_s n \tau_s = \tau_s(n)$. Let $\alpha_s \neq \beta \in \Phi_+$ be a root such that $\{\alpha_s, \beta\} \notin \mathcal{P}$. Then $-\alpha_s \subseteq \beta$. Let $w \in W$ with $\ell(sw) = \ell(w) - 1$ and let $G \in \mathrm{Min}_s(w)$ with $s\beta \in \Phi(G)$. By Proposition (2.2.9) we have the following in P_s:

$$\begin{aligned}
\tau_s(u_s u_\beta u_s) &= u_s \left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} u_{s\gamma} \right) u_\beta u_s \\
&= u_s \left(\prod_{\gamma \in M^G_{\alpha_s, s\beta}} u_\gamma \right)^{\tau_s} u_\beta u_s \\
&= u_s [u_s, u_{s\beta}]^{\tau_s} u_\beta u_s \\
&= u_s u_\beta [u_{s\beta}, u_s]^{\tau_s} u_s
\end{aligned}$$

Moreover, if $-\alpha_s \subseteq \beta_1, \dots, \beta_k \in \Phi_+$, then we have

$$\begin{aligned}
\tau_s(u_s u_{\beta_1} \cdots u_{\beta_k} u_s) &= u_s \left(u_{s\beta_1}[u_{s\beta_1}, u_s] \cdots u_{s\beta_k}[u_{s\beta_k}, u_s] \right)^{\tau_s} u_s \\
&= u_s \left(u_s u_{s\beta_1} \cdots u_{s\beta_k} u_s \right)^{\tau_s} u_s = u_s u_{\beta_1} \cdots u_{\beta_k} [u_{s\beta_1} \cdots u_{s\beta_k}, u_s]^{\tau_s} u_s
\end{aligned}$$

Note that $[(\tau_s\tau_t)^{m_{st}}, n] = 1$ implies $[(\tau_t\tau_s)^{m_{st}}, n] = 1$. We remark that for each $\alpha \in \Phi_+ \backslash \Phi_+^{\{s,t\}}$, we have $(\tau_s\tau_t)^{m_{st}}.u_\alpha = u_\alpha$.

(3.1.2) Remark. Let $s \neq t \in S$ be such that $6 \neq m_{st} < \infty$. In order to show that $[(\tau_t\tau_s)^{m_{st}}, n] = 1$, we use the fact $m_{ru} \neq 6$ for all $r,t \in S$ only in a few cases. If we do, we will explicitly state it in the hypothesis.

3.2. The case $m_{st} = 2$

(3.2.1) Lemma. *We have* $[(\tau_s\tau_t)^2, n] = 1$ *for all* $n \in N_{s,t}$ *in the group* $P_s \star_{U_+} P_t$.

Proof. Let $\alpha \in \Phi_+ \backslash \Phi_+^{\{s,t\}}$. Assume $-\alpha_s \subseteq \alpha$. Then the following hold:

$$\begin{aligned}
(\tau_s\tau_t)^2.u_s u_\alpha u_s &= \tau_s\tau_t\tau_s.u_s u_{t\alpha} u_s \\
&= \tau_s\tau_t.u_s u_{t\alpha}[u_{st\alpha}, u_s]^{\tau_s} u_s \\
&= \tau_s.u_s u_\alpha [u_{st\alpha}, u_s]^{\tau_s\tau_t} u_s \\
&= \tau_s.u_s u_\alpha [u_{st\alpha}, u_s]^{\tau_t\tau_s} u_s \\
&= \tau_s.u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} u_s \\
&= \tau_s^2.u_s u_\alpha u_s \\
&= u_s u_\alpha u_s
\end{aligned}$$

Interchanging s and t, we deduce $(\tau_t\tau_s)^2.u_t u_\alpha u_t = u_t u_\alpha u_t$ for each $-\alpha_t \subseteq \alpha \in \Phi_+$ and, in particular, $(\tau_s\tau_t)^2.u_t u_\alpha u_t = u_t u_\alpha u_t$.

Now we assume $-\alpha_s, -\alpha_t \subseteq \alpha$. Then the following hold:

$$\begin{aligned}
(\tau_s\tau_t)^2.u_t u_s u_\alpha u_s u_t &= (\tau_s\tau_t)^2.u_s u_t u_\alpha u_t u_s \\
&= \tau_s\tau_t\tau_s.u_s u_t u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_t u_s \\
&= \tau_s\tau_t\tau_s.u_t u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_s u_t \\
&= \tau_s\tau_t.u_t \tau_s (u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_s) u_t \\
&\overset{(2.2.10)}{=} \tau_s\tau_t.u_t u_s \tau_s (u_s \tau_s (u_\alpha [u_{t\alpha}, u_t]^{\tau_t}) u_s) u_s u_t \\
&= \tau_s\tau_t.u_t u_s \tau_s (u_s u_{s\alpha} [u_{t\alpha}, u_t]^{\tau_t\tau_s} u_s) u_s u_t \\
&= \tau_s\tau_t.u_t u_s \tau_s (u_s u_{s\alpha} [u_{st\alpha}, u_t]^{\tau_t} u_s) u_s u_t \\
&= \tau_s\tau_t.u_t u_s \tau_s (\tau_t (u_s u_t u_{st\alpha} u_t u_s)) u_s u_t \\
&= \tau_s\tau_t.u_t u_s \tau_s (\tau_t (u_t u_{st\alpha} [u_{st\alpha}, u_s] u_t)) u_s u_t \\
&= \tau_s\tau_t.u_t u_s \left(u_{st\alpha} [u_{st\alpha}, u_t][u_{st\alpha}, u_s][[u_{st\alpha}, u_s], u_t]\right)^{\tau_t\tau_s} u_s u_t \\
&= \tau_s\tau_t.u_t u_s \left(u_{st\alpha} [u_{st\alpha}, u_t][u_{st\alpha}, u_s][[u_{st\alpha}, u_s], u_t]\right)^{\tau_s\tau_t} u_s u_t \\
&= \tau_s\tau_t.u_t u_s \tau_t (\tau_s (u_t u_{st\alpha} [u_{st\alpha}, u_s] u_t)) u_s u_t \\
&= \tau_s\tau_t.u_s u_t \tau_t (u_t u_{t\alpha} [u_{st\alpha}, u_s]^{\tau_s} u_t) u_t u_s \\
&= \tau_s.u_s \tau_t (u_t \tau_t (u_t u_{t\alpha} [u_{st\alpha}, u_s]^{\tau_s} u_t) u_t) u_s \\
&\overset{(2.2.10)}{=} \tau_s.u_s u_t \tau_t (u_{t\alpha} [u_{st\alpha}, u_s]^{\tau_s}) u_t u_s \\
&= \tau_s.u_t u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} u_s u_t \\
&= \tau_s^2.u_t u_s u_\alpha u_s u_t \\
&= u_t u_s u_\alpha u_s u_t
\end{aligned}$$

□

3.3. The case $m_{st} = 3$

In this case we assume that the groups U_w are of nilpotency class at most 2 and that the commutator blueprint $\mathcal{M}$ satisfies (CR1) and (CR2) (cf. Theorem (3.5.1)). We note that the root β in (CR1) and (CR2) is not necessarily a positive root. Later if we refer to one of these conditions, we will not go into detail. E.g. if $o(r_{\alpha_t} r_\alpha) < \infty$, Condition (CR2) implies $M^G_{\alpha_s,\alpha} = \emptyset$. In particular, we will not state w and G.

(3.3.1) Lemma. *Let $G = \langle g_1, \ldots, g_n \rangle$ be a group of nilpotency class at most 2 such that $g_i^2 = 1$ for all i. Then $[g,h]^2 = 1$ for all $g,h \in G$.*

Proof. Let $f_1, \ldots, f_r, h_1, \ldots, h_m \in \{g_1, \ldots, g_n\}$ be such that $g = f_1 \cdots f_r, h = h_1 \cdots h_m$. We show the claim via induction on $r+m$. If $r+m \in \{0,1\}$ the claim follows directly. Thus we assume $r+m \geq 2$. Again, for $0 \in \{r,m\}$ the claim follows directly. Thus we can assume $r,m \geq 1$. Using the nilpotency class we obtain

$$\begin{aligned}
[g,h]^2 &= \left([g,h_m][g,hh_m^{-1}]^{h_m}\right)^2 \\
&= \left([gf_r^{-1},h_m]^{f_r}[f_r,h_m][g,hh_m^{-1}]\right)^2 \\
&= [gf_r^{-1},h_m]^2[f_r,h_m]^2[g,hh_m^{-1}]^2
\end{aligned}$$

Using the nilpotency class and the fact that $[f_r,h_m]^2 = [f_r,[f_r,h_m]]$, the claim follows by induction. □

(3.3.2) Lemma. *We have $[(\tau_s\tau_t)^3, u^{-1}u_\alpha u] = 1$ for all $\alpha \in \Phi_+ \backslash \Phi_+^{\{s,t\}}$ and $u \in \{u_s, u_{st} = u_{ts}, u_t\}$ in the group $P_s \star_{U_+} P_t$.*

Proof. At first we consider the case $u = u_s$. If $\{\alpha_s, \alpha\} \in \mathcal{P}$ the claim clearly holds. Thus we assume $-\alpha_s \subseteq \alpha$. Then we compute

$$\begin{aligned}
(\tau_s\tau_t)^3.u_su_\alpha u_s &= (\tau_s\tau_t)^2.u_tu_{st\alpha}u_t \\
&= \tau_s\tau_t\tau_s.u_tu_{st\alpha}[u_{tst\alpha},u_t]^{\tau_t}u_t \\
&= \tau_s.u_su_\alpha[u_{tst\alpha},u_t]^{\tau_t\tau_s\tau_t}u_s \\
&= \tau_s.u_su_\alpha[u_{s\alpha},u_s]^{\tau_s}u_s \\
&= \tau_s^2.u_su_\alpha u_s \\
&= u_su_\alpha u_s
\end{aligned}$$

Interchanging s and t, we deduce $(\tau_t\tau_s)^3.u_tu_\alpha u_t = u_tu_\alpha u_t$ and, in particular, $(\tau_s\tau_t)^3.u_tu_\alpha u_t = u_tu_\alpha u_t$. Now we consider the case $u = u_{ts}$. Again, if $\{t\alpha_s, \alpha\} \in \mathcal{P}$, the claim is trivial. Thus we assume $-t\alpha_s \subseteq \alpha$. Using the case $u = u_s$, we deduce

$$(\tau_s\tau_t)^3.u_{st}u_\alpha u_{st} = (\tau_s\tau_t)^2\tau_s.u_su_{t\alpha}u_s = \tau_t(\tau_t\tau_s)^3.u_su_{t\alpha}u_s = \tau_t.u_su_{t\alpha}u_s = u_{st}u_\alpha u_{st}$$ □

(3.3.3) Lemma. *We have $[(\tau_s\tau_t)^3, u^{-1}u_\alpha u] = 1$ for all $\alpha \in \Phi_+$ with $-\alpha_s, -\alpha_t \subseteq \alpha$ and all $u \in \{u_su_t, u_su_{st}u_t, u_su_{st}\}$ in the group $P_s \star_{U_+} P_t$.*

Proof. We deduce from the nilpotency class of the U_w the following (note that s and t are interchangeable in the following equations):

$$\begin{aligned}
[[u_{t\alpha},u_t]^{\tau_t\tau_s},u_s] &= [[u_{t\alpha},u_t]^{\tau_s\tau_t},u_t]^{\tau_s\tau_t} = [[u_{tst\alpha},u_s],u_t]^{\tau_s\tau_t} = 1 \\
[u_{st\alpha},u_s] &= [u_{tst\alpha},u_{ts}]^{\tau_t} = [u_{tst\alpha},[u_s,u_t]]^{\tau_t} = 1 \\
[[u_{s\alpha},u_s]^{\tau_s\tau_t\tau_s},u_s] &= [[u_{s\alpha},u_s]^{\tau_t},u_t]^{\tau_s\tau_t} = [[u_{ts\alpha},u_{ts}],u_t]^{\tau_s\tau_t} = 1 \\
[[u_{t\alpha},u_t]^{\tau_s},u_s] &= [[u_{st\alpha},u_{st}],u_s] = 1 \\
[[u_{s\alpha},u_s]^{\tau_s},[u_{t\alpha},u_t]^{\tau_t}] &= [[u_{sts\alpha},u_t],[u_{tst\alpha},u_s]]^{\tau_s\tau_t\tau_s} = 1
\end{aligned}$$

Case 1: $u = u_su_t$: Note that by (CR1) there exist $w \in W$ with $\ell(tw) = \ell(w)-1$ and $G \in \mathrm{Min}_t(w)$ with $t\alpha \in \Phi(G)$ such that $-t\alpha_s \subseteq \gamma$ and, in particular, $-\alpha_s \subseteq t\gamma$ for all $\gamma \in M_{\alpha_t,t\alpha}^G$. Using (CR1) again, we deduce $-\alpha_t \subseteq \gamma$ and hence $-\alpha_s \subseteq ts\gamma$ for all $\gamma \in M_{\alpha_s,s\alpha}^G$ and

$-\alpha_s \subseteq \gamma$ for all $\gamma \in M^G_{\alpha t, t\alpha}$ (cf. also Remark (1.3.3)). Using the previous computations we compute the following:

$$\begin{aligned}
(\tau_s\tau_t)^3.u_t u_s u_\alpha u_s u_t &= (\tau_s\tau_t)^3.u_s u_{st} u_t u_\alpha u_t u_{st} u_s \\
&= (\tau_s\tau_t)^2\tau_s.u_{ts} u_s u_t u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_t u_s u_{ts} \\
&= (\tau_s\tau_t)^2\tau_s.u_t u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_s u_t \\
&= (\tau_s\tau_t)^2.u_{st} u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} [u_{t\alpha}, u_t]^{\tau_t} [[u_{t\alpha}, u_t]^{\tau_t\tau_s}, u_s]^{\tau_s} u_s u_{st} \\
&= (\tau_s\tau_t)^2.u_{st} u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} [u_{t\alpha}, u_t]^{\tau_t} u_s u_{st} \\
&= \tau_s\tau_t\tau_s.u_s u_{ts} u_{t\alpha} [u_{s\alpha}, u_s]^{\tau_s\tau_t} [u_{t\alpha}, u_t] u_{ts} u_s \\
&= \tau_s\tau_t\tau_s.u_{ts} u_s u_{t\alpha} [u_{s\alpha}, u_s]^{\tau_s\tau_t} [u_{t\alpha}, u_t] u_s u_{ts} \\
&= \tau_s\tau_t.u_t u_s u_{t\alpha} [u_{st\alpha}, u_s]^{\tau_s} [u_{s\alpha}, u_s]^{\tau_s\tau_t} [[u_{s\alpha}, u_s]^{\tau_s\tau_t\tau_s}, u_s]^{\tau_s} \\
&\quad \cdot [u_{t\alpha}, u_t][[u_{t\alpha}, u_t]^{\tau_s}, u_s]]^{\tau_s} u_s u_t \\
&= \tau_s\tau_t.u_s u_{ts} u_t u_{t\alpha} [u_{s\alpha}, u_s]^{\tau_s\tau_t} [u_{t\alpha}, u_t] u_t u_{ts} u_s \\
&= \tau_s\tau_t.u_s u_{ts} u_t u_{t\alpha} [u_{t\alpha}, u_t] u_t [u_{s\alpha}, u_s]^{\tau_s\tau_t} u_{ts} u_s \\
&= \tau_s\tau_t.u_s u_{ts} u_{t\alpha} [u_{s\alpha}, u_s]^{\tau_s\tau_t} u_{ts} u_s \\
&= \tau_s.u_{ts} u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} u_s u_{ts} \\
&= \tau_s^2.u_t u_s u_\alpha u_s u_t \\
&= u_t u_s u_\alpha u_s u_t
\end{aligned}$$

Case 2: $u = u_s u_{st} u_t$: Interchanging s and t, we deduce the following:

$$(\tau_t\tau_s)^3.u_t u_{st} u_s u_\alpha u_s u_{st} u_t = (\tau_t\tau_s)^3.u_s u_t u_\alpha u_t u_s = u_s u_t u_\alpha u_t u_s = u_t u_{st} u_s u_\alpha u_s u_{st} u_t$$

Case 3: $u = u_s u_{st}$: Using (CR1) we deduce $-t\alpha_s \subseteq \gamma$ and hence $-\alpha_s \subseteq t\gamma$ for all $\gamma \in M^G_{\alpha t, t\alpha}$. Moreover, we deduce $-\alpha_t \subseteq s\gamma, t\delta$ for all $\gamma \in M_{\alpha_s, s\alpha}$ and all $\delta \in M_{\alpha_t, t\alpha}$ by (CR1). We compute the following:

$$\begin{aligned}
(\tau_s\tau_t)^3.u_{st} u_s u_\alpha u_s u_{st} &= (\tau_s\tau_t)^2\tau_s.u_s u_{ts} u_{t\alpha} u_{ts} u_s \\
&= (\tau_s\tau_t)^2\tau_s.u_{ts} u_s u_{t\alpha} u_s u_{ts} \\
&= (\tau_s\tau_t)^2.u_t u_s u_{t\alpha} [u_{st\alpha}, u_s]^{\tau_s} u_s u_t \\
&= (\tau_s\tau_t)^2.u_s u_{st} u_{t\alpha} [u_{t\alpha}, u_t] u_{st} u_s \\
&= \tau_s\tau_t\tau_s.u_{ts} u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_s u_{ts} \\
&= \tau_s\tau_t.u_t u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} [u_{t\alpha}, u_t]^{\tau_t} [[u_{t\alpha}, u_t]^{\tau_t\tau_s}, u_s]^{\tau_s} u_s u_t \\
&= \tau_s\tau_t.u_s u_{st} u_t u_\alpha [u_{s\alpha}, u_s]^{\tau_s} [u_{t\alpha}, u_t]^{\tau_t} u_t u_{st} u_s \\
&= \tau_s.u_{ts} u_s u_t u_\alpha [u_{t\alpha}, u_t]^{\tau_t} [u_{s\alpha}, u_s]^{\tau_s} [[u_{s\alpha}, u_s]^{\tau_s\tau_t}, u_t]^{\tau_t} \\
&\quad \cdot [u_{t\alpha}, u_t]^{\tau_t} [[u_{t\alpha}, u_t], u_t]^{\tau_t} u_t u_s u_{ts} \\
&\overset{(3.3.1)}{=} \tau_s.u_t u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} u_s u_t \\
&= \tau_s^2.u_{st} u_s u_\alpha u_s u_{st} \\
&= u_{st} u_s u_\alpha u_s u_{st}
\end{aligned}$$

□

(3.3.4) Lemma. *Let $\alpha \in \Phi_+$ be a root such that $-\alpha_s, -s\alpha_t \subseteq \alpha$ hold. Then we have $[(\tau_s\tau_t)^3, u_{st} u_s u_\alpha u_s u_{st}] = 1$ in the group $P_s \star_{U_+} P_t$, if $m_{rt} \neq 6$ for all $r \in S$.*

Proof. We distinguish the following cases:

(a) $-\alpha_t \subseteq \alpha$: Then the claim follows from the previous lemma.

(b) $\alpha_t \subseteq \alpha$: Then $-\alpha_s, -\alpha_t \subseteq t\alpha$ and the previous lemma implies:

$$(\tau_s\tau_t)^3.u_{st}u_su_\alpha u_su_{st} = (\tau_s\tau_t)^2\tau_s.u_su_{ts}u_{t\alpha}u_{ts}u_s = \tau_t.u_su_{ts}u_{t\alpha}u_{ts}u_s = u_{st}u_su_\alpha u_su_{st}$$

(c) $o(r_{\alpha_t}r_\alpha) < \infty$: Using (CR2) we deduce:

$$[u_{st\alpha}, u_s] = 1$$
$$[u_{s\alpha}, u_s] = 1$$

We compute the following:

$$\begin{aligned}(\tau_s\tau_t)^3.u_{st}u_su_\alpha u_su_{st} &= (\tau_s\tau_t)^2\tau_s.u_su_{ts}u_{t\alpha}u_{ts}u_s \\ &= (\tau_s\tau_t)^2\tau_s.u_{ts}u_su_{t\alpha}u_su_{ts} \\ &= (\tau_s\tau_t)^2.u_tu_su_{t\alpha}[u_{st\alpha}, u_s]^{\tau_s}u_su_t \\ &= (\tau_s\tau_t)^2.u_su_{st}u_{t\alpha}[u_{t\alpha}, u_t]u_{st}u_s \\ &= \tau_s\tau_t\tau_s.u_{ts}u_su_\alpha[u_{t\alpha}, u_t]^{\tau_t}u_su_{ts}\end{aligned}$$

Note that $u_\alpha[u_{t\alpha}, u_t]^{\tau_t} = \tau_tu_t\tau_t.u_\alpha = u_t\tau_tu_t.u_\alpha = u_tu_{t\alpha}[u_\alpha, u_t]^{\tau_t}u_t$. Since $m_{rt} \neq 6$ for all $r \in S$, we have $1 \in \{[u_{t\alpha}, u_t], [u_\alpha, u_t]\}$ (because of the Weyl-invariance). We distinguish the following cases:

(i) $[u_\alpha, u_t] = 1$: Then we have the following:

$$\begin{aligned}\tau_s\tau_t\tau_s.u_{ts}u_su_\alpha[u_{t\alpha}, u_t]^{\tau_t}u_su_{ts} &= \tau_s\tau_t\tau_s.u_{ts}u_su_tu_{t\alpha}u_tu_su_{ts} \\ &= \tau_s\tau_t\tau_s.u_tu_su_{t\alpha}u_su_t \\ &= \tau_s\tau_t.u_{st}u_su_{t\alpha}[u_{st\alpha}, u_s]^{\tau_s}u_su_{st} \\ &= \tau_s\tau_t.u_su_{st}u_{t\alpha}[u_\alpha, u_t]^{\tau_t}u_{st}u_s\end{aligned}$$

(ii) $[u_{t\alpha}, u_t] = 1$: Then we have the following:

$$\begin{aligned}\tau_s\tau_t\tau_s.u_{ts}u_su_\alpha[u_{t\alpha}, u_t]^{\tau_t}u_su_{ts} &= \tau_s\tau_t\tau_s.u_{ts}u_su_\alpha u_su_{ts} \\ &= \tau_s\tau_t.u_tu_su_\alpha[u_{s\alpha}, u_s]^{\tau_s}u_su_t \\ &= \tau_s\tau_t.u_su_{st}u_tu_\alpha[u_{t\alpha}, u_t]^{\tau_t}u_tu_{st}u_s \\ &= \tau_s\tau_t.u_su_{st}u_{t\alpha}[u_\alpha, u_t]^{\tau_t}u_{st}u_s\end{aligned}$$

In both cases we obtain the same result. This implies:

$$\begin{aligned}\tau_s\tau_t\tau_s.u_{ts}u_su_\alpha[u_{t\alpha}, u_t]^{\tau_t}u_su_{ts} &= \tau_s\tau_t.u_su_{st}u_{t\alpha}[u_\alpha, u_t]^{\tau_t}u_{st}u_s \\ &= \tau_s.u_{ts}u_su_\alpha[u_\alpha, u_t]u_su_{ts} \\ &= \tau_s.u_tu_su_\alpha u_su_t \\ &= u_{st}u_su_\alpha[u_{s\alpha}, u_s]^{\tau_s}u_su_{st} \\ &= u_{st}u_su_\alpha u_su_{st}\end{aligned}$$

□

(3.3.5) Remark. We note that in almost all cases we have $1 \in \{[u_{t\alpha}, u_t], [u_\alpha, u_t]\}$ if $o(r_{\alpha_t}r_\alpha) < \infty$. But in a hexagon, we have $[u_1, u_3] \neq 1 \neq [u_1, u_5]$. This is the only example of commutators, where none of these two commutators is trivial.

(3.3.6) Lemma. *We have $[(\tau_s\tau_t)^3, n] = 1$ for all $n \in N_{s,t}$ in the group $P_s \star_{U_+} P_t$, if the groups U_w are of nilpotency class at most 2, $m_{ru} \neq 6$ for all $r, u \in S$ and $\mathcal{M}$ satisfies (CR1) and (CR2).*

Proof. Using the previous lemmas in this subsection, it suffices to consider $n = u_t u_{st} u_\alpha u_{st} u_t$. Let $\alpha \in \Phi_+$ be a root such that $-\alpha_t, -s\alpha_t = -t\alpha_s \subseteq \alpha$. Interchanging s and t in the previous lemma we deduce

$$(\tau_t\tau_s)^3.u_t u_{st} u_\alpha u_{st} u_t = (\tau_t\tau_s)^3.u_{ts} u_t u_\alpha u_t u_{ts} = u_{ts} u_t u_\alpha u_t u_{ts} = u_t u_{st} u_\alpha u_{st} u_t$$

In particular, we have $(\tau_s\tau_t)^3.u_t u_{st} u_\alpha u_{st} u_t = u_t u_{st} u_\alpha u_{st} u_t$. □

3.4. The case $m_{st} = 4$

In this case we again assume that the groups U_w are of nilpotency class at most 2 and that the commutator blueprint $\mathcal{M}$ satisfies the additional Conditions (CR1) and (CR2).

(3.4.1) Lemma. *For $\alpha \in \Phi_+$ we have $[(\tau_s\tau_t)^4, u^{-1}u_\alpha u] = 1$ for $u \in \{u_s, u_t, u_{st}, u_{ts}\}$ in the group $P_s \star_{U_+} P_t$.*

Proof. Let $u = u_s$. We can assume that $-\alpha_s \subseteq \alpha$. Otherwise the claim is obvious. Using the nilpotency class of the groups U_w we obtain:

$$\begin{aligned}(\tau_s\tau_t)^4.u_s u_\alpha u_s &= (\tau_s\tau_t)^2\tau_s.u_s u_{ts t\alpha} u_s \\ &= (\tau_s\tau_t)^2.u_s u_{tst\alpha}[u_{stst\alpha}, u_s]^{\tau_s} u_s \\ &= \tau_s.u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} u_s \\ &= \tau_s^2.u_s u_\alpha u_s \\ &= u_s u_\alpha u_s\end{aligned}$$

This also implies $[(\tau_t\tau_s)^4, u_s u_\alpha u_s] = 1$. Interchanging s and t we deduce the claim for u_t. Now let $u = u_{st}$ and assume $-s\alpha_t \subseteq \alpha$. Then $-\alpha_t \subseteq st\alpha$ and the case $u = u_t$ implies

$$(\tau_s\tau_t)^4.u_{st} u_\alpha u_{st} = (\tau_s\tau_t)^3.u_t u_{st\alpha} u_t = (\tau_s\tau_t)^{-1}.u_t u_{st\alpha} u_t = u_{st} u_\alpha u_{st}$$

Interchanging s and t the claim does also hold for $u = u_{ts}$. □

(3.4.2) Lemma. *Let $\alpha \in \Phi_+$ be such that $-\alpha_s, -t\alpha_s \subseteq \alpha$. Then $[(\tau_s\tau_t)^4, u_{ts} u_s u_\alpha u_s u_{ts}] = 1$ in the group $P_s \star_{U_+} P_t$.*

Proof. Let $\beta \in \{st\alpha, s\alpha\}$. Then we have $\alpha_s \subseteq \beta$ as well as $\alpha_s, t\alpha_s \subseteq tst\beta$. Using the nilpotency class of the groups U_w we deduce:

$$\begin{aligned}[[u_\beta, u_s]^{\tau_s\tau_t\tau_s}, u_s] &= [[u_{tst\beta}, u_s], u_{ts}]^{\tau_s\tau_t} = 1 \\ [[u_{st\alpha}, u_s]^{\tau_s}, [u_{s\alpha}, u_s]^{\tau_s\tau_t}] &= [[u_{st\alpha}, u_s]^{\tau_t\tau_s\tau_t}, [u_{s\alpha}, u_s]^{\tau_t\tau_s}]^{\tau_t\tau_s\tau_t\tau_s} \\ &= [[u_{sts\alpha}, u_s], [u_{sts\alpha}, u_{ts}]]^{\tau_t\tau_s\tau_t\tau_s} = 1\end{aligned}$$

The last equation follows from the fact that $u_{ts}, u_{sts\alpha}$ commute with the first commutator. Note that $-t\alpha_s \subseteq st\alpha$ and hence by (CR1) we obtain $-t\alpha_s \subseteq \gamma$ for all $\gamma \in M^G_{\alpha_s, st\alpha}$. In particular, $-\alpha_s \subseteq ts\gamma$ for all $\gamma \in M^G_{\alpha_s, st\alpha}$. Using (CR1) again, we have $-t\alpha_s \subseteq \gamma$ and hence $-\alpha_s \subseteq ts\gamma$ for all $\gamma \in M^G_{\alpha_s, s\alpha}$. We compute the following:

$$(\tau_s\tau_t)^4.u_{ts} u_s u_\alpha u_s u_{ts} = (\tau_s\tau_t)^3\tau_s.u_s u_{ts} u_{t\alpha} u_{ts} u_s$$

$$\begin{aligned}
&= (\tau_s\tau_t)^3\tau_s.u_{ts}u_su_{t\alpha}u_su_{ts} \\
&= (\tau_s\tau_t)^3.u_{ts}u_su_{t\alpha}[u_{st\alpha},u_s]^{\tau_s}u_su_{ts} \\
&= (\tau_s\tau_t)^2\tau_s.u_su_{ts}u_\alpha[u_{st\alpha},u_s]^{\tau_s\tau_t}u_{ts}u_s \\
&= (\tau_s\tau_t)^2\tau_s.u_{ts}u_su_\alpha[u_{st\alpha},u_s]^{\tau_s\tau_t}u_su_{ts} \\
&= (\tau_s\tau_t)^2.u_{ts}u_su_\alpha[u_{s\alpha},u_s]^{\tau_s}[u_{st\alpha},u_s]^{\tau_s\tau_t}[[u_{st\alpha},u_s]^{\tau_s\tau_t\tau_s},u_s]^{\tau_s}u_su_{ts} \\
&= (\tau_s\tau_t)^2.u_{ts}u_su_\alpha[u_{s\alpha},u_s]^{\tau_s}[u_{st\alpha},u_s]^{\tau_s\tau_t}u_su_{ts} \\
&= \tau_s\tau_t\tau_s.u_su_{ts}u_{t\alpha}[u_{s\alpha},u_s]^{\tau_s\tau_t}[u_{st\alpha},u_s]^{\tau_s}u_{ts}u_s \\
&= \tau_s\tau_t\tau_s.u_{ts}u_su_{t\alpha}[u_{s\alpha},u_s]^{\tau_s\tau_t}[u_{st\alpha},u_s]^{\tau_s}u_su_{ts} \\
&= \tau_s\tau_t.u_{ts}u_su_{t\alpha}[u_{st\alpha},u_s]^{\tau_s}[u_{s\alpha},u_s]^{\tau_s\tau_t}[[u_{s\alpha},u_s]^{\tau_s\tau_t\tau_s},u_s]^{\tau_s} \\
&\qquad\cdot[u_{st\alpha},u_s]^{\tau_s}[[u_{st\alpha},u_s],u_s]^{\tau_s}u_su_{ts} \\
&\overset{(3.3.1)}{=} \tau_s\tau_t.u_{ts}u_su_{t\alpha}[u_{s\alpha},u_s]^{\tau_s\tau_t}u_su_{ts} \\
&= \tau_s.u_su_{ts}u_\alpha[u_{s\alpha},u_s]^{\tau_s}u_{ts}u_s \\
&= \tau_s.u_{ts}u_su_\alpha[u_{s\alpha},u_s]^{\tau_s}u_su_{ts} \\
&= \tau_s^2.u_{ts}u_su_\alpha u_su_{ts} \\
&= u_{ts}u_su_\alpha u_su_{ts}
\end{aligned}$$

□

(3.4.3) Lemma. *Let $\alpha \in \Phi_+$ be such that $-\alpha_t, -s\alpha_t \subseteq \alpha$. Then $[(\tau_s\tau_t)^4, u_tu_{st}u_\alpha u_{st}u_t] = 1$ in the group $P_s \star_{U_+} P_t$.*

Proof. Interchanging s and t in the previous lemma, it follows that $(\tau_t\tau_s)^4.u_tu_{st}u_\alpha u_{st}u_t = (\tau_t\tau_s)^4.u_{st}u_tu_\alpha u_tu_{st} = u_{st}u_tu_\alpha u_tu_{st} = u_tu_{st}u_\alpha u_{st}u_t$. This finishes the claim. □

(3.4.4) Lemma. *Let $\alpha \in \Phi_+$ be such that $-\alpha_s, -\alpha_t \subseteq \alpha$. Then we have $[(\tau_s\tau_t)^4, u^{-1}u_\alpha u] = 1$ for $u \in \{u_su_{st}u_t, u_su_{st}u_{ts}u_t, u_su_t, u_su_{ts}u_t, u_su_{st}u_{ts}, u_su_{st}, u_{ts}u_t, u_{st}u_{ts}\}$ in the group $P_s \star_{U_+} P_t$.*

Proof. Note that $\alpha_s, \alpha_t \subseteq stst\alpha$. Using the nilpotency class of the groups U_w, we obtain the following (note that s and t are interchangeable in the following equations; cf. also Remark (1.3.3)):

$$\begin{aligned}
[[u_{t\alpha},u_t]^{\tau_t\tau_s},u_s] &= [[u_{stst\alpha},u_t]^{\tau_t\tau_s\tau_t},u_s] = [[u_{stst\alpha},u_t],u_s]^{\tau_t\tau_s\tau_t} = 1 \\
[[u_{t\alpha},u_t]^{\tau_t},[u_{s\alpha},u_s]^{\tau_s}] &= [[u_{stst\alpha},u_t],[u_{tsts\alpha},u_s]]^{\tau_t\tau_s\tau_t\tau_s} = 1 \\
[[u_{t\alpha},u_t]^{\tau_t\tau_s\tau_t\tau_s},u_s] &= [[u_{stst\alpha},u_t],u_{ts}]^{\tau_t} = 1 \\
[[u_{t\alpha},u_t]^{\tau_t\tau_s\tau_t},u_t] &= [[u_{stst\alpha},u_t],u_{st}]^{\tau_t\tau_s} = 1 \\
[[u_{sts\alpha},u_s]^{\tau_s},u_t] &= [[u_{tsts\alpha},u_{ts}],u_{st}]^{\tau_t\tau_s} = 1 \\
[u_{ts\alpha},u_t][u_{sts\alpha},u_s]^{\tau_s} &= ([u_{tsts\alpha},u_{st}][u_{tsts\alpha},u_{ts}])^{\tau_t\tau_s} \\
&= ([u_{tsts\alpha},u_{st}][u_{tsts\alpha},[u_s,u_t]u_{st}])^{\tau_t\tau_s} \\
&= ([u_{tsts\alpha},u_{st}][u_{tsts\alpha},u_{st}][u_{tsts\alpha},[u_s,u_t]]^{u_{st}})^{\tau_t\tau_s} \overset{(3.3.1)}{=} 1 \\
[[u_{ts\alpha},u_t]^{\tau_t},u_s] &= [[u_{tsts\alpha},u_{st}],u_s]^{\tau_t\tau_s\tau_t} = 1 \\
[[u_{st\alpha},u_s]^{\tau_s},[u_{s\alpha},u_s]^{\tau_s\tau_t}] &= ([[u_{st\alpha},u_s]^{\tau_t\tau_s},[u_{s\alpha},u_s]^{\tau_t\tau_s\tau_t}])^{\tau_s\tau_t\tau_s} \\
&= ([[u_{stst\alpha},u_{ts}],[u_{tsts\alpha},u_s]])^{\tau_s\tau_t\tau_s} = 1 \\
[u_{st\alpha},u_s]^{\tau_s}[u_{ts\alpha},u_t]^{\tau_t\tau_s\tau_t} &= ([u_{sts\alpha},u_s]^{\tau_s}[u_{ts\alpha},u_t])^{\tau_t\tau_s\tau_t} = 1 \\
[[u_{ts\alpha},u_t]^{\tau_t\tau_s\tau_t\tau_s},u_s] &= [[u_{stst\alpha},u_{st}],u_{ts}]^{\tau_s\tau_t} = 1 \\
[[u_{t\alpha},u_t]^{\tau_s},u_s] &= [[u_{t\alpha},u_t]^{\tau_s\tau_t\tau_s},u_{ts}]^{\tau_s\tau_t} = [[u_{stst\alpha},u_t],u_{ts}]^{\tau_s\tau_t} = 1 \\
[[u_{st\alpha},u_s]^{\tau_s\tau_t\tau_s},u_s] &= [[u_{st\alpha},u_s]^{\tau_t\tau_s},u_s]^{\tau_t\tau_s\tau_t} = [[u_{stst\alpha},u_{ts}],u_s]^{\tau_t\tau_s\tau_t} = 1
\end{aligned}$$

Case 1: $u = u_s u_{st} u_t$: We note that $-t\alpha_s \subseteq t\alpha$ and hence $-t\alpha_s \subseteq \gamma$ for all $\gamma \in M^G_{\alpha_t,t\alpha}$ by (CR1). This implies $-\alpha_s \subseteq t\gamma$ for all $\gamma \in M^G_{\alpha_t,t\alpha}$. Moreover, we have $-\alpha_t \subseteq s\gamma$ for all $\gamma \in M^G_{\alpha_s,s\alpha}$ by (CR1). We obtain the following:

$$\begin{aligned}
(\tau_s\tau_t)^2.u_t u_{st} u_s u_\alpha u_s u_{st} u_t &= (\tau_s\tau_t)^2.u_s u_{ts} u_t u_\alpha u_t u_{ts} u_s \\
&= \tau_s\tau_t\tau_s.u_{ts} u_s u_t u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_t u_s u_{ts} \\
&= \tau_s\tau_t\tau_s.u_{st} u_t u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_s u_t u_{st} \\
&= \tau_s\tau_t.u_t u_{st} u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} [u_{t\alpha}, u_t]^{\tau_t} [[u_{t\alpha}, u_t]^{\tau_t\tau_s}, u_s]^{\tau_s} u_s u_{st} u_t \\
&= \tau_s\tau_t.u_{ts} u_s u_t u_\alpha [u_{s\alpha}, u_s]^{\tau_s} [u_{t\alpha}, u_t]^{\tau_t} u_t u_s u_{ts} \\
&= \tau_s.u_s u_{ts} u_t u_\alpha [u_{t\alpha}, u_t]^{\tau_t} [u_{s\alpha}, u_s]^{\tau_s} [[u_{s\alpha}, u_s]^{\tau_s\tau_t}, u_t]^{\tau_t} \\
&\qquad \cdot [u_{t\alpha}, u_t]^{\tau_t} [[u_{t\alpha}, u_t], u_t]^{\tau_t} u_t u_{ts} u_s \\
&\overset{(3.3.1)}{=} \tau_s.u_{st} u_t u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} u_s u_t u_{st} \\
&= \tau_s^2.u_t u_{st} u_s u_\alpha u_s u_{st} u_t \\
&= u_t u_{st} u_s u_\alpha u_s u_{st} u_t
\end{aligned}$$

Case 2: $u = u_s u_{st} u_{ts} u_t$: We note that we have $-\alpha_s \subseteq tst\gamma$ for all $\gamma \in M^G_{\alpha_t,t\alpha}$ by (CR1). Moreover, we have $-\alpha_t \subseteq s\gamma, t\delta$ for all $\gamma \in M^G_{\alpha_s,s\alpha}$ and all $\delta \in M^G_{\alpha_t,t\alpha}$ by (CR1). We compute the following:

$$\begin{aligned}
(\tau_s\tau_t)^4.u_t u_{ts} u_{st} u_s u_\alpha u_s u_{st} u_{ts} u_t &= (\tau_s\tau_t)^4.u_s u_t u_\alpha u_t u_s \\
&= (\tau_s\tau_t)^3\tau_s.u_{ts} u_t u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_t u_{ts} \\
&= (\tau_s\tau_t)^3.u_{ts} u_{st} u_{s\alpha} [u_{t\alpha}, u_t]^{\tau_t\tau_s} u_{st} u_{ts} \\
&= (\tau_s\tau_t)^2\tau_s.u_s u_{st} u_{ts\alpha} [u_{t\alpha}, u_t]^{\tau_t\tau_s\tau_t} u_{st} u_s \\
&= (\tau_s\tau_t)^2\tau_s.u_{st} u_s u_{ts\alpha} [u_{t\alpha}, u_t]^{\tau_t\tau_s\tau_t} u_s u_{st} \\
&= (\tau_s\tau_t)^2.u_t u_s u_{ts\alpha} [u_{sts\alpha}, u_s]^{\tau_s} [u_{t\alpha}, u_t]^{\tau_t\tau_s\tau_t} [[u_{t\alpha}, u_t]^{\tau_t\tau_s\tau_t\tau_s}, u_s]^{\tau_s} u_s u_t \\
&= (\tau_s\tau_t)^2.u_s u_{st} u_{ts} u_t u_{ts\alpha} [u_{sts\alpha}, u_s]^{\tau_s} [u_{t\alpha}, u_t]^{\tau_t\tau_s\tau_t} u_t u_{ts} u_{st} u_s \\
&= (\tau_s\tau_t)^2.u_s u_{st} u_{ts} u_{ts\alpha} [u_{ts\alpha}, u_t][u_{sts\alpha}, u_s]^{\tau_s} [u_{t\alpha}, u_t]^{\tau_t\tau_s\tau_t} u_{ts} u_{st} u_s \\
&= (\tau_s\tau_t)^2.u_s u_{st} u_{ts} u_{ts\alpha} [u_{t\alpha}, u_t]^{\tau_t\tau_s\tau_t} u_{ts} u_{st} u_s \\
&= \tau_s\tau_t\tau_s.u_{ts} u_{st} u_s u_{s\alpha} [u_{t\alpha}, u_t]^{\tau_t\tau_s} u_s u_{st} u_{ts} \\
&= \tau_s\tau_t\tau_s.u_{ts} u_{st} u_{s\alpha} [u_{s\alpha}, u_s][u_{t\alpha}, u_t]^{\tau_t\tau_s} u_{st} u_{ts} \\
&= \tau_s\tau_t.u_{ts} u_t u_\alpha [u_{s\alpha}, u_s]^{\tau_s} [u_{t\alpha}, u_t]^{\tau_t} u_t u_{ts} \\
&= \tau_s.u_s u_t u_\alpha [u_{t\alpha}, u_t]^{\tau_t} [u_{s\alpha}, u_s]^{\tau_s} [[u_{s\alpha}, u_s]^{\tau_s\tau_t}, u_t]^{\tau_t} \\
&\qquad \cdot [u_{t\alpha}, u_t]^{\tau_t} [[u_{t\alpha}, u_t], u_t]^{\tau_t} u_t u_s \\
&\overset{(3.3.1)}{=} \tau_s.u_t u_{ts} u_{st} u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} u_s u_{st} u_{ts} u_t \\
&= \tau_s^2.u_{st} u_{ts} u_t u_s u_\alpha u_s u_t u_{ts} u_{st} \\
&= u_{st} u_{ts} u_t u_s u_\alpha u_s u_t u_{ts} u_{st} \\
&= u_t u_{ts} u_{st} u_s u_\alpha u_s u_{st} u_{ts} u_t
\end{aligned}$$

Case 3: $u = u_s u_t$: Interchanging s and t in the previous case, we deduce the following:

$$\begin{aligned}
(\tau_t\tau_s)^4.u_t u_s u_\alpha u_s u_t &= (\tau_t\tau_s)^4.u_s u_{st} u_{ts} u_t u_\alpha u_t u_{ts} u_{st} u_s \\
&= u_s u_{st} u_{ts} u_t u_\alpha u_t u_{ts} u_{st} u_s
\end{aligned}$$

$$= u_t u_s u_\alpha u_s u_t$$

In particular, this yields $(\tau_s\tau_t)^4.u_t u_s u_\alpha u_s u_t = u_t u_s u_\alpha u_s u_t$.

Case 4: $u = u_s u_{ts} u_t$: Note that we have $-\alpha_t \subseteq st\gamma$ for all $\gamma \in M^G_{\alpha_t,t\alpha}$ by (CR1). Similarly, we have $-\alpha_s \subseteq ts\gamma_1, tst\gamma_2, \gamma_3$ for all $\gamma_1 \in M^G_{\alpha_s,s\alpha}, \gamma_2 \in M^G_{\alpha_t,ts\alpha}, \gamma_3 \in M^G_{\alpha_t,t\alpha}$. We compute the following:

$$\begin{aligned}
(\tau_s\tau_t)^4.u_t u_{ts} u_s u_\alpha u_s u_{ts} u_t &= (\tau_s\tau_t)^4.u_s u_{st} u_t u_\alpha u_t u_{st} u_s \\
&= (\tau_s\tau_t)^3\tau_s.u_{ts} u_{st} u_t u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_t u_{st} u_{ts} \\
&= (\tau_s\tau_t)^3.u_{ts} u_t u_{st} u_{s\alpha} [u_{t\alpha}, u_t]^{\tau_t\tau_s} u_{st} u_t u_{ts} \\
&= (\tau_s\tau_t)^3.u_{ts} u_{st} u_t u_{s\alpha} [u_{t\alpha}, u_t]^{\tau_t\tau_s} u_t u_{st} u_{ts} \\
&= (\tau_s\tau_t)^2\tau_s.u_s u_{st} u_t u_{s\alpha} [u_{ts\alpha}, u_t]^{\tau_t} [u_{t\alpha}, u_t]^{\tau_t\tau_s} [[u_{t\alpha}, u_t]^{\tau_t\tau_s\tau_t}, u_t]^{\tau_t} u_t u_{st} u_s \\
&= (\tau_s\tau_t)^2\tau_s.u_{ts} u_t u_s u_{s\alpha} [u_{ts\alpha}, u_t]^{\tau_t} [u_{t\alpha}, u_t]^{\tau_t\tau_s} u_s u_t u_{ts} \\
&= (\tau_s\tau_t)^2\tau_s.u_{ts} u_t u_{s\alpha} [u_{s\alpha}, u_s][u_{ts\alpha}, u_t]^{\tau_t} [u_{t\alpha}, u_t]^{\tau_t\tau_s} u_t u_{ts} \\
&= (\tau_s\tau_t)^2.u_{ts} u_{st} u_\alpha [u_{s\alpha}, u_s]^{\tau_s} [u_{ts\alpha}, u_t]^{\tau_t\tau_s} [u_{t\alpha}, u_t]^{\tau_t} u_{st} u_{ts} \\
&= \tau_s\tau_t\tau_s.u_s u_{st} u_{t\alpha} [u_{s\alpha}, u_s]^{\tau_s\tau_t} [u_{ts\alpha}, u_t]^{\tau_t\tau_s\tau_t} [u_{t\alpha}, u_t] u_{st} u_s \\
&= \tau_s\tau_t\tau_s.u_{st} u_s u_{t\alpha} [u_{s\alpha}, u_s]^{\tau_s\tau_t} [u_{ts\alpha}, u_t]^{\tau_t\tau_s\tau_t} [u_{t\alpha}, u_t] u_s u_{st} \\
&= \tau_s\tau_t.u_t u_s u_{t\alpha} [u_{st\alpha}, u_s]^{\tau_s} [u_{s\alpha}, u_s]^{\tau_s\tau_t} [[u_{s\alpha}, u_s]^{\tau_s\tau_t\tau_s}, u_s]^{\tau_s} \\
&\qquad \cdot [u_{ts\alpha}, u_t]^{\tau_t\tau_s\tau_t} [[u_{ts\alpha}, u_t]^{\tau_t\tau_s\tau_t\tau_s}, u_s]^{\tau_s} [u_{t\alpha}, u_t][[u_{t\alpha}, u_t]^{\tau_s}, u_s]^{\tau_s} u_s u_t \\
&= \tau_s\tau_t.u_s u_{st} u_{ts} u_t u_{t\alpha} [u_{s\alpha}, u_s]^{\tau_s\tau_t} [u_{t\alpha}, u_t] u_t u_{ts} u_{st} u_s \\
&= \tau_s\tau_t.u_s u_{st} u_{ts} u_t u_{t\alpha} [u_{t\alpha}, u_t] u_t [u_{s\alpha}, u_s]^{\tau_s\tau_t} u_{ts} u_{st} u_s \\
&= \tau_s\tau_t.u_s u_{st} u_{ts} u_{t\alpha} [u_{s\alpha}, u_s]^{\tau_s\tau_t} u_{ts} u_{st} u_s \\
&= \tau_s.u_{ts} u_{st} u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} u_s u_{st} u_{ts} \\
&= \tau_s^2.u_{ts} u_t u_s u_\alpha u_s u_t u_{ts} \\
&= u_{ts} u_t u_s u_\alpha u_s u_t u_{ts} \\
&= u_t u_{ts} u_s u_\alpha u_s u_{ts} u_t
\end{aligned}$$

Case 5: $u = u_s u_{st} u_{ts}$: Note that $-\alpha_t \subseteq st\gamma_1, sts\gamma_2$ for all $\gamma_1 \in M^G_{\alpha_t,t\alpha}, \gamma_2 \in M^G_{\alpha_s,st\alpha}$ by (CR1). As before, we deduce $-\alpha_t \subseteq s\gamma, t\delta$ for all $\gamma \in M^G_{\alpha_s,s\alpha}$ and all $\delta \in M^G_{\alpha_t,t\alpha}$ by (CR1). We obtain the following:

$$\begin{aligned}
(\tau_s\tau_t)^4.u_{ts} u_{st} u_s u_\alpha u_s u_{st} u_{ts} &= (\tau_s\tau_t)^3\tau_s.u_s u_{st} u_{ts} u_{t\alpha} u_{ts} u_{st} u_s \\
&= (\tau_s\tau_t)^3\tau_s.u_{st} u_{ts} u_s u_{t\alpha} u_s u_{ts} u_{st} \\
&= (\tau_s\tau_t)^3.u_t u_{ts} u_s u_{t\alpha} [u_{st\alpha}, u_s]^{\tau_s} u_s u_{ts} u_t \\
&= (\tau_s\tau_t)^3.u_{st} u_s u_t u_{t\alpha} [u_{st\alpha}, u_s]^{\tau_s} u_t u_s u_{st} \\
&= (\tau_s\tau_t)^3.u_{st} u_s u_{t\alpha} [u_{t\alpha}, u_t][u_{st\alpha}, u_s]^{\tau_s} u_s u_{st} \\
&= (\tau_s\tau_t)^2\tau_s.u_{st} u_{ts} u_\alpha [u_{t\alpha}, u_t]^{\tau_t} [u_{st\alpha}, u_s]^{\tau_s\tau_t} u_{ts} u_{st} \\
&= (\tau_s\tau_t)^2.u_t u_{ts} u_{s\alpha} [u_{t\alpha}, u_t]^{\tau_t\tau_s} [u_{st\alpha}, u_s]^{\tau_s\tau_t\tau_s} u_{ts} u_t \\
&= (\tau_s\tau_t)^2.u_{ts} u_t u_{s\alpha} [u_{t\alpha}, u_t]^{\tau_t\tau_s} [u_{st\alpha}, u_s]^{\tau_s\tau_t\tau_s} u_t u_{ts} \\
&= \tau_s\tau_t\tau_s.u_s u_t u_{s\alpha} [u_{ts\alpha}, u_t]^{\tau_t} [u_{t\alpha}, u_t]^{\tau_t\tau_s} [[u_{t\alpha}, u_t]^{\tau_t\tau_s\tau_t}, u_t]^{\tau_t} \\
&\qquad \cdot [u_{st\alpha}, u_s]^{\tau_s\tau_t\tau_s} [[u_{st\alpha}, u_s]^{\tau_s\tau_t\tau_s\tau_t}, u_t]^{\tau_t} u_t u_s \\
&= \tau_s\tau_t\tau_s.u_t u_{ts} u_{st} u_s u_{s\alpha} [u_{ts\alpha}, u_t]^{\tau_t} [u_{t\alpha}, u_t]^{\tau_t\tau_s} [u_{st\alpha}, u_s]^{\tau_s\tau_t\tau_s} u_s u_{st} u_{ts} u_t
\end{aligned}$$

$$
\begin{aligned}
&= \tau_s\tau_t\tau_s.u_t u_{ts} u_{st} u_{s\alpha}[u_{s\alpha}, u_s][u_{ts\alpha}, u_t]^{\tau_t}[u_{t\alpha}, u_t]^{\tau_t\tau_s}[u_{st\alpha}, u_s]^{\tau_s\tau_t\tau_s} u_{st} u_{ts} u_t \\
&= \tau_s\tau_t\tau_s.u_t u_{ts} u_{st} u_{s\alpha}[u_{s\alpha}, u_s][u_{t\alpha}, u_t]^{\tau_t\tau_s} u_{st} u_{ts} u_t \\
&= \tau_s\tau_t.u_{st} u_{ts} u_t u_\alpha [u_{s\alpha}, u_s]^{\tau_s}[u_{t\alpha}, u_t]^{\tau_t} u_t u_{ts} u_{st} \\
&= \tau_s.u_{st} u_s u_t u_\alpha [u_{t\alpha}, u_t]^{\tau_t}[u_{s\alpha}, u_s]^{\tau_s}[[u_{s\alpha}, u_s]^{\tau_s\tau_t}, u_t]^{\tau_t} \\
&\qquad \cdot [u_{t\alpha}, u_t]^{\tau_t}[[u_{t\alpha}, u_t], u_t]^{\tau_t} u_t u_s u_{st} \\
&\overset{(3.3.1)}{=} \tau_s.u_{ts} u_t u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} u_s u_t u_{ts} \\
&= \tau_s^2.u_{ts} u_{st} u_s u_\alpha u_s u_{st} u_{ts} \\
&= u_{ts} u_{st} u_s u_\alpha u_s u_{st} u_{ts}
\end{aligned}
$$

Case 6: $u = u_s u_{st}$: Note that $-\alpha_s, -\alpha_t \subseteq t\gamma_1, s\gamma_2$ for all $\gamma_1 \in M^G_{\alpha_t, t\alpha}, \gamma_2 \in M^G_{\alpha_s, s\alpha}$ by (CR1). We compute the following:

$$
\begin{aligned}
(\tau_s\tau_t)^4.u_{st} u_s u_\alpha u_s u_{st} &= (\tau_s\tau_t)^3\tau_s.u_{st} u_{ts} u_{t\alpha} u_{ts} u_{st} \\
&= (\tau_s\tau_t)^3.u_t u_{ts} u_{st\alpha} u_{ts} u_t \\
&= (\tau_s\tau_t)^3.u_{ts} u_t u_{st\alpha} u_t u_{ts} \\
&= (\tau_s\tau_t)^2\tau_s.u_s u_t u_{st\alpha}[u_{tst\alpha}, u_t]^{\tau_t} u_t u_s \\
&= (\tau_s\tau_t)^2\tau_s.u_t u_{ts} u_{st} u_s u_{st\alpha}[u_{tst\alpha}, u_t]^{\tau_t} u_s u_{st} u_{ts} u_t \\
&= (\tau_s\tau_t)^2\tau_s.u_t u_{ts} u_{st} u_{st\alpha}[u_{st\alpha}, u_s][u_{tst\alpha}, u_t]^{\tau_t} u_{st} u_{ts} u_t \\
&= (\tau_s\tau_t)^2\tau_s.u_t u_{ts} u_{st} u_{st\alpha} u_{st} u_{ts} u_t \\
&= (\tau_s\tau_t)^2.u_{st} u_{ts} u_t u_{t\alpha} u_t u_{ts} u_{st} \\
&= (\tau_s\tau_t)^2.u_{st} u_{ts} u_{t\alpha}[u_{t\alpha}, u_t] u_{ts} u_{st} \\
&= \tau_s\tau_t\tau_s.u_{st} u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_s u_{st} \\
&= \tau_s\tau_t.u_t u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s}[u_{t\alpha}, u_t]^{\tau_t}[[u_{t\alpha}, u_t]^{\tau_t\tau_s}, u_s]^{\tau_s} u_s u_t \\
&= \tau_s\tau_t.u_s u_{st} u_{ts} u_t u_\alpha [u_{s\alpha}, u_s]^{\tau_s}[u_{t\alpha}, u_t]^{\tau_t} u_t u_{ts} u_{st} u_s \\
&= \tau_s.u_{ts} u_{st} u_s u_t u_\alpha [u_{t\alpha}, u_t]^{\tau_t}[u_{s\alpha}, u_s]^{\tau_s}[[u_{s\alpha}, u_s]^{\tau_s\tau_t}, u_t]^{\tau_t} \\
&\qquad \cdot [u_{t\alpha}, u_t]^{\tau_t}[[u_{t\alpha}, u_t], u_t]^{\tau_t} u_t u_s u_{st} u_{ts} \\
&\overset{(3.3.1)}{=} \tau_s.u_t u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} u_s u_t \\
&= \tau_s^2.u_{st} u_s u_\alpha u_s u_{st} \\
&= u_{st} u_s u_\alpha u_s u_{st}
\end{aligned}
$$

Case 7: $u = u_{ts} u_t$: Interchanging s and t in the previous case we deduce

$$
(\tau_t\tau_s)^4.u_t u_{ts} u_\alpha u_{ts} u_t = (\tau_t\tau_s)^4.u_{ts} u_t u_\alpha u_t u_{ts} = u_{ts} u_t u_\alpha u_t u_{ts} = u_t u_{ts} u_\alpha u_{ts} u_t
$$

In particular, this yields $(\tau_s\tau_t)^4.u_t u_{ts} u_\alpha u_{ts} u_t = u_t u_{ts} u_\alpha u_{ts} u_t$.

Case 8: $u = u_{st} u_{ts}$: We obtain the following:

$$
\begin{aligned}
(\tau_s\tau_t)^4.u_{ts} u_{st} u_\alpha u_{st} u_{ts} &= (\tau_s\tau_t)^3\tau_s.u_s u_{st} u_{t\alpha} u_{st} u_s \\
&= (\tau_s\tau_t)^3\tau_s.u_{st} u_s u_{t\alpha} u_s u_{st} \\
&= (\tau_s\tau_t)^3.u_t u_s u_{t\alpha}[u_{st\alpha}, u_s]^{\tau_s} u_s u_t \\
&= (\tau_s\tau_t)^3.u_s u_{st} u_{ts} u_t u_{t\alpha}[u_{st\alpha}, u_s]^{\tau_s} u_t u_{ts} u_{st} u_s \\
&= (\tau_s\tau_t)^3.u_s u_{st} u_{ts} u_{t\alpha}[u_{t\alpha}, u_t][u_{st\alpha}, u_s]^{\tau_s} u_{ts} u_{st} u_s
\end{aligned}
$$

$$= (\tau_s\tau_t)^2\tau_s.u_{ts}u_{st}u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t}[u_{st\alpha}, u_s]^{\tau_s\tau_t} u_s u_{st} u_{ts}$$

Note that $-\alpha_s, -\alpha_t \subseteq t\delta, ts\gamma$ for all $\delta \in M^G_{\alpha_t, t\alpha}$ and $\gamma \in M^G_{\alpha_s, st\alpha}$ by (CR1). Using Case 5 we deduce the following:

$$\begin{aligned}(\tau_s\tau_t)^4.u_{ts}u_{st}u_\alpha u_{st}u_{ts} &= (\tau_s\tau_t)^2\tau_s.u_{ts}u_{st}u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t}[u_{st\alpha}, u_s]^{\tau_s\tau_t} u_s u_{st} u_{ts} \\ &= \tau_t\tau_s\tau_t(\tau_t\tau_s)^4.u_{ts}u_{st}u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t}[u_{st\alpha}, u_s]^{\tau_s\tau_t} u_s u_{st} u_{ts} \\ &= \tau_t\tau_s\tau_t.u_{ts}u_{st}u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t}[u_{st\alpha}, u_s]^{\tau_s\tau_t} u_s u_{st} u_{ts} \\ &= \tau_t\tau_s\tau_t.\left(\tau_t\tau_s\tau_t.u_{ts}u_{st}u_\alpha u_{st}u_{ts}\right) \\ &= u_{ts}u_{st}u_\alpha u_{st}u_{ts}\end{aligned}$$

□

(3.4.5) Lemma. *Let $\alpha \in \Phi_+$ be a root such that $-\alpha_s, -s\alpha_t, -t\alpha_s \subseteq \alpha$. Then we have $[(\tau_s\tau_t)^4, u_{ts}u_{st}u_s u_\alpha u_s u_{st} u_{ts}] = 1$ in the group $P_s \star_{U_+} P_t$.*

Proof. If $\{\alpha_t, \alpha\} \notin \mathcal{P}$, then $\{-\alpha_t, \alpha\}$ is a prenilpotent pair by [2, Lemma 8.42(3)]. As $(-\alpha_t) \notin 1_W \in \alpha$, we deduce $(-\alpha_t) \subseteq \alpha$ and the claim follows from Lemma (3.4.4). Thus we can assume that $\{\alpha_t, \alpha\} \in \mathcal{P}$. We distinguish the following cases:

(a) $\alpha_t \subseteq \alpha$: Then we have $-\alpha_t, -\alpha_s \subseteq t\alpha$. Using Lemma (3.4.4) again we obtain that $(\tau_t\tau_s)^4.u_{ts}u_{st}u_s u_{t\alpha} u_s u_{st} u_{ts} = u_{ts}u_{st}u_s u_{t\alpha} u_s u_{st} u_{ts}$. This implies:

$$\begin{aligned}(\tau_s\tau_t)^4.u_{ts}u_{st}u_s u_\alpha u_s u_{st} u_{ts} &= \tau_t^2(\tau_s\tau_t)^3\tau_s.u_s u_{st} u_{ts} u_{t\alpha} u_{ts} u_{st} u_s \\ &= \tau_t(\tau_t\tau_s)^4.u_{ts}u_{st}u_s u_{t\alpha} u_s u_{st} u_{ts} \\ &= \tau_t.u_{ts}u_{st}u_s u_{t\alpha} u_s u_{st} u_{ts} \\ &= u_s u_{st} u_{ts} u_\alpha u_{ts} u_{st} u_s \\ &= u_{ts}u_{st}u_s u_\alpha u_s u_{st} u_{ts}\end{aligned}$$

(b) $o(r_{\alpha_t} r_\alpha) < \infty$: Using the nilpotency class of the groups U_w and (CR2), we deduce the following:

$$\begin{aligned}[u_{st\alpha}, u_s] &= 1 \\ [[u_{t\alpha}, u_t]^{\tau_t\tau_s\tau_t}, u_t] &= [[u_{t\alpha}, u_t]^{\tau_s\tau_t}, u_t]^{\tau_s\tau_t\tau_s} = [[u_{tst\alpha}, u_{st}], u_t]^{\tau_s\tau_t\tau_s} = 1 \\ \tau_t.u_t[u_{t\alpha}, u_t]^{\tau_t\tau_s}u_t &= \tau_t u_t.[u_{t\alpha}, u_t]^{\tau_t\tau_s} \\ &= u_t\tau_t u_t\tau_t.[u_{t\alpha}, u_t]^{\tau_t\tau_s} \\ &= u_t\tau_t.[u_{t\alpha}, u_t]^{\tau_t\tau_s\tau_t}[[u_{t\alpha}, u_t]^{\tau_t\tau_s\tau_t}, u_t] \\ &= u_t.[u_{t\alpha}, u_t]^{\tau_t\tau_s} \\ &= u_t[u_{t\alpha}, u_t]^{\tau_t\tau_s}u_t \\ [u_{ts\alpha}, u_t] &= 1 \\ [[u_{t\alpha}, u_t]^{\tau_t\tau_s}, u_s] &= [[u_{t\alpha}, u_t]^{\tau_s\tau_t\tau_s}, u_s]^{\tau_t\tau_s\tau_t} = [[u_{stst\alpha}, u_t], u_s]^{\tau_t\tau_s\tau_t} = 1 \\ [u_{s\alpha}, u_s] &= 1 \\ u_t u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_t &= (u_t\tau_t)^2.u_\alpha = \tau_t u_t.u_\alpha = u_{t\alpha}[u_\alpha, u_t]^{\tau_t}\end{aligned}$$

We compute the following:

$$\begin{aligned}(\tau_s\tau_t)^4.u_{ts}u_{st}u_s u_\alpha u_s u_{st} u_{ts} &= (\tau_s\tau_t)^3\tau_s.u_s u_{st} u_{ts} u_{t\alpha} u_{ts} u_{st} u_s \\ &= (\tau_s\tau_t)^3\tau_s.u_{st} u_{ts} u_s u_{t\alpha} u_s u_{ts} u_{st} \\ &= (\tau_s\tau_t)^3.u_t u_{ts} u_s u_{t\alpha}[u_{st\alpha}, u_s]^{\tau_s} u_s u_{ts} u_t\end{aligned}$$

$$
\begin{aligned}
&= (\tau_s\tau_t)^3.u_{st}u_s u_{t\alpha}[u_{t\alpha}, u_t]u_s u_{st} \\
&= (\tau_s\tau_t)^2\tau_s.u_{st}u_{ts}u_\alpha[u_{t\alpha}, u_t]^{\tau_t}u_{ts}u_{st} \\
&= (\tau_s\tau_t)^2.u_t u_{ts}u_{s\alpha}[u_{t\alpha}, u_t]^{\tau_t\tau_s}u_{ts}u_t \\
&= (\tau_s\tau_t)^2.u_{ts}u_t u_{s\alpha}[u_{t\alpha}, u_t]^{\tau_t\tau_s}u_t u_{ts} \\
&= \tau_s\tau_t\tau_s.u_s u_t u_{s\alpha}[u_{ts\alpha}, u_t]^{\tau_t}[u_{t\alpha}, u_t]^{\tau_t\tau_s}u_t u_s \\
&= \tau_s\tau_t\tau_s.u_t u_{ts}u_{st}u_{s\alpha}[u_{s\alpha}, u_s][u_{t\alpha}, u_t]^{\tau_t\tau_s}u_{st}u_{ts}u_t \\
&= \tau_s\tau_t\tau_s.u_t u_{ts}u_{st}u_{s\alpha}[u_{t\alpha}, u_t]^{\tau_t\tau_s}u_{st}u_{ts}u_t \\
&= \tau_s\tau_t.u_{st}u_{ts}u_t u_\alpha[u_{t\alpha}, u_t]^{\tau_t}u_t u_{ts}u_{st} \\
&= \tau_s\tau_t.u_{st}u_{ts}u_{t\alpha}[u_\alpha, u_t]^{\tau_t}u_{ts}u_{st} \\
&= \tau_s.u_{st}u_s u_\alpha[u_\alpha, u_t]u_s u_{st} \\
&= \tau_s.u_{ts}u_t u_s u_\alpha u_s u_t u_{ts} \\
&= u_{ts}u_{st}u_s u_\alpha[u_{s\alpha}, u_s]^{\tau_s}u_s u_{st}u_{ts} \\
&= u_{ts}u_{st}u_s u_\alpha u_s u_{st}u_{ts}
\end{aligned}
$$

□

(3.4.6) Lemma. *Let $\alpha \in \Phi_+$ be a root such that $-s\alpha_t, -t\alpha_s, -\alpha_t \subseteq \alpha$. Then we have $[(\tau_s\tau_t)^4, u_t u_{ts}u_{st}u_\alpha u_{st}u_{ts}u_t] = 1$ in the group $P_s \star_{U_+} P_t$.*

Proof. Interchanging s and t in the previous lemma we deduce

$$
\begin{aligned}
(\tau_t\tau_s)^4.u_t u_{ts}u_{st}u_\alpha u_{st}u_{ts}u_t &= (\tau_t\tau_s)^4.u_{st}u_{ts}u_t u_\alpha u_t u_{ts}u_{st} \\
&= u_{st}u_{ts}u_t u_\alpha u_t u_{ts}u_{st} \\
&= u_t u_{ts}u_{st}u_\alpha u_{st}u_{ts}u_t
\end{aligned}
$$

This implies $(\tau_s\tau_t)^4.u_t u_{ts}u_{st}u_\alpha u_{st}u_{ts}u_t = u_t u_{ts}u_{st}u_\alpha u_{st}u_{ts}u_t$. □

(3.4.7) Lemma. *Let $\alpha \in \Phi_+$ be a root such that $-\alpha_s, -s\alpha_t \subseteq \alpha$ holds. Then we have $[(\tau_s\tau_t)^4, u_{st}u_s u_\alpha u_s u_{st}] = 1$ in the group $P_s \star_{U_+} P_t$, if $m_{rt} \neq 6 \neq m_{sr}$ for all $r \in S$.*

Proof. We distinguish the following cases:

(A) $-\alpha_t \subseteq \alpha$: This is covered by Lemma (3.4.4).

