Bifurcation from Periodic Solutions in Functional Differential Equations

Hans-Otto Walther

Mathematisches Institut der Universität, Theresienstraße 39,
D-8000 München 2, Federal Republic of Germany

Introduction

In parameterized functional differential equations

$$\dot{x}(t)=\alpha f(x(t-1)), \qquad \alpha>0, \tag{α}$$

with a continuous function f: $\mathbb{R}\to\mathbb{R}$ satisfying $f(0)=0$ and $f'(0)=-1$, the trivial solution $t\to 0$ becomes unstable at $\alpha=\pi/2$, and a continuum of periodic solutions bifurcates. This was shown by Nussbaum [20] for nonlinearities f, bounded from below or from above, for which the negative feedback condition

$$x f(x)<0 \tag{NF}$$

holds true for all $x\neq 0$.

Here we shall study nonlinearities which satisfy (NF) only locally, and which have a humped graph. Nonlinearities of this type occur in models for physiological control processes, see Mackey and Glass [6, 17], Lasota and Wazewska [16, 15]. Equation (α) itself with $f(x)=\delta-\sin(x+\xi)$, $\delta=0=\xi$ allowed, stands for a prototype of a phaselocked loop as they are used in communication systems. For a study of this case, see Furumochi [5].

Numerical results by Hadeler [8] and by Jürgens, Peitgen and Saupe [12] substantiated the conjecture that for humped nonlinearities periodic solutions undergo a series of bifurcations as α increases. The first proven theorem in this direction was Nussbaum's nonuniqueness result. He obtained two disjoint continua over an unbounded interval [21]. – For nonlinearities which are step functions or close to step functions bifurcation from periodic solutions, and also existence of chaotic motion, can be shown essentially by explicit computation of certain solutions and by introducing coordinates on a finite-dimensional locally invariant subset in state space, compare [1, 22, 25].

In the present paper we prove that bifurcation from a curve of periodic solutions does exist for a class of nonlinearities which includes, for example, the odd continuation of $0\leqq x\to -x(1-x)$ and $f=-\sin$ mentioned before. The precise result is stated in Theorem 6.2 in the final section.

Let us briefly describe the organization of the paper, and give some comments. Section 1 deals with "slowly oscillating" periodic solutions. For any solution x of this type we define a map P by translation along trajectories with $P(\eta)=\eta$ for the initial value η of x. Domain and range of P belong to a fixed hyperplane C^* in state space which does not depend on $\alpha f=g$, x. This is done as indicated by Hadeler [8]. It is shown that P is equal to a Poincaré map constructed by means of the implicit function theorem so that the index of the fixed point η can be computed from the spectrum of the period map U associated with the linear variational equation along x (Theorem 1.1).

In Sect. 2 we modify results of Kaplan and Yorke [13] about slowly oscillating periodic solutions with the symmetry

$$x(t)=-x(t-2) \quad \text{for all } t\in\mathbb{R} \tag{S}$$

for nonlinearities with zeros $X>0$, as in the examples above. Obviously, these "symmetric" periodic solutions have period 4. From Sect. 2 on the nonlinearity is required to be an odd function. The symmetry (S) then allows to write the time-4-map U as the iterate of the time-2-map W. This will make arguments in Sects. 3, 5 and 6 considerably easier.

After these preliminaries the proof of bifurcation begins with the basic Lemma 3.1. It tells how to compute spectrum and resolvent of W if a symmetric periodic solution is given. Let us point out that similar results hold in more general situations, too. f need not be odd, the period may be any integer $p>1$. This should become clear from the easy proof of Lemma 3.1.

From Sect. 4 on we consider continua of symmetric periodic solutions. The investigation is restricted to nonlinearities which guarantee uniqueness of the corresponding orbits so that we obtain a curve of initial values $x_0^\alpha\in C^*$, $\alpha>\pi/2$. Sufficient conditions for uniqueness were given by Nussbaum [21]. Studying this particular situation means that we avoid bifurcation within the symmetric periodic solutions. The latter problem is easier and may be investigated using ideas from [13, 21]. By restricting the class of nonlinearities further we derive all the a priori information about the solutions x^α which is necessary to show that $\lambda=-1$ is always a simple eigenvalue of W_α (Sect. 5), and that an odd number of eigenvalues crosses at $\lambda=1$ if α increases beyond a critical value. Here, the idea how to prove the comparison result Lemma 6.1 is due to my colleague H. Steinlein. – Section 6 shows in particular that the symmetric periodic solutions x^α become unstable. Attractivity of x^α for $\alpha>\pi/2$ not too large follows from a result of Kaplan and Yorke [14] and from Nussbaum's uniqueness result Theorem 2.2 [21].

In Theorem 6.1 we obtain a jump of the fixed point index along the curve x_0^α, $\alpha>\pi/2$, and bifurcation follows.

The nonlinearities f in Theorems 6.1 and 6.2 are certainly not the only ones for which symmetry-breaking bifurcation from periodic solutions occurs, compare the numerical results [8, 12]. But they seem to constitute a class for which proofs are not too involved. We hope that Sects. 3–6 will convince the reader that our approach might also be used for other nonlinearities.

Another objective of further study should be to obtain Hopf bifurcation for the Poincaré maps $P(\alpha, \cdot)$.

1. Poincaré Maps for Slowly Oscillating Periodic Solutions

Let a C^1-function $g:\mathbb{R}\to\mathbb{R}$ be given with $xg(x)<0$ for all $x\neq 0$ in an open neighborhood N of $0\in\mathbb{R}$. Consider equation

$$\dot{x}(t)=g(x(t-1)) \tag{g}$$

Every initial value ϕ in the Banach space C of continuous real functions on $[0, 1]$, $|\phi|=\sup|\phi(t)|$, defines a unique continuous solution $x^\phi: [0, \infty)\to\mathbb{R}$ satisfying Eq. (g) for $t>1$ and $x^\phi|[0, 1]=\phi$. This is proved by means of the formulas

$$x(n+a)=x(n)+\int_{n-1}^{n-1+a} g(x(s))\,ds \qquad \text{for } a\in[0, 1],\ n\in\mathbb{N}. \tag{1}$$

They also imply that on compact intervals solutions depend continuously on initial data with respect to supremum-norms.

Let $x:\mathbb{R}\to N$ denote a periodic solution with minimal period $p>0$ and with $x(0)=0$, $0<\dot{x}$ in $[0, 1)$, $\dot{x}<0$ in $(1, z+1)$ for some zero $z>1$ of x, $0<\dot{x}$ in $(z+1, p)$. Theorems on existence of such "slowly oscillating" periodic solutions may be found in e.g. [11, 7, 19, 20, 13, 14, 24]. We associate a map P with x. Set $C^*:=\{\phi\in C\,|\,\phi(0)=0\}$. Continuous dependence in our initial value problem and $x(\mathbb{R})\subset N$ imply that there is an open neighborhood Ω of $\eta:=x|[0, 1]$ in C^* such that every x^ϕ, $\phi\in\Omega$, has the following properties: There exist zeros $z_1^\phi\in(1, p-1)$ and $z_2^\phi\in(p-1,\ p+1)$ with $z_2^\phi>z_1^\phi+1$, $0<x^\phi$ in $[1, z_1)$, $\dot{x}^\phi<0$ in $[z_1^\phi, z_1^\phi+1)$, $0<\dot{x}^\phi$ in $(z_1^\phi+1,\ z_2^\phi+1)$, $x^\phi([0, z_2^\phi+1])\subset N$.

For $\phi\in\Omega$, set $P\phi(a):=x^\phi(z_2^\phi+a)$, $a\in[0, 1]$. P maps Ω into C^*, and $P(\eta)=\eta$. – P is continuous and compact (bounded sets become precompact): Continuous dependence of solutions on initial data gives continuity of the maps $\phi\to z_i^\phi$ $(i=1, 2)$, and continuity of P follows easily. – The maps z_i send bounded sets into bounded sets. This may be used together with Ascoli's theorem and formula (1) in order to derive compactness. For details, see e.g. [19]. – Let $U: C_\mathbb{C}\to C_\mathbb{C}$ denote the time-p-map for the linear variational equation along x,

$$\dot{u}(t)=g'(x(t-1))\,u(t-1); \tag{x}$$

$U\psi(a)=u^\psi(p+a)$ for $a\in[0, 1]$, with the continuous solution $u^\psi: [0, \infty)\to C_\mathbb{C}$ satisfying Eq. (x) for $t>1$ and $u^\psi|[0, 1]=\psi$ in the space $C_\mathbb{C}$ of complex-valued continuous functions on the unit interval. U is linear, continuous and compact. Let $M(\lambda)$, $0\neq\lambda\in\mathbb{C}$, denote the algebraic multiplicity if λ is an eigenvalue of U, and set $M(\lambda):=0$ if not. Recall $U\dot{\eta}=\dot{\eta}$ from Eqs. (g) and (x) so that $M(1)\geqq 1$.

Theorem 1.1. *Suppose $M(1)=1$. Then η is an isolated fixed point of P with index given by*

$$(-1)^{\Sigma_{\lambda>1}M(\lambda)}.$$

Proof. For Poincaré maps associated with O.D.E's [9], or more generally with flows of C^1-mappings in Banach spaces [18], the index formula is certainly well known. We prefer to give the proof, however, since in case of F.D.E's one only has semiflows, with less smoothness, so that one has to be a little careful.

– Also, we must show that the a priori given map P is a Poincaré map in the usual sense, differentiable and defined on a domain transversal to the flow.

i) Consider the continuous semiflow $T: [0, \infty) \times C \to C$ of Eq. (g); $T_t \phi = T(t, \phi) = x_t^\phi$ with $x_t^\phi(a) = x^\phi(t+a)$ for $a \in [0, 1]$. For every compact interval $I \subset (1, \infty)$ and for every bounded set $B \subset C$, cl $T(I \times B)$ is compact. It follows from Theorem 4.1, p. 46 in [9] that T is C^1 on $(1, \infty) \times C$, with partial derivatives given by $D_1 T(t, \phi) s = s\dot{x}_t^\phi$ and $D_2 T(t, \phi) \psi = w_t$. Here $w_0 = \psi$ and $\dot{w}(s) = g'(x^\phi(s-1))\, w(s-1)$ for $s > 1$. As $p > 1$, $D_1 T(p, \eta)$ is defined and equals multiplication by $\dot\eta$, and $D_2 T(p, \eta)\psi = U\psi$ for all $\psi \in C$.

ii) We have $C = \mathbb{R}\dot\eta \oplus C^*$ since C^* has codimension 1 and $\dot\eta(0) > 0$. The corresponding projections are $p_\eta: \phi \to (\phi(0)/\dot\eta(0))\,\dot\eta$ and $p^* = \mathrm{id} - p_\eta$.

P is differentiable on a closed ball $B \subset \Omega$ with $\eta \in \operatorname{int} B$, and $DP(\eta) = p^* \circ U \circ (C^* \subset C) =: U^*$. Proof: Note $z_2^\eta = p$ and $0 = p_\eta(T(z_2^\phi, \phi))$ for all $\phi \in \Omega$. By $p > 1$ the semiflow T is C^1 in a neighborhood of (p, η), and $D_1(p_\eta \circ T)(p, \eta): \mathbb{R} \ni s \to s\dot\eta \in \mathbb{R}\dot\eta$ is an isomorphism. By continuity of $\phi \to z_2^\phi$, the implicit function theorem [3, p. 265] guarantees that $\phi \to z_2^\phi$ is C^1 on an open ball $B' \subset \Omega$ around η. It follows that $P: \phi \to T(z_2^\phi, \phi) = p^* \circ T(z_2^\phi, \phi)$ is C^1 on B' (provided B' is chosen small enough), and $DP(\eta) = p^* \circ (D_1 T(p, \eta) \circ D z_2(\eta) + D_2 T(p, \eta) \circ (C^* \subset C)) = 0 + U^*$.

