H.O. WALTHER, A Uniqueness Problem for a Nonlinear Differential Delay Equation.

The class of autonomous retarded functional differential equations which is relatively best understood today is given by

$$\dot{x}(t) = \alpha f(x(t-1)) \qquad (\alpha f)$$

with $f : R \to R$ satisfying $xf(x) < 0$ for $x \neq 0$, $f(0) = 0$. If positive parameters α are considered then (αf) represents a simple case of delayed negative feedback: A deviation $x(t-1) \lessgtr 0$ from the equilibrium solution $t \to 0$ is followed by a move $\dot{x}(t) \gtrless 0$ in the opposite direction. The dynamics of (αf) depend in a very subtile way on the graph of f, and many of the bifurcation phenomena which are of interest in dissipative O.D.E.s may be found in this small class of F.D.E.s. Understanding (αf) will also help to get a better feeling for the equations

$$\dot{x}(t) = p(x_t) - d(x_t) \qquad (p,d)$$

where the functionals p,d describe autocatalytic production and destruction respectively. p and d are defined on $C([-1,0],R)$ and $x_t \in C$ is given by $x_t(a) = x(t+a)$ for $[t-1,t]$ in the domain of x - as usual in F.D.E.s [3]. The interplay of autocatalytic production and destruction is common to many control processes in living systems [4].

A special and also historically important case is Hutchinson's equation for delayed logistic growth of a single species

$$\dot{n}(t) = rn(t)[1 - \frac{n(t-\tau)}{K}] \quad , \quad r,\tau,K \text{ positive} \qquad (n)$$

[5]. The positive solutions (which are the biologically meaningful ones) correspond to the set of all solutions of (αf) with $\alpha = r\tau$, $f = f_H$, $f_H(x) = 1-e^x$, via $x(t) = \log \frac{n(\tau t)}{K}$. It is well known that for every $\alpha > \pi/2$ equation (αf_H) has a periodic solution x with $x(-1) = 0$, $\dot{x} > 0$ on $[-1,0)$, $\dot{x} < 0$ on an interval $(0,z_1+1)$ with $x(z_1) = 0$, $\dot{x} > 0$ on (z_1+1,z_2), and $x(t) = x(t+z_2+1)$ for all real t. Numerical results strongly suggest that x has a stable and attractive orbit in C, and

that for $\alpha < \pi/2$ no periodic solutions exist.

Problem: Prove uniqueness and stability properties of x!
The tools might be available, compare [6,8]. Related results are contained in [15,11,12,9]. Solving this problem will be instructive for the investigation of a larger set of equations: f_H is not an odd function - one can give reasons that most of the nonlinearities f related to applications are far from being odd - most of the more detailed results on bifurcation of periodic solutions were obtained for odd functions f only.

Another promising, possibly harder problem is to show for all equations (αf) that the set of initial conditions $\phi \in C$ which define slowly oscillating solutions - i.e. solutions x with $|z-z'| > 1$ for every pair of zeros $z \neq z'$ in some unbounded interval $[t_x,\infty)$ - is open and dense. For a partial result, see [13]. The first statement of the conjecture is in [6].

Suggestions for numerical analysis: It is desirable to improve and develop algorithms for the computation of bifurcation diagrams for slowly oscillating solutions of (αf) [2,10]. In particular stability properties (Floquet multipliers) of these periodic solutions should be studied, compare [14,1].

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