(B) $o(r_{\alpha_t}r_\alpha) < \infty$: By definition we have $-\alpha_t \subseteq st\alpha$. Assume that $t\alpha_s \subseteq \alpha$. Then Lemma (1.3.2) would imply $\alpha_t \subseteq (-\alpha_s) \cup t\alpha_s \subseteq \alpha$, which is a contradiction. Now assume that $\{\alpha_s, st\alpha\} \notin \mathcal{P}$. Then [2, Lemma 8.42(3)] would imply that $\{-\alpha_s, st\alpha\}$ is a pair of nested roots and, as $(-\alpha_s) \notin 1_W \in st\alpha$, we obtain $-\alpha_s \subseteq st\alpha$. But then $t\alpha_s \subseteq \alpha$, which is a contradiction. Thus we have $\{\alpha_s, st\alpha\} \in \mathcal{P}$. Using the nilpotency class of the groups U_w and (CR2), we have the following:

$$
\begin{aligned}
[u_{ts t\alpha}, u_t] &= 1 \\
[u_{s\alpha}, u_s] &= 1 \\
u_t u_\alpha[u_{t\alpha}, u_t]^{\tau_t}u_t &= u_{t\alpha}[u_\alpha, u_t]^{\tau_t} \\
u_s u_{t\alpha}[u_{st\alpha}, u_s]^{\tau_s}u_s &= u_{st\alpha}[u_{t\alpha}, u_s]^{\tau_s} \\
[[u_{st\alpha}, u_s]^{\tau_s}, u_t] &= [[u_{st\alpha}, u_s], u_{st}]^{\tau_s} = 1
\end{aligned}
$$

We compute the following:

$$
(\tau_s\tau_t)^4.u_{st}u_s u_\alpha u_s u_{st} = (\tau_s\tau_t)^3\tau_s.u_{st}u_{ts}u_{t\alpha}u_{ts}u_{st}
$$

$$\begin{aligned}
&= (\tau_s\tau_t)^3 . u_t u_{ts} u_{st\alpha} u_{ts} u_t \\
&= (\tau_s\tau_t)^3 . u_{ts} u_t u_{st\alpha} u_t u_{ts} \\
&= (\tau_s\tau_t)^2 \tau_s . u_s u_t u_{st\alpha} [u_{tst\alpha}, u_t]^{\tau_t} u_t u_s \\
&= (\tau_s\tau_t)^2 \tau_s . u_t u_{ts} u_{st} u_s u_{st\alpha} u_s u_{st} u_{ts} u_t \\
&= (\tau_s\tau_t)^2 \tau_s . u_t u_{ts} u_{st} u_{st\alpha} [u_{st\alpha}, u_s] u_{st} u_{ts} u_t \\
&= (\tau_s\tau_t)^2 . u_{st} u_{ts} u_t u_{t\alpha} [u_{st\alpha}, u_s]^{\tau_s} u_t u_{ts} u_{st}
\end{aligned}$$

Later we will do a case distinction and two cases are similar. Thus we will assume for the moment that $[u_{st\alpha}, u_s] = 1$. Then we compute

$$\begin{aligned}
(\tau_s\tau_t)^2 . u_{st} u_{ts} u_t u_{t\alpha} [u_{st\alpha}, u_s]^{\tau_s} u_t u_{ts} u_{st} &= (\tau_s\tau_t)^2 . u_{st} u_{ts} u_{t\alpha} [u_{t\alpha}, u_t] u_{ts} u_{st} \\
&= \tau_s\tau_t\tau_s . u_{st} u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_s u_{st}
\end{aligned}$$

If, furthermore, $[u_{t\alpha}, u_t] = 1$, we deduce the following:

$$\begin{aligned}
\tau_s\tau_t\tau_s . u_{st} u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_s u_{st} &= \tau_s\tau_t\tau_s . u_{st} u_s u_\alpha u_s u_{st} \\
&= \tau_s\tau_t . u_t u_s u_\alpha [u_{s\alpha}, u_s]^{\tau_s} u_s u_t \\
&= \tau_s\tau_t . u_s u_{st} u_{ts} u_t u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_t u_{ts} u_{st} u_s \\
&= \tau_s\tau_t . u_s u_{st} u_{ts} u_{t\alpha} [u_\alpha, u_t]^{\tau_t} u_{ts} u_{st} u_s
\end{aligned}$$

Now we distinguish the following cases:

(a) $-t\alpha_s \subseteq \alpha$: Then $[u_{st\alpha}, u_s] = 1$ by (CR2) and the previous computation yields

$$(\tau_s\tau_t)^2 . u_{st} u_{ts} u_t u_{t\alpha} [u_{st\alpha}, u_s]^{\tau_s} u_t u_{ts} u_{st} = \tau_s\tau_t\tau_s . u_{st} u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_s u_{st}$$

Since $m_{rt} \neq 6$, we deduce $1 \in \{[u_{t\alpha}, u_t], [u_\alpha, u_t]\}$. We distinguish these two cases:

(I) $[u_{t\alpha}, u_t] = 1$: Then again the previous computations yield:

$$\tau_s\tau_t\tau_s . u_{st} u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_s u_{st} = \tau_s\tau_t . u_s u_{st} u_{ts} u_{t\alpha} [u_\alpha, u_t]^{\tau_t} u_{ts} u_{st} u_s$$

(II) $[u_\alpha, u_t] = 1$: Then we have the following:

$$\begin{aligned}
\tau_s\tau_t\tau_s . u_{st} u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_s u_{st} &= \tau_s\tau_t\tau_s . u_{st} u_s u_t u_{t\alpha} [u_\alpha, u_t]^{\tau_t} u_t u_s u_{st} \\
&= \tau_s\tau_t\tau_s . u_t u_{ts} u_s u_{t\alpha} u_s u_{ts} u_t \\
&= \tau_s\tau_t . u_{st} u_{ts} u_s u_{t\alpha} [u_{st\alpha}, u_s]^{\tau_s} u_s u_{ts} u_{st} \\
&= \tau_s\tau_t . u_s u_{st} u_{ts} u_{t\alpha} [u_\alpha, u_t]^{\tau_t} u_{ts} u_{st} u_s
\end{aligned}$$

(b) $\{t\alpha_s, \alpha\} \in \mathcal{P}$: As $t\alpha_s \not\subseteq \alpha$, we have $o(r_{t\alpha_s} r_\alpha) < \infty$ and hence $o(r_{\alpha_s} r_{t\alpha}) < \infty$. Since $m_{sr} \neq 6$ for all $r \in S$, we have $1 \in \{[u_{st\alpha}, u_s], [u_{t\alpha}, u_s]\}$. We distinguish these two cases:

(aa) $[u_{st\alpha}, u_s] = 1$: Then the previous computations yield:

$$(\tau_s\tau_t)^2 . u_{st} u_{ts} u_t u_{t\alpha} [u_{st\alpha}, u_s]^{\tau_s} u_t u_{ts} u_{st} = \tau_s\tau_t\tau_s . u_{st} u_s u_\alpha [u_{t\alpha}, u_t]^{\tau_t} u_s u_{st}$$

Since $m_{rt} \neq 6$, we deduce $1 \in \{[u_{t\alpha}, u_t], [u_\alpha, u_t]\}$. We distinguish these two cases:

(i) $[u_{t\alpha}, u_t] = 1$: Then again the previous computations yield:

$$\tau_s\tau_t\tau_s.u_{st}u_su_\alpha[u_{t\alpha}, u_t]^{\tau_t}u_su_{st} = \tau_s\tau_t.u_su_{st}u_{ts}u_{t\alpha}[u_\alpha, u_t]^{\tau_t}u_{ts}u_{st}u_s$$

(ii) $[u_\alpha, u_t] = 1$: Then we have the following:

$$\begin{aligned}\tau_s\tau_t\tau_s.u_{st}u_su_\alpha[u_{t\alpha}, u_t]^{\tau_t}u_su_{st} &= \tau_s\tau_t\tau_s.u_{st}u_su_tu_{t\alpha}[u_\alpha, u_t]^{\tau_t}u_tu_su_{st}\\ &= \tau_s\tau_t\tau_s.u_tu_{ts}u_su_{t\alpha}u_su_{ts}u_t\\ &= \tau_s\tau_t\tau_s.u_tu_{ts}u_{t\alpha}[u_{t\alpha}, u_s]u_{ts}u_t\\ &= \tau_s\tau_t.u_{st}u_{ts}u_{st\alpha}[u_{t\alpha}, u_s]^{\tau_s}u_{ts}u_{st}\\ &= \tau_s\tau_t.u_{st}u_{ts}u_su_{t\alpha}[u_{st\alpha}, u_s]^{\tau_s}u_su_{ts}u_{st}\\ &= \tau_s\tau_t.u_su_{st}u_{ts}u_{t\alpha}[u_\alpha, u_t]^{\tau_t}u_{ts}u_{st}u_s\end{aligned}$$

(bb) $[u_{t\alpha}, u_s] = 1$: Then we compute the following:

$$\begin{aligned}(\tau_s\tau_t)^2.u_{st}u_{ts}u_tu_{t\alpha}[u_{st\alpha}, u_s]^{\tau_s}u_tu_{ts}u_{st} &= (\tau_s\tau_t)^2.u_{st}u_{ts}u_tu_su_{st\alpha}[u_{t\alpha}, u_s]^{\tau_s}u_su_tu_{ts}u_{st}\\ &= (\tau_s\tau_t)^2.u_su_tu_{st\alpha}u_tu_s\\ &= \tau_s\tau_t\tau_s.u_{ts}u_tu_{st\alpha}[u_{tst\alpha}, u_t]^{\tau_t}u_tu_{ts}\\ &= \tau_s\tau_t\tau_s.u_{ts}u_tu_{st\alpha}u_tu_{ts}\\ &= \tau_s\tau_t.u_{ts}u_{st}u_{t\alpha}u_{st}u_{ts}\\ &= \tau_s\tau_t.u_{ts}u_{st}u_su_{t\alpha}u_su_{st}u_{ts}\\ &= \tau_s\tau_t.u_su_{st}u_{ts}u_{t\alpha}u_{ts}u_{st}u_s\end{aligned}$$

Since $o(r_{\alpha_s}r_{t\alpha}) < \infty$ and $-\alpha_s \subseteq \alpha$, we have $\alpha \neq t\alpha$. Clearly, we have

$$-\alpha_s = (-\alpha_s) \cap W = ((-\alpha_s) \cap (-s\alpha_t)) \cup ((-\alpha_s) \cap s\alpha_t) \subseteq (-s\alpha_t) \cup ((-\alpha_s) \cap s\alpha_t)$$

Note that there exists $R \in \partial^2\alpha_t \cap \partial^2\alpha \cap \partial^2 t\alpha$. Lemma (1.4.8) now implies (as $-t\alpha \notin (\alpha_t, \alpha) \cup (-\alpha_t, \alpha)$) $\alpha \in (\alpha_t, t\alpha)$ or $t\alpha \in (\alpha_t, \alpha)$. Assume $t\alpha \in (\alpha_t, \alpha)$. Then $\alpha_t \cap \alpha \subseteq t\alpha$ by definition. Since $s\alpha_t \in (\alpha_s, \alpha_t)$, we deduce $\alpha_t \in (-\alpha_s, s\alpha_t)$ and hence $(-\alpha_s) \cap s\alpha_t \subseteq \alpha_t$. But then we would have the following:

$$-\alpha_s \subseteq (-s\alpha_t) \cup ((-\alpha_s) \cap s\alpha_t) \subseteq t\alpha \cup (\alpha_t \cap \alpha) \subseteq t\alpha$$

This is a contradiction as $o(r_{\alpha_s}r_{t\alpha}) < \infty$ and hence we deduce $\alpha \in (\alpha_t, t\alpha)$. Since $m_{rt} \neq 6$ for all $r \in S$, the commutator relations imply $[u_{t\alpha}, u_t] \neq 1$ and hence $[u_\alpha, u_t] = 1$ because $1 \in \{[u_{t\alpha}, u_t], [u_\alpha, u_t]\}$ as before. We infer:

$$\tau_s\tau_t.u_su_{st}u_{ts}u_{t\alpha}u_{ts}u_{st}u_s = \tau_s\tau_t.u_su_{st}u_{ts}u_{t\alpha}[u_\alpha, u_t]^{\tau_t}u_{ts}u_{st}u_s$$

We see that in both cases (a) and (b) we have the same result. Thus we compute further:

$$\begin{aligned}(\tau_s\tau_t)^2.u_{st}u_{ts}u_tu_{t\alpha}[u_{st\alpha}, u_s]^{\tau_s}u_tu_{ts}u_{st} &= \tau_s\tau_t.u_su_{st}u_{ts}u_{t\alpha}[u_\alpha, u_t]^{\tau_t}u_{ts}u_{st}u_s\\ &= \tau_s.u_{ts}u_{st}u_su_\alpha[u_\alpha, u_t]u_su_{st}u_{ts}\\ &= \tau_s.u_{ts}u_{st}u_su_tu_\alpha u_tu_su_{st}u_{ts}\\ &= \tau_s.u_tu_su_\alpha u_su_t\\ &= u_{st}u_su_\alpha[u_{s\alpha}, u_s]^{\tau_s}u_su_{st}\\ &= u_{st}u_su_\alpha u_su_{st}\end{aligned}$$

(C) $\alpha_t \subseteq \alpha$: We distinguish the following cases:

(aaa) $-t\alpha_s \subseteq \alpha$: Then $-\alpha_s, -\alpha_t \subseteq t\alpha$ and we can apply again Lemma (3.4.4) with $u = u_{st}u_{ts}$ to deduce the following:

$$\begin{aligned}(\tau_s\tau_t)^4.u_{st}u_su_\alpha u_su_{st} &= \tau_t^2(\tau_s\tau_t)^3\tau_s.u_{st}u_{ts}u_{t\alpha}u_{ts}u_{st}\\ &= \tau_t(\tau_t\tau_s)^4.u_{ts}u_{st}u_{t\alpha}u_{st}u_{ts}\\ &= \tau_t.u_{ts}u_{st}u_{t\alpha}u_{st}u_{ts}\\ &= \tau_t.u_{st}u_{ts}u_{t\alpha}u_{ts}u_{st}\\ &= u_{st}u_su_\alpha u_su_{st}\end{aligned}$$

(bbb) $t\alpha_s \subseteq \alpha$: Then $-\alpha_s, -\alpha_t \subseteq st\alpha$ and we deduce from Lemma (3.4.4):

$$\begin{aligned}(\tau_s\tau_t)^4.u_{st}u_su_\alpha u_su_{st} &= (\tau_s\tau_t)^{-1}(\tau_s\tau_t)(\tau_s\tau_t)^3.u_tu_{ts}u_{st\alpha}u_{ts}u_t\\ &= (\tau_s\tau_t)^{-1}.u_tu_{ts}u_{st\alpha}u_{ts}u_t\\ &= u_{st}u_su_\alpha u_su_{st}\end{aligned}$$

(ccc) $o(r_{t\alpha_s}r_\alpha) < \infty$: Note that $-t\alpha_s, -\alpha_t \subseteq st\alpha$ and $o(r_{\alpha_s}r_{st\alpha}) < \infty$. Interchanging s and t in Case (B) yields $(\tau_t\tau_s)^4.u_{ts}u_tu_{st\alpha}u_tu_{ts} = u_{ts}u_tu_{st\alpha}u_tu_{ts}$ and hence

$$\begin{aligned}(\tau_s\tau_t)^4.u_{st}u_su_\alpha u_su_{st} &= (\tau_s\tau_t)^3.u_tu_{ts}u_{st\alpha}u_{ts}u_t\\ &= (\tau_s\tau_t)^{-1}\cdot(\tau_s\tau_t)^4.u_{ts}u_tu_{st\alpha}u_tu_{ts}\\ &= \tau_t\tau_s.u_{ts}u_tu_{st\alpha}u_tu_{ts}\\ &= \tau_t\tau_s.u_tu_{ts}u_{st\alpha}u_{ts}u_t\\ &= u_{st}u_su_\alpha u_su_{st}\end{aligned}$$ □

(3.4.8) Lemma. *Assume that $m_{rt} \neq 6 \neq m_{sr}$ for all $r \in S$. Then for $\alpha \in \Phi_+ \backslash \Phi_+^{\{s,t\}}$ and $u \in U_{s,t}$ we have $[(\tau_s\tau_t)^4, u^{-1}u_\alpha u] = 1$ in the group $P_s \star_{U_+} P_t$, if one of the following hold:*

(a) $u = u_{st}u_{ts}$ and $-s\alpha_t, -t\alpha_s \subseteq \alpha$,

(b) $u = u_{ts}u_t$ and $-t\alpha_s, -\alpha_t \subseteq \alpha$;

Proof. Assume that (a) holds. Then we have $-s\alpha_t, -\alpha_s \subseteq t\alpha$ and the previous lemma yields

$$\begin{aligned}(\tau_s\tau_t)^4.u_{ts}u_{st}u_\alpha u_{st}u_{ts} &= \tau_t^2(\tau_s\tau_t)^3\tau_s.u_su_{st}u_{t\alpha}u_{st}u_s\\ &= \tau_t(\tau_t\tau_s)^4.u_{st}u_su_{t\alpha}u_su_{st}\\ &= \tau_t.u_{st}u_su_{t\alpha}u_su_{st}\\ &= u_{st}u_{ts}u_\alpha u_{ts}u_{st}\\ &= u_{ts}u_{st}u_\alpha u_{st}u_{ts}\end{aligned}$$

Now assume that (b) holds. Then we have $-s\alpha_t, -t\alpha_s \subseteq s\alpha$ and we infer the following from Assertion (a):

$$(\tau_s\tau_t)^4.u_tu_{ts}u_\alpha u_{ts}u_t = (\tau_s\tau_t)^4\tau_s.u_{st}u_{ts}u_{s\alpha}u_{ts}u_{st} = \tau_s.u_{st}u_{ts}u_{s\alpha}u_{ts}u_{st} = u_tu_{ts}u_\alpha u_{ts}u_t$$ □

(3.4.9) Lemma. *We have $[(\tau_s\tau_t)^4, n] = 1$ for all $n \in N_{s,t}$ in the group $P_s \star_{U_+} P_t$, if the groups U_w are of nilpotency class at most 2, $m_{ru} \neq 6$ for all $r, u \in S$ and $\mathcal{M}$ satisfies (CR1) and (CR2).*

Proof. Since $N_{s,t}$ is generated by the elements $u^{-1}u_\alpha u$ with $u \in U_{s,t}$ and $\alpha \in \Phi_+ \backslash \Phi_+^{\{s,t\}}$ it suffices to show the claim for $n = u^{-1}u_\alpha u$. Since $U_{s,t}$ is a group of order 16, we have to distinguish these 16 cases. The claim is trivial for $u = 1$. The other cases follow from the Lemmas (3.4.1) - (3.4.8). □

3.5. First main result

(3.5.1) Theorem. *Suppose $m_{st} \neq 6$ for all $s, t \in S$. Let $\mathcal{M} = \left(M_{\alpha,\beta}^G\right)_{(G,\alpha,\beta)\in\mathcal{I}}$ be a commutator blueprint of type (W, S), which is faithful and Weyl-invariant. Suppose $s \neq t \in S$ with $m_{st} < \infty$ and let $n \in N_{s,t}$. Then $[(\tau_s\tau_t)^{m_{st}}, n] = 1$ in $P_s \star_{U_+} P_t$ if the following hold:*

(a) $m_{st} = 2$;

(b) $m_{st} \in \{3, 4\}$, *the groups* U_w *are of nilpotency class at most* 2 *and* $\mathcal{M}$ *satisfies the following two additional conditions, where* $\alpha \in \Phi_+$ *is such that* $\alpha_s \subseteq \alpha$.

(CR1) If $\beta \in \Phi^{\{s,t\}}$ is such that $\beta \subseteq \alpha$, then there exist $w \in W$ with $\ell(sw) = \ell(w) - 1$ and $G \in \mathrm{Min}_s(w)$ with $\alpha \in \Phi(G)$ such that $\beta \subseteq \gamma$ for all $\gamma \in M_{\alpha_s,\alpha}^G$.

(CR2) If $\beta \in \Phi^{\{s,t\}}$ satisfies $o(r_\beta r_\alpha) < \infty$, then there exist $w \in W$ with $\ell(sw) = \ell(w)-1$ and $G \in \mathrm{Min}_s(w)$ with $\alpha \in \Phi(G)$ such that $M_{\alpha_s,\alpha}^G = \emptyset$.

In particular, $\mathcal{M}$ is integrable.

Proof. The first part is a consequence of Lemma (3.2.1), Lemma (3.3.6) and Lemma (3.4.9). Now we deduce from Theorem (2.2.14) and Theorem (2.4.3) that $\mathcal{M}$ is integrable. □

(3.5.2) Remark. We remark that (CR2) is not always satisfied. To see this one may consider (W, S) to be of affine type and an RGD-system of type (W, S) with non-abelian root groups at infinity. We do not know whether (CR1) and the nilpotency class assumption are necessary.

4. Construction of the groups U_w

In this chapter we assume $m_{st} \neq 6$ for all $s \neq t \in S$. We will discuss the nilpotency class assumption of the last chapter. We show that each family $\mathcal{M} = \left(M_{\alpha,\beta}^G\right)_{(G,\alpha,\beta)\in\mathcal{I}}$ of subsets $M_{\alpha,\beta}^G \subseteq (\alpha,\beta)$ ordered via $\leq_G$ which induces (roughly speaking) nilpotency class 2 groups U_w and satisfies (CB1) and (CB2) satisfies automatically (CB3). Hence such a family is a commutator blueprint of type (W,S).

4.1. Auxiliary results

(4.1.1) Lemma. *Let $G = \langle g_1,\dots,g_n\rangle$ be a group such that $[g_i,[g_j,g_k]] = 1$ for all $i,j,k \in \{1,\dots,n\}$. Then G is of nilpotency class at most 2.*

Proof. Let $x,y,z \in G$ and let $x_1,\dots,x_k,y_1,\dots,y_l,z_1,\dots,z_m \in \{g_1,\dots,g_n\}$ be such that $x = x_1\cdots x_k, y = y_1\cdots y_l, z = z_1\cdots z_m$. We will show that $[x,[y,z]] = 1$. Assume first $l = 1 = m$. Induction on k yields $[x,[y,z]] = [xx_k^{-1},[y,z]]^{x_k}[x_k,[y,z]] = 1$. Now we assume $l = 1$. Induction on m implies $[x,[y,z]] = [x,[y,z_m][y,zz_m^{-1}]^{z_m}] = [x^{(z_m^{-1})},[y,zz_m^{-1}]]^{z_m}[x,[y,z_m]]^{[y,zz_m^{-1}]^{z_m}} = 1$. Now induction on l yields

$$[x,[y,z]] = [x,[yy_l^{-1},z]^{y_l}[y_l,z]] = [x,[y_l,z]][x^{(y_l^{-1})},[yy_l^{-1},z]]^{y_l[y_l,z]} = 1 \qquad \square$$

(4.1.2) Proposition. *Let N be a group and let $g,h \in \mathrm{Aut}(N)$ be two involutions with $[g,h] = \mathrm{id}_N$. Assume that there exists $u \in Z(N)$ such that $u^2 = 1$ and $g(u) = u = h(u)$. Let $G = \mathbb{Z}_2 \ltimes_g N$ (i.e. $\mathbb{Z}_2$ acts on N via g) and $H = \mathbb{Z}_2 \ltimes_h N$. Moreover, we let x_g (resp. x_h) be the generator of $\mathbb{Z}_2 \leq G$ (resp. $\mathbb{Z}_2 \leq H$) and we let $\varphi : G \star_N H \to G \star_N H/\langle\langle [x_g,x_h]u^{-1}\rangle\rangle$. Then*

$$\ker\varphi = \{[x_g,x_h]^k u^l \mid k,l \in \mathbb{Z}, k+l \equiv 0 \mod 2\}$$

In particular, the product map $\langle x_g\rangle \times N \times \langle x_h\rangle \to G \star_N H/\langle\langle [x_g,x_h]u^{-1}\rangle\rangle, (g',n,h') \mapsto g'nh'$ is a bijection.

Proof. Let $n \in N$. By assumption we have $[g,h](n) = n$. We note that in G (resp. H) we have $x_g^{-1}nx_g = g(n)$ (resp. $x_h^{-1}nx_h = h(n)$) for all $n \in N$. We consider a conjugate of $[x_g,x_h]u^{-1}$ in $G \star_N H$. For $n \in N$ we obtain:

$$n^{-1}\left([x_g,x_h]u^{-1}\right)n = n^{-1}[x_g,x_h]n[x_g,x_h]^{-1}[x_g,x_h]u^{-1} = n^{-1}[g,h](n)[x_g,x_h]u^{-1} = [x_g,x_h]u^{-1}$$

Since $g,h \in \mathrm{Aut}(N)$, we have $g(u^{-1}) = g(u)^{-1} = u^{-1}$ and $h(u^{-1}) = u^{-1}$. Thus we obtain:

$$x_g^{-1}\left([x_g,x_h]u^{-1}\right)x_g = x_gx_gx_hx_gx_hu^{-1}x_g = x_hx_gx_hx_gx_gu^{-1}x_g = [x_h,x_g]g(u^{-1}) = [x_g,x_h]^{-1}u^{-1}$$
$$x_h^{-1}\left([x_g,x_h]u^{-1}\right)x_h = x_hx_gx_hx_gx_hu^{-1}x_h = [x_h,x_g]h(u^{-1}) = [x_g,x_h]^{-1}u^{-1}$$

We also note that $[[x_g,x_h]^{\pm1},u^{\pm1}] = [g,h]^{\mp1}(u^{\mp1})u^{\pm1} = 1$. Thus we conclude that $\ker\varphi = \langle\langle [x_g,x_h]u^{-1}\rangle\rangle = \langle [x_g,x_h]^{\varepsilon}u^{-1} \mid \varepsilon \in \{1,-1\}\rangle = \{[x_g,x_h]^k u^l \mid k,l \in \mathbb{Z}, k+l \equiv 0 \mod 2\}$. For the second assertion we note at first that the mapping is surjective. We denote the product map by p. Let $a \in \langle x_g\rangle, b \in \langle x_h\rangle$ and $n \in N$ be such that $p((a,n,b)) = anb = 1$.

Then $anb = abn' \in \ker\varphi$, where $n' = b^{-1}nb = b(n) \in N$. Considering normal forms in amalgamated products we obtain $a = 1 = b$ and $n' = u^{2l}$. Since $u^2 = 1$, we obtain $n' = 1$ and hence $n = 1$. Now we show that p is injective. Let $a, a' \in \langle x_g\rangle, b, b' \in \langle x_h\rangle$ and $n, n' \in N$ be such that $anb = p((a,n,b)) = p((a',n',b')) = a'n'b'$. Then

$$1 = a^{-1}a'n'b'b^{-1}n^{-1} = a^{-1}a'n'\left(b'b^{-1}n^{-1}b(b')^{-1}\right)b'b^{-1}$$

and hence $p((a^{-1}a', n'\left(b'b^{-1}n^{-1}b(b')^{-1}\right), b'b^{-1})) = 1$. We have already shown that this implies $a = a', b = b'$ and $1 = n'\left(b'b^{-1}n^{-1}b(b')^{-1}\right) = n'n^{-1}$. This finishes the claim. □

(4.1.3) Corollary. *Let N be a group and let $g, h \in \mathrm{Aut}(N)$ be two involutions. Assume that $G = \mathbb{Z}_2 \ltimes_g N$ and $H = \mathbb{Z}_2 \ltimes_h N$ are of nilpotency class at most 2 and that $h(n)n^{-1} \in \mathrm{Stab}_N(g), g(n)n^{-1} \in \mathrm{Stab}_N(h)$ for all $n \in N$. Let $u \in N$ be such that $u \in Z(G), u \in Z(H)$ and $u^2 = 1$. Let x_g (resp. x_h) be the generator of $\mathbb{Z}_2$ in G (resp. H). Then the mapping*

$$\langle x_g\rangle \times N \times \langle x_h\rangle \to G \star_N H/\langle\langle [x_g, x_h]u^{-1}\rangle\rangle$$

is a bijection. Furthermore, the latter group is of nilpotency class at most 2.

Proof. Since $u \in Z(G), u \in Z(H)$, we have $1 = [x_g, u^{-1}] = g(u)u^{-1}$ and hence $g(u) = u$. Similarly, we have $h(u) = u$. Moreover, we have $u \in Z(N)$. In view of the previous proposition it suffices to show that $[g,h] = \mathrm{id}_N$. As G is of nilpotency class at most 2, we have $[g(n)n^{-1}, n'] = [[x_g, n^{-1}], n'] = 1$ for all $n, n' \in N$. We compute the following:

$$\begin{aligned}
[g,h](n) &= ghg\left(h(n)n^{-1}n\right) \\
&= gh\left(g\left(h(n)n^{-1}\right)g(n)n^{-1}n\right) = gh\left(h(n)g(n)n^{-1}\right) \\
&= g\left(nh\left(g(n)n^{-1}\right)\right) = g\left(ng(n)n^{-1}\right) = g\left(g(n)\right) \\
&= n
\end{aligned}$$

Thus $[g,h] = \mathrm{id}_N$. For the nilpotency class it suffices to show $[a,[b,c]] = 1$ for $a,b,c \in \{x_g, x_h\} \cup N$ by Lemma (4.1.1). If $x_g \notin \{a,b,c\}$ the claim follows by the nilpotency class of H. Using similar arguments we obtain the result if $x_h \notin \{a,b,c\}$. Thus we can assume $x_g, x_h \in \{a,b,c\}$. If $\{x_g, x_h\} \neq \{b,c\}$, the claim follows from the fact that $[x_h, n^{-1}] \in \mathrm{Stab}_N(g)$ and $[x_g, n^{-1}] \in \mathrm{Stab}_N(h)$ for all $n \in N$. Thus we assume $\{b,c\} = \{x_g, x_h\}$. Since $[b,c] = u^{\pm 1}$ is contained in $Z(G)$ and in $Z(H)$ the claim follows. □

4.2. Pre-commutator blueprints

A *pre-commutator blueprint of type* (W,S) is a family $\mathcal{M} = \left(M_{\alpha,\beta}^G\right)_{(G,\alpha,\beta)\in\mathcal{I}}$ of subsets $M_{\alpha,\beta}^G \subseteq (\alpha,\beta)$ ordered via $\leq_G$ satisfying (CB1), (CB2) and the following axiom:

(PCB) For every $G \in \mathrm{Min}(w)$ the canonical homomorphism $U_G \to U_w$ is an isomorphism.

Let $G = (d_0, \ldots, d_n = c_0, \ldots, c_k = e_0, \ldots, e_m) \in \mathrm{Min}$ and let $(\alpha_1, \ldots, \alpha_{n+k+m})$ be the sequence of roots crossed by G. We define the group $U_{(c_0,\ldots,c_k),G}$ via the presentation

$$U_{(c_0,\ldots,c_k),G} := \left\langle u_{\alpha_{n+1}}, \ldots, u_{\alpha_{n+k}} \mid \begin{cases} \forall 1 \leq i \leq k : u_{\alpha_{n+i}}^2 = 1, \\ \forall 1 \leq i < j \leq k : [u_{\alpha_{n+i}}, u_{\alpha_{n+j}}] = \prod_{\gamma \in M^G_{\alpha_{n+i},\alpha_{n+j}}} u_\gamma \end{cases} \right\rangle$$

(4.2.1) Remark. In Axiom (PCB) we do not require that $|U_w| = 2^{\ell(w)}$. We will see in Lemma (4.2.2) that under some mild conditions, a pre-commutator blueprint is a commutator blueprint. Moreover, we remark that $U_{G,G} = U_G$ (cf. Lemma (2.1.4)).

For every $\alpha_{n+1} \leq_G \alpha \leq_G \beta \leq \alpha_{n+k} \in \Phi(G)$ we have $M^G_{\alpha,\beta} \subseteq (\alpha, \beta) \subseteq \{\alpha_{n+2}, \ldots, \alpha_{n+k-1}\}$. We call a pre-commutator blueprint 2-*nilpotent*, if for all $G = (d_0, \ldots, d_n = c_0, \ldots, c_k = e_0, \ldots, e_m) \in \text{Min}, \alpha_{n+2} \leq_G \alpha \leq_G \alpha_{n+(k-1)}$ the following hold in $U_{(c_1,\ldots,c_{k-1}),G}$:

(2-n1) $\prod_{\gamma \in M^G_{\alpha,\alpha_{n+k}}} \left(\prod_{\delta \in M^G_{\alpha_{n+1},\gamma}} u_\delta \right) u_\gamma = \prod_{\gamma \in M^G_{\alpha,\alpha_{n+k}}} u_\gamma$;

(2-n2) $\prod_{\gamma \in M^G_{\alpha_{n+1},\alpha}} \left(u_\gamma \prod_{\delta \in M^G_{\gamma,\alpha_{n+k}}} u_\delta \right) = \prod_{\gamma \in M^G_{\alpha_{n+1},\alpha}} u_\gamma$;

(2-n3) $\left(\prod_{\gamma \in M^G_{\alpha_{n+1},\alpha_{n+k}}} u_\gamma \right)^2 = 1$ and $\prod_{\gamma \in M^G_{\alpha_{n+1},\alpha_{n+k}}} u_\gamma \in Z(U_{(c_1,\ldots,c_{k-1}),G})$;

(2-n4) $\prod_{\gamma \in M^G_{\alpha_{n+1},\alpha_{n+k}}} \left(\prod_{\delta \in M^G_{\alpha_{n+1},\gamma}} u_\delta \right) u_\gamma = \prod_{\gamma \in M^G_{\alpha_{n+1},\alpha_{n+k}}} u_\gamma$;

(2-n5) $\prod_{\gamma \in M^G_{\alpha_{n+1},\alpha_{n+k}}} \left(u_\gamma \prod_{\delta \in M^G_{\gamma,\alpha_{n+k}}} u_\delta \right) = \prod_{\gamma \in M^G_{\alpha_{n+1},\alpha_{n+k}}} u_\gamma$.

Condition (2-n1) will imply that $[u_{\alpha_{n+1}}, [u_\alpha, u_{\alpha_{n+k}}]] = 1$ holds and Condition (2-n2) that $[[u_{\alpha_{n+1}}, u_\alpha], u_{\alpha_{n+k}}] = 1$ holds. Conditions (2-n4) and (2-n5) imply that $[u_{\alpha_{n+1}}, u_{\alpha_{n+k}}]$ commutes with $u_{\alpha_{n+1}}$ and $u_{\alpha_{n+k}}$. Let $\mathcal{M}$ be a commutator blueprint of type (W, S). Then $\mathcal{M}$ is a pre-commutator blueprint of type (W, S) by Lemma (2.1.4). It is not hard to see that if the groups U_w of a commutator blueprint are of nilpotency class at most 2, then the pre-commutator blueprint is 2-nilpotent (cf. Lemma (3.3.1)).

(4.2.2) Lemma. *Let $\mathcal{M}$ be a 2-nilpotent pre-commutator blueprint of type (W, S). Then $\mathcal{M}$ is a commutator blueprint of type (W, S) and the groups U_w are of nilpotency class at most 2.*

Proof. Let $w \in W, G = (d_0, \ldots, d_n = c_0, \ldots, c_k) \in \text{Min}(w)$ and $H = (c_0, \ldots, c_k)$. We show by induction on $k \geq 0$, that $|U_{H,G}| = 2^k$ and $U_{H,G}$ is of nilpotency class at most 2. This will finish the claim as $U_{G,G} \cong U_G \cong U_w$ by (PCB). We remark that the induction is on the length of the gallery H and not on the length of the gallery G.

If $k \leq 2$, the claim follows as $U_{H,G} \cong \begin{cases} \{1\} & k = 0 \\ \mathbb{Z}_2 & k = 1 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & k = 2 \end{cases}$. Thus we assume $k > 2$. Let $G' = (d_0, \ldots, d_n = c_0, \ldots, c_{k-1}), G_1 = (c_0, \ldots, c_{k-1}), G_2 = (c_1, \ldots, c_k)$ and $K = (c_1, \ldots, c_{k-1})$. Using induction, the groups $U_{G_1,G'}, U_{G_2,G}$ are of nilpotency class at most 2 and we have

$$|U_{G_1,G'}| = 2^{k-1}, \qquad |U_{K,G'}| = 2^{k-2}, \qquad |U_{G_2,G}| = 2^{k-1}.$$

Because of (CB1) we have $U_{G_1,G'} = U_{G_1,G}$ as well as $U_{K,G'} = U_{K,G}$. Clearly, $U_{K,G} \to U_{G_1,G}, U_{G_2,G}$ are injective and $U_{G_1,G} \cong \langle u_{\alpha_{n+1}} \rangle \ltimes U_{K,G}, U_{G_2,G} \cong \langle u_{\alpha_{n+k}} \rangle \ltimes U_{K,G}$. In particular, $u_{\alpha_{n+1}}, u_{\alpha_{n+k}}$ act on $U_{K,G}$ via conjugation. Using (2-n2) and (2-n1) we deduce

$$u_{\alpha_{n+1}}(u_\alpha)u_\alpha = \prod_{\gamma \in M^G_{\alpha_{n+1},\alpha}} u_\gamma \in \text{Stab}_{U_{K,G}}(u_{\alpha_{n+k}})$$

$$u_{\alpha_{n+k}}(u_\alpha)u_\alpha = \left(u_\alpha \cdot u_{\alpha_{n+k}}(u_\alpha)\right)^{-1} = \left(\prod_{\gamma \in M^G_{\alpha,\alpha_{n+k}}} u_\gamma \right)^{-1} \in \text{Stab}_{U_{K,G}}(u_{\alpha_{n+1}})$$

for all $\alpha = \alpha_{n+i}$ with $2 \leq i \leq k-1$. Since $U_{K,G}$ is generated by these u_α and since $U_{G_1,G}, U_{G_2,G}$ are of nilpotency class at most 2, it follows by induction that for $n, n', u_\alpha \in U_{K,G}$ with $n = n'u_\alpha$ we have

$$u_{\alpha_{n+1}}(n)n^{-1} = u_{\alpha_{n+1}}(n')u_{\alpha_{n+1}}(u_\alpha)u_\alpha(n')^{-1}$$

$$
\begin{aligned}
&= u_{\alpha_{n+1}}(n')[u_{\alpha_{n+1}}, u_\alpha](n')^{-1} \\
&= u_{\alpha_{n+1}}(n')(n')^{-1}u_{\alpha_{n+1}}(u_\alpha)u_\alpha \in \mathrm{Stab}_{U_{K,G}}(u_{\alpha_{n+k}}) \\
nu_{\alpha_{n+k}}(n^{-1}) &= n'u_\alpha u_{\alpha_{n+k}}(u_\alpha)u_{\alpha_{n+k}}((n')^{-1}) \\
&= n'[u_\alpha, u_{\alpha_{n+k}}]u_{\alpha_{n+k}}((n')^{-1}) \\
&= u_\alpha u_{\alpha_{n+k}}(u_\alpha)n'u_{\alpha_{n+k}}((n')^{-1}) \in \mathrm{Stab}_{U_{K,G}}(u_{\alpha_{n+k}})
\end{aligned}
$$

In particular, $u_{\alpha_{n+k}}(n)n^{-1} = \left(nu_{\alpha_{n+k}}(n^{-1})\right)^{-1} \in \mathrm{Stab}_{U_{K,G}}(u_{\alpha_{n+k}})$. Using (2-n3), (2-n4) and (2-n5), Corollary (4.1.3) implies that the mapping

$$\mathbb{Z}_2 \times U_{K,G} \times \mathbb{Z}_2 \to U_{G_1,G} \star_{U_{K,G}} U_{G_2,G}/\langle\langle [u_{\alpha_{n+1}}, u_{\alpha_{n+k}}] = \prod_{\gamma \in M^G_{\alpha_{n+1},\alpha_{n+k}}} u_\gamma \rangle\rangle$$

is a bijection and the latter group is of nilpotency class at most 2. Moreover, the latter group is isomorphic to $U_{(c_0,\ldots,c_k),G}$ and we are done. □

(4.2.3) Theorem. *Let (W, S) be right-angled (i.e. $m_{st} \in \{2, \infty\}$ for all $s \neq t \in S$) such that every connected component of the Coxeter diagram of (W, S) is the complete graph. Then each Weyl-invariant 2-nilpotent pre-commutator blueprint of type (W, S) is integrable.*

Proof. The previous lemma implies that $\mathcal{M}$ is a commutator blueprint of type (W, S). Let k be the number of connected components of the Coxeter diagram of (W, S) and let $J_1, \ldots, J_k \subseteq S$ be the vertex sets of the connected components. Then $W \cong \langle J_1 \rangle \times \cdots \times \langle J_k \rangle$. Let $\{\alpha, \beta\} \in \mathcal{P}$. If $\alpha = w\alpha_s, \beta = v\alpha_t$ and $m_{st} = 2$, then $(\alpha, \beta) = \emptyset$. Consider the commutator blueprint $\mathcal{M}_i = \left(M^G_{\alpha,\beta}\right)_{(G,\alpha,\beta)\in\mathcal{I}_i}$, where $\mathcal{I}_i = \{(G, \alpha, \beta) \in \mathcal{I} \mid G \in \bigcup_{w\in\langle J_i\rangle} \mathrm{Min}(w)\}$ of type $(\langle J_i \rangle, J_i)$. Then $\mathcal{M}_i$ is integrable by Corollary (2.2.15). Let $\mathcal{D}_i = \left(G_i, (U^i_\alpha)_{\alpha\in\Phi^{J_i}}\right)$ be an RGD-system of type $(\langle J_i \rangle, J_i)$ over $\mathbb{F}_2$ such that $\mathcal{M}_{\mathcal{D}_i} = \mathcal{M}_i$. Then $G_1 \times \cdots \times G_k$ yields an RGD-system $\mathcal{D}$ such that $\mathcal{M} = \mathcal{M}_\mathcal{D}$ and hence $\mathcal{M}$ is integrable. □

(4.2.4) Theorem. *Let (W, S) be a union of $\tilde{A}_1$ diagrams. Let $\mathcal{M}$ be a Weyl-invariant pre-commutator blueprint of type (W, S). Then the following are equivalent:*

(i) $\mathcal{M}$ is integrable.

(ii) $\mathcal{M}$ is 2-nilpotent.

Proof. Let $\mathcal{M}$ be a Weyl-invariant pre-commutator blueprint of type (W, S). Assume that $\mathcal{M}$ is integrable. Then there exists an RGD-system $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ of type (W, S) over $\mathbb{F}_2$ such that $\mathcal{M} = \mathcal{M}_\mathcal{D}$. Let k be the number of connected components of the Coxeter diagram of (W, S) and let $J_1, \ldots, J_k \subseteq S$ be the vertex sets of the connected components. Then $W \cong \langle J_1 \rangle \times \cdots \langle J_k \rangle$ and we can write every $w \in W$ as a product $v_1 \cdots v_k$, where each v_i is contained in $\langle J_i \rangle$. In particular, $U_w \cong U_{v_1} \times \cdots \times U_{v_k}$. It is a direct consequence of [21, Theorem A] that each U_{v_i} and hence U_w is of nilpotency class at most 2. In particular, $\mathcal{M}_\mathcal{D}$ is 2-nilpotent. The other implication follows from the previous theorem. □

(4.2.5) Definition. Suppose that $m_{st} = \infty$ for all $s \neq t \in S$. Let $s \neq t \in S$.

(a) Let $1 \leq k \in \mathbb{N}, J \subseteq \{1, \ldots, k\}$, let $\alpha \neq \beta \in \Phi_+$ and let $G \in \mathrm{Min}$ be such that $\alpha, \beta \in \Phi(G)$. Assume that there exists a minimal gallery $H = (c_0, \ldots, c_k)$ of type $(s, t, \ldots, s, t, s)$ such that $\{c_0, c_1\} \in \partial\alpha, c_0 \in \alpha, \{c_{k-1}, c_k\} \in \partial\beta, c_{k-1} \in \beta$ and s appears $k + 1$ times and t appears k times in the type of H. Let $(\alpha_1, \ldots, \alpha_k)$ be the sequence of roots crossed by H. Then we define

$$M(k, J, (s, t))^G_{\alpha,\beta} := \{\alpha_{2j} \mid j \in J\}$$

(b) Let $\emptyset \neq K \subseteq \mathbb{N}$ and let $\mathcal{J} = (J_k)_{k\in K}$ be a family of subsets $J_k \subseteq \{1,\dots,k\}$. For $\alpha \neq \beta \in \Phi_+$ and $G \in \mathrm{Min}$ with $\alpha, \beta \in \Phi(G)$ we define

$$M(K, \mathcal{J}, (s,t))^G_{\alpha,\beta} := \bigcup_{k\in K} M(k, J_k, (s,t))^G_{\alpha,\beta}$$

Moreover, we define $\mathcal{M}(K,\mathcal{J},(s,t)) := \left(M(K,\mathcal{J},(s,t))^G_{\alpha,\beta} \right)_{(G,\alpha,\beta)\in\mathcal{I}}$.

(4.2.6) Theorem. *Let $s \neq t \in S$, let $\emptyset \neq K \subseteq \mathbb{N}$ and let $\mathcal{J} = (J_k)_{k\in K}$ be a family of subsets $J_k \subseteq \{1,\dots,k\}$. Then $\mathcal{M}(K,\mathcal{J},(s,t))$ is an integrable commutator blueprint.*

Proof. We abbreviate $\mathcal{M} := \mathcal{M}(K,\mathcal{J},(s,t))$. By definition, $\mathcal{M}$ satisfies (CB1) and (CB2). As $|\mathrm{Min}(w)| = 1$ for every $w \in W$, (PCB) is also satisfied and $\mathcal{M}$ is a pre-commutator blueprint. Let $\alpha \in \Phi$ be a root. Because of the type of (W,S) we deduce that $|\partial\alpha| = 1$ (cf. Lemma (1.4.2)), and we call $\delta(c,d) \in S$ the *type* of α, where $\{c,d\} \in \partial\alpha$. Now let $\alpha \neq \beta \in \Phi_+$ be such that $M^G_{\alpha,\beta} \neq \emptyset$. Then α, β are roots of type s and every $\gamma \in M^G_{\alpha,\beta}$ is a root of type t. Now it is straight forward to verify that $\mathcal{M}$ is 2-nilpotent. Moreover, $\mathcal{M}$ is Weyl-invariant, as $M^G_{\alpha,\beta}$ does only depend on the existence of a suitable gallery H and not on G. Now Theorem (4.2.3) yields the claim. □

(4.2.7) Remark. It is mentioned in [16, Remark before Lemma 5] that Abramenko and Mühlherr constructed an example of an RGD-system of right-angled type and of rank 3 which does not satisfy property (FPRS). The author of this thesis is not aware of any publication that provides the existence of RGD-systems of rank at least 3 which do not satisfy property (FPRS). We have defined this property in Section 1.7.

(4.2.8) Corollary. *Let $s \neq t \in S$ and for every $n \in \mathbb{N}$ we let $J_n \subseteq \{1,\dots,n\}$ with $1 \in J_n$. Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be the RGD-system associated with $\mathcal{M}\left(\mathbb{N}, (J_n)_{n\in\mathbb{N}}, (s,t)\right)$. Then $\mathcal{D}$ does not satisfy property (FPRS).*

Proof. Assume $\mathcal{D}$ would have property (FPRS). Let $G_n = (c_0,\dots,c_n) \in \mathrm{Min}(w)$ be of type $(s,t,s,t,\dots)$ with $\ell(w) = n$ (i.e. G_1 has type (s) and G_2 has type (s,t)) and we define $\alpha_n := \alpha_{G_n}$. Then $\lim_{i\to\infty} \ell(1_W, \alpha_{2i-1}) = \infty$. As $\mathcal{D}$ has property (FPRS), there exists $n_0 \in \mathbb{N}$ such that for all $i \geq n_0$ the root group $U_{\alpha_{2i-1}}$ fixes the ball $B(c,2)$ pointwise. But then $[u_{\alpha_1}, u_{\alpha_{2i-1}}] = \prod_{j\in J_i} u_{\alpha_{2j}}$ would also fix $B(c,2)$ pointwise, which is a contradiction, as $1 \in J_i$. □

Part III.

Faithful commutator blueprints of type $(4, 4, 4)$

5. Buildings of type $(4,4,4)$

In Chapter 5 we assume that (W,S) is of type $(4,4,4)$ and that $S=\{r,s,t\}$. This chapter contains many auxiliary results and proofs about roots in the Coxeter buildings of type $(4,4,4)$. Moreover, we prove that any RGD-system of type $(4,4,4)$ over $\mathbb{F}_2$ contains a suitable subgroup, which is a sequence of groups.

5.1. Coxeter buildings of type $(4,4,4)$

(5.1.1) Lemma. *Suppose $w\in W$ with $\ell(ws)=\ell(w)+1=\ell(wt)$. Then $\ell(w)+2\in\{\ell(wsr),\ell(wtr)\}$. Moreover, if $\ell(wsr)=\ell(w)$, then $\ell(wsrt)=\ell(w)+1$.*

Proof. Let $(c_0=1_W,\dots,c_{k-2}=w,c_{k-1}=ws,c_k=wst)$ be a minimal gallery of type $(s_1,\dots,s_{k-2},s,t)$. Then we have $s_{k-2}=r$. We assume that $\ell(s_1\cdots s_{k-2}sr)=k-2=\ell(s_1\cdots s_{k-2}tr)$. Then $\ell(s_1\cdots s_{k-3}s)=k-4=\ell(s_1\cdots s_{k-3}t)$. Let R be the $\{s,t\}$ residue containing c_{k-3}, let T be the $\{t,r\}$-residue containing c_{k-3} and let P be the t-panel containing c_{k-3}. Then $P=R\cap T$ and Proposition (1.5.4) yields $\operatorname{proj}_T 1_W=\operatorname{proj}_P 1_W$, which is a contradiction to the type $(4,4,4)$. Thus the first claim follows.

Now suppose that $\ell(wsr)=\ell(w)$. Assume that $\ell(wsrt)=\ell(w)-1$. Then $\mathcal{P}_t(c_{k-1})=R_{\{s,t\}}(c_{k-1})\cap R_{\{r,t\}}(c_{k-1})$ and $\ell(1_W,\operatorname{proj}_{R_{\{r,t\}}(c_{k-1})}1_W)<\ell(w)=\ell(1_W,\operatorname{proj}_{R_{\{s,t\}}(c_{k-1})}1_W)$. Again Proposition (1.5.4) yields a contradiction and we have $\ell(wsrt)=\ell(w)+1$. □

(5.1.2) Lemma. *Suppose $w\in W$ such that $\ell(ws)=\ell(w)+1=\ell(wt)$ and suppose $w'\in\langle s,t\rangle$ with $\ell(w')\geq 2$. Then $\ell(ww'rf)=\ell(w)+\ell(w')+1+\ell(f)$ for each $f\in\{1_W,s,t\}$.*

Proof. At first we show the claim for $w'=st$. If $\ell(wsr)=\ell(ws)+1$, then $\ell(wstrt)=\ell(wst)+2$, as $\ell(wst)=\ell(ws)+1$. As $\ell(wstr)=\ell(wsts)=\ell(wst)+1$, we deduce $\ell(wstrf)=\ell(w)+3+\ell(f)$ for $f\in\{1_W,s,t\}$. Thus we can assume $\ell(wsr)=\ell(w)$. By Lemma (5.1.1) we have $\ell(wsrt)=\ell(w)+1=\ell((wsr)r)$. This implies $\ell(wstrt)=\ell(w)+4$. Moreover, we have $\ell(wstr)=\ell(wsts)=\ell(w)+3$ and hence $\ell(wstrs)=\ell(w)+4$. Using similar arguments, the claim follows for all $w'\in\langle s,t\rangle$ with $\ell(w')\geq 2$. □

(5.1.3) Lemma. *We have $tstr\alpha_s\cap stsr\alpha_t\cap(W\backslash\{r_{\{s,t\}}r\})\subseteq r_{\{s,t\}}\alpha_r$.*

Proof. Let $r_{\{s,t\}}r\neq w\in tstr\alpha_s\cap stsr\alpha_t$ be an element. We have to show that $r_{\{s,t\}}w\in\alpha_r$, i.e. $\ell(rr_{\{s,t\}}w)=\ell(r_{\{s,t\}}w)+1$. We distinguish the following cases:

(i) $\ell(w^{-1})+2\in\{\ell(w^{-1}ts),\ell(w^{-1}st)\}$: Then $\ell(w^{-1}r_{\{s,t\}})\geq\ell(\operatorname{proj}_{R_{\{s,t\}(w^{-1})}}1_W)+2$ and we deduce $\ell(rr_{\{s,t\}}w)=\ell(w^{-1}r_{\{s,t\}}r)=\ell(w^{-1}r_{\{s,t\}})+1=\ell(r_{\{s,t\}}w)+1$ from Lemma (5.1.2).

(ii) $\ell(w^{-1}s)=\ell(w^{-1})+1$ and $\ell(w^{-1}st)=\ell(w^{-1})$: By assumption, we have $w\in tstr\alpha_s$ and hence $\ell\left(s(rtstw)\right)=\ell(rtstw)+1$. This implies $\ell(w^{-1}tstrs)=\ell(w^{-1}tstr)+1$ and, in particular, $\ell(w^{-1}r_{\{s,t\}}r)=\ell(w^{-1}r_{\{s,t\}})+1$.

(iii) $\ell(w^{-1}t)=\ell(w^{-1})+1$ and $\ell(w^{-1}ts)=\ell(w^{-1})$: This follows similar as in the previous case.

(iv) $\ell(w^{-1}s) = \ell(w^{-1}) - 1 = \ell(w^{-1}t)$: If $\ell(rr_{\{s,t\}}w) = \ell(r_{\{s,t\}}w) + 1$ there is nothing to show. Thus we suppose $\ell(rr_{\{s,t\}}w) = \ell(r_{\{s,t\}}w) - 1$. Assume that $\ell(w^{-1}stsr) = \ell(w^{-1}sts) - 1$. Then we would have $\ell(w^{-1}stsrt) = \ell(w^{-1}sts) - 2$, which is a contradiction to the assumption $\ell(trstsw) = \ell(rstsw) + 1$. Thus we have $\ell(w^{-1}stsr) = \ell(w^{-1}sts) + 1$ and $\ell(w^{-1}stsrt) = \ell(w^{-1}sts) + 2$. Similarly, we deduce $\ell(w^{-1}tstrs) = \ell(w^{-1}tst) + 2$. This yields $\ell(w^{-1}r_{\{s,t\}}ru) = \ell(w^{-1}r_{\{s,t\}}r) + 1$ for each $u \in S = \{r, s, t\}$ and hence $w^{-1}r_{\{s,t\}}r = 1$. Since $w \neq r_{\{s,t\}}r$ by assumption, we have a contradiction and we are done. □

(5.1.4) Lemma. *Let $H = (d_0, \ldots, d_4)$ be a minimal gallery of type (r, s, t, r) and let $\beta \in \Phi$ with $\{d_0, d_1\} \in \partial\beta$ and $d_0 \in \beta$. Then $\beta \subsetneq \gamma$ for each $\gamma \in \{\alpha_{(d_0,\ldots,d_3)}, \alpha_{(d_0,\ldots,d_4)}\}$.*

Proof. We use the canonical linear representation of (W, S) (cf. [2, §2.5]). Let $V := \mathbb{R}^S$ be the vector space over $\mathbb{R}$ with standard basis $(e_s)_{s\in S}$ and let $(\cdot, \cdot)$ be the symmetric bilinear form on V given by

$$(e_s, e_t) := -\cos\left(\frac{\pi}{m_{st}}\right) = \begin{cases} 1 & \text{if s=t,} \\ -\frac{\sqrt{2}}{2} & \text{else.} \end{cases}$$

Then W acts on V via $\sigma : W \to \mathrm{GL}(V), s \mapsto (\sigma_s : V \to V, x \mapsto x - 2(x, e_s)e_s)$ and $(\cdot, \cdot)$ is invariant under this action. Let β and γ be as in the statement. Without loss of generality we can assume $\beta = \alpha_r$ and $\gamma \in \{rs\alpha_t, rst\alpha_r\}$. At first, we consider the case $\gamma = \alpha_{(d_0,\ldots,d_3)}$. Then $\gamma = rs\alpha_t$. We compute:

$$(e_r, \sigma(rs)(e_t)) = (e_r, \sigma_r(\sigma_s(e_t))) = (\sigma_r(e_r), \sigma_s(e_t)) = (-e_r, e_t + \sqrt{2}e_s) = \frac{\sqrt{2}}{2} + 1 > 1$$

Now we assume $\gamma = \alpha_{(d_0,\ldots,d_4)}$. Then $\gamma = rst\alpha_r$ and we compute:

$$\begin{aligned}
(e_r, \sigma(rst)(e_r)) &= (e_r, \sigma_r(\sigma_s(\sigma_t(e_r)))) \\
&= (\sigma_s(-e_r), \sigma_t(e_r)) \\
&= (-e_r - 2(-e_r, e_s)e_s, e_r - 2(e_r, e_t)e_t) \\
&= -(e_r, e_r) + 2(e_r, e_t)(e_r, e_t) + 2(e_r, e_s)(e_s, e_r) - 4(e_r, e_s)(e_r, e_t)(e_s, e_t) \\
&= -1 + 2 \cdot \frac{-\sqrt{2}}{2} \cdot \frac{-\sqrt{2}}{2} + 2 \cdot \frac{-\sqrt{2}}{2} \cdot \frac{-\sqrt{2}}{2} - 4 \cdot \frac{-\sqrt{2}}{2} \cdot \frac{-\sqrt{2}}{2} \cdot \frac{-\sqrt{2}}{2} \\
&= -1 + 1 + 1 + \sqrt{2} > 1
\end{aligned}$$

Using [2, Lemma 2.77] we obtain that $o(r_\beta r_\gamma) = \infty$. As $\{\beta, \gamma\} \in \mathcal{P}$, Lemma (1.4.7) and [2, Lemma 8.42(3)] yield that $\{\beta, \gamma\}$ is a pair of nested roots and hence $\beta \subsetneq \gamma$. □

(5.1.5) Lemma. *Let $s \neq t \in S$, let $P := \{1_W, s\} \neq Q \in \partial\alpha_s$ and let $P_0 := P, \ldots, P_n = Q, R_1, \ldots, R_n$ be as in Lemma (1.4.2). If $n > 1$, then there exists $\varepsilon \in \{+, -\}$ such that for every root $\beta \in \{\varepsilon\alpha_t, \varepsilon s\alpha_t, \varepsilon t\alpha_s\}$ there exists a non-simple root γ of R_n with $\beta \subseteq \gamma$.*

Proof. We prove the hypothesis by induction on n. Suppose first $n = 2$. At first we observe by Lemma (5.1.4) that for each root $\alpha_s \neq \beta \in \Phi_+$ with $R_1 \in \partial^2\beta$ there exists a non-simple root γ_β of R_2 such that $\beta \subseteq \gamma_\beta$. If $R = R_1$, the claim follows with $\varepsilon := +$. If $R \neq R_1$, we apply our observation twice and the claim follows with $\varepsilon := -$. Thus we can assume $n > 2$. Using our observation there exists $\varepsilon \in \{+, -\}$ such that for every root $\alpha_s \neq \beta \in \Phi_+$ with $R \in \partial^2\beta$ there exists a non-simple root γ'_β of R_{n-1} such that $\varepsilon\beta \subseteq \gamma'_\beta$. Using induction again, there exists a non-simple root γ_β of R_n such that $\gamma'_\beta \subseteq \gamma_\beta$. In particular, we have $\varepsilon\beta \subseteq \gamma'_\beta \subseteq \gamma_\beta$ and the claim follows. □

5.2. Roots in Coxeter systems of type $(4, 4, 4)$

Let $\alpha \in \Phi_+$ be a root such that $k_\alpha > 1$, i.e. α is not a simple root. Let $R \in \partial^2\alpha$ be a residue such that α is not a simple root of R (for the existence of such a residue see the next remark). Let $P \neq P' \in \partial\alpha$ be contained in R. Then $\ell(1_W, \operatorname{proj}_P 1_W) \neq \ell(1_W, \operatorname{proj}_{P'} 1_W)$ and we can assume that $\ell(1_W, \operatorname{proj}_P 1_W) < \ell(1_W, \operatorname{proj}_{P'} 1_W)$. Let $G = (c_0, \ldots, c_k) \in \mathrm{Min}$ be of type $(s_1, \ldots, s_k)$ such that $c_{k-2} = \operatorname{proj}_R 1_W, c_{k-1} = \operatorname{proj}_P 1_W$ and $c_k \in P \backslash \{c_{k-1}\}$. For $P \neq Q := \{x, y\} \in \partial\alpha$ with $x \in \alpha$ and $y \notin \alpha$ we let $P_0 = P, \ldots, P_n = Q$ and $R_1, \ldots, R_n$ be as in Lemma (1.4.2). We assume that $r \notin \{s_{k-1}, s_k\}$.

(5.2.1) Remark. Let $\alpha \in \Phi_+$ be a positive root such that $k_\alpha > 1$. Let $G = (c_0, \ldots, c_{k_\alpha}) \in \mathrm{Min}$ be a minimal gallery with $\{c_{k_\alpha - 1}, c_{k_\alpha}\} \in \partial\alpha$. Then α is not a simple root of the rank 2 residue containing $c_{k_\alpha-2}, c_{k_\alpha-1}, c_{k_\alpha}$. In particular, there exists $R \in \partial^2\alpha$ such that α is not a simple root of R.