For eigenvalues $\lambda \in \mathbb{C} \setminus \{0\}$ of the complexification $U_{\mathbb{C}}^*$ of U^* let $M^*(\lambda)$ denote the algebraic multiplicity. Set $M^*(\lambda) := 0$ for $\lambda \neq 0$ not in the spectrum of $U_{\mathbb{C}}^*$.

$\lambda = 1$ is not an eigenvalue of $U_{\mathbb{C}}^*$. Proof: $\psi = U_{\mathbb{C}}^* \psi$ implies $\phi = U^* \phi$ for $\phi \in \{\operatorname{Re}\psi, \operatorname{Im}\psi\}$. As $\phi \in C^*$, $p^* \phi = U^* \phi = p^* U \phi$ and $p^*(U - \mathrm{id})\phi = 0$. Hence $(U - \mathrm{id})\phi = s\dot\eta$ with some $s \in \mathbb{R}$. Assume $\phi \neq 0$. Since $\phi \in C^*$, ϕ and $\dot\eta$ are linearly independent, and $(U - \mathrm{id})^2 \phi = 0$. This contradicts $M(1) = 1$.

Since P is compact we may apply e.g. Satz 1, p. 91 from [2]. It follows that η is an isolated fixed point with index defined and given by the degree

$$\deg(\eta, \mathrm{id} - P \mid \hat{\Omega}, 0) = (-1)^{\Sigma_{\lambda > 1} M^*(\lambda)}$$

for a sufficiently small neighborhood $\hat{\Omega}$ of η.

iii) Proof of $M^*(\lambda) = M(\lambda)$ for $\lambda \neq 1$. $M(1) = 1$ and $U\dot\eta = \dot\eta$ imply a spectral decomposition $C_{\mathbb{C}} = \mathbb{C}\dot\eta \oplus E$ into U-invariant closed subspaces. Let Pr_η, Pr_E denote the spectral projections onto $\mathbb{C}\dot\eta$, E respectively. For details, consult [4]. – Since U is a real operator ($UC \subset C$) the projections are real operators, too, and we obtain $C = \mathbb{R}\dot\eta \oplus F$ for $F := \{\operatorname{Re}\psi \mid \psi \in E\}$. pr_η, pr_F: $C \ni \phi \to \operatorname{Re} Pr_\eta \phi$, $\operatorname{Re} Pr_E \phi$ are projections onto $\mathbb{R}\dot\eta$ and F respectively with $pr_\eta + pr_F = \mathrm{id}$. We have $E = F + iF$ and $UF \subset F$. $U_E: E \ni \psi \to U\psi \in E$ is the complexification of $U_F: F \ni \phi \to U\phi \in F$. The spectrum of U_E is $\sigma(U) \setminus \{1\}$. For eigenvalues λ, $0 \neq \lambda \neq 1$, $M(\lambda)$ coincides with the algebraic multiplicity $M_E(\lambda)$ of λ as an eigenvalue of U_E. Set $M_E(\lambda) := 0$ if $\lambda \neq 0$ is not an eigenvalue of U_E.

In order to derive $M^*(\lambda) = M_E(\lambda)$ one may now use the complexification of the diagram

$$\begin{array}{ccc} C^* & \xrightarrow{\;U^*\;} & C^* \\ {\scriptstyle pr_F \mid C^* =: p_F^*}\big\downarrow & & \big\uparrow{\scriptstyle p_*^F := p^* \mid F} \\ F & \xrightarrow{\;U_F\;} & F \end{array}$$

Proof of commutativity: $p_*^F \circ U_F \circ p_F^*(\phi) = p^*(U(pr_F(\phi))) = p^*(U(pr_F(\phi) - \phi) + U(\phi)) = 0 + U^*\phi$ since $pr_F\phi - \phi \in \mathbb{R}\dot{\eta}$. p_F^* and p_*^F are inverse to each other: $\phi \in F$ implies $\phi = pr_F\phi = pr_F(p^*\phi + (\phi - p^*\phi)) = pr_F(p^*\phi) + 0$. Also, $\phi = p^*(pr_F\phi)$ for $\phi \in C^*$.

2. Symmetric Periodic Solutions for Odd Nonlinearities

In this section g is a continuous real function which is odd: $g(x) = -g(-x)$ for all $x \in \mathbb{R}$. Furthermore we assume that g is differentiable at $x=0$ with $g'(0) < 0$, and that for some $X > 0$, $g(X) = 0$ and $g < 0$ in $(0, X)$. Kaplan and Yorke [13] showed how to obtain slowly oscillating periodic solutions with symmetry (S) to Eq. (g) under slightly different assumptions on g, including $xg(x) < 0$ for all $x \neq 0$. The following is a version of their approach fitted to our situation.

Consider the Hamiltonian system

$$\dot{x} = g(y), \qquad \dot{y} = -g(x) \tag{(g)}$$

with first integral $G: (x, y) \to \int_0^x g + \int_0^y g$.

Proposition 2.1. *There is a solution (x, y): $\mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}^-$ with $\dot{x} > 0$, $\dot{y} > 0$ and $(x, y)(t) \to (0, -X)$ as $t \to -\infty$, $(x, y)(t) \to (X, 0)$ as $t \to +\infty$.*

Proof. Choose $x_0 \in (0, X)$ with $G(x_0, -x_0) = G(X, 0)$ and consider the maximal solution $(x, y): I \to \mathbb{R}^2$ with value $(x_0, -x_0)$ at $t=0$. On $([x_0, X] \times [-x_0, 0]) \setminus \{(X, 0)\}$, both components of the vectorfield are positive while $(X, 0)$ is a stationary point. It follows that (x, y) either crosses the part $A := (x_0, X) \times \{0\} \cup \{X\} \times (-x_0, 0)$ of the boundary of $[x_0, X] \times [-x_0, 0]$ at a first finite time $s > 0$, or $\sup I = +\infty$ and $(x, y)(t) \to (X, 0)$ as $t \to +\infty$. By $G(x_0, -x_0) = G(X, 0) \notin G(A)$, $A \cap (x, y)(I) = \emptyset$. Hence $(x, y)(t) \to (X, 0)$ as $t \to +\infty$. $\lim_{t \to -\infty} (x, y)(t) = (0, -X)$ follows from $(x, y)(t) = (-y, -x)(-t)$ for all $t > 0$.

Connecting solutions between the stationary points $(X, 0)$ and $(0, X)$, $(0, X)$ and $(-X, 0)$, $(-X, 0)$ and $(0, -X)$ in the other quadrants are most easily obtained by considering the solutions $(-y, x)$, $(-x, -y)$, $(y, -x)$.

Corollary 2.1. *For every $\eta \in (-X, 0)$ there is a first $t = T(\eta) > 0$ such that the solution (x, y) of Eq. ((g)) with value $(0, \eta)$ for $t=0$ intersects with $(0, X) \times \{0\}$. The map $\eta \to T(\eta)$ is continuous.*

Proposition 2.2. *The solution (x, y) of Eq. ((g)) with $x(0) = 0$, $y(0) = \eta \in (-X, 0)$ satisfies $x = -y(T(\eta) - \cdot)$, $y = -x(T(\eta) - \cdot)$ on $[0, T(\eta)]$. In particular, $x(T(\eta)) = -\eta$.*

Proof. The map $[0, T(\eta)] \ni t \to (-y(T(\eta) - t), -x(T(\eta) - t))$ is a solution with values $(0, -x(T(\eta))$ at $t=0$, $(-\eta, 0)$ at $t = T(\eta)$. Hence $G(0, -x(T(\eta))) = G(-\eta, 0) = G(0, \eta)$. $G(0, \cdot)$ is strictly monotone on $(-X, 0)$. Therefore $\eta = -x(T(\eta))$, and the solution above coincides with (x, y).

Corollary 2.2. *The solution (x, y) of Eq. ((g)) with $x(0) = 0$, $y(0) = \eta \in (-X, 0)$ is periodic with minimal period $4T(\eta)$. On $(0, T(\eta))$, $\dot{x} > 0$, $\dot{y} > 0$, and $x = y(\cdot + T(\eta))$*

$=-x(\cdot+2T(\eta))=-y(\cdot+3T(\eta)),\ y=-x(\cdot+T(\eta))=-y(\cdot+2T(\eta))=x(\cdot+3T(\eta))$. *Also*, $(x, y)(\mathbb{R})\subset[\eta, -\eta]^2$.

Proof. $x(T(\eta))=-\eta$ implies that the formulas in Corollary 2.2 define a solution on $[0, 4T(\eta)]$.

Proposition 2.3. $\lim_{\eta\to -X} T(\eta)=+\infty$, $\lim_{\eta\to 0} T(\eta)=-\pi/2g'(0)$.

Proof. The first assertion is clear from $(0, -X)$ being a stationary point. To obtain the second one use the argument in the proof of Theorem 1.1 [13]. Note that for, say, $-X/2<\eta<0$ all periodic orbits lie in $[-X/2, X/2]^2$, and that $xg(x)<0$ for $-X/2\leqq x\leqq X/2$, $x\neq 0$, which means that the assumptions of [13] hold true.

Corollary 2.3. *For* $g'(0)<-\pi/2$ *there exists* $\eta\in(-X, 0)$ *with* $T(\eta)=1$, *and there is a slowly oscillating periodic solution to Eq.* (g) *with* $x(\mathbb{R})\subset(-X, X)$ *and with symmetry* (S).

Proof. The first component of the solution of ((g)) with period $4T(\eta)=4$ satisfies (g) since $y=x(\cdot-1)$.

3. Spectra of Symmetric Periodic Solutions

We assume that the nonlinearity g satisfies the hypotheses of Sects. 1 and 2, and we consider a periodic solution x to Eq. (g) as given by Corollary 2.3. It follows that the map $t\to(x(t), y(t))$ with $y(t):=x(t-1)$ is a solution to Eq. ((g)).

The coefficient $t\to g'(x(t-1))$ in the linear variational equation (x) has period 2 since g' is an even function and $x(t)=-x(t-2)$ for all t. Therefore $U=W\circ W$ with the linear compact continuous operator $W\colon C_{\mathbb{C}}\to C_{\mathbb{C}}$, $W\psi(a):=w^{\psi}(2+a)$ for $a\in[0, 1]$, w^{ψ} the solution to equation

$$\dot{w}(t)=g'(x(t-1))\,w(t-1) \tag{x}$$

with $w_0=w\,|\,[0, 1]=\psi$. The spectral mapping theorem, or more easily the decomposition $U-\lambda^2\,\mathrm{id}=(W-\lambda\,\mathrm{id})\circ(W+\lambda\,\mathrm{id})$, implies $\sigma(U)=\{\lambda^2\in\mathbb{C}\,|\,\lambda\in\sigma(W)\}$ for the spectra of U, W. Define $m(\lambda)$ to be the algebraic multiplicity if $\lambda\neq 0$ is an eigenvalue of W, and set $m(\lambda):=0$ if not. We may reduce the investigation of multiplicities with respect to U to a study of the operator W:

Proposition 3.1. $M(\lambda^2)=m(\lambda)+m(-\lambda)$ *for* $0\neq\lambda\in\mathbb{C}$.

Proof. See e.g. Chaps. VII.3, VII.4 in [4].

Proposition 3.2. $\dot{x}_0$ *is an eigenvector of the eigenvalue* $\lambda=-1$ *of* W.

Proof. Differentiate Eq. (g) and observe (S).

In order to compute spectrum and resolvent of W we need the fundamental matrix solution

$$S^{\lambda}=\begin{pmatrix} u_1^{\lambda} & u_2^{\lambda} \\ z_1^{\lambda} & z_2^{\lambda} \end{pmatrix}$$

of equation

$$\dot{u}=(g'\circ y)z, \qquad \dot{z}=\lambda^{-1}(g'\circ x)u; \qquad \lambda\neq 0 \tag{λ}$$

with $S^{\lambda}(1)$ the unit matrix. We set

$$Q(\lambda):=\begin{pmatrix}\lambda z_1^{\lambda}(0)-1 & \lambda z_2^{\lambda}(0)\\ u_1^{\lambda}(0) & u_2^{\lambda}(0)-1\end{pmatrix} \quad \text{and} \quad q(\lambda):=\det Q(\lambda).$$

The Wronskian $\det S^{\lambda}$ is constant $(=\det S^{\lambda}(0)=1)$. Therefore $q(\lambda)=1-\lambda-u_2^{\lambda}(0)-\lambda z_1^{\lambda}(0)$ for $0\neq\lambda\in\mathbb{C}$. q is analytic in $\mathbb{C}\setminus\{0\}$, compare e.g. Chap. X.7, p. 295 in [3].