(5.2.2) Lemma. *Assume that one of the following hold:*

(a) *$R_1 \neq R$ and $\ell(s_1 \cdots s_{k-1} r) = k$;*

(b) *$n > 1$.*

Then $\operatorname{proj}_{R_n} 1_W = \operatorname{proj}_{P_{n-1}} 1_W$.

Proof. Suppose $R_1 \neq R$ and $\ell(s_1 \cdots s_{k-1} r) = k$. Then $\operatorname{proj}_{R_1} c_0 = \operatorname{proj}_{P_0} c_0$ and the claim follows from Corollary (1.5.5). Now suppose that $n > 1$. Assume that $R_1 = R$. Then Lemma (5.1.2) implies $\operatorname{proj}_{R_2} 1_W = \operatorname{proj}_{P_1} 1_W$ and the claim follows from Corollary (1.5.5). Now we suppose $R_1 \neq R$. If $\ell(s_1 \cdots s_{k-1} r) = k$, the claim follows by Assertion (a). Thus we can assume that $\ell(s_1 \cdots s_{k-1} r) = k - 2$. Let $d := \operatorname{proj}_{R_{\{s_k, r\}}(c_k)} c_0$ be and replace G by a minimal gallery $(d_0 = c_0, \ldots, d, c_{k-1}, c_k)$. Now we are in the situation of $R_1 = R$ and the claim follows. □

(5.2.3) Lemma. *We have $k = k_\alpha$ and the panel $P_\alpha := P$ is the unique panel in $\partial\alpha$ with the property that $\ell(1_W, \operatorname{proj}_{P_\alpha} 1_W) = k_\alpha - 1$.*

Proof. We have $\ell(1_W, \operatorname{proj}_P 1_W) = k - 1$. Thus it suffices to show that $\ell(1_W, \operatorname{proj}_Q 1_W) > k - 1$. For $n = 1$ we obtain $\ell(1_W, \operatorname{proj}_Q 1_W) \in \{k, k + 2\}$. Now we assume $n > 1$. Using the previous lemma we obtain $\operatorname{proj}_{R_n} 1_W = \operatorname{proj}_{P_{n-1}} 1_W$. Since $Q \subseteq R_n$ we obtain $\ell(1_W, \operatorname{proj}_Q 1_W) \geq \ell(1_W, \operatorname{proj}_{R_n} 1_W) = \ell(1_W, \operatorname{proj}_{P_{n-1}} 1_W)$. Now the claim follows by induction. □

(5.2.4) Lemma. *Let $\gamma \in \Phi_+$ be the simple root of R containing P_α and let $\delta \in \Phi_+$ be the simple root of R which does not contain P_α. Then the following hold:*

(a) *If $R \neq R_1$ and $\ell(s_1 \cdots s_{k-1} r) = k$, then α is a simple root of R_1 and $-\gamma$ is contained in all roots $\alpha \neq \rho \in \Phi_+$ with $R_n \in \partial^2\rho$.*

(b) *If $R \neq R_1$ and $\ell(s_1 \cdots s_{k-1} r) = k - 2$, then α is a non-simple root of R_1 and $-\gamma$ is contained in the non-simple root of R_1 different from α and in the simple root of R_1 which contains P_α. If in addition $n > 1$, then $-\gamma$ is contained in all roots $\alpha \neq \rho \in \Phi_+$ with $R_n \in \partial^2\rho$.*

(c) *If $R = R_1$ and $n > 1$, then $-\delta$ is contained in the simple root of R_2 different from α and in the non-simple root ε of R_2, where P_ε and P_α have the same type. If in addition $n > 2$, then $-\delta$ is contained in all roots $\alpha \neq \rho \in \Phi_+$ with $R_n \in \partial^2\rho$.*

In particular, if $R \neq R_1$ or if $n > 1$, then there exists a simple root of R, say ω, and a non-simple root of R_n, say ω_n, such that $-\omega \subseteq \omega_n$.

Proof. Suppose we are in situation of Assertion (a). It follows from Lemma (5.1.4) that $-\gamma$ is contained in all roots $\alpha \neq \rho \in \Phi_+$ with $R_1 \in \partial^2\rho$. Now it follows by induction, that for every root $\alpha \neq \rho \in \Phi_+$ with $R_n \in \partial^2\rho$, there exists a root $\alpha \neq \rho' \in \Phi_+$ with $R_1 \in \partial^2\rho'$ with $\rho' \subseteq \rho$. Thus (a) follows.

The first part of the Assertions (b) and (c) follows from Lemma (5.1.4). The second part follows similarly as in the proof of Assertion (a) by induction. □

(5.2.5) Lemma. *We define $R_{\alpha,Q}$ to be the residue R_1 if $R \neq R_1$ and $\ell(s_1 \cdots s_{k-1}r) = k-2$. In all other cases, we define $R_{\alpha,Q} := R$. Then there exists a minimal gallery $H = (d_0 = c_0, \dots, d_m = \operatorname{proj}_Q c_0, y)$ with the following properties:*

- *$\alpha_H = \alpha$;*
- *There exists $0 \leq i \leq m$ such that $d_i = \operatorname{proj}_{R_{\alpha,Q}} 1_W$.*
- *For each $i+1 \leq j \leq m$ there exists $L_j \in \partial^2\alpha$ with $\{c_{j-1}, c_j\} \subseteq L_j$. In particular, we have $d_j \in \mathcal{C}(\partial^2\alpha)$.*

Proof. We define

$$d := \begin{cases} \operatorname{proj}_{P_0} c_0 & \text{if } R \neq R_1 \text{ and } \ell(s_1 \cdots s_{k-1}r) = k, \\ \operatorname{proj}_{P_1} c_0 & \text{else.} \end{cases}$$

We first show that $\ell(c_0, \operatorname{proj}_Q c_0) = \ell(c_0, \operatorname{proj}_{R_{\alpha,Q}} c_0) + \ell(\operatorname{proj}_{R_{\alpha,Q}} c_0, d) + \ell(d, \operatorname{proj}_Q c_0)$. By definition we have $R_{\alpha,Q} = R_{\alpha,P_i}$ for all $1 \leq i \leq n$. We prove the hypothesis by induction on n. Suppose first $n = 1$ and that one of the following hold:

- $R = R_1$;
- $R \neq R_1$ and $\ell(s_1 \cdots s_{k-1}r) = k-2$;

Then $Q = P_1 \subseteq R_{\alpha,Q}, d = \operatorname{proj}_Q c_0$ and the claim follows. We prove the case $\ell(s_1 \cdots s_{k-1}r) = k$ and $R \neq R_1$ together with the case $n > 1$ simultaneously. Lemma (5.2.2) provides in both cases $\operatorname{proj}_{R_n} c_0 = \operatorname{proj}_{P_{n-1}} c_0$. If $n > 1$, we have $R_{\alpha,Q} = R_{\alpha,P_{n-1}}$; if $n = 1$ we have $P_{n-1} = P_0 \subseteq R_{\alpha,Q}$ and $d = \operatorname{proj}_{P_{n-1}} c_0$. This is used in the third equation below. We compute the following:

$$\begin{aligned} \ell(c_0, \operatorname{proj}_Q c_0) &= \ell(c_0, \operatorname{proj}_{R_n} c_0) + \ell(\operatorname{proj}_{R_n} c_0, \operatorname{proj}_Q c_0) \\ &= \ell(c_0, \operatorname{proj}_{P_{n-1}} c_0) + \ell(\operatorname{proj}_{P_{n-1}} c_0, \operatorname{proj}_Q c_0) \\ &= \ell(c_0, \operatorname{proj}_{R_{\alpha,Q}} c_0) + \ell(\operatorname{proj}_{R_{\alpha,Q}} c_0, d) + \ell(d, \operatorname{proj}_{P_{n-1}} c_0) \\ &\quad\; + \ell(\operatorname{proj}_{P_{n-1}} c_0, \operatorname{proj}_Q c_0) \\ &\geq \ell(c_0, \operatorname{proj}_{R_{\alpha,Q}} c_0) + \ell(\operatorname{proj}_{R_{\alpha,Q}} c_0, d) + \ell(d, \operatorname{proj}_Q c_0) \\ &\geq \ell(c_0, \operatorname{proj}_Q c_0) \end{aligned}$$

Thus concatenating a minimal gallery from c_0 to $\operatorname{proj}_{R_{\alpha,Q}} c_0$, a minimal gallery from $\operatorname{proj}_{R_{\alpha,Q}} c_0$ to d and a minimal gallery from d to $\operatorname{proj}_Q c_0$ yields a minimal gallery from c_0 to $\operatorname{proj}_Q c_0$. Using Lemma (1.4.3) there exists a minimal gallery from d to $\operatorname{proj}_Q c_0$ such that every chamber of this gallery is contained in $\mathcal{C}(\partial^2\alpha)$ and for two adjacent chambers there exists a residue in $\partial^2\alpha$ containing both. Since $R_{\alpha,Q} \in \{R, R_1\} \subseteq \partial^2\alpha$ and, as $R_{\alpha,Q}$ is convex, each chamber of a minimal gallery from $\operatorname{proj}_{R_{\alpha,Q}} c_0$ to d is contained in $R_{\alpha,Q}$ the claim follows. □

(5.2.6) Lemma. *Let $\beta \in \Phi(k)\backslash\{\alpha_s \mid s \in S\}$ be a root such that $o(r_\alpha r_\beta) < \infty$ and $R \notin \partial^2\beta$. Moreover, we assume that $\ell(s_1 \cdots s_{k-1} r) = k$. Then one of the following hold:*

(a) $\beta = \alpha_F$, where F is the minimal gallery of type $(s_1, \ldots, s_{k-1}, r)$;

(b) $\beta = \alpha_F$, where F is the minimal gallery of type $(s_1, \ldots, s_{k-2}, s_k, s_{k-1}, r)$, and we have $\ell(s_1 \cdots s_{k-2} s_k r) = k-2$.

Proof. Recall that $\alpha = \alpha_G$. As $R \in \partial^2\alpha$, we have $\alpha \neq \pm\beta$. By Lemma (1.4.7) there exists $C \in \partial^2\alpha \cap \partial^2\beta$. By Remark (1.4.4) there exists a panel $Q' \in \partial\alpha$ which is contained in C. We let $\operatorname{proj}_{Q'} c_0 \neq y \in Q'$. Let P_i, R_i as before (with $P_n = Q'$), let $G' := (c_0, \ldots, c_{k-1})$ and let $G'' := (c_0, \ldots, c_k, c_{k+1})$ be the minimal gallery of type $(s_1, \ldots, s_k, s_{k-1})$. Let E be a minimal gallery from c_0 to y as in Lemma (5.2.5). We can extend this minimal gallery (if necessary) to a minimal gallery from c_0 to $e \in C$, where $\ell(e) = \ell(\operatorname{proj}_C c_0) + 4$. Let $Q'' \in \partial\beta$ be a panel contained in C and let $\operatorname{proj}_{Q''} c_0 \neq y' \in Q''$. Let $H = (d_0 = c_0, \ldots, d_{m-2} = \operatorname{proj}_{R_{\beta,Q''}} c_0, \ldots, d_q := \operatorname{proj}_{Q''} d_0, d_{q+1} := y')$ be a minimal gallery as in Lemma (5.2.5). Then $m = k_\beta \leq k$. As before, we can extend H (if necessary) to a minimal gallery from d_0 to e. Note that $R \neq C$ by assumption, and since $R \in \partial^2\alpha_{G'} \cap \partial^2\alpha_{G''}, R \notin \partial^2\beta$ we have $\alpha_{G'} \neq \pm\beta \neq \alpha_{G''}$.

(i) Assume that $R = R_1$: Since $R \in \partial^2\alpha_{G''} \cap \partial^2\alpha, C \in \partial^2\alpha$ and $\alpha_{G''} \neq \pm\beta$, Lemma (1.4.6) implies $C \notin \partial^2\alpha_{G''}$ and hence the gallery H has to cross the wall $\partial\alpha_{G''}$. Assume that $(d_0, \ldots, d_{m-2})$ crosses the wall $\partial\alpha_{G''}$. Let $1 \leq j \leq m-2$ be such that $\{d_{j-1}, d_j\} \in \partial\alpha_{G''}$. Then $k = k_{\alpha_{G''}} \leq j \leq m-2 \leq k-2$ which is a contradiction. Thus the gallery $(d_0, \ldots, d_{m-2})$ does not cross the wall $\partial\alpha_{G''}$ and hence $(d_{m-1}, \ldots, d_{q+1})$ has to cross the wall $\partial\alpha_{G''}$. Let $m \leq j \leq q+1$ be such that $\{d_{j-1}, d_j\} \in \partial\alpha_{G''}$. By Lemma (5.2.5) there exists $L \in \partial^2\beta$ such that $\{d_{j-1}, d_j\} \subseteq L$. Then $L \in \partial^2\beta \cap \partial^2\alpha_{G''}$ and hence $o(r_{\alpha_{G''}} r_\beta) < \infty$. As $\partial^2\alpha \cap \partial^2\alpha_{G''} = \{R\} \neq \{C\} = \partial^2\alpha \cap \partial^2\beta$ (cf. Lemma (1.4.6)), Lemma (1.4.8)(a) yields $\partial^2\alpha \cap \partial^2\beta \cap \partial^2\alpha_{G''} = \emptyset$ and hence $\{r_\alpha, r_{\alpha_{G''}}, r_\beta\}$ is a reflection triangle. As $\operatorname{proj}_R c_0 \in R \cap \beta \neq \emptyset$ and $R \notin \partial^2\beta$, we deduce $R \subseteq \beta$. As $e \in C \cap (-\alpha_{G''}) \neq \emptyset$ and $C \notin \partial^2\alpha_{G''}$, we deduce $C \subseteq (-\alpha_{G''})$. As $L \in \partial^2\alpha_{G''} \cap \partial^2\beta, \{d_{j-1}, d_j\} \subseteq L \cap \alpha$ and $L \notin \partial^2\alpha$, we deduce $L \subseteq \alpha$. Thus $T := \{\alpha, -\alpha_{G''}, \beta\}$ is a triangle. For $d \in W$ with $\delta(c_{k-2}, d) = s_k s_{k-1}$ we have $d \in \bigcap_{\gamma \in T} \gamma$ and Lemma (1.5.3) implies $\bigcap_{\gamma \in T} \gamma = \{d\}$. If $\ell(s_1 \cdots s_{k-2} s_k r) = k$, then $k_\beta = k+1$. Thus $\ell(s_1 \cdots s_{k-2} s_k r) = k-2$ and (b) follows.

(ii) Assume that $R \neq R_1$: Since $R \in \partial^2\alpha_{G'} \cap \partial^2\alpha$ and $R \neq C \in \partial^2\alpha$, Lemma (1.4.6) implies $C \notin \partial^2\alpha_{G'}$ and hence H has to cross the wall $\partial\alpha_{G'}$. Suppose that $(d_0, \ldots, d_{m-2})$ does not cross the wall $\partial^2\alpha_{G'}$. Replacing $\alpha_{G''}$ by $\alpha_{G'}$ in (i) we obtain that $T := \{\alpha, -\alpha_{G'}, \beta\}$ is a triangle. Using Lemma (1.5.3), we have $\bigcap_{\gamma \in T} \gamma = \{c_{k-1}\}$ and hence (a) follows. Now we suppose that $(d_0, \ldots, d_{m-2})$ crosses the wall $\partial\alpha_{G'}$ and let $1 \leq j \leq m-2$ be such that $P' := \{d_{j-1}, d_j\} \in \partial\alpha_{G'}$. Note that $1 \leq m-2 \leq k-2$ and hence $k \geq 3$. Let Z be the $\{s_{k-1}, r\}$-residue containing c_{k-2}. Then $\alpha_{G'}$ is not a simple root of Z and hence $k_{\alpha_{G'}} \in \{k-2, k-1\}$. This implies $k-2 \leq k_{\alpha_{G'}} \leq j \leq m-2 \leq k-2$. Lemma (5.2.3) implies $P' = P_{\alpha_{G'}}$ and hence P' is contained in Z. Moreover, we have $j = m-2$ and $R_{\beta,Q''} = R_{\{r,s_k\}}(d_j)$. Both non-simple roots of $R_{\beta,Q''}$ contain $-\alpha$ by Lemma (5.1.4). As one of them is equal to β, we have a contradiction. □

(5.2.7) Remark. Let $\gamma \in \Phi(k)\backslash\{\alpha_s \mid s \in S\}$ be a root such that $\{\alpha, \gamma\}$ is prenilpotent. If $o(r_\alpha r_\gamma) = \infty$, we have $\gamma \subseteq \alpha$, since $k_\gamma \leq k = k_\alpha$. This implies $\gamma = \alpha_{(c_0, \ldots, c_i)}$ for some $1 \leq i \leq k$. If $o(r_\alpha r_\gamma) < \infty$, then γ is known by the previous theorem.

5.3. RGD-systems of type $(4,4,4)$ over $\mathbb{F}_2$

In this section we let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD-system of type (W,S) over $\mathbb{F}_2$ (e.g. the one in Example (5.3.1)). Furthermore, we let $V_{r_{\{s,t\}}} := \langle U_s \cup U_t \rangle \leq U_{r_{\{s,t\}}}$ for all $s \neq t \in S$. By Example (1.7.1) and [2, Corollary 8.34(1)] this subgroup has index 2 in $U_{r_{\{s,t\}}}$. Moreover, we let $\Delta(\mathcal{D}) = (\Delta(\mathcal{D})_+, \Delta(\mathcal{D})_-, \delta_*)$ be the twin building associated with $\mathcal{D}$ and let (c_+, c_-) be the distinguished pair of opposite chambers. We denote for every $s \in S$ the unique chamber contained in $A(c_+, c_-)$ which is s-adjacent to c_- by c_s. Then U_+ acts on $\Delta_- := \Delta(\mathcal{D})_-$. We abbreviate $c := c_-$ (for more information we refer to Section 1.7 and [2, Section 8.9]).

(5.3.1) Example. Let $\mathcal{D} = (\mathcal{G}, (U_\alpha)_{\alpha\in\Phi})$ be the RGD-system associated with the split Kac-Moody group of type $(4,4,4)$ over $\mathbb{F}_2$ (for the definition of Kac-Moody groups we refer to [33]). Then $\mathcal{D}$ is over $\mathbb{F}_2$. Let $\{\alpha,\beta\}$ be a prenilpotent pair. We will determine the commutator relations $[u_\alpha, u_\beta] \leq \langle U_\gamma \mid \gamma \in (\alpha,\beta)\rangle$. For $o(r_\alpha r_\beta) < \infty$, the commutator relations follow from Example (1.7.1). For $o(r_\alpha r_\beta) = \infty$ we use the functoriality of Kac-Moody groups: Let $(\mathcal{G}, (\varphi_i)_{i\in I}, \eta)$ be the system as in [33, Ch. 2]. For every field $\mathbb{K}$ we let $U_{\alpha_i}(\mathbb{K}) := \varphi_i\left(\left\{\begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix} \mid k \in \mathbb{K}\right\}\right)$ and $U_{-\alpha_i}(\mathbb{K}) := \varphi_i\left(\left\{\begin{pmatrix}1 & 0\\ -k & 1\end{pmatrix} \mid k \in \mathbb{K}\right\}\right)$ be the root groups corresponding to the simple roots. For every i and any two fields $\mathbb{F}$ and $\mathbb{K}$ with a homomorphism $f : \mathbb{F} \to \mathbb{K}$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{SL}_2(\mathbb{F}) & \xrightarrow{\mathrm{SL}_2(f)} & \mathrm{SL}_2(\mathbb{K}) \\ \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_i} \\ \mathcal{G}(\mathbb{F}) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(\mathbb{K}) \end{array}$$

In particular, we have $\mathcal{G}(f)(U_{\alpha_i}(\mathbb{F})) \leq U_{\alpha_i}(\mathbb{K})$ and hence $\mathcal{G}(f)(U_\alpha(\mathbb{F})) \leq U_\alpha(\mathbb{K})$ for each root $\alpha \in \Phi$ by using (RGD2). Moreover, if f is injective, then $\mathcal{G}(f)$ is injective by the axiom (KMG4) (cf. [33]). Let $f : \mathbb{F}_2 \to \mathbb{F}_4$ be the canonical inclusion. We have $[U_\alpha(\mathbb{F}_4), U_\beta(\mathbb{F}_4)] = 1$ by [7, Theorem A]. This implies $\mathcal{G}(f)\left([U_\alpha(\mathbb{F}_2), U_\beta(\mathbb{F}_2)]\right) \leq [U_\alpha(\mathbb{F}_4), U_\beta(\mathbb{F}_4)] = 1$ and, as $\mathcal{G}(f)$ is injective, we deduce $[U_\alpha(\mathbb{F}_2), U_\beta(\mathbb{F}_2)] = 1$. All in all we have the following commutator relations, where $U_\alpha = \langle u_\alpha \rangle$ for all $\alpha \in \Phi$:

$$[u_\alpha, u_\beta] = \begin{cases} \prod_{\gamma\in(\alpha,\beta)} u_\gamma & \text{if } o(r_\alpha r_\beta) < \infty, |(\alpha,\beta)| = 2 \\ 1 & \text{else.} \end{cases}$$

(5.3.2) Lemma. *The following hold:*

(a) $\ell(c_s, c_s.h) \geq 3$ *for all* $h \in V_{r_{\{s,t\}}} \backslash \{1, u_s\}$;

(b) $\ell(c_s, c_t.h) \geq 2$ *for all* $h \in V_{r_{\{s,t\}}}$;

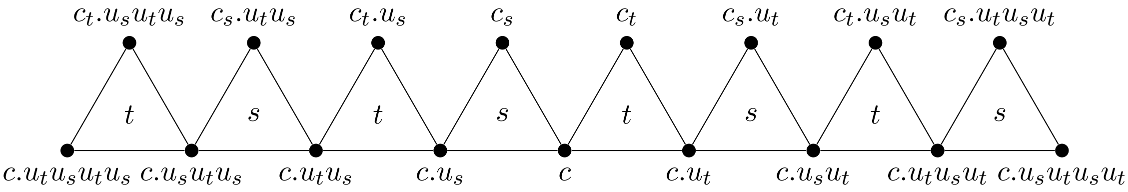
(c) $\ell(c_t.h, p) \geq 2$ *for all* $p \in \mathcal{P}_t(c)$ *and* $h \in V_{r_{\{s,t\}}} \backslash \{1, u_t\}$;

(d) $\ell(c_s.h, p) \geq 2$ *or* $\delta(c_s.h, p) = s$ *for all* $p \in \mathcal{P}_t(c)$ *and* $h \in V_{r_{\{s,t\}}}$;

(e) $\ell(p,q) \geq 2$ *or* $\delta(p,q) = s$ *for all* $p \in \mathcal{P}_t(c.h), q \in \mathcal{P}_t(c)$ *and* $h \in V_{r_{\{s,t\}}} \backslash \{1, u_t\}$.

(f) $\ell(p, c_s) \geq 2$ *or* $\delta(p, c_s) = s$ *for all* $p \in \mathcal{P}_t(c.h)$ *and* $h \in V_{r_{\{s,t\}}}$.

Proof. Before we prove the claim, we consider the following picture, where the lower chambers are all opposite to B_+ and the upper chambers d satisfy $\ell_\star(B_+, d) = 1$ and the letter in the triangles denotes the type of the panels:



We first show Assertion (a). As $c_s = c_s.u_s$ and $u_tu_su_tu_s = u_su_tu_su_t$, we can assume $h \in \{u_t, u_tu_s, u_tu_su_t\}$. Now we deduce the following:

(i) $\ell(c_s, c_s.u_t) = 3$;

(ii) $\ell(c_s, c_s.u_tu_s) = \ell(c_s, c_s.u_t) = 3$;

(iii) $\ell(c_s, c_s.u_tu_su_t) \geq 3$.

To show Assertion (b) we can similarly assume that $h \in \{1, u_s, u_su_t, u_su_tu_s\}$. We deduce the following:

(i) $\ell(c_s, c_t) = 2$;

(ii) $\ell(c_s, c_t.u_s) = \ell(c_s, c_t) = 2$;

(iii) $\ell(c_s, c_t.u_su_t) = 4$;

(iv) $\ell(c_s, c_t.u_su_tu_s) = \ell(c_s, c_t.u_su_t) = 4$

For Assertion (c) we can again assume $h \in \{u_s, u_su_t, u_su_tu_s\}$. We deduce the following:

(i) $\ell(c_t.u_s, p) \in \{2, 3\}$;

(ii) $\ell(c_t.u_su_t, p) = \ell(c_t.u_s, p.u_t) \in \{2, 3\}$;

(iii) $\ell(c_t.u_su_tu_s, p) \in \{3, 4\}$;

For Assertion (d) we can again assume that $h \in \{1, u_t, u_tu_s, u_tu_su_t\}$. We deduce the following:

(i) $\delta(c_s, p) \in \{s, st\}$;

(ii) $\delta(c_s.u_t, p) = \delta(c_s, p.u_t) \in \{s, st\}$;

(iii) $\ell(c_s.u_tu_s, p) \geq 3$;

(iv) $\ell(c_s.u_tu_su_t, p) = \ell(c_s.u_tu_s, p.u_t) \geq 3$.

For Assertion (e) we can assume that $h \in \{u_s, u_su_t, u_su_tu_s\}$, as $\mathcal{P}_t(c.u_t) = \mathcal{P}_t(c)$. We deduce the following:

(i) $h = u_s$: We have $\delta(p, q) = s$ or $\ell(p, q) \in \{2, 3\}$.

(ii) $h = u_su_t$: This follows similar as in the case $h = u_s$.

(iii) $h = u_su_tu_s$: Then we have $\ell(p, q) \geq 3$.

For Assertion (f) we can assume $h \in \{1, u_s, u_su_t, u_su_tu_s\}$, as $\mathcal{P}_t(c.u_t) = \mathcal{P}_t(c)$. We deduce the following:

(i) $h = 1$: Then $\delta(p, c_s) = s$ or $\ell(p, c_s) = 2$.

(ii) $h = u_s$: We have $\delta(p, c_s) = \delta(p.u_s, c_s)$. As $p \in \mathcal{P}_t(c.u_s)$ if and only if $p.u_s \in \mathcal{P}_t(c)$, the claim follows from (i).

(iii) $h = u_s u_t$: In this case we have $\ell(p, c_s) \geq 3$.

(iv) $h = u_s u_t u_s$: We have $\delta(p, c_s) = \delta(p.u_s, c_s)$. As $p \in \mathcal{P}_t(c.h)$ if and only if $p.u_s \in \mathcal{P}_t(c.u_s u_t)$, the claim follows from (iii). □

(5.3.3) Remark. For each root $\alpha \in \Phi_+$ there exist $w \in W, s \in S$ with $\alpha = w\alpha_s$. For short we will write u_{ws} to be the generator of $U_{w\alpha_s}$.

(5.3.4) Lemma. *Let $n > 0$, let $g_1, \dots, g_n \in \{u_{sr}, u_{tr}, u_{rt}, u_{rt}u_{tr}\}$ and let $h_1, \dots, h_n \in V_{r_{\{s,t\}}}$ be such that $h_i \notin \{1, u_s\}$ if $g_i = g_{i+1} = u_{sr}$ and such that $h_i \notin \{1, u_t\}$ if $g_i, g_{i+1} \in \{u_{tr}, u_{rt}, u_{rt}u_{tr}\}$. Then $g_1 h_1 \cdots g_n h_n \neq 1$ holds in G.*

Proof. Note that $h_i \in V_{r_{\{s,t\}}} = \{1, u_s, u_t, u_s u_t, u_t u_s, u_s u_t u_s, u_t u_s u_t, u_s u_t u_s u_t = u_t u_s u_t u_s\}$ as well as $g_n^{-1} = g_n$. For the proof we consider the action of U_+ on Δ_- as in Lemma (5.3.2). We abbreviate $\delta := \delta_-, c := B_-, R_{ef} := R_{\{e,f\}}(c)$ for any $e \neq f \in S$ and we let $g := g_1 h_1 \cdots g_n h_n$. We show the following via induction on $n \geq 1$:

- If $g_n = u_{fr}$ for some $f \in \{s, t\}$, then the following hold:
  - (a) $\mathrm{proj}_{R_{st}}(c.g) = c_f.h_n$;
  - (b) $\ell(c.g, \mathrm{proj}_{R_{st}}(c.g)) > 0$.
- If $g_n \in \{u_{rt}, u_{rt}u_{tr}\}$, then the following hold:
  - (a) $\mathrm{proj}_{R_{st}}(c.g) = \mathrm{proj}_{\mathcal{P}_t(c.h_n)}(c.g)$;
  - (b) $\ell(\delta(c.g, \mathrm{proj}_{R_{st}}(c.g))srs) = \ell(c.g, \mathrm{proj}_{R_{st}}(c.g)) + 3$;
  - (c) $\ell(c.g, \mathrm{proj}_{R_{st}}(c.g)) > 0$.

Once this is shown, the claim follows, since $g = 1$ would imply $\ell(c.g, \mathrm{proj}_{R_{st}}(c.g)) = 0$. Let $n = 1$ and suppose $g_1 \in \{u_{sr}, u_{tr}\}$. Then we have $\mathrm{proj}_{R_{st}}(c.u_{fr}) = c_f$ and, in particular, $\mathrm{proj}_{R_{st}}(c.g) = \left(\mathrm{proj}_{R_{st}}(c.g_1)\right).h_1 = c_f.h_1$. Moreover, we have $\ell(c.g, \mathrm{proj}_{R_{st}}(c.g)) = \ell(c.g, c_f.h_1) = \ell(c.g_1, c_f) > 0$. Now we suppose $g_1 \in \{u_{rt}, u_{rt}u_{tr}\}$. Note that $\delta(c, c.u_{rt}u_{tr}) = \delta(c, c.u_r u_t u_r u_t) = r_{\{r,t\}}$ and $\delta(c, c.u_{rt}) = rtr$. In particular, we have $\delta(c.g_1, c) \in \{rtr, r_{\{r,t\}}\}$. Let $q = \mathrm{proj}_{\mathcal{P}_t(c)}(c.g_1)$. Then, by [2, Lemma 2.15], $\ell(\delta(c.g_1, q)s) = \ell(\delta(c.g_1, q)) + 1$ and hence $q = \mathrm{proj}_{R_{st}}(c.g_1)$. This implies $\mathrm{proj}_{R_{st}}(c.g) = q.h_1 = \mathrm{proj}_{\mathcal{P}_t(c.h_1)}(c.g)$. Since $\delta(c.g_1, q) \in \langle r, t \rangle$, we infer (again by [2, Lemma 2.15]) $\mathrm{proj}_{\mathcal{P}_r(q)}(c.g_1) = \mathrm{proj}_{R_{\{s,r\}}(q)}(c.g_1)$. Thus we have $\delta(\mathrm{proj}_{R_{\{r,s\}}(q)}(c.g_1), q) \in \langle r \rangle$ and hence $\ell(\delta(c.g, q.h_1)srs) = \ell(\delta(c.g_1, q)srs) = \ell(c.g_1, q) + 3 = \ell(c.g, q.h_1)$. Moreover, $\ell(c.g, \mathrm{proj}_{R_{st}}(c.g)) = \ell(c.g_1, q) > 0$.

Now we assume that $n > 1$. We define $h := g_1 h_1 \cdots g_{n-1} h_{n-1}$. In both cases we will show that $\ell(c.h, \mathrm{proj}_{R_{st}.g_n}(c.h)) > \ell(c.h, \mathrm{proj}_{R_{st}}(c.h))$. Once this is done it follows $\ell(c.g, \mathrm{proj}_{R_{st}}(c.g)) = \ell(c.h, \mathrm{proj}_{R_{st}.g_n}(c.h)) > \ell(c.h, \mathrm{proj}_{R_{st}}(c.h)) > 0$ by induction. We distinguish the following cases, where the first case is a special case which we will use in the other cases:

(a) $g_n \in \{u_{rt}, u_{rt}u_{tr}\}$ and $\mathrm{proj}_{\mathcal{P}_t(c)}(c.h) = \mathrm{proj}_{R_{rt}}(c.h)$: As above, we deduce $\delta(c, c.g_n) \in \{rtr, r_{\{r,t\}}\}$. We define $p := \mathrm{proj}_{\mathcal{P}_t(c)}(c.h) = \mathrm{proj}_{R_{rt}}(c.h)$. As $p \in \mathcal{P}_t(c)$, we deduce $\delta(p, c.g_n) \in \{rtr, r_{\{r,t\}}\}$. We define $q := \mathrm{proj}_{\mathcal{P}_t(c.g_n)}(c.h)$ and note that $q \in R_{rt}$. Then $q = \mathrm{proj}_{\mathcal{P}_t(c.g_n)} \mathrm{proj}_{R_{rt}}(c.h), \delta(p, q) = rtr$ and Lemma (5.1.2) implies $\ell(\delta(c.h, q)s) =$

$\ell(c.h,q)+1$. Thus $q = \text{proj}_{R_{\{s,t\}}(c.g_n)}(c.h) = \text{proj}_{R_{st}.g_n}(c.h)$ and hence $\text{proj}_{\mathcal{P}_t(c.h_n)}(c.g) = q.g_nh_n = \text{proj}_{R_{st}}(c.g)$. Since $\delta(p,q) = rtr$ and $p \in \mathcal{P}_t(c) \subseteq R_{st}$, we deduce

$$\ell(c.h, \text{proj}_{R_{st}.g_n}(c.h)) = \ell(c.h,q) \overset{q\in R_{rt}}{=} \ell(c.h,p) + 3 \overset{p\in R_{st}}{>} \ell(c.h, \text{proj}_{R_{st}}(c.h))$$

Moreover, as $\ell(p, \text{proj}_{\mathcal{P}_r(q)}(c.h)) = 2$, Lemma (5.1.2) implies that $\text{proj}_{\mathcal{P}_r(q)}(c.h) = \text{proj}_{R_{\{r,s\}}(q)}(c.h)$ and hence $\ell(\delta(c.g, q.g_nh_n)srs) = \ell(\delta(c.h,q)srs) = \ell(c.h,q)+3 = \ell(c.g, q.g_nh_n)+3$.

(b) $g_{n-1} = u_{fr}$ for some $f \in \{s,t\}$: Then we have $\text{proj}_{R_{st}}(c.h) = c_f.h_{n-1}$ by induction. We distinguish the following two cases:

  (i) $g_n \notin \{u_{rt}, u_{rt}u_{tr}\}$: Then there exists $e \in \{s,t\}$ with $g_n = u_{er}$. If $e = f$, we have $h_{n-1} \notin \{1, u_f\}$ by assumption and $\ell(c_f.h_{n-1}, c_e) \geq 3$ by Lemma (5.3.2)(*a*). If $e \neq f$, we have $\ell(c_f.h_{n-1}, c_e) \geq 2$ by Lemma (5.3.2)(*b*). Note that in both cases we have $\delta(c_f.h_{n-1}, c_e) \in \langle s,t\rangle$. Using Lemma (5.1.2) we obtain $\ell(\delta(c.h,c_e)ru) = \ell(\delta(c.h,c_f.h_{n-1})\delta(c_f.h_{n-1},c_e)ru) = \ell(c.h,c_f.h_{n-1})+\ell(c_f.h_{n-1},c_e)+2 = \ell(c.h,c_e)+2$ for each $u \in \{s,t\}$. Since $\delta(c_e, c_e.u_{er}) = r$, the previous computations imply that $c_e.u_{er} = \text{proj}_{R_{\{s,t\}}(c_e.u_{er})}(c.h) = \text{proj}_{R_{st}.u_{er}}(c.h)$ and hence $c_e.h_n = \text{proj}_{R_{st}}(c.g)$. In particular, we have $\ell(c.h, \text{proj}_{R_{st}.g_n}(c.h)) = \ell(c.h, c_e.u_{er}) = \ell(c.h,c_e)+1 > \ell(c.h, c_f.h_{n-1}) = \ell(c.h, \text{proj}_{R_{st}}(c.h))$.

  (ii) $g_n \in \{u_{rt}, u_{rt}u_{tr}\}$: We define $p := \text{proj}_{\mathcal{P}_t(c)}(c.h)$. If $f = t$, we have $h_{n-1} \notin \{1, u_t\}$ and hence $\ell(c_t.h_{n-1}, p) \geq 2$ by Lemma (5.3.2)(*c*). By Lemma (5.1.2) we obtain that $\ell(\delta(c.h,p)r) = \ell(c.h,p)+1$ and hence $p = \text{proj}_{R_{rt}}(c.h)$. The claim follows now from Case (a). If $f = s$, Lemma (5.3.2)(*d*) yields $\ell(c_s.h_{n-1},p) \geq 2$ or $\delta(c_s.h_{n-1},p) = s$. If $\ell(c_s.h_{n-1},p) \geq 2$, we obtain $p = \text{proj}_{R_{rt}}(c.h)$ as before and the claim follows again from Case (a). Thus we suppose that $\delta(c_s.h_{n-1},p) = s$. Note that we have $\delta(c, c.u_{rt}u_{tr}) = r_{\{r,t\}}$ and $\delta(c, c.u_{rt}) = rtr$. In particular, we have $\delta(p, c.g_n) \in \{rtr, r_{\{r,t\}}\}$. If $\ell(\delta(c.h,p)r) = \ell(c.h,p)+1$, we have $p = \text{proj}_{R_{rt}}(c.h)$ and the claim follows as before. Thus we assume $\ell(\delta(c.h,p)r) = \ell(c.h,p)-1$. Then $\ell(\delta(c.h,p)rt) = \ell(c.h,p)$ by Lemma (5.1.1) and hence $\ell(wu) = \ell(w)+1$ for $w = \delta(c.h,p)r$ and each $u \in \{r,t\}$. Since $p \in \mathcal{P}_t(c) \subseteq R_{rt}$ and hence $\mathcal{P}_r(p) \subseteq R_{rt}$, we infer $\text{proj}_{\mathcal{P}_r(p)}(c.h) = \text{proj}_{R_{rt}}(c.h)$. By definition we have $p \in \mathcal{P}_t(c)$. We define $q := \text{proj}_{\mathcal{P}_t(c.g_n)}(c.h)$. Since $\delta(p, c.g_n) \in \{rtr, r_{\{r,t\}}\}$ we have $\ell(\text{proj}_{R_{rt}}(c.h), q) = 2$ and hence $\ell(c.h,q) = \ell(c.h, \text{proj}_{R_{rt}}(c.h)) + 2 > \ell(c.h, \text{proj}_{R_{rt}}(c.h)) + 1 = \ell(c.h,p)$. Lemma (5.1.2) implies $q = \text{proj}_{R_{st}(c.g_n)}(c.h) = \text{proj}_{R_{st}.g_n}(c.h)$ and hence, as $p \in R_{st}$, we deduce

$$\ell(c.h, \text{proj}_{R_{st}.g_n}(c.h)) = \ell(c.h,q) > \ell(c.h,p) > \ell(c.h, \text{proj}_{R_{st}}(c.h)).$$

  Moreover, we have $\text{proj}_{\mathcal{P}_t(c.h_n)}(c.g) = q.g_nh_n = \text{proj}_{R_{st}}(c.g)$. We have already mentioned that $\ell(\text{proj}_{R_{rt}}(c.h), q) = 2$. Using Lemma (5.1.1), Lemma (5.1.2) and the fact that $\ell(\delta(c.h, \text{proj}_{R_{rt}}(c.h))rs) = \ell(c.h, \text{proj}_{R_{rt}}(c.h))$, we deduce $\ell(\delta(c.h,z)s) = \ell(c.h,z)+1$ for all $z \in R_{rt}\backslash\mathcal{P}_r(p)$. Note that $\mathcal{P}_r(\text{proj}_{R_{rt}}(c.h)) = \mathcal{P}_r(p)$. In particular, as $\delta(\text{proj}_{R_{rt}}(c.h), \text{proj}_{\mathcal{P}_r(q)}(c.h)) = t$, we deduce $\text{proj}_{\mathcal{P}_r(q)}(c.h) \in R_{rt}\backslash\mathcal{P}_r(p)$. Thus we have $\text{proj}_{R_{\{r,s\}}(q)}(c.h) = \text{proj}_{\mathcal{P}_r(q)}(c.h)$ and hence $\ell(\delta(c.g, q.g_nh_n)srs) = \ell(\delta(c.h,q)srs) = \ell(c.h,q)+3 = \ell(c.g, qg_nh_n)+3$.

(c) $g_{n-1} \in \{u_{rt}, u_{rt}u_{tr}\}$: We define $p := \text{proj}_{R_{st}}(c.h)$. Using induction, we have $p = \text{proj}_{\mathcal{P}_t(c.h_{n-1})}(c.h)$ and $\ell(\delta(c.h,p)srs) = \ell(c.h,p)+3$. We distinguish the following three cases:

(i) $g_n = u_{tr}$: Then we have $h_{n-1} \notin \{1, u_t\}$ by assumption. As $h_{n-1}^{-1} \notin \{1, u_t\}$ and $p.h_{n-1}^{-1} \in \mathcal{P}_t(c)$, Lemma (5.3.2)($c$) yields $\ell(p, c_t) = \ell(p.h_{n-1}^{-1}, c_t.h_{n-1}^{-1}) \geq 2$. Using Lemma (5.1.2) we obtain $\ell(\delta(c.h, c_t)ru) = \ell(c.h, c_t) + 2$ for each $u \in \{s, t\}$ and hence $c_t.u_{tr} = \mathrm{proj}_{R_{st}.u_{tr}}(c.h)$, as $\delta(c_t, c_t.u_{tr}) = r$. This implies $c_t.h_n = \mathrm{proj}_{R_{st}}(c.g)$. Moreover, we have $\ell(c.h, \mathrm{proj}_{R_{st}.g_n}(c.h)) = \ell(c.h, c_t.g_n) > \ell(c.h, p) = \ell(c.h, \mathrm{proj}_{R_{st}}(c.h))$.

(ii) $g_n \in \{u_{rt}, u_{rt}u_{tr}\}$: Then we have $h_{n-1} \notin \{1, u_t\}$ by assumption. We define $q := \mathrm{proj}_{\mathcal{P}_t(c)}(c.h)$. Using Lemma (5.3.2)($e$) we have either $\ell(p, q) \geq 2$ or $\delta(p, q) = s$. If $\ell(p, q) \geq 2$, we obtain $q = \mathrm{proj}_{R_{rt}}(c.h)$ by Lemma (5.1.2). If $\delta(p, q) = s$, we have $\ell(\delta(c.h, p)srs) = \ell(c.h, p) + 3$ by induction and hence $\ell(\delta(c.h, q)r) = \ell(c.h, q) + 1$. In particular, $q = \mathrm{proj}_{R_{rt}}(c.h)$. Both cases yield $q = \mathrm{proj}_{R_{rt}}(c.h)$ and the claim follows from Case (a).

(iii) $g_n = u_{sr}$: Using Lemma (5.3.2)(f) we have either $\ell(p, c_s) \geq 2$ or $\delta(p, c_s) = s$. If $\ell(p, c_s) \geq 2$, we obtain $\ell(\delta(c.h, c_s)ru) = \ell(c.h, c_s) + 2$ for each $u \in \{s, t\}$ by Lemma (5.1.2) and hence $c_s.u_{sr} = \mathrm{proj}_{R_{st}.u_{sr}}(c.h)$. This implies $c_s.h_n = \mathrm{proj}_{R_{st}}(c.g)$ as well as $\ell(c.h, \mathrm{proj}_{R_{st}.g_n}(c.h)) = \ell(c.h, c_s.g_n) > \ell(c.h, p) = \ell(c.h, \mathrm{proj}_{R_{st}}(c.h))$.

Suppose now that $\delta(p, c_s) = s$. By induction we have $\ell(\delta(c.h, p)srs) = \ell(c.h, p) + 3$. Since $\delta(p, c_s) = s$, we have $p \in R_{rs}$ and hence $\mathrm{proj}_{R_{rs}}(c.h) \in \mathcal{P}_r(p)$. By Lemma (5.1.2) we obtain $\ell(\delta(c.h, p)srt) = \ell(c.h, p) + 3$. Since $\delta(p, c_s) = s$ and $\delta(c_s, c_s.u_{sr}) = r$, we have $\delta(p, c_s.u_{sr}) = sr$ and $c_s.u_{sr} = \mathrm{proj}_{R_{st}.u_{sr}}(c.h)$. This implies $c_s.h_n = \mathrm{proj}_{R_{st}}(c.g)$ and, in particular, $\ell(c.h, \mathrm{proj}_{R_{st}.g_n}(c.h)) = \ell(c.h, c_s.g_n) > \ell(c.h, p) = \ell(c.h, \mathrm{proj}_{R_{st}}(c.h))$. □

(5.3.5) Theorem. *The canonical homomorphism* $\varphi : U_{sr} \star_{U_s} V_{r_{\{s,t\}}} \star_{U_t} U_{trt} \to G$ *is injective.*

Proof. We abbreviate $H := U_{sr} \star_{U_s} V_{r_{\{s,t\}}} \star_{U_t} U_{trt}$. We note that any $g \in H$ can be written in the form $h_0 g_1 h_1 \cdots g_n h_n$, where $g_i \in \{u_{sr}, u_{tr}, u_{rt}, u_{rt}u_{tr}\}$, $h_i \in V_{r_{\{s,t\}}}$ and $n \geq 0$. We reduce the product as follows:

(a) Suppose that $g_i = g_{i+1} = u_{sr}$ and $h_i \in \{1, u_s\}$ for some $1 \leq i \leq n-1$. Then $g_i h_i g_{i+1} = h_i$, as $[g_i, h_i] = 1$. Thus

$$g = h_0 g_1 h_1 \cdots g_{i-1}(h_{i-1} h_i h_{i+1}) g_{i+2} h_{h+2} \cdots g_n h_n$$

(b) Suppose that $g_i, g_{i+1} \in \{u_{tr}, u_{rt}, u_{rt}u_{tr}\}$ and $h_i \in \{1, u_t\}$ for some $1 \leq i \leq n-1$. Then $g_i h_i g_{i+1} = h_i g_i g_{i+1}$, as $[g_i, h_i] = 1$. We distinguish the following two cases:

(i) $g_i = g_{i+1}$: Then we can write g as before as

$$g = h_0 g_1 h_1 \cdots g_{i-1}(h_{i-1} h_i h_{i+1}) g_{i+2} h_{i+2} \cdots g_n h_n$$

(ii) $g_i \neq g_{i+1}$: Then $g_i g_{i+1} \in \{u_{tr}, u_{rt}, u_{rt}u_{tr}\}$ and we can write g as follows:

$$g = h_0 g_1 h_1 \cdots g_{i-1}(h_{i-1} h_i)(g_i g_{i+1}) h_{i+1} \cdots g_n h_n$$

In each step we reduce the number of generators n and hence we can only reduce finitely many times. At some point we can not apply (a) or (b). In particular, any $g \in H$ can be written as $h_0 g_1 h_1 \cdots g_n h_n$, where $g_i \in \{u_{sr}, u_{tr}, u_{rt}, u_{rt}u_{tr}\}$, $h_i \in V_{r_{\{s,t\}}}$ and if $g_i = g_{i+1} = u_{sr}$ for some $1 \leq i \leq n-1$, then $h_i \notin \{1, u_s\}$ and if $g_i, g_{i+1} \in \{u_{tr}, u_{rt}, u_{rt}u_{tr}\}$ for some $1 \leq i \leq n-1$, then $h_i \notin \{1, u_t\}$.

Now we assume that there exists $1 \neq g \in \ker(\varphi)$. Then there exist g_i, h_i as before such that $g = h_0 g_1 h_1 \cdots g_n h_n$. As $V_{r_{\{s,t\}}} \cap \ker(\varphi) = \{1\}$, we have $n > 0$. Since $\ker(\varphi)$ is a normal subgroup of H, we have also $g_1 h_1 \cdots g_n (h_n h_0) = h_0^{-1} g h_0 \in \ker(\varphi)$. But the previous lemma says that $g_1 h_1 \cdots g_n (h_n h_0)$ is non-trivial in G, which yields a contradiction. Thus φ is injective. □

6. Commutator blueprints of type $(4,4,4)$

In this chapter we let $\mathcal{M} = \left(M^G_{\alpha,\beta}\right)_{(G,\alpha,\beta)\in\mathcal{I}}$ be a locally Weyl-invariant commutator blueprint of type $(4,4,4)$. Moreover, we let $S = \{r,s,t\}$. We will show that $\mathcal{M}$ is faithful. For this purpose we introduce several tree products.

For a residue R of $\Sigma(W,S)$ we put $w_R := \mathrm{proj}_R 1_W$. Let $s \neq t \in S$ and let R be a residue of type $\{s,t\}$. Then we have $\ell(w_R s) = \ell(w_R) + 1 = \ell(w_R t)$. We define the group $V_{w_R r_{\{s,t\}}} := \langle U_{w_R s} \cup U_{w_R t}\rangle \leq U_{w_R r_{\{s,t\}}}$. Using (CB3) and fact that $\mathcal{M}$ is locally Weyl-invariant, the group $V_{w_R r_{\{s,t\}}}$ is an index 2 subgroup of $U_{w_R r_{\{s,t\}}}$ (cf. Remark (2.1.2)). For each $i \in \mathbb{N}$ we let $\mathcal{R}_i$ be the set of all rank 2 residues R with $\ell(w_R) = i$ (e.g. $\mathcal{R}_0 = \{R_{\{s,t\}}(1_W) \mid s \neq t \in S\}$). We let $\mathcal{T}_{i,1}$ be the set of all residues $R \in \mathcal{R}_i$ with $\ell(w_R sr) = \ell(w_R)+2 = \ell(w_R tr)$, where $\{s,t\}$ is the type of R. Let $R \in \mathcal{R}_i \backslash \mathcal{T}_{i,1}$ be of type $\{s,t\}$. Then we have $\ell(w_R) \in \{\ell(w_R sr), \ell(w_R tr)\}$. By Lemma (5.1.1) we have $\{\ell(w_R), \ell(w_R)+2\} = \{\ell(w_R sr), \ell(w_R tr)\}$. Let $u \neq v \in \{s,t\}$ be such that $\ell(w_R ur) = \ell(w_R)$. Then $T_R := R_{\{v,r\}}(w_R u) \neq R$ and $T_R \in \mathcal{R}_i$ by Lemma (5.1.1). In particular, $T_R \in \mathcal{R}_i \backslash \mathcal{T}_{i,1}$ and we have $T_{(T_R)} = R$. We define $\mathcal{T}_{i,2} := \{\{R, T_R\} \mid R \in \mathcal{R}_i \backslash \mathcal{T}_{i,1}\}$. Moreover, we let $\mathcal{T}_i := \mathcal{T}_{i,1} \cup \mathcal{T}_{i,2}$.

In order to prove that $\mathcal{M}$ is faithful, we need to introduce several sequences of groups. The groups in the sequences of groups will always be generated by elements u_α for suitable $\alpha \in \Phi_+$. Let $\Phi_A, \Phi_B \subseteq \Phi_+$ be such that $A = \langle u_\alpha \mid \alpha \in \Phi_A\rangle$ and $B = \langle u_\alpha \mid \alpha \in \Phi_B\rangle$. Let $C = \langle u_\alpha \mid \alpha \in \Phi_A \cap \Phi_B\rangle$ and assume that $C \to A, C \to B$ are injective. Then we define $A \hat{\star} B := A \star_C B$. We note that in all cases the group C will be such that $C \to A$ and $C \to B$ are injective by definition. Furthermore, we implicitly assume that every edge group C between two vertex groups $A = \langle u_\alpha \mid \alpha \in \Phi_A\rangle$ and $B = \langle u_\alpha \mid \alpha \in \Phi_B\rangle$ in a sequence of groups is given by $C = \langle u_\alpha \mid \alpha \in \Phi_A \cap \Phi_B\rangle$.

6.1. The groups V_R and O_R

For a residue $R \in \mathcal{T}_{i,1}$ of type $\{s,t\}$ we define the group V_R to be the tree product of the sequence of groups with vertex groups

$$U_{w_R sr}, V_{w_R r_{\{s,t\}}}, U_{w_R tr}$$

Furthermore, we define the group O_R to be the tree product of the sequence of groups with vertex groups

$$V_{w_R sr_{\{r,t\}}}, U_{w_R r_{\{s,t\}}}, V_{w_R tr_{\{r,s\}}}$$

(6.1.1) Remark. For V_R we consider $\alpha := w_R s \alpha_r$. Using Lemma (5.1.4) we see that $-w_R\alpha_t \subseteq \alpha$. As $w_R t \in (-w_R\alpha_t)$, we deduce $w_R tr, w_R r_{\{s,t\}} \in \alpha$ and hence u_α is neither a generator of $V_{w_R r_{\{s,t\}}}$ nor of $U_{w_R tr}$. Now we consider $w_R\alpha_s$. As $-w_R t\alpha_r \subseteq w_R\alpha_s$ by Lemma (5.1.4) we deduce that $u_{w_R\alpha_s}$ is not a generator of $U_{w_R tr}$. Using similar methods we infer that V_R is generated by $\{u_\alpha \mid \exists v \in \{w_R sr, w_R s, w_R t, w_R tr\} : v \notin \alpha\}$. A similar result holds for O_R.

(6.1.2) Lemma. *Let $R \in \mathcal{T}_{i,1}$. Then the canonical homomorphism $V_R \to O_R$ is injective.*

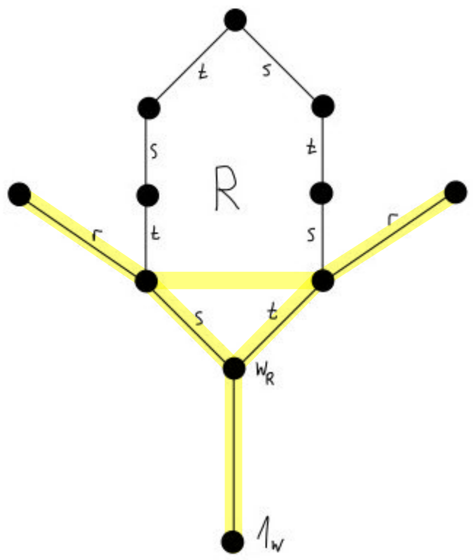


Figure 6.1.: Illustration of the group V_R

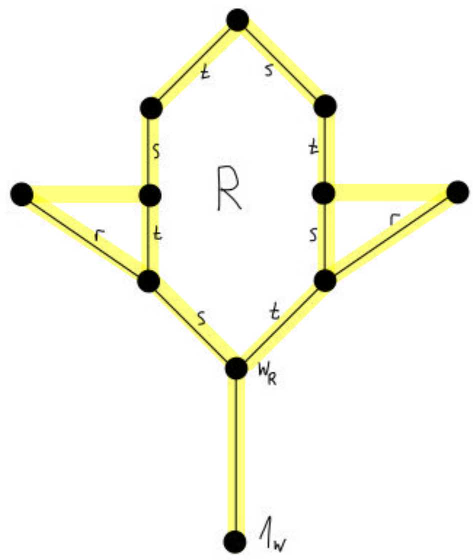


Figure 6.2.: Illustration of the group O_R

Proof. Let R be of type $\{s,t\}$. We will apply Proposition (1.8.3). Therefore we first see that each vertex group of V_R is contained in the corresponding vertex group of O_R, e.g. $U_{w_R sr} \leq V_{w_R sr_{\{r,t\}}}$. Next we have to show that the preimages of the boundary monomorphisms are equal and coincide with the edge groups of V_R. For this we compute $\alpha_e(G_e) \cap H_{o(e)}$ and $\omega_e(G_e) \cap H_{t(e)}$, as $\alpha_e^{-1}(H_{o(e)}) = \alpha_e^{-1}(\alpha(G_e) \cap H_{o(e)})$ and $\omega_e^{-1}(H_{t(e)}) = \omega_e^{-1}(\omega(G_e) \cap H_{t(e)})$. We compute the following:

$$U_{w_R sr} \cap U_{w_R st} = U_{w_R s} = V_{w_R r_{\{s,t\}}} \cap U_{w_R st}$$
$$V_{w_R r_{\{s,t\}}} \cap U_{w_R ts} = U_{w_R t} = U_{w_R tr} \cap U_{w_R ts}$$

Now the claim follows from Proposition (1.8.3). □

6.2. The groups $V_{R,s}$ and $O_{R,s}$

Let $R \in \mathcal{T}_{i,1}$ be a residue of type $\{s,t\}$ such that $\ell(w_R srs) = \ell(w_R) + 3$. Then we define the group $V_{R,s}$ to be the tree product of the sequence of groups with vertex groups

$$U_{w_R srs}, V_{w_R r_{\{s,t\}}}, U_{w_R tr}$$

Moreover, we define the group $O_{R,s}$ to be the tree product of the sequence of groups with vertex groups

$$U_{w_R srs}, V_{w_R sr_{\{r,t\}}}, U_{w_R r_{\{s,t\}}}, V_{w_R tr_{\{r,s\}}}$$

Using similar arguments as in Remark (6.1.1) it follows that $V_{R,s}$ and $O_{R,s}$ are generated by suitable u_α.

(6.2.1) Lemma. *Let $R \in \mathcal{T}_{i,1}$ be a residue of type $\{s,t\}$ such that $\ell(w_R srs) = \ell(w_R) + 3$. Then the canonical homomorphisms $V_R \to V_{R,s}, O_R \to O_{R,s}$ and $V_{R,s} \to O_{R,s}$ are injective. Moreover, we have $V_{R,s} \star_{V_R} O_R \cong U_{w_R srs} \star_{U_{w_R sr}} O_R \cong O_{R,s}$.*

Proof. Note that $V_{R,s} \cong U_{w_R srs} \star_{U_{w_R sr}} V_R$ and $O_{R,s} \cong U_{w_R srs} \star_{U_{w_R sr}} O_R$ by Proposition (1.8.1) and Lemma (1.8.7). Using Proposition (1.8.3) and Lemma (6.1.2) the claim follows. In particular, using Remark (1.8.6) we deduce

$$V_{R,s} \star_{V_R} O_R \cong (U_{w_R srs} \star_{U_{w_R sr}} V_R) \star_{V_R} O_R \cong U_{w_R srs} \star_{U_{w_R sr}} O_R \cong O_{R,s}$$ □

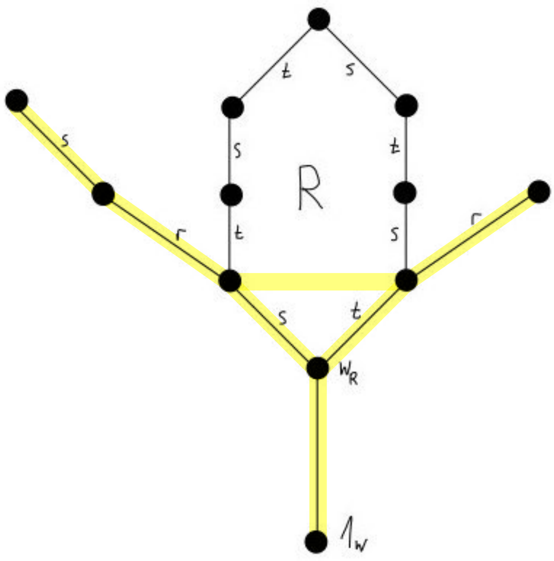


Figure 6.3.: Illustration of the group $V_{R,s}$

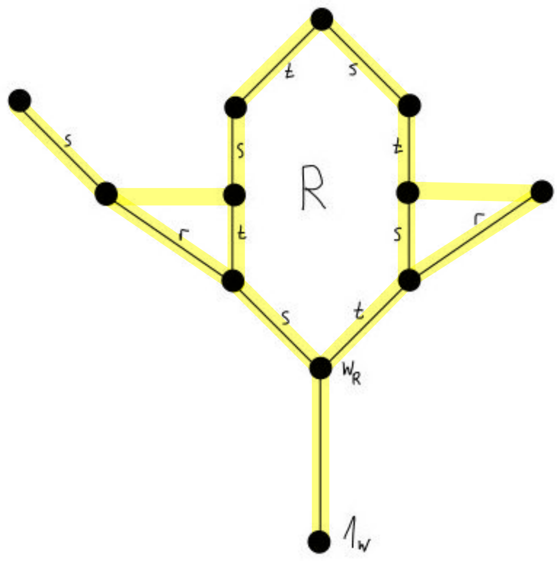


Figure 6.4.: Illustration of the group $O_{R,s}$

6.3. The groups H_R, G_R and $J_{R,t}$

Let $R \in \mathcal{T}_{i,1}$ be of type $J = \{s,t\}$. We define the group H_R to be the tree product of the sequence of groups with vertex groups

$$U_{w_R s r_{\{r,t\}}}, V_{w_R s t r_{\{r,s\}}}, U_{w_R r_{\{s,t\}}}, V_{w_R t s r_{\{r,t\}}}, U_{w_R t r_{\{r,s\}}}$$

We define the group $J_{R,t}$ to be the tree product of the sequence of groups with vertex groups

$$U_{w_R s r_{\{r,t\}}}, V_{w_R s t r_{\{r,s\}}}, V_{w_R t s t r_{\{r,s\}}}, U_{w_R t s r_{\{r,t\}}}, V_{w_R t s r r_{\{s,t\}}}, U_{w_R t r_{\{r,s\}}}$$

Furthermore, we define the group G_R to be the tree product of the sequence of groups with vertex groups

$$U_{w_R s r_{\{r,t\}}}, V_{w_R s t r r_{\{s,t\}}}, U_{w_R s t r_{\{r,s\}}}, V_{w_R s t s r r_{\{s,t\}}},$$
$$U_{w_R s t s r_{\{r,t\}}}, V_{w_R r_{\{s,t\}} r r_{\{s,t\}}}, U_{w_R t s t r_{\{r,s\}}},$$
$$V_{w_R t s t r r_{\{s,t\}}}, U_{w_R t s r_{\{r,t\}}}, V_{w_R t s r r_{\{s,t\}}}, U_{w_R t r_{\{r,s\}}}$$

Using similar arguments as in Remark (6.1.1) it follows that H_R, $J_{R,t}$ and G_R are generated by suitable u_α.

(6.3.1) Lemma. *Let $R \in \mathcal{T}_{i,1}$ be of type $\{s,t\}$. Then the canonical homomorphisms $H_R \to J_{R,t}$ and $J_{R,t} \to G_R$ are injective. In particular, the canonical homomorphism $H_R \to G_R$ is injective.*

Proof. At first we show that $H_R \to J_{R,t}$ is injective. Using Proposition (1.8.1) the group $J_{R,t}$ is isomorphic to the tree product of the sequence of groups with vertex groups

$$U_{w_R s r_{\{r,t\}}}, V_{w_R s t r_{\{r,s\}}}, V_{w_R t s t r_{\{r,s\}}}, U_{w_R t s r_{\{r,t\}}}, V_{w_R t s r r_{\{s,t\}}} \hat{\star} U_{w_R t r_{\{r,s\}}}$$

We will apply Proposition (1.8.3). Therefore we first see that each vertex group of H_R is contained in the corresponding vertex group of the previous tree product, e.g. $U_{w_R t r_{\{r,s\}}} \leq$

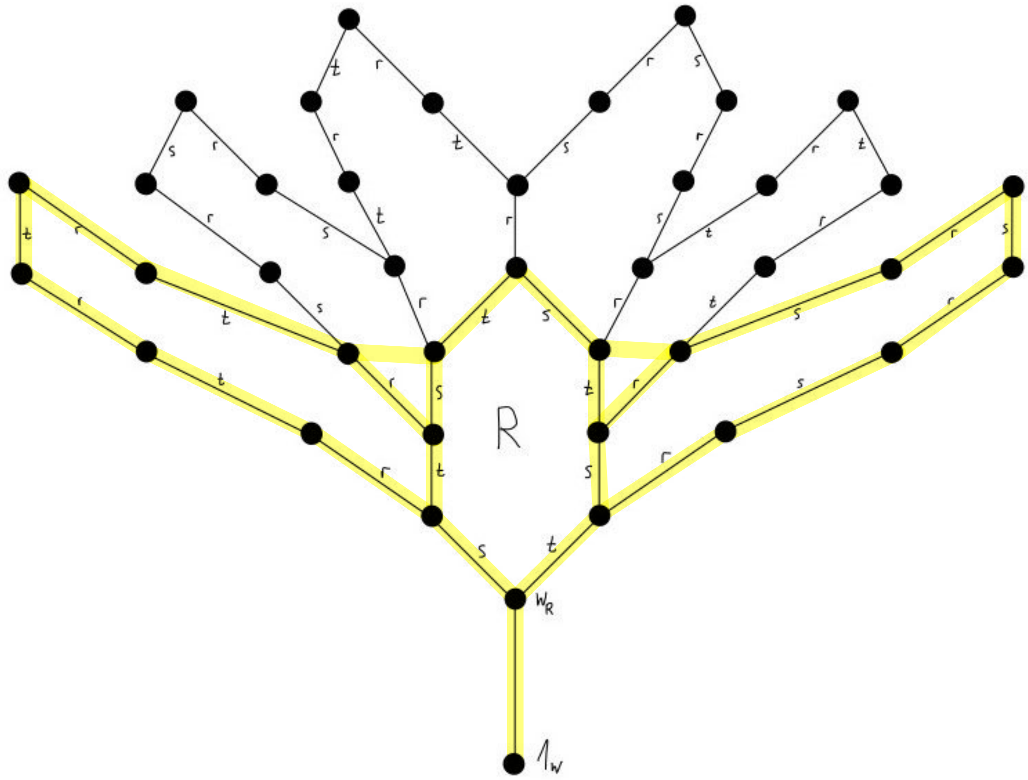


Figure 6.5.: Illustration of the group H_R

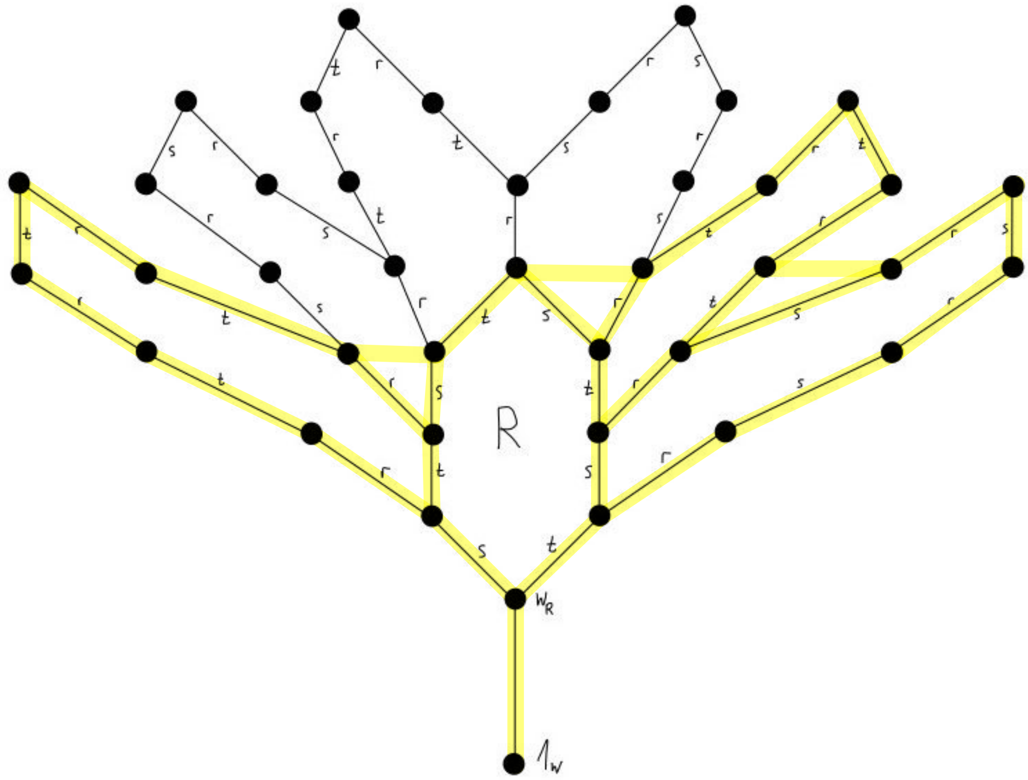


Figure 6.6.: Illustration of the group $J_{R,t}$

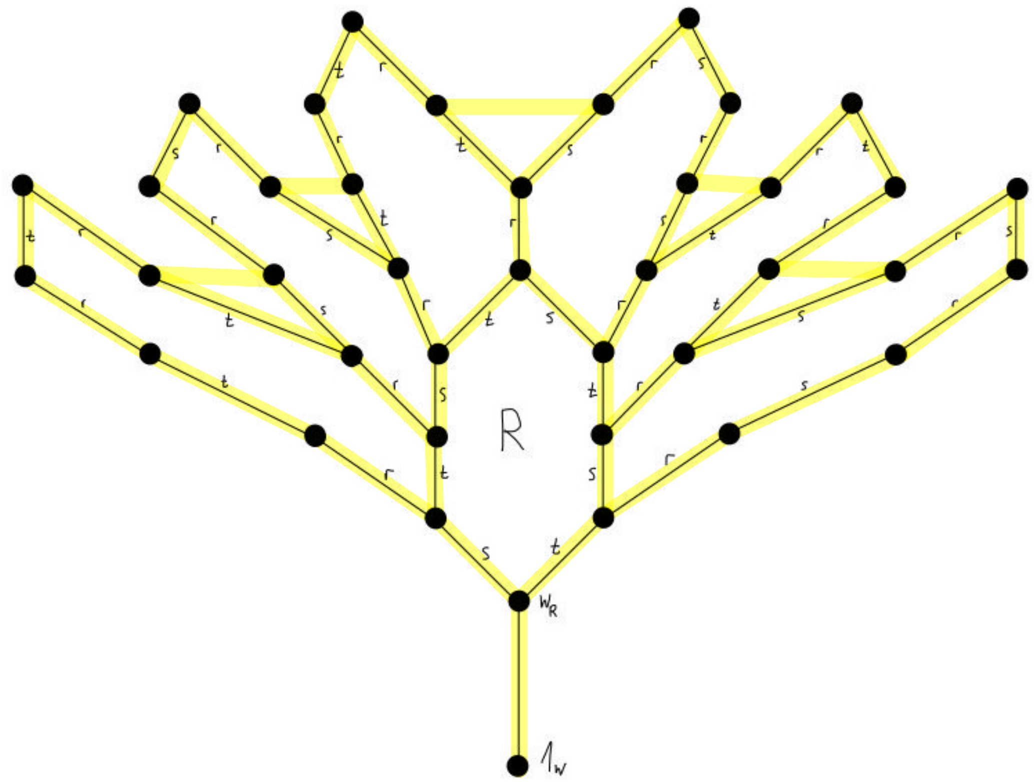


Figure 6.7.: Illustration of the group G_R

$V_{w_R tsrr_{\{s,t\}}} \hat{\star} U_{w_R tr_{\{r,s\}}}$. Next we have to show that the preimages of the boundary monomorphisms are equal and coincide with the edge groups of H_R. As before, we compute $\alpha_e(G_e) \cap H_{o(e)}$ and $\omega_e(G_e) \cap H_{t(e)}$. Note that if the vertex groups H_v and G_v coincide, we do not have to compute the intersection. We compute the following:

$$V_{w_R str_{\{r,s\}}} \cap U_{w_R sts} = U_{w_R sts} = U_{w_R r_{\{s,t\}}} \cap U_{w_R sts}$$
$$U_{w_R r_{\{s,t\}}} \cap U_{w_R tstr} = U_{w_R tst} = V_{w_R tsr_{\{r,t\}}} \cap U_{w_R tstr}$$
$$V_{w_R tsr_{\{r,t\}}} \cap U_{w_R tsrt} = U_{w_R tsr} = U_{w_R tr_{\{r,s\}}} \cap U_{w_R tsrt}$$

We determine two preimages in detail. The others will follow similarly. It is easy to see that $U_{w_R tsr} \subseteq V_{w_R tsr_{\{r,t\}}} \cap U_{w_R tsrt}$. For the other inclusion we note that $V_{w_R tsr_{\{r,t\}}} \cap U_{w_R tsrt} \subseteq U_{w_R tsr}$, as this inclusion holds in $U_{w_R tsr_{\{r,t\}}}$. Again, it is easy to see that $U_{w_R tsr} \subseteq U_{w_R tr_{\{r,s\}}} \cap U_{w_R tsrt}$. For the other inclusion we have to compute the intersection in $V_{w_R tsrr_{\{s,t\}}} \hat{\star} U_{w_R tr_{\{r,s\}}}$. Using Lemma (1.8.5), we deduce $U_{w_R tsrt} \cap U_{w_R tr_{\{r,s\}}} \subseteq V_{w_R tsrr_{\{s,t\}}} \hat{\star} U_{w_R tr_{\{r,s\}}} = U_{w_R tsrs}$. This yields $U_{w_R tsrt} \cap U_{w_R tr_{\{r,s\}}} = U_{w_R tsrt} \cap U_{w_R tr_{\{r,s\}}} \cap U_{w_R tsrs} = U_{w_R tsr} \cap U_{w_R tr_{\{r,s\}}} = U_{w_R tsr}$. We deduce that $H_R \to J_{R,t}$ is injective by Proposition (1.8.3).

Now we will show that $J_{R,t} \to G_R$ is injective. Using Proposition (1.8.1) the group G_R is isomorphic to the tree product of the following sequence of groups with vertex groups

$$U_{w_R sr_{\{r,t\}}} \hat{\star} V_{w_R strr_{\{s,t\}}}, U_{w_R str_{\{r,s\}}} \hat{\star} V_{w_R stsrr_{\{s,t\}}}, U_{w_R stsr_{\{r,t\}}} \hat{\star} V_{w_R r_{\{s,t\}} rr_{\{s,t\}}} \hat{\star} U_{w_R tstr_{\{r,s\}}},$$
$$V_{w_R tstrr_{\{s,t\}}} \hat{\star} U_{w_R tsr_{\{r,t\}}}, V_{w_R tsrr_{\{s,t\}}}, U_{w_R tr_{\{r,s\}}}$$

One easily sees that each vertex group of $J_{R,t}$ is contained in the corresponding vertex group of the previous tree product. Considering the preimage of the boundary monomorphisms the following hold:

$$U_{w_R sr_{\{r,t\}}} \cap U_{w_R strs} = U_{w_R str} = V_{w_R str_{\{r,s\}}} \cap U_{w_R strs}$$
$$V_{w_R str_{\{r,s\}}} \cap U_{w_R stsrt} = U_{w_R sts} = V_{w_R tstr_{\{r,s\}}} \cap U_{w_R stsrt}$$
$$V_{w_R tstr_{\{r,s\}}} \cap U_{w_R tstrs} = U_{w_R tstr} = U_{w_R tsr_{\{r,t\}}} \cap U_{w_R tstrs}$$

We comment on the equation $U_{w_R sts} = V_{w_R tstr_{\{r,s\}}} \cap U_{w_R stsrt}$. The inclusion $\subseteq$ is clear. Now we consider $\supseteq$. Using Proposition (1.8.1) and Corollary (1.8.5) twice it follows that $V_{w_R tstr_{\{r,s\}}} \cap U_{w_R stsrt} \subseteq U_{w_R r_{\{s,t\}} rt} \cap U_{w_R r_{\{s,t\}} rs} = U_{w_R r_{\{s,t\}} r}$. Thus we obtain

$$V_{w_R tstr_{\{r,s\}}} \cap U_{w_R stsrt} = V_{w_R tstr_{\{r,s\}}} \cap U_{w_R stsrt} \cap U_{w_R r_{\{s,t\}} r} = V_{w_R tstr_{\{r,s\}}} \cap U_{w_R sts} = U_{w_R sts}$$

As before, $J_{R,t} \to G_R$ is injective by Proposition (1.8.3). □

(6.3.2) Lemma. *Let $R \in \mathcal{T}_{i,1}$ be a residue of type $\{s,t\}$ and let $T = R_{\{r,t\}}(w_R ts)$. Then $T \in \mathcal{T}_{i+2,1}$, the canonical homomorphism $V_T \to H_R$ is injective and we have $J_{R,t} \cong H_R \star_{V_T} O_T$.*

Proof. Note that $T \in \mathcal{T}_{i+2,1}$. By Proposition (1.8.1), $U_{w_R r_{\{s,t\}}} \hat{\star} V_{w_R tsr_{\{r,t\}}} \hat{\star} U_{w_R tr_{\{r,s\}}} \to H_R$ is injective. Using Proposition (1.8.3), we deduce that

$$V_T = U_{w_R tsts} \hat{\star} V_{w_R tsr_{\{r,t\}}} \hat{\star} U_{w_R tsrs} \to U_{w_R r_{\{s,t\}}} \hat{\star} V_{w_R tsr_{\{r,t\}}} \hat{\star} U_{w_R tr_{\{r,s\}}}$$

is injective and hence also the concatenation $V_T \to H_R$. Using Proposition (1.8.1), Proposition (1.8.3), Remark (1.8.6), Lemma (1.8.7) and Lemma (6.1.2) we obtain the following isomorphisms (we abbreviate $K := V_T \star_{U_{w_R tsrs}} U_{w_R tr_{\{r,s\}}}$):

$$\begin{aligned} J_{R,t} &\cong U_{w_R sr_{\{r,t\}}} \hat{\star} V_{w_R str_{\{r,s\}}} \star_{U_{w_R sts}} \left(O_T \star_{U_{w_R tsrs}} U_{w_R tr_{\{r,s\}}} \right) \\ &\cong U_{w_R sr_{\{r,t\}}} \hat{\star} V_{w_R str_{\{r,s\}}} \star_{U_{w_R sts}} K \star_K \left(O_T \star_{U_{w_R tsrs}} U_{w_R tr_{\{r,s\}}} \right) \\ &\cong H_R \star_K \left(O_T \star_{U_{w_R tsrs}} U_{w_R tr_{\{r,s\}}} \right) \\ &\cong H_R \star_K \left(U_{w_R tr_{\{r,s\}}} \star_{U_{w_R tsrs}} V_T \star_{V_T} O_T \right) \\ &\cong H_R \star_K \left(U_{w_R tr_{\{r,s\}}} \star_{U_{w_R tsrs}} V_T \right) \star_{V_T} O_T \\ &\cong H_R \star_{V_T} O_T \end{aligned}$$

□

6.4. The group $K_{R,s}$

For a residue $R \in \mathcal{T}_{i,1}$ of type $\{s,t\}$ we define the group $K_{R,s}$ to be the tree product of the sequence of groups with vertex groups

$$U_{w_R sr_{\{r,t\}}}, V_{w_R str_{\{r,s\}}}, U_{w_R r_{\{s,t\}}}, V_{w_R tr_{\{r,s\}}}$$

Using similar arguments as in Remark (6.1.1) it follows that $K_{R,s}$ is generated by suitable u_α.

(6.4.1) Lemma. *Let $R \in \mathcal{T}_{i,1}$ be of type $\{s,t\}$. Then the canonical homomorphisms $O_R \to K_{R,s}, K_{R,t}$ are injective and we have $H_R \cong K_{R,s} \star_{O_R} K_{R,t}$.*

Proof. Using Proposition (1.8.1) the group $K_{R,s}$ is isomorphic to the tree product of the sequence of groups with vertex groups

$$U_{w_R sr_{\{r,t\}}}, V_{w_R str_{\{r,s\}}} \hat{\star} U_{w_R r_{\{s,t\}}}, V_{w_R tr_{\{r,s\}}}$$

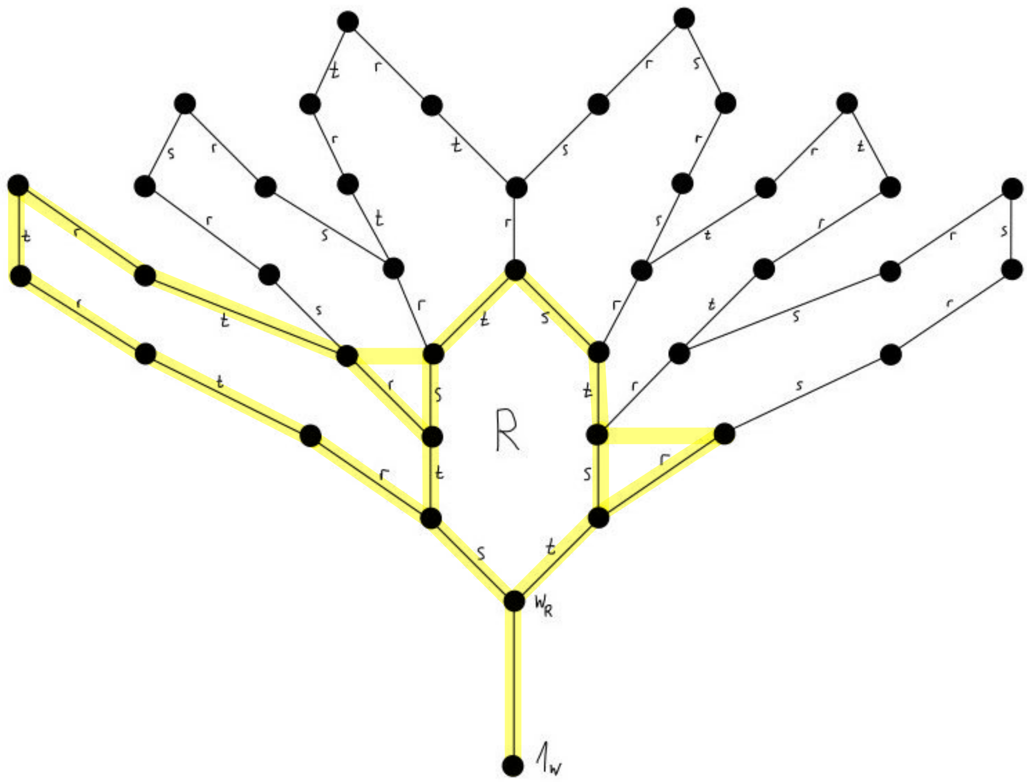


Figure 6.8.: Illustration of the group $K_{R,s}$

One easily sees that each vertex group of O_R is contained in the corresponding vertex group of the previous tree product. Considering the preimage of the boundary monomorphisms the following holds:

$$V_{w_R sr_{\{r,t\}}} \cap U_{w_R str} = U_{w_R st} = U_{w_R r_{\{s,t\}}} \cap U_{w_R str}$$

As before, Proposition (1.8.3) yields that the canonical homomorphism $O_R \to K_{R,s}$ is injective. Using similar arguments, we obtain that $O_R \to K_{R,t}$ is injective. We define $C_0 := V_{w_R sr_{\{r,t\}}} \hat{\star} U_{w_R r_{\{s,t\}}}$ and note that $U_{w_R tst} \to C_0$ and $C_0 \to O_R$ are injective. Moreover, the computations above imply that $C_0 \to U_{w_R sr_{\{r,t\}}} \hat{\star} V_{w_R str_{\{r,s\}}} \hat{\star} U_{w_R r_{\{s,t\}}}$ is injective. Now the following isomorphisms follow from Proposition (1.8.1), Remark (1.8.6) and Lemma (1.8.7):

$$\begin{aligned}
H_R &\cong \left(U_{w_R sr_{\{r,t\}}} \hat{\star} V_{w_R str_{\{r,s\}}} \hat{\star} U_{w_R r_{\{s,t\}}} \right) \star_{U_{w_R tst}} \left(V_{w_R tsr_{\{r,t\}}} \hat{\star} U_{w_R tr_{\{r,s\}}} \right) \\
&\cong \left(U_{w_R sr_{\{r,t\}}} \hat{\star} V_{w_R str_{\{r,s\}}} \hat{\star} U_{w_R r_{\{s,t\}}} \right) \star_{C_0} C_0 \star_{U_{w_R tst}} \left(V_{w_R tsr_{\{r,t\}}} \hat{\star} U_{w_R tr_{\{r,s\}}} \right) \\
&\cong \left(U_{w_R sr_{\{r,t\}}} \hat{\star} V_{w_R str_{\{r,s\}}} \hat{\star} U_{w_R r_{\{s,t\}}} \right) \star_{C_0} K_{R,t} \\
&\cong \left(U_{w_R sr_{\{r,t\}}} \hat{\star} V_{w_R str_{\{r,s\}}} \hat{\star} U_{w_R r_{\{s,t\}}} \right) \star_{C_0} O_R \star_{O_R} K_{R,t} \\
&\cong \left(\left(U_{w_R sr_{\{r,t\}}} \hat{\star} V_{w_R str_{\{r,s\}}} \hat{\star} U_{w_R r_{\{s,t\}}} \right) \star_{C_0} O_R \right) \star_{O_R} K_{R,t} \\
&\cong \left(\left(U_{w_R sr_{\{r,t\}}} \hat{\star} V_{w_R str_{\{r,s\}}} \hat{\star} U_{w_R r_{\{s,t\}}} \right) \star_{C_0} C_0 \star_{U_{w_R ts}} V_{w_R tr_{\{r,s\}}} \right) \star_{O_R} K_{R,t} \\
&\cong \left(\left(U_{w_R sr_{\{r,t\}}} \hat{\star} V_{w_R str_{\{r,s\}}} \hat{\star} U_{w_R r_{\{s,t\}}} \right) \star_{U_{w_R ts}} V_{w_R tr_{\{r,s\}}} \right) \star_{O_R} K_{R,t} \\
&\cong K_{R,s} \star_{O_R} K_{R,t}
\end{aligned}$$

□

(6.4.2) Remark. Let $R \in \mathcal{T}_{i,1}$ be of type $\{s,t\}$ such that $\ell(w_R srs) = \ell(w_R) + 3$ and let $T = R_{\{r,t\}}(w_R s)$. In the next lemma we consider $O_{R,s} \star_{V_T} O_T$. Similar as in Remark (6.1.1) we will show that if x_α is a generator of $O_{R,s}$ and y_α is a generator of O_T, then $x_\alpha = y_\alpha$ holds in $O_{R,s} \star_{V_T} O_T$. It suffices to consider $w_R\alpha_t$ and $w_R t\alpha_r$. As $-w_R\alpha_s \subseteq w_R t\alpha_r$ and $-w_R s\alpha_r, -w_R st\alpha_r \subseteq w_R\alpha_t$, we deduce that x_α is not a generator of O_T for $\alpha \in \{w_R\alpha_t, w_R t\alpha_r\}$.

(6.4.3) Lemma. *Let $R \in \mathcal{T}_{i,1}$ be of type $\{s,t\}$ such that $\ell(w_R srs) = \ell(w_R) + 3$ and let $T = R_{\{r,t\}}(w_R s)$. Then the canonical homomorphisms $V_T \to O_{R,s}$ and $K_{R,s} \to O_{R,s} \star_{V_T} O_T$ are injective and we have $K_{R,s} \cap O_{R,s} = O_R$ in $O_{R,s} \star_{V_T} O_T$.*

Proof. We have $O_{R,s} \cong V_T \star_{U_{w_R sts}} U_{w_R r_{\{s,t\}}} \hat{\star} V_{w_R t r_{\{r,s\}}}$ by Lemma (1.8.7) and Proposition (1.8.1). Now Proposition (1.8.1) yields that the mapping $V_T \to O_{R,s}$ is injective. This, together with Proposition (1.8.1), Remark (1.8.6), Lemma (1.8.7) and Lemma (6.1.2) yields the following isomorphisms:

$$\begin{aligned}
O_{R,s} \star_{V_T} O_T &\cong \left(V_T \star_{U_{w_R sts}} U_{w_R r_{\{s,t\}}} \hat{\star} V_{w_R t r_{\{r,s\}}} \right) \star_{V_T} O_T \\
&\cong V_{w_R t r_{\{r,s\}}} \hat{\star} U_{w_R r_{\{s,t\}}} \star_{U_{w_R sts}} V_T \star_{V_T} O_T \\
&\cong V_{w_R t r_{\{r,s\}}} \hat{\star} U_{w_R r_{\{s,t\}}} \star_{U_{w_R sts}} \left(V_{w_R st r_{\{r,s\}}} \hat{\star} U_{w_R s r_{\{r,t\}}} \hat{\star} V_{w_R sr r_{\{s,t\}}} \right) \\
&\cong K_{R,s} \star_{U_{w_R srt}} V_{w_R sr r_{\{s,t\}}}
\end{aligned}$$

For the second claim we note that $O_{R,s} \cong O_R \star_{U_{w_R sr}} U_{w_R srs}$ by Lemma (6.2.1). By Lemma (6.4.1) we have that $O_R \to K_{R,s}$ is injective and, moreover, $U_{w_R srs} \leq V_{w_R sr r_{\{s,t\}}}$. Considering the preimage of the boundary monomorphisms the following hold:

$$O_R \cap U_{w_R srt} = U_{w_R sr} = U_{w_R srs} \cap U_{w_R srt}$$

Note that the first equation follows from the following: Proposition (1.8.3) implies $O_R \cap U_{w_R s r_{\{r,t\}}} = V_{w_R s r_{\{r,t\}}}$ and hence $O_R \cap U_{w_R srt} = O_R \cap U_{w_R srt} \cap V_{w_R s r_{\{r,t\}}} = O_R \cap U_{w_R sr} = U_{w_R sr}$. As before, Proposition (1.8.3) implies that $O_{R,s} \cong O_R \star_{U_{w_R sr}} U_{w_R srs} \to K_{R,s} \star_{U_{w_R srt}} V_{w_R sr r_{\{s,t\}}}$ is injective and that $O_{R,s} \cap K_{R,s} = O_R$. This finishes the claim. □

6.5. The groups $E_{R,s}$ and $U_{R,s}$

Let $R \in \mathcal{T}_{i,1}$ be of type $\{s,t\}$ and assume that $\ell(w_R rs) = \ell(w_R) - 2$. We put $R' = R_{\{r,s\}}(w_R)$ and $w' = w_{R'}$. We define the group $E_{R,s}$ to be the tree product of the sequence of groups with vertex groups

$$U_{w' rs r_{\{r,t\}}}, V_{w' rsrt r_{\{r,s\}}}, U_{w' rsr r_{\{s,t\}}}, V_{w_R srt r_{\{r,s\}}}, U_{w_R s r_{\{r,t\}}},$$
$$V_{w_R st r_{\{r,s\}}}, U_{w_R r_{\{s,t\}}}, V_{w_R ts r_{\{r,t\}}}, U_{w_R t r_{\{r,s\}}}$$

Furthermore, we define the group $U_{R,s}$ to be the tree product of the sequence of groups with vertex groups

$$U_{w' rs r_{\{r,t\}}}, V_{w' rsrt r_{\{r,s\}}}, U_{w' rsr r_{\{s,t\}}}, V_{w_R srt r_{\{r,s\}}}, U_{w_R s r_{\{r,t\}}}, V_{w_R str r_{\{s,t\}}}, U_{w_R st r_{\{r,s\}}}, V_{w_R stsr r_{\{s,t\}}},$$
$$U_{w_R sts r_{\{r,t\}}}, V_{w_R r_{\{s,t\}} r r_{\{s,t\}}}, U_{w_R tst r_{\{r,s\}}}, V_{w_R tstr r_{\{s,t\}}}, U_{w_R ts r_{\{r,t\}}}, V_{w_R tsr r_{\{s,t\}}}, U_{w_R t r_{\{r,s\}}}$$

Using similar arguments as in Remark (6.1.1) it follows that $E_{R,s}$ and $U_{R,s}$ are generated by suitable u_α.

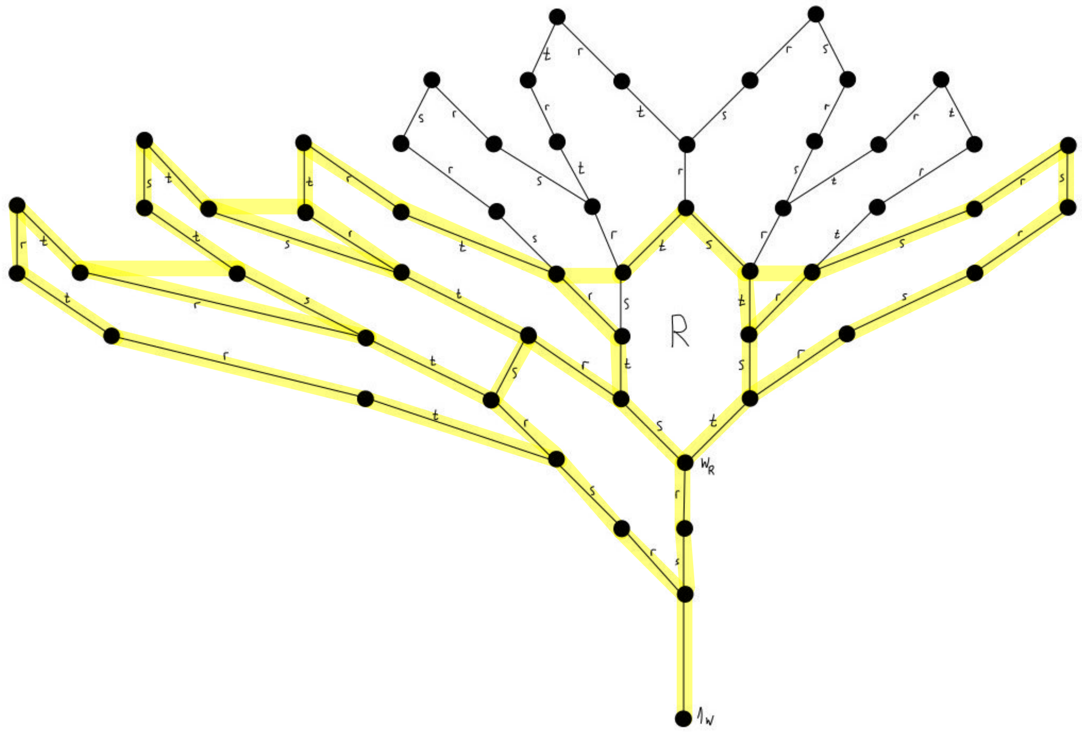


Figure 6.9.: Illustration of the group $E_{R,s}$

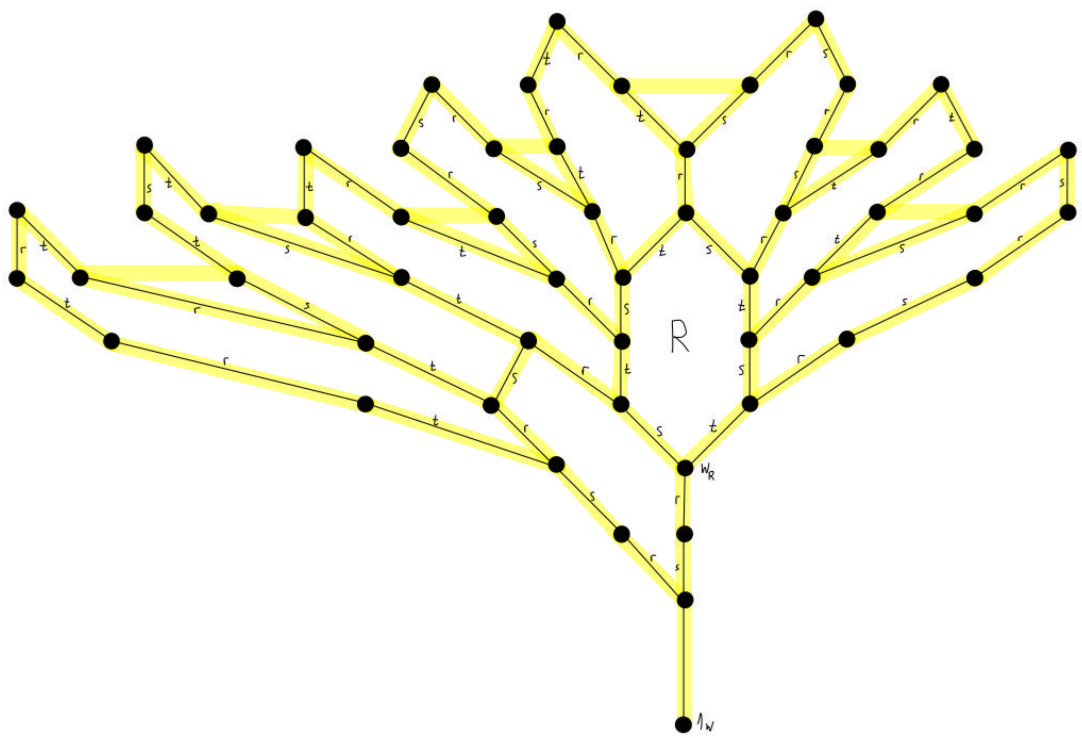


Figure 6.10.: Illustration of the group $U_{R,s}$

(6.5.1) Lemma. *Let $R \in \mathcal{T}_{i,1}$ be of type $\{s,t\}$ such that $\ell(w_R rs) = \ell(w_R) - 2$. Then the canonical homomorphisms $H_R \to E_{R,s}$ and $E_{R,s} \to U_{R,s}$ are injective and we have $E_{R,s} \star_{H_R} G_R \cong U_{R,s}$.*

Proof. The first four vertex groups of the underlying sequences of groups of $E_{R,s}$ and $U_{R,s}$ coincide. Thus we denote the tree product of these first four vertex groups by F_4. Using Proposition (1.8.1) we deduce $E_{R,s} \cong F_4 \star_{U_{w_R srtr}} H_R$ and $U_{R,s} \cong F_4 \star_{U_{w_R srtr}} G_R$. In particular, $H_R \to E_{R,s}$ is injective. Using Lemma (6.3.1), Proposition (1.8.1), Remark (1.8.6) and Lemma (1.8.7) we infer

$$U_{R,s} \cong F_4 \star_{U_{w_R srtr}} G_R \cong F_4 \star_{U_{w_R srtr}} H_R \star_{H_R} G_R \cong E_{R,s} \star_{H_R} G_R$$

Proposition (1.8.1) yields that $E_{R,s} \to U_{R,s}$ is injective and the claim follows. □

6.6. The group X_R

Let $R \in \mathcal{T}_{i,1}$ be a residue of type $\{s,t\}$ and assume that $\ell(w_R rs) = \ell(w_R) - 2$ and $\ell(w_R rt) = \ell(w_R)$. Let $R' = R_{\{r,s\}}(w_R)$ and let $w' = w_{R'}$. Let X_R be the tree product of the sequence of groups with vertex groups

$$U_{w'rsr_{\{r,t\}}}, V_{w'rsrtr_{\{r,s\}}}, U_{w'rsrr_{\{s,t\}}}, V_{w_R srtr_{\{r,s\}}}, U_{w_R sr_{\{r,t\}}}, \\ V_{w_R str_{\{r,s\}}}, U_{w_R r_{\{s,t\}}}, V_{w_R tr_{\{r,s\}}}, U_{w' sr_{\{r,t\}}}$$

Using similar arguments as in Remark (6.1.1) it follows that X_R is generated by suitable u_α.

(6.6.1) Remark. Let $R \in \mathcal{T}_{i,1}$ be a residue of type $\{s,t\}$ such that $\ell(w_R rs) = \ell(w_R) - 2$ and $\ell(w_R rt) = \ell(w_R)$ and let $T := R_{\{r,s\}}(w_R t)$. In the next lemma we consider $X_R \star_{V_T} O_T$. Similar as in Remark (6.1.1) we have to show that if x_α is a generator of X_R and y_α is a generator of O_T, then $x_\alpha = y_\alpha$ holds in $X_R \star_{V_T} O_T$. It suffices to consider $w_R tr\alpha_s$ and $w_R ts\alpha_r$. As $-w_R\alpha_s \subseteq w_R tr\alpha_s, w_R ts\alpha_r$, we deduce that x_α is not a generator of X_R for $\alpha \in \{w_R tr\alpha_s, w_R ts\alpha_r\}$.

(6.6.2) Lemma. *Let $R \in \mathcal{T}_{i,1}$ be a residue of type $\{s,t\}$ such that $\ell(w_R rs) = \ell(w_R) - 2$ and $\ell(w_R rt) = \ell(w_R)$ and let $T := R_{\{r,s\}}(w_R t)$. Then the canonical homomorphisms $V_T \to X_R$ and $E_{R,s} \to X_R \star_{V_T} O_T$ are injective.*

Proof. The first part follows from Proposition (1.8.1) and Proposition (1.8.3). Let F_6 be the tree product of the first six vertex groups of the underlying sequence of groups of X_R. Using Proposition (1.8.1), Remark (1.8.6), Lemma (1.8.7) and Lemma (6.1.2) we obtain the following isomorphisms:

$$\begin{aligned} X_R \star_{V_T} O_T &\cong \left(F_6 \star_{U_{w_R sts}} U_{w_R r_{\{s,t\}}} \hat{\star} V_{w_R tr_{\{r,s\}}} \hat{\star} U_{w' sr_{\{r,t\}}} \right) \star_{V_T} O_T \\ &\cong \left(F_6 \star_{U_{w_R sts}} U_{w_R r_{\{s,t\}}} \star_{U_{w_R tst}} U_{w_R tst} \hat{\star} V_{w_R tr_{\{r,s\}}} \hat{\star} U_{w' sr_{\{r,t\}}} \right) \star_{V_T} O_T \\ &\cong F_6 \star_{U_{w_R sts}} U_{w_R r_{\{s,t\}}} \star_{U_{w_R tst}} V_T \star_{V_T} O_T \\ &\cong F_6 \star_{U_{w_R sts}} U_{w_R r_{\{s,t\}}} \star_{U_{w_R tst}} O_T \\ &\cong E_{R,s} \star_{U_{w_R trs}} V_{w_R trr_{\{s,t\}}} \end{aligned}$$

□

(6.6.3) Lemma. *Let $R \in \mathcal{T}_{i,1}$ be a residue of type $\{s,t\}$ such that $\ell(w_R rs) = \ell(w_R) - 2$ and $\ell(w_R rt) = \ell(w_R)$. Let $Z := R_{\{r,s\}}(w_R)$ be and suppose that $Z \in \mathcal{T}_{i-1,1}$. Then $X_R \to G_Z$ is injective.*

Proof. As the last nine vertex groups of the underlying sequence of groups of G_Z coincide with the vertex groups of the underlying sequence of groups of X_R, the claim follows from Proposition (1.8.1). □

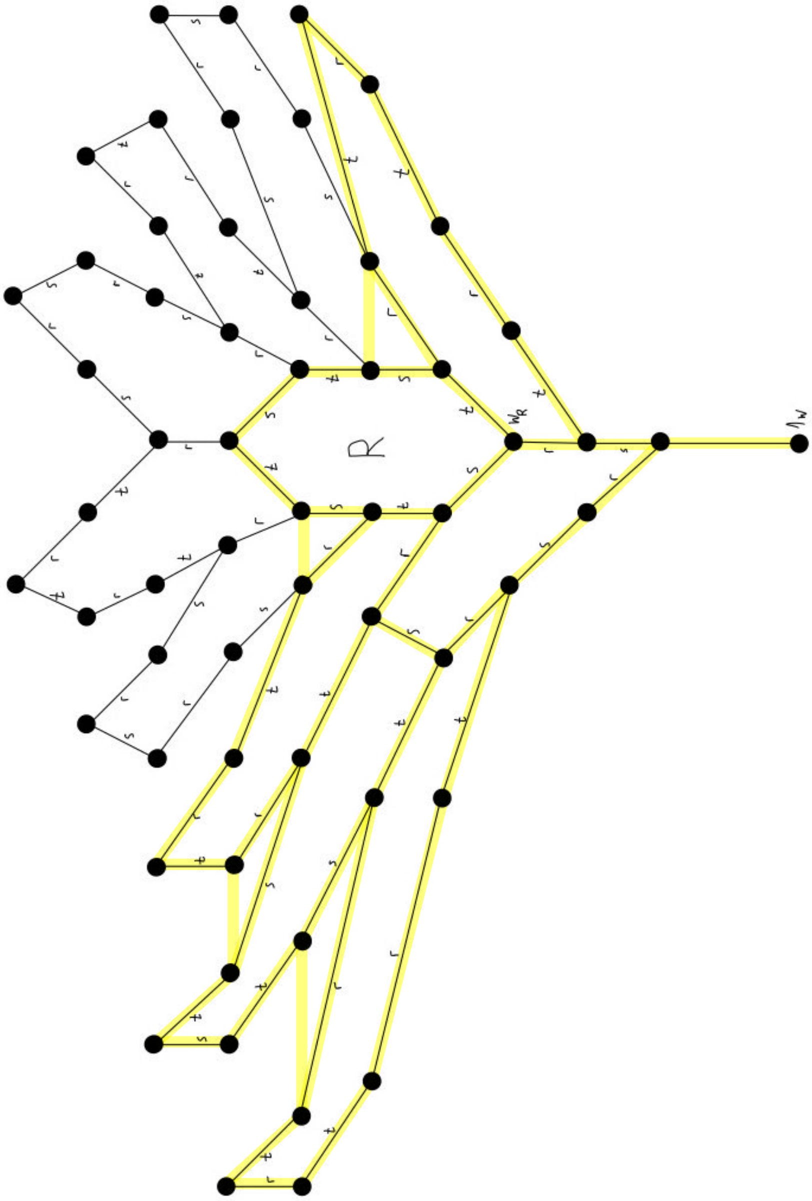


Figure 6.11.: Illustration of the group X_R

6.7. The groups $H_{\{R,R'\}}, G_{\{R,R'\}}$ and $J_{(R,R')}$

Let $\{R, R'\} \in \mathcal{T}_{i,2}$. Let $w = w_R, w' = w_{R'}$ and let $\{r, s\}$ (resp. $\{r, t\}$) be the type of R (resp. R'). Let $T = R_{\{r,t\}}(w)$ and $T' = R_{\{r,s\}}(w')$. Then we define the group $H_{\{R,R'\}}$ to be the tree product of the sequence of groups with vertex groups

$$U_{w_T rtrr_{\{s,t\}}}, V_{w_T r_{\{r,t\}} sr_{\{r,t\}}}, U_{w_T trtr_{\{r,s\}}}, V_{w_T trtsr_{\{r,t\}}}, U_{w_T trr_{\{s,t\}}},$$
$$V_{wrsr_{\{r,t\}}}, U_{wr_{\{r,s\}}}, V_{wsrr_{\{s,t\}}}, U_{w'r_{\{r,t\}}}, V_{w'rtr_{\{r,s\}}},$$
$$U_{w_{T'} srr_{\{s,t\}}}, V_{w_{T'} srstr_{\{r,s\}}}, U_{w_{T'} srsr_{\{r,t\}}}, V_{w_{T'} r_{\{r,s\}} tr_{\{r,s\}}}, U_{w_{T'} rsrr_{\{s,t\}}}$$

We define the group $J_{(R,R')}$ to be the tree product of the sequence of groups with vertex groups

$$U_{w_T rtrr_{\{s,t\}}}, V_{w_T r_{\{r,t\}} sr_{\{r,t\}}}, U_{w_T trtr_{\{r,s\}}}, V_{w_T trtsr_{\{r,t\}}},$$
$$U_{w_T trr_{\{s,t\}}}, V_{wrstr_{\{r,s\}}}, U_{wrsr_{\{r,t\}}}, V_{wrsrr_{\{s,t\}}}, V_{wsrr_{\{s,t\}}}, U_{w'r_{\{r,t\}}}, V_{w'rtr_{\{r,s\}}},$$
$$U_{w_{T'} srr_{\{s,t\}}}, V_{w_{T'} srstr_{\{r,s\}}}, U_{w_{T'} srsr_{\{r,t\}}}, V_{w_{T'} r_{\{r,s\}} tr_{\{r,s\}}}, U_{w_{T'} rsrr_{\{s,t\}}}$$

Furthermore, we define the group $G_{\{R,R'\}}$ to be the tree product of the sequence of groups with vertex groups

$$U_{w_T rtrr_{\{s,t\}}}, V_{w_T r_{\{r,t\}} sr_{\{r,t\}}}, U_{w_T trtr_{\{r,s\}}}, V_{w_T trtsr_{\{r,t\}}},$$
$$U_{w_T trr_{\{s,t\}}}, V_{wrstr_{\{r,s\}}}, U_{wrsr_{\{r,t\}}}, V_{wrsrtr_{\{r,s\}}}, U_{wrsrr_{\{s,t\}}}, V_{wr_{\{r,s\}} tr_{\{r,s\}}}, U_{wsrsr_{\{r,t\}}},$$
$$V_{wsrstr_{\{r,s\}}}, U_{wsrr_{\{s,t\}}}, V_{w'trtsr_{\{r,t\}}},$$
$$U_{w'trtr_{\{r,s\}}}, V_{w'r_{\{r,t\}} sr_{\{r,t\}}}, U_{w'rtrr_{\{s,t\}}}, V_{w'rtrsr_{\{r,t\}}}, U_{w'rtr_{\{r,s\}}}, V_{w'rtsr_{\{r,t\}}}, U_{w_{T'} srr_{\{s,t\}}},$$
$$V_{w_{T'} srstr_{\{r,s\}}}, U_{w_{T'} srsr_{\{r,t\}}}, V_{w_{T'} r_{\{r,s\}} tr_{\{r,s\}}}, U_{w_{T'} rsrr_{\{s,t\}}}$$

Using similar arguments as in Remark (6.1.1) it follows that $H_{\{R,R'\}}, G_{\{R,R'\}}$ and $J_{(R,R')}$ are generated by suitable u_α.

(6.7.1) Lemma. *Let $\{R, R'\} \in \mathcal{T}_{i,2}$, let $\{r, s\}$ be the type of R and let $\{r, t\}$ be the type of R'. Then the canonical homomorphisms $H_{\{R,R'\}} \to J_{(R,R')}$ and $J_{(R,R')} \to G_{\{R,R'\}}$ are injective. In particular, the canonical homomorphism $H_{\{R,R'\}} \to G_{\{R,R'\}}$ is injective.*

Proof. We first show that the homomorphism $H_{\{R,R'\}} \to J_{(R,R')}$ is injective. Using Proposition (1.8.1) the group $J_{(R,R')}$ is isomorphic to the tree product of the following sequence of groups with vertex groups

$$U_{w_T rtrr_{\{s,t\}}}, V_{w_T r_{\{r,t\}} sr_{\{r,t\}}}, U_{w_T trtr_{\{r,s\}}}, V_{w_T trtsr_{\{r,t\}}},$$
$$U_{w_T trr_{\{s,t\}}} \hat{\star} V_{wrstr_{\{r,s\}}}, U_{wrsr_{\{r,t\}}}, V_{wrsrr_{\{s,t\}}}, V_{wsrr_{\{s,t\}}}, U_{w'r_{\{r,t\}}}, V_{w'rtr_{\{r,s\}}},$$
$$U_{w_{T'} srr_{\{s,t\}}}, V_{w_{T'} srstr_{\{r,s\}}}, U_{w_{T'} srsr_{\{r,t\}}}, V_{w_{T'} r_{\{r,s\}} tr_{\{r,s\}}}, U_{w_{T'} rsrr_{\{s,t\}}}$$

One easily sees that each vertex groups of $H_{\{R,R'\}}$ is contained in the corresponding vertex group of the previous tree product. Note that the first five and the last eight vertex groups of the underlying sequence of groups of $H_{\{R,R'\}}$ and $J_{(R,R')}$ coincide. Thus we only have to consider the preimage of the other boundary monomorphisms. We compute the following:

$$U_{w_T trr_{\{s,t\}}} \cap U_{w_T trstr} = U_{w_T trst} = V_{w_R rsr_{\{r,t\}}} \cap U_{w_T trstr}$$
$$V_{w_R rsr_{\{r,t\}}} \cap U_{w_R rsrt} = U_{w_R rsr} = U_{w_R r_{\{r,s\}}} \cap U_{w_R rsrt}$$

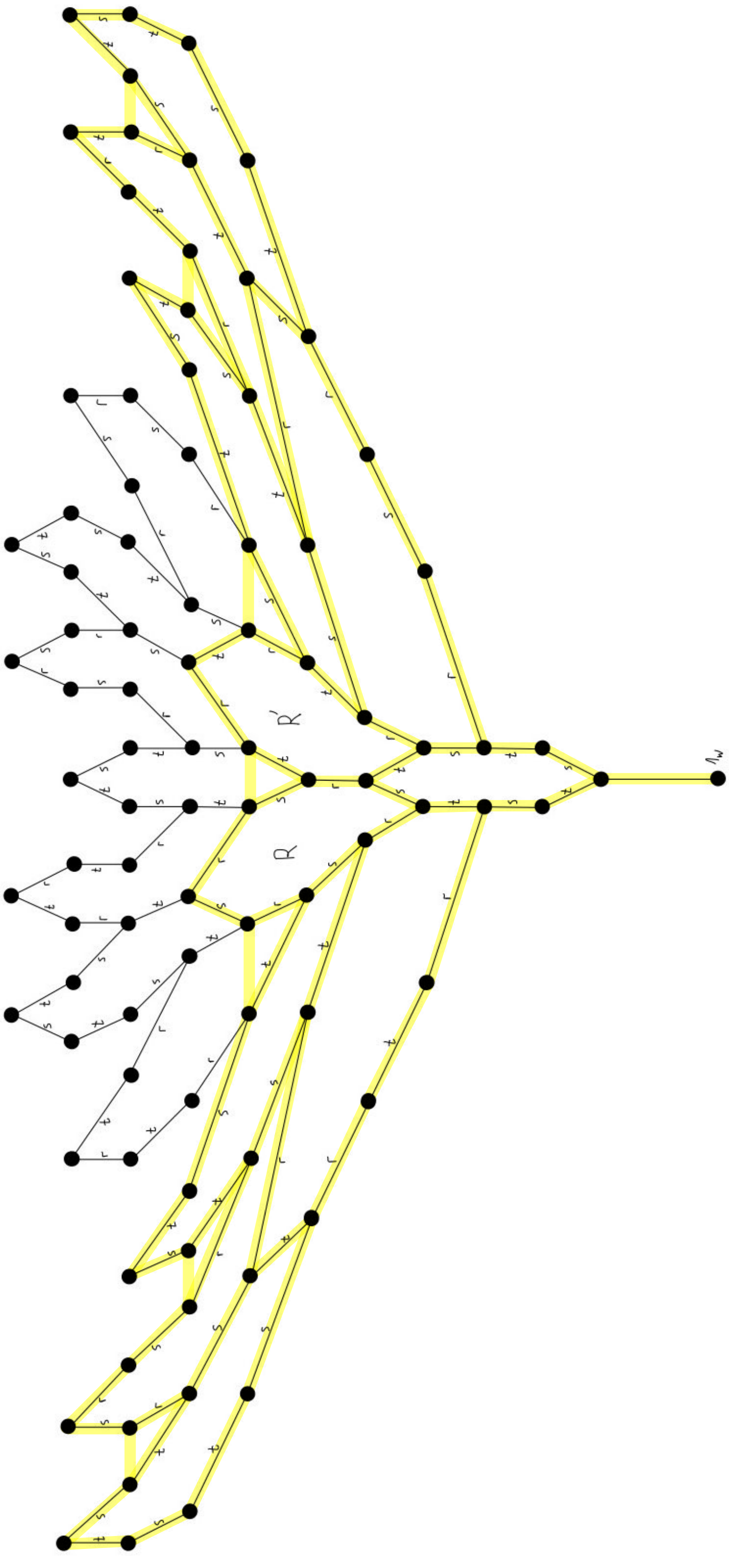


Figure 6.12.: Illustration of the group $H_{\{R,R'\}}$

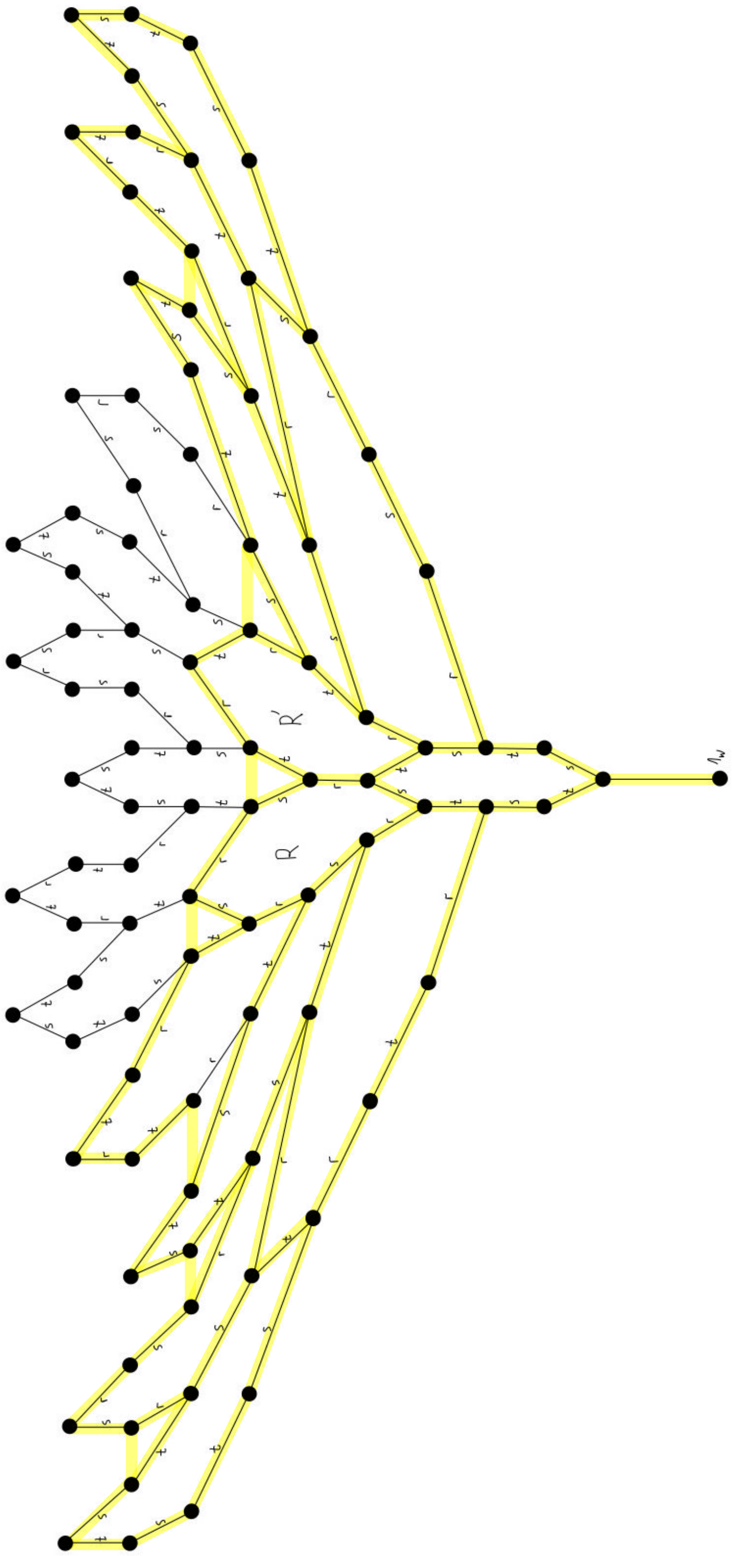


Figure 6.13.: Illustration of the group $J_{(R,R')}$

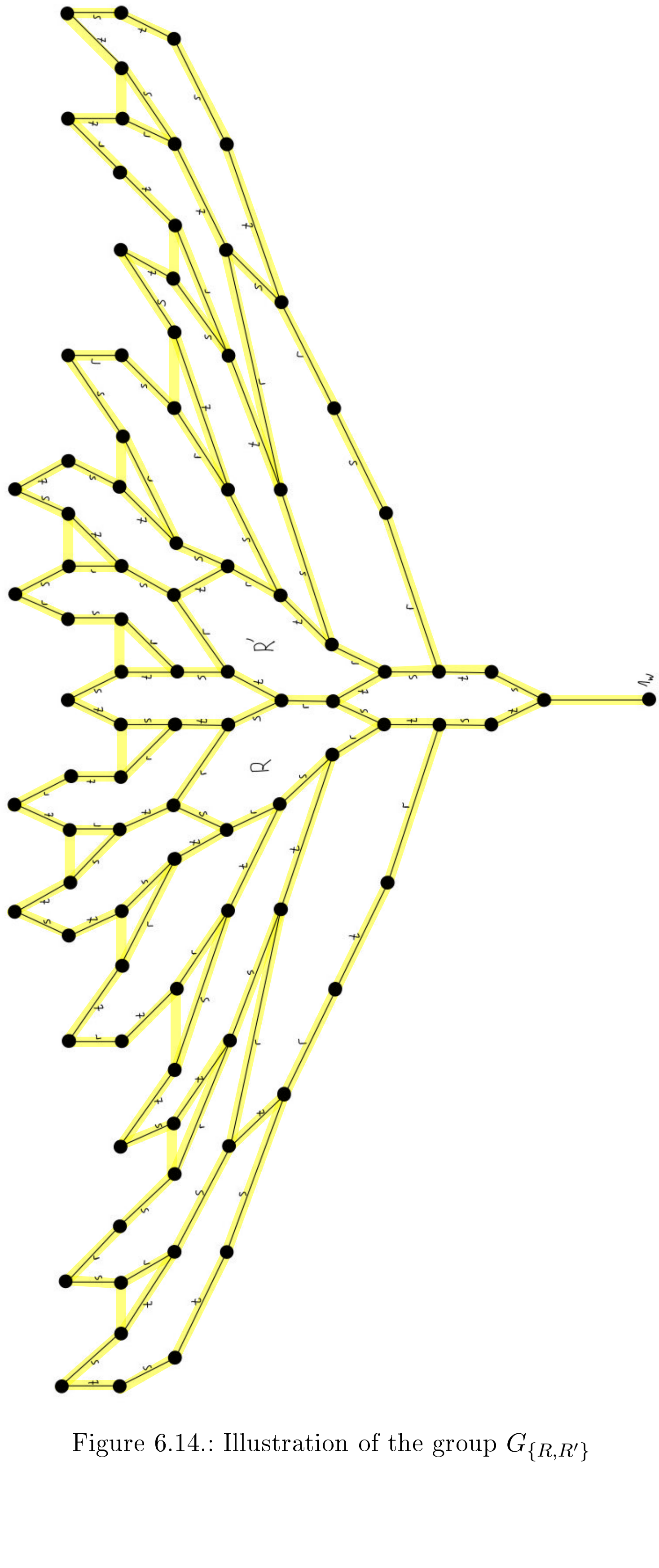


Figure 6.14.: Illustration of the group $G_{\{R,R'\}}$

$$U_{w_R r_{\{r,s\}}} \cap U_{w_R srs} = U_{w_R srs} = V_{w_R srr_{\{s,t\}}} \cap U_{w_R srs}$$

As before, $H_{\{R,R'\}} \to J_{(R,R')}$ is injective by Proposition (1.8.3).

Now we show that $J_{(R,R')} \to G_{\{R,R'\}}$ is injective. Using Proposition (1.8.1) the group $G_{\{R,R'\}}$ is isomorphic to the tree product of the following sequence of groups with vertex groups

$$\begin{aligned} U_{w_T rtrr_{\{s,t\}}}, V_{w_T r_{\{r,t\}} sr_{\{r,t\}}}, U_{w_T trtr_{\{r,s\}}}, V_{w_T trtsr_{\{r,t\}}}, U_{w_T trr_{\{s,t\}}}, V_{wrstr_{\{r,s\}}}, \\ U_{wrsr_{\{r,t\}}} \hat{\star} V_{wrsrtr_{\{r,s\}}}, U_{wrsrr_{\{s,t\}}} \hat{\star} V_{wr_{\{r,s\}} tr_{\{r,s\}}} \hat{\star} U_{wsrsr_{\{r,t\}}}, \\ V_{wsrstr_{\{r,s\}}} \hat{\star} U_{wsrr_{\{s,t\}}} \hat{\star} V_{w'trtsr_{\{r,t\}}}, \\ U_{w'trtr_{\{r,s\}}} \hat{\star} V_{w'r_{\{r,t\}} sr_{\{r,t\}}} \hat{\star} U_{w'rtrr_{\{s,t\}}}, V_{w'rtrsr_{\{r,t\}}} \hat{\star} U_{w'rtr_{\{r,s\}}}, V_{w'rtsr_{\{r,t\}}} \hat{\star} U_{w_{T'} srr_{\{s,t\}}}, \\ V_{w_{T'} srstr_{\{r,s\}}}, U_{w_{T'} srsr_{\{r,t\}}}, V_{w_{T'} r_{\{r,s\}} tr_{\{r,s\}}}, U_{w_{T'} rsrr_{\{s,t\}}} \end{aligned}$$

One easily sees that each vertex group of $J_{(R,R')}$ is contained in the corresponding vertex group of the previous tree product. Note that the first seven and the last five vertex groups of the underlying sequence of groups of $J_{(R,R')}$ and $G_{\{R,R'\}}$ coincide. Thus it suffices to consider the following preimages of the boundary monomorphisms:

$$\begin{aligned} U_{wrsr_{\{r,t\}}} \cap U_{wrsrts} &= U_{wrsrt} = V_{wrsrr_{\{s,t\}}} \cap U_{wrsrts} \\ V_{wrsrr_{\{s,t\}}} \cap U_{wsrstr} &= U_{wsrs} = V_{wsrr_{\{s,t\}}} \cap U_{wsrstr} \\ V_{w'trr_{\{s,t\}}} \cap U_{w'trtsr} &= U_{w'trt} = U_{w'r_{\{r,t\}}} \cap U_{w'trtsr} \\ U_{w'r_{\{r,t\}}} \cap U_{w'rtrst} &= U_{w'rtr} = V_{w'rtr_{\{r,s\}}} \cap U_{w'rtrst} \\ V_{w'rtr_{\{r,s\}}} \cap U_{w'rtsr} &= U_{w'rts} = U_{w'rr_{\{s,t\}}} \cap U_{w'rtsr} \end{aligned}$$

We should say something to the equation $V_{wrsrr_{\{s,t\}}} \cap U_{wsrstr} = U_{wsrs}$. Clearly, $\supseteq$ holds. For the other inclusion we obtain similar as in Lemma (6.3.1) that

$$V_{wrsrr_{\{s,t\}}} \cap U_{wsrstr} \subseteq U_{wr_{\{r,s\}} ts} \cap U_{wr_{\{r,s\}} tr} = U_{wr_{\{r,s\}} t}$$

and hence $V_{wrsrr_{\{s,t\}}} \cap U_{wsrstr} = V_{wrsrr_{\{s,t\}}} \cap U_{wsrstr} \cap U_{wr_{\{r,s\}} t} = V_{wrsrr_{\{s,t\}}} \cap U_{wsrs} = U_{wsrs}$. As before, $J_{(R,R')} \to G_{\{R,R'\}}$ is injective by Proposition (1.8.3). □

(6.7.2) Lemma. *Let $R \in \mathcal{T}_{i,1}$ be of type $\{s,t\}$ and assume that $\ell(w_R rs) = \ell(w_R) - 2 = \ell(w_R rt)$. Let $T = R_{\{r,s\}}(w_R)$ and $T' = R_{\{r,t\}}(w_R)$. Then $\{T, T'\} \in \mathcal{T}_{i-2,2}$ and the canonical homomorphism $E_{R,s} \to G_{\{T,T'\}}$ is injective.*

Proof. Since $R \in \mathcal{T}_{i,1}$, we have $\{T, T'\} \in \mathcal{T}_{i-2,2}$. The second assertion follows directly from Proposition (1.8.1), as the vertex groups of $E_{R,s}$ and the vertex groups $7 - 15$ of $G_{\{T,T'\}}$ coincide. □

(6.7.3) Lemma. *Let $\{R, R'\} \in \mathcal{T}_{i,2}$, let $\{r,s\}$ be the type of R, let $\{r,t\}$ be the type of R', and let $Z = R_{\{r,t\}}(w_R rs)$. Then $Z \in \mathcal{T}_{i+1,1}$, the canonical homomorphism $V_Z \to H_{\{R,R'\}}$ is injective and we have $J_{(R,R')} \cong H_{\{R,R'\}} \star_{V_Z} O_Z$.*

Proof. Note that $Z \in \mathcal{T}_{i+1,1}$. By Proposition (1.8.1), $U_{w_R rr_{\{s,t\}}} \hat{\star} V_{w_R rsr_{\{r,t\}}} \hat{\star} U_{w_R rsrs} \to H_{\{R,R'\}}$ is injective. Using Proposition (1.8.3), we deduce that

$$V_Z = U_{w_R rsts} \hat{\star} V_{w_R rsr_{\{r,t\}}} \hat{\star} U_{w_R rsrs} \to U_{w_R rr_{\{s,t\}}} \hat{\star} V_{w_R rsr_{\{r,t\}}} \hat{\star} U_{w_R rsrs}$$

is injective and hence also the concatenation $V_T \to H_{\{R,R'\}}$. Let F_i be the tree product of the first i vertex groups and let L_j be the tree product of the last j vertex groups of the underlying sequence of groups of $J_{(R,R')}$. Note that by Proposition (1.8.3) and Lemma (6.1.2) the homomorphism $F_5 \star_{U_{w_R rsts}} V_Z \to F_5 \star_{U_{w_R rsts}} O_Z$ is injective. We deduce from Proposition (1.8.1) and Lemma (1.8.7) that $F_5 \star_{U_{w_R rsts}} V_Z \star_{U_{w_R srs}} L_8 \cong H_{\{R,R'\}}$. Note also, that $U_{w_R srs} \to V_Z$ is injective. Using Proposition (1.8.1), Remark (1.8.6), Lemma (1.8.7) and Lemma (6.1.2) we obtain the following isomorphisms:

$$\begin{aligned}
J_{(R,R')} &\cong F_5 \star_{U_{w_R rsts}} V_{w_R rstr_{\{r,s\}}} \hat{\star} U_{w_R rsr_{\{r,t\}}} \hat{\star} V_{w_R rsrr_{\{s,t\}}} \star_{U_R srs} L_8 \\
&\cong F_5 \star_{U_{w_R rsts}} O_Z \star_{U_{w_R srs}} L_8 \\
&\cong \left(F_5 \star_{U_{w_R rsts}} O_Z\right) \star_{\left(F_5 \star_{U_{w_R rsts}} V_Z\right)} \left(F_5 \star_{U_{w_R rsts}} V_Z\right) \star_{U_{w_R srs}} L_8 \\
&\cong \left(F_5 \star_{U_{w_R rsts}} V_Z \star_{V_Z} O_Z\right) \star_{\left(F_5 \star_{U_{w_R rsts}} V_Z\right)} \left(F_5 \star_{U_{w_R rsts}} V_Z \star_{U_{w_R srs}} L_8\right) \\
&\cong \left(O_Z \star_{V_Z} \left(F_5 \star_{U_{w_R rsts}} V_Z\right)\right) \star_{\left(F_5 \star_{U_{w_R rsts}} V_Z\right)} H_{\{R,R'\}} \\
&\cong O_Z \star_{V_Z} H_{\{R,R'\}}
\end{aligned}$$

□

(6.7.4) Lemma. *Let $R \in \mathcal{T}_{i,1}$ be a residue of type $\{s,t\}$ and assume that $\ell(w_R rs) = \ell(w_R) - 2$ and $\ell(w_R rt) = \ell(w_R)$. Let $Z := R_{\{r,s\}}(w_R)$ be and suppose that $Z \notin \mathcal{T}_{i-1,1}$. Let $P_Z \in \mathcal{T}_{i-2,2}$ be the unique element with $Z \in P_Z$. Then $X_R \to G_{P_Z}$ is injective.*

Proof. As the vertex groups $13-21$ of the underlying sequence of groups of G_{P_Z} coincide with the vertex groups of the underlying sequence of groups of X_R, the claim follows from Proposition (1.8.1). □

6.8. The groups C and $C_{(R,R')}$

Let $\{R,R'\} \in \mathcal{T}_{i,2}$. Let R be of type $\{r,s\}$ and let R' be of type $\{r,t\}$. We let $T = R_{\{r,t\}}(w_R)$ and $T' = R_{\{r,s\}}(w_{R'})$. We define the group C to be the tree product of the following sequence of groups with vertex groups

$$U_{w_T r_{\{r,t\}}}, V_{w_T trr_{\{s,t\}}}, U_{w_R r_{\{r,s\}}}, V_{w_R srr_{\{s,t\}}}, U_{w_{R'} r_{\{r,t\}}}, V_{w_{T'} srr_{\{s,t\}}}, U_{w_{T'} r_{\{r,s\}}}$$

We let $C_{(R,R')}$ be the tree product of the following sequence of groups with vertex groups

$$U_{w_T rtrr_{\{s,t\}}}, V_{w_T r_{\{r,t\}} sr_{\{r,t\}}}, U_{w_R rtr_{\{r,s\}}}, V_{w_R rtsr_{\{r,t\}}}, U_{w_R rr_{\{s,t\}}}, V_{w_R rsr_{\{r,t\}}}, \\ U_{w_R r_{\{r,s\}}}, V_{w_R srr_{\{s,t\}}}, U_{w_{R'} r_{\{r,t\}}}, V_{w_{R'} rr_{\{s,t\}}}, U_{w_{T'} r_{\{r,s\}}}$$

For completeness, the group $C_{(R',R)}$ is the tree product of the following sequence of groups with vertex groups

$$U_{w_T r_{\{r,t\}}}, V_{w_R rr_{\{s,t\}}}, U_{w_R r_{\{r,s\}}}, V_{w_R srr_{\{s,t\}}}, U_{w_{R'} r_{\{r,t\}}}, \\ V_{w_{R'} rtr_{\{r,s\}}}, U_{w_{R'} rr_{\{s,t\}}}, V_{w_{R'} rstr_{\{r,s\}}}, U_{w_{R'} rsr_{\{r,t\}}}, V_{w_{T'} r_{\{r,s\}} tr_{\{r,s\}}}, U_{w_{T'} rsrr_{\{s,t\}}}$$

Using similar arguments as in Remark (6.1.1) it follows that $C_R, C_{(R,R')}$ are generated by suitable u_α.