Lemma 3.1. *Let $\lambda\in\mathbb{C}\setminus\{0\}$. There exists a surjective linear operator $L(\lambda)$: $C_{\mathbb{C}}\to\mathbb{C}^2$ such that $(W-\lambda\,\mathrm{id})\chi=\psi$ implies*

$$\begin{pmatrix}W\chi\\ w_1^{\chi}\end{pmatrix}=S^{\lambda}c+S^{\lambda}\int_1^{\bullet}(S^{\lambda}(s))^{-1}\begin{pmatrix}0\\ -\lambda^{-1}g'(x(s))\,\psi(s)\end{pmatrix}ds \tag{1}$$

with $c\in\mathbb{C}^2$ and

$$Q(\lambda)\,c=L(\lambda)\,\psi. \tag{2}$$

Proof. Set $w:=w^{\chi}$ for the solution of Eq. (x) with initial value χ. On the interval $[0,1]$, $w_2=W\chi$ and w_1 satisfy $\dot{w}_2=\dot{w}(2+\cdot)=g'(x(2+\cdot-1))\,w(1+\cdot)=g'(-x(\cdot-1))\,w_1=g'(x(\cdot-1))\,w_1=(g'\circ y)\,w_1$ and $\dot{w}_1=\dot{w}(1+\cdot)=g'(x(1+\cdot-1))\,w_0=(g'\circ x)\,\lambda^{-1}(w_2-\psi)$ with $\psi=(W-\lambda\,\mathrm{id})\chi=w_2-\lambda w_0$. By the variation-of-constants formula we obtain Eq. (1) with some $c\in\mathbb{C}^2$. Define $L(\lambda)\psi$ by

$$\begin{pmatrix}-\psi(1)-\left\{\text{second compoent of } S^{\lambda}(0)\int_1^0(S^{\lambda}(s))^{-1}\begin{pmatrix}0\\ -g'(x(s))\,\psi(s)\end{pmatrix}ds\right\}\\ \left\{\text{first component of } S^{\lambda}(0)\int_1^0(S^{\lambda}(s))^{-1}\begin{pmatrix}0\\ -\lambda^{-1}g'(x(s))\,\psi(s)\end{pmatrix}ds\right\}\end{pmatrix}$$

The first component of Eq. (2) follows from $W\chi(1)=\psi(1)+\lambda\chi(1)=\psi(1)+\lambda w_1(0)$ and from Eq. (1) with $t=1$, $t=0$. To derive the second component of Eq. (2) note $c_2=w_1(1)$ (see Eq. (1) with $t=1$) and $w_1(1)=w(2)=W\chi(0)$, and substitute the right hand side of Eq. (1) with $t=0$ for $W\chi(0)$.

It is easy to see that $L(\lambda)\,C_{\mathbb{C}}$ contains a basis of $\mathbb{C}^2$.

Remark. Lemma 3.1 may be generalized. For example, if x is any periodic solution of Eq. (g), $g:\mathbb{R}\to\mathbb{R}$ C^1, with integer period $p>1$ one would compute the solution χ to $(U_p-\lambda\,\mathrm{id})\chi=\psi$, U_p the time-p-map of (x), by means of a fundamental matrix for a system of p linear nonautonomous O.D.E's.

Corollary 3.1. *For $0\neq\lambda\in\mathbb{C}$, $q(\lambda)=0$ is equivalent to $\lambda\in\sigma(W)$.*

Proof. $(W-\lambda\,\mathrm{id})\chi=0$ and $\chi\neq 0$, $\lambda\neq 0$ yield $W\chi\neq 0$. By the preceding lemma, there exists $c\neq 0$ with $Q(\lambda)\,c=0$. Hence $q(\lambda)=0$. – If $\lambda\neq 0$ and $\lambda\notin\sigma(W)$ then for every $\psi\in C_{\mathbb{C}}$ there exists $c\in\mathbb{C}^2$ such that $Q(\lambda)\,c=L(\lambda)\,\psi$. Since $L(\lambda)$ is surjectice the rank of $Q(\lambda)$ must be 2, and $q(\lambda)\neq 0$.

We have seen that $Q(\lambda)$ is invertible if $0 \neq \lambda \notin \sigma(W)$. This means that we can compute the resolvent $(W-\lambda\,\mathrm{id})^{-1}$: $\chi=(W-\lambda\,id)^{-1}\psi$ implies $(W-\lambda\,\mathrm{id})\chi=\psi$, or $\chi=\lambda^{-1}(W\chi-\psi)$ with $W\chi$ given by (1), (2).

Let $L(C_{\mathbb{C}})$ denote the space of bounded linear operators from $C_{\mathbb{C}}$ to $C_{\mathbb{C}}$.

Corollary 3.2. *The analytic mapping* $\mathbb{C}\setminus(\sigma(W)\cup\{0\})\ni\lambda\to q(\lambda)(W-\lambda\,\mathrm{id})^{-1} \in L(C_{\mathbb{C}})$ *admits a continuous extension* H *to* $\mathbb{C}\setminus\{0\}$.

Proof. For $\psi\in C_{\mathbb{C}}$ and $\lambda\neq 0$, $\lambda\notin\sigma(W)$, we have $q(\lambda)(W-\lambda\,\mathrm{id})^{-1}\psi=q(\lambda)\,\lambda^{-1}(W\chi-\psi)$ with $W\chi$ given by (1) and (2). It follows that $q(\lambda)\,W\chi$ is the first component of

$$q(\lambda)\left(S^{\lambda}(\cdot)(q(\lambda))^{-1}\,\tilde{Q}(\lambda)\,L(\lambda)\,\psi+S^{\lambda}(\cdot)\int_{1}^{\bullet}(S^{\lambda}(s))^{-1}\begin{pmatrix}0\\ -\lambda^{-1}g'(x(s))\,\psi(s)\end{pmatrix}ds\right)$$

with

$$\tilde{Q}(\lambda)=\begin{pmatrix}u_2^{\lambda}(0)-1 & -\lambda z_2^{\lambda}(0)\\ -u_1^{\lambda}(0) & \lambda z_1^{\lambda}(0)-1\end{pmatrix}=q(\lambda)(Q(\lambda))^{-1}.$$

Now the assertion is easily derived from continuity of the maps

$$0\neq\lambda\to u_i^{\lambda}\,|\,[0,1]\in C_{\mathbb{C}},\qquad 0\neq\lambda\to z_i^{\lambda}\,|\,[0,1]\in C_{\mathbb{C}},\qquad i\in\{1,2\}.$$

Proposition 3.3. $0\neq\lambda\in\sigma(W)$ *and* $Q(\lambda)\neq 0$ *imply* $H(\lambda)\neq 0$.

Proof. Let $0\neq\lambda\in\sigma(W)$. By $q(\lambda)=0$ we see that $H(\lambda)\,\psi$ is the first component of $\lambda^{-1}\,S^{\lambda}(\cdot)\,\tilde{Q}(\lambda)\,L(\lambda)\,\psi$, compare the proof of Corollary 3.2. By $Q(\lambda)\neq 0$, $\tilde{Q}(\lambda)\neq 0$. By $L(\lambda)\,C_{\mathbb{C}}=\mathbb{C}^2$ there exists $\psi\in C_{\mathbb{C}}$ such that $c:=(c_1,c_2)^{\mathrm{tr}}:=\tilde{Q}(\lambda)\,L(\lambda)\,\psi\neq 0$. In case $c_1\neq 0$ we find $H(\lambda)\,\psi(1)=\lambda^{-1}\,c_1\,u_1^{\lambda}(1)+0=c_1/\lambda\neq 0$. In case $c_1=0$ we find $H(\lambda)\,\psi=\lambda^{-1}\,c_2\,u_2^{\lambda}$ with $c_2\neq 0$, and $\dot{u}_2^{\lambda}(1)=g'(y(1))\,z_2^{\lambda}(1)=g'(0)<0$ yields $H(\lambda)\,\psi \neq 0$.

For a zero λ of q let $j(\lambda)$ denote the order, and set $j(\lambda):=0$ if $q(\lambda)\neq 0$.

Corollary 3.3. *For every* $\lambda\in\mathbb{C}\setminus\{0\}$ *with* $Q(\lambda)\neq 0$, $m(\lambda)=j(\lambda)$.

Proof. Corollary 3.1 gives $m(\lambda)=0=j(\lambda)$ if $0\neq\lambda\notin\sigma(W)$. Furthermore, $0 \neq\lambda\in\sigma(W)$ and $Q(\lambda)\neq 0$ imply that $Q(\lambda)$ has rank 1. By Lemma 3.1 the geometric eigenspace $\ker(W-\lambda\,\mathrm{id})$ is contained in the set

$$\{c_1(u_1^{\lambda}\,|\,[0,1])+c_2(u_2^{\lambda}\,|\,[0,1])\,|\,Q(\lambda)(c_1,c_2)^{\mathrm{tr}}=0\}.$$

Hence $\dim\ker(W-\lambda\,\mathrm{id})=1$. It follows that the algebraic multiplicity $m(\lambda)$ coincides with the stabilizing exponent

$$J(\lambda):=\min\{k\in\mathbb{N}\,|\,\ker(W-\lambda\,\mathrm{id})^k=\ker(W-\lambda\,\mathrm{id})^{k+1}\}-$$

since

$$m(\lambda)=\dim\ker(W-\lambda\,\mathrm{id})^{J(\lambda)}\leq\sum_{k}{}_{1}^{J(\lambda)}\dim\ker(W-\lambda\,\mathrm{id})=J(\lambda)$$

and

$$\dim\ker(W-\lambda\,\mathrm{id})<\ldots<\dim\ker(W-\lambda\,\mathrm{id})^{J(\lambda)}.$$

On the other hand, $J(\lambda)$ is equal to the order of the pole of the resolvent of W at $z=\lambda$. Therefore $J(\lambda)=\min\{k\in\mathbb{N}\,|\,\mu\to(\mu-\lambda)^k(W-\mu\,\mathrm{id})^{-1}$ admits a continuous extension to $(\mathbb{C} \setminus \sigma(W)) \cup \{\lambda\}\} =: K(\lambda)$. By definition of $j(\lambda)$, $q = (\cdot - \lambda)^{j(\lambda)} h$ with h analytic in a neighborhood of $z = \lambda$ and $h(\lambda) \neq 0$. Now use Corollary 3.2 and Proposition 3.3 to deduce $K(\lambda) = j(\lambda)$.

Proposition 3.4. *$Q(\lambda) \neq 0$ for every $\lambda \in \sigma(W) \setminus \{0, -1\}$.*

Proof. $Q(\lambda) = 0$ and $\lambda \neq 0$ imply $u_1^\lambda(0) = 0 = z_2^\lambda(0)$, $u_2^\lambda(0) = 1 = \lambda z_1^\lambda(0)$. By $1 = \det S^\lambda(0)$, $\lambda = -1$.

Corollary 3.4. *Suppose $q(1) \neq 0$ and $m(-1) = 1$. Then the index of the fixed point x_0 of P is given by*

$$(-1)^{\sum_{\lambda < -1} j(\lambda)} (-1)^{\sum_{1 < \lambda} j(\lambda)}.$$

Proof. Proposition 3.1 and Corollary 3.1 give $M(1) = 1$. Use Proposition 3.1, Theorem 1.1, Proposition 3.4 and Corollary 3.3.