(6.8.1) Remark. We note that the vertex groups of $C_{(R',R)}$ can be obtained from $C_{(R,R')}$ by interchanging s and t and starting with the last vertex group of $C_{(R,R')}$. Interchanging s and t and the order of the vertex groups of C does not change the group C.

(6.8.2) Lemma. *Let $\{R,R'\} \in \mathcal{T}_{i,2}$. Then the canonical homomorphisms $C \to C_{(R,R')}, C_{(R',R)}$ are injective and we have $H_{\{R,R'\}} \cong C_{(R,R')} \star_C C_{(R',R)}$.*

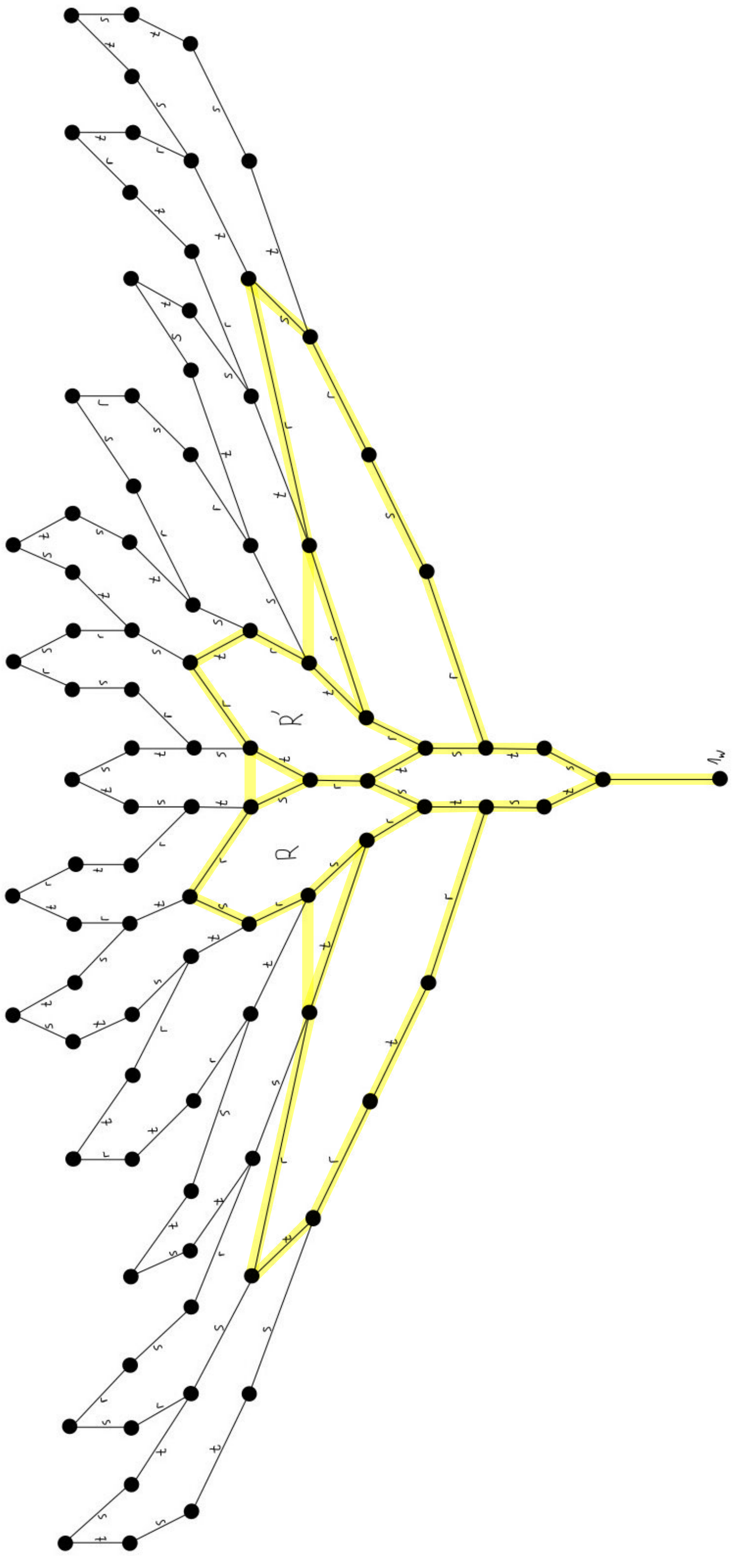


Figure 6.15.: Illustration of the group C

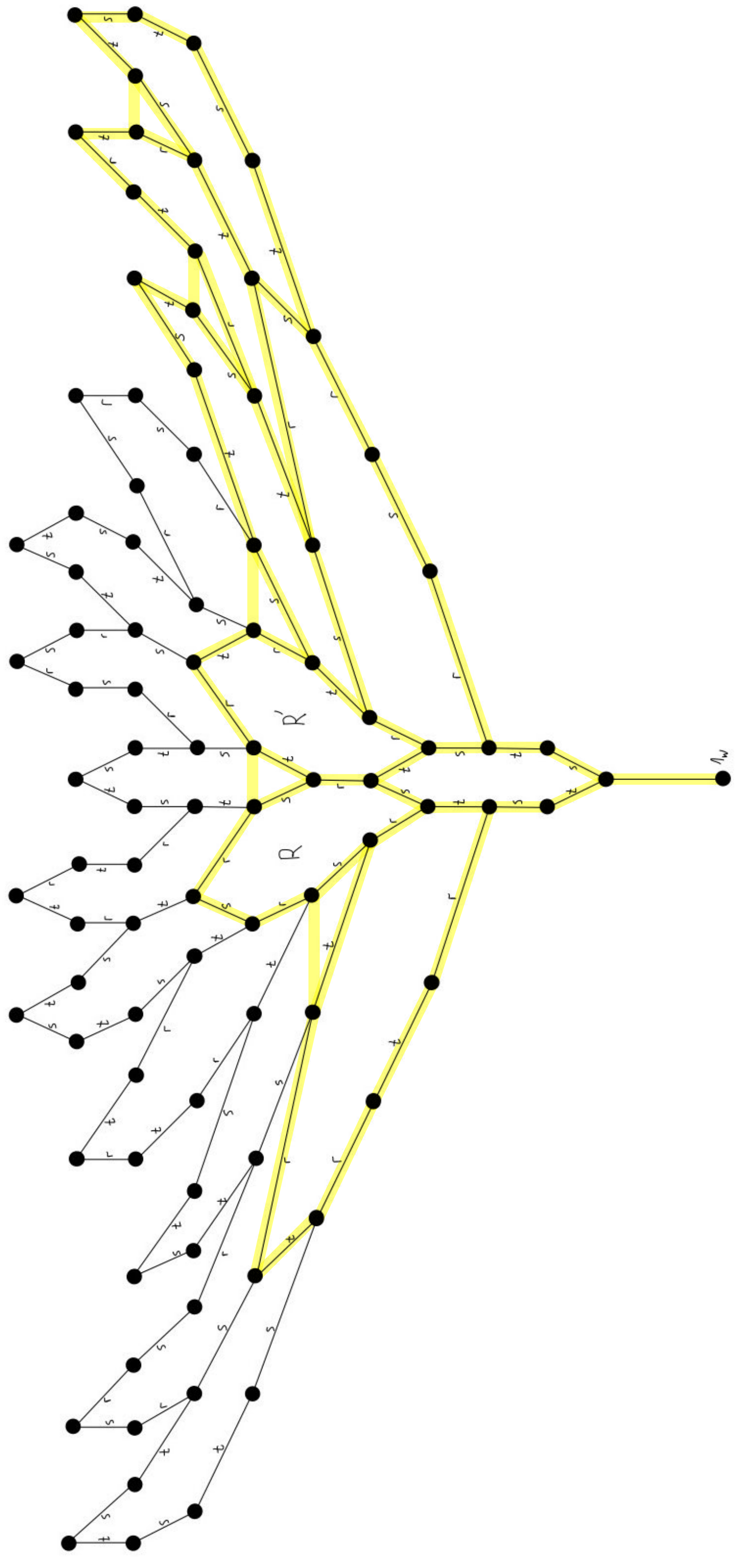


Figure 6.16.: Illustration of the group $C_{(R',R)}$

Proof. We first show that $C \to C_{(R,R')}$ is injective. Let $\{r,s\}$ be the type of R and let $\{r,t\}$ be the type of R'. Using Proposition (1.8.1) the group $C_{(R,R')}$ is isomorphic to the tree product of the following sequence of groups with vertex groups

$$U_{w_T rtrr_{\{s,t\}}} \hat{\star} V_{w_T r_{\{r,t\}} sr_{\{r,t\}}} \hat{\star} U_{w_R rtr_{\{r,s\}}}, V_{w_R rtsr_{\{r,t\}}} \hat{\star} U_{w_R rr_{\{s,t\}}},$$
$$V_{w_R rsr_{\{r,t\}}} \hat{\star} U_{w_R r_{\{r,s\}}}, V_{w_R srr_{\{s,t\}}}, U_{w_{R'} r_{\{r,t\}}}, V_{w_{R'} rr_{\{s,t\}}}, U_{w_{T'} r_{\{r,s\}}}$$

One easily sees that each vertex group of C is contained in the corresponding vertex group of the previous tree product. Considering the preimage of the boundary monomorphisms the following hold:

$$U_{w_T r_{\{r,t\}}} \cap U_{w_R rtsr} = U_{w_R rt} = V_{w_T trr_{\{s,t\}}} \cap U_{w_R rtsr}$$
$$V_{w_T trr_{\{s,t\}}} \cap U_{w_R rst} = U_{w_R rs} = U_{w_R r_{\{r,s\}}} \cap U_{w_R rst}$$

As before, the claim follows from Proposition (1.8.3). Interchanging s and t and the order of the vertex groups of $C_{(R,R')}$ and C, we obtain that $C \to C_{(R',R)}$ is injective. Let F_7 be the tree product of the first seven vertex groups of the underlying sequence of groups of $H_{\{R,R'\}}$ and let L_7 be the tree product of the last seven vertex groups of the underlying sequence of groups of $H_{\{R,R'\}}$. It follows from the computations above that $U_{left} := U_{w_T r_{\{r,t\}}} \hat{\star} V_{w_R rr_{\{s,t\}}} \hat{\star} U_{w_R r_{\{r,s\}}} \to F_7$ and $U_{right} := U_{w_{R'} r_{\{r,t\}}} \hat{\star} V_{w_{R'} rr_{\{s,t\}}} \hat{\star} U_{w_{T'} r_{\{r,s\}}} \to L_7$ are injective. Moreover, $U_{right} \to C$ is injective by Proposition (1.8.1). Using Proposition (1.8.1), Lemma (1.8.7) and Remark (1.8.6) we obtain the following isomorphisms:

$$\begin{aligned}
H_{\{R,R'\}} &\cong F_7 \star_{U_{w_R srs}} V_{w_R srr_{\{s,t\}}} \star_{U_{w_{R'} trt}} L_7 \\
&\cong F_7 \star_{U_{w_R srs}} V_{w_R srr_{\{s,t\}}} \star_{U_{w_{R'} trt}} U_{right} \star_{U_{right}} L_7 \\
&\cong C_{(R,R')} \star_{U_{right}} L_7 \\
&\cong C_{(R,R')} \star_C C \star_{U_{right}} L_7 \\
&\cong C_{(R,R')} \star_C \left(C \star_{U_{right}} L_7\right) \\
&\cong C_{(R,R')} \star_C \left(U_{left} \star_{U_{w_R srs}} V_{w_R srr_{\{s,t\}}} \star_{U_{w_{R'} trt}} U_{right} \star_{U_{right}} L_7\right) \\
&\cong C_{(R,R')} \star_C \left(U_{left} \star_{U_{w_R srs}} V_{w_R srr_{\{s,t\}}} \star_{U_{w_{R'} trt}} L_7\right) \\
&\cong C_{(R,R')} \star_C C_{(R',R)}
\end{aligned}$$ ☐

(6.8.3) Lemma. *Let $\{R,R'\} \in \mathcal{T}_{i,2}$. Let R be of type $\{r,s\}$, let R' be of type $\{r,t\}$ and let $T' := R_{\{r,s\}}(w_{R'})$. Then $T' \in \mathcal{T}_{i-1,1}$, the canonical homomorphism $C_{(R',R)} \to U_{T',s}$ is injective and we have $C_{(R',R)} \cap E_{T',s} = C$ in $U_{T',s}$. In particular, for $T := R_{\{r,t\}}(w_R)$ we have $T \in \mathcal{T}_{i-1,1}$, the canonical homomorphism $C_{(R,R')} \to U_{T,t}$ is injective and we have $C_{(R,R')} \cap E_{T,t} = C$ in $U_{T,t}$.*

Proof. The claim $T, T' \in \mathcal{T}_{i-1,1}$ follows from Lemma (5.1.1), as for $Z := R_{\{s,t\}}(w_R)$ we have $\ell(w_Z trs), \ell(w_Z srt) \geq \ell(w_Z) + 1$. We note that $\ell(w_{T'} ts) = \ell(w_{T'}) - 2$. We let $w' = w_Z$. For completeness we recall that $U_{T',s}$ is the tree product of the underlying sequence of groups with vertex groups

$$U_{w' tsr_{\{r,t\}}}, V_{w' tstrr_{\{s,t\}}}, U_{w' tstr_{\{r,s\}}}, V_{w_{T'} strr_{\{s,t\}}}, U_{w_{T'} sr_{\{r,t\}}}, V_{w_{T'} srtr_{\{r,s\}}},$$
$$U_{w_{T'} srr_{\{s,t\}}}, V_{w_{T'} srstr_{\{r,s\}}}, U_{w_{T'} srsr_{\{r,t\}}}, V_{w_{T'} r_{\{r,s\}} tr_{\{r,s\}}}, U_{w_{T'} rsrr_{\{s,t\}}},$$
$$V_{w_{T'} rsrtr_{\{r,s\}}}, U_{w_{T'} rsr_{\{r,t\}}}, V_{w_{T'} rstr_{\{r,s\}}}, U_{w_{T'} rr_{\{s,t\}}}$$

As the first eleven vertex groups of $U_{T',s}$ coincide with the vertex groups of $C_{(R',R)}$, Proposition (1.8.1) implies that $C_{(R',R)} \to U_{T',s}$ is injective. Before we show the claim, we have to analyse the embedding $E_{T',s} \to U_{T',s}$ from Lemma (6.5.1) in more detail. Using Proposition (1.8.1) the group $U_{T',s}$ is isomorphic to the tree product of the following sequence of groups with vertex groups

$$U_{w'tsr_{\{r,t\}}}, V_{w'tstrr_{\{s,t\}}}, U_{w'tstr_{\{r,s\}}}, V_{w_{T'}strr_{\{s,t\}}}, U_{w_{T'}sr_{\{r,t\}}} \hat{\star} V_{w_{T'}srtr_{\{r,s\}}},$$
$$U_{w_{T'}srr_{\{s,t\}}} \hat{\star} V_{w_{T'}srstr_{\{r,s\}}}, U_{w_{T'}srsr_{\{r,t\}}} \hat{\star} V_{w_{T'}r_{\{r,s\}}tr_{\{r,s\}}} \hat{\star} U_{w_{T'}rsrr_{\{s,t\}}},$$
$$V_{w_{T'}rsrtr_{\{r,s\}}} \hat{\star} U_{w_{T'}rsr_{\{r,t\}}}, V_{w_{T'}rstr_{\{r,s\}}} \hat{\star} U_{w_{T'}rr_{\{s,t\}}}$$

One easily sees that each vertex group of $E_{T',s}$ is contained in the corresponding vertex group of the previous tree product. As the first four vertex groups of $E_{T',s}$ and $U_{T',s}$ coincide, it suffices to consider the following preimages of the boundary monomorphisms:

$$U_{w_{T'}sr_{\{r,t\}}} \cap U_{w_{T'}srts} = U_{w_{T'}srt} = V_{w_{T'}srr_{\{s,t\}}} \cap U_{w_{T'}srts}$$
$$V_{w_{T'}srr_{\{s,t\}}} \cap U_{w_{T'}srstr} = U_{w_{T'}srs} = U_{w_{T'}r_{\{r,s\}}} \cap U_{w_{T'}srstr}$$
$$U_{w_{T'}r_{\{r,s\}}} \cap U_{w_{T'}rsrts} = U_{w_{T'}rsr} = V_{w_{T'}rsr_{\{r,t\}}} \cap U_{w_{T'}rsrts}$$
$$V_{w_{T'}rsr_{\{r,t\}}} \cap U_{w_{T'}rstr} = U_{w_{T'}rst} = U_{w_{T'}rr_{\{s,t\}}} \cap U_{w_{T'}rstr}$$

As before, $E_{T',s} \to U_{T',s}$ is injective by Proposition (1.8.3). We have known this already before, but this time we know how the embedding looks like and we can apply Corollary (1.8.4). We deduce from it that in $U_{T',s}$ the intersection $C_{(R',R)} \cap E_{T',s}$ is equal to the tree product of the first seven vertex groups of the underlying sequence of groups of $E_{T',s}$, which is isomorphic to C. □

6.9. Faithful commutator blueprints

For two elements $w_1, w_2 \in W$ we define $w_1 \prec w_2$ if $\ell(w_1)+\ell(w_1^{-1}w_2) = \ell(w_2)$. For any $w \in W$ we put $C(w) := \{w' \in W \mid w' \prec w\}$. We now define for every $i \in \mathbb{N}$ a subset $C_i \subseteq W$ as follows:

$$C_0 := \bigcup_{S=\{r,s,t\}} \left(C(r_{\{s,t\}}) \cup C(rr_{\{s,t\}})\right)$$

For every $R \in \mathcal{R}_i$ of type $J = \{s,t\}$ we let

$$C(R) := C(w_R str_{\{r,s\}}) \cup C(w_R r_J rtr) \cup C(w_R r_J rsr) \cup C(w_R tsr_{\{r,t\}}).$$

For every $\{R, R'\} \in \mathcal{T}_{i,2}$ we let $C(\{R,R'\}) := C(R) \cup C(R')$. We note that this union is not disjoint. For $i \geq 1$ we define

$$C_i := C_{i-1} \cup \bigcup_{R \in \mathcal{R}_{i-1}} C(R) = C_{i-1} \cup \bigcup_{R \in \mathcal{T}_{i-1,1}} C(R) \cup \bigcup_{\{R,R'\} \in \mathcal{T}_{i-1,2}} C(\{R,R'\}).$$

Moreover, we define $D_i := \{w_R r_{\{s,t\}} \mid R \text{ is of type } \{s,t\}, w_R s, w_R t \in C_i\}$.

(6.9.1) Definition. We denote by G_i the direct limit of the inductive system formed by the groups U_w and $V_{w'}$ for $w \in C_i, w' \in D_i$, together with the natural inclusions $U_w \to U_{ws}$ if $\ell(ws) = \ell(w) + 1$ and $U_{w_R s} \to V_{w_R r_{\{s,t\}}}$.

(6.9.2) Remark. Let $i \in \mathbb{N}$. Then G_i is generated by elements $x_{\alpha,w}$ and $y_{\alpha,w'}$ for $w \in C_i, w' \in D_i$, $x_{\alpha,w}$ is a generator of U_w and $y_{\alpha,w'}$ is a generator of $V_{w'}$. We first note that for every $w' = w_R r_{\{s,t\}}$ and every $\alpha \in \Phi_+$ with $w_R s \notin \alpha$, we have $x_{\alpha,w_R s} = y_{\alpha,w'}$ in G_i. Thus $G_i = \langle x_{\alpha,w} \mid \alpha \in \Phi_+, \exists w \in C_i : w \notin \alpha \rangle$.

Suppose $s \in S$ and $w \in W$ with $w \notin \alpha_s$. Then $\ell(sw) = \ell(w) - 1$. Let $k := \ell(w)$ and let $s_2, \ldots, s_k \in S$ be such that $w = ss_2 \cdots s_k$. Then, as $U_{ss_2 \ldots s_m} \to U_{ss_2 \cdots s_{m+1}}$ are the canonical inclusions for any $1 \leq m \leq k-1$, we deduce $x_{\alpha_s,s} = x_{\alpha_s,w}$ in G_i. Let $\alpha \in \Phi_+$ be a non-simple root and let $\mathrm{proj}_{P_\alpha} 1_W \neq d \in P_\alpha$. It is a consequence of Lemma (5.2.5) that $x_{\alpha,d} = x_{\alpha,w}$ for every $w \in W$ with $w \notin \alpha$. Thus G_i is generated by $\{x_\alpha \mid \alpha \in \Phi_+, \exists w \in C_i : w \notin \alpha\}$.

By the definition of the direct limit we have canonical homomorphisms $G_i \to G_{i+1}$ extending the identities $U_w \to U_w$ and $V_{w'} \to V_{w'}$. Let G be the direct limit of the inductive system formed by the groups $(G_i)_{i \in \mathbb{N}}$ with the canonical homomorphisms $G_i \to G_{i+1}$. Then the following diagram commutes for every $i \in \mathbb{N}$ by definition:

$$\begin{array}{ccc} G_i & \xrightarrow{U_w \to U_w} & G_{i+1} \\ & \searrow & \downarrow \\ & & G \end{array}$$

Furthermore, the universal property of direct limits yields a unique homomorphism $f_i : G_i \to U_+$ extending the identities $U_w \to U_w$ and $V_{w'} \to V_{w'} \leq U_{w'}$. Thus the following diagram commutes:

$$\begin{array}{ccc} G_i & \xrightarrow{U_w \to U_w} & G_{i+1} \\ & \overset{f_i}{\searrow} & \downarrow{\scriptstyle f_{i+1}} \\ & & U_+ \end{array}$$

Again, the universal property of direct limits yields a unique homomorphism $f : G \to U_+$ such that the following diagram commutes for every $i \in \mathbb{N}$:

$$\begin{array}{ccc} G_i & \longrightarrow & G \\ & \overset{f_i}{\searrow} & \downarrow{\scriptstyle f} \\ & & U_+ \end{array}$$

(6.9.3) Remark. By Remark (6.9.2), the group G_i is generated by the set $\{x_\alpha \mid \alpha \in \Phi_+, \exists w \in C_i : w \notin \alpha\}$. We let $x_{\alpha,i}$ be the elements in G under the homomorphism $G_i \to G$. Then G is generated by $\{x_{\alpha,i} \mid i \in \mathbb{N}, \alpha \in \Phi_+, \exists w \in C_i : w \notin \alpha\}$. By construction we have $x_{\alpha,i} = x_{\alpha,i+1}$ in G for every $i \in \mathbb{N}$. Thus G is generated by $\{x_\alpha \mid \alpha \in \Phi_+\}$.

(6.9.4) Lemma. *The homomorphism $f : G \to U_+$ is an isomorphism.*

Proof. By Remark (6.9.3) we have $G = \langle x_\alpha \mid \alpha \in \Phi_+ \rangle$. We will construct a homomorphism $U_+ \to G$ which extends $U_w \to U_w$. For every $w \in W$ we have a canonical homomorphism $U_w \to G$. Suppose $w \in W$ and $s \in S$ with $\ell(ws) = \ell(w) + 1$. Then the following diagram commutes:

$$\begin{array}{ccc} U_w & \longrightarrow & U_{ws} \\ & \searrow & \downarrow \\ & & G \end{array}$$

The universal property of direct limits yields a homomorphism $h : U_+ \to G$ extending the identities on $U_w \to U_w$. As both concatenations $f \circ h$ and $h \circ f$ are the identities on each generator x_α, the uniqueness of such a homomorphism implies that $f \circ h = \mathrm{id}_{U_+}$ and $h \circ f = \mathrm{id}_G$. In particular, f is an isomorphism. □

(6.9.5) Lemma. *For any $P \in \mathcal{T}_i$ we have a canonical homomorphism $H_P \to G_i$.*

Proof. Suppose $S = \{r, s, t\}$. We distinguish the following cases:

$P \in \mathcal{T}_{i,1}$: Let $\{s, t\}$ be the type of P. By Remark (6.9.2) it suffices to show that C_i contains the elements $w_P sr_{\{r,t\}}, w_P r_{\{s,t\}}, w_P tr_{\{r,s\}}$. Note that $\ell(w_P) = i$. If $i = 0$, the claim follows. Thus we can assume $i > 0$ and hence $\ell(w_P r) = i - 1$. But then $w_P sr_{\{r,t\}} \in C(R_{\{r,s\}}(w_P)) \subseteq C_i$ and $w_P tr_{\{r,s\}} \in C(R_{\{r,t\}}(w_P)) \subseteq C_i$. If $i = 1$, we have $w_P r_{\{s,t\}} \in C_0 \subseteq C_1$ and we are done. If $i > 1$, we have $i - 2 \in \{\ell(w_P rs), \ell(w_P rt)\}$. Without loss of generality we assume $\ell(w_P rs) = i - 2$. Then $w_P r_{\{s,t\}} \in C(R_{\{r,s\}}(w_P)) \subseteq C_{i-1} \subseteq C_i$ and the claim follows.

$P \in \mathcal{T}_{i,2}$: Suppose $P = \{R, R'\}$, where R is of type $\{r, s\}$ and R' is of type $\{r, t\}$. Moreover, we define $T := R_{\{r,t\}}(w_R)$ and $T' := R_{\{r,s\}}(w_{R'})$. Again, and using symmetry, it suffices to show that $w_T rtrr_{\{s,t\}}, w_T trtr_{\{r,s\}}, w_T trr_{\{s,t\}}, w_R r_{\{r,s\}} \in C_i$. We define $Z := R_{\{s,t\}}(w_R)$. Note that $\ell(w_Z) = i - 3$ and hence $w_R r_{\{s,t\}} \in C(Z) \subseteq C_{i-2} \subseteq C_i$. Moreover, we have $\ell(w_T) = i - 1$ and hence $w_T rtrr_{\{s,t\}}, w_T trtr_{\{r,s\}}, w_T trr_{\{s,t\}} \in C(T) \subseteq C_i$. This finishes the claim. □

(6.9.6) Definition. (a) The group G_i is called *natural* if the following axioms are satisfied:

(N1) For all $w \in C_i, w' \in D_i$ the canonical homomorphisms $U_w, V_{w'} \to G_i$ are injective.

(N2) For every $P \in \mathcal{T}_i$ the homomorphism $H_P \to G_i$ from Lemma (6.9.5) is injective.

(b) If G_i is natural, then we define the tree product $B_P := G_i \star_{H_P} G_P$ for every $P \in \mathcal{T}_i$ (cf. (N2), Lemma (6.3.1) and Lemma (6.7.1)).

(6.9.7) Lemma. *For $i \in \{0, 1\}$ the group G_i satisfies (N1). Moreover, for all $s \neq t \in S$ the canonical homomorphism $V_{R_{\{s,t\}}(1_W),s} \to G_i$ is injective.*

Proof. We abbreviate $R := R_{\{s,t\}}(1_W)$. Before we prove the claim we show that we have a canonical homomorphism $V_{R,s} \to G_i$. By Remark (6.9.2) it suffices to show that $srs, tr \in C_i$. But this is true, as $srs, tr \in C_0 \subseteq C_i$.

Now we prove the claim. Let $\mathcal{D} = (\mathcal{G}, (U_\alpha)_{\alpha \in \Phi})$ be the RGD-system associated with the split Kac-Moody group of type $(4, 4, 4)$ over $\mathbb{F}_2$ as in Example (5.3.1). We first show that we have canonical homomorphisms $U_w \to \mathcal{G}$ for each $w \in C_i$. Suppose $\alpha \in \Phi_+$ with $w \notin \alpha$. We show that the canonical mappings $x_\alpha \mapsto x_\alpha \in U_\alpha$ extend to homomorphisms $U_w \to \mathcal{G}$. Let $\{\alpha, \beta\}$ be a pair of prenilpotent positive roots, let $w \in C_i$ and let $G \in \mathrm{Min}(w)$ be such that $\alpha \leq_G \beta \in \Phi(G)$. Suppose $o(r_\alpha r_\beta) < \infty$. As $\mathcal{M}$ is locally Weyl-invariant, we have

$$M^G_{\alpha,\beta} = \begin{cases} (\alpha, \beta) & \text{if } |(\alpha, \beta)| = 2, \\ \emptyset & \text{else.} \end{cases}$$

We have seen in Example (5.3.1) that $[x_\alpha, x_\beta] = \prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma$ is also a relation in $\mathcal{G}$. Suppose now $o(r_\alpha r_\beta) = \infty$ and hence $\alpha \subsetneq \beta$. As $w \in C_i$ and $i \in \{0, 1\}$, we deduce $(\alpha, \beta) = \emptyset$ and hence $[x_\alpha, x_\beta] = \prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma = 1$ does also hold in $\mathcal{G}$ by Example (5.3.1). This implies that the mappings $x_\alpha \mapsto x_\alpha$ extend to a homomorphism $U_w \to \mathcal{G}$. To show that the mappings $x_\alpha \mapsto x_\alpha$

do also extend to a homomorphism $V_{w_R r_{\{u,v\}}} \to \mathcal{G}$, we have to show that the subgroup in $\mathcal{G}$ generated by $x_{w_R\alpha_u}, x_{w_R\alpha_v}$ has at most 8 elements. As this is true, $x_\alpha \mapsto x_\alpha$ extend to a homomorphism $V_{w_R r_{\{u,v\}}} \to \mathcal{G}$. By definition the following diagrams commute:

$$\begin{array}{ccc} U_w & \longrightarrow & U_{wu} \\ & \searrow & \downarrow \\ & & \mathcal{G} \end{array} \qquad \begin{array}{ccc} U_{w_R u} & \longrightarrow & V_{w_R r_{\{u,v\}}} \\ & \searrow & \downarrow \\ & & \mathcal{G} \end{array}$$

The universal property of direct limits yields a unique homomorphism $G_i \to \mathcal{G}$ extending $U_w, V_{w'} \to \mathcal{G}$. Note that $V_{R,s} \to \mathcal{G}$ is an injective homomorphism by Theorem (5.3.5). The following diagram commutes:

$$\begin{array}{ccc} V_{R,s} & \longrightarrow & G_i \\ & \searrow & \downarrow \\ & & \mathcal{G} \end{array}$$

As $V_{R,s} \to \mathcal{G}$, the homomorphism $V_{R,s} \to G_i$ is also injective and we are done. □

(6.9.8) Definition. (a) We let

$$C_{-1} = \bigcup_{s \neq t \in S} C(r_{\{s,t\}}) \quad \text{and} \quad D_{-1} := \{w_R r_{\{s,t\}} \mid R \text{ is of type } \{s,t\}, w_R s, w_R t \in C_{-1}\}$$

and define G_{-1} to be the direct limit of the groups $U_w, V_{w'}$ with $w \in C_{-1}, w' \in D_{-1}$ as in Definition (6.9.1).

(b) For $S = \{r, s, t\}$ we let

$$C_r := C(r_{\{r,s\}}) \cup C(r_{\{r,t\}}) \quad \text{and} \quad D_r := \{w_R r_{\{s,t\}} \mid R \text{ is of type } \{s,t\}, w_R s, w_R t \in C_r\}$$

and define $G_{\{s,t\}}$ to be the direct limit of the groups $U_w, V_{w'}$ with $w \in C_r, w' \in D_r$ as in Definition (6.9.1).

(6.9.9) Remark. We note that there are nine roots $\alpha \in \Phi_+$ with the property that there exists $w \in C_{-1}$ such that $w \notin \alpha$. Moreover, G_{-1} is generated by $x_{\alpha,\{s,t\}}$ where $\alpha \in \Phi_+$ and $r_{\{s,t\}} \notin \alpha$. Thus G_{-1} is generated by twelve elements. As $x_{\alpha_s,\{r,s\}} = x_{\alpha_s,\{s,t\}}$ in G_{-1} for $S = \{r, s, t\}$, we deduce that G_{-1} is generated by nine elements. In particular, the generator $x_{\alpha,w}$ does not depend on w. A similar result holds for $G_{\{s,t\}}$, which is generated by seven elements.

(6.9.10) Lemma. *Let $s \neq t \in S$ and let $R := R_{\{s,t\}}(1_W)$. Then $V_{R,s} \to G_{\{s,t\}}$ is injective and $G_{-1} \cong G_{\{s,t\}} \star_{V_{R,s}} O_{R,s}$.*

Proof. As before, the assignments $x_\alpha \mapsto x_\alpha$ extend to homomorphisms $\pi : G_{\{s,t\}} \to G_0$ and $G_{\{s,t\}} \to G_{-1}$. Note that $srs, tr \in C_r \subseteq C_0$ and hence we have canonical homomorphisms $\varphi : V_{R,s} \to G_{\{s,t\}}$ and $\psi : V_{R,s} \to G_0$. As $\psi = \pi \circ \varphi$, Lemma (6.9.7) implies that φ is injective. We abbreviate $H := G_{\{s,t\}} \star_{V_{R,s}} O_{R,s}$ (cf. Lemma (6.2.1)). Note that for each $w \in C(srs) \cup C(tr)$ the following diagram commutes:

$$\begin{array}{ccc} U_w & \longrightarrow & O_{R,s} \\ \downarrow & & \downarrow \\ G_{\{s,t\}} & \longrightarrow & G_{-1} \end{array}$$

The universal property of direct limits implies that there exists a unique homomorphism $H \to G_{-1}$. Now we want to construct a homomorphism $G_{-1} \to H$. Suppose that $S = \{r, s, t\}$. At first we recall that $G_{\{s,t\}}$ is generated by the seven elements $\{x_{\alpha,G} \mid \alpha \in \Phi_+, \exists w \in C_r : w \notin \alpha\}$ and $O_{R,s}$ is generated by the seven elements $\{x_{\alpha,O} \mid \alpha \in \Phi_+, \exists w \in \{srs, r_{\{s,t\}}, tr\} : w \notin \alpha\}$. In H we have $x_{\alpha,G} = x_{\alpha,O}$ for $\alpha \in \{\alpha_s, \alpha_t, s\alpha_r, sr\alpha_s, t\alpha_r\}$. Thus H is generated by nine elements and we have a bijection between the set of generators of H and the set of roots contained in $\{\alpha \in \Phi_+ \mid \exists w \in C_{-1} : w \notin \alpha\}$. For $w \in C_r, w' \in D_r$ we have canonical homomorphisms $U_w, V_{w'} \to G_{\{s,t\}} \to H$. For $w \in C_{-1} \backslash C_r, w' \in D_{-1} \backslash D_r$ we have canonical homomorphisms $U_w, V_{w'} \to O_{R,s} \to H$. The universal property of direct limits yields a unique homomorphism $G_{-1} \to H$ extending the identities $U_w \to U_w \leq H$ and $V_{w'} \to V_{w'} \leq H$.

Note that the concatenations of $H \to G_{-1}$ and $G_{-1} \to H$ fix all generators and hence they must be the identities. In particular, $H \to G_{-1}$ is an isomorphism. □

(6.9.11) Lemma. *For $R := R_{\{s,t\}}(r)$ the canonical homomorphisms $V_R, V_{R,s} \to G_{-1}$ are injective. For $D_R := G_{-1} \star_{V_R} O_R$ we obtain $D_R \cong G_{-1} \star_{V_{R,s}} O_{R,s}$. Moreover, we have $G_0 \cong \star_{G_{-1}} D_T$, where T runs over $\mathcal{R}_1$.*

Proof. To show that we have canonical homomorphisms $V_R, V_{R,s} \to G_{-1}$ it suffices to check that $rsrs, rtr \in C_{-1}$. But this holds by definition.

Let $T := R_{\{r,s\}}(1_W)$. By definition we have $O_{T,r} = V_{sr_{\{r,t\}}} \hat{\star} U_{r_{\{r,s\}}} \hat{\star} V_{rr_{\{s,t\}}} \hat{\star} U_{rtr}$ and $V_{R,s} = U_{r_{\{r,s\}}} \hat{\star} V_{rr_{\{s,t\}}} \hat{\star} U_{rtr}$. Using Proposition (1.8.1) we obtain that $V_{R,s} \to O_{T,r}$ is injective. Using Lemma (6.2.1) and Lemma (6.9.10), we obtain that each of the canonical homomorphisms $V_R \to V_{R,s} \to O_{T,r} \to G_{-1}$ is injective. Using Proposition (1.8.1), Remark (1.8.6), Lemma (1.8.7), Lemma (6.1.2) and Lemma (6.2.1) we obtain the following isomorphisms:

$$G_{-1} \star_{V_R} O_R \cong G_{-1} \star_{V_{R,s}} V_{R,s} \star_{V_R} O_R \cong G_{-1} \star_{V_{R,s}} (V_{R,s} \star_{V_R} O_R) \cong G_{-1} \star_{V_{R,s}} O_{R,s}$$

It remains to show that $G_0 \cong \star_{G_{-1}} D_T$. Let $R \in \mathcal{R}_1$ be of type $\{s,t\}$. To see that we have a canonical homomorphism $O_R \to G_0$, it suffices to show that $rsr, rr_{\{s,t\}}, rtr \in C_0$. But this holds by definition. Using Remark (6.9.2) and Remark (6.9.9), we obtain a canonical homomorphism $\star_{G_{-1}} D_T \to G_0$, where T runs over $\mathcal{R}_1$. Note that $\star_{G_{-1}} D_T$ is generated by the elements $x_\alpha, x_{\beta,T}$, where $C_{-1} \not\subseteq \alpha \in \Phi_+$ and $T \in \mathcal{R}_1$ is such that $T = R_{\{s,t\}}(r)$ and $\{rsr, rr_{\{s,t\}}, rtr\} \not\subseteq \beta \in \Phi_+$. Note that if $\{rsr, rtr\} \not\subseteq \beta \in \Phi_+$, then $x_\beta = x_{\beta,T}$ holds in $\star_{G_{-1}} D_T$. Thus $\star_{G_{-1}} D_T$ is generated by the elements $x_\alpha, x_{\beta,T}$, where $C_{-1} \not\subseteq \alpha \in \Phi_+$ and $T \in \mathcal{R}_1$ is such that β is a non-simple root of T.

Let $T := R_{\{r,s\}}(t)$ and $T' := R_{\{s,t\}}(r)$. Then $-\alpha_r$ is contained in both non-simple roots of T by Lemma (5.1.4) and, moreover, α_r is contained in both non-simple roots of T'. In particular, let $T \neq T' \in \mathcal{R}_1$, let α be a non-simple root of T and let β be a non-simple root of T', then $-\alpha \subsetneq \beta$. This implies that the group $\star_{G_{-1}} D_T$, where T runs over $\mathcal{R}_1$, is generated by 15 elements and there is a bijection between the generators of $\star_{G_{-1}} D_T$ and the set $\{\alpha \in \Phi_+ \mid C_0 \not\subseteq \alpha\}$. Hence the mappings $x_\alpha \mapsto x_\alpha$ extend to homomorphisms $U_w, V_{w'} \to \star_{G_{-1}} D_T$ for $w \in C_0$ and $w' \in D_0$. Note that the following diagrams commute:

$$\begin{array}{ccc} U_w & \longrightarrow & U_{ws} \\ & \searrow & \downarrow \\ & & \star_{G_{-1}} D_T \end{array} \qquad \begin{array}{ccc} U_{w_R s} & \longrightarrow & V_{w_R r_{\{s,t\}}} \\ & \searrow & \downarrow \\ & & \star_{G_{-1}} D_T \end{array}$$

The universal property of direct limits yields a unique homomorphism $G_0 \to \star_{G_{-1}} D_T$ extending $U_w, V_{w'} \to \star_{G_{-1}} D_T$. As the concatenations of $\star_{G_{-1}} D_T \to G_0$ and $G_0 \to \star_{G_{-1}} D_T$ fix x_α, both concatenations are the identities and hence both homomorphisms are isomorphisms inverse to each other. □

(6.9.12) Lemma. *For all $i \in \mathbb{N}$ and $w \in C_{i+1} \backslash C_i$ there exists a unique $P \in \mathcal{T}_i$ with $w \in C(P)$.*

Proof. The existence follows by definition of C_{i+1}. Before we prove the uniqueness, suppose $P \in \mathcal{T}_i$ with $w \in C(P) \backslash C_i$. We distinguish the following two cases:

$P \in \mathcal{T}_{i,1}$: Let P be of type $\{s,t\}$ and let γ, δ be the non-simple roots of P. As $C(P) = C(w_P str_{\{r,s\}}) \cup C(w_P r_{\{s,t\}} rtr) \cup C(w_P r_{\{s,t\}} rsr) \cup C(w_P tsr_{\{r,t\}})$ and $w_P r_{\{s,t\}} \in C_i$ by induction, we infer $C(w) \cap \{w_P st, w_P ts\} \neq \emptyset$. But this implies $w \in (-\gamma) \cup (-\delta)$. Moreover, for $\varepsilon \in \{\gamma, \delta\}$ we have a unique rank 2 residue R_ε containing P_ε.

$P \in \mathcal{T}_{i,2}$: Suppose $P = \{R, R'\}$, where R is of type $\{r,s\}$ and R' is of type $\{r,t\}$, and we let γ, ε (resp. δ, ε) be the non-simple roots of R (resp. R'). As $C(P) = C(R) \cup C(R')$ and $w_R r_{\{r,s\}}, w_{R'} r_{\{r,t\}} \in C_i$ by induction, it follows similarly as in (i) that $C(w) \cap \{w_R rs, w_R sr = w_{R'} tr, w_{R'} rt\} \neq \emptyset$. Again, this implies $w \in (-\gamma) \cup (-\varepsilon) \cap (-\delta)$. Since $\gamma = w_R r\alpha_s, \delta = w_{R'} r\alpha_t = w_R str\alpha_t$ and $\varepsilon = w_R s\alpha_r$, it follows

$$\begin{aligned}\gamma \cap \delta \cap (W \backslash \{w_R sr\}) \subseteq \varepsilon &\Leftrightarrow r\alpha_s \cap str\alpha_t \cap (W \backslash \{sr\}) \subseteq s\alpha_r \\ &\Leftrightarrow tstr\alpha_s \cap stsr\alpha_t \cap (W \backslash \{r_{\{s,t\}} r\}) \subseteq r_{\{s,t\}} \alpha_r\end{aligned}$$

Now Lemma (5.1.3) implies $W \backslash ((-\gamma) \cup (-\delta) \cup \{w_R sr\}) = \gamma \cap \delta \cap (W \backslash \{w_R sr\}) \subseteq \varepsilon$. But this implies $(-\varepsilon) \subseteq (-\gamma) \cup (-\delta) \cup \{w_R sr\}$ and, as $w_R sr \in C_i$, we have $w \neq w_R sr$ and hence $w \in (-\gamma) \cup (-\delta)$. Moreover, for $\varepsilon \in \{\gamma, \delta\}$ we have a unique rank 2 residue R_ε containing P_ε.

In both cases we have $w \in (-\gamma) \cup (-\delta)$ and hence $w \notin \gamma \cap \delta$. Now we will show that P is unique with the required property. Assume that $P \neq Q \in \mathcal{T}_i$ does also satisfy the property. Let δ_P, γ_P and δ_Q, γ_Q be the non-simple roots as before. We note that for each $\varepsilon \in \{\delta_P, \gamma_P, \delta_Q, \gamma_Q\}$ there is a unique residue rank 2 residue R_ε such that ε is a non-simple root of R_ε.

Assume $\delta_P = \delta_Q$. Then we have $R_{\delta_P} = R_{\delta_Q}$. If $P \in \mathcal{T}_{i,1}$, then $P = R_{\delta_P} = R_{\delta_Q}$. Moreover, $Q \in \mathcal{T}_{i,2}$ would imply $R_{\delta_Q} \in Q$, which is a contradiction to $R_{\delta_Q} \in \mathcal{T}_{i,1}$. Thus $Q \in \mathcal{T}_{i,1}$ and $P = R_{\delta_Q} = Q$. But this is a contradiction. If $P \in \mathcal{T}_{i,2}$, then $R_{\delta_P} = R_{\delta_Q} \in P$. In particular, we have $R_{\delta_Q} \notin \mathcal{T}_{i,1}$. As $Q \in \mathcal{T}_{i,1}$ would imply $Q = R_{\delta_Q}$, we deduce $Q \in \mathcal{T}_{i,2}$ and $R_{\delta_Q} \in Q$. But $R_{\delta_Q} \in P \cap Q \neq \emptyset$ implies $P = Q$, which is again a contradiction. We infer that $|\{\delta_P, \gamma_P, \delta_Q, \gamma_Q\}| = 4$.

We have $w \in (-\delta_P) \cup (-\gamma_P)$ and $w \in (-\delta_Q) \cup (-\gamma_Q)$. Assume that non of $\{\delta_Q, \delta_P\}$, $\{\delta_Q, \gamma_P\}$, $\{\gamma_Q, \delta_P\}$, $\{\gamma_Q, \gamma_P\}$ is prenilpotent. Then [2, Lemma 8.42(3)] yields that each of $\{(-\delta_Q), \delta_P\}$, $\{(-\delta_Q), \gamma_P\}$, $\{(-\gamma_Q), \delta_P\}$, $\{(-\gamma_Q), \gamma_P\}$ is a pair of nested roots. Since $o(r_{\delta_P} r_{\gamma_P}), o(r_{\delta_Q} r_{\gamma_Q}) < \infty$, we deduce either $(-\delta_Q) \subseteq \delta_P, \gamma_P$, or else $\delta_P, \gamma_P \subseteq (-\delta_Q)$ (resp. $(-\gamma_Q) \subseteq \delta_P, \gamma_P$ or $\delta_P, \gamma_P \subseteq (-\gamma_Q)$). As $1_W \in \delta_P \cap \gamma_P \cap \delta_Q \cap \gamma_Q$, we cannot have $\delta_P, \gamma_P \subseteq (-\delta_Q), (-\gamma_Q)$ and hence $(-\delta_Q), (-\gamma_Q) \subseteq \delta_P, \gamma_P$. But this implies $w \in (-\delta_Q) \cup (-\gamma_Q) \subseteq \delta_P \cap \gamma_P$, which is a contradiction. Thus one of the previous pairs of roots must be prenilpotent. Without loss of generality we can assume that $\{\delta_P, \delta_Q\}$ is prenilpotent. Note that $P, Q \in \mathcal{T}_i$ and hence $k_\varepsilon = i + 2$ for every $\varepsilon \in \{\delta_Q, \delta_P, \gamma_Q, \gamma_P\}$ and $\{\delta_P, \delta_Q\}$ is not nested. If $\{-\delta_P, \delta_Q\}$ would be nested, [2, Lemma 8.42(3)] implies that $\{\delta_P, \delta_Q\}$ is not prenilpotent which is a contradiction. Thus $\{-\delta_P, \delta_Q\}$ is not nested and Lemma (1.4.7) yields $o(r_{\delta_P} r_{\delta_Q}) < \infty$.

Assume that $R_{\delta_P} \in \partial^2 \delta_Q$. We recall $k_{\delta_P} = k_{\delta_Q}$. If $R_{\delta_P} \in \mathcal{T}_{i,1}$, then $\delta_Q \in \{\delta_P, \gamma_P\}$, which is a contradiction. If $R_{\delta_P} \notin \mathcal{T}_{i,1}$, then we have $\delta_Q = \delta_P$ by definition of the roots δ_Q, γ_Q, which is again a contradiction. Thus we have $R_{\delta_P} \notin \partial^2 \delta_Q$.

Recall that δ_Q is a non-simple root by definition. Now we can apply Lemma (5.2.6). Assertion (b) would imply $\delta_Q = \gamma_P$, which is a contradiction. Thus we are in Case (a). Then $k_{\delta_P} = k_{\delta_Q}$ implies $i = 0$. Let $\{s,t\}$ be the type of P and let $\{r,s\}$ be the type of Q. Then we have $P = R_{\{s,t\}}(1_W)$ and $Q = R_{\{r,s\}}(1_W)$ as well we $\delta_Q = s\alpha_r, \gamma_Q = r\alpha_s$. It follows from Lemma (5.1.4) that $(-\alpha_r) \subseteq \delta_P \cap \gamma_P$ and hence $w \in (-\delta_P) \cup (-\gamma_P) \subseteq \alpha_r$. Note that $C(P) \subseteq (-t\alpha_s) \cup C(strsr) \cup \{t\} \subseteq \delta_Q$. Lemma (1.3.2) yields $\alpha_s \subseteq (-\alpha_r) \cup s\alpha_r$ and as (W, S) is of type $(4, 4, 4)$, we deduce $(-r\alpha_s) \subseteq (-s\alpha_r) \cup (-\alpha_r)$. This implies $\alpha_r \cap s\alpha_r \subseteq r\alpha_s$. But then $w \in \alpha_r \cap \delta_Q \subseteq \gamma_Q$, which is a contradiction to $w \notin \delta_Q \cap \gamma_Q$. Thus P is unique with the required property. ☐

(6.9.13) Lemma. *Let $i \in \mathbb{N}, P \in \mathcal{T}_i$ and $w \in C(P)$. Then there is a canonical homomorphism $U_w \to G_P$. In particular, this homomorphism is injective.*

Proof. We distinguish the following two cases:

$P \in \mathcal{T}_{i,1}$ Suppose that P is of type $\{s, t\}$. Then $C(P) = C(w_P str_{\{r,s\}}) \cup C(w_P r_{\{s,t\}} rtr) \cup C(w_P r_{\{s,t\}} rsr) \cup C(w_P tsr_{\{r,t\}})$. As $U_v \to U_{vs}$ is injective, we can assume $w \in \{w_P str_{\{r,s\}}, w_P r_{\{s,t\}} rtr, w_P r_{\{s,t\}} rsr, w_P tsr_{\{r,t\}}\}$. By definition of G_P and Proposition (1.8.1) we see that $U_w \to G_P$ is injective.

$P \in \mathcal{T}_{i,2}$ Suppose $P = \{R, R'\}$, where R is of type $\{r, s\}$ and R' is of type $\{r, t\}$. As in the previous case we can assume that $w \in \{w_R rsr_{\{r,t\}}, w_R r_{\{r,s\}} tst, w_R r_{\{r,s\}} trt, w_R srr_{s,t}\} \cup \{w_{R'} trr_{\{s,t\}}, w_{R'} r_{\{r,t\}} srs, w_{R'} r_{\{r,t\}} sts, w_{R'} rtr_{\{r,s\}}\}$. Again, the claim follows from the definition of G_P together with Proposition (1.8.1). ☐

(6.9.14) Definition. For $i \in \mathbb{N}$ and $P \in \mathcal{T}_i$ we let $C'(P) \subseteq W$ be the set of all $w \in W$ such that U_w is a vertex group of G_P.

(6.9.15) Lemma. *For $i \in \mathbb{N}$ and $P \in \mathcal{T}_i$, we have $C'(P) \subseteq C_{i+1}$.*

Proof. We distinguish the following two cases:

$P \in \mathcal{T}_{i,1}$: Suppose that P is of type $\{s, t\}$. Then $C'(P) \subseteq C(P) \cup \{w_P sr_{\{r,t\}}, w_P tr_{\{r,s\}}\}$. By definition, we have $C(P) \subseteq C_{i+1}$ and (using symmetry) it suffices to show that $w_P sr_{\{r,t\}} \in C_{i+1}$. For $i = 0$ we have $w_P sr_{\{r,t\}} \in C_0 \subseteq C_1$ and we are done. For $i > 0$ we have $w_P sr_{\{r,t\}} \in C(R_{\{r,s\}}(w_P)) \subseteq C_i \subseteq C_{i+1}$ and the claim follows.

$P \in \mathcal{T}_{i,2}$: Suppose $P = \{R, R'\}$, where R is of type $\{r, s\}$ and R' is of type $\{r, t\}$. As in the previous case it suffices to show that $\{w_R rtrsts, w_R rtrsrs, w_R rr_{\{s,t\}}\} \subseteq C_{i+1}$. As $R_{\{r,t\}}(w_R) \in \mathcal{R}_{i-1}$, it follows that $\{w_R rtrsts, w_R rtrsrs, w_R rr_{\{s,t\}}\} \subseteq C_i \subseteq C_{i+1}$ and the claim follows. ☐

(6.9.16) Lemma. *For $P \in \mathcal{T}_i$ we let δ_P, γ_P be the roots as in Lemma (6.9.12). Moreover, we let R_ε be the unique residue of rank 2 containing P_ε for $\varepsilon \in \{\delta_P, \gamma_P, \delta_Q, \gamma_Q\}$. Then following hold:*

(a) For $i > 0$ and $P \neq Q \in \mathcal{T}_i$, we have $(-\delta_P), (-\gamma_P) \subseteq \delta_Q, \gamma_Q$.

(b) Suppose $P \in \mathcal{T}_i$ and $Q \in \mathcal{T}_{i-1}$. For $\varepsilon \in \{\delta_Q, \gamma_Q\}, \varepsilon' \in \{\delta_P, \gamma_P\}$ we have $(-\varepsilon) \subseteq \varepsilon'$ or $R_\varepsilon \cap R_{\varepsilon'}$ is a panel containing w_{R_ε}.

Proof. To prove (a) it suffices to show $(-\delta_P) \subseteq \delta_Q$. We see as in Lemma (6.9.12) that $|\{\delta_P, \gamma_P, \delta_Q, \gamma_Q\}| = 4$. Assume $(-\delta_P) \not\subseteq \delta_Q$. Then $\{\delta_P, \delta_Q\} \in \mathcal{P}$. As $k_{\delta_P} = k_{\delta_Q}$, we have $o(r_{\delta_P} r_{\delta_Q}) < \infty$. As $R_{\delta_P} \notin \partial^2 \delta_Q$, Lemma (5.2.6)$(b)$ would imply $\delta_Q = \gamma_P$, which is a contradiction. Lemma (5.2.6)(a) implies $i = 0$ because of $k_{\delta_P} = k_{\delta_Q}$, which is also a contradiction.

To show (b) we argue similar as in (a). Assume that $(-\delta_Q) \not\subseteq \delta_P$. Then $\{\delta_Q, \delta_P\} \in \mathcal{P}$ and as $k_{\delta_Q} = k_{\delta_P} - 1$, we deduce $o(r_{\delta_Q} r_{\delta_P}) < \infty$. If $R_{\delta_P} \in \partial^2 \delta_Q$, then (as $k_{\delta_Q} = k_{\delta_P} - 1$)

$P_{\delta_Q} = R_{\delta_P} \cap R_{\delta_Q}$ is a panel. Thus we can assume $R_{\delta_P} \notin \partial^2 \delta_Q$. Then we can apply Lemma (5.2.6). As (b) does not apply, we obtain again (using $k_{\delta_Q} = k_{\delta_P} - 1$) that $R_{\delta_P} \cap R_{\delta_Q}$ is a panel. $\square$

(6.9.17) Definition. Let $i \in \mathbb{N}$ and let $R \in \mathcal{R}_i$ be a residue of type $\{s, t\}$. We let $\hat{\Phi}_R$ be the set of all non-simple roots of $R_{\{r,s\}}(w_R st), R_{\{r,t\}}(w_R r_{\{s,t\}}), R_{\{r,s\}}(w_R r_{\{s,t\}}), R_{\{r,t\}}(w_R ts)$. If $P := \{R, R'\} \in \mathcal{T}_i$, then we define $\hat{\Phi}_P := \hat{\Phi}_R \cup \hat{\Phi}_{R'}$.

(6.9.18) Lemma. *Let $i \in \mathbb{N}, R \in \mathcal{R}_i$ and let $\alpha \in \hat{\Phi}_R$ be a root. Then we have $C_i \subseteq \alpha$.*

Proof. Let R be of type $\{s, t\}$ and suppose $S = \{r, s, t\}$. We note that $C(P) \subseteq C_i \cup (-\delta_P) \cup (-\gamma_P)$ for $P \in \mathcal{T}_i$, where δ_P, γ_P are as in Lemma (6.9.12). For a residue $T \in \mathcal{R}_i$ we denote by $P_T \in \mathcal{T}_i$ the unique element with $P_T = T$ or $T \in P_T$. We prove the hypothesis by induction on i. For $i = 0$ it is not hard to see that

$$C_0 = \bigcup_{S=\{r,s,t\}} C(r_{\{s,t\}}) \cup C(rr_{\{s,t\}}) \subseteq \alpha$$

Thus we can assume $i > 0$ and hence $\ell(w_R r) = \ell(w_R) - 1$. We have $C_i = C_{i-1} \cup \bigcup_{P \in \mathcal{T}_{i-1}} C(P)$. We denote by α_R, β_R the two non-simple roots of R and note that $\alpha_R \subseteq \alpha$ or $\beta_R \subseteq \alpha$ holds. We distinguish the following two cases:

(a) $\ell(w_R rs) = \ell(w_R) - 2 = \ell(w_R rt)$: Let $P := \{T, T'\} \in \mathcal{T}_{i-2,2}$, where $T := R_{\{r,s\}}(w_R)$ and $T' := R_{\{r,t\}}(w_R)$. As $\alpha_R, \beta_R \in \hat{\Phi}_T$, the induction hypothesis yields $C_{i-2} \subseteq \alpha_R \cap \beta_R \subseteq \alpha$. We observe the following:

$$C_i = C_{i-1} \cup \bigcup_{Z \in \mathcal{T}_{i-1}} C(Z) = C_{i-2} \cup \bigcup_{Z \in \mathcal{T}_{i-2}} C(Z) \cup \bigcup_{Z \in \mathcal{T}_{i-1}} C(Z) \subseteq \alpha \cup \bigcup_{Z \in \mathcal{T}_{i-1} \cup \mathcal{T}_{i-2}} C(Z)$$

- $Z \in \mathcal{T}_{i-2}$: If $Z \neq P$, then Lemma (6.9.16)(a) and Lemma (5.1.3) imply that $C(Z) \subseteq C_{i-2} \cup (\delta_P \cap \gamma_P) \subseteq \alpha \cup w_R r \alpha_r \cup C(w_R) \subseteq \alpha$. Now we consider $Z = P$. Note that $w_T \alpha_s, w_{T'} \alpha_t, (-w_R srt \alpha_s), (-w_R trs \alpha_t) \subseteq \alpha_R, \beta_R$ and it suffices to show that $w_R sr_{\{r,t\}}, w_R r_{\{s,t\}}, w_R tr_{\{r,s\}} \in \alpha$. As $-w_R strt \alpha_r, -w_R tsrs \alpha_r \subseteq \alpha, w_R r_{\{s,t\}} \in \alpha$ and roots are convex, we deduce $C(P) \subseteq \alpha$.
- $Z \in \mathcal{T}_{i-1}$: Then Lemma (6.9.16)(b) implies $C(Z) \subseteq C_{i-1} \cup (\alpha_R \cap \beta_R) \subseteq C_{i-1} \cup \alpha$.

We conclude the following:

$$C_i = C_{i-1} \cup \bigcup_{Z \in \mathcal{T}_{i-1}} C(Z) \subseteq C_{i-1} \cup \alpha = C_{i-2} \cup \bigcup_{Z \in \mathcal{T}_{i-2}} C(Z) \cup \alpha \subseteq \alpha$$

(b) $\ell(w_R) \in \{\ell(w_R rs), \ell(w_R rt)\}$: Without loss of generality we can assume $\ell(w_R rt) = \ell(w_R)$. We distinguish the following two cases:

(i) $\ell(w_R rs) = \ell(w_R) = \ell(w_R rt)$: Then $\ell(w_R) = 1$ and $R = R_{\{s,t\}}(r)$. Clearly, $rr_{\{s,t\}} \in \alpha$. Using Lemma (5.1.4) we see that $\alpha_r, -\alpha_s, -\alpha_t \subseteq \alpha_R, \beta_R$ and, as roots are convex, we deduce $C_0 \subseteq \alpha$. For $T := R_{\{s,t\}}(1_W)$ and $\beta \in \hat{\Phi}_T$ we have $-\beta \subseteq (-\delta_T) \cup (-\gamma_T) \subseteq \alpha_r \subseteq \alpha$ and hence $C(T) \subseteq \alpha$. Using symmetry it suffices to show that $C(R_{\{r,s\}}(1_W)) \subseteq \alpha$. As $srr_{\{s,t\}}, srsr_{\{r,t\}}, rsrr_{\{s,t\}} \in (-\alpha_s) \subseteq \alpha$, it suffices to show that $rsr_{\{r,t\}} \in \alpha$. It follows from Lemma (5.1.4) that $rsr_{\{r,t\}} \in (-s\alpha_r) \subseteq \alpha$.

(ii) $\ell(w_R rs) = \ell(w_R) - 2$: Define $T := R_{\{r,t\}}(w_R)$ and $T' := R_{\{r,s\}}(w_R)$. Lemma (6.9.16)(b) implies for $P \in \mathcal{T}_{i-1} \backslash \{P_T\}$:

$$C(P) \subseteq C_{i-1} \cup (-\delta_P) \cup (-\gamma_P) \subseteq C_{i-1} \cup (\alpha_R \cap \beta_R) \subseteq C_{i-1} \cup \alpha$$

Note that $-w_T\alpha_t, \subseteq \alpha_R, \beta_R$ and $C(w_R t r_{\{r,s\}}) \subseteq \alpha$. As roots are convex and $\alpha_R \cap \beta_R \subseteq \alpha$, this yields $C(P_T) \subseteq \alpha$. We deduce

$$C_i \subseteq C_{i-1} \cup \alpha = C_{i-2} \cup \bigcup_{P \in \mathcal{T}_{i-2}} C(P) \cup \alpha$$

We distinguish the following cases:

(1) $\ell(w_R rsr) = \ell(w_R) - 1$: If $i - 2 = 0$, then $C(P) \subseteq \alpha$ for all $P \in \mathcal{T}_0$ and $C_0 \subseteq \alpha_R \cap \beta_R \subseteq \alpha$ by induction. Thus we assume $i - 2 > 0$ and Lemma (6.9.16)(a) implies for $P \in \mathcal{T}_{i-2} \backslash \{P_{T'}\}$ (as $w_R r \alpha_r \in \{\delta_{T'}, \gamma_{T'}\}$):

$$C(P) \subseteq C_{i-2} \cup (-\delta_P) \cup (-\gamma_P) \subseteq C_{i-2} \cup w_R r \alpha_r \subseteq C_{i-2} \cup \alpha$$

As $\alpha_R, \beta_R \in \hat{\Phi}_{T'}$, we deduce $C_{i-2} \subseteq \alpha_R \cap \beta_R \subseteq \alpha$ by induction. Moreover, we have $w_{T'}\alpha_s, \left(-w_{T'} r_{\{r,s\}} t \alpha_s\right) \subseteq \alpha_R, \beta_R$ as well as $w_R s r_{\{r,t\}}, w_R r_{\{s,t\}} \in \alpha$. As roots are convex, we conclude $C(P_{T'}) \subseteq (\alpha_R \cap \beta_R) \cup \alpha \subseteq \alpha$. This yields the following:

$$C_i \subseteq C_{i-2} \cup \bigcup_{P \in \mathcal{T}_{i-2}} C(P) \cup \alpha \subseteq \alpha$$

(2) $\ell(w_R rsr) = \ell(w_R) - 3$: We let $X := R_{\{r,t\}}(w_{T'} s)$ and $Y := R_{\{s,t\}}(w_{T'} r)$.

- Suppose that $\ell(w_{T'} st) = \ell(w_{T'}) + 2$. We will show that $C(P_X) \subseteq \alpha$. Note that $w_{T'} r \alpha_s \subseteq \alpha_R, \beta_R$. This yields $C(P_X) \subseteq w_{T'} r \alpha_s \subseteq \alpha_R \cap \beta_R \subseteq \alpha$.
- Suppose that $\ell(w_{T'} rt) = \ell(w_{T'}) + 2$. Again we will show that $C(P_Y) \subseteq \alpha$. Note that $w_{T'} s \alpha_r \subseteq \alpha_R, \beta_R$. This yields $C(P_Y) \subseteq w_{T'} s \alpha_r \subseteq \alpha_R \cap \beta_R \subseteq \alpha$.

Note that $P_{T'} \in \mathcal{T}_{i-3}$ and $\alpha_R, \beta_R \in \hat{\Phi}_{T'}$. Thus the induction hypothesis implies $C_{i-3} \subseteq \alpha_R \cap \beta_R \subseteq \alpha$. We distinguish the following cases:

(aa) $T' \in \mathcal{T}_{i-3,1}$: Lemma (6.9.16)$(b)$ implies $C(Z) \subseteq C_{i-2} \cup (\delta_{T'} \cap \gamma_{T'}) \subseteq C_{i-2} \cup \alpha$ for all $Z \in \mathcal{T}_{i-2} \backslash \{P_X, P_Y\}$. We conclude

$$\bigcup_{P \in \mathcal{T}_{i-2}} C(P) \subseteq C_{i-2} \cup \alpha$$

We show now that $C(T') \subseteq \alpha$. First note that $w_{T'} s r r_{\{s,t\}}, w_{T'} r s r_{\{r,t\}} \subseteq \alpha_R, \beta_R$ and $w_{T'} s r s r_{\{r,t\}}, w_R r_{\{s,t\}} \subseteq \alpha$. This yields $C(T') \subseteq (\alpha_R \cap \beta_R) \cup \alpha \subseteq \alpha$. Now Lemma (6.9.16)$(a)$ yields the following for $P \in \mathcal{T}_{i-3} \backslash \{T'\}$:

$$C(P) \subseteq C_{i-3} \cup (\delta_{T'} \cap \gamma_{T'}) \subseteq \alpha$$
$$C_i \subseteq C_{i-2} \cup \alpha \subseteq C_{i-3} \cup \bigcup_{P \in \mathcal{T}_{i-3}} C(P) \cup \alpha \subseteq \alpha$$

(bb) $\ell(w_R rst) = \ell(w_R) - 3$: Define $Z := R_{\{r,t\}}(w_R rsts)$ and note that $X, Z \in \mathcal{T}_{i-2,1}$. We have already shown that $C(X) \subseteq \alpha$. Note that $-w_{T'} rt \alpha_s \subseteq \alpha_R, \beta_R$. As roots are convex, this implies $C(Z) \subseteq \alpha$. Lemma (6.9.16)(a) implies for $P \in \mathcal{T}_{i-3} \backslash \{P_{T'}\}$:

$$C(P) \subseteq C_{i-3} \cup w_R r \alpha_r \subseteq C_{i-3} \cup (\alpha_R \cap \beta_R) \subseteq \alpha$$

Now we consider $P = P_{T'}$. Note that $w_R r\alpha_r, (-w_R r\alpha_t), (-w_R srt\alpha_s) \subseteq \alpha_R, \beta_R$. Moreover, $w_R strt, w_R r_{\{s,t\}} \in \alpha$. As roots are convex, we obtain $C(P_{T'}) \subseteq \alpha$.

Lemma (6.9.16)(b) implies for $P \in \mathcal{T}_{i-2}\backslash\{X, Z\}$:

$$C(P) \subseteq C_{i-2} \cup w_R r\alpha_r \subseteq C_{i-2} \cup (\alpha_R \cap \beta_R) \subseteq C_{i-2} \cup \alpha$$

$$C_i \subseteq C_{i-2} \cup \bigcup_{P\in\mathcal{T}_{i-2}} C(P) \cup \alpha \subseteq C_{i-3} \cup \bigcup_{P\in\mathcal{T}_{i-3}} C(P) \cup \alpha \subseteq \alpha$$

(cc) $\ell(w_R rsrst) = \ell(w_R) - 3$: Define $Z := R_{\{s,t\}}(w_R rsrstr)$ and note that $Y, Z \in \mathcal{T}_{i-2,1}$. We have already shown that $C(Y) \subseteq \alpha$. As before, we note that $-w_{T'}st\alpha_r \subseteq w_{T'}r\alpha_s \subseteq \alpha_R, \beta_R$. This yields as before $C(Z) \subseteq \alpha$, as roots are convex. Lemma (6.9.16)(a) implies for $P \in \mathcal{T}_{i-3}\backslash\{P_{T'}\}$:

$$C(P) \subseteq C_{i-3} \cup w_R rs\alpha_s \subseteq C_{i-3} \cup (\alpha_R \cap \beta_R) \subseteq \alpha$$

Now we consider $P = P_{T'}$. Note that $w_R rs\alpha_s, (-w_R srt\alpha_s), (-w_R tr\alpha_t) \subseteq \alpha_R, \beta_R$ and $w_R strt \in \alpha$ as before. As roots are convex, we obtain $C(P_{T'}) \subseteq \alpha$.

Moreover, Lemma (6.9.16)(b) implies for $P \in \mathcal{T}_{i-2}\backslash\{Y, Z\}$:

$$C(P) \subseteq C_{i-2} \cup w_R rs\alpha_s \subseteq C_{i-2} \cup (\alpha_R \cap \beta_R) \subseteq C_{i-2} \cup \alpha$$

$$C_i \subseteq C_{i-2} \cup \bigcup_{P\in\mathcal{T}_{i-2}} C(P) \cup \alpha \subseteq C_{i-3} \cup \bigcup_{P\in\mathcal{T}_{i-3}} C(P) \cup \alpha \subseteq \alpha \qquad \square$$

(6.9.19) Lemma. *Let $i \in \mathbb{N}$ and $w' = w_R r_{\{s,t\}} \in D_{i+1}\backslash D_i$. Then there exists a unique $P \in \mathcal{T}_i$ with $w_R s, w_R t \in C'(P)$ and the canonical homomorphism $V_{w'} \to G_P$ is injective.*

Proof. We use in the proof a different notation than in the statement. We let $w' = w_T r_{\{u,v\}}$. As $w' \in D_{i+1}\backslash D_i$, we have $\{w_T u, w_T v\} \not\subseteq C_i$. Without loss of generality we assume $w_T u \notin C_i$. Using Lemma (6.9.12), we obtain a unique $P \in \mathcal{T}_i$ with $w_T u \in C(P)\backslash C_i$. Let $\beta \in \Phi_+$ be the root with $\{w_T, w_T v\} \in \partial\beta$. Assume that there exists $i < j \in \mathbb{N}$ and $Z \in \mathcal{R}_j$ with $\beta \in \hat{\Phi}_Z$. Then the previous lemma implies $C_{i+1} \subseteq C_j \subseteq \beta$, which is a contradiction to our assumption, as $w_T v \in C_{i+1} \notin \beta$. We distinguish the following cases:

$P \in \mathcal{T}_{i,1}$ Suppose that P is of type $\{s, t\}$. It suffices to consider the following cases:

$$w_T u \in \{w_R strsr, w_R str_{\{r,s\}}, w_R stsrtr, w_R r_{\{s,t\}} rtr\}$$

The symmetric case (interchanging s and t) follows similarly. The other cases follow from Proposition (1.8.1), as $V_{w'}$ is either a vertex group of G_P, or else is contained in the vertex group $U_{w'}$ of G_P. If $w_T u = w_R str_{\{r,s\}}$, then $\beta \in \hat{\Phi}_Z$ for $Z = R_{\{r,s\}}(w_R st)$. If $w_T u = w_R r_{\{s,t\}} rtr$, then $\beta \in \hat{\Phi}_Z$ for $Z = R_{\{r,t\}}(w_R sts)$. If $w_T u = w_R stsrtr$, then $\beta \in \hat{\Phi}_Z$ for $Z = R_{\{r,s\}}(w_R st)$. If $w_T u = w_R strsr$, then $\beta \in \hat{\Phi}_Z$, where $Z = R_{\{r,t\}}(w_R s)$.

$P \in \mathcal{T}_{i,2}$ Suppose $P = \{R, R'\}$, where R is of type $\{r, s\}$ and R' is of type $\{r, t\}$. Using exactly the same arguments, the claim follows as in the case $P \in \mathcal{T}_{i,1}$. $\square$

(6.9.20) Proposition. *Assume that G_i is natural for some $i \in \mathbb{N}$. Then $G_{i+1} \cong \star_{G_i} B_P$, where P runs over $\mathcal{T}_i$. In particular, the mappings $G_i \to G_{i+1}$ and $B_P \to G_{i+1}$ are injective for each $P \in \mathcal{T}_i$.*

Proof. Recall from Definition (6.9.6) that $B_P = G_i \star_{H_P} G_P$ for every $P \in \mathcal{T}_i$ and note that G_i, G_P are subgroups of B_P by Proposition (1.8.1). The second part follows from Proposition (1.8.1) and the first part. We let x_α be the generators of G_i, where $C_i \not\subseteq \alpha \in \Phi_+$, and we let $x_{\alpha,P}$ be the generators of G_P, where $C'(P) \not\subseteq \alpha \in \Phi_+$. We define $H_i := \star_{G_i} B_P$, where P runs over $\mathcal{T}_i$. Since we have canonical homomorphisms $G_i, G_P \to G_{i+1}$ extending $x_\alpha \mapsto x_\alpha$ and $x_{\alpha,P} \to x_\alpha$ (cf. Lemma (6.9.15)) which agree on H_P (cf. Remark (6.9.2)), we obtain a unique homomorphism $B_P \to G_{i+1}$. Moreover, we obtain a (surjective) homomorphism $H_i \to G_{i+1}$. Now we will construct a homomorphism $G_{i+1} \to H_i$. Before we do that, we consider the generators of H_i.

Let $\alpha \in \Phi_+$ and suppose $P \in \mathcal{T}_i$ with $C'(P) \not\subseteq \alpha$ and $C_i \not\subseteq \alpha$. Then x_α is a generator of G_i and $x_{\alpha,P}$ is a generator of G_P. Lemma (6.9.18) implies that $\alpha \notin \hat{\Phi}_P$ and by definition of H_P we have $x_\alpha = x_{\alpha,P}$ in G_P. Thus H_i is generated by the set $\{x_\alpha, x_{\beta,P} \mid C_i \not\subseteq \alpha \in \Phi_+, P \in \mathcal{T}_i, \beta \in \hat{\Phi}_P\}$. Note that if $P, Q \in \mathcal{T}_i$ and $\alpha \in \Phi_+$ are such that $C'(P) \not\subseteq \alpha, C'(Q) \not\subseteq \alpha$, then $P = Q$. This can be seen by using Lemma (6.9.16) for $i > 0$. In the case $i = 0$ it follows from Lemma (5.1.4) that if $P \neq Q$, then $-\beta \subsetneq \alpha$ for all $\beta \in \hat{\Phi}_P, \alpha \in \hat{\Phi}_Q$.

We need to construct for each $w \in W$ a homomorphism $U_w \to H_i$. We start by defining a mapping from the generators $x_{\alpha,w}$ of U_w to H_i. Let $\alpha \in \Phi_+$ be a root and let $w \in C_{i+1}$ with $w \notin \alpha$. If $C_i \not\subseteq \alpha$, we define $x_{\alpha,w} \mapsto x_\alpha$. If $C_i \subseteq \alpha$, then $w \notin C_i$ and there exists a unique $P \in \mathcal{T}_i$ with $w \in C(P)$ by Lemma (6.9.12). We define $x_{\alpha,w} \mapsto x_{\alpha,P}$.

If $w \in C_i$, then we have a canonical homomorphism $U_w \to G_i \to H_i$. Thus we assume $w \notin C_i$. As before, there exists a unique $P \in \mathcal{T}_i$ such that $w \in C(P)$. We have already shown that for each $\alpha \in \Phi_+$ with $w \notin \alpha$ and $C_i \not\subseteq \alpha$, we have $x_\alpha = x_{\alpha,P}$ in B_P. Thus these mappings extend to homomorphisms $U_w \to G_P \to H_i$. Now suppose $w' = w_R r_{\{s,t\}} \in D_{i+1}$ for some R of type $\{s,t\}$. We have to show that the homomorphisms $U_{w_R s}, U_{w_R t} \to H_i$ extend to a homomorphism $V_{w'} \to H_i$. If $w' \in D_i$, this holds by definition of G_i. If $w' \notin D_i$, then Lemma (6.9.19) implies that there exists a unique $P \in \mathcal{T}_i$ with $\{w_R s, w_T t\} \subseteq C'(P)$ and $V_{w'} \to G_P$ is injective. In particular, $V_{w'} \to H_i$ is an injective homomorphism. Moreover, following diagrams commute, where R is a residue of type $\{s,t\}$:

$$\begin{array}{ccc} U_w & \longrightarrow & U_{ws} \\ & \searrow & \downarrow \\ & & H_i \end{array} \qquad\qquad \begin{array}{ccc} U_{w_R s} & \longrightarrow & V_{w_R r_{\{s,t\}}} \\ & \searrow & \downarrow \\ & & H_i \end{array}$$

The universal property of direct limits yields a homomorphism $G_{i+1} \to H_i$. It is clear that the concatenations of the two homomorphisms $G_{i+1} \to H_i$ and $H_i \to G_{i+1}$ take x_α to itself. Thus both concatenations are equal to the identities and both homomorphisms are isomorphisms. □

6.10. Second main result

(6.10.1) Remark. (a) In the next lemma we use the following basic fact about intersections of subgroups and monomorphisms. Let G, H be groups, let $U, V \leq G$ be subgroups of G and let $\varphi : G \to H$ be a monomorphism. Then $\varphi(U \cap V) = \varphi(U) \cap \varphi(V)$.

(b) In the next lemma we consider $D_R = G_{-1} \star_{V_R} O_R$ for $R := R_{\{r,t\}}(s)$ (cf. Lemma (6.9.11)). Similar as in Remark (6.1.1) we have to show that if x_α is a generator of G_{-1} and y_α is a generator of O_R, then $x_\alpha = y_\alpha$ holds in D_R. It suffices to consider $\alpha \in \{st\alpha_r, sr\alpha_t\}$. We deduce from Lemma (5.1.4) that $-\alpha_t, -\alpha_r \subseteq \alpha$ and hence $C_{-1} \subseteq \alpha$. Thus x_α is not a generator of G_{-1}.

(6.10.2) Lemma. *Let $R \in \mathcal{T}_{0,1}$ be a residue of type J. For $s \in J$ the canonical homomorphism $K_{R,s} \to D_{R_{\{r,t\}}(s)}$ is injective and we have $K_{R,s} \cap G_{-1} = O_R$ in $D_{R_{\{r,t\}}(s)}$.*

Proof. We suppose $J = \{s,t\}$. Note that $R = R_J(1_W)$. Since $O_{R,s} \cong U_{srs} \star_{U_{sr}} O_R$ by Lemma (6.2.1), we obtain that both homomorphisms $O_R \to O_{R,s} \to G_{-1}$ are injective by Lemma (6.9.10) and Proposition (1.8.1). For $T := R_{\{r,t\}}(s)$ we have that $X := U_{sr_{\{r,t\}}} \hat{\star} V_{str_{\{r,s\}}} \to O_T$ is injective by Proposition (1.8.1). Using Corollary (1.8.4) we obtain $X \cap V_T = V_{sr_{\{r,t\}}} \hat{\star} U_{sts}$ in O_T. Note that $V_T \to O_{R,s}$ is injective by Lemma (6.4.3). Recall that

$$O_{R,s} = U_{srs} \hat{\star} V_{sr_{\{r,t\}}} \hat{\star} U_{r_{\{s,t\}}} \hat{\star} V_{tr_{\{r,s\}}} \qquad \text{and} \qquad V_T = U_{srs} \hat{\star} V_{sr_{\{r,t\}}} \hat{\star} U_{sts}$$

As O_R corresponds to the last three vertex groups and V_T is a subgroup of the first three vertex groups of $O_{R,s}$, Corollary (1.8.4) implies that $O_R \cap V_T = V_{sr_{\{r,t\}}} \hat{\star} U_{sts}$ in $O_{R,s}$. We define $Y := V_{sr_{\{r,t\}}} \hat{\star} U_{sts}$. Applying Proposition (1.8.3) and Remark (6.10.1), the canonical homomorphism $X \star_Y O_R \to O_T \star_{V_T} G_{-1} = D_T$ is injective. In particular, Proposition (1.8.1), Remark (1.8.6) and Lemma (1.8.7) yield

$$X \star_Y O_R \cong X \star_Y \left(Y \star_{U_{sts}} U_{r_{\{s,t\}}} \hat{\star} V_{tr_{\{r,s\}}} \right) \cong U_{sr_{\{r,t\}}} \hat{\star} V_{str_{\{r,s\}}} \hat{\star} U_{r_{\{s,t\}}} \hat{\star} V_{tr_{\{r,s\}}} = K_{R,s}$$

This implies that $K_{R,s} \cong X \star_Y O_R \to O_T \star_{V_T} G_{-1} = D_T$ is injective. Applying Proposition (1.8.3), we obtain $K_{R,s} \cap G_{-1} = O_R$ in D_T. This finishes the claim. □

(6.10.3) Theorem. *The groups G_0 and G_1 are natural.*

Proof. Suppose $j \in \{0,1\}$. Then G_j satisfies (N1) by Lemma (6.9.7). Note that $\mathcal{T}_{j,2} = \emptyset$ and hence $\mathcal{T}_j = \mathcal{T}_{j,1}$. Thus G_j is natural, if $H_R \to G_j$ is injective for each $R \in \mathcal{T}_{j,1}$. Let $R \in \mathcal{T}_{j,1}$ be of type $\{s,t\}$. Then Lemma (6.4.1) implies that $H_R \cong K_{R,s} \star_{O_R} K_{R,t}$. Thus it suffices to show that $K_{R,s} \star_{O_R} K_{R,t} \to G_j$ is injective. We distinguish the cases $j = 0$ and $j = 1$.

$j = 0$: Then $R = R_{\{s,t\}}(1_W)$. By Proposition (1.8.1) and Lemma (6.9.11) it follows that $D_{R_{\{r,t\}}(s)} \star_{G_{-1}} D_{R_{\{r,s\}}(t)} \to G_0$ is injective. Using Lemma (6.10.2) we obtain that $K_{R,s} \to D_{R_{\{r,t\}}(s)}$ and $K_{R,t} \to D_{R_{\{r,s\}}(t)}$ are injective and that $K_{R,s} \cap G_{-1} = O_R$ (resp. $K_{R,t} \cap G_{-1} = O_R$) in $D_{R_{\{r,t\}}(s)}$ (resp. $D_{R_{\{r,s\}}(t)}$). Now Proposition (1.8.3) implies that the following homomorphism is injective:

$$K_{R,s} \star_{O_R} K_{R,t} \to D_{R_{\{r,t\}}(s)} \star_{G_{-1}} D_{R_{\{r,s\}}(t)} \to G_0$$

$j = 1$: Then $R = R_{\{s,t\}}(r)$. We abbreviate $T = R_{\{r,t\}}(rs)$ and $Z = R_{\{r,s\}}(1_W)$. Since G_0 is natural, the mapping $H_Z \to G_0$ is injective. Using Proposition (1.8.1), Remark (1.8.6), Lemma (1.8.7), Lemma (6.3.1) and Lemma (6.3.2) we infer

$$\begin{aligned} B_Z &= G_0 \star_{H_Z} G_Z \\ &\cong G_0 \star_{H_Z} J_{Z,r} \star_{J_{Z,r}} G_Z \\ &\cong (G_0 \star_{H_Z} J_{Z,r}) \star_{J_{Z,r}} G_Z \\ &\cong (G_0 \star_{H_Z} H_Z \star_{V_T} O_T) \star_{J_{Z,r}} G_Z \\ &\cong (G_0 \star_{V_T} O_T) \star_{J_{Z,r}} G_Z \end{aligned}$$

Thus the homomorphism $G_0 \star_{V_T} O_T \to B_Z$ is injective. By Lemma (6.4.3) the mappings $V_T \to O_{R,s}$ and, in particular, $K_{R,s} \to O_{R,s} \star_{V_T} O_T$ are injective. Lemma (6.9.11) implies that the canonical homomorphisms $V_T \to O_{R,s} \to G_0$ are injective. Using Proposition (1.8.3) the homomorphisms $O_{R,s} \star_{V_T} O_T \to G_0 \star_{V_T} O_T \to B_Z$ are injective. Using

Proposition (1.8.3) again, we deduce $(O_{R,s} \star_{V_T} O_T) \cap G_0 = O_{R,s}$ in $G_0 \star_{V_T} O_T$ and hence $K_{R,s} \cap G_0 \leq O_{R,s}$ in $G_0 \star_{V_T} O_T$. By Lemma (6.4.3) we have $K_{R,s} \cap O_{R,s} = O_R$ in $O_{R,s} \star_{V_T} O_T$ and by Remark (6.10.1)(a) all the previous intersections do also hold in B_Z. Thus we obtain the following in B_Z:

$$K_{R,s} \cap G_0 = K_{R,s} \cap G_0 \cap O_{R,s} = K_{R,s} \cap O_{R,s} = O_R$$

Let $T' = R_{\{r,s\}}(rt)$. Replacing s and t, we deduce that the homomorphisms $K_{R,t} \to O_{R,t} \star_{V_{T'}} O_{T'} \to B_{R_{\{r,t\}}(1_W)}$ are injective and $K_{R,t} \cap G_0 = K_{R,t} \cap O_{R,t} = O_R$. Now Proposition (1.8.3) yields that $K_{R,s} \star_{O_R} K_{R,t} \to B_{R_{\{r,s\}}(1_W)} \star_{G_0} B_{R_{\{r,t\}}(1_W)}$ is injective. Since G_0 is natural, Proposition (1.8.1) and Proposition (6.9.20) imply that $B_{R_{\{r,s\}}(1_W)} \star_{G_0} B_{R_{\{r,t\}}(1_W)} \to G_1$ is injective and the claim follows. □

(6.10.4) Lemma. *Suppose $2 \leq i \in \mathbb{N}$ is such that G_{i-2} and G_{i-1} are natural. Then for each $R \in \mathcal{T}_{i,1}$ of type $\{s,t\}$ with $\ell(w_R rs) = \ell(w_R) - 2$ the canonical homomorphism $E_{R,s} \to G_i$ is injective.*

Proof. Let $R \in \mathcal{T}_{i,1}$ be of type $\{s,t\}$ with $\ell(w_R rs) = \ell(w_R) - 2$, let $T = R_{\{r,t\}}(w_R)$ and $T' = R_{\{r,s\}}(w_R)$. Suppose $\ell(w_R rt) = \ell(w_R) - 2$. Using Lemma (6.7.2), we have $\{T, T'\} \in \mathcal{T}_{i-2,2}$ and $E_{R,s} \to G_{\{T,T'\}}$ is injective. As G_{i-2} is natural, the homomorphism $G_{\{T,T'\}} \to G_{i-2} \star_{H_{\{T,T'\}}} G_{\{T,T'\}} = B_{\{T,T'\}}$ is injective by Proposition (1.8.1). Moreover, as G_{i-2} and G_{i-1} are natural, the homomorphisms $B_{\{T,T'\}} \to G_{i-1}$ and $G_{i-1} \to G_i$ are injective by Proposition (6.9.20). This finishes the claim. Thus we can assume that $\ell(w_R rt) = \ell(w_R)$. We abbreviate $Z := R_{\{r,s\}}(w_R t)$ and distinguish the following two cases:

(i) $T \in \mathcal{T}_{i-1,1}$: As G_{i-1} is natural, we deduce from Proposition (6.9.20) that $B_T \to G_i$ is injective. Using Proposition (1.8.1), Remark (1.8.6), Lemma (1.8.7), Lemma (6.3.1) and Lemma (6.3.2) infer

$$\begin{aligned} B_T &= G_{i-1} \star_{H_T} G_T \\ &\cong G_{i-1} \star_{H_T} J_{T,r} \star_{J_{T,r}} G_T \\ &\cong (G_{i-1} \star_{H_T} J_{T,r}) \star_{J_{T,r}} G_T \\ &\cong (G_{i-1} \star_{H_T} H_T \star_{V_Z} O_Z) \star_{J_{T,r}} G_T \\ &\cong (G_{i-1} \star_{V_Z} O_Z) \star_{J_{T,r}} G_T \end{aligned}$$

In particular, each of the mappings $G_{i-1} \star_{V_Z} O_Z \to B_T \to G_i$ is injective.

(ii) $T \notin \mathcal{T}_{i-1,1}$: Then there exists a unique $P_T \in \mathcal{T}_{i-1,2}$ with $T \in P_T$. Suppose $P_T = \{T, T''\}$. As G_{i-1} is natural, we deduce from Proposition (6.9.20) that $B_{P_T} \to G_i$ is injective. Then Proposition (1.8.1), Remark (1.8.6), Lemma (1.8.7), Lemma (6.7.1) and Lemma (6.7.3) imply that

$$\begin{aligned} B_{P_T} &= G_{i-1} \star_{H_{\{T,T''\}}} G_{\{T,T''\}} \\ &\cong G_{i-1} \star_{H_{\{T,T''\}}} J_{(T,T'')} \star_{J_{(T,T'')}} G_{\{T,T''\}} \\ &\cong \left(G_{i-1} \star_{H_{\{T,T''\}}} J_{(T,T'')}\right) \star_{J_{(T,T'')}} G_{\{T,T''\}} \\ &\cong \left(G_{i-1} \star_{H_{\{T,T''\}}} H_{\{T,T''\}} \star_{V_Z} O_Z\right) \star_{J_{(T,T'')}} G_{\{T,T''\}} \\ &\cong (G_{i-1} \star_{V_Z} O_Z) \star_{J_{(T,T'')}} G_{\{T,T''\}} \end{aligned}$$

and hence each of the mappings $G_{i-1} \star_{V_Z} O_Z \to B_{P_T} \to G_i$ is injective.

We conclude that $G_{i-1} \star_{V_Z} O_Z \to G_i$ is injective. We will show now that $X_R \to G_{i-1}$ is injective. We distinguish the following two cases:

(i) $T' \in \mathcal{T}_{i-2,1}$: As G_{i-2} is natural by assumption, the mapping $G_{T'} \to B_{T'} \to G_{i-1}$ is injective by Proposition (6.9.20) and by Lemma (6.6.3) the homomorphism $X_R \to G_{T'}$ is injective.

(ii) $T' \notin \mathcal{T}_{i-2,1}$: Then there exists a unique $P_{T'} \in \mathcal{T}_{i-2,2}$ with $T' \in P_{T'}$. As G_{i-2} is natural by assumption, the mapping $G_{P_{T'}} \to B_{P_{T'}} \to G_{i-1}$ is injective by Proposition (6.9.20) and by Lemma (6.7.4) the homomorphism $X_R \to G_{P_{T'}}$ is injective.

We conclude that $X_R \to G_{i-1}$ is injective. Moreover, $V_Z \to X_R$ is injective by Lemma (6.6.2) and hence $X_R \star_{V_Z} O_Z \to G_{i-1} \star_{V_Z} O_Z \to G_i$ is injective by Proposition (1.8.3). Using Lemma (6.6.2) again, we infer that $E_{R,s} \to X_R \star_{V_Z} O_Z$ and, in particular, $E_{R,s} \to G_i$ is injective. □

(6.10.5) Theorem. *For each $i \geq 0$ the group G_i is natural.*

Proof. We show the claim via induction on $i \geq 0$. If $i \leq 1$, claim follows from Theorem (6.10.3). Thus we can assume that $i \geq 2$ and that G_k is natural for all $0 \leq k < i$. We have to show that G_i satisfies (N1) and (N2).

(N1) Let $w \in C_i$. If $w \in C_{i-1}$, then each of the homomorphisms $U_w \to G_{i-1} \to G_i$ is injective by induction and Proposition (6.9.20). If $w \notin C_{i-1}$, then there exists $P \in \mathcal{T}_{i-1}$ with $w \in C(P)$ by definition of C_i. Using Lemma (6.9.13) and Proposition (6.9.20), each of the homomorphisms $U_w \to G_P \to G_i$ is injective. Now we consider $w' \in D_i$. If $w' \in D_{i-1}$, induction and Proposition (6.9.20) imply that each of the homomorphisms $V_{w'} \to G_{i-1} \to G_i$ is injective. Thus we can assume that $w' \notin D_{i-1}$. As $w' = w_R r_{\{s,t\}}$ for some residue R of type $\{s,t\}$ with $w_R s, w_R t \in C_i$, we deduce $\{w_R s, w_R t\} \cap (C_i \backslash C_{i-1}) \neq \emptyset$. By definition of C_i there exists $P \in \mathcal{T}_{i-1}$ such that $\{w_R s, w_R t\} \cap (C(P) \backslash C_{i-1}) \neq \emptyset$. But then Lemma (6.9.19), induction and Proposition (6.9.20) imply that each of the homomorphisms $V_{w'} \to G_P \to G_i$ is injective and (N1) is satisfied.

(N2) To prove that (N2) holds we have to show that $H_P \to G_i$ is injective for every $P \in \mathcal{T}_i$. Suppose $P \in \mathcal{T}_{i,1}$ is of type $\{s,t\}$. As $i \geq 2$, we can assume that $\ell(w_P rs) = \ell(w_P) - 2$. Since $H_P \to E_{P,s}$ is injective by Lemma (6.5.1) and $E_{P,s} \to G_i$ is injective by Lemma (6.10.4), the claim follows. Now suppose that $P \in \mathcal{T}_{i,2}$. Let $P = \{R, R'\}$, where R is of type $\{r,s\}$ and R' is of type $\{r,t\}$. Let T be the $\{r,t\}$-residue containing w_R and let T' be the $\{r,s\}$-residue containing $w_{R'}$. By Lemma (6.8.3) we have $T, T' \in \mathcal{T}_{i-1,1}$. As G_{i-1} is natural, Proposition (6.9.20) and Proposition (1.8.1) imply that the mapping $B_T \star_{G_{i-1}} B_{T'} \to G_i$ is injective. By Lemma (6.8.2) we have $H_{\{R,R'\}} \cong C_{(R,R')} \star_C C_{(R',R)}$. Thus it suffices to show that $C_{(R,R')} \star_C C_{(R',R)} \to B_T \star_{G_{i-1}} B_{T'}$ is injective and we will prove it by using Proposition (1.8.3).

Using Lemma (6.10.4), the mappings $E_{T,t}, E_{T',s} \to G_{i-1}$ are injective. Then Lemma (6.5.1), Proposition (1.8.1), Remark (1.8.6) and Lemma (1.8.7) yield

$$B_T = G_{i-1} \star_{H_T} G_T \cong G_{i-1} \star_{E_{T,t}} E_{T,t} \star_{H_T} G_T \cong G_{i-1} \star_{E_{T,t}} U_{T,t}$$
$$B_{T'} = G_{i-1} \star_{H_{T'}} G_{T'} \cong G_{i-1} \star_{E_{T',s}} E_{T',s} \star_{H_{T'}} G_{T'} \cong G_{i-1} \star_{E_{T',s}} U_{T',s}$$

Lemma (6.8.3) shows that $C_{(R,R')} \to U_{T,t}, C_{(R',R)} \to U_{T',s}$ are injective and, in particular, $C_{(R,R')} \to B_T, C_{(R',R)} \to B_{T'}$ are injective. Moreover, Lemma (6.8.3) implies that $C_{(R,R')} \cap E_{T,t} = C$ holds in $U_{T,t}$ and $C_{(R',R)} \cap E_{T',s} = C$ holds in $U_{T',s}$. Remark

(6.10.1)(a) implies that these intersections do also hold in B_T and $B_{T'}$, respectively. Corollary (1.8.5) now yields:

$$\begin{aligned} C_{(R,R')} \cap G_{i-1} &= C_{(R,R')} \cap G_{-1} \cap E_{T,t} = C_{(R,R')} \cap E_{T,t} = C && \text{in } B_T \\ C_{(R',R)} \cap G_{i-1} &= C_{(R',R)} \cap G_{-1} \cap E_{T',s} = C_{(R',R)} \cap E_{T',s} = C && \text{in } B_{T'} \end{aligned}$$

Now Proposition (1.8.3) implies that the canonical homomorphism $C_{(R,R')} \star_C C_{(R',R)} \to B_T \star_{G_{i-1}} B_{T'}$ is injective. This finishes the proof. □

(6.10.6) Corollary. *$\mathcal{M}$ is a faithful commutator blueprint of type* $(4,4,4)$.

Proof. By Lemma (6.9.4) we have $G \cong U_+$. We have to show that for each $w \in W$ the canonical homomorphism $U_w \to G \cong U_+$ is injective. Note that the following diagram commutes for every $i \in \mathbb{N}$ with $w \in C_i$ (cf. Remark (6.9.2) and Remark (6.9.3)):

$$\begin{array}{ccc} U_w & \longrightarrow & G_i \\ & \searrow & \downarrow \\ & & G \end{array}$$

By Theorem (6.10.5) the group G_i is natural for each $i \geq 0$. Proposition (6.9.20) implies that the canonical homomorphisms $G_i \to G_{i+1}$ are injective. It follows from [30, 1.4.9(iii)] that the canonical homomorphisms $G_i \to G$ are injective. Since for each $w \in W$ there exists $i \in \mathbb{N}$ such that $w \in C_i$, we infer that $U_w \to G$ is injective. This finishes the proof. □

(6.10.7) Corollary. *Let $\mathcal{M}$ be a* 2*-nilpotent pre-commutator blueprint of type* $(4,4,4)$*, which is Weyl-invariant and satisfies (CR*1*) and (CR*2*). Then $\mathcal{M}$ is integrable.*

Proof. By Lemma (4.2.2), $\mathcal{M}$ is a commutator blueprint and the groups U_w are of nilpotency class at most 2. By Corollary (6.10.6), $\mathcal{M}$ is faithful and by Theorem (3.5.1), $\mathcal{M}$ is integrable. □

7. Applications

We first construct new examples of integrable commutator blueprints of type $(4,4,4)$. Then we discuss several applications.

7.1. New RGD-systems

Let $\mathcal{D} = (\mathcal{G}, (U_\alpha)_{\alpha\in\Phi})$ be the RGD-system associated with the split Kac-Moody group of type $(4,4,4)$ over $\mathbb{F}_2$ as in Example (5.3.1). Then $\mathcal{M}_\mathcal{D}$ is an integrable commutator blueprint of type $(4,4,4)$ by Example (2.1.8). In this section we will construct new examples of integrable commutator blueprints of type $(4,4,4)$.

(7.1.1) Proposition. *Let $\mathcal{M} = \left(M_{\alpha,\beta}^G\right)_{(G,\alpha,\beta)\in\mathcal{I}}$ be a pre-commutator blueprint of type $(4,4,4)$, which is locally Weyl-invariant. Let $G \in$ Min and let $\alpha,\beta \in \Phi(G)$ be two roots such that $\alpha \neq \beta$ and $\alpha \leq_G \beta$. Assume that the following hold:*

(a) Suppose that $o(r_\alpha r_\beta) < \infty$ and let $\varepsilon \in \Phi(G)$.

(i) If for each $\gamma \in M_{\alpha,\beta}^G$ we have $\varepsilon \subsetneq \gamma$, then $M_{\varepsilon,\gamma}^G = M_{\varepsilon,\delta}^G$ holds for all $\gamma,\delta \in M_{\alpha,\beta}^G$.

(ii) If for each $\gamma \in M_{\alpha,\beta}^G$ we have $\gamma \subsetneq \varepsilon$, then $M_{\gamma,\varepsilon}^G = M_{\delta,\varepsilon}^G$ holds for all $\gamma,\delta \in M_{\alpha,\beta}^G$.

(b) Suppose that $o(r_\alpha r_\beta) = \infty$ and suppose $G = (d_0,\ldots,d_n = c_0,\ldots,c_k = e_0,\ldots,e_m)$ such that $\{c_0,c_1\} \in \partial\alpha$ and $\{c_{k-1},c_k\} \in \partial\beta$. Then the following hold:

(i) We have $\prod_{\gamma\in M_{\alpha,\beta}^G} u_\gamma \in Z(U_{(d_i,\ldots,d_n=c_0,\ldots,c_k=e_0,\ldots,e_j),G})$ for each $0 \leq i \leq n$ and each $0 \leq j \leq m$. Moreover, we have $\prod_{\gamma\in M_{\alpha,\beta}^G} u_\gamma \in Z(U_{(c_1,\ldots,c_{k-1}),G})$.

(ii) We have $\left(\prod_{\gamma\in M_{\alpha,\beta}^G} u_\gamma\right)^2 = 1$ in $U_{(c_1,\ldots,c_{k-1}),G}$.

Then $\mathcal{M}$ is 2-nilpotent (cf. Section 4.2).

Proof. Let $G = (d_0,\ldots,d_n = c_0,\ldots,c_k) \in$ Min and let $(\alpha_1',\ldots,\alpha_{n+k}')$ be the sequence of roots crossed by G. We abbreviate $\alpha_i := \alpha_{n+i}'$ as well as $u_i := u_{\alpha_i}$ for all $1 \leq i \leq k$.

(2-n1) Let $1 \leq i \leq k-1$. We have to show that $[u_1,[u_i,u_k]] = 1$. If $R \in \partial^2\alpha_1 \cap \partial^2\alpha_i \cap \partial^2\alpha_k$, then the claim follows. Thus we can assume that $\partial^2\alpha_1 \cap \partial^2\alpha_i \cap \partial^2\alpha_k = \emptyset$. Moreover, we can assume that $M_{\alpha_i,\alpha_k}^G \neq \emptyset$. If $o(r_{\alpha_i}r_{\alpha_k}) = \infty$, then $[u_i,u_k]$ commutes with u_1 by Condition $(b)(i)$ and the claim follows. Thus we assume $o(r_{\alpha_i}r_{\alpha_k}) < \infty$ and hence $|(\alpha_i,\alpha_k)| = 2$. We let $M_{\alpha_i,\alpha_k}^G = \{\delta,\gamma\}$ be with $\delta \leq_G \gamma$. Suppose that $o(r_{\alpha_1}r_\rho) = \infty$ for each $\rho \in (\alpha_i,\alpha_k)$. Then $\alpha_1 \subseteq \rho$ and we have $M_{\alpha_1,\delta}^G = M_{\alpha_1,\gamma}^G$ by Condition $(a)(i)$ and we infer

$$\prod_{\varepsilon\in M_{\alpha_i,\alpha_k}^G}\left(\prod_{\omega\in M_{\alpha_1,\varepsilon}^G} u_\omega\right)u_\varepsilon = \left(\prod_{\omega\in M_{\alpha_1,\delta}^G} u_\omega\right)u_\delta\left(\prod_{\omega\in M_{\alpha_1,\gamma}^G} u_\omega\right)u_\gamma$$

$$\overset{(b)(i)}{=} u_\delta \left(\prod_{\omega \in M^G_{\alpha_1,\gamma}} u_\omega \right)^2 u_\gamma$$

$$\overset{(b)(ii)}{=} u_\delta u_\gamma = \prod_{\varepsilon \in M^G_{\alpha_i,\alpha_k}} u_\varepsilon$$

Now we suppose that there exists $\rho \in (\alpha_i, \alpha_k)$ with $o(r_{\alpha_1} r_\rho) < \infty$. Since $\alpha_i \cap \alpha_k \subseteq \rho$, we deduce that $\alpha_1 \not\subseteq \alpha_i$ or $\alpha_1 \not\subseteq \alpha_k$ and hence $o(r_{\alpha_1} r_{\alpha_i}) < \infty$ or $o(r_{\alpha_1} r_{\alpha_k}) < \infty$. Let $\varepsilon \in \{\alpha_i, \alpha_k\}$ be a root such that $o(r_{\alpha_1} r_\varepsilon) < \infty$. Then $o(r_\varepsilon r_\rho) < \infty$ by Lemma (1.5.2). As $\partial^2\rho \cap \partial^2\varepsilon = \partial^2\alpha_i \cap \partial^2\alpha_k$ by Lemma (1.5.2) and Lemma (1.4.8)(a) and $\partial^2\alpha_1 \cap \partial^2\alpha_i \cap \partial^2\alpha_k = \emptyset$, we infer that $\{r_{\alpha_1}, r_\varepsilon, r_\rho\}$ is a reflection triangle. Using Remark (1.5.1) there exist $\beta_1 \in \{\alpha_1, -\alpha_1\}, \beta_\varepsilon \in \{\varepsilon, -\varepsilon\}$ and $\beta_\rho \in \{\rho, -\rho\}$ such that $\{\beta_1, \beta_\varepsilon, \beta_\rho\}$ is a triangle. Note that $\partial^2\alpha_i \cap \partial^2\alpha_k \cap \partial^2\rho \neq \emptyset$ by Lemma (1.5.2). We let $\varepsilon' \in \{\alpha_i, \alpha_k\} \backslash \{\varepsilon\}$. By Lemma (1.4.8)$(b)$ we have $((\beta_\varepsilon, \beta_\rho) \cup (-\beta_\varepsilon, \beta_\rho)) \cap \{\varepsilon', -\varepsilon'\} \neq \emptyset$. As $(-\beta_\varepsilon, \beta_\rho) = \emptyset$ by Lemma (1.5.3), there exists $\beta_{\varepsilon'} \in \{\varepsilon', -\varepsilon'\}$ such that $\beta_{\varepsilon'} \in (\beta_\varepsilon, \beta_\rho)$. By Lemma (1.5.6) we have $o(r_{\alpha_1} r_{\varepsilon'}) = \infty$ and hence (as $\{\alpha_1, \varepsilon'\} \in \mathcal{P}$) $\alpha_1 \subseteq \varepsilon'$. Recall that $\partial^2\rho \cap \partial^2\varepsilon \cap \partial^2\varepsilon' = \partial^2\rho \cap \partial^2\alpha_i \cap \partial^2\alpha_k \neq \emptyset$. For $R \in \partial^2\varepsilon' \cap \partial^2\rho = \partial^2\alpha_i \cap \partial^2\alpha_k$ (cf. Lemma (1.4.8)(a)), we deduce $\emptyset \neq R \cap (-\varepsilon') \subseteq (-\alpha_1)$ and, as $R \notin \partial^2\alpha_1$, we have $R \subseteq (-\alpha_1)$. This yields $\beta_1 = -\alpha_1$. For $R \in \partial^2\alpha_1 \cap \partial^2\varepsilon$ we have $\emptyset \neq \alpha_1 \cap R \subseteq \varepsilon'$. As $\partial^2\alpha_1 \cap \partial^2\varepsilon \cap \partial^2\varepsilon' = \partial^2\alpha_1 \cap \partial^2\alpha_i \cap \partial^2\alpha_k = \emptyset$, we deduce $R \notin \partial^2\varepsilon'$ and hence $R \subseteq \varepsilon'$. In particular, we have $\emptyset \neq \varepsilon \cap R \subseteq \varepsilon \cap \varepsilon' = \alpha_i \cap \alpha_k \subseteq \rho$. As $R \notin \partial^2\rho$ ($\{r_{\alpha_1}, r_\varepsilon, r_\rho\}$ is a reflection triangle), we infer $R \subseteq \rho$ and hence $\beta_\rho = \rho$. Lemma (1.5.3) implies $(\alpha_1, \rho) = \emptyset$. Now let $\rho \neq \sigma \in (\alpha_i, \alpha_k)$. Using Lemma (1.5.2) and Lemma (1.4.8)(b), we deduce $((\beta_\varepsilon, \beta_\rho) \cup (-\beta_\varepsilon, \beta_\rho)) \cap \{\sigma, -\sigma\} \neq \emptyset$. Using Lemma (1.5.3), there exists $\beta_\sigma \in \{\sigma, -\sigma\}$ such that $\beta_\sigma \in (\beta_\varepsilon, \beta_\rho)$ as before. Using Lemma (1.5.6), we deduce $o(r_{\alpha_1} r_\sigma) = \infty$ and hence (as $\{\alpha_1, \sigma\} \in \mathcal{P}$) $\alpha_1 \subseteq \sigma$. Applying Lemma (1.5.6) again, we deduce $\alpha_1 = -\beta_1 \subseteq \beta_\sigma$ and $(\alpha_1, \beta_\sigma) = \emptyset$. In particular, as $\alpha_1 \subseteq \beta_\sigma \cap \sigma$, we have $\beta_\sigma = \sigma$. Since $(\alpha_1, \delta) = (\alpha_1, \gamma) = \emptyset$, we compute

$$\prod_{\varepsilon \in M^G_{\alpha_i,\alpha_k}} \left(\prod_{\omega \in M^G_{\alpha_1,\varepsilon}} u_\omega \right) u_\varepsilon = \prod_{\varepsilon \in M^G_{\alpha_i,\alpha_k}} u_\varepsilon$$

(2-n2) Let $2 \leq i \leq k-1$. We have to show that $[[u_1, u_i], u_k] = 1$. As in (2-n1) we can assume $\partial^2\alpha_1 \cap \partial^2\alpha_i \cap \partial^2\alpha_k = \emptyset$ and $M^G_{\alpha_1,\alpha_i} \neq \emptyset$. If $o(r_{\alpha_1} r_{\alpha_i}) = \infty$, then $[u_1, u_i]$ commutes with u_k by Condition $(b)(i)$ and the claim follows. Thus we assume $o(r_{\alpha_1} r_{\alpha_i}) < \infty$ and hence $|(\alpha_1, \alpha_i)| = 2$. We let $M^G_{\alpha_1,\alpha_i} = \{\delta, \gamma\}$ be with $\delta \leq_G \gamma$. Suppose that $o(r_\rho r_{\alpha_k}) = \infty$ for each $\rho \in (\alpha_1, \alpha_i)$. Then $\rho \subseteq \alpha_k$ and we have $M^G_{\delta,\alpha_k} = M^G_{\gamma,\alpha_k}$ by Condition $(a)(ii)$ and we infer

$$\prod_{\varepsilon \in M^G_{\alpha_1,\alpha_i}} \left(u_\varepsilon \prod_{\omega \in M^G_{\varepsilon,\alpha_k}} u_\omega \right) = u_\delta \left(\prod_{\omega \in M^G_{\delta,\alpha_k}} u_\omega \right) u_\gamma \left(\prod_{\omega \in M^G_{\gamma,\alpha_k}} u_\omega \right)$$

$$\overset{(b)(i)}{=} u_\delta \left(\prod_{\omega \in M^G_{\gamma,\alpha_k}} u_\omega \right)^2 u_\gamma$$

$$\overset{(b)(ii)}{=} u_\delta u_\gamma = \prod_{\varepsilon \in M^G_{\alpha_1,\alpha_i}} u_\varepsilon$$

Now we suppose that there exists $\rho \in (\alpha_1, \alpha_i)$ with $o(r_\rho r_{\alpha_k}) < \infty$. Since $(-\alpha_1) \cap (-\alpha_i) \subseteq (-\rho)$, we deduce $o(r_{\alpha_1} r_{\alpha_k}) < \infty$ or $o(r_{\alpha_i} r_{\alpha_k}) < \infty$, as otherwise we would have $(-\alpha_k) \subseteq (-\alpha_1) \cap (-\alpha_i) \subseteq (-\rho)$. Let $\varepsilon \in \{\alpha_1, \alpha_i\}$ be a root with $o(r_\varepsilon r_{\alpha_k}) < \infty$. Then $o(r_\varepsilon r_\rho) < \infty$ by Lemma (1.5.2). As $\partial^2\rho \cap \partial^2\varepsilon = \partial^2\alpha_1 \cap \partial^2\alpha_i$ by Lemma (1.5.2) and Lemma (1.4.8)(a), and $\partial^2\alpha_1 \cap \partial^2\alpha_i \cap \partial^2\alpha_k = \emptyset$, we infer that $\{r_\varepsilon, r_\rho, r_{\alpha_k}\}$ is a reflection triangle. Using Remark (1.5.1) there exist $\beta_\varepsilon \in \{\varepsilon, -\varepsilon\}, \beta_\rho \in \{\rho, -\rho\}$ and $\beta_k \in \{\alpha_k, -\alpha_k\}$ such that $\{\beta_\varepsilon, \beta_k, \beta_\rho\}$ is a triangle. Note that $\partial^2\alpha_1 \cap \partial^2\alpha_i \cap \partial^2\rho \neq \emptyset$ by Lemma (1.5.2). We let $\varepsilon' \in \{\alpha_1, \alpha_i\}\backslash\{\varepsilon\}$. By Lemma (1.4.8)$(b)$ we have $((\beta_\rho, \beta_\varepsilon) \cup (-\beta_\rho, \beta_\varepsilon)) \cap \{\varepsilon', -\varepsilon'\} \neq \emptyset$. As $(-\beta_\rho, \beta_\varepsilon) = \emptyset$ by Lemma (1.5.3), there exists $\beta_{\varepsilon'} \in \{\varepsilon', -\varepsilon'\}$ such that $\beta_{\varepsilon'} \in (\beta_\rho, \beta_\varepsilon)$. By Lemma (1.5.6) we have $o(r_{\varepsilon'} r_{\alpha_k}) = \infty$ and hence (as $\{\varepsilon', \alpha_k\} \in \mathcal{P}$) $\varepsilon' \subseteq \alpha_k$. For $R \in \partial^2\varepsilon \cap \partial^2\rho = \partial^2\alpha_i \cap \partial^2\alpha_1$ (cf. Lemma (1.4.8)(a)), we have (as $\varepsilon' \in \{\alpha_1, \alpha_i\}$) $\emptyset \neq R \cap \varepsilon' \subseteq \alpha_k$. As $R \notin \partial^2\alpha_k$, we infer $R \subseteq \alpha_k$ and hence $\beta_k = \alpha_k$. For $R \in \partial^2\varepsilon \cap \partial^2\alpha_k$ we have $R \notin \partial^2\varepsilon'$ and $\emptyset \neq R \cap (-\alpha_k) \subseteq (-\varepsilon')$. This implies $R \subseteq (-\varepsilon')$ and hence $\emptyset \neq (-\varepsilon) \cap R \subseteq (-\varepsilon) \cap (-\varepsilon') \subseteq (-\rho)$. As $R \notin \partial^2\rho$, we deduce $\beta_\rho = -\rho$ and Lemma (1.5.3) implies $(\rho, \alpha_k) = \emptyset$. Now let $\rho \neq \sigma \in (\alpha_1, \alpha_i)$. Again by Lemma (1.5.2), Lemma (1.4.8)(b) and Lemma (1.5.3) there exists $\beta_\sigma \in \{\sigma, -\sigma\}$ such that $\beta_\sigma \in (\beta_\rho, \beta_\varepsilon)$. Using Lemma (1.5.6) we deduce $o(r_\sigma r_{\alpha_k}) = \infty$ and hence (as $\{\delta, \alpha_k\} \in \mathcal{P}$) $\sigma \subseteq \alpha_k$. Applying Lemma (1.5.6) again, we deduce $-\alpha_k = -\beta_k \subseteq \beta_\sigma$ and $(-\alpha_k, \beta_\sigma) = \emptyset$. In particular, as $-\alpha_k \subseteq \beta_\sigma \cap (-\sigma)$, we have $\beta_\sigma = -\sigma$. Since $(\sigma, \alpha_k) = (\rho, \alpha_k) = \emptyset$, we compute

$$\prod_{\varepsilon \in M^G_{\alpha_1, \alpha_i}} \left(u_\varepsilon \prod_{\omega \in M^G_{\varepsilon, \alpha_k}} u_\omega \right) = \prod_{\varepsilon \in M^G_{\alpha_1, \alpha_i}} u_\varepsilon$$

(2-n3) At first we assume $o(r_{\alpha_1} r_{\alpha_k}) = \infty$. Then $[u_1, u_k]$ commutes with u_i by Condition $(b)(i)$ and $[u_1, u_k]^2 = 1$ by Condition $(b)(ii)$. Thus we can assume $o(r_{\alpha_1} r_{\alpha_k}) < \infty$. If $|(\alpha_1, \alpha_k)| < 2$, then $M^G_{\alpha_1, \alpha_k} = \emptyset$ and the claim follows directly. Thus we can assume $(\alpha_1, \alpha_k) = \{\delta, \gamma\}$ and $\delta \leq_G \gamma$. The first claim is obvious, as $M^G_{\delta, \gamma} = \emptyset$. For the second claim we let $2 \leq i \leq k-1$. If $\alpha_i \in (\alpha_1, \alpha_k)$, the claim follows directly. Thus we can assume $\alpha_i \notin (\alpha_1, \alpha_k)$. In particular, Lemma (1.4.6) implies $\partial^2\alpha_1 \cap \partial^2\alpha_i \cap \partial^2\alpha_k = \emptyset$.

At first we suppose $o(r_{\alpha_i} r_\varepsilon) = \infty$ for both $\varepsilon \in \{\delta, \gamma\}$. As $\{\delta, \alpha_i\}, \{\gamma, \alpha_i\} \in \mathcal{P}$, we infer that $\{\delta, \alpha_i\}$ and $\{\gamma, \alpha_i\}$ are pairs of nested roots. The fact that $o(r_\delta r_\gamma) < \infty$ implies that either $\alpha_i \subseteq \delta, \gamma$ or else $\delta, \gamma \subseteq \alpha_i$. If $\alpha_i \subseteq \delta, \gamma$, then we have $M^G_{\alpha_i, \delta} = M^G_{\alpha_i, \gamma}$ by Condition $(a)(i)$ and we deduce

$$[u_i, u_\delta] = \prod_{\omega \in M^G_{\alpha_i, \delta}} u_\omega = \prod_{\omega \in M^G_{\alpha_i, \gamma}} u_\omega = [u_i, u_\gamma]$$

In particular, we obtain $[u_i, u_\delta u_\gamma] = [u_i, u_\gamma][u_i, u_\delta]^{u_\gamma} = [u_i, u_\gamma][u_i, u_\gamma]^{u_\gamma} = [u_i, u_\gamma^2] = 1$. Similarly, if $\delta, \gamma \subseteq \alpha_i$, we obtain $[u_\delta, u_i] = [u_\gamma, u_i]$ and $[u_\delta u_\gamma, u_i] = [u_\delta, u_i]^{u_\gamma}[u_\gamma, u_i] = [u_\gamma, u_i]^{u_\gamma}[u_\gamma, u_i] = [u_\gamma^2, u_i] = 1$.

Now we can assume that $o(r_\varepsilon r_{\alpha_i}) < \infty$ for some $\varepsilon \in \{\delta, \gamma\}$. We deduce from $\alpha_1 \not\subseteq \alpha_k$ that we have $\alpha_1 \not\subseteq \alpha_i$ or $\alpha_i \not\subseteq \alpha_k$, i.e. we have $o(r_{\alpha_1} r_{\alpha_i}) < \infty$ or $o(r_{\alpha_i} r_{\alpha_k}) < \infty$. Let $\omega \in \{\alpha_1, \alpha_k\}$ be a root such that $o(r_{\alpha_i} r_\omega) < \infty$. Note that $\partial^2\alpha_1 \cap \partial^2\alpha_i \cap \partial^2\alpha_k = \emptyset$ and by Lemma (1.4.8)(a) and Lemma (1.5.2) we have $\partial^2\alpha_1 \cap \partial^2\alpha_k = \partial^2\omega \cap \partial^2\varepsilon$. This implies that $\{r_\omega, r_{\alpha_i}, r_\varepsilon\}$ is a reflection triangle. By Remark (1.5.1) there exist $\beta_\omega \in \{\omega, -\omega\}, \beta_i \in \{\alpha_i, -\alpha_i\}, \beta_\varepsilon \in \{\varepsilon, -\varepsilon\}$ such that $\{\beta_\omega, \beta_i, \beta_\varepsilon\}$ is a triangle. By Lemma (1.5.3) we have $(-\beta_\omega, \beta_\varepsilon) = \emptyset$. Let $\omega \neq \omega' \in \{\alpha_1, \alpha_k\}$ and let $\varepsilon \neq \varepsilon' \in \{\delta, \gamma\}$.

Using Lemma (1.4.8)(b) there exists $\beta_{\omega'} \in \{\omega', -\omega'\}$ and $\beta_{\varepsilon'} \in \{\varepsilon', -\varepsilon'\}$ such that $\beta_{\omega'}, \beta_{\varepsilon'} \in (\beta_\omega, \beta_\varepsilon)$. It follows from Lemma (1.5.6) that $o(r_{\omega'} r_{\alpha_i}) = \infty, o(r_{\alpha_i} r_{\varepsilon'}) = \infty, -\beta_i \subseteq \beta_{\varepsilon'}$ and $(-\beta_i, \beta_{\varepsilon'}) = \emptyset$. Now we distinguish the following cases:

(a) $\omega = \alpha_1$: Then $\omega' = \alpha_k$ and (as $\{\alpha_i, \alpha_k\} \in \mathcal{P}$) $\alpha_i \subseteq \alpha_k$. Assume that $\varepsilon' \subseteq \alpha_i$. Then we would have $\varepsilon' \subseteq \alpha_i \subseteq \alpha_k$ which is a contradiction. As $\{\alpha_i, \varepsilon'\} \in \mathcal{P}$, we deduce $\alpha_i \subseteq \varepsilon'$. For $R \in \partial^2\alpha_1 \cap \partial^2\varepsilon = \partial^2\alpha_1 \cap \partial^2\alpha_k$ (cf. Lemma (1.5.2) and Lemma (1.4.8)(a)) we deduce $\emptyset \neq R \cap (-\alpha_k) \subseteq (-\alpha_i)$ and hence, as $R \notin \partial^2\alpha_i$ because $\{r_{\alpha_1}, r_{\alpha_i}, r_\varepsilon\}$ is a reflection triangle, that $R \subseteq (-\alpha_i)$ and $\beta_i = -\alpha_i$. As $\alpha_i = -\beta_i \subseteq \beta_{\varepsilon'} \cap \varepsilon'$, we deduce $\beta_{\varepsilon'} = \varepsilon'$ and hence $(\alpha_i, \varepsilon') = (-\beta_i, \beta_{\varepsilon'}) = \emptyset$. For $R \in \partial^2\alpha_1 \cap \partial^2\alpha_i$ we have $R \notin \partial^2\varepsilon' \cup \partial^2\varepsilon$, as $\partial^2\alpha_1 \cap \partial^2\varepsilon = \partial^2\alpha_1 \cap \partial^2\alpha_k = \partial^2\alpha_1 \cap \partial^2\varepsilon'$ and $\partial^2\alpha_1 \cap \partial^2\alpha_i \cap \partial^2\alpha_k = \emptyset$. Assume that $R \subseteq (-\varepsilon)$. As $(\varepsilon, \varepsilon') = \emptyset$, Lemma (1.4.8)$(b)$ yields $(-\varepsilon, \varepsilon') \cap \{\alpha_1, -\alpha_1\} \neq \emptyset$. In particular, we have $(-\varepsilon) \cap \varepsilon' \subseteq \beta_1$ for some $\beta_1 \in \{\alpha_1, -\alpha_1\}$. We deduce from $R \in \partial^2\alpha_1, R \subseteq (-\varepsilon)$, that $R \not\subseteq \varepsilon'$ and hence $R \subseteq (-\varepsilon')$. But this would imply $\emptyset \neq \alpha_i \cap R \subseteq \alpha_i \cap (-\varepsilon') = \emptyset$, which is a contradiction. As $R \notin \partial^2\varepsilon$, we deduce $R \subseteq \varepsilon$ and hence $\beta_\varepsilon = \varepsilon$. In particular, we have $(\alpha_i, \varepsilon) = (-\beta_i, \beta_\varepsilon) = \emptyset$. Thus u_i commutes with u_ε and $u_{\varepsilon'}$ and hence with $\prod_{\gamma \in M^G_{\alpha_1, \alpha_k}} u_\gamma$.

(b) $\omega = \alpha_k$: Then $\omega' = \alpha_1$ and (as $\{\alpha_1, \alpha_i\} \in \mathcal{P}$) $\alpha_1 \subseteq \alpha_i$. Assume that $\alpha_i \subseteq \varepsilon'$. Then we would have $\alpha_1 \subseteq \alpha_i \subseteq \varepsilon'$ which is a contradiction. As $\{\varepsilon', \alpha_i\} \in \mathcal{P}$, we deduce $\varepsilon' \subseteq \alpha_i$. For $R \in \partial^2\alpha_k \cap \partial^2\varepsilon = \partial^2\alpha_1 \cap \partial^2\alpha_k$ (cf. Lemma (1.5.2) and Lemma (1.4.8)(a)) we deduce $\emptyset \neq R \cap \alpha_1 \subseteq \alpha_i$ and hence, as $R \notin \partial^2\alpha_i$ because $\{r_{\alpha_i}, r_{\alpha_k}, r_\varepsilon\}$ is a reflection triangle, that $R \subseteq \alpha_i$ and $\beta_i = \alpha_i$. As $-\alpha_i = -\beta_i \subseteq \beta_{\varepsilon'} \cap (-\varepsilon')$, we deduce $\beta_{\varepsilon'} = -\varepsilon'$ and hence $(\varepsilon', \alpha_i) = (-\beta_{\varepsilon'}, \beta_i) = (\beta_{\varepsilon'}, -\beta_i) = \emptyset$. For $R \in \partial^2\alpha_k \cap \partial^2\alpha_i$ we have $R \notin \partial^2\varepsilon \cup \partial^2\varepsilon'$, as $\partial^2\varepsilon \cap \partial^2\alpha_k = \partial^2\alpha_1 \cap \partial^2\alpha_k = \partial^2\varepsilon' \cap \partial^2\alpha_k$ and $\partial^2\alpha_1 \cap \partial^2\alpha_i \cap \partial^2\alpha_k = \emptyset$. Assume that $R \subseteq \varepsilon$. As $(\varepsilon, \varepsilon') = \emptyset$, Lemma (1.4.8)$(b)$ yields $(\varepsilon, -\varepsilon') \cap \{\alpha_k, -\alpha_k\} \neq \emptyset$. In particular, we have $(-\varepsilon') \cap \varepsilon \subseteq \beta_k$ for some $\beta_k \in \{\alpha_k, -\alpha_k\}$. We deduce from $R \in \partial^2\alpha_k, R \subseteq \varepsilon$, that $R \not\subseteq (-\varepsilon')$ and hence $R \subseteq \varepsilon'$. But this would imply that $\emptyset \neq (-\alpha_i) \cap R \subseteq (-\alpha_i) \cap \varepsilon' = \emptyset$, which is a contradiction. As $R \notin \partial^2\varepsilon$, we deduce $R \subseteq (-\varepsilon)$ and hence $\beta_\varepsilon = -\varepsilon$. In particular, we have $(\alpha_i, \varepsilon) = (\beta_i, -\beta_\varepsilon) = \emptyset$. Thus u_i commutes with u_ε and $u_{\varepsilon'}$ and hence with $\prod_{\gamma \in M^G_{\alpha_1, \alpha_k}} u_\gamma$.