Proposition 3.5. *$q(\lambda)/\lambda \to -1$ as $|\lambda| \to +\infty$.*

Proof. For $|\lambda| \to +\infty$ S^λ converges to the matrix solution S^∞ of $\dot{u} = (g' \circ y) z$, $\dot{z} = 0$ with $S^\infty(1)$ the unit matrix. Hence $z_1^\lambda(0) \to 0$ and $u_2^\lambda(0) \to -\int_0^1 g'(y(s))\, ds$. This implies $q(\lambda)/\lambda = \lambda^{-1} - 1 - u_2^\lambda(0)/\lambda - z_1^\lambda(0) \to -1$.

4. Continua of Symmetric Periodic Solutions

From now on we consider families of equations

$$\dot{x}(t) = \alpha f(x(t-1)), \qquad \alpha > 0, \tag{α}$$

with odd C^1-functions $f: \mathbb{R} \to \mathbb{R}$ satisfying

$$f'(0) = -1, \quad f(X) = 0 \quad \text{for some } X > 0 \text{ and } f < 0 \text{ in } (0, X), \tag{H1}$$

$$f(x)/x < 0 \text{ strictly increasing on } (0, X). \tag{H2}$$

Let us write $((\alpha))$ for the system $((g))$ with $g = \alpha f$. We use a result of Nussbaum [21] to obtain uniqueness of symmetric periodic solutions from condition (H2):

Proposition 4.1. (i) *The maps $T_\alpha: (-X, 0) \to \mathbb{R}^+$ defined by Eq. $((\alpha))$ in the sense of Corollary 2.1 are strictly monotonic decreasing.*

(ii) *T_1 maps $(-X, 0)$ onto $(\pi/2, \infty)$.*

(iii) *For all $\alpha > 0$ and all $\eta \in (-X, 0)$, $\alpha T_\alpha(\eta) = T_1(\eta)$.*

Proof. (i) Let $\alpha > 0$, $-X < \eta' < \eta < 0$. The orbits of Eq. $((\alpha))$ through $(0, \eta')$, $(0, \eta)$ lie in a square $[\eta' - \varepsilon, -\eta' + \varepsilon]^2 \subset (-X, X)^2$, $\varepsilon > 0$, see Corollary 2.2. Since $f < 0$ on $(0, -\eta' + \varepsilon)$ Nussbaum's proof of his Theorem 1.3 [21] applies. - Note that the period map in [21] is defined on the values of the Hamiltonian, not on initial values as T_α. - Proposition 2.3 implies (ii). To prove (iii) observe that $t \to (x(\alpha t), y(\alpha t))$ is a solution of Eq. $((\alpha))$ if and only if (x, y) is a solution to Eq. $((1))$.

Corollary 4.1. *For every $\alpha > \pi/2$ there exists a unique periodic solution $x^\alpha: \mathbb{R} \to (-X, X)$ of Eq. (α) with symmetry* (S) *and $x^\alpha(0)=0$, $\dot{x}^\alpha > 0$ on $[0, 1)$. (x^α, y^α) with $y^\alpha := x^\alpha(\cdot - 1)$ is the unique solution of Eq.* ((α)) *with the properties $x^\alpha(0)=0$, $-X < y^\alpha(0) = \eta < 0$, $T_\alpha(\eta)=1$.*

Proof. Corollary 2.3 gives existence. For every periodic solution $x: \mathbb{R} \to \mathbb{R}$ to Eq. (α) with $x(\mathbb{R}) \subset (-X, X)$, with (S) and $x(0)=0$, $\dot{x} > 0$ on $[0, 1)$, x and $y := x(\cdot - 1)$ solve Eq. ((α)) with $x(0)=0$, $y(0)=\eta$ in $(-X, 0)$, $T_\alpha(\eta)=1$. η is unique by the preceding proposition.

We have to adapt the notation of Sect. 3: Let $m_\alpha(-1)$ denote the algebraic multiplicity of the eigenvalue -1 of the time-2-map W_α which belongs to the linearization (x^α) of Eq. (α) along x^α. $j_\alpha(\lambda) \geqq 0$ is the order of $\lambda \in \mathbb{C} \setminus \{0\}$ as a zero of the map

$$q_\alpha: 0 \neq \lambda \to 1 - \lambda - u_2^{\lambda,\alpha}(0) - \lambda z_1^{\lambda,\alpha}(0) = \det Q_\alpha(\lambda)$$

where

$$Q_\alpha(\lambda) = \begin{pmatrix} \lambda z_1^{\lambda,\alpha}(0) - 1 & \lambda z_2^{\lambda,\alpha}(0) \\ u_1^{\lambda,\alpha}(0) & u_2^{\lambda,\alpha}(0) - 1 \end{pmatrix}$$

with the solutions $(u_i^{\lambda,\alpha}, z_i^{\lambda,\alpha})$, $i \in \{1, 2\}$, of equation

$$\dot{u} = \alpha (f' \circ y^\alpha) z, \qquad \dot{z} = \alpha \lambda^{-1} (f' \circ x^\alpha) u \tag{α, λ}$$

with initial conditions $u_1^{\lambda,\alpha}(1) = 1 = z_2^{\lambda,\alpha}(1)$, $z_1^{\lambda,\alpha}(1) = 0 = u_2^{\lambda,\alpha}(1)$.

Proposition 4.2. *The map $\pi/2 < \alpha \to x_0^\alpha \in C$ is continuous. There is an open set $\Omega \supset \{(\alpha, x_0^\alpha) \mid \alpha > \pi/2\}$ in $R^+ \times C^*$ with the following properties.*

(i) *For every $\alpha > \pi/2$ and all $\phi \in \Omega_\alpha := \{\phi \in C^* \mid (\alpha, \phi) \in \Omega\}$ there exist zeros $z_1 = z_1^{\phi,\alpha} > 1$ and $z_2 = z_2^{\phi,\alpha} > z_1 + 1$ of the solution $x = x^{\phi,\alpha}$ of Eq.* (α), *$x_0 = \phi$, with $0 < x$ on $[1, z_1)$, $\dot{x} < 0$ on $[z_1, z_1+1)$ $0 < \dot{x}$ on (z_1+1, z_2+1), $x([0, z_2+1]) \subset (-X, X)$.*

(ii) *The map $P: \Omega \ni (\alpha, \phi) \to x^{\phi,\alpha}(z_2^{\phi,\alpha} + \cdot) \in C^*$ is continuous and compact with $P(\alpha, x_0^\alpha) = x_0^\alpha$ for all $\alpha > \pi/2$.*

(iii) *Suppose $m_\alpha(-1)=1$ and $q_\alpha(1) \neq 0$. Then x_0^α is an isolated fixed point of $P(\alpha, \cdot)$ with index given by*

$$(-1)^{\sum_{\lambda < -1} j_\alpha(\lambda)} (-1)^{\sum_{1 < \lambda} j_\alpha(\lambda)}.$$

Sketch of Proof. The first assertion follows from continuity of the map $\pi/2 < \alpha \to T_\alpha^{-1}(1)$. Existence and properties of Ω are derived as in Sect. 1, essentially by continuity of $x^{\phi,\alpha}$ on bounded intervals with respect to (ϕ, α). (iii) is a consequence of Corollary 3.4.

Proposition 4.3. (i) $(x^\alpha, y^\alpha)(\mathbb{R}) \to (0, 0)$ *as* $\alpha \overset{>}{\to} \pi/2$.

(ii) $y^\alpha(0) \to -X$ *as* $\alpha \to +\infty$.

Proof. (i) It is enough to show $y^\alpha(0) \to 0$ since we have continuous dependence and all periods are equal to 4. Proposition 2.3 gives $1 = \lim_{\eta \to 0} T_{\pi/2}(\eta)$. By Proposition 4.1 $T_{\pi/2}$ is strictly monotonic decreasing. Therefore $1 < T_{\pi/2}$ on $(-X, 0)$. Let $\varepsilon \in (-X, 0)$. Continuity of the map $0 < \alpha \to T_\alpha(\varepsilon)$ implies $1 < T_\alpha(\varepsilon)$ for α in a neighborhood N of $\pi/2$. It follows that for $\alpha \in N \cap (\pi/2, \infty)$ the unique $\eta \in (-X, 0)$ with $1 = T_\alpha(\eta)$ must lie in $(\varepsilon, 0)$.

(ii) Let $-X < -X+\varepsilon < 0$. We have $0 < T_1(-X+\varepsilon) = \alpha T_\alpha(-X+\varepsilon)$. Hence there exists $\alpha_\varepsilon > \pi/2$ with $T_\alpha(-X+\varepsilon) < 1$ for $\alpha > \alpha_\varepsilon$. Monotonicity of T_α yields $y^\alpha(0) = T_\alpha^{-1}(1) \in (-X, -X+\varepsilon)$.

Proposition 4.4. *For odd C^1-functions $f: \mathbb{R} \to \mathbb{R}$ with (H1), (H2) and*

$$f(x) = f(X-x) \quad \text{for all} \quad x \in (0, X) \tag{H3}$$

there is a solution to Eq. ((α)) which connects $(0, -X)$ and $(X, 0)$ in the sense of Proposition 2.1 and satisfies $y(t) = x(t) - X$ for all t.

Proof. The solution x of $\dot{x} = \alpha f(x-X)$, $x(0) = X/2$, satisfies $x(t) \to 0$ as $t \to -\infty$, $x(t) \to X$ as $t \to +\infty$, $0 < x < X$ on $\mathbb{R}$. Together with $d(x-X)/dt = \dot{x} = \alpha f(x-X) = -\alpha f(X-x) = -\alpha f(x)$ we find that $(x, x-X)$ is a solution to Eq. ((α)).

Corollary 4.2. *$y^\alpha > x^\alpha - X$ in $[0, 1]$ for all $\alpha > \pi/2$.*

Proposition 4.5. *Let f be given as in Proposition 4.4.*

(i) *For $\alpha > \pi/2$ with $-X/2 < y^\alpha(0)$ we have $-X/2 < y^\alpha$ in $[0, 1]$.*

(ii) *For $\alpha > \pi/2$ and $y^\alpha(0) \leqq -X/2$ there exists a unique $a_\alpha \in [0, 1/2)$ with $y^\alpha(a_\alpha) = -X/2$. We have $x^\alpha(1-a_\alpha) = X/2$, and $0 < a_\alpha$ if $y^\alpha(0) < -X/2$.*

(iii) $\lim_{\alpha \to \infty} a_\alpha = 1/2$.

(iv) *For $\alpha > \pi/2$ with $y^\alpha(0) \leqq -3X/4$ there exists a unique b_α in $[0, 1/2)$ with $y^\alpha(b_\alpha) = -3X/4$. We have* $\lim_{\alpha \to \infty} b_\alpha = 1/2$.

Proof. Assertion (i), existence and uniqueness of $a_\alpha \in [0, 1/2)$ follow from $\dot{y}^\alpha > 0$ in $(0, 1]$ and from $y^\alpha(1/2) > x^\alpha(1/2) - X = -y^\alpha(1/2) - X$ (Proposition 2.2). Also, $x^\alpha(1-a_\alpha) = -y^\alpha(a_\alpha) = X/2$. $0 < a_\alpha$ for $y^\alpha(0) < -X/2$ is clear.

Proof of (iii). Proposition 4.3(ii) implies that there is $\alpha' > \pi/2$ such that for $\alpha \geqq \alpha'$, $-X < y^\alpha(0) \leqq -3X/4$. Orbits of solutions to Eq. ((α)) through a given point do not depend on α. In particular, $(x^\alpha, y^\alpha)([0, 1])$, $\alpha \geqq \alpha'$, is contained in the region in the fourth quadrant between the line $y = x - X$ and the orbit through $(0, -3X/4)$. The time which (x^α, y^α), $\alpha \geqq \alpha'$, spends in the compact set R bounded by $y = -X/2$, $x = X/2$ and by the orbit through $(0, -3X/4)$ is just $1 - 2a_\alpha$. Both components of the vectorfield $(x, y) \to (\alpha f(y), -\alpha f(x))$ tend to $+\infty$ uniformly on R as $\alpha \to +\infty$. Hence $1 - 2a_\alpha \to 0$ as $\alpha \to +\infty$. (iv) is proved in a similar way.

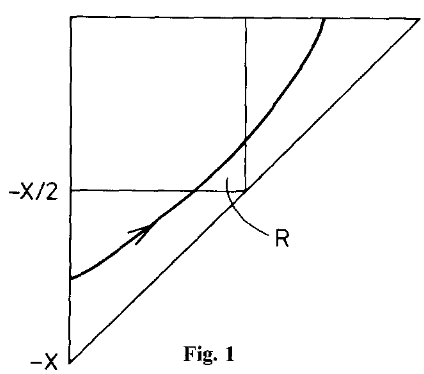

Fig. 1

The investigation of W_α and q_α requires information on the right hand side of Eq. (α, λ):

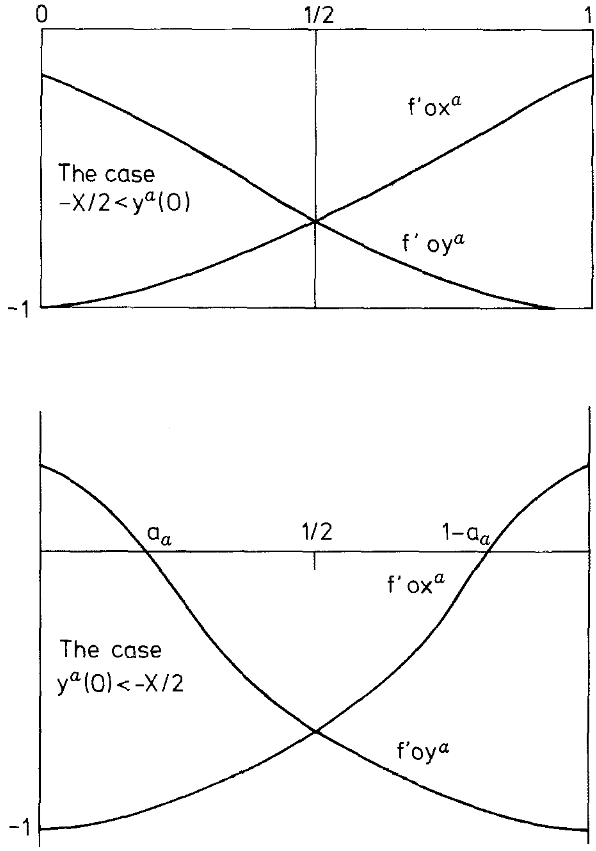

Fig. 2

Proposition 4.6. *Let* $f: \mathbb{R} \to \mathbb{R}$ *be an odd* C^1*-function with* (H1), (H2), (H3). *Assume in addition that*

$$f'<0 \text{ on } (0, X/2) \quad \text{and} \quad f \text{ is } C^2 \text{ on } (0, X/2) \text{ with } f \cdot f'/f'' \tag{H4}$$

negative and strictly decreasing on $(X/2, X)$.

It follows that for all $\alpha > \pi/2$ *with* $y^\alpha(0) < -X/2$, $f' \circ x^\alpha$ *has no zero on* $(1-a_\alpha, 1]$, *and* $-f' \circ y^\alpha / f' \circ x^\alpha$ *is strictly decreasing on* $(1-a_\alpha, 1)$.

Proof. On $(1-a_\alpha, 1]$, $-X/2 < x^\alpha < X$. Hence $0 < f' \circ x^\alpha$ by (H3), (H4). $a_\alpha < 1/2 < 1 - a_\alpha$ yields $0 < -y^\alpha < X/2 < x^\alpha < X$ on $(1-a_\alpha, 1)$ so that $f' < 0$ in $(0, X/2)$ and the symmetries of f imply $0 < f' \circ x^\alpha$, $f \circ y^\alpha$ and $f \circ x^\alpha$, $f' \circ y^\alpha < 0$ on $(1-a_\alpha, 1)$. Since $f \cdot f'/f'' < 0$ on $(X/2, X)$, $0 < f'' \circ x^\alpha$ on $(1-a_\alpha, 1)$. For $1-a_\alpha < t < 1$, $X + y^\alpha(t) \in (X/2, X)$. Since $X + y^\alpha > x^\alpha$ and $f \cdot f'/f''$ is decreasing on $(X/2, X)$ we obtain $(f \cdot f'/f'')(X + y^\alpha(t)) < (f \cdot f'/f'')(x^\alpha(t))$ for these t.

Note $0 < f(y^\alpha(t)) = -f(-y^\alpha(t)) = -f(X - (-y^\alpha(t))) = -f(X + y^\alpha(t))$, $0 > f'(y^\alpha(t)) = -f'(X + y^\alpha(t))$, $f''(y^\alpha(t)) = -f''(X + y^\alpha(t))$. With $X + y^\alpha(t) \in (X/2, X)$, this yields $0 > -f''(X + y^\alpha(t))$.

Altogether: $f''\circ y^\alpha\cdot f\circ x^\alpha\cdot f'\circ x^\alpha+f''\circ x^\alpha\cdot f\circ y^\alpha\cdot f'\circ y^\alpha<0$ in $(1-a_\alpha, 1)$. Therefore $d(-f'\circ y^\alpha/f'\circ x^\alpha)/dt=(f'\circ x^\alpha)^{-2}(-f''\circ y^\alpha\cdot\dot{y}^\alpha\cdot f'\circ x^\alpha+f''\circ x^\alpha\cdot\dot{x}^\alpha\cdot f'\circ y^\alpha)<0$ in $(1-a_\alpha, 1)$.

Corollary 4.3. *Let* $\alpha>\pi/2$.

(i) *In case* $-X/2\leqq y^\alpha(0)$, $f'\circ x^\alpha$ *is strictly increasing in* $[0, 1]$ *from* -1 *to* $f'(x^\alpha(1))\in(-1, 0]$, *and* $(f'\circ x^\alpha-f'\circ y^\alpha)(t)\neq 0$ *for some* $t\in(1/2, 1)$.

(ii) *In case* $y^\alpha(0)<-X/2$, $f'\circ x^\alpha$ *is strictly increasing in* $[0, 1]$ *from* -1 *to* $f'(x^\alpha(1))\in(0, 1)$, *and* $f'\circ x^\alpha(1-a_\alpha)=0$. *We have* $-f'\circ x^\alpha-f'\circ y^\alpha\geqq 0$ *in* $[0, a_\alpha]$.

Proof. (i) (H3) and $x^\alpha(1)=-y^\alpha(0)$, $\dot{x}^\alpha>0$ in $[0, 1)$, $f'<0$ in $(0, X/2)$, $f\cdot f'/f''<0$ in $(X/2, X)$ imply that $f'\circ x^\alpha$ is strictly increasing on $[0, 1]$. Also, $f'(x^\alpha(1))>f'(x^\alpha(0))=f'(-x^\alpha(0))=f'(y^\alpha(1))$.

(ii) The first assertion follows as in (i). $f'(x^\alpha(1-a_\alpha))=f'(X/2)=0$ is obvious from (H2), (H3). Proposition 2.2 and f' even give $(-f'\circ x^\alpha-f'\circ y^\alpha)(t)=(-f'\circ y^\alpha-f'\circ x^\alpha)(1-t)$ for $t\in[0, 1]$. $t\in[0, a_\alpha]$ implies $1-t\in[1-a_\alpha, 1]$. $-f'\circ y^\alpha/f'\circ x^\alpha$ decreases on $(1-a_\alpha, 1)$, and $-f'\circ y^\alpha(1)/f'\circ x^\alpha(1)=1/f'\circ x^\alpha(1)>1$. This implies the last claim.

Examples. $f=-\sin$ and the odd continuation of $0\leqq x\to -x(1-x)$ satisfy the hypotheses (H1)-(H4).

5. The Eigenvalue $\lambda=-1$ of W_α is Simple

Proposition 5.1. *We have* $z_1^{-1,\alpha}(0)=-1$, $u_2^{-1,\alpha}(0)=1$, $z_2^{-1,\alpha}(0)=0$ *for every* $\alpha>\pi/2$, *and* $\dot{x}_0^\alpha$ *is a multiple of* $u_2^{-1,\alpha}$.

Proof. By Proposition 3.2 and Lemma 3.1, $\dot{x}_0^\alpha=c_1u_1^{-1,\alpha}+c_2u_2^{-1,\alpha}$ with $Q_\alpha(-1)(c_1, c_2)^{tr}=0$. $\dot{x}_0^\alpha(1)=\dot{x}^\alpha(1)=0$, $u_2^{-1,\alpha}(1)=0$ and $u_1^{-1,\alpha}(1)=1$ imply $c_1=0$, $c_2\neq 0$ and $Q_\alpha(-1)(0, 1)^{tr}=0$. This yields $z_2^{-1,\alpha}(0)=0$, $u_2^{-1,\alpha}(0)=1$. Finally, $z_1^{-1,\alpha}(0)=-1$ follows since the Wronskian of the solutions $(u_i^{-1,\alpha}, z_i^{-1,\alpha})$, $i\in\{1, 2\}$, has constant value 1.

It follows that $Q_\alpha(-1)\neq 0$ is equivalent to $u_1^{-1,\alpha}(0)\neq 0$. By Lemma 3.1 we obtain

Corollary 5.1. $\dim\ker(W_\alpha+\mathrm{id})=1$ *is equivalent to* $u_1^{-1,\alpha}(0)\neq 0$.

For real $\lambda\neq 0$ we introduce polar coordinates and find $(u_i^{\lambda,\alpha}, z_i^{\lambda,\alpha})=r_i^{\lambda,\alpha}(\cos\theta_i^{\lambda,\alpha}, \sin\theta_i^{\lambda,\alpha})$ for the solutions $r=r_i^{\lambda,\alpha}$, $\theta=\theta_i^{\lambda,\alpha}$ of the equations

$$\dot{r}=\alpha(f'\circ y^\alpha+\lambda^{-1}f'\circ x^\alpha)r\cos\theta\sin\theta \qquad \text{(r)}$$

$$\dot{\theta}=\alpha\lambda^{-1}f'\circ x^\alpha\cos^2\theta-\alpha f'\circ y^\alpha\sin^2\theta \qquad (\theta)$$

with $r_i^{\lambda,\alpha}(1)=1$ for $i\in\{1, 2\}$, $\theta_1^{\lambda,\alpha}(1)=0$, $\theta_2^{\lambda,\alpha}(1)=\pi/2$. Clearly $r_i^{\lambda,\alpha}>0$ on $\mathbb{R}$, and $\dim\ker(W_\alpha+\mathrm{id})=1$ provided that $\theta_1^{-1,\alpha}(0)\notin\pi/2+\mathbb{Z}\pi$.

Proposition 5.2. *For all* $\alpha>\pi/2$ *we have*

(i) $\theta_2^{-1,\alpha}(0)=0$

(ii) $\theta_2^{-1,\alpha}=\pi/2-\theta_2^{-1,\alpha}(1-\cdot)$

(iii) $\theta_2^{-1,\alpha}(1/2)=\pi/4$

(iv) $\theta_2^{-1,\alpha}((0,1/2))\subset(0,\pi/4)$

(v) *In case* $X/2<x^\alpha(1)$, $0<\dot\theta_2^{-1,\alpha}$ *on* $[1/2, 1-a_\alpha)$ *and*

$$0<\theta_2^{-1,\alpha}(1-a_\alpha)<\pi/2.$$

Proof. (i) $z_2^{-1,\alpha}(0)=0$ yields $\theta_2^{-1,\alpha}(0)\in\mathbb{Z}\pi$. Continuity of the map $\pi/2<\alpha\to\theta_2^{-1,\alpha}(0)$ implies that $\theta_2^{-1,\alpha}(0)$ is constant. For $\alpha>\pi/2$ close to $\pi/2$, $x^\alpha(1)<X/2$. By Corollary 4.3(i), $0<\dot\theta_2^{-1,\alpha}$ in $[0,1]$. Also, $|f'|\leqq 1$ so that $|\dot\theta_2^{-1,\alpha}|\leqq\alpha$. Since $\theta_2^{-1,\alpha}(1)=\pi/2$ we find $0=\theta_2^{-1,\alpha}(0)$ for α close to $\pi/2$.