(2-n4) The claim is obvious if $o(r_{\alpha_1} r_{\alpha_k}) < \infty$. If $o(r_{\alpha_1} r_{\alpha_k}) = \infty$, then $[u_1, u_k]$ commutes with u_1 by Condition $(b)(i)$.

(2-n5) This follows similar as in (2-n4). □

(7.1.2) Definition. Let $H = (c_0, \ldots, c_k)$ be a gallery in $\Sigma(W, S)$.

(a) H is said to be *of type* $(n, r) \in \mathbb{N}^\star \times S$, if $S = \{r, s, t\}$ and the gallery H is of type $(u, r, r_{\{s,t\}}, \ldots, r, r_{\{s,t\}}, v)$ for some $u, v \in \{1_W, s, t\}$, where $r_{\{s,t\}}$ appears n times in the type of H. We note that $(1_W, c_0^{-1}c_1, \ldots, c_0^{-1}c_k)$ is a minimal gallery by Lemma (5.1.2) and [2, Lemma 2.15] and so is H.

(b) Let H be of type $(n, r) \in \mathbb{N}^\star \times S$ and let $\alpha, \beta \in \Phi$. We say that H is *between* α and β, if $c_0 \in \alpha, c_{k-1} \in \beta$ and $\{c_0, c_1\} \in \partial\alpha, \{c_{k-1}, c_k\} \in \partial\beta$. In this case we let for each $1 \leq i \leq n$ the roots $\omega_i \neq \omega_i' \in \Phi$ be the two roots with $\{c_{k_i+1}, c_{k_i+2}\} \in \partial\omega_i, c_{k_i+1} \in \omega_i$ and $\{c_{k_i+2}, c_{k_i+3}\} \in \partial\omega_i', c_{k_i+2} \in \omega_i'$, where $k_i = \ell\left(ur(r_{\{s,t\}}r)^{i-1}\right)$. Note that if R_i is the $\{s,t\}$-reside containing c_{k_i}, then $c_{k_i} = \operatorname{proj}_{R_i} c_0$. Using Lemma (5.1.4), we deduce $\alpha \subsetneq \omega_1, \omega_1' \subsetneq \cdots \subsetneq \omega_n, \omega_n' \subsetneq \beta$. We should remark that if $\alpha, \beta \in \Phi_+$, then not all of the roots crossed by H are necessarily positive roots. But the roots ω_i, ω'_i are. Consider for example the case $c_0 = trt$ and H is of type $(r, r_{\{s,t\}}, r, \dots, r_{\{s,t\}}, r)$.

In the next definition we will define subsets $M(n,r,L)^G_{\alpha,\beta} \subseteq (\alpha,\beta)$, where $3 \leq n \in \mathbb{N}, r \in S$ and $L \subseteq \{2,\dots,n-1\}$. To have an intuition in mind, we will describe these symbols here: n and r mean that there exists a minimal gallery of type (n,r) between α and β. The subset L indicates, which of the ω_i, ω'_i are contained in the set $M(n,r,L)^G_{\alpha,\beta}$.

(7.1.3) Definition. (a) Let $S = \{r,s,t\}$, let $3 \leq n \in \mathbb{N}$ and let $L \subseteq \{2,\dots,n-1\}$. Let $G \in \mathrm{Min}$ and suppose $\alpha, \beta \in \Phi(G)$ with $\alpha \leq_G \beta$. If $o(r_\alpha r_\beta) < \infty$, then we define

$$M(n,r,L)^G_{\alpha,\beta} := \begin{cases} (\alpha,\beta) & \text{if } |(\alpha,\beta)| = 2; \\ \emptyset & \text{else.} \end{cases}$$

Now we consider the case $o(r_\alpha r_\beta) = \infty$. Suppose that there exists a minimal gallery $H = (c_0, \dots, c_k)$ of type (n,r) between α and β. Let $\omega_i \neq \omega'_i$ be as in Definition (7.1.2)(b). As $\alpha, \beta \in \Phi(G)$ and $\alpha \subseteq \omega_i, \omega'_i \subseteq \beta$, we also have $\omega_i, \omega'_i \in \Phi(G)$ and we define

$$M(n,r,L)^G_{\alpha,\beta} := \{\omega_i, \omega'_i \mid i \in L\}$$

Note that $\omega_i, \omega'_i \subsetneq \omega_{i+1}, \omega'_{i+1}$ and hence $\omega_i, \omega'_i \leq_G \omega_{i+1}, \omega'_{i+1}$, but the order on $\{\omega_i, \omega'_i\}$ depends on G. For all other prenilpotent pairs of positive roots we put $M(n,r,L)^G_{\alpha,\beta} := \emptyset$.

(b) Let $\emptyset \neq K \subseteq \mathbb{N}_{\geq 3}$, let $\mathcal{J} = (J_k)_{k \in K}$ be a family of subsets $\emptyset \neq J_k \subseteq S$ and let $\mathcal{L} = \left(L^j_k\right)_{k \in K, j \in J_k}$ be a family of subsets $L^j_k \subseteq \{2,\dots,k-1\}$. Let $G \in \mathrm{Min}$ and suppose $\alpha, \beta \in \Phi(G)$ with $\alpha \leq_G \beta$. Then we define

$$M(K, \mathcal{J}, \mathcal{L})^G_{\alpha,\beta} := \bigcup_{k \in K, j \in J_k} M\left(k, j, L^j_k\right)^G_{\alpha,\beta}.$$

Moreover, we let $\mathcal{M}(K, \mathcal{J}, \mathcal{L}) := \left(M(K, \mathcal{J}, \mathcal{L})^G_{\alpha,\beta}\right)_{(G,\alpha,\beta) \in \mathcal{I}}$.

(7.1.4) Remark. In Definition (7.1.3) we have defined the sets $M(n,r,L)^G_{\alpha,\beta}$. Note that this set does actually not depend on G: in the case $o(r_\alpha r_\beta) < \infty$, the subset $M(n,r,L)^G_{\alpha,\beta}$ depends only on $|(\alpha,\beta)|$; in the case $o(r_\alpha r_\beta) = \infty$, the subset $M(n,r,L)^G_{\alpha,\beta}$ depends only on the existence of a suitable minimal gallery which crosses $\partial\alpha$ and $\partial\beta$.

(7.1.5) Lemma. *Let $\alpha, \beta \in \Phi_+$ be two roots, let $n \in \mathbb{N}_{\geq 2}$ and $S = \{r,s,t\}$. Suppose that there exists a minimal gallery $H = (c_0, \dots, c_k)$ of type (n,r) between α and β. Then the following hold:*

(a) We can extend $(c_6, \dots, c_k)$ to a minimal gallery contained in Min.

(b) We have $R_{\{s,t\}}(c_7) \in \bigcup_{i \in \mathbb{N}} \mathcal{T}_{i,1}$.

(c) Let $R \in \partial^2\alpha$ be a residue such that α is a non-simple root of R. If $\{c_0, c_1\} \not\subseteq R$, then there exists a simple root of R, say $\gamma \in \Phi_+$, such that $-\gamma \subseteq \beta$.

Proof. We prove (a) and (b) simultaneously. We define $T := R_{\{s,t\}}(c_7)$ and $j := \ell(\mathrm{proj}_T 1_W)$. Recall that the type of H is given by $(u, r, r_{\{s,t\}}, \dots, r, r_{\{s,t\}}, r, v)$, where $u, v \in \{1_W, s, t\}$ and $r_{\{s,t\}}$ appears n times. Suppose $u = 1_W$. Then $\ell(c_0 r) = \ell(c_0) + 1$. If $\ell(c_0 rs) = \ell(c_0 r) + 1 =$

$\ell(c_0rt)$, we can extend H to a gallery contained in Min. Moreover, Lemma (5.1.2) implies $T \in \mathcal{T}_{j,1}$. If $\ell(c_0rs) = \ell(c_0)$, we deduce from Lemma (5.1.2) that $\ell(c_0rst) = \ell(c_0) + 1$. Hence we can extend $(c_5, \ldots, c_k)$ to a gallery contained in Min. Moreover, Lemma (5.1.2) implies $T \in \mathcal{T}_{j,1}$. The same holds if $\ell(c_0rt) = \ell(c_0)$. Now we suppose $u = s$ (the case $u = t$ is symmetric). Again we note that $\ell(c_0s) = \ell(c_0) + 1$. If $\ell(c_0sr) = \ell(c_0)$, Lemma (5.1.2) yields $\ell(c_0srt) = \ell(c_0) + 1$. Thus we can extend $(c_6, \ldots, c_k)$ to a gallery contained in Min. Moreover, Lemma (5.1.2) implies $T \in \mathcal{T}_{j,1}$. Suppose that $\ell(c_0sr) = \ell(c_0) + 2$. Note that Lemma (5.1.2) implies that $\ell(c_0srt) = \ell(c_0) + 3$. If s increases the length of c_0sr, then we can extend H to a gallery contained in Min. Moreover, Lemma (5.1.2) implies $T \in \mathcal{T}_{j,1}$. Otherwise, Lemma (5.1.2) again implies $\ell(c_0srst) = \ell(c_0) + 2$ and we can extend $(c_6, \ldots, c_k)$ to a gallery contained in Min. Moreover, Lemma (5.1.2) implies $T \in \mathcal{T}_{j,1}$. In any case we can extend $(c_6, \ldots, c_k)$ to a gallery $\Gamma \in$ Min and we have $T \in \mathcal{T}_{j,1}$. This proves the Assertions (a) and (b).

To prove Assertion (c), we suppose $\{c_0, c_1\} \not\subseteq R$. As $P_\alpha \subseteq R$, we have $P_\alpha \neq \{c_0, c_1\}$. Let $P_0 = P_\alpha, \ldots, P_n = \{c_0, c_1\}$ and $R_1, \ldots, R_n$ be as in Lemma (1.4.2). For every $1 \leq i \leq n$ we define $w_i := \mathrm{proj}_{R_i} 1_W$, we let $\{x, y\}$ be the type of R_n, we let $\{x\}$ be the type of $\{c_0, c_1\}$ and we let $S = \{x, y, z\}$. We note the following:

(i) $\mathrm{proj}_{R_n} 1_W = \mathrm{proj}_{P_{n-1}} 1_W$: Depending on H one of the following roots is contained in β by Lemma (5.1.4): α_K, where $K = (w_n, \ldots, w)$ is of type $(x, y, z, x), (x, y, x, z, y)$ or (x, y, x, y, z). Note that if K is of type (x, y), then it is contained in the three previous roots by Lemma (5.1.4).

(ii) $\mathrm{proj}_{R_n} 1_W \neq \mathrm{proj}_{P_{n-1}} 1_W$: Depending on H one of the following roots is contained in β by Lemma (5.1.4): α_K, where $K = (w_n, \ldots, w)$ is of type $(x, y, x, y, z), (x, y, x, z)$ or (y, x, y, z, x). Note that if K is of type (x, y), then is is contained in the previous three roots by Lemma (5.1.4)

Thus it suffices to show that there exists a simple root γ of R such that $-\gamma \subseteq \alpha_K$, where $K = (w_n, \ldots, w)$ is of type (x, y). We distinguish the following cases:

(a) $R = R_1$: Then we have $n \geq 2$ (as $P_n \not\subseteq R$) and $\mathrm{proj}_{R_n} 1_W = \mathrm{proj}_{P_{n-1}} 1_W$ by Lemma (5.2.2). Let $\gamma \in \Phi_+$ be the simple root of R which does not contain P_α. We first suppose $n = 2$. Using Lemma (5.1.4) we deduce that $-\gamma$ is contained in all three roots α_K mentioned in (i). Moreover, $-\gamma \subseteq \alpha_K$ holds, where $K = (w_2, \cdots, w)$ is of type (y, x). Now we assume $n \geq 3$. Using induction, $-\gamma$ is contained in a non-simple root of R_{n-1}. As such a root is contained in both non-simple roots of R_n by Lemma (5.1.4), it follows that $-\gamma \subseteq \alpha_K$ holds, where $K = (w_n, \ldots, w)$ is of type (x, y).

(b) $R \neq R_1$: Let $\gamma \in \Phi_+$ be the simple root of R containing P_α. We prove by induction on n, that $-\gamma \subseteq \alpha_K$, where $K = (w_n, \ldots, w)$ is of type (x, y). We first suppose $n = 1$. If $\mathrm{proj}_{R_1} 1_W \neq \mathrm{proj}_{P_0} 1_W$, then $-\gamma$ is contained in α_K, where $K = (w_1, \ldots, w)$ is of type (x, y), by Lemma (5.1.4). Thus we can assume that $\mathrm{proj}_{R_1} 1_W = \mathrm{proj}_{P_0} 1_W$. We see again, that $-\gamma \subseteq \alpha_K$ holds, where $K = (w_1, \ldots, w)$ is of type (x, y). Now we suppose $n > 1$. Using induction, $-\gamma$ is contained in a non-simple root of R_{n-1}. As such a root is contained both non-simple roots of R_n by Lemma (5.1.4), it follows that $-\gamma \subseteq \alpha_K$ holds, where $K = (w_n, \ldots, w)$ is of type (x, y). □

(7.1.6) Lemma. *Let* $\emptyset \neq K \subseteq \mathbb{N}_{\geq 3}$, *let* $\mathcal{J} = (J_k)_{k \in K}$ *be a family of subsets* $\emptyset \neq J_k \subseteq S$ *and let* $\mathcal{L} = \left(L_k^j\right)_{k \in K, j \in J_k}$ *be a family of subsets* $L_k^j \subseteq \{2, \ldots, k-1\}$. *Then* $\mathcal{M}(K, \mathcal{J}, \mathcal{L})$ *is a Weyl-invariant,* 2*-nilpotent pre-commutator blueprint of type* $(4, 4, 4)$.

Proof. We abbreviate $M^G_{\alpha,\beta} := M(K, \mathcal{J}, \mathcal{L})^G_{\alpha,\beta}$ for all $(G, \alpha, \beta) \in \mathcal{I}$. By definition, we have $M^G_{\alpha,\beta} \subseteq (\alpha, \beta)$. Clearly, (CB1) and (CB2) hold. To show that (PCB) holds, we let $w \in W$ and $G \in \mathrm{Min}(w)$. Then we have a homomorphism $U_G \to U_w$. It suffices to show that we have a homomorphism $U_w \to U_G$ extending $u_\alpha \mapsto u_\alpha$. Let $F \in \mathrm{Min}(w)$ and let $\alpha \leq_F \beta \in \Phi(F)$. At first we assume $o(r_\alpha r_\beta) < \infty$. We distinguish the following two cases:

(i) $\alpha \leq_G \beta$: Then we have $M^F_{\alpha,\beta} = M^G_{\alpha,\beta}$ by definition and we are done.

(ii) $\beta \leq_G \alpha$: If $|(\alpha, \beta)| < 2$, then $M^F_{\alpha,\beta} = \emptyset = M^G_{\beta,\alpha}$ and we are done. Thus we assume $(\alpha, \beta) = \{\delta, \gamma\}$ and $\delta \leq_F \gamma$. Then $\gamma \leq_G \delta$ and we have the following relation in U_G:

$$[u_\alpha, u_\beta] = [u_\beta, u_\alpha]^{-1} = (u_\gamma u_\delta)^{-1} = u_\delta u_\gamma$$

Thus we can consider the case $o(r_\alpha r_\beta) = \infty$. Then we have $\alpha \leq_G \beta$. If there is no gallery H of type (n, r) between α and β with $n \in K$ and $r \in J_n$, then $M^F_{\alpha,\beta} = \emptyset = M^G_{\alpha,\beta}$. Suppose that there exists a gallery H of type (n, r) between α and β for some $n \in K$ and $r \in J_n$. Then $M^F_{\alpha,\beta} = \{\omega_i, \omega'_i \mid i \in L^r_n\} = M^G_{\alpha,\beta}$ as sets. Note that $\omega_i, \omega'_i \leq_G \omega_{i+1}, \omega'_{i+1} \geq_F \omega_i, \omega'_i$. As $M^G_{\omega_i,\omega'_i} = \emptyset$, we deduce that $[u_\alpha, u_\beta] = \prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma = \prod_{\gamma \in M^F_{\alpha,\beta}} u_\gamma$ is a relation in U_G. Thus we obtain a homomorphism $U_w \to U_G$ and the universal property implies that (PCB) holds. In particular, $\mathcal{M}(K, \mathcal{J}, \mathcal{L})$ is a pre-commutator blueprint of type $(4, 4, 4)$.

Now we show that $\mathcal{M}(K, \mathcal{J}, \mathcal{L})$ is Weyl-invariant. Let $1 \neq w \in W, s \in S, G \in \mathrm{Min}(w)$ and let $\alpha \leq_G \beta \in \Phi(G)$. We distinguish the following cases:

- $\ell(sw) = \ell(w) + 1$: If $o(r_\alpha r_\beta) < \infty$, then $o(r_{s\alpha} r_{s\beta}) < \infty$ and, as $(s\alpha, s\beta) = \{s\gamma \mid \gamma \in (\alpha, \beta)\}$, we infer $M^{sG}_{s\alpha,s\beta} = sM^G_{\alpha,\beta}$. Thus we can assume $o(r_\alpha r_\beta) = \infty$. Suppose that there exists a gallery $H = (c_0, \ldots, c_k)$ of type (n, r) between α and β for some $n \in K, r \in J_n$. Then $(sc_0, \ldots, sc_k)$ is a gallery of type (n, r) between the roots $s\alpha, s\beta$. This implies that a gallery of type (n, r) exists between the roots α and β if and only if a gallery of type (n, r) exists between the roots $s\alpha$ and $s\beta$. This finishes the claim.

- $\ell(sw) = \ell(w) - 1$ and $G \in \mathrm{Min}_s(w)$. Moreover, we assume $\alpha_s \neq \alpha \leq_G \beta$. Using the same arguments as above, the claim follows.

We will apply Proposition (7.1.1) to show that $\mathcal{M}(K, \mathcal{J}, \mathcal{L})$ is 2-nilpotent. Let $G \in \mathrm{Min}$ and let $\alpha, \beta \in \Phi(G)$ be two roots such that $\alpha \neq \beta, \alpha \leq_G \beta$ and $o(r_\alpha r_\beta) < \infty$ hold. Without loss of generality we can assume $M^G_{\alpha,\beta} \neq \emptyset$. Suppose that $\varepsilon \in \Phi(G)$ is such that $\varepsilon \subsetneq \gamma$ holds for all $\gamma \in M^G_{\alpha,\beta}$. If $M^G_{\varepsilon,\gamma} = \emptyset$ for all $\gamma \in M^G_{\alpha,\beta}$, we are done. Thus we can assume that there exists $\gamma \in M^G_{\alpha,\beta}$ with $M^G_{\varepsilon,\gamma} \neq \emptyset$. Then there exists a minimal gallery $H = (c_0, \ldots, c_k)$ of type (n, r) between the roots ε and γ for some $n \in K$ and $r \in J_n$, i.e. the type of H is given by $(u, r, r_{\{s,t\}}, r, \ldots, r_{\{s,t\}}, r, v)$, where $u, v \in \{1_W, s, t\}$ and $r_{\{s,t\}}$ appears n times. Using Lemma (7.1.5)(a), we can extend $(c_6, \ldots, c_k)$ to a gallery $\Gamma \in \mathrm{Min}$. Let R be the residue of rank 2 containing c_{k-2}, c_{k-1} and c_k. Using Lemma (5.1.2) we deduce that $\gamma = \alpha_\Gamma$ is a non-simple root of R. We distinguish the following cases:

(a) $v = 1$: Then we have $P_\gamma = \{c_{k-1}, c_k\}$ and $P_\gamma \subseteq R$. Let $R' \neq R$ be the other residue of rank 2 containing P_γ. If T is a rank 2 residue such that γ is a non-simple root of T, then $T \in \{R, R'\}$ (cf. Lemma (5.2.3)). Let Γ_x be the gallery Γ extended by an x-adjacent chamber for $x \in \{s, t\}$. Then $(\alpha, \beta) \cap \{\alpha_{\Gamma_s}, \alpha_{\Gamma_t}\} \neq \emptyset$ and we have $M^G_{\alpha,\beta} \in \{\{\gamma, \alpha_{\Gamma_s}\}, \{\gamma, \alpha_{\Gamma_t}\}\}$. As Γ_s, Γ_t are galleries of type (n, r) as well, we deduce $M^G_{\varepsilon,\gamma} = M^G_{\varepsilon,\alpha_{\Gamma_x}}$, where $x \in \{s, t\}$ is such that $\alpha_{\Gamma_x} \in \Phi(G)$.

(b) $v \neq 1$: Using Lemma (5.1.1) and Lemma (5.1.2) we deduce that R is the only residue such that γ is a non-simple root of R. For $K := (c_0, \dots, c_{k-1})$ the root α_K is also a non-simple root of R and K is a gallery of type (n, r). This implies $(\alpha, \beta) = \{\gamma, \alpha_K\}$ and hence $M^G_{\varepsilon,\gamma} = M^G_{\varepsilon,\alpha_K}$.

Now we assume that $\varepsilon \in \Phi(G)$ is such that $\gamma \subsetneq \varepsilon$ holds for all $\gamma \in M^G_{\alpha,\beta}$. If $M^G_{\gamma,\varepsilon} = \emptyset$ holds for all $\gamma \in M^G_{\alpha,\beta}$, we are done. Thus we can assume that there exists $\gamma \in M^G_{\alpha,\beta}$ with $M^G_{\gamma,\varepsilon} \neq \emptyset$. Then there exists a minimal gallery $H = (c_0, \dots, c_k)$ of type (n, r) between γ and ε for some $n \in K$ and $r \in J_n$, i.e. the type of H is given by $(u, r, r_{\{s,t\}}, r, \dots, r_{\{s,t\}}, r, v)$, where $u, v \in \{1_W, s, t\}$ and $r_{\{s,t\}}$ appears n times. Let R be the unique rank 2 residue contained in $\partial^2\alpha \cap \partial^2\beta$ (cf. Lemma (1.4.6)). Then γ is a non-simple root of R. Note that $\alpha, \beta, \varepsilon \in \Phi(G)$ and hence $\{\alpha, \varepsilon\}, \{\beta, \varepsilon\} \in \mathcal{P}$. Then Lemma (7.1.5)$(c)$ yields $\{c_0, c_1\} \subseteq R$ and we distinguish the following two cases:

(a) $u = 1_W$: Assume that $\ell(\mathrm{proj}_R 1_W, c_0) = 1$. Then Lemma (5.1.4) would imply that one of $-\alpha, -\beta$ is contained in one non-simple root of the $\{s,t\}$-residue containing c_1 (i.e. ω_1 or ω_1') and hence one of $-\alpha, -\beta$ is contained in ε. As this is a contradiction, we deduce $\ell(\mathrm{proj}_R 1_W, c_0) = 2$. Let d be the chamber in R adjacent to both $\mathrm{proj}_R 1_W$ and c_0. Then the gallery $(d, c_0, \dots, c_k)$ is of type (n, r) and we have $M^G_{\alpha_K,\varepsilon} = M^G_{\gamma,\varepsilon}$, where $K = (d, c_0)$ (note that $\alpha_K \in (\alpha, \beta)$ and hence $\alpha_K \in \Phi(G)$).

(b) $u \neq 1_W$: Assume $\ell(\mathrm{proj}_R 1_W, c_0) = 2$. In both cases ($c_2 \in R$ and $c_2 \notin R$) Lemma (5.1.4) implies that one of $-\alpha, -\beta$ would be contained in ω_2, ω_2' and hence in ε, which is a contradiction. Thus $\ell(\mathrm{proj}_R 1_W, c_0) = 1$. Again, if $c_2 \notin R$, then Lemma (5.1.4) would imply that one of $-\alpha, -\beta$ is contained in ω_1, ω_1', which is a contradiction. Note that α_K with $K = (c_1, c_2)$ is also a non-simple root of R and hence $(\alpha, \beta) = \{\gamma, \alpha_K\}$. As $(c_1, \dots, c_k)$ is gallery of type (n, r) and α_K is the first root which is crossed by this gallery, we deduce $M^G_{\alpha_K,\varepsilon} = M^G_{\gamma,\varepsilon}$ and the claim follows.

Thus Condition (a) holds. Now we will show that Condition $(b)(i)$ holds. Let $G \in \mathrm{Min}$ and let $\alpha \neq \beta \in \Phi(G)$ be two roots with $o(r_\alpha r_\beta) = \infty$, let $G = (d_0, \dots, d_n = c_0, \dots, c_k = e_0, \dots, e_m)$ and suppose that $\{c_0, c_1\} \in \partial\alpha$ and $\{c_{k-1}, c_k\} \in \partial\beta$. If $M^G_{\alpha,\beta} = \emptyset$, we are done. Thus we assume can assume that $M^G_{\alpha,\beta} \neq \emptyset$. Then there exists a gallery of type (n, r) between α and β for some $n \in K$ and $r \in J_n$. In particular, we have $M^G_{\alpha,\beta} = \{\omega_i, \omega_i' \mid i \in L^r_n\}$. Note that $\{\omega_i, \omega_i'\} = M^G_{\gamma_i,\gamma_i'}$ for some $\gamma_i \leq_G \gamma_i' \in \Phi(G)$ with $\alpha \subseteq \gamma_i \leq_G \gamma_i' \subseteq \beta$, as $L^r_n \subseteq \{2, \dots, n-1\}$. We show that $u_{\omega_p} u_{\omega_p'} \in Z(U_{(d_i,\dots,d_n=c_0,\dots,c_k=e_0,\dots,e_j)})$ for all $0 \leq i \leq n, 0 \leq j \leq m$ and $u_{\omega_p} u_{\omega_p'} \in Z(U_{(c_1,\dots,c_{k-1}),G})$ for all $p \in L^r_n$. This will imply that $\prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma = \prod_{p \in L^r_n} u_{\omega_p} u_{\omega_p'}$ is contained in the center and we are done. Note that the order on $\{\omega_p, \omega_p'\}$ depends on G.

Let $(\beta_1, \dots, \beta_{n+k+m})$ be the sequence of roots crossed by G and let $\varepsilon \in \Phi(G)$. Then it suffices to show that $u_{\omega_p} u_{\omega_p'}$ commutes with u_ε in

- $U_{(c_1,\dots,c_{k-1}),G}$, if $\varepsilon = \beta_{n+q}$ for some $1 \leq q \leq n-1$;
- $U_{(d_i,\dots,d_n=c_0,\dots,c_k=e_0,\dots,e_j)}$, where $(i,j) = \begin{cases} (q, 0) & \text{if } 0 \leq q \leq n, \varepsilon = \beta_q \\ (n, n+k+q) & \text{if } 0 \leq q \leq m, \varepsilon = \beta_{n+k+q} \end{cases}$

If $\varepsilon \in \{\omega_p, \omega_p'\}$, then clearly $u_{\omega_p} u_{\omega_p'}$ commutes with u_ε and we can assume $\varepsilon \notin \{\omega_p, \omega_p'\}$. We distinguish the following cases:

(a) $\{\varepsilon, \omega_p\}$ and $\{\varepsilon, \omega_p'\}$ are nested: At first we assume $\varepsilon \subseteq \omega_p$. As $o(r_{\omega_p} r_{\omega_p'}) < \infty$, we deduce $\varepsilon \subseteq \omega_p'$. Now Condition $(a)(i)$ implies $M^G_{\varepsilon,\omega_p} = M^G_{\varepsilon,\omega_p'}$ and hence

$$[u_\varepsilon, u_{\omega_p} u_{\omega_p'}] = [u_\varepsilon, u_{\omega_p'}][u_\varepsilon, u_{\omega_p}]^{u_{\omega_p'}}$$

$$= \left(\prod_{\gamma \in M^G_{\varepsilon,\omega'_p}} u_\gamma \right) \left(\prod_{\gamma \in M^G_{\varepsilon,\omega_p}} u_\gamma \right)^{u_{\omega'_p}}$$
$$= \left(\prod_{\gamma \in M^G_{\varepsilon,\omega'_p}} u_\gamma \right) \left(\prod_{\gamma \in M^G_{\varepsilon,\omega'_p}} u_\gamma \right)^{u_{\omega'_p}}$$
$$= [u_\varepsilon, u_{\omega'_p}][u_\varepsilon, u_{\omega'_p}]^{u_{\omega'_p}}$$
$$= [u_\varepsilon, u^2_{\omega'_p}] = 1$$

If $\omega_p \subseteq \varepsilon$, we infer $\omega'_p \subseteq \varepsilon$ similarly. Condition $(a)(ii)$ implies $M^G_{\omega_p,\varepsilon} = M^G_{\omega'_p,\varepsilon}$ and hence $[u_{\omega_p} u_{\omega'_p}, u_\varepsilon] = [u_{\omega_p}, u_\varepsilon]^{u_{\omega'_p}} [u_{\omega'_p}, u_\varepsilon] = [u_{\omega'_p}, u_\varepsilon]^{u_{\omega'_p}} [u_{\omega'_p}, u_\varepsilon] = 1$.

(b) One of $\{\varepsilon, \omega_p\}$ and $\{\varepsilon, \omega'_p\}$ is not nested: As $\{\varepsilon, \omega_p\}, \{\varepsilon, \omega'_p\} \in \mathcal{P}$, Lemma (1.4.7) and [2, Lemma 8.42(3)] yield $R \in \left(\partial^2\varepsilon \cap \partial^2\omega_p\right) \cup \left(\partial^2\varepsilon \cap \partial^2\omega'_p\right)$. Let T be the residue of rank 2 with $T \in \partial^2\omega_p \cap \partial^2\omega'_p$. If $T \in \partial^2\varepsilon$, then $\varepsilon \in \{\gamma_i, \gamma'_i\}$ and the claim follows. Thus we can assume $T \notin \partial^2\varepsilon$ and hence $T \neq R$. Recall that $\{\varepsilon, \gamma_i\}, \{\varepsilon, \gamma'_i\} \in \mathcal{P}$, as $\varepsilon, \gamma_i, \gamma'_i \in \Phi(G)$. Without loss of generality we can assume that $R \in \partial^2\varepsilon \cap \partial^2\omega_p$. Using Remark (1.4.4) there exists $Q' \in \partial\omega_p$ with $Q' \subseteq R$. Using Lemma (1.4.2) and the fact that $Q' = \operatorname{proj}_{Q'} P_{\omega_p} = \operatorname{proj}_{Q'} \operatorname{proj}_R P_{\omega_p}$, [18, Lemma 13] yields that P_{ω_p} and $\operatorname{proj}_R P_{\omega_p}$ are parallel. Lemma (1.4.2) implies $\operatorname{proj}_R P_{\omega_p} \in \partial\omega_p$. Let $Q \subseteq R$ be opposite to $\operatorname{proj}_R P_{\omega_p}$ in R and let $P_0 := P_{\omega_p}, \dots, P_n := \operatorname{proj}_R P_0$ and $R_1, \dots, R_n$ be as in Lemma (1.4.2). Note that $\operatorname{proj}_R P_{\omega_p}$ and Q are parallel by [2, Proposition 5.114] and, in particular, P_{ω_p} and Q are parallel. It follows from [18, Lemma 17] that $P_0, \dots, P_n, Q$ and $R_1, \dots, R_n, R$ is as in Lemma (1.4.2). If $T \neq R_1$, Lemma (5.2.4)(a) yields a contradiction, as $T \in \mathcal{T}_{j,1}$ for some j. If $T = R_1$, we have $n > 1$. If $n > 2$, Lemma (5.2.4)(c) yields a contradiction. For $n = 2$ Lemma (5.2.4)(c) either yields directly a contradiction, or else yields that $(\varepsilon, \omega_p) = \emptyset = (\varepsilon, \omega'_p)$ and hence u_ε commutes with $u_{\omega_p} u'_{\omega_p}$ (one can show that even this case does not occur).

We have seen that $u_{\omega_i} u_{\omega'_i} \in Z(U_{(c_1,\dots,c_{k-1}),G})$ for every $i \in L^r_n$. In particular, $u_{\omega_i} u_{\omega'_i}$ and $u_{\omega_j} u_{\omega'_j}$ do commute for $i, j \in L^r_n$. We infer the following in $U_{(c_1,\dots,c_{k-1}),G}$:

$$\left(\prod_{\gamma \in M^G_{\alpha,\beta}} u_\gamma \right)^2 = \left(\prod_{i \in L^r_n} u_{\omega_i} u_{\omega'_i} \right)^2 = \prod_{i \in L^r_n} (u_{\omega_i} u_{\omega'_i})^2 = 1. \qquad \square$$

(7.1.7) Theorem. *Let $\emptyset \neq K \subseteq \mathbb{N}_{\geq 3}$, let $\mathcal{J} = (J_k)_{k \in K}$ be a family of subsets $\emptyset \neq J_k \subseteq S$ and let $\mathcal{L} = \left(L^j_k\right)_{k \in K, j \in J_k}$ be a family of subsets $L^j_k \subseteq \{2, \dots, k-1\}$. Then $\mathcal{M}(K, \mathcal{J}, \mathcal{L})$ is an integrable commutator blueprint of type $(4,4,4)$ and the groups U_w are of nilpotency class at most 2.*

Proof. By Lemma (7.1.6), $\mathcal{M}(K, \mathcal{J}, \mathcal{L})$ is a Weyl-invariant and 2-nilpotent pre-commutator blueprint of type $(4,4,4)$. By Lemma (4.2.2), $\mathcal{M}(K, \mathcal{J}, \mathcal{L})$ is a commutator blueprint of type $(4,4,4)$ and the groups U_w are of nilpotency class at most 2. By Corollary (6.10.6) it is faithful. By Theorem (3.5.1) it suffices to show that $\mathcal{M}(K, \mathcal{J}, \mathcal{K})$ satisfies (CR1) and (CR2).

Let $s \neq t \in S, \beta \in \Phi^{\{s,t\}}, \alpha_s \subsetneq \alpha \in \Phi_+, w \in W$ with $\ell(sw) = \ell(w) - 1$, let $G \in \mathrm{Min}_s(w)$ with $\alpha \in \Phi(G)$ and assume $M^G_{\alpha_s,\alpha} \neq \emptyset$. Then there exists a minimal gallery $H = (c_0, \dots, c_k)$

of type (k,r) between α_s and α for some $k \in K$ and $r \in J_k$ (note that $r = s$ is possible). Let ω_i, ω_i' be the roots as in Definition (7.1.2)(b). We show that either $\beta \subseteq \omega_2, \omega_2'$ or else $-\beta \subseteq \omega_2, \omega_2'$. We distinguish the following cases:

(i) H is of type $(s, r_{\{r,t\}}, s, \ldots, r_{\{r,t\}}, s, v)$: Using Lemma (5.1.4) we deduce $\alpha_{H_2}, \alpha_{H_5}, \alpha_{H_6} \subseteq \omega_2, \omega_2'$. If $c_0 = 1_W$, then $-t\alpha_s, -\alpha_t \subseteq s\alpha_r \in \{\alpha_{H_2}, \alpha_{H_5}\}$ and hence they are all contained in ω_2, ω_2'. Moreover, we have $s\alpha_t \in \{\alpha_{H_2}, \alpha_{H_5}\}$ and hence the claim follows, as $\Phi^{\{s,t\}} = \{\pm\alpha_s, \pm\alpha_t, \pm s\alpha_t, \pm t\alpha_s\}$. Now we suppose $c_0 \neq 1_W$. Let $P = \{1_W, s\}$ and $Q = \{c_0, c_1\}$. Then $P, Q \in \partial\alpha_s$. Let $P_0 = P, \ldots, P_n = Q$ and $R_1, \ldots, R_n$ be two sequences as in Lemma (1.4.2). As $\mathrm{proj}_{R_1} 1_W = 1_W = \mathrm{proj}_{P_0} 1_W$, Corollary (1.5.5) implies $\mathrm{proj}_{R_n} 1_W = \mathrm{proj}_{P_{n-1}} 1_W$. As $1_W, c_0 \in \alpha$ and roots are convex, we deduce $c_0 = \mathrm{proj}_{P_n} 1_W \in \alpha$. It follows from Lemma (1.4.2) that P_{n-1} and P_n are opposite in R_n. Thus there exists $d \in P_n$ such that $\mathrm{proj}_{R_n} 1_W, d$ are opposite in R_n. As $\mathrm{proj}_{R_n} 1_W$ and $c_0 = \mathrm{proj}_{P_n} 1_W = \mathrm{proj}_{P_n} \mathrm{proj}_{R_n} 1_W$ are not opposite in R_n, we deduce $d = c_1$ and hence $\ell(c_1 s) = \ell(c_1) - 1 = \ell(c_1 x)$, where $s \neq x \in S$ is such that $\{s, x\}$ is the type of R_n. Let $i \in \{3, 4\}$ be such that c_{i-1}, c_i are contained in an x-panel. Using Lemma (5.1.4) we see that both non-simple roots of $R := R_n = R_{\{s,x\}}(c_1)$ are contained in α_{H_i}. Applying Lemma (5.1.4) again we deduce that α_{H_i} is contained in α_{H_6} and this root is already known to be contained in ω_2 and ω_2'. Thus the non-simple roots of R are contained in ω_2, ω_2'. If $n = 1$, two things can happen. If R_1 has type $\{s, t\}$, then we have $-\alpha_t = \alpha_{H_5}$ and the claim follows. If R_1 does not have type $\{s, t\}$, then R_1 has type $\{r, s\}$ and each root in $\{-\alpha_t, -s\alpha_t, -t\alpha_s\}$ is contained in a non-simple root of R_1. This finishes the claim. If $n > 1$ it follows from Lemma (5.1.5) that there exists $\varepsilon \in \{+, -\}$ such that for every root $\delta \in \{\varepsilon\alpha_t, \varepsilon s\alpha_t, \varepsilon t\alpha_s\}$ there exists a non-simple root γ of R_n with $\delta \subseteq \gamma$. As those are contained in ω_2, ω_2', the claim follows.

(ii) H is of type $(s, x, r_{\{s,y\}}, r, \ldots, r_{\{s,y\}}, r, v)$, where $S = \{s, x, y\}$: Using Lemma (5.1.4) we deduce that $\alpha_{H_2}, \alpha_{H_3}, \alpha_{H_6}, \alpha_{H_7} \subseteq \omega_2, \omega_2'$. Without loss of generality we assume that c_2, c_3 are contained in an s-panel and c_5, c_6 are contained in a y-panel. At first we suppose $c_0 = 1_W$. If $(x, y) = (r, t)$, it follows from Lemma (5.1.4) that $-\alpha_t \subseteq \alpha_{H_6}$ and $-s\alpha_t, -t\alpha_s \subseteq \alpha_{H_3}$. If $(x, y) = (t, r)$, it follows from Lemma (5.1.4) that $-\alpha_t \subseteq \alpha_{H_6}, s\alpha_t = \alpha_{H_2}$ and $t\alpha_s = \alpha_{H_3}$. Thus we can assume $c_0 \neq 1_W$. Let $P = \{1_W, s\}$ and let $Q = \{c_0, c_1\}$. As in the previous case we let $P_0 = P, \ldots, P_n = Q$ and $R_1, \ldots, R_n$ be as in Lemma (1.4.2). Let $q \in S$ be such that $\{s, q\}$ is the type of R_n. We distinguish the following cases:

(a) $q \neq x = r$ and $n = 1$: Then we have $q = y = t$. We deduce from Lemma (5.1.4) that $\alpha_t, t\alpha_s \subseteq \alpha_{H_3}$ and $s\alpha_t \subseteq \alpha_{H_6}$.

(b) $q \neq x = t$ and $n = 1$: Then we have $q = y = r$. We deduce from Lemma (5.1.4) that every root $\delta \in \{-\alpha_t, -s\alpha_t, -t\alpha_s\}$ is contained in $q\alpha_s$ and hence in α_{H_2}.

(c) $q = x = t$ and $n = 1$: Then $-\alpha_t = \alpha_{H_2}, -t\alpha_s = \alpha_{H_3}$ and $s\alpha_t \subseteq \alpha_{H_6}$ by Lemma (5.1.4). This finishes the claim.

(d) $q = x = r$ and $n = 1$: Then $-s\alpha_t$ is contained in α_{H_6} by applying Lemma (5.1.4) and $-\alpha_t, -t\alpha_s$ are contained in α_{H_6} by applying Lemma (5.1.4) twice.

(e) $q \neq x$ and $n = 2$: It follows from Lemma (5.1.5) that there exists $\varepsilon \in \{+, -\}$ such that for every root $\delta \in \{\varepsilon\alpha_t, \varepsilon s\alpha_t, \varepsilon t\alpha_s\}$ there exists a non-simple root γ of R_n with $\delta \subseteq \gamma$. As γ is contained in α_{H_2}, the claim follows.

(f) $q = x = r$ and $n = 2$: Then it follows from Lemma (5.1.4) that $\alpha_t, s\alpha_t, t\alpha_s$ are contained in α_{H_6} and the claim follows.

(g) $q = x = t$ and $n = 2$: Using similar arguments as in the case $q \neq x = t$ and $n = 1$, we deduce that $-\alpha_t, -s\alpha_t, -t\alpha_s$ are contained in α_{H_6} and the claim follows.

(h) $n > 2$: It follows from Lemma (5.1.5) that there exists $\varepsilon \in \{+, -\}$ such that for every root $\delta \in \{\varepsilon\alpha_t, \varepsilon s\alpha_t, \varepsilon t\alpha_s\}$ there exists a non-simple root γ of R_{n-1} with $\delta \subseteq \gamma$. As γ is contained in both non-simple roots of R_n by Lemma (5.1.4), the claim follows.

As $\omega_2 \subseteq \alpha$, we infer $o(r_\beta r_\alpha) = \infty$ and hence (CR2) is satisfied. Moreover, we have $\omega_2 \subseteq \omega_i, \omega_i'$ for all $3 \leq i \leq k$. Suppose that $\beta \subseteq \alpha$. Then we have shown that either $\beta \subseteq \omega_2, \omega_2'$ or else $-\beta \subseteq \omega_2, \omega_2'$. But the latter one would imply $W = \beta \cup (-\beta) \subseteq \alpha$, which is a contradiction. Thus $\beta \subseteq \omega_2, \omega_2'$ and by the above we have $\beta \subseteq \omega_2 \subseteq \omega_i, \omega_i'$ for every $3 \leq i \leq k$. In particular, we have $\beta \subseteq \gamma$ for each $\gamma \in M^G_{\alpha_s,\alpha}$ and (CR1) is satisfied. This finishes the proof. □

(7.1.8) Remark. Let $\emptyset \neq K \subseteq \mathbb{N}_{\geq 3}$, let $\mathcal{J} = (J_k)_{k\in K}$ be a family of subsets $\emptyset \neq J_k \subseteq S$ and let $\mathcal{L} = \left(L_k^j\right)_{k\in K, f\in J_k}$ be a family with $L_k^j = \emptyset$ for all $k \in K, j \in J_k$. Then we have $M(K, \mathcal{J}, \mathcal{L})^G_{\alpha,\beta} = \emptyset$ for all $(G, \alpha, \beta) \in \mathcal{I}$ with $o(r_\alpha r_\beta) = \infty$. Hence this is the commutator blueprint associated with the split Kac-Moody group of type $(4, 4, 4)$ over $\mathbb{F}_2$ (cf. Example (5.3.1)).

(7.1.9) Corollary. *For each $n \in \mathbb{N}$ there exists an RGD-system $\mathcal{D}_n = \left(G_n, \left(U_\alpha^{(n)}\right)_{\alpha\in\Phi}\right)$ of type $(4, 4, 4)$ over $\mathbb{F}_2$ with the following properties:*

(i) If $w \in W$ is such that $\ell(w) \leq n$ and if $\alpha, \beta \in \Phi_+$ are such that $w \in (-\alpha) \cap (-\beta)$ and $\alpha \subseteq \beta$, then $\left[U_\alpha^{(n)}, U_\beta^{(n)}\right] = 1$.

(ii) There exist $\alpha, \beta \in \Phi_+$ such that $\alpha \subsetneq \beta$ and $\left[U_\alpha^{(n)}, U_\beta^{(n)}\right] \neq 1$.

Proof. Note that it suffices to show the claim for $n \in \mathbb{N}_{\geq 3}$. We fix $n \in \mathbb{N}_{\geq 3}$. Let $\emptyset \neq J_n \subseteq S$ and $L_n^j \subseteq \{2, \ldots, n-1\}$ for each $j \in J_n$. Moreover, we assume that $L_n^j \neq \emptyset$ for some $j \in J_n$. We define $\mathcal{J} = (J_k)_{k\in\{n\}}$ and $\mathcal{L} := \left(L_k^j\right)_{k\in\{n\}, j\in J_k}$. Then $\mathcal{M}(\{n\}, \mathcal{J}, \mathcal{L})$ is an integrable commutator blueprint by Theorem (7.1.7). Let $\mathcal{D} = \left(G, (U_\alpha)_{\alpha\in\Phi}\right)$ be its associated RGD-system. We claim that $\mathcal{D}$ is as required. As $L_n^j \neq \emptyset$ for some $j \in J_n$, it suffices to show that (i) holds. Let $w \in W$ and let $\alpha, \beta \in \Phi_+$ be such that $w \in (-\alpha) \cap (-\beta)$ and $\alpha \subseteq \beta$. This means that $U_\alpha, U_\beta \leq U_w$. Suppose that $[U_\alpha, U_\beta] \neq 1$ and that $r \in J_n$. Then there exists a minimal gallery $H = (c_0, \ldots, c_k)$ of type (n, r) between α and β. By Lemma (7.1.5)(a) we can extend $(c_6, \ldots, c_k)$ to a gallery $E = (c_0', \ldots, c_{k'}') \in \mathrm{Min}$. In particular, we have $k' \geq k - 6$.

Let $(e_0, \ldots, e_m) \in \mathrm{Min}(w)$ be a minimal gallery. As $e_0 = 1_W \in \beta$ and $e_m = w \in (-\beta)$, there exists $0 \leq j \leq m - 1$ with $\{e_j, e_{j+1}\} \in \partial\beta$. Using Lemma (5.2.5) there exists a minimal gallery $(d_0 = e_0, \ldots, d_q = e_{j+1})$ such that $d_i = \mathrm{proj}_{R_{\beta,\{e_j,e_{j+1}\}}} 1_W$ for some $0 \leq i \leq q - 1$. As $\{c_{k-1}, c_k\} \subseteq R_{\beta,\{e_j,e_{j+1}\}}$, we deduce that $\ell(d_i) \geq k' - 3 \geq (k - 6) - 3$ and hence $\ell(w) \geq \ell(d_i) \geq k - 9$. By definition, we have $k \geq 5n$. But then $\ell(w) \geq k - 9 \geq 5n - 9 > n$. Thus $\mathcal{D}$ satisfies (i) and we are done. □

(7.1.10) Remark. Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD-system of type $(4, 4, 4)$. It is shown in [7, Theorem A] that if every root group contains at least 3 elements, then $[U_\alpha, U_\beta] = 1$ for all pairs $\{\alpha, \beta\}$ of nested roots. The previous corollary shows that the assumption on the cardinality of the root groups is necessary in order to prove that root groups corresponding to nested roots do commute.

7.2. Extension theorem for twin buildings

The *extension problem* for twin buildings asks whether a given local isometry can be extended to the whole twin building. For more details we refer to the introduction and to [25].

(7.2.1) Theorem. *The extension theorem does not hold for arbitrary thick 2-spherical twin buildings.*

Proof. Let $\mathcal{M}, \mathcal{M}'$ be two different integrable commutator blueprints as constructed in Theorem (7.1.7) and let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi}), \mathcal{D}' = (G', (U'_\alpha)_{\alpha\in\Phi})$ be their associated RGD-systems. We let $\Delta = \Delta(\mathcal{D})$ and $\Delta' = \Delta(\mathcal{D}')$ be the corresponding twin buildings and let $\Sigma = \Sigma(c_+, c_-)$ and $\Sigma' = \Sigma(c'_+, c'_-)$ be the distinguished twin apartments in Δ and Δ'. Let $\{\alpha, \beta\} \in \mathcal{P}$ and $H \in \mathrm{Min}$ be such that $\alpha, \beta \in \Phi(H), \alpha \leq_H \beta$ and $M(\mathcal{D})^H_{\alpha,\beta} \neq M(\mathcal{D}')^H_{\alpha,\beta}$.

Every residue R of Δ or of Δ' of rank 2 is isomorphic to the generalized quadrangle of order $(2, 2)$, i.e. to the building which is associated with the group $C_2(2)$. For each $s \in S$ we fix an order on $\mathcal{P}_s(c_+) = \{c_0 := c_+, c_1, c_2\}$ and on $\mathcal{P}_s(c'_+) = \{c'_0 := c'_+, c'_1, c'_2\}$. Note that the mapping $\varphi_s : \mathcal{P}_s(c_+) \to \mathcal{P}_s(c'_+), c_i \mapsto c'_i$ is a bijection and hence an isometry. We will show that for all $s \neq t \in S$ there exists an isometry $\varphi_{\{s,t\}} : R_{\{s,t\}}(c_+) \to R_{\{s,t\}}(c'_+)$ with $\varphi_{\{s,t\}}|_{\mathcal{P}_s(c_+)} = \varphi_s$.

Let $s \neq t \in S$ and define $J := \{s, t\}$. Using the fact that the automorphism group of the generalized quadrangle of order $(2, 2)$ acts transitive on the chambers, we obtain an isometry $R_J(c_+) \to R_J(c'_+)$ mapping c_+ onto c'_+. Using the *root automorphisms* (if necessary), we obtain an isometry $\varphi_J : R_J(c_+) \to R_J(c'_+)$ with $\varphi_J|_{\mathcal{P}_s(c_+)} = \varphi_s$. Thus we obtain a bijection $\varphi : E_2(c_+) \to E_2(c'_+)$ such that for all $s \neq t \in S$ and $x \in R_{\{s,t\}}(c_+)$ we have $\varphi(x) = \varphi_{\{s,t\}}(x)$. Note that φ is an isometry by [38, Proposition 4.2.4]. Using [38, Proposition 7.1.6] there exist $d \in c_+^{\mathrm{op}}, d' \in (c'_+)^{\mathrm{op}}$ such that φ extends to an isometry $E_2(c_+) \cup \{d\} \to E_2(c'_+) \cup \{d'\}$. Assume that the extension theorem would hold for Δ. Then we can extend this isometry to an isometry $\Phi : \Delta \to \Delta'$. Moreover, $\Psi : \mathrm{Aut}(\Delta) \to \mathrm{Aut}(\Delta'), f \mapsto \Phi \circ f \circ \Phi^{-1}$ is an isomorphism. Let $g \in G$ be such that $g(\Sigma) = A(c_+, d)$ and let $g' \in G'$ be such that $g'(\Sigma') = A(c'_+, d')$. Then the isomorphism $\Psi_0 : \mathrm{Aut}(\Delta) \to \mathrm{Aut}(\Delta'), f \mapsto \gamma_{(g')^{-1}} \circ \Psi \circ \gamma_g$ maps U_α onto U'_α for every $\alpha \in \Phi$. We deduce

$$\prod_{\gamma\in M(\mathcal{D})^H_{\alpha,\beta}} u'_\gamma = \Psi_0\left(\prod_{\gamma\in M(\mathcal{D})^H_{\alpha,\beta}} u_\gamma\right) = \Psi_0([u_\alpha, u_\beta]) = [u'_\alpha, u'_\beta] = \prod_{\gamma\in M(\mathcal{D}')^H_{\alpha,\beta}} u'_\gamma$$

As $M(\mathcal{D})^H_{\alpha,\beta} \neq M(\mathcal{D}')^H_{\alpha,\beta}$, [2, Corollary 8.34(1)] yields a contradiction. Thus, such an isometry can not exist and the extension theorem does not hold for these two twin buildings. □

7.3. Finiteness properties

Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD-system of irreducible 2-spherical type (W, S) and of rank at least 2. The *Steinberg group* associated with $\mathcal{D}$ is the group $\widehat{G}$ which is the direct limit of the inductive system formed by the groups U_α and $U_{[\alpha,\beta]} := \langle U_\gamma \mid \gamma \in [\alpha, \beta]\rangle$ for all prenilpotent pairs $\{\alpha, \beta\} \subseteq \Phi$. For each $\alpha \in \Phi$ we denote the canonical image of U_α in $\widehat{G}$ by $\widehat{U}_\alpha$. It follows from [11, Theorem 3.10] that $\widehat{D} = (\widehat{G}, (\widehat{U}_\alpha)_{\alpha\in\Phi})$ is an RGD-system and the kernel of $\widehat{G} \to G$ is contained in the center of $\widehat{G}$.

(7.3.1) Lemma. *Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD-system of irreducible 2-spherical type and rank at least 2 over $\mathbb{F}_2$ such that G is generated by the root groups. Then $\bigcap_{\alpha\in\Phi} N_G(U_\alpha) = 1$. In particular, the homomorphism $\widehat{G} \to G$ from the Steinberg group associated with $\mathcal{D}$ to G is an isomorphism.*

Proof. As $\mathcal{D}$ is an RGD-system such that G is generated by the root groups it follows from [2, Corollary 8.79 and remark thereafter] that $\bigcap_{\alpha\in\Phi} N_G(U_\alpha) = \langle m(u)^{-1}m(v) \mid u,v \in U_{\alpha_s}\backslash\{1\}, s\in S\rangle$. As $\mathcal{D}$ is over $\mathbb{F}_2$, we have $U_{\alpha_s}\backslash\{1\} = \{u_s\}$. Moreover, $m(u_s) = u_{-s}u_su_{-s}$, where $U_{-\alpha_s}\backslash\{1\} = \{u_{-s}\}$. This implies $m(u_s)^{-1}m(u_s) = (u_{-s}u_su_{-s})^{-1}u_{-s}u_su_{-s} = 1$ and hence $\bigcap_{\alpha\in\Phi} N_G(U_\alpha) = 1$. As $Z(G) \leq \bigcap_{\alpha\in\Phi} N_G(U_\alpha) = 1$, the claim follows. □

(7.3.2) Lemma. *Let $G = \langle X \mid R\rangle$ be a finitely presented group with $|X| < \infty$. Then there exists a finite subset $F \subseteq R$ with $G = \langle X \mid F\rangle$.*

Proof. Since G is finitely presented, there exist finite sets Y, E such that $G = \langle Y \mid E\rangle$. Since $G = \langle Y\rangle$, we have $x = \prod y_i$ in G for each $x \in X$. Thus $G = \langle X\cup Y \mid E'\rangle$, where $E' = E \cup \{x = \prod y_i \mid x \in X\}$ and $X \cup Y$ is finite. Since $G = \langle X\rangle$, we have $y = \prod x_j$ and $G = \langle X \cup Y \mid E''\rangle$, where $E'' = E' \cup \{y = \prod x_j \mid y \in Y\}$. Then we can replace in every relation y by the corresponding product $\prod x_j$ (if $y = \prod x_j$) and we can remove the generators $y \in Y$ together with the relations $y = \prod x_j$. We denote this set of relations by E''' and we have $G = \langle X \mid E'''\rangle$. Note that E''' is finite.

Now for each $e \in E'''$ there exists a finite subset $F_e \subseteq R$ such that $e \in \langle\langle F_e\rangle\rangle$. For $F := \bigcup_{e\in E'''} F_e \subseteq R$ we have $E''' \subseteq \langle\langle F_e \mid e \in E'''\rangle\rangle$. Clearly, we have the following epimorphisms:

$$\langle X \mid R\rangle \overset{\cong}{\to} \langle X \mid E'''\rangle \twoheadrightarrow \langle X \mid F\rangle \twoheadrightarrow \langle X \mid R\rangle$$

Since the concatenation maps each $x \in X$ to itself, all epimorphisms must be isomorphisms and the claim follows. □

(7.3.3) Theorem. *The split Kac-Moody group over $\mathbb{F}_2$ of type $(4,4,4)$ is not finitely presented.*

Proof. Let $\mathcal{G}$ be the split Kac-Moody group of type $(4,4,4)$ over $\mathbb{F}_2$. Using Lemma (7.3.1), we deduce that $\mathcal{G} = \langle X \mid R\rangle$, where $X = \{u_\alpha \mid \alpha\in\Phi\}$ and $R = \{\{u_\alpha^2 \mid \alpha \in \Phi\} \cup \{[u_\alpha, u_\beta]v \mid \{\alpha,\beta\} \text{ prenilpotent pair}, v \in U_{(\alpha,\beta)}\}\}$. We apply Tietze-transformations to slightly modify the given presentation. We add τ_s to the set of generators and $\tau_s = u_{-\alpha_s}u_{\alpha_s}u_{-\alpha_s}$ to the set of relations. Note that $\mathcal{G} = \langle u_{\alpha_s}, \tau_s \mid s \in S\rangle$. Since $\tau_s^2 = 1$ in $\mathcal{G}$, we add this relation to the set of relations. For $\alpha \in \Phi$ there exist $w \in W, s \in S$ with $\alpha = w\alpha_s$. For $w \in W$ there exist $s_1,\dots,s_k \in S$ with $w = s_1\cdots s_k$. Note that $u_\alpha = u_{\alpha_s}^{\tau_k\cdots\tau_1}$ is a relation in $\mathcal{G}$, where $\tau_i = \tau_{s_i}$. Thus we can add these relations to the set of relations. We modify the relations further and delete all commutator relations $[u_\alpha, u_\beta] = v$, where $\{\alpha,\beta\} \notin \mathcal{P}$ (for every prenilpotent pair $\{\alpha,\beta\}$ there exists $w \in W$ such that $\{w\alpha, w\beta\} \in \mathcal{P}$). This is possible because the commutator relations are Weyl-invariant. We replace in each relation every u_α by the corresponding element $u_{\alpha_s}^{\tau_k\cdots\tau_1}$. Now we delete all generators u_α with $\alpha \in \Phi\backslash\{\alpha_s \mid s \in S\}$ and the corresponding relations $u_\alpha = u_{\alpha_s}^{\tau_k\cdots\tau_1}$. We note that we have the *same* relations as before plus the relations $\tau_s = u_{\alpha_s}^{\tau_s}u_{\alpha_s}u_{\alpha_s}^{\tau_s}$ and $\tau_s^2 = 1$. But the former relation is equivalent to the relation $(u_{\alpha_s}\tau_s)^3 = 1$.

Now we assume that $\mathcal{G}$ is finitely presented. Then, by the previous lemma, there exists a finite set F of the set of relations such that $\mathcal{G} = \langle\{u_{\alpha_s}, \tau_s \mid s \in S\} \mid F\rangle$. Now we let $k := \max\{k_\alpha \mid u_\alpha \text{ appears in some } f \in F\}$ (u_α seen as conjugate of u_{α_s} by a product of τ_{s_i} for suitable $s, s_i \in S$). We consider the RGD-systems $\mathcal{D}_k = (G, (U_\alpha)_{\alpha\in\Phi})$ obtained from Corollary (7.1.9). Then $[U_\alpha, U_\beta] = 1$, where $\alpha \subseteq \beta$ are such that there exists $w \in W$ of length $\leq k$ with $w \in (-\alpha)\cap(-\beta)$ and $[U_\delta, U_\gamma] \neq 1$ for some $\delta \subsetneq \gamma \in \Phi_+$. It is not hard to see that we obtain a homomorphism $\varphi : \mathcal{G} \to \mathcal{D}_k$ from the finite presentation to $\mathcal{D}_k$ such that $u_{\alpha_s} \mapsto u_{\alpha_s}, \tau_s \mapsto \tau_s$ (note that for $\alpha \subsetneq \beta$ we have $[U_\alpha, U_\beta] = 1$ in $\mathcal{G}$ by Example (5.3.1)). The commutator relations of $\mathcal{G}$ and $\mathcal{D}_k$ yields us $1 = \varphi(1) = \varphi([U_\delta, U_\gamma]) = [\varphi(U_\delta), \varphi(U_\gamma)] = [U_\delta, U_\gamma] \neq 1$. This yields a contradiction and hence the Kac-Moody group is not finitely presented. □

(7.3.4) Theorem. *Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD-system of type $(4,4,4)$ over $\mathbb{F}_2$. Then the group U_+ is not finitely generated.*

Proof. The group U_+ is isomorphic to the direct limit of its subgroups U_w for all $w \in W$ by [2, Theorem 8.85]. We have shown in Lemma (6.9.4) that U_+ is isomorphic to the direct limit G of the inductive system formed by the groups G_i. By definition the following diagram commutes:

$$\begin{array}{ccc} G_i & \longrightarrow & G_{i+1} \\ & \searrow & \downarrow \\ & & G \end{array}$$

Moreover, the homomorphisms $G_i \to G_{i+1}$ are injective by Proposition (6.9.20) and Theorem (6.10.5), and hence the homomorphisms $G_i \to G$ are injective by [30, 1.4.9(*iii*)]. By construction, the canonical homomorphism $G_i \to G_{i+1}$ is not surjective and hence $G_i \to G$ are not surjective as well. Assume that U_+ is finitely generated, i.e. $U_+ = \langle g_1, \ldots, g_n \rangle$. Since $U_+ = \langle u_\alpha \mid \alpha \in \Phi_+ \rangle$, there exists $i \in \mathbb{N}$ such that $U_+ = \langle U_w \mid w \in C_i \rangle$. This implies that G is also finitely generated and we have $G = \langle U_w \mid w \in C_i \rangle = G_i$, i.e. the canonical homomorphism $G_i \to G$ is surjective. This is a contradiction and hence U_+ is not finitely generated. □

7.4. Locally compact groups

Haar measure and modular function

Let G be a locally compact group. Then there exists a (left) *Haar measure* μ on G. For every measurable $U \subseteq G$ and $g \in G$ we have $\mu(gU) = \mu(U)$ and $\mu(Ug) = \mu(U)\Delta(g)$, where $\Delta : G \to \mathbb{R}^\star$ is the *modular function* of G. The group G is called *unimodular*, if $\Delta \equiv 1$. For details we refer to [17, Chapter 9].

Lattices

Let G be a locally compact group which is unimodular, and let X be a left G-set such that the stabilisers G_x are compact and open for each $x \in X$ and such that $G\backslash X$ is finite. Then a subgroup $\Gamma \leq G$ is called a *lattice*, if it is discrete and if

$$\mathrm{Vol}(\Gamma\backslash\backslash X) := \sum_{x\in\Gamma\backslash X} \frac{1}{|\Gamma_x|} < \infty.$$

We note that as Γ is discrete, the stabilisers Γ_x are compact and discrete and hence finite. In the literature this is not the definition of a general lattice in a locally compact group. But using [4, Ch. 1] and, in particular, [4, Corollary 1.6], it follows that a discrete subgroup of the group G is a lattice in the general sense if and only if it is a lattice in our sense.

Permutation topology

Let $\Delta = (\mathcal{C}, \delta)$ be a building of type (W, S). Then we endow the automorphism group $\mathrm{Aut}(\Delta)$ of Δ with the permutation topology (i.e. fixators of finitely many chambers form a basis of neighbourhoods of the identity). It is well-known that $\mathrm{Aut}(\Delta)$ is locally compact and totally disconnected, if Δ is locally finite. For details we refer to [39, Theorem 1.24] or [40]. In particular, stabilizers of chambers are compact open subgroups. Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$

be an RGD-system of type (W, S) such that every root group is finite, and let $\Delta(\mathcal{D}) = (\Delta(\mathcal{D})_+, \Delta(\mathcal{D})_-, \delta_*)$ be its associated twin building. Then for $\varepsilon \in \{+,-\}$ the building $\Delta(\mathcal{D})_\varepsilon$ is locally finite and $\mathrm{Aut}(\Delta_\varepsilon)$ is a totally disconnected locally compact group. If $G \leq \mathrm{Aut}(\Delta_\varepsilon)$, then we call $\overline{G} \leq \mathrm{Aut}(\Delta_\varepsilon)$ *the geometric completion* of G in $\mathrm{Aut}(\Delta_\varepsilon)$. Moreover, any closed subgroup $K \leq \mathrm{Aut}(\Delta_\varepsilon)$ containing G is unimodular (cf. [5, Corollary 5]).

(Twin building) lattices and property (T)

(7.4.1) Definition. Let $(G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD-system of type (W, S) such that all root groups are finite and (W, S) is not spherical. Let $W(t) = \sum_{i=0}^{\infty} c_i t^i$ be the growth series of W (i.e. $c_i = |\{w \in W \mid \ell(w) = i\}|$) and let $q_{\min} = \min\{|U_\alpha| \mid \alpha \in \Phi\}$. If $W(1/q_{\min}) < \infty$ and $Z_G(\langle U_\alpha \mid \alpha \in \Phi\rangle)$ is finite, then G is called a *twin building lattice*. For more details about twin building lattices we refer to [16].