(ii) Assertion (i) and $f'\circ x^\alpha(1-\cdot)=f'\circ y^\alpha$, $f'\circ y^\alpha(1-\cdot)=f'\circ x^\alpha$ show that the function on the right hand side satisfies both Eq. (θ) on $[0,1]$ and the same initial condition as $\theta_2^{-1,\alpha}$ at $t=1$.

(iii) Follows from (i) and (ii).

(iv) Is an easy consequence of (i), (iii), Eq. (θ) and the assertions of Corollary 4.3 on $f'\circ x^\alpha$, $f'\circ y^\alpha$. The vectorfield $(t,\theta)\to(1, -\alpha f'\circ x^\alpha(t)\cos^2\theta -\alpha f'\circ y^\alpha(t)\sin^2\theta)$ has positive components for $0\leqq t\leqq 1/2$, $\theta=0$, and nonnegative components for $0\leqq t\leqq 1/2$, $\theta=\pi/4$, and a positive second component at $(1/2,\pi/4)$.

(v) Observe $0<-f'\circ x^\alpha$, $-f'\circ y^\alpha$ in $[1/2, 1-a_\alpha)$ from Corollary 4.3 and note that the vectorfield above has a positive second component for $\theta=\pi/2$, $1-a_\alpha\leqq t\leqq 1$.

Corollary 5.2. $\dim\ker(W_\alpha+\mathrm{id})=1$ *for every* $\alpha>\pi/2$.

Proof. We have to show $\theta_1^{-1,\alpha}(0)\notin\pi/2+\mathbb{Z}\pi$. Equation (θ) has period π with respect to the variable θ so that $\theta_2^{1,\alpha}-\pi$ is another solution and $\theta_2^{-1,\alpha}-\pi<\theta_1^{-1,\alpha}<\theta_2^{-1,\alpha}$. By Proposition 5.2(i) it remains to exclude $\theta_1^{-1,\alpha}(0)=-\pi/2$. Assume this equation holds true. As above we infer that $\theta_1^{-1,\alpha}=-\theta_1^{-1,\alpha}(1-\cdot)-\pi/2$. Hence $\theta_1^{-1,\alpha}(1/2)=-\pi/4$.

(i) The case $x^\alpha(1)\leqq X/2$. Corollary 4.3(i) implies that $\theta_2^{-1,\alpha}$ increases from $\pi/4$ to $\pi/2$ on $[1/2,1]$, and $\theta_1^{-1,\alpha}$ from $-\pi/4$ to 0. On the other hand, $\sin^2 v<\sin^2 w$ for $-\pi/4<v<0$, $\pi/4<w<\pi/2$. Rewrite Eq. (θ) as $\dot\theta=-\alpha f'\circ x^\alpha+\sin^2\theta\,[\alpha f'\circ x^\alpha-\alpha f'\circ y^\alpha]$. The last factor is strictly positive on $(1/2,1)$. It follows that $\dot\theta_1^{-1,\alpha}\leqq\dot\theta_2^{-1,\alpha}$ on $[1/2,1]$, and $\pi/4=\theta_2^{-1,\alpha}(1)-\theta_2^{-1,\alpha}(1/2)>\theta_1^{-1,\alpha}(1)-\theta_1^{-1,\alpha}(1/2)=\pi/4$, contradiction.

(ii) The case $X/2<x^\alpha(1)$. The second component of the vectorfield above is negative for $1-a_\alpha<t\leqq 1$, $\theta=0$. Hence $0<\theta_1^{-1,\alpha}(1-a_\alpha)$, by $\theta_1^{-1,\alpha}(1)=0$. With Proposition 5.2(v), $0<\theta_1^{-1,\alpha}(1-a_\alpha)<\theta_2^{-1,\alpha}(1-a_\alpha)<\pi/2$. Corollary 4.3(ii) and $f'\circ y^\alpha(1-\cdot)=f'\circ x^\alpha$, $f'\circ x^\alpha(1-\cdot)=f'\circ y^\alpha$ imply that both $\theta_i^{-1,\alpha}$ are increasing on $[1/2,1-a_\alpha]$. For some $t^*\in(1/2,1-a_\alpha)$, $\theta_1^{-1,\alpha}(t^*)=0$ and $\theta_1^{-1,\alpha}([1/2,t^*])\subset[-\pi/4,0]$, $\theta_2^{-1,\alpha}(t^*)<\pi/2$ and $\theta_2^{-1,\alpha}([1/2,t^*])\subset[\pi/4,\pi/2]$. Now we can proceed as in case (i): $\theta_2^{-1,\alpha}$ increases faster than $\theta_1^{-1,\alpha}$ in $[1/2,t^*]$, and we arrive at a contradiction to $\theta_1^{-1,\alpha}(t^*)-\theta_1^{-1,\alpha}(1/2)=\pi/4>\theta_2^{-1,\alpha}(t^*)-\theta_2^{-1,\alpha}(1/2)$.

Corollary 5.3. *For every* $\alpha>\pi/2$, $m_\alpha(-1)=1$.

Proof. It is enough to show $\ker(W_\alpha+\mathrm{id})=\ker(W_\alpha+\mathrm{id})^2$. By the preceding corollary this holds true provided there is no $\chi\in C_{\mathbb{C}}$ with $W_\alpha\chi+\chi=\dot x_0^\alpha$. Assume the contrary. Let S_α denote the matrix with columns $(u_i^{-1,\alpha}, z_i^{-1,\alpha})^{\mathrm{tr}}$, $i\in\{1,2\}$.

Proposition 5.1, $\dot{x}^\alpha(1)=0$ and Eq. (2) in Lemma 3.1 yield

$$0=\text{second component of } S_\alpha(0)\int_1^0 (S_\alpha(s))^{-1}\begin{pmatrix}0\\ -\alpha f'(x^\alpha(s))\, c_2\, u_2^{-1,\alpha}(s)\end{pmatrix} ds$$
$$=-\alpha c_2\int_1^0 f'(x^\alpha(s))(u_2^{-1,\alpha}(s))^2\, ds, \qquad (\#)$$

with $c_2 \neq 0$.

We have $r_2^{-1,\alpha}=r_2^{-1,\alpha}(1-\cdot)$: Both functions have the value 1 at $t=0$, see Proposition 5.1 and Proposition 5.2(i), and they satisfy Eq. (r) since

$$\begin{aligned} d(r_2^{-1,\alpha}(1-\cdot))/dt &= -\dot{r}_2^{-1,\alpha}(1-\cdot)\\ &= -\alpha(f'\circ y^\alpha(1-\cdot)-f'\circ x^\alpha(1-\cdot))\, r_2^{-1,\alpha}(1-\cdot)\cos\theta_2^{-1,\alpha}(1-\cdot)\sin\theta_2^{-1,\alpha}(1-\cdot)\\ &= \alpha(f'\circ y^\alpha - f'\circ x^\alpha)\, r_2^{-1,\alpha}(1-\cdot)\cos(\pi/2-\theta_2^{-1,\alpha})\sin(\pi/2-\theta_2^{-1,\alpha})\\ &= \alpha(f'\circ y^\alpha - f'\circ x^\alpha)\, r_2^{-1,\alpha}(1-\cdot)\sin\theta_2^{-1,\alpha}\cos\theta_2^{-1,\alpha}. \end{aligned}$$

We conclude

$$\begin{aligned} &\int_{1/2}^{1} f'(x^\alpha(s))(r_2^{-1,\alpha}(s))^2(\cos\theta_2^{-1,\alpha}(s))^2\, ds\\ &\quad = -\int_{1/2}^{0} f'(x^\alpha(1-s))(r_2^{-1,\alpha}(1-s))^2(\cos\theta_2^{-1,\alpha}(1-s))^2\, ds\\ &\quad = \int_0^{1/2} f'(y^\alpha(s))(r_2^{-1,\alpha}(s))^2(\cos(\pi/2-\theta_2^{-1,\alpha}(s)))^2\, ds. \qquad (\textstyle\int) \end{aligned}$$

(i) In case $x^\alpha(1)\leqq X/2$ Proposition 5.2(iv) and $f'\circ x^\alpha < f'\circ y^\alpha$ on $[0, 1/2)$ imply that the last integrand is negative on $(0, 1/2)$ and strictly greater than $f'\circ x^\alpha (r_2^{-1,\alpha})^2(\cos\theta_2^{-1,\alpha})^2$. Therefore the absolute value of the first integral in $(\int)$ is strictly smaller than the absolute value of $\int_0^{1/2} f'(x^\alpha(s))(r_2^{-1,\alpha}(s))^2(\cos\theta_2^{-1,\alpha}(s))^2\, ds$, contradiction to $(\#)$.

(ii) The case $X/2 < x^\alpha(1)$. Proposition 5.2(iv) and $f'\circ x^\alpha<0$ in $[0, 1/2]$ imply the estimate

$$\begin{aligned} I :&= \left|\int_0^{1/2} f'(x^\alpha(s))(r_2^{-1,\alpha}(s))^2(\cos\theta_2^{-1,\alpha}(s))^2\, ds\right|\\ &= -\int_0^{1/2}\ldots > -\int_0^{1/2} f'(x^\alpha(s))(r_2^{-1,\alpha}(s))^2(\cos(\pi/2-\theta_2^{-1,\alpha}(s)))^2\, ds. \end{aligned}$$

From Corollary 4.3(ii), applied to $-\int_0^{a_\alpha}\ldots$, and from $f'\circ x^\alpha < f'\circ y^\alpha < 0$ on $(a_\alpha, 1/2)$ we infer that the last integral is not smaller than

$$\begin{aligned} &\int_0^{a_\alpha} f'(y^\alpha(s))\ldots ds - \int_{a_\alpha}^{1/2} f'(y^\alpha(s))\ldots ds > \left|\int_0^{1/2} f'(y^\alpha(s))\ldots ds\right|\\ &= \left|\int_{1/2}^{1} f'(x^\alpha(s))(r_2^{-1,\alpha}(s))^2(\cos\theta_2^{-1,\alpha}(s))^2\, ds\right| =: J. \end{aligned}$$

Here we have used $0<f'\circ y^\alpha$ on $(0, a_\alpha)$, $f'\circ y^\alpha<0$ on $(a_\alpha, 1/2)$, and ($\int$). The estimate $I>J$ contradicts (#).

Corollary 5.4. *For every* $\alpha>\pi/2$, $(\mathrm{d}q_\alpha/\mathrm{d}\lambda)(-1)\neq 0$.

Proof. $Q_\alpha(-1)\neq 0$ gives $j_\alpha(-1)=m_\alpha(-1)$, see Corollary 3.3. Apply Corollary 5.3.

6. Bifurcation

Proposition 6.1. $\lim\limits_{\alpha \to \pi/2} q_\alpha(1)<0$.

Proof. For $\alpha \to \pi/2$ the coefficients $f'\circ y^\alpha$ and $f'\circ x^\alpha$ in Eq. (α, 1) tend to -1 uniformly on $[0, 1]$, see Proposition 4.3. It follows that the matrix with columns $(u_i^{1,\alpha}(0), z_i^{1,\alpha}(0))^{tr}$, $i\in\{1, 2\}$, tends to

$$\exp\left((0-1)\begin{pmatrix} 0 & -\pi/2 \\ -\pi/2 & 0\end{pmatrix}\right).$$

Hence $q_\alpha(1)=-u_2^{1,\alpha}(0)-z_1^{1,\alpha}(0)\to-(e^{\pi/2}-e^{-\pi/2})$.

Proposition 6.2. (i) *There exists* $\tilde{\alpha}>\pi/2$ *with* $\theta_1^{1,\tilde{\alpha}}(1-a_{\tilde{\alpha}})=-\pi/2$. (ii) *We have* $q_{\tilde{\alpha}}(1)>0$ *provided that* $\theta_2^{1,\tilde{\alpha}}(1-a_{\tilde{\alpha}})\geqq 0$.