(7.4.2) Remark. Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD-system of type (W, S) such that G is generated by the root groups, all root groups are finite, W is infinite and $Z(G)$ is finite. By [15, Theorem 6.8] the condition $|S| < q_{\min}$ implies that $\mathcal{D}$ is a twin building lattice. We will show that if $\mathcal{D}$ is of type $(4,4,4)$ then $\mathcal{D}$ is a twin building lattice. In particular, we enlarge the result to RGD-systems of type $(4,4,4)$ with $q_{\min} \in \{2,3\}$. We note that the arguments in [15, Theorem 6.8] can be enlarged to the case $|S| = q_{\min}$.

(7.4.3) Proposition. *Let (W, S) be of type $(4,4,4)$. For $2 \leq q \in \mathbb{N}$ we have $W(1/q) < \infty$.*

Proof. We will apply the quotient criterion in order to show the claim. For this we need a few (in-)equalities. For $i \in \mathbb{N}$ we put $C_i := \{w \in W \mid \ell(w) = i\}, D_i := \{w \in C_i \mid \exists s \neq t \in S : \ell(ws) = \ell(w) + 1 = \ell(wt)\}$ and $d_i := |D_i|$. For $i \geq 5$ we establish the following (in-)equalities:

Claim 1: $c_i - d_i = d_{i-4}$: Let $w \in C_i \backslash D_i$. Then there exist $s \neq t \in S$ such that $\ell(ws) = \ell(w) - 1 = \ell(wt)$. This implies that the mapping $C_i \backslash D_i \ni w \mapsto \mathrm{proj}_R 1_W \in D_{i-4}$ is a bijection, where $R = R_{\{s,t\}}(w)$. Here we use the fact that $\mathrm{proj}_R 1_W \neq 1_W$ and hence that there exist unique $s \neq t \in S$ with $\ell((\mathrm{proj}_R 1_W)s) = \ell(\mathrm{proj}_R 1_W) + 1 = \ell((\mathrm{proj}_R 1_W)t)$.

Claim 2: $d_i \leq d_{i+1}$: Let $w \in D_i$. Then there exist $s \neq t \in S$ with $\ell(ws) = \ell(w) + 1 = \ell(wt)$. Lemma (5.1.1) implies $\{ws, wt\} \cap D_{i+1} \neq \emptyset$. Let $w, w' \in D_i$ and let $s, t \in S$ with $ws = w't \in D_{i+1}$. If $s \neq t$, then there would be only one $r \in S$ with $\ell(wsr) = \ell(ws) + 1$, which is a contradiction. Thus $s = t$ and hence $w = w'$. This finishes the claim.

Claim 3: $\frac{1}{2} \leq \frac{d_i}{c_i} \leq 1$: As $D_i \subseteq C_i$, it follows directly that $d_i \leq c_i$ and hence $\frac{d_i}{c_i} \leq 1$. For the other inequality we use Claim 1 and 2 and compute

$$1 = \frac{c_i - d_i + d_i}{c_i} = \frac{d_{i-4} + d_i}{c_i} \leq 2\frac{d_i}{c_i}$$

Claim 4: $c_{i+1} \leq c_i + d_i - d_{i-3}$: Let $M_i := \{(w, s) \in C_i \times S \mid ws \in C_{i+1}\}$. Then $|M_i| = 2d_i + (c_i - d_i)$. We consider the mapping $f : M_i \to C_{i+1}, (w, s) \mapsto ws$. Then f is surjective and $c_i + d_i = |M_i| = \sum_{w\in C_{i+1}} |f^{-1}(w)|$. We define $C_{i+1}^1 = \{w \in C_{i+1} \mid |f^{-1}(w)| = 1\}$ and let $C_{i+1}^{>1} = \{w \in C_{i+1} \mid |f^{-1}(w)| > 1\}$. We show that $C_{i+1}^{>1} = C_{i+1} \backslash D_{i+1}$. Let $\overline{w} \in C_{i+1}^{>1}$ and let $(w, s) \neq (w', s') \in f^{-1}(\overline{w})$ be. Then $s \neq s'$ and hence $w \neq w'$. This implies $\overline{w} \in C_{i+1} \backslash D_{i+1}$. Similarly, for each $w \in C_{i+1} \backslash D_{i+1}$

there exist $s \neq t \in S$ with $ws, wt \in C_i$ and hence $(ws, s) \neq (wt, t) \in f^{-1}(w)$. Thus $C_{i+1}^{>1} = C_{i+1} \backslash D_{i+1}, C_{i+1}^{1} = D_{i+1}$ and we compute the following:

$$\sum_{w \in C_{i+1}} |f^{-1}(w)| = \sum_{w \in D_{i+1}} |f^{-1}(w)| + \sum_{w \in C_{i+1} \backslash D_{i+1}} |f^{-1}(w)| \geq d_{i+1} + 2(c_{i+1} - d_{i+1})$$

This implies $c_i + d_i \geq c_{i+1} + (c_{i+1} - d_{i+1}) = c_{i+1} + d_{i-3}$ and the claim follows.

Claim 5: $c_{i+1} \leq 2c_i$: This readily follows from Claim 3 and 4, as $c_{i+1} \leq c_i + d_i - d_{i-3} \leq 2c_i$.

Now we are in the position to apply the quotient criterion. For $i \geq 6$ and $t = \frac{1}{q_{\min}} \leq \frac{1}{2}$ we compute

$$\frac{c_{i+1} t^{i+1}}{c_i t^i} = \frac{c_{i+1}}{c_i} t \leq \frac{c_i + d_i - d_{i-3}}{c_i} t \leq (2 - \frac{d_{i-3}}{c_i}) t \leq (2 - \frac{d_{i-3}}{8c_{i-3}}) t \leq \frac{31}{16} t \leq \frac{31}{32} < 1 \qquad \square$$

(7.4.4) Corollary. *Let $(G, (U_\alpha)_{\alpha \in \Phi})$ be an RGD-system of type $(4,4,4)$ with finite root groups and $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$ such that $Z(G)$ is finite. Then the following hold:*

(a) G is a twin building lattice.

(b) Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be the associated twin building and let K be a closed subgroup of $\mathrm{Aut}(\Delta_-)$ containing G. Then U_+ is a lattice in K.

Proof. For Assertion (a) it suffices to show that $W(1/q_{\min}) < \infty$. For Assertion (b) we note that U_+ is discrete in $\mathrm{Aut}(\Delta_-)$, as $U_+ \cap \mathrm{Stab}(c_-) = \{1\}$. Thus it is discrete in K. Recall that stabilizers of chambers are compact and open and K is unimodular. By definition it suffices to show that $\mathrm{Vol}(U_+ \backslash\backslash \Delta_-) < \infty$. As explained in [15, Proof of Theorem 6.8], we have $\mathrm{Vol}(U_+ \backslash\backslash \Delta_-) \leq W(1/q_{\min})$ and it also suffices to show $W(1/q_{\min}) < \infty$ (cf. also [28, Théorème 1]). But this follows from the previous proposition. $\square$

(7.4.5) Remark. For the definition and more details about property (T) we refer to [6].

(7.4.6) Lemma. *Let $\mathcal{D} = (G, (U_\alpha)_{\alpha \in \Phi})$ be an RGD-system of type $(4,4,4)$ over $\mathbb{F}_2$ and let $\Delta := \Delta(\mathcal{D})_-$. Then $\mathrm{Aut}(\Delta)$ does not satisfy property (T).*

Proof. By Theorem (7.3.4) and Corollary (7.4.4)(b), the subgroup U_+ is a lattice in $\mathrm{Aut}(\Delta)$ which is not finitely generated. By [6, Theorem 1.7.1] the group $\mathrm{Aut}(\Delta)$ has property (T) if and only if U_+ has property (T). As discrete groups with property (T) are finitely generated by [6, Theorem 1.3.1], U_+ can not have property (T) and hence $\mathrm{Aut}(\Delta)$ does not have property (T). $\square$

(7.4.7) Remark. Let $\mathcal{D} = (G, (U_\alpha)_{\alpha \in \Phi})$ be an RGD-system of type $(4,4,4)$ with finite root groups and let $\Delta := \Delta(\mathcal{D})_-$. For $s \in S$ we let $q_s + 1$ be the order of the s-panels. Using [26, Theorem 1 and 4.1(3)], a sufficient condition for $\mathrm{Aut}(\Delta)$ to have property (T) is that for every $s \neq t \in S$ the following inequality is satisfied:

$$\begin{aligned} 1 - \sqrt{\frac{q_s + q_t}{(q_s+1)(q_t+1)}} > \frac{1}{2} &\Leftrightarrow \frac{1}{4} > \frac{q_s + q_t}{(q_s+1)(q_t+1)} \\ &\Leftrightarrow q_s q_t + q_s + q_t + 1 = (q_s+1)(q_t+1) > 4(q_s + q_t) \\ &\Leftrightarrow q_s q_t + 1 > 3(q_s + q_t) \end{aligned}$$

If $7 \leq q_{\min}$ and if $q_s \leq q_t$, we have $3(q_s + q_t) \leq 3(q_t + q_t) = 6q_t \leq q_s q_t < q_s q_t + 1$ and $\mathrm{Aut}(\Delta)$ satisfies property (T).

7.5. Property (FPRS)

Although we have defined property (FPRS) in Section 1.7, we recall the definition here. Let $(G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD-system and let $\Delta(\mathcal{D}) = (\Delta(\mathcal{D})_+, \Delta(\mathcal{D})_-, \delta_*)$ be the associated twin building. For $\Gamma \leq G$ we define $r(\Gamma)$ to be the supremum of the set of all non-negative real numbers r such that Γ fixes pointwise the *closed ball* $B(c_+, r) := \{d \in \mathcal{C}_+ \mid \ell_+(c_+, d) \leq r\}$, where $\mathcal{C}_+$ is the set of chambers of $\Delta(\mathcal{D})_+$. Then $\mathcal{D}$ has property (FPRS), if the following holds, where $\ell(1_W, \alpha) := \min\{k \in \mathbb{N} \mid \exists d \in \alpha : \ell(1_W, d) = k\}$ for all roots $\alpha \in \Phi$:

(FPRS) Given any sequence of roots $(\alpha_n)_{n\geq 0}$ of Φ such that $\lim_{n\to\infty} \ell(1_W, \alpha_n) = \infty$, we have $\lim_{n\to\infty} r(U_{-\alpha_n}) = \infty$.

(7.5.1) Lemma. *Let (W, S) be irreducible and non-spherical. Let $(G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD-system of type (W, S) with finite and solvable root groups such that G is generated by the root groups and satisfies (FPRS). We endow $\mathrm{Aut}(\Delta_+)$ with the permutation topology. We define $G^\dagger := \langle U_\alpha \mid \alpha \in \Phi\rangle \leq \mathrm{Aut}(\Delta(\mathcal{D})_+)$. Then $\overline{G^\dagger} \leq \mathrm{Aut}(\Delta(\mathcal{D})_+)$ is topologically simple, i.e. if $N \trianglelefteq \overline{G^\dagger}$ is a dense normal subgroup, then $\overline{N} = G$.*

Proof. This is a consequence of [16, Lemma 9 and Proposition 11]. □

(7.5.2) Remark. Let $\mathcal{M}$ be a commutator blueprint of type $(4, 4, 4)$ which is integrable. If the corresponding RGD-system satisfies (FPRS), then $\overline{G} \leq \mathrm{Aut}(\Delta_+)$ is a topologically simple, non-discrete, compactly generated t.d.l.c. group. Caprace, Reid and Willis initiated a systematic study of such groups in [14].

Next we generalize [16, Lemma 5]. Recall that for every RGD-system $\mathcal{D}$ we have a distinguished pair (c_+, c_-) of opposite chambers in $\Delta(\mathcal{D})$. We define $\Sigma_+ := A(c_+, c_-) \cap \mathcal{C}_+$ and $\ell(c, \alpha) := \min\{k \in \mathbb{N} \mid \exists d \in \alpha : \ell(c, d) = k\}$ for any $c \in \Sigma_+$.

(7.5.3) Proposition. *Let $\mathcal{D} = (G, (U_\alpha)_{\alpha\in\Phi})$ be an RGD-system of type (W, S) over $\mathbb{F}_2$ such that for every $w \in W$ the group U_w is of nilpotency class at most 2. Suppose $4 \leq k \in \mathbb{N}$ such that for all $\alpha \subsetneq \beta \in \Phi_+$ there exists $H \in \mathrm{Min}$ with $\alpha, \beta \in \Phi(H)$ such that for each $\gamma \in M^H_{\alpha,\beta}$ we have $\ell(1_W, -\gamma) \geq \ell(1_W, -\beta) - (k - 1)$. Then for each $m \in \mathbb{N}$, each root $\alpha \in \Phi$ and each $c \in \Sigma_+$, if $d(c, \alpha) \geq \frac{(4k)^{m+1}-1}{3}$, then $U_{-\alpha}$ fixes $B(c, m)$ pointwise. In particular, $\mathcal{D}$ satisfies property (FPRS).*

Proof. In this proof we use more or less the same arguments as in [16, Lemma 5]. Thus large parts of the proof are just copied from the proof of [16, Lemma 5].

We prove the claim by induction on m. If $\ell(c, \alpha) \geq \frac{4k-1}{3} \geq 1$, then $c \notin \alpha$ whence $c \in -\alpha$. In particular, c is fixed by $U_{-\alpha}$. Thus the desired property holds for $m = 0$.

Assume now $m > 0$ and let α be a root such that $\ell(c, \alpha) \geq \frac{(4k)^{m+1}-1}{3}$. Note that

$$\frac{(4k)^{m+1} - 1}{3} - 1 > \frac{4((4k)^m - 1) + 3}{3} - 1 > \frac{(4k)^m - 1}{3}$$

The induction hypothesis implies that the group $U_{-\alpha}$ fixes the ball $B(c, m-1)$ pointwise. Furthermore, if c' is a chamber contained in Σ_+ and adjacent to c, then $\ell(c', \alpha) \geq \ell(c, \alpha) - 1 \geq \frac{(4k)^m-1}{3}$ and the induction hypothesis also implies that $U_{-\alpha}$ fixes $B(c', m-1)$ pointwise.

Let now x be a chamber at distance m from c. Let $(c_0 = c, \ldots, c_m = x)$ be a minimal gallery from c to x. We must prove that $U_{-\alpha}$ fixes x. If c_1 is contained in Σ_+, then we are done by the above. Thus we may assume that c_1 is not in Σ_+. Let c' be the unique chamber of Σ_+ such that c, c_1, c' share a panel. Let $\beta \in \Phi$ be one of the two roots such that $\partial\beta$ separates c from c'. Upon replacing β by its opposite if necessary, we may - and shall - assume that the pair $\{-\alpha, \beta\}$ is prenilpotent (cf. [2, Lemma 8.42(3)]). Let $u := u_\beta \in U_\beta^\star$ be the unique element such that $u(c_1)$ belongs to Σ_+; thus we have $u(c_1) \in \{c, c'\}$. Since $u(c_1), u(c_2), \dots, u(c_m)$ is a minimal gallery, it follows that $u(x)$ is contained in $B(c, m-1) \cup B(c', m-1)$. We distinguish the following three cases:

(i) Suppose first that $[U_{-\alpha}, U_\beta] = 1$. For any $g \in U_{-\alpha}$ we have $g = u^{-1}gu$ whence $g(x) = u^{-1}gu(x) = x$ because $g \in U_{-\alpha}$ fixes $B(c, m-1) \cup B(c', m-1)$ pointwise by the above.

(ii) Suppose now that $[U_{-\alpha}, U_\beta] \neq 1$ and that $\langle r_\alpha, r_\beta \rangle$ is infinite. Let $\{d, d'\} = \{c, c'\}$ and assume that $d \in \beta$. Then, as $\{-\alpha, \beta\}$ is a pair of prenilpotent roots and $d' \in (-\alpha)\backslash\beta$, we have $\beta \subseteq (-\alpha)$. Moreover, $d \in \beta \cap (-\alpha)$ and hence $\{d^{-1}\beta, -d^{-1}\alpha\} \in \mathcal{P}$. Suppose $H \in$ Min with $d^{-1}\beta, -d^{-1}\alpha \in \Phi(H)$. By assumption, we have $\ell(1_W, -\gamma) \geq \ell(1_W, d^{-1}\alpha) - (k-1)$ for all $\gamma \in M^H_{d^{-1}\beta, -d^{-1}\alpha}$. In particular, we have $\ell(d, -d\gamma) \geq \ell(d, \alpha) - (k-1)$. Note that $\ell(d', -d\gamma) \geq \ell(d, \alpha) - k$ and hence $\ell(c, -d\gamma), \ell(c', -d\gamma) \geq \ell(d, \alpha) - k$. Note that

$$\ell(d, \alpha) - k \geq \ell(c, \alpha) - (k+1) \geq \frac{(4k)^{m+1} - 1}{3} - (k+1) \geq \frac{4k(4k)^m - 4k}{3} \geq \frac{(4k)^m - 1}{3}$$

Using induction we deduce that $U_{d\gamma}$ fixes $B(c, m-1) \cup B(c', m-1)$ for all $\gamma \in M^H_{d^{-1}\beta, -d^{-1}\alpha}$. Note that $[u_\beta, u_{-\alpha}] = \prod_{\gamma \in M^H_{d^{-1}\beta, -d^{-1}\alpha}} u_{d\gamma}$ and $g(x) = [g^{-1}, u](x)$ as before for any $g \in U_{-\alpha}$. Using the nilpotency class assumption, we know that $[g^{-1}, u]$ commutes with u and, using the fact that $U_{d\gamma}$ fixes $B(c, m-1) \cup B(c', m-1)$ pointwise, we compute

$$g(x) = [g^{-1}, u](x) = u^{-1}[g^{-1}, u]u(x) = u^{-1}u(x) = x$$

(iii) Suppose finally that $[U_{-\alpha}, U_\beta] \neq 1$ and that $\langle r_\alpha, r_\beta \rangle$ is finite. The first part goes through unchanged until the inequality, which has to be modified to the following:

$$\ell(c, -\beta_i) \geq \ell(c, -\beta_1) \geq \frac{\ell(c, \alpha) - 1}{4} \geq \frac{\frac{(4k)^{m+1} - 1}{3} - 1}{4} = \frac{4k(4k)^m - 4}{12} \geq \frac{(4k)^m - 1}{3}$$

By the induction hypothesis, it follows that for each $\gamma \in (-\alpha, \beta)$, the root subgroup U_γ fixes $B(c, m-1)$ pointwise. As before, we obtain $g(x) = [g, u^{-1}](x)$ for any $g \in U_\alpha$ and $[g, u^{-1}]$ fixes $u(x)$ pointwise. Using the nilpotency class assumption of the groups U_w, we infer $[g, u^{-1}](x) = u^{-1}[g, u^{-1}]u(x) = x$. □

(7.5.4) Corollary. *Let $\emptyset \neq K \subseteq \mathbb{N}_{\geq 3}$ be a finite set, let $\mathcal{J} = (J_k)_{k \in K}$ be a family of subsets $\emptyset \neq J_k \subseteq S$ and let $L_k^j \subseteq \{2, \dots, k-1\}$. We define $\mathcal{L} := \left(L_k^j\right)_{k \in K, j \in J_k}$, $\mathcal{M} := \mathcal{M}(K, \mathcal{J}, \mathcal{L})$ and let $\mathcal{D}(\mathcal{M}) = (G, (U_\alpha)_{\alpha \in \Phi})$ be the RGD-system associated with the commutator blueprint $\mathcal{M}$. Then $\mathcal{D}(\mathcal{M})$ satisfies property (FPRS).*

Proof. Recall from Theorem (7.1.7) that $\mathcal{M}$ is integrable and the groups U_w are of nilpotency class at most 2. We will apply the previous proposition. Let $\alpha \subsetneq \beta \in \Phi_+$ be two positive roots, let $G \in$ Min such that $\alpha, \beta \in \Phi(G)$ and $M^G_{\alpha,\beta} \neq \emptyset$. Then there exists a gallery $H = (c_0, \dots, c_k)$ of type (n, r) between α and β for some $n \in K$ and $r \in J_n$. Using Lemma (7.1.5)(a) we can extend $(c_6, \dots, c_k)$ to a gallery $(d_0, \dots, d_m)$ contained in Min. Let $\gamma \in M^G_{\alpha,\beta}$ be a root. Then $\gamma = \gamma_i \in \{\omega_i, \omega_i'\}$ for some $i \in L_n^r$ and ω_i, ω_i' are non-simple roots of the corresponding residue R_i. Using Lemma (5.2.3) we deduce that $\ell(1_W, \mathrm{proj}_{R_i} 1_W) \leq \ell(1_W, -\gamma)$. In particular, we have $\ell(1_W, -\beta) \leq m \leq \ell(1_W, \mathrm{proj}_{R_i} 1_W) + k \leq \ell(1_W, -\gamma) + k$. Let $n := \max K$. By definition of H we see that in the type of H there appear $r, r_{\{s,t\}}$ at most n times plus $u, v \in \{1_W, s, t\}$ and an additional r. Thus we deduce $k \leq 5n + 3$. For $K := 5n + 4 \in \mathbb{N}$ we have $4 \leq K$ and we infer

$$\ell(1_W, -\gamma) \geq \ell(1_W, -\beta) - (K - 1)$$ □

(7.5.5) Theorem. *Let $\mathcal{J} = (J_n)_{n \in \mathbb{N}_{\geq 3}}$ be a family of subsets $\emptyset \neq J_n \subseteq S$ and let $L_n^j := \{2\}$ for every $n \in \mathbb{N}_{\geq 3}$ and $j \in J_n$. We define $\mathcal{L} := \left(L_n^j \right)_{n \in \mathbb{N}_{\geq 3}, j \in J_n}$. Then the RGD-system associated with the commutator blueprint $\mathcal{M}(\mathbb{N}_{\geq 3}, \mathcal{J}, \mathcal{L})$ does not satisfy condition (FPRS). In particular, there exists an RGD-system of 2-spherical type, which does not satisfy Condition (FPRS).*

Proof. We abbreviate $M_{\alpha,\beta}^G := M(\mathbb{N}_{\geq 3}, \mathcal{J}, \mathcal{L})_{\alpha,\beta}^G$. We let $G_n \in \mathrm{Min}$ be a minimal gallery of type $(r, r_{\{s,t\}}, r, \ldots, r_{\{s,t\}}, r)$, where $r_{\{s,t\}}$ appears n times in the type and we let $\alpha_n := \alpha_{G_n}$. We recall that α_n is the last root which is crossed by G_n. We note that α_n is a non-simple root of the $\{r, s\}$ residue R containing $(r r_{\{s,t\}})^n r$. Using Lemma (5.2.3) we have $\ell(1_W, -\alpha_n) \geq \ell(1_W, \mathrm{proj}_R 1_W) = 5n - 2$. In particular, we have $\lim_{n \to \infty} \ell(1_W, -\alpha_n) = \infty$.

Let $\mathcal{D} = (G, (U_\alpha)_{\alpha \in \Phi})$ be the RGD-system associated with $\mathcal{M}(\mathbb{N}_{\geq 3}, \mathcal{J}, \mathcal{L})$ and assume that $\mathcal{D}$ satisfies property (FPRS). Then there would exist $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $r(U_{\alpha_n}) \geq 10$. In particular, U_{α_n} fixes $B(c_+, 10)$ pointwise. We deduce that $u_{\alpha_0}^{-1} u_{\alpha_n} u_{\alpha_0}$ and hence also $[u_{\alpha_0}, u_{\alpha_n}]$ fixes $B(c_+, 10)$ pointwise. But $[u_{\alpha_0}, u_{\alpha_n}] = u_{\omega_2} u_{\omega_2'}$, which does not fix $B(c_+, 10)$. Thus $\mathcal{D}$ does not have property (FPRS). $\square$

Part IV.

Appendix

For the sake of clarity, we have decided to reproduce all the figures from Chapter 6:

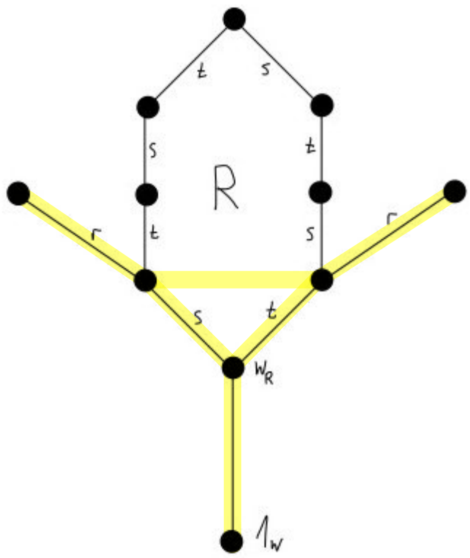


Figure 7.1.: Illustration of the group V_R

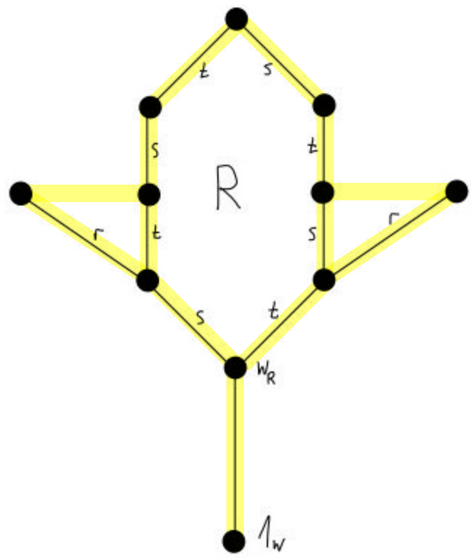


Figure 7.2.: Illustration of the group O_R

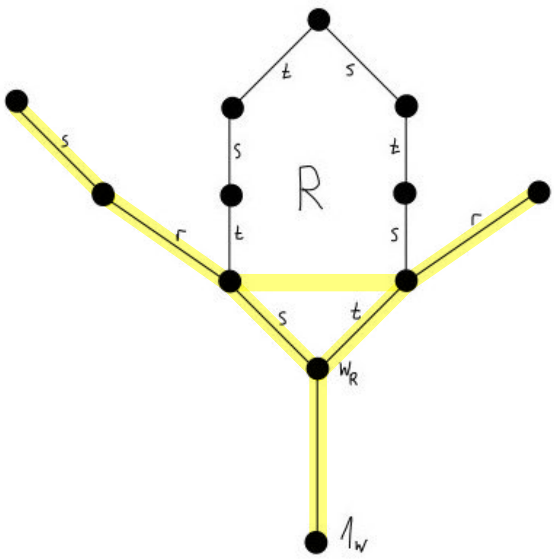


Figure 7.3.: Illustration of the group $V_{R,s}$

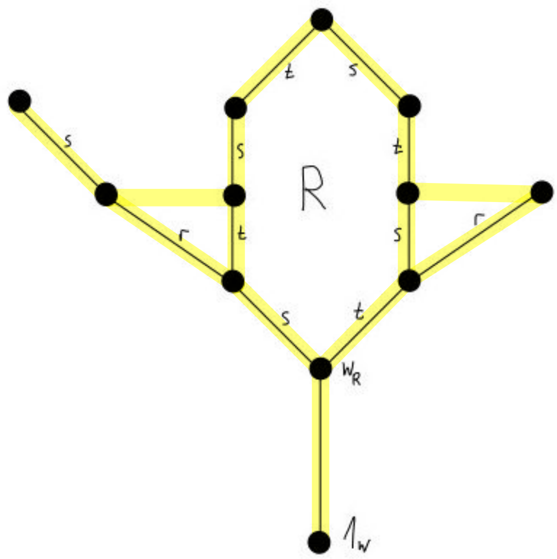


Figure 7.4.: Illustration of the group $O_{R,s}$

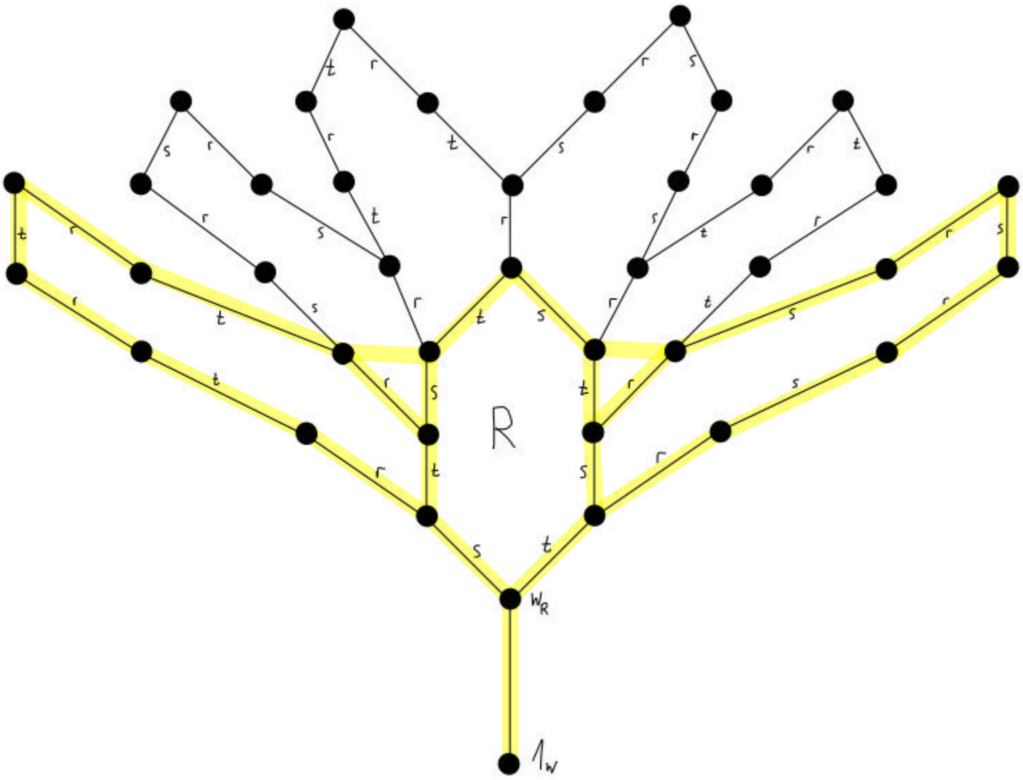


Figure 7.5.: Illustration of the group H_R

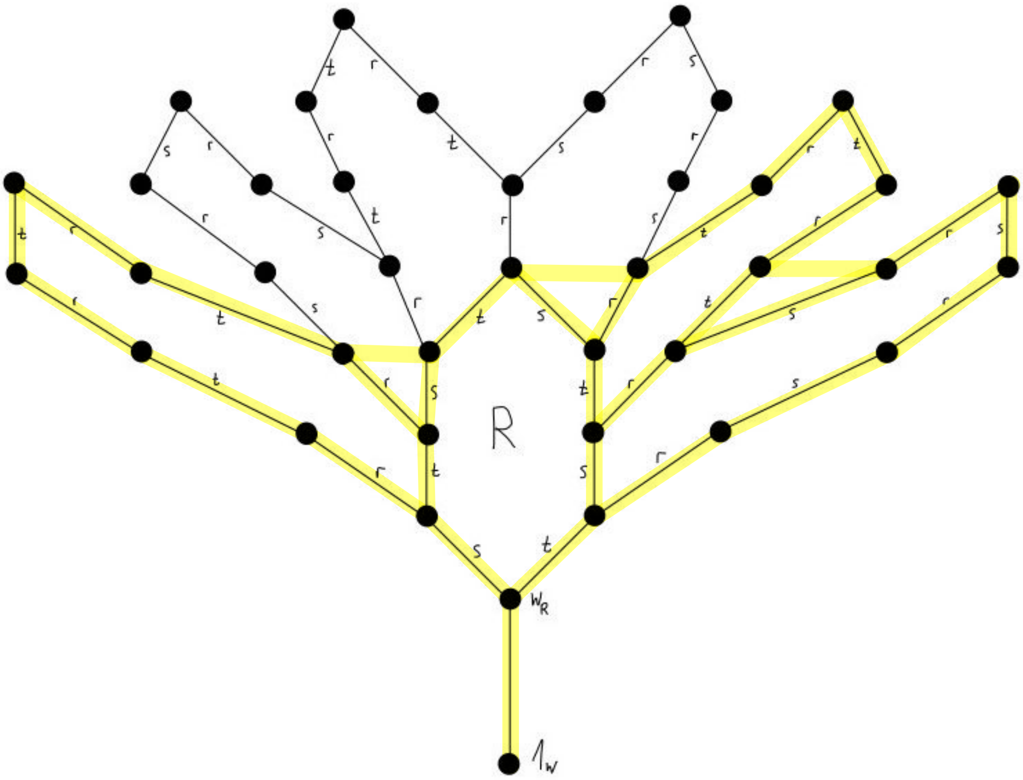


Figure 7.6.: Illustration of the group $J_{R,t}$

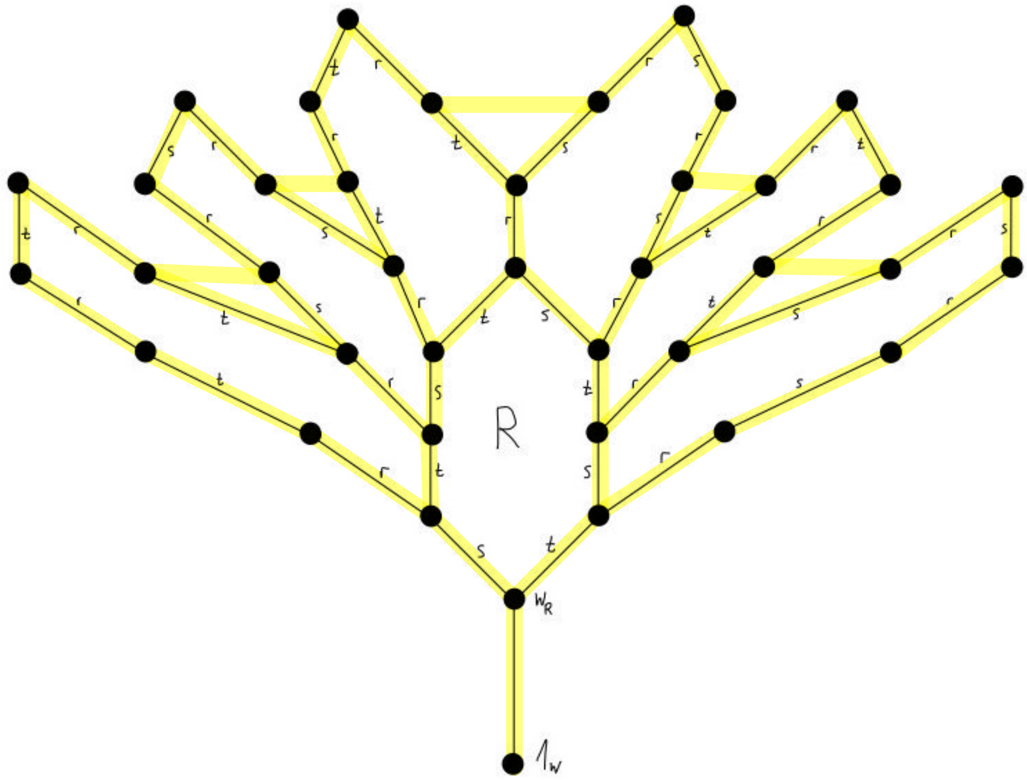


Figure 7.7.: Illustration of the group G_R

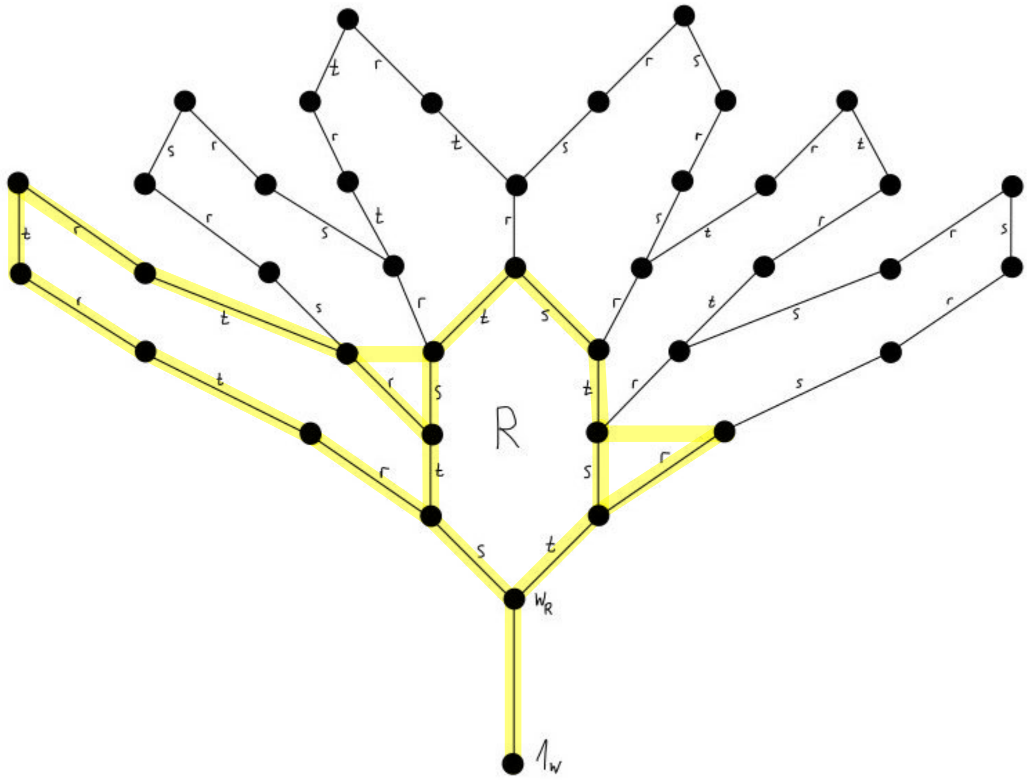


Figure 7.8.: Illustration of the group $K_{R,s}$

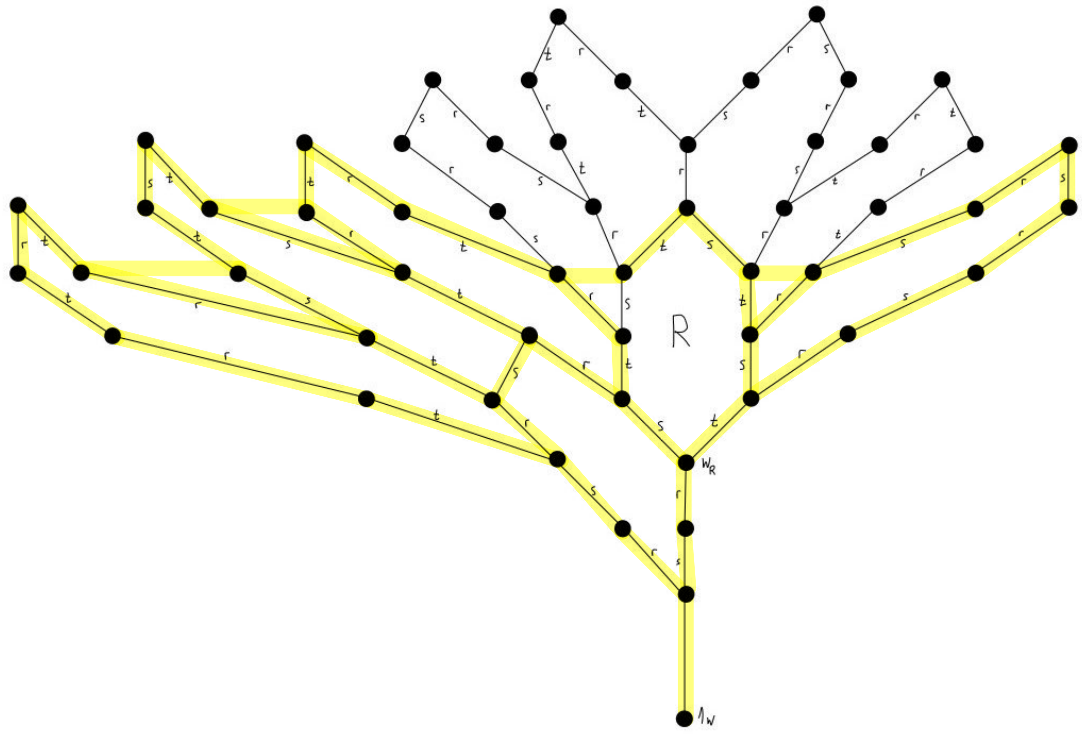


Figure 7.9.: Illustration of the group $E_{R,s}$

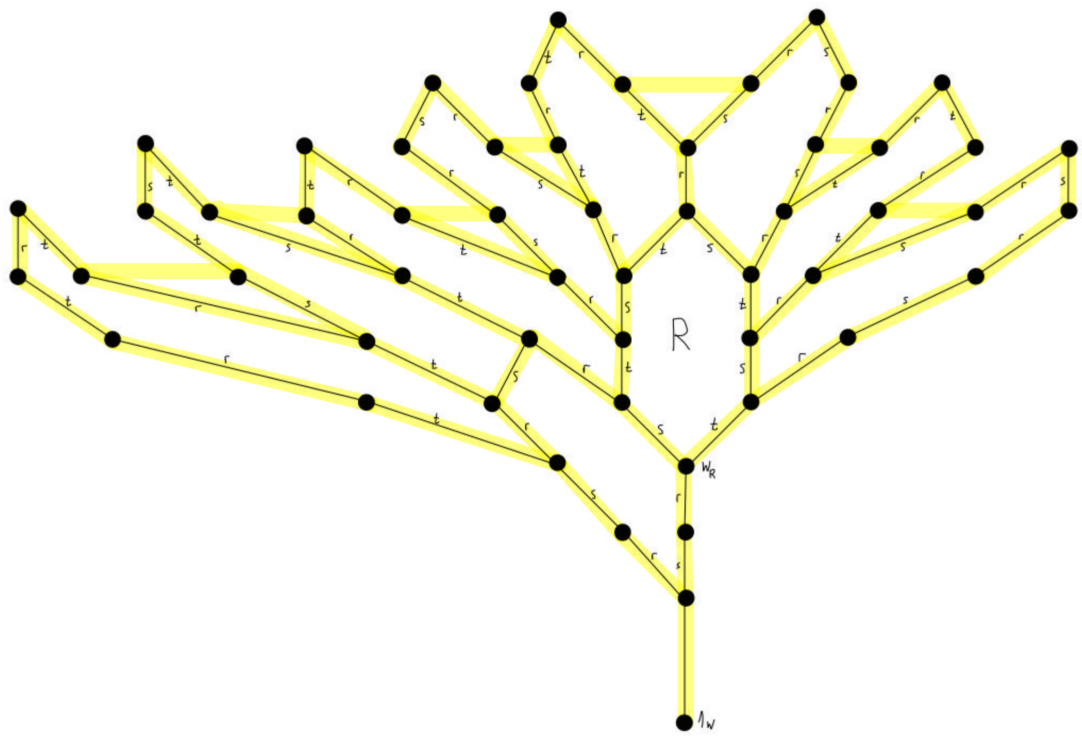


Figure 7.10.: Illustration of the group $U_{R,s}$

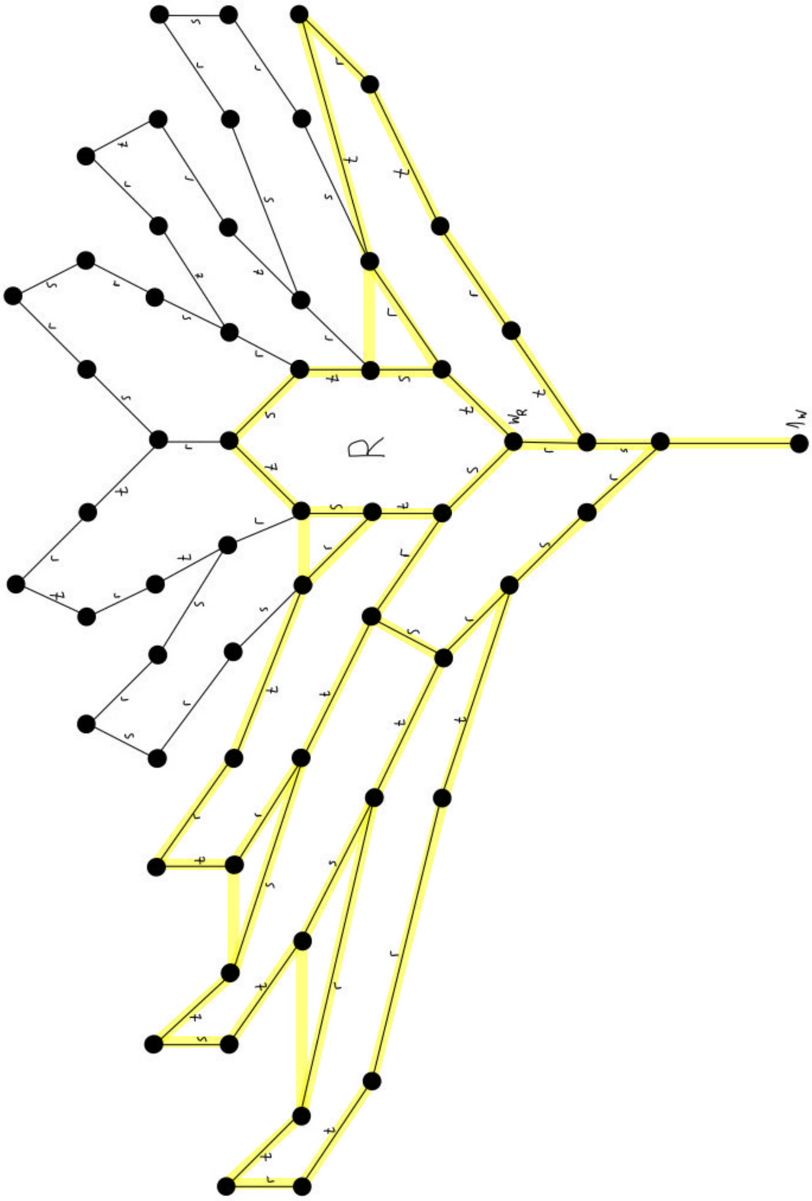


Figure 7.11.: Illustration of the group X_R

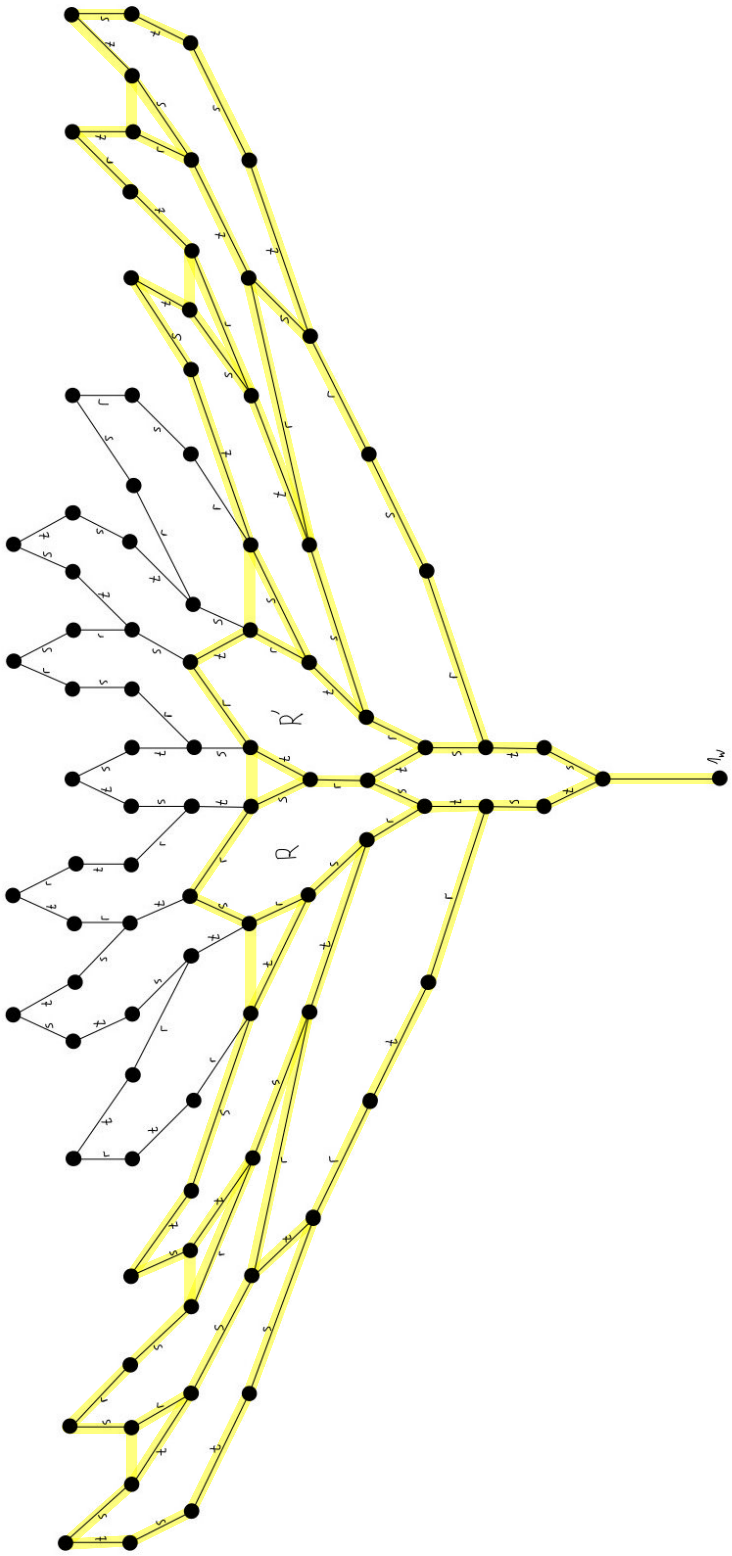


Figure 7.12.: Illustration of the group $H_{\{R,R'\}}$

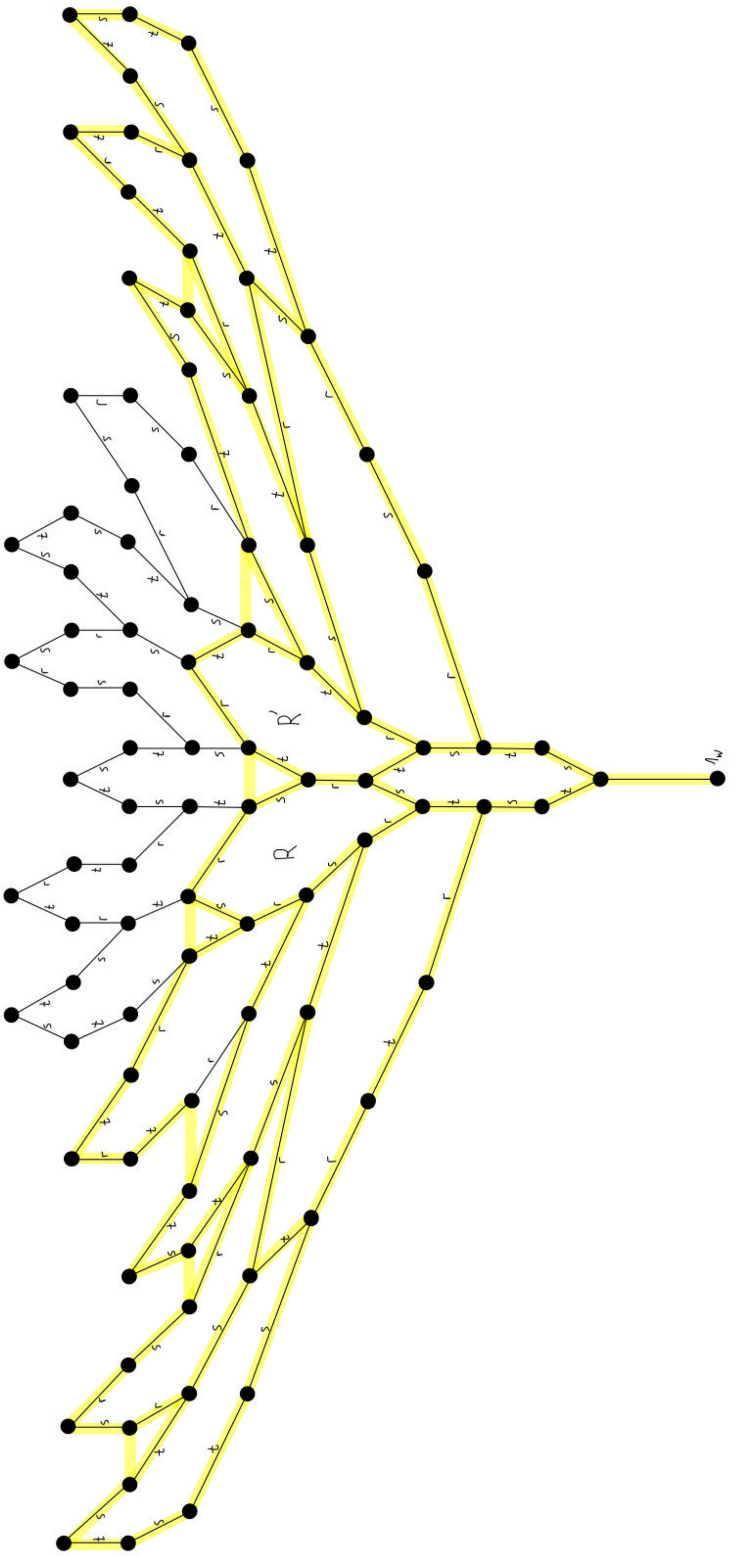


Figure 7.13.: Illustration of the group $J_{(R,R')}$

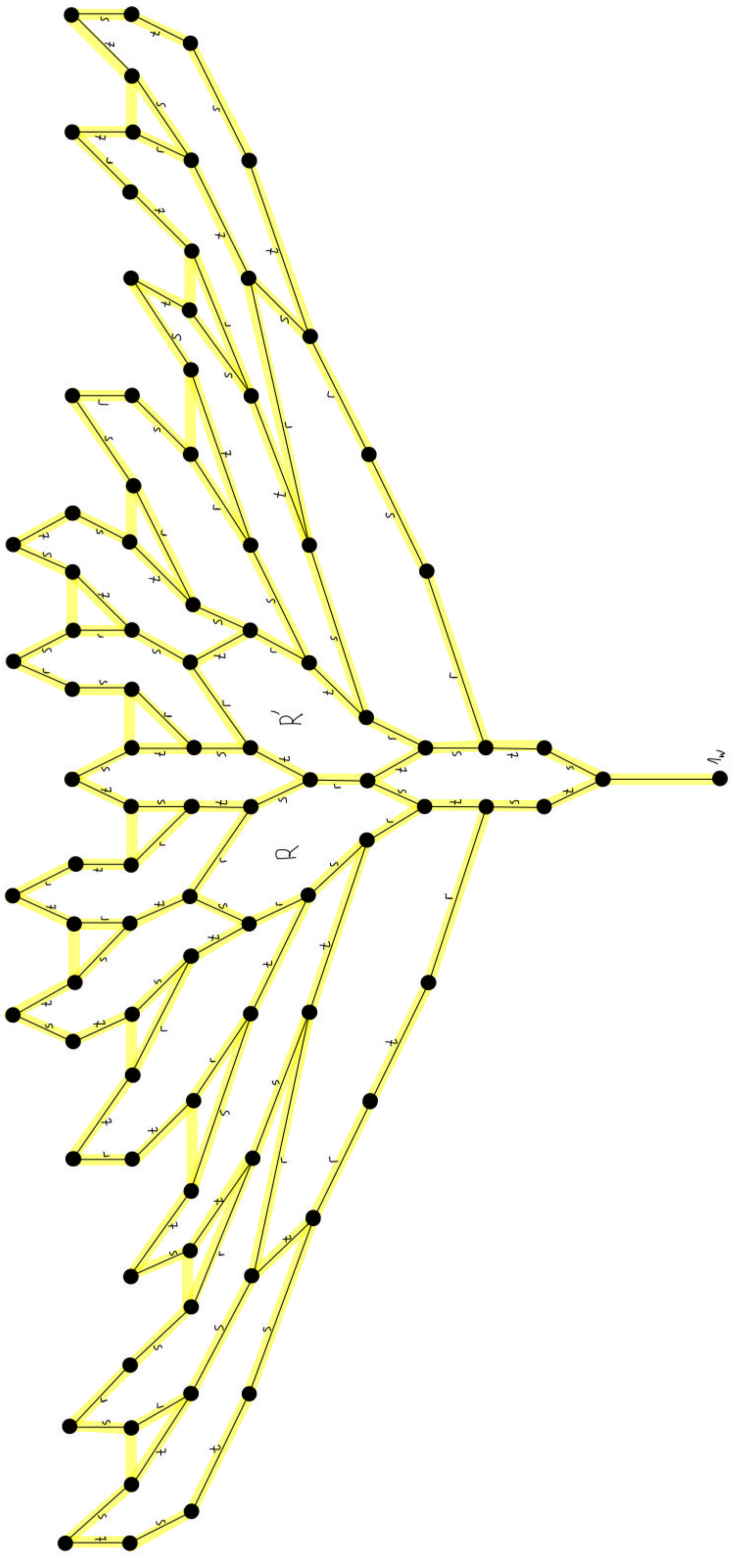


Figure 7.14.: Illustration of the group $G_{\{R,R'\}}$

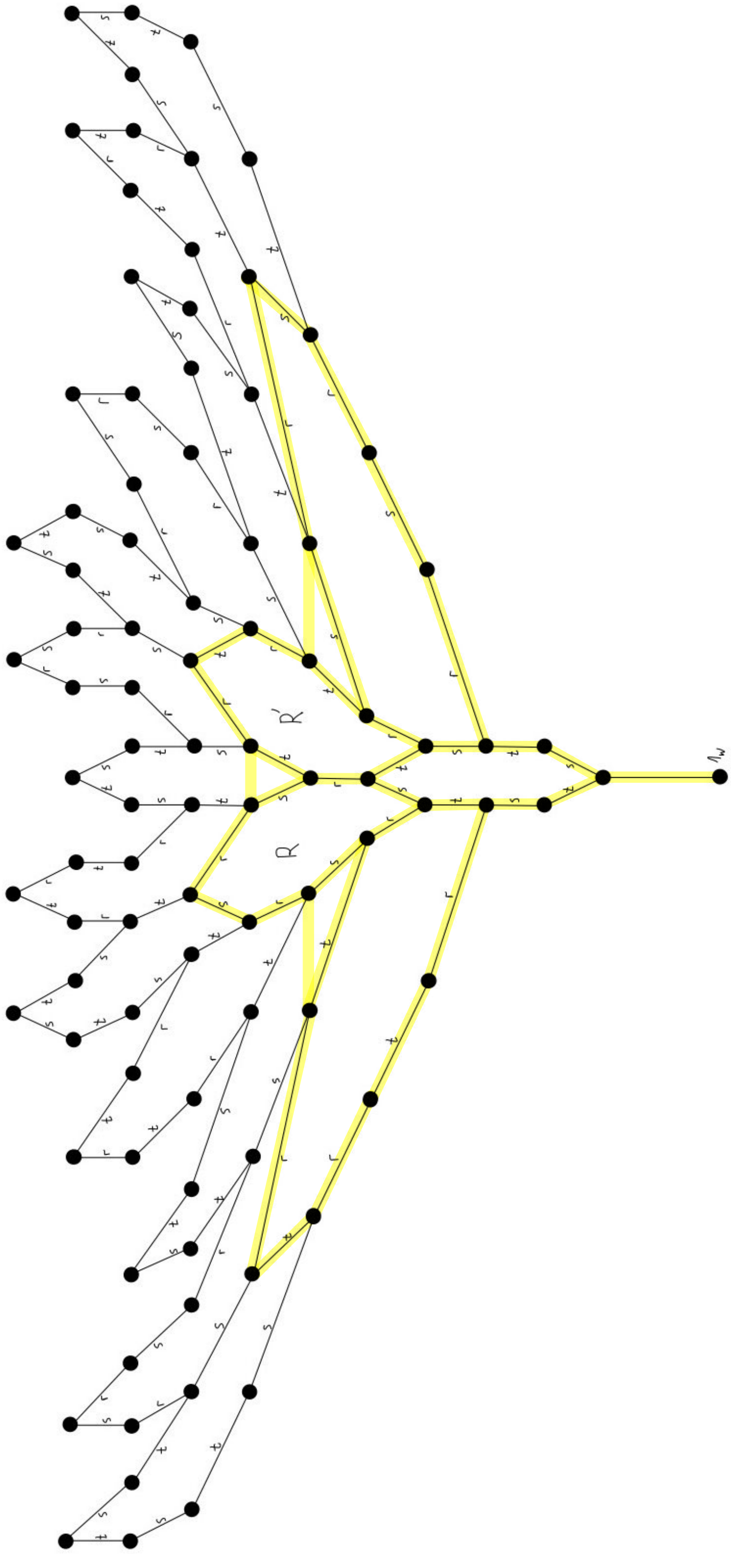


Figure 7.15.: Illustration of the group C

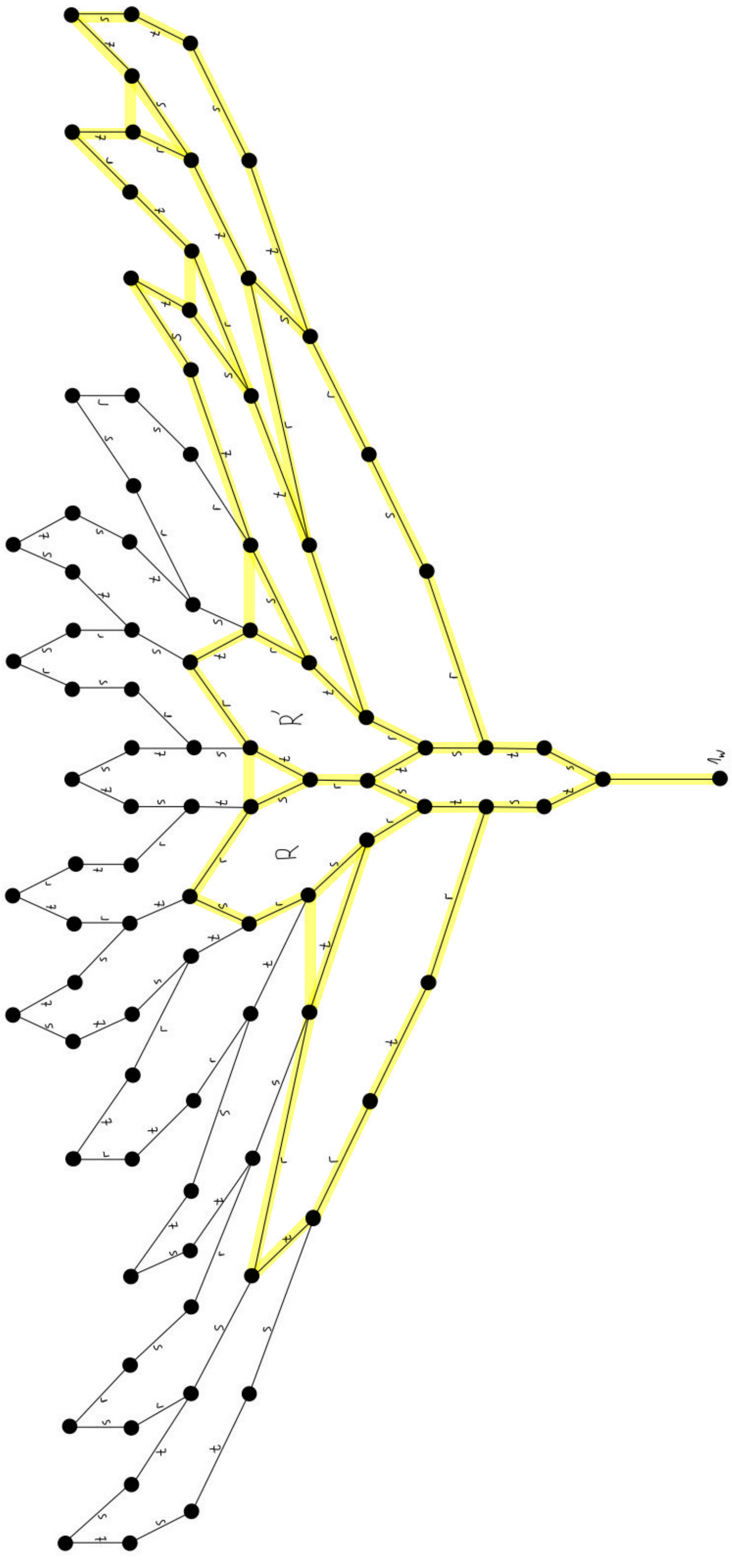


Figure 7.16.: Illustration of the group $C_{(R',R)}$

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Selbstständigkeitserklärung

Ich erkläre: Ich habe die vorgelegte Dissertation selbstständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt, die ich in der Dissertation angegeben habe. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben, die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht. Ich stimme einer evtl. Überprüfung meiner Dissertation durch eine Antiplagiat-Software zu. Bei den von mir durchgeführten und in der Dissertation erwähnten Untersuchungen habe ich die Grundsätze guter wissenschaftlicher Praxis, wie sie in der „Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis“ niedergelegt sind, eingehalten.

Gießen, Juni 2023