Proof. (i) By Proposition 4.3 there exists $\bar{\alpha}>\pi/2$ with $y^{\bar{\alpha}}(0)=-X/2$ and $y^\alpha(0)<-X/2$ for $\alpha>\bar{\alpha}$, or $a_{\bar{\alpha}}=0$ and $a_\alpha\in(0, 1/2)$ for $\alpha>\bar{\alpha}$. Clearly $\theta_1^{1,\bar{\alpha}}(1-a_{\bar{\alpha}})=\theta_1^{1,\bar{\alpha}}(1)=0$.

We have $\theta_1^{1,\alpha}(1-a_\alpha)\to-\infty$ as $\alpha\to+\infty$. Proof: By Proposition 4.5(iv) there exists $\alpha^*>\pi/2$ such that $b_\alpha>1/4$ for $\alpha>\alpha^*$. Hence $-X/4\leqq y^\alpha$ and $3X/4\leqq x^\alpha$ on $[3/4, 1]$, $f'\circ x^\alpha\geqq f'(3X/4)>0$ and $-f'\circ y^\alpha\geqq -f'(X/4)=f'(3X/4)$ and $\dot{\theta}=\alpha(f'\circ x^\alpha\cos^2\theta-f'\circ y^\alpha\sin^2\theta)\geqq\alpha f'(3X/4)>0$ for $\theta=\theta_1^{1,\alpha}$ on $[3/4, 1]$, and $\theta_1^{1,\alpha}(3X/4)\to-\infty$ as $\alpha\to+\infty$. By Proposition 4.5(iii), $1-a_\alpha<3/4$ for $\alpha>\hat{\alpha}\geqq\alpha^*$, with $\hat{\alpha}$ sufficiently large. From Eq. (θ) with $\lambda=1$ and from Corollary 4.3(ii) together with $f'\circ x^\alpha=f'\circ y^\alpha(1-\cdot)$ we obtain $\dot{\theta}_1^{1,\alpha}\geqq 0$ on $[1-a_\alpha, 3/4]$, $\alpha>\hat{\alpha}$. It follows that $\theta_1^{1,\alpha}(1-a_\alpha)\to-\infty$ for $\alpha\to+\infty$. Finally, the intermediate value theorem implies assertion (i).

(ii) We have $q_\alpha(1)=-u_2^{1,\alpha}(0)-z_1^{1,\alpha}(0)=-r_2^{1,\alpha}(0)\cos\theta_2^{1,\alpha}(0)-r_1^{1,\alpha}(0)\sin\theta_1^{1,\alpha}(0)$. Therefore it is enough to show $\theta_2^{1,\alpha}(0)\in(\pi/2, \pi)$ and $\theta_1^{1,\alpha}(0)\in(-\pi, 0)$ for $\alpha=\tilde{\alpha}$. Corollary 4.3(ii) says that the right hand side in Eq. (θ) is positive for $1-a_{\tilde{\alpha}}<t\leqq 1$ and $\theta\in\mathbb{R}$, hence $0\leqq\theta_2^{1,\tilde{\alpha}}(1-a_{\tilde{\alpha}})<\pi/2$. It is also positive for $a_{\tilde{\alpha}}<t\leqq 1-a_{\tilde{\alpha}}$ and $\theta\in\pi/2+\mathbb{Z}\pi$, and negative for $a_{\tilde{\alpha}}\leqq t<1-a_{\tilde{\alpha}}$ and $\theta\in\mathbb{Z}\pi$. With $\theta_1^{1,\tilde{\alpha}}(1-a_{\tilde{\alpha}})=-\pi/2$, we conclude $\theta_1^{1,\tilde{\alpha}}(a_{\tilde{\alpha}})\in(-\pi, -\pi/2)$ and $\theta_2^{1,\tilde{\alpha}}(a_{\tilde{\alpha}})\in(0, \pi/2)$. – Note that for a solution θ to Eq. (θ) the functions $\pi/2+j\pi+\theta(1-\cdot)$, $j\in\mathbb{Z}$, are solutions to Eq. (θ), too. Therefore $\theta_2^{1,\tilde{\alpha}}(1-a_{\tilde{\alpha}})\in[0, \pi/2]$ implies that the solution with $\theta(0)=\pi$, that is $\pi/2+\theta_2^{1,\tilde{\alpha}}(1-\cdot)$, satisfies $\pi/2\leqq\theta(a_{\tilde{\alpha}})\leqq\pi$, and $\theta_1^{1,\tilde{\alpha}}(1-a_{\tilde{\alpha}})=-\pi/2$ implies that the solution with $\theta(0)=\pi/2$ satisfies $\theta(a_{\tilde{\alpha}})=0$. It follows from $\theta_2^{1,\tilde{\alpha}}(a_{\tilde{\alpha}})\in(0, \pi/2)$ that $\theta_2^{1,\tilde{\alpha}}(0)$ is in $(\pi/2, \pi)$. Similarly we find $\theta_1^{1,\tilde{\alpha}}(0)\in(-\pi/2, 0)$.

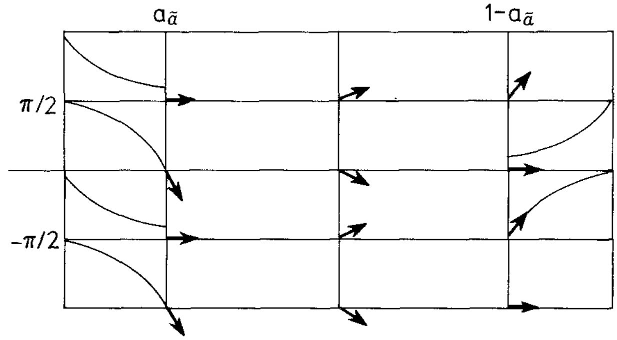

Fig. 3

The proof of the next result uses an idea which I owe to H. Steinlein [23].

Lemma 6.1. *Let continuous functions* $a, b\colon [c, 1] \to \mathbb{R}$ *be given with* $a > 0$, $a \geqq b$, $b(c) = 0$, $b > 0$ *on* $(c, 1]$, a *and* b *increasing,* a/b *decreasing on* $(c, 1]$. *If the solution* θ_1 *of equation*

$$\dot{\theta} = a \sin^2 \theta + b \cos^2 \theta \qquad ((\theta))$$

with $\theta_1(1) = 0$ *satisfies* $\theta_1(c) = -\pi/2$ *then* $\theta_2(c) \geqq 0$ *for the solution* θ_2 *with* $\theta_2(1) = \pi/2$.

Proof. (i) We have

$$\dot{\theta}_i > 0 \text{ on } (c, 1] \text{ for } i \in \{1, 2\} \quad \text{and} \quad \dot{\theta}_1 > 0 \text{ on } [c, 1]. \qquad (3)$$

It follows that there is a unique $\tau \in (c, 1)$ with $-\theta_1(\tau) = \theta_2(\tau)$ since $(\theta_1 + \theta_2)(1) = \pi/2$, $(\theta_1 + \theta_2)(c) < -\pi/2 + \pi/2 = 0$. Set $A := a(\tau)$, $B := b(\tau)$. Note $A \geqq B > 0$. There exist $T > c$ and solutions y_i, $i \in \{1, 2\}$, of equation

$$\dot{y} = A \sin^2 y + B \cos^2 y \qquad (\mathrm{y})$$

on $[c, T]$ with $y_1(c) = -\pi/2$, $y_1(T) = 0$, $y_2(c) = 0$, $y_2(T) = \pi/2$. This is easily seen from symmetries in the equivalent equation $\dot{y} = A + (B - A) \cos^2 y$ with $-A < B - A < A$, $B - A \leqq 0$. Both y_i are strictly increasing. θ_1 and y_1 map $[c, 1]$ and $[c, T]$ respectively one-to-one onto $[-\pi/2, 0]$, and $\theta_2 \circ \theta_1^{-1}$, $y_2 \circ y_1^{-1}$ are defined. We have

$$\begin{aligned}(\theta_2 \circ \theta_1^{-1})' &= \theta_2' \circ \theta_1^{-1} / \theta_1' \circ \theta_1^{-1} \\ &= \frac{(a \circ \theta_1^{-1}) \sin^2 \theta_2 \circ \theta_1^{-1} + (b \circ \theta_1^{-1}) \cos^2 \theta_2 \circ \theta_1^{-1}}{(a \circ \theta_1^{-1}) \sin^2 + (b \circ \theta_1^{-1}) \cos^2} \quad \text{on } [-\pi/2, 0]. \end{aligned} \qquad (4)$$

(ii) We have

$$(\theta_2 \circ \theta_1^{-1})' \leqq \frac{A \sin^2 \theta_2 \circ \theta_1^{-1} + B \cos^2 \theta_2 \circ \theta_1^{-1}}{A \sin^2 + B \cos^2}$$

on $[-\pi/2, 0]$. Proof: Case I: $\tau \leqq \theta_1^{-1}(s)$ with $s \in [-\pi/2, 0]$. The definition of τ and $-\theta_1 < \theta_2$ on $(\tau, 1]$ give

$$0 \leqq -s = -\theta_1(\theta_1^{-1}(s)) \leqq \theta_2(\theta_1^{-1}(s)) \leqq \pi/2. \qquad (5)$$

Also, $\tilde{a}:=a(\theta_1^{-1}(s))\geqq a(\tau)=A$ and $\tilde{b}:=b(\theta_1^{-1}(s))\geqq b(\tau)=B$. From (5), $R:=\sin^2 s\leqq\sin^2\theta_2\circ\theta_1^{-1}(s)=:\rho$ and $\sigma:=\cos^2\theta_2\circ\theta_1^{-1}(s)\leqq\cos^2 s=:S$. We have to show $(AR+BS)\ (\tilde{a}\rho+\tilde{b}\sigma)\leqq(A\rho+B\sigma)\ (\tilde{a}R+\tilde{b}S)$. The assumption that a/b decreases on $(c, 1]$ implies $B\tilde{a}\leqq A\tilde{b}$. With $\sigma\leqq S$, $A\tilde{b}(\sigma-S)\leqq B\tilde{a}(\sigma-S)$. Hence $A\tilde{b}\sigma+BS\tilde{a}\leqq A\tilde{b}S+B\sigma\tilde{a}$, $A(1-S)\tilde{b}\sigma+BS\tilde{a}(1-\sigma)\leqq A(1-\sigma)\tilde{b}S+B\sigma\tilde{a}(1-S)$. Now use $R+S=1$, $\rho+\sigma=1$.

Case II: $c\leqq\theta_1^{-1}(s)\leqq\tau<1$. Then $\theta_2(\theta_1^{-1}(s))\leqq-\theta_1(\theta_1^{-1}(s))=-s$, as $\dot{\theta}_1+\dot{\theta}_2>0$ and $\theta_1(\tau)+\theta_2(\tau)=0$. In case $0\leqq\theta_2(\theta_1^{-1}(s))$ this means

$$0\leqq|\theta_2(\theta_1^{-1}(s))|\leqq-s. \tag{6}$$

If $\theta_2(\theta_1^{-1}(s))<0$ then $-\pi/2\leqq s=\theta_1(\theta_1^{-1}(s))\leqq\theta_2(\theta_1^{-1}(s))<0$ as $\theta_1(1)<\theta_2(1)$, and we have (6) again. In the present case, $\tilde{a}:=a(\theta_1^{-1}(s))\leqq a(\tau)=A$, $\tilde{b}:=b(\theta_1^{-1}(s))\leqq b(\tau)=B$. Define R, S, ρ, and σ as in case I. Then $R+S=1$, $\rho+\sigma=1$, $\tilde{a}\geqq\tilde{b}$, $A\geqq B$. (6) yields $\rho\leqq R$, $S\leqq\sigma$. Also, $\tilde{b}=0$ if $\theta_1^{-1}(s)=c$, and $\tilde{a}/\tilde{b}\geqq A/B$ otherwise. Hence $A\tilde{b}\leqq B\tilde{a}$, $A\tilde{b}(\sigma-S)\leqq B\tilde{a}(\sigma-S)$. This is equivalent to $(\tilde{a}\rho+\tilde{b}\sigma)\ (AR+BS)\leqq(\tilde{a}R+\tilde{b}S)\ (A\rho+B\sigma)$, and the assertion follows.

(iii) $y_2\circ y_1^{-1}\leqq\theta_2\circ\theta_1^{-1}$ in $[-\pi/2, 0]$. Proof: Assume

$$\theta_2\circ\theta_1^{-1}(s)<y_2\circ y_1^{-1}(s)\quad\text{for some } s\in(-\pi/2, 0). \tag{7}$$

Since $y_2\circ y_1^{-1}(0)=\pi/2=\theta_2\circ\theta_1^{-1}(0)$ there exists $s'>s$ with

$$\theta_2\circ\theta_1^{-1}<y_2\circ y_1^{-1}\quad\text{in } [s, s') \tag{8}$$

and

$$\theta_2\circ\theta_1^{-1}(s')=y_2\circ y_1^{-1}(s')>0. \tag{9}$$

It follows that there exists $s''\in(s, s')$ with

$$0<\theta_2\circ\theta_1^{-1}<y_2\circ y_1^{-1}\leqq\pi/2\quad\text{in } [s'', s'). \tag{8'}$$

As in the proof of (4) we find

$$(y_2\circ y_1^{-1})'=\frac{A\sin^2 y_2\circ y_1^{-1}+B\cos^2 y_2\circ y_1^{-1}}{A\sin^2+B\cos^2} \tag{10}$$

$A\geqq B$, $\cos^2=1-\sin^2$, (ii) and (8'), and (10) show $(\theta_2\circ\theta_1^{-1})'\leqq(y_2\circ y_1^{-1})'$ in $[s'', s')$. This contradicts (8') and (9).

(iv) From (iii), $\theta_2(c)=\theta_2(\theta_1^{-1}(-\pi/2))\geqq y_2(y_1^{-1}(-\pi/2))=y_2(c)=0$.

Theorem 6.1. *For every odd* C^1*-function* $f:\mathbb{R}\to\mathbb{R}$ *with properties* (H1)–(H4) *there exists* $\tilde{\alpha}>\pi/2$ *such that*

$$\operatorname{ind}(x_0^{\tilde{\alpha}}, P(\tilde{\alpha}, \cdot), C^*)\neq\lim_{\alpha\underset{>}{\to}\pi/2}\operatorname{ind}(x_0^{\alpha}, P(\alpha, \cdot), C^*).$$

Proof. (i) We have $q_{\tilde{\alpha}}(1)>0$. Proof: By Proposition 4.6 and by Corollary 4.3(ii) the functions $a:=-\tilde{\alpha}f'\circ y^{\tilde{\alpha}}$ and $b:=\tilde{\alpha}f'\circ x^{\tilde{\alpha}}$ on $[c, 1]:=[1-a_{\tilde{\alpha}}, 1]$ satisfy the hypotheses of Lemma 6.1. By Proposition 6.2, $\theta_1^{1,\tilde{\alpha}}(1-a_{\tilde{\alpha}})=-\pi/2$. Recall the initial conditions $\theta_1^{1,\tilde{\alpha}}(1)=0$, $\theta_2^{1,\tilde{\alpha}}(1)=\pi/2$. Lemma 6.1 gives $\theta_2^{1,\tilde{\alpha}}(1-a_{\tilde{\alpha}})\geqq 0$. By Proposition 6.2(ii), $q_{\tilde{\alpha}}(1)>0$.

(ii) Proposition 3.5 and Corollary 5.4 imply that $(-1)^{\sum_{\lambda<-1} j_\alpha(\lambda)}$ is independent of $\alpha>\pi/2$. Consider $(-1)^{\sum_{\lambda<1} j_\alpha(\lambda)}$. Proposition 3.5 says that for all $\alpha>\pi/2$, $q_\alpha(\lambda)<0$ for λ sufficiently large ($\lambda \geqq \lambda_\alpha$). Therefore (i) and Proposition 6.1 yield

$$(-1)^{\sum_{1<\lambda} j_{\tilde{\alpha}}(\lambda)} \neq \lim_{\alpha \underset{>}{\to} \pi/2} (-1)^{\sum_{1<\lambda} j_\alpha(\lambda)}.$$

Since $\lim_{\alpha \underset{>}{\to} \pi/2} q_\alpha(1) \neq 0 \neq q_{\tilde{\alpha}}(1)$ and $m_\alpha(-1)=1$ for all $\alpha>\pi/2$, we may now use Proposition 4.2 (iii).

In order to produce a change of the index at an isolated parameter value we need more assumptions on f. A strong one is analyticity:

Theorem 6.2. *Let $f: \mathbb{R} \to \mathbb{R}$ be an analytic odd function which satisfies* (H1) – (H4). *Then there exists $\beta>\pi/2$ such that every neighborhood of (β, x_0^β) contains fixed points $(\alpha, \phi) \in (\mathbb{R}^+ \times C^*) \setminus \{(\alpha, x_0^\alpha) \mid \alpha>\pi/2\}$ of P. These fixed points define periodic solutions of equation*

$$\dot{x}(t) = \alpha f(x(t-1)) \tag{α}$$

with $x(\mathbb{R}) \subset (-X, X)$, $x(0)=0$, $\dot{x}>0$ on $[0, 1)$, $\dot{x}<0$ on $(1, z+1)$ for some $z>1$, $0<\dot{x}$ on $(z+1, p+1)$ for the minimal period p. The periodic solutions $x^{\phi,\alpha}$ do not satisfy the symmetry condition

$$x(t) = -x(t-2) \quad \textit{for all} \quad t \in \mathbb{R}. \tag{S}$$

Proof. (i) $q_{\tilde{\alpha}}(1)>0$ (part (i) of the last proof) and $\lim_{\alpha \underset{>}{\to} \pi/2} q_\alpha(1)<0$ (Proposition 6.1) imply $\pi/2 < \inf\{\alpha>\pi/2 \mid q_\alpha(1)>0\} =: \beta$. Clearly, $q_\alpha(1) \leqq 0$ for $\pi/2<\alpha<\beta$ and $0<q_{\alpha_n}(1)$ for a sequence $\alpha_n \underset{>}{\to} \beta$. We would like to find $\varepsilon>0$ such that $q_\alpha(1)<0$ on $(\beta-\varepsilon, \beta)$, $q_\alpha(1)>0$ on $(\beta, \beta+\varepsilon)$. Unfortunately we do not know whether the map $\pi/2<\alpha \to q_\alpha(1)$ is analytic. It is defined by means of the solution η to $1 = T_\alpha(\eta) = T_1(\eta)/\alpha$, hence by T_1^{-1}. We shall see that T_1 is analytic but it remains unclear whether its inverse is analytic, too. A proof would require $T_1' \neq 0$ everywhere on $(-X, 0)$. –

We shall employ the composition $(-X, 0) \ni \eta \to q_{T_1(\eta)}(1)$ instead of $\pi/2<\alpha \to q_\alpha(1)$. Recall that T_1 maps its domain continuously and strictly decreasing onto $(\pi/2, \infty)$.

(ii) T_1 is analytic. Proof: T_1 is continuous and satisfies $0=y(T_1(\eta), \eta)$ for all $\eta \in (-X, 0)$, with the map $(t, \eta) \to (x(t, \eta), y(t, \eta))$ given by the solutions of the initial value problems $\dot{x}=f(y)$, $\dot{y}=-f(x)$, $x(0, \eta)=0$, $y(0, \eta)=\eta$. The latter map is analytic on $\mathbb{R} \times (-X, 0)$, and $(\partial y/\partial t)\,(T_1(\eta), \eta) = -f(x(T_1(\eta), \eta))>0$ for all $\eta \in (-X, 0)$. The implicit function theorems of chapter X.2 in [3] now imply the assertion.

(iii) The map $(-X, 0) \ni \eta \to q_{T_1(\eta)}(1)$ is analytic. Proof: The map $(t, \eta, \alpha) \to (x(t, \eta, \alpha), y(t, \eta, \alpha))$ defined by the solution to Eq. ((α)) and $x(0, \eta, \alpha) = 0$, $y(0, \eta, \alpha)=\eta$ is analytic on $\mathbb{R} \times (-X, 0) \times (\pi/2, \infty)$. Therefore the coefficients of the system $\dot{u}=\alpha f'(y(t, \eta, \alpha))\, z$, $\dot{z}=\alpha f'(x(t, \eta, \alpha))\, u$ are analytic on $\mathbb{R} \times (-X, 0) \times (\pi/2, \infty)$. By (ii) the coefficients of the system

$$\dot{u} = T_1(\eta)\, f'(y(t, \eta, T_1(\eta)))\, z, \qquad \dot{z} = T_1(\eta)\, f'(x(t, \eta, T_1(\eta)))\, u \tag{§}$$

are analytic on $\mathbb{R}\times(-X,0)$. By Proposition 4.1, $\alpha=T_1(\eta)$ implies $T_\alpha(\eta)=1$. Hence $x(\cdot,\eta,T_1(\eta))=x^{T_1(\eta)}$, $y(\cdot,\eta,T_1(\eta))=y^{T_1(\eta)}$. It follows that the parameterized flow of Eq. (§) is analytic on $\mathbb{R}\times\mathbb{R}^2\times(-X,0)$, and that the map $(-X,0)\ni\eta\to -u_2^{1,T_1(\eta)}(0)-z_1^{1,T_1(\eta)}(0)=q_{T_1(\eta)}(1)$ is analytic, too.

(iv) Set $\eta_\beta:=T_1^{-1}(\beta)$. We have $q_{T_1(\eta)}(1)\leqq 0$ for $\eta_\beta<\eta<0$ and $q_{T_1(\eta_n)}(1)>0$ for a sequence $\eta_n=T_1^{-1}(\alpha_n)\underset{<}{\to}\eta_\beta$ since T_1^{-1} is continuous. By analyticity there exists $\delta>0$ such that $q_{T_1(\eta)}(1)<0$ in $(\eta_\beta,\eta_\beta+\delta)$, $q_{T_1(\eta)}(1)>0$ in $(\eta_\beta-\delta,\eta_\beta)$. This implies $q_\alpha(1)<0$ in $(\beta-\varepsilon,\beta)$ and $q_\alpha(1)>0$ in $(\beta,\beta+\varepsilon)$ for some $\varepsilon>0$.

(v) As in the proof of Theorem 6.1 we obtain that the index of the fixed point x_0^α is, say, $(-1)^k$ on $(\beta-\varepsilon,\beta)$ and $(-1)^{k+1}$ on $(\beta,\beta+\varepsilon)$ with an integer k. The homotopy property of the degree implies that every neighborhood of (β,x_0^β) contains fixed points $(\alpha,\phi)\in(\mathbb{R}^+\times C^*)\setminus\{(\alpha,x_0^\alpha)\,|\,\alpha>\pi/2\}$. For such a fixed point, set $x:=x^{\phi,\alpha}$ and $z_i:=z_i^{\phi,\alpha}$, $i\in\{1,2\}$, in Proposition 4.2. We obtain $\dot{x}>0$ on $[z_2,z_2+1)$. Hence $\phi=\dot{x}(z_2+\cdot)>0$ on $[0,1)$. x maps $[0,z_2+1]$ into $(-X,X)$. Now $xf(x)<0$ on $(0,X)$ gives $\dot{x}<0$ on $(1,z_1+1)$, and x has all the monotonicity properties as claimed, compare Proposition 4.2. x does not satisfy (S) since this would imply period 4 and $x=x^\alpha$, by Corollary 4.1, or $\phi=x_0^\alpha$, a contradiction.

Example. Theorem 6.2 applies to $f=-\sin$.

Remark. The assertion of Theorem 6.2 remains true if the analyticity assumption is replaced by the weaker hypothesis that the restriction of f to $[0,X)$ admits an analytic continuation to some open interval.

Example. The odd continuation of $0\leqq x\to -x(1-x)$.